CSCE 222 Discrete Structures for Computing – Fall 2023

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Homework 4 Solutions

Total $100 + 5$ (bonus) points.

Problem 1. (15 points) Section 4.1, Exercise 4.3

Solution. We prove this by induction on n. Let $P(n)$ denote the claimed equation, Eqn (4.3).

Base case. The equation $P(n)$ holds for $n = 1$, since

$$\sum_{k=1}^{1} k^2 = 1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}.$$

Inductive step. We show that $[P(n) \to P(n+1)]$ holds for all $n \geq 1$. As the induction hypothesis, suppose that $P(n)$ is true. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 && \text{by the definition of } \textstyle\sum \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 && \text{by the induction hypothesis} \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} && \text{by a common denominator} \\
&= \frac{(n+1)(2n^2+n+6n+6)}{6} && \text{by factoring out } (n+1) \\
&= \frac{(n+1)(2n^2+7n+6)}{6} && \text{simplifying} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} && \text{to have the right-hand side of } P(n+1) \\
&= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6} && \text{exactly the right-hand side of } P(n+1)
\end{aligned}$$

so $P(n+1)$ holds. Therefore, the implication $P(n) \to P(n+1)$ is true for all $n \geq 1$.

It follows by induction that $P(n)$ is true for all $n \geq 1$.

Problem 2. (15 points) Section 4.1, Exercise 4.4

Solution. Let $P(n)$ denote the claimed equation, Eqn (4.4), that is,

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2 = \frac{n^2(n+1)^2}{4}.$$

We prove the claim by induction on n.

Base case. The equation $P(n)$ holds for $n = 1$, since

$$\sum_{k=1}^{1} k^3 = 1^3 = (1)^2 = \frac{1^2 \cdot (1+1)^2}{4}.$$

Inductive step. We need to show that the implication $[P(n) \to P(n+1)]$ holds for all $n \geq 1$. As the induction hypothesis, suppose that $P(n)$ is true. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 && \text{by definition of } \textstyle\sum \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 && \text{by the induction hypothesis} \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} && \text{by a common denominator} \\
&= \frac{(n+1)^2(n^2 + 4(n+1))}{4} && \text{by factoring out } (n+1)^2 \\
&= \frac{(n+1)^2(n^2 + 4n + 4)}{4} = \frac{(n+1)^2(n+2)^2}{4} && \text{by simplifying} \\
&= \frac{(n+1)^2((n+1)+1)^2}{4}.
\end{aligned}$$

Also, since $(1 + 2 + \cdots + n + (n+1)) = (n+1)(n+2)/2$, we have

$$\sum_{k=1}^{n+1} k^3 = (1 + 2 + \cdots + n + n + 1)^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2 = \frac{(n+1)^2(n+2)^2}{4}.$$

Therefore, $P(n+1)$ is true. Hence, $P(n) \to P(n+1)$ holds for all $n \geq 1$.

Therefore, $P(n)$ holds for all $n \geq 1$ by induction.

Problem 3. (15 points) Section 4.1, Exercise 4.5

Solution. We prove this by induction on n. Let $P(n)$ denote the claimed equation, Eqn (4.5).

Base case. The equation $P(n)$ holds for $n = 1$, since

$$\sum_{k=1}^{1}(2k-1)^2 = 1^2 = 1 = \frac{1}{3}(3) = \frac{1}{3}(4-1) = \frac{1}{3}(4\cdot 1^3 - 1).$$

Inductive step. We show that $[P(n) \to P(n+1)]$ holds for all $n \geq 1$. As the induction hypothesis, suppose that $P(n)$ is true. Then

$$\begin{aligned}
\sum_{k=1}^{n+1}(2k-1)^2 &= \sum_{k=1}^{n}(2k-1)^2 + (2(n+1)-1)^2 && \text{by the definition of } \textstyle\sum \\
&= \frac{1}{3}(4n^3-n) + (2(n+1)-1)^2 && \text{by the induction hypothesis} \\
&= \frac{1}{3}(4n^3-n) + (2n+1)^2 && \text{since } 2(n+1)-1 = 2n+2-1 = 2n+1 \\
&= \frac{1}{3}(4n^3-n+3(2n+1)^2) && \text{by a common denominator} \\
&= \frac{1}{3}(4n^3-n+3(4n^2+4n+1)) && \text{by expanding } (2n+1)^2 \\
&= \frac{1}{3}(4n^3-n+12n^2+12n+3) && \text{by distributing 3} \\
&= \frac{1}{3}(4n^3+12n^2+12n+4-n-1) && \text{by rearranging terms} \\
&= \frac{1}{3}(4(n^3+3n^2+3n+1)-(n+1)) && \text{regroup, factor out 4, and } -1\text{, resp.} \\
&= \frac{1}{3}(4(n+1)^3-(n+1)) && \text{exactly the right-hand side of } P(n+1)
\end{aligned}$$

so $P(n+1)$ holds. Therefore, the implication $P(n) \to P(n+1)$ is true for all positive integers n.

It follows by induction that $P(n)$ is true for all positive integers n.

Problem 4. (20 points) Section 4.1, Exercise 4.6

Solution. *Base case.* For $n = 1$, the integer $2^{2\cdot 1} - 1 = 3$ is divisible by 3.

Inductive step. As the induction hypothesis, suppose that $2^{2n} - 1$ is divisible by 3, that is, $2^{2n} - 1 = 3k$ for some integer $k \geq 1$. We need to show that this implies that $2^{2(n+1)} - 1$ is divisible by 3. Indeed, expanding the exponent and rewriting yields

$$\begin{aligned} 2^{2(n+1)} - 1 &= 2^{2n+2} - 1 = 2^{2n} \cdot 2^2 - 1 = 4 \cdot 2^{2n} - 1 = (3+1)2^{2n} - 1 \\ &= 3 \cdot 2^{2n} + (2^{2n} - 1) \\ &= 3 \cdot 2^{2n} + 3k \quad \text{by the induction hypothesis} \\ &= 3(2^{2n} + k) \quad \text{by factoring out 3} \end{aligned}$$

which is divisible by 3 since $2^{2n} + k$ is an integer ≥ 1, thus the inductive step holds.

Thus, the claim holds by induction on n.

Problem 5. (20 points) Section 4.3, Exercise 4.15

Solution. Let $P(n)$ denote the claimed equation. We prove the claim by induction on n.
Base case. The equation $P(n)$ is true for $n = 1$, since

$$\sum_{k=1}^{1} f_{2k} = f_{2\cdot 1} = 1 = 2 - 1 = f_3 - 1 = f_{2\cdot 1+1} - 1.$$

Inductive step. We claim that $P(n) \to P(n+1)$ holds for all $n \geq 1$. Indeed, as the induction hypothesis, suppose that $P(n)$ is true. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} f_{2k} &= \sum_{k=1}^{n} f_{2k} + f_{2n+2} && \text{by definition of } \sum \\
&= f_{2n+1} - 1 + f_{2n+2} && \text{by the induction hypothesis} \\
&= f_{2n+3} - 1 && \text{by definition of } f_{2n+3} = f_{2n+2} + f_{2n+1} \\
&= f_{2(n+1)+1} - 1.
\end{aligned}$$

Therefore, $P(n+1)$ holds.

Thus, we can conclude by induction that $P(n)$ holds for all $n \geq 1$.

Problem 6. (20 points) Section 4.6, Exercise 4.31

Solution. Let $P(n)$ denote the predicate $f_n = n!$.
Base cases. There are three base cases: $P(1)$, $P(2)$, and $P(3)$ hold since

$$f_1 = 1 = 1!, \quad f_2 = 2 = 2!, \quad \text{and} \quad f_3 = 6 = 3!.$$

Inductive step. Suppose that $n > 3$ and that $P(k)$ holds for all k in the range $1 \leq k < n$. By definition and induction hypothesis, we have

$$f_n = (n^3 - 3n^2 + 2n)f_{n-3} = (n^3 - 3n^2 + 2n)(n-3)!$$

Since $n^3 - 3n^2 + 2n = n(n^2 - 3n + 2) = n(n-1)(n-2)$, we can conclude that

$$f_n = (n^3 - 3n^2 + 2n)f_{n-3} = n(n-1)(n-2) \cdot (n-3)! = n!,$$

so $P(n)$ holds. Therefore, the claim follows by strong induction.