5th Homework — MATH 304 — Fall 2023
— Due October 19th —

1. Consider the following collections of polynomials in P_2:

   a. $p_1(x) = 1,\ p_2(x) = x + 1,\ p_3(x) = x^2.$

   b. $p_1(x) = x - 1,\ p_2(x) = x + 1,\ p_3(x) = x^2 - 1$

   c. $p_1(x) = x^2 - 1,\ p_2(x) = x^2 + 1,\ p_3(x) = x^2.$

   Decide in each case if these vectors are linearly independent.

   Write the dimension of the subspace $S := \operatorname{span}\{p_1, p_2, p_3\}$ in each case.

   In which case(s) we have that $S = P_2$? Explain your answer.

2. Consider the following collections of smooth functions $[0, 1]$:

   a. $f_1(x) = x^2,\ f_2(x) = x^{\frac{1}{2}}$

   b. $f_1(x) = \cos(x),\ f_2(x) = \sin(x).$

   c. $f_1(x) = 1,\ f_2(x) = \frac{e^x + e^{-x}}{2},\ f_3(x) = \frac{e^x - e^{-x}}{2}.$

   Decide in each case if these vectors (functions) are linearly independent.

3. Find the dimension of the space spanned by the functions

$$1,\ \cos(2x)\ \cos^2(x).$$

4. For each of the following find the transition matrix corresponding to the change of basis from $\{u_1, u_2\}$ to the standard one $\{e_1, e_2\}$.

   a. $u_1 = (1, 1)^T,\ u_2 = (-1, 1)^T.$

   b. $u_1 = (1, 2)^T,\ u_2 = (2, 5)^T.$

   c. $u_1 = (0, 1)^T,\ u_2 = (1, 0)^T.$

   Let

$$v_1 = (3, 2)^T,\ v_2 = (4, 3)^T$$

   For each of the basis above find the transition matrix from $[v_1, v_2]$ to $[u_1, u_2]$.

   Let

$$x = (2, 4)^T,\ y = (1, 1)^T,\ z = (0, 10)$$

   Find the coordinates of x, y, z with respect to each of the basis mentioned above.

Show your work in each exercise.