Chapter 5
One-Factor Risk Metrics and Hedges

Overview

- Measure of risk
  - DV01
  - Duration
  - Convexity
- Yield-based risk management
  - Hedging

Definition of DV01, cont'd

- Suppose the price of a security is the function of a yield, y, then DV01 of the security is the change in price for one-basis point of change in yield:

$$\begin{aligned} DV01 &= -\Delta P(y) \\ &= -P(y+0.01\%) + P(y) \end{aligned}$$

Example

- Ex: The yield of 5s of 2/15/2011 at 2/15/2001 is y=5%. What is its DV01?

$$P(y) = 100$$

$$P(y + 0.01\%) = 100\left[\frac{c}{y} + \left(1 - \frac{c}{y}\right)\left(1 + \frac{y}{2}\right)^{-2T}\right]\Bigg|_{\substack{c=y=5\% \\ T=10}} = 99.9221$$

$$\Rightarrow DV01 = -99.9221 + 100 = 0.0779$$

Example, cont'd

- When yield drops by one basis point, there is

$$P(y) = 100$$

$$P(y - 0.01\%) = 100\left[\frac{c}{y} + \left(1 - \frac{c}{y}\right)\left(1 + \frac{y}{2}\right)^{-2T}\right]\Bigg|_{\substack{c=y=5\% \\ T=10}} = 100.0780$$

$$\Rightarrow DV01 = 100.0780 - 100 = 0.0780$$

- So

$$P(y - 0.01\%) - P(y) \neq P(y) - P(y + 0.01\%)$$

DV01 Redefined

- For symmetry and higher accuracy, we redefine

$$DV01 = -\frac{P(y+0.01\%) - P(y-0.01\%)}{2}$$

- In the last example

$$DV01 = 0.07794$$

Price curve

- For bond

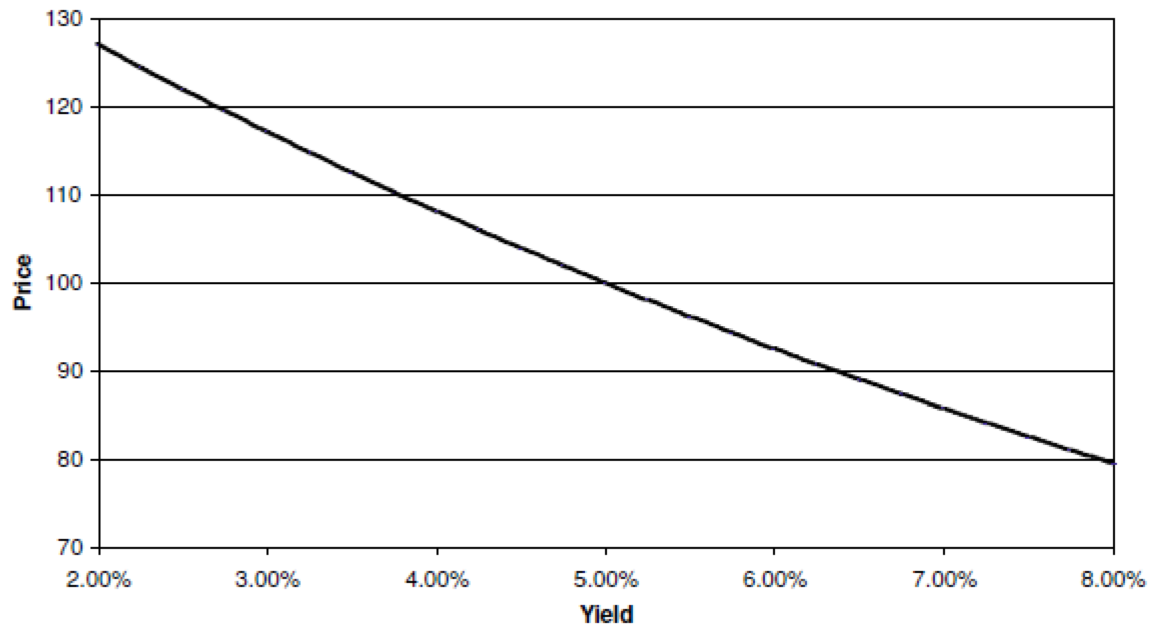


FIGURE 5.1 The Price-Rate Function of the 5s of February 15, 2011

Bond Option Value vs. Yields

- For option

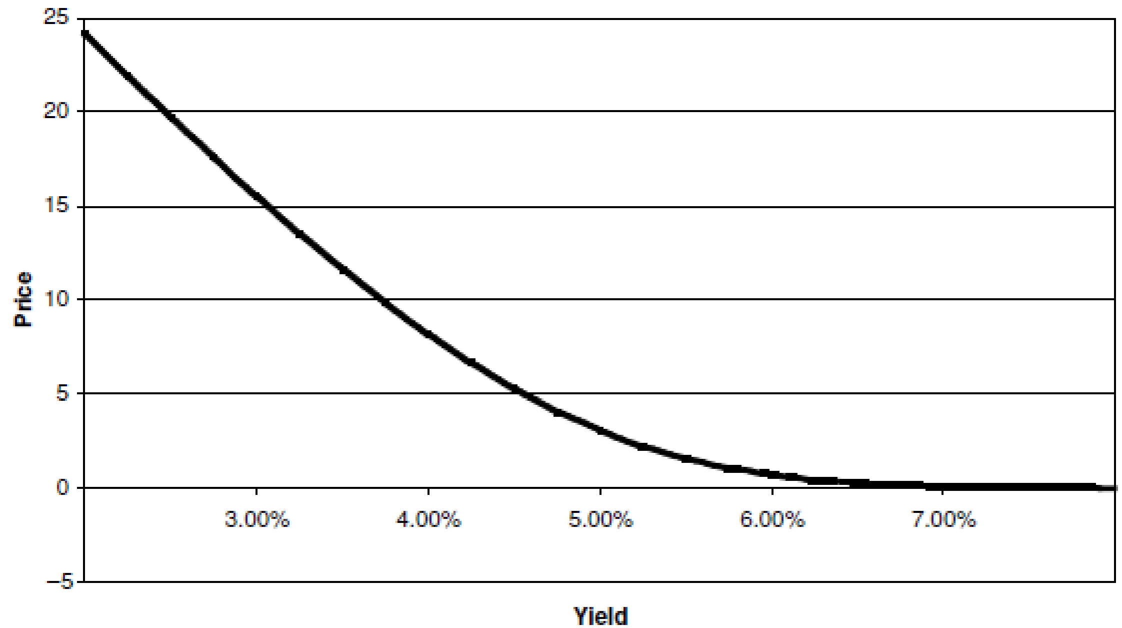


FIGURE 5.2 The Price-Rate Function of a One-Year European Call Option Struck at Par on the 5s of February 15, 2011

Option Definition

- A call option on a bond is a right to buy the bond in a future date for a specified price.
  - Ex: At 2/15/2001, you acquire the right to buy 5s 2/15/2011 at par in 2/15/2006.
- A put option on a bond is a right a sell the bond in a future date for a specified price
  - Ex: At 2/15/2001, you acquire the right to sell 5s 2/15/2011 at par in 2/15/2006.

DV01 of bond & option

- Ex: In 2/15/2002, the price of a call option to buy 5s of 2/15/2011 at PAR is

TABLE 5.1 Selected Option Prices, Underlying Bond Prices, and DV01s at Various Rate Levels

Rate Level	Option Price	Option DV01	Bond Price	Bond DV01
3.99%	8.2148		108.2615	
4.00%	8.1506	0.0641	108.1757	0.0857
4.01%	8.0866		108.0901	
4.99%	3.0871		100.0780	
5.00%	3.0501	0.0369	100.0000	0.0779
5.01%	3.0134	0.03685	99.9221	0.07795
5.99%	0.7003		92.6322	
6.00%	0.6879	0.0124	92.5613	0.0709
6.01%	0.6756		92.4903	

DV01 based hedging

- Question: a market maker sells \$100m face value of the call option when the yield is 5%. How should the market maker hedge the market risk?
- Answer: Long the underlying bond of certain face value to make the total DV01 vanishes, or make the hedge “DV01 neutral”.

The hedge

- Let F be the face value for bond buying, then F is subject to

$$F \times DV01_{bond} = \$100m \times DV01_{option}$$

- If follows that

$$F = \$100m \times \frac{DV01_{option}}{DV01_{bond}} = \$100m \times \frac{0.0369}{0.0779}$$

$$= \$47,370,000$$

Reality Check

- If yield decreases by one bps, then the value of the portfolio (of short call and long bond) is

$$\begin{aligned} \mathit{Before} &= -\$100m \times \frac{3.0501}{100} + 47,370,000 \times \frac{100}{100} \\ &= \$44,319,900 \\ \mathit{After} &= -\$100m \times \frac{3.0871}{100} + 47,370,000 \times \frac{100.078}{100} \\ &= \$44,319,847 \\ \mathit{Change} &= \$53 \qquad \Rightarrow \qquad \text{negligible!} \end{aligned}$$

General DV01 hedge

- If we hedge security A with security B, the face amount of B for hedging is given by

$$F_B = F_A \times \frac{DV01_A}{DV01_B}$$

Duration

- We often need to calculate price change by percentage:

$$\frac{\Delta P}{P} = \frac{P(y+\Delta y) - P(y)}{P(y)}$$

- There is

$$\frac{\Delta P}{P} = \frac{1}{P}\frac{\Delta P}{\Delta y}\Delta y \triangleq -D\Delta y$$

where $D = -\frac{1}{P}\frac{\Delta P}{\Delta y}$ is called *duration*.

Duration, cont'd

- Duration measures the rate of change per dollar value of a security.
- Duration value is not sensitive to the size of $\Delta y \ll 1$. When an explicit formula for the price-rate function is available, we calculate the duration by

$$D = -\frac{1}{P}\frac{\Delta P}{\Delta y} \approx -\frac{1}{P}\frac{dP}{dy}$$

Example of duration

TABLE 5.2 Selected Option Prices, Underlying Bond Prices, and Durations at Various Rate Levels

Rate Level	Option Price	Option Duration	Bond Price	Bond Duration
3.99%	8.2148		108.2615	
4.00%	8.1506	78.60	108.1757	7.92
4.01%	8.0866		108.0901	
4.99%	3.0871		100.0780	
5.00%	3.0501	120.82	100.0000	7.79
5.01%	3.0134		99.9221	
5.99%	0.7003		92.6322	
6.00%	0.6879	179.70	92.5613	7.67
6.01%	0.6756		92.4903	

Example

- Ex: (see table 5.2) Central differencing is used to calculate the duration. At $y = 4\%$, we have, for bond

$$D = -\frac{1}{108.1751}\frac{(108.0901 - 108.2615)}{0.02\%} = 7.92$$

- While for the option

$$D = -\frac{1}{8.1506}\frac{(8.0866 - 8.2148)}{0.02\%} = 78.60$$

Origin of Duration

- MacCaulay duration (1932): let $T_i = i \times \Delta T = 0.5i$, then

$$D_{mac} = \frac{1}{P}\left(\sum_{i=1}^{2T} \Delta T c \times d(T_i) \times T_i + d(T) \times T\right)$$

--- the weighted average of cash flow (coupon and/or principal) dates.

- $D = -\frac{1}{P}\frac{\Delta P}{\Delta y}$ is called the modified duration.

Duration of a portfolio

- Consider a portfolio of two securities, $P = \{P_1, P_2\}$, assume $P_1 + P_2 \neq 0$. By definition

$$
\begin{aligned}
D_p &= -\frac{1}{P_1 + P_2} \frac{\Delta(P_1 + P_2)}{\Delta y} \\
&= -\frac{1}{P_1 + P_2} \left(P_1 \frac{1}{P_1} \frac{\Delta P_1}{\Delta y} + P_2 \frac{1}{P_2} \frac{\Delta P_2}{\Delta y} \right) \\
&= \frac{P_1}{P_1 + P_2} \left(-\frac{1}{P_1} \frac{\Delta P_1}{\Delta y} \right) + \frac{P_2}{P_1 + P_2} \left(-\frac{1}{P_2} \frac{\Delta P_2}{\Delta y} \right) \\
&= \frac{P_1}{P_1 + P_2} D_1 + \frac{P_2}{P_1 + P_2} D_2
\end{aligned}
$$

Duration neutral hedge

- If we try to hedge P_1 using P_2 , we long or short P_2 in the amount

$$P_2 = -\frac{D_1}{D_2} P_1$$

Example

- To hedge one unit of LONG option in Table 5.1, we need to short

$$P_2 = -\frac{78.60}{7.92} \times \$8.1506 = -\$80.88$$

or $80.88/108.1751 = 0.74$ unit of the bond.

Duration vs. DV01

- Duration links to DV01 *linearly*

$$
\begin{aligned}
D &= -\frac{1}{P}\frac{\Delta P}{\Delta y} \\
&= \frac{1}{P}\frac{DV01}{0.0001} \\
&= \frac{10000}{P} DV01
\end{aligned}
$$

Duration neutral vs. DV01 neutral

- The same.
- Ex: To hedge a unit of LONG option in Table 5.1, we need to short

$$F_2 = \frac{DV01_{option}}{DV01_{bond}} F_1$$

$$= \frac{0.0641}{0.0857} \times \$100m = \$74,795,799$$

- The same answer.

2nd-order approximation of price change

- A *second-order Taylor approximation* of the price-rate function

$$P(y+\Delta y) \approx P(y) + \frac{dP}{dy}\Delta y + \frac{1}{2}\frac{d^2P}{dy^2}\Delta y^2$$

$$\Delta P \approx \frac{dP}{dy}\Delta y + \frac{1}{2}\frac{d^2P}{dy^2}\Delta y^2$$

- So,

$$\frac{\Delta P}{P} \approx \frac{1}{P}\frac{dP}{dy}\Delta y + \frac{1}{2}\frac{1}{P}\frac{d^2P}{dy^2}\Delta y^2$$

$$\frac{\Delta P}{P} \approx -D\Delta y + \frac{1}{2}C\Delta y^2$$

Convexity

where

$$C \equiv \frac{1}{P}\frac{d^2 P}{dy^2}$$

is called the convexity or convexity measure.

Quality of approximations: bond

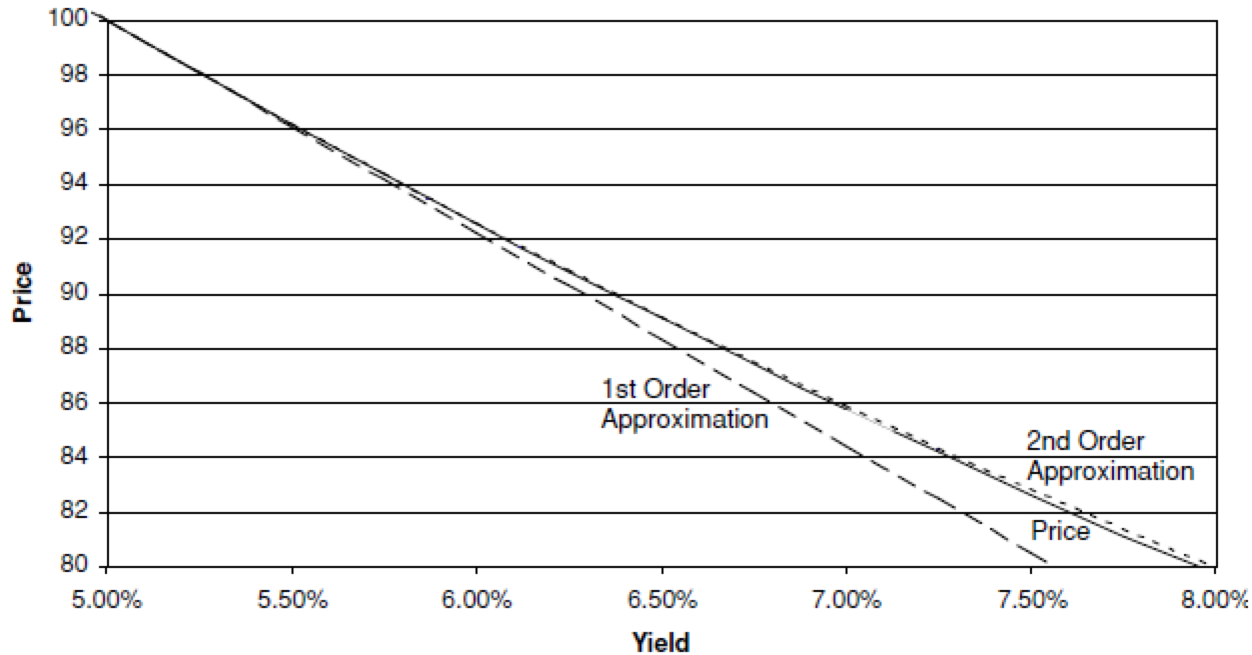


FIGURE 5.7 First and Second Order Approximations to Price of 5s of February 15, 2011

Quality of approximations: option

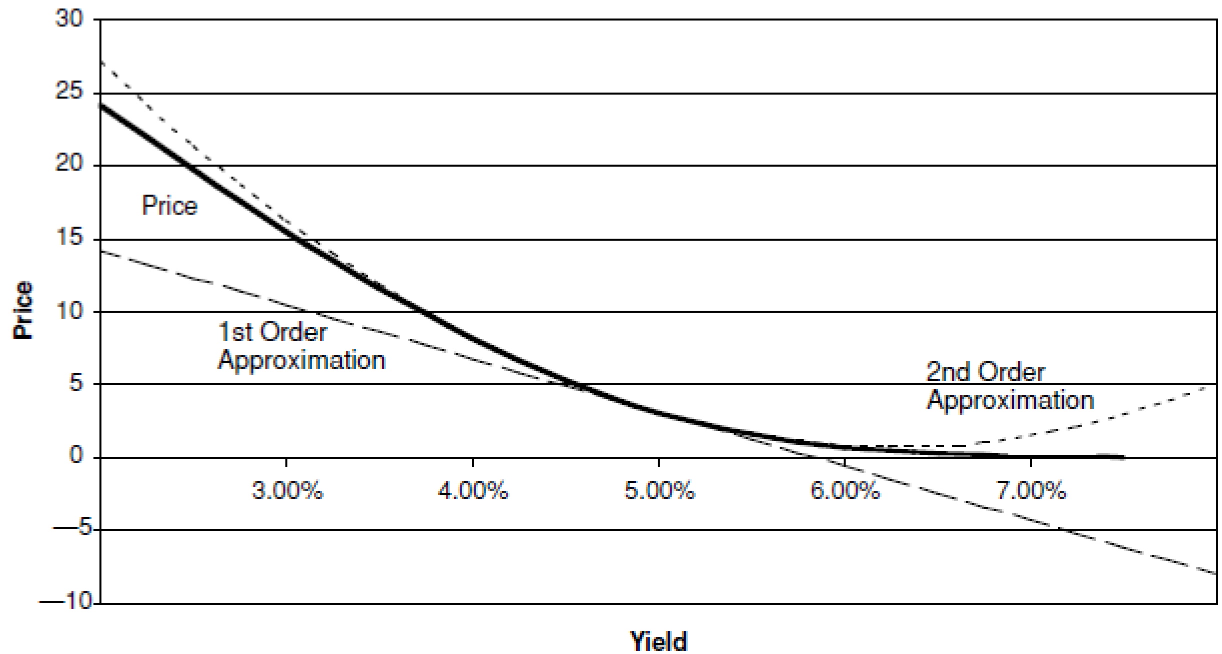
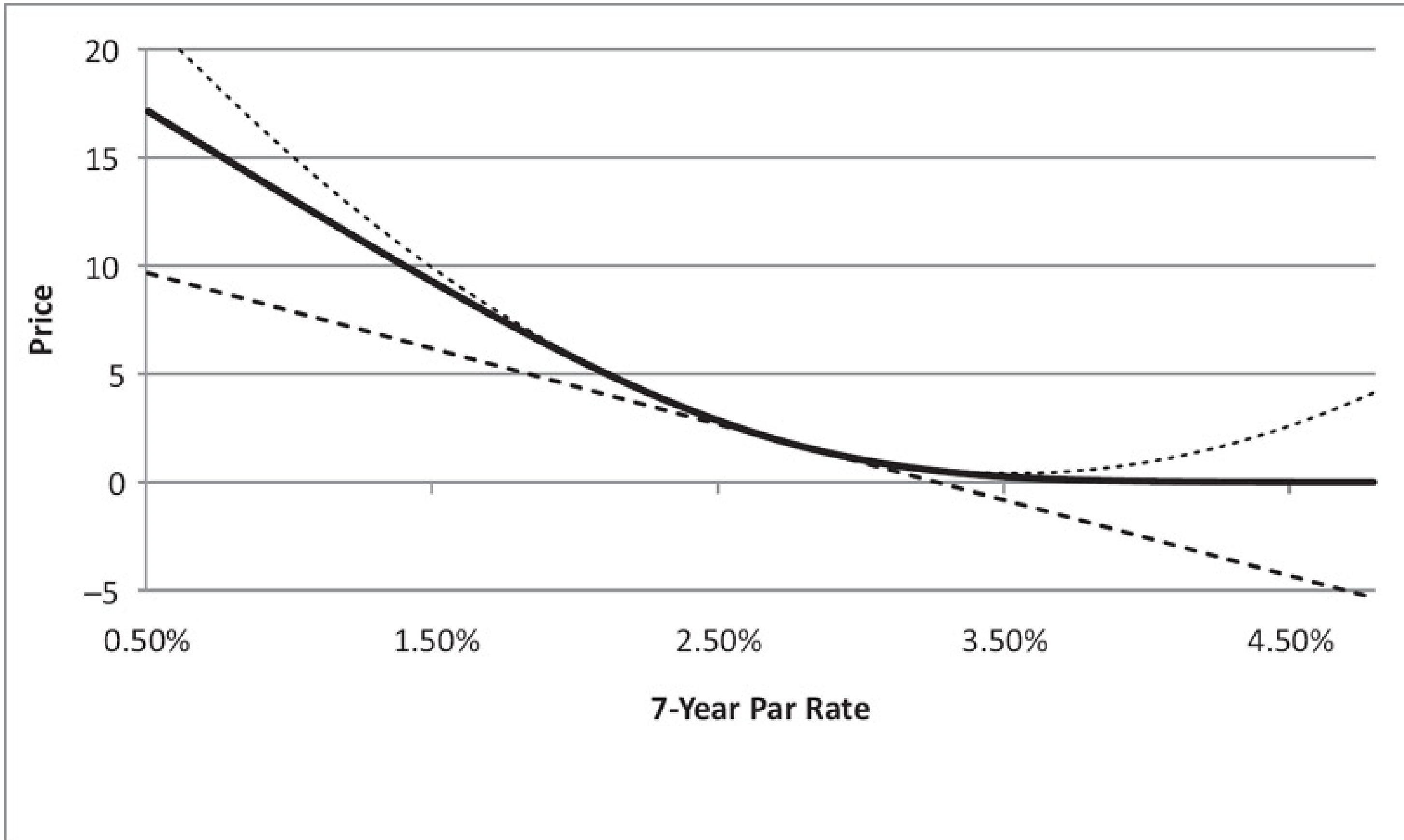


FIGURE 5.6 First and Second Order Approximations to Call Option Price

Approximation in general



Calculation of Convexity

- The central-differencing formula

$$C = \frac{1}{P}\frac{P(y+\Delta y) - 2P(y) + P(y-\Delta y)}{\Delta y^2}$$

- Table 4.3

TABLE 5.3 Convexity Calculations for the Bond and Option at Various Rates

Rate Level	Bond Price	First Derivative	Convexity	Option Price	First Derivative	Convexity
3.99%	108.2615			8.2148		
4.00%	108.1757	–857.4290	75.4725	8.1506	–641.8096	2,800.9970
4.01%	108.0901	–856.6126		8.0866	–639.5266	
4.99%	100.0780			3.0871		
5.00%	100.0000	–779.8264	73.6287	3.0501	–369.9550	9,503.3302
5.01%	99.9221	–779.0901		3.0134	–367.0564	
5.99%	92.6322			0.7003		
6.00%	92.5613	–709.8187	71.7854	0.6879	–124.4984	25,627.6335
6.01%	92.4903	–709.1542		0.6756	–122.7355	

Convexity In Asset-liability Management

- Positive convexity increases return so long as interest rates move.
- The bigger the move in either direction, the greater the gains from positive convexity.
- Negative convexity works in the reverse.
- Asset-liability managers (or hedgers, more generally) can achieve greater protection against interest rate changes by hedging duration and maintaining a non-negative convexity.

Duration of Portfolios

- The duration of a portfolio is the weighted average of those of its components, since

$$P = \sum P^i, \quad \frac{dP}{dy} = \sum \frac{dP^i}{dy}$$

$$-\frac{1}{P}\frac{dP}{dy} = \sum -\frac{1}{P}\frac{dP^i}{dy}$$

$$-\frac{1}{P}\frac{dP}{dy} = \sum -\frac{P^i}{P}\frac{1}{P^i}\frac{dP^i}{dy}$$

$$D = \sum \frac{P^i}{P} D^i$$

Convexity Of Portfolios

- Similarly, the convexity of a portfolio is the weighted average of those of its components:

$$C = \sum \frac{P^i}{P} C^i$$