Exercise Set 1:
Due: 2/1/22
MATH 4061
Professor Florian Johnne

Problem 1: Implication and Contraposition (10 points)

Suppose A and B are assertions. Please find the truth table of the following two expressions:

$$(1)(\neg A \Rightarrow \neg B)$$

Apply the definition of implies/right arrow and simplify:

$$\neg\neg A \vee \neg B = A \vee \neg B$$

A	B	$A \vee \neg B$
T	T	T
T	F	T
F	T	F
F	F	T

$$(2)(\neg B \Rightarrow \neg A)$$

Apply the definition of implies/right arrow and simplify:

$$\neg\neg B \vee \neg A = B \vee \neg A$$

A	B	$B \vee \neg A$
T	T	T
T	F	F
F	T	T
F	F	T

Compare them with the truth table for the expression $(A \Rightarrow B)$. What can you conclude?

We saw the truth table for $(A \Rightarrow B)$ to be:

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

We see that the truth table for $(A \Rightarrow B)$ is the same as for $\neg B \Rightarrow \neg A$, but is different from $\neg A \Rightarrow \neg B$

2. Relations for inclusion, union, and intersections (10 points)

Suppose $A, B, C \subset M$ are subsets. Please prove the following relations:

$$(1)\ (A \subset C) \wedge (B \subset C) \Leftrightarrow \big((A \cup B) \subset C\big)$$

We must prove the relation both ways. First we prove that

$$\big((A \cup B) \subset C\big) \Rightarrow (A \subset C) \wedge (B \subset C)$$

Start by using the definition of union on the LHS, and defining L as the larger space over which A, B exist. Then

$$(\{x \in L | (x \in A) \vee (x \in B)\}) \subset C$$

Since L is a subset of C, we can leave the expression more simply below as

$$\big((x \in A) \vee (x \in B)\big) \subset C$$

Next we take the definition of the subset:

$$\big((x \in A) \vee (x \in B)\big) \Rightarrow (x \in C)$$

So for $\forall x$, if x is either an element of A or an element of B then it implies x is in C. That is equivalent to saying that A is a subset of C and B is a subset of C, or $(A \subset C \wedge B \subset C)$

So we have shown that $\big((A \cup B) \subset C\big) \Rightarrow (A \subset C) \wedge (B \subset C)$.

Next we prove $(A \subset C) \wedge (B \subset C) \Rightarrow \big((A \cup B) \subset C\big)$:

By definition of subset: $A \subset C \coloneqq \forall x, (x \in A) \Rightarrow (x \in C)$

The RHS states by definition of union that:

$$\{x \in L |\ (x \in A) \vee (x \in B)\}$$

Where L is the space over which A and B exist. The RHS subset tells us that we can also write this as $(x \in A) \vee (x \in B) \Rightarrow (x \in C)$. So if x is in either A or B then it implies x is in C. Since the only two cases are when either $x \in A$ or $x \in B$, these are equivalent. Thus we have shown that:

$$(A \subset C) \wedge (B \subset C) \Rightarrow \big((A \cup B) \subset C\big)$$

Combined with the first part:

$$\big((A \cup B) \subset C\big) \Rightarrow (A \subset C) \wedge (B \subset C)$$

So we can say:

$$\big((A \cup B) \subset C\big) \Leftrightarrow (A \subset C) \wedge (B \subset C)$$

$$(2)\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(Distributive law for unions over intersections).

I prove by showing if $A \cap (B \cup C)$, then $(A \cap B) \cup (A \cap C)$ or

$$x \in A \cap (B \cup C) \Rightarrow x \in (A \cap B) \cup (A \cap C)$$

And if $x \notin (A \cap (B \cup C))$, then $x \notin ((A \cap B) \cup (A \cap C))$
you
Which we showed in class is logically equivalent to the contrapositive, or

$$(A \cap B) \cup (A \cap C) \Rightarrow A \cap (B \cup C)$$

Consider the over the space of M, some value $\forall x$:

Case 1 [LHS is true - $x \in A \cap (B \cup C)$]: x is in the subset A. In this case, x is also either in B or C, because the LHS tells us that it is the intersection between A and the union of B, C.

Case1.1: x is in the set of B as well as the set A. In this case, on the RHS, $A \cap B$ is true, and so the entire RHS is true. $x \in A \cap (B \cup C)$

Case1.2: x is in the set of C as well as the set A. In this case, on the RHS, $A \cap C$ is true, and so the entire RHS is true. $x \in A \cap (B \cup C)$

Case 2 [LHS is false - $x \notin (A \cap (B \cup C))$]: x is either not in the subset A or not in the subset $B \cup C$.

Case2.1: x is not in subset A. In this case, the LHS is false always, and the RHS is false always since intersection $A \cap B$ or $A \cap C$ both require an x to be in the set A as well.

$$x \notin ((A \cap B) \cup (A \cap C))$$

Case2.2: x is not in $B \cup C$ In this case, the LHS is false, and the RHS is false always since intersection $A \cap B$ require an x to be in the set B or intersection $A \cap C$ would require x to be in C.

$$x \notin ((A \cap B) \cup (A \cap C))$$

These cases comprise the $\forall x$ statement and we have shown that whenever the $A \cap (B \cup C)$ would evaluate to true, so would $(A \cap B) \cup (A \cap C)$, and vice versa, so we can say they are equivalent.

$$(3)\ M \setminus (A \cup B) = (M \setminus A) \cap (M \setminus B)$$

We do the same proof structure as above, using contrapositive. First show:

$$x \in M \setminus (A \cup B) \Rightarrow x \in (M \setminus A) \cap (M \setminus B)$$

By dealing with the cases where LHS is true, then show:

$$(M \setminus A) \cap (M \setminus B) \Rightarrow M \setminus (A \cup B)$$

By dealing with the cases where LHS is false, which gives us the contrapositive.

Consider the over the space of M, some value $\forall x$:

Case 1 [LHS evaluates to true - $x \in M \setminus (A \cup B)$]:

- x is in the set M and not in $A \cup B$. Since $A, B \subset M$, any point in $A \cup B$ would also be in M from that fact.
- Then we can say for the right side to be equivalent we would expect $M \setminus A$ to evaluate to true as well as $M \setminus B$ to be true.
    - We know that both will be true since x is not in A, so $M \setminus A$ includes x and x is not in B so $M \setminus B$ also includes x. So: $x \in (M \setminus A) \cap (M \setminus B)$

Case 2 [LHS evaluates to false - $x \notin M \setminus (A \cup B)$]:

- x is in the set A or B.
    - Case 2.1: x is in set A, but not in B. The RHS will evaluate to false since $M \setminus A$ would not contain the point while with $M \setminus B$ it would, and therefore the point is not in the intersection of these two. $x \notin (M \setminus A) \cap (M \setminus B)$
    - Case 2.2: x is in set B, but not in A. The RHS will evaluate to false since $M \setminus B$ would not contain the point while with $M \setminus A$, it would, and therefore the point is not in the intersection of these two . $x \notin (M \setminus A) \cap (M \setminus B)$
    - Case 2.3: x is in set B and in A. The RHS will evaluate to false since $M \setminus B$ would contain x and $M \setminus A$ as well. $x \notin (M \setminus A) \cap (M \setminus B)$

These cases comprise the $\forall x$ statement and we have shown that whenever the $M \setminus (A \cup B)$ would evaluate to true, so would $x \in (M \setminus A) \cap (M \setminus B)$ and vice versa, so we can say they are equivalent.

3. Cartesian Product (12 points)

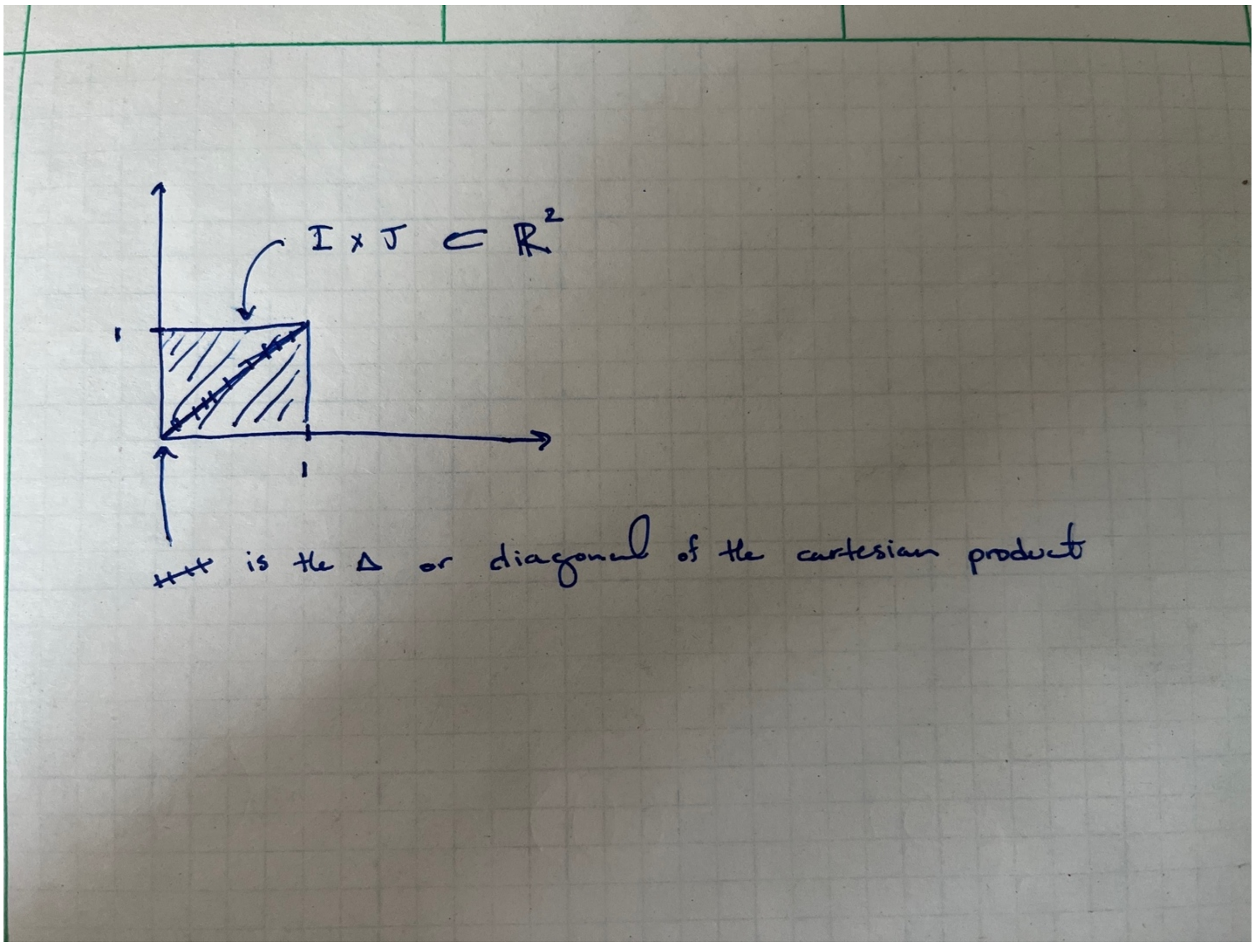
Please give a geometric representation of the following objects:

1. The cartesian product $I \times J$ of the two line segments $I = J = [0,1] \subset \mathbb{R}$
2. The cartesian product of a line $\mathbb{R}$ and a circle $\mathbb{S}^1$

3. The cartesian product of two circles.
   Hint: Donuts are delicious!

1.

$$I \times J = ((x, y)|0 \leq x \leq 1, 0 \leq y \leq 1\}$$



2.

$$\mathbb{R} \times \mathbb{S}^1 = \{(x, (y, z))|x \in \mathbb{R}, (y, z) \in \mathbb{S}^1\}$$

If we imagine x, y, z as cardinal directions, in 3-D, for every x value, the object would be defined by a circle with with coordinates y, z. So it would resemble a cylinder along the x-axis.

Or, if you define the circle by only a radius, then you could also represent as:

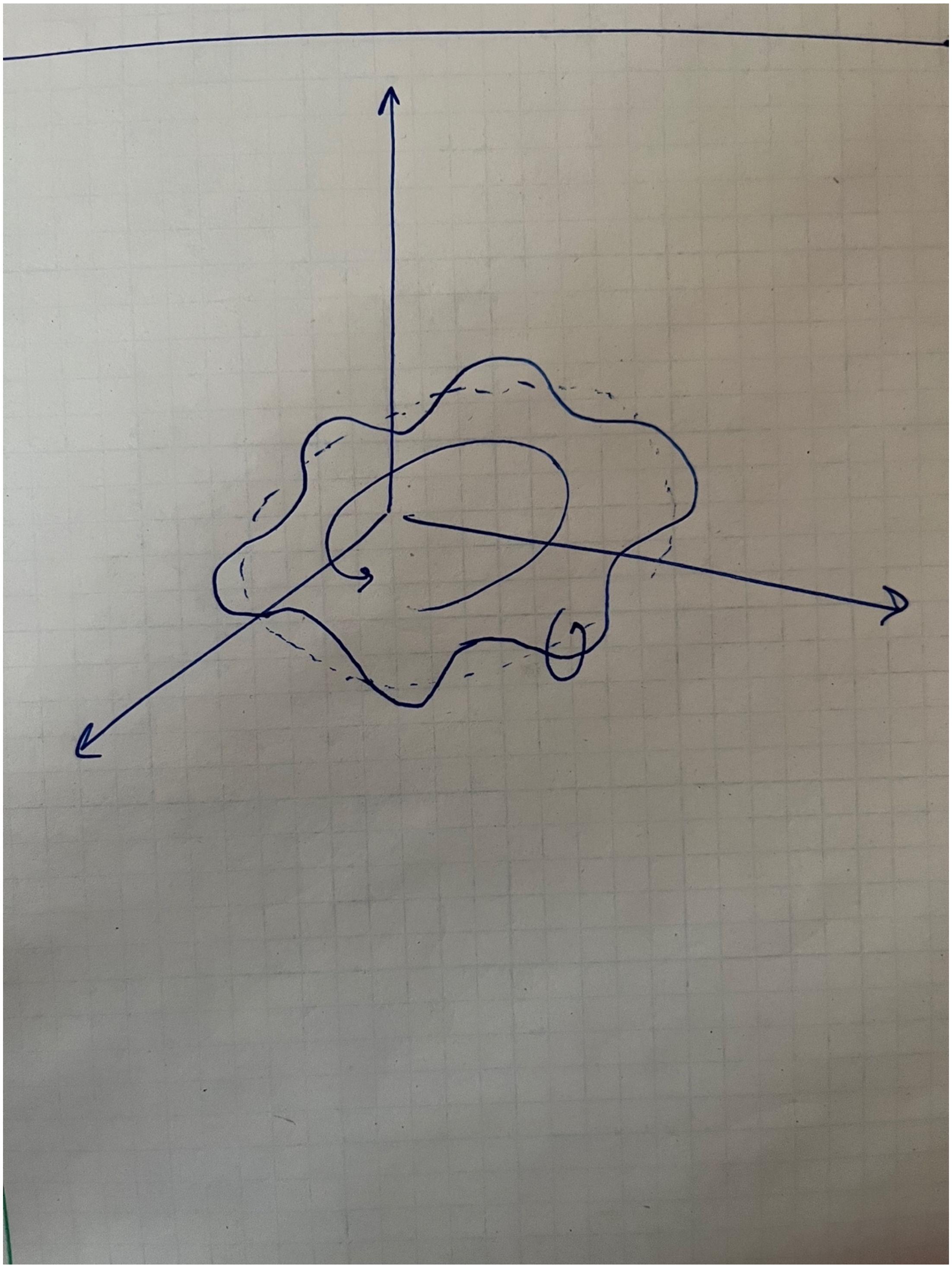
$$\mathbb{R} \times \mathbb{S}^1 = \{(x, r)|x \in \mathbb{R}, (r) \in \mathbb{S}^1\}$$

3.

$$\mathbb{S}^1 \times \mathbb{S}^2 = \{(x, y)|x \in \mathbb{S}^1, y \in \mathbb{S}^2\}$$

So x values make a circle shape ($\mathbb{S}^1$), and for each value of x, there is a corresponding second circle ($\mathbb{S}^2$). So the cartesian product is a torus/donut.

The subset Δ can be thought of as a line tracing out both the larger and smaller circles. For each point on the first circle, there is one corresponding point on the second circle. It can be thought of as like a helix. With one axis being the first circle, and the second axis being the second circle.



The set $\Delta = \{(x_1, x_2) \in X \times X \mid x_1 = x_2\}$ is called the diagonal of the cartesian product. Visualize the diagonal in the first and third example.

4. Properties of Multiplication in a field (12 points)

Please prove the following proposition:

Proposition 1.

Suppose $(F, +, \cdot)$ is a field. Then we have:

1. If $x \in F \setminus \{0\}$, and $x \cdot y = x \cdot z$, then $y = z$
2. If $x \in F \setminus \{0\}$, and $x \cdot y = x$, then $y = 1$
3. If $x \in F \setminus \{0\}$, and $x \cdot y = 1$, then $y = x^{-1}$
4. If $x \in F \setminus \{0\}$, and $(x^{-1})^{-1} = x$

(1.)

If $x \in F \setminus \{0\}$, and $x \cdot y = x \cdot z$, then $y = z$

$$y = 1 \cdot y \ \ (M4)$$
$$y = \left(x \cdot \frac{1}{x}\right) \cdot y \ \ (M5)$$
$$y = (x \cdot y) \cdot \frac{1}{x} \ \ (M3)$$
$$y = (x \cdot z) \cdot \frac{1}{x} \ \ (Assumption)$$
$$y = \left(x \cdot \frac{1}{x}\right) \cdot z \ \ (M3)$$
$$y = 1 \cdot z \ \ (M5)$$
$$y = z \ \ (M4)$$
$$y = z$$

(2.)

If $x \in F \setminus \{0\}$, and $x \cdot y = x$, then $y = 1$

$$x \cdot y = x = x \cdot 1 \ \ (M4)$$
$$x \cdot y = x \cdot 1 \ \ (By\ 1, let\ z = 1)$$
$$x \cdot y = x \cdot z$$
$$By\ 1, y = z = 1$$

(3.)

If $x \in F \setminus \{0\}$, and $x \cdot y = 1$, then $y = x^{-1}$

$$x \cdot y = x \cdot \frac{1}{x} \ (M5)$$
$$By\ 1, let\ z = \frac{1}{x}$$
$$x \cdot y = x \cdot z$$
$$By\ 1, y = z = \frac{1}{x} = x^{-1}$$

(4.)

If $x \in F \setminus \{0\}$, and $(x^{-1})^{-1} = x$

$$let\ (x^{-1})\ \cdot z = 1$$

$$By\ part\ 3, z = (x^{-1})^{-1}$$

$$(x^{-1}) \cdot (x^{-1})^{-1} = 1$$

$$multiply\ both\ sides\ by\ x$$

$$(x) \cdot (x^{-1} \cdot (x^{-1})^{-1}) = 1 \cdot x$$

$$x \cdot x^{-1} \cdot (x^{-1})^{-1} = 1 \cdot x \ \ (M3\ on\ LHS)$$

$$1 \cdot (x^{-1})^{-1} = 1 \cdot x \ \ (M5\ on\ LHS)$$

$$(x^{-1})^{-1} = x \ \ (M4\ on\ LHS\ and\ RHS)$$