Non-Cooperative Game Theory

Informally, non-cooperative games are those in which the actions of individual players form the action/strategy primitives while in cooperative games, joint actions of groups of players form the action/strategy primitives.

Some Notations

- Game theory may be defined as the study of mathematical models of interaction between rational, intelligent decision makers.
- The decision maker is usually referred to as players or agents.
- The interaction may involve conflict as well as cooperation.
- In the student coordination problem, let SFU be represented as A and Club be represented as B. We have two players (student 1 and 2). Each of them can choose any action or strategy from the set {A,B}. They choose their individual actions simultaneously, independent of each other.

Depending on the strategies chosen, the two players obtain payoffs as shown in the table:

1	2	
	A	B
A	10,10	0,0
B	0,0	1,1

- Definition: Game Γ is a tuple $\langle N, (S_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where:
- N={1,2,….,n} is a set of players
- $S_1, S_2, \cdots, S_n$ are sets called the strategy sets of players 1, …., n, respectively. S is $S_1 \times S_2 \times \cdots \times S_n$ which is a Cartesian product (collection of all strategy profiles).
- $u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$ for i=1,2, …, n are mappings called the utility functions or payoff functions.
- In the above example, we can write:

$$N = \{1, 2\};\ S_1 = S_2 = \{A, B\};$$

$$u_1(A, A) = 10;\ u_1(A, B) = 0; u_1(B, A) = 0;\ u_1(B, B) = 1;$$
$$u_2(A, A) = 10;\ u_2(A, B) = 0; u_2(B, A) = 0;\ u_2(B, B) = 1.$$

- Preferences: the student coordination game has four outcomes, namely (A,A), (A,B), (B,A), and (B,B) which are also four strategy profile.
- Each student has certain preferences over these outcomes.
- Clearly, in this case, each student prefers outcome (A,A) over (B,B); prefers outcome (B,B) over outcomes (A,B) and (B,A); and has no preference between (A,B) and (B,A).
- The preferences that a player has over outcomes can be formularized as a *preference relation* over the set of outcomes S.

Utilities: the utility function or payoff function of a player is a real valued function defined on the set of all outcomes or strategy profiles.

The utility function of each player maps multi-dimensional information (strategy profiles) into real numbers to capture preferences.

Rationality: one of the key assumptions in game theory is that the players are rational. An agent is said to be rational if the agent always makes decisions in pursuit of her own objectives.

Intelligence: another key notion in game theory is that of intelligence of the players. This notion means that each player in the game knows everything about the game and player is competent enough to make inferences about the game.

Common knowledge: in a game with complete information, the set N, the strategy set S, and the utility functions u are common knowledge, that is every player knows them, every player knows that every player knows them, and so on.

Classification of Games

- *non-cooperative games and cooperative games*: non-cooperative games are those in which the actions of individual players are the primitives; in cooperative games, joint actions of groups of players are the primitives.

  For example, a game is cooperative if commitments (agreements, promises, threats) among players are enforceable and that a game becomes non-cooperative if the commitments are nor enforceable.

- *Static games and dynamic games*: in static games, players choose their actions simultaneously and no information is received during the play (e.g. SFU or Club example).

In dynamic game which is often called multi-stage game, there is a temporal order in which actions are played by the players. For example, a certain player chooses an action before other players do and the player knows that the choice of actions by other players will be influenced by her/his action (e.g. Chess game).

- Games with perfect information and games with imperfect information: when the players are fully informed about the entire past history (each player, before making a move, knows the past moves of all other players as well as his own past moves), the game is said to be perfect information. Otherwise it is called game with imperfect information.
- Complete information and incomplete information games: a game with incomplete information is one in which, at the first point in time when the players can begin to plan their moves, some players have private information about the game that other players do not know. In a game with complete information, every aspect of the game is common knowledge.

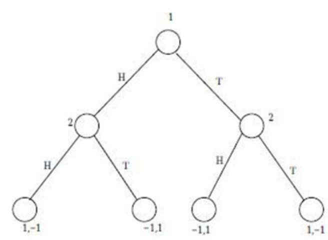
Extensive Form Games

- The extensive form representation of a game provides a detailed and richly structured way to describe a game.

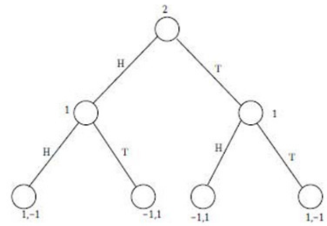
Illustrative Examples

(Matching Loonies with Observation)

- There are two players, 1 and 2, each with a dollar coin.
- One players puts down his coin heads or tails up.
- The other player sees the outcome and puts down her coin heads up or tails up.
- If both coins show heads or both coins show tails, player 2 gives one dollar to player 1 who thus becomes richer by one dollar.
- If one of the coins shows heads and the other coin shows tails, then player 1 pays one dollar to player 2 who becomes richer by one dollar.
- Depending on whether player 1 or player 2 moves first, there are two versions of this game.



Player 1 moves first.

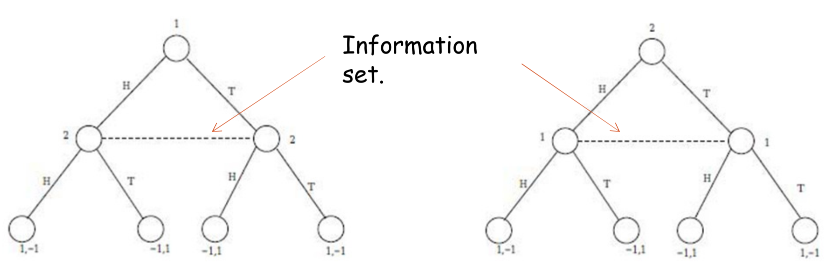


Player 2 moves first.

- In the game tree representation, the nodes are of three types:
- Root node (initial decision node); internal nodes (which are decision nodes); and leaf nodes or terminal nodes (which are outcome nodes).
- When the game is played, the path that represents the sequence of events is called the path of play.
- Each decision node is labeled with the player who makes the decision at that node.
- Each decision node can be uniquely identified by a sequence of actions leading to that decision node from the root node.
- Note that each node represents not only the current position in the game but also how it was reached.

Matching Loonies without Observation

- In this case, one of the players places her/his dollar coin heads or tails up.
- The other player does not observe the outcome and only puts down her/his dollar coin heads up or tails up.
- Depending on whether player 1 moves first or player 2 moves first, we obtain the following game trees.



Matching Loonies with Simultaneous Play

- A set of nodes that are connected with dashed lines is called an information set.
- When the game reaches a decision node in an information set, the player involved at the node does not know the node in the information set she is in.
- The reason for this is that the player cannot observe the previous moves in the game.

- Best response strategy (Strategic Form Games):

We denote by S, the set of alll strategy profiles or strategy vectors, which is the Cartisian product $S_1 \times S_2 \times \cdots \times S_n$.
A typical stretegy profile is represented by $(s_1, s_2, \cdots, s_n)$ where s_i is the strategy of player i $(i = 1, \cdots, n)$.
We denote by S_{-i} the Cartisian product $S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$ of strategy sets of all players other than player i.
We denote by s_{-i} is a typical strategy profile in S_{-i}.
When we are focusing on a particular player i, a convenient way of representing a stretegy profile is (s_i, s_{-i}) where $s_i \in S_i$ and $s_{-i} \in S_{-i}$.

- The idea behind strategic form representation is that a player's decision problem is essentially choose a strategy that will counter most effectively the strategies adopted by the other players.
- Such strategy is called a best response strategy which is formally defined as:

Give a strategic form game $\Gamma = \langle N,(S_i),(u_i)\rangle$ and a strategy profile $s_{-i} \in S_{-i}$, we say $s_i \in S_i$ is a best response strategy of player i with respect to s_{-i} if $u_i(s_i,s_{-i}) \geq u_i(s_i',s_{-i}) \forall s_i' \in S_i$.

- Given a strategy profile $s_{-i} \in S_{-i}$, of all players except player i, there could exist multiple best response strategies for player i.
- In a strategic form game, each player is faced with the problem of choosing her/his best response strategy and the players can be thought of as simultaneously choosing their strategies from the respective sets $S_1,\cdots\cdots,S_n$.

Matching Loonies with Simultaneous Moves

- In this game, two players 1 and 2 put down their respective loonie coins, heads or tails up.
- If both coins match, the player 2 pays one loonie to player 1.
- Otherwise, player 1 pays one loonie to player 2.
- Let A denote heads and B denotes tails.

$$N = \{1,2\}$$
$$S_1 = S_2 = \{A, B\}$$
$$S = S_1 \times S_2 = \{(A,A),(A,B),(B,A),(B,B)\}$$

- The payoff matrix is given by:

1	2	
	A	B
A	1,-1	-1,1
B	-1,1	1,-1

- This is what is referred to as zero sum games (so called because the sum of utilities in every outcome is equal to zero).
- In the above game, it is easy to see that the best response strategy of player 1 is A (B) when player 2 plays A (B).
- On the other hand, the best response strategy of player 2 is A (B) when player 1 is B (A).

Practical example of the above simple game

- There are two companies 1 and 2. Each can produce two products A and B, but at a given time, a company can only produce one product.
- Company 1 is known to produce superior quality products but company 2 scores over company 1 in terms of marketing and advertisement.
- If both companies produce the same product (A or B), it turns out that company 1 makes all the profits and company B looses out (i.e. payoff +1 for 1 and a payoff of -1 for company 2).

- However, if one company produces A and the other produces B, it may turn out that due to aggressive marking of 2, company 2 captures all the market (i.e. payoff of -1 for 1 and payoff of +1 for 2).
- Two companies must simultaneously decide and each one does not know about the strategies of the other.

1	2	
	A	B
A	1,-1	-1,1
B	-1,1	1,-1

Rock-Paper-Scissors Game

- Another example of zero sum game where each player has three strategies, called *rock, paper*, and *scissors*.
- The rock symbol beats scissors symbol; scissors symbol beats paper symbol; paper symbol beats rock symbol.
- The payoff matrix for this game is given as follows:

1	2		
	Rock	Paper	Scissors
Rock	0,0	-1,1	1,-1
Paper	1,-1	0,0	-1,1
Scissors	-1,1	1,-1	0,0

Battle of Sexes Game

- This is a game where players want to coordinate with each other; however, they have disagreement about which of the outcome is better.
- Let us say two players 1 and 2 wish to go out together to an event A or to an alternative event B.
- Player 1 prefers to go to event A and player 2 prefers to go to event B. The payoff matrix:

1	2	
	A	B
A	2,1	0,0
B	0,0	1,2

- Clearly, this game captures a situation where the players want to coordinate but they have conflicting interests.
- The outcomes (A,B) and (B,A) are unfavorable to either player.
- The choice is essentially between (A,A) and (B,B).

A Practical example of the above simple game

- Recalling the company analogy, suppose we have two companies, 1 and 2.
- Each company can produce only one of two competing products A and B, but at any given time, a company can only produce one type of product.
- Assume A is a niche product for company 1 while product B is a niche product to company 2.
- For example, if both companies produce product A, the consumers are compelled to buy product A and would naturally prefer to buy it from company 1 rather 2.

- If both companies produce product B, the reverse situation will prevail.
- On the other hand, if two companies decide to produce different products, then the market gets segmented and each company tries to outwit the other through increased spending on advertisement.
- In fact, their competition may actually benefit.

A Coordination Game (student coordination game)

- This game is similar to the Battle of Sexes Game. But the two players now have a preference for the same option, namely event A. The payoff matrix is shown:

1	2	
	A	B
A	10,10	0,0
B	0,0	1,1

- Continuing our analogy of companies, the above game corresponds to a situation wherein the two companies produce the same product, and they have equal market share.
- The market share is ten times as much for product A as for product B. If two companies produce different products, a third company may capture all the market share.
- Since the payoff is the same for both companies, this is called common payoff games.

Prisoner's Dilemma Game

- Two individuals are arrested for allegedly committing a crime and are lodged in separate cells.
- The interrogator questions them separately.
- The interrogator privately tells each prisoner that if he/she is the only one to confess, he will get a light sentence of 1 year in jail while the other would be sentences to 10 years in jail.
- If both players confess, they would get 5 years each in jail.
- If neither confesses, then each would get 2 years in jail.
- The interrogator also informs each prisoner what has been told to each prisoner.

1	2	
	NC	C
NC	-2,-2	-10,-1
C	-1,-10	-5,-5

- They would like to play a strategy that offers a best response to a best response strategy that the other player may adopt, the latter player also would like to play a strategy that offers a best response to the other player's best response strategy, and so on.
- First observe that *C* is each player's best strategy regardless of what the other player plays:

$$u_1(C,C) = -5 > u_1(NC,C) = -10; \;\; u_1(C,NC) = -1 > u_1(NC,NC) = -2.$$
$$u_2(C,C) = -5 > u_2(C,NC) = -10; \;\; u_2(NC,C) = -1 > u_2(NC,NC) = -2.$$

- Thus (*C*,*C*) is a natural prediction for this game.
- However, the outcome (NC,NC) is the best outcome jointly for the players.