Question 1

(a) $\bigcup_{i=1}^{\infty} A_i = \mathbb{R}$

(b) $\bigcap_{i=1}^{\infty} A_i = (-1, 1)$

Question 2

(a) $\bigcup_{i\in\mathbb{Z}^+} A_i = [0, 3)$

(b) $\bigcap_{i\in\mathbb{Z}^+} A_i = [1, 2)$

Question 3

1. This is a function. The range of this function is $\{p, q, r\}$.
2. This is not a function; the element a in the domain is mapped to two different elements in the codomain.
3. This is not a function; the element b in the domain is not mapped to any element in the codomain.
4. This is a function. The range of this function is $\{p, q, r\}$.

Question 4

Proof. Let $y \in [0, \infty)$. In order for y to be in the range of f, there must exist an $x \in [0, \infty)$ such that $y = f(x)$. Consider the case $x = y^4 - 2$. By the completeness of the real numbers, $y^4 \geq 0$, so $y^4 - 2 \geq -2$. Therefore, x is in the domain of f. Also,

$$\begin{aligned} f(x) &= f(y^4 - 2) \\ &= (y^4 - 2 + 2)^{1/4} \\ &= |y| \\ &= y \text{ (since } y \geq 0). \end{aligned}$$

Therefore, $y \in \mathrm{Ran}(f)$. Hence, $[0, \infty) \subseteq \mathrm{Ran}(f)$. □

Question 5

Proof. Let $y \in \mathrm{Ran}(f)$. Then, there exists an $x \in \mathbb{R}\backslash\{3\}$ such that $y = \frac{x}{x-3}$. We want to show that $y \in \mathbb{R}\backslash\{1\}$, or in other words, $y \neq 1$. Seeking a contradiction, assume that $y = 1$. Then,

$$\begin{aligned} 1 &= \frac{x}{x-3} \\ x - 3 &= x \\ -3 &= 0. \end{aligned}$$

This is a contradiction, so $y \neq 1$, or $y \in \mathbb{R} \setminus \{1\}$, and $\mathrm{Ran}(f) \subseteq \mathbb{R} \setminus \{1\}$. Now, let $y \in \mathbb{R} \setminus \{1\}$, and we want to show that $y \in \mathrm{Ran}(f)$. That means we want to find some $x \in \mathbb{R} \setminus \{3\}$ such that $y = \frac{x}{x-3}$. Consider $x = \frac{-3y}{1-y}$. We must show that x is in the domain of f, or in other words, $x \neq 3$. Seeking a contradiction, assume that $x = 3$. Then,

$$\begin{aligned} 3 &= \frac{-3y}{1-y} \\ 3 - 3y &= -3y \\ 3 &= 0. \end{aligned}$$

This is a contradiction, so $x \neq 3$, and $x \in \mathbb{R} \setminus \{3\}$, meaning that x is in the domain of f. Also,

$$\begin{aligned} f(x) &= f\left(\frac{-3y}{1-y}\right) \\ &= \frac{\frac{-3y}{1-y}}{\frac{-3y}{1-y} - 3} \\ &= \frac{-3y}{-3y - 3(1-y)} \\ &= \frac{-3y}{-3y - 3 + 3y} \\ &= \frac{-3y}{-3} \\ &= y. \end{aligned}$$

Therefore, $y \in \mathrm{Ran}(f)$, and $\mathbb{R} \setminus \{1\} \subseteq \mathrm{Ran}(f)$. Hence, $\mathrm{Ran}(f) = \mathbb{R} \setminus \{1\}$. $\square$

Question 6

Proof. Let $y \in \mathrm{Ran}(f)$. We want to show that $y \in (-\infty, 0]$. In other words, $y \leq 0$. Seeking a contradiction, assume that $y > 0$. That means there exists an $x \in \mathbb{R}$ such that $y = f(x)$. In other words, we want to find some real number whose square is negative. However, the square of any real number is nonnegative, so there is no such x, a contradiction. Therefore, $y \leq 0$, and $\mathrm{Ran}(f) \subseteq (-\infty, 0]$. Now let $y \in (-\infty, 0]$. We want to show that $y \in \mathrm{Ran}(f)$. In other words, we want to show that there exists some $x \in \mathbb{R}$ such that $y = f(x)$.

Consider $x = \sqrt{-y}$. We must show that x is in the domain of f, or in other words, $x \in \mathbb{R}$. Since $y \leq 0$, $-y \geq 0$, so the square root is real. Also,

$$\begin{aligned} f(x) &= f(\sqrt{-y}) \\ &= -(-\sqrt{-y})^2 \\ &= -(-y) \\ &= y. \end{aligned}$$

Therefore, $y \in \text{Ran}(f)$, and $(-\infty, 0] \subseteq \text{Ran}(f)$. Hence, $\text{Ran}(f) = (-\infty, 0]$. $\square$

Question 7

1. $G_f = \{(a, 0), (b, 1), (c, 3), (d, 6), (e, 10)\}$
2. $G_g = \{(w, 10), (x, 1), (y, 3), (z, 0)\}$