7th Homework — MATH 304 — Fall 2023
— Due November 9th —

1. Determine whether the following transformations are linear: Explain your answer.
   a. $F((x_1, x_2, x_3)^T) = (x_1 - x_2, x_2 - x_1)^T$
   b. $F((x_1, x_2, x_3)^T) = (1, 2, x_1 + x_2 + x_3)^T$
   c. $F((x_1)) = (x_1, 2x_1, 3x_1)^T$.
   d. $F((x_1, x_2, x_3, x_4)^T) = (x_1, 0, 0, 0, x_2^2 + x_3^2 + x_4^2)^T$
2. Determine whether the following transformations are linear from $C([0,1])$ to $\mathbb{R}$.
   a. $L(f) = f(0)$, $(L := C([0,1]) \to \mathbb{R})$
   b. $L(f) = |f(0)|$, $(L := C([0,1]) \to \mathbb{R})$
   c. $L(f) = f'(0) + f(0)$. $(L := C^1([0,1]) \to \mathbb{R})$.
   d. $L(f)(x) = x^2 + f(x)$, $(L := C([0,1]) \to C([0,1]))$.
3. For each of the following transformations, find a matrix A such that $L(x) = Ax$.
   a. $L((x_1, x_2, x_3)^T) = (x_1 + x_2)^T$
   b. $L((x_1, x_2, x_3)^T) = (x_1 + x_2, x_2 + x_3, x_1 + x_2 + x_3)^T$
   c. $L((x_1)) = (x_1, 2x_1, 3x_1)^T$.
   d. $L((x_1, x_2, x_3, x_4)^T) = (x_1 + x_2 + x_3 + 2x_4)^T$
4. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that
$$L((x_1, x_2, x_3)^T) = (2x_1, x_1 + x_2).$$
   a. Find A that represents L with respect to the standard basis of $\mathbb{R}^3$.
   b. Find B that represents L with respect to the following basis of $\mathbb{R}^3$. $E := [v_1, v_2, v_3]$, where,
$$v_1 = (1,1,1)^T,\ v_2 = (1,1,0)^T,\ v_3 = (1,0,0)^T.$$
5. In the vector space $C[-\pi, \pi]$ we define inner product
$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi,\pi} f(x)g(x)dx.$$
   a. Show the the above is indeed an inner product.
   b. Show that $f(x) = \cos(x), g(x) = \sin(x)$ are orthogonal and that they have length 1.

Show your work in each exercise.