We have in this case also $q_{rs} = q_{sr}$, for

$$q_{rs} = \frac{de_r}{dV_s} = \frac{d}{dV_s}\frac{dW_V}{dV_r} = \frac{d}{dV_r}\frac{dW_V}{dV_s} = \frac{de_s}{dV_r} = q_{sr}. \tag{19}$$

By substituting the values of the charges in the equation for the electric energy

$$W = \tfrac{1}{2}\left[e_1 V_1 + \dots + e_r V_r \dots + e_n V_n\right], \tag{20}$$

we obtain an expression for the energy in terms of the potentials

$$\begin{aligned} W_V = {} & \tfrac{1}{2}\, q_{11} V_1^2 + q_{12} V_1 V_2 + \tfrac{1}{2}\, q_{22} V_2^2 \\ & + q_{13} V_1 V_3 + q_{23} V_2 V_3 + \tfrac{1}{2}\, q_{33} V_3^2 + \text{\&c.} \end{aligned} \tag{21}$$

A coefficient in which the two suffixes are the same is called the Electric Capacity of the conductor to which it belongs.

Definition. The Capacity of a conductor is its charge when its own potential is unity, and that of all the other conductors is zero.

This is the proper definition of the capacity of a conductor when no further specification is made. But it is sometimes convenient to specify the condition of some or all of the other conductors in a different manner, as for instance to suppose that the charge of certain of them is zero, and we may then define the capacity of the conductor under these conditions as its charge when its potential is unity.

The other coefficients are called coefficients of induction. Any one of them, as q_{rs}, denotes the charge of A_r when A_s is raised to potential unity, the potential of all the conductors except A_s being zero.

The mathematical calculation of the coefficients of potential and of capacity is in general difficult. We shall afterwards prove that they have always determinate values, and in certain special cases we shall calculate these values. We shall also show how they may be determined by experiment.

When the capacity of a conductor is spoken of without specifying the form and position of any other conductor in the same system, it is to be interpreted as the capacity of the conductor when no other conductor or electrified body is within a finite distance of the conductor referred to.

It is sometimes convenient, when we are dealing with capacities and coefficients of induction only, to write them in the form $[A.P]$, this symbol being understood to denote the charge on A when

P is raised to unit potential {the other conductors being all at zero potential}.

In like manner $[(A+B) \, . \, (P+Q)]$ would denote the charge on $A+B$ when P and Q are both raised to potential 1; and it is manifest that since

$$[(A+B) \, . \, (P+Q)] = [A \, . \, P] + [A \, . \, Q] + [B \, . \, P] + [B \, . \, Q]$$
$$= [(P+Q) \, . \, (A+B)],$$

the compound symbols may be combined by addition and multiplication as if they were symbols of quantity.

The symbol $[A \, . \, A]$ denotes the charge on A when the potential of A is 1, that is to say, the capacity of A.

In like manner $[(A+B) \, . \, (A+Q)]$ denotes the sum of the charges on A and B when A and Q are raised to potential 1, the potential of all the conductors except A and Q being zero.

It may be decomposed into

$$[A \, . \, A] + [A \, . \, B] + [A \, . \, Q] + [B \, . \, Q].$$

The coefficients of potential cannot be dealt with in this way. The coefficients of induction represent charges, and these charges can be combined by addition, but the coefficients of potential represent potentials, and if the potential of A is V_1 and that of B is V_2, the sum $V_1 + V_2$ has no physical meaning bearing on the phenomena, though $V_1 - V_2$ represents the electromotive force from A to B.

The coefficients of induction between two conductors may be expressed in terms of the capacities of the conductors and that of the two conductors together, thus:

$$[A \, . \, B] = \tfrac{1}{2}[(A+B) \, . \, (A+B)] - \tfrac{1}{2}[A \, . \, A] - \tfrac{1}{2}[B \, . \, B].$$

Dimensions of the coefficients.

88.] Since the potential of a charge e at a distance r is $\frac{e}{r}$, the dimensions of a charge of electricity are equal to those of the product of a potential into a line.

The coefficients of capacity and induction have therefore the same dimensions as a line, and each of them may be represented by a straight line, the length of which is independent of the system of units which we employ.

For the same reason, any coefficient of potential may be represented as the reciprocal of a line.

On certain conditions which the coefficients must satisfy.

89 *a*.] In the first place, since the electric energy of a system is an essentially positive quantity, its expression as a quadratic function of the charges or of the potentials must be positive, whatever values, positive or negative, are given to the charges or the potentials.

Now the conditions that a homogeneous quadratic function of n variables shall be always positive are n in number, and may be written

$$\left.\begin{array}{l} p_{11} > 0, \\ \begin{vmatrix} p_{11}, p_{12} \\ p_{21}, p_{22} \end{vmatrix} > 0, \\ - - - - - - - \\ \begin{vmatrix} p_{11} \cdots p_{1n} \\ - - - - - \\ p_{n1} \cdots p_{nn} \end{vmatrix} > 0. \end{array}\right\} \qquad (22)$$

These n conditions are necessary and sufficient to ensure that W_e shall be essentially positive *.

But since in equation (16) we may arrange the conductors in any order, every determinant must be positive which is formed symmetrically from the coefficients belonging to any combination of the n conductors, and the number of these combinations is $2^n - 1$.

Only n, however, of the conditions so found can be independent.

The coefficients of capacity and induction are subject to conditions of the same form.

89 *b*.] *The coefficients of potential are all positive, but none of the coefficients* p_{rs} *is greater than* p_{rr} *or* p_{ss}.

For let a charge unity be communicated to A_r, the other conductors being uncharged. A system of equipotential surfaces will be formed. Of these one will be the surface of A_r, and its potential will be p_{rr}. If A_s is placed in a hollow excavated in A_r so as to be completely enclosed by it, then the potential of A_s will also be p_{rr}.

If, however, A_s is outside of A_r its potential p_{rs} will lie between p_{rr} and zero.

* See Williamson's *Differential Calculus*, 3rd edition, p. 407.

For consider the lines of force issuing from the charged conductor A_r. The charge is measured by the excess of the number of lines which issue from it over those which terminate in it. Hence, if the conductor has no charge, the number of lines which enter the conductor must be equal to the number which issue from it. The lines which enter the conductor come from places of greater potential, and those which issue from it go to places of less potential. Hence the potential of an uncharged conductor must be intermediate between the highest and lowest potentials in the field, and therefore the highest and lowest potentials cannot belong to any of the uncharged bodies.

The highest potential must therefore be p_{rr}, that of the charged body A_r, the lowest must be that of space at an infinite distance, which is zero, and all the other potentials such as p_{rs} must lie between p_{rr} and zero.

If A_s completely surrounds A_t, then $p_{rs} = p_{rt}$.

89 *c.*] *None of the coefficients of induction are positive, and the sum of all those belonging to a single conductor is not numerically greater than the coefficient of capacity of that conductor, which is always positive.*

For let A_r be maintained at potential unity while all the other conductors are kept at potential zero, then the charge on A_r is q_{rr}, and that on any other conductor A_s is q_{rs}.

The number of lines of force which issue from A_r is q_{rr}. Of these some terminate in the other conductors, and some may proceed to infinity, but no lines of force can pass between any of the other conductors or from them to infinity, because they are all at potential zero.

No line of force can issue from any of the other conductors such as A_s, because no part of the field has a lower potential than A_s. If A_s is completely cut off from A_r by the closed surface of one of the conductors, then q_{rs} is zero. If A_s is not thus cut off, q_{rs} is a negative quantity.

If one of the conductors A_t completely surrounds A_r, then all the lines of force from A_r fall on A_t and the conductors within it, and the sum of the coefficients of induction of these conductors with respect to A_r will be equal to q_{rr} with its sign changed. But if A_r is not completely surrounded by a conductor the arithmetical sum of the coefficients of induction q_{rs}, &c. will be less than q_{rr}.

We have deduced these two theorems independently by means of electrical considerations. We may leave it to the mathematical student to determine whether one is a mathematical consequence of the other.

89 *d.*] When there is only one conductor in the field its coefficient of potential on itself is the reciprocal of its capacity.

The centre of mass of the electricity when there are no external forces is called the electric centre of the conductor. If the conductor is symmetrical about a centre of figure, this point is the electric centre. If the dimensions of the conductor are small compared with the distances considered, the position of the electric centre may be estimated sufficiently nearly by conjecture.

The potential at a distance c from the electric centre must be between

$$\frac{e}{c}\left(1+\frac{a^2}{c^2}\right) \text{ and } \frac{e}{c}\left(1-\frac{1}{2}\frac{a^2}{c^2}\right)^*;$$

where e is the charge, and a is the greatest distance of any part of the surface of the body from the electric centre.

For if the charge be concentrated in two points at distances a on opposite sides of the electric centre, the first of these expressions is the potential at a point in the line joining the charges, and the second at a point in a line perpendicular to the line joining the charges. For all other distributions within the sphere whose radius is a the potential is intermediate between those values.

If there are two conductors in the field, their mutual coefficient of potential is $\frac{1}{c'}$, where c' cannot differ from c, the distance between the electric centres, by more than $\frac{a^2+b^2}{c}$; a and b being the greatest distances of any part of the surfaces of the bodies from their respective electric centres.

* {For let ρ be the density of the electricity at any point, then if we take the line joining the electric centre to P as the axis of z, the potential at P is

$$\iiint\frac{\rho\,dx\,dy\,dz}{r}=\iiint\rho\left\{\frac{1}{c}+\frac{z}{c^2}+\frac{2z^2-(x^2+y^2)}{2c^3}+\dots\right\}dx\,dy\,dz,$$

where c is the distance of P from the electric centre. The first term equals e/c, the second vanishes since the origin is the electric centre, and the greatest value of the

89 *e*.] If a new conductor is brought into the field the coefficient of potential of any one of the others on itself is diminished.

For let the new body, B, be supposed at first to be a non-conductor {having the same specific inductive capacity as air} free from charge in any part, then when one of the conductors, A_1, receives a charge e_1, the distribution of the electricity on the conductors of the system will not be disturbed by B, as B is still without charge in any part, and the electric energy of the system will be simply

$$\tfrac{1}{2} e_1 V_1 = \tfrac{1}{2} e_1^2 p_{11}.$$

Now let B become a conductor. Electricity will flow from places of higher to places of lower potential, and in so doing will diminish the electric energy of the system, so that the quantity $\frac{1}{2} e_1^2 p_{11}$ must diminish.

But e_1 remains constant, therefore p_{11} must diminish.

Also if B increases by another body b being placed in contact with it, p_{11} will be further diminished.

For let us first suppose that there is no electric communication between B and b; the introduction of the new body b will diminish p_{11}. Now let a communication be opened between B and b. If any electricity flows through it, it flows from a place of higher to a place of lower potential, and therefore, as we have shewn, still further diminishes p_{11}.

third is when the electricity is concentrated at the points for which the third term inside the bracket has its greatest value, which is a^2/c^3, thus the greatest value of the third term is ea^2/c^3; the least value of this term is when the electricity is concentrated at the points for which the third term inside the bracket has its greatest negative value which is $-\frac{1}{2}a^2/c^3$; thus the least value of the third term is $-\frac{1}{2}ea^2/c^3$.

The result at the end of Art. 89 *d* may be deduced as follows. Suppose the charge is on the first conductor, then the potential due to the electricity on this conductor by the above is less than

$$\frac{e}{R} + \frac{ea^2}{R^3},$$

where R is the distance of the point from the electric centre of the first conductor; in the second term if we are only proceeding as far as c^{-3}, we may put $R = c$ for any point on the second conductor. The first term represents the potential to which the second conductor is raised by a charge e at the electric centre of the first, but by Art. 86, this is the same as the potential at the electric centre of the first due to a charge e on the second conductor, but we have just seen that this must be less than

$$\frac{e}{c} + \frac{eb^2}{c^3};$$

thus the potential of the second conductor due to a charge e on the first must be less than

$$\frac{e}{c} + \frac{e(a^2+b^2)}{c^3}.$$

This however is not in general a very close approximation to the mutual potential of two conductors.}

Hence the diminution of p_{11} by the body B is greater than that which would be produced by any conductor the surface of which can be inscribed in B, and less than that produced by any conductor the surface of which can be described about B.

We shall shew in Chapter XI, that a sphere of diameter b at a distance r, great compared with b, diminishes the value of p_{11} by a quantity which is approximately $\frac{1}{8}\frac{b^3}{r^4}$ *.

Hence if the body B is of any other figure, and if b is its greatest diameter, the diminution of the value of p_{11} must be less than $\frac{1}{8}\frac{b^3}{r^4}$.

Hence if the greatest diameter of B is so small compared with its distance from A_1 that we may neglect quantities of the order $\frac{1}{8}\frac{b^3}{r^4}$, we may consider the reciprocal of the capacity of A_1 when alone in the field as a sufficient approximation to p_{11}.

90 *a*.] Let us therefore suppose that the capacity of A_1 when alone in the field is K_1, and that of A_2, K_2, and let the mean distance between A_1 and A_2 be r, where r is very great compared with the greatest dimensions of A_1 and A_2, then we may write

$$p_{11} = \frac{1}{K_1}, \quad p_{12} = \frac{1}{r}, \quad p_{22} = \frac{1}{K_2};$$

$$V_1 = e_1 K_1^{-1} + e_2 r^{-1},$$

$$V_2 = e_1 r^{-1} \quad + e_2 K_2^{-1}.$$

Hence

$$q_{11} = \quad K_1 (1 - K_1 K_2 r^{-2})^{-1},$$

$$q_{12} = -K_1 K_2 r^{-1} (1 - K_1 K_2 r^{-2})^{-1},$$

$$q_{22} = \quad K_2 (1 - K_1 K_2 r^{-2})^{-1}.$$

Of these coefficients q_{11} and q_{22} are the capacities of A_1 and A_2 when, instead of being each alone at an infinite distance from any other body, they are brought so as to be at a distance r from each other.

90 *b*.] When two conductors are placed so near together that their coefficient of mutual induction is large, the combination is called a Condenser.

Let A and B be the two conductors or electrodes of a condenser.

* {See equation (43), Art. 146.}