Practice for the Final exam — MATH 304 — Fall 2023
— No Due Date —

1. Find the general solution of each of the following systems:

   a. $\begin{cases} y_1 + y_2 = y_1' \\ -2y_1 + 4y_2 = y_2' \end{cases}$

   b. $\begin{cases} y_1 - y_2 = y_1' \\ y_1 + y_2 = y_2' \end{cases}$

   c. $\begin{cases} y_1 + y_3 = y_1' \\ 2y_2 + 6y_3 = y_2' \\ y_2 + 3y_3 = y_3' \end{cases}$

2. Find the general solution of each of the following systems:

   a. $\begin{cases} -y_1 + 2y_2 = y_1' \\ 2y_1 - y_2 = y_2' \end{cases},$

   b. $\begin{cases} y_1 - 2y_2 = y_1' \\ 2y_1 + y_2 = y_2' \end{cases},$

3. In each of the following, "diagonalize" the matrix X and use it to compute A^{-1}, A^4, e^A.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix}$$

4. Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}.$$

   a. Use the Gram-Scmidt process to find an orthonormal basis for the column space of A.

   b. Solve the least square problem $Ax = b$.

5. Let $\{x_1, x_2, x_3\} := \{(0,1,0), (2,1,2), (0,0,1)\}$, be a basis of $\mathbb{R}^3$. Use the Gram-Scmidt process to obtain an orthonormal basis.

6. Consider the vector space $C[0,1]$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

a. Find an orthonormal basis of the subspace E spanned by $1, x, x^2$.

c. Compute the length of $2x^2 + 3$.

c. Compute the projection of e^x onto E

7. Find the orthogonal complement of the subspace of $\mathbb{R}^4$ spanned by $(1, 1, 1, 1), (1, -1, 1, -1)$.

8. For each of the following systems $Ax = b$ find all least squares solutions:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

9. Decide if the following statements are True or False:

    1. There is only one inner product in $\mathbb{R}^2$, the dot product.
    2. Product of orthogonal matrices is also orthogonal.
    3. Sum of orthogonal matrices is also orthogonal.
    4. The inverse of an orthogonal matrix is its transpose.
    5. Let V be an vector space with an inner product. Then
    $$\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$$
    for all vectors v_1, v_2.
    6. In we have a norm that satisfies
    $$\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$$
    for all vectors v_1, v_2 then is a norm that is induced by an inner product.
    7. $|\langle x, y\rangle|$ is always greater than the products of the norms of x and y.
    8. The matrices $A^T A$ and AAT always have the same rank.
    9. Let u_1, u_2 be two orthogonal matrices in $\mathbb{R}^n$. Let V be the matrix that has these vectors as columns. Then $U^T U$ is the the 2×2 identity matrix.
    10. The projection of a vector x in a subspace S is the closest point in the subspace to the vector x.
    11. If $\lambda \in \mathbb{R}$ and $\|\cdot\|$ is a norm, then
    $$\|\lambda x\| = \lambda \|x\|.$$
    12. The functions $\cos x$ and $\sin x$ are orthogonal with respect to the inner product
    $$\langle f, g\rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) dx.$$

13. If v, u are two orthogonal vectors then $\|v + 2u\| = \sqrt{5}$.
14. A defective matrix can not be diagonalized.
15. The characteristic polynomial of an $n \times n$ matrix A has n distinct roots.
16. The product of the eigenvalues of a $n \times n$ matrix is always a real number.
17. Similar matrices have the same eigenvalues.
18. Similar matrices have the same eigenvectors.
19. If a matrix is singular then at least one of the eigenvalues is the 0 one.
20. If a matrix is singular then all the eigenvalues are 0.
21. If a 3×3 has eigenvalues $1, 2, 0$ then it is diagonalizable.
22. If λ is an eigenvalues of A then e^{λ} is an eigenvalue for e^{A}.

This file covers the material after the second mid-term. Use the "practice" files for the first and second midterm exams to practice on problems on the previous chapters. Work on the Homework assignments.

The final exam will be cumulative.

Final exam Date/Time:

1. **Section 509:** December 11, 1:00 till 3:00 pm.
2. **Section 508:** December 12, 1:00 till 3:00 pm.
3. **Section 510:** December 12, 3:30 till 5:30 pm.