Swaptions

- A swaption is an option to enter into a swap for a pre-specified swap rate in the future.
- Let the

$$T_0 \text{ –maturity of the option}$$

$$T_N - T_0 \text{ –life of the underlying payer's swap}$$

$$k \text{ –the strike rate}$$

- Payoff of swaption

$$swtn(T_0;k,T_N) = \max(swap(T_0;k,T_N),0)$$

Swaption Pricing Under an Interest-rate tree

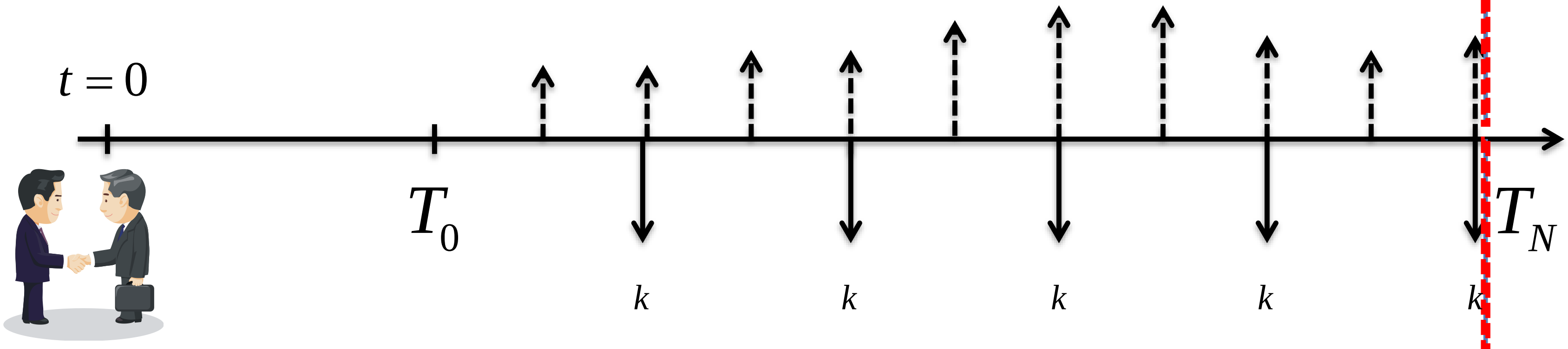
- The payoff had been treated as call or option on the bond with coupon rate *k* and with the par strike.
- But this is not a popular approach.
- Two of the reasons are
  - interest rate can be negative on the tree.
  - It is not intuitive how to hedge the swaption.

Popular approach

- The more popular approach is to treat swaption as an option on the swap rate for a forward-starting swap.
- An then we treat the forward swap rate as a forward price;
- and apply the Black's formula for the option on the forward swap rate.

Forward–starting Swaps

- A swap can start in the future, T_0, so-called the forward-starting swap, with cash flows:



Forward-starting swaps

- The swap is again priced as the difference in value between the floating leg and the fixed leg.
- The market prevailing swap rate is the fixed rate that makes the value of the forward swap equal to zero.

Determination of the swap rate

– Floating leg: par at T_0, so at $t = 0$, it is

$$V_{float} = d(0,T_0)$$

– Fixed leg: let $s(0;T_0,T_N)$ be the forward swap rate, then

$$V_{fixed} = \sum_{i=1}^{N} \Delta T \times s(0;T_0,T_N) d(0,T_i) + d(0,T_N)$$

- Set

$$0 = V_{float} - V_{fix}$$

- We obtain

$$s(0;T_0,T_N) = \frac{d(0,T_0) - d(0,T_N)}{\sum_{i=1}^{N} \Delta T \times d(0,T_i)}$$

- The above swap rate is often taken as the strike rate for swaption (So-called ATM swaption).

Swaption on payer's swap

- Payoff of the swaption on a payer's swap:

$$swap(T_0;k,T_N)^+ = \left(V_{float} - V_{fix}\right)^+$$

$$= \left(1 - \sum_{i=1}^{N} \Delta T d(T_0,T_i)k - d(T_0,T_N)\right)^+$$

- At T_0, the market prevailing swap rate of maturity $T_N - T_0$ is

$$s(T_0;T_0,T_N) = \frac{d(T_0,T_0) - d(T_0,T_N)}{\sum_{i=1}^{N} \Delta T \times d(T_0,T_i)} = \frac{1 - d(T_0,T_N)}{\sum_{i=1}^{N} \Delta T \times d(T_0,T_i)}$$

- It follows that

$$
\begin{aligned}
1 &= s(T_0;T_0,T_N)\sum_{i=1}^{N}\Delta T \times d(T_0,T_i) + d(T_0,T_N) \\
&= s(T_0;T_0,T_N)A(T_0;T_0,T_N) + d(T_0,T_N)
\end{aligned}
$$

- We then obtain an alternative expression of the payoff of a payer's swaption:

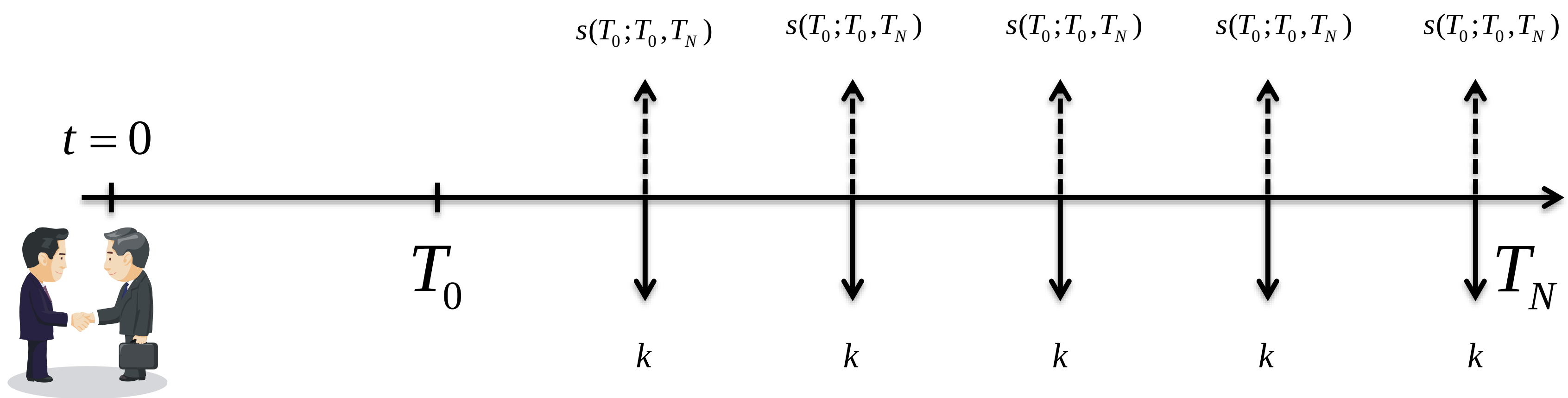
$$swap(T_0;k,T_N)^+ = \left(V_{float} - V_{fix}\right)^+$$

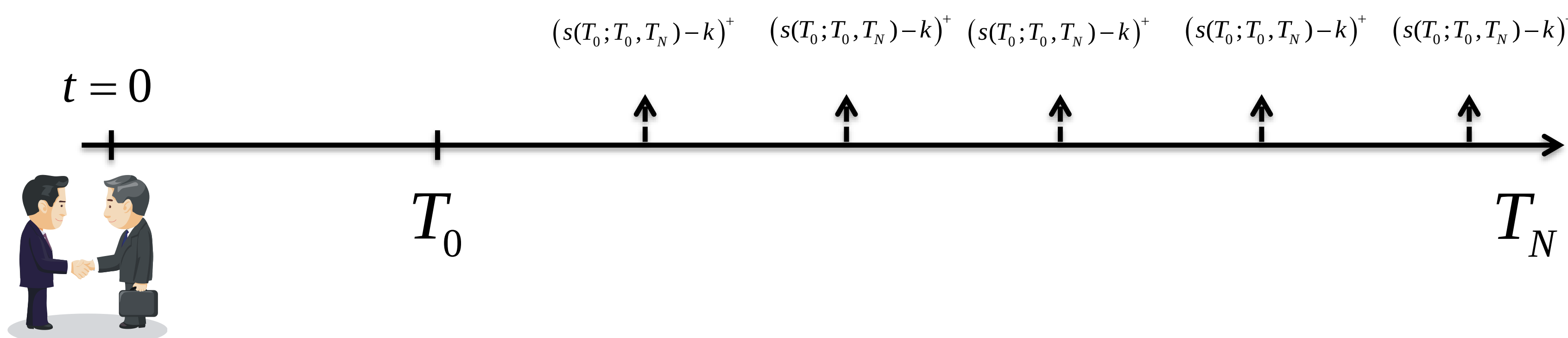
$$= \left(s(T_0;T_0,T_N)\sum_{i=1}^{N} \Delta T \times d(T_0,T_i) + d(T_0,T_N) \right.$$

$$\left. - \sum_{i=1}^{N} \Delta T d(T_0,T_i)k - d(T_0,T_N) \right)^+$$

$$= A(T_0;T_0,T_N)\left(s(T_0;T_0,T_N) - k\right)^+$$

Equal-value cash flows





Black's formula for A Swaplet

- The Black's formula for the j^th swaplet is

$$swlt(0;k,T_j) \\ = \Delta T P(0,T_j)\left(s(0;T_0,T_N)\Phi(d_1) - k\Phi(d_2)\right)$$

where

$$d_1 = \frac{\ln\frac{s(0;T_0,T_N)}{k} + \frac{1}{2}\sigma^2 T_0}{\sigma\sqrt{T_0}}, \quad d_2 = d_1 - \sigma\sqrt{T_0}$$

Black's formula for Swaptions

- The Black's formula for payer's swaption

$$swtn(0;k,T_N)$$

$$=\sum_{j=1}^{N} swlt(0;k,T_j)$$

$$=A(0;T_0,T_N)\left(s(0;T_0,T_N)\Phi(d_1)-k\Phi(d_2)\right)$$

where

$$d_1=\frac{\ln\frac{s(0;T_0,T_N)}{k}+\frac{1}{2}\sigma^2 T_0}{\sigma\sqrt{T_0}},\quad d_2=d_1-\sigma\sqrt{T_0}$$

Black's formula for Swaptions

- The Black's formula for payer's swaption at *t*

$$
\begin{aligned}
& swtn(t;k,T_N) \\
&= \sum_{j=1}^{N} swlt(t;k,T_j) \\
&= A(t;T_0,T_N)\left(s(t;T_0,T_N)\Phi(d_1) - k\Phi(d_2)\right)
\end{aligned}
$$

where

$$
d_1 = \frac{\ln \frac{s(t;T_0,T_N)}{k} + \frac{1}{2}\sigma^2(T_0-t)}{\sigma\sqrt{T_0-t}}, \quad d_2 = d_1 - \sigma\sqrt{T_0-t}
$$

Hedging with ATM swap and Annuity

- For the seller of the payer's swaption, the swaption can be hedged by
  - entering $\Phi(d_1)$ unit of (T_0, T_N) swap to pay fixed.
  - being long $\left(s(t;T_0,T_N)\Phi(d_1) - k\Phi(d_2)\right)$ unit of $A(t;T_0,T_N)$, the annuity.

Example

- The Black's model with forward-rate curve

$$f_j(0) = 0.01 + 0.0005 \times (j - 1),\ j = 1, \cdots, 120.$$

- Swaption and swap maturities: TC=2, TS=10
- ATM swap rate and volatility : k=2.157%, $\sigma = 0.3$
- Swaption value for $1m notional: $26,139.32.
- Swaption calculation

Swaption on receiver's swap

- Payoff of the swaption on a receiver's swap:

$$swtn_R(T_0;k,T_N) = swap_R(T_0;k,T_N)^+$$
$$= \left(V_{fix} - V_{float}\right)^+$$

- Payoff of the swaption on a payer's swap:

$$swtn_P(T_0;k,T_N) = swap_P(T_0;k,T_N)^+$$
$$= \left(V_{float} - V_{fix}\right)^+$$

Call-Put Parity, again

- Payoff of the swaption on a receiver's swap:

$$swtn_R(T_0;k,T_N) = \left(V_{fix} - V_{float}\right)^+$$

- Payoff of the swaption on a payer's swap:

$$swtn_P(T_0;k,T_N) = \left(V_{float} - V_{fix}\right)^+$$

- Portfolio of long p-swtn and short r-swtn equals to swap:

$$swtn_P(T_0;k,T_N) - swtn_R(T_0;k,T_N)$$

$$= \left(V_{float} - V_{fix}\right)^+ - \left(V_{fix} - V_{float}\right)^+ = V_{float} - V_{fix} = swap_P(T_0;k,T_N)$$

Value of the Receiver's Swaption

- By the call-put parity,

$$
\begin{aligned}
& swtn_R(0,k,T_N) \\
&= swtn_P(0,k,T_N) - swap_P(0,k,T_N) \\
&= A(t;T_0,T_N)\left(s(t;T_0,T_N)\Phi(d_1) - k\Phi(d_2)\right) \\
&\qquad\qquad\qquad\qquad - A(t;T_0,T_N)\left(s(t;T_0,T_N) - k\right) \\
&= A(t;T_0,T_N)\left[s(t;T_0,T_N)(\Phi(d_1)-1) - k(\Phi(d_2)-1)\right] \\
&= A(t;T_0,T_N)\left[k\Phi(-d_2) - s(t;T_0,T_N)\Phi(-d_1)\right]
\end{aligned}
$$

Black's formula for Receiver's Swaptions

- The Black's formula for receiver's swaption

$$
\begin{aligned}
& swtn(0;k,T_N) \\
&= A(0;T_0,T_N)\left(k\Phi(-d_2) - s(0;T_0,T_N)\Phi(-d_1)\right)
\end{aligned}
$$

where

$$
d_1 = \frac{\ln\frac{s(0;T_0,T_N)}{k} + \frac{1}{2}\sigma^2 T_0}{\sigma\sqrt{T_0}}, \quad d_2 = d_1 - \sigma\sqrt{T_0}
$$

Hedging with ATM swap and Annuity

- For the seller of the receiver's swaption, the swaption can be hedged by, at time *t*
  - entering $\Phi(-d_1)$ unit of (T_0, T_N) receiver's swap to pay float.
  - being long $\left(k\Phi(-d_2) - s(t;T_0,T_N)\Phi(-d_1)\right)$ unit of the annuity $A(t;T_0,T_N)$.

Examples

- Black's formula for swaption.

  http://www.math.ust.hk/~malwu/math4511/Matlab codes/Black_swaption_call.zip

The General Pricing Principle

- Under both deterministic and stochastic interest rates, option can be priced by
  - Taking the expectation of the terminal payoff,
  - followed by discounting

$$V_t = d(t,T)E\left[V_T(F_T)\right]$$

- Example:

$$V_T(F_T) = \sqrt{\left(F_T - K\right)^+}$$

Monte Carlo Simulations

- Recall that

$$F_T = F_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T} z}, \quad z \sim N(0,1)$$

- Let $z_1, z_2, \cdots, z_n$ be *n* randomly drawn random variable from $N(0,1)$.
- Define

$$F_T^{(i)} = F_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T} z_i}, \quad i = 1, 2, \cdots, n.$$

Monte Carlo Simulations

- Let $V(F_T, T)$ be the option payoff at time *T*.
- Define

$$V_n(F_0,0) = d(0,T)\frac{1}{n}\sum_{i=1}^{n} V(F_T^{(i)},T)$$

- By the Large Number Theorem, there is

$$\lim_{n\to\infty} V_n(F_0,0) = V(F_0,0)$$

- Example: Monte Carlo simulations for call.

  (http://www.math.ust.hk/~malwu/math4511/Matlab%20codes/MC_call.m)

What is more?

- Path dependent options, which normally require monte Carlo simulations.
- We need stochastic models on the intertemporal evolution of the underlying securities $(S_t \text{ or } F_t)$.
- This will require Stochastic Calculus.