MATH105.
HW5

Ex 8.2.7 Let $p > 2$ and $c > 0$. Using the Borel Cantelli lemma, show that the set $\{x \in [0,1] : |x - a/q| \le c/q^p$ for infinitely many positive integers $a, q\}$ has measure 0.

Since $p > 2$, if q is large enough, we can get $\frac{c}{q^p} < \frac{1}{2q}$

So we just need to consider $0 \le a \le q$, as $|x - \frac{a}{q}| \ge \frac{1}{2q} > \frac{c}{q^p}$ for $x \in [0,1]$.

Write E as $(\frac{a}{q} - \frac{c}{q^p}, \frac{a}{q} + \frac{c}{q^p})$.

then the intervals are: $(\frac{1}{q} - \frac{c}{q^p}, \frac{1}{q} + \frac{c}{q^p}), (\frac{2}{q} - \frac{c}{q^p}, \frac{2}{q} + \frac{c}{q^p}), \dots, (\frac{q}{q} - \frac{c}{q^p}, \frac{q}{q}$ ...

So the measure of the given set is $(q-1) \times \frac{2c}{q^p} + \frac{c}{q^p}$

$$= (2q-1)\frac{c}{q^p}.$$

$\sum_{q=1}^{\infty} (2q-1)\frac{c}{q^p}$ converges by p-test, $p > 2$.

So by Borel Cantelli lemma, the given set has measure $= 0$.

Ex 8.2.9. $\forall n \in \mathbb{N}$, let $f_n : \mathbb{R} \to [0, \infty)$ be nonnegative measurable function s.t. $\int_{\mathbb{R}} f_n \le \frac{1}{4^n}$, show $\forall \varepsilon > 0$, $\exists$ a set E of $m(E) \le \varepsilon$ s.t. $f_n(x)$ converges pointwise to 0 $\forall x \in \mathbb{R} \setminus E$.

Proof: ~~Let $E = \{x \in \mathbb{R} \mid f_n(x)$ not tends to 0 when $n \to \infty\}$.~~
~~Showing E has measure 0:~~
~~according to Archimedian principle, $\forall x \in E$, $\forall N \in \mathbb{N}$, $\exists \varepsilon > 0$ s.t. $f_n(x) \ge \varepsilon$.~~ ~~$n > N$~~

Given $\int_R f_n \le \frac{1}{4^n}$, so f_n is measurable on R.
$m(f(R)) \le \frac{1}{4^n}$

There exist $\varepsilon > 0$ and $n \in \mathbb{N}$ let $A = \{x \in \mathbb{R} : f_n(x) > \frac{1}{\varepsilon 2^n}\}$.

Then $A_n = f_n^{-1}((\frac{1}{\varepsilon 2^n}, +\infty])$, $\int_R f_n(A_n) = m(f_n(A_n)) \le \int_{\mathbb{R}} f_n \le \frac{1}{4^n}$

(Cr. to shuqike): Then, define $s_n = \begin{cases} \frac{1}{\varepsilon 2^n} & x \in A_n \\ 0 & x \notin A_n \end{cases}$

Note that for all $x \in A_n$, $s_n \le f_n$.

$\int_{\mathbb{R}} s_n = m(A_n) \cdot \frac{1}{\varepsilon 2^n} \le \frac{1}{4^n} \iff m(A_n) \le \frac{\varepsilon}{2^n}$

So, we have shown that for $\varepsilon > 0$, $\forall n \ge 1$,
$m(\{x \in \mathbb{R} : f_n(x) > \frac{1}{\varepsilon 2^n}\}) \le \frac{\varepsilon}{2^n}$.

Consider A now. $A = \bigcup_{n=1}^{\infty} A_n$ for $\varepsilon > 0$.

Since A_n is measurable, A is measurable.

By sub-additivity, $m(A) \leq \sum_{n=1}^{\infty} m(A_n) \leq \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \varepsilon$.

Then, consider $\mathbb{R} \setminus A$. for all $x' \in \mathbb{R} \setminus A$, $x' \notin A$,

there exist $N \geq 1$ s.t. $\forall n \geq N$, $f_n(x') \leq \frac{1}{\varepsilon 2^n} < \varepsilon$, $N \in \mathbb{N}$.

and we have $|f_n(x') - 0| = f_n(x) < \varepsilon$.

so $f_n(x)$ converges pointwise to 0 for $x \in \mathbb{R} \setminus A$

8.2.10. Show that $\forall \varepsilon > 0$, $\exists E$ of $m(E) \leq \varepsilon$ s.t. $f_n(x)$ converges uniformly to 0 $\forall x \in [0,1] \setminus E$.

Proof: For $\varepsilon > 0$, let $A_n = \{x \in [0,1] \mid f_n(x) \leq \varepsilon\}$

where $A_1 \subset A_2 \subset A_3 \ldots$

$\Rightarrow$ by monotonicity, $m(A_1) \leq m(A_2) \leq m(A_3) \leq \cdots$

Given f_n converges pointwise to 0.

$\forall x \in [0,1]$, $\exists N$ s.t. $\forall n > N$, $f_n(x) \leq \varepsilon$.

$\Rightarrow \lim_{n \to \infty} A_n = [0,1]$

$\Rightarrow \lim_{n \to \infty} m(A_n) = m([0,1]) = 1$.

Rewrite the above statement using def. of limit:

$\exists M \in \mathbb{N}$ s.t. $|m(A_m) - 1| < \varepsilon$.

$\Rightarrow$ since $\varepsilon > 0$, $m(A_m) > 1 - \varepsilon$.

Define $A_m' = \{x \in [0,1] \mid f_n(x) > \varepsilon \ \forall n > M\}$.

$m(A_m') + m(A_m) = 1$. (as $\lim m(A_n) = 1$)

$\Rightarrow 1 - \varepsilon < 1 - m(A_m')$

$\Rightarrow m(A_m') < \varepsilon$.

$\forall x \in [0,1] \setminus A_m'$, $n > M \Rightarrow f_n(x) \leq \varepsilon$.

$\therefore$ Proved $f_n(x)$ converges uniformly to 0 $\forall x \in [0,1] \setminus E$. QED.