1. Let

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Compute A^2, A^3 and A^n for $n \geq 4$.

$$A^2 = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+\frac{1}{4} & \frac{-1}{4}-\frac{1}{4} \\ \frac{-1}{4}-\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}$$

$$A^3 = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+\frac{1}{4} & \frac{-1}{4}-\frac{1}{4} \\ \frac{-1}{4}-\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}+\frac{1}{4} & \frac{-1}{4}-\frac{1}{4} \\ \frac{-1}{4}-\frac{1}{4} & \frac{1}{4}+\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}$$

$$A^n = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{pmatrix}$$

2. Let A, B are singular matrices. Is it true that $A + B$ will be singular? If yes, prove it. If no, provide a counterexample.

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\det(A) = 0$, $\det(B) = 0$

$A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\det(A+B) = 1$

The statement is false

3. For each of the following pairs, find E elementary matrix such that $EA = B$.

   - a]

$$A = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}, \ B = \begin{pmatrix} -4 & 2 \\ 5 & 3 \end{pmatrix} \qquad E = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

   - b]

supposed to be 1?

$$A = \begin{pmatrix} 2 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

4. Find the inverse of the following matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 0 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \qquad A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 5 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 2 & 0 & 5 & 1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & -1 & 1 & 0 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -5 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & -1 & 1 & 0 & -2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 0 & -5 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & -1 & 1 & 0 & -2 \end{array}\right)$$

$$B^{-1} = \begin{pmatrix} 3 & 0 & -5 \\ 0 & \frac{1}{3} & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

5. Find the LU factorization of the matrix

$$A = \begin{pmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{pmatrix} \overset{r_1 \to \frac{1}{2}r_1}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{pmatrix} \overset{\substack{r_2 \to r_2 - r_1 \\ r_3 \to r_3 + r_1}}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & 3 \end{pmatrix} \overset{r_2 \to -r_2}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 3 \end{pmatrix} \overset{r_3 \to r_3 - 2r_2}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{pmatrix} = U$$

$$E_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix} = L$$

$$\underset{L}{\begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}} \underset{U}{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{pmatrix}} = \underset{A}{\begin{pmatrix} 2 & 4 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & 2 \end{pmatrix}}$$

6. Compute the determinant of the following matrices:

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 4 & 2 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{vmatrix} = 4\begin{vmatrix} 1 & 2 \\ -1 & -4 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 5 & -4 \end{vmatrix} + 0\begin{vmatrix} 3 & 1 \\ 5 & -1 \end{vmatrix} = 4(-4-(-2)) - 3(-12-10) = -8+66 = 58$$

$$\det(B) = \begin{vmatrix} 3 & 0 & 0 \\ 2 & 1 & 1 \\ 4 & 2 & 2 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 3(2-2) = 0$$

$$\det(C) = \begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{vmatrix} = -0\begin{vmatrix} 0 & 0 & 1 \\ 6 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 1 & -2 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 & 1 \\ 1 & 6 & 0 \\ 1 & 1 & 3 \end{vmatrix} + 0\begin{vmatrix} 2 & 0 & 0 \\ 1 & 6 & 2 \\ 1 & 1 & -2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 0 \\ -2 & 3 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= 2(6-0) + (-2-2)$$

$$= 8$$