Question 1

Let $\{u_1, u_2, u_3\}$ be an orthonormal set of vectors in some vector space with inner product. Let

$$u := u_1 + 2u_2 + 3u_3 \text{ and } v := u_1 - u_3$$

Compute $\langle u, v\rangle$, $\|u\|$, and $\|v\|$.

Solution: Since the basis is orthonormal, the inner product of any two vectors in the basis is 0, and the inner product of a vector in the basis with itself is 1.

$$\begin{aligned}
\langle u, v\rangle &= \langle u_1 + 2u_2 + 3u_3, u_1 - u_3\rangle \\
&= \langle u_1, u_1 - u_3\rangle + \langle 2u_2, u_1 - u_3\rangle + \langle 3u_3, u_1 - u_3\rangle \\
&= \langle u_1, u_1\rangle - \langle u_1, u_3\rangle + \langle 2u_2, u_1\rangle - \langle 2u_2, u_3\rangle + \langle 3u_3, u_1\rangle - \langle 3u_3, u_3\rangle \\
&= 1 - 0 + 0 - 0 + 0 - 3 \\
&= -2
\end{aligned}$$

$$\begin{aligned}
\|u\|^2 &= \langle u, u\rangle \\
&= \langle u_1 + 2u_2 + 3u_3, u_1 + 2u_2 + 3u_3\rangle \\
&= \langle u_1, u_1 + 2u_2 + 3u_3\rangle + \langle 2u_2, u_1 + 2u_2 + 3u_3\rangle + \langle 3u_3, u_1 + 2u_2 + 3u_3\rangle \\
&= \langle u_1, u_1\rangle + \langle 2u_2, 2u_2\rangle + \langle 3u_3, 3u_3\rangle \\
&= 1 + 4 + 9 \\
&= 14 \implies \|u\| = \sqrt{14}
\end{aligned}$$

$$\begin{aligned}
\|v\|^2 &= \langle v, v\rangle \\
&= \langle u_1 - u_3, u_1 - u_3\rangle \\
&= \langle u_1, u_1 - u_3\rangle + \langle -u_3, u_1 - u_3\rangle \\
&= \langle u_1, u_1\rangle - \langle u_1, u_3\rangle - \langle u_3, u_1\rangle + \langle u_3, u_3\rangle \\
&= 1 - 0 - 0 + 1 \\
&= 2 \implies \|v\| = \sqrt{2}
\end{aligned}$$

Question 2

Consider the vector space $C[-1,1]$ equipped with the inner product:

$$\langle f,g\rangle := \int_{-1}^{1} f(x)g(x)dx$$

1. Show that $1, x$ are orthogonal.
2. Compute the norms $\|1\|$, $\|x\|$.

Solution: Two vectors are orthogonal if their inner product is 0.

$$\langle 1,x\rangle = \int_{-1}^{1} 1\cdot x\ dx = \int_{-1}^{1} x\ dx = \frac{1}{2}x^2\Big|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$$

Thus, $1, x$ are orthogonal. The norms of 1 and x are the square root of their inner product with themselves.

$$\begin{aligned}
\|1\|^2 = \langle 1,1\rangle &= \int_{-1}^{1} 1\cdot 1\ dx = \int_{-1}^{1} 1\ dx = x\Big|_{-1}^{1} = 1-(-1) = 2 \\
&\implies \|1\| = \sqrt{2} \\
\|x\|^2 = \langle x,x\rangle &= \int_{-1}^{1} x\cdot x\ dx = \int_{-1}^{1} x^2\ dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{1}{3} - \frac{1}{3} = \frac{2}{3} \\
&\implies \|x\| = \sqrt{\frac{2}{3}}
\end{aligned}$$

Question 3

Let

$$u_1 = \left(\frac{1}{3\sqrt{2}}, \frac{1}{3\sqrt{2}}, -\frac{4}{3\sqrt{2}}\right)^T, \ u_2 = \frac{1}{3}(2,2,1)^T, \ u_3 = \frac{1}{\sqrt{2}}(1,-1,0)^T$$

1. Show that u_1, u_2, u_3 is an orthonormal basis for $\mathbb{R}^3$.
2. Let $x = (1,2,2)^T$. Find the projection of p of x onto $S := \text{span}\{u_2, u_3\}$.

Solution: A set of vectors form an orthonormal basis if they are orthogonal and their norms are 1. Since we are working in $\mathbb{R}^3$, we can use the dot product to check if the vectors are orthogonal.

$$\begin{aligned}
\langle u_1, u_2 \rangle &= \frac{1}{3\sqrt{2}} \cdot \frac{2}{3} + \frac{1}{3\sqrt{2}} \cdot \frac{2}{3} + -\frac{4}{3\sqrt{2}} \cdot \frac{1}{3} = \frac{2}{9\sqrt{2}} + \frac{2}{9\sqrt{2}} - \frac{4}{9\sqrt{2}} = 0 \\
\langle u_1, u_3 \rangle &= \frac{1}{3\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{3\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} + -\frac{4}{3\sqrt{2}} \cdot 0 = \frac{1}{6} - \frac{1}{6} = 0 \\
\langle u_2, u_3 \rangle &= \frac{2}{3} \cdot \frac{1}{\sqrt{2}} + \frac{2}{3} \cdot -\frac{1}{\sqrt{2}} + \frac{1}{3} \cdot 0 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} = 0
\end{aligned}$$

Thus, u_1, u_2, u_3 are orthogonal. To check if their norms are 1, we can use the formula $\|u\|^2 = \langle u, u \rangle$.

$$\begin{aligned}
\|u_1\|^2 &= \langle u_1, u_1 \rangle = \frac{1}{3\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} + \frac{1}{3\sqrt{2}} \cdot \frac{1}{3\sqrt{2}} + -\frac{4}{3\sqrt{2}} \cdot -\frac{4}{3\sqrt{2}} = \frac{1}{18} + \frac{1}{18} + \frac{16}{18} = 1 \\
\|u_2\|^2 &= \langle u_2, u_2 \rangle = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} + 1 \cdot 1 = \frac{1}{9} + \frac{1}{9} + 1 = 1 \\
\|u_3\|^2 &= \langle u_3, u_3 \rangle = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}} \cdot -\frac{1}{\sqrt{2}} + 0 \cdot 0 = \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

Since the squares of norms are 1, the norms are 1. Thus, u_1, u_2, u_3 are orthonormal. To find the projection of x onto S, we can use the projection matrix:

$$p = \text{proj}_S(x) = A(A^T A)^{-1} A^T x$$

We form the matrix A by taking the basis vectors of S and using them as columns. Since we have that S is spanned by u_2 and u_3, we have the following:

$$A = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} \\ \frac{1}{3} & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Now, we need to find $A^T A$ and $(A^T A)^{-1}$.

$$A^T A = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} \\ \frac{1}{3} & 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} + \frac{4}{9} + \frac{1}{9} & \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} \\ \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} & \frac{1}{2} + \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $A^T A$ is just the identity matrix, its inverse is also the identity matrix.

$$(A^T A)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, we have:

$$\begin{aligned}
p &= A(A^T A)^{-1} A^T x = AIA^T x = AA^T x \\
&= \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{9} + \frac{1}{2} & \frac{4}{9} - \frac{1}{2} & \frac{2}{9} \\ \frac{4}{9} - \frac{1}{2} & \frac{4}{9} + \frac{1}{2} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{17}{18} & \frac{1}{18} & \frac{2}{9} \\ \frac{1}{18} & \frac{17}{18} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 17 & 1 & 4 \\ 1 & 17 & 4 \\ 4 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\
&= \frac{1}{18} \begin{bmatrix} 17 + 2 + 8 \\ 1 + 34 + 8 \\ 4 + 8 + 4 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 27 \\ 43 \\ 16 \end{bmatrix}
\end{aligned}$$

Simplifying further:

$$p = \frac{1}{18} \begin{bmatrix} 27 \\ 43 \\ 16 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{43}{18} \\ \frac{8}{9} \end{bmatrix}$$

Question 4

Let $v_1 := (1, 2, 0, -1)^T$ $v_2 := (1, -1, 0, 0)^T$ $v_3 := (0, 1, 0, -1)^T$. Find the angle between v_1, v_2, v_2, v_3, and v_1, v_3. Find the norm of each of these vectors. Find the projection of v_1 onto v_2 and onto v_3.

Solution: For two vectors v_1 and v_2 in a vector space with inner product, the angle θ between them is given by:

$$\cos\theta = \frac{\langle v_1, v_2 \rangle}{\|v_1\|\|v_2\|}$$

Finding the angles between v_1, v_2, v_2, v_3, and v_1, v_3:

$$\begin{aligned}
\cos\theta_{v_1,v_2} &= \frac{\langle v_1, v_2 \rangle}{\|v_1\|\|v_2\|} = \frac{1 \cdot 1 + 2 \cdot -1 + 0 \cdot 0 + -1 \cdot 0}{\sqrt{1^2 + 2^2 + 0^2 + (-1)^2}\sqrt{1^2 + (-1)^2 + 0^2 + 0^2}} = \frac{1-2}{\sqrt{6}\sqrt{2}} = -\frac{1}{\sqrt{12}} \\
\theta_{v_1,v_2} &= \cos^{-1}\left(-\frac{1}{\sqrt{12}}\right)
\end{aligned}$$

$$\begin{aligned}
\cos\theta_{v_2,v_3} &= \frac{\langle v_2, v_3 \rangle}{\|v_2\|\|v_3\|} = \frac{1 \cdot 0 + -1 \cdot 1 + 0 \cdot 0 + 0 \cdot -1}{\sqrt{1^2 + (-1)^2 + 0^2 + 0^2}\sqrt{0^2 + 1^2 + 0^2 + (-1)^2}} = \frac{-1}{\sqrt{2}\sqrt{2}} = -\frac{1}{2} \\
\theta_{v_2,v_3} &= \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}
\end{aligned}$$

$$\begin{aligned}
\cos\theta_{v_1,v_3} &= \frac{\langle v_1, v_3 \rangle}{\|v_1\|\|v_3\|} = \frac{1 \cdot 0 + 2 \cdot 1 + 0 \cdot 0 + -1 \cdot -1}{\sqrt{1^2 + 2^2 + 0^2 + (-1)^2}\sqrt{0^2 + 1^2 + 0^2 + (-1)^2}} = \frac{2+1}{\sqrt{6}\sqrt{2}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2} \\
\theta_{v_1,v_3} &= \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}
\end{aligned}$$

The norm of these vectors is given by $\|v\| = \sqrt{\langle v, v \rangle}$.

$$\begin{aligned}
\|v_1\| &= \sqrt{\langle v_1, v_1 \rangle} = \sqrt{1^2 + 2^2 + 0^2 + (-1)^2} = \sqrt{6} \\
\|v_2\| &= \sqrt{\langle v_2, v_2 \rangle} = \sqrt{1^2 + (-1)^2 + 0^2 + 0^2} = \sqrt{2} \\
\|v_3\| &= \sqrt{\langle v_3, v_3 \rangle} = \sqrt{0^2 + 1^2 + 0^2 + (-1)^2} = \sqrt{2}
\end{aligned}$$

The projection of one vector onto another is given by:

$$\mathrm{proj}_u(v) = \frac{\langle u, v \rangle}{\|u\|^2} u$$

Thus we have:

$$
\begin{aligned}
\operatorname{proj}_{v_2}(v_1) &= \frac{\langle v_2, v_1\rangle}{\|v_2\|^2} v_2 \\
&= \frac{1\cdot 1 + 2\cdot -1 + 0\cdot 0 + -1\cdot 0}{\sqrt{1^2+(-1)^2+0^2+0^2}^2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{proj}_{v_3}(v_1) &= \frac{\langle v_3, v_1\rangle}{\|v_3\|^2} v_3 \\
&= \frac{1\cdot 0 + 2\cdot 1 + 0\cdot 0 + -1\cdot -1}{\sqrt{0^2+1^2+0^2+(-1)^2}^2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \frac{3}{2} \\ 0 \\ -\frac{3}{2} \end{bmatrix}
\end{aligned}
$$

Question 5

Let A be an $m \times n$ matrix. Show that $A^T A$ and AA^T is a symmetric matrix. Assume that $m \geq n$ and $\text{rank}(A) = n$. Show that if $P = A(A^T A)^{-1} A^T$ then

$$P^2 = P$$

Solution: A matrix is symmetric if it is equal to its transpose. Using the properties of the matrix transpose, we can show that $A^T A$ and AA^T are symmetric.

$$\text{Transpose of } A^T A\text{: } (A^T A)^T = (A^T)^T A^T = AA^T = A^T A$$
$$\text{Transpose of } AA^T\text{: } (AA^T)^T = A^T (A^T)^T = A^T A = AA^T$$

As we can see here, applying the transpose to either matrix results in the same matrix. Proving that $P^2 = P$ is a bit more involved.

Proof. We have that $P = A(A^T A)^{-1} A^T$. We need to find

$$P^2 = A(A^T A)^{-1} A^T A (A^T A)^{-1} A^T$$

and show that it is equal to P. Using the associative property of matrix multiplication, we can change our order of multiplication to get

$$P^2 = A(A^T A)^{-1} (A^T A)(A^T A)^{-1} A^T$$

Since it is given that $\text{rank}(A) = n$, and $A^T A$ must be and $n \times n$ matrix, $A^T A$ is invertible. Since $A^T A$ is invertible, we can multiply it by its inverse to get the identity matrix. Now, we have

$$P^2 = AI(A^T A)^{-1} A^T = A(A^T A)^{-1} A^T = P$$

Thus, $P^2 = P$. $\square$