Question 1

Consider the following collections of polynomials in $\mathbb{P}_2$:

(a) $p_1(x) = 1, \quad p_2(x) = x + 1, \quad p_3(x) = x^2.$

(b) $p_1(x) = x - 1, \quad p_2(x) = x + 1, \quad p_3(x) = x^2 - 1.$

(c) $p_1(x) = x^2 - 1, \quad p_2(x) = x^2 + 1, \quad p_3(x) = x^2.$

Decide in each case if these vectors are linearly independent. Write the dimension of the subspace $S := span\{p_1, p_2, p_3\}$ in each case. In which case(s) would we have that $S = \mathbb{P}_2$? Explain your answer.

Answer: To check for linear independence, we can use the Wronskian determinant.

(a)

$$W(p_1, p_2, p_3) = \begin{vmatrix} 1 & x+1 & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & x+1 \\ 0 & 1 \end{vmatrix} = 2$$

Since $W(p_1, p_2, p_3) \neq 0$, the vectors are linearly independent. The dimension of the subspace $S := span\{p_1, p_2, p_3\}$ is 3, since there are 3 linearly independent vectors in its basis.
In this case, we have that $S = \mathbb{P}_2$.

Proof. To show that $S = \mathbb{P}_2$, we need to show that $span\{p_1, p_2, p_3\} = \mathbb{P}_2$. Let $p(x) = ax^2 + bx + c$ be an arbitrary polynomial in $\mathbb{P}_2$. Our goal is to show that $p(x)$ can be shown as a linear combination of p_1, p_2, p_3. That means we must find α, β, γ such that:

$$\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) = ax^2 + bx + c$$

Substituting in p_1, p_2, p_3, we have:

$$\alpha + \beta(x + 1) + \gamma x^2 = ax^2 + bx + c$$
$$\alpha + \beta x + \beta + \gamma x^2 = ax^2 + bx + c$$
$$\gamma x^2 + \beta x + (\alpha + \beta) = ax^2 + bx + c$$

Equating coefficients, we have:

$$\gamma = a, \quad \beta = b, \quad \alpha + \beta = c$$

Since $\beta = b$, we have that $\alpha = c - b$. Substituting everyting back into the original equation, we have:

$$(c - b) + b(x + 1) + ax^2 = ax^2 + bx + c$$

Now we can see that $p(x)$ can be written as a linear combination of p_1, p_2, p_3. Therefore, $span\{p_1, p_2, p_3\} = \mathbb{P}_2$, and $S = \mathbb{P}_2$. $\square$

(b)

$$W(p_1,p_2,p_3) = \begin{vmatrix} x-1 & x+1 & x^2-1 \\ 1 & 1 & 2x \\ 0 & 0 & 2x \end{vmatrix} = 2x \begin{vmatrix} x-1 & x+1 \\ 1 & 1 \end{vmatrix}$$
$$= 2x(x-1-x-1) = -4x$$

Since $W(p_1,p_2,p_3) \neq 0$, the vectors are linearly independent. The dimension of the subspace $S := span\{p_1,p_2,p_3\}$ is 3, since there are 3 linearly independent vectors in its basis.
In this case, we have that $S = \mathbb{P}_2$.

Proof. To show that $S = \mathbb{P}_2$, we need to show that $span\{p_1,p_2,p_3\} = \mathbb{P}_2$. Let $p(x) = ax^2 + bx + c$ be an arbitrary polynomial in $\mathbb{P}_2$. Our goal is to show that $p(x)$ can be shown as a linear combination of p_1,p_2,p_3. That means we must find α,β,γ such that:

$$\alpha p_1(x) + \beta p_2(x) + \gamma p_3(x) = ax^2 + bx + c$$

Substituting in p_1,p_2,p_3, we have:

$$\alpha(x-1) + \beta(x+1) + \gamma(x^2-1) = ax^2 + bx + c$$
$$\alpha x - \alpha + \beta x + \beta + \gamma x^2 - \gamma = ax^2 + bx + c$$
$$\gamma x^2 + (\alpha+\beta)x + (\beta-\alpha-\gamma) = ax^2 + bx + c$$

Equating coefficients, we have:

$$\gamma = a, \quad \alpha+\beta = b, \quad \beta-\alpha-\gamma = c$$

Now we have arbitrary elements a,b,c in terms of α,β,γ. To solve for α,β,γ in terms of a,b,c, we can use the augmented matrix:

$$\left[\begin{array}{ccc|ccc} 0 & 0 & \gamma & a & 0 & 0 \\ \alpha & \beta & 0 & 0 & b & 0 \\ -\alpha & \beta & -\gamma & 0 & 0 & c \end{array}\right]$$

Using row operations, we have the following equivalent matricies:

□

(c)

$$W(p_1,p_2,p_3) = \begin{vmatrix} x^2-1 & x^2+1 & x^2 \\ 2x & 2x & 2x \\ 2 & 2 & 0 \end{vmatrix} = 2 \begin{vmatrix} x^2-1 & x^2+1 \\ 2x & 2x \end{vmatrix}$$
$$= 2(2x^3 - 2x - 2x^3 - 2x) = -8x$$

Since $W(p_1, p_2, p_3) \neq 0$, the vectors are linearly independent. The dimension of the subspace $S := span\{p_1, p_2, p_3\}$ is 3, since there are 3 linearly independent vectors in its basis.
In this case, $S \neq \mathbb{P}_2$, since none of the polynomials in S have a term of degree 1. Therefore, there is no way to represent a polynomial of degree 1 as a linear combination of p_1, p_2, p_3.

Question 2

Consider the following collections of smooth functions $[0, 1]$:

1. $f_1(x) = x^2$, $f_2(x) = \frac{1}{x^2}$
2. $f_1(x) = \cos(x)$, $f_2(x) = \sin(x)$
3. $f_1(x) = 1$, $f_2(x) = \frac{e^x+e^{-x}}{2}$, $f_3(x) = \frac{e^x-e^{-x}}{2}$

Decide in each case if these vectors (functions) are linearly independent.

Question 3

Find the dimension of the space spanned by the functions

$$1,\ \cos(2x),\ \cos^2(x)$$

Question 4

For each of the following find the transition matrix corresponding to the change of basis from $\{u_1, u_2\}$ to the standard one $\{e_1, e_2\}$:

(a) $u_1 = (1, 1)^T$, $u_2 = (-1, 1)^T$

(b) $u_1 = (1, 2)^T$, $u_2 = (2, 5)^T$

(c) $u_1 = (0, 1)^T$, $u_2 = (1, 0)^T$

Let

$$v_1 = (3, 2)^T, \quad v_2 = (4, 3)^T$$

For each of the basis above find the transition matrix from $[v_1, v_2]$ to $[u_1, u_2]$.
Let

$$x = (2, 4)^T, \quad y = (1, 1)^T, \quad z = (0, 10)$$

Find the coordinates of x, y, z with respect to each of the basis mentioned above.