MATH300 Homework 3 (Due Friday, 2/9)

1. (9 pts) Prove that if m is even and n is odd, then $m + n$ is odd.

2. (9 pts) Let $a, b, c, k, \ell \in \mathbb{Z}$. Prove that if $a \mid b$ and $a \mid c$, then $a \mid (bk + c\ell)$.

3. (14 pts) In mathematics, a **lemma** is a generally minor, proven proposition used to prove another, usually more substantial, theorem. It is sometimes called a "helping theorem" or "pre-theorem". Use the following lemma to prove that for all integers m, if m is odd, then there exists $k \in \mathbb{Z}$ such that $m^2 = 8k + 1$.

   Lemma 1. For all integers n, $n^2 + n$ is even.

   (Note that we proved this lemma in class; you can use this result without reproving it. Somewhere in your proof, you should have "By Lemma 1, ..." or "... by Lemma 1.".)

4. (14 pts) Prove that for every integer n, n^2+n-9 is odd. Give a formal proof (use only definitions, axioms, and logic).

5. (6 pts) Consider

   Lemma 2 (from Problem 1). If m is even and n is odd, then $m + n$ is odd. (simply put: "even plus odd is odd")

   We can use Lemmas 1 and 2 to give another proof that for every integer n, $n^2 + n - 9$ is odd.

   Proof. Let $n \in \mathbb{Z}$. By Lemma 1, $n^2 + n$ is even. Since -9 is odd, Lemma 2 gives $(n^2 + n) + (-9)$ is odd. □

   Share your thoughts when comparing the two proofs of "For every integer n, $n^2 + n - 9$ is odd".

6. (48 pts) Prove or disprove the following. (Hint: exactly three statements are true.)

   (a) Let $a, b \in \mathbb{Z}$. If $a \mid b$, then $a^2 \mid b^2$.

   (b) For all positive integers a, b, and c, the equality $a^{b^c} = (a^b)^c$ holds.

(c) If $x, y \in \mathbb{R}$, then $|x - y| = |x| - |y|$.

(d) For all even integers m and n, $4 \mid mn$.

(e) For all integers a, b, c, if $a \mid (b + c)$, then $a \mid b$ or $a \mid c$.

(f) There exist integers m and n such that $15m + 12n = -6$.