

The Wonder That Is Pi

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This blog began life more than two decades ago, as part of a series of lectures I delivered to very bright first-year engineering students at an Australian university.

The number π (pronounced “pie”) has been recognized from time immemorial because its physical significance can be grasped easily: it is the ratio of the circumference of a circle to its diameter. But who would have thought that such an innocent ratio would exercise such endless fascination because of the complexities enfolded into it?

Not surprisingly, some students I met recently wanted to know more about π . Accordingly, I have substantially recast and refreshed my original presentation to better accord with the form and substance of a blog. The online references have also been updated to keep up with a rapidly changing Web.

If there are any errors or omissions, please [email](#) me your feedback.

Circumference, diameter, and π

The straight line or [geodesic](#) is the shortest distance between any two points on a plane, sphere, or other space. The circle is the [locus](#) traversed by a moving point that is [equidistant](#) from another fixed point on a two-dimensional plane. It is the most [symmetrical](#) figure on the plane. The [diameter](#) is the name given both to any straight line passing through the centre of the circle—intersecting it at two points—as well as to its length. When we divide the [perimeter](#) of circle, more properly called its [circumference](#), C , by its diameter, d , we get the enigmatic constant π , which has a value between 3.141 and 3.142:

$$\frac{C}{d} = \pi. \quad (1)$$

The diameter d is twice the radius r , and substituting for d into Equation (1), we get the well-known school formula:

$$C = \pi d = 2\pi r \approx 2 \left[\frac{22}{7} \right] r \approx 6.28r. \quad (2)$$

Note, however, that π is *not exactly equal* to $\frac{22}{7}$. This value is a convenient *rational fraction approximation* for π that serves well in elementary contexts.¹

You might reasonably wonder whether the ratio of the circumference to the diameter of *any* circle is *always* π . The answer is “Yes”, because *all circles are similar*. The ratios of corresponding lengths of similar figures are equal. This idea is also covered in my blog [“A tale of two measures: degrees and radians”](#).

The symbol π is the lowercase version of the sixteenth letter of the Greek alphabet. For the history of its use in mathematics, see [adoption of the symbol \$\pi\$ in Wikipedia](#).

¹See [“A tale of two measures: degrees and radians”](#).

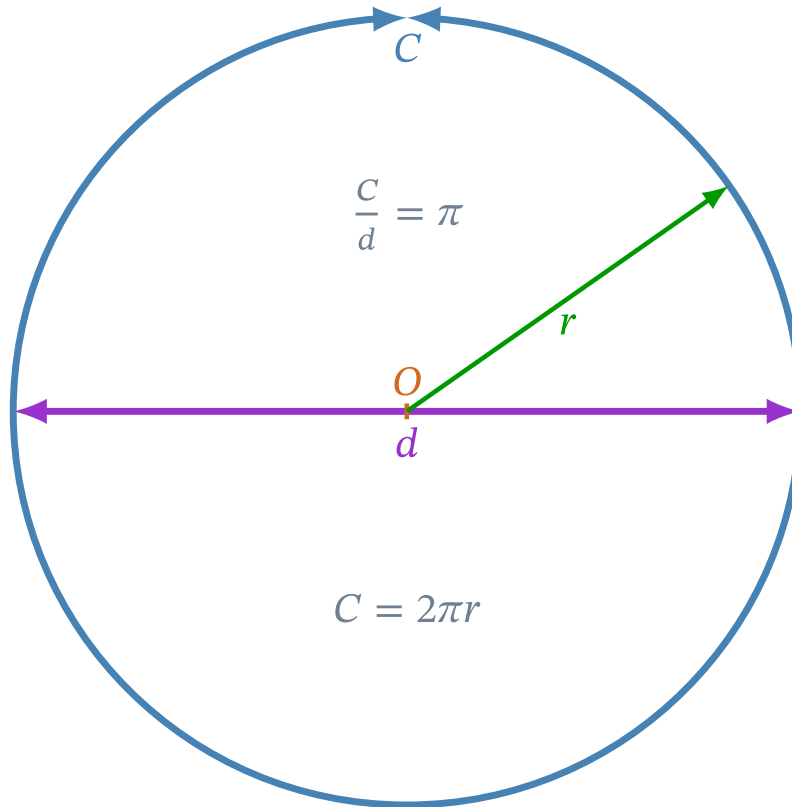


Figure 1: The ratio of the circumference to the diameter of *any* circle is π .

Figure 1 shows the relationships in Equation (1) and Equation (2) pictorially. The circumference of a circle is about 6.28 times its radius. Why this should be so is a secret, a mystery of Nature.

Is π really equal to $22/7$?

Is π really equal to $\frac{22}{7}$, as it has been drummed into our heads at school? A wonderfully revealing story lies behind this mysterious relationship, and it is due to the **labours** of one man, in the days when calculators could not be dreamed of, and when neither the decimal system of numbers nor trigonometry were known. That is the story we look at next.

Archimedes of Syracuse

Archimedes of Syracuse (Ἀρχιμήδης, 287–212 BCE) was a **polymath** and genius of the ancient world. He was one of the greatest mathematicians the world has ever known. By today's standards, he would be called a mathematician, physicist, engineer, and astronomer, **all rolled into one**. He is perhaps most famous for running out of his bathtub naked exclaiming “**Eureka**”—Greek for “I have found it”—oblivious of those around him. The principle that he had then discovered—that the upthrust on a body submerged in a fluid is equal to the weight of fluid displaced—is known as **Archimedes' Principle**.

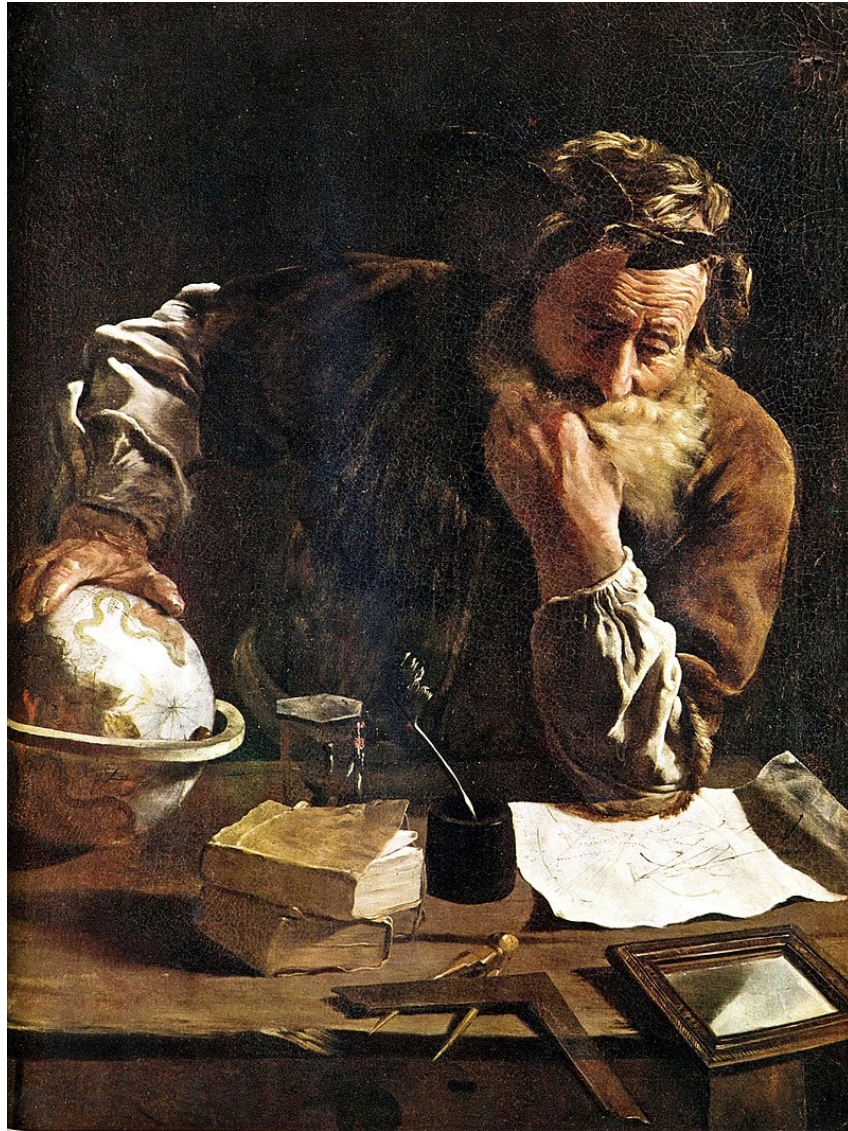


Figure 2: Archimedes of Syracuse.

Among the many accomplishments of Archimedes is his method for estimating π , which was the best approximation for almost 1900 years. What is even more remarkable is that Archimedes made his discovery without the benefit of either trigonometry, decimal (positional) notation, or calculators. He extracted square roots laboriously by hand. His method is also an excellent geometrical illustration of the idea of a limit, with which he was doubtless familiar. It is known that Archimedes was familiar with what we now know as integral calculus, and it is possible that he may have anticipated differential calculus as well.

The algorithm as the foundation

The method that Archimedes used is instructive because it is a synthesis of several principles by which the greatest human minds have furthered scientific progress over time. The abstract principles that Archimedes used were these:

1. Start with the known and progress to the unknown;
2. Devise a method of increasing the accuracy of the estimate and iterate over it;
3. Stop when the desired accuracy is reached.

These three principles constitute what is known as an **algorithm**. Once such a systematic framework has been put in place, it can be applied in many research domains to aid rapid scientific progress.

Of polygons and circles

Archimedes realized that when a circle is inscribed by a regular polygon with n sides and circumscribed by a regular polygon with the same n sides, the circumference of the circle must lie between the perimeter of the inscribed polygon (limit from below) and the circumscribed polygon (limit from above). This visually obvious fact, illustrated in Figure 3, is the first instance of “starting with the known.”

The second application of the same principle was his decision to initiate his algorithm using a regular hexagon, which is a mosaic of six juxtaposed equilateral triangles. We know from symmetry that each angle of an equilateral triangle is 60° . When an equilateral triangle is bisected, we get two right angled triangles with angles of thirty and sixty degrees, as shown in Figure 4.

The inscribed hexagon, within a circle of radius one unit, also has a side of one unit. Thus, the hypotenuse of the circle OAP in Figure 4 has unit length. Moreover, the base OP , resulting from a bisected side, has a length of half a unit. By applying the theorem of Pythagoras, the third side, AP is

$$\sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}}.$$

Archimedes started with regular hexagons and successively doubled the number of sides until he had the circle sandwiched between two 96-gons. He repeatedly calculated rational approximations to π until he was satisfied with the accuracy. The principle of the method is clearly seen in ?? to ??.

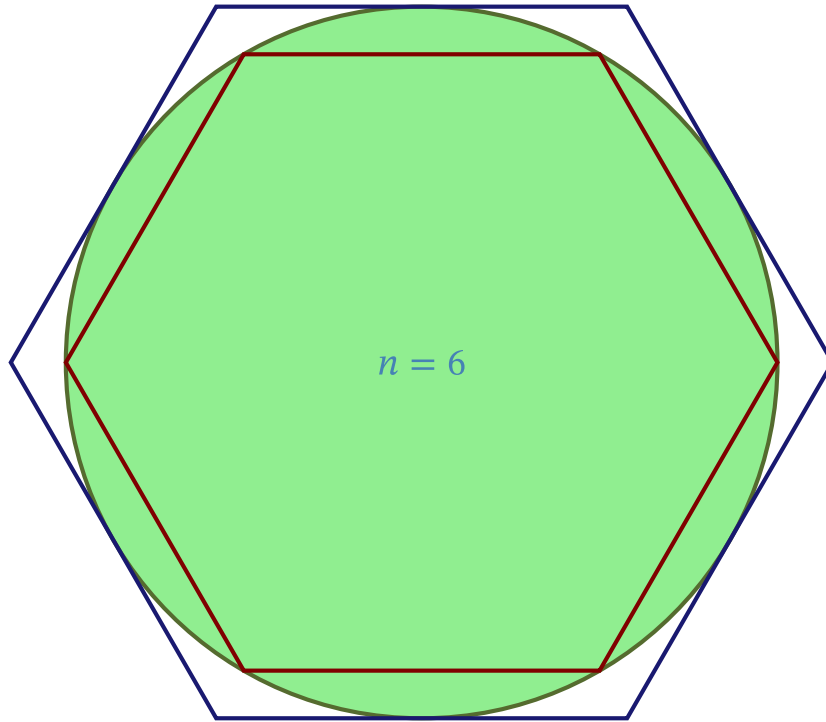


Figure 3: The length of the circle is bounded from below by the inscribed heagon in maroon and bounded from above by the inscribed hexagon in midnight blue. The circumference of the circle must lie between the perimeters of these two hexagons.

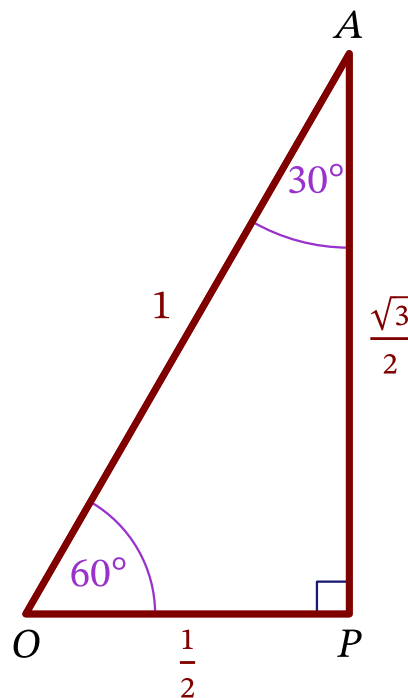


Figure 4: This right-angled, obtained by bisecting an equilateral triangle, must be familiar to all school students. These lengths—obtainable from symmetry and the theorem of Pythagoras—allowed Archimedes to start off his process for estimating π .

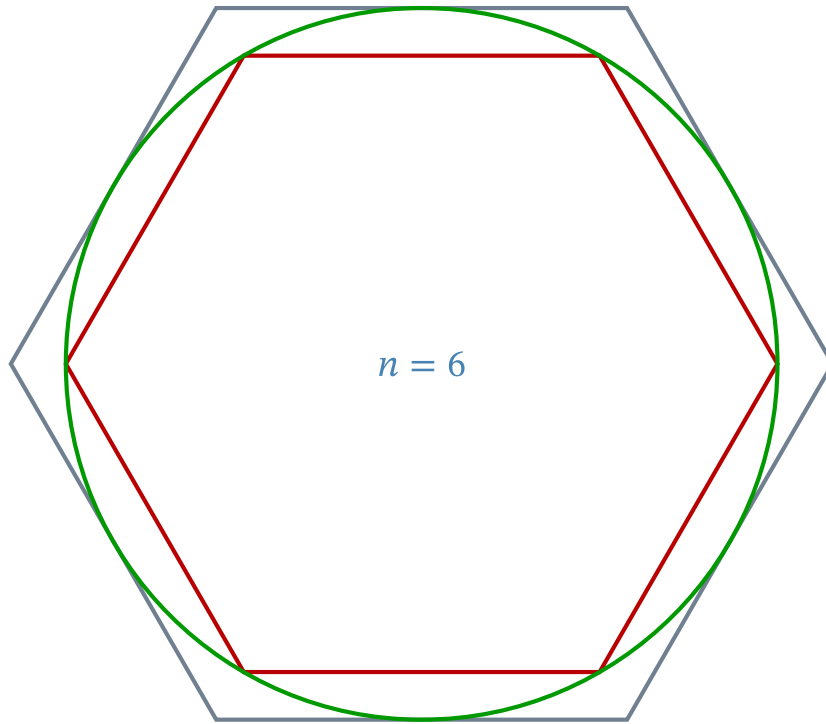


Figure 5: The estimate for π lies between $C_i = 3.000 < \pi C_c \leq 3.4641$.

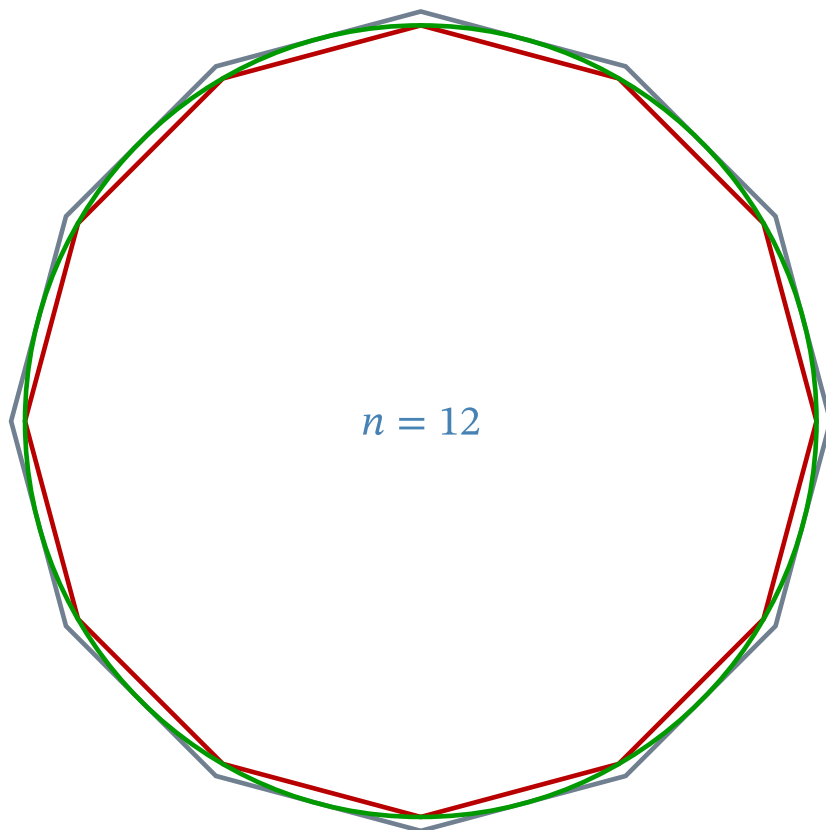


Figure 6: The estimate for π lies between $C_i = 3.000 < \pi C_c \leq 3.4641$.

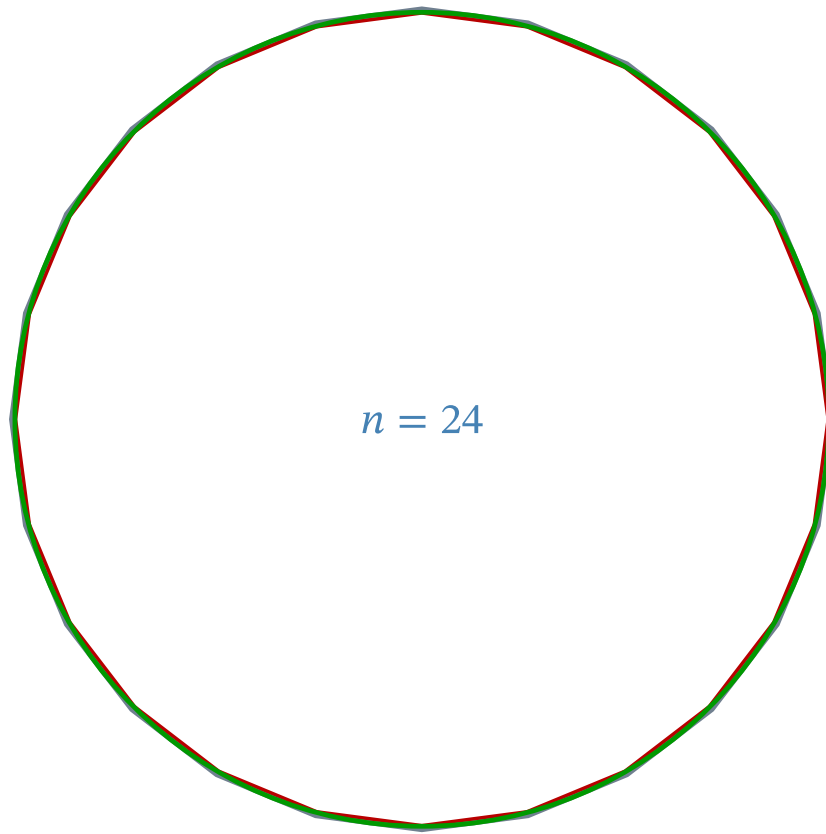


Figure 7: The estimate for π lies between $C_i = 3.000 < \pi C_c \leq 3.4641$.

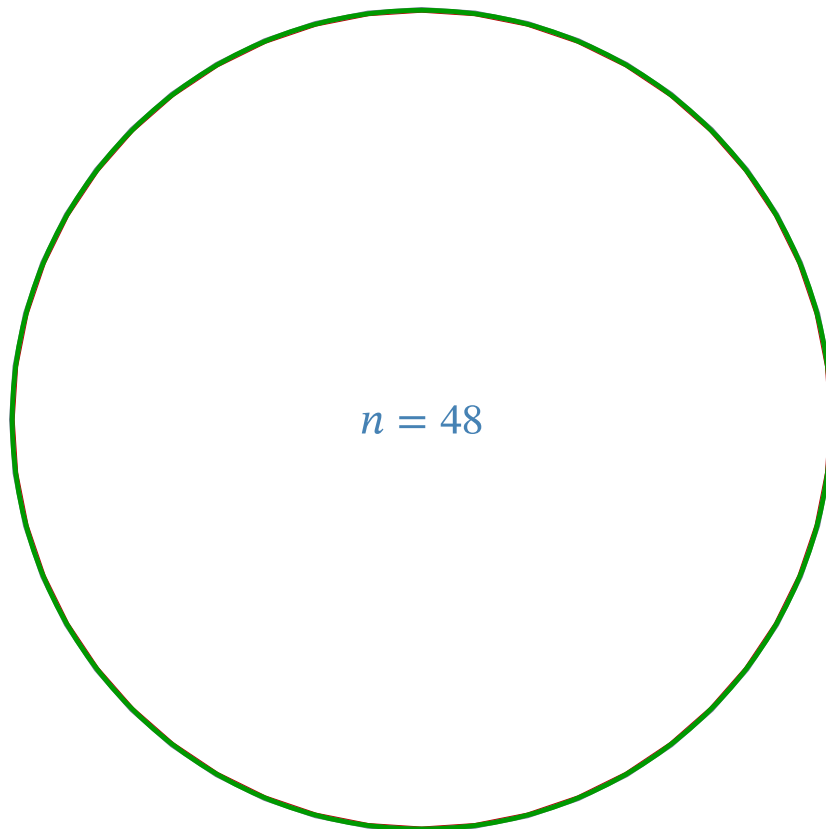


Figure 8: The estimate for π lies between $C_i = 3.000 < \pi C_c \leq 3.4641$.

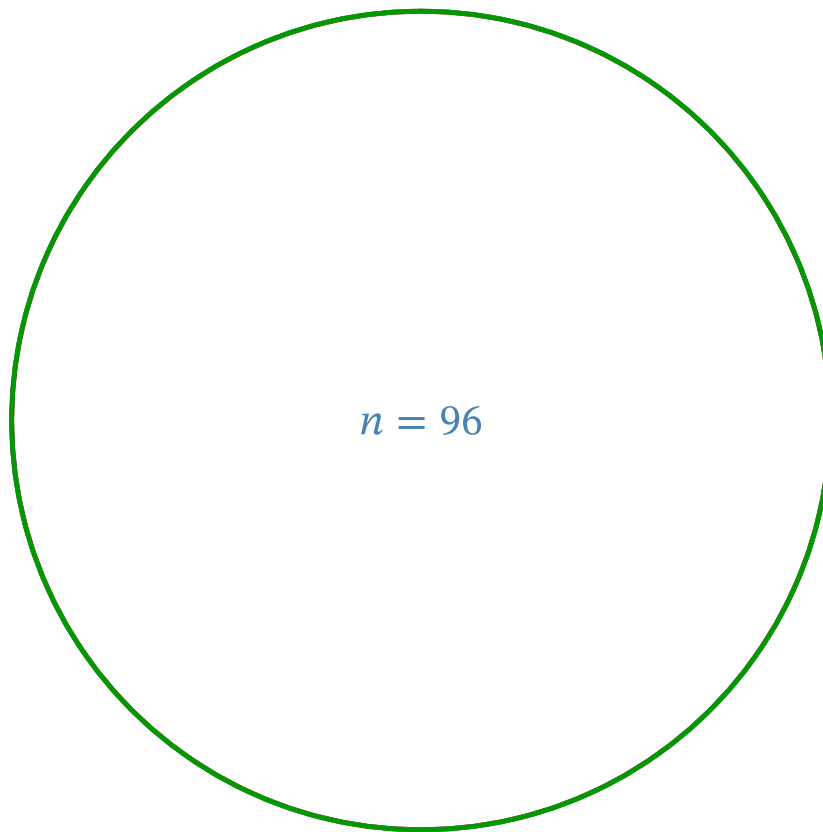


Figure 9: The estimate for π lies between $C_i = 3.000 < \pi C_c \leq 3.4641$.

He devised an ingenious method for estimating the circumference of a circle. It was not based on using a length of string, superimposing it on a circle, and getting an estimate. He used a sophisticated algorithm that allowed him to obtain successively more accurate values for the circumference of a circle, and therefore of π .

A well-written, accessible article on how Archimedes estimated that π is approximately $\frac{22}{7}$ is available online: “**How Archimedes showed that pi is approximately 22 by 7**”. I urge you to read it.² You will then appreciate for yourselves how arduous the process must have been in an age without the benefit of: #. Trigonometry; he used the theorem of pythagors instead; #. Algebra; he used geometry and the ratios of the lengths of well-known triangles; #. Decimal numbers for division; he used fractions instead;

Iteration, recursion, bisection, squeeze, etc.

<https://www.pbs.org/wgbh/nova/physics/approximating-pi.html>

<https://demonstrations.wolfram.com/ArchimedesApproximationOfPi/> John Tucker “Archimedes’ Approximation of Pi” <http://demonstrations.wolfram.com/ArchimedesApproximationOfPi/> Wolfram Demonstrations Project Published: March 5 2009

<https://math.stackexchange.com/questions/4851929/archimedes-method-to-estimate-pi>

<http://arxiv.org/pdf/2008.07995>

²This article is all the more remarkable because its first author is a Grade 8 student: proof that deep mathematics is not beyond the school student.

<https://mathsciencehistory.com/2019/10/01/archimedes-and-his-pi-the-great-numerical-hope/>

<https://carmamaths.org/resources/jon/pi-culture.pdf>

<https://nonagon.org/ExLibris/archimedes-pi>

<https://www.exploratorium.edu/pi/history-of-pi>

https://en.wikipedia.org/wiki/Approximations_of_%CF%80

In the series Figure 5 to Figure 8 below, which illustrate the approach Archimedes took to estimate π , we see very clearly that the perimeter of the *inscribed polygon* c_n and the perimeter of the *circumscribed polygon* C_n represent respectively the *lower bound* and *upper bound* of the estimated value of π . As the number of sides, n , of the polygon increases, the estimates become increasingly accurate.

https://publications.azimpremjiuniversity.edu.in/3356/1/02-DaminiAndAbhishek_PiIs22By7_Final.pdf

<https://azimpremjiuniversity.edu.in/at-right-angles>

How did Archimedes arrive at $\pi = 22/7$?

$22/7 = 3.142857\ 142857\ 142857$ (recurring decimal)

A closer look at π

π is both an **irrational** and a **transcendental** number. Let us see what each of these **appellations** mean.

Rational, Irrational, Algebraic, and Transcendental Numbers

Recurring decimals.

Formulae involving π

Quest for the endless digits of π

Buffon's Needle

π Trivia

Book References

Web resources

Appendix: Circumscribed and inscribed polygons of circle

Archimedes devised his ingenious *squeeze* method for computing the upper and lower bounds of the perimeter of a circle by computing instead the perimeters of the polygons that inscribe and circumscribe the circle. The approximations become more accurate as the number of sides, n , of the polygon is increased. **This YouTube presentation** might help you understand the algorithm of Archimedes better, but remember that he did not have trigonometry to aid him.

Feedback

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