1 Main Idea

In this problem we are given a road where we define the western point to be at position 0 and the eastern point to be at the road's length of $L > 0$. There are n houses at positions $x_1, x_2, \ldots, x_n$ where $0 \leq x_1 < x_2 < \ldots < x_n \leq L$. The goal is to place the minimum amount of phone stations such that each house is within k distance of a phone station.

The main idea of the algorithm that solves this efficiently is to find the position of the westernmost house, and then place a phone station as far east as possible while still being within k distance of the westernmost house. Then, move to the next house that is not within k distance of the phone station and repeat the process. The upshot of this algorithm is that a greedy approach is appropriate because the globally optimal solution contains the locally optimal solutions. Specifically, the set of phone stations that cover all houses must contain the phone station that is as far east as possible while still being within k distance of the westernmost house.

2 Pseudocode

Algorithm 1: Phone Station Placement

Input: Array of house positions $x_1, x_2, \ldots, x_n$; road length L; distance k

Output: Array of phone station positions $p_1, p_2, \ldots, p_m$

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P = ∅ ;                                  // Set of phone station positions
i = 1;
while i ≤ n do
    p_i = min(x_i + k, L);
    P = P ∪ {p_i};
    while i ≤ n and x_i ≤ p_i + k do
        i = i + 1 ;                      // Find the next house
    end
end
return P;
```

3 Proof of Correctness

We will use induction to prove the correctness of the algorithm.

3.1 Base Case

For $n = 1$, the algorithm will place a phone station at $\min(x_1 + k, L)$. This is clearly optimal, as there is only one house to cover.

3.2 Inductive Hypothesis

Assume that the algorithm is correct for all $n \leq m$.

3.3 Inductive Step

We want to show that the algorithm is correct for $n = m + 1$. This algorithm will place the first phone station p_1 at $\min(x_1 + k, L)$. Thus, this will cover all the houses between x_1 and x_i where $x_i \leq p_1 + k < x_{i+1}$. At this point, there are two cases to consider:

1. If $x_{i+1} > L$, then p_1 is the only phone station needed. This is clearly optimal, as no placement further east can cover the first house.
2. If $x_{i+1} \leq L$, then we need to cover the houses from x_{i+1} to x_{m+1}. By the inductive hypothesis, the algorithm will place the phone stations optimally to cover these houses.

Now, seeking a contradiction, suppose that the algorithm is not optimal. In other words, there exists a solution that uses fewer phone stations. However, that would require either of the following:

1. The first phone station is places further west of p_1. This would cover fewer houses with the same number of phone stations, thus a contradiction.
2. The phone stations are placed further east of p_1. This would not cover the first house at all, making the solution incorrect.

Therefore, the algorithm is correct and optimal for any number of houses.

4 Runtime Analysis

Initializing the set of phone stations P and index i takes $O(1)$ time. The outer loop runs at most n times. Calculating the position of the phone station p_i and adding it to the set P takes $O(1)$ time. The inner loop runs at most n times across all iterations of the outer loop, since they share the same index i, meaning that each house is visited at most once. Therefore, this algorithm runs in $O(n)$ time.