Chapter 12

Pi

Pi is probably the most famous number known to man. The ancient Greeks bequeathed us their lowercase letter π—corresponding to the Roman letter p—as its unique and enduring symbol. π is the ratio of the circumference of a circle to its diameter. The symmetry of the circle has bestowed on π a lasting fascination and an almost mystic significance. Through the trigonometric functions, π pervades almost all of mathematics, and turns up in the most unexpected places, as illustrated in this chapter.

We begin with the notion of radian measure and its natural relationship with π. Next, we take a look at the method Archimedes used to estimate π: surely an ancient, elegant, and geometric exposition of the squeeze theorem known to contemporary students of analysis. The close link between π and power series is considered next. An unlikely but interesting method of probabilistically estimating π through the famous Buffon's needle experiment is then described.

Because π is an irrational transcendental number, it has no closed form decimal representation. This, together with the natural fascination π exercises on the human imagination, has led aficionados to estimate its decimal representation to ever larger numbers of digits. The computer algebra system `Maple`®*, for example, can calculate the first 1,000 or 10,000 digits of π in a trice. The chapter concludes with suggestions to explore π further through books and Web resources.*

12.1 Introduction

It has been said that

$$e^{i\pi} + 1 = 0 \tag{12.1}$$

is the most poetic of all mathematical equations because it embodies the five most important mathematical symbols in one expression. It is a special case of Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{12.2}$$

that relates the exponential and trigonometric functions.

Pi, written as the lowercase Greek letter, π, is the most famous of all numbers. It is the ratio of the circumference of a circle to its diameter. The second most famous number e is the base of the natural logarithms. The symbol i satisfies $i^2 = -1$, and although it represents imaginary numbers, its applications in science and engineering are very real indeed. Zero and one are familiar from arithmetic. They are simple numbers with profound consequences: the world of digital computers and communications has its foundations in symbols of zeros and ones.

Other mathematically important numbers include the golden ratio ϕ, Euler's constant γ, and the Feigenbaum number δ. Important and useful number patterns include Pascal's Triangle and the Fibonacci Sequence.

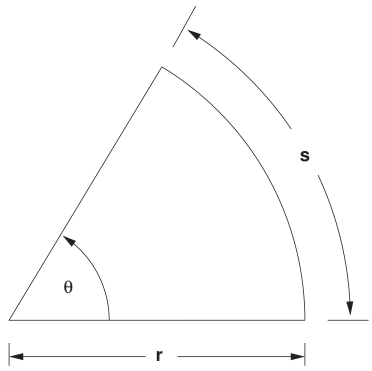

FIGURE 12.1: Radian measure is a normalized, less arbitrary way of measuring angles than degrees. The angle θ subtended at the centre of a circle of radius r by an arc of length s is defined to $\theta = \frac{s}{r}$ radians.

In this series of lectures, we will look at some important numbers, and number patterns, beginning with π.

12.2 Why is Pi Important?

It is clear that in day-to-day life, we need to measure lengths, areas, and volumes. The number π enters naturally into the formulae for these measurements for the circle, cylinder and the sphere. But, surely, this cannot be the only reason why π is pre-eminent among numbers.

12.2.1 Radians, Trigonometric Functions, and Pi

Although degrees are generally used to measure angles with a protractor, the number of degrees in one revolution, or a full circle, is an arbitrary number: 360, which most likely has its origins in the approximate number of days in a year, as determined by the ancient Babylonians.

A less arbitrary measure of an angle was required for serious mathematics. You may recall by looking at Figure 12.1 that an angle θ may also be measured in *radians,* as the ratio of the arc length s subtended at the circumference of a circle, to the radius r of the circle:

$$\theta = \frac{s}{r} \tag{12.3}$$

The "unit" radians is not really a unit, though. Being the ratio of two lengths, it possesses no dimensions in the sense of Physics. Since the circumference of a circle of radius r is $2\pi r$, one full revolution of 360° is actually 2π radians. Therefore, the number π very naturally enters *angular measure* in radians, and this is one of the principal reasons for its importance.

When trigonometric functions like $\sin x$ are differentiated or integrated, the variable x, representing the angle, must be measured in radians. Moreover, when oscillatory phenomena, whether of water or electricity or light, are studied in the physical sciences and engineering, trigonometric functions usually form part of the solution. The importance of π is therefore entrenched in the importance of trigonometric functions and radian measure, in solving a variety of physical problems.

12.2.2 Pi is Irrational and Transcendental

Pi is not only important, it is also tantalizing. It is like a beautiful butterfly that cannot be caught. It is not a rational number, which means that it cannot be expressed as the ratio of two whole numbers, the denominator being non-zero. Its decimal representation is neither finite nor does it contain a recurring segment. It is also not the root to any polynomial equation whose coefficients are integers or rational numbers. This earns for π the rather exalted title of a transcendental number, which must also be irrational by definition.

The unpredictability of successive decimal places of π has enchanted mathematicians and still continues to engross them. Pi has been calculated to an unprecedented number of decimal places, and such a quest is certainly driven not by practical necessity but possibly by the need for aesthetic satisfaction.

12.2.3 Geometry, Algebra, Probability, Limit, and Convergence

We use pictures and words to communicate. In mathematics, *geometry* corresponds to pictures, and *algebra* to words. The interplay between geometry and algebra has been responsible for many mathematical advances. For example, the development of co-ordinate geometry laid the foundations for calculus.

The search for increasingly more accurate values for π has resulted in many approaches to solve the problem. Geometric and algebraic approaches to estimate π have both borne fruit. Interestingly, π may also be estimated by repeatedly performing a random, or probabilistic, experiment, whose precise outcome cannot be predicted, but whose average behaviour may be estimated. Such an experiment is called a *Monte Carlo* simulation. Thus the quest for π brings together geometry, algebra and probabilistic simulation.

Two of the most important ideas in calculus are those of a *limit* and of *convergence*. We shall become acquainted with both as we find out how π may be estimated by these three approaches.

12.3 The Method of Archimedes

Archimedes of Syracuse was one of the greatest mathematicians the world has known. By today's standards, he would be called a mathematician, physicist and engineer, all rolled into one. He is perhaps most famous for running out of his bathtub naked exclaiming "Eureka" (Greek for "I have found it") oblivious of those around him. The principle that he had then discovered, that the upthrust on a body submerged in a fluid is equal to the weight of fluid displaced, is known as Archimedes' Principle.

Among the many accomplishments of Archimedes is his method for estimating π, which was the best approximation for almost 1900 years. What is even more remarkable is that Archimedes made his discovery without the benefit of either trigonometry or decimal (positional) notation. His method is also an excellent geometrical illustration of the idea of a limit, with which he was doubtless familiar. It is known that Archimedes was familiar with what we now know as integral calculus, and it is possible that he may have anticipated differential calculus as well.

Archimedes realized that when a circle is inscribed by a regular polygon with n sides and circumscribed by a regular polygon with the same n sides, the circumference of the circle must lie between the perimeter of the inscribed polygon (limit from below) and the circumscribed polygon (limit from above).

You may recall that if the limit from below exists, and if the limit from above exists, and if they are both equal, the limit itself exists, and is equal to this unique value.

Archimedes started with regular hexagons and successively doubled the number of sides until he had the circle sandwiched between two 96-gons. The principle of the method is clearly seen in Figure 12.2.

We will cheat a little and use trigonometry to derive the perimeters of the inscribed and circumscribed polygons. For a polygon of side n, the angle subtended by the any one side at the centre of the circle is $\frac{2\pi}{n}$. The half-angle θ subtended by one side is therefore $\theta = \frac{\pi}{n}$. If the radius of the circle is r, the length of one side of the inscribed polygon is

$$s_i = 2r\sin\theta \tag{12.4}$$

and likewise, the length of one side of the circumscribed polygon is

$$s_c = 2r\tan\theta \tag{12.5}$$

Figure 12.3 illustrates in one picture what equations (12.4) and (12.5) state in "words".

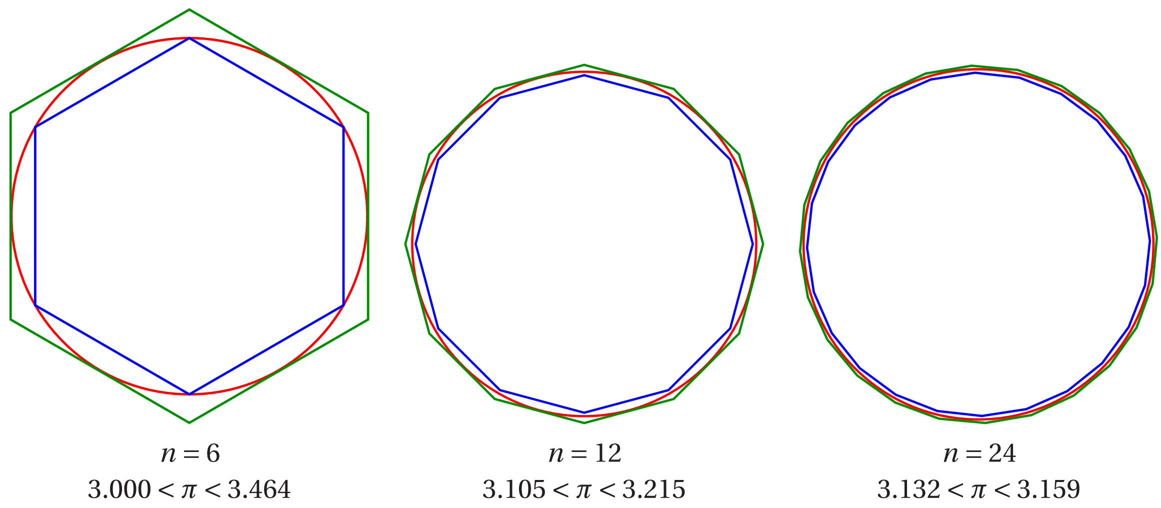

FIGURE 12.2: Illustration of the method used by Archimedes to estimate π. The perimeter of the circle is bounded from below by the perimeter of the blue inscribed regular polygon, and bounded from above by the green circumscribed regular polygon. Archimedes started out with the number of sides, $n = 6$, and stopped at $n = 96$, after four doublings. Note that with $n = 24$ it is already becoming difficult to distinguish the three figures.

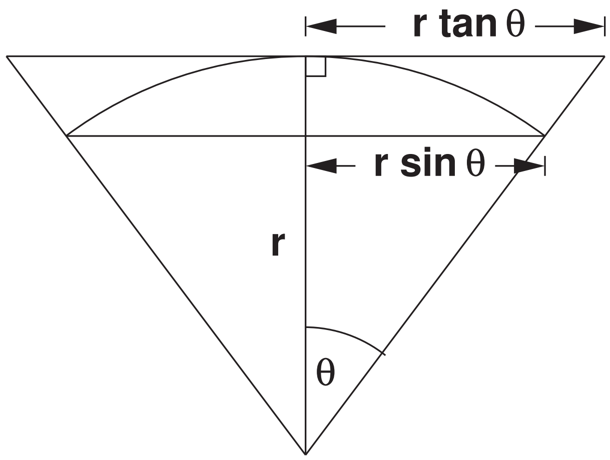

FIGURE 12.3: A sector of a circle and parts of the inscribed and circumscribed polygons are shown above. The half-angle at the centre is $\theta = \frac{\pi}{n}$ and the semi-lengths of the sides of the inscribed and circumscribed polygons are $r \sin\theta$ and $r \tan\theta$ respectively.

The circumference of the circle $C = 2\pi r$ is bounded by the perimeters of these two polygons giving us the inequality

$$\begin{aligned} ns_i < 2\pi r < ns_c \\ 2nr\sin\theta < 2\pi r < 2nr\tan\theta \\ n\sin\theta < \pi < n\tan\theta \end{aligned} \tag{12.6}$$

If the number of sides is doubled, θ is halved, n is doubled, and the equation (12.6) becomes

$$2n\sin\frac{\theta}{2} < \pi < 2n\tan\frac{\theta}{2} \tag{12.7}$$

and if the doubling is done k times, we have

$$2^k n\sin\frac{\theta}{2^k} < \pi < 2^k n\tan\frac{\theta}{2^k} \tag{12.8}$$

By making k sufficiently large, we could tighten the bounds on π and approximate it arbitrarily closely. However, if you looked closely at equation (12.8) and bear in mind that $\theta = \frac{\pi}{n}$ it seems like we have cheated again because equation (12.8) really states

$$2^k n\sin\frac{\pi}{2^k n} < \pi < 2^k n\tan\frac{\pi}{2^k n} \tag{12.9}$$

The way out of the circularity into which we have landed is to invoke the Theorem of Pythagoras and the half-angle formulae, which are independent of any estimates of π, but which rely on the roots of polynomial equations. The following exercise shows how to go about this.

Exercise 12.1 Estimating π using half angle formulae iteratively

In this exercise, you will estimate π, using the method of Archimedes as the basis, but employing trigonometric identities to simplify the work. This is an iterative exercise which you may try out with paper and pencil, or with a calculator, or using `Maple`®, the last of which is recommended.

Let n be the number of sides of the regular polygons that inscribe and circumscribe a circle. Also, let $\theta = \frac{\pi}{n}$ be the half-angle subtended at the centre by one side. We have already shown that after k doublings of the original number of sides,

$$2^k n\sin\frac{\theta}{2^k} < \pi < 2^k n\tan\frac{\theta}{2^k}$$

1. Set $n = 6$, $\theta = \frac{\pi}{6}$, and $k = 0$. Note that $\sin\theta = \frac{1}{2}$, $\cos\theta = \frac{\sqrt{3}}{2}$, and $\tan\theta = \frac{\sqrt{3}}{3}$ from Pythagoras' Theorem.

2. Increment k by one.

3. Use the half-angle formulae

$$\sin\frac{\theta}{2^k} = \left[\frac{1-\cos\left[\frac{\theta}{2^{k-1}}\right]}{2}\right]^{\frac{1}{2}}$$

$$\cos\frac{\theta}{2^k} = \left[\frac{1+\cos\left[\frac{\theta}{2^{k-1}}\right]}{2}\right]^{\frac{1}{2}}$$

$$\tan\frac{\theta}{2^k} = \left[\frac{1-\cos\left[\frac{\theta}{2^{k-1}}\right]}{1+\cos\left[\frac{\theta}{2^{k-1}}\right]}\right]^{\frac{1}{2}}$$

to compute the values of $\sin\frac{\theta}{2^k}$, $\cos\frac{\theta}{2^k}$, and $\tan\frac{\theta}{2^k}$ from the value of $\cos\frac{\theta}{2^{k-1}}$.

4. If $k = 4$ stop; otherwise repeat steps 2 to 3.

5. Tabulate your results showing the values of k, $2^k n \sin\frac{\theta}{2^k}$, $2^k n \tan\frac{\theta}{2^k}$ and estimate the percentage error of the upper and lower estimates from the last iteration, using the stored value of π in your calculator, or `Pi` in `Maple`.

If you found the previous exercise to be demanding, spare a thought for Archimedes, who attacked and solved the problem without trigonometry, and without decimals. He must have devised rational number approximations to the square roots he encountered, and been careful enough to keep the rational approximations slightly lower than the exact values for the lower bound and slightly larger than the exact values for the upper bound. As an engineer, you will one day have to develop the same skills when you perform "best case" and "worst case" numerical simulations.

Archimedes reported his result as

$$\begin{aligned} \tfrac{223}{71} < \pi < \tfrac{220}{70} \\ 3\tfrac{10}{71} < \pi < 3\tfrac{1}{7} \end{aligned} \tag{12.10}$$

and most of you will remember this last upper bound as the famous approximation

$$\pi \approx \frac{22}{7} \tag{12.11}$$

from your schooldays.

12.4 Gregory-Leibniz Series

It must be obvious by now that trigonometry and the number π are inextricably entwined. The quest for π continued to fascinate mathematicians in the centuries after Archimedes. This time though, rather than geometric iteration, sums of successive terms were used to approximate π.

For our purposes, a *sequence* is an *ordered* procession of numbers, and a *series* is a sum of successive terms that obey some specific rule. If the summation stops at some particular term, we have a *partial sum*; if the summation goes on indefinitely, we have an *infinite series.* If this infinite sum approaches ever closer to a finite value, the series is said to *converge.* To see what all this means in practice, let us look at the Gregory-Leibniz series.

James Gregory was the first Professor of Mathematics at the University of Edinburgh and in 1671, he published the series now known by his name. Rather than draw Gregory's formula out of a hat, we will sketch its derivation, and show its origins in integral calculus. Gregory found that:

$$\int_0^x \frac{1}{1+t^2}\,dt = \arctan x \tag{12.12}$$

This integral should be familiar to you. If it is not, try substituting $t = \tan\theta$:

$$\begin{aligned} t &= \tan\theta \quad \text{which gives} \\ \frac{dt}{d\theta} &= \frac{d}{d\theta}[\tan\theta] \\ &= \sec^2\theta \\ &= 1+\tan^2\theta \\ &= 1+t^2 \\ \text{Therefore} \quad \tfrac{1}{1+t^2}\,dt &= d\theta \end{aligned}$$

The integral of equation (12.12) now becomes

$$\begin{aligned} \int_0^x \frac{1}{1+t^2}\,dt &= \int_{\arctan 0}^{\arctan x} d\theta \\ &= [\theta]_{\arctan 0}^{\arctan x} \\ &= \arctan x \end{aligned} \tag{12.13}$$

This takes care of the right hand side of equation (12.12). If we performed long division on the left hand side of the same equation, we get

$$\begin{aligned} \int_0^x \frac{1}{1+t^2}\,dt &= \int_0^x \left[1-t^2+t^4-t^6+\ldots\right]dt \\ &= \left[t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\ldots\right]_0^x \\ &= x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots \end{aligned} \tag{12.14}$$

Using equations (12.13) and (12.14), we get the Gregory series

$$\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots \tag{12.15}$$

Notice that it is only a small step from here to substitute $x = 1$ to get the equation

$$\begin{aligned} \arctan 1 = \tfrac{\pi}{4} &= 1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots \\ \pi &= 4(1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\ldots) \end{aligned} \tag{12.16}$$

Strangely, Gregory did not publish the special case of equation (12.16), and it was Gottfried Wilhelm Leibniz who discovered both equations (12.15) and (12.16) in 1674, and published them in 1682. Thus the dual names by which these series are known.

It is noteworthy that equation (12.16) was the first infinite series ever found for π. However, it converges rather slowly, and one needs many terms before a reasonable approximation emerges.

Over the last 350 years, by far the most effort has been expended in discovering series that *converge rapidly* to π, so that even a partial sum of only a few terms will provide an accurate estimate of π.

We wrap up this section with a selection of formulae from famous mathematicians who have bequeathed other series for calculating π.

Newton used the binomial theorem to derive:

$$\begin{aligned} \arcsin x &= x+\tfrac{1}{2}\tfrac{x^3}{3}+\tfrac{1\cdot 3}{2\cdot 4}\tfrac{x^5}{5}+\ldots \\ \pi &= 6\left(\tfrac{1}{2}+\tfrac{1}{2}\tfrac{1}{3\cdot 2^3}+\tfrac{1\cdot 3}{2\cdot 4}\tfrac{1}{5\cdot 2^5}+\ldots\right) \end{aligned} \tag{12.17}$$

John Machin gave the formula:

$$\frac{\pi}{4} = 4\arctan\left[\frac{1}{5}\right]-\arctan\left[\frac{1}{239}\right] \tag{12.18}$$

where $\arctan x$ may be approximated by equation (12.15).

Karl Friedrich Gauss gave the elegant formula:

$$\frac{\pi}{4} = \arctan\left[\frac{1}{2}\right]+\arctan\left[\frac{1}{5}\right]+\arctan\left[\frac{1}{8}\right] \tag{12.19}$$

which again may be computed using equation (12.15).

Among the countless formulae for π may be mentioned one, due to Srinivasa Ramanujan, which is so *unusual* that one wonders how it was ever derived in the first place:

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!\,[1103+26390n]}{(n!)^4 396^{4n}} \tag{12.20}$$

We conclude this section with an exercise that explores convergence and gives a practical measure for estimating its rate.

Exercise 12.2 Rates of convergence for different formulae for π

You will explore the different rates of convergence for four of the formulae that we have encountered: the Gregory-Leibniz series, Newton's formula, Machin's formula, and Gauss's formula. Proceed as follows, using `Maple`:

1. Write expressions for the general terms for each formula and assign them to four different variables.
2. Write expressions for the sum to n terms for each formula.
3. Compute this sum for $n = 10^k$; $k = 0, 1, \ldots, 10$.
4. For each value of k, calculate the percentage error in the respective partial sums by subtracting them from π, dividing by π, and multiplying by 100. Use the `Maple` constant `Pi` for this.
5. Plot the percentage error as dependent variable against $\log_{10} n = k$ as dependent variable. Use semi-log axes.
6. Comment on the different rates of convergence of the four formulae. If you had to choose a formula, which would you choose, and why?

12.5 Buffon's Needle

In the preceding two sections, we have seen the connection between π and geometry, and between π and trigonometry. Such a relationship is not unexpected, for we may mataphorically say that π has its "home" in the circle, and is "neighbours with" trigonometric functions, which are also called *circular functions*. But how indeed does π enter the domain of probability?

Georges Louis Leclerc Comte de Buffon was a French naturalist of the eighteenth century, who is probably most well-remembered for proposing and solving the problem that goes by his name. It is a probabilistic method for determining the value of π, and antedates modern *Monte Carlo* simulations on computers.

Buffon's Needle problem may be posed thus. Consider a needle of length ℓ that is thrown at random on a floor that has parallel lines spaced $d > \ell$ apart. What is the probability that the needle will touch or cross one of the lines? We may assume that the needle's position and its orientation, when it lands, are both independent, and random.

This problem may be solved elegantly using knowledge of trigonometry and the integral calculus. First we draw a diagram of how the needle may fall with respect to a *single* line, as shown in Figure 12.4. It is important to realize that the analysis with respect to a single line suffices. This sort of *problem abstraction* or *modelling* is an important skill you should acquire as engineers, because it helps focus on the essentials, and in the process usually simplifies the solution of the problem.

With reference to Figure 12.4, let x be the perpendicular distance from the centre of the needle to the nearest parallel line. If the needle makes an angle ϕ measured in the conventional sense with respect to the nearest line, the distance of the tip of the needle from the centre, in the direction perpendicular to the parallel lines, is $\frac{1}{2}\ell\sin\phi$. Clearly, if $x > \frac{1}{2}\ell\sin\phi$, the needle cannot touch the nearest line. Therefore the condition for the needle to touch the nearest line is

$$x \le \frac{1}{2}\ell\sin\phi \tag{12.21}$$

Because of symmetry, we may restrict our consideration to angles $0 \le \phi \le \pi$ and for $0 \le x \le \frac{d}{2}$. We may now define the space of all events as the rectangle on the ϕ-x plane bounded by the lines $x = 0$, $\phi = \pi$, $x = \frac{d}{2}$ and $\phi = 0$, as shown in Figure 12.5.

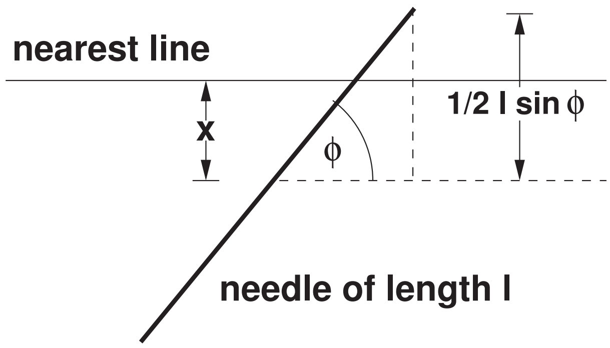

FIGURE 12.4: Illustration of *problem abstraction* for solving the problem of Buffon's Needle. Because the spacing between the parallel lines d is greater than the length of the needle ℓ, we know that the needle can *at most* intersect *one* line. We have shown that one line here, and called it the nearest line. The perpendicular distance from the centre of the needle to the nearest parallel line is denoted by x. By relating x to the angle ϕ and needle length ℓ, we derive the condition for intersection or non-intersection as a simple inequality.

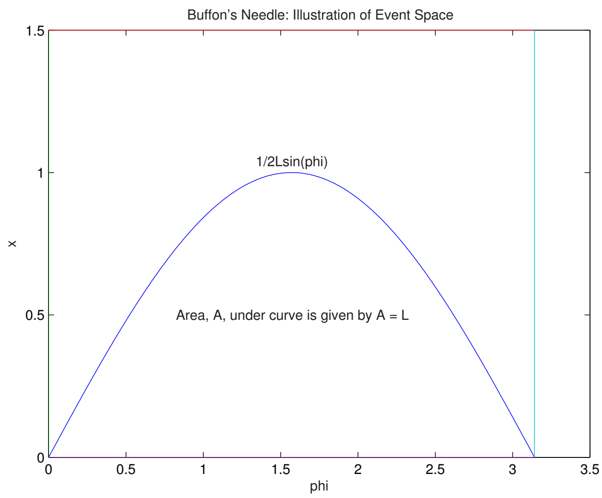

FIGURE 12.5: Graphical depiction of the event space for the Buffon Needle experiment. The area under the curve satisfies the inequality $x \leq \frac{1}{2}\ell \sin\phi$. Let this area be called A. We show in equation (12.22) that $A = \ell$. The universal set is the rectangle bounded by the ϕ and x axes and the lines $x = \frac{d}{2}$ and $\phi = \pi$ as shown. Its area is simply $\frac{\pi d}{2}$. The probability that the needle will touch or cross one of the parallel lines is therefore $\frac{A}{\frac{\pi d}{2}}$. (The values of d and ℓ have been arbitrarily set to 3 and 2 respectively in this diagram, with no loss of generality.)

The set corresponding to the needle touching or crossing a line is the set of all points for which inequality (12.21) is satisfied.

The area A under the curve in Figure 12.5 is given by

$$\begin{aligned} A &= \frac{1}{2}\ell \int_0^{\pi} \sin\phi \, d\phi \\ &= \left[-\cos\phi\right]_0^{\pi} \\ &= \tfrac{1}{2}\ell[2] \\ &= \ell \end{aligned} \tag{12.22}$$

The probability that the needle touches or crosses a parallel line is therefore equal to

$$\begin{aligned} P(\ell, d) &= \left[\frac{A}{\frac{\pi d}{2}}\right] \\ &= \left[\frac{\ell}{\frac{\pi d}{2}}\right] \\ &= \frac{2\ell}{\pi d} \end{aligned} \tag{12.23}$$

If a probabilistic experiment is repeated independently a great many times, the relative frequency of the event whose probability we are trying to measure, approaches the true probability. Using this principle, it is possible to simulate Buffon's needle experiment on computer, calculate the relative frequency, associate it with the theoretical probability, and thereby derive π.

Incidentally, the derivation of the probability $P(\ell, d)$ *does* involve geometry and trigonometry, and the appearance of π is not so inexplicable after all.

12.6 The Quest for Ever Greater Precision

In 1560, π was known only to 6 decimal places. By 1706, it had been computed to 100 decimal places. This jumped to 707 by 1874. In 1947, π was known to 808 decimal places. The advent of computers meant that by 1957, the number of decimal places had grown to 10,000. By 1967, this number had climbed up to 500,000. In 1997, researchers in Japan computed π to a mind-boggling 51.5 *billion* decimal places.

One may wonder what drives this quest for ever greater precision. As we have already observed, it is not driven by practical need. Because π is ultimately unknowable as a decimal number, it has a mystery to it. Each successive attempt to improve the precision to which it is known, is a quest to tame the untameable.

But apart from aesthetic motives, the decimal expansion of π may reveal new knowledge about number sequences, randomness, and how to generate random numbers.

12.7 Maple Notes

In `Maple`, the constant π is designated by the mathematical constant `Pi`, which is therefore a reserved keyword. In keeping with Maple's philosophy of retaining precision in calculations, typing `Pi;` at the `Maple` prompt [> only displays the symbol π.

What does typing `pi;` do? It also simply displays the symbol π. The difference between the two may be discovered by typing `sin(Pi);` which gives the result 0 whereas typing `sin(pi);` simply echoes back the expression $\sin(\pi)$. To be sure, try typing `beta` and see whether you get β echoed. Evidently, `Maple` knows its Greek lowercase letters!

If you try to assign a value to `Pi`, by typing, say, `Pi := 22/7;` you will get an error message saying that you tried to assign a value to `Pi` which is protected.

If you want a numerical value for π from `Maple`, you need to evaluate the constant by typing `evalf(Pi);` which gives the number 3.141592654. If you wanted π to be evaluated to 50 *digits,* not decimal places, you would type `evalf(Pi, 50);` and you world see the result to 50 glorious digits.

`Maple` uses an environment variable called `Digits` that controls the number of digits when `Maple` calculates with floating point numbers. To find the default value of this variable, type `Digits;`. Very likely, you saw the answer 10 which is the default setting.

Now, type `kernelopts(maxdigits);` and you will see an answer that depends on your platform. You may see a number close to 2^{28}. This is the theoretical maximum number of digits that Maple can display.

Now for some questions to ponder about.

How does `Maple` perform arbitrary precision arithmetic that enables results to be output, correct to, say, 50 digits?

Also, how does `Maple` respond so promptly when you type `evalf(Pi, 50);`?

What algorithm(s) does `Maple` use to calculate π?

Finally, some information about useful Maple-related websites. The Maple Application Center is a very useful web site from which to learn about `Maple`. You should also visit the Maple Student Center which contains many resources for students.

Gregory Moore has an introductory web site on π using `Maple`. It is part of the PowerTools Geometry package.

12.8 Book References

The material in this chapter has been compiled mainly from four books. The first is the encyclopaedic source book on π by Berggren et al. [1]. It contains a wealth of historical information and has facsimiles of many original publications or their translations. Interestingly, the next two references are written by engineers. Beckmann's book [2], although somewhat dated and opinionated, gives detailed historical accounts of efforts at computing π. The book by Banks [3] is a delightful, instructive and entertaining romp through some areas of applied mathematics. It is easy to read and devotes three chapters to famous numbers and number sequences. Another interesting and accessible popular exposition, exclusively on π, is the book by Posamentier and Lehmann [4]. Lastly, the popular book by Blatner [5] is historically informative and instructive.

12.9 Web Resources

If you are unsure about a mathematical term, or definition, I would recommend, as first port of call, Eric Weisstein's World of Mathematics [6]. It is a searchable, authoritative and encyclopaedic web site. Although Weisstein is himself an astronomer, his enduring love of Mathematics has resulted in this treasure trove of mathematical information on the web, from which all can benefit.

The lives of mathematicians have been chronicled at several places on the Web. One of the most comprehensive and scholarly—fully searchable, and with many related links—is the MacTutor History of Mathematics archive [7].

12.10 To Probe Further

If you have in any way been intrigued by what is in this chapter, and you have the time and interest to pursue these ideas further, you may wish to look at some of the books or web sites discussed in this section.

The development in Sections 12.3, 12.4 and 12.5 follows that of Beckmann [2]. The numbers quoted in Section 12.6 are from Banks [3].

A faithful account of how Archimedes used the geometry of his day to arrive at his estimates of π is given in the translation of his original works by Heath [8]. Chuck Lindsey's web site gives a web-based account of the same.

The web site by Peter Alfeld is particularly interesting because it has a Java applet that illustrates the method of Archimedes and allows the user to progressively change the number of sides in the polygons and view the corresponding upper and lower limits on π.

The life of Archimedes makes fascinating reading. He comes across as the archetypical absent-minded scientist, totally engrossed in his work, oblivious of everything else. You can find out more about him from one of these web sites [7, 9].

You may wish to find out more about the formulae for computing π at Eric Weisstein's Pi Formulas. One particularly interesting formula is that of Wallis: it is composed of an infinite product of rational numbers to yield the irrational number π. The book by Blatner [5] gives a brief account of the astounding achievement of the brothers Chudnovsky in developing efficient algorithms for calculating π.

There are two web sites with simulations of the Buffon's Needle experiment. George Reese's site has a discussion and simulation of the experiment. Michael Hurben's site not only has a simulation, but also tracks and displays how close the estimate of π approaches the true value as the experiment is repeated.

If you wish to explore more about π, the Fibonacci sequence, and other numbers, you may want to visit Ron Knott's information-packed site.

Although π is known to more digits than we care to count, such is its allure that the quest for computing π is not over yet and programming enthusiasts still have active projects for this purpose [10].

REFERENCES

[1] Lennart Berggren, Jonathan Borwein and Peter Borwein. *Pi: A Source Book*. New York, NY, USA: Springer-Verlag, 1997.

[2] Petr Beckmann. *A History of Pi*. 2nd ed. Boulder, CO, USA: The Golem Press, 1997.

[3] Robert B Banks. *Slicing pizzas, racing turtles and further adventures in applied mathematics*. Princeton, NJ, USA: Princeton University Press, 1999.

[4] Alfred S Posamentier and Ingmar Lehmann. *Pi: A Biography of the World's Most Mysterious Number*. Amherst, NY, USA: Prometheus Books, 2004.

[5] David Blatner. *The Joy of Pi*. London, UK: Penguin Books, 1997.

[6] Eric W. Weisstein. *Eric Weisstein's World of Mathematics*. Published electronically and as a book with ISBN 1584883472. Available from `http://mathworld.wolfram.com/` [cited 17 February 2004]. Feb. 2004.

[7] John J O'Connor and Edmund F Robertson. *The MacTutor History of Mathematics archive*. Published electronically. Available from `http://www-history.mcs.st-andrews.ac.uk/history/index.html` [cited 17 February 2004]. Jan. 2004.

[8] Thomas Heath. *The Works of Archimedes*. New York, NY, USA: Dover, 2002.

[9] Paul Golba. *Archimedes (287? -212 B.C.)* Published electronically. Available from `http://www.shu.edu/projects/reals/history/archimed.html` [cited 17 February 2004]. Sept. 2000.

[10] Sayan Chakraborti. *Calculating Pi :: The open source project for calculating Pi*. Published electronically. Available from `http://projectpi.sourceforge.net/` [cited 17 February 2004]. Jan. 2003.