Voting systems for multiple choice Schelling games

Abstract

In this work we consider social choice problems motivated by blockchain oracles. Here participants are asked to respond to semi-objective questions and are economically incentivized to provide responses that agree with each other, with the idea that the true response is then a "Schelling point", i.e. a "focal point" in the sense of Thomas Schelling. This approach, as it attempts to select "true" alternatives that participants see with some noise, builds on previous work that views voting rules as statistical estimators, albeit with a different incentive structure. We define a notion of honest behaviour in this framework, and explore the roles of the voting rule used, as well as the structure of payments made to participants in the incentivization of such a system. However, we show an impossibility theorem that demonstrates that, under minimal assumptions on the underlying voting rule and payoff structure chosen, if there are three or more possible alternatives being voted upon, there will always exist some situations where participants are incentivized to participate "dishonestly". Hence, this situation mirrors phenomena in preference aggregation, such as Arrow's Theorem and the Gibbard-Satterthwaite Theorem.

1 Introduction

In recent years blockchains have offered new possibilities for economic relationships in the absence of trust between parties. However, useful applications often need access to information that is not verified as part of the internal blockchain consensus protocol. For example, a blockchain-based prediction market platform might need to know which candidate won an election in order to make payouts afterwards. Similarly, a crop insurance contract might need to know whether it rained in Saskatchewan. However, this information is generally not part of the blockchain consensus protocol and, as a result, it is not natively available on-chain. The problem of making such information available to blockchain applications without adding additional trust assumptions on parties, is known as the "blockchain oracle problem" [1], [30].

One solution to this problem has been to make use of Schelling games [30], [7]. The idea of this structure is to set up a coordination game [11], namely a game in which members of the community of users of a blockchain application

Figure 1: Each voter expects the other voters to submit the "honest" outcome as it is a distinguished choice. Hence as the voters are incentivized to vote with the majority of other voters, they have an incentive to vote honestly. Here, as part of a blockchain oracle, three voters are asked who won a high profile election. The voter on the left can see that Alice is reported in newspapers as having won, so she says that Alice won expecting the other voters to do the same.

	X wins	Y wins
$\mathcal{USR}$ votes X	p	0
$\mathcal{USR}$ votes Y	0	p

Figure 2: An example payoff matrix for a voter $\mathcal{USR}$ in a symmetrical, binary coordination game. If one of X or Y is "true", then one might expect that response to be a Schelling point for voters.

are asked to vote on the answer to some external question and they are rewarded or penalized based on whether they agree with the votes of others. Then, in many contexts, one expects that the true answer will be a Schelling point (or focal point) in the sense of Thomas Schelling [27], namely "a solution that people will tend to use [in attempts to coordinate] in the absence of communication, because it seems natural, special, or relevant to them". Thus, a "Schelling game" is a coordination game where one option is such a focal point, that is distinguished for reasons external to the payoff structure of the game[1].

One might attempt to generalize this approach to semi-objective questions with more than two possible outcomes. For example, a decentralized, blockchain-based marketplace might use a Schelling game system to determine whether a seller respected the conditions of a sale with possible outcomes of no refund, a partial refund, or a full refund to the buyer. A decentralized social media platform might use a Schelling game to determine whether a post violated its terms

[1] This is in contrast to approaches based on participants coordinating based on asymmetries in the payoff structure of the game, such as when equilibria are Pareto-ranked; see, for example, [33], [13], [11] for work in this direction.

and conditions, with possible outcomes including the post remaining available, the post being removed but the user who posted it not being banned, or the post being removed and the user who posted it being banned [19].

This framework raises a number of interesting research questions. Particularly, one asks what voting rules should be used to convert user responses into a collective outcome, and what payoff structure should be put in place to determine the rewards and penalties for different responses. For semi-objective questions, one might take the point of view that there exists some "true" response to the question that voters see with noise. This is similar to the approach of the analysis of [10]. Already in that work interesting questions are raised with respect to what types of voting rules can have desirable properties in aggregating user responses so that the collective outcome is likely to be the underlying "true" value; this has natural applications to the design of voting rules for crowdsourcing or micro-task platforms. However, in [10], the incentivization of voters is not considered, rather they are assumed to report the noisy truth as they observe it. Such a perspective is not sufficient for the adversarial setting of blockchain applications. Hence, in our setting, we are interested in the degree to which it is possible to choose a voting rule and payoff structure that both *actually* incentivize participants to submit votes that reflect the noisy truth as they observe it *and* then aggregate these votes effectively. Such attempts to incentivize particular voting behaviour in participants raises parallels to some of the impossibility theorems that have been seen in standard elections [3], [15], [26].

In non-blockchain platforms, trusted third parties can act as moderators and exclude voters who act malicious in visible ways. Nevertheless, these services have issues regarding spam responses [2], [18]. Thus, the framework of Schelling game based mechanisms discussed in this work could potentially be implemented for semi-objective questions on such platforms as an additional tool to improve response quality.

2 Related work

This work, as it ultimately deals with mechanisms by which the information provided by many individuals can be aggregated into a collective choice, draws on a wide variety of ideas in social choice theory. In traditional elections, one typically imagines voters as having preferred choices; namely that they receive more or less utility depending on which candidate is chosen by the group. Then the motivation of participants is to vote in such a way that the resulting collective choice will provide them with as much utility as possible. Already this problem gives rise to a rich set of complexities; particularly, it has been seen that many natural properties that one would want electoral systems to have are incompatible [3], [15], [26], [21]. Remark, for our purposes that the classic impossibility theorems of social choice theory such as Arrow's Impossibility Theorem [3] and the Gibbard-Satterthwaite Theorem [15], [26], to the degree that they make statements about the positions of alternatives in submitted lists, still apply directly in a setting where the voters have another incentive structure, while they no longer necessarily have the same implications in terms of incentive compatibility. Also, of particular note for our purposes is work in which voters are rewarded or penalized financially based on how they vote, which is the perspective of [29] which reconsiders the Gibbard-Satterthwaite Theorem in the context where one allows for divisible private goods (money) to be distributed among the voters in a way that is determined by the outcome of the vote.

An alternative framework, relevant to the design of crowdsourcing platforms, is to suppose that there exists some "truth" that voters observe with noise. This perspective already dates back to Condorcet's Jury Theorm [12]. Then one can pose questions about which voting rules, if any, have good properties in terms of converting the information provided by voters into collective outcomes that are likely to recover the "truth". Theoretical work along these lines is considered in [10], [23], [9], and behavioural experiments are considered in [20]. However, this perspective generally assumes that voters honestly report the truth as they observe it. This point of view is not realistic in the adversarially, pseudo-anonymous environment of blockchain applications. Existing work that reintroduces strategic voting and individual incentives into the perspective of this model is [4] where it is noted that Condorcet's Jury Theorem implicitly assumes that votes reflect the truth as observed by participants, and indeed the strategy of providing such a vote does not generally give a Nash equilibrium. In [17], results along the lines of the Jury Theorem are recovered with potentially strategic voters in the context of a set of alternatives that is strictly ordered. However, in these works voters' incentives when considering to cast a strategic vote come from an intrinsic desire to have the "true" alternative (as well as they are able to evaluate it through the noise) adopted.

Meanwhile, many prominent blockchain oracles [30], [7], [1], [22], [19] rely on some version of a Schelling game idea. This setting also uses crowdsourcing ideas, where participants are often holders of some "token". These voters are then incentivized purely through the payouts of the game and only have a weak intrinsic interest in the system producing honest results. Note that in a more general sense, Schelling (or focal) points have a long tradition in social choice theory as a model for collaboration [28].

Past work considering such Schelling games often focuses on situations where voters are presented with a binary choice (see for example [1]), avoiding the classical problems of social choice theory in multi-candidate elections. In some applications such as [30], to the degree that voters are presented with non-binary choices, the "honest" choice is typically so clear (i.e. reporting the winner of a well-known election), that it is not problematic to choose the winner via a Plurality voting system. However, in other applications, such as the blockchain-based dispute resolution system of Kleros [19] which uses a Schelling game to incentivize a sort of "dispute resolution" oracle, there is less likely to be unanimity around winning choices. In these cases, use of a Plurality voting system may be less appropriate in the presence of non-binary alternatives.

Finally in [14], a Schelling point based system that allows for non-binary answers, specifically that allows for real value answers, is considered, but this is done by binarizing the choices provided to voters. (Note that taking a binarization approach to multiple alternative decisions more generally has limitations, compare to the discussion on discursive dilemmas in 17.1 of [6].)

3 Contribution of this work

In this work we will explore questions of incentive compatibility in multiple choice Schelling games. Particularly, in Section 5 we define a natural concept of "Schelling-honesty" for what an "honest vote" might mean in this context. In Section 7, under very minimal assumptions on voting and payoff systems, we prove in Theorem 1 that there will always be some situations where participants will be incentivized to deviate from Schelling-honesty. This theorem draws heavily upon the Gibbard-Satterthwaite Theorem, and could naturally be seen as an analog of that result in this context. In Section 8 we show to what degree these (minimal) assumptions on the voting system and payoff system in Theorem 1 are necessary by providing counterexamples when these criteria are not satisfied. In Section 9, we consider some specific examples where Schelling-honest voting is not economically optimal, and we explore the question of under what conditions on a voting system is it always possible to even provide a Schelling-honest vote. In Section 10, we conclude by considering how this work might fit with broader goals in designing Schelling games with multiple choices.

4 Notation and basic notions

We use the following notation throughout: A is a set of alternatives that can be a possible outcome for a voting system. $\mathcal{L}$ is the set of possible orderings of the alternatives in A. N is the number of voters[2]. We denote the number of alternatives by $\#A = n$. A social choice function is a function $f : \mathcal{L}^N \to A$. Often we will consider the perspective of a single voter $\mathcal{USR}_i$. Then if we denote $\mathcal{USR}_i$'s vote by $v_i \in \mathcal{L}$, we denote the votes of the other users by $v_{-i} \in \mathcal{L}^{N-1}$, and one can compute the outcome given these votes as $f(v_i, v_{-i})$. If the vote v_i ranks a higher than b for $a, b \in A$, we write $a >_{v_i} b$.

Note that for a permutation σ on the set of alternatives A, one has natural maps on $\mathcal{L}$, $\mathcal{L}^{N-1}$, and $\mathcal{L}^N$ given by applying σ to each element of each ranked list. We denote these maps also by σ when doing so does not create ambiguity.

We will make use of certain standard ideas from social choice theory. Of particular relevance are the following concepts:

Definition 1. *A social choice function f is said to be neutral if for any permutation σ on A and any $l_1 \ldots, l_N \in \mathcal{L}$,*

$$f(\sigma(l_1), \ldots, \sigma(l_N)) = f(l_1, \ldots, l_N).$$

[2]In the Schelling games employed by blockchain oracles [30], [7], [22], [19] these voters are often holders of some "token" or "coin", and can be thought of as representing a crowd of users of a blockchain application. However, our results are independent of how voters are chosen, so this selection process is beyond the scope of this article.

Definition 2. *A social choice function f is said to have the absence of veto (AV) property if, for any choice $a \in A$, if a is at the top of all but at most one rankings l_i, then $f(l_1, \ldots l_N) = a$.*

We will consider as needed several other standard concepts of social choice functions such as dictators, monotonicity, the Condorcet criterion, Condorcet winners, and Pareto efficiency. See, for example, [6] for definitions of these properties. Furthermore, while it is possible that there are new, not yet considered, voting systems that are particularly adapted to the context of Schelling games, most of the examples that we consider in this work are drawn from standard voting systems (combined with some payoff system) such as Plurality, Instant-Runoff, and Ranked Pairs; see [6] for descriptions of these systems. In this work, our results will apply to resolute voting systems, namely on any given input of $\mathcal{L}^N$ the corresponding social choice function returns some outcome in A even in the event of a tie. Hence, when we consider the above mentioned standard voting systems, we will assume that there is some unspecified deterministic tie-break system being used[3].

Finally, we will have a notion of a payoff system; namely to each voter $\mathcal{USR}_i$ will be paid an amount given by a function

$$G_i : \mathcal{L}^N \to \mathbb{R}.$$

Considerations around payoff systems will be further discussed in Section 6.

5 Model

In this work, participants will be assumed to submit as a vote an element of $\mathcal{L}$, namely a strict ranking of the alternatives.

We consider a setting where there is some "true" answer that observers see with noise. Then, based on this observation, each participant $\mathcal{USR}_i$ assigns some probability to each alternative $a \in A$ being the "true" answer. Namely, $\mathcal{USR}_i$ see the "true" answer as following a probability distribution given by a probability mass function that we call her *assessment*:

$$S_i : A \to [0, 1].$$

Then, for each $a \in A$, $S_i(a)$ is the probability that $\mathcal{USR}_i$ believes a has of being the "true" answer based on her observation.

Then, in order to specify a *noise model* that yields this situation, one needs the values of

$$\text{prob}\left((S_i) = (h_i) | \text{“true” value} = a_k\right),$$

where (S_i) is the tuple of assessments over the voters $\mathcal{USR}_i$, for all $a_k \in A$ and for all (h_i) tuples of probability mass functions $A \to [0, 1]$. Namely, the noise

[3]Note that we do not need to assume that voting rules are anonymous in this work, so this deterministic tie-break system can be as simple as taking the first candidate alphabetically.

model determines the probabilities that each voter obtains a given assessment for each possible value of the “truth”[4].

A participant ideally should be incentivized to cast a vote consisting of a ranking of the alternatives that reflects her assessment. In this sense, we define:

Definition 3. *Given an assessment S_i, $\mathcal{USR}_i$'s vote $v_i = a_1 > a_2 > \ldots > a_M$ is said to be observation-honest if*

$$S_i(a_1) \geq S_i(a_2) \geq \ldots \geq S_i(a_M).$$

However, the incentive system does not have direct access to the “true” value. Hence, the basic idea of a Schelling game is that the winning outcome of the vote should be used by the incentive system as a proxy for this “true” answer. Then, instead of directly incentivizing voters to rank the alternatives in terms of their chances of being “true”, one wants to design an incentive systems such that voters are incentivized to rank the alternatives in terms of their probability of winning. Indeed, if voters believe that other participants will arrive at similar assessments as themselves, they might believe that there will be an equilibrium where these two perspectives coincide and where their best strategy is also to make an observation-honest vote[5].

Then, we suppose that a given voter $\mathcal{USR}_i$ has some probabilistic expectations for the behaviour of the other voters. We formalize this by attaching to each voter a probability distribution that captures how other voters will vote that is given by a probability mass function that we call her *evaluation*:

$$V_{-i} : \mathcal{L}^{N-1} \to [0, 1].$$

Namely, for each $v_{-i} \in \mathcal{L}^{N-1}$, $V_{-i}(v_{-i})$ is the probability that the other voters will submit v_{-i} as their votes, from the perspective of $\mathcal{USR}_i$ given her assessment.

A participant will generally obtain her evaluation based (partially) on her assessment, combined with priors she might have about the other voters. The other voters' rankings are not necessarily identically distributed. For example, $\mathcal{USR}_i$ might know exactly how a given other voter will vote with probability one

[4]Choosing a probability mass function $h_i : A \to [0, 1]$ is equivalent to choosing $h_i(a_j)$ for all $a_j \in A$ such that all $h(a_j) \in [0, 1]$ and $\sum_j h_i(a_j) = 1$. Hence, choosing h_i is equivalent to choosing a point in a simplex. Then choosing a tuple of (h_i) is equivalent to choosing a point in a finite product of simplices, so the set of such tuples is measurable.

[5]How good this proxy is, namely how closely the ranking of alternatives in order of winning aligns with the ranking of alternatives in order of being “true”, is then a question that builds on the viewpoint of voting systems as statistical estimators [10], [9], [23]. For example, if one is in an equilibrium where all voters submit observation-honest votes and the noise model satisfies reasonable properties, then heuristically if $\mathcal{USR}_i$'s observation-honest vote is $a_1 > a_2 > \ldots > a_M$, over many draws of the votes of the other participants in $\mathcal{L}^{N-1}$ she might expect a_1 to be the most likely alternative to be the maximum likelihood estimate with respect to producing the observed $\mathcal{L}^{N-1}$, a_2 to be the second most likely maximum likelihood estimate, etc. Then, if the voting rule given by holding $\mathcal{USR}_i$'s vote constant and restricting to the votes of the other participants is a MLEW in the sense of [10], a_1 is also the most likely alternative to win, a_2 is the second most likely alternative to win, etc. So the vote $a_1 > a_2 > \ldots > a_M$ is also ranking of the alternatives in order of their probability of winning.

while only having a probabilistic knowledge of others. Nor are the other voters' rankings necessarily independent of each other. Indeed $\mathcal{USR}_i$'s evaluation for their votes could include knowledge that voter j_1 will always vote exactly the same way as voter j_2 even if $\mathcal{USR}_i$ does not know either of their votes in advance. However, we assume that all of the other voters' rankings are independent of $\mathcal{USR}_i$'s vote[6].

Hence, for any given vote v_i by $\mathcal{USR}_i$, $\mathcal{USR}_i$ can consider the probability of each given value v_{-i} giving the votes of the other participants, and she can thus estimate the probability of each outcome $a \in A$ winning as

$$\sum_{\substack{v_{-i} \in \mathcal{L}^{N-1} \text{ such that} \\ f(v_i, v_{-i}) = a}} V_{-i}(v_{-i}).$$

Suppose $\mathcal{USR}_i$ receives an assessment of $S_i = h_i$, for some probability mass function $h_i : A \to [0,1]$, and she believes that other voters will all cast observation-honest votes. Then, for $v_{-i} \in \mathcal{L}^{N-1}$,

$$V_{-i}(v_{-i}) = \text{prob}\left(\begin{array}{c} \text{other voters} \\ \text{vote } v_{-i} \end{array} |S_i = h_i \right)$$

$$= \text{prob}\left(\begin{array}{c} S_r(a_j) > S_r(a_l) \forall a_j, a_l, \\ r \neq i \text{ such that } a_j >_{v_r} a_l \end{array} |S_i = h_i \right) + T_i(v_{-i})$$

$$= \frac{\text{prob}\left(\begin{array}{c} S_r(a_j) > S_r(a_l) \forall a_j, a_l, r \neq i \\ \text{such that } a_j >_{v_r} a_l, \text{ and } S_i = h_i \end{array} \right)}{\text{prob}(S_i = h_i)} + T_i(v_{-i})$$

$$= \frac{\sum_{a_t \in A} \text{prob}\left(\begin{array}{c} \text{“true”} \\ \text{value} = a_t \end{array} \right) \text{prob}\left(\begin{array}{c} S_r(a_j) > S_r(a_l) \\ \forall a_j, a_l, r \neq i \text{ such that} \\ a_j >_{v_r} a_l, \text{ and } S_i = h_i \end{array} | \begin{array}{c} \text{“true”} \\ \text{value } = a_t \end{array} \right)}{\text{prob}(S_i = h_i)} + T_i(v_{-i}), \tag{1}$$

where

$$T_i(v_{-i}) = \text{prob}\left(\begin{array}{c} \text{other voters vote } v_{-i}, \\ \text{and } S_r(a_j) = S_r(a_l) \text{ for} \\ \text{some } r \neq i, a_j, a_l \in A \end{array} |S_i = h_i \right).$$

Thus, if $\mathcal{USR}_i$ believes that she is in an equilibrium where everyone casts observation-honest votes, $\mathcal{USR}_i$ can derive her evaluation from her assessment,

[6]This does not preclude the other voters adjusting their votes to their *expectations* for $\mathcal{USR}_i$'s vote, the effect of which can be captured in $\mathcal{USR}_i$'s evaluation, such as if the system is in an equilibrium and we are analyzing whether a distinguished voter $\mathcal{USR}_i$ is incentivized to deviate or not. This assumption of independence is reasonable in situations where the votes are cast simultaneously. While this assumption would also be reasonable if $\mathcal{USR}_i$ votes last, such as if $\mathcal{USR}_i$ is a follower in a Stackelberg game [34], [16]; this assumption would not be appropriate if other participants cast their votes after $\mathcal{USR}_i$, such as if $\mathcal{USR}_i$ is a leader in a Stackelberg game. Hence, our setting does not necessarily apply to such staggered voting. Particularly, as we do not in general assume that our voting rule is anonymous, we cannot make arguments where the relevant voters are assumed to vote last as these voters do not necessarily have the same role in the voting rule as voters who vote earlier.

the noise model, the values prob (“true” value $= a_t$), and her expectations of how other voters rank alternatives which they think are equally likely[7].

We consider participants that are economically rational in the sense that they take actions that maximize their expected return, where the uncertainty in their return is captured in their evaluations[8].

It seems inevitable that any system based on a Schelling-game structure will have situations where some voters will be incentivized to cast votes that are not observation-honest, as similar to the situations considered in [24], there can be scenarios where a minority of participants that hold specialized knowledge about a decision will believe that the majority will cast incorrect votes due to not having this knowledge[9].

Instead, one might nevertheless hope for the system to usually produce good outcomes if voters are incentivized to vote for alternatives that are likely to win when they believe that *other voters* will cast observation-honest votes. This motivates us to define “Schelling-honest” behaviour as the following:

Definition 4. *Given an estimation V_{-i} for how a user believes that other voters will vote which is independent from her own vote, $\mathcal{USR}_i$'s vote $v_i = a_1 > a_2 > \ldots > a_n$ is said to be Schelling-honest if*

$$p(a_1) \geq p(a_2) \geq \ldots \geq p(a_n),$$

where $p(a_j) = prob(a_j \text{ wins} | \mathcal{USR}_i \text{ votes } v_i)$ is the probability of a_j winning based on V_{-i} and assuming that $\mathcal{USR}_i$ votes $v_i = a_1 > a_2 > \ldots > a_n$.

Note that there may be several Schelling-honest votes possible to a user, to the degree that the user's vote changes the ordering of the most likely alternatives to win.

Similarly, we define

Definition 5. *An evaluation V_{-i} is said to be Schelling-dishonest if the highest possible expected payoff for $\mathcal{USR}_i$ according to this evaluation is not realized*

[7] In particular, if S_t is such that $S_t(a_j) = S_t(a_l)$ with probability zero for $t = 1, \ldots, N$, $a_j \neq a_l$, namely if voters only estimate two alternatives as being *exactly* equally likely to be true with probability zero, then $T_i(v_{-i}) = 0$ and voters do not have to make further assumptions about other participants behaviour to calculate $V_{-i}(v_{-i})$ beyond that they make observation-honest votes.

[8] We do not consider bribes or attempts to warp the payoff matrices of participants in this work, though such considerations already create novel effects in the case of Schelling games having only two alternatives [8]. In the multiple alternative case, this is a potential avenue for future work, see Section 10.

[9] The solutions to this issue presented by [24] that give more weight to subpopulations of voters based on their reported expectations of how others will vote do not fit into the model given above of each voter only submitting an ordered ranking in $\mathcal{L}$. Moreover, this approach is generally not appropriate for applications to blockchain oracles as a blockchain consensus algorithm cannot distinguish an informed minority that is participating honestly from an attacker that has a minority of the votes and is voting in such a way to manipulate the system. Hence, such approaches risk creating new attack vectors by reducing the threshold for the number of votes that an adversary would need to corrupt to attack the system.

by a Schelling-honest vote. Similarly, a noise model is said to be dishonest[10] *if, with positive probability, a participant receives an assessment such that the evaluation given by equation 1 is Schelling-dishonest. Namely, that there exists some positive probability that a voter would be incentivized to make a vote that is not Schelling-honest, even if she believes that all other voters make observation-honest votes.*

6 Payoff systems

We begin this section with a natural criterion that a payoff system might satisfy in the context of Schelling games. This criterion will particularly capture the property of paying participants higher rewards for ranking the winning choice higher than other alternatives. Then we discuss several examples of payoff systems from the perspective of this criterion.

Definition 6. *A payoff system that pays to $\mathcal{USR}_i$ an amount given by*

$$G_i : \mathcal{L}^N \to \mathbb{R}$$

is said to be reasonable if:

- *Suppose $v_{-i} \in \mathcal{L}^{N-1}$ is some fixed set of choices for all votes other than those of $\mathcal{USR}_i$. Suppose $r_i,\ s_i \in \mathcal{L}$ are such that the winner $f(r_i, v_{-i})$ is ranked in the k_1st place of r_i and the winner $f(s_i, v_{-i})$ is ranked in the k_2nd place of s_i. Then if, $k_1 \leq k_2 \Rightarrow G_i(r_i, v_{-i}) \geq G_i(s_i, v_{-i})$.*
- *For any $v_i \in \mathcal{L}$, $v_{-i} \in \mathcal{L}^{N-1}$ and permutation σ of the alternatives, if $f(v_i, v_{-i}) = a$ and $f(\sigma(v_i), \sigma(v_{-i})) = \sigma(a)$, then $G(v_i, v_{-i}) = G(\sigma(v_i), \sigma(v_{-i}))$.*

Furthermore, G_i is said to be strictly reasonable if:

- *G_i is reasonable.*
- *Suppose $r_i,\ s_i \in \mathcal{L}$ are such that the winner $f(r_i, v_{-i})$ is ranked in the k_1st place of r_i and the winner $f(s_i, v_{-i})$ is ranked in the k_2nd place of s_i. Then if, $k_1 < k_2 \Rightarrow G_i(r_i, v_{-i}) > G_i(s_i, v_{-i})$.*

When the relevant voter $\mathcal{USR}_i$ is clear from context, we will sometimes simply denote $G_i = G$. Also, for $v_i \in \mathcal{L}$ and V_{-i} an evaluation, we denote by $E[G(v_i, V_{-i})]$ the expected value of the payoff of voting v_i where the distribution of the votes of the participants other than $\mathcal{USR}_i$ is given by the evaluation V_i. Namely,

$$E[G(v_i, V_{-i})] = \sum_{x \in \mathcal{L}^{N-1}} V_{-i}(x) G(v, x).$$

Now we consider a few specific payoff systems. In all cases, we suppose that all voters submit an ordering in $\mathcal{L}$ along with a deposit D. (Hence the minimal possible payoff for a given vote is $-D$.)

First we consider:

[10] For given choices of the values prob (“true”value $= a_t$) and participant expectations of how other voters rank alternatives which they think are equally likely.

Definition 7. *(All or nothing incentive system)*

- *Determine the winner $w \in A$ via an underlying voting system.*
- *Any voter that places w first in her list is returned her deposit D. All other deposits are burned.*

Now we consider a system that is still entirely based on penalties; however it gives higher or lower penalties based on where the winning choice is placed in a participant's vote:

Definition 8. *(Penalty based incentive system)*

- *Determine the winner $w \in A$ via an underlying voting system.*
- *For each pair of the winner versus another choice in A, e.g. w versus a, w versus b, etc any voter who did not rank w ahead of the other choice loses a deposit $d = \frac{D}{\#A-1}$.*
- *These lost deposits are burnt.*

Finally, we consider a more realistic payoff system that distributes amounts from lost from penalized voters to other participants who ranked the winning choice higher:

Definition 9 (ϵ-redistributive payoff system)**.** *Let $\epsilon > 0$. Then*

- *Determine the winner $w \in A$ via an underlying voting system.*
- *For each pair of the winner versus another alternative in A, e.g. w versus a, w versus b, etc any voter who did not rank w ahead of the other choice loses a deposit $d = \frac{D}{\#A-1}$.*
- *Voters are paid a reward, that measures the number of pairs where $\mathcal{USR}_i$ ranks w ahead of other alternatives $a \in A$ with respect to the number of such pairs over all voters:*

$$\frac{Lost\ deposits}{\sum_{voter_t} \sum_{a_j \in A, a_j \neq w} \mathbb{1}_{voter_t\ voted\ a_j < w} + \epsilon} \sum_{a_j \in A, a_j \neq w} \mathbb{1}_{\mathcal{USR}_i\ voted\ a_j < w}$$

$$= d \cdot \frac{\sum_{voter_t} \sum_{a_j \in A, a_j \neq w} \mathbb{1}_{voter_t\ voted\ a_j > w}}{\sum_{voter_t} \sum_{a_j \in A, a_j \neq w} \mathbb{1}_{voter_t\ voted\ a_j < w} + \epsilon} \sum_{a_j \in A, a_j \neq w} \mathbb{1}_{\mathcal{USR}_i\ voted\ a_j < w}.$$

Similarly, a platform can provide a total net reward R that is distributed to voters according to:

$$\frac{R}{\sum_{voter_t} \sum_{a_j \in A, a_j \neq w} \mathbb{1}_{voter_t\ voted\ a_j < w} + \epsilon} \sum_{a_j \in A, a_j \neq w} \mathbb{1}_{\mathcal{USR}\ voted\ a_j < w}.$$

The presence of ϵ in the ϵ-redistributive payoff system prevents pathological situations where one would divide by zero, such as if a voting system was used that favors alternatives that are placed "last" and all candidates rank the winner "last".

Proposition 1. *Let $D > 0$. Then the all or nothing, penalty based, and ϵ-redistributive payoff systems are all reasonable. Furthermore, the penalty based and ϵ-redistributive payoff systems are strictly reasonable.*

Proof. For all three systems, suppose σ is a permutation of A, $v_i \in \mathcal{L}$, $v_{-i} \in \mathcal{L}^{N-1}$, such that $f(v_i, v_{-i}) = w$ and $f(\sigma(v_i), \sigma(v_{-i})) = \sigma(w)$. Then, for the all or nothing system and the penalty based system, the position of $\sigma(w)$ in $\sigma(v_i)$ is the same as the position of w in v_i. So these systems give the same payout after permutation. Moreover, when computing $G(\sigma(v_i), \sigma(v_{-i}))$, as $\sigma(w)$ has the same place in each participant's vote for $\sigma(v_{-i})$ as w does for v_{-i}, the number of pairs where the winner is ranked ahead or behind other alternatives is the same before and after the permutation. So the redistributive payoff system also preserves payouts under σ.

For the all or nothing and the penalty based payoff systems, the payoff monotonically decreases with the position that $\mathcal{USR}_i$ places w by fixed amounts regardless of v_{-i}. For the penalty based payoff system this amount is non-zero in each place. For the redistributive system, note that for a fixed v_{-i}, if $\mathcal{USR}_i$ places w in the k_ith place, then she places w ahead of other alternatives in $\#A - k_i$ pairs and behind other alternatives in $k_i - 1$ pairs. Then there exist some values c_1, $c_2 \geq 0$, that are constant with respect to $\mathcal{USR}_i$'s vote, such that

$$\sum_{\text{voter}_t} \sum_{a_j \in A, a_j \neq w} \mathbb{1}_{\text{voter}_t \text{ voted } a_j > w} = c_1 + k_1 - 1$$

and

$$\sum_{\text{voter}_t} \sum_{a_j \in A, a_j \neq w} \mathbb{1}_{\text{voter}_t \text{ voted } a_j < w} = c_2 + \#A - k_i.$$

Then $\mathcal{USR}_i$'s total return (i.e. her payoff minus her lost deposit) is

$$\frac{c_1 + k_i - 1}{c_2 + \#A - k_i + \epsilon}(\#A - k_i)d + \frac{R}{c_2 + \#A - k_i + \epsilon}(\#A - k_i) - (k_i - 1)d.$$

However, a standard first derivative argument, noting that $D > 0$, $\#A \geq 1$, and $\epsilon > 0$, shows that this quantity is strictly decreasing in $k_i > 0$.

□

7 An impossibility result

We will see that, under very limited assumptions on a social choice function and payoff structure, that inevitably there will be situations where participants are incentivized to deviate from the notions of honesty that we have described in Section 5.

Theorem 1. *Consider $n = \#A \geq 3$. Suppose $f : \mathcal{L}^N \to A$ is a neutral social choice function satisfying absence of veto (AV). Then, for any strictly reasonable payoff structure, there exists a Schelling-dishonest evaluation for some voter $\mathcal{USR}_i$.*

Namely, we have that under minimal conditions the goal of incentivizing participants to vote according to a "Schelling principle" is already self-incompatible, at least in some rare cases. Before continuing with the proof of this result, we note the following corollary:

Corollary 1. *Consider $n = \#A \geq 3$. Suppose $f : \mathcal{L}^N \to A$ is a neutral social choice function satisfying absence of veto (AV). Then, for any strictly reasonable payoff structure, there exists a dishonest noise model.*

Proof. Let V_{-i} be the Schelling-dishonest evaluation given by Theorem 1, and let v_i be the Schelling-dishonest vote that maximizes the payoff for $\mathcal{USR}_i$ according to V_{-i}.

For each $v \in \mathcal{L}$, take a fixed probability mass function h_v such that $h_v(a) > h_v(b) \Leftrightarrow a >_v b$ for all $a, b \in A$.

Then suppose the noise model is such that $\mathcal{USR}_i$ receives h_{v_i} as her assessment with probability one, namely $\text{prob}(S_i = h_{v_i}) = 1$, and

$$\text{prob}\left(\begin{array}{c} S_r = h_{v_r} \forall r \neq i, \\ \text{where } v_r \text{ is } \mathcal{USR}_r\text{'s vote in } v_{-i} \end{array} \mid \begin{array}{c} \text{“true”} \\ \text{value } = a_t \end{array} \right) = V_{-i}(v_{-i})$$

for all $v_{-i} \in \mathcal{L}^{N-1}$ and $a_t \in A$.

Then,

$$\frac{\sum_{a_t \in A} \text{prob}\left(\begin{array}{c} \text{“true”} \\ \text{value} = a_t \end{array} \right) \text{prob}\left(\begin{array}{c} S_r(a_j) > S_r(a_l) \\ \forall a_j, a_l, r \neq i \text{ such that} \\ a_j >_{v_r} a_l, \text{ and } S_i = h_{v_i} \end{array} \mid \begin{array}{c} \text{“true”} \\ \text{value } = a_t \end{array} \right)}{\text{prob}(S_i = h_{v_i})} \geq V_{-i}(v_{-i}), \tag{2}$$

for all $v_{-i} \in \mathcal{L}^{N-1}$. As V_{-i} is a probability mass function, we already have $\sum_{v_{-i} \in \mathcal{L}^{N-1}} V_{-i}(v_{-i}) = 1$, so we must have that inequality 2 is actually an equality and $T_i(v_{-i}) = 0$ for all $v_{-i} \in \mathcal{L}^{N-1}$. Then the evaluation given by equation 1 is exactly V_{-i}[11]. □

In the remainder of this section we develop the tools allowing us to eventually prove Theorem 1.

Lemma 1. *Suppose f is reasonable, and let $\mathcal{USR}_i$ be a voter. Take $v, v' \in \mathcal{L}$ and a probability mass function V_{-i} for the votes of other users. Then*

$$E[G(v, V_{-i}) - G(v', V_{-i})] = \sum_{\substack{x \in \mathcal{L}^{N-1} \\ \textit{position of } f(v,x) \\ \textit{in } v \textit{ differs from} \\ \textit{position of } f(v',x) \textit{ in } v'}} V_{-i}(x)\left(G(v,x) - G(v',x)\right).$$

[11]In future work, one might consider further to what degree there exist more application-realistic noise models that satisfy these conditions.

Proof. We compute

$$E[G(v,V_{-i})-G(v',V_{-i})]=\sum_{x\in\mathcal{L}^{N-1}} V_{-i}(x)\left(G(v,x)-G(v',x)\right)$$

Consider some fixed value $x\in\mathcal{L}^{N-1}$, and take $j\in\mathbb{N}$. Suppose that $f(v,x)$ is ranked in the jth place of v and that $f(v',x)$ is ranked in the jth place of v'. Then, by applying the inequality of Definition 6 in both directions, $G(v,x)=G(v',x)$. Hence all terms corresponding to x for which v and v' agree on the position of the winner cancel, and we are left with

$$E[G(v,V_{-i})-G(v',V_{-i})]=\sum_{\substack{x\in\mathcal{L}^{N-1}\\ \text{the position of } f(v,x)\\ \text{in } v \text{ differs from the}\\ \text{position of } f(v',x) \text{ in } v'}} V_{-i}(x)\left(G(v,x)-G(v',x)\right).$$

□

In the remaining lemmas, we see that certain structures for what outcomes are given by a social choice function under votes that are modified in specific ways imply the existence of Schelling-dishonest evaluations.

Lemma 2. *Consider $n=\#A\geq 3$. Suppose $f:\mathcal{L}^N\to A$ is such that for all $w\in A$ and for all voters $\mathcal{USR}_i$ there exists some choice of votes $v_{-i,const=w}\in\mathcal{L}^{N-1}$ such that for all $v_i\in\mathcal{L}$, the resulting social choice is $f(v_i,v_{-i,const=w})=w$. Consider two votes by $\mathcal{USR}_i$: v_i and v_i' that differ only by a single transposition of adjacent elements: $v_i=[...]cd[...]$, $v_i'=[...]dc[...]$. Suppose that there exists a set of votes $v_{-i,var}$ for the other participants such that $f(v_i,v_{-i,var})=a\neq f(v_i',v_{-i,var})=b$. Then, if*

- *$a=d$ or*
- *$b=c$,*

there exists a Schelling-dishonest evaluation where all votes available to $\mathcal{USR}_i$ are Schelling-dishonest.

Proof. Denote by n the number of alternatives and j the position of the alternative d in v_i. Define $\epsilon=\frac{1}{6\left(\frac{n(n-1)}{2}+j-\frac{1}{2}\right)}$ and $z=2\epsilon=\frac{1}{3\left(\frac{n(n-1)}{2}+j-\frac{1}{2}\right)}$.

Consider the case where $a=d$ (with either $b=c$ or $b\neq c$ possible). Then consider an evaluation V_{-i} composed of:

- $v_{-i,var}$ with probability z.
- $v_{-i,const=w}$, corresponding to w the rth choice of v_i, with constant probability $2(n-r)(z+\epsilon)$ for $r>j$.
- $v_{-i,const=w}$, corresponding to w the rth choice of v_i, with constant probability $2(n-r+1)(z+\epsilon)$ for $r<j-1$.

- $v_{-i,const=c}$ with constant probability $2(n-j+1)(z+\epsilon)+\epsilon$.
- $v_{-i,const=d}$ with constant probability $2(n-j+1)(z+\epsilon)$.

Then note that

$$2(z+\epsilon)\left(\sum_{r=j+1}^{n}(n-r)+\sum_{r=1}^{j-2}(n-r+1)+2(n-j+1)\right)+\epsilon+z$$

$$=2(z+\epsilon)\left(\frac{n(n-1)}{2}+j-\frac{1}{2}\right)=1,$$

namely the probabilities of the different cases add to 1. (Furthermore, as these probabilities are each clearly positive, they are all in $(0,1)$.)

However, as $z=2\epsilon>\epsilon$, we note in each case that the total probability of d winning (between the constant and variable v_{-i}) is greater than the total probability of c winning for v_i, and the total probability of c winning is greater than the total probability of d winning for the v_i' (whether $b=c$ or not). Hence, neither of these two votes is Schelling-honest. However, as all other pairs of alternatives differ in their probability of winning by at least $2(z+\epsilon)$, and at most a probability of z can change between alternatives in different votes, no other vote can be Schelling-honest. See Figure 3 for an illustration of this argument.

For the case $b=c$, we employ the same argument using an evaluation with the same probabilities as above only where the probabilities of the constant $v_{-i,const=w}$ for $w=c$ and $w=d$ are reversed. Namely: the choice d with constant probability $2(n-j+1)(z+\epsilon)+\epsilon$ and the choice c with constant probability $2(n-j+1)(z+\epsilon)$.

$\square$

Lemma 3. *Consider $n=\#A\geq 3$. Suppose $f:\mathcal{L}^N\to A$ is neutral and G is strictly reasonable. Further suppose that for some $w\in A$ and for all voters $\mathcal{USR}_i$ there exists some choice of votes $v_{-i,const=w}\in\mathcal{L}^{N-1}$ such that for all $v_i\in\mathcal{L}$, the resulting social choice is $f(v_i,v_{-i,const=w})=w$. Consider two votes by $\mathcal{USR}_i$: v_i and v_i' that differ only by a single transposition of adjacent elements: $v_i=[...]cd[...]$, $v_i'=[...]dc[...]$. Suppose that there exists a set of votes $v_{-i,var}$ for the other participants such that $f(v_i,v_{-i,var})=a\neq f(v_i',v_{-i,var})=b$. Then, if*

- *$c\neq a,b$, $d\neq a,b$ or*
- *$d=b$, $c\neq a$, and $a>_{v_i}d$,*

there exists a Schelling-dishonest evaluation where $\mathcal{USR}_i$ can make both a Schelling-honest and a Schelling-dishonest vote, with the highest payoff available to her coming from a Schelling-dishonest vote.

Proof. Suppose we are in one of the cases:

- $c\neq a,b$, $d\neq a,b$, $a>_{v_i}b$ or

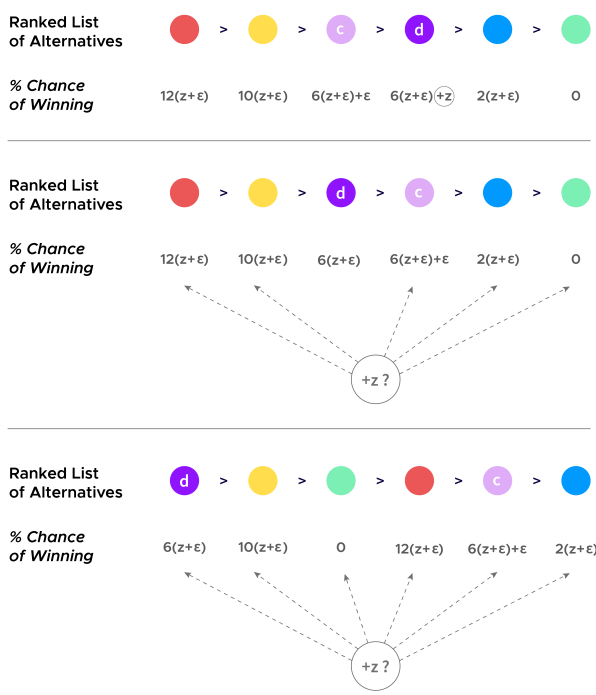

Figure 3: This figure represents the winning chances of each alternative for different choices of $\mathcal{USR}_i$'s vote when she has an evaluation constructed as in Lemma 2. If $\mathcal{USR}_i$ provides a vote as in the first line, then the alternative d wins whenever the other voters vote $v_{-i,var}$, contributing a chance of z to d's overall winning chances. Hence, as $z > \epsilon$, d is more likely to win than c and $\mathcal{USR}_i$'s vote is Schelling-dishonest. However, if $\mathcal{USR}_i$ alters this vote by transposing c and d as in the second line, d *does not* win when the other voters vote $v_{-i,var}$, so this extra chance of z goes to some other option and c is more likely to win than d. If $\mathcal{USR}_i$ attempts other votes, such as the vote in the third line, any alternative could à priori win when the other voters vote $v_{-i,var}$. However, as this only implies a difference of z in the chances of that alternative to win and all options other than c and d differ by at least $2(z + \epsilon) > z$ in their chances to win, all votes must be Schelling-dishonest.

- $d = b$, $c \neq a$, $a >_{v_i} b$.

As f is neutral we can permute $v_{-i,const=w}$ interchanging w with other alternatives; hence we know that there exists some choice of the votes of other participants so that any given alternative wins regardless of the vote of $\mathcal{USR}_i$. Namely, we can take w to be any alternative in A without loss of generality and find a corresponding $v_{-i,const=w}$.

Take σ as the transposition that swaps c and d. As $\sigma(v_i) = v_i'$ and $\sigma(v_i') = v_i$, by the symmetry assumptions in Definition 6, we have $G(v_i, \sigma(v_{-i,const=d})) = G(v_i', v_{-i,const=d})$ and $G(v_i, v_{-i,const=d}) = G(v_i', \sigma(v_{-i,const=d}))$.

By strict reasonableness, as v_i' ranks d higher than v_i does, $G(v_i', v_{-i,const=d}) - G(v_i, v_{-i,const=d}) > 0$. Also, as we have assumed $a >_{v_i} b$ in these cases, by strict reasonableness $G(v_i, v_{-i,var}) - G(v_i', v_{-i,var}) > 0$.

Then, denote by n the number of alternatives and j the position of the alternative d in v_i. Define

$$m = \max\left\{1, \frac{G(v_i', v_{-i,const=d}) - G(v_i, v_{-i,const=d})}{G(v_i, v_{-i,var}) - G(v_i', v_{-i,var})}\right\},$$

$$\epsilon = \frac{1}{(2m+1)(n(n-1)+2j-1)},$$

and

$$z = 2\epsilon m.$$

Note, in particular, $z > \epsilon$ and

$$z > \epsilon \frac{G(v_i', v_{-i,const=d}) - G(v_i, v_{-i,const=d})}{G(v_i, v_{-i,var}) - G(v_i', v_{-i,var})}.$$

Then consider an evaluation composed V_{-i} of:

- $v_{-i,var}$ with probability z.
- $v_{-i,const=w}$, corresponding to w the rth choice of v_i, with constant probability $2(n-r)(z+\epsilon)$ for $r > j$.
- $v_{-i,const=w}$, corresponding to w the rth choice of v_i, with constant probability $2(n-r+1)(z+\epsilon)$ for $r < j-1$.
- $\sigma(v_{-i,const=d})$ with constant probability $2(n-j+1)(z+\epsilon)$.
- $v_{-i,const=d}$ with constant probability $2(n-j+1)(z+\epsilon)+\epsilon$.

Then note that

$$2(z+\epsilon)\left(\sum_{r=j+1}^{n}(n-r) + \sum_{r=1}^{j-2}(n-r+1) + 2(n-j+1)\right) + \epsilon + z$$

$$= 2(z+\epsilon)\left(\frac{n(n-1)}{2} + j - \frac{1}{2}\right) = 1,$$

by our choices of ϵ and z. Namely the probabilities of the different cases add to 1. (Furthermore, as these probabilities are each clearly positive, they are all in $(0,1)$.)

As $\epsilon > 0$, the vote v_i' is Schelling-honest by construction. However, using Lemma 1 and noting that only the $j-1$ and jth places differ between v_i and v_i', we compute

$$E[G(v_i, V_{-i}) - G(v_i', V_{-i})]$$
$$= z\left(G(v_i, v_{-i,var}) - G(v_i', v_{-i,var})\right) + \epsilon\left(G(v_i, v_{-i,const=d}) - G(v_i', v_{-i,const=d})\right) > 0,$$

by our choices of z and ϵ.

Hence, v_i has a higher expected payoff than v_i', but d has a greater total probability of winning than c when $\mathcal{USR}_i$ votes v_i. Thus, v_i is Schelling-dishonest. Moreover, as all other pairs of alternatives differ in their probability of winning by at least $2(z+\epsilon)$, and at most a probability of z can change between alternatives in different votes, no other vote can be Schelling-honest. So the vote with the highest payoff must be Schelling-dishonest.

The case where $c \neq a, b$, $d \neq a, b$, $a <_{v_i} b$ is similar where c has constant probability $2(n-r+1)(z+\epsilon)+\epsilon$, d has constant probability $2(n-r+1)(z+\epsilon)$, we take $m = \frac{G(v_i, v_{-i,const=c}) - G(v_i', v_{-i,const=c})}{G(v_i', v_{-i,var}) - G(v_i, v_{-i,var})}$, and v_i' is the Schelling-dishonest but higher payoff vote.

$\square$

Lemma 4. *Consider $n = \#A \geq 3$. Suppose $f : \mathcal{L}^N \to A$ is neutral and G is strictly reasonable. Further suppose that for some $w \in A$ and for all voters $\mathcal{USR}_i$ there exists some choice of votes $v_{-i,const} \in \mathcal{L}^{N-1}$ such that for all $v_i \in \mathcal{L}$, the resulting social choice is $f(v_i, v_{-i,const}) = w$. Consider two votes by $\mathcal{USR}_i$: v_i and v_i' that differ only by a single transposition of adjacent elements: $v_i = [...]cd[...]$, $v_i' = [...]dc[...]$. Suppose that there exists a set of votes $v_{-i,var}$ for the other participants such that $f(v_i, v_{-i,var}) = a \neq f(v_i', v_{-i,var}) = d$. Then, if $c \neq a$, and $a <_{v_i} d$, there exists a Schelling-dishonest evaluation.*

Proof. Denote by n the number of alternatives and j the position of the alternative d in v_i. We consider two cases. First suppose that there exists some $v_i'' \in \mathcal{L}$ such that $f(v_i'', v_{-i,var}) = h$ is in the jth position of v_i''. Then v_i'' is of the form $[...]gh[...]$, with the alternative h being in the jth position. We consider the vote given by transposing g and h, $v_i''' : [...]hg[...]$. Then if $f(v_i''', v_{-i,var}) \neq h$, there must exist an evaluation where there are no possible Schelling-honest votes by Lemma 2. Hence, we can assume that $f(v_i''', v_{-i,var}) = h$.

Take σ to be the permutation of the alternatives such that $\sigma(v_i'') = v_i'$. In particular, $\sigma(h) = c$ and $\sigma(g) = d$. Denote the kth element of a vote v by $v(k)$. Then, thinking of σ as a function $\sigma : A \to A$, we have that for $k \neq j-1, j$,

$$\sigma(v_i'''(k)) = \sigma(v_i''(k)) = v_i'(k) = v_i(k).$$

Hence, the same permutation σ also gives $\sigma(v_i''') = v_i$.

Then by neutrality, $f(\sigma(v_i''), \sigma(v_{-i,var})) = \sigma(h) = c$ and $f(\sigma(v_i'''), \sigma(v_{-i,var})) = \sigma(h) = c$. We define $v_{-i,var2} = \sigma(v_{-i,var})$. Now we compute

$$G(v_i', v_{-i,var2}) = G(\sigma(v_i''), \sigma(v_{-i,var})) = G(v_i'', v_{-i,var}) > G(v_i, v_{-i,var}),$$

where we use the symmetry properties of reasonableness in Definition 6 for the second equality, and we use strict reasonableness and the fact that $f(v_i'', v_{-i,var}) = v_i''(j)$, whereas $v_i(j) = d >_{v_i} a = f(v_i, v_{-i,var})$ for the inequality. Similarly, we see

$$G(v_i, v_{-i,var2}) = G(\sigma(v_i'''), \sigma(v_{-i,var})) = G(v_i''', v_{-i,var}) = G(v_i', v_{-i,var}),$$

where we again use both properties of reasonableness, this time with the fact that $f(v_i''', v_{-i,var}) = v_i'''(j-1)$ and $f(v_i', v_{-i,var}) = v_i'(j-1)$.

We now follow an argument similar to that of Lemma 3. By neutrality, the existence of some $w \in A$ such that exists $v_{-i,const=w}$ such that for all $v_i \in \mathcal{L}$, $f(v_i, v_{-i,const}) = w$ implies that for all $w \in A$ there is some corresponding $v_{-i,const=w}$ such that this holds. Take σ_2 to be the permutation that transposes c and d. Furthermore, as $\sigma_2(v_i) = v_i'$ and $\sigma_2(v_i') = v_i$, by the symmetry assumptions in Definition 6, we have $G(v_i, \sigma_2(v_{-i,const=c})) = G(v_i', v_{-i,const=c})$ and $G(v_i, v_{-i,const=c}) = G(v_i', \sigma_2(v_{-i,const=c}))$.

By strict reasonableness, as v_i ranks c higher than v_i' does, $G(v_i, v_{-i,const=c}) - G(v_i', v_{-i,const=c}) > 0$.

Define

$$m = 2\max\left\{1, \frac{G(v_i, v_{-i,const=c}) - G(v_i', v_{-i,const=c})}{G(v_i', v_{-i,var2}) - G(v_i, v_{-i,var})}\right\},$$

$$\epsilon = \frac{1}{(2m+1)(n(n-1)+2j-1)},$$

and

$$z = 2\epsilon m.$$

Note, in particular, $z > 2\epsilon$ and

$$z > 2\epsilon \frac{G(v_i, v_{-i,const=c}) - G(v_i', v_{-i,const=c})}{G(v_i', v_{-i,var2}) - G(v_i, v_{-i,var})}.$$

Then consider an evaluation V_{-i} composed of:

- $v_{-i,var}$ with probability $\frac{z}{2}$.
- $v_{-i,var2}$ with probability $\frac{z}{2}$.
- $v_{-i,const=w}$, corresponding to w the rth choice of v_i, with constant probability $2(n-r)(z+\epsilon)$ for $r > j$.
- $v_{-i,const=w}$, corresponding to w the rth choice of v_i, with constant probability $2(n-r+1)(z+\epsilon)$ for $r < j-1$.
- $v_{-i,const=c}$ with constant probability $2(n-j+1)(z+\epsilon)+\epsilon$.
- $\sigma_2(v_{-i,const=c})$ with constant probability $2(n-j+1)(z+\epsilon)$.

Then note that

$$2(z+\epsilon)\left(\sum_{r=j+1}^{n}(n-r)+\sum_{r=1}^{j-2}(n-r+1)+2(n-j+1)\right)+\epsilon+z$$

$$=2(z+\epsilon)\left(\frac{n(n-1)}{2}+j-\frac{1}{2}\right)=1,$$

Namely the probabilities of the different cases add to 1. (Furthermore, as these probabilities are each clearly positive, they are all in $(0,1)$.)

We use Lemma 1 and note that only the $j-1$ and jth places differ between v_i and v_i'. Furthermore, we use the equality and inequalities found above on payoffs for different votes to compute

$$E[G(v_i',V_{-i})-G(v_i,V_{-i})]$$

$$=\frac{z}{2}\left(G(v_i',v_{-i,var2})-G(v_i,v_{-i,var})\right)+\epsilon\left(G(v_i',v_{-i,const=c})-G(v_i,v_{-i,const=c})\right).$$

Then $E[G(v_i',V_{-i})-G(v_i,V_{-i})]>0$ by our choices of z and ϵ.

Hence, v_i' has a higher expected payoff than v_i, but c has a greater total probability of winning than d when $\mathcal{USR}_i$ votes v_i'. Thus, v_i' is Schelling-dishonest. Moreover, as all other pairs of alternatives differ in their probability of winning by at least $2(z+\epsilon)$, and at most a probability of z can change between alternatives in different votes, no other vote can be Schelling-honest. So the vote with the highest payoff must be Schelling-dishonest in this case.

Now suppose that there does not exist any $v_i''\in\mathcal{L}$ such that $f(v_i'',v_{-i,var})$ is in the jth position of v_i''. As $a<_{v_i}d$, v_i has the form $[...]cd[...a...]$. Consider the transposition of v_i that moves a up one place. Then if the resulting vote does not go to a we have a pair of votes that correspond to (the first case of) Lemma 2, implying the existence of a dishonest evaluation. However, as long as this process does not produce a Schelling-dishonest evaluation, it can be continued until eventually we transpose a with d to produce a vote of the form $v_i''':[...]cad[...]$. As a is then in the jth position of this vote, by assumption, $f(v_i''',v_{-i,var})\neq a$. Then, finally, we must have a Schelling-dishonest evaluation by applying Lemma 2 to this last transposition.

$\square$

We now have the tools to prove the above result.

Proof of Theorem 1. As we assume that f has AV, it is also Pareto efficient and non-dictatorial. Hence by Theorem A of [25], we know that f is not strongly monotonic[12]. Here we are essentially applying a version of the Gibbard-Satterthwaite Theorem.

[12]This strong sense of monotonicity used in [25], which includes properties related to independence of irrelevant alternatives and differs from definitions of monotonicity sometimes considered. See Definition 2.11 of [6] where this property is also called Maskin monotonicity.

Namely, there exist $l_1, \ldots, l_N$ such that $f(l_1, \ldots, l_N) = a$, $l'_1, \ldots, l'_N$ such that for all b each l'_i ranks $a > b$ if l_i does, but $f(l'_1, \ldots, l'_N) \neq a$.

Then we have a sequence

$$a = f(l_1, \ldots l_N), f(l_1, \ldots l_{N-1}, l'_N), \ldots, f(l_1, l'_2 \ldots, l'_N), f(l'_1, \ldots, l'_N) \neq a$$

formed by replacing the l_j by the l'_j one at a time. Take i to the smallest value at which this replacement results in a choice choice other than a.

Note that as $a >_{l'_i} b$ for all b such that $a >_{l_i} b$, there exists a series of elements in $\mathcal{L}$: $t_1, \ldots t_r$ from l_i to l'_i each differing by a transposition of adjacent elements σ_j such that none of the σ_j move a down.

Similarly to above, as

$$f(l_1, \ldots, l_{i-1}, l_i, l'_{i+1}, \ldots, l'_N) = a$$

and

$$f(l_1, \ldots, l_{i-1}, l'_i, l'_{i+1}, \ldots, l'_N) \neq a,$$

we can think of $\mathcal{USR}_i$'s vote being changed one transposition at a time and take k as the smallest value such that

$$f(l_1, \ldots, l_{i-1}, t_k, l'_{i+1}, \ldots, l'_N) \neq a.$$

Hence, we have $l_1, \ldots, l_{i-1}, l'_{i+1}, \ldots, l'_N$, t_{k-1}, and t_k,

- where t_{k-1} and t_k differ by a transposition σ_k of adjacent elements that either moves a up or fixes a, and changes the result from a to b
- $f(l_1, \ldots, l_{i-1}, t_{k-1}, l'_{i+1}, \ldots, l'_N) = a$
- $f(l_1, \ldots, l_{i-1}, t_k, l'_{i+1}, \ldots, l'_N) = b \neq a$

Denote the two elements being transposed by σ_k as c and d, so that t_{k-1} has the form [...]cd[...] and t_k has the form [...]dc[...] (where as above terms in brackets are fixed between the two orders). Then we have several cases:

- $a = d$ OR $b = c$. Here we construct an evaluation where there is no Schelling-honest vote using Lemma 2,
- $c \neq a, b$, $d \neq a, b$, OR $b = d$, $a \neq c$, $a >_{v_i} b$. Here we construct an evaluation where there is a Schelling-honest vote, but the highest payoff comes from a Schelling-dishonest vote using Lemma 3.
- $b = d$, $a \neq c$, $a <_{v_i} d$. Here we construct a Schelling-dishonest evaluation (in which a Schelling-honest vote may or may not exist) using Lemma 4.

For the applications of the lemmas in each case, we use our assumed absence of veto to establish the required existence of constant evaluations.

Hence, in each case there exists some Schelling-dishonest evaluation. (Note that any cases where $a = c$ are already excluded as we know that σ_k does not move a down.)

$\square$

8 Situations where incentive compatible systems are possible

In this section, we see that there are counterexamples when one attempts to relax some of the assumptions of Theorem 1. Towards this end, we will first prove the following proposition.

Proposition 2. *Suppose we have a monotonic voting system*[13]*. Let V_{-i} be any evaluation for the voter $\mathcal{USR}_i$. Let v be a vote that maximizes the probability*

$$prob(\text{first choice of } v \text{ wins} : \mathcal{USR}_i \text{ votes } v)$$

over the finite set $v \in \mathcal{L}$, where this probability is defined in terms of V_{-i}. Take $a_1 \in A$ to be the alternative ranked highest by v. Then,

$$prob(a_1 \text{ wins} : \mathcal{USR}_i \text{ votes } v) \geq prob(a_j \text{ wins} : \mathcal{USR}_i \text{ votes } v) \tag{3}$$

for all a_j. In particular, for any evaluation, $\mathcal{USR}_i$ can cast a vote v such that the highest ranked choice is the alternative with the highest probability of winning.

Proof. We write $v = a_1 > a_2 > \ldots > a_n$. Suppose that inequality 3 does not hold for some a_j. Then, consider the alternative vote $v' = a_j > a_1 > a_2 > \ldots > a_{j-1} > a_{j+1} > \ldots > a_n$. For any $v_{-i} \in \mathcal{L}^{N-1}$, if $f(v, v_{-i}) = a_j$, then we also have $f(v', v_{-i}) = a_j$ by monotonicity. Then, over the entire evaluation V_{-i}, we have

$$\text{prob}(a_j \text{ wins} : \mathcal{USR} \text{ votes } v') \geq \text{prob}(a_j \text{ wins} : \mathcal{USR} \text{ votes } v).$$

So using the assumption that inequality 3 does not hold for a_j gives

$$\text{prob}(a_j \text{ wins} : \mathcal{USR} \text{ votes } v') > \text{prob}(a_1 \text{ wins} : \mathcal{USR} \text{ votes } v),$$

which contradicts the maximality assumption on v.

□

Now, we consider an example where the payoff function G is only assumed to be reasonable rather than strictly reasonable.

Example 1. *Suppose participants provide a vote $v \in \mathcal{L}$ and place a deposit D. In this example, we take the winning outcome to be decided by Plurality voting and payoffs given by the the "all or nothing" payoff system of Definition 7.*

Then, for a given vote $v = a_1 > a_2 > \ldots > a_n$, the user's expected return is

$$-D \cdot (1 - prob(a_1 \text{ wins} | \mathcal{USR}_i \text{ votes } v)).$$

[13] Here, and in the remainder of this work by monotonicity we mean the "standard" version of monotonicity, see Definition 2.10 of [6], in contrast to the Maskin or strong monotonicity used in the proof of Theorem 1.

Note that this payoff is optimized by maximizing $prob(a_1\ wins|\mathcal{USR}_i\ votes\ v)$. Hence, by Proposition 2 and the monotonicity of Plurality, one can take a vote v that optimizes the payoff such that

$$prob(a_1\ wins : \mathcal{USR}_i\ votes\ v) \geq prob(a_j\ wins : \mathcal{USR}_i\ votes\ v)$$

for all a_j, where a_1 is the first choice of v.

Then, as according to the payoff scheme of Definition 7 only the first choice vote of a user has any effect on either the winner or the payout, $\mathcal{USR}_i$ does no worse by choosing $a_2, a_3, \ldots$ such that

$$prob(a_j\ wins : \mathcal{USR}\ votes\ v) \geq prob(a_{j+1}\ wins : \mathcal{USR}\ votes\ v)$$

for all j. Hence, for any evaluation, the highest expected payoff can be achieved with a Schelling-honest vote.

Now we consider an example where the social choice function is Pareto efficient and non-dictatorial, but fails neutrality and absence of vetos. This example can be performed with any (strictly reasonable) payoff function.

Example 2. *Consider the following voting system when there are three alternatives $\{a, b, c\}$, and participants submit votes in $\mathcal{L}$. If the first choices of voters are unanimous, their unanimous first choice is selected. Otherwise, the outcome is a. This system is clearly Pareto efficient and non-dictatorial, but it does not satisfy neutrality or absence of veto. Then, under a strictly reasonable payoff system, voting $a > b > c$ (or voting $a > c > b$ which by reasonableness has the same payoff) is always a Schelling-honest vote that maximes a user's payoff.*

Note that as Example 2 gives a non-dictatorial game with three alternatives that is incentive compatible in the sense of Section 5, we have that Schelling-honesty cannot be made to fit into the notion of "straightforwardness" as expressed in [15], even though that notion is very general. Indeed, in the context of Schelling games the honest vote of a participant *depends* on the user's beliefs about the votes of others, which is not covered in the framework of straightforwardness.

9 Existence of Schelling-honest votes

In Section 7 we saw that, when certain natural requirements are placed on a Schelling game voting and payoff system, there will inevitably be situations where it is in the economic interest of participants to cast Schelling-dishonest votes. As part of that argument, in Lemma 2, we considered cases where no Schelling-honest vote is available. However, in the proof of Theorem 1, it is not shown that the situation of Lemma 2 is ever necessarily realized. In this section we will consider explicit situations where it is impossible for a voter to cast a Schelling-honest vote in several specific voting systems.

Example 3. *Consider an Instant-runoff voting system that is attempting to decide between three alternatives* $\{a, b, c\}$.

Suppose $\mathcal{USR}_i$ *has two votes (this avoids having situations with ties). Before voting, her evaluation* V_{-i} *is given as follows*

- *There is a* 49% *chance that there will be*
  - 21 *votes for* $a > b > c$
  - 22 *votes for* $b > c > a$
  - 20 *votes for* $c > a > b$
- *There is a* 49% *chance that there will be*
  - 20 *votes for* $a > b > c$
  - 21 *votes for* $b > c > a$
  - 22 *votes for* $c > a > b$
- *There is a* 2% *chance that there will be*
  - 22 *votes for* $a > b > c$
  - 20 *votes for* $b > c > a$
  - 21 *votes for* $c > a > b$

Then, regardless of which of the possibilities in $\mathcal{L}^{N-1}$ *occurs and regardless of how* $\mathcal{USR}_i$ *votes,* a *wins any eventual duels against* b*,* b *wins any eventual duels against* c*, and* c *wins any eventual duels against* a.

If $\mathcal{USR}_i$ *ranks* a *first with her two votes, then there is a* 49% *chance that the runoff will be between* a *and* b *and a* 51% *chance that the runoff will be between* a *and* c*. Hence* a *has a* 49% *chance of winning,* b *a* 0% *chance of winning, and* c *a* 51% *chance of winning.*

If $\mathcal{USR}_i$ *ranks* b *or* c *first with her two votes we have similar results:*

$\mathcal{USR}_i$ \vote outcome	a	b	c
a *first*	49%	0%	51%
b *first*	51%	49%	0%
c *first*	0%	98%	2%

Hence, in any case, $\mathcal{USR}_i$ *does not produce a vote in which the first choice is the alternative with the highest chance of winning. Then, for a user in this situation, the strategy of producing a Schelling-honest vote cannot be incentive compatible because it is impossible to follow. This is true regardless of the system of rewards and punishments applied.*

We already saw in Example 1 that in some systems it is possible to always have a Schelling-honest vote. We might ask if this property is compatible with other properties of voting systems, such as the Condorcet criterion.

Example 4. *Consider the voting system that*

- *Checks to see if there is a Condorcet winner, if so this choice is declared the winner*
- *If there is no Condorcet winner, the Plurality winner is selected.*

One might hope that this would be a reasonable candidate for a system that would always have a Schelling-honest vote considering its relation to the Plurality voting system and Example 1. However, this is not the case. Consider the situation where $\mathcal{USR}_i$ *has three votes and has an evaluation of*

- *Scenario 1: There is a* 34% *chance that*
  - 24 *votes for* $b > c > a$
  - 22 *votes for* $b > a > c$
  - 27 *votes for* $a > b > c$
  - 27 *votes for* $c > a > b$

$\mathcal{USR}_i$ *vote*	$a > b > c$	$a > c > b$	$b > a > c$	$b > c > a$	$c > a > b$	$c > b > a$
outcome	a	a	a	b	b	b

- *Scenario 2: There is a* 33% *chance that one has the appropriate permutation of Scenario 1 so that we have*

$\mathcal{USR}_i$ *vote*	$a > b > c$	$a > c > b$	$b > a > c$	$b > c > a$	$c > a > b$	$c > b > a$
outcome	a	c	a	a	c	c

- *Scenario 3: There is a* 33% *chance that one has the appropriate permutation of Scenario 1 so that we have*

$\mathcal{USR}_i$ *vote*	$a > b > c$	$a > c > b$	$b > a > c$	$b > c > a$	$c > a > b$	$c > b > a$
outcome	c	c	b	b	c	b

This leads to percent chances of winning of

$\mathcal{USR}_i$ *vote outcome*	a	b	c
$a > b > c$	67%	0%	33%
$a > c > b$	34%	0%	66%
$b > a > c$	67%	33%	0%
$b > c > a$	33%	67%	0%
$c > a > b$	0%	34%	66%
$c > b > a$	0%	67%	33%

Here there are votes where the first choice has the highest chance of winning, but there is no vote where all alternatives are ranked in order by chance of winning.

We see that even with a monotonic voting system, even though we saw in Proposition 2 that one can find a vote whose first choice is more likely to win than any other alternative, it is still possible to have situations where there is no Schelling-honest vote.

Example 5. *Suppose that $\mathcal{USR}_i$ controls three votes (again this is assumed to avoid dealing with ties), and her evaluation is as given below. Then, all of the Ranked Pairs, Minmax, Kemeny-Young, and Schulze systems give the same results in each scenario for each choice of $\mathcal{USR}_i$'s vote; these results are shown in the tables below. These systems are all Condorcet and monotonic, (see for example [32] for a discussion of why Ranked Pairs possesses these properties); however, we see that under these assumptions, none of the possible votes are Schelling-honest.*

- *Scenario 1: There is a 10% chance that there will be*
  - 14 *votes for* $a > b > c$
  - 3 *votes for* $a > c > b$
  - 15 *votes for* $b > c > a$
  - 16 *votes for* $c > a > b$

$\mathcal{USR}_i$ *vote*	$a > b > c$	$a > c > b$	$b > a > c$	$b > c > a$	$c > a > b$	$c > b > a$
outcome	a	c	a	c	c	c

- *Scenario 2: There is a 32% chance that there will be*
  - 12 *votes for* $a > b > c$
  - 3 *votes for* $a > c > b$
  - 17 *votes for* $b > c > a$
  - 16 *votes for* $c > a > b$

$\mathcal{USR}_i$ *vote*	$a > b > c$	$a > c > b$	$b > a > c$	$b > c > a$	$c > a > b$	$c > b > a$
outcome	c	c	b	b	c	c

- *Scenario 3: There is a 17% chance that one is in a scenario where b wins regardless of the vote of $\mathcal{USR}_i$, such as*
  - 48 *votes for* $b > a > c$
- *Scenario 4: There is a 26% chance that there will be*
  - 13 *votes for* $a > b > c$
  - 6 *votes for* $a > c > b$
  - 18 *votes for* $b > c > a$
  - 11 *votes for* $c > a > b$

$\mathcal{USR}_i$ *vote*	$a > b > c$	$a > c > b$	$b > a > c$	$b > c > a$	$c > a > b$	$c > b > a$
outcome	a	a	a	b	c	b

- *Scenario 5: There is a 15% chance that one is in a scenario where a wins regardless of the vote of $\mathcal{USR}_i$, such as*
  - 48 *votes for* $a > b > c$

Then, we summarize which scenarios and which user votes give each outcome:

$\mathcal{USR}_i$ \vote outcome	*a*	*b*	*c*
$a > b > c$	*1,4,5*	*3*	*2*
$a > c > b$	*4,5*	*3*	*1,2*
$b > a > c$	*1,4,5*	*2,3*	
$b > c > a$	*5*	*2,3,4*	*1*
$c > a > b$	*5*	*3*	*1,2,4*
$c > b > a$	*5*	*3,4*	*1,2*

In percent chance of winning:

$\mathcal{USR}_i$ \vote outcome	*a*	*b*	*c*
$a > b > c$	51%	17%	32%
$a > c > b$	41%	17%	42%
$b > a > c$	51%	49%	0%
$b > c > a$	15%	75%	10%
$c > a > b$	15%	17%	68%
$c > b > a$	15%	43%	42%

Remark 1. *Consider a monotonic voting system and suppose one begins with $a > b > c$ as the vote guaranteed by Proposition 2 that places the highest ranked choice first. Then one could attempt to obtain a Schelling-honest vote by permuting two alternatives at a time: e.g. if $a > b > c$ yields prob(c wins) $>$ prob(b wins) then try $a > c > b$. By monotonicity, it is still the case that prob(c wins) $>$ prob(b wins), but it is possible that prob(c wins) $>$ prob(a wins). If this is the case one can try $c > a > b$. By monotonicity, it is then the case that prob(c wins) $>$ prob(a wins), but it is possible that prob(b wins) $>$ prob(a wins), etc. Continuing like this any evaluation that $\mathcal{USR}_i$ might have that results in no possible Schelling-honest vote in a monotonic voting system (at least with three alternatives) must have a particular form. Then, we see in Example 5 that that form can be realized in the Ranked Pairs, Minmax, Kemeny-Young, and Schulze systems.*

Given these examples, we ask:

Question 1. *Does there exist any Condorcet voting system in which it is always possible to provide a Schelling-honest vote, i.e. a ranking in order by the probability that each alternative wins, once your vote is taken into account and for an arbitrary evaluation of the distribution of the other votes?*

10 Conclusion and future work

In this work we have begun to look at a version of social choice theory in the context where participants in an election are motivated by the economic incentives of a Schelling game rather than by their own candidate preferences.

We have provided definitions for what "honesty" can mean in this framework, and we have observed that many of the subtleties of traditional social choice theory regarding tactical voting have analogs here.

In particular, we saw an impossibility result in Theorem 1 that shows that some of the properties that one would hope an ideal Schelling game based voting system to have are incompatible. Ultimately this result is similar to Arrow's Impossibility Theorem and the Gibbard-Satterthwaite Theorem in that they only necessarily apply to a fairly narrow range of potential voting scenarios. With many voting systems, honest behaviour is incentivized most of the time. Moreover, just as the classic impossibility theorems of social choice theory do not render standard elections unusable for determining collective decisions, Schelling game based voting is still likely to be useful as a tool in applications to blockchain oracles, and perhaps beyond to other crowdsourcing platforms.

Thus, in the absence of ideal voting systems[14], one is left with the task of considering which partial properties one can establish for such systems and the potential tradeoffs between goals that are potentially incompatible (e.g. such as the incompatibility between the Condorcet criteria and escaping the no-show paradox, [21]). We conclude by considering more broadly some the properties that one would want Schelling game voting systems to have.

- We want voters to be incentivized as much as possible to *actually* provide an "honest" list of choices. For example, we would want:
  - Some partial guarantees[15] limiting the failures of incentive compatibility for Schelling-honesty.
  - Clone independence. Note that while alternatives may have identical implications when implemented (on a blockchain platform or otherwise) leading them to be reasonably thought of as clones, as the voters themselves do not have intrinsic preferences the standard definition of clones [32] no longer makes sense. Thus, notions of clone independence are not included in our notion of Schelling-honesty, in contrast to their inclusion in non-manipuability in standard elections. Nonetheless, to the degree that a collection of essentially identical choices reduces the chances of any given choice to win, their presence and how they are treated by a voting system may have implications for parties interested in the outcome of a given vote. More generally, questions arise in this context that touch on the concept of "control", see Chapter 7 of [6].

[14] Future work may consider voting systems such as that considered in [5] where participants submit information other than a ranking of the alternatives in $\mathcal{L}$.

[15] One potential such approach is the following: In some Schelling game based blockchain oracle applications there is a notion of appeal, where payoffs are determined based on an ultimate outcome decided in a later round. In such situations, a voter $\mathcal{USR}_i$ that expected an appeal might belief the outcome of such a process to be independent of her vote. Under this (rather strong) assumption, one can see that a wider range of voting and payoff systems can be seen to incentivize Schelling-honesty in the presence of appeals. Compare Theorem 1 to Proposition 1 of [19].

  - Attack resistance based on the idea that it should be expensive to bribe sufficient numbers of voters to change the outcome of a decision, at least relative to the values that can be gained by successful attacks on a given crowdsourcing platform. Here as voters are assumed to be incentivized purely economically, analyzing the bribes required for them to change their votes is tractable. Also, one can clearly apply work in computational social choice theory that estimates the computational difficulty of finding which sets of voters are sufficient to be bribed, see Section 7 of [6]. Note that the framework of Schelling games opens up new possibilities for game-theoretic attacks [8] that must be considered.

- Then we want the lists voters provide to be converted into a collective outcome in such a way that if voters provide "honest" lists, the collective choice will also be "honest". For this we might want:

  - A voting rule that is a Maximum Likelihood Estimator under some reasonable noise model, following the work of [10]. Indeed, in the ideal where the above considerations manage to incentivize participants to submit observation-honest votes, one is back in the framework of the kind of crowdsourced decision making considered by [10]. Note that while [10] shows that, under a model of there being a single "true" alternative and with i.i.d. noise, the only voting rules that are Maximum Likelihood Estimators are scoring rules, one might nonetheless explore whether we we can find such a voting rule under less rigid, but still realistic and non-trivial, noise models. Alternatively, one can explore whether our voting rules could satisfy other weaker properties in this sense.

- Finally, use on a blockchain platform may impose additional constraints such as:

  - New running time considerations. Already in standard voting systems, there are interesting questions related to the running time of tabulating votes, with implications on the viability of different voting systems, see for example discussions related to "winner problem" complexity for various voting system in [6]. However, in the domain of blockchains, as on-chain computation is performed by a large number of verifiers and consequently has elevated cost, it is often advantageous to minimized such on-chain computation by using processes that allow for parties to submit an answer, computed off-chain, that can be efficiently *verified* on-chain, in the style of [31]. Then one asks for which voting rules can vote tabulation be effectively adapted to such a framework.

Then, future work considering to what degree these goals are compatible seems fruitful.

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