Pure Strategy Nash Equilibria

- Dominant strategy equilibria if they exist, are very desirable, however, rarely do they exist because the conditions to be satisfied are quite demanding.
- A dominant strategy equilibrium requires that each player's strategy be a best response strategy against all possible strategy choices of the other players.
- We get the notion of Nash equilibrium, a central notion in game theory, if we only insist that each player's strategy offers a best response against the Nash equilibrium strategies of the other players.
- This solution concept is named after John Nash, one of the most celebrated game theorist of our time.

Pure Strategy Nash Equilibrium

- Given a strategic from game

$$\Gamma = \langle N, (S_i), (u_i) \rangle$$

The strategy profile $s^* = (s_1^*, s_2^*, \cdots, s_n^*)$ is called a pure strategy Nash equilibrium of Γ if

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \quad \forall s_i \in S_i, \quad \forall i = 1, 2, \ldots, n.$$

Another way of stating the above is

$$u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} u_i(s_i, s_{-i}^*) \ , \ \forall i = 1, 2, \ldots, n.$$

That is, each player's Nash equilibrium strategy is a best response to the Nash equilibrium strategies of the other players.

In the above definition, we have implicitly assumed that the utilities represented benefits or profits to the players and therefore the players always seek to maximize their utilities.

Example: (The BOS game)

- Recall the two player BOS game with the following payoff matrix:

1	2	
	A	B
A	2,1	0,0
B	0,0	1,2

- There are two Nash equilibrium here, namely (A,A) and (B,B). The profile (A,A) is a Nash equilibrium because:

$$u_1(A,A) > u_1(B,A)$$
$$u_2(A,A) > u_2(A,B)$$

- The profile (B,B) is a Nash equilibrium because

$$u_1(B,B) > u_1(A,B)$$
$$u_2(B,B) > u_2(B,A)$$

- Example (Prisoner's Dilemma)

1	2	
	NC	C
NC	-2,-2	-10,-1
C	-1,-10	-5,-5

- Note that (C,C) is the unique Nash Equilibrium here.

Example: The Stag Hunt, Pure Strategy Nash Equilibrium, and Best Responses

- Two hunters enter a field filled with hares and a single stag.
- Hares are unintelligent and easy to capture.
- The stag, on the other hand, is cunning - the hunters can only catch it by working together.
- Without communicating, the hunters independently choose whether to hunt hares or the stag.
- If they both hunt hares, they each capture half of the hares in the field.

- If one hunts the stag and the other hunts hares, the stag hunter goes home empty-handed while the hare hunter captures all of the hares.
- Finally, if both hunt the stag, then each of their shares of the stag is greater than the value of all of the hares.

	Stag	Hare
Stag	3,3	0,2
Hare	2,0	1,1

- Each player most prefers the <stag, stag> outcome.
- From that, we might assume that the <stag, stag> is the only sensible outcome of the game.
- However, the players could rationally wind-up at a different outcome.
- We have learned on how to solve games using iterated elimination of strictly dominated strategies.

First, suppose player 1 knew that player 2 will hunt the stag:

	Stag
Stag	3,?
Hare	2,?

In this case, hunting the stag is optimal for player 1 as well: doing so nets him 3, whereas chasing hares gives him only 2.

Now suppose player 1 knew player 2 will hunt hares:

	Hare
Stag	0,?
Hare	1,?

- Hunting the stag is no longer optimal for player 1; hare has stag beat by a 1 to 0 margin.
- Thus player 1 has no strictly (or weakly) dominated strategy.
- In fact, player 1's optimal strategy is completely dependent on what player 2 selects. If she hunts the stag, so should he; but if she hunts for hares, he ought to as well.
- Given the symmetry of the game, the same is true for player 2: she should also play whichever strategy player 1 selects.

To verify this, suppose player 1 hunted the stag:

	Stag	Hare
Stag	?,3	?,2

Player 2 should choose stag, since 3 beats 2.

But suppose player 1 hunted for hares:

	Stag	Hare
Hare	?,0	?,1

Then she ought to opt for hares, as 1 beats 0.

How do we solve games lacking dominated strategies?
We look for **Nash equilibria**.

- A Nash equilibrium is a set of strategies, one for each player, such that no player has incentive to change his or her strategy given what the other players are doing.
- For example, first, consider the set of strategies <stag, stag>. Does either player have incentive to change his or her strategy?
- Let's look at this from the perspective of player 1. First, we have to hold player 2's strategy constant; that is, we assume that player 2 will stick to her strategy of stag. Should player 1 switch his strategy?

	Stag
Stag	3,?
Hare	2,?

- We just saw this image, so we know he should not: 3 still beats 2.
- What about player 2?
- Similarly, we must hold player 1's strategy constant and ask whether player 2 would want to deviate from her strategy:

	Stag	Hare
Stag	?,3	?,2

- Once more, she would not want to: 3 remains greater than 2. so <stag, stag> is a Nash equilibrium.
- Specifically, we call it a pure strategy Nash equilibrium (PSNE) because both players are playing deterministic strategies. That is in this equilibrium, player 1 *always* plays stag and player 2 *always* plays stag.
- Are there any other PSNE?

- Let's start by seeing if player 1 would want to switch his strategy in the <stag, hare> outcome:

	Hare
Stag	0,?
Hare	1,?

- He should alter his strategy. If he keeps hunting a stag, he will end up with 0. But if he switches his strategy to hare, he can profitably deviate to 1.
- If even a single player would want to deviate, a set of strategies is not a Nash equilibrium. So without even checking player 2's move, we can throw out <stag, hare> as an equilibrium candidate. Or:

	Stag	**Hare**
Stag	?,3	?,2

- So we should know player 2 has a profitable deviation: she should switch to stag.
- Now let's check whether <hare, stag> is a Nash equilibrium. Given the game's symmetry and how <stag, hare> is not a Nash equilibrium, it should be obvious that <hare, stag> is not either:

	Stag
Stag	3,?
Hare	2,?

- Currently, he earns 2; if he switches to stag, he receives 3. since that is a profitable deviation, <hare, stag> is not a Nash equilibrium.
- Once more we could have also verifies that <hare, stag> is not a Nash equilibrium by looking at player 2's choice:

	Stag	Hare
Hare	?,0	?,1

- Optimally, player 2 should switch from hunting the stag and earning 0 to chasing hares and earning 1.
- Finally, let's check whether <hare, hare> is a Nash equilibrium. We will begin with player 1's choice:

	Hare
Stag	0,?
Hare	1,?

- Hare remains optimal for player 1; switching to stag decreases his payoff from 1 to 0. So the only way for <hare, hare> to not be a Nash equilibrium is if player 2 would want to switch. Let's check if that is the case:

	Stag	Hare
Hare	?,0	?,1

- She should not switch - deviation also decreases her payoff from 1 to 0.
- Since neither player has incentive to change his or her strategy, <hare, hare> is a Nash equilibrium.
- Therefore, the stag hunt has two pure strategy Nash equilibria: <stag, stag> and <hare, hare>.

Maximin Values and Minmax Values

- Given a Nash equilibrium, we have seen that the equilibrium strategy of a player provides a best response strategy assuming optimistically that the other payers do not deviate from their equilibrium strategies.
- If a player wants to play so as to protect her payoff against any possible irrationality of other players, then she has to plan for a worst case satiation.
- Such situations lead to maxmin strategies.
- The notion of maxmin strategy of a player looks at the best possible payoff the player can guarantee herself even in the worst case when the other players are free to choose any strategies.

- Example: consider the non-symmetric company's dilemma

1	2	
	A	B
A	4,1	0,4
B	1,5	1,1

- If player 1 chooses strategy A, the minimum payoff he could get is 0 (when player 2 chooses strategy B).
- If player 1 chooses B, then the minimum he could get is 1 (when player 2 chooses A or B).
- Thus player 1 could decide to play strategy B and he is guaranteed to get a minimum payoff of 1, regardless of the strategy played by player 2.
- This payoff is called the maxmin value of player 1 and the strategy B which assures him this payoff is called a maxmin strategy.

- Similarly, if player 2 chooses strategy A, the minimum payoff she could get is 1 (when player 1 chooses strategy A).

1	2	
	A	B
A	4,1	0,4
B	1,5	1,1

- On the other and, if player 2 chooses B, then the minimum she could get is again 1 (when player 1 chooses B).
- Thus whether player 2 plays strategy A or strategy B, she is guaranteed to get a minimum payoff of 1, regardless of player 1's strategy.
- Here, the payoff 1 is called maxmin value and either of the strategies of A, B, is called a maxmin strategy of player 2.

- Suppose $\Gamma = \langle N, (S_i), (u_i) \rangle$ is any strategic form game. If player i chooses a strategy s_i , then the minimum payoff for this player would be:

$$\min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

- Player i can choose a strategy in s_i that would maximize the above to obtain a payoff that she is guaranteed to obtain, irrespective of the strategies adopted by the rest if the players.
- Maxmin value and maxmin strategy – Given a strategic form game, $\Gamma = \langle N, (S_i), (u_i) \rangle$, the maxmin value or security value of a player i (i=1,…,n) is given by:

$$\underline{v}_i = \max_{s_i \in S_i} \min_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}).$$

Any strategy $s_i^* \in S_i$ that guarantees this payoff to player i is called a maxmin strategy or security strategy of player i.

Inroduction to Game theory 21

- In the above example, the maxmin value of player 1 is 1 and that of player 2 is also 1.
- Strategy B is a maxmin strategy of player 1 while strategy A is not a maxmin strategy for him.
- Strategies A and B are both maxmin strategies for player 2.

This shows that a player may have multiple maxmin strategies.

1	2	
	A	B
A	4,1	0,4
B	1,5	1,1

Inroduction to Game theory 22

Minmax Value

- The minmax value of a player i is the lowest payoff that can be forced on the player i when the other players choose strategies that hurt player i the most.
- Given $\Gamma = \langle N, (S_i), (u_i) \rangle$, the minmax value of a player i (i=1,..n) is given by:

$$\overline{v}_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

Any strategy $s^*_{-i} \in S_{-i}$ of other players that forces this payoff on player i is called a minmax strategy profile against player i.

Example

- On the non-symmetric company's example dilemma game, suppose we want to compute the minmax value of player 1.
- If player 2 plays strategy A, the maximum that player 1 could get is 4 (by playing strategy A).
- If player 2 plays strategy B, the maximum that player 1 could get is 1 (by playing strategy B).
- Thus if player 2 plays strategy B, player 1 is forced to get a maximum payoff of 1, so the minmax value of player 1 is 1.

1	2	
	A	B
A	4,1	0,4
B	1,5	1,1

The minmax strategy of player 2 against player 1 is clearly the strategy B.

Similarly, the minmax value of player 2 is 4 and the minmax strategy of player 1 against 2 is strategy A.

1	2	
	A	B
A	4,1	0,4
B	1,5	1,1