The Wonder That Is Pi

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This is a sequel to the blog "The Pi of Archimedes". Here, we look at π as a number—without explicit reference to its geometric tethering—and explore its remarkable ubiquity in mathematics. As an appetizer, see Figure 1, where the symbol for Pi is surmounted by two very disparate equations defining it. How in all the world could these two different-looking equations be true? But they are indeed!

$$\pi = 4\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots)$$

$$\pi$$

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!\,[1103 + 26390n]}{(n!)^4 396^{4n}}$$

Figure 1: Pi expressed by two very different equations. Note that both are sums to infinity of expressions involving integers.

The Number Menagerie

Numbers may be compared to animals in a zoo. Each is different, and yet they all share some attributes in common. The variety and diversity of zoo animals can be challenging. That is why the big cats are grouped together, the herbivores live in another part of the zoo, etc.

Numbers, like animals, have evolved over many centuries into what I call the *number menagerie*. A very elementary picture of this zoo is outlined in my blog "The Two Most Important Numbers: Zero and One" in case you need to review some definitions.

To appreciate π as a number, we need to be aware of the taxonomy in the zoo of numbers. It turns out that π *is a real number that is transcendental and therefore also irrational*. Let us make a short detour to better understand what this means.

Real and Complex Numbers

There are two major sets of numbers: real numbers, denoted by the set $\mathbb{R}$, and complex numbers, denoted by the set $\mathbb{C}$. The difference between the two is that while a real number is a single number, a complex number is a pair, composed of two real numbers, conjoined by the imaginary unit i, where $i^2 = -1$. In set-theoretic notation, we write

$$\mathbb{C} = \{a + bi \,:\, a, b \in \mathbb{R}\}.$$

Sometimes, the complex number $a + bi$ is written as the ordered pair (a, b), provided the context is clear.

What then are the reals? The real numbers are the union of the set of rational numbers and the irrational numbers. Alternatively, the reals are the union of the algebraic numbers and the transcendental numbers.[1]

We will define each of these terms below and how they relate to one another. As always, we start with the known and proceed to the unknown.

The Integers and Friends

The set $\mathbb{N}$ of *natural or counting numbers* is defined as

$$\mathbb{N} = \{1, 2, 3, \dots, n, n + 1, \dots\}.$$

It is a countably infinite set whose members begin with 1 and progress by the addition of 1 to the predecessor. It is an infinite set, which means it never ends, as denoted by the ellipsis or dots at the end of the definition.

Zero is not a natural number and is assigned its own, unnamed set, $\{0\}$.[2]

The set of *integers* $\mathbb{Z}$ includes the negative numbers, zero, and the positive numbers:

$$\mathbb{Z} = \{\dots - 3, -2, -1, 0, 1, 2, 3, \dots\}$$

Like $\mathbb{N}$, $\mathbb{Z}$ is also a countably infinite set.

A first dichotomy

The real numbers may be partitioned into subsets in different ways: one way is into the *rational* and *irrational* numbers.

Every real number is *either* rational or irrational. If the universe of discourse is the real number set, the rational and irrational numbers are complements of each other. In other words, the *union* of the set of rational numbers and the set of irrational numbers *is* the set of real numbers.

[1] Since both algebraic and transcendental numbers can be complex, we need the added condition that these do not involve the imaginary unit, i. For example, $(1 + \frac{\sqrt{(-7)}}{2}) = (1 + \frac{\sqrt{7}}{2}i)$, and πi are examples of algebraic and transcendental numbers respectively that involve i.

[2] Some folks include zero in $\mathbb{N}$.

Rational Numbers

The *rational numbers* are denoted by the set $\mathbb{Q}$ defined to be:

$$\mathbb{Q} = \{\tfrac{a}{b} \text{ where } a, b \in \mathbb{Z} \text{ and } b \neq 0\}.$$

The condition imposed on b arises from the stricture that division by zero is not permitted among the integers and reals.[3]

Let us amplify the consequences of these definitions. Is the number 25 rational? Yes, indeed. But where is the denominator? It is *implicit* and equals 1. The fact that

$$25 = \frac{25}{1}$$

makes it clear that 25 is a rational number. Every integer is a rational number.

And it is obvious from the definition that $\frac{2}{3}$ is a rational number. But is $-\frac{11}{16}$ a rational number? Yes, indeed, because the definition depends upon the *integer* a and the *non-zero integer* b, where both integers—being drawn from $\mathbb{Z}$—can be signed.

When a rational number is expressed as a decimal, that decimal can either terminate or recur without end.

For example, the fraction $\frac{1}{3} = 0.\overline{3}$ has a recurring decimal representation as revealed by division. The line on top indicates the portion of the decimal which recurs—in this case, it is the single digit 3.

When we look at the fraction $\frac{1}{2} = 0.5$, we have an example of a terminating decimal. We could, however, pad zeros after the first decimal place, and claim that even a terminating decimal is recurring; witness that $\frac{1}{2} = 0.5 = 0.5000 \cdots = 0.5\overline{0}$. But that is not the whole story.

We can further show that:

$$\frac{1}{2} = 0.5 = 0.5\overline{0} = 0.4\overline{9}.$$

It does seem strange to claim that two different decimals can express the *same* rational number $\frac{1}{2}$.

To see why, let us rewrite $0.4\overline{9}$ as

$$\begin{aligned} 0.4\overline{9} = 0.4999 \ldots &= \frac{4}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} \ldots \\ &= \frac{4}{10} + 9\left[\frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} \ldots\right] \end{aligned} \tag{1}$$

[3] See "The Two Most Important Numbers: Zero and One" for the reason why.

Consider now the expression in square brackets on the right hand side (RHS) of Equation (1). We can recognize it as a geometric series with first term $a = \frac{1}{100}$ and common ratio $r = \frac{1}{10}$. Since $r < 1$, the series is *convergent* and its sum to infinity [1] is given by:

$$\begin{aligned} \frac{a}{1-r} &= \frac{\frac{1}{100}}{[1 - \frac{1}{10}]} \\ &= \frac{\left[\frac{1}{100}\right]}{\left[\frac{9}{10}\right]} \\ &= \left[\frac{1}{100}\right]\left[\frac{10}{9}\right] \\ &= \frac{1}{90}. \end{aligned} \tag{2}$$

Substituting for the terms in square brackets in Equation (1), we get

$$0.4\overline{9} = \frac{4}{10} + 9\left[\frac{1}{90}\right] = \frac{4}{10} + \frac{1}{10} = \frac{5}{10} = \frac{1}{2}.$$

Even if it seems counter-intuitive that $0.4\overline{9} = 0.5 = 0.5\overline{0} = \frac{1}{2}$, it is mathematically consistent and correct. One may therefore hazard a guess, and correctly so, that *every rational number may be expressed as a recurring decimal.*[4]

Infinite sums have this property of upending our "intuition" about what is correct. So, we have to be extra careful when dealing with the value of a limit as some variable goes to infinity. Moreover, infinity, represented by ∞ is *not* a number and cannot be treated as one. It is simply a convenient shorthand symbol. This caveat should be kept in mind when we encounter infinite sums involving π, as shown for example, in Figure 1.

Irrational Numbers

Irrational numbers are numbers which are *not rational.* The discovery that $\sqrt{2}$—which is the length of the diagonal of a unit square—was not rational [2,3], caused the first ripples of disquiet in the ancient mathematical world, because it upset the prevailing philosophy that ratios of whole numbers alone ruled the world.

There are many celebrated proofs that $\sqrt{2}$ is not the ratio of two integers and is therefore irrational [4]. Nevertheless, it took almost two millennia for $\sqrt{2}$ to be accepted into the fold of properly defined numbers [5].

An irrational number like $\sqrt{2}$ does not have any recurring sequence of digits when expressed as a decimal. But the absence of recurring sequences in the decimal representation of a number should not solely be used to identify a number as irrational, because some rationals with large denominators can and do have very long recurring sequences, which may be difficult to detect by visual inspection . For example, $\frac{8119}{5741}$—which incidentally is a rational approximation to $\sqrt{2}$—has a recurring sequence of length 5740.[5]

[4] In this case either the digit 9 or the digit 0 recurs.
[5] Also called the *period* of a repeating decimal. See https://www.wolframalpha.com/input?i=8119%2F5741.

The irrationals exceed in number the rationals

If you are curious, you might wonder which are the more numerous: the rationals or the irrationals. You might guess that the familiar rationals are more numerous than the obscure irrationals. But you would be mistaken.

In fact, the irrationals far exceed in number the rational numbers [6]. This fact is stated baldly here, because going into the whys and wherefores of this claim will lead us too far astray from our focus on π. It is an interesting fact, though, that you should stash away for future use.

A second dichotomy

The real numbers may also be split another way into two mutually exclusive sets: the *algebraic numbers* and the *transcendental numbers*. Every real number is *either* an algebraic number or a transcendental number; it cannot be both.

It bears noting though, that both the algebraic and the transcendental numbers may be complex, i.e, have an imaginary part. But in this blog, we have restricted our universe to the real numbers. In this blog, we will not consider algebraic or transcendental numbers that embody the imaginary unit.

The Algebraic Numbers

An algebraic number is the root of a non-zero polynomial with integer or rational coefficients. Things have gotten abstract enough thus far for eyes to be glazed. So, let us invoke some examples to revive attention.

The simplest algebraic number is an integer. Let us take 5 as an example. If the polynomial $p(x) = x - 5$, its root is when $p(x) = 0$, i.e., when $x - 5 = 0$. This implies $x = 5$ and we have shown that 5 is algebraic *by definition*.

Note that we could have used any other polynomial with the same root, such as $q(x) = 2x - 10$. All we need do is find *one* polynomial whose root equals the number and we have shown that the number is algebraic.

Likewise, the rational number $-(\frac{2}{3})$ is the root of the polynomial $3x + 2$ and is therefore algebraic.

We may assert that *every rational number is algebraic and therefore not transcendental.*

But what about an irrational number like $\sqrt{2}$? Is it algebraic? The polynomial $(x^2 - 2)$ has a zeros at $\pm\sqrt{2}$, thereby demonstrating that both $\pm\sqrt{2}$ are algebraic.

Can an algebraic number be a complex root of a real polynomial? Let us find the roots for the real polynomial $x^2 - 10x + 34$:

$$
\begin{aligned}
x^2 - 10x + 34 &= 0 \\
(x^2 - 10x + 25) + 9 &= 0 \\
(x - 5)^2 + 9 &= 0 \\
(x - 5)^2 &= -9 \\
(x - 5) &= \pm 3i \\
x &= 5 \pm 3i
\end{aligned}
$$

We have just shown that an algebraic number can be a complex root of a real polynomial. While we will not consider complex algebraic numbers in this blog, it is useful to know that they do exist.

The Transcendental Numbers

Numbers which are *not algebraic* are assigned the rather exalted title of transcendental numbers. Numbers like π, e, and $\ln 2$ are transcendental. But proving that a particular number is transcendental is no mean task. We will accept π as transcendental if it has been proved to be so by professional mathematicians [7–9].

All transcendental numbers are perforce irrational.

Transcendental numbers can also be complex, e.g., e^i, but we will steer clear of that category here, because we don't want to get more dizzy (mathematically) than we already are right now! ☺ .

Taxonomy via Tetrachotomy

We have established a tetrachotomy among the real numbers. But the four parts are not mutually exclusive. They overlap. There are two non-overlapping dichotomies: the rationals and irrationals as one pair, and the algebraic and transcendental numbers as the other.

It is noteworthy that irrational numbers like $\sqrt{2}$ and transcendental numbers like π and e are denoted, not by values, but by *symbols*.

This classification of the real numbers seems to be crying out for a Venn diagram to depict it visually. But before we do that, let us marshal the facts we have gathered so far:

1. The real numbers are represented by the standard set $\mathbb{R}$.
2. The rationals are represented by the standard set $\mathbb{Q}$.
3. There is no assigned symbol for the set of irrationals. Because it is the set difference between the reals and the rationals, it is often denoted as $\mathbb{R} \setminus \mathbb{Q}$. But this notation is cumbersome. So, let us define a non-standard set $\mathbb{I}$ and let it stand for the irrationals: $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$.
4. Let us introduce the non-standard symbol $\mathbb{A}_\mathbb{R}$ for the set of real algebraic numbers.
5. Let us introduce the non-standard symbol $\mathbb{T}_\mathbb{R}$ for the set of real transcendental numbers.
6. The real numbers are the union of the rational and the irrational numbers: $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$.
7. The real numbers are also the union of the algebraic and transcendental numbers that do not embody the imaginary unit i: $\mathbb{R} = \mathbb{A}_\mathbb{R} \cup \mathbb{T}_\mathbb{R}$
8. Algebraic numbers can be either rational or irrational: $\mathbb{A}_\mathbb{R} \subseteq (\mathbb{Q} \cup \mathbb{I})$.
9. All rational numbers are algebraic: $\mathbb{Q} \subset \mathbb{A}_\mathbb{R}$
10. No rational number is transcendental: $\mathbb{Q} \cap \mathbb{T}_\mathbb{R} = \varnothing$
11. All real transcendental numbers are irrational: $\mathbb{T}_\mathbb{R} \subset \mathbb{I}$.
12. The irrational numbers contain *all* transcendental numbers and a subset of the algebraic numbers, again excluding those that embody i: $(\mathbb{T}_\mathbb{R} \subset \mathbb{I}) \wedge (\mathbb{A}_\mathbb{R} \cap \mathbb{I} \neq \varnothing)$.

That was quite mouthful even with mathematical symbols. We are now ready to draw the Venn diagram for the tetrachotomy of the real numbers.

And surprise! surprise! There are only *three* regions in the Venn diagram that are populated. So, taking mathematical liberties, we may say that our tetrachotomy was not “linearly independent”.

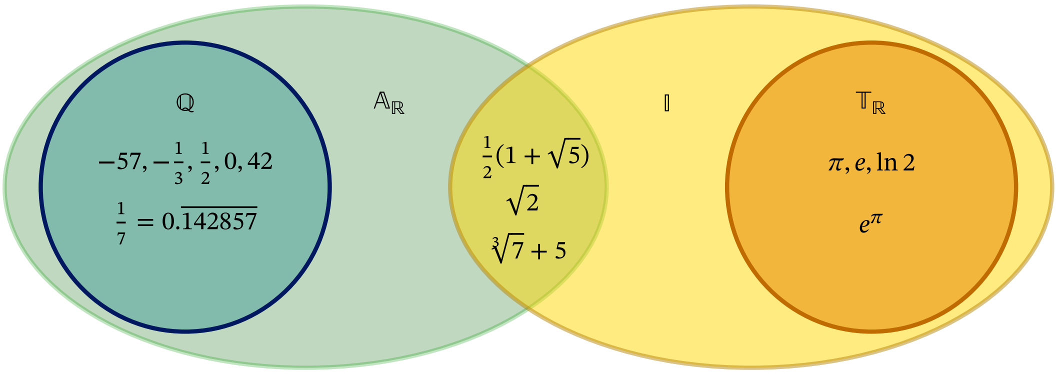

Figure 2: Venn diagram showing the rationals, $\mathbb{Q}$, the irrationals, $\mathbb{I}$, the real algebraics, $\mathbb{A}_{\mathbb{R}}$, and the real transcendentals $\mathbb{T}_{\mathbb{R}}$. From this diagram, we may assert that $\mathbb{R} = \mathbb{A}_{\mathbb{R}} \cup \mathbb{I}$. Note where π resides, and also that there are only *three* populated regions in the Venn diagram: $\mathbb{Q}$, $(\mathbb{A}_{\mathbb{R}} \cap \mathbb{I})$, and $\mathbb{T}_{\mathbb{R}}$.

Enter π

We have gone through all this huffing and puffing to place π contextually among the real numbers. Let us list its characteristics:

1. It is not a rational number, which means that it cannot be expressed as the ratio of two whole numbers, the denominator being non-zero.
2. Its decimal representation is neither finite nor does it contain a recurring segment, regardless of how long the decimal is.
3. It is also not the root to any non-zero polynomial equation whose coefficients are integers or rational numbers.
4. When Pi is used in equations, the placeholder symbol π is used.

These properties have earned for π the rather exalted title of transcendental number, which it shares with other pivotal numbers like e. Pi is not only important, it is also tantalizing. Pi is like a beautiful butterfly that cannot be caught in the net of finitude. It is like a rainbow that is beautiful to behold from afar, but can never be reached.

One could almost say that π is not numerically friendly. And you would not be too wrong. Rational approximations for π, like $\frac{22}{7}$, are used in practice. And the matter would have rested there were it not for the human quest for beauty.

The unpredictability of successive decimal places of π has enchanted mathematicians and still continues to engross them. Pi has been calculated to an unprecedented number of decimal places, and such a quest is certainly driven, not by practical necessity, but possibly by the need for aesthetic fulfilment.

The search for increasingly more accurate values for π has resulted in many approaches to solve the problem. Geometric[6] and analytical approaches to estimate π have both borne fruit. Interestingly, π may also be estimated by repeatedly performing a random—or probabilistic—experiment, whose precise outcome cannot be predicted, but whose average behaviour may be estimated. Such an experiment is called a Monte Carlo simulation. Thus the quest for π brings together the mathematical sub-fields of geometry, analysis, and probabilistic simulation.

This quest for the unattainable—but supremely beautiful—has engaged human minds to seek π in countless infinite sums, such are shown in Figure 1. These equations are sometimes starkly simple and at other times thoroughly mystifying, and embody the paradox that is π more succinctly than all the words in the world.

Ludolph van Ceulen and François Viète

Before we set sail to explore π further, let us indulge in one last glance at The Pi of Archimedes and how the work of two mathematicians—who lived in the latter half of the 1500s—formed a bridge from the polygon method of Archimedes to newer infinite series methods for estimating this transcendental number.

Ludolph van Ceulen [10] is one of the unsung heroes in the perennial quest to calculate ever more digits of π. He faithfully and heroically followed in the footsteps of Archimedes and spent almost *twenty-five years* of his life to evaluate π to 35 decimal places[7] [10,11]. It is fitting that his restored tombstone in Leiden is inscribed with the upper and lower bounds of π [12] that he so painstakingly computed:

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π > 3.14159265358979323846264338327950288
π < 3.14159265358979323846264338327950289
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François Viète not only emulated the polygonal approach of Archimedes to estimate π, but also introduced algebraic notation [13,14] to allow for greater abstraction. Even more significantly, he introduced—for the first time—an explicit, infinite product formula for for the *exact* value of π, now known as Viète's formula [14,15], consisting of a product of nested radicals:

$$\begin{aligned}\frac{2}{\pi} &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2+\sqrt{2}}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots \\ \frac{2}{\pi} &= \prod_{n=1}^{\infty}\cos\frac{\pi}{2^{n+1}}.\end{aligned} \tag{3}$$

Eli Maor has observed that:

> "Viète's formula marks a milestone in the history of mathematics: it was the first time an infinite process was explicitly written as a succession of algebraic operations. … By adding the three dots at the end of his product, Viète, in one bold stroke, declared the infinite a bona fide part of mathematics. This marked the beginning of mathematical analysis in the modern sense of the word." [14].

[6] See the "Pi of Archimedes".
[7] The final polygon he used had almost 500 million sides!

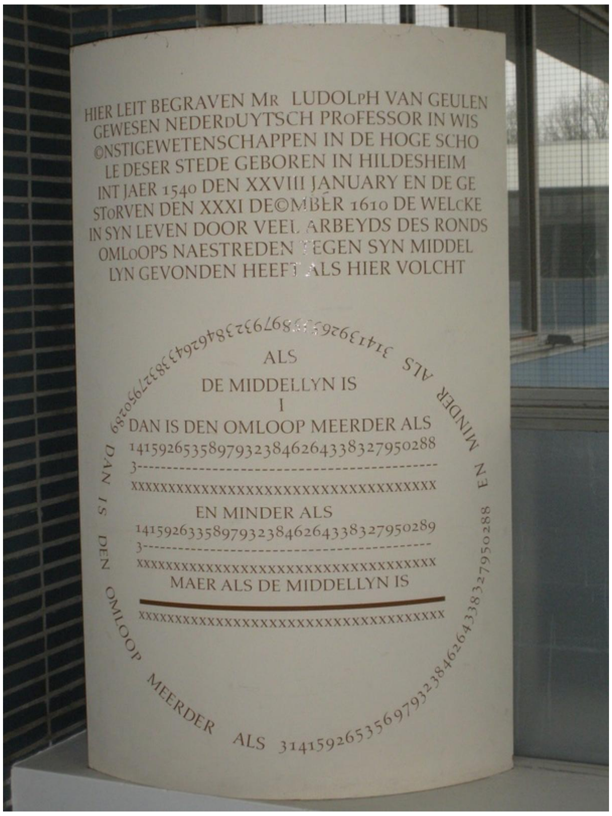

Figure 3: An image of the restored tombstone in Leiden celebrating Ludolph van Cuelen’s extraordinary achievement in calculating π to 35 decimal places. Image is taken from https://www.history-of-mathematics.org/artifacts/pi-tombstone [12].

While van Cuelen's work displayed superhuman effort and dedication, it also demonstrated that the method of Archimedes did not converge rapidly to π. Viète's formula bridges the divide between the ancient and the modern by embodying π and infinity for the first time in an explicit and exact equation, allowing more and more efficient estimates of π to be uncovered in due course.

The Madhava-Gregory-Leibniz (MGL) series

It must be obvious by now that trigonometry, circles, and the number π are inextricably entwined.[8] The quest for more accurate values of π continued to fascinate mathematicians in the centuries after Archimedes. This time though, rather than geometric iteration, *sums of successive terms* were used to approximate π.

For our purposes, a *sequence* is an *ordered* procession of numbers, and a *series* is a sum of successive terms that obey some specific rule. If the summation stops at some particular term, we have a *partial sum*; if the summation goes on indefinitely, we have an *infinite series*. If this infinite sum approaches ever closer to a finite value, the series is said to *converge*. To see what all this means in practice, let us look at the Madhava-Gregory-Leibniz series.

Why a triple-barrelled name?

The series we are about to look at was originally called the *Gregory series*. Leibniz evaluated the Gregory series for a specific value and came up with a formula for π, and that series was called the *Leibniz series*.

The accomplishments of medieval Indian mathematicians—whose discoveries antedated those of Gregory and Leibniz—remained unknown to the larger world. But recent scholarship has accorded priority to the leading Indian mathematician-astronomer of that period, Madhava, who anticipated both the Gregory series and the Leibniz series by more than 250 years [16–20]. This explains the triple-barrelled name for the series. Thumbnail sketches are given in the links below for all three mathematicians.

James Gregory was the first Professor of Mathematics at the University of Edinburgh and in 1671, he published the series that was called the the arctangent series, or the Gregory series.

Gottfried Wilhelm Leibniz evaluated the arctangent series at $\frac{\pi}{4}$ to get an estimate of $\frac{\pi}{4}$; the result was known as the Gregory-Leibniz series or the Leibniz Formula.

Madhava of Sangamagrama was a mathematician-astronomer who pursued research in trigonometric power series. In this, he showed remarkable prescience in defining angular measure as the ratio of arc length s to radius, r, thus establishing the *naturalness* of radian measure for serious work in trigonometry.[9]

[8] If this sounds unfamiliar, I invite you to read my blogs "A tale of two measures: degrees and radians" and "The Pi of Archimedes".

[9] See also "A tale of two measures: degrees and radians". Some papers attribute the results of Madhava to Nilakantha—a student in the lineage of Madhava—but more recent papers cite Madhava correctly as the fountainhead of this research.

Derivation

Rather than draw the Madhava-Gregory-Leibniz (here abbreviated as the MGL) series out of a hat, we will sketch its derivation, according to Gregory, and show its origins in integral calculus.

We assert that

$$\int_0^x \frac{1}{1+t^2}\mathrm{d}t = \arctan x \tag{4}$$

This integral should be familiar to most high school students. If it is not, try substituting $t = \tan\theta$:

$$\begin{aligned} t &= \tan\theta \quad \text{which gives} \\ \frac{\mathrm{d}t}{\mathrm{d}\theta} &= \frac{\mathrm{d}}{\mathrm{d}\theta}[\tan\theta] \\ &= \sec^2\theta \\ &= 1+\tan^2\theta \\ &= 1+t^2 \\ \text{Therefore} \quad \tfrac{1}{1+t^2}\mathrm{d}t &= \mathrm{d}\theta \end{aligned}$$

The integral of Equation (4) now becomes

$$\begin{aligned} \int_0^x \frac{1}{1+t^2}\mathrm{d}t &= \int_{\arctan 0}^{\arctan x} \mathrm{d}\theta \\ &= \big[\theta\big]_{\arctan 0}^{\arctan x} \\ &= \arctan x \end{aligned} \tag{5}$$

This takes care of the right hand side of Equation (4). If we performed long division on the left hand side of the same equation, we get:

$$\begin{aligned} \int_0^x \frac{1}{1+t^2}\mathrm{d}t &= \int_0^x \big[1 - t^2 + t^4 - t^6 + \dots\big]\,\mathrm{d}t \\ &= \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots\right]_0^x \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \end{aligned} \tag{6}$$

Using Equations (5) and (6), we get the Madhava-Gregory series

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \tag{7}$$

Notice that it is only a small step from here to substitute $x = 1$—because $\tan\frac{\pi}{4} = 1$—to get the equation

$$\begin{aligned} \arctan 1 = \frac{\pi}{4} &= 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \dots \\ \pi &= 4(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \dots) \end{aligned} \tag{8}$$

which is the MGL series, that is also shown at the top of Figure 1. Strangely, Gregory did not publish the special case of Equation (8), and it was Leibniz who discovered both Equations (7) and (8) in 1674, and published them in 1682. For details of Madhava's terminology and approach, do consult the literature [16–19]. It is noteworthy that Equation (8) was the first infinite series ever found for π. However, it converges very slowly. "Calculating π to 10 correct decimal places using direct summation of the series requires precisely five billion terms..." [21].

The Quest for faster convergence

Over the last 370 years, by far the most effort has been expended in discovering series that *converge rapidly* to π, so that even a partial sum of only a few terms will provide an accurate estimate of π. We now consider a selection of famous formulae from mathematicians who have bequeathed series for calculating π efficiently.

Machin's Formula

John Machin followed in the footsteps of the Madhava-Gregory-Leibniz series, but he used the difference in the arctangents of *two* values to arrive at a more rapidly convergent series for π. To better understand his method, let us recall that if $\tan A = \frac{a_1}{b_1}$ and $\tan B = \frac{a_2}{b_2}$, then [22]:

$$\begin{aligned}
\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \\
&= \frac{\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 - \frac{a_1 a_2}{b_1 b_2}} \\
&= \frac{a_1 b_2 + a_2 b_1}{b_1 b_2 - a_1 a_2}
\end{aligned}$$

Notice that

$$\begin{gathered}
\arctan \tan(A+B) = (A+B) \text{ which implies} \\
\arctan \frac{a_1}{b_1} + \arctan \frac{a_2}{b_2} = \arctan \left[\frac{a_1 b_2 + a_2 b_1}{b_1 b_2 - a_1 a_2} \right]
\end{gathered} \tag{9}$$

Suppose we set $a_1 = a_2 = 1$, then, Equation (9) we get these sum and difference formulae:

$$\begin{aligned}
\arctan \frac{1}{b_1} + \arctan \frac{1}{b_2} &= \arctan \left[\frac{b_1 + b_2}{b_1 b_2 - 1} \right] \\
\arctan \frac{1}{b_1} - \arctan \frac{1}{b_2} &= \arctan \left[\frac{b_1 - b_2}{b_1 b_2 + 1} \right]
\end{aligned} \tag{10}$$

Machin knew all *four* arctan formulae shown below:

$$\begin{aligned}
\frac{\pi}{4} &= \arctan \frac{1}{2} + \arctan \frac{1}{3} \\
\frac{\pi}{4} &= 2 \arctan \frac{1}{2} - \arctan \frac{1}{7} \\
\frac{\pi}{4} &= 2 \arctan \frac{1}{2} - \arctan \frac{1}{7} \\
\frac{\pi}{4} &= 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}
\end{aligned} \tag{11}$$

Note that the rational arguments of the arctan functions on the RHS of these e quations all have a numerator of 1. Henceforth, whenever we talk of the *two-term Machin-like formulae* we implictly mean these rational fractions with a numerator of one.

Specifically, Equation (12) is referred to as the Machin-formula:

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239} \tag{12}$$

Because $\frac{1}{5} = \frac{2}{10}$, the first term on the RHS, and its powers, are well suited for decimal calculation by hand. And because $\frac{1}{239}$ has a large denominator, it converges rapidly. Accordingly, Machin was able to compute π to one hundred decimal places [23] using the first 21 terms.

But what made Machin choose these particular numbers in Equation (12)? I have sought the answer(s) to this vital question from many quarters [24] without much success.

My questions

Was the historical process of discovery serendipitous, or was it directed by knowledge that led straight to it? Even if historically serendipitous, is there a systematic and simple route that can today deliver the four two-term Machin-like formulae, much like a can of Coke is delivered from a Coke machine when the requisite coins are inserted?

How many ways are there of looking at this one problem? Pythagorean Triples? Gaussian Integers? Nested Square Roots? Trial and Error in a restricted domain?

What was the *unifying thread* that enabled Størmer to claim in his 1899 paper [25] that there were *four and only four* Machin-like formulae with two terms?

Some books and papers

The following books were consulted: *Elements of Number Theory* [26] and Conway and Guy's *The Number Book* [27]. As also the following papers by Calcut [28,29], Todd [30] and Wetherfield [31]. In the Todd paper, where *four* two-term formulae were expected to drop out of Table 1, like a *Deus ex machina*, only *three* were tabulated. Why?

Historical Treasure Hunt

The most exhaustive treasure hunt through the Machin fomula history is the paper by Rickey [32] in which he carefully unearths how the somewhat unusual term $\arctan \frac{1}{239}$ makes its appearance. To quote him:

> Now we have a full explanation of the numbers in Machin's formula. The 5 was a product of insight, guesswork, and experimentation. Using it, the 239 falls right out. This is Machin's proof. It has real explanatory value. It is good mathematics [32].

The exposition by Nishiyama on the use of Pythagorean triples to justify Machin's formula [33] states tantalizingly:

> Machin's formula was explained above, but regarding its derivation, just how the formula was discovered seems to be unknown. Perhaps Machin's formula was discovered by accident. Or perhaps it was obtained by building on a mathematical concept.

Finally, the paper by Nimbran [34] was consulted, but it did not deliver the *stroke of insight* or the systematic method by which such a formula could have been derived for a two-term Machin-like formula.

If and when I find satisfying answers to my questions, I will write about them in a separate, dedicated blog. Meanwhile, if any reader of this blog can throw light on the answers to my questions, I kindly request him or her to email me.

Newton's estimate of π via the Binomial Theorem

We now look at how Newton, Euler, Gauss, and Ramanujan each approached the problem of estimating π. Like all self-driven geniuses, each of them hewed his own independent path, and the fact that the same destination was reached each time is testimony to the unimaginable mathematical riches that lie buried, waiting to be explored by prepared minds in the future.

The mathematician Steven Strogatz has written a charming essay in Quanta Magazine [35]. It recounts how a young Newton made an inspired and imaginative leap of faith, and gingerly attempted to extend his own pathbreaking binomial theorem to non-integral powers, to derive the binomial series. When the results justified his extrapolation, he decided to apply it to estimate π more efficiently.

Once again, this episode exemplifies how mathematics is at heart an exploratory science, that does admit of experimentation, and in which logical correctness grants the ultimate seal of approval and acceptance.

Euler's solution to the Basel Problem

Leonhard Euler is an illustrious polymath among mathematician-polymaths [36]. One of his less celebrated contributions is his solution to the Basel Problem [37] in 1734—eighty-four years after it was posed—when Euler was a mere twenty-eight years old.

The Basel Problem asked for the exact sum of the infinite series of the squares of the reciprocals of the natural numbers. It is perhaps much better expressed and understood in mathematical notation. What is the value of the sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \ ?$$

Euler's answer was:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}. \tag{13}$$

His proof was challenged when first presented, but accepted later, in 1741.

What I find fascinating about Equation (13) is that the left-hand side (LHS) is entirely the sum of rational numbers while the sum on the right-hand side (RHS) is irrational. And yet we have *exact equality* of both sides, not to mention the *unexpected* closed form of the sum being $\frac{\pi^2}{6}$. How come?

This is the mind-twisting paradox of infinity. I like to think that *infinity is where the rationals meet the irrationals*. And this equation is not unique in displaying this characteristic. Countless other identities exhibit this same paradoxical property of an infinite sum of rationals *exactly* equalling an irrational number.

Thus Euler not only gave us another way of computing π, but he also showed how elegantly the rationals could dovetail into the irrationals in entirely surprising ways, whenever infinity is involved.

Gauss and the AGM

Arithmetic-Geometric Mean AGM

Ramanujan

The Chudnovskys

Resources for Enrichment

The Basel Problem

How Augustin-Louis Cauchy solved the Basel Problem is clearly laid out and explained in this mesmerizing Rise to the Equation YouTube video [38]. The explanation in this video should be clear to a high school student who has encountered trigonometry but not calculus.

Those of you who are puzzled by the appearance of π^2 in the solution to the Basel problem should also view this 3Blue1Brown YouTube vdeo.

Veritasium and 3Blue1Brown put out quality educational videos on Mathematics and are an authoritative source of enrichment. Do benefit from them.

Conclusion

This story of how π was extracted from the ore of geometry and refined into an enigmatic number which cannot be trapped within finite digits embodies the thrill of the chase. Mathematics is not merely a logical edifice built from the granite of unassailable logic, but is also the fruit of a pliant but disciplined imagination fuelled by inspiration to continually expand the boundaries of its domain.

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Feedback

Please email me your comments and corrections.

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