HW6

Ex 8.3.2. Prove Proposition 8.3.3. ($\Omega$ = measurable set, $f, g : \Omega \to \mathbb{R}$ are absolutely integrable functions)

a). W.T.S. For any real number $c$ (+ve, 0, −ve), we have $cf$ is absolutely integrable and $\int_\Omega cf = c\int_\Omega f$.

Proof. First, we can write $\int_\Omega f$ as $\int_\Omega f^+ - \int_\Omega f^-$ and $\int_\Omega g = \int_\Omega g^+ - \int_\Omega g^-$, where $f^+ = \max(f, 0)$ and $f^-$ is $-\min(f, 0)$, same as $g$.

From the definition of $\int_\Omega cf$, $\int_\Omega cf = \int_\Omega (cf)^+ - \int_\Omega (cf)^-$

Case 1: $c > 0$. then $(cf)^+ = cf^+$ and $(cf)^- = cf^-$

$\Rightarrow \int_\Omega (cf)^+ = \int_\Omega cf^+ = c\int_\Omega f^+$

& $\int_\Omega (cf)^- = \int_\Omega cf^- = c\int_\Omega f^-$

$$\begin{aligned}\Rightarrow \int_\Omega cf &= \int_\Omega (cf)^+ - \int_\Omega (cf)^- \\ &= c\left(\int_\Omega f^+ - \int_\Omega f^-\right) \\ &= c\int_\Omega f\end{aligned}$$

Case 2: $c = 0$. then $\int_\Omega cf = c\int_\Omega f = 0$ thus equality holds.

Case 3: $c < 0$. then let $c = -d$, where $d > 0$.

We have $\int_\Omega cf = \int_\Omega (-df)^+ - \int_\Omega (-df)^-$

Since $(-df)^+ = \max(-df, 0) = -d\min(f, 0) = df^-$

and $(-df)^- = \min(-df, 0) = -d\max(f, 0) = df^+$

We have:

$$\begin{aligned}&\int_\Omega cf \\ &= \int_\Omega (-df)^+ - \int_\Omega (-df)^- \\ &= \int_\Omega df^- - \int_\Omega df^+ \\ &= d\left(\int_\Omega f^- - \int_\Omega f^+\right) \\ &= -d\int_\Omega f \\ &= c\int_\Omega f\end{aligned}$$

b). W.T.S. The function $f+g$ is absolutely integrable, and $\int_\Omega (f+g) = \int_\Omega f + \int_\Omega g$.

Proof: $\int_\Omega (f+g) = \int_\Omega (f+g)^+ - \int_\Omega (f+g)^-$

W.T.S. $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$ —(*).

For $\forall x \in \Omega$, either we have $f(x)+g(x) \geq 0$ or $f(x)+g(x) < 0$, so we have:

Case 1: $f(x)+g(x) \geq 0$.

Case 1.1: $f(x) \geq 0, g(x) < 0$, then

LHS $((f+g)^+ + f^- + g^-)(x) = (f+g)(x) - g(x) = f(x)$

RHS $((f+g)^- + f^+ + g^+)(x) = f(x)$

Case 1.2: $f(x) < 0, g(x) \geq 0$, then

LHS $((f+g)^+ + f^- + g^-)(x) = (f+g)(x) - f(x) = g(x)$

RHS $((f+g)^- + f^+ + g^+)(x) = g(x)$

Case 1.3: $f(x) \geq 0, g(x) \geq 0$, then

LHS: $((f+g)^+ + f^- + g^-)(x) = (f+g)(x) = f(x) + g(x)$

RHS: $((f+g)^- + f^+ + g^+)(x) = f(x) + g(x)$.

Case 2: $f(x) + g(x) < 0$.

Case 2.1: $f(x) \geq 0, g(x) < 0$, then

LHS: $((f+g)^+ + f^- + g^-)(x) = -g(x)$

RHS: $((f+g)^- + f^+ + g^+)(x) = -(f+g)(x) + f(x) = -g(x)$

Case 2.2: $f(x) < 0, g(x) \geq 0$, then

LHS: $((f+g)^+ + f^- + g^-)(x) = -f(x)$

RHS: $((f+g)^- + f^+ + g^+)(x) = -(f+g)(x) + g(x) = -f(x)$

Case 2.3: $f(x) < 0, g(x) < 0$, then

LHS: $((f+g)^+ + f^- + g^-)(x) = -f(x) - g(x)$

RHS: $((f+g)^- + f^+ + g^+)(x) = -f(x) - g(x)$

So $(*)$ is proven. Using lemma 8.2.10, we have:

$$\int_\Omega ((f+g)^+ + f^- + g^-) = \int_\Omega ((f+g)^- + f^+ + g^+)$$

$$\int_\Omega (f+g)^+ + \int_\Omega f^- + \int_\Omega g^- = \int_\Omega (f+g)^- + \int_\Omega f^+ + \int_\Omega g^+$$

$$\int_\Omega (f+g)^+ - \int_\Omega (f+g)^- = \int_\Omega f^+ - \int_\Omega f^- + \int_\Omega g^+ - \int_\Omega g^-$$

$$\int_\Omega (f+g) = \int_\Omega f + \int_\Omega g$$

hence proved.

c). W.T.S. if $f(x) \leq g(x)$ $\forall x \in \Omega$, we have $\int_\Omega f \leq \int_\Omega g$.

Proof: if $f(x) \leq g(x)$, $f^+ \leq g^+$ and $f^- \geq g^-$.

by proposition 8.2.6 (c), we have

$\int_\Omega f^+ \leq \int_\Omega g^+$, $\int_\Omega f^- \geq \int_\Omega g^-$

So proved.

d). W.T.S. if $f(x) = g(x)$ for almost every $x \in \Omega$, then $\int_\Omega f = \int_\Omega g$.

Proof: if $f(x) = g(x)$, $f^+ = g^+$ and $f^- = g^-$.

by proposition 8.2.6 (d), we have

$\int_\Omega f^+ = \int_\Omega g^+$, $\int_\Omega f^- = \int_\Omega g^-$.

So proved

Q8.3.3.

Proof: Using the result from 8.3.2, we know that $g-f$ is also absolutely integrable, and

$$\int_\Omega g = \int_\Omega f + \int_\Omega g-f.$$

since we are given $f(x) \leq g(x) \ \forall x \in \mathbb{R}$,

$$g(x)-f(x) \geq 0 \Rightarrow \int_\Omega g-f \geq 0.$$

also, we have

$$\int_{\mathbb{R}} g = \int_{\mathbb{R}} f \Rightarrow \int_{\mathbb{R}} g-f = 0$$

by proposition 8.2.6,

$$\int_{\mathbb{R}} g-f = 0 \Rightarrow (g-f)(x) = 0 \text{ for almost every } x \in \mathbb{R}$$

$$\Rightarrow g(x) = f(x) \text{ for almost every } x \in \mathbb{R}.$$

$\therefore f(x) = g(x)$ for almost every $x \in \mathbb{R}$, proved.

3. Refer to student area

4. done.