Question 1

Proof. Let A be a set. We want to show that $f : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by $f(X) = \overline{X}$ is a bijection. First, we must show that f is injective. Assume that $X_1, X_2 \in \mathcal{P}(A)$ and $f(X_1) = f(X_2)$. Then,

$$\begin{aligned} f(X_1) &= f(X_2) \\ \overline{X_1} &= \overline{X_2} \\ \overline{\overline{X_1}} &= \overline{\overline{X_2}} \\ X_1 &= X_2. \end{aligned}$$

Thus, f is injective. Now, we must show that f is surjective, or in other words, that $\text{Ran}(f) = \mathcal{P}(A)$. The subset relation $\text{Ran}(f) \subseteq \mathcal{P}(A)$ is trivial. To show that $\mathcal{P}(A) \subseteq \text{Ran}(f)$, let $Y \in \mathcal{P}(A)$. Consider the set $\overline{Y}$. Since $Y \in \mathcal{P}(A)$, $\overline{Y} \in \mathcal{P}(A)$. Then, $f(\overline{Y}) = \overline{\overline{Y}} = Y$. Thus, f is surjective. Since f is both injective and surjective, f is a bijective. $\square$

Question 2

Part a

Proof. We want to show that the function $f : (-\infty, 1) \to \mathbb{R}$ defined by $f(x) = x^3$ is not bijective. Seeking a contradiction, assume that f is bijective. Then, f is surjective. Consider the element $8 \in \mathbb{R}$. Since f is surjective, there exists an element $x \in (-\infty, 1)$ such that $f(x) = x^3 = 8$. Thus,

$$\begin{aligned} x^3 &= 8 \\ x &= 2. \end{aligned}$$

However, $2 \notin (-\infty, 1)$, a contradiction. Therefore, f is not bijective. $\square$

Part b

Proof. We want to show that the function $D : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $D(f(x)) = f'(x)$ is not bijective. Consider the elements $x, x + 1 \in \mathbb{R}[x]$. Indeed, $\frac{d}{dx}(x) = 1 = \frac{d}{dx}(x + 1)$, but $x \neq x + 1$. Therefore, D is not injective, and thus not bijective. $\square$

Part c

Proof. We want to show that the function $s : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $s(m, n) = m + n$ is not bijective. Consider the elements $(1, 1), (2, 0) \in \mathbb{N} \times \mathbb{N}$. Indeed, $s(1, 1) = 2 = s(2, 0)$, but $(1, 1) \neq (2, 0)$. Therefore, s is not injective, and thus not bijective. $\square$

Question 3

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be functions. We want to show that if $g \circ f$ is injective, then f is injective, but g need not be injective. Assume that $g \circ f$ is injective. To show that f is injective, assume that $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$. Then, $(g \circ f)(x_1) = (g \circ f)(x_2)$. Since $g \circ f$ is injective, $x_1 = x_2$. To show that g need not be injective, consider the following counterexample. Let $X = \{1, 2\}$, $Y = \{3, 4\}$, and $Z = \{5\}$. Let the functions $f : X \to Y$ and $g : Y \to Z$ be defined by the graphs $G_f = \{(1, 3), (2, 4)\}$ and $G_g = \{(3, 5), (4, 5)\}$. Then, the graph of $g \circ f$ is $G_{g \circ f} = \{(1, 5), (2, 5)\}$. □

Question 4

(a) $f(\{-3, 2, 7\}) = \{10, 5, 50\}$

(b) $f([-1, 3]) = [1, 10]$

(c) $f((-\infty, -2)) = (-3, \infty)$

Question 5

Part a

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be invertible functions. Since f and g are invertible, they are also bijective. Since f and g are bijective, the composition $g \circ f$ is also bijective. Since $g \circ f$ is bijective, it is invertible. □

Part b

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be invertible functions. Then,

$$\begin{aligned}(f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ (g^{-1} \circ g) \circ f \\ &= f^{-1} \circ \mathrm{id}_Y \circ f \\ &= f^{-1} \circ f \\ &= \mathrm{id}_X.\end{aligned}$$

Similarly,

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ (f \circ f^{-1}) \circ g^{-1} \\ &= g \circ \mathrm{id}_Y \circ g^{-1} \\ &= g \circ g^{-1} \\ &= \mathrm{id}_Z.\end{aligned}$$

Thus, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

Question 6

Proof. $\square$

Question 7

Proof. $\square$

Question 8

Proof. $\square$