Question 1

Determine if the vectors are linearly independent.

(a) $v_1 = (1, 2, 3)^T, v_2 = (2, 3, 4)^T, v_3 = (3, 4, 5)^T$

(b) $v_2 = (0, 1, 0, 1)^T, v_2 = (1, 0, 1, 0)^T, v_3 = (2, 0, 2, 0)^T, v_4 = (0, 2, 0, 2)^T$

(c) $v_3 = (-1, 1, -1, 1)^T, v_2 = (1, -1, 1, -1)^T, v_3 = (-1, 1, 1, -1)^T, v_4 = (1, 1, 1, 1)^T$

Solution: Use the determinant of the matrix formed by the vectors to determine if they are linearly independent. If the determinant is non-zero, then the vectors are linearly independent. If the determinant is zero, then the vectors are linearly dependent.

(a)

$$V = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

$$\begin{aligned} \det(V) &= 1 \times \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + 3 \times \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ &= 1 \times (15 - 16) - 2 \times (10 - 12) + 3 \times (8 - 9) \\ &= 1 \times (-1) - 2 \times (-2) + 3 \times (-1) \\ &= -1 + 4 - 3 \\ &= 0 \end{aligned}$$

$(1, 2, 3)^T, (2, 3, 4)^T, (3, 4, 5)^T$ are linearly dependent.

(b)

$$V = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}$$

Rows 1 and 3 are identical, so the determinant is zero, and $(0, 1, 0, 1)^T, (1, 0, 1, 0)^T, (2, 0, 2, 0)^T, (0, 2, 0, 2)^T$ are linearly dependent.

(c)

$$V = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Columns 1 and 2 are scalar multiples (by -1) of each other, so the determinant is zero, and $(-1, 1, -1, 1)^T, (1, -1, 1, -1)^T, (-1, 1, 1, -1)^T, (1, 1, 1, 1)^T$ are linearly dependent.

Question 2

Determine if the following vectors in the vector space of smooth functions in $[0, 1]$ are linearly independent.

(a) $p_1(x) := x^2, \quad p_2(x) := x^3, \quad p_3(x) := x^{99}$

(b) $f_1(x) := e^x, \quad f_2(x) := e^{3x}, \quad f_3(x) := e^{5x}, f_4(x) := e^{7x}$

(c) $f_1(x) := \cos(x), \quad f_2(x) := \sin(x), \quad f_3(x) := x$

Solution: We can use the Wronskian to determine if the vectors of functions are linearly independent.

(a) Since x^2, x^3, x^{99} cannot be written as a linear combination of each other, they are linearly independent.

(b) Since $e^x, e^{3x}, e^{5x}, e^{7x}$ cannot be written as a linear combination of each other, they are linearly independent.

(c)

$$\begin{aligned}
W(f_1, f_2, f_3) &= \begin{vmatrix} \cos(x) & \sin(x) & x \\ -\sin(x) & \cos(x) & 1 \\ -\cos(x) & -\sin(x) & 0 \end{vmatrix} \\
&= x \times \begin{vmatrix} -\sin(x) & \cos(x) \\ -\cos(x) & -\sin(x) \end{vmatrix} - 1 \times \begin{vmatrix} \cos(x) & \sin(x) \\ -\cos(x) & -\sin(x) \end{vmatrix} \\
&= x \times (\sin^2(x) + \cos^2(x)) - 1 \times (-\sin(x)\cos(x) + \sin(x)\cos(x)) \\
&= x \times 1 - 1 \times 0 \\
&= x
\end{aligned}$$

Since x is not identically zero, $f_1(x) := \cos(x), \quad f_2(x) := \sin(x), \quad f_3(x) := x$ are linearly independent.

Question 3

Find a basis for the row space, column space, and null space for the following matrices.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 4 \\ 2 & 3 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 4 & -7 & -1 \\ 0 & -7 & 8 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 2 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix}$$

Solution: Use Gaussian elimination to find the reduced row echelon form of each matrix.

Matrix A

$$\begin{bmatrix} 1 & 1 & 0 \\ 3 & 1 & 4 \\ 2 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 14 \end{bmatrix}$$

Here we can see that all three rows are linearly independent, so the row space is the span of the three rows.

$$\text{rowspace}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \right\}$$

Since dim(rowspace) = dim(colspace), the column space is the span of the three columns.

$$\text{colspace}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \right\}$$

By the Rank-Nullity Theorem:

$$\text{nullspace}(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Matrix B

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 4 & -7 & -1 \\ 0 & -7 & 8 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & -2 & 3 \\ 0 & -3 & 1 & 0 \\ 0 & -7 & 8 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & -2 & 3 \\ 0 & -1 & -1 & 3 \\ 0 & 7 & -8 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & -4 & 9 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & -15 & 20 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & -15 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 1 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -16 \\ 0 & 0 & 0 & -55 \end{bmatrix}$$

Since rank $= 4$:

$$\text{rowspace} = \text{span}\left\{\begin{bmatrix}2\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\4\\-7\\-1\end{bmatrix}, \begin{bmatrix}0\\-7\\8\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\right\}, \quad \text{colspace} = \text{span}\left\{\begin{bmatrix}2\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\4\\-7\\-1\end{bmatrix}, \begin{bmatrix}0\\-7\\8\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}\right\}$$

$$\text{nullspace} = \text{span}\left\{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right\}$$

Matrix C

$$\begin{bmatrix}1 & 1\\-1 & 1\\2 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 1\\0 & 2\\0 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 1\\0 & 2\\0 & 0\end{bmatrix}$$

Since row 3 can be written as row 1 - row 2, the row space is the span of the first two rows.

$$\text{rowspace}(C) = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\}$$

Since the column space is the span of the vectors in the columns of the original matrix corresponding to the columns with leading ones in the reduced row echelon form, the column space is the span of the first two (or both) columns.

$$\text{colspace}(C) = \text{span}\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$$

By the Rank-Nullity Theorem:

$$\text{nullspace}(C) = \text{span}\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$$

Matrix D

$$\begin{bmatrix}0 & 1 & 2\\-1 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & -1 & -2\\0 & 1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\0 & 1 & 2\end{bmatrix}$$

Thus, we have:

$$\text{rowspace}(D) = \text{span}\left\{\begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\2\end{bmatrix}\right\}, \quad \text{colspace}(D) = \text{span}\left\{\begin{bmatrix}0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$$

Solving for the null space:

$$\begin{cases}x_1 = 0\\x_2 + 2x_3 = 0\end{cases} \rightarrow \begin{cases}x_1 = 0\\x_2 = -2x_3\end{cases}$$

$$\text{nullspace}(D) = \{(0, -2a, a)^T \mid a \in \mathbb{R}\} = \text{span}\left\{\begin{bmatrix}0\\-2\\1\end{bmatrix}\right\}$$

Question 4

Let

$$u_1 := (1,0,2)^T,\ u_2 := (-1,1,0)^T,\ u_3 := (1,0,1)^T$$
$$v_1 := (1,1,-1)^T,\ v_2 := (-1,0,0)^T,\ v_3 := (-1,1,1)^T$$

a. Find the transition matrix corresponding to the change of basis from $\{e_1, e_2, e_3\}$ to $\{u_1, u_2, u_3\}$.

b. Find the transition matrix corresponding to the change of basis from $\{v_1, v_2, v_3\}$ to $\{e_1, e_2, e_3\}$.

c. Find the transition matrix from $\{v_1, v_2, v_3\}$ to $\{u_1, u_2, u_3\}$.

d. Let $x = 1v_1 + 0v_2 - v_3$. Find the coordinates of x with respect to $\{u_1, u_2, u_3\}$.

e. Verify your answer to the previous part by computing the coordinates in each case with respect to the standard basis.

Solution:
a. Let the transition matrix from $\{e_1, e_2, e_3\}$ to $\{u_1, u_2, u_3\}$ be U^{-1}. If U is the transition matrix from $\{u_1, u_2, u_3\}$ to $\{e_1, e_2, e_3\}$, then we just need to find the inverse of U.

$$U = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{bmatrix}$$

b. Let the transition matrix from $\{v_1, v_2, v_3\}$ to $\{e_1, e_2, e_3\}$ be V.

$$V = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

c. Let the transition matrix from $\{v_1, v_2, v_3\}$ to $\{u_1, u_2, u_3\}$ be $U^{-1}V$.

$$U^{-1}V = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 1 \\ 1 & 0 & 1 \\ 5 & -2 & -1 \end{bmatrix}$$

d. x with respect to $\{u_1, u_2, u_3\}$ is $U^{-1}Vx$.

$$U^{-1}Vx = \begin{bmatrix} -3 & 1 & 1 \\ 1 & 0 & 1 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix}$$

e. x with respect to the standard basis:

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

Question 5

For each of the following choises of A, b, determine whether b is in the column space of A and state whether the system Ax = b is consistent or not.

a. $A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

b. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$

c. $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Question 6

Let

$$u_1 := (1,1,0)^T,\ u_2 := (1,0,1)^T,\ u_3 := (0,1,1)^T$$

be a basis of $\mathbb{R}^3$. Define : $\mathbb{R}^2 \to \mathbb{R}^3$ as

$$L(x) := x_1 u_1 + x_2 u_2 + (x_1 + x_2) u_3$$

Find a matrix A representing L with respect to the ordered basis e_1, e_2 and u_1, u_2, u_3.

Question 7

For each of the following transformations, find a matrix A such that $L(x) = Ax$.

a. $L((x_1, x_2)^T) = (x_1 + x_2, x_2 + x_1, x_1)^T$

b. $L((x_1, x_2, x_3)^T) = (x_3, x_2, x_1)^T$

c. $L((x_1, x_2, x_3)^T) = (x_1)$

d. $L : P_3 \to P_3,\ P(a + bx + cx^2) = (c + bx + ax^2)$. (Consider the standard basis).

Question 8

Determine which of the following sentences are true or false

1. If $\{v_1, ..., v_n\}$ are linearly independent, then at least one of them can be written as a linear combination of the rest.
   False. If they are linearly independent, then none of them can be written as a linear combination of the rest.

2. If $L : V \to W$ is a lienar map then maps 0_v to 0_w.
3. Let S be a subspace of V and $\dim(S) = n = \dim(V)$. Then $S = V$.
4. If $\{v_1, ..., v_n\} \subseteq V$ are linearly independent and $\dim(V) = n+1$, then one can always find a v_{n+1} such that $\{v_1, ..., v_{n+1}\}$ form a bsis for V.
5. The only linear maps $L : \mathbb{R} \to \mathbb{R}$ are of the form $f(x) = ax$ for some $a \in \mathbb{R}$.
6. The map that reflects a point through the origin is a linear map.
7. The are vector spaces with infinite dimensions.
8. The every vector space has either only 1 element of infinitely many.
9. If 0 is inside a subset S of V then S is a subspace.
10. If S is a subspace then 0 is inside S.
11. If $L : V \to W$ is a linear map then $2L$ is also a linear map.
12. If $L : V \to W$ is a linear map then $L + 2$ is also a linear map.
13. If A is a singular 3×3 matrix then the map $L(x) = A(x)$ is a linear map.
14. If $Ax = b$ has a solution then b lies inside the row space of A.
15. IF A^T is row equivalent with B^T, then A, B have the same column space.