HW7

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Homework 6. Quantitative Methods for fixed Income Securities

Chapter 13 and 31, Hull

1. Prove that the solutions to the equations

$$\alpha S_{\Delta t}^{d} + (1 + r\Delta t)\beta = C_{\Delta t}^{d}$$
$$\alpha S_{\Delta t}^{u} + (1 + r\Delta t)\beta = C_{\Delta t}^{u}$$

are

$$\alpha = \frac{C_{\Delta t}^{u} - C_{\Delta t}^{d}}{S_{\Delta t}^{u} - S_{\Delta t}^{d}}, \quad \text{and} \quad \beta = \frac{S_{\Delta t}^{u} C_{\Delta t}^{d} - S_{\Delta t}^{d} C_{\Delta t}^{u}}{(1 + r\Delta t)(S_{\Delta t}^{u} - S_{\Delta t}^{d})}.$$

2. Let

$$S_{t+\Delta t}^{d} = S_t D$$
$$S_{t+\Delta t}^{u} = S_t U$$

with $0 < D < (1 + r\Delta t) < U$, and let

$$q = \frac{U - (1 + r\Delta t)}{U - D},$$

Verify that

$$S_t = \frac{1}{1 + r\Delta t}\left(q S_{t+\Delta t}^{d} + (1 - q) S_{t+\Delta t}^{u}\right).$$

3. For the one-period interest-rate model, the general option pricing formula is

$$C_{0,0} = (1 + r_{0,0}\Delta t)^{-1}\left(q_0 C_{0,1} + (1 - q_0) C_{1,1}\right)$$

for

$$q_0 = \frac{P_{1,1} - P_{0,0}(1 + r_{0,0}\Delta t)}{P_{1,1} - P_{0,1}},$$

with

$$P_{0,1} = \frac{1000}{1 + r_{0,1}\Delta t} \quad \text{and} \quad P_{1,1} = \frac{1000}{1 + r_{1,1}\Delta t},$$

Verify that

$$P_{0,0} = (1 + r_{0,0}\Delta t)^{-1}\left(q_0 P_{0,1} + (1 - q_0) P_{1,1}\right).$$

You can solve the following problems manually, but you are encouraged to use MATLAB, Excel or any other programming languages.

4. Consider the pricing of a put option on AAPL. The spot price is \$160, the strike price is \$160, and the maturity is six months. The growth rate of the AAPL is 2% and its annual volatility is 25%. The annualized interest rate for monthly compounding is fixed at 0.75%.

4.1. Calculate the option value and hedge ratio (i.e., alpha) using a six-step binomial tree.

4.2. If the path of the stock price for the next six months is {(0,0), (0,1), (1,2), (2,3), (3,4), (3,5), (4,6)}, provide the alpha's for delta hedging along the path.

5. Consider pricing a 6m-maturity call option on one unit of the 1yr-maturity zero-coupon bond for the strike price of \$970 under the Ho-Lee model

$$\Delta r_t = \theta \Delta t + \sigma \sqrt{\Delta t}\ \varepsilon_B,$$

with $\Delta t = 0.5$, θ=0.01, σ=0.01 and $r_0 = \hat{r}(\frac{1}{2}) = 0.05$. Here, ε_B takes ± 1 with 50% equal probabilities. In addition, the one-year spot rate (for semi-annual compounding) is $\hat{r}(1) = 0.055$. Calculate the option price.

1. Prove that the solutions to the equations

$$\alpha S_{\Delta t}^{d} + (1 + r\Delta t)\beta = C_{\Delta t}^{d} \quad -①$$
$$\alpha S_{\Delta t}^{u} + (1 + r\Delta t)\beta = C_{\Delta t}^{u} \quad -②$$

are

$$\alpha = \frac{C_{\Delta t}^{u} - C_{\Delta t}^{d}}{S_{\Delta t}^{u} - S_{\Delta t}^{d}}, \quad \text{and} \quad \beta = \frac{S_{\Delta t}^{u} C_{\Delta t}^{d} - S_{\Delta t}^{d} C_{\Delta t}^{u}}{(1 + r\Delta t)(S_{\Delta t}^{u} - S_{\Delta t}^{d})}.$$

$$② - ①). \quad \alpha S_{\Delta t}^{u} - \alpha S_{\Delta t}^{d} = C_{\Delta t}^{u} - C_{\Delta t}^{d}$$

$$\alpha(S^u_{\Delta t} - S^d_{\Delta t}) = C^u_{\Delta t} - C^d_{\Delta t}$$

$$\alpha = \frac{C^u_{\Delta t} - C^d_{\Delta t}}{S^u_{\Delta t} - S^d_{\Delta t}}$$

From (1). $\alpha S^d_{\Delta t} + (1 + r\Delta t)\beta = C^d_{\Delta t}$

$$\frac{C^u_{\Delta t} - C^d_{\Delta t}}{S^u_{\Delta t} - S^d_{\Delta t}} S^d_{\Delta t} + (1+r\Delta t)\beta = C^d_{\Delta t}$$

$$(1+r\Delta t)\beta = C^d_{\Delta t} - \frac{C^u_{\Delta t} - C^d_{\Delta t}}{S^u_{\Delta t} - S^d_{\Delta t}} S^d_{\Delta t}$$

$$(1+r\Delta t)\beta = \frac{S^u_{\Delta t} C^d_{\Delta t} - S^d_{\Delta t} C^d_{\Delta t} - S^d_{\Delta t} C^u_{\Delta t} + S^d_{\Delta t} C^d_{\Delta t}}{S^u_{\Delta t} - S^d_{\Delta t}}$$

$$\beta = \frac{S^u_{\Delta t} C^d_{\Delta t} - S^d_{\Delta t} C^u_{\Delta t}}{(1+r\Delta t)(S^u_{\Delta t} - S^d_{\Delta t})}$$

2. Let

$$S^d_{t+\Delta t} = S_t D \quad - (1)$$
$$S^u_{t+\Delta t} = S_t U \quad - (2)$$

with $0 < D < (1 + r\Delta t) < U$, and let

$$q = \frac{U - (1 + r\Delta t)}{U - D}, \quad - (3)$$

Verify that

$$S_t = \frac{1}{1 + r\Delta t}\left(qS^d_{t+\Delta t} + (1-q)S^u_{t+\Delta t}\right).$$

From (3), $U - D = \frac{U - (1+r\Delta t)}{q}$

② -①). $S^u_{t+\Delta t} - S^d_{t+\Delta t} = S_t U - S_t D$ 1

$$S^u_{t+\Delta t} - S^d_{t+\Delta t} = S_t (U - D)$$

$$S^u_{t+\Delta t} - S^d_{t+\Delta t} = S_t \left(\frac{U - (1+r\Delta t)}{q} \right)$$

$$S^u_{t+\Delta t} - S^d_{t+\Delta t} = S_t \left(\frac{S^u_{t+\Delta t}/S_t - (1+r\Delta t)}{q} \right)$$

$$S^u_{t+\Delta t} - S^d_{t+\Delta t} = \frac{S^u_{t+\Delta t} - S_t(1+r\Delta t)}{q}$$

$$qS^u_{t+\Delta t} - qS^d_{t+\Delta t} = S^u_{t+\Delta t} - S_t(1+r\Delta t)$$

$$S^u_{t+\Delta t} - qS^u_{t+\Delta t} + qS^d_{t+\Delta t} = S_t(1+r\Delta t)$$

$$S_t = \frac{1}{(1+r\Delta t)} \left(qS^d_{t+\Delta t} + (1-q) S^u_{t+\Delta t} \right).$$

3. For the one-period interest-rate model, the general option pricing formula is

$$C_{0,0} = (1 + r_{0,0}\Delta t)^{-1} \left(q_0 C_{0,1} + (1 - q_0) C_{1,1} \right)$$

for

$$q_0 = \frac{P_{1,1} - P_{0,0}(1 + r_{0,0}\Delta t)}{P_{1,1} - P_{0,1}}, \qquad -①$$

with

$$P_{0,1} = \frac{1000}{1 + r_{0,1}\Delta t} \quad \text{and} \quad P_{1,1} = \frac{1000}{1 + r_{1,1}\Delta t},$$

Verify that

$$P_{0,0} = (1 + r_{0,0}\Delta t)^{-1} \left(q_0 P_{0,1} + (1 - q_0) P_{1,1} \right).$$

From ①).

$$q_0 (P_{1,1} - P_{0,1}) = P_{1,1} - P_{0,0}(1 + r_{0,0}\Delta t)$$

$$q_0 (P_{1,1} - P_{0,1}) = P_{1,1} - P_{0,0}(1 + r_{0,0}\Delta t)$$

$$P_{1,1} - q_0 (P_{1,1} - P_{0,1}) = P_{0,0}(1 + r_{0,0}\Delta t)$$

$$q_0 P_{0,1} + (1-q_0) P_{1,1} = P_{0,0}(1 + r_{0,0}\Delta t)$$

$$P_{0,0} = (1 + r_{0,0}\Delta t)^{-1}(q_0 P_{0,1} + (1-q_0) P_{1,1})$$

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4.2. If the path of the stock price for the next six months is {(0,0), (0,1), (1,2), (2,3), (3,4), (3,5), (4,6)}, provide the alpha's for delta hedging along the path.

given $\mu = 0.02$, $\sigma = 0.25$, $S_0 = 160$,

$K = 160$, $r = 0.0075$, $\Delta t = \frac{1}{12}$

$$q = \frac{(\mu - r)\Delta t + \sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}}$$

$$q = 0.5072168 \approx 0.5072$$

$$1 - q = 0.4928$$

$$S^u_{\Delta t} = S_0(1 + \mu\Delta t + \sigma\sqrt{\Delta t})$$

$$S^d_{\Delta t} = S_0(1 + \mu\Delta t - \sigma\sqrt{\Delta t})$$

$$S_{\Delta t}^{d} = S_0\left(1+\mu\Delta t - \sigma\sqrt{\Delta t}\right)$$

∴ The six step binomial tree for S is:

S						
0	1	2	3	4	5	6
						245.3276
					228.4592	
				212.7507		212.3524
			198.1222		197.7513	
		184.4996		184.1542		183.8094
	171.8137		171.4920		171.1709	
160.00		159.7004		159.4014		159.1030
	148.7197		148.4412		148.1633	
		138.2346		137.9758		137.7175
			128.4888		128.2482	
				119.4300		119.2064
					111.0100	
						103.1835

$$C_{i,j} = \frac{1}{1+r\Delta t}\left((1-q)\,C_{i,j+1} + q\,C_{i+1,j+1}\right)$$

∴ The six step binomial tree for C is:

Put Option						
						0.0000
					0	
				0		0.0000
			0.11682945		0	
		1.70128124		0.23047827		0.0000
	5.2711		3.24274098		0.45468186	
10.6289329		8.74576598		6.17328016		0.8970
	15.8474		14.1029693		11.7367524	
		22.7665249		21.8243829		22.2825
			31.211597		31.6518443	
				40.3701405		40.7936
					48.8900868	
						56.8165

$$\alpha_{i,j} = \frac{C_{i+1,j+1} - C_{i,j+1}}{S_{i+1,j+1} - S_{i,j+1}}$$

∴ The six step binomial tree for alpha is:

alpha					
					0
				0	
			-0.0080597		0
		-0.1173821		-0.0171059	
	-0.2841		-0.2400866		-0.0363058
-0.4579698		-0.4711438		-0.4903624	
	-0.6532		-0.7304845		-1
		-0.8574701		-1	
			-1		-1
				-1	
					-1

path.

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Given $\Delta r_t = \theta \Delta t + \sigma \sqrt{t}\ \varepsilon_B$

$\varepsilon_B = 1$ or -1 each with 50% probability.

$$\Delta_t^u = \frac{0.01}{2} + 0.01\sqrt{\frac{1}{2}} = 0.01207$$

$$\Delta t^d = \frac{0.01}{2} - 0.01\sqrt{\frac{1}{2}} = -0.00207$$

$$r_{0,1} = 0.05 - 0.00207 = 0.04793$$

$$r_{1,1} = 0.05 + 0.00207 = 0.06207$$

bond price $P_0 = \dfrac{1000}{\left(1+\frac{5.5\%}{2}\right)^2} = 947.18833$

$$P_{0,1} = \frac{1000}{1+r_{0,1}/2} = 976.59588$$

$$P_{1,1} = \frac{1000}{1+r_{1,1}/2} = 969.89918$$

$$q = \frac{P_{1,1} - P_{0,0}(1+r_{0,0}\Delta t)}{P_{1,1} - P_{0,1}} = 0.14473$$

option price $= \dfrac{q \times (P_{0,1} - 970)}{1 + r_{0,0} \times 0.5} = 0.9314.$