Discrete Structures

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Preface

Discrete mathematics is concerned with the study of finite or countably infinite objects. It is a mélange of topics from logic, set theory, algebra, combinatorics, number theory, and other areas of mathematics rather than a mathematical discipline itself. Computer scientists pragmatically characterize it as the subject that provides the mathematical foundation for computing. In particular, it supplies the tools to analyze algorithms and data structures.

Since discrete mathematics consists of many different subjects, it uses a great variety of methods to solve problems. The aim of this book is to bring some coherency into these seemingly incongruent subjects. We begin by laying a solid foundation for the study of discrete structures by discussing logic, sets, and mathematical proofs. We systematically develop methods to find the closed form for finite sums. We deduce combinatorial methods from the foundations that we have laid.

The methods of discrete mathematics are often seemingly simple. For example, anyone can immediately grasp the pigeonhole principle, but may be dazzled by the sophisticated applications of the principle. Generally, it requires a lot of practice until one is able to master the methods. We have included more than 750 exercises of various levels of difficulty in this book. We encourage the reader to study the examples in the text and solve many of these exercises.

The first chapter of this book discusses a few recreational problems that are meant to excite students. The solutions to these recreational problems illustrate the virtue of abstract methods. The second chapter discusses propositional logic and a little bit of predicate logic, before exposing the student to the main principles of mathematical arguments.

The axiomatic method is illustrated by giving a brief introduction to Zermelo-Fraenkel set theory. As an application, we show how set theory can be used to derive profound limitations of computing.

The book contains an entire chapter devoted to proofs by induction, giving a thorough exposition to this fundamental proof technique.

In Chapter 5, we discuss equivalence relations and their application to the construction of number systems. We construct the set of integers, the set of rational numbers, and the set of real numbers. Furthermore, we discuss the basics of modular arithmetic.

In Chapter 6, we discuss partial orders, strict orders, and cover relations. We give an introduction to lower and upper bounds, infima and suprema, and lattices.

The ubiquitous floor and ceiling functions are discussed in Chapter 7. These functions allow us for example to derive the number of digits of a positive integer that is written in base b, among other applications. This chapter eases the reader into topics in number theory.

Chapter 8 contains a more thorough discussion of topics in number theory. We discuss greatest common divisors and their applications in solving linear Diophantine equations. We discuss the RSA public key cryptosystem after introducing linear congruence equations and the Chinese remainder theoren.

The second part of the book concerns sums. We discuss the calculus of finite differences in Chapter 9. We will see that many sums can be turned into telescoping sums that are easy to evaluate.

The next chapter is concerned with the estimation of sums. We discuss the estimation of sums by integrals. We first derive formulas for monotonic functions. We then develop Euler's summation formula for sums of more general functions. As an application, we prove Stirling's approximation formula for the factorial function.

We discuss asymptotic notations in Chapter 11. These notations allow us to characterize the limiting behavior of a function in terms of simpler functions. The main benefit is that bounds on the growth of function can be expressed in terms of functions that are easier to understand.

The last part of the book is concerned with combinatorial methods. In Chapter 12, we discuss fundamental counting principles. The chapter introduces combinatorial proofs that are often more insightful than inductive arguments. The combinatorial interpretation of falling factorials, binomial coefficients, Stirling numbers and other counting coefficients gives the student a deeper appreciation of their significance.

The next two chapters are concerned with generating functions and recurrence relations. Chapter 13 introduces the basic operations on generating functions and illustrates how they can simplify the solution to some counting problems. In Chapter 14, generating functions are used to solve recurrence relations.

Chapter 15 is concerned with undirected graphs. We discuss basic notions of graph theory and common examples of graphs. Simple graph-theoretic arguments are illustrated for connected graphs, trees, and planar graphs. A section on graph coloring provides an interesting problem that inspired many developments in graph theory.

In Chapter 16 we give an introduction of probability theory. We begin with classical probability theory which is basically a variation on counting. Then we develop the basics of probability theory in earnest, discussing σ-algebras and probability measures, random variables, and their expectation. A brief discussion of the probabilistic method concludes this chapter.

As an audience, we had undergraduate students in mind that have at least mastered precalculus. Some more advanced material is contained in starred sections. The starred sections can be skipped at first reading without interrupting the flow.

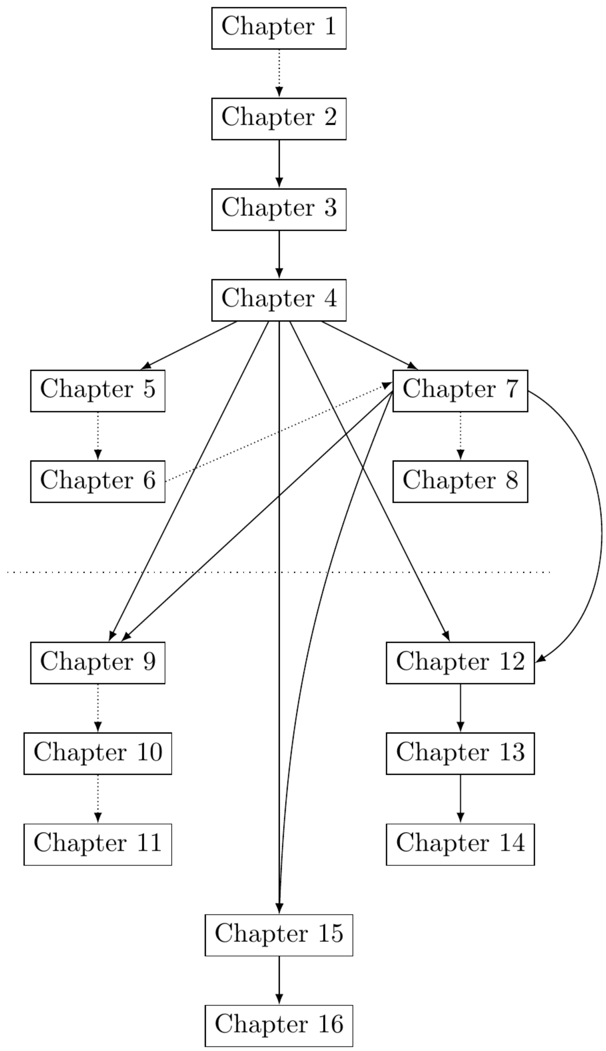

Figure 0.1: The main chapter dependencies are shown in this graph. A solid arrow Chapter A → Chapter B indicates that Chapter A should be studied before Chapter B. A dotted arrow means that it would be our recommendation to read the chapters in this order. We recommend to study Chapters 1-8 from the first part of the book before the later chapters.

The general outline of chapter dependencies is given in Figure 0.1. The order of the first four chapters is fairly fixed. Chapters 5, 6, and 7 are fairly independent, so they can be studied in any order. However, we recommend studying Chapter 5 before Chapter 6, as this is a natural progression in difficulty. Also, it should be noted that some exercises in Chapter 7 require material from Chapter 6. Chapter 8 is optional, as no other chapter depends on it. It should be evident from Figure 0.1 that there is more flexibility in choosing the order of study for the later parts of the book.

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