Question 1

Determine whether the following subsets are subspaces:

Part a

$S_1 := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 = \sqrt{123}x_2\}$

Answer: This is a subspace of $\mathbb{R}^2$ since it is a straight line that passes through the origin.

Proof. Let (a, b) and (c, d) be two elements of S_1. We want to show that $(a, b) + (c, d) \in S_1$ and $n(a, b) \in S_1 \forall n \in \mathbb{R}$. Using the definition of S_1, we have that $a = \sqrt{123}b$ and $c = \sqrt{123}d$. Adding elements (a, b) and (c, d), we have that $(a, b) + (c, d) = (a + c, b + d)$ We can then substitute in the values of a and c to get $(a + c, b + d) = (\sqrt{123}b + \sqrt{123}d, b + d)$ This can then be factored to $(\sqrt{123}(b + d), b + d)$ Since this satisfies the definition of S_1, we have that $(a, b) + (c, d) \in S_1$. To show that $n(a, b) \in S_1 \forall n \in \mathbb{R}$, we can use the definition of S_1 again. The element (a, b) can be written as $(\sqrt{123}b, b)$. Multiplying this by n gives us $(n\sqrt{123}b, nb)$. Since this satisfies the definition of S_1, we have that $n(a, b) \in S_1 \forall n \in \mathbb{R}$. □

Part b

$S_2 := \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 x_2 = 1\}$

Answer: This is not a subspace of $\mathbb{R}^2$ since it does not satisfy the addition property.

Proof. Let (a, b) and (c, d) be two elements of S_2. Seeking a contradiction, lets assume that $(a, b) + (c, d) \in S_2$. Since we can write $(a, b) + (c, d)$ as $(a + c, b + d)$, our assumption would imply that $(a + c)(b + d) = 1$. Expanding this, we get $ab + ad + bc + cd = 1$. It is given that $ab = 1$ and $cd = 1$, so we can substitute these in to get $1 + ad + bc + 1 = 1$. This can be simplified to $ad + bc = -1$. However, a and b multiply to a positive number, and c and d multiply to a positive number. This implies that $ad + bc$ must be positive, so we have reached a contradiction. Therefore, $(a, b) + (c, d) \notin S_2$, so S_2 is not a subspace of $\mathbb{R}^2$. □

Part c

$S_3 := \{\text{the set of singular } 2 \times 2 \text{ matrices}\}$

Answer: This is not a subspace of $\mathbb{R}^{2\times 2}$ since it does not satisfy the addition property.

Counterexample: Let matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ where A, B $\in S_3$. $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is not a singular matrix.

Part d

Let A be a fixed (but arbitrary) 2×2 matrix.

$$S_4 := \{B \in \mathbb{R}^{2\times 2} : BA = 0\}$$

Answer:

Part e

$$S_5 := \{\text{the set of all polynomials of degree 2 or 4}\}$$

Answer:

Part f

$$S_6 := \{\text{the set of upper triangular } 2 \times 2 \text{ matrices}\}$$

Answer:

Part g

$$S_7 := \{p \in \mathbb{P}_4 : p(0) = 0\}$$

Answer:

Question 2

Find the null space of the following matrices:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix},$$
$$B = \begin{bmatrix} -1 & -2 & 2 & 1 \\ 2 & 4 & -4 & -2 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & 3 \\ 4 & 3 & 0 \end{bmatrix}.$$

Answer:

Question 3

Show that the following matrices form a spanning set for $\mathbb{R}^{2\times 2}$. Also, show that these matrices are linearly independent.

Answer:

Question 4

Let x_1, x_2, and x_3 be linearly independent vectors in $\mathbb{R}^n$. Define:

$$\begin{aligned} y_1 &= x_1 + x_2, \\ y_2 &= x_2 + x_3, \\ y_3 &= x_3 + x_1. \end{aligned}$$

Decide if y_1, y_2, and y_3 are linearly independent or not.

Answer: