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Report – Numerical Simulation of Light-Ray Deflection in Schwarzschild Space-Time

1. Introduction

Gravitational lensing - the bending of light by massive bodies - provides one of the most striking confirmations of general relativity. In this project, I simulate photon trajectories around a non-rotating (Schwarzschild) black hole by numerically integrating the null geodesic equations. The primary goal is to reproduce the classic result that light passing at impact parameter b is deflected by an angle

$$\Delta\phi \approx \frac{4M}{b}$$

in geometrized units ($c = G = 1$). I build an interactive dashboard (using Dash and Plotly) that allows the user to vary the black-hole mass M and visualize both individual light-ray trajectories and the dependence of $\Delta\phi$ on b.

2. Mathematical Formulation

2.1. Schwarzschild Metric and Null Geodesics

In Schwarzschild coordinates (t, r, θ, ϕ), a spherically symmetric, static black hole of mass M has line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Restricting motion to the equatorial plane ($\theta = \pi/2$), null geodesics ($ds^2 = 0$) satisfy two first integrals:

$$E = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\lambda}, \qquad L = r^2\frac{d\phi}{d\lambda},$$

where E, L are the conserved energy and angular momentum per unit mass, and λ is just a parameter that increases along the light ray's path. Eliminating $dt/d\lambda$ and using $ds^2 = 0$ yields the radial equation (see [theconfused.me, 2017]):

$$\frac{d^2r}{d\lambda^2} = \frac{L^2}{r^4}(r - 3M).$$

Thus we have the system of three first-order ODEs for variables $\{r, \dot{r}, \phi\}$:

$$\begin{cases} \dfrac{dr}{d\lambda} = \dot{r}, \\ \dfrac{d\dot{r}}{d\lambda} = \dfrac{L^2}{r^4}(r - 3M), \\ \dfrac{d\phi}{d\lambda} = \dfrac{L}{r^2}. \end{cases}$$

We adopt geometrized units throughout, so M and all lengths are in the same unit.

2.2. Impact Parameter and Initial Conditions

The impact parameter b is the asymptotic closest-approach distance for a straight-line light ray. We launch photons from a point $(x_0, y_0) = (-x_{\text{init}}, b)$ with an initial velocity directed along $+x$. Converting to polar coordinates,

$$r_0 = \sqrt{x_{\text{init}}^2 + b^2}, \qquad \phi_0 = \arccos(x_{\text{init}}/r_0),$$

and components of the velocity in (r, ϕ) form are chosen so that the ray is initially moving purely in the $+x$ direction. One solves

$$\dot{r_0} = \cos\phi_0, \qquad \dot{\phi_0} = -\frac{\sin\phi_0}{r_0},$$

then sets $L = r_0^2 \dot{\phi}_0$.

3. Numerical Methods

3.1. `solve_ivp` (RK45 by default)

I use SciPy's solve_ivp routine, which implements an adaptive Runge-Kutta method (RK45), to integrate the ODE system from from $\lambda = 0$ to $\lambda = \lambda_{\max}$. The trajectory is sampled uniformly in λ with step $\Delta\lambda = 0.05$.

3.2. Range-Kutta 4-th Order (`rk4`)

This classic fixed-step method evaluates the derivative four times per step to compute a weighted average. It provides higher accuracy than Euler while remaining efficient for smooth trajectories. Step size is also set to $\Delta\lambda = 0.05$.

3.3. Explicit Euler Method (`euler`)

This first-order method updates the solution using a single slope estimate at each step. It is computationally simple but less accurate and less stable, especially near the black hole's photon sphere.

Trajectories are computed for a range of impact parameters b, then transformed from polar coordinates (r, ϕ) to Cartesian coordinates (x, y) using:

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

4. Results and Error Analysis

4.1. Photon Trajectories

For a representative mass $M = 1.0$, trajectories with various b values from 4 to 30 clearly show increasing deflection for smaller b, and capture (plunge into the horizon) when $b < b_{\text{crit}} = 3\sqrt{3}M \approx 5.20$ (see Fig.1,2).

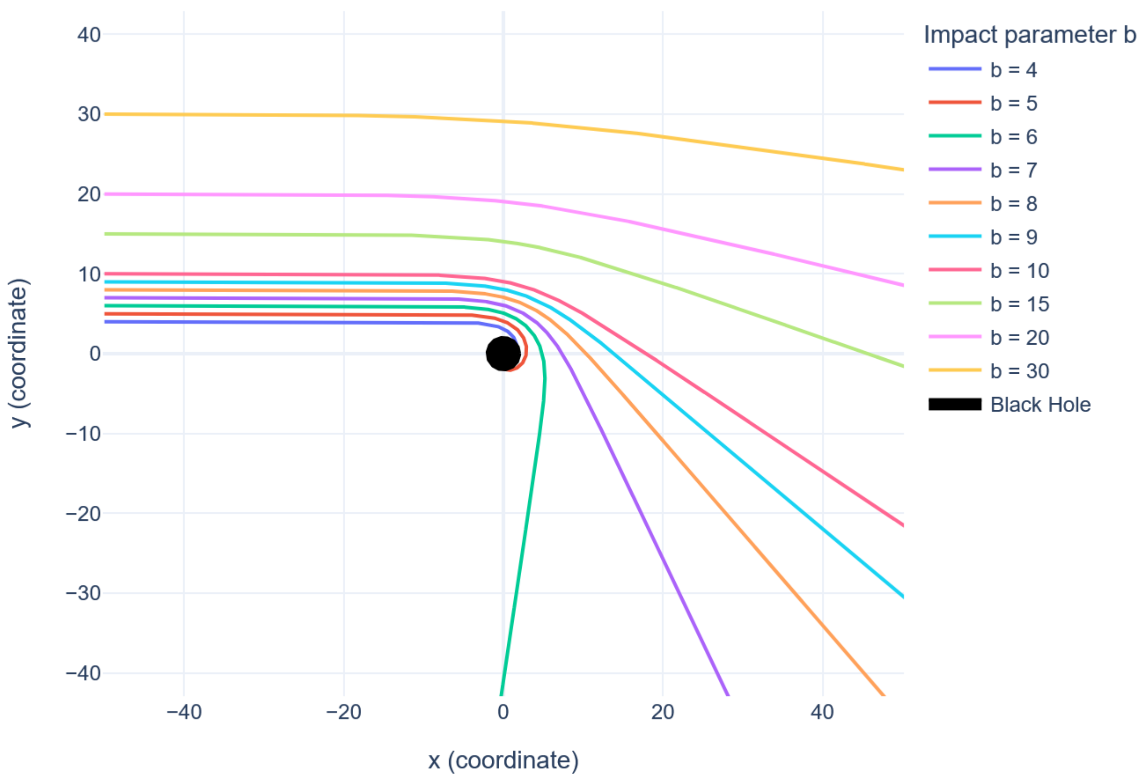

Fig.1. The plot of Light Rays Around a Schwarzschild Black Hole ($M = 1.0$, `rk4` method)

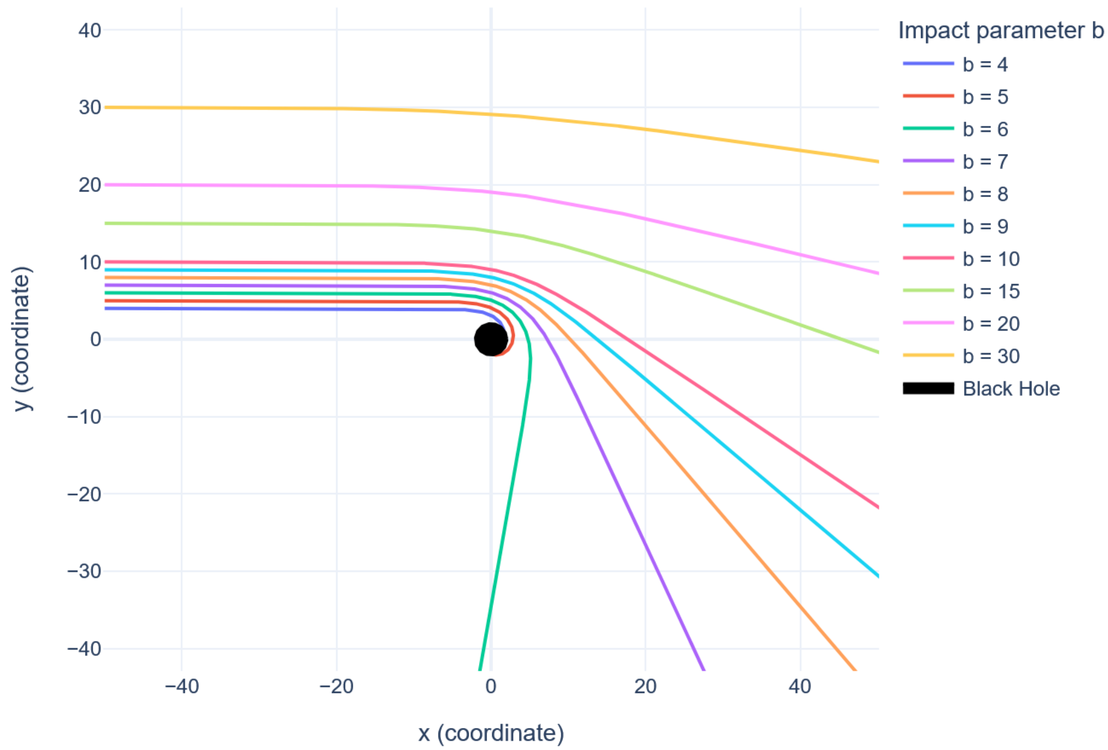

Fig.2. The plot of Light Rays Around a Schwarzschild Black Hole ($M = 1.0$, `euler` method)

4.2. Deflection Angle vs. Impact Parameter

I compare:

- **Theoretical:** $\Delta\phi_{\text{th}} = 4M/b$.
- **Numerical:** estimated by measuring the asymptotic incoming and outgoing slopes of the trajectory and computing

$$\Delta\phi_{\text{num}} = \arctan(-\text{slope}).$$

For $b \gtrsim 10$, numerical and theoretical agree within $< 1\%$. As b decreases toward the critical value, nonlinear effects introduce larger deviations; at $b = 6$ the relative error is a few percent, growing rapidly for $b < 5.5$.

4.3. Sources of Error

- **Integration step size** $(\Delta\lambda)$: fixed-step sampling of output can poorly resolve rapid curvature near periapsis, leading to slope-estimation error.
- **Slope estimation:** using only two points near the end of the trajectory amplifies noise; a least-squares fit to the outgoing leg would reduce variance.

5. Conclusions

I have successfully implemented and visualized light bending around a Schwarzschild black hole, reproducing the classical $4M/b$ law in the weak-field regime. The interactive dashboard highlights how deflection grows with mass and falls off with b, and illustrates the capture threshold at $b = 3\sqrt{3}M$.

Possible Improvements:

- Extended domain: increase $\lambda_{\max}$ until the angle converges to a desired precision.
- Robust asymptotic fitting: perform linear regression on the far-field segment of (x, y) to extract the deflection angle more accurately.
- Higher-order methods: implement higher-order integrators tailored to geodesic equations.

Such improvements would reduce numerical error, extend validity closer to the photon-sphere, and provide even more faithful simulations of strong-field lensing.

6. References

- Numerical integration of light paths in a Schwarzschild metric, *theconfused.me* (2017), available at https://theconfused.me/blog/numerical-integration-of-light-paths-in-a-schwarzschild-metric/