Topic 12

Binomial Tree Models

Outline

Outline

1. Binomial Trees
2. Replicating Portfolios
3. Risk Neutral Pricing

- Binomial trees are a **convenient tool** for introducing a fundamental technique to price derivative securities, called **Risk Neutral Pricing**
- Besides being simple to understand, they also offer a pricing methodology for complicated problems
- The use of binomial trees is a standard numerical procedure to solve for American options and other complex options in practice.

NPV Rule:

Value of an asset = Net present value of future cash flows discounted at the risk-adjusted discount rate

That is,

$$\mathrm{PV}_0(S_T) = e^{-\mu T}\,\mathrm{E}_0[S_T]$$

How to determine the risk-adjusted discount rate μ?

One step binomial trees

- Consider some stock at $t = 0$
- Analysts believe that the stock price at $t = 1$ can either be

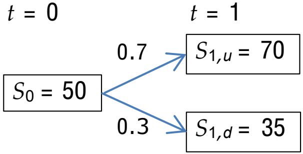
  $S_{1,u} = 70$ with probability $q = 0.7$, or $S_{1,d} = 35$ with probability $1 - q = 0.3$

- Suppose risk-adjusted discount rate $\mu = 17.4\%$. The stock price at $t = 0$ is

$$S_0 = e^{-\mu}\mathrm{E}_0\left[S_1\right] = e^{-\mu}\left(q \times S_{1,u} + (1 - q) \times S_{1,d}\right)$$

$$= e^{-0.174}(.7 \times 70 + .3 \times 35) = 50$$

- We can summarize the uncertainties using a stock tree:



Option prices on a binomial tree

- Consider a call option with $T = 1$ and $K = 50$
- According to the tree, what is the payoff of the call option at $T = 1$?

$$\text{In the Up Node} = c_{1,u} = \max(S_{1,u} - K,\ 0) = \max(70 - 50,\ 0) = 20$$
$$\text{In the Down Node} = c_{1,d} = \max(S_{1,d} - K,\ 0) = \max(35 - 50,\ 0) = 0$$

- On the tree:

$t = 0$	$t = 1$
	$S_{1,u} = 70$ $c_{1,u} = \max(70 - 50, 0) = 20$
$S_0 = 50$ $c_0 = ??$	
	$S_{1,d} = 35$ $c_{1,d} = \max(35 - 50, 0) = 0$

What is price of the call option at $t = 0$?

- The risk-adjusted discount rate is not μ for the call option...

higher than that of stock

↑risk → ↑ rate

Outline

A replicating portfolio

- Recall: In the put-call parity, we learned how to replicate a call with a put, stock and bond

$$C = S_0 + P - Ke^{-rT}$$

- Now, we can replicate a call with just a stock and a bond!

$$C = \Delta S_0 - B_0$$

  - a self-financed portfolio to borrow money at the risk free rate and then buy stocks.

- Δ (delta) is a key concept
- the call is a leveraged position in the stock

$\rightarrow$ expected return on the call > expected return on the stock

A replicating portfolio: Example

- Let the (continuously compounded) risk free interest rate be $r = 11\%$
- Consider a portfolio of stocks and bonds, with
  - Position $\Delta = 0.5714$ (shares) in stocks, with value $\Delta \times S_0 = 28.5714$
  - Position of $B_0 = -17.9167$ in bonds (negative = short bonds)
  - The value of the portfolio today is $V_0 = 28.5714 - 17.9167 = 10.6547$
- What is the value of the portfolio at time $t = 1$?

$$\text{In the Up Node} = V_{1,u} = \Delta \times S_{1,u} + B_0 \times e^r = 0.5714 \times 70 - 20 = 20$$
$$\text{In the Down Node} = V_{1,d} = \Delta \times S_{1,d} + B_0 \times e^r = 0.5714 \times 35 - 20 = 0$$

- **This is the payoff of the call option!**

$$\textbf{No Arbitrage} \Rightarrow c_0 = V_0 = \Delta \times S_0 + B_0 = 10.6547$$

  - If not, “buy low and sell high”
  - e.g. if $c_0 > V_0$, sell the call option, and buy the replicating portfolio

Where does the replicating portfolio come from?

- Consider using Δ shares of stock and B_0 bonds to replicate the call option.

  $\Rightarrow$ choose Δ and B_0 at time 0 so that

  Value of Portfolio in Up/Down Node = Call Option's Payoff in Up/Down Node

$$\Longleftrightarrow \Delta \times S_{1,u} + B_0 \times e^{r} = c_{1,u}$$

$$\Longleftrightarrow \Delta \times S_{1,d} + B_0 \times e^{r} = c_{1,d}$$

- Two equations with two unknowns (Δ and B_0):

$$\Delta = \frac{c_{1,u} - c_{1,d}}{S_{1,u} - S_{1,d}}$$

$$B_0 = e^{-r}(c_{1,u} - \Delta \times S_{1,u})$$

- **Interpretation:** Δ = sensitivity of the call's price to changes in the stock price

  = implicit leverage in the option

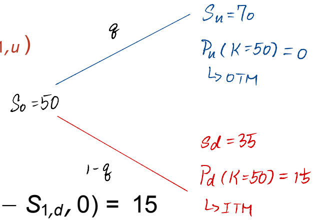
- By law of one price, $c_0 = \Delta \times S_0 + B_0$

- To summarize, in order to price any derivative security with payoff $V_{1,u}$ and $V_{1,d}$ on the tree, we proceed as follows:

  (1) Compute Δ shares of the stock: $\Delta = \dfrac{V_{1,u} - V_{1,d}}{S_{1,u} - S_{1,d}}$

  (2) Compute amount of bonds: $B_0 = e^{-rT} \times (V_{1,u} - \Delta \times S_{1,u})$

  (3) Compute the value of security: $V_0 = \Delta \times S_0 + B_0$



- Example: Put option with strike price $K = 50$

  $\Rightarrow p_{1,u} = \max(K - S_{1,u}, 0) = 0$ and $p_{1,d} = \max(K - S_{1,d}, 0) = 15$

  (1) Delta: $\Delta = \dfrac{p_{1,u} - p_{1,d}}{S_{1,u} - S_{1,d}} = \dfrac{0 - 15}{70 - 35} = -0.4285$

  +ve → Call
  (↑P → ↑Call)
  −ve → Put
  (↓P → ↑Put)

  (2) Bonds: $B_0 = e^{-rT} \times (p_{1,u} - \Delta \times S_{1,u})$

  $= e^{-0.11} \times (0 + 0.4285 \times 70) = 26.8750$

  (3) Value: $p_0 = \Delta \times S_0 + B_0 = -0.4285 \times 50 + 26.8750 = 5.4464$

Where did the probabilities go?

- The pricing formula does not include the probability of moving up or down $(q, 1-q)$
- **Question**: Does this imply that these probabilities do not impact option prices?
- **Answer**: yes and no
  - Given S_0, $S_{1,u}$ and $S_{1,d}$, options' payoffs can be replicated without reference to probabilities

    $\Rightarrow$ No impact of q on option prices
  - However, q determines the expected future stock price as well as the risk of the stock return. Thus, q affects S_0 by affecting the risk-adjusted discount rate μ.

    $\Rightarrow$ The current value of S_0 already depends on q !
  - Since option values depend on S_0, the probability q does impact the value of options
  - But, given S_0, the value of options is independent of q !

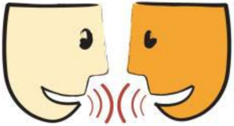
Exercise

Among the following factors, which does the option value depend on?

- I. Current stock price
- II. Future stock prices
- III. Probabilities of future stock prices
- IV. Strike price

1. I and II only
2. I, II and III only
3. I, II and IV only
4. I, II, III and IV

TAPPS



$$S_0 = 40 \quad \begin{cases} 42, & C_{1,u}(K=39) = 3 \\ 38, & C_{1,d}(K=39) = 0 \end{cases}$$

A stock price is currently \$40. It is known that at the end of one month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding.

What is the value of a one-month European call option with a strike price of \$39?

$$\Delta = \frac{C_{1,u} - C_{1,d}}{S_{1,u} - S_{1,d}} = \frac{3 - 0}{42 - 38} = 0.75$$

$$\text{Bond} = e^{-0.08/12}(3 - 0.75 \times 42)$$
$$= -28.3106$$

$$\text{Value} = 0.75 \times 40 - 28.3106 = 1.689$$

Risk neutral pricing

- The above procedure is cumbersome
- There is an alternative procedure that is much easier to work with
- Since given S_0, $S_{1,u}$ and $S_{1,d}$, the probability q does not impact the price of the option, we can choose a fictitious probability q^* that simplifies our computations

Risk Neutral Pricing

Choose q^* so that all risky assets yield the risk free rate

- **Find q^*** such that

$$q^* \times S_{1,u} \times e^{-rT} + (1 - q^*) \times S_{1,d} \times e^{-rT} = S_0$$

(notice the discount rate is r, not μ)

$$\Rightarrow \qquad q^* = \frac{S_0 \times e^{rT} - S_{1,d}}{S_{1,u} - S_{1,d}}$$

- In other words,

$$S_0 = e^{-rT} \mathrm{E}_0^*(S_1)$$

Risk neutral pricing

- We can now price any derivative security with a payoff function $G(S_T)$ using the "fictitious" probability q^* and the risk-free discount rate r :

$$\text{Price of any derivative security} = e^{-rT}\,\mathrm{E}^*_0[G(S_T)] \tag{1}$$

- The star $*$ on $\mathrm{E}^*_0[\cdot]$ denotes the fact that we use the fictitious probability q^*
- Does it work? (use the same example in previous slides)
  - Risk Neutral Probability: $q^* = \dfrac{50 \times e^{11\%} - 35}{70 - 35} = 0.5947$
  - Call Price:

$$c_0 = e^{-rT} \times \mathrm{E}^*[c_1] = e^{-11\%} \times [q^* \times 20 + (1 - q^*) \times 0] = 10.6547$$

  - Put Price:

$$p_0 = e^{-rT} \times \mathrm{E}^*[p_1] = e^{-11\%} \times [q^* \times 0 + (1 - q^*) \times 15] = 5.4464$$

Risk neutral pricing: a general recipe

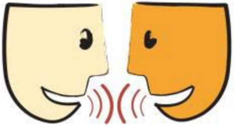
- The recipe to price derivative securities is as follows:
  - Assume everyone is risk neutral
  - Compute the risk-neutral probabilities with which the underlying asset S can be evaluated using the risk-free rate as the discount factor
  - Price any <u>derivative</u> security with payoff function $G(S_T)$ as

$$\text{Price of Derivative Security} = e^{-r \times T} \times \mathrm{E}*_0 [G(S_T)]$$

- This methodology works even outside of the binomial tree model
  - It is the implication of no arbitrage.

$$q^* = \frac{S_0 \times e^{rT} - S_{1,d}}{S_{1,u} - S_{1,d}}$$

TAPPS



$$S_0 = e^{-rT} E_0^*(S_1)$$

$$c_0 = e^{-rT} \times E^*[c_1]$$

A stock price is currently \$40. It is known that at the end of one month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding.

Using risk neutral pricing, what is the value of a one-month European call option with a strike price of \$39?

$$q^* = \frac{40 \times e^{\frac{0.08}{12}} - 38}{42 - 38} = 0.566889$$

$$c_0 = e^{-rT} \times E^*[c_1] = e^{-\frac{0.08}{12}} \times [0.566889 \times 3 + 0]$$

$$= 1.689$$

Connection: risk neutral pricing and discount rates

- The previous examples show the convenience of risk neutral pricing technique
- **Fundamental theory**: This technique is backed by no arbitrage conditions
- It is a convenient pricing device, and it DOES NOT imply that market participants are risk neutral!
  - Market participants *are* risk averse in our setting
  - We can account for risk aversion in two different ways:
    (1) by adding a risk premium to the risk-free discount rate
    (2) by "distorting" the probabilities towards the bad states
  - **Key Idea**: With these distorted probabilities, we can *pretend* that market participants are risk neutral and discount future payoffs with the risk-free rate

Connection: risk neutral pricing and discount rates

Let's revisit the stock example:

$$
\begin{array}{cc}
t = 0 & t = 1 \\
 & \boxed{S_{1,u} = 70} \\
\boxed{S_0 = 50} & \\
 & \boxed{S_{1,d} = 35}
\end{array}
$$

- Pricing the stock under actual probabilities:

$$S_0 = e^{-\mu T} \times \mathrm{E}_0[S_1] = e^{-\mu T} \times [q \times S_{1,u} + (1 - q) \times S_{1,d}]$$
$$= e^{-17.4\%}(.7 \times 70 + .3 \times 35) = 50$$

- Pricing the stock under distorted probabilities:

$$S_0 = e^{-rT} \times \mathrm{E}^*{}_0[S_1] = e^{-rT}(q^* \times S_{1,u} + (1 - q^*) \times S_{1,d})$$
$$= e^{-11\%}(0.5947 \times 70 + 0.4053 \times 35) = 50$$

- **Risk adjustment:** $\mu > r \Longleftrightarrow q^* < q$ (intuitions?)

Risk neutral pricing: revisit forward prices

- The payoff at T from a forward contract is

$$G(S_T) \equiv S_T - F_{0,T}$$

- It costs nothing to enter into a forward contract $\Rightarrow$ the value today is 0

$$e^{-rT} \times \mathrm{E}^*_0[S_T - F_{0,T}] = 0 \qquad \Rightarrow F_{0,T} = \mathrm{E}^*_0[S_T]$$

- If everybody is risk neutral, what should be the return on stocks?

  $\Rightarrow$ the risk free rate

$$S_0 = e^{-r \times T}\mathrm{E}^*_0[S_T]$$

- Thus, we find

$$F_{0,T} = \mathrm{E}^*_0[S_T] = S_0 \times e^{r \times T}$$

Forward Pricing:

Forward price is the risk-neutral expectation of the underling asset value at T

Conclusion

- Binomial Trees
- Replicating Portfolios
- Risk Neutral Pricing

Appendix:

Risk neutral pricing: revisit forward prices

- As a final comment, we found that the forward price is

$$F_{0,T} = \mathrm{E}^*[S_T] \neq \mathrm{E}[S_T]$$

  - The forward price is the risk neutral expected future stock price

- There is little information in forward prices about the true market expectations of stock prices in the future

  - A risk adjustment has to be made. In fact, we have

$$F_{0,T} = S_0 \times e^{r \times T}$$

  - However, using the true expectations and expected returns for the stock, we have:

$$\mathrm{E}[S_T] = S_0 \times e^{\mu \times T}$$

  - Substitute, to obtain the relation

$$F_{0,T} = e^{-(\mu - r) \times T} \mathrm{E}[S_T]$$

⇒ The forward price is the expected future stock price, discounted at the excess rate of return $(\mu - r)$: risk premium

Prep for Next Class

- Next topic: Mutli-step Binomial tree models
- Read Ch. 13
- Assignment 4 to be due on Apr 10 (Wed)
- Attend tutorials on Wed/Thur