CSCE 222 Discrete Structures for Computing – Fall 2023
Hyunyoung Lee

Problem Set 4

Due dates: Electronic submission of *yourLastName-yourFirstName-hw4.tex* and *yourLastName-yourFirstName-hw4.pdf* files of this homework is due on **Friday, 10/13/2023 before 11:59 p.m.** on `https://canvas.tamu.edu`. You will see two separate links to turn in the .tex file and the .pdf file separately. Please do not archive or compress the files. **If any of the two files are missing, you will receive zero points for this homework.**

Name: Kevin Lei **UIN: 432009232**

Resources. (All people, books, articles, web pages, etc. that have been consulted when producing your answers to this homework)

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to answer this homework.

Electronic signature: <u>**Kevin Lei**</u>

Total $100 + 5$ (bonus) points.

The intended formatting is that this first page is a cover page and each problem solved on a new page. You only need to fill in your solution between the `\begin{solution}` and `\end{solution}` environment. Please do not change this overall formatting.

Make sure that you strictly follow the structure of induction proof as shown in the lecture notes and how I solved in my videos.

Checklist:
- ☐ Did you type in your name and UIN?
- ☐ Did you disclose all resources that you have used?
  (This includes all people, books, websites, etc. that you have consulted)
- ☐ Did you sign that you followed the Aggie Honor Code?
- ☐ Did you solve all problems?
- ☐ Did you submit both the .tex and .pdf files of your homework to each correct link on Canvas?

Problem 1. (15 points) Section 4.1, Exercise 4.3

Solution. We want to prove that the sum of the first n squares is given by

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \geq 1$. For the base case, the claim $P(n)$ holds for $n = 1$ since $\sum_{k=1}^{1} k^2 = 1^2 = \frac{1(1+1)(2\times1+1)}{6} = \frac{1\times2\times3}{3} = 1$. In the inductive step, it is our goal to show that $P(n) \to P(n+1)\ \forall n \geq 1$, which means we need to prove that

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

Using our base case, we can substitute the left side of the equation with $\sum_{i=1}^{k} i^2 + (k+1)^2$. We can then substitute $\sum_{i=1}^{k} i^2$ with $\frac{k(k+1)(2k+1)}{6}$, so our equation then becomes

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

Multiply the equation by 6:

$$k(k+1)(2k+1) + 6(k+1)^2 = (k+1)(k+2)(2k+3)$$

Expand both sides:

$$2k^3 + k^2 + 2k^2 + k + 6k^2 + 12k + 6 = 2k^3 + 3k^2 + 6k^2 + 9k + 4k + 6$$

Collect like terms:

$$2k^3 + 9k^2 + 13k + 6 = 2k^3 + 9k^2 + 13k + 6$$

And thus both sides of the equation are equal, so $P(n) \to P(n+1)\ \forall n \geq 1$. Therefore, we have proven that the sum of the first n squares is given by

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all $n \geq 1$.

Problem 2. (15 points) Section 4.1, Exercise 4.4

Solution. We want to prove by induction that the sum of the first n cubes is given by

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + ... + n^3 = (1 + 2 + .. + n)^2 = \frac{n^2(n+1)^2}{4}$$

for all $n \geq 1$. To do this we must prove this in two parts, since there are two claims in the equation. In the first part we will prove that

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$$

for all $n \geq 1$. For the base case where $n = 1$, the formula holds true since:

$$\sum_{k=1}^{1} k^3 = 1^3 = (1)^2 = \frac{1^2(1+1)^2}{4} = 1$$

In the inductive step, we want to show that $P(n) \to P(n+1)\ \forall n \geq 1$, which means we need to prove the following equation:

$$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$$

Using the base case, we can substitute the left side of the equation with $\sum_{i=1}^{k} i^3 + (k+1)^3$, and then substitute $\sum_{i=1}^{k} i^3$ with $\frac{k^2(k+1)^2}{4}$, so our equation then becomes

$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

Multiply the equation by 4:

$$k^2(k+1)^2 + 4(k+1)^3 = (k+1)^2(k+2)^2$$

Expand both sides:

$$k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4k^2 + 12k + 12 = k^4 + 4k^3 + 6k^2 + 4k + k^2 + 4k + 4$$

Collect like terms:

$$k^4 + 6k^3 + 19k^2 + 28k + 12 = k^4 + 6k^3 + 19k^2 + 28k + 12$$

And thus both sides of the equation are equal, so $P(n) \to P(n+1)\ \forall n \geq 1$. Therefore, we have proven that the sum of the first n cubes is given by

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$$

for all $n \geq 1$. In the second part, we want to prove that

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + ... + n^3 = (1 + 2 + .. + n)^2$$

for all $n \geq 1$. For the base case where $n = 1$, the formula holds true since:

$$\sum_{k=1}^{1} k^3 = 1^3 = (1)^2 = 1$$

In the inductive step, we need to prove the following equation:

$$\sum_{i=1}^{k+1} i^3 = (1 + 2 + .. + k + (k+1))^2$$

Using the base case, we can substitute the left side of the equation with $\sum_{i=1}^{k} i^3 + (k+1)^3$, and then substitute $\sum_{i=1}^{k} i^3$ with $(1+2+..+k)^2$, so our equation then becomes

$$(1 + 2 + .. + k)^2 + (k+1)^3 = (1 + 2 + .. + k + (k+1))^2$$

Using the formula for the first n natural numbers, we can substitute $(1+2+..+k)$ with $\frac{k(k+1)}{2}$, so our equation then becomes

$$\frac{k(k+1)}{2}^2 + (k+1)^3 = (\frac{k(k+1)}{2} + (k+1))^2$$

Expanding both sides yields:

$$\frac{k^4 + 2k^3 + k^2}{4} + k^3 + 3k^2 + 3k + 1 = \frac{k^4 + 2k^3 + k^2}{4} + k^3 + 3k^2 + 3k + 1$$

And thus both sides of the equation are equal, so we have proven that

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + ... + n^3 = (1 + 2 + .. + n)^2$$

for all $n \geq 1$. Since we have proven both parts of the equation, we have proven that the sum of the first n cubes is given by

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + ... + n^3 = (1 + 2 + .. + n)^2 = \frac{n^2(n+1)^2}{4}$$

for all $n \geq 1$.

Problem 3. (15 points) Section 4.1, Exercise 4.5

Solution. We want to prove by induction that the squares of the first n odd positive integers is given by

$$\sum_{k=1}^{n}(2k-1)^2 = 1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{1}{3}(4n^3 - n)$$

for all positive integers n. In the base step, we want to show that the equation holds true for $n = 1$, which it does since:

$$\sum_{k=1}^{1}(2k-1)^2 = 1^2 = \frac{1}{3}(4(1)^3 - 1) = \frac{1}{3}(4-1) = \frac{3}{3} = 1$$

In the inductive step, we want to show that $P(n) \to P(n+1)\ \forall n \geq 1$, which means we need to prove the following equation:

$$\sum_{i=1}^{k+1}(2i-1)^2 = \frac{1}{3}(4(k+1)^3 - (k+1))$$

Starting with the left side, we can substitute the summation with $\sum_{i=1}^{k}(2i-1)^2 + (2(k+1)-1)^2$, and then substitute $\sum_{i=1}^{k}(2i-1)^2$ with $\frac{1}{3}(4k^3 - k)$. Our equation then becomes

$$\frac{1}{3}(4k^3 - k) + (2(k+1)-1)^2 = \frac{1}{3}(4(k+1)^3 - (k+1))$$

Expanding both sides yields:

$$\frac{4k^3 - k}{3} + 4k^2 + 4k + 1 = \frac{4k^3 + 12k^2 + 12k + 3}{3}$$

Collect like terms:

$$\frac{4k^3 + 12k^2 + 12k + 3}{3} = \frac{4k^3 + 12k^2 + 12k + 3}{3}$$

And thus, both sides of the equation are equal, so we have proven by induction that the squares of the first n odd positive integers is given by

$$\sum_{k=1}^{n}(2k-1)^2 = 1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{1}{3}(4n^3 - n)$$

for all positive integers n.

Problem 4. (20 points) Section 4.1, Exercise 4.6

Solution. We want to prove using induction that for all integers $n \geq 1$, the integer $2^{2n} - 1$ is divisible by 3. In the base step, we want to show that the equation holds true for $n = 1$. Substituting $n = 1$ into the equation yields: $2^{2(1)} - 1 = 2^2 - 1 = 4 - 1 = 3$, which is divisible by 3, so the base case holds true. In the inductive step, we want to show that $P(n) \rightarrow P(n+1)$ $\forall n \geq 1$, so we need to prove that $2^{2(n+1)} - 1$ is also divisible by 3. To show this, lets first assume that the statement holds true for some integer n, so we can say that $2^{2n} - 1 = 3m$ which is divisible by 3. Algebraically, $2^{2(n+1)} - 1 = 2^{2n+2} - 1$ can be rewritten as $2^{2n} \times 2^2 - 1$ which is $2^{2n} \times 4 - 1$. From our assumption, we can say that $2^{2n} = 3m + 1$, so we can substitute that into our equation to get $(3m + 1) \times 4 - 1$. Simplyfing the equation yields $12m + 3$, which is divisible by 3. Thus, we have shown that $2^{2(n+1)} - 1$ is divisible by 3, so $P(n) \rightarrow P(n+1)$ $\forall n \geq 1$. Therefore, we have proven by induction that for all integers $n \geq 1$, the integer $2^{2n} - 1$ is divisible by 3.

Problem 5. (20 points) Section 4.3, Exercise 4.15

Solution. We want to prove by induction that the sum of the first n terms of the fibonacci sequence that have an even index is given by

$$\sum_{k=1}^{n} f_{2k} = f_2 + f_4 + ... + f_{2n} = f_{2n+1} - 1$$

In the base step, we need to show that the equation holds for $n = 1$, which it does since: $\sum_{k=1}^{1} f_{2k} = f_2 = f_{2(1)+1} - 1 = f_3 - 1 = 2 - 1 = 1$, and the second fibonacci number is 1. In the inductive step, we shall assume that the proposition holds true for some arbitray number n, so we need to prove that it also holds true for $n + 1$. This means we need to prove the following equality:

$$f_2 + f_4 + ... + f_{2n} + f_{2(n+1)} = f_{2(n+1)+1} - 1$$

Using the induction hypothesis, we can substitute the left side of the equation with $f_{2n+1} - 1 + f_{2(n+1)}$. Simplifying the subscripts and adding 1 to both sides, the equation then becomes

$$f_{2n+1} + f_{2n+2} = f_{2n+3}$$

This equation now matches the definition of the fibonacci sequence, since we have $f_n + f_{n+1} = f_{n+2}$. Therefore, we have proven the hypothesis using induction.

Problem 6. (20 points) Section 4.6, Exercise 4.31

Solution. Given that f_n is a sequence of nonnegative integers satisfying the recurrence relation $f_n = (n^3 - 3n^2 + 2n)f_{n-3}$, and $f_1 = 1$, $f_2 = 2$, and $f_3 = 6$, we want to prove by strong induction that $f_n = n!$ holds for all $n \geq 1$. We have three base cases: $n = 1$, $n = 2$, and $n = 3$. Since $f_1 = 1$ and $1! = 1$, $f_2 = 2$ and $2! = 2$, and $f_3 = 6$ and $3! = 6$, the base cases hold true. In the inductive step, we shall assume that the proposition holds true for some arbitrary number k, so we need to prove that it also holds true for $k+1$. That means we are assuming that $f_k = k!$ holds true, and we need to prove that $f_{k+1} = (k+1)!$ also holds true. The following equation:

$$f_{k+1} = (k+1)!$$

Can be rewritten as:

$$(k+1)! = ((k+1)^3 - 3(k+1)^2 + 2(k+1))f_{k-2}$$

using the induction hypothesis. f_{k-2} can be rewritten as $(k-2)!$. We can also simplify the inside of the parenthesis on the right side, so our equation turns into

$$(k+1)! = (k^3 - k)(k-2)!$$

$k^3 - k$ can be rewritten as $k(k^2 - 1)$, which can be further simplified to $k(k+1)(k-1)$. We can then substitute $k(k+1)(k-1)$ into our equation to get

$$(k+1)! = k(k+1)(k-1)(k-2)!$$

At this point, we can rearrange the terms to see the equality better:

$$(k+1)! = (k+1)k(k-1)(k-2)!$$

More clearly, using the definition of the factorial, the left side of the equation can be rewritten as $(k+1)k(k-1)(k-2)!$. Since both sides of the equation are equal, we have proven the hypothesis using strong induction.