The Wonder That Is Pi: Part One

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This blog began life more than two decades ago, as part of a series of lectures I delivered to very bright first-year engineering students at an Australian university.

The number π (pronounced "pie") has been recognized from time immemorial because its physical significance can be grasped easily: it is the ratio of the circumference of a circle to its diameter. But who would have thought that such an innocent ratio would exercise such endless fascination because of the complexities enfolded into it?

Not surprisingly, some students I met recently wanted to know more about π. Accordingly, I have substantially recast and refreshed my original presentation to better accord with the form and substance of a blog. The online references have also been updated to keep up with a rapidly changing Web.

My original intention was to write a single blog on π. But because I did not want it to become another overly long *slog*, I have decided to divide it into two parts.

If there are any errors or omissions, please email me your feedback.

Circumference, diameter, and π

The straight line or geodesic is the shortest distance between any two points on a plane, sphere, or other space. The circle is the locus traversed by a moving point that is equidistant from another fixed point on a two-dimensional plane. It is the most symmetrical figure on the plane. The diameter is the name given both to any straight line passing through the centre of the circle—intersecting it at two points—as well as to its length. When we divide the perimeter of circle, more properly called its circumference, C, by its diameter, d, we get the enigmatic constant π, which has a value between 3.141 and 3.142:

$$\frac{C}{d} = \pi. \tag{1}$$

The diameter d is twice the radius r, and substituting for d into Equation (1), we get the well-known school formula:

$$C = \pi d = 2\pi r \approx 2\left[\frac{22}{7}\right] r \approx 6.28r. \tag{2}$$

Note, however, that π is *not exactly equal* to $\frac{22}{7}$. This value is a convenient *rational fraction approximation* for π that serves well in elementary contexts.[1]

You might reasonably wonder whether the ratio of the circumference to the diameter of *any* circle is *always* π. The answer is "Yes", because *all circles are similar*. The ratios of corresponding lengths of similar figures are equal. This idea is also covered in my blog "A tale of two measures: degrees and radians".

[1] See "A tale of two measures: degrees and radians".

The symbol π is the lowercase version of the sixteenth letter of the Greek alphabet. For the history of its use in mathematics, see adoption of the symbol π in Wikipedia.

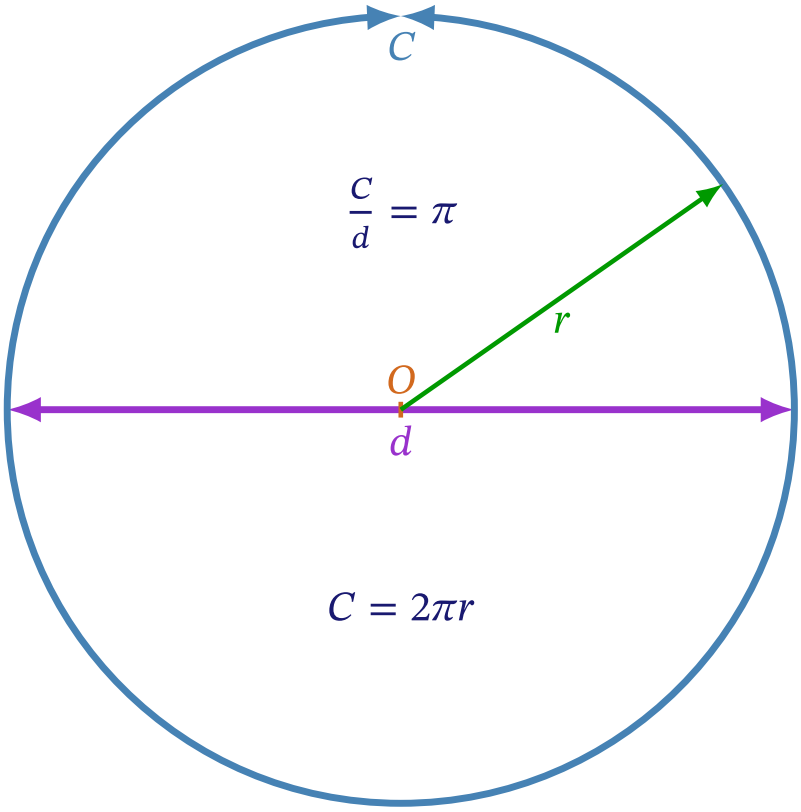


Figure 1: The ratio of the circumference to the diameter of *any* circle is π.

Figure 1 shows the relationships in Equation (1) and Equation (2) pictorially. The circumference of a circle is about 6.28 times its radius. Why this should be so is a secret, a mystery of Nature.

Is π really equal to 22/7?

Is π really equal to $\frac{22}{7}$, as it has been drummed into our heads at school? A wonderfully revealing story lies behind this mysterious relationship, and it is due to the labours of one man, in the days when calculators could not be dreamed of, and when neither the decimal system of numbers nor trigonometry were known. That is the story we look at next.

Archimedes of Syracuse

Archimedes of Syracuse[2] (Ἀρχιμήδης, 287–212 BCE) was a polymath and genius of the ancient world. He was one of the greatest mathematicians the world has ever known. By today's standards, he would be called a mathematician, physicist, engineer, and astronomer, all rolled into one. He is perhaps most famous for running out of his bathtub naked exclaiming "Eureka"—Greek for "I have found it"—oblivious of those around him. The principle that he had then discovered—that the upthrust on a body submerged in a fluid is equal to the weight of fluid displaced—is known as Archimedes' Principle.

[2]His very name, Archimedes, means "master thinker" in Greek.

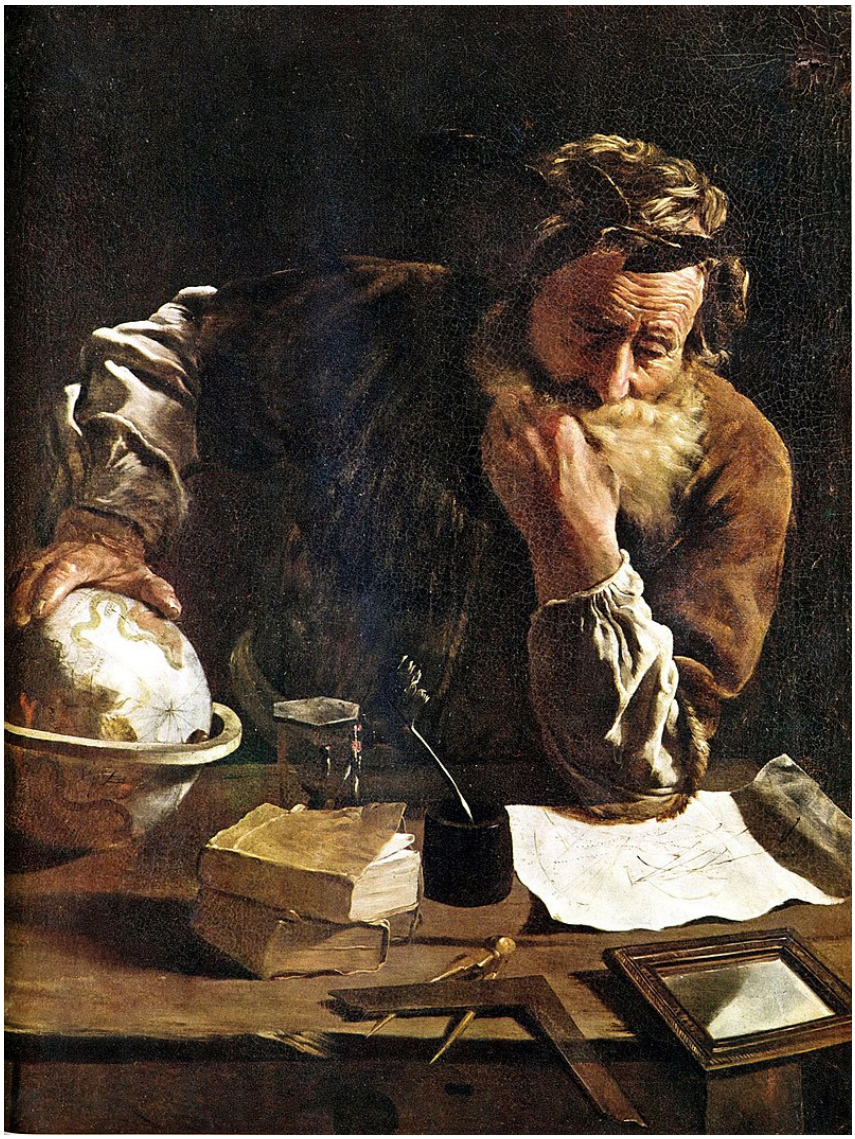


Figure 2: Archimedes of Syracuse.[3]

Among the many accomplishments of Archimedes is his method for estimating π, which was the best approximation for almost 1900 years. And it was not based on using a length of string, superimposing it on a circle, and getting an estimate! ☺

What is even more remarkable is that Archimedes made his discovery *without* the benefit of:

(a) algebra;

(b) trigonometry;

(c) decimal (positional) notation; and

(d) calculators.

Instead he applied the theorem of Pythagoras and extracted square roots laboriously by hand. His method is also an excellent geometrical illustration of the idea of a *limit*, with which he was doubtless familiar. It is known that Archimedes was familiar with what we now know as integral calculus, and it is possible that he may have anticipated differential calculus as well.

Archimedes devised an ingenious method for estimating the circumference of a circle. He used a sophisticated algorithm that allowed him to obtain successively more accurate values for the circumference of a circle, and therefore of π.

Principles used by Archimedes

The method that Archimedes devised is instructive because it is a synthesis of several principles by which the greatest human minds have furthered scientific progress over time. The abstract principles that Archimedes used to estimate π were these:

1. Start with the known and progress to the unknown;
2. Initialize variables;
3. Devise a method of increasing the accuracy of the estimate by recursion or iteration;
4. Stop when the desired accuracy is reached.

These principles constitute what is known as an algorithm. Once such a systematic framework has been put in place, it can be applied in many research domains to aid rapid scientific progress. The algorithm is the basis of modern computing.

Of polygons and circles

Archimedes considered a circle, containing an inscribed regular polygon with n sides, and circumscribed by a regular polygon with the same n sides. Figure 3 illustrates this for the case $n = 6$, i.e., with a regular hexagon.

Let us tabulate below some variables and their values for this geometric configuration. These values will be better understood when viewed in conjunction with Figure 4.

[3] Domenico Fetti's 1620 painting entitled *Archimedes Thoughtful*. Public domain.

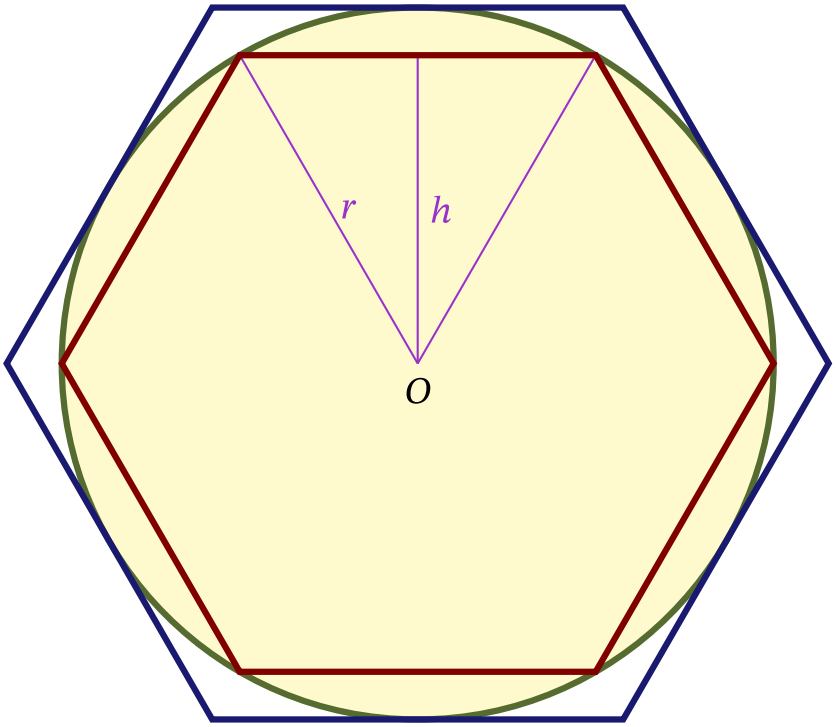
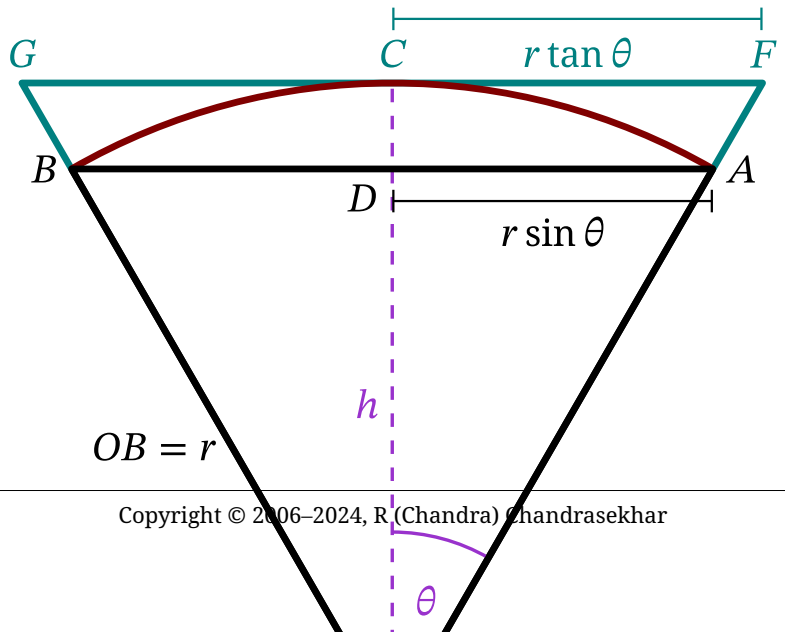


Figure 3: The circumference of the circle in darkolivegreen is bounded from below by the perimeter of the inscribed regular hexagon in maroon and bounded from above by the perimeter of the circumscribed regular hexagon in midnightblue. The circumference of the circle must lie between the perimeters of these two hexagons. The value r is the radius of the circle and the height h—from the centre to the mid-point of a side—is called the apothem.

Table 1: Circle, inscribed, and circumscribed regular polygons (n-gons).

Parameter	Circle	Inscribed Polygon	Circumscribed Polygon
Radius	r		
Side		$2r \sin\theta$	$2r \tan\theta$
Angle		$\theta = \frac{\pi}{n} = \frac{180^\circ}{n}$	$\theta = \frac{\pi}{n} = \frac{180^\circ}{n}$
Apothem		$h = r \cos\theta$	$h = r$
Area	$A = \pi r^2$	$A_i = n \sin\theta \cos\theta r^2$	$A_c = n \tan\theta r^2$
Perimeter	$C = 2\pi r$	$C_i = 2n \sin\theta r$	$C_c = 2n \tan\theta r$



In the case of regular hexagons, we have this triple of equations:

$$C_i = 2(6)r\sin 30° = 12(1)(0.5) = 6.C = 2\pi r = 2\pi = 2(1)\pi = 6.2381.C_c = 2(6)r\tan 30° = 12(1)(\frac{\sqrt{3}}{3}) \approx 6.9282.$$

{eq:triple-6}

As n the three values approach one another. With $n = 96$, we have $\theta = \frac{180}{n} = 1.875°$, we get:

$$C_i = 2(96)r\sin 1.875° \approx 192(1)(0.0327) = 6.2820C = 2\pi r = 2\pi = 2(1)\pi = 6.2381.C_c = 2(96)r\tan 1.875° \approx 192($$

{eq:triple-96}

Note that in the case of 96 sides, we have a very small angle θ whose sin and tan are almost equal. This is what gives us tight bounds on the estimate of π.

Remember this equation because it helps us to estimate lower and upper bounds for the value of the circumference. Archimedes's application of the squeeze theorem nineteen centuries before the calculus was invented is illustrated in Figure 3 and later figures.

Sanity check: Does $2\pi = 6.283$, from a calculator, lie within the bounds of **??** and **??**? Yes, indeed, and we are home and dry.

The thirty, sixty, ninety right triangle

Archimedes applied the same principle "of starting from the known" to initiate his algorithm using a regular hexagon, which is a mosaic of six juxtaposed equilateral triangles. We know from symmetry that each angle of an equilateral triangle is 60°. When an equilateral triangle is bisected, we get two right angled triangles with angles of thirty and sixty degrees, as shown in Figure 5.

The inscribed hexagon, within a circle of radius one unit, also has a side of one unit. Thus, the hypotenuse of the circle OAP in Figure 5 has unit length. Moreover, the base OP, resulting from a bisected side, has a length of half a unit. By applying the theorem of Pythagoras, the third side, AP is

$$\sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}.$$

Extracting square roots by hand

Archimedes must have known how to extract square roots by hand. Perhaps, he used one of the methods described in my blog "How Are Numbers Built?". He should have known the value of $\sqrt{3}$ as a rational fraction. With remarkable accuracy, he stated that:

$$\sqrt{3} \approx \frac{265}{153} \approx 1.732. \tag{4}$$

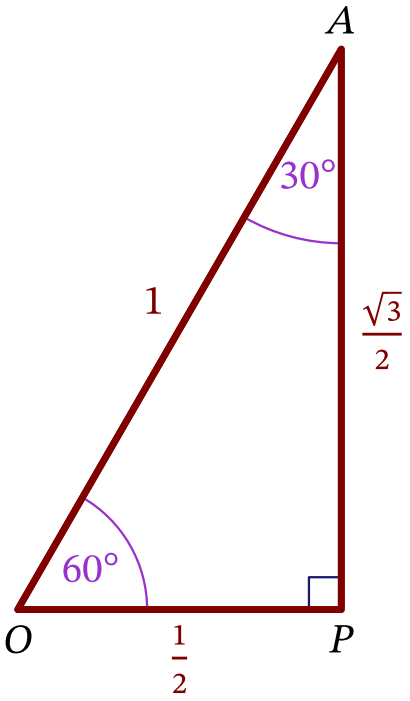


Figure 5: This right-angled, obtained by bisecting an equilateral triangle, must be familiar to all school students. These lengths—obtainable from symmetry and the theorem of Pythagoras—allowed Archimedes to start off his process for estimating π.

Recursion and Iteration

Archimedes started with regular hexagons and successively doubled the number of sides, until he had the circle closely sandwiched between two 96-gons. Successively doubling or halving is a fast-converging technique used in numerical estimation when mathematics is applied to solving a variety of problems. That Archimedes was aware of it shows how far ahead of his time his thinking was.

He repeatedly calculated rational approximations to π until he was satisfied with the accuracy. The principle of the method is clearly seen in Figure 6 to Figure 10.

Lessons from this derivation

Iteration, recursion, bisection, squeeze, etc.

A closer look at π

Pi is both an irrational and a transcendental number. Let us see what each of these appelations mean.

Recurring decimals.

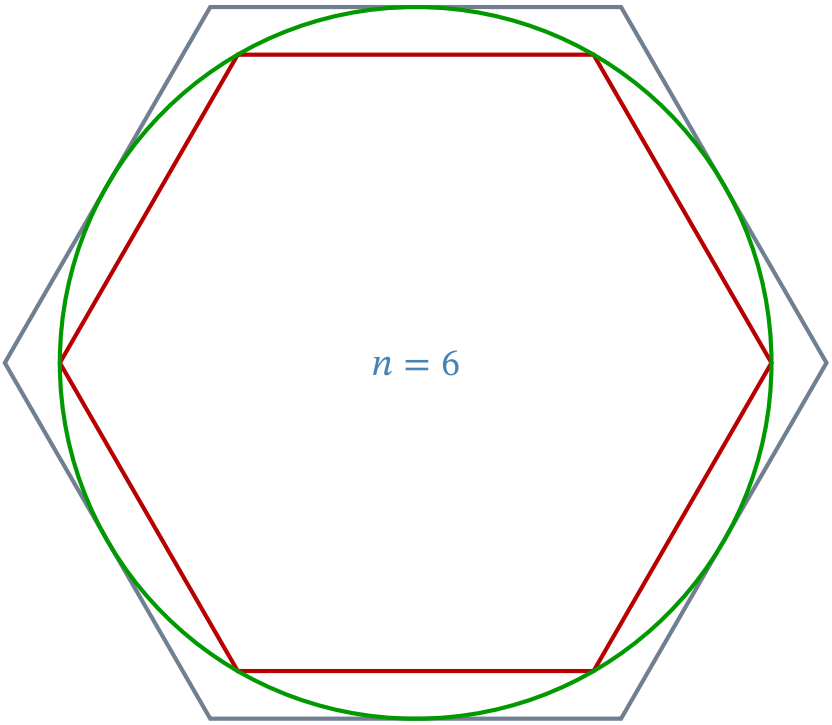


Figure 6: The estimate for π lies between $C_i = 3.0000 < \pi < C_c = 3.4641$.

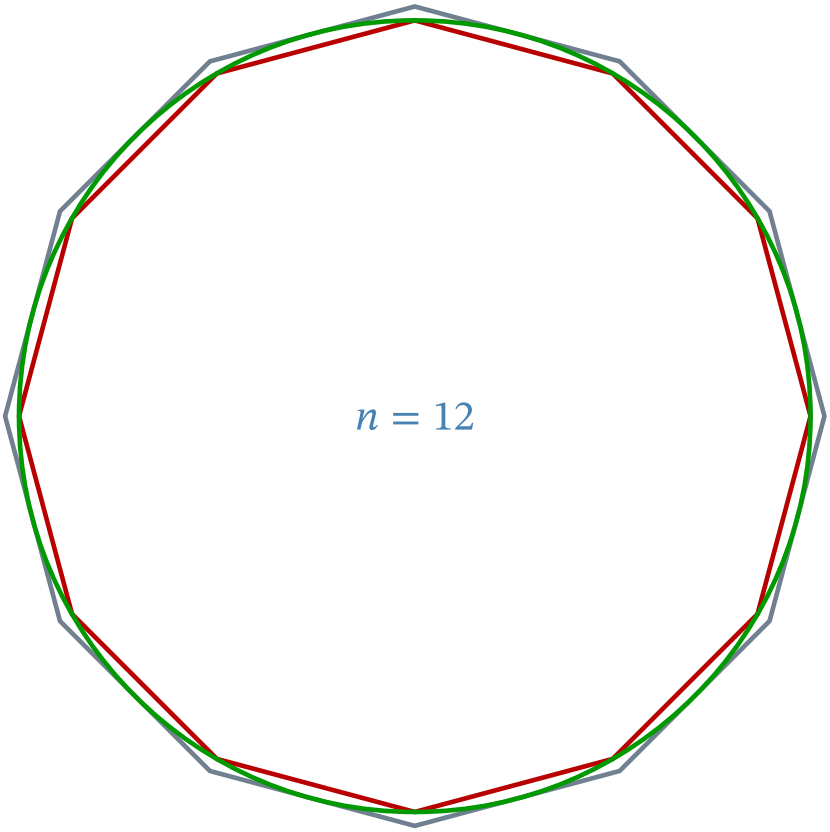


Figure 7: The estimate for π lies between $C_i = 3.1058 < \pi < C_c = 3.2153$.

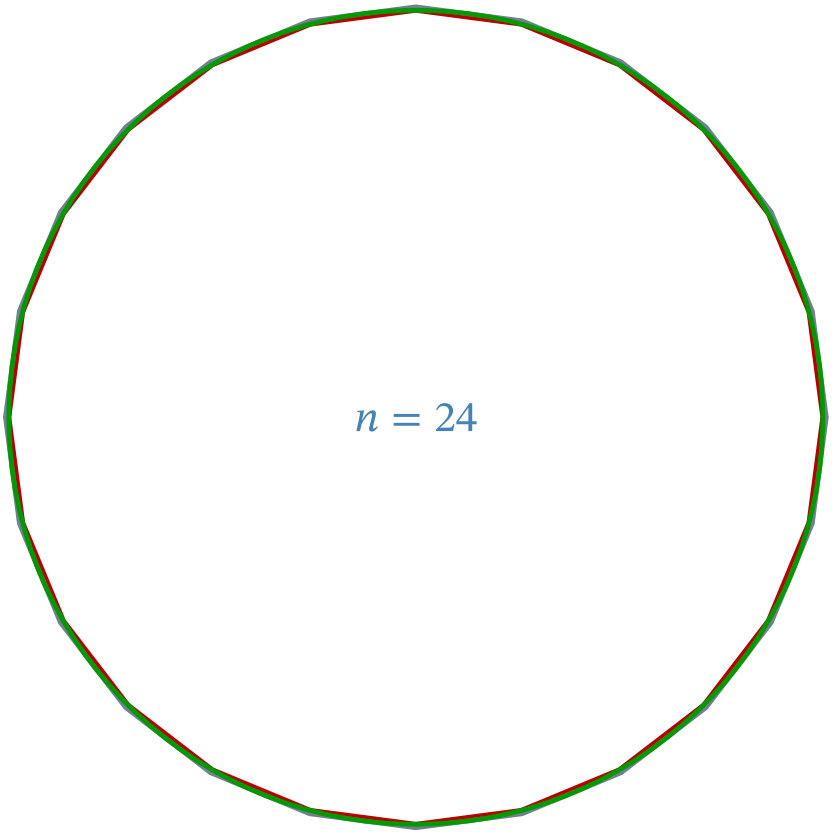


Figure 8: The estimate for π lies between $C_i = 3.1326 < \pi < C_c = 3.1596$.

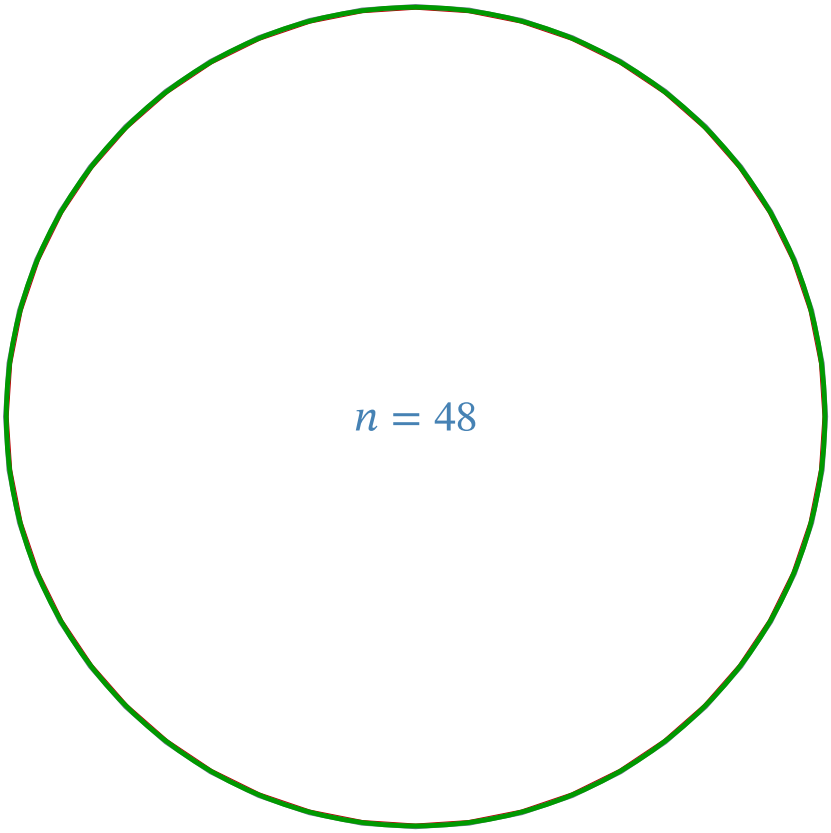


Figure 9: The estimate for π lies between $C_i = 3.1393 < \pi < C_c = 3.1460$.

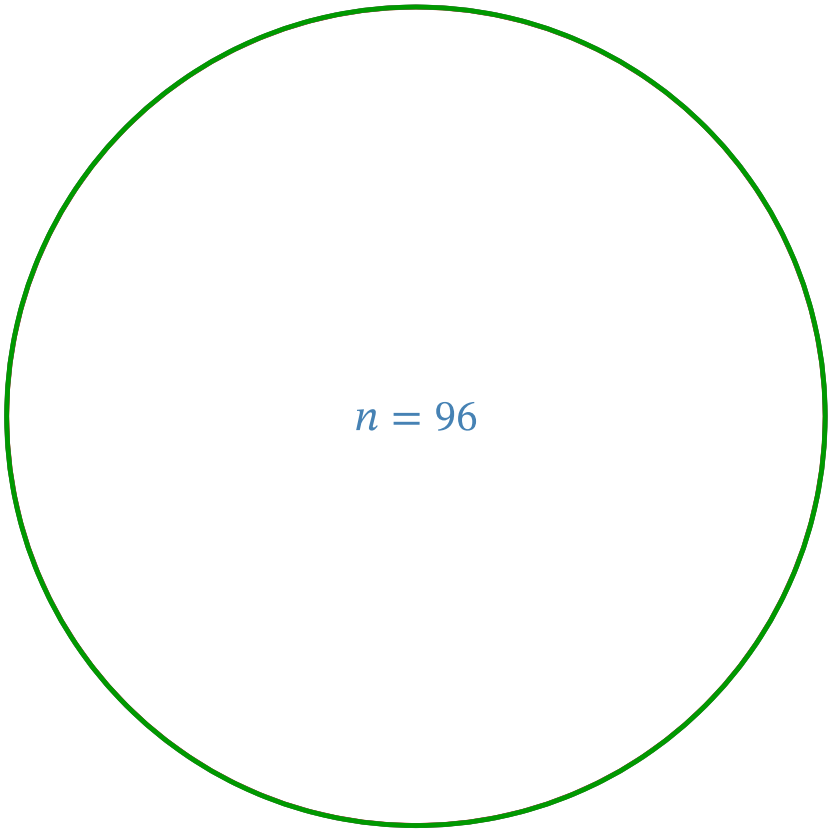


Figure 10: The estimate for π lies between $C_i = 3.1410 < \pi < C_c = 3.1427$. Notice in this sequence of images how the circumference of the circle approaches the perimeter of the inscribed and circumscribed heaxgons to the point of being indistinguishable from them.*The final estimate of Archimedes was* $\frac{223}{71} < \pi < \frac{22}{7}$.

To explore further

A well-written, accessible article on how Archimedes estimated that π is approximately $\frac{22}{7}$ is available online: "How Archimedes showed that pi is approximately 22 by 7". I urge you to read it.[4] You will then appreciate for yourselves how arduous the process must have been in an age without the benefit of: #. Trigonometry; he used the theorem of pythagors instead; #. Algebra; he used geometry and the ratios of the lengths of well-known triangles; #. Decimal numbers for division; he used fractions instead; square roots by hand; similar and congruent figures; bisection theorems; exhaustion methods

In the series **??** to **??** below, which illustrate the approach Archimedes took to estimate π, we see very clearly that the perimeter of the *inscribed polygon* c_n and the perimeter of the *circumscribed polygon* C_n represent respectively the *lower bound* and *upper bound* of the estimated value of π. As the number of sides, n, of the polygon increases, the estimates become increasingly accurate.

https://publications.azimpremjiuniversity.edu.in/3356/1/02-DaminiAndAbhishek_PiIs22By7_Final.pdf

https://azimpremjiuniversity.edu.in/at-right-angles

How did Archimedes arrive at π = 22/7?

22/7 = 3.142857 142857 142857 (recurring decimal)

Formulae involving π

Quest for the endless digits of π

Buffon's Needle

π Trivia

Web links

https://www.pbs.org/wgbh/nova/physics/approximating-pi.html

https://demonstrations.wolfram.com/ArchimedesApproximationOfPi/ John Tucker "Archimedes' Approximation of Pi" http://demonstrations.wolfram.com/ArchimedesApproximationOfPi/ Wolfram Demonstrations Project Published: March 5 2009

https://math.stackexchange.com/questions/4851929/archimedes-method-to-estimate-pi

http://arxiv.org/pdf/2008.07995

https://mathsciencehistory.com/2019/10/01/archimedes-and-his-pi-the-great-numerical-hope/

https://carmamaths.org/resources/jon/pi-culture.pdf

https://nonagon.org/ExLibris/archimedes-pi

https://www.exploratorium.edu/pi/history-of-pi

https://en.wikipedia.org/wiki/Approximations_of_%CF%80

[4] This article is all the more remarkable because its first author is a Grade 8 student: proof that deep mathematics is not beyond the school student.

Book References

Web resources

Teaser: are of circle half radius multiplied by circumference. How?

Appendix: Circumscribed and inscribed polygons of circle

Archimedes devised his ingenious *squeeze* method for computing the upper and lower bounds of the perimeter of a circle by computing instead the perimeters of the polygons that inscribe and circumscribe the circle. The approximations become more accurate as the number of sides, n, of the polygon is increased. This YouTube presentation might help you understand the algorithm of Archimedes better, but remember that he did not have trigonometry to aid hm.

Feedback

Please email me your comments and corrections.

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