Final Review

What Have We Learned?

I. Static hedging and data-based hedging
II. Term structures of interest rates
III. Interest-rate or fixed-income derivatives
IV. Binomial or Ho-Lee model
V. Black's model

I. Static hedging and data-based hedging

Risk measures

1. DV01
2. Duration (MaCaulay or modified)
3. Kr01
4. Key-rate duration

Static Hedging

- Static hedging means to nullify any of these risk measures
  - DV01 or duration neutral
  - Kr01 or key-rate duration neutral
- We hedge security B using a more liquid security A such that

$$\begin{cases} F_A XV01_A + F_B XV01_B = 0, \\ F_A = -\dfrac{XV01_B}{XV01_A} F_B, \end{cases} \quad \text{or} \quad \begin{cases} P_A D_A + P_B D_B = 0, \\ P_A = -\dfrac{D_B}{D_A} P_B. \end{cases}$$

Regression of yields

- Regression method between Δy_t^{20} and Δy_t^{30}:

$$\min_{\alpha,\beta} \sum_t \left(\Delta y_t^{20} - \alpha - \beta \Delta y_t^{30} \right)^2$$

yielding

$$\beta = \rho \frac{\sigma_{20}}{\sigma_{30}}, \quad \alpha = 0.$$

and projection of yield change:

$$\Delta y_t^{20} = \alpha + \beta \Delta y_t^{30} + \varepsilon_t$$

Data-Based Hedging

- Let F^{20} be the face amount of the 20-year bond, then for hedging we choose

$$F^{30} = -F^{20} \frac{DV01^{20}}{DV01^{30}} \beta$$

- Let P^{20} be the value of the 20-year bond, then for hedging we choose

$$P^{30} = -P^{20} \frac{D_{20}}{D_{30}} \beta$$

II. Term structures of interest rates

- Term structures
  i. Discount curve
  ii. Zero-coupon yield curve
  iii. Forward rate curve
  iv. Swap rate or par yield curve
- These curves are equivalent (from one we can derive the other).

III. Interest-rate derivatives

- Linear derivatives
  - FRA
  - Swaps
- Nonlinear derivatives
  - Bond options
  - Caps, floors
  - Swaptions

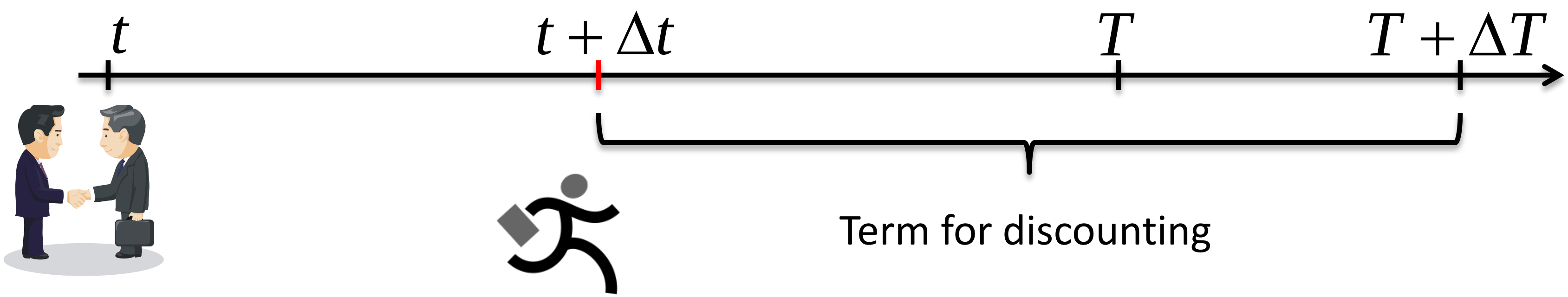
FRA

- A bet on interest rates b/w two parties, fixed for floating, indexed to LIBOR.
- Initially the value of the FRA is zero.
- Typically,
  - Three month LIBOR (or CD rates)
  - At least $1m notional.
- The arbitrage free fixed rate is

$$f_t = \frac{1}{\Delta T}\left(\frac{d(t,T)}{d(t,T+\Delta T)} - 1\right)$$

MtM Value of FRA

- Let A long the FRA (i.e. pays fixed) at *t*.

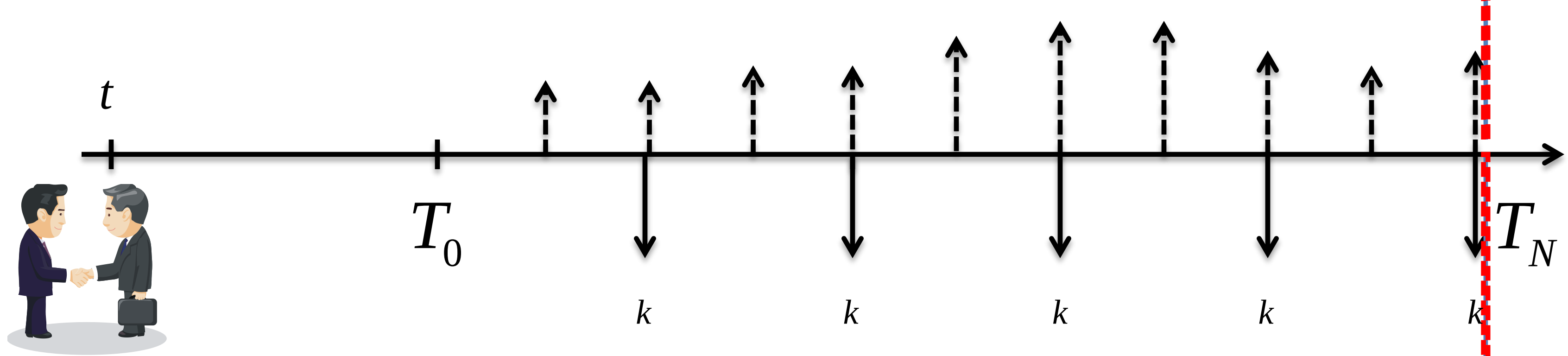


- At a later time $t+\Delta t$, then the MtM value of the FRA to the party who longs

$$\text{P\&L to A} = d(t+\Delta t, T+\Delta T)\left[\$1m \times \Delta T \times (f_{t+\Delta t} - f_t)\right]$$

Swaps

- Let a payer's swap start in time T_0 and end at T_N.



- $T_0 = t$, spot starting
- $T_0 > t$, forward starting

Determination of the swap rate

– Floating leg: par at T_0, so at $t \leq T_0$, it is

$$V_{float} = d(t, T_0)$$

– Fixed leg: let $s(t; T_0, T_N)$ be the swap rate, then

$$V_{fixed} = \sum_{i=1}^{N} \Delta T \times s(t; T_0, T_N) d(t, T_i) + d(t, T_N)$$

- Set

$$0 = V_{float} - V_{fix}$$

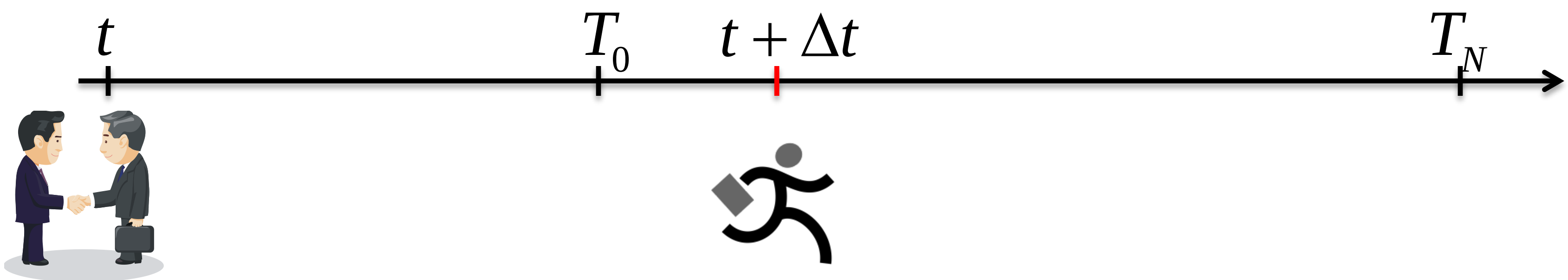
- We obtain

$$s(t;T_0,T_N) = \frac{d(t,T_0) - d(t,T_N)}{\sum_{i=1}^{N} \Delta T \times d(t,T_i)}$$

the prevailing or ATM swap rate.

MtM Value of a swap

- Let A pays fixed and B pays LIBOR.



- At time $t+\Delta t$, the MtM value of the swap is

$$\text{MtM to A} = \left(s(t+\Delta t; t, T_N) - s(t; T_0, T_N) \right) \sum_{T_i > t+\Delta t}^{T_N} \Delta T d(t, T_i)$$

Bond Options

- A bond option allows its holder to buy/sell a bond for a pre-specified price in a future date.
- Let the

$$T_0 \ - \ \text{maturity of the option}$$

$$T_N - T_0 \ - \ \text{life of the underlying bond at } T_0$$

$$K \ - \ \text{the strike price}$$

$$c \ - \ \text{coupon rate of the bond}$$

Bond Price at T_0

- Payoff of the call option on the bond

$$Option(T_0; K, T_N) = \max(Bond(T_0; K, T_N) - K, 0)$$

- Value of the bond option T_0 :

$$Bond(T_0; K, T_N) = \sum_{i=1}^{N} \Delta T d(T_0, T_i) c + d(T_0, T_N)$$

Swaptions

- A swaption is an option to enter into a swap for a pre-specified swap rate in the future.
- Let the

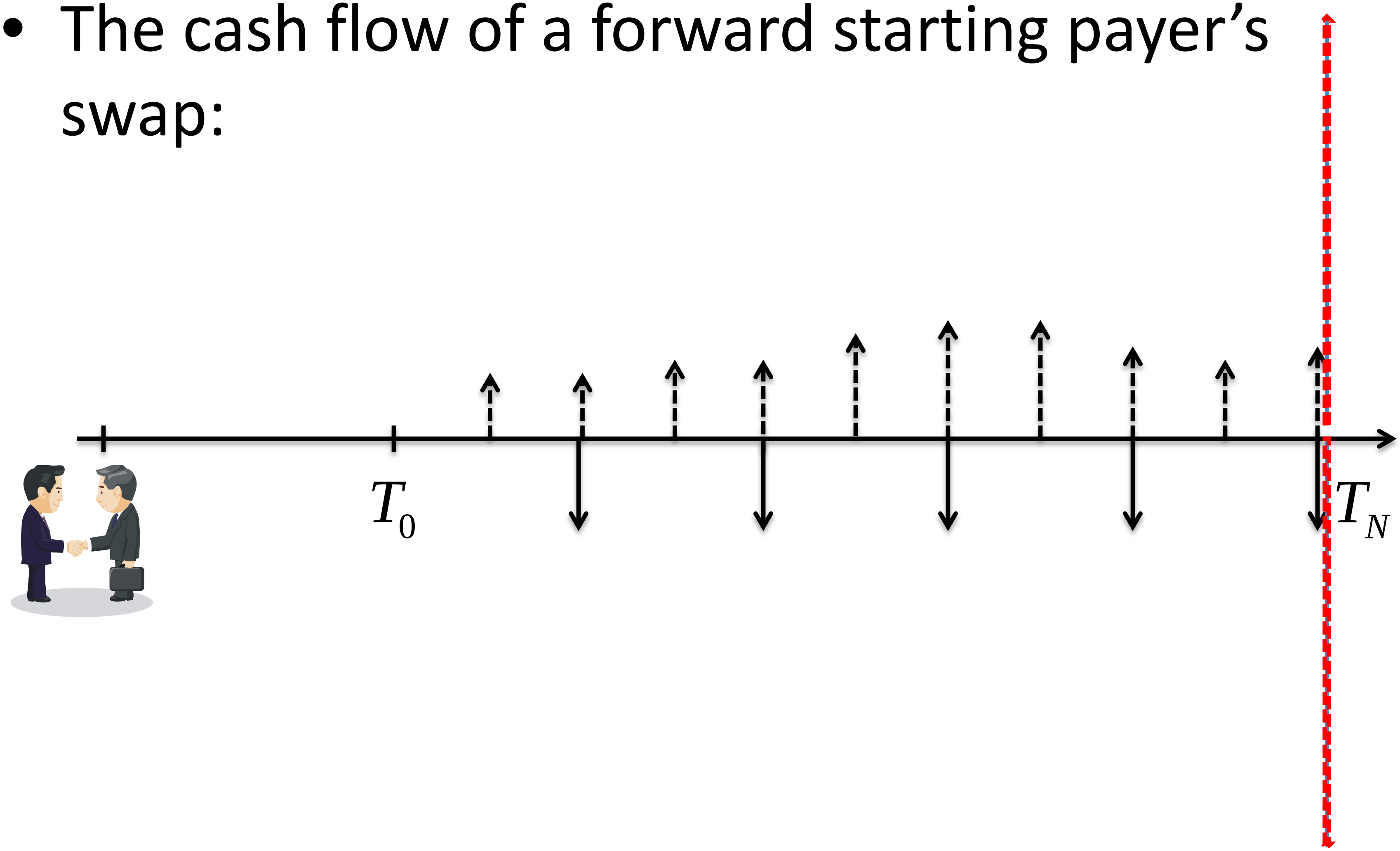
$$T_0 \;-\; \text{maturity of the option}$$
$$T_N - T_0 \;-\; \text{life of the underlying swap}$$
$$k \;-\; \text{the strike rate}$$

- Payoff of swaption

$$swtn(T_0;k,T_N) = \max(swap(T_0;k,T_N),0)$$

Cash Flow of The Underling Swap

- The cash flow of a forward starting payer's swap:



Swaption Turned Bond Options

- Payoff of the swaption on a payer's swap:

$$swap(T_0;k,T_N)^+ = \left(V_{float} - V_{fix}\right)^+$$
$$= \left(1 - \sum_{i=1}^{N} \Delta T d(T_0,T_i)k - d(T_0,T_N)\right)^+$$

- A payer's swaption can be priced as a put option on a coupon bond with PAR strike!

IV: Binomial or Ho-Lee model

- Binomial model for equity options (omitted)
- Ho-Lee model for interest-rate options.

Ho-Lee model Interest-Rate Models

- The discrete version of the Ho-Lee model is

$$\Delta r_t = \theta_t \Delta t + \sigma \sqrt{\Delta t}\ \varepsilon_B$$

where θ_t is fitted to the discount curve.

- Rate tree

$$r_0 \quad \begin{array}{l} \overset{1/2}{\nearrow} \quad r_{\Delta t}^{u} = r_0 + \theta_0 \Delta t + \sigma \sqrt{\Delta t} \\ \underset{1/2}{\searrow} \quad r_{\Delta t}^{d} = r_0 + \theta_0 \Delta t - \sigma \sqrt{\Delta t} \end{array}$$

Bond and option trees

- Price tree

$$P_{0,0} \quad \begin{array}{l} \xrightarrow{1/2} \; P_{1,1} = \dfrac{1000}{1+[r_{0,0}+\theta_0\Delta t+\sigma\sqrt{\Delta t}]\Delta t} \longrightarrow 1000 \\[2em] \xrightarrow{1/2} \; P_{0,1} = \dfrac{1000}{1+[r_{0,0}+\theta_0\Delta t-\sigma\sqrt{\Delta t}]\Delta t} \longrightarrow 1000 \end{array}$$

- Option price tree

$$C_{0,0} = ? \quad \begin{array}{l} C_{1,1} = \left(P_{1,1}-K\right)^+ \\[2em] C_{0,1} = \left(P_{0,1}-K\right)^+ \end{array}$$

Arbitrage pricing, cont'd

- Consider replicating the payoffs with (α, β) units of bond and cash, such that

$$\alpha P_{0,1} + \beta(1 + r_{0,0}\Delta t) = C_{0,1}$$
$$\alpha P_{1,1} + \beta(1 + r_{0,0}\Delta t) = C_{1,1}$$

- Solution

$$\alpha = \frac{C_{1,1} - C_{0,1}}{P_{1,1} - P_{0,1}}, \qquad \beta = \frac{P_{1,1}C_{0,1} - P_{1,0}C_{1,1}}{(1 + r_{0,0}\Delta t)(P_{1,1} - P_{1,0})}$$

- Arbitrage-free value: $C_{0,0} = \alpha P_{0,0} + \beta,$

Linear pricing rule

- Rewrite the option formula into

$$
\begin{aligned}
&C_{0,0} = \alpha P_{0,0} + \beta \\
&= \frac{C_{1,1} - C_{0,1}}{P_{1,1} - P_{0,1}} P_{0,0} + \frac{\left(P_{1,1}C_{0,1} - P_{1,0}C_{1,1}\right)}{(1 + r_{0,0}\Delta t)(P_{1,1} - P_{0,1})} \\
&= (1 + r_{0,0}\Delta t)^{-1} \left(\frac{P_{1,1} - P_{0,0}(1 + r_{0,0}\Delta t)}{P_{1,1} - P_{0,1}} C_{0,1} + \frac{P_{0,0}(1 + r_{0,0}\Delta t) - P_{0,1}}{P_{1,1} - P_{0,1}} C_{1,1} \right) \\
&= (1 + r_{0,0}\Delta t)^{-1} \left(q_0 C_{0,1} + (1 - q_0) C_{1,1} \right) \quad !!!
\end{aligned}
$$

Linear pricing rule

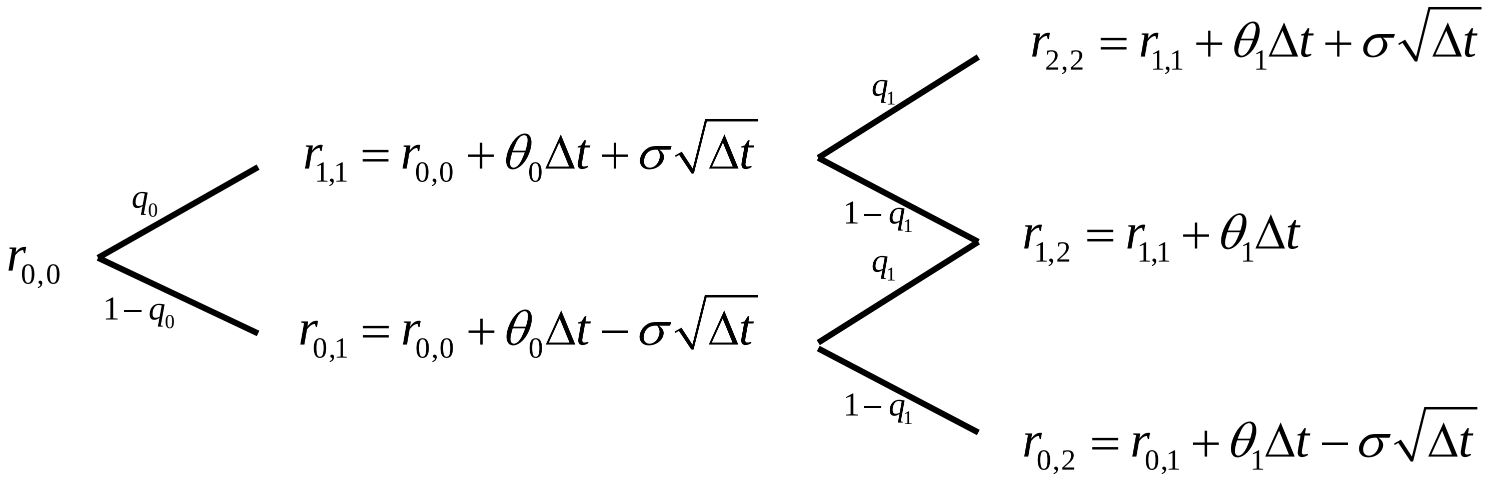
- There is

$$P_{0,0} = (1 + r_{0,0}\Delta t)^{-1}\left(q_0 P_{0,1} + (1 - q_0) P_{1,1}\right)$$

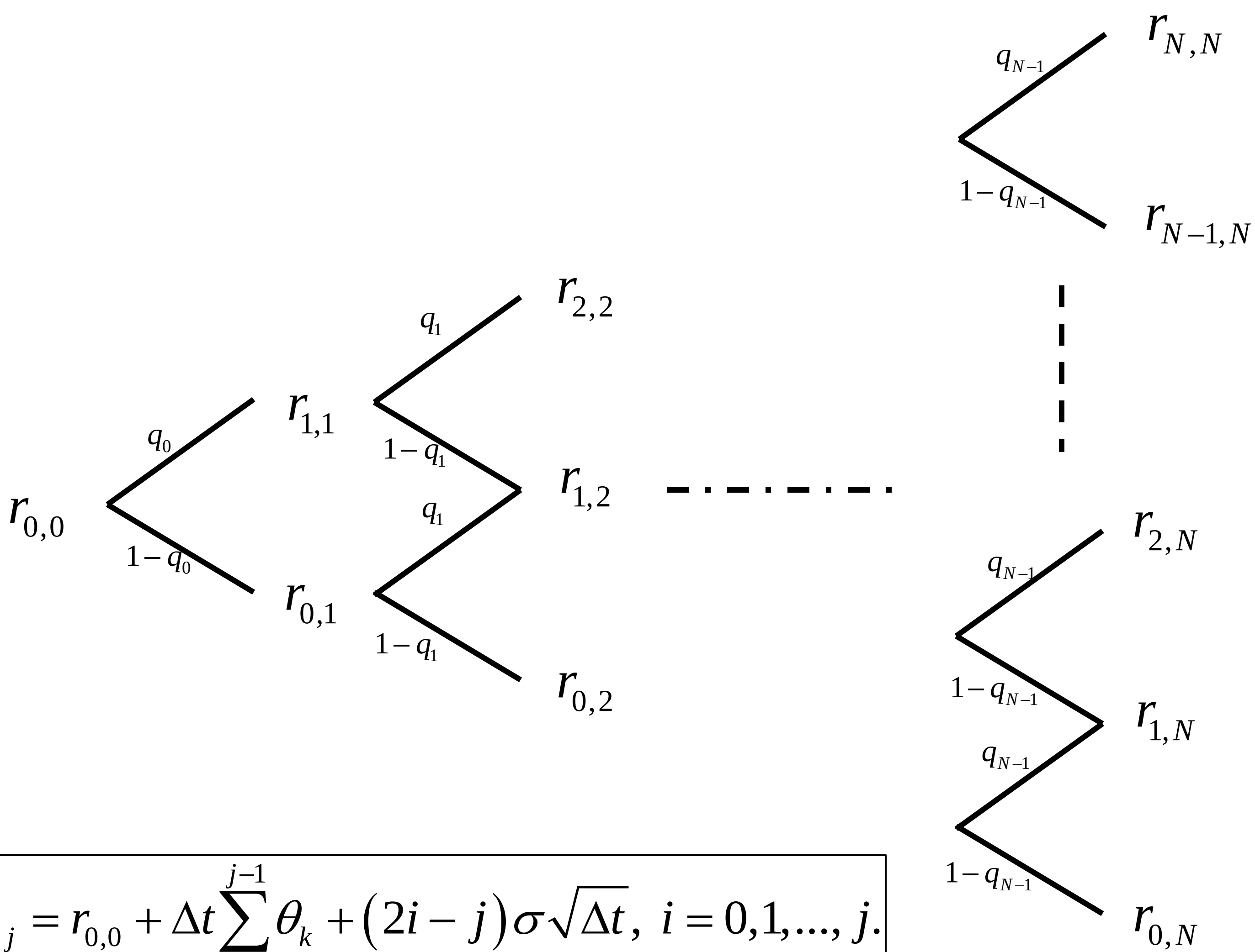
--- the pricing probabilities reproduce the bond price!

Two-period risk-neutral tree

- By duplication, we obtain



Extension to multi-period tree



$$\boxed{r_{i,j} = r_{0,0} + \Delta t \sum_{k=1}^{j-1} \theta_k + (2i - j)\sigma\sqrt{\Delta t},\ i = 0,1,\ldots,j.}$$

Inputs for tree building

- The inputs include
  - Growth rate θ_t and volatility σ
  - The term structure of discount curve, $P(j\Delta t)$
    $j = 1, \cdots, N+1$
  - Chosen step size, Δt
- Determine q_{j-1} by fitting to the price of
  $P((j+1)\Delta t), j = 1, \cdots, N.$

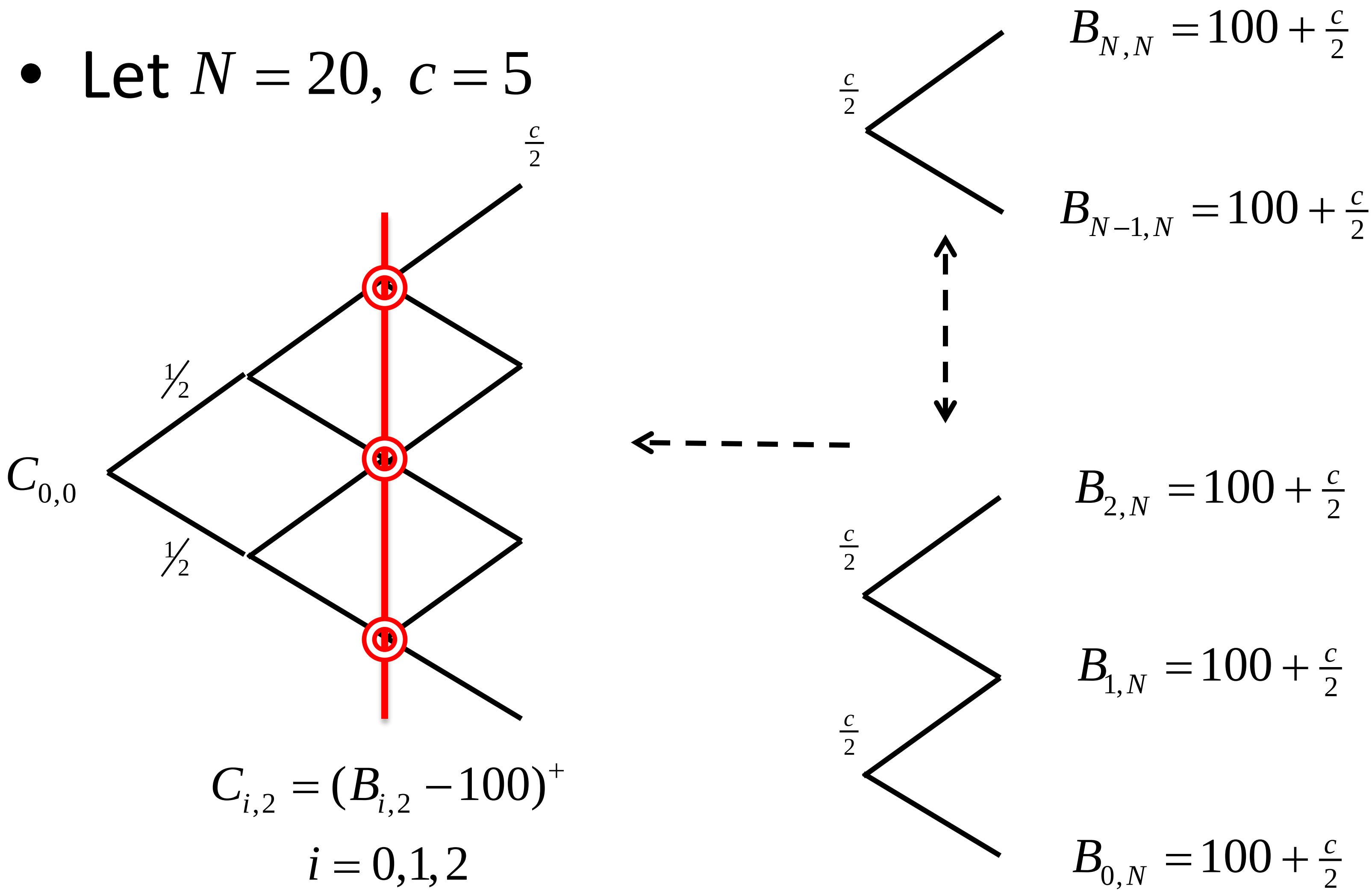
Pricing options on coupon bonds

1. Build a tree for ten years that reproduce the discount curve between 0 to TB years.
2. Obtain the price distribution of the coupon bond in TC years.
3. Calculate option's payoff.
4. Calculate the PV of the option.

Ex: $TC=1$, $TB=10$, $c=5$.

Strategy, cont'd

- Let $N = 20,\ c = 5$



$$C_{i,2} = (B_{i,2} - 100)^{+}$$
$$i = 0, 1, 2$$

V. The Black's formula

- The Black's formula (1976) applies to call and put options on
  - Equity derivatives
  - Interest-rate derivatives;
  - Forex derivatives
  - Commodity derivatives;
  - All derivatives;

Forward Contract

- A forward contract allows its counterparties to buy/sell certain asset for certain price *K* in certain date *T* in the future.
- The value of the long forward price is

$$V_t = S_t - d(t,T)K$$

- The forward price

$$F_t = \frac{S_t}{d(t,T)}, \qquad \text{for } t \le T,$$

MtM a forward Contract

- If you enter a T-maturity forward contract at time t and close it out at time t+dt, then the P&L is

$$P\,\&\,L = d(t+dt,T)\left[F_{t+dt} - F_t\right]$$
$$= d(t+dt,T)dF_t$$

- This is the PV if you hold the two contracts till the maturity.

Features of the Forward Price

- The forward price satisfies
  1. At an option's maturity, forward = spot:

$$F_T = \frac{S_T}{d(T,T)} = S_T$$

  2. It is assumes that

$$\ln \frac{F_T}{F_t} = -\frac{1}{2}\sigma^2 (T-t) + \sigma\sqrt{T-t}z$$

or

$$F_T = F_t e^{-\frac{1}{2}\sigma^2(T-t)+\sigma\sqrt{T-t}z}$$

Options on Forward

- Black's model for call option.

$$C_t = d(t,T)E\left[\left(F_T - K\right)^+\right]$$

$$= d(t,T)E\left[\left(F_t e^{-\frac{\sigma^2}{2}(T-t)+\sigma\sqrt{T-t}\times z} - K\right)^+\right]$$

$$= d(t,T)\left[F_t\Phi(d_1) - K\Phi(d_2)\right]$$

where

$$d_1 = \frac{\ln\frac{F_t}{K} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Black's formula for call option

- Let

$$g(F_t) = \left[F_t\Phi(d_1) - K\Phi(d_2)\right]$$

- There is

$$\frac{\partial g(F_t)}{\partial F_t} = \Phi(d_1)$$

- We write

$$C_t = d(t,T)g(F_t)$$

The differentiation

- The change over $(t, t+dt)$ is

$$
\begin{aligned}
dC_t &= C_{t+\mathrm{d}t} - C_t \\
&= d(t+\mathrm{d}t, T)g(F_{t+\mathrm{d}t}) - d(t,T)g(F_t) \\
&= d(t+\mathrm{d}t, T)\left[g(F_{t+\mathrm{d}t}) - g(F_t)\right] \\
&\qquad\qquad + \left[d(t+\mathrm{d}t, T) - d(t,T)\right]g(F_t) \\
&= d(t+\mathrm{d}t, T)\frac{\partial g(F_t)}{\partial F_t}\mathrm{d}F_t(t) + g(F_t)\mathrm{d}d(t,T) \\
&= \underbrace{\Phi(d_1)\times d(t+\mathrm{d}t, T)\mathrm{d}F_t}_{\text{Change in Forward contract}} + \underbrace{g(F_t)\mathrm{d}d(t,T)}_{\text{Change in ZCB}}
\end{aligned}
$$

Hedging Strategy

- For the call seller, the call can be hedged by
  - Long $\Phi(d_1)$ unit of T-maturity forward contract.
  - Long $g(F_t)$ unit of the T-maturity zero-coupon bond (ZCB).

Call-Put Parity

- For the same strike and maturity,

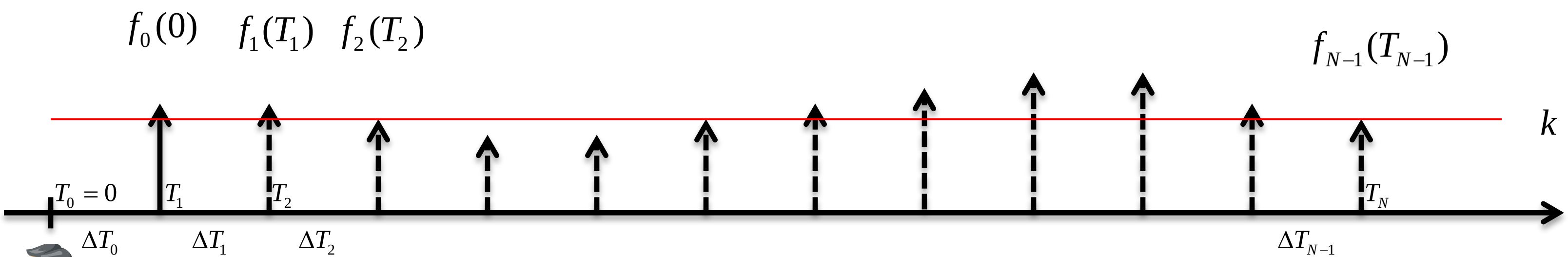
$$\text{Call} \; - \; \text{Put} \; = \; \text{Forward contract}$$

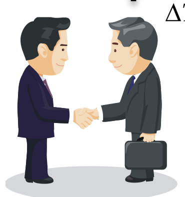
- i.e.,

$$\begin{aligned} P_t &= C_t - d(t,T)(F_t - K) \\ &= d(t,T)\left[\Phi(-d_2)K - \Phi(-d_1)F_t\right] \end{aligned}$$

Interest rate caps

- The cash flows of an interest rate cap:





where

$$f_{j-1}(T_{j-1}) = \frac{1}{\Delta T_j}\left(\frac{d(T_{j-1}, T_{j-1})}{d(T_{j-1}, T_j)} - 1\right)$$

- Cash flow of an interest-rate cap:

$$Not. \times \Delta T_j (f_{j-1}(T_{j-1}) - k)^+, \qquad j = 1, 2, \cdots, N.$$

- The Black's formula for cap:

$$Cap = \sum_{j=1}^{N} c_j$$

where

$$c_j = Not. \times \Delta T_j \times d(t, T_j) \left[f_{j-1}(t) \Phi(d_1^{(j)}) - k \Phi(d_2^{(j)}) \right],$$

$$d_1^{(j)} = \frac{\ln \frac{f_{j-1}(t)}{k} + \frac{1}{2}\sigma^2 (T_{j-1} - t)}{\sigma \sqrt{T_{j-1} - t}}, \quad d_2^{(j)} = d_1^{(j)} - \sigma \sqrt{T_{j-1} - t}$$

$$j = 1, 2, \cdots, N$$

Hedging with FRA and ZCB

- Let
$$g_j(t) = Not. \times \Delta T_j \times \left[f_{j-1}(t)\Phi(d_1^{(j)}) - k\Phi(d_2^{(j)}) \right]$$
- For the cap underwritter, the cap can be hedged by
  - Enter $\Phi(d_1^{(j)})$ unit of T_{j-1}-maturity FRA for the term (T_{j-1}, T_j) to long rate or pay fixed
  - being long or $g_j(t)$ unit of $P(t, T_j)$, the T_j -maturity zero-coupon bond (ZCB).

Interest rate floors

- Cash flow of an interest-rate floor: at T_j,

$$Not.\times \Delta T_j (k - f_{j-1}(T_{j-1}))^{+}, \qquad j = 1, 2, \cdots, N.$$

- The Black's formula

$$floor = \sum_{j=1}^{N} p_j$$

$$p_j = Not.\times \Delta T_j \times d(t, T_j)\left[k\Phi(-d_2^{(j)}) - f_{j-1}(t)\Phi(-d_1^{(j)}) \right],$$

$$d_1^{(j)} = \frac{\ln \frac{f_{j-1}(t)}{k} + \frac{1}{2}\sigma^2 (T_{j-1} - t)}{\sigma\sqrt{T_{j-1} - t}}, \quad d_2^{(j)} = d_1^{(j)} - \sigma\sqrt{T_{j-1} - t}$$

$$j = 1, 2, \cdots, N$$

Hedging with FRA and ZCB

- Let
$$g_j(t) = Not. \times \Delta T_j \times \left[k\Phi(-d_2^{(j)}) - f_{j-1}(t)\Phi(-d_1^{(j)}) \right]$$
- For the cap underwritter, the cap can be hedged by
  - enter $\Phi(-d_1^{(j)})$ unit of T_{j-1}-maturity FRA for the term (T_{j-1}, T_j) to short rate or pay float
  - being long $g_j(t)$ unit of $P(0, T_j)$, the T_j-maturity zero-coupon bond (ZCB).

Swaption pricing revisited

- Payoff of the swaption on a payer's swap:

$$swap(T_0;k,T_N)^+ = \left(V_{float} - V_{fix}\right)^+$$

$$= \left(1 - \sum_{i=1}^{N} \Delta T d(T_0,T_i)k - d(T_0,T_N)\right)^+$$

- At T_0, the swap rate of maturity $T_N - T_0$ is

$$s(T_0;T_0,T_N) = \frac{d(T_0,T_0) - d(T_0,T_N)}{\sum_{i=1}^{N} \Delta T \times d(T_0,T_i)} = \frac{1 - d(T_0,T_N)}{A(T_0;T_0,T_N)}$$

- Swap rate = par yield

$$1 = s(T_0;T_0,T)A(T_0;T_0,T_N) + d(T_0,T_N)$$

- Swaption payoff

$$swap(T_0;k,T_N)^+ = \left(V_{float} - V_{fix}\right)^+$$

$$= \Big(s(T_0;T_0,T_N)A(T_0;T_0,T_N) + d(T_0,T_N)$$

$$- kA(T_0;T_0,T_N) - d(T_0,T_N) \Big)^+$$

$$= A(T_0;T_0,T_N)\left(s(T_0;T_0,T_N) - k\right)^+$$

Black's formula for Swaptions

- The Black's formula for swaption

$$
\begin{aligned}
& swtn(t;k,T_N) \\
&= A(t;T_0,T_N)\left(s(t;T_0,T_N)\Phi(d_1) - k\Phi(d_2)\right)
\end{aligned}
$$

where

$$
d_1 = \frac{\ln \frac{s(t;T_0,T_N)}{k} + \frac{1}{2}\sigma^2(T_0 - t)}{\sigma\sqrt{T_0 - t}}, \quad d_2 = d_1 - \sigma\sqrt{T_0 - t}
$$

- Let

$$
g(s) = s\Phi(d_1) - k\Phi(d_2)
$$

Hedging with ATM swap and Annuity

- For the seller of the payer's swaption, the swaption can be hedged by
  - entering $\Phi(d_1)$ unit of (T_0, T_N) swap to pay fixed.
  - being long $g(s(t;T_0,T_N))$ unit of $A(t;T_0,T_N)$, the annuity.