$\dot{\omega} = \{\psi, \omega\}$ (Euler 2D)

$\Updownarrow$

$\dot{W} = [P, W]$ (Zeitlin–Euler)

- Long-time behaviour
- connection w. integrability
- Rigorous convergence
- Interpretation (rep. theory)
- High-performance implementation
- Canonical splitting
- statistical mechanics
- Lagrangian coordinates
- Shape analysis
- completion of orbits
- Optimal transport
- gradient flows: IPM, Toda, etc.
- More general models:
  - compressible
  - MHD
  - GFD
  - Stochastic versions

LONG-TIME BEHAVIOUR & INTEGRABILITY

$$\dot{W} = [P, W], \quad W = \underbrace{\sum_{i=1}^{3} [X_i, [X_i, P]]}_{\Delta_N P}$$

stream matrix ↗   ↖ vorticity matrix in $\mathfrak{su}(N)$

Numerical observations with Zeitlin's model:

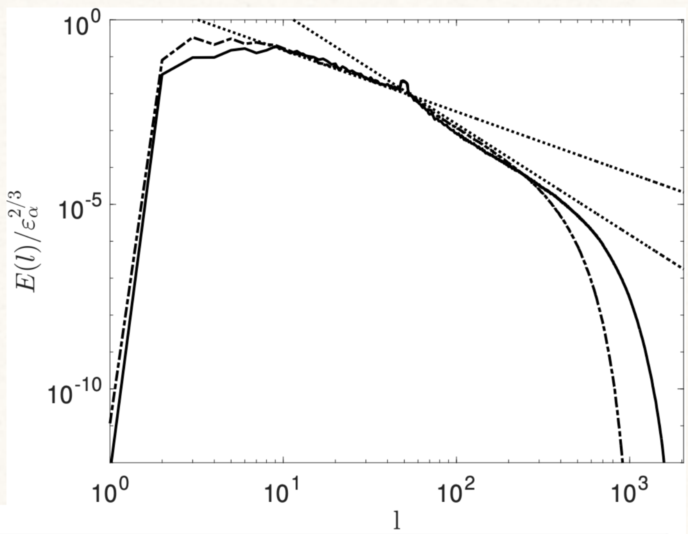
1. Qualitative behaviour largely independent of N

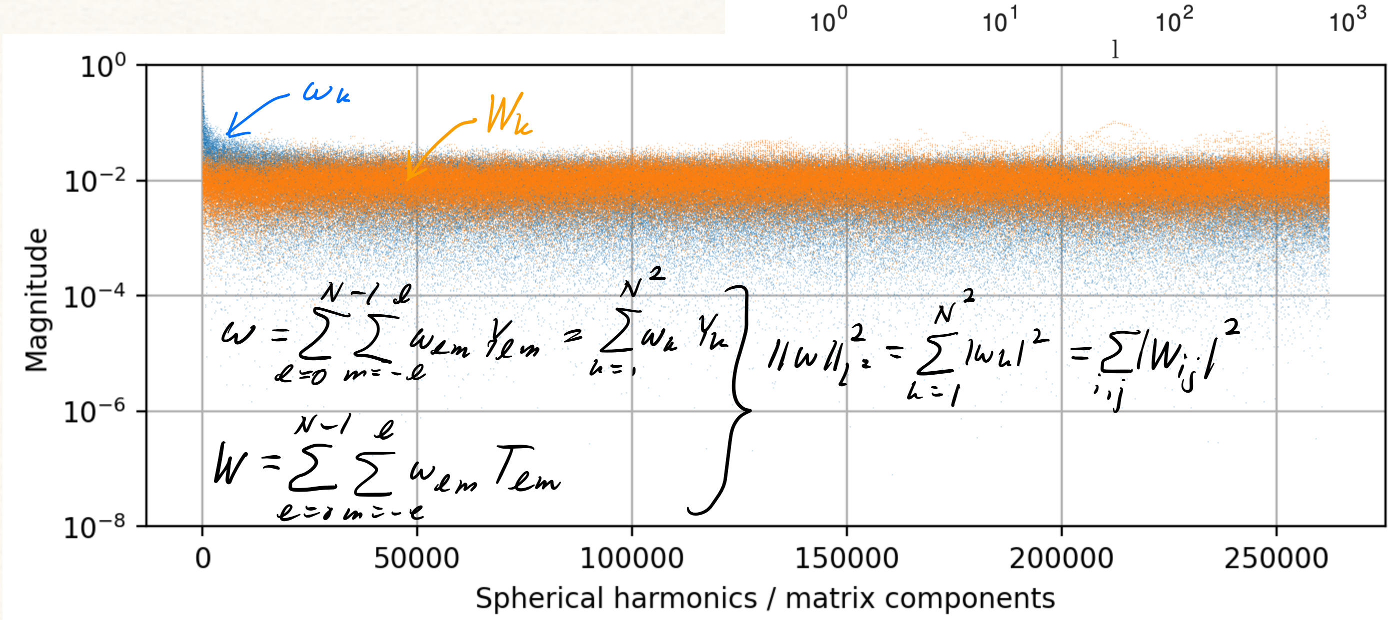
   ... but why?

   - isospectral ⟺ co-adjoint orbits ⟹ Casimirs
   - Hamiltonian ⟺ symplectic ⟺ Lie–Poisson

2. Conservation laws

   - Energy (H^{-1}-norm)
   - Enstrophy (L^2-norm)
   - Higher order Casimirs (L^p-norms)





$$\left.\begin{aligned} \omega &= \sum_{\ell=0}^{N-1} \sum_{m=-\ell}^{\ell} \omega_{\ell m} Y_{\ell m} = \sum_{k=1}^{N^2} \omega_k Y_k \\ W &= \sum_{\ell=0}^{N-1} \sum_{m=-\ell}^{\ell} \omega_{\ell m} T_{\ell m} \end{aligned}\right\} \quad \|\omega\|_{L^2}^2 = \sum_{k=1}^{N^2} |\omega_k|^2 = \sum_{i,j} |W_{ij}|^2$$

Flow-map: $S_t : \omega_0 \longmapsto \omega(t)$

Arnold (1966): $\{S_t(\omega_0) \mid t \geq 0\} \subset \underbrace{\{\omega_0 \circ \eta \mid \eta \in \mathrm{Diff}^0_\mu(\mathbb{S}^2)\}}_{\text{co-adjoint orbit } \mathcal{O}_{\omega_0}}$

$$\boxed{\Omega_+(\omega_0) = \bigcap_{s \geq 0} \overline{\{S_t(\omega_0) \mid t \geq s\}}^{*} \subseteq \overline{\mathcal{O}}^{*}_{\omega_0}}$$ (persisting set)

BIG question: what is contained in $\Omega_+(\omega_0)$?

PROGRAM

1. Study corresponding (finite-dim) problems in Zeitlin's model

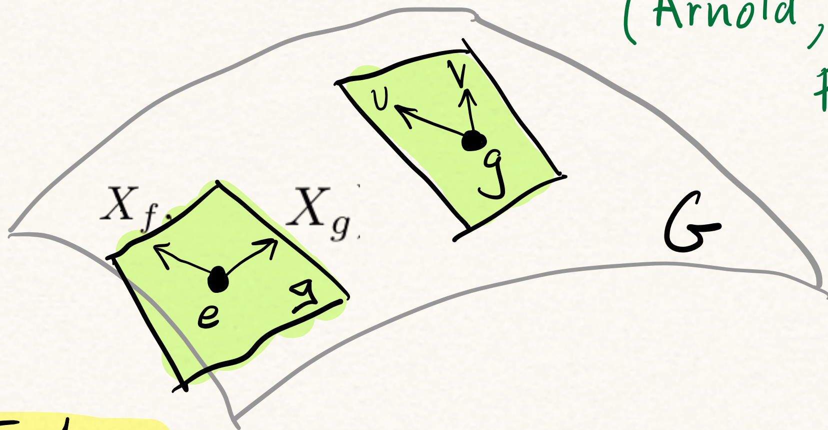
2. Use quantization theory to obtain results for $N \to \infty$

TOPIC OF TODAY: SECTIONAL CURVATURE

Euler $\iff$ geodesic eq. on $G = \mathrm{Diff}_\mu(M)$

$\implies$ study sectional curvature (e.g. stability)

(Arnold, Khesin, Misiolek, Preston, etc)



2D Euler

$$
\begin{aligned}
&C(X_f, X_g) = \\
&-\frac{1}{4}\Big\langle \{\Delta f, g\} + \{\Delta g, f\}, \Delta^{-1}\Big(\{\Delta f, g\} + \{\Delta g, f\}\Big)\Big\rangle_{L^2} \\
&-\frac{1}{2}\langle \{f,g\}, \{\Delta f, g\} - \{\Delta g, f\}\rangle_{L^2} \\
&+\frac{3}{4}\langle \{f,g\}, \Delta\{f,g\}\rangle_{L^2} \\
&+\big\langle \{\Delta f, f\}, \Delta^{-1}\{\Delta g, g\}\big\rangle_{L^2}.
\end{aligned}
$$

Zeitlin

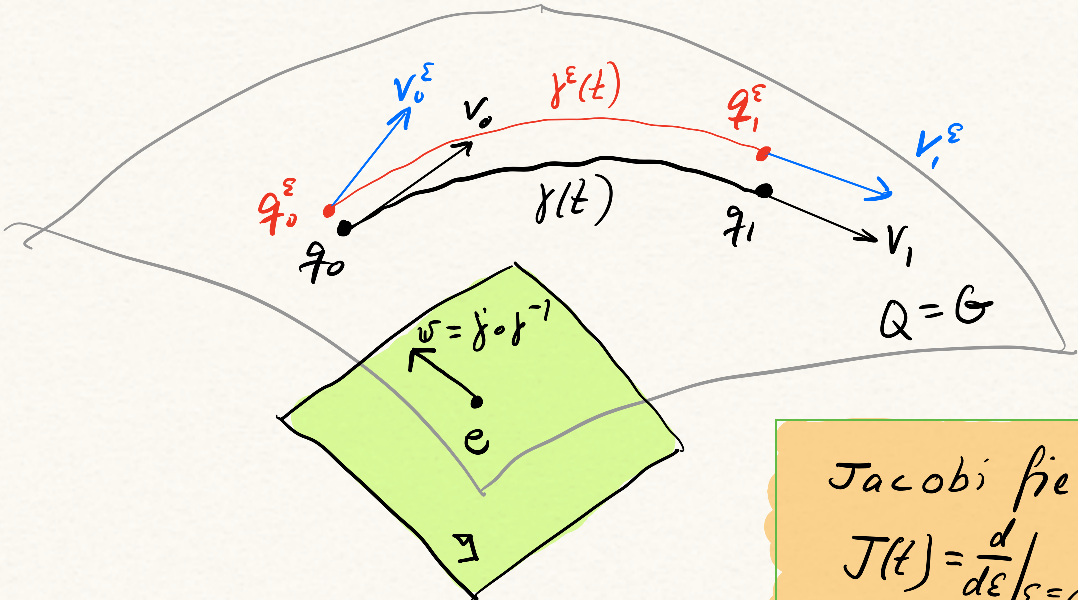
$$
\begin{aligned}
&C_N(F, G) = \\
&-\frac{1}{4\hbar_N^2}\Big\langle [\Delta_N F, G] + [\Delta_N G, F], \Delta_N^{-1}\Big([\Delta_N F, G] + [\Delta_N G, F]\Big)\Big\rangle_{L^2_N} \\
&-\frac{1}{2\hbar_N^2}\langle [F,G], [\Delta_N F, G] - [\Delta_N G, F]\rangle_{L^2_N} \\
&+\frac{3}{4\hbar_N^2}\langle [F,G], \Delta_N[F,G]\rangle_{L^2_N} \\
&+\frac{1}{\hbar_N^2}\big\langle [\Delta_N F, F], \Delta_N^{-1}[\Delta_N G, G]\big\rangle_{L^2_N}.
\end{aligned}
$$

Theorem 1. *The sectional curvature of* $\mathrm{SU}(N)$ *(with Zeitlin's metric) converges to the sectional curvature of* $\mathrm{Diff}_\mu(\mathbb{S}^2)$ *(with Arnold's metric) as follows: for any* $f, g \in H^7(\mathbb{S}^2)$

$$
|C_N(p_N f, p_N g) - C(X_f, X_g)| \le \hbar_N c_0 \, \|f\|^2_{H^7} \, \|g\|^2_{H^7},
$$

where the constant $c_0 > 0$ *is independent of* f, g, N.

Eulerian and Lagrangian stability



Jacobi field:

$$J(t)=\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\gamma^\varepsilon(t)\in T_{\gamma(t)}G$$

2D Euler

$$T^3G\simeq \mathfrak{g}\times\mathfrak{g}\times\mathfrak{g}\times G \Rightarrow$$

$$\dot\omega=\{\psi,\omega\},\quad \Delta\psi=\omega \qquad (\psi\in\mathfrak{g},\ \omega\in\mathfrak{g}^*)$$

reduced Jacobi eq.

$$\left.\begin{cases}\dot v=\{\psi,v\} \quad \leftarrow \text{transport of} & (v\in\mathfrak{g})\\ \dot\zeta=\{\Delta^{-1}\zeta,\omega\}+\{\psi,\zeta\} & (\zeta\in\mathfrak{g}^*)\end{cases}\right\}\ (13)$$

Zeitlin model

$$\Delta_N\dot P=\frac{1}{\hbar}[P,\Delta_N P]$$

$$\left.\begin{aligned}\dot Y&=\frac{1}{\hbar}[P,Y]\\ \dot Z&=\frac{1}{\hbar}[\Delta_N^{-1}Z,W]+\frac{1}{\hbar}[P,Z]\end{aligned}\right\}\ (14)$$

Special case: stability of stationary solutions

$$\{\psi, \omega\} = 0, \quad \Delta\psi = \omega \qquad (2)$$

$$[P, W] = 0, \quad \Delta_N P = W, \quad W = p_N \omega \qquad (8)$$

Theorem 2. *Let ω be a stationary solution of the vorticity equation (2) such that $W = p_N\omega$ is a stationary solution of the Euler–Zeitlin equation (8). Let $v(t)$ and $\zeta(t)$ be solutions of the reduced Jacobi equations (13). Furthermore, let $Y(t)$ and $Z(t)$ be corresponding solutions of the finite-dimensional reduced Jacobi equations (14) with $Y(0) = p_N v(0)$ and $Z(0) = p_N\zeta(0)$. Then, for any fixed t,*

$$\begin{aligned} \|\iota_N Y(t) - v(t)\|_{L^2} &\longrightarrow 0 \\ \|\iota_N Z(t) - \zeta(t)\|_{L^2} &\longrightarrow 0 \end{aligned} \quad \text{as} \quad N \to \infty.$$

Moreover, the convergence is uniform on bounded intervals of t.

Conclusion: Zeitlin model preseves

1. Lie-Poisson structure
2. Energy (nearly, BEA)
3. Casimirs (exactly)
4. Sectional curvature (as $O(\frac{1}{N})$)
5. Lagrangian and Eulerian stability (as $O(\frac{1}{N})$)

[more info: klasmodin.github.io]