In this exercise, we discuss the multiway cut problem, which is defined as follows. Given an undirected graph $G = (V, E)$, a set of terminal nodes $s_1, s_2, \dots, s_k \in V$, find a minimum set of edges in E whose removal disconnects all k terminal nodes from each other. We want to show that when $k = 3$, this problem has a polynomial-time 2-approximation algorithm.

Theorem 1. *The multiway cut problem with $k = 3$ has a polynomial-time 2-approximation algorithm.*

Proof. Let $G = (V, E)$ be an undirected graph with a set of terminal nodes $s_1, s_2, s_3 \in V$, and assume that there exists an algorithm that solves this problem in polynomial time when $k = 2$. Consider the following algorithm.

Algorithm 1: 2-approximation algorithm for multiway cut with $k = 3$

Input: An undirected graph $G = (V, E)$ and terminal nodes $s_1, s_2, s_3 \in V$

Output: A minimum set of edges in E whose removal disconnects all terminal nodes from each other

C_{12} = minimum cut between s_1 and s_2 in G;
C_{13} = minimum cut between s_1 and s_3 in G;
C_{23} = minimum cut between s_2 and s_3 in G;
if $|C_{12}| \leq |C_{23}|$ *and* $|C_{13}| \leq |C_{23}|$ **then**
  return $C_{12} \cup C_{13}$;
end
if $|C_{12}| \leq |C_{13}|$ *and* $|C_{23}| \leq |C_{13}|$ **then**
  return $C_{12} \cup C_{23}$;
end
return $C_{13} \cup C_{23}$;

First we prove that this algorithm is a 2-approximation algorithm for the multiway cut problem with $k = 3$. Let C^* be the optimal solution to the multiway cut problem with $k = 3$. We have that $|C_{12}| \leq |C^*|$, $|C_{13}| \leq |C^*|$, and $|C_{23}| \leq |C^*|$, since a three terminal cut must be at least as large as the minimum cut between any two terminal nodes. Thus we have

$$|C_{12}| + |C_{13}| + |C_{23}| \leq 3|C^*|.$$

Let C be the set of edges returned by the algorithm. Since the algorithm returns the minimum two of the three cuts, we have that

$$C \leq \frac{2}{3}(|C_{12}| + |C_{13}| + |C_{23}|) \leq \frac{2}{3} \cdot |3C^*| = 2|C^*|$$

and the algorithm is a 2-approximation algorithm for the multiway cut problem with $k = 3$. Now, we show that the algorithm runs in polynomial time. The algorithm computes the minimum cut between each pair of terminal nodes, which is assumed to be solvable in polynomial time. After that, the comparison and union operations are done in constant time. Thus the algorithm runs in polynomial time. Therefore, the multiway cut problem with $k = 3$ has a polynomial-time 2-approximation algorithm. ☐