Chapter 8 of Tuckman
Regression-based Hedging

Overview

- Yields may not shift the same amount.
- Scenario:
  - When 30-year yield moves by 1bps,
  - the 20-year yield is most likely to move by 1.1 bps.

The volatility ratio, $\Delta y_t^{20} \big/ \Delta y_t^{30}$

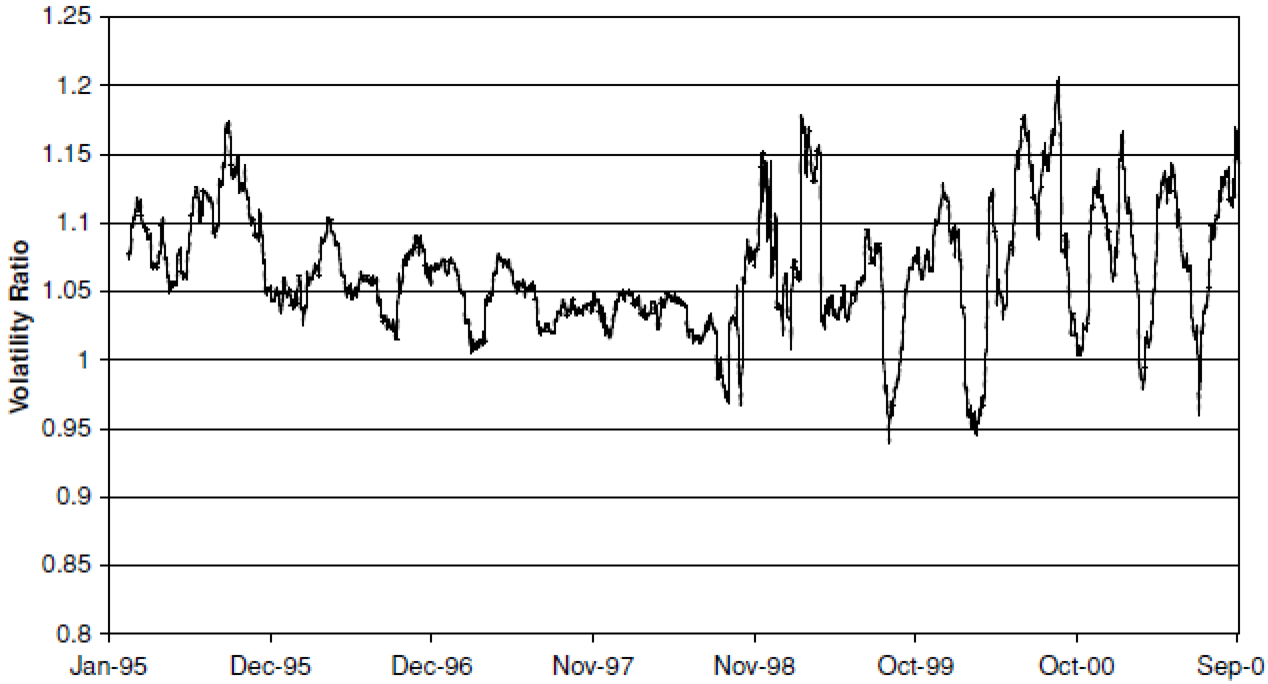

FIGURE 8.1 Ratio of 20-Year Yield Volatility to 30-Year Yield Volatility

Dot plot for $\left(\Delta y_t^{30}, \Delta y_t^{20}\right)$

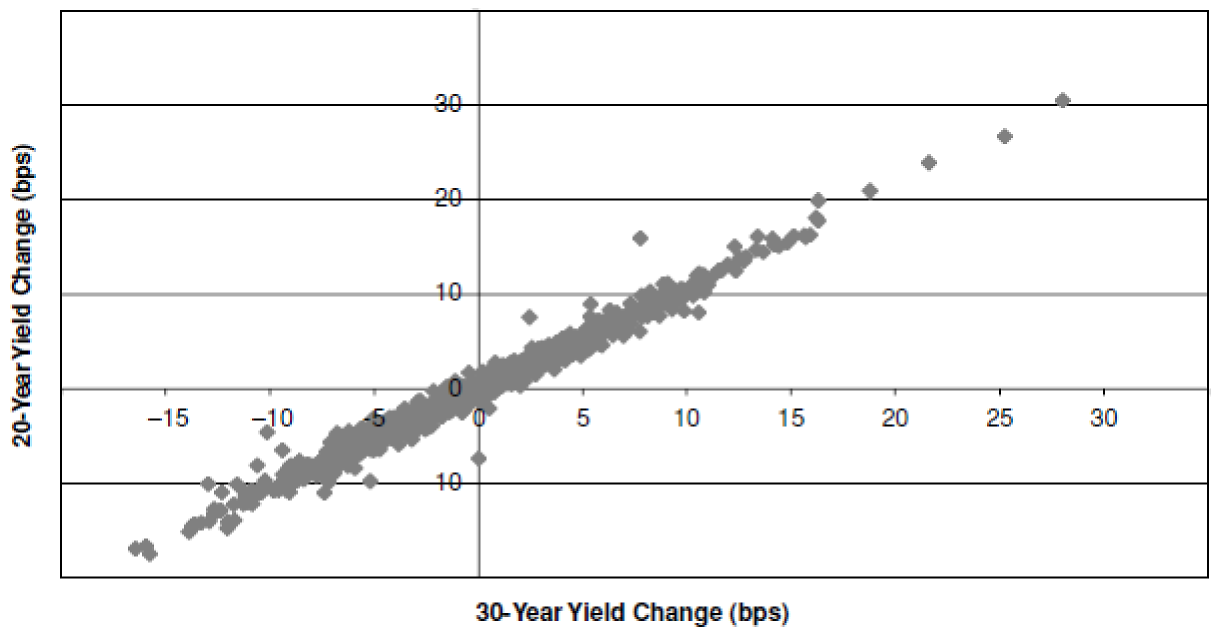

FIGURE 8.2 20-Year Yield Changes versus 30-Year Yield Changes

Better Hedging?

- Suppose a market maker shorts a 20-year bond. How much of 30-year bond he should buy for hedging?
- Let P_{20} and P_{30} be the dollar values of the 20- and 30-yr bond, and let $\Delta y = 1bps = 0.01\%$. Then,

$$\Delta P_{30} = -D_{30} P_{30} \Delta y$$
$$\Delta P_{20} = -D_{20} P_{20} \Delta y \times 1.1$$

Volatility ratio

- We choose P_{30} so that

$$-\Delta P_{20} + \Delta P_{30} = 0$$
$$\Downarrow$$
$$\left(-D_{20}P_{20} \times 1.1 + D_{30}P_{30}\right)\Delta y = 0$$
$$\Downarrow$$
$$P_{30} = P_{20} \times \frac{D_{20}}{D_{30}} \times 1.1$$

- Volatility ratio, 1.1, is taken into account.

- Ex: Let

$$P_{20} = \$10,000,000$$
$$D_{20} = 11.8428$$
$$D_{30} = 14.2940$$

- Then

$$P_{30} = \$9,113,670$$

Perfect hedging not guaranteed

- Remark: If Δy_{20} does not move by 1.1bps, we get hedging error.
- Ex: If it turns out that

$$\Delta y_{30} = 1bps, \quad \text{yet} \quad \Delta y_{20} = 1.3\text{bps}$$

- Then

$$\begin{aligned} \text{Hedging error} &= \$10,000,000 \times 11.84 \times 0.00013 \\ &\quad - \$9,113,670 \times 14.294 \times 0.0001 \\ &= +\$2,369 \end{aligned}$$ 😀

20-yr Yields vs. 30-yr Yields

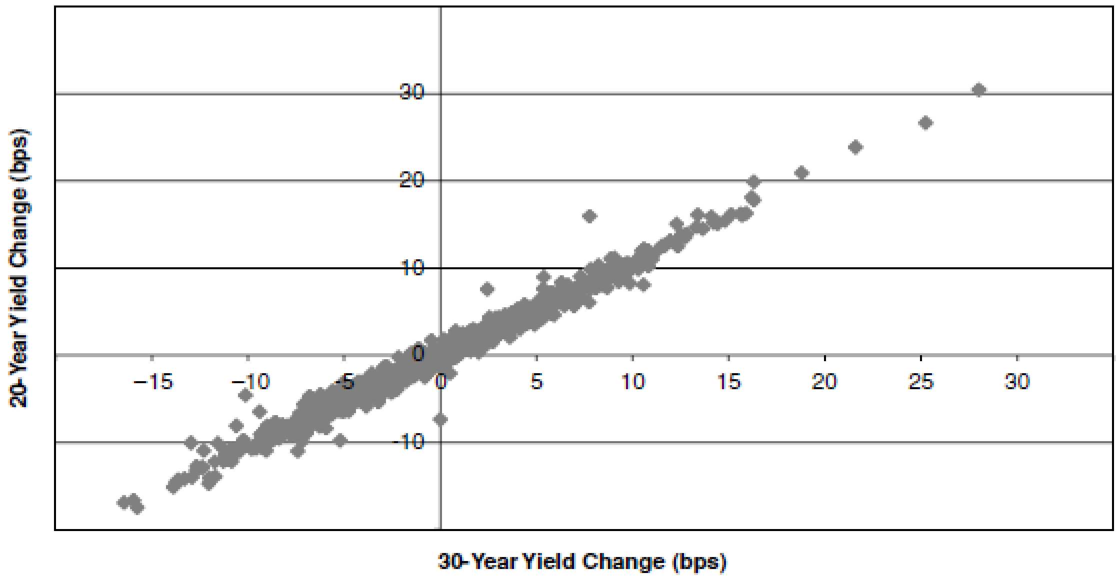

FIGURE 8.2 20-Year Yield Changes versus 30-Year Yield Changes

Tuckman, B, 2nd edition

1-V Least-Squares Regression

- Assume

$$\Delta y_t^{20} = \alpha + \beta \Delta y_t^{30} + \varepsilon_t$$

- The intercept, α, and the slope, β, need to be estimated from data.

1-V Least-Squares Regression, cont'd

- The error term ϵ_t is the deviation of the 20yr yield change on a particular day from the change predicted by the model.
- *Least-squares estimation* requires that the model be a true description of the dynamics in question and that the errors are *iid* and uncorrelated with the independent variable.
- The Least-squares estimation method is also called linear regression.

Least-squares estimation

- Least-squares estimation of α and β finds the estimates $\hat{\alpha}$ and $\hat{\beta}$ that minimize the sum of the squares of the realized error terms over the observation period,

$$\sum_t \varepsilon_t^2 = \sum_t \left(\Delta y_t^{20} - \hat{\alpha} - \hat{\beta} \Delta y_t^{30} \right)^2$$

- Least-squares estimation is available through many statistical packages and spreadsheet add-ins.

1-V Regression

- Consider the problem of

$$\min_{\hat{\alpha},\hat{\beta}} \sum_t \varepsilon_t^2 = \min_{\hat{\alpha},\hat{\beta}} \sum_t \left(\Delta y_t^{20} - \hat{\alpha} - \hat{\beta} \Delta y_t^{30} \right)^2$$

- Let

$$f(\hat{\alpha},\hat{\beta}) = \sum_t \left(\Delta y_t^{20} - \hat{\alpha} - \hat{\beta} \Delta y_t^{30} \right)^2$$

- Set

$$0 = \frac{\partial f}{\partial \hat{\alpha}} = -2\sum_t \left(\Delta y_t^{20} - \hat{\alpha} - \hat{\beta} \Delta y_t^{30} \right)$$

$$= -2N \left(\overline{\Delta y^{20}} - \hat{\alpha} - \hat{\beta} \overline{\Delta y^{30}} \right)$$

$$0 = \frac{\partial f}{\partial \hat{\beta}} = -2\sum_t \left(\Delta y_t^{20} - \hat{\alpha} - \hat{\beta} \Delta y_t^{30} \right) \Delta y_t^{30}$$

$$= -2N \left(\overline{\Delta y^{20}}\,\overline{\Delta y^{30}} - \hat{\alpha} \overline{\Delta y^{30}} - \hat{\beta} \overline{\left(\Delta y^{30} \right)^2} \right)$$

- where

$$\overline{\Delta y^{30}} = \frac{1}{N}\sum_t \Delta y_t^{30}$$

$$\overline{\Delta y^{20}} = \frac{1}{N}\sum_t \Delta y_t^{20}$$

$$\overline{\Delta y^{20}\Delta y^{30}} = \frac{1}{N}\sum_t \Delta y_t^{20}\Delta y_t^{30}$$

$$\overline{\left(\Delta y^{30}\right)^2} = \frac{1}{N}\sum_t \left(\Delta y_t^{30}\right)^2$$

- then

$$\hat{\alpha} + \hat{\beta}\,\overline{\Delta y^{30}} = \overline{\Delta y^{20}}$$

$$\hat{\alpha}\,\overline{\Delta y^{30}} + \hat{\beta}\,\overline{\left(\Delta y^{30}\right)^{2}} = \overline{\Delta y^{20}\Delta y^{30}}$$

- Solve the equations we obtain

$$\hat{\alpha} = \frac{\overline{\Delta y^{20}}\,\overline{\left(\Delta y^{30}\right)^{2}} - \overline{\Delta y^{30}}\;\;\overline{\Delta y^{30}\Delta y^{20}}}{\overline{\left(\Delta y^{30}\right)^{2}} - \left(\overline{\Delta y^{30}}\right)^{2}},$$

$$\hat{\beta} = \frac{\overline{\Delta y^{20}\Delta y^{30}} - \overline{\Delta y^{20}}\;\;\overline{\Delta y^{30}}}{\overline{\left(\Delta y^{30}\right)^{2}} - \left(\overline{\Delta y^{30}}\right)^{2}}$$

- Introducing

$$\mu = \frac{1}{N}\sum_t \Delta y_t \qquad \text{------ mean}$$

$$\sigma = \sqrt{\frac{1}{N}\sum_t \left(\Delta y_t - \mu\right)^2} = \sqrt{Var\left(\Delta y\right)} \qquad \text{------- volatility}$$

- Note that

$$\sigma^2 = \frac{1}{N}\sum_t \left(\Delta y_t\right)^2 - \mu^2 = \overline{\left(\Delta y\right)^2} - \mu^2$$

Correlation

- Define the correlation between Δy_t^{20} and Δy_t^{30} by

$$\rho = \frac{Cov\left(\Delta y^{20}, \Delta y^{30}\right)}{\sqrt{Var(\Delta y^{20})}\sqrt{Var(\Delta y^{30})}} = \frac{Cov\left(\Delta y^{20}, \Delta y^{30}\right)}{\sigma_{20}\sigma_{30}}$$

where

$$Cov\left(\Delta y^{20}, \Delta y^{30}\right) = \frac{1}{N}\sum_t \left(\Delta y_t^{20} - \mu_{20}\right)\left(\Delta y_t^{30} - \mu_{30}\right)$$
$$= \overline{\Delta y^{20}\Delta y^{30}} - \mu_{20}\mu_{30}$$

Solution to the 1-V Regression

- We then have

$$\alpha = \frac{\mu_{20}\left(\sigma_{30}^2 + \mu_{30}^2\right) - \mu_{30}\left(\rho\sigma_{20}\sigma_{30} + \mu_{20}\mu_{30}\right)}{\sigma_{30}^2}$$

$$\beta = \rho\frac{\sigma_{20}}{\sigma_{30}}$$

where β is the volatility ratio adjusted by correlation.

The Numerical Results

- Results of regression

TABLE 8.1 Regression Analysis of Changes in 20-Year Yields on 30-Year Yields

Number of observations	1,680	
R-squared	98.25%	
Standard error	0.6973	
Regression Coefficients	**Value**	**t-Statistic**
Constant	0.0007	0.0438
Change in 30-year yield	1.0570	306.9951

- Also,

$$\sigma_{20} = 5.27, \quad \sigma_{30} = 4.94, \quad \rho = 0.9912$$

Terminologies Explained

- R squared: two possibilities
  - The square of the correlation, ρ^2
  - The explanation ratio

$$R^2 = 1 - \sum \varepsilon_t^2 \Big/ Var(\Delta y^{20})$$

- Standard error: $\mathrm{std}\left(\Delta y^{20} - \hat{\alpha} - \hat{\beta}\Delta y^{30}\right)$

Terminologies Explained

- *t*-statistics:

$$t_{\hat{\alpha}} = \frac{\hat{\alpha}}{\mathrm{std}(\hat{\alpha})} \quad \text{and} \quad t_{\hat{\beta}} = \frac{\hat{\beta}}{\mathrm{std}(\hat{\beta})}$$

which are the significant indicator s.t. when |t|>2, we trust the results.

The Prediction and the Error

- Then, if $\Delta y_t^{30} = 3$ basis points on a particular day, the predicted change in the nominal yield, $\Delta \hat{y}_t^{20}$, is

$$\Delta \hat{y}_t^{20} = \hat{\alpha} + \hat{\beta} \Delta y_t^{30} = 0.0007 + 1.057 \times 3 = 3.1717$$

- Should it turn out that the 20yr yield changes by 4 basis points on that day, then the realized error that day is

$$\begin{aligned} \varepsilon_t &= \Delta y_t^{20} - \alpha - \beta \Delta y_t^{30} \\ &= 4 - 3.1717 = 0.8283 \end{aligned}$$

Application for Hedging

- Hedging: choose an appropriate amount of P_{30} to minimize the absolute value of P&L:

$$
\begin{aligned}
\text{P\&L} &= P_{20}D_{20}\Delta y_t^{20} - P_{30}D_{30}\Delta y_t^{30} \\
&= P_{20}D_{20}\left(\hat{\alpha} + \hat{\beta}\Delta y_t^{30} + \varepsilon_t\right) - P_{30}D_{30}\Delta y_t^{30} \\
&= \left(P_{20}D_{20}\hat{\beta} - P_{30}D_{30}\right)\Delta y_t^{30} + P_{20}D_{20}(\hat{\alpha} + \varepsilon_t)
\end{aligned}
$$

- We take

$$P_{30} = P_{20}\frac{D_{20}}{D_{30}}\hat{\beta}$$

Example, cont'd

- Let

$$
\begin{aligned}
P_{20} &= \$10,000,000 \\
D_{20} &= 11.8428 \\
D_{30} &= 14.2940 \\
\beta &= 1.0507
\end{aligned}
$$

- Then,

$$P_{30} = \$10,000,000 \times \frac{11.8428}{14.2940} \times 1.0507 = \$8,757,410$$

Face value and DV01

- The hedging equation

$$P_{30} \times D_{30} = P_{20} \times D_{20} \times \hat{\beta}$$

- Since

$$-\Delta P = D \times P \times 0.01\% = F \times \frac{DV01}{100}$$

- We have

$$D \times P = F \times \frac{DV01}{100} \times 10000 = F \times DV01 \times 100$$

- In terms of F and $DV01$, there is, after 100 is cancelled

$$F_{30} = F_{20} \times \frac{DV01_{20}}{DV01_{30}} \times \hat{\beta}$$

2-V Regression-Based Hedging

- The market maker has bought a 20-year receiver’s swap, relatively illiquid and needs to hedge the resulting interest rate exposure.
- The market maker chooses instead to sell a combination of 10- and 30-year swaps.

2-V Linear Regression

- The market maker relies on a two-variable regression model to describe the relationship between changes in 20-year swap rates and changes in 10- and 30-year swap rates.

$$\Delta y_t^{20} = \alpha + \beta^{10}\Delta y_t^{10} + \beta^{30}\Delta y_t^{30} + \epsilon_t$$

- α and β's can be estimated by least squares, by minimizing

$$\sum_t \left(\Delta y_t^{20} - \widehat{\alpha} - \widehat{\beta}^{10}\Delta y_t^{10} - \widehat{\beta}^{30}\Delta y_t^{30}\right)^2$$

System of Linear Equations

$$-2\sum_t \left(\Delta y_t^{20} - \hat{\alpha} - \hat{\beta}^{10} \Delta y_t^{10} - \hat{\beta}^{30} \Delta y_t^{30} \right) = 0$$

$$-2\sum_t \left(\Delta y_t^{20} - \hat{\alpha} - \hat{\beta}^{10} \Delta y_t^{10} - \hat{\beta}^{30} \Delta y_t^{30} \right) \Delta y_t^{10} = 0$$

$$-2\sum_t \left(\Delta y_t^{20} - \hat{\alpha} - \hat{\beta}^{10} \Delta y_t^{10} - \hat{\beta}^{30} \Delta y_t^{30} \right) \Delta y_t^{30} = 0$$

- Or

$$+\hat{\alpha} + \hat{\beta}^{10}\,\overline{\Delta y^{10}} + \hat{\beta}^{30}\,\overline{\Delta y^{30}} = \overline{\Delta y^{20}}$$

$$\hat{\alpha}\,\overline{\Delta y_t^{10}} + \hat{\beta}^{10}\,\overline{(\Delta y^{10})^2} + \hat{\beta}^{30}\,\overline{\Delta y^{30}\Delta y^{10}} = \overline{\Delta y^{10}\Delta y^{20}}$$

$$\hat{\alpha}\,\overline{\Delta y^{30}} + \hat{\beta}^{10}\,\overline{\Delta y_t^{10}\Delta y^{30}} + \hat{\beta}^{30}\,\overline{(\Delta y_t^{30})^2} = \overline{\Delta y^{20}\Delta y^{30}}$$

- Solve them numerically we obtain $\hat{\alpha}$, $\hat{\beta}^{10}$ and $\hat{\beta}^{30}$.

Prediction

- The estimation of these parameters then provides a predicted change for the 20-year swap rate:

$$\Delta\widehat{y_t^{20}} = \widehat{\alpha} + \widehat{\beta}^{10}\Delta y_t^{10} + \widehat{\beta}^{30}\Delta y_t^{30}$$

Enhanced Duration Hedging

- To hedge the 20-year bond of value P_{20}, we need P_{10} and P_{30} dollars of the 10- and 30-year bonds, such that

$$P_{10} = P_{20} \frac{D_{20}}{D_{10}} \beta_{10}$$

$$P_{30} = P_{20} \frac{D_{20}}{D_{30}} \beta_{30}$$

- Hedge error

$$\text{P\&L} = P_{20} D_{20} (\hat{\alpha} + \varepsilon_t)$$

Enhanced DV01 Hedging

- To hedge the 20-year bond of face value F_{20}, we need F_{10} and F_{30} face values the 10- and 30-year bonds, such that

$$F_{20}\frac{DV01_{20}}{100}\Delta y_t^{20} - F_{10}\frac{DV01_{10}}{100}\Delta y_t^{10} - F_{30}\frac{DV01_{30}}{100}\Delta y_t^{30}$$

$$=\left(F_{20}\frac{DV01_{20}}{100}\beta_{10} - F_{10}\frac{DV01_{10}}{100}\right)\Delta y_t^{10}$$

$$+\left(F_{20}\frac{DV01_{20}}{100}\beta_{30} - F_{30}\frac{DV01_{30}}{100}\right)\Delta y_t^{30} + F_{20}\frac{DV01_{20}}{100}\left(\alpha+\varepsilon_t\right)$$

- The hedge

$$F_{10} = F_{20} \frac{DV01_{20}}{DV01_{10}} \beta_{10}$$

$$F_{30} = F_{20} \frac{DV01_{20}}{DV01_{30}} \beta_{30}$$

2V regression for swap rates

TABLE 6.4 Regression Analysis of Changes in the Yield of the 20-Year EUR Swap Rate on Changes in the 10- and 30-Year EUR Swap Rates From July 2, 2001, to July 3, 2006

No. of Observations	**1281**	
R-Squared	**99.8%**	
Standard Error	**.14**	
Regression Coefficients	**Value**	**Std. Error**
Constant ($\widehat{\alpha}$)	−.0014	.0040
Change in 10-Year Swap Rate ($\widehat{\beta}^{10}$)	.2221	.0034
Change in 30-Year Swap Rate ($\widehat{\beta}^{30}$)	.7765	.0037

- The results in Table 6.4 say that 22.21% of the *DV*01 of the 20-year swap should be hedged with a 10-year swap and 77.65% with a 30-year swap.
- The sum of these weights, 99.86%, happens to be very close to one, meaning that the *DV*01 of the regression hedge very nearly matches the *DV*01 of the 20-year swap.

Hedging Errors

- Errors are computed as the realized change in the 20-year yield minus the predicted change for that yield based on the estimated regression in Table 6.4:

$$\widehat{\epsilon}_t = \Delta y_t^{20} - \left(-.0014 + .2221\,\Delta y_t^{10} + .7765\,\Delta y_t^{30}\right)$$

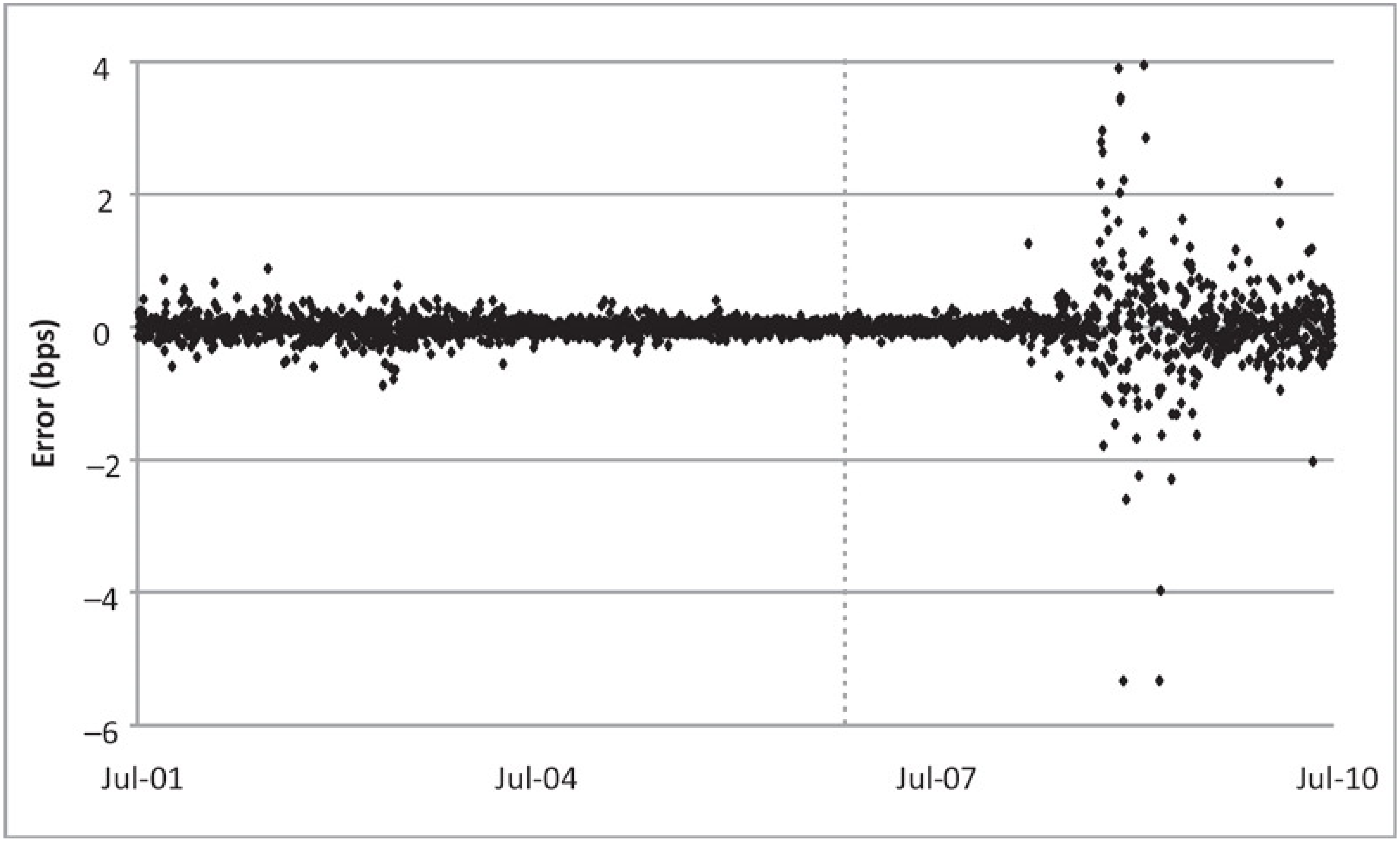

Figure 6.3: In- and Out-of-Sample Errors for a Regression of Changes of 20-Year and 10- and 30-Year EUR Swap Rates with Estimation Period July 2, 2001, to July 3, 2006

Error and Crisis

- The errors to the left of the vertical dotted line are in-sample, while, the errors to the right of the dotted line are out-of-sample.
- The size and behavior of these out-of-sample errors that provide evidence as to the stability for the estimated coefficients over time.
- The out-of-sample errors are small for the most part, until August and September 2008, a peak in the financial crisis of 2007–2009.

Puzzles

- It is obvious and easy to say that the market maker, during the turbulence of a financial crisis, should have replaced the regression of Table 6.4 and the resulting hedging rule.
- But replace these with what?
- What does the market maker do at that time, before there exist sufficient post-crisis data points?

Art or Science

- And what does the market maker do after the worst of the crisis: estimate a regression from data during the crisis or revert to some earlier, more stable period?
- These are the kinds of issues that make regression hedging an art rather than a science.