Two issues

- For the binomial option pricing model, as $N\Delta t = T$ and $N \to \infty$ and $\Delta t \to 0$,
  - What is the limiting distribution of $S_{\cdot,N}$?
  - What is the limiting value of the option price?

The Binomial Model Revisited

- The binomial model for equity option is

$$S_t \begin{cases} \xrightarrow{1-q} \; S^u_{t+\Delta t} = S_t(1+\mu\Delta t + \sigma\sqrt{\Delta t}) = S_t u \\[2ex] \xrightarrow{q} \; S^d_{t+\Delta t} = S_t(1+\mu\Delta t - \sigma\sqrt{\Delta t}) = S_t d \end{cases}$$

for

$$q = \frac{u-R}{u-d}$$

with

$$R = 1 + r\Delta t$$

The Binomial Model Revisited

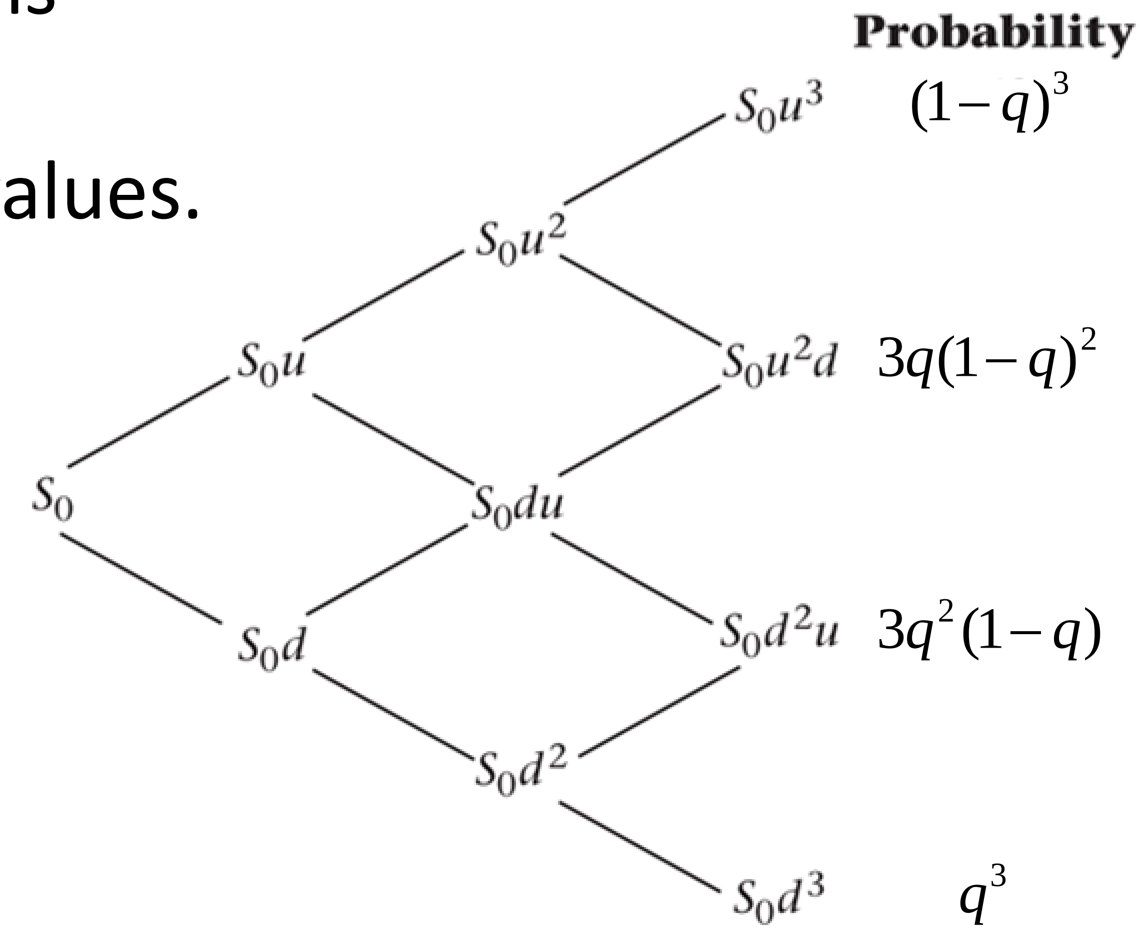
- We write

$$S_{t+\Delta t} = S_t(1 + \mu\Delta t + \sigma\sqrt{\Delta t}\varepsilon_t)$$

$$\varepsilon_t = \begin{cases} +1 & \text{with prob. } 1-q \\ -1 & \text{with prob. } q \end{cases}$$

Nodal values and nodal probabilities

- The binomial model implicitly assigns probabilities to various nodal values.



$$q = \frac{u-R}{u-d}$$

$$R = 1 + r\Delta t$$

The Binomial distribution

- The number of different paths that reach $S_0 u^i d^{n-i}$ is

$$C_n^i = \binom{n}{i} = \frac{n!}{(n-i)! \times i!}$$

- The probability to travel along each path is

$$(1-q)^i q^{n-i}$$

- So the binomial probability of reaching the node $S_0 u^i d^{n-i}$ at step n is given by

$$\frac{n!}{(n-i)! \times i!}(1-q)^i q^{n-i}$$

Binomial induction scheme

- Payoff:

$$C_{i,N} = \max(S_{i,N} - K, 0) = \max(S_0 u^i d^{N-i} - K, 0)$$

- Let $R = 1 + r\Delta t$. Backward induction yield

$$C_{i,N-1} = R^{-1}\left(qC_{i,N} + (1-q)C_{i+1,N}\right)$$

$$\begin{aligned} C_{i,N-2} &= R^{-1}\left(qC_{i,N-1} + (1-q)C_{i+1,N-1}\right) \\ &= R^{-2}\left(q^2 C_{i,N} + 2q(1-q)C_{i+1,N} + (1-q)^2 C_{i+2,N}\right) \end{aligned}$$

- All the way back to the root:

$$C_{0,0} = R^{-N} \sum_{i=0}^{N} \binom{N}{i} q^{N-i} (1-q)^i C_{i,N}$$

$$= R^{-N} \sum_{i=0}^{N} \binom{N}{i} q^{N-i} (1-q)^i \left(S_0 u^i d^{N-i} - K \right)^+$$

$$= R^{-N} \sum_{i=0}^{N} \binom{N}{i} q^{N-i} (1-q)^i S_0 u^i d^{N-i} 1_{\left\{ S_0 u^i d^{N-i} > K \right\}}$$

$$- R^{-N} K \sum_{i=0}^{N} \binom{N}{i} q^{N-i} (1-q)^i 1_{\left\{ S_0 u^i d^{N-i} > K \right\}}$$

- Let $S_T = S_0 u^i d^{N-i}$ and write

$$E^{q,N}\left[S_T 1_{\{S_T > K\}}\right] = \sum_{i=0}^{N} \binom{N}{i} q^{N-i}(1-q)^i S_0 u^i d^{N-i} 1_{\left\{S_0 u^i d^{N-i} > K\right\}}$$

$$E^{q,N}\left[1_{\{S_T > K\}}\right] = \sum_{i=0}^{N} \binom{N}{i} q^{N-i}(1-q)^i 1_{\left\{S_0 u^i d^{N-i} > K\right\}}$$

- Then

$$C_{0,0} = R^{-N} E^{q,N}\left[S_T 1_{\{S_T > K\}}\right] - K R^{-N} E^{q,N}\left[1_{\{S_T > K\}}\right]$$

Convergence of the probabilities

- When $N \to \infty$, there is

$$E^{q,N}\left[S_T 1_{\{S_T>K\}}\right] \to e^{rT} S_0 \Phi(d_1)$$

$$E^{q,N}\left[1_{\{S_T>K\}}\right] \to \Phi(d_2), \qquad R^{-N} \to e^{-rT}$$

where $\Phi(x)$ is the normal accumulative function:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du, \qquad d_{1,2} = \frac{\ln \frac{S_0}{K} + rT}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}$$

- Resulting

$$C_{0,0} \to S_0 \Phi(d_1) - e^{-rT} K \Phi(d_2)$$

The Black-Scholes formula (1973).

- The Black-Scholes formula is a limiting case of the binomial formula (infinitely many periods) for the price of a European option.

Convergence of the binomial model

TABLE 12.1 Binomial option prices for different numbers of binomial steps. As in Figure 10.3, all calculations assume that the stock price $S = \$41$, the strike price $K = \$40$, volatility $\sigma = 0.30$, risk-free rate $r = 0.08$, time to expiration $T = 1$, and dividend yield $\delta = 0$.

Number of Steps (n)	Binomial Call Price ($)
1	7.839
4	7.160
10	7.065
50	6.969
100	6.966
500	6.960
∞	6.961

Convergence of distribution

- The binomial model is

$$S_{t+\Delta t} = S_t(1 + \mu\Delta t + \sigma\sqrt{\Delta t}\varepsilon_B)$$

where

$$\varepsilon_B = \begin{cases} +1 & \text{with probability } 1 - q \\ -1 & \text{with probability } q \end{cases}$$

- Let $\Delta t = T/N$, then

$$S_T = S_0 \prod_{i=1}^{N} (1 + \mu\Delta t + \sigma\sqrt{\Delta t}\varepsilon_i)$$

where $\{\varepsilon_i\}$ are iid random variables.

- Taking log, we obtain

$$\ln\frac{S_T}{S_0} = \sum_{i=1}^{N} \ln(1+\mu\Delta t + \sigma\sqrt{\Delta t}\varepsilon_i)$$

$$= \sum_{i=1}^{N}\left(\mu\Delta t + \sigma\sqrt{\Delta t}\varepsilon_i - \frac{1}{2}\sigma^2\Delta t\varepsilon_i^2 + O(\Delta t^{3/2})\right)$$

$$= \mu T + \sigma\sqrt{\Delta t}\sum_{i=1}^{N}\varepsilon_i - \frac{1}{2}\sigma^2\Delta t\sum_{i=1}^{N}\varepsilon_i^2 + O(\Delta t^{1/2})$$

- Here, $O(\Delta t^{\delta})$ represents a term A such that

$$\left|E[A]\right| \le C\Delta t^{\delta}, \quad VaR(A) \le C\Delta t^{2\delta}$$

for some constant $C > 0.$

- Note that

$$E[\varepsilon_i] = q(-1) + (1-q)(+1) = 1 - 2q$$

$$E[\varepsilon_i^2] = q(-1)^2 + (1-q)(+1)^2 = 1$$

$$VaR(\varepsilon_i) = E[\varepsilon_i^2] - \left(E[\varepsilon_i]\right)^2 = 4q(1-q)$$

$$VaR(\varepsilon_i^2) = 0$$

- **The Lundeberg-Levi Central Limit Theorem (1922)**: let ε_i be i.i.d. random variables and

$$\bar{\varepsilon}_N = \frac{1}{N}\sum_{i=1}^{N}\varepsilon_i$$

then

$$\frac{\bar{\varepsilon}_N - E[\bar{\varepsilon}_N]}{\sqrt{VaR(\bar{\varepsilon}_N)}} \quad \rightarrow \quad z \sim N(0,1) \qquad \square$$

- $z \sim N(0,1)$ means

$$\Pr ob(z \leq a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

where

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

is the density function for z.

- We have

$$E[\overline{\varepsilon}_N] = E[\varepsilon_i] = 1 - 2q$$

$$VaR(\overline{\varepsilon}_N) = \frac{1}{N^2} \sum_{i=1}^{N} VaR(\varepsilon_i) = \frac{4q(1-q)}{N}$$

- Then

$$\ln\frac{S_T}{S_0} = \mu T + \sigma\sqrt{N\Delta t}\,\frac{\frac{1}{N}\sum_{i=1}^{N}\varepsilon_i - (1-2q) + (1-2q)}{\frac{1}{\sqrt{N}}} - \frac{1}{2}\sigma^2 \Delta t N + O(\Delta t^{1/2})$$

$$= \mu T + \sigma\sqrt{T}\left(2\sqrt{q(1-q)}\,\frac{\bar{\varepsilon}_N - (1-2q)}{\frac{2\sqrt{q(1-q)}}{\sqrt{N}}} + \frac{(1-2q)}{\frac{1}{\sqrt{N}}}\right)$$

$$-\frac{1}{2}\sigma^2 \Delta t N + O(\Delta t^{1/2})$$

- Since

$$q = \frac{1}{2} + \frac{(\mu - r)\sqrt{\Delta t}}{2\sigma}$$

- There is

$$\frac{1-2q}{\frac{1}{\sqrt{N}}} = -\frac{(\mu - r)\sqrt{N\Delta t}}{\sigma} = \frac{(r-\mu)\sqrt{T}}{\sigma}$$

$$2\sqrt{q(1-q)} = 2\sqrt{\left(\frac{1}{2} + \frac{(\mu-r)\sqrt{\Delta t}}{2\sigma}\right)\left(\frac{1}{2} - \frac{(\mu-r)\sqrt{\Delta t}}{2\sigma}\right)} = \sqrt{1-\left(\frac{(\mu-r)}{\sigma}\right)^2 \Delta t} \to 1$$

- It follows that

$$\ln\frac{S_T}{S_0} = \mu T + \sigma\sqrt{T}\left(2\sqrt{q(1-q)}\,\frac{\bar{\varepsilon}_N - (1-2q)}{\frac{2\sqrt{q(1-q)}}{\sqrt{N}}} + \frac{(1-2q)}{\frac{1}{\sqrt{N}}}\right)$$

$$-\frac{1}{2}\sigma^2 T + O(\Delta t^{1/2})$$

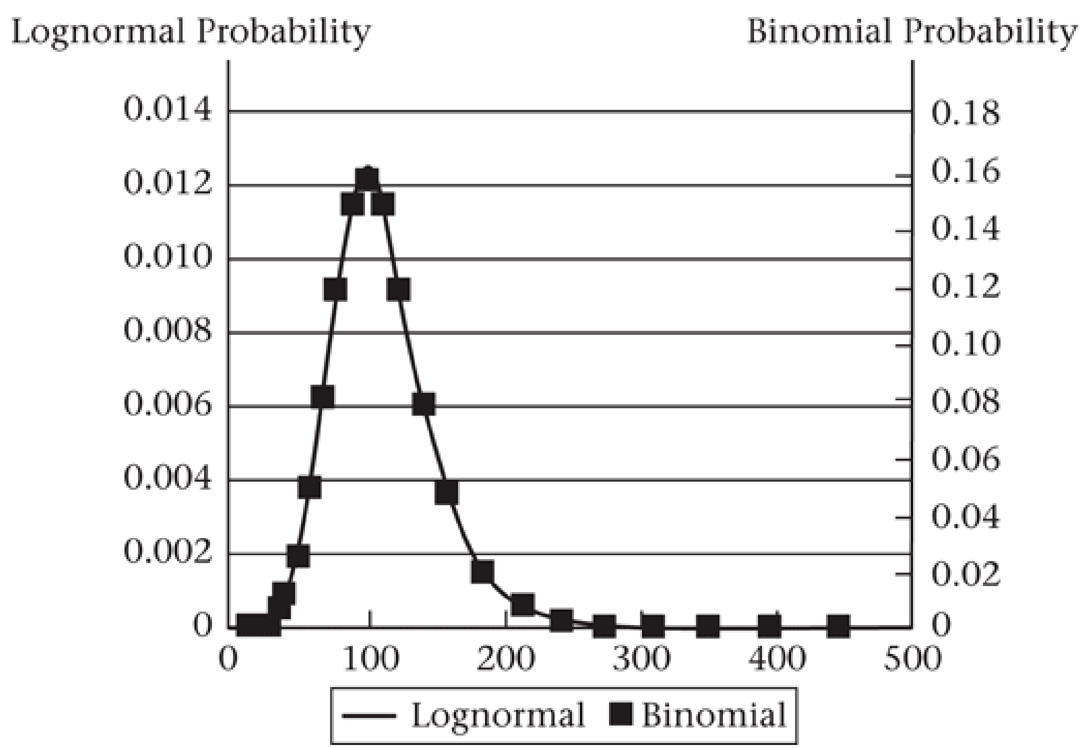
$$= \mu T + \sigma\sqrt{T}\left(z + \frac{(r-\mu)\sqrt{T}}{\sigma}\right) - \frac{1}{2}\sigma^2 T + O(\Delta t^{1/2})$$

$$\to rT + \sigma\sqrt{T}z - \frac{1}{2}\sigma^2 T \quad \text{as} \quad \Delta t \to 0$$

- Demonstration

Limit of the Binomial distribution

- The following graph compares the probability distribution for a 25-period binomial tree with the corresponding lognormal distribution



The Black-Scholes formula model

- Black and Scholes (1973) showed that the options can be priced by "taking the expectation of the payoff function and then discounting":

$$
\begin{aligned}
C_0 &= e^{-rT} E\left[\left(S_T - K \right)^+ \right] \\
&= e^{-rT} E\left[\left(S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma\sqrt{T} \times z} - K \right)^+ \right]
\end{aligned}
$$

- Here, the interest rate r is for continuous compounding and remains constant.

- Note that

$$S_T \geq K$$

$$\Updownarrow$$

$$z \geq \frac{\ln \frac{K}{S_0 e^{rT}} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = -\frac{\ln \frac{S_0 e^{rT}}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = -d_2$$

- It follows that

$$E\left[\left(S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}\times z}-K\right)^+\right]$$

$$=\int_{-d_2}^{+\infty}\left(S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}\times x}-K\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$=S_0 e^{rT}\int_{-d_2}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\sigma^2 T+\sigma\sqrt{T}\times x-\frac{x^2}{2}}dx-K\int_{-d_2}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$= S_0 e^{rT} \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x-\sigma\sqrt{T}\right)^2} dx - K \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= S_0 e^{rT} \int_{-d_2-\sigma\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy - K\Phi(d_2)$$

$$= S_0 e^{rT} \int_{-\infty}^{d_2+\sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy - K\Phi(d_2)$$

$$= S_0 e^{rT} \Phi(d_1) - K\Phi(d_2)$$

where $d_2 = \dfrac{\ln \frac{S_0 e^{rT}}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \; d_1 = d_2 + \sigma\sqrt{T}$