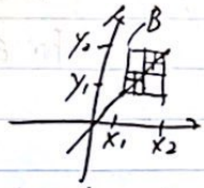
MATH 105
HW3
Q1. Ex3



Given a line segment $\{x_1, y_1\}, \{x_2, y_2\}$,
We can use an open box B to cover this line segment.
Then, we can divide the box into 4 parts, and reject the parts where it does not contain the line segment.
(In the right figure, top left and bottom right parts are rejected).
We can make the unrejected boxes has total area $\le \frac{\varepsilon}{2}$ for arbitrary $\varepsilon > 0$, and it still covers all points on the line segment.

We can repeat this process, and the boxes is still covering the line segment, with a total area $\le \frac{\varepsilon}{2^{n+1}}|B|$
$\because \sum \frac{\varepsilon}{2^{n+1}}|B| = \varepsilon$, so $m^*(\text{line segment } \{x_1, y_1\}, \{x_2, y_2\}) \le \varepsilon$ for all n, $\varepsilon > 0$.
thus $m^*(\text{line segment } \{x_1, y_1\}, \{x_2, y_2\}) = 0$, since ε is arbitrary.
Then extend $\{x_1, y_1\}, \{x_2, y_2\}$ to infinite long segment.
whatever how small/large x_1, x_2, y_1, y_2 are, we can still apply the same strategy as above. It has measure 0.

By zero slice theorem, if $E \subset \mathbb{R}^n \times \mathbb{R}^k$ is measurable, then E is a zero set if and only if almost every slice E_x is measure 0. So, we can write E as $\mathbb{R} \times \mathbb{R}^{n-1}$. Any straight line can be represented as an affine transformation on the $\mathbb{R}$ plane, which is measure 0.

Q2. Complete the proofs of theorems 16 and 21 in the unbounded, n-dimensional case.
~~Theorem 16: Every open set in n-space is a countable disjoint union of open cubes plus a zero~~ set.
Let ~~$U \subset \mathbb{R}^n$ be an open set, then we can divide~~ E in E_i, with i from dimension 1 to n. so, E is a disjoint union of all E_i.
Theorem 16: Lebesgue measure is regular : each measurable set E can be written as: $F \subset E \subset G$, F: F_σ set, G: G_δ set, and $G \setminus F$ is zero set.
"$\Rightarrow$" Now E is an unbounded, measurable set.
E can be divided into countably many bounded set E_n, by cutting E using unit open boxes, each E_n corresponds to an F_σ set F_n, and a G_δ set G_n, where $m^*(G_n \setminus F_n) = 0$.

Q2 cont'd). G_δ: countable intersection of open sets). (F_σ: countable union of closed sets).

Now, we first prove countable union of F_σ set is a F_σ set.

Proof: Intuitively, countable union of a countable union is still countable, so union of F_σ set is F_σ set.

For G_δ set, if $x \in \bigcup_m^\infty (\bigcap_n^\infty O_n^m)$, O is open set,

$\exists m_0$ s.t. $x \in \bigcap_n^\infty O_n^{m_0}$ (x belongs to one of the sets in union).

then $x \in O_n^{m_0}$ for all n (from intersection).

$\Rightarrow x \in \bigcup_m^\infty O_n^m$ for all n (taking union over m)

$\Rightarrow x \in \bigcap_n^\infty (\bigcup_m^\infty O_n^m)$

$\Rightarrow \bigcup_m^\infty (\bigcap_n^\infty O_n^m) \subseteq \bigcap_n^\infty (\bigcup_m^\infty O_n^m)$

$\forall x \in \bigcap_n^\infty (\bigcup_m^\infty O_n^m)$,

$\Rightarrow x \in \bigcup_m^\infty O_n^m$ for all n (since x belongs to an intersection)

$\Rightarrow \exists m_0$ s.t. $x \in O_n^{m_0}$ (choose a specific m_0)

$\Rightarrow x \in \bigcap_n^\infty O_n^{m_0}$

$\Rightarrow x \in \bigcup_m^\infty (\bigcap_n^\infty O_n^m)$

$\Rightarrow \bigcup_m^\infty (\bigcap_n^\infty O_n^m) \supseteq \bigcap_n^\infty (\bigcup_m^\infty O_n^m)$

combining both, $\bigcup_m^\infty (\bigcap_n^\infty O_n^m) = \bigcap_n^\infty (\bigcup_m^\infty O_n^m)$.

i.e. a countable union of G_δ set is G_δ set.

Having these, I note that each E_i can be sandwiched between F_σ and G_δ.

$$\underbrace{\bigcup_i F_i}_{\text{still } F_\sigma \text{ set}} \subset \bigsqcup_i E_i \subset \underbrace{\bigcup_i G_i}_{\text{still } G_\delta \text{ set}}.$$

$\Rightarrow \bigsqcup_i E_i$ measurable.

$\Rightarrow \because E = F \cup Z$ where Z is $E \cap (G \setminus F)$ is zero set.

$\Rightarrow E$ is measurable.

Theorem 21: If $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^k$ are measurable, then $A \times B$ is measurable and $m(A \times B) = m(A) \cdot m(B)$.

Proof: Now, A and B are unbounded, measurable set.

We can partition A, B into countably many bounded and measurable sets, denote as A_n and B_m. (using similar approach in thm 16).

Then, ~~$A\times B = \left(\bigcup_{n\in\mathbb{N}} A_n\right)\times\left(\bigcup_{m\in\mathbb{N}} B_m\right)$~~

$$m(A\times B) = m\left(\bigsqcup_{i,j\in\mathbb{N}} A_i\times B_j\right)$$

$$= \sum_{i,j\in\mathbb{N}} m(A_i\times B_j)$$

$$= \sum_{i,j\in\mathbb{N}} m(A_i)\, m(B_j)$$

$$= \sum_{i\in\mathbb{N}} m(A_i) \sum_{j\in\mathbb{N}} m(B_j)$$

$$= m(A)\cdot m(B). _{//}$$

Q3. Ex 12. Prove $J^*A = J^*\bar{A} = m\bar{A}$, $\bar{A}$ is closure of A.

($J^*A = \inf\{\sum_{k=1}^{n} |I_k| : \text{each } I_k \text{ is an open interval and } A\subset \bigcup_{k=1}^{n} I_k\}$).

-(Assume A is bounded)

Proof. $\because \bar{A}$ is a closed and bounded set.

claim 1: If A is compact then $mA = J^*A$ (Ex 11 c).

Def. of closure: $\bar{A} = \{x\in X : \text{for all } N(x), N(x)\cap A \neq \emptyset\}$.

claim 2: $\overline{\bigcup_{j=1}^{n} B_j} = \bigcup_{j=1}^{n} \bar{B_j}$

Part 1: $J^*A = J^*\bar{A}$

"$\leq$": by monotonicity, $J^*A \leq J^*\bar{A}$, (properties of outer Jordan measure).
$\because A\subseteq \bar{A}$

"$\geq$": Let $\{B_j\}_{j\in J}$ be a countable collection of open covers of A.

$$A\subset \bigcup_{j=1}^{n} B_j$$

$$\bar{A}\subset \overline{\bigcup_{j=1}^{n} B_j}$$

$$\Rightarrow \bar{A}\subset \bigcup_{j=1}^{n} \bar{B_j} \quad \text{(claim 2)}.$$

so, by monotonicity,

$$J^*\bar{A} \leq J^*\left(\bigcup_{j=1}^{n} \bar{B_j}\right)$$

$$\leq \sum_{j=1}^{n} J^*(\bar{B_j}) \quad \text{(finite subadditivity)}$$

$$\leq \sum_{j=1}^{n} m(\bar{B_j}) \quad \text{(by claim 1)}$$

$$= \sum_{j=1}^{n} m(B_j)$$

~~Proof of cla~~

- Proof of claim 1: If $A \subset \mathbb{R}^n$ is compact, then $J^*(A) = m(A)$.

" $\le$ ": let R be a rectangle containing A, $\varepsilon > 0$ be given.

We can find a countable open covering $\{B_k\}_{k\geq 1}$ of A s.t. $\sum_{k\geq 1} vol(B_k) < m(A) + \varepsilon$

Note that finite subcover of K determines a partition P of R so $J(A) \le m(A) + \varepsilon$

$\Rightarrow J(A) \le mA$

" $\geq$ ". let R be a rectangle containing A.

let P be any partition of R.

Consider the rectangle R_k that intersect A, by definition,

$$J^*(A) = \sum_{k\geq 1} vol(B_k)$$

let $\varepsilon > 0$ be arbitrary. then, by enlarging B_k slightly,

(i.e. $B_k = \prod_{i=1}^{k} (a_i, b_i)$, $B_k' = \prod_{i=1}^{k} (a_i - \frac{\varepsilon}{2^i}, b_i + \frac{\varepsilon}{2^i}))$

R_k' satisfies:

$$\sum_{k\geq 1} vol(B_k') = J^*(A) + \varepsilon$$

$\Rightarrow$ $m(A) \le J^*(A)$. since ε is arbitrary.

Combining both, we have $J^*(A) = m(A)$.

Proof of claim 2: $\overline{\cup B_j} = \cup \overline{B_j}$

Proof: $\because \bigcup_{j=1}^{n} B_j \subset \bigcup_{j=1}^{n} \overline{B_j}$ ($\because B_j \subset \overline{B_j}$). — ①

$\because$ $B_j \subset \bigcup_{j=1}^{n} B_j \subset \overline{\bigcup_{j=1}^{n} B_j}$

$\therefore$ $\overline{B_j} \subset \overline{\bigcup_{j=1}^{n} B_j}$ (closure = smallest closed set including the original one).

$\Rightarrow \bigcup_{j=1}^{n} \overline{B_j} \subset \overline{\bigcup_{j=1}^{n} B_j}$ — ②

~~$\because \bigcup_{j=1}^{n} B_j \subset \bigcup_{j=1}^{n} B_j$~~

From ① and definition of closure,

$\overline{\bigcup_{j=1}^{n} B_j} \subset \bigcup_{j=1}^{n} \overline{B_j}$ — ③

Combining ② & ③, $\overline{\cup B_j} = \cup \overline{B_j}$