MATH300 Homework 10 (due 4/19)

1. (10 pts) Let A be a set. Prove that $f : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by $f(X) = \overline{X}$ is bijective.

2. (27 pts) Prove that each of the following functions is not bijective.

   (a) $f : (-\infty, 1) \to \mathbb{R}$, $f(x) = x^3$.

(b) $D : \mathbb{R}[x] \to \mathbb{R}[x],\ D(f) = f'$ (where $\mathbb{R}[x] = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \mid n \geq 0, a_i \in \mathbb{R}\}$, the set of all polynomials over x with real coefficients. Simply put, D sends a polynomial to its derivative, e.g., $D(3x^2 + \pi x) = 6x + \pi$.)

(c) $s : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $s(m, n) = m + n$. (Simply put, s sends an ordered pair of natural numbers to their sum.)

3. (14 pts) Let $f : X \to Y$ and $g : Y \to Z$. Prove that if $g \circ f$ is one-to-one, then f is one-to-one, but g need not be. (For "g need not be", you may use a diagram.)

4. (9 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$. Find each of the following (you do not need to prove your answers.)

   (a) $f(\{-3, 2, 7\})$

   (b) $f([-1, 3])$

   (c) $f((-\infty, -2))$

5. (18 pts) Let $f : X \to Y$ and $g : Y \to Z$.

   (a) Prove that if f and g are invertible, then $g \circ f$ is invertible. (Hint: use Theorem 4 of Section 5.4 notes.)

   (b) ("Socks and Shoes Theorem") It turns out that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Prove this by showing

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = \mathrm{id}_X \quad \text{and} \quad (g \circ f) \circ (f^{-1} \circ g^{-1}) = \mathrm{id}_Z.$$

6. (6 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2024 - 2x$. Compute $f\big([-3, 5]\big)$. Justify your answer.

7. (6 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^4$. Compute $f((0, 2))$. Justify your answer.

8. (10 pts) Disprove: Let $f : X \to Y$ and $A_1, A_2 \subseteq X$. If $f(A_1) \subseteq f(A_2)$, then $A_1 \subseteq A_2$.