Replication Pricing turned Expectation Pricing

- Rewrite the option formula into

$$
\begin{aligned}
C_0 &= \alpha S_0 + \beta \\
&= \frac{C^u_{\Delta t} - C^d_{\Delta t}}{S^u_{\Delta t} - S^d_{\Delta t}} S_0 + \frac{S^u_{\Delta t} C^d_{\Delta t} - S^d_{\Delta t} C^u_{\Delta t}}{(1 + r\Delta t)(S^u_{\Delta t} - S^d_{\Delta t})} \\
&= \frac{1}{1 + r\Delta t} \left(\frac{S^u_{\Delta t} - S_0(1 + r\Delta t)}{S^u_{\Delta t} - S^d_{\Delta t}} C^d_{\Delta t} + \frac{S_0(1 + r_0\Delta t) - S^d_{\Delta t}}{S^u_{\Delta t} - S^d_{\Delta t}} C^u_{\Delta t} \right) \\
&= \frac{1}{1 + r\Delta t} \left(q C^d_{\Delta t} + (1 - q) C^u_{\Delta t} \right), \quad q = \frac{(\mu - r)\Delta t + \sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}}
\end{aligned}
$$

where $0 \le q \le 1$ is independent of the option.

Risk-Neutral Valuation

- The general pricing formula

$$C_0 = \frac{1}{1+r\Delta t}\left(qC_{\Delta t}^d + (1-q)C_{\Delta t}^u\right)$$

applies to any one-period options.

- $\{Q_\mu(S_{\Delta t}^d), Q_\mu(S_{\Delta t}^u)\} = \{q, 1-q\}$ is called the pricing measure.
- And it also called the risk-neutral measure.

Another basic feature

- Another basic feature of pricing measure is that it prices the underlying correctly:

$$S_0 = \frac{1}{1+r\Delta t}\left(qS_{\Delta t}^{d} + (1-q)S_{\Delta t}^{u}\right).$$

- The original binomial model has been changed to

$$S_t \begin{cases} \xrightarrow{1-q} & S_{t+\Delta t}^{u} = S_t(1+\mu\Delta t + \sigma\sqrt{\Delta t}) \\ \xrightarrow{q} & S_{t+\Delta t}^{d} = S_t(1+\mu\Delta t - \sigma\sqrt{\Delta t}) \end{cases}$$

Risk Premiums

- In markets, investors put
  - preferences, or
  - risk premiums,

  into pricing, resulted in
  - either twist probabilities
  - or changed nodal values
- The CRR model twists the probabilities.

Multi-Step CRR Tree

- The dynamics of equity price

$$\frac{\Delta S_t}{S_t} = \mu_t \Delta t + \sigma \sqrt{\Delta t} \ \varepsilon_B$$

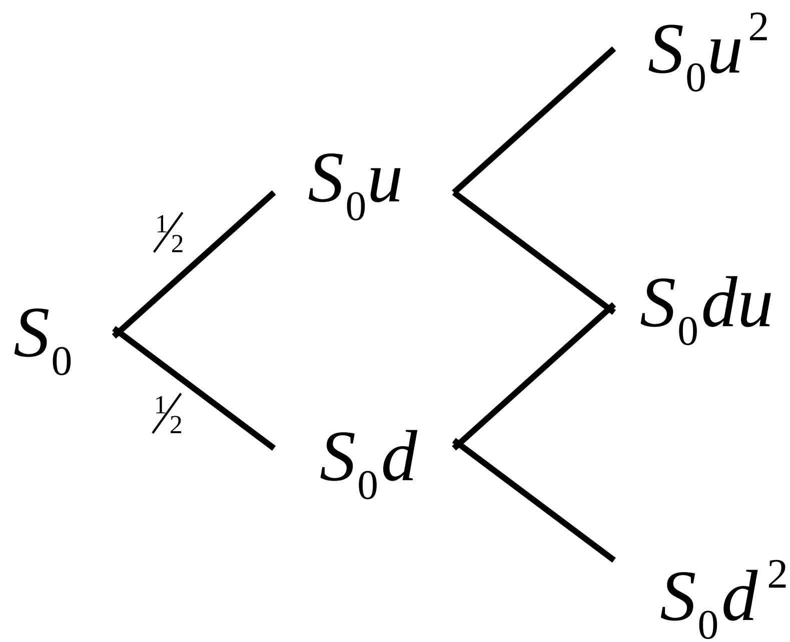
- Assume $\mu_t = \mu$, a constant, then

$$S_{t+\Delta t}^{d} = S_t \left[1 + \mu \Delta t - \sigma \sqrt{\Delta t} \right] = S_t d$$

$$S_{t+\Delta t}^{u} = S_t \left[1 + \mu \Delta t + \sigma \sqrt{\Delta t} \right] = S_t u$$

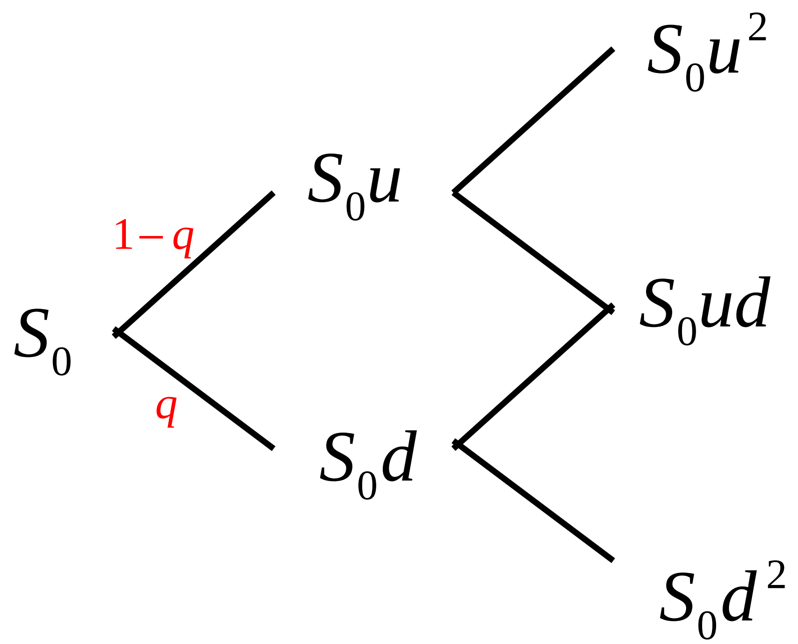
Two-period tree

- By duplication, we obtain

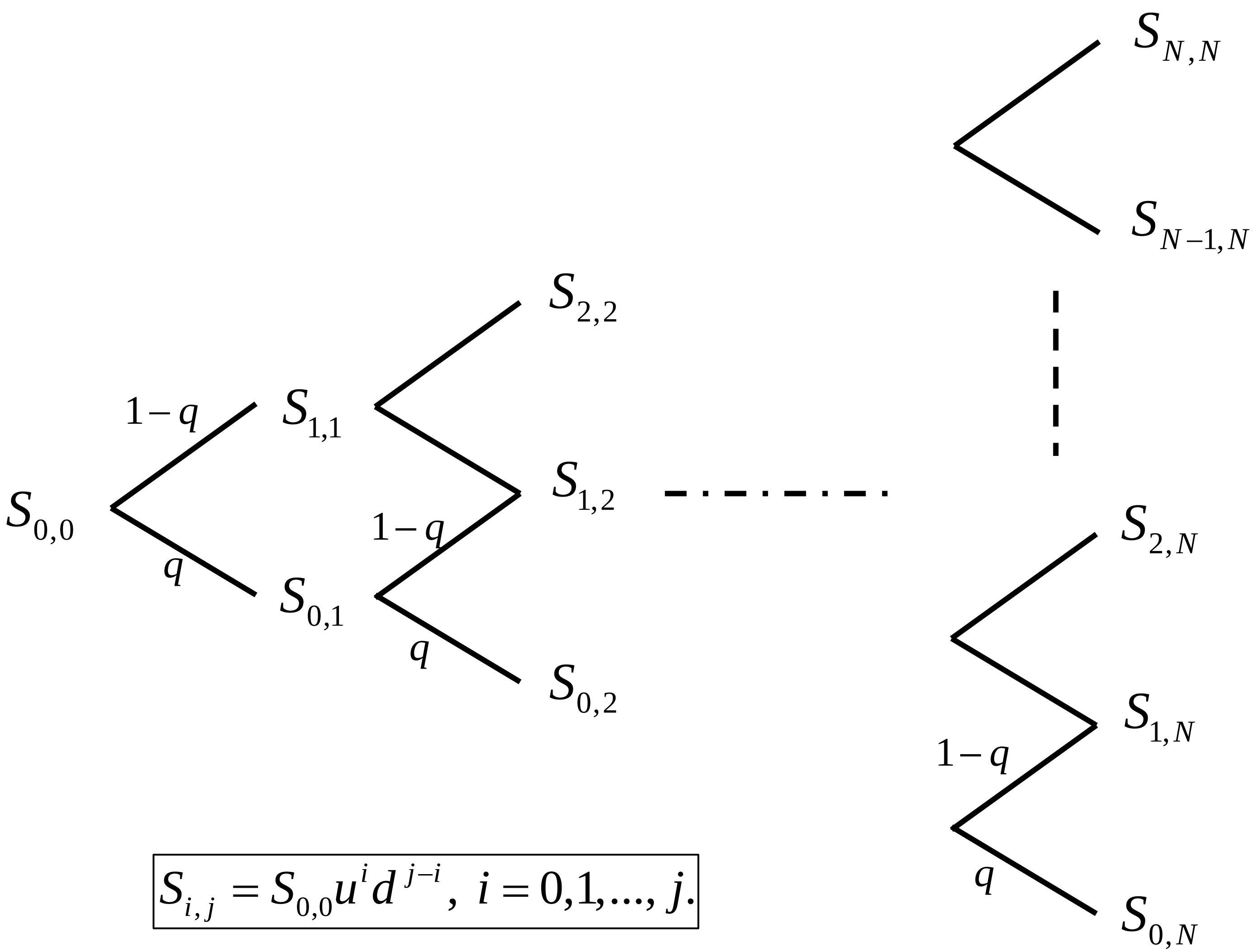


Risk-neutral two-period tree

- The risk-neutral tree is



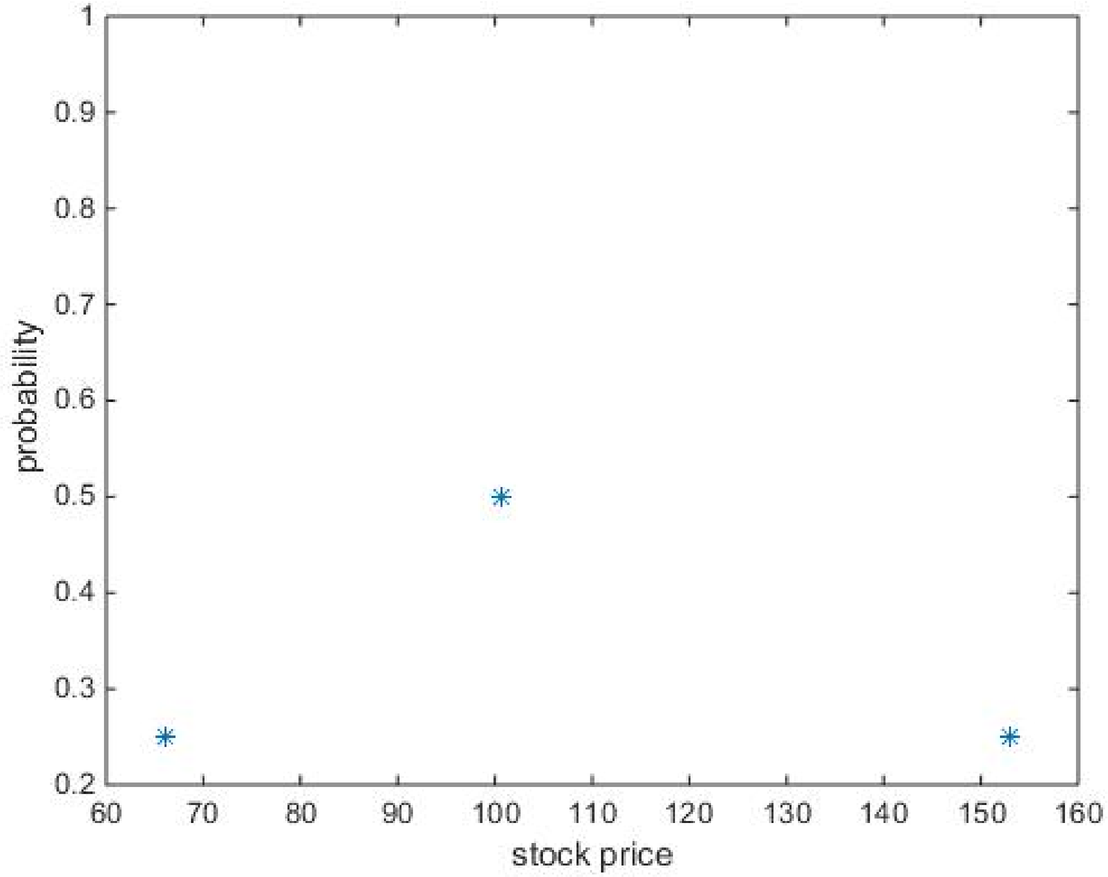
Multi-period risk-neutral tree



$$\boxed{S_{i,j} = S_{0,0} u^i d^{j-i},\ i = 0, 1, \ldots, j.}$$

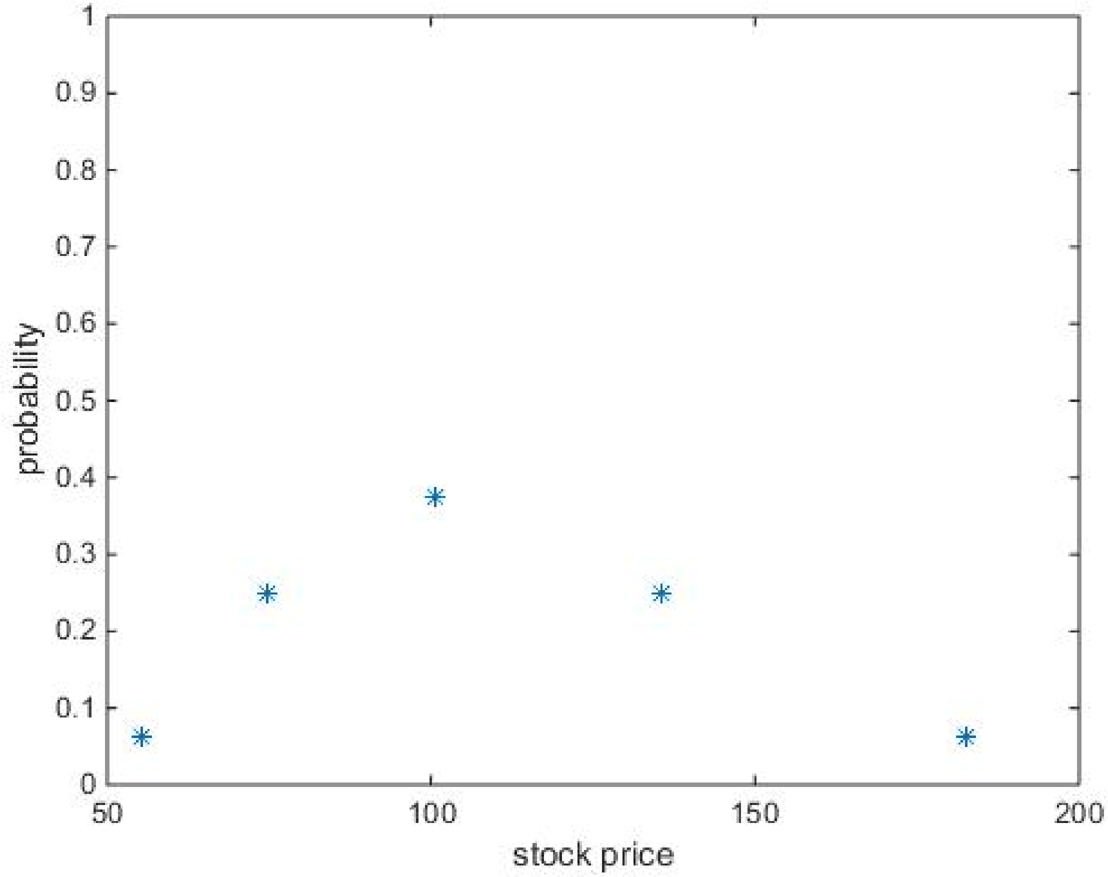
Why we need more steps?

- $S_0 = 100, \mu = 5\%, \sigma = 30\%, T = 1, \Delta t = {}^{T}\!/_{2^n}, n = 1$

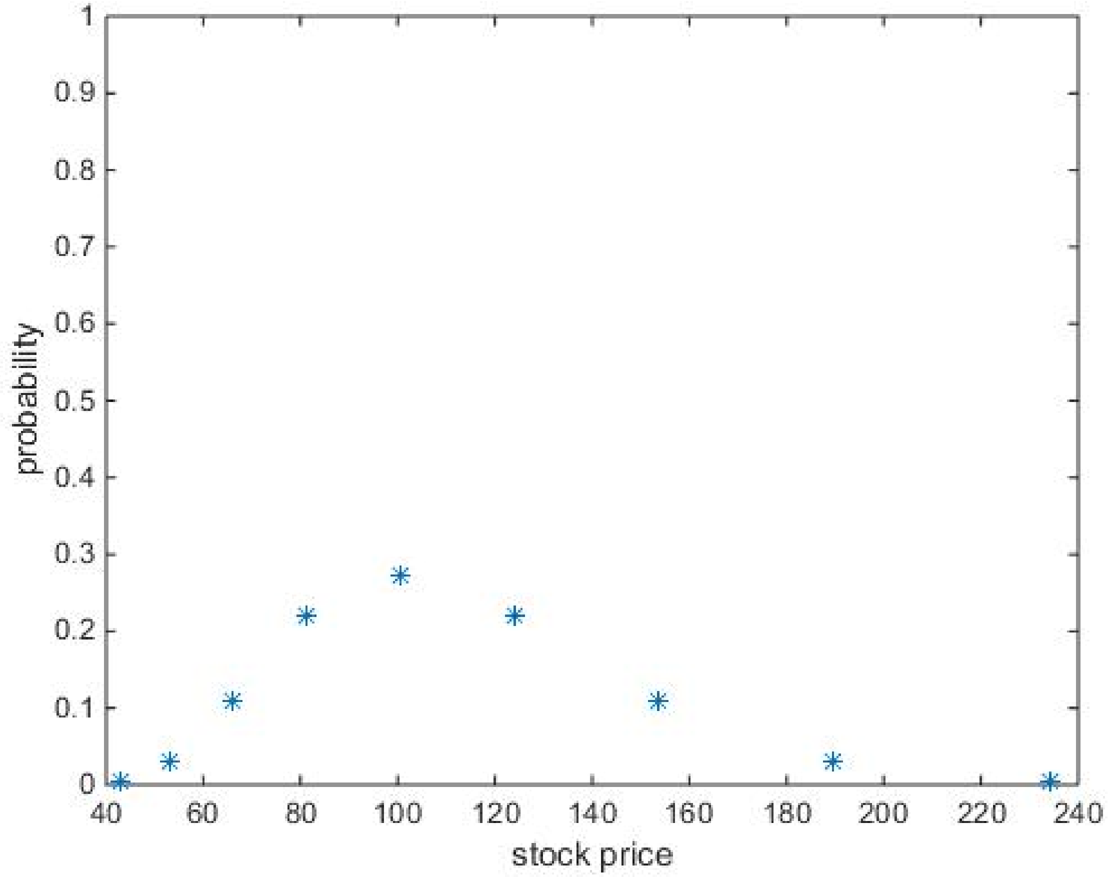


For finer distribution

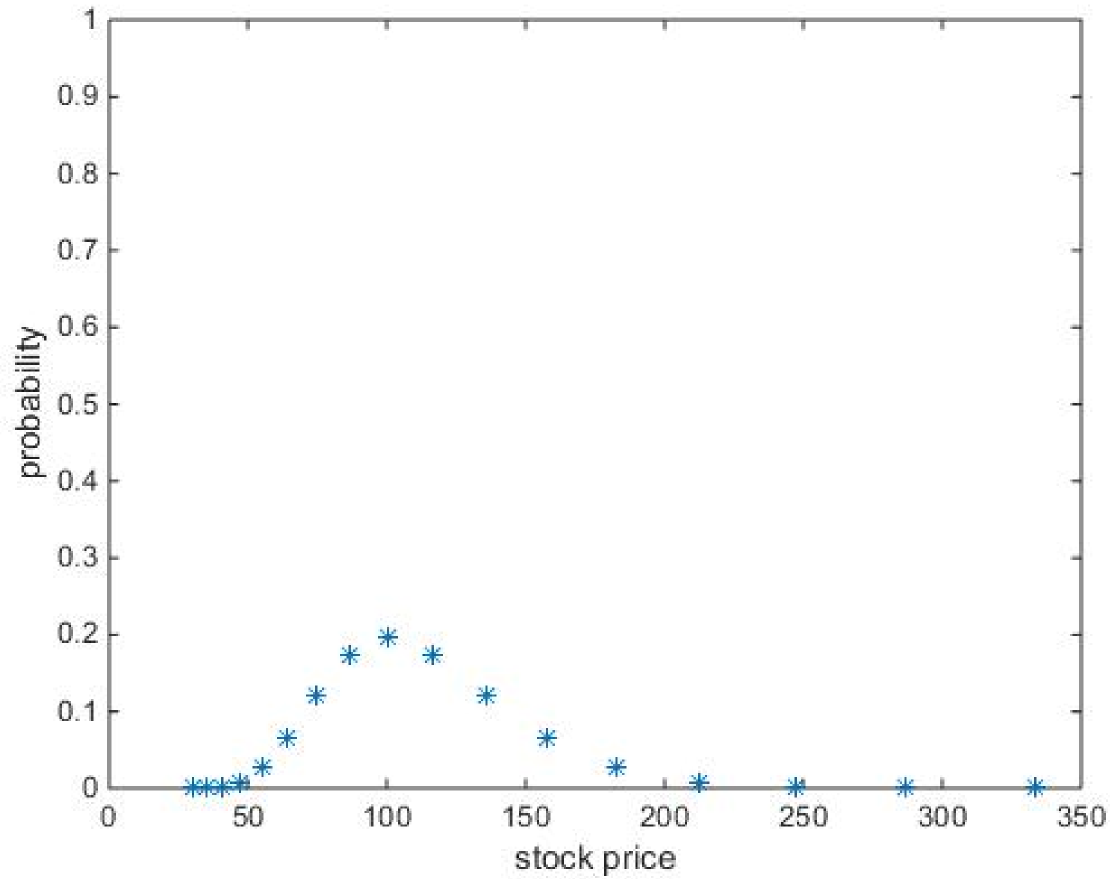
- $S_0 = 100, \mu = 5\%, \sigma = 30\%, T = 1, \Delta t = {}^{T}\!/_{2^n}, n = 2$



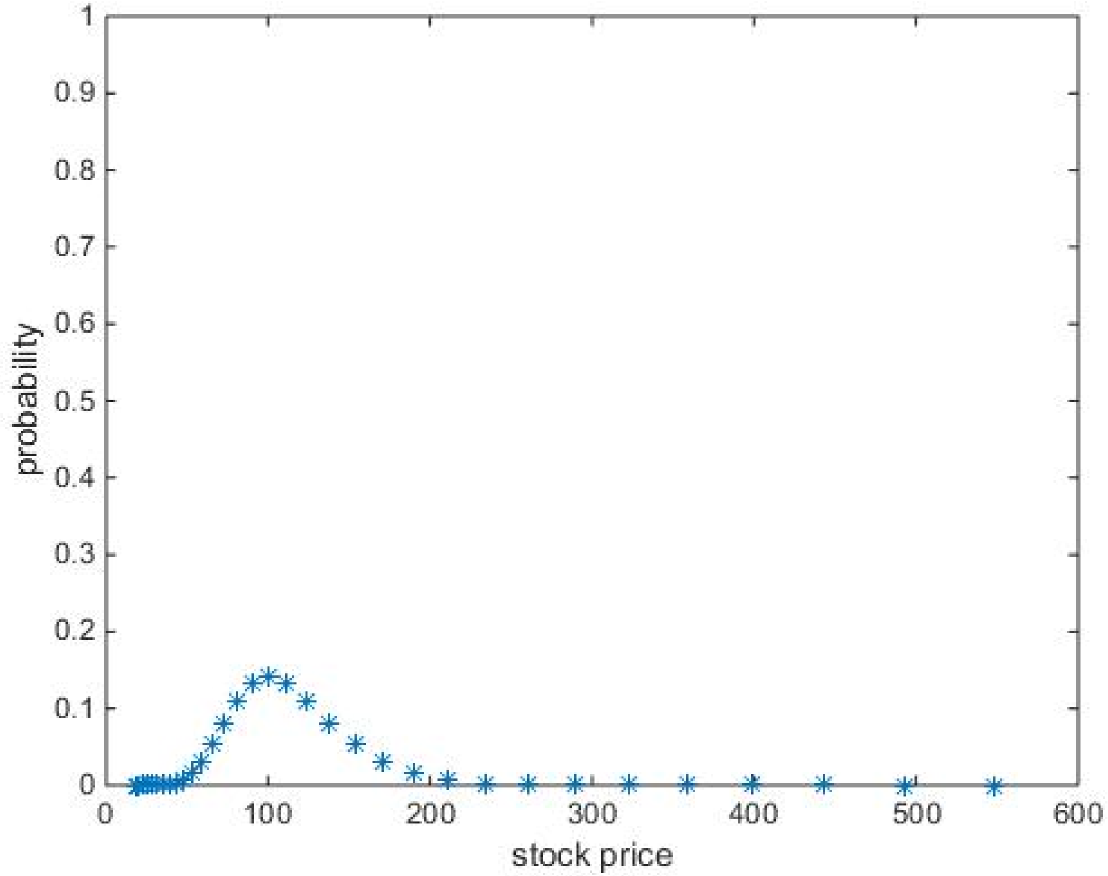
- $S_0 = 100, \mu = 5\%, \sigma = 30\%, T = 1, \Delta t = T/2^n, n = 3$



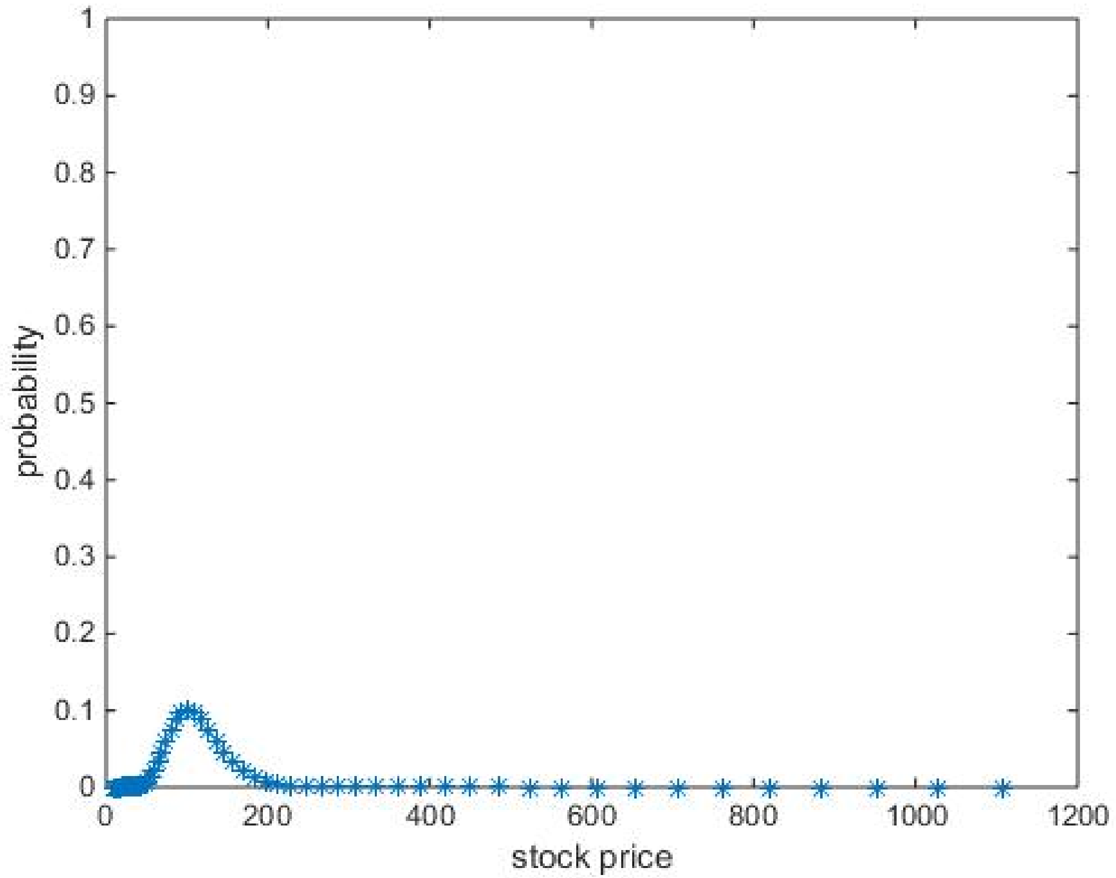
- $S_0 = 100, \mu = 5\%, \sigma = 30\%, T = 1, \Delta t = T/2^n, n = 4$



- $S_0 = 100, \mu = 5\%, \sigma = 30\%, T = 1, \Delta t = T/2^n, n = 5$

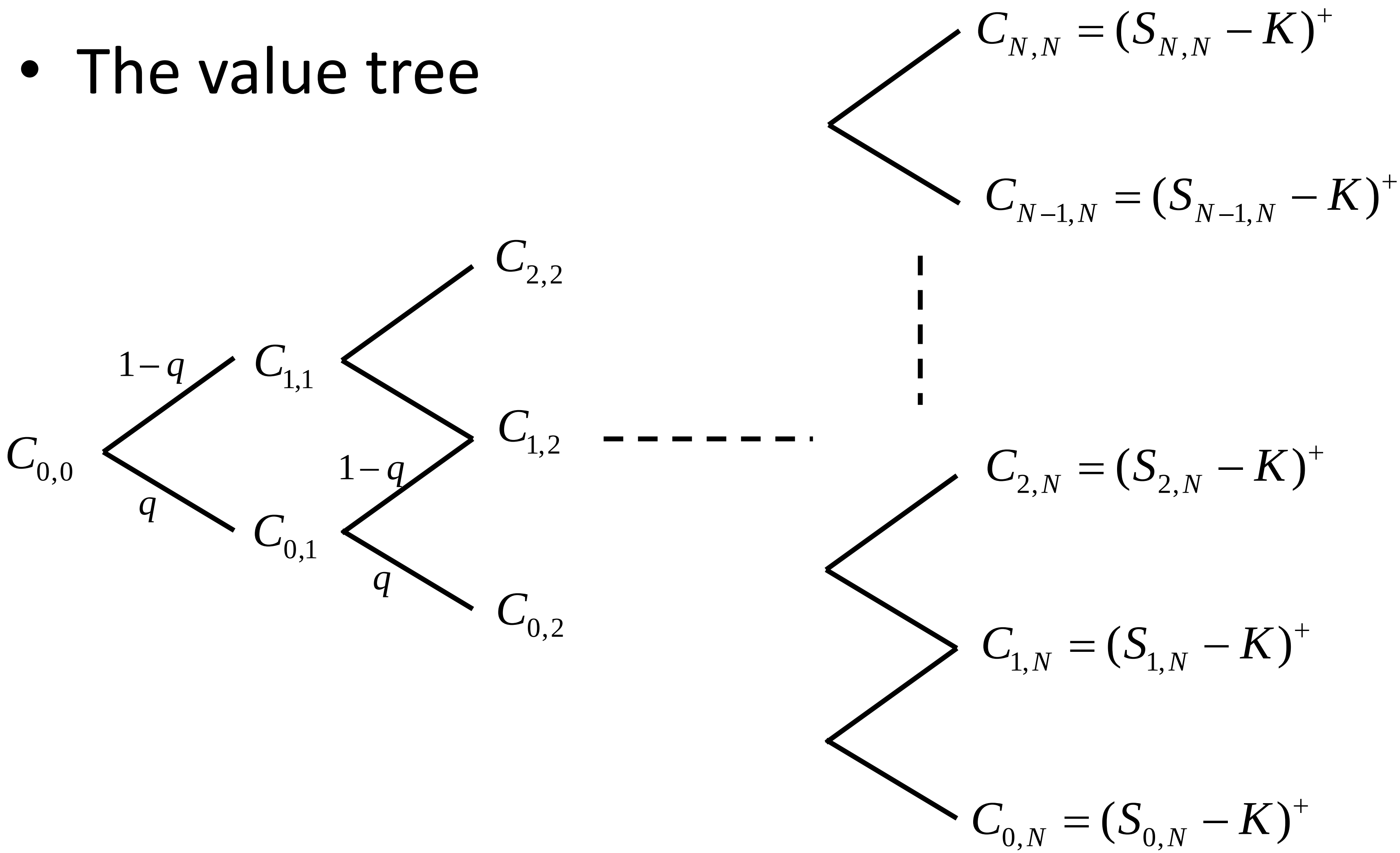


- $S_0 = 100, \mu = 5\%, \sigma = 30\%, T = 1, \Delta t = T/2^n, n = 6$



Multi-period option tree

- The value tree



CRR model

- Backward induction scheme

$$
\begin{aligned}
&\text{For } j = N-1:-1:0\\
&\qquad \text{For } i = 0:1:j\\
&\qquad\qquad C_{i,j} = \tfrac{1}{1+r\Delta t}\Big((1-q)C_{i,j+1} + qC_{i+1,j+1}\Big)\\
&\qquad \text{End}\\
&\text{End}
\end{aligned}
$$

- Remark: Cannot be simpler.

Dynamical hedging

- In reality, options are hedged with dynamical hedging strategy: at state (i, j), the seller of the option need to keep

$$\alpha_{i,j} = \frac{C_{i+1,j+1} - C_{i,j+1}}{S_{i+1,j+1} - S_{i,j+1}}$$

unit of share for hedging.

Self-financing strategy

- By the backward induction, there is, first,

$$\begin{aligned} C_{i,j+1} &= \alpha_{i,j+1} S_{i,j+1} + \beta_{i,j+1} \\ C_{i+1,j+1} &= \alpha_{i+1,j+1} S_{i+1,j+1} + \beta_{i+1,j+1} \end{aligned}$$

- Then, there is

$$\alpha_{i,j} S_{i,j+1} + (1 + r_j \Delta t)\beta_{i,j} = C_{i,j+1}$$

$$\alpha_{i,j} S_{i+1,j+1} + (1 + r_j \Delta t)\beta_{i,j} = C_{i+1,j+1}$$

Self-financing strategy

- It follows that

Before hedge rebalancing

After hedge rebalancing

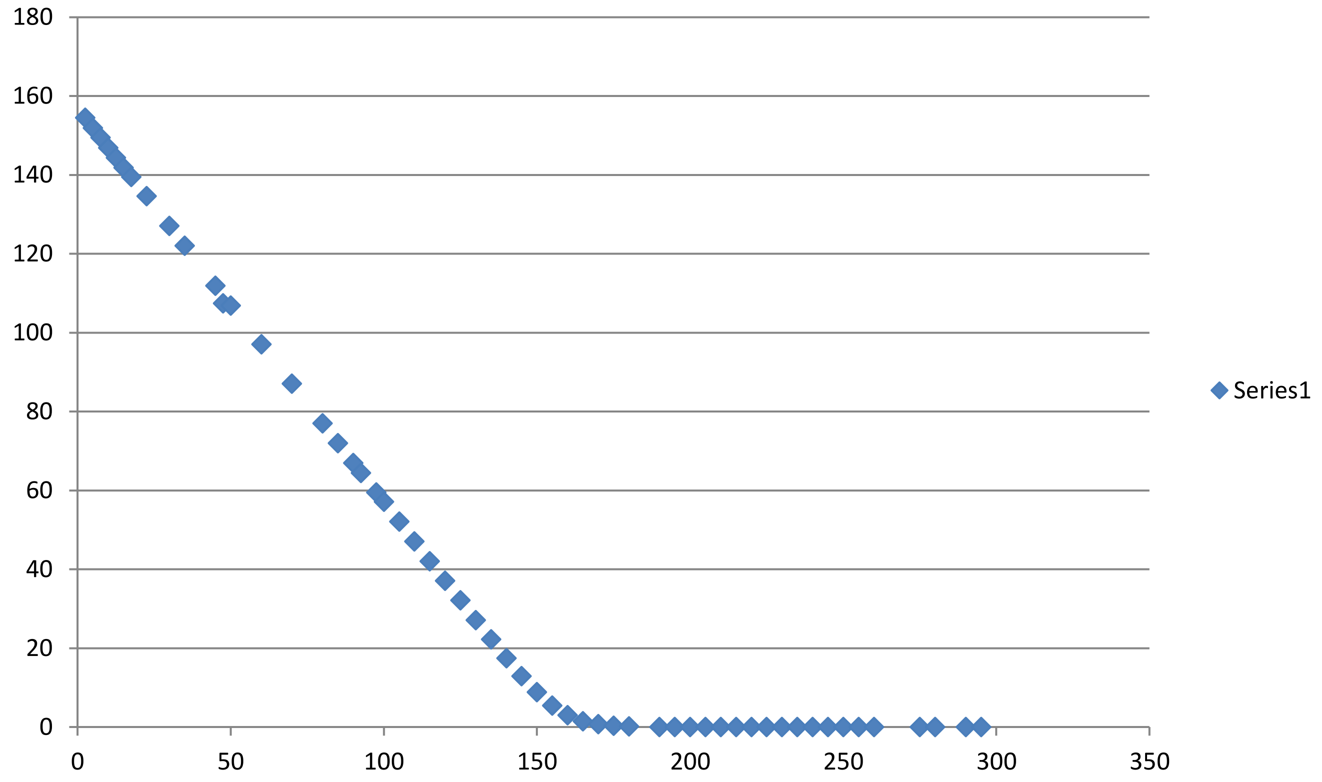
$$\alpha_{i,j}S_{i,j+1} + (1+r_j\Delta t)\beta_{i,j} = C_{i,j+1} = \alpha_{i,j+1}S_{i,j+1} + \beta_{i,j+1}$$

$$\alpha_{i,j}S_{i+1,j+1} + (1+r_j\Delta t)\beta_{i,j} = C_{i+1,j+1} = \alpha_{i+1,j+1}S_{i+1,j+1} + \beta_{i+1,j+1}$$

Self-Financing Strategy

- Meaning that: before and after the hedge revision at $j+1$, the value of the replicating portfolio stays unchanged.
- Such strategies are called self-financing strategy.

17/11/2017		bid	ask	mid
AAPL171117C00002500	100	56.95	57.25	57.1
AAPL171117C00005000	105	51.9	52.25	52.075
AAPL171117C00007500	110	46.95	47.25	47.1
AAPL171117C00010000	115	41.85	42.3	42.075
AAPL171117C00012500	120	37	37.25	37.125
AAPL171117C00015000	125	32	32.25	32.125
AAPL171117C00017500	130	27	27.3	27.15
AAPL171117C00022500	135	22.1	22.4	22.25
AAPL171117C00030000	140	17.3	17.55	17.425
AAPL171117C00035000	145	12.8	13	12.9
AAPL171117C00045000	150	8.75	8.9	8.825
AAPL171117C00047500	155	5.4	5.5	5.45
AAPL171117C00050000	160	2.99	3.05	3.02
AAPL171117C00055000	165	1.49	1.52	1.505
AAPL171117C00060000	170	0.71	0.74	0.725
AAPL171117C00065000	175	0.33	0.36	0.345
AAPL171117C00070000	180	0.17	0.18	0.175
AAPL171117C00075000	185	0.08	0.1	0.09
AAPL171117C00080000	190	0.04	0.06	0.05
AAPL171117C00085000	195	0.02	0.04	0.03
AAPL171117C00090000	200	0.01	0.03	0.02
AAPL171117C00092500	205	0	0.02	0.01
AAPL171117C00095000	210	0	0.01	0.005
AAPL171117C00097500	215	0	0.01	0.005


180
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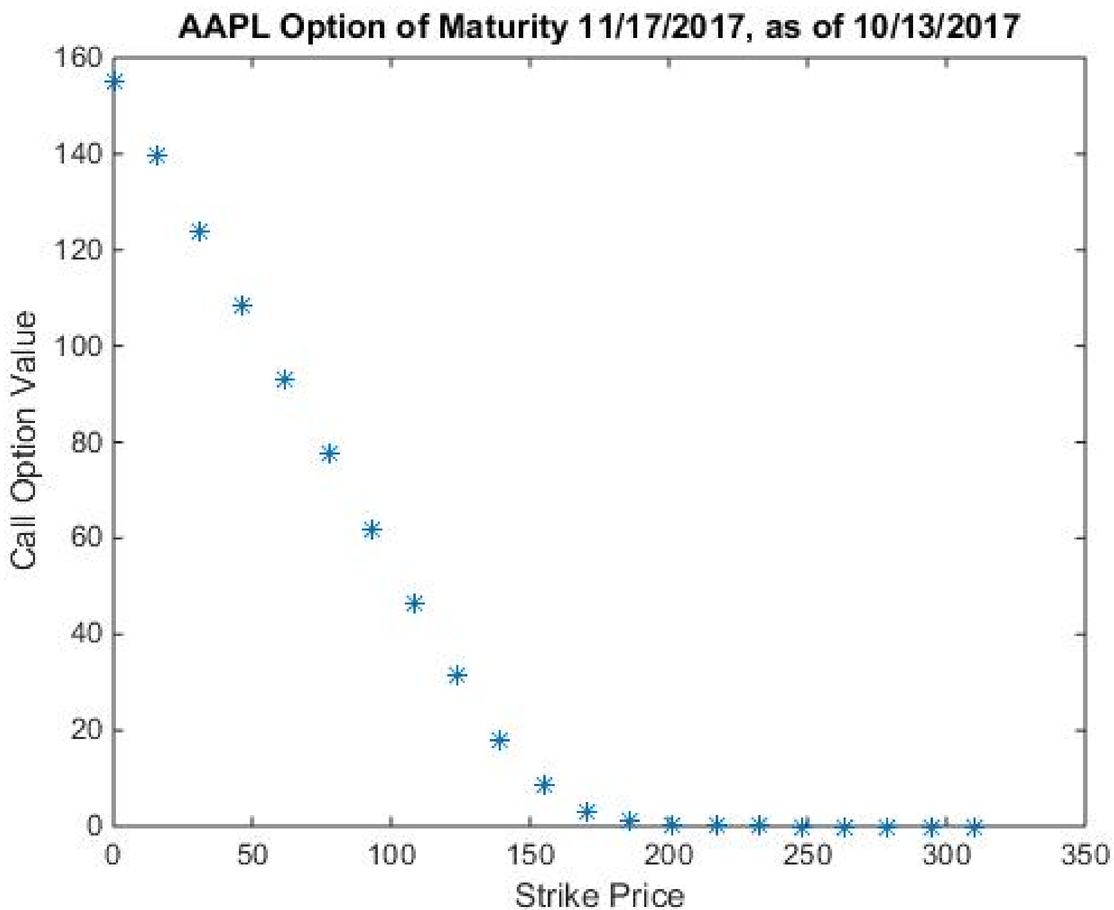
350
Series1


AAPL Option of Maturity 11/17/2017, as of 10/13/2017
Call Option Value
Strike Price
0
20
40
60
80
100
120
140
160
0
50
100
150
200
250
300
350