The Two Most Important Numbers: Zero and One

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The unique properties of the numbers zero and one make them mathematically indispensable. In this slow-paced stroll though the ideas streaming out of these two numbers. we uncover well-known as well as relatively obscure facts about them. It is hoped that in the process we may discover how they cement disparate areas of Mathematics.

Starting at the beginning

I thought I would skirt around the idea of the well-known sets of numbers, but found that each time I tried, I would have to sneak in furtively a paragraph here or a footnote there as explanation. So, I have decided to start at the beginning and work my way, the natural numbers, the integers, the rationals, etc., again using intuitively obvious justification rather than a purely mathematical approach.

Counting

Initially, a shepherd with five sheep used to count "one sheep, two sheep, three sheep, four sheep, and five sheep." But wait! Since he did not have the names for numbers—nor indeed the abstract concept of a number—he could have used either of two slightly different methods.

Naming sheep

He could have given *unique* names to each of his five sheep and developed enough familiarity with them to identify them by name. Then all he needed to do was to check that all his named sheep were home by sundown.

One-to-one correspondence

The later, and more likely, alternative was to used a stone to correspond to a sheep. He could have taken a leather bag and dropped a stone in it for each sheep that he owned. He did not need to learn counting. All he needed to do was to establish a one-to-one correspondence[1] between sheep and stone. As long as he had the right number of stones in his bag, he could account for each one of his sheep.

The Latin word for *stone* is calculus and from the stone has come the whole science of *calculation*.

[1] One-to-one correspondence is a simple but extremely powerful idea which guided Georg Cantor to develop his radical but consistent ideas about types of infinity.

Measurement

When we *count*, as with sheep, where do we start? We start with one. We do not start with zero, because we cannot point to any sheep or other object and say “zero”.

Nevertheless, zero has fundamental importance when we start *measuring*. When the petrol tank in a car is empty, we can fill it up and measure the volume of petrol for which we have to pay.

When we count, we start with 1*; when we measure, we start with* 0*.*

Sets of numbers

Although Mathematics has rigorous foundations, at the very bottom, notions are not defined explicitly. One such notion is that of a _set, which is loosely defined as a collection of objects that can either be enumerated or described clearly. The *sets of numbers* we will deal with have names, symbols, and definitions as shown below.

Name	Symbol	Definition
Natural numbers	$\mathbb{N}$	$\{1, 2, 3, 4, \ldots\}$
Integers	$\mathbb{Z}$	$\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Rational numbers	$\mathbb{Q}$	$\{x : x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0\}$
Irrational numbers		Numbers which are not rational
Real numbers	$\mathbb{R}$	The rationals and irrationals
Complex numbers	$\mathbb{C}$	$\{a + ib : a, b \in \mathbb{R}$ and $i^2 = -1\}$

While it is premature to talk about them and their peculiarities just now, it is worth making some points about these sets.

1. There is no agreement on whether or not to include zero as a member of $\mathbb{N}$. I have chosen not to because I consider $\mathbb{N}$ the set of *counting numbers*.
2. Zero is neither positive nor negative. It is itself and, as a set, is often denoted $\{0\}$.
3. The *integers* are named $\mathbb{Z}$ after the German word *Zahlen* which stands for integers. The integers include positive and negative whole numbers as well as zero.
4. The *rational numbers* are so named because they are really *ratios* of whole numbers with the proviso that the denominator cannot be zero. More about this in a later section. The symbol $\mathbb{Q}$ is used because it denotes *quotient* the result of *division*.
5. There is no symbol for the irrationals, which are simply defined as numbers which are not rational. In fact, the set of irrationals may shown using set notation only indirectly as $\mathbb{R} \setminus \mathbb{Q}$, which means the set of real numbers, excluding the rational numbers.
6. The *complex* numbers

The shy one

The number one is often implicit in mathematical notation. While we may write $2x$ to denote $2 \times x$, or two multiplied by x, we *do not* write $1x$, even if it is literally correct, because of convention. In instances like this, the number one is implicit, and assumed to be understood by those who know. If you happen to be one of those *not* in the know, here's your chance to join the other side.

When we write a fraction as $\frac{3}{4}$ we mean the decimal 0.75 and matters are clear. But all whole numbers are also fractions with the denominator being 1. So, the fraction $\frac{3}{1}$ is rarely written in that form, even if syntactically correct, because usage dictates that whole numbers are written to stand on their own, as 3, in this case. Again, the 1 in the denominator is assumed to be unobtrusively present: out of sight but *not* out of mind.

When we write 4^2, spoken out as "four squared" we mean the number obtained by multiplying 4 by itself. This nomenclature arose because, if 4 was associated with the *length* of, say, a piece of string, the number "four squared" was used to denote the *area* of a square that had a side of length 4. So, $4^2 = 4 \times 4 = 16$.

Likewise, the expression 7^3 or "seven cubed" denoted the volume of a cube of side 7. Beyond the third dimension, this naming scheme faded out, because we cannot percieve dimensions higher than three.

Therefore, 6^4 is spoken as "six raised to the fourth (power)" or "six to the four". In such statements, the number 6 is called the *base* and the number 4 is called the *exponent*.

Following this logic, we might assert that $5^1 = 5$ and that is perfectly correct. But again, convention intrudes to say that we write it simply as 5. *Any number raised to the power of* 1 *equals itself.*

The notation making 1 implicit in these scenarios reduces clutter and simplifies notation. The absence of the implicit 1 might trouble the heart of the sincere young mathematician, but familiarity with these conventions will make for comfort in using them.

The additive and multiplicative identities

When mathematicians in the nineteenth century contemplated the then extant mathematical systems, they recognized certain commonalities. Whether it was arithmetic or geometry, or some other branch of mathematics, they were able to distil certain underlying principles behind the common practices of mathematics. By systematizing and classifying what they observed, they were able to *invent* names for the *classes of objects* they discerned, along with their properties. Thus was born abstract algebra. The ideas of the additive and multiplicative identity were born from this exercise in classification.

The numbers we use for counting, starting from 1, and never ending, are called the *natural numbers*. The collection or *set* of these numbers is denoted $\mathbb{N}$. We can add and multiply these numbers.

For example, we have seen that multiplying 5 by 1 gives the original number 5. The number 1 is called the *multiplicative identity* because multiplying any natural number by 1 preserves the original number.

What do you think is the additive equivalent, the *additive identity*? We know that if we add 0 to any number, we get the original number again. So, 0 *preserves* the original number intact after addition. But is 0 in the set $\mathbb{N}$? Not as we have defined it here.[2] Nevertheless, we may posit that under appropriate conditions, the additive identity is 0.

Zero and one enjoy their coign of vantage because they are the additive and multiplicative identities respectively:

$$a + 0 = a$$

$$a \times 1 = a$$

where a is an arbitrary number, of the sort we are familiar with.[3]

Mathematics as a discipline tends to generalize and extend simple ideas to increasing levels of complexity, while at the same time maintaining consistency in definition and behaviour across these disparate domains.

It should come as no surprise that some objects called matrices[4] (singular matrix) also have their additive and multiplicative identities, *where applicable*. We will consider an arbitrary *square matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ having four elements, and called a 2×2 matrix. Then,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1}$$

and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the multiplicative identity for matrix multiplication.[5]

Likewise,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2}$$

and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the additive identity for matrix addition.

Do you see how the seed ideas of the additive and multiplicative identities, sown far and wide, germinate into shoots that are surprisingly similar to the original ones. The numbers 0 and 1 do indeed rule the roost. Obviously, the identity matrices will change with the matrix sizes, but the principles remain the same.

Zero, one, and repeated addition

All natural numbers may be generated by zero, one, and repeated addition:

$$0 + 1 = 1 \tag{3}$$

$$1 + 1 = 2 \tag{4}$$

[2] Some folks do include 0 in the set $\mathbb{N}$.

[3] This is an easy-to-read blog; so I will not belabour the reader with the finer points of different types of numbers, but will reserve them for a later blog.

[4] I facetiously call them numbers in teabags

[5] The rules of matrix multiplication are a little involved and will not detain us here. The interested reader is referred to another blog of mine for details.

The *next natural number* is obtained by adding in the sequence is generated by adding 1 to the current number._ By repeating this process, we can generate any desired natural number.

Multiplication

Division

Why we cannot divide by zero

Exponentiation

Exponentiation may also be called *raising (something) to a power*. It is a short form for repeated multiplication by the *same* number. For example, if we multiply 5 by itself three times, we write it so:

$$5 \times 5 \times 5 = 5^1 \times 5^1 \times 5^1 = 5^{(1+1+1)} = 5^3 = 125 \tag{5}$$

The number 5 is called the *base* and the power 3 is called the *exponent*. As noted before, $5^1 = 5$ and the exponent 1 is omitted.

Acknowledgements

Feedback

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