Ho-Lee model for bond options

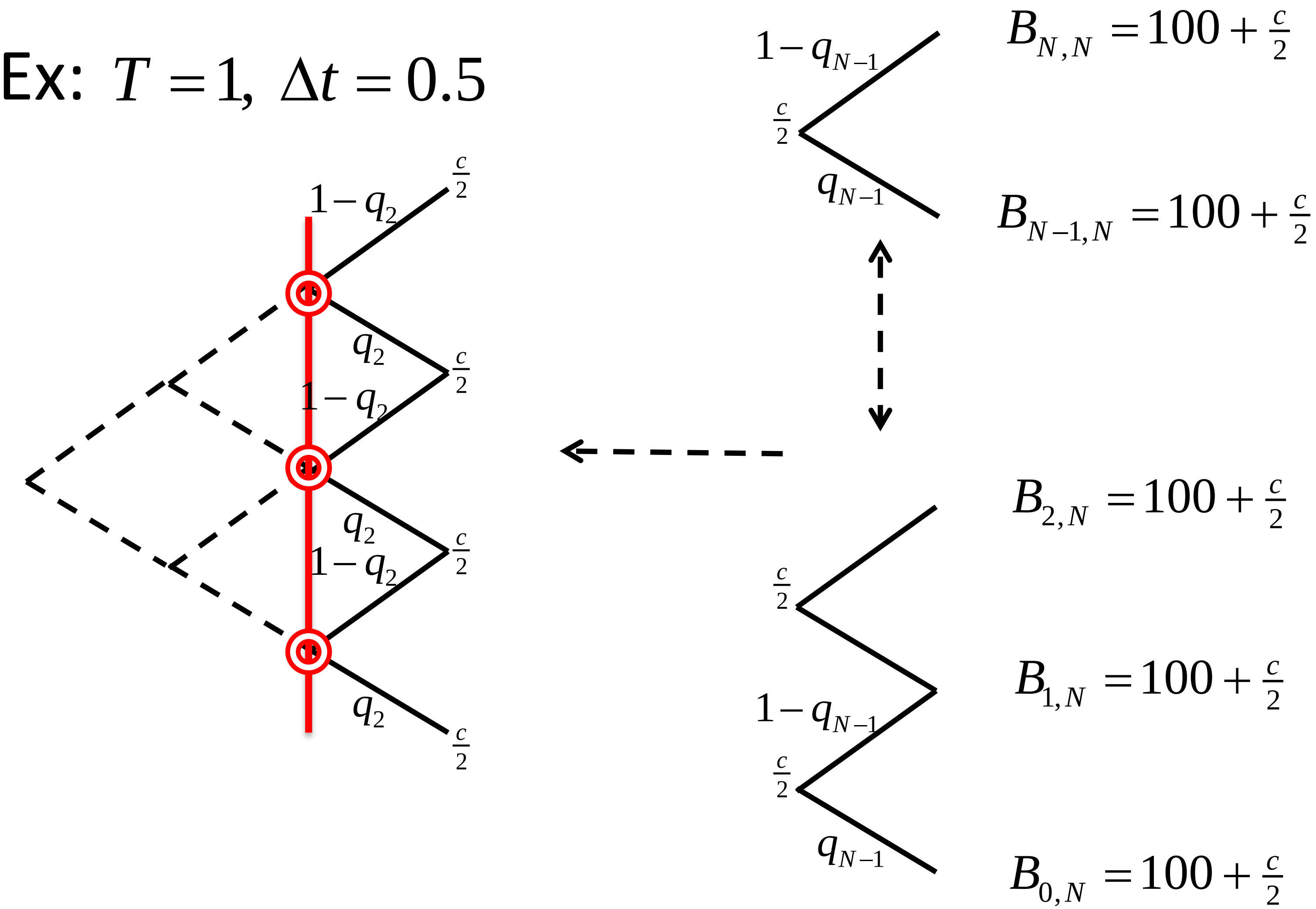
- Ho-Lee (1986) model consists of three steps:
  1. Build the interest-rate tree by matching to the discount factors.
  2. Calculate option's payoff.
  3. Backward induction to pricing options.
- We have just finished step 1.

The 2nd step: bond pricing

- We need to obtain the risk-neutral distribution of bond price for the time when an option matures.
- A bond can be regarded as a portfolio of zero-coupon bonds.
- The bond pricing can be done through backward induction with the interest-rate tree.

The 2nd Step: pricing the underlying

- Ex: $T=1,\ \Delta t=0.5$



Backward induction for bond pricing- version 1

For $i = 0 : N$

$B_{i,N} = 100 + c/2$ % we work with clean prices

end

For $j = N - 1 : M$

For $i = 0 : j$

$$B_{i,j} = \left(q_j B_{i,j+1} + (1 - q_j) B_{i+1,j+1} \right) / (1 + r_{i,j} \Delta t) + c/2;$$

end

end

For $i = 0 : M$

$B_{i,M} = B_{i,M} - c/2$ % we work with clean prices

end

% $B_{.,M}$ are clean prices

Backward induction for bond pricing – version 2

For $i = 0 : N$

$B_{i,N} = 100$ % we work with clean prices

end

For $j = N\text{-}1 : M$

for $i = 0 : j$

$$B_{i,j} = \left(q_j B_{i,j+1} + (1 - q_j) B_{i+1,j+1} + {}^{c}\!/_{2} \right) \Big/ (1 + r_{i,j} \Delta t);$$

end

end

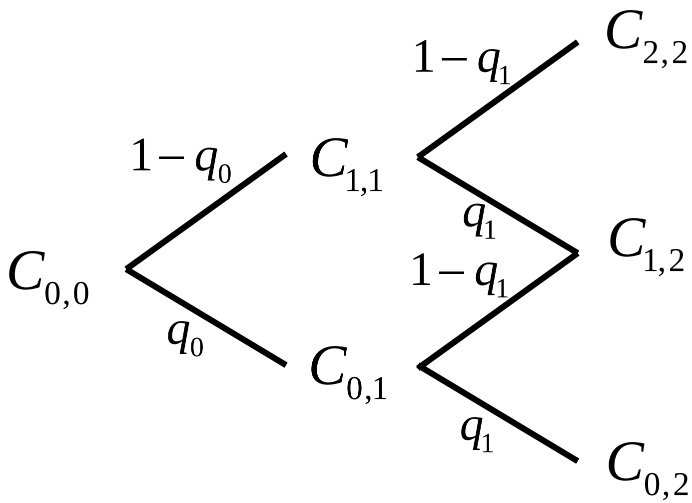
% $B_{.,M}$ are clean prices

The 3rd step: option pricing

- Once we obtain the (distribution of the) payoff function of the option, we can calculate the option value via backward induction.
- At the root of the tree, we usually need to alpha for hedging purpose.

The third step: backward induction

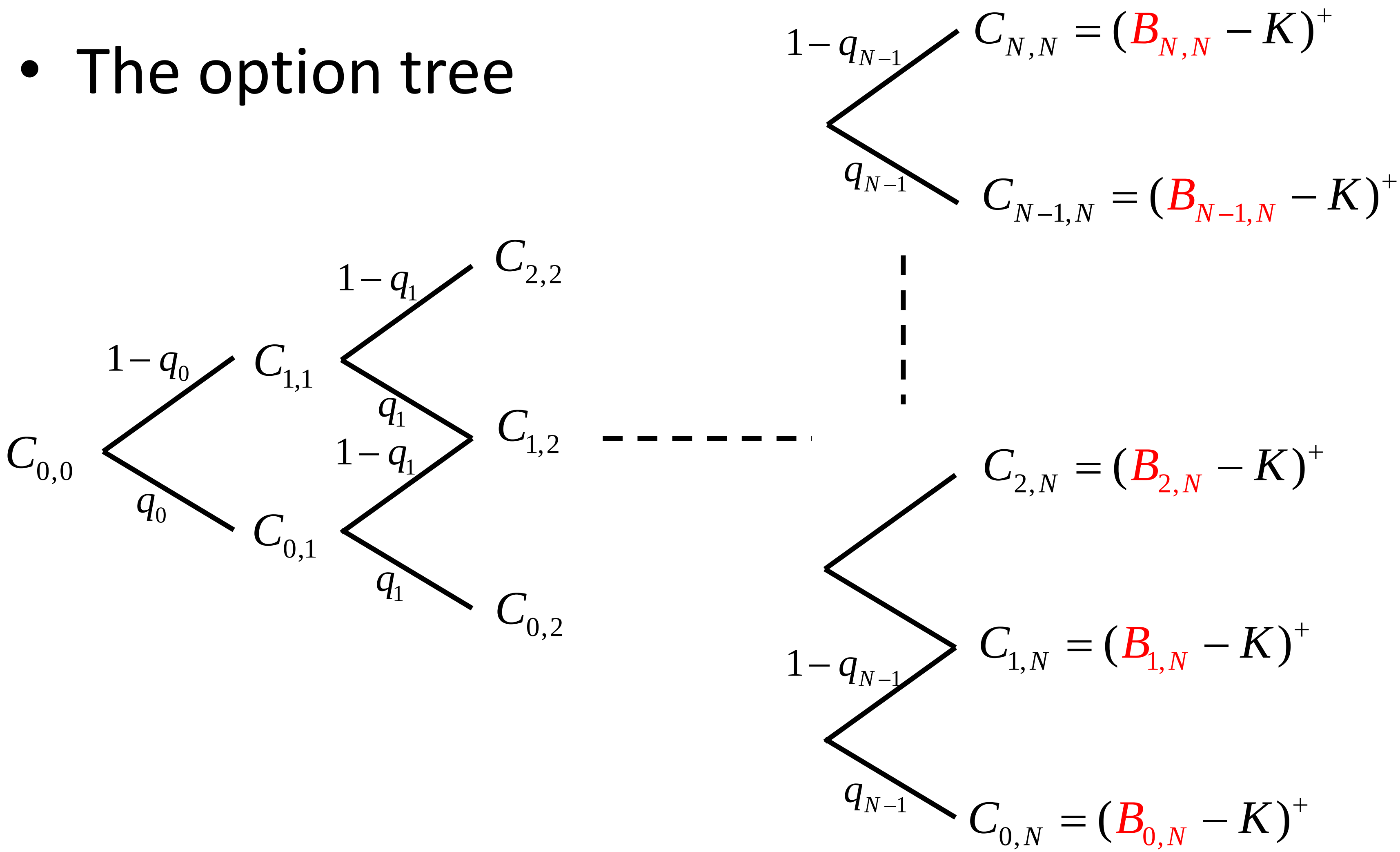
- The option tree for T=1



$$C_{i,2} = (B_{i,2} - K)^+$$
$$i = 0,1,2$$

Option tree in general

- The option tree



The backward induction for option

For $j = N - 1 : -1 : 0$

For $i = 0 : 1 : j$

$$C_{i,j} = \left(q_j C_{i,j+1} + (1 - q_j) C_{i+1,j+1} \right) \Big/ (1 + r_{i,j} \Delta t)$$

End

End

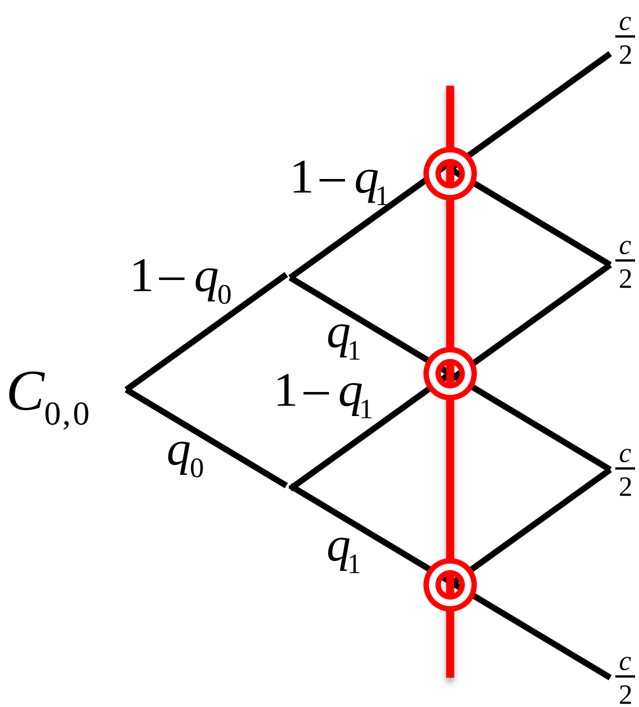
$$\alpha = \frac{C_{1,1} - C_{0,1}}{B_{1,1} - B_{0,1}}$$

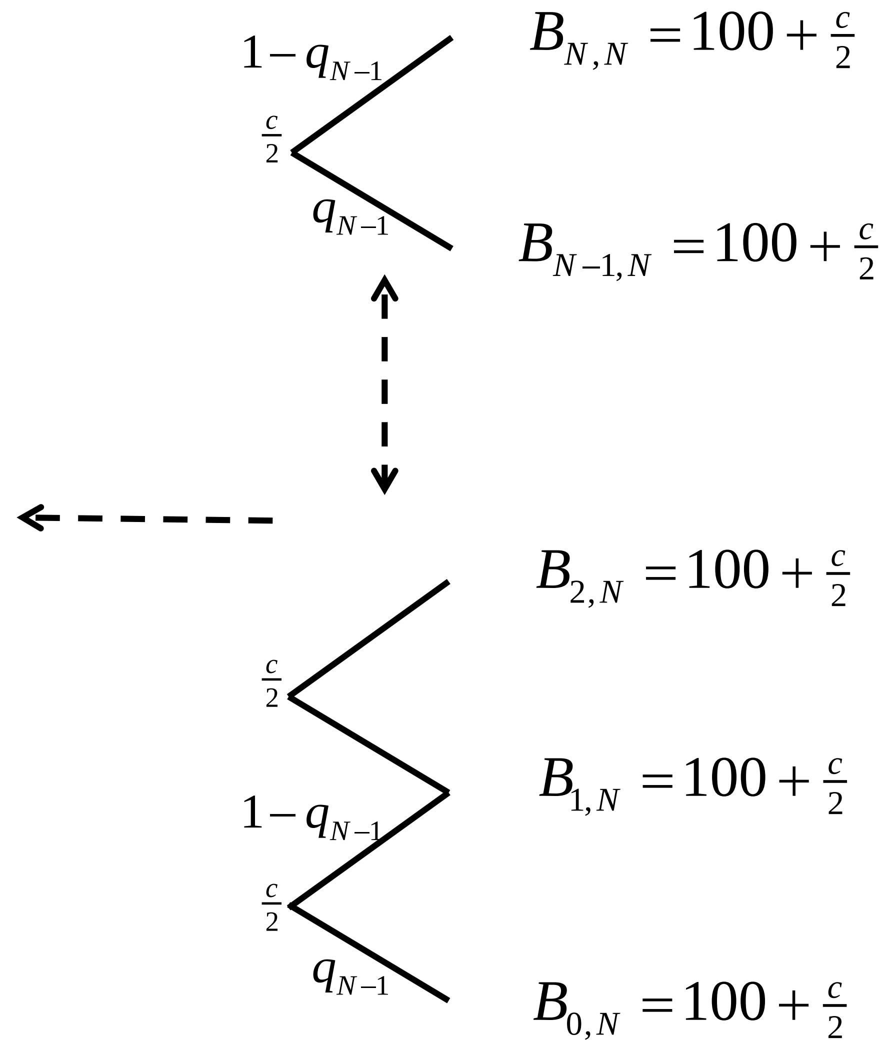
Pricing options on coupon bonds

- Today is February 15, 2001.
- Consider the call option with
  - Underlying: 5s of February 15, 2011
  - Maturity: one year
  - Strike: PAR
- Assume the par-yield curve is flat at 5%

The option tree

- Let $N = 20,\ c = 5$





$$C_{i,2} = (B_{i,2} - 100)^{+}$$
$$i = 0,1,2$$

The price curve

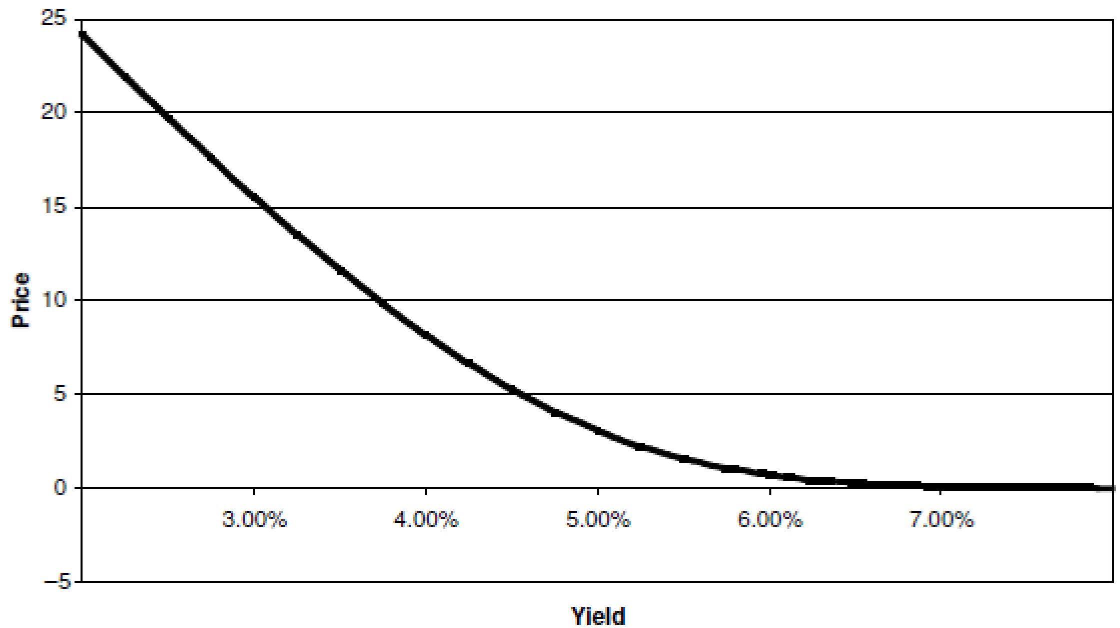


FIGURE 5.2 The Price-Rate Function of a One-Year European Call Option Struck at Par on the 5s of February 15, 2011

Dynamical hedging

- In reality, options are hedged with dynamical hedging strategy: at state (i, j), the seller of the option need to keep $\alpha_{i,j}$ unit of the bond for hedging, such that

$$\alpha_{i,j}(B_{i,j+1} + \tfrac{c}{2}) + \beta_{i,j}(1 + r_{i,j}\Delta t) = C_{i,j+1}$$
$$\alpha_{i,j}(B_{i+1,j+1} + \tfrac{c}{2}) + \beta_{i,j}(1 + r_{i,j}\Delta t) = C_{i+1,j+1}$$

i.e, replication, where $B_{i,j}$ is the clean price at state (i, j).

Dynamical hedging, cont'd

- The solution:

$$\alpha_{i,j} = \frac{C_{i+1,j+1} - C_{i,j+1}}{B_{i+1,j+1} - B_{i,j+1}}$$

$$\beta_{i,j} = \frac{C_{i,j+1}(B_{i+1,j+1} + \frac{c}{2}) - C_{i+1,j+1}(B_{i,j+1} + \frac{c}{2})}{(1 + r_{i,j}\Delta t)(B_{i+1,j+1} - B_{i,j+1})}$$

Self-financing strategy

- By construction, there is

$$
\begin{aligned}
\alpha_{i,j}(B_{i,j+1}+\tfrac{c}{2})+\beta_{i,j}(1+r_{i,j}\Delta t) &= C_{i,j+1} &&= \alpha_{i,j+1}B_{i,j+1}+\beta_{i,j+1} \\
\alpha_{i,j}(B_{i+1,j+1}+\tfrac{c}{2})+\beta_{i,j}(1+r_{i,j}\Delta t) &= C_{i+1,j+1} &&= \alpha_{i+1,j+1}B_{i+1,j+1}+\beta_{i+1,j+1}
\end{aligned}
$$

- Meaning that: before and after the hedge revision at $j+1$, the value of the replicating portfolio stays unchanged.
- Such strategies are called self-financing strategy.

Swaption Pricing

- A swaption is an option to enter into a swap in the future.
- A swap starts in the future is called the forward-starting swap.
- The swaption can be treated as a bond option with a special strike price: par.

Swaptions

- A swaption is an option to enter into a swap for a pre-specified swap rate in the future.
- Let the

  T_0 –maturity of the option

  τ –life of the underlying swap

  K –the strike rate

- Payoff of swaption

$$\max(swap(T_0;k,\tau),0)$$

Payoff of swaption analyzed

- When the underlying is a receiver's swap,

$$swap(T_0;k,\tau) = V_{fix} - V_{float}$$
$$= \underbrace{\sum_{i=1}^{2\tau} \Delta T d(T_0,T_i)k + d(T_0,T_0+\tau)}_{\text{bond price at } T_0} - \underbrace{1}_{\text{strike price}}$$

- This is the payoff of a bond option with par strike!
- So, swaption is a just a bond option with par strike!