Question 1

Find the general solution of each of the following systems:

a. $\begin{cases} y_1 + y_2 = y_1' \\ -2y_1 + 4y_2 = y_2' \end{cases}$

b. $\begin{cases} y_1 - y_2 = y_1' \\ y_1 + y_2 = y_2' \end{cases}$

c. $\begin{cases} y_1 + y_3 = y_1' \\ 2y_2 + 6y_3 = y_2' \\ y_2 + 3y_3 = y_3' \end{cases}$

Solution: For systems in the form $y' = Ay$, we can find y as e^{At}, where $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$. Additionally, when A is diagonalizable, we can write $A = Xe^DX^{-1}$, where X is the matrix of eigenvectors of A and D is the diagonal matrix of eigenvalues of A. Then, $e^{At} = Xe^{Dt}X^{-1}$.

System a:

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \\
P(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} \\
&= (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 \\
&= (\lambda - 2)(\lambda - 3)
\end{aligned}
$$

$$\lambda_1 = 2, \quad \lambda_2 = 3$$

$$
\begin{aligned}
N(A - \lambda_1 I) &= N\left(\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\right) \\
&= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -x_1 + x_2 = 0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \\
N(A - \lambda_2 I) &= N\left(\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}\right) \\
&= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -2x_1 + x_2 = 0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}
\end{aligned}
$$

$$
\begin{aligned}
X &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \\
e^{Dt} &= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
y = e^{At} &= X e^{Dt} X^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2e^{2t} & -e^{2t} \\ -e^{3t} & e^{3t} \end{pmatrix} \\
&= \begin{pmatrix} 2e^{2t} - e^{3t} & -e^{2t} + e^{3t} \\ 2e^{2t} - 2e^{3t} & -e^{2t} + 2e^{3t} \end{pmatrix}
\end{aligned}
$$

System b:

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\
P(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2
\end{aligned}
$$

$$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i$$

$$
\begin{aligned}
N(A - \lambda_1 I) &= N\left(\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -ix_1 - x_2 = 0\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}\right\} \\
N(A - \lambda_2 I) &= N\left(\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : ix_1 - x_2 = 0\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix}\right\}
\end{aligned}
$$

$$X = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad X^{-1} = \frac{1}{2i}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

Using the fact that $e^{a+bi} = e^a(\cos b + i \sin b)$

$$e^{Dt} = \begin{pmatrix} e^t(\cos t + i\sin t) & 0 \\ 0 & e^t(\cos t - i \sin t) \end{pmatrix}$$

$$
\begin{aligned}
y = e^{At} &= Xe^{Dt}X^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^t(\cos t + i\sin t) & 0 \\ 0 & e^t(\cos t - i \sin t) \end{pmatrix}\frac{1}{2i}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \\
&= \frac{1}{2i}\begin{pmatrix} e^t(\cos t + i\sin t) & e^t(\cos t - i\sin t) \\ -ie^t(\cos t + i\sin t) & ie^t(\cos t - i\sin t) \end{pmatrix}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \\
&= \frac{1}{2i}\begin{pmatrix} ie^t(\cos t + i\sin t) + ie^t(\cos t - i\sin t) & -e^t(\cos t + i\sin t) + e^t(\cos t - i\sin t) \\ e^t(\cos t + i\sin t) - e^t(\cos t - i\sin t) & ie^t(\cos t + i\sin t) + ie^t(\cos t - i\sin t) \end{pmatrix} \\
&= \frac{1}{2i}\begin{pmatrix} 2ie^t\cos t & -2ie^t\sin t \\ 2ie^t\sin t & 2ie^t\cos t \end{pmatrix} \\
&= \begin{pmatrix} e^t\cos t & -e^t\sin t \\ e^t\sin t & e^t\cos t \end{pmatrix}
\end{aligned}
$$

System c:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 6 \\ 0 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 6 \\ 1 & 3-\lambda \end{vmatrix}$$
$$= (1-\lambda)((2-\lambda)(3-\lambda)-6) = (1-\lambda)(\lambda^2 - 5\lambda) = -\lambda^3 + 6\lambda^2 - 5\lambda$$
$$= -\lambda(\lambda^2 - 6\lambda + 5) = -\lambda(\lambda-1)(\lambda-5)$$

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 5$$

$$N(A - \lambda_1 I) = N\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{pmatrix}\right) = N\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}\right)$$
$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 + x_3 = 0,\ x_2 + 3x_3 = 0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \right\}$$
$$N(A - \lambda_2 I) = N\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 6 \\ 0 & 1 & 2 \end{pmatrix}\right) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$
$$N(A - \lambda_3 I) = N\left(\begin{pmatrix} -4 & 0 & 1 \\ 0 & -3 & 6 \\ 0 & 1 & -2 \end{pmatrix}\right) = N\left(\begin{pmatrix} 4 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\right)$$
$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 4x_1 - x_3 = 0,\ x_2 - 2x_3 = 0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 8 \\ 4 \end{pmatrix} \right\}$$

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 8 \\ -1 & 0 & 4 \end{pmatrix}, \quad X^{-1} = \frac{1}{20}\begin{pmatrix} 0 & 4 & -8 \\ 20 & -5 & 5 \\ 0 & 1 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$
$$e^{Dt} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}$$

$$y = e^{At} = Xe^{Dt}X^{-1}$$
$$= \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 8 \\ -1 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}\frac{1}{20}\begin{pmatrix} 0 & 4 & -8 \\ 20 & -5 & 5 \\ 0 & 1 & 3 \end{pmatrix}$$
$$= \frac{1}{20}\begin{pmatrix} 20e^t & e^{5t} - 5e^t + 4 & 3e^{5t} + 5e^t - 8 \\ 0 & 8e^{5t} + 12 & 24e^{5t} - 24 \\ 0 & 4e^{5t} - 4 & 12e^{5t} + 8 \end{pmatrix}$$

Question 2

Solve the following initial value problems:

a. $\begin{cases} -y_1 + 2y_2 = y_1' \\ \;\;2y_1 - \;\;y_2 = y_2' \end{cases}, \quad y_1(0) = 3, \quad y_2(0) = 1.$

b. $\begin{cases} \;\;y_1 - 2y_2 = y_1' \\ 2y_1 + \;\;y_2 = y_2' \end{cases}, \quad y_1(0) = 1, \quad y_2(0) = -2.$

Solution: Again, we use the fact that $y = e^{At}$, where A is the matrix of coefficients of the system. To solve the initial conditions, we use the fact that $y = e^{At}c$, where c is a vector of constants. Then, we can solve for c using the initial conditions.

System a:

$$
\begin{aligned}
A &= \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \\
P(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} \\
&= (-1-\lambda)^2 - 4 = \lambda^2 + 2\lambda - 3 \\
&= (\lambda+3)(\lambda-1) \\
\\
\lambda_1 &= -3, \quad \lambda_2 = 1
\end{aligned}
$$

$$
\begin{aligned}
N(A - \lambda_1 I) &= N\left(\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 + x_2 = 0\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} \\
N(A - \lambda_2 I) &= N\left(\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -x_1 + x_2 = 0\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}
\end{aligned}
$$

$$
\begin{aligned}
X &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} \\
e^{Dt} &= \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^t \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
y &= e^{At} = Xe^{Dt}X^{-1}c \\
&= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-3t} & 0 \\ 0 & e^t \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} c \\
&= \frac{1}{2}\begin{pmatrix} e^{-3t} & e^t \\ -e^{-3t} & e^t \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} c \\
&= \frac{1}{2}\begin{pmatrix} e^{-3t} + e^t & -e^{-3t} + e^t \\ -e^{-3t} + e^t & e^{-3t} + e^t \end{pmatrix} c
\end{aligned}
$$

$$
\begin{aligned}
y(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} c \\
\implies c &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= \frac{1}{2}\begin{pmatrix} e^{-3t} + e^t & -e^{-3t} + e^t \\ -e^{-3t} + e^t & e^{-3t} + e^t \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{-3t} + 2e^t \\ -e^{-3t} + 2e^t \end{pmatrix}
\end{aligned}
$$

System b:

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \\
P(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 5
\end{aligned}
$$

$$
\lambda_1,\ \lambda_2 = \frac{2 \pm \sqrt{4-20}}{2} \to \lambda_1 = 1 + 2i, \quad \lambda_2 = 1 - 2i
$$

$$
\begin{aligned}
N(A - \lambda_1 I) &= N\left(\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -ix_1 - x_2 = 0\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}\right\} \\
N(A - \lambda_2 I) &= N\left(\begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : ix_1 - x_2 = 0\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix}\right\}
\end{aligned}
$$

$$
X = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad X^{-1} = \frac{1}{2i}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix}
$$

Using the fact that $e^{a+bi} = e^a(\cos b + i\sin b)$

$$
e^{Dt} = \begin{pmatrix} e^t(\cos 2t + i\sin 2t) & 0 \\ 0 & e^t(\cos 2t - i\sin 2t) \end{pmatrix}
$$

Let $\alpha = \cos 2t + i\sin 2t,\ \beta = \cos 2t - i\sin 2t$

$$
\begin{aligned}
y = e^{At} &= Xe^{Dt}X^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^t\alpha & 0 \\ 0 & e^t\beta \end{pmatrix}\frac{1}{2i}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \\
&= \frac{1}{2i}\begin{pmatrix} e^t\alpha & e^t\beta \\ -ie^t\alpha & ie^t\beta \end{pmatrix}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \\
&= \frac{1}{2i}\begin{pmatrix} ie^t\alpha + ie^t\beta & -e^t\alpha + e^t\beta \\ e^t\alpha - e^t\beta & ie^t\alpha + ie^t\beta \end{pmatrix} \\
&= \frac{1}{2i}\begin{pmatrix} ie^t(\cos 2t + i\sin 2t) + ie^t(\cos 2t - i\sin 2t) & -e^t(\cos 2t + i\sin 2t) + e^t(\cos 2t - i\sin 2t) \\ e^t(\cos 2t + i\sin 2t) - e^t(\cos 2t - i\sin 2t) & ie^t(\cos 2t + i\sin 2t) + ie^t(\cos 2t - i\sin 2t) \end{pmatrix} \\
&= \frac{1}{2i}\begin{pmatrix} 2ie^t\cos(2t) & -2ie^t\sin(2t) \\ 2ie^t\sin(2t) & 2ie^t\cos(2t) \end{pmatrix} \\
&= e^t\begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
y(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} &= e^0\begin{pmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{pmatrix}c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}c \\
&\implies c = \begin{pmatrix} 1 \\ -2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= e^t\begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \\
&= e^t\begin{pmatrix} \cos(2t) + 2\sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}
\end{aligned}
$$

Question 3

In each of the following, "diagonalize" the matrix X and use it to compute A^{-1}, A^4, e^A.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix}$$

Solution: Use the fact that $A = XDX^{-1}$, where X is the matrix of eigenvectors of A and D is the diagonal matrix of eigenvalues of A. Then, $A^{-1} = XD^{-1}X^{-1}$, $A^4 = XD^4X^{-1}$, and $e^A = Xe^DX^{-1}$.

First matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

$$N(A - \lambda_1 I) = N\left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right) = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -x_1 + x_2 = 0\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

$$N(A - \lambda_2 I) = N\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 + x_2 = 0\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

$$X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad X^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$A^{-1} = XD^{-1}X^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^4 = XD^4X^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$e^A = Xe^DX^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e & e^{-1} \\ e & -e^{-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e + e^{-1} & e - e^{-1} \\ e - e^{-1} & e + e^{-1} \end{pmatrix}$$

Second matrix:

$$
\begin{aligned}
A &= \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \\
P(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} 2-\lambda & 2 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{vmatrix} \\
&= (-1-\lambda)(2-\lambda)(1-\lambda) \\
&\lambda_1 = -1, \quad \lambda_2 = 2, \quad \lambda_3 = 1
\end{aligned}
$$

$$
\begin{aligned}
N(A-\lambda_1 I) &= N\left(\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\right) = N\left(\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right) \\
&= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 3x_1 + x_2 = 0,\ x_2 + x_3 = 0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} \right\} \\
N(A-\lambda_2 I) &= N\left(\begin{pmatrix} 0 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{pmatrix}\right) = N\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \\
N(A-\lambda_3 I) &= N\left(\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}\right) = N\left(\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right) = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}
\end{aligned}
$$

$$
X = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 0 & -1 \\ 3 & 0 & 0 \end{pmatrix}, \quad X^{-1} = \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 6 & 5 \\ 0 & -3 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
A^{-1} = XD^{-1}X^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 0 & -1 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 6 & 5 \\ 0 & -3 & -3 \end{pmatrix}
$$

$$
A^4 = XD^4X^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 0 & -1 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 6 & 5 \\ 0 & -3 & -3 \end{pmatrix}
$$

$$
e^A = Xe^DX^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 0 & -1 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-1} & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e \end{pmatrix} \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 6 & 5 \\ 0 & -3 & -3 \end{pmatrix}
$$

Third matrix:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 4-\lambda & -2 \\ 3 & 6 & -3-\lambda \end{vmatrix}$$

Question 4

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$$

a. Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.

b. Factor A into QR.

c. Use the above to solve the system $Ax = b$.

Solution:

Question 5

Let $\{x_1, x_2, x_3\} = \{(0,1,0), (2,1,2), (0,0,1)\}$, be a basis of $\mathbb{R}^3$.

a. Use the Gram-Schmidt process to obtain an orthonormal basis.

b. Let $b := (1,1,1)$. Compute the projection of b onto $\text{span}\{x_1, x_2\}$ and to $\text{span}\{x_3, x_2\}$.

Solution:

Question 6

Consider the vector space $C[0,1]$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

a. Find an orthonormal basis of the subspace E spanned by $1, x, x^2$.

b. Compute the length of $2x^2 + 3$.

c. Compute the projection of e^x onto E

Solution:

Question 7

Find the orthogonal complement of the subspace of $\mathbb{R}^4$ spanned by $(1,1,1,1), (1,-1,1,-1)$.

Solution:

Question 8

For each of the following systems $Ax = b$ find all least squares solutions:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Solution: