Question 1

Part a

Disproof. Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{1, 3\}$. Indeed, $A \subseteq B \cup C$, but $A \nsubseteq B$ and $A \nsubseteq C$. Thus, the statement is false. $\square$

Part b

Proof. Let A, B, and C be sets. Assume that $A \subseteq B \cap C$. Let x be an element of A. Since A is a subset of $B \cap C$, $x \in B \cap C$. By definition of intersection, $x \in B$ and $x \in C$. Since x was arbitrary, we have that $A \subseteq B$ and $A \subseteq C$. $\square$

Question 2

Proof. Let A and B be subsets of a universal set $\mathcal{U}$. Then,

$$\begin{aligned} (A \cap B) \cup (A - B) &= (A \cap B) \cup (A \cap \overline{B}) \quad \text{(by definition of set difference)} \\ &= A \cap (B \cup \overline{B}) \quad \text{(by distributive law)} \\ &= A \cap \mathcal{U} \\ &= A. \end{aligned}$$

Thus, $(A \cap B) \cup (A - B) = A$. $\square$

Question 3

Proof. Let A be a set. Seeking a contradiction, assume that $A - A \neq \emptyset$. Then, there exists an element $x \in A - A$. By definition of set difference, $A - A$ is equivalent to $A \cap \overline{A}$. Thus, $x \in A$ and $x \in \overline{A}$. However, this is a contradiction, as A and $\overline{A}$ are disjoint. Therefore, $A - A = \emptyset$. $\square$

Question 4

$R = \{2, 8, J, Q, A\}$, $S = \{\heartsuit, \diamondsuit\}$

$R \times S = \{(2, \heartsuit), (2, \diamondsuit), (8, \heartsuit), (8, \diamondsuit), (J, \heartsuit), (J, \diamondsuit), (Q, \heartsuit), (Q, \diamondsuit), (A, \heartsuit), (A, \diamondsuit)\}$

Question 5

Proof. Let A, B, and C be sets. Then,

$$\begin{aligned}(x,y) \in A \times (B \cap C) &\iff x \in A \land y \in B \cap C \\ &\iff x \in A \land y \in B \land y \in C \\ &\iff (x,y) \in A \times B \land (x,y) \in A \times C\end{aligned}$$

Therefore, $(x,y) \in (A \times B) \cap (A \times C)$. $\square$

Question 6

Part a

The cardinality of the cartesian product of two sets is the product of the cardinalities of the two sets.

$$\begin{aligned}|\{2,4,6,\dots,20\} \times \{a,b,c,d,e,f\}| &= |\{2,4,6,\dots,20\}| \cdot |\{a,b,c,d,e,f\}| \\ &= 10 \cdot 6 \\ &= 60.\end{aligned}$$

Part b

The cardinality of the power set of a set A is $2^{|A|}$.

$$\begin{aligned}|\mathcal{P}(\mathcal{P}(A))| &= 2^{|\mathcal{P}(A)|} \\ &= 2^{2^{|A|}} \\ &= 2^{2^3} \\ &= 2^8 \\ &= 256.\end{aligned}$$

Question 7

Let $A = \{1, \{2, \{3\}\}\}$.

(a) The elements of A are 1 and $\{2, \{3\}\}$.

(b) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{\{2, \{3\}\}\}, \{1, \{2, \{3\}\}\}\}$.

(c) True.

Question 8

Proof. Let A and B be sets, and assume that $A \subseteq B$. Let x be an element of A. Then $\{x\} \subseteq A$, so $\{x\} \in \mathcal{P}(A)$. Since $A \subseteq B$, $\{x\} \subseteq B$, so $\{x\} \in \mathcal{P}(B)$. Therefore, $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. $\square$

Question 9

For $i \in \mathbb{Z}^+$, let $A_i = [i-4, i]$.

Part a

$$\begin{aligned}
\bigcup_{i=4}^{7} A_i &= A_4 \cup A_5 \cup A_6 \cup A_7 \\
&= [4-4, 4] \cup [5-4, 5] \cup [6-4, 6] \cup [7-4, 7] \\
&= [0, 4] \cup [1, 5] \cup [2, 6] \cup [3, 7] \\
&= [0, 7].
\end{aligned}$$

Part b

$$\begin{aligned}
\bigcap_{i=4}^{7} A_i &= [0, 4] \cap [1, 5] \cap [2, 6] \cap [3, 7] \\
&= [1, 4] \cap [2, 6] \cap [3, 7] \\
&= [2, 4] \cap [3, 7] \\
&= [3, 4].
\end{aligned}$$