Chapter 7

Floor and Ceiling Functions

Are you really sure that a floor can't also be a ceiling?
— Maurits C. Escher, *On Being a Graphic Artist*

In this chapter, we learn about functions that round real numbers to an integer. The floor function rounds the numbers down and the ceiling function rounds the numbers up. We will explore a few elementary properties of these functions. Despite their simplicity, the floor and ceiling functions find many applications.

7.1 Rounding Up and Down

The **floor** is a function from the set $\mathbf{R}$ of real numbers to the set $\mathbf{Z}$ of integers that maps a real number x to the greatest integer n not exceeding x. The floor function of a real number x is denoted by $\lfloor x \rfloor$. In other words, the floor function of a real number x is defined to be equal to the integer $n = \lfloor x \rfloor$ if and only if $n \leqslant x < n+1$.

The floor function $\lfloor x \rfloor$ rounds the real number x down to the greatest integer that is smaller or equal to x. If we restrict the floor function to the set of integers, then it is the identity; thus, $\lfloor n \rfloor = n$ for all integers n. The graph of the floor function is shown in Figure 7.1.

Example 7.1. We have $\lfloor 0.3 \rfloor = 0$, $\lfloor 0.5 \rfloor = 0$, $\lfloor 0.9999 \rfloor = 0$, and $\lfloor 1.01 \rfloor = 1$. For negative numbers, one should pay attention to the fact that rounding down does not mean rounding towards 0. Indeed, $\lfloor -0.3 \rfloor = -1$, $\lfloor -0.7 \rfloor = -1$, and $\lfloor -1.1 \rfloor = -2$.

Similarly, the **ceiling** is a function from the set $\mathbf{R}$ of real numbers to the set $\mathbf{Z}$ of integers that maps a real number x to the smallest integer n greater than or equal to x. The ceiling function of a real number x is denoted by $\lceil x \rceil$. In other words, the ceiling of a real number x is defined to be the integer $n = \lceil x \rceil$ if and only if $n-1 < x \leqslant n$. If we restrict the ceiling function to the set of integers, then it is the identity, so $\lceil n \rceil = n$ for all integers n. The graph of the ceiling function is shown in Figure 7.1.

Example 7.2. We have $\lceil 0.1 \rceil = 1$, $\lceil 0.5 \rceil = 1$, $\lceil 0.999 \rceil = 1$, and $\lceil 1.1 \rceil = 2$. For negative numbers, we have $\lceil -0.3 \rceil = 0$, $\lceil -0.7 \rceil = 0$, and $\lceil -1.1 \rceil = -1$.

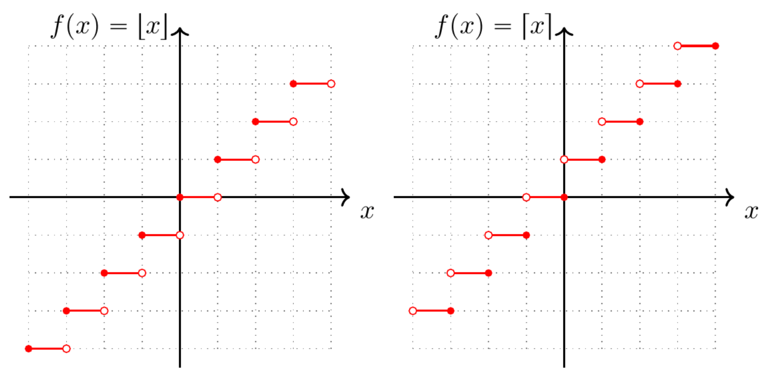

Figure 7.1: The graph on the left depicts the floor function and the graph on the right depicts the ceiling function.

The floor and ceiling functions are closely related. The next proposition shows that you can use the floor function to express the ceiling function and vice versa[1]. You should pay close attention to the style of proof that we use here in the argument, since it is typical for proofs of many other properties of these two functions.

Proposition 7.3. *For all real numbers x, we have*

$$\lfloor -x \rfloor = -\lceil x \rceil .$$

Proof. By definition, the floor of the real number $-x$ is equal to the integer $n = \lfloor -x \rfloor$ if and only if $n \leqslant -x < n+1$. Multiplying the terms in these inequalities by -1 yields $-n \geqslant x > -(n+1)$, which we can express in terms of the ceiling function as $-n = \lceil x \rceil$. It follows that $\lfloor -x \rfloor = n = -\lceil x \rceil$ holds, as claimed. ☐

Proposition 7.4. *For any real number x and any integer k, we have*

$$\lfloor x+k \rfloor = \lfloor x \rfloor + k, \quad \textit{and} \quad \lceil x+k \rceil = \lceil x \rceil + k.$$

Proof. By definition of the floor function, we have $\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + 1$. Adding the integer k to each term yields

$$\lfloor x \rfloor + k \leqslant x + k < \lfloor x \rfloor + k + 1.$$

Therefore, we can conclude that $\lfloor x+k \rfloor$ is equal to the integer $\lfloor x \rfloor + k$, which proves our first claim.

By the previous proposition and the first claim, we have

$$\lceil x+k \rceil = -\lfloor -x-k \rfloor = -(\lfloor -x \rfloor - k) = -\lfloor -x \rfloor + k = \lceil x \rceil + k,$$

which proves the second claim. ☐

[1] In this case, the coincidence of floor and ceiling is even more real than in the optical illusions of M.C. Escher.

You should memorize the inequalities characterizing the floor and ceiling functions. If you compare a floor or ceiling with an integer, then the floor or ceiling function might not be needed, as the following proposition shows. It is a simple but instructive exercise to prove this proposition.

Proposition 7.5. *Suppose that x is a real number and n an integer. Then*
(a) $\lfloor x \rfloor < n$ if and only if $x < n$,
(b) $n \leqslant \lfloor x \rfloor$ if and only if $n \leqslant x$,
(c) $n < \lceil x \rceil$ if and only if $n < x$,
(d) $\lceil x \rceil \leqslant n$ if and only if $x \leqslant n$.

Proof. See Exercise 7.4. □

When programming, we often have to split an array into two parts that are nearly of equal length. The next proposition shows that taking the first $\lfloor n/2 \rfloor$ elements and the last $\lceil n/2 \rceil$ elements of the array will do the trick.

Proposition 7.6. *For any integer n, we have*

$$n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil.$$

Proof. If n is an even integer, then there exists an integer k such that $n = 2k$. It follows that $\lfloor n/2 \rfloor = k = \lceil n/2 \rceil$, so $n = 2k = \lfloor n/2 \rfloor + \lceil n/2 \rceil$.

If n is an odd integer, then there exists an integer k such that $n = 2k + 1$. It follows that $\lfloor n/2 \rfloor = \lfloor k + 1/2 \rfloor = k$ and $\lceil n/2 \rceil = \lceil k + 1/2 \rceil = k + 1$, so

$$n = 2k + 1 = k + (k + 1) = \lfloor n/2 \rfloor + \lceil n/2 \rceil.$$

Since an integer is either even or odd, this proves the claim. □

We conclude this section with a useful result concerning the number of digits in base b representation. Recall that a positive integer a can be represented in base b in the form

$$a = \sum_{k=0}^{n} a_k b^k,$$

where the digits a_k are integers in the range $0 \leqslant a_k < b$, and the leading coefficient $a_n \neq 0$. In base b, the number a can be represented by its digits $a_n a_{n-1} \cdots a_1 a_0$. If the base b is not understood from the context, then it is customary to add the base b as a subscript.

For example, the number $1000 = 10^3$ requires 4 digits in base $b = 10$. The next proposition shows that a representation of 1000 in base 2 requires

$$\lfloor \log_2 1000 \rfloor + 1 = \lfloor 9.9657 \cdots \rfloor + 1 = 10$$

digits. Indeed, the integer 1000 is represented in base 2 by the 10-digit number 1111101000_2. The next proposition allows you to accurately determine the number of digits in base b without calculating a base change.

Proposition 7.7. *The number d of digits of a positive integer n in base b is given by*

$$d = \lfloor \log_b n \rfloor + 1 \quad \textit{and} \quad d = \lceil \log_b(n+1) \rceil .$$

Proof. Suppose that a positive integer n has d digits in base b. Then it is of the form

$$n = \sum_{k=0}^{d-1} a_k b^k,$$

where a_k is some integer in the range $0 \leqslant a_k < b$. It follows that

$$b^{d-1} \leqslant n < b^d, \tag{7.1}$$

as b^{d-1} is the smallest d-digit integer in base b, and $b^d - 1$ is the largest d-digit integer. Taking the logarithm in base b yields

$$d - 1 \leqslant \log_b n < d.$$

Therefore, $d - 1 = \lfloor \log_b n \rfloor$ or $d = \lfloor \log_b n \rfloor + 1$, as claimed.

For the version with the ceiling function, we add 1 to the terms in (7.1) and deduce the inequalities

$$b^{d-1} < n + 1 \leqslant b^d$$

as a consequence. Applying the logarithm to base b yields

$$d - 1 < \log_b(n+1) \leqslant d.$$

Therefore, $d = \lceil \log_b(n+1) \rceil$, which proves our second claim. ☐

EXERCISES

7.1. Determine the values of the following floor and ceiling functions.
(a) $\lfloor (2^{32} - 1)/8 \rfloor$
(b) $\lfloor e^{(e^e)} \rfloor$
(c) $\lfloor -1.234 \rfloor$
(d) $\lceil \pi^\pi \rceil$
(e) $\lceil (2^{32} - 1)/33 \rceil$
(f) $\lceil -1.92 \rceil$

7.2. Let x be a real number and n an integer. Show that the floor and ceiling function satisfy

(a) $\lfloor x \rfloor = n$ if and only if $x - 1 < n \leqslant x$.

(b) $\lceil x \rceil = n$ if and only if $x \leqslant n < x + 1$.

7.3. Characterize the set

$$\{x \in \mathbf{R} \mid \lfloor x \rfloor = \lceil x \rceil\}$$

of arguments where the floor and ceiling functions coincide. Prove your result.

7.4. Prove Proposition 7.5. In other words, show that if x is a real number and n an integer, then
(a) $\lfloor x \rfloor < n$ if and only if $x < n$,
(b) $n \leqslant \lfloor x \rfloor$ if and only if $n \leqslant x$,
(c) $n < \lceil x \rceil$ if and only if $n < x$,
(d) $\lceil x \rceil \leqslant n$ if and only if $x \leqslant n$.
As a cautionary tale, find **counterexamples** to each of the following statements:
(e) $\lfloor x \rfloor \leqslant n$ if and only if $x \leqslant n$,
(f) $n < \lfloor x \rfloor$ if and only if $n < x$,
(g) $n \leqslant \lceil x \rceil$ if and only if $n \leqslant x$,
(h) $\lceil x \rceil < n$ if and only if $x < n$.
The moral of the story is that you need to be very careful when you omit floor or ceiling functions in integer comparisons.

7.5. Graph the function $f(x) = \sin\left(\frac{\pi}{2}\left\lceil\frac{x}{3}\right\rceil\right)$ on the interval $[-5, 5]$.

7.6. Show that for all real numbers x, we have

$$2\lfloor x \rfloor \leqslant \lfloor 2x \rfloor \leqslant 2\lfloor x \rfloor + 1.$$

7.7. Let n be an integer and m a positive integer. Show that

$$\left\lfloor \frac{n+m-1}{m} \right\rfloor = \left\lceil \frac{n}{m} \right\rceil.$$

7.8. Rep. Quattro Sei plays around with his calculator and observes that

$$\left\lfloor \sqrt{44} \right\rfloor = 6, \quad \left\lfloor \sqrt{4444} \right\rfloor = 66, \quad \left\lfloor \sqrt{444444} \right\rfloor = 666.$$

Show that in general the floor of the square root of the integer consisting of $2n$ repeated digits of 4s is given by the integer consisting of n repeated digits of 6s, so that

$$\underbrace{\left\lfloor \sqrt{44\cdots4} \right\rfloor}_{\text{4 is repeated } 2n \text{ times}} = \underbrace{66\cdots6}_{\text{6 is repeated } n \text{ times}}$$

holds for all positive integers n.

7.9. Show that for any real number x, we have

$$0 \leqslant \lfloor 2x \rfloor - 2\lfloor x \rfloor \leqslant 1.$$

7.10. Let $f\colon \mathbf{N}_1 \to \mathbf{N}_1$ be the function on the set of positive integers given by

$$f(x) = \left\lfloor n + \sqrt{n} + \frac{1}{2} \right\rfloor.$$

Show that if $m \notin \operatorname{ran}(f)$, then m must be a perfect square.

7.11. In January 2016, the largest known prime was

$$p = 2^{74,207,281} - 1,$$

a number with 22,338,618 decimal digits. You join a team building a computer based on ternary arithmetic that is supposed to find even larger primes. How many ternary digits are needed to represent p?

7.12. Show that for any real number x and integer n, we have

$$\lfloor nx \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor.$$

7.13. Evaluate the sum

$$\sum_{k=1}^{n^2-1} \left\lfloor \sqrt{k} \right\rfloor.$$

Hint: First determine the range of positive integers k satisfying $a = \left\lfloor \sqrt{k} \right\rfloor$.

7.14. (a) Prove by induction that

$$\sum_{k=0}^{n} k2^k = (n-1)2^{n+1} + 2$$

holds for all nonnegative integers n.

(b) Evaluate the sum

$$\sum_{k=1}^{n} \lfloor \log_2 k \rfloor.$$

7.15. Show that for all nonneegative integers n, we have

$$\sum_{k=0}^{\infty} \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor = n.$$

(Source: 1968 IMO Problem 6)

7.16. Show that the function $f\colon \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \lim_{n\to\infty} \left\lfloor (\cos(n!x\pi))^2 \right\rfloor$$

satisfies

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \in \mathbf{R}\backslash\mathbf{Q}. \end{cases}$$

(Source: de Koninck, Mercier [19])

7.2 Divisibility and Primes

The next proposition determines the number of positive integer multiples of a positive integer k that do not exceed a bound x.

Proposition 7.8. *Let x be a real number and k a positive integer such that $k \leqslant x$. Then the number m of positive integers that are divisible by k and do not exceed x is given by*

$$m = \left\lfloor \frac{x}{k} \right\rfloor.$$

Proof. By definition of the floor function, we have

$$\left\lfloor \frac{x}{k} \right\rfloor \leqslant \frac{x}{k} < \left\lfloor \frac{x}{k} \right\rfloor + 1.$$

Multiplying the terms of these inequalities by k, we get

$$\left\lfloor \frac{x}{k} \right\rfloor k \leqslant x < \left(\left\lfloor \frac{x}{k} \right\rfloor + 1 \right) k.$$

In other words, multiplying k by $m = \lfloor x/k \rfloor$ does not exceed x, but $(m+1)k$ does. Therefore, the number of positive integers that are divisible by k and do not exceed x are given by m, as claimed. $\square$

For example, the number of integers in the range from 1 to 100 that are multiples of the prime 13 is given by $\lfloor 100/13 \rfloor = 7$. Indeed, the multiples of 13 in this range are explicitly given by

$$13, 26, 39, 52, 65, 78, \text{and } 91.$$

Let us now turn to a different question. Recall that each positive integer n can be written as a product of primes

$$n = \prod_p p^{v_p(n)}$$

where p ranges over the set of primes, and $k = v_p(n)$ denotes the highest power of p such that p^k divides n, but p^{k+1} does not divide n. We call $v_p(n)$ the **p-adic valuation** of n. If a prime p does not divide n, then $v_p(n) = 0$, hence $p^{v_p(n)} = 1$, so this product really consists of a finite number of factors that are not equal to 1.

Example 7.9. The number $n = 12$ has the factorization $12 = 2^2 \cdot 3$. Therefore, $v_2(12) = 2$, $v_3(12) = 1$, and $v_p(12) = 0$ for all other primes $p > 3$.

For a positive integer n, we now want to focus on the question to determine the highest power of a prime p that divides $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$. So we would like to determine

$$v_p(n!) = \sum_{k=1}^{n} v_p(k).$$

It is a good idea to study an example before trying to settle this question.

Example 7.10. Let us determine the 3-adic valuation of $13! = 6,227,020,800$. Fortunately, there is no need to factorize such large numbers, as

$$v_3(13!) = v_3(1) + v_3(2) + v_3(3) + \cdots + v_3(13).$$

Since $v_p(k) = 0$ unless k is a multiple of 3, we have

$$v_3(13!) = v_3(3) + v_3(6) + v_3(9) + v_3(12) = 1 + 1 + 2 + 1 = 5.$$

The 3-adic valuation of $9 = 3^2$ is greater than 1, since it is divisible by 3^2. We do not have any 3-adic valuations greater than 2, since none of the numbers k in the range $1 \leqslant k \leqslant 13$ is divisible by $3^3 = 27$.

Proposition 7.11. *Let n be a positive integer and p a prime. Then*

$$v_p(n!) = \sum_{k=1}^{\lfloor \log_p n \rfloor} \left\lfloor \frac{n}{p^k} \right\rfloor.$$

Proof. Since $n!$ is the product of the integers k in the range $1 \leqslant k \leqslant n$, we have

$$v_p(n!) = \sum_{k=1}^{n} v_p(k).$$

By Proposition 7.8, the number of integers that contribute a factor of p is given by $\lfloor n/p \rfloor$. Among those numbers, $\lfloor n/p^2 \rfloor$ contribute a second prime factor p. Among those numbers $\lfloor n/p^3 \rfloor$ contribute a third factor, etc.

We have $p^m > n$ if and only if $m > \log n / \log p = \log_p n$ if and only if $m > \lfloor \log_p n \rfloor$. Therefore, the sum

$$\sum_{k=1}^{\lfloor \log_p n \rfloor} \left\lfloor \frac{n}{p^k} \right\rfloor$$

counts the number of times the prime factor p occurs in the product of the numbers from 1 to n. ☐

We can even use the floor function to construct a function that allows us to characterize primes.

Proposition 7.12. *A positive integer n is prime if and only if*

$$\sum_{k=2}^{\lfloor \sqrt{n} \rfloor} \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor \right) = 0. \tag{7.2}$$

Proof. We can express n in the form $n = qk + r$ with $0 \leqslant r < k$. If k does not divide n, then the remainder r satisfies $1 \leqslant r < k$ and we have

$$\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor = \left\lfloor \frac{qk+r}{k} \right\rfloor - \left\lfloor \frac{qk+r-1}{k} \right\rfloor = q - q = 0.$$

If k divides n, then the remainder $r = 0$ and we have

$$\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor = \left\lfloor \frac{qk}{k} \right\rfloor - \left\lfloor \frac{(q-1)k + k - 1}{k} \right\rfloor = q - (q-1) = 1.$$

Therefore, the sum (7.2) counts the number of divisors k of n in the range from 2 to $\lfloor \sqrt{n} \rfloor$. The positive integer n is composite if and only if it has a divisor k in the range from $2 \leqslant k \leqslant \lfloor \sqrt{n} \rfloor$. Thus, n is prime if and only if the sum (7.2) is equal to 0. □

EXERCISES

7.17. Show that every real number $x \in \mathbf{R}$ has a *unique* representation in the form

$$x = q + r$$

where q is an integer and r is a real number in the range $0 \leqslant r < 1$. The number $q = \lfloor x \rfloor$ and $r = x - \lfloor x \rfloor$. [You can interpret this as a division by 1 with remainder $r \equiv x \pmod 1$.]

7.18. Let d be a positive integer. We know that every integer n can be uniquely expressed in the form

$$n = qd + r,$$

where r is an integer in the range $0 \leqslant r < d$. Show that the quotient q is given by $q = \lfloor n/d \rfloor$ and the remainder r by $r = n - \lfloor n/d \rfloor d$.

7.19. Let n and k be nonnegative integers such that $n \geqslant k$, m an integer such that $m > n^k$, and ℓ be the integer given by $\ell = \lfloor (m+1)^n / m^k \rfloor$. Show that the division of ℓ by m yields $\binom{n}{k}$ as a remainder. In other words, if we express ℓ in the form $\ell = qm + r$ with $0 \leqslant r < m$, then $r = \binom{n}{k}$. [Hint: Use the binomial theorem to expand $(m+1)^n$.]

7.20. Determine the number of terminating zeros of 1000!.

7.21. Does there exist a prime p such that $v_p(46!) = 42$?

7.22. Let n be a positive integer and p a prime. Suppose that $n = \sum_{k=0}^{m} n_k p^k$ is the base p expression of n, so the digits n_k are in the range $0 \leqslant n_k < p$ for all indices k, and $m = \lfloor \log_p(n) \rfloor$. Show that

$$v_p(n!) = \frac{n - (n_0 + n_1 + \cdots + n_m)}{p - 1}.$$

7.23. Let n be a positive integer. Show that 2^n does not divide $n!$. [Hint: You can use Exercise 7.22.]

7.24. Let n be an integer such that $n \geqslant 2$. Show that if 2^{n-1} divides $n!$, then n must be a power of 2. [Hint: You can use Exercise 7.22.]

7.25. Let n be an integer such that $n > 1$.
(a) Show that $\binom{2n}{n} \equiv 0 \pmod 2$
(b) Show that $\binom{2n}{n} \equiv 0 \pmod 4$ unless n is a power of 2.
[Hint: You can use Exercise 7.22.]

7.26. Use Proposition 7.12 to show that 113 is a prime number.

7.27. Explain why Proposition 7.12 does not lead to a fast check for primality.

7.28. Let n be a positive integer. Show that

$$\lfloor\sqrt{n}\rfloor - \lfloor\sqrt{n-1}\rfloor = \begin{cases} 1 & \text{if } n \text{ is a perfect square,} \\ 0 & \text{otherwise.} \end{cases}$$

7.3 Functions of Floors and Ceilings

If a function has a floor or ceiling applied to its argument and value, then it requires sometimes delicate arguments to fully analyse the behavior of the resulting function. In other words, if we are given a function $f(x)$, we might be interested in the behavior of

$$\lfloor f(\lfloor x\rfloor)\rfloor, \quad \lfloor f(\lceil x\rceil)\rfloor, \quad \lceil f(\lfloor x\rfloor)\rceil, \quad \text{or} \quad \lceil f(\lceil x\rceil)\rceil.$$

Fortunately, applying the floor (or ceiling) to the argument is sometimes redundant when a floor (or ceiling) is applied to the function value. The next proposition gives a simple example.

Proposition 7.13. *For all nonnegative real numbers, we have*

$$\left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor = \lfloor\sqrt{x}\rfloor.$$

Proof. Suppose that $n = \left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor$. This means that

$$n \leqslant \sqrt{\lfloor x\rfloor} < n+1.$$

Squaring the terms of these inequalities yields

$$n^2 \leqslant \lfloor x\rfloor < (n+1)^2.$$

Since the bounds are integers, it follows from Proposition 7.5 that

$$n^2 \leqslant x < (n+1)^2$$

holds. Taking square roots yields

$$n \leqslant \sqrt{x} < n+1,$$

so $n = \lfloor\sqrt{x}\rfloor$. We can conclude that

$$\left\lfloor\sqrt{\lfloor x\rfloor}\right\rfloor = n = \lfloor\sqrt{x}\rfloor,$$

as claimed. □

The next theorem is a vast generalization of the previous proposition. This result is due to McEliece. The domain of the function is an interval $I \subseteq \mathbf{R}$ that is closed under taking floors for the first claim and closed under taking ceilings in the second claim.

Theorem 7.14. *Let f be a continuous, monotonically strictly increasing real-valued function such that $f_{-1}(\mathbf{Z}) \subseteq \mathbf{Z}$. Then*

$$\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor \quad \textit{and} \quad \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil .$$

Proof. Let us prove the first claim. If $x = \lfloor x \rfloor$, then the claim evidently holds. If $\lfloor x \rfloor < x$, then $f(\lfloor x \rfloor) < f(x)$, as f is monotonically strictly increasing. Since the floor function is a non-decreasing function, it follows that

$$\lfloor f(\lfloor x \rfloor) \rfloor \leqslant \lfloor f(x) \rfloor .$$

Seeking a contradiction, let us suppose that $\lfloor f(\lfloor x \rfloor) \rfloor < \lfloor f(x) \rfloor$, so equality does not hold. This implies the inequality $f(\lfloor x \rfloor) < \lfloor f(x) \rfloor$, whence we have $f(\lfloor x \rfloor) < \lfloor f(x) \rfloor \leqslant f(x)$. Since f is a continuous function, it follows by the intermediate value theorem that there must exist an element y in the range $\lfloor x \rfloor \leqslant y < x$ such that $\lfloor f(x) \rfloor = f(y)$. As $f(y)$ is an integer, it follows from the hypothesis that y must be an integer. We must have $\lfloor x \rfloor \neq y$, since $\lfloor f(\lfloor x \rfloor) \rfloor$ is supposed to be different from $f(y) = \lfloor f(x) \rfloor$. However, there cannot exist an integer y satisfying $\lfloor x \rfloor < y < x$, which is our desired contradiction. Therefore, we can conclude that we must have $\lfloor f(\lfloor x \rfloor) \rfloor = \lfloor f(x) \rfloor$.

The argument for the second claim is similar, so we omit it here. □

We collect some consequences of this theorem.

Corollary 7.15. *Let n be an integer satisfying $n \geqslant 2$. Then*

$$\left\lfloor \sqrt[n]{\lfloor x \rfloor} \right\rfloor = \left\lfloor \sqrt[n]{x} \right\rfloor \quad \textit{and} \quad \left\lceil \sqrt[n]{\lceil x \rceil} \right\rceil = \left\lceil \sqrt[n]{x} \right\rceil .$$

Proof. The function $f(x) = \sqrt[n]{x}$ is continuous and monotonically strictly increasing. If the value $m = \sqrt[n]{x}$ is an integer, then its argument $x = m^n$ is an integer as well. Therefore, the claim follows from the previous theorem. □

Corollary 7.16. *Let b be an integer satisfying $b > 1$. Then*

$$\lfloor \log_b \lfloor x \rfloor \rfloor = \lfloor \log_b x \rfloor \quad \textit{and} \quad \lceil \log_b \lceil x \rceil \rceil = \lceil \log_b x \rceil .$$

Proof. The function $f(x) = \log_b x$ is continuous and monotonically strictly increasing. If the value $m = \log_b x$ is an integer, then the argument $x = b^m$ is an integer as well. Therefore, the claim follows from the previous theorem. □

Corollary 7.17. *Suppose that n is a positive integer. Then*

$$\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor \quad \textit{and} \quad \left\lceil \frac{\lceil x \rceil}{n} \right\rceil = \left\lceil \frac{x}{n} \right\rceil .$$

Proof. The function $f(x) = x/n$ is continuous and monotonically strictly increasing. If the function value $f(x) = m$ is an integer, then the argument $x = mn$ is an integer as well. Therefore, the claim follows from the previous theorem. $\square$

We can slightly generalize the previous corollary.

Corollary 7.18. *Suppose that n is a positive integer and m an arbitrary integer. Then*

$$\left\lfloor \frac{\lfloor x \rfloor + m}{n} \right\rfloor = \left\lfloor \frac{x+m}{n} \right\rfloor \quad \textit{and} \quad \left\lceil \frac{\lceil x \rceil + m}{n} \right\rceil = \left\lceil \frac{x+m}{n} \right\rceil$$

Proof. The function $f(x) = (x+m)/n$ is continuous and monotonically strictly increasing. If the function value $f(x) = k$ is an integer, then the argument $x = kn-m$ is an integer as well. Therefore, the claim follows from the previous theorem. $\square$

We can extend the previous theorem to monotonically strictly decreasing functions. We assume that the domain of the function is an interval $I \subseteq \mathbf{R}$ that is closed under taking floors for the first claim and closed under taking ceilings in the second claim.

Theorem 7.19. *Let f be a continuous, monotonically strictly decreasing real-valued function such that $f_{-1}(\mathbf{Z}) \subseteq \mathbf{Z}$. Then*

$$\lceil f(x) \rceil = \lceil f(\lfloor x \rfloor) \rceil \quad \textit{and} \quad \lfloor f(x) \rfloor = \lfloor f(\lceil x \rceil) \rfloor .$$

Proof. Let us prove the first claim. If $x = \lfloor x \rfloor$, then the claim evidently holds. If $\lfloor x \rfloor < x$, then $f(\lfloor x \rfloor) > f(x)$, as f is a monotonically strictly decreasing function. Since $y > y'$ implies $\lceil y \rceil \geqslant \lceil y' \rceil$, we can conclude that

$$\lceil f(\lfloor x \rfloor) \rceil \geqslant \lceil f(x) \rceil .$$

Seeking a contradiction, let us suppose that $\lceil f(\lfloor x \rfloor) \rceil > \lceil f(x) \rceil$, so equality does not hold. This implies the inequality $f(\lfloor x \rfloor) > \lceil f(x) \rceil$, whence we have $f(\lfloor x \rfloor) > \lceil f(x) \rceil \geqslant f(x)$. Since f is a continuous function, it follows by the intermediate value theorem that there must exist an element y in the range $\lfloor x \rfloor \leqslant y < x$ such that $\lceil f(x) \rceil = f(y)$. As $f(y)$ is an integer, it follows from the hypothesis that y must be an integer. We must have $\lfloor x \rfloor \neq y$, since $\lceil f(\lfloor x \rfloor) \rceil$ is supposed to be different from $f(y) = \lceil f(x) \rceil$. However, there cannot exist an integer y satisfying $\lfloor x \rfloor < y < x$, which is our desired contradiction. Therefore, we can conclude that we must have $\lceil f(\lfloor x \rfloor) \rceil = \lceil f(x) \rceil$.

The argument for the second claim is similar, so we omit it here. $\square$

EXERCISES

7.29. Carl was floored when he saw the following expression in a program:

$$\lfloor\lfloor\lfloor\lfloor x/10\rfloor/10\rfloor/10\rfloor/10\rfloor.$$

Help him rewrite it using a single floor function.

7.30. Let n be an integer and ξ a real number in the range $0 \leqslant \xi < 1$. The function $r_+ : \mathbf{R} \to \mathbf{R}$ rounds a real number towards infinity, so

$$r_+(n+\xi) = \begin{cases} n & \text{if } 0 \leqslant \xi < 1/2, \\ n+1 & \text{if } 1/2 \leqslant \xi < 1. \end{cases}$$

Show that the rounding function $r_+(x)$ can also be expressed in the following two ways:
(a) $r_+(x) = \left\lfloor x + \frac{1}{2} \right\rfloor$ for all $x \in \mathbf{R}$,
(b) $r_+(x) = \left\lceil \frac{\lfloor 2x \rfloor}{2} \right\rceil$ for all $x \in \mathbf{R}$.

7.31. The function $r_+(x) = \lfloor x + 1/2 \rfloor$ rounds a real number x to the nearest integer. If two integers are equally near to x, then $r_+(x) = \lfloor x + 1/2 \rfloor$ rounds to the larger of the two integers. Use the floor function to define a function $r_-(x)$ that rounds a real number x to the nearest integer. If two integers are equally near to x, then $r_-(x)$ rounds to the smaller of the two integers.

7.32. Emmy is playing with a calculator. She enters an integer, and takes its square root. Then she repeats the process with the integer part of the answer. After the third repetition, the integer part equals 1 for the first time. What is the difference between the largest and the smallest number Emmy could have started with? (Source: Norwegian Math Olympiad 2014–15)

7.33. Show that for all integers m and n, and all real numbers x, we have

$$\left\lfloor \frac{1}{m} \left\lfloor \frac{1}{n} x \right\rfloor \right\rfloor = \left\lfloor \frac{1}{n} \left\lfloor \frac{1}{m} x \right\rfloor \right\rfloor = \left\lfloor \frac{1}{mn} x \right\rfloor.$$

The functions $L_m(x) = \lfloor x/m \rfloor$ are digital lines that are relevant in computer graphics. This exercise claims that the composition of the digital line functions L_m and L_n is commutative, so

$$L_m \circ L_n = L_n \circ L_m = L_{mn}.$$

7.4 Notes

The notations for the floor and ceiling functions were introduced by Iverson and popularized by Knuth. A rich source of information on properties of floor and ceiling functions is Knuth [50] and Graham, Knuth, Patashnik [31]. The book by Herman, Kučera, and Šimša [36] contains some interesting problems about floor and ceiling functions.