MATH 605 Jan 27 Lecture 4

one remark about last homework: we need

$m^*(\text{any box}) = \text{volume of a box}$

we know $m^*(\text{closed box}) = \text{volume}$

$m^*(\text{open box}) = \text{volume}$

If we have any "half open half closed" box, B.

$B^\circ \subset B \subset \bar{B}$, then $vol(B^\circ) = m^*(B^\circ) \le m^*(B) \le m^*(\bar{B}) = vol(\bar{B})$

($\because vol(B^\circ) = vol(\bar{B})$, so $m^*(B) = vol(\bar{B}) = vol(B^\circ) = \prod_i (b_i - a_i)$)

Lemma 7.4.7: If $A, B$ are measurable, $A \subset B$, then $B \setminus A$ is measurable, and

$m^*(B\setminus A) = m^*(B) - m^*(A)$.

Proof: $B\setminus A = B \cap A^c$

$\because$ A is measurable $\Rightarrow \therefore A^c$ is measurable.

$\because$ B and $A^c$ are measurable, $\therefore B\cap A^c$ are measurable)

W.T.S. $m^*(B) = m^*(A) + m^*(B\setminus A)$

this follows from measurability of A, applied to test set B.

$\because m^*(B) = m^*(\underbrace{B\cap A}_{=A}) + m^*(B\cap A^c))$ #

Lemma 7.4.8 (Countable additivity). Let $\{E_j\}_{j=1}^\infty$ be a countable collection of disjoint measurable set. W.T.S.

$E = \bigcup_{j=1}^\infty E_j$ is measurable

$m^*(E) = \sum_{j=1}^\infty m^*(E_j)$

Proof: To prove measurability, we want to show, $\forall A \subset \mathbb{R}^n$, $m^*(A) = m^*(A\cap E) + m^*(A\setminus E)$

· Define $F_N = \bigcup_{j=1}^N E_j$. We know $F_N$ is measurable (finite union of meas. set)

$m^*(F_N) = \sum m^*(E_j)$

If we replace E by $F_N$, $\because E \supset F_N$, $E^c \subset F_N^c$.

$\therefore m^*(A\cap E) \geq m^*(A\cap F_N)$ (need fixing)

$m^*(A\cap E^c) \le m^*(A\cap F_N^c)$

To prove (*), we need "$\le$" and "$\geq$"

sub-additivity

$m^*(A\cap E) \le \sum_{j=1}^\infty m^*(A\cap E_j)$ by countable sub-additivity

$m^*(A \cap E) = \sup_N \left(\sum_{j=1}^{N} m^*(A \cap E_j)\right)$

$m^*(A \cap E) = \sup_N (m^*(A \cap F_N))$ (by finite additivity)

Thus: $m^*(A \cap E) + m^*(A \cap E^c) \le [\sup_N m^*(A \cap F_N)] + m^*(A \cap E^c)$

$\le \sup_N (m^*(A \cap F_N) + m^*(A \cap E^c))$

$\le \sup_N (m^*(A \cap F_N) + m^*(A \cap F_N^c))$

$= \sup_N [m^*(A)] = m^*(A)$

"$\le$":

$\cdot\, m^*(E) \le \sum_j m^*(E_j)$ by sub-additivity

$m^*(E) \geq m^*(F_N) = \sum_{j=1}^{N} m^*(E_j)$

↑ by monotonicity

sup over $N$, one have $m^*(E) \geq \sum_{j=1}^{\infty} m^*(E_j)$

$\therefore m^*(E) = \sum_{j=1}^{\infty} m^*(E_j)$ #

Prop. 7.4.9: The set of measurable sets forms a $\sigma$-algebra. i.e. given any collection $\Omega_j$ of measurable sets, $\bigcap_{j=1}^{\infty} \Omega_j$ and $\bigcup_{j=1}^{\infty} \Omega_j$ are measurable.

Proof: consider $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$, let $\Omega_0 = \phi$, $\Omega_N = \bigcup_{j=1}^{N} \Omega_j$.

let $E_N = \Omega_N \setminus \Omega_{N-1}$

then, $\{\Omega_N\}$ are measurable $\Rightarrow$ $\{E_N\}$ are measurable.

Since $\Omega = \bigcup_{j=1}^{\infty} E_j$, $\Omega$ is measurable.

$\bigcap_{j=1}^{\infty} \Omega_j = \left(\bigcup_{j=1}^{\infty} \Omega_j^c\right)^c$ $\because$ Complement & countable union preserve measurability

$\therefore$ this is measurable.

Lemma 7.4.10: All open sets in $\mathbb{R}^n$ can be written as a countable union of open boxes.

[recall some topology:

$\cdot$ topology for a metric space $(X, d)$.

- open ball $x \in X$, $r > 0$ real $B(x,r) = \{y \in X \mid d(y,x) < r\}$

$\cdot$ open sets in $X$ are generated from open balls, by taking finite intersection and arbitrary union.

- Equivalently, $U \subset X$ is open, iff $\forall x \in U$, $\exists r > 0$, s.t. $B(x,r) \subset U$.

$\cdot$ topology for product space: if $X, Y$ are top. spaces, then $X \times Y$ can be endowed with product topology, i.e. $W \subset X \times Y$, is open.

if $\forall (x,y) \in W$, $\exists U \subset X$, $V \subset Y$ is open. s.t. $(x,y) \in U \times V \subset W$.

topology on $\mathbb{R}^2$: can be generated by open balls (using Euclidean metric on $\mathbb{R}^2$)

can be generated by open box.

Proof: Consider the set of "rational boxes". A box $\prod_{i=1}^{n}(a_i,b_i)$ is rational if $a_1,b_1,\dots,a_n,b_n \in \mathbb{Q}$

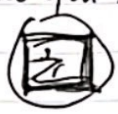
{the collection of rational open boxes} $\subset \mathbb{Q}^{2n}$ is countable

(∵ $\mathbb{Q}$ is countable, finite product of countable set is countable).

and subset of countable set is countable.

· Suffice to show that, every open set in $\mathbb{R}^n$ is a union of rational boxes, i.e. if $U$ is open, $x \in U$, we want to find a rational open box $B$, s.t. $x \in B(x,r) \subset U$

Let $s$ be a rational number, s.t.



$$s^2 \cdot n < r^2 \Leftrightarrow s < \frac{r}{\sqrt{n}}.$$

claim: $\exists$ rational box $B$, s.t. $x \in B \subset B(x,r)$

Prop: Open sets in $\mathbb{R}^n$ are measurable.

Proof: · open boxes are measurable

· a open set is a countable union of open boxes

Alternative definition of measurable set

Def 2: A subset $E \subset \mathbb{R}^n$ is measurable if $\forall \varepsilon > 0$, there exist an open set $U$, s.t. $U \supset$

$m^*(U \setminus E) < \varepsilon$.

In discussion: prove that all the properties of measurable sets can be derived using Def 2.