A tale of two measures: degrees and radians

R (Chandra) Chandrasekhar

2023-10-17 | 2023-11-02

The transition from degrees to radians is often the most traumatic mathematical change that the student has to endure when moving from elementary to intermediate mathematics. The simplicity of 360° seems so much more welcoming than the equivalent of 2π radians for a full circle. π is forbidding, because it is not the convenient fractional fiction $\frac{22}{7}$, but rather a number which is both transcendental and irrational, and therefore somewhat "untidy". Surely this tradeoff between simplicity and complexity must have been worth it, or it would not have been so ordained. Here we attempt to fathom the method in the madness.

What is an angle?

For most of us, the idea of an *angle* first arose when we studied geometry in elementary or primary school. We then encountered *triangles*, which are closed figures with three straight sides and three enclosed angles. An *equilateral triangle* is particularly symmetric, with three equal sides and three equal angles, as shown in Figure 1.

The point at which a line meets another line is called a *vertex*, which means a "turning point". By convention, vertices (plural of vertex) are labelled with uppercase letters like A, B, and C. The lengths of the sides opposite the vertices are assigned the lowercase labels a, b, and c respectively. The angles have been labelled with the Greek letters α, β, and γ. For all equilateral triangles, $a = b = c$, by definition, and by symmetry, $\alpha = \beta = \gamma$.

Degrees

On encountering geometry, we very likely proudly trotted out our set of mathematical instruments, which would include a pair of compasses, a protractor, one or two set squares, and a ruler or straight edge. Of these, the protractor—that plastic semi-circle marked out in *degrees*—was the proud badge that proclaimed that we had left behind arithmetic and progressed to geometry.

After we had learned to construct an equilateral triangle, using only compasses and a straight edge—*without measurement* by ruler—we would take out the protractor to verify that each angle of an equilateral triangle was indeed 60°. That small circle ° at the top—the superscript—was called the *degree sign*, and we could then jubilantly celebrate our first rite of passage into geometry and mathematical symbols.

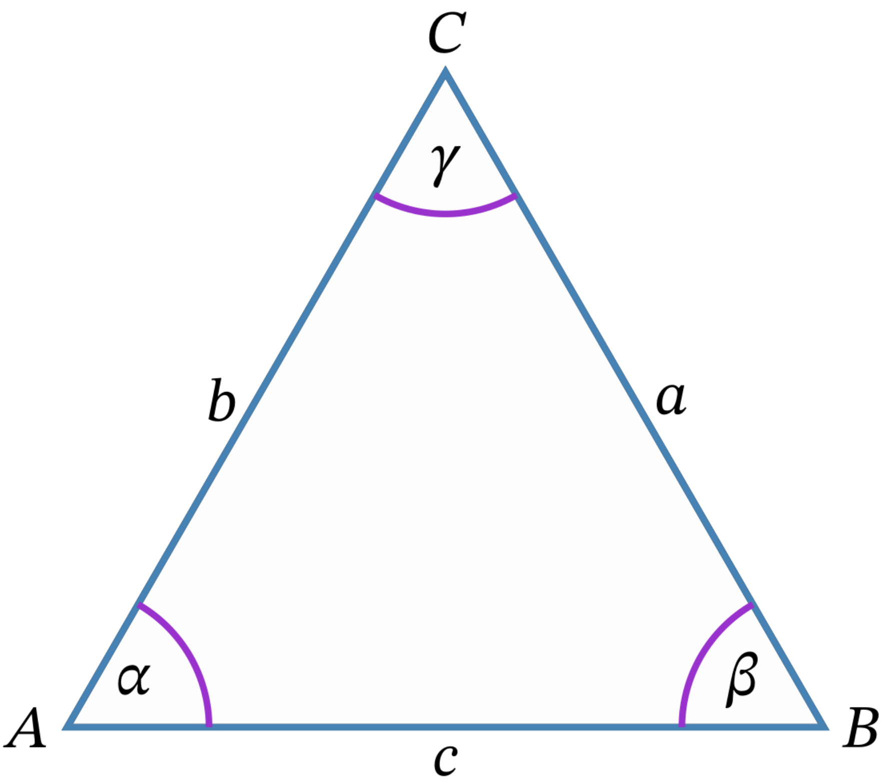

Figure 1: An equilateral triangle is one in which the three sides and three angles are equal.

Where did degrees come from?

Surely, degrees did not come from a protractor, although we use one to measure angles. How did degrees come about? With sixty degrees each in an equilateral triangle, ninety in a right angle , 180° in a straight line, and 360° in a full circle, how did degrees come to rule the roost of angular measure in elementary school?

Why not 100° in a full circle, or half circle, or even a quarter circle, also known as a right angle? Who imposed this measure upon us and what is its basis?

My favourite explanation for 360° degrees equalling a full circle is that the ancients estimated a solar year at around 360 days, and assigned one degree for each day of the year. Even if inexact, the number 360 had some sexagesimal[1] charm as it could be divided by the first three primes 2, 3, 5, and by their products. Indeed, $360 = 2^3 \times 3^2 \times 5$. Accordingly, 360 has a large family of factors: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, and 360.

But beyond the approximation of a solar year, and the convenience of ready division by its factors, the use of degrees as a unit of angular measure is, to me, arbitrary. Who deemed the circle to be 360°, despite it being very factor-friendly?

[1] It appears that all measures of time, from seconds, minutes, and hours, to months and days in a year, are based on 60 or its factors or multiples.

From triangles to circles

What is the root concept behind the idea of an angle? Harking back to the etymology of the word vertex—and applying it to the equilateral triangle—when one line *changes direction* by sixty degrees, we get the second line. These two lines form the angle. Therefore, a change of direction may also be called *turning* or *rotation*.

The quintessential two-dimensional geometric figure that is associated with rotation is of course the *circle*. It is the most simple and symmetrical two-dimensional figure we can construct. It is the path or *locus* traced out by a point that remains the *same* distance from a fixed point called the *centre*. When a protractor is centred on the centre of a circle, we can measure out degrees on the circumference of the circle. So far so good. But what about that magic number 360? Well, we are about to exorcise it now. ☺

Radians as an alternative to degrees

One traumatic transition for the student of elementary mathematics is when he or she is forced to abandon the warm comfort of degrees as angular measure, and compulsorily made to embrace the cold and cruel radian as *the* angular measure forever afterward. Why this unfair compulsion?

Using circles to measure angles

Because the idea of an angle is related to rotation, it seems natural that we should define angles using the circle as a basis, rather than the triangles that we encountered at first.

It is a fact that the *length* of a circle, or its *perimeter*, or its *circumference*, C, is always related to its radius, r, through the formula:

$$C = 2\pi r. \tag{1}$$

And π is not $\frac{22}{7}$ as we were originally taught, but really a number whose precise expression cannot be predicted or exhausted. The digits simply keep rolling on, without pattern or end. But the beauty is that π is nevertheless a unique number, a universal mathematical constant. It seems that Nature has played a game on us by making the simple symmetrical circle have a circumference that can only be approximated but never entirely known to an unlimited precision.[2]

One radian

So, how does one define a radian? If, on the basis of its name, you guessed that it very likely involves the radius of a circle, your suspicion is well-founded. *One radian is the angle subtended at the centre of a circle of radius one unit by an arc that is also one unit long*. This is illustrated in Figure 2.

But what happens when our circle has a radius larger or smaller than one unit? We will take up this case, after a short detour.

[2] π, e the base of natural logarithms, ϕ the golden ratio, along with a large pantheon of mathematical constants are irrational, and some are even possibly transcendental. Why Nature has this preference for the irrational is an intriguing question that needs an answer.

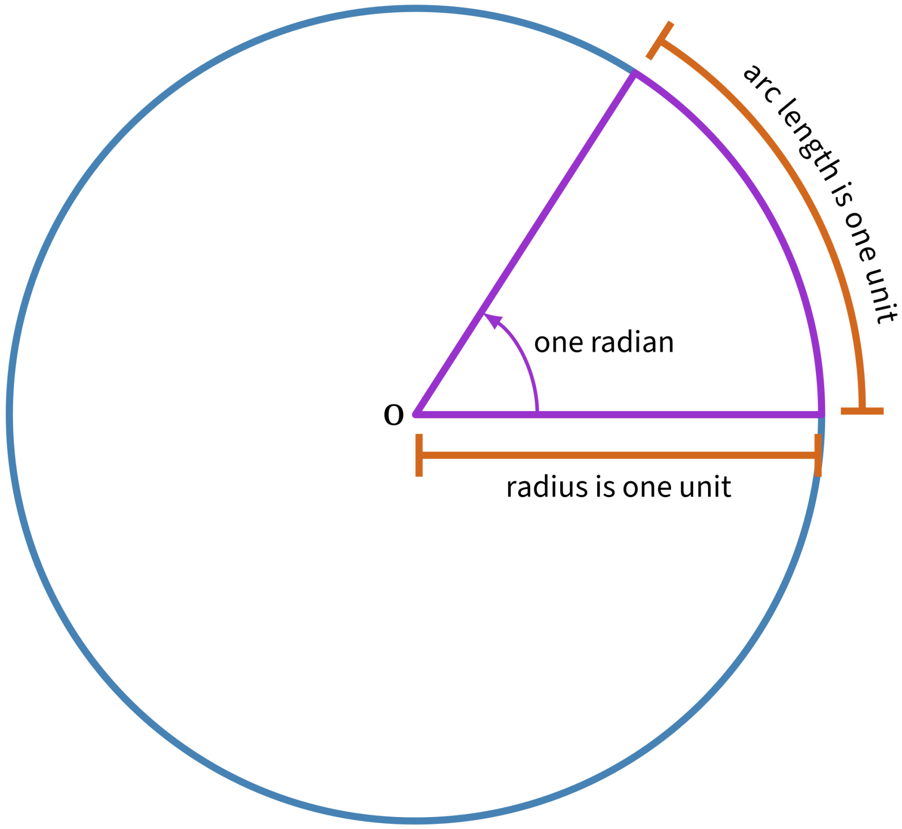

Figure 2: One radian is the angle subtended at the centre of a unit circle by an arc of length equal to one unit.

Congruence and similarity

This is a mathematically non-rigorous digression on congruence and similarity, both of which are first encountered in school in the context of triangles, as shown in Figure 3.

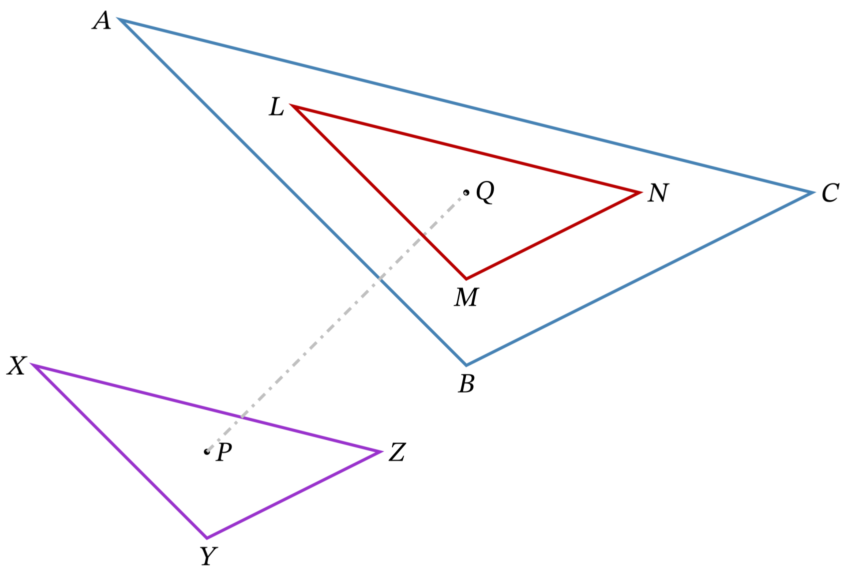

Figure 3: Similarity and congruence in the context of triangles. See the text for the explanation.

Consider the triangle XYZ, and ignore for a moment triangle LMN. Suppose that XYZ is moved in the direction of the line PQ for a distance equal to the length of PQ. We would *then* have the triangle LMN.

Triangle LMN, being a shifted version of triangle XYZ, is identical with it, having identical respective angles and sides. Indeed, if triangle XYZ were laid on top of LNM, we could not tell them apart. We say that triangle XYZ is *congruent* to triangle LMN.

Any two-dimensional geometrical shape is congruent to another if the two shapes may be superimposed[3] on each other to visually demonstrate that they are indistinguishable.

Similarity is less restrictive than congruence and applies to geometric objects that have the same shape but not necessarily the same size. In Figure 3, triangle ABC is similar to triangles XYZ and LMN.

Intuitively, if two objects are similar, one may *zoom in* or *zoom out* on one object of the pair—without distortion—to obtain a version that may be superimposed on the other object to demonstrate that they are identical or congruent. In this case, we may *enlarge* triangle LMN until it attains the same size as triangle ABC. It will then be congruent to ABC.

The ratios of the respective lengths of corresponding sides of similar triangles are the same. In like fashion, the ratio of any arc length to the radius of a circle is the same for all arcs subtending the *same* angle at the centre. For example, the ratio of the circumference to the radius for two circles of radii r_1 and r_2 will be $\frac{2\pi r_1}{r_1} = \frac{2\pi r_2}{r_2} = 2\pi$, which is a constant.[4] This is a consequence of the fact that *all circles are similar to each other*.

[3]After any necessary translation and rotation.
[4]This also demonstrates that a full circle corresponds to an angle of 360° or 2π radians.

What other classes of geometrical objects can you think of that are similar to each other within their class?[5]

Radians as angular measure

Consider Figure 4 in which two circles having different radii are shown.

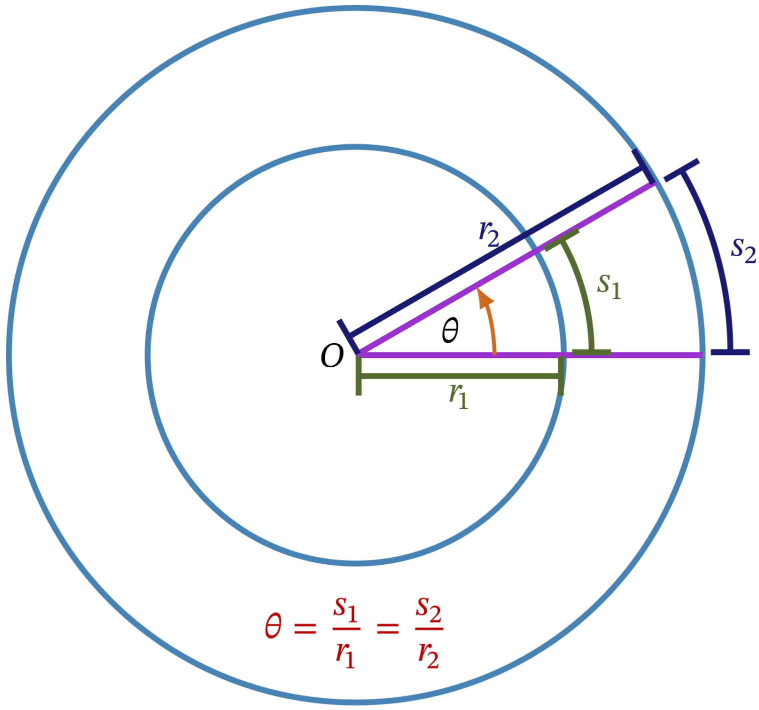

Figure 4: Generalized measure of an angle in radians. The arbitrary angle θ in radians is defined as $\theta = \frac{s_1}{r_1} = \frac{s_2}{r_2}$. The equality is valid because all circles are similar to each other.

We are now ready to define the value of any angle in radians. Consider a circle of radius r. Let an arc of arbitrary length s subtend an angle of θ at the centre. The angle θ in radians is *defined* to be:

$$\theta \triangleq \frac{\text{arc length}}{\text{radius}} = \frac{s}{r}. \tag{2}$$

By dividing the arc length by the radius, we have in effect *normalized* radian measure, and removed any trace of arbitrariness[6] in its definition. And that is why we started out with Figure 2, which dealt with a circle of radius one unit. We know from Figure 4 that similarity guarantees that the value $\frac{s}{r}$ for any given θ is constant for all circles regardless of radius.

Note that the value of θ is a ratio of two lengths and is therefore dimensionless in the sense of Physics. Although it may be considered a unitless *pure number* the SI units do define the *radian* as the *SI unit of angular measure*.

In summary, we have the following:

1. Radian angular measure is directly proportional to arc length on the circle.

[5] All circles are similar, as are all equilateral triangles, all squares, and indeed, all regular n-gons, and all parabolas.
[6] Such as dependence on the radius of the circle.

2. This measure is independent of the radius of the circle.
3. The resulting “unit” is really a unitless ratio of two lengths.
4. Since 360° equals 2π radians, 1° approximately equals 0.017453292 radians. Likewise, 1 radian equals approximately 57.29577951°.[7]

By defining radians as above, we remove the arbitrariness associated with 360° for a full circle. But the mathematical elegance and rigour conferred by radians comes at a cost. The angle of a full circle is 2π, which is a computationally inconvenient number to say the least.

If you think about it, the size of an angle in radians is expressed as a *ratio of two lengths*. But we have encountered angles being associated with ratios of lengths elsewhere in mathematics as well. Such ratios are familiar to us from *trigonometry* where the sin, cos, and tan functions are expressed as the ratios of lengths in a right-angled triangle. We review this relationship next.

Trigonometric functions

Trigonometric functions are one of the workhorses of applied mathematics. They arose from the study of right-angled triangles. The three standard trigonometric functions are called *sine*, *cosine*, and *tangent*. They are represented by the abbreviated functional names sin, cos, and tan when used in mathematics. Figure 5 shows the pictorial definitions of these three trigonometric functions. Notice particularly how these function values are the unitless *ratios of two lengths*, just as with radians.

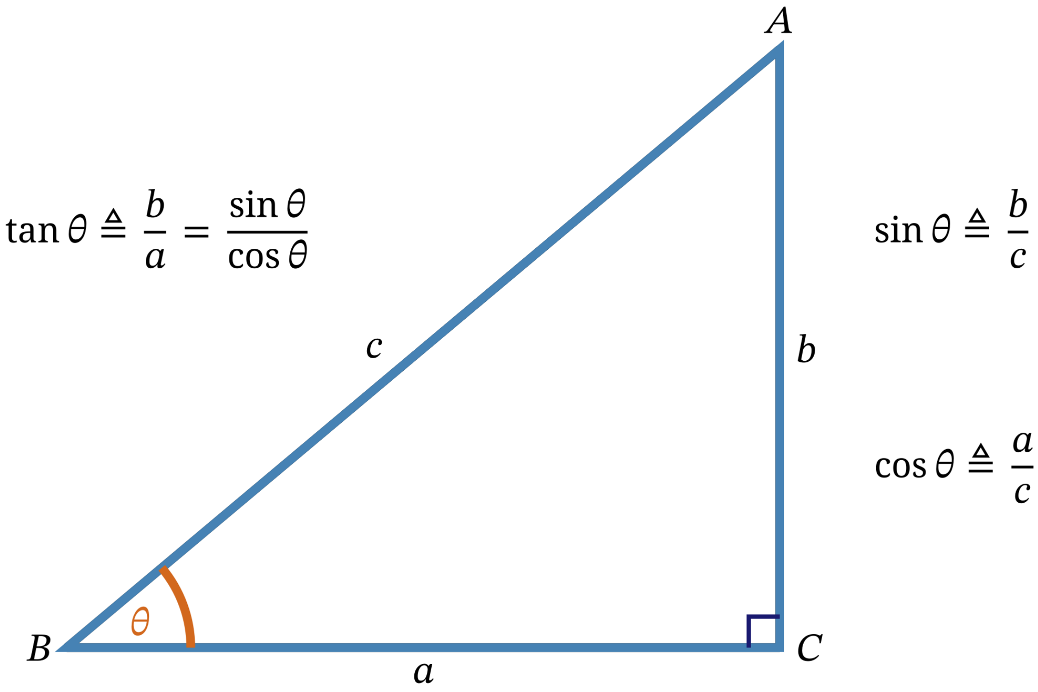

Figure 5: Trigonometric functions defined as ratios of lengths of sides in a right-angled triangle.

[7] These conversion factors must dispel any mystique attached to radians vis-a-vis degrees.

The Circular Functions

The trigonometric functions are also called the *circular* trigonometric functions, uniting the circle *and* the triangle as their progenitors. We will briefly review that relationship here, to better understand not only the terminology but also the hidden relationships between the triangle and the circle.[8]

I used to wonder why the word *tangent* was used for the name of a trigonometric function because a circle was not involved in its definition; a triangle was. But when the three standard trigonometric functions are viewed vis-a-vis a unit circle, the mystery behind the nomenclature is revealed.

The radian was introduced here using a *unit circle*. The same helpful unit circle will serve to relate the triangle and the circle to the trigonometric functions, as illustrated in Figure 6 below.

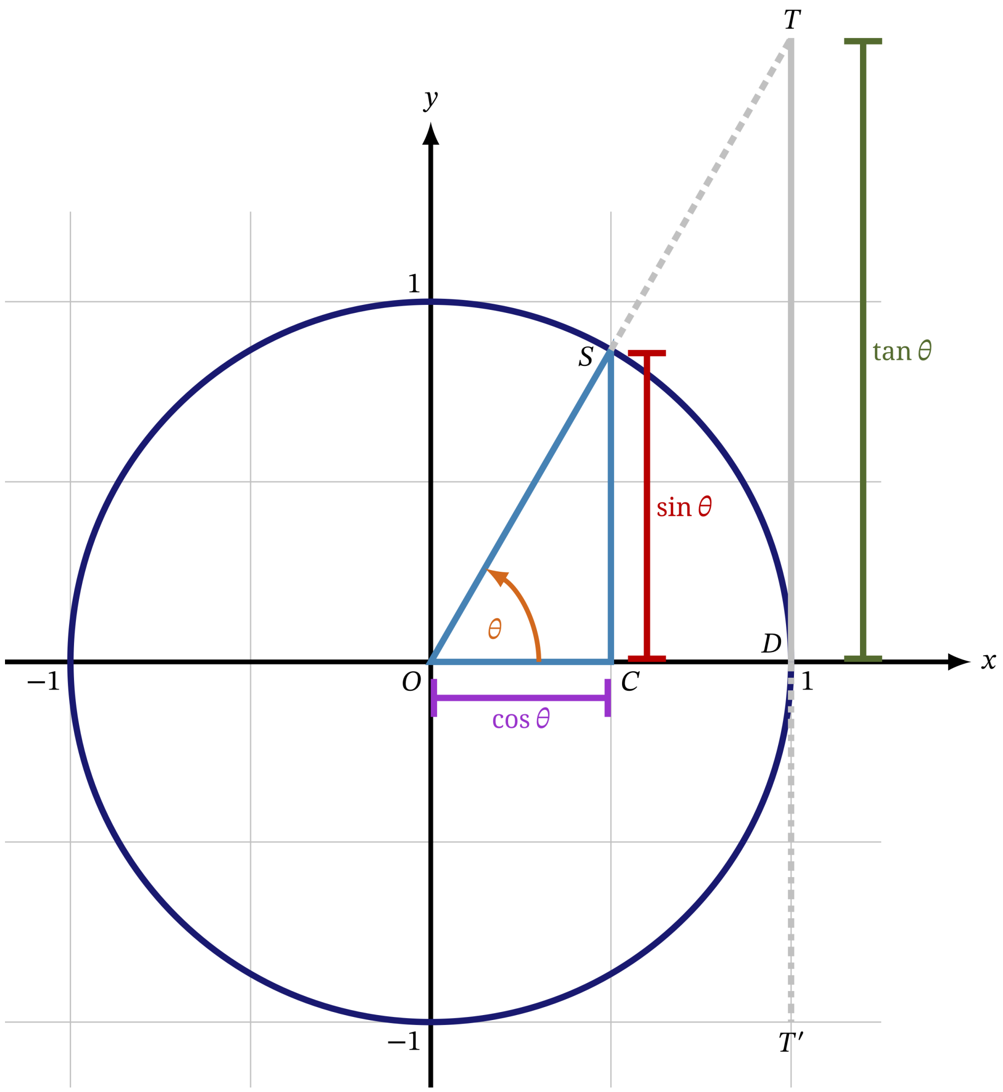

Figure 6: A pictorial representation of the unit circle, the three standard trigonometric functions, and their inter-relationships. See the text for a full explanation.

[8]The equilateral triangle is the regular n-gon with the smallest number of sides and the circle is the limiting case of an n-gon when n tends to infinity. The trigonometric functions are the children of these unlikely parents, at the extreme ends of the n-gon spectrum.

Figure 6 shows a unit circle drawn on the two-dimensional coordinate plane with x and y axes and grid markings. The centre of the circle is O and S is a variable point on the circumference of the circle, that makes a counter-clockwise angle θ with the positive x-axis. As θ varies, so does the position of S on the circle.

The line OS is a radius and therefore one unit in length. The perpendicular from S to the x-axis meets it at C. Referring to Figure 5, we may say $\cos\theta = \frac{OC}{OS} = OC$ since $OS = 1$. Accordingly, the x-coordinate of S is $\cos\theta$. Likewise, $\sin\theta = \frac{SC}{OS} = SC$. Thus, the y-coordinate of S is $\sin\theta$.

S is therefore the point with coordinates $(\cos\theta, \sin\theta)$.

Note that since the coordinates of S are confined to the unit circle, the values of $\sin\theta$ and $\cos\theta$ are confined to the closed interval $[-1, 1]$, i.e. they perforce have values lying between -1 and 1, both inclusive. From Figure 6, we see that $(\cos\theta, \sin\theta)$—which represent the coordinates of S—take on signed values in accordance with the signs of x and y in the respective quadrants. One could also view the associated *lengths* as signed values.

The tangent

The really insightful revelation from Figure 6 comes from looking at $\tan\theta$. Have you ever wondered why the ratio $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is called the *tangent*?

Recall from geometry that a straight line and a circle might lie relative to each other in three different ways as shown in Figure 7.

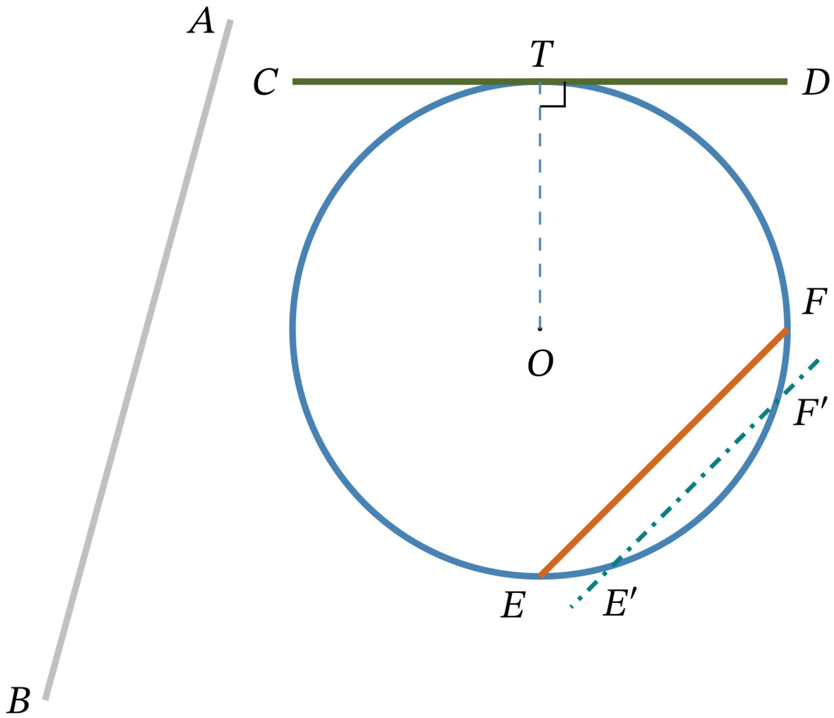

Figure 7: The three different ways in which a circle and line may lie relative to each other. See the text for the explanation.

The line AB does not cut or *intersect* the circle at all. The line EF cuts the circle at *two* points, E and F, and is called a chord. When the line EF moves parallel to itself toward the circumference of the circle, we get the chord $E'F'$. Eventually the points E' and F' will coincide and the line will cut the circle at one and only one point.

This case is illustrated by the line CTD which cuts the circle at only one point, T. CTD is called a tangent (line) to a circle and T is the point of tangency.[9] Note that the radius OT is perpendicular to the tangent CDT.

With that out of the way, we know from Figure 5, that the tangent is the ratio of the lengths of the opposite side to the adjacent side, which in Figure 6 translates to $\tan\theta = \frac{SC}{OC}$. But the denominator in this case, OC, is not 1 like it was for the other two trigonometric ratios. To work around this, with reference to Figure 6, we construct the triangle ODT thus:

a. Extend[10] the line OC to intersect the circle at the point D which is $(1, 0)$. OD, being a radius, has unit length.
b. Draw a tangent $T'DT$ to the circle at D.
c. Extend the radius OS to intersect the tangent at the point T.

Because the triangle ODT is similar to triangle OCS, we can assert that the ratios of corresponding sides are equal. Thus,

$$\tan\theta = \frac{CS}{OC} = \frac{DT}{OD} = DT \tag{3}$$

bearing in mind that, like OS, OD is also a radius of unit length. We resorted to this construction for the following reasons:

1. Because $T'DT$ is tangent to the circle, the angle ODT is a right angle.
2. The triangles OCS and ODT are therefore similar.
3. The length of OC is not one unit, but that of OD is one unit.

From Figure 6 the length of the *tangent line* DT is equal to $\tan\theta$, explaining the nomenclature.

Therefore, the value of the tangent function for an angle θ may be determined geometrically by extrapolating the radius OS until it intersects the tangent to the circle at D at a point called T.[11] The length DT is the value of $\tan\theta$.

Note, though, that $\tan\theta$ is a length *outside the unit circle* and is therefore not constrained to take on values between -1 and 1. Indeed, as θ starts increasing in the first quadrant of the unit circle, you will notice that as OS approaches the y-axis and as θ approaches $\frac{\pi}{2}$ (or 90°, if you are still attached to degrees), the line OS is increasingly aligned with DT. At $\theta = \frac{\pi}{2}$, OS is parallel to DT and "never the twain shall meet". Loosely speaking, parallel lines are only supposed to "meet at infinity" and that is why $\tan\frac{\pi}{2}$ is said to be "infinite" at that point. I find this geometric explanation—of why $\tan\theta$ does not assume a finite value at $\theta = \frac{\pi}{2}$—most fulfilling.

[9] Therefore, the tangent is a limiting case of a chord.
[10] Extending a line used to be called *producing a line* but that usage has now slipped into relative obscurity.
[11] One could also extraolate in the *opposite* direction to intersect at, say, the point T'.

When θ is in the third quadrant, for instance, OS extrapolated in the negative y direction will not intersect the tangent DT' in the negative y direction as they diverge. So, the line SO must be produced in the positive y direction to once more intersect the tangent $T'DT$ at T. That explains why tangents of angles in the third quadrant are positive.

The trigonometric functions

By moving from triangles to the unit circle on coordinate axes, we have enabled θ to take on any value between 0 and 2π radians. The *trigonometric ratios* have been unshackled from the triangle to become the *trigonometric functions* which can take on *any* real number as arguments. The graphs of the three standard trigonometric functions are shown in Figure 8.

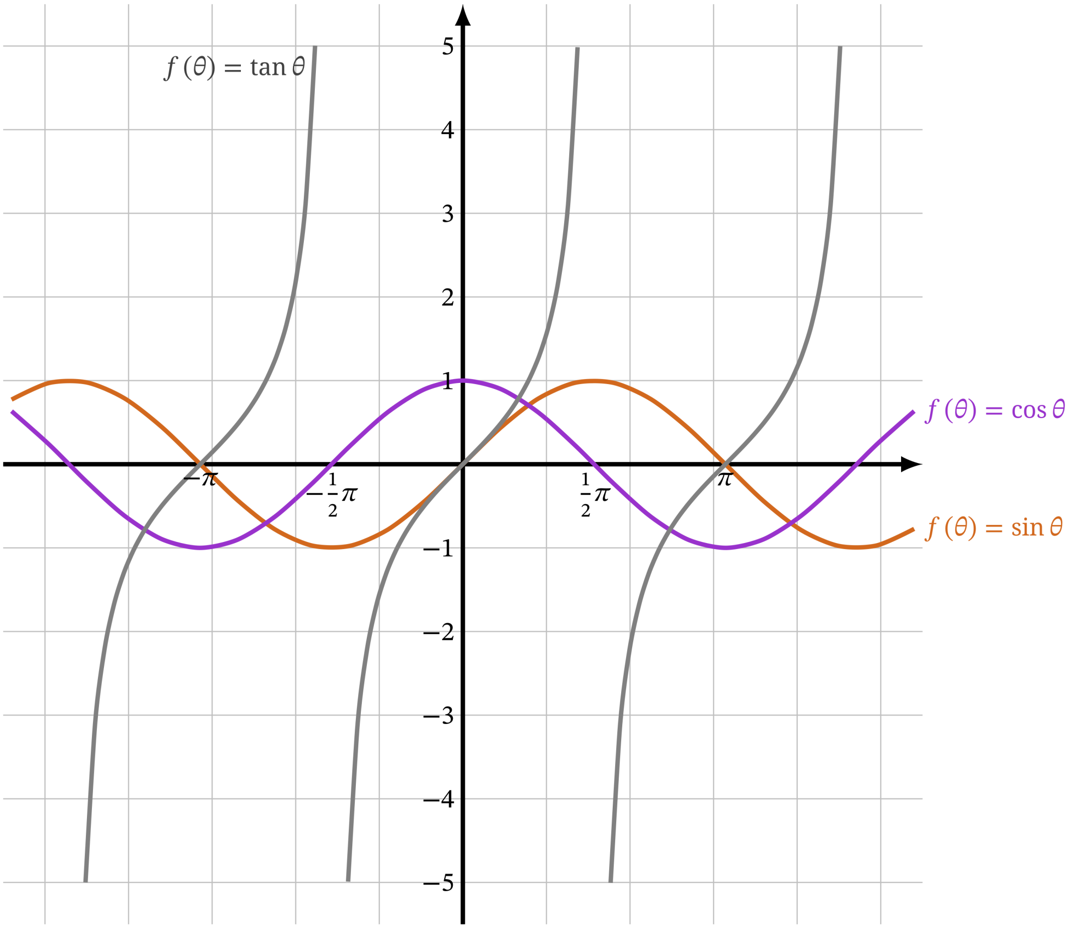

Figure 8: Graphs of the three trigonometric functions. Notice how sin and cos are bounded in their values, but tan is not. There are discontinuities for $\theta = \frac{2n+1}{2}\pi$.

Note that when $\theta = 2\pi$ radians, we cannot really distinguish it from $\theta = 0$ radians. The trigonometric functions therefore repeat themselves every time the point S in Figure 6 completes a full circle: they are *periodic* with a period of 2π. So, one angle may masquerade as another unless we have accounting devices to optionally add $2n\pi$ to it with the proviso that n is an integer. And this concept is a segue to power series expansions of trigonometric functions, their use in calculus, and later on, in Fourier series.

Power series for trigonometric functions

This is where the plot really thickens.

Both the statements $\sin(30°) = 0.5$ and $\sin(\frac{\pi}{6}) = 0.5$ are factually correct and perfectly acceptable. We will not be committing any mathematical heresies through either statement.[12]

But it is possible to express trigonometric functions in terms of power series in which an argument in degrees would be inadmissible. It is only after we cross this threshold in mathematics that radians truly come into their own, after which there is "no going back to the old ways".

I will now do a bit of hand-waving and say that it has been proved[13] that:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \tag{4}$$

where the dots at the end of Equation (4) tell us to imagine that this series *never ends but goes on forever* following the pattern shown. Equation (4) demonstrates a paradox: a trigonometric function is not a polynomial; yet a trigonometric function may be expressed as an infinite polynomial. Infinity has this beguiling attribute of "enabling the impossible."

Recall that if a number is less than one, raising it to a power greater than one makes it smaller than it originally was. So, when θ is very close to zero, the higher powers on the right hand side (RHS) of Equation (4) become smaller and smaller, and may be ignored without much loss in accuracy. In this case, we may assert that:

$$\sin\theta \approx \theta \text{ for } |\theta| \to 0. \tag{5}$$

In English this expression means that for vanishingly small values of θ—whether positive or negative—$\sin\theta$ is approximately equal to θ.

From Figure 5 we know that the number on the left hand side (LHS) of Equation (5) is a unitless ratio of two lengths and thus a "pure" number. This requires the right hand side to be also expressed in a similar unitless measure, and the radian fits the bill.

The validity of Equation (5) may also be seen from Figure 9 where the closeness of the curve $f(\theta) = \theta$ and $f(\theta) = \sin\theta$ near the origin is evident. Indeed, right up to a value of $|\theta| \approx 0.3$, the two curves track each other closely.

Let us now illustrate the reasonableness of Equation (5) by evaluation. Set θ to 0.005 radians, which is a small value close to zero. Then $\sin\theta = \sin(0.005) = 0.004999979167$, which demonstrates the validity of Equation (5). However, if one were to interpret the number 0.005 as degrees rather than radians, we then have $\sin(0.005°) = 0.00008726646249$ which is almost 57 times smaller than the number 0.005.

[12]Note that while it is mandatory to affix the degree sign as a superscript, radians being pure numbers do not require any special identification.

[13]Search the web for Taylor Series or Maclaurin series, thinking of it as a treasure hunt and enrich yourself with that knowledge!

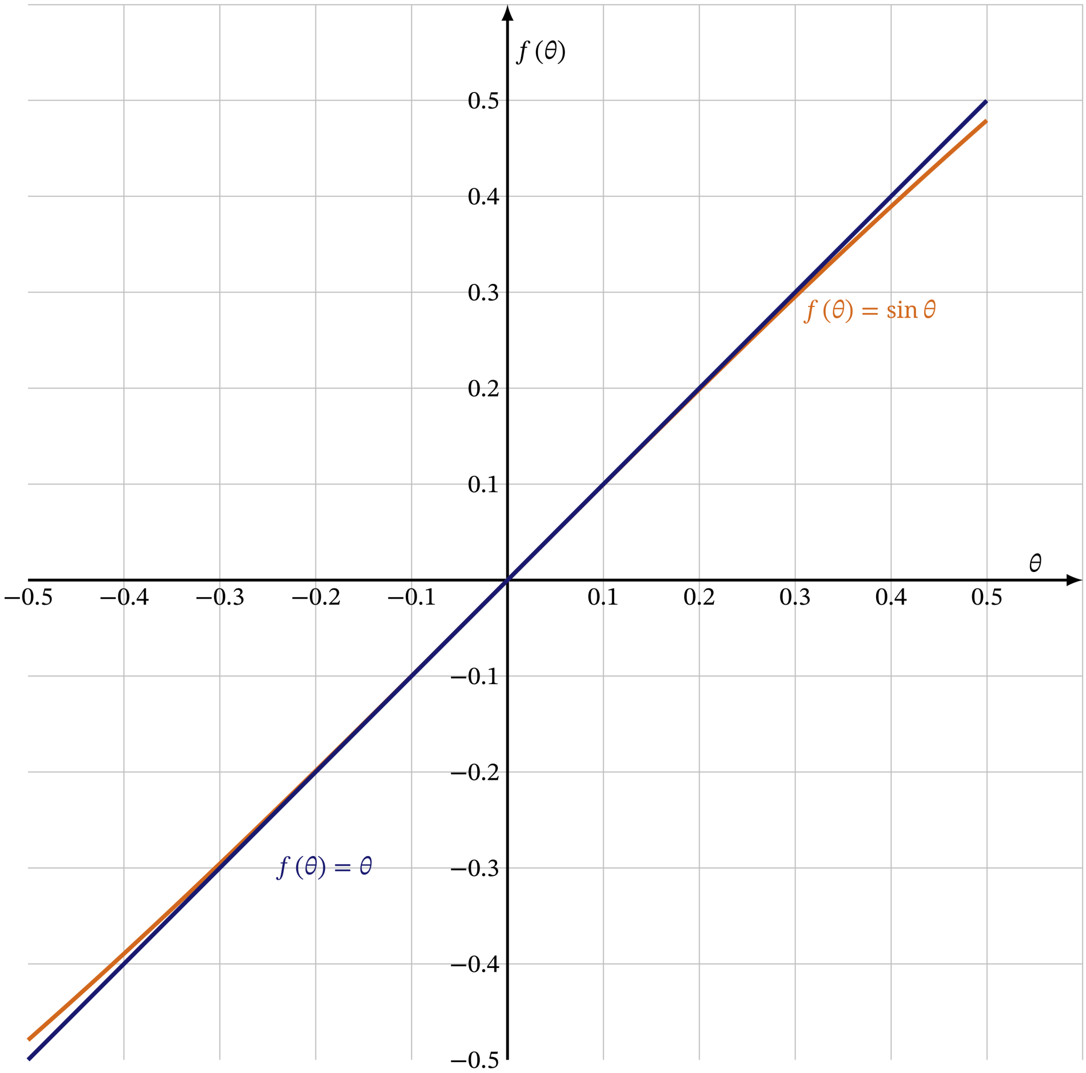

Figure 9: Graphs of $f(\theta) = \sin\theta$ and $f(\theta) = \theta$ for $|\theta| \approx 0$.

The moral of this example is that when we evaluate trigonometric functions in degrees in the context of their power series, we must apply a correction factor of $\frac{\pi}{180}$ to implicitly convert the function argument on the LHS from degrees to radians. Otherwise, keeping the degree argument, we have to apply a factor of $\frac{180}{\pi}$ to *each term* on the RHS. This is a layer of bookkeeping we may easily avoid by using radians on both sides of the equation.

Fourier series

Periodic waveforms repeat themselves indefinitely. The sinusoids—which are trigonometric functions like sin and cos—are one such example. Such waveforms arise frequently in signal analysis and synthesis in electrical engineering. In that context, the independent variable is *time*.

A *periodic function* with certain properties may be represented by an infinite sum of sinusoids. This was the great insight of Jean-Baptiste Joseph Fourier from whom Fourier series derive their name.

For example, let the original signal be a square waveform denoted by the function $s(t)$ in Figure 10.[14] Imagine that the single cycle of the square wave—shown below— is repeated periodically forever.

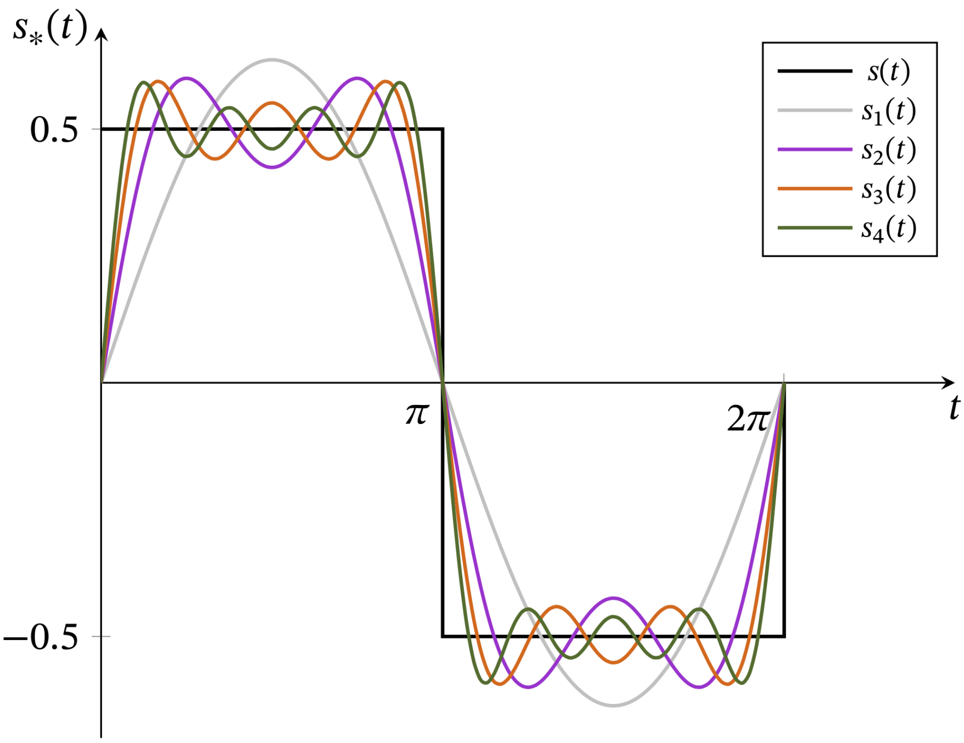

Figure 10: A square waveform $s(t)$ and the sequential sums of its first four terms.

[14]Strictly speaking, there are *point discontinuities* in $s(t)$, at $t = \pi$ and $t = 2\pi$, where the function changes value. The graph of the waveform is shown as a vertical line at these points because that is what an oscilloscope trace of the waveform will show. This is convenient but inaccurate because a *function* cannot be multi-valued at one point. Nevertheless, the theory behind Fourier series is still applicable to the square wave. The Fourier series will converge at these points to zero—the average value at these discontinuities—which is what the partial sums show.

The Fourier series for such a square waveform is an infinite sum of sinusoids that collectively represent the waveform. This might seem like a tall order but it is nevertheless true. The Fourier series representation of the square wave $s(t)$ is given by:

$$s(t) = \frac{2}{\pi}\left[\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \frac{\sin(7t)}{7} + \cdots\right] \tag{6}$$

where again, the dots mean that the pattern repeats forever. Note that this is no approximation, but an equality.

The successive *partial sums* of the RHS of Equation (6) are termed $s_1(t)$, $s_2(t)$, etc., with $s_1(t)$ denoting the first term, $s_2(t)$, the sum of the first and second term etc. It is evident from Figure 10 that the more terms we add, the better the match between the original signal and its approximation, denoted generically by $s_*(t)$ in the graph.[15]

On the surface, Equation (6) seems a remarkable claim. How could a square wave with right angle corners be the result of sums of sine waves which have no corners? The answer lies in the fact that the series goes on forever and "infinity confers the equality".

Where do radians feature here? As with the power series for the sine function, it is on the RHS where the variable t should be expressed in radians. Note that while t may have units of time, the fact that radians are unitless does not intrude into the equation as an extraneous factor.

We will close with one more example where radians make mathematical life much easier.

Euler's formula and identity

The prodigiously productive Swiss mathematician, Leonhard Euler, gave us many equations, one of which is known as *Euler's formula*, shown below:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{7}$$

The letter i is called the imaginary unit and its definition as $i^2 = -1$, takes us into the field of complex numbers.[16]

When we substitute $\theta = \pi$ into Equation (7), and transpose terms, we get what is called *Euler's identity*:

$$\begin{aligned} e^{i\pi} &= \cos\pi + i\sin\pi \\ e^{i\pi} + 1 &= 0 \end{aligned} \tag{8}$$

Equation (8) has been described as the most poetic mathematical equation because it unites in one equation the five most fundamental quantities in all of mathematics: $e, i, \pi, 1$ and 0. And it could not have come about without radians. With that epiphany on beauty, we shall conclude our tale of two measures.

[15]This property—where the larger the number of terms in the partial sum, the better the approximation to the original function—is one reason why Fourier series are widely applied.

[16]The Fundamental Theorem of Algebra states that all non-constant polynomials with complex coefficients contain at least one complex root, to express which i is necessary.

Acknowledgements

All illustrations in this blog were generated using the TikZ-PGF and PGFPlots packages with LaTeX and Pandoc. To the authors of these packages, and to others who posted numerous examples of their usage on the Web, my humble gratitude. For Figure 10, I have drawn upon and modified the example of caverac at the TeX StackExchange forum.

Feedback

Please email me your comments and corrections.

A PDF version of this article is available for download here:

https://swanlotus.netlify.app/blogs/radians.pdf