Hw8

Pugh Ex 83. Question: Suppose that f_k is a seq. of measurable functions that converge a.e to f as $k \to \infty$. (a) Formulate and prove Egoroff's theorem if functions are on a box in n space.

Proof: Let $X(k,l) = \{x \in B : \forall n \geq k, |f_n(x) - f(x)| < \frac{1}{l}\}$. where $B = \prod_{i=1}^{n}(a_i, b_i)$, $B \in \mathbb{R}^n$

We are given that $f_n(x) \to f(x)$ almost everywhere. for any $N \geq k$, $|f_n(x) - f(x)| < \frac{1}{l}$

$\Rightarrow f_n(x) \in (f(x) - \frac{1}{l}, f(x) + \frac{1}{l})$

measurable $\Rightarrow$ measurable for its inverse. $\Rightarrow \bigcap_{n=k}^{\infty} f_n^{-1}((f(x) - \frac{1}{l}, f(x) + \frac{1}{l}))$ is measurable

$\forall x$ in $X(k,l)$, $\forall n \geq k$, $x \in f_n^{-1}(f(x) - \frac{1}{l}, f(x) + \frac{1}{l}))$

$\Rightarrow x \in \bigcap_{n \geq k}^{\infty} f_n^{-1}((f(x) - \frac{1}{l}, f(x) + \frac{1}{l}))$ satisfies $|f_n(x) - f(x)| < \frac{1}{l}$

So $x \in X(k,l) \Rightarrow X(k,l)$ is measurable

Let $\varepsilon > 0$, Measure continuity implies $m(X(k,l)) \to b-a$ as k approaches to ∞.

because $X(k,l) \to [a,b]$.

Then we can choose $k_1 < k_2 < \dots$ s.t. for $X_l = X(k_l, l)$, we have $m(X_l^c) < \frac{\varepsilon}{2^l}$.

Let $X = \bigcap_l X_l$, we have $m(X^c) < \varepsilon$.

$\Rightarrow f_n$ converges uniformly on X. Let $\varepsilon > 0$, we can choose l s.t. $\frac{1}{l} < \varepsilon$.

then $\forall n \geq k_l$, $x \in X$ implies $x \in X_l = X(k_l, l) \Rightarrow |f_n(x) - f(x)| < 1/l < \varepsilon$

b). True, following the similar approach as above, we can define M to be the finite measure of the unbounded domain, ~~and~~ and each $X(k,l)$ is subset of the domain hence tends to M.

c). Consider the moving bump function $f_i(x) = \begin{cases} 1 & x \in (i, i+1) \\ 0 & \text{else} \end{cases}$

Then, (f_n) converges pointwise to 0, but on $\mathbb{R} \setminus B$ for any finite subinterval $B \subset \mathbb{R}$ does not converges uniformly to 0. ($f_n(x) = 1$ for $x = \lfloor n \rfloor$).

d). We can still apply Egoroff's theorem, by assuming the function converges pointwise on $\mathbb{R}^k$, then the domain has finite measure as the set K is bounded.

3. Let $(\mathbb{R}^n, |\cdot|_1)$ be the normed vector space where $|(x_1, \cdots, x_n)|_1 := \sum_i |x_i|$. Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator, given by the matrix T_{ij}, that sends (x_i) to (y_j), where $y_i = \sum_j T_{ij} x_j$. How to compute $\|T\|$?

Proof: Expand $\|Tv\|_1 = \left\| \begin{bmatrix} | & | & & | \\ T_1 & T_2 & \cdots & T_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \right\|_1 = \| v_1 [T_1] + v_2 [T_2] + \cdots + v_n [T_n] \|_1$

$$\leq \| v_1 [T_1] \|_1 + \| v_2 [T_2] \|_1 + \cdots + \| v_n [T_n] \|_1$$

$$= |v_1| [T_1] + |v_2| [T_2]_1 + \cdots + |v_n| [T_n]$$

$$\leq (|v_1| + |v_2| + \cdots + |v_n|) \max(T_1, T_2, \cdots T_n)$$

$$= \|v\|_1 \times \max(|T_1|, |T_2|, \cdots |T_n|)$$

$$= \max(|T_1|_1, |T_2|_1, \cdots, |T_n|_1)$$

so, $\|T\| = \max \|t_i\|_1$ which is the max L_1 norm of columns.

1, 4, done