1 Main Idea

In this problem, we have athletes from n countries participating in m games at the 2020 Olympics. We denote $C = \{c_1, c_2, \ldots, c_n\}$ as the set of n countries and $H = \{h_1, h_2, \ldots, h_m\}$ as the set of m games. We assume for simplicity that each country sends at most one athlete to each game, and each game has at least three countries participating. Let $A_i \subseteq C$ for $i = 1, 2, \ldots, m$ be the set of countries participating in game h_i. Each game produces three medals: gold, silver, and bronze. Let x_j for $j = 1, 2, \ldots, n$ be the number of medals won by country c_j. The goal is to determine if some arbitrary vector $X = (x_1, x_2, \ldots, x_n)$ is a possible outcome of the 2020 Olympics.

This problem can be modeled as a maximum flow problem. We can construct a flow network where the maximum flow corresponds to a valid medal distribution. Let there be a source node s and a sink node t. We create a node c_j for each country, and it is connected to the source with a capacity of x_j. Create nodes h_i for each game, and connect it to each country in A_i with a capacity of 1. Finally, connect each game node to the sink with a capacity of 3.

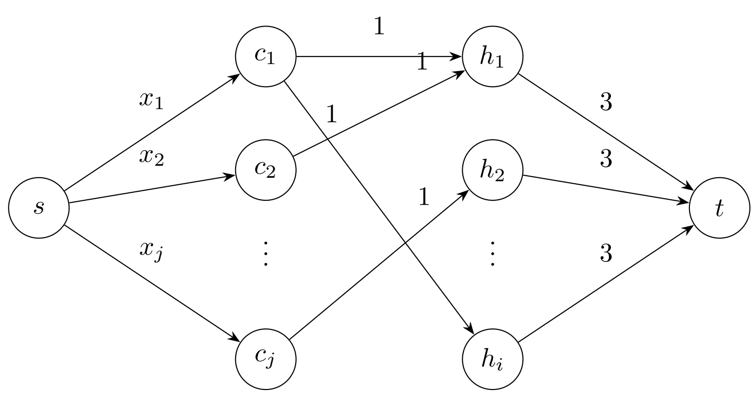

Figure 1: Flow network for the 2020 Olympics

We then find the maximum flow using the Ford-Fulkerson algorithm. If the maximum flow is equal to the total number of medals (i.e. $3m$), then X is a possible outcome of the 2020 Olympics.

2 Pseudocode

Algorithm 1: Check possible outcome

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Input: Vector of outcomes X, set of n countries C, set of m games H
Output: True if X is a possible outcome, False otherwise

Initialize flow network G = (V, E) as described above;
totalMedals = 3m;
maxFlow = 0;
while there exists an augmenting path p from s to t in the residual
  graph G_f do
    c_f = min{c_f(u, v) | (u, v) ∈ p};
    for each edge (u, v) ∈ p do
        if (u, v) is a forward edge then
            f(u, v) = f(u, v) + c_f;
        end
        else
            f(v, u) = f(v, u) − c_f;
        end
    end
    maxFlow = maxFlow + c_f;
end
if maxFlow = totalMedals then
    return True;
end
return False;
```

3 Proof of Correctness

Proof. ($\Rightarrow$) Assume the algorithm retuns true. That means we have found a maximum flow equal to the total number of medals. That implies the following:

- All game nodes h_i both send and receive 3 units of flow.
- Each country node c_j sends x_j units of flow.
- Country nodes send either 0 or 1 unit of flow to game nodes.

Each game node sending 3 units of flow implies that the game results in 3 medals. Each country sending x_j units of flow implies that country c_j won exactly x_j medals. Finally, since each country sends either 0 or 1 unit of flow to game nodes, each country participates in at most one game. Therefore, when the algorithm returns true, the flow network corresponds to a valid medal distribution.

($\Leftarrow$) Assume that X is a possible outcome of the 2020 Olympics. Then there exists a flow network that corresponds to the medal distribution with the constraints. For each medal won by country c_j, we send x_j units of flow from s to

c_j. By construction, this flow satisfies the capacity constraints. The total flow will be the maximum flow, which is equal to the total number of medals. Thus the algorithm is correct. $\square$

4 Runtime Analysis

The majority of this algorithm is the Ford-Fulkerson algorithm. However, first we need to construct the flow network. Graph construction can be done in $O(n + m + |A|)$ time, where $|A|$ is the total number of participations in all games. In the Ford-Fulkerson algorithm, the number of iterations is bounded by the maximum flow value, i.e. $3m$. In each iteration, finding an augmenting path takes $O(V + E)$ time. Since the flow network has $|V| = n + m + 2$ and $|E| \leq n + |A| + m$, the time complexity of the algorithm is $O(3m(n + m + |A|))$.