Homotopy techniques for analytic combinatorics in several variables

(joint work with Stephen Melczer[†] and Josip Smolčić[†])

[†] University of Waterloo

Kisun Lee (UC San Diego) - kisunlee@ucsd.edu

Joint Mathematics Meetings 2023 - AMS Special Session on Applied Enumerative Geometry

Acknowledgements

MATHEMATICS
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This work was started and supported in the

AMS Math Research Community 2021

Combinatorial Applications of Computational Geometry and Algebraic Topology

Analytic combinatorics

$(f_n) = f_1, f_2, \ldots$: a sequence of complex numbers.

$F(z) = \sum_{i=1}^{\infty} f_i \, z^i$: **the generating function** of the sequence.

- The generating function can be considered as the power series expansion of a complex valued function.

Q. Can we study the asymptotic behavior of (f_n) using the (analytic) behavior of F ?

- (Cauchy integral formula). $f_n = \dfrac{1}{2\pi i} \displaystyle\int_\gamma F(z) \frac{dz}{z^{n+1}}$

Analytic combinatorics in several variables

$F(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^n} f_{\mathbf{i}} \, \mathbf{z}^{\mathbf{i}}$: the **multivariate** generating function of the sequence.

- Specifically, the $\mathbf{r}$-diagonal sequence $(f_{n\mathbf{r}})$ for any $\mathbf{r} \in \mathbb{R}^n$ is considered.
- The common situation to arise in practice is the main-diagonal $\mathbf{r} = \mathbf{1}$. **Ex) (Furstenberg 1967), (Christol 1984), (Chudnovsky-Chudnovsky 1985), (André 2000)**
- Applications : Lattice path enumeration, random walks, and so on.

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(Need a system of equations)

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- It is important to find where the singularity locates.

- Define Abs : $(z_1, \ldots, z_n) \mapsto |z_1 \cdots z_n|$.
- We are interested in critical points of Abs.
- $\mathscr{V} := \{\mathbf{z} \in \mathbb{C}^n \mid H(\mathbf{z}) = 0\}$: the **singular variety** of $F = \dfrac{G}{H}$
- The smooth critical points for Abs on $\mathscr{V}$ are obtained by solving a polynomial system
- $H(\mathbf{z}) = 0, \quad z_1 \dfrac{\partial H}{\partial z_1} = \cdots = z_n \dfrac{\partial H}{\partial z_n}$ **(critical-point equations)**
- Especially, we are interested in **minimal** critical points (i.e. critical points lie in $\partial\mathscr{D} \cap \mathscr{V}$).

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- $H(\mathbf{z}) = 0, \quad z_1 \dfrac{\partial H}{\partial z_1} = \cdots = z_n \dfrac{\partial H}{\partial z_n}$ **(critical-point equations)**
- Especially, we are interested in **minimal** critical points (i.e. critical points lie in $\partial\mathscr{D} \cap \mathscr{V}$).

ACSV + Numerical algebraic geometry (how to construct a system of equations)

- $F(\mathbf{z}) = \dfrac{G(\mathbf{z})}{H(\mathbf{z})}$ where G and H are co-prime polynomials with $H(\mathbf{0}) \neq 0$.

- $F(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^n} f_{\mathbf{i}}\, \mathbf{z}^{\mathbf{i}}$: The Taylor expansion of F centered at the origin with a nonempty open domain of convergence $\mathscr{D} \subset \mathbb{C}^n$.

- Interested in computing the asymptotic behavior of coefficients of $(f_{\mathbf{i}})_{\mathbf{i}}$.

$$f_{i_1,\ldots,i_n} = \frac{1}{(2\pi i)^n} \int_\gamma \frac{F(\mathbf{z})}{z_1^{i_1} \cdots z_n^{i_n}} \cdot \frac{dz_1 \ldots dz_n}{z_1 \cdots z_n}$$

- It is important to find where the singularity locates.

- Define Abs $: (z_1, \ldots, z_n) \mapsto |z_1 \cdots z_n|$.

- We are interested in critical points of Abs.

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ACSV for combinatorial case (Melczer-Salvy 2021)

$F(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^n} f_{\mathbf{i}}\, \mathbf{z}^{\mathbf{i}}$ is called **combinatorial** if all coefficients $f_{\mathbf{i}}$ of the Taylor expansion are non-negative.

1. Determine the set $\mathcal{S}$ of zeros $(\mathbf{z}, \lambda, t)$ of the system $\left[H, z_1 \frac{\partial H}{\partial z_1} - \lambda, \ldots, z_n \frac{\partial H}{\partial z_n} - \lambda, H(tz_1, \ldots, tz_n)\right]$. If $\mathcal{S}$ is not finite, then FAIL.

2. Construct the subset points $(\boldsymbol{\zeta}, \lambda, t) \in \mathcal{S}$ which are candidates for minimal critical points.
   - $\mathbf{z}$ is minimal if and only if the line segment $\{(t|z_1|, \ldots, t|z_n|) \mid 0 < t < 1\}$ doesn't intersect $\mathcal{V}$.

3. Identify $\boldsymbol{\zeta}$ among the elements of $\mathcal{C}$ (critical points Abs on $\mathcal{V}$).

4. Return $\mathcal{U} := \{\mathbf{z} \in \mathbb{C}^n \mid |z_1| = |\zeta_1|, \ldots, |z_n| = |\zeta_n| \text{ for some } (\mathbf{z}, \lambda) \in \mathcal{C}\}$

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Doesn't hold if F is not combinatorial

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Q. How to deal with non-combinatorial case?

Lemma (Melczer-Salvy 2021)

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Q. How to deal with polydisk using polynomial equations?

A. Decompose polynomials into the real and imaginary part.

$$f(\mathbf{x} + i\mathbf{y}) = f^{(R)}(\mathbf{x}, \mathbf{y}) + if^{(I)}(\mathbf{x}, \mathbf{y})$$

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1. Using real and imaginary part decomposition, critical-point equations are given as

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Want to have no solutions with $\mathbf{x}, \mathbf{y}, t$ real and $0 < t < 1$.

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$$a_j \frac{\partial H^{(R)}}{\partial x_j}(\mathbf{a}, \mathbf{b}) + b_j \frac{\partial H^{(R)}}{\partial y_j}(\mathbf{a}, \mathbf{b}) - \lambda_R = 0 \qquad j = 1, \ldots, n$$

$$a_j \frac{\partial H^{(I)}}{\partial x_j}(\mathbf{a}, \mathbf{b}) + b_j \frac{\partial H^{(I)}}{\partial y_j}(\mathbf{a}, \mathbf{b}) - \lambda_I = 0 \qquad j = 1, \ldots, n$$

2. For checking emptiness of $\mathscr{V} \cap D(\mathbf{z}) = \emptyset$, we consider equations

$$H^{(R)}(\mathbf{x}, \mathbf{y}) = H^{(I)}(\mathbf{x}, \mathbf{y}) = 0$$

$$x_j^2 + y_j^2 - t(a_j^2 + b_j^2) = 0 \qquad j = 1, \ldots, n$$

Want to have no solutions with $\mathbf{x}, \mathbf{y}, t$ real and $0 < t < 1$.

3. For checking extremity for values of t, we add equations

$$(y_j - \nu x_j)\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{x}, \mathbf{y}) - (x_j + \nu y_j)\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{x}, \mathbf{y}) = 0, \qquad j = 1, \ldots, n$$

Want to have no solutions with $\mathbf{x}, \mathbf{y}, \nu, t$ real and $0 < t < 1$.

ACSV for non-combinatorial case (Melczer-Salvy 2021)

Q. How to deal with non-combinatorial case?

Lemma (Melczer-Salvy 2021)

Let $D(\mathbf{z}) := \{\mathbf{w} \in \mathbb{C}^n \mid |w_i| < |z_i|, i = 1,\ldots,n\}$ be the open polydisk. If $\mathbf{z} \in \mathscr{V}$ and $\mathscr{V} \cap D(\mathbf{z}) = \emptyset$, then $\mathbf{z} \in \partial\mathscr{D}$.

Q. How to deal with polydisk using polynomial equations?

A. Decompose polynomials into the real and imaginary part.

$$f(\mathbf{x} + i\mathbf{y}) = f^{(R)}(\mathbf{x}, \mathbf{y}) + if^{(I)}(\mathbf{x}, \mathbf{y})$$

For derivatives, applying Cauchy-Riemann equations,

$$\frac{\partial f}{\partial z_j}(\mathbf{x} + i\mathbf{y}) = \frac{1}{2} \cdot \frac{\partial}{\partial x_j}\left(f^{(R)}(\mathbf{x}, \mathbf{y}) + if^{(I)}(\mathbf{x}, \mathbf{y})\right) - \frac{i}{2} \cdot \frac{\partial}{\partial y_j}\left(f^{(R)}(\mathbf{x}, \mathbf{y}) + if^{(I)}(\mathbf{x}, \mathbf{y})\right)$$

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$$
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(y_j - \nu x_j)\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{x},\mathbf{y}) - (x_j + \nu y_j)\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{x},\mathbf{y}) &= 0 && j = 1,\ldots,n
\end{aligned}\right\} \quad (\star)
$$

$(\star)$ is a square polynomial system with $4n + 4$ variables $(\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}, \lambda_R, \lambda_I, \nu, t)$.

ACSV for combinatorial case

1. Determine the set $\mathcal{S}$ of zeros $(\mathbf{z}, \lambda, t)$ of the system $\left[H, z_1\frac{\partial H}{\partial z_1} - \lambda, \ldots, z_n\frac{\partial H}{\partial z_n} - \lambda, H(tz_1, \ldots, tz_n)\right]$. If $\mathcal{S}$ is not finite, then FAIL.

2. Construct the subset points $(\boldsymbol{\zeta}, \lambda, t) \in \mathcal{S}$ which are candidates for minimal critical points.

   - $\mathbf{z}$ is minimal if and only if the line segment $\{(t|z_1|, \ldots, t|z_n|) \mid 0 < t < 1\}$ doesn't intersect $\mathcal{V}$.

3. Identify $\boldsymbol{\zeta}$ among the elements of $\mathcal{C}$ (critical points Abs on $\mathcal{V}$).

4. Return $\mathcal{U} := \{\mathbf{z} \in \mathbb{C}^n \mid |z_1| = |\zeta_1|, \ldots, |z_n| = |\zeta_n|$ for some $(\mathbf{z}, \lambda) \in \mathcal{C}\}$

ACSV for non-combinatorial case

1. Determine the set $\mathcal{S}$ of zeros $(\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}, \lambda_R, \lambda_I, \nu, t)$ of the system ($\star$). If $\mathcal{S}$ is not finite, then FAIL.

2. Construct the set of minimal critical points $\mathcal{U} := \left\{\mathbf{a} + i\mathbf{b} \mid (\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}, \lambda_R, \lambda_I, \nu, t) \in \mathcal{S}_{\mathbb{R}},\ t \notin (0,1)\right\} \subset \mathcal{S}$

3. If $\mathcal{U} = \emptyset$ or $\lambda_R = \lambda_I = 0$ or the elements of $\mathcal{U}$ do not all belong to the same torus, then FAIL.

4. Identify elements of $\mathcal{U}$ from $\mathcal{C}$ and return them.

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ACSVHomotopy.jl (L.-Melczer-Smolčić 2022)

- Implemented using `HomotopyContinuation.jl` **(Breiding-Timme 2018)**
- Available at github.com/ACSVMath/ACSVHomotopy

- Competitive to other ACSV software for combinatorial cases.
  - Solve the critical-point equations using the polyhedral homotopy
- The first software of ACSV for non-combinatorial cases.
  - Solve the decomposed critical-point equations ($\star$) using the polyhedral homotopy
  - Provide faster heuristics including the monodromy method.

Implementation details

The polyhedral homotopy is the default for solving critical point equations.

- Returns reliable results via interval arithmetic certification.
- Effective for combinatorial case compared to other software based on symbolic algorithm.
- May be slow for non-combinatorial.
  - Faster heuristics used.

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The polyhedral homotopy is the default for solving critical point equations.

- Returns reliable results via interval arithmetic certification.

- Effective for combinatorial case compared to other software based on symbolic algorithm.

- May be slow for non-combinatorial.
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Implementation details

Heuristics for non-combinatorial case.

1. Approximating critical points

$$\left.\begin{aligned}
H^{(R)}(\mathbf{a},\mathbf{b}) = H^{(I)}(\mathbf{a},\mathbf{b}) &= 0 \\
a_j\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{a},\mathbf{b}) + b_j\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{a},\mathbf{b}) - \lambda_R &= 0 \qquad j = 1,\ldots,n \\
a_j\frac{\partial H^{(I)}}{\partial x_j}(\mathbf{a},\mathbf{b}) + b_j\frac{\partial H^{(I)}}{\partial y_j}(\mathbf{a},\mathbf{b}) - \lambda_I &= 0 \qquad j = 1,\ldots,n \\
H^{(R)}(\mathbf{x},\mathbf{y}) = H^{(I)}(\mathbf{x},\mathbf{y}) &= 0 \\
x_j^2 + y_j^2 - t(a_j{}^2 + b_j{}^2) &= 0 \qquad j = 1,\ldots,n \\
(y_j - \nu x_j)\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{x},\mathbf{y}) - (x_j + \nu y_j)\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{x},\mathbf{y}) &= 0 \qquad j = 1,\ldots,n
\end{aligned}\right\} \quad (\star)$$

Implementation details

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a_j\dfrac{\partial H^{(I)}}{\partial x_j}(\mathbf{a},\mathbf{b}) + b_j\dfrac{\partial H^{(I)}}{\partial y_j}(\mathbf{a},\mathbf{b}) - \lambda_I = 0 & j = 1,\dots,n \\
H^{(R)}(\mathbf{x},\mathbf{y}) = H^{(I)}(\mathbf{x},\mathbf{y}) = 0 & \\
x_j^2 + y_j^2 - t(a_j^2 + b_j^2) = 0 & j = 1,\dots,n \\
(y_j - \nu x_j)\dfrac{\partial H^{(R)}}{\partial x_j}(\mathbf{x},\mathbf{y}) - (x_j + \nu y_j)\dfrac{\partial H^{(R)}}{\partial y_j}(\mathbf{x},\mathbf{y}) = 0 & j = 1,\dots,n
\end{array}\right\} \quad (\star)$$

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$$\left.\begin{array}{rr} H^{(R)}(\mathbf{x},\mathbf{y}) = H^{(I)}(\mathbf{x},\mathbf{y}) = 0 & \\ x_j^2 + y_j^2 - t(a_j{}^2 + b_j{}^2) = 0 & j = 1,\dots,n \\ (y_j - \nu x_j)\dfrac{\partial H^{(R)}}{\partial x_j}(\mathbf{x},\mathbf{y}) - (x_j + \nu y_j)\dfrac{\partial H^{(R)}}{\partial y_j}(\mathbf{x},\mathbf{y}) = 0 & j = 1,\dots,n \end{array}\right\} \quad (B)$$

1. Approximating critical points
   - Solve the subsystem (A) to get an approximation of $(\mathbf{a},\mathbf{b})$.
   - Using the approximations, solve the subsystem (B).

Implementation details

Heuristics for non-combinatorial case.

$$
\left.\begin{aligned}
&H^{(R)}(\mathbf{a},\mathbf{b}) = H^{(I)}(\mathbf{a},\mathbf{b}) = 0 \\
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\end{aligned}\right\} (A)
$$

$$
\left.\begin{aligned}
&H^{(R)}(\mathbf{x},\mathbf{y}) = H^{(I)}(\mathbf{x},\mathbf{y}) = 0 \\
&x_j^2 + y_j^2 - t(a_j^2 + b_j^2) = 0 \qquad j = 1,\ldots,n \\
&(y_j - \nu x_j)\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{x},\mathbf{y}) - (x_j + \nu y_j)\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{x},\mathbf{y}) = 0 \qquad j = 1,\ldots,n
\end{aligned}\right\} (B)
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H^{(R)}(\mathbf{x},\mathbf{y}) = H^{(I)}(\mathbf{x},\mathbf{y}) &= 0 \\
x_j^2 + y_j^2 - t(a_j^2 + b_j^2) &= 0 \qquad j = 1,\ldots,n \\
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2. Monodromy

- Solve the subsystem (A) to get an approximation of $(\mathbf{a},\mathbf{b})$.
- Solve the subsystem (B) using monodromy with $(\mathbf{x},\mathbf{y},t) = (\mathbf{a},\mathbf{b},1)$.

  (ν can be found from the Jacobian)
- Caveat : the subsystem (B) may have several irreducible components. (Fail to find all critical points)

Implementation details

Heuristics for non-combinatorial case.

$$\left.\begin{array}{rr} H^{(R)}(\mathbf{a},\mathbf{b}) = H^{(I)}(\mathbf{a},\mathbf{b}) = 0 & \\ a_j \dfrac{\partial H^{(R)}}{\partial x_j}(\mathbf{a},\mathbf{b}) + b_j \dfrac{\partial H^{(R)}}{\partial y_j}(\mathbf{a},\mathbf{b}) - \lambda_R = 0 & j = 1,\ldots,n \\ a_j \dfrac{\partial H^{(I)}}{\partial x_j}(\mathbf{a},\mathbf{b}) + b_j \dfrac{\partial H^{(I)}}{\partial y_j}(\mathbf{a},\mathbf{b}) - \lambda_I = 0 & j = 1,\ldots,n \end{array}\right\} \quad (A)$$

$$\left.\begin{array}{rr} H^{(R)}(\mathbf{x},\mathbf{y}) = H^{(I)}(\mathbf{x},\mathbf{y}) = 0 & \\ x_j^2 + y_j^2 - t(a_j{}^2 + b_j{}^2) = 0 & j = 1,\ldots,n \\ (y_j - \nu x_j)\dfrac{\partial H^{(R)}}{\partial x_j}(\mathbf{x},\mathbf{y}) - (x_j + \nu y_j)\dfrac{\partial H^{(R)}}{\partial y_j}(\mathbf{x},\mathbf{y}) = 0 & j = 1,\ldots,n \end{array}\right\} \quad (B)$$

2. Monodromy
   - Solve the subsystem (A) to get an approximation of $(\mathbf{a},\mathbf{b})$.
   - Solve the subsystem (B) using monodromy with $(\mathbf{x},\mathbf{y},t) = (\mathbf{a},\mathbf{b},1)$.

     (ν can be found from the Jacobian)
   - Caveat : the subsystem (B) may have several irreducible components. (Fail to find all critical points)