Forward Rates and Forward Rate Agreement

Forward Rates and Forward Loans

- A *forward* rate is the rate on a *forward loan*, which is an agreement to lend money at some time in the future to be repaid some time after that.
- A previous example revised:
  - A 100,000,000 1.5-year loan, six months *forward* (i.e., a loan made in six months for 1.5 years) will pay 103,797,070 in two years.
  - The loan rate is 2.5%

Forward Rates

- There are many possible forward rates, e.g.,
  - the rate on a loan given in six months for a subsequent term of 1.5 years;

  - the rate in five years for six months; etc.

Forward Rates

- To connect with zero-coupon rates, we focus on forward rates over sequential, six-month periods.
- Let $f(t)$ denote the forward rate on a loan from year t−.5 to year t. Then, investing 1 unit of currency from year t−.5 for six months generates proceeds, at year t, of

$$\left(1+\frac{f(t)}{2}\right)$$

Forward and Spot Rates

- Forward rates are implied by spot rates.
- The two investment alternatives
  - A *t*-year loan
  - A *t*-0.5 year loan combined with a forward loan over *t*-0.5 to *t* years

  should produce the same return, otherwise arbitrage occurs.

Forward, Spot Rates & ZCB

- So there is

$$\left(1+\frac{\widehat{r}(t)}{2}\right)^{2t} = \left(1+\frac{\widehat{r}(t-.5)}{2}\right)^{2(t-.5)}\left(1+\frac{f(t)}{2}\right)$$

$$= \left(1+\frac{\widehat{r}(t-.5)}{2}\right)^{2t-1}\left(1+\frac{f(t)}{2}\right)$$

so

$$\left(1+\frac{f(t)}{2}\right) = \frac{d(t-.5)}{d(t)}$$

i.e.

$$f(t) = 2\left(\frac{d(t-.5)}{d(t)} - 1\right)$$

Discount Factors and Forward rates

TABLE 2.1 Discount Factors, Spot Rates, and Forward Rates Implied by Par USD Swap Rates as of May 28, 2010

Term in Years	Swap Rate	Discount Factor	Spot Rate	Forward Rate
0.5	.705%	.996489	.705%	.705%
1.0	.875%	.991306	.875%	1.046%
1.5	1.043%	.984494	1.045%	1.384%
2.0	1.235%	.975616	1.238%	1.820%
2.5	1.445%	.964519	1.450%	2.301%

- Table 2.1.

Spot Rates & Forward Rates

- Spot rates can be expressed in terms of forward rates:

$$\left(1+\frac{\widehat{r}(t)}{2}\right)^{2t}=\left(1+\frac{f(.5)}{2}\right)\left(1+\frac{f(1)}{2}\right)\cdots\left(1+\frac{f(t)}{2}\right)$$

- In particular, $f(0.5)=\hat{r}(0.5)$!

Calculating a Bond Price, cont'd

- In terms of spot rates

$$P = \frac{c}{2}\left[\frac{1}{\left(1+\frac{\widehat{r}(.5)}{2}\right)} + \frac{1}{\left(1+\frac{\widehat{r}(1)}{2}\right)^2} + \cdots + \frac{1}{\left(1+\frac{\widehat{r}(T)}{2}\right)^{2T}}\right]$$

$$+ \frac{1}{\left(1+\frac{\widehat{r}(T)}{2}\right)^{2T}}$$

Calculating a Bond Price, cont'd

- In terms of forward rates

$$P = \frac{c}{2}\left[\frac{1}{\left(1+\frac{f(.5)}{2}\right)} + \frac{1}{\left(1+\frac{f(.5)}{2}\right)\left(1+\frac{f(1)}{2}\right)} + \cdots \right.$$

$$\left. + \frac{1}{\left(1+\frac{f(.5)}{2}\right)\left(1+\frac{f(1)}{2}\right)\cdots\left(1+\frac{f(T)}{2}\right)}\right]$$

$$+ \frac{1}{\left(1+\frac{f(.5)}{2}\right)\left(1+\frac{f(1)}{2}\right)\cdots\left(1+\frac{f(T)}{2}\right)}$$

CHARACTERISTICS OF SPOT, FORWARD, AND PAR RATES

- Proposition: When any of the spot-rate curve, forward-rate curve and swap-rate curve is flat, the other two curves are flat and identical.

- We give a proof for the case of flat spot-rate curve.

Proof:

- On forward rates.

  Let $\hat{r}(t) = r,\ \forall t$. Then,

$$f(t) = 2\left(\frac{d(t-.5)}{d(t)} - 1\right)$$

$$= 2\left(\frac{\left(1+\frac{r}{2}\right)^{-2t+1}}{\left(1+\frac{r}{2}\right)^{-2t}} - 1\right)$$

$$= 2\left(1 + \frac{r}{2} - 1\right) = r$$

- On swap rates. We have

$$C(t)=\frac{1-\left(1+\frac{r}{2}\right)^{-2t}}{\sum_{i=1}^{2t}\frac{1}{2}\left(1+\frac{r}{2}\right)^{-i}},\quad \text{let}\ \ z=\left(1+\frac{r}{2}\right)^{-1}$$

$$=\frac{1-z^{2t}}{\frac{1}{2}z\,\frac{1-z^{2t}}{1-z}}=\frac{2(1-z)}{z}$$

$$=\frac{2\left(1-\left(1+\frac{r}{2}\right)^{-1}\right)}{\left(1+\frac{r}{2}\right)^{-1}}=2\left(\left(1+\frac{r}{2}\right)-1\right)=r$$