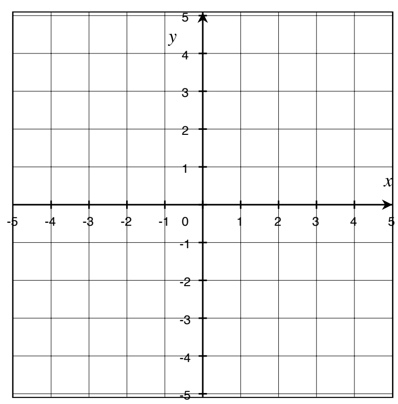
Appendix J.1: Vectors

A **vector** is a quantity that has both magnitude and direction (such as velocity and force). We will focus on **two-dimensional** vectors, which can be represented several ways.

Algebraically: As an ordered pair of real numbers, denoted either $\mathbf{a} = \langle a_1, a_2 \rangle$ or $\overrightarrow{a} = \langle a_1, a_2 \rangle$. The numbers a_1 and a_2 are called the **components** of $\mathbf{a}$. In particular, a_1 is the x-component and a_2 is the y-component.

Geometrically: As an arrow with an **initial point** and **terminal point**, such that starting at the initial point, if we move a_1 units in the x direction and a_2 units in the y direction, we end up at the terminal point. There are infinitely many geometric **representations** of the vector $\mathbf{a} = \langle a_1, a_2 \rangle$.



The representation of $\mathbf{a} = \langle a_1, a_2 \rangle$ with the origin as the initial point is called the **position vector** of the point (a_1, a_2).

Given the points $A(x_1, y_1)$ and $B(x_2, y_2)$, the vector with initial point A and terminal point B is

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

The **magnitude** or **length** of the vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

In particular, the length of the vector from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\left|\overrightarrow{AB}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note: the length $\left|\overrightarrow{AB}\right|$ is precisely the distance between A and B.

Example 1: Given the points $A(-2,1)$ and $B(3,4)$, find the vector $\overrightarrow{AB}$ and its magnitude.

Vector Algebra:

Vector Addition: If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then we define

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

Geometrically, vector addition is represented by the Triangle Law or Parallelogram Law:

Scalar Multiplication: If c is a scalar (real number) and $\mathbf{a} = \langle a_1, a_2 \rangle$, then we define

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Geometrically, scalar multiplication is represented by scaling the length of $\mathbf{a}$ by a factor of $|c|$. If c is positive, then $c\mathbf{a}$ points in the same direction as $\mathbf{a}$. If c is negative, then $c\mathbf{a}$ points in the opposite direction.

Vector Difference:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = \langle a_1 - b_1, a_2 - b_2 \rangle$$

Example 2: If $\mathbf{a} = \langle -1, -2 \rangle$ and $\mathbf{b} = \langle 0, 5 \rangle$, find $|3\mathbf{a} + 2\mathbf{b}|$.

A **unit vector** is a vector whose length is 1. If $\mathbf{a} \neq \mathbf{0}$, then the unit vector that has the same direction as $\mathbf{a}$ is

$$\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

Example 3: Find the unit vector in the direction of the vector $\mathbf{a} = \langle 2, -3 \rangle$. Also, find a vector that has the same direction as $\mathbf{a}$ but has length 4.

STANDARD BASIS VECTORS: The following two unit vectors play a special role:

$$\mathbf{i} = \langle 1, 0 \rangle \qquad \mathbf{j} = \langle 0, 1 \rangle$$

Note that $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the direction of the positive x- and y-axes. They lead to another algebraic representation of a vector (sometimes called **ij notation**):

$$\mathbf{a} = \langle a_1, a_2 \rangle =$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are called **parallel** if they are multiples of each other.

Example 4: Are the vectors $\mathbf{a} = 3\mathbf{i} - 5\mathbf{j}$ and $\mathbf{b} = -6\mathbf{i} + 10\mathbf{j}$ parallel?

<u>DESCRIBING DIRECTION WITH AN ANGLE</u>: The direction of a vector is sometimes described by an angle θ formed by the vector and a positive or negative coordinate axis. We can then use simple trigonometry to determine the components of the vector.

It is often convenient to work with a **reference angle**, that is, the acute angle formed by the vector and the x-axis.

<u>Example 5</u>: A vector $\mathbf{a}$ has a magnitude of 5 and makes a 150° angle (counterclockwise) with the positive x-axis. What are the components of $\mathbf{a}$?

<u>Example 6</u>: Find the reference angle for the vector $\mathbf{a} = -2\mathbf{i} - \mathbf{j}$.

Applications to Physics and Engineering:

Force: A force is represented by a vector because it has both a magnitude and direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

Example 7: Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on an object. The force $\mathbf{F}_1$ has a magnitude of 20 lbs and a direction of $60°$ from the positive x-axis, and $\mathbf{F}_2$ has a magnitude of 10 lbs and a direction of $330°$ from the positive x-axis. Find the resultant force $\mathbf{F}$ along with both its magnitude and direction.

Velocity: The velocity of an object is represented by a vector, where the magnitude of the vector is the speed, and the direction of the vector is the direction of motion.

Velocity direction is sometimes given in **bearings**:

Example 8: Suppose that a wind is blowing from the direction N45°W at a speed of 50 km/h. A pilot is steering a plane in the direction N60°E at an airspeed (speed in still air) of 250 km/h. Find the *true course* (direction of the resultant of the velocity vectors of the plane and the wind) and *ground speed* (magnitude of the resultant) of the plane.