MATH 105 HW 7

Pugh Ex 39. Prove fg is measurable integrable and $\int fg \le \sqrt{\int f^2}\sqrt{\int g^2}$

let c be a constant >0.

$$\int (cf+g)^2 \geq 0$$
$$\Rightarrow \int ((cf)^2 + g^2 + (2cfg)) \geq 0$$
$$\Rightarrow c^2\int f^2 + \int g^2 + 2c\int fg \geq 0$$
$$\Rightarrow (2\int fg)^2 - 4\int f^2 \int g^2 \leq 0 \quad \text{(from the determinant).}$$
$$\Rightarrow 4(\int fg)^2 \leq 4\int f^2 \int g^2$$
$$\int fg \leq \sqrt{\int f}\sqrt{\int g}$$

Q 48. Devil's ski slope.

First, show J is continuous.

We are given that H is nondecreasing.

$\Rightarrow \hat{H}$ is nondecreasing –

$\forall x \in [0,1]$, $3^k(x) \leq 3^k \quad \forall k \in \mathbb{Z}$, therefore

$\hat{H}(3^k x) \leq \hat{H}(3^k) = 3^k \quad \forall k \in \mathbb{Z}$

We note that $\sum_{k=0}^{\infty} (\frac{3}{4})^k$ converges implies $\sum_{k=0}^{\infty} \frac{\hat{H}(3^k x)}{4^k}$ converges.

Given H is continuous, $\hat{H}$ is also continuous since $\forall x \notin \mathbb{Z}$ and $\forall x \in \mathbb{Z}$, the limit from both directions are equal, then H_k is continuous.

$J = \sum_{k=0}^{\infty} \frac{H_k(x)}{4^k}$ is continuous for all k, for all $x \in [0,1]$.

so J is continuous.

Second, show J is strictly increasing.

$$J(x) = \sum_{k=0}^{\infty} \frac{H_k(x)}{4^k} = \sum_{k=0}^{\infty} \frac{\hat{H}(3^k x)}{4^k} = \sum_{k=0}^{\infty} \frac{H(3^k x - \lfloor 3^k x \rfloor) + \lfloor 3^k x \rfloor}{4^k}$$

then, we note that $\hat{H}(x)$ is a non decreasing function, as derived from H.

then, if for $x, y \in [0,1]$, $y - x > \frac{1}{3^n}$, y and x would lie in different intervals of values where $H(y) > H(x)$.

So $\hat{H}(y) > \hat{H}(x) \Rightarrow \hat{H}(3^k y) > \hat{H}(3^k x) \Rightarrow J(y) > J(x)$.

$\Rightarrow J$ is strictly increasing.

Show $J' = 0$ a.e. (don't know how to proceed).

My desired approach: (credit to Shuqi Ke):

Use differentiable limit theorem: let $f_n \to f$ pointwise on the closed interval $[a,b]$, and assume that each f_n is differentiable. If (f_n') converges uniformly on $[a,b]$ to a function g, then the function f is differentiable and $f' = g$.

Let $J_n'(x) := \sum_{k=0}^{n} \frac{H_k'(x)}{4^k}$

Proof: $J_n'(x) = 0$ a.e.

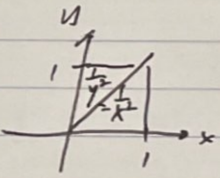
$J(x) = \lim_{n\to\infty} J_n(x)$, uniform convergence.

We need to prove if $H_0'(x) = 0$ a.e., then there is a null set W s.t. $\forall x \in \mathbb{R}\setminus W$, $\forall k \in \mathbb{N}$, $H_k'(x) = 0$.

and then use the above statement + differential limit theorem to give the final result.

Q13. Consider $f: \mathbb{R}^2 \to \mathbb{R}$: $f(x,y) = \begin{cases} \frac{1}{y^2} & \text{if } 0<x<y<1 \\ -\frac{1}{x^2} & \text{if } 0<y<x<1 \\ 0 & \text{else} \end{cases}$

a). Show that the iterated integrals exist and are finite but the double integral does not exist.



$$\begin{aligned}
&\int_0^1 \left[\int_0^1 f(x,y)\,dx\right] dy \\
&= \int_0^1 \left[\int_0^y f(x,y)\,dx + \int_y^1 f(x,y)\,dx\right] dy \\
&= \int_0^1 \left[\int_0^y \frac{1}{y^2}\,dx + \int_y^1 -\frac{1}{x^2}\,dx\right] dy \\
&= \int_0^1 \left[\frac{x}{y^2}\right]_0^y + \left[\frac{1}{x}\right]_y^1 dy \\
&= \int_0^1 \frac{1}{y} + 1 - \frac{1}{y}\,dy \\
&= \int_0^1 1\,dy = [y]_0^1 = 1
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \left[\int_0^1 f(x,y)\,dy\right] dx &= \int_0^1 \left[\int_0^x -\frac{1}{x^2}\,dy + \int_x^1 \frac{1}{y^2}\,dy\right] dx \\
&= \int_0^1 \left[\left[-\frac{y}{x^2}\right]_0^x - \left[\frac{1}{y}\right]_x^1\right] dx \\
&= \int_0^1 \left(-\frac{1}{x} - 1 + \frac{1}{x}\right) dx = \int_0^1 -1\,dx = -1
\end{aligned}$$

We know that if double integral exists, it will equal to $\int f(x,y)dxdy$ and $\int f(x,y)dydx$. But if one of the integration part does not exist, then the double integral does not exist.

b). In corollary 43, it assumed $f: \mathbb{R}^{2} \to [0,\infty)$. Therefore, the setting of this question sets output of f to be both positive and negative. So it does not contradict 43.

Q48. a). The density of E at x is defined as $\delta_B(x, E) = \lim_{r\to 0} \frac{m(E\cap Q)}{m(Q)}$

$\forall x \in dp(E)$, for every cube that contains x, we can find a ball Q that is contained in the cube, where the cubes need not to be centered at x.

Then, almost every point $x \in E$ is a balanced density point.

b). not sure how to proceed...

c). from balanced to a more general situation, we can take note that Q is not needed to be centered at x, so we can relax this requirement.

Q66. Construct a monotone function $f:[0,1]\to\mathbb{R}$ whose discontinuity set is exactly the set $\mathbb{Q}\cap[0,1]$, or prove that such a function does not exist.

Proof.

Credit to math.stackexchange.com/questions/1727513.

Let $\mathbb{Q}=\{q_n : n\in\mathbb{N}\}$ be an enumeration of the rational numbers, and define

$$f:\mathbb{R}\to\mathbb{R} : x\mapsto \sum_{q_n\le x}\frac{1}{2^n}$$

Note that f is strictly increasing, since:

if $x<y$, $\exists$ rational $q_n\in(x,y)$

and $f(y)\geq f(x)+\frac{1}{2^n}>f(x)$.

However, this function is discontinuous at every rational point:

$$\lim_{x\to q_n^-} f(x)=\sum_{q_k<q_n}\frac{1}{2^k}<\sum_{q_k\le q_n}\frac{1}{2^k}=f(q_n)$$

So, the function is discontinuous at every rational number.