Question 1

Let

$$u_1 := (1,1,1)^T, u_2 := (1,2,2)^T, u_3 := (3,2,4)^T$$
$$v_1 := (4,6,7)^T, v_2 := (0,1,1)^T, v_3 := (0,1,2)^T$$

(a) Find the transition matrix corresponding to the change of basis from e_1, e_2, e_3 to u_1, u_2, u_3.

Solution: Let U be the transition matrix from the standard basis to the basis u_1, u_2, u_3.

$$U = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

To find the transition matrix from e_1, e_2, e_3 to u_1, u_2, u_3, we need to find the inverse of U.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -2 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Thus, we have U^{-1} as:

$$U^{-1} = \begin{bmatrix} 2 & 1 & -2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

which is the transition matrix from e_1, e_2, e_3 to u_1, u_2, u_3.

(b) Find the transition matrix corresponding to the change of basis from v_1, v_2, v_3 to e_1, e_2, e_3.

Solution: Let V be the transition matrix from the basis v_1, v_2, v_3 to the standard basis. V is simply the matrix of the basis vectors v_1, v_2, v_3. Thus, we have:

$$V = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix}$$

(c) Find the transition matrix from v_1, v_2, v_3 to u_1, u_2, u_3.

Solution: We want to find the transition matrix from v_1, v_2, v_3 to u_1, u_2, u_3. This can be done by first finding the transition matrix from v_1, v_2, v_3 to the standard basis, and then multiplying it by the transition matrix from the standard basis to u_1, u_2, u_3. This means the transition matrix from v_1, v_2, v_3 to u_1, u_2, u_3 is simply $U^{-1}V$. Since we have already found U^{-1} and V in previous parts, we have:

$$U^{-1}V = \begin{bmatrix} 2 & 1 & -2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -3 \\ \frac{5}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(d) Let $x = 2v_1 + 3v_2 - 4v_3$. Find the coordinates of x with respect to u_1, u_2, u_3.

Solution: The vector x can be written as:

$$\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$$

with respect to the basis v_1, v_2, v_3. To find the coordinates of x with respect to u_1, u_2, u_3, we can multiply the vector by the transition matrix from v_1, v_2, v_3 to u_1, u_2, u_3. Thus, we have:

$$U^{-1}Vx = \begin{bmatrix} 0 & -1 & -3 \\ \frac{5}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ -1 \end{bmatrix}$$

Thus, the coordinates of the vector x with respect to u_1, u_2, u_3 is $9u_1 + 2u_2 - u_3$.

(e) Verify your answer to previous one, by computing the coordinates in each case with respect to the standard basis.

The vector x can be written with respect to the basis v_1, v_2, v_3 as:

$$x = 2\begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 14 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix}$$

Here we have that the coordinates of x with respect to the standard basis is $8e_1 + 11e_2 + 9e_3$. Writing x with respect to the basis u_1, u_2, u_3 gives:

$$x = 9\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix}$$

Since the two coordinates are the same, we have verified our answer.

Question 2

Find a basis for the row space, column space and null space of the following matrices.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} B = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} C = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

Solution:
Matrix A
To find the row space, column space, and null space of the matrix, we can use gaussian elimination. Using row operations, we have the following equivalent matrices:

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we have two linearly independent rows in row echelon form. Thus, the row space of the matrix is spanned by the vectors:

$$\text{rowspace}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix} \right\}$$

Identifying the columns of the leading 1's in the row echelon form, we have that the column space is spanned by the vectors:

$$\text{colspace}(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} \right\}$$

The null space can be found by solving the homogeneous system:

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -5 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \to \begin{cases} x_1 = -2x_3 \\ x_2 = 0 \\ x_3 = x_3 \end{cases}$$

Thus, the null space is the following:

$$N(A) = \{(-2a, 0, a)^T \mid a \in \mathbb{R}\} = \text{span} \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Matrix B
First, row reduce the matrix to row echelon form:

$$\begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 14 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & -\frac{2}{7} \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{10}{7} \\ 0 & 1 & 0 & -\frac{2}{7} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

From here we can observe the following:

$$\text{rowspace}(B) = \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ 4 \\ 2 \end{bmatrix} \right\}, \quad \text{colspace}(B) = \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \right\}$$

Solving for the null space:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{10}{7} & 0 \\ 0 & 1 & 0 & -\frac{2}{7} & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right] \to \begin{cases} x_1 = \frac{10}{7}x_4 \\ x_2 = \frac{2}{7}x_4 \\ x_3 = 0 \\ x_4 = x_4 \end{cases}$$

$$N(B) = \left\{ \left(\frac{10}{7}a, \frac{2}{7}a, 0, a \right)^T \mid a \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 10 \\ 2 \\ 0 \\ 7 \end{bmatrix} \right\}$$

Matrix C

Performing row reduction:

$$\begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & -5 & 11 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 0 & -\frac{21}{4} \\ 0 & 0 & 1 & \frac{3}{4} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & \frac{10}{4} \\ 0 & 1 & 0 & \frac{21}{20} \\ 0 & 0 & 1 & \frac{3}{4} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{13}{20} \\ 0 & 1 & 0 & \frac{21}{20} \\ 0 & 0 & 1 & \frac{3}{4} \end{bmatrix}$$

The row and column space are spanned by the following basis vectors:

$$\text{rowspace}(C) = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \right\}, \quad \text{colspace}(C) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} \right\}$$

The null space is given by:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{13}{20} & 0 \\ 0 & 1 & 0 & \frac{21}{20} & 0 \\ 0 & 0 & 1 & \frac{3}{4} & 0 \end{array}\right] \to \begin{cases} x_1 = \frac{13}{20}x_4 \\ x_2 = -\frac{21}{20}x_4 \\ x_3 = -\frac{3}{4}x_4 \\ x_4 = x_4 \end{cases}$$

$$N(C) = \left\{ \left(\frac{13}{20}a, -\frac{21}{20}a, -\frac{3}{4}a, a \right)^T \mid a \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 13 \\ -21 \\ -15 \\ 20 \end{bmatrix} \right\}$$

Question 3

Let $E := [p_1(x) = 1, p_2(x) = x + 1, p_3(x) = x^2 - 1]$ and $F := q_1(x) = 1, q_2(x) = x, q_3(x) = x^2$. These are two basis of the vector space P_2 of all polynomials of degree at least 2. Find the transition matrix from E to F and the transition matrix from F to E. Express the polynomial

$$p(x) = 11x^2 - 2x + 5$$

with respect to the basis E.

Solution:
To find the transition matrix from E to F, we can write the basis vectors of E as linear combinations of the basis vectors of F.

$$\begin{cases} 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ x + 1 = 1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \\ x^2 - 1 = -1 \cdot 1 + 0 \cdot x + 1 \cdot x^2 \end{cases}$$

Thus, the transition matrix from E to F is:

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find the transition matrix from F to E, we can do the same process.

$$\begin{cases} 1 = 1 \cdot 1 + 0 \cdot (x + 1) + 0 \cdot (x^2 - 1) \\ x = -1 \cdot 1 + 1 \cdot (x + 1) + 0 \cdot (x^2 - 1) \\ x^2 = 1 \cdot 1 + 0 \cdot (x + 1) + 1 \cdot (x^2 - 1) \end{cases}$$

Thus, the transition matrix from F to E is:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The polynomial $p(x) = 11x^2 - 2x + 5$ can be written as a vector with respect to the standard basis as:

$$p(x) = \begin{bmatrix} 5 \\ -2 \\ 11 \end{bmatrix}$$

To write $p(x)$ with respect to the basis E, we can multiply the vector by the transition matrix from the standard basis to E. Since the basis F is actually the standard basis, we can use the transition matrix from F to E as the transition matrix from the standard basis to E. That means we have:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 11 \end{bmatrix} = \begin{bmatrix} 18 \\ -2 \\ 11 \end{bmatrix}, \quad p(x) = 18p_1(x) - 2p_2(x) + 11p_3(x)$$