Solutions to Final Exam for MATH4511

December 11, 2017

(The problem sheet is required to be returned with your booklet)

Problems (Numbers in brackets are credits, totaled to 66):

1. **Solution**:
   1.1. (4) The floating leg is equal to par immediately after coupon payments, and is equal or close to par at all times, hence is insensitive to changes in interest rates and has tiny DV01 of kr01. Once the DV01 or kr01 of the floating leg is neglected, the DV01 or kr01s of the swap equal to those of the fixed leg, a coupon bond with coupon rate equal to the swap rate.
   1.2. (4) Given an *i*-year key-rate shift, the *j*-year par yield remains unperturbed or unchanged, so that the price remains unchanged, and the kr01 is thus zero.
   1.3. (2) For a par bond, we can use the DV01 formula for calculating kr01.

2. (6) **Solution**: The change in the bond price is estimated to be

$$\begin{aligned}
&\Delta B_{20} + \Delta B_{10} + \Delta B_{30} \\
&= -D_{20}B_{20}\Delta y_t^{20} - D_{10}B_{10}\Delta y_t^{10} - D_{30}B_{30}\Delta y_t^{30} \\
&= -D_{20}B_{20}\left(\alpha + \beta_{10}\Delta y_t^{10} + \beta_{30}\Delta y_t^{30} + \varepsilon_t\right) \\
&\qquad - D_{10}B_{10}\Delta y_t^{10} - D_{30}B_{30}\Delta y_t^{30} \\
&= -D_{20}B_{20}\left(\alpha + \varepsilon_t\right) - \left(D_{20}B_{20}\beta_{10} + D_{10}B_{10}\right)\Delta y_t^{10} \\
&\qquad - \left(D_{20}B_{20}\beta_{30} + D_{30}B_{30}\right)\Delta y_t^{30}
\end{aligned}$$

   It follows that

$$B_{10} = -\frac{D_{20}}{D_{10}}B_{20}\beta_{10} = -\$252,500,$$

$$B_{30} = -\frac{D_{20}}{D_{30}}B_{20}\beta_{30} = -\$707,661.29.$$

3. (6) **Solution**: Do the following transactions:
   1.1. Short the FRA (to receive fixed and pay float);
   1.2. Long one unit of T-maturity zero-coupon bond (ZCB);
   1.3. short $d(0,T)/d(0,T+\Delta T)$ units of $(T+\Delta T)$-unit ZCB.

   At *T*,

$$\begin{aligned}
\text{P\&L} &= d(T,T+\Delta T)\Delta T\left(f_0 - f_T\right) + 1 - \frac{d(0,T)}{d(0,T+\Delta T)}d(T,T+\Delta T) \\
&= d(T,T+\Delta T)\Delta T\left\{f_0 - \frac{1}{\Delta T}\left(\frac{d(0,T)}{d(0,T+\Delta T)} - 1\right)\right\} > 0
\end{aligned}$$

4. **Solution**:
   4.1. (4) Four steps:
      a. Find the risk-neutral probabilities by fitting to the discount curve.

b. Calculate the bond price at the option's maturity.
c. Calculate the payoff of the option's payoff.
d. Backward induction for the option.

4.2. (4) Use the underlying bond and cash with the amount

$$\alpha_{0,0} = \frac{C_{1,1} - C_{0,1}}{B_{1,1} - B_{0,1}},\ \beta_{0,0} = \frac{C_{0,1}(B_{1,1} + \frac{c}{2}) - C_{1,1}(B_{0,1} + \frac{c}{2})}{(1 + r_{0,0}\Delta t)(B_{1,1} - B_{0,1})}$$

4.3. (4) It is self-financing because the value of the hedging portfolio remains unchanged before and after the hedge revision.

$$\alpha_{i,j}(B_{i,j+1} + \tfrac{c}{2}) + \beta_{i,j}(1 + r_{i,j}\Delta t) = C_{i,j+1} \quad = \alpha_{i,j+1}B_{i,j+1} + \beta_{i,j+1}$$
$$\alpha_{i,j}(B_{i+1,j+1} + \tfrac{c}{2}) + \beta_{i,j}(1 + r_{i,j}\Delta t) = C_{i+1,j+1} = \alpha_{i+1,j+1}B_{i+1,j+1} + \beta_{i+1,j+1}$$

5. **Solution**:

5.1. (4) The value of the swap is

$$\begin{aligned} V_t &= 1 - A(t,T_0,T_N)k - d(t,T_N) \\ &= A(t,T_0,T_N)\left(\frac{1 - d(t,T_N)}{A(t,T_0,T_N)} - k\right) \\ &= A(t,T_0,T_N)\left(s(t;T_0,T_N) - k\right) \end{aligned}$$

5.2. (4) One way to show is to compare the payoffs: at maturity, the payoff is

$$\begin{aligned} &A(T_0,T_0,T_N)(s(T_0;T_0,T_N) - k)^+ - A(T_0,T_0,T_N)(k - s(T_0;T_0,T_N))^+ \\ &= A(T_0,T_0,T_N)(s(T_0;T_0,T_N) - k) \end{aligned}$$

The conclusion follows.

6. **Solution**:

6.1.(4)

$$\begin{aligned} dV_t &= V_{t+dt} - V_t \\ &= A(t+dt,T_0,T_N)G(s(t+dt;T_0,T_N)) - A(t,T_0,T_N)G(s(t;T_0,T_N)) \\ &= A(t+dt,T_0,T_N)\left[G(s(t+dt;T_0,T_N)) - G(s(t;T_0,T_N))\right] \\ &\qquad + G(s(t;T_0,T_N))\left[A(t+dt,T_0,T_N) - A(t,T_0,T_N)\right] \\ &= A(t+dt,T_0,T_N)dG(s(t;T_0,T_N)) + G(s(t;T_0,T_N))dA(t,T_0,T_N) \end{aligned}$$

6.2.(4)

$$\begin{aligned} \frac{dG(s)}{ds} &= \Phi(d_1) + \frac{sn(d_1) - kn(d_2)}{s\sigma\sqrt{T_0 - t}} \\ &= \Phi(d_1) + \frac{n(d_1) - \frac{k}{s}n(d_2)}{\sigma\sqrt{T_0 - t}} \end{aligned}$$

where the last term is zero because

$$
\begin{aligned}
n(d_1) &= n(d_2 + \sigma\sqrt{T_0 - t}) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{(d_2+\sigma\sqrt{T_0-t})^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2+2d_2\sigma\sqrt{T_0-t}+\sigma^2(T_0-t)}{2}} \\
&= n(d_2)\times e^{-(\ln\frac{s}{k}-\frac{1}{2}\sigma^2(T_0-t)+\frac{1}{2}\sigma^2(T_0-t))} \\
&= e^{-\ln\frac{s}{k}} n(d_2) = \tfrac{k}{s} n(d_2) \quad \square
\end{aligned}
$$

6.3.(4) Long $\Phi(d_1)$ unit of ATM payer's swap and long $G(s(t;T_0,T_N))$ unit of the annuity.

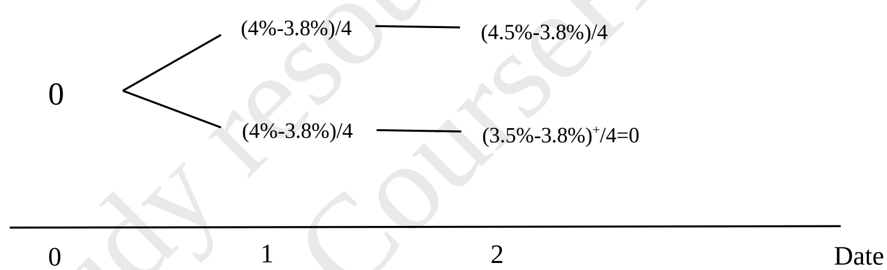
7. **Solution**:

7.1. (4) The equation for q_0 is

$$\frac{1}{(1+4\%/4)^2} = q_0 \frac{1}{(1+3.6\%/4)(1+4\%/4)} + (1-q_0)\frac{1}{(1+4.4\%/4)(1+4\%/4)}$$

Solve it, we obtain $q_0 = 0.4994$

7.2. (4) The cash flows at step 0, 1 and 2 are shown below:



7.3. (4) There are two caplets,

$$c_1 = 0.25\times(4\%-3.8\%)^+ \big/ (1+4\%/4) = \$495.04$$

$$c_2 = \frac{0.25}{(1+4\%/4)}\left(q_0\frac{(3.5\%-3.8\%)^+}{(1+3.5\%/4)} + (1-q_0)\frac{(4.5\%-3.8\%)^+}{(1+4.5\%/4)}\right) = \$855.76$$

It follows that $c = c_1 + c_2 = \$1,352.80.$

====================== Good Luck Everyone! ========================