Hw10

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Q1. Rudin Ch9 Q12(a,b,c).

Fix two real numbers a, b, where $0<a<b$.

Define $f=(f_1,f_2,f_3)$ of $\mathbb{R}^2$ into $\mathbb{R}^3$ by:

$f_1(s,t)=(b+a\cos s)\cos t$

$f_2(s,t)=(b+a\cos s)\sin t$

$f_3(s,t)=a\sin s$, describe the range K of f.

(a) show that there are exactly 4 points $\vec{p}\in K$ s.t.

$(\nabla f_1)(f^{-1}(p))=0$

Proof: first, compute ∇f_1.

$$\nabla f_1=\begin{pmatrix}-a\sin s\cos t\\ -(b+a\cos s)\sin t\end{pmatrix}$$

setting $\nabla f_1=0$, we have

$-a\sin s\cos t=0$ or $-(b+a\cos s)\sin t=0$

$$\Rightarrow\begin{cases}\sin t=0 \text{ and } \sin s=0\\ \cos t=0 \text{ and } \cos s=-\frac{b}{a}\end{cases}$$

However, since $b>a$, $\cos s$ can't be $-\frac{b}{a}$,

so we have $s=n_1\pi$, $t=n_2\pi$

$f^{-1}(p)=(n_1\pi, n_2\pi)$

$$f(s,t)=\begin{pmatrix}(b+a\cos s)\cos t\\ 0\\ 0\end{pmatrix}$$

Therefore, the four fixed points are:

$$\begin{pmatrix}b+a\\0\\0\end{pmatrix},\begin{pmatrix}b-a\\0\\0\end{pmatrix},\begin{pmatrix}-b+a\\0\\0\end{pmatrix},\begin{pmatrix}-b-a\\0\\0\end{pmatrix}.$$

b). Determine the set of all $q\in K$ s.t.

$(\nabla f_3)(f^{-1}(q))=0$

First, compute ∇f_3.

$\nabla f_3=\Big($ $a\cos s$ $\Big)$

$$\nabla f_3 = \begin{pmatrix} a\cos s \\ 0 \end{pmatrix}$$

Setting $\nabla f_3 = 0$ we have

$$s = (\tfrac{1}{2}+n)\pi$$

$$f^{-1}(q) = \begin{pmatrix} (\frac{1}{2}+n)\pi \\ t \end{pmatrix}$$

$\Rightarrow$ the set is: $\left\{ \begin{pmatrix} b\cos t \\ b\sin t \\ \pm a \end{pmatrix} \right\}$.

c). Show that one of the points p found in part (a) correspond to a local maximum of f_1, one correspond to a local minimum, and other two are saddle points.

Proof: To study the maximum/minimum/saddle points, we first calculate $\nabla^2(f_1)$.

$$\nabla f_1 = \begin{pmatrix} -a\sin s\cos t \\ -(b+a\cos s)\sin t \end{pmatrix}$$

$$\nabla^2 f_1 = \begin{pmatrix} -a\cos s\cos t & a\sin s\sin t \\ a\sin s\sin t & -(b+a\cos s)\cos t \end{pmatrix}$$

To acquire local minimum, $\nabla^2 f_1$ needs to be P.S.D.

We have $|\nabla^2 f_1| \geq 0$ and $(\nabla^2 f_1)_{UL} \geq 0$

where UL is the upper left of $\nabla^2 f_1$.

i.e. $\begin{cases} -a\cos s\cos t \geq 0 \\ |\nabla^2 f_1| \geq 0 \end{cases}$ $-(*_1)$

$$|\nabla^2 f_1| = (-a\cos s\cos t)(-(b+a\cos s)\cos t) - (a\sin s\sin t)^2$$

$$= a(b+a\cos s)(\cos s\cos^2 t) - a^2\sin^2 s\sin^2 t$$

$$= ab\cos s\cos^2 t + a^2\cos^2 s\cos^2 t - a^2\sin^2 s\sin^2 t$$

$$- ab\cos s\cos^2 t + a^2\cos^2 s\cos^2 t - a^2\sin^2 s\sin^2 t$$

the only combination to fulfill $(\divideontimes_1)$ is $\begin{cases} \cos s = 1 \\ \cos t = -1 \end{cases}$

$\Rightarrow$ The minimum occurs at $\begin{pmatrix} -a-b \\ 0 \\ 0 \end{pmatrix}$.

To acquire local maximum, $\nabla^2 f_1$ needs to be negative semi definite., i.e.

$$\begin{cases} -a\cos s\cos t \leq 0 \\ |\nabla^2 f_1| \geq 0 \end{cases}$$

combining the above, the only combination to satisfy the above is $\begin{cases} \cos s = 1 \\ \cos t = 1 \end{cases}$

$\Rightarrow$ The maximum occurs at $\begin{pmatrix} a+b \\ 0 \\ 0 \end{pmatrix}$.

And for $\begin{pmatrix} a-b \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} b-a \\ 0 \\ 0 \end{pmatrix}$, since $|\nabla^2 f_1| < 0$, so they are saddle points.

Q9.13. Suppose f is a differentiable mapping of $\mathbb{R}^1$ into $\mathbb{R}^3$ s.t. $|f(t)| = 1$ for every t. Prove that $f'(t) \cdot f(t) = 0$

let $f = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix}$,

We have $|f(t)| = 1$

$$\sqrt{\sum_{i=1}^{3} f_i^2(t)} = 1$$

Taking partial derivatives w.r.t. to t on

[illegible] have:

Taking partial derivatives ...
both sides, we have:

$$2\sum_{i=1}^{3} f_i(t) f_i'(t) = 0$$

$$\text{so } f_i(t) \cdot f_i'(t) = 0$$

Q19. Show that the system of eqt.

$$3x + y - z + u^2 = 0 \quad - ①$$

$$x - y + 2z + u = 0 \quad - ②$$

$$2x + 2y - 3z + 2u = 0 \quad - ③$$

① - ② - ③ => $u^2 - 3u = 0 \Rightarrow u = 0$ or $u = 3$

Rewrite the system of eqt as matrix:

$$\text{then } \nabla f = \begin{bmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix}$$

Solving x, we have $\begin{bmatrix} 1 & -1 & 2u \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix} \overset{\text{row reduction}}{=} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2u+1 \\ 0 & 1 & 4 \end{bmatrix}$

It is invertible when $u = 0$ or $u = 3$, hence solvable.

Solving y, we have $\begin{bmatrix} 3 & -1 & 2u \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & 0 \\ 0 & -7 & 2u-3 \end{bmatrix}$

It is invertible when $u = 0$ or $u = 3$, hence solvable.

Solving z, we have $\begin{bmatrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & 0 \\ 0 & 4 & 2u-3 \end{bmatrix}$

It is invertible when $u = 0$ or $u = 3$, hence solvable.

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Solving u, we have $\begin{bmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & -7 \\ 0 & 0 & 0 \end{bmatrix}$

It is not invertible, hence not solvable

Ch5 Ex24 $f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{(3x^2y-y^3)}{x^2+y^2} - \frac{(x^3y-xy^3)(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x^3-2xy^2}{x^2+y^2} - \frac{(x^3y-xy^3)(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^5-4x^3y^2+x^3y^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$$

$$\frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0$$

$$\frac{\partial f}{\partial x\partial y} = \lim_{h\to 0} \frac{\frac{\partial f(0,h)}{\partial x} - \frac{\partial f(0,0)}{\partial x}}{h} = -1$$

$$\frac{\cdots}{\partial x \partial y} = \lim_{h \to 0} \frac{\frac{\cdots}{\partial x} - \frac{\cdots}{\partial x}}{h} = -1$$

$$\frac{\partial f}{\partial y \partial x} = \lim_{h \to 0} \frac{\frac{\partial f(h,0)}{\partial x} - \frac{\partial f(0,0)}{\partial x}}{h} = 1$$

therefore, partial derivatives exist but different.