Math 4511 Final

1.1. payoff function for forward rate agreement

is :

notional $V_1 \times \Delta T \times (f_T - f_0)$

1.2 $P\&L = d(T, T+\Delta T) [Not. \times \Delta T \times [f_t(T+\Delta T) - f_0(T+\Delta T)]$

1.3. Long 1 unit of T-maturity zero coupon bond, short $\frac{d(0,T)}{d(0,T+\Delta T)}$ units of $T+\Delta T$-maturity zero coupon bond

Q 2.1 :

$$U = 1+\mu\Delta t + \sigma\sqrt{\Delta t}$$

$$D = 1+\mu\Delta t - \sigma\sqrt{\Delta t}$$

$\because \mu, \sigma > 0,$

$\therefore U > D.$

$$0 < q = \frac{U-R}{U-D} < 1$$

$$\Rightarrow \quad D < 1+r\Delta t < U$$

22.2. Input : $\theta, \Delta t, r$

$S_t^u = 1 + \mu \Delta t - \sigma\sqrt{\Delta t}$

$S_t^d = 1 + \mu \Delta t + \sigma\sqrt{\Delta t}$.

So $\mu = \frac{D+U-2}{2\Delta t}$ $\sigma = \frac{U-D}{2\sqrt{\Delta t}}$

Tree Construction:

for $j := 1$ to n :

$S_{0,j} = S_{0,j-1} \times D$.

for $i := 1$ to j:

$S_{i,j} = S_{i-1,j-1} \times U$

end

end

Backward Induction:

for $i := 0$ to n:

$C_{i,n} = \max(K - S_{i,n}, 0)$

end

$$q = \frac{(\mu - r)\sqrt{\Delta t}}{2\sigma} + 0.5$$

$$\left(q = \frac{\left[\frac{D+U-2}{2\Delta t} - r\right]\sqrt{\Delta t}}{\frac{U-D}{2\sqrt{\Delta t}}} + 0.5 \right)$$

$$\left(q = \frac{\left[\frac{U+D-2}{2\Delta t} - r\right] t}{U-D} + 0.5 \right)$$

for $j = N-1$ to 0:
    for $i = 0$ to j:
        $C_{i,j} = \frac{1}{1+r\Delta t}\left(q C_{i,j+1} + (1-q) C_{i+1,j+1}\right)$
    end
end

output

$$\alpha_{0,0} = \frac{C_{1,1} - C_{0,1}}{S_{1,1} - S_{0,1}}$$

$$\beta_{0,0} = \frac{P_{1,1} C_{0,1} - P_{0,1} C_{1,1}}{(1 + r_{0,0}\Delta t)(P_{1,1} - P_{0,1})}$$

for replication pricing, where $C_{0,0} = \alpha S_{0,0} + \beta$.

3). We could use dynamic hedging,

$$\alpha_{i,j} = \frac{C_{i+1,j+1} - C_{i,j+1}}{S_{i+1,j+1} - S_{i,j+1}}, \quad \beta_{i,j} = \frac{\left(P_{i+1,j+1} C_{i,j+1} - P_{i,j+1} C_{i+1,j+1}\right)}{(1 + r_{i,j}\Delta t)(P_{i+1,j+1} - P_{i,j+1})}$$

Q3.

The payoff of a call option on a coupon bond is

$$\left(\sum_{i=1}^{2\tau} \delta T\, d(T_0, T_i)\, c + d(T_0, T_N) - 1\right)^+$$

$$= \Big(\sum_{i=1}^{2\tau} \Delta T\, d(T_0,T_i)\, c + d(T_0,T_N) - S(T_0; T_0, T_N) \sum_{i=1}^{2T} \Delta T \times d(T_0,T_i) - d(T_0,T_N)\Big)^+$$

$$= A(T_0; T_0, T_N)\,(c - S(T_0;T_0,T_N))^+$$

which is the payoff of a receiver's swaption.

And we could observe that:

$$\Big(\underbrace{\sum_{i=1}^{2\tau} \delta T\, d(T_0, T_i)\, c + d(T_0, T_N)}_{\text{bond price at } T} - \underbrace{1}_{\text{strike price}}\Big)^+$$

∴ swaption is equal to bond option with par strike

Q4.1

$$\text{payoff} = A(t;T_0,T_N)\,E[K - S(t;T_0,T_N)^+]$$

$$= A(t;T_0,T_N)[K\Phi(-d_2) - S(t;T_0,T_N)\,\Phi(-d_1)]$$

$$= A(t;T_0,T_N)\,G(S(t;T_0,T_N))$$

4.2

$$dV_t = V_{t+dt} - V_t$$

$$= A(t+dt, T_0, T_N)\, G(s(t+dt; T_0, T_N)) - A(t, T_0, T_N)\, G(s(t; T_0, T_N))$$

$$= A(t+dt, T_0, T_N)\, [G(s(t+dt; T_0, T_N)) - G(s(t; T_0, T_N))] + [A(t+dt, T_0, T_N) - A(t, T_0, T_N)]$$

$$= A(t+dt, T_0, T_N)\, dG(s(t; T_0, T_N)) + G(s(t; T_0, T_N))\, dA(t, T_0, T_N)$$

4.3 Given $G(S) = K\Phi(-d_2) - S\Phi(-d_1)$

$$\frac{dG(S)}{dS} = -\Phi(-d_1) + \frac{Kn(-d_2) - Sn(-d_1)}{S\sigma\sqrt{T_0-t}}$$

$$= \Phi(d_1) + \frac{\frac{K}{S}n(-d_2) - n(-d_1)}{\sigma\sqrt{T_0-t}}$$

$$n(-d_1) = n(-d_2 + \sigma\sqrt{T_0-t})$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(-d_2+\sigma\sqrt{T_0-t})^2/2}$$

$$= n(-d_2) \cdot e^{-\ln\frac{S}{K} - \frac{1}{2}\sigma^2(T_0-t) + \frac{1}{2}\sigma^2(T_0-t)}$$

$$= \frac{K}{S}n(-d_2)$$

$$\Rightarrow \frac{K}{S}n(-d_2) - n(-d_1) = 0$$

$$\frac{dG(S)}{dS} = -\Phi(-d_1).$$

4.4. long $\Phi(-d_1)$ unit of ATM payer's swap,

long $G(S(t; T_0, T_N))$ unit of the annuity.

5.

$$r_t^u = r_t + \theta_t \Delta t + \sigma\sqrt{\Delta t}$$

$$r_t^d = r_t + \theta_t \Delta t - \sigma\sqrt{\Delta t}$$

5.1 $\Delta r_t = 0 \pm \sqrt{0.25} \times 1\% = 0.5\%$

As the forward-rate is flat at 4%, spot rate also flat at 4%.

Thus, $r_0 = 4\%$.

$$r_0 = 4\% \begin{cases} 4.5\% \\ 3.5\% \end{cases}$$

By risk neutral property,

$$\frac{1}{(1+4\%/4)^2} = \frac{1}{1+4\%/4}\left(q_0\left(\frac{1}{1+\frac{3.5\%}{4}}\right) + (1-q_0)\left(\frac{1}{1+\frac{4.5\%}{4}}\right)\right)$$

$$0.98029608 = q_0\left(\frac{1}{1+4\%/4}\right)\left(\frac{1}{1+3.5\%/4}\right) + (1-q_0)\left(\frac{1}{1+4\%/4}\right)\frac{1}{(1+4.5\%/4)}$$

$\Rightarrow q_0 = 0.499361$

5.2

$$\begin{tikzcd}[column sep=small]
 & & \$100 \times (4.5\% - 3.8\%)^+ = 0.7 \\
0.3469 \arrow[r, dash] & 0.3504 \arrow[ur, dash, "\frac{1}{2}"] \arrow[dr, dash, "\frac{1}{2}"'] & \\
 & & \$100 \times (3.5\% - 3.8\%)^+ = 0
\end{tikzcd}$$

∴ The price is 0.3469.