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$\boxed{\text{EXAM 1 STARTS HERE}}$

Q1 The statement is true since the two numbers can always be equal

Q2 "Is child of" does not satisfy reflexivity since a person cannot be their own child. It is also not symmetric or transitive so it is not an equivalence relation

Q3 The proper subset is not a partial order since no set can be a partial order of itself

Q4 $((p \to q) \wedge p) \to q$

$= ((\neg p \vee q) \wedge p) \to q = (\cancel{\neg p \wedge p}^{F}) \vee (q \wedge p) \to q$

$= \neg(q \wedge p) \vee q = \neg q \vee \neg p \vee q = \text{True}$

Q5 $f(x) = \lceil x \rceil$ is surjective but not injective because there exists at least one $x \in \mathbb{R}$ such that $\lceil x \rceil = z \ \forall z \in \mathbb{Z}$. Not injective since multiple reals can map to the same integer.

Q6 $\forall s \exists c\, H(s, c)$ means for all students there exists a computer such that the student s owns the computer c

Q7 The sets $\{b\}$ and $\emptyset$ are lower bounds for X since it is less than or equal to every element in X according to the partial order.

Q8 The empty set is a member of $\{\emptyset\}$.
The empty set is a subset of all sets.
1 and 2 both true.

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Q9 Assume the opposite of what we are trying to prove:
"There exist some integers m and n such that mn is even but m is odd and n is odd" is the negation since $\neg\forall$ is $\exists$, or gets negated to and, and use the opposite of the predicates.

Q10 $f(x) = x - 42$ is bijective since there is only output per input and covers the entire codomain

Q11 $(1,4)$ is not part of the cover relation since 1 is not directly covered by 4

Q12 Antisymmetric - two sets cannot be proper subsets of each other
Irreflexive - no set is a proper subset of itself
Transitive - If $A \subset B$, $B \subset C$ then $A \subset C$
Asymmetric - $A \subset B$ means $B \subset A$ is impossible

Q13 $\{a\}$ is not in the power set: $\{\{a\}\}$ is

Q14 $\cap S$ is the set of integers ≤ 2 since k starts at 1 and every A_k include integers less than or equal to $2k$

Q15 This is modus Tollens since Tim having the password implies he can log in. Since he cannot log in, he has no password.

Q16 False since there is no single positive integer such that it is equal to all positive integers

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Q17 $\mathbb{N}_0$ and $\mathbb{Z}$ have the same cardinality, so there's a bijective function between them. Real numbers between any interval is an uncountable infinity

Q18 Negate both the subject and predicate, so it becomes "There exist some integers m and n such that mn is even but m is odd and n is odd.

Q19 False. Counterexample: $\lfloor 1.4 + 0.4 \rfloor \neq \lfloor 1.4 \rfloor + \lceil 0.4 \rceil$

$$\underset{1}{\overset{\shortparallel}{\,}} \neq \underset{1}{\overset{\shortparallel}{\,}} + \underset{1}{\overset{\shortparallel}{\,}}$$

Q20 Negate the hypothesis: the equation has an integer solution means there exist integers m_0, n_0 where $42m_0 + 70n_0 = 1000$

Q21 The equivalence $A \leftrightarrow B$ means A is true iff B is true and B is true iff A is true. Symbolically it means $A \rightarrow B$ and $B \rightarrow A$, so A and B must have the same value for $A \leftrightarrow B$ to hold.

Q22 $\neg(\neg(A \leftrightarrow B)) \vee C$ is equivalent to $(\neg A \leftrightarrow B) \rightarrow C$ because $p \rightarrow q = \neg p \vee q$ and $\neg(A \leftrightarrow B) = (\neg A \leftrightarrow B)$

Q23 Numbers in the equivalence class $[-1]$ divide by 3 to have a remainder of -1. $31 \div 3$ leaves a remainder of 2 so it's not in the equivalence class

Q24 $\neg(A \rightarrow B) = \neg(\neg A \vee B) = A \wedge \neg B$

Q25 $\neg \exists x \forall y \, (P(x,y) \rightarrow Q(x,y)) = \forall x \exists y \neg(\neg P(x,y) \vee Q(x,y))$

$= \forall x \exists y \, (P(x,y) \wedge \neg Q(x,y))$

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Q26 The natural order $\leq$ on the set of integers is a total order since it is comparable on the entire set

Q27 There exists an open but not closed knights tour on a 3×4 board

8	11	6	3
1	4	9	12
10	7	2	5

EXAM 2 STARTS HERE

Q1 Taking the logarithm of both functions:

$$\ln(\ln n)^n = n \ln(\ln n)$$

$$\ln(n^{\ln n}) = (\ln n)(\ln n) = (\ln(n))^2$$

$$\lim_{n\to\infty} \frac{n \ln(\ln(n))}{(\ln(n))^2} < \lim_{n\to\infty} \frac{n}{\ln n} \quad \text{since } \ln(\ln(n)) < \ln(n)$$

$$\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \infty \quad \text{therefore } (\ln n)^n \notin O(n^{\ln n})$$

Q2 True since $\lim_{n\to\infty} \frac{11n^2 + 7n\log n}{n^3} = 0$

Q3 True since $((-1)^n + 5)$ alternates between 4 and 6 which are the bounds in the definition of Θ, and $n^3 \in \Theta(n^3)$

Q4 True since $\lim_{n\to\infty} \frac{41n^2 + 13n\log n}{n^2 \log n} = \lim_{n\to\infty} \frac{41}{\log n} + \frac{13}{n} = 0$

Q5 True since $\lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \infty$

Q6 True since $\lim_{n\to\infty} \frac{n\log n}{5n^2 + 4n\log n} = \lim_{n\to\infty} \frac{n \log n}{5n^2} = \lim_{n\to\infty} \frac{\log n}{n} = \infty$

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Q7 $g(n)$ is an upper and lower bound for $f(n)$ since both are polynomials of highest degree 2. They are not strict bounds since you can choose constants such that $c|g(n)| \nleq f(n)$ and $|f(n)| \nleq c|g(n)|$. Since $g(n)$ is both an upper and lower bound, it is also a strict bound.

Q8 int i starts at 1 and doubles until $i \leq 99$.
i goes in the order of $1 \to 2 \to 4 \to 8 \to 16 \to 32 \to 64$
so the block is executed 7 times

Q9 Θ estimate for $f(n)$ is $\Theta(\log_2 n)$ since the index i doubles with each iteration which is consistent with a base 2 logarithm. The logarithm operates on the input size.

Q10 The outer loop runs 99 times and the inner loop runs $a+1$ times for each iteration of the outer loop.
That means the block is run $\sum_{a=1}^{99}(a+1) = \frac{99 \times 100}{2} = 5049$ times

Q11 $f(n)$ is in $\Theta(n^2)$ since it is the sum of the first n natural numbers which is $\frac{n(n+1)}{2}$

Q12 The number of multiplications required is given by
$(1+(k-1)) + (1+(k-2)) + \ldots + 0$ for each term of $p(x)$.
this is the sum of the first n natural numbers which is $\frac{n(n+1)}{2}$ which is $\Theta(n^2)$

Q13 In this algorithm, only one multiplication is done per coefficient, so it is $\Theta(n)$

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Q14 Each of the 10 elements in the domain has two options, so there are $2^{10} = 1024$ functions

Q15 Let $b_0 = 1$, $b_1 = 2$, $b_2 = 3$ and $b_n = b_{n-1} + b_{n-2} + b_{n-3}$ for all integers $n \geq 3$. We want to prove using strong induction that $b_n \leq 2^n$.

Base cases: $b_0 = 1 \leq 2^0 = 1$ $\quad b_1 = 2 \leq 2^1 = 2$ $\quad b_2 = 3 \leq 2^2 = 4$

Now assume $b_k \leq 2^k$ holds for all k $0 \leq k < n$, $n \geq 3$

$$\begin{aligned} b_n &= b_{n-1} + b_{n-2} + b_{n-3} && \text{by definition} \\ &\leq 2^{n-1} + 2^{n-2} + 2^{n-3} && \text{by induction hypothesis} \\ &= 2^n 2^{-1} + 2^n 2^{-2} + 2^n 2^{-3} \\ &= 2^n \left(2^{-1} + 2^{-2} + 2^{-3}\right) \end{aligned}$$

Therefore $b_n \leq 2^n \left(2^{-1} + 2^{-2} + 2^{-3}\right) < 2^n$

Q16 We want to prove by induction that the sum of squares of the first n positive integers is given by

$$\sum_{j=1}^{n} (2j-1)^2 = \frac{1}{3}(4n^3 - n) \quad \text{for all positive integers } n$$

Base case: $\sum_{j=1}^{1} (2j-1)^2 = 1^2 = \frac{1}{3}(4 \cdot 1^3 - 1)$

Now assume $\sum_{j=1}^{n} (2j-1)^2 = \frac{1}{3}(4n^3 - n)$ holds

Then $\sum_{j=1}^{n+1} (2j-1)^2 = \sum_{j=1}^{n} (2j-1)^2 + (2(n+1)-1)^2 = \sum_{j=1}^{n} (2j-1)^2 + (2n+1)^2$ by definition

$$= \frac{1}{3}(4n^3 - n) + (2n+1)^2 \quad \text{by the induction hypothesis}$$

$$= \frac{1}{3}\left(4(n+1)^3 - (n+1)\right)$$

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Q17 Using the pigeonhole principle, we have 240000 pigeonholes and 540000 pigeons, so there will be at least

$\left\lceil \frac{540000}{240000} \right\rceil = 3$ people in Asgleland who have the same number of hairs on their head.

Q18 The coefficient is given by $\binom{7}{3} = \frac{7!}{3!(7-3)!} = 35$

Q19 The number of bit strings of length 10 that contain exactly one 0 or exactly three 0s is given by

$\binom{10}{1} + \binom{10}{3} = 10 + 120 = 130$ bit strings

Q20 Binomial theorem states that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$,

which means $\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = (2+1)^n = 3^n$

Q21 For the domain of the set with three elements, each element must be mapped to a distinct element in the codomain, so there are $10 \times 9 \times 8$ options for the three elements in the domain.

Q22 The number of bit strings that start with two ones is 2^6 since 6 digits have two options each.

Bit strings that end with three zeros is 2^5

since now there are 5 free digits.

However, some strings can satisfy both conditions, which is given by 2^3 since 5 digits are predetermined.

Therefore there are $2^6 + 2^5 - 2^3 = 88$ bitstrings that start with two ones or end with three zeros.

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Q23 using the pigeonhole principle, we want to find n pigeons for 12 pigeonholes such that 5 will be in the same hole.
$\lceil \frac{n}{12} \rceil = 5$ the smallest value of n is 49, since $\frac{48}{12} = 4$

Q24 To count the number of permutations of ABCDEF that contain the substring ACD, we can treat the substring ACD as a single letter since it must stay together in that order. There are now three other elements, so the permutations is given by $4! = 24$