Method of least squares

Simple linear regression $p = 1$

Model $\quad y_i = \boxed{\beta_0} + \boxed{\beta_1} x_{i1} + e_i \qquad i = 1, \text{—}, n$

Find $\hat\beta_0, \hat\beta_1$ s.t.

$$\boxed{\underbrace{\sum_{i=1}^{n} (\underbrace{y_i - \hat y_i}_{\text{Residual}})^2}_{\text{Res S.S.}}} \text{ is minimized} \qquad \hat y_i = \hat\beta_0 + \hat\beta_1 x_{i1}$$

$$\frac{\partial}{\partial \hat\beta_0} \sum_{i=1}^{n} \left(y_i - (\hat\beta_0 + \hat\beta_1 x_{i1})\right)^2 = 0 \qquad \text{———} \ (1)$$

$$\frac{\partial}{\partial \hat\beta_1} \sum_{i=1}^{n} \left(y_i - (\hat\beta_0 + \hat\beta_1 x_{i1})\right)^2 = 0 \qquad \text{———} \ (2)$$

$(1) \Rightarrow \quad 2 \sum_{i=1}^{n} \left(y_i - (\hat\beta_0 + \hat\beta_1 x_{i1})\right)(-1) = 0$

$$\sum_{i=1}^{n} \left(y_i - (\hat\beta_0 + \hat\beta_1 x_{i1})\right) = 0$$

$$\sum_{i=1}^{n} y_i - n\hat\beta_0 - \hat\beta_1 \sum_{i=1}^{n} x_{i1} = 0$$

$$n\hat\beta_0 + \hat\beta_1 \sum_{i=1}^{n} x_{i1} = \sum_{i=1}^{n} y_i \qquad \text{———} \ (3)$$

$$\hat\beta_0 + \hat\beta_1 \bar x_1 = \bar y$$

$$\hat\beta_0 = \bar y - \hat\beta_1 \bar x_1$$

$(2) \Rightarrow \quad 2 \sum_{i=1}^{n} \left(y_i - (\hat\beta_0 + \hat\beta_1 x_{i1})\right)(-x_{i1}) = 0$

$$\sum_{i=1}^{n} y_i x_{i1} - \hat\beta_0 \sum_{i=1}^{n} x_{i1} - \hat\beta_1 \sum_{i=1}^{n} x_{i1}^2 = 0$$

$$\hat\beta_0 \sum_{i=1}^{n} x_{i1} + \hat\beta_1 \sum_{i=1}^{n} x_{i1}^2 = \sum_{i=1}^{n} y_i x_{i1} \qquad \text{———} \ (4)$$

$\Rightarrow \quad \overset{n\bar x_1}{\sum_{i=1}^{n} x_{i1}} (\bar y - \hat\beta_1 \bar x_1) + \hat\beta_1 \sum_{i=1}^{n} x_{i1}^2 = \sum_{i=1}^{n} y_i x_{i1}$

$\Rightarrow \quad \cancel{\sum_{i=1}^{n}} \, n \bar x_1 \bar y - n\hat\beta_1 \bar x_1^2 + \hat\beta_1 \sum_{i=1}^{n} x_{i1}^2 = \sum_{i=1}^{n} y_i x_{i1}$

$\Rightarrow \quad \hat\beta_1 \left(\sum_{i=1}^{n} x_{i1}^2 - n\bar x_1^2 \right) = \sum_{i=1}^{n} y_i x_{i1} - n \bar x_1 \bar y$

$\Rightarrow \quad \hat\beta_1 = \dfrac{\sum_{i=1}^{n} y_i x_{i1} - n\bar x_1 \bar y}{\sum_{i=1}^{n} x_{i1}^2 - n\bar x_1^2} = \dfrac{\sum_{i=1}^{n} (y_i - \bar y)(x_{i1} - \bar x_1)}{\sum_{i=1}^{n} (x_{i1} - \bar x_1)^2} = \dfrac{S_{x_1 y}}{S_{x_1 x_1}} \qquad (I)$

$$\sum_{i=1}^{n}(x_{i1}-\bar{x}_1)^2 = \sum_{i=1}^{n}x_{i1}^2 - 2\bar{x}_1\underbrace{\left(\sum_{i=1}^{n}x_{i1}\right)}_{=n\bar{x}_1} + n\bar{x}_1^2$$

$$= \sum_{i=1}^{n}x_{i1}^2 - n\bar{x}_1^2$$

$$\sum_{i=1}^{n}(y_i-\bar{y})(x_{i1}-\bar{x}_1) = \sum_{i=1}^{n}(y_i x_{i1} - \bar{y}x_{i1} - \bar{x}_1 y_i + \bar{x}_1\bar{y})$$

$$= \sum_{i=1}^{n}y_i x_{i1} - \bar{y}\underbrace{\sum_{i=1}^{n}x_{i1}}_{n\bar{x}_1} - \bar{x}_1\underbrace{\sum_{i=1}^{n}y_i}_{n\bar{y}} + n\bar{x}_1\bar{y}$$

$$= \sum_{i=1}^{n}y_i x_{i1} - n\bar{x}_1\bar{y}$$

$$\Downarrow$$

$$\sum_{i=1}^{n}(x_{i1}-\bar{x}_1)y_i - \boxed{\sum_{i=1}^{n}(x_{i1}-\bar{x}_1)\bar{y}}$$

$$= \bar{y}\boxed{\sum_{i=1}^{n}(x_{i1}-\bar{x}_1)} = 0$$

$$\bar{x}_1 = \frac{\sum_{i=1}^{n}x_{i1}}{n}$$

$$(3) \Rightarrow n\hat{\beta}_0 + \hat{\beta}_1\sum_{i=1}^{n}x_{i1} = \sum_{i=1}^{n}y_i$$

$$(4) \Rightarrow \hat{\beta}_0\sum_{i=1}^{n}x_{i1} + \hat{\beta}_1\sum_{i=1}^{n}x_{i1}^2 = \sum_{i=1}^{n}y_i x_{i1}$$

$$\Rightarrow \begin{pmatrix} n & \sum_{i=1}^{n}x_{i1} \\ \sum_{i=1}^{n}x_{i1} & \sum_{i=1}^{n}x_{i1}^2 \end{pmatrix}\begin{pmatrix}\hat{\beta}_0 \\ \hat{\beta}_1\end{pmatrix} = \begin{pmatrix}\sum_{i=1}^{n}y_i \\ \sum_{i=1}^{n}y_i x_{i1}\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\hat{\beta}_0 \\ \hat{\beta}_1\end{pmatrix} = \begin{pmatrix} n & \sum_{i=1}^{n}x_{i1} \\ \sum_{i=1}^{n}x_{i1} & \sum_{i=1}^{n}x_{i1}^2 \end{pmatrix}^{-1}\begin{pmatrix}\sum_{i=1}^{n}y_i \\ \sum_{i=1}^{n}y_i x_{i1}\end{pmatrix}$$

$$\det = n\sum_{i=1}^{n}x_{i1}^2 - \left(\sum_{i=1}^{n}x_{i1}\right)^2$$

$$= n S_{x_1x_1}$$

For any p $\quad \sum_{i=1}^{n} \left(y_i - (\hat\beta_0 + \hat\beta_1 x_{i1} + \cdots + \hat\beta_p x_{ip}) \right)^2$

$$\frac{\partial}{\partial \hat\beta_0} \sum_{i=1}^{n} \left(y_i - (\hat\beta_0 + \hat\beta_1 x_{i1} + \cdots + \hat\beta_p x_{ip}) \right)^2 = 0 \Rightarrow \sum_{i=1}^{n} \left(y_i - \underbrace{(\hat\beta_0 + \hat\beta_1 x_{i1} + \cdots + \hat\beta_p x_{ip})}_{\hat y_i} \right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \hat e_i = 0$$

$$\frac{\partial}{\partial \hat\beta_1} \sum_{i=1}^{n} \left(y_i - (\hat\beta_0 + \hat\beta_1 x_{i1} + \cdots + \hat\beta_p x_{ip}) \right)^2 = 0 \Rightarrow \sum_{i=1}^{n} \left(y_i - \underbrace{(\hat\beta_0 + \hat\beta_1 x_{i1} + \cdots + \hat\beta_p x_{ip})}_{\hat y_i} \right)(x_{i1}) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \hat e_i x_{i1} = 0$$

$$\vdots$$

$$\frac{\partial}{\partial \hat\beta_p} \sum_{i=1}^{n} \left(y_i - (\hat\beta_0 + \hat\beta_1 x_{i1} + \cdots + \hat\beta_p x_{ip}) \right)^2 = 0 \Rightarrow \sum_{i=1}^{n} \left(y_i - (\hat\beta_0 + \hat\beta_1 x_{i1} + \cdots + \hat\beta_p x_{ip}) \right)(x_{ip}) = 0$$

$$\Rightarrow \sum_{i=1}^{n} \hat e_i x_{ip} = 0$$

Properties of $\hat e_i$

$$\boxed{\begin{array}{l} \sum_{i=1}^{n} \hat e_i = 0 \\ \sum_{i=1}^{n} x_{ij} \hat e_i = 0 \qquad j = 1, \ldots, p \end{array}} \quad *$$

$$\Rightarrow \begin{array}{l} \sum_{i=1}^{n} \hat e_i = 0 \quad 1* \\ \sum_{i=1}^{n} x_{i1} \hat e_i = 0 \\ \vdots \\ \sum_{i=1}^{n} x_{ip} \hat e_i = 0 \end{array} \Rightarrow \underbrace{\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_{11} & x_{21} & \cdots & x_{n1} \\ \vdots & & & \vdots \\ x_{1p} & x_{2p} & \cdots & x_{np} \end{pmatrix}}_{X^T} \begin{pmatrix} \hat e_1 \\ \hat e_2 \\ \vdots \\ \hat e_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$(p+1) * n \qquad n \times 1 \qquad (p+1) \times 1$

Model

$$\underset{n\times 1}{\underset{\sim}{Y}} = \underset{n\times(p+1)}{\underset{\sim}{X}} \underset{(p+1)\times 1}{\underset{\sim}{\beta}} + \underset{n\times 1}{\underset{\sim}{e}}$$

design matrix

values of X_1 $\qquad$ values of X_p

$$\begin{pmatrix} 1 & x_{11} & \cdots & x_{1p} \\ 1 & x_{21} & & x_{2p} \\ \vdots & \vdots & & \vdots \\ 1 & x_{n1} & \cdots & x_{np} \end{pmatrix}$$

coeff. of intercept, β_0

$\Rightarrow \underset{\sim}{X}^T \hat{\underset{\sim}{e}} = \underset{\sim}{0}$ obs. fitted value $\hat{e}_i = y_i - \hat{y}_i$

$\Rightarrow \underset{\sim}{X}^T (\underset{\sim}{Y} - \hat{\underset{\sim}{Y}}) = 0$ $\hat{\underset{\sim}{Y}} = \underset{\sim}{X} \hat{\underset{\sim}{\beta}}$

$\Rightarrow \underset{\sim}{X}^T (\underset{\sim}{Y} - \underset{\sim}{X} \hat{\underset{\sim}{\beta}}) = 0$

$\Rightarrow \underset{\sim}{X}^T \underset{\sim}{Y} - \underset{\sim}{X}^T \underset{\sim}{X} \hat{\underset{\sim}{\beta}} = 0$

$\Rightarrow \hat{\underset{\sim}{\beta}} = (\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \underset{\sim}{X}^T \underset{\sim}{Y}$

OR $\sum_{i=1}^{n} \hat{e}_i^2 = (\hat{e}_1, \cdots, \hat{e}_n) \begin{pmatrix} \hat{e}_1 \\ \vdots \\ \hat{e}_n \end{pmatrix}$

$\hat{e}_i = y_i - \hat{y}_i$

$\hat{\underset{\sim}{e}} = \underset{\sim}{Y} - \hat{\underset{\sim}{Y}}$

$\hat{\underset{\sim}{Y}} = \underset{\sim}{X} \hat{\underset{\sim}{\beta}}$

$$= (\underset{\sim}{Y} - \hat{\underset{\sim}{Y}})^T (\underset{\sim}{Y} - \hat{\underset{\sim}{Y}})$$

$$= (\underset{\sim}{Y} - \underset{\sim}{X} \hat{\underset{\sim}{\beta}})^T (\underset{\sim}{Y} - \underset{\sim}{X} \hat{\underset{\sim}{\beta}})$$

$$= (\underset{\sim}{Y}^T - \hat{\underset{\sim}{\beta}}^T \underset{\sim}{X}^T)(\underset{\sim}{Y} - \underset{\sim}{X} \hat{\underset{\sim}{\beta}})$$

$\underset{n \times (p+1)}{\underset{\sim}{X}}$ $\underset{n \times 1}{\underset{\sim}{Y}}$

$$= \underset{\sim}{Y}^T \underset{\sim}{Y} - \hat{\underset{\sim}{\beta}}^T \boxed{\underset{\sim}{X}^T \underset{\sim}{Y}} - \underset{\sim}{Y}^T \underset{\sim}{X} \hat{\underset{\sim}{\beta}} + \hat{\underset{\sim}{\beta}}^T \underset{\sim}{X}^T \underset{\sim}{X} \hat{\underset{\sim}{\beta}}$$

(a) Use p.4 for the formula $(p+1) * 1$

(b)
2nd & 3rd terms

$\underset{\sim}{A} = (a_0, a_1, \cdots, a_p)^T$

$\frac{\partial (\ )}{\partial \hat{\beta}_0}$ $a_0 \hat{\beta}_1 + a_1 \hat{\beta}_1 + \cdots + a_p \hat{\beta}_p$

$\vdots$

$\frac{\partial (\ )}{\partial \hat{\beta}_p}$ $\Rightarrow \frac{\partial (\ )}{\partial \hat{\underset{\sim}{\beta}}} \Rightarrow \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_p \end{pmatrix} \Rightarrow \underset{\sim}{X}^T \underset{\sim}{Y}$

4th term $\hat{\underset{\sim}{\beta}}^T \boxed{\underset{\sim}{X}^T \underset{\sim}{X}} \hat{\underset{\sim}{\beta}}$ $\underset{\sim}{C} = \begin{pmatrix} C_{00} & C_{01} & \cdots & C_{0p} \\ \vdots & & & \vdots \\ C_{p0} & C_{p1} & \cdots & C_{pp} \end{pmatrix}$

$(p+1) * (p+1)$

Is it symmetric? Yes $\Leftarrow (\underset{\sim}{X}^T \underset{\sim}{X})^T = \underset{\sim}{X}^T \underset{\sim}{X}$

$$\frac{\partial(\quad)}{\partial\hat\beta_0} = \qquad (\hat\beta_0, \hat\beta_1, \dots, \hat\beta_p)\, \underset{\sim}{C} \begin{pmatrix}\hat\beta_0\\ \hat\beta_1\\ \vdots\\ \hat\beta_p\end{pmatrix}$$

$$= (\hat\beta_0 \boxed{C_{00}} + \hat\beta_1 \boxed{C_{10}} + \dots + \hat\beta_p C_{p0})\hat\beta_0 + (\hat\beta_0 \boxed{C_{01}} + \dots + \hat\beta_p C_{p1})\hat\beta_1$$

$$+ \dots + (\hat\beta_0 C_{0p} + \dots + \hat\beta_p C_{pp})\hat\beta_p$$

$$\frac{\partial(\quad)}{\partial\hat\beta_0} = 2\hat\beta_0 C_{00} + \hat\beta_1 C_{10} + \dots + \hat\beta_p C_{p0} + \hat\beta_1 C_{01} + \dots$$

$$= 2(C_{00} \quad C_{01} \quad \dots \quad C_{0p}) \begin{pmatrix}\hat\beta_0\\ \hat\beta_1\\ \vdots\\ \hat\beta_p\end{pmatrix}$$

$$\frac{\partial(\quad)}{\partial\hat\beta_1} =$$

$$\vdots$$

$$\Rightarrow \frac{\partial(\quad)}{\partial\hat{\underset{\sim}{\beta}}} = 2\underbrace{\boxed{(\underset{\sim}{X}^T\underset{\sim}{X})}}_{}\overset{\underset{\sim}{C}}{}\hat{\underset{\sim}{\beta}}$$

$$\frac{\partial}{\partial\hat{\underset{\sim}{\beta}}}(\underset{\sim}{Y} - \underset{\sim}{X}\hat{\underset{\sim}{\beta}})^T(\underset{\sim}{Y} - \underset{\sim}{X}\hat{\underset{\sim}{\beta}}) = 0$$

$$\Rightarrow 2\underset{\sim}{X}^T\underset{\sim}{Y} + 2(\underset{\sim}{X}^T\underset{\sim}{X})\hat{\underset{\sim}{\beta}} = 0$$

$$\Rightarrow \hat{\underset{\sim}{\beta}} = (\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\underset{\sim}{X}^T\underset{\sim}{Y} \quad (*)$$

Example 3 in p.5 Centered model

$$x'_{ij} = x_{ij} - \bar{x}_j \qquad i = 1, \dots, n \quad j = 1, \dots, p$$

$$z_{ij} = \frac{x_{ij} - \bar{x}_j}{s_j}$$

std. deviation of $x_{1j}, \dots, x_{nj}$

z-score

Fit y_i on $x'_{i1}, \dots, x'_{ip}$

Centred model $y_i = \beta'_0 + \beta'_1(x_{i1} - \bar{x}_1) + \dots + \beta'_p(x_{ip} - \bar{x}_p) + e_i$

intercept

$$= \underline{\beta'_0 - \beta'_1\bar{x}_1} + \beta'_1 x_{i1} + \dots \underline{- \beta'_p\bar{x}_p} + \beta'_p x_{ip} + e_i$$

Original model $\quad y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip} + e_i$

$$\Rightarrow \beta_1 = \beta_1'$$
$$\vdots$$
$$\beta_p = \beta_p'$$
$$\beta_0 = \beta_0' - \beta_1'\bar{X}_1 - \cdots - \beta_p'\bar{X}_p$$

$$\underset{\sim}{X}^T\underset{\sim}{X} = \begin{pmatrix} n & \Sigma X_{i1} & \cdots & \Sigma X_{ip} \\ \Sigma X_{i1} & \Sigma X_{i1}^2 & \cdots & \Sigma X_{i1}X_{ip} \\ \vdots & & & \\ \Sigma X_{ip} & \Sigma X_{i1}\cancel{X}_{ip} & \cdots & \Sigma X_{ip}^2 \end{pmatrix} \qquad \underset{\sim}{X}^T\underset{\sim}{Y} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n X_{i1}y_i \\ \vdots \\ \sum_{i=1}^n X_{ip}y_i \end{pmatrix}$$

Centered model $\quad X_{ij}' = X_{ij} - \bar{X}_j$

$\Sigma(X_{i1} - \bar{X}_1) = 0$

$=0$

$$\underset{\sim}{X}^T\underset{\sim}{X} = \begin{pmatrix} n & \boxed{\Sigma X_{i1}'} & \cdots & \boxed{\Sigma X_{ip}'} \\ \boxed{\Sigma X_{i1}'} & \Sigma X_{i1}'^2 & \cdots & \Sigma X_{i1}'X_{ip}' \\ \vdots & & & \vdots \\ \boxed{\Sigma X_{ip}'} & \Sigma X_{i1}'X_{ip}' & \cdots & \Sigma X_{ip}'^2 \end{pmatrix}$$

$(p+1)*(p+1)$

$\Sigma(X_{i1} - \bar{X}_1)^2 = S_{x_1x_1}$

sum square

ΣX_{i1}^2 – uncorrected ~~SS~~ sum of squares

$\Sigma(X_{i1} - \boxed{\bar{X}_1})^2$ – Corrected sum of squares

corrected by mean

$$= \begin{pmatrix} S_{x_1x_1} & S_{x_1x_2} & \cdots & S_{x_1x_p} \\ \vdots & & & \\ S_{x_1x_p} & S_{x_2x_p} & \cdots & S_{x_px_p} \end{pmatrix} \quad p \times p$$

$$= \begin{pmatrix} n & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{X_c}^T\underset{\sim}{X_c} \end{pmatrix}$$

$$\underset{\sim}{X}^T\underset{\sim}{Y} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n X_{i1}' y_i \\ \vdots \\ \sum_{i=1}^n X_{ip}' y_i \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n y_i \\ S_{x_1y} \\ \vdots \\ S_{x_py} \end{pmatrix}$$

$\sum_{i=1}^n (X_{i1} - \bar{X}_1) y_i$

$$(\underset{\sim}{X}^T\underset{\sim}{X})^{-1} = \begin{pmatrix} \frac{1}{n} & \underset{\sim}{0} \\ \underset{\sim}{0} & (\underset{\sim}{X_c}^T\underset{\sim}{X_c})^{-1} \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{\beta} = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & (\underset{\sim}{X_c}^T\underset{\sim}{X_c})^{-1} \end{pmatrix} * \begin{pmatrix} \sum_{i=1}^n y_i \\ S_{x_1y} \\ \vdots \\ S_{x_py} \end{pmatrix}$$

$$\Rightarrow \hat{\beta}_0' = \frac{\sum_{i=1}^{n} y_i}{n} = \bar{y}$$

$$\Rightarrow \hat{\beta}_0 = \underset{\bar{y}}{\underset{\shortparallel}{\hat{\beta}_0'}} - \hat{\beta}_1 \bar{x}_1 - \cdots - \hat{\beta}_p \bar{x}_p \Leftarrow$$

For $p = 1$

$$\Rightarrow \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}_1$$

Example 5: Example in Multiple Linear Regression

The percent survival of a certain type of animal semen after storage was measured at various combinations of concentrations of three materials used to increase chance of survival. The data are as follows:

$p = 3$

y (% survival)	x_1 (weight %)	x_2 (weight %)	x_3 (weight %)
25.5	1.74	5.30	10.80
31.2	6.32	5.42	9.40
25.9	6.22	8.41	7.20
38.4	10.52	4.63	8.50
18.4	1.19	11.60	9.40
26.7	1.22	5.85	9.90
26.4	4.10	6.62	8.00
25.9	6.32	8.72	9.10
32.0	4.08	4.42	8.70
25.2	4.15	7.60	9.20
39.7	10.15	4.83	9.40
35.7	1.72	3.12	7.60
26.5	1.70	5.30	8.20

$\underset{\sim}{X}$ $n \times 4$ $\quad$ $\underset{\sim}{X}^T \underset{\sim}{X}$ 4×4

Summary statistics:

$$\sum_{i=1}^{13} y_i = 377.5 \qquad \sum_{i=1}^{13} y_i^2 = 11,400.15 \qquad \sum_{i=1}^{13} x_{i1} = 59.43$$

$$\sum_{i=1}^{13} x_{i2} = 81.82 \qquad \sum_{i=1}^{13} x_{i3} = 115.40 \qquad \sum_{i=1}^{13} x_{i1}^2 = 394.7255$$

$$\sum_{i=1}^{13} x_{i2}^2 = 576.7264 \qquad \sum_{i=1}^{13} x_{i3}^2 = 1035.96 \qquad \sum_{i=1}^{13} x_{i1}y_i = 1877.567$$

$$\sum_{i=1}^{13} x_{i2}y_i = 2246.661 \qquad \sum_{i=1}^{13} x_{i3}y_i = 3337.78 \qquad \sum_{i=1}^{13} x_{i1}x_{i2} = 360.6621$$

$$\sum_{i=1}^{13} x_{i1}x_{i3} = 522.078 \qquad \sum_{i=1}^{13} x_{i2}x_{i3} = 728.31 \qquad n = 13$$

$$\begin{pmatrix} 377.5 \\ 1877.567 \\ 2246.661 \\ 3337.78 \end{pmatrix}$$

$$\begin{pmatrix} 13 & 59.43 & 81.82 & 115.40 \\ 59.43 & 394.7255 & 360.6621 & 522.078 \\ 81.82 & 360.6621 & 576.7264 & 728.31 \\ 115.40 & 522.078 & 728.31 & 1035.96 \end{pmatrix}^{-1} = \begin{pmatrix} 8.06479 & -0.0825927 & -0.0941951 & -0.790527 \\ -0.0825927 & 0.00847982 & 0.00171669 & 0.00372002 \\ -0.0941951 & 0.00171669 & 0.0166294 & -0.00206331 \\ -0.790527 & 0.00372002 & -0.00206331 & 0.0886013 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

Or

$$S_{x_1x_1} = \sum_{i=1}^{n} (x_{i1} - \bar{x}_1)^2 = \sum_{i=1}^{n} x_{i1}^2 - n\bar{x}_1^2 = \sum_{i=1}^{n} x_{i1}^2 - \frac{\left(\sum_{i=1}^{n} x_{i1}\right)^2}{n}$$

$S_{x_1x_2}$ $\quad$ $S_{x_1x_3}$

$$(\underset{\sim}{X}_c^T \underset{\sim}{X}_c)^{-1} = \begin{pmatrix} 13 & 0 & 0 & 0 \\ 0 & 123.039 & -13.3812 & -5.4775 \\ 0 & -13.3812 & 61.7639 & 2.0002 \\ 0 & -5.4775 & 2.0002 & 11.5631 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0.0769231 & 0 & 0 & 0 \\ 0 & 0.00847981 & 0.00171669 & 0.00371998 \\ 0 & 0.00171669 & 0.0166294 & -0.00206338 \\ 0 & 0.00371998 & -0.00206338 & 0.0886011 \end{pmatrix}$$

$$\underset{\sim}{X}_c^T \underset{\sim}{Y} = \begin{pmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} (x_{i1} - \bar{x}_1) y_i \\ \sum_{i=1}^{n} (x_{i2} - \bar{x}_2) y_i \\ \sum_{i=1}^{n} (x_{i3} - \bar{x}_3) y_i \end{pmatrix} = \begin{pmatrix} 377.5 \\ 151.81123 \\ -129.265923 \\ -13.258462 \end{pmatrix}$$

$\Rightarrow \hat{\beta}_0 = 39.1574,\ \hat{\beta}_1 = 1.0161,\ \hat{\beta}_2 = -1.8616,\ \hat{\beta}_3 = -0.3433.$

Fitted ~~fit~~ line / estimated regression curve

$$\hat{y} = 39.1574 + 1.0161 X_1 - 1.8616 X_2 - 0.3433 X_3$$