Hw11

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1a Ω_{n-1} defined on $\mathbb{R}^n \setminus 0$

For $n=2$, $\Omega_1 = |x|^{-2}(x_1 dx_2 - x_2 dx_1) = (x_1^2+x_2^2)^{-1}(x_1 dx_2) - (x_1^2+x_2^2)^{-1}(x_2 dx_1)$

For $n=3$, $\Omega_2 = |x|^{-3}(x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2)$

a). compute $d\Omega_1$.

$$d\Omega_1 = \frac{\partial}{\partial x_1}|x|^{-2}(x_1 dx_2 - x_2 dx_1) - \frac{\partial}{\partial x_2}|x|^{-2}(x_1 dx_2 - x_2 dx_1)$$

$$= \{-(x_1^2+x_2^2)^{-2}\cdot 2x_1^2 + (x_1^2+x_2^2)^{-1}\} dx_1 \wedge dx_2 - \{-(x_1^2+x_2^2)^{-2}\cdot 2x_2^2 + (x_1^2+x_2^2)^{-1}\} dx_2 \wedge dx_1$$

$$= \frac{x_1^2 - x_2^2}{(x_1^2+x_2^2)^2} dx_1 \wedge dx_2 - \frac{x_1^2 - x_2^2}{(x_1^2+x_2^2)^2} dx_2 \wedge dx_1$$

$$= dx_1 \wedge dx_2 \left(\frac{x_2^2 - x_1^2}{(x_1^2+x_2^2)^2} \right) = 0$$

b). $\Omega_2 = |x|^{-3}(x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2)$

$$\Omega_2 = \frac{x_1}{(x_1^2+x_2^2+x_3^2)^{3/2}} dx_2 \wedge dx_3 - \frac{x_2}{(x_1^2+x_2^2+x_3^2)^{3/2}} dx_1 \wedge dx_3 + \frac{x_3}{(x_1^2+x_2^2+x_3^2)^{3/2}} dx_1 \wedge dx_2$$

$$d\Omega_2 = \frac{\partial}{\partial x_1}\left(\frac{x_1}{(x_1^2+x_3^2+x_3^2)^{3/2}}\right) dx_1 \wedge dx_2 \wedge dx_3 - \frac{\partial}{\partial x_2}\left(\frac{x_2}{(x_1^2+x_2^2+x_3^2)^{3/2}}\right) dx_2 \wedge dx_1 \wedge dx_3 + \frac{\partial}{\partial x_3}\left(\frac{x_3}{(x_1^2+x_2^2+x_3^2)^{3/2}}\right) dx_3 \wedge dx_2 \wedge dx_1$$

$$= \frac{x_2^2+x_3^2-2x_1^2}{(x_1^2+x_2^2+x_3^2)^{3/2}} dx_1 \wedge dx_2 \wedge dx_3 - \frac{x_1^2+x_3^2-2x_2^2}{(x_1^2+x_2^2+x_3^2)^{3/2}} dx_2 \wedge dx_1 \wedge dx_3 + \frac{x_1^2+x_2^2-2x_3^2}{(x_1^2+x_2^2+x_3^2)^{3/2}} dx_3 \wedge dx_2 \wedge dx_1 = 0\, dx_1 \wedge dx_2 \wedge dx_3 = 0$$

b). Expression for general n:

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$$\Omega_n = |x|^{-(n+1)}\left(\sum_{i=1}^{n}(-1)^{i+1} x_i\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots dx_n\right)$$

$$= \sum_{i=1}^{n+1} \frac{x_i(-1)^{i+1}}{|x|^{n+1}}\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots dx_n$$

Prove that $d\Omega_n = 0$.

$$\Rightarrow d\Omega_n = \sum_{i=1}^{n+1} \frac{\partial |x|^{-(n+1)} x_i}{\partial x_i} \cdot dx_1 \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^{n+1} \frac{|x|^{n+1} - x_i\left(\frac{n+1}{2}\right)|x|^{n-1} \cdot 2x_i}{\left(|x|^{2n+2}\right)}\Bigg)\, dx_1 \wedge \cdots \wedge dx_n$$

$$= \frac{1}{|x|^{2n+2}} \sum_{i=1}^{n+1}\left(|x|^{n+1} - (n+1)x_i^2|x|^{n-1}\right) dx_1 \wedge \cdots \wedge dx_n$$

$$= \frac{1}{|x|^{2n+2}}\left[(n+1)|x|^{n+1} - (n+1)\sum_{i=1}^{n+1} x_i^2|x|^{n-1}\Big)\right] dx_1 \wedge \cdots \wedge dx_n$$

$$= \frac{1}{|x|^{2n+2}}\left[(n+1)|x|^{n+1} - (n+1)|x|^{n+1}\right] dx_1 \wedge \cdots \wedge dx_n$$

$$= 0$$

Hence proved.

c). Given following 2-cell in $\mathbb{R}^3$:

$$\gamma : [0,1]^2 \to \mathbb{R}^3, \quad \gamma(s,t) = \begin{pmatrix} \sin(\pi s)\cos(2\pi t) \\ \sin(\pi s)\cos(2\pi t) \\ \cos(\pi s) \end{pmatrix}$$

$$\int_\gamma \Omega_2 = \int_\gamma \left\{ \frac{x_1}{|x|^3}\, dx_2 \wedge d3 - \frac{x_2}{|x|^3} dx_1 \wedge dx_3 + \frac{x_3}{|x|^3}\, dx_1 \wedge dx_2 \right\}$$

$$= \int_\gamma \left\{ x_1\, dx_2 \wedge dx_3 - x_2\, dx_1 \wedge dx_3 + x_3\, dx_1 \wedge dx_2 \right\} \quad — (*)$$

Let F is a vector field, $\vec{F} = (x_1, x_2, x_3)$ over a unique sphere

let σ be the same parametrization as $\sigma(u,v) \to (x_1, x_2, x_3)$,

the normal vector $\vec{n} = \sigma_u \times \sigma_v$

$$= \left(\begin{vmatrix} \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\ \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix}, -\begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_3}{\partial u} \\ \frac{\partial x_1}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix}, \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} \\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} \end{vmatrix} \right)$$

$$= \left(\begin{vmatrix} \frac{\partial x_2}{\partial u} & \frac{\partial x_3}{\partial u} \\ \frac{\partial x_2}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix}, - \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_3}{\partial u} \\ \frac{\partial x_1}{\partial v} & \frac{\partial x_3}{\partial v} \end{vmatrix}, \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} \\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} \end{vmatrix} \right)$$

then (*) can be rewritten as:

$$\int_u \int_v \vec{F} \cdot \vec{n} \, du \, dv$$

By divergence theorem,

$$= \iiint \operatorname{div} \vec{F} \, dx \wedge dy \wedge dz$$

$$= \iiint 3 \, dx \wedge dy \wedge dz$$

$$= 3 \iiint dV$$

$$= 3 \times \text{volume (sphere)}$$

$$= 3 \times \frac{4}{3} \pi$$

$$= 4\pi.$$