Forward price as a random variable

- Any European options can be treated as option on forward prices.
- Forward price is defined by

$$F_t = \frac{S_t}{d(t,T)}, \qquad \text{for } t \leq T,$$

which is also a random variable.

Features of the Forward Price

- The forward price satisfies
  1. At an option's maturity, forward = spot:

$$F_T = \frac{S_T}{d(T,T)} = S_T$$

  2. It is assumes that

$$\ln \frac{F_T}{F_0} = -\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}z$$

or

$$F_T = F_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}z}$$

Options on Forward

- Any European options can be treated as option on forward prices.
- E.g., call option.

$$
\begin{aligned}
V_0 &= e^{-rT} E\left[\left(S_T - K\right)^+\right] \\
&= d(0,T) E\left[\left(F_T - K\right)^+\right] \\
&= d(0,T) E\left[\left(F_0 e^{-\frac{\sigma^2}{2}T + \sigma\sqrt{T}\times z} - K\right)^+\right]
\end{aligned}
$$

- Note that

$$F_T \geq K$$

$$\Updownarrow$$

$$z \geq \frac{\ln \frac{K}{F_0} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = -\frac{\ln \frac{F_0}{K} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = -d_2$$

- It follows that

$$E\left[\left(F_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\times\tilde{\varepsilon}} - K\right)^+\right]$$

$$= \int_{-d_2}^{+\infty}\left(F_0 e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\times x} - K\right)\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= F_0 \int_{-d_2}^{+\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2 T + \sigma\sqrt{T}\times x - \frac{x^2}{2}} dx - K\int_{-d_2}^{+\infty}\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= F_0 \int_{-d_2}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x - \sigma\sqrt{T}\right)^2} dx - K \int_{-\infty}^{d_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= F_0 \int_{-d_2 - \sigma\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} dy - K\Phi(d_2)$$

$$= F_0 \int_{-\infty}^{d_2 + \sigma\sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} dy - K\Phi(d_2)$$

$$= F_0 \Phi(d_1) - K\Phi(d_2)$$

where $d_1 = d_2 + \sigma\sqrt{T}$.

Black's formula for call option

- Black formula (worth at least multi-billion dollars):

$$C_0 = d(0,T)\left[F_0\Phi(d_1) - K\Phi(d_2)\right]$$

where

$$d_1 = \frac{\ln\frac{F_0}{K} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

- Hedge ratio with the underlying

$$\alpha = \Phi(d_1)$$

Black's formula for put option

- For put options, the Black's formula is

$$P_0 = d(0,T)\left[K\Phi(-d_2) - F_0\Phi(-d_1)\right]$$

- Hedge ratio with the underlying

$$\alpha = -\Phi(-d_1)$$

The Roadmap

- Discrete binomial model for equity options.
- Discrete binomial model (or Ho-Lee model) for interest rate options.
- Continuous-time limit of the binomial model.
- Forward contract and forward price.
- From Black-Scholes formula to Black formula.
- The Black's formula for all markets, including interest-rate markets.

Broad Applications of the Black's formula

- The markets use the Black's formula to price
  - Interest-rate derivatives;
  - Commodity derivatives;
  - European options.
- Because
  - At maturity forward = spot; and
  - any forward prices or forward rates are assumed to satisfy $\ln \frac{F_T}{F_0} = -\frac{1}{2}\sigma^2 T + \sigma\sqrt{T} z$

The General Pricing Principle

- Under both deterministic and stochastic interest rates, option can be priced by
  - Taking the expectation of the terminal payoff,
  - followed by discounting

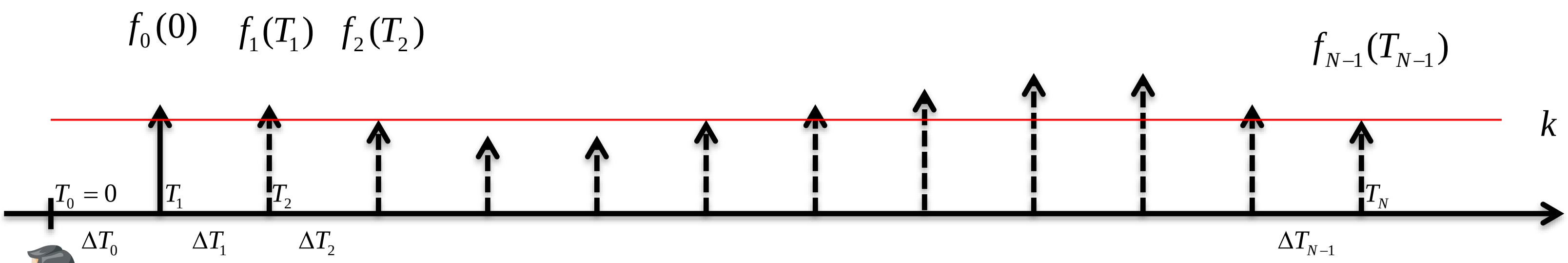
$$V_0 = d(0,T)E\left[V_T(F_T)\right]$$

Ho-Lee model vs. the Black model

- Ho-Lee is a normal model.
- Black is a lognormal model.
- Under a lognormal model, interest rates (zero-coupon yields, forward rates, swap rates, and etc.) are kept positive.
- For American options (when early exercise is allowed), Ho-Lee or tree models are more convenient.

Interest rate caps

- The cash flows of an interest rate cap:



where $f_{j-1}(T_{j-1})$ is the CD rate for the term set in time T_{j-1} :

$$f_{j-1}(T_{j-1}) = \frac{1}{\Delta T_j}\left(\frac{1}{d(T_{j-1}, T_j)} - 1\right)$$

Interest rate caps, cont'd

- Cash flow of an interest-rate cap: at T_j ,

$$Not.\times \Delta T_j \left(f_{j-1}(T_{j-1}) - K \right)^{+}, \qquad j = 1, 2, \cdots, N.$$

  $Not.$ means Notional, like \$1m or \$10m.

- It is a series of options, each is called a caplet.

- Let $f_{j-1}(0)$ be the forward rate for the period (T_{j-1}, T_j) observed at time 0.
- We use the Black's formula for pricing the caplet:

$$c_j = Not.\times \Delta T_j \times d(0,T_j)\left[f_{j-1}(0)\Phi(d_1^{(j)}) - k\Phi(d_2^{(j)}) \right],$$

$$d_1^{(j)} = \frac{\ln \frac{f_{j-1}(0)}{k} + \frac{1}{2}\sigma^2 T_{j-1}}{\sigma\sqrt{T_{j-1}}}, \quad d_2^{(j)} = d_1^{(j)} - \sigma\sqrt{T_{j-1}}$$

$$j = 1, 2, \cdots, N$$

Hedging of Caps

- Value of a cap:

$$Cap = \sum_{j=1}^{N} c_j$$

$$= Not.\times \sum_{j=1}^{N} \Delta T_j \times d(0,T_j)\left[f_{j-1}(0)\Phi(d_1^{(j)}) - k\Phi(d_2^{(j)})\right]$$

- Let

$$g(f_{j-1}(0)) = Not.\times \Delta T_j \times \left[f_{j-1}(0)\Phi(d_1^{(j)}) - k\Phi(d_2^{(j)})\right]$$

Hedging with FRA and ZCB

- For the cap underwritter, the cap can be hedged by
  - Enter $\Phi(d_1^{(j)})$ unit of T_{j-1}-maturity FRA for the term (T_{j-1}, T_j) to long rate or pay fixed
  - being long or $g(f_{j-1}(0))$ unit of $P(0, T_j)$, the T_j-maturity zero-coupon bond (ZCB).
- Here $P(0, T_j) = d(0, T_j)$.

Proof:

- Write the time-t value of the j^{th} caplet as

$$c_j(t) = d(t, T_j)\,g(f_{j-1}(t)),$$

- We consider $\mathrm{d}c_j(t) = c_j(t + \mathrm{d}t) - c_j(t)$ for clues of hedging.

- There is

$$c_j(t+\mathrm{d}t)-c_j(t)=d(t+\mathrm{d}t,T_j)g(f_{j-1}(t+\mathrm{d}t))-d(t,T_j)g(f_{j-1}(t))$$

$$=d(t+\mathrm{d}t,T_j)\left[g(f_{j-1}(t+\mathrm{d}t))-g(f_{j-1}(t))\right]$$

$$+\left[d(t+\mathrm{d}t,T_j)-d(t,T_j)\right]g(f_{j-1}(t))$$

$$=d(t+\mathrm{d}t,T_j)\frac{\partial g(f_{j-1}(t))}{\partial f_{j-1}}\mathrm{d}f_{j-1}(t)+g(f_{j-1}(t))\mathrm{d}d(t,T_j)$$

$$=\underbrace{\Phi(d_1^{(j)})\times Not.\times\Delta T_j\times d(t+\mathrm{d}t,T_j)\mathrm{d}f_{j-1}(t)}_{\text{Change in FRA}}+g(f_{j-1}(t))\underbrace{\mathrm{d}d(t,T_j)}_{\text{Change in ZCB}}$$

Perfect Hedging

- The perfect hedging is achieved through

$$
\begin{aligned}
&\text{Change of the hedged portfolio} \\
&= -\mathrm{d}C_t + \Phi(d_1) \times d(t + \mathrm{d}t, T)\mathrm{d}F_t + g(F_t)\mathrm{d}d(t, T) \\
&= 0
\end{aligned}
$$

Example

- Black's model with 3m forward-rate curve

$$f_j(0) = 0.01 + 0.0005 \times (j-1), \; j = 1, \cdots, 120.$$

- Forward rate volatility $\sigma = 0.3$
- Cap maturity and strike: T=10, k=1.9467%
- Value of cap for $1m notional: $51,418.17.
- [Cap calculation]

Interest rate floors

- Cash flow of an interest-rate floor: at T_j,

$$Not. \times \Delta T_j \left(K - f_{j-1}(T_{j-1})\right)^{+}, \qquad j = 1, 2, \cdots, N.$$

- It is a series of options, each is called a floorlet.

- We use the Black's formula for pricing the floorlet at *t*:

$$p_j(t) = Not. \times \Delta T_j \times d(t,T_j)\left[k\Phi(-d_2^{(j)}) - f_{j-1}(t)\Phi(-d_1^{(j)})\right],$$

$$d_1^{(j)} = \frac{\ln\frac{f_{j-1}(t)}{k} + \frac{1}{2}\sigma^2(T_{j-1}-t)}{\sigma\sqrt{T_{j-1}-t}}, \quad d_2^{(j)} = d_1^{(j)} - \sigma\sqrt{T_{j-1}-t}$$

$$j = 1, 2, \cdots, N$$

- Value of a floor at time t:

$$floor = \sum_{j=1}^{N} p_j$$

$$= Not.\times \sum_{j=1}^{N} \Delta T_j \times d(t, T_j)\left[k\Phi(-d_2^{(j)}) - f_{j-1}(t)\Phi(-d_1^{(j)})\right]$$

- Let

$$g(f_{j-1}(t)) = Not.\times \Delta T_j \times \left[k\Phi(-d_2^{(j)}) - f_{j-1}(t)\Phi(-d_1^{(j)})\right]$$

Hedging with FRA and ZCB

- For the cap underwritter, the cap can be hedged by
  - Short $\Phi(-d_1^{(j)})$ unit of T_{j-1}-maturity FRA for the term (T_{j-1}, T_j) .
  - long $g(f_{j-1}(t))$ unit of $P(t, T_j)$, the T_j -maturity zero-coupon bond (ZCB).