Question 1

Consider the following collections of polynomials in $\mathbb{P}_2$:

(a) $p_1(x) = 1, \quad p_2(x) = x + 1, \quad p_3(x) = x^2.$

(b) $p_1(x) = x - 1, \quad p_2(x) = x + 1, \quad p_3(x) = x^2 - 1.$

(c) $p_1(x) = x^2 - 1, \quad p_2(x) = x^2 + 1, \quad p_3(x) = x^2.$

Decide in each case if these vectors are linearly independent. Write the dimension of the subspace $S := span\{p_1, p_2, p_3\}$ in each case. In which case(s) would we have that $S = \mathbb{P}_2$? Explain your answer.

Answer: To check for linear independence, we can use the Wronskian determinant.

(a)

$$\begin{aligned} W(p_1, p_2, p_3) &= \begin{vmatrix} 1 & x+1 & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} \\ &= 2 \neq 0 \end{aligned}$$

(b)

(c)

Question 2

Consider the following collections of smooth functions $[0, 1]$:

1. $f_1(x) = x^2$, $f_2(x) = \frac{1}{x^2}$
2. $f_1(x) = \cos(x)$, $f_2(x) = \sin(x)$
3. $f_1(x) = 1$, $f_2(x) = \frac{e^x + e^{-x}}{2}$, $f_3(x) = \frac{e^x - e^{-x}}{2}$

Decide in each case if these vectors (functions) are linearly independent.

Question 3

Find the dimension of the space spanned by the functions

$$1, \, \cos(2x), \, \cos^2(x)$$

Question 4

For each of the following find the transition matrix corresponding to the change of basis from $\{u_1, u_2\}$ to the standard one $\{e_1, e_2\}$:

(a) $u_1 = (1, 1)^T$, $u_2 = (-1, 1)^T$

(b) $u_1 = (1, 2)^T$, $u_2 = (2, 5)^T$

(c) $u_1 = (0, 1)^T$, $u_2 = (1, 0)^T$

Let
$$v_1 = (3, 2)^T, \quad v_2 = (4, 3)^T$$

For each of the basis above find the transition matrix from $[v_1, v_2]$ to $[u_1, u_2]$.
Let
$$x = (2, 4)^T, \quad y = (1, 1)^T, \quad z = (0, 10)$$

Find the coordinates of x, y, z with respect to each of the basis mentioned above.