Question 1

Determine whether the following transformations are linear: Explain your answer.

a. $F((x_1, x_2, x_3)^T) = (x_1 - x_2, x_2 - x_1)^T$

b. $F((x_1, x_2, x_3)^T) = (1, 2, x_1 + x_2 + x_3)^T$

c. $F((x_1)) = (x_1, 2x_1, 3x_1)^T$

d. $F((x_1, x_2, x_3, x_4)^T) = (x_1, 0, 0, 0, x_2^2 + x_3^2 + x_4^2)^T$

Solution: To check for linearity, we need to check for additivity and homogeneity, which implies that the zero vector is preserved. That means a map is linear if $L(c(u + v)) = cL(u + v) = cL(u) + cL(v)$ for all u, v in the domain and $c \in \mathbb{R}$.

a. $F((x_1, x_2, x_3)^T) = (x_1 - x_2, x_2 - x_1)^T$
Additivity:

$$\begin{aligned} F((x_1, x_2, x_3)^T + (y_1, y_2, y_3)^T) &= F((x_1 + y_1, x_2 + y_2, x_3 + y_3)^T) \\ &= (x_1 + y_1 - x_2 - y_2, x_2 + y_2 - x_1 - y_1)^T \\ &= (x_1 - x_2, x_2 - x_1)^T + (y_1 - y_2, y_2 - y_1)^T \\ &= F((x_1, x_2, x_3)^T) + F((y_1, y_2, y_3)^T) \end{aligned}$$

Homogeneity:

$$\begin{aligned} F(c(x_1, x_2, x_3)^T) &= F((cx_1, cx_2, cx_3)^T) \\ &= (cx_1 - cx_2, cx_2 - cx_1)^T \\ &= c(x_1 - x_2, x_2 - x_1)^T \\ &= cF((x_1, x_2, x_3)^T) \end{aligned}$$

Therefore, $F((x_1, x_2, x_3)^T) = (x_1 - x_2, x_2 - x_1)^T$ is linear.

b. $F((x_1, x_2, x_3)^T) = (1, 2, x_1 + x_2 + x_3)^T$
Homogeneity:

$$F((0, 0, 0)^T) = (1, 2, 0)^T$$

Since this transformation does not preserve the zero vector, it is not linear.

c. $F((x_1)) = (x_1, 2x_1, 3x_1)^T$
Additivity:

$$\begin{aligned} F((x_1) + (y_1)) &= F((x_1 + y_1)) \\ &= (x_1 + y_1, 2(x_1 + y_1), 3(x_1 + y_1))^T \\ &= (x_1, 2x_1, 3x_1)^T + (y_1, 2y_1, 3y_1)^T \\ &= F((x_1)) + F((y_1)) \end{aligned}$$

Homogeneity:

$$\begin{aligned} F(c(x_1)) &= F((cx_1)) \\ &= (cx_1, 2(cx_1), 3(cx_1))^T \\ &= c(x_1, 2x_1, 3x_1)^T \\ &= cF((x_1)) \end{aligned}$$

Therefore, $F((x_1)) = (x_1, 2x_1, 3x_1)^T$ is linear.

d. $F((x_1, x_2, x_3, x_4)^T) = (x_1, 0, 0, 0, x_2^2 + x_3^2 + x_4^2)^T$
Since this transformation includes squared terms, it cannot satisfy additivity, and therefore is not linear.

Question 2

Determine whether the following transformations are linear from $C([0,1])$ to $\mathbb{R}$.

a. $L(f) = f(0), (L := C([0,1]) \to \mathbb{R})$

b. $L(f) = |f(0)|, (L := C([0,1]) \to \mathbb{R})$

c. $L(f) = f'(0) + f(0)$. $(L := C^1([0,1]) \to \mathbb{R})$

d. $L(f)(x) = x^2 + f(x),\ (L := C([0,1]) \to C([0,1]))$

Solution: Linear maps must satisfy additivity and homogeneity.

a. $L(f) = f(0), (L := C([0,1]) \to \mathbb{R})$
Additivity:

$$\begin{aligned} L(f+g) &= (f+g)(0) \\ &= f(0) + g(0) \\ &= L(f) + L(g) \end{aligned}$$

Homogeneity:

$$\begin{aligned} L(cf) &= (cf)(0) \\ &= cf(0) \\ &= cL(f) \end{aligned}$$

The transformation $L(f) = f(0)$ preserves additivity and homogeneity, and therefore is linear.

b. $L(f) = |f(0)|, (L := C([0,1]) \to \mathbb{R})$
Additivity:

$$\begin{aligned} L(f+g) &= |(f+g)(0)| \\ &= |f(0) + g(0)| \\ &\neq |f(0)| + |g(0)| \end{aligned}$$

Counterexample: $f(x) = x + 1,\ g(x) = x - 1$

$$\begin{aligned} L(f+g) &= |(f+g)(0)| \\ &= |f(0) + g(0)| \\ &= |1 + (-1)| \\ &= 0 \end{aligned}$$

$$\begin{aligned} L(f) + L(g) &= |f(0)| + |g(0)| \\ &= |1| + |-1| \\ &= 2 \end{aligned}$$

Since additivity is not preserved, this transformation is not linear.

c. $L(f) = f'(0) + f(0)$. $(L := C^1([0,1]) \to \mathbb{R})$
Additivity:

$$\begin{aligned} L(f+g) &= (f+g)'(0) + (f+g)(0) \\ &= f'(0) + g'(0) + f(0) + g(0) \\ &= f'(0) + f(0) + g'(0) + g(0) \\ &= L(f) + L(g) \end{aligned}$$

Homogeneity:

$$\begin{aligned} L(cf) &= (cf)'(0) + (cf)(0) \\ &= cf'(0) + cf(0) \\ &= c(f'(0) + f(0)) \\ &= cL(f) \end{aligned}$$

The transformation $L(f) = f'(0) + f(0)$ preserves additivity and homogeneity, and therefore is linear.

d. $L(f)(x) = x^2 + f(x),\ (L := C([0,1]) \to C([0,1]))$
Since x^2 is a constant in the domain, that means the zero vector cannot be preserved, and therefore this transformation is not linear.

Question 3

For each of the following transformations, find a matrix A such that $L(x) = Ax$.

a. $L((x_1, x_2, x_3)^T) = (x_1 + x_2)^T$

b. $L((x_1, x_2, x_3)^T) = (x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3)^T$

c. $L((x_1)) = (x_1, 2x_1, 3x_1)^T$

d. $L((x_1, x_2, x_3, x_4)^T) = (x_1 + x_2 + x_3 + 2x_4)^T$

Solution: To find a matrix A such that $L(x) = Ax$, we need to find the image of the standard basis vectors.

a. $L((x_1, x_2, x_3)^T) = (x_1 + x_2)^T$

$$L((1,0,0)^T) = (1,0)^T, \quad L((0,1,0)^T) = (1,0)^T, \quad L((0,0,1)^T) = (0,0)^T$$
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b. $L((x_1, x_2, x_3)^T) = (x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3)^T$

$$L((1,0,0)^T) = (1,0,1)^T, \quad L((0,1,0)^T) = (1,1,1)^T, \quad L((0,0,1)^T) = (0,1,1)^T$$
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

c. $L((x_1)) = (x_1, 2x_1, 3x_1)^T$

$$L((1)) = (1,2,3)^T$$
$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

d. $L((x_1, x_2, x_3, x_4)^T) = (x_1 + x_2 + x_3 + 2x_4)^T$

$$L((1,0,0,0)^T) = (1)^T, \quad L((0,1,0,0)^T) = (1)^T$$
$$L((0,0,1,0)^T) = (1)^T, \quad L((0,0,0,1)^T) = (2)^T$$
$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \end{bmatrix}$$

Question 4

Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$L((x_1, x_2, x_3)^T) = (2x_1, x_1 + x_2).$$

a. Find A that represents L with respect to the standard basis of $\mathbb{R}^3$.

b. Find B that represents L with respect to the following basis of $\mathbb{R}^3$.
$E := [v_1, v_2, v_3]$, where,

$$v_1 = (1, 1, 1)^T, \quad v_2 = (1, 1, 0)^T, \quad v_3 = (1, 0, 0)^T.$$

Solution:
a. To find a matrix A such that $L(x) = Ax$, we need to find the image of the standard basis vectors.

$$L((1, 0, 0)^T) = (2, 1)^T, \quad L((0, 1, 0)^T) = (0, 1)^T, \quad L((0, 0, 1)^T) = (0, 0)^T$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

b. To find a matrix B such that $L(x) = Bx$ where E is a basis of $\mathbb{R}^3$, we need to find the image of v_1, v_2, v_3.

$$L((1, 1, 1)^T) = (2, 2)^T, \quad L((1, 1, 0)^T) = (2, 2)^T, \quad L((1, 0, 0)^T) = (2, 1)^T$$

$$B = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Question 5

In the vector space $C[-\pi, \pi]$ we define inner product

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx.$$

a. Show that the above is indeed an inner product.

b. Show that $f(x) = \cos(x),\ g(x) = \sin(x)$ are orthogonal and that they have length 1.

Solution:

a. To show that the above is an inner product, we need to show that it satisfies the following properties:

i. $\langle av_1 + bv_2, v_3\rangle = a\langle v_1, v_3\rangle + b\langle v_2, v_3\rangle$

ii. $\langle v_1, v_2\rangle = \langle v_2, v_1\rangle$

iii. $\langle v_1, v_1\rangle \geq 0$ and $\langle v_1, v_1\rangle = 0$ if and only if $v_1 = 0$

For the first property:

$$\begin{aligned}
\langle af + bg, h\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} (af(x) + bg(x))h(x)\ dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} af(x)h(x) + bg(x)h(x)\ dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} af(x)h(x)\ dx + \frac{1}{\pi}\int_{-\pi}^{\pi} bg(x)h(x)\ dx \\
&= a\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)h(x)\ dx + b\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)h(x)\ dx
\end{aligned}$$

$$a\langle f, h\rangle + b\langle g, h\rangle = a\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)h(x)\ dx + b\frac{1}{\pi}\int_{-\pi}^{\pi} g(x)h(x)\ dx$$

For the second property:

$$\begin{aligned}
\langle f, g\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\ dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)f(x)\ dx \\
&= \langle g, f\rangle
\end{aligned}$$

For the third property:

$$\begin{aligned}\langle f, f\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)f(x)\ dx\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\ dx\end{aligned}$$

We want to show that $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\ dx$ is positive for all $f \in \mathbb{C}[-\pi, \pi]$ and that it is zero if and only if $f = 0$. The integrand $f(x)^2$ must be greater than or equal to zero, since it is the square of a real number. Additionally, we have that $f(x)^2 = 0$ if and only if $f(x) = 0$. Since the integrand is greater than or equal to zero, and is zero if and only if $f(x) = 0$, then the integral must be greater than or equal to zero, and is zero if and only if $f(x) = 0$. Therefore, $\langle f, f\rangle \geq 0$ and $\langle f, f\rangle = 0$ if and only if $f = 0$, and the above is an inner product.

b. To show that $f(x) = \cos(x),\ g(x) = \sin(x)$ are orthogonal, we need to show that $\langle f, g\rangle = 0$. The inner product of f and g is:

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(x)\sin(x)\ dx\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sin(x)\ dx\int_{-\pi}^{\pi}\cos(x)\ dx\\ &= \frac{1}{\pi}\left[-\cos(x)\right]_{-\pi}^{\pi}\left[\sin(x)\right]_{-\pi}^{\pi}\\ &= \frac{1}{\pi}(-1-(-1))(0-0)\\ &= 0\end{aligned}$$

Since the inner product of f and g is zero, f and g are orthogonal.
To show that f and g have length 1, we can take the euclidian norm of f and g, which is just the square root of their inner product with themselves.

$$\begin{aligned}\langle f, f\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)f(x)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(x)^2\ dx\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1+\cos(2x)\ dx\\ &= \frac{1}{2\pi}\left[x+\frac{1}{2}\sin(2x)\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\pi+\frac{1}{2}\sin(2\pi)-(-\pi+\frac{1}{2}\sin(-2\pi))\right)\\ &= \frac{1}{2\pi}\left(\pi+0-(-\pi+0)\right) = \frac{1}{2\pi}(2\pi) = 1\\ \sqrt{\langle f, f\rangle} &= \sqrt{1} = 1\end{aligned}$$

The inner product of f and f is 1, so the euclidian norm of f is 1.

$$\begin{aligned}
\langle g, g\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)g(x)\ dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(x)^2\ dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} 1 - \cos(2x)\ dx \\
&= \frac{1}{2\pi}\left[x - \frac{1}{2}\sin(2x)\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\pi - \frac{1}{2}\sin(2\pi) - (-\pi - \frac{1}{2}\sin(-2\pi))\right) \\
&= \frac{1}{2\pi}\left(\pi + 0 - (-\pi - 0)\right) = \frac{1}{2\pi}(2\pi) = 1 \\
\sqrt{\langle g, g\rangle} &= \sqrt{1} = 1
\end{aligned}$$

The inner product of g and g is 1, so the euclidian norm of g is 1. Therefore, f and g have length 1.