10.2 Derivatives & Integrals.

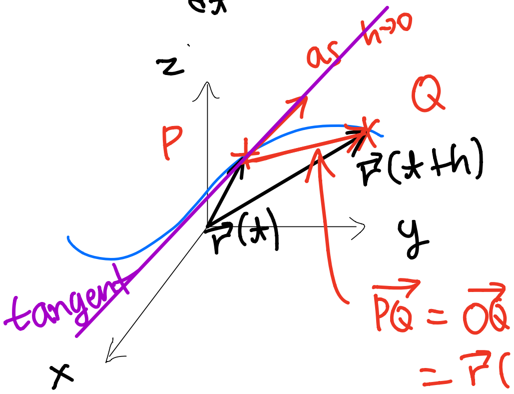
* Derivatives

<u>**Recall**</u> (1 - variable derivatives)

$$\frac{d f(t)}{dt} = f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \quad \text{if limit exists}$$

$\vec{r}(t) = (f(t), g(t), h(t))$: a vector function

$$\Rightarrow \frac{d\vec{r}}{dt}(t) = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \quad \text{if limit exists}$$



$\vec{PQ} = \vec{OQ} - \vec{OP}$
$= \vec{r}(t+h) - \vec{r}(t)$

$\vec{r}\,'(t)$ means that the direction vector of the tangent line of the curve at P such that $\vec{OP} = \vec{r}(t)$

(line passing thru P & parallel to $\vec{r}\,'(t)$)

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \quad : \text{the unit tangent vector.}$$

$\vec{r}(t) = (f(t), g(t), h(t))$

$\Rightarrow \vec{r}\,'(t) = (f'(t), g'(t), h'(t))$

ex) $\vec{r}(t) = (1+t^2, te^{-t}, \sin 2t)$

$\Rightarrow \vec{r}\,'(t) = (2t, e^{-t} + te^{-t}(-1), 2\cos 2t)$

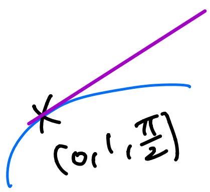
$= (2t, e^{-t} - te^{-t}, 2\cos 2t)$

Find the unit tangent vector at $t=0$

$$\vec{T}(0) = \frac{\vec{r}'(0)}{|\vec{r}'(0)|} = \frac{1}{\sqrt{0^2+1^2+2^2}}(0, 1, 2) = \left(0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$

Example) Find parametric equations for the tangent line

to $\begin{cases} x = 2\cos t \\ y = \sin t \\ z = t \end{cases}$ at $\left(0, 1, \frac{\pi}{2}\right)$



① $\vec{r}'(t) = (-2\sin t, \cos t, 1)$

② Find t such that $\vec{r}(t) = \left(0, 1, \frac{\pi}{2}\right)$

if $t = \frac{\pi}{2}$, $\vec{r}\left(\frac{\pi}{2}\right) = \left(0, 1, \frac{\pi}{2}\right)$.

③ $\vec{r}'\left(\frac{\pi}{2}\right) = (-2, 0, 1)$

④ tangent line : $\begin{cases} \text{parallel to } (-2, 0, 1) \\ \text{passes thru } \left(0, 1, \frac{\pi}{2}\right) \end{cases}$

$$\Rightarrow \begin{cases} x = -2t \\ y = 1 \\ z = \frac{\pi}{2} + t \end{cases}$$

* Differentiation Rules

① $\frac{d}{dt}(\vec{u}(t)+\vec{v}(t)) = \vec{u}'(t)+\vec{v}'(t).$

② $\frac{d}{dt}(f(t)\,.\,\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\,.\,\vec{u}'(t)$

③ $\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

④ $\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$

⑤ $\frac{d}{dt}(\vec{u}(f(t))) = f'(t)\,.\,\vec{u}'(f(t))$ (Chain Rule)

* Integrals

$\vec{r}(t) = (f(t), g(t), h(t))$

$\Rightarrow \int_a^b \vec{r}(t)\,dt = \left(\int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt\right)$

Example) $\vec{r}(t) = (2\cos t, \sin t, 2t)$

$\Rightarrow \int_0^{\frac{\pi}{2}} \vec{r}(t)dt = \left(2\sin t\Big]_0^{\frac{\pi}{2}}, -\cos t\Big]_0^{\frac{\pi}{2}}, t^2\Big]_0^{\frac{\pi}{2}}\right)$

$= \left(2, 1, \frac{\pi^2}{4}\right)$