Question 1

Let

$$u_1 := (1,1,1)^T, u_2 := (1,2,2)^T, u_3 := (3,2,4)^T$$
$$v_1 := (4,6,7)^T, v_2 := (0,1,1)^T, v_3 := (0,1,2)^T$$

(a) Find the transition matrix corresponding to the change of basis from e_1, e_2, e_3 to u_1, u_2, u_3.

Solution: Let U be the transition matrix from the standard basis to the basis u_1, u_2, u_3.

$$U = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

To find the transition matrix from e_1, e_2, e_3 to u_1, u_2, u_3, we need to find the inverse of U.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 2 & -1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -2 \\ 0 & 1 & 0 & -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Thus, we have U^{-1} as:

$$U^{-1} = \begin{bmatrix} 2 & 1 & -2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

which is the transition matrix from e_1, e_2, e_3 to u_1, u_2, u_3.

(b) Find the transition matrix corresponding to the change of basis from v_1, v_2, v_3 to e_1, e_2, e_3.

Solution: Let V be the transition matrix from the basis v_1, v_2, v_3 to the standard basis. V is simply the matrix of the basis vectors v_1, v_2, v_3. Thus, we have:

$$V = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix}$$

(c) Find the transition matrix from v_1, v_2, v_3 to u_1, u_2, u_3.

Solution: We want to find the transition matrix from v_1, v_2, v_3 to u_1, u_2, u_3. This can be done by first finding the transition matrix from v_1, v_2, v_3 to the standard basis, and then multiplying it by the transition matrix from the standard basis to u_1, u_2, u_3. This means the transition matrix from v_1, v_2, v_3 to u_1, u_2, u_3 is simply $U^{-1}V$. Since we have already found U^{-1} and V in previous parts, we have:

$$U^{-1}V = \begin{bmatrix} 2 & 1 & -2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 1 \\ 7 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -3 \\ \frac{5}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(d) Let $x = 2v_1 + 3v_2 - 4v_3$. Find the coordinates of x with respect to u_1, u_2, u_3.

Solution: The vector x can be written as:

$$\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$$

with respect to the basis v_1, v_2, v_3. To find the coordinates of x with respect to u_1, u_2, u_3, we can multiply the vector by the transition matrix from v_1, v_2, v_3 to u_1, u_2, u_3. Thus, we have:

$$U^{-1}Vx = \begin{bmatrix} 0 & -1 & -3 \\ \frac{5}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ -1 \end{bmatrix}$$

Thus, the coordinates of the vector x with respect to u_1, u_2, u_3 is $9u_1 + 2u_2 - u_3$.

(e) Verify your answer to previous one, by computing the coordinates in each case with respect to the standard basis.

The vector x can be written with respect to the basis v_1, v_2, v_3 as:

$$x = 2\begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 14 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix}$$

Here we have that the coordinates of x with respect to the standard basis is $8e_1 + 11e_2 + 9e_3$. Writing x with respect to the basis u_1, u_2, u_3 gives:

$$x = 9\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ 9 \end{bmatrix}$$

Since the two coordinates are the same, we have verified our answer.

Question 2

Find a basis for the row space, column space and null space of the following matrices.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} B = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} C = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

Question 3

Let $E := [p_1(x) = 1, p_2(x) = x + 1, p_3(x) = x^2 - 1]$ and $F := q_1(x) = 1, q_2(x) = x, q_3(x) = x^2$. These are two basis of the vector space P_2 of all polynomials of degree at least 2. Find the transition matrix from E to F and the transition matrix from F to E. Express the polynomial

$$p(x) = 11x^2 - 2x + 5$$

with respect to the basis E.