29 October 2020

Two categorical variables

Test "interaction" effect

$H_0 : \gamma_{11} = \gamma_{12} = \ldots = \gamma_{a-1,b-1} = 0$ is equivalent to

$$\begin{aligned} H_0 : \quad \mu_{11} - \mu_{21} \quad &= \mu_{12} - \mu_{22} = \ldots = \mu_{1b} - \mu_{2b}; \\ \mu_{11} - \mu_{31} \quad &= \mu_{12} - \mu_{32} = \ldots = \mu_{1b} - \mu_{3b}; \\ &\vdots \\ \mu_{11} - \mu_{a1} \quad &= \mu_{12} - \mu_{a2} = \ldots = \mu_{1b} - \mu_{ab} \end{aligned}$$

- For example, $H_0 : \mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}$

$$\underset{\sim}{C} = \begin{pmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{pmatrix}$$

- Sum of squares is calculated as $(\underset{\sim}{C}\underset{\sim}{\hat{\beta}})^T[\underset{\sim}{C}(\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\underset{\sim}{C}^T]^{-1}(\underset{\sim}{C}\underset{\sim}{\hat{\beta}})$ by choosing an appropriate $\underset{\sim}{C}$.

- $$F = \frac{(\underset{\sim}{C}\underset{\sim}{\hat{\beta}} - \underset{\sim}{d})^T[\underset{\sim}{C}(\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\underset{\sim}{C}^T]^{-1}(\underset{\sim}{C}\underset{\sim}{\hat{\beta}} - \underset{\sim}{d})}{r\hat{\sigma}^2} \sim \mathrm{F}_{((a-1)(b-1),N-ab)}$$

  where $\hat{\sigma}^2 = \dfrac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}(y_{ijk} - \bar{y}_{ij.})^2}{N - ab}$.

  Then, reject H_0 if $f_{obs} > F_\alpha((a-1)(b-1), N - ab)$.

  "Interaction" is insignificant

  - "error" S.S. for the model without interaction is equal to the sum of "interaction" S.S. & "error" S.S. for the model with interaction, i.e.,

$$\begin{aligned} \text{SSE} \quad &= \quad \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}(y_{ijk} - \bar{y}_{ij.})^2 \ + \ SS(AB) \\ \text{d.f.} \quad &= \quad \sum_{i=1}^{a}\sum_{j=1}^{b}(n_{ij} - 1) + (a-1) * (b-1) \\ \hat{\sigma}^2_{\text{no int}} \quad &= \quad \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}(y_{ijk} - \bar{y}_{ij.})^2 \ + \ SS(AB)}{\sum_{i=1}^{a}\sum_{j=1}^{b}(n_{ij} - 1) + (a-1) * (b-1)} \end{aligned}$$

Tests on the main effects are meaningful, i.e., test

1. No Difference in Means Due to Factor A
   $H_0^1 : \mu_{1.} = \mu_{2.} = \ldots = \mu_{a.}$
2. No Difference in Means Due to Factor B
   $H_0^2 : \mu_{.1} = \mu_{.2} = \ldots = \mu_{.b}$

For example,

- Test "method" effect is equivalent to test $H_0 : \mu_{1.} = \mu_{2.} \;\Rightarrow\; H_0 : \alpha_1 = 0$. Then,

$$\underset{\sim}{C} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \quad \hat{\underset{\sim}{\beta}} = \begin{pmatrix} \mu \\ \alpha_1 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

- Test "variety" effect is equivalent to test $H_0 : \mu_{.1} = \mu_{.2} = \mu_{.3} \;\Rightarrow\; H_0 : \beta_1 = \beta_2 = 0$. Then,

$$\underset{\sim}{C} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \hat{\underset{\sim}{\beta}} = \begin{pmatrix} \mu \\ \alpha_1 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$