Final Exam for MATH4511

December 11, 2017

(The problem sheet is required to be returned with your booklet)

Problems (Numbers in brackets are credits, totaled to 60):

1. Answer the following questions.
   1.1. (2) Why we say that the DV01 or kr01s of a payer's swap are essentially equal to those of a coupon bond?
   1.2. (4) Let $i, j \in \{2,5,10,30\}$. For $i \neq j$, why the i-year kr01 of a j-year par bond is zero?
   1.3. (2) Can you suggest an efficient way to calculate the i-year kr01 of the i-year par bond?

2. (4) Suppose that by the regression analysis we have already obtained

$$\Delta y_t^{20} = \alpha + \beta_{10}\Delta y_t^{10} + \beta_{30}\Delta y_t^{30} + \varepsilon_t \ ,$$

where $\beta_{10} = 0.1613$, $\beta_{30} = 0.8774$ and ε_t is the noise term. If the dollar value of the 20-year bond is one million, how much 10-year and 30-year bonds in dollar terms should be purchased for hedging? Assume the modified duration of the three bonds are $D_{10} = 7.89$, $D_{20} = 12.8$ and $D_{30} = 15.9$, respectively.

3. (8) At time 0, if the fixed rate for the FRA of the term $(T, T+\Delta T)$ is taken to be f_0 and there is

$$f_0 > \frac{1}{\Delta T}\left(\frac{d(0,T)}{d(0,T+\Delta T)} - 1\right),$$

which will create arbitrage opportunities. Explain how to arbitrage.

4. Given the model for the six-month interest rate for simple compounding:

$$\Delta r_t = \theta_t \Delta t + \sigma\sqrt{\Delta t}\ \ \varepsilon_B$$

where $\Delta t = 0.5$ and ε_B takes $+1$ or -1 with equal probabilities.
   4.1. (4) Describe in **words** how to price an option on a coupon bond using the corresponding binomial interest-rate tree.
   4.2. (4) Describe how to hedge the option.
   4.3. (4) Explain why the hedging strategy is a self-financing one?

5. Prove the following results regarding swaps and swaptions.
   5.1. (4) Let $t \le T_0$. The value for a forward starting payer's swap for the term (T_0, T_N) and fixed rate k is

$$V_t = A(t, T_0, T_N)[s(t; T_0, T_N) - k],$$

where $s(t; T_0, T_N)$ is the market prevailing swap rate for term (T_0, T_N) and

$$A(t, T_0, T_N) = \sum_{i=1}^{N} \tfrac{1}{2} d(t, T_i).$$

5.2. (4) With the same term (T_0, T_N) and the same strike rate *k*, there is the call-put parity:

$$\text{Payer's swaption} - \text{Receiver's swaption} = \text{Payer's swap}.$$

6. According to the Black's formula, the value of a swaption at $t \le T_0$ is given by

$$V(t) = A(t, T_0, T_N)G(s(t; T_0, T_N)),$$

where

$$G(s) = s\Phi(d_1) - k\Phi(d_2),$$

with

$$d_1 = \frac{\ln \frac{s}{k} + \frac{1}{2}\sigma^2(T_0 - t)}{\sigma\sqrt{T_0 - t}}, \; d_2 = d_1 - \sigma\sqrt{T_0 - t}.$$

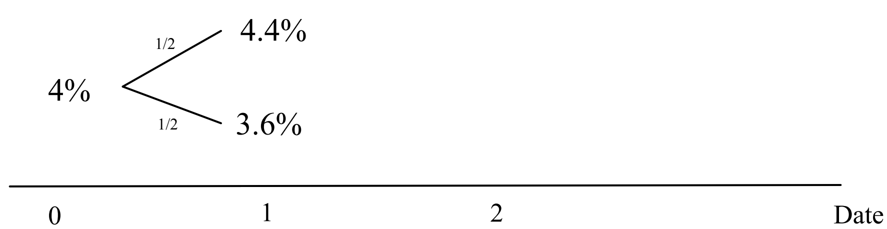
Show that

6.1. (4) $dV_t = A(t + dt, T_0, T_N)dG(s(t; T_0, T_N)) + G(s(t; T_0, T_N))dA(t, T_0, T_N)$.

6.2. (4) $\dfrac{dG(s)}{ds} = \Phi(d_1)$

6.3. (4) Provide a hedging strategy for the swaption.

7. Binomial models can also be used to price interest-rate options like caps. Let the (annualized) three-month CD rate for **simple compounding** evolves according the following tree:



The size of time step is $\Delta t = 0.25$ year. Let the forward-rate curve for quarterly compounding be flat at 4%.

7.1. (4) Find out the **risk neutral probabilities** $\{q_0, 1 - q_0\}$.

7.2. (4) Calculate the value of a cap with **six** months maturity, 3.8% strike rate and \$1m notional value.

====================== Good Luck Everyone! ========================