Project for the course "Numerical Integration for Stochastic Differential Equations"

Modeling asset pricing via SDEs

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In finance, an asset is a resource with economic value that an investor owns with the expectation that it will give a future benefit. Specifically, the price of an asset $S(t)$ is a quantity whose current value is known, but the future one is not. Its evolution depends on the expected rate of return $I(t)$, i.e., the estimated profit that an investor expects to achieve, and on the volatility $\sigma(t)$, which is a measure of the risk in the market. By their nature, $I(t)$ and $\sigma(t)$ should be nonnegative. All of these quantities can vary in time and are supposed to be random. Therefore, one can see S, I, and σ as stochastic processes.

Let us consider the following stochastic basis $(\Omega, \mathcal{A}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ and assume that $S(t)$ is described by the following one-dimensional equation

$$\mathrm{d}S(t) = I(t)S(t)\mathrm{d}t + \sigma(t)S(t)\mathrm{d}W(t), \tag{1}$$

where W is a one-dimensional Brownian-motion adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We are interested in knowing the stochastic properties of (1) and numerically quantifying the price of an asset.

- **(Q1)** In order to determine the price of an asset over time, we need to solve the SDE (1).
  1. Derive a closed form expression for the solution of (1) assuming constant $I(t) \equiv I$, $\sigma(t) \equiv \sigma$, with $\sigma, I \in \mathbb{R}_+$.
  2. Show that the process $G(t) = e^{\sigma W(t)}$ is a submartingale, but not a martingale.
  3. Find for which values of I $S(t)$ in **(Q1)-1** is a martingale.
  4. Find the solution of (1) in closed form assuming generic $I(t)$, $\sigma(t)$.

- **(Q2)** Assume that the interest rate $I(t)$ is stochastic and it is determined by some Itô differential equation. Consider the following equation
$$\mathrm{d}I(t) = (a - bI(t))\mathrm{d}t + c\mathrm{d}W_t^{(2)}, \tag{2}$$
with $I(0) > 0$, $a, b, c > 0$ and $W_t^{(2)}$ a one-dimensional Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$, independent of W_t.
  1. Find a closed form expression for the solution of (2).
  2. What is the distribution of $I(t)$? Compute the probability that $I(t) < 0$ with $a = b = c = 1$ for $t = 1, 5, 10, 15, 20$. Could this behavior be a problem from the financial point of view?
  3. Let us consider the following alternative interest model
  $$\mathrm{d}I(t) = (a - bI(t))\mathrm{d}t + c\sqrt{I(t)}\mathrm{d}W_t^{(2)}, \tag{3}$$
  It is known that (3) admits a unique strong solution I. Prove that $I(t)$ is nonnegative for all $t \in \mathbb{R}$.
  Hint: you can consider the following ausiliary process (which also have a unique strong solution)
  $$\mathrm{d}I(t) = (a - bI(t))\mathrm{d}t + c\sqrt{(I(t) \vee 0)}\mathrm{d}W_t^{(2)}, \tag{4}$$
  and the stopping time $\tau_\epsilon = \inf\{t \ : \ I(t) = -\varepsilon\}$ for any ε small enough and show that $\mathbb{P}(\tau_\epsilon < \infty) = 0$.
  4. (2) is a mean-reverting process. What does it mean? If $I(0) = \frac{a}{b}$, what can we say about $\mathbb{E}[I(t)]$? And if $I(0) \neq \frac{a}{b}$? Is (3) mean-reverting?

- **(Q3)** Let us suppose that $\sigma = \sqrt{v(t)}$, where $v(t)$ is non-negative process defined as $v(t) = \tilde{\sigma}^2(t)$ with
$$\mathrm{d}\tilde{\sigma}(t) = -\lambda\tilde{\sigma}(t)\mathrm{d}t + f\mathrm{d}W_t^{(3)}, \tag{5}$$
where $\lambda, f > 0$ and $W_t^{(3)}$ is a one-dimensional Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$, independent of W_t and $W_t^{(2)}$ (notice that in this model $\sigma(t) = |\tilde{\sigma}(t)|$).

1. Derive a closed form expression for the solution of (5).
2. The process $\tilde{\sigma}(t)$ of (5) has the property of being ergodic, i.e., its distribution μ_t tends for $t \to \infty$ to an invariant measure, which we denote by μ_∞ . Such limit distribution admits a probability density function ρ_∞ with respect to the Lebesgue measure on $\mathbb{R}$. Derive the invariant measure of (5) and verify that the corresponding probability density function satisfies the Fokker-Planck equation.
3. Show that the following equation holds for the square volatility process $v(t) = \tilde{\sigma}^2(t)$ with suitable values of k, μ, η

$$\mathrm{d}v(t) = k\,[\mu - v(t)]\,\mathrm{d}t + \eta\sqrt{v(t)}\mathrm{d}W_t^{(3)}, \tag{6}$$

and specify k, μ, η as functions of λ and f. Briefly argument whether (6) is a good model from the financial point of view or not.

- **(Q4)** Suppose to approximate (1) with $I(t) = I$ and $\sigma(t) = \sigma$ with the Euler-Maruyama method using a uniform time-step Δt, and denote by $\hat{S}(t)$ its numerical solution (continuously interpolated process) on $[0, T]$.

1. Which is the order of convergence of $\sqrt{\mathbb{E}[\sup_{0\le t\le T}(S(t) - \hat{S}(t))^2]}$?
2. Suppose now that $I(t)$ in (1) is described by (2) and consider an approximation of $I(t)$ by the Euler-Maruyama method with a uniform time-step Δt (the same as in 1.). Which is the order of convergence of $\sqrt{\mathbb{E}[\sup_{0\le t\le T}(S(t) - \hat{S}(t))^2]}$? Provide a rigorous argument.

Hint: To simplify your argument, you can consider the following modification of (1)

$$dS = \varphi_M(I(t))S(t)dt + \varphi_M(\sigma(t))S(t)dW_t,$$

where $\varphi_M(x) = \max\{-M, \min\{M, x\}\}$ and $M > 0$ a large enough constant. Notice that φ_M is bounded and Lipschitz function with Lipschitz constant 1. You might need to derive uniform bounds on L^p norms ($p \ge 2$) of the exact solution $\mathbb{E}[|S(t)|^p]$ and numerical one $\mathbb{E}[|\hat{S}(t)|^p]$.

3. Suppose that $I(t)$ and $\sigma(t)$ in (1) are described by (2) and (5), respectively. Consider a discretization of both of them by the Euler-Maruyama method with a uniform time-step Δt (the same as 1.). Which is the order of convergence of $\sqrt{\mathbb{E}[\sup_{0\le t\le T}(S(t) - \hat{S}(t))^2]}$? Which is the order of convergence of the Euler-Maruyama method applied to $I(t)$ and $\sigma(t)$? Provide rigorous arguments.
4. Let us assume that $a = 0.1$, $b = c = 0.5$, $d = f = 1$. Consider $T = 2$, and simulate the Brownian motion $W_t, W_t^{(2)}, W_t^{(3)}$ with a uniform time-step $\mathrm{d}t = 10^{-5}$ (i.e. 200001 points) and $M = 100$ paths. Moreover, assume $S(0) = 2$, $I(0) = 0.5$, and $\sigma(0) = 2$ for all paths. Compute the solution $S(t)$ of (1) with interest rate $I(t)$ and $\sigma(t)$ defined by the closed formulae obtained in **(Q3)**-1. and **(Q4)**-1 (you can approximate the integrals present therein with a left point quadrature formula as in the Euler method, and using the generated paths of $W_t, W_t^{(2)}, W_t^{(3)}$), and their Euler-Maruyama approximation for uniform $\Delta t = 5\cdot 10^{-4}, 10^{-3}, 4\cdot 10^{-3}, 8\cdot 10^{-3}, 10^{-2}$. Plot the error $\sqrt{\mathbb{E}[(S(T) - \hat{S}(T))^2]}$ (you can estimate the expectation with a Monte-Carlo procedure). What order do you observe? Is it consistent with your conclusions of the previous points?

(If your machine does not sustain computations with these time-steps, you can change them accordingly.)

- **(Q5)** Assume that in (1), $I(t) = I$ and $\sigma(t) = \sigma$ are <u>real</u> constants, and also assume $S(0) = 2$. Consider $T = 5$ and $M = 1000$ paths.

1. Show that

$$\lim_{t\to+\infty} \frac{1}{t}\log(S(t)) = I - \frac{\sigma^2}{2}, \text{ a.s. if } I \neq \frac{\sigma^2}{2} \tag{7}$$

and

$$\limsup_{t\to+\infty} \frac{\log(S(t)}{\sqrt{2t\log\log t}} = \sigma \text{ and } \liminf_{t\to+\infty} \frac{\log(S(t)}{\sqrt{2t\log\log t}} = -\sigma, \text{ a.s. if } I = \frac{\sigma^2}{2} \tag{8}$$

Deduce a condition of mean-square stability of the price S, i.e. $\lim_{t\to+\infty} \mathbb{E}[|S_t|^2] = 0$.

2. Assume $\sigma = 0.12$. For $\Delta t = 0.01, 0.02, 0.05, 0.1$, plot $\mathbb{E}[|S(t)|^2]$ for the interest rates $I = 0.002, 0.006, 0.0072, 0.012, 0.024$ (*you can approximate it with a forward Euler method*) and comment the results. Is this performance a good asset behavior? Comment.
3. Suppose to apply the stochastic-θ method to (1), denoting its solution by $(S_n^\theta)_n$. Discuss the mean-square stability of the stochastic-θ method applied to (1), i.e. $\lim_n \mathbb{E}[|S_n^\theta|^2] = 0$, and derive sufficient conditions of mean-square stability for $\theta = 0, \frac{1}{2}, 1$. Verify numerically the conditions for $\theta = 0, \frac{1}{2}, 1$, with the data of **(Q5)-2**.
4. Assume now that $\sigma(t)$ is no more constant and satisfies equation (5), in **(Q3)**. Consider an approximation of $\sigma(t)$ by the Euler-Maruyama method, denoting its solution by $(\sigma_n)_n$. Find a condition on Δt so that there exists a constant $M > 0$ such that $\mathbb{E}[|\sigma_n|^2] \leq M$ for all n. Do we have mean-square stability of $(\sigma_n)_n$ under such condition on Δt? Verify these properties numerically for $f = 0.2, \lambda = 0.1, T = 5$, and $\sigma_0 = 1$.

- **(Q6)** Suppose to have a *basket of options*, i.e. a collection of multiple financial securities, composed only by assets whose prices are described by the following system of SDEs

$$\mathrm{d}S(t) = IS(t)\mathrm{d}t + \sigma K(S(t) \circ \mathrm{d}W_t), \tag{9}$$

where, $S(t,\omega) \in \mathbb{R}^n$ fixed $\omega \in \Omega$, $(S(t) \circ \mathrm{d}W_t)$ denote the Hadamard product between $S(t)$ and $\mathrm{d}W_t$, i.e. $A \circ B = [a_{ij}b_{ij}]_{ij}$ for $A, B \in \mathbb{R}^{m,n}$, $K \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, I is a symmetric positive definite matrix, σ is a nonnegative constants and W is a n-dimensional Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$.

1. Derive an equation for $G(t) := \mathbb{E}[S(t)S(t)^\top]$.
2. Assume $T = 1$, $I = 0.01 \cdot I_{d\times d}$, $K = RR^\top$, with R a matrix whose entries are sampled from a $\mathrm{Unif}(0,1)$ r.v., and $\sigma = 10^{-3}$.
   Approximate (9) with a Euler-Maruyama method. Denote $(S_n)_n$ the approximate solution and compute the approximate matrix $(\mathbb{E}[S_nS_n^\top])_n$ at discrete times $t_n = n\Delta t$, approximating the expectation $\mathbb{E}$ with a Monte-Carlo method $\hat{\mathbb{E}}$. Compute the error

$$\sqrt{\sum_{n=0}^{N} \|\hat{\mathbb{E}}[S_nS_n^\top] - G(t_n)\|_F^2 \Delta t} \tag{10}$$

   for $\Delta t = 0.001, 0.01, 0.05, 0.1$, $M = 100$, $d = 5$. Which are the sources of errors in computing (10)? Repeat the same computatations but with $M = 2000$ and comment the results.

- **(Q7)** Consider equation (6). Let $k = 1$, $\mu = 1$, $v(0) = 1$ for all the $M = 500$ paths.

1. In [1], a Euler-type method based on the Lamperti transformation $Y_t = \sqrt{v_t}$ is proposed, obtaining the accuracy result of Theorem 1.1 therein. Derive the same discretization scheme for (6) and show with a simulation that Theorem 1.1 holds, fixing $\eta = 0.5, 1.5, 1.75$. Compare the convergence rate with the one of the Stochastic θ-method with $\theta = 1$ applied to the same problem (you can compute the true solution using [1] and a very fine mesh (at least $\Delta t \approx 10^{-5}$)). Comment the results.
2. Why can't we use the formula obtained in **(Q3)-1** in point **(Q7)-1**?

- **(Q8)** The theory of asset pricing is the standard tool in the evaluation of *options*. An option is a contract between two parties, a "writer" and a "holder". Both sides agree that a fixed time $t = T$, called the *expire date*, the writer will pay an amount of money to the holder, called the *payoff*, whereas the contract is assumed to be stipulated at time $t = 0$. The value of the *payoff* is determined by the behavior of the price $S(t)$ of the chosen asset in the interval of time $[0, T]$. On the other hand, the holder does have to pay a small amount, called "premium", to the writer at time $t = 0$ by contract. For example, let us consider a "European call" option, which is defined by the following payoff function

$$P(S(T)) = \max\{S(T) - K, 0\}, \tag{11}$$

where $K > 0$ is called the exercise price decided in the option. If $S(T) > K$, the holder of the option will receive a positive payoff amount $P(S(T))$ from the writer, otherwise no money will be received. Therefore, option pricing is essentially betting at time $t = 0$ about what will be the value $P(S(T))$, i.e. the random variable $S(T)$, so that the holder and writer can deal the best option for themselves.

Suppose that now $I(t)$ and $\sigma(t)$ are real positive constants. Consider $S(0)$ the initial condition of the asset.

Under the no-arbitrage assumption, the fair value of the premium is given by

$$C(S_0, T) = e^{-rt}\mathbb{E}[P(S(T))]. \tag{12}$$

where r is the risk free interest rate. A closed formula for (12) is

$$C(s,t) = s\mathcal{N}(x_1) - Ke^{-rt}\mathcal{N}(x_2) \tag{13}$$

where

$$x_1 = \frac{log(s/K) + (r + \frac{1}{2}\sigma^2 t)}{\sigma\sqrt{T}}, \quad x_2 = \frac{log(s/K) + (r - \frac{1}{2}\sigma^2 t)}{\sigma\sqrt{T}}$$

and $\mathcal{N}$ denotes the standard normal cumulative distribution function. Approximate (12) with the Euler-Maruyama algorithm, where I is replace by r in the dynamics (1), computing the expectation in (12) by Monte-Carlo with $M = 100$ realizations, considering $S(0) = 8$, $K = 10$, $r = 0.05$, $\sigma = 0.5$ and $T = 1$. Plot the error between the approximation and true solution at time T commenting the results. Is the result consistent with the theory?

References

[1] Steffen Dereich, Andreas Neuenkirch, and Lukasz Szpruch. "An Euler-type method for the strong approximation of the Cox–Ingersoll–Ross process". In: *Proceedings of the royal society A: mathematical, physical and engineering sciences* 468.2140 (2012), pp. 1105–1115.

Asset Pricing: NISDE Project

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1 Q1

1.1 : We have that:

$$dS_t = IS_t dt + \sigma S_t dW_t \Leftrightarrow \int_0^t \frac{dS_u}{S_u} = It + \sigma W_t$$

To evaluate the integral on the LHS we use the Itô formula for function: $f(t, x) = \ln(x)$ and obtain:

$$\begin{aligned} d(\ln S_t) &= \frac{1}{S_t} \cdot dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2 \\ &= \frac{dS_t}{S_t} - \frac{1}{2S_t^2}\left(I^2 S_t^2 (dt)^2 + 2I\sigma S_t^2 dt dW_t + \sigma^2 S_t^2 (dW_t)^2\right) = \frac{dS_t}{S_t} - \frac{1}{2}\sigma^2 dt \end{aligned}$$

Where we have used the rules of Itô calculus: $dt \cdot dt = dt \cdot dW_t = dW_t \cdot dt = 0$ and $dW_t \cdot dW_t = dt$. Hence, we can conclude:

$$\begin{aligned} \frac{dS_t}{S_t} &= d(\ln S_t) + \frac{1}{2}\sigma^2 dt \Leftrightarrow \\ \ln\left(\frac{S_t}{S_0}\right) &= \left(I - \frac{1}{2}\sigma^2\right)t + \sigma W_t \Leftrightarrow S_t = S_0 \exp\left(\left(I - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right) \end{aligned}$$

1.2 : For $G(t)$ to be a martingale, it must satisfy $\mathbb{E}[G(t) \mid \mathcal{F}_s] \geq G(s)$ for all $0 \leq s \leq t$. Let's calculate $\mathbb{E}[G(t) \mid \mathcal{F}_s]$. Given that $W(t) = W(s) + (W(t) - W(s))$ where $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ is independent of $\mathcal{F}_s$ and the expectation of the exponential of a normally distributed random variable can be computed using the moment-generating function of the normal distribution, $\mathbb{E}[e^{\sigma(W(t)-W(s))}] = \exp\left(\frac{1}{2}\sigma^2(t-s)\right)$ we obtain:

$$\begin{aligned} \mathbb{E}[G(t) \mid \mathcal{F}_s] &= \mathbb{E}[e^{\sigma W(t)} \mid \mathcal{F}_s] = e^{\sigma W(s)} \cdot \mathbb{E}[e^{\sigma(W(t)-W(s))} \mid \mathcal{F}_s] \\ &= e^{\sigma W(s)} \cdot \mathbb{E}[e^{\sigma(W(t)-W(s))}] = e^{\sigma W(s)} e^{\frac{1}{2}\sigma^2(t-s)} \end{aligned}$$

Since $\exp\left(\frac{1}{2}\sigma^2(t-s)\right) > 1$ for $t > s$, it follows that:

$$\mathbb{E}[G(t) \mid \mathcal{F}_s] > G(s) \quad \text{for } t > s$$

1.3 : Using the same reasoning and properties as before:

$$\begin{aligned} \mathbb{E}[S_t \mid \mathcal{F}_s] &= \mathbb{E}\left[S_0 \cdot \exp\left(\left(I - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \mid \mathcal{F}_s\right] \\ &= S_0 \cdot \exp\left(\left(I - \frac{\sigma^2}{2}\right)t\right) \cdot \mathbb{E}\left[\exp(\sigma W_s) \mid \mathcal{F}_s\right] \cdot \mathbb{E}\left[\exp(\sigma(W_t - W_s)) \mid \mathcal{F}_s\right] \\ &= S_0 \cdot \exp\left(\left(I - \frac{\sigma^2}{2}\right)t + \sigma W_s\right) \cdot \mathbb{E}\left[\exp(\sigma(W_t - W_s))\right] \\ &= S_0 \cdot \exp\left(\left(I - \frac{\sigma^2}{2}\right)t + \sigma W_s\right) \cdot \exp\left(\frac{\sigma^2(t-s)}{2}\right) \\ &= S_0 \cdot \exp\left(\left(I - \frac{\sigma^2}{2}\right)t + \sigma W_s + \frac{\sigma^2(t-s)}{2}\right) \end{aligned}$$

Knowing that for S_t to be a martingale we must have $\mathbb{E}[S_t \mid \mathcal{F}_s] = S_s$ meaning that $\exp(I(t-s)) = 1$ for all $t > s$ which holds if and only if $I = 0$.

1.4 : Given the SDE: $dS_t = I_t S_t \, dt + \sigma_t S_t \, dW_t$, we can factor out S_t on the right-hand side and then integrate over the interval $[0, t]$:

$$\begin{aligned} \frac{dS_t}{S_t} &= I_t \, dt + \sigma_t \, dW_t \Leftrightarrow \int_0^t \frac{dS_u}{S_u} = \int_0^t I_u \, du + \int_0^t \sigma_u \, dW_u \\ \ln\left(\frac{S_t}{S_0}\right) &= \int_0^t I_u \, du + \int_0^t \sigma_u \, dW_u \Leftrightarrow S_t = S_0 \cdot \exp\left(\int_0^t I_u \, du + \int_0^t \sigma_u \, dW_u\right) \end{aligned}$$

2 Q2

2.1 : We notice that this process is very similar to the *Ornstein-Uhlenbeck* process seen in the course [3], with the addition of the $a \cdot dt$ deterministic term. We take the integrating factor $\mu(t) = \exp(bt)$ and multiply both sides of the SDE by it:

$$\exp(bt)\, dI_t = \exp(bt)(a\, dt - bI_t\, dt + c\, dW_t)$$

We use the Itô formula and the previous relation and then integrate to obtain the wanted result:

$$\exp(bt)\, dI_t = d(I_t \exp(bt)) - bI_t \exp(bt)\, dt \Rightarrow d(I_t \exp(bt)) = a\exp(bt)\, dt + c\exp(bt)\, dW_t$$

$$\begin{aligned} I_t \exp(bt) &= I_0 + \int_0^t a\exp(bu)\, du + \int_0^t c\exp(bu)\, dW_u, \\ &= I_0 + \frac{a}{b}(\exp(bt) - 1) + \int_0^t c\exp(bu)\, dW_u \end{aligned}$$

And we finally obtain:$I_t = I_0 \exp(-bt) + \frac{a}{b}(1 - \exp(-bt)) + \int_0^t c\exp(-b(t-u))\, dW_u$.

2.2 : Taking $a = b = c = 1$ and $I_0 \geq 0$ we obtain $I_t = \exp(-t)I_0 + 1 - \exp(-t) + \int_0^t \exp(-b(t-u))\, dW_u$. Given that I_t is a linear transformation of a normally distributed Brownian motion term $\int_0^t \exp(-b(t-u))\, dW_u$, I_t itself is normally distributed. We first determine the expectation of I_t using the fact that the stochastic integral has expectation 0:

$$\mathbb{E}[I_t] = \mathbb{E}[\exp(-t)I_0 + 1 - \exp(-t)] + \mathbb{E}[\int_0^t \exp(-b(t-u))\, dW_u] = \exp(-t)I_0 + 1 - \exp(-t)$$

For the variance, we use the Itô isometry for the stochastic integral term:

$$\begin{aligned} \mathrm{Var}(I_t) = \mathrm{Var}\left(\int_0^t \exp(s-t)\, dW_s\right) &= \mathbb{E}\left[\left(\int_0^t \exp(s-t)\, dW_s\right)^2\right] + \mathbb{E}\left[\int_0^t \exp(s-t)\, dW_s\right]^2 \\ &= \mathbb{E}\left[\int_0^t \exp(2(s-t))\, ds\right] = \exp(-2t)\int_0^t \exp(2s)\, ds = \exp(-2t)\left[\frac{\exp(2s)}{2}\right]_0^t = \frac{1}{2}(1 - \exp(-2t)) \end{aligned}$$

Therefore, the distribution of I_t is $I_t \sim \mathcal{N}\left(\exp(-t)I_0 + 1 - \exp(-t), \frac{1}{2}(1 - \exp(-2t))\right)$. From Table 1 it is clear that for any strictly positive value of I_0 we find that there is a non-negligible probability for the expected return to become negative. Financially, this is a bad sign for an investor as the anticipated amount that an investment or portfolio is expected to generate over a specified period can be negative, meaning that he will lose money.

Time t	$I_0 = 0.1$	$I_0 = 1$	$I_0 = 10$
1	0.1545	0.0641	0.0000
5	0.0799	0.0786	0.0668
10	0.0787	0.0786	0.0786
15	0.0786	0.0786	0.0786
20	0.0786	0.0786	0.0786

Table 1: Values of I_t for different initial conditions I_0 at various times.

2.3 : The comparison principle for solutions of SDEs (see [6]) states that if we have two SDEs of the form:

$$\begin{cases} dX_t = f(t, X_t)\, dt + g(t, X_t)\, dW_t, \\ dY_t = h(t, Y_t)\, dt + k(t, Y_t)\, dW_t, \end{cases}$$

with initial conditions $X_0 = Y_0$. And we assume that the following conditions are satisfied: $X_0 \leq Y_0$, $f(t,x) \leq h(t,x)$ for all x and for all $t \geq 0$, and $g(t,x) \leq k(t,x)$ for all x and for all $t \geq 0$; then for all $t \geq 0$ we have $X_t \leq Y_t$ almost surely.

In our case, let $I(t)$ be the original process governed by $dI(t) = (a - bI(t))\, dt + c\sqrt{I(t)}\, dW_t$ and let $J(t)$ be the auxiliary process governed by $dJ(t) = (a - bJ(t))\, dt + c\sqrt{J(t) \vee 0}\, dW_t$. We can see that the conditions of the theorem are verified: we assume $I(0) = J(0) > 0$, so $I_0 = J_0$; drift terms

$(a-bI(t))$ and $(a-bJ(t))$ are the same for both processes and for all $J(t)\geq 0$, $\sqrt{J(t)\vee 0}=\sqrt{J(t)}$, so the processes are identical. For $J(t)<0$, the diffusion term for the auxiliary process vanishes, ensuring non-negativity. Since the auxiliary process $J(t)$ cannot become negative and $I(t)$ coincides with $J(t)$ as long as both are nonnegative, we have:$\mathbb{P}(I(t)<0)=0 \quad \text{for all } t\geq 0$.

2.4 : The term $(a-bI(t))\,dt$ causes the process $I(t)$ to move towards $\frac{a}{b}$: when $I(t)>\frac{a}{b}$, the term $(a-bI(t))$ becomes negative, causing the process to decrease towards the mean) and conversely, when $I(t)<\frac{a}{b}$, the term is positive causing the process to increase towards the mean. Over time, the process oscillates around $\frac{a}{b}$, exhibiting a mean-reverting tendency. If the initial condition $I(0)=\frac{a}{b}$, then the process is already at its long-term mean, meaning that the expected value of $I(t)$ remains constant over time: $\mathbb{E}[I(t)]=\frac{a}{b}$ for all $t\geq 0$ (we can see this by replacing I_0 in the solution).If $I(0)\neq\frac{a}{b}$, the expected value of $I(t)$ can be shown to evolve according to $\mathbb{E}[I(t)]=\frac{a}{b}+\left(I(0)-\frac{a}{b}\right)e^{-bt}$: as $t\to\infty$, the term $\left(I(0)-\frac{a}{b}\right)e^{-bt}$ decays to zero, and $\mathbb{E}[I(t)]$ converges to $\frac{a}{b}$, demonstrating the mean-reverting behavior.

3 Q3

3.1 : We clearly notice that this is an Ornstein-Uhlenbech process and therefore we can find the solution explicitly using Itôs formula (see Lecture notes [3]). We suppose $\tilde{\sigma}_0=\eta$ Let $X_t=\exp\{\lambda t\}\tilde{\sigma}_t$ then $dX_t=\lambda\exp\{\lambda t\}\tilde{\sigma}_t dt+\exp\{\lambda t\}d\tilde{\sigma}_t=\exp\{\lambda t\}f dW_t^{(3)}$ and therefore:

$$X_t=\eta+\int_0^t \exp\{\lambda s\}f\,dW_t^{(3)}\Rightarrow\tilde{\sigma}_t=\exp\{-\lambda t\}\eta+\int_0^t\exp\{-\lambda(t-s)\}f\,dW_t^{(3)}$$

3.2 : We can assert that the solution $\tilde{\sigma}_t$ follows a Gaussian distribution as it is the sum of a deterministic term $\exp(-\lambda t)\eta$ and a stochastic integral which $\int_0^t\exp(-\lambda(t-s))f\,dW_t^{(3)}$ is a weighted sum of increments $dW_t^{(3)}$ which we have proven in the course (see Lecture Notes [2]) to be gaussian. We first compute the expectation:

$$\mathbb{E}[\tilde{\sigma}_t]=\mathbb{E}[\exp(-\lambda t)\eta]+\mathbb{E}[\int_0^t\exp(-\lambda(t-s))f\,dW_t^{(3)}]=\exp(-\lambda t)\eta\Rightarrow\lim_{t\to\infty}\mathbb{E}[\tilde{\sigma}_t]=0$$

For the variance:

$$\begin{aligned}\mathrm{Var}[\tilde{\sigma}_t]&=\mathrm{Var}\left[\int_0^t\exp(-\lambda(t-s))f\,dW_t^{(3)}\right]=\mathbb{E}\left[\left(\int_0^t\exp(-\lambda(t-s))f\,dW_t^{(3)}\right)^2\right]-\mathbb{E}\left[\int_0^t\exp(-\lambda(t-s))f\,dW_t^{(3)}\right]^2\\&=\mathbb{E}\left[\int_0^t f^2\exp(-2\lambda(t-s))\,ds\right]=f^2\int_0^t\left(e^{-\lambda(t-s)}\right)^2ds=\frac{f^2}{2\lambda}\left(1-e^{-2\lambda t}\right)\Rightarrow\lim_{t\to\infty}\mathrm{Var}[\tilde{\sigma}_t]=\frac{f^2}{2\lambda}\end{aligned}$$

Therefore the invariant density function is : $\rho_\infty(x)=\sqrt{\frac{\lambda}{\pi f^2}}\exp\left(-\frac{\lambda}{f^2}x^2\right)$. The Fokker-Planck equation is given by:$\frac{\partial\rho(t,x)}{\partial t}=-\frac{\partial}{\partial x}\left(A(x)\rho(t,x)\right)+\frac{\partial^2}{\partial x^2}\left(\frac{1}{2}B^2(x)\rho(t,x)\right)$ where in this case $A(x)=-\lambda x$ and $B(x)=f$. For the steady-state distribution $\rho_\infty(x)$, we have $\frac{\partial\rho(t,x)}{\partial t}=0$. We need to verify the following equality: $\frac{\partial}{\partial x}\left(\lambda x\rho_\infty(x)\right)=-\frac{\partial^2}{\partial x^2}\left(\frac{f^2}{2}\rho_\infty(x)\right)$. We develop both sides:

$$\frac{\partial}{\partial x}\left(\lambda x\rho_\infty(x)\right)=\lambda\rho_\infty(x)+\lambda x\frac{\partial\rho_\infty(x)}{\partial x}=\lambda\rho_\infty(x)-\frac{2\lambda^2}{f^2}x^2\rho_\infty(x)=\lambda\rho_\infty(x)(1-\frac{2\lambda}{f^2}x^2)$$

$$-\frac{\partial^2}{\partial x^2}\left(\frac{f^2}{2}\rho_\infty(x)\right)=-\frac{f^2}{2}\cdot\left(\frac{4\lambda^2}{f^4}x^2-\frac{2\lambda}{f^2}\right)\rho_\infty(x)=\lambda\rho_\infty(x)(1-\frac{2\lambda}{f^2}x^2)$$

3.3 : Using Itô's formula, we obtain: $dv(t)=\frac{\partial\tilde{\sigma}}{\partial v}d\tilde{\sigma}(t)+\frac{1}{2}\frac{\partial^2\tilde{\sigma}}{\partial v^2}\left(d\tilde{\sigma}(t)\right)^2$. We develop each term:

$$\frac{\partial\tilde{\sigma}}{\partial v}d\tilde{\sigma}(t)=-2\lambda\tilde{\sigma}^2\,dt+2f\tilde{\sigma}\,dW_t^{(3)},\quad\frac{1}{2}\frac{\partial^2\tilde{\sigma}}{\partial v^2}\left(d\tilde{\sigma}(t)\right)^2=\frac{1}{2}\cdot 2\cdot f^2\,dt=f^2\,dt$$

We substitute these terms:

$$dv(t)=-2\lambda\tilde{\sigma}^2\,dt+2f\tilde{\sigma}\,dW_t^{(3)}+f^2\,dt=-2\lambda v(t)\,dt+2f\sqrt{v(t)}\,dW_t^{(3)}+f^2\,dt=\left(f^2-2\lambda v(t)\right)dt+2f\sqrt{v(t)}\,dW_t^{(3)}$$

And we conclude therefore that we have the demanded form with $k = 2\lambda$, $\mu = \frac{f^2}{2\lambda}$ and $\eta = 2f$ (η should not be confused with the initial condition in the previous question). This is an interesting model from a financial point of view, as the drift term $k[\mu - v_t]$ ensures that v_t reverts to its long-term mean $\mu = \frac{f^2}{2\lambda}$ and $\eta\sqrt{v_t}dW_t^{(3)}$ introduces randomness in volatility. The model ensures $v_t \geq 0$ if started at $v_0 \geq 0$, but there is a possibility of hitting zero due to $\sqrt{v_t}$: the possibility of $v_t \to 0$ implies no price movement which is not realistic in actual markets.

4 Q4

4.1 : From theorem 1 of the Lecture Notes (see [4]) we know that $\sqrt{\mathbb{E}\left[\sup_{t\in[0,T]}\left(S(t) - \hat{S}(t)\right)^2\right]}$ is of order $\Delta t^{\frac{1}{2}}$ if the following conditions are respected by $b(t,x)$ and $\sigma(t,x)$, where in our case $b(t,S) = IS_t$, $\sigma(t,S) = \sigma S_t$ and $S_t = S_0 \exp\left(\left(I - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$. Firstly, we check that $b(S)$ and $\sigma(S)$ are Lipschitz continuous:

$$|b(S_1) - b(S_2)| = |IS_1 - IS_2| = |I||S_1 - S_2|, \quad |\sigma(S_1) - \sigma(S_2)| = |\sigma S_1 - \sigma S_2| = |\sigma||S_1 - S_2|$$
$$\Rightarrow |b(S_1) - b(S_2)| + |\sigma(S_1) - \sigma(S_2)| = (|I| + |\sigma|)|S_1 - S_2| = L|S_1 - S_2|$$

Then, we check that they verify a linear growth bound $|b(S)|^2 + |\sigma(S)|^2 \leq C(1+|S|^2)$ for some constant $C > 0$:

$$|b(S)|^2 = |IS|^2 = I^2|S|^2 \Rightarrow |b(S)|^2 \leq I^2|S|^2 + I^2 = C_1(1+|S|^2)$$
$$|\sigma(S)|^2 = |\sigma S|^2 = \sigma^2|S|^2 \Rightarrow |\sigma(S)|^2 \leq \sigma^2|S|^2 + \sigma^2 = C_2(1+|S|^2) \Rightarrow |b(S)|^2 + |\sigma(S)|^2 \leq C(1+|S|^2)$$

where $C_1 = I^2$, $C_2 = \sigma^2$ and $C = C_1 + C_2$. Now we check for local Hölder continuity for two times $t_1, t_2 \in [0,T]$ where $|b(t_1,S) - b(t_2,S)| + |\sigma(t_1,S) - \sigma(t_2,S)| \leq C|t_1 - t_2|^\alpha$ for some $C > 0$ and $\alpha > \frac{1}{2}$ by using the mean-value theorem for the exponential function. We also use $f(t) = \exp\left(\left(I - \frac{1}{2}\sigma^2\right)t\right)$:

$$\exp\left(\left(I - \frac{1}{2}\sigma^2\right)t_1\right) - \exp\left(\left(I - \frac{1}{2}\sigma^2\right)t_2\right) \leq \sup_{t\in[t_1,t_2]}|f'(t)| \cdot |t_1 - t_2| = K_1 \cdot |t_1 - t_2|$$

$$\exists c \in (W_{t_1}, W_{t_2}) \quad \text{such that} \quad \exp(\sigma W_{t_1}) - \exp(\sigma W_{t_2}) = \sigma\exp(\sigma c)(W_{t_1} - W_{t_2}) \leq \sigma\exp(\sigma c)K|t_1 - t_2|^{\frac{1}{2}} = K_2|t_1 - t_2|^{\frac{1}{2}}.$$

$$|b(t_1,S) - b(t_2,S)| = IS_0\left|\exp\left(\left(I - \frac{1}{2}\sigma^2\right)t_1 + \sigma W_{t_1}\right) - \exp\left(\left(I - \frac{1}{2}\sigma^2\right)t_2 + \sigma W_{t_2}\right)\right| \leq IS_0K_1K_2|t_1 - t_2|^{\frac{3}{2}}$$

The same reasoning follows for $|\sigma(t_1,S) - \sigma(t_2,S)| \leq \sigma S_0K_1K_2|t_1 - t_2|^{\frac{3}{2}}$ and we obtain the final result:

$$|b(t_1,S) - b(t_2,S)| + |\sigma(t_1,S) - \sigma(t_2,S)| \leq C|t_1 - t_2|^{\frac{3}{2}} \quad \text{where} \quad C = (I+\sigma)S_0K_1K_2$$

5 Q5

5.1 : By the law of large numbers for a Brownian Motion we know that $\lim_{t\to\infty}\frac{W_t}{t} = 0$ and we can prove directly, and knowing $S_t = S_0\exp\left(\left(I - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right)$:

$$\frac{\log(S_t)}{t} = \frac{\log(S_0)}{t} + I - \frac{1}{2}\sigma^2 + \frac{\sigma W_t}{t} \Rightarrow \lim_{t\to\infty}\frac{\log(S_t)}{t} = I - \frac{1}{2}\sigma^2, \quad \text{a.s. if } I \neq \frac{\sigma^2}{2}$$

Now we compute the infimum and supremum (considering $I = \frac{\sigma^2}{2}$, by using the law of the iterated logarithm for Brownian motion $\limsup_{t\to\infty}\frac{W_t}{\sqrt{2t\log\log t}} = 1, \liminf_{t\to\infty}\frac{W_t}{\sqrt{2t\log\log t}} = -1$ a.s. (see [1]) :

$$\log(S_t) = \log(S_0) + \sigma W_t \Rightarrow \frac{\log(S_t)}{\sqrt{2t\log\log t}} = \frac{\log(S_0)}{\sqrt{2t\log\log t}} + \frac{\sigma W_t}{\sqrt{2t\log\log t}}$$
$$\limsup_{t\to\infty}\frac{\log(S_t)}{\sqrt{2t\log\log t}} = \sigma, \quad \liminf_{t\to\infty}\frac{\log(S_t)}{\sqrt{2t\log\log t}} = -\sigma, \quad \text{a.s.}$$

We deduce a stability condition for $\lim_{t\to\infty} \mathbb{E}[|S_t|^2] = 0$ using the fact that the moment generating function for a normal distribution $X \sim \mathcal{N}(\mu, \sigma^2)$ is $\mathbb{E}[\exp(uX)] = \exp\left(u\mu + \frac{1}{2}u^2\sigma^2\right)$ which in this case for $W_t \sim \mathcal{N}(0,t)$ yields $\mathbb{E}[\exp(2\sigma W_t)] = \exp\left(\frac{1}{2}(2\sigma)^2 t\right) = \exp\bigl(2\sigma^2 t\bigr)$:

$$\mathbb{E}[|S_t|^2] = \mathbb{E}\left[S_0^2 \exp\left(2\left(I - \frac{1}{2}\sigma^2\right)t + 2\sigma W_t\right)\right] = S_0^2 \exp\left(2\left(I - \frac{1}{2}\sigma^2\right)t\right) \cdot \mathbb{E}\left[\exp(2\sigma W_t)\right]$$

$$= S_0^2 \exp\left(2\left(I - \frac{1}{2}\sigma^2\right)t\right) \cdot \exp\bigl(2\sigma^2 t\bigr) = S_0^2 \exp\left(2It + \sigma^2 t\right)$$

Therefore we can deduce: $\lim_{t\to\infty} \mathbb{E}[|S_t|^2] = 0 \Leftrightarrow 2I + \sigma^2 < 0$

5.2 : As we can observe from Fig. 1(left), higher interest rates lead to a strict increase in $\mathbb{E}[|S_t|^2]$, which is consistent with the theoretical condition $I < -\frac{\sigma^2}{2}$ as the values of I are all strictly positive and therefore we are not in the case $\lim_{t\to\infty} \mathbb{E}[|S_t|^2] = 0$. This is a good asset behavior as we can achieve higher asset pricing and therefore better returns.

5.3 : The stochastic-θ method with $\Delta W_n \sim \mathcal{N}(0, \Delta t)$ and $\theta \in (0;1]$, is expressed as follows:

$$S_{n+1} = S_n + \theta S_{n+1} I\Delta t + (1-\theta) S_n I \Delta t + \sigma S_n \Delta W_n \Rightarrow \mathbb{E}[|S_{n+1}^2|] = \mathbb{E}\left[\frac{\bigl(S_n\bigl(1 + (1-\theta)I\Delta t\bigr) + \sigma S_n \Delta W_n\bigr)^2}{(1-\theta I \Delta t)^2}\right]$$

If we expand the numerator:

$$\bigl(S_n\bigl(1+(1-\theta)I\Delta t\bigr) + \sigma S_n \Delta W_n\bigr)^2 = S_n^2\bigl(1+(1-\theta)I\Delta t\bigr)^2 + \sigma^2 S_n^2 (\Delta W_n)^2 + 2S_n^2\bigl(1+(1-\theta)I\Delta t\bigr)\sigma\Delta W_n$$

Knowing that $\mathbb{E}[\Delta W_n] = 0$ and $\mathbb{E}[\Delta W_n^2] = \Delta t$ (also the independence of ΔW_n with S_n), the cross-term vanishes $\mathbb{E}[2S_n^2\bigl(1+(1-\theta)I\Delta t\bigr)\sigma\Delta W_n] = 0$ and we obtain:

$$\mathbb{E}[|S_{n+1}|^2] = \frac{\mathbb{E}[S_n^2]\bigl(1 + 2(1-\theta)I\Delta t + ((1-\theta)I\Delta t)^2 + \sigma^2\Delta t\bigr)}{(1-\theta I\Delta t)^2}$$

Finally, using D'Alembert's criterion $\lim_{t\to\infty} \mathbb{E}[|S_n|^2] = 0 \Leftrightarrow \mathbb{E}[|S_{n+1}|^2] < \mathbb{E}[|S_n|^2]$ we obtain the condition:

$$\frac{(1 + 2(1-\theta)I\Delta t + ((1-\theta)I\Delta t)^2 + \sigma^2\Delta t)}{(1-\theta I\Delta t)^2} < 1$$

For $\theta = 0$ (explicit method) we obtain:$1 + 2I\Delta t + (I\Delta t)^2 + \sigma^2\Delta t < 1 \Rightarrow \Delta t\bigl(2I + I^2\Delta t + \sigma^2\bigr) < 0$. And neglecting the small second-order term $I^2\Delta t$ for small Δt we obtain $2I + \sigma^2 < 0 \Rightarrow I < -\frac{\sigma^2}{2}$ and we observe that we find the same condition as in subsection 5.1.

For $\theta = 0.5$ we obtain: $\frac{(1+2(1-0.5)I\Delta t + ((1-0.5)I\Delta t)^2 + \sigma^2\Delta t)}{(1-0.5I\Delta t)^2} < 1$, which yields the same condition as before:

$$1 + I\Delta t + (0.5I\Delta t)^2 + \sigma^2\Delta t < (1 - 0.5I\Delta t)^2 \Rightarrow 1 + I\Delta t + (0.5I\Delta t)^2 + \sigma^2\Delta t < 1 - I\Delta t + (0.5I\Delta t)^2$$

$$\Rightarrow I\Delta t + \sigma^2\Delta t < -I\Delta t \Rightarrow I < -\frac{\sigma^2}{2}$$

For $\theta = 1$ the condition reduces to: $1 + \sigma^2\Delta t < 1 \Rightarrow \sigma^2 < 0$. And since $\sigma^2 > 0$ and $\Delta t > 0$, this condition is never satisfied meaning that the method is not mean-square stable for $\theta = 1$. All these conditions are also satisfied numerically in Fig. 1(right), as for all θs we have growth in $\mathbb{E}[|S_t|^2]$ as expected.

5.4 : We proceed in the same way as for the last question. The scheme is written as (knowing that $\sigma_n^2 = \tilde{\sigma}_n^2$ and knowing $\mathbb{E}[\Delta W_n] = 0$ and $\mathbb{E}[\Delta W_n^2] = \Delta t$ as in subsection 5.3):

$$\sigma_{n+1} = \sigma_n - \lambda\sigma_n\Delta t + f\Delta W_n \Rightarrow \sigma_{n+1}^2 = \sigma_n^2(1-\lambda\Delta t)^2 + f^2(\Delta W_n)^2 + 2\sigma_n(1-\lambda\Delta t)f\Delta W_n$$

$$\mathbb{E}[\sigma_{n+1}^2] = \mathbb{E}[\sigma_n^2](1-\lambda\Delta t)^2 + f^2\Delta t \Rightarrow (1-\lambda\Delta t)^2 < 1 \Rightarrow 0 < \Delta t < \frac{2}{\lambda}$$

Let $\mathbb{E}[\sigma_n^2] = x_n$. For stability, we require x_n to remain bounded as n goes to infinity. The steady-state value is given by solving the recursion at equilibrium, where $x_{n+1} = x_n = x_\infty$:

$$x_\infty = x_\infty(1-\lambda\Delta t)^2 + f^2\Delta t \Rightarrow M = x_\infty = \frac{f^2\Delta t}{1-(1-\lambda\Delta t)^2}$$

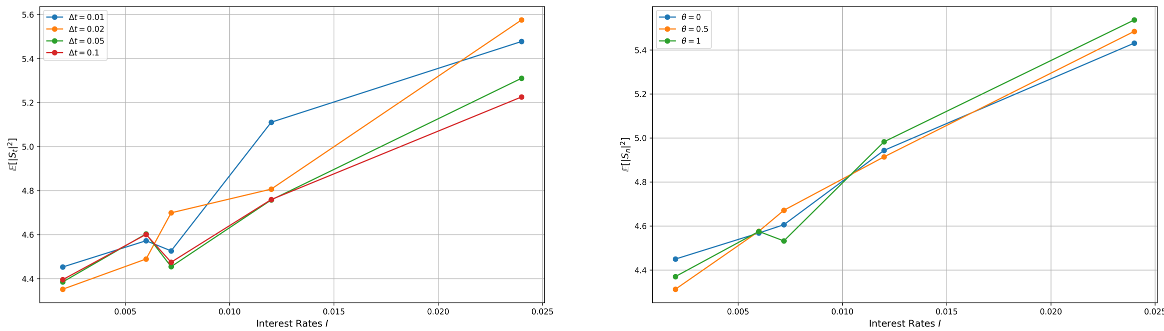


Figure 1: Left: forward euler method for 5.2. Right: stochastic θ-method for 5.3.

6 Q6

6.1 : Using Itô's product rule: $d[S_t S_t^\top] = S_t dS_t^\top + dS_t S_t^\top + dS_t dS_t^\top$, and using the same rules as in subsection 1.1 we obtain: $dS_t dS_t^\top = \sigma^2 K(S_t \circ dW_t)(S_t \circ dW_t)^\top K^\top$ we have that:

$$d[S_t S_t^\top] = S_t S_t^\top I^\top dt + I S_t S_t^\top dt + \sigma S_t [K(S_t \circ dW_t)]^\top + \sigma [K(S_t \circ dW_t)] S_t^\top + \sigma^2 K(S_t \circ dW_t)(S_t \circ dW_t)^\top K^\top dt$$

Now we take the expectation for all the terms. For the last term, using $\mathbb{E}[S_i(t)^2 (dW_i(t))^2] = \mathbb{E}[S_i(t)^2]\mathbb{E}[(dW_i(t))^2] = \mathbb{E}[S_i(t)^2] \cdot dt$ and the independence of $dW_i(t)$ and $dW_j(t)$ for $i \neq j$:

$$\mathbb{E}[(S(t) \circ dW_t)(S(t) \circ dW_t)^\top] = \begin{bmatrix} \mathbb{E}[S_1(t)^2]dt & 0 & \cdots & 0 \\ 0 & \mathbb{E}[S_2(t)^2]dt & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbb{E}[S_n(t)^2]dt \end{bmatrix}$$

And if we also have $\mathbb{E}[\sigma S_t [K(S_t \circ dW_t)]^\top] = \mathbb{E}[\sigma [K(S_t \circ dW_t)] S_t^\top] = 0$ because $\mathbb{E}[dW_t] = 0$ we can conclude: $\frac{dG(t)}{dt} = G(t) I^\top + I G(t) + \sigma^2 K \text{diag}(G(t)) K^\top$.

6.2 : From Fig. 2(right), we can observe that increasing M from 100 to 2000 reduces the Monte Carlo estimation error from 10^{-7} to 10^{-8} as expected from Monte Carlo methods. On the other hand, as Δt becomes smaller we don't observe a decrease in the error as it should be expected: the smaller Δt the smaller also the discretization error.

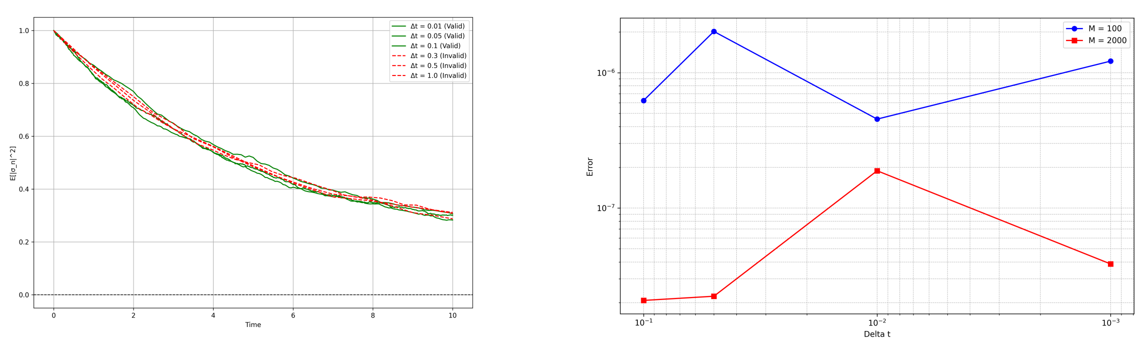


Figure 2: Left: Sigma validation for various time steps as discussed in 5.4. Right: Euler-maruyama and error for varying M and Δt presented in 6.2.

7 Q7

7.1 : According to [7], we know that the following Cox-Ingersoll-Ross process (CIR): $\frac{dv(t)}{dt} = \kappa[\mu - v(t)] + \eta v(t)dW_t$ admits the following Lamperti transformation, where $Y(t) = \sqrt{v(t)}$:

$$Y(t) = \left(\frac{\alpha}{Y(t)} + \beta Y(t) \right) dt + \gamma dW(t)$$

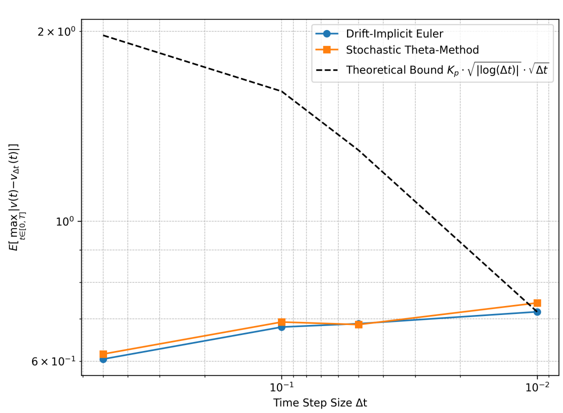
with $\alpha = \frac{4\kappa\mu - \eta^2}{8}$, $\beta = -\frac{\kappa}{2}$ and $\gamma = \frac{\eta}{2}$. The drift-implicit square-root Euler method for the Lamperti-transformed process is:

$$Y_{k+1} = \frac{Y_k + \gamma \Delta W_k}{2(1 - \beta \Delta t)} + \sqrt{\frac{(Y_k + \gamma \Delta W_k)^2}{4(1 - \beta \Delta t)^2} + \frac{\alpha \Delta t}{1 - \beta \Delta t}}$$

The exact solution of the CIR process at time t, given $v(u)$, can be found in [8] which is basically sampling from a non-central χ^2 distribution:$V_t \sim \frac{\eta^2(1-e^{-\kappa\Delta t})}{4\kappa} \cdot \chi^2 \left(\frac{4\kappa\mu}{\eta^2}, \frac{4\kappa e^{-\kappa\Delta t} V_{t-\Delta t}}{\eta^2(1-e^{-\kappa\Delta t})} \right)$. So in our program the $v(t)$s are determined in a markovian-like process where each one of them is used to determine $v(t)$ at the next time-step. From Theorem 1.1 we should obtain:

$$E\left[\max_{t\in[0,T]} |v(t) - v_{\Delta t}(t)| \right] \leq K_p \cdot \sqrt{|\log(\Delta t)|} \cdot \sqrt{\Delta t} \quad \text{if} \quad 2\kappa\mu > \eta^2$$

where $v(t)$ is the exact solution and $v_{\Delta t}(t)$ is the piecewise linear interpolation obtained from the numerical scheme. From Fig. 3 we observe that the result of the numerical scheme follows closely the trend of the scheme obtained with the stochastic-θ with a smaller error, but does not follow the theoretical trend $\sqrt{|\log(\Delta t)|} \cdot \sqrt{\Delta t}$. This should have been the case normally for $\eta = 0.5$ as we respect the convergence criterion $2\kappa\mu > \eta^2$, and we should observe stable and controlled convergence, but not in the case where $\eta = 1.5, 1.75$ where the criterion is not respected and therefore we should observe larger errors and potentially less steep convergence slopes. The only difference that is noticeable, is that for $\eta = 0.5$ we obtain a smaller error at an order of magnitude of 10^{-1} compared to the other values of η which are at an order of magnitude of 10^0.



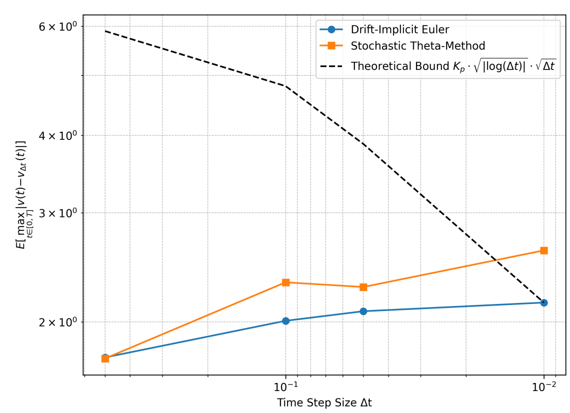


Figure 3: Left: Drift-implicit Euler-type method for approximating Cox-Ingersoll-Ross (CIR) process for $\eta = 0.5$. Right: $\eta = 1.5$

7.2 : The solution that was found in Subsection 3.1 can not be used in this case because it corresponds to a different SDE. Firstly, the noise term is different as for the CIR it is multiplicative $\sqrt{v(t)}dW_t$ while for 3.1 it's additive $f dW_t$. Then also the drift term is different as for CIR it is non-linear $\kappa[\mu - v(t)]$ while for 3.1 it is linear $-\lambda\sigma(t)$. The Lamperti transformation changes therefore the structure of the SDE, and we need to determine the exact solution by sampling as we did before rather than using the closed-form solution from 3.1.

8 Q8

Figure 4 shows the absolute error between the numerical approximation (C_{approx}) and the theoretical Black-Scholes solution (C_{true}) as a function of the time step size (Δt).

As Δt decreases, the error is expected to decrease monotonically. This behavior is predicted by the Euler-Maruyama method, which has an expected error convergence rate of $\mathcal{O}(\sqrt{\Delta t})$. This is not seen from the result as the error fluctuates quite significantly. This might most possibly due to an error in the coding implementation. Other possible reasons are the following.

With each run of the simulation, the random numbers used to simulate the Brownian motion (dW_t) are different. This introduces variability in the computed payoff and hence in the error. This stochastic nature explains why repeated runs yield different results. Another reason might be to increase even more the number of runs for the Monte Carlo simulation but this is not likely to be the issue in this case.

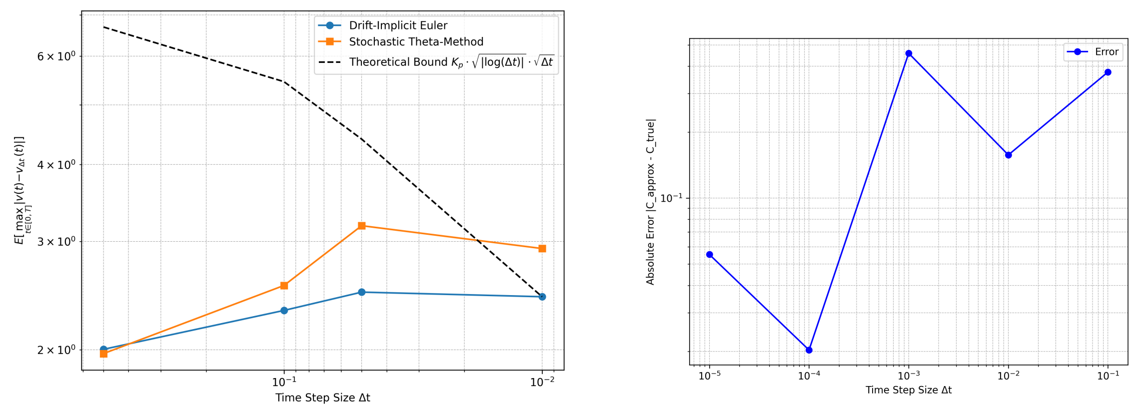


Figure 4: Left: Drift-implicit Euler-type method for approximating Cox-Ingersoll-Ross (CIR) process with $\eta = 0.5$. Right: Error between approximation and real value of asset price from Section 8

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