MATH 105 HW4

0. Summary about Lebesgue Integral, how we define it, how does it compare with Riemann integrals.

- Lebesgue Integrals are "opposite" approach to Riemann integrals, and it can be divided into the following steps:

1. Subdivide the range of function into infinitely many intervals
2. Construct a simple function by taking a function whose values are those finitely many numbers
3. take limit of these simple functions, when more points are added in the range of original functions.

Then, we defined characteristic function, to distinguish whether a given value x is in the measurable set A_i. And define simple function as the linear combination of characteristic functions.

And Lebesgue integral of $\varphi(x)$ is defined as $\int_E \varphi(x)dx = \sum_{i=1}^{n} a_i m(A_i)$

We define upper and lower Lebesgue integrals as:

$$I^*(f)_L = \int_E \inf\{\varphi(x)dx : \varphi \text{ is simple and } \varphi \geq f\} \text{ and}$$
$$I_*(f)_L = \int_E \sup\{\varphi(x)dx : \varphi \text{ is simple and } \varphi \leq f\} \text{ respectively.}$$

If $I^*(f)_L = I_*(f)_L$, then f is Lebesgue integrable over E.

Compared with Riemann integral, the steps for constructing Riemann integral are different.

1. Subdivide the domain of the function (usually a closed & bounded interval) into finitely many subintervals (the partition).
2. Construct a step function that has a constant value on each of the subintervals of the partition (the upper and lower sums)
3. take the limit of these step functions as adding more & more points to the partition.

1 done.

2. Ex25. Pugh..

Q25. a). Let $f: \mathbb{R} \to [0,\infty)$ be given.
If f is measurable why the graph of f a zero set?

Assume $f: \mathbb{R}^n \to \mathbb{R}$ is measurable.

W.T.S. $\{(x, f(x)) : x \in \mathbb{R}^n\}$ is zero set.

suffices to show when $f: B \to \mathbb{R}$ is measurable, and $B \subset \mathbb{R}^n$ is a box.
(this is true since $\mathbb{R}^n$ is a countable union of boxes).

Let $K := \{(x, f(x)) : x \in B\}$.

Let $F_n := K \cap (B \times [n, n+1])$, $n \in \mathbb{Z}$

suffices to show that F_n is zero set for every $n \in \mathbb{Z}$, so WLOG, suffice to show F_0 case.

We fix some $k \in \mathbb{N}$, let $I_j := [\frac{j}{k}, \frac{j+1}{k}]$, where $j = 0, \dots, k-1$.

then, we have:

$$m(K \cap (B \times I_j)) \le m(f^{-1}(I_j) \times I_j) \le \frac{1}{k} m(f^{-1}(I_j))$$

$$\Rightarrow m(F_0) = m(\bigcup_j K \cap (B \times I_j)) = \sum_j m(K \cap (B \times I_j)) \le \frac{1}{k} \sum_j m(f^{-1}(I_j)) = \frac{1}{k} m(\bigcup_j f^{-1}(I_j)) = \frac{1}{k} m(f^{-1}([0,1])) \le \frac{1}{k} m(B)$$

Since k is arbitrary chosen, $m(F_0) = 0$, and we proved that the graph of f is a zero set.

b). No. Consider indicator function $f: \mathbb{R} \to \{0,1\}$ where $f(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$, E is a nonmeasurable set.

So we have measure of graph of $f = 0$. (Since we only have to cover $f(x)=1$ and $f(x)=0$, but f is not measurable).

c). Sketch: Build a function using axiom of choice, whose graph is not contained in ~~a~~ any G_δ set with less than full measure.
So the graph has full outer measure. Meanwhile, the inner measure must always be 0, since there are countably many disjoint vertical

~~#(d)~~ translations. (stackexchange: Lebesgue Measure of the graph of function).

25d). Infer that the measurability hypothesis in ZST is necessary.

"If the measurability assumption is dropped, then ~~the slices~~ we can suppose that there exist a slice that of E is ~~non~~ zero set, iff ~~and~~ E is zero set., E does not have to be measurable.

So we can have a slice of nonmeasurable set E with ~~zero~~ measure 0.

Consider a function f, and the graph of function is nonmeasurable.

$\Rightarrow$ ~~the~~ measure of the graph is non zero

$\Leftrightarrow$ Contradict zero slice theorem's conclusion of measure $=0$.

e). A graph can never have positive inner measure, since every function graph has countably many disjoint vertical translations, which cover the plane.

If a function graph G of positive inner measure, we can pick a ~~set~~ measurable set of ~~positive~~ inner measure >0, and it is the subset of G. Considering the translate of this set, it is impossible to have many disjoint measurable sets of ~~zero~~ positive measure in ~~finite~~ measure space. Contradiction occurred.

Q28.

a). Show that total undergraph is measurable iff +ve, -ve of f are meas.

Pf: If P is the total undergraph of positive f, and N be the total undergraph of negative f. Then total undergraph of f is the union of P and N.

"$\Rightarrow$" If P & N are measurable, Uf is measurable.

"$\Leftarrow$" If Uf is measurable, we can find two half planes to cut Uf into P and N. Since Uf, half ~~mea~~ planes $(\{(x,y)\in\mathbb{R}^2 \mid y(>/\leq)0\})$ are measurable, P and N are both measurable.

b). Suppose $f:\mathbb{R}\to(0,\infty)$ measurable, $\frac{1}{f}$ is measurable.

Pf: Let $(x,y)\in U(f)$. , then $y<f(x)$.

$\Rightarrow \frac{1}{y} > \frac{1}{f(x)}$

$\Rightarrow T(x,y) \in (\hat{U}(\frac{1}{f}))^c$

Since T is a diffeomorphism, from exercise 23, it preserves measurability

$U(f)$ measurable $\Rightarrow (\hat{U}(\frac{1}{f}))^c$ measurable $\Rightarrow \hat{U}(\frac{1}{f})$ measurable $\Rightarrow U(\frac{1}{f})$ measurable, since the boundary is a measure zero set.

Q28(c) Suppose $f, g: \mathbb{R} \to (0, \infty)$ are measurable. Prove $f \cdot g$ is measurable.

We have f, g measurable functions.

$\Rightarrow$ $\mathcal{U}f$ and $\mathcal{U}g$ are measurable.

Consider $T: (x, y) \to (x, \log y)$

so $T_1: \mathcal{U}f \to \mathcal{U}\log f$, $T_2: \mathcal{U}g \to \mathcal{U}\log g$

Since T is diffeomorphism, it preserves measurability

So $\mathcal{U}\log f$ and $\mathcal{U}\log g$ are measurable.

Now, since $\log f, \log g: \mathbb{R} \to [0, \infty)$, so

$$\int \log f + \log g = \int \log f + \int \log g$$

$$\text{so } \mathcal{U}\log f + \mathcal{U}\log g = \mathcal{U}(\log f + \log g)$$
$$= \mathcal{U}(\log(f \cdot g)), \text{ so } \mathcal{U}(\log(f \cdot g)) \text{ measurable.}$$

Let $T^{-1}: (x, y) \to (x, e^y)$, it is a diffeomorphism, it also preserves measurability.

So $\mathcal{U}(\log(f \cdot g))$ measurable $\Rightarrow$ $\mathcal{U}(f \cdot g)$ measurable.

d). All statements for (a,b,c) did not depend on the fact that $x \in \mathbb{R}$, so conclusions are still valid.

e). Generalize (c) to the case that f, g have both signs.

Pf.: Define f^+ as $\mathbb{R} \to (0, \infty)$, f^- as $\mathbb{R} \to (-\infty, 0]$, for g similarly.

from (c), we know that $f^+ g^+$ is measurable.

So we can split $\mathcal{U}(fg)$ into $\mathcal{U}(f^+g^+) \cup \mathcal{U}(f^+g^-) \cup \mathcal{U}(f^-g^+) \cup \mathcal{U}(f^-g^-)$

Note that the domain for these four sliced pieces are measurable, adopting the similar approach in (c), by considering the sum of $\pm|f| \pm |g|$.

So, $\int f^+g^+ + f^+g^- + f^-g^+ + f^-g^- = \int fg$ is measurable

$\therefore$ fg is measurable for both signs.