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SHARP CONCENTRATION OF THE CHROMATIC NUMBER ON RANDOM GRAPHS $G_{n,p}$

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The distribution of the chromatic number on random graphs $G_{n,p}$ is quite sharply concentrated. For fixed p it concentrates almost surely in $\sqrt{n}\,\omega(n)$ consecutive integers where $\omega(n)$ approaches infinity arbitrarily slowly. If the average degree pn is less than $n^{1/6}$, it concentrates almost surely in five consecutive integers. Large deviation estimates for martingales are used in the proof.

1. Graphs

Given parameters n, p let $G_{n,p}$ denote the random graph with vertex set $[n]=\{1, \ldots, n\}$ and edge probability p. More precisely, let (Ω, P) denote the discrete probability space with Ω equal to the set of all graphs on $[n]$ and probability P generated by

$$P[\{i,j\}\in G] = p \tag{1}$$

these events mutually independent over all $\{i,j\}\in[n]^2$. For convenience we set

$$d = pn \tag{2}$$

so that d is the average degree. We shall be particularly interested in the cases p constant and $p=n^{-\alpha}$, $0<\alpha<1$, though our results are more general.

The chromatic number $\chi(G)$ has been the object of much study. (See, e.g., [1—4, 7]). For constant p

$$(1-o(1))\frac{n\ln(1/(1-p))}{2\ln n} \leqq \chi(G) \leqq (1+o(1))\frac{n\ln(1/(1-p))}{\ln n} \tag{3}$$

while for $p\to 0$

$$(1-o(1))\frac{d}{2\ln d} \leqq \chi(G) \leqq (1+o(1))\frac{d}{\ln d} \tag{4}$$

almost surely. The factor of two between the bounds is most vexing. We do not here contribute directly to its elimination.

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We shall show that the distribution of $\chi(G)$ is tightly concentrated. We say a sequence $X(G)$ with $p=p(n)$ is *concentrated in width* $s=s(n,p)$ if there exist $u=u(n;p)$ so that

$$P[u \leqq X(G) \leqq u+s] \to 1. \tag{5}$$

We show that s may be taken relatively small. It is both a strength and weakness of the method that u is not given explicitly.

We let $\omega(n)$ denote a function of n approaching infinity arbitrarily slowly.

Theorem 1. *For any p, $\chi(G)$ is concentrated in width $\sqrt{n}\,\omega(n)$.*

Theorem 2. *Let $p=n^{-\alpha}$*

(i) For $0<\alpha<0.5$ $\chi(G)$ is concentrated in width $s=n^{0.5-\alpha}(\ln n)\omega(n)$

(ii) For $0.5<\alpha<1$ $\chi(G)$ is concentrated in width s where s is the minimal integer greater than $(2\alpha+1)/(2\alpha-1)$.

An extreme case occurs when $\alpha>5/6$. Then five consecutive numbers suffice in that almost surely $\chi(G)=u, u+1, u+2, u+3$ or $u+4$. This holds also for $p=d(n)/n$ provided $d(n)<n^{1/6}$.

Note. By Chebyschev's Inequality any random variable X is concentrated in width $\sqrt{\mathrm{Var}(X)}\,\omega(n)$. While using of this inequality, often called the *second-moment method*, has worked in some related questions it does not appear applicable to the study of the chromatic number.

Our basic tool is a *martingale* known as *Doob's Martingale Process*. (See, e.g., [5, 6].) In general, let $X:\Omega\to\mathbf{R}$ be a random variable and let $\equiv_i$, $0\leqq i\leqq n$, be a sequence of equivalence relations on Ω reverse ordered by refinement. (That is, $G\equiv_{i+1}G'$ implies $G\equiv_i G'$.) Now define $X_0, X_1, \ldots, X_n$ by

$$X_i(G) = E[X(H)|H \equiv_i G]. \tag{6}$$

This forms a margingale—i.e.

$$E[X_{i+1}|X_i] = X_i. \tag{7}$$

In this work we define $\equiv_i$ by setting $G\equiv_i H$ if for all $1\leqq x\leqq i$, $1\leqq y\leqq n$, $\{x,y\}\in G$ iff $\{x,y\}\in H$. Note that $\equiv_0$ is universal so that

$$X_0(G) = E[X(G)], \tag{8}$$

a constant, whereas $\equiv_n$ is equality (*not* just isomorphism, these are labelled graphs!) so that

$$X_n = X. \tag{9}$$

Thus $X_i(G)$ is the expected value of X on the random graph whose first i vertices are precisely like G's and where all edges $\{x,y\}$, $i+1\leqq x,y$, are in the graph with probability p. The sequence $X_0, \ldots, X_n$ moves from no information to full information in small steps.

The following result bounds large deviations in any martingale.

Theorem 3. *Let $X_0, \dots, X_n$ be a martingale with X_0 constant and $|X_{i+1}-X_i|\leqq c$ for $0\leqq i\leqq n$ always. Then*

$$P[|X_n-X_0| > \lambda c n^{1/2}] < 2e^{-\lambda^2/2}. \tag{10}$$

Lemma 4. *If $E(Y)=0$, $|Y|\leqq 1$, $\alpha>0$ then $E[e^{\alpha Y}]\leqq (1/2)(e^{\alpha}+e^{-\alpha})<e^{\alpha^2/2}$.*

Proof. Set $f(z)=e^{\alpha z}$. If Y has values $z_1, \dots, z_s$ with probabilities $p_1, \dots, p_s$ then

$$\begin{aligned} E[e^{\alpha Y}] &= \sum_{i=1}^{s} p_i f(z_i) \leqq \sum_{i=1}^{s} p_i\left[\left(\frac{1+z_i}{2}\right)f(1)+\left(\frac{1-z_i}{2}\right)f(-1)\right] \quad (\textit{as } f \textit{ convex}) \\ &= \frac{f(1)+f(-1)}{2} < e^{\alpha^2/2} \end{aligned} \tag{11}$$

by examining the Taylor series. ∎

Proof of Theorem 3. Take $c=1$, $X_0=0$ for convenience. Set $Y_i=X_i-X_{i-1}$, $1\leqq i\leqq n$. Let $\alpha>0$. Now

$$\begin{aligned} E[e^{\alpha X_i}] &= E[e^{\alpha Y_i}e^{\alpha X_{i-1}}] \\ &= E\big[E[e^{\alpha Y_i}|X_{i-1}]e^{\alpha X_{i-1}}\big] \\ &\leqq e^{\alpha^2/2}E[e^{\alpha X_{i-1}}] \end{aligned} \tag{12}$$

as for any X_{i-1} the Lemma assures $E[e^{\alpha Y_i}|X_{i-1}]\leqq e^{\alpha^2/2}$. By induction

$$E[e^{\alpha X_n}] \leqq e^{\alpha^2 n/2}. \tag{13}$$

By Markov's Inequality

$$\begin{aligned} P\big[X_n > \lambda\sqrt{n}\big] &= P\big[e^{\alpha X_n} > e^{\alpha\lambda\sqrt{n}}\big] < e^{-\alpha\lambda\sqrt{n}}E[e^{\alpha X_n}] \\ &= \exp\big[\alpha^2 n/2-\alpha\lambda\sqrt{n}\big] = e^{-\lambda^2/2} \end{aligned} \tag{14}$$

by setting $\alpha=\lambda n^{-1/2}$. By symmetry the absolute value gives at most a factor of two. ∎

It will be convenient to define

$$I_i(G) = \{H\colon H \equiv_i G\} \tag{15}$$

$$P^i(H) = \frac{P(H)}{\sum_{J\in I_i(H)} P(J)}. \tag{16}$$

Here P^i represents a conditional probability with edges involving $1, \dots, i$ fixed. Given G, H with $G\equiv_i H$ we define a bijection

$$\varphi\colon I_{i+1}(G) \to I_{i+1}(H) \tag{17}$$

by setting $\varphi(J)=K$ if for all $1\leqq x<y\leqq n$ with $x\neq i+1$, $\{x,y\}\in J$ iff $\{x,y\}\in K$. That is, $\varphi(J)$ is given by changing those edges $\{i+1,y\}$, $i+1<y$, to conform to H. The bijection φ preserves conditional probability

$$\varphi(J) = K \Rightarrow P^{i+1}(J) = P^{i+1}(K) \tag{18}$$

as P^{i+1} depends only on edges $\{x, y\}$ with $i+1<x<y$.

Theorem 5. *Let* X: $\Omega \to \mathbf{R}$ *be the chromatic number. Then*

$$|X_{i+1}(G)-X_i(G)| \leqq 1. \tag{19}$$

An intuitive view may be useful. $X_i(G)$ is the expected value of $X(G)$ given knowledge of points $1, \ldots, i$. Now we are told about point $i+1$. But a single point can only affect chromatic number by at most one!

Proof. Doing average of averages

$$X_i(G) = \sum_{H \equiv_i G} X_{i+1}(H) P^i(H) \tag{20}$$

so

$$X_{i+1}(G)-X_i(G) = \sum_{H \equiv_i G} [X_{i+1}(G)-X_{i+1}(H)] P^i(H). \tag{21}$$

Now

$$X_{i+1}(G) = \sum_{J \equiv_{i+1} G} X(J) P^{i+1}(J) \tag{22}$$

$$X_{i+1}(H) = \sum_{K \equiv_{i+1} H} X(K) P^{i+1}(K). \tag{23}$$

Employing the bijection φ and (18)

$$X_{i+1}(H)-X_{i+1}(G) = \sum_{J \equiv_{i+1} G} \big(X(\varphi J)-X(J)\big) P^{i+1}(J). \tag{24}$$

Suppose $X(J)=c$. We may c-color $\varphi J-\{i+1\}=J-\{i+1\}$. We then may $(c+1)$-color φJ using the extra color on $i+1$, if necessary. Hence $X(\varphi J) \leqq c+1$. By symmetry

$$|X(\varphi J)-X(J)| \leqq 1. \tag{25}$$

Applying this to (24)

$$|X_{i+1}(H)-X_{i+1}(G)| \leqq 1 \tag{26}$$

and applying this to (21)

$$|X_{i+1}(G)-X_i(G)| \leqq 1 \tag{27}$$

as desired. ∎

Proof of Theorem 1. Let p arbitrary, $\lambda>0$. Theorem 3 gives

$$P\big[|X(G)-E[X(G)]| > \lambda \sqrt{n}\big] < 2e^{-\lambda^2/2}. \tag{28}$$ ∎

Note. A concentration result similar to Theorem 1 can be obtained for a random function X whenever $X(G)-X(\varphi G)$ is small.

2. Sparse graphs

While Theorem 1 holds for all p its usefulness declines as p decreases in view of the known bounds (3). For sparse graphs we modify the function X. We set $p = n^{-\alpha}$ with $0<\alpha<1$.

Let $\varepsilon>0$ be arbitrarily small and fixed. Let $u=u(n,p)$ be the minimal integer such that

$$P[\chi(G) \leqq u] > \varepsilon. \tag{29}$$

From (4)

$$u > \frac{(1-o(1))d}{2\ln d}. \tag{30}$$

Our object is to show that with s given by Theorem 2

$$P[\chi(G) \leqq u+s] \to 1. \tag{31}$$

For any G and any map $\psi\colon [n]\to[u]$ we let flaws(G,ψ) denote the number of edges $\{i,j\}\in G$ with $\psi(i)=\psi(j)$. Set

$$X(G) = \min \text{flaws}\,(G,\psi), \tag{32}$$

the minimum over all $\psi\colon [n]\to[u]$. Then $X(G)=0$ if and only if $\chi(G)\leqq u$ so

$$P[X \leqq u] \geqq \varepsilon. \tag{33}$$

Let $X_0, X_1, \dots, X_n$ denote the Doob Martingale generated by X.

Let $\deg(G,i)$ denote the degree of point i in graph G. Call G *normal* if

$$\deg(G,i) \leqq 5d, \quad \text{all} \quad i\in G. \tag{34}$$

Theorem 6. *If G is normal*

$$|X_{i+1}(G)-X_i(G)| \leqq 11d/u. \tag{35}$$

Proof. Equations (21), (24) and bijection φ are as before. We must bound $|X(J)-X(\varphi J)|$, noting that J, φJ differ only at point $i+1$ where they are identical to G and H respectively. Let ψ be the coloring of J with $X(J)$ flaws. Deleting $i+1$, ψ is a coloring of $J-\{i+1\}=\varphi J-\{i+1\}$ with at most $X(J)$ flaws. Point $i+1$ is adjacent to $\deg(H,i+1)$ other vertices in J. Color $i+1$ so that at most $\deg(H,i+1)/u$ of these points have that color. This gives a ψ^* with

$$X(\varphi J) \leqq \text{flaws}\,(\varphi J,\psi^*) \leqq X(J)+[\deg(H,i+1)/u]. \tag{36}$$

Similarly

$$X(J) \leqq X(\varphi J)+[\deg(G,i+1)/u] \tag{37}$$

and thus

$$|X(\varphi J)-X(J)| \leqq u^{-1}[\deg(G,i+1)+\deg(H,i+1)]. \tag{38}$$

As this holds for all $J={}_{i+1}G$, (24) gives

$$|X_{i+1}(J)-X_{i+1}(G)| \leqq u^{-1}[\deg(G,i+1)+\deg(H,i+1)] \tag{39}$$

and now (21) gives

(40)
$$|X_{i+1}(G)-X_i(G)| \leqq u^{-1}\Big[\sum_{H\equiv_i G}\deg(G, i+1)P^i(H)+\sum_{H\equiv_i G}\deg(H, i+1)P^i(H)\Big].$$

The first sum is simply $\deg(G, i+1)\leqq 5d$. The second sum is the conditional expected value

(41)
$$E[\deg(H, i+1)|H\equiv_i G]$$

which is the number of $j\leqq i$ with $\{j, i+1\}\in G$ plus $p(n-i-1)$ (i.e., the random edges) so at most $\deg(G, i+1)+pn\leqq 6pn$. Together

(42)
$$|X_{i+1}(G)-X_i(G)| \leqq 11pn/u = 11d/u. \quad \blacksquare$$

Most G are normal and for these G, $X_{i+1}-X_i$ is never large. We modify Theorem 3 to allow for a small probability of a large change. In application the r below shall be extremely small.

Theorem 7. *Let $X_0, \ldots, X_n$ be a martingale with X_0 constant and such that*
(i) With probability $\geqq 1-r$, $|X_{i+1}-X_i|\leqq c$ for $0\leqq i<n$.
(ii) Always $|X_{i+1}-X_i|\leqq n$.
Suppose $nr^{1/2}\leqq c$. Then

(43)
$$P[|X_n-X_0| > (2c)\lambda n^{1/2}+n^2r^{1/2}] < 2e^{-\lambda^2/2}+2nr^{1/2}.$$

Proof. Let F_i be the event "$|X_{i+1}-X_i|>c$". We define a new martingale $Y_0, \ldots$ $\ldots, Y_n$. Set $Y_0=X_0$. Assume Y_i has been defined and let $p=P[F_i|X_i]$.

Case 1. $p\geqq r^{1/2}$. We "terminate" the martingale by setting $Y_{i+1}=Y_i$ and $Y_j=Y_i$ for all j, $i+1\leqq j\leqq n$.

Case 2. $p<r^{1/2}$ *and the martingale has not been previously terminated.* Define

(44)
$$X^\bullet_{i+1} = \begin{cases} X_i & \text{if } F_i \\ X_{i+1} & \text{if } \neg F_i. \end{cases}$$

Then

(45)
$$E[X^\bullet_{i+1}|X_i] = E[X_{i+1}|X_i]+E[X^\bullet_{i+1}-X_{i+1}|X_i] = X_i+A_i$$

where $A_i=E[X_i-X_{i+1}|X_i, F_i]\cdot P[F_i|X_i]$ so that

(46)
$$|A_i| \leqq E[|X_i-X_{i+1}|\,|X_i, F_i]\,P[F_i|X_i] \leqq nr^{1/2}$$

by (ii) and the assumption of Case 2. Finally set

(47)
$$Y_{i+1} = Y_i+[X^\bullet_{i+1}-X_i-A_i].$$

The expectation $Y_{i+1}-Y_i$ conditioned upon Y_i (which is determined by $X_{\leqq i}$) is zero, in view of Case 1 and (45). Thus $Y_0, \ldots, Y_n$ is indeed a martingale. Moreover

(48)
$$|Y_{i+1}-Y_i| \leqq c+A_i \leqq 2c \quad \text{always.}$$

[The gambling analogy may be useful here. A gambler plays a series of fair games and $X_{i+1}-X_i$ is the amount he wins in the $(i+1)$-st game. If he is about to play a "wild" game (with $P[|X_{i+1}-X_i|>c]>r^{1/2}$) then we adjust by quitting. If the game is not wild we change all big wins and losses ($|X_{i+1}-X_i|>c$) to zero ($X^{*}_{i+1}=X_i$) and making an adjustment A_i on all payoffs to keep the game fair.] Call G supernormal if Case 1 never occurs and F_i never occurs. For supernormal G

(49) $$Y_n = X_n-(A_0+\ldots+A_{n-1})$$

so that

(50) $$|Y_n-X_n| \leqq n^2r^{1/2}.$$

We bound

(51) $$P[G \text{ is not supernormal}] \leqq \sum_{i=1}^{n} P[F_i]+\sum_{i=1}^{n} P\big[P[F_i|X_i]>r^{1/2}\big].$$

Now

(52) $$r \geqq P[F_i] \geqq r^{1/2}P\big[P[F_i|X_i]>r^{1/2}\big]$$

so

(53) $$P\big[P[F_i|X_i]>r^{1/2}\big]<r^{1/2}$$

and

(54) $$P[G \text{ is not supernormal}] \leqq nr+nr^{1/2} \leqq 2nr^{1/2}.$$

We apply Theorem 3 to the martingale $Y_0, \ldots, Y_n$. Using (48),

(55) $$P[|Y_n-Y_0|>(2c)\lambda n^{1/2}]<2e^{-\lambda^2/2}.$$

Now $X_0=Y_0$ so using (50)

(56) $$\begin{aligned} &P[|X_n-X_0|>(2c)\lambda n^{1/2}+2n^2r^{1/2}] \ll \\ &\ll P[G \text{ is not supernormal}]+P[|Y_n-Y_0|>2c\lambda n^{1/2}] < \\ &< 2e^{-\lambda^2/2}+2nr^{1/2} \end{aligned}$$

as desired.

In our case

(57) $$c = 11d/n \leqq 23\ln d \leqq 23\ln n$$

(58) $$r = P[G \text{ is not normal}] \leqq nP[\deg(G,i) \geqq 5pn].$$

But $\deg(G,i)$ has Binomial Distribution $Z\sim B(n,p)$. We use the result

(59) $$P[Z>pn+K]<e^{-K^2/2pn}.$$

With $k=4pn$

(60) $$r \leqq ne^{-8pn} = ne^{-8n^{1-\alpha}}$$

with $p=n^{-\alpha}$. As r is so small $3n^3r^{1/2}\ll 1$ and so, as $X_n=X$ and $X_0=E(X)$

(61) $$P[|X-EX|>\omega(n)n^{1/2}\ln n] \to 0.$$

Combining (61) and the defining property (33) of u:

(62) $$EX < n^{1/2}(\ln n)\omega(n).$$

Applying (61) once more in the opposite direction

$$P[X > 2n^{1/2}(\ln n)\omega(n)] \to 0. \tag{63}$$

That is, almost surely G may be u-colored with less than $n^{1/2}(\ln n)\omega(n)$ flaws.

Lemma 8. *Almost surely every subset of G with $k \leqq n^{1/2}(\ln n)\omega(n)$ vertices may be s-colored where s is given by Theorem 2.*

Proof. If G does not have this property there will be a minimal set V of vertices which is not s-colorable with $|V| = k < n^{1/2}(\ln n)\omega(n)$. Then each $x \in V$ must be degree in V of a least $s-1$ as otherwise $V - \{x\}$ may be s-colored and then x can be given one of these colors. Hence V must have at least $k(s-1)/2$ edges. Then

$$P[G \text{ does not have property}] < \sum \binom{n}{k} \binom{\binom{k}{2}}{\frac{k(s-1)}{2}} p^{k(s-1)/2}. \tag{64}$$

The sum is over all $k < n^{1/2}(\ln n)\omega(n)$. The factors are the number of k-sets, the choices for the $k(s-1)/2$ edges, and the probability that those edges are in G. We bound

$$\binom{n}{k} < \left(\frac{ne}{k}\right)^k, \quad \binom{\binom{k}{2}}{\frac{k(s-1)}{2}} \sim \left[\frac{k}{s-1}\right]^{k(s-1)/2}. \tag{65}$$

If s is such that

$$\frac{ne}{k}\left[\frac{kp}{s-1}\right]^{(s-1)/2} < 1 \tag{66}$$

then the sum will be small. It suffices to have $k = n^{1/2+o(1)}$, i.e.,

$$n^{1/2+o(1)}\left[\frac{n^{1/2+o(1)}p}{s-1}\right]^{(s-1)/2} < 1. \tag{67}$$

For $p = n^{-\alpha}$ with $\alpha < 1/2$, (67) holds with $s = n^{1/2-\alpha+o(1)}$, or, more precisely, with $s = n^{1/2}(\ln n)\omega(n)p$. For $p = n^{-\alpha}$ with $\alpha > 1/2$, (67) gives that: $(1/2-\alpha)((s-1)/2) + 1/2 < 0$ and the minimal integer s with that property will suffice.

What about $\chi(G)$. Almost surely G can be u-colored with $\leqq n^{1/2}(\ln n)\omega(n)$ flaws. Let V be the set of vertices in flaws so that $G - V$ is u-colored. Now V can be s-colored, together this gives a $(u+s)$-coloring of G, completing the proof of Theorem 2. ∎

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