Question 1

Proof. Let A be a set. We want to show that $f : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by $f(X) = \overline{X}$ is a bijection. First, we must show that f is injective. Assume that $X_1, X_2 \in \mathcal{P}(A)$ and $f(X_1) = f(X_2)$. Then,

$$\begin{aligned} f(X_1) &= f(X_2) \\ \overline{X_1} &= \overline{X_2} \\ \overline{\overline{X_1}} &= \overline{\overline{X_2}} \\ X_1 &= X_2. \end{aligned}$$

Thus, f is injective. Now, we must show that f is surjective, or in other words, that $\operatorname{Ran}(f) = \mathcal{P}(A)$. The subset relation $\operatorname{Ran}(f) \subseteq \mathcal{P}(A)$ is trivial. To show that $\mathcal{P}(A) \subseteq \operatorname{Ran}(f)$, let $Y \in \mathcal{P}(A)$. Consider the set $\overline{Y}$. Since $Y \in \mathcal{P}(A)$, $\overline{Y} \in \mathcal{P}(A)$. Then, $f(\overline{Y}) = \overline{\overline{Y}} = Y$. Thus, f is surjective. Since f is both injective and surjective, f is a bijective. $\square$

Question 2

Part a

Proof. We want to show that the function $f : (-\infty, 1) \to \mathbb{R}$ defined by $f(x) = x^3$ is not bijective. Seeking a contradiction, assume that f is bijective. Then, f is surjective. Consider the element $8 \in \mathbb{R}$. Since f is surjective, there exists an element $x \in (-\infty, 1)$ such that $f(x) = x^3 = 8$. Thus,

$$\begin{aligned} x^3 &= 8 \\ x &= 2. \end{aligned}$$

However, $2 \notin (-\infty, 1)$, a contradiction. Therefore, f is not bijective. $\square$

Part b

Proof. We want to show that the function $D : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $D(f(x)) = f'(x)$ is not bijective. Consider the elements $x, x+1 \in \mathbb{R}[x]$. Indeed, $\frac{d}{dx}(x) = 1 = \frac{d}{dx}(x+1)$, but $x \neq x+1$. Therefore, D is not injective, and thus not bijective. $\square$

Part c

Proof. We want to show that the function $s : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $s(m, n) = m + n$ is not bijective. Consider the elements $(1, 1), (2, 0) \in \mathbb{N} \times \mathbb{N}$. Indeed, $s(1, 1) = 2 = s(2, 0)$, but $(1, 1) \neq (2, 0)$. Therefore, s is not injective, and thus not bijective. $\square$

Question 3

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be functions. We want to show that if $g \circ f$ is injective, then f is injective, but g need not be injective. Assume that $g \circ f$ is injective. To show that f is injective, assume that $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$. Then, $(g \circ f)(x_1) = (g \circ f)(x_2)$. Since $g \circ f$ is injective, $x_1 = x_2$. To show that g need not be injective, consider the following example. Let $X = \{1, 2\}$, $Y = \{a, b, c\}$, and $Z = \{0, 1\}$. Additionally, let $f : X \to Y$ be defined by its graph

$$G_f = \{(1, a), (2, b)\}$$

and $g : Y \to Z$ be defined by its graph

$$G_g = \{(a, 0), (b, 1), (c, 1)\}.$$

Now consider the composition $g \circ f$, defined by the graph

$$G_{g \circ f} = \{(1, 0), (2, 1)\}.$$

Here it is obvious that $g \circ f$ is injective, since all elements in the range have no common preimage. However, g is not injective, since $g(b) = g(c) = 1$. Thus, f is injective, but g need not be injective. □

Question 4

(a) $f(\{-3, 2, 7\}) = \{10, 5, 50\}$

(b) $f([-1, 3]) = [1, 10]$

(c) $f((-\infty, -2)) = (-3, \infty)$

Question 5

Part a

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be invertible functions. Since f and g are invertible, they are also bijective. Since f and g are bijective, the composition $g \circ f$ is also bijective. Since $g \circ f$ is bijective, it is invertible. □

Part b

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be invertible functions. Then,

$$\begin{aligned}
(f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ (g^{-1} \circ g) \circ f \\
&= f^{-1} \circ \mathrm{id}_Y \circ f \\
&= f^{-1} \circ f \\
&= \mathrm{id}_X.
\end{aligned}$$

Similarly,

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ (f \circ f^{-1}) \circ g^{-1} \\ &= g \circ \mathrm{id}_Y \circ g^{-1} \\ &= g \circ g^{-1} \\ &= \mathrm{id}_Z.\end{aligned}$$

Thus, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. $\square$

Question 6

Given that $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 2024 - 2x$, we claim that $f([-3, 5]) = [2030, 2014]$.

Proof. Let $x \in [-3, 5]$. Then, $-3 \leq x \leq 5$. Plugging these bounds into $f(x)$, we have

$$\begin{aligned}f(-3) &= 2024 - 2(-3) = 2030, \\ f(5) &= 2024 - 2(5) = 2014.\end{aligned}$$

Since f is a linear function, it is continuous and also strictly decreasing by the negative slope. Thus, for any $x \in [-3, 5]$, $f(x) \in [2030, 2014]$. $\square$

Question 7

Given that $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^4$, we claim that $f((0, 2)) = (0, 16)$.

Proof. Let $x \in (0, 2)$. Then, $0 < x < 2$. Plugging these bounds into $f(x)$, we have

$$\begin{aligned}f(0) &= 0^4 = 0, \\ f(2) &= 2^4 = 16.\end{aligned}$$

We know that f is a 4th degree polynomial, so it is continuous. Since $f'(x) = 4x^3 > 0$ for all $x \in (0, 2)$, f is strictly increasing. Thus, for any $x \in (0, 2)$, $f(x) \in (0, 16)$. $\square$

Question 8

Disproof. Consider the following counterexample. Let $X = \{1, 2\}$, $Y = \{a\}$ and $G_f = \{(1, a), (2, a)\}$. Also, let $A_1 = \{1\}$ and $A_2 = \{2\}$. Then, $f(A_1) = f(A_2) = \{a\}$, but $A_1 \not\subseteq A_2$. Thus, the statement is false. $\square$