The Pi of Archimedes

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This blog began life more than two decades ago, as part of a series of lectures I delivered to very bright first-year engineering students at an Australian university.

The number π (pronounced "pie") has been recognized from time immemorial because its physical significance can be grasped easily: it is the ratio of the circumference of a circle to its diameter. But who would have thought that such an innocent ratio would exercise such endless fascination because of the complexities it enfolds?

Not surprisingly, some high students I met recently wanted to know more about π. Accordingly, I have substantially recast and refreshed my original presentation to better accord with the form and substance of a blog. The online references have also been updated to keep up with a rapidly changing Web.

My original intention was to write a single blog on π. But because I did not want it to become yet another overly long *slog*, I have decided to divide the material into two parts.

If there are any errors or omissions, please email me your feedback.

Circumference, diameter, and π

The straight line or geodesic is the shortest distance between any two points on a plane, sphere, or other space. The circle is the locus traversed by a moving point that is equidistant from another fixed point on a two-dimensional plane. It is the most symmetrical figure on the plane. The diameter is the name given both to any straight line passing through the centre of the circle—intersecting it at two points—as well as to its length. When we divide the perimeter of a circle, more properly called its circumference, C, by its diameter, d, we get the enigmatic constant π, which has a value between 3.141 and 3.142:

$$\frac{C}{d} = \pi. \tag{1}$$

The diameter d is twice the radius r, and substituting for d into Equation (1), we get the well-known school formula:

$$C = \pi d = 2\pi r \approx 2\left[\frac{22}{7}\right] r \approx 6.28r. \tag{2}$$

Note, however, that π is *not exactly equal* to $\frac{22}{7}$. This value is a convenient *rational fraction approximation* for π that serves well in elementary contexts.[1]

You might reasonably wonder whether the ratio of the circumference to the diameter of *any* circle is *always* π. The answer is "Yes", because *all circles are similar*. The ratios of corresponding lengths of similar figures are equal. This idea is also covered in my blog "A tale of two measures: degrees and radians".

[1] See "A tale of two measures: degrees and radians".

The symbol π is the lowercase version of the sixteenth letter of the Greek alphabet. For the history of its use in mathematics, see adoption of the symbol π in Wikipedia.

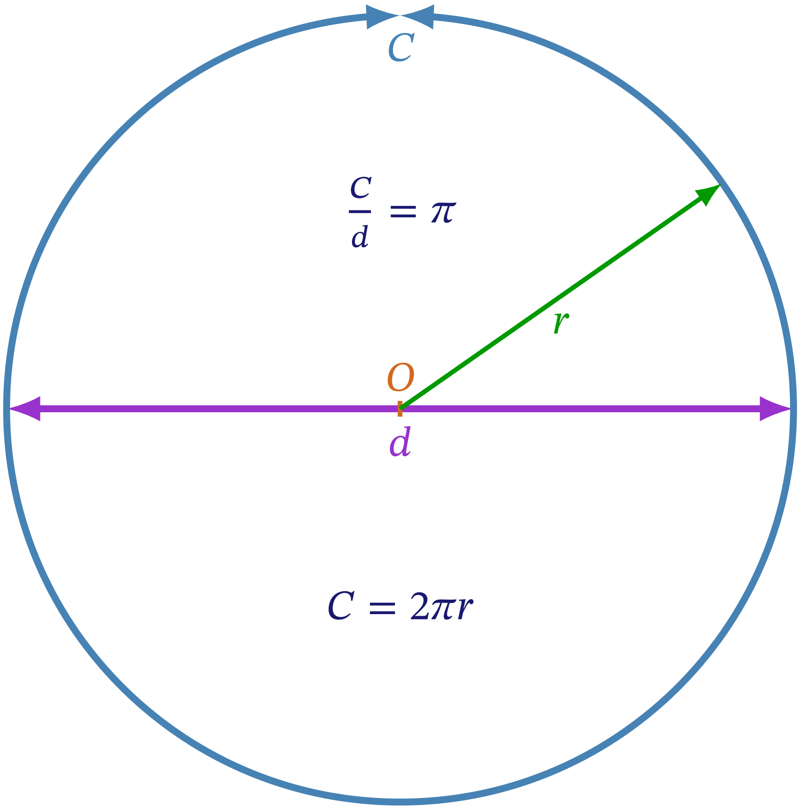

Figure 1: The ratio of the circumference to the diameter of *any* circle is π.

Figure 1 shows the relationships in Equations (1) and (2) pictorially. The circumference of a circle is about 6.28 times its radius. Why this should be so is a secret of Nature, a mystery of the space we inhabit.

A wonderfully revealing story lies behind this mysterious relationship, and it is due to the labours of one man, in the days when calculators could not be dreamed of, and when neither the decimal system of numbers, nor trigonometry were known. That is the story we look at next.

Archimedes of Syracuse

Archimedes of Syracuse[2] (Ἀρχιμήδης, 287–212 BCE) was a polymath and genius of the ancient world. He was one of the greatest mathematicians the world has ever known. By today's standards, he would be called a mathematician, physicist, engineer, and astronomer, all rolled into one. He is perhaps most famous for running out of his bathtub naked exclaiming "Eureka"—Greek for "I have found it"—oblivious of those around him. The principle that he had then discovered—that the upthrust on a body submerged in a fluid is equal to the weight of fluid displaced—is known as Archimedes' Principle.

[2]His very name, Archimedes, means "master thinker" in Greek.
[3]Domenico Fetti's 1620 painting entitled *Archimedes Thoughtful*. Public domain.

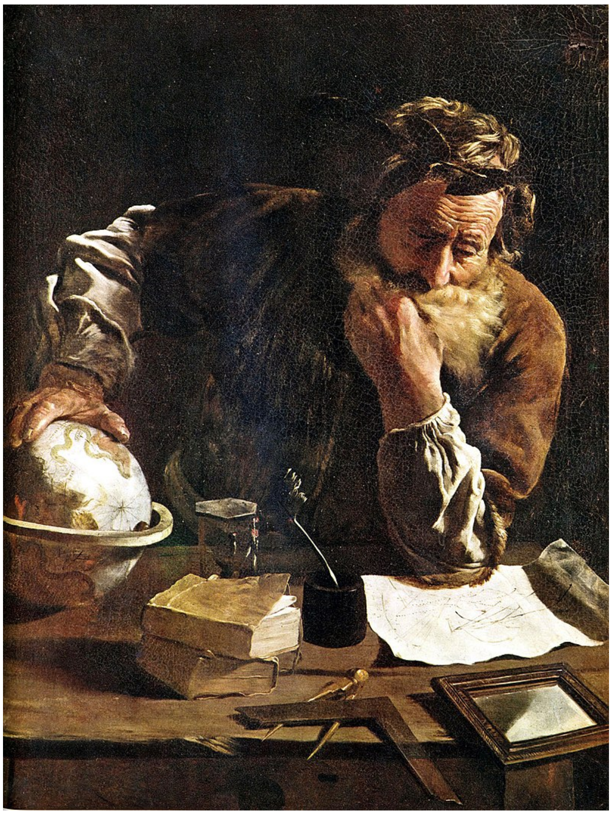

Figure 2: Archimedes of Syracuse.[3]

Among the many accomplishments of Archimedes is his method for estimating π, which was the best approximation for almost 1900 years. And it was not based on using a length of string, superimposing it on a circle, and getting an estimate! ☺

What is even more remarkable is that Archimedes made his discovery *without* the benefit of:

(a) the real numbers;

(b) algebra;

(c) trigonometry;

(d) decimal notation; and

(e) devices like logarithm tables, slide rules, calculators, or computers.

Instead he applied geometry—including the theorem of Pythagoras—and extracted rational values for square roots, laboriously by hand.

His method is also an excellent geometrical illustration of the idea of a *limit*, with which he was doubtless familiar. It is known that Archimedes was familiar with what we now know as integral calculus, and it is possible that he may have anticipated differential calculus as well.

Archimedes devised an ingenious method for estimating π by obtaining successively more accurate values for the circumference of a circle.

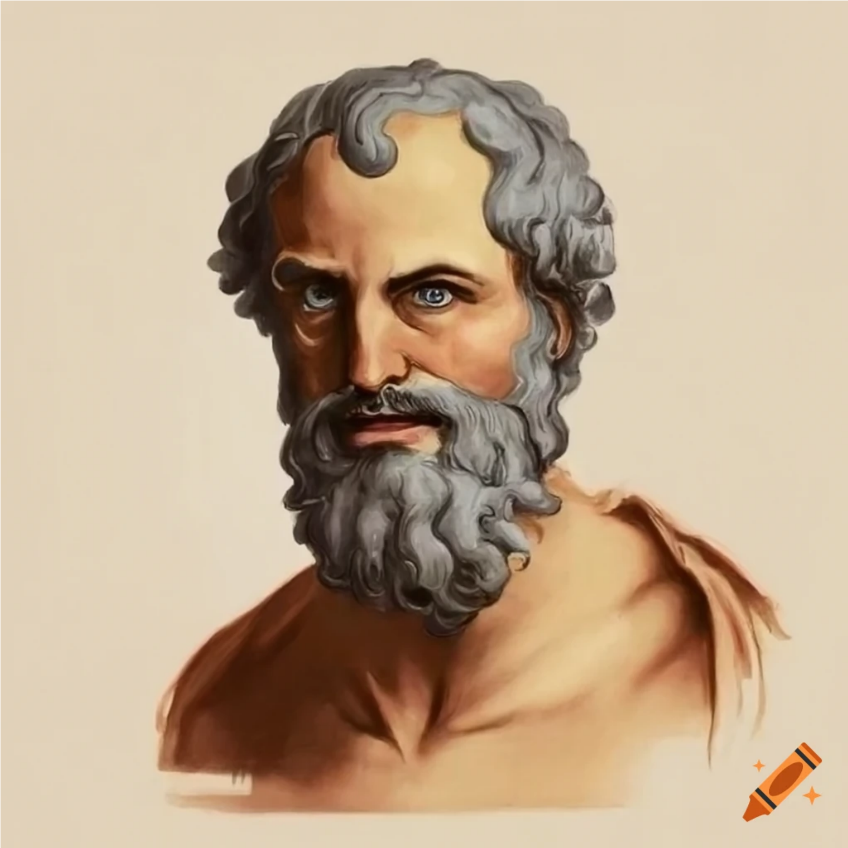

Figure 3: Portrait of Archimedes generated by AI and available at the Craiyon website here. All portraits of Archimedes are flights of fancy rather than true likenesses.

Principles used by Archimedes

The method that Archimedes devised is instructive because it is a synthesis of several principles by which the greatest human minds have furthered scientific progress over time. The abstract principles that Archimedes used to estimate π include:

1. Start with the known and progress to the unknown;
2. Initialize variables;
3. Devise a method of increasing the accuracy of the estimate by *looping through or repeating the same process*;[4]
4. Stop when the desired accuracy is reached.

These steps constitute what is known as an algorithm. Once such a systematic framework has been put in place, it can be applied in many research domains to aid rapid scientific progress. The algorithm is the basis of modern computing.

Of polygons and circles

Archimedes wanted to estimate the *circumference* and *area of a circle* by a systematic and logical method. That π is involved in both of these is somewhat incidental. Nevertheless, his approach is the first well documented account of how to estimate π with reasonable accuracy, and is the focus of our blog.

Archimedes considered a circle, containing an inscribed regular polygon with n sides, and circumscribed by a regular polygon with the same n sides. Figure 4 illustrates this for the case $n = 6$, i.e., with a regular hexagon.

Let us tabulate below the variables arising from Figures 4 and 5.

Table 1: Circle, inscribed, and circumscribed regular polygons (n-gons).

Parameter	Circle	Inscribed	Circumscribed
Radius	r		
Sides		n	n
Length		$2r\sin\theta$	$2r\tan\theta$
Angle		$\theta(n) = \frac{\pi}{n} = \frac{180^\circ}{n}$	$\theta(n) = \frac{\pi}{n} = \frac{180^\circ}{n}$
Apothem		$h = r\cos\theta$	$h = r$
Area	$A = \pi r^2$	$a(n) = n\sin\theta\cos\theta r^2$	$A(n) = n\tan\theta r^2$
Perimeter	$C = 2\pi r$	$c(n) = 2n\sin\theta r$	$C(n) = 2n\tan\theta r$

When n varies, so do the values of θ and the areas and perimeters; they are therefore shown as functions of n in Table 1.

[4] These are called recursion or iteration—as the case may be—in computer science.
[5] Recall that the area of a triangle is half the product of its base and perpendicular height

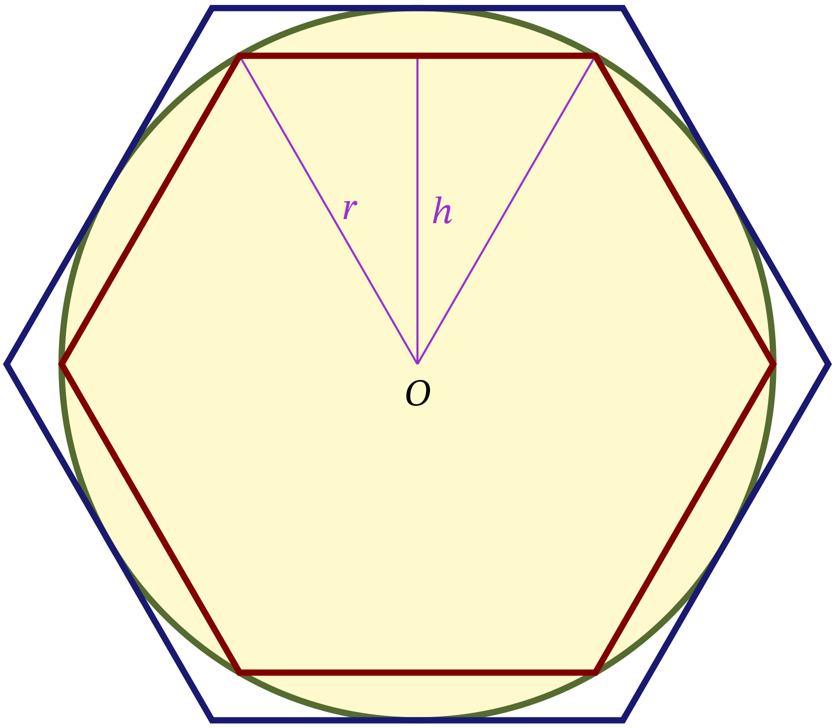

Figure 4: The circumference of the circle in darkolivegreen is bounded from below by the perimeter of the inscribed regular hexagon in maroon and bounded from above by the perimeter of the circumscribed regular hexagon in midnightblue. The circumference of the circle must lie between the perimeters of these two hexagons. The value r is the radius of the circle and the height h—from the centre to the mid-point of a side—is called the apothem.

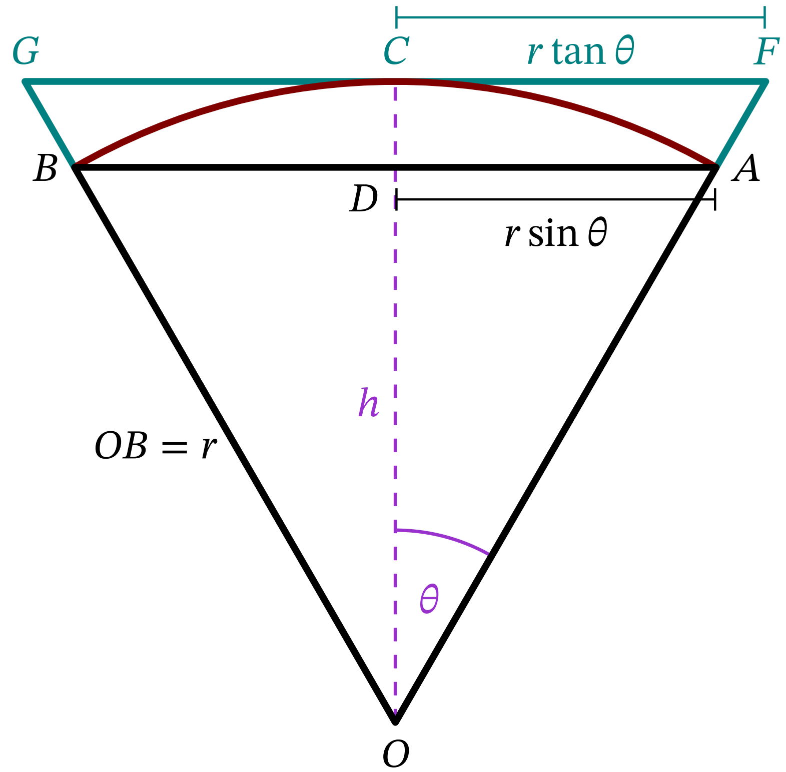

Figure 5: The relationship between the circle and its inscribed and circumscribed regular polygons. The symbol h is used for the apothem in both cases. Note that $OD = h = r\cos\theta$ for the inscribed polygon, whereas $OC = h = r$ for the circumscribed polygon.[5]

Looping

Archimedes started with regular hexagons and successively *doubled* the number of sides, until he had the circle closely sandwiched between two 96-sided-regular polygons—one inscribed; the other circumscribed.

Successively doubling or halving is a fast-converging technique used in numerical estimation, called the bisection method, that is applied to solving a variety of problems. That Archimedes was aware of it, shows how far ahead of his time his thinking was.

When he moved from $n = 6$ to $n = 12$ sides, how did Archimedes estimate the respective perimeters without the aid of trigonometry? He used geometry and the Pythagorean theorem, as described online here [1] and here [2] to obtain recurrence relations that gave the current perimeter from the previous one.

For an English translation of the book *Measurement of a Circle* by Archimedes click on this link. It is the original source material from the man himself, and will give you a sense of completeness in your understanding of his method.

He repeatedly calculated *rational approximations* to π until he was satisfied with the accuracy. The principle of the method is clearly illustrated in Figures 6 to 10.

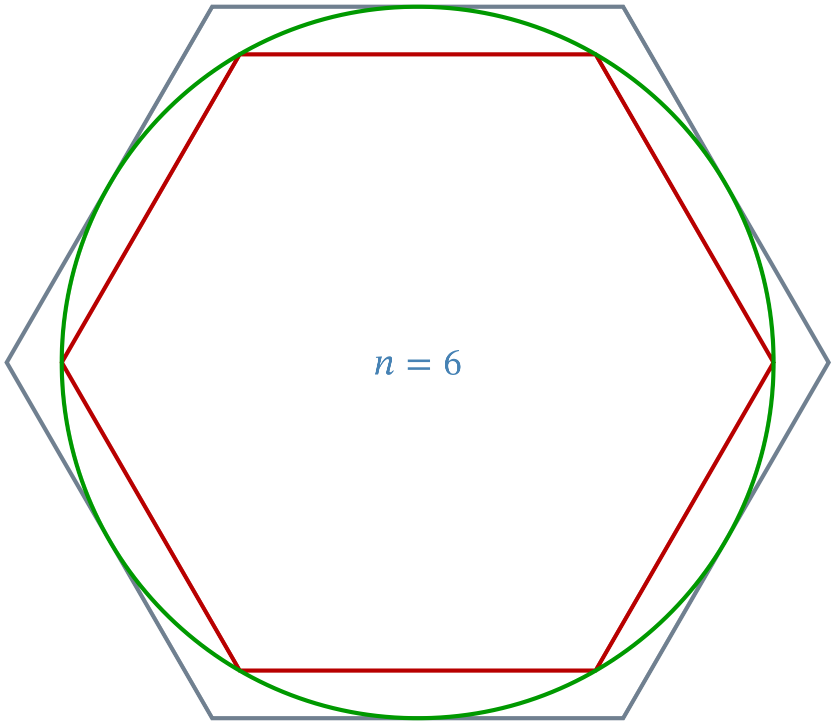

Figure 6: The estimate for π lies between $c(6) = 3.0000 < \pi < C(6) = 3.4641$.

The final estimate of Archimedes was $\frac{223}{71} < \pi < \frac{22}{7}$.

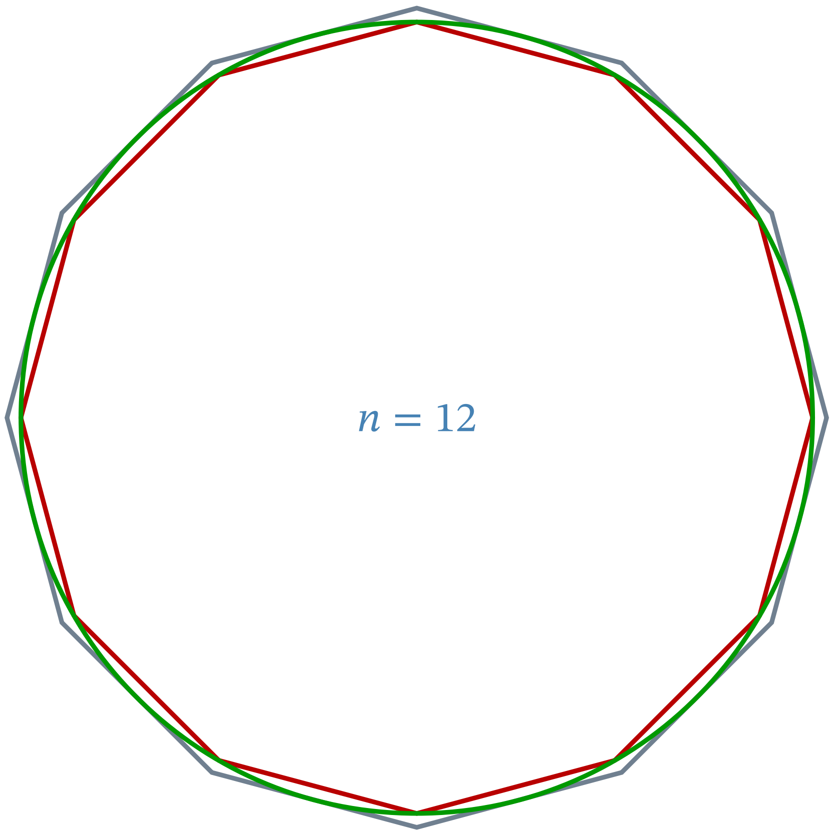

Figure 7: The estimate for π lies between $c(12) = 3.1058 < \pi < C(12) = 3.2153$.

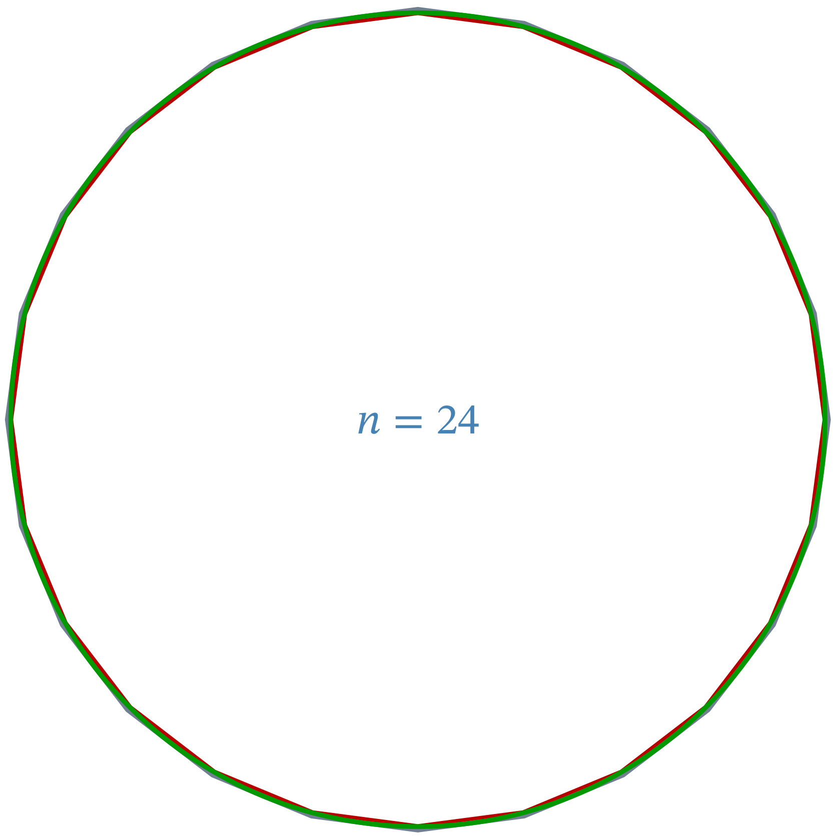

Figure 8: The estimate for π lies between $c(24) = 3.1326 < \pi < C(24) = 3.1596$.

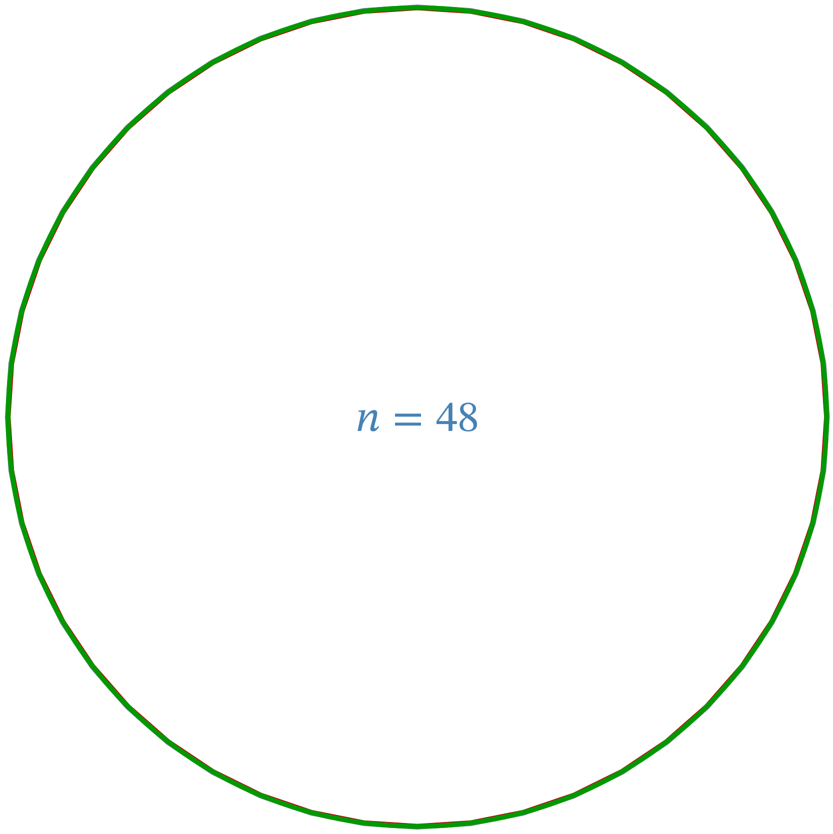

Figure 9: The estimate for π lies between $c(48) = 3.1393 < \pi < C(48) = 3.1460$.

Calculus before it was discovered

Evaluating the bounds given in Table 1 and Equation (5) by setting $r = 1, n = 6$, and $\theta = \frac{180}{n} = 30°$[6] gives us these values, expressed to four decimal places:

$$\begin{aligned}
C_i &= 2n \sin\theta r = 12(\sin 30°) = 12(0.5) && = 6.0000. \\
C &= 2\pi r && = 6.2381. \\
C_c &= 2n \tan\theta r = 12(\tan 30°) = 12\left(\frac{\sqrt{3}}{3}\right) = 4\sqrt{3} && \approx 6.9282.
\end{aligned} \tag{3}$$

Archimedes doubled n four times to compute values for regular polygons with 12, 24, 48, and 96 sides. For his last calculation with $n = 96$ and $\theta = \frac{180}{96}° \approx 1.875°$, we have:

$$\begin{aligned}
C_i &= 2n \sin\theta r = 2(96) \sin 1.875° \approx 192(0.0327) && \approx 6.2820. \\
C &= 2\pi r && = 6.2381. \\
C_c &= 2n \tan\theta r = 2(96) \tan 1.875° \approx 192(0.0327) && \approx 6.2854.
\end{aligned} \tag{4}$$

[6]Rather than use radians with π entering the proceedings, I decided to stick with degrees as units to avoid confusion. If one uses power series to probe further, of course, radians are called for.

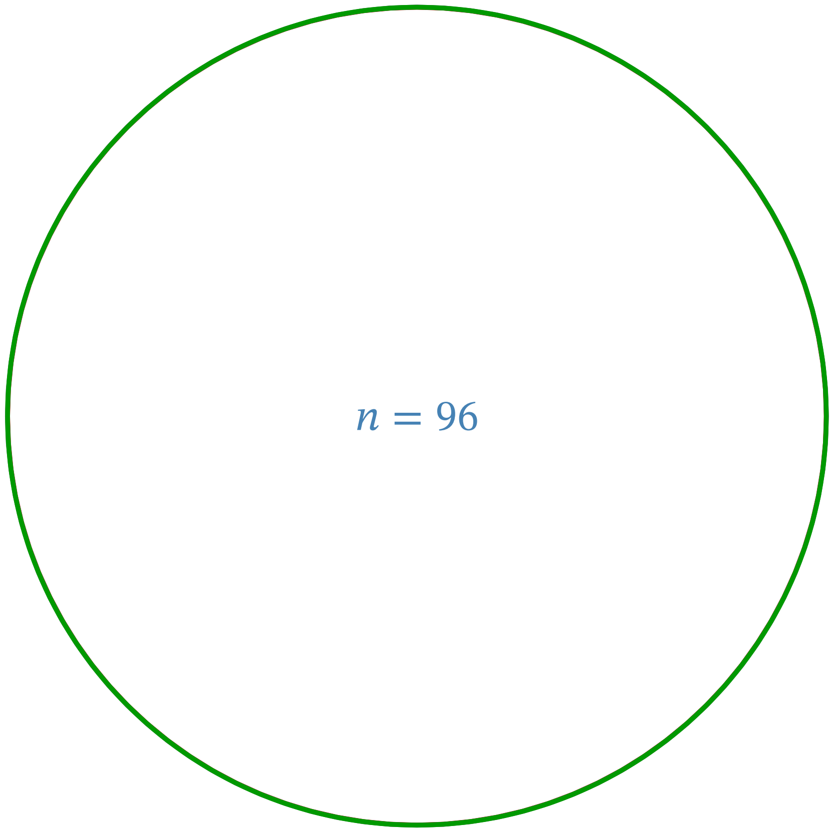

Figure 10: The estimate for π lies between $c(96) = 3.1410 < \pi < C(96) = 3.1427$. Notice in this sequence of images how the circumference of the circle approaches the perimeter of the inscribed and circumscribed heaxgons to the point of being indistinguishable from either of them. *The final estimate of Archimedes was* $\frac{223}{71} < \pi < \frac{22}{7}$.

Note that in the case of 96 sides, we have a *very small angle* θ whose sin and tan are almost equal. This is what gives us tight bounds on the estimate of π. If you know the power series for $\sin\theta$ and $\tan\theta$, you will appreciate even better how the value of π is trapped and squeezed between these two rather close limits.

Remember Equation (4) because it helps us to estimate lower and upper bounds for the value of the circumference. Archimedes's application of the squeeze theorem nineteen centuries before the calculus was invented is illustrated in the series of Figures 6 to 10.

If you study the calculus or analysis later on, and encounter the epsilon-delta ($\epsilon - \delta$) definition of a limit hark back to this example of Archimedes for a graphic and concrete example of how a value may be bounded from below and above and how it may be squeezed into the limit.

Conclusions from initial results

If we divide the last row of entries in Table 1 by $2r$, we get the entries π, $n\sin\theta$, and $n\tan\theta$. We will use these values henceforth as they are directly comparable and relatable to π.

$$\begin{aligned} a(n) < A < A(n) &\implies n\sin\theta\cos\theta < \pi < n\tan\theta \\ c(n) < C < C(n) &\implies n\sin\theta < \pi < n\tan\theta \end{aligned} \tag{5}$$

From the right hand side of Equation (5), using the inequalities for perimeters, we have

$$n\sin\frac{180^\circ}{n} = n\sin\frac{\pi}{n}, \text{ for the lower bound.} \tag{6}$$

and

$$n\tan\frac{180^\circ}{n} = n\tan\frac{\pi}{n}, \text{ for the upper bound.} \tag{7}$$

Equations (6) and (7) represent respectively the lower and upper bounds on the value of π obtained through the method of Archimedes using *polygon perimeter*.

If, instead, we were to use *polygon area*, the relevant equations will be obtained by dividing the second last row of Table 1 by r^2. The resulting equations will be:

$$n\sin\frac{180^\circ}{n}\cos\frac{180^\circ}{n} = n\sin\frac{\pi}{n}\cos\frac{\pi}{n}, \text{ for the lower bound.} \tag{8}$$

and

$$n\tan\frac{180^\circ}{n} = n\tan\frac{\pi}{n}, \text{ for the upper bound.} \tag{9}$$

Note that Equation (7) and Equation (9) are equal. Therefore, the upper bound is the same, regardless of whether we consider polygon area or perimeter.

Obviously, the circle may be viewed as a regular polygon whose number of sides, n, has become exceedingly large, or *infinite*. So, as n is increased, we should expect the two bounds to converge to the limiting value of π.

Table 2: Estimates of π from the perimeters and areas of inscribed and circumscribed polygons of n sides.

n	$n \sin \frac{180°}{n}$	$n \tan \frac{180°}{n}$	$n \sin \frac{180°}{n} \cos \frac{180°}{n}$	$n \tan \frac{180°}{n}$
6	3.0000000000	3.4641016151	2.5980762114	3.4641016151
12	3.1058285412	3.2153903092	3.0000000000	3.2153903092
24	3.1326286133	3.1596599421	3.1058285412	3.1596599421
48	3.1393502030	3.1460862151	3.1326286133	3.1460862151
96	3.1410319509	3.1427145996	3.1393502030	3.1427145996
100	3.1410759078	3.1426266043	3.1395259765	3.1426266043
1000	3.1415874859	3.1416029891	3.1415719828	3.1416029891
10000	3.1415926019	3.1415927569	3.1415924469	3.1415927569
100000	3.1415926531	3.1415926546	3.1415926515	3.1415926546
1000000	3.1415926536	3.1415926536	3.1415926536	3.1415926536

The upper and lower bounds are equal up to ten decimal digits when $n = 10^6$, and we might as well declare the problem of estimating π solved.

But in the time of Archimedes, trigonometry was not known; only geometry was. Moreover, the decimal system and calculators were also in the future. So, we are not done yet!

Figures 6 to 10 together present a compelling case for why the estimate for π is sandwiched between two values and becomes ever closer to the true value of π. It is the engine of logic on which the algorithm runs.

We can view Archimedes' approach through the lens of a mathematical function as well. We could plot discrete values of n against $n \sin \frac{180°}{n}$, and $n \tan \frac{180°}{n}$. However, if we relax the conditions, and move from integers to real values, i.e., from discrete n to continuous x; from $n \sin \frac{180°}{n}$ to $x \sin \frac{180°}{x}$, and from $n \tan \frac{180°}{n}$ to $x \tan \frac{180°}{x}$, we may plot these two curves against x to better visualize the functional relationship. This is shown in Figure 11.

Sanity checks

Sanity checks help nip errors in the bud, and are an essential part of problem solving. We perform two of them here.

1. Does $2\pi = 6.2820$, from a calculator, lie within the bounds of Equation (4)? Yes, indeed, and we are home and dry.
2. When n is very large, we expect $n \sin \frac{180°}{n}$ and $n \tan \frac{180°}{n}$ to be closer and closer to the true value of π. Setting $n = 10^6$ and evaluating on a calculator we get $10^6 \sin \frac{180°}{10^6} = 3.14159$ which is reassuring. Likewise, $10^6 \tan \frac{180°}{10^6} = 3.14159$. This means that to five decimals places, the two bounds are equal to each other and to the actual value of $\pi = 3.14159$. All is well again.

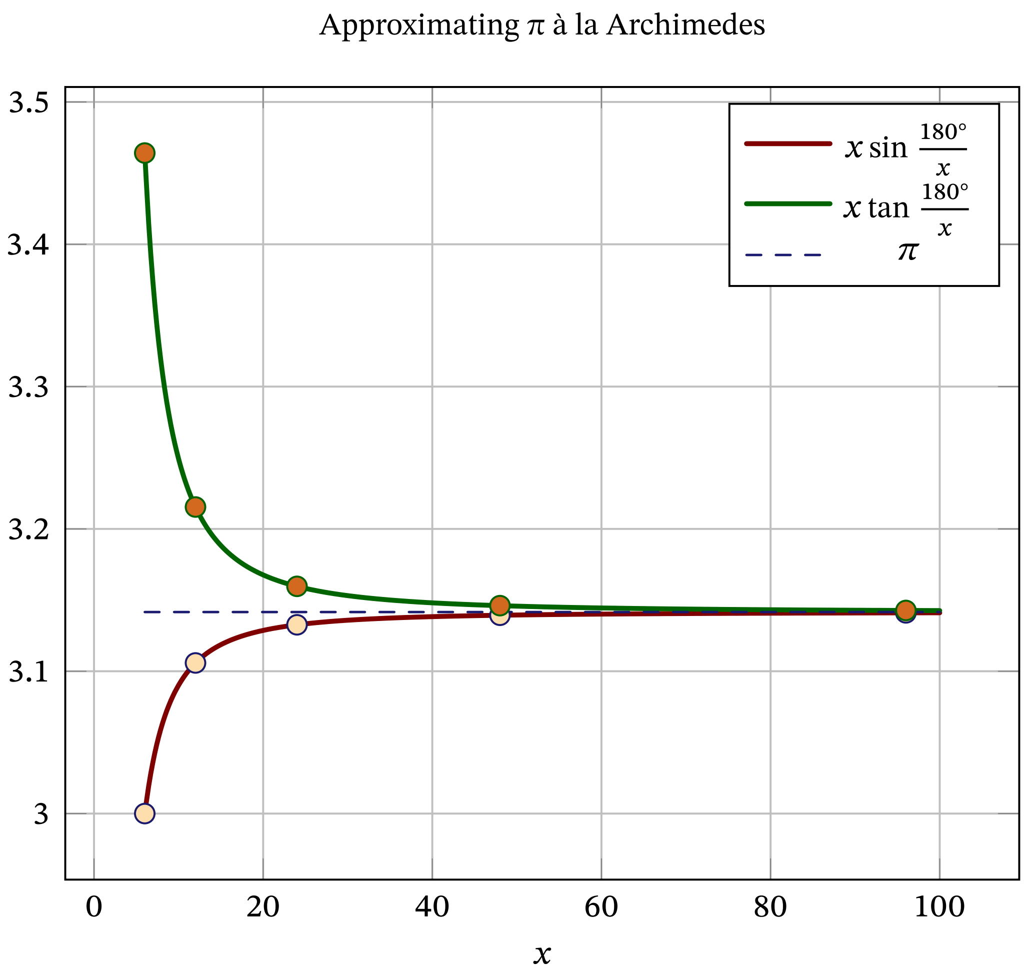

Figure 11: Plot of $x \sin \frac{180°}{x}$ and $x \tan \frac{180°}{x}$ versus x in the domain $[6 : 100]$. The actual data points obtained by Archimedes are shown as coloured circles. As x becomes large, the values of the functions approach π—indicated by a dashed horizontal line which is also a horizontal asymptote to the two curves. The shapes and positions of the two curves themselves eloquently explain why they are called the upper and lower bounds.

A reflection on triangles and circles

It is interesting that the method of Archimedes leverages the properties of the equilateral triangle, which is the regular polygon with the smallest number of sides. And it ends with the circle, which is the regular polygon with an infinite number of sides. Linking both these extremes is trigonometry, which we have used extensively thus far. This deep connection between the triangle, the circle, and the trigonometric functions also explains why they are sometimes called the *circular functions*.[7]

We now have to backtrack and attempt to retrace the steps Archimedes used to estimate π—*without trigonometry*—to better appreciate his heroic efforts.

The thirty, sixty, ninety right triangle

Archimedes applied the principle "of starting from the known" to initiate his algorithm using a *regular hexagon*, which is a mosaic of six juxtaposed equilateral triangles. We know from symmetry that each angle of an equilateral triangle is 60°. When an equilateral triangle is bisected, we get two right-angled triangles with angles of thirty and sixty degrees, as shown in Figure 12.

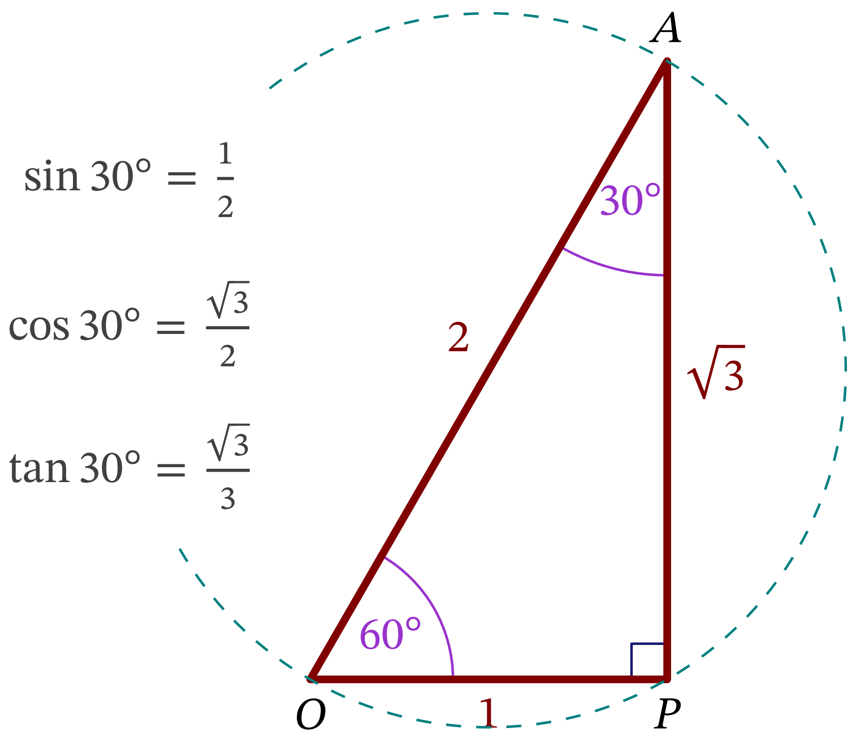

Figure 12: This right-angled triangle, obtained by bisecting an equilateral triangle, must be familiar to all school students. The lengths shown—obtainable from symmetry and the theorem of Pythagoras—allowed Archimedes to start off his process for estimating π.

[7] See my blog A tale of two measures: degrees and radians.

The inscribed hexagon, within a circle of *radius* one unit, also has a *side* of one unit. Thus, the hypotenuse of the circle OAP in Figure 12 has a length of 2 units. Moreover, the base OP, resulting from a bisected side, has a length of one a unit. By applying the theorem of Pythagoras, the third side, AP is

$$\sqrt{2^2 - 1^2} = \sqrt{3}. \tag{10}$$

Extracting square roots by hand

The next thing Archimedes needed—and knew how to do—was to compute $\sqrt{3}$, which figures in Equation (10). Finding square roots is a tedious process, not unlike long division, and prone to human error. The patience and doggedness of Archimedes that must have gone into the effort is astounding.

Archimedes must have known how to extract square roots by hand. Perhaps, he used one of the methods described in my blog "How Are Numbers Built?". He should have known the value of $\sqrt{3}$ as a rational fraction. With remarkable accuracy, he claimed [3] that:

$$1.73\overline{205128} = \frac{1351}{780} > \sqrt{3} > \frac{265}{153} = 1.\overline{7320261437908496} \tag{11}$$

Trigonometry and half angles

Archimedes had no trigonometric tables to aid him. But he did know the square root of three, and the geometric properties of triangles whose angles were repeatedly bisected. He used a previous result to feed values into the next result, as he successively doubled the sides of the regular hexagon. We will look at his method a little later, but for now, we will try to simulate what he did using trigonometry. He repeated the same algorithmic step—with previous values feeding into current values—which is a bit like a snake eating its own tail.

From Figure 12, we know:

$$\begin{aligned} \sin 30^\circ &= \tfrac{1}{2} \\ \cos 30^\circ &= \tfrac{\sqrt{3}}{2} \\ \tan 30^\circ &= \tfrac{1}{\sqrt{3}} = \tfrac{\sqrt{3}}{3} \end{aligned} \tag{12}$$

The half-angle formulae

The whole trick is to

a. move from one estimate to the next, more accurate estimate of π; and

b. use a known value of a trigonometric function to estimate the next unknown value in the chain, *without* resorting to tables of values, or calculators.

The trigonometry of half angles in terms of the full angle [4] helps relate the successive values of θ:[8]

$$\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$$
$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$$

Let us step through this:

1. We know from Figure 12 and Equation (12) that $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$.
2. We calculate the trigonometric ratios for 15° from cos 30° using the half-angle formula:

$$\begin{aligned}
\sin 15° &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} \\
&= \sqrt{\frac{2-\sqrt{3}}{4}} \\
&= \frac{1}{2}\sqrt{2-\sqrt{3}} \\
\cos 15° &= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} \\
&= \sqrt{\frac{2+\sqrt{3}}{4}} \\
&= \frac{1}{2}\sqrt{2+\sqrt{3}} \\
\tan 15° &= \frac{\sqrt{2-\sqrt{3}}}{\sqrt{2+\sqrt{3}}}
\end{aligned}$$

   For comparison with another method we will use later on—in The angle bisector theorem—the value of sin 15° we get from the equation above is 0.2588190451025208.

3. Using the value of cos 15°, we get, for 7.5°[9]

$$\begin{aligned}
\sin 7.5° &= \sqrt{\frac{1-\frac{1}{2}\sqrt{2+\sqrt{3}}}{2}} \\
&= \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}} \\
\cos 7.5° &= \sqrt{\frac{1+\frac{1}{2}\sqrt{2+\sqrt{3}}}{2}} \\
&= \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}}
\end{aligned}$$

[8] All angles are in the first quadrant.

[9] Since $\tan\theta = \frac{sin\theta}{\cos\theta}$ we can save ourselves some mathematical labour by leaving out the calculation for $\tan\theta$.

4. Using the value of $\cos 7.5°$, we get for $3.75°$

$$\sin 3.75° = \sqrt{\frac{1 - \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}}}{2}}$$
$$= \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$$
$$\cos 3.75° = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$$

5. We can see a pattern emerge and we *guess* that the values for $\theta = 1.875°$ corresponding to $n = 96$ *should* be:

$$\sin 1.875° = \frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$$
$$\cos 1.875° = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$$

Because we guessed, we checked the value we obtained above—expressed as a decimal—with a calculator, and it checked out.

We went through this somewhat painful process for the reasons outlined below because we wanted to simulate the steps Archimedes took [1,2]. It is a proof of concept: we have only evaluated the sine and cosine values, and not estimated the two perimeters. The following points bear noting:

(a) Archimedes knew the sine of 30° and had to work out all other values by hand, without using decimals. That was why we started with a regular hexagon, and retained surds, along with their awkward algebraic manipulation.

(b) Archimedes only knew rational numbers of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. So, his approximations for $\sqrt{2}$ and $\sqrt{3}$ were expressed as improper fractions that approximated those numbers.

(c) Archimedes did not have positional notation for his calculations and he had to rely on an arithmetical system that we would find forbidding [3].

(d) We have demonstrated how Archimedes used looping in his estimate of π. He started with $n = 6$ and stopped at $n = 96$. He was justified in doing so, as we have seen the upper and lower bounds converge, as shown in Figure 11.

(e) We cheated when we used the trigonometric half-angle formulae. Archimedes did not have them, but he used right-angled triangles in a semi-circle and leveraged his knowledge of similar triangles and Pythagoras' theorem. We use a slightly different approach, considered next, to get the results he did, without using trigonometry.

The angle bisector theorem

Without using the half-angle formulae of trigonometry, how can we successively obtain expressions for the values of $c(n)$ and $C(n)$ as we halve the angles and double the sides each time? We have to rely on something called the angle bisector theorem from geometry.

This derivation might seem tedious, but it is closer to what Archimedes did in order to establish the recurrence relation that tied the current value to the previous value.

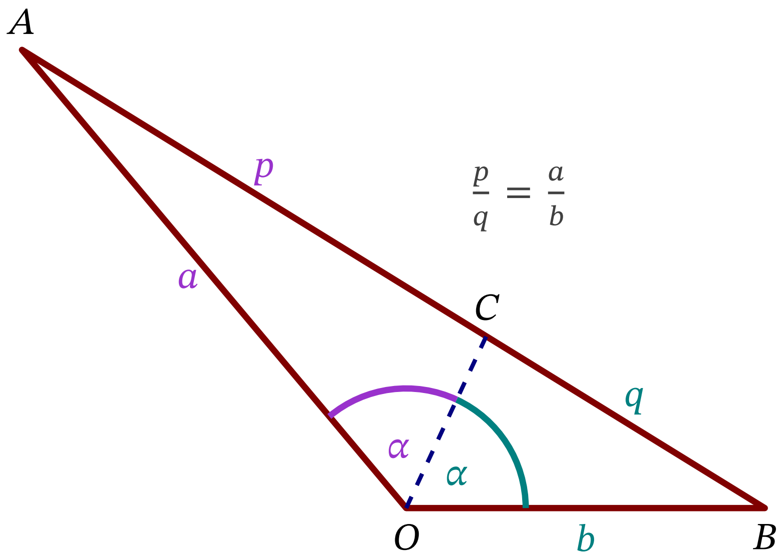

Figure 13: The angle bisector theorem. The relative lengths of the two segments that a triangle's side is divided into by a line that bisects the opposite angle equals the relative lengths of the other two sides of the triangle, as shown on the diagram.

Referring to Figure 13, if the line OC bisects the angle BOA, then the base AB is divided in the same ratio as the corresponding sides. This means

$$\begin{aligned} \frac{AO}{OB} &= \frac{AC}{CB} \text{ which in turn means that} \\ \frac{a}{b} &= \frac{p}{q} \end{aligned} \tag{13}$$

Applying the theorem to a thirty-sixty-ninety right-angled triangle, we get Figure 14 shown below.

Since OQ is one unit,

$$a + b = 1. \tag{14}$$

Also,

$$\begin{aligned} \frac{a}{b} &= \frac{2}{\sqrt{3}} \\ a &= \frac{2}{\sqrt{3}} b \\ &= \frac{2\sqrt{3}}{3} b \end{aligned} \tag{15}$$

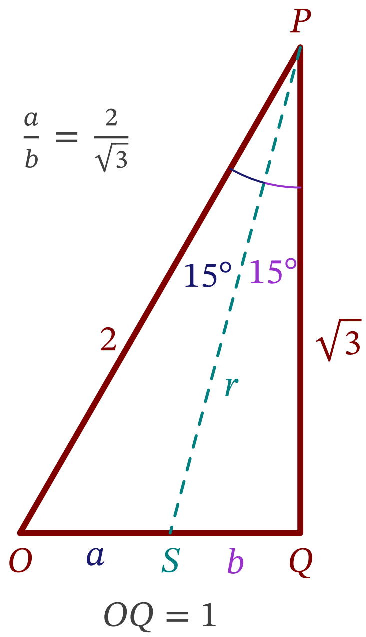

Figure 14: The angle bisector theorem applied to a thirty-sixty-ninety right triangle. The ratio of a to b is that same as the ratio of 2 to $\sqrt{3}$.

Substituting for a from Equation (15) into Equation (14) gives us

$$\begin{aligned} \frac{2\sqrt{3}}{3}b + b &= 1 \\ \left[\frac{2\sqrt{3}}{3} + 1\right] b &= 1 \\ \left[\frac{2\sqrt{3}+3}{3}\right] b &= 1 \\ b &= 2\sqrt{3} - 3 \end{aligned} \tag{16}$$

Pythagoras' theorem, applied to right triangle PQS, gives us

$$\begin{aligned} PS^2 &= SQ^2 + QP^2 \\ r^2 &= b^2 + \sqrt{3}^2 \\ &= b^2 + 3 \implies r = \sqrt{b^2+3} \end{aligned} \tag{17}$$

Now,

$$\begin{aligned} b^2 + 3 &= \left[2\sqrt{3} - 3\right]^2 + 3 \\ &= 12 - 12\sqrt{3} + 9 + 3 \\ &= 12(2 - \sqrt{3}) \end{aligned} \tag{18}$$

Therefore,

$$\begin{aligned} r &= \sqrt{b^2+3} \\ &= \sqrt{12(2-\sqrt{3})} \\ &= 2\sqrt{6-3\sqrt{3}} \end{aligned} \tag{19}$$

Putting together Equations (16) and (18), we get

$$\begin{aligned} \sin 15^\circ &= \frac{b}{r} \\ &= \frac{1}{2}\left[\frac{2\sqrt{3}-3}{\sqrt{6-3\sqrt{3}}}\right] \\ &\approx 0.2588190451025207 \end{aligned}$$

And we are done! I do not intend to pursue this tedious process any more. I have pressed on thus far only because I wanted to convey to you an appreciation of the travails that Archimedes must have undergone without any of the modern mathematical conveniences we enjoy, like calculators and computers. *Gauging by the heroic effort he put in to estimate π, Archimedes must have loved mathematics very dearly.*

The fact that we have obtained the same value of $\sin 15^\circ$ by using two different approaches—one using trigonometry, and the other using pure geometry— illustrates the richness that lies ahead, waiting to be explored by some intrepid student of mathematics.

Digression: Denesting Surds

But wait a minute. How do we simplify expressions containing square roots within square roots? Such expressions are called nested surds. Is there an easy way to confirm—without using calculators—that the two results we got are indeed the same number? How do we unpack surds within surds? Because calculators have finite precision, how do we know that the two exact expressions involving surds, on either side of the equality sign below, are indeed equal?

$$\frac{1}{2}\sqrt{2-\sqrt{3}} = \frac{1}{2}\left[\frac{2\sqrt{3}-3}{\sqrt{6-3\sqrt{3}}}\right] \text{ or more simply that}$$

$$\sqrt{2-\sqrt{3}} = \left[\frac{2\sqrt{3}-3}{\sqrt{6-3\sqrt{3}}}\right]$$

We want to reduce nested surds to their simplest forms so that two dissimilar surds may be compared and declared equal if the they both equal another, possibly third, simpler surd.

Fortunately, there are many resources on the Web, from book chapters, to dedicated web pages, to video presentations, that deal with this interesting, but seldom discussed topic—*denesting surds* [5–8]. Choose any one, or even all, references to learn from, and then tackle the above problem.

For starters, I will outline how to denest $\sqrt{6-3\sqrt{3}}$. Let $\sqrt{6-3\sqrt{3}} = \sqrt{a} - \sqrt{b}$ where $0 \leq b \leq a$. We square both sides and obtain expressions for $a+b$ and ab. This will result in a quadratic equation with two solutions. We choose the larger value for a. The two solutions in this case are $a = \frac{9}{2}$ and $b = \frac{3}{2}$ leading to

$$\sqrt{6-3\sqrt{3}} = \frac{3-\sqrt{3}}{\sqrt{2}} = \frac{3\sqrt{2}-\sqrt{6}}{2}.$$

Note that there are no nested square roots on the right hand side (RHS). The salient point is that, since we are dealing with surds, we should get identical, closed form, exact expressions for both $\sqrt{2-\sqrt{3}}$ and $\left[\frac{2\sqrt{3}-3}{\sqrt{6-3\sqrt{3}}}\right]$ *without using decimals*. And that takes some effort, using paper and pencil, or software like Geogebra [9].

Is π really 22/7?

Is π really equal to $\frac{22}{7}$, as has been drummed into our heads at school?

The answer is a qualified “Yes and No”.

“Yes”, because, thanks to Archimedes, $\frac{22}{7}$ is an overestimate for π that has survived for nineteen centuries, and served us well in all this time. Whenever, more accuracy was desired, we could always press into service a slightly better rational approximation, like $\frac{355}{113}$.

“No”, because an irrational number like π can never be expressed *exactly* within the confines of a finite numerical representation. Consequently, we use rational approximations, or a decimal representation, at the accuracy desired for our practical purposes.

Philosophically speaking, π can only be represented, truly as it is, by a *symbol*, not by digits.

Geometry might have given birth to π, but does not confine π exclusively. This wondrous number is a free citizen of all of mathematics, and can roam the entire domain. How mathematicians became aware of the ubiquity of π, and what riches have accrued as a result, will engage us in our second blog on π, entitled “The Wonder that is Pi”.

To explore further

A well-written, accessible article on how Archimedes estimated that π is approximately $\frac{22}{7}$ is available online: “How Archimedes showed that pi is approximately 22 by 7”. I urge you to read it.[10] You will then appreciate for yourselves how arduous the process must have been in an age without the benefit of:

1. Trigonometry; he used geometry and the theorem of Pythagoras instead;
2. Algebra; he used geometry and the ratios of the lengths of well-known triangles;
3. Decimal numbers for division; he used fractions instead;

[10] This article is all the more remarkable because its first author is a Grade 8 student: proof that deep mathematics is not beyond the school student.

4. Calculators for evaluating square roots.

Another recommended online article is Archimedes and Pi [bertrand2014] at a website interestingly named https://nonagon.org/.

There is an online demonstration showing how estimates of π vary with n, $a(n)$, and $A(n)$ [10]. The development in our blog uses the perimeters, $c(n)$ and $C(n)$, rather than the areas, and gives faster convergence.

Another article on Archimedes' estimation of π is available on this PBS website [11]. Unfortunately, the interactive demonstration, using Macromedia Flash is no longer live.[11]

This online article [12] recounts, with facsimile reproductions from Archimedes' own writings, how he went about estimating π.

Acknowledgements

Some computations for this blog were performed using a program written by Nandakumar Chandrasekhar in the Julia programming Language. The output was formatted so that it could be easily cut and pasted into the blog document. The source code is available here.

Feedback

Please email me your comments and corrections.

A PDF version of this article is available for download here:

<https://swanlotus.netlify.app/blogs/the-pi-of-archimedespdf-blog .pdf>

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