MTH208: Coursework II Solution Sheet (15 %)

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Instructions on the Coursework

- Total marks for the coursework are 15. There are 2 comprehensive questions in the exam. Please write down detailed solution process and submit your completed solution via a submission link provided on the LMO. You may complete the work by using the provided solution sheet in word or tex.
- All the learning materials on the LMO can be referenced during the exam including lecture notes, Lab codes, lecture videos etc. However, you must complete the coursework independently.
- Please name your submission in the form MTH017Final+ID+ZhangSan.pdf
- The coursework will be available on 9:00AM May 10th and deadline for submission is 9:00AM May 19th.

1. Question I: Numerical Quadrature.

Solution: We call the following integrals:

$$f_c(x) = \frac{\cos(x)}{\sqrt{x}}, and \quad f_s(x) = \frac{\sin(x)}{\sqrt{x}}$$

$$I_c := \int_0^1 \frac{\cos(x)}{\sqrt{x}}\,dx = \int_0^1 f_c(x)\,dx, and \quad I_d := \int_0^1 \frac{\sin(x)}{\sqrt{x}}\,dx = \int_0^1 f_s(x)\,dx,$$

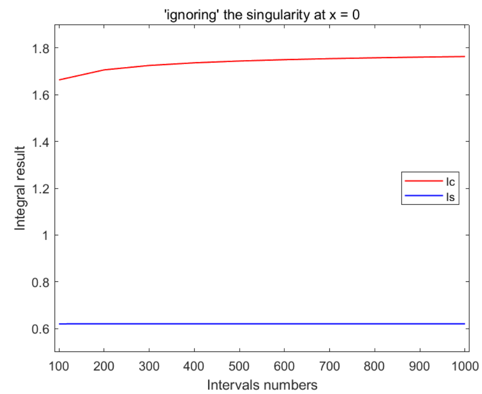
(1-a)
Firstly, let's 'ignoring' the singularity at $x = 0$, which means let $f(0) = g(0) = 0$ in the trapezonidal rule method calculation. And use this function with n intervals of equal length $h = 1/n$

```
function I = trapez_1a(f,n)
% assume f is vectorized
h = 1/n;
x = linspace(0,1,n+1);
%let f(0)=0
I = 0 + f(1);
I = I + 2*sum(f(x(2:end-1)));
I = I*h/2;
end
```

```
n = 100:10:1000;
fc = @(x) cos(x)./sqrt(x);
fs = @(x) sin(x)./sqrt(x);
Ic_a = zeros(1,length(n));
Is_a = zeros(1,length(n));
index = 1;
for i = n
    Ic_a(index) = trapez_1a(fc,i);
    Is_a(index) = trapez_1a(fs,i);
    index = index+1;
end
```

Then we plot I_c I_s depend on n (n can be hundreds to 1000)

```
n = 100:10:1000;
fc = @(x) cos(x)./sqrt(x);
fs = @(x) sin(x)./sqrt(x);
Ic_a = zeros(1,length(n));
Is_a = zeros(1,length(n));
index = 1;
for i = n
    Ic_a(index) = trapez_1a(fc,i);
    Is_a(index) = trapez_1a(fs,i);
    index = index+1;
end
save("1_a","Is_a","Ic_a");
```




```
fprintf('For n = 1000:\nIc_a is %.4f\nIs_a is %.4f\n',Ic_a(end),Is_a(end))
```

```
>> Q1_a
For n = 1000:
Ic_a is 1.7629
Is_a is 0.6205
```

We can observe that the error on [0,h] is $\int_0^1 f(x)\,dx - \frac{1}{2}h \cdot f(h)$. It will decrease by increasing n.
And the error on [h,1] will decrease by increasing n too.
For Ic, the initial error is large because $\lim_{x\to 0} f_c(x) = \inf$, but we consider its singularity as 0. So the error on [0,h] will be big. For Is, the initial error is small because $\lim_{x\to 0} f_s(x) = 0$, so we consider its singularity as 0 is reasonable.
So it is a good way to approximate calculate Is, but not work good for Ic.

(1-b)
Secondly, we will treat the function on [0,h] as $w(x) = x^{-1/2}$, and preserve the function on [h,1]. Notice that we can calculate directly that $\int_0^h = 2\sqrt{h}$.

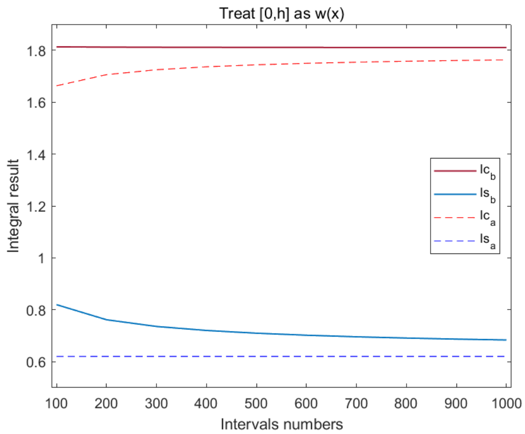
```
function I = trapez_2a(f,n)
% assume f is vectorized
h = 1/n;
% on [h,1]
x = linspace(0,1,n+1);
x = x(2:end);
I1 = f(x(1)) + f(x(end));
I1 = I1 + 2*sum(f(x(2:end-1)));
I1 = I1*h/2;
% on [0,h]
I0 = 2.*sqrt(h);
I = I0+I1;
end
```

```
n = 100:10:1000;
fc = @(x) cos(x)./sqrt(x);
fs = @(x) sin(x)./sqrt(x);
Ic_b = zeros(1,length(n));
Is_b = zeros(1,length(n));
index = 1;
for i = n
    Ic_b(index) = trapez_2a(fc,i);
    Is_b(index) = trapez_2a(fs,i);
    index = index+1;
end
save("1_b","Is_b","Ic_b");
```

Then we plot I_c I_s depend on n (n can be hundreds to 1000), And compare with the data from (1-a)

```
plot(n,Ic_b,'Color',"#A2142F","LineWidth",1);
hold on
plot(n,Is_b,'Color',"#0072BD","LineWidth",1)
hold on
load 1_a.mat
plot(n,Ic_a,'--r',n,Is_a,'--b',"LineWidth",0.6);
```

```
axis([90 1010 0.5 1.9]);
title('Treat [0,h] as w(x)');
xlabel('Intervals numbers');
ylabel('Integral result');
legend('Ic_b','Is_b','Ic_a','Is_a','Location','East')
```




```
fprintf('For n = 1000:\n...
Ic_b is %.4f\n...
Is_b is %.4f\n'...
,Ic_b(end),Is_b(end))
```

```
>> Q1_b
For n = 1000:
Ic_b is 1.8103
Is_b is 0.6838
```

We can obverse that the error on [0,1] is $\int_0^1 w(x) - f(x)\, dx$ (Notice that $w(x)$ is larger than $f(x)$). is will decrease by increasing n. And the error on [h,1] will decrease by increasing n too. For Ic, the initial error is small because $\lim_{x\to 0} \frac{f_c(x)}{w(x)} = 1$, So the error is small on [0,h]. But for Is, the situation is the opposite.
For Ic, Compare with data in (1-a) we can see Ic squeezed by Ic in (a) and (b). So when n is large enough, Ic_a, Ic_b will converge to the real Ic. (Here we don't considerate the rounding).
So it is a good way to approximate calculate Ic, but not work good for Is.

(1-c)
Third, let's make a change of variables $x = t^2$ and apply the composite trapezoidal rule. Now,

$$f_c(x) = g_c(t) = 2\cos(t^2) \quad and \quad f_s(x) = g_s(t) = 2\sin(t^2)$$

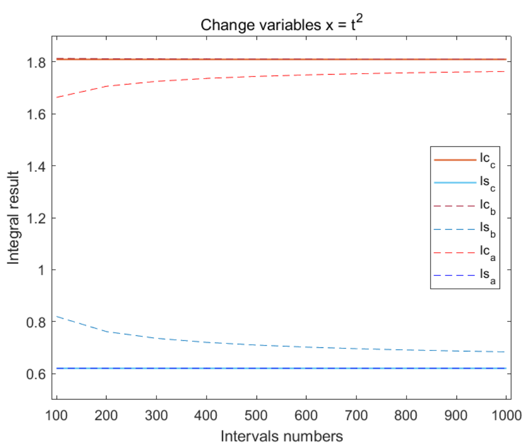
$$I_c = \int_0^1 2\cos(t^2)\, dt = \int_0^1 f_c(x)\, dt \quad and \quad I_s = \int_0^1 2\sin(t^2)\, dt = \int_0^1 g_s(t)\, dt$$

```
function I = trapez_3a(f,n)
% assume f is vectorized
h = 1/n;
x = linspace(0,1,n+1);
I = f(x(1)) + f(x(end));
I = I + 2*sum(f(x(2:end-1)));
I = I*h/2;
end
```

```
n = 100:10:1000;
fc = @(t) 2.*cos(t.^2);
fs = @(t) 2.*sin(t.^2);
Ic_c = zeros(1,length(n));
Is_c = zeros(1,length(n));
index = 1;
for i = n
    Ic_c(index) = trapez_3a(fc,i);
    Is_c(index) = trapez_3a(fs,i);
    index = index+1;
end
save("1_c","Is_c","Ic_c");
```

Then we plot I_c I_s depend on n (n can be hundreds to 1000), And compare with the data from (1-a),(1-b)

```
n = 10:100;
Ic_c = zeros(1,length(n));
Is_c = zeros(1,length(n));
for i = n
    [Ic_c(i-9),Is_c(i-9)] = Q1_c(i.*10);
end
load("1_b.mat")
load("1_a.mat")
%plot
plot(n,Ic_c,'Color',"#D95319","LineWidth",1)
hold on
plot(n,Is_c,'Color',"#4DBEEE","LineWidth",1)
hold on
plot(n,Ic_b,'--','Color',"#A2142F","LineWidth",0.5);
hold on
plot(n,Is_b,'--','Color',"#0072BD","LineWidth",0.5)
hold on
plot(n,Ic_a,'--r',n,Is_a,'--b',"LineWidth",0.5);
axis([5 105 0.5 1.9]);
title('Change variables x = t^2');
xlabel('Intervals numbers');
ylabel('Integral result');
legend('Ic_c','Is_c','Ic_b','Is_b','Ic_a','Is_a','Location','East')
%save data for next question
save("1_c","Is_c","Ic_c");
```




```
fprintf('For n = 1000:\n...
Ic_c is %.4f\n...
Is_c is %.4f\n'...
,Ic_c(end),Is_c(end))
```

```
>> Q1_c
For n = 1000:
Ic_c is 1.8090
Is_c is 0.6205
```

Because we don't have singularity now, so the integrals do not need to consider the error near the singularity.
It seems converge immediately. And is squeezed by the Integrals we calculate in (a) and (b).
It is a better way to remove the singularity for Is and Ic.

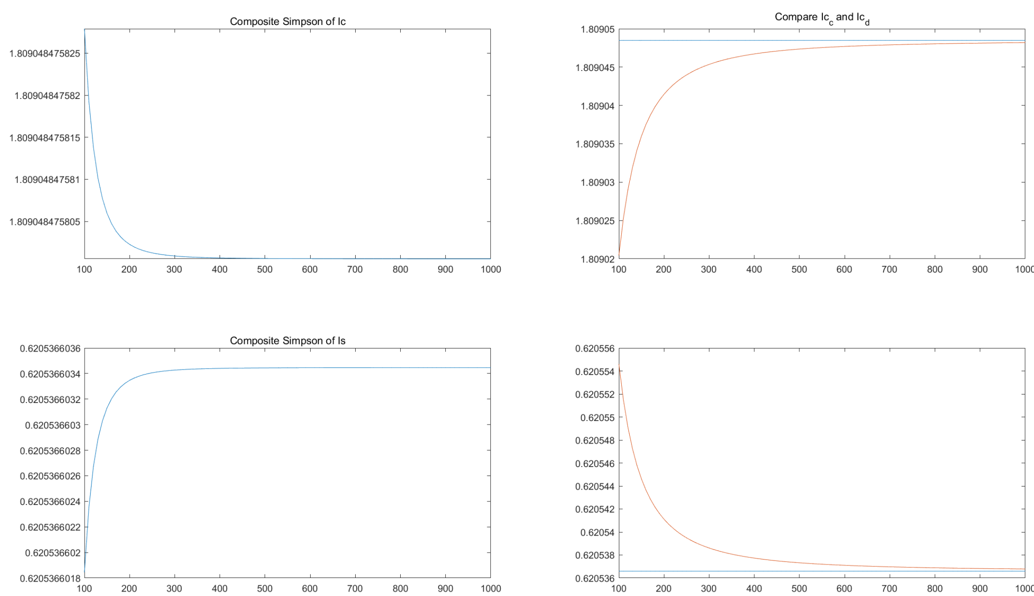
(1-d)
Here I will use composite Simpson method, and then use Adapt Simpson method writing by loop. Firstly, use the function in (C) and Composite Simpson

```
function I = simpson(f,n)
% n must be an even number
% f must be a vectorized
if mod(n,2)~=0
    warning...
    ("n must be a even number");
end
h = 1/n;
x = linspace(0,1,n+1);
I = f(0)+f(1)...
    +2*sum(f(x(3:2:n-1)))...
    +4*sum(f(x(2:2:n)));
I = I*h/3;
end
```

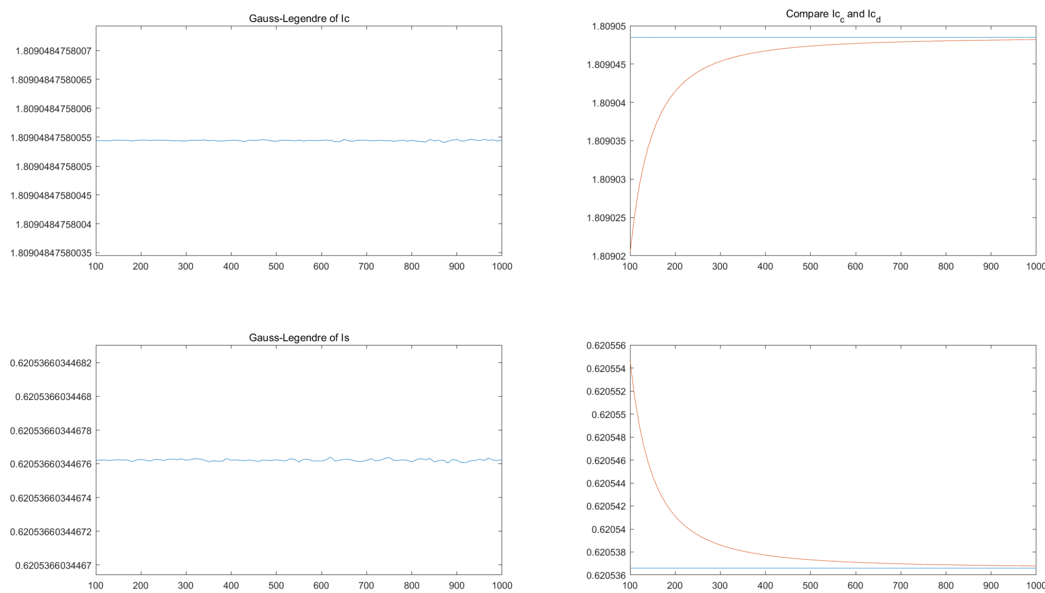
```
n = 100:10:1000;
fc = @(t) 2.*cos(t.^2);
fs = @(t) 2.*sin(t.^2);
Ic_d = zeros(1,length(n));
Is_d = zeros(1,length(n));
index = 1;
for i = n
    Ic_d(index) = simpson(fc,i);
    Is_d(index) = simpson(fs,i);
    index = index+1;
end
```

Then plot and compare with (c) Similiarly:

```
load("1_c.mat");
subplot(2,2,1);plot(n,Ic_d);title("Composite Simpson of Ic");
subplot(2,2,2);plot(n,Ic_d,n,Ic_c);title("Compare Ic_c and Ic_d");
subplot(2,2,3);plot(n,Is_d);title("Composite Simpson of Is");
subplot(2,2,4);title("Compare Is_c and Is_d");plot(n,Is_d,n,Is_c);
```



The picture has showed that the converge rate of Composite Simpson is much faster than the way we used in (c).
Then we will show the Gauss-Legendre method to calculate it. The main function is similarity to simpson, you just need to change '`simpson`' to '`gauss`' in loop 'for'. And the total Gauss-Legendre function is too long, I will not post it here. But you can find it in the LAB codes. Similar to the method we mentioned earlier, we can plot it:



It is obvisely that integral converge when n is just 100. So Gauss-Legendre seems better than Compose Simpson, especially n is not too big.

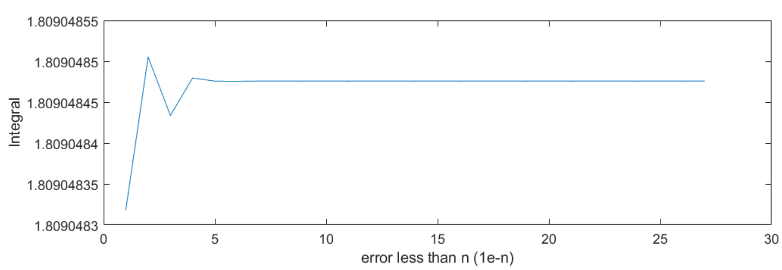
(Appendx of Q(1-d)) More explaination but maybe not useful to get marks.
But it still can't 'control' the Approximate error we want. We don't know how big 'n' is enough. And if we just increasing n, computer may cost much more time to calculate it. So we can chooose adapt method, like Sdapt Simpson method. Here we just write it in loop way.

```
function [Q,err,sort_interval] = loopsimp(f,tinterval,tol)
subs = [tinterval(1:end-1);tinterval(2:end)];midpt = sum(subs)/2;
dif = diff(subs);subs_mid = [subs(1,:); midpt; subs(2,:)];
Ssub = 1/6.*dif.*([1,4,1]*f(subs_mid));Q = 0;interval = [];i=1;
while 1
    %calculate the error
    if isempty(subs)
        break;
    end
    midpt = sum(subs)/2;
    subs = reshape([subs(1,:); midpt; midpt; subs(2,:)],2,[]);
    dif = diff(subs);Sbig = Ssub;
    subs_mid = [subs(1,:); sum(subs)/2; subs(2,:)];
    Ssub = 1/6.*dif.*([1,4,1]*f(subs_mid));
    Ssub_accb = sum(reshape(Ssub,2,[]));err = abs(Sbig - Ssub_accb)/15;
    tol = tol/2;err_index = err < tol;
    suberr_index = reshape([err_index;err_index],1,[]);
    Q = Q + sum(Ssub_accb(err_index),'all');
    interval(:,i:i+sum(suberr_index)-1) = [subs(:,suberr_index)];
    i = i+sum(suberr_index);
    subs(:,suberr_index) = [];Ssub(:,suberr_index) = [];
end
[sort_interval(1,:),I] = sort(interval(1,:));
sort_interval(2,:) = interval(2,I);
end
```

```
function [Ic,Is] = Q1_d_AS(tol)
% Adapt Simpson with loop
fc = @(t) 2.*cos(t.^2);
fs = @(t) 2.*sin(t.^2);
[Ic,~,~] = loopsimp(fc,[0,1],tol);
[Is,~,~] = loopsimp(fs,[0,1],tol);
end
```

```
i = 4:30;Q = 0.*i;err = 0.1.^i;
for j = i
    Q(j-3) = Q1_d_AS(err(j-3));
end
plot(Q);
xlabel("error less than n (1e-n)");
ylabel("Integral");
```

For example, we want to calculate Ic, we can choose a not too big 'Approximate Error', and directly get the integral:



We can see it will converge immediately if we choose a not 'too big' error.

2. Question II: IVP.

We can substitute the value to get the differential equation, and ignore the units of these numbers

$$v' = -9.8 - 0.002/0.11 \cdot v|v| = f(t, y), \quad v(0) = 80$$

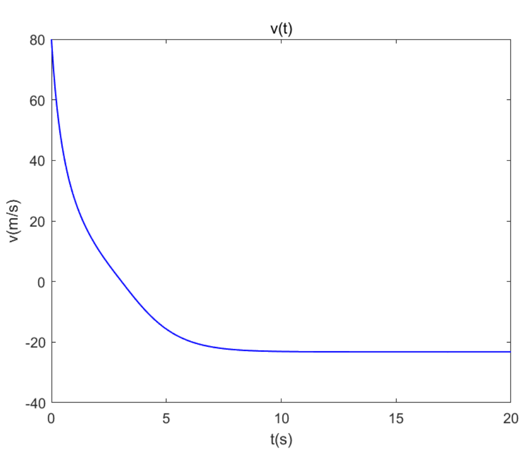
(2-a)
Use Euler's method, and decrease the step sizes:

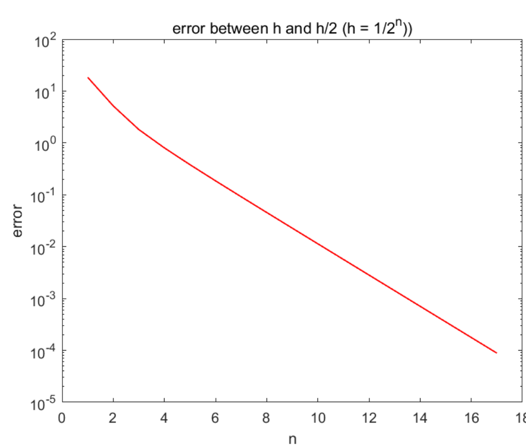
```
f = @(t,v) -9.8-0.002/0.11*v.*abs(v);
k = 1:100;n = 20.*2.^k;
err = inf(1,length(n));
FIRSTTIME = true;
for i = k
    [t, y] = eulerstep(f,0,20,n(i),80);
    if FIRSTTIME == true
        last_t=t;last_y=y;
        FIRSTTIME = false;
        continue
    else
        err(i-1) = max(abs( ...
            y(1:2:end)- last_y));
        if(err(i-1)<1e-4)
            t=last_t;y=last_y;
            err(i:end) = [];
            break
        end
        last_t=t;last_y=y;
    end
end
save("Q2_a",'t','y','err')
```

```
function [t, y] = eulerstep(f,a,b,n,ya)
% Euler explicit time-stepping
% y'=f(t,y), y(a) = ya
t = zeros(n+1,1); t(1) = a;
h = (b-a)/n; d = numel(ya);
y = zeros(n+1,d); y(1,:) = ya;
for i = 1:n
    y(i+1,:) = y(i,:) + h*f(t(i),y(i,:));
    t(i+1) = t(i) + h;
end
```

```
%% load data
load("Q2_a")
%% plot v(t)
plot(t,y,'b',"LineWidth",1);
xlabel("t(s)");ylabel("v(m/s)")
title("v(t)")
```

```
%% plot error
plot(err,'r',"LineWidth",1)
ax = gca;
ax.YScale = 'log';
xlabel("n");
ylabel("error")
title("error between h " + ...
    "and h/2 (h = 1/2^n)")
```





(2-b)
Runge-Kutta method is a family of iterative methods. In general, an explicit s-stage Runge-Kutta method is of the form:

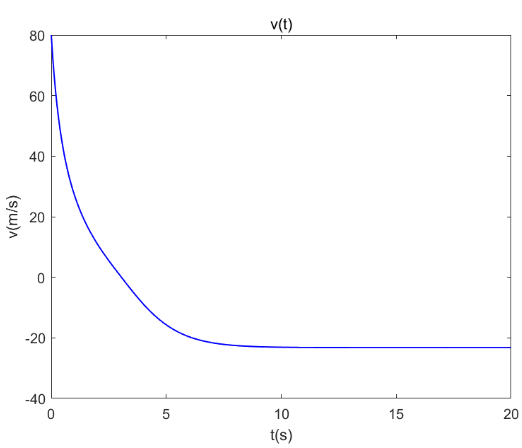
$$\begin{aligned}
w_0 &= \alpha \\
k_1 &= f(ti, w_i) \\
k_2 &= f(ti + hc_2, w_i + ha_{21}k_1) \\
k_3 &= f(ti + hc_3, w_i + h(a_{31}k_1 + a_{32}k_2)) \\
&\vdots \\
k_s &= f(ti + hc_s, w_i + h(a_{s1}k_1 + a_{s2}k_2 + \cdots a_{s,s-1}k_{s-1})) \\
w_{i+1} &= w_i + h(b_1k_1 + b_2k_2 + \cdots + b_sk_s)
\end{aligned}$$

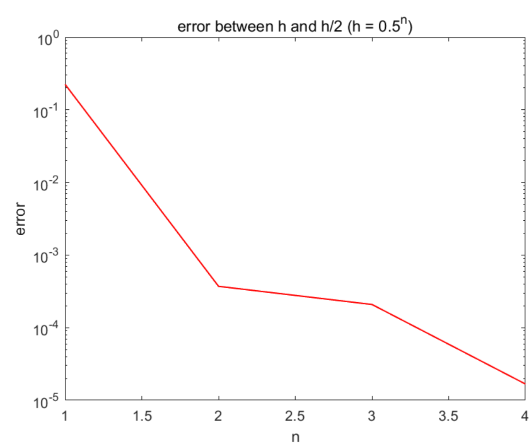
In this question, we will use 4-stage Runge-Kutta method, and control the error between h and h/2 just like Euler's mehod used in (2-a)

```
f = @(t,v) -9.8-0.002/0.11*v.*abs(v);
k = 1:100;n = 20.*2.^k;
err = inf(1,length(n));
FIRSTTIME = true;
for i = k
[t, y] = rkstep(f,0,20,n(i),80);
if FIRSTTIME == true
    last_t=t;last_y=y;
    FIRSTTIME = false;
    continue
else
    err(i-1) = max(abs( ...
        y(1:2:end)- last_y));
    if(err(i-1)<1e-4)
        t=last_t;y=last_y;
        err(i:end) = [];
        break
    end
    last_t=t;last_y=y;
end
end
save("Q2_b",'t','y','err')
```

```
function [t, y] = rkstep(f,a,b,n,ya)
% 4-stage Runge-Kutta
% y'=f(t,y), y(a) = ya
t = zeros(n+1,1); t(1) = a;
h = (b-a)/n; d = numel(ya);
y = zeros(n+1,d); y(1,:) = ya;
for i = 1:n
    k1 = h*f(t(i),y(i,:));
    k2 = h*f(t(i)+h/2,y(i,:)+k1/2);
    k3 = h*f(t(i)+h/2,y(i,:)+k2/2);
    k4 = h*f(t(i)+h,y(i,:)+k3);
    y(i+1,:)=y(i,:)+(k1+2*k2+2*k3+k4)/6;
    t(i+1) = t(i) + h;
end
```

Then you can plot it similarity in Q(2-a) plot codes. Just change `load "Q2_a"` to `load("Q2_b")`.





In the same error control, Comparing from the output:
In Euler's method, we can see that final $h = 7.629394531250000e - 06$, which is 2^{-17}, it means it iterated many times, and the length of t and y calculated will be very big, is 5242881. The data will use more memory to store.
but in 4-stage Runge Kutta, the final $h = 0.0625$, which is 2^{-4}, is means it iterated shorter times, and the length of t and y calcuulated will be smaller, is 641. The data will use less memory to store.
Comparing from the running time:
In Euler's method, running time for error <1e-4 in my computer is 0.637615s.
In in 4-stage Runge Kutta, running time for error <1e-4 in the same computer is 0.00989s. Is much shorter. (notice test it for 100 times and take average)

(2-c)
It is obviously that the "easy way" is connect the distrete 'y' to be a function. For example use `construct = griddedInterpolant(t,y);` and `f = @(t) construct(t);` (Notice that I first construct the construct to avoid creating it too each time when you want to use the function). And then use the function in LAB code 'bisect' and 'trapeze'. But I think it is boring and not the clever idea. Because you need to use the 'not basic' function provided by matlab. So I rewrite the `bisect.m` and `trapeze.m` to make it work for discrete data.
The projectile reaches its maximum height when its 'v' is zero. So we can use bisection to find the smallest interval [a,b], and y(a)>0, y(b)<0. Then use the linear interpolation to find the approximate 't', $v(t) = 0$

```
load("Q2_a.mat")
[a,b] = bisect(y);
tt = -y(a).*(t(b)-t(a))./...
    (y(b)-y(a))+t(a);
```

Here we can get tt = 3.05212 which is the approximate time the projectile reachs its maximum height.(Function bisection is showed below).

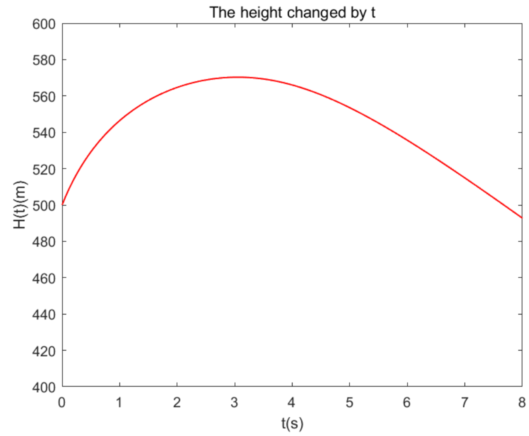
Then we need to integrals it to find the maximum height. Here we will use composite trapez method. Firstly, we will calculate it peace by peace, and then add them step by step:

```
function [a,b] = bisect(y)
% find root boundary
% ab is the index through 0
a = 1;b = length(y);
if y(a)*y(b) > 0
    error(['f(a) and ' ...
        'f(b) must have ' ...
        'different signs']);
end
while b-a > 1
    if y(a)==0||y(b)==0
        break
    end
    mid = floor((a+b)/2);
    if y(a)*y(mid) > 0
        a = mid;
    else
        b = mid;
    end
end
end
```

```
function I = trapeze_eacht(y,t)
y = y(:)';
h = t(2)-t(1);
subinterval = [y(1:end-1);y(2:end)];
I = [1,1]*subinterval*h/2;
end

function int_I = sigema(I)
int_I = 0.*I;
Q = 0;
for i = 1:length(I)
  Q = Q+I(i);
  int_I(i) = Q;
end
end
```

```
load("Q2_a.mat")
[a,b] = bisect(y);
tt = -y(a).*(t(b)-t(a))./...
    (y(b)-y(a))+t(a);
I = trapeze_eacht(y,t);
int_I = sigema(I)+500;
plot(t,int_I,'r','LineWidth',1)
title("The height changed by t")
xlabel("t(s)")
ylabel("H(t)(m)")
axis([0 8 400 600]);
max_height=max(int_I(a),int_I(b));
```



Here we can get maximum height, which is 570.276869(m) for tt = 3.05212(s)

Then we can know the projectile reaches the maximum height 570.276869(m) at the time 3.05212(s). Then lets make sure the errors are less than 10^{-4}:
Let's assume the our approximate for v is very approximate, it means not consider the error created by Euler's method. In the bisect method, because we use the data calculated in (2-a), the h here is 2^{-17}. So our error boundary is $(0+20)/2^{17} = 1.5259e-04$ which is less than 10^{-4}. So computed time is less than 10^{-4}. In the error term for the composite Trapezoidal rule:

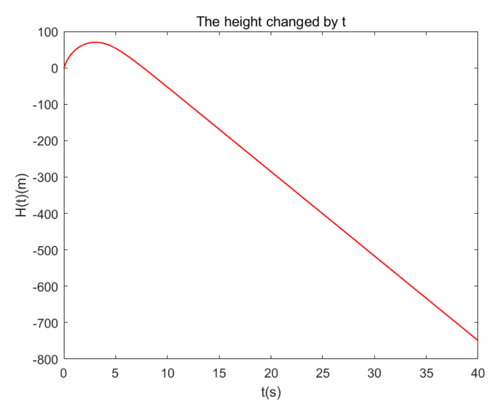
$$v^{''} = -0.002 \cdot 2v \cdot v^{'} = -0.004v \cdot (-9.8 - 0.002v^2) < 10, \quad 80 > v > 0$$

$$\frac{b-a}{12} h^2 v^{''}(\mu) < \frac{20}{12} \cdot 2^{-34} \cdot 10 = 9.7e-10 \ll 1e-4$$

So the height are less than 10^{-4}.

(2-d)
Now we need to extend the simulation time from [0,20] to [0,40]. The code is similarity to the (Q2-b) just change the `[t, y] = rkstep(f,0,20,n,80);` to `[t, y] = rkstep(f,0,40,n,80);`. and then we can use the functions we written in (Q2-c):

```
load("Q2_b.mat")
I = trapeze_eacht(y,t);
int_I = sigema(I);
[a,b] = bisect(int_I+500);
t_between=t([a,b]);
t_middle =sum(t_between)./2;
plot(t,int_I,'r','LineWidth',1)
title("The height changed by t")
xlabel("t(s)")
ylabel("H(t)(m)")
```



Here we can get the time it hits the earth is between 29.25 and 29.28125. We can choose the middle time as approximate, which is 29.265625(s).