Example:
Matching Loonies and Mixed Strategies Nash Equilibrium

• You and I simultaneously choose whether to put our loonie on the table with heads or tails facing up.

•If both of the loonies show heads or both loonies show tails (that is they match), then you have to give me a dollar.

•But if one shows heads and the other shows tails (that is, they do not match), then I have to pay you a dollar.

•What is the optimal way to play the matching loonies game?

•Let us look at the payoff matrix:

	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

•Matching loonies is an example of a strictly competitive (zero sum) game.

•Many real world examples fit these payoffs.

•For example, on a soccer penalty kick, a kicker can kick left or right; the goalkeeper can dive left or right.

•In football, the offense can choose a running play or passing play; the defense can choose to protect against the pass or the run.

•In baseball, a pitcher can throw a fastball or a curveball; a batter can guess fastball or curveball.

- Without even marking the best responses, clearly there are no pure strategy Nash equilibria.
- Suppose player 1 always played heads. Then player 2 should play tails, since that will win her a dollar.
- But if player 2 is playing tails as a pure strategy, player 1 should play tails as well, as that will win him the dollar.
- But given that player 2 is playing heads, player 1 should go back to playing heads.
- The process now begins to loop, as now player 2's best response is to play tails and we have returned to our starting point.

If we analyze the game by marking best responses, we would windup with the following which shows no mutual best response exists in pure strategies. The game has no pure strategy Nash equilibrium.

	Head	Tail
Head	1*,-1	-1,1*
Tail	-1,1*	1*,-1

Unsurprisingly, every 1 has an asterisk and every -1 does not. No outcome has an asterisk over each player's payoff, meaning no mutual best response exists in pure strategies. Therefore, the game has no pure strategy Nash equilibrium.

- Does that mean this game has no solution? Not nearly.
- As it turns out, every finite game has at least one Nash equilibrium.
- What is a Mixed Strategy?
- Suppose I can read your mind. That is, as you were deciding whether to place your coin with heads or tails showing, I could perfectly anticipate what you were about to do.
- Is there a way you could minimize your losses?
- The easiest solution is to flip the coin and cover it with your hand as it is about to land on the table.
- If you did, my mind reading skills would be rendered harmless.
- No matter what I chose, my net expected gain from playing the game would be $0.
- If I guessed heads every time, I would win $1 half of the time and lose $1 half the time, for an average of $0.
- Likewise, if I guessed tails every time, I would win $1 half of the time and lose $1 half of the time, again for an average of $0.

- Moreover, if I flipped my coin in the same manner you do, I would still net $0.
- A quarter of the time they would both land on heads, and I would earn $1.
- Another quarter of the time they would both land on tails, and I would earn $1.
- In yet another quarter of the time, mine would land on heads and yours would land on tails, and I would lose $1.
- In the final quarter of the time, mine would land on tails and your would land on heads, and I would again lose $1.
- On average, all of these cancel out, and my expected value of the game is equal to $0.

- We can extend this further and say that <u>any</u> probability distribution over playing heads and tails would net me $0 on average as long as you kept flipping your coin.
- For example, if I played heads 1/3 of the time and tails 2/3 of the time, I would earn $0.
- Likewise, if I played heads 7/16 of the time and tails 9/16 of the time, I would still be stuck earning $0.
- Effectively, your coin flipping strategy is unbeatable and unexplainable.
- From my perspective, if I cannot beat you, then I might as well flip my coin as well.

- As we just saw, I will make $0 from this game.
- But interestingly, if I am flipping my coin, then you can not beat me from the same reason I could not beat you.
- So when we both flip our coins, neither one of us has a profitable deviation from our strategies.
- Thus, this pair of coin flipping strategies is a <u>mixed strategy Nash equilibrium</u>.
- The "<u>mixed strategy</u>" part of the term refers to how we are randomizing over multiple strategies rather then playing a single "pure" strategy.

- Consider the following game:

	Left	Right
Up	3,-3	-2,2
Down	-1,1	0,0

- This game is very similar to matching loonies, except the values exchanged differ depending on the outcome; anywhere from 0 or 3 points changes hands depending on the outcome.
- There are no pure strategy Nash equilibria.
- If player 1 is playing up, then player 2 will want to play right and earn 2.
- But if player 2 is playing right, then player 1 will want to play down and earn 0 instead of -2.
- Yet if player 1 goes down, then player 2 can deviate to left and earn 1.
- However, that causes player 1 to switch to up and earn 3.
- Now player 2 wants to go right, and we have a new cycle.

- We can see this with the best responses marked:

	Left	Right
Up	3*,-3	-2,2*
Down	-1,1*	0*,0

- Since no pure strategy Nash equilibria exist, there must be an equilibrium in mixed strategies.
- But is merely flipping a coin sufficient?
- Unfortunately, no. Suppose player 1 flipped a fair coin and played up on heads and down on tails.
- Then player 2's expected utility for playing left is weighted combination of her left column payoffs. We can write it as:

$$EU_{left} = (0.5)(-3) + (0.5)(1) = -1$$

- To see how we arrived at this equation, let's break down the game a step further.

- We know that player 1 is playing both up and down some portion of the time, and we also know player 2 is always playing left.

	Left
Up	?,-3
Down	?,1

- Half of the time, or (.5) of the time, player 1 plays up, and player 2 earns -3.
- That gives us the (.5)(-3) part of the equation.
- The second half follows similarly. Half the time, player 1 plays down, at which point player 2 earn 1. Thus, we get the (.5)(1) part. Summing those two parts together yields the answer of -1.

- Let's do the same for player 2 playing right. Blocking out the irrelevant information gives us the following:

	Right
Up	?,2
Down	?,0

- Half of the time, she earns 2. The other half of the time, she earns 0. so her expected utility of playing right in response to player 1 flipping a coin equals:

$$EU_{right} = (0.5)(2) + (0.5)(0) = 1$$

- This is a problem. If player 1 flips a coin, player 2 ought to play right as pure strategy since it generates a higher expected payoff for her.

- Consequently, player 2 would not want to randomize between left and right in response to player 1 flipping, as doing so feeds her more of the negative payoff from left and less of the positive payoff from right.
- But if she plays right, we already know player 1's best response is to play down as a pure strategy.
- Now player 1 has abandoned his mixtures, and we are back to where we started.

The Mixed Strategy Algorithm

- We can specify an algorithm to find how a player can induce his or her opponent to be indifferent between the opponent's two pure strategies.
- With mixed strategies, we have to introduce unknown variables and the necessary algebra to solve for them.
- Let σ represent the probability that a player plays a particular pure strategy. For example, we use σ_{up} as the probability player 1 plays up.
- Using this notation, we can write player 2's expected utility of playing left as a pure strategy as a function of player 1's mixed strategy:

$$EU_{left} = (\sigma_{up})(-3) + (\sigma_{down})(1)$$

- and player 2's expected utility of playing right as a pure strategy is:

$$EU_{right} = (\sigma_{up})(2) + (\sigma_{down})(0)$$

- We are looking for a mixed strategy from player 1 that leaves player 2 indifferent between her pure strategies. Or:

$$EU_{left} = EU_{right}$$
$$(\sigma_{up})(-3) + (\sigma_{down})(1) = (\sigma_{up})(2) + (\sigma_{down})(0)$$
$$\sigma_{down} = 1 - \sigma_{up}$$

$$(\sigma_{up})(-3) + (1-\sigma_{up})(1) = (\sigma_{up})(2) + (1-\sigma_{up})(0)$$
$$-3\sigma_{up} + 1 - \sigma_{up} = 2\sigma_{up}$$
$$\sigma_{up} = 1/6$$

•So if player 1 plays up with probability of 1/6 and down with probability of 5/6, player 2 earns the same payoff for selecting either left or right as a pure strategy.
•Player 2 also has to play a mixed strategy in the Nash equilibrium of this game.

• Let's calculate the mixed strategy for player 2 that leaves player 1 indifferent between his two strategies.
•First we need to find player 1's payoff for playing up as a function of player 2's mixed strategy σ_{left}

	Left	Right
Up	3,?	-2,?

$$EU_{up} = (\sigma_{left})(3) + (1-\sigma_{left})(-2)$$

- Now we move to player 1's payoff for playing down as a function of that same mixed strategy of player 2's:

	Left	Right
Down	-1,?	0,?

$$EU_{down} = (\sigma_{left})(-1) + (1-\sigma_{left})(0)$$

- We need to find a mixed strategy for player 2 that leaves player 1 indifferent between his pure strategies.

$$EU_{up} = EU_{down}$$
$$(\sigma_{left})(3) + (1-\sigma_{left})(-2) = (\sigma_{left})(-1) + (1-\sigma_{left})(0)$$
$$3\sigma_{left} - 2 + \sigma_{left} = -\sigma_{left}$$
$$\sigma_{left} = 1/3$$

- So if player 2 plays left with probability 1/3 and right with probability 2/3, player 1 is indifferent between playing up and down as pure strategies.

- Connecting the mixed strategies of both players together, we see they are a best response to each other and therefore a Nash equilibrium.
- That is, if player 1 plays up with probability 1/6 and down with probability 5/6, then any strategy of player 2's produces the same payoff, so any mixture between her two strategies is a best response.
- This includes mixing left with probability 1/3 and down with probability 2/3.
- Likewise, if player 2 plays left with probability 1/3 and right with probability 2/3, any strategy of player 1's generates the same payoff, so any mixture between his two strategies is a best response.
- This includes mixing up with probability 1/6 and down with probability 5/6
- Since neither player can profitably change his or her strategy, those mixture are a mixed strategy Nash equilibrium.

Example: Tennis

Server

	F	B
Receiver F	90,10	20,80
Receiver B	30,70	60,40

- Here the payoffs to the Receiver is the probability of saving and the payoffs to the Server is the probability of scoring.

- Let's consider the potential strategies for the server:
  - if the Server always aims Forehands then the Receiver (anticipating the Forehand serve) will always move Forehands and the payoffs will be (90,10) to Receiver and Server respectively.
  - if the Server always aims Backhands then the receiver (anticipating the Backhand serve) will always move Backhands and the payoffs will be (60,40).

- How can the Server do better than that? The Server can increase his performance by mixing Forehands and Backhands.

- For example suppose the Server aims Forehand with 50% chance and backhands with 50% chance (or simply mixes 50-50). Then Receiver's payoff is:

  0.5x90+.5x20=55 if she moves Forehands and

  0.5x30+0.5x60=45 if she moves backhands.

- Since it is better to move Forehands, she will do that and her payoff will be 55. Therefore if Server mixes 50-50 her payoff will be 45. (Note that the payoffs add up to 100). This is already an improvement for the Server's performance.

- The next step is searching for the best mix for the Server. How can he get the best performance?

- The player mix between strategies if the expected payoffs are the same.

- Suppose the Server aims Forehands with q probability and Backhands with 1-q probability. Then the Receiver's payoff is:
  $q \times 90+(1-q) \times 20=20+70q$ if she moves Forehands and
  $q \times 30+(1-q) \times 60=60-30q$ if she moves Backhands.

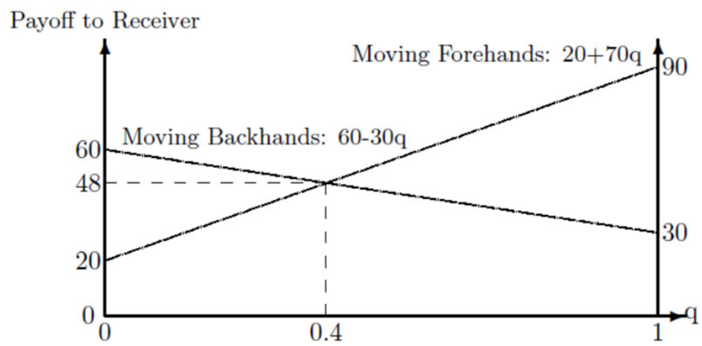
- The receiver will move towards the side that maximizes her payoff. Therefore she will move
  - Forehands if $20 + 70q > 60-30q$,
  - Backhands if $20 + 70q < 60 - 30q$, and
  - either one if $20 + 70q = 60 - 30q$.
- That is the Receiver's payoff is the larger of $20 + 70q$ and $60 - 30q$.

- In order for the Server to make the Receiver indifferent between Forehands and Backhands plays, he can set $20 + 70q$ and $60 + 30q$ equal:

  $20 + 70q = 60 - 30q \rightarrow 100q = 40 \rightarrow q=0.4$

- In order to maximize his payoff the Server should aim Forehands 40% of the time Forehands and Backhands 60% of the time. In this case the Receiver's payoff will be $20 + 70 \times 0.4 = 60 - 30 \times 0.4 = 48$.

- In other words if the Server mixes 40-60 then the Receiver's payoff will be 48 whether she moves Forehands or backhands (or mixes between them). Therefore the Server's payoff will be 100-48 = 52.



Next let us carry out a similar analysis for the Receiver.

- If the Receiver does not mix, then the Server will aim the other side.
- Suppose the Receiver moves with p probability. Then her payoff is:

The payoffs at the mixed strategy Nash equilibrium can be computed as:

Server

Joint probability

Receiver

	F(0.4)	B(0.6)
F(0.3)	90,10(0.12)	20,80(0.18)
B(0.7)	30,70(0.28)	60,40(0.42)

Payoff for the player 1 (Receiver):

90(0.12)+20(0.18)+30(0.28)+60(0.42) = 48

Payoff for the player 2 (Server):

10(0.12)+80(0.18)+70(0.28)+40(0.42) = 52

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Mixed Strategies and Mixed Strategy Nash Equilibrium

•A mixed strategy of a player associates a probability distribution over the pure strategies of the player.

•Consider a strategic form game: $\Gamma = \langle N, (S_i), (u_i) \rangle$.

•Recall that the elements of S_i are called action or pure strategy of player i (i=1,…,n).

•If player i chooses a strategy S_i according to a probability distribution, we have a mixed strategy or randomized strategy.

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• Given a player i with S_i as the set of pure strategies, a mixed strategy σ_i of player i is a probability distribution over S_i .

• That is, $\sigma_i : S_i \to [0,1]$ is a mapping that assigns to each pure strategies $s_i \in S_i$ as probability $\sigma_i(s_i)$ such that:

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1.$$

•If $S_i = \{s_{i1}, s_{i2}, \ldots, s_{im}\}$, then clearly, the set of all mixed strategies of player i is the set of all probability distribution on the set S_i . In other words, it is the set:

$$\Delta(S_i) = \left\{ \left(\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{im}\right) \in R^m : \sigma_{ij} \geq 0 \text{ for } j = 1, \ldots, m \text{ and } \sum_{j=1}^{m} \sigma_{ij} = 1 \right\}.$$

A mixed extension of pure strategy game $\Gamma = \langle N, (S_i), (u_i) \rangle$ is defined as:

$$\Gamma_{ME} = \langle N, (\Delta(S_i)), (U_i) \rangle$$

Where U_i is a mapping that maps mixed strategy profiles to real numbers:

$$U_i : \Delta(S_1) \times \cdots \times \Delta(S_n) \to \mathbb{R}$$

The joint probability of a pure strategy profile $(s_1, s_2, \cdots, s_n)$ is given by:

$$\sigma(s_1, \cdots, s_n) = \prod_{i \in N} \sigma_i(s_i).$$

The payoff functions U_i are defined as:

$$U_i(\sigma_1, \cdots, \sigma_n) = \sum_{(s_1, \cdots, s_n) \in S} \sigma(s_1, \cdots, s_n)\, u(s_1, \cdots, s_n)$$

Example: Mixed Strategies in the BOS Problem

- Recall the payoff matrix:

1	2	
	A	B
A	2,1	0,0
B	0,0	1,2

- Suppose (σ_1, σ_2) is a mixed strategy profile. This means that σ_1 is a probability distribution on $S_1 = \{A, B\}$, and σ_2 is a probability distribution on $S_2 = \{A, B\}$, or:

$$\sigma_1 = (\sigma_1(A), \sigma_1(B)); \quad \sigma_2 = (\sigma_2(A), \sigma_2(B)).$$

- We have $S = S_1 \times S_2 = \{(A,A),(A,B),(B,A),(B,B)\}.$
- We now computer the payoff function u_1 and u_2.

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- For i=1,2 the payoff function is defined as:

$$u_i(\sigma_1, \sigma_2) = \sum_{(s_1,s_2)\in S} \sigma(s_1, s_2)\, u_i(s_1, s_2).$$

- The function u_1 can be computed as:

$$\begin{aligned} u_1(\sigma_1,\sigma_2) &= \sigma_1(A)\sigma_2(A)u_1(A,A) + \sigma_1(A)\sigma_2(B)u_1(A,B) \\ &\quad + \sigma_1(B)\sigma_2(A)u_1(B,A) + \sigma_1(B)\sigma_2(B)u_1(B,B) \\ &= 2\sigma_1(A)\sigma_2(A) + \sigma_1(B)\sigma_2(B) \\ &= 2\sigma_1(A)\sigma_2(A) + (1-\sigma_1(A))(1-\sigma_2(A)) \end{aligned}$$

- The above leads to: $u_1(\sigma_1, \sigma_2) = 1 + 3\sigma_1(A)\sigma_2(A) - \sigma_1(A) - \sigma_2(A).$

- Similarly, we can show that $u_2(\sigma_1, \sigma_2) = 2 + 3\sigma_1(A)\sigma_2(A) - 2\sigma_1(A) - 2\sigma_2(A).$

- Suppose $\sigma_1 = \left(\frac{2}{3}, \frac{1}{3}\right)$ and $\sigma_2 = \left(\frac{1}{3}, \frac{2}{3}\right)$. Then it is easy to see that:

$$u_1(\sigma_1, \sigma_2) = \frac{2}{3} \text{ and } u_2(\sigma_1, \sigma_2) = \frac{2}{3}.$$

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