Chapter 6 of Tuckman
Measures of Price Sensitivity Based on Parallel Yield Shifts

Parallel Yield Shifts

- Bonds are functions of their yields.
- Assume that yields shifts in parallel, we can calculate DV01 or duration and find out hedging ratio efficiently.

Bonds as functions of yields

- Yield-based measures assume that the yield of a security is the risk factor

$$P(y) = \frac{100c}{2} \sum_{t=1}^{2T} \frac{1}{\left(1+\frac{y}{2}\right)^t} + \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}$$

- or

$$P(y) = \frac{100c}{y} \left(1 - \frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right) + \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}$$

Duration and DV01

- Recall that

$$D = -\frac{1}{P}\frac{dP}{dy}$$

- and

$$DV01 = \frac{-P}{10000}\frac{1}{P}\frac{dP}{dy} = \frac{-1}{10000}\frac{dP}{dy}$$

Yield-Based *Duration*

- Taking the negation of the derivative of the above two price expressions, we obtain

$$D = \frac{1}{P}\frac{1}{1+\frac{y}{2}}\left[\frac{100c}{2}\sum_{t=1}^{2T}\frac{t}{2}\frac{1}{\left(1+\frac{y}{2}\right)^{t}} + T\frac{100}{\left(1+\frac{y}{2}\right)^{2T}}\right]$$

- or

$$D = \frac{1}{P}\left[\frac{100c}{y^2}\left(1-\frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right) + T\left(1-\frac{c}{y}\right)\frac{100}{\left(1+\frac{y}{2}\right)^{2T+1}}\right]$$

An Example

- Table 4.6 calculates the *DV*01 and duration of the U.S. Treasury 2s due May 31, 2015, as of May 28, 2010, for the market yield of 2.092%.

TABLE 4.6 *DV*01 and Duration Calculations for the $2\frac{1}{8}$s of May 31, 2015, as of May 28, 2010, at a Yield of 2.092 Percent

Date	Term	Cash Flow	Present Value	Time-Wtd. PV	% of Wtd. Sum
11/30/10	0.5	1.0625	1.0515	.5258	.1%
5/31/11	1.0	1.0625	1.0406	1.0406	.2%
11/30/11	1.5	1.0625	1.0298	1.5448	.3%
5/31/12	2.0	1.0625	1.0192	2.0384	.4%
11/30/12	2.5	1.0625	1.0086	2.5216	.5%
5/31/13	3.0	1.0625	.9982	2.9946	.6%
11/30/13	3.5	1.0625	.9879	3.4575	.7%
5/31/14	4.0	1.0625	.9776	3.9105	.8%
11/30/14	4.5	1.0625	.9675	4.3538	.9%
5/31/15	5.0	101.0625	91.0749	455.3746	95.3%
Total			100.1559	477.7621	
*DV*01		.04728			
Duration		4.7208			

Table 4.6

DV01 and Duration for Zero Coupon Bonds

- Let c=0 in

$$D = \frac{1}{P}\left[\frac{100c}{y^2}\left(1 - \frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right) + T\left(1 - \frac{c}{y}\right)\frac{100}{\left(1+\frac{y}{2}\right)^{2T+1}}\right]$$

$$DV01 = \frac{1}{10{,}000}\left[\frac{100c}{y^2}\left(1 - \frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right) + T\left(1 - \frac{c}{y}\right)\frac{100}{\left(1+\frac{y}{2}\right)^{2T+1}}\right]$$

- We obtain

$$D_{c=0} = \frac{T}{\left(1+\frac{y}{2}\right)}$$

$$DV01_{c=0} = \frac{T}{100\left(1+\frac{y}{2}\right)^{2T+1}} = \frac{TP}{10{,}000\left(1+\frac{y}{2}\right)}$$

Duration and DV01 for Par Bonds

- For a par bond, take P=100 and c=y, we have

$$D_{c=y} = \frac{1}{y}\left(1 - \frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right)$$

$$DV01_{c=y} = \frac{1}{100y}\left(1 - \frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right)$$

Duration and DV01 for Perpetuities

- Let $T \to +\infty$, we obtain

$$D_{T=\infty} = \frac{1}{y}$$

$$DV01_{T=\infty} = \frac{1}{100} \frac{c}{y^2}$$

Yield-Based Convexity

- The general formula for convexity is

$$C = \frac{1}{P}\frac{1}{\left(1+\frac{y}{2}\right)^2}\left[\frac{100c}{2}\sum_{t=1}^{2T}\frac{t}{2}\frac{t+1}{2}\frac{1}{\left(1+\frac{y}{2}\right)^t} + T\,(T+.5)\,\frac{100}{\left(1+\frac{y}{2}\right)^{2T}}\right]$$

- Table 4.6

Yield-Based Convexity

- Setting c=0 and $P = 100\left(1+\frac{y}{2}\right)^{-2T}$, we obtain

$$C_{c=0} = \frac{T\,(T+.5)}{\left(1+\frac{y}{2}\right)^2}$$

Convexity of Par Bonds

- Differentiate

$$P(y) = \frac{100c}{y}\left(1 - \frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right) + \frac{100}{\left(1+\frac{y}{2}\right)^{2T}}$$

twice and set $c=y$, we obtain

$$\mathrm{C}_{c=y} = \frac{2}{y^2}\left[1 - \frac{1}{\left(1+\frac{y}{2}\right)^{2T}}\right] - \frac{2T}{y\left(1+\frac{y}{2}\right)^{2T+1}}$$

THE BARBELL VERSUS THE BULLET

TABLE 4.7 Data on Three U.S. Treasury Bonds as of May 28, 2010

Coupon	Maturity	Price	Yield	Duration	Convexity
$2\frac{1}{2}$	3/31/15	102.5954	2.025%	4.520	23.4
$3\frac{3}{8}$	11/15/19	100.8590	3.288%	8.033	74.8
$4\frac{3}{8}$	11/15/39	102.7802	4.221%	16.611	389.7

- The three bonds in the table have maturities of approximately 5 years, 10 years, and 30 years, respectively.

BARBELL VS. BULLET, cont'd

- Two strategies
  - *bullet* investment in the 10-year 3⅜s
  - *barbell* portfolio of the shorter maturity, 5-year 2½s, and the longer maturity, 30-year 4⅜s.
- In particular, the barbell portfolio would be constructed to cost the same and have the same duration as the bullet investment.

BARBELL VS. BULLET, cont'd

- Let V^5 and V^{30} be the value in the barbell portfolio of the 5-year and 30-year bonds, respectively. Then, by matching the value

$$V^5 + V^{30} = 100{,}859{,}000$$

and the duration

$$\frac{V^5}{100{,}859{,}000} \times 4.520 + \frac{V^{30}}{100{,}859{,}000} \times 16.611 = 8.033$$

- We have V^5 = \$71.555m and V^{30} =\$29.304 m (70.95% and 29.05% of the portfolio).

BARBELL VS. BULLET, cont'd

- Finally, the convexity of the portfolio

$$70.95\% \times 23.4 + 29.05\% \times 389.7 = 129.8$$

- An insight is that spreading out the cash flows of a portfolio, without changing duration, raises convexity.

BARBELL VS. BULLET, cont'd

- What then is the disadvantage of the barbell portfolio? The weighted yield of the barbell portfolio is

$$70.95\% \times 2.025\% + 29.05\% \times 4.221\% = 2.663\%$$

- Hence, the barbell will not do as well as the bullet portfolio if yields remain at current levels.

Limitations of Duration and Convexity

- Two major limitations
  - First, they are defined only for securities with fixed cash flows.
  - Second, as will be seen shortly, their use implicitly assumes parallel shifts in yield, which is not a particularly good assumption.

Yield-based risk metrics

- Yet there are several reasons fixed income professionals must understand these measures.
  - First, these measures of price sensitivity are simple to compute, easy to understand, and, in many situations, reasonable to use.
  - Second, these measures are already widely used in the financial industry.
  - Third, much of the intuition gained from a full understanding of these measures carries over to more general measures of price sensitivity.