Homotopy techniques for analytic combinatorics in several variables

(joint work with Stephen Melczer[†] and Josip Smolčić[†])

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AMS Math Research Community 2021

Combinatorial Applications of Computational

Geometry and Algebraic Topology

Analytic combinatorics

$(f_n) = f_1, f_2, \ldots$: a sequence of complex numbers.

$F(z) = \sum_{i=1}^{\infty} f_i \, z^i$: **the generating function** of the sequence.

- The generating function can be considered as the power series expansion of a complex valued function.

Q. Can we study the asymptotic behavior of (f_n) using the (analytic) behavior of F?

- (Cauchy integral formula). $f_n = \dfrac{1}{2\pi i} \displaystyle\int_{\gamma} F(z) \dfrac{dz}{z^{n+1}}$

Analytic combinatorics in several variables

$F(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^n} f_{\mathbf{i}} \, \mathbf{z}^{\mathbf{i}}$: the **multivariate** generating function of the sequence.

- Specifically, the $\mathbf{r}$-diagonal sequence $(f_{n\mathbf{r}})$ for any $\mathbf{r} \in \mathbb{R}^n$ is considered.

- The common situation to arise in practice is the main-diagonal $\mathbf{r} = \mathbf{1}$. **Ex) (Furstenberg 1967), (Christol 1984), (Chudnovsky-Chudnovsky 1985), (André 2000)**

- Applications : Lattice path enumeration, random walks, and so on.

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(Need a system of equations)

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- It is important to find where the singularity locates.

- Define Abs $: (z_1, \ldots, z_n) \mapsto |z_1 \cdots z_n|$.

- We are interested in critical points of Abs.

- $\mathscr{V} := \{\mathbf{z} \in \mathbb{C}^n \mid H(\mathbf{z}) = 0\}$: the **singular variety** of $F = \dfrac{G}{H}$

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- $H(\mathbf{z}) = 0, \quad z_1\dfrac{\partial H}{\partial z_1} = \cdots = z_n\dfrac{\partial H}{\partial z_n}$ **(critical-point equations)**

- Especially, we are interested in **minimal** critical points

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- $F(\mathbf{z}) = \dfrac{G(\mathbf{z})}{H(\mathbf{z})}$ where G and H are co-prime

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- Interested in computing the asymptotic behavior of
  coefficients of $(f_{\mathbf{i}})_{\mathbf{i}}$.

$$f_{i_1, \ldots, i_n} = \frac{1}{(2\pi i)^n} \int_{\gamma} \frac{F(\mathbf{z})}{z_1^{i_1} \cdots z_n^{i_n}} \cdot \frac{dz_1 \ldots dz_n}{z_1 \cdots z_n}$$

- It is important to find where the singularity locates.

- Define Abs : $(z_1, \ldots, z_n) \mapsto |z_1 \cdots z_n|$.

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1. Determine the set $\mathcal{S}$ of zeros $(\mathbf{z}, \lambda, t)$ of the system
   $$\left[H, z_1 \frac{\partial H}{\partial z_1} - \lambda, \ldots, z_n \frac{\partial H}{\partial z_n} - \lambda, H(t z_1, \ldots, t z_n) \right].$$ If $\mathcal{S}$ is not finite, then FAIL.

2. Construct the subset points $(\boldsymbol{\zeta}, \lambda, t) \in \mathcal{S}$ which are candidates for minimal critical points.

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Doesn't hold if F is not combinatorial

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Q. How to deal with non-combinatorial case?

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Q. How to deal with polydisk using polynomial equations?

A. Decompose polynomials into the real and imaginary part.

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ACSV for non-combinatorial case (Melczer-Salvy 2021)

Q. How to deal with non-combinatorial case?

Lemma (Melczer-Salvy 2021)

Let $D(\mathbf{z}) := \{\mathbf{w} \in \mathbb{C}^n \mid |w_i| < |z_i|, i = 1, \ldots, n\}$ be the open polydisk. If $\mathbf{z} \in \mathscr{V}$ and $\mathscr{V} \cap D(\mathbf{z}) = \varnothing$, then $\mathbf{z} \in \partial \mathscr{D}$.

Q. How to deal with polydisk using polynomial equations?

A. Decompose polynomials into the real and imaginary part.

$$f(\mathbf{x} + i\mathbf{y}) = f^{(R)}(\mathbf{x}, \mathbf{y}) + if^{(I)}(\mathbf{x}, \mathbf{y})$$

For derivatives, applying Cauchy-Riemann equations,

$$\frac{\partial f}{\partial z_j}(\mathbf{x} + i\mathbf{y}) = \frac{1}{2} \cdot \frac{\partial}{\partial x_j}\left(f^{(R)}(\mathbf{x}, \mathbf{y}) + if^{(I)}(\mathbf{x}, \mathbf{y})\right) - \frac{i}{2} \cdot \frac{\partial}{\partial y_j}\left(f^{(R)}(\mathbf{x}, \mathbf{y}) + if^{(I)}(\mathbf{x}, \mathbf{y})\right)$$

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$$H^{(R)}(\mathbf{x}, \mathbf{y}) = H^{(I)}(\mathbf{x}, \mathbf{y}) = 0$$

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$$(\star)$$

$(\star)$ is a square polynomial system with $4n + 4$ variables $(\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}, \lambda_R, \lambda_I, \nu, t)$.

ACSV for combinatorial case

1. Determine the set $\mathcal{S}$ of zeros $(\mathbf{z}, \lambda, t)$ of the system
$$\left[H, z_1 \frac{\partial H}{\partial z_1} - \lambda, \ldots, z_n \frac{\partial H}{\partial z_n} - \lambda, H(tz_1, \ldots, tz_n) \right].$$ If $\mathcal{S}$ is not finite, then FAIL.

2. Construct the subset points $(\boldsymbol{\zeta}, \lambda, t) \in \mathcal{S}$ which are candidates for minimal critical points.

   - $\mathbf{z}$ is minimal if and only if the line segment
     $$\{(t|z_1|, \ldots, t|z_n|) \mid 0 < t < 1\} \text{ doesn't}$$
     intersect $\mathcal{V}$.

3. Identify $\boldsymbol{\zeta}$ among the elements of $\mathscr{C}$ (critical points Abs on $\mathcal{V}$).

4. Return
   $$\mathcal{U} := \{\mathbf{z} \in \mathbb{C}^n \mid |z_1| = |\zeta_1|, \ldots, |z_n| = |\zeta_n| \text{ for some } (\mathbf{z}, \lambda) \in \mathscr{C}\}$$

ACSV for non-combinatorial case

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2. Construct the set of minimal critical points
   $$\mathcal{U} := \{\mathbf{a} + i\mathbf{b} \mid (\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}, \lambda_R, \lambda_I, \nu, t) \in \mathcal{S}_{\mathbb{R}}, t \notin (0,1)\} \subset \mathcal{S}$$

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ACSVHomotopy.jl (Melczer-Lee-Smolčić 2022)

- Implemented using `HomotopyContinuation.jl` **(Breiding-Timme 2018)**

- Available at
  `github.com/ACSVMath/ACSVHomotopy`

- Competitive to other ACSV software for combinatorial cases.
  - Solve the critical-point equations using the polyhedral homotopy

- The first software of ACSV for non-combinatorial cases.
  - Solve the decomposed critical-point equations (★) using the polyhedral homotopy
  - Provide faster heuristics including the monodromy method.

Implementation details

The polyhedral homotopy is the default for solving critical point equations.

- Returns reliable results via interval arithmetic certification.

- Effective for combinatorial case compared to other software based on symbolic algorithm.

- May be slow for non-combinatorial.
  - Faster heuristics used.

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Implementation details

Heuristics for non-combinatorial case.

$$
\left.
\begin{aligned}
&H^{(R)}(\mathbf{a}, \mathbf{b}) = H^{(I)}(\mathbf{a}, \mathbf{b}) = 0 \\[1em]
&a_j \frac{\partial H^{(R)}}{\partial x_j}(\mathbf{a}, \mathbf{b}) + b_j \frac{\partial H^{(R)}}{\partial y_j}(\mathbf{a}, \mathbf{b}) - \lambda_R = 0 \qquad j = 1,\ldots,n \\[1em]
&a_j \frac{\partial H^{(I)}}{\partial x_j}(\mathbf{a}, \mathbf{b}) + b_j \frac{\partial H^{(I)}}{\partial y_j}(\mathbf{a}, \mathbf{b}) - \lambda_I = 0 \qquad j = 1,\ldots,n \\[1em]
&H^{(R)}(\mathbf{x}, \mathbf{y}) = H^{(I)}(\mathbf{x}, \mathbf{y}) = 0 \\[1em]
&x_j^2 + y_j^2 - t(a_j^2 + b_j^2) = 0 \qquad j = 1,\ldots,n \\[1em]
&(y_j - \nu x_j)\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{x}, \mathbf{y}) - (x_j + \nu y_j)\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{x}, \mathbf{y}) = 0 \qquad j = 1,\ldots,n
\end{aligned}
\right\} \quad (\star)
$$

Implementation details

Heuristics for non-combinatorial case.

1. Approximating critical points

$$
\left.
\begin{aligned}
H^{(R)}(\mathbf{a}, \mathbf{b}) = H^{(I)}(\mathbf{a}, \mathbf{b}) = 0 \\[1em]
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(A)

$$H^{(R)}(\mathbf{x},\mathbf{y}) = H^{(I)}(\mathbf{x},\mathbf{y}) = 0$$

$$x_j^2 + y_j^2 - t(a_j^2 + b_j^2) = 0 \qquad j = 1,\dots,n$$

$$(y_j - \nu x_j)\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{x},\mathbf{y}) - (x_j + \nu y_j)\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{x},\mathbf{y}) = 0 \qquad j = 1,\dots,n$$

(B)

1. Approximating critical points

- Solve the subsystem (A) to get an approximation of $(\mathbf{a},\mathbf{b})$.

- Using the approximations, solve the subsystem (B).

Implementation details

Heuristics for non-combinatorial case.

$$H^{(R)}(\mathbf{a},\mathbf{b}) = H^{(I)}(\mathbf{a},\mathbf{b}) = 0$$

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(A)

$$H^{(R)}(\mathbf{x},\mathbf{y}) = H^{(I)}(\mathbf{x},\mathbf{y}) = 0$$

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Implementation details

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Heuristics for non-combinatorial case.

$$H^{(R)}(\mathbf{a},\mathbf{b}) = H^{(I)}(\mathbf{a},\mathbf{b}) = 0$$

$$a_j\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{a},\mathbf{b}) + b_j\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{a},\mathbf{b}) - \lambda_R = 0 \qquad j = 1,\ldots,n$$

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(A)

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$$(y_j - \nu x_j)\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{x},\mathbf{y}) - (x_j + \nu y_j)\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{x},\mathbf{y}) = 0 \qquad j = 1,\ldots,n$$

(B)

2. Monodromy

- Solve the subsystem (A) to get an approximation of $(\mathbf{a},\mathbf{b})$.

- Solve the subsystem (B) using monodromy with $(\mathbf{x},\mathbf{y},t) = (\mathbf{a},\mathbf{b},1)$.

  (ν can be found from the Jacobian)

- Caveat : the subsystem (B) may have several irreducible components. (Fail to find all critical points)

Implementation details

Heuristics for non-combinatorial case.

$$
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Experiments : Combinatorial examples

Elapsed time (s) \ Examples	square-root	Apéry ζ(2)	Apéry ζ(3)	Random	3D Walk
ACSVHomotopy.jl	0.01	0.025	0.7	0.9	0.08
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Polyhedral homotopy	29.5	670	INC	236	INC
Approximating Crits	0.72	3.8	15.3	3.6	189.4
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INC indicates the code did not complete after running for an hour.

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Future directions

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$(\star)$

- Geometric understanding of ($\star$)

  - What is degree for generic H ?

- Numerical techniques for ($\star$)

  - How to improve the performance of monodromy?

  - Solving equation-by-equation (i.e. regeneration)

- How to verify the completeness?

Thank you for your attention

The paper is available at (https://arxiv.org/pdf/2208.04490.pdf)