The Wonder That Is Pi

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This blog began life more than two decades ago, as part of a series of lectures I delivered to very bright first-year engineering students at an Australian university.

The number π (pronounced "pie") has been recognized from time immemorial because its physical significance can be grasped easily: it is the ratio of the circumference of a circle to its diameter. But who would have thought that such an innocent ratio would exercise such endless fascination because of the complexities enfolded into it?

Not surprisingly, some students I met recently wanted to know more about π. Accordingly, I have substantially recast and refreshed my original presentation to better accord with the form and substance of a blog. The online references have also been updated to keep up with a rapidly changing Web.

If there are any errors or omissions, please email me your feedback.

Circumference, diameter, and π

The straight line or geodesic is the shortest distance between any two points on a plane, sphere, or other space. The circle is the locus traversed by a moving point that is equidistant from another fixed point on a two-dimensional plane. It is the most symmetrical figure on the plane. The diameter is the name given both to any straight line passing through the centre of the circle—intersecting it at two points—as well as to its length. When we divide the perimeter of circle, more properly called its circumference, C, by its diameter, d, we get the enigmatic constant π, which has a value between 3.141 and 3.142:

$$\frac{C}{d} = \pi. \tag{1}$$

The diameter d is twice the radius r, and substituting for d into Equation (1), we get the well-known school formula:

$$C = \pi d = 2\pi r \approx 2\left[\frac{22}{7}\right] r \approx 6.28r. \tag{2}$$

Note, however, that π is *not exactly equal* to $\frac{22}{7}$. This value is a convenient *rational fraction approximation* for π that serves well in elementary contexts.[1]

You might reasonably wonder whether the ratio of the circumference to the diameter of *any* circle is *always* π. The answer is "Yes", because *all circles are similar*. The ratios of corresponding lengths of similar figures are equal. This idea is also covered in my blog "A tale of two measures: degrees and radians".

The symbol π is the lowercase version of the sixteenth letter of the Greek alphabet. For the history of its use in mathematics, see adoption of the symbol π in Wikipedia.

[1] See "A tale of two measures: degrees and radians".

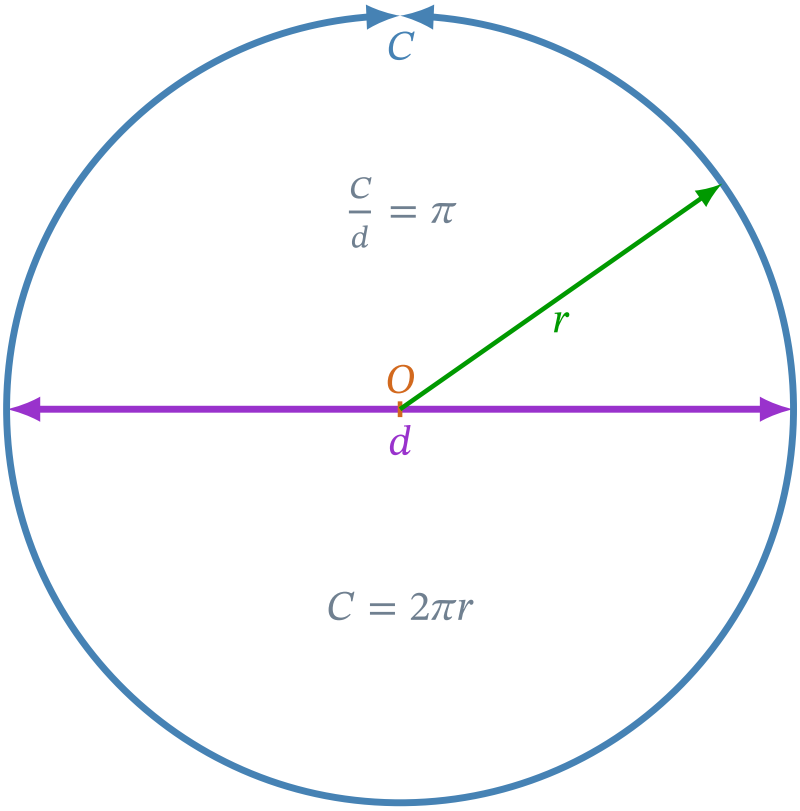

Figure 1: The ratio of the circumference to the diameter of *any* circle is π.

Figure 1 shows the relationships in Equation (1) and Equation (2) pictorially. The circumference of a circle is about 6.28 times its radius. Why this should be so is a secret, a mystery of Nature.

Is π really equal to 22/7?

Is π really equal to $\frac{22}{7}$, as it has been drummed into our heads at school? A wonderfully revealing story lies behind this mysterious relationship, and it is due to the labours of one man, in the days when calculators could not be dreamed of, and when neither the decimal system of numbers nor trigonometry were known. That is the story we look at next.

Archimedes of Syracuse

Archimedes of Syracuse (Ἀρχιμήδης, 287–212 BCE) was a polymath and genius of the ancient world. He was one of the greatest mathematicians the world has ever known. By today’s standards, he would be called a mathematician, physicist, engineer, and astronomer, all rolled into one. He is perhaps most famous for running out of his bathtub naked exclaiming “Eureka”—Greek for “I have found it”—oblivious of those around him. The principle that he had then discovered—that the upthrust on a body submerged in a fluid is equal to the weight of fluid displaced—is known as Archimedes’ Principle.

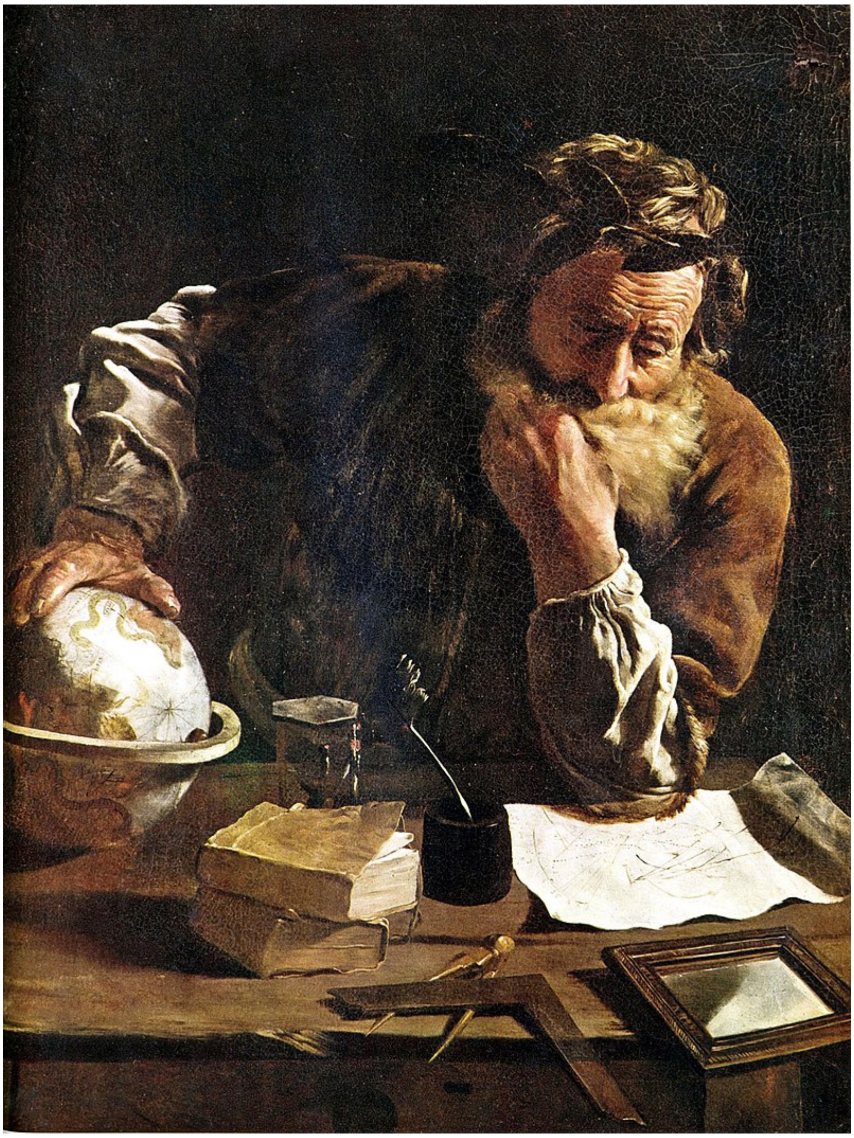

Figure 2: Archimedes of Syracuse.

Among the many accomplishments of Archimedes is his method for estimating π, which was the best approximation for almost 1900 years. What is even more remarkable is that Archimedes made his discovery without the benefit of either trigonometry, decimal (positional) notation, or calculators. He extracted square roots laboriously by hand. His method is also an excellent geometrical illustration of the idea of a limit, with which he was doubtless familiar. It is known that Archimedes was familiar with what we now know as integral calculus, and it is possible that he may have anticipated differential calculus as well.

Principles used by Archimedes

The method that Archimedes devised is instructive because it is a synthesis of several principles by which the greatest human minds have furthered scientific progress over time. The abstract principles that Archimedes used to estimate π were these:

1. Start with the known and progress to the unknown;
2. Initialize variables;
3. Devise a method of increasing the accuracy of the estimate by recursion or iteration;
4. Stop when the desired accuracy is reached.

These principles constitute what is known as an algorithm. Once such a systematic framework has been put in place, it can be applied in many research domains to aid rapid scientific progress.

Of polygons and circles

Archimedes considered a circle with an inscribed by a regular polygon with n sides along with a circumscribed regular polygon with the same n sides. Figure 3 illustrates this for the case $n = 6$, i.e., with a regular hexagon.

It is obvious that the *area* of the inscribed hexagon is smaller than the *area* of the circle, while the *area* of the circumscribed hexagon exceeds that of the circle. In symbols, with A_i representing the area of the inscribed hexagon, A representing the area of the circle, and A_c representing the area of the circumscribed hexagon, we may say:

$$A_i < A < A_c. \tag{3}$$

But can we say the same thing about the *perimeters* of these three objects? This is where the choice of *regular* hexagons makes matters more tractable. A regular hexagon is composed of six equilateral triangles, where the length of each side equals the radius. And each triangle has an area that is the product of half its base multiplied by its height. But the base equals the radius. Therefore the area of one triangle $A_\Delta = \frac{1}{2}rh$. Now six such triangles sum up to $\frac{1}{2}6rh$. But $6r$ is the perimeter of the hexagon. *Therefore, the area of the hexagon is $\frac{1}{2}Ph$ where P is the perimeter of the hexagon.*[2]

[2]This statement is true of any *inscribed polygon* as well.

Since the area is proportional to the perimeter, we are justified in claiming that the magnitudes of the perimeters accord with the magnitudes of the areas. Therefore, the circumference of the circle must lie between the perimeter of the inscribed polygon (limit from below) and the circumscribed polygon (limit from above). This visually obvious fact, illustrated in Figure 3, is an example of "starting with the known."

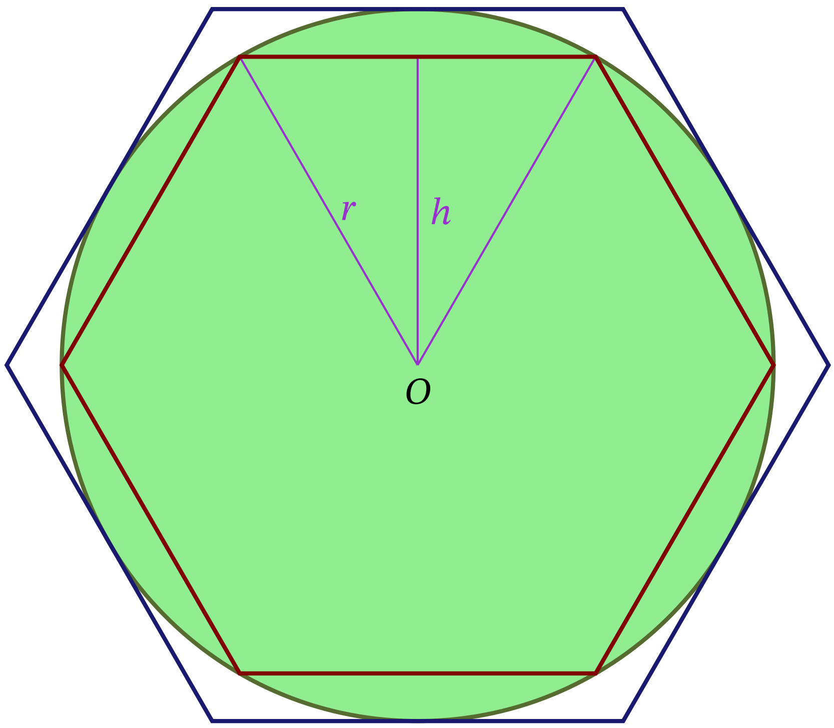

Figure 3: The length of the circle is bounded from below by the inscribed hexagon in maroon and bounded from above by the inscribed hexagon in midnight blue. The circumference of the circle must lie bween the perimeters of these two hexagons.

The second application of the same principle was his decision to initiate his algorithm using a regular hexagon, which is a mosaic of six juxtaposed equilateral triangles. We know from symmetry that each angle of an equilateral triangle is 60°. When an equilateral triangle is bisected, we get two right angled triangles with angles of thirty and sixty degrees, as shown in Figure 4.

The inscribed hexagon, within a circle of radius one unit, also has a side of one unit. Thus, the hypotenuse of the circle *OAP* in Figure 4 has unit length. Moreover, the base *OP*, resulting from a bisected side, has a length of half a unit. By applying the theorem of Pythagoras, the third side, *AP* is

$$\sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{2}}.$$

Archimedes started with regular hexagons and successively doubled the number of sides until he had the circle sandwiched between two 96-gons. He repeatedly calculated rational approximations to π until he was satisfied with the accuracy. The principle of the method is clearly seen in **??** to **??**.

$\sqrt{3} \approx 265/153$

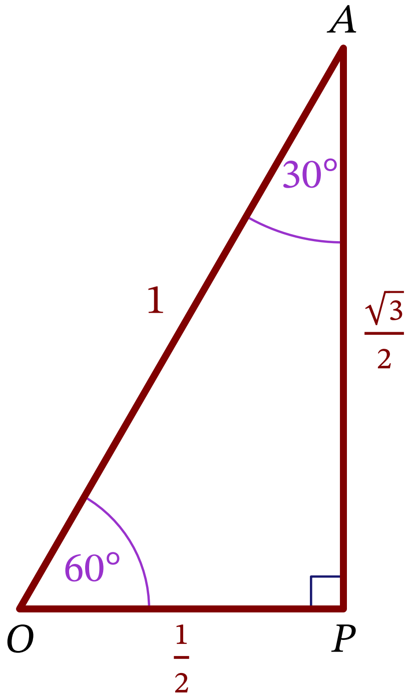

Figure 4: This right-angled, obtained by bisecting an equilateral triangle, must be familiar to all school students. These lengths—obtainable from symmetry and the theorem of Pythagoras—allowed Archimedes to start off his process for estimating π.

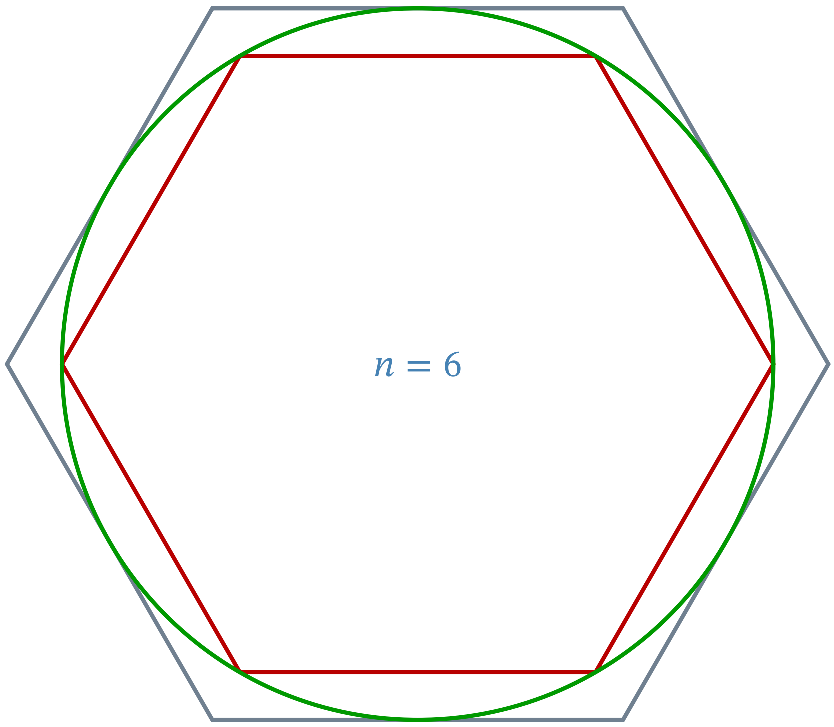

Figure 5: The estimate for π lies between $C_i = 3.000 < \pi C_c <= 3.4641$.

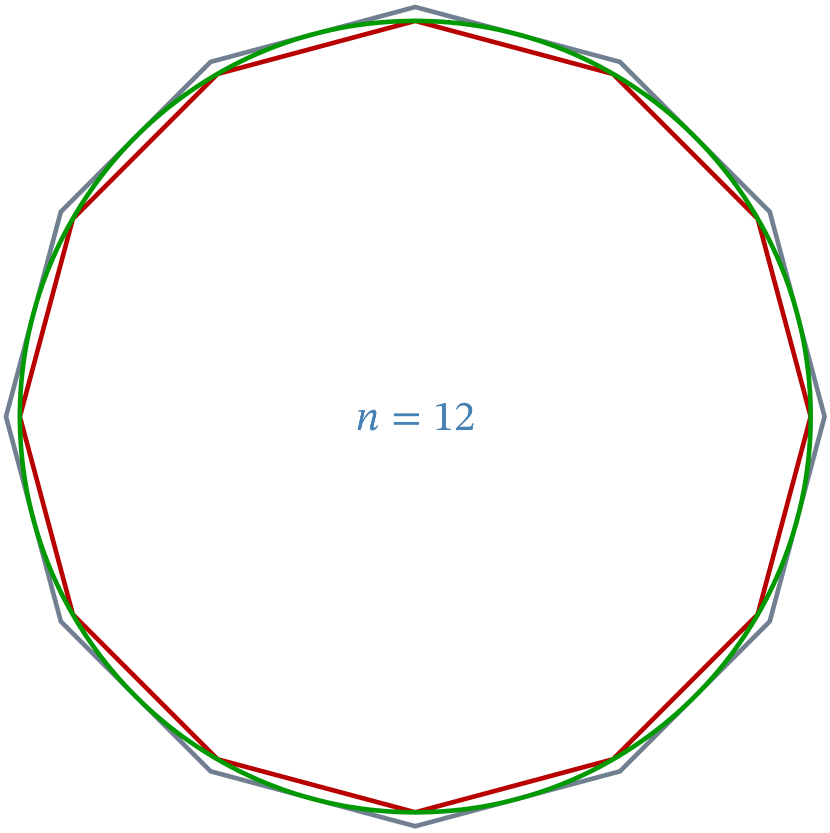

Figure 6: The estimate for π lies between $C_i = 3.000 < \pi C_c <= 3.4641$.

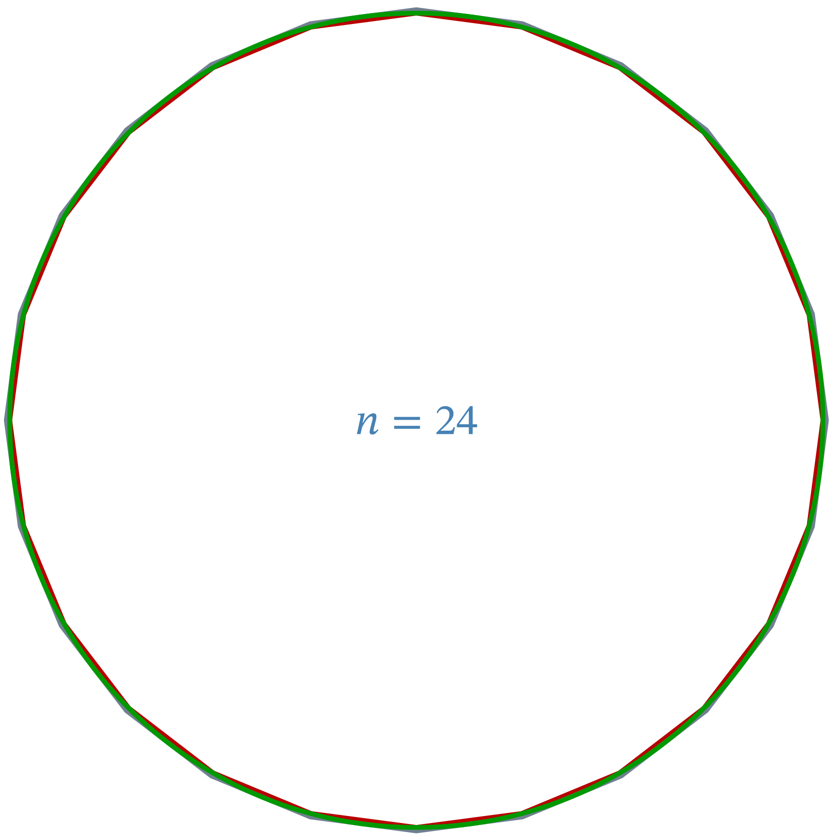

Figure 7: The estimate for π lies between $C_i = 3.000 < \pi C_c <= 3.4641$.

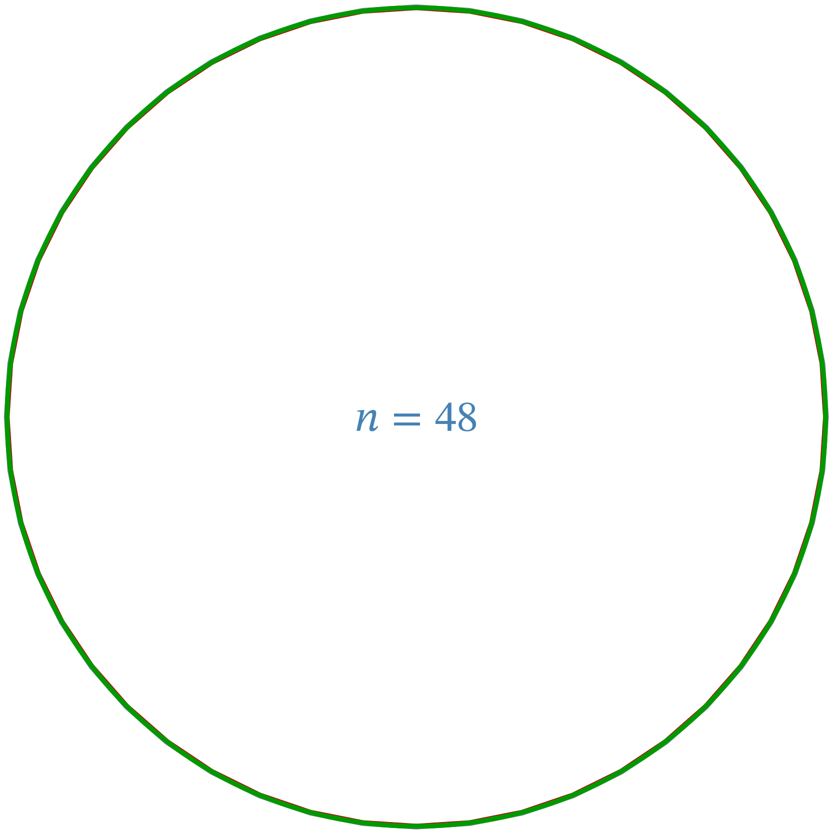

Figure 8: The estimate for π lies between $C_i = 3.000 < \pi C_c <= 3.4641$.

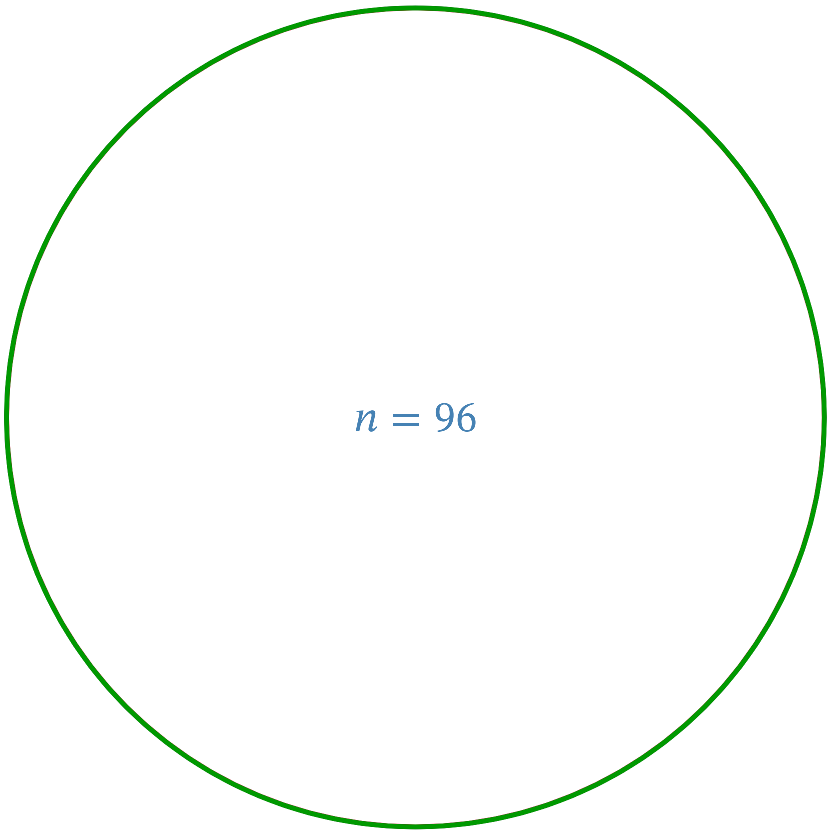

Figure 9: The estimate for π lies between $C_i = 3.000 < \pi C_c <= 3.4641$.

He devised an ingenious method for estimating the circumference of a circle. It was not based on using a length of string, superimposing it on a circle, and getting an estimate. He used a sophisticated algorithm that allowed him to obtain successively more accurate values for the circumference of a circle, and therefore of π.

A well-written, accessible article on how Archimedes estimated that π is approximately $\frac{22}{7}$ is available online: "How Archimedes showed that pi is approximately 22 by 7". I urge you to read it.[3] You will then appreciate for yourselves how arduous the process must have been in an age without the benefit of: #. Trigonometry; he used the theorem of pythagors instead; #. Algebra; he used geometry and the ratios of the lengths of well-known triangles; #. Decimal numbers for division; he used fractions instead; square roots by hand; similar and congruent figures; bisection theorems; exhaustion methods

Iteration, recursion, bisection, squeeze, etc.

https://www.pbs.org/wgbh/nova/physics/approximating-pi.html

https://demonstrations.wolfram.com/ArchimedesApproximationOfPi/ John Tucker "Archimedes' Approximation of Pi" http://demonstrations.wolfram.com/ArchimedesApproximationOfPi/ Wolfram Demonstrations Project Published: March 5 2009

https://math.stackexchange.com/questions/4851929/archimedes-method-to-estimate-pi

http://arxiv.org/pdf/2008.07995

https://mathsciencehistory.com/2019/10/01/archimedes-and-his-pi-the-great-numerical-hope/

https://carmamaths.org/resources/jon/pi-culture.pdf

https://nonagon.org/ExLibris/archimedes-pi

https://www.exploratorium.edu/pi/history-of-pi

https://en.wikipedia.org/wiki/Approximations_of_%CF%80

In the series Figure 5 to Figure 8 below, which illustrate the approach Archimedes took to estimate π, we see very clearly that the perimeter of the *inscribed polygon* c_n and the perimeter of the *circumscribed polygon* C_n represent respectively the *lower bound* and *upper bound* of the estimated value of π. As the number of sides, n, of the polygon increases, the estimates become increasingly accurate.

https://publications.azimpremjiuniversity.edu.in/3356/1/02-DaminiAndAbhishek_PiIs22By7_Final.pdf

https://azimpremjiuniversity.edu.in/at-right-angles

How did Archimedes arrive at π = 22/7?

22/7 = 3.142857 142857 142857 (recurring decimal)

[3] This article is all the more remarkable because its first author is a Grade 8 student: proof that deep mathematics is not beyond the school student.

A closer look at π

Pi is both an irrational and a transcendental number. Let us see what each of these appelations mean.

Rational, Irrational, Algebraic, and Transcedental Numbers

Recurring decimals.

Formulae involving π

Quest for the endless digits of π

Buffon's Needle

π Trivia

Book References

Web resources

Appendix: Circumscribed and inscribed polygons of circle

Archimedes devised his ingenious *squeeze* method for computing the upper and lower bounds of the perimeter of a circle by computing instead the perimeters of the polygons that inscribe and circumscribe the circle. The approximations become more accurate as the number of sides, n, of the polygon is increased. This YouTube presentation might help you understand the algorithm of Archimedes better, but remember that he did not have trigonometry to sid hm.

Feedback

Please email me your comments and corrections.

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