Chapter 13

Generating Functions

> *A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag.*
>
> — George Pólya, *Mathematics and Plausible Reasoning, Volume 1*

Some counting problems are difficult to solve by a direct approach. For example, we often want to count the number of elements in a set that have a certain property. Euler and Laplace introduced generating functions that can often help. At first sight it might appear as a mere change of representation, but solutions using generating functions can be surprisingly effective. In the next chapter, we will see how generating functions can help solving recurrence relations.

13.1 The Basic Concept

Given a sequence of real numbers $a = (a_0, a_1, a_2, a_3, \ldots)$, the **ordinary generating function** of a, or **generating function** for short, is given by the power series

$$A(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots .$$

Informally, one can view a power series as a generalization of a polynomial with an infinite number of terms. A power series is a polynomial if all but a finite number of the coefficients a_k are equal to zero.

We will denote by $[z^k]$ the operator that extracts the k-th coefficient from the generating function,

$$[z^k]A(z) = a_k.$$

What is the benefit of using a generating function when it apparently encodes the same information as the sequence? One advantage of generating functions is that they allow us to do algebraic manipulations. Viewing the sequences in this way can give us considerable insight, as we will see. Let us have a look at an example to give you a first taste of this tool.

Example 13.1. A six-sided normal die has the face values

⚀, ⚁, ⚂, ⚃, ⚄, ⚅.

So each value from 1 to 6 occurs precisely once, and all other values do not occur. The generating function of the normal die is

$$D(z) = z + z^2 + z^3 + z^4 + z^5 + z^6.$$

The exponents represent the face values. As each face value occurs only once, $[z^k]D(z) = 1$ for all values k in the range $1 \leqslant k \leqslant 6$. Suppose that we want to know in how many different ways a pair of normal dice will yield a sum of 7. Since the problem is so small, we could simply list all possible cases. So here are the only combinations of the pair of dice that sum to 7:

⚀⚅, ⚁⚄, ⚂⚃, ⚃⚂, ⚄⚁, ⚅⚀.

However, it might be easy to overlook a combination, so this approach does not scale well to larger problems. Let's see how we can use generating functions to solve the problem. For each pair (a, b) of values of the dice, the product $z^a z^b = z^{a+b}$ yields the sum $a + b$ of the values in the exponent. Therefore, if we form the product of the generating polynomials of the two dice, $D(z)^2 = D(z)D(z)$, then the coefficient of $z^{a+b} = z^7$ of the resulting polynomial counts the number of pairs that sum to 7. In this case, we have

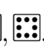
$$D(z)^2 = z^2 + 2z^3 + 3z^4 + 4z^5 + 5z^6 + 6z^7 + 5z^8 + 4z^9 + 3z^{10} + 2z^{11} + z^{12}.$$

Therefore, we have $[z^7]D(z)^2 = 6$ different pairs of die values that sum to 7, confirming that our above enumeration of such pairs is complete.

Example 13.2. Sam has a pair of six-sided crazy dice. The first die has the following six sides

⚀, ⚁, ⚁, ⚂, ⚂, ⚃,

so it has the 2 and 3 repeated twice, and the largest value is a 4. The second die has the following six sides

⚀, ⚂, ⚃, ⚄, ⚅, .

The generating functions of the face values of the first and second die are given by

$$C(z) = z + 2z^2 + 2z^3 + z^4 \quad \text{and} \quad W(z) = z + z^3 + z^4 + z^5 + z^6 + z^8.$$

The coefficient $[z^k]C(z)$ counts the number of times the value k occurs on the first die. For instance $[z^3]C(z) = 2$, since the value 3 occurs twice. Multiplying the two generating functions yields

$$C(z)W(z) = z^2 + 2z^3 + 3z^4 + 4z^5 + 5z^6 + 6z^7 + 5z^8 + 4z^9 + 3z^{10} + 2z^{11} + z^{12},$$

which is exactly the same distribution of values as for a pair of standard dice. In other words, even though each die is a bit wacky, their sums behaves exactly in the same way as the sum of a pair of normal dice. It is quite remarkable that one can obtain a fair sum of dice values from dice that are not all fair. The generating function allowed us to quickly check this fact without much hassle.

EXERCISES

13.1. Show that apart from the pair of normal dice and the pair of crazy dice there cannot exist any other pair of dice such that the product of their generating functions is equal to

$$P(z) = z^2 + 2z^3 + 3z^4 + 4z^5 + 5z^6 + 6z^7 + 5z^8 + 4z^9 + 3z^{10} + 2z^{11} + z^{12},$$

assuming that (a) the face values are at least 1 and (b) each die has 6 faces. [Hint: Factor the polynomial $P(z)$ over the rational numbers and inspect all possible combinations of factors.]

13.2. Let us use the notations of Example 13.2. Find all terms z^a and z^b in $C(z)$ and $W(z)$, respectively, such that $z^{a+b} = z^6$. Explain in your own words why there must be 5 such pairs.

13.3. Determine the number of ways to obtain a sum of 12 when rolling 3 six-sided normal (fair) dice using generating functions.

13.4. Grandpa Dell found 20 collectible baseball cards that he wants to divide among his three grandchildren. Since the oldest grandson Albert just helped with the dishes, he wants to give him him an even number of baseball cards, so that Albert receives at least 8 but not more than 14 baseball cards. The younger grandchildren Bella and Clara should each receive an odd number of baseball cards; they should get at least 3 and at most 9 cards, but not necessarily the same number of cards. In how many different ways can grandpa Dell distribute the 20 cards subject to these rules?

13.2 Operations on Generating Functions

Let $A(z)$ and $B(z)$ denote the power series

$$A(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad B(z) = \sum_{k=0}^{\infty} b_k z^k,$$

where a_k and b_k are real numbers for all nonnegative integer indices k.

The **sum** $A(z) + B(z)$ is defined as

$$A(z) + B(z) = \sum_{k=0}^{\infty} (a_k + b_k) z^k.$$

The **product** $A(z)B(z)$ of two power series is defined as

$$A(z)B(z) = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{k} a_{k-\ell} b_\ell \right) z^k.$$

We denote by $\mathbf{R}[[z]]$ the set of all formal power series

$$\mathbf{R}[[z]] = \left\{ \sum_{k=0}^{\infty} a_k z^k \,\middle|\, a_k \in \mathbf{R} \text{ for all } k \in \mathbf{N}_0 \right\}.$$

A formal power series does not need to be convergent. A power series that is not convergent cannot be used as a function. However, we can still algebraically manipulate the formal power series, since for instance sum and product are defined for all formal power series, convergent or not. We can often prove remarkable facts about the sequences using formal power series, even though they might lack any convergence properties.

We say that a power series $B(z)$ is a **multiplicative inverse** of the power series $A(z)$ if and only if their product satisfies

$$A(z)B(z) = 1,$$

so all terms in the product vanish except the constant term. If a multiplicative inverse $B(z)$ of the power series $A(z)$ in $\mathbf{R}[[z]]$ exists, then we can express $A(z)$ also in the form

$$A(z) = \frac{1}{B(z)}.$$

This form is particularly useful when $B(z)$ is a polynomial. Among other things, it will allow us to find a closed form expression for the Fibonacci numbers.

Not all formal power series have a multiplicative inverse. The next proposition gives a simple criterion for the existence of a multiplicative inverse.

Proposition 13.3. *A formal power series $A(z) = \sum_{k=0}^{\infty} a_k z^k$ has a multiplicative inverse if and only if $a_0 \neq 0$.*

Proof. Suppose that $B(z) = \sum_{k=0}^{\infty} b_k z^k$ is a multiplicative inverse of $A(z)$. Then we must have

$$[z^0]A(z)B(z) = a_0 b_0 = 1,$$

which implies that $a_0 \neq 0$.

Conversely, suppose that $A(z)$ is a formal power series with nonzero constant coefficient $[z^0]A(z) = a_0 \neq 0$. Then the putative inverse power series $B(z) = \sum_{k=0}^{\infty} b_k z^k$ must satisfy $b_0 = 1/a_0$. Let us now consider the coefficients b_k with nonzero index. Suppose that we have already determined the coefficients $b_0, \ldots, b_{k-1}$. Then we can define b_k by the expression

$$b_k = -\frac{1}{a_0} \sum_{\ell=1}^{k} a_\ell b_{k-\ell}.$$

Indeed, all coefficients in the sum on the right-hand side are defined, and $a_0 \neq 0$, so forming the quotient $-1/a_0$ is valid. If we multiply both sides by $-a_0$, then we obtain

$$-a_0 b_k = \sum_{\ell=1}^{k} a_\ell b_{k-\ell}.$$

Adding $a_0 b_k$ on both sides yields

$$\sum_{\ell=0}^{k} a_\ell b_{k-\ell} = 0,$$

as it should. It follows by induction that all coefficients b_k are defined and satisfy $A(z)B(z) = 1$. □

The next example shows how to determine the multiplicative inverse of the constant 1 sequence.

Example 13.4. Let $A(z)$ denote the generating function of the constant sequence $a = (1, 1, 1, \ldots)$, so

$$A(z) = \sum_{k=0}^{\infty} z^k.$$

Then

$$zA(z) = \sum_{k=0}^{\infty} z^{k+1} = \sum_{k=1}^{\infty} z^k = A(z) - 1.$$

It follows that

$$1 = A(z) - zA(z) = A(z)(1 - z).$$

Therefore $B(z) = (1 - z)$ is the multiplicative inverse of $A(z)$. For this reason, we often write

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k.$$

In this case, we even have convergence for all z satisfying $|z| < 1$, since this is a geometric series.

The next example illustrates the product of two formal power series.

Example 13.5. Let $A(z)$ denote the generating function of the constant sequence $a = (1, 1, 1, \ldots)$, so

$$A(z) = \sum_{k=0}^{\infty} z^k.$$

Then

$$A(z)A(z) = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{k} 1 \cdot 1 \right) z^k = \sum_{k=0}^{\infty} (k+1) z^k.$$

In view of the previous example, we have

$$\frac{1}{(1-z)^2} = \sum_{k=0}^{\infty} (k+1)z^k.$$

Thus, $(1-z)^{-2}$ is the generating function of the sequence $(1, 2, 3, \ldots)$.

We can generalize the previous example as follows.

Proposition 13.6. *Let $A(z)$ denote the generating function of the sequence $(a_0, a_1, a_2, \ldots)$. Then*

$$\frac{1}{1-z}A(z)$$

is the generating function of the associated sequence of partial sums

$$(a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \ldots).$$

Proof. By definition, the product of $1/(1-z)$ and $A(z)$ yields

$$\begin{aligned}
\frac{1}{1-z}A(z) &= \left(\sum_{k=0}^{\infty} z^k\right)\left(\sum_{\ell=0}^{\infty} a_\ell z^\ell\right) \\
&= \sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{k} 1 \cdot a_\ell\right) z^k \\
&= \sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{k} a_\ell\right) z^k.
\end{aligned}$$

The right-hand side is the generating function of

$$(a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \ldots),$$

as claimed. ☐

Another useful operation is the multiplication of coefficients by powers of a real number r.

Proposition 13.7. *Let $A(z)$ denote the generating function of the sequence $(a_0, a_1, a_2, a_3, \ldots)$ and let r denote a real number. Then $A(rz)$ is the generating function of the sequence $(a_0, ra_1, r^2a_2, r^3a_3, \ldots)$.*

Proof. If we expand the formal power series $A(rz)$, then we get

$$A(rz) = \sum_{k=0}^{\infty} a_k (rz)^k = \sum_{k=0}^{\infty} a_k r^k \, z^k,$$

which is the generating function of $(a_0, ra_1, r^2a_2, r^3a_3, \ldots)$, as claimed. ☐

Suppose that the sequence $(a_0, a_1, a_2, \ldots)$ has the generating function $A(z)$, then the **right-shifted sequence**

$$(\underbrace{0, \ldots, 0}_{k \text{ zeros}}, a_0, a_1, a_2, \ldots)$$

has the generating function $z^k A(z)$.

Similarly, the **left-shifted sequence** $(a_k, a_{k+1}, a_{k+2}, \ldots)$ has the generating function

$$\frac{A(z) - a_{k-1}z^{k-1} - \cdots - a_1 z - a_0}{z^k}.$$

Example 13.8. The generating function of the sequence $(1, 2, 3, \ldots)$ is given by

$$A(z) = \frac{1}{(1-z)^2}.$$

The generating function of the left-shifted sequence is given by

$$\frac{A(z) - 1}{z} = \frac{1 - (1-z)^2}{z(1-z)^2} = \frac{2}{(1-z)^2} - \frac{z}{(1-z)^2}.$$

This equation asserts that the doubled sequence $(2, 4, 6, \ldots)$ minus the right-shifted sequence $(0, 1, 2, \ldots)$ is equal to the left-shifted sequence $(2, 3, 4, \ldots)$.

Given a formal power series

$$A(z) = \sum_{k=0}^{\infty} a_k z^k,$$

we can define its **formal derivative** by

$$\frac{d}{dz} A(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k.$$

In other words, if a sequence $(a_0, a_1, a_2, \ldots)$ has the generating function $A(z)$, then the sequence

$$b = (b_0, b_1, b_2, \ldots) = (a_1, 2a_2, 3a_3, \ldots)$$

has the generating function $\frac{d}{dz} A(z)$.

Let $A(z)$ denote the generating function of a sequence

$$(a_0, a_1, a_2, \ldots)$$

Then $\frac{d}{dz} A(z)$ is the generating function of the sequence $(a_1, 2a_2, 3a_3, \ldots)$ that is multiplied by a proportionality factor. However, this also **shifts** the sequence to the left. We can compensate for the left-shift of the differential operator by multiplying it with z. Thus $(z\frac{d}{dz})A(z)$ is the generating function of the sequence

$$(0a_0, 1a_1, 2a_2, 3a_3, \ldots)$$

that multiplies each term by its index.

Let D denote the differential operation $D = \frac{d}{dz}$.

Proposition 13.9. *Let $A(z)$ denote the generating function of the sequence $(a_0, a_1, a_2, \ldots)$. Let $P(x)$ denote a polynomial with real coefficients. Then the sequence $(P(0)a_0, P(1)a_1, P(2)a_2, \ldots)$ has the generating function $P(zD)A(z)$. In particular, $zDA(z)$ is the generating function of $(0a_0, 1a_1, 2a_2, 3a_3, \ldots)$.*

Proof. The generating function of $(a_0, a_1, a_2, \ldots)$ is given by $A(z) = \sum_{k=0}^{\infty} a_k z^k$. Then

$$zDA(z) = z\sum_{k=0}^{\infty}(k+1)a_{k+1}z^k = \sum_{k=0}^{\infty} ka_k z^k$$

is the generating function of the sequence $(na_n)_{n\geqslant 0}$. It follows that

$$(zD)^m A(z) = \sum_{k=0}^{\infty} k^m a_k z^k$$

is the generating function of the sequence $(n^m a_n)_{n\geqslant 0}$.

Consequently, if we are given a polynomial $P(x) = \sum_{k=0}^{m} c_k x^k$, then

$$P(zD)A(z) = \sum_{k=0}^{m} c_k (zD)^k A(z)$$

is the generating function of the sequence

$$\left(\sum_{k=0}^{m} c_k n^k a_n\right)_{n\geqslant 0} = (P(n)a_n)_{n\geqslant 0},$$

as claimed. $\square$

Given a formal power series

$$A(z) = \sum_{k=0}^{\infty} a_k z^k,$$

we can define its **formal integral** by

$$\int_0^z A(x)dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1} = \sum_{k=1}^{\infty} \frac{a_{k-1}}{k} z^k$$

In other words, if a sequence $(a_0, a_1, a_2, \ldots)$ has the generating function $A(z)$, then the sequence

$$b = (b_0, b_1, b_2, b_3, \ldots) = (0, \frac{a_0}{1}, \frac{a_1}{2}, \frac{a_2}{3}, \ldots)$$

has the generating function $\int_0^z A(x)dx$. This shifts the sequence to the right and divides it by a proportionality factor such that

$$b_k = \frac{a_{k-1}}{k}$$

holds for all $k \geqslant 1$ and $b_0 = 0$.

Example 13.10. If we formally integrate the generating function

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

of the constant one sequence $(1, 1, 1, \ldots)$, then we obtain the generating function

$$\ln\left(\frac{1}{1-z}\right) = \sum_{k=1}^{\infty} \frac{z^k}{k}.$$

of the sequence $(0, 1, 1/2, 1/3, \ldots)$.

If we are given a formal power series $A(z) = \sum_{k=0}^{\infty} a_k z^k$ and a positive integer m, then the **power** $A(z)^m$ is given by

$$\sum_{k=0}^{\infty} \left(\sum_{k_1+k_2+\cdots+k_m=k} a_{k_1} a_{k_2} \cdots a_{k_m} \right) z^k.$$

The inner sum ranges over all m-tuples $(k_1, k_2, \ldots, k_m)$ of nonnegative integers such that $k_1 + k_2 + \cdots + k_m = k$.

As an application, we record the following important consequence.

Proposition 13.11. *If m is a positive integer, then*

$$\frac{1}{(1-z)^m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} z^k.$$

Proof. Since

$$A(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k,$$

is the generating function of the constant one sequence, its m-th power is given by

$$\begin{aligned} A(z)^m &= \sum_{k=0}^{\infty} \left(\sum_{k_1+k_2+\cdots+k_m=k} a_{k_1} a_{k_2} \cdots a_{k_m} \right) z^k \\ &= \sum_{k=0}^{\infty} \left(\sum_{k_1+k_2+\cdots+k_m=k} 1 \right) z^k \end{aligned}$$

By Corollary 12.26, the number of nonnegative integer solutions to the equation $k_1 + k_2 + \cdots + k_m = k$ is given by $\binom{m+k-1}{k}$. Therefore, we can conclude that

$$A(z)^m = \sum_{k=0}^{\infty} \binom{m+k-1}{k} z^k,$$

as claimed. □

EXERCISES

13.5. Let $A(z) = 1/(1-z) = \sum_{k=0}^{\infty} z^k$ be the generating function of the constant one sequence $(1,1,1,\ldots)$. Then $A(z)^2$ is the generating function of the sequence $(1,2,3,4,\ldots)$. Determine the power series of $A(z)^3$ using the product of $A(z)^2$ and $A(z)$. The coefficients of the formal power series of $A(z)^3 = (1-z)^{-3}$ should look familiar. Identify them.

13.6. Let $A(z)$ be the generating function of the sequence $(a_0, a_1, a_2, a_3, \ldots)$. Determine the generating functions of the sequences

$$(a_0, \underbrace{0,\ldots,0}_{m \text{ zeros}}, a_1, \underbrace{0,\ldots,0}_{m \text{ zeros}}, a_2, \underbrace{0,\ldots,0}_{m \text{ zeros}}, a_3 \ldots)$$

and

$$(a_0, \underbrace{0,\ldots,0}_{m \text{ zeros}}, ra_1, \underbrace{0,\ldots,0}_{m \text{ zeros}}, r^2a_2, \underbrace{0,\ldots,0}_{m \text{ zeros}}, r^3a_3 \ldots),$$

where r is a real number.

13.7. Determine the generating function of the sequence

$$(1,0,1,0,1,0,\cdots)$$

in closed form (that is, as a rational function) and find its multiplicative inverse.

13.8. Determine the generating function of the sequence

$$(1,-1,1,-1,1,-1,\cdots)$$

in closed form (that is, the generating function should be given as a rational function).

13.9. Let $A(z) = \sum_{k=0}^{\infty} a_k z^k$ denote the generating function of the sequence $(a_0, a_1, a_2, \ldots)$. Determine the generating function of the sequences

$$(a_0, 0, a_2, 0, a_4, 0, \ldots) \quad \text{and} \quad (0, a_1, 0, a_3, 0, a_5, \ldots)$$

as a linear combination of terms of the form $A(rz)$, where r is a real number.

13.10. Suppose that $A(z)$ denotes the generating function of the sequence $(a_0, a_1, a_2, \ldots)$. Determine the generating function of the sequence

$$(a_0, a_1 - a_0, a_2 - a_1, a_3 - a_2, \ldots).$$

13.3 Elementary Generating Functions

We will need a little dictionary that collects the generating functions for a few well-known sequences. We have already derived some, so it will be good to collect them in one place for better reference.

Example 13.12. We have seen that the constant sequence $(1, 1, 1, 1, \ldots)$ has the generating function

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k.$$

The left-hand side can be interpreted as a convenient shorthand.

Example 13.13. The sequence $(0, 1, 2, 3, 4, \ldots)$ has the generating function

$$\frac{z}{(1-z)^2} = z\sum_{k=0}^{\infty}(k+1)z^k = \sum_{k=0}^{\infty} kz^k.$$

We obtained this result from right-shifting the sequence $(1, 2, 3, 4, \ldots)$ that has the generating function $1/(1-z)^2$, as we have seen in the previous section.

Example 13.14. The sequence $s = (0, 1, 4, 9, 16, \ldots)$ of squares has the generating function

$$\frac{z(1+z)}{(1-z)^3} = \sum_{k=0}^{\infty} k^2 z^k.$$

Indeed, the sequence $(0, 1, 2, 3, 4, \ldots)$ has the generating function $z/(1-z)^2$. Taking the formal derivative yields the sequence $(1, 4, 9, 16, \ldots)$ with generating function

$$\frac{d}{dz}\frac{z}{(1-z)^2} = \frac{1+z}{(1-z)^3}.$$

Right-shifting yields the generating function of the sequence s.

Example 13.15. The sequence $(0, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots)$ of reciprocals of n has generating function

$$\ln\left(\frac{1}{1-z}\right) = \sum_{k=1}^{\infty}\frac{z^k}{k}.$$

Example 13.16. Let n be a nonnegative integer. The binomial sequence $s = \left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots\right)$ has the generating function

$$S(z) = \sum_{k=0}^{\infty}\binom{n}{k}z^k.$$

Since the binomial coefficient $\binom{n}{k} = 0$ when k exceeds n, the power series $S(z)$ is equal to the polynomial

$$S(z) = \sum_{k=0}^{n}\binom{n}{k}z^k.$$

By the binomial theorem, we have

$$S(z) = (1+z)^n.$$

This fact is particularly useful. Many binomial coefficient identities can be derived using the fact that $(1+z)^n$ is the generating function of the binomial coefficient sequence.

There exists a useful generalization of the previous example to noninteger exponents n.

Example 13.17. Let x be a real number. The function $f(z) = (1+z)^x$ can be differentiated arbitrarily often. The first derivative of $f(z)$ is given by $f^{(1)}(z) = x(1+z)^{x-1}$, the second derivative by $f^{(2)}(z) = x(x-1)(1+z)^{x-2}$, and in general the k-th derivative is given by $f^{(k)}(z) = x^{\underline{k}}(1+z)^{x-k}$. Therefore, the Taylor series of $f(z)$ about $z = 0$ is given by

$$(1+z)^x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=0}^{\infty} \frac{x^{\underline{k}}}{k!} z^k.$$

The falling power of x divided by $k!$ equals the generalized binomial coefficient $\frac{x^{\underline{k}}}{k!} = \binom{x}{k}$. Therefore, we have

$$(1+z)^x = \sum_{k=0}^{\infty} \binom{x}{k} z^k.$$

This **binomial series** has in general infinitely many nonzero terms, but it converges for all z satisfying $|z| < 1$. The binomial series allows one to derive the formal power series for expressions such as $\sqrt{1+z} = (1+z)^{1/2}$.

We can use results from calculus to expand our repertoire of generating functions.

Example 13.18. The sequence $(\frac{1}{0!}, \frac{1}{1!}, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \cdots)$ of reciprocals of the factorials has the **exponential function**

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

as a generating function.

Example 13.19. The sequence $(1, 0, -\frac{1}{2!}, 0, \frac{1}{4!}, 0, \ldots)$ has the **cosine function**

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

as a generating function.

Example 13.20. The sequence $(0, 1, 0, -\frac{1}{3!}, 0, \frac{1}{5!}, 0, \ldots)$ has the **sine function**

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

as a generating function.

Example 13.21. The sequence $(0, 1, 0, \frac{1}{3}, 0, \frac{2}{15}, 0, \frac{17}{315}, 0, \frac{62}{2835}, \ldots)$ has the generating function

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^9 + \cdots$$

EXERCISES

13.11. The generating function of the sequence $(1, 2, 4, 8, 16, \ldots)$ of powers of two is given by

$$A(z) = \sum_{k=0}^{\infty} 2^k z^k.$$

Find a closed form of the generating function $A(z)$ that does not use a power series.

13.12. Let a and $d > 0$ be real numbers. Find a closed form of the generating function of the arithmetic progression

$$(a, a+d, a+2d, a+3d, \ldots).$$

The closed form is an expression for the generating function that does not use a power series.

13.13. Let a and $r \neq 0$ be real numbers. Find a closed form of the generating function of the geometric progression

$$(a, ar, ar^2, ar^3, \ldots).$$

The closed form is an expression for the generating function that does not use a power series.

13.14. Use the binomial series to prove that the generating function of the central binomial coefficients $\binom{2n}{n}$ is given by

$$\frac{1}{\sqrt{1+4z}} = \sum_{k=0}^{\infty} \binom{2k}{k} z^k.$$

13.15. Let n be a nonnegative integer. Use the generating function of the binomial sequence

$$\left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}\right)$$

to prove that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

13.16. Let n be a nonnegative integer. Use the generating function of the binomial sequence

$$\left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}\right)$$

to prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

13.17. Let n be a nonnegative integer. Use the generating function of the binomial sequence

$$\left(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}\right)$$

to prove that

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n.$$

13.18. Deduce Vandermonde's identity

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$$

from $[z^k](1+z)^{m+n} = [z^k](1+z)^m(1+z)^n$.

13.19. Show that the sum of the Fibonacci numbers is given by

$$f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$$

using generating functions. You can use the fact that the Fibonacci numbers have the generating function

$$\sum_{k=0}^{\infty} f_k z^k = \frac{z}{1-z-z^2},$$

as we will show in the next chapter.

13.20. Use generating functions to show that the sum of the first n Harmonic numbers is given by

$$H_1 + H_2 + \cdots + H_n = (n+1)H_n - n.$$

13.4 Giving Change

Let us conclude this chapter with a classical application of generating functions. Suppose that you want to determine in how many different ways one can give change for an amount of c cents using pennies, nickels, dimes, and quarters.

In other words, we need to find the number of nonnegative integer solutions to the equation

$$c = p + 5n + 10d + 25q,$$

where p denote the number of pennies, n the number of nickels, d the number of dimes, and q the number of quarters. Counting the number of such solutions can be a bit tedious if we approach it directly.

Let us reformulate the problem using generating functions. The generating function listing the different possibilities for the number of pennies is given by

$$P(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k.$$

The generating functions for the number of nickels, dimes, and quarters are respectively given by

$$N(z) = \frac{1}{1-z^5}, \quad D(z) = \frac{1}{1-z^{10}}, \quad Q(z) = \frac{1}{1-z^{25}}.$$

In principle, the answer to the counting problem is

$$[z^c]P(z)N(z)D(z)Q(z).$$

But how can we extract the coefficient of z^c in

$$\frac{1}{1-z}\,\frac{1}{1-z^5}\,\frac{1}{1-z^{10}}\,\frac{1}{1-z^{25}}.$$

We do not want to expand the terms directly, as this leads to a mess when c is large. Instead, we follow a slightly more principled approach and solve simpler problems by restricting the available denominations.

If we only have pennies available, then the problem is very simple. The number of ways to give change for the amount of c cents when only pennies are available is given by

$$[z^c]P(z) = [z^c]\frac{1}{1-z} = [z^c]\sum_{k=0}^{\infty} z^k = 1.$$

This is obvious, since there are no other options than giving c pennies.

If we have pennies and nickels available, then the problem becomes a bit more interesting. Let us write the product $P(z)N(z)$ in the form

$$P(z)N(z) = \frac{1}{1-z}\,\frac{1}{1-z^5} = \sum_{k=0}^{\infty} n_k z^k.$$

Then

$$P(z) = (1-z^5)P(z)N(z) = (1-z^5)\sum_{k=0}^{\infty} n_k z^k.$$

Applying $[z^k]$ to both sides yields $1 = n_k - n_{k-5}$ or $n_k = n_{k-5} + 1$ when $k \geqslant 5$, and $n_k = 1$ when k is in the range $0 \leqslant k < 5$. Therefore, $n_k = \lfloor k/5 \rfloor + 1$ for all $k \geqslant 0$.

Example 13.22. The number of ways to give change for $c = 11$ cents using pennies and nickels is given by $\lfloor 11/5 \rfloor + 1 = 3$. This makes sense, as you have three choices: either use two nickels, merely one nickel, or no nickels at all. Once you have decided which of the three options you would like to choose, then there is no other option than giving the remainder in pennies and there is just one way to do that. By the multiplication principle, there are 3×1 solutions. The solutions are explicitly given by (a) 5, 5, 1, (b) 5, 1, 1, 1, 1, 1, 1, and (c) 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1.

If we have pennies, nickels, and dimes available, then the solutions become quite a bit more varied, but our approach remains the same. We simply try to reduce the problem to smaller cases. We write the product $P(z)N(z)D(z)$ in the form

$$P(z)N(z)D(z) = \frac{1}{1-z}\,\frac{1}{1-z^5}\,\frac{1}{1-z^{10}} = \sum_{k=0}^{\infty} d_k z^k.$$

Thus, the coefficient d_k denotes the number of ways to give change to k cents using pennies, nickels, and dimes. Multiplying by $1 - z^{10}$ yields

$$P(z)N(z) = (1 - z^{10}) \sum_{k=0}^{\infty} d_k z^k.$$

Applying $[z^k]$ to both sides yields $n_k = d_k - d_{k-10}$ or $d_k = d_{k-10} + n_k$ when $k \geqslant 10$, and $d_k = n_k$ when k is in the range $0 \leqslant k < 10$.

Finally, if we have pennies, nickels, dimes, and quarters available, then the number of ways q_k to give change for k cents has the generating function

$$P(z)N(z)D(z)Q(z) = \sum_{k=0}^{\infty} q_k z^k$$

By now it should be entirely routine. We multiply by $1 - z^{25}$ to obtain

$$P(z)N(z)D(z) = (1 - z^{25}) \sum_{k=0}^{\infty} q_k z^k.$$

Applying $[z^k]$ to both sides yields $d_k = q_k - q_{k-25}$ or $q_k = d_k + q_{k-25}$ when $k \geqslant 25$, and $q_k = d_k$ when $0 \leqslant k < 25$.

Example 13.23. Suppose that we want to know the number of ways to give change for 75 cents using pennies, nickels, dimes, and quarters. In other words, we would like to calculate q_{75}. We have

$$q_{75} = d_{75} + q_{50} = d_{75} + d_{50} + q_{25} = d_{75} + d_{50} + d_{25} + d_0.$$

We can expand the d_k's in terms of n_k terms

$$\begin{aligned}
d_{75} &= n_{75} + n_{65} + n_{55} + n_{45} + n_{35} + n_{25} + n_{15} + n_5\\
d_{50} &= n_{50} + n_{40} + n_{30} + n_{20} + n_{10} + n_0\\
d_{25} &= n_{25} + n_{15} + n_5\\
d_0 &= n_0
\end{aligned}$$

Since $n_k = \lfloor k/5 \rfloor + 1$, we get

$$\begin{aligned} d_{75} &= 16 + 14 + 12 + 10 + 8 + 6 + 4 + 2 = 72 \\ d_{50} &= 11 + 9 + 7 + 5 + 3 + 1 = 36 \\ d_{25} &= 6 + 4 + 2 = 12 \\ d_0 &= 1 \end{aligned}$$

Therefore, we can conclude that there are

$$q_{75} = d_{75} + d_{50} + d_{25} + d_0 = 72 + 36 + 12 + 1 = 121$$

ways to make change for 75 cents using pennies, nickels, dimes, and quarters.

EXERCISES

13.21. In how many different ways can you make change for 89 cents using pennies, nickels, dimes, and quarters?

13.22. Let q_k denote the number of ways to give change for k cents using pennies, nickels, dimes, and quarters. Find a structure in the generating function of (q_k) which implies that if k is a nonnegative integer such that $k \equiv 0 \pmod 5$, then $q_k = q_{k+1} = q_{k+2} = q_{k+3} = q_{k+4}$.

13.23. The country of Binoria uses 1, 2, 4, and 8 cent coins. In how many different ways can you give change for 75 cents using the Binorian denominations.