HW9

Saturday, 26 March 2022 7:50 PM

8.6

If $f(0,0) = 0$ and

$$f(x,y) = \frac{xy}{x^2+y^2} \quad \text{if } (x,y) \neq (0,0)$$

prove that $(D_1 f)(x,y)$ and $(D_2)(x,y)$ exist at every point of $\mathbf{R}^2$, although f is not continuous at $(0,0)$.

Fix $y_0 \in \mathbb{R}$, let $x \in \mathbb{R}$.

We can discuss only $y_0 \neq 0$ in the following, since $f(x,0)=0$ has derivative 0 everywhere, so $\frac{\partial f(x_0,y_0)}{\partial x y)} = 0$ exists

also $\frac{\partial f(x_0,y_0)}{\partial y} = 0$ exists.

For $y_0 \neq 0$, $\frac{\partial f(x,y)}{\partial x} = \frac{xy_0}{x^2+y_0^2}$

$$= \frac{y_0(x^2+y_0^2)}{(x^2+y_0^2)^2} - \frac{2x^2y_0}{(x^2+y_0^2)^2}$$

$$= \frac{y_0^3 - x^2y_0}{(x^2+y_0^2)^2}$$

$\frac{\partial f(}{\partial y} = \frac{x^3-xy^2}{(x^2+y^2)^2}$

When $x=0$, $\frac{\partial f(0,y_0)}{\partial x} = 0$ so it exists, and it is not continuous at $(0,0)$.

When $x=y=a$, $\lim_{a\to 0} f(x,y) = \frac{a^2}{a^2+a^2} = \frac{1}{2}$,

and $\frac{1}{2} \neq 0$, so $f(x,y)$ is not continuous at $(0,0)$,

but $D_1f(x,y)$, $D_2f(x,y)$ exist.

Q.8. 7. Suppose that f is a real-valued function defined in an open set $E \subset R^n$, and that the partial derivatives $D_1f, \ldots, D_nf$ are bounded in E. Prove that f is continuous in E.

Hint: Proceed as in the proof of Theorem 9.21.

Given the proof in thm 9.21, we can do the following.

Given the proof in thm 9.21, we can do the following.

Let M be the given bound of $(D_i f)_p$, i.e. $\exists M \in \mathbb{R}, M \geq 0$, s.t. $\forall p \in E, 1 \leq i \leq n$, $|(D_i f)_p| \leq M$.

Let $\varepsilon > 0$, we can choose $\delta = \frac{\varepsilon}{Mn}$ where $M = \sup_i M_i$

Define $h = \sum_{i=1}^{n} h_i e_i$ s.t. $\|h\| < \delta$. Define $v_0 = \vec{0}$, $v_k = h_1 e_1 + \dots + h_k e_k$.

$$\Rightarrow \|f(x+h) - f(x)\| = \left\| \sum_{j=1}^{n} f(x+v_j) - f(x+v_{j-1}) \right\|$$

$$\leq \sum_{j=1}^{n} \| f(x+v_j) - f(x+v_{j-1}) \| \quad \text{(triangle inequality)}$$

by MVT, $\theta_j \in (0,1)$

$$= \sum_{j=1}^{n} \| h_j (D_j f)(x + v_{j-1} + \theta_j h_j e_j) \|$$

$$\leq \sum_{j=1}^{n} |h_j M| \quad \text{(triangle inequality)}$$

$$\leq \sum_{j=1}^{n} |h| |M| = \varepsilon/2 < \varepsilon$$

so f is cont. on E.

Q3. Show that, for any closed subset $E \subset \mathbf{R}^2$, there is a continuous function $f : \mathbf{R}^2 \to \mathbf{R}$, such that $f^{-1}(0) = E$.

Proof: First define $dist(x, E)$ to denote the distance between a point and closet subset.

$dist(x,E) = \inf(|x-y| \mid y \in E)$

W.T.S. $f: \mathbb{R}^2 \to \mathbb{R}$ s.t. $f^{-1}(0) = E$ is continuous,

equivalent to show: $\forall x_0 \in \mathbb{R}^2$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

[illegible]

$\forall x \in \mathbb{R}^2$, $|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon$.

the requirement of $f^{-1}(0)=E$ is same as $\forall r \in E, f(r)=0$,

$\forall r \in \mathbb{R}^2 \setminus E$, $\inf\{|r-y| \mid y \in E\} \geq r$

$\Rightarrow f(r) \neq 0$

$\therefore f^{-1}(0)=E$.

showing f is continuous,

If $f(x_0) < f(x)$, let $s = \min(\delta, \frac{f(x)-f(x_0)}{2})$

$\exists x_1 \in E$ s.t. $|x_0 - x_1| < f(x_0) + s$

$\Rightarrow 0 < f(x)-f(x_0) < |x-x_1| - f(x_0) < |x-x_0| + |x_0-x_1| - f(x_0)$

$< \delta + f(x_0) + s - f(x_0) < \frac{1}{2}\varepsilon < \varepsilon$ (if $\delta = \frac{1}{4}\varepsilon$)

if $f(x_0) > f(x)$, let $s = \min(\delta, \frac{f(x_0)-f(x)}{2})$

$\exists x_1 \in E$ s.t. $|x - x_1| < f(x) + s$

$\Rightarrow 0 < f(x_0) - f(x) < |x_0 - x_1| - f(x) < |x_0 - x| + |x - x_1| + f(x)$

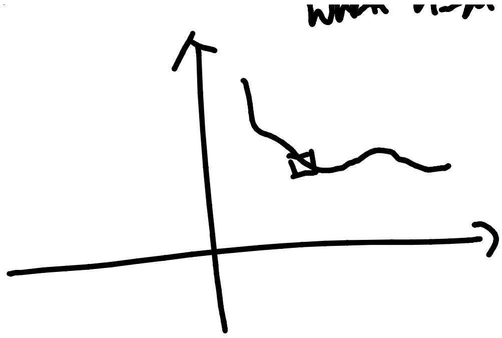
$< \delta + s < \frac{1}{2}\varepsilon < \varepsilon$

$\therefore |f(x)-f(x_0)| < \varepsilon \Rightarrow f$ is continuous.

Q4. For the implicit function theorem, take n=m=1, and interpret it graphically and intuitively.

when $n=m=1$,

$\uparrow$,



Implicit function theorem claims that for a continuously differentiable function $F: \mathbb{R}^2 \to \mathbb{R}$ and a point $(x_0, y_0) \in \mathbb{R}^2$, so that $F(x_0, y_0) = c$, if $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$, then there is a neighbourhood of (x_0, y_0) s.t. that whenever x is close to x_0, $\exists$ unique y s.t. $F(x, y) = c$.

So, if provided $\frac{\partial f(x_0, y_0)}{\partial x} \neq 0$ and $\frac{\partial f(x_0, y_0)}{\partial x} \neq 0$, if we take a close enough box on the function, it behaves like derivative, and shall approximate by a linear function.