ØV8 — DFT

Oppgave 1 - DFT fra definisjon

a) Vi har $x(n) = \{1,2,1,2\}$

$$x(0) = 1$$
$$x(1) = 2$$
$$x(2) = 1$$
$$x(3) = 2$$

DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi}{N}kn}, \qquad N = 4$$

$k = 0$ gir

$$X(0) = \sum_{n=0}^{3} x(n)e^{0} = x(0)e^{0} + x(1)e^{0} + x(2)e^{0} + x(3)e^{0}$$

$$X(0) = 1 + 2 + 1 + 2 = 6$$

$k = 1$ gir

$$X(1) = \sum_{n=0}^{3} x(n)e^{-\frac{j\pi}{2}n} = x(0)e^{0} + x(1)e^{-\frac{j\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-\frac{3j\pi}{2}}$$

Bruker Eulers identitet

$$e^{-\frac{j\pi}{2}} = \cos\left(-\frac{\pi}{2}\right) + jsin\left(-\frac{\pi}{2}\right) = 0 - j = -j$$
$$e^{-j\pi} = \cos(-\pi) + jsin(-\pi) = -1 + 0 = -1$$
$$e^{-\frac{3j\pi}{2}} = \cos\left(-\frac{3\pi}{2}\right) + jsin\left(-\frac{3\pi}{2}\right) = 0 + j = j$$
$$X(1) = 1 - 2j - 1 + 2j = 0$$

$k = 2$ gir

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0)e^{0} + x(1)e^{-j\pi} + x(2)e^{-j\pi 2} + x(3)e^{-j\pi 3}$$

Bruker Eulers identitet

$$e^{-j\pi} = \cos(-\pi) + jsin(-\pi) = -1 + 0 = -1$$
$$e^{-j\pi 2} = \cos(-2\pi) + jsin(-2\pi) = 1 + 0 = 1$$
$$e^{-j\pi 3} = \cos(-3\pi) + jsin(-3\pi) = -1 + 0 = -1$$
$$X(2) = 1 - 2 + 1 - 2 = -2$$

$k = 3$ gir

$$X(3) = \sum_{n=0}^{3} x(n)e^{-\frac{3j\pi}{2}n} = x(0)e^{0} + x(1)e^{-\frac{3j\pi}{2}} + x(2)e^{-3j\pi} + x(3)e^{-\frac{9j\pi}{2}}$$

Bruker Eulers identitet

$$e^{-\frac{3j\pi}{2}} = \cos\left(-\frac{3\pi}{2}\right) + jsin\left(-\frac{3\pi}{2}\right) = 0 + j = j$$

$$e^{-j\pi 3} = \cos(-3\pi) + jsin(-3\pi) = -1 + 0 = -1$$

$$e^{-\frac{9j\pi}{2}} = \cos\left(-\frac{9\pi}{2}\right) + jsin\left(-\frac{9\pi}{2}\right) = 0 - j = -j$$

$$X(3) = 1 + 2j - 1 - 2j = 0$$

b) Vi har $x(n) = \{2,1,3,0,4\}$

$$x(0) = 2$$

$$x(1) = 1$$

$$x(2) = 3$$

$$x(3) = 0$$

$$x(4) = 4$$

DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi}{N}kn}, \qquad N = 5$$

$k = 0$ gir

$$X(0) = \sum_{n=0}^{4} x(n)e^{0} = x(0)e^{0} + x(1)e^{0} + x(2)e^{0} + x(3)e^{0} + x(4)e^{0}$$

$$X(0) = 2 + 1 + 3 + 0 + 4 = 10$$

$k = 1$ gir

$$X(1) = \sum_{n=0}^{4} x(n)e^{-\frac{2j\pi}{5}n}$$

$$X(1) = x(0)e^{0} + x(1)e^{-\frac{2j\pi}{5}n} + x(2)e^{-\frac{2j\pi}{5}n} + x(3)e^{-\frac{2j\pi}{5}n} + x(4)e^{-\frac{2j\pi}{5}n}$$

$$X(1) = 2 + e^{-\frac{2j\pi}{5}} + 3e^{-\frac{4j\pi}{5}} + 4e^{-\frac{8j\pi}{5}}$$

Bruker Eulers identitet

$$e^{-\frac{2j\pi}{5}} = \cos\left(-\frac{2\pi}{5}\right) + jsin\left(-\frac{2\pi}{5}\right) = \frac{\sqrt{2}\sqrt{3-\sqrt{5}} - \sqrt{2}\sqrt{5+\sqrt{5}}j}{4}$$

$$e^{-\frac{4j\pi}{5}} = \cos\left(-\frac{4\pi}{5}\right) + jsin\left(-\frac{4\pi}{5}\right) = \frac{-1-\sqrt{5} - \sqrt{2}\sqrt{5+\sqrt{5}}j}{4}$$

$$e^{-\frac{8j\pi}{5}} = \cos\left(-\frac{8\pi}{5}\right) + jsin\left(-\frac{8\pi}{5}\right) = \frac{\sqrt{2}\sqrt{3-\sqrt{5}} + \sqrt{2}\sqrt{5+\sqrt{5}}j}{4}$$

$$X(1) = \frac{\sqrt{5}}{2} + \frac{3\sqrt{2}\left(\sqrt{5-\sqrt{5}} - \sqrt{5-\sqrt{5}}\right)}{4}j$$

$k = 2$ gir

$$X(2)=\sum_{n=0}^{4}x(n)e^{-\frac{4j\pi}{5}n}=x(0)e^{0}+x(1)e^{-\frac{4j\pi}{5}}+x(2)e^{-\frac{8j\pi}{5}}+x(3)e^{-\frac{12j\pi}{5}}+x(4)e^{-\frac{16j\pi}{5}n}$$

$$X(2)=\sum_{n=0}^{4}x(n)e^{-\frac{4j\pi}{5}n}=2+e^{-\frac{4j\pi}{5}}+3e^{-\frac{8j\pi}{5}}+4e^{-\frac{16j\pi}{5}}$$

Bruker Eulers identitet

$$e^{-\frac{4j\pi}{5}}=\cos\left(-\frac{4\pi}{5}\right)+j\sin\left(-\frac{4\pi}{5}\right)=\frac{-1-\sqrt{5}}{4}-j\frac{\sqrt{10-2\sqrt{5}}}{4}$$

$$e^{-\frac{8j\pi}{5}}=\cos\left(-\frac{8\pi}{5}\right)+j\sin\left(-\frac{8\pi}{5}\right)=\frac{-1+\sqrt{5}}{4}+j\frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$e^{-\frac{16j\pi}{5}}=\cos\left(-\frac{16\pi}{5}\right)+j\sin\left(-\frac{16\pi}{5}\right)=\frac{-1-\sqrt{5}}{4}+j\frac{\sqrt{10-2\sqrt{5}}}{4}$$

$$X(2)=\frac{\sqrt{5}}{2}+\frac{\sqrt{2}\left(-\sqrt{5-\sqrt{5}}+3\sqrt{5-\sqrt{5}}\right)+4\sqrt{10-2\sqrt{5}}}{4}j$$

$k=3$ gir

$$X(3)=\sum_{n=0}^{4}x(n)e^{-\frac{6j\pi}{5}}=x(0)e^{0}+x(1)e^{-\frac{6j\pi}{5}}+x(2)e^{-\frac{12j\pi}{5}}+x(3)e^{-\frac{18j\pi}{5}}+x(4)e^{-\frac{24j\pi}{5}}$$

$$X(3)=2+e^{-\frac{6j\pi}{5}}+3e^{-\frac{12j\pi}{5}}+4e^{-\frac{24j\pi}{5}}$$

Bruker Eulers identitet

$$e^{-\frac{6j\pi}{5}}=\cos\left(-\frac{6\pi}{5}\right)+j\sin\left(-\frac{6\pi}{5}\right)=\frac{-1-\sqrt{5}}{4}+j\frac{\sqrt{10-2\sqrt{5}}}{4}$$

$$e^{-\frac{12j\pi}{5}}=\cos\left(-\frac{12\pi}{5}\right)+j\sin\left(-\frac{12\pi}{5}\right)=\frac{-1+\sqrt{5}}{4}+j\frac{-\sqrt{10-2\sqrt{5}}-\sqrt{50-10\sqrt{5}}}{8}$$

$$e^{-\frac{24j\pi}{5}n}=\cos\left(-\frac{24\pi}{5}\right)+j\sin\left(-\frac{24\pi}{5}\right)=\frac{-1-\sqrt{5}}{4}-j\frac{\sqrt{10-2\sqrt{5}}}{4}$$

$$X(3)=-2\sqrt{5}+j\frac{-6\sqrt{10-2\sqrt{5}}-3\sqrt{50-10\sqrt{5}}}{4}$$

$k=4$ gir

$$X(4)=\sum_{n=0}^{4}x(n)e^{-\frac{8j\pi}{5}}=x(0)e^{0}+x(1)e^{-\frac{8j\pi}{5}}+x(2)e^{-\frac{16j\pi}{5}}+x(3)e^{-\frac{24j\pi}{5}}+x(4)e^{-\frac{32j\pi}{5}}$$

$$X(3)=2+e^{-\frac{8j\pi}{5}}+3e^{-\frac{16j\pi}{5}}+4e^{-\frac{32j\pi}{5}}$$

Bruker Eulers identitet

$$e^{-\frac{8j\pi}{5}}=\cos\left(-\frac{8\pi}{5}\right)+j\sin\left(-\frac{8\pi}{5}\right)=\frac{\sqrt{2}\sqrt{3-\sqrt{5}}+\sqrt{2}\sqrt{5+\sqrt{5}}j}{4}$$

$$e^{-\frac{16j\pi}{5}}=\cos\left(-\frac{16\pi}{5}\right)+j\sin\left(-\frac{16\pi}{5}\right)=\frac{-1-\sqrt{5}}{4}+\frac{\sqrt{10-2\sqrt{5}}}{4}j$$

$$e^{-\frac{32j\pi}{5}n} = \cos\left(-\frac{32\pi}{5}\right) + jsin\left(-\frac{32\pi}{5}\right) = \frac{-1+\sqrt{5}}{4} - \frac{-\sqrt{10-2\sqrt{5}} - \sqrt{50-10\sqrt{5}}}{8}j$$

$$X(4) = \frac{\sqrt{5}}{2} + \frac{\sqrt{2}\left(\sqrt{5+\sqrt{5}} + \sqrt{5-\sqrt{5}} - 2\sqrt{5}\sqrt{5-\sqrt{5}}\right)}{4}j$$

c) Vi har $x(n) = \{2,2,2,2\}$

$$x(0) = 2$$

$$x(1) = 2$$

$$x(2) = 2$$

$$x(3) = 2$$

DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi}{N}kn}, \qquad N = 4$$

$k = 0$ gir

$$X(0) = \sum_{n=0}^{3} x(n)e^{0} = x(0)e^{0} + x(1)e^{0} + x(2)e^{0} + x(3)e^{0}$$

$$X(0) = 2+2+2+2 = 8$$

$k = 1$ gir

$$X(1) = \sum_{n=0}^{3} x(n)e^{-\frac{j\pi}{2}n} = x(0)e^{0} + x(1)e^{-\frac{j\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-\frac{3j\pi}{2}}$$

Bruker Eulers identitet

$$e^{-\frac{j\pi}{2}} = \cos\left(-\frac{\pi}{2}\right) + jsin\left(-\frac{\pi}{2}\right) = 0 - j = -j$$

$$e^{-j\pi} = \cos(-\pi) + jsin(-\pi) = -1 + 0 = -1$$

$$e^{-\frac{3j\pi}{2}} = \cos\left(-\frac{3\pi}{2}\right) + jsin\left(-\frac{3\pi}{2}\right) = 0 + j = j$$

$$X(1) = 2 - 2j - 2 + 2j = 0$$

$k = 2$ gir

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0)e^{0} + x(1)e^{-j\pi} + x(2)e^{-j\pi 2} + x(3)e^{-j\pi 3}$$

Bruker Eulers identitet

$$e^{-j\pi} = \cos(-\pi) + jsin(-\pi) = -1 + 0 = -1$$

$$e^{-j\pi 2} = \cos(-2\pi) + jsin(-2\pi) = 1 + 0 = 1$$

$$e^{-j\pi 3} = \cos(-3\pi) + jsin(-3\pi) = -1 + 0 = -1$$

$$X(2) = 2 - 2 + 2 - 2 = 0$$

$k = 3$ gir

$$X(3) = \sum_{n=0}^{3} x(n)e^{-\frac{3j\pi}{2}n} = x(0)e^{0} + x(1)e^{-\frac{3j\pi}{2}} + x(2)e^{-3j\pi} + x(3)e^{-\frac{9j\pi}{2}}$$

Bruker Eulers identitet

$$e^{-\frac{3j\pi}{2}} = \cos\left(-\frac{3\pi}{2}\right) + jsin\left(-\frac{3\pi}{2}\right) = 0 + j = j$$

$$e^{-j\pi 3} = \cos(-3\pi) + jsin(-3\pi) = -1 + 0 = -1$$

$$e^{-\frac{9j\pi}{2}} = \cos\left(-\frac{9\pi}{2}\right) + jsin\left(-\frac{9\pi}{2}\right) = 0 - j = -j$$

$$X(3) = 2 + 2j - 2 - 2j = 0$$

d) Vi har $x(n) = \{1,0,0,0,0,0,0,0\}$

$$x(0) = 1$$
$$x(1) = 0$$
$$x(2) = 0$$
$$x(3) = 0$$
$$x(4) = 0$$
$$x(5) = 0$$
$$x(6) = 0$$
$$x(7) = 0$$

DFT:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi}{N}kn}, \qquad N = 8$$

$k = 0$ gir

$$X(0) = \sum_{n=0}^{8} x(n)e^{0}$$

$$X(0) = x(0)e^{0} + x(1)e^{0} + x(2)e^{0} + x(3)e^{0} + x(4)e^{0} + x(5)e^{0} + x(6)e^{0} + x(7)e^{0}$$

$$X(0) = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 1$$

$k = 1$ gir

$$X(1) = \sum_{n=0}^{8} x(n)e^{-\frac{j\pi}{8}n}$$

$$X(1) = x(0)e^{0} + x(1)e^{-\frac{j\pi}{8}} + x(2)e^{-\frac{j\pi}{2}} + x(3)e^{-\frac{3j\pi}{8}} + x(4)e^{-\frac{j\pi}{2}} + x(5)e^{-\frac{5j\pi}{8}} + x(6)e^{-\frac{3j\pi}{4}} + x(7)e^{-\frac{7j\pi}{8}}$$

$$X(1) = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 1$$

$k = 2$ gir

$$X(2) = \sum_{n=0}^{8} x(n)e^{-\frac{j\pi}{4}n}$$

$$X(2) = x(0)e^{0} + x(1)e^{-\frac{j\pi}{4}} + x(2)e^{-\frac{j\pi}{2}} + x(3)e^{-\frac{3j\pi}{4}} + x(4)e^{-j\pi} + x(5)e^{-\frac{5j\pi}{4}} + x(6)e^{-\frac{3j\pi}{2}} + x(7)e^{-\frac{7j\pi}{4}}$$

$$X(2) = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 1$$

$k = 3$ gir

$$X(3) = \sum_{n=0}^{8} x(n)e^{-\frac{3j\pi}{8}n}$$

$$X(3) = x(0)e^{0} + x(1)e^{-\frac{3j\pi}{8}} + x(2)e^{-\frac{3j\pi}{4}} + x(3)e^{-\frac{9j\pi}{8}} + x(4)e^{-\frac{3j\pi}{2}} + x(5)e^{-\frac{15j\pi}{8}} + x(6)e^{-\frac{18j\pi}{8}} + x(7)e^{-\frac{24j\pi}{8}}$$

$$X(3) = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 1$$

$k = 4$ gir

$$X(4) = \sum_{n=0}^{8} x(n)e^{-\frac{j\pi}{2}n}$$

$$X(4) = x(0)e^{0} + x(1)e^{-\frac{j\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-\frac{3j\pi}{2}} + x(4)e^{-2j\pi} + x(5)e^{-\frac{5j\pi}{2}} + x(6)e^{-3j\pi} + x(7)e^{-\frac{7j\pi}{2}}$$

$$X(4) = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 1$$

$k = 5$ gir

$$X(5) = \sum_{n=0}^{8} x(n)e^{-\frac{5j\pi}{8}n}$$

$$X(5) = x(0)e^{0} + x(1)e^{-\frac{5j\pi}{8}} + x(2)e^{-\frac{5j\pi}{4}} + x(3)e^{-\frac{15j\pi}{8}} + x(4)e^{-\frac{20j\pi}{8}} + x(5)e^{-\frac{25j\pi}{8}} + x(6)e^{-\frac{30j\pi}{8}} + x(7)e^{-\frac{35j\pi}{8}}$$

$$X(5) = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 1$$

$k = 6$ gir

$$X(2) = \sum_{n=0}^{8} x(n)e^{-\frac{3j\pi}{4}n}$$

$$X(2) = x(0)e^{0} + x(1)e^{-\frac{3j\pi}{4}} + x(2)e^{-\frac{3j\pi}{2}} + x(3)e^{-\frac{9j\pi}{4}} + x(4)e^{-3j\pi} + x(5)e^{-\frac{15j\pi}{4}} + x(6)e^{-\frac{18j\pi}{4}} + x(7)e^{-\frac{21j\pi}{4}}$$

$$X(2) = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 1$$

$k = 7$ gir

$$X(7) = \sum_{n=0}^{8} x(n)e^{-\frac{7j\pi}{8}n}$$

$$X(7)=x(0)e^{0}+x(1)e^{-\frac{7j\pi}{8}}+x(2)e^{-\frac{7j\pi}{4}}+x(3)e^{-\frac{21j\pi}{8}}+x(4)e^{-\frac{28j\pi}{8}}+x(5)e^{-\frac{35j\pi}{8}}+x(6)e^{-\frac{42j\pi}{8}}+x(7)e^{-\frac{49j\pi}{8}}$$

$$X(7)=1+0+0+0+0+0+0+0=1$$

Oppgave 2 – DFT

a) Vi har $x_1(n)=4-n,\quad N=8$:

$$x(k)=\sum_{n=0}^{N-1}(4-n)e^{-\frac{j2\pi}{N}kn},\quad N=8$$

$$x(k)=4e^{0}+3\left(e^{-\frac{j2\pi}{N}k}-e^{-\frac{j2\pi}{N}(N-1)k}\right)+2\left(e^{-\frac{j2\pi}{N}2k}-e^{-\frac{j2\pi}{N}(N-2)k}\right)+\left(e^{-\frac{j2\pi}{N}3k}-e^{\frac{-j2\pi}{N}(N-3)k}\right)$$

Vi vett at $e^{0}=1$ og $e^{-\frac{j2\pi}{N}(N-n)k}=e^{-j2\pi k}\cdot e^{\frac{j2\pi}{N}nk}$, derfor

$$x(k)=4+3\left(e^{-\frac{j2\pi}{N}k}-e^{-j2\pi k}\cdot e^{\frac{j2\pi}{N}k}\right)+2\left(e^{-\frac{j2\pi}{N}2k}-e^{-j2\pi k}\cdot e^{\frac{j2\pi}{N}2k}\right)+\left(e^{-\frac{j2\pi}{N}3k}-e^{-j2\pi k}\cdot e^{\frac{j2\pi}{N}3k}\right)$$

Vi vett at $k=0,1,2,\dots,(N-1)$, derfor blir $e^{-j2\pi k}=1$, altså

$$x(k)=4+3\left(e^{-\frac{j2\pi}{N}k}-e^{\frac{j2\pi}{N}k}\right)+2\left(e^{-\frac{j2\pi}{N}2k}-e^{\frac{j2\pi}{N}2k}\right)+\left(e^{-\frac{j2\pi}{N}3k}-e^{\frac{j2\pi}{N}3k}\right)$$

Vi bruker Eulers formel, $sin\theta=\frac{e^{j\theta}-e^{-j\theta}}{2j}$, og får

$$x(k)=4-6j\,sin\left(\frac{2\pi}{N}k\right)-4j\,sin\left(\frac{2\pi}{N}2k\right)-2j\,sin\left(\frac{2\pi}{N}3k\right)$$

Setter at $N=8$

$$x(k)=4-6j\,sin\left(\frac{2\pi}{8}k\right)-4j\,sin\left(\frac{2\pi}{8}2k\right)-2j\,sin\left(\frac{2\pi}{8}3k\right)$$

$$x(k)=4-6j\,sin\left(\frac{\pi}{4}k\right)-4j\,sin\left(\frac{\pi}{2}k\right)-2j\,sin\left(\frac{3\pi}{4}k\right)$$

Der $k=0,1,2,\dots,(N-1)=0,1,2,\dots,7$.

c) Vi har $x_3(n)=6\cos^2(0.2\pi n)$, $N=10$, vi bruker formelen for dobbelte vinkler $cos^2\theta=\frac{1+cos2\theta}{2}$, og får

$$x_3(n)=\frac{6}{2}(1+\cos(2\cdot 0.2\pi n))=3+3\cos(0.4\pi n)$$

Videre bruker vi Eulers formel, $cos\theta=\frac{e^{j\theta}+e^{-j\theta}}{2}$, og får

$$x_3(n)=3+\frac{3}{2}\left(e^{j0.4\pi n}+e^{-j0.4\pi n}\right)$$

$$DFT(3)=3\big(N\delta(k)\big)$$

$$DFT(e^{-\frac{2j\pi mn}{N}}) = N\delta(k+m)$$

Da får vi

$$x_3(n) = 3\big(10\delta(k)\big) + 5(10)\delta(k-2) + 15\delta(k-8)$$

$15\delta(k+2) \rightarrow 15\delta(k-8)$ because $x(n)$ is periodic.

d) Vi har $x_4(n) = 5(0.8)^n$, $N = 16$:

$$x(k) = \sum_{n=0}^{N-1} 5(0.8)^n e^{-\frac{j2\pi}{N}kn} = 5\sum_{n=0}^{N-1}\left(0.8e^{-\frac{j2\pi}{N}k}\right)^n$$

Bruker at $\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$

$$x(k) = \frac{1-(0.8e^{-\frac{j2\pi}{N}k})^N}{1-0.8e^{-\frac{j2\pi}{N}k}}$$

Bruker at $\left(e^{-\frac{j2\pi}{N}k}\right)^N = e^{-j2\pi k} = 1$

$$x(k) = \frac{1-(0.8)^N}{1-0.8e^{-\frac{j2\pi}{N}k}}$$

Setter $N = 16$

$$x(k) = \frac{1-(0.8)^{16}}{1-0.8e^{-\frac{j\pi}{8}k}}$$

Der $k = 0,1,2,\dots,(N-1) = 0,1,2,\dots,15$.

Oppgave 3 – Sampling og aliansing

a)