Final Exam for MATH4511
December 17, 2015
(The problem sheet is required to be returned with your booklet)

Problems (Numbers in brackets are credits, total to 80):

1. **Solution:**

1.1. By differentiation,

$$DV01 = \frac{F}{10,000}\left[\frac{c}{y^2}\left(1-\frac{1}{(1+\frac{y}{2})^{2T}}\right)+\left(1-\frac{c}{y}\right)\frac{T}{(1+\frac{y}{2})^{2T+1}}\right]$$

Plug in $c = 2.5\%$, $y = 2\%$ and $T = 6$ we obtain $DV01 = 0.05327.$

1.2. In a swap contract, DV01 of the floating leg is neglected, so the DV01 of the swap is the DV01 of the fixed leg, which is a par bond. For the par bond we have

$$DV01 = \frac{F}{10,000}\frac{1}{y}\left(1-\frac{1}{(1+\frac{y}{2})^{2T}}\right)$$

Plug in $y = 2\%$ and $T = 5$ we obtain $DV01 = 0.04735$. The notional value for the swap is

$$F_{swap} = F_{bond} \times \frac{DV01_{bond}}{DV01_{swap}} = \$100000000 \times \frac{0.05327}{0.04375} = \$112,493,130$$

1.3. For the ED futures, DV01=$25, so the number of contract for hedging is

$$No. = F_{bond} \times \frac{DV01_{bond}}{100} \Big/ 25 = 2134$$

2. **Solution:**

We assume that the interest rate for the term of Δt is a constant r over time.

2.1. q and λ must be chosen so that the market price of the share is reproduced.

2.2.

$$q = \frac{S_{t+\Delta t}^{u} - (1+r\Delta t)S_t}{S_{t+\Delta t}^{u} - S_{t+\Delta t}^{d}}$$
$$\lambda = r - \mu$$

2.3. The second approach is more advantageous, as the first approach can break down if $(1+r\Delta t)S_t \notin (S_{t+\Delta t}^{d}, S_{t+\Delta t}^{u})$.

3. **Solution**:
   3.1. θ_t should satisfy the condition that the market prices of zero-coupon bonds are reproduced.
   3.2. To build an N-period interest-rate tree, we use a forward inductive procedure to determine θ_j by fitting to $P((j+2)\Delta t)$, the price of $(j+2)\Delta t$-maturity zero-coupon bond. Once we have θ_j, we can calculate $r_{i,j+1}$ and $r_{i+1,j+1}$ out of $r_{i,j}$, with $j = 0,1,\cdots,N-1$.

4. **Solution**:

   4.1. $\hat{r}(\Delta t)$ is naturally fit. For $\hat{r}(2\Delta t) = 5.1237\%$, the corresponding ZCB price is $P(2\Delta t) = 100\left(1+{}^{\hat{r}(2\Delta t)}\!/_2\right)^2 = 95.0666$. By backward induction with the tree, we reproduce the price:

	5.9571%	
	97.1076	100
5%		
95.0666		
	4.5429%	
	97.7790	100

   4.2. The pricing tree for the coupon bond and option is

	5.9571%	
	99.5353	102.5
	0.4647	
5%		
0.2267		
	4.5429%	
	100.2235	102.5
	0.0000	

   4.3. We short the underlying bond for the unit of

delta=	-0.675267626

5. **Solution**:

5.1. (4) Calculate the fair swap rate for the in-2-to-2 swap.

1	0.0180	0.991080278
2	0.0185	0.981996807
3	0.0190	0.972755629
4	0.0195	0.963362841
5	0.0200	0.953824595
6	0.0205	0.944147087
7	0.0210	0.934336553
8	0.0215	0.924399261
	$\sum_{i=5}^{8} \Delta TP(T_i)$	1.878353749
	Swap rate(0)=	0.020743473

5.2. (4)

1		
2		
3	0.0200	0.99009901
4	0.0245	0.978117076
5	0.0245	0.966280144
6	0.0245	0.95458646
7	0.0245	0.94303429
8	0.0245	0.931621921
	$\sum_{i=5}^{8} \Delta TP(T_i)$	1.897761407
	Swap rate(1)=	0.0245
	P&L=	712899.2777

6. Solution:

6.1 A swaption is an option to enter into a swap with pre-specified swap rate in the future.

6.2 Consider the swaption on a payer's swap. The payoff of a payer's swap is

$$\max(swap(T;k,\tau),0) = \max(V_{float} - V_{fix}, 0)$$
$$= \max\left\{1 - \left(\sum_{i=1}^{2\tau} \Delta T d(T,T_i)k + d(T,\tau)\right)\right\}$$

which is a put option on coupon bond with par strike. Similarly, we can show that the swaption on receiver's swap is a call option with par strike.

7. Solution:

7.1. Price of date-2 maturity zero-coupon bond

	4.40%	
	98.91196835	100
4%		
98.02970104		
	3.60%	
	99.10802775	100

Price of date-3 maturity zero-coupon bond

		4.84%	
		98.80446596	100
	4.40%		
	97.83589064		
4%		3.96%	
97.0594		99.01970492	100
	3.60%		
	98.22408979		
		3.24%	
		99.19650828	100

The fair rate for FRA is the forward LIBOR rate:

$$f_0 = \frac{1}{\Delta t}\left(\frac{d(2\Delta t)}{d(3\Delta t)} - 1\right) = \frac{1}{0.25}\left(\frac{98.0297)}{97.0594)} - 1\right) = 3.9988\%$$

7.2. The fair rate for the futures is 4%

		4.84%
	4.40%	
4.00%		3.96%
	3.60%	
		3.24%

7.3.

P&L for FRA		
		Date 2
		4.84%
		-2077.837582
	4.40%	
	0	
4%		3.96%
0		96.06949522
	3.60%	
	0	
		3.24%
		1881.77818

P&L for futures		
		Date 2
		4.84%
		-2111
	4.40%	
	-1000	
4%		3.96%
0		89
		3.96%
		109
	3.60%	
	1000	
		3.24%
		1909

====================== Good Luck Everyone! =======================