Homework 9. Quantitative Methods for fixed Income Securities

1. Prove the following call-put parities:
   1.1. For caplet and floorlet of the same maturity and strike rate:

   long caplet + short floorlet = FRA.

   1.2. For a cap and floor of the same maturity and strike rate

   long cap + short floor = Swap.

2. At time t, the value of equity put option (for maturity T and strike K) is given by

$$P_t = d(t,T)\left[K\Phi(-d_2) - F_t\Phi(-d_1)\right],$$

with

$$F_t = \frac{S_t}{d(t,T)},\quad d_1 = \frac{\ln\frac{F_t}{K} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

The hedging strategy for the option is

a. Long $g(F_t) = \left[K\Phi(-d_2) - F_t\Phi(-d_1)\right]$ unit of zero-coupon bond; and

b. Short $\Phi(-d_1)$ unit of forward contract with strike $K = F_t$.

Show that

2.1. The value of the hedging portfolio is equal to P_t.

2.2. The hedging strategy is a self-financing one, by proving

$$dP_t = -\Phi(-d_1)[d(t+dt,T)\mathrm{d}F_t] + g(F_t)\mathrm{d}d(t,T)$$

3. The current term structure of **quarterly** forward rates is given by

$$f_j(0) = 0.01 + 0.0003\times(j-1),\ j = 1,\cdots,120.$$

Use the **Black's formula** to price a cap and a swaption with the same notional value of \$1m. Note that the payment frequency is a quarter year for caps and half a year for swaps.

3.1. Calculate the 10-year swap rate (The fixed leg has semi-annual payment).

3.2. Price the 10-year maturity cap with the strike rate to be the 10-year swap rate (such a cap is called *at-the-money-forward cap*). Take the cap volatility to be 25%.

3.3. Calculate the market prevailing swap rate for in-5-to-10 forward swap.

3.4. Price the in-5-to-10 ATM swaption on the payer's swap (i.e., maturity of the option: 5 years; tenor of the underlying swap: 10 years, strike rate: market prevailing swap rate for the in-5-to-10 forward swap). Assume a 25% swap-rate volatility.