Homotopy techniques for analytic combinatorics in several variables

(joint work with Stephen Melczer[†] and Josip Smolčić[†])

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MATHEMATICS
RESEARCH COMMUNITIES

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Combinatorial Applications of Computational Geometry and Algebraic Topology

Analytic combinatorics

$(f_n) = f_1, f_2, \ldots$: a sequence of complex numbers.

$F(z) = \sum_{i=1}^{\infty} f_i\, z^i$: **the generating function** of the sequence.

- The generating function can be considered as the power series expansion of a complex valued function.

Q. Can we study the asymptotic behavior of (f_n) using the (analytic) behavior of F ?

- (Cauchy integral formula). $f_n = \frac{1}{2\pi i} \int_\gamma F(z) \frac{dz}{z^{n+1}}$

Analytic combinatorics in several variables

$F(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^n} f_{\mathbf{i}}\, \mathbf{z}^{\mathbf{i}}$: the **multivariate** generating function of the sequence.

- Specifically, the $\mathbf{r}$-diagonal sequence $(f_{n\mathbf{r}})$ for any $\mathbf{r} \in \mathbb{R}^n$ is considered.

- The common situation to arise in practice is the main-diagonal $\mathbf{r} = \mathbf{1}$. **Ex) (Furstenberg 1967), (Christol 1984), (Chudnovsky-Chudnovsky 1985), (André 2000)**

- Applications : Lattice path enumeration, random walks, and so on.

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(Need a system of equations)

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- $F(\mathbf{z}) = \dfrac{G(\mathbf{z})}{H(\mathbf{z})}$ where G and H are co-prime polynomials with $H(\mathbf{0}) \neq 0$.
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- It is important to find where the singularity locates.
- Define Abs : $(z_1, \ldots, z_n) \mapsto |z_1 \cdots z_n|$.
- We are interested in critical points of Abs.
- $\mathscr{V} := \{\mathbf{z} \in \mathbb{C}^n \mid H(\mathbf{z}) = 0\}$: the **singular variety** of $F = \dfrac{G}{H}$
- The critical points for Abs on $\mathscr{V}$ are obtained by solving a polynomial system
- $H(\mathbf{z}) = 0, \quad z_1 \dfrac{\partial H}{\partial z_1} = \cdots = z_n \dfrac{\partial H}{\partial z_n}$ **(critical-point equations)**
- We assume that all critical points are smooth.
- Especially, we are interested in **minimal** critical points (i.e. critical points lie in $\partial\mathscr{D} \cap \mathscr{V}$).

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- Define Abs $: (z_1, \ldots, z_n) \mapsto |z_1 \cdots z_n|$.
- We are interested in critical points of Abs.
- $\mathscr{V} := \{\mathbf{z} \in \mathbb{C}^n \mid H(\mathbf{z}) = 0\}$: the **singular variety** of $F = \dfrac{G}{H}$
- The critical points for Abs on $\mathscr{V}$ are obtained by solving a polynomial system
- $H(\mathbf{z}) = 0, \quad z_1 \dfrac{\partial H}{\partial z_1} = \cdots = z_n \dfrac{\partial H}{\partial z_n}$ **(critical-point equations)**
- We assume that all critical points are smooth.
- Especially, we are interested in **minimal** critical points (i.e. critical points lie in $\partial\mathscr{D} \cap \mathscr{V}$).

ACSV + Numerical algebraic geometry (how to construct a system of equations)

- $F(\mathbf{z}) = \dfrac{G(\mathbf{z})}{H(\mathbf{z})}$ where G and H are co-prime polynomials with $H(\mathbf{0}) \neq 0$.
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$F(\mathbf{z}) = \sum_{\mathbf{i} \in \mathbb{N}^n} f_{\mathbf{i}}\, \mathbf{z}^{\mathbf{i}}$ is called **combinatorial** if all coefficients $f_{\mathbf{i}}$ of the Taylor expansion are non-negative.

1. Determine the set $\mathcal{S}$ of zeros $(\mathbf{z}, \lambda, t)$ of the system $\left[H, z_1 \frac{\partial H}{\partial z_1} - \lambda, \ldots, z_n \frac{\partial H}{\partial z_n} - \lambda, H(tz_1, \ldots, tz_n)\right]$. If $\mathcal{S}$ is not finite, then FAIL.
2. Construct the subset points $(\zeta, \lambda, t) \in \mathcal{S}$ which are candidates for minimal critical points.
   - $\mathbf{z}$ is minimal if and only if the line segment $\{(t|z_1|, \ldots, t|z_n|) \mid 0 < t < 1\}$ doesn't intersect $\mathcal{V}$.
3. Identify ζ among the elements of $\mathcal{C}$ (critical points Abs on $\mathcal{V}$).
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Doesn't hold if F is not combinatorial

ACSV for non-combinatorial case (Melczer-Salvy 2021)

Q. How to deal with non-combinatorial case?

Lemma (Melczer-Salvy 2021)

Let $D(\mathbf{z}) := \{\mathbf{w} \in \mathbb{C}^n \mid |w_i| < |z_i|, i = 1,\ldots,n\}$ be the open polydisk. If $\mathbf{z} \in \mathscr{V}$ and $\mathscr{V} \cap D(\mathbf{z}) = \varnothing$, then $\mathbf{z} \in \partial\mathscr{D}$

Q. How to deal with polydisk using polynomial equations?

A. Decompose polynomials into the real and imaginary part.

$$f(\mathbf{x} + i\mathbf{y}) = f^{(R)}(\mathbf{x},\mathbf{y}) + if^{(I)}(\mathbf{x},\mathbf{y})$$

For derivatives, applying Cauchy-Riemann equations,

$$\frac{\partial f}{\partial z_j}(\mathbf{x}+i\mathbf{y}) = \frac{1}{2}\cdot\frac{\partial}{\partial x_j}\left(f^{(R)}(\mathbf{x},\mathbf{y}) + if^{(I)}(\mathbf{x},\mathbf{y})\right) - \frac{i}{2}\cdot\frac{\partial}{\partial y_j}\left(f^{(R)}(\mathbf{x},\mathbf{y}) + if^{(I)}(\mathbf{x},\mathbf{y})\right)$$

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ACSV for non-combinatorial case (Melczer-Salvy 2021)

Q. How to deal with non-combinatorial case?

<u>Lemma</u> (Melczer-Salvy 2021)

Let $D(\mathbf{z}) := \{\mathbf{w} \in \mathbb{C}^n \mid |w_i| < |z_i|, i = 1,\ldots,n\}$ be the open polydisk. If $\mathbf{z} \in \mathscr{V}$ and $\mathscr{V} \cap D(\mathbf{z}) = \emptyset$, then $\mathbf{z} \in \partial\mathscr{D}$.

Q. How to deal with polydisk using polynomial equations?

A. Decompose polynomials into the real and imaginary part.

$$f(\mathbf{x} + i\mathbf{y}) = f^{(R)}(\mathbf{x}, \mathbf{y}) + if^{(I)}(\mathbf{x}, \mathbf{y})$$

For derivatives, applying Cauchy-Riemann equations,

$$\frac{\partial f}{\partial z_j}(\mathbf{x} + i\mathbf{y}) = \frac{1}{2} \cdot \frac{\partial}{\partial x_j}\left(f^{(R)}(\mathbf{x}, \mathbf{y}) + if^{(I)}(\mathbf{x}, \mathbf{y})\right) - \frac{i}{2} \cdot \frac{\partial}{\partial y_j}\left(f^{(R)}(\mathbf{x}, \mathbf{y}) + if^{(I)}(\mathbf{x}, \mathbf{y})\right)$$

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\end{array}\right\} \quad (\star)$$

$(\star)$ is a square polynomial system with $4n+4$ variables $(\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}, \lambda_R, \lambda_I, \nu, t)$.

ACSV for combinatorial case

1. Determine the set $\mathscr{S}$ of zeros $(\mathbf{z}, \lambda, t)$ of the system $\left[H, z_1\frac{\partial H}{\partial z_1} - \lambda, \ldots, z_n\frac{\partial H}{\partial z_n} - \lambda, H(tz_1, \ldots, tz_n)\right]$. If $\mathscr{S}$ is not finite, then FAIL.

2. Construct the subset points $(\boldsymbol{\zeta}, \lambda, t) \in \mathscr{S}$ which are candidates for minimal critical points.

   - $\mathbf{z}$ is minimal if and only if the line segment $\{(t|z_1|, \ldots, t|z_n|) \mid 0 < t < 1\}$ doesn't intersect $\mathscr{V}$.

3. Identify $\boldsymbol{\zeta}$ among the elements of $\mathscr{C}$ (critical points Abs on $\mathscr{V}$).

4. Return $\mathscr{U} := \{\mathbf{z} \in \mathbb{C}^n \mid |z_1| = |\zeta_1|, \ldots, |z_n| = |\zeta_n| \text{ for some } (\mathbf{z}, \lambda) \in \mathscr{C}\}$

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ACSVHomotopy.jl (L.-Melczer-Smolčić 2022)

- Implemented using `HomotopyContinuation.jl` **(Breiding-Timme 2018)**
- Available at `github.com/ACSVMath/ACSVHomotopy`

- Competitive to other ACSV software for combinatorial cases.
  - Solve the critical-point equations using the polyhedral homotopy
- The first software of ACSV for non-combinatorial cases.
  - Solve the decomposed critical-point equations ($\star$) using the polyhedral homotopy
  - Provide faster heuristics including the monodromy method.

Implementation details

The polyhedral homotopy is the default for solving critical point equations.

- Returns reliable results via interval arithmetic certification.
- Effective for combinatorial case compared to other software based on symbolic algorithm.
- May be slow for non-combinatorial.
  - Faster heuristics used.

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Implementation details

Heuristics for non-combinatorial case.

1. Approximating critical points

$$\left.\begin{aligned}
H^{(R)}(\mathbf{a},\mathbf{b}) = H^{(I)}(\mathbf{a},\mathbf{b}) &= 0 \\
a_j\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{a},\mathbf{b}) + b_j\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{a},\mathbf{b}) - \lambda_R &= 0 \qquad j = 1,\ldots,n \\
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H^{(R)}(\mathbf{x},\mathbf{y}) = H^{(I)}(\mathbf{x},\mathbf{y}) &= 0 \\
x_j^2 + y_j^2 - t(a_j^2 + b_j^2) &= 0 \qquad j = 1,\ldots,n \\
(y_j - \nu x_j)\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{x},\mathbf{y}) - (x_j + \nu y_j)\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{x},\mathbf{y}) &= 0 \qquad j = 1,\ldots,n
\end{aligned}\right\} \quad (\star)$$

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$$
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\end{aligned}\right\} \quad (B)
$$

1. Approximating critical points
   - Solve the subsystem (A) to get an approximation of $(\mathbf{a},\mathbf{b})$.
   - Using the approximations, solve the subsystem (B).

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$$\left.\begin{aligned} H^{(R)}(\mathbf{a},\mathbf{b}) = H^{(I)}(\mathbf{a},\mathbf{b}) &= 0 \\ a_j\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{a},\mathbf{b}) + b_j\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{a},\mathbf{b}) - \lambda_R &= 0 \qquad j=1,\ldots,n \\ a_j\frac{\partial H^{(I)}}{\partial x_j}(\mathbf{a},\mathbf{b}) + b_j\frac{\partial H^{(I)}}{\partial y_j}(\mathbf{a},\mathbf{b}) - \lambda_I &= 0 \qquad j=1,\ldots,n \end{aligned}\right\} \quad (A)$$

$$\left.\begin{aligned} H^{(R)}(\mathbf{x},\mathbf{y}) = H^{(I)}(\mathbf{x},\mathbf{y}) &= 0 \\ x_j^2 + y_j^2 - t(a_j^2 + b_j^2) &= 0 \qquad j=1,\ldots,n \\ (y_j - \nu x_j)\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{x},\mathbf{y}) - (x_j + \nu y_j)\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{x},\mathbf{y}) &= 0 \qquad j=1,\ldots,n \end{aligned}\right\} \quad (B)$$

2. Monodromy

- Solve the subsystem (A) to get an approximation of $(\mathbf{a},\mathbf{b})$.
- Solve the subsystem (B) using monodromy with $(\mathbf{a},\mathbf{b})$ and $t=1$.
- Caveat : the subsystem (B) may have several irreducible components. (Fail to find all critical points)

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&a_j \frac{\partial H^{(I)}}{\partial x_j}(\mathbf{a},\mathbf{b}) + b_j \frac{\partial H^{(I)}}{\partial y_j}(\mathbf{a},\mathbf{b}) - \lambda_I = 0 && j = 1,\ldots,n
\end{aligned}\right\} \quad (A)$$

$$\left.\begin{aligned}
&H^{(R)}(\mathbf{x},\mathbf{y}) = H^{(I)}(\mathbf{x},\mathbf{y}) = 0 && \\
&x_j^2 + y_j^2 - t(a_j^2 + b_j^2) = 0 && j = 1,\ldots,n \\
&(y_j - \nu x_j)\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{x},\mathbf{y}) - (x_j + \nu y_j)\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{x},\mathbf{y}) = 0 && j = 1,\ldots,n
\end{aligned}\right\} \quad (B)$$

2. Monodromy
   - Solve the subsystem (A) to get an approximation of $(\mathbf{a},\mathbf{b})$.
   - Solve the subsystem (B) using monodromy with $(\mathbf{x},\mathbf{y},t) = (\mathbf{a},\mathbf{b},1)$.
   - Caveat : the subsystem (B) may have several irreducible components. (Fail to find all critical points)

Experiments : Combinatorial examples

Elapsed time (s) \ Examples	square-root	Apéry ζ(2)	Apéry ζ(3)	Random	3D Walk
ACSVHomotopy.jl	0.01	0.025	0.7	0.9	0.08
Maple implementation	0.06	0.06	0.3	840	2.7

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Polyhedral homotopy	29.5	670	INC	236	INC
Approximating Crits	0.72	3.8	15.3	3.6	189.4
Monodromy	14.9	8.5	31.9	3.8	583.1

INC indicates the code did not complete after running for an hour.

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Future directions

$$\left.\begin{aligned}
H^{(R)}(\mathbf{a},\mathbf{b}) = H^{(I)}(\mathbf{a},\mathbf{b}) &= 0 && \\
a_j\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{a},\mathbf{b}) + b_j\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{a},\mathbf{b}) - \lambda_R &= 0 && j = 1,\ldots,n \\
a_j\frac{\partial H^{(I)}}{\partial x_j}(\mathbf{a},\mathbf{b}) + b_j\frac{\partial H^{(I)}}{\partial y_j}(\mathbf{a},\mathbf{b}) - \lambda_I &= 0 && j = 1,\ldots,n \\
H^{(R)}(\mathbf{x},\mathbf{y}) = H^{(I)}(\mathbf{x},\mathbf{y}) &= 0 && \\
x_j^2 + y_j^2 - t(a_j^2 + b_j^2) &= 0 && j = 1,\ldots,n \\
(y_j - \nu x_j)\frac{\partial H^{(R)}}{\partial x_j}(\mathbf{x},\mathbf{y}) - (x_j + \nu y_j)\frac{\partial H^{(R)}}{\partial y_j}(\mathbf{x},\mathbf{y}) &= 0 && j = 1,\ldots,n
\end{aligned}\right\} \quad (\star)$$

- Geometric understanding of $(\star)$
  - What is degree for generic H?
- Numerical techniques for $(\star)$
  - How to improve the performance of monodromy?
  - Solving equation-by-equation (i.e. regeneration)
- How to verify the completeness?

Thank you for your attention

The paper is available at
(https://arxiv.org/pdf/2208.04490.pdf)