Research Statement

Kisun Lee

I study *applied algebraic geometry*, also called *nonlinear algebra*. The primary object in algebraic geometry is an *algebraic variety*, defined by solutions of polynomial equations. The term 'applied' implies that its methods are applied to and integrated with other mathematical disciplines, including numerical and complex analysis, combinatorics, and commutative algebra. Primarily, my research has been greatly assisted by tools from numerical analysis or combinatorics. I aim to utilize these tools to facilitate the usefulness of algebraic geometry hidden from the classical algebraic geometry point of view.

The following questions motivate me:

1. How do we develop and apply efficient numerical methods to benefit non-experts in various fields?
2. How do we verify the correctness of the output generated by these methods?
3. How do we use combinatorial viewpoints to understand problems in algebraic geometry?

I have briefly introduced my research interest and the projects I am involved in. What follows is a detailed explanation of these projects. To enhance readability, **boldface is used to highlight some of my results** in the sections below instead of providing formal theorem statements.

Solving a polynomial system and its applications

Computations on algebraic varieties are performed using symbolic methods such as Gröbner bases, which can be expensive. *Numerical algebraic geometry* [26], on the other hand, involves approximates that may produce results faster. Often, problems in algebraic geometry are related to the solutions of zero-dimensional polynomial systems, i.e. a system with finitely many solutions. Recent advancements in methods for solving these systems have resulted in numerous applications in numerical algebraic geometry.

The *homotopy continuation* is a popular method for computing numerical solutions. The main idea is to introduce a system G (called a *start system*) whose solutions are known in advance and track these solutions to the solutions to a system F (called a *target system*) that is desired to solve. It involves the homotopy $H(x, t)$ such that $H(x, 0) = G(x)$ and $H(x, 1) = F(x)$.

The performance of the homotopy continuation is significantly influenced by the choice of a start system and a target system. The typical choice of these systems is given by Bézout's theorem or Bernstein's theorem. The *polyhedral homotopy continuation* induced by Bernstein's theorem is known to be "optimal" for various families of polynomial systems with parameters, so-called *Bernstein generic* systems. In cases where these options are unavailable, *monodromy* can be an alternative heuristic. On the other hand, the *real polyhedral homotopy*, designed to find only real solutions, may be more efficient for problems arising in engineering or economics.

Past work

The paper [8] establishes a framework of the monodromy method to solve a system $F(x; p)$ with parameters p. For an incidence variety $\{(p, x) \mid F(x; p) = 0\}$, all systems $F(x; p)$ share the same monomials but has different coefficients represented by p. A generic choice of p induces the *monodromy action* which permutes the solutions of the system $F(x; p)$. The algorithm proposed in the paper finds all solutions of $F(x; p)$ under assumptions that the monodromy action is transitive and the number of solutions is finite. It is implemented as Macaulay2 [13] package `MonodromySolver`. This package has been extensively used in various publications covering a range of topics, including enumerative geometry [3], combinatorics [21], algebraic vision [9], and more.

In joint work [20] with Lindberg and Rodriguez, we implement the real polyhedral homotopy method for finding real roots as the Julia package `RealPolyhedralHomotopy.jl`. The theory for the real polyhedral homotopy was established in [11], which combines the polyhedral homotopy [15] and Viro's patchworking for complete intersections [27]. It is motivated by the fact that the number of real roots remains unchanged as long as a homotopy path does not cross the discriminant locus. **The current implementation is designed to find all real solutions for a family of polynomial systems that meet a specific combinatorial condition.**

I am interested in finding a moment when numerical algebraic geometry may significantly impact other fields. The project started in 2021 [21] establishes a numerical method for problems in *analytic combinatorics in several variables*. For a sequence $(f_i)_{i\in\mathbb{N}^d}$ of complex numbers with generating function $F(z) = \sum_{i\in\mathbb{N}^d} f_i z^i$, the field of analytic combinatorics studies the asymptotic behavior of f_n through an analytic property of $F(z)$. Motivated by the paper [25], **we implement a Julia package `ACSVHomotopy.jl` computing this asymptotic behavior using the polyhedral homotopy and monodromy methods.** Our implementation performs more effectively than existing software, especially when the denominator of F has a higher degree or random coefficients. Also, it is the first known software for dealing with the non-combinatorial case.

In addition, the paper [22] studies the *generalized Nash equilibrium problem* (GNEP). A GNEP for N-player game can be considered as a problem for finding an optimizers (so called generalized Nash equilibria) $u = (u_1, \ldots, u_N)$ such that for given $u_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N)$, each $u_i \in \mathbb{R}^{n_i}$ is a minimizer for

$$\begin{cases} \min\limits_{x_i\in\mathbb{R}^{n_i}} & f_i(x_i, u_{-i}) \\ \text{s.t} & x_i \in X_i(u_{-i}) \end{cases}$$

where X_i is a feasible set consists of equality and inequality constraints. Under some constraint qualifications, each optimizer satisfies a square system called the *KKT system*. **We prove that the KKT system deduced from a generic GNEP is Bernstein generic.** Using the polyhedral homotopy, we devise the algorithm for solving GNEPs. When, the polyhedral homotopy finds the mixed volume many solutions for the KKT system, then all GNEs can be found or we can detect the non-existence of GNEs. Our algorithm can be more effective than existing methods, especially when the feasible sets X_i are non-convex.

Future projects

Extending the work [22], I currently study the requirements regarding *sparsity* for the KKT system to exhibit Bernstein genericity. Considering the sparsity of polynomial systems, a tighter mixed volume may be feasible compared to dense systems. Through the lens of toric geometry, I aim to clarify which monomials are needed for constraints and objective functions in GNEPs to attain Bernstein genericity. This work is anticipated to theoretically enhance the necessity of polyhedral homotopy methods for GNEPs.

Certified algorithms in algebraic geometry

Certified algorithms produce both a solution and a certificate of correctness, i.e. a computer-generated proof that ensures the correctness of their outcome. In particular, my work focuses on such algorithms for isolated solutions to a square polynomial (or analytic) system. They are designed either to prove the correctness of a given approximate solution to the system (*a posteriori* certification) or to generate such approximation in a certified way that the correctness is guaranteed (a certified homotopy tracking). It increases the utility of numerical methods in algebraic geometry since the reliability of numerical results is not generally granted.

For a polynomial system $F : \mathbb{C}^n \to \mathbb{C}^n$, let $x \in \mathbb{C}^n$ be a numerical solution approximating an exact solution ξ. We say that x is *certified* if there is a region that contains ξ uniquely deduced from a given x. In this case, x can be refined to ξ by applying a finite iteration of some contraction maps (e.g. Newton operator) on this region.

Algorithms of *a posteriori* certification depend on the singularity of a solution to certify. If the exact solution ξ is nonsingular (i.e. the Jacobian $JF(\xi)$ is invertible), *Newton's method* is a pivotal tool for certifying a numerical solution. The iterative application of the Newton operator $N_F(x) := x - JF(x)^{-1}F(x)$ refines x when it is "close" to ξ. Therefore, proving if x is "close enough" to ξ is the main task for certification. On the other hand, when ξ is singular, the algorithm for certifying x greatly hinges on the algebraic characterizations of ξ, such as the multiplicity, local standard monomial basis, etc. The study of commutative algebra becomes essential to classify singular solutions according to these characterizations.

The methods described earlier carry out post-processing once numerical solutions are obtained, while certified homotopy tracking rigorously produces numerical solutions for the system. One possible pathological issue of heuristic implementation of the homotopy tracking is *path jumping*, when a solution from another solution path is produced due to a misguided intermediate path tracking. Certain computations using homotopy continuation, such as monodromy, necessitate not only accurate output but also rigorous path tracking to ensure path jumping does not occur.

Past work

The paper [4] **introduces frameworks to certify a nonsingular solution of a system of analytic functions using (1) the *Krawczyk method* and (2) *Smale's alpha theory*.** Both approaches are in common in that they are about refining x to ξ up to arbitrary precision by an iterative method. We develop the implementation for certifying a nonsingular solution to a system with *D-finite* functions based on these frameworks. In addition, the paper [17] **introduces `Macaulay2` [13] implementation for these frameworks for systems with polynomial equations**. More recently, **the paper [18] implements alpha theory certification using interval arithmetic to avoid expensive exact arithmetic**. This paper establishes *alpha theory over regions*, which enhances efficiency by performing alpha theory computations with interval arithmetic.

For a singular solution, different approaches for certification are considered; isolating the singular solution ξ from other solutions or modifying Newton's method to guarantee convergence toward the singular solution. The work [5] proposes an algorithm for isolating the exact singular solution ξ uniquely. It is achieved by the method of the inflation [6] which involves a more amenable system induced by the original system F. **We prove that the certification using the inflation is available for a special family of singular solutions called *regular zeros*, and characterize the regular zeros by their Hilbert series**. The paper [19] proposes the modified Newton's method for refining a numerical solution x to an exact singular solution ξ. It defines a linear operator by combining the Jacobian and a projection of Hessian to the direction of $\ker JF(\xi)$. When ξ makes the linear operator invertible, we call it a *deflation-one singular solution*. By replacing the Jacobian JF with this operator, **we propose the improved Newton's method for deflation-one singular solutions and prove its quadratic convergence**.

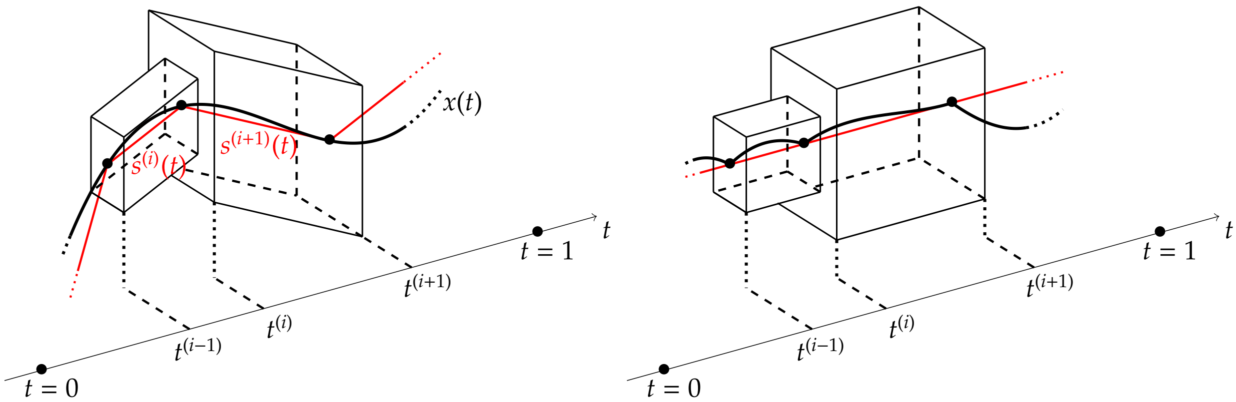

Figure 1: An illustration of the certified homotopy tracking.

The paper [10] establishes the certified homotopy tracking using the Krawczyk method. For a homotopy $H(x,t)$ with $H(x,1) = F(x)$ and $H(x,0) = G(x)$, it tracks the solution path $\xi(t)$ from $t = 0$ to $t = 1$ in without path jumping. The algorithm consists of iterative process constructing discrete time-step $0 = t^{(0)} < t^{(1)} < \cdots < t^{(k)} = 1$ and compact region that contains a solution path $\xi(t)$ uniquely along each time interval $[t^{(i-1)}, t^{(i)}]$. Within each $[t^{(i-1)}, t^{(i)}]$, we define the line segment $s^{(i)}(t)$ approximating $\xi(t)$. The Krawczyk method is applied on the (tilted) interval box whose center is $s^{(i)}(t)$ (See Figure 1). **Despite its conceptually simple approach, the algorithm proposed in the paper applies in the general parameter homotopies setting.** In addition, we prove that the algorithm is terminated in finitely many iterations. **The experimental results based on a preliminary**

implementation indicate that our approach is competitive compared to known certified homotopy tracking algorithms [1, 14].

Future projects

I am excited when I apply the theory from commutative algebra to improve the certified algorithm. To investigate the limit of the inflation method [5, 6], I am characterizing the singular solutions that can be certified by this method. This characterization requires the study of dual space of isolated singular solutions. From observing their dual space, I expect to describe the collection of "inflatable" singular solutions using the integer lattice of the standard monomial basis.

Motivated by my work [19] studying the deflation-one singular solutions, I am focusing on the ability of the deflation method to desingularize algebraic varieties. The paper [23] establishes the primary decomposition algorithm for a variety using higher-order deflation. It was questioned whether iterative first-order deflation may execute the same task since it has more computational efficiency than higher deflation. I conjecture this is false and attempt to clarify the conditions on algebraic varieties that can be decomposed by iterative deflation.

Combinatorial understanding of Symmetric matrix completion

My interest on applied algebraic geometry reaches to tropical geometry or combinatorics. It enables to understand the algebraic variety using the polyhedral geometric point of view. In particular, I have tackled problems in the *matrix completion* using these tools. Given a partially specified matrix, matrix completion is the problem of completing the matrix to achieve the lowest possible rank. Recently, interest has been gained from algebraic geometers because matrices of a certain rank can be described as vanishing loci of a determinantal variety.

For a given size $m \times n$ of a matrix, it is typical to characterize which partial matrices can be completed to rank r. Assuming known entries of the matrix are chosen generically, the lowest possible rank only depends on the locations of known entries, not the values of them. This lowest possible rank is called *generic completion rank*. There are several variations of matrix completion problems based on the types of matrices (e.g. symmetric matrices).

After a noteworthy contribution in [16], there have been various combinatorial approaches to the matrix completion problem. In particular, representing partial matrices using graphs is an accessible and usual way to deal with the problem. Given a partial matrix, we consider a graph whose known entries are an edge of the graph. For instance, symmetric partial matrices are encoded using *semisimple* graphs (graphs allowing a loop on each vertex).

Past work

In [2], we considered a real symmetric matrix completion. For a partial symmetric matrix over $\mathbb{R}$, it allows the completion only with real numbers. In this case, even if the entries are chosen generically, the lowest possible rank can be higher than the generic completion rank. This rank is called the *typical rank* of a matrix. By encoding partial symmetric matrices to semisimple graphs, **we characterize semisimple graphs having their number of vertices as a typical rank**. In addition, **for a semisimple graph of generic completion rank** 1 **or** 2**, we classify entirely all of them according to their possible typical ranks**.

More recently, I studied the *tropicalization* of the space of symmetric rank 2 matrices [7]. The tropicalization of an algebraic variety is obtained by replacing the usual sum and product with their tropical counterpart, min and sum. As the resulting algebraic variety has a polyhedral structure, the tropicalization enables understanding the algebraic variety using the combinatorial point of view. Tropicalizing the symmetric matrices induces the notion of *symmetric tropical rank*, and matrices with symmetric tropical rank 2 can be represented by the *symmetric bicolored tree*. Inspired by the result from [24] studied the tropicalization of general rank 2 matrices, **we show that the space of the symmetric tropical rank** 2 **matrices is shellable.**

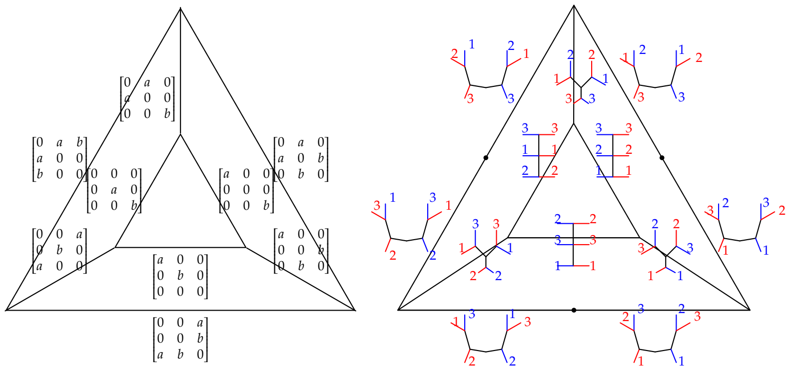

Figure 2: The space of 3×3 symmetric tropical rank 2 matrices.

An *algebraic matroid* is another fundamental tool for understanding the tropicalization of the space of matrices with a certain rank. An algebraic matroid is a combinatorial object that expresses the relation of algebraic independence from the coordinate ring of a given irreducible algebraic variety. Since the algebraic matroid is preserved under the tropicalization [28], we may define an algebraic matroid of a given symmetric bicolored tree. **The main result in this direction from [7] is that these matroids can be represented by a Cayley sum of point configurations corresponding to the nodes of the given tree.**

Future projects

For a given partial matrix, there are only finitely many ways to complete this matrix to achieve the generic completion rank. This finite number of completions is called *the algebraic degree of the matrix completion*. Classifying the partial matrices according to their algebraic degree is desirable for a given matrix size and rank. It requires understanding the changes in algebraic degrees regarding the patterns of known matrix entries. I expect that the combinatorial representations (e.g. graphs) of partial matrices will be helpful to visualize these changes.

The coding theory has been known as one of the applications of the matrix completion problem [12]. It requires tackling this problem with a positive characteristic to apply known results of the matrix completion problem to the coding theory. As there are active studies about the algebraic matroid of determinantal varieties, proving if the algebraic matroid is preserved in finite characteristics enlarges the utility of these results. Even though it is known that 3×3 minors of matrices do not form a tropical basis when the characteristic is 2, there is still a lack of evidence on this question to answer in either direction.

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