MATH 3424 Tutorial

October 16, 2020

1 Review

Chapter 1 Sec 4.2 F test.

2 Exercises

1. Using the following data set

x_1	x_2	y_1	x_1	x_2	y_1
2	-2	2	1	-1	2
0	1	3	1	3	5
2	3	7	3	3	11
1	4	6	0	5	6
1	6	9	1	8	11

with summary statistics:

Overall

$n = 10$, $\sum_{i=1}^{10} x_{i1} = 12$, $\sum_{i=1}^{10} x_{i2} = 30$, $\sum_{i=1}^{10} y_i = 62$,

$\sum_{i=1}^{10} x_{i1}^2 = 22$, $\sum_{i=1}^{10} x_{i1}x_{i2} = 31$, $\sum_{i=1}^{10} x_{i2}^2 = 174$, $\sum_{i=1}^{10} x_{i1}y_i = 84$,

$\sum_{i=1}^{10} x_{i2}y_i = 262$, $\sum_{i=1}^{10} y_i^2 = 486$,

$S_{x_1x_1} = 7.6000$, $S_{x_1x_2} = -5.0000$, $S_{x_2x_2} = 84.0000$, $S_{x_1y} = 9.6$,

$S_{x_2y} = 76.0000$, $S_{yy} = 101.6000$.

$Cov(\hat{\beta}) = \sigma^2(X^TX)^{-1}$

and

$$\begin{pmatrix} 7.6000 & -5.0000 \\ -5.0000 & 84.0000 \end{pmatrix}^{-1} = \begin{pmatrix} 0.136942 & 0.008151 \\ 0.008151 & 0.012390 \end{pmatrix},$$

When β_0, β_1 and β_2 are all unknown, to fit the following model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + e_i, \qquad e_i \sim_{iid} N(0, \sigma^2)$$

Note that estimations of **centered model** are $\hat{\beta}_0' = 6.20000$, $\hat{\beta}_1' = 1.93414$, $\hat{\beta}_2' = 1.01989$ Give **all** your answers in **4** decimal points.

(a) Find the Regression Sum of Squares, Residual Sum of Squares, Total Sum of Squares and the unbiased estimate of the unknown parameter σ^2

$y_i - \bar{y} = \beta_0' + \beta_1'(x_{i1} - \bar{x}_1) + \beta_2'(x_{i2} - \bar{x}_2) + e_i'$.

$y_i = \underbrace{\beta_0' + \bar{y} - \beta_1'\bar{x}_1 - \beta_2'\bar{x}_2} + \underbrace{\beta_1'} x_{i1} + \underbrace{\beta_2'} x_{i2} + e_i'$.

$\hat{\beta}_0 = \hat{\beta}_0' + \bar{y} - \hat{\beta}_1'\bar{x}_1 - \hat{\beta}_2'\bar{x}_2 = 0.8195$

$\hat{\beta}_1 = \hat{\beta}_1'$, $\hat{\beta}_2 = \hat{\beta}_2'$.

$Reg.SS = \hat{\beta}_1 S_{x_1y} + \hat{\beta}_2 S_{x_2y} = 96.07938$

$Res.SS = S_{yy} - \hat{\beta}_1 S_{x_1y} - \hat{\beta}_2 S_{x_2y} = 5.5206$.

$T.SS = S_{yy} = 101.6$

$\hat{\sigma}^2 = \frac{Res.SS}{n-p-1} = 0.78866$.

(b) Fill the ANOVA table for H_0: $\beta_1 = \beta_2 = 0$ at significance level $\alpha = 0.05$. Write down your conclusion clearly.

Source	Sum of Squares	D.F.	Mean Squares	F value
Regression	96.0794	2	48.0397	60.9100
Residual	5.5206	7	0.7887	-
Total	101.6000	9	-	-

$F_{obs.} > F_{0.05}(2,7) = 4.74$

Reject H_0.

(c) Test the hypothesis $H_0 : \beta_1 = 1.5$ vs $H_1 : \beta_1 > 1.5$ at the significant level of $\alpha = 0.05$

$\hat{\beta}_1 - 1.5 = 0.43402$.

$s.e.(\hat{\beta}_1) = \sqrt{0.78866 \times 0.13694} = 0.32863$.

$t = 1.3207$.

$t_{0.05}(7) = 1.8946$.

$t < t_{0.05}(7)$. $\Rightarrow$ Cannot reject H_0.

* Note that $\hat{\beta}_1 = \hat{\beta}_1'$,
so no need of the complicated computation as in (d).

(d) Find a 95% confidence interval of β_0.

$$\hat{\beta}_0 = \hat{\beta}_0' - \hat{\beta}_1' \bar{x}_1 - \hat{\beta}_2' \bar{x}_2 .$$

$$Var(\hat{\beta}_0) = Var(\hat{\beta}_0') + \bar{x}_1^2 Var(\hat{\beta}_1') + \bar{x}_2^2 Var(\hat{\beta}_2') - 2\bar{x}_1 Cov(\hat{\beta}_0', \hat{\beta}_1') - 2\bar{x}_2 Cov(\hat{\beta}_1, \hat{\beta}_2) + 2\bar{x}_1\bar{x}_2 Cov(\hat{\beta}_1, \hat{\beta}_2).$$

We have $\hat{Var}(\hat{\beta}_0') = \hat{\sigma}^2 \cdot \frac{1}{10} = 0.078866$, $\hat{Var}(\hat{\beta}_1') = \hat{\sigma}^2 \cdot 0.13694 = 0.108001$,

$\hat{Var}(\hat{\beta}_2') = \hat{\sigma}^2 \cdot 0.012390 = 0.009771497$, $\hat{Cov}(\hat{\beta}_1', \hat{\beta}_2') = \hat{\sigma}^2 \cdot 0.008151 = 0.006428$

$\hat{Cov}(\hat{\beta}_0, \hat{\beta}_1) = Cov(\hat{\beta}_0', \hat{\beta}_2') = 0.$

$\Rightarrow \hat{Var}(\hat{\beta}_0) = 0.3686147.$

$s.e.(\hat{\beta}_0) = 0.6071.$

$t_{0.025}(7) = 2.365.$

C.I. of β_0 is $\hat{\beta}_0 \mp t_{0.025}(7)\, s.e.(\hat{\beta}_0)$.

$\beta_1 - \beta_2 = 0$

(e) Test the hull hypothesis $H_0 : \beta_1 = \beta_2$ *vs* $H_\alpha : \beta_1 \neq \beta_2$ at the significant level of $\alpha = 0.05$.

i. t-test. Write down the test statistic, the critical value and your conclusion clearly.

$\hat{\beta}_1 - \hat{\beta}_2 = 0.91425$

$Var(\hat{\beta}_1 - \hat{\beta}_2) = Var(\hat{\beta}_1) + Var(\hat{\beta}_2) - 2Cov(\hat{\beta}_1, \hat{\beta}_2)$

$\hat{Var}(\hat{\beta}_1 - \hat{\beta}_2) = 0.7887 \times (0.1369 + 0.01239 - 2 \times 0.00815) = 0.10492$

$s.e.(\hat{\beta}_1 - \hat{\beta}_2) = 0.32391$

$$t = \frac{\hat{\beta}_1 - \hat{\beta}_2}{s.e.(\hat{\beta}_1 - \hat{\beta}_2)} = 2.822543$$

$t_{0.025}(7) = 2.365$

$|t| > t_{0.025}(7) \Rightarrow$ Reject H_0.

ii. F test for testing $H_0 : \underset{\sim}{C}\underset{\sim}{\beta} = \underset{\sim}{d}$. Write down the test statistic, the critical value and your conclusion clearly.

$C = (0 \ 1 \ -1)$, $d = 0$.

$C\hat{\beta} - d = 0.91425$.

$[C(X^TX)^{-1}C^T]^{-1} = 7.517101$.

$r = \text{rank}(C) = 1$.

$$F = \frac{0.91425 \times 7.517101 \times 0.91425/1}{0.78866} = 7.966921.$$

$F_{0.05}(1,7) = 5.59$.

$\Rightarrow$ Reject H_0.

iii. F test in terms of "Increase in Regression Sum of Squares". Write down the test statistic, the critical value and your conclusion clearly.

Reduced model: $y = \beta_0 + \beta_1 \underbrace{(x_1 + x_2)}_{x'} + e_i$

Under $\downarrow$: $\hat{\beta}_1\Big|_R = \dfrac{S_{x'y}}{S_{x'x'}} = \dfrac{S_{x_1y} + S_{x_2y}}{S_{x_1x_1} + 2S_{x_1x_2} + S_{x_2x_2}} = 1.04902$

$\text{Reg.SS}|_R = S_{yy} - \hat{\beta}_1|_R \cdot S_{x'y} = 11.80389$.

$\text{Inc. Reg.SS} = \text{RegSS}|_F - \text{Reg.SS}|_R = 6.283288$.

$$F = \frac{\text{Inc. Reg.SS}}{\hat{\sigma}^2} = 7.967045 > F_{0.05}(1,7) = 5.59$$

Reject H_0.