Question 1

Find the general solution of each of the following systems:

a. $$\begin{cases} y_1 + y_2 = y_1' \\ -2y_1 + 4y_2 = y_2' \end{cases}$$

b. $$\begin{cases} y_1 - y_2 = y_1' \\ y_1 + y_2 = y_2' \end{cases}$$

c. $$\begin{cases} y_1 + y_3 = y_1' \\ 2y_2 + 6y_3 = y_2' \\ y_2 + 3y_3 = y_3' \end{cases}$$

Solution: For systems in the form $y' = Ay$, we can find y as e^{At}, where $e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$. Additionally, when A is diagonalizable, we can write $A = Xe^DX^{-1}$, where X is the matrix of eigenvectors of A and D is the diagonal matrix of eigenvalues of A. Then, $e^{At} = Xe^{Dt}X^{-1}$.

System a:

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \\
P(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} \\
&= (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 \\
&= (\lambda - 2)(\lambda - 3)
\end{aligned}$$

$$\lambda_1 = 2, \quad \lambda_2 = 3$$

$$\begin{aligned}
N(A - \lambda_1 I) &= N\left(\begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -x_1 + x_2 = 0\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \\
N(A - \lambda_2 I) &= N\left(\begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -2x_1 + x_2 = 0\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}
\end{aligned}$$

$$\begin{aligned}
X &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad X^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \\
e^{Dt} &= \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
y = e^{At} &= Xe^{Dt}X^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2e^{2t} & -e^{2t} \\ -e^{3t} & e^{3t} \end{pmatrix} \\
&= \begin{pmatrix} 2e^{2t} - e^{3t} & -e^{2t} + e^{3t} \\ 2e^{2t} - 2e^{3t} & -e^{2t} + 2e^{3t} \end{pmatrix}
\end{aligned}$$

System b:

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\
P(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2
\end{aligned}
$$

$$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i$$

$$
\begin{aligned}
N(A - \lambda_1 I) &= N\left(\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -ix_1 - x_2 = 0\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}\right\} \\
N(A - \lambda_2 I) &= N\left(\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : ix_1 - x_2 = 0\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ i \end{pmatrix}\right\}
\end{aligned}
$$

$$X = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad X^{-1} = \frac{1}{2i}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

Using the fact that $e^{a+bi} = e^a(\cos b + i \sin b)$

$$e^{Dt} = \begin{pmatrix} e^t(\cos t + i \sin t) & 0 \\ 0 & e^t(\cos t - i \sin t) \end{pmatrix}$$

$$
\begin{aligned}
y = e^{At} &= X e^{Dt} X^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^t(\cos t + i \sin t) & 0 \\ 0 & e^t(\cos t - i \sin t) \end{pmatrix} \frac{1}{2i} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} e^t(\cos t + i \sin t) & e^t(\cos t - i \sin t) \\ -ie^t(\cos t + i \sin t) & ie^t(\cos t - i \sin t) \end{pmatrix} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} ie^t(\cos t + i \sin t) + ie^t(\cos t - i \sin t) & -e^t(\cos t + i \sin t) + e^t(\cos t - i \sin t) \\ e^t(\cos t + i \sin t) - e^t(\cos t - i \sin t) & ie^t(\cos t + i \sin t) + ie^t(\cos t - i \sin t) \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} 2ie^t \cos t & -2ie^t \sin t \\ 2ie^t \sin t & 2ie^t \cos t \end{pmatrix} \\
&= \begin{pmatrix} e^t \cos t & -e^t \sin t \\ e^t \sin t & e^t \cos t \end{pmatrix}
\end{aligned}
$$

System c:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\begin{aligned} P(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 6 \\ 0 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 6 \\ 1 & 3-\lambda \end{vmatrix} \\ &= (1-\lambda)((2-\lambda)(3-\lambda)-6) = (1-\lambda)(\lambda^2 - 5\lambda) = -\lambda^3 + 6\lambda^2 - 5\lambda \\ &= -\lambda(\lambda^2 - 6\lambda + 5) = -\lambda(\lambda-1)(\lambda-5) \end{aligned}$$

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 5$$

$$\begin{aligned} N(A - \lambda_1 I) &= N\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{pmatrix}\right) = N\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}\right) \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 + x_3 = 0,\ x_2 + 3x_3 = 0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \right\} \\ N(A - \lambda_2 I) &= N\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 6 \\ 0 & 1 & 2 \end{pmatrix}\right) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \\ N(A - \lambda_3 I) &= N\left(\begin{pmatrix} -4 & 0 & 1 \\ 0 & -3 & 6 \\ 0 & 1 & -2 \end{pmatrix}\right) = N\left(\begin{pmatrix} 4 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}\right) \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 4x_1 - x_3 = 0,\ x_2 - 2x_3 = 0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 8 \\ 4 \end{pmatrix} \right\} \end{aligned}$$

$$X = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 8 \\ -1 & 0 & 4 \end{pmatrix}, \quad X^{-1} = \frac{1}{20}\begin{pmatrix} 0 & 4 & -8 \\ 20 & -5 & 5 \\ 0 & 1 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$e^{Dt} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}$$

$$\begin{aligned} y = e^{At} &= Xe^{Dt}X^{-1} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & 8 \\ -1 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}\frac{1}{20}\begin{pmatrix} 0 & 4 & -8 \\ 20 & -5 & 5 \\ 0 & 1 & 3 \end{pmatrix} \\ &= \frac{1}{20}\begin{pmatrix} 20e^t & e^{5t} - 5e^t + 4 & 3e^{5t} + 5e^t - 8 \\ 0 & 8e^{5t} + 12 & 24e^{5t} - 24 \\ 0 & 4e^{5t} - 4 & 12e^{5t} + 8 \end{pmatrix} \end{aligned}$$

Question 2

Solve the following initial value problems:

a. $$\begin{cases} -y_1 + 2y_2 = y_1' \\ 2y_1 - y_2 = y_2' \end{cases}, \quad y_1(0) = 3, \quad y_2(0) = 1.$$

b. $$\begin{cases} y_1 - 2y_2 = y_1' \\ 2y_1 + y_2 = y_2' \end{cases}, \quad y_1(0) = 1, \quad y_2(0) = -2.$$

Solution: Again, we use the fact that $y = e^{At}$, where A is the matrix of coefficients of the system. To solve the initial conditions, we use the fact that $y = e^{At}c$, where c is a vector of constants. Then, we can solve for c using the initial conditions.

System a:

$$
\begin{aligned}
A &= \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \\
P(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} \\
&= (-1-\lambda)^2 - 4 = \lambda^2 + 2\lambda - 3 \\
&= (\lambda+3)(\lambda-1) \\
\\
\lambda_1 &= -3, \quad \lambda_2 = 1
\end{aligned}
$$

$$
\begin{aligned}
N(A - \lambda_1 I) &= N\left(\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 + x_2 = 0\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\} \\
N(A - \lambda_2 I) &= N\left(\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}\right) \\
&= \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -x_1 + x_2 = 0\right\} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}
\end{aligned}
$$

$$
\begin{aligned}
X &= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} \\
e^{Dt} &= \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
y = e^{At} &= Xe^{Dt}X^{-1}c \\
&= \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}c \\
&= \frac{1}{2}\begin{pmatrix} e^{-3t} & e^{t} \\ -e^{-3t} & e^{t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}c \\
&= \frac{1}{2}\begin{pmatrix} e^{-3t} + e^{t} & -e^{-3t} + e^{t} \\ -e^{-3t} + e^{t} & e^{-3t} + e^{t} \end{pmatrix}c
\end{aligned}
$$

$$
\begin{aligned}
y(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}c \\
\implies c &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= \frac{1}{2}\begin{pmatrix} e^{-3t} + e^{t} & -e^{-3t} + e^{t} \\ -e^{-3t} + e^{t} & e^{-3t} + e^{t} \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} e^{-3t} + 2e^{t} \\ -e^{-3t} + 2e^{t} \end{pmatrix}
\end{aligned}
$$

System b:

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \\
P(\lambda) = \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 5
\end{aligned}
$$

$$
\lambda_1,\ \lambda_2 = \frac{2 \pm \sqrt{4-20}}{2} \rightarrow \lambda_1 = 1 + 2i, \quad \lambda_2 = 1 - 2i
$$

$$
\begin{aligned}
N(A - \lambda_1 I) &= N\left(\begin{pmatrix} -2i & -2 \\ 2 & -2i \end{pmatrix}\right) \\
&= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -ix_1 - x_2 = 0 \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} \\
N(A - \lambda_2 I) &= N\left(\begin{pmatrix} 2i & -2 \\ 2 & 2i \end{pmatrix}\right) \\
&= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : ix_1 - x_2 = 0 \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} \right\}
\end{aligned}
$$

$$
X = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad X^{-1} = \frac{1}{2i}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix}
$$

Using the fact that $e^{a+bi} = e^a(\cos b + i \sin b)$

$$
e^{Dt} = \begin{pmatrix} e^t(\cos 2t + i\sin 2t) & 0 \\ 0 & e^t(\cos 2t - i \sin 2t) \end{pmatrix}
$$

Let $\alpha = \cos 2t + i \sin 2t,\ \beta = \cos 2t - i \sin 2t$

$$
\begin{aligned}
y = e^{At} &= Xe^{Dt}X^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^t\alpha & 0 \\ 0 & e^t\beta \end{pmatrix} \frac{1}{2i} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} e^t\alpha & e^t\beta \\ -ie^t\alpha & ie^t\beta \end{pmatrix} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} ie^t\alpha + ie^t\beta & -e^t\alpha + e^t\beta \\ e^t\alpha - e^t\beta & ie^t\alpha + ie^t\beta \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} ie^t(\cos 2t + i\sin 2t) + ie^t(\cos 2t - i \sin 2t) & -e^t(\cos 2t + i \sin 2t) + e^t(\cos 2t - i\sin 2t) \\ e^t(\cos 2t + i \sin 2t) - e^t(\cos 2t - i \sin 2t) & ie^t(\cos 2t + i \sin 2t) + ie^t(\cos 2t - i\sin 2t) \end{pmatrix} \\
&= \frac{1}{2i} \begin{pmatrix} 2ie^t\cos(2t) & -2ie^t\sin(2t) \\ 2ie^t\sin(2t) & 2ie^t\cos(2t) \end{pmatrix} \\
&= e^t \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
y(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} &= e^0 \begin{pmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{pmatrix} c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} c \\
\implies c &= \begin{pmatrix} 1 \\ -2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= e^t \begin{pmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\
&= e^t \begin{pmatrix} \cos(2t) + 2\sin(2t) \\ \sin(2t) - 2\cos(2t) \end{pmatrix}
\end{aligned}
$$

Question 3

In each of the following, "diagonalize" the matrix X and use it to compute A^{-1}, A^4, e^A.

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix}$$

Solution: Use the fact that $A = XDX^{-1}$, where X is the matrix of eigenvectors of A and D is the diagonal matrix of eigenvalues of A. Then, $A^{-1} = XD^{-1}X^{-1}$, $A^4 = XD^4X^{-1}$, and $e^A = Xe^DX^{-1}$.

First matrix:

$$\begin{aligned}
A &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
P(\lambda) &= \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \\
&= \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)
\end{aligned}$$

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

$$N(A - \lambda_1 I) = N\left(\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\right) = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : -x_1 + x_2 = 0\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$$

$$N(A - \lambda_2 I) = N\left(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 + x_2 = 0\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

$$X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad X^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{aligned}
A^{-1} &= XD^{-1}X^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
A^4 &= XD^4X^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
e^A &= Xe^DX^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} e & e^{-1} \\ e & -e^{-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} e + e^{-1} & e - e^{-1} \\ e - e^{-1} & e + e^{-1} \end{pmatrix}
\end{aligned}$$

Second matrix:

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{vmatrix}$$

$$= (-1-\lambda)(2-\lambda)(1-\lambda)$$

$$\lambda_1 = -1, \quad \lambda_2 = 2, \quad \lambda_3 = 1$$

$$N(A - \lambda_1 I) = N\left(\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\right) = N\left(\begin{pmatrix} 3 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 3x_1 + x_2 = 0,\ x_2 + x_3 = 0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} \right\}$$

$$N(A - \lambda_2 I) = N\left(\begin{pmatrix} 0 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{pmatrix}\right) = N\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$N(A - \lambda_3 I) = N\left(\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}\right) = N\left(\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\right) = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$X = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 0 & -1 \\ 3 & 0 & 0 \end{pmatrix}, \quad X^{-1} = \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 6 & 5 \\ 0 & -3 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = XD^{-1}X^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 0 & -1 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 6 & 5 \\ 0 & -3 & -3 \end{pmatrix}$$

$$A^4 = XD^4X^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 0 & -1 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 6 & 5 \\ 0 & -3 & -3 \end{pmatrix}$$

$$e^A = Xe^DX^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 0 & -1 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-1} & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e \end{pmatrix} \frac{1}{3}\begin{pmatrix} 0 & 0 & 1 \\ 3 & 6 & 5 \\ 0 & -3 & -3 \end{pmatrix}$$

Third matrix:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix}$$

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 2 & 4-\lambda & -2 \\ 3 & 6 & -3-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 & -\lambda \\ 2 & 4-\lambda & 0 \\ 3 & 6 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2\lambda & -\lambda \\ 2 & -\lambda & 0 \\ 3 & 0 & -\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1-\lambda & 2\lambda & -\lambda \\ 2 & -\lambda & 0 \\ 2+\lambda & -2\lambda & 0 \end{vmatrix} = \begin{vmatrix} 3 & 0 & -\lambda \\ 2 & -\lambda & 0 \\ 2+\lambda & -2\lambda & 0 \end{vmatrix} = -\lambda \begin{vmatrix} 2 & -\lambda \\ 2+\lambda & -2\lambda \end{vmatrix} = \lambda \begin{vmatrix} 2 & \lambda \\ 2+\lambda & 2\lambda \end{vmatrix}$$

$$= \lambda(2\lambda^2 - 2\lambda - \lambda^2) = \lambda(\lambda^2 - 2\lambda)$$

$$= \lambda^2(\lambda - 2)$$

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = 2$$

$$N(A - \lambda_1 I) = N(A - \lambda_2 I) = N(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 + 2x_2 - x_3 = 0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$$

$$N(A - \lambda_3 I) = N\left(\begin{pmatrix} -1 & 2 & -1 \\ 2 & 2 & -2 \\ 3 & 6 & -5 \end{pmatrix} \right) = N\left(\begin{pmatrix} -1 & 2 & -1 \\ 0 & 6 & -4 \\ 0 & 12 & -8 \end{pmatrix} \right) = N\left(\begin{pmatrix} -1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : -x_1 + 2x_2 - x_3 = 0,\ 3x_2 - 2x_3 = 0 \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$$

$$X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad X^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -2 & 1 \\ -2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Question 4

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$$

a. Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.

b. Factor A into QR.

c. Use the above to solve the system $Ax = b$.

Solution:
Part a: The column space of A has the basis $v_1 := (2,1,2)^T, v_2 := (1,1,1)^T$. To find an orthonormal basis, we use the Gram-Schmidt process.

$$\begin{aligned} u_1 &= \frac{v_1}{\|v_1\|_2} = \frac{1}{3}(2,1,2)^T \\ p_1 &= \langle v_2, u_1 \rangle u_1 = \frac{5}{9}(2,1,2)^T \\ u_2 &= \frac{v_2 - p_1}{\|v_2 - p_1\|_2} = \frac{1}{3}(-\frac{1}{\sqrt{2}}, 2\sqrt{2}, -\frac{1}{\sqrt{2}})^T \end{aligned}$$

Part b: The orthogonal matrix Q is formed by the orthonormal basis vectors u_1, u_2.
and the upper triangular matrix R is formed by $Q^T A$.
Part c: We can now solve the system $Ax = b$ by first solving $Rx = Q^T b$.

Question 5

Let $\{x_1, x_2, x_3\} = \{(0,1,0), (2,1,2), (0,0,1)\}$, be a basis of $\mathbb{R}^3$.

a. Use the Gram-Schmidt process to obtain an orthonormal basis.

b. Let $b := (1,1,1)$. Compute the projection of b onto $\text{span}\{x_1, x_2\}$ and to $\text{span}\{x_3, x_2\}$.

Solution: *Part a:* We have that $v_1 = (0,1,0)^T, v_2 = (2,1,2)^T, v_3 = (0,0,1)^T$.

$$
\begin{aligned}
u_1 &= \frac{v_1}{\|v_1\|_2} = (0,1,0)^T \\
p_1 &= \langle v_2, u_1 \rangle u_1 = (0,1,0)^T \\
u_2 &= \frac{v_2 - p_1}{\|v_2 - p_1\|_2} = \frac{1}{\sqrt{2}}(1,0,1)^T \\
p_2 &= \langle v_3, u_1 \rangle u_1 + \langle v_3, u_2 \rangle u_2 = \frac{1}{2}(1,0,1)^T \\
u_3 &= \frac{v_3 - p_2}{\|v_3 - p_2\|_2} = \frac{1}{2\sqrt{2}}(-1,0,1)^T
\end{aligned}
$$

Part b: We have that $b = (1,1,1)^T$.

$$
\begin{aligned}
\text{proj}_{\text{span}\{x_1,x_2\}}(b) &= \langle b, u_1 \rangle u_1 + \langle b, u_2 \rangle u_2 \\
&= \frac{1}{\sqrt{2}}(1,1,1)^T \\
\text{proj}_{\text{span}\{x_3,x_2\}}(b) &= \langle b, u_2 \rangle u_2 + \langle b, u_3 \rangle u_3 \\
&= \frac{1}{2\sqrt{2}}(1,0,1)^T + \frac{1}{2\sqrt{2}}(-1,0,1)^T \\
&= \frac{1}{\sqrt{2}}(0,0,1)^T
\end{aligned}
$$

Question 6

Consider the vector space $C[0,1]$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

a. Find an orthonormal basis of the subspace E spanned by $1, x, x^2$.

b. Compute the length of $2x^2 + 3$.

c. Compute the projection of e^x onto E

Solution:

Question 7

Find the orthogonal complement of the subspace of $\mathbb{R}^4$ spanned by $(1,1,1,1), (1,-1,1,-1)$.

Solution:

Question 8

For each of the following systems $Ax = b$ find all least squares solutions:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Solution:

Question 9

Decide if the following statements are true or false:

1. There is only one inner product in $\mathbb{R}^2$, the dot product.
2. Product of orthogonal matricies is also orthogonal.
3. Sum of orthogonal matricies is also orthogonal.
4. The inverse of an orthogonal matrix is its transpose.
5. Let V be a vector space with an inner product. Then

$$\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$$

for all vectors v_1, v_2.

6. In we have a norm that satisfies

$$\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$$

for all vectors v_1, v_2, then there is a norm that is induced by an inner product.

7. $|\langle x, y \rangle|$ is always greater than the products of the norms of x and y.
8. The matricies $A^T A$ and AA^T always have the same rank.
9. Let u_1, u_2 be two orthogonal matricies in $\mathbb{R}^n$. Let V be the matrix that has these vectors as columns. Then $U^T U$ is the 2×2 identity matrix.
10. The projection of a vector x in a subspace S is the closest point in the subspace to the vector x.
11. If $\lambda \in \mathbb{R}$ and $\|\cdot\|$ is a norm, then

$$\|\lambda x\| = \lambda \|x\|$$

12. The functions $\cos x$ and $\sin x$ are orthogonal with respect to the inner product
$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$$
13. If u, v are two orthogonal vectors then $\|v + 2u\| = \sqrt{5}$
14. A defective matrix cannot be diagonalized.
15. The characteristic polynomial of an $n \times n$ matrix A has n distinct roots.
16. The product of the eigenvalues of an $n \times n$ matrix is always a real number.
17. Similar matricies have the same eigenvalues.
18. Similar matricies have the same eigenvectors.
19. If a matrix is singular then at least one of the eigenvalues is the 0 one.
20. If a matrix is singular then all the eigenvalues are 0.
21. If a 3×3 jas eigenvalues 1, 2, 0 then it is diagonalizable.
22. If λ is an eigenvalues of A then e^{λ} is an eigenvalue for e^{A}.