Chapter 2

Mathematical Arguments

My dear friend, I'd advise, in sum,
First, the Collegium Logicum.
There your mind will be trained,
As if in Spanish boots, constrained,
So that painfully, as it ought,
It creeps along the way of thought, ...

— Johann Wolfgang von Goethe, *Faust I*

In this chapter, we will discuss the structure of mathematical arguments. We begin with an overview of propositional logic. We first introduce propositions and logical operators. Then we discuss tautologies, satisfiability, and logical equivalence. We discuss a propositional calculus that give us a first glimpse into the method of formal proofs. We sketch the rudiments of predicate logic including universal and existential quantifiers, predicates, and their interpretations. We conclude this chapter with a discussion of basic proof techniques. Furthermore, we will learn how to properly negate statements and how to leverage this knowledge in proofs by contradiction.

2.1 Statements

In mathematics, a **statement** is a sentence that is either true or false. Statements are also called **assertions** or **propositions**. For example, the sentence

There exists a closed knight's tour on an 8×8 *chessboard*

is a true statement. We showed in Chapter 1, Proposition 1.1, that the sentence

There exists a closed knight's tour on a 4×4 *chessboard*

is a false statement.

In general, how do we go about proving a mathematical statement? We assume some assertions that are believed to be true as **axioms**. Then we use logical reasoning to derive the statement from these axioms. Euclid nicely illustrated this deductive method in his *Elements*, a most influential textbook written around 300BC that developed the foundations of Euclidean geometry and the beginnings of number theory.

It should be self-evident that one needs to understand the rules of logic to construct a valid mathematical argument. Therefore, we give a brief discussion of symbolic logic. We conclude this chapter by discussing some proof strategies that illustrate how to apply the logical reasoning in mathematical arguments.

EXERCISES

2.1. Which of the following sentences are mathematical statements? Justify your answers.
(a) The number π is the smallest irrational real number.
(b) The number π is an irrational number.
(c) The quadratic equation $x^2 - 4x + 2 = 0$ does not have a solution in the integers.
(d) The inequality $123 - 100 > 23$ holds.
(e) x is a prime number.

2.2. Which of the following sentences are statements? Justify your answers.
(a) Is 17 a prime?
(b) $12x + 4 = 10$.
(c) Does there exist a knight's tour on a 31×31 chessboard?
(d) x is the sum of two squares.
(e) Every even integer greater than 2 can be expressed as the sum of two primes.

2.3. For each of the following statements, determine whether the statement is true or false. Explain why.
(a) The real number $0.111\cdots$ is equal to $1/9$.
(b) The real number $0.1212121212\cdots$ is not a rational number.
(c) The greatest common divisor of 1111 and 11111111 is equal to 1.
(d) $(-1)(-1) = 1$.
(e) The set of positive integers $\{1, 2, 3, \ldots\}$ is infinite.

2.4. Research the following statement: There exist infinitely many pairs of prime numbers that differ by 2. It is evidently a statement, but is it true?

2.5. Consider the statement "This statement is false". Is it true or false? Discuss.

2.2 Logical Operations

Logical operations allow one to combine several statements into a new compound statement. We discuss the disjunction, conjunction, negation, implication, and equivalence operations. You need to be aware of the precise meaning of these operations so that you can properly follow a mathematical argument.

Disjunction. Let A and B be statements. When is "A or B" true? In English, the word "or" is sometimes used in the inclusive sense "A or B, or both" and sometimes in the exclusive sense "either A or B, but not both". For example, if a restaurant offers complimentary coffee or tea after a meal, they don't expect you to choose both. If they offer milk or sugar with your coffee, then it is meant that you can select milk, sugar, or both.

In mathematics, one cannot have this kind of ambiguity. Mathematicians agreed that the **disjunction** "A or B" is always understood in the inclusive way "A or B, or both". For instance, if we consider a real number x, then

$$x \leqslant 0 \quad \text{or} \quad x \geqslant 0$$

is a true statement, even though $x = 0$ fulfills both $x \leqslant 0$ and $x \geqslant 0$.

In symbolic logic, we write $A \vee B$ for the mathematical statement A or B. The statements A and B can each be true or false, so there are four possibilities in all for the arguments of the disjunction. Writing T for true and F for false, we can express the behavior of the disjunction by the truth table

A	B	$A \vee B$
F	F	F
F	T	T
T	F	T
T	T	T

Note that "A or B" is false only when both the statement A and the statement B are false.

You might be wondering why we are concerned with statements that are false. Well, one proof technique first negates the statement and then tries to find a contradiction. So false statements occur quite often within mathematical arguments, even though we are ultimately concerned with establishing the truth of mathematical assertions that are stated in theorems, propositions, and lemmas.

Conjunction. The **conjunction** "A and B" is true if and only if both A is true and B is true. The conjunction is also written as $A \wedge B$. The truth table is given by

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

For example, one might want to establish that $2 < e < 3$, which amounts to prove the conjunction of the inequalities

$$2 < e \quad \text{and} \quad e < 3.$$

Negation. Let A be a statement. Then "not A" or "$\neg A$" is the **negation** of the statement. For instance, if A is the statement "We do serve breakfast until 11:00am", then the negation $\neg A$ is "We do not serve breakfast until 11:00am". The statement A is true if and only if the negated statement $\neg A$ is false. The truth table is given by

A	$\neg A$
F	T
T	F

Implication. The **implication** "A implies B" is particularly important in proofs. It is often written in the form "If A, then B". We will also denote it as $A \to B$. We call A the **hypothesis** and B the **conclusion**.

If the hypothesis A is true and the conclusion B is true, then "A implies B" ought to be true. If the hypothesis A is true and the conclusion B is false, then "A implies B" ought to be false, since we do not want to conclude something that is false from a true hypothesis.

However, it is not so clear what the truth value of the implication should be if the hypothesis is false. Mathematicians agreed on the convention that an implication is true if its hypothesis is false. If the hypothesis A is false, then we say that the implication $A \to B$ is **vacuously true**. Thus, the truth table of the implication is given by

A	B	$A \to B$
F	F	T
F	T	T
T	F	F
T	T	T

It might seem strange that one can conclude anything from a false statement. However, even in everyday English we sometimes make a point by drawing an even more absurd conclusion from an absurd statement, such as *If this mafioso is innocent, then I will eat my hat.* Exercise 2.14 explains why mathematicians did not really have the option to define the implication in a different way.

Another important point is that an implication is not based on cause and effect. For example, if A is the statement "1+1=2" and B is the statement "blueberries are blue", then the implication $A \to B$ is true. Evidently, the correctness of $1 + 1 = 2$ does not cause blueberries to be blue. So merely the truth of the statements A and B determine whether the implication $A \to B$ is true.

Equivalence. If an implication $A \to B$ and its converse $B \to A$ are both true, then we say that A and B are **equivalent** and we write $A \leftrightarrow B$. The truth table of an equivalence is given by

A	B	$A \leftrightarrow B$
F	F	T
F	T	F
T	F	F
T	T	T

We see that $A \leftrightarrow B$ is true if and only if A and B have the same truth value. In the literature, an equivalence is sometimes also called a **bi-implication**.

Boolean Formulas. A **Boolean variable** is a variable that can take on the values T for true and F for false. We can construct **Boolean formulas** as follows.

B1. Every Boolean variable is a Boolean formula.

B2. If A is a Boolean formula, then so is $\neg A$.

B3. If A and B are Boolean formulas, then $(A \vee B)$, $(A \wedge B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$ are Boolean formulas.

A Boolean formula is formed by applying **B1**–**B3** a finite number of times.

Example 2.1. We claim that

$$((\neg A \wedge B) \rightarrow (\neg A \vee B))$$

is a Boolean formula. Indeed, the Boolean variables A and B are Boolean formulas by **B1**. It follows that $\neg A$ is a Boolean formula by **B2**. By **B3**, $(\neg A \wedge B)$ and $(\neg A \vee B)$ are Boolean formulas. Using **B3** once more, we can conclude that $((\neg A \wedge B) \rightarrow (\neg A \vee B))$ is a Boolean formula.

By building up Boolean formulas starting from Boolean variables, their negations, and more and more complicated compound formulas, we can unravel the syntactic structure of the Boolean formula. However, this also allows us to form the truth table that assigns truth values to the Boolean variables. After a truth value is assigned to a Boolean variable, we can deduce the value of its negation. If the truth values are known for subformulas A and B, then we can deduce the truth value for expressions such as $(A \vee B)$, $(A \wedge B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$.

Let us illustrate how to derive a truth table for the Boolean formula $((\neg A \wedge B) \rightarrow (\neg A \vee B))$ of the above example. The formula contains two Boolean variables, so there are a total for four assignments of truth values to these two Boolean variables A and B. The value of A determines $\neg A$. Knowing the truth values of $\neg A$ and B, we can deduce the truth values of $(\neg A \wedge B)$ and $(\neg A \vee B)$. After determining the truth values of $(\neg A \wedge B)$ and $(\neg A \vee B)$, we can deduce the truth value of the entire formula

$$((\neg A \wedge B) \rightarrow (\neg A \vee B)).$$

integers $a, b > 1$. Since a and b are numbers that are smaller than m, we can write them in the form

$$a = a_1 a_2 \cdots a_k, \quad \text{and} \quad b = b_1 b_2 \cdots b_\ell$$

such that $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_\ell$ are primes. This would mean that $m = a_1 a_2 \cdots a_k b_1 b_2 \cdots b_\ell$ is a product of primes, contradicting our assumption. ❑

Remark 2.49. Let $P(m)$ denote the predicate that is true if and only if m is an integer that can be factored into a product of primes. The claim of the previous proposition is that $\forall m P(m)$, where m extends over all integers that are greater than 1. Then the negation is the statement $\neg\forall m\, P(m) \equiv \exists m\, \neg P(m)$. In words, we begin the proof by assuming that there exists an integer $m > 1$ that cannot be factored into a product of primes. However, we used an important trick. We assumed that m is the smallest integer $m > 1$ that cannot be factored into primes. This made it easier to find a contradiction.

Proofs by contradiction go back to antiquity. The next theorem is contained in Euclid's elements, but we give Kummer's elegant variant of Euclid's proof.

Theorem 2.50 (Euclid)**.** *There are infinitely many prime numbers.*

Proof. Seeking a contradiction, suppose that there exist only a finite number of primes $p_1, p_2, \ldots, p_k$. Let $P = p_1 p_2 \cdots p_k$ be their product. The integer $P - 1$ can be factored into primes by Proposition 2.48. Each prime p_m dividing $P-1$ also divides P, hence by Proposition 2.40 it must divide the difference

$$P - (P - 1) = 1,$$

which is absurd. ❑

Let us conclude this chapter with a particularly striking application of a proof by contradiction. The game of hex was invented by the Danish mathematician, scientist and poet Piet Hein and later rediscovered by John Nash. It is a board game that is played by two players on an $n \times n$ rhombus-shaped grid with hexagonal fields. Two opposing sides of the game board are labeled white and the other two are labeled black. The four corner hexagons belong to either side. A 4×4 game board is depicted on the left of Figure 2.3.

One player plays white playing pieces and the other plays black. The players take turns placing a single playing piece of their color into a empty hexagon cell anywhere on the game board. The goal of each player is to connect the two sides of their color with a connected path of playing pieces of their color. The first player to connect the two regions of her color wins. Figure 2.3 on the right shows a game that ends with White winning after a total of 7 moves.

One can show that the game of hex cannot end in a tie. Nash proved in 1952 that the first player has a winning strategy assuming optimal play. We give his argument here.

Theorem 2.51. *The first player in a hex game has a winning strategy.*

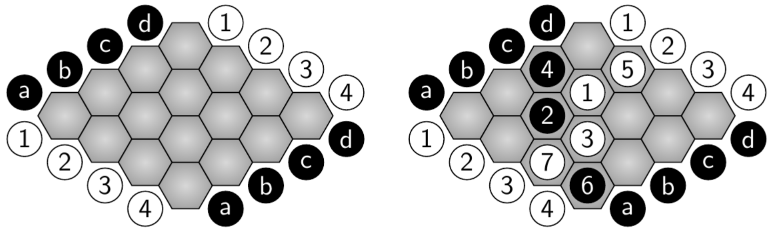

Figure 2.3: The figure on the left shows a 4×4 hex board with the lower left and upper right sides in white, and the upper left and lower right in black. We can use algebraic notation to specify the hexagons of the game board. For instance, a1 is the left-most hexagon and d4 is the rightmost hexagon. The figure on the right illustrates one game. White begins with c2, then Black plays b2, White plays b3, then Black c1. White plays d2 and Black responds with a4. White's final move a3 completes the path between the two white regions, so White wins.

Proof. Since the hex game cannot end in a tie, one of the players must have a winning strategy. Seeking a contradiction, let us assume that the second player has a winning strategy. Then the first player can place a playing piece T anywhere on the board and from then on pretend to be the second player; this means that he will play according to the winning strategy. If the winning strategy requires the first player to place a playing piece on the field where T is placed, then he can place during this move his piece on any other field, as this cannot hurt his chances. This means that the second player cannot win anymore, contradicting our assumption that the second player has a winning strategy. □

This strategy-stealing argument works because the first move cannot hurt the first player. What does it really mean that the first player can steal the strategy? The nexus of the argument is that the second player simply cannot have a winning strategy!

The theorem proves the existence of a winning strategy for the first player, but it does not provide a recipe for such a strategy. Working out an explicit winning strategy turns out to be difficult. Even six decades later all explicitly known winning strategies for hex are confined to small boards of size $n \leqslant 10$, despite a significant amount of research. In fact, we cannot hope that good algorithms for hex will be developed, since it is know that hex is PSPACE-complete (which means that is ranks among the most difficult problems that can be solved on a computer with a polynomial amount of space, and it is unlikely that such problems can be solved in polynomial time). You can learn more about PSPACE-complete problems in a course on algorithms.

EXERCISES

2.73. Suppose that m and n are consecutive integers. Use a direct proof to show that their sum $m + n$ is an odd integer.

2.74. Suppose that n is an odd integer. Use a direct proof to show that n^2-1 is divisible by 8.

2.75. Use a direct proof to show that n^3-n is divisible by 6 for all integers n.

2.76. Let n be an integer. Give a direct proof that $n(n+1)(n+2)(n+3)$ is divisible by 24.

2.77. Let n be a positive even integer. Show that

$$\frac{1!2!3!\cdots(2n)!}{n!}$$

is a perfect square.

2.78. Using a direct proof, show that if a and b are real numbers, then the reverse triangle inequality

$$|a-b| \geqslant \big||a|-|b|\big|$$

holds.

2.79. Using a direct proof, show that if a and b are real numbers, then

$$|a|+|b| \leqslant |a+b|+|a-b|.$$

2.80. Let m and n be integers. Use a proof by contraposition to show that if their sum $m+n>100$, then $m>40$ or $n>60$.

2.81. Let n be a positive integer. Use a proof by contraposition to show that if $n \leqslant ab$ for some positive integers a and b, then $a \leqslant n^{1/2}$ or $b \leqslant n^{1/2}$.

2.82. Let m be an integer. Use a proof by contraposition to show that if $5m+7$ is even, then m is odd.

2.83. Let n be a positive integer. Use a proof by contraposition to show that if 2^n-1 is prime, then n must be prime. [A prime M_n of the form $M_n=2^n-1$ is called a **Mersenne prime**.]

2.84. Prove by contradiction that the equation $42m+70n=1000$ does not have an integer solution.

2.85. Proof by contradiction that $\sin x+\cos x \geqslant 1$ holds for all real numbers x in the range $0 \leqslant x \leqslant \pi/2$.

2.10 Notes

You will need to read many proofs and practice regularly to master the art of proofs. The books by Eccles [25] and Velleman [80] are recommended for further reading about proof techniques. We gave a brief introduction to symbolic logic, and we hope that you want to study this important topic in more depth. Fortunately, there are many excellent texts on first-order predicate logic, see for example Ebbinghaus, Flum, and Thomas [24], Enderton [27], Gallier [29], Goldrei [30], Huth and Ryan [38], Mendelson [62], Smullyan [71] and [72]. We recommend these texts for further reading. A delightful collection of knight and knave puzzles is given in [70]. There is a wide range of applications of symbolic logic and a very long history that starts with Aristotle's logical works. The formalization of arguments that started with Aristotle were brought to perfection through Frege, Hilbert, Łukasiewicz, Russell, and Whitehead and many others. We strongly recommend that you explore `metamath.org` to see many more example of formal proofs. It would be very remiss of us not to mention at least some limitations of formal methods. For example, it is undecidable whether a formula of first order logic is true under all possible interpretations.