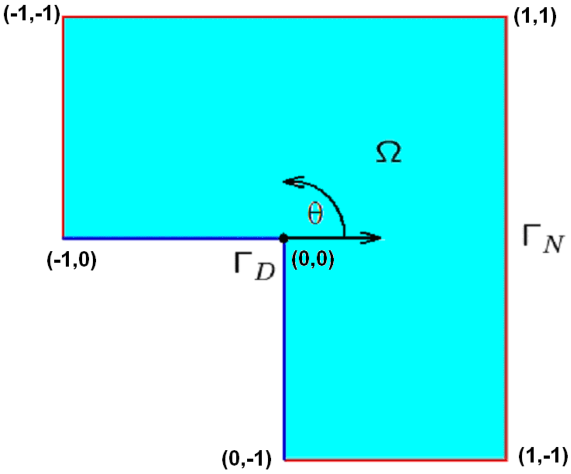
L-shape domain

Strong formulation



Ω is defined as [-1,1]x[-1,1] \ [-1,0]x[-1,0] (L-shape domain)
Dirichlet boundary Γ_D is denoted by blue color
Neumann boundary Γ_N is denoted by red colour

We seek for temperature scalar field
$R^2 \ni (x_1, x_2) \to u(x_1, x_2) \in R$ where $u(x_1, x_2)$ denotes temperature at point (x_1, x_2).
The strong formulation concerns the heat transfer equation

$$\sum_{i=1}^{2} \frac{\partial^2 u}{\partial x_i^{\,2}} = 0 \text{ over } \Omega$$

(equivalent short notation $\Delta u = 0$)
We introduce the following boundary conditions:
$u = 0$ on Γ_D (zero temperature oper Dirichlet boundary)
$\frac{\partial u}{\partial n} = g$ on Γ_N (derivative in the norma direction is prescribed by *g* function)
where

$$\frac{\partial u}{\partial n} = \nabla u \circ n = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right) \circ (n_1, n_2) = \frac{\partial u}{\partial x_1} n_1 + \frac{\partial u}{\partial x_2} n_2 = g$$

where $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right)$ is a gradient and (n_1, n_2) are components of the normal vector;
for example on the bottom of the domain – over (0,-1)-(1,-1) we have

$$\frac{\partial u}{\partial n} = \nabla u \circ n = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2} \right) \circ (0, -1) = -\frac{\partial u}{\partial x_2} = g$$

where *g* is a function defined in polar system of coordinates with the origin at point (0,0)

$$R \times (0, 2\Pi) \ni (r, \theta) \to g(r, \theta) = r^{\frac{2}{3}} \sin^{\frac{2}{3}} \left(\theta + \frac{\Pi}{2} \right)$$

Weak formulation

We get the weak formulation by taking L2 scalar produkt with so called test functions v

$$b(u,v) = l(v) \quad \forall v \in V$$

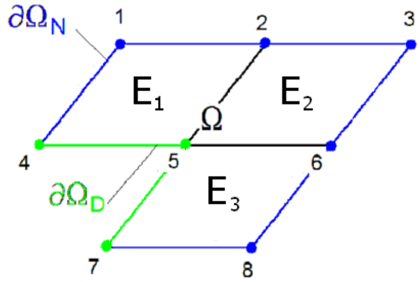
$$b(u,v) = \int_\Omega \nabla u \circ \nabla v \, dx$$ (in other words $b(u,v) = \int_\Omega \sum_{i=1}^{2} \frac{\partial u_i}{\partial x_i} \frac{\partial v_i}{\partial x_i} dx$)

$$l(v) = \int_{\Gamma_N} g \, v \, dS$$

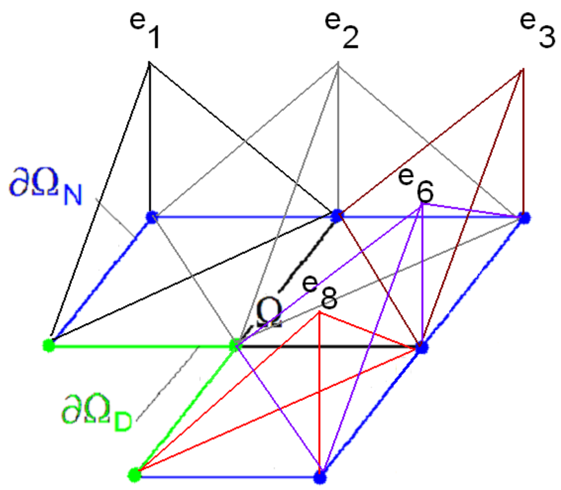
$$V = \{ v \in H^1(\Omega) : trv = 0 \text{ on } \Gamma_D \}$$

Finie element method

We partition Ω into Finie elements (in this example into three elements E1,E2,E3)



with the following basis functions (p=1)



we also generate the system of equations:

$$u \approx u_h = \sum_{i=1}^{N} a_i e_i$$

$$\sum_{i=1}^{N} a_i b(e_i, e_j) = l(e_j) \quad j = 1, \ldots, N$$

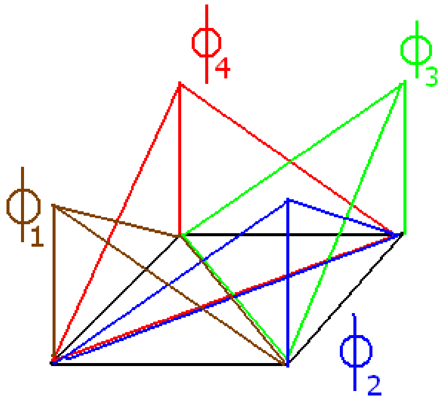
$$b(e_i, e_j) = \int_\Omega \nabla e_i \nabla e_j \, dx = \int_\Omega \sum_{k=1}^{2} \frac{\partial e_i}{\partial x_k} \frac{\partial e_j}{\partial x_k} dx$$

$$l(e_j) = \int_{\Gamma_N} e_j g \, dS$$

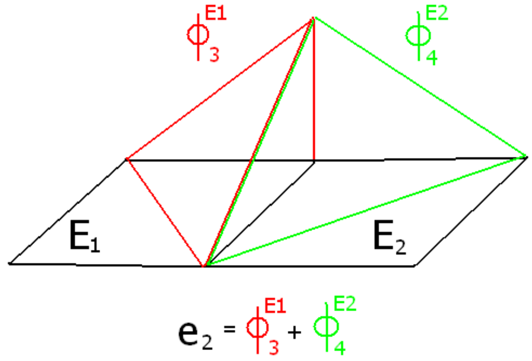
The Dirichlet boundary condition is enforced by setting rows and columns related to notes 4, 5 and 7 to zero.

Some observations necessary for efficient implementation of the algorithm

1. We introduce the following four shape functions over each element (here for p=1)

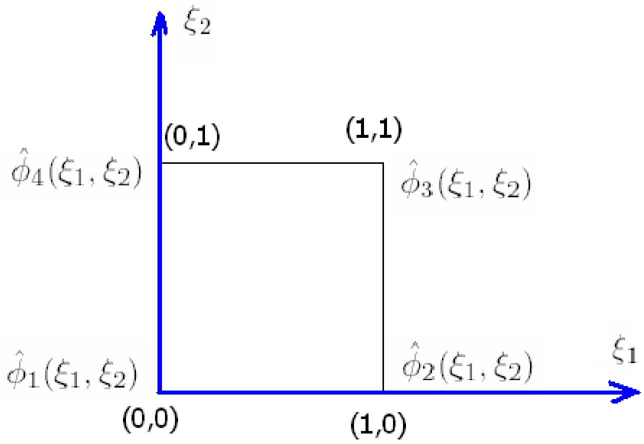


Each basis functions is a sum of one or two shape functions ϕ_i



(for example basis function e2 consists of the third shape function over element E1, and fourth basis function over element E2)

2. How can we compute the formula for such arbitrary shape function ϕ_i ?
We introduce so called master element $\hat{E}$ = [0,1]x[0,1] and define template shape functions over the element



$$
\begin{aligned}
\hat{\phi}_1(\xi_1,\xi_2) &= \hat{\chi}_1(\xi_1)\hat{\chi}_1(\xi_2) &&= (1-\xi_1)(1-\xi_2)\\
\hat{\phi}_2(\xi_1,\xi_2) &= \hat{\chi}_2(\xi_1)\hat{\chi}_1(\xi_2) &&= \xi_1(1-\xi_2)\\
\hat{\phi}_3(\xi_1,\xi_2) &= \hat{\chi}_2(\xi_1)\hat{\chi}_2(\xi_2) &&= \xi_1\xi_2\\
\hat{\phi}_4(\xi_1,\xi_2) &= \hat{\chi}_1(\xi_1)\hat{\chi}_2(\xi_2) &&= (1-\xi_1)\xi_2\,,
\end{aligned}
$$

where

$$\hat{\chi}_1(\xi) = 1-\xi$$

$$\hat{\chi}_2(\xi) = \xi$$

Each rectangular element E can be prescribed by its location (b_1,b_2) as well as its length and height (a_1,a_2)

For example the three exemplary elements E1, E2, E3 are defined as

E1: $(b_1,b_2) = (-1,0)$; $(a_1,a_2)=(1,1)$

E2: $(b_1,b_2) = (0,0)$; $(a_1,a_2)=(1,1)$

E3: $(b_1,b_2) = (0,-1)$; $(a_1,a_2)=(1,1)$

We define the mapping from master element $\hat{E}$ into arbitrary element E

$$\hat{E} \ni (\xi_1,\xi_2) \to x_E(\xi_1,\xi_2) = (b_1 + a_1\xi_1, b_2 + a_2\xi_2) = (x_1,x_2) \in E$$

and the reverse mapping

$$E \ni (x_1,x_2) \to x_E^{-1}(x_1,x_2) = \left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}\right) = (\xi_1,\xi_2) \in \hat{E}$$

In other words

$$x_1 = b_1 + a_1\xi_1; \quad x_2 = b_2 + a_2\xi_2$$

and

$$\xi_1 = \frac{x_1-b_1}{a_1}; \quad \xi_2 = \frac{x_2-b_2}{a_2}$$

We can prescribe formule for arbitrary shape function ϕ_i , i=1,2,3,4 by using the map x_E^{-1}

$$\phi_1(x_1,x_2) = \hat{\phi}_1\left(x_E^{-1}(x_1,x_2)\right) = \hat{\phi}_1\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}\right) =$$

$$\hat{\chi}_1\left(\frac{x_1-b_1}{a_1}\right)\hat{\chi}_1\left(\frac{x_2-b_2}{a_2}\right) = \left(1-\frac{x_1-b_1}{a_1}\right)\left(1-\frac{x_2-b_2}{a_2}\right)$$

$$\phi_2(x_1,x_2) = \hat{\phi}_2\left(x_E^{-1}(x_1,x_2)\right) = \hat{\phi}_2\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}\right) =$$

$$\hat{\chi}_2\left(\frac{x_1-b_1}{a_1}\right)\hat{\chi}_1\left(\frac{x_2-b_2}{a_2}\right) = \left(\frac{x_1-b_1}{a_1}\right)\left(1-\frac{x_2-b_2}{a_2}\right)$$

$$\phi_3(x_1,x_2)=\hat{\phi}_3\left(x_E^{-1}(x_1,x_2)\right)=\hat{\phi}_3\left(\frac{x_1-b_1}{a_1},\frac{x_2-b_2}{a_2}\right)=$$

$$\hat{\chi}_2\left(\frac{x_1-b_1}{a_1}\right)\hat{\chi}_2\left(\frac{x_2-b_2}{a_2}\right)=\left(\frac{x_1-b_1}{a_1}\right)\left(\frac{x_2-b_2}{a_2}\right)$$

$$\phi_4(x_1,x_2)=\hat{\phi}_4\left(x_E^{-1}(x_1,x_2)\right)=\hat{\phi}_4\left(\frac{x_1-b_1}{a_1},\frac{x_2-b_2}{a_2}\right)=$$

$$\hat{\chi}_1\left(\frac{x_1-b_1}{a_1}\right)\hat{\chi}_2\left(\frac{x_2-b_2}{a_2}\right)=\left(1-\frac{x_1-b_1}{a_1}\right)\left(\frac{x_2-b_2}{a_2}\right)$$

3. The integrals can be partitions according to elements

$$b\left(\phi_i^k,\phi_j^k\right)=\int_{E_k}\frac{\partial\phi_i^k}{\partial x_1}(x_1,x_2)\frac{\partial\phi_j^k}{\partial x_1}(x_1,x_2)\,dx_1dx_2+\int_{E_k}\frac{\partial\phi_i^k}{\partial x_2}(x_1,x_2)\frac{\partial\phi_j^k}{\partial x_2}(x_1,x_2)\,dx_1dx_2$$

For the first order approximation it is only necessary to take the value at the center of element and the area of the element (a_1*a_2)

In other words

$$b\left(\phi_i^k,\phi_j^k\right)=\left(\frac{\partial\phi_i^k}{\partial x_1}\left(b_1+\frac{a_1}{2},b_2+\frac{a_2}{2}\right)\frac{\partial\phi_j^k}{\partial x_1}\left(b_1+\frac{a_1}{2},b_2+\frac{a_2}{2}\right)\right)(a_1a_2)+$$

$$\left(\frac{\partial\phi_i^k}{\partial x_2}\left(b_1+\frac{a_1}{2},b_2+\frac{a_2}{2}\right)\frac{\partial\phi_j^k}{\partial x_2}\left(b_1+\frac{a_1}{2},b_2+\frac{a_2}{2}\right)\right)(a_1a_2)$$

The derivatives $\frac{\partial\phi_i^k}{\partial x_1},\frac{\partial\phi_i^k}{\partial x_2},\frac{\partial\phi_j^k}{\partial x_1},\frac{\partial\phi_j^k}{\partial x_2}$ are constant and equal to +/- $\frac{1}{a_1}$ or +/- $\frac{1}{a_2}$, depending on the function and the direction of the integration.

4. The integral

$$b\left(\phi_i^k,\phi_j^k\right)=\int_{E_k}\frac{\partial\phi_i^k}{\partial x_1}(x_1,x_2)\frac{\partial\phi_j^k}{\partial x_1}(x_1,x_2)\,dx_1dx_2+\int_{E_k}\frac{\partial\phi_i^k}{\partial x_2}(x_1,x_2)\frac{\partial\phi_j^k}{\partial x_2}(x_1,x_2)\,dx_1dx_2$$

is assembled into proper row and column of the global matrix.

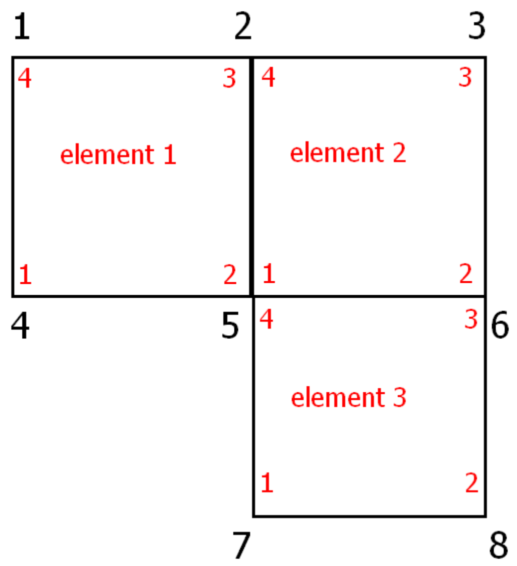
`B(i1,j1)+=` $\int_{E_k}\frac{\partial\phi_i^k}{\partial x_1}(x_1,x_2)\frac{\partial\phi_j^k}{\partial x_1}(x_1,x_2)\,dx_1dx_2+\int_{E_k}\frac{\partial\phi_i^k}{\partial x_2}(x_1,x_2)\frac{\partial\phi_j^k}{\partial x_2}(x_1,x_2)\,dx_1dx_2$

How can we translate i and j into the global row i1 and column j1?

According to the scheme – each row (column) of the matrix is related to one coefficient a_i of one basis function e_i

In other words (red color denotes the shape functions ϕ_i^k over elements k=1,2,3,4,

black color denotes coresponding basis functions e_i = row number / column number in the matrix)



5. Integral over the boundary $\int_{E_k \cap \Gamma_N} g(x_1,x_2)\phi_i^k(x_1,x_2)\,dx_1dx_2$

We need to check whether edges of a given element E_k are located on the Neumann boundary Γ_N .

If the given edge is located on the Neumann boundary, then we need to add the integral over the edge to the right hand side

$$\int_{edge} g(x_1,x_2)\phi_i^k(x_1,x_2)\,dx_1dx_2 = g(x_1^*,x_2^*)\phi_i^k(x_1^*,x_2^*)|edge|$$

where

(x_1^*,x_2^*) is the point from the centem of the edge

$g(x_1^*,x_2^*)$ is the function value at the point

$\phi_i^k(x_1^*,x_2^*)$ is the value of the shape function ϕ_i^k at the point (always equal to ½ or 0)

$|edge|$ is the length of the edge

Sequential algorithm for global system generation

B(1:8,1:8)=0 (creation of the matrix)
L(1:8)=0 (creation of the right hand side)
Loop with respekt to elements $E_k \in \{E_1,E_2,E_3\}$
  Loop wrt functions ϕ_i^k of element E_k , $\phi_i^k \in \{\phi_1^k,\phi_2^k,\phi_3^k,\phi_4^k\}$
    i1 = row of the matrix related with ϕ_i^k
    L(i1)+= $\int_{E_k \cap \Gamma_N} g(x_1,x_2)\phi_i^k(x_1,x_2)\,dx_1dx_2$
    Loop wrt functions ϕ_j^k of element E_k , $\phi_j^k \in \{\phi_1^k,\phi_2^k,\phi_3^k,\phi_4^k\}$
      j1 = column of the matrix related with ϕ_j^k
      B(i1,j1)+= $\int_{E_k} \frac{\partial \phi_i^k}{\partial x_1}(x_1,x_2)\frac{\partial \phi_j^k}{\partial x_1}(x_1,x_2)\,dx_1dx_2 + \int_{E_k} \frac{\partial \phi_i^k}{\partial x_2}(x_1,x_2)\frac{\partial \phi_j^k}{\partial x_2}(x_1,x_2)\,dx_1dx_2$
    End of loop over functions ϕ_j^k
  End of loop over functions ϕ_i^k
End of loop over elements E_k

B(4,1:8)=0 (enforcing Dirichlet b.c. at node 4)
B(5,1:8)=0 (enforcing Dirichlet b.c. at node 5)
B(7,1:8)=0 (enforcing Dirichlet b.c. at node 7)
L(4)=0 (enforcing Dirichlet b.c. at node 4)
L(5)=0 (enforcing Dirichlet b.c. at node 5)
L(7)=0 (enforcing Dirichlet b.c. at node 7)
B(4,4)=1 (1 on diagonal at row 4)
B(5,5)=1 (1 on diagonal at row 5)
B(7,7)=1 (1 on diagonal at row 7)

Call frontal solver algorithm for
Ba=L

Get the solution **a**={a1,...,a8} for $u \approx u_h = \sum_{i=1}^{N} a_i e_i$