

Final Review

Regarding the Final Exam

- To be held in Dec. 18, 2019, offsite, from 8:30-11:30am.
- Exam papers will be available for download from 8:15am in the course webpage:
www.math.ust.hk/~malwu/math4511/m4511.html.
- Submit the **photo copy** of your answer via email to malwu@ust.hk by 11:45am of the day.
- There are three versions of the exam papers, to be adopted according to the ending digit of your **student ID**:

	Paper A	Paper B	Paper C
Ending digit of ID	1, 2, 5, 8	3, 6, 9	4, 7, 0

- For questions during the exam, call me at 6675-4769.

What Have We Learned?

- I. Static hedging and data-based hedging
- II. Term structures of interest rates
- III. Interest-rate or fixed-income derivatives
- IV. Binomial or Ho-Lee model
- V. Black's model

I. Static hedging and data-based hedging

Risk measures

1. DV01
2. Duration (Macaulay or modified)
3. Kr01
4. Key-rate duration

Static Hedging

- Static hedging means to nullify any of these risk measures
 - DV01 or duration neutral
 - Kr01 or key-rate duration neutral
- We hedge security B using a more liquid security A such that

$$\left\{ \begin{array}{l} F_A XV01_A + F_B XV01_B = 0, \\ F_A = -\frac{XV01_B}{XV01_A} F_B, \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} P_A D_A + P_B D_B = 0, \\ P_A = -\frac{D_B}{D_A} P_B. \end{array} \right.$$

Regression of yields

- Regression method between Δy_t^{20} and Δy_t^{30} :

$$\min_{\alpha, \beta} \sum_t \left(\Delta y_t^{20} - \alpha - \beta \Delta y_t^{30} \right)^2$$

yielding

$$\beta = \rho \frac{\sigma_{20}}{\sigma_{30}}, \quad \alpha = 0.$$

and projection of yield change:

$$\Delta y_t^{20} = \alpha + \beta \Delta y_t^{30} + \varepsilon_t$$

Data-Based Hedging

- Let F^{20} be the face amount of the 20-year bond, then for hedging we choose

$$F^{30} = -F^{20} \frac{DV01^{20}}{DV01^{30}} \beta$$

- Let P^{20} be the value of the 20-year bond, then for hedging we choose

$$P^{30} = -P^{20} \frac{D_{20}}{D_{30}} \beta$$

II. Term structures of interest rates

- Term structures
 - i. Discount curve
 - ii. Zero-coupon yield curve
 - iii. Forward rate curve
 - iv. Swap rate or par yield curve
- These curves are equivalent (from one we can derive the other).

III. Interest-rate derivatives

- Linear derivatives
 - FRA
 - Swaps
- Nonlinear derivatives
 - Bond options
 - Caps, floors
 - Swaptions

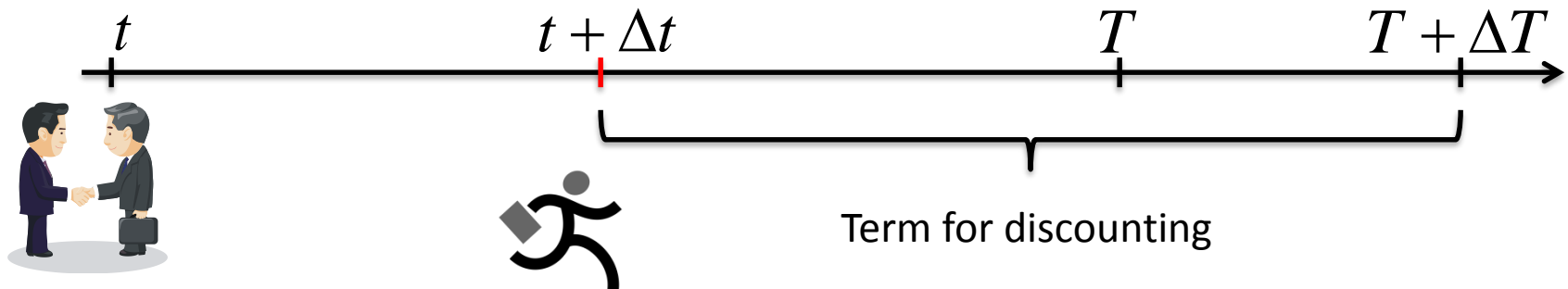
FRA

- A bet on interest rates b/w two parties, fixed for floating, indexed to LIBOR.
- Initially the value of the FRA is zero.
- Typically,
 - Three month LIBOR (or CD rates)
 - At least \$1m notional.
- The arbitrage free fixed rate is

$$f_t = \frac{1}{\Delta T} \left(\frac{d(t, T)}{d(t, T + \Delta T)} - 1 \right)$$

MtM Value of FRA

- Let A long the FRA (i.e. pays fixed) at t .

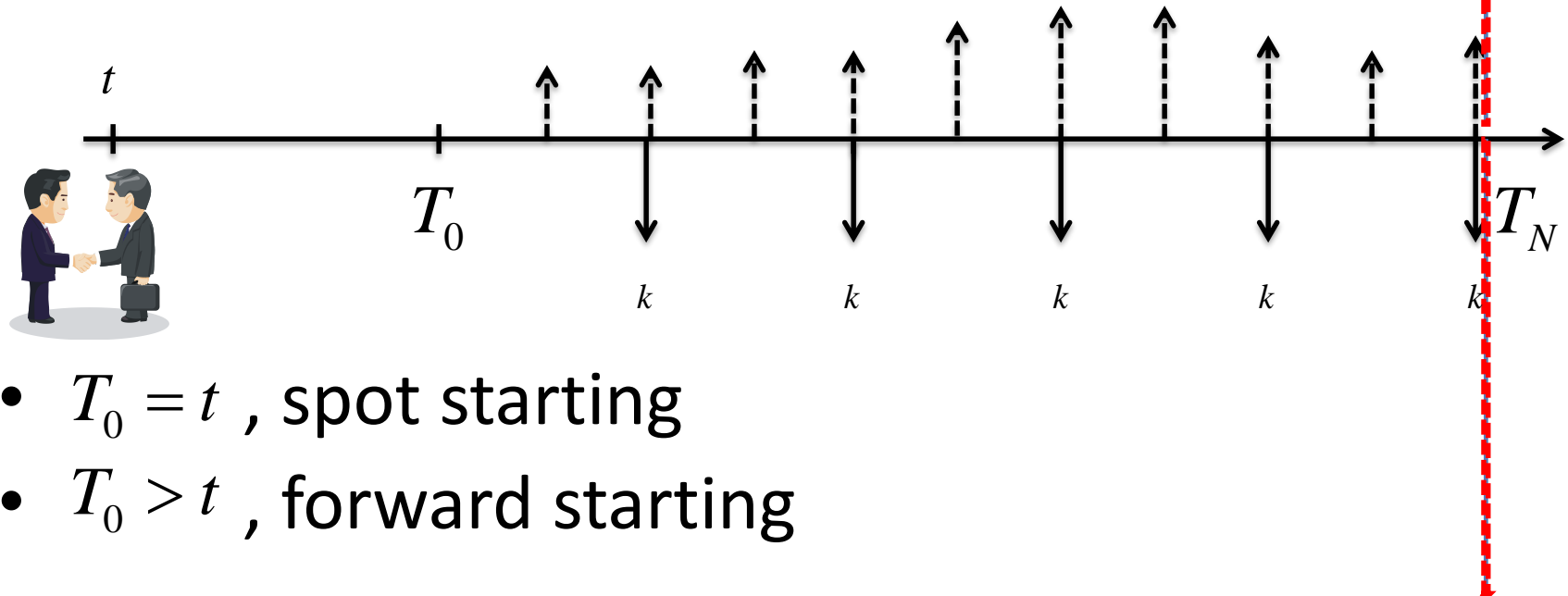


- At a later time $t + \Delta t$, then the MtM value of the FRA to the party who longs

$$\text{P\&L to A} = d(t + \Delta t, T + \Delta T) [\$1m \times \Delta T \times (f_{t+\Delta t} - f_t)]$$

Swaps

- Let a payer's swap start in time T_0 and end at T_N .



- $T_0 = t$, spot starting
- $T_0 > t$, forward starting

Determination of the swap rate

- Floating leg: par at T_0 , so at $t \leq T_0$, it is

$$V_{float} = d(t, T_0)$$

- Fixed leg: let $s(t; T_0, T_N)$ be the swap rate, then

$$V_{fixed} = \sum_{i=1}^N \Delta T \times s(t; T_0, T_N) d(t, T_i) + d(t, T_N)$$

- Set

$$0 = V_{float} - V_{fix}$$

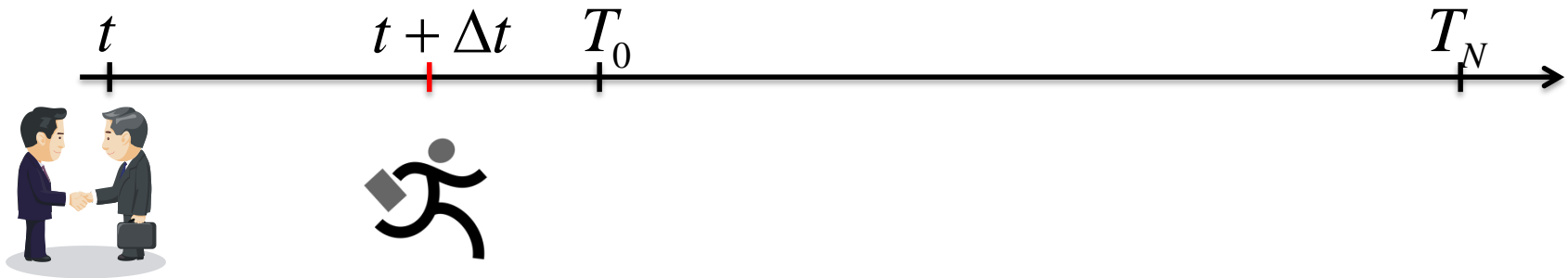
- We obtain

$$s(t; T_0, T_N) = \frac{d(t, T_0) - d(t, T_N)}{\sum_{i=1}^N \Delta T \times d(t, T_i)}$$

the prevailing or ATM swap rate.

MtM Value of a swap

- Let A pays fixed and B pays LIBOR.

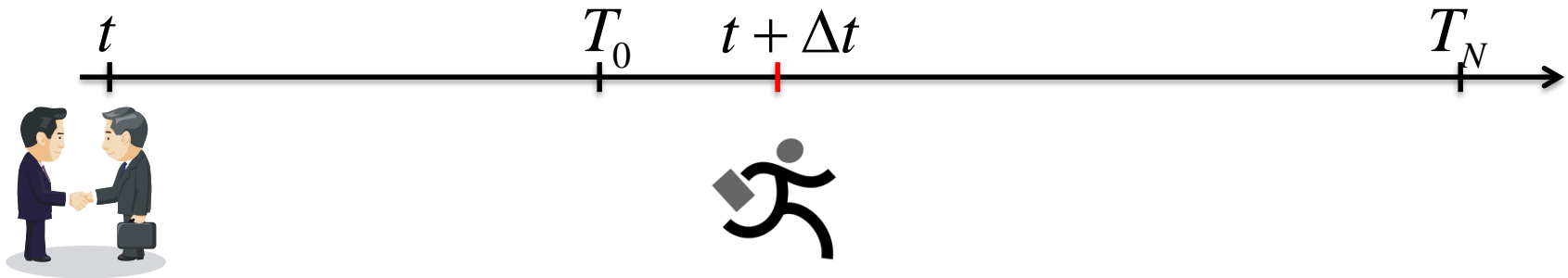


- At time $t + \Delta t$, the MtM value of the swap is

$$\text{MtM to A} = \left(s(t + \Delta t; \textcolor{red}{T}_0, T_N) - s(t; T_0, T_N) \right) \sum_{\textcolor{red}{T}_i, i > 0}^{T_N} \Delta T d(t + \Delta t, T_i)$$

MtM Value of a swap

- Let A pays fixed and B pays LIBOR.



- At time $t + \Delta t$, the MtM value of the swap is

$$\text{MtM to A} = \left(s(t + \Delta t; \textcolor{red}{t} + \textcolor{red}{\Delta t}, T_N) - s(t; T_0, T_N) \right) \sum_{\textcolor{red}{T_i} > t + \Delta t}^{T_N} \Delta T d(t + \Delta t, T_i)$$

Bond Options

- A bond option allows its holder to buy/sell a bond for a pre-specified price in a future date.

- Let the

T_0 – maturity of the option

$T_N - T_0$ – life of the underlying bond at T_0

K – the strike price

c – coupon rate of the bond

Bond Price at T_0

- Payoff of the call option on the bond

$$Option(T_0; K, T_N) = \max(Bond(T_0, T_N; c) - K, 0)$$

- Value of the bond option T_0 :

$$Bond(T_0, T_N; c) = \sum_{i=1}^N \Delta T d(T_0, T_i) c + d(T_0, T_N)$$

Swaptions

- A swaption is an option to enter into a swap for a pre-specified swap rate in the future.

- Let the

T_0 – maturity of the option

$T_N - T_0$ – life of the underlying swap

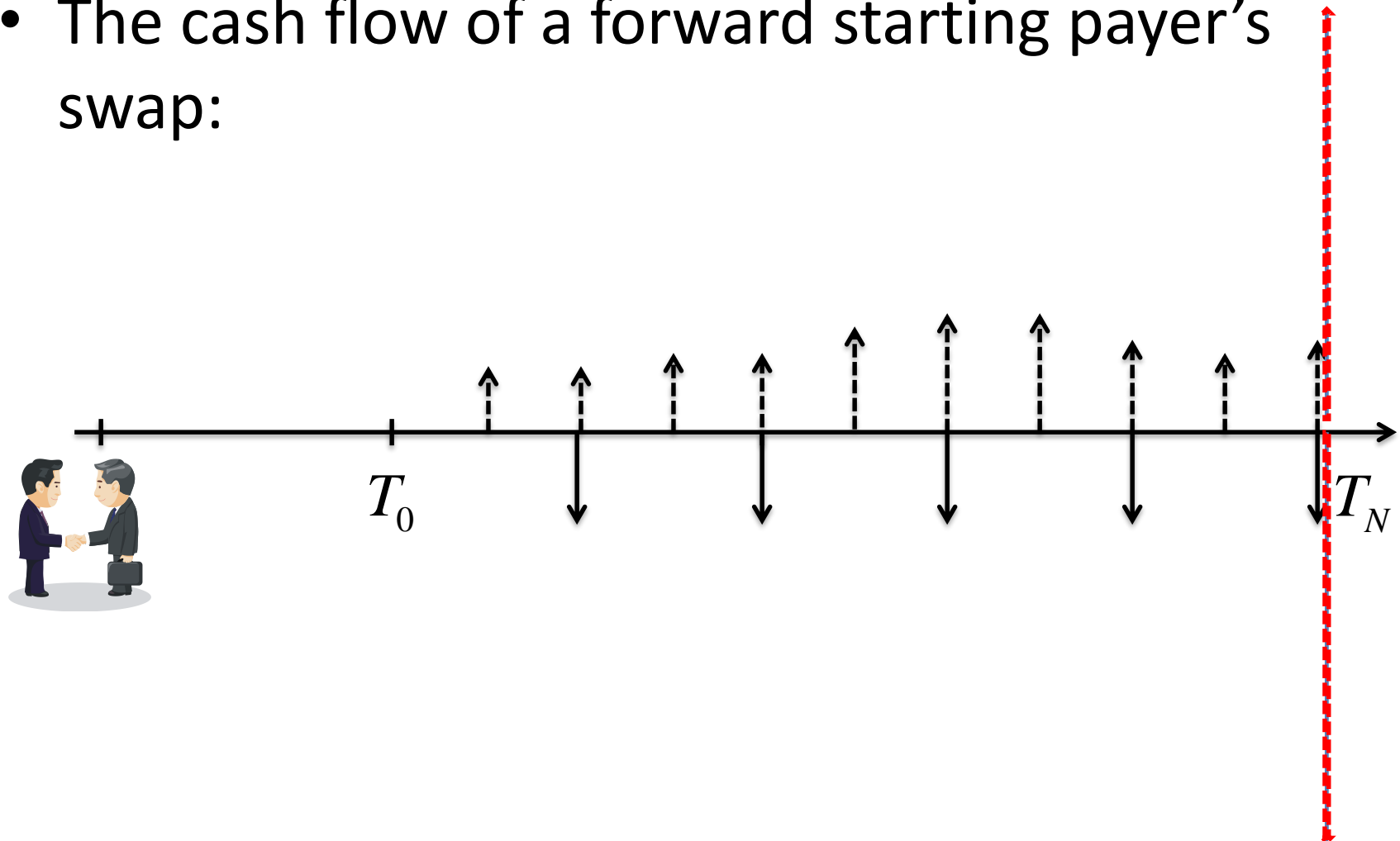
k – the strike rate

- Payoff of swaption

$$swtn(T_0; k, T_N) = \max(\text{swap}(T_0; k, T_N), 0)$$

Cash Flow of The Underling Swap

- The cash flow of a forward starting payer's swap:



Swaption Turned Bond Options

- Payoff of the swaption on a payer's swap:

$$\begin{aligned} swap(T_0; k, T_N)^+ &= (V_{float} - V_{fix})^+ \\ &= \left(1 - \sum_{i=1}^N \Delta T d(T_0, T_i) k - d(T_0, T_N) \right)^+ \end{aligned}$$

- A payer's swaption can be priced as a put option on a coupon bond with **PAR** strike!

IV: Binomial or Ho-Lee model

- Binomial model for equity options (omitted)
- Ho-Lee model for interest-rate options.

Ho-Lee model Interest-Rate Models

- The discrete version of the Ho-Lee model is

$$\Delta r_t = \theta_t \Delta t + \sigma \sqrt{\Delta t} \varepsilon_B$$

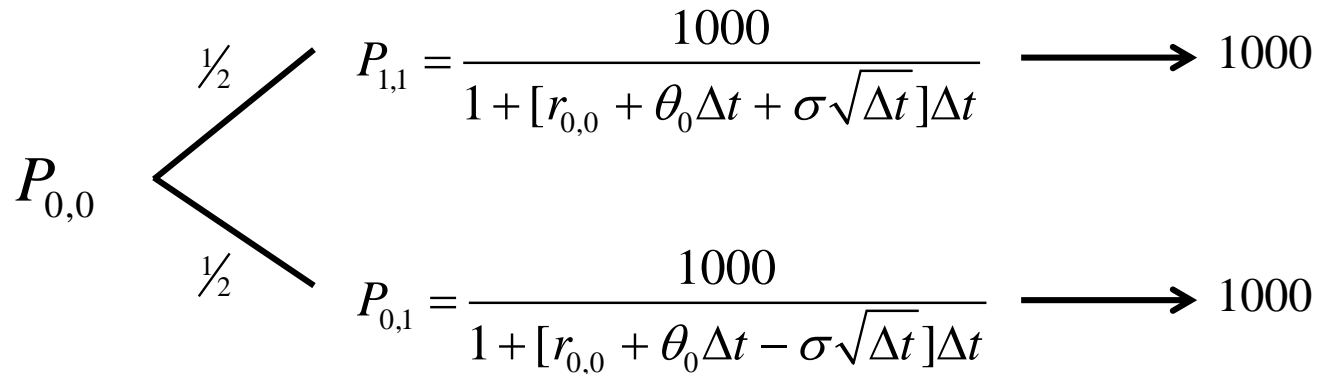
where θ_t is fitted to the discount curve.

- Rate tree

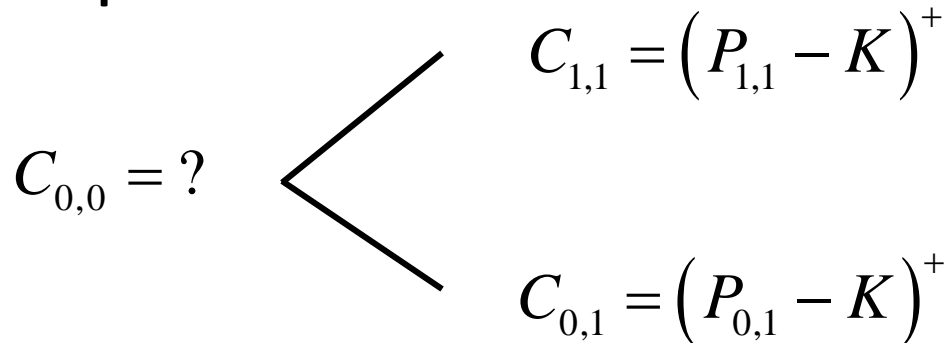
$$\begin{array}{lcl} & \nearrow \frac{1}{2} & r_{\Delta t}^u = r_0 + \theta_0 \Delta t + \sigma \sqrt{\Delta t} \\ r_0 & & \\ & \searrow \frac{1}{2} & r_{\Delta t}^d = r_0 + \theta_0 \Delta t - \sigma \sqrt{\Delta t} \end{array}$$

Bond and option trees

- Price tree



- Option price tree



Arbitrage pricing, cont'd

- Consider replicating the payoffs with (α, β) units of bond and cash, such that

$$\alpha P_{0,1} + \beta(1 + r_{0,0}\Delta t) = C_{0,1}$$

$$\alpha P_{1,1} + \beta(1 + r_{0,0}\Delta t) = C_{1,1}$$

- Solution

$$\alpha = \frac{C_{1,1} - C_{0,1}}{P_{1,1} - P_{0,1}}, \quad \beta = \frac{P_{1,1}C_{0,1} - P_{1,0}C_{1,1}}{(1 + r_{0,0}\Delta t)(P_{1,1} - P_{1,0})}$$

- Arbitrage-free value: $C_{0,0} = \alpha P_{0,0} + \beta,$

Linear pricing rule

- Rewrite the option formula into

$$\begin{aligned}C_{0,0} &= \alpha P_{0,0} + \beta \\&= \frac{C_{1,1} - C_{0,1}}{P_{1,1} - P_{0,1}} P_{0,0} + \frac{(P_{1,1} C_{0,1} - P_{1,0} C_{1,1})}{(1 + r_{0,0} \Delta t)(P_{1,1} - P_{0,1})} \\&= (1 + r_{0,0} \Delta t)^{-1} \left(\frac{P_{1,1} - P_{0,0}(1 + r_{0,0} \Delta t)}{P_{1,1} - P_{0,1}} C_{0,1} + \frac{P_{0,0}(1 + r_{0,0} \Delta t) - P_{0,1}}{P_{1,1} - P_{0,1}} C_{1,1} \right) \\&= (1 + r_{0,0} \Delta t)^{-1} (q_0 C_{0,1} + (1 - q_0) C_{1,1}) \quad !!!\end{aligned}$$

Linear pricing rule

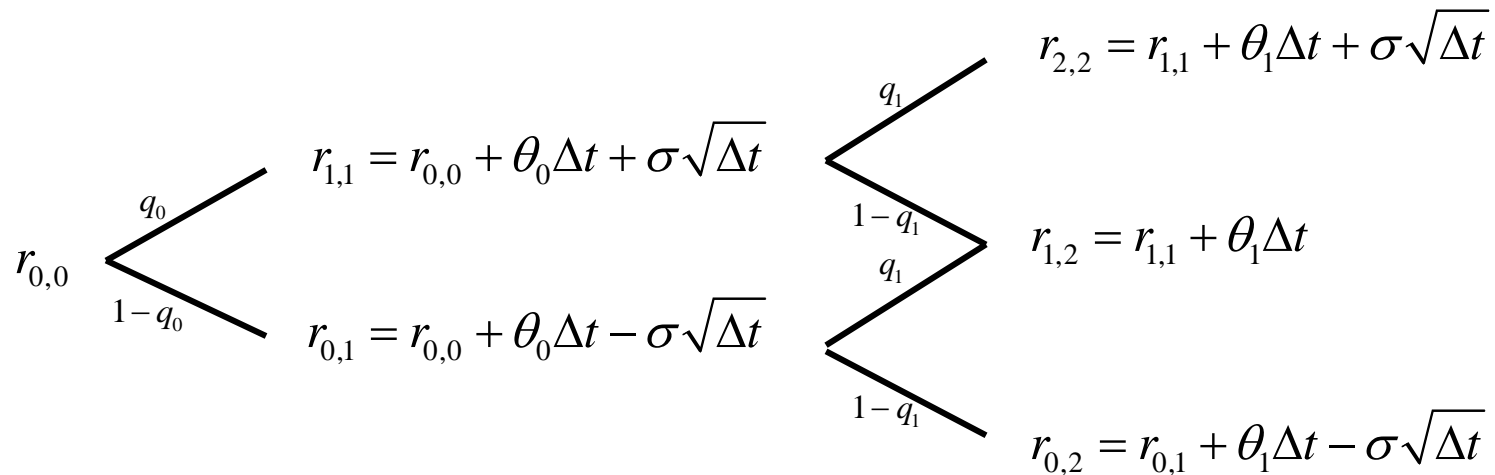
- There is

$$P_{0,0} = (1 + r_{0,0}\Delta t)^{-1} (q_0 P_{0,1} + (1 - q_0) P_{1,1})$$

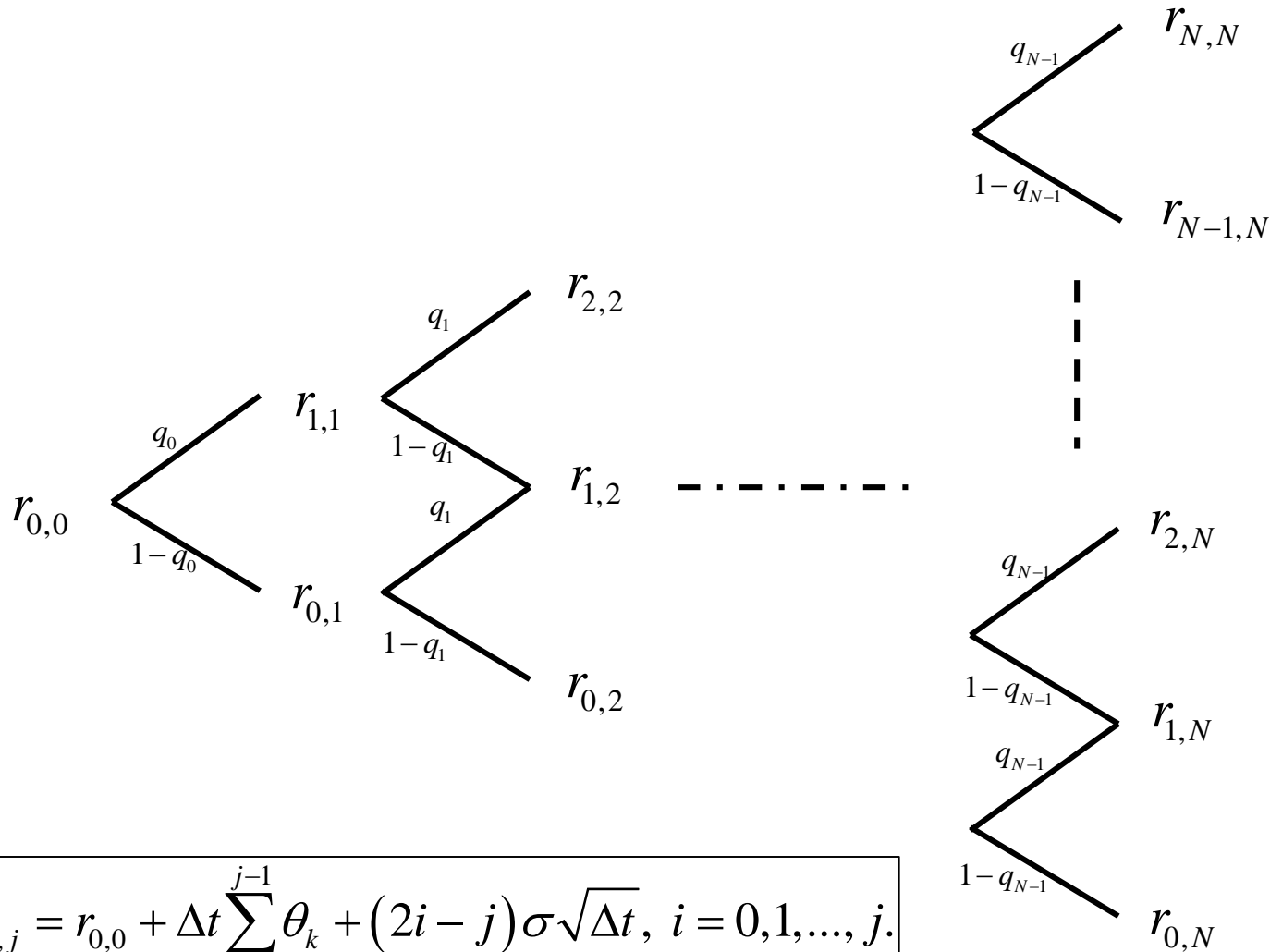
--- the pricing probabilities reproduce the bond price!

Two-period risk-neutral tree

- By duplication, we obtain



Extension to multi-period tree



$$r_{i,j} = r_{0,0} + \Delta t \sum_{k=0}^{j-1} \theta_k + (2i - j) \sigma \sqrt{\Delta t}, \quad i = 0, 1, \dots, j.$$

Inputs for tree building

- The inputs include
 - Growth rate θ_t and volatility σ
 - The term structure of discount curve, $P(j\Delta t)$
 $j = 1, \dots, N + 1$
 - Chosen step size, Δt
- Determine q_{j-1} by fitting to the price of $P((j+1)\Delta t)$, $j = 1, \dots, N$.

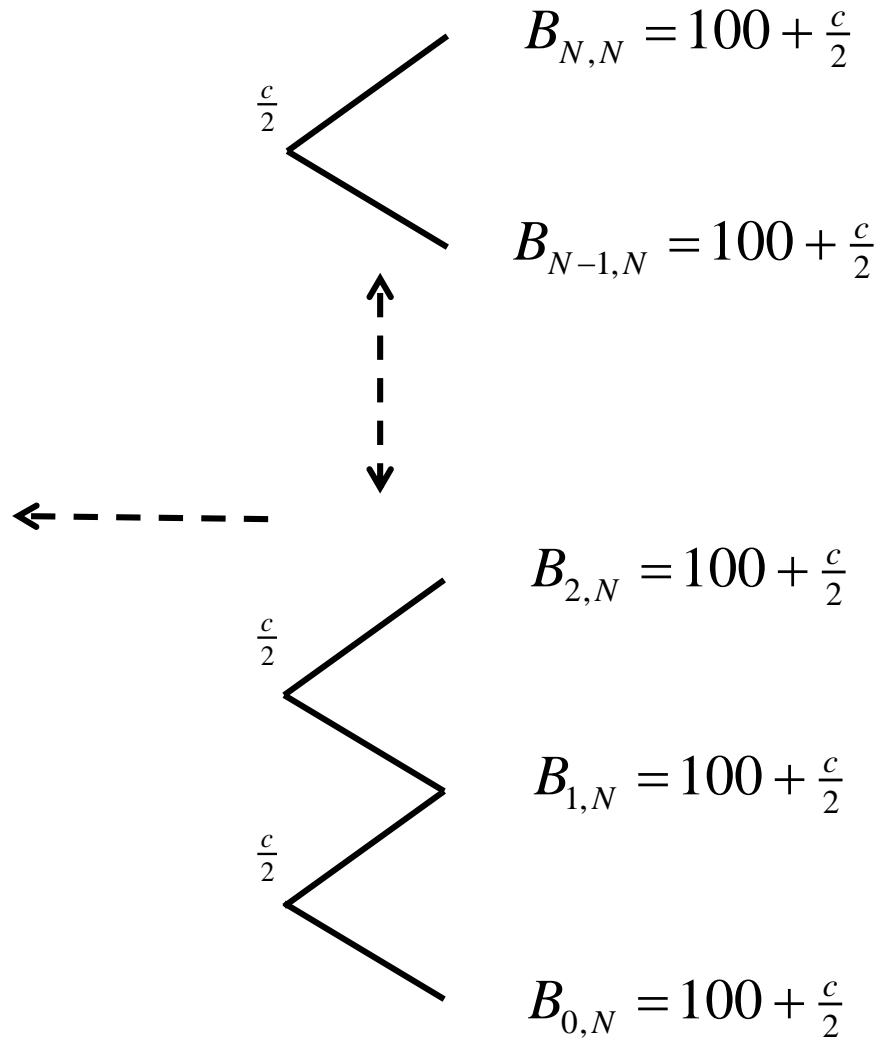
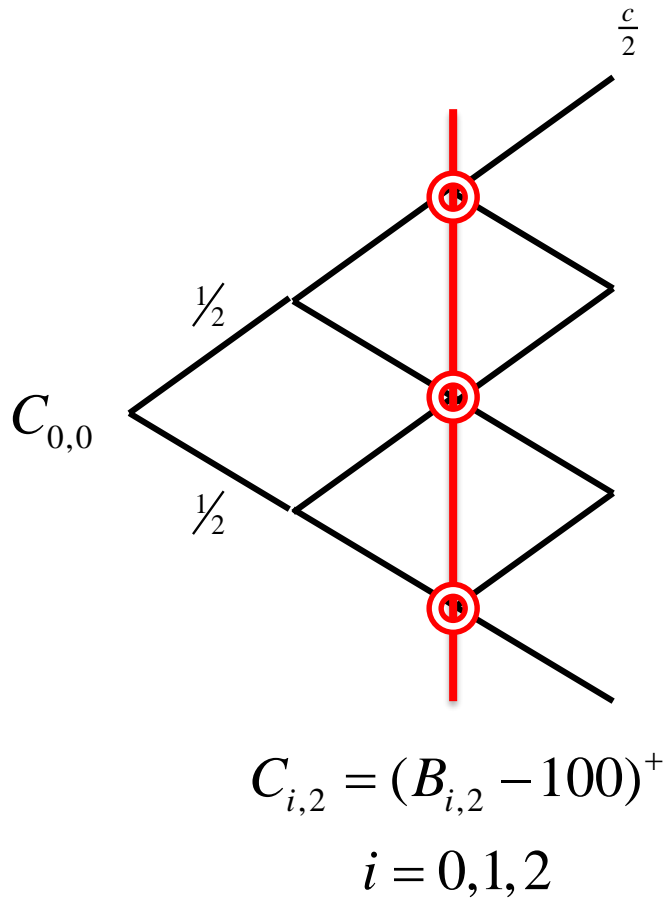
Pricing options on coupon bonds

1. Build a tree for ten years that reproduce the discount curve between 0 to TB years.
2. Obtain the price distribution of the coupon bond in TC years.
3. Calculate option's payoff.
4. Calculate the PV of the option.

Ex: $TC=1$, $TB=10$, $c=5$.

Strategy, cont'd

- Let $N = 20$, $c = 5$



V. The Black's formula

- The Black's formula (1976) applies to call and put options on
 - Equity derivatives
 - Interest-rate derivatives;
 - Forex derivatives
 - Commodity derivatives;
 - All derivatives;

Forward Contract

- A forward contract allows its counterparties to buy/sell certain asset for certain price K in certain date T in the future.
- The value of the long forward price is

$$V_t = S_t - d(t, T)K$$

- The forward price

$$F_t = \frac{S_t}{d(t, T)}, \quad \text{for } t \leq T,$$

MtM a forward Contract

- If you enter a T -maturity forward contract at time t and close it out at time $t+dt$, then the P&L is

$$\begin{aligned} P \ \& \ L &= d(t + dt, T) [F_{t+dt} - F_t] \\ &= d(t + dt, T) dF_t \end{aligned}$$

- This is the PV if you hold the two contracts till the maturity.

Features of the Forward Price

- The forward price satisfies
 1. At an option's maturity, forward = spot:

$$F_T = \frac{S_T}{d(T, T)} = S_T$$

2. It is assumed that

or
$$\ln \frac{F_T}{F_t} = -\frac{1}{2} \sigma^2 (T - t) + \sigma \sqrt{T - t} z$$

$$F_T = F_t e^{-\frac{1}{2} \sigma^2 (T - t) + \sigma \sqrt{T - t} z}$$

Options on Forward

- Black's model for call option.

$$\begin{aligned}C_t &= d(t, T) E \left[\left(F_T - K \right)^+ \right] \\&= d(t, T) E \left[\left(F_t e^{-\frac{\sigma^2}{2}(T-t) + \sigma \sqrt{T-t} z} - K \right)^+ \right] \\&= d(t, T) \left[F_t \Phi(d_1) - K \Phi(d_2) \right]\end{aligned}$$

where

$$d_1 = \frac{\ln \frac{F_t}{K} + \frac{1}{2} \sigma^2 (T-t)}{\sigma \sqrt{T-t}}, \quad d_2 = d_1 - \sigma \sqrt{T-t}.$$

Black's formula for call option

- Let

$$g(F_t) = [F_t \Phi(d_1) - K \Phi(d_2)]$$

- There is

$$\frac{\partial g(F_t)}{\partial F_t} = \Phi(d_1)$$

- We write

$$C_t = d(t, T) g(F_t)$$

The differentiation

- The change over $(t, t + dt)$ is

$$\begin{aligned}
 dC_t &= C_{t+dt} - C_t \\
 &= d(t + dt, T)g(F_{t+dt}) - d(t, T)g(F_t) \\
 &= d(t + dt, T)[g(F_{t+dt}) - g(F_t)] \\
 &\quad + [d(t + dt, T) - d(t, T)]g(F_t) \\
 &= d(t + dt, T)\frac{\partial g(F_t)}{\partial F_t}dF_t(t) + g(F_t)dd(t, T) \\
 &= \underbrace{\Phi(d_1) \times d(t + dt, T)dF_t}_{\text{Change in Forward contract}} + \underbrace{g(F_t)dd(t, T)}_{\text{Change in ZCB}}
 \end{aligned}$$

Hedging Strategy

- For the call seller, the call can be hedged by
 - Long $\Phi(d_1)$ unit of T -maturity forward contract.
 - Long $g(F_t)$ unit of the T -maturity zero-coupon bond (ZCB).

Call-Put Parity

- For the same strike and maturity,

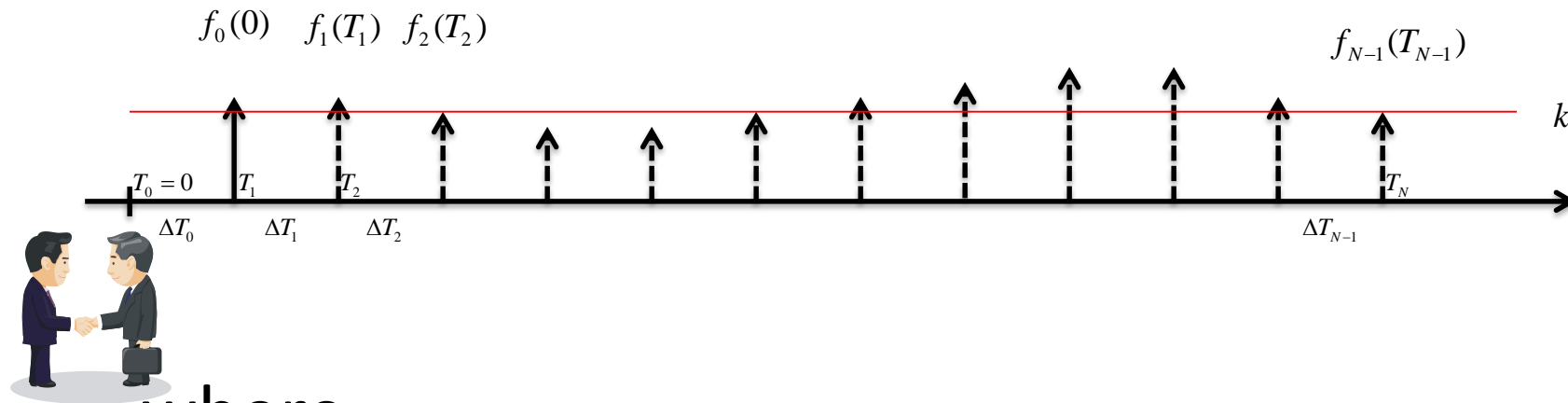
$$\text{Call} - \text{Put} = \text{Forward contract}$$

- i.e.,

$$\begin{aligned} P_t &= C_t - d(t, T)(F_t - K) \\ &= d(t, T) [\Phi(-d_2)K - \Phi(-d_1)F_t] \end{aligned}$$

Interest rate caps

- The cash flows of an interest rate cap:



where

$$f_{j-1}(T_{j-1}) = \frac{1}{\Delta T_j} \left(\frac{d(T_{j-1}, T_{j-1}) - d(T_{j-1}, T_j)}{d(T_{j-1}, T_j)} \right)$$

- Cash flow of an interest-rate cap:

$$Not. \times \Delta T_j (f_{j-1}(T_{j-1}) - k)^+, \quad j = 1, 2, \dots, N.$$

- The Black's formula for cap:

$$Cap = \sum_{j=1}^N c_j$$

where

$$c_j = Not. \times \Delta T_j \times d(t, T_j) \left[f_{j-1}(t) \Phi(d_1^{(j)}) - k \Phi(d_2^{(j)}) \right],$$

$$d_1^{(j)} = \frac{\ln \frac{f_{j-1}(t)}{k} + \frac{1}{2} \sigma^2 (T_{j-1} - t)}{\sigma \sqrt{T_{j-1} - t}}, \quad d_2^{(j)} = d_1^{(j)} - \sigma \sqrt{T_{j-1} - t}$$

$$j = 1, 2, \dots, N$$

Hedging with FRA and ZCB

- Let

$$g_j(t) = \text{Not.} \times \Delta T_j \times \left[f_{j-1}(t) \Phi(d_1^{(j)}) - k \Phi(d_2^{(j)}) \right]$$

- For the cap underwriter, the cap can be hedged by
 - Enter $\Phi(d_1^{(j)})$ unit of T_{j-1} -maturity FRA for the term (T_{j-1}, T_j) to long rate or pay fixed
 - being long or $g_j(t)$ unit of $P(t, T_j)$, the T_j -maturity zero-coupon bond (ZCB).

Interest rate floors

- Cash flow of an interest-rate floor: at T_j ,

$$Not. \times \Delta T_j (k - f_{j-1}(T_{j-1}))^+, \quad j = 1, 2, \dots, N.$$
- The Black's formula

$$floor = \sum_{j=1}^N p_j$$

$$p_j = Not. \times \Delta T_j \times d(t, T_j) \left[k \Phi(-d_2^{(j)}) - f_{j-1}(t) \Phi(-d_1^{(j)}) \right],$$

$$d_1^{(j)} = \frac{\ln \frac{f_{j-1}(t)}{k} + \frac{1}{2} \sigma^2 (T_{j-1} - t)}{\sigma \sqrt{T_{j-1} - t}}, \quad d_2^{(j)} = d_1^{(j)} - \sigma \sqrt{T_{j-1} - t}$$

$$j = 1, 2, \dots, N$$

Hedging with FRA and ZCB

- Let

$$g_j(t) = Not. \times \Delta T_j \times \left[k\Phi(-d_2^{(j)}) - f_{j-1}(t)\Phi(-d_1^{(j)}) \right]$$

- For the cap underwriter, the cap can be hedged by
 - enter $\Phi(-d_1^{(j)})$ unit of T_{j-1} -maturity FRA for the term (T_{j-1}, T_j) to short rate or pay float
 - being long $g_j(t)$ unit of $P(0, T_j)$, the T_j -maturity zero-coupon bond (ZCB).

Swaption pricing revisited

- Payoff of the swaption on a payer's swap:

$$\begin{aligned} swap(T_0; k, T_N)^+ &= (V_{float} - V_{fix})^+ \\ &= \left(\textcolor{red}{1} - \sum_{i=1}^N \Delta T d(T_0, T_i) k - d(T_0, T_N) \right)^+ \end{aligned}$$

- At T_0 , the swap rate of maturity $T_N - T_0$ is

$$s(T_0; T_0, T_N) = \frac{d(T_0, T_0) - d(T_0, T_N)}{\sum_{i=1}^N \Delta T \times d(T_0, T_i)} = \frac{\textcolor{red}{1} - d(T_0, T_N)}{A(T_0; T_0, T_N)}$$

- Swap rate = par yield

$$1 = s(T_0; T_0, T)A(T_0; T_0, T_N) + d(T_0, T_N)$$

- Swaption payoff

$$\begin{aligned} swap(T_0; k, T_N)^+ &= (V_{float} - V_{fix})^+ \\ &= (s(T_0; T_0, T_N)A(T_0; T_0, T_N) + d(T_0, T_N) \\ &\quad - kA(T_0; T_0, T_N) - d(T_0, T_N))^+ \\ &= A(T_0; T_0, T_N)(s(T_0; T_0, T_N) - k)^+ \end{aligned}$$

Black's formula for P-Swaptions

- The Black's formula for the payer's swaption

$$\begin{aligned} & swtn_P(t; k, T_N) \\ &= A(t; T_0, T_N) \left(s(t; T_0, T_N) \Phi(d_1) - k \Phi(d_2) \right) \end{aligned}$$

where

$$d_1 = \frac{\ln \frac{s(t; T_0, T_N)}{k} + \frac{1}{2} \sigma^2 (T_0 - t)}{\sigma \sqrt{T_0 - t}}, \quad d_2 = d_1 - \sigma \sqrt{T_0 - t}$$

- Let

$$g(s) = s \Phi(d_1) - k \Phi(d_2)$$

Hedging with ATM swap and Annuity

- For the seller of the payer's swaption, the swaption can be hedged by
 - entering $\Phi(d_1)$ unit of (T_0, T_N) swap to pay fixed.
 - being long $g(s(t; T_0, T_N))$ unit of $A(t; T_0, T_N)$, the annuity.

Black's formula for R-Swaptions

- The Black's formula for the receiver's swaption

$$\begin{aligned} & swtn_R(t; k, T_N) \\ &= A(t; T_0, T_N) \left(k \Phi(-d_2) - s(t; T_0, T_N) \Phi(-d_1) \right) \end{aligned}$$

where

$$d_1 = \frac{\ln \frac{s(t; T_0, T_N)}{k} + \frac{1}{2} \sigma^2 (T_0 - t)}{\sigma \sqrt{T_0 - t}}, \quad d_2 = d_1 - \sigma \sqrt{T_0 - t}$$