Question 1

Proof. Let m and n be integers. We want to show that if m is even and n is odd, then $m+n$ is odd. Since m is even, we can write $m = 2k$ for some integer k. Likewise, since n is odd, we can write $n = 2l + 1$ for some integer l. Then,

$$\begin{aligned} m + n &= 2k + 2l + 1 \\ &= 2(k + l) + 1. \end{aligned}$$

Since $k + l$ is an integer, $m + n$ is odd. $\square$

Question 2

Proof. Let a, b, c, k and l be integers. We want to show that if $a \mid b$ and $a \mid c$, then $a \mid (bk+cl)$. Since $a \mid b$, we can write $b = am$ for some integer m. Likewise, since $a \mid c$, we can write $c = an$ for some integer n. Thus, we can write

$$\begin{aligned} bk + cl &= amk + anl \\ &= a(mk + nl). \end{aligned}$$

Since $mk + nl$ is an integer, we have shown that a divides $bk + cl$. $\square$

Question 3

Proof. Let m be an odd integer. We want to show that for all integers m, if m is odd, then there exists some integer k such that $m^2 = 8k + 1$. By definition, m can be written as $m = 2n + 1$ for some integer n. We can then substitute this into m^2 to get

$$\begin{aligned} m^2 &= (2n + 1)^2 \\ &= 4n^2 + 4n + 1 \\ &= 4(n^2 + n) + 1. \end{aligned}$$

By Lemma 1, $n^2 + n$ is even for all integers n. Thus, we can write $n^2 + n = 2k$ for some integer k. Substituting this into m^2, we get

$$\begin{aligned} m^2 &= 4(2k) + 1 \\ &= 8k + 1. \end{aligned}$$

Now we have found an integer k such that $m^2 = 8k + 1$. $\square$

Question 4

Proof. Let n be an integer. We want to show that for any integer n, $n^2 + n - 9$ is odd. In the case that n is even, we can write $n = 2k$ for some integer k. Thus, we can write

$$\begin{aligned} n^2 + n - 9 &= (2k)^2 + 2k - 9 \\ &= 4k^2 + 2k - 9 \\ &= 4k^2 + 2k - 10 + 1 \\ &= 2(2k^2 + k - 5) + 1. \end{aligned}$$

Since $2k^2 + k - 5$ is an integer, $n^2 + n - 9$ is odd for even n. In the case that n is odd, we can write $n = 2k + 1$ for some integer k. Then, we can substitute as follows:

$$\begin{aligned} n^2 + n - 9 &= (2k + 1)^2 + 2k + 1 - 9 \\ &= 4k^2 + 4k + 1 + 2k + 1 - 9 \\ &= 4k^2 + 6k - 8 + 1 \\ &= 2(2k^2 + 3k - 4) + 1. \end{aligned}$$

Since $2k^2 + 3k - 4$ is an integer, $n^2 + n - 9$ is odd for odd n. Therefore, we have shown that $n^2 + n - 9$ is odd for all integers n. $\square$