MATH 105 HW2

Def 2: A subset E is measurable, if for any $\varepsilon>0$, $\exists$ an open set $U \supset E$, s.t. $m^*(U\setminus E)<\varepsilon$

lemma 0: (Finite additivity for separated sets). Let $E, F \subset \mathbb{R}^d$ be such that $\operatorname{dist}(E,F)>0$, where $\operatorname{dist}(E,F) := \inf\{|x-y| : x\in E, y\in F\}$ is the distance between E and F.
Then $m^*(E\cup F) = m^*(E)+m^*(F)$

Proof: W.T.S. $m^*(E\cup F) \le m^*(E)+m^*(F)$ ①

$m^*(E\cup F) \geq m^*(E)+m^*(F)$. ②

① is obvious, by finite sub-additivity.

proving ②: let $\{B_j\}_{j\in J}$ be a countable collection of open boxes that covers $E\cup F$.

$m^*(E\cup F) = \inf\{\sum|B_j| : \{B_j\}_{j\in J} \text{ covers } E\cup F\}$

then. $\forall \varepsilon>0$, $\sum_{j\in J}|B_j| < m^*(E\cup F)+\varepsilon$

Then, we can split B_j into many subboxes, namely B_{jE} and B_{jF}, which represents the intersections with E and F respectively.

Then, $E \subseteq \bigcup_j\{B_{jE}\}$, $F \subseteq \bigcup_j\{B_{jF}\}$.

$$m^*(E)+m^*(F) \le m^*(\bigcup_j\{B_{jE}\}) + m^*(\bigcup_j\{B_{jF}\})$$
$$\le \sum_j m^*(B_{jE}) + \sum_j m^*(B_{jF})$$
$$\le \sum_j m^*(B_j)$$
$$\le m^*(E\cup F)+\varepsilon.$$

so $m^*(E\cup F) \geq m^*(E)+m^*(F)$

$\Rightarrow$ Combining ① and ②, we have $m^*(E\cup F) = m^*(E)+m^*(F)$.

lemma 1: let A be any subset of $\mathbb{R}^n$, then $m^*(A) = \inf\{m^*(U) \mid U\supset A, U \text{ is open}\}$.

Proof: let $\{B_j\}_{j\in J}$ be a countable collection of open boxes that covers U,

i.e. $m^*(U) = \inf\{\sum_{j\in J}|B_j|, B_j \text{ covers } U\}$.

$\Rightarrow m^*(U) \le \sum_{j\in J}|B_j|$

$\forall\varepsilon>0$,

$< m^*(A)+\varepsilon$

So, $m^*(A) \geq \inf\{m^*(U) \mid U\supset A, U \text{ is open}\}$

Furthermore, since $U\supset A$, by monotonicity,

$m^*(A) \le \inf\{m^*(U) \mid U\supset A, U \text{ open}\}$.

Combining both directions, $m^*(A) = \inf\{m^*(U) \mid U\supset A, U \text{ open}\}$. Q.E.D.

Lemma 2. W.T.S. If $\{E_i\}$ is a countable collection of measurable set, then $\bigcup_i E_i$ is measurable

Proof: E_i measurable $\xrightarrow[\text{Def 2.}]{\forall \varepsilon > 0,}$ $\exists$ open sets U_i s.t. $U_i \supset E_i$ and $m^*(U_i \setminus E_i) < \frac{\varepsilon}{2^i}$

Let $U = \bigcup_i U_i$, we have

$$U \setminus \bigcup_j E_j = \left(\bigcup_i U_i\right) \setminus \left(\bigcup_j E_j\right)$$

Consider $x \in \left(\bigcup_i U_i\right) \setminus \left(\bigcup_j E_j\right)$.

Note $x \in U_i \setminus E_j$ for some j.

So, $\left(\bigcup_i U_i\right) \setminus \left(\bigcup_j E_j\right) \subset \bigcup_{i,j} (U_i \setminus E_j)$.

By monotonicity,

$$m^*\left(\left(\bigcup_i U_i\right) \setminus \left(\bigcup_j E_j\right)\right) \leq m^*\left(\bigcup_{i,j} (U_i \setminus E_j)\right)$$

$$\leq \sum m^*(U_i \setminus E_j) \quad \text{(finite subadditivity)}$$

$$< \sum \frac{\varepsilon}{2^i} = \varepsilon.$$

Since ε is arbitrary, $\bigcup_i E_i$ is measurable.

Lemma 3: W.T.S. Every closed subset $A \subset \mathbb{R}^n$ is measurable.

(Cr. to Shuqi Ke, we can first prove ① A in $\mathbb{R}^n$ can be divided into a countable union of bounded closed subsets

② A is closed set $\Rightarrow$ A is measurable under original definition

③ Any closed and bounded set is measurable under original definition $\Rightarrow$ any closed and bounded set is measurable under new definition.

Proving ①: Partition A by each dimension. In dimension k, divide A_k as $\prod_{i=1}^{k} [n_i, n_i+1]$, note that it is also closed (it is the intersection of two closed set).

So, A can be written as $\bigcup_{n_i \in \mathbb{Z}} \left(A \cap \prod_{i=1}^{k} [n_i, n_i+1]\right)$

$\therefore$ Any closed set is a countable union of bounded closed subsets.

Proving ①.

From homework 1, E is measurable $\Rightarrow E^c$ is measurable (under the original definition),

From lecture 4, E is an open set $\Rightarrow E$ is measurable (under the original definition).

so, E is a closed set

$\Rightarrow E^c$ is open set

$\Rightarrow E^c$ is measurable

$\Rightarrow E$ is measurable.

Proving ③, W.T.S. if a bounded set E is measurable under original definition, E is also measurable under def 2.

Let $\{B_j\}_{j\in J}$ be a collection of open boxes covering E,

so we have, $\forall \varepsilon > 0$,

$$\sum_{j\in J} |B_j| < m^*(E) + \varepsilon.$$

Let $U = \bigcup_{j\in J} B_j$, by monotonicity.

$$m^*(U) \le \sum_{j\in J} |B_j| < m^*(E) + \varepsilon. \quad \text{—①}$$

$\because E$ is measurable under original definition.

$$\therefore m^*(E) = m^*(U\cap E) + m^*(U\cap E^c) \quad (E \subseteq U)$$
$$= m^*(E) + m^*(U\setminus E) \quad \text{—②}$$

From ①, ②, we have:

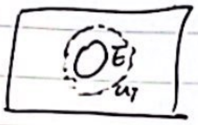
$$m^*(E) + m^*(U\setminus E) < m^*(E) + \varepsilon$$
$$m^*(U\setminus E) < \varepsilon.$$

$\Rightarrow E$ is measurable under def 2. QED.

Lemma 4. W.T.S. 'If E is measurable, then E^c is measurable.' (in def 2).

Proof: First (W.T.S) divide E^c as a countable union of closed sets. (Cr. to Jenchi)

by def 2, E is measurable $\Rightarrow \forall \varepsilon > 0$, $\exists U$ open set s.t. $m^*(U \setminus E) < \varepsilon$.



Note that $\forall x \notin \bigcup_{i=1}^{\infty} U_i^c$,

$\Rightarrow x \in \bigcap_{i=1}^{\infty} U_i$,

$\Rightarrow x \in (\bigcap_{i=1}^{\infty} U_i) \cap E^c$

$\Rightarrow x \in (\bigcap_{i=1}^{\infty} U_i) \setminus E$.

$$\therefore \quad E^c = (\underbrace{\bigcup_{i=1}^{\infty} U_i^c}_{}) \cup ((\bigcap_{i=1}^{\infty} U_i) \setminus E). \qquad -(*)$$

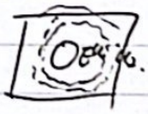
measurable since closed subsets are measurable

For $(\bigcap_{i=1}^{\infty} U_i) \setminus E$,

we can pick $i \in \mathbb{N}$, using def 2, we can find an open set $U_i \supset E$ s.t. $m^*(U_i \setminus E) < \frac{1}{i} < \varepsilon$. (valid by archimedean principle).

$$\bigcap_{i=1}^{\infty} (U_i) \setminus E = \bigcap_{i=1}^{\infty} (U_i \setminus E)$$

and $\bigcap_{i=1}^{\infty} (U_i \setminus E) \subset U_i \setminus E \quad \forall i$



by monotonicity, $m^*(\bigcap_{i=1}^{\infty} (U_i \setminus E)) \le m^*(U_i \setminus E)$

$< \frac{1}{i}$

$= 0$ when $i \to \infty$.

$\Rightarrow m^*((\bigcap_{i=1}^{\infty} U_i) \setminus E) = 0$.

$\Rightarrow (\bigcap_{i=1}^{\infty} U_i) \setminus E$ is measurable.

Back to $(*)$, since E^c is a (countable) union of measurable subsets, E^c is measurable. QED.