How Are Numbers Made?

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"Is mathematics part of nature?" This was the question that my polymath friend Solus "Sol" Simkin asked me when we met at an opening ceremony.

"Sol, have a sense of occasion, of time and place, please! This is a social event at a Music School, not the Agora of Athens! Your question is too profound to be discussed here and now. We are planning to go on a tour of Santorini in one month's time. Let our thoughts mingle with those of the profound, ancient Greeks. Until then, I will take a raincheck on your question," I remonstrated.

And so it was that Sol next resumed this conversational thread while we gazed upon the azure sea, from under the shade of an olive grove, atop a hillock on Santorini.

How would you build a universe?

"If were given the power to build a world, how would you do it?" Sol fired away, without any warning.

"Why so outlandish a question? Enjoy the sun and the breeze, and the bleats of the sheep," I replied.

"Have you heard of the Worldbuilding Stackexchange? It 'is a question and answer site for writers/artists using science, geography and culture to construct imaginary worlds and settings'.

"No," I said.

"It is a serious site on the Web where imaginary worlds with negative gravity and entropy may be conceived, discussed, and constructed. My question is not a flippant one."

"I stand educated. But what has mathematics to do with those flights of fancy?" I queried.

Sol said, "Everything". One cannot build a universe without the laws of physics, or the laws of mind. Or the laws of cause and effect. As long as structure, consistency, repeatability, and durability are concerned, one cannot do without numbers. More than atoms, numbers are the building blocks of the world."

We had launched at last into the ocean of discussion proper. And what a majestic premise: that the world is built upon numbers before it could be built upon atoms. I asked Sol to let his canons of unassailable argument boom, waiting passively to be informed and entertained.

Lessons from observing life

"You must have heard of my paternal cousin, once removed, Hieronymus Septimus Simkin, whom I affectionately call Seven. He it was who opened my eyes first to the unguarded secrets staring at us from Nature. He introduced me to books like D'Arcy Wentworth Thompson's classic *On Growth and Form* [1] and the interestingly titled *The Parsimonious Universe* [2]. These books postulate, with incontrovertible evidence, that the Book of Nature derives its intelligence from adaptation, powered by mathematics.

"If Nature is constructed from, or using, mathematics, who constructed mathematics? Are numbers some self-evident LEGO® blocks kindly provided by the Creator of the Universe for human beings to amuse themselves with, or were numbers themselves the very first creation of a colossal intelligence? Numbers. Before light, before atoms, before cause and effect?

"Nature is varied and variegated in a way that defies monotony. There is pattern but also variation. Fractals typify what I am trying to convey. Perhaps, you will remember that Leopold Kronecker was reputed to have said *'Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk'*, meaning that 'God made the integers, all else is the work of man' [3].

Sol waited for his exposition to sink in. Given our idyllic surroundings, it was hard not to day dream and slip silently into slumber. He ordered two glasses of Frappé to keep me from slipping into somnolence.

The integers have their place

Sol told me that he started off with the integers that Kronecker had so exalted. "The integers are fundamental because all mathematics begins with counting. The quantitative fields are all founded on the natural numbers we count with. And zero and one are the two most important integers—that I grant you," he had told Seven. "But we cannot stop with integers and exclude everything else."

The square and the circle

Sol had told Seven that the square is *the* four-sided regular polygon. If we consider a square with a side length equal to one unit, by the theorem of Pythagoras, we know that its diagonal has a length equal to $\sqrt{1^2 + 1^2} = \sqrt{2}$ units. And there are proofs aplenty on the Web that this number is in no way an integer. Indeed, it is not even the ratio of two integers. How could something as basic as the diagonal of a square cause the first chink in Kronecker's armour?

Moving from the finite to the infinite, the circle may be viewed as the limiting case of a regular polygon of n sides as $n \to \infty$. And if we tried to find out how many diameters would fit into the circumference of a circle, we do not get an integer, or even an exact fraction, but rather a number that sits between 3 and 4, having decimal places without end, namely, $\pi \approx 3.141592654$. And that number is not an integer by a country mile.

"The natural numbers, the integers, and the rationals—all of these come under Kronecker's integers, but where do we stash $\sqrt{2}$ and 2π amongst them?" Sol had asked Seven.

He was met with bemused silence.

How about the number e?

Encouraged that he had stupefied Seven right at the start, Sol had mounted his next hobby horse, and expounded on e.

"The number e is probably *the* most important number after 0 and 1. And do you know what it is? It is both irrational and transcendental. If you differentiate or integrate, you will find that the exponential function $\exp(x) = e^x$ is an eigenfunction of each operation. If you look into Nature, e holds the pride of place in the normal distribution. If you are into linear system theory you cannot escape e.

But what exactly is the value of e? Again it cannot be confined like an integer: $e \approx 2.718281828$ again in a never ending decimal sequence. This number pervades all of Nature and yet it cannot be bottled into a finite number of digits! Where are the legions of integers to duel with this puny expeditionary force of three numbers? *It appears that Nature prefers the non-integers!*"

"Very poetic and ably put," I nodded in appreciation.

Open secrets

"Helen Keller is reputed to have exclaimed, when she felt the warm glow of a wood-fire, that it was the release of sunbeams that had been trapped long ago in the wood. Her statement is remarkably perceptive, poetic, and precise," Sol continued.

"Unlike ancient sunlight trapped in wood, $\sqrt{2}$, π, and e, cannot be caged in a finite box. These three numbers—that pervade Nature—have decimal forms that clearly announce that they are *not* integers. Their value defies finite expression; only with symbols may we do them justice.

"Do you know why they are open secrets? They are public, staring at us from every square, circle, and signal, and yet, their full form is never revealed. They cannot be contained except in infinity. To know the next decimal place of $\sqrt{2}$, or π, or e, one needs to *compute it* using some formula. Or one could look up a table. But there is no *knowing* that sought after next decimal place, as we know $\frac{1}{2} = 0.5$, with as many zeros stacked at the end as we wish. That sort of closed form is not baked into nature. She prefers the indescribable exactitude of numbers like e." Sol had told Seven.

The rest of Sol's dialogue with Seven was intricately mathematical. I have recorded it here, not as a dialogue, but as logical exposition—complete with references—for the benefit of the casual reader.

The square root of two

Of the triad—$\sqrt{2}$, π, and e—we first consider $\sqrt{2}$. It is the most within our everyday grasp. It evokes geometry rather than number for its precise expression. It is the diagonal of a unit square. And we know that its square root must lie between 1 and 1.5, as the latter squared is 2.25. It may be evaluated painstakingly using algorithms from the age-before-calculators. So, let us look at one of those first.

Manual extraction of √2

The manual extraction of square roots is analogous to long division. The process is both tedious and error-prone. The algorithm uses the fact that the factor 2 figures in any square, witness: $(x + a) = x^2 + 2x + a^2$. So, this particular method makes use of this fact at each step in the "long division" that is done. To see the end result and the working, please see this [4]. For a deeper explanation, read this blog [5]. "I consider this form of working, with pencil and paper, a sophisticated form of torture. Euler of Gauss might have revelled in such pursuits, but count me out!" Sol added as a snide aside.

Two ways of expressing the same number

"I came perilously close to losing the bet to Seven," Sol continued. "You see, I had forgotten that the decimal system was not the only way to represent irrationals and transcendentals in never-ending glory. And I don't mean a change of base. Can you guess what I had forgotten?" Sol asked me during our conversation.

"Nothing from me to egg you on," I said in a sleepy tone. The time, place, and weather had lulled me into a restful somnolence that was ill-suited to mathematical head-scratching.

"It is something that we learn at high school, more as a curiosity than as useful mathematics," Sol continued by way of enticing me with a clue. "Can you guess what it is?"

When I shook my head with a dazed stare, Sol said, "Come on. One last clue. It has to do with fractions."

When I refused to be drawn into guessing what it was, Sol exclaimed, "Continued Fractions!" [10] rousing me into full wakefulness with his thunderous voice.

"Apart from a change of base, there are basically *two* ways of representing real numbers: decimals, and continued fractions. Patterns not discernible in the decimal representation suddenly pop out with pellucid clarity when the same number is expressed as a continued fraction. The advent of computers and 64-bit computation has diverted our attention away from experiencing the periodic beauty of a quadratic irrational, expressed as a continued fraction," Sol went on, lyrically.

"Practically, every irrational, when pressed to computational use, is really a rational approximation to the irrational, to an accuracy that serves the purpose. In that sense, Kronecker was not far from the truth. But the full glory of $\sqrt{2}$, or π, or e can only be encapsulated by the symbols we use for them. Every other, rational expression is but a costumed appearance, not the true persona." Sol was in his element as he expounded.

The charm of continued fractions

Sol then went on to demonstrate his preferred method of evaluating $\sqrt{2}$, using continued fractions. The method seemed like sleight of hand, but it is well-founded, and is also an example of how integers are used to tame the irrationals.

Continued fractions are curious mathematical entities that have surprising properties. They are an alternative rational number representation of real numbers. No finite continued fraction can equate to an irrational number. But a never-ending continued fraction can indeed represent an irrational number. “this is why I say that the rationals and the irrationals meet at infinity,” Sol said with panache.

Continued fraction expansion of a rational number

“Let us start modestly and try to expand a *rational* number using continued fractions,” said Sol. “Give me a scary or hairy rational number, preferably larger than one,” he said.

“What about $\frac{3257}{106}$?” I answered, choosing the two numbers that randomly came to mind.

“Taken,” replied Sol. “Here is a little program that I have written to do it for me. I will then explain the algorithm. We start off by doing plain long division to get:”

$$\frac{3257}{106} = 30 + \frac{77}{106} = 30 + \frac{1}{\frac{106}{77}}$$

“Why do we write it like this? we want to get whole number quotients and whole number remainders and the trick is to always divide the larger number by the smaller, by inverting the remainder fraction,” Sol said. “If you keep in mind that our goal is an improper fraction, you are good to go.”

Continued fraction expansion of √2

The number $\sqrt{2}$ is particularly amenable to a simply derived continued fraction expansion. Consider:

$$\begin{aligned}
\sqrt{2} &= \sqrt{2} && \text{[add and subtract 1 on the RHS]} \\
&= 1 + \sqrt{2} - 1 && \text{[multiply second term on RHS by } \frac{\sqrt{2}+1}{\sqrt{2}+1} = 1\text{]} \\
&= 1 + \frac{(\sqrt{2}-1)(\sqrt{2}+1)}{\sqrt{2}+1} && \text{[difference of two squares]} \\
&= 1 + \frac{1}{1+\sqrt{2}}
\end{aligned} \tag{1}$$

Since the LHS in Equation (1) is $\sqrt{2}$, we may substitute the entire RHS in place of the term $\sqrt{2}$ on the RHS. So doing, we get the following descending staircase of continued fractions:

$$\begin{aligned}
\sqrt{2} &= 1 + \cfrac{1}{1+\sqrt{2}} \\
&= 1 + \cfrac{1}{1 + 1 + \cfrac{1}{1+\sqrt{2}}} \\
&= 1 + \cfrac{1}{2 + \cfrac{1}{1+\sqrt{2}}} \qquad \text{[and recursively substituting for } \sqrt{2} \text{ again]} \\
&= 1 + \cfrac{1}{2 + \cfrac{1}{1 + 1 + \cfrac{1}{1+\sqrt{2}}}} \\
&= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1+\sqrt{2}}}} \\
&= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1+\sqrt{2}}}}} \\
& \qquad\qquad \ddots
\end{aligned} \tag{2}$$

The *congruents* or *approximants* from a continued fraction are partial sums that we may accumulate as approximations to the irrational number, in our case, that we seek to represent. Unfurling the continued fractions into partial sums is a tricky exercise. There are also recurrence relations for them. In our particular case, we ignore the $\frac{1}{1+\sqrt{2}}$ terms that occur in the *denominator* of Equation (2) but count the numerator terms to get a sequence of fractions.

In this way, we start off with 1, followed by $1 + \frac{1}{2} = \frac{3}{2}$. Working our way down, we encounter $1 + \frac{1}{2+\frac{1}{2}} = 1 + \frac{1}{\frac{5}{2}} = 1 + \frac{2}{5} = \frac{7}{5}$. The next convergent after this, when simplified, is $1 + \frac{1}{2+\frac{2}{5}} = 1 + \frac{5}{12} = \frac{17}{12}$.

Sol said that working out these fractions could be a form of torture unless you are particularly fond of or adept at computing them. He himself did not relish such hand computations but preferred to program to get a solution. As it turns out, he was able to get a sequence of the first eight successive convergents from the `Julia` code below:

```
using Pkg
Pkg.add("RealContinuedFractions")
#
# Use the above only for the first time.
#
using RealContinuedFractions
convergents(contfrac(sqrt(2), 15))
```

which gave the following results:

```
15-element Vector{Rational{Int64}}:
      1//1
      3//2
      7//5
     17//12
     41//29
     99//70
    239//169
    577//408
   1393//985
   3363//2378
   8119//5741
  19601//13860
  47321//33461
 114243//80782
 275807//195025
```

The rational fractions above are tabulated with their decimal versions to provide an idea of how the convergents do indeed converge to the “benchmark” decimal value of $\sqrt{2}$ as available on a `Julia` REPL, which is shown below:

```
sqrt(big(2))
1.414213562373095048801688724209698078569671875376948073176679737990732478462102
```

SEE Wolfram Alpha for repeating digits

Table 1: The first fifteen convergents for $\sqrt{2}$.

Convergent	Decimal Value	Period
$\frac{1}{1}$	1.0	0
$\frac{3}{2}$	1.5	0
$\frac{7}{5}$	1.4	0
$\frac{17}{12}$	$1.41\overline{6}$	1

Convergent	Decimal Value	Period
$\frac{41}{29}$	$1.\overline{4137931034482758620689655172}$	28
$\frac{99}{70}$	$1.4\overline{142857}$	6
$\frac{239}{169}$	$1.\overline{414201183431952662721893491124260355029585798816568047337278106508875739}$	78
$\frac{577}{408}$	$1.414\overline{2156862745098039}$	16
$\frac{1393}{985}$	1.41421319796954314...	98
$\frac{3363}{2378}$	1.4142136248948696	140
$\frac{8119}{5741}$	1.414213551646054778387906480929814279079437255859375...	5740
$\frac{19601}{13860}$	$1.41\overline{421356}$	6
$\frac{47321}{33461}$	1.414213562057320405784821559791453182697296142578125...	4780
$\frac{114243}{80782}$	1.414213562427273402490653858532841474585922606521254734965[illegible]01829...	[illegible]
$\frac{275807}{195025}$	1.4142135623637994701340403480571694672107696533203125...	1876

The "benchmark" decimal avlue of $\sqrt{2}$ as available on a `Julia` REPL is shown below. There is agreement at best to four decimal places.

```
sqrt(big(2))
1.414213562373095048801688724209698078569671875376948073176679737990732478462102
```

The results are tabulated for comparison with the decimal value of $\sqrt{2}$ which, to 15 decimal places, is 1.414213562373095. 1.4142135623730951454746218587388284504413604736328125 Julia 1.41421356237309504880168872420969807856967187537694807317667973799073247846210 Julia Big Int $1.414213562373095048801688724209698078569671875376948073176679737990732478462102 Julia Big Int $1.414213562373095048801688724209698078569671875376948073176679737990732478462107038850 Wolfram Alpha $1.4142135623730951454746218587388284504413604736328125 ## Where the rationals and the irrationals meet

"Infinity is where the rationals and irrationals meet," Sol had continues in his discussion with Seven. "And as far as I know, infinity s not an integer. What it is, I do not precisely know." Take

https://r-knott.surrey.ac.uk/Fibonacci/cfINTRO.html#section3.1 https://xlinux.nist.gov/dads/HTML/squareRoot.html https://math.stackexchange.com/questions/2916718/calculating-the-square-root-of-2 https://medium.com/not-zero/how-to-calculate-square-roots-by-hand-21a78b6da9ae https://www.cantorsparadise.com/the-square-root-algorithm-f97ab5c29d6d https://nebus-research.wordpress.com/2014/10/17/how-richard-feynman-got-from-the-square-root-of-2-to-e/ https://medium.com/i-math/how-to-find-square-roots-by-hand-f3f7cadf94bb https://en.wikipedia.org/wiki/List_of_formulae_involving_%CF%80

https://mathworld.wolfram.com/PiFormulas.html

https://math.stackexchange.com/questions/2153619/where-do-mathematicians-get-inspiration-for-pi-formulas

Synesthtetic people can see grayness blurring a beautiful landscape if there is wrong digit in the digit in the decimal expression for pi.

Where the rationals and the irrationals meet

What is the next digit?

Fibonacci has closed form but ...

Continues fractions pi

How to construct a rational from two irrationals

Acknowledgements

Feedback

Please email me your comments and corrections.

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https://swanlotus.netlify.app/blogs/open-secrets.pdf

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Euler and Lagrange proved that periodic continued fractions represent quadratic irrational numbers. https://qr.ae/pKUeyc

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Lists of pi and e expansions

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