Expressions, Equations, and Formulae

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An unforeseen challenge

My dear friend, Solus "Sol" Simkin, casually asked me one summer day if I would write a blog demystifying the meanings and uses of four mathematical terms: expression, equation, formula, and differential equation. I thought he spoke in jest, and let his request lie in a dusty corner of my mind, as a memento to his humour.

Imagine my surprise when he accosted me again after two months and asked if I had put pen to paper to explain the four mathematical terms.

"Surely, you cannot be serious, Sol", I said. "Who would want to know something as fundamental as this? With the exception of differential equations, it should have been mostly taught by the fourth year of elementary school mathematics."

"You would be astounded to know how many so-called STEM *graduates* and *postgraduates*—who have passed through the degree mill—are ignorant of these *definitions*, let alone their *purpose*," replied Sol. "As an added bonus, write your blog so that it is also perfectly clear to elementary school students, going on to middle school. It will serve as a valuable review for them."

"Differential equations are in an entirely different class of mathematical sophistication from the other three topics. They should be excluded from the elementary treatment you are proposing," I said to Sol.

Never one to be pig-headed, Sol agreed.

Somewhat diffidently, I took up his challenge, complete with its stipulations. This blog was born after much cogitation, and is really my first attempt at presenting and exemplifying fundamental definitions, usually taught in elementary school.[1] Any reader who still finds it conceptually muddy or murky is cordially invited to write to me.

I have borrowed liberally from material contained in my book-manuscript *Secrets of Academic Success*, henceforth referred to as *SAS*. Perhaps the earnest student will be inspired to look for clarification there as well. ☺

[1] I usually find it easier to explain concepts to students in middle school and beyond. rather than to elementary school students.

Starting at the beginning

My decades of muddling in matters scholastic have convinced me that there are *four* stages in all learning, as shown in Figure 1. These have been explained *in extenso* in my *SAS* book, and the interested reader is directed to the first chapter of that book [1] for a more substantial discussion.

THE FOUR STAGES OF LEARNING

1. Naming
2. Knowing
3. Doing
4. Being

Figure 1: Learning any subject involves four stages as shown above [1].

All knowledge begins with *naming*. You cannot analyze or understand what you cannot name. In specialized subject areas, names are called *definitions*. In this blog, we have the following *four* mathematical names to define, understand, analyze, and apply:

1. expression;
2. equation; and
3. formula.

After naming, we move to *knowing*. At this stage, we systematically study the subject that has been defined to the extent that we are familiar with it *ourselves*, without recourse to a teacher, a textbook, or other reference material.

The third stage, *doing*, involves *application* of the newfound concept that has already been defined and studied. If you were learning to fly an aircraft, you could not claim to be a pilot, based on mere theoretical knowledge. You must practise flying—first under supervision, and later solo—so that you accumulate enough experience to claim competence in that art.

Once the doing stage has been mastered, it becomes effortless: this is the *being* stage of knowledge. You are now a master at what you started out to learn. You can start teaching others.

Every subject of study—whether academic like mathematics, or practical like surgery—involves these four steps and their mastery. By steadily moving from one stage to another, and finally by graduating to the being stage, you achieve mastery of your subject.

This blog is mainly concerned with the naming stage, but our discussion will not be complete without a modicum of knowing and doing as well. Let us set to.

Expressions

The word expression literally means "(something) that is pressed out". In the context of mathematics, an expression is a collection of numbers or symbols that are written out or expressed.

Sometimes, the expression might seem complicated, but it might also be amenable to simplification.

Let us start with something basic:

$$2 + 4 \tag{1}$$

It is a mathematical expression for adding four to two. But is that not 6? So, is the expression $2+4$ or is it 6? The *expression* itself is *two plus four*. Its *value* is six.

But if we know that $2+4=6$, why can't we say that the expression is 6? We *may* if we were asked to *simplify* the expression. But the expression itself remains as it was originally written.

Let us move up a notch. Look at:

$$\sqrt{25} \tag{2}$$

What does it mean? Now you need to know the language of mathematics. What does $\sqrt{}$ stand for? It is a stylized letter "r" for the word radix which stands for the positive square root of the number inside the symbol. What number multiplied by itself will give us 25? Well, 5×5 equals 25.

But is that all? What about $(-5) \times (-5) = 25$? That too is correct. So, what does $\sqrt{25}$ really stand for? It is *defined* to be the *positive* square root of 25 which is 5.

The case of -5 is catered for by the expression $-\sqrt{25}$. We may write $-\sqrt{25} = -5$; thus, we do not have notational ambiguity.

Simplifying an expression

In school, you might have been asked to *simplify an expression*. In that case, you are being asked to produce a result that is the same as the original expression but is simpler in form and appearance. For example, we could write:

$$2 + 4 = 6 \tag{3}$$

Look! What have we done? We have produced an *equation*. The sum of the two numbers on the left hand side (LHS) equals the single number on the right hand side (RHS).

We will consider *equations* a little later, but for now, bear in mind, that to simplify an expression, we need to find a *mathematical alias* for it that *equals* the original expression, but is simpler in form.

Enter algebra

After we mature a little more mathematically, we start dealing with numbers whose values are not known. We use *letters* to denote these unknown quantities, much like we use *pronouns* instead of *proper nouns* for the names of people we do not know. Let us take a look at a potentially confusing expression:

$$\frac{\frac{a}{b}}{c} \tag{4}$$

What does it mean? Can it be simplified? If so, what is its simplified form? Does it convey any meaning?

Mathematics is a language in which ambiguity is prohibited by strictly enforced conventions. We already saw that with the $\sqrt{}$ sign.

Does Equation (4)[2] mean more than one thing? Not if we know our conventions. The expression consists of a value on top divided by a value at the bottom. But the value at the top is itself a fraction:

$$\frac{a}{b} \tag{5}$$

This is now divided by the value c. We know that *dividing* by c amounts to *multiplying* by $\frac{1}{c}$. The expression may therefore be simplified so[3]:

$$\begin{aligned}\frac{\frac{a}{b}}{c} &= \frac{a}{b} \times \frac{1}{c}\\ &= \frac{a \times 1}{b \times c}\\ &= \frac{a}{bc}\end{aligned} \tag{6}$$

Note that the horizontal line separating the numerator and the denominator is called the *vinculum* and it is long enough to cover *both* b and c in the denominator.

If we did not have access to mathematical typesetting, this fraction would be written unambiguously as $a/(bc)$ where the two terms in the denominator must be grouped together by parentheses. If instead, this was written as a/bc the expression could also be correctly read as $(a/b)\times c = (ac)/b$ which is different from $a/(bc)$. This is reason enough to justify the use of brackets in mathematical expressions, which we take a look at next.

BIDMAS

When a mathematical expression is evaluated, we work from right to left, respecting operator precedence. This is a convention that lays down a hierarchy or protocol about which operation is performed before which. It is often reduced to the mnemonic BIDMAS.

The initial *B* stands for brackets, or parentheses. Bracketed expressions are evaluated first. Then we evaluate *I* or indices: powers and square roots. The *DMAS* stands for division, multiplication, addition, and subtraction in that order.

This *convention* ensures that everyone is on the same page when evaluating mathematical expressions. All will get the same result. Ambiguity is hence exiled from the mathematical landscape.

If you love mathematical symbols, you might wish to remember this unpronounceable visual mnemonic instead:

$$()x^y \div \times + -$$

Choose whichever mnemonic appeals more to you.

[2] It is not an equation but an expression; my software did not allow that degree of customization. Please excuse this inaccuracy.

[3] Refer to the chapter "Arithmetic Revisited" in the *SAS* book [1] if you are still unclear about what follows.

A visual metaphor for mathematical expressions

My preferred visual image for a mathematical expression is a tied-up bundle of clothes:

Figure 2: Bundle of clothes as a vsiual metaphor for a mathematical expression.

Equations

We now look at *equations*. All equations embody the $=$ symbol, which is called an *equals sign*. It is a mathematical shorthand to denote that what is on the LHS of this symbol is equal to what is on the RHS, however different they may appear to be. We have previously encountered this symbol in the very simple equation

$$2 + 4 = 6.$$

Operations and relations

Before venturing further, we need to distinguish between *operations* and *relations*.[4]

Addition, multiplication, exponentiation, etc., are familiar binary operations, which take two inputs or *operands* and produce a single output or result, as in

$$2 + 4 = 6.$$

[4] I have avoided a set theoretic framework and notation to keep this blog within the grasp of young students.

Equality is an equivalence relation that is reflexive, symmetric, and transitive.

1. Reflexivity means that every number is equal to itself: $2 + 4 = 2 + 4$ and $6 = 6$.
2. Symmetry means that if $2 + 4 = 6$, then $6 = 2 + 4$. Note that this is *not* commutativity which applies to operands, not to relations.
3. Transitivity means that if $2 + 4 = 6$ and if $6 = 3 + 3$, then $2 + 4 = 3 + 3$.

You might think that these definitions and explanations are absurd and were probably invented by crazies because they state the obvious—and you would not be far wrong. But the power of these ideas lies in their ability to be generalized beyond the immediate context in which they arose: something you would appreciate as you plumb the deeper depths and higher heights of mathematics.

In sum, a binary operation works on two inputs to produce a third output. A relation, like equality, on the other hand, establishes a relationship—sameness in this case—between two mathematical entities.

A visual metaphor for equality

A two-pan balance is an excellent visual metaphor for equality. Even though the material in each pan might be different, when the pans balance, we have equality. This means each pan contains the same weight or mass. It is the principle behind how we buy foodstuffs. And it is identical to the principle of equality as a mathematical relation.

Simple equations

Quadratic equations

Polynomials

Binomial theorem

Trigonometric identities

Formulae

Completing the square and the quadratic formula

Circumference of a circle

Area of a triangle

Volume of a cylinder

Acknowledgements

Feedback

Please email me your comments and corrections.

A PDF version of this article is available for download here:

https://swanlotus.netlify.app/blogs/math-fundamentals.pdf

Figure 3: A two-pan balance in equilibrium, indicating that the mass on the left hand side equals that on the right hand side, even though the contents differ.

References

[1] R (Chandra) Chandrasekhar. 2025. Secrets of Academic Success: Timeless Principles for Lifelong Learning. Retrieved 11 February 2025 from https://swanlotus.netlify.app/sas-manuscript/SAS-partial.pdf