# Extension to two-period tree

- By duplication, we obtain

$$r_0 + (\theta_0 + \theta_1)\Delta t + 2\sigma\sqrt{\Delta t}$$

$q_1$

$$r_0 + \theta_0 \Delta t + \sigma\sqrt{\Delta t}$$

$q_0$

$1-q_1$

$$r_0$$

$$r_0 + (\theta_0 + \theta_1)\Delta t$$

$q_1$

$1-q_0$

$$r_0 + \theta_0 \Delta t - \sigma\sqrt{\Delta t}$$

$1-q_1$

$$r_0 + (\theta_0 + \theta_1)\Delta t - 2\sigma\sqrt{\Delta t}$$

# Double indexes

- Using double indexes, we obtain

$$r_{2,2} = r_{1,1} + \theta_1 \Delta t + \sigma \sqrt{\Delta t}$$

$q_1$

$$r_{1,1} = r_{0,0} + \theta_0 \Delta t + \sigma \sqrt{\Delta t}$$

$q_0$

$1 - q_1$

$$r_{0,0}$$

$$r_{1,2} = r_{1,1} + \theta_1 \Delta t$$

$q_1$

$1 - q_0$

$$r_{0,1} = r_{0,0} + \theta_0 \Delta t - \sigma \sqrt{\Delta t}$$

$1 - q_1$

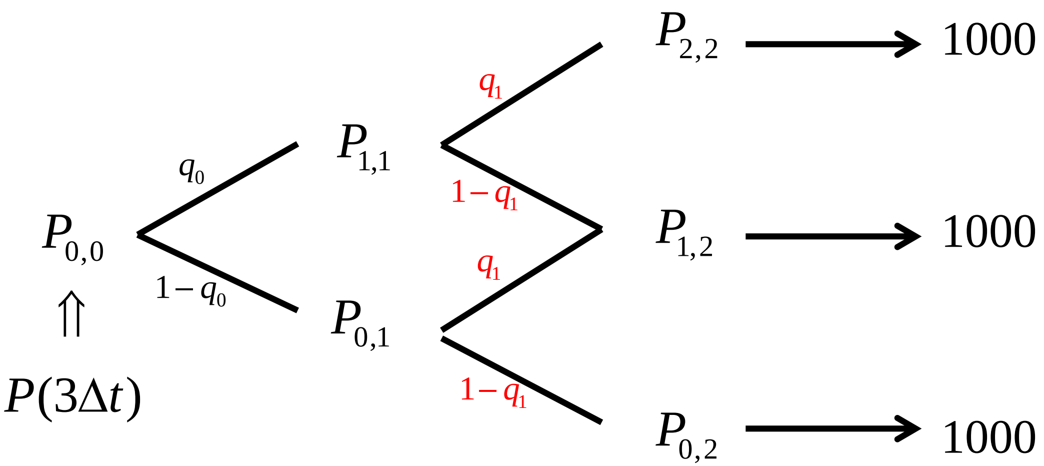
$$r_{0,2} = r_{0,1} + \theta_1 \Delta t - \sigma \sqrt{\Delta t}$$

# The principle for determining $q_1$

- The principle is to choose $q_1$ so that we can reproduce $P(3\Delta t)$, the price of $3\Delta t$-maturity zero-coupon bond.

# Construction of the bond price tree

- The tree must price $3\Delta t$-maturity zero-coupon bond correctly.



- The induction scheme in details

$$P_{0,0} = \frac{(1-q_0)P_{0,1} + q_0 P_{1,1}}{1 + r_{0,0}\Delta t}$$

$$P_{1,1} = \frac{(1-q_1)P_{1,2} + q_1 P_{2,2}}{1 + r_{1,1}\Delta t}$$

$$P_{0,1} = \frac{(1-q_1)P_{0,2} + q_1 P_{1,2}}{1 + r_{0,1}\Delta t}$$

$$P_{2,2} = \frac{1,000}{1 + r_{2,2}\Delta t}$$

$$P_{1,2} = \frac{1,000}{1 + r_{1,2}\Delta t}$$

$$P_{0,2} = \frac{1,000}{1 + r_{0,2}\Delta t}$$

- In these equations, $q_1$ is the only unknown!

# Matching to the price of $P(3\Delta t)$

- We write

$$P_{0,1} = \frac{P_{0,2}}{1+r_{0,1}\Delta t} + \frac{P_{1,2}-P_{0,2}}{1+r_{0,1}\Delta t} q_1 = P_{0,1}^{(1)} + P_{0,1}^{(2)} q_1$$

$$P_{1,1} = \frac{P_{1,2}}{1+r_{1,1}\Delta t} + \frac{P_{2,2}-P_{1,2}}{1+r_{1,1}\Delta t} q_1 = P_{1,1}^{(1)} + P_{1,1}^{(2)} q_1$$

- And set

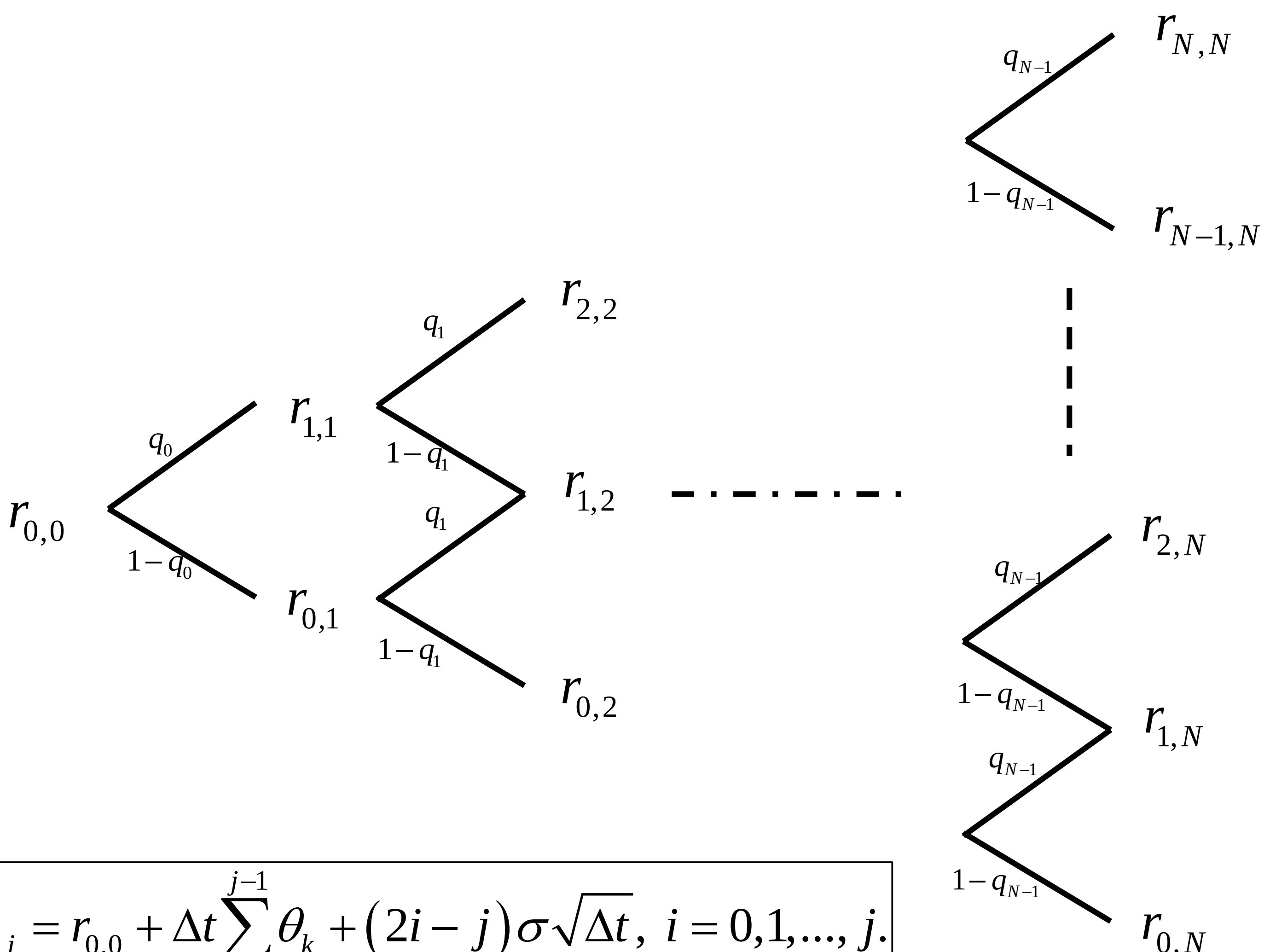
$$P(3\Delta t) = P_{0,0} = \frac{(1-q_0)[P_{0,1}^{(1)} + P_{0,1}^{(2)} q_1] + q_0[P_{1,1}^{(1)} + P_{1,1}^{(2)} q_1]}{1+r_{0,0}\Delta t}$$

$$= \frac{(1-q_0)P_{0,1}^{(1)} + q_0 P_{1,1}^{(1)}}{1+r_{0,0}\Delta t} + \frac{(1-q_0)P_{0,1}^{(2)} + q_0 P_{1,1}^{(2)}}{1+r_{0,0}\Delta t} q_1 = P_{0,0}^{(1)} + P_{0,0}^{(2)} q_1$$

- Then

$$q_1 = \frac{P(3\Delta t) - P_{0,0}^{(1)}}{P_{0,0}^{(2)}}$$

# Extension to multi-period tree




$$\boxed{r_{i,j} = r_{0,0} + \Delta t \sum_{k=1}^{j-1} \theta_k + (2i - j)\sigma\sqrt{\Delta t},\ i = 0,1,\ldots,j.}$$

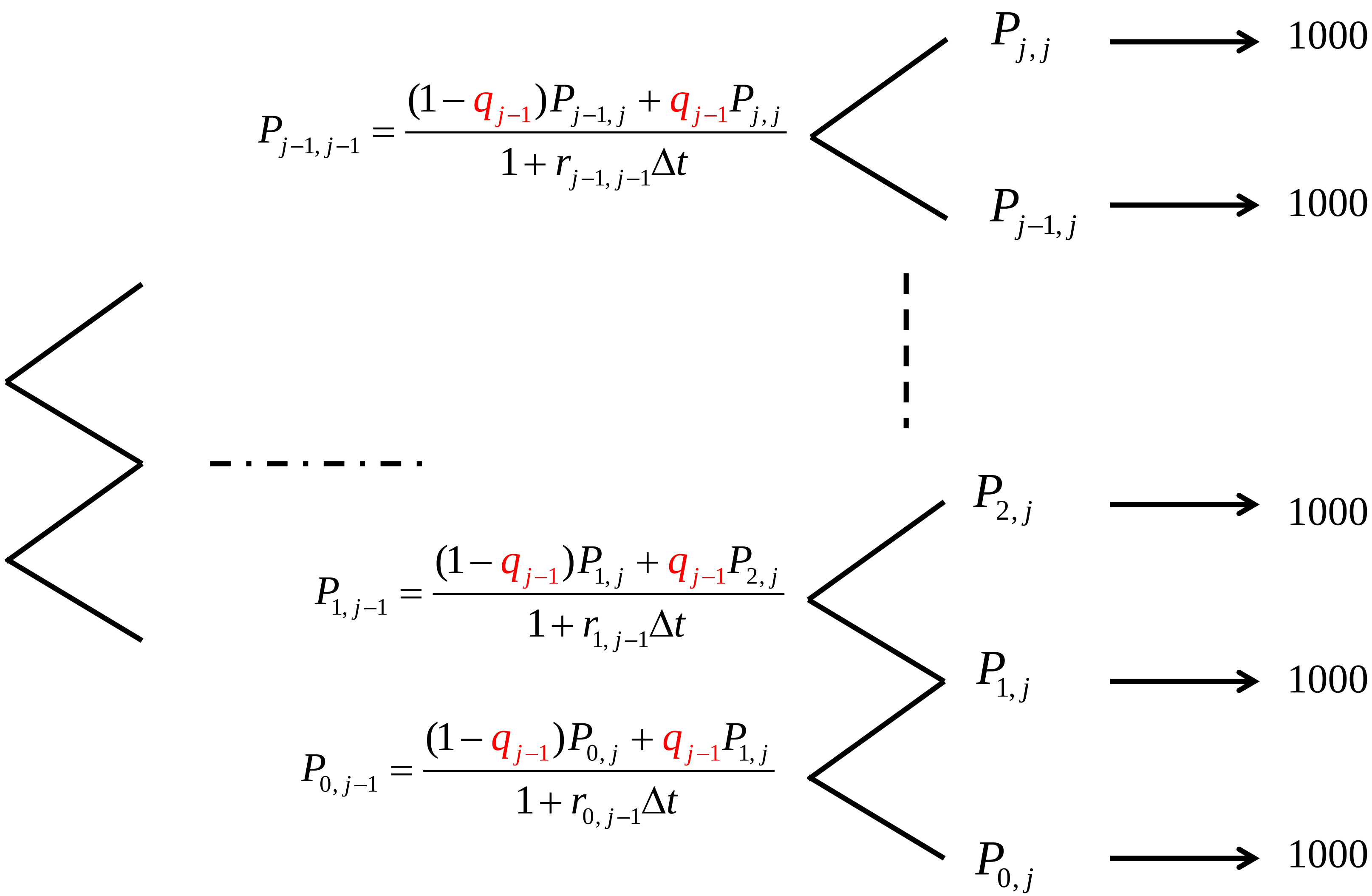
# Inputs for tree building

- The inputs include
  - Growth rate $\theta_t$ and volatility $\sigma$
  - The term structure of discount curve, $P(j\Delta t)$ $j=1,\cdots,N+1$
  - Chosen step size, $\Delta t$
- Determine $q_{j-1}$ by fitting to the price of $P((j+1)\Delta t), j=1,\cdots,N.$

# The principle for determining $q_{j-1}$

- The principle is to choose $q_{j-1}$ so that we can reproduce $P((j+1)\Delta t)$ , the price of $(j+1)\Delta t$ -maturity zero-coupon bond.

# Matching the price of




$$P_{j-1,j-1} = \frac{(1-q_{j-1})P_{j-1,j} + q_{j-1}P_{j,j}}{1 + r_{j-1,j-1}\Delta t}$$

$$P_{1,j-1} = \frac{(1-q_{j-1})P_{1,j} + q_{j-1}P_{2,j}}{1 + r_{1,j-1}\Delta t}$$

$$P_{0,j-1} = \frac{(1-q_{j-1})P_{0,j} + q_{j-1}P_{1,j}}{1 + r_{0,j-1}\Delta t}$$

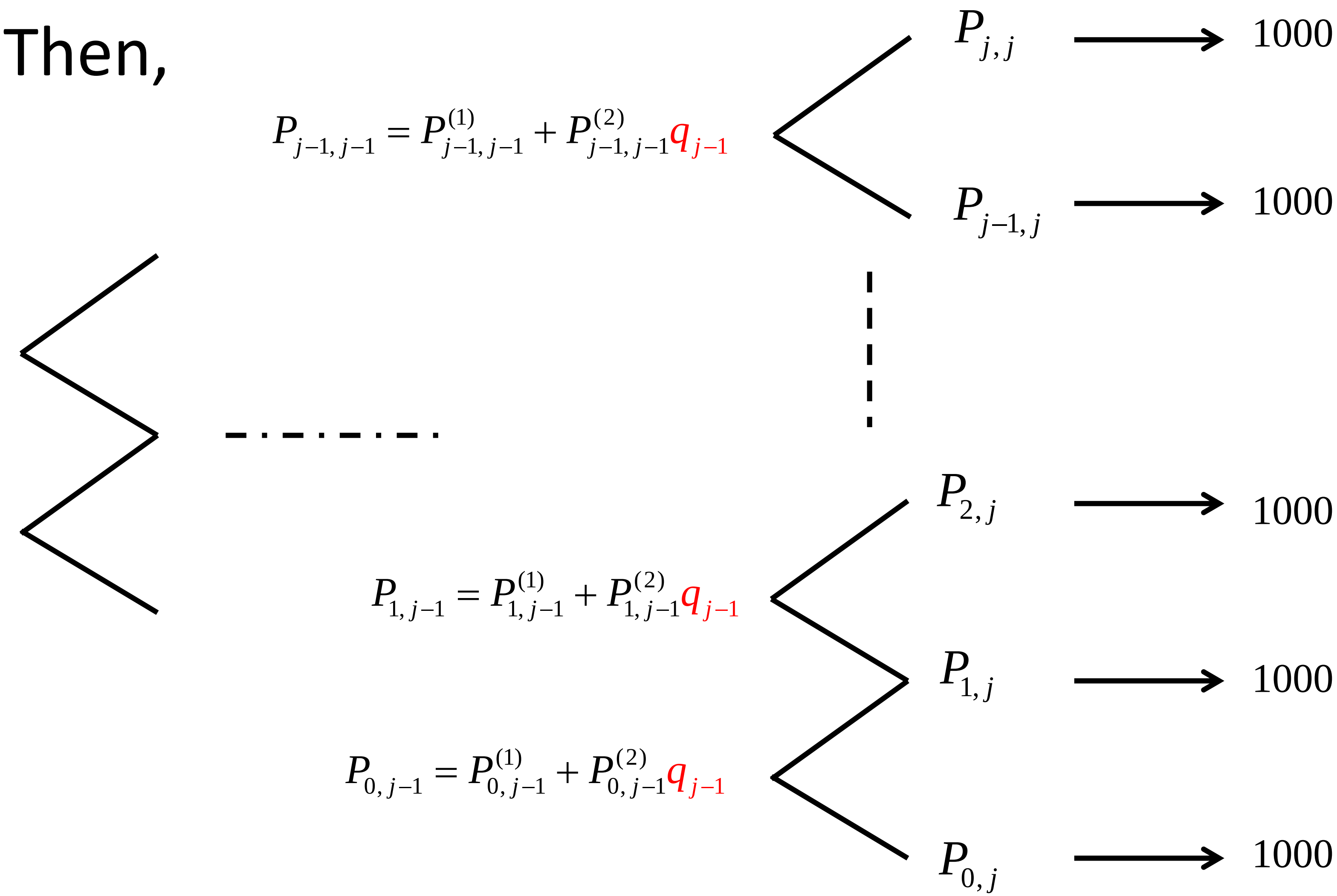
# Construction of Bond Price Tree

- Let

$$P_{i,j-1}^{(1)} = \frac{P_{i,j}}{1 + r_{i,j-1}\Delta t}, \qquad P_{i,j-1}^{(2)} = \frac{P_{i+1,j} - P_{i,j}}{1 + r_{i,j-1}\Delta t}$$

# Bond price tree

- Then,



- Backward induction for, separately, payoffs at time $(j-1)\Delta t$:

$$P_{i,j-1}^{(1)} = \frac{P_{i,j}}{1+r_{i,j-1}\Delta t} \quad \text{and} \quad P_{i,j-1}^{(2)} = \frac{P_{i+1,j} - P_{i,j}}{1+r_{i,j-1}\Delta t}$$

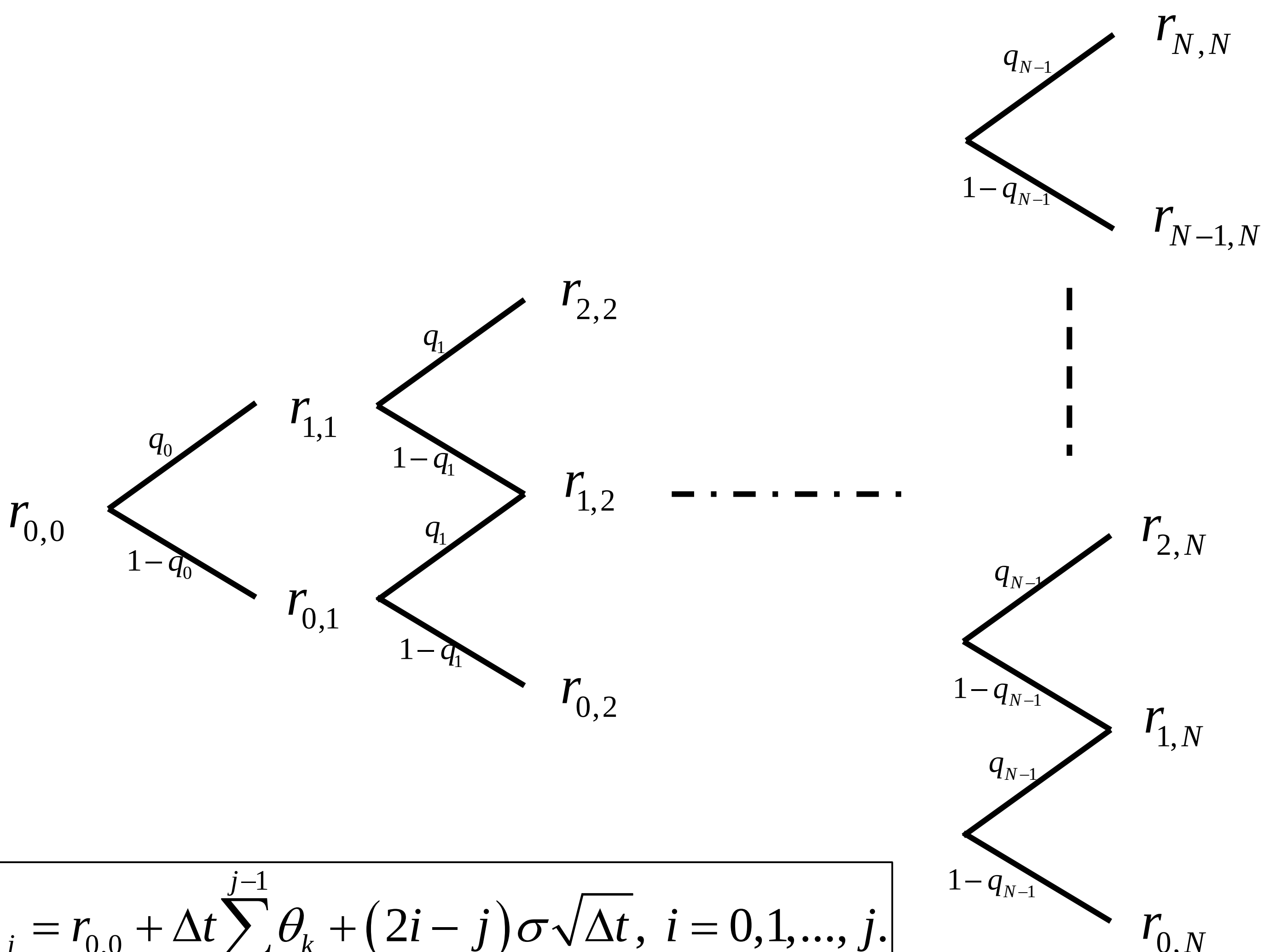
all the way back to root, we obtain

$$P_{0,0}^{(1)} \quad \text{and} \quad P_{0,0}^{(2)}$$

- We set

$$q_{j-1} = \frac{P((j+1)\Delta t) - P_{0,0}^{(1)}}{P_{0,0}^{(2)}}$$

# The constructed tree




$$r_{i,j} = r_{0,0} + \Delta t \sum_{k=1}^{j-1} \theta_k + (2i - j)\sigma\sqrt{\Delta t},\ i = 0,1,\ldots, j.$$

# The pseudo code

INPUT: $\sigma, \Delta t, \left\{P(j\Delta t)\right\}_{j=1}^{2T}$

For $j = 1 : N-1$

Set $r_{i,j} = r_{i,j-1} + \theta_{j-1}\Delta t - \sigma\sqrt{\Delta t}, \quad i = 0 : j-1$

$r_{j,j} = r_{j-1,j-1} + \theta_{j-1}\Delta t + \sigma\sqrt{\Delta t}$

Find $q_{j-1},$ by the bisection method, so that the

$(j+1)\Delta t$-maturity ZCB matches $P((j+1)\Delta t)$

End