1. If it is possible, perform the multiplication:

a.

$$\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 & -1 \\ -3 & 0 & -6 & 3 \\ 2 & 0 & 4 & -2 \end{pmatrix}$$

b.

$$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1+0 & 0-1 & 3+1 \\ 2+0 & 0+1 & 6-1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 4 \\ 2 & 1 & 5 \end{pmatrix}$$

c.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & -3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \end{pmatrix} \quad \text{not possible}$$

2. Let A, B be two $n \times n$ matrices. Is it true that

$$A^2 - B^2 = (A - B)(A + B) \ ?$$

If yes, prove it. If no, provide a counterexample.

$$A^2 - B^2 = (A-B)(A+B)$$

$$A^2 - B^2 = A^2 + AB - BA - B^2$$

$$0 = AB - BA$$

$$BA = AB$$

let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$AB = \begin{pmatrix} 1+2 & 1+2 \\ 3+4 & 3+4 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 7 & 7 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1+3 & 2+4 \\ 1+3 & 2+4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 4 & 6 \end{pmatrix}$$

Thus, $BA \neq AB$ and the statement is false

3. Let A be an $n \times n$ matrix such that $A^2 = A$. Is it true that $A = I_n$? If yes, prove it. If no, provide a counterexample.

$$A^2 = A$$

$$A^2 - A = 0$$

$$A(A - I) = 0$$

$$A = I, 0$$

The statement is false, A can be an $n \times n$ zero matrix

4. Let $0 < \theta < \frac{\pi}{2}$. Let R the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Show that R^{-1} exists and $R^{-1} = R^T$.

$\det(R) = (\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta) = \cos^2\theta + \sin^2\theta = 1$

Since the determinant of matrix R is non-zero, we know that R is non-singular and has an inverse

$R^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

For 2×2 matricies $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore, $R^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, which is equivalent to R^T

5. For all the matrices below, compute the inverse (if exists):

$$\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$\det\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} = 4(1) - 3(1) = 1 \neq 0 \quad \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$

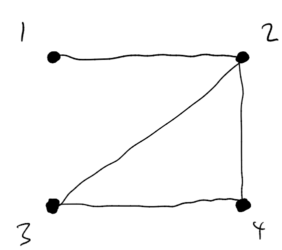
$\det\begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix} = 2(6) - (-3)(-4) = 0 \quad \begin{pmatrix} 2 & -3 \\ -4 & 6 \end{pmatrix}$ is a singular matrix

$\det\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = 1(-1) - 2(2) = -5 \neq 0 \quad \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -2 \\ -2 & 1 \end{pmatrix}$

6. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Draw a graph that has A as its adjacency matrix. Compute the number of walks of length 3 from the vertex "2" to the vertex "4" in two ways: By counting on the graph that you draw and by computing the matrix A^3.



$A^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 1 & 1 \\ 3 & 2 & 4 & 4 \\ 1 & 4 & 2 & 3 \\ 1 & 4 & 3 & 2 \end{pmatrix}$

There are 4 walks of length 3 from vertex 2 to vertex 4