MATH4511 Quantitive Methods for Fixed Income Derivatives, 2015-16 Fall
Quiz 04(T1C)

Name: ____________________ ID No.: ____________________ Tutorial Section:______

1. (20 points) Assume the spot rate under the risk neutral measure follows the dynamic:

$$\Delta r_t = (\mu + \lambda_t)\Delta t + \sigma\sqrt{\Delta t}\epsilon_B,$$

where ϵ_B takes +1 or -1 with equal probability.The current 1-year spot rate is r(0) and the 2-year zero coupon bond price is P(2) and the 3-year zero-coupon bond price is P(3). Use a two step binomial tree to describe the evolution of the 1-year spot rate. Here, μ, σ are constant, risk neutral probability is $\{\frac{1}{2}, \frac{1}{2}\}$. State the procedure of calculating λ_t's.

Tree:

$r(0) = r_{0,0}$

$\frac{1}{2}$: $r_{1,1} = (\mu+\lambda_1)\Delta t + \sigma\sqrt{\Delta t} + r_{0,0}$

$\frac{1}{2}$: $r_{0,1} = (\mu+\lambda_1)\Delta t - \sigma\sqrt{\Delta t} - r_{0,0}$

$\frac{1}{2}$: $r_{2,2} = r_{1,1} + (\mu+\lambda_2)\Delta t + \sigma\sqrt{\Delta t}$

$\frac{1}{2}$: $r_{1,2} = r_{1,1} + (\mu+\lambda_2)\Delta t - \sigma\sqrt{\Delta t} = r_{0,1} + (\mu+\lambda_2)\Delta t + \sigma\sqrt{\Delta t}$

$\frac{1}{2}$: $r_{0,2} = r_{0,1} + (\mu+\lambda_2)\Delta t - \sigma\sqrt{\Delta t}$.

To calculate λ_1:

Tree of 2-y zero-coupon bond (face = 1).

$$\frac{\frac{1}{2}\left(\frac{1}{1+r_{1,1}\Delta t}\right)+\frac{1}{2}\left(\frac{1}{1+r_{0,1}\Delta t}\right)}{1+r_{0,0}\Delta t} \quad \begin{cases} \frac{1}{1+r_{1,1}\Delta t} & \begin{cases} 1 \\ 1 \end{cases} \\ \frac{1}{1+r_{0,1}\Delta t} & \begin{cases} 1 \\ 1 \end{cases} \end{cases}$$

$r_{1,1}$, $r_{0,1}$ are function of λ_1 only.

$$\Rightarrow \frac{\frac{1}{2}\left(\frac{1}{1+r_{1,1}\Delta t}\right)+\frac{1}{2}\left(\frac{1}{1+r_{0,1}\Delta t}\right)}{1+r_{0,0}\Delta t} = P(2) \qquad (eq1)$$

solve λ_1. from (eq1)

To calculate λ_2:

Tree of 3-y zero-coupon bond. (now λ_1 is known)

$$\frac{\frac{1}{2}(A)+\frac{1}{2}(B)}{1+r_{0,0}\Delta t} \quad \begin{cases} A \triangleq \frac{\frac{1}{2}\left(\frac{1}{1+r_{2,2}\Delta t}\right)+\frac{1}{2}\left(\frac{1}{1+r_{1,2}\Delta t}\right)}{1+r_{1,1}\Delta t} & \begin{cases} \frac{1}{1+r_{2,2}\Delta t} & \begin{cases} 1 \\ 1 \end{cases} \\ \frac{1}{1+r_{1,2}\Delta t} & \begin{cases} 1 \\ 1 \end{cases} \end{cases} \\ B \triangleq \frac{\frac{1}{2}\left(\frac{1}{1+r_{1,2}\Delta t}\right)+\frac{1}{2}\left(\frac{1}{1+r_{0,2}\Delta t}\right)}{1+r_{0,1}\Delta t} & \begin{cases} \frac{1}{1+r_{1,2}\Delta t} \\ \frac{1}{1+r_{0,2}\Delta t} & \begin{cases} 1 \\ 1 \end{cases} \end{cases} \end{cases}$$

Since λ_1 is known, A, B are functions of λ_2.

$$\Rightarrow \frac{\frac{1}{2}A+\frac{1}{2}B}{1+r_{0,0}\Delta t} = P(3) \qquad (eq2)$$

solve λ_2. from (eq2).

2. (20 points) Given a risk-neutralized interest-rate tree

$$\Delta r_t = 0.0025\Delta t + 0.005\sqrt{\Delta t}\epsilon_B,$$

where Δt=1, r_0=5% and ϵ_B takes +1 or -1 with equal probability. Consider an interest rate product has a payoff \$1000 when the 1-year spot rate exceeds 5.5% every year. Assume the maturity is 2 years. Use the tree to price this product.

$$\frac{\frac{1}{2}\left(\frac{1000}{1+5.75\%}+1000\right)+\frac{1}{2}\left(\frac{500}{1+4.75\%}\right)}{1+5\%}$$

5%

5.75% — $\frac{1000}{1+5.75\%}+1000$

4.75% — $\frac{500}{1+4.75\%}$

6.5% — 1000

5.5% — 1000

4.5% — 0

$$\| $$

$$\$1153.8.$$

3. (10 points) The current forward-rate curve for semi-annual compounding is

$$f(\frac{i}{2}) = .02 + .001 * \frac{i}{2}, \quad i = 0, 1, 2, ...$$

Calculate the in-5-to-10 swap rate (i.e., the swap rate for the swap for the period from 5 to 15 years).

$$S(0;5,15) = \frac{d(0,5)-d(0,15)}{\frac{1}{2}\sum_{i=11}^{30} d(0,\frac{i}{2})}.$$

calculate all the $d(0,\frac{i}{2})$ from the curve of forward rate. □.