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Simple dynamics for plurality consensus

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Abstract We study a *plurality-consensus* process in which
each of n anonymous agents of a communication network
initially supports a color chosen from the set $[k]$. Then,
in every round, each agent can revise his color accord-
ing to the colors currently held by a random sample of
his neighbors. It is assumed that the initial color config-
uration exhibits a sufficiently large *bias s* towards a fixed
plurality color, that is, the number of nodes supporting the
plurality color exceeds the number of nodes supporting any
other color by s additional nodes. The goal is having the
process to converge to the *stable* configuration in which
all nodes support the initial plurality. We consider a basic
model in which the network is a clique and the update
rule (called here the *3-majority dynamics*) of the process
is the following: each agent looks at the colors of three
random neighbors and then applies the majority rule (break-
ing ties uniformly). We prove that the process converges
in time $\mathcal{O}(\min\{k, (n/\log n)^{1/3}\} \log n)$ with high probabil-
ity, provided that $s \geqslant c\sqrt{\min\{2k, (n/\log n)^{1/3}\}} \, n \log n$. We
then prove that our upper bound above is tight as long as
$k \leqslant (n/\log n)^{1/4}$. This fact implies an exponential time-
gap between the plurality-consensus process and the *median*
process (see Doerr et al. in Proceedings of the 23rd annual
ACM symposium on parallelism in algorithms and archi-
tectures (SPAA'11), pp 149–158. ACM, 2011). A natural
question is whether looking at more (than three) random
neighbors can significantly speed up the process. We provide
a negative answer to this question: in particular, we show that
samples of polylogarithmic size can speed up the process by
a polylogarithmic factor only.

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1 Introduction

We consider a communication network in which each of n
anonymous nodes supports an initial opinion (a color chosen
from a finite set $[k]$). In the *Plurality Consensus* problem, it
is assumed that the initial (color) configuration has a suffi-
ciently large *bias s* towards a fixed color $m \in [k]$—that is, the
number c_m of nodes supporting the plurality color (in short,
the *initial plurality size*) exceeds the number c_j of nodes
supporting any other color j by an additive value s—and the

goal is to design an efficient fully-distributed protocol that lets the network converge to the *plurality consensus*, i.e., to the monochromatic configuration in which all nodes support the plurality color.

Reaching plurality consensus in a distributed system is a fundamental problem arising in several areas such as Distributed Computing [7,20], Communication Networks [21], and Social Networks [6,17,18]. Inspired by some recent works analyzing simple updating-rules (called *dynamics*) for this problem [1,7], we study a discrete-time, synchronous process in which, in every round, each of the n anonymous nodes revises his color according to a (small) random sample of neighbors. We consider one of the simplest models, in which the network is a clique, and the updating rule, called here *3-majority dynamics*, is the following simple one: Each node samples independently and uniformly at random three nodes, including itself and with repetitions, and adopts the plurality color among those three (breaking ties uniformly at random). We remark that looking at only two random nodes and breaking ties uniformly at random would yield a coloring process equivalent to the *polling process* [12], which is known to converge to a *minority* color with constant probability even for $k = 2$ and large initial bias (i.e., $s = \Theta(n)$) [12].

In [7], a tight analysis of a 3-input dynamics for the *median* problem on the clique was presented: the goal there is to converge to a stable configuration where all nodes support a value which is a good approximation of the *median* of the initial configuration. It turns out that, in the binary case (i.e., $k = 2$), the median problem is equivalent to plurality consensus and the 3-input dynamics for the median is equivalent to the 3-majority dynamics: As a result, they obtain, for any bias $s \geqslant c\sqrt{n \log n}$ for some constant $c > 0$, an optimal bound $\Theta(\log n)$ on the convergence time of the 3-majority dynamics for the binary case of the problem considered in this paper.

However, for any $k \geqslant 3$, it is easy to see that the two problems above differ significantly (in particular, the median may be very different from the plurality) and thus, the two dynamics are different from each other as well. Moreover, the analysis in [7]—strongly based on the properties of the median function—cannot be adapted to bound the convergence time of the 3-majority dynamics. The role of parameter $k = k(n)$ in the convergence time of this dynamics is currently unknown and, more generally, the existence of efficient dynamics reaching plurality consensus for $k \geqslant 3$ is left as an important open issue in [2,3,7].

Our contribution. We present a new analysis of the 3-majority dynamics in the general case (i.e., for any $k \in [n]$). Our analysis shows that, with high probability (in short, w.h.p.,[1]) the process converges to plurality consensus within

time $\mathcal{O}\left(\min\{k, (n/\log n)^{1/3}\} \log n\right)$, provided that the initial bias is $s \geqslant 72\sqrt{2 \min\{2k, (n/\log n)^{1/3}\} n \log n}$ (see Corollary 1).

Our proof technique is accurate enough to get another interesting form of the above upper bound that does not depend on k. Indeed, when the initial plurality size c_m is larger than $n/\lambda(n)$ for any function $\lambda(n)$ such that $1 \leqslant \lambda(n) < \sqrt[3]{n}$ and $s \geqslant 72\sqrt{2\lambda(n) n \log n}$, then the process converges in time $\mathcal{O}(\lambda(n) \log n)$ w.h.p., no matter how large k is (see Theorem 1). Hence, when $c_m \geqslant n/\text{polylog}(n)$ and $s \geqslant \sqrt{n}\,\text{polylog}(n)$, the convergence time is polylogaritmic (see Corollary 2).

We then show that our upper bound is tight for a wide range of the input parameters. When $k \leqslant (n/\log n)^{1/4}$, we prove a lower bound $\Omega(k \log n)$ on the convergence time of the 3-majority dynamics starting from some configurations with bias $s \leqslant (n/k)^{1-\epsilon}$, for an arbitrarily small constant $\epsilon > 0$ (see Theorem 2). Observe that this range largely includes the initial bias required by our upper bound when $k \leqslant (n/\log n)^{1/4}$. So, the *linear-in-k* dependence of the convergence time cannot be removed for a wide range of the parameter k.

Our analysis also provides a clear picture of the 3-majority dynamic process. Informally speaking, the larger the initial value of c_m is (w.r.t. n), the smaller the required initial bias s and the faster the convergence time are. On the other hand, our lower-bound argument shows, as a by-product, that the initial plurality size c_m needs $\Omega(k \log n)$ rounds just to increase from $n/k + o(n/k)$ to $2n/k$. Another natural issue is to analyze the process under weaker assumptions on the initial bias. We show that there are initial configurations with bias $s = O(\sqrt{kn})$ for which the bias decreases in a single round with constant probability (see Lemma 10). This implies that our assumption on the magnitude of the initial bias is in a sense (almost) tight if one wants to prove a monotonic increase of the bias in every round with high probability, as we did in proving our upper bounds on the convergence time.

We then prove a general negative result: Under the distributed model we consider, within the class of dynamics using no additional state other than the initial colors, no dynamics with at most 3 inputs (other than 3-majority) converges w.h.p. to plurality consensus starting from any initial configuration with bias $s = o(n)$ (see Theorem 3). In other words, within the class above, not only there is no 3-input dynamics that achieves convergence to plurality consensus in $o(k \log n)$ rounds, but the 3-majority dynamics is the only one that eventually achieves this goal at all, no matter how long the process takes. Rather interestingly, by comparing the $\mathcal{O}(\log n)$ bound for the median [7] to our negative results for the plurality on the same distributed model, we get an exponential time-gap between the task of computing the median and the one of

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[1] We say that a family of events $\{\mathcal{E}_n\}_n$ holds w.h.p. if a positive constant c exists such that $\mathbf{P}(\mathcal{E}_n) \geqslant 1 - n^{-c}$ for sufficiently large n.

computing plurality (this happens for instance when $k = n^a$, for any constant $0 < a < 1/4$).

A natural question suggested by our findings is whether (slightly) larger random samples of nodes' neighborhoods might lead to significant improvements in convergence time to plurality consensus. We provide a negative answer to this question. To this purpose, we consider h-plurality, i.e., the natural generalization of the 3-majority dynamics in which every node, in each round, updates his color according to the plurality of the colors supported by h randomly sampled neighbors. We prove a lower bound $\Omega\left(k/h^2\right)$ on the convergence time of the h-plurality dynamics, for integers k and h such that $k/h = \mathcal{O}\left(n^{1/4-\epsilon}\right)$, with ϵ an arbitrarily-small positive constant (see Theorem 4). We emphasize that scalable and efficient protocols must yield low communication complexity and small node congestion in every round. These properties are guaranteed by the h-plurality dynamics only when h is small, say $h = \mathcal{O}(\mathrm{polylog}(n))$: In this case, our lower bound implies that the resulting speed-up is only polylogarithmic with respect to the 3-majority dynamics.

One motivation for adopting dynamics in reaching (*simple*) consensus[2] (such as the median dynamics shown in [7]) lies in their provably-good *self-stabilizing* properties against *dynamic adversary corruptions*: It turns out that the 3-majority dynamics has good self-stabilizing properties for the *plurality consensus* problem. More formally, a *T-bounded adversary* knows the state of every node at the end of each round and, based on this knowledge, it can corrupt the color of up to T nodes in an arbitrary way, just before the next round begins. In this case, the goal is to achieve an almost-stable phase where all but at most $\mathcal{O}(T)$ nodes agree on the plurality value. This "almost-stability" phase must have $\mathrm{poly}(n)$ length, with high probability. Our analysis implicitly shows that the 3-majority dynamics guarantees the self-stabilization property for plurality consensus for any k and for $T = o(s/k)$ if the initial bias is $s \geqslant c \cdot \sqrt{2\min\{2k, (n/\log n)^{1/3}\} \, n \log n}$ for a sufficiently large constant c (see Corollary 4).

Related work. The plurality consensus problem arises in several applications such as distributed database management, where data redundancy or replication and majority rules are used to manage the presence of unknown faulty processors [7,20]. The goal here is to converge to the version of the data supported by the plurality of the initial distributed copies (it is reasonable that a sufficiently strong plurality of the nodes are not faulty and thus possess the correct data). Another application is distributed item ranking, in which every node initially selects some item and the goal is to agree on the most popular item according to the initial plurality

opinion [21]. Further applications of majority updating rules in networks can be found in [10,20].

The results most related to our contribution are those in [7] which have been already discussed above. Several variants of binary majority consensus have been studied in different distributed models [2,18]. The simplest protocol is the polling rule, i.e., the 1-majority dynamics, which has been extensively studied on several classes of graphs (see [20]).

As for the *population model*, where there is only one random node-pair interaction per round (so the dynamics are strictly sequential), the binary case on the clique has been studied in [2] where the *undecided-state* protocol has been introduced. Their generalization to the multivalued case ($k \geqslant 3$) does not converge to plurality even starting with a large bias $s = \Theta(n)$. Following [2,16] has analyzed a similar protocol on general graphs which solves the binary majority consensus deterministically.

More expensive and complex protocols have been considered in order to speed up the process. For instance, in [13], a protocol for the sequential-interaction model is presented that requires $\Theta(\log n)$ memory per node and converges in time $\mathcal{O}(n^7)$. Other protocols for the sequential-interaction model have been analyzed in [5,14] (with no time bound).

In [3,8,21], the undecided-state protocol on the continuous-time population model is proved to converge in $\mathcal{O}(n \log n)$ expected time only for $k = \Theta(1)$ and $s = \Theta(n)$: Even assuming such strong restrictions, the bound does not hold in "high probability" and, moreover, their analysis, based on real-valued differential-equations, do not work for the discrete-time parallel model considered in this paper. The simple rule of the *undecided-state* [2,21] is to "add" one extra state to somewhat account for the "previous" opinion supported by an agent. We remark that, contrary to the undecided-state protocol, in the class of protocols considered here the nodes do not make use of any additional state apart from those present in the initial configuration (i.e., the initial colors).

In a recent work [4] (appeared after the conference version of this paper), the undecided-state protocol has been analyzed on the discrete-time parallel model for any $k = \mathcal{O}((n/\log n)^{1/3})$ and for initial configurations $\mathbf{c} = (c_1, \ldots, c_k)$ such that the (multiplicative) bias is $c_m/c_j = \Omega(1)$. There, it is shown that this dynamics has a convergence time which is w.h.p. linear in the *monochromatic* distance of the initial configuration $\mathbf{c}$. The monochromatic distance of a configuration $\mathbf{c} = (c_1, \ldots, c_k)$ is defined as

$$\sum_{i=1}^{k} \left(\frac{c_i}{c_m}\right)^2.$$

It turns out that there are initial configurations (in particular, those having "almost all" nodes supporting only a poly-

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[2] In the (simple) consensus problem the goal is to reach any stable monochromatic configuration (any color is accepted) starting from any initial configuration.

logarithmic number of colors) from which the undecided-state dynamics is exponentially faster than the 3-majority. On the other hand, in addition to the above condition on the multiplicative bias, we note that the undecided-state dynamics may fail to reach plurality consensus when $k = \omega(\sqrt{n})$. Indeed, for such parameter range there are initial configurations in which the undecided-state dynamics makes the plurality color disappear after one round with constant probability (see, e.g., Section 3 in [4]).

Finally, protocols for specific network topologies and some "social-based" communities have been studied in [1,8, 17,21].

Roadmap. Section 2 formalizes the basic concepts and gives some preliminary results. Section 3 is devoted to the proofs of the upper bounds on the convergence time of the 3-majority dynamics. In Sect. 4, the lower bounds for the studied dynamics are described. Section 5 discusses some interesting open questions such as the tightness of the initial bias.

2 Preliminaries

A (*k-color*) *configuration* (for short *k-cd*) is any *k*-tuple $\mathbf{c} = (c_1, \ldots, c_k)$ such that c_js are non negative integers and $\sum_{j=1,\ldots,k} c_j = n$. In what follows, we will always assume wlog $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_k$. So c_1 is the *plurality color* and $s(\mathbf{c}) = c_1 - c_2$ is the *bias* of $\mathbf{c}$.
The 3-majority dynamics works as follows:

At every round, every node samples three nodes (including itself and with repetitions) independently and uniformly at random and recolors itself according to the majority of the colors it sees. If it sees three different colors, it chooses the first one.

Clearly, in the case of three different colors, choosing the second or the third one would not make any difference. The same holds even if the choice would be uniformly at random among the three colors.

For any round t and for any $j \in [k]$, let $C_j^{(t)}$ be the r.v. counting the number of nodes colored j at round t and let $\mathbf{C}^{(t)} = (C_1^{(t)}, \ldots, C_k^{(t)})$ denote the random variable indicating the *k-cd* at time t of the execution of the 3-majority dynamics.

For every $j \in [k]$ let $\mu_j(\mathbf{c})$ be the expected number of nodes with color j at the next round when the current *k-cd* is $\mathbf{c}$, i.e.,

$$\mu_j(\mathbf{c}) = \mathbf{E}\left[C_j^{(t+1)} \,\middle|\, \mathbf{C}^{(t)} = \mathbf{c} \right].$$

To simplify the notation, in all the technical proofs we will write μ_j and s instead of $\mu_j(\mathbf{c})$ and $s(\mathbf{c})$ when the dependence on configuration $\mathbf{c}$ will be clear from the context.

Lemma 1 (**Next expected coloring**) *For any k-cd* $\mathbf{c}$ *and for every color* $j \in [k]$, *it holds that*

$$\mu_j(\mathbf{c}) = c_j \left[1 + \frac{1}{n^2} \left(nc_j - \sum_{h \in [k]} c_h^2 \right) \right]. \tag{1}$$

Proof According to the 3-majority dynamics, a node i gets color j if it chooses three times color j, or if it chooses two times j and one time a different color, or if it chooses the first time color j and then, the second and third time, two different distinct colors. Hence if we name $X_{i,j}^{(t)}$ the indicator random variable of the event "Node i gets color j at time t", we have that

$$\mathbf{P}\left(X_{i,j}^{(t+1)} = 1 \mid \mathbf{C}^{(t)} = \mathbf{c} \right)$$

$$= \left(\frac{c_j}{n}\right)^3 + 3\left(\frac{c_j}{n}\right)^2 \left(\frac{n - c_j}{n}\right) +$$

$$+ \left(\frac{c_j}{n}\right)\left[1 - \left(\frac{\sum_{h=1}^k c_h^2}{n^2} + 2\left(\frac{c_j}{n}\right)\left(\frac{n - c_j}{n}\right) \right) \right]$$

$$= \left(\frac{c_j}{n^3}\right)\left(n^2 + c_j n - \sum_{h=1}^k c_h^2 \right).$$

$\square$

Lemma 2 (**Next expected bias**) *For any k-cd* $\mathbf{c}$ *and for every color* $j \in [k]$ *with* $j \neq 1$, *it holds that*

$$\mu_1(\mathbf{c}) - \mu_j(\mathbf{c}) \geqslant s(\mathbf{c})\left(1 + \frac{c_1}{n}\left(1 - \frac{c_1}{n}\right)\right).$$

Proof Observe that, when we assume $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_k$, we can give the following upper bound on the sum of squares in Lemma 1

$$\sum_{h \in [k]} c_h^2 = c_1^2 + \sum_{h=2}^k c_h^2 \leqslant c_1^2 + c_2 \sum_{h=2}^k c_h$$

$$= c_1^2 + c_2(n - c_1). \tag{2}$$

From Lemma 1 it thus follows that, for any $j \neq 1$,

$$\mu_1 - \mu_j \geqslant \mu_1 - \mu_2$$

$$= (c_1 - c_2) + \frac{(c_1^2 - c_2^2)}{n} - \frac{c_1 - c_2}{n^2}\sum_{h \in k} c_h^2$$

$$= s \cdot \left(1 + \frac{c_1 + c_2}{n} - \frac{1}{n^2}\sum_{h \in k} c_h^2 \right)$$

$$\geqslant s \cdot \left(1 + \frac{c_1 + c_2}{n} - \frac{c_1^2 + nc_2}{n^2} \right)$$

$$= s \cdot \left(1 + \frac{c_1}{n}\left(1 - \frac{c_1}{n}\right) \right),$$

where in the inequality we used (2) and the fact that $c_1 - c_2 \geqslant 0$. □

3 Upper bounds for 3-majority

In this section, we provide the following upper bound on the convergence time of the 3-majority dynamics which clarifies the roles played by the plurality color and by the initial bias.

Theorem 1 (General upper bound) *Let λ be any value such that $\lambda < \sqrt[3]{n}$ and let $\mathbf{c}$ be any initial k-cd, with $c_1 \geqslant n/\lambda$ and $s(\mathbf{c}) \geqslant 72\sqrt{2\lambda n \log n}$. Then the 3-majority protocol converges to the plurality color in $\mathcal{O}(\lambda \log n)$ time w.h.p.*

The next three corollaries of Theorem 1 address three relevant special cases. Corollary 1 is obtained by setting $\lambda = \min\{2k, \sqrt[3]{n/\log n}\}$ and it provides a bound which does not assume any condition on c_m.

Corollary 1 *Let $\mathbf{c}$ be any initial k-cd with*

$$s(\mathbf{c}) \geqslant 72\sqrt{2\min\left\{2k, \sqrt[3]{\frac{n}{\log n}}\right\} n \log n}.$$

Then, the 3-majority protocol converges to the plurality color in $\mathcal{O}\left(\min\{2k, \sqrt[3]{n/\log n}\} \log n\right)$ time w.h.p.

Corollaries 2 and 3 are obtained by setting $\lambda = \text{poly} \log(n)$ and $\lambda = \Theta(1)$, respectively. They provide sufficient conditions for a polylogarithmic convergence time.

Corollary 2 *Let $\mathbf{c}$ be any initial k-cd with $c_1 \geqslant n/\log^\ell n$ and $s(\mathbf{c}) \geqslant 72\sqrt{2n \log^{\ell+1} n}$. Then, the 3-majority protocol converges to the plurality color in $\mathcal{O}(\log^{\ell+1} n)$ time w.h.p.*

Corollary 3 *Let $\mathbf{c}$ be any k-cd with $c_1 \geqslant n/\beta$ and $s(\mathbf{c}) \geqslant 72\sqrt{2\beta n \log n}$, for some constant $\beta \geqslant 1$. Then, the 3-majority protocol converges to the plurality color in $\mathcal{O}(\log n)$ time w.h.p.*

In order to prove Theorem 1, we need the following three technical lemmas that essentially characterize three different phases of the process analysis. Each of them concerns a different range assumed by the plurality c_1. The first lemma considers configurations in which c_1 is less than a suitable constant fraction of n: in this case, it shows that the bias between the plurality size and the size of any other color increases by a factor $1 + \Omega(c_1/n) = 1 + \Omega(1/\lambda)$.

Lemma 3 (From plurality to majority) *Let $\mathbf{c}$ be any k-cd with $n/\lambda \leqslant c_1 \leqslant 2n/3$ and $s(\mathbf{c}) \geqslant 72\sqrt{2\lambda n \log n}$ where $\lambda < \sqrt[3]{n}$. Then, for any other color $j \neq 1$ it holds that*

$$\mathbf{P}\left(C_1^{(t+1)} - C_j^{(t+1)} \geqslant s(\mathbf{c})\left(1 + \frac{c_1}{4n}\right) \middle| \mathbf{C}^{(t)} = \mathbf{c}\right)$$

$$\geqslant 1 - \frac{1}{n^3}.$$

Proof Conditional on any configuration $\mathbf{C}^{(t)} = \mathbf{c}$, from the Chernoff bounds in Lemma 11 (bounds 1 with $\delta = 3\sqrt{\log n/\mu}$ if $\mu > \log n$, bound 2 with $\delta = 4 \log n/\mu$ otherwise), it follows that w.h.p.

$$C_j^{(t+1)} \leqslant \max\left\{\mu_j + 3\sqrt{\mu_j \log n}, 5 \log n\right\},$$
$$C_1^{(t+1)} \geqslant \mu_1 - 3\sqrt{\mu_1 \log n}.$$

Thus, if $\mu_j + 3\sqrt{\mu_j \log n} \geqslant 5 \log n$, w.h.p. it holds that[3]

$$C_1^{(t+1)} - C_j^{(t+1)}$$
$$\geqslant \mu_1 - \mu_j - 3\sqrt{\mu_1 \log n} - 3\sqrt{\mu_j \log n}$$
$$\geqslant \mu_1 - \mu_j - 6\sqrt{\mu_1 \log n}.$$

Otherwise, if $\mu_j + 3\sqrt{\mu_j \log n} < 5 \log n$, then w.h.p. it holds that

$$C_1^{(t+1)} - C_j^{(t+1)}$$
$$\geqslant \mu_1 - 3\sqrt{\mu_1 \log n} - 5 \log n$$
$$\geqslant \mu_1 - \mu_j - 6\sqrt{\mu_1 \log n},$$

where in the last inequality we used that $\mu_1 \geqslant c_1 \geqslant n/\lambda \geqslant n^{\frac{2}{3}}$. From Lemma 2 and the hypothesis $c_1 \leqslant 2n/3$ we get that

$$\mu_1 - \mu_j \geqslant (c_1 - c_j)\left(1 + \frac{c_1}{3n}\right),$$

and from (1) we also have that $\mu_1 \leqslant 2c_1$. Thus, in (3) and (3) we get

$$\mu_1 - \mu_j - 6\sqrt{\mu_1 \log n}$$
$$\geqslant (c_1 - c_j)\left(1 + \frac{c_1}{3n}\right) - 6\sqrt{2c_1 \log n}$$
$$\geqslant (c_1 - c_j)\left(1 + \frac{c_1}{3n} - 6\frac{\sqrt{2c_1 \log n}}{(c_1 - c_j)}\right)$$
$$\overset{(a)}{\geqslant} (c_1 - c_j)\left(1 + \frac{c_1}{3n} - \frac{1}{12}\sqrt{\frac{c_1}{\lambda n}}\right)$$
$$\geqslant (c_1 - c_j)\left(1 + \frac{c_1}{3n}\left(1 - \frac{1}{4}\sqrt{\frac{n}{c_1 \lambda}}\right)\right)$$
$$\overset{(b)}{\geqslant} (c_1 - c_j)\left(1 + \frac{c_1}{4n}\right),$$

where in (a) we used that $c_1 - c_j \geqslant s \geqslant 72\sqrt{2\lambda n \log n}$ and in (b) we used that $c_1 \geqslant n/\lambda$, concluding the proof. □

Once c_1 becomes larger than $2n/3$ the negative occurrence of c_1 in (2) does not allow to directly show a drift towards plurality. We thus consider another useful "drift" of the process:

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[3] We are using the fact that $\mathbf{P}(A \cap B) \geqslant 1 - \mathbf{P}(A^C) - \mathbf{P}(B^C)$.

The sum of all the other color sizes decreases exponentially w.h.p., as long as this sum is enough large to apply concentration bounds. This result is formalized in the next lemma.

Lemma 4 (From majority to almost all) *Let* **c** *be any k-cd with* $2n/3 \leqslant c_1 \leqslant n - \omega (\log n)$. *Then, it holds that*

$$\mathbf{P} \left(\sum_{i \neq 1} C_i^{(t+1)} \leqslant \frac{8}{9} \sum_{i \neq 1} c_i \,\Big|\, \mathbf{C}^{(t)} = \mathbf{c} \right) \geqslant 1 - \frac{1}{n^3}.$$

Proof Let us define $\mu_{-1} = \sum_{i \neq 1} \mu_i$. From (1) we have

$$
\begin{aligned}
\frac{\mu_{-1}}{n} &= \sum_{i \neq 1} \frac{c_i}{n} \left(1 + \frac{c_i}{n} - \sum_j \left(\frac{c_j}{n} \right)^2 \right) \\
&= 1 - \frac{c_1}{n} + \sum_{i \neq 1} \left(\frac{c_i}{n} \right)^2 - \left(1 - \frac{c_1}{n} \right) \sum_j \left(\frac{c_j}{n} \right)^2 \\
&= 1 - \frac{c_1}{n} - \left(\frac{c_1}{n} \right)^2 + \frac{c_1}{n} \sum_j \left(\frac{c_j}{n} \right)^2 \\
&\overset{(a)}{\leqslant} 1 - \frac{c_1}{n} - \left(\frac{c_1}{n} \right)^2 \\
&\quad + \frac{c_1}{n} \left(\left(\frac{c_1}{n} \right)^2 + \frac{c_2}{n} \left(1 - \frac{c_1}{n} \right) \right) \\
&= \left(1 - \frac{c_1}{n} \right) \left(1 - \left(\frac{c_1}{n} \right)^2 + \frac{c_1}{n} \frac{c_2}{n} \right) \\
&= \left(1 - \frac{c_1}{n} \right) \left(1 - \frac{c_1}{n} \left(\frac{c_1}{n} - \frac{c_2}{n} \right) \right),
\end{aligned}
\tag{3}
$$

where in (a) we used (2). Using the hypothesis $c_1/n \geqslant 2/3$ (hence $c_2/n \leqslant 1/3$), from (3) we obtain

$$
\begin{aligned}
&\left(1 - \frac{c_1}{n} \right) \left(1 - \frac{c_1}{n} \left(\frac{c_1}{n} - \frac{c_2}{n} \right) \right) \\
&\leqslant \left(1 - \frac{c_1}{n} \right) \left(1 - \frac{c_1}{3n} \right) \leqslant \frac{7}{9} \sum_{i \neq 1} \frac{c_i}{n}.
\end{aligned}
$$

Now observe that, from the Chernoff bound, as long as $\mu_{-1} \in \omega (\log n)$, w.h.p. it holds

$$
\begin{aligned}
\sum_{i \neq 1} C_i^{(t+1)} &\leqslant \mu_{-1} + \sqrt{\mu_{-1} \log n} \\
&= \mu_{-1} \left(1 + \sqrt{\frac{\log n}{\mu_{-1}}} \right) \\
&= \mu_{-1} (1 + o(1)).
\end{aligned}
\tag{4}
$$

Thus, by replacing (3) in (3), we get that w.h.p. it holds

$$\sum_{i \neq 1} C_i^{(t+1)} \leqslant \mu_{-1} (1 + o(1)) \leqslant \frac{8}{9} \sum_{i \neq 1} c_i,$$

concluding the proof. □

Finally, when the sum of all the minority colors is not larger than a polylogarithmic function, the probability that they all disappear in one round is high. This is shown in the new, final lemma.

Lemma 5 (Last step) *Let* $\alpha > 0$ *and let* **c** *be any k-cd with* $c_1 \geqslant n - \log^\alpha n$. *Then, it holds that*

$$\mathbf{P} \left(\sum_{i \neq 1} C_i^{(t+1)} = 0 \,|\, \mathbf{C}^{(t)} = \mathbf{c} \right) \geqslant 1 - \frac{3 \log^{2\alpha} n}{n}.$$

Proof As in the previous proof let we name $\mu_{-1} = \sum_{i \neq 1} \mu_i$. Note that $c_1 \geqslant n - \log^\alpha n$ implies $\sum_{i \neq 1} c_i \leqslant \log^\alpha n$. Thus, from (1) we have

$$
\begin{aligned}
\mu_{-1} &= \sum_{i \neq 1} c_i \left(1 + \frac{c_i}{n} - \sum_j \left(\frac{c_j}{n} \right)^2 \right) \\
&\leqslant \sum_{i \neq 1} c_i \left(1 + \frac{c_i}{n} - \left(\frac{c_1}{n} \right)^2 \right) \\
&= \sum_{i \neq 1} c_i \left(1 + \frac{c_i}{n} - \left(1 - \sum_{j \neq 1} \frac{c_j}{n} \right)^2 \right) \\
&\leqslant \sum_{i \neq 1} c_i \left(\frac{c_i}{n} + 2 \sum_{j \neq 1} \frac{c_j}{n} \right) \\
&\leqslant \sum_{i \neq 1} c_i \left(\frac{3 \log^\alpha n}{n} \right) = \frac{3 \log^{2\alpha} n}{n}.
\end{aligned}
$$

Finally, (5) follows from Markov's inequality on the event "$\sum_{i \neq 1} C_i^{(t+1)} \geqslant 1$" and, since $\sum_{i \neq 1} C_i^{(t+1)}$ is a non-negative integer-valued r.v., this is equivalent as "$\sum_{i \neq 1} C_i^{(t+1)} > 0$". □

Proof of Theorem 1 From Lemma 3 it follows that, as long as the number of nodes with the plurality color c_1 is smaller than a constant fraction of n, the bias between c_1 and c_2 increases by a factor $\left(1 + \frac{1}{4\lambda} \right)$, w.h.p. (ii) In Lemma 4 it follows that, when the plurality color reaches a suitable constant fraction of n, then the number of nodes with non-plurality colors decreases at exponential rate, w.h.p. Finally, (iii) in Lemma 5 we consider separately the last round of the protocol, where all colors but the plurality one disappear w.h.p. □

3.1 Plurality consensus with adversary

In this section we show that the 3-majority dynamics is robust against Byzantine adversaries. Let $F \leqslant n$, we consider an *F-bounded dynamic adversary* that, at every round, can change

the color of up to F nodes with the goal of preventing the system to converge to the plurality color. Clearly, reaching complete plurality consensus is not possible in this framework. In presence of an F-bounded dynamic adversary we thus consider the *M-plurality consensus*, in which all but M nodes have to agree on the plurality color.

Notice that it is not possible to reach M-plurality consensus against an F-bounded dynamic adversary if $F > M$. Our previous analysis of the 3-majority dynamics can be easily adapted to show that it achieves $o(s/\lambda)$-plurality consensus against any F-bounded adversary for $F = o(s/\lambda)$, where s is the initial bias and $\lambda < \sqrt[3]{n}$.

Corollary 4 (Upper bound with adversary) *Let λ be any value such that $\lambda < \sqrt[3]{n}$ and let $\mathbf{c}$ be any initial configuration, with $c_1 \geq n/\lambda$ and $s(\mathbf{c}) \geq 24\sqrt{2\lambda n \log n}$. The 3-majority dynamics achieves $\mathcal{O}(s(\mathbf{c})/\lambda)$-plurality consensus against any F-bounded adversary with $F = o(s(\mathbf{c})/\lambda)$, and the convergence time is $\mathcal{O}(\lambda \log n)$ w.h.p.*

Proof In order to formalize the analysis of the process with an adversary, we split each round in two consecutive steps: In the first step nodes apply the 3-majority updating rule while, in the second step, the adversary can change the color of up to F arbitrary nodes. Hence, if the configuration of the system at some round t is $\mathbf{C}^{(t)} = \hat{\mathbf{c}}$, we name $\mathbf{H}^{(t+1)}$ the random variable indicating the configuration after the first step of round $t + 1$ and $\mathbf{C}^{(t+1)}$ the configuration after the second step of round $t + 1$, i.e.,

$$\mathbf{C}^{(t)} = \hat{\mathbf{c}} \overset{\mathcal{R}andom}{\longrightarrow} \mathbf{H}^{(t+1)} \overset{Adversary}{\longrightarrow} \mathbf{C}^{(t+1)}.$$

Notice that $\mathbf{C}^{(t+1)}$ is a function of $\mathbf{H}^{(t+1)}$ arbitrarily determined by the adversary within its constraints.

If the configuration at some round t is $\mathbf{C}^{(t)} = \hat{\mathbf{c}}$, with $\hat{c}_1 \leq 2n/3$, then from Lemma 3 it follows that w.h.p.

$$H_1^{(t+1)} - H_j^{(t+1)} \geq s(\hat{\mathbf{c}}) + \frac{s(\hat{\mathbf{c}})}{4\lambda}.$$

The bias after the adversarial step thus will be $C_1^{(t+1)} - C_j^{(t+1)} \geq s(\hat{\mathbf{c}}) + s(\hat{\mathbf{c}})/(4\lambda) - F$ w.h.p. Since by hypothesis $F = \mathcal{O}(s(\mathbf{c})/\lambda)$, as long as the bias $s(\hat{\mathbf{c}})$ of the current configuration is at least as large as the bias $s(\mathbf{c})$ of the initial configuration, we have that w.h.p.

$$\begin{aligned} C_1^{(t+1)} - C_j^{(t+1)} &\geq s(\hat{\mathbf{c}}) + \frac{s(\hat{\mathbf{c}})}{4\lambda} - F \\ &\geq s(\hat{\mathbf{c}}) + \frac{s(\hat{\mathbf{c}})}{5\lambda}. \end{aligned} \tag{5}$$

Notice that the requirement $s(\hat{\mathbf{c}}) \geq s(\mathbf{c})$ trivially holds in the initial configuration, when $\hat{\mathbf{c}} = \mathbf{c}$, and from (5) by induction it will hold in all the following rounds w.h.p.

Equation (5) guarantees that, as long as the plurality color is supported by at most $2n/3$ nodes (see hypothesis of Lemma 3) the bias increases by a factor $1 + \Theta(1/\lambda)$ at each round w.h.p., even in the presence of the adversary. Hence, after $\mathcal{O}(\lambda \log n)$ rounds the plurality color will be supported by at least $2n/3$ nodes w.h.p.

When the system reaches, at some round t, a configuration $\mathbf{C}^{(t)} = \hat{\mathbf{c}}$ such that the plurality color is supported by $2n/3 \leq \hat{c}_1 \leq n - \omega(\log n)$ nodes, then Lemma 4 guarantees that the total number of nodes supporting the other colors in configuration $\mathbf{H}^{(t+1)}$ after the 3-majority step of the next round is w.h.p.

$$\sum_{i \neq 1} H_i^{(t+1)} \leq \frac{8}{9} \sum_{i \neq 1} \hat{c}_1.$$

Hence, as long as $\sum_{i \neq 1} \hat{c}_1 = \Omega(s(\mathbf{c})/\lambda)$, the total number of nodes supporting the other colors in configuration $\mathbf{C}^{(t+1)}$ (after the adversarial step of the next round) is w.h.p.

$$\sum_{i \neq 1} C_i^{(t+1)} \leq \frac{8}{9} \sum_{i \neq 1} \hat{c}_1 + F \leq \frac{9}{10} \sum_{i \neq 1} \hat{c}_1.$$

Thus, when the plurality color reaches $2n/3$ nodes, after further $\mathcal{O}(\log n)$ rounds all but $o(s(\mathbf{c})/\lambda)$ nodes will support the plurality color w.h.p. Notice that (5) also guarantees that, once reached M-plurality consensus, the system will only take on configurations that satisfy M-plurality consensus w.h.p. $\square$

4 Lower bounds

This section is organized into four subsections: in the first one, we prove a lower bound on the convergence time of the 3-majority dynamics; in the second subsection, we show that 3-majority is essentially the only 3-input dynamics that converges to plurality consensus; in the third subsection, we provide a lower bound on the convergence time of the h-plurality dynamics for $h > 3$; finally, in the fourth subsection we show that our assumption on the magnitude of the initial bias is in a sense (almost) tight if one wants to prove a monotonic increase of the bias in every round with high probability, as we did in Sect. 3.

4.1 Lower bound for 3-majority

In this section we show that if the 3-majority dynamics starts from a sufficiently balanced configuration (i.e., at the beginning there are $n/k \pm o(n/k)$ nodes of every color) then it will take $\Omega(k \log n)$ rounds w.h.p. to reach one of the absorbing configurations where all nodes have the same color. In what

follows, all events and random variables thus concern the Markov process yielded by the 3-majority dynamics.

In the next lemma we show that if there are at most $n/k+b$ nodes of a specific color, where b is smaller than n/k, then at the next round there are at most $n/k + (1 + 3/k)b$ nodes of that color w.h.p.

Lemma 6 *Let* $k \leqslant (n/\log n)^{1/4}$ *denote the number of colours. Let b be any number such that*
$k\sqrt{n \log n} \leqslant b \leqslant n/k$ *and let* $\mathbf{c} = (c_1, \ldots, c_k)$ *be a configuration. If* $c_j = n/k + a$ *for some color* $j \in [k]$ *and for some* $a \leqslant b$, *then the number of nodes with color j at the next round are at most* $n/k + (1 + 3/k)b$ *w.h.p.; more precisely, for any* $a \leqslant b$ *and for any configuration* $\mathbf{c}$ *such that* $c_j = n/k + a$ *it holds that*

$$\mathbf{P}\left(C_j^{(t+1)} \geqslant \frac{n}{k} + \left(1 + \frac{3}{k}\right)b \mid \mathbf{C}^{(t)} = \mathbf{c}\right) \leqslant \frac{1}{n^2}.$$

Proof For any configuration $\mathbf{c} = (c_1, \ldots, c_k)$ with $\sum_{j=1}^{k} c_j = n$ and any color $j \in [k]$, the expected value of the number of nodes colored j at round $t + 1$ conditional on $\{\mathbf{C}^{(t)} = \mathbf{c}\}$ is (see Lemma 1)

$$\mathbf{E}\left[C_j^{(t+1)} \mid \mathbf{C}^{(t)} = \mathbf{c}\right] = c_j \left(1 + \frac{c_j}{n} - \frac{1}{n^2}\sum_{j=1}^{k} c_j^2 \right).$$

Observe that, since $\sum_{j=1}^{k} c_j = n$, Jensen's inequality (see Lemma 12) implies $(1/n^2)\sum_{j=1}^{k} c_j^2 \geqslant 1/k$. Hence, we can give an upper bound on the expectation of $C_j^{(t+1)}$ that only depends on c_j and not on the whole configuration $\mathbf{c}$ at round t, namely

$$\mathbf{E}\left[C_j^{(t+1)} \mid \mathbf{C}^{(t)} = \mathbf{c}\right] \leqslant c_j \left(1 + \frac{c_j}{n} - \frac{1}{k} \right).$$

If we condition on the number of nodes of color j being $c_j = n/k + a$ in configuration $\mathbf{c}$, for some $a \leqslant b$, we get

$$\mathbf{E}\left[C_j^{(t+1)} \mid \mathbf{C}^{(t)} = \mathbf{c}\right]$$
$$\leqslant \left(\frac{n}{k} + a\right)\left(1 + \frac{n/k + a}{n} - \frac{1}{k}\right)$$
$$= \frac{n}{k} + \left(1 + \frac{1}{k}\right)a + \frac{a^2}{n} \leqslant \frac{n}{k} + \left(1 + \frac{1}{k}\right)b + \frac{b^2}{n}$$
$$\leqslant \frac{n}{k} + \left(1 + \frac{2}{k}\right)b,$$

where in the last two inequalities we used that $a \leqslant b$ and $b \leqslant n/k$.[4] Since $C_j^{(t+1)}$ conditional on $\{\mathbf{C}^{(t)} = \mathbf{c}\}$ can be written

---

[4] Notice that the inequality holds in particular for negative a as well.

as a sum of n independent Bernoulli random variables, from Chernoff's bound (see Lemma 11) we thus get that for every $a \leqslant b$ it holds that

$$\mathbf{P}\left(C_j^{(t+1)} \geqslant \frac{n}{k} + \left(1 + \frac{3}{k}\right)b \mid \mathbf{C}^{(t)} = \mathbf{c}\right)$$
$$\leqslant e^{-2(b/k)^2/n} \leqslant \frac{1}{n^2},$$

where in the last inequality we used that $b \geqslant k\sqrt{n \log n}$. $\square$

We say that a configuration $\mathbf{c} = (c_1, \ldots, c_k) \in \{0, 1, \ldots, n\}^k$ with $\sum_{j=1}^{k} c_j = n$ is *monochromatic* if there is an $j \in [k]$ such that $c_j = n$. The next theorem shows that, if we start from a sufficiently *balanced* configuration, the 3-majority protocol requires $\Omega(k \log n)$ rounds w.h.p. to reach a monochromatic configuration. More specifically, denote by $\tau = \inf\{t \in \mathbb{N} : \mathbf{C}^{(t)}$ is monochromatic$\}$ the random variable indicating the first round in which the system is in a monochromatic configuration.

Theorem 2 *If the initial number of colors is* $k \leqslant (n/\log n)^{1/4}$ *and the initial configuration is* $\mathbf{c} = (c_1, \ldots, c_k)$ *with* $\max\{c_j : j = 1, \ldots, k\} \leqslant \frac{n}{k} + \left(\frac{n}{k}\right)^{1-\varepsilon}$, *for some* $\varepsilon > 0$, *then* $\tau = \Omega(k \log n)$ *w.h.p.*

Idea of the proof. For a color $j \in [k]$ let us denote the difference $C_j - n/k$ as the *positive imbalance*. In Lemma 6 we proved that, as long as the positive imbalance of a color is smaller than n/k, this will increase by a factor smaller than $(1+3/k)$ at every round (w.h.p.). Hence, if a color starts with a positive imbalance smaller than $(n/k)^{1-\varepsilon}$, for some $\varepsilon > 0$, then it will take $\Omega(k \log n)$ rounds to reach an imbalance of n/k w.h.p. By union bounding on all the colors, we can get the stated lower bound. $\square$

Proof Observe that for any $T \leqslant \alpha k \log n$, where α is a suitable positive constant, it holds that

$$(1 + 3/k)^T (n/k)^{1-\varepsilon} \leqslant \frac{n}{k}$$

Since in the initial configuration $\mathbf{c}$ for any color $j \in [k]$ we have that $c_j \leqslant n/k + (n/k)^{1-\varepsilon}$, for any round $T \leqslant c k \log n$ it holds that

$$\mathbf{P_c}\left(C_j^{(T)} = n\right) \leqslant \mathbf{P_c}\left(C_j^{(T)} \geqslant 2\frac{n}{k}\right)$$
$$\leqslant \mathbf{P_c}\left(C_j^{(T)} \geqslant \frac{n}{k} + \left(1 + \frac{3}{k}\right)^T \left(\frac{n}{k}\right)^{1-\varepsilon}\right)$$

where, for an event $\mathcal{E}$, we use notation $\mathbf{P_c}(\mathcal{E})$ as a shortcut for $\mathbf{P}\left(\mathcal{E} \mid \mathbf{C}^{(0)} = \mathbf{c}\right)$.

Let us name $Z_j^{(t)} = C_j(t) - n/k$ and $z_j = c_j - n/k$. By hypothesis we have that $z_j \leqslant (n/k)^{1-\varepsilon}$ and the above inequality can be rewritten as

$$\mathbf{P_c}\left(C_j^{(T)} = n\right) \leqslant \mathbf{P_c}\left(Z_j^T \geqslant \left(1 + \frac{3}{k}\right)^T z_j\right) \tag{6}$$

Observe that if we have $Z_j^{(T)} \geqslant (1 + 3/k)^T z_j$, then a round t with $0 \leqslant t \leqslant T - 1$ must exist such that $Z_j^{(t)} \leqslant b$ and $Z_j^{(t+1)} \geqslant (1 + 3/k)b$ for some value b, with $k\sqrt{n \log n} \leqslant b \leqslant n/k$, thus

$$\mathbf{P_c}\left(Z_j^{(T)} \geqslant \left(1 + \frac{3}{k}\right)^T z_j\right) \tag{7}$$

$$\leqslant \mathbf{P_c}\left(\exists t \leqslant T - 1 : \begin{array}{c} Z_j^{(t)} \leqslant b \text{ and} \\[6pt] Z_j^{(t+1)} \geqslant \left(1 + \frac{3}{k}\right) b \end{array}\right) \tag{8}$$

$$\leqslant \sum_{t=0}^{T-1} \mathbf{P_c}\left(\begin{array}{c} Z_j^{(t)} \leqslant b_t \text{ and} \\[6pt] Z_j^{(t+1)} \geqslant \left(1 + \frac{3}{k}\right) b_t \end{array}\right) \tag{9}$$

where the inequality from (4.1) to (4.1) holds for some b with $k\sqrt{n \log n} \leqslant b \leqslant n/k$, and the inequality from (4.1) to (4.1) holds for some $b_0, \ldots, b_{T-1}$ with $k\sqrt{n \log n} \leqslant b_t \leqslant n/k$ for every $t = 0, \ldots, T - 1$. Now observe that

$$\begin{aligned} &\mathbf{P_c}\left(\begin{array}{c} Z_j^{(t)} \leqslant b_t \text{ and} \\[6pt] Z_j^{(t+1)} \geqslant \left(1 + \frac{3}{k}\right) b_t \end{array}\right) \\[8pt] &= \sum_{a \leqslant b_t} \mathbf{P_c}\left(\begin{array}{c} Z_j^{(t)} = a \text{ and} \\[6pt] Z_j^{(t+1)} \geqslant \left(1 + \frac{3}{k}\right) b_t \end{array}\right) \\[8pt] &= \sum_{a \leqslant b_t} \mathbf{P_c}\left(Z_j^{(t+1)} \geqslant \left(1 + \frac{3}{k}\right) b_t \mid Z_j^{(t)} = a\right) \cdot \\[8pt] &\qquad\qquad \cdot \mathbf{P_c}\left(Z_j^{(t)} = a\right) \\[8pt] &\leqslant \frac{1}{n^2} \sum_{a \leqslant b_t} \mathbf{P_c}\left(Z_j^{(t)} = a\right) \leqslant \frac{1}{n^2}, \end{aligned} \tag{10}$$

where in the last line we used Lemma 6.

By combining (4.1), (4.1), and (4.1) we get that, for every color $j \in [k]$, if the initial number of nodes colored j is $c_j \leqslant n/k + (n/k)^{1-\varepsilon}$ at any round $T \leqslant c\, k \log n$ the probability that all nodes are colored j is at most T/n^2. The probability that $\mathbf{C}^{(T)}$ is monochromatic is thus at most $(kT)/n^2 \leqslant n^{-\alpha}$ for some positive constant α. $\qquad\square$

It may be worth noticing that what we actually prove in Theorem 2 is that $\Omega(k \log n)$ rounds are required in order to go from a configuration where the majority color has at most $n/k + (n/k)^{1-\varepsilon}$ nodes to a configuration where it has $2n/k$ colors.

4.2 A negative result for 3-input dynamics

In order to prove that dynamics that differ from the majority ones do not solve plurality consensus, we first give some formal definitions of the dynamics we are considering.

Definition 1 (*h*-**input dynamics**) An *h*-dynamics is a synchronous protocol where at each round every node picks h random neighbors (including itself and with repetition) and recolors itself according to some deterministic rule that depends only on the colors it sees. Let $\mathcal{D}_h(k)$ be the class of *h*-dynamics and observe that a dynamics $\mathcal{P} \in \mathcal{D}_h$ can be specified by a function

$$f : [k]^h \to [k], \text{ s.t. } f(x_1, \ldots, x_h) \in \{x_1, \ldots, x_h\},$$

where $f(x_1, \ldots, x_h)$ is the color chosen by a node that sees the (ordered) sequence $(x_1, \ldots, x_h)$ of colors.

In the class $\mathcal{D}_3(k)$, there is a subset $\mathcal{M}^3$ of equivalent protocols called 3-majority dynamics having two key-properties described below: the clear-majority and the uniform one.

Definition 2 (**Clear-majority property**) Let $(x_1, x_2, x_3) \in [k]^3$ be a triple of colors. We say that (x_1, x_2, x_3) has a *clear majority* if at least two of the three entries have the same value. A dynamics $\mathcal{P} \in \mathcal{D}_3(k)$ has the *clear-majority* property if whenever its f sees a clear majority it returns the majority color.

Given any 3-input dynamics function $f(x_1, x_2, x_3)$, for any triple of distinct colors $r, g, b \in [k]$, let $\Pi(r, g, b)$ be the subset of permutations of the colors r, g, b and define the following "counters":

$$\delta_r = |\{(z_1, z_2, z_3) \in \Pi(r, g, b), \, s.t. \, f(z_1, z_2, z_3) = r\}|,$$
$$\delta_g = |\{(z_1, z_2, z_3) \in \Pi(r, g, b), \, s.t. \, f(z_1, z_2, z_3) = g\}|,$$
$$\delta_b = |\{(z_1, z_2, z_3) \in \Pi(r, g, b), \, s.t. \, f(z_1, z_2, z_3) = b\}|.$$

Observe that for any 3-input dynamics it must hold $\delta_g + \delta_r + \delta_b = 6$.

Definition 3 (**Uniform property**) A dynamics $\mathcal{P} \in \mathcal{D}_3(k)$ has the *uniform* property if, for any triple of distinct colors $r, g, b \in [k]$, it holds that $\delta_r = \delta_g = \delta_b (= 2)$.

Informally speaking, the clear-majority and the uniform properties provide a clean characterization of those dynamics that are good solvers for plurality consensus. This fact is formalized in the next definitions and in the final theorem.

Definition 4 (3-**majority dynamics**) A protocol $\mathcal{P} \in \mathcal{D}_3(k)$ belongs to the class $\mathcal{M}^3 \subset \mathcal{D}_3(k)$ of *3-majority dynamics* if its function $f(x_1, x_2, x_3)$ has the clear-majority and the uniform properties.

Definition 5 ((s, ε)-**plurality consensus solver**) We say that a protocol $\mathcal{P}$ is an (s, ε)-*solver* (for the plurality consensus problem) if for every initial s-biased configuration **c**, when running $\mathcal{P}$, with probability at least $1 - \varepsilon$ there is a round t by which all nodes gets the plurality color of c.

Let us observe that, by definition of h-dynamics (see Definition 1), any monochromatic configuration is an absorbing state of the relative Markov process. Moreover, the smaller s and ε the better an (s, ε)-solver is; in other words, if a dynamics is an (s, ε)-solver then it is also an (s', ε')-solver for every $s' \geqslant s$ and $\varepsilon' \geqslant \varepsilon$. In Sect. 3, we showed that any dynamics in $\mathcal{M}^3$ is a $(\Theta(\sqrt{\min\{2k, (n/\log n)^{1/3}\}n\log n}), \Theta(1/n))$-solver in $\mathcal{D}_3$. We can now state the main result of this section.

Theorem 3 (properties of good solvers) *Given a protocol $\mathcal{P}$, the following hold:*

(a) *If $\mathcal{P}$ is an $(n/4, 1/4)$-solver in $\mathcal{D}_3$, then its f must have the clear-majority property.*
(b) *A constant $\eta > 0$ exists such that, if $\mathcal{P}$ is an $(\eta \cdot n, 1/4)$-solver, then its f must have the uniform property.*

The above theorem also provides the clear reason why some dynamics can solve consensus but cannot solve plurality consensus in the non-binary case. A relevant example is the *median* dynamics studied in [7]: it has the clear-majority property but not the uniform one.

For readability sake, we split the proof of the above theorem in two technical lemmas: in the first one, we show the first claim about clear majority while in the second lemma we show the second claim about the uniform property.

Lemma 7 (Clear majority) *If a protocol $\mathcal{P} \in \mathcal{D}_3$ is an $(n/4, 1/4)$-solver, then it chooses the majority color every time there is a triple with a clear majority.*

Proof For every triple of colors $(x_1, x_2, x_3) \in [k]^3$ that has a clear majority, let us define $\delta(x_1, x_2, x_3)$ to be 1 if protocol $\mathcal{P}$ behaves like the majority protocol over triple (x_1, x_2, x_3) and 0 otherwise. Consider an initial configuration with only two colors, say red (r) and blue (b), with c_r red nodes and $c_b = n - c_r$ blue nodes. Let us define Δ_r and Δ_b as follows

$$\Delta_r = \delta(r, r, b) + \delta(r, b, r) + \delta(b, r, r),$$
$$\Delta_b = \delta(b, b, r) + \delta(b, r, b) + \delta(r, b, b).$$

We can write the probability that a node chooses color red as

$$p(r) = \left(\frac{c_r}{n}\right)^3 + \left(\frac{c_r}{n}\right)^2 \frac{c_b}{n} \cdot \Delta_r + \left(\frac{c_b}{n}\right)^2 \frac{c_r}{n} (3 - \Delta_b)$$
$$= \frac{c_r}{n^3} \left(c_r^2 + c_b (c_r \Delta_r - c_b \Delta_b) + 3c_b^2\right).$$

Observe that for a majority protocol we have that $\Delta_r = \Delta_b = 3$. In what follows we show that if this is not the case then there are configurations where the majority color does not increase in expectation. We distinguish two cases, case $\Delta_r \neq \Delta_b$ and case $\Delta_r = \Delta_b$.

Case $\Delta_r \neq \Delta_b$: Suppose w.l.o.g. that $\Delta_r < \Delta_b$, and observe that since they have integer values it means $\Delta_r \leqslant \Delta_b - 1$. Now we show that, if we start from a configuration where the red color has the majority of nodes, the number of red nodes decreases in expectation. By using $\Delta_r \leqslant \Delta_b - 1$ in (4.2) we get

$$p(r) \leqslant \frac{c_r}{n^3} \left(c_r^2 + c_b(c_r - c_b)\Delta_b - c_r c_b + 3c_b^2\right).$$

If the majority of nodes is red then $c_r - c_b$ is positive, and since Δ_b can be at most 3 from (4.2) we get

$$p(r) \leqslant \frac{c_r}{n^3} \left(c_r^2 + 2c_r c_b\right).$$

Finally, if we put $c_r = n/2 + s$ and $c_b = n/2 - s$, for some positive s, in (4.2), we get that

$$p(r) \leqslant \frac{c_r}{n^3} \left(\frac{3}{4}n^2 + (n - s)s\right) \leqslant \frac{c_r}{n}.$$

Case $\Delta_r = \Delta_b$: When $\Delta_r = \Delta_b$, observe that if the protocol is not a majority protocol then it must be $\Delta_r = \Delta_b \leqslant 2$. Hence, if we start again from a configuration where $c_r \geqslant c_b$, from (4.2) we get that

$$p(r) \leqslant \frac{c_r}{n^3} \left(c_r^2 + 2c_b(c_r - c_b) + 3c_b^2\right) = \frac{c_r}{n}.$$

In both cases, for any protocol $\mathcal{P}$ that does not behave like a majority protocol on triples with a clear majority, if we name X_t the random variable indicating the number of red nodes at round t, from (4.2) and (4.2) we get that $\mathbf{E}\left[X_{t+1} \mid X_t\right] \leqslant X_t$, hence X_t is a supermartingale. Now let τ be the random variable indicating the first time the chain hits one of the two absorbing states, i.e.,

$$\tau = \inf\{t \in \mathbb{N} : X_t \in \{0, n\}\}.$$

Since $\mathbf{P}(\tau < \infty) = 1$ and all X_t's have values bounded between 0 and n, from the martingale stopping theorem[5] we get that $\mathbf{E}[X_\tau] \leqslant \mathbf{E}[X_0]$. If we start from a configuration that is $n/4$-imbalanced in favor of the red color, we have that $X_0 = n/2 + n/8$, and if we call ε is the probability that the process ends up with all blue nodes we have that $\mathbf{E}[X_\tau] = (1 - \varepsilon)n$. Hence it must be $(1 - \varepsilon)n \leqslant n/2 + n/8$ and the

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[5] See e.g., Chapter 17 in [15] for a summary of martingales and related results.

probability to end up with all blue nodes is $\varepsilon \geqslant 5/8 > 1/4$. Thus the protocol is not a $(n/4, 1/4)$-solver. □

Lemma 8 (Uniform property) *A constant $\eta > 0$ exists such that, if $\mathcal{P}$ is an $(\eta n, 1/4)$-solver, then its f must have the uniform property.*

Proof Thanks to the previous lemma, we can assume that f has the clear-majority property but a triple (r, g, b) exists such that $\delta_r < \max\{\delta_g, \delta_b\}$. Let us start the process with the following initial configuration having only the above 3 colors and then show that the process w.h.p. will not converge to the plurality color r. Let

$$\mathbf{c} = (c_r, c_g, c_b) = (n/3 + s, \; n/3, \; n/3 - s)$$

where $s = \Theta(\sqrt{n \log n})$. We consider the "hardest" case where $\delta_r = 1$: the case $\delta_r = 0$ is simpler since in this case, no matter how the other $\delta's$ are distributed, it is easy to see that the r.v. c_r will decrease exponentially to 0 starting from the above configuration.

- Case $\delta_r = 1$, $\delta_g = 3$, and $\delta_b = 2$ (and color-symmetric cases). Starting from the above initial configuration, we can compute the probability $p(r) = \mathbf{P}(X_v = r \mid C = \mathbf{c})$ that a node gets the color r.

$$p(r) = \left(\frac{c_r}{n}\right)^3 + 3\left(\frac{c_r}{n}\right)^2 \frac{n - c_r}{n} + \frac{c_r c_g c_b}{n^3}$$
$$= \frac{n + 3s}{3n^3}\left(\left(\frac{n}{3} + s\right)^2 + 3\left(\frac{n}{3} + s\right)\left(\frac{2}{3}n - s\right)\right.$$
$$\left. + \left(\frac{n}{3}\right)\left(\frac{n}{3} - s\right)\right).$$

After some easy calculations, we get

$$p(r) = \frac{8}{27}\left(1 + O\left(\frac{s}{n}\right)\right).$$

As for $p(g)$, by similar calculations, we obtain the following bound

$$p(g) = \frac{10}{27}\left(1 - O\left(\frac{s^2}{n^2}\right)\right).$$

From the above two equations, we get the following bounds on the expectation of the r.v.'s X^r and X^g counting the nodes colored with r and g, respectively (at the next round).

$$\mathbf{E}\left[X^r \mid C = \mathbf{c}\right] \leqslant \frac{8}{27}n + O(s) \text{ and}$$

$$\mathbf{E}\left[X^g \mid C = \mathbf{c}\right] \geqslant \frac{10}{27}n - O\left(\frac{s^2}{n}\right).$$

By a standard application of Chernoff's Bound, we can prove that, if $s \leqslant \eta n$ for a sufficiently small $\eta > 0$, the initial value

c_r will w.h.p. decreases by a constant factor, going much below the new plurality c_g. Then, by applying iteratively the above reasoning we get that the process will not converge to r, w.h.p.

- Case $\delta_r = 1$, $\delta_g = 4$, and $\delta_b = 1$ (and color-symmetric cases). In this case it is even simpler to show that w.h.p., starting from the same initial configuration considered in the previous case, the process will not converge to color r. □

4.3 A lower bound for h-plurality

In Sect. 4.1, we have shown that the 3-majority protocol takes $\Theta(k \log n)$ rounds w.h.p. to converge in the worst case. A natural question is whether by using the h-plurality protocol, with h slightly larger than 3, it is possible to significantly speed-up the process. We prove that this is not the case.

Let us consider a set of n nodes, each node colored with one out of k colors. The h-plurality protocol works as follows:

> At every round, every node picks h nodes uniformly at random (including itself and with repetitions) and recolors itself according to the plurality of the colors it sees (breaking ties u.a.r.)

Let $j \in [k]$ be an arbitrary color, in the next lemma we prove that, if the number of j-colored nodes is smaller than $2n/k$ and if $k/h = \mathcal{O}(n^{(1-\varepsilon)/4})$, then the probability that the number of j-nodes increases by a factor $(1 + h^2/k)$ is exponentially small.

Lemma 9 *Let $\mathbf{c} = (c_1, \ldots, c_k)$ be a configuration and let $j \in [k]$ be a color such that $(n/k) \leqslant c_j \leqslant 2(n/k)$. If $k/h = \mathcal{O}(n^{(1-\varepsilon)/4})$ then it holds that*

$$\mathbf{P}\left(C_j^{(t+1)} \geqslant \left(1 + \frac{h^2}{k}\right)c_j \;\middle|\; \mathbf{C}^{(t)} = \mathbf{c}\right) \leqslant e^{-\Theta(n^\varepsilon)}.$$

Proof Consider a specific node, say $u \in [n]$, let N_j be the number of j-colored nodes picked by u during the sampling stage of the tth round and let Y be the indicator random variable of the event that node u chooses color j at round $t + 1$. We give an upper bound on the probability of the event $Y = 1$ by conditioning it on $N_j = 1$ and $N_j \geqslant 2$ (observe that if $N_j = 0$ node u cannot choose j as its color at the next round)

$$\mathbf{P}(Y_u = 1)$$
$$\leqslant \mathbf{P}(Y_u = 1 \mid N_j = 1)\mathbf{P}(N_j = 1) + \mathbf{P}(N_j \geqslant 2).$$

Now observe that

- $\mathbf{P}(Y_u = 1 \mid N_j(u) = 1) \leqslant 1/h$ since it is exactly $1/h$ if all other sampled nodes have distinct colors and it is 0 otherwise;

– $\mathbf{P}\left(N_j = 1\right) \leqslant hc_j/n$ since it can be bounded by the probability that at least one of the h samples gives color j;

– $\mathbf{P}\left(N_j \geqslant 2\right) \leqslant \binom{h}{2}c_j^2/n^2$ since it is the probability that a pair of sampled nodes exist with the same color j.

Hence, in (4.3) we have that

$$\mathbf{P}\left(Y = 1\right) \leqslant \frac{c_j}{n} + \frac{h^2}{2} \cdot \frac{c_j^2}{n^2}.$$

Thus, for the expected number of j-colored nodes at the next round we get

$$\mathbf{E}\left[C_j^{(t+1)} \mid \mathbf{C}^{(t)} = \mathbf{c}\right]$$
$$\leqslant c_j + \frac{h^2}{2n}c_j^2 = c_j\left(1 + \frac{h^2}{2n}c_j\right) \leqslant c_j\left(1 + \frac{h^2}{k}\right),$$

where in the last inequality we used the hypothesis $c_j \leqslant 2(n/k)$. Since $C_j^{(t+1)}$ conditional on $\{\mathbf{C}^{(t)} = \mathbf{c}\}$ is a sum of n independent Bernoulli random variables, from the Chernoff bound (Lemma 11 with $\lambda = c_j h^2/k$), we finally get

$$\mathbf{P}\left(C_j^{(t+1)} \geqslant c_j\left(1 + 2\frac{h^2}{k}\right) \,\middle|\, \mathbf{C}^{(t)} = \mathbf{c}\right)$$
$$\leqslant \exp\left(-\frac{2(c_j h^2/k)^2}{n}\right) \leqslant \exp\left(-\Omega(n^\varepsilon)\right),$$

where in the last inequality we used $c_j \geqslant n/k$ and $k/h = \mathcal{O}(n^{(1-\varepsilon)/4})$. $\qquad\square$

By adopting a similar argument to that used for proving Theorem 2, we can get a lower bound $\Omega\left(k/h^2\right)$ on the completion time of the h-plurality.

Theorem 4 *Let $\mathbf{C}^{(t)}$ be the random variable indicating the configuration at round t according to the h-plurality protocol and let $\tau = \inf\{t \in \mathbb{N} : \mathbf{C}^{(t)}$ is monochromatic$\}$. If the initial configuration $\mathbf{c} = (c_1, \ldots, c_k)$ is such that $\max\{c_j : j = 1, \ldots, k\} \leqslant 3n/(2k)$ then $\tau = \Omega(k/h^2)$ w.h.p.*

Proof Since in the initial configuration for any color $j \in [k]$ we have that $c_j \leqslant 3n/(2k)$, from Lemma 9 it follows that the number of nodes supporting the plurality color increases at a rate smaller than $(1 + 2h^2/k)$ with probability exponentially close to 1. This easily implies a recursive relation of the form $C_j^{(t+1)} \leqslant \left(1 + 2h^2/k\right) C_j^{(t)}$ which, in turn, gives

$$C_j^{(t)} \leqslant \left(1 + \frac{2h^2}{k}\right)^t C_j^{(0)} \leqslant \left(1 + \frac{2h^2}{k}\right)^t \frac{3n}{2k}.$$

Thus, for $t < k/h^2 \log(4/3)$, w.h.p. we have that

$$C_j^{(t)} \leqslant \frac{3n}{2k}\left(1 + \frac{2h^2}{k}\right)^t < \frac{2n}{k},$$

concluding the proof. $\qquad\square$

4.4 On the initial bias

In this section, we show that there are initial configurations with bias $s = \mathcal{O}(\sqrt{kn})$ for which the bias decreases in a single round with constant probability. This shows that under initial imbalances of this magnitude, it seems unlikely that one can prove bounds as those shown in Sect. 3, at least with high probability.

Lemma 10 *Assume $k \geqslant 4$. For any value $s \leqslant \sqrt{kn}/6$ of the initial bias, there are initial configurations $\mathbf{c}$ such that, for any fixed color $j \neq 1$ we have:*

$$\mathbf{P}\left(C_1^{(1)} - C_j^{(1)} < s \mid \mathbf{C}^{(0)} = \mathbf{c}\right) \geqslant \frac{1}{16e}.$$

Proof We consider an initial configuration $\mathbf{c}$ in which we have k colors. Let $x = (n - s)/k$ (we neglect integer parts for the sake of the analysis). We let $c_1^{(0)} = x + s$ and $c_j^{(0)} = x$, for $j \neq 1$ and we further assume that $s \leqslant x$. Considered any fixed $j \neq 1$, we next prove that $C_1^{(1)} - C_j^{(1)} < s$ with constant probability.

The outline of the proof is as follows: We first show that

$$\mathbf{E}\left[C_1^{(1)} \mid \mathbf{C}^{(0)} = \mathbf{c}\right] - \mathbf{E}\left[C_j^{(1)} \mid \mathbf{C}^{(0)} = \mathbf{c}\right]$$
$$\leqslant s + 3xs/n,$$

then we observe that with constant probablity $C_1^{(1)}$ is not above its expectation. Finally, we prove that

$$C_j^{(1)} > \mathbf{E}\left[C_j^{(1)} \mid \mathbf{C}^{(0)} = \mathbf{c}\right] + 3xs/n$$

with constant probability, whenever $s \leqslant \sqrt{kn}/6$, which concludes the proof of the lemma.

To begin with, from Lemma 1, we easily get the following derivations:

$$\mathbf{E}\left[C_1^{(1)} \mid \mathbf{C}^{(0)} = \mathbf{c}\right]$$
$$= x + s + \frac{x^2}{n} + \frac{2xs + s^2}{n} - \frac{x + s}{n^2}\gamma$$
$$\geqslant x - \frac{2x^2 s}{n^2},$$
$$\mathbf{E}\left[C_j^{(1)} \mid \mathbf{C}^{(0)} = \mathbf{c}\right] = x + \frac{x^2}{n} - \frac{x}{n^2}\gamma, \qquad (11)$$

where $\gamma = \sum_h c_h^2 = nx + xs + s^2$.

We next have:

$$\mathbf{E}\left[C_1^{(1)} \mid \mathbf{C}^{(0)} = \mathbf{c}\right] - \mathbf{E}\left[C_j^{(1)} \mid \mathbf{C}^{(0)} = \mathbf{c}\right]$$

$$= s + \frac{2xs + s^2}{n} - \frac{x+s}{n^2}\gamma + \frac{x}{n^2}\gamma$$

$$\leqslant s + \frac{2xs + s^2}{n} - \frac{s}{n^2}\gamma \leqslant s + \frac{3xs}{n},$$

where the last inequality follows from the assumption that $s \leqslant x$.

For convenience sake, let us name for any $j \in [k]$

$$\mu_j := \mathbf{E}\left[C_j^{(1)} \mid \mathbf{C}^{(0)} = \mathbf{c}\right]$$

We note that

$$\mathbf{P}\left(C_1^{(1)} - C_j^{(1)} < s \mid \mathbf{C}^{(0)} = \mathbf{c}\right)$$

$$\geqslant \mathbf{P}\left(C_1^{(1)} < \mu_1 \wedge C_j^{(1)} \geqslant \mu_j + \frac{3xs}{n} \mid \mathbf{C}^{(0)} = \mathbf{c}\right).$$

Fact 1 *The following holds:*

$$\mathbf{P}\left(C_1^{(1)} < \mu_1 \wedge C_j^{(1)} \geqslant \mu_j + \frac{3xs}{n} \,\middle|\, \mathbf{C}^{(0)} = \mathbf{c}\right)$$

$$\geqslant \mathbf{P}\left(C_1^{(1)} < \mu_1 \,\middle|\, \mathbf{C}^{(0)} = \mathbf{c}\right)$$

$$\cdot \mathbf{P}\left(C_j^{(1)} \geqslant \mu_j + \frac{3xs}{n} \,\middle|\, \mathbf{C}^{(0)} = \mathbf{c}\right).$$

Proof We use the following observation: if A, B and C are events over the same probability space, we have

$$\mathbf{P}\left(\overline{A} \wedge B \mid C\right) \geqslant \mathbf{P}\left(\overline{A} \mid C\right)\mathbf{P}\left(B \mid C\right)$$

$$\iff$$

$$\mathbf{P}\left(A \wedge B \mid C\right) \leqslant \mathbf{P}\left(A \mid C\right)\mathbf{P}\left(B \mid C\right).$$

In our case we have:

$$\mathbf{P}\left(C_1^{(1)} \geqslant \mu_1 \wedge C_j^{(1)} \geqslant \mu_j + \frac{3xs}{n} \,\middle|\, \mathbf{C}^{(0)} = \mathbf{c}\right)$$

$$\leqslant \mathbf{P}\left(C_1^{(1)} \geqslant \mu_1 \,\middle|\, \mathbf{C}^{(0)} = \mathbf{c}\right)$$

$$\cdot \mathbf{P}\left(C_j^{(1)} \geqslant \mu_j + \frac{3xs}{n} \,\middle|\, \mathbf{C}^{(0)} = \mathbf{c}\right),$$

where the inequality follows from [9, Proposition 3, claim (-OD)]. In particular, $\left\{C_1^{(1)} \geqslant \mu_1\right\}$ and $\left\{C_j^{(1)} \geqslant \mu_j + 3xs/n\right\}$ are the events that the numbers of balls thrown independently at random into two distinct bins both exceed some given thresholds. This and the observation above immediately imply the claim. □

Fact 2 *The following holds:*

$$\mathbf{P}\left(C_1^{(1)} < \mu_1 \mid \mathbf{C}^{(0)} = \mathbf{c}\right) > \frac{1}{4}.$$

Proof Set $X = n - C_1^{(1)}$. Clearly, X is distributed as a binomial $B(n, p)$, where $p = 1 - p_1$, with p_1 the probability that the generic node recolors itself with color 1. Clearly, $p > 1/n$ as long as the number of colors is not too large (in the order of n). Then we have:

$$\mathbf{P}\left(C_1^{(1)} < \mu_1 \mid \mathbf{C}^{(0)} = \mathbf{c}\right)$$

$$= \mathbf{P}\left(X \geqslant \mathbf{E}[X] \mid \mathbf{C}^{(0)} = \mathbf{c}\right) > \frac{1}{4},$$

where the second inequality follows from [11, Theorem 1]. □

We finally apply Theorem 5 to $C_j^{(1)}$ and we have:

$$\mathbf{P}\left(C_j^{(1)} > \mu_j + \frac{3xs}{n} \mid \mathbf{C}^{(0)} = \mathbf{c}\right)$$

$$\geqslant \frac{1}{4}e^{-\frac{18x^2s^2}{n^2\mu_j}} \geqslant \frac{1}{4}e^{-\frac{18xs^2}{n^2-2xs}} \geqslant \frac{1}{4e}, \tag{12}$$

where the second inequality follows from (4.4) and the third one holds since $s \leqslant \sqrt{kn}/6$ and recalling that $x \leqslant n/k$. Finally, from Fact 2 and (4.4), we get the claim. □

5 Open questions

A general open question on the plurality consensus problem is whether a *dynamics* exists that achieves plurality consensus in polylogarithmic time for any function $k = k(n)$. By *dynamics* we mean synchronous distributed algorithms with a very simple structure, whereby every node, at every round, updates its state according to its previous state and a symmetric function of the states of its neighbors; namely, the state of a node at round t depends only on its state and the multiset of states of its neighbors at round $t-1$, while the update rule is the same for every graph and every node at every round. Note that this definition implies that the update rule cannot depend on n and the network is *anonymous*, that is, nodes do not possess distinguished identities and the

A more specific question about our simple distributed model is to explore the case in which the initial bias s is smaller than the lower bound assumed in our analysis (i.e., $s \geqslant c\sqrt{\min\{2k, (n/\log n)^{1/3}\}n\log n}$).

Notice that when k is polylogarithmic, we required a bias which is only a polylogarithmic factor larger than the standard deviation $\Omega(\sqrt{n})$: the latter is a lower bound for the initial bias to converge (w.h.p.) to the plurality color. As for

larger k, we did not derive any stronger bound on the required bias, however, in Sect. 4.4, we have shown some initial configurations with bias $s = \mathcal{O}(\sqrt{kn})$ for which the initial bias *decreases* in a single round with constant probability. This result implies that, when the initial bias s is "slightly" smaller than "ours", the process may be *non-monotone* w.r.t. the bias function $s(t)$. The fact that $s(t)$ is an increasing function played a key-role in the proof of our upper bound. So, under such a weaker assumption, if any upper bound similar to ours might be proved then a much more complex argument (departing from ours) seems to be necessary.

In this work, we were interested in deriving sufficient conditions under which the h-plurality dynamics converges in polylogarithmic time. A further interesting open question is to derive conditions on the parameters k, s, and h under which this dynamics converges very fast, i.e., in sublogarithmic time.

Useful bounds

Lemma 11 (Chernoff bounds) *Let $X = \sum_{i=1}^n X_i$ where X_i's are independent Bernoulli random variables and let $\mu = \mathbf{E}[X]$. Then,*

1. *For any $0 < \delta \leqslant 4$, $\mathbf{P}(X > (1+\delta)\mu) < e^{-\frac{\delta^2 \mu}{4}}$;*
2. *For any $\delta \geqslant 4$, $\mathbf{P}(X > (1+\delta)\mu) < e^{-\delta\mu}$;*
3. *For any $\lambda > 0$, $\mathbf{P}(X \geqslant \mu + \lambda) \leqslant e^{-2\lambda^2/n}$.*

Lemma 12 (Jensen inequality) *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function and $x_1, \ldots, x_k \in \mathbb{R}$ be k real numbers, then*

$$\phi\left(\frac{1}{k}\sum_{i=1}^k x_i\right) \leqslant \frac{1}{k}\sum_{i=1}^k \phi(x_i).$$

In Sect. 4.4, we use the following "reverse"-Chernoff bound [19, Theorem 2] [6]

Theorem 5 (Reverse Chernoff bound) *Let X be the sum of m independent Bernoulli variables with probability $p \leqslant 1/4$ and let $\mu = pm$. Then, for any $t \in (0, m - \mu)$:*

$$\mathbf{P}(X - \mu > t) \geqslant \frac{1}{4}e^{-\frac{2t^2}{\mu}}.$$

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[6] A number of pretty similar "folklore" results can be found in specialized mathematical forums, for example http://cstheory.stackexchange.com/questions/14471/reverse-chernoff-bound.

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