Practice for the 2nd Mid-term exam — MATH 304 — Fall 2023
— No Due Date —

1. Determine if the following vectors are linear independent:

    a. $v_1 := (1,2,3)^T, v_2 := (2,3,4), v_3 := (3,4,5)$.

    b. $v_1 := (0,1,0,1), v_2 := (1,0,1,0), v_3 := (2,0,2,0), v_4 := (0,2,0,2)$.

    c. $v_1 := (-1,1,-1,1), v_2 := (1,-1,1,-1), v_3 := (-1,1,1,-1), v_4 := (1,1,1,1)$.

2. Determine if the following vectors in the vector space of smooth functions in $[0,1]$ are linear independent

    a. $p_1(x) := x^2, p_2(x) := x^3, p_3(x) := x^{99}$.

    b. $f_1(x) := e^x, f_2(x) := e^{3x}, f_3(x) := e^{5x}, f_4(x) := e^{7x}$.

    c. $f_1(x) := \cos x, f_2(x) := \sin x, f_3(x) = x$.

3. Find a basis for the row space, column space and null space for the following matrices:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & 4 \\ 2 & 3 & 5 \end{pmatrix},\ B = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 4 & -7 & -1 \\ 0 & -7 & 8 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix},$$

4. Let

$$u_1 := (1,0,2)^T,\ u_2 := (-1,1,0)^T,\ u_3 = (1,0,1)^T \text{ and}$$
$$v_1 := (1,1,-1)^T,\ v_2 := (-1,0,0)^T,\ v_3 = (-1,1,1)^T.$$

    a. Find the transition matrix corresponding to the change of basis from $\{e_1, e_2, e_3\}$ to $\{u_1, u_2, u_3\}$.

    b. Find the transition matrix corresponding to the change of basis from $\{v_1, v_2, v_3\}$ to $\{e_1, e_2, e_3\}$.

    c. Find the transition matrix from $\{v_1, v_2, v_3\}$ to $\{u_1, u_2, u_3\}$

    d. Let $x = 1v_1 + 0v_2 - v_3$. Find the coordinates of x with respect to $\{u_1, u_2, u_3\}$.

    e. Verify your answer to previous one, by computing the coordinates in each case with respect to the standard basis.

5. For each of the following choices of A, b, determine whether b is in the column space of A and state whether the system $Ax = b$ is consistent or not.

    a. $A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

b. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$.

c. $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

6. Let

$$u_1 := (1,1,0)^T, \ u_2 := (1,0,1)^T, \ u_3 = (0,0,1)^T$$

be a basis of $\mathbb{R}^3$. Define : $\mathbb{R}^2 \to \mathbb{R}^3$ as

$$L(x) := x_1 u_1 + x_2 u_2 + (x_1 + x_2) u_3.$$

Find a matrix A representing L with respect to the ordered basis e_1, e_2 and u_1, u_2, u_3.

7. For each of the following transformations, find a matrix A such that $L(x) = Ax$.

a. $L((x_1, x_2)^T) = (x_1 + x_2, x_2 + x_1, x_1)^T$

b. $L((x_1, x_2, x_3)^T) = (x_3, x_2, x_1)^T$

c. $L((x_1, x_2, x_3)) = (x_1)$.

d. $L : P_3 \to P_3, P(a + bx + cx^2) = (c + bx + ax^2)$. (Consider the standard basis).

8. Determine which of the follow sentences are true or false

1. If $\{v_1, \cdots, v_n\}$ are linearly dependent, then at least one of them can be written as a linear combination of the rest.
2. If $L : V \to W$ is a linear map then maps 0_V to 0_W.
3. Let S be a subspace of V and $\dim(S) = n = \dim(V)$. Then $S = V$.
4. If $\{v_1, \cdots, v_n\} \subseteq V$ are linearly independent and $\dim(V) = n+1$, then one can always find a v_{n+1} such that $\{v_1, \cdots, v_{n+1}\}$ form a basis for V.
5. The only linear maps $L : \mathbb{R} \to \mathbb{R}$ are of the form $f(x) = ax$ for some $a \in \mathbb{R}$.
6. The map that reflect a point through the origin is a linear map.
7. The are vector spaces with infinite dimensions.
8. The every vector space has either only 1 element of infinite many.
9. If 0 is inside a subset S of V then S is a subspace.
10. If S is a subspace then 0 is inside S.
11. If $L : V \to W$ is a linear map then $2L$ is also a linear map.
12. If $L : V \to W$ is a linear map then $L + 2$ is also a linear map.

13. If A is a singular 3×3 matrix then the map $L(x) = A(x)$ is a linear map.
14. If $Ax = b$ has a solution then b lies inside the row space of A.
15. If A^T is row equivalent with B^T, then A, B have the same column space.

The exam will be in class: Tuesday October 31. The exam will be on Chapters 3 and 4 of the book.