Question 1

Proof. Let $f : X \to Y$ and $A_1, A_2 \subseteq X$. Assume $y \in f[A_1 \cup A_2]$. Then,

$$\begin{aligned} y \in f[A_1 \cup A_2] &\iff y = f(x), x \in A_1 \cup A_2 \\ &\iff y = f(x), x \in A_1 \vee x \in A_2 \\ &\iff y \in f[A_1] \vee y \in f[A_2] \\ &\iff y \in f[A_1] \cup f[A_2]. \end{aligned}$$

Thus, $f[A_1 \cup A_2] = f[A_1] \cup f[A_2]$. □

Question 2

$f^{-1}([-3, 5]) = [1009.5, 1013.5]$

Question 3

$D^{-1}(\{4x^3\}) = \{x^4\}$

Question 4

1. $s^{-1}(\{4\}) = \{(0, 4), (1, 3), (2, 2), (3, 1), (4, 0)\}$
2. $s^{-1}(\{1\}) = \{(0, 1), (1, 0)\}$

Question 5

Proof. Let $f : X \to Y$ and $B_1, B_2 \subseteq Y$. Assume $x \in f^{-1}[B_1 \cup B_2]$. Then,

$$\begin{aligned} x \in f^{-1}[B_1 \cup B_2] &\iff f(x) \in B_1 \cup B_2 \\ &\iff f(x) \in B_1 \vee f(x) \in B_2 \\ &\iff x \in f^{-1}[B_1] \vee x \in f^{-1}[B_2] \\ &\iff x \in f^{-1}[B_1] \cup f^{-1}[B_2]. \end{aligned}$$

Thus, $f^{-1}[B_1 \cup B_2] = f^{-1}[B_1] \cup f^{-1}[B_2]$. □

Question 6

Part a

Proof. Let $f : X \to Y$ and $A \subseteq X$. Assume $x \in A$. Then $f(x) \in f[A]$. Since $x \in f^{-1}[f[A]]$, $A \subseteq f^{-1}[f[A]]$. □

Part b

Proof. Let $f : X \to Y$ and $A \subseteq X$. Assume that f is injective. From part a, we know that $A \subseteq f^{-1}[f[A]]$. Now let $x \in f^{-1}[f[A]]$. Then, $f(x) \in f[A]$. Thus, there exists $a \in A$ such that $f(x) = f(a)$. Since f is injective, $x = a$. Since $x = a$, $x \in A$. Therefore, $f^{-1}[f[A]] \subseteq A$, so $f^{-1}[f[A]] = A$. $\square$

Question 7

Part a

$$\begin{aligned} 1207 &= 569 \cdot 2 + 69 \\ 569 &= 69 \cdot 8 + 17 \\ 69 &= 17 \cdot 4 + 1 \end{aligned}$$

Thus, $\gcd(1207, 569) = 1$.

Part b

$$\begin{aligned} \gcd(1207, 569) &= 1 \\ &= 69 - 17 \cdot 4 \\ &= 69 - (569 - 69 \cdot 8) \cdot 4 \\ &= 69 - 569 \cdot 4 + 69 \cdot 32 \\ &= 69 \cdot 33 - 569 \cdot 4 \\ &= (1207 - 569 \cdot 2) \cdot 33 - 569 \cdot 4 \\ &= 1207 \cdot 33 - 569 \cdot 66 - 569 \cdot 4 \\ &= 1207 \cdot 33 - 569 \cdot 70. \end{aligned}$$

Thus, $x = 33$ and $y = -70$.