Question 1

Proof. Let m and n be integers. We want to show that if m is even and n is odd, then $m+n$ is odd. Since m is even, we can write $m = 2k$ for some integer k. Likewise, since n is odd, we can write $n = 2l + 1$ for some integer l. Then,

$$\begin{aligned} m + n &= 2k + 2l + 1 \\ &= 2(k + l) + 1. \end{aligned}$$

Since $k + l$ is an integer, $m + n$ is odd. □

Question 2

Proof. Let a, b, c, k and l be integers. We want to show that if $a \mid b$ and $a \mid c$, then $a \mid (bk+cl)$. Since $a \mid b$, we can write $b = am$ for some integer m. Likewise, since $a \mid c$, we can write $c = an$ for some integer n. Thus, we can write

$$\begin{aligned} bk + cl &= amk + anl \\ &= a(mk + nl). \end{aligned}$$

Since $mk + nl$ is an integer, we have shown that a divides $bk + cl$. □

Question 3

Proof. Let m be an odd integer. We want to show that for all integers m, if m is odd, then there exists some integer k such that $m^2 = 8k + 1$. By definition, m can be written as $m = 2n + 1$ for some integer n. We can then substitute this into m^2 to get

$$\begin{aligned} m^2 &= (2n + 1)^2 \\ &= 4n^2 + 4n + 1 \\ &= 4(n^2 + n) + 1. \end{aligned}$$

By Lemma 1, $n^2 + n$ is even for all integers n. Thus, we can write $n^2 + n = 2k$ for some integer k. Substituting this into m^2, we get

$$\begin{aligned} m^2 &= 4(2k) + 1 \\ &= 8k + 1. \end{aligned}$$

Now we have found an integer k such that $m^2 = 8k + 1$. □

Question 4

Proof. Let n be an integer. We want to show that for any integer n, $n^2 + n - 9$ is odd. In the case that n is even, we can write $n = 2k$ for some integer k. Thus, we can write

$$\begin{aligned} n^2 + n - 9 &= (2k)^2 + 2k - 9 \\ &= 4k^2 + 2k - 9 \\ &= 4k^2 + 2k - 10 + 1 \\ &= 2(2k^2 + k - 5) + 1. \end{aligned}$$

Since $2k^2 + k - 5$ is an integer, $n^2 + n - 9$ is odd for even n. In the case that n is odd, we can write $n = 2k + 1$ for some integer k. Then, we can substitute as follows:

$$\begin{aligned} n^2 + n - 9 &= (2k + 1)^2 + 2k + 1 - 9 \\ &= 4k^2 + 4k + 1 + 2k + 1 - 9 \\ &= 4k^2 + 6k - 8 + 1 \\ &= 2(2k^2 + 3k - 4) + 1. \end{aligned}$$

Since $2k^2 + 3k - 4$ is an integer, $n^2 + n - 9$ is odd for odd n. Therefore, we have shown that $n^2 + n - 9$ is odd for all integers n. □

Question 5

Although the formal proof and the proof using lemmas are both valid, they have some differences that are worth noting. For starters, the formal proof is easier to follow since you can see everything step by step. This also makes the formal proof longer than the proof using lemmas. On the other hand, the proof using lemmas is a lot more concise while still being logically equivalent to the formal proof. The downside to the proof using lemmas is that it abstracts away a lot of the details, so it sounds a lot more like "trust me bro" than the formal proof.

Question 6

a)

Proof. Let a and b be integers. We want to show that if $a \mid b$, then $a^2 \mid b^2$. Since $a \mid b$, we can write $b = ak$ for some integer k. Then, we can square both sides to get

$$\begin{aligned} b^2 &= (ak)^2 \\ &= a^2k^2. \end{aligned}$$

Since k^2 is an integer, we have shown that $a^2 \mid b^2$. □

b)

Disproof. Letting $a = 2, b = 2, c = 3$, we have a counterexample since

$$2^{2^3} = 2^8 = 256,\ (2^2)^3 = 4^3 = 64.$$

Thus, the equality $a^{bc} = (a^b)^c$ does not hold for integers a, b, and c. □

c)

Disproof. Letting $x = 1, y = -1$, we have a counterexample since

$$|1 - (-1)| = 2,\ |1| - |-1| = 0.$$

Thus, the equality $|x-y| = |x|-|y|$ does not hold for real numbers x and y. □

d)

Proof. Let m and n be even integers. We want to show that for all m and n, 4 is a factor of mn. Since m and n are even, we can write $m = 2k$ and $n = 2l$ for some integers k and l. Therefore, we can write

$$\begin{aligned} mn &= (2k)(2l) \\ &= 4kl. \end{aligned}$$

Since kl is an integer, we have shown that 4 is a factor of mn. □

e)

Disproof. If we let $a = 3, b = 2, c = 4$, we have a counterexample since $b + c$ is 6 and 3 divides 6, but 3 does not divide 2 or 4. □

f)

Proof. If we let $m = 2, n = -3$, we have found two integers m and n such that $15m + 12n = -6$. □