

Question 1

Find the general solution of each of the following systems:

a.
$$\begin{cases} y_1 + y_2 = y_1' \\ -2y_1 + 4y_2 = y_2' \end{cases}$$

b.
$$\begin{cases} y_1 - y_2 = y_1' \\ y_1 + y_2 = y_2' \end{cases}$$

c.
$$\begin{cases} y_1 + y_3 = y_1' \\ 2y_2 + 6y_3 = y_2' \\ y_2 + 3y_3 = y_3' \end{cases}$$

Solution: For systems in the form $y' = Ay$, we can find y as e^{At} , where $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots$. Additionally, when A is diagonalizable, we can write $A = Xe^DX^{-1}$, where X is the matrix of eigenvectors of A and D is the diagonal matrix of eigenvalues of A . Then, $e^{At} = Xe^{Dt}X^{-1}$.

System a:

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

$$\begin{aligned} P(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 \\ -2 & 4 - \lambda \end{vmatrix} \\ &= (1 - \lambda)(4 - \lambda) + 2 = \lambda^2 - 5\lambda + 6 \\ &= (\lambda - 2)(\lambda - 3) \end{aligned}$$

$$\lambda_1 = 2, \quad \lambda_2 = 3$$

$$\begin{aligned} N(A - \lambda_1 I) &= N\left(\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\right) \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -x_1 + x_2 = 0 \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

$$\begin{aligned} N(A - \lambda_2 I) &= N\left(\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}\right) \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -2x_1 + x_2 = 0 \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \end{aligned}$$

$$\begin{aligned} X &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \\ e^{Dt} &= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} y &= e^{At} = X e^{Dt} X^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2e^{2t} & -e^{2t} \\ -e^{3t} & e^{3t} \end{pmatrix} \\ &= \begin{pmatrix} 2e^{2t} - e^{3t} & -e^{2t} + e^{3t} \\ 2e^{2t} - 2e^{3t} & -e^{2t} + 2e^{3t} \end{pmatrix} \end{aligned}$$

System b:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{aligned} P(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & -1 \\ 1 & 1 - \lambda \end{vmatrix} \\ &= (1 - \lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 \end{aligned}$$

$$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i$$

$$\begin{aligned} N(A - \lambda_1 I) &= N\left(\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\right) \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -ix_1 - x_2 = 0 \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} \end{aligned}$$

$$\begin{aligned} N(A - \lambda_2 I) &= N\left(\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\right) \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : ix_1 - x_2 = 0 \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} \right\} \end{aligned}$$

$$X = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad X^{-1} = \frac{1}{2i} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

Using the fact that $e^{a+bi} = e^a(\cos b + i \sin b)$

$$e^{Dt} = \begin{pmatrix} e^t(\cos t + i \sin t) & 0 \\ 0 & e^t(\cos t - i \sin t) \end{pmatrix}$$

$$\begin{aligned} y &= e^{At} = X e^{Dt} X^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^t(\cos t + i \sin t) & 0 \\ 0 & e^t(\cos t - i \sin t) \end{pmatrix} \frac{1}{2i} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \\ &= \frac{1}{2i} \begin{pmatrix} e^t(\cos t + i \sin t) & e^t(\cos t - i \sin t) \\ -ie^t(\cos t + i \sin t) & ie^t(\cos t - i \sin t) \end{pmatrix} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \\ &= \frac{1}{2i} \begin{pmatrix} ie^t(\cos t + i \sin t) + ie^t(\cos t - i \sin t) & -e^t(\cos t + i \sin t) + e^t(\cos t - i \sin t) \\ e^t(\cos t + i \sin t) - e^t(\cos t - i \sin t) & ie^t(\cos t + i \sin t) + ie^t(\cos t - i \sin t) \end{pmatrix} \\ &= \frac{1}{2i} \begin{pmatrix} 2ie^t \cos t & -2ie^t \sin t \\ 2ie^t \sin t & 2ie^t \cos t \end{pmatrix} \\ &= \begin{pmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{pmatrix} \end{aligned}$$

System c:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\begin{aligned} P(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 6 \\ 0 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 2-\lambda & 6 \\ 1 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)((2-\lambda)(3-\lambda) - 6) = (1-\lambda)(\lambda^2 - 5\lambda) = -\lambda^3 + 6\lambda^2 - 5\lambda \\ &= -\lambda(\lambda^2 - 6\lambda + 5) = -\lambda(\lambda - 1)(\lambda - 5) \end{aligned}$$

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 5$$

$$\begin{aligned} N(A - \lambda_1 I) &= N\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{pmatrix}\right) = N\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}\right) \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 + x_3 = 0, x_2 + 3x_3 = 0 \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \right\} \end{aligned}$$

$$N(A - \lambda_2 I) = N\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 6 \\ 0 & 1 & 2 \end{pmatrix}\right) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\begin{aligned} N(A - \lambda_3 I) &= N\left(\begin{pmatrix} -4 & 0 & 1 \\ 0 & -3 & 6 \\ 0 & 1 & -2 \end{pmatrix}\right) = N\left(\begin{pmatrix} 4 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\right) \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 4x_1 - x_3 = 0, x_2 - 2x_3 = 0 \right\} = \text{span} \left\{ \begin{pmatrix} 1 \\ 8 \\ 4 \end{pmatrix} \right\} \end{aligned}$$

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 8 \\ -1 & 0 & 4 \end{pmatrix}, \quad X^{-1} = \frac{1}{20} \begin{pmatrix} 0 & 4 & -8 \\ 20 & -5 & 5 \\ 0 & 1 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$e^{Dt} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}$$

$$\begin{aligned} y &= e^{At} = X e^{Dt} X^{-1} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 8 \\ -1 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{5t} \end{pmatrix} \frac{1}{20} \begin{pmatrix} 0 & 4 & -8 \\ 20 & -5 & 5 \\ 0 & 1 & 3 \end{pmatrix} \\ &= \frac{1}{20} \begin{pmatrix} 20e^t & e^{5t} - 5e^t + 4 & 3e^{5t} + 5e^t - 8 \\ 0 & 8e^{5t} + 12 & 24e^{5t} - 24 \\ 0 & 4e^{5t} - 4 & 12e^{5t} + 8 \end{pmatrix} \end{aligned}$$

Question 2

Solve the following initial value problems:

a. $\begin{cases} -y_1 + 2y_2 = y_1' \\ 2y_1 - y_2 = y_2' \end{cases}, \quad y_1(0) = 3, \quad y_2(0) = 1.$

b. $\begin{cases} y_1 - 2y_2 = y_1' \\ 2y_1 + y_2 = y_2' \end{cases}, \quad y_1(0) = 1, \quad y_2(0) = -2.$

Solution:

Question 3

In each of the following, "diagonalize" the matrix X and use it to compute A^{-1} , A^4 , e^A .

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix}$$

Solution:

Question 4

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$$

- Use the Gram-Schmidt process to find an orthonormal basis for the column space of A .
- Factor A into QR .
- Use the above to solve the system $Ax = b$.

Solution:

Question 5

Let $\{x_1, x_2, x_3\} = \{(0, 1, 0), (2, 1, 2), (0, 0, 1)\}$, be a basis of \mathbb{R}^3 .

- Use the Gram-Schmidt process to obtain an orthonormal basis.

- b. Let $b := (1, 1, 1)$. Compute the projection of b onto $\text{span}\{x_1, x_2\}$ and to $\text{span}\{x_3, x_2\}$.

Solution:

Question 6

Consider the vector space $C[0, 1]$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

- a. Find an orthonormal basis of the subspace E spanned by $1, x, x^2$.
b. Compute the length of $2x^2 + 3$.
c. Compute the projection of e^x onto E

Solution:

Question 7

Find the orthogonal complement of the subspace of \mathbb{R}^4 spanned by $(1, 1, 1, 1), (1, -1, 1, -1)$.

Solution:

Question 8

For each of the following systems $Ax = b$ find all least squares solutions:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Solution: