Question 1

Proof. Let n be an integer. We want to show that if $n^2 - 3n + 5$ is even, then n is odd. We will prove this by proving the contrapositive. That is, we will show that if n is even, then $n^2 - 3n + 5$ is odd. By definition, if n is even, then

$$n = 2k$$

for some integer k. We can substitute this into the expression for $n^2 - 3n + 5$ to get

$$\begin{aligned} n^2 - 3n + 5 &= (2k)^2 - 3(2k) + 5 \\ &= 4k^2 - 6k + 5 \\ &= 2(2k^2 - 3k + 2) + 1. \end{aligned}$$

Since $2k^2 - 3k + 2$ is an integer, we have shown that $n^2 - 3n + 5$ is odd. Thus, the contrapositive is true, and this also shows that the original statement is true. □

Question 2

Proof. Let n be an integer. We want to show that n is odd if and only if $n + 2$ is odd. To prove this, we will need to show that if n is odd, then $n + 2$ is odd, and if $n + 2$ is odd, then n is odd. By definition, if n is odd, then

$$n = 2k + 1$$

for some integer k. We can substitute this into the expression for $n + 2$ to get

$$\begin{aligned} n + 2 &= (2k + 1) + 2 \\ &= 2k + 3 \\ &= 2(k + 1) + 1. \end{aligned}$$

Since $k + 1$ is an integer, we have shown that $n + 2$ is odd when n is odd. Now, we will show that if $n + 2$ is odd, then n is odd using the contrapositive. That is, we will show that if n is even, then $n + 2$ is even. By definition, if n is even, then

$$n = 2k$$

for some integer k. We can substitute this into the expression for $n + 2$ to get

$$\begin{aligned} n + 2 &= 2k + 2 \\ &= 2(k + 1). \end{aligned}$$

Since $k + 1$ is an integer, we have shown that $n + 2$ is even when n is even. Thus, the contrapositive is true, and this also shows that $n + 2$ is odd when n is odd. Now we have proven both directions of the biconditional, so we have shown that n is odd if and only if $n + 2$ is odd. □

Question 3

Proof. Let m and n be integers. We want to show that mn is even if and only if m is even or n is even. We will need to prove both ways of the biconditional, so we will first show that if mn is even, then m is even or n is even. To do this, we will prove the contrapositive, which is that if m is odd and n is odd, then mn is odd. By definition, if m and n are odd, then we can write $m = 2k + 1$ and $n = 2l+1$ for some integers k and l. We can substitute these into the expression for mn to get

$$\begin{aligned} mn &= (2k+1)(2l+1) \\ &= 4kl + 2k + 2l + 1 \\ &= 2(2kl + k + l) + 1. \end{aligned}$$

Since $2kl + k + l$ is an integer, we have shown that mn is odd when m and n are odd, and thus the contrapositive is true. Since the contrapositive is true, this also shows that if mn is even, then m is even or n is even. Now, proving the other direction of the biconditional, we will show that if m is even or n is even, then mn is even. By definition, if m is even, then we can write $m = 2k$ for some integer k. We can substitute this into mn to get $mn = 2kn$, which is even. Without loss of generality, we can also show that if n is even, then mn is even. Thus, both directions of the biconditional have been proven, and we have shown that mn is even if and only if m is even or n is even. $\square$

Question 4

Proof. We want to show that if integers a and b are odd, then $4 \nmid (a^2 + b^2)$. Seeking a contradiction, suppose that there exist odd integers a and b such that $4 \mid (a^2 + b^2)$. Since a and b are odd, we can write $a = 2m + 1$ and $b = 2n + 1$ for some integers m and n. Additionally, since $4 \mid (a^2+b^2)$, we can write $a^2+b^2 = 4k$ for some integer k. Substituting these expressions into the equation, we get

$$\begin{aligned} (2m+1)^2 + (2n+1)^2 &= 4k \\ 4m^2 + 4m + 1 + 4n^2 + 4n + 1 &= 4k \\ m^2 + m + n^2 + n + \frac{1}{2} &= k. \end{aligned}$$

We know that k is an integer by the closure axioms, but $\frac{1}{2}$ is not reducible to an integer. Thus, we have reached a contradiction, and we have shown that if integers a and b are odd, then $4 \nmid (a^2 + b^2)$. $\square$

Question 5

Proof. We want to show that there do not exist integers m and n such that $8m + 26n = 1$. Seeking a contradiction, suppose that there exist integers m and n such that $8m + 26n = 1$. Then, we can write the following:

$$\begin{aligned} 8m + 26n &= 1 \\ 2(4m + 13n) &= 1 \\ 4m + 13n &= \frac{1}{2}. \end{aligned}$$

We know that $4m+13n$ is an integer by the closure axioms, but $\frac{1}{2}$ is not reducible an integer. Thus, we have reached a contradiction, and we have shown that there do not exist integers m and n such that $8m + 26n = 1$. □

Question 6

Lemma 1. An integer n is not divisible by 3 if and only if there exists an integer k such that $n = 3k + 1$ or $n = 3k + 2$.

Proof. Let n be an integer. We want to show that if $3 \mid n^2$, then $3 \mid n$. We will prove this by proving the contrapositive, which is that if $3 \nmid n$, then $3 \nmid n^2$. By lemma 1, we know that if $3 \nmid n$, then we can write $n = 3k + 1$ or $n = 3k + 2$ for some integer k. In the case that $n = 3k + 1$, we can write

$$n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1,$$

which is not divisible by 3. In the case that $n = 3k + 2$, we can write

$$n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1,$$

which is also not divisible by 3. Thus, the contrapositive is true, and this also shows that if $3 \mid n^2$, then $3 \mid n$. □

Question 7

Proof. We want to show that there do not exist integers m and n such that $m^2 = 4n + 3$. Seeking a contradiction, suppose that there does exist integers m and n such that $m^2 = 4n + 3$. Since m^2 can be written as $2(2n + 2) + 1$, we know that m^2 is odd. By corollary 2.2.5, m is also odd. Then, we can write $m = 2k + 1$ for some integer k. Substituting this into the equation, we get

$$\begin{aligned} (2k + 1)^2 &= 4n + 3 \\ 4k^2 + 4k + 1 &= 4n + 3 \\ 4k^2 + 4k - 2 &= 4n \\ 2k^2 + 2k - \frac{1}{2} &= n. \end{aligned}$$

We know that n, k^2, and k are all integers by the closure axioms, but $\frac{1}{2}$ is not reducible to an integer. Thus, we have reached a contradiction, and we have shown that there do not exist integers m and n such that $m^2 = 4n + 3$. $\square$