Question 1

Determine whether the following subsets are subspaces:

Part a

$S_1 := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 = \sqrt{123}x_2\}$

Answer: This is a subspace of $\mathbb{R}^2$ since it is a straight line that passes through the origin.

Proof. Let (a, b) and (c, d) be two elements of S_1. We want to show that $(a, b) + (c, d) \in S_1$ and $n(a, b) \in S_1 \forall n \in \mathbb{R}$. Using the definition of S_1, we have that $a = \sqrt{123}b$ and $c = \sqrt{123}d$. Adding elements (a, b) and (c, d), we have that $(a, b) + (c, d) = (a + c, b + d)$ We can then substitute in the values of a and c to get $(a + c, b + d) = (\sqrt{123}b + \sqrt{123}d, b + d)$ This can then be factored to $(\sqrt{123}(b + d), b + d)$ Since this satisfies the definition of S_1, we have that $(a, b) + (c, d) \in S_1$. To show that $n(a, b) \in S_1 \forall n \in \mathbb{R}$, we can use the definition of S_1 again. The element (a, b) can be written as $(\sqrt{123}b, b)$. Multiplying this by n gives us $(n\sqrt{123}b, nb)$. Since this satisfies the definition of S_1, we have that $n(a, b) \in S_1 \forall n \in \mathbb{R}$. □

Part b

$S_2 := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1x_2 = 1\}$

Answer: This is not a subspace of $\mathbb{R}^2$ since it does not satisfy the addition property.

Proof. Let (a, b) and (c, d) be two elements of S_2. Seeking a contradiction, lets assume that $(a, b)+(c, d) \in S_2$. Since we can write $(a, b)+(c, d)$ as $(a+c, b+d)$, our assumption would imply that $(a + c)(b + d) = 1$. Expanding this, we get $ab + ad + bc + cd = 1$. It is given that $ab = 1$ and $cd = 1$, so we can substitute these in to get $1 + ad + bc + 1 = 1$. This can be simplified to $ad + bc = -1$. Since a can be rewritten as $\frac{1}{b}$ and c can be rewritten as $\frac{1}{d}$, we can substitute these in to get $\frac{d}{b} + \frac{b}{d} = -1$. Multiplying the entire equation by bd gives us $d^2 + b^2 = -bd$. Since d^2 and b^2 must be positive, this equation cannot be true, and thus we have reached a contradiction. Therefore, $(a, b) + (c, d) \notin S_2$, so S_2 is not a subspace of $\mathbb{R}^2$. □

Part c

$S_3 := \{\text{the set of singular } 2 \times 2 \text{ matrices}\}$

Answer: This is not a subspace of $\mathbb{R}^{2\times 2}$ since it does not satisfy the addition property.

Counterexample: Let matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ where A, B $\in S_3$. $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is not a singular matrix.

Part d

Let A be a fixed (but arbitrary) 2×2 matrix.
$S_4 := \{B \in \mathbb{R}^{2\times 2} : BA = 0\}$
Answer: This is a subspace of $\mathbb{R}^{2\times 2}$.

Proof. Let J and K be two arbitrary elements in S_4. We want to show that J, $K \in S_4 \to J + K \in S_4$ and $aJ \in S_4 \forall a \in \mathbb{R}$. In order for $J + K$ to be in S_4, we must have that $(J + K)A = 0$. We can rewrite this as $JA + KA = 0$. Since J and K are in S_4, we know that $JA = 0$ and $KA = 0$. Substituting these in, we get $0 + 0 = 0$, which is true. Therefore, $J + K \in S_4$. Additionally, in order for aJ to be in S_4, we must show that $\forall a \in \mathbb{R}, J \in S_4 \to aJ \in S_4$. To do this, we must verify the validity of $(aJ)A = 0$. Since matrix multiplication is associative, we can rewrite this as $a(JA) = 0$. Since J is in S_4, we know that $JA = 0$. Substituting this in, we get $a(0) = 0$, which is true. Therefore, $aJ \in S_4$, so S_4 is a subspace of $\mathbb{R}^{2\times 2}$. □

Part e

$S_5 := \{$the set of all polynomials of degree 2 or 4$\}$
Answer: This is not a subspace of $\mathbb{P}_n$.
Counterexample:
Let $p(x), q(x) \in S_5$. Suppose that $p(x) = x^2$ and $q(x) = -x^2$.
Then $p(x) + q(x) = 0$, which is not a polynomial of degree 2 or 4.

Part f

$S_6 := \{$the set of upper triangular 2×2 matrices$\}$
Answer: This is a subspace of $\mathbb{R}^{2\times 2}$.

Proof. Let A and B be two arbitrary elements in S_6. To prove that S_6 is a subspace of $\mathbb{R}^{2\times 2}$, we must show that A, $B \in S_6 \to A + B \in S_6$ and $\forall n \in \mathbb{R}, nA \in S_6$. Given that A and B are in S_6, we know that A and B are upper triangular matrices. Writing these out, we have:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$
$$B = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}$$

Since matrix addition is element-wise, we can write $A + B$ as:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ 0 & a_{22} + b_{22} \end{bmatrix}$$

It is clear in this form that $A + B$ is an upper triangular matrix, so $A + B \in S_6$.

To show that $\forall n \in \mathbb{R}, nA \in S_6$, we must show that nA is an upper triangular matrix. We know that nA is an upper triangular matrix if nA is of the form:

$$nA = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

Since scalar multiplication is distributed element-wise in matricies, we can write nA as:

$$nA = \begin{bmatrix} na_{11} & na_{12} \\ 0 & na_{22} \end{bmatrix}$$

In this form, it is clear that nA is an upper triangular matrix. Therefore, $nA \in S_6$, so S_6 is a subspace of $\mathbb{R}^{2\times 2}$. $\square$

Part g

$S_7 := \{p \in \mathbb{P}_4 : p(0) = 0\}$

Answer: This is a subspace of $\mathbb{P}_4$.

Proof. Let $f(x)$ and $g(x)$ be two arbitrary elements in S_7. To prove that S_7 is a subspace of $\mathbb{P}_4$, we must show that $f(x),\ g(x) \in S_7 \to f(x) + g(x) \in S_7$ and $\forall n \in \mathbb{R}, nf(x) \in S_7$. Given that $f(x)$ and $g(x)$ are in S_7, we know that $f(0) = 0$ and $g(0) = 0$, which is by definition of S_7. This means that $f(x)$ and $g(x)$ are of the form:

$$f(x) = a_1x^4 + a_2x^3 + a_3x^2 + a_4x + 0$$
$$g(x) = b_1x^4 + b_2x^3 + b_3x^2 + b_4x + 0$$

This means we can write $f(x) + g(x)$ as:

$$f(x) + g(x) = (a_1 + b_1)x^4 + (a_2 + b_2)x^3 + (a_3 + b_3)x^2 + (a_4 + b_4)x + 0$$

In this form it is clear that $f(x) + g(x)$ satisfies the condition that $p(0) = 0$, so $f(x) + g(x) \in S_7$.

To show that $\forall n \in \mathbb{R}, nf(x) \in S_7$, we can follow a similar process. We know that $nf(x)$ is of the form:

$$nf(x) = na_1x^4 + na_2x^3 + na_3x^2 + na_4x + 0$$

Regardless of the value of n, $nf(x)$ will always satisfy the condition that $p(0) = 0$, so $nf(x) \in S_7$. Therefore, S_7 is a subspace of $\mathbb{P}_4$. $\square$

Question 2

Find the null space of the following matrices:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -2 & 2 & 1 \\ 2 & 4 & -4 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 3 \\ 4 & 3 & 0 \end{bmatrix}$$

Matrix A:

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{6} \\ 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

$$\begin{cases} x_1 + \frac{1}{6}x_3 = 0 \\ x_2 - \frac{1}{3}x_3 = 0 \end{cases} \rightarrow \begin{cases} x_1 = -\frac{1}{6}x_3 \\ x_2 = \frac{1}{3}x_3 \end{cases} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6}x_3 \\ \frac{1}{3}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

$$N(A) = \left\{ a \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ 1 \end{bmatrix} : a \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -\frac{1}{6} \\ \frac{1}{3} \\ 1 \end{bmatrix} \right\}$$

Matrix B:

$$\begin{bmatrix} -1 & -2 & 2 & 1 \\ 2 & 4 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim -x_1 - 2x_2 + 2x_3 + x_4 = 0$$

$$\begin{cases} x_1 = -2x_2 + 2x_3 + x_4 \\ x_2 = x_2 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 + 2x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$N(B) = \left\{ a \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Matrix C:

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 3 \\ 4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 3 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -24 \end{bmatrix}$$

$$\begin{cases} x_1 + 3x_3 = 0 \\ x_2 + 4x_3 = 0 \\ -24x_3 = 0 \end{cases} \rightarrow x_1 = x_2 = x_3 = 0$$

$$N(C) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Question 3

Show that the following matrices form a spanning set for $\mathbb{R}^{2\times 2}$. Also, show that these matrices are linearly independent.

$$A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Proposition: The matricies A_{11}, A_{12}, A_{21}, and A_{22} form a spanning set for $\mathbb{R}^{2\times 2}$

Proof. Suppose we have some arbitrary A in $\mathbb{R}^{2\times 2}$. To show that the matricies A_{11}, A_{12}, A_{21}, and A_{22} form a spanning set for $\mathbb{R}^{2\times 2}$, we must show that A can be written as a linear combination of A_{11}, A_{12}, A_{21}, and A_{22}. We can write A as:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Which means we need to find scalars x_1, x_2, x_3, and x_4 such that:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

From here we can see that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

Therefore, $x_1 = a$, $x_2 = b$, $x_3 = c$, and $x_4 = d$. Since we can write A as a linear combination of A_{11}, A_{12}, A_{21}, and A_{22}, we have proven that the matricies A_{11}, A_{12}, A_{21}, and A_{22} form a spanning set for $\mathbb{R}^{2\times 2}$. $\square$

Proposition: The matricies A_{11}, A_{12}, A_{21}, and A_{22} are linearly independent.

Proof. A set of vectors is linearly independent if and only if the only solution to $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ is $c_1 = c_2 = \cdots = c_n = 0$. This means we have:

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

From this we have the following system:

$$\begin{cases} c_1 \times 1 = 0 \\ c_2 \times 1 = 0 \\ c_3 \times 1 = 0 \\ c_4 \times 1 = 0 \end{cases}$$

From this it is clear that the only solution to the system is $c_1 = c_2 = c_3 = c_4 = 0$. This is exactly the definition of linear independence, so we have proven that the matricies A_{11}, A_{12}, A_{21}, and A_{22} are linearly independent. $\square$

Question 4

Let x_1, x_2, and x_3 be linearly independent vectors in $\mathbb{R}^n$. Let

$$y_1 = x_1 + x_2, \quad y_2 = x_2 + x_3, \quad y_3 = x_3 + x_1.$$

Decide if y_1, y_2, and y_3 are linearly independent or not.
Answer: The vectors y_1, y_2, and y_3 are linearly independent.

Proof. A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is linearly independent if and only if the only solution to $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ is $c_1 = c_2 = \cdots = c_n = 0$. This means we have the following equation:

$$c_1y_1 + c_2y_2 + c_3y_3 = 0$$

For the set of vectors $\{y_1, y_2, y_3\}$ to be linearly independent, we must show that the only solution to this equation is $c_1 = c_2 = c_3 = 0$. Using the definitions of y_1, y_2, and y_3, we can rewrite this equation as:

$$c_1(x_1 + x_2) + c_2(x_2 + x_3) + c_3(x_3 + x_1) = 0$$

Distributing the coefficients, we get:

$$c_1x_1 + c_1x_2 + c_2x_2 + c_2x_3 + c_3x_3 + c_3x_1 = 0$$

Factoring out the x_i terms, we get:

$$(c_1 + c_3)x_1 + (c_1 + c_2)x_2 + (c_2 + c_3)x_3 = 0$$

Since it is given that x_1, x_2, and x_3 are linearly independent, the only way for this equation to equal zero is if $c_1 + c_3 = 0$, $c_1 + c_2 = 0$, and $c_2 + c_3 = 0$. This means we have the following system:

$$\begin{cases} c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \end{cases}$$

To solve this system we can use an augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Using row operations, we get the following equivalent matricies:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

From this we can see that the only solution to the system is $c_1 = c_2 = c_3 = 0$. Since this is the only solution to the system, we have proven that the vectors y_1, y_2, and y_3 are linearly independent. $\square$