Black-Scholes call formula

- The Black-Scholes call formula

$$
\begin{aligned}
C_0 &= e^{-rT}\left[S_0 e^{rT}\Phi(d_1) - K\Phi(d_2)\right] \\
&= S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)
\end{aligned}
$$

- Two more questions:
  - Can it reprise the underlying?
  - What should be the limit of *alpha,* the hedge ratio?

Repricing the stock

$$E\left[e^{-rt}S_t\right]=E\left[S_0 e^{-\frac{1}{2}\sigma^2 t+\sigma\sqrt{t}\times z}\right]$$

$$=S_0\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\sigma^2 t+\sigma\sqrt{t}\times x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$=S_0\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\sigma^2 t+\sigma\sqrt{t}\times x-\frac{x^2}{2}}dx$$

$$=S_0\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(x-\sigma\sqrt{t}\right)^2}dx$$

$$=S_0\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy=S_0!!!$$

Alternative Proof

- As $K \to 0$, there are

$$d_1 \to +\infty \ \text{ and } \ \Phi(d_1) \to 1,$$

so that we have

$$C_0 = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2) \to S_0 !$$

The *alpha*

- Under the binomial model,

$$\alpha = \frac{C(S_{\Delta t}^{u}, \Delta t) - C(S_{\Delta t}^{d}, \Delta t)}{S_{\Delta t}^{u} - S_{\Delta t}^{d}}$$

$$\to \frac{\partial C(S_0, 0)}{\partial S} \quad \text{as} \quad \Delta t \to 0$$

The *alpha*, cont'd

- Proposition: for the Black-Scholes formula

$$C_0 = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2),$$

there is

$$\frac{\partial C(S_0, 0)}{\partial S_0} = \Phi(d_1) \qquad \square$$

- $z \sim N(0,1)$ means

$$\Pr ob(z \le a) = \Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

where

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

is the density function of z.

Proof:

- Note that

$$\frac{\partial \Phi(d_i)}{\partial S_0} = \frac{\partial \Phi(d_i)}{\partial d} \frac{\partial d_i}{\partial S_0} = n(d_i) \frac{1}{S_0 \sigma \sqrt{T}}$$

- It follows that

$$\frac{\partial C(S_0, 0)}{\partial S_0} = \Phi(d_1) + \frac{S_0 n(d_1) - e^{-rT} K n(d_2)}{S_0 \sigma \sqrt{T}}$$

$$= \Phi(d_1) + \frac{n(d_1) - e^{-rT} \frac{K}{S_0} n(d_2)}{\sigma \sqrt{T}}$$

- The last term equals to zero since

$$
\begin{aligned}
n(d_1) &= n(d_2 + \sigma\sqrt{T}) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{(d_2+\sigma\sqrt{T})^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2+2d_2\sigma\sqrt{T}+\sigma^2 T}{2}} \\
&= n(d_2) \times e^{-(\ln\frac{S_0}{K}+(r-\frac{1}{2}\sigma^2)T+\frac{1}{2}\sigma^2 T)} \\
&= e^{-(\ln\frac{S_0}{K}+rT)} n(d_2) = e^{-rT}\,\frac{K}{S_0} n(d_2) \quad \square
\end{aligned}
$$

Black-Scholes formula for put options

- It can be proved by brute force that

$$
\begin{aligned}
P_0 &= e^{-rT} E\left[\left(K - S_T\right)^+\right] \\
&= K e^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1)
\end{aligned}
$$

and

$$
\frac{\partial P_0}{\partial S_0} = -\Phi(-d_1)
$$

- But we want to take an alternative approach.

Forward contract – constant interest rate

- Forward contract: a deal entered at time 0 to buy/sell an asset at a future time for certain price.
- What should be the fair value for the forward contract?
- We try arbitrage pricing.
- To ensure delivery, the seller must acquire the asset at time 0.

Arbitrage Pricing

- Denote
  - the current price of the asset by S_0 ,
  - The strike price by K
  - the delivery time by T ,
- To be able to deliver one unit of the asset, the seller then does the following transactions:
  - Borrow cash for the amount of S_0 .
  - Long 1 unit of the asset.
- This is set of of zero-net transactions at time 0.

Arbitrage Pricing, cont'd

- At the delivery time, T , the seller will deliver the asset to the buyer for the price of K, and thus ends up with the following profit/loss value,

$$\text{P\&L} = K - S_0 e^{rT}.$$

- If the absence of arbitrage, the fair value of the forward contract must be

$$V_0 = -e^{-rT}\text{P\&L} = S_0 - e^{-rT}K.$$

Forward Price

- Forward price is the special strike price K at which the value of the forward contract is zero:

$$F_0 = S_0 e^{rT}$$

Call-Put Parity

- Call and put parity means that: for the same strike and maturity,

$$\text{Long Call} - \text{Short Put} = \text{Forward Contract}$$

- Proof: The future value at T equal:

$$\left(S_T - K\right)^+ - \left(K - S_T\right)^+ = S_T - K$$

The PV must be equal as well:

$$e^{-rT}E\left[\left(S_T - K\right)^+\right] - e^{-rT}E\left[\left(K - S_T\right)^+\right] = e^{-rT}E\left[S_T\right] - e^{-rT}K$$

Black's formula for put option

- Put-call parity:

$$
\begin{aligned}
P_0 &= C_0 - \left[S_0 - Ke^{-rT} \right] \\
&= S_0 \Phi(d_1) - Ke^{-rT} \Phi(d_2) - \left[S_0 - Ke^{-rT} \right] \\
&= S_0 \left[\Phi(d_1) - 1 \right] - Ke^{-rT} \left[\Phi(d_2) - 1 \right] \\
&= Ke^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1)
\end{aligned}
$$

- Hedge ratio

$$\alpha = -\Phi(-d_1)$$

Black-Scholes Formulae

- Call option price:

$$C(S,K,\sigma,r,T) = S\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

- Put option price:

$$P(S,K,\sigma,r,T) = Ke^{-rT}\Phi(-d_2) - S\Phi(-d_1)$$

where

$$d_1 = \frac{\ln\frac{S}{K} + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T}$$

Limitations of the BS formula

- Constant interest rate
- Unacceptable for fixed-income markets
- We look for fixes.

Hiding the interest rate

- To price
  - Interest-rate derivatives,
  - Commodity derivatives, and
  - European options,
- Black (1976) revised the Black-Scholes formula by replacing

$$e^{-rT} \Rightarrow d(0,T) \qquad \text{the discount factor}$$

$$S_0 e^{rT} \Rightarrow F_0 = \frac{S_0}{d(0,T)} \qquad \text{the forward price}$$

From Black-Scholes formula to Black formula

- Black formula (worth at least multi-billion dollars):

$$C_0 = d(0,T)\left[F_0\Phi(d_1) - K\Phi(d_2)\right]$$
$$= S_0\Phi(d_1) - d(0,T)K\Phi(d_2)$$

where

$$d_1 = \frac{\ln\frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

- Hedge ratio

$$\alpha = \Phi(d_1), \qquad \beta = -K\Phi(d_2)$$

Zero-coupon Bond Prices as Discount Factors

- We now rewrite $d(0,T)$ as $P(0,T)$, the price of zero-coupon bond with $1 notional.
- The Black's formula:

$$\begin{aligned} C_0 &= P(0,T)\left[F_0\Phi(d_1) - K\Phi(d_2)\right] \\ &= S_0\Phi(d_1) - P(0,T)K\Phi(d_2) \end{aligned}$$

- with

$$d_{1,2} = \frac{\ln\frac{F_0}{K} \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad F_0 = \frac{S_0}{P(0,T)}.$$

Regarding Hedging

- Hedge ratio:

$$\alpha = \Phi(d_1), \qquad \beta = -K\Phi(d_2)$$

- To hedge the option, the option seller
  - Long $\alpha = \Phi(d_1)$ unit of share.
  - Long $\beta = -K\Phi(d_2)$ unit of zero-coupon bond (with \$1 notional) .

Black's formula for put option

- Multi-million formula:

$$P_0 = P(0,T)\left[K\Phi(-d_2) - F_0\Phi(-d_1)\right]$$
$$= P(0,T)K\Phi(-d_2) - S_0\Phi(-d_1)$$

- Hedge ratio

$$\alpha = -\Phi(-d_1), \qquad \beta = K\Phi(-d_2)$$

Forward Contracts

- For using the call-put parity I need to price forward contracts under stochastic interest rates?
- We try arbitrage pricing again.
- To ensure delivery, the seller must acquire certain units of the asset using capital obtained through shorting a zero-coupon bond.

Arbitrage Pricing

- Denote
  - the current price of the asset by S_0 ,
  - The strike price by K
  - the delivery time by T
- To be able to deliver one unit of the asset, the seller then does the following transactions:
  - Short $S_0 / P(0,T)$ units of -maturity zero-coupon bond for \$1 notional, and
  - Long 1 unit of the asset.
- This is set of of zero-net transactions at time 0.

Arbitrage Pricing, cont'd

- At the delivery time, T, the seller will deliver the asset to the buyer for the price of K, and thus ends up with the following profit/loss value,

$$\text{P\&L}(T) = K - \frac{S_0}{P(0,T)}.$$

- The value of the forward contract is thus

$$\begin{aligned} V_0 &= \text{negative PV of the P\&L} \\ &= -P(0,T) \times \text{P\&L}(T) = S_0 - P(0,T)K \end{aligned}$$

Forward Price

- The special strike price that makes the value of the forward contract equal to zero is

$$F_0 = \frac{S_0}{P(0,T)}.$$

which is called the forward price.

Put-Call Parity

- Long a call and short a put with the same strike equals to a forward contract:

$$
\begin{aligned}
C_0 - P_0 &= S_0\left[\Phi(d_1) + \Phi(-d_1)\right] \\
&\qquad - P(0,T)K\left[\Phi(d_2) + \Phi(-d_2)\right] \\
&= S_0 - P(0,T)K
\end{aligned}
$$

Here we have used the equality

$$
\begin{aligned}
\Phi(a) &= \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= 1 - \int_{a}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1 - \int_{-\infty}^{-a} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1 - \Phi(-a)
\end{aligned}
$$

The Black's formula

- The Black's formulae for call and put options are

$$
\begin{aligned}
C_0 &= P(0,T)\left[F_0\Phi(d_1) - K\Phi(d_2)\right] \\
&= S_0\Phi(d_1) - P(0,T)K\Phi(d_2) \\
P_0 &= P(0,T)\left[K\Phi(-d_2) - F_0\Phi(-d_1)\right] \\
&= P(0,T)K\Phi(-d_2) - S_0\Phi(-d_1)
\end{aligned}
$$

- for

$$d_1 = \frac{\ln\frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Alpha for Hedging

- For alpha, we have

$$\frac{\partial C(S_0,0)}{\partial S_0} = \frac{\partial C(F_0,0)}{\partial F_0}\frac{\partial F_0}{\partial S_0} = \Phi(d_1)$$

$$\frac{\partial P(S_0,0)}{\partial S_0} = \frac{\partial P(F_0,0)}{\partial F_0}\frac{\partial F_0}{\partial S_0} = -\Phi(-d_1)$$

- Examples: Black's formula for call options
  (http://www.math.ust.hk/~malwu/math4511/Matlab codes/Black_call.zip)