Question 1

Consider the vectors $\{x_1, x_2, x_3\}$ of $\mathbb{R}^4$ where

$$x_1 = (4, 2, 2, 1)^T,\ x_2 = (2, 0, 0, 2)^T,\ x_3 = (1, 1, -1, 1)^T$$

Let $S := \text{span}\{x_1, x_2, x_3\}$. Use the Gram-Schmidt process to obtain an orthonormal basis for S.

Solution:

$$\begin{aligned}
u_1 &= \frac{x_1}{\|x_1\|_2} = \frac{1}{5}(4, 2, 2, 1)^T \\
p_1 &= \langle x_2, u_1 \rangle u_1 = 2 \times \frac{1}{5}(4, 2, 2, 1)^T = \frac{2}{5}(4, 2, 2, 1)^T \\
u_2 &= \frac{x_2 - p_1}{\|x_2 - p_1\|_2} = \frac{(2 - \frac{8}{5}, 0 - \frac{4}{5}, 0 - \frac{4}{5}, 2 - \frac{2}{5})^T}{\sqrt{(2 - \frac{8}{5})^2 + (0 - \frac{4}{5})^2 + (0 - \frac{4}{5})^2 + (2 - \frac{2}{5})^2}} \\
&= \frac{1}{2}\left(\frac{2}{5}, -\frac{4}{5}, -\frac{4}{5}, \frac{8}{5}\right)^T = \frac{1}{5}(1, -2, -2, 4)^T \\
p_2 &= \langle x_3, u_1 \rangle u_1 + \langle x_3, u_2 \rangle u_2 \\
&= \frac{1}{5}(4 + 2 - 2 + 1) \times \frac{1}{5}(4, 2, 2, 1)^T + \frac{1}{5}(1 - 2 + 2 + 4) \times \frac{1}{5}(1, -2, -2, 4)^T \\
&= \frac{1}{5}(4, 2, 2, 1)^T + \frac{1}{5}(1, -2, -2, 4)^T = \frac{1}{5}(5, 0, 0, 5)^T \\
u_3 &= \frac{x_3 - p_2}{\|x_3 - p_2\|_2} = \frac{(1 - \frac{5}{5}, 1 - 0, -1 - 0, 1 - \frac{5}{5})^T}{\sqrt{(1 - \frac{5}{5})^2 + (1 - 0)^2 + (-1 - 0)^2 + (1 - \frac{5}{5})^2}} \\
&= \frac{1}{\sqrt{2}}(0, 1, -1, 0)^T
\end{aligned}$$

Thus we have $S := \text{span}\{x_1, x_2, x_3\} = \text{span}\{u_1, u_2, u_3\}$ where

$$u_1 = \frac{1}{5}(4, 2, 2, 1)^T,\ u_2 = \frac{1}{5}(1, -2, -2, 4)^T,\ u_3 = \frac{1}{\sqrt{2}}(0, 1, -1, 0)^T$$

Question 2

Find the orthogonal complement of the subspace of $\mathbb{R}^3$ spanned by $(1,2,1)^T$, $(1,-1,2)^T$.

Solution: Given the subspace $S := \text{span}\{(1,2,1)^T, (1,-1,2)^T\}$, we want to find the orthogonal complement $S^\perp$. The orthogonal complement is the set of all vectors in the vector space that are orthogonal to every vector in S. Thus, by definition, we have

$$S^\perp = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;\middle|\; \left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle = 0 \,,\; \left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\rangle = 0 \right\}$$

To find x_1, x_2, x_3, we can solve the system of equations

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_1 - x_2 + 2x_3 = 0 \end{cases}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 1 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \end{array} \right)$$

$$\sim \begin{cases} x_1 = -\frac{5}{3}x_3 \\ x_2 = \frac{1}{3}x_3 \end{cases}$$

Thus, we have

$$S^\perp = \left\{ \begin{pmatrix} -\frac{5}{3}x_3 \\ \frac{1}{3}x_3 \\ x_3 \end{pmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} -\frac{5}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} \right\}$$

Question 3

Let A be an $m \times n$ matrix. Show that A and $A^T A$ have the same rank. Show that

$$N(A^T A) = N(A)$$

Solution:

Proof. We have that A is an $m \times n$ matrix. To show that $N(A^T A) = N(A)$, we need to show that $N(A^T A) \subseteq N(A)$ and $N(A) \subseteq N(A^T A)$.

Let $x \in N(A)$. By definition of the null space, we have $Ax = 0$. Then we have $A^T Ax = A^T 0 = 0$. Again, by definition of the null space, we have $x \in N(A^T A)$. Since $x \in N(A) \implies x \in N(A^T A)$, we have $N(A) \subseteq N(A^T A)$.

Now let $x \in N(A^T A)$. By definition of the null space, $A^T Ax = A^T(Ax) = 0$. Therefore, $Ax \in N(A^T)$. However, Ax is also in the column space of A, which means $Ax \in R(A)$. The Fundamental Theorem of Linear Algebra tells us that $N(A^T) \perp R(A)$. The only way for Ax to be in both $N(A^T)$ and $R(A)$ is if $Ax = 0$, since the only vector that is orthogonal to itself is the zero vector. Since we have that $Ax = 0$, that implies that $x \in N(A)$. Since $x \in N(A^T A) \implies x \in N(A)$, we have $N(A^T A) \subseteq N(A)$.

Now that we have shown that $N(A^T A) \subseteq N(A)$ and $N(A) \subseteq N(A^T A)$, we can conclude that $N(A^T A) = N(A)$.

To show that A and $A^T A$ have the same rank, we can use the rank-nullity theorem. We have that $\text{rank}(A) + \text{nullity}(A) = n$, so $\text{rank}(A) = n - \text{nullity}(A)$. Matrix A is an $m \times n$ matrix, so its rank is given by $n - \text{nullity}(A)$. Similarly, matrix $A^T A$ is an $n \times n$ matrix, so its rank is given by $n - \text{nullity}(A^T A)$. We have already shown that $N(A^T A) = N(A)$, so $\text{nullity}(A^T A) = \text{nullity}(A)$. Therefore, we have that $\text{rank}(A) = \text{rank}(A^T A)$. $\square$

Question 4

Let A be an $m \times n$ matrix and $\text{rank}(A) = r$. What are the dimensions of $N(A)$ and $N(A^T)$?

Solution: By the rank-nullity theorem, we know that $\text{rank}(A) + \text{nullity}(A) = n$. Thus, we have

$$\text{nullity}(A) = n - \text{rank}(A) = n - r$$

Since taking the transpose of a matrix does not change the number of linearly independent rows or columns, A and A^T have the same rank. Therefore,

$$\text{nullity}(A^T) = m - \text{rank}(A^T) = m - r$$

Question 5

For each of the following systems $Ax = b$ find all least squares solutions.

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 4 \\ -1 & 0 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \ b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Solution: According to the Least Squares Theorem, the solution to the least squares problem is given by the solution to the normal equations

$$A^T A\hat{x} = A^T b$$

where $\hat{x}$ is the least squares solution. If A has full rank, then $A^T A$ is invertible and the solution is given by

$$\hat{x} = (A^T A)^{-1} A^T b$$

For the first system, we can do the following:

$$\begin{aligned}
\hat{x} &= \left(\begin{pmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 4 \\ -1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 11 & 13 \\ 13 & 17 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\
&= \frac{1}{18} \begin{pmatrix} 17 & -13 \\ -13 & 11 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} \frac{8}{9} \\ \frac{-4}{9} \end{pmatrix}
\end{aligned}$$

For the second system, we have:

$$\begin{aligned}
\hat{x} &= \left(\begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 5 & 5 & 2 \\ 5 & 6 & 4 \\ 2 & 4 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \\
&= \frac{1}{6} \begin{pmatrix} 20 & -22 & 8 \\ -22 & 26 & -10 \\ 8 & -10 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{-1}{3} \\ \frac{2}{3} \\ \frac{1}{6} \end{pmatrix}
\end{aligned}$$

Question 6

Consider the basis $\{x_1, x_2, x_3\}$ of $\mathbb{R}^3$ where

$$x_1 = (1, 2, -2)^T,\ x_2 = (4, 3, 2)^T,\ x_3 = (1, 2, 1)^T$$

Use the Gram-Schmidt process to obtain an orthonormal basis.

Solution:

$$u_1 = \frac{x_1}{\|x_1\|_2} = \frac{1}{3}(1, 2, -2)^T$$
$$p_1 = \langle x_2, u_1\rangle u_1 = (4 \times \frac{1}{3} + 3 \times \frac{2}{3} + 2 \times \frac{-2}{3}) \times \frac{1}{3}(1, 2, -2)^T = \frac{1}{3}(2, 4, -4)^T$$
$$u_2 = \frac{x_2 - p_1}{\|x_2 - p_1\|_2} = \frac{(4 - \frac{2}{3}, 3 - \frac{4}{3}, 2 - \frac{-4}{3})^T}{\sqrt{(4 - \frac{2}{3})^2 + (3 - \frac{4}{3})^2 + (2 - \frac{-4}{3})^2}}$$
$$= \frac{1}{5}(\frac{10}{3}, \frac{5}{3}, \frac{10}{3})^T = \frac{1}{3}(2, 1, 2)^T$$
$$p_2 = \langle x_3, u_1\rangle u_1 + \langle x_3, u_2\rangle u_2$$
$$= (1 \times \frac{1}{3} + 2 \times \frac{2}{3} + 1 \times \frac{-2}{3}) \times \frac{1}{3}(1, 2, -2)^T + (1 \times \frac{2}{3} + 2 \times \frac{1}{3} + 1 \times \frac{2}{3}) \times \frac{1}{3}(2, 1, 2)^T$$
$$= \frac{1}{3}(1, 2, -2)^T + \frac{1}{3}(4, 2, 4)^T = \frac{1}{3}(5, 4, 2)^T$$
$$u_3 = \frac{x_3 - p_2}{\|x_3 - p_2\|_2} = \frac{(1 - \frac{5}{3}, 2 - \frac{4}{3}, 1 - \frac{2}{3})^T}{\sqrt{(1 - \frac{5}{3})^2 + (2 - \frac{4}{3})^2 + (1 - \frac{2}{3})^2}} = \left(-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)^T$$
$$= \frac{1}{3}(-2, 2, 1)^T$$

Thus, an orthonormal basis for the span of x_1, x_2, x_3 is given by u_1, u_2, u_3, where

$$u_1 = \frac{1}{3}(1, 2, -2)^T,\ u_2 = \frac{1}{3}(2, 1, 2)^T,\ u_3 = \frac{1}{3}(-2, 2, 1)^T$$