MATRIX HYDRODYNAMICS & MIXING

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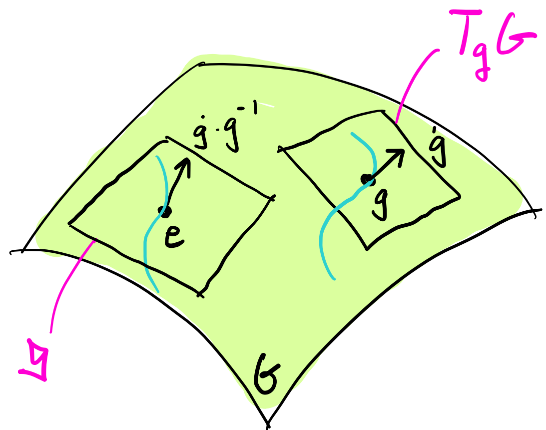
Isle of Skye 2025

Euler–Arnold equations

Lie group G ← geodesics?

Lie algebra $\mathfrak{g} = T_e G$



Riemannian metric on G:

$$\langle \dot g, \dot g \rangle_g = \langle \underbrace{\dot g \cdot g^{-1}}_{v}, \underbrace{\dot g \cdot g^{-1}}_{v} \rangle_e$$

Right-invariant: $\langle \dot g, \dot g \rangle_g = \langle \dot g \cdot h, \dot g \cdot h \rangle_{g \cdot h}$

Euler–Arnold equation

$$\underbrace{A\dot v}_{\dot m} + ad_v^* \underbrace{Av}_{m \in \mathfrak{g}^*} = 0$$

momentum

$$\begin{cases} A : \mathfrak{g} \to \mathfrak{g}^*, & Av = \langle v, \cdot \rangle_e \\ ad_v^* : \mathfrak{g}^* \to \mathfrak{g}^*, & \langle ad_v^* m, u \rangle = \langle m, [v,u] \rangle \end{cases}$$

inertia operator

Hamiltonian formulation (Lie–Poisson system)

$$\dot m + ad^*_{dH(m)} m = 0 \qquad H(m) = \langle m, A^{-1} m \rangle$$

Reconstruction equation: $\dot g(t) = v(t) \cdot g(t)$

geodesic curves on G

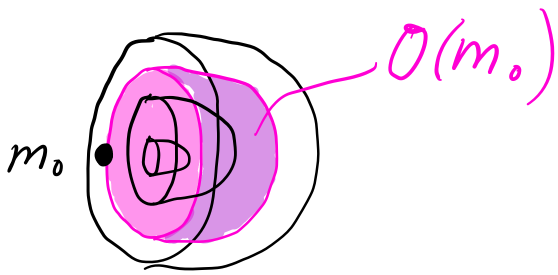
"Abstract" transport:

momentum $m = Av$ transported by g (or v):

$$\boxed{m(t) = Ad^*_{g(t)^{-1}} m(0)} \quad \begin{cases} \text{"}m = g m_0 g^{-1}\text{"} \\ \text{"}\omega = \omega_0 \circ \Phi^{-1}\text{"} \end{cases}$$

Consequence: co-adjoint orbits as "kinematic" transport barrier

$$\mathcal{O}(m_0) = \{ Ad^*_g m_0 \mid g \in G \} \subset \mathfrak{g}^*$$

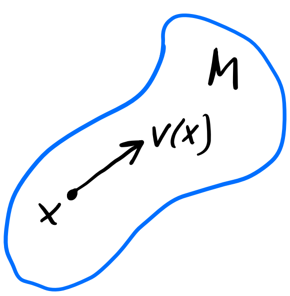


NOTE: $C : \mathfrak{g}^* \to \mathbb{R}$ constant on co-adjoint orbit

$\Longrightarrow$ C conserved (Casimir function)

Example: inviscid hydrodynamics [Arnold 1966]

Fluid domain: compact Riemannian manifold M



$$G = Diff_\mu(M), \quad \mathfrak{g} = \mathfrak{X}_\mu(M)$$

$$\langle v, v \rangle_e = \int_M |v(x)|^2 \, d\mu(x)$$

divergence free vector fields

Euler–Arnold eq. = incompressible Euler

$$\boxed{\dot v + \nabla_v v = -\nabla p}$$

Lagrange multiplier

Reconstruction:

$$\dot\Phi(t) = v(t)\circ\Phi(t)$$

Special case: 2D ($M = M^2$ two-manifold)

$$G = \mathrm{Diff}_\mu(M^2) = \text{symplectomorphisms}$$

symplectic form

$$\mathfrak{g} = \mathfrak{X}_\mu(M^2) = \text{symplectic vector fields}$$

(locally $v = X_\psi = \nabla^\perp\psi$)

$$\mathfrak{g}^* \simeq C_0^\infty(M^2) = \text{vorticity functions}$$

Inertia operator $A : \mathfrak{g} \to \mathfrak{g}^*$

$$v \longmapsto \underbrace{\mathrm{curl}(v)}_{\omega}$$

Hamiltonian:

$$H(\omega) = \frac{1}{2}\int_{M^2} \omega\, \Delta^{-1}\omega \, d\mu$$

Euler–Arnold (Hamiltonian form)

$$\boxed{\dot\omega + \{\psi, \omega\} = 0, \quad \Delta\psi = \omega}$$

$ad_v^*\omega$ stream function

QUESTION: Spatial discretization by sequence of $\begin{cases} \text{groups } G_1, G_2, \ldots, G_N, \ldots \\ \text{algebras } \mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_N, \ldots \end{cases}$

such that $\begin{cases} G_N \approx \mathrm{Diff}_\mu(M) \\ \mathfrak{g}_N \approx \mathfrak{X}_\mu(M) \end{cases}$ as $N \to \infty$

YES! Via quantization [Zeitlin 1991]

in 2D case

Quantization in a nutshell:

complex skew-Hermitian $N \times N$

CLASSICAL $\omega \longmapsto T_N(\omega)$	QUANTUM
function $\omega \in C^\infty$	operator (matrix) $W \in \mathfrak{u}(N)$
Poisson bracket $\{\psi, \omega\}$	(scaled) commutator $\frac{1}{\hbar}[P, W]$
$\lVert\omega\rVert_{L^\infty}$	spectral norm $\lVert W\rVert_\infty$
	$\max_k \lvert\lambda_k\rvert$
2D Euler	**Euler–Zeitlin**
$\dot\omega + \{\psi, \omega\} = 0$	$\dot W + \frac{1}{\hbar}[P, W] = 0$
$\Delta\psi = \omega$	$\Delta_N P = W$

Simplest case : $M = S^2$ (sphere)

Cartesian coordinate functions x_1, x_2, x_3

$$\Delta \psi = \sum_{\alpha=1}^{3} \{x_\alpha, \{x_\alpha, \psi\}\}$$

Unitary representation X_1, X_2, X_3 of $so(3)$ in $u(N)$

$$\Delta_N P = \frac{1}{\hbar^2} \sum_{\alpha=1}^{3} [X_\alpha, [X_\alpha, P]]$$

[DEMO]

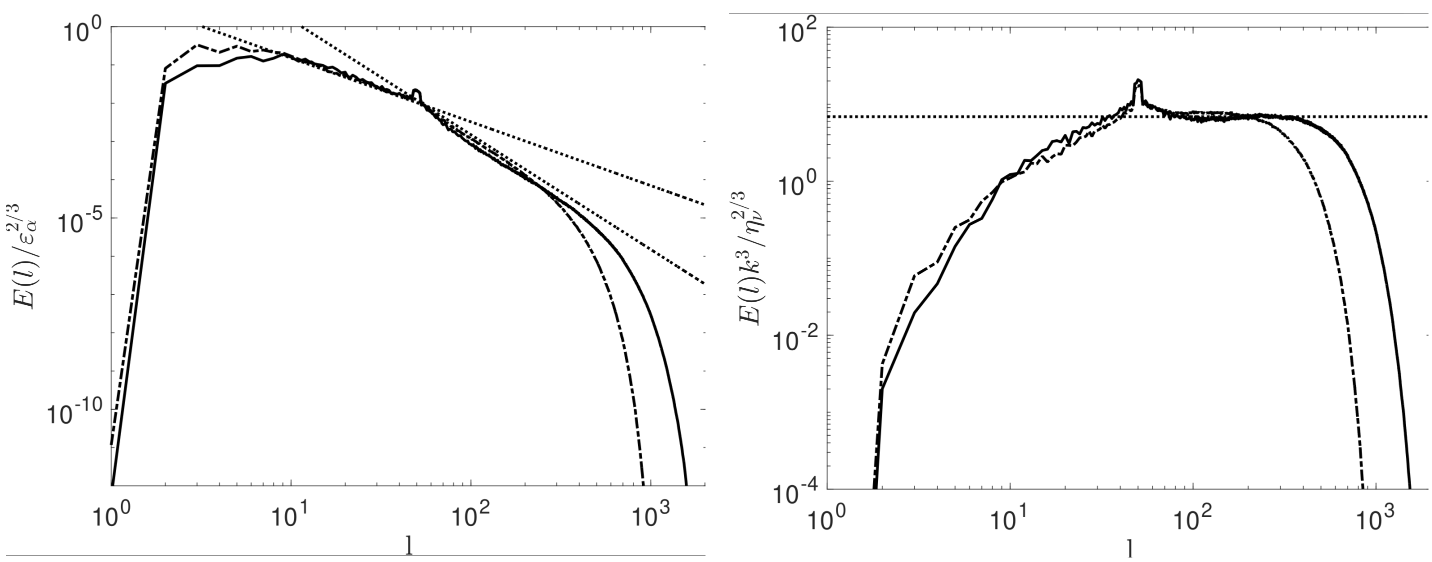
CONS

- Low spatial order of convergence: $\mathcal{O}\left(\frac{1}{N}\right)$ for L^2-error of vorticity
- "Noisy"

PROS

- All geometric structure preserved

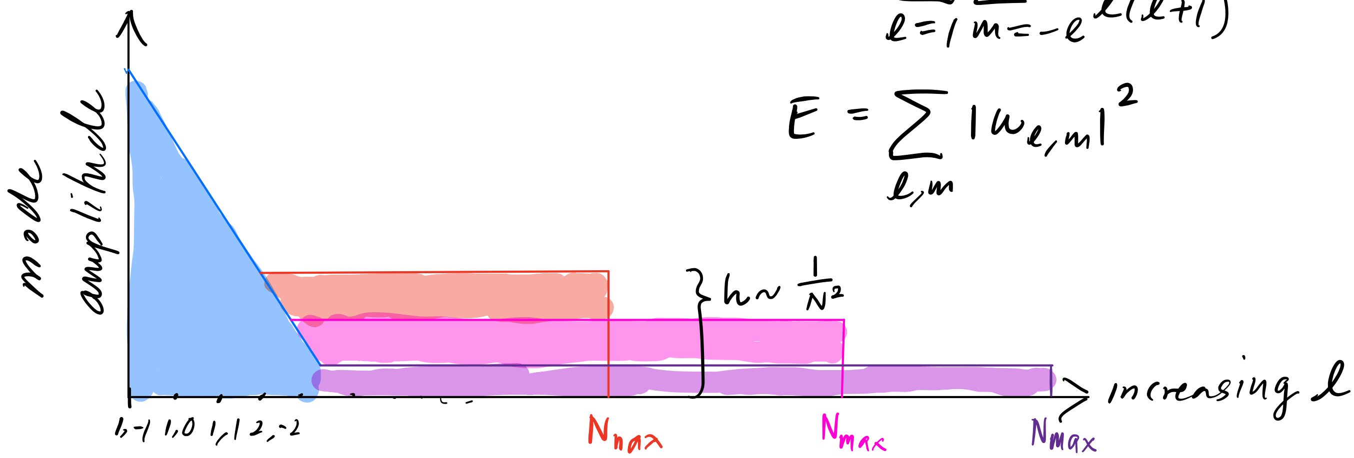
$\Longrightarrow$ good qualitative statistics



The noise must be there!

$$H = \sum_{\ell=1}^{N} \sum_{m=-\ell}^{\ell} \frac{|w_{\ell,m}|^2}{\ell(\ell+1)}$$

$$E = \sum_{\ell,m} |w_{\ell,m}|^2$$

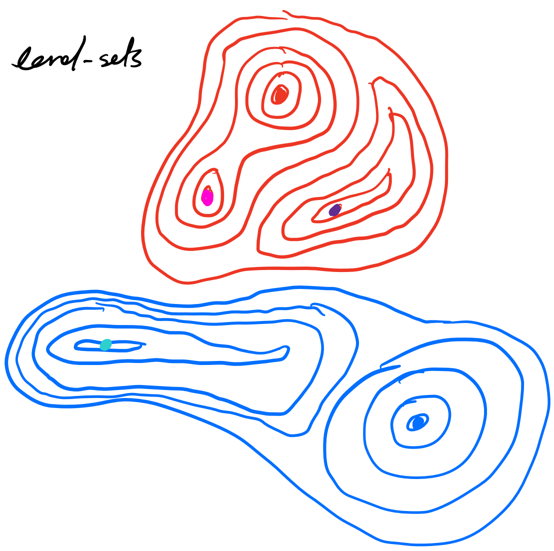


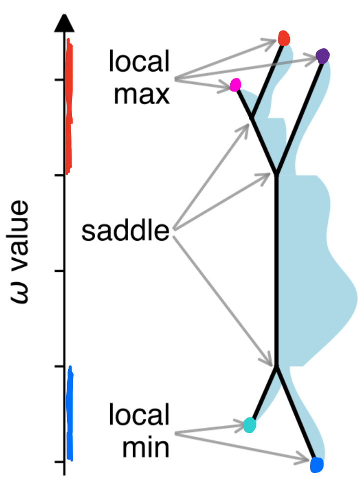
What is mixing in matrix hydrodynamics?

	CLASSICAL	MATRIX HYDRODYNAMICS
Casimirs	$C_f(\omega) = \int f(\omega)\, d\mu$	$C_f^N(W) = \frac{4\pi}{N} \operatorname{tr}(f(W))$
Transport	$\omega(t) = \omega_0 \circ \Phi(t)^{-1}$	$W(t) = F(t) W_0 F(t)^{-1}$
Conserved	values of ω	eigenvalues of W
Transported	level-sets of ω	eigenvectors of W

Classical (smooth) co-adjoint orbits:

ω level-sets





Measured Reeb graph:

- measure on each edge

Thm [Izosimov, Khesin, Mousavi, 2016]

$\omega \in \mathcal{O}(\omega_0) \iff \omega$ and ω_0 have same measured Reeb graph

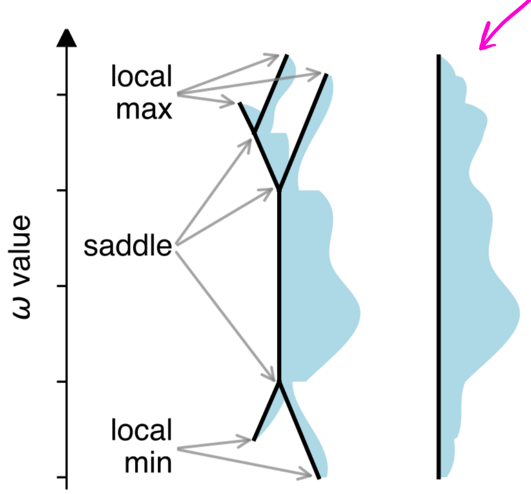
BUT:

L^∞-indistinguishable from 2-blob configuration

(A) ω_0 (B) $\omega_0 \circ \Phi_k^{-1}$ (C) $\lim_{k\to\infty} \omega_0 \circ \Phi_k^{-1}$

$\Longrightarrow$ measured Reeb graph not stable in L^∞

collapsed Reeb graph



Def Level-set measure

$\lambda_\omega([a,b]) = C_{\chi_{[a,b]}}(\omega)$

Corresponding measure in Zeitlin's model:

Def Empirical spectral measure for $W \in \mathfrak{u}(N)$

$\lambda_W^N = \frac{1}{N}\sum_{k=1}^{N} \delta_{\lambda_k}$ eigenvalues of iW

Thm [M. & Viviani, 2024]

$$\omega \in C^2(S^2) \Rightarrow \lambda^N_{T_N(\omega)} \longrightarrow \lambda_\omega \quad \text{as} \quad N \to \infty$$

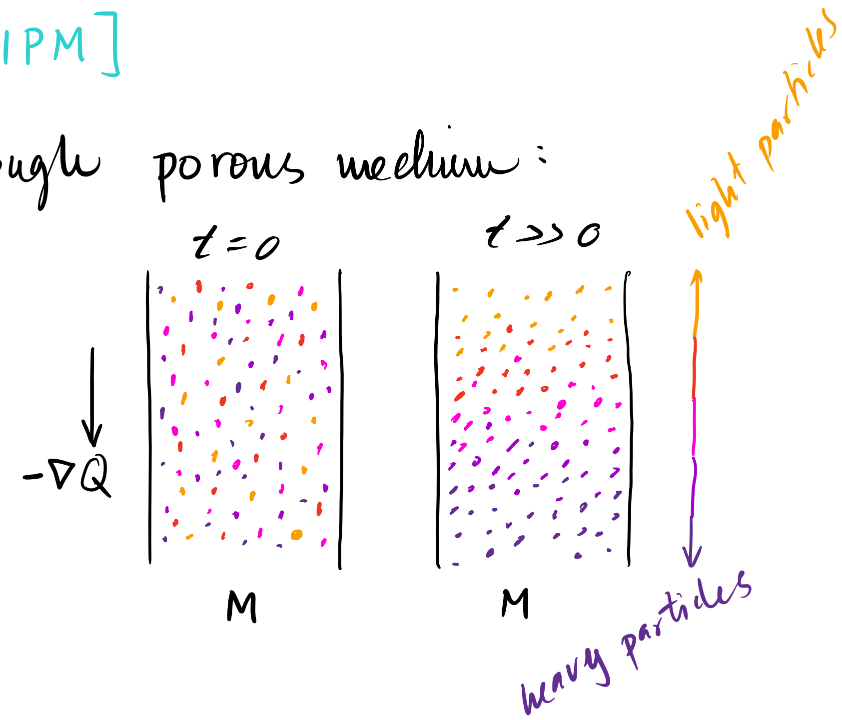
Consequence: Euler–Zeitlin is a generalized model because it allows **weak mixing**:

$$W \in \mathcal{O}(W_0) \iff \lambda^N_W = \lambda^N_{W_0} \quad \text{(same spectrum)}$$

[DEMO: IPM]

Incompressible flow through porous medium:

$$\begin{cases} \dot{\rho} + \operatorname{div}(\rho v) = 0 \\ v + \rho \nabla Q = -\nabla p \\ \operatorname{div} v = 0 \end{cases}$$



SUMMARY: "MIXING" OF CO-ADJOINT ORBITS

Smooth (no mixing)	Yudovich L^∞ (weak mixing)	Shnirelman L^∞-weak* (full mixing)
same measured Reeb graph	same level-set measure	same max and min

matrix hydrodynamics with $N \to \infty$

Schur–Horn–Kostant convexity & closure of Diff_μ

[Bloch, Flaschka, Ratiu, 1993]:

1 Introduction

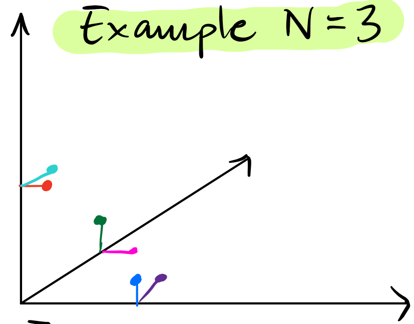
Diffeomorphism groups are huge, infinite-dimensional Lie groups, but in some respects they can be strikingly similar to finite-dimensional semisimple groups.

Kostant's thm (special case $G = SU(N)$):

$$\pi_N : SU(N) \longrightarrow \mathbb{R}^N$$

$$W \longmapsto \begin{bmatrix} iW_{11} \\ iW_{22} \\ \vdots \\ iW_{NN} \end{bmatrix}$$

Example N=3



$\pi_N(\mathcal{O}(W))$ is convex set

whose extremal points are $S^N \cdot \begin{bmatrix} i\lambda_1 \\ i\lambda_2 \\ \vdots \\ i\lambda_N \end{bmatrix}$

symmetric group = permutations (Weyl group)

Infinite dimensional case $G = \mathrm{Diff}_\mu(S^2)$

Projection via averaging along longitudes

$$\pi(\omega) : [-1,1] \longrightarrow \mathbb{R}$$

$$\omega \longmapsto \int_0^{2\pi} \omega(\cdot,\phi)\, d\phi$$

Consider: $\bar{\mathcal{O}}(\omega) := \pi^{-1}(\pi(\omega))$

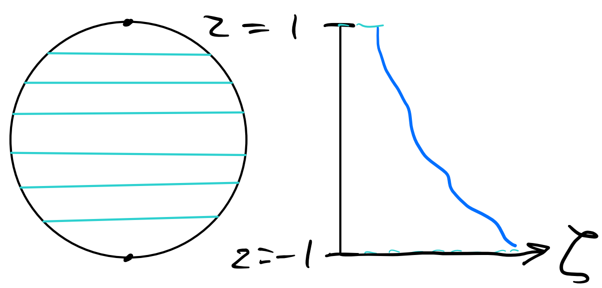
Clearly: $\bar{\mathcal{O}}(\omega) \neq \mathcal{O}(\omega) := \omega \circ \mathrm{Diff}_\mu(S^2)$

But: If $\overline{\mathrm{Diff}_\mu(S^2)} = \{$measure-preserving maps $S^2 \to S^2\}$

semi-group

Thm [BRF] Let $\omega_0 \in L^\infty(S^2)$. $\exists!\ \zeta \in L^2(S^2)$ non-increasing, right-continuous zonal function such that:

$$\overline{\mathcal{O}}(\omega_0) = \{\omega \in L^\infty(S^2) \mid \lambda_\omega = \lambda_{\omega_0}\} = \zeta \circ \overline{\mathrm{Diff}_\mu(S^2)}$$



The closure $\overline{\mathcal{O}}(\omega_0)$ above is located strictly in between the smooth coadjoint orbits $\mathcal{O}(\omega_0)$ and Shnirelman's weak closure $\overline{\mathcal{O}(\omega_0)}^*$, i.e.,

$$\mathcal{O}(\omega_0) \subset \overline{\mathcal{O}}(\omega_0) \subset \overline{\mathcal{O}(\omega_0)}^*.$$

In a sense, $\overline{\mathcal{O}}(\omega_0)$ is the smallest closure that allows the matrix theory results to remain intact. It is also compatible with Yudovich's global well-posedness theory, which gives a transport map $\Phi_t \in \overline{\mathrm{Diff}_\mu(\mathbb{S}^2)}$ for initial data $\omega_0 \in L^\infty(\mathbb{S}^2)$.

matrix hydrodynamics as $N \to \infty$

NOTE: $\pi_N(W) \simeq$ projection onto:

$$X_3 = \begin{bmatrix} i & & 0 \\ & \ddots & \\ 0 & & -i \end{bmatrix} \qquad \mathrm{stab}_{X_3} = \{A \in \mathfrak{su}(N) \mid [X_3, A] = 0\}$$

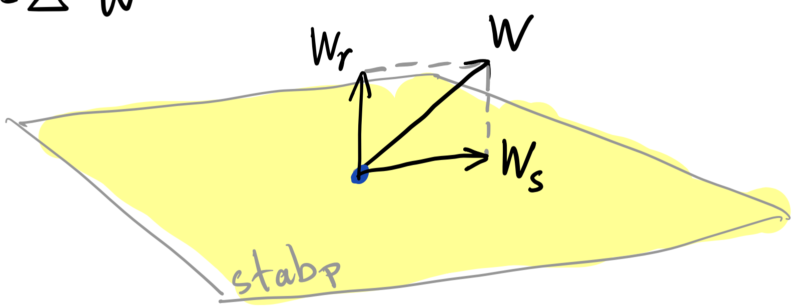
$\Rightarrow$ gives decomposition

$$W = \underset{\mathrm{stab}_{X_3}}{\underset{\cup\!\!\!\!\,\shortmid}{W_s}} + \underset{\mathrm{stab}_{X_3}^\perp}{\underset{\cup\!\!\!\!\,\shortmid}{W_r}}$$

Generalization: canonical decomposition

[M. & Viviani, 2022]

Def: Let $W \in \mathfrak{u}(N)$. $W = W_s + W_r$
where $W_s = \Pi_P W$ Frobenius projection (2-norm)
onto $\mathrm{stab}_P = \{ W_s \in \mathfrak{u}(N) \mid [W_s, P] = 0 \}$ for
$P = \Delta^{-1} W$



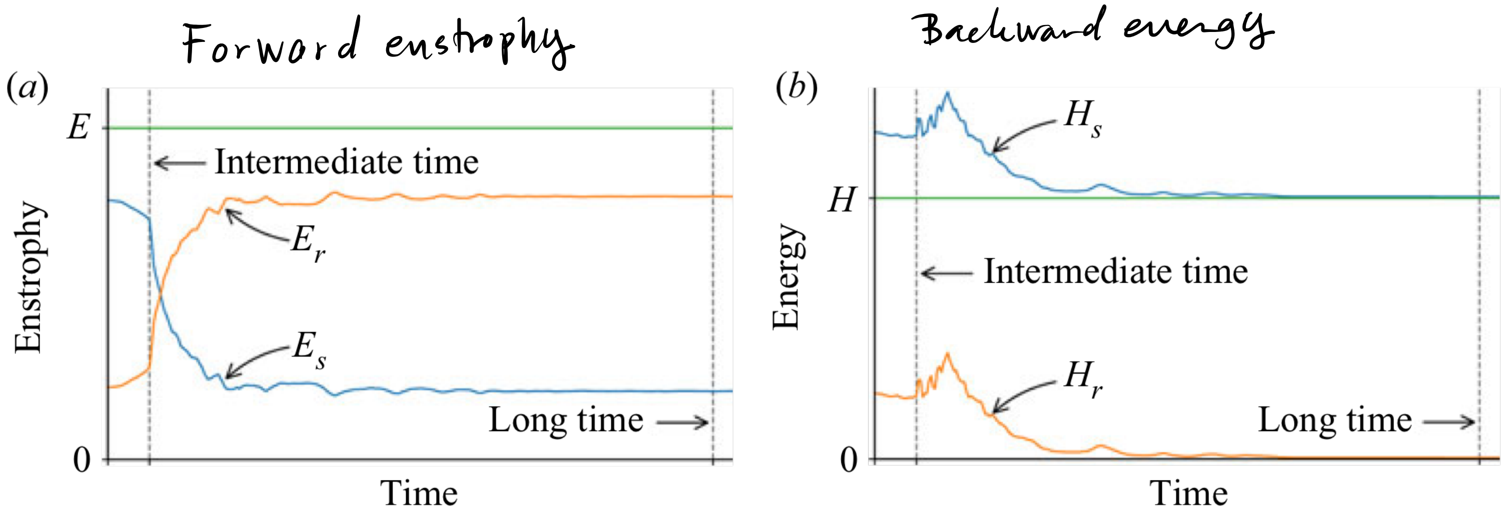
Observation: in long-time, W_r is noisy and W_s smooth
$\Longrightarrow$ separation of scales

Properties of canonical decomposition:

$\langle W_1, W_2 \rangle_2 := \mathrm{tr}(W_1 W_2^\dagger)$ (Frobenius inner product)

$\langle W_1, W_2 \rangle_H := -\mathrm{tr}(\Delta_1^{-1} W_1 W_2^\dagger)$ (energy inner product)

- $\langle W_s, W_r \rangle_2 = 0 \quad \Rightarrow \quad I_2^N(W_s) \leq I_2^N(W)$
- $\langle W, W_r \rangle_H = 0$ (!) $\Rightarrow \quad H(W_s) \geq H(W)$
- f convex $\Rightarrow \quad I_f^N(W_s) \leq I_f^N(W)$
- W stationary $\Longleftrightarrow \quad W_r = 0$



[DEMO: scale separation]

$\Pi_p W$

$\mathrm{stab}_p = \{ W \mid \{P, W\} = 0 \}$