MATH300 Homework 5 (Not for a grade)

1. (0 pts) We know that the set of integers and the set of real numbers are closed under addition and multiplication. The same is also true for rational numbers:
   Proof Sketch.
   $$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \in \mathbb{Q} \qquad \text{and} \qquad \frac{a}{b}\frac{c}{d} = \frac{ac}{bd} \in \mathbb{Q}.$$
   However, irrational numbers are *not* closed under addition nor multiplication. We verify this in (b) and (c) below.

   (a) Let $x \in \mathbb{R}$. Prove that x is irrational if and only if $-x$ is irrational.

   (b) Use part (a) and the fact that we know $\sqrt{2}$ is irrational to disprove: the sum of two irrational numbers is irrational.

   (c) Use the fact that we know $\sqrt{2}$ is irrational to disprove: the product of two irrational numbers is irrational.

   (d) Prove that if x is a nonzero rational number and y is an irrational number, then xy is irrational.

2. (0 pts) Prove that for all positive integers n,
   $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3. (0 pts) Prove that for all positive integers n, $8 \mid (9^n - 1)$.

4. (0 pts) Prove that for all integers $n \geq 0$,
   $$\sum_{k=0}^{n} 2^k = 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

5. (0 pts) Section 3.1, problem 20 of the textbook.