Question 1

Let $\{u_1, u_2, u_3\}$ be an orthonormal set of vectors in some vector space with inner product. Let

$$u := u_1 + 2u_2 + 3u_3 \text{ and } v := u_1 - u_3$$

Compute $\langle u, v\rangle$, $\|u\|$, and $\|v\|$.

Solution: Since the basis is orthonormal, the inner product of any two vectors in the basis is 0, and the inner product of a vector in the basis with itself is 1.

$$\begin{aligned}
\langle u, v\rangle &= \langle u_1 + 2u_2 + 3u_3, u_1 - u_3\rangle \\
&= \langle u_1, u_1 - u_3\rangle + \langle 2u_2, u_1 - u_3\rangle + \langle 3u_3, u_1 - u_3\rangle \\
&= \langle u_1, u_1\rangle - \langle u_1, u_3\rangle + \langle 2u_2, u_1\rangle - \langle 2u_2, u_3\rangle + \langle 3u_3, u_1\rangle - \langle 3u_3, u_3\rangle \\
&= 1 - 0 + 0 - 0 + 0 - 3 \\
&= -2
\end{aligned}$$

$$\begin{aligned}
\|u\|^2 &= \langle u, u\rangle \\
&= \langle u_1 + 2u_2 + 3u_3, u_1 + 2u_2 + 3u_3\rangle \\
&= \langle u_1, u_1 + 2u_2 + 3u_3\rangle + \langle 2u_2, u_1 + 2u_2 + 3u_3\rangle + \langle 3u_3, u_1 + 2u_2 + 3u_3\rangle \\
&= \langle u_1, u_1\rangle + \langle 2u_2, 2u_2\rangle + \langle 3u_3, 3u_3\rangle \\
&= 1 + 4 + 9 \\
&= 14 \implies \|u\| = \sqrt{14}
\end{aligned}$$

$$\begin{aligned}
\|v\|^2 &= \langle v, v\rangle \\
&= \langle u_1 - u_3, u_1 - u_3\rangle \\
&= \langle u_1, u_1 - u_3\rangle + \langle -u_3, u_1 - u_3\rangle \\
&= \langle u_1, u_1\rangle - \langle u_1, u_3\rangle - \langle u_3, u_1\rangle + \langle u_3, u_3\rangle \\
&= 1 - 0 - 0 + 1 \\
&= 2 \implies \|v\| = \sqrt{2}
\end{aligned}$$

Question 2

Consider the vector space $C[-1, 1]$ equipped with the inner product:

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)dx$$

1. Show that $1, x$ are orthogonal.
2. Compute the norms $\|1\|$, $\|x\|$.

Solution: Two vectors are orthogonal if their inner product is 0.

$$\langle 1, x \rangle = \int_{-1}^{1} 1 \cdot x \, dx = \int_{-1}^{1} x \, dx = \frac{1}{2}x^2 \Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$$

Thus, $1, x$ are orthogonal. The norms of 1 and x are the square root of their inner product with themselves.

$$\begin{aligned}
\|1\|^2 &= \langle 1, 1 \rangle = \int_{-1}^{1} 1 \cdot 1 \, dx = \int_{-1}^{1} 1 \, dx = x \Big|_{-1}^{1} = 1 - (-1) = 2 \\
&\implies \|1\| = \sqrt{2} \\
\|x\|^2 &= \langle x, x \rangle = \int_{-1}^{1} x \cdot x \, dx = \int_{-1}^{1} x^2 \, dx = \frac{1}{3}x^3 \Big|_{-1}^{1} = \frac{1}{3} - \frac{1}{3} = \frac{2}{3} \\
&\implies \|x\| = \sqrt{\frac{2}{3}}
\end{aligned}$$

Question 3

Let

$$u_1 = \left(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{4}{3\sqrt{2}}\right)^T,\ u_2 = \frac{1}{3}(2,2,1)^T,\ u_3 = \frac{1}{\sqrt{2}}(1,-1,0)^T$$

1. Show that u_1, u_2, u_3 is an orthonormal basis for $\mathbb{R}^3$.
2. Let $x = (1,2,2)^T$. Find the projection of p of x onto $S := \text{span}\{u_2, u_3\}$.

Solution: A set of vectors form an orthonormal basis if they are orthogonal and their norms are 1. Since we are working in $\mathbb{R}^3$, we can use the dot product to check if the vectors are orthogonal.

$$\begin{aligned}
\langle u_1, u_2\rangle &= \frac{1}{3\sqrt{2}} \cdot \frac{2}{3} + \frac{1}{3\sqrt{2}} \cdot \frac{2}{3} + -\frac{4}{3\sqrt{2}} \cdot \frac{1}{3} = \frac{2}{9\sqrt{2}} + \frac{2}{9\sqrt{2}} - \frac{4}{9\sqrt{2}} = 0 \\
\langle u_1, u_3\rangle &= \frac{1}{3\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} + -\frac{4}{3\sqrt{2}} \cdot 0 = \frac{1}{6} - \frac{1}{6} = 0 \\
\langle u_2, u_3\rangle &= \frac{2}{3} \cdot \frac{1}{\sqrt{2}} + \frac{2}{3} \cdot -\frac{1}{\sqrt{2}} + \frac{1}{3} \cdot 0 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} = 0
\end{aligned}$$

Thus, u_1, u_2, u_3 are orthogonal. To check if their norms are 1, we can use the formula $\|u\|^2 = \langle u, u\rangle$.

$$\begin{aligned}
\|u_1\|^2 = \langle u_1, u_1\rangle &= \frac{1}{3\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} + -\frac{4}{3\sqrt{2}} \cdot -\frac{4}{3\sqrt{2}} = \frac{1}{18} + \frac{1}{18} + \frac{16}{18} = 1 \\
\|u_2\|^2 = \langle u_2, u_2\rangle &= \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} + 1 \cdot 1 = \frac{1}{9} + \frac{1}{9} + 1 = 1 \\
\|u_3\|^2 = \langle u_3, u_3\rangle &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} + 0 \cdot 0 = \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

Since the squares of norms are 1, the norms are 1. Thus, u_1, u_2, u_3 are orthonormal. To find the projection of x onto S...

Question 4

Let $v_1 := (1, 2, 0, -1)^T$ $v_2 := (1, -1, 0, 0)^T$ $v_3 := (0, 1, 0, -1)^T$. Find the angle between v_1, v_2, v_2, v_3, and v_1, v_3. Find the norm of each of these vectors. Find the projection of v_1 onto v_2 and onto v_3.

Question 5

Let A be an $m \times n$ matrix. Show that A^TA and AA^T is a symmetric matrix. Assume that $m \geq n$ and $\text{rank}(A) = n$. Show that if $P = A(A^TA)^{-1}A^T$ then

$$P^2 = P$$

Solution: A matrix is symmetric if it is equal to its transpose. Using the properties of the matrix transpose, we can show that A^TA and AA^T are symmetric.

$$\text{Transpose of } A^TA\text{: } (A^TA)^T = (A^T)^TA^T = AA^T = A^TA$$
$$\text{Transpose of } AA^T\text{: } (AA^T)^T = A^T(A^T)^T = A^TA = AA^T$$

As we can see here, applying the transpose to either matrix results in the same matrix. Proving that $P^2 = P$ is a bit more involved.

Proof. We have that $P = A(A^TA)^{-1}A^T$. We need to find

$$P^2 = A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T$$

and show that it is equal to P. Using the associative property of matrix multiplication, we can change our order of multiplication to get

$$P^2 = A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T$$

Since it is given that $\text{rank}(A) = n$, and A^TA must be and $n \times n$ matrix, A^TA is invertible. Since A^TA is invertible, we can multiply it by its inverse to get the identity matrix. Now, we have

$$P^2 = AI(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T = P$$

Thus, $P^2 = P$. $\square$