

The Wonder That Is Pi

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2004-01-14 | 2024-07-25

This is a sequel to the blog “[The Pi of Archimedes](#)”. Here, we look at π as a number—without explicit reference to its geometric tethering—and explore its remarkable ubiquity in mathematics. As an appetizer, see Figure 1, where the symbol for Pi is surmounted by two very disparate equations defining it. How in all the world could these two different-looking equations be true? But they are indeed!

$$\pi = 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots)$$

π

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)! [1103 + 26390n]}{(n!)^4 396^{4n}}$$

Figure 1: Pi expressed by two very different equations.

The Number Menagerie

Numbers may be compared to animals in a zoo. Each is different, and yet they all share some things in common. The variety and diversity of zoo animals can be challenging. That is why the big cats are grouped together, the herbivores live in another part of the zoo, etc.

Numbers, like animals, have evolved over many centuries into what I call the *number menagerie*. A very elementary picture of this zoo is outlined in my blog “[The Two Most Important Numbers: Zero and One](#)” in case you need to review some definitions.

To appreciate π as a number, we need to be aware of the [taxonomy](#) in the zoo of numbers. It turns out that π is a real number that is transcendental and therefore also irrational. Let us pause for a while to better understand what this means.

Real and Complex Numbers

Broadly speaking, there are two classes of numbers: real numbers, denoted by the set \mathbb{R} , and complex numbers, denoted by the set \mathbb{C} . The difference between the two is that while a real number is a single number, a complex number is a pair, composed of two real numbers, conjoined by the **imaginary unit** i , where $i^2 = -1$. In set-theoretic notation, we write

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}.$$

What then are the reals? We will leave that question aside for the moment, and look instead into what undivided or whole numbers are.

The Integers and Friends

The set \mathbb{N} of *natural or counting numbers* is defined as

$$\mathbb{N} = \{1, 2, 3, \dots, n, n + 1, \dots\}.$$

It is a *countably infinite* set whose members begin with 1 and progress by the addition of 1 to the predecessor. It goes on without end.

Zero is not a natural number and is assigned its own, unnamed set, $\{0\}$.¹

The set of *integers* \mathbb{Z} includes the negative numbers, zero, and the positive numbers:

$$\mathbb{Z} = \{\dots - 3, -2, -1, 0, 1, 2, 3, \dots\}$$

Like \mathbb{N} , \mathbb{Z} is also a countably infinite set.

Rational and Irrational Numbers

Every real number is *either* rational or irrational. If the universe of discourse is the real number set, the rational and irrational numbers are complements of each other. In other words, the *union* of the set of rational numbers and the set of irrational numbers is the set of real numbers.

Rational Numbers

The *rational numbers* are denoted by the set \mathbb{Q} defined to be:

$$\mathbb{Q} = \left\{\frac{a}{b} \text{ where } a, b \in \mathbb{Z} \text{ and } b \neq 0\right\}.$$

The condition imposed on b arises from the stricture that division by zero is not permitted among the integers and reals.²

Let us amplify the consequences of these definitions. Is the number 25 rational? Yes, indeed. But where is the denominator? It is *implicit* and equals 1. The fact that

$$25 = \frac{25}{1}$$

¹Some folks include zero in \mathbb{N} .

²See “**The Two Most Important Numbers: Zero and One**” for the reason why.

makes it clear that 25 is a rational number. Every integer is a rational number.

And it is obvious from the definition that $\frac{2}{3}$ is a rational number. But is $-\frac{11}{16}$ a rational number? Yes, indeed, because the definition depends upon the *integer* a and the *non-zero integer* b , where both integers, being drawn from \mathbb{Z} , can be signed.

When a rational number is expressed as a decimal, that decimal can either terminate or recur without end.

For example, the fraction $\frac{1}{3} = 0.\overline{3}$ has a recurring decimal representation as revealed by division. The line on top indicates the portion of the decimal which recurs—in this case, it is the single digit 3.

When we look at the fraction $\frac{1}{2} = 0.5$, we have an example of a terminating decimal. We could, however, pad zeros after the first decimal place, and claim that even a terminating decimal is recurring; witness that $\frac{1}{2} = 0.5 = 0.5000 \dots = 0.5\overline{0}$. But that is not the whole story.

We can further show that:

$$\frac{1}{2} = 0.5 = 0.5\overline{0} = 0.4\overline{9}.$$

It does seem strange to claim that two different decimals can express the *same* rational number $\frac{1}{2}$.

To see why, let us rewrite $0.4\overline{9}$ as

$$\begin{aligned} 0.4\overline{9} = 0.4999 \dots &= \frac{4}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} \dots \\ &= \frac{4}{10} + 9 \left[\frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} \dots \right] \end{aligned} \quad (1)$$

Consider now the expression in square brackets on the right hand side (RHS) of Equation (1). We can recognize it as a **geometric series** with first term $a = \frac{1}{100}$ and common ratio $r = \frac{1}{10}$. Since $r < 1$, the series is *convergent* and its **sum to infinity** [1] is given by:

$$\begin{aligned} \frac{a}{1-r} &= \frac{\frac{1}{100}}{\left[1 - \frac{1}{10}\right]} \\ &= \frac{\left[\frac{1}{100}\right]}{\left[\frac{9}{10}\right]} \\ &= \left[\frac{1}{100}\right] \left[\frac{10}{9}\right] \\ &= \frac{1}{90}. \end{aligned} \quad (2)$$

Substituting for the terms in square brackets in Equation (1), we get

$$0.4\overline{9} = \frac{4}{10} + 9 \left[\frac{1}{90} \right] = \frac{4}{10} + \frac{1}{10} = \frac{5}{10} = \frac{1}{2}.$$

Even if it seems counter-intuitive that $0.4\overline{9} = 0.5 = 0.5\overline{0} = \frac{1}{2}$, it is mathematically consistent and correct. One may therefore hazard a guess, and correctly so, that *every rational number may be expressed as a recurring decimal*.³

³In this case either the digit 9 or the digit 0 recurs.

Infinite sums have this property of upending our “intuition” about what is correct. So, we have to be extra careful when dealing with the value of a limit as some variable goes to infinity. Moreover, infinity, represented by ∞ is *not* a number and cannot be treated as one. It is simply a convenient shorthand symbol. This caveat should be kept in mind when we encounter infinite sums involving π , as shown for example, in Figure 1.

Irrational Numbers

Irrational numbers are numbers which are *not rational*. The discovery that $\sqrt{2}$ —which is the length of the diagonal of a unit square—was not rational, caused the first ripples of disquiet in the ancient mathematical world, because it upset the prevailing philosophy that whole numbers alone ruled the world.

An irrational number like $\sqrt{2}$ does not have any recurring sequence of digits when expressed as a decimal. But the absence of recurring sequences in the decimal representation of a number should not solely be used to identify a number as irrational, because some rationals with large denominators can and do have very long recurring sequences, which may be difficult to detect by visual inspection. For example, $\frac{8119}{5741}$, which incidentally is a rational approximation to $\sqrt{2}$, has a recurring sequence of length 5740.⁴

The irrationals exceed in number the rationals

The irrationals exceed in number the rational numbers [2]. This fact is stated baldly here because going into the whys and wherefores of this claim will lead us too far astray from our focus in this blog.

Algebraic and Transcendental Numbers

Another dichotomy that may be applied to the real numbers is to segregate them into two classes of numbers: the algebraic numbers and the transcendental numbers. There is one complication: both algebraic and transcendental numbers may be complex. But if we restrict our universe to the real numbers, then these two sets are disjoint, i.e., they do not overlap.

An algebraic number is the root of a non-zero polynomial with integer or rational coefficients. Things have gotten abstract enough thus far for eyes to be glazed. So, let us invoke some examples to revive attention.

We will steer clear of transcendental numbers that are complex because we don’t want to get dizzy right at the beginning 😊 .

Real Rational Integer Irrational Algebraic Transcendental Complex etc.

Hark back to

Venn diagram showing the number taxonomy can be challenging [Venn diagram is shown here](#). This Venn diagram is flawed.

⁴Also called the *period* of a repeating decimal. See <https://www.wolframalpha.com/input?i=8119%2F5741>.

Taxonomy

1. The real numbers are a union of the algebraic and transcendental numbers.
2. The algebraic numbers can be either rational or irrational.
3. All rational numbers are algebraic.
4. No rational number is transcendental.
5. All real transcendental numbers are irrational.
6. The irrational numbers contain *all* transcendental numbers and a subset of the algebraic numbers.

Madhava-Gregory series

What does the equality sign mean for an infinite sum? How can a sum of rationals equal an irrational number? A transcendental number? The meaning of $=$ is therefore not in the sense of $2 + 2 = 4$ or even of $x - 5 = 0$. It is something similar and yet something different, because, despite Cantor, we have not tamed the idea of infinity yet.

Acknowledgements

Feedback

Please **email me** your comments and corrections.

<https://math.stackexchange.com/questions/4675933/is-the-equal-symbol-in-an-infinite-series-misleading-notation>

<https://mathmonks.com/transcendental-numbers>

<https://gfredericks.com/blog/>

<https://www.quora.com/How-do-you-draw-a-Venn-diagram-showing-the-relationship-of-the-set-of-real-rational-irrational-integers-and-non-integer-numbers>

https://www.reddit.com/r/math/comments/725nxu/how_would_you_improve_this_types_of_numbers_venn/

<https://i.pinimg.com/736x/57/db/7f/57db7fb6dd9a4f2649b0d8ae5689ff98-math-teacher-math-class.jpg>

<https://study.com/skill/learn/how-to-construct-a-venn-diagram-to-classify-real-numbers-explanation.html>

References

- [1] Sum to Infinity of a Geometric Series—Maths: Edexcel A Level Pure Maths. Retrieved 28 July 2024 from <https://senecalearning.com/en-GB/revision-notes/a-level/maths/edexcel/pure-maths/4-2-9-sum-to-infinity-of-a-geometric-series>

- [2] George C. 2017. Are there more rational numbers than irrational numbers? | Socratic. Retrieved 28 July 2024 from <https://socratic.org/questions/58c80a37b72cff29df40c794>