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MATH4511 Quantitive Methods for Fixed Income Derivatives, 2015-16 Fall

Quiz 04(T1D)

Name: Li Xtudi ID No.: 20175277 Tutorial Section: T1D

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1. (20 points) Assume the spot rate under the risk neutral measure follows the dynamic:

$$\Delta r_t = \kappa(\theta_t - r_t)\Delta t + \sigma\sqrt{\Delta t}\epsilon_B,$$

where ϵ_B takes +1 or -1 with equal probability. The current 1-year spot rate is r(0) and the 2-year zero coupon bond price is P(2) and the 3-year zero-coupon bond price is P(3). Use a two step binomial tree to describe the evolution of the 1-year spot rate. Here, κ, σ are constant, risk neutral probability is $\{\frac{1}{2}, \frac{1}{2}\}$. State the procedure of calculating the θ_t's. $\Delta t = 1$.

$r(0)$

$r^u(1) = r(0) + k(\theta_0 - r_0)\Delta t + \sigma\sqrt{\Delta t} = r(0) + k(\theta_0 - r_0) + \sigma$

$r_1^d = r(0) + k(\theta_0 - r_0)\Delta t - \sigma\sqrt{\Delta t} = r_0 + k(\theta_0 - r_0) - \sigma$

use this to price the two year zero coupon bond.

P_0: P_1^u — 100, P_1^d — 100

$P_1^u = \frac{100}{1+r_1^u}$, $P_1^d = \frac{100}{1+r_1^d}$.

$P_0 = \frac{\frac{1}{2}(P_1^u + \frac{1}{2}P_1^d)}{1+r_0}$ the only unknown in this equation is θ_0 ~~θ_0~~.

so we can solve θ_0 by matching P_0 with the true price $P(2)$

then we proceed to next step. using θ_0 from above. we can calculate r_1^u. r_1^d.

r_0 — r_1^u, r_1^d

$r_2^{uu} = r_1^u + k(\theta_1 - r_1^u) + \sigma$

$r_2^{ud} = r_1^u + k(\theta_1 - r_1^u) - \sigma$

$r_2^{du} = r_1^d + k(\theta_1 - r_1^d) + \sigma$

$r_2^{dd} = r_1^d + k(\theta_1 - r_1^d) - \sigma$.

using this tree to price bond (3-year zero coupon bond).

P_0 — P_1^u (P_2^{uu} — 100, P_2^{ud} — 100), P_1^d (P_2^{du} — 100, P_2^{dd} — 100)

$P_2^{uu} = \frac{100}{1+r_2^{uu}}$ $P_2^{ud} = \frac{100}{1+r_2^{ud}}$

$P_1^u = \frac{\frac{1}{2}(P_2^{uu} + \frac{1}{2}P_2^{ud})}{1+r_1^u}$

$P_2^{du} = \frac{100}{1+r_2^{du}}$ $P_2^{dd} = \frac{100}{1+r_2^{dd}}$

$P_1^d = \frac{\frac{1}{2}(P_2^{du} + P_2^{dd})}{1+r_1^d}$.

$P_0 = \frac{\frac{1}{2}(P_1^d + P_1^u)}{1+r_0}$.

so P_0 is $P_0(\theta_1)$. r_0. r_1^d. r_1^u are known. only θ_1 is unknown

matching $P_0(\theta_1)$ with true price $P(3)$. we can solve θ_1.

put θ_1 back to the tree. we have r_2^{uu}. r_2^{ud}. r_2^{du}. r_2^{dd}.

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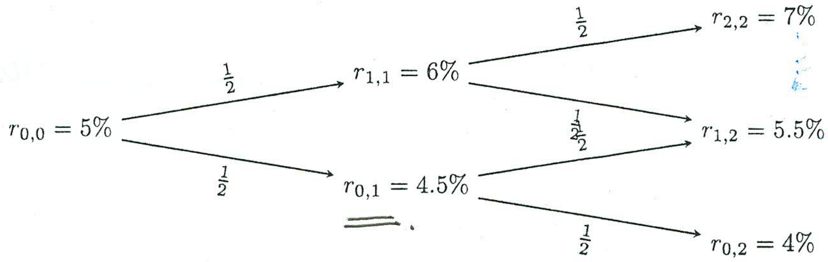
2. (20 points) Assume we have built the spot rate tree under the risk neutral measure:



maturity = 0.5y.

coupon payment is $\frac{5\%}{2} \times 1000 = 25$.

Each time step $\Delta t = 0.5$. Use this tree to price a call option of a 1.5-year coupon bond. The coupon rate is 5% and face value is 1000 and the strike price of this option is 1000.

$P_{0,0} \to P_{1,1},\ P_{0,1}$; $P_{1,1} \to P_{2,2},\ P_{1,2}$; $P_{0,1} \to P_{1,2},\ P_{0,2}$

$$P_{2,2} = \frac{1000+25}{1+r_{2,2}\cdot 0.5} = \frac{1025}{1+0.035} = 990.3382.$$

$$P_{1,2} = \frac{1025}{1+r_{1,2}\cdot 0.5} = \frac{1025}{1+0.055\div 2} = 997.5669$$

$$P_{0,2} = \frac{1025}{1+r_{0,2}\cdot 0.5} = \frac{1025}{1+0.04\times 0.5} = 1004.9020.$$

} clean price.

$$P_{1,1} = \frac{\frac{1}{2}(P_{2,2}+P_{1,2})+25}{1+\frac{1}{2}r_{1,1}} = \frac{\frac{1}{2}(990.3382+997.5669)+25}{1+0.5\times 6\%} = 989.2743$$

$$P_{0,1} = \frac{\frac{1}{2}(P_{1,2}+P_{0,2})+25}{1+\frac{1}{2}r_{0,1}} = \frac{\frac{1}{2}(997.5669+1004.9020)+25}{1+0.5\times 4.5\%} = 1003.652274$$

} clean price.

$C_{0,0} \to C_{1,1},\ C_{0,1}$

$$C_{1,1} = (P_{1,1} - K)^+ = 0$$

$$C_{0,1} = (P_{0,1} - K)^+ = 3.652274.$$

$$C_{0,0} = \frac{\frac{1}{2}(C_{1,1}+C_{0,1})}{1+r_{0,0}\cdot\frac{1}{2}} = \frac{\frac{1}{2}\times 3.652274}{1+5\%\cdot 0.5} = 1.7816.$$

so price is 1.7816.

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3. (10 points) Given a two step risk-neutralized interest-rate tree (which reproduces market price of zero-coupon bonds of maturity $2\Delta t$ and $3\Delta t$):

$$\Delta r_t = 0.0025\Delta t + 0.005\sqrt{\Delta t}\epsilon_B,$$

where Δt=1, r_0=4% and ϵ_B takes +1 or -1 with equal probability. Calculate (a) the forward rate for the period ($2\Delta t$, $3\Delta t$) and (b) the futures rate for the same period.

build the interest rate tree.

$r_0 = 4\%$, $\Delta t = 1$ $\sqrt{\Delta t} = 1$. $\Delta r_t = 0.0025 + 0.005\,\epsilon_B$.

$r_{1,1} = r_0 + 0.0025 + 0.005\times 1 = r_0 + 0.0075 = 4.75\%$.

$r_{0,1} = r_0 + 0.0025 - 0.005\times 1 = r_0 - 0.0025 = 3.75\%$.

$r_{2,2} = r_{1,1} + 0.0075 = 5.5\%$. $r_{1,2} = r_{1,1} - 0.0025 = 4.5\%$

$r_{0,2} = r_{0,1} - 0.0025 = 3.5\%$. calculate $d(2)$ first

r_0 4% $\to$ $r_{1,1}$ 4.75%, $r_{0,1}$ 3.75%; $r_{1,1} \to r_{2,2}$ 5.5%, $r_{1,2}$ 4.5%; $r_{0,1} \to r_{1,2}$ 4.5%, $r_{0,2}$ 3.5%

$P_0 \to P_{1,1} - 1,\ P_{0,1} - 1$.

$$P_{1,1} = \frac{1}{1+r_{1,1}} = 0.954654$$

$$P_{0,1} = \frac{1}{1+r_{0,1}} = 0.963855$$

$$P_0 = \frac{\frac{1}{2}P_{1,1} + \frac{1}{2}P_{0,1}}{1+r_0} = 0.92236.$$

$d(2) = 0.92236.$

then calculate $d(3)$

(b) futures rate $= E(r_2)$.

$$= \frac{1}{2}\left(\frac{1}{2}\times 5.5\% + \frac{1}{2}\times 4.5\%\right) + \frac{1}{2}\left(\frac{1}{2}\times 4.5\% + \frac{1}{2}\times 3.5\%\right)$$

$$= \frac{1}{4}\times 0.055 + \frac{1}{2}\times 0.045 + \frac{1}{4}\times 0.035$$

$= 0.045 = 4.5\%$.

(a) $P_0 \to P_{1,1},\ P_{0,1}$; $P_{1,1} \to P_{2,2} - 1,\ P_{1,2} - 1$; $P_{0,1} \to P_{1,2} - 1,\ P_{0,2} - 1$

$$P_{2,2} = \frac{1}{1+r_{2,2}} = 0.9478673$$

$$P_{1,2} = \frac{1}{1+r_{1,2}} = 0.958378$$

$$P_{0,2} = \frac{1}{1+r_{0,2}} = 0.966183575$$

$$P_{0,1} = \frac{\frac{1}{2}(P_{1,2}+P_{0,2})}{1+r_{0,1}} = 0.9268$$

$$P_{1,1} = \frac{1}{2}(P_{1,2}+P_{2,2})/(1+r_{1,1}) = 0.909214845$$

$$d(3) = P_{0,0} = \frac{\frac{1}{2}(P_{0,1}+P_{1,1})}{1+r_0} = 0.8827.$$

$$\text{forward rate} = \frac{1}{1}\left(\frac{d(2)}{d(3)} - 1\right) = 4.493\%.$$