Quiz3 T1A

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2. (15 points) 2-v regression-based hedging

$$\Delta y_t^{20} = \alpha + \beta^{10}\Delta y_t^{10} + \beta^{30}\Delta y_t^{30} + \epsilon_t$$

$$\Delta \hat{y}_t^{20} = \hat{\alpha} + \hat{\beta}^{10}\Delta \hat{y}_t^{10} + \hat{\beta}^{30}\Delta \hat{y}_t^{30}$$

What is the face amount for 10- and 30-year bonds (F_{10} and F_{30}) if you want to hedge for the 20-year bond of value F_{20}? And what is the hedge error? Here we assume $DV01_{20}$, $DV01_{10}$ and $DV01_{30}$ are known, and you can use notations $\hat{\alpha}$, $\hat{\beta}^{10}$, $\hat{\beta}^{30}$ and ϵ_t to represent the DV01 hedging and corresponding error.

We want to choose an appropriate amount of F_{10} and F_{30} to minimize the absolute of P&L:

$$P\&L = F_{20} \times \frac{DV01_{20}}{100} \times \Delta y_t^{20} - F_{10} \times \frac{DV01_{10}}{100}\Delta y_t^{10} - F_{30} \times \frac{DV01_{30}}{100}\Delta y_t^{30}$$

$$= F_{20} \times \frac{DV01_{20}}{100}\left(\hat{\alpha} + \hat{\beta}_{10}\Delta y_t^{10} + \hat{\beta}_{30}\Delta y_t^{30} + \epsilon_t\right) - F_{10} \times \frac{DV01_{10}}{100}\Delta y_t^{10} - F_{30} \times \frac{DV01_{30}}{100}\Delta y_t^{30}$$

$$= \left(F_{20} \times \frac{DV01_{20}}{100}\hat{\beta}_{10} - F_{10} \times \frac{DV01_{10}}{100}\right)\Delta y_t^{10} + \left(F_{20} \times \frac{DV01_{20}}{100}\hat{\beta}_{30} - F_{30} \times \frac{DV01_{30}}{100}\right)\Delta y_t^{30} + F_{20} \times \frac{DV01_{20}}{100}\left(\hat{\alpha} + \epsilon_t\right)$$

To hedge 20-year bond using 10 & 30 year bonds, we want to set coefficient of Δy_t^{10}, Δy_t^{30} to 0.

$$\Rightarrow F_{20} \times \frac{DV01_{20}}{100}\hat{\beta}_{10} - F_{10} \times \frac{DV01_{10}}{100} = 0$$

$$F_{20} \times \frac{DV01_{20}}{100}\hat{\beta}_{10} = F_{10} \times \frac{DV01_{10}}{100}$$

$$F_{20} \times \frac{DV01_{20}}{DV01_{10}}\hat{\beta}_{10} = F_{10}$$

$DVol_{10}$

$$F_{10} = F_{20} \times \frac{DVol_{20}}{DVol_{10}} \hat{\beta}_{10}$$

Similarly, $F_{30} = F_{20} \times \frac{DVol_{20}}{DVol_{30}} \hat{\beta}_{30}$

and the hedging error would be

$$F_{20} \times \frac{DVol_{20}}{100} (\hat{\alpha} + \varepsilon_t)$$

3. (20 points) Consider the model

$$\Delta r_t = \mu(t)\Delta t + \sigma\sqrt{\Delta t}\epsilon_B,$$

where ϵ_B takes +1 or -1 with equal probability. Here $r_0 = 5\%$, $\sigma = 2\%$, $\mu(0.5) = 0.02008$, $\mu(1) = 0.02017$ and $dt = 1/2$.
(a) Build a two-step binomial tree for the short rate process.
(b) price a one-year call option on a 1.5 year zero-coupon bond with face value \$1,000, with strike price \$960. Explain how to hedge the call.

Time	0	0.5	1	1.5
Spot		5%	5.5%	6%
μ		0.02008	0.02017	

$\sigma = 0.02$. $\Delta t = 0.5$

$$r_{0,0} \left\langle \begin{array}{l} r_{1,1} \left\langle \begin{array}{l} r_{2,2} \\ r_{1,2} \end{array} \right. \\ r_{0,1} \left\langle \begin{array}{l} r_{1,2} \\ r_{0,2} \end{array} \right. \end{array} \right.$$

$r_{1,1} = r_{0,0} + \mu(0.5)\Delta t + \sigma\sqrt{\Delta t}$

$r_{0,1} = r_{0,0} + \mu(0.5)\Delta t - \sigma\sqrt{\Delta t}$

$\therefore r_{1,1} = 0.0742, \quad r_{0,1} = 0.0459$

$r_{2,2} = r_{1,1} + \mu(1)\Delta t + \sigma\sqrt{\Delta t}$

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$r_{1,2} = r_{1,1} + \mu(1)\Delta t + \sigma\sqrt{\Delta t}$

$r_{0,2} = r_{0,1} + \mu(1)\Delta t - \sigma\sqrt{\Delta t}$

$r_{2,2} = 0.0984$, $r_{1,2} = 0.0701$, $r_{0,2} = 0.0418$

$\therefore$ Rate tree is:

$$\begin{tikzcd}[row sep=small]
 & & 0.0984 \\
 & 0.0742 \arrow[ur] \arrow[dr] & \\
0.05 \arrow[ur] \arrow[dr] & & 0.0701 \\
 & 0.0459 \arrow[ur] \arrow[dr] & \\
 & & 0.0418
\end{tikzcd}$$

0.5y ZCB:
tree

$$\begin{tikzcd}[row sep=small]
 & 1 \\
P(0.5) \arrow[ur] \arrow[dr] & \\
 & 1
\end{tikzcd}$$

$$P(0.5) = \frac{\frac{1+1}{2}}{(1+5\%/2)} = 0.975610$$

1y ZCB:
tree

$$\begin{tikzcd}[row sep=small]
 & & 1 \\
 & P_{1,1} \arrow[ur] \arrow[dr] & \\
P_{0,0} \arrow[ur] \arrow[dr] & & 1 \\
 & P_{0,1} \arrow[ur] \arrow[dr] & \\
 & & 1
\end{tikzcd}$$

where $P_{1,1} = \frac{1+1}{2} \times \left(\frac{1}{1+r_{1,1}/2}\right)$

$P_{1,1} = 0.964235$

$P_{0,1} = \frac{1+1}{2} \times \left(\frac{1}{1+r_{0,1}/2}\right)$

$P_{0,1} = 0.977566$

$$P_{0,1} = 0.977566$$

$$P_{0,0} = \frac{P_{0,1} + P_{1,1}}{2} \times \left(\frac{1}{1 + r_{0,0}/2}\right)$$

$$P_{0,0} = 0.947220$$

Tree:

$$0.947220 \begin{array}{l} \nearrow 0.964235 \begin{array}{l} \nearrow 1 \\ \searrow 1 \end{array} \\ \searrow 0.977566 \begin{array}{l} \nearrow 1 \\ \searrow 1 \end{array} \end{array}$$

For 1.5 year ZCB:

$$P_{0,0} \begin{array}{l} \nearrow P_{1,1} \begin{array}{l} \nearrow P_{2,2} \begin{array}{l} \nearrow 1 \\ \searrow 1 \end{array} \\ \searrow P_{1,2} \begin{array}{l} \nearrow 1 \\ \searrow 1 \end{array} \end{array} \\ \searrow P_{0,1} \begin{array}{l} \nearrow P_{1,2} \\ \searrow P_{0,2} \begin{array}{l} \nearrow 1 \\ \searrow 1 \end{array} \end{array} \end{array}$$

where $P_{2,2} = \frac{1+1}{2} \times \left(\frac{1}{1 + r_{2,2}/2}\right)$

$$P_{2,2} = 0.953103$$

$$P_{1,2} = \frac{1+1}{2} \times \left(\frac{1}{1 + r_{1,2}/2}\right)$$

$$P_{1,2} = 0.966125$$

$$P_{0,2} = \frac{1+1}{2} \times \left(\frac{1}{1 + r_{0,2}/2}\right)$$

$$P_{0,2} = 0.979508$$

$$P_{1,1} = \frac{P_{2,2} + P_{1,2}}{2} \times \left(\frac{1}{1 + r_{1,1}/2}\right)$$

$$P_{1,1} = 0.925294$$

$$P_{0,1} = \frac{P_{0,2} + P_{1,2}}{2} \times \left(\frac{1}{1 + r_{0,1}/2}\right)$$

$$P_{0,1} = 0.950993$$

$P_{0,0} = (P_{1,1} + P_{0,1}) \times \left(\frac{1}{\quad}\right)$

$$P_{0,0} = \left(\frac{P_{0,1}+P_{1,1}}{2}\right) \times \left(\frac{1}{1+r_{0,0}/2}\right)$$

$$P_{0,0} = 0.915262$$

$\therefore$ 1.5 year ZCB tree:

$$\begin{tikzcd}[column sep=small]
 & & & 1 \\
 & & 0.953103 \arrow[ur] \arrow[r] & 1 \\
 & 0.92594 \arrow[ur] \arrow[dr] & & \\
0.915262 \arrow[ur] \arrow[dr] & & 0.966125 \arrow[r] \arrow[uur, phantom] & 1 \\
 & 0.950993 \arrow[ur] \arrow[dr] & & \\
 & & 0.979508 \arrow[r] \arrow[dr] & 1 \\
 & & & 1
\end{tikzcd}$$

6). $K = 960$, $F = 1000$

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & C_{2,2} \\
 & C_{1,1} \arrow[ur] \arrow[dr] & \\
C_{0,0} \arrow[ur] \arrow[dr] & & C_{1,2} \\
 & C_{0,1} \arrow[ur] \arrow[dr] & \\
 & & C_{0,2}
\end{tikzcd}$$

where $C_{i,2} = \max(F \times P_{i,2} - K, 0)$

$\therefore$ option tree:

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & 0 \\
 & C_{1,1} \arrow[ur] \arrow[dr] & \\
C_{0,0} \arrow[ur] \arrow[dr] & & 6.125234 \\
 & C_{0,1} \arrow[ur] \arrow[dr] & \\
 & & 19.50833
\end{tikzcd}$$

$$C_{1,1} = \left(\frac{C_{2,2}+C_{1,2}}{2}\right) \times \left(\frac{1}{1+r_{1,1}/2}\right)$$

$$C_{0,1} = \left(\frac{C_{0,2}+C_{1,2}}{2}\right) \times \left(\frac{1}{1+r_{0,1}/2}\right)$$

$$C_{0,0} = \left(\frac{C_{0,1}+C_{1,1}}{2}\right) \times \left(\frac{1}{1+r_{0,0}/2}\right)$$

$\Rightarrow C_{1,1} = 2.953084$

$$\Rightarrow C_{1,1} = 2.755007$$

$$C_{0,1} = 12.549249$$

$$C_{0,0} = 7.55236$$

$$\therefore \text{Hedge ratio} = \frac{C_{1,1} - C_{0,1}}{P_{1,1} - P_{0,1}} \times \left(\frac{1}{1000}\right)$$

$$\text{Hedge ratio} = 0.37624.$$

$\therefore$ one could long 0.37624 unit of bond and short 1 unit of option.