**Solutions Final Exam for MATH4511**

**Problems** (Numbers in brackets are credits, totaled to 80):

1. **Solution**:

$$F_{30} = F_{20} \times \frac{DV01_{20}}{DV01_{30}} \times \frac{\sigma_{20}}{\sigma_{30}} \times \rho$$
$$= \$1m \times \frac{0.12}{0.16} \times \frac{1\%}{0.95\%} \times 0.9 = \$711,424.95$$

2. **Solution**:

2.1. $C = (1+r\Delta t)^{-1} \times \frac{1}{2}\left(C_d + C_u\right)$.

2.2. The arbitrage pricing is

$$\begin{aligned} C &= \alpha S + \beta \\ &= \frac{C_u - C_d}{S_u - S_d} S_{0,0} + \frac{S_u C_d - S_d C_u}{(1+r\Delta t)(S_u - S_d)} \\ &= \frac{1}{1+r\Delta t}\left(\frac{S_u - S(1+r\Delta t)}{S_u - S_d} C_d + \frac{S(1+r\Delta t) - S_d}{S_u - S_d} C_u\right) \end{aligned}$$

2.3. The tree reprice the underlying:

$$S = (1+r\Delta t)^{-1} \tfrac{1}{2}\left(S_d + S_u\right)$$

3. **Solution**:

3.1. The hedging strategy that generates no cash inflow or outflow.

3.2. By the backward induction, there is, first,

$$\begin{aligned} C_{i,j+1} &= \alpha_{i,j+1} S_{i,j+1} + \beta_{i,j+1} \\ C_{i+1,j+1} &= \alpha_{i+1,j+1} S_{i+1,j+1} + \beta_{i+1,j+1} \end{aligned}$$

Then, by the construction of $\alpha_{i,j}$ and $\beta_{i,j}$, there is

$$\begin{aligned} \alpha_{i,j} S_{i,j+1} + (1+r_j\Delta t)\beta_{i,j} &= C_{i,j+1} \\ \alpha_{i,j} S_{i+1,j+1} + (1+r_j\Delta t)\beta_{i,j} &= C_{i+1,j+1} \end{aligned}$$

It follows that

$$\begin{aligned} \alpha_{i,j} S_{i,j+1} + (1+r_j\Delta t)\beta_{i,j} &= C_{i,j+1} = \alpha_{i,j+1} S_{i,j+1} + \beta_{i,j+1} \\ \alpha_{i,j} S_{i+1,j+1} + (1+r_j\Delta t)\beta_{i,j} &= C_{i+1,j+1} = \alpha_{i+1,j+1} S_{i+1,j+1} + \beta_{i+1,j+1} \end{aligned}$$

4. **Solution**:

4.1. The *T*-maturity forward contract is a contract to buy/sell an asset at time *T* for a specified price. The *T*-forward price is the price at which the value of the forward contract equals to zero. The *T*-forward price is given by $F_0 = S_0 / d(0,T)$.

4.2. The Black's formula:

$$C_0 = d(0,T)\left[F_0\Phi(d_1) - K\Phi(d_2)\right]$$
$$d_1 = \frac{\ln\frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

4.3. When the interest rate for instantaneous compounding, *r*, is a constant, we have $d(0,T) = e^{-rT}$, then the Black's formula reduces to the Black-Scholes formula.

5. **Solution**:
   5.1. The value is difference between the floating leg and the fixed leg:

$$V = V_{float} - V_{fixed} = d(0,T_0) - \left[ \sum_{i=1}^{N} \Delta T \times k d(0,T_i) + d(0,T_N) \right]$$

   5.2. A swaption is an option to enter into a swap in a future date. The ATM swaption has the strike rate equal to the market prevailing swap rate at initiation.
   5.3. The payoff of a call option on a coupon bond is

$$\left( \sum_{i=1}^{2\tau} \Delta T d(T_0,T_i)c + d(T_0,T_N) - 1 \right)^+$$
$$= \left( \sum_{i=1}^{2\tau} \Delta T d(T_0,T_i)c + d(T_0,T_N) - s(T_0;T_0,T_N) \sum_{i=1}^{2T} \Delta T \times d(T_0,T_i) - d(T_0,T_N) \right)^+$$
$$= A(T_0;T_0,T_N)\left(c - s(T_0;T_0,T_N)\right)^+$$

   which is the payoff of a receiver's swaption.
6. **Solution**:
   6.1. The MtM value of FRA is

$$FRA(k) = Not. \times \Delta T_j \times d(0,T_j)\left[ f_{j-1}(0) - k \right]$$

   Here *Not.* Is the notional value of the trade.
   6.2. The caplet formula

$$Caplet_j(k) = Not. \times \Delta T_j \times d(0,T_j)\left[ f_{j-1}(0)\Phi(d_1^{(j)}) - k\Phi(d_2^{(j)}) \right]$$
$$d_1^{(j)} = \frac{\ln \frac{f_{j-1}(0)}{k} + \frac{1}{2}\sigma^2 T_{j-1}}{\sigma\sqrt{T_{j-1}}}, \quad d_2^{(j)} = d_1^{(j)} - \sigma\sqrt{T_{j-1}}$$

   6.3. Long $\Phi(d_1^{(j)})$ unit of $T_{j-1}$-maturity FRA, $j = 1,2,\cdots,N$.
   6.4. The parity is

$$caplet(k) - floorlet(k) = FRA(k)$$

   The formula is

$$floorlet_j(k) = Not. \times \Delta T_j \times d(0,T_j)\left[ k\Phi(-d_2^{(j)}) - f_{j-1}(0)\Phi(-d_1^{(j)}) \right]$$

7. **Solution**:

7.1. The Black formula

$$swtn(0;k,T_N) = Not.\times A(0;T_0,T_N)\left(k\Phi(-d_2) - s(0;T_0,T_N)\Phi(-d_1)\right)$$

$$d_1 = \frac{\ln\frac{s(0;T_0,T_N)}{k} + \frac{1}{2}\sigma^2 T_0}{\sigma\sqrt{T_0}}, \quad d_2 = d_1 - \sigma\sqrt{T_0}$$

7.2. Let $m = 2T_0, n = 2T_N$ . Then the annuity is

$$\begin{aligned}
A(0;T_0,T_N) &= \sum_{i=m+1}^{n} \tfrac{1}{2} d(0,T_i) \\
&= \sum_{i=m+1}^{n} \tfrac{1}{2}(1+f/2)^{-i} \\
&= \tfrac{1}{2}(1+f/2)^{-m-1}\sum_{i=0}^{n-m-1}(1+f/2)^{-i} \\
&= \tfrac{1}{2}(1+f/2)^{-m-1}\frac{1-(1+f/2)^{-(n-m)}}{1-(1+f/2)^{-1}} \\
&= \frac{1}{f}\left((1+f/2)^{-m} - (1+f/2)^{-n}\right)
\end{aligned}$$

7.3. Due to the flat forward-rate curve, swap-rate curve is also flat and,

$$s(0,5,10) = f = 0.03.$$

7.4. For ATM swaptions, $k = s(0;T_0,T_N),$ and

$$\begin{aligned}
&A(0,T_0,T_N) = 7.39682 \\
&d_1 = \frac{\ln\frac{s(0;T_0,T_N)}{s(0;T_0,T_N)} + \frac{1}{2}\sigma^2 T_0}{\sigma\sqrt{T_0}} = \tfrac{1}{2}\sigma\sqrt{T_0} = \tfrac{1}{2}\cdot 0.2\sqrt{5}, \\
&d_2 = d_1 - \sigma\sqrt{T_0} = -\tfrac{1}{2}\cdot 0.2\sqrt{5}, \\
&\Phi(-d_2) = \Phi(\tfrac{1}{2}\cdot 0.2\sqrt{5}) = 0.588468 \\
&\Phi(-d_1) = \Phi(d_2) \\
&\qquad = 1 - \Phi(-d_2) \\
&\qquad = 1 - \Phi(\tfrac{1}{2}\cdot 0.2\sqrt{5}) = 1 - 0.588468 = 0.411532
\end{aligned}$$

It follows that $swtn = \$39{,}263.11.$

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