Decision Under Risk

- Decisions under risk differ from decisions under ignorance in that the decision maker knows the probabilities of the outcomes.
- If you play roulette, you are making a decision under risk, since you then know the probability of winning and thus how much you should expect to lose.
- However, knowing the probability need not mean that you are in a position to immediately determine the probability of expected outcome.
- It is sufficient that you have enough information for figuring out the answer after having performed a series of calculations, which may be very complex.
- Principle of maximizing expected value is a reasonable decision rule to use in decision under risk.

In this process your 'tacit knowledge' is made explicit.

When you play roulette and bet on a single number, the probability of winning is 1/38:

There are 38 equally probable outcomes of the game, viz. 1–36 and 0 and 00, and if the ball lands on the number you have betted on the croupier will pay you 35 times the amount betted, and return your bet.

Hence, if you bet $1, the expected payout is:

$$\$(35+1)(\frac{1}{38})+\$(0)(\frac{37}{38})=\$(\frac{36}{38})\approx \$0.947$$

This means that you can expect to lose about $1 – $0.947 = $.053 for every dollar betted. While you might win several times in a row in the short term, in the long run you will lose over 5 cents on average each time you play.

The principle of maximizing can be applied to:

1. Maximizing expected monetary value
2. Maximizing expected value
3. Maximizing expected utility

According to the principle of maximizing expected monetary value it would obviously be a mistake to play roulette in Las Vegas.

However, this does not show that it is irrational to play there, all things considered.

Let us investigate this Principle further.

Imagine that you are offered a choice between receiving a million dollars for sure, and receiving a lottery ticket that entitled you to a fifty per cent chance of winning either three million dollars or nothing.

	1/2	1/2
Lottery A	$1 M	$1 M
Lottery B	$3 M	$ 0

The expected monetary value (EMV) of these lotteries can be computed by applying the following general formula, in which p1 is the probability of the first value and m1 the monetary value of the corresponding outcome:

$$EMV = p_1 m_1 + p_2 m_2 + \cdots + p_n m_n$$

$$\text{EMV(Lottery A)} = \frac{1}{2}\cdot \$1M + \frac{1}{2}\cdot \$1M = \$1M.$$

$$\text{EMV(Lottery B)} = \frac{1}{2}\cdot \$3M + \frac{1}{2}\cdot \$0M = \$1.5M.$$

However, even though EMV(Lottery B)> EMV(Lottery A), many of us would prefer a million for sure.

The principle of maximizing expected value makes more sense from a normative point of view (i.e. normative means relating to an ideal standard or model) than the principle of maximizing expected monetary value. In this case, we can write:

$$EV = p_1 v_1 + p_2 v_2 + \cdots + p_n v_n$$

Unfortunately, not all concepts of value are reliable guides to rational decision making.

For example, let us take moral value, for instance.

If a billionaire decides to donate his entire fortune to charity, the expected moral value of doing so might be very high. However, this is because many poor people would benefit from the money, not because the billionaire himself would be any happier.

Therefore, in order to single out the kind of value that is the primary object of study in decision theory – the value of an outcome as evaluated from the decision maker's point of view –
it is helpful to introduce the concept of utility.

Utility is an abstract entity that cannot be directly observed.

By definition, the utility of an outcome depends on how valuable the outcome is from the decision maker's point of view.

The principle of maximizing expected utility is obtained from the principle of maximizing expected value by replacing v for u, or:

$$EU = p_1 u_1 + p_2 u_2 + \cdots + p_n u_n$$

Decision theorists have proposed a number of fundamentally different arguments for the expected utility principle.

The first argument is based on the law of large numbers; it seeks to show that in the long run you will be better off if you maximize expected utility.

Example - Entropy

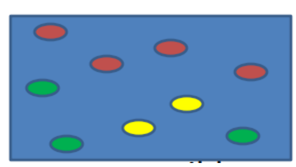
The average (expected) amount of the information from the event.

$$Entropy = -\sum_{i=1}^{n} p_i \log_b(p_i)$$

n = number of different outcomes and b is the chosen base.

Calculate the entropy of three possible outcomes when you choose a ball which can be either red, yellow or green, i.e. for n=3:

$$Entropy = -(\frac{4}{9})\log(\frac{4}{9}) + (-)(\frac{2}{9})\log(\frac{2}{9}) + (-)(\frac{3}{9})\log(\frac{3}{9})$$
$$= 1.53$$



Therefore, you are expected to get 1.53 information each time you choose a ball from the bin.

Utility

It is hard to think of any minimally reasonable decision rule that totally ignores the concept of utility.

However, the concept of utility has many different technical meanings, and it is important to keep these different meanings separate.

Previously we distinguished three fundamentally different kinds of measurement scales. All scales are numerical, i.e. utility is represented by real numbers, but the information conveyed by the numbers depends on which type of scale is being used.

1. Ordinal scales ('10 is better than 5')
2. Interval scales ('the difference between 10 and 5 equals that between 5 and 0')
3. Ratio scales ('10 is twice as valuable as 5')

How to construct an ordinal scale.

Preference relationships in ordinal scale among finite choices can be shown as:

$$x > y$$

Your preferences are revealed in your choice behavior.

Therefore, you prefer x to y if and only if you choose x over y whenever given the opportunity.

The main advantage of this proposal is that it links preference to directly observable behavior, which entails that the concept of preference (and hence utility) becomes firmly connected with empirical observations.

However, it is of course easy to question this alleged connection between choice and preference.

Perhaps you actually preferred x over y, but chose y by mistake, or did not know that y was available.

Furthermore, using the behaviorist interpretation of preferences it becomes difficult to distinguish between strict preference ('strictly better than') and indifference ('equally good as').

The observation that you repeatedly choose x over y is equally compatible with the hypothesis that you strictly prefer x over y as with the hypothesis that you are indifferent between the two.

Indifference will be represented by the symbol ~. For future reference we also introduce the symbol $\succeq$, which represents the relation 'at least as preferred as'.

$x \succeq y$ if and only if $x \succ y$ or $x \sim y$

$x \sim y$ if and only if $x \succeq y$ and $y \succeq x$

$x \succ y$ if and only if $x \succeq y$ and not $x \sim y$

Asymmetry
if $x \succ y$, then it is false that $y \succ x$.

Transitivity
If $x \succ y$ and $y \succ z$, then $x \succ z$.

Negative transitivity
If it is false that $x \succ y$ and false that $y \succ z$, then it is false that $x \succ z$.

- What assumptions must we make about preference relations for this to be possible?
- That is, what must we assume about preferences for there to exist a function u that assigns real numbers to your selections, i.e.

$$x > y \text{ if and only if } u(x) > u(y)$$

However, it can be shown that such numbers do exist, i.e. that there is a real-valued function u such that the above equation holds true, if and only if the relation $>$ is complete, asymmetric and negatively transitive.

Von Neumann and Morgenstern's interval scale

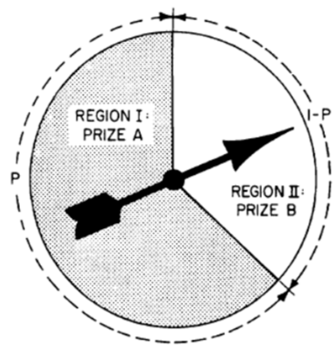
In many cases ordinal utility scales do not provide the information required for analyzing a decision problem.

The expected utility principle as well as several other decision rules presuppose that utility is measured on an interval scale.

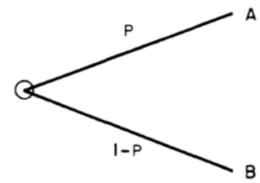
The key idea in von Neumann and Morgenstern's theory is to ask the decision maker to state a set of preferences over risky acts.

These acts are called lotteries, because the outcome of each act is assumed to be randomly determined by events (with known probabilities) that cannot be controlled by the decision maker.

The set of preferences over lotteries is then used for calculating utilities by reasoning 'backwards'.



A lottery



A lottery diagram (tree)

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Illustrative example:

Suppose that Rachelle has decided to go to a movie, and that there are three movies are playing in the theater. Rachelle thinks that movie A is better than B, which is better than C.

For some reason, never mind why, it is not possible to get a ticket that entitles her to watch movie A or C with 100% certainty.

However, she is offered a ticket that entitles her to a 70% chance of watching movie A and a 30% chance of watching movie C.

The only other ticket available entitles her to watch movie B with 100% certainty.

For simplicity, we shall refer to both options as lotteries, even though only the first involves a genuine element of chance.

After considering her two options carefully, she declares them to be equally attractive.

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That is, to watch movie B with 100% certainty is, for Rachelle, exactly as valuable as a 70% chance of watching movie A and a 30% chance of watching movie C.

Now suppose we happen to know that Rachelle always acts in accordance with the principle of maximizing expected utility.

By reasoning backwards, we can then figure out what her utility for movie is, i.e. we can determine the values of u(A), u(B) and u(C).

Consider the following equation, which formalizes the hypothesis that Rachelle accepts the expected utility principle and thinks that the two options are equally desirable:

$$0.7 \cdot u(\mathrm{A}) + 0.3 \cdot u(\mathrm{C}) = 1.0 \cdot u(\mathrm{B})$$

The above equation has three unknown variables, so it has infinitely many solutions. However, if utility is measured on an interval scale, the above equation nevertheless provides all information we need, since the unit and end points of an interval scale can be chosen arbitrarily.

We may therefore stipulate that the utility of the best outcome is 100 (that is, u(A) = 100), and that the utility of the worst outcome is 0 (that is, u(C) = 0). By inserting these arbitrarily chosen end points into the above, we get the following equation.

$$0.7 \cdot 100 + 0.3 \cdot 0 = 1.0 \cdot u(\mathrm{B})$$

The above equation has only one unknown variable, and it can be easily solved: As you can see, u(B) = 70. Hence, we now know the following.

u(A) = 100
u(B) = 70
u(C) = 0

Now suppose that Rachelle is told that another movie will play in theater tonight, namely movie D, which she thinks is slightly better than B.

What would Rachelle's numerical utility of watching movie D be?

For some reason, never mind why, Rachelle is offered a ticket that entitles her to watch movie D with probability p and movie C with probability 1 – p, where p is a variable that she is free to fix herself.

To figure out what her utility for D is, she asks herself the following question: "Which value of p would make me feel totally indifferent between watching movie B with 100% certainty, and D with probability p and C with probability 1 – p?"

After considering her preferences carefully, Rachelle finds that she is indifferent between a 100% chance of watching movie B and a 78% chance of watching D combined with a 22% chance of watching C. Since we know that u(B) = 70 and u(C) = 0, it follows that:

$$1.0 \cdot 70 = 0.78 \cdot u(\mathrm{D}) + 0.22 \cdot 0$$

By solving this equation, we find that u(D)= 70/0.78 =89.7.

Of course, the same method could be applied for determining the utility of every type of good. For instance, if Rachelle is indifferent between a 100% chance of watching movie B and a 95% chance of winning a holiday in Malibu combined with a 5% chance of watching movie C, then her utility of a holiday in Malibu is 70/0.95 = 73.7.

An obvious problem with this preliminary version of von Neumann and Morgenstern's theory is that it presupposes that the decision maker chooses in accordance with the principle of maximizing expected utility.

We seem to have no reason for thinking that the decision maker will apply the expected utility principle, rather than some other principle, for evaluating lotteries. This very strong assumption needs to be justified in one way or another. Von Neumann and Morgenstern proposed a clever way of doing that.

Instead of directly assuming that the decision maker will always apply the expected utility principle (as we did above) they proposed a set of constraints on rational preferences which imply that the decision maker behaves as if she or he is making decisions by calculating expected utilities.

More precisely put, von Neumann and Morgenstern were able to prove that if a decision maker's preferences over the sort of lotteries exemplified above satisfy a number of formal constraints, or axioms, then the decision maker's choices can be represented by a function that assigns utilities to lotteries (including lotteries comprising no uncertainty), such that one lottery is preferred to another in the case that the expected utility of the first lottery exceeds that of the latter.

- We assume that Z is a finite set of basic prizes, which may include a holiday in Malibu, a ticket to a movie, as well as almost any kind of good. That is, the elements of Z are the kind of things that typically constitute outcomes of risky decisions.
- We furthermore assume that L is the set of lotteries that can be constructed from Z by applying the following inductive definition. (Note that even a 100% chance of winning a basic prize counts as a 'lottery' in this theory.)

1. Every basic prize in Z is a lottery.
2. If A and B are lotteries, then so is the prospect of getting A with probability p and B with probability 1 – p, for every p between 0 and 1.
3. Nothing else is a lottery .

For simplicity, ApB will be used as an abbreviation for a lottery in which one wins A with probability p and B with probability 1 – p.

Thus, the second condition stated above could equally well be formulated as follows: If A and B are lotteries, then so is ApB, for every p greater than and equal to 0 and less than or equal to 1.

Furthermore, since ApB is a lottery it follows that also Cq(ApB) (for every q greater than and equal to 0 and less than or equal to 1) is a lottery, given that q is a probability and C is a lottery. And so on and so forth.

The next assumption introduced by von Neumann and Morgenstern holds that the decision maker is able to state pairwise preferences between lotteries. The formula A>B means that lottery A is preferred over lottery B, and A ~ B means that lottery A and B are equally preferred.

Now, it should be obvious that preferences have to satisfy some structural conditions.

(Completness) $A \succ B$ or $A \sim B$ or $B \succ A$

(Transitivity) if $A \succ B$ and $B \succ C$, then $A \succ C$

(Independence) $A \succ B$ if and only if $ApC \succ BpC$

The independence axiom is best illustrated by considering an example. Imagine that you are offered a choice between lotteries A and B in the following table.

Each ticket is equally likely to be drawn, so the probability of winning, say, \$5M if lottery B is chosen is 10/11.

	Ticket no. 1	Ticket no. 2–11
A	\$1M	\$1M
B	\$0	\$5M

Suppose that you prefer lottery A to lottery B. Then, according to the independence axiom it must also hold that you prefer the first lottery to the second in the situation illustrated in the following table. That is, you must prefer ApC to BpC.

	Ticket no. 1	Ticket no. 2–11	Ticket no. 12–100
ApC	\$1M	\$1M	\$1M
BpC	\$0	\$5M	\$1M

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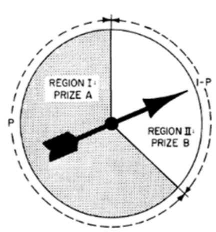
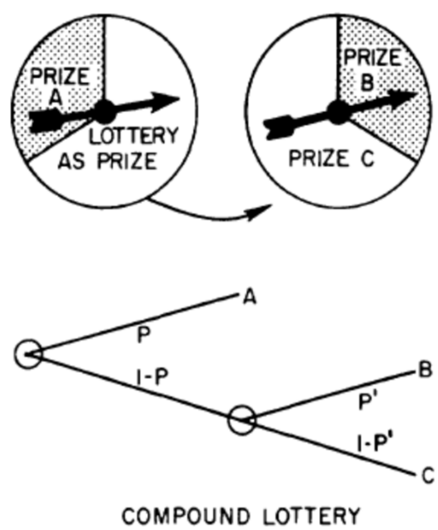
Consider the lottery:

$$(p, A; (1-p), (p', B; (1-p'), C)).$$

The assumption is that subdividing region II into two parts whose proportions correspond to the probabilities p' and $1-p'$ of the second lottery creates an equivalent simple lottery in which all of the prizes are outcomes.





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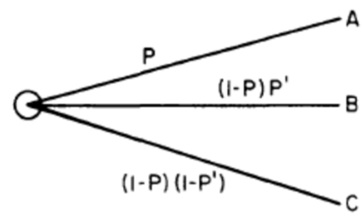
You can decompose a compound lottery by multiplying the probabilities of the individual prizes in the second lottery; you should be indifferent between

$$(p, A; 1-p, (p', B; 1-p', C))$$

and

$$(p, A; p' - pp', B; 1 - p - p' + pp', C).$$

In another words, your preferences are not affected by the way in which the uncertainty is resolved-bit by bit, or all at once.



EQUIVALENT SIMPLE LOTTERY

The fourth and last axiom proposed by von Neumann and Morgenstern is a continuity condition. Let p and q be some probabilities strictly greater than 0 and strictly smaller than 1.

(Continuity) If $A \succ B \succ C$ then there exist some p and q such that $ApC \succ B \succ AqC$

The following example explains the assumption articulated by the continuity axiom.

Suppose that A is a prize worth \$10M, B a prize worth \$9M and C a prize worth \$0. Now, according to the continuity axiom, it holds that if you prefer \$10M to \$9M and \$9M to \$0, then there must be some probability p, which may be very close to 1, such that you prefer \$10M with probability p and \$0 with probability 1 – p over \$9M for certain.

Furthermore, there must be some probability q such that you prefer \$9M for certain over \$10M with probability q and \$0M with probability 1 – q.

Of course, some people might feel tempted to deny that these probabilities exist; perhaps it could be argued that there is no probability p simply because \$9M for certain is always better than a lottery yielding either \$10M or \$0, no matter how small the probability of getting \$0 is. The standard reply to this complaint is that p might lie very close to 1.

In addition to the four axioms stated above, we also need to make an additional technical assumption, saying that the probability calculus applies to lotteries.

The essence of this assumption is that it does not matter if you are awarded prize A if you first roll a die and then roll it again, or make a double roll, provided that you only get the prize if you get two sixes.

Put into mathematical vocabulary, compound lotteries can always be reduced to simple ones, involving only basic prizes. Hence, if p, q, r and s are probabilities such that $pq + (1 – p)r = s$, then $(AqB)p(ArB) \sim AsB$.

The axioms stated above imply the following theorem, which is frequently referred to as von Neumann and Morgenstern's theorem. It consists of two parts, a representation part and a uniqueness part.

Theorem

The preference relation > satisfies four conditions stated above if and only if there exists a function u that takes a lottery as its argument and returns a real number between 0 and 1, which has the following properties:

(1) $A \succ B$ if and only if $u(A) > u(B)$.

(2) $u(ApB) = pu(A) + (1-p)u(B)$.

(3) For every other function u' satusfying (1) and (2), there are numbers $c > 0$ and d such that $u' = c.u + d$.