Question 1

Determine whether the following subsets are subspaces:

Part a

$S_1 := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 = \sqrt{123}x_2\}$

Answer: This is a subspace of $\mathbb{R}^2$ since it is a straight line that passes through the origin.

Proof. Let (a, b) and (c, d) be two elements of S_1. We want to show that $(a, b) + (c, d) \in S_1$ and $n(a, b) \in S_1 \forall n \in \mathbb{R}$. Using the definition of S_1, we have that $a = \sqrt{123}b$ and $c = \sqrt{123}d$. Adding elements (a, b) and (c, d), we have that $(a, b) + (c, d) = (a + c, b + d)$ We can then substitute in the values of a and c to get $(a + c, b + d) = (\sqrt{123}b + \sqrt{123}d, b + d)$ This can then be factored to $(\sqrt{123}(b + d), b + d)$ Since this satisfies the definition of S_1, we have that $(a, b) + (c, d) \in S_1$. To show that $n(a, b) \in S_1 \forall n \in \mathbb{R}$, we can use the definition of S_1 again. The element (a, b) can be written as $(\sqrt{123}b, b)$. Multiplying this by n gives us $(n\sqrt{123}b, nb)$. Since this satisfies the definition of S_1, we have that $n(a, b) \in S_1 \forall n \in \mathbb{R}$. □

Part b

$S_2 := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1x_2 = 1\}$

Answer: This is not a subspace of $\mathbb{R}^2$ since it does not satisfy the addition property.

Proof. Let (a, b) and (c, d) be two elements of S_2. Seeking a contradiction, lets assume that $(a, b) + (c, d) \in S_2$. Since we can write $(a, b) + (c, d)$ as $(a + c, b + d)$, our assumption would imply that $(a + c)(b + d) = 1$. Expanding this, we get $ab + ad + bc + cd = 1$. It is given that $ab = 1$ and $cd = 1$, so we can substitute these in to get $1 + ad + bc + 1 = 1$. This can be simplified to $ad + bc = -1$. However, a and b multiply to a positive number, and c and d multiply to a positive number. This implies that $ad + bc$ must be positive, so we have reached a contradiction. Therefore, $(a, b) + (c, d) \notin S_2$, so S_2 is not a subspace of $\mathbb{R}^2$. □

Part c

$S_3 := \{\text{the set of singular } 2 \times 2 \text{ matrices}\}$

Answer: This is not a subspace of $\mathbb{R}^{2\times 2}$ since it does not satisfy the addition property.

Counterexample: Let matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ where A, B $\in S_3$. $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is not a singular matrix.

Part d

Let A be a fixed (but arbitrary) 2×2 matrix.

$$S_4 := \{B \in \mathbb{R}^{2\times 2} : BA = 0\}$$

Answer: This is a subspace of $\mathbb{R}^{2\times 2}$.

Proof. Let J and K be two arbitrary elements in S_4. We want to show that J, $K \in S_4 \rightarrow J + K \in S_4$ and $aJ \in S_4 \forall a \in \mathbb{R}$. In order for $J + K$ to be in S_4, we must have that $(J + K)A = 0$. We can rewrite this as $JA + KA = 0$. Since J and K are in S_4, we know that $JA = 0$ and $KA = 0$. Substituting these in, we get $0 + 0 = 0$, which is true. Therefore, $J + K \in S_4$. Additionally, in order for aJ to be in S_4, we must show that $\forall n \in \mathbb{R}, J \in S_4 \rightarrow aJ \in S_4$. To do this, we must verify the validity of $(aJ)A = 0$. Since scalar multiplication is commutative, we can rewrite this as $a(JA) = 0$. Since J is in S_4, we know that $JA = 0$. Substituting this in, we get $a(0) = 0$, which is true. Therefore, $aJ \in S_4$, so S_4 is a subspace of $\mathbb{R}^{2\times 2}$. □

Part e

$$S_5 := \{\text{the set of all polynomials of degree 2 or 4}\}$$

Answer: This is not a subspace of $\mathbb{P}_n$.
Counterexample:
Let $p(x), q(x) \in S_5$. Suppose that $p(x) = x^2$ and $q(x) = -x^2$.
Then $p(x) + q(x) = 0$, which is not a polynomial of degree 2 or 4.

Part f

$$S_6 := \{\text{the set of upper triangular } 2 \times 2 \text{ matrices}\}$$

Answer: This is a subspace of $\mathbb{R}^{2\times 2}$.

Proof. Let A and B be two arbitrary elements in S_6. To prove that S_6 is a subspace of $\mathbb{R}^{2\times 2}$, we must show that A, $B \in S_6 \rightarrow A + B \in S_6$ and $\forall n \in \mathbb{R}, nA \in S_6$. Given that A and B are in S_6, we know that A and B are upper triangular matrices. Writing these out, we have:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$
$$B = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}$$

Since matrix addition is element-wise, we can write $A + B$ as:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ 0 & a_{22} + b_{22} \end{bmatrix}$$

It is clear in this form that $A+B$ is an upper triangular matrix, so $A+B \in S_6$.

To show that $\forall n \in \mathbb{R}, nA \in S_6$, we must show that nA is an upper triangular matrix. We know that nA is an upper triangular matrix if nA is of the form:

$$nA = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

Since scalar multiplication is element-wise, we can write nA as:

$$nA = \begin{bmatrix} na_{11} & na_{12} \\ 0 & na_{22} \end{bmatrix}$$

In this form, it is clear that nA is an upper triangular matrix. Therefore, $nA \in S_6$, so S_6 is a subspace of $\mathbb{R}^{2\times 2}$. $\square$

Part g

$$S_7 := \{p \in \mathbb{P}_4 : p(0) = 0\}$$

Answer: This is a subspace of $\mathbb{P}_4$.

Proof. Let $f(x)$ and $g(x)$ be two arbitrary elements in S_7. To prove that S_7 is a subspace of $\mathbb{P}_4$, we must show that $f(x),\ g(x) \in S_7 \to f(x) + g(x) \in S_7$ and $\forall n \in \mathbb{R}, nf(x) \in S_7$. Given that $f(x)$ and $g(x)$ are in S_7, we know that $f(0) = 0$ and $g(0) = 0$, which is by definition of S_7. This means that $f(x)$ and $g(x)$ are of the form:

$$f(x) = a_1x^4 + a_2x^3 + a_3x^2 + a_4x + 0$$
$$g(x) = b_1x^4 + b_2x^3 + b_3x^2 + b_4x + 0$$

This means we can write $f(x) + g(x)$ as:

$$f(x) + g(x) = (a_1 + b_1)x^4 + (a_2 + b_2)x^3 + (a_3 + b_3)x^2 + (a_4 + b_4)x + 0$$

In this form it is clear that $f(x) + g(x)$ satisfies the condition that $p(0) = 0$, so $f(x) + g(x) \in S_7$.

To show that $\forall n \in \mathbb{R}, nf(x) \in S_7$, we can follow a similar process. We know that $nf(x)$ is of the form:

$$nf(x) = na_1x^4 + na_2x^3 + na_3x^2 + na_4x + 0$$

Regardless of the value of n, $nf(x)$ will always satisfy the condition that $p(0) = 0$, so $nf(x) \in S_7$. Therefore, S_7 is a subspace of $\mathbb{P}_4$. $\square$

Question 2

Find the null space of the following matrices:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -2 & 2 & 1 \\ 2 & 4 & -4 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 3 \\ 4 & 3 & 0 \end{bmatrix}$$

Answer:

Question 3

Show that the following matrices form a spanning set for $\mathbb{R}^{2\times 2}$. Also, show that these matrices are linearly independent.

$$A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Answer:

Question 4

Let x_1, x_2, and x_3 be linearly independent vectors in $\mathbb{R}^n$. Let

$$y_1 = x_1 + x_2, \quad y_2 = x_2 + x_3, \quad y_3 = x_3 + x_1.$$

Decide if y_1, y_2, and y_3 are linearly independent or not.

Answer: