Varieties of Multiplication

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This was my first blog—written in 2012—under the *Mathemtical Musings* tag. The intention was to re-visit topics in mathematics that trigger concern or disquiet in the earnest student of the subject. My hope was that ideas that appeared puzzling or forbidding at first sight could be coaxed to become friendly and helpful. Unhurried explanations and a different perspective would underpin the approach. I have retained, substantially unchanged, what I first wrote, to maintain the freshness, flavour, and vintage of the original blog, even if it is a little rough around the edges.

Prologue

This blog is experimental in three ways.

First, this is my maiden attempt to display mathematics on a web page. It might look simple, but believe me, it is no mean task. Thanks to the concerted efforts of many generous people, I am using MathJax to render the mathematics via Pandoc, and its flavour of Markdown.

The second experimental feature is what I have called "slicing the orange of knowledge with a different cut" in my book *Secrets of Academic Success*. The idea of multiplication runs like a thread through much of mathematics, from the most elementary stages of counting to what constitutes cutting edge research. Unfortunately, in the way mathematics is taught at present, multiplication is bundled with each stage of mathematics and viewed separately as an operation in that context.

By concentrating on the single unifying *idea* of multiplication, and viewing it across the whole of mathematics, we are indeed "slicing the orange of knowledge with a different cut". Even if you have not encountered some of the varieties of multiplication mentioned here, this exposure will help you grasp those varieties better when you do encounter them. Please send me feedback on whether this approach works for you.

Third, this is an *extremely long* blog. In fact, I call it a *slog* ☺ . It took me some weeks to write it. So, take your time reading it. It is unlikely that you will finish it in one sitting. *Read it in parts at your own pace.* After having read it once, cast your eyes and mind over the whole to get an overall view of the main ideas.

I thought of splitting the blog into three or four manageable parts, but decided against it because I wanted the evolution of multiplication as an idea to be left whole in a single post. Tell me if it put you to sleep ☺ .

With that out of the way, let us begin. I want to look at some of the varieties of multiplication that mathematicians have developed over time. It is a survey that will serve as a pinhole through which we can view how a single, simple mathematical idea has been expanded and elaborated into uses far beyond its historical moorings.

Multiplication as a binary operation

Consistency is valued more in mathematics than in other disciplines. *The idea is not to upset the apple cart but to expand it.* Definitions, conventions, rules, facts, and fallacies—once established—are usually above dispute, and do not vary with time or place. So, let us start by defining some terms.

Multiplication is a *binary* operation: it is something that we do with *two* mathematical objects, whatever they might be. Usually, the two are *similar* objects or at least *compatible* objects. The whole numbers are an example. We can and do multiply two whole numbers.

Multiplication as repeated addition

Practically and historically, multiplication arose as an arithmetic convenience for repeated addition. If we add the number 3 four times, we have

$$3 + 3 + 3 + 3 = (3 + 3) + (3 + 3) = 6 + 6 = 12$$

The reason for adding 3 in *pairs*, as shown above, is that addition is a binary operation, just like multiplication. Using the shorthand of multiplication, we write this as $4 \times 3 = 12$. *So, multiplication is a shorthand for repeated addition.*

When we see the arithmetic expression 4×3, we say "four times three" in English. Or, we could equally well say "four threes", as I was taught at school, which is less ambiguous and much clearer. Think of four lots of three being added together like we have seen above:

$$
\begin{array}{ccccc}
4 & \times & 3 & = 3 + 3 + 3 + 3 = (3 + 3) + (3 + 3) = 6 + 6 = 12 & \\
\uparrow & & \uparrow & \uparrow & \text{(1)} \\
\text{multiplier} & & \text{multiplicand} & \text{product} &
\end{array}
$$

The number 4 is the multiplier and the number 3 is the multiplicand. This is the standard definition.

We say that the "something" which is repeatedly added, is the *multiplicand.* The number of times that "something" is added is the *multiplier.* And the result of this operation is the *product.* Thus far we are in perfect harmony with accepted usage.

Commutativity and multiplication

Multiplication of numbers is commutative, i.e., the multiplier and multiplicand can change roles without affecting the result.

$$4 \times 3 = 3 \times 4 = 12. \tag{2}$$

Note that at the very left of Equation (2), 4 is the multiplier, and 3 the multiplicand, whereas in the middle of Equation (2), 3 is the multiplier and 4 the multiplicand. To labour the point,

$$3 \quad \times \quad 4 \quad = 4 + 4 + 4 = (4 + 4) + 4 = 4 + (4 + 4) = 12 \qquad (3)$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \uparrow$$

multiplier   multiplicand   product

Although the two numbers have changed their names and roles from Equation (1) to Equation (3), the result is the same because the multiplication of numbers is commutative.

To accommodate our zeitgeist, the distinction between multiplier and multiplicand is fading away, in favour of the symmetrical and neutral term *factor*. The result of multiplying two *factors* is still the *product*, as before.

Rectangular numbers

Historically, stones were used to count. And the stones representing any number may be arranged in geometric shapes, like lines, triangles, rectangles, and so on. This gives us a geometrical or pictorial representation of numbers.

All numbers which are the products of two whole numbers, neither of which is one, may be expressed as *rectangular numbers*. The symbolic operation 4×3 may be shown pictorially as a series of 12 icons arranged four across and three high. Because we may swap the order of multiplication, we may also show it as 12 rectangles three across and four high. As we have seen, multiplication of numbers is commutative. The image below shows this equivalence.
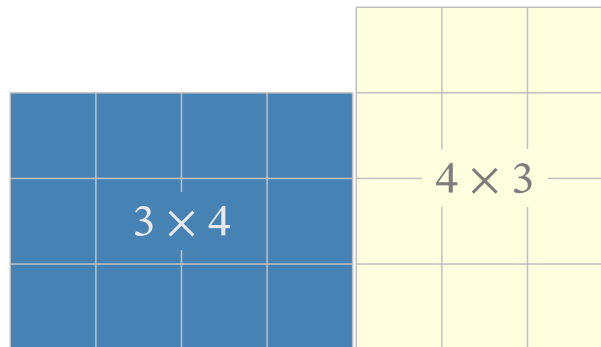

Figure 1: The multiplier is the number of rows, and the multiplicand is the number of columns.

Factorization is not unique

There is a subtle but important point to grasp here. The product 12 is called a *composite* number and its *factors* in the illustrated representation are 4 and 3. *But this factorization is not unique.* We could have equally correctly claimed that $2 \times 6 = 12$ leading to a different factorization. While we may assert that both 4×3 and 2×6 lead to the same unique composite number as the product, we cannot reverse the process and claim uniqueness of factors for any particular composite number.
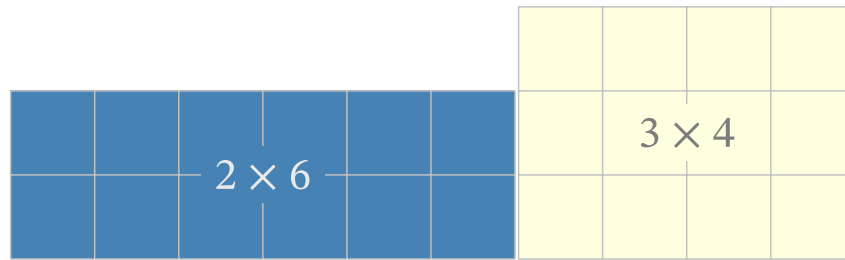
Figure 2: Twelve is a composite number. It may be factorized as 2×6, 4×3, or as the trivial 1×12.

Square numbers

A square is a special case of a rectangle whose length and width are equal. When we write $3 \times 3 = 9$, and we arrange the resulting nine squares in a rectangle, we get a *square number* of three squares by three squares. This is why we say that we are *squaring* a number when we multiply it by itself.
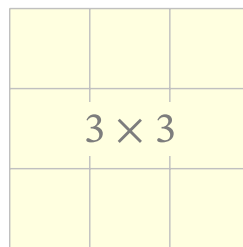

Figure 3: Nine is a square number. Because $3 \times 3 = 9$, it may be so portrayed geometrically.

One could carry this analogy further and move into three dimensions to represent a number like $3 \times 3 \times 3 = 27$ with small cubes arranged in a large *cube*.[1] This is why we say we are *cubing* a number when we multiply it by itself twice.

Prime numbers

A number which cannot be expressed as the product of two numbers other than one and itself is called a *prime number*. Prime numbers can only be arranged in a line, never in a rectangle. Seven is a prime number as illustrated below.
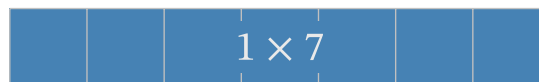

Figure 4: Seven is a prime number. Its seven icons cannot be re-arranged in a rectangle.

Try to rearrange the seven icons into a rectangle to convince yourself that it is not possible and that seven is prime. Experimenting like this will help you better understand what testing for primality entails.

Prime numbers are like building blocks that may be used to build larger numbers by multiplication.

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[1]Try this with toy blocks to convince yourself of its truth.

Prime factorization is unique

Let us look at the number 12 again, breaking it down this time into its *prime factors* so: $12 = 2 \times 2 \times 3$. There are two instances of the number 2 and one instance of the number 3. These numbers cannot be decomposed any further because they are prime. If we disregard the order of arrangement of these prime factors, i.e., we do not distinguish between $2 \times 2 \times 3$ and $2 \times 3 \times 2$ and so on, we may assert that *the prime factors of a composite number are unique.* This statement is known as the Fundamental Theorem of Arithmetic. It is also sometimes called the Prime Factorization Theorem.

Symbols for multiplication

If you are sharp-eyed, you might have come across the multiplication of two negative numbers by enclosing each of them in parentheses: (). The same symbols are also used to define associativity and distributivity later in this blog. We now look at the chequered history of how the notation for multiplication has changed with time, need, and context.

Times sign

The symbol for multiplication that we first learn is the rotated plus sign "+" that looks like $\times$. It is called the "multiplication sign" and is usually read as "times", as we have already seen. It serves reasonably well even when we outgrow the whole numbers and move onto fractions.

Parentheses

Parentheses, written in pairs as (), have traditionally denoted *precedence* during evaluation of an expression. Division and multiplication are evaluated *before* subtraction and addition. This order may be altered by including terms in parentheses, which are accorded highest priority during evaluation. So, $5 \times 4 + 1 = 20 + 1 = 21$, whereas $5 \times (4 + 1) = 5 \times 5 = 25$.

When our discourse embraces negative quantities, in order to avoid ambiguity, we need something to enclose both the negative sign, $-$, and the number to which it applies. The expression 5×-4 is ambiguous and therefore never written so when we actually mean $5 \times (-4)$. This notation led to two pairs of parentheses *without any explicit multiplication symbol in between* to denote the multiplication of the two enclosed numbers thus: $(5)(-4) = 5 \times (-4) = -20$.

Juxtaposition without any symbol

The archetypal symbol for the unknown in algebra is x, which looks a little too much like the multiplication symbol $\times$, especially when handwritten.

To avoid confusion, a convention was adopted that when two *algebraic variables,* representing unknown quantities, were written next to each other or *juxtaposed,* it indicated the multiplication of the two quantities. There is no intervening symbol between the two variables.

Thus, $x \times y = (x)(y) = xy$. Note that this convention is for algebraic variables only. We cannot use this convention with digits representing numbers because of *place value* in our decimal system. The number 45 does *not* represent the product of 4 and 5 but actually means $40 + 5 = 45$.

Centred dot

As more exotic objects entered the mathematical collection, yet another symbol was devised to show (at least one form of) multiplication. The vertically centred dot · was used to indicate one of the several products of *vectors,* which we shall discuss later in this blog. Again, the dot is not useful in the context of digits because it could be confused with a decimal point.

So, there is both a rationale and a mathematical context for the introduction of each symbol for multiplication, according to time, need, and circumstance.

Asterisk

The latter half of the twentieth century saw the introduction of yet another symbol for multiplication, this time for use in programming languages. Because the ASCII character set, devised during the era of teleprinters, did not include the symbol $\times$ for multiplication, another available symbol had to be chosen for multiplication. The winner was the asterisk, denoted by *. Note that even with the symbol *, multiplying 3 by -4 still requires one to type $3 * (-4)$ to avoid ambiguity.

Repeated multiplication—or exponentiation—is usually represented by a double asterisk ** in most computing languages, although a caret ^ assumes this function in some languages.

Laws of arithmetic

We now return to the assertion, made at the start of this blog, that multiplication is a *binary* operation: something that happens between *two* mathematical objects. You might object that we can and do multiply three numbers. For example, $2 \times 3 \times 5 = 30$ and no one would doubt the veracity of that assertion. Why then is multiplication classified as a binary operation and how might it be reconciled with what we know about the multiplication of three or more numbers?

Early mathematics was eminently practical, concerned with computing areas and volumes, profit and loss, and so on. In the course of time, mathematicians started to systematize their body of knowledge by introducing logic and rigour into their subject. They wanted to move beyond ad hoc methodology and construct an intellectual edifice that was stable, durable, and generalizable.

The real numbers as a field

Some of the most unpleasant experiences of school mathematics are the sudden and unexpected changes that intrude into the familiar arithmetic of primary school. Division, fractions, negative numbers, multiplication by zero, product of two negatives being positive, etc. are a few examples. When rule upon unanticipated rule is foisted on the young student, with no rhyme or reason, the student gets overwhelmed and develops a distaste for mathematics or even a reflex fear of it. This need not be so.

One way out is a quick but easy introduction to some ideas of abstract algebra which lay the foundation for all subsequent mathematics. This way, all the rules are bunched together as unquestioned assumptions or axioms. Then, based on those assumptions, we build a mathematical edifice that is logical, consistent, and extensible. Mathematics will then be changed from a mysterious game with ever-changing rules into a trustworthy friend who can be relied upon.

The numbers we use every day are drawn from a set called the real numbers denoted by $\mathbb{R}$. Numbers such as $0, 1, -200, 50004, \frac{1}{2}, -\frac{3}{4}, 0.3333\cdots, -0.5, \sqrt{2}, \pi$, and countless others belong to this grab bag set.

The set $\mathbb{R}$ comes bundled with *two* binary operations: addition, denoted by $+$, and multiplication denoted by $\times$ or by other means as outlined above. One property that makes real numbers so useful is that any addition or multiplication of real numbers results in another real number. This is called *closure* and is an important aspect of real arithmetic. In addition, the reals also exhibit other familiar behaviours—with fancy-sounding names—which are tabulated below, using arbitrary real numbers, a, b, c.

Table 1: Axioms for the real numbers

Property	Addition	Multiplication
Associativity	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Commutativity	$a + b = b + a$	$ab = ba$
Identity	$a + 0 = a = 0 + a$	$a \cdot 1 = a = 1 \cdot a$
Inverse	$a + (-a) = 0 = (-a) + a$	$aa^{-1} = 1 = a^{-1}a$ for $a \neq 0$

For the record, formal definitions for the above terms are available on the Web from reputable sites whose links are listed below:

1. Associativity.

2. Commutativity.

3. Distributivity. Multiplication is distributive over addition. For completeness, we define

$$
\begin{array}{ll}
\text{Left Distributivity} & a(b + c) = ab + ac \\
\text{Right Distributivity} & (a + b)c = ac + bc
\end{array}
$$

   In our case, both conditions hold, and we may simply say that *multiplication is distributive over addition for the reals.*

A mathematical object consisting of a set with two binary operations having the above properties is called a field. The real numbers constitute a field.

Associativity of multiplication

Because multiplication is binary, we can only multiply two numbers at any one time. If we need to multiply together three or more numbers, we have to multiply two of them first to get a single product before we can multiply it with next number, and so on.

The associative law simply states that when we multiply three numbers, it does not matter which two of the three we multiply first; the result will always be the same. It is this property that allows us to write something like $2 \times 3 \times 5$ or abc and still make sense—because it denotes something unique—even though we know that multiplication is a binary operation.

In addition to the three properties of associativity, commutativity, and distributivity, the real numbers we use every day have an additive identity and inverse in Table 1. These are considered next.

The additive identity and inverse in $\mathbb{R}$

The additive identity in $\mathbb{R}$ is 0. This means that for any $a \in \mathbb{R}$, $a + 0 = 0 + a = a$. The additive inverse of a is $(-a)$ and this means $a + (-a) = (-a) + a = 0$.

When we add together a number and its inverse, we get the additive identity.

Another way of understanding the additive inverse is to look at it geometrically as a reflection in a double-sided mirror placed perpendicular to the real line at the position of 0. Why at zero? *Because zero is its own additive inverse. The reflection of zero in the mirror gives us back zero.*

Reflect $-a$ in the blue silvered mirror to get a Reflect a in the red silvered mirror to get $-a$
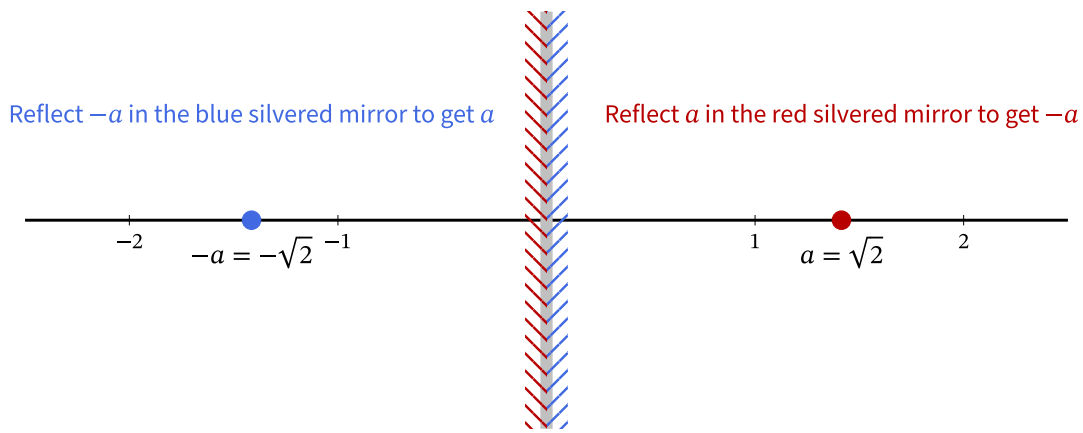
$-a = -\sqrt{2}$ $a = \sqrt{2}$

Figure 5: Geometric interpretation of the additive inverse.

With reference to Figure 5, the mirror is the silver-colored line placed at zero. The irrational number $a = \sqrt{2}$, represented by the red dot, is reflected in the red silvered side of the mirror to give the blue dot, which is $-a = -\sqrt{2}$. A second reflection of $-a = -\sqrt{2}$ on the blue silvered side gives us back the original number $a = \sqrt{2}$. This means that *the additive inverse of the additive inverse gives us back the original number.*

At the risk of expounding the obvious, let us take another look at a pictorial representation of how to obtain the additive inverse of a number $x \in \mathbb{R}$. This is illustrated graphically in Figure 6. Every ordered pair of coordinates lying on the straight line $y = -x$ represents a number and its additive inverse. So, if $x = a$, $y = -a$, and the ordered pair $(a, -a)$ represents a number and its additive inverse. Moreover, the ordered pair $(-a, a)$ also lies on this line and again represents a number and its additive inverse, this time for $x = -a$ and $y = a$. This view might be helpful for those who think in terms of pictures rather than symbols.

An even more concrete algorithm to obtain the additive inverse is now given. Suppose we want the additive inverse of 1.5. We draw a vertical line from 1.5 on the x-axis to cut the line $y = x$ and then extend the line horizontally until it cuts the y-axis at -1.5. This last number is the additive inverse. It is clear from Figure 6 that the *only* point that is invariant to this operation is 0 which maps to itself. *The additive inverse of the additive identity is itself.*

We show later in Equation (7) that multiplying a by (-1) also gives us its additive inverse.
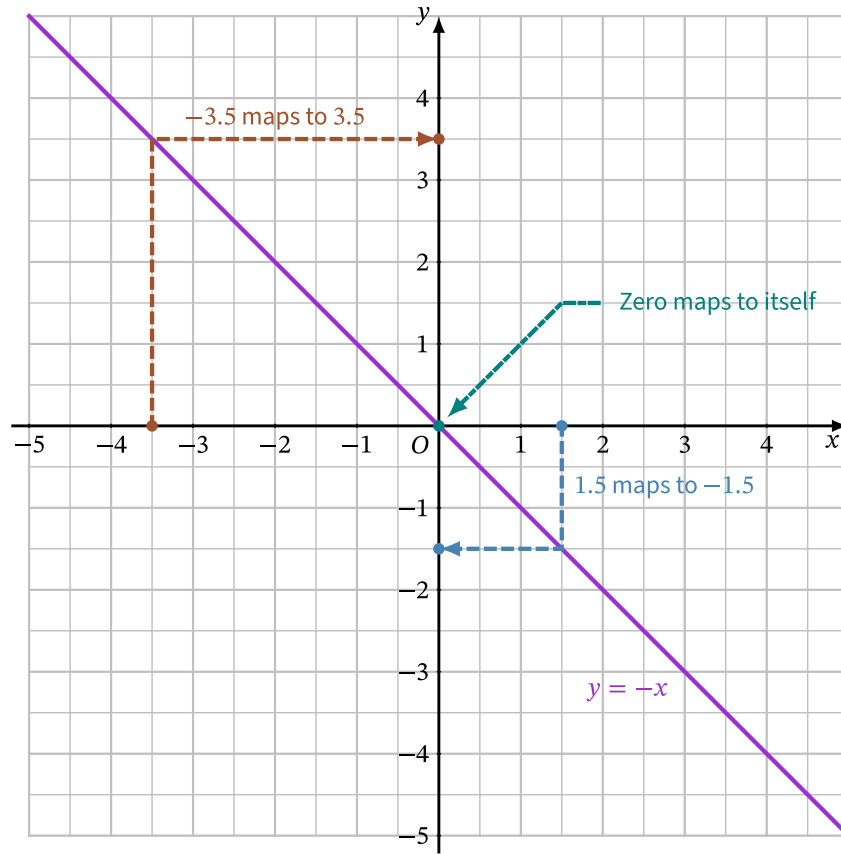

Figure 6: A graphical construction to obtain the additive inverse.

The multiplicative identity and inverse in $\mathbb{R}$

We next consider the multiplicative identity and inverse in $\mathbb{R}$ which are also shown in Table 1.

The multiplicative inverse for arbitrary $a \in \mathbb{R}$ is defined as $\frac{1}{a}$ for $a \neq 0$. Why the exclusion of zero? Let us look for a graphical reason.

If we plot x against its multiplicative inverse $\frac{1}{x}$, we would get a rectangular hyperbola, as shown in Figure 7.[2] We have coloured the two "arms" of the hyperbola in the first and third quadrants in blue and red respectively, although they are part of the same curve.

A construction to get the multiplicative inverse of some point a (not equal to zero) is to locate the point with the x coordinate a on the curve and to find out its y coordinate. Every ordered pair $(a, \frac{1}{a})$ therefore represents a number and its multiplicative inverse, with the proviso that $a \neq 0$. Indeed, we cannot ever get to $a = 0$ on a graph of the hyperbola.

Note the following insights from Figure 7:

a. The function becomes unbounded as x approaches 0. This happens both from the positive and negative sides. Symbolically, $\lim_{x \to 0^+} \frac{1}{x} = \infty$ and equally, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.[3] *This is why* 0 *has no multiplicative inverse.*

b. The multiplicative inverse has the same sign as the original number, since the hyperbola has two "arms".

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[2]We use the familiar x instead of a in the graphical context.
[3]There is no limit as $x \to 0$.

c. There are only two values of x for which the multiplicative inverse is also the original number. They are at $x = 1$ and $x = -1$. This is because the line $y = x$ intersects the hyperbola at two points: $(1, 1)$ and $(-1, -1)$.
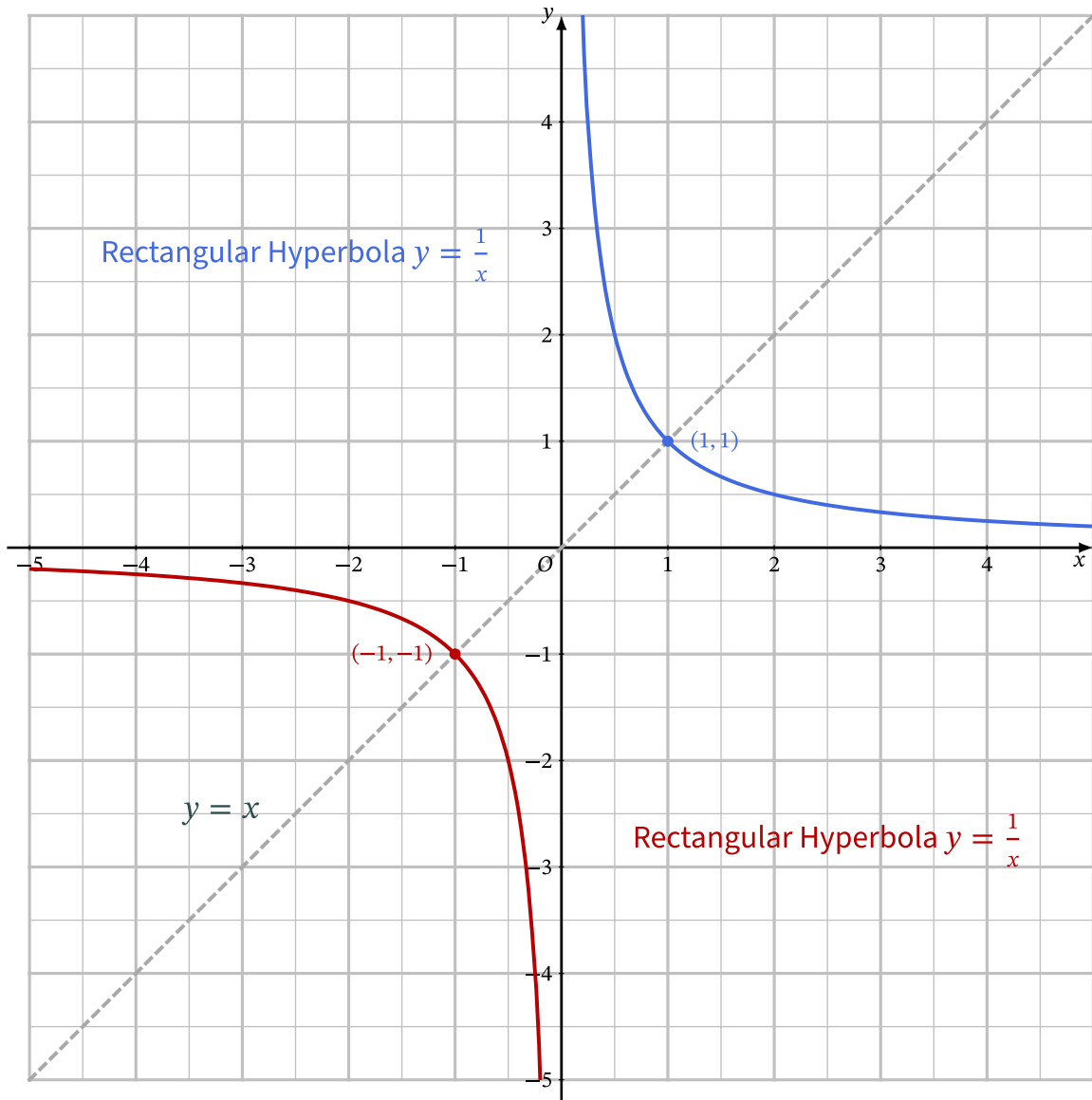

Figure 7: The multiplicative inverse in $\mathbb{R}$ plotted as $y = \frac{1}{x}$.

Where have subtraction and division disappeared?

If you are wondering where subtraction and division have disappeared, they are hiding in plain sight. Subtracting b from a amounts to adding the additive inverse of b to a. So,

$$a - b = a + (-b)$$

Likewise, dividing a by $b \neq 0$ amounts to multiplying a by the multiplicative inverse of b which equals $\frac{1}{b}$:

$$a \div b = a \times \frac{1}{b} = \frac{a}{b} \text{ for } b \neq 0.$$

Multiplying any number by zero always gives zero

Recall from Table 1 that 0 is the unique *additive* identity. Zero's claim to notoriety is that it does *not* have a *multiplicative inverse*. And this is because any number *multiplied* by zero gives zero.

Consider an arbitrary number $a \in \mathbb{R}$. Recall also that because 0 is the additive identity, $0 + 0 = 0$. We may then claim that

$$\begin{aligned}
a \cdot 0 &= a \cdot (0 + 0) && \text{0 is the identity for addition} \\
a \cdot 0 &= a \cdot 0 + a \cdot 0 && \text{distributive law; subtract } a \cdot 0 \\
0 &= a \cdot 0
\end{aligned} \tag{4}$$

Note that if we take $a = 0$, the product of zero with itself is also zero—something which might be dismissed as trivially obvious, but which could be deceptively difficult to prove.

We have carefully tiptoed our way to justify each step with one of the field axioms. This is the power of the axiomatic approach. There is no "wasted" axiom; there is no "missing" axiom. The set of axioms are the minimum necessary for the numbers to rest on a stable foundation.

This *minimal sufficiency* is at the heart of why mathematics is so strong. It has no extra fat. There is also no deficiency. It is frugal but sufficient. We will encounter this same idea in the statement that a *basis is a minimal spanning set in a vector space.*[4] Any other algebraic structures, like the complex numbers $\mathbb{C}$, that obey the same axioms as $\mathbb{R}$, also have the same properties.

If you feel that Equation (4) is too much sleight of hand, and you would like something more concrete, you can always console yourself with the convenient but specific example of 5×0. Here 5 is the multiplier and 0 the multiplicand. So, we may write:

$$\begin{aligned}
5 \times 0 &= 0 + 0 + 0 + 0 + 0; && \text{multiplication is repeated addition} \\
&= (0 + 0) + (0 + 0) + 0; && \text{addition is a binary operation} \\
&= (0 + 0) + 0; && \text{0 is the additive inverse} \\
&= 0. && \text{0 is the additive inverse}
\end{aligned}$$

And the choice of a whole number made it easy to see the multiplication as repeated addition. What about 0×5? Because multiplication is commutative, we may assert that 0×5 is also 0, without doing any additional work. I hope you are getting to see mathematics as treasure hunt with clues and short cuts that lead to an exciting finale. If you are interested in pursuing these ideas further, please read this excellent article online [1].

Product of a positive and a negative number

Negative numbers arose when loans had to be given and taken. They also find use in describing the depth of an ocean trench as being so much *below* sea level. Other applications arise naturally with temperatures below the freezing point or with electric charges of a negative type, etc.

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[4]Which is the subject for another blog.

The signs of products featuring negative numbers are not intuitively comprehensible. So, we have to rely on the axioms to guide us to consistent results. What is the sign of the product of a positive and a negative number? To find out, we first prove that *multiplying a number by the additive inverse of another number gives the additive inverse of their product*, i.e.,

$$a(-b) = -(ab) \tag{5}$$

To prove Equation (5), we use the fact that zero multiplied by any number is zero, as shown in Equation (4).

$$
\begin{aligned}
a \cdot 0 &= a(b + (-b)) && \text{additive inverse} \\
0 &= ab + a(-b) && \text{distributive law; add } -(ab) \\
-(ab) &= a(-b)
\end{aligned}
\tag{6}
$$

If we now assume that a and b are both positive, then ab is positive, and $(-b)$ is negative. The product of a and $(-b)$ is the negative number $-(ab)$. So, *the product of a positive and a negative number is a negative number*. Conversely, a negative number may be split into the product of a positive number and a negative number.

One interesting corollary from Equation (5) and Equation (6) is:

$$
\begin{aligned}
(-1)(a) &= -(1 \cdot a). \\
&= -a
\end{aligned}
\tag{7}
$$

This means that *the additive inverse of a number is obtained by multiplying the original number by* (-1). This we also already know from Figure 6.

These slow but careful explanations might seem contrived, but they provide guardrails against falling off when future mathematical objects are encountered. And it is a whole lot more satisfying than hand-waving or saying "Just take it on faith."

Why is the product of two negative numbers positive?

Let us use the field axioms to navigate our way to this result as well.

Let $a > 0, b > 0 \in \mathbb{R}$. Then, both $(-a)$ and $(-b)$ are negative. Note that ab is positive and its additive inverse is $-(ab) = (-a)(b) = (a)(-b)$ which is negative from our previous result.

The scheme we have followed so far is to add something to the object of interest and use the axioms to prove that the sum is zero. The result we are after will then pop out. Let us apply that method again, using the final result fron Equation (6):

$$
\begin{aligned}
-(ab) + (-a)(-b) &= (a)(-b) + (-a)(-b); && \text{result from previous section} \\
&= (a + (-a))(-b) && \text{distributive law} \\
&= (0)(-b) && \text{additive inverse} \\
&= 0. && \text{zero mutiplied by anything is zero}
\end{aligned}
$$

The upshot is that we have $-(ab) + (-a)(-b) = 0$ which means that they are additive inverses. Since the additive inverse of $-(ab)$ is ab, we conclude that $(-a)(-b) = ab$ implying that the product of two negative numbers is a positive number. With the axioms to guide us, we can move with the sure-footedness of a mountain goat as we scale the heights and depths of each proof.

I have dealt with the arithmetic of fractions and negative numbers in my freely downloadable chapter "Arithmetic Revisited" from *Secrets of Academic Success*. I urge you to read it if you feel the need.

Exponentiation

Just as multiplication with whole numbers is repeated addition, exponentiation is repeated multiplication. A new notation is used to indicate repeated multiplication. We denote it by a *superscript* indicating how many times the number is multiplied: $5 \times 5 \times 5 = 5^3$, and so on. The number 5 is called the *base* and 3 the *exponent*. We call the number 5^3 as "five cubed" for reasons already explained, or as "five (raised) to the (power of) three".

What is the exponent in the number written plainly as 5? The base is 5 but the exponent is *implicit* or understood but not written. If we take $5 = 5^1$, the $5 \times 5 \times 5 = 5^1 \times 5^1 \times 5^1 = 5^3$. When we multiply numbers with the same base, we can simply add the exponents. And this fact underlies a very powerful computational device—logarithms.

Logarithms: multiplying by adding

We may reduce multiplication to addition if we focused on the exponents of a common base. This is exactly what was done by an eccentric Scottish laird called John Napier whose labours have made all our lives much less tedious. The books of logarithmic tables, affectionately called "log books" when I was at school, along with the slide rule were the mainstay of physicists and engineers before the advent of the electronic calculator in the mid-1970s. And they all relied on Napier's scheme of reducing multiplication to addition.

Multiplying by adding is simple. Suppose we want to multiply 2 by 3. We follow this algorithm:

1. Express the number 2 as a power of 10: $2 \approx 10^{0.30103}$.

2. Express the number 3 as a power of 10: $3 \approx 10^{0.47712}$.

3. Add the two powers or exponents: $0.30103 + 0.47712 = 0.77815$.

4. Find out what number equals $10^{0.77815}$. In our case, $10^{0.77815} \approx 5.99998$.

"Aha!" you might say. "But the answer is not exactly 6 which is the correct answer." You are right. With logarithms and the limited number of digits of precision, we can at best obtain a very good approximation to a computation.

If you had to compute 23.589×459.1213 what would you do? You would run to a calculator and punch in the digits to get 10830.212 with very little effort today. But if you lived in a period without electronic calculators, you would be very glad that logarithms existed, and you would happy to chirp out the answer as 10830, thanks to Napier.

Square roots

Taking a square root is a form of *exponentiation*. If I gave you a number like 9 and asked you to find a number such that when it was *added* to itself, you would get 9, you would almost instinctively divide 9 by 2 to give me 4.5 and indeed $4.5 + 4.5 = 9$.

If instead, I asked you to find that number, which when *multiplied* by itself gives 9, how would you go about solving it? We know that in this case, the answer is 3 because $3 \times 3 = 9$. To approach it systematically, we need a symbol for the operation, which when performed on 9 gives us 3. For historical reasons it was called *taking the square root* and the symbol $\sqrt{\ }$ is a stylized letter "r" for "radix" meaning "root" in Latin [2]. So, we write $\sqrt{9} = 3$.

But is that the whole story? Recall that the product of two negative numbers is positive. So, $(-3)(-3) = 9$ as well. So, which is the "true" or "correct" square root? By convention, we associate the positive square root with the symbol $\sqrt{\ }$. So, without any ambiguity, $\sqrt{9} = 3$ and $-\sqrt{9} = -3$.

Complex numbers

Talking about square roots and negative numbers, can we ever take the square root of a negative number? Do such numbers exist? If so, where? And are they useful?

The squares of real numbers give rise to only two possibilities. The square of zero is zero. The square of any non-zero real number, whether positive or negative, is always positive, as we have just seen. So, the possibility of a negative number being the square of a real number does not ever arise.

But we do encounter a different situation when solving an algebraic equation like $x^2 + 4 = 0$. When such equations were first encountered, mathematicians simply said that they had *no solutions.* This is true even when you initially encounter them at school today. *This equation has no solution in the field of real numbers.*

In the course of time, these pesky numbers—whose square is a negative number—kept popping up insistently in unlikely places. They were then reluctantly assigned mathematical existence with the somewhat pejorative term *imaginary numbers.* They were after all *not real numbers!*

In the course of time, a sandwich number was invented which was composed of the sum of a real number and an imaginary number. This number was called a *complex number.* It is the first of several mathematical objects we will encounter in this blog that are different from the real numbers of everyday life.

The set of complex numbers is denoted by $\mathbb{C}$ and is defined as

$$\mathbb{C} = \{a + bi \ : \ a, b, \in \mathbb{R} \text{ and } i^2 = -1\}$$

In plain English, this means that the set of complex numbers, denoted by $\mathbb{C}$, is defined as all numbers of the form $a + bi$ where a and b are real numbers and i, is called the *imaginary unit* and is the positive of the two roots of the equation $z^2 + 1 = 0$.[5]

---

[5]Because i is often associated with current, electrical engineers often use the symbol j instead.

But there is a little asymmetry in the expression $a + bi$. While b is multiplied by i, the a stands alone. Or does it? We could always write $a(1) + bi$ to restore symmetry into the expression. Then 1 is the unit for the real part and i is the unit for the imaginary part. I find this latter way of writing the complex number very reassuring because it displays the symmetry that I was looking for. In the interests of cutting through the clutter though, the (1) is dropped by convention, and we write $a + bi$.

Multiplication of complex numbers

When we multiply complex numbers, we are really performing multiplication on *two pairs* of real numbers with the imaginary unit sandwiched in between. Because of the property $i^2 = -1$, the rules of multiplication with terms involving i must be modified to honour this equation. Let us see how, with a step-by-step example:

$$\begin{aligned}
(2 + 3i)(4 - 5i) &= 2(4) + 2(-5i) + (3i)(4) + (3i)(-5i) \\
&= 8 - 10i + 12i + (-15)(i^2) \\
&= 8 + 2i + (-15)(-1) \\
&= 23 + 2i.
\end{aligned}$$

Note that we use $i^2 = -1$ and group like terms, but otherwise proceed as normal. If we were to generalize this to a pair of complex numbers, we get

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Although the rules of multiplication of complex numbers differ from those of real numbers, the *field axioms* still hold. This is the purpose behind the development of abstract algebra. The principle is DRY (Don't Repeat Yourself) or, equivalently, do not re-invent the wheel.

Polar form of complex numbers

We have resorted to the unit circle to unravel the meaning of the tangent ratio and function in a previous blog.

We now take recourse to the same unit circle to better understand the multiplication of complex numbers. Let us press the xy coordinate plane to represent complex numbers of the form $a + ib$ where the real part a is the x coordinate and the imaginary part ib is the y coordinate. This use of the coordinate plane is referred to as an Argand diagram. The x axis is labelled $\Re$ to represent the real axis and the y axis is labelled $\Im$ to represent the real coefficient of the imaginary part, shown in Figure 8.

The formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{8}$$

is called the Euler Formula. This particular equation has been called "the unification of algebra and geometry" by the legendary physicist, Richard Feynman [3]. Do read the transcript of his lecture to better appreciate what exactly he means by the above quote.
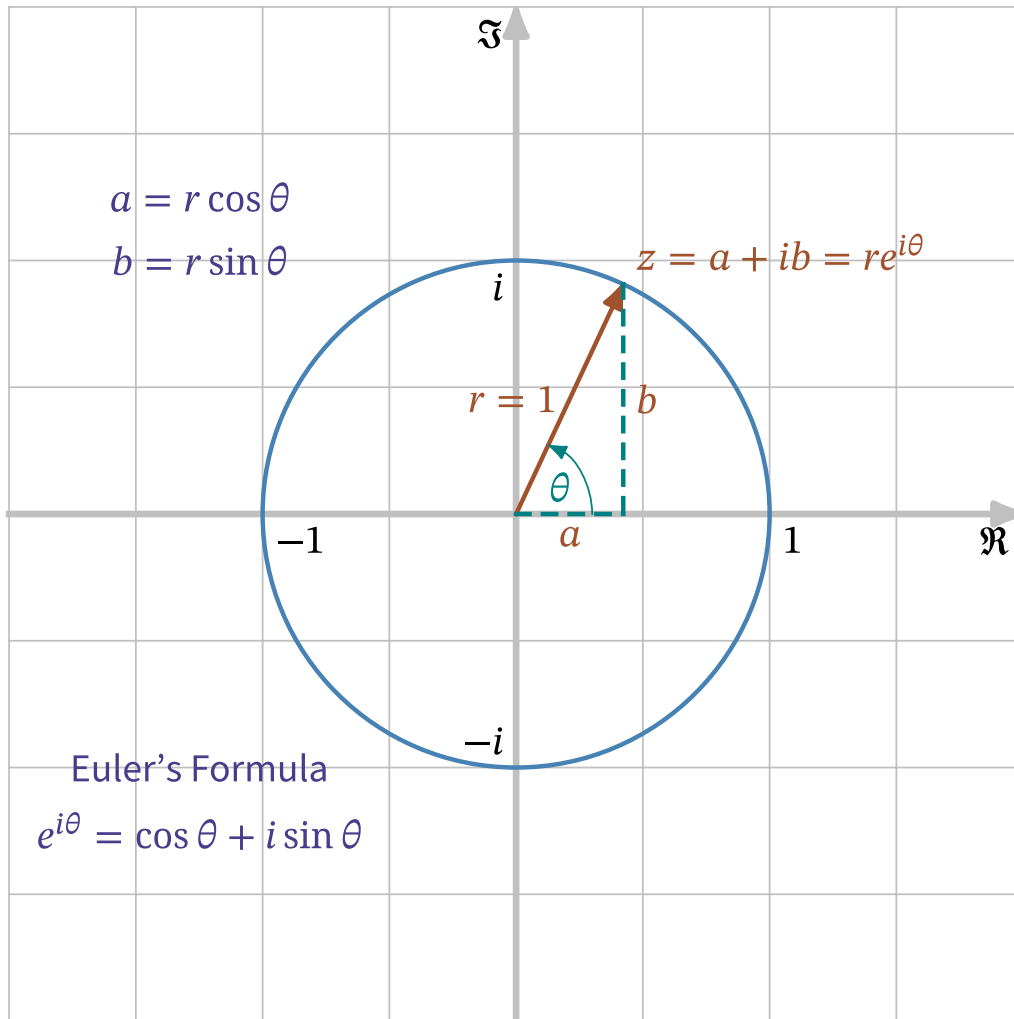
Figure 8: The complex plane with an arbitrary point $z = a + ib$, represented by the point (a, ib) on the unit circle. The same number may also be expressed as $re^{i\theta}$ where $r = 1$.

Referring to Figure 8, we may say:

$$a = r\cos\theta$$

$$ib = ir\sin\theta$$

$$r = \sqrt{a^2 + b^2} \text{ since } \cos^2\theta + \sin^2\theta = 1$$

r is called the *modulus* and θ the *argument* form of the complex number. Alternatively, it may be called the *polar* representation of the complex number.

What exactly is the advantage of all this jiggery-pokery? It makes multiplication easier because exponents are added when the numbers they represent are multiplied.
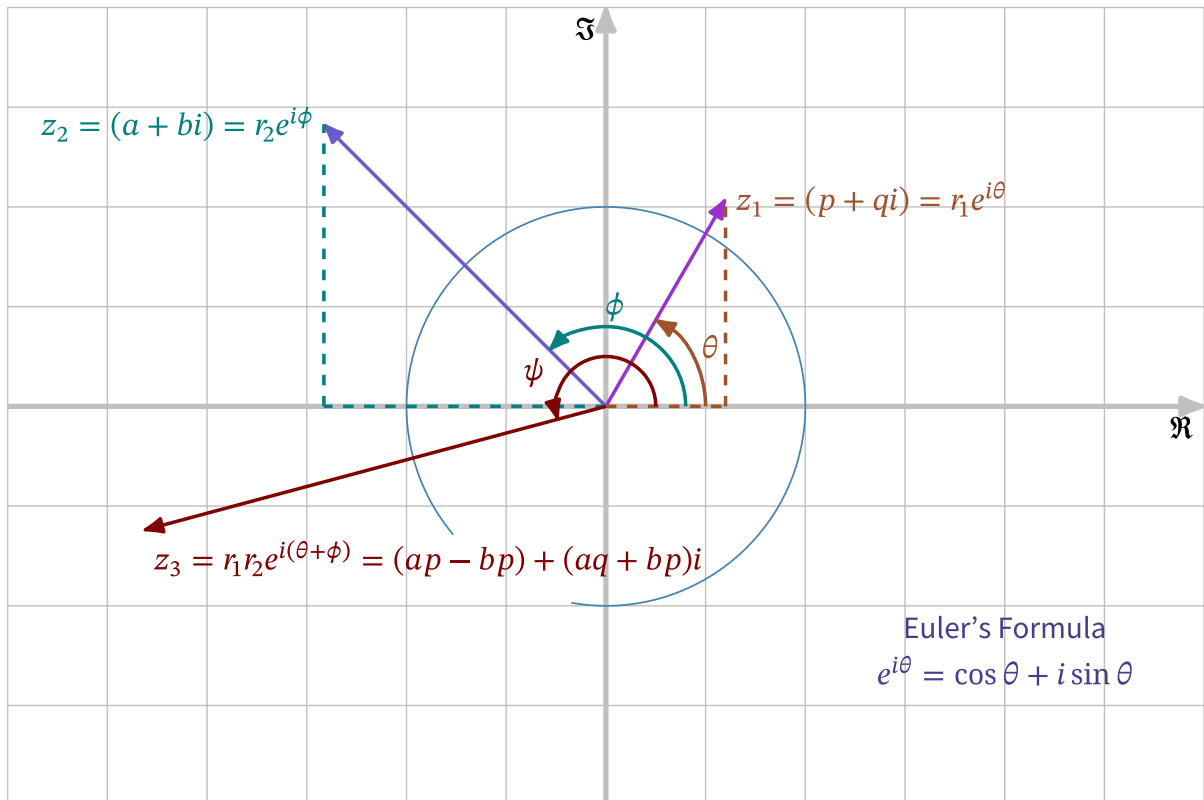

Figure 9: The multiplication of two complex numbers is more easily performed when they are expressed in polar form. The x coordinate is the real part and the y coordinate the imaginary part.

If we have two complex numbers $z_1 = r_1 e^{i\theta}$ and $z_2 = r_2 e^{i\phi}$, the modulus of their product is simply the product of their moduli, and the argument is simply the sum of their arguments, as illustrated in **??**. This means:

$$
\begin{aligned}
z_1 z_2 &= r_1 e^{i\theta} \cdot r_2 e^{i\phi} \\
&= r_3 e^{i\psi} \text{ where} \\
r_3 &= r_1 r_2 \text{ and} \\
\psi &= \theta + \phi.
\end{aligned}
\tag{9}
$$

Consistency between real and complex multiplication

What happens if we used the rule for complex multiplication above but set the imaginary parts to zero so that complex multiplication reverts to the multiplication of two real numbers? Do we get consistent results? Let us try it by substituting $b = d = 0$ in the above case. We then have:

$$(a + 0i)(c + 0i) = [ac - (0)(0)] + [a(0) + (0)c]i$$
$$= ac.$$

It is a great relief that the result is as expected. The multiplication of two complex numbers gives us the same result as the earlier definition for multiplication of two real numbers when we set to zero the imaginary parts of the two complex numbers. The definitions of real and complex multiplication are therefore consistent.

It is a primary requirement in mathematics that when we extend the definition of an operation on a simpler object to encompass a more complicated mathematical object, the new definition should revert to the accepted definition for the simpler object when the complicated object reverts into the simpler object. This is the point about consistency that I made at the start of this blog.

Complex numbers as ordered pairs

Arithmetic operations on complex numbers result only in complex numbers and do not give rise to new types of numbers. And all complex numbers consist of two parts: a real part and an imaginary part.

Hence, if we develop a new notation for this two-part number for purposes of arithmetic, we may dispense altogether with the symbol i, which after all, seems somewhat out of place in a numerical context.

The simplest notation is to represent complex numbers as *ordered pairs* of real numbers with the convention that the first number is the real part and the second number is the imaginary part. Once we have done this, we need to re-define all the arithmetic operations for these ordered pairs.

We may thus use the ordered pair (a, b) to denote the complex number $a + bi$ and likewise, (c, d) for the complex number $c + di$.

Drawing upon our previous results, we may then *define* multiplication for this ordered pair as being

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

Note that order matters here: the first number of the ordered pair is the real part of the complex number and the second is the imaginary part. We cannot swap them willy nilly. TO REMOVE

$$\begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Vectors

Ordered pairs lead rather neatly to the idea of [*vectors*.][https://en.wikipedia.org/wiki/Euclidean_vector][6] Indeed, there is more than a passing resemblance between complex numbers and two-dimensional vectors.

Both may be represented by ordered pairs of real numbers and the rules for addition and subtraction of these ordered pairs are identical. Moreover, they may both be represented as points on a Cartesian plane. Vectors are the second new mathematical object, after complex numbers, that we are meeting in this blog.
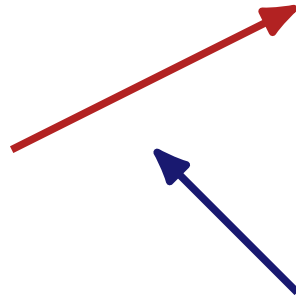

Figure 10: Two vectors shown as directed line segments with different lengths and directions.

A *vector* is traditionally defined as a quantity having two attributes: *magnitude* and *direction.* A simple everyday example is the wind which has both a speed and direction, and may therefore be represented by a vector. Indeed, if you have already encountered vectors, you might think of them exclusively as *directed line segments* or lines of specific lengths with arrow tips as shown in Figure 10.

Addition and subtraction of vectors: geometric viewpoint

How do we add and subtract these geometric entities, let alone multiply and divide them? If you have done physics at high school, you will know that we use something called the *parallelogram law.*

We draw a pair of two-dimensional vectors so that both originate from the same point. We then complete the parallelogram formed by these two vectors by drawing in the other two sides. The *sum* of the two vectors, or their *resultant,* is the diagonal of the parallelogram that starts at the same origin as the two vectors. This is something best grasped from a picture: see Figure 11.

The origin of the Cartesian plane is labelled *O.* We have three named vectors and three ordered pairs indicate the positions of their arrow tips:

1. **u** : the vector from *O* to the point $(8, 4)$;
2. **v** : the vector from *O* to the point $(2, 6)$; and
3. **w** : the vector from *O* to the point $(10, 10)$.

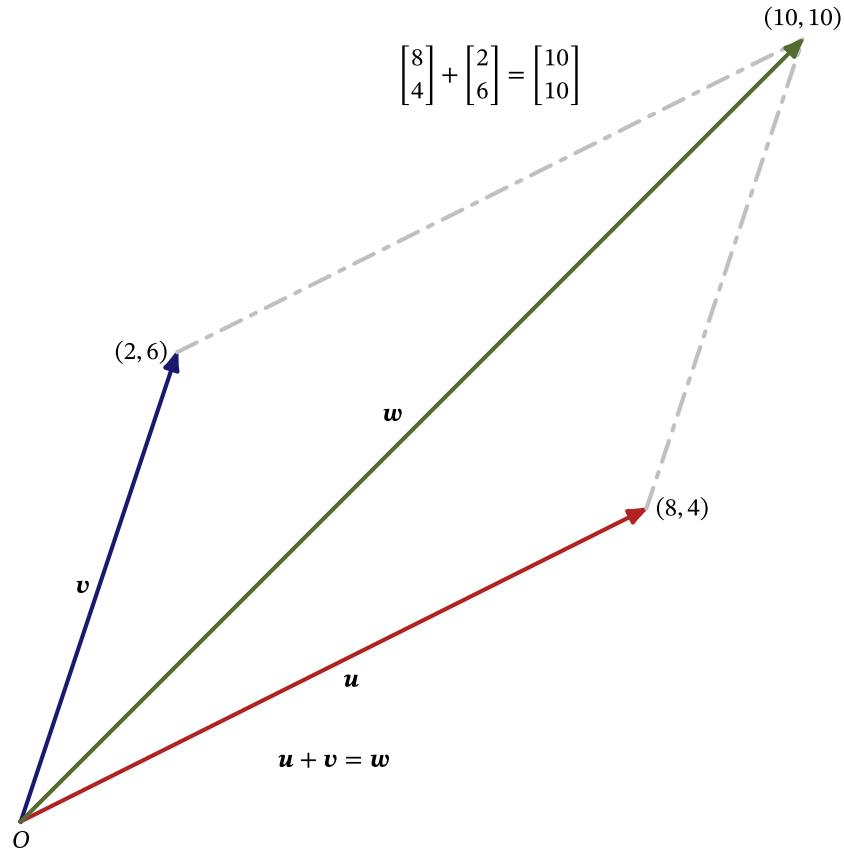[6]Properly called Euclidean vectors in our context.

Figure 11: The parallelogram law for the addition of two vectors.

The dotted grey lines indicate the two sides of the parallelogram that we draw to close the figure. The vector **w** is the diagonal that represents the sum of **u** and **v**. This sum is written as

$$\mathbf{u} + \mathbf{v} = \mathbf{w}$$

If you look at the illustration carefully, you would note that if we added the first co-ordinate of **u** and the first co-ordinate of **v** we get the first co-ordinate of **w**. And likewise for the second co-ordinate.

So, we may represent the addition of **u** and **v** as *row* vectors, so

$$(8, 4) + (2, 6) = ((8 + 2), (4 + 6)) = (10, 10)$$

Likewise, we could treat the vectors as *column* vectors, and write, equally correctly, as:

$$\begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

This is no accident. Nor is it a special case. It works for all parallelograms.[7]

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[7]I leave it to you to convince yourself of this. (Hint: use graph paper, draw the co-ordinate axes, and use algebraic variables for the co-ordinates of **u**, **v**, and **w**.)

The parallelogram law is a geometric statement of what happens when we add two ordered pairs the way we would two complex numbers:

$$(a, b) + (c, d) = (a + c, b + d)$$

We may therefore dispense with line segments and arrow tips and abstract vectors as ordered pairs even as we rescued complex numbers from explicit association with *i*.

Addition and subtraction of vectors: algebraic viewpoint

We may identify two-dimensional vectors *uniquely* by an ordered pair representing their co-ordinates on the Cartesian plane. This is the algebraic viewpoint. It is less cumbersome and more powerful as we have already seen from the addition of two vectors.

Subtraction is equally simple. We may add the *additive inverse* of each component of the vector being subtracted to get:

$$(8, 4) + (-(2), -(6)) = (8, 4) + (-2, -6) = (6, -2)$$

Geometrically, this is tantamount to reversing the second vector and adding it to the first. Of course, plain old subtraction also works, provided we allow for negative numbers:

$$(8, 4) - (2, 6) = (6, -2)$$

Both results are the same and again underline the consistency that runs through mathematics like a golden thread. 😁

Vectors as algebraic entities

We have just seen that two-dimensional vectors may be represented by ordered pairs on the Cartesian plane. This representation might be extrapolated to include vectors of dimensions greater than two. Obviously, we would then be moving from ordered pairs to ordered triples, etc.[8]

To generalize, we may think of vectors as a list of "numbers in a slim teabag" where their order matters. Formally, an *n*-dimensional vector is an *ordered list* of *n* numbers written within enclosing parentheses or brackets and treated as a *single entity.* This is also called an *ordered n-tuple.* Each individual number is called a *component* of the vector.

---

[8]An ordered triple, for example, would live in our familiar three-dimensional space.

The components of a vector may be written within parentheses or brackets. They may be arranged vertically as a *column vector* of a single column and n rows or horizontally as a *row vector* of a single row and n columns. A column vector may be *transposed* into a row vector. Transposition is indicated by the superscript T. For example, with $n = 4$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}^T = \begin{bmatrix} a & b & c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b & c & d \end{bmatrix}^T = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Note that there are no commas separating the elements of a row vector unlike in ordered pairs. Also, we could just as well have used parentheses as brackets.

It is conventional to assume that an arbitrary vector is a column vector. Row vectors are then the transposes of the column vectors. This is the convention we will follow.

Row-column nomenclature

The size of a vector is denoted by writing down the number of rows followed by a $\times$ sign followed by the number of columns. Thus a column vector with four rows is a 4×1 *column vector* while a row vector with four columns is a 1×4 *row vector*.

By definition, a vector must have at least one dimension equal to 1. When *both* dimensions are equal to one, the vector degenerates into a single component which, in the context of vectors, is called a *scalar* to distinguish it from a vector.

Addition and subtraction of vectors

Addition and subtraction for the ordered n-tuples representing two vectors may be defined as the addition or subtraction of their *respective components.*

Just to free ourselves from geometrical thinking about vectors, let us add two *four-dimensional* vectors whose components are given by algebraic variables representing real numbers.

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} + \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} a + p \\ b + q \\ c + r \\ d + s \end{bmatrix}$$

Subtraction is equally "commonsensical":

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = \begin{bmatrix} a - p \\ b - q \\ c - r \\ d - s \end{bmatrix}$$

These vector sums and differences would be difficult to visualize geometrically, but they are trivially routine algebraically.

Multiplication of vectors

It is easy to think of the addition or subtraction of vectors, say in the context of "wind speed" and "air speed" of an aircraft. But what does the multiplication of vectors consist of and what meaning could we extract from the operation?

Vector multiplication is a strange, many-headed beast. It is important to know what it is and what it is not. Here is a quick run down:

1. Vector multiplication is different from real number multiplication.

2. Vector multiplication is different from complex number multiplication.

3. There are several varieties of vector multiplication, some of which give us scalars, others vectors, and still others matrices:[9]

    (a) multiplication of a vector by a scalar to yield a vector
    (b) dot product or scalar product or inner product of two vectors to yield a scalar
    (c) cross product of two vectors to yield a third vector orthogonal to the other two
    (d) tensor product or outer product of two vectors to yield a matrix

4. Each variety of vector product was devised as an operation because it is useful and has a ready meaning in a particular context.

5. When it comes to multiplication, vectors reveal their nature as a class of mathematical object quite different from real or complex numbers.

Let us consider each type of multiplication in turn.

Multiplication by a scalar

Multiplication of a vector by a scalar is the easiest to understand. In this operation, we see the original arithmetic definition of real multiplication at play. We are magnifying or diminishing the magnitude of the vector by multiplying it with a scalar, while the direction of the vector is either reversed or remains unchanged.

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[9]Ignore any unfamiliar terms for now.

If we have a vector **u** and we multiply it by a scalar k the result is the vector $k\mathbf{u}$. This may be easily visualized geometrically. If we associate **u** with an arbitrary four-dimensional column vector, we may write

$$\mathbf{u} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Multiplication of **u** by a real scalar k gives us

$$k\mathbf{u} = k\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} ka \\ kb \\ kc \\ kd \end{bmatrix}$$

When we multiply a vector by a scalar k, each component of the vector is individually multiplied by k. The magnitude of the resulting vector takes on a different meanings depending on the value and sign of k:

1. $k = 1$: the vector is unchanged in magnitude and direction.

2. $k = -1$: the vector is unchanged in magnitude but reversed in direction.

3. $k < -1$: the vector is enlarged in magnitude and reversed in direction.

4. $k > 1$: the vector is enlarged in magnitude and unchanged in direction.

5. $-1 < k < 0$: the vector is diminished in magnitude and reversed in direction.

6. $0 < k < 1$: the vector is diminished in magnitude and unchanged in direction.

7. $k = 0$: the vector has zero magnitude and its direction is undefined.

While this might seem quite a mouthful, it is really quite simple:

- a negative k reverses the direction;
- a positive k keeps the direction unchanged;
- value of k that lies between -1 and 0 or between 0 and 1 diminishes the magnitude; and
- a value of k less than -1 or greater than $+1$ increases the magnitude of the vector.

The special cases pertaining to $k = \pm 1$ and $k = 0$ underscore the paramount importance of 0 and 1 in the whole of mathematics. They are keystone numbers.

Did you pick up the fact that after uncoupling geometry and vectors, we finally resorted to geometry when talking about the meaning of scalar multiplication? This dual viewpoint runs through much of mathematical thinking.

Scalar division of vectors by $k \neq 0$ is really multiplication by $\frac{1}{k}$ and is therefore not considered separately.

If we view scalar multiplication as a black box, it takes in one n-tuple and gives out another n-tuple. Like the merchant in *Aladdin and the Wonderful Lamp,* we are simply exchanging old vectors for new. There is no difference in *kind* between the input and output mathematical objects.

The zero vector is not the number zero

Distinguish carefully between the *real number* 0 and the *n-dimensional zero vector* which results from scalar multiplication with $k = 0$. The latter is an n-tuple of zeros and does *not* equal the real number zero. For example, with $n = 4$, the zero vector is:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \neq 0$$

As and when new mathematical objects are invented (or discovered?) new definitions for the equivalents of zero and one for these new objects may also be necessary. The n-dimensional zero vector is unique as shown above.

The four-dimensional column vector with all entries equal to 1 exists and is, of course, unique:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \neq 1$$

But it is not special enough to merit its own name, as it does not function as the vector analogue to the number 1. But it does have its uses in data analysis and might get its own name in the future! [4]

We now turn to other varieties of multiplication that may be applied to vectors.

Dot or scalar product

The centred dot · as a symbol for multiplication makes its appearance here. The first departure from conventional multiplication was with complex numbers. The *dot product,* also called the *scalar product,* or *inner product*[10] is the next variety of unconventional multiplication.

Existence of the dot product

The dot product is defined only between vectors of the same dimension. This is important to grasp. When we deal with real numbers, the multiplicand, multiplier, and product are all real numbers. They are mathematical objects of the *same* kind. So, we may afford to be a little careless in multiplying them together without performing any check.

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[10]An inner product is something more general, of which the dot product is a special case.

We cannot afford to be equally lackadaisical with vectors. We have to respect the fact that they are not numbers per se, but a different type of mathematical object. A product of some sort might not exist between any two arbitrary vectors.

Example of dot product

It is helpful to begin with a concrete example. Let $\mathbf{u}^T = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ and let $\mathbf{v}^T = \begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix}$. Their dot product is written as:

$$
\begin{aligned}
\mathbf{u}^T \cdot \mathbf{v} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \\
= (1)(5) + (2)(6) + (3)(7) + (4)(8) \\
= 5 + 12 + 21 + 32 \\
= 70
\end{aligned}
$$

If you look at the dot product carefully, you will see the following:

1. the first component of $\mathbf{u}$ is multiplied by the first component of $\mathbf{v}$ and likewise for the other components;

2. the individual products are then added together; and

3. the sum is the dot or scalar product.

It is now clear why the two vectors must have the same dimensions. If not, we will run out of either multiplicand or multiplier for pairwise multiplication.

The result, being a sum of products, is a single number, or scalar, explaining the name *scalar product* for this operation. We prefer the term dot product to avoid confusion with multiplication by a scalar.

It is easy to verify by direct evaluation that the dot product is commutative and therefore symmetrical. Verify if you please that $\mathbf{u}^T \cdot \mathbf{v} = \mathbf{v}^T \cdot \mathbf{u}$ for the above case.

Why did we need to write the dot product as being between a row vector and a column vector? One reason is that the product of a column vector with a row vector is actually a different type of multiplication which we will meet later. Another reason is that the row-column product mirrors matrix multiplication as explained later.

General case and formula

These results for the dot product may be generalized by taking $\mathbf{u}$ and $\mathbf{v}$ to be arbitrary n-dimensional vectors whose components may be referred to by the *subscripts* 1 to n.

It is conventional to write the vector itself in boldface as **u** or with an arrow on top as $\vec{u}$ (or using some other mark of distinction when written by hand or printed) whereas the individual components are always written normally. They are after all scalars.

$$\mathbf{u}^T = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\mathbf{u}^T \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$= \sum_{i=1}^{n} u_i v_i$$

Observe that the vector $\mathbf{u}^T$ is a $1 \times n$ vector whereas **v** is an $n \times 1$ vector. Their dot product—between a $1 \times n$ vector and an $n \times 1$ vector—yields a 1×1 "vector" which is really a scalar. In a manner of speaking, we may "cancel out" the two inner dimensions n to get the dimension of the product as being 1×1. This mnemonic will prove useful later on as well.

Consistency with real multiplication

What happens if our two vectors degenerate into scalars having single components u_1 and v_1? The dot product then collapses into plain multiplication between two numbers and equals $u_1 v_1$, entirely consistent with the product of two numbers. Consistency rules once again!

Geometric viewpoint

What does the dot product mean? What does it signify given that vectors originated as physical abstractions? We need to put on our "geometric glasses" and view the dot product geometrically. We will need a little bit of trigonometry on the way.[11]

[caption id="attachment_83" align="alignleft" width="227"] Vector components as values of projections on orthogonal axes.[/caption]

Let us consider a two-dimensional vector **u** from the origin O to a point U at (u_x, u_y) on the Cartesian plane. Let OU make an angle α with the positive x axis as shown. Then, we have:

1. By the Theorem of Pythagoras, the magnitude of the vector **u**, denoted by $\|\mathbf{u}\|$, is given by $\sqrt{u_x^2 + u_y^2}$. The symbol $\|\ \|$ denoting a pair of double vertical lines represents the norm or magnitude of the vector written within it.

2. The magnitudes of the projections of **u** in the directions of the x and y axes are respectively

$$u_x = \|\mathbf{u}\| \cos \alpha$$
$$u_y = \|\mathbf{u}\| \sin \alpha$$
$$= \|\mathbf{u}\| \cos(90° - \alpha)$$

---

[11]Which you might have to take on trust if it is unfamiliar or forgotten.

The magnitude of the projection of a vector in a particular direction is equal the magnitude of the vector multiplied by the *cosine* of the angle made by the vector with that direction.

We could make similar claims for a vector $\mathbf{v}$ at (v_x, v_y) that makes an angle β with the positive x axis.

Let the angle between the two vectors be denoted by $\alpha - \beta = \theta$. The dot product of the two vectors may then be written as:

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= u_x v_x + u_y v_y \\
&= \|\mathbf{u}\| \cos\alpha \|\mathbf{v}\| \cos\beta + \|\mathbf{u}\| \sin\alpha \|\mathbf{v}\| \sin\beta \\
&= \|\mathbf{u}\|\|\mathbf{v}\|(\cos\alpha \cos\beta + \sin\alpha \sin\beta) \\
&= \|\mathbf{u}\|\|\mathbf{v}\| \cos(\alpha - \beta) \\
&= \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta
\end{aligned}$$

[caption id="attachment_79" align="alignright" width="230"] Dot product of two vectors.[/caption]

The dot product of two vectors is equal to the product of their magnitudes and the cosine of the angle between them. It is a scalar.

Applications of the dot product

Unlike straightforward multiplication of real or complex numbers, the dot product seems a little contrived. Why is it so defined? And is it useful?

The answers to both questions lie in the practical utility of the dot product. Vectors are used to represent forces, displacements, momenta, and a whole host of other abstractions that are the bread and butter of physics. And the dot product neatly dovetails with a recurring pattern of relationships in physics where two vectors give rise to a scalar in a multiplicative fashion.

For example, mechanical work W is a scalar defined as the dot product of the vector representing force $\mathbf{F}$ and the vector representing displacement $\mathbf{s}$ through the equation $W = \mathbf{F} \cdot \mathbf{s}$. Alternatively, work is the projection of the force in the direction of the displacement, multiplied by the displacement. Both definitions are equivalent.

The dot product is succinct, precise, notationally crisp, and practically useful. That is why it has been defined and that is why it still exists.

The cosine and sine functions

The cosine and sine of an angle are trigonometric ratios from right-angled triangles that were later expanded in scope to become mathematical functions. Graphs of $\cos x$ and $\sin x$ against x in degrees are shown below.[12] If these are concepts new to you, you might want to look them up as suggested in the sidebar.

For now, we only need to focus on these facts:

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[12]These functions are usually plotted against x in the "unitless unit" radians. I am using degrees here because the reader might be unfamiliar with radians.

1. The values of $\cos x$ and $\sin x$ lie only between -1 and 1.

2. $\cos 0° = 1$ and $\cos 90° = 0$.

3. $\sin 0° = 0$ and $\sin 90° = 1$.

4. $\sin x = \cos(90° - x)$.

5. $\cos x = \sin(90° - x)$.

[caption id="attachment_86" align="aligncenter" width="482"] Graphs of cos x and sin x.[/caption]

Oddness and evenness

Observe from these graphs that $\cos x$ is an even function that is symmetrical about the y axis whereas $\sin x$ is an odd function.[13]

Although not apparent from the graphs, both the cosine and sine functions are periodic and repeat themselves every 360° for all values of the independent variable.

Orthogonality

To recapitulate, the cosine of an angle is 1 at zero degrees and 0 at ninety degrees. Therefore if two vectors are orthogonal—or perpendicular or at ninety degrees—to each other, their dot product is zero.

For example, a force vector **F** has no effect perpendicular to the direction in which it acts, and this is because its *component* or projection in that direction is $\|\mathbf{F}\| \cos 90° = 0$.

The dot product therefore measures the *degree of alignment* or *similarity* between two vectors. When the angle between them is zero degrees, this alignment is at its greatest. When the vectors are orthogonal, each vector has no component in the direction of the other; so they are *independent*. When the two vectors make an angle greater than 90° the sine of their angle is negative. The two vectors act in opposition when they are at an angle of 180° with each other.

Orthogonality—and the independence of vectors it implies—is a very powerful property that finds application daily whenever we talk over the telephone or download a compressed image from the Web.

The idea of projecting some mathematical object onto another and the idea of one mathematical object being orthogonal to another are both fundamental to many areas of mathematics and are well worth keeping in mind.

We now move on to the next type of vector product.

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[13]These properties make the dot product a commutative operation and the cross product an anti-commutative operation.

Cross product

The third type of vector product is the *cross product.* Because the dot product gave a *scalar* result that involved the *cosine* function, you might ask tongue in cheek, whether the cross product produces a *vector* result that involves the *sine* function in its definition. And facetious or not, you are actually right. :-)

The cross product is a vector and it does involve the sine of the angle between the two vectors. In addition, just as in the dot product, orthogonality peeps at us again through the cross product.

We will consider three-dimensional vectors. Any pair of three-dimensional vectors **u** and **v** between them define a two-dimensional plane. Just think of two rulers arranged in any orientation on a flat table to visualize and understand why.

The result of a cross product is orthogonal to the two vectors giving rise to it. There are *two* directions orthogonal to the plane. Think of the flat table again. An arrow at right angles to the table coming *out* of it and pointing *upwards* is in one direction. Now reverse the direction of the arrow so that it goes *into* the table pointing *downwards.* This is the other orthogonal direction. They both lie along the same straight line but are oriented in opposite directions.

We are now ready to define the cross product as

$$\mathbf{w} = \mathbf{u} \times \mathbf{v}$$
$$= (\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta)\,\mathbf{n}$$

[caption id="attachment_78" align="alignright" width="220"] Cross-product of two vectors.[/caption]

Both the vectors **u** and **v** point outward from the same origin *O*. The cross product vector **w** is perpendicular or orthogonal to *both* **u** and **v** and again points outward from the same origin *O*.

The expression $(\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta)$ is a scalar. **n** is a *unit vector* perpendicular to both **u** and **v** with a magnitude of one. Its direction can only be one of two as we have seen. To determine which, we use a convention called the right-hand rule or right hand corkscrew rule.

Imagine that you are rotating a corkscrew starting at **u** and moving toward **v**. The direction in which the corkscrew advances is the positive direction for the unit vector **n**.[14]

Since the corkscrew would then move *upwards,* that is the direction of both **w** and **n**. The sole purpose of **n** is to indicate the direction of **w** as determined by the right-hand rule. The letter **n** is used to indicate that it is *normal* or *perpendicular* to the plane.

In the cross product, we have just met the × sign for multiplication again, but so far afield from its original use and meaning that it is almost unrecognizable except for form. Many mathematical terms and symbols are reused in different contexts with completely different meanings.

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[14]This is a convenient mathematical convention which is also in accord with actual physical situations.

Anti-commutativity

If we were to compute $\mathbf{v} \times \mathbf{u}$ we would turn the right-handed corkscrew from $\mathbf{v}$ to $\mathbf{u}$ and the cross-product vector would then point *downwards.* Its magnitude would however be the same as before. We therefore write:

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$$

The cross product is said to be anti-commutative. This means that if we reverse the order of the operands, there is a change in the sign of the result. In contrast, the dot product is commutative. So, each variety of vector multiplication has its own well-delineated properties.

Applications of the cross product

Like the dot product, the cross product owes its definition and existence to its usefulness in physics. For example, the torque vector τ[15] is defined by $\tau = \mathbf{r} \times \mathbf{F}$. Torque is what makes the wheels of a car rotate. Visit the torque hyperlink to see an animation and explanation of what the term "torque" means.

Outer product of two vectors

The outer product is the last of the four varieties of multiplication for vectors that we will consider here.

Recall that the dot product is defined as the scalar resulting from the multiplication of a $1 \times n$ row vector with an $n \times 1$ column vector. The result was a 1×1 "vector" which is really a scalar. The two "inner dimensions" n cancel out.

What happens if we swap the order and start multiplying an $n \times 1$ column vector with a $1 \times n$ row vector? Would the two "inner dimensions," both equal to 1, cancel out? And would we get an $n \times n$ "vector" of some sort?

Indeed we do. And the resulting product is a different mathematical object called a matrix. This is an example of a mathematical operation involving two known mathematical objects whose result gives rise to a new kind of mathematical object which then acquires a life and personality of its own.

This type of multiplication is called an outer product, in contrast to the dot product which is a type of inner product. It is also sometimes called a tensor product in honour of the fact that we are ascending a hierarchy in linear algebra that starts with scalars and moves on to vectors and then to matrices and on to tensors with progressive generalizations at each step.

Outer product: symbol and example

The symbol for the outer product is $\otimes$ which is a "circled times" sign.

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[15]Pronounced tau.

As already presaged, the outer product results from the multiplication of an $m \times 1$ column vector and a $1 \times n$ row vector to give an $m \times n$ matrix. As with the dot product, the "inner dimensions" of the two vectors, both equal to one here, "cancel out" in a manner of speaking, to yield a matrix of dimension $m \times n$.

In contrast to the dot product, however, the two vectors may have different numbers of elements. This is why the resulting matrix is not necessarily a square matrix with equal numbers of rows and columns, but rather has m rows and n columns. The values of m and n may be equal but are not required to be so.

Here is an example that will help you decipher how the outer product is computed:

$$
\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \otimes \begin{bmatrix} 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 \\ 10 & 12 & 14 \\ 15 & 18 & 21 \\ 20 & 24 & 28 \end{bmatrix}
$$

Each element of the outer product matrix is the product of a pair of real numbers. What is new here is their position in the product matrix and their meaning in this context.

The outer product is non-commutative

The outer product is not commutative. To see why, consider $\mathbf{u}$ as an $m \times 1$ column vector and $\mathbf{v}^T$ as a $1 \times n$ row vector. Then, $\mathbf{u} \otimes \mathbf{v}^T$ gives an $m \times n$ matrix. But $\mathbf{v} \otimes \mathbf{u}^T$ gives an $n \times m$ matrix. The two are obviously not constrained to be equal.

Applications of the outer product

The outer product finds application in fields like physics, electrical engineering, and statistics. Whether application precedes or follows the original mathematical development, whenever a new mathematical object persists, it is almost always due to its usefulness for some purpose or other.

The outer product of two vectors leads to matrices and the *multiplication of matrices* is yet another variety of multiplication. It is the last we will consider in this blog.

Matrices

We have already seen that a matrix consists of a lot more "numbers in a teabag" in which order is respected. An $m \times n$ matrix is an array of m rows and n columns of numbers. A row vector is just a degenerate matrix with one row, i.e., $m = 1$, whereas a column vector is a degenerate matrix with a single column, i.e., $n = 1$. The plural of *matrix* is *matrices.*

Hark back to complex numbers and remember how the real numbers are merely complex numbers whose imaginary parts are zero. We hear a similar refrain with matrices, vectors, and scalars. Vectors are matrices with one column or one row. A matrix with a single column and row is a scalar.

Each time a mathematical object is generalized, we will see a previously defined object appearing as a degenerate case of the new object. This provides a link between the new and the old and also ensures that consistency is maintained in this evolutionary spiral.

It is customary to refer to a matrix by an uppercase letter. The individual numbers, or *elements*, of a matrix are usually denoted by a lowercase letter and given double subscripts denoting their position in the matrix.

The element a_{ij} in a matrix A is the element occupying the ith row and the jth column in the matrix. For the subscript, the row number is written first followed by the column number. If all this seems too abstract, here is a concrete example of a 2×3 matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

where we have $a_{12} = 3$ and $a_{21} = 2$.

Applications of matrices

Matrices arose naturally from the study and solution of systems of linear equations. They are also useful in succinctly embodying geometric transformations of points in the two-dimensional Cartesian plane. They are profoundly useful in electrical engineering, physics, economics, and many other fields.

Indeed, if one considers matrices as a class of mathematical object, what we do with them and the meanings we assign to these actions are largely limited only by our imagination and the mathematical consistency of the results. This is how new mathematics is built up from the old, and constantly expanded in scope, variety, and application.

Matrix multiplication

The product of matrix A with matrix B is denoted by AB with no intervening symbol. If matrix A has dimensions $m \times n$, and matrix B has dimensions $p \times q$, and we wish to multiply them *in that order,* we first need to ensure that the "inner dimensions," n and p in this case, are indeed equal. If $n = p$ the two matrices A and B can be multiplied together to yield the $m \times q$ matrix AB, and the two matrices are said to be conformable. Otherwise, the product AB does not exist. Conversely, if $n \neq p$ but $q = m$, the matrix product BA exists and the result is a $p \times n$ matrix.

Any two real numbers may be multiplied together, but the product of any two matrices need not necessarily be defined. As the mathematical objects that we deal with become increasingly complex, additional constraints often apply to operations on them.

Example of matrix multiplication

[caption id="attachment_81" align="alignleft" width="298"] How the product of two matrices is computed.[/caption]

Here is an example of matrix multiplication. We group a whole row on the left matrix and multiply it element-wise with a whole column on the right matrix and add all the products. In this case, we compute $(1)(7) + (2)(9) + (3)(1) = 7 + 18 + 3 = 28$.

This is reminiscent of the dot product. Indeed, matrix multiplication may be viewed as a generalization of the dot product for matrices and the dot product as a degenerate case of matrix multiplication in which the left matrix is a row vector and the right matrix is a column vector.

Non-commutativity

For any two matrices A and B, the matrix product AB exists only if the matrices are conformable, i.e., the number of columns in A equals the number of rows in B. Likewise, the product BA exists only if the number of columns in B equals the number of rows in A.

If both matrices are square and of the same dimensions, is their multiplication commutative? In other words, does $AB = BA$?

In mathematics a single exception falsifies the rule. Let us consider the following simple example:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 23 & 31 \\ 34 & 46 \end{bmatrix} \quad BA = \begin{bmatrix} 19 & 43 \\ 22 & 50 \end{bmatrix}$$

Clearly matrix multiplication is not commutative. In any matrix product, the matrix on the left *pre-multiplies* the matrix on the right. Conversely, the matrix on the right *post-multiplies* the matrix on the left.

Geometric effects of matrix multiplication: 2D case

A 2×2 matrix may be interpreted as a geometric transformation of points on the Cartesian plane. Indeed, this is often how you might have first encountered matrices as mathematical objects. Suppose we wish to reflect an arbitrary point (a, b) using the y axis as a mirror. With a little visualization, you will agree that image point is $(-a, b)$.

How might a matrix accomplish this? If we *post-multiply* a matrix by a vector, we will get another vector. We need to transform a to $-a$ while leaving b unchanged. A little thought or tinkering with matrices will show that required matrix is as shown below:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ b \end{bmatrix}$$

Because a and b are arbitrary, we may associate a reflection across the y axis with the above matrix. Likewise, it may be shown that a counter-clockwise rotation about the origin O through an angle θ with the positive x-axis is associated with the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

We can then chain together such transformations by multiplying the relevant matrices in the correct order. Those who author video games use concepts such as these to program their games.

Pay attention to the interplay between the symbolic and the pictorial, between the algebraic and the geometric aspects of the one operation. If you develop the ability to maintain this "dual vision" as you study mathematics, it will be helpful for your own unfolding understanding. A strange algebraic object correctly used might work geometric miracles right under your nose, and vice versa.

And that completes my survey of varieties of multiplication. I do not know if you are heaving a sigh of relief but I certainly am! We have only scratched the surface here. There are many more varieties of multiplication and each serves a purpose. You will discover them in the course of your studies.

Summary

This blog has been a journey through mathematics tracking multiplication as the single theme.

Multiplication happens between two mathematical objects to yield a third. In this survey, we have encountered *four* different mathematical objects:

1. Real numbers
2. Complex numbers
3. Vectors
4. Matrices

The way the multiplication is accomplished as well as its meaning differ with context. We have met *seven* different varieties of multiplication here:

1. Real multiplication

    - product is real
    - commutative

2. Complex multiplication

    - product is complex
    - commutative

3. Multiplication of a vector by a scalar

    - product is a vector
    - magnitude and direction depend on value of scalar

4. Dot product of two vectors

    - product is a scalar
    - commutative
    - measures "similarity" or "alignment" between the two vectors
    - involves cosine of angle between the two vectors

5. Cross product of two vectors

- product is a third vector orthogonal to the two vectors
- anti-commutative
- involves the sine of the angle between the two vectors

6. Outer product of two vectors

   - product is a matrix
   - not commutative

7. Matrix product

   - product is another matrix
   - not commutative

We have made glancing acquaintance with logarithms and how they transform multiplication into addition. We have also skimmed over the trigonometric functions, given their place in the theory of vectors.

If you carry away nothing else from this blog than a few *qualitative ideas,* they should include some of these:

1. Multiplication is a binary operation: it takes place between two compatible mathematical objects.

2. Mathematical objects are more varied than animals in a zoo. Each has its own nature, diet, habitat etc. Apart from the real numbers, we have encountered complex numbers, vectors, and matrices here.

3. Multiplication is commutative for the real and complex numbers but not for necessarily for vectors or matrices.

4. The meaning of a product has evolved a long way from the original "three lots of four" in the context of whole numbers. The product of a multiplication might yield an object that is quite different from the multiplicand and multiplier. We have seen scalars popping out of dot products of two vectors and matrices issuing from the outer product of two vectors.

5. The ideas of zero and one, of symmetry, of commutativity, of consistency of definitions, of projections, and of orthogonality, are worth remembering because they pervade much if not all of mathematics.

May the product be with you!

To explore further

Nahin's books

Feynman's lectures etc.

Afterword

This blog started off as something that promised to be short and fizzy, tangy and piquant. But it soon became a little like hot treacle: too hot to swallow and too sticky to spit out. It transmogrified into a jumboblog. If you have stuck with me this far, I applaud and thank you.

The thought crossed my mind that I could split this blog into several sub-blogs. But I soon gave up that idea because the connectedness of the thread will be lost in the segmentation. So, here you have the whole hog and the whole blog.

Mathematics is like a pastry puff: only the layers never seem to end and neither does the puff! I needed to cap the well at some point, and matrices seemed as good a place to stop as any. The pleasures of many other types of multiplication await your future explorations! :-)

As an independent scholar, I work in isolation without the benefits of a university environment or consultation with peers. So, an error of fact or fancy is all the more likely in what I write. If you are mathematically inclined, and have spotted any mistakes here, please let me know.

If you wish to read this blog as a single PDF document you may get it from the downloads page.

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