Final for Math 361

Quantitative Methods for Fixed-Income Securities

May 24, 2005

Solution:

1.

1.1 The risk-neutralized interest-rate tree is

$$\begin{array}{ccc} & & 5.21\% \\ & \overset{p}{\nearrow} & \\ 4.5\% & & \\ & \underset{1-p}{\searrow} & \\ & & 3.79\% \end{array}$$

where $\{p, 1-p\}$ are the risk-neutral probabilities. The corresponding price tree is

$$\begin{array}{ccc} & & P_u = 974.63 \\ & \overset{p}{\nearrow} & \\ P = 955 & & \\ & \underset{1-p}{\searrow} & \\ & & P_d = 981.39 \end{array}$$

which satisfy

$$\begin{aligned} p &= \frac{P(1+r_0\Delta t) - P_d}{P_u - P_d} \\ &= \frac{955(1+4.5\%/2) - 981.39}{974.63 - 981.39} = 0.725 \end{aligned} \tag{1.1}$$

1.2 The option tree and the price are

$$\begin{array}{ccc} & & C_u = 0 \\ & \overset{0.725}{\nearrow} & \\ C = 0.913 & & \\ & \underset{0.275}{\searrow} & \\ & & C_d = 3.388 \end{array}$$

1.3 Buy

$$\begin{aligned} \alpha &= \frac{C_u - C_d}{P_u - P_d} \\ &= \frac{0 - 3.388}{974.63 - 981.39} = 0.5 \end{aligned}$$

unit of the $2\Delta t$ - maturity zero-coupon bond, and ``buy”

$$\begin{aligned} \beta &= \frac{C - \alpha P(2\Delta t)}{P(\Delta t)} \\ &= \frac{0.913 - 0.5 \times 955}{\frac{1000}{1+0.045\%/2}} = -0.48 \end{aligned} \tag{1.2}$$

The negative sign means to short.

2. By considering the following interest-rate tree in p.252,

$$
\begin{tikzcd}[column sep=large]
 & r_0 e^{a_1 dt+\sigma\sqrt{dt}} \\
r_0 \arrow[ur, "0.5"] \arrow[dr, "0.5"'] & \\
 & r_0 e^{a_1 dt-\sigma\sqrt{dt}}
\end{tikzcd}
$$

Since $r(2\Delta t) = 5.25\%$, we have

$$\frac{1}{\left(1+\frac{5.25\%}{2}\right)^2} = \frac{0.5\left[\left(\frac{1}{1+\frac{r_0 e^{a_1 dt+\sigma\sqrt{dt}}}{2}}\right)+\left(\frac{1}{1+\frac{r_0 e^{a_1 dt-\sigma\sqrt{dt}}}{2}}\right)\right]}{\left(1+\frac{5\%}{2}\right)}$$

$$\frac{2\left(1+\frac{5\%}{2}\right)}{\left(1+\frac{5.25\%}{2}\right)^2} = \left(\frac{1}{1+\frac{r_0 e^{a_1 dt+\sigma\sqrt{dt}}}{2}}\right)+\left(\frac{1}{1+\frac{r_0 e^{a_1 dt-\sigma\sqrt{dt}}}{2}}\right)$$

Let $x = \frac{r_0 e^{a_1 dt}}{2}$

$$1.94647 = \left(\frac{1}{1+e^{\sigma\sqrt{dt}}x}\right)+\left(\frac{1}{1+e^{-\sigma\sqrt{dt}}x}\right)$$

$$1.94647 = \frac{2+\left(e^{\sigma\sqrt{dt}}+e^{-\sigma\sqrt{dt}}\right)x}{1+\left(e^{\sigma\sqrt{dt}}+e^{-\sigma\sqrt{dt}}\right)x+x^2}$$

By substituting $\sigma = 20\%$ and $dt = 0.5$, we have

$$1.94647x^2 + 1.9119x - 0.05353 = 0$$

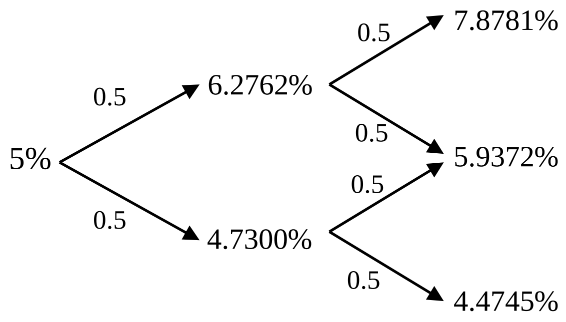
So, x = 0.027243 or x = - 1.00948 (rejected).

Therefore, a_1 = 0.1718.

The resulting 1-period interest-rate tree is given by

$$
\begin{tikzcd}[column sep=large]
 & 6.2762\% \\
5\% \arrow[ur, "0.5"] \arrow[dr, "0.5"'] & \\
 & 4.7300\%
\end{tikzcd}
$$

3. By referring to the 2-period interest rate tree in p.252 and taking $a_2 = a_1 = 0.1718$, we have



On date1, the state 1 and state 0 payoffs are, respectively,

$$\frac{\$1{,}000{,}000}{2}\max(5\% - 6.2762\%, 0) = \$0$$
$$\frac{\$1{,}000{,}000}{2}\max(5\% - 4.7300\%, 0) = \$1{,}350$$

Similarly on date 2, the state 2, 1, and 0 payoffs are, respectively,

$$\frac{\$1{,}000{,}000}{2}\max(5\% - 7.8781\%, 0) = \$0$$
$$\frac{\$1{,}000{,}000}{2}\max(5\% - 5.9372\%, 0) = \$0$$
$$\frac{\$1{,}000{,}000}{2}\max(5\% - 4.4745\%, 0) = \$2{,}627.5$$

The resulting date 1 values in states 1 and 0, respectively, are

$$\frac{0.5 \cdot \$0 + 0.5 \cdot \$0}{\left(1 + \frac{6.2762\%}{2}\right)} + \$0 = \$0$$
$$\frac{0.5 \cdot \$0 + 0.5 \cdot \$2{,}627.5}{\left(1 + \frac{4.73\%}{2}\right)} + \$1{,}350 = \$2{,}633.40$$

Finally, the price of an interest-rate floor on date 0 is

$$\frac{0.5 \cdot \$0 + 0.5 \cdot \$2{,}633.4}{\left(1 + \frac{5\%}{2}\right)} = \$1{,}284.58$$

4. For this bond,

$$c = 3, \qquad D = 181, \qquad AI(0) = c\frac{97}{181} = 3 \times \frac{97}{181} = 1.60. \tag{1.3}$$

4.5 The P&L is

$$
\begin{aligned}
P\&L &= \$10,000 \times \left(-P(d) + P(0) - \frac{cd}{D} + \left(P(0) + AI(0)\right)\frac{rd}{360} \right) \\
&= \$10,000 \times \left(-104\tfrac{12}{32} + 104\tfrac{13}{32} - \frac{3\times 1}{181} + \left(104\tfrac{13}{32} + 1.6\right)\frac{4.5\%}{360} \right) \qquad (1.4) \\
&= \$10,000 \times \left(0.03125 - 0.01657 + 0.01325\right) \\
&= \$279.26
\end{aligned}
$$

4.6 The cost of carry is

$$
\begin{aligned}
Carry &= \$10,000 \times \left(-0.01657 + 0.01325\right) \\
&= \$10,000 \times \left(-0.00332\right) \\
&= -\$33.2
\end{aligned}
$$

4.7 The breakeven price is the price that makes $P\&L = 0$. So it is

$$
\begin{aligned}
P(d) &= P(0) - \frac{cd}{D} + \left(P(0) + AI(0)\right)\frac{rd}{360} \\
&= 104\tfrac{13}{32} - \frac{3\times 1}{181} + \left(104\tfrac{13}{32} + 1.6\right)\frac{4.5\%}{360} \\
&= 104.4029 \approx 104 - 13
\end{aligned}
$$