The Pi of Archimedes

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This blog began life more than two decades ago, as part of a series of lectures I delivered to some very bright first-year engineering students at an Australian university.

The number π (pronounced "pie") has been recognized from time immemorial because its physical significance can be grasped easily: it is the ratio of the circumference of a circle to its diameter. But who would have thought that such an innocent ratio would exercise such endless fascination because of the complexities it enfolds?

Not surprisingly, some high school students I met recently wanted to know more about π and how it got its unusual value of $\frac{22}{7}$. Accordingly, I have substantially recast and refreshed my original presentation to better accord with the form and substance of a blog. The online references have also been updated to keep up with a rapidly changing Web.

My original intention was to write a single blog on π. But because I did not want it to become yet another overly long *slog*, I have decided to divide the material into two parts.

If there are any errors or omissions, please email me your feedback.

Circumference, diameter, and π

The straight line or geodesic is the shortest distance between any two points on a plane, sphere, or other space. The circle is the locus traversed by a moving point that is equidistant from another, fixed point on a two-dimensional plane. It is the most symmetrical figure on the plane. The diameter is the name given both to any straight line passing through the centre of the circle—intersecting it at two points—as well as to its length. When we divide the perimeter of a circle, more properly called its circumference, C, by its diameter, d, we get the enigmatic constant π, which has a value between 3.141 and 3.142:[1]

$$\frac{C}{d} = \pi. \tag{1}$$

The diameter d is twice the radius r, and substituting for d into Equation (1), we get the well-known school formula:

$$C = \pi d = 2\pi r \approx 2\left[\frac{22}{7}\right] r \approx 6.28r. \tag{2}$$

Note, however, that π is *not exactly equal* to $\frac{22}{7}$. This value is a convenient *rational fraction approximation* for π that serves well in elementary contexts.[2]

[1] The analogous expression for a unit square with a perimeter of 4 units and a diagonal of $\sqrt{2}$ units is $2\sqrt{2} \approx 2.8284$.

[2] See "A tale of two measures: degrees and radians".

You might reasonably wonder whether the ratio of the circumference to the diameter of *any* circle is *always* π. The answer is "Yes", because *all circles are similar*. The ratios of corresponding lengths of similar figures are equal. This idea is also covered in my blog "A tale of two measures: degrees and radians".

The symbol π is the lowercase version of the sixteenth letter of the Greek alphabet. For the history of its use in mathematics, see adoption of the symbol π in Wikipedia.

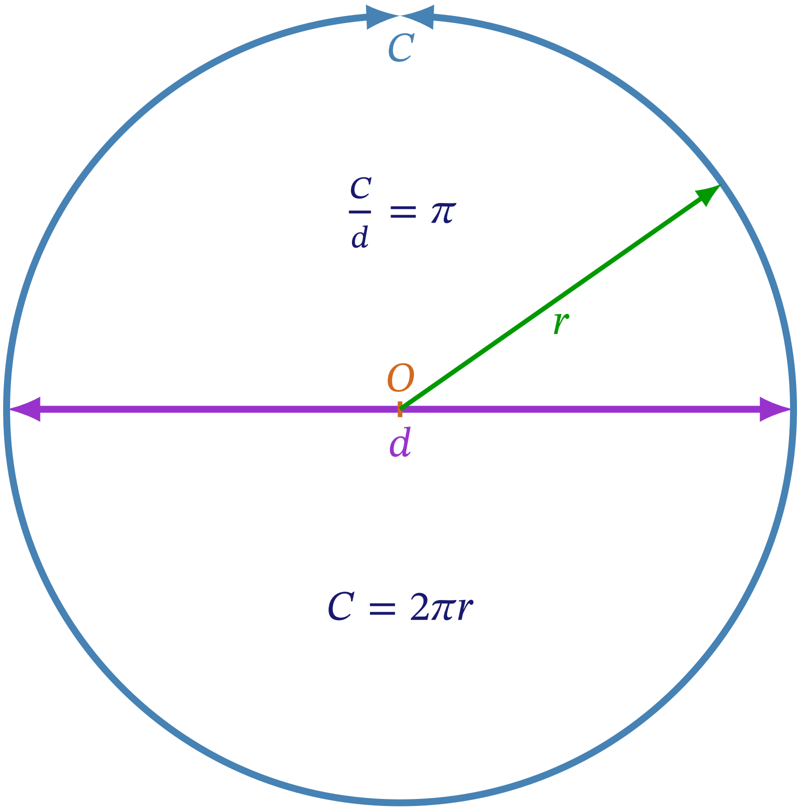

Figure 1: The ratio of the circumference to the diameter of *any* circle is π.

Figure 1 shows the relationships in Equations (1) and (2) pictorially. The circumference of a circle is about 6.28 times its radius. Why this should be so is a secret of Nature, a conundrum of the spacetime [1] we inhabit.[3]

A wonderfully revealing story lies behind this mysterious relationship—between the circumference of a circle and its diameter—and it is due to the labours of one man, in the days when calculators could not be dreamed of, and when neither the decimal system of numbers, nor trigonometry were known. It is the story of Archimedes and his estimate of π.

[3]This article is well worth reading; it will help enlarge your idea of what space is.

Archimedes of Syracuse

Archimedes of Syracuse[4] (Ἀρχιμήδης, 287–212 BCE) was a polymath and genius of the ancient world. He was one of the greatest mathematicians the world has ever known. By today's standards, he would be called a mathematician, physicist, engineer, and astronomer, all rolled into one. He is perhaps most famous for running out of his bathtub naked exclaiming "Eureka"—Greek for "I have found it"—oblivious of those around him. The principle that he had then discovered—that the upthrust on a body submerged in a fluid is equal to the weight of fluid displaced—is known as Archimedes' Principle.

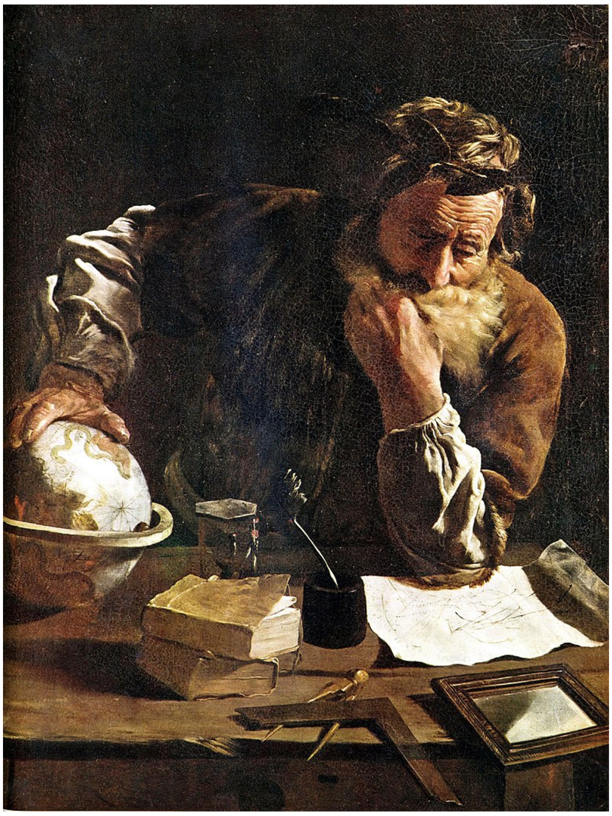

Figure 2: Archimedes of Syracuse.[5]

Among the many accomplishments of Archimedes is his method for estimating π, which was the best approximation for almost 1900 years. And it was not based on using a length of string, superimposing it on a circle, and getting an estimate! ☺

What is even more remarkable is that Archimedes made his discovery *without* the benefit of:

(a) the real numbers;

(b) algebra;

(c) trigonometry;

(d) decimal notation; and

(e) devices like logarithm tables, slide rules, calculators, or computers.

[4]His very name, Archimedes, means "master thinker" in Greek.
[5]Domenico Fetti's 1620 painting entitled *Archimedes Thoughtful*. Public domain.

Instead he applied geometry—including the theorem of Pythagoras—and extracted rational values for square roots, laboriously by hand.

His method is also an excellent geometrical illustration of the idea of a *limit*, with which he was doubtless familiar. It is known that Archimedes was aware of what we now know as integral calculus, and it is possible that he may have anticipated differential calculus as well.

Archimedes devised an ingenious method for estimating π by obtaining successively more accurate values for the circumference of a circle.

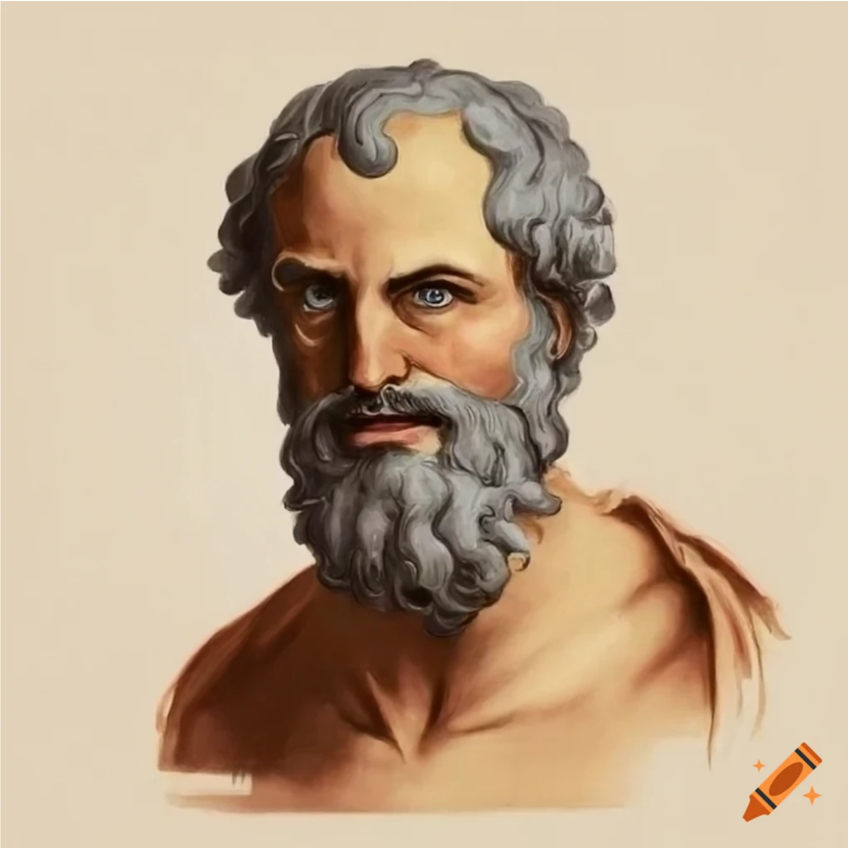

Figure 3: Portrait of Archimedes generated by AI and available at the Craiyon website here. All portraits of Archimedes are flights of fancy rather than true likenesses.

Principles used by Archimedes

The method that Archimedes devised is instructive because it is a synthesis of several principles by which the greatest human minds have furthered scientific progress over time. The abstract principles that Archimedes used to estimate π include:

1. Start with the known and progress to the unknown;
2. Initialize variables;
3. Devise a method of increasing the accuracy through repetition;
4. Stop when the desired accuracy is reached.

These steps constitute what is known as an algorithm. Once such a systematic framework has been put in place, it can be applied in many research domains to aid rapid scientific progress. Algorithms are the basis of modern computing.

Of polygons and circles

The goal of Archimedes was to estimate the *circumference* and *area of a circle* by a systematic and logical method. That π is involved in both of these measurements was somewhat incidental to Archimedes.[6] Nevertheless, his approach is the first well documented account of how to estimate π with reasonable accuracy, and is the focus of this blog.

Archimedes considered a circle, containing an inscribed regular polygon with n sides, and circumscribed by a regular polygon with the same n sides. Figure 4 illustrates this for the case $n = 6$, i.e., with a regular hexagon.

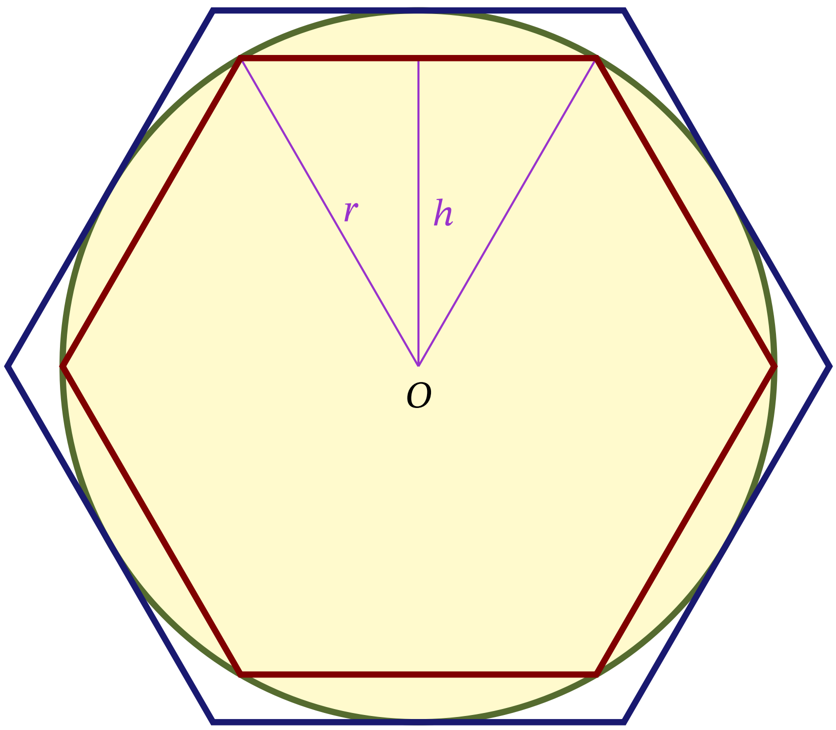

Figure 4: The circumference of the circle in darkolivegreen is bounded from below by the perimeter of the inscribed regular hexagon in maroon and bounded from above by the perimeter of the circumscribed regular hexagon in midnightblue. The circumference of the circle must lie between the perimeters of these two hexagons. The value r is the radius of the circle and the height h—from the centre to the mid-point of a side—is called the apothem.

Archimedes "started with the known" perimeter and area of a regular hexagon. A hexagon of side r has a perimeter of $6r$, and its area is the area of six equilateral triangles of side r, which is $\frac{3\sqrt{3}r^2}{2}$.[8]

Let us tabulate below the variables arising from Figures 4 and 5.

[6]The area of a circle is given by the complementary formula $A = \pi r^2$.
[7]Recall that the area of a triangle is half the product of its base and perpendicular height
[8]See later in this blog for how these numbers are obtained.

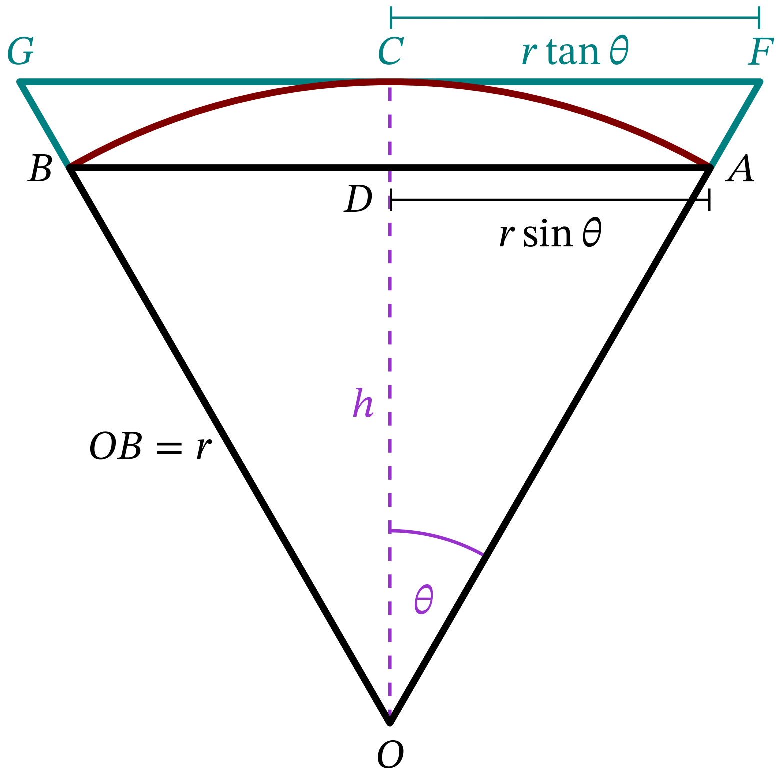

Figure 5: The relationship between the circle and its inscribed and circumscribed regular polygons. The symbol h is used for the apothem in both cases. Note that $OD = h = r\cos\theta$ for the inscribed polygon, whereas $OC = h = r$ for the circumscribed polygon.[7]

Table 1: Circle, inscribed, and circumscribed regular polygons (n-gons).

Parameter	Circle	Inscribed	Circumscribed
Radius	r		
Sides		n	n
Length		$2r\sin\theta$	$2r\tan\theta$
Angle		$\theta(n) = \frac{\pi}{n} = \frac{180°}{n}$	$\theta(n) = \frac{\pi}{n} = \frac{180°}{n}$
Apothem		$h = r\cos\theta$	$h = r$
Area	$A = \pi r^2$	$a(n) = n\sin\theta\cos\theta r^2$	$A(n) = n\tan\theta r^2$
Perimeter	$C = 2\pi r$	$c(n) = 2n\sin\theta r$	$C(n) = 2n\tan\theta r$

When n varies, so do the values of θ and the areas and perimeters; they are therefore shown as functions of n in **??** .

The power of repetition

Archimedes started with regular hexagons and successively *doubled* the number of sides, until he had the circle closely sandwiched between two 96-sided-regular polygons—one inscribed; the other circumscribed.

Successively doubling or halving is a fast-converging technique used in numerical estimation, called the bisection method, that is applied to solving a variety of problems. That Archimedes was aware of it, shows how far ahead of his time his thinking was.

When he moved from $n = 6$ to $n = 12$ sides, how did Archimedes estimate the respective perimeters without the aid of trigonometry? He used geometry and the Pythagorean theorem, as described later in this blog. This online presentation [2] explains his method with fidelity. A more recent account [3] is more accessible, but relies to some extent on trigonometry. He thereby obtained recurrence relations that gave the current perimeter from the previous one.

Archimedes repeatedly calculated *rational approximations* to π until he was satisfied with the accuracy. The principle of the method is clearly illustrated in Figures 6 to 10.

The original source material from the man himself, is the book, *Measurement of a Circle* by Archimedes. For an English translation of the book click on this link. It will give you a sense of completeness in your understanding of his method.

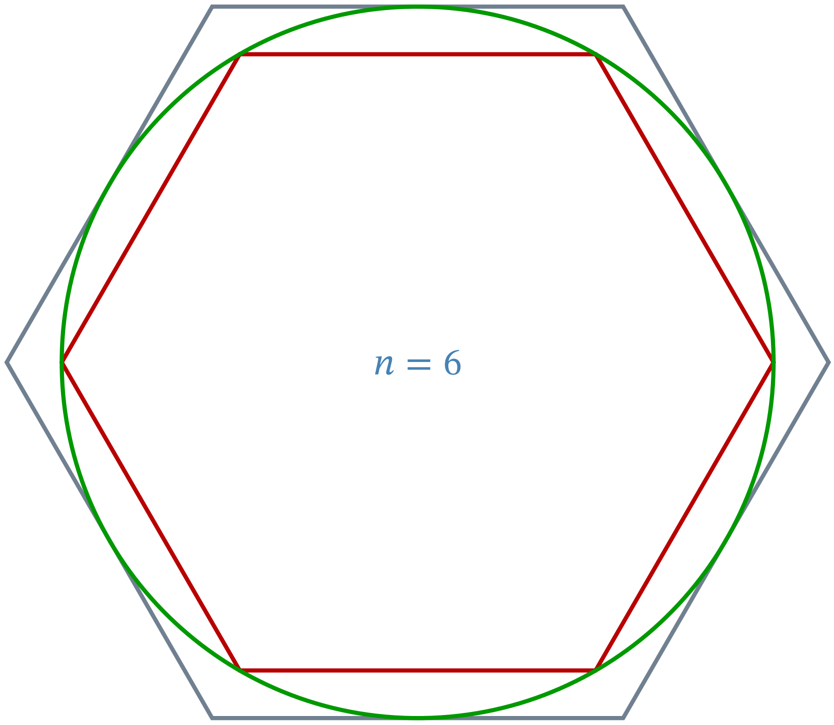

Figure 6: The estimate for π lies between $c(6) = 3.0000 < \pi < C(6) = 3.4641$.

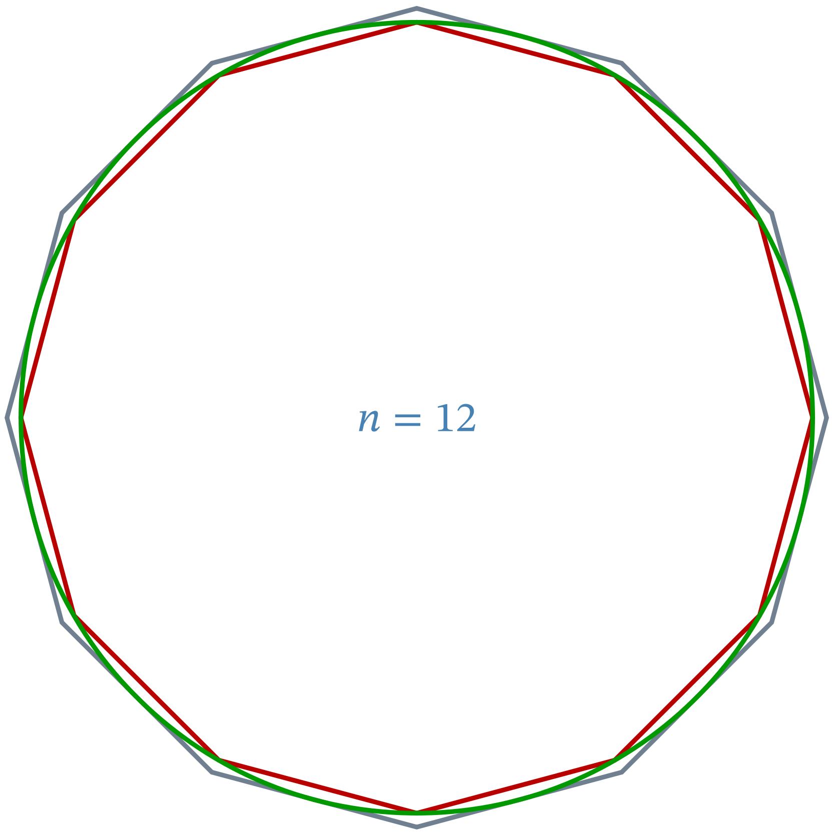

Figure 7: The estimate for π lies between $c(12) = 3.1058 < \pi < C(12) = 3.2153$.

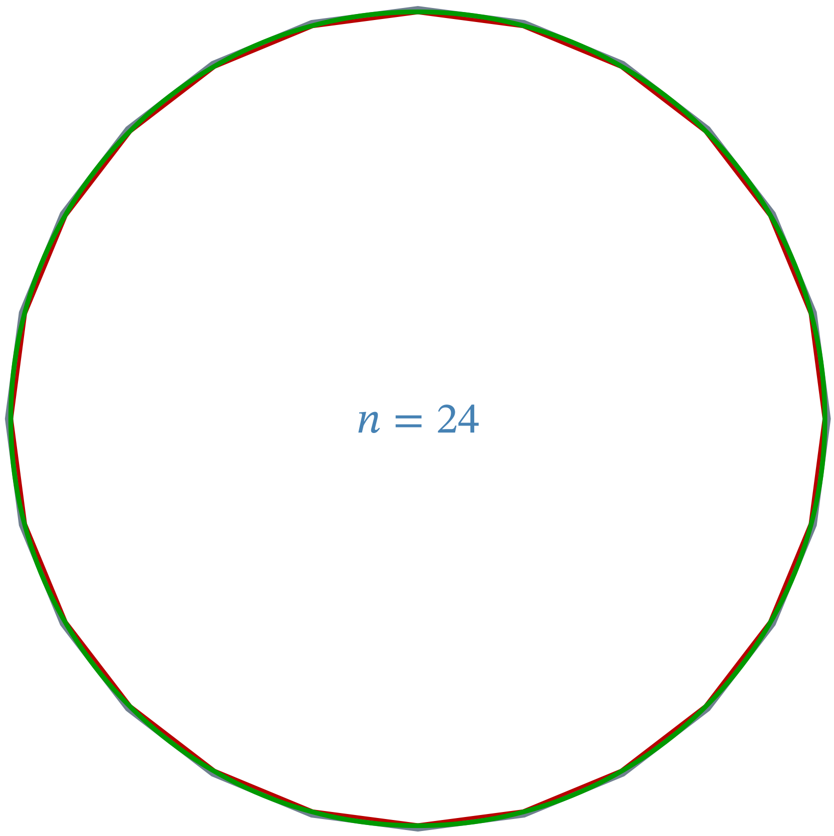

Figure 8: The estimate for π lies between $c(24) = 3.1326 < \pi < C(24) = 3.1596$.

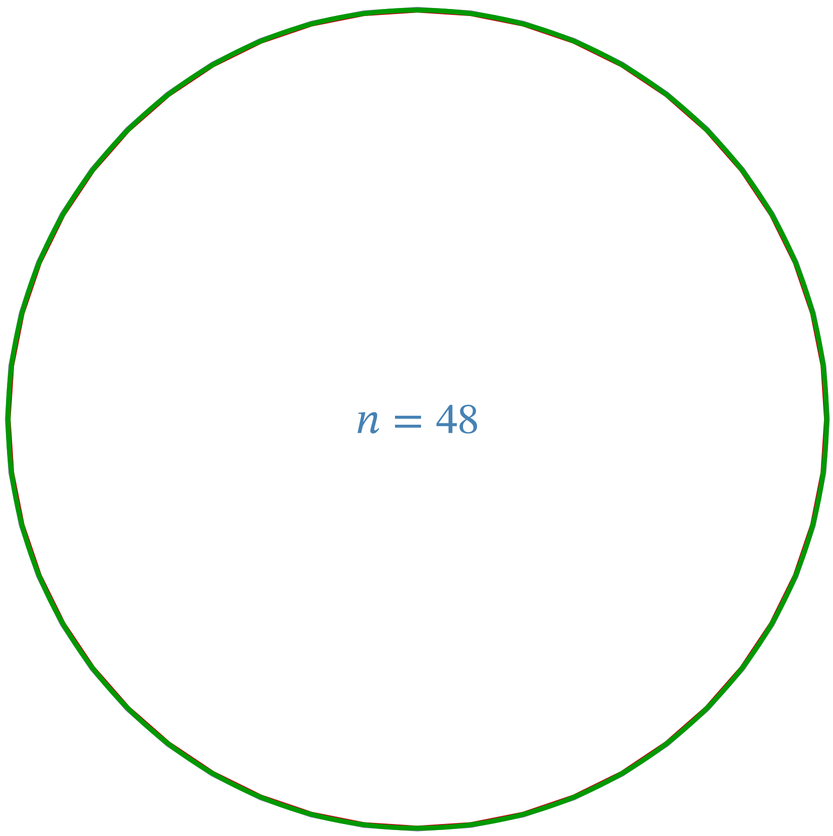

Figure 9: The estimate for π lies between $c(48) = 3.1393 < \pi < C(48) = 3.1460$.

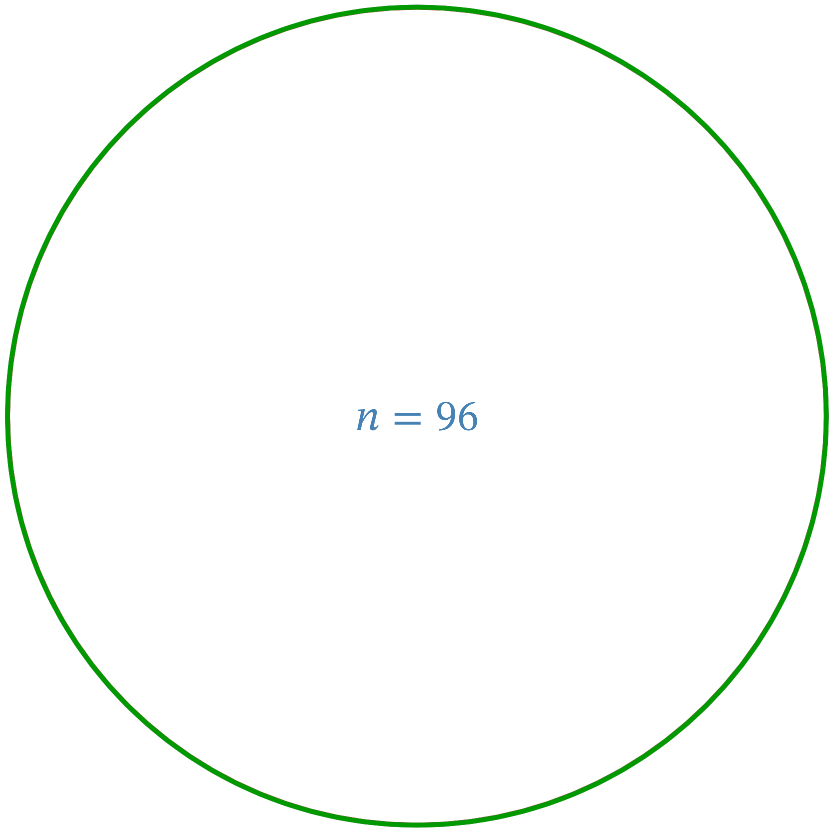

Figure 10: The estimate for π lies between $c(96) = 3.1410 < \pi < C(96) = 3.1427$. Notice in this sequence of images how the circumference of the circle approaches the perimeter of the inscribed and circumscribed hexagons to the point of being indistinguishable from either of them. *The final estimate of Archimedes was* $\frac{223}{71} < \pi < \frac{22}{7}$.

Calculus, before it was discovered

Evaluating the bounds given in **??** and Equation (6) by setting $r = 1$, $n = 6$, and $\theta = \frac{180}{n} = 30°$[9] gives us these values, expressed to four decimal places:

$$\begin{aligned} C_i &= 2n \sin\theta r = 12(\sin 30°) = 12(0.5) && = 6.0000. \\ C &= 2\pi r && = 6.2381. \\ C_c &= 2n \tan\theta r = 12(\tan 30°) = 12\left(\frac{\sqrt{3}}{3}\right) = 4\sqrt{3} && \approx 6.9282. \end{aligned} \tag{3}$$

Archimedes doubled n four times to compute values for regular polygons with 12, 24, 48, and 96 sides. For his last calculation with $n = 96$ and $\theta = \frac{180}{96}° \approx 1.875°$, we have:

$$\begin{aligned} C_i &= 2n \sin\theta r = 2(96)\sin 1.875° \approx 192(0.0327) && \approx 6.2820. \\ C &= 2\pi r && = 6.2381. \\ C_c &= 2n \tan\theta r = 2(96)\tan 1.875° \approx 192(0.0327) && \approx 6.2854. \end{aligned} \tag{4}$$

Note that in the case of 96 sides, we have a *very small angle* θ whose sin and tan are almost equal. This is what gives us tight bounds on the estimate of π. If you know the power series for $\sin\theta$ and $\tan\theta$, you will appreciate even better how the value of π is trapped and squeezed between these two rather close limits.

Remember Equation (4) because it helps us to estimate lower and upper bounds for the value of the circumference.

Archimedes stated his final estimate of π as [4]:

$$\frac{223}{71} < \pi < \frac{22}{7}. \tag{5}$$

Archimedes' application of the squeeze theorem nineteen centuries before the calculus was invented is illustrated in the series of Figures 6 to 10.

If you study the calculus or analysis later on, and encounter the epsilon-delta ($\epsilon - \delta$) definition of a limit, hark back to this example of Archimedes for a graphic and concrete example of how a value may be bounded from below and above, and how it may be squeezed into the limit.

Initial results

If we divide the last row of entries in **??** by $2r$, we get the entries π, $n\sin\theta$, and $n\tan\theta$. We will use these values henceforth as they are directly comparable and relatable to π.

$$\begin{aligned} a(n) < A < A(n) &\implies n\sin\theta\cos\theta < \pi < n\tan\theta \\ c(n) < C < C(n) &\implies n\sin\theta < \pi < n\tan\theta \end{aligned} \tag{6}$$

[9]Rather than use radians with π entering the proceedings, I decided to stick with degrees as units to avoid confusion. If one uses power series to probe further, of course, radians are called for.

From the right hand side of Equation (6), using the inequalities for perimeters, we have

$$n \sin \frac{180°}{n} = n \sin \frac{\pi}{n}, \text{ for the lower bound.} \tag{7}$$

and

$$n \tan \frac{180°}{n} = n \tan \frac{\pi}{n}, \text{ for the upper bound.} \tag{8}$$

Equations (7) and (8) represent respectively the lower and upper bounds on the value of π obtained through the method of Archimedes using *polygon perimeter*.

If, instead, we were to use *polygon area*, the relevant equations will be obtained by dividing the second last row of **??** by r^2. The resulting equations will be:

$$n \sin \frac{180°}{n} \cos \frac{180°}{n} = n \sin \frac{\pi}{n} \cos \frac{\pi}{n}, \text{ for the lower bound.} \tag{9}$$

and

$$n \tan \frac{180°}{n} = n \tan \frac{\pi}{n}, \text{ for the upper bound.} \tag{10}$$

Note that Equation (8) and Equation (10) are equal. Therefore, the upper bound is the same, regardless of whether we consider polygon area or perimeter.

Obviously, the circle may be viewed as a regular polygon whose number of sides, n, has become exceedingly large, or *infinite*. So, as n is increased, we should expect the two bounds to converge to the limiting value of π.

Table 2: Estimates of π from the perimeters and areas of inscribed and circumscribed polygons of n sides.

n	$n \sin \frac{180°}{n}$	$n \tan \frac{180°}{n}$	$n \sin \frac{180°}{n} \cos \frac{180°}{n}$	$n \tan \frac{180°}{n}$
6	3.0000000000	3.4641016151	2.5980762114	3.4641016151
12	3.1058285412	3.2153903092	3.0000000000	3.2153903092
24	3.1326286133	3.1596599421	3.1058285412	3.1596599421
48	3.1393502030	3.1460862151	3.1326286133	3.1460862151
96	3.1410319509	3.1427145996	3.1393502030	3.1427145996
100	3.1410759078	3.1426266043	3.1395259765	3.1426266043
1000	3.1415874859	3.1416029891	3.1415719828	3.1416029891
10000	3.1415926019	3.1415927569	3.1415924469	3.1415927569
100000	3.1415926531	3.1415926546	3.1415926515	3.1415926546
1000000	3.1415926536	3.1415926536	3.1415926536	3.1415926536

The upper and lower bounds are equal up to ten decimal digits when $n = 10^6$, and we might as well declare the problem of estimating π solved. But in the time of Archimedes, trigonometry was not known; only geometry was. Moreover, the decimal system and calculators were far off in the future. We are not done yet!

Figures 6 to 10 together present a compelling case for why the estimate for π is sandwiched between two values and becomes ever closer to the true value of π. It is this engine of logic on which the algorithm runs.

We can view Archimedes' approach through the lens of a mathematical function as well. We could plot discrete values of n against $n \sin \frac{180°}{n}$, and $n \tan \frac{180°}{n}$. However, if we relax the conditions, and move from integers to real values, i.e., from discrete n to continuous x; from $n \sin \frac{180°}{n}$ to $x \sin \frac{180°}{x}$, and from $n \tan \frac{180°}{n}$ to $x \tan \frac{180°}{x}$, we may plot these two curves against x to better visualize the functional relationship. This is shown in Figure 11.

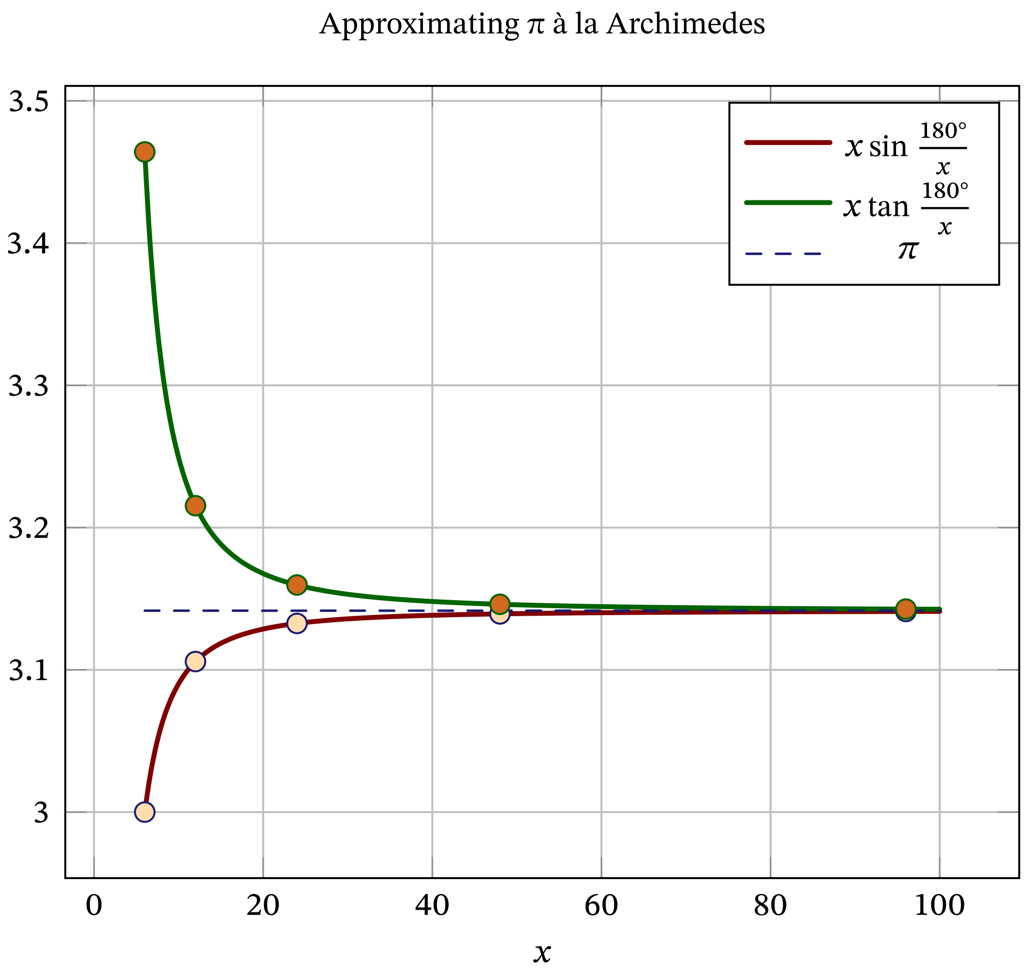

Figure 11: Plot of $x \sin \frac{180°}{x}$ and $x \tan \frac{180°}{x}$ versus x in the domain $[6 : 100]$. The actual data points obtained by Archimedes are shown as coloured circles. As x becomes large, the values of the functions approach π—indicated by a dashed horizontal line which is also a horizontal asymptote to the two curves. The shapes and positions of the two curves themselves eloquently explain why they are called the upper and lower bounds.

Sanity checks

Sanity checks help nip errors in the bud, and are an essential part of problem solving. We perform two of them here.

1. Does $2\pi = 6.2820$, from a calculator, lie within the bounds of Equation (4)? Yes, indeed, and we are home and dry.
2. When n is very large, we expect $n \sin \frac{180^\circ}{n}$ and $n \tan \frac{180^\circ}{n}$ to be closer and closer to the true value of π. This is apparent from **??** . But if we need to be doubly sure, we can set $n = 10^6$ and evaluating on a calculator we get $10^6 \sin \frac{180^\circ}{10^6} = 3.14159$ which is reassuring. Likewise, $10^6 \tan \frac{180^\circ}{10^6} = 3.14159$. This means that to five decimals places, the two bounds are equal to each other and to the actual value of $\pi = 3.14159$. All is well again.

A reflection on triangles and circles

It is interesting that the method of Archimedes leverages the properties of the equilateral triangle, which is the regular polygon with the smallest number of sides. And it ends with the circle, which is the regular polygon with an infinite number of sides. Linking both these extremes is trigonometry, which we have used extensively thus far. This deep connection between the triangle, the circle, and the trigonometric functions also explains why they are sometimes called the *circular functions*.[10]

We now have to backtrack and attempt to retrace the steps Archimedes used to estimate π—*without trigonometry*—to better appreciate his heroic efforts.

The thirty-sixty-ninety right triangle

Archimedes applied the principle "of starting with the known" to initiate his algorithm using a *regular hexagon*, which is a mosaic of six juxtaposed equilateral triangles. We know from symmetry that each angle of an equilateral triangle is 60°. When an equilateral triangle is bisected, we get two right-angled triangles with angles of thirty and sixty degrees, as shown in Figure 12.

The inscribed hexagon, within a circle of *radius* one unit, also has a *side* of one unit. Thus, the hypotenuse of the circle OAP in Figure 12 has a length of 2 units. Moreover, the base OP, resulting from a bisected side, has a length of one a unit. By applying the theorem of Pythagoras, the third side, AP is

$$\sqrt{2^2 - 1^2} = \sqrt{3}. \tag{11}$$

Extracting square roots by hand

The next thing Archimedes needed—and knew how to do—was to compute $\sqrt{3}$, which figures in Equation (11). Finding square roots is a tedious process, not unlike long division, and prone to human error. The patience and doggedness of Archimedes that must have gone into the effort is astounding.

[10] See my blog A tale of two measures: degrees and radians.

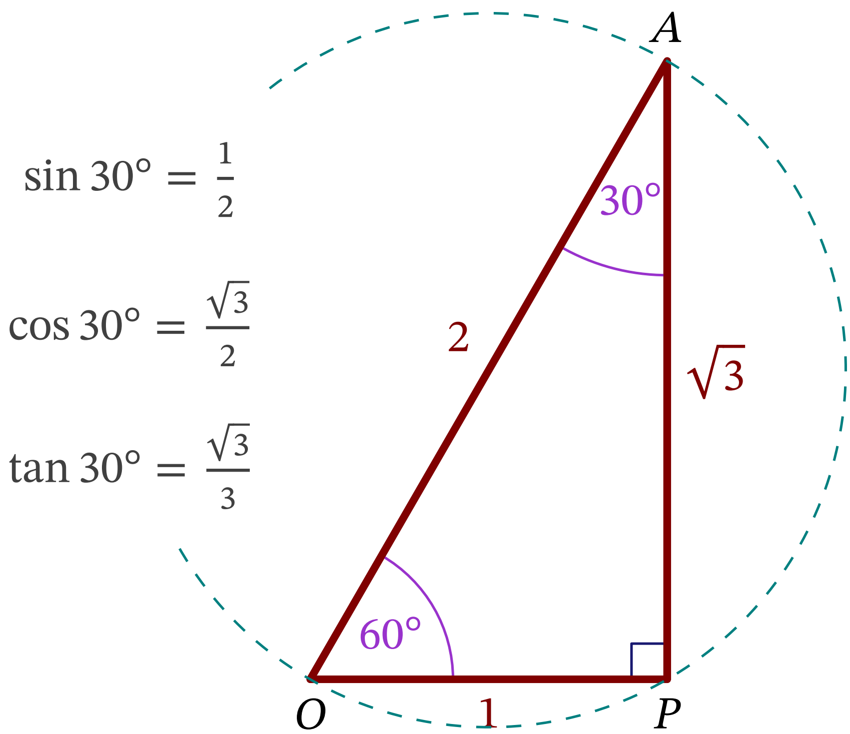

Figure 12: This right-angled triangle, obtained by bisecting an equilateral triangle, must be familiar to all school students. The lengths shown—obtainable from symmetry and the theorem of Pythagoras—allowed Archimedes to start off his process for estimating π. The dotted circle is strictly not necessary in our approach, but pays homage to Archimedes, who relied on triangles within semicircles to enforce right angles.

Archimedes must have known how to extract square roots by hand. Perhaps, he used one of the methods described in my blog "How Are Numbers Built?". He should have known the value of $\sqrt{3}$ as a rational fraction. With remarkable accuracy, he claimed [4] that:[11]

$$1.73\overline{205128} = \frac{1351}{780} > \sqrt{3} > \frac{265}{153} = 1.7\overline{3202614379084 96} \tag{12}$$

Trigonometry and half-angles

Although Archimedes had no trigonometric tables to aid him, he did know the square root of three, and the geometric properties of triangles whose angles were repeatedly bisected.

For example, he calculated the length of the side of a regular dodecagon using the *known* length of the side of a regular hexagon, as he successively doubled the sides of the regular hexagon. He repeated the same algorithmic step—with previous values feeding into calculations for current values—which is a bit like a snake eating its own tail [5].

Figure 13: Ouroboros: a snake eating its own tail.[12]

We will look at the method of Archimedes a little later, but for now, we will try to simulate what he did using trigonometry.

From Figure 12, we know:

$$\begin{aligned} \sin 30^\circ &= \frac{1}{2} \\ \cos 30^\circ &= \frac{\sqrt{3}}{2} \\ \tan 30^\circ &= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{(\sqrt{3})(\sqrt{3})} = \frac{\sqrt{3}}{3} \end{aligned} \tag{13}$$

[11]The rule above the digits indicates the sequence that recurs in the decimal representation. The value of $\sqrt{3}$ rounded to ten decimal places is 1.7320508076.

[12]https://openclipart.org/user-detail/xoxoxo, CC0, via Wikimedia Commons

The half-angle formulae

The whole trick is to

a. move from one estimate to the next, more accurate estimate of π; and

b. use a known value of a trigonometric function to estimate the next unknown value in the chain, *without* resorting to tables of values, or calculators.

The trigonometry of half angles in terms of the full angle [6] helps relate the successive values of θ:[13]

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$$
$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$$

Let us step through this:

1. We know from Figure 12 and Equation (13) that $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$.
2. We calculate the trigonometric ratios for 15° from cos 30° using the half-angle formula:

$$\begin{aligned}
\sin 15° &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} \\
&= \sqrt{\frac{2-\sqrt{3}}{4}} \\
&= \frac{1}{2}\sqrt{2-\sqrt{3}} \\
\cos 15° &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\
&= \sqrt{\frac{2+\sqrt{3}}{4}} \\
&= \frac{1}{2}\sqrt{2+\sqrt{3}} \\
\tan 15° &= \frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}}
\end{aligned}$$

   For comparison with another method we will use later on—in The angle bisector theorem—the value of sin 15° from the equation above is 0.2588190451025208.

[13] All angles are in the first quadrant.

3. Using the value of $\cos 15°$, for $7.5°$[14] we get

$$\begin{aligned}\sin 7.5° &= \sqrt{\frac{1-\frac{1}{2}\sqrt{2+\sqrt{3}}}{2}}\\ &= \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}}\\ \cos 7.5° &= \sqrt{\frac{1+\frac{1}{2}\sqrt{2+\sqrt{3}}}{2}}\\ &= \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}}\end{aligned}$$

4. Using the value of $\cos 7.5°$, for $3.75°$, we get

$$\begin{aligned}\sin 3.75° &= \sqrt{\frac{1-\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}}\\ &= \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}\\ \cos 3.75° &= \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}\end{aligned}$$

5. A pattern can be seen, and a guess may be hazarded that the values for $\theta = 1.875°$ corresponding to $n = 96$ *should* be:

$$\begin{aligned}\sin 1.875° &= \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}\\ \cos 1.875° &= \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}\end{aligned}$$

   Because we guessed, we checked the value we obtained above—expressed as a decimal—with a calculator, and it checked out.

We went through this somewhat painful process for the reasons outlined below because we wanted to simulate the steps Archimedes took [2,3]. It is a proof of concept: we have only evaluated the sine and cosine values, and not estimated the two perimeters. The following points bear noting:

(a) Archimedes knew the sine of 30° and had to work out all other values by hand, without using decimals. That was why we started with a regular hexagon, and retained surds, along with their awkward algebraic manipulation.

[14] Since $\tan\theta = \frac{sin\theta}{\cos\theta}$ we can save ourselves some mathematical labour by leaving out the calculation for $\tan\theta$.

(b) Archimedes only knew rational numbers of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. So, his approximations for $\sqrt{2}$ and $\sqrt{3}$ were expressed as improper fractions that approximated those numbers.

(c) Archimedes did not have positional notation for his calculations and he had to rely on an arithmetical system that we would find forbidding [4].

(d) We have demonstrated how Archimedes used repetition in his estimate of π. He started with $n = 6$ and stopped at $n = 96$. He was justified in doing so, as we have seen the upper and lower bounds converge, as shown in Figure 11.

(e) We cheated when we used the trigonometric half-angle formulae. Archimedes did not have them, but he used right-angled triangles in a semi-circle and leveraged his knowledge of similar triangles and Pythagoras' theorem. We use a slightly different approach, considered next, to get the results he did, without using trigonometry.

The angle bisector theorem

Without using the half-angle formulae of trigonometry, how can we successively obtain expressions for the values of $c(n)$ and $C(n)$ as we halve the angles and double the sides each time? We have to rely on something called the angle bisector theorem from geometry.

This derivation might seem tedious, but it is closer to what Archimedes did in order to establish the recurrence relation that tied the current value to the previous value.

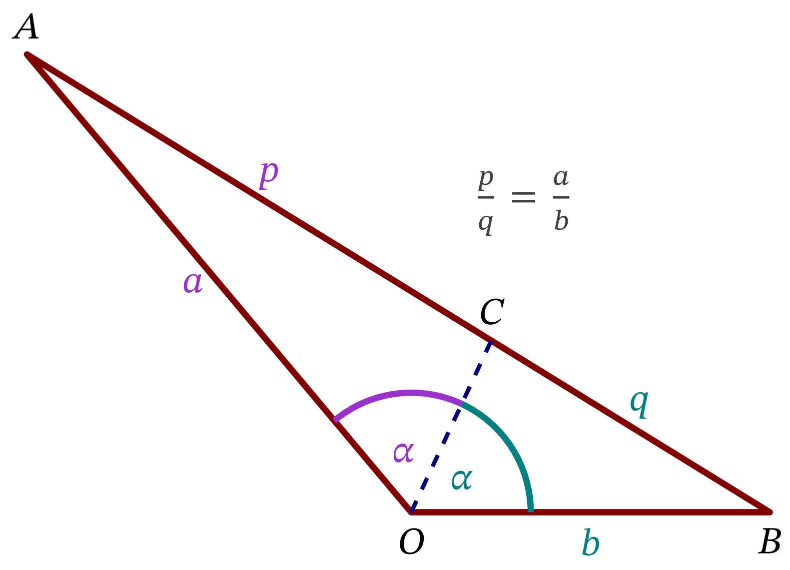

Figure 14: The angle bisector theorem. The relative lengths of the two segments that a triangle's side is divided into by a line that bisects the opposite angle equals the relative lengths of the other two sides of the triangle, as shown on the diagram.

Referring to Figure 14, if the line OC bisects the angle BOA, then the base AB is divided in the same ratio as the corresponding sides. This means

$$\begin{aligned} \frac{AO}{OB} &= \frac{AC}{CB} \text{ which in turn means that} \\ \frac{a}{b} &= \frac{p}{q} \end{aligned} \tag{14}$$

Applying the theorem to a thirty-sixty-ninety right-angled triangle, we get Figure 15 shown below.

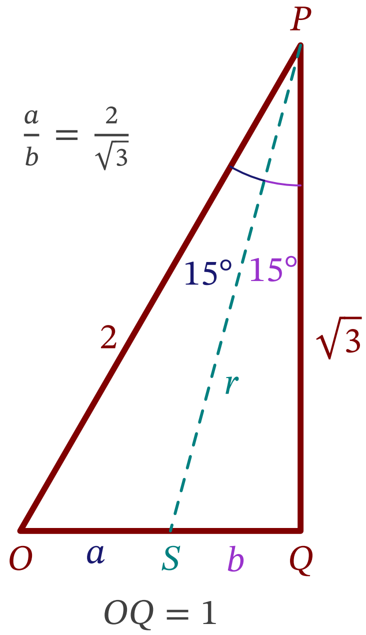

Figure 15: The angle bisector theorem applied to a thirty-sixty-ninety right triangle. The ratio of a to b is the same as the ratio of 2 to $\sqrt{3}$.

Since OQ is one unit,

$$a + b = 1. \tag{15}$$

Also,

$$\begin{aligned} \frac{a}{b} &= \frac{2}{\sqrt{3}} \\ a &= \frac{2}{\sqrt{3}}b \\ &= \frac{2\sqrt{3}}{3}b \end{aligned} \tag{16}$$

Substituting for a from Equation (16) into Equation (15) gives us

$$
\begin{aligned}
\frac{2\sqrt{3}}{3}b + b &= 1 \\
\left[\frac{2\sqrt{3}}{3} + 1\right]b &= 1 \\
\left[\frac{2\sqrt{3}+3}{3}\right]b &= 1 \\
b &= \left[\frac{3}{2\sqrt{3}+3}\right] \\
&= \left[\frac{3(2\sqrt{3}-3)}{(2\sqrt{3}+3)(2\sqrt{3}-3)}\right] \\
&= \left[\frac{3(2\sqrt{3}-3)}{(12-9)}\right] \\
&= 2\sqrt{3}-3
\end{aligned}
\tag{17}
$$

Pythagoras' theorem, applied to right triangle PQS, gives us

$$
\begin{aligned}
PS^2 &= SQ^2 + QP^2 \\
r^2 &= b^2 + \sqrt{3}^2 \\
&= b^2 + 3 \implies r = \sqrt{b^2+3}
\end{aligned}
\tag{18}
$$

Now,

$$
\begin{aligned}
b^2 + 3 &= \left[2\sqrt{3}-3\right]^2 + 3 \\
&= 12 - 12\sqrt{3} + 9 + 3 \\
&= 12(2-\sqrt{3})
\end{aligned}
\tag{19}
$$

Therefore,

$$
\begin{aligned}
r &= \sqrt{b^2+3} \\
&= \sqrt{12(2-\sqrt{3})} \\
&= 2\sqrt{6-3\sqrt{3}}
\end{aligned}
\tag{20}
$$

Putting together Equations (17) and (19), we get

$$
\begin{aligned}
\sin 15^\circ &= \frac{b}{r} \\
&= \frac{1}{2}\left[\frac{2\sqrt{3}-3}{\sqrt{6-3\sqrt{3}}}\right] \\
&\approx 0.2588190451025207
\end{aligned}
$$

And we are done! I do not intend to pursue this tedious process any more. I have pressed on thus far only because I wanted to convey to you an appreciation of the travails that Archimedes must have undergone without any of the modern mathematical conveniences we enjoy, like calculators and computers. *Gauging by the heroic effort he put in to estimate π, Archimedes must have loved mathematics very dearly*.

The fact that we have obtained the same value of $\sin 15°$ by using two different approaches—one using trigonometry, and the other using pure geometry— illustrates the richness that lies ahead, waiting to be explored by some intrepid student of mathematics.

Digression: Denesting Surds

But wait a minute. How do we simplify expressions containing square roots within square roots? Such expressions are called nested surds. Is there an easy way to confirm—without using calculators—that the two results we got are indeed the same number? How do we unpack surds within surds? Because calculators have finite precision, how do we know that the two exact expressions involving surds, on either side of the equality sign below, are indeed *exactly* equal?

$$\frac{1}{2}\sqrt{2-\sqrt{3}} \stackrel{?}{=} \frac{1}{2}\left[\frac{2\sqrt{3}-3}{\sqrt{6-3\sqrt{3}}}\right] \text{ or more simply that}$$
$$\sqrt{2-\sqrt{3}} \stackrel{?}{=} \left[\frac{2\sqrt{3}-3}{\sqrt{6-3\sqrt{3}}}\right]$$

We want to reduce nested surds to their simplest forms so that two dissimilar surds may be compared and declared equal if the they both equal another, possibly third, simpler surd.

Fortunately, there are many resources on the Web, from book chapters, to dedicated web pages, to video presentations, that deal with this interesting, but seldom discussed topic—*denesting surds* [7–10]. Choose any one, or even all, references to learn from, and then tackle the above problem.

For starters, I will go through how to denest $\sqrt{6-3\sqrt{3}}$. Let

$$\sqrt{6-3\sqrt{3}} = \sqrt{a}-\sqrt{b} \text{ where } 0 \le b \le a. \tag{21}$$

We now square both sides:

$$\left(\sqrt{6-3\sqrt{3}}\right)^2 = \left(\sqrt{a}-\sqrt{b}\right)^2$$
$$6-3\sqrt{3} = a-2\sqrt{ab}+b$$

Equating like terms, we get

$$\begin{aligned} a+b &= 6;\ \text{and} \\ \sqrt{27} &= \sqrt{4ab} \\ 4ab &= 27. \end{aligned} \tag{22}$$

Obtaining b in terms of a from Equation (22), and substituting, we get

$$\begin{aligned}
4a(6-a) &= 27 \\
-4a^2 + 24a &= 27 \\
4a^2 - 24a + 27 &= 0 \\
(2a-3)(2a-9) &= 0 \\
a &= \tfrac{3}{2} \text{ or } a = \tfrac{9}{2}
\end{aligned}$$

Because $a \geq b$, we choose $a = \frac{9}{2}$ and $b = \frac{3}{2}$.

Substituting into Equation (21), we get

$$\sqrt{6-3\sqrt{3}} = \sqrt{\frac{9}{2}} - \sqrt{\frac{3}{2}} = \frac{3-\sqrt{3}}{\sqrt{2}} = \frac{3\sqrt{2}-\sqrt{6}}{2}.$$

Note that there are no nested square roots on the right hand side (RHS). The salient point is that, since we are dealing with surds, we should get identical, closed form, exact expressions for both $\sqrt{2-\sqrt{3}}$ and $\left[\frac{2\sqrt{3}-3}{\sqrt{6-3\sqrt{3}}}\right]$ *without using decimals*. And, indeed we do:

$$\sqrt{2-\sqrt{3}} = \left[\frac{2\sqrt{3}-3}{\sqrt{6-3\sqrt{3}}}\right] = \frac{\sqrt{6}-\sqrt{2}}{2}.$$

And that takes some effort, using paper and pencil, or less effort, using software like Geogebra [11] ☺ .

The moral of this section is that π may be approximated as a potpourri of expressions involving surds. Imagine mathematics as a kitchen blender into which selected terms involving surds are put in as ingredients and blended into a smoothie that tastes like π! I find that image mind-boggling.

This completes the modern guided tour of the method Archimedes—used to estimate π—that I had envisaged for this blog. We now turn to the question of its oft-quoted value.

Is π really 22/7?

Is π really equal to $\frac{22}{7}$, as has been drummed into our heads at school?

The answer is a qualified "Yes and No".

"Yes", because, thanks to Archimedes, $\frac{22}{7}$ is an overestimate for π that has survived for nineteen centuries, and served us well in all this time (see Equation (5)). Whenever more accuracy is desired, we can always press into service a slightly better rational approximation, like $\frac{355}{113}$.

"No", because an irrational number like π can never be expressed *exactly* within the confines of a finite numerical representation. Consequently, we use rational approximations, or a decimal representation, at the accuracy desired for our practical purposes.

Philosophically speaking, π can only be represented, truly as it is, by a *symbol*, not by digits.

Geometry might have given birth to π, but it does not confine π exclusively. This wondrous number is a free citizen of all mathematics, and can roam the entire domain. How mathematicians became aware of the ubiquity of π, and what riches have accrued as a result, will engage us in our second blog on π, entitled "The Wonder that is Pi".

To explore further

A well-written, accessible article on the subject of this blog is available online: "How Archimedes showed that pi is approximately 22 by 7" [3]. I urge you to read it.[15] You will then appreciate for yourselves how arduous the process must have been in an age without the benefit of:

1. Trigonometry; he used geometry and the theorem of Pythagoras instead;
2. Algebra; he used geometry and the ratios of the lengths of well-known triangles;
3. Decimal numbers for division; he used fractions instead;
4. Calculators for evaluating square roots.

Another recommended online article is Archimedes and Pi [2] at a website interestingly named https://nonagon.org/.

There is an online Wolfram demonstration [12] showing how estimates of π vary with the *areas* of the inscribed and circumscribed polygons, as n changes. I have used the *perimeters*, $c(n)$ and $C(n)$, rather than the areas $a(n)$, and $A(n)$, in this blog.

Another article on Archimedes' estimation of π is available on this PBS website [13]. Unfortunately, the interactive demonstration, using Macromedia Flash is no longer live.[16]

This online article [14] recounts, with facsimile reproductions from Archimedes' own writings, how he went about estimating π.

Acknowledgements

The computations for **??** were performed using a program written by Nandakumar Chandrasekhar in the Julia programming Language. The output was formatted so that it could be easily cut and pasted into the blog itself, to avoid transcription errors. The source code is available here.

Thanks are also due to Geogebra and Wolfram Alpha for free online mathematical support.

Feedback

Please email me your comments and corrections.

A PDF version of this article is available for download here:

https://swanlotus.netlify.app/blogs/pi-of-archimedes.pdf

[15] This article is all the more remarkable because its first author is a Grade 8 student: proof that deep mathematics is not beyond the school student.

[16] It is paradoxical that modern, newer media age and die faster than old-fashioned manuscripts written on papyrus, or palm leaves, or clay tablets.

References

[1] George Musser. 2018. What Is Spacetime? Retrieved 14 July 2024 from https://www.nature.com/articles/d41586-018-05095-z

[2] Mike Bertrand. 2014. Archimedes and Pi. Ex libris. Retrieved 5 July 2024 from https://nonagon.org/ExLibris/archimedes-pi

[3] Damini D B and Abhishek Dhar. 2022. How archimedes showed that pi is approximately 22 by 7. *Azim Premji University At Right Angles*. Retrieved 2 July 2024 from https://publications.azimpremjiuniversity.edu.in/3356/1/02-DaminiAndAbhishek_PiIs22By7_Final.pdf

[4] Thomas Heath. 2002. *The Works of Archimedes*. Dover, New York, NY, USA.

[5] Joobin Bekhrad. 2017. The ancient symbol that spanned millennia. Retrieved 25 July 2024 from https://www.bbc.com/culture/article/20171204-the-ancient-symbol-that-spanned-millennia

[6] OpenStax. Algebra and Trigonometry. Double-Angle, Half-Angle, and Reduction Formulas. Retrieved 6 July 2024 from https://math.libretexts.org/Bookshelves/Algebra/Algebra_and_Trigonometry_1e_(OpenStax)/09:_Trigonometric_Identities_and_Equations/9.03:_Double-Angle_Half-Angle_and_Reduction_Formulas

[7] Underground Mathematics. 2016. Solution | Nested surds | Thinking about Algebra | Underground Mathematics. Rich resources for teaching A level mathematics. Retrieved 11 July 2024 from https://undergroundmathematics.org/thinking-about-algebra/nested-surds/solution

[8] Stan Brown. 2023. Denesting Radicals (or Unnesting Radicals). Retrieved 11 July 2024 from https://brownmath.com/alge/nestrad.htm

[9] David J Jeffrey and Albert D Rich. 2003. Simplifying square roots of square roots by denesting. From Computer Algebra Systems: A Practical Guide, M J Wester, Ed., Wiley 1999. Retrieved 11 July 2024 from https://cybertester.com/data/denest.pdf

[10] MathSolvingChannel. 2022. A universal method to solve nested radicals (nested square root). Retrieved 11 July 2024 from https://www.youtube.com/watch?v=K5W0DMMukZQ

[11] Markus Hohenwarter. 2001. GeoGebra. The world's favorite, free math tools used by over 100 million students and teachers. Retrieved 13 July 2024 from https://www.geogebra.org/

[12] John Tucker. 2009. Archimedes' Approximation of Pi. Wolfram demonstrations project. Retrieved 13 July 2024 from https://demonstrations.wolfram.com/ArchimedesApproximationOfPi/#more

[13] Rick Groleau. 2003. Approximating Pi. Infinite secrets. Retrieved 13 July 2024 from https://www.pbs.org/wgbh/nova/physics/approximating-pi.html

[14] Gabrielle Birchak. 2019. Archimedes and his Pi: The Great Numerical Hope. Math! Science! History! with Gabrielle Birchak. Retrieved from https://mathsciencehistory.com/2019/10/01/archimedes-and-his-pi-the-great-numerical-hope/

