

Question 1

Determine whether the following transformations are linear: Explain your answer.

a. $F((x_1, x_2, x_3)^T) = (x_1 - x_2, x_2 - x_1)^T$

b. $F((x_1, x_2, x_3)^T) = (1, 2, x_1 + x_2 + x_3)^T$

c. $F((x_1)) = (x_1, 2x_1, 3x_1)^T$

d. $F((x_1, x_2, x_3, x_4)^T) = (x_1, 0, 0, 0, x_2^2 + x_3^2 + x_4^2)^T$

Solution: To check for linearity, we need to check for additivity and homogeneity, which implies that the zero vector is preserved. That means a map is linear if $L(c(u + v)) = cL(u + v) = cL(u) + cL(v)$ for all u, v in the domain and $c \in \mathbb{R}$.

a. $F((x_1, x_2, x_3)^T) = (x_1 - x_2, x_2 - x_1)^T$

Additivity:

$$\begin{aligned} F((x_1, x_2, x_3)^T + (y_1, y_2, y_3)^T) &= F((x_1 + y_1, x_2 + y_2, x_3 + y_3)^T) \\ &= (x_1 + y_1 - x_2 - y_2, x_2 + y_2 - x_1 - y_1)^T \\ &= (x_1 - x_2, x_2 - x_1)^T + (y_1 - y_2, y_2 - y_1)^T \\ &= F((x_1, x_2, x_3)^T) + F((y_1, y_2, y_3)^T) \end{aligned}$$

Homogeneity:

$$\begin{aligned} F(c(x_1, x_2, x_3)^T) &= F((cx_1, cx_2, cx_3)^T) \\ &= (cx_1 - cx_2, cx_2 - cx_1)^T \\ &= c(x_1 - x_2, x_2 - x_1)^T \\ &= cF((x_1, x_2, x_3)^T) \end{aligned}$$

Therefore, $F((x_1, x_2, x_3)^T) = (x_1 - x_2, x_2 - x_1)^T$ is linear.

b. $F((x_1, x_2, x_3)^T) = (1, 2, x_1 + x_2 + x_3)^T$

Homogeneity:

$$F((0, 0, 0)^T) = (1, 2, 0)^T$$

Since this transformation does not preserve the zero vector, it is not linear.

c. $F((x_1)) = (x_1, 2x_1, 3x_1)^T$

Additivity:

$$\begin{aligned} F((x_1) + (y_1)) &= F((x_1 + y_1)) \\ &= (x_1 + y_1, 2(x_1 + y_1), 3(x_1 + y_1))^T \\ &= (x_1, 2x_1, 3x_1)^T + (y_1, 2y_1, 3y_1)^T \\ &= F((x_1)) + F((y_1)) \end{aligned}$$

Homogeneity:

$$\begin{aligned}
 F(c(x_1)) &= F((cx_1)) \\
 &= (cx_1, 2(cx_1), 3(cx_1))^T \\
 &= c(x_1, 2x_1, 3x_1)^T \\
 &= cF((x_1))
 \end{aligned}$$

Therefore, $F((x_1)) = (x_1, 2x_1, 3x_1)^T$ is linear.

d. $F((x_1, x_2, x_3, x_4)^T) = (x_1, 0, 0, 0, x_2^2 + x_3^2 + x_4^2)^T$

Since this transformation includes squared terms, it cannot satisfy additivity, and therefore is not linear.

Question 2

Determine whether the following transformations are linear from $C([0, 1])$ to \mathbb{R} .

- a. $L(f) = f(0), (L := C([0, 1]) \rightarrow \mathbb{R})$
- b. $L(f) = |f(0)|, (L := C([0, 1]) \rightarrow \mathbb{R})$
- c. $L(f) = f'(0) + f(0), (L := C^1([0, 1]) \rightarrow \mathbb{R})$
- d. $L(f)(x) = x^2 + f(x), (L := C([0, 1]) \rightarrow C([0, 1]))$

Solution:

Question 3

For each of the following transformations, find a matrix A such that $L(x) = Ax$.

- a. $L((x_1, x_2, x_3)^T) = (x_1 + x_2)^T$
- b. $L((x_1, x_2, x_3)^T) = (x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3)^T$
- c. $L((x_1)) = (x_1, 2x_1, 3x_1)^T$
- d. $L((x_1, x_2, x_3, x_4)^T) = (x_1 + x_2 + x_3 + 2x_4)^T$

Solution: To find a matrix A such that $L(x) = Ax$, we need to find the image of the standard basis vectors.

a. $L((x_1, x_2, x_3)^T) = (x_1 + x_2)^T$

$$L((1, 0, 0)^T) = (1, 0)^T, \quad L((0, 1, 0)^T) = (1, 0)^T, \quad L((0, 0, 1)^T) = (0, 0)^T$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b. $L((x_1, x_2, x_3)^T) = (x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3)^T$

$$L((1, 0, 0)^T) = (1, 0, 1)^T, \quad L((0, 1, 0)^T) = (1, 1, 1)^T, \quad L((0, 0, 1)^T) = (0, 1, 1)^T$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

c. $L((x_1)) = (x_1, 2x_1, 3x_1)^T$

$$L((1)) = (1, 2, 3)^T$$

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

d. $L((x_1, x_2, x_3, x_4)^T) = (x_1 + x_2 + x_3 + 2x_4)^T$

$$L((1, 0, 0, 0)^T) = (1)^T, \quad L((0, 1, 0, 0)^T) = (1)^T$$

$$L((0, 0, 1, 0)^T) = (1)^T, \quad L((0, 0, 0, 1)^T) = (2)^T$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \end{bmatrix}$$

Question 4

Let $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that

$$L((x_1, x_2, x_3)^T) = (2x_1, x_1 + x_2).$$

a. Find A that represents L with respect to the standard basis of \mathbb{R}^3 .

b. Find B that represents L with respect to the following basis of \mathbb{R}^3 .

$E := [v_1, v_2, v_3]$, where,

$$v_1 = (1, 1, 1)^T, \quad v_2 = (1, 1, 0)^T, \quad v_3 = (1, 0, 0)^T.$$

Solution:

a. To find a matrix A such that $L(x) = Ax$, we need to find the image of the standard basis vectors.

$$L((1, 0, 0)^T) = (2, 1)^T, \quad L((0, 1, 0)^T) = (0, 1)^T, \quad L((0, 0, 1)^T) = (0, 0)^T$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

b. To find a matrix B such that $L(x) = Bx$ where E is a basis of \mathbb{R}^3 , we need to find the image of v_1, v_2, v_3 .

$$L((1, 1, 1)^T) = (2, 2)^T, \quad L((1, 1, 0)^T) = (2, 2)^T, \quad L((1, 0, 0)^T) = (2, 1)^T$$

$$B = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Question 5

In the vector space $C[-\pi, \pi]$ we define inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi, \pi} f(x)g(x)dx.$$

- a. Show that the above is indeed an inner product.
- b. Show that $f(x) = \cos(x)$, $g(x) = \sin(x)$ are orthogonal and that they have length 1.

Solution: