

## Report – Numerical Simulation of Light-Ray Deflection in Schwarzschild Space-Time

### 1. Introduction

Gravitational lensing - the bending of light by massive bodies - provides one of the most striking confirmations of general relativity. In this project, I simulate photon trajectories around a non-rotating (Schwarzschild) black hole by numerically integrating the null geodesic equations. The primary goal is to reproduce the classic result that light passing at impact parameter  $b$  is deflected by an angle

$$\Delta\phi \approx \frac{4M}{b}$$

in geometrized units ( $c = G = 1$ ). I build an interactive dashboard (using Dash and Plotly) that allows the user to vary the black-hole mass  $M$  and visualize both individual light-ray trajectories and the dependence of  $\Delta\phi$  on  $b$ .

### 2. Mathematical Formulation

#### 2.1. Schwarzschild Metric and Null Geodesics

In Schwarzschild coordinates  $(t, r, \theta, \phi)$ , a spherically symmetric, static black hole of mass  $M$  has line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

Restricting motion to the equatorial plane ( $\theta = \pi/2$ ), null geodesics ( $ds^2 = 0$ ) satisfy two first integrals:

$$E = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\lambda}, \quad L = r^2 \frac{d\phi}{d\lambda},$$

where  $E, L$  are the conserved energy and angular momentum per unit mass, and  $\lambda$  is just a parameter that increases along the light ray's path. Eliminating  $dt/d\lambda$  and using  $ds^2 = 0$  yields the radial equation (see [theconfused.me, 2017]):

$$\frac{d^2 r}{d\lambda^2} = \frac{L^2}{r^4} (r - 3M).$$

Thus we have the system of three first-order ODEs for variables  $\{r, \dot{r}, \phi\}$ :

$$\begin{cases} \frac{dr}{d\lambda} = \dot{r}, \\ \frac{d\dot{r}}{d\lambda} = \frac{L^2}{r^4} (r - 3M), \\ \frac{d\phi}{d\lambda} = \frac{L}{r^2}. \end{cases}$$

We adopt geometrized units throughout, so  $M$  and all lengths are in the same unit.

## 2.2. Impact Parameter and Initial Conditions

The impact parameter  $b$  is the asymptotic closest-approach distance for a straight-line light ray. We launch photons from a point  $(x_0, y_0) = (-x_{\text{init}}, b)$  with an initial velocity directed along  $+x$ . Converting to polar coordinates,

$$r_0 = \sqrt{x_{\text{init}}^2 + b^2}, \quad \phi_0 = \arccos(x_{\text{init}}/r_0),$$

and components of the velocity in  $(r, \phi)$  form are chosen so that the ray is initially moving purely in the  $+x$  direction. One solves

$$\dot{r}_0 = \cos \phi_0, \quad \dot{\phi}_0 = -\frac{\sin \phi_0}{r_0},$$

then sets  $L = r_0^2 \dot{\phi}_0$ .

## 3. Numerical Method

I use SciPy's `solve_ivp` routine (RK45 by default) to integrate the ODE system from  $\lambda = 0$  to  $\lambda = \lambda_{\text{max}}$ . The trajectory is sampled uniformly in  $\lambda$  with step  $\Delta\lambda = 0.05$ . Trajectories for various  $b$  values are computed and then transformed back to Cartesian  $(x, y)$  via

$$x = r \cos \phi, \quad y = r \sin \phi.$$

## 4. Results and Error Analysis

### 4.1. Photon Trajectories

For a representative mass  $M = 1.0$ , trajectories with various  $b$  values from 4 to 30 clearly show increasing deflection for smaller  $b$ , and capture (plunge into the horizon) when  $b < b_{\text{crit}} = 3\sqrt{3}M \approx 5.20$  (see Fig.1).

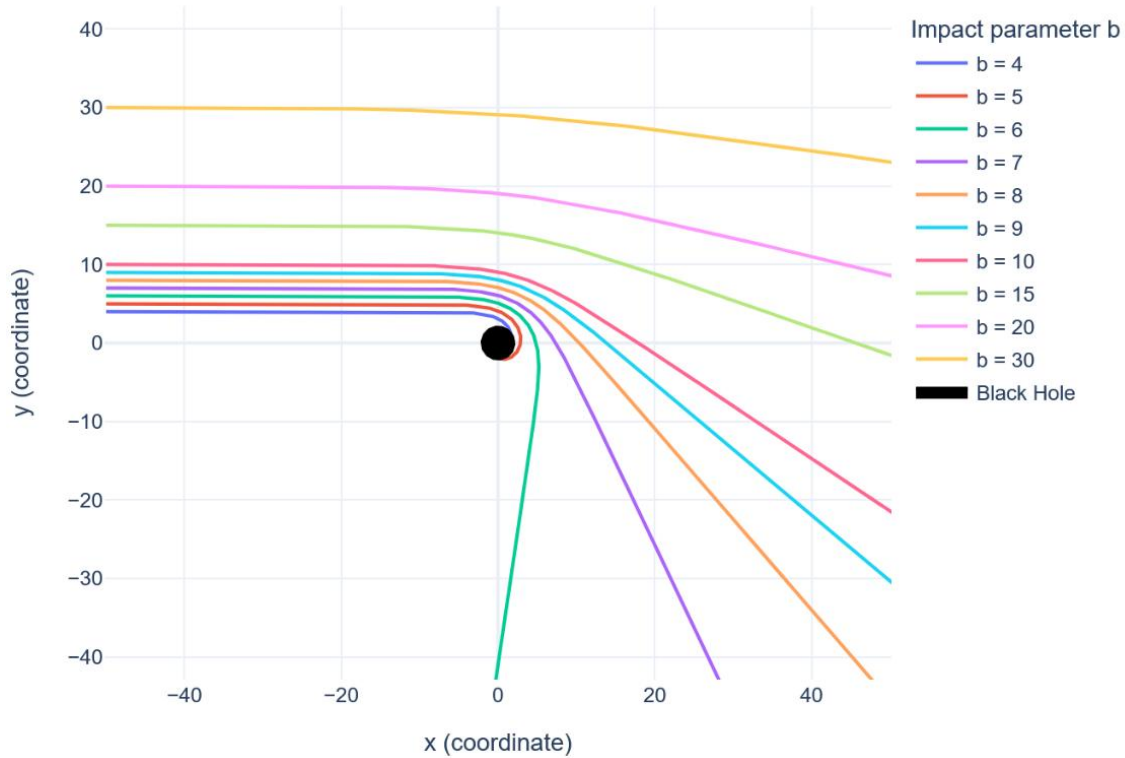


Fig.1. The plot of Light Rays Around a Schwarzschild Black Hole ( $M = 1.0$ )

### 4.2. Deflection Angle vs. Impact Parameter

I compare:

- **Theoretical:**  $\Delta\phi_{\text{th}} = 4M/b$ .
- **Numerical:** estimated by measuring the asymptotic incoming and outgoing slopes of the trajectory and computing

$$\Delta\phi_{\text{num}} = \arctan(-\text{slope}).$$

For  $b \gtrsim 10$ , numerical and theoretical agree within  $< 1\%$ . As  $b$  decreases toward the critical value, nonlinear effects introduce larger deviations; at  $b = 6$  the relative error is a few percent, growing rapidly for  $b < 5.5$ .

### 4.3. Sources of Error

- **Integration step size** ( $\Delta\lambda$ ): fixed-step sampling of output can poorly resolve rapid curvature near periapsis, leading to slope-estimation error.
- **Slope estimation**: using only two points near the end of the trajectory amplifies noise; a least-squares fit to the outgoing leg would reduce variance.

## 5. Conclusions

I have successfully implemented and visualized light bending around a Schwarzschild black hole, reproducing the classical  $4M/b$  law in the weak-field regime. The interactive dashboard highlights how deflection grows with mass and falls off with  $b$ , and illustrates the capture threshold at  $b = 3\sqrt{3}M$ .

### Possible Improvements:

- **Extended domain**: increase  $\lambda_{\max}$  until the angle converges to a desired precision.
- **Robust asymptotic fitting**: perform linear regression on the far-field segment of  $(x, y)$  to extract the deflection angle more accurately.
- **Higher-order methods**: implement higher-order integrators tailored to geodesic equations.

Such improvements would reduce numerical error, extend validity closer to the photon-sphere, and provide even more faithful simulations of strong-field lensing.

## 6. References

- Numerical integration of light paths in a Schwarzschild metric, *theconfused.me* (2017), available at <https://theconfused.me/blog/numerical-integration-of-light-paths-in-a-schwarzschild-metric/>