Final Exam for MATH4511

December 17, 2015

(The problem sheet is required to be returned with your booklet)

Problems (Numbers in brackets are credits, totaled to 80):

1. Consider the hedging of "2.5s of December 17, 2021" using the concept of "DV01 neutral". Let the notional value be \$100m and let the par yield be flat at 2%.
   1.1. (4) Calculate the DV01 of the bond for \$100 face or notional value (using the formula $\mathrm{DV01} = \frac{-1}{10000}\frac{dP}{dy}$).
   1.2. (4) Calculate the face value of the 5-year maturity payer's swap for hedging.
   1.3. (4) Calculate the number of units of 2-year maturity Eurodollar futures contracts (which is for \$1m notional value) for hedging. Indicate long or short futures.

2. For the equity derivatives model,

$$\frac{\Delta S_t}{S_t} = \mu \Delta t + \sigma \sqrt{\Delta t}\ \varepsilon_B,$$

where ε_B takes values of ± 1 with equal probabilities, there are two methods to build a binomial tree. The first is to change the branching probabilities, from $\{\frac{1}{2}, \frac{1}{2}\}$ to $\{q, 1-q\}$:

$$S_t \quad \begin{cases} \xrightarrow{1-q} & S_{t+\Delta t}^{u} = S_t\left(1+\mu\Delta t + \sigma\sqrt{\Delta t}\right) \\ \xrightarrow{q} & S_{t+\Delta t}^{d} = S_t\left(1+\mu\Delta t - \sigma\sqrt{\Delta t}\right) \end{cases}$$

The second is to add a risk premium, λ, to μ such that

$$S_t \quad \begin{cases} \xrightarrow{0.5} & S_{t+\Delta t}^{u} = S_t\left(1+(\mu+\lambda)\Delta t + \sigma\sqrt{\Delta t}\right) \\ \xrightarrow{0.5} & S_{t+\Delta t}^{d} = S_t\left(1+(\mu+\lambda)\Delta t - \sigma\sqrt{\Delta t}\right) \end{cases}$$

   2.1. (4) Explain when the trees are arbitrage free, and which method is more advantageous?
   2.2. (4) What are the principles for determining q and λ?
   2.3. (8) Write down the expressions for q and λ.

3. Consider the normal model for interest rates:

$$\Delta r_t = \theta_t \Delta t + \sigma\sqrt{\Delta t}\ \varepsilon_B.$$

Answer the following questions:
   3.1. (4) For pricing purposes, what conditions should θ_t satisfy?
   3.2. (4) Explain how to build a tree that satisfies the above conditions.

4. Price a European **put option** on a **coupon bond** with an interest-rate tree.

P.T.O.

4.1. (4) Let $\Delta t = 0.5$, σ=1% and $r_{0,0} = 5\%$. Verify that the one-period interest-rate tree with $\theta = 0.5\%$ fits to the zero-coupon yields $\hat{r}(\Delta t) = 5\%$ and $\hat{r}(2\Delta t) = 5.1237\%$.

4.2. (4) With the tree built using the parameters above, price a European put option on the **clean price** of a one-year maturity bond with 5% **coupon rate** (and \$100 face value). The option maturity is half a year and the strike price is par.

4.3. (4) Explain how to hedge the option and provide the hedge ratio at date 0.

5. Consider being long an in-2-to-2 payer's swap (i.e., the swap period is from 2 to 4 years) with \$100m notional value. Let the current forward-rate curve for **semi-annual compounding** be

$$f\left(\tfrac{i}{2}\right) = 0.018 + 0.0005 \times (i-1),\ i = 1, \cdots, 60.$$

Do the following.

5.1. (4) Calculate the fair swap rate for the in-2-to-2 swap.

5.2. (4) One year later, the forward-rate curve becomes

$$f\left(\tfrac{i}{2}\right) = 0.02 + 0.00045 \times (i-1),\ i = 1, \cdots, 60.$$

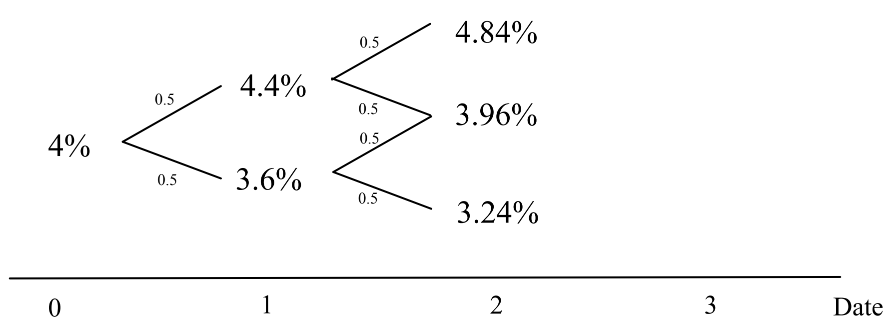
Calculate the P&L of the payer's swap.

6. This problem concerns with swaptions.

6.1 (4) Give the definition of swaptions.

6.2 (4) Show that swaptions can be priced as either call or put options on coupon bonds with the PAR strike.

7. Under the **risk-neutral measure**, (annualized) three-month CD rate for **simple compounding** evolves according the following tree:



The size of time step is $\Delta t = 0.25$ year. At date 0, long both FRA and futures for the maturity of date 2 (i.e., you short rate), for a notional value of \$1m.

7.1. (4) Calculate the fair rate for the FRA.

7.2. (4) Calculate the fair rate for the futures.

7.3. (8) At date 2, calculate the cumulated P&L of both FRA and futures, separately, for all paths of the CD rate, supposing that the futures contract is **marked to market quarterly.**

===================== Good Luck Everyone! =====================