HYD 142 HW #2

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10.

$$Z = X + Y$$

$$Var(X+Y) = Var(Z) = \mathbb{E}[(Z-\mu_Z)^2] = \mathbb{E}[Z^2 - 2Z\mu_Z + \mu_Z^2] = \mathbb{E}[Z^2] - 2\mu_Z\mathbb{E}[Z] + \mu_Z^2$$

Since the expectation value operator is linear. Substituting $\mu_Z = \mu_X + \mu_Y$ and $Z = X + Y$ yields

$$\mathbb{E}[X^2 + 2XY + Y^2] - 2(\mu_X + \mu_Y)\mathbb{E}[X+Y] + (\mu_X + \mu_Y)^2$$

$$= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - 2\mu_X^2 - 2\mu_Y^2 - 4\mu_X\mu_Y + \mu_X^2 + \mu_Y^2 + 2\mu_X\mu_Y$$

$$= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mu_X^2 + \mu_Y^2 + 2\mu_X\mu_Y)$$

$$= (\mathbb{E}[X^2] - \mu_X^2) + (\mathbb{E}[Y^2] - \mu_Y^2) + 2(\mathbb{E}[XY] - \mu_X\mu_Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

Since X and Y are independent, $Cov(X,Y) = 0$, therefore $\boxed{Var(X+Y) = Var(X) + Var(Y)}$.

11.

a. Take R = it rains, F = rain forecasted, U = takes umbrella. We are given that:

$$P(R) = P(R^c) = \frac{1}{2}$$

$$P(R|F) = P(R^c|F^c) = \frac{2}{3}, \quad \text{thus } P(R^c|F) = P(R|F^c) = \frac{1}{3}$$

$$P(U|F) = 1, \quad P(U|F^c) = \frac{1}{3}; \quad \text{thus } P(U^c|F) = 0, \quad P(U^c|F^c) = \frac{2}{3}$$

Find $P(U^c|R)$.

$$P(R) = P(R|F)P(F) + P(R|F^c)(1 - P(F)) = \frac{2}{3}P(F) + \frac{1}{3}(1 - P(F)) = \frac{1}{2}$$

$$\frac{1}{3}P(F) = \frac{1}{6} \quad \rightarrow \quad P(F) = \frac{1}{2}, \quad P(F^c) = \frac{1}{2}$$

$$P(U^c|R)P(R) = P(U^c \cap R) = P(U^c \cap R \cap F) + P(U^c \cap R \cap F^c)$$

$$P(U^c|R)P(R) = P(U^c|R \cap F)P(R|F)P(F) + P(U^c|R \cap F^c)P(R|F^c)P(F^c)$$

Since $P(R) = P(F) = P(F^c) = \frac{1}{2}$, we can divide by $\frac{1}{2}$ on both sides, yielding:

$$P(U^c|R) = P(U^c|R \cap F)P(R|F) + P(U^c|R \cap F^c)P(R|F^c)$$

$$P(U^c|R) = 0 \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{3} = \boxed{\frac{2}{9}}$$

b. Find $P(U|R^c)$.

Similarly,

$$P(U|R^c)P(R^c) = P(U|R^c \cap F)P(R^c|F)P(F) + P(U|R^c \cap F^c)P(R^c|F^c)P(F^c)$$

Dividing both sides by $P(R^c) = P(F) = P(F^c) = \frac{1}{2}$:

$$P(U|R^c) = P(U|R^c \cap F)P(R^c|F) + P(U|R^c \cap F^c)P(R^c|F^c)$$

$$P(U|R^c) = 1 \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \boxed{\frac{5}{9}}$$

12. Exercise 2.10 parts (a) - (d)

a. Take M_1 = insufficient water supply, D_i = water demand level. Then:

$$P(M_1) = \sum_{i=1}^{3} P(M_1|D_i)P(D_i) = 0 \cdot 0.6 + 0.1 \cdot 0.3 + 0.5 \cdot 0.1 = \boxed{0.21}$$

b. For $D_2 = 150,000$ gd, we want $P(D_2|M_1)$, the probability that D_2 "caused" insufficient water supply.

$$P(D_2|M_1) = \frac{P(M_1|D_2)P(D_2)}{P(M_1)} = \frac{0.1 \cdot 0.3}{0.21} = \boxed{0.143}$$

c. Take M_2 = pump failure occurs. Then

$$P(M_2) = \sum_{i=1}^{3} P(M_2|D_i)P(D_i) = \sum_{i=1}^{3} P(M_2)P(D_i) = P(M_2)\sum_{i=1}^{3} P(D_i) = P(M_2) \cdot 1 = \boxed{0.02}$$

d. Take M_3 = overload of purification plant, F = failure occured. For each M_i, we want $P(M_i|F \cap D_2)$. Applying Bayes' Theorem:

$$P(M_i|F \cap D_2) = \frac{P(F|M_i \cap D_2)P(M_i|D_2)}{P(F|D_2)} = \frac{1 \cdot P(M_i|D_2)}{P(F|D_2)}$$

$$P(F|D_2) = P(F|M_1 \cap D_2)P(M_1|D_2) + P(F|M_2 \cap D_2)P(M_2|D_2) + P(F|M_3 \cap D_2)P(M_3|D_2)$$

$$P(F|D_2) = P(M_1|D_2) + P(M_2|D_2) + P(M_3|D_2) = 0.1 + 0.02 + 0 = 0.12$$

$$P(M_1|F \cap D_2) = \frac{P(F|M_1 \cap D_2)P(M_1|D_2)}{P(F|D_2)} = \frac{0.1}{0.12} = \boxed{\frac{5}{6}}$$

$$P(M_2|F \cap D_2) = \frac{P(F|M_2 \cap D_2)P(M_2|D_2)}{P(F|D_2)} = \frac{0.02}{0.12} = \boxed{\frac{1}{6}}$$

$$P(M_3|F \cap D_2) = \frac{P(F|M_3 \cap D_2)P(M_3|D_2)}{P(F|D_2)} = \frac{1 \cdot 0}{0.12} = \boxed{0}$$

13.

a.

$$\boxed{P(X \leq 1) = F_X(1) = 1 - e^{-4} = 0.98168}$$

$$\boxed{P(X \geq 2) = 1 - F_X(2) = e^{-8} = 0.000335}$$

$$\boxed{P(X = 2) = 0}$$

since the distribution is continuous.

b.

$$\boxed{f_X(x) = \frac{\partial F_X(x)}{\partial x} = 4e^{-4x}, \quad x \geq 0}$$

c.

$$\boxed{p_Y(0) = F_X(2) = 1 - e^{-8} = 0.999665}$$

$$\boxed{p_Y(1) = 1 - F_X(2) = e^{-8} = 0.000335}$$

d.

$$Z = X_1 + X_2$$

$$\boxed{f_Z(z) = \int_0^z f_X(x) f_X(z-x) dx = 16 \int_0^z e^{-4x} e^{-4(z-x)} dx = 16 \int_0^z e^{-4z} dx = 16\, z\, e^{-4z}, \quad z \geq 0}$$

$$\boxed{F_Z(z) = \int_0^z 16\, z\, e^{-4z} dz = -4ze^{-4z} + \int_0^z 4e^{-4z} dz = 1 - e^{-4z}(4z+1), \quad z \geq 0}$$

$$\varphi_Z(u) = \int_{-\infty}^{\infty} e^{iuz} f_Z(z) dz = 16 \int_0^{\infty} z e^{(iu-4)z} dz = \frac{16}{iu-4} z e^{(iu-4)z} \Big|_0^{\infty} - \frac{16}{iu-4} \int_0^{\infty} e^{(iu-4)z} dz$$

$$\varphi_Z(u) = \frac{16}{(iu-4)^2}$$

$$\mu = \mathbb{E}[Z] = \frac{1}{i}\frac{\partial \varphi_Z(u)}{\partial u}\Big|_{u=0} = \frac{-32}{(iu-4)^3}\Big|_{u=0} = \frac{1}{2}$$

We see the derivatives of $\varphi_Z(u)$ follow a pattern:

$$\frac{\partial^n \varphi_Z(u)}{\partial u^n}\Big|_{u=0} = \frac{(-1)^n i^n (n+1)!16}{(iu-4)^{n+2}}\Big|_{u=0} = \frac{i^n (n+1)!}{4^n}$$

Thus the moments become:

$$\mathbb{E}[Z^n] = \frac{1}{i^n}\frac{\partial^n \varphi_Z(u)}{\partial u^n}\Big|_{u=0} = \frac{(n+1)!}{4^n}$$

The first central moment is always zero:

$$\mathbb{E}[(Z-\mu)^1] = \mathbb{E}[Z] - \mathbb{E}[\mu] = \mu - \mu = \boxed{0}$$

Second central moment:

$$\mathbb{E}[(Z-\mu)^2] = \mathbb{E}[Z^2] - 2\mu\mathbb{E}[Z] + \mu^2\mathbb{E}[1] = \mathbb{E}[Z^2] - \mu^2$$

$$\mathbb{E}[(Z-\mu)^2] = \frac{(2+1)!}{4^2} - \frac{1}{4} = \boxed{\frac{1}{8}}$$

Third central moment:

$$\mathbb{E}[(Z-\mu)^3] = \sum_{n=0}^{3}\binom{3}{n}(-\mu)^{3-n}\mathbb{E}[Z^n] = \sum_{n=0}^{3}\frac{3!}{n!(3-n)!}\left(\frac{-1}{2}\right)^{3-n}\frac{(n+1)!}{4^n} =$$

$$\sum_{n=0}^{3}\frac{3!}{(3-n)!}\left(\frac{-1}{2}\right)^{3-n}\frac{(n+1)}{4^n} = \sum_{n=0}^{3}(-1)^{n+1}\frac{3!}{(3-n)!}\frac{n+1}{2^{n+3}}$$

$$\mathbb{E}[(Z-\mu)^3] = -1\cdot\frac{1}{8} + 3\cdot\frac{2}{16} - 3\cdot\frac{3}{32} + 1\cdot\frac{4}{64} = \boxed{\frac{1}{16}}$$

Fourth central moment, similarly:

$$\mathbb{E}[(Z-\mu)^4] = \sum_{n=0}^{4}\binom{4}{n}(-\mu)^{4-n}\mathbb{E}[Z^n] = \sum_{n=0}^{4}(-1)^n\frac{4!}{(4-n)!}\frac{n+1}{2^{n+4}} = \boxed{\frac{3}{32}}$$

e.

Let $Z_N = X_1 + X_2 + ... + X_N$, where the X_i's are i.i.d. with pdf $f_X(x)$.

We already know $f_{Z_1}(x) = 4e^{-4x}$ and $f_{Z_2}(x) = 16xe^{-4x}$.

Then when $n = 3$:

$$f_{Z_3}(z) = \int_{-\infty}^{\infty} f_X(x) f_{Z_2}(z-x)dx = \int_0^z 4e^{-4x}16(z-x)e^{-4(z-x)}dx = \frac{4^3}{2!}z^2e^{-4z}, \quad z \geq 0$$

This pattern continues, such that:

$$\boxed{f_{Z_N}(x) = \frac{4^N}{(N-1)!}x^{N-1}e^{-4x} = \frac{4^N}{\Gamma(N)}x^{N-1}e^{-4x}, \quad x \geq 0}$$

This is a Gamma distribution: $\boxed{Z_N \sim \text{Gamma}(N,4)}$

CDF:

$$\boxed{F_{Z_N}(z) = \frac{1}{\Gamma(N)}\gamma(N,4z)}$$

where $\gamma a, b$ is the lower incomplete gamma function.

moments:

$$\mathbb{E}[Z_N] = \mu_{Z_N} = \frac{N}{4}$$

$$\mathbb{E}[(Z_N - \mu_{Z_N})^1] = \boxed{0}$$

$$\mathbb{E}[(Z_N - \mu_{Z_N})^2] = \boxed{\frac{N}{16}}$$

$$\mathbb{E}[(Z_N - \mu_{Z_N})^3] = \boxed{\frac{2N^{5/2}}{16^3}}$$

$$\mathbb{E}[(Z_N - \mu_{Z_N})^4] = \boxed{(\frac{6}{N}+3)(\frac{N}{16})^4}$$

15.

$$h_0 = a - bi = \text{ retention capacity}, \quad 0 \leq i \leq \frac{a}{b}; \quad h_0 = 0, \quad \text{o.w.}$$

$$h_A = \text{ actual retention}$$

Assume $i \sim Exp$, $h \sim Gamma$.

When $h < h_0(i)$, $h_A = h$ and when $h \geq h_0(i)$, $h_A = h_0(i)$.

$$\mathbb{E}[h_A] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_A\, f_H(h) f_I(i)\, dh\, di = \int_0^{\frac{a}{b}}\int_0^{a-bi} h\, f_H(h) f_I(i)\, dh\, di + \int_0^{\frac{a}{b}}\int_{a-bi}^{\infty} (a-bi)\, f_H(h) f_I(i)\, dh\, di$$

Plugging in the pdfs $f_I(i) = \lambda e^{-\lambda i}$ and $f_H(h) = \frac{\beta^\alpha h^{\alpha-1}e^{-\beta h}}{\Gamma(\alpha)}$:

$$\mathbb{E}[h_A] = \frac{\lambda\beta^\alpha}{\Gamma(\alpha)}\left(\int_0^{\frac{a}{b}}\int_0^{a-bi} h\, h^{\alpha-1}e^{-(\lambda i+\beta h)}\, dh\, di + \int_0^{\frac{a}{b}}\int_{a-bi}^{\infty}(a-bi)\, h^{\alpha-1}e^{-(\lambda i+\beta h)}\, dh\, di\right)$$

$$\mathbb{E}[h_A] = \frac{\lambda\beta^\alpha}{\Gamma(\alpha)}\left(\int_0^a h^\alpha e^{-\beta h}\int_0^{\frac{a-h}{b}} e^{-\lambda i}\,di\,dh + \int_0^\infty h^{\alpha-1}e^{-\beta h}\int_{\frac{a-h}{b}}^{\frac{a}{b}}(a-bi)\,e^{-\lambda i}\,di\,dh + \int_0^\infty h^{\alpha-1}e^{-\beta h}\int_{\frac{a}{b}}^{\infty}(a-bi)\,e^{-\lambda i}\,di\,dh\right)$$

16.

a.

$$\varphi_X(u) = \frac{\sin(u)}{u} = \mathbb{E}[e^{iux}] = \int_{-\infty}^{\infty} e^{iux} f_X(x)dx$$

Then $f_X(x)$ is the inverse Fourier transform of $\varphi_X(u)$:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\frac{\sin(u)}{u}du = \boxed{\begin{cases} \frac{1}{2} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

b. Find $\varphi_X(u)$ for exponential, uniform, and Gaussian r.v.'s.

expoential: $f_X(x) = \lambda e^{-\lambda x}$, $x \ge 0$

$$\varphi_X(u) = \int_0^\infty e^{iux}\lambda e^{-\lambda x}dx = \frac{\lambda}{iu-\lambda}e^{(iu-\lambda)x}\Big|_{x=0}^{\infty} = \boxed{\frac{\lambda}{\lambda - iu}}$$

uniform: $f_X(x) = \frac{1}{b-a}$, $a \le x \le b$

$$\varphi_X(u) = \int_a^b e^{iux}\frac{1}{b-a}dx = \boxed{\frac{1}{iu(b-a)}e^{iu(b-a)}}$$

Gaussian: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $x \in \mathbb{R}$

$$\varphi_X(u) = \int_{-\infty}^{\infty} e^{iux}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx = \boxed{e^{i\mu u - \frac{\sigma^2 u^2}{2}}}$$

c.

$$\varphi_X(u) = \frac{\lambda^2}{\lambda^2 + u^2}$$

Calculating the first four derivatives for finding central moments:

$$\frac{\partial \varphi_X(u)}{\partial u} = \frac{-2\lambda^2 u}{(\lambda^2+u^2)^2}$$

$$\frac{\partial^2 \varphi_X(u)}{\partial u^2} = \frac{-2\lambda^2(\lambda^2 - 3u^2)}{(\lambda^2+u^2)^3}$$

$$\frac{\partial^3 \varphi_X(u)}{\partial u^3} = \frac{-24\lambda^2 u(u^2-\lambda^2)}{(\lambda^2+u^2)^4}$$

$$\frac{\partial^4 \varphi_X(u)}{\partial u^4} = \frac{24\lambda^2(\lambda^4 + 5u^4 - 10\lambda^2 u^2)}{(\lambda^2 + u^2)^5}$$

$$\mu = \mathbb{E}[X] = \frac{1}{i}\frac{\partial \varphi_X(u)}{\partial u}\Big|_{u=0} = \frac{-2\lambda^2 u}{i(\lambda^2 + u^2)^2}\Big|_{u=0} = 0$$

First central moment is always zero: $\mathbb{E}[(X-\mu)^1] = \boxed{0}$

Second central moment:

$$\mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2] = -\frac{\partial^2 \varphi_X(u)}{\partial u^2}\Big|_{u=0} = \boxed{\frac{2}{\lambda^2}}$$

Third central moment:

$$\mathbb{E}[(X-\mu)^3] = \mathbb{E}[X^3] = \boxed{0}$$

Fourth central moment:

$$\mathbb{E}[(X-\mu)^4] = \mathbb{E}[X^4] = \frac{\partial^4 \varphi_X(u)}{\partial u^4}\Big|_{u=0} = \boxed{\frac{24}{\lambda^4}}$$

17.

a.

$$\int_{-\infty}^{\infty} f_X(x)dx = \frac{1}{2}k + k = \frac{3}{2}k = 1 \quad \rightarrow \quad \boxed{k = \frac{2}{3}}$$

b.

$$F_X(x) = \int_{-\infty}^{x} f_X(x)dx$$

$$\boxed{F_X(x) = \begin{cases} \frac{x^2}{3} & 0 \le x \le 1 \\ \frac{1}{3}(2x-1) & 1 \le x \le 2 \\ 0 & \text{o.w.} \end{cases}}$$

c. The median m satisfies the following:

$$\int_{-\infty}^{m} f_X(x)dx = F_X(m) = \frac{1}{2} \quad \rightarrow \quad \boxed{m = \frac{5}{4}}$$

Similarly, the lower quartile q_l satisfies:

$$F_X(q_l) = \frac{1}{4} \quad \rightarrow \quad \boxed{q_l = \frac{\sqrt{3}}{2}}$$

And for the upper quartile q_u:

$$F_X(q_u) = \frac{3}{4} \quad \rightarrow \quad \boxed{q_u = \frac{13}{8}}$$

18. Exercise 2.32

a. We are given the pdf of Z:

$$f_Z(z) = \begin{cases} \frac{2\lambda}{d(\lambda d+2)} z & 0 < z < d \\ \frac{2\lambda}{\lambda d+2} e^{-\lambda(z-d)} & z \geq d \end{cases}$$

We are also given that Y is a function of Z, namely:

$$Y = Z, \quad 0 < Z < d$$

$$Y = d, \quad d \leq Z < c + d$$

$$Y = Z - c, \quad Z \geq c + d$$

To show that $f_Z(z)$ is properly defined, the following statements must be true:

$$f_Z(z) \geq 0 \quad \forall z \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} f_Z(z) dz = 1$$

Let's check:

$$\int_{-\infty}^{\infty} f_Z(z) dz = \int_0^d \frac{2\lambda}{d(\lambda d + 2)} z dz + \int_d^{\infty} \frac{2\lambda}{\lambda d + 2} e^{-\lambda(z-d)} dz$$

$$= \frac{\lambda d}{\lambda d + 2} + \left[\frac{-2}{\lambda d + 2} e^{-\lambda(z-d)} \right] \Big|_{z=d}^{\infty} = \frac{\lambda d}{\lambda d + 2} + \frac{2}{\lambda d + 2} = 1$$

So it checks out.

A plot of the distribution for $\lambda = d = 1$:

In [1]:

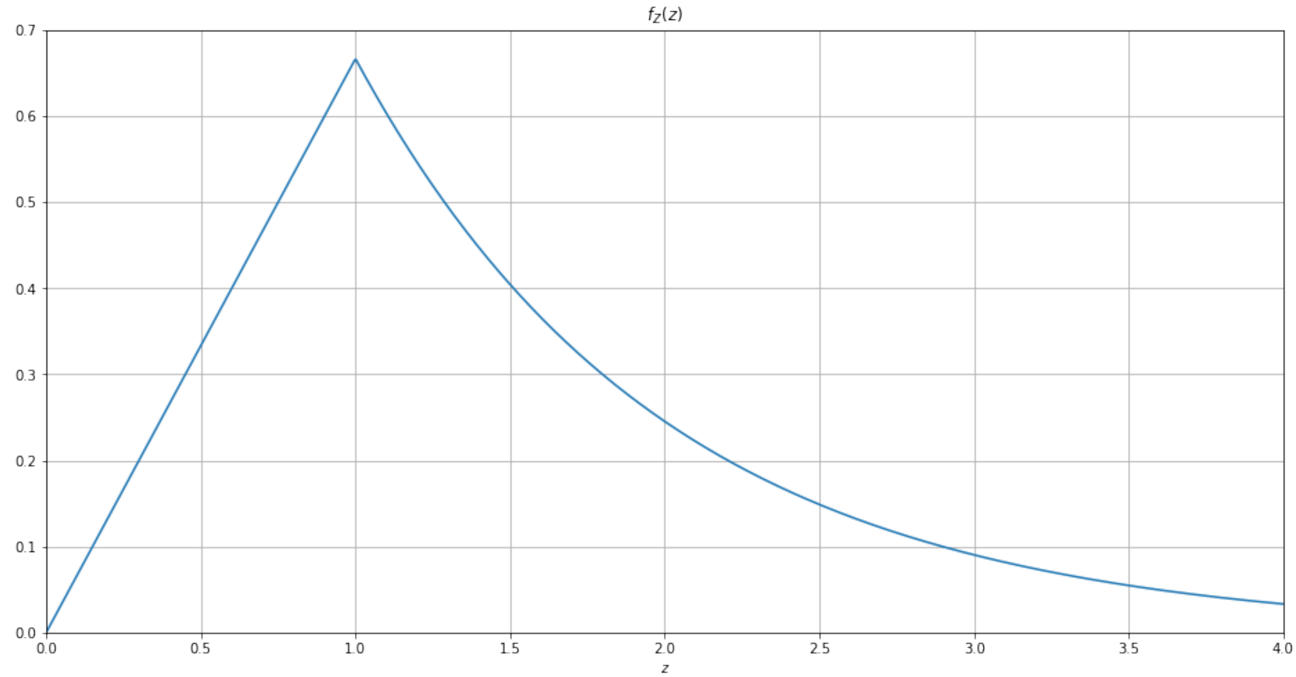
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

fig, ax = plt.subplots(figsize=(16,8))

x = np.linspace(0,4, 1000)
y = np.piecewise(x, [x<1, x >= 1], [lambda x: 2.0/3*x, lambda x: 2.0/3*np.e**(1-x)])

ax.plot(x, y)
```

```python
ax.grid()
ax.set_title(r'$f_Z(z)$')
ax.set_xlabel(r'$z$')
ax.set_xlim(0, 4)
ax.set_ylim(0, 0.7)
plt.show()
```



b. Find $F_Y(y)$.

For $0 < y < d$:

$$F_Y(y) = \int_0^y \frac{2\lambda}{d(\lambda d + 2)} z dz = \boxed{\frac{\lambda y^2}{d(\lambda d + 2)}}$$

For $y = d$:

$$F_Y(y) = \frac{\lambda d}{\lambda d + 2} + \int_d^{c+d} \frac{2\lambda}{\lambda d + 2} e^{-\lambda(z-d)} dz = \boxed{1 - \frac{2}{\lambda d + 2} e^{-\lambda c}}$$

For $d < y < \infty$:

$$F_Y(y) = \frac{\lambda d}{\lambda d + 2} + \int_d^{y+c} \frac{2\lambda}{\lambda d + 2} e^{-\lambda(z-d)} dz = \frac{\lambda d}{\lambda d + 2} + \left[\frac{-2}{\lambda d + 2} e^{-\lambda(z-d)} \right] \Big|_{z=d}^{y+c} = \boxed{1 - \frac{2}{\lambda d + 2} e^{-\lambda(y+c-d)}}$$