A Model for Term Rates

- Let r_t be the annualized rate for term Δt, we have a similar model

$$\Delta r_t = \theta_t \Delta t + \sigma \sqrt{\Delta t}\ \varepsilon_B$$

where

$$\varepsilon_B = \begin{cases} 1, & \text{with probability } 1/2 \\ -1, & \text{with probability } 1/2 \end{cases}$$

Mean and Variance

- The mean and variance of Δr_t is

$$E\left[\Delta r_t\right] = \theta_t \Delta t$$

$$Var\left(\Delta r_t\right) = \sigma^2 \Delta t$$

Binomial model for interest rates

- The tree

$$
r_t \begin{cases} \overset{1/2}{\nearrow} & r^{u}_{t+\Delta t} = r_t + \theta_t \Delta t + \sigma\sqrt{\Delta t} \\[2em] \underset{1/2}{\searrow} & r^{d}_{t+\Delta t} = r_t + \theta_t \Delta t - \sigma\sqrt{\Delta t} \end{cases}
$$

Bond Pricing

- Model parameters: $\theta = 0.01$ and $\sigma = 0.01$
- Market information: $r_0 = 0.05$
- Consider the pricing of a bond with
  - \$1,000 notional value
  - Maturity: 1 year
  - Spot price: $P_{0,0} = 950.42$

One-step interest-rate tree

- Rate tree

$$r_0 \begin{cases} \tfrac{1}{2}: & r_{\Delta t}^{u} = r_0 + \theta_0 \Delta t + \sigma\sqrt{\Delta t} \\ \tfrac{1}{2}: & r_{\Delta t}^{d} = r_0 + \theta_0 \Delta t - \sigma\sqrt{\Delta t} \end{cases}$$

- Price tree

$$P_{0,0} \begin{cases} \tfrac{1}{2}: & P_{1,1} \longrightarrow 1000 \\ \tfrac{1}{2}: & P_{0,1} \longrightarrow 1000 \end{cases}$$

Bond price tree

- Backward induction formula for bond price

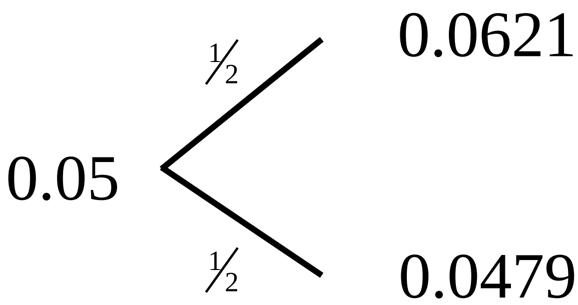
$$P_{0,1} = \frac{1000}{1 + [r_{0,0} + \theta_0 \Delta t - \sigma \sqrt{\Delta t}\,] \Delta t}$$

$$P_{1,1} = \frac{1000}{1 + [r_{0,0} + \theta_0 \Delta t + \sigma \sqrt{\Delta t}\,] \Delta t}$$

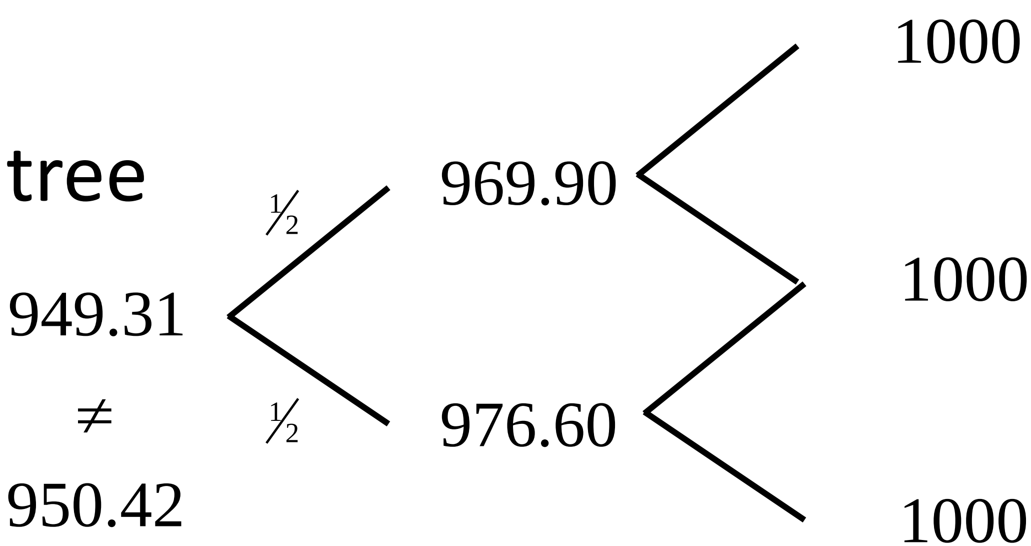
$$P_{0,0} = \frac{\frac{1}{2}(P_{0,1} + P_{1,1})}{1 + ?\,\Delta t} = 950.42.$$

Rate Tree vs. Bond Tree

- Rate tree: take $\Delta t = 0.5.$



- Bond tree



Arbitrage pricing, again

- Option price tree

$$C_{1,1} = \left(P_{1,1} - 970\right)^{+} = 0.0$$

$$C_{0,0} = ?$$

$$C_{0,1} = \left(P_{0,1} - 970\right)^{+} = 6.6$$

Arbitrage pricing, cont'd

- Consider replicating the payoffs with a portfolio of α and β units of bond and cash, such that

$$\alpha P_{0,1} + \beta(1 + r_{0,0}\Delta t) = C_{0,1}$$

$$\alpha P_{1,1} + \beta(1 + r_{0,0}\Delta t) = C_{1,1}$$

- Solution

$$\alpha = \frac{C_{1,1} - C_{0,1}}{P_{1,1} - P_{0,1}}, \qquad \beta = \frac{\left(P_{1,1}C_{0,1} - P_{1,0}C_{1,1}\right)}{(1 + r_{0,0}\Delta t)(P_{1,1} - P_{1,0})}$$

Arbitrage pricing, cont'd

- The value of the option is thus

$$C_{0,0} = \alpha P_{0,0} + \beta,$$

  which is unique.

- Arbitrage arises if the option value differs from above.
- Actual numbers

$$P_{0,0} = \$950.42, \quad \alpha = 0.9849, \quad \beta = -\$931.92$$

$$C_{0,0} = \$4.11$$

Linear pricing rule

- Rewrite the option formula into

$$C_{0,0} = \alpha P_{0,0} + \beta$$

$$= \frac{C_{1,1} - C_{0,1}}{P_{1,1} - P_{0,1}} P_{0,0} + \frac{\left(P_{1,1}C_{0,1} - P_{1,0}C_{1,1}\right)}{(1 + r_{0,0}\Delta t)(P_{1,1} - P_{0,1})}$$

$$= (1 + r_{0,0}\Delta t)^{-1}\left(\frac{P_{1,1} - P_{0,0}(1 + r_{0,0}\Delta t)}{P_{1,1} - P_{0,1}} C_{0,1} + \frac{P_{0,0}(1 + r_{0,0}\Delta t) - P_{0,1}}{P_{1,1} - P_{0,1}} C_{1,1} \right)$$

$$= (1 + r_{0,0}\Delta t)^{-1}\left(q_0 C_{0,1} + (1 - q_0) C_{1,1} \right) \quad !!!$$

$$q_0 = 0.6393$$

Linear pricing rule

- General pricing formula

$$C_{0,0} = (1 + r_{0,0}\Delta t)^{-1}\left(q_0 C_{0,1} + (1 - q_0)C_{1,1}\right)$$

for any one-period options.

- There is

$$P_{0,0} = (1 + r_{0,0}\Delta t)^{-1}\left(q_0 P_{0,1} + (1 - q_0)P_{1,1}\right)$$

--- the pricing probabilities reproduce the bond price!