Question 1

Consider the vectors $\{x_1, x_2, x_3\}$ of $\mathbb{R}^4$ where

$$x_1 = (4,2,2,1)^T,\ x_2 = (2,0,0,2)^T,\ x_3 = (1,1,-1,1)^T$$

Let $S := \text{span}\{x_1, x_2, x_3\}$. Use the Gram-Schmidt process to obtain an orthonormal basis for S.

Solution:

$$\begin{aligned} u_1 &= \frac{x_1}{\|x_1\|_2} = \frac{1}{5}(4,2,2,1)^T \\ p_1 &= \langle x_2, u_1 \rangle u_1 = \frac{1}{5} \times 10 \times \frac{1}{5}(4,2,2,1)^T = \frac{2}{5}(4,2,2,1)^T \end{aligned}$$

Question 2

Find the orthogonal complement of the subspace of $\mathbb{R}^3$ spanned by $(1,2,1)^T$, $(1,-1,2)^T$.

Solution: Given the subspace $S := \text{span}\{(1,2,1)^T, (1,-1,2)^T\}$, we want to find the orthogonal complement $S^\perp$. The orthogonal complement is the set of all vectors in the vector space that are orthogonal to every vector in S. Thus, by definition, we have

$$S^\perp = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle = 0 \ , \ \left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\rangle = 0 \right\}$$

To find x_1, x_2, x_3, we can solve the system of equations

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_1 - x_2 + 2x_3 = 0 \end{cases}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 1 & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{5}{3} & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \end{array} \right)$$

$$\sim \begin{cases} x_1 = -\frac{5}{3}x_3 \\ x_2 = \frac{1}{3}x_3 \end{cases}$$

Thus, we have

$$S^\perp = \left\{ \begin{pmatrix} -\frac{5}{3}x_3 \\ \frac{1}{3}x_3 \\ x_3 \end{pmatrix} \middle| x_3 \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} -\frac{5}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix} \right\}$$

Question 3

Let A be an $m \times n$ matrix. Show that A and A^TA have the same rank. Show that

$$N(A^TA) = N(A)$$

Solution:

Proof. We have that A is an $m \times n$ matrix. To show that $N(A^TA) = N(A)$, we need to show that $N(A^TA) \subseteq N(A)$ and $N(A) \subseteq N(A^TA)$. Let $x \in N(A)$. By definition of the null space, we have $Ax = 0$. Then we have $A^TAx = A^T0 = 0$. Again, by definition of the null space, we have $x \in N(A^TA)$. Since $x \in N(A) \implies x \in N(A^TA)$, we have $N(A) \subseteq N(A^TA)$. Now let $x \in N(A^TA)$. By definition of the null space, $A^TAx = A^T(Ax) = 0$. Therefore, $Ax \in N(A^T)$. However, Ax is also in the column space of A, which means $Ax \in R(A)$. The Fundamental Theorem of Linear Algebra tells us that $N(A^T) \perp R(A)$. □

Question 4

Let A be an $m \times n$ matrix and $\text{rank}(A) = r$. What are the dimensions of $N(A)$ and $N(A^T)$?

Solution: By the rank-nullity theorem, we know that $\text{rank}(A) + \text{nullity}(A) = n$. Thus, we have

$$\text{nullity}(A) = n - \text{rank}(A) = n - r$$

Since taking the transpose of a matrix does not change the number of linearly independent rows or columns, A and A^T have the same rank. Therefore,

$$\text{nullity}(A^T) = m - \text{rank}(A^T) = m - r$$

Question 5

For each of the following systems $Ax = b$ find all least squares solutions.

$$A = \begin{pmatrix} 1 & 1 \\ 3 & 4 \\ -1 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \text{ and } A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Solution: According to the Least Squares Theorem, the solution to the least squares problem is given by the solution to the normal equations

$$A^TA\hat{x} = A^Tb$$

where $\hat{x}$ is the least squares solution. If A has full rank, then A^TA is invertible and the solution is given by

$$\hat{x} = (A^TA)^{-1}A^Tb$$

For the first system, we can do the following:

$$\begin{aligned}
\hat{x} &= \left(\begin{pmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 3 & 4 \\ -1 & 0 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \\
&= \begin{pmatrix} 11 & 13 \\ 13 & 17 \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\
&= \frac{1}{18} \begin{pmatrix} 17 & -13 \\ -13 & 11 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\
&= \begin{pmatrix} \frac{8}{9} \\ \frac{-4}{9} \end{pmatrix}
\end{aligned}$$

For the second system, we have:

$$\begin{aligned}
\hat{x} &= \left(\begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 5 & 5 & 2 \\ 5 & 6 & 4 \\ 2 & 4 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \\
&= \frac{1}{6} \begin{pmatrix} 20 & -22 & 8 \\ -22 & 26 & -10 \\ 8 & -10 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \\
&= \begin{pmatrix} \frac{-1}{3} \\ \frac{2}{3} \\ \frac{1}{6} \end{pmatrix}
\end{aligned}$$

Question 6

Consider the basis $\{x_1, x_2, x_3\}$ of $\mathbb{R}^3$ where

$$x_1 = (1, 2, -2)^T,\ x_2 = (4, 3, 2)^T,\ x_3 = (1, 2, 1)^T$$

Use the Gram-Schmidt process to obtain an orthonormal basis.

Solution: