Chapter 5

Equivalence Relations

> *Equivalence relations are so ubiquitous in everyday life that we often forget about their proactive existence.*
>
> — T. Britz, M. Mainetti, L. Pezzoli

The elements of a set may differ in many ways. Sometimes we want to single out some of their properties or attributes. An equivalence relation allows one to relate and "identify" elements that have the same properties. For instance, the relation "has the same model year" is an equivalence relation on the set of cars. In this chapter, we derive the most common properties of equivalence relations and give some applications.

5.1 Generalities

Let S be a set. An **equivalence relation** on the set S is a reflexive, symmetric, and transitive relation[1] $\sim$ on S. In other words, $\sim$ is an equivalence relation if and only if it satisfies the following three properties:

E1. For all x in S, we have $x \sim x$ (reflexivity).

E2. For all x, y in S, if $x \sim y$, then $y \sim x$ (symmetry).

E3. For all x, y, z in S, if $x \sim y$ and $y \sim z$, then $x \sim z$ (transitivity).

An equivalence relation allows us to describe what elements of a set we consider to be "essentially the same", so it broadens the concept of an equality.

Example 5.1. The identity relation $=$ on a set S is an equivalence relation. It is given by the set of pairs $\{(x, x) \mid x \in S\}$.

Example 5.2. The relation "has the same birthday as" is an equivalence relation on the set of people.

Example 5.3. The relation "has the same absolute value" on the set of real numbers is an equivalence relation.

[1] You might recall that we introduced these notions in Chapter 3. In this chapter, we will study this important trio of properties in more detail.

Example 5.4. Let S be the set of strings over an alphabet A. Then the relation "has the same string length as" is an equivalence relation on the set S of strings.

Example 5.5. Let S and T be nonempty sets and $f\colon S \to T$. Then the relation $\sim$ on S defined by $x \sim y$ if and only if $f(x) = f(y)$ is an equivalence relation. All previous examples are special cases of this example.

Let us verify that $\sim$ is indeed an equivalence relation. Since $f(x) = f(x)$ holds for all x in S, we can deduce that $x \sim x$ for all x in S; thus, the relation $\sim$ is reflexive. If $x \sim y$, then $f(x) = f(y)$, so $f(y) = f(x)$, which implies $y \sim x$; therefore, the relation $\sim$ is symmetric. Finally, if $x \sim y$ and $y \sim z$, then $f(x) = f(y)$ and $f(y) = f(z)$, which implies $f(x) = f(z)$, whence $x \sim z$; thus, the relation $\sim$ is transitive.

For an element x in a set S, we can define its **equivalence class** $[x]$ under an equivalence relation $\sim$ as the set

$$[x] = \{y \in S \mid x \sim y\}.$$

In other words, $[x]$ is the set of all elements in S that are equivalent to x.

Lemma 5.6. *Two equivalence classes are either the same or are disjoint.*

Proof. Suppose that the equivalence classes $[x]$ and $[y]$ have an element z in common. Then $x \sim z$ and $y \sim z$. It follows that $z \sim y$ by symmetry. Since $x \sim z$ and $z \sim y$, we have $x \sim y$. Thus, $y \in [x]$ and the transitivity of $\sim$ allows us to conclude $[y] \subseteq [x]$. As $x \sim y$ implies by symmetry that $y \sim x$, we can conclude in the same vein that $[x] \subseteq [y]$. Therefore, if two equivalence classes of $\sim$ contain a common element, then they are the same. ❑

Example 5.7. We can define a relation $\sim$ on Euclidean plane $\mathbf{R}^2$ such that $(x_1, y_1) \sim (x_2, y_2)$ if and only if $\sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$. This relation is reflexive, symmetric, and transitive, so it is an equivalence relation.

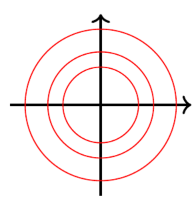


Figure 5.1: The equivalence classes $[(0, 0.5)]$, $[(0, 0.7)]$, and $[(0, 1)]$.

Since $\sqrt{x_1^2 + y_1^2}$ denotes the distance of the point (x_1, y_1) from the origin $(0, 0)$, all points that are at the same distance from the origin belong to the same equivalence class. So an equivalence class of points forms a circle about the origin. Figure 5.1 shows three equivalence classes.

A **partition** of a set S is a family P of nonempty subsets of S such that $S = \bigcup P$ and the sets in P are pairwise disjoint. The elements of P are called **blocks**. For instance, the partition $P = \{\{1,2\},\{3\}\}$ of the set $S = \{1,2,3\}$ consists of two blocks, namely $\{1,2\}$ and $\{3\}$.

Proposition 5.8. *Let S be a nonempty set and $\sim$ an equivalence relation on S. Then the equivalence classes of $\sim$ partition the set S.*

Proof. Let $P = \{[x] \mid x \in S\}$. The sets in the family P are pairwise disjoint by Lemma 5.6. Since the relation $\sim$ is reflexive, we have $x \in [x]$, so the equivalence classes are not empty. Furthermore, it follows that $\bigcup P = S$. □

Proposition 5.9. *Let P be a partition of a nonempty set S. For x, y in S, we define $x \equiv y$ if and only if x in C and y in C for some C in P. Then $\equiv$ is an equivalence relation on S and P is its set of equivalence classes.*

Proof. It follows from that definition that the relation $\equiv$ is reflexive and symmetric. If x, y, z are elements of S such that $x \equiv y$ and $y \equiv z$, then there exist sets C and D in P such that x, y in C and y, z in D. Since C and D are elements of a partition and the element y is contained in both C and D, we must have $C = D$. Thus, $x \equiv z$, which proves that the relation is transitive. We can conclude that $\equiv$ is indeed an equivalence relation.

Let C be a set in the partition P. For all x, y in C, we have by definition $[x] = C = [y]$. This implies the second claim. □

Example 5.10. Consider the partition

$$P = \{\{1,2\},\{3\},\{4,5\}\}.$$

of the set $S = \{1,2,3,4,5\}$. Each block of the partition corresponds to elements of the set S that are identified under the equivalence relation $\equiv$. The equivalence relation $\equiv$ on S corresponding to P is given by

$$\equiv \, = \{(1,1),(2,2),(1,2),(2,1)\} \cup \{(3,3)\} \cup \{(4,4),(5,5),(4,5),(5,4)\}.$$

For instance, $1 \equiv 2$ in the equivalence relation $\equiv$, since 1 and 2 belong to the same block in the partition P.

Given an equivalence relation $\sim$ on a set S, we denote by $S/{\sim}$ the set

$$S/{\sim} = \{[x] \mid x \in S\}$$

of all equivalence classes. The set $S/{\sim}$ is called the **quotient set** of the set S under $\sim$. The map $x \mapsto [x]$ from S to its quotient set $S/{\sim}$ is called a **natural map**.

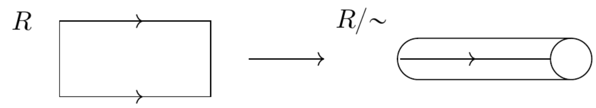
Example 5.11. Consider the set

$$R = \{(x,y) \in \mathbf{R}^2 \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1\}$$

of points that form a rectangle in the Euclidean plane. We can form an equivalence relation $\sim$ on R that identifies points on the lower and upper border of the rectangle. In other words, the equivalence classes of a point (x, y) is given by

$$[(x,y)] = \begin{cases} \{(x,y)\} & \text{if } 0 < y < 1, \\ \{(x,0),(x,1)\} & \text{if } y = 0 \text{ or } y = 1. \end{cases}$$

The equivalence relation identifies the line segment $L_0 = \{(x, 0) \mid 0 \leqslant x \leqslant 2\}$ with the line segment $L_1 = \{(x, 1) \mid 0 \leqslant x \leqslant 2\}$ by identifying the points $(x, 0) \sim (x, 1)$ for all x in the range $0 \leqslant x \leqslant 2$. We can think of the quotient space $R/\sim$ as a cylinder that is obtained from the rectangle R by gluing the line segments L_0 and L_1 together.



Of course, we cannot take this representation too literally, but it conveys the idea of the identification of the points on the top and bottom border of the rectangle. In topologie, this identification is actually made precise.

Equivalence relations permeate all areas of computer science and mathematics. In the subsequent sections, we show how to obtain integers, rational numbers, real numbers and more using equivalence relations.

EXERCISES

5.1. Let n be a positive integer. Define on the set $\mathbf{R}$ of real numbers an approximate equality $\approx_n$ such that $x \approx_n y$ if and only if $|x - y| \leqslant 10^{-n}$. Either show that $\approx_n$ is an equivalence relation or give a counterexample.

5.2. Let L denote the set of all lines in the Euclidean plane $\mathbf{R}^2$. Let ℓ_1 and ℓ_2 be lines in L. We say that $\ell_1 \sim \ell_2$ if and only if the lines ℓ_1 and ℓ_2 either coincide or have no point in common. In other words, $\ell_1 \sim \ell_2$ if and only if the lines ℓ_1 and ℓ_2 are parallel. (a) Show that $\sim$ is an equivalence relation. (b) Describe the equivalence class $[\ell]$ of a line ℓ in L.

5.3. Dr. S. Marty Pants claims that any symmetric and transitive relation $\sim$ is an equivalence relation. He argues as follows: Given x, y in S, $x \sim y$ implies $y \sim x$ by symmetry. Then transitivity yields $x \sim x$, so $\sim$ is reflexive. What is the flaw in this argument?

5.4. We define on the set $\mathbf{N}_1 = \{1, 2, 3, \cdots\}$ of positive integers a relation $\sim$ such that two positive integers x and y satisfy $x \sim y$ if and only if $x/y = 2^k$ for some integer k. Show that $\sim$ is an equivalence relation.

5.5. Consider the set $S = \mathbf{R}^2 \backslash \{(0,0)\}$ consisting of the points in the Euclidean plane without the origin. We define a relation $\sim$ on S by $(x_1, y_1) \sim (x_2, y_2)$ if and only if there exists a nonzero real number λ such that $(x_2, y_2) = (\lambda x_1, \lambda y_1)$. (a) Show that the relation $\sim$ is an equivalence relation on S. (b) Describe the equivalence class $[(x, y)]$ of a point (x, y) in S.

5.6. The set of integers $\mathbf{Z}$ can be partitioned in the following three subsets: the set of positive integers, the singleton set $\{0\}$ containing 0, and the set of negative integers. Describe the equivalence relation $\equiv$ on $\mathbf{Z}$ that corresponds to this partition.

5.7. Show that every equivalence relation $\sim$ on a set S can be defined by a function $f \colon S \to S$ such that for all x, y in S, we have

$$x \sim y \quad \text{if and only if} \quad f(x) = f(y).$$

5.8. Let $\equiv_c$ and $\equiv_s$ denote the equivalence relations on the set of real numbers respectively induced by the cosine function and the sine function. In other words, $x \equiv_c y$ if and only if $\cos(x) = \cos(y)$, and $x \equiv_s y$ if and only if $\sin(x) = \sin(y)$. Are the equivalence relations $\equiv_c$ and $\equiv_s$ the same or different? Prove your claim.

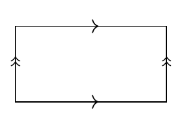
5.9. Suppose that an equivalence relation $\equiv_f$ on the set of real numbers is induced by a function f, meaning that $x \equiv_f y$ if and only if $f(x) = f(y)$. Determine the equivalence classes of $\equiv_f$, when
(a) $f(x) = x^2$,
(b) $f(x) = x^3$,
(c) f is an injective function,
(d) f is a constant function.

5.10. Consider the set $S = \{1, 2, \ldots, 6\}$. Find the equivalence relations that correspond to the following partitions of S:
(a) $P_1 = \{\{1, 2\}, \{3\}, \{4, 5, 6\}\}$,
(b) $P_2 = \{\{1, 4\}, \{3, 5\}, \{2, 6\}\}$.

5.11. Consider the set

$$R = \{(x, y) \in \mathbf{R}^2 \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 1\}$$

of points that form a rectangle in the Euclidean plane, as in Example 5.11. Define an equivalence relation $\equiv$ on R that models gluing of the left and right border line segments, and gluing of the top and bottom line segments.



So the equivalence relation identifies the corresponding points on the left and right border line segments, and identifies points on the top and bottom border line segments. (a) Explicitly give the equivalence class $[(x, y)]$ for each point (x, y). (b) Find a geometrically inspired interpretation of the quotient space $R/\!\equiv$.

5.12. Determine all equivalence relations on the set $\{a, b\}$ with two distinct elements a and b.

5.13. Determine all equivalence relations on the set $\{a, b, c\}$.

5.14. Determine the number of different equivalence relations on $\{a, b, c, d\}$. You do not need to give the equivalence relations themselves, but merely count their number.

5.15. Let S be a set and I an arbitrary nonempty index set. For every $k \in I$, let R_k be an equivalence relation on S. Show that

$$M = \bigcap_{k \in I} R_k$$

is an equivalence relation on S.

5.16. Let R_1 and R_2 be equivalence relations on a nonempty set S. Is their union $R_1 \cup R_2$ an equivalence relation? Prove your claim or given a counter example.

5.17. Find the smallest equivalence relation $\equiv$ on the set $S = \{1, 2, 3, 4, 5\}$ that contains the relation $R = \{(1, 2), (1, 3), (4, 5)\}$.

5.18. Let R be a relation on a set S. Show that the intersection of all equivalence relations containing R is the uniquely determined minimal equivalence relation on S containing R.

5.19. Let R be a relation on a nonempty set S. Then the relation given by

$$M_R = \bigcup_{k=0}^{\infty} (R \cup R^{-1})^k$$

is equal to the minimal equivalence relation containing R.

Here we follow the convention that $(R \cup R^{-1})^0$ is the identity relation on S. For $k \geqslant 1$, the relation $(R \cup R^{-1})^k$ is the k-fold composition of $R \cup R^{-1}$ with itself.

5.20. Show that the relation $\sim$ on the set $\mathbf{Z}$ of integers given by

$$x \sim y \text{ if and only if } x + 2y \text{ is divisible by } 3$$

is an equivalence relation.

5.2 Integers

Given two nonnegative integers a and b, we can form the difference

$$a - b$$

when $a \geqslant b$. Since we would like to form the difference between any two nonnegative integers, we need to extend the set of nonnegative integers to the set of integers.

How do we form the set of integers? Naturally, the guiding principle is that we want to define an integer as a difference of two nonnegative integers, but we have to do this without relying on the undefined difference $a-b$ when $a < b$. We circumvent this problem by letting pairs (a, b) of nonnegative integers represent the elusive value $a - b$.

Since a pair (c, d) with the same difference $c - d = a - b$ should represent the same number, we will define an equivalence relation $\sim$ on the set of pairs of nonnegative integers

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad a + d = b + c. \tag{5.1}$$

Since $a - b$ is undefined when $a < b$, we simply rewrote $c - d = a - b$ into the equivalent form $a + d = b + c$. Let us verify that the relation $\sim$ is indeed an equivalence relation.

Proposition 5.12. *The relation $\sim$ defined by (5.1) is an equivalence relation.*

Proof. The relation is reflexive, since $a + b = b + a$, hence $(a, b) \sim (a, b)$.

The relation is symmetric. Indeed, since $(a, b) \sim (c, d)$ implies $a+d = b+c$, we obtain by commutativity of addition the equation $c + b = d + a$. It follows that $(c, d) \sim (a, b)$.

Finally, we are going to prove that the relation $\sim$ is transitive. Suppose that $(a_1, b_1) \sim (a_2, b_2)$ and $(a_2, b_2) \sim (a_3, b_3)$ hold. The first relation implies $a_1 + b_2 = b_1 + a_2$. Adding b_3 to both sides yields

$$(a_1 + b_3) + b_2 = b_1 + (a_2 + b_3). \tag{5.2}$$

Since the second relation $(a_2, b_2) \sim (a_3, b_3)$ implies $a_2+b_3 = b_2+a_3$, this allows us to deduce from (5.2) the equation

$$(a_1 + b_3) + b_2 = b_1 + (b_2 + a_3) = (b_1 + a_3) + b_2.$$

Subtracting b_2 from both sides yields $a_1 + b_3 = b_1 + a_3$, which implies $(a_1, b_1) \sim (a_3, b_3)$. Therefore, the relation is transitive. □

We define the set $\mathbf{Z}$ of integers as the quotient set

$$\mathbf{Z} = \mathbf{N}_0 \times \mathbf{N}_0 / \sim .$$

The motivation for introducing the equivalence relation should now be clear. For example, 0 is represented by $(0, 0) \sim (1, 1) \sim (2, 2) \sim \cdots$. We defined the value 0 in **Z** as the equivalence class that combines all the various representations of 0, namely

$$0 = \{(n, n) \mid n \in \mathbf{N}_0\}.$$

A positive integer k is represented in **Z** by the equivalence class

$$k = \{(n + k, n) \mid n \in \mathbf{N}_0\}.$$

Our goal was to obtain a representation of negative integers. We accomplished this goal by representing -2 by any pair where the second coordinate is two more than the first coordinate, so $(0, 2)$ and $(5, 7)$ both represent the value -2. The equivalence class of $-k$ is given by

$$-k = \{(n, n + k) \mid n \in \mathbf{N}_0\}.$$

Perhaps you find it bewildering to represent a single integer value by an infinite number of pairs of nonnegative integers. However, most people have no qualms about the fact that the rational number $2/3$ really represents an infinite set

$$\{2k/3k | k \in \mathbf{Z}, k \neq 0\}$$

of equivalent representations. In both cases, we use equivalence classes of numbers. Of course, the representative $2/3$ in lowest terms is in many ways preferable, but allowing non-reduced representations such as $165/935$ instead of $3/17$ can save time when conversions to the representatives are somewhat costly. Exercise 5.21 shows how to identify canonical representatives of integers.

EXERCISES

5.21. Show that each equivalence class $[(a, b)]$ of $\mathbf{Z} = \mathbf{N}_0 \times \mathbf{N}_0/\sim$ contains a unique element that has at least one zero coordinate.

5.22. Explain how to define addition and multiplication when the set of integers **Z** is given by $\mathbf{N}_0 \times \mathbf{N}_0/\sim$. The definition should work for any representative.

5.3 Modular Arithmetic

Consider the set **Z** of integers and a positive integer n. We follow Gauss and write

$$x \equiv y \pmod{n},$$

if and only if x and y are two integers such that $x - y$ is an integer multiple of n. We call this relation the **congruence modulo** n on the set of integers. The notation is a bit idiosyncratic but turns out to be convenient, since the modulus is more readable than in the alternate subscript form $\equiv_n$.

Proposition 5.13. *Let n be a positive integer. The congruence modulo n is an equivalence relation on the set of integers.*