Binomial Model
for Interest Rate derivatives

Regarding bond option

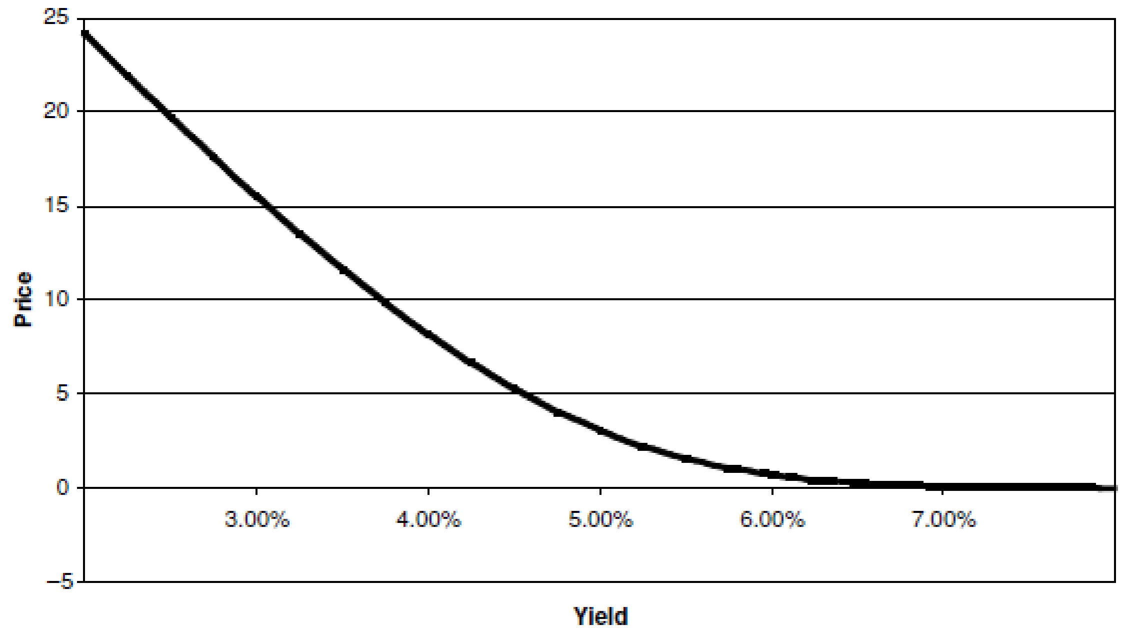


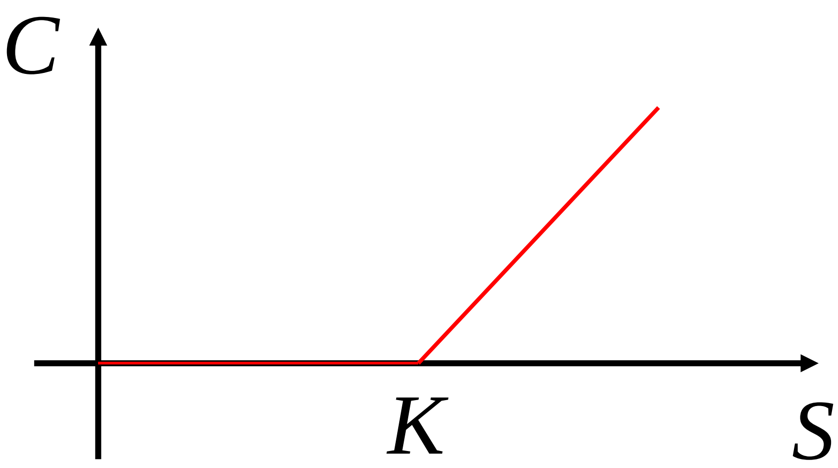
FIGURE 5.2 The Price-Rate Function of a One-Year European Call Option Struck at Par on the 5s of February 15, 2011

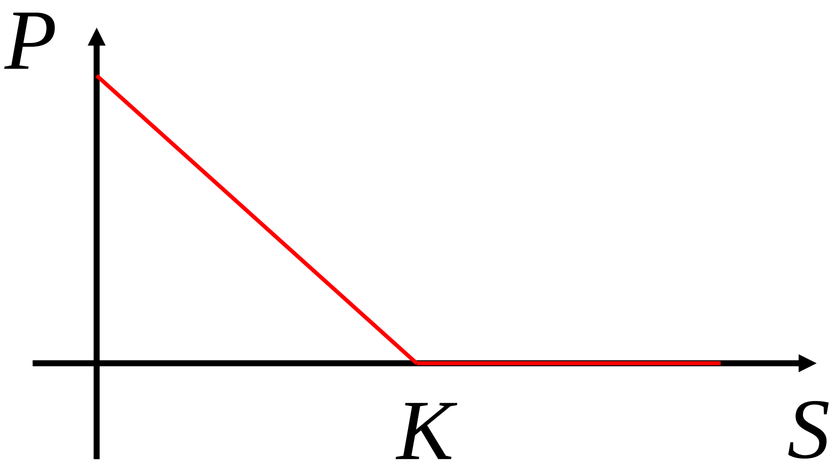
Why we need a model?

- When pricing by static replications fails, we need a dynamical model for the state variable.
- Such a model
  - Describes the marginal distribution of the state variables.
  - Allows us to price by dynamical replication.
  - Dynamical replication implies hedging.

Definition of Options

- A call/put option is a right, not an obligation, to buy/sell certain asset for certain price at certain time in the future.
- Payoff functions for call and put:





Starting from Equity Option Model

Basic Features

- Let S_t denote the time-t price of a stock.
- The basic feature of a model is to prescribed by the mean and variance of increment or change, such as

$$E\left[\frac{\Delta S_t}{S_t}\right] = \mu \Delta t, \quad Var\left(\frac{\Delta S_t}{S_t}\right) = \sigma^2 \Delta t,$$

where $\Delta S_t = S_{t+\Delta t} - S_t$.

A Binomial Model

- The simplest model is the Cox-Ross-Rubinstein model (1976)

$$\frac{\Delta S_t}{S_t} = \mu \Delta t + \sigma \sqrt{\Delta t}\ \varepsilon_B$$

where

$$\varepsilon_B = \begin{cases} 1, & \text{with probability } 1/2 \\ -1, & \text{with probability } 1/2 \end{cases}$$

is the Bernoulli random variable.

The binomial tree

- Under the binomial model, stock price evolves as

$$
\begin{tikzcd}
 & S^u_{t+\Delta t} = S_t(1+\mu\Delta t + \sigma\sqrt{\Delta t}) \\
S_t \arrow[ur, "1/2"] \arrow[dr, "1/2"'] & \\
 & S^d_{t+\Delta t} = S_t(1+\mu\Delta t - \sigma\sqrt{\Delta t})
\end{tikzcd}
$$

μ_t as a discount rate for the stock

- Can the binomial model price the stock?
- Yes when μ_t is taken as the discount rate:

$$
E\left[S_{t+\Delta t}\right] \Big/ 1+\mu_t \Delta t = \frac{1}{1+\mu_t \Delta t}\left(\frac{1}{2} S_{t+\Delta t}^{d} + \frac{1}{2} S_{t+\Delta t}^{u}\right)
$$

$$
= \frac{1}{1+\mu_t \Delta t}\left(\frac{1}{2} S_t\left[1+\mu \Delta t - \sigma\sqrt{\Delta t}\right] + \frac{1}{2} S_t\left[1+\mu \Delta t + \sigma\sqrt{\Delta t}\right]\right)
$$

$$
= \frac{1+\mu_t \Delta t}{1+\mu_t \Delta t} S_t = S_t
$$

Risk Free Security --- the Cash Bond

- The growth rate for saving account (or cash bond) is r, such that

$$\beta \quad \rightarrow \quad (1+r\Delta t)\beta$$

$$t \qquad\qquad\qquad t+\Delta t$$

Risk Premium

- The difference $\mu - r$ is called risk-premium in finance.
- It is the excess of return investors demand for taking risky.
- It is typically positive.

Option pricing --- one period

- Consider the option of one-period maturity

$$S_0 \quad \begin{cases} \xrightarrow{1/2} S^u_{\Delta t} = S_0(1+\mu\Delta t + \sigma\sqrt{\Delta t}) \\ \xrightarrow{1/2} S^d_{\Delta t} = S_0(1+\mu\Delta t - \sigma\sqrt{\Delta t}) \end{cases}$$

$$C_0 = ? \quad \begin{cases} \xrightarrow{1/2} C^u_{\Delta t} = \left(S^u_{\Delta t} - K\right)^+ \\ \xrightarrow{1/2} C^d_{\Delta t} = \left(S^d_{\Delta t} - K\right)^+ \end{cases}$$

- Expectation pricing,

$$C_0 = \frac{1}{1+?\Delta t} \times \left(\tfrac{1}{2} C^u_{\Delta t} + \tfrac{1}{2} C^d_{\Delta t}\right).$$

- What should be taken as the discount rate?

Arbitrage pricing: the alternative way

- Consider replicating the payoffs with a portfolio of α and β units of shares and cash, such that

$$\alpha S_{\Delta t}^{d} + (1 + r\Delta t)\beta = C_{\Delta t}^{d}$$

$$\alpha S_{\Delta t}^{u} + (1 + r\Delta t)\beta = C_{\Delta t}^{u}$$

Solution

$$\alpha = \frac{C_{\Delta t}^{u} - C_{\Delta t}^{d}}{S_{\Delta t}^{u} - S_{\Delta t}^{d}}, \qquad \beta = \frac{S_{\Delta t}^{u} C_{\Delta t}^{d} - S_{\Delta t}^{d} C_{\Delta t}^{u}}{(1 + r\Delta t)(S_{\Delta t}^{u} - S_{\Delta t}^{d})}$$

Arbitrage pricing, cont'd

- The value of the option is thus

$$C_0 = \alpha S_0 + \beta,$$

  which is the no-arbitrage price.

- Arbitrage opportunity arises if the option value differs from value above.

The discount rate for the option

- The discount rate for the option is the weighted average of the discount rates for the share and cash.
- The weighted average of the discount rate is

$$\gamma = \frac{\alpha S_0}{\alpha S_0 + \beta}\mu + \frac{\beta}{\alpha S_0 + \beta} r$$

or

$$1 + \gamma \Delta t = \frac{\alpha S_0}{\alpha S_0 + \beta}\left(1 + \mu \Delta t\right) + \frac{\beta}{\alpha S_0 + \beta}\left(1 + r \Delta t\right)$$

Expected Pricing

- Let us try the expectation pricing:

$$
\begin{aligned}
\frac{1}{1+\gamma\Delta t}E\left[C_{\Delta t}\right] &= \frac{1}{1+\gamma\Delta t}\left(\frac{1}{2}C_{\Delta t}^{d}+\frac{1}{2}C_{\Delta t}^{u}\right) \\
&= \frac{1}{1+\gamma\Delta t}\left(\frac{1}{2}\left(\alpha S_{\Delta t}^{u}+(1+r\Delta t)\beta\right)+\frac{1}{2}\left(\alpha S_{\Delta t}^{d}+(1+r\Delta t)\beta\right)\right) \\
&= \frac{\alpha S_0+\beta}{\alpha(1+\mu\Delta t)S_0+(1+r\Delta t)\beta}\left(\alpha(1+\mu\Delta t)S_0+(1+r\Delta t)\beta\right) \\
&= \alpha S_0+\beta \quad \text{!!!}
\end{aligned}
$$

which yields the right price!

Replication pricing and discount rate

- To find out the replication price is equivalent to find out the proper discount rate.
- This is a labor intensive process.
- There is, however, a short cut.