CSCE 222 Discrete Structures for Computing – Fall 2023

Hyunyoung Lee

Homework 5 Solutions

Total $100 + 10$ (bonus) points.

Problem 1. (10 points) Section 11.1, Exercise 11.3

Solution. Ernie is right. We have $f \sim g$, since

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^2 + 2n}{n^2} = \lim_{n\to\infty} \left(1 + \frac{2}{n}\right) = 1.$$

Bert's argument that the absolute error $f(n) - g(n) \geqslant 2n$ is irrelevant, since the notation $f \sim g$ only claims that the relative error $(f(n) - g(n))/g(n)$ vanishes. The relative error $(f(n) - g(n))/g(n) = 2n/n^2 = 2/n$ does vanish as $n \to \infty$, even though the absolute error gets arbitrarily large.

Problem 2. (20 points) Section 11.3, Exercise 11.14. [Requirement: Study the definition of $\asymp$ involving the inequalities carefully and use the definition to answer the questions.]

Solution. (The detailed explanation in blue and red fonts in (iii) is to explain why we take the maximum between n_0 and m_0 as the threshold for n for $f \asymp h$, which may be omitted.)

(i) Property $f \asymp f$ obviously holds since

$$1 \cdot |f(n)| \leqslant |f(n)| \leqslant 1 \cdot |f(n)|$$

holds for all $n \geqslant 1$.

(ii) If $f \asymp g$, then by definition there exist positive constants c and C and a positive integer n_0 such that

$$c|g(n)| \leqslant |f(n)| \leqslant C|g(n)|$$

holds for all $n \geqslant n_0$. Therefore,

$$\frac{1}{C}|f(n)| \leqslant |g(n)| \leqslant \frac{1}{c}|f(n)|$$

holds for all $n \geqslant n_0$, which implies $g \asymp f$. Therefore, property (ii) follows.

(iii) If (1) $f \asymp g$ and (2) $g \asymp h$, then this means that

(1) there exist positive constants c, C and a positive integer n_0 such that

$$c|g(n)| \leqslant |f(n)| \leqslant C|g(n)| \text{ hold for all } n \geqslant n_0$$

that is,

$$c|g(n)| \leqslant |f(n)| \text{ and } |f(n)| \leqslant C|g(n)| \text{ hold for all } n \geqslant n_0$$

that is,

(1a) $c|g(n)| \leqslant |f(n)|$ holds for all $n \geqslant n_0$ and

(1b) $|f(n)| \leqslant C|g(n)|$ holds for all $n \geqslant n_0$,

and

(2) there exist positive constants d, D and a positive integer m_0 such that

$$d|h(m)| \leqslant |g(m)| \leqslant D|h(m)| \text{ hold for all } m \geqslant m_0$$

that is,

$$\text{(2a)} \quad d|h(m)| \leqslant |g(m)| \text{ holds for all } m \geqslant m_0 \text{ and}$$

$$\text{(2b)} \quad |g(m)| \leqslant D|h(m)| \text{ holds for all } m \geqslant m_0.$$

Now, if $n_0 \geqslant m_0$ (i.e., n_0 is the $\max(n_0, m_0)$), then (2a) and (1a) can be combined as:

(2a) $d|h(n)| \leqslant |g(n)|$ and (1a) $c|g(n)| \leqslant |f(n)|$ holds for all $n \geqslant n_0 \geqslant m_0$

and by substituting $|g(n)|$ in (1a) with $d|h(n)|$, we get

$$cd|h(n)| \leqslant |f(n)| \text{ holds for all } n \geqslant n_0 \geqslant m_0. \quad (3)$$

Similarly, for (1b) and (2b) if $n_0 \geqslant m_0$ (i.e., n_0 is the $\max(n_0, m_0)$):

(1b) $|f(n)| \leqslant C|g(n)|$ and (2b) $|g(n)| \leqslant D|h(n)|$ holds for all $n \geqslant n_0 \geqslant m_0$

and by substituting $|g(n)|$ in (1b) with $D|h(n)|$, we get

$$|f(n)| \leqslant CD|h(n)| \text{ holds for all } n \geqslant n_0 \geqslant m_0. \quad (4)$$

The conjunction of (3) and (4) yields

$$cd|h(n)| \leqslant |f(n)| \leqslant CD|h(n)| \text{ holds for all } n \geqslant n_0 \geqslant m_0.$$

In the case of $m_0 \geqslant n_0$ (i.e., m_0 is the $\max(n_0, m_0)$), (2a) and (1a) can be combined as:

$$d|h(n)| \leqslant |g(n)| \text{ and } c|g(n)| \leqslant |f(n)| \text{ holds for all } n \geqslant m_0 \geqslant n_0$$

and we get

$$cd|h(n)| \leqslant |f(n)| \text{ holds for all } n \geqslant m_0 \geqslant n_0. \quad (5)$$

Also, for (1b) and (2b) if $m_0 \geqslant n_0$ (i.e., m_0 is the $\max(n_0, m_0)$):

$$|f(n)| \leqslant C|g(n)| \text{ and } |g(n)| \leqslant D|h(n)| \text{ holds for all } n \geqslant m_0 \geqslant n_0$$

and we get

$$|f(n)| \leqslant CD|h(n)| \text{ holds for all } n \geqslant m_0 \geqslant n_0. \quad (6)$$

The conjunction of (5) and (6) yields

$$cd|h(n)| \leqslant |f(n)| \leqslant CD|h(n)| \text{ holds for all } n \geqslant m_0 \geqslant n_0.$$

It follows that

$$cd|h(n)| \leqslant |f(n)| \leqslant CD|h(n)|$$

holds for all $n \geqslant \max\{n_0, m_0\}$. This shows that $f \asymp h$ holds.

Problem 3. (15 points) Prove that $3n^2 + 41 \in O(n^3)$ by giving a direct proof based on the definition of big-O involving the inequalities and absolute values, as given in the lecture notes Section 11.4.

To do so, first write out what $3n^2 + 41 \in O(n^3)$ means according to the definition. Then, you need to find a positive real constant C and a positive integer n_0 that satisfy the definition.

Solution. By substituting $f(n) = 3n^2 + 41$ and $g(n) = n^3$ in the definition of big-O, we prove the claim: There exist a positive real constant C and a positive integer n_0 such that

$$|3n^2 + 41| \leqslant C\,|n^3|$$

holds for all $n \geqslant n_0$.

Since $n > 0$ and both f and g are positive valued functions, we can remove the absolute value functions and divide both sides of the inequality by n^3 to get

$$\frac{3n^2 + 41}{n^3} = \frac{3}{n} + \frac{41}{n^3} \leqslant C$$

which holds for all $n \geqslant 4$ with $C = 2$. Thus, by taking $(C, n_0) = (2, 4)$, the claim holds.

Note that there are infinitely many such value pairs for (C, n_0) that make the inequality holds for all $n \geqslant n_0$.

Problem 4. (15 points) Prove that $\frac{1}{2}n^2 + 5 \in \Omega(n)$ by giving a direct proof based on the definition of big-Ω involving the inequalities and absolute values, as given in the lecture notes Section 11.5.

To do so, first write out what $\frac{1}{2}n^2 + 5 \in \Omega(n)$ means according to the definition. Then, you need to find a positive real constant c and a positive integer n_0 that satisfy the definition.

Solution. By substituting $f(n) = \frac{1}{2}n^2 + 5$ and $g(n) = n$ in the definition of Ω, we show that there exist a positive real constant c and a positive integer n_0 such that

$$c\,|n| \leqslant \left|\frac{1}{2}n^2 + 5\right|$$

holds for all $n \geqslant n_0$.

Since $n > 0$ and both f and g are positive valued functions, we can remove the absolute value functions and divide both sides of the inequality by n to get

$$c \leqslant \left(\frac{1}{2}n^2 + 5\right)\frac{1}{n} = \frac{1}{2}n + \frac{5}{n}$$

which holds for all $n \geqslant 1$ with $c = 1$. Thus, by taking $(c, n_0) = (1, 1)$, the claim holds.

Again, note that there are infinitely many such value pairs for (c, n_0) that make the inequality holds for all $n \geqslant n_0$.

Problem 5. (10 + 10 = 20 points) Read Section 11.6 carefully before attempting this problem. Analyze the running time of the following algorithm using a step count analysis as shown in the Horner scheme (Example 11.40).

```
// search a key in an array a[1..n] of length n
search(a, n, key)          cost   times
  for k in (1..n) do        c1    [ n+1 ]
    if a[k]=key then        c2    [ n ]
       return k             c3    [ 1 ]
  endfor                    c4    [ n ]
  return false              c5    [ 1 ]
```

(a) Fill in the `[ ]`s in the above code each with a number or an expression involving `n` that expresses the step count for the line of code.

(b) Determine the worst-case complexity of this algorithm and give it in the Θ notation. Show your work and explain using the definition of Θ involving the inequalities.

Solution. (For part (b))
Let $f(n)$ be the worst-case time complexity of this algorithm. Then, we have

$$\begin{aligned} f(n) &= c1(n+1) + c2(n) + c3 + c4(n) + c5 \\ &= (c1 + c2 + c4)n + (c1 + c3 + c5) \\ &\in \Theta(n). \end{aligned}$$

Indeed, by the fact that $c1, c2, c3, c4$, and $c5$ are all positive real constants, if we let $p = c1 + c2 + c4 > 0$ and $q = c1 + c3 + c5 > 0$, then by taking $(c, C, n_0) = (p, p+q, 1)$ for instance, we have

$$p \cdot |n| \leqslant |pn + q| \leqslant p|n| + q|n| = (p+q)|n| \text{ hold for all } n \geqslant 1.$$

Problem 6. (15 + 15 = 30 points) Read Section 11.6 carefully before attempting this problem. Analyze the running time of the following algorithm using a step count analysis as shown in the Horner scheme (Example 11.40).

```
// determine the number of digits of an integer n
binary_digits(n)              cost  times
  int cnt = 1                  c1   [ 1 ]
  while (n > 1) do             c2   [ floor(log_2 (n)) + 1 ]
    cnt = cnt + 1              c3   [ floor(log_2 (n)) ]
    n = floor( n/2.0 )         c4   [ floor(log_2 (n)) ]
  endwhile                     c5   [ floor(log_2 (n)) ]
  return cnt                   c6   [ 1 ]
```

(a) Fill in the `[   ]`s in the above code each with a number or an expression involving `n` that expresses the step count for the line of code.

(b) Determine the worst-case complexity of this algorithm as a function of n and give it in the Θ notation. Show your work and explain using the definition of Θ involving the inequalities.

Solution. (For part (b))

Let $f(n)$ be the worst-case time complexity of this algorithm. Then, we have

$$\begin{aligned} f(n) &= c1 + c2(\lfloor \log_2 n \rfloor + 1) + c3(\lfloor \log_2 n \rfloor) + c4(\lfloor \log_2 n \rfloor) + c5(\lfloor \log_2 n \rfloor) + c6 \\ &= (c2 + c3 + c4 + c5)\lfloor \log_2 n \rfloor + (c1 + c2 + c6) \\ &\in \Theta(\log_2 n). \end{aligned}$$

Indeed, by the fact that $c1, c2, c3, c4, c5$, and $c6$ are all positive real constants, if we let $p = c2 + c3 + c4 + c5 > 0$ and $q = c1 + c2 + c6 > 0$, then by taking $(c, C, n_0) = (\frac{p}{2}, p + q, 2)$ for instance, we have

$$\frac{p}{2} \cdot |\log_2 n| \leqslant |p\lfloor \log_2 n \rfloor + q| \leqslant p|\log_2 n| + q|\log_2 n| = (p + q)|\log_2 n|$$

holds for all $n \geqslant 2$. Taking $p + q$ for C is straightforward. Justification of taking $\frac{p}{2}$ for c with $n_0 = 2$ is as follows. All the terms involved have nonnegative values, so we can remove the absolute value functions. Let $\lfloor \log_2 n \rfloor = m$ for a positive integer m (since $\log_2 n \geqslant 1$ for all $n \geqslant 2$). Then by the definition of $\lfloor\,\rfloor$, we have

$$m \leqslant \log_2 n < m + 1. \qquad (1)$$

Since $\frac{p}{2} > 0$, multiplying $\frac{p}{2}$ to all sides of (1) yields

$$\frac{p}{2}m \leqslant \frac{p}{2}\log_2 n < \frac{p}{2}(m+1) = \frac{p}{2}m + \frac{p}{2}. \qquad (2)$$

Since $m \geqslant 1$, by extracting the blue parts from (2), we get

$$\frac{p}{2}\log_2 n < \frac{p}{2}m + \frac{p}{2} \leqslant \frac{p}{2}m + \frac{p}{2}m$$

and by substituting m back with $\lfloor \log_2 n \rfloor$, we get

$$\frac{p}{2}\log_2 n < \frac{p}{2}\lfloor \log_2 n \rfloor + \frac{p}{2}\lfloor \log_2 n \rfloor = 2 \cdot \frac{p}{2}\lfloor \log_2 n \rfloor = p\lfloor \log_2 n \rfloor < p\lfloor \log_2 n \rfloor + q$$

holds for all $n \geqslant 2$.

(The exact values for (c, C, n_0) may not be necessary. As long as the student gives some explanation why $f(n)$ is in $\Theta(\log_2 n)$, it should be fine.)