CSCE 222 Discrete Structures for Computing – Fall 2023
Hyunyoung Lee

Problem Set 6

Due dates: Electronic submission of *yourLastName-yourFirstName-hw6.tex* and *yourLastName-yourFirstName-hw6.pdf* files of this homework is due on **Monday, 11/6/2023 before 11:59 p.m.** on `https://canvas.tamu.edu`. You will see two separate links to turn in the .tex file and the .pdf file separately. Please do not archive or compress the files. **If any of the two files are missing, you will receive zero points for this homework.** Your files must contain your first and last names and UIN in the given spaces and the electronic signature (your full name) correctly.

Name: Kevin Lei **UIN: 432009232**

Resources. (All people, books, articles, web pages, etc. that have been consulted when producing your answers to this homework)

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to answer this homework.

Electronic signature: Kevin Lei

Total 100 points. For a counting problem, *careful, detailed explanation* will be worth majority (about 80%) of your grade.

The intended formatting is that this first page is a cover page and each problem solved on a new page. You only need to fill in your solution between the `\begin{solution}` and `\end{solution}` environment. Please do not change this overall formatting.

Checklist:
- ☐ Did you type in your name and UIN?
- ☐ Did you disclose all resources that you have used?
  (This includes all people, books, websites, etc. that you have consulted)
- ☐ Did you sign that you followed the Aggie Honor Code?
- ☐ Did you solve all problems?
- ☐ Did you submit both the .tex and .pdf files of your homework to each correct link on Canvas?

Problem 1. (10 points) Section 12.1, Exercise 12.4. Specify what counting principle(s) you are using. Also explain carefully how you got your final answer.

Solution. We can use the multiplication principle here. Since there are 26 ways to choose a letter and 10 ways to choose a digit, we can multiply them together to get the total number of license plates. Thus we have the following:

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000 \text{ license plates}$$

Problem 2. (10 points) Section 12.1, Exercise 12.5. Specify what counting principle(s) you are using. Also explain carefully how you got your final answer.

Solution. For this we need to use the multiplication principle and summation principle. First we need to find the number of password combinations for password lengths 6, 7, and 8. Then we can add them together to get the total number of password combinations. For each character, there are 36 choices since there are 26 lowercase letters and 10 digits. However, for the first character, there are only 26 choices since the first character must be a digit. Thus we have the following:

$$26 \times 36^5 + 26 \times 36^6 + 26 \times 36^7 = 2,095,636,727,808 \text{ password combinations}$$

Problem 3. (15 points) Section 12.1, Exercise 12.9. Specify what counting principle(s) you are using. Also explain carefully how you got your final answer.

Solution. Here we can use the subtraction principle by first finding the total number of words of length n over an alphabet of k letters, then subtracting all palindromes of length n. The number of words of length n over an alphabet of k letters is given by k^n, since there are k choices for each of the n characters. This is the multiplication principle. A word is a palindrome if it is the same forwards and backwards. This means that the first half of the word must be the same as the second half of the word. Since the word length can be either odd or even, we need to consider both cases. For both cases, the number of letters in the first half of the word is $\lceil n/2 \rceil$. Therefore, the number of palindromes of length n is given by $k^{\lceil n/2 \rceil}$. Once again, this is the multiplication principle. Finally, the total number of non-palindrome words of length n over an alphabet of k letters is given by $k^n - k^{\lceil n/2 \rceil}$.

Problem 4. (10 points) Section 12.2, Exercise 12.19. Specify what counting principle(s) you are using. Also explain carefully how you got your final answer.

Solution. Since Tom wants to keep the math books together, they can be treated as a single book in calculating the number of arrangements. We can use the multiplication principle by first finding the permutations of the math books within the novels, then multiplying by the number of permutations of the novels. The number of permutations of the novels and math books (math books stick together) is 21!, since there are 20 novels, and we are treating the math books as a single book, since they must stick together. Now, within the math books, there are 7! permutations, since there are 7 math books. Thus, the total number of ways to arrange the bookshelf is $21! \times 7!$.

Problem 5. (10 + 10 = 20 points) Section 12.3, Exercise 12.27. For (a), use the formula involving the factorials. For (b), use the hint given in the problem statement, and explain carefully your double counting (combinatorial) proof in your own words.

Solution. **(a)** Using the definition with factorials:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We need to show that the following is true:

$$\binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1}$$

Starting with the right hand side, we have:

$$n \cdot \binom{n-1}{k-1} = n \cdot \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = \frac{n!}{(k-1)!(n-k)!}$$

The left hand side can be rewritten as follows:

$$\binom{n}{k} \cdot k = \frac{n!}{k!(n-k)!} \cdot k = \frac{k \cdot n!}{k(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$$

Now we can see that both sides of the equation are the same, so the equation is true.

(b) Using a combinatorial proof:
We can use two methods to choose a team of fixed size and their captain to show this is true. One way is to first choose a team of size k from n people, and then choose a captain from the team of size k. That means we have $\binom{n}{k} \cdot k$ ways to choose a team and their captain. The second way is to first choose the captain from n people, and then choose the rest of the team of size $k-1$ from the remaining $n-1$ people. This means there are $n \cdot \binom{n-1}{k-1}$ ways to choose a team and their captain. Both methods count the same exact scenario, and since the outcome is the same, the two expressions must be equal.

Problem 6. (15 points) Section 12.6, Exercise 12.50. Explain your reasoning carefully in your own words and show your work step-by-step. [Hint: Consider the three sets: Set 1 with page numbers that contain a 1 in the least significant (1s) digit; set 2 with those that contain a 1 in the middle (10s) digit, and set 3 with those that contain a 1 in the most significant (100s) digit.]

Solution. To use the inclusion-exclusion principle, we first define the following sets:

- S_1: The set of all page numbers that contain a 1 in the least significant digit.
- S_2: The set of all page numbers that contain a 1 in the middle digit.
- S_3: The set of all page numbers that contain a 1 in the most significant digit.

We want to find the number of page numbers that do not contain a 1 in any of the digits. If the set S is the set of all page numbers, then the number of pages that do not contain a 1 in any of the digits is given by $|S \setminus (S_1 \cup S_2 \cup S_3)|$, which is the inclusion-exclusion principle. Since this is equivalent to $|S| - |S_1 \cup S_2 \cup S_3|$, we need to find the cardinality of $S_1 \cup S_2 \cup S_3$, which is given by:

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|$$

Every ten pages, there is a page that contains a 1 in the least significant digit. That means the cardinality of S_1 is given by:

$$|S_1| = \frac{500}{10} = 50$$

Each set of 100s has the numbers 10-19, 110-119, 210-219, etc. This means the cardinality of S_2 is given by:

$$|S_2| = \frac{500}{100} \times 10 = 50$$

The number of pages with 1 in the most significant digit is simply the pages from 100-199. Therefore, the cardinality of S_3 is given by:

$$|S_3| = 100$$

The set $S_1 \cap S_2$ is the set of all page numbers that end in 11. Since this only happens once every 100 pages, the cardinality of $S_1 \cap S_2$ is given by:

$$|S_1 \cap S_2| = \frac{500}{100} = 5$$

$S_1 \cap S_3$ are all the pages in the 100s that end in 1. Since this happens once every 10 pages and can only happen in the 100s, the cardinality of $S_1 \cap S_3$ is given by:

$$|S_1 \cap S_3| = 10$$

$S_2 \cap S_3$ are all the pages in the 100s with a 1 in the middle digit, such as 110, 111, 112, etc. Since this only happens 10 times in the 100s, the cardinality of $S_2 \cap S_3$ is given by:

$$|S_2 \cap S_3| = 10$$

Finally, $S_1 \cap S_2 \cap S_3$ is the set of all page numbers that are 111. Obviously, there is only one page number that is 111, so the cardinality of $S_1 \cap S_2 \cap S_3$ is 1. Now we can plug in all the values to get the cardinality of $S_1 \cup S_2 \cup S_3$. Thus, we have:

$$\begin{aligned}
|S \setminus (S_1 \cup S_2 \cup S_3)| &= |S| - |S_1 \cup S_2 \cup S_3| \\
&= |S| - (|S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|) \\
&= 500 - (50 + 50 + 100 - 5 - 10 - 10 + 1) \\
&= 500 - 176 \\
&= 324
\end{aligned}$$

There are 324 pages from 1-500 that contain a 1 in at least one of the digits.

Problem 7. (20 points) What is the smallest number of ordered pairs of integers (x, y) that are needed to guarantee that there are three ordered pairs $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) such that $x_1 \textbf{ mod } 5 = x_2 \textbf{ mod } 5 = x_3 \textbf{ mod } 5$ and $y_1 \textbf{ mod } 4 = y_2 \textbf{ mod } 4 = y_3 \textbf{ mod } 4$? Explain your reasoning carefully. [Hint: This problem is about Pigeonhole Principle (Section 12.7). Carefully think what are the pigeonholes and what are the pigeons here.]

Solution. In this problem we need to guarantee that there are three ordered pairs with the same $x \textbf{ mod } 5$ and $y \textbf{ mod } 4$. To do this, we should consider the worst case scenario, where the pairs are distributed as evenly as possible across the pigeonholes. $x \textbf{ mod } 5$ can be any number from 0 to 4, so it has 5 possible values. Similarly, $y \textbf{ mod } 4$ can be any number from 0 to 3, so there are 4 possible values for it. This means there are $5 \times 4 = 20$ possible combinations of $x \textbf{ mod } 5$ and $y \textbf{ mod } 4$. We need to find the number of pairs n such that $\lceil n/20 \rceil \geq 3$. Since n must be an integer, the ceiling function is unnecessary. Thus, we have:

$$n \geq 3 \times 20 = 60$$

Therefore, we need at least 60 ordered pairs to guarantee that there are three ordered pairs with the same $x \textbf{ mod } 5$ and $y \textbf{ mod } 4$.