A tale of two measures: degrees and radians

R (Chandra) Chandrasekhar

2023-10-17 | 2023-10-17

The transition from degrees to radians is often the most traumatic mathematical change that the student has to endure when moving from elementary to intermediate mathematics. The simplicity of 360° seems so much more welcoming than the equivalent of 2π radians for a full circle. π is forbidding, because it is not the convenient fractional fiction $\frac{22}{7}$, but rather a number which is both transcendental and irrational, and therefore somewhat "untidy". Surely this tradeoff between simplicity and complexity must have been worth it, or it would not have been so ordained. Here we attempt to fathom the method behind the madness for this change.

What is an angle?

We first came across the idea of an *angle* when we studied geometry in elementary or primary school. We then encountered *triangles*, which are closed figures with three straight sides and three enclosed angles. An *equilateral triangle* is particularly symmetric, with three equal sides and three equal angles, as shown in Figure 1.

Degrees

When we started on geometry, we would proudly trot out our set of mathematical instruments, which would include a pair of compasses, a protractor, one or two set squares, and a ruler or straight edge. Of these, the protractor—that plastic semi-circle marked out in *degrees*—was the proud badge that proclaimed that we had left behind arithmetic and progressed onto geometry.

After we had learned to construct an equilateral triangle, using only compasses and a straight edge—*without measurement* by ruler—we would take out the protractor to verify that each angle of an equilateral triangle was indeed 60°. That small circle ° at the top—the superscript—was called the *degree sign*, and we could then jubilantly celebrate our first rite of passage into geometry and mathematical symbols.

Where did degrees come from?

Surely, degrees did not come from a protractor, although we use one to measure angles. How did degrees come about? With sixty degrees each in an equilateral triangle, ninety in a right angle , 180° in a straight line, and 360° in a full circle, how did degrees come to rule the roost of angular measure in elementary school?

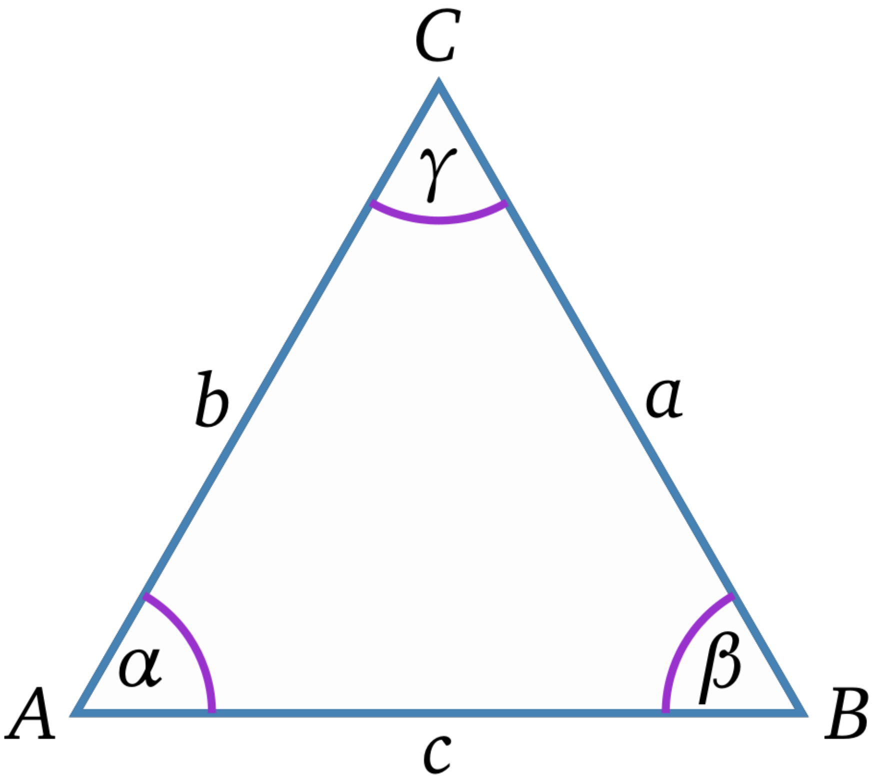

Figure 1: An equilateral triangle is one in which the three sides are equal. The points at which the lines meet to form the triangle are called *vertices* (singular *vertex*). By convention, vertices are labelled with uppercase letters like A, B, and C. The lengths of the sides opposite the vertices are assigned the lowercase labels a, b, and c respectively. The angles are here labelled α, β, and γ. In this triangle, $a = b = c$, by definition, and by symmetry, $\alpha = \beta = \gamma$.

Why not 100° in a full circle, or half circle, or even a quarter circle, also known as a right angle? Who imposed this measure upon us and what is its basis?

My favourite explanation for 360° degrees equalling a full circle is that the ancients estimated a solar year at around 360 days, and assigned one degree for each day of the year. Even if inexact, the number 360 had some sexagesimal[1] charm as it could be divided by the first three primes 2, 3, 5, and by their products. Indeed, $360 = 2^3 \times 3^2 \times 5$. Accordingly, 360 has a large family of factors: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, and 360.

But beyond the approximation of a solar year, and the convenience of ready division by its factors, the use of degrees as a unit of angular measure is, to me, arbitrary. Who deemed the circle to be 360°, despite it being very factor-friendly?

From triangles to circles

What is the root concept behind the idea of an angle? Harking back to the equilateral triangle, when one line *changes direction* by sixty degrees, we get the second line. These two lines form the angle. Change of direction may also be called *turning*. And when something turns, we may also say that it *rotates*.

The quintessential two-dimensional geometric figure that is associated with rotation is of course the *circle*. It is the most simple and symmetrical two-dimensional figure we can construct. It is the path or *locus* traced out by a point that remains the *same* distance from a fixed point called the *centre*. When a protractor is centred on the centre of a circle, we can measure out degrees on the circumference of the circle. So far so good. But what about that magic number 360? Well, we are about to exorcise it now. ☺

Radians as an alternative to degrees

One traumatic transition for the student of elementary mathematics is when he or she is forced to abandon the warm comfort of degrees as angular measure, and compulsorily made to embrace the cold and cruel radian as *the* angular measure forever afterward. Why this unfair compulsion?

Using circles to measure angles

Because the idea of an angle is related to rotation, it seems natural that we should define angles using the circle as a basis, rather the triangles that we encountered at first.

It is a fact that the length of a circle, or its *perimeter*, or its *circumference*, C, is always related to its radius, r, through the formula $C = 2\pi r$.

[1] It appears that all measures of time, from seconds, minutes, and hours, to months and days in a year, are based on 60 or its factors or multiples.

And π is not $\frac{22}{7}$ as we were originally taught, but really a number whose precise expression cannot be predicted or exhausted. The digits simply keep rolling on, without pattern or end. But the beauty is that π is nevertheless a unique number, a universal mathematical constant. It seems that Nature has played a game on us by making the simple symmetrical circle have a length that can only be approximated but never entirely known to an unlimited precision.[2]

One radian

So, how does one define a radian? If, on the basis of its name, you guessed that it very likely involves the radius of a circle, your suspicion is well-founded. *One radian is the angle subtended at the centre of a circle of radius one unit by an arc that is also one unit long*. This is illustrated in Figure 2.

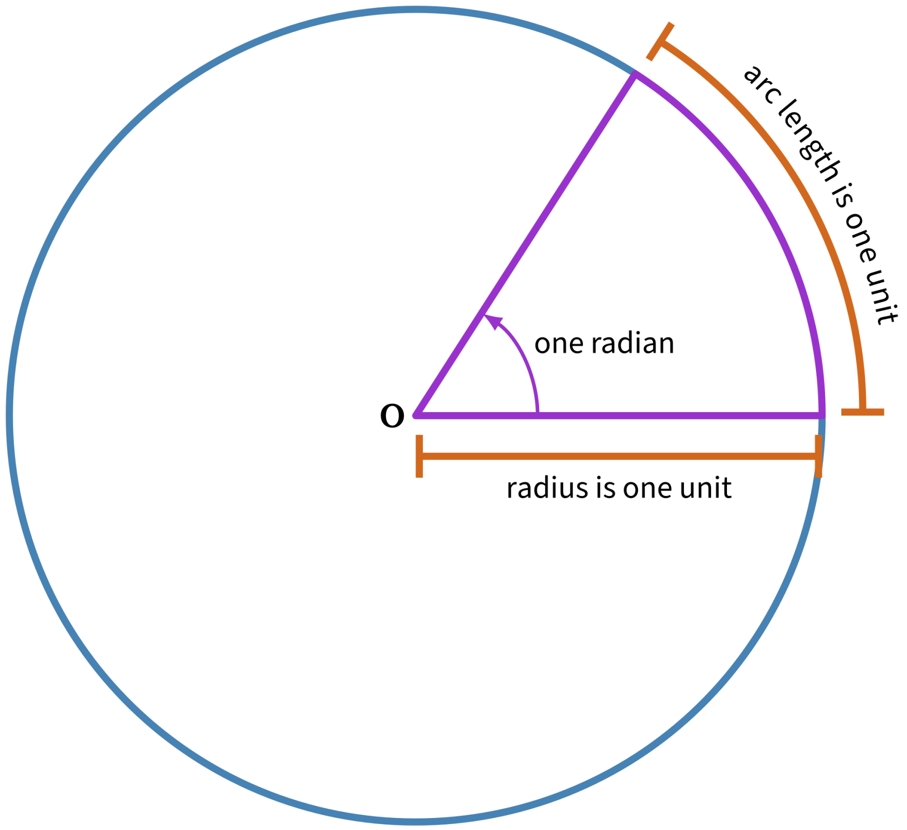

Figure 2: One radian is the angle subtended at the centre of a unit circle by an arc of length equal to one unit.

But what happens when our circle has a radius larger or smaller than one unit? The generalized definition of a radian is shown in Figure 3.

[2] π, e the base or natural logarithms, ϕ the golden ratio, along with a large pantheon of mathematical constants are irrational, and some are even possibly transcendental. Why Nature has this preference for the irrational is an intriguing question that needs an answer.

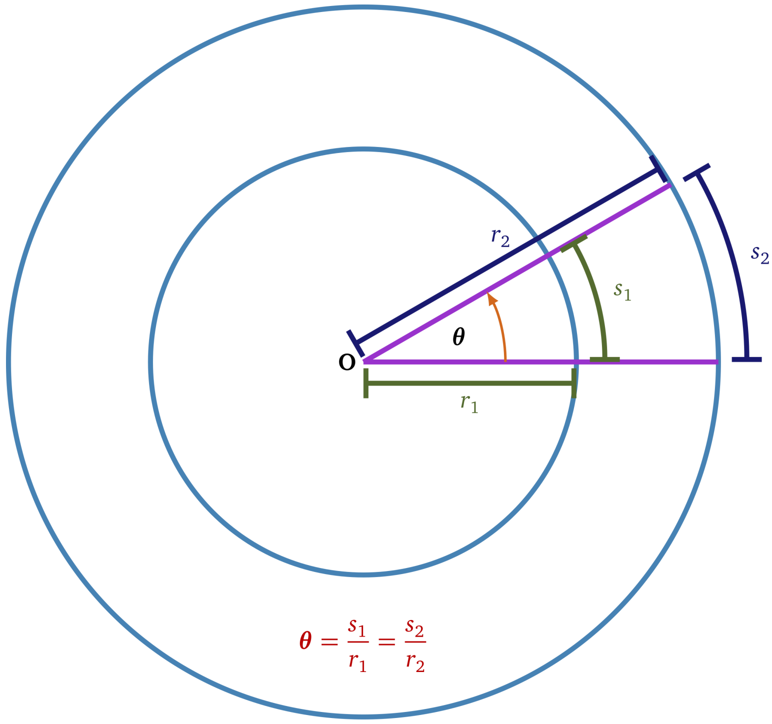

Figure 3: Generalized measure of an angle in radians. The angle in radians is defined as $\theta = \frac{s_1}{r_1} = \frac{s_2}{r_2}$. The equality is valid because all circles are similar to each other.

Congruence and similarity

This is a mathematically non-rigorous digression on congruence and similarity. One two-dimensional geometrical shape is *congruent* to another if the two shapes may be rotated and translated, if necessary, and superimposed on each other to visually demonstrate that they are identical in size and shape and cannot then be told apart.

Similarity is less restrictive than congruence and applies to geometric objects that have the same shape but not necessarily the same size. Intuitively, if two objects are similar, one may *zoom in* or *zoom out* on one object of the pair—without distortion—to obtain a version that may be superimposed on the other object to demonstrate that they are identical or congruent.

The ratios of the respective lengths of corresponding sides of similar triangles are the same. In like fashion, the ratio of the circumference to the radius of two circles of radii r_1 and r_2 will be $\frac{2\pi r_1}{r_1} = \frac{2\pi r_2}{r_2} = 2\pi$.[3] This is a consequence of the fact that *all circles are similar to each other*.

What other classes of geometrical objects can you think of that are similar to each other within their class?[4]

[3]This is also a demonstration that a full circle corresponds to an angle of 360° or 2π radians.

[4]All circles are similar, as are all equilateral triangles, all squares, and indeed, all regular n-gons, and all parabolas. What exactly does similarity mean? It means that the shape remains the same. Imagine a square or a circle. If you zoomed on the figure to enlarge or diminish it without distortion, and you could not see a change of shape, that figure exhibits similarity to every other figure in that class.

Radians as angular measure

We are now ready to define any angle θ in radians. The ratio of the length of the arc that subtends an angle of θ at the centre, to its radius, is the value of θ in radians. Succinctly, with reference to Figure 3, $\theta = \frac{s}{r}$. By dividing by the radius, we have in effect *normalized* the definition of the radian, and removed any trace of arbitrariness in the definition. And that is why we started out slowly with Figure 2, which dealt with a circle with unit radius.

Note that the value of θ is a ratio of two lengths and is therefore dimensionless in the sense of Physics. It may be considered a unitless *pure number* although the SI units do define the radian as the SI unit of angular measure.

With radians, we have the following:

1. Angular measure is directly proportional to arc length on a circle for all angles less than or equal to a full circle.
2. This measure is independent of the radius of the circle.
3. The resulting "unit" is really a unitless ratio of two lengths.

By defining radians as above, we remove the arbitrariness associated with 360° for a full circle. But the mathematical elegance and rigour conferred by radians comes at a cost. The angle of a full circle is 2π, which is a computationally inconvenient number to say the least.

If you think about it, with radians the size of an angle is expressed as a ratio of two lengths. But we have encountered angles being associated with ratios of lengths elsewhere in mathematics as well. Such ratios are familiar to us from trigonometry where the sin, cos, and tan functions are expressed as the ratios of lengths in a right-angled triangle. We review this relationship next.

Trigonometric functions

Trigonometric functions are one of the workhorses of applied mathematics. They arose from the study of right-angled triangles. The three standard trigonometric functions are the *sine*, *cosine*, and *tangent* functions. They are represented by the abbreviated functional names sin, cos, and tan when used in mathematics. Figure 4 shows the pictorial definitions of these three trigonometric functions. Notice particularly how these function values are the unitless *ratios of two lengths*, just as with radians.

The Circular Functions

The trigonometric functions are also called the *circular* trigonometric functions, uniting the circle *and* the triangle as their progenitors. We will briefly review that relationship here, to better understand not only the terminology but also the hidden relationships between the triangle and the circle.[5]

[5] The equilateral triangle is the regular n-gon with the smallest number of sides and the circle is the limiting case of an n-gon when n tends to infinity. The trigonometric functions are the children of these unlikely parents, at the extreme ends of the n-gon spectrum.

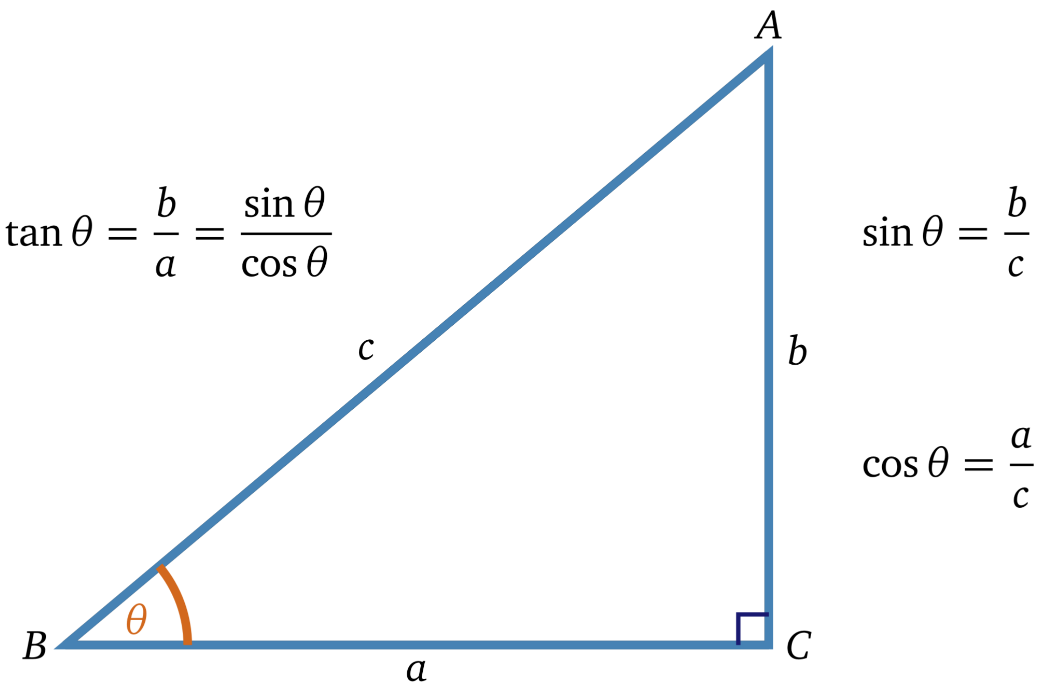

Figure 4: Trigonometric functions defined as ratios of lengths of sides in a right-angled triangle.

I used to wonder why the word *tangent* was used for the name of a trigonometric function because a circle was not involved in its definition; a triangle was. But when the three standard trigonometric functions are viewed vis-a-vis a unit circle, the mystery behind the nomenclature is revealed.

The radian was introduced here using a *unit circle*. The same helpful unit circle will serve to relate the triangle and the circle to the trigonometric functions, as illustrated in Figure 5 below.

Figure 5 shows a unit circle drawn on the two-dimensional coordinate plane with x and y axes and grid markings. The centre of the circle is O and S is a variable point on the circumference of the circle, that makes a counter-clockwise angle θ with the positive x-axis. As θ varies, so does the position of S on the circle.

The line OS is a radius and therefore one unit in length. The perpendicular from S to the x-axis meets it at C. Referring to Figure 4, we may say $\cos\theta = \frac{OC}{OS} = OC$ since $OS = 1$. Accordingly, the x-coordinate of S is $\cos\theta$. Likewise, $\sin\theta = \frac{SC}{OS} = SC$. Thus, the y-coordinate of S is $\sin\theta$.

S is therefore the point with coordinates $(\cos\theta, \sin\theta)$.

The tangent function

The really insightful revelation from Figure 5 comes from looking at $\tan\theta$. Have you ever wondered why the function $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is called the *tangent* function? Take a look at Figure 5 to see that the line DT is *tangent* to the circle at the point $(1, 0)$, which is D. In this case, $\tan\theta = \frac{DT}{OD} = DT$ since OD is equal to one unit.

The value of the tangent function for an angle θ may be determined geometrically by extrapolating the radius OS until it intersects the tangent to the circle at D at a point called T. The length DT is the value of $\tan\theta$.

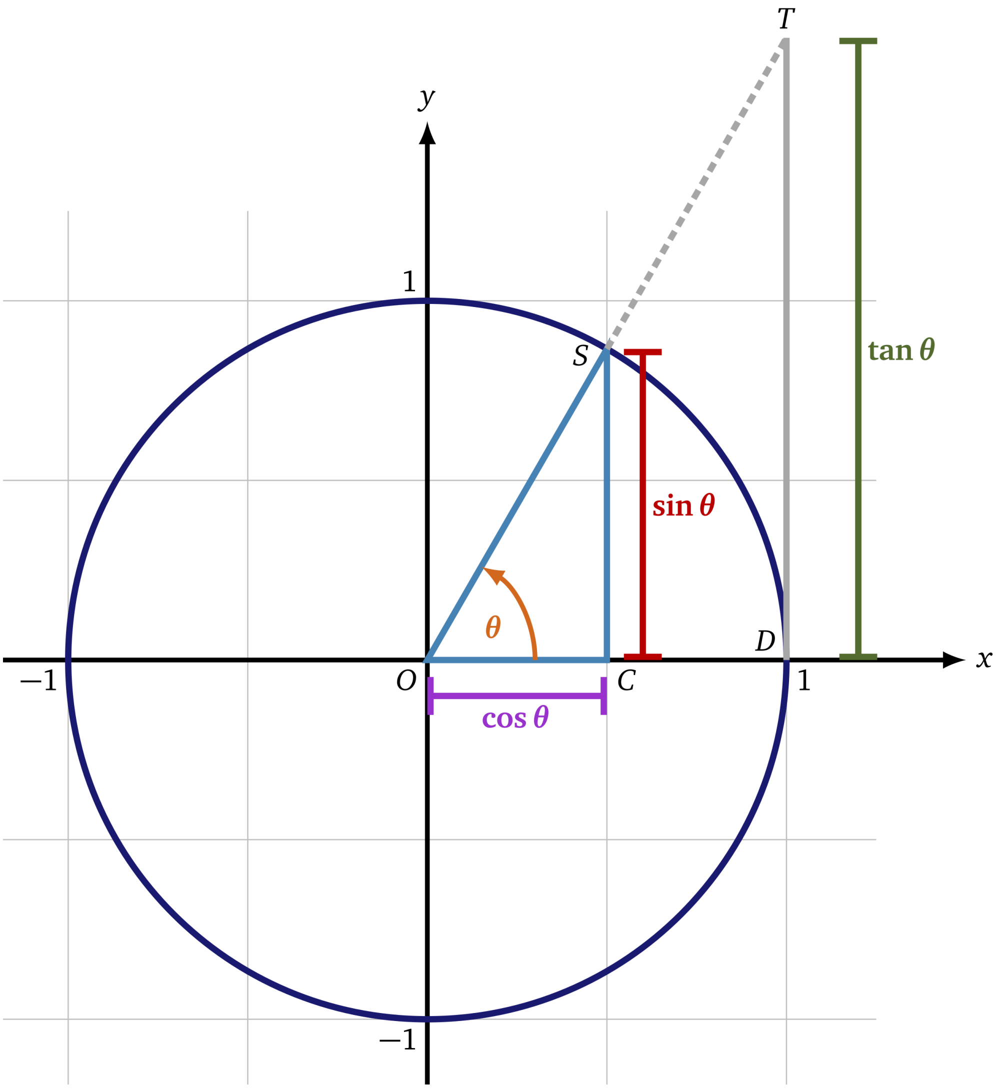

Figure 5: A pictorial representation of the unit circle, the three standard trigonometric functions, and their inter-relationships. See the text for a full explanation.

Note that since the coordinates of S are confined to the unit circle, the values of $\sin\theta$ and $\cos\theta$ are confined to the closed interval $[-1, 1]$, i.e. they perforce have values lying between -1 and 1, both inclusive.

However, $\tan\theta$ is a length *outside the unit circle* and is therefore not similarly constrained. Indeed, as θ starts increasing in the first quadrant of the unit circle, you will notice that as OS approaches the y-axis and as θ approaches $\frac{\pi}{2}$ (or 90°, if you are still attached to degrees), the line OS is increasingly aligned with DT. At $\theta = \frac{\pi}{2}$, OS is parallel to DT and "never the twain shall meet". Loosely speaking, parallel lines are only supposed to "meet at infinity" and that is why $\tan\frac{\pi}{2}$ is said to be "infinite" at that point. This is why $\tan\theta$ does not assume a finite value at $\theta = \frac{\pi}{2}$. I find this geometric explanation most fulfilling.

In triangle OSC, note that the hypotenuse OS, being the radius, is one unit long. Therefore the length of OC equals $\cos\theta$ and the length of CS equals $\sin\theta$. The point T denotes the intersection of OS produced and the tangent to the circle at D. Given that OD, being a radius, is also one unit long, the length DT is equal to $\tan\theta$. The fact that DT is a *tangent* to the circle should explain why this function is called the tan.

The simple device of unit radius helps us understand how the cos and sin functions takeo n the are the x and y taken on the signed x and y co-ordinates of S.

The signed trigonometric functions

From Figure 5, we see that the $(\cos\theta, \sin\theta)$—which represent the coordinates of S—take on signed values in accordance with the signs of x and y in the respective quadrants. One could also view the associated *lengths* as signed values.

For example, when θ is in the third quadrant, OS extrapolated in the negative direction will not intersect the tangent DT in the negative y direction. So, the line SO must be produced in the positive y direction to once more intersect intersect the tangent at T. That explains why tangents of angles in the third quadrant are positive.

By moving from triangles to the unit circle, we have enabled the trigonometric functions to take on any value between 0 and 2π radians. Note that when $\theta = 2\pi$ radians, we cannot really distinguish θ from 0 radians. So, one angle may masquerade as another unless we have accounting devices to optionally add $2n\pi$ to angles with the proviso that n is an integer. And this concept is a segue to power series expansions of trigonometric functions, their use in calculus, and later on, in Fourier series.

Power series expansions of trigonometric functions

This is where the plot thickens.

Both the statements $\sin(30°) = 0.5$ and $\sin(\frac{\pi}{6}) = 0.5$ are factually correct and perfectly acceptable. We will not be committing any mathematical heresies through either statement.

But there are some infinite series expansions or power series expansions for the trigonometric functions in which an argument in degrees would be inadmissible. It is after we cross this threshold in mathematics that radians truly come into their own, and after which there is "no going back to the old ways".

The Taylor and Maclaurin series for $\sin\theta$ are given below:

$\sin\theta \approx \theta$ for θ close to zero. The number on the left hand side is a unitless ratio of lengths or "pure" number. This requires the right hand side to also be expressed in a similar unitless measure, and radians fits the bill.

When we deal with trigonometric functions in calculus or when applying Fourier series, we will fully realize the boon conferred by the unitless radians as an angular measure since they will be compatible with the other *pure numbers* (real or complex) that we will be dealing with.

Acknowledgements

Feedback

Please email me your comments and corrections.

A PDF version of this article is available for download here:

https://swanlotus.netlify.app/blogs/radians.pdf