CSCE 222 Discrete Structures for Computing – Fall 2023

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Homework 3 Solutions

Total $100 + 10$ (bonus) points.

Problem 1. (20 points) Section 3.4, Exercise 3.26. [Hint: Use the definition of set difference, the distributive laws, and de Morgan's laws involving the set complement. Starting from the right side of the equal sign may be easier.]

Solution. We have

$$\begin{aligned}
(A \cap B) - (A \cap C) &= (A \cap B) \cap (A \cap C)^{\complement} \quad \text{def. of set difference} \\
&= (A \cap B) \cap (A^{\complement} \cup C^{\complement}) \quad \text{de Morgan's law} \\
&= (A \cap B \cap A^{\complement}) \cup (A \cap B \cap C^{\complement}) \quad \text{distributive law} \\
&= \emptyset \cup (A \cap (B \cap C^{\complement})) \quad \text{since } A \cap A^{\complement} = \emptyset \\
&= A \cap (B - C) \quad \text{def. of set difference}
\end{aligned}$$

Problem 2. (20 points) Section 3.5, Exercise 3.33. [Hint: To show two sets S_1 and S_2 are equal ($S_1 = S_2$), you need to show that (1) $S_1 \subseteq S_2$ *and* (2) $S_2 \subseteq S_1$. Here, for each direction, you need to argue based on the definition of $\subseteq$.]

Solution. (1) First, prove $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$.
If $(u, v) \in (A \cup B) \times C$, then $u \in A \cup B$ and $v \in C$, so $u \in A$ or $u \in B$ and $v \in C$. Therefore, (u, v) belongs to $A \times C$ or to $B \times C$, so $(u, v) \in (A \times C) \cup (B \times C)$.

(2) Second, prove the converse: $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$.
Conversely, if $(u, v) \in (A \times C) \cup (B \times C)$, then (u, v) is an element of $A \times C$ or $B \times C$, so u is an element of A or of B, and v is an element of C. Therefore, (u, v) is an element of $(A \cup B) \times C$.

Thus, we have $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

Problem 3. (20 points) Section 3.6, Exercise 3.37. *Justify your answers.*

Solution. The relation "is child of" on the set of people is irreflexive, asymmetric, and antisymmetric. Indeed, we can see this as follows.

(a) The relation is not reflexive, as no person can be their own child.

(b) The relation is irreflexive, for the same reason as in (a).

(c) If A is the child of B, then B cannot be the child of A; therefore, the relation is asymmetric.

(d) The relation is antisymmetric, since the relations "A is a child of B" and "B is a child of A" can never hold at the same time (thus the definition of antisymmetry holds by vacuously true).

(e) The relation is not symmetric, since A is a child of B does not imply that B is a child of A.

(f) The relation is not transitive. Indeed, if A is the child of B, and B is the child of C, then A is a grandchild of C, but not a child.

Problem 4. (30 points) Section 3.9, Exercise 3.60. Proving your function is bijective by showing that it is injective and surjective is required. [Hint: Define a bijective function $f\colon \mathbf{N}_0 \to \mathbf{Z}$ by considering the argument being even or odd. Then prove that your function is indeed bijective by showing that it is injective and surjective.]

Solution. [16 points] (Note: Many different functions are possible.)
We claim that the function $f\colon \mathbf{N}_0 \to \mathbf{Z}$ given by

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ -(n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

is a bijection.

[7 points] Indeed, if $f(n)$ and $f(m)$ have the same nonnegative value, then $n/2 = f(n) = f(m) = m/2$ implies that $n = m$. If $f(n)$ and $f(m)$ have the same negative value, then $-(n+1)/2 = f(n) = f(m) = -(m+1)/2$ implies that $n = m$. Therefore, the function f is injective.

[7 points] If k is a nonnegative integer, then $f(2k) = k$; if $-k$ is a negative integer, then $f(2k-1) = -k$, so f is surjective as well.

Thus, we can conclude that $|\mathbf{N}_0| = |\mathbf{Z}|$.

Problem 5. (20 points) Section 5.1, Exercise 5.4.

Solution. Since $x/x = 1 = 2^0$ holds for all x in $\mathbf{N}_1$, we have $x \sim x$ for all x in $\mathbf{N}_1$. Therefore, the relation is reflexive.

Suppose that x and y are positive integers such that $x \sim y$. This means that $x/y = 2^k$ for some integer k; hence, $y/x = 2^{-k}$, which implies $y \sim x$. Therefore, $\sim$ is symmetric.

Suppose that $x \sim y$ and $y \sim z$, so there exist integers k and ℓ such that $x/y = 2^k$ and $y/z = 2^\ell$. It follows that $x/z = (x/y)(y/z) = 2^k \cdot 2^\ell = 2^{k+\ell}$; hence, $x \sim z$. Therefore, $\sim$ is transitive.

Therefore, the relation $\sim$ on the set of positive integers is an equivalence relation.