1. Yes. we can assumpt the noise as zero-mean Gaussian because according to the two error plot, the range error and Speed error, their means are very close to 0, and the noise of motion model comes ~~fro~~ mostly from speed error ($\to$ mean = -0.000451 m/s), and ~~mea~~ the noise of measurement model comes mostly from range error ($\to$ mean = -0.0000 m).

$\sigma_q^2 = 0.04755 4^2$ $\qquad$ $\sigma_r^2 = 0.019155^2$.

2.
$$\begin{cases} x_k = x_{k-1} + \underbrace{T}_{\text{scaler}} \cdot u_k + w_k & \to \text{motion model} \\ y_k = x_k + n_k & \to \text{observation model} \end{cases}$$

$$\vec{x} = (x_1, x_2, \dots x_K) = x_{1:K}$$
$$\vec{u} = (u_1, u_2, \dots u_K) = u_{1:K}$$
$$\vec{y} = (y_1, y_2, \dots y_K) = y_{1:K}$$

using MAP method:

$$\hat{x} = \arg\max_x p(\vec{y}\,|\,\vec{x})\, p(\vec{x}\,|\,\vec{u})$$

$$\begin{cases} p(\vec{y}\,|\,\vec{x}) = \prod_{k=1}^{K} p(y_k\,|\,x_k) \\ p(\vec{x}\,|\,\vec{u}) = \prod_{k=1}^{K} p(x_k\,|\,x_{k-1}, u_k) \end{cases}$$

$$\hat{x} = \arg\min_x \left(-\ln p(\vec{y}\,|\,\vec{x})\, p(\vec{x}\,|\,\vec{u})\right)$$
$$= \arg\min_x \left(-\ln \sum_{k=1}^{K} p(y_k\,|\,x_k) - \ln \sum_{k=1}^{K} p(x_k\,|\,x_{k-1}, u_k)\right)$$

$$\begin{cases} p(y_k\,|\,x_k) = \cancel{\dfrac{1}{\sqrt{2\pi\sigma_q^2}}} \exp\left(-\frac{1}{2}\dfrac{(y_k - x_k)^2}{\sigma_q^2}\right) \\ p(x_k\,|\,x_{k-1}, u_k) = \cancel{\dfrac{1}{\sqrt{2\pi\sigma_r^2}}} \exp\left(-\frac{1}{2}\dfrac{(x_k - x_{k-1} - Tu_k)^2}{\sigma_r^2}\right) \end{cases}$$

$$\begin{cases} -\ln p(y_k\,|\,x_k) = \cancel{\frac{1}{2}\ln(2\pi\sigma_q^2)} + \frac{1}{2}(y_k - x_k)^2/\sigma_q^2 \\ -\ln p(x_k\,|\,x_{k-1}, u_k) = \cancel{\frac{1}{2}\ln(2\pi\sigma_r^2)} + \frac{1}{2}(x_k - x_{k-1} - Tu_k)^2/\sigma_r^2 \end{cases}$$

$$\hat{x} = \arg\min_x J(x)$$

$$J(x) = \frac{1}{2}\sum\left[(x_k - x_{k-1} - T\cdot u_k)^2 \cdot \sigma_r^{-2} + (y_k - x_k)^2 \cdot \sigma_q^{-2}\right].$$

litted form.

$$\underset{2K\times K}{z = \begin{bmatrix} T\cdot\vec{u} \\ \vec{y} \end{bmatrix}} \quad H = \begin{bmatrix} A^{-1} \\ C \end{bmatrix} \quad x = \vec{x} \quad \underset{2K \times 2K}{W = \begin{bmatrix} Q & \\ & R \end{bmatrix}}$$

$$\underset{K\times K}{A^{-1}} = \begin{bmatrix} 1 & & \\ -1 & \ddots & \\ & -1 & 1 \end{bmatrix} \quad \underset{K\times K}{C} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \quad \underset{K\times K}{\text{diag}(\sigma_q^2)} \quad \underset{K\times K}{\text{diag}(\sigma_r^2)}$$

$$J(x) = \frac{1}{2}(z - Hx)^T W^{-1}(z - Hx)$$

3.
$$\left.\frac{\partial J(x)}{\partial x^T}\right|_{\hat{x}} = -H^T W^{-1}(z - H\hat{x}) = 0 \Rightarrow (H^T W^{-1} H)\, x^*_{1:K} = H^T W^{-1} z.$$

4. Motion model: $x_{\delta k} = x_{\delta(k-1)} + T \sum_{i=\delta(k-1)}^{\delta k} u_i + \delta w_k$.

Observation model: $y_{\delta k} = x_{\delta k} + \delta n_k$

$$J(x) = \frac{1}{2}(z - Hx)^T W^{-1}(z - Hx)$$

$$\underset{2n\times 1}{z = \begin{bmatrix} v \\ y \end{bmatrix}} \qquad \underset{n\times 1}{v = \begin{bmatrix} T\cdot\sum_{i=0}^{\delta} u_i \\ T\cdot\sum_{i=\delta}^{2\delta} u_i \\ \vdots \\ T\cdot\sum_{i=n\delta}^{k} u_i \end{bmatrix}} \qquad \underset{n\times 1}{y = \begin{bmatrix} y_\delta \\ y_{2\delta} \\ \vdots \\ y_{n\delta} \end{bmatrix}}$$

$$\underset{2n\times n}{H = \begin{bmatrix} A^{-1} \\ C \end{bmatrix}} \qquad \underset{n\times n}{A^{-1} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & \\ & & -1 & 1 \end{bmatrix}} \qquad \underset{n\times n}{C = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}}$$

$$\underset{n\times 1}{x = \begin{bmatrix} x_\delta \\ x_{2\delta} \\ \vdots \\ x_{n\delta} \end{bmatrix}}$$

$$W = \begin{bmatrix} Q & \\ & R \end{bmatrix} \qquad \underset{n\times n}{Q = \begin{bmatrix} \delta\sigma_q^2 & & \\ & \ddots & \\ & & \delta\sigma_q^2 \end{bmatrix}} \qquad R = \begin{bmatrix} \delta\sigma_r^2 & & \\ & \ddots & \\ & & \delta\sigma_r^2 \end{bmatrix}$$

$$\frac{\partial J(x)}{\partial x^T} = -H^T W^{-1}(z - H\hat{x}) = 0$$

$$(H^T W^{-1} H)\hat{x} = H^T W^{-1} z$$