Option pricing:

* Down-and-out call (T, K, B) pays $(S_T - K)_+$ if $S_u > B$ $\forall u \in [t, T]$
* S_t follows GBM $\begin{cases} dS_t = rS_t\,dt + \sigma S_t\,dW_t & \text{with constant } r \\ S_t = X & \text{Start point} \end{cases}$
* In numerical experiments, we take $t = 0$. $T = 1$ year. $r = 0.02\%$, $\sigma = 0.20$ (20% per year)
  $X = \$100$ $B = \$80$ $K = \$110$

Part I Monte Carlo

(a) WTS: $V_t = e^{-r(T-t)}\,\mathbb{E}[(S_T - K)_+ \mathbb{1}_{S_u \geq B, \forall u \in [t,T)} \mid \mathcal{F}_t]$

Proof: As our call pays $\begin{cases} (S_T - K)_+ & \text{if } S_u > B \ \forall u \in (t, T) \\ 0 & \text{else} \end{cases}$

Define Indicator $\mathbb{1}_{S_u \in B, \forall u \in [t,T)} = \begin{cases} 1, & \text{if } S_u \in B\ \forall u \in [t,T) \\ 0 & \text{else.} \end{cases}$

Payoff of our call becomes $V_T = (S_T - K)_+ \mathbb{1}_{S_u \in B, \forall u \in [t,T)}$ at T

Define Π_t to be portfolio at t that uses delta hedging Strategy at time t to replicate buying an option V_t

$$\Pi_t = \underset{\text{in risky}}{\Delta_t S_t} + \underset{\text{in risk free}}{(\Pi_t - \Delta_t S_t)} \qquad \text{where } \Delta_t = \frac{\partial C}{\partial S}(t)$$

$$d\Pi_t = \Delta_t\,dS_t + (\Pi_t - \Delta_t S_t)\,r\,dt$$

Let $D_t = e^{-rt}$ be the discount factor

$$d(D_t\Pi_t) = dD_t \cdot \Pi_t + D_t\,d\Pi_t + \cancel{dD_t\,d\Pi_t}^{\,0} \quad \text{(by Ito chain)}$$

$[dD_t = -re^{-rt}dt \Rightarrow dD_t\,d\Pi_t \sim O(dt^{3/2}) \to 0]$

$$= -r\cancel{D_t\,dt\,\Pi_t} + D_t\Delta_t(rS_t\,dt + \sigma S_t\,dW_t) + (\cancel{\Pi_t} - \Delta_t S_t)\,r\,dt\,D_t$$

$$= (D_t\cancel{\Delta_t}\,rS_t - D_t\cancel{\Delta_t}S_t r)\,dt + D_t\Delta_t\sigma S_t\,dW_t$$

$$= D_t\Delta_t\sigma S_t\,dW_t \qquad \text{Follows non-arbitrage in risk-neutral measure } (\mu_t = r)$$

$(D_t\Pi_t)$ is a martingale under current $\mathbb{Q}$ measure

By non-arbitrage we need $D_T \Pi_T = D_t \Pi_t$

So $\mathbb{E}_Q[D_T \Pi_T \mid F_t] = \mathbb{E}_Q[D_T V_T \mid F_t]$

$$D_t \Pi_t = \mathbb{E}_Q[D_T V_T \mid F_t] \quad \text{(by martingale } D_t \Pi_t\text{)}$$

$$V_t = \Pi_t = \mathbb{E}_Q\left[\frac{D_T}{D_t} V_T \mid F_t\right]$$

$$V_t = \mathbb{E}_Q[e^{-r(T-t)} V_T \mid F_t]$$

$$= e^{-r(T-t)} \mathbb{E}_Q[(S_T - K)^+ \mathbb{1}_{S_u \in B, \forall u \in [t,T]} \mid F_t]$$

(b)(c)(d) See attached .ipynb file

The price of option from 100000 Monte Carlo simulation is 4.924328

Part II PDE

(e) $\tau = \min\{\inf\{u \geq t \mid S_u = B\}, T\}$

the first time you hit the barrier if it's less than T

$$\phi(y,s) = \begin{cases} (y-K)_+ & \text{if } s = T \\ 0 & \text{if } s < T \end{cases}$$

Claim $\quad V_t = \mathbb{E}[e^{-r(\tau-t)}\phi(S_\tau,\tau) \mid S_t = x]$

Explanation: $\phi(S_\tau,\tau)$ is the pay off of the option evaluated at τ

$$\phi(S_\tau,\tau) = \begin{cases} (S_\tau-K)_+ & \text{if } \tau = T \\ 0 & \text{if } \tau < T \end{cases}$$

$$= \begin{cases} (S_T-K)_+ & \text{if not hit barrier before } T \\ 0 & \text{if hit before } T \end{cases}$$

$$= (S_T-K)_+ \mathbb{1}_{S_u \geq B \ \forall u \in (t,T)}$$

Because $\phi(S_\tau,\tau)$ is evaluated at τ our discount factor changes accordingly to $e^{-r(\tau-t)}$

According to (a), with the same reasoning

$$V_t = \mathbb{E}[e^{-r(T-t)}(S_T-K)_+ \mathbb{1}_{S_u \geq B \ \forall u \in (t,T)} \mid S_t = x]$$

$$= \mathbb{E}[e^{-r(\tau-t)} \phi(S_\tau,\tau) \mid S_t = x] \quad \blacksquare$$

In this definition, $\phi(S_\tau,\tau)$ = payoff at T of the option.

(f) V Solves

$$\begin{cases} -rV + \frac{\partial}{\partial t}V + rx\frac{\partial}{\partial x}V + \frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}V = 0 \\ V(T,x) = (x-K)_+ \quad \forall\ x > B \\ V(t,B) = 0 \quad \forall\ t < T \\ t < T \quad B < x \end{cases}$$

Here $V = V(t,x)$

Show that $\quad d(e^{-rt}V(t,S_t)) = e^{-rt}\sigma S_t \frac{\partial}{\partial x}V(t,S_t)\,dW_t$

Proof: By Ito chain rule:

$$d(e^{-rt}V(t,S_t)) = de^{-rt}\cdot V + e^{-rt}dV + de^{-rt}dV \quad (*)$$

By Ito lemma, $V = V(t,S_t)$

$$dV(t,S_t) = \partial_t V dt + \partial_s V ds + \tfrac{1}{2}\partial_{ss}V ds^2 \quad \text{(plug in ds)}$$

$$= (\partial_t V + rS\partial_s V + \tfrac{1}{2}\sigma^2S^2\partial_{ss}V)dt + \sigma S\partial_s V dW_t$$

Then we plug dV into (*) :

$$d(e^{-rt}V(t,S_t)) = -re^{-rt}dt \cdot V + e^{-rt}\left(\partial_t V + rS\partial_s V + \tfrac{1}{2}\sigma^2 s^2 \partial_{ss} V\right)dt$$

$$+\, e^{-rt}\sigma S \partial_s V dW_t + \cancel{-re^{-rt}dt\,[(\quad)dt + (\quad)dW_t]}$$

$\sim O(dt^2)$ $\sim O(dt^{3/2})$

$\to 0$ by PDE

$$= e^{-rt}\left(\cancel{-rV + \partial_t V + rS\partial_s V + \tfrac{1}{2}\sigma^2 s^2 \partial_{ss}V}\right)dt$$

$$+\, e^{-rt}\sigma S \partial_s V dW_t$$

☆ Because we are solving PDE here, there should not be any stochastic differential. What I meant by ∂_s is actually ∂_x, not stochastic

$$= e^{-rt}\sigma S_t \partial_s V(t,S_t)dW_t \quad (\Delta) \quad ∎$$

$$(= e^{-rt}\sigma S_t V_x(t,S_t)dW_t)$$

(g) Deduce $e^{-r\tau}V(\tau,S_\tau) - e^{-rt}V(t,S_t) = \int_t^\tau \sigma e^{-rs} S_s \partial_x V(s,S_s)dW_s$

Derivation: From (Δ) take integral $\int_t^\tau$ on both side

$$\int_t^\tau d(e^{-rt}V(t,S_t)) = \int_t^\tau e^{-rs}\sigma S_s \partial_s V(s,S_s)dW_s$$

That is, $e^{-r\tau}V(\tau,S_\tau) - e^{-rt}V(t,S_t) = \int_t^\tau e^{-rs}\sigma S_s \partial_s V(s,S_s)dW_s$

$$(e^{-r\tau}V(\tau,S_\tau) - e^{-rt}V(t,S_t) = \int_t^\tau e^{-rs}\sigma S_s V_x(s,S_s)dW_s) \quad ∎$$

$(\square)$

(h) Dynkin's theorem / Doob's optional stopping time thm.

$$\mathbb{E}\left[\int_t^\tau \sigma e^{-rs} S_s V_x(s,S_s)dW_s \,\middle|\, S_t = x\right] = 0$$

Show

$$V(t,x) = \mathbb{E}\left[e^{-r(\tau-t)}\phi(S_\tau,\tau) \,\middle|\, S_t = x\right]$$

Proof: Take expectation $\mathbb{E}[\,\cdot \mid S_t = x]$ on both side of $(\square)$

$$\mathbb{E}\left[e^{-r\tau}V(\tau,S_\tau) - e^{-rt}V(t,S_t) \,\middle|\, S_t = x\right]$$

$$= \mathbb{E}\left[\int_t^\tau \sigma e^{-rs} S_s V_x(s,S_s)dW_s \,\middle|\, S_t = x\right] = 0$$

$$\mathbb{E}\left[e^{-r\tau}V(\tau,S_\tau) \mid S_t = x\right] - e^{-rt}V(t,x) = 0$$

$$V(t,x) = e^{rt}\,\mathbb{E}\left[e^{-r\tau}V(\tau,S_\tau) \mid S_t = x\right]$$

$$= \mathbb{E}\left[e^{-r(\tau-t)}V(\tau,S_\tau) \mid S_t = x\right] \quad ∎$$

(Rmk): We showed that $V(t,x) = \mathbb{E}[e^{-r(\tau-t)}V(\tau,S_\tau) \mid S_t = x]$ solve the PDE. To show uniqueness, we need $V \underset{x\to\infty}{\sim} x - e^{-r(T-t)}K$

(i) Explain the growth condition $\lim_{x \to +\infty} \frac{V(t,x)}{x - e^{-r(T-t)}K} = 1$

which we need for uniqueness of solution

Explanation: As $S_t = x \to \infty$

$$V(t,x) \sim x - e^{-r(T-t)}K$$

(1) Buying the option's value is approximately replicated by a portfolio in which the investor borrow present value of K from bank and holding the stock.

(2) In other words. as $x \to \infty$, initially the stock price is really high, so any option with strike K finite would be far "in the money". The value of that option $V(t,x) \to \infty$ as well.

(3) we had from (h) that

$$V(t,x) = \mathbb{E}[e^{-r(\tau-t)} V(\tau, S_\tau) | S_t = x]$$

Now when $x \to \infty$, for finite barrier B, it is unlikely that $\tau < T$. So:

$$V(t,x) = \mathbb{E}[e^{-r(T-t)} (S_T - K)_+ | S_t = x]$$

With finite strike K, $S_T - K > 0$ with probability 1

$$V(t,x) = e^{-r(T-t)} \mathbb{E}[S_T | S_t = x] - e^{-r(T-t)} K$$

$$= x - e^{-r(T-t)} K$$

that is why the growth condition make sense.

Numerical solution of PDE

Solve for $X \in (B, R)$ for $R = 300$, $N_x = 2200$ = # of points of x

$N_t = 252$ = # points of t , $\Delta x = \frac{R-B}{N_x}$, $\Delta t = \frac{T-t}{N_t}$

$x_k = B + k\Delta x$ for $k = 0 \ldots N_x$. $t_j = t + j\Delta t$ for $j = 0 \ldots N_t$

$V(t_j, x_k) = V_j^k$ our notation

(j) Check BC of PDE
$$\begin{cases} V(T, x) = (x-K)_+ & ① \\ V(t, B) = 0 \quad \forall t < T & ② \end{cases}$$

Can be written as
$$\begin{cases} V_{N_t}^{k} = (x_k - K)_+ = V(t_{N_t}, x_k),\ k = 0 \ldots N_x & (1) \\ V_j^0 = 0 = V(t_j, x_0) \quad \forall j = 0 \ldots N_t & (2) \\ V_j^{N_x} = R - e^{-r(T-t_j)}K \quad \forall j = 0 \ldots N_t & (3) \\ \quad = V(t_j, x_{N_x}) \end{cases}$$

(1) $t_{N_t} = T$. $x_k = x$ (at time T)

$\Rightarrow V(t_{N_t}, x_k) = V(T, x) = (x-K)_+$ Correspond to ①

(2) $x_0 = B$, $t_j = t + j\Delta t \in [t, T]$

$\Rightarrow V(t_j, x_0) = \underset{t' \in (t,T)}{V(t', B)} = 0$ Correspond to ②

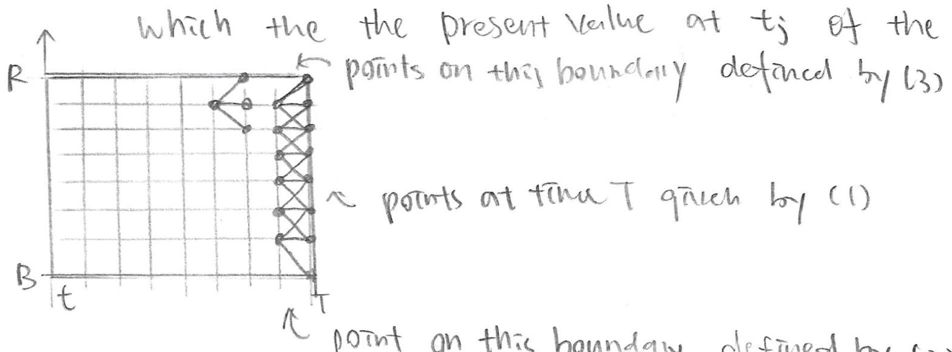
(3) $x_{N_x} = R$

$\Rightarrow V(t_j, x_{N_x}) = \underset{t' \in (t,T]}{V(t', R)} = R - e^{-r(T-t')}K$.

this is the additional boundary condition we set for the numerical system : the values on the upper boundary of the grid will be set according to

$V(t_j, R) = R - e^{-r(T-t_j)}K$. $\forall t_j \in [t, T]$

which the the present value at t_j of the value of option



(k) Show the descretization of PDE gives

$$V_j^k = V(t_j, x_k) = \left(1 + r\Delta t + \sigma^2 \frac{\Delta t}{(\Delta x)^2} x_k^2\right) V_{j-1}^k - \Delta t\left(\frac{r x_k}{2\Delta x} + \frac{1}{2(\Delta x)^2}\sigma^2 x_k^2\right) V_{j-1}^{k+1}$$

$$- \Delta t\left(-\frac{r x_k}{2\Delta x} + \frac{1}{2(\Delta x)^2}\sigma^2 x_k^2\right) V_{j-1}^{k-1} \qquad \forall\ k = 1 \cdots N_x - 1$$

$$j = 0 \cdots N_t - 1$$

Proof. By Taylor expansion, we have:

$$\partial_t V(t_{j-1}, x_k) = \frac{V_j^k - V_{j-1}^k}{\Delta t}$$

(Implicit Euler scheme)

$$\partial_x V(t_{j-1}, x_k) = \frac{V_{j-1}^{k+1} - V_{j-1}^{k-1}}{2\Delta x}$$

$$\partial_{xx} V(t_{j-1}, x_k) = \frac{V_{j-1}^{k+1} - 2V_{j-1}^k + V_{j-1}^{k-1}}{\Delta x^2}$$

plug those into PDE we have in (f):

$$0 = -r V_{j-1}^k + \frac{V_j^k - V_{j-1}^k}{\Delta t} + r x_k \frac{V_{j-1}^{k+1} - V_{j-1}^{k-1}}{2\Delta x} + \frac{1}{2}\sigma^2 x_k^2 \frac{V_{j-1}^{k+1} - 2V_{j-1}^k + V_{j-1}^{k-1}}{\Delta x^2}$$

where $x_k = B + k\Delta x$

$$-\frac{1}{\Delta t} V_j^k = \left(-r - \frac{1}{\Delta t} + \frac{1}{2}\sigma^2 x_k^2 \frac{-2}{\Delta x^2}\right) V_{j-1}^k + \left(r x_k \frac{-1}{2\Delta x} + \frac{1}{2}\sigma^2 x_k^2 \frac{1}{\Delta x^2}\right) V_{j-1}^{k-1}$$

$$+ \left(r x_k \frac{1}{2\Delta x} + \frac{1}{2}\sigma^2 x_k^2 \frac{1}{\Delta x^2}\right) V_{j-1}^{k+1}$$

Multiply both side by $(-\Delta t)$

$$V_j^k = \left(r\Delta t + 1 + \sigma^2 x_k^2 \frac{\Delta t}{(\Delta x)^2}\right) V_{j-1}^k - \Delta t\left(-\frac{r x_k}{2\Delta x} + \frac{\sigma^2 x_k^2}{2(\Delta x)^2}\right) V_{j-1}^{k-1}$$

$$- \Delta t\left(\frac{r x_k}{2\Delta x} + \frac{\sigma^2 x_k^2}{2(\Delta x)^2}\right) V_{j-1}^{k+1} \qquad (\Diamond) \quad \blacksquare$$

(l) Define $(N_x+1)\times 1$ vectors $V_j = \begin{pmatrix} V_j^0 \\ \vdots \\ V_j^{N_x} \end{pmatrix}$

Write the system in the form of $V_j = M(V_{j-1} - C_{j-1})$
explicitly define M. C_{j-1} is a correction term for the first and last entry of the vector V_{j-1}

$\Rightarrow$ Write M as by $(\Diamond)$

$$\begin{pmatrix} r\Delta t + 1 + \sigma^2 x_0^2 \frac{\Delta t}{(\Delta x)^2} & -\Delta t\left(-\frac{r x_0}{2\Delta x} + \frac{\sigma^2 x_0^2}{2(\Delta x)^2}\right) & & 0 \\ -\Delta t\left(-\frac{r x_1^2}{2\Delta x} + \frac{\sigma^2 x_1^2}{2(\Delta x)}\right) & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}$$

But here since we want to satisfy the BC

$$\begin{cases} V_j^0 = 0 \\ V_j^{N_x} = R - e^{-r(T-t_j)} K. \end{cases}$$

We let the first row of M to be $(1, 0, \cdots 0)$

last row of M to be $(0, 0, \cdots, 1)$

Then if we denote for $k = 1 \cdots N_x - 1$

$$a_k = -\Delta t \left(-\frac{r x_k}{2\Delta x} + \frac{\sigma^2 x_k^2}{2(\Delta x)^2}\right)$$

$$b_k = 1 + r\Delta t + \sigma^2 x_k^2 \frac{\Delta t}{(\Delta x)^2}$$

$$c_k = -\Delta t \cdot \left(\frac{r x_k}{2\Delta x} + \frac{\sigma^2 x_k^2}{2(\Delta x)^2}\right)$$

we can rewrite (Δ) as:

$$V_j^k = a_k V_{j-1}^{k-1} + b_k V_{j-1}^{k} + c_k V_{j-1}^{k+1} \qquad (\Delta)$$

Then M can be defined as

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & & 0 & 0 \\ a_1 & b_1 & c_1 & 0 & \cdots & & 0 \\ 0 & a_2 & b_2 & c_2 & \cdots & & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & \cdots & & 0 & a_{N_x-1} & b_{N_x-1} & c_{N_x-1} \\ 0 & \cdots & & & 0 & 0 & 1 \end{pmatrix}$$

With $(1, b_1 \cdots b_{N_x-1}, 1)$ on diagonal

$(0, c_1 \cdots c_{N_x-1})$ on upper band

$(a_1 \cdots a_{N_x-1}, 0)$ on lower band

Then we write $V_j = M \cdot (V_{j-1} - C_{j-1})$

That is,

$$\begin{pmatrix} V_j^0 \\ \vdots \\ V_j^{N_x} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & & 0 \\ a_1 & b_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & a_{N_x-1} & b_{N_x-1} & c_{N_x-1} \\ & & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_{j-1}^0 - C_{j-1}^0 \\ V_{j-1}^1 \\ \vdots \\ V_{j-1}^{N_x-1} \\ V_{j-1}^{N_x} - C_{j-1}^{N_x} \end{pmatrix}$$

Boundaries: (1) $V_{j-1}^{0} - C_{j-1}^{0} = V_j^{0} = 0$ for $j = 1 \cdots N_t$

$$\Rightarrow C_{j-1}^{0} = V_{j-1}^{0} = 0$$

(2) $V_{j-1}^{N_x} - C_{j-1}^{N_x} = V_j^{N_x} = R - e^{-r(T-t_j)}K.$ for $j = 1 \cdots N_t$

$$\begin{aligned}\Rightarrow C_{j-1}^{N_x} &= V_{j-1}^{N_x} - V_j^{N_x} \\ &= (\cancel{R} - e^{-r(T-t_{j-1})}K) - (\cancel{R} - e^{-r(T-t_j)}K) \\ &= -e^{-rT} \cdot e^{rt_{j-1}}K + e^{-rT} \cdot e^{rt_j}K \\ &= e^{-rT}K(e^{rt_j} - e^{rt_{j-1}})\end{aligned}$$

☆ So we can write the PDE as

$$V_j = M(V_{j-1} - C_{j-1})$$

where
$$M = \begin{pmatrix} 1 & 0 & 0 & & & 0 \\ a_1 & b_1 & c_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & & & \\ 0 & & a_{N_x-1} & b_{N_x-1} & c_{N_x-1} \\ & & 0 & 0 & 1 \end{pmatrix} \qquad C_{j-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ e^{-rT}K(e^{rt_j} - e^{rt_{j-1}}) \end{pmatrix}$$

In which
$$\begin{cases} a_k = -\Delta t\left(-\dfrac{r x_k}{2\Delta x} + \dfrac{\sigma^2 x_k^2}{2(\Delta x)^2}\right) \\ b_k = 1 + r\Delta t + \sigma^2 x_k^2 \dfrac{\Delta t}{(\Delta x)^2} \\ c_k = -\Delta t\left(\dfrac{r x_k}{2\Delta x} + \dfrac{\sigma^2 x_k^2}{2(\Delta x)^2}\right) \end{cases}$$
for $k = 1 \cdots N_x - 1$

Alternatively we can write $V_{j-1} = M^{-1}V_j + C_{j-1}$

(m) See .ipynb file. the price of option at time t is 4.916733 by numerical method.

Analytical Solution of PDE

(n) Claim : $V(t,x) = C_K(t,x) - \left(\frac{x}{B}\right)^{2\partial} C_K\left(\frac{B^2}{x}, t\right)$

is the solution to the PDE

where $C_K(t,x)$ = Black_Sholes price for call (T,K)

$$\partial = \frac{1}{2}\left(1 - \frac{2r}{\sigma^2}\right)$$

Check : (1) $-rV(t,x) + V_t(t,x) + rxV_x(t,x) + \frac{1}{2}\sigma^2 x^2 V_{xx}(t,x) = 0$

(2) $V(T,x) = (x-K)_+ \quad \forall\, x > B$

(3) $V(t,B) = 0 \quad \forall\, t < T$

(1) As we know :

$$C_K(t,x) = xN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where $d_1 = \dfrac{\log(x/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$\Rightarrow \quad \partial_t V = \partial_t C(t,x) - \left(\frac{x}{B}\right)^{2\partial} \partial_t C\left(t, \frac{B^2}{x}\right)$

Here $\partial_t C(t,x) = x\,\partial_t N(d_1) - (Ke^{-r(T-t)}\partial_t N(d_2)$

$\qquad + Kre^{-r(T-t)} N(d_2))$

where $\partial_t N(d_1) = N'(d_1) \cdot \frac{dd_1}{dt}$ (chain rule)

$\partial_t N(d_2) = N'(d_2) \frac{dd_2}{dt}$

$\qquad = N'(d_2)\left(\frac{dd_1}{dt} - \frac{\sigma}{2\sqrt{T-t}}\right)$

because we have below:

$\frac{dd_2}{dt} = \frac{dd_1}{dt} + \frac{\sigma}{2\sqrt{T-t}}$, $\partial_t N(d_2) = N'(d_1)\frac{dd_1}{dt} - \frac{\sigma}{2\sqrt{T-t}}$

$\Rightarrow \quad \partial_t C(t,x) = x N'(d_1)\frac{dd_1}{dt} - Ke^{-r(T-t)}N'(d_2)\left(\frac{dd_1}{dt} + \frac{\sigma}{2\sqrt{T-t}}\right)$

$\qquad = \frac{dd_1}{dt}\left(xN'(d_1) - Ke^{-r(T-t)}N'(d_2)\right)$

$\qquad - \frac{\sigma Ke^{-r(T-t)}}{2\sqrt{T-t}} N'(d_2) - Kre^{-r(T-t)}N(d_2)$

△ Introduce Lemma : $X N'(d_1) = K e^{-r(T-t)} N'(d_2)$

Proof. $d_2^2 - d_1^2 = (d_2 - d_1)(d_2 + d_1)$

$$= (-\sigma\sqrt{T-t})(2d_1 - \sigma\sqrt{T-t})$$

$$= (-\sigma\sqrt{T-t})\left(\frac{\log(\frac{X}{K}) + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\cdot 2 - \sigma\sqrt{T-t}\right)$$

$$= -2\left(\log\left(\frac{X}{K}\right) + r(T-t)\right)$$

$$\log\left(\frac{N'(d_1)}{N'(d_2)}\right) = -\frac{d_1^2}{2} + \frac{d_2^2}{2}$$

$$= \frac{1}{2}(d_2^2 - d_1^2)$$

$$= -\left(\log\frac{X}{K} + r(T-t)\right)$$

$$\Rightarrow \frac{N'(d_1)}{N'(d_2)} = e^{-(\log(\frac{X}{K}) + r(T-t))}$$

$$= e^{-r(T-t)}\frac{K}{X}$$

(△)

$$\Rightarrow X N'(d_1) = e^{-r(T-t)} K N'(d_2) \quad \blacksquare$$

$$\therefore \partial_t C(t,X) = \frac{dd_1}{dt}\cdot\overset{\text{by }(\triangle)}{(0)} - \frac{Ke^{-r(T-t)}\sigma}{2\sqrt{T-t}}N'(d_2) - Kre^{-r(T-t)}N(d_2)$$

$$= -\frac{\sigma K e^{-r(T-t)}N'(d_2)}{2\sqrt{T-t}} - Kre^{-r(T-t)}N(d_2)$$

by (△) ↓

$$= -\frac{XN'(d_1)\sigma}{2\sqrt{T-t}} - Kre^{-r(T-t)}N(d_2) \qquad (H)$$

$$\partial_X C(t,X) = N(d_1) + XN'(d_1)\frac{dd_1}{dX} - Ke^{-r(T-t)}N'(d_2)\frac{dd_2}{dX}$$

(by $\frac{dd_1}{dX} = \frac{dd_2}{dX}$) ↓

$$= N(d_1) + \frac{dd_1}{dX}\left(XN'(d_1) - Ke^{-r(T-t)}N'(d_2)\right)$$

by (△) ↓

$$= N(d_1) + 0 = N(d_1) \qquad (\triangle)$$

$$\partial_{XX} C(t,X) = \partial_X N(d_1) = N(d_1)\frac{dd_1}{dX} = \frac{N'(d_1)\frac{K}{X}\cdot\frac{1}{K}}{\sigma\sqrt{T-t}} = \frac{N'(d_1)}{\sigma\sqrt{T-t}\,X}$$

Now we check that PDE(I)' holds for $V = C(t,X)$ (P)

$$-\cancel{rXN(d_1)} + \cancel{rKe^{-r(T-t)}N(d_2)} - \cancel{\frac{XN'(d_1)\sigma}{2\sqrt{T-t}}} - \cancel{Kre^{-r(T-t)}N(d_2)}$$
$$+ \cancel{rXN(d_1)} + \cancel{\frac{1}{2}\sigma^2X^2\frac{N'(d_1)}{\sigma\sqrt{T-t}\,X}} = 0 \quad \text{(B-S PDE)}$$

So we just showed that PDE (1) holds for $V = C(t,x)$

Now we want to show it also holds for $V = \left(\frac{x}{B}\right)^{2\alpha} C\left(t, \frac{B^2}{x}\right)$

let $y = \frac{B^2}{x}$

$$\partial_x C\left(t, \frac{B^2}{x}\right) = \partial_{\frac{B^2}{x}} C(t, B^2/x) \cdot \frac{d\frac{B^2}{x}}{dx} = -\frac{B^2}{x^2} \partial_y C(t,y)$$

$$\partial_{xx} C\left(t, \frac{B^2}{x}\right) = \partial_x\left(-\frac{B^2}{x^2} \partial_y C(t,y)\right)$$

$$= 2\frac{B^2}{x^3} \partial_y C(t,y) + \frac{B^2}{x^2} \partial_x(\partial_y C(t,y))$$

$$= 2B^2 x^{-3} \partial_y C(t,y) + B^2 x^{-2}\left(\frac{B^2}{x^2}\right) \partial_y^2 C(t,y)$$

$$= 2\frac{B^2}{x^3} \partial_y C(t,y) + \frac{B^4}{x^4} \partial_y^2 C(t,y)$$

$\partial_t C\left(t, \frac{B^2}{x}\right)$ will not be affected.

Then show that $\left(\frac{x}{B}\right)^{2\alpha} C\left(t, \frac{B^2}{x}\right)$ solves PDE (1)

$$(*) \quad -r\cdot\left(\frac{x}{B}\right)^{2\alpha} C(t,y) + \left(\frac{x}{B}\right)^{2\alpha} \partial_t C(t,y) + rx\left(\frac{x}{B}\right)^{2\alpha-1}(2\alpha)\cdot\frac{1}{B}\cdot C(t,y)$$
$$- rx\left(\frac{x}{B}\right)^{2\alpha} \frac{B^2}{x^2} \partial_y C(t,y) + \frac{1}{2}\sigma^2 x^2\, 2\alpha(2\alpha-1)\left(\frac{x}{B}\right)^{2\alpha-2} \frac{1}{B^2} C(t,y)$$
$$+ \frac{1}{2}\sigma^2 x^2 \cdot 4\alpha\left(\frac{x}{B}\right)^{2\alpha-1} \frac{1}{B}\left(-\frac{B^2}{x^2} \partial_y C(t,y)\right) + \frac{1}{2}\sigma^2 x^2\left(\frac{x}{B}\right)^{2\alpha}\left(2\frac{B^2}{x^3} \partial_y C(t,y) + \frac{B^4}{x^4} \partial_y^2 C(t,y)\right)$$

$$\Big[\partial_{xx}\left(\frac{x}{B}\right)^{2\alpha} C(t,y) = \partial_x\left(\partial_x\left(\frac{x}{B}\right)^{2\alpha} C(t,y)\right)$$
$$= \partial_x\left(2\alpha\left(\frac{x}{B}\right)^{2\alpha-1}\frac{1}{B} C(t,y) + \left(\frac{x}{B}\right)^{2\alpha}\partial_x C(t,y)\right)$$
$$= 2\alpha(2\alpha-1)\left(\frac{x}{B}\right)^{2\alpha-2}\frac{1}{B^2} C(t,y) + 2\alpha\left(\frac{x}{B}\right)^{2\alpha-1}\frac{1}{B}\partial_x C(t,y)$$
$$+ 2\alpha\left(\frac{x}{B}\right)^{2\alpha-1}\frac{1}{B}\partial_x C(t,y) + \left(\frac{x}{B}\right)^{2\alpha}\partial_{xx} C(t,y)\Big]$$

$C(t,y)$ term: $\left(\frac{x}{B}\right)^{2\alpha}\left[-r + 2\alpha r + \frac{1}{2}\sigma^2(2\alpha-1)2\alpha\right]$

Substitute $\alpha = \frac{1}{2}\left(1 - \frac{2r}{\sigma^2}\right)$

$$-\cancel{r} + \cancel{r} - \frac{2r^2}{\sigma^2} + \cancel{\tfrac{1}{2}}\cancel{\sigma^2}\left(-\frac{\cancel{2}r}{\cancel{\sigma^2}}\right)\left(1 - \frac{2r}{\sigma^2}\right)$$
$$= -\frac{2r^2}{\sigma^2} - r + \frac{2r^2}{\sigma^2}$$
$$= -r$$

$\partial_t C(t,y)$ term $\quad \left(\frac{x}{B}\right)^{2\alpha}\cdot 1$

$\partial_y C(t,y)$ term $\quad \left(\frac{x}{B}\right)^{2\alpha}\left[-r\cancel{x}\frac{B^2}{x^{\cancel{2}}} - \frac{1}{2}\sigma^2\cancel{x}(4\alpha)\frac{B^2}{x^{\cancel{2}}} + \sigma^2\cancel{x^2}\frac{B^2}{x^{\cancel{3}}}\right]$

$$\left(\frac{x}{B}\right)^{2\alpha}\left(\frac{B^2}{x}\right)\left(-r - 2\sigma^2\alpha + \sigma^2\right)$$

As $-r - \sigma^2\left(1 - \frac{2r}{\sigma^2}\right) + \sigma^2 = -r - \sigma^2 + 2r + \sigma^2 = r$

$(\frac{X}{B})^{2\alpha} \cdot yr$

$\partial_y^2 \mathbb{C}(t,y)$ term $\quad \frac{1}{2}\sigma^2 x^2 (\frac{x}{B})^{2\alpha} \frac{B^4}{x^4} \partial_y^2 \mathbb{C}(t,y)$

$= (\frac{X}{B})^{2\alpha} (\frac{1}{2}\sigma^2 y^2)$

$\therefore$ (*) becomes

$$(\frac{X}{B})^{2\alpha} \underbrace{[-r\mathbb{C}(t,y) + \partial_t \mathbb{C}(t,y) + yr\,\partial_y \mathbb{C}(t,y) + \frac{1}{2}\sigma^2 y^2 \partial_y^2 \mathbb{C}(t,y)]}_{=0 \text{ by (B-S PDE)}}$$

$= 0$ (2nd-term PDE)

By combining (2nd term PDE) and (B-S PDE), because (I).(II) satisfy PDE (1)

$V(t,x) = \underbrace{\mathbb{C}(t,X)}_{I} - \underbrace{(\frac{X}{B})^{2\alpha} \mathbb{C}(\frac{B^2}{x}, t)}_{II}$ also satisfies PDE (1)

(2) Check $V(T,K) = (X-K)_+ \quad \forall X > B$

$V(T,k) = \mathbb{C}(T,X) - (\frac{X}{B})^{2\alpha} \mathbb{C}(T, \frac{B^2}{X})$

$= (X-K)_+ - (\frac{X}{B})^{2\alpha} (\frac{B^2}{X} - K)_+$

for $X > B$. $\frac{B^2}{X} < B < K$ so $(\frac{B^2}{X} - K)_+ = 0$

$= (X-K)_+ \quad \forall X > B$

(3) Check $V(t,B) = 0$

$V(t,B) = \mathbb{C}(t,B) - (\frac{B}{B})^{2\alpha} \mathbb{C}(t, \frac{B^2}{B})$

$= (1 - 1^{2\alpha}) \mathbb{C}(t,B) = 0$

By (1) (2) (3) results, $V(t,x) = \mathbb{C}_K(t,x) - (\frac{X}{B})^{2\alpha} \mathbb{C}_K(\frac{B^2}{X}, t)$

Satisfies PDE in (j)

(0) I wrote a function in Ipynb to compute the value of analytical solution. The price from analytical solution is 4.920256. From Monte Carlo we had 4.924328. From numerical we had 4.916733

They are very close.