Assume U stored in substrate:

$$U_j = \frac{qb}{2} \sum_j^n \left(eq\text{-}pos_{j \to n} - (beam\text{-}pos_{j \to n} + \delta_j) \right)^2$$

eq-pos is the position of the beam at $\varepsilon = 0$

beam-pos is the current position of the beam

δ_j is the beam displacement

$$\frac{\partial U_j}{\partial \delta_j} = 0 = \partial/\partial\delta \left(\sum_j^n \left(eq\text{-}pos_{j \to n} - (beam\text{-}pos_{j \to n} + \delta_j) \right)^2 \right)$$

$$\frac{\partial U_j}{\partial \delta_j} = \partial/\partial\delta \left(\sum_j^n \left(eq\text{-}pos_{j \to n} - (beam\text{-}pos)_{j \to n} - \delta_j \right) \right)^2 = \partial/\partial\delta \left(\sum_j^n (eq\,pos_{j \to n} - beam\text{-}pos_{j \to n}) - (n+1-j)\,\delta_j \right)^2$$

$$\frac{\partial U_j}{\partial \delta_j} = -2 \left(\sum_j^n (eq\text{-}pos - beam\text{-}pos) \right)(n+1-j) + 2(n+1-j)^2 \delta_j = 0$$

$$\delta_j = \frac{\sum_j^n (eq\text{-}pos - beam\,pos)}{n+1-j}$$

The beam tip position that yields minimum energy, taking the position of the following (further from symmetry) beams is

$$\delta_j = \frac{\sum_j^n (eq\text{-}pos_{j \to n} - beam\text{-}pos_{jn})}{n+1-j}$$

Assume beam has point force displacement δ w/ fixed guided conditions

$$\delta(x) = \frac{F}{EI}\left(\frac{-x^3}{6} + \frac{Lx^2}{4}\right)$$, where x = position on beam & L is beam length

if the displacement is some known value δ_j

$$\frac{12\delta_j EI}{3L^3 - 2L^3} = F \quad \Rightarrow \quad F = \frac{12EI\delta_j}{L^3}$$

substituting back into Euler beam bending eqn

$$\delta(x) = \frac{12\delta_j}{L^3}\left(\frac{-x^3}{6} + \frac{Lx^2}{4}\right) = \frac{\delta_j}{L^3}\left(3Lx^2 - 2x^3\right)$$

$$M(x) = EI\,\delta''(x) = \frac{EI\delta_j}{L^3}(6L - 12x) = \frac{2\cdot 6EI\delta_j}{L^3}\left(\frac{L}{2} - x\right)$$

$$U_{beam} = \int_0^L \frac{M^2}{2EI} = \frac{2^2(6EI)^2}{2EI}\left(\frac{\delta_j}{L^3}\right)^2 \int_0^L \left(\frac{L}{2} - x\right)^2 dx$$

$$U_{beam} = 2^2\cdot 18EI\left(\frac{\delta_j}{L^3}\right)^2 \left[\frac{L^2}{4}x - \frac{Lx^2}{2} + \frac{x^3}{3}\right]\Bigg|_0^L$$

$$U_{beam} = 2^2\cdot 18EI\left(\frac{\delta_j}{L^3}\right)^2 \left[L^3\right]\left(\frac{1}{4} - \frac{1}{2} + \frac{1}{3}\right) = \frac{3}{2}EI\,\frac{\delta_j^2}{L^3}(2^2) = \frac{6EI\delta_j^2}{L^3}$$

A

(1)

U substrate

$$= \frac{q}{2}\int_0^L \left(eq\text{-}pos - [\delta(x) + D]\right)^2 dx$$, where D is the base of the beam

$$= \frac{q}{2}\int_0^L \left(eq\text{-}pos^2 - 2\,eq\,pos\,(\delta(x)+D) + (\delta(x)+D)^2\right) dx$$

$$= \frac{q}{2}\int_0^L \left(eq\text{-}pos^2 - 2eqpos\left(\frac{\delta_j}{L^3}(3Lx^2-2x^3)^2+D\right) + \left(\left(\frac{\delta_j}{L^3}\right)^2(3Lx^2-2x^3)^2 + 2\left(\frac{\delta_j}{L^3}\right)(3Lx^2-2x^3)D + D^2\right)\right) dx$$

$$= \frac{q}{2}\int_0^L \left(eq\text{-}pos^2 - 2eq\text{-}pos\left(\frac{\delta_j}{L^3}(3Lx^2-2x^3)+D\right) + \left(\left(\frac{\delta_j}{L^3}\right)^2(9L^2x^4-12Lx^5+4x^6) + 2\left(\frac{\delta_j}{L^3}\right)(3Lx^2-2x^3)D + D^2\right)\right) dx$$

$$= \frac{q}{2}\left(eq\text{-}pos^2x - 2eq\text{-}pos\left(\frac{\delta_j}{L^3}\left(Lx^3-\frac{x^4}{2}\right)+Dx\right) + \left(\left(\frac{\delta_j}{L^3}\right)^2\left(\frac{9L^2x^5}{5} - 2Lx^6 + \frac{4x^7}{7}\right) + 2\frac{\delta_j}{L^3}\left(Lx^3-\frac{x^4}{2}\right)D + D^2x\right)\right)\Big|_0^L$$

$$= \frac{q}{2}\left(eq\text{-}pos^2L - 2eq\text{-}pos\left(\frac{\delta_j}{L^3}\left(\frac{L^4}{2}\right)+DL\right) + \left(\left(\frac{\delta_j}{L^3}\right)^2(L^7)\left(\frac{13}{35}\right) + \delta_j LD + D^2L\right)\right)$$

$$= \frac{q}{2}\left(eq\text{-}pos^2L - 2eq\text{-}pos\left(\frac{\delta_jL}{2}+DL\right) + \frac{13\,\delta_j^2}{35}L + \delta_j LD + D^2L\right)$$

$$U_{tot} = U_{subs} + U_{beam} = \frac{q}{2}\left(eq\,pos^2L - 2\,eq\,pos\left(\frac{\delta_jL}{2}+DL\right) + L\left(\frac{13\delta_j^2}{35} + \delta_jD + D^2\right)\right) + \frac{6EI\,\delta_j^2}{L^3}$$

$$\frac{\partial U_{tot}}{\partial L} = \frac{q}{2}\left(eqpos^2 - 2eqpos\left(D+\frac{\delta_j}{2}\right) + \left(\frac{13\delta_j^2}{35} + \delta_jD + D^2\right)\right) - \frac{18EI\,\delta_j^2}{L^4} = 0$$

$$L = \left(\left[\frac{q}{2}\left(eqpos^2 - 2eqpos\left(D+\frac{\delta_j}{2}\right) + \left(\frac{13\delta_j^2}{35} + \delta_jD + D^2\right)\right)\right]^{-1} \cdot 18EI\,\delta_j^2\right)^{1/4}$$

The effective beam length L is determined by the minimum energy as given above

Assume beam has distributed force displacement δ w/ fixed-guided conditions

$$\delta(x) = \frac{-qx^4}{24EI} + \frac{qLx^3}{6EI} \Rightarrow \frac{q}{6EI}\left(-\frac{x^4}{4} + Lx^3\right) \Rightarrow \frac{q}{24EI}\left(4Lx^3 - x^4\right)$$

$$\delta_j = \left(\frac{-q}{24EI} + \frac{q}{6EI}\right)L^4 = \frac{qL^4}{8EI}$$

$$q = \frac{8\delta_j EI}{L^4}$$

$$\delta''(x)EI = M = \frac{-qx^2}{2EI} + \frac{qLx}{EI} = \frac{q}{EI}\left(Lx - \frac{x^2}{2}\right)\cdot EI$$

$$U_{beam} = \frac{1}{2EI}\int_0^L q^2\left(L^2x^2 - Lx^3 + \frac{x^4}{2}\right)dx$$

$$U_{beam} = \frac{q^2}{2EI}\left[\frac{L^2x^3}{3} - \frac{Lx^4}{4} + \frac{x^5}{10}\right]\Big|_0^L$$

$$U_{beam} = \frac{q^2}{2EI}\left[L^5\right]\left(\frac{1}{3} - \frac{1}{4} + \frac{1}{10}\right) = \frac{q^2L^5}{2EI}\left(\frac{20}{60} - \frac{15}{60} + \frac{6}{60}\right) = \frac{q^2L^5}{2EI}\left(\frac{11}{60}\right)$$

$$U_{beam} = \frac{64\,\delta_j^2 EI}{L^8}\left(\frac{11L^5}{60}\right) = \frac{(16)(11)\,\delta_j^2 EI}{15L^3}$$

B

15

$$U_{subs} = \frac{q}{2}\int_0^L \left(eqp - (\delta_j + D)\right)^2 dx$$

$$U_{subs} = \frac{q}{2}\int_0^L eqp^2 - 2eqp(\delta_j + D) + \delta_j^2 + 2D\delta_j + D^2\, dx$$

$$U_{subs} = \frac{q}{2}\int_0^L eqp^2 - 2eqp\left(\frac{\delta_j}{3L^4}(4Lx^3 - x^4) + D\right) + \left(\frac{\delta_j}{3L^4}\right)^2\left(16L^2x^6 - 8Lx^7 + x^8\right) + \frac{2D\delta_j}{3L^4}(4Lx^3 - x^4) + D^2\, dx$$

$$U_{subs} = \frac{q}{2}\left[eqp^2 x - 2eqp\left(\frac{\delta_j}{3L^4}\left(Lx^4 - \frac{x^5}{5}\right) + Dx\right) + \left(\frac{\delta_j}{3L^4}\right)^2\left(\frac{16L^2x^7}{7} - Lx^8 + \frac{x^9}{9}\right) + \frac{2D\delta_j}{3L^4}\left(Lx^4 - \frac{x^5}{5}\right) + D^2 x\right]_0^L$$

$$U_{subs} = \frac{q}{2}\left[eqp^2 L - 2eqp\left(\frac{\delta_j}{3L^4}\left(\frac{4L^5}{5}\right) + DL\right) + \left(\frac{\delta_j}{3L^4}\right)^2 (L^9)\left(\frac{16}{7} - 1 + \frac{1}{9}\right) + \frac{2D\delta_j}{3L^4}\left(\frac{4L^5}{5}\right) + D^2 L\right]$$

$$U_{subs} = \frac{q}{2}\left[eqp^2 L - 2eqp\left(\frac{\delta_j}{3L^4}(4L^5/5) + DL\right) + \left(\frac{\delta_j}{3L^4}\right)^2 (L^9)\left(\frac{88}{63}\right) + \frac{2D\delta_j}{3L^4}\left(\frac{4L^5}{5}\right) + D^2 L\right]$$

$$U_{subs} = \frac{q}{2}\left[eqp^2 L - 2eqp\left(\frac{4\delta_j}{3L^4}\cdot\frac{L^5}{5} + DL\right) + \left(\frac{\delta_j}{3L^4}\right)\right.$$

$$U_{tot} = U_{subs} + U_{beam}$$

$$\frac{\partial U_{tot}}{\partial L} = \frac{q}{2}\left[eqp^2 - 2eqp\left(\frac{4\delta_j}{15} + D\right) + \left(\frac{\delta_j^2}{9}\right)\left(\frac{88}{63}\right) + \frac{2D\delta_j}{3}\left(\frac{4}{5}\right) + D^2\right] - \frac{3\cdot 16\cdot 11\,\delta_j^2 EI}{15L^4} = 0$$

$$\frac{q}{2}\left[eqp^2 - 2eqp\left(\frac{4\delta_j}{15} + D\right) + \left(\frac{\delta_j}{3}\right)^2\left(\frac{88}{63}\right) + \frac{8D\delta_j}{15} + D^2\right]L^4 = \frac{16\cdot 11\,\delta_j^2 EI}{5}$$

B (2)