Hw1

2022年1月26日 15:01

7.2.1 – 7.2.4, 7.4.1 – 7.4.4

Exercise 7.2.1: Prove lemma 7.2.5.

Definition of outer measure: $\forall E \subset \mathbb{R}^n$,

$m^*(\Omega) = \inf\{\sum_{i=1}^{\infty} |B_i|, \bigcup_{i=1}^{\infty} B_i \supset E, B_i \subset \mathbb{R}^n$ are open boxes$\}$

v). Empty set: The empty set ϕ has outer measure $m^*(\phi) = 0$.

It is intuitive as no box is needed to cover an empty set.

vi). Positivity: Since each of the $\sum_{i=1}^{\infty} |B_i|$ has positive value, inf over $\sum_{i=1}^{\infty} |B_i|$ is also positive.

vii). Monotonicity: if $A \subseteq B \subseteq \mathbb{R}^n$, then $m^*(A) \leq m^*(B)$.

Proof: $\forall$ open cover $\{B_i\}$ of B, is also an open cover of A.

So $m^*(A) \leq \sum_{i=1}^{\infty} |B_i| < m^*(B) + \varepsilon \quad \forall \varepsilon > 0,$

$\Rightarrow m^*(A) \leq m^*(B)$

x). Countable sub-additivity: If $(A_j)_{j \in J}$ are a countable collection of subsets of $\mathbb{R}^n$, then $m^*(\bigcup_{j \in J} A_j) \leq \sum_{j \in J} m^*(A_j)$.

Proof: If $m^*(A_j) = +\infty$, then $\sum_{j \in J} m^*(A_j) = +\infty$, the inequality holds.

Now, if $m^*(A_j) < +\infty$, we can find a collection of open covers $\{B_i^{(j)}\}$ for A_j s.t.

$m^*(A_j) + \frac{\varepsilon}{\ } \geq \sum^{\infty} |B_i^{(j)}|$

covers $\{B_i^{(j)}\}$ ...

$$m^*(A_j) + \frac{\varepsilon}{2^j} \geq \sum_{i=1}^{\infty} |B_i^{(j)}|$$

$$m^*(\bigcup_{j\in J} A_j) \leq \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |B_i^{(j)}| < \sum_{k=1}^{\infty} m^*(A_{j_k}) + \sum_{k=1}^{\infty} \left(\frac{\varepsilon}{2^k}\right)$$

$$= \sum_{j\in J} m^*(A_j) + \varepsilon$$

$$\Rightarrow m^*(\bigcup_{j\in J} A_j) \leq \sum_{j\in J} m^*(A_j).$$

viii). Finite sub-additivity : If $(A_j)_{j\in J}$ are finite collection of subsets of $\mathbb{R}^n$, then $m^*(\bigcup_{j\in J} A_j) \leq \sum_{j\in J} m^*(A_j)$

Proof: We can prove the base case first, which is:

$$m^*(A\cup B) \leq m^*(A) + m^*(B).$$

Consider total area of covering of A and covering of B:

$M = \sum |C_i| + \sum |C_j|$ where C_i, C_j are covering for A and B respectively.

then $M \geq m^*(A\cup B)$.

so $\forall \varepsilon > 0$, $m^*(A) + m^*(B) \geq m^*(A\cup B) - \varepsilon$

$$\Rightarrow m^*(A) + m^*(B) \geq m^*(A\cup B)$$

$\because m^*(A) = \inf\{|C_i| \mid \{C_i\} \text{ cover } A\}$

$\therefore \forall \varepsilon > 0$, $\exists$ a covering $\{C_i\}$ s.t. $\sum_i C_i \leq m^*(A) + \varepsilon$

Similarly $\forall \varepsilon > 0$, $\exists$ a covering $\{C_j\}$ s.t. $\sum_j C_j \leq m^*(B) + \varepsilon$

$\therefore \forall \varepsilon > 0$, $\exists$ a covering $\{C_i\}$ s.t. ...

similarly, $\forall \varepsilon > 0$, $\exists$ a covering $\{C_j\}$ s.t. $\sum_j C_j \leq m^*(B) + \varepsilon$.

$\Rightarrow \{C_i\} \cup \{C_j\}$ is a cover of $A \cup B$.

Hence $m^*(A \cup B) \leq m^*(A) + m^*(B)$

Repeat the above for all $\bigcup_{j \in J} A_j$ and we are done.

XIII) Translation invariance:

If Ω is a subset of $\mathbb{R}^n$, and $x \in \mathbb{R}^n$, then $m^*(x + \Omega)$ $= m^*(\Omega)$

Proof: $\forall \varepsilon > 0$, we can find $B_i^{(i)}$ covers Ω, satisfying

$$\sum_{j \in J} |B_j| < m^*(\Omega) + \varepsilon$$

then, write B_j as $\prod_{i=1}^{n} (a_i^j, b_i^j)$

and we define $B_j' = \prod_{i=1}^{n} (a_i^j + x_i, b_i^j + x_i)$

$\Rightarrow |B_j| = |B_j'|$

Consider $V = \{x + \Omega \mid x \in \mathbb{R}^n, \Omega \in \mathbb{R}^n\}$ and $v \in V$

$\exists u \in \Omega$, we can find a B_k s.t.

$u \in B_k$, $v \in B_k'$

since B_j' covers V, so

$$m^*(V) \leq \sum_{j \in J} |B_j'| = \sum_{j \in J} |B_j| < m^*(\Omega) + \varepsilon$$

$m^* \cdots \sum_{j\in J} |\cdots| \quad \sum_{j\in J} |\cdots| \cdots$

$\Rightarrow m^*(x+\Omega) \leq m^*(\Omega) < +\infty$ — ①

note that we can swap V and Ω and yield the same result, namely:

$$m^*(\Omega) \leq \sum_{j\in J} |B_j| = \sum_{j\in J} |B_j'| < m^*(V) + \varepsilon.$$

$$\Rightarrow m^*(\Omega) \leq m^*(x+\Omega) < +\infty \quad ②$$

By ① & ②, $m^*(x+\Omega) = m^*(\Omega)$.

Ex 7.2.2. Let A be a subset of $\mathbb{R}^n$. Let B be a subset of $\mathbb{R}^m$. Want to show that $m^*_{n+m}(A\times B) \leq m^*_n(A)\, m^*_m(B)$.

Proof: First, we can assume $0 < m^*(A) < +\infty$ and $0 < m^*(B) < +\infty$,

because in $m^*(A) = m^*(B) = 0$, or $m^*(A) = m^*(B) = +\infty$, the inequality holds.

We can find collections of open covers $\{A_k\} \in \mathbb{R}^n$, $\{B_j\} \in \mathbb{R}^m$,

$$\sum_{k=1}^{\infty} |A_k| < m^*_n(A) + \frac{\varepsilon}{2}$$

$$\sum_{j=1}^{\infty} |B_j| < m^*_m(B) + \frac{\varepsilon}{2}$$

Since $A_k \times B_j$ is open box in $\mathbb{R}^{n+m}$, it is covers for

Since $A_k \times B_j$ is open box in $\mathbb{R}^{n+m}$, it is covers for $A \times B$.

We have $|A_k \times B_j| = |A_k||B_j|$

$$m^*_{n+m}(A \times B) \le \sum_{k,j=1}^{\infty} |A_k \times B_j|$$

$$= \sum_{k,j=1}^{\infty} |A_k|\,|B_j|$$

$$= \left(\sum_{k=1}^{\infty} |A_k|\right)\left(\sum_{j=1}^{\infty} |B_j|\right)$$

$$< m^*_n(A)\, m^*_m\, 3) - \quad T$$

$$< m^* + m^*_m(B) + \varepsilon$$

Since ε is arbitrary small, we have

$$m^*_{n+m}(A+B) \le m^*_n(A)\, m^*_m(1) \quad \text{Q.E.D.}$$

Ex .2.3 a). Show that if $A_1 \subseteq A_2 \subseteq A_3 \dots$ is an increasing sequence of measurable sets, then

$$m^*\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} m^*(A_j)$$

Proof: We can prove $m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \lim_{j\to\infty} m^*(A_j)$ and $m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \ge \lim_{j\to\infty} m^*(A_j)$, then equality holds.

For "$\leq$" direction, we have:

$$m^*\left(\bigcup_{j=1}^{\infty} A_j\right) = m^*\left(\bigcup_{j=1}^{n} A_j\right) + m^*\left(\bigcup_{j=n+1}^{\infty} A_j\right)$$

$$= \underbrace{m^*(A_n)}_{(\because\, A_1 \subseteq A_2 \subseteq A_3 \ldots)} + m^*\left(\bigcup_{j=n+1}^{\infty} A_j\right) \quad - (*)$$

Taking $\lim_{n\to\infty}$ to $(*)$,

$$m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \lim_{j\to\infty} m^*(A_j) + \lim_{n\to\infty} \underbrace{\left(\bigcup_{j=n+1}^{\infty} A_j\right)}_{\text{this is } \emptyset \text{ as } n \text{ approach } \infty}$$

$$\Rightarrow m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \lim_{j\to\infty} m^*(A_j) + 0$$

$$m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \lim_{j\to\infty} m^*(A_j)$$

For "$\geq$" direction, we have:

frall n, $A_n \subseteq \bigcup_{j=1}^{\infty} A_j$

$$m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \geq m^*(A_n) \quad - (**)$$

taking $\lim_{n\to\infty}$ to $(**)$,

$$m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \geq \lim_{n\to\infty} m^*(A_n)$$

$$m^*\left(\bigcup_{j=1}^{\infty} A_j\right) \geq \lim_{n\to\infty} m^*(A_n)$$

So, $m^*\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} m^*(A_j)$.

b). Show that if $A_1 \supseteq A_2 \supseteq A_3 \dots$ is a decreasing seq. of measurable sets, $m(A_1) < +\infty$, then $m^*\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} m^*(A_j)$

Proof: Basic Idea is to use $B_j = \mathbb{R}^n \setminus A_j$,

then $\bigcup_{i=1}^{\infty} B_i = \mathbb{R}^n \setminus \bigcap_{i=1}^{\infty} A_i$

$$m^*\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{i\to\infty} m^*(B_i) \quad \text{(Result in (a))}$$

$$\Leftrightarrow \quad m^*\left(\mathbb{R}^n \setminus \bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} m^*(\mathbb{R} \setminus A_i) \quad -(**)$$

Using the fact that

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E),$$

we have $m^*(\mathbb{R}^n) = m^*\left(\mathbb{R}^n \setminus \bigcap_{i=1}^{\infty} A_i\right) + m^*\left(\bigcap_{i=1}^{\infty} A_i\right)$

$$m^*(\mathbb{R}^n) = m^*(\mathbb{R}^n \setminus A_i) + m^*(A_i) \quad -(*)$$

notice that $\lim_{i\to\infty} m^*(\mathbb{R}^n) = m^*(\mathbb{R}^n)$ (no difference)

so we can take $\lim_{i\to\infty}$ to $(*)$,

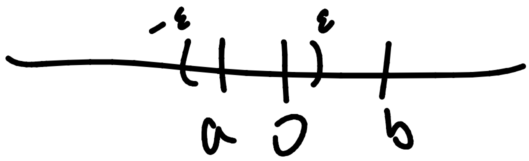
$$m^*(\mathbb{R}^n) = \lim_{i\to\infty} m^*(\mathbb{R}^n \setminus A_i) + \lim_{i\to\infty} m^*(A_i)$$

By (**),

$$m^*\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} m^*(A_i).$$

Ex 7.4.1. If A is an open interval in $\mathbb{R}$, show that
$$m^*(A) = m^*(A \cap (0,\infty)) + m^*(A \setminus (0,\infty))$$

Proof:



$A = (a,b)$.

(We omit the case of $0 \le a < b$, $a < b \le 0$, and $a = b$ here, as they are trivial as if $0 \le a < b$, then $A \setminus (0,\infty)$ is $\emptyset$ so $m^*(A \setminus (0,\infty)) = 0$, the equality holds).

In the case of $a < 0 < b$, notice $A \cap (0,\infty) = (0,b)$ and $A \setminus (0,\infty) = (a,0]$

Since $A \cap (0,\infty)$ is an open box,
$$m^*(A \cap (0,\infty)) = b - 0 = b$$
To handle a half-open box, we use ε-trick,

$$(a, -\varepsilon) \subseteq [a, 0] \subseteq (a, \varepsilon)$$

$$-\varepsilon - a \leq m^*((a, 0]) \leq \varepsilon - a$$

Since ε is arbitrary,

$$m^*((a, 0]) = -a.$$

Hence, $m^*(A) = b - a = m^*(A \cap (0, \infty)) + m^*(A \setminus (0, \infty))$

Ex 7.4.2 If A is an open box in $\mathbb{R}^n$, and E is the half plane, $E := \{(x_1, x_2, \dots x_n), x_n > 0\}$, show that $m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$

Proof. Consider the case of $n=1$ first.

Define $A_+ = A \cap (0, \infty)$, $A_- = A \cap (-\infty, 0]$

then $A \cap E = A_+$, $A \setminus E = A_-$.

① By sub-additivity, since $A = A_+ \sqcup A_-$,

$$m^*(A) \leq m^*(A_+) + m^*(A_-).$$

② w.t.s. $\forall \varepsilon > 0$, $m^*(A) + \varepsilon \geq m^*(A_+) + m^*(A_-)$

(then we'll have $m^*(A) \geq m^*(A_+) + m^*(A_-)$)

Consider an open cover of A by $\{B_j\}$ open boxes s.t.

Consider an open cover of A by $\{B_j\}$ open boxes s.t.

$$\sum |B_j| \le m^*(A) + \varepsilon/2$$

Define $B_j^+ = B_j \cap (0, \infty)$, $B_j^- = B_j \cap (-\infty, \frac{\varepsilon}{2^{j+1}})$

then $B_j = B_j^- \cup B_j^+$ and $|B_j| + \frac{\varepsilon}{2^{j+1}} \geq |B_j^+| + |B_j^-| \geq |B_j|$

$$\bigcup B_j^+ \supset A_+ \,,\quad \bigcup B_j^- \supset A_-$$

Thus $m^*(A^+) + m^*(A^-) \le \sum |B^+| + \sum |B_j^-| \le \sum_{j=1}^{\infty} \left(|B_j| + \frac{\varepsilon}{2^{j+1}}\right)$

$$\le \sum_{j=1}^{\infty} |B_j| + \frac{\varepsilon}{2} \le m^*(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = m^*(A) + \varepsilon$$

so $\quad m^*(A) \geq m^*(A_+) + m^*(A_-)$

Combine ① & ②, $m^*(A) = m^*(A_+) + m^*(A_-)$

Extending to Higher dimensional :

For "$\leq$" direction,

We can do the similar operation,

define a cut only on the n-th dimension space,

$n-1$

define a cut only on the n-th ...

So $A_{n-1} := \prod_{i=1}^{n-1} (a_i, b_i)$, $A' := (a_n, b_n)$

then $A \cap E = A_{n-1} \times (A' \cap (0, +\infty))$

$A \setminus E = A_{n-1} \times (A' \setminus (0, +\infty))$

we have $m^*(A \cap E) \leq m^*_{n-1}(A_{n-1})\, m_1^*(A' \cap (0, +\infty))$

$m^*(A \setminus E) \leq m^*_{n-1}(A_{n-1})\, m_1^*(A' \setminus (0, +\infty))$

$\Rightarrow m^*(A \cap E) + m^*(A \setminus E) \leq m^*_{n-1}(A_{n-1}) \big(m_1^*(A' \cap (0, +\infty))$

$+ m_1^*(A' \setminus (0, +\infty))\big)$

$= m^*_{n-1}(A_{n-1})\, m_1^*(A') = m^*(A)$

So $m^*(A \cap E) + m^*(A \setminus E) \leq m^*(A)$.

for "$\geq$" direction, just apply finite sub-additivity.

$\Rightarrow m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E)$

Combine both directions, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E).$$

Ex 7.4.3.

... are measurable :

Ex 1.1.7.

Prove half spaces are measurable :

half space : $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$.

proof: For A = open box in $\mathbb{R}^n$,

by finite sub-additivity,

$m^*(A) \le m^*(A \cap E) + m^*(A \setminus E)$

Proving "$\geq$" direction,

we can find open box B_k where

$$m^*\left(\bigcup_{k=1}^{\infty} B_k\right) \le m^*(A) + \varepsilon \quad \text{for all } \varepsilon > 0.$$

by exercise 1.4.2,

$$m^*(B_k) = m^*(B_k \cap E) + m^*(B_k \setminus E)$$

since $A \cap E \subseteq \bigcup_{k=1}^{\infty} (B_k \cap E)$,

$A \setminus E \subseteq \bigcup_{k=1}^{\infty} (B_k \setminus E)$,

$\Rightarrow m^*(A \cap E) \le \sum_k m^*(B_k \cap E)$

$m^*(A \setminus E) \le \sum_k m^*(B_k \setminus E)$ (countable subadditivity)

So, $m^*(A \cap E) + m^*(A \setminus E)$

$$\le \sum_k m^*(B_k \cap E) + \sum_k m^*(B_k \setminus E)$$

$$= \sum_k m^*(B_k \cap E) + \sum_k m^*(B_k \setminus E)$$

$$= \sum_k \left(m^*(B_k \cap E) + m^*(B_k \setminus E) \right)$$

$$= m^*(B_k)$$

$$\leq m^*(A) + \varepsilon .$$

Since ε is arbitrary,

$$m^*(A \cap E) + m^*(A \setminus E) \leq m^*(A)$$

Combining both directions, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E).$$

Ex 7.4.4, see another notes.

Lemma 7.4.4: (Properties of measurable set)

(a) If $E \subset \mathbb{R}^n$ is measurable, then $\mathbb{R}^n \setminus E$ is also measurable (by definition, same tests for E, E^c)

(b) Translation Invariance if E measurable, then $\forall x \in \mathbb{R}^n$, $x+E$ is also measurable. $(m(E)=m(x+E))$

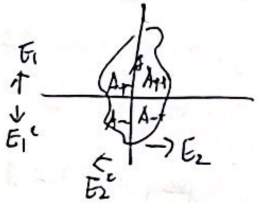
(Pf: outer measurable is translation invariance, more precisely,

$\forall A \subset \mathbb{R}^n,\ m^*(A) \overset{?}{=} m^*(A \cap (x+E)) + m^*(A \cap (x+E)^c)$

$\iff m^*(A-x) = m^*((A-x)\cap E) + m^*((A-x)\cap E^c)$ this holds.

(c) If E_1 and E_2 are measurable, then $E_1 \cap E_2$, $E_1 \cup E_2$ are measurable.

(Pf. WTS $\forall A \subset \mathbb{R}^n,\ m^*(A) = m^*(A \cap (E_1 \cap E_2)) + m^*(A \setminus (E_1 \cap E_2))$ (*)



Define $A_{++} = A \cap E_1 \cap E_2$, $A_{+-} = A \cap E_1 \cap E_2^c$

$A_{-+} = A \cap E_1^c \cap E_2^c$ $A_{--} = A \cap E_1^c \cap E_2^c$

$A = A_{++} \sqcup A_{+-} \sqcup A_{--} \sqcup A_{-+}$

(*) $\iff m^*(A) = m^*(A_{++}) + m^*(A_{+-} \cup A_{-+} \cup A_{--})$

We can show: $m^*(A) = m^*(\underbrace{A_{+-} \cup A_{++}}_{A \cap E_1}) + m^*(\underbrace{A_{-+} \cup A_{--}}_{A \cap E_1^c})$ (using E_1 measurable)

$m^*(A_{+-} \cup A_{++}) = m^*(A_{+-}) + m^*(A_{++})$. $\because E_2$ measurable

$m^*(A_{-+} \cup A_{--}) = m^*(A_{--}) + m^*(A_{-+})$

exercise $m^*(A_{+-} \cup A_{-+} \cup A_{--}) = m^*(A_{+-}) + m^*(A_{-+}) + m^*(A_{--})$

d). Boolean algebra: finite intersection/union preserve measurability

(prove using induction on number of operand)

e). Every box (open or closed) is measurable

($[a,b] = [a,\infty) \cap (-\infty, b]$)

boxes are intersections of translated half spaces

f). if $m^*(E) = 0$, then E is measurable.

(pf: $\forall A \subset \mathbb{R}^n$, $\overset{\text{wts}}{m^*(A)} = m^*(A \cap E) + m^*(A \setminus E)$

only need $m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E)$ (*)

$\because m^*(A \cap E) \leq m^*(E) = 0$, $\therefore m^*(A \cap E) = 0$

(*) $\Leftrightarrow$ $m^*(A) \geq m^*(A \setminus E)$ which is true by monotonicity

lemma 7.4.5 (finite additivity) if $E_1, \dots, E_n$ are disjoint measurable set, then

$\forall A \subset \mathbb{R}^n$, $E = \bigcup_{i=1}^{n} E_i$

$m^*(A \cap E) = \sum_{i=1}^{n} m^*(A \cap E_i)$

c). ~~A~~ Use a splitting several times

7.3 Jan 25 Presentation:

A is coset of $\mathbb{Q}$ if $A = x + \mathbb{Q}$ for some $x \in \mathbb{R}$.

① $x+\mathbb{Q}$ and $y+\mathbb{Q}$ are either disjoint or entirely the same

② Every coset has a non-empty intersection with $[0,1]$

Construction.

$x_A \in [0,1]$: respresentative of set of all cosets

$E = \{x_A\}$, $X = \bigcup_{q \in \mathbb{Q} \cap [-1,1]} (q+E)$

w.t.s. $[0,1] \subset X \subset [-1,2]$.

First, $X \subset [-1,2]$, $x \in X$ can be expressed as $x = q+e$, $q \in \mathbb{Q} \cap [-1,1]$, $e \in E$, so $x \in [-1,2]$.

$m^*(X) < 3$ by monotonicity

Second, for $[0,1] \subset X$, for any $x \in [0,1]$, $x \in X + \mathbb{Q}$, so $\exists y \in [0,1]$, $y \in E$ st. $x = y + q$ when $q \in \mathbb{Q}$.

Since $|x-y| \le 1$, $q \in [-1,1]$, so $x \in X$, Since $[0,1] \subset X$, $m^*(X) \ge 1$ by monotonicity.

by later we can show contradiction showing that $m^*(\bigcup_{j \in J} A_j) \neq \sum_{j \in J} m^*(A_j)$.

Lecture Measurable set

$A \subset \mathbb{R}^n$

Recall: $m^*(A) := \inf\{\sum_i |B_i|, \{B_i\}$ is countable collection of open boxes covering $A\}$

Properties of outer measure

- $m^*(\emptyset) = 0$
- $A \subset B \Rightarrow m^*(A) \le m^*(B)$ (monotonicity)
- countable sub-additivity if $A = \bigcup_{i=1}^{\infty} A_i$, then $m^*(A) \le \sum_{i=1}^{\infty} m^*(A_i)$

· Defn of measurable set: $E \subset \mathbb{R}^n$ is measurable iff $\forall A \subset \mathbb{R}^n$, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

Today:

Lemma (7.4.2) (half spaces in $\mathbb{R}^n$ are measurable)

i.e. $\{(x_1, \dots, x_n) \mid x_n > 0\}$ is measurable in $\mathbb{R}^n$.

Pf for $n=1$: w.t.s., $\forall A \subset \mathbb{R}$, $m^*(A) = m^*(A_+) + m^*(A_-)$ where $A_+ = A \cap (0,\infty)$, $A_- = A \cap (-\infty, 0]$

(1) first, $\because A = A_+ \sqcup A_-$, by sub-additivity, $m^*(A) \leq m^*(A_+) + m^*(A_-)$

(2) To show $m^*(A) \geq m^*(A_+) + m^*(A_-)$, it suffices to show

$\forall \varepsilon > 0, \quad m^*(A) + \varepsilon \geq m^*(A_+) + m^*(A_-)$

Consider an open cover of A by $\{B_j\}$ open boxes, s.t.

$\Sigma |B_j| \leq m^*(A) + \varepsilon/2$

Define $B_j^+ = B_j \cap (0, \infty)$, $B_j^- = B_j \cap (-\infty, \frac{\varepsilon}{2^{j+1}})$

then $B_j = B_j^- \cup B_j^+$, and $|B_j| + \frac{\varepsilon}{2^{j+1}} \geq |B_j^+| + |B_j^-| \geq |B_j|$

$\because \bigcup B_j^+ \supset A_+$, $\bigcup B_j^- \supset A_-$

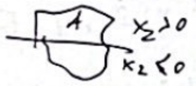
Thus, $m^*(A_+) + m^*(A_-) \leq \Sigma |B_j^+| + \Sigma |B_j^-| \leq \sum_{j=1}^{\infty} \left(|B_j| + \frac{\varepsilon}{2^{j+1}}\right)$

$\leq (\Sigma |B_j|) + \frac{\varepsilon}{2} \leq m^*(A) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = m^*(A) + \varepsilon$

Q.E.D.

Higher dimensional:

$A^+ = A \cap \{x_2 > 0\}$
$A^- = A \cap \{x_2 \leq 0\}$



Try $n=2$, w.t.s. $m^*(A) + \varepsilon \geq m^*(A_+) + m^*(A_-)$

One can do the same as above:

① get $\{B_j\}$ cover of A, with $m^*(A) + \frac{\varepsilon}{2} \geq \Sigma |B_j|$

② $B_j^+ = B_j \cap$

There is a better, more systematic approach Tao Ex 7.4.3.:

For $A =$ open box in $\mathbb{R}^n$, prove $m^*(A) = m^*(A_+) + m^*(A_-)$

$\because m^*(A) = |A|$, $m^*(A_+) = |A_+|$, by direct computation.

For general A, for any $\varepsilon > 0$, find $\{B_j\}$ cover of A

s.t. $m^*(A) + \varepsilon \geq \Sigma |B_j| = \sum_j |B_j^+| + |B_j^-| = (\Sigma |B_j^+|) + (\Sigma |B_j^-|)$

define $B_j^+ = B_j \cap \{x_n > 0\}$, $B_j^- = B_j \cap \{x_n \leq 0\}$ (may not be open)

$\because A_+ \subset \bigcup B_j^+$, $\therefore m^*(A_+) \leq \Sigma m^*(B_j^+) = \Sigma |B_j^+|$

$A_- \subset \bigcup B_j^-$, $\therefore m^*(A_-) \leq \Sigma m^*(B_j^-) = \Sigma |B_j^-|$

$\therefore m^*(A_+) + m^*(A_-) \leq m^*(A) + \varepsilon$

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