$\underline{n_{ij} = n \quad \forall i, j}$ e.g. $n_{ij} = 6$ in our example (balanced design)

- corr. coeff. of $\hat{\alpha}_1$ & $\hat{\beta}_1$ ~~as~~ is equal to 0

corr. coeff of $\hat{\alpha}_1$ & $\hat{\beta}_2$ is equal to 0

3 Nov

- $$(X_c^T X_c)^{-1} = \begin{pmatrix} 36 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 8 & -4 \\ 0 & 0 & -4 & 8 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{36} & 0 & 0 & 0 \\ 0 & \frac{1}{9} & \boxed{0 \quad 0} \\ 0 & 0 & \frac{1}{6} & \frac{1}{12} \\ 0 & 0 & \frac{1}{12} & \frac{1}{6} \end{pmatrix}$$

~~Var(X^T~~ $\mathrm{Var}(\hat{\beta}) = (X_c^T X_c)^{-1} \sigma^2$

- $\hat{\beta} = (X^T X)^{-1} X^T Y$

$$\Rightarrow \hat{\beta}_0' = \frac{1}{36} \boxed{\sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk}} = \text{overall mean}$$

$$\hat{\alpha}_1 = \frac{1}{9} \left[\frac{1}{2} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{1jk} - \frac{1}{2} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{2jk} \right]$$

$$= \frac{1}{18} \boxed{\sum_{j=1}^{b} \sum_{k=1}^{n} y_{1jk}} - \frac{1}{18} \boxed{\sum_{j=1}^{b} \sum_{k=1}^{n} y_{2jk}}$$

= (mean of y when method = 1) − (mean of y when $\boxed{\text{method} = 2}$) reference group

$$\hat{\beta}_1 = \frac{1}{12} \sum_{i=1}^{a} \sum_{k=1}^{n} y_{i1k} - \frac{1}{12} \sum_{i=1}^{a} \sum_{k=1}^{n} y_{i3k}$$

= (mean of y when variety = 1) − (mean of y when $\boxed{\text{variety} = 3}$) reference group

$$\hat{\beta}_2 = \frac{1}{12} \sum_{i=1}^{a} \sum_{k=1}^{n} y_{i2k} - \frac{1}{12} \sum_{i=1}^{a} \sum_{k=1}^{n} y_{i3k}$$

= (mean of y when variety = 2) − (mean of y when variety = 3) when $n_{ij} \neq n$ (unbalanced design)

$$(\underset{\sim}{X}_c^T \underset{\sim}{X}_c)^{-1} = \left(\begin{array}{cc|cc} b & 0 & 0 & 0 \\ 0 & a & b & c \\ \hline 0 & & & \\ 0 & & & \end{array}\right) \qquad \cancel{\underset{\sim}{X}_c^T \underset{\sim}{X}_c} \quad \underset{\sim}{X}_c^T \underset{\sim}{Y} = \begin{pmatrix} \boxed{d} \\ \boxed{e} \\ \boxed{f} \\ \boxed{g} \end{pmatrix}$$

$$\hat{\alpha}_1 = a * e + b * f + c * g$$

dep. on mean of y for ~~method~~ Factor A

dep. on mean of y for ~~variety~~ Factor B

② Total S.S. = Reg S.S.$|_{int}$ + Res S.S.$|_{int}$

$$\| $$

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}(y_{ijk}-\bar{y}_{...})^2 = R(\underset{\sim}{\alpha}, \underset{\sim}{\beta}, \underset{\sim}{\gamma} \mid \beta_0) + \text{Res S.S.}|_{int}.$$

Factor B

increase in Reg.S.S. Factor A interaction

SS (interaction)

$$= R(\underset{\sim}{\alpha}, \underset{\sim}{\beta} \mid \beta_0) + \boxed{R(\underset{\sim}{\gamma} \mid \underset{\sim}{\alpha}, \underset{\sim}{\beta}, \beta_0)} + \text{Res S.S}|_{int}$$

β_0

$+ \underset{\sim}{\alpha}, \underset{\sim}{\beta}$

$R(\underset{\sim}{\alpha}, \underset{\sim}{\beta} \mid \beta_0)$ $\Downarrow$

$+ \underset{\sim}{\gamma}$

$R(\underset{\sim}{\gamma} \mid \underset{\sim}{\alpha}, \underset{\sim}{\beta}, \beta_0)$

sequential S.S.

β_0

$\Downarrow$

$+ \underset{\sim}{\alpha}, \underset{\sim}{\beta}, \underset{\sim}{\gamma}$

$R(\underset{\sim}{\alpha}, \underset{\sim}{\beta}, \underset{\sim}{\gamma} \mid \beta_0)$

$$= \boxed{R(\underset{\sim}{\alpha}, \underset{\sim}{\beta} \mid \beta_0)} + \text{Res S.S.}|_{\text{no int}}$$

$$\|$$

Reg S.S.$|_{\text{no int}}$.

$\Downarrow$ $\hat{\sigma}^2_{\text{no int}}$ to est. σ^2

$\Rightarrow$ Reg S.S.$|_{\text{no int}}$ = Total S.S. − Res S.S.$|_{\text{no int}}$.

$\downarrow$

$R(\underset{\sim}{\alpha}, \underset{\sim}{\beta} \mid \beta_0)$

$H_0: \alpha_1 = \alpha_2 = \cdots = \alpha_{a-1} = 0 \leftarrow R(\underset{\sim}{\alpha} \mid \underset{\sim}{\beta}, \beta_0)$ Need to calculate

β_0 $\qquad\qquad\qquad\qquad$ β_0

$+\ \underset{\sim}{\beta}$

$\overline{R(\underset{\sim}{\beta} \mid \beta_0)}$

$+\ \underset{\sim}{\alpha}$ $\qquad\qquad$ $+\ \underset{\sim}{\alpha}, \underset{\sim}{\beta}$

$\overline{R(\underset{\sim}{\alpha} \mid \underset{\sim}{\beta}, \beta_0)}$ $\qquad$ $\overline{R(\underset{\sim}{\alpha}, \underset{\sim}{\beta} \mid \beta_0)}$

$$R(\underset{\sim}{\alpha}, \underset{\sim}{\beta} \mid \beta_0) = \boxed{R(\underset{\sim}{\alpha} \mid \underset{\sim}{\beta}, \beta_0)} + R(\underset{\sim}{\beta} \mid \beta_0)$$

Reg S.S, when the model has the 2nd categoral variable (Factor B)

Example in p.11

~~Reg S.S. | no int = Total S.S. − Res S.S | no int~~

~~$R(\underset{\sim}{\beta} \mid \beta_0)$ = Total S.S. − Res S.S |~~

$$R(\underset{\sim}{\beta} \mid \beta_0) = \text{Reg S.S.}\,|_{\text{Factor B}}$$

$$= \text{Total S.S.} - \underbrace{\text{Res S.S.}\,|_{\text{Factor B}}}$$

$$= 1408.53 - \sum_{j=1}^{b} n_{\cdot j}(\bar{y}_{\cdot j \cdot} - \bar{y}_{\cdots})^2$$

$$\sum_{j=1}^{b} n_{\cdot j}(\bar{y}_{\cdot j \cdot} - \bar{y}_{\cdots})^2 = (260.0425 + 583.02 + 499.349167) = 1342.42$$

$$= 66.118$$

$$\Rightarrow R(\underset{\sim}{\alpha} \mid \underset{\sim}{\beta}, \beta_0) = R(\underset{\sim}{\alpha}, \underset{\sim}{\beta} \mid \beta_0) - R(\underset{\sim}{\beta} \mid \beta_0)$$

$$= \text{Total S.S.} - \text{Res S.S.}\,|_{\text{no int.}} - R(\underset{\sim}{\beta} \mid \beta_0)$$

$$= 1408.53 - (627.74) - 66.118$$

$$= 780.79 - 66.118$$

$$= 714.67$$

Example

	Variety				
Method	1	2	3	Sum	CSS
1	22.3	19.8	20		
	25.8	28.3	17		
	22.8	26.8	24		
	28.3	27.3	22.5		
	21.3	26.8	28		
	18.3	26.8	22.5		
Sum	138.8	155.8	134	428.6	
Corrected S.S.	61.333333	47.333333	68.833333	221.237778	
2	16.4	24.5	11.8		
	14.4	16	14.3		
	21.4	11	21.3		
	19.9	7.5	6.3		
	10.4	14.5	7.8		
	21.4	15.5	13.8		
Sum	103.9	89	75.3	268.2	
Corrected S.S.	97.208333	163.833333	143.375	472.62	
Sum	242.7	244.8	209.3	696.8	
Corrected S.S.	260.0425 +	583.02 +	499.349167	1408.53	

\+ Res S.S. | Factor A

Res S.S. | Factor B

Total S.S.

Source of Variation of	Sum of Squares	Degrees of freedom	Mean Square	Computed f
Method	714.671111	1	714.671111	36.84
Variety	66.117222	2	33.058611	1.71
Interaction	45.823889	2	22.911944	1.18
Error	581.916667	30	19.397222	
Total	1408.528889	35		

Res S.S. | no int

= Res S.S. | int + SS(interaction)

= (581.916667 + 45.823889)

= 627.74

⇒ Reg S.S. | no int

= 1408.53 − 627.74

= 780.79

Test "interaction" effect is equivalent to test $H_0 : \mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}$.

As the interaction terms are not significant, we re-construct the ANOVA table.

R

$H_0 = \beta_1 = \beta_2 = \cdots = \beta_{b-1} = 0$

$\cancel{R(\underset{\sim}{\alpha}} \; R(\underset{\sim}{\beta} \mid \underset{\sim}{\alpha}, \beta_0) = R(\underset{\sim}{\alpha}, \underset{\sim}{\beta} \mid \beta_0) - R(\underset{\sim}{\alpha} \mid \beta_0)$

$1408.53 - 627.74$

$\shortparallel$

780.79

Reg S.S. when the model has Factor A only

$\sum_{i=1}^{a} n_{i\cdot}(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot\cdot\cdot})^2$

$\shortparallel$

$(221.2378 + 472.62)$

$\shortparallel$

714.67

$\Rightarrow R(\underset{\sim}{\beta} \mid \underset{\sim}{\alpha}, \beta_0) = 66.118$

$H_0 : \beta_1 = \cdots = \beta_{b-1} = 0$

$\rightarrow \underline{R(\underset{\sim}{\beta} \mid \underset{\sim}{\alpha}, \beta_0) = R(\underset{\sim}{\beta} \mid \beta_0)} = 66.118$

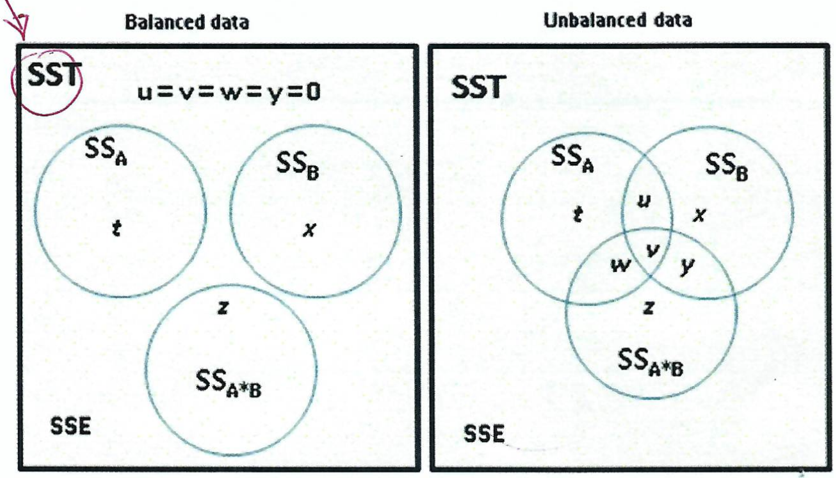
$H_0 : \alpha_1 = \cdots = \alpha_{a-1} = 0$

$\rightarrow \underline{R(\underset{\sim}{\alpha} \mid \underset{\sim}{\beta}, \beta_0) = R(\underset{\sim}{\alpha} \mid \beta_0)} = 714.67$

When $n_{ij} = n$ (balanced design)

Total S.S.

Balanced data vs Unbalanced data



Chapter 1

$$\text{Total S.S.} = \text{Reg S.S.} + \text{Res S.S.} \leftarrow$$

$$\text{Total S.S.} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{...})^2$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij.}+\bar{y}_{ij.}-\bar{y}_{i..}+\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{.j.}-\bar{y}_{...}-\bar{y}_{...}+\bar{y}_{...})^2$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\Big[\overset{1}{(y_{ijk}-\bar{y}_{ij.})}+\overset{2}{(\bar{y}_{i..}-\bar{y}_{...})}+\overset{3}{(\bar{y}_{.j.}-\bar{y}_{...})}+\overset{4}{(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})}\Big]^2$$

$$\text{Total SS} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\bar{y}_{ij.})^2 \;* \quad \text{— Res S.S.}$$

$$+\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(\bar{y}_{i..}-\bar{y}_{...})^2 \quad \text{— SSA} \quad = bn\sum_{i=1}^{a}(\bar{y}_{i..}-\bar{y}_{...})^2$$

$$+\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(\bar{y}_{.j.}-\bar{y}_{...})^2 \quad \text{— SSB} \quad = an\sum_{j=1}^{b}(\bar{y}_{.j.}-\bar{y}_{...})^2$$

$$+\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2 \quad \text{— for interaction term SS (interaction) SS AB}$$

$$= n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2$$

$$+ \underline{\text{6 cross-product terms}} = 0 \text{ when } n_{ij} = n \text{ (balanced design)}$$

Example

	Variety				
Method	1	2	3	Sum	CSS
1	22.3	19.8	20		
	25.8	28.3	17		
	22.8	26.8	24		
	28.3	27.3	22.5		
	21.3	26.8	28		
	18.3	26.8	22.5		
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Corrected S.S.	61.333333	47.333333	68.833333	221.237778	
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	21.4	11	21.3		
	19.9	7.5	6.3		
	10.4	14.5	7.8		
	21.4	15.5	13.8		
Sum	103.9	89	75.3	268.2	
Corrected S.S.	97.208333	163.833333	143.375	472.62	
Sum	242.7	244.8	209.3	696.8	
Corrected S.S.	260.0425	583.02	499.349167	1408.53	

balanced Design (when $\gamma_{ij} = 0$)

$bn \sum_{i=1}^{a} (\bar{y}_{i..} - \bar{y}_{...})^2$

$an \sum_{j=1}^{b} (\bar{y}_{.j.} - \bar{y}_{...})^2$

Source of Variation of	Sum of Squares	Degrees of freedom	Mean Square	Computed f
Method	714.671111	1	714.671111	36.84
Variety	66.117222	2	33.058611	0.1990
Interaction	45.823889 +	2 +	~~22.911944~~	1.18
Error	581.916667	30	~~19.397222~~	
Total	1408.528889	35		

32

627.74

$\frac{627.74}{32} = 19.6168$

Test "interaction" effect is equivalent to test $H_0 : \mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}$.

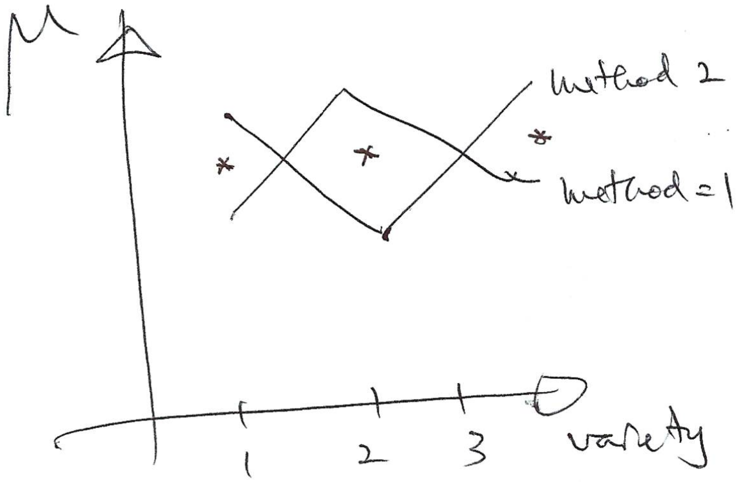
As the interaction terms are not significant, we re-construct the ANOVA table.

Reject $H_0 : \gamma_{ij} = 0$

$$\boxed{\begin{array}{l} H_0 : \mu_{1\cdot} = \mu_{2\cdot} \\ H_0 : \mu_{\cdot 1} = \mu_{\cdot 2} = \mu_{\cdot 3} \end{array}}$$

$\Downarrow$

Result is misleading



$\Rightarrow$ We can only consider single effect Model

~~Sig~~ single effect of ~~method~~ variety :

method = 1 $\quad H_0 : \mu_{11} = \mu_{12} = \mu_{13}$

= 2 $\quad H_0 : \mu_{21} = \mu_{22} = \mu_{23}$

$H_0 : \underset{\sim}{c}\, \underset{\sim}{\beta} = \underset{\sim}{d}$

single effect of ~~variety~~ method :

variety = 1 $\quad H_0 : \mu_{11} = \mu_{21}$

= 2 $\quad H_0 : \mu_{12} = \mu_{22}$

= 3 $\quad H_0 : \mu_{13} = \mu_{23}$

t - test

~~$H_0 : \mu_{11} = \mu_{21}$~~

<u>ANOVA Model</u> : $y_{ijk} = \mu_{ij} + e_{ijk}$ $\quad i = 1, \ldots, a$; $j = 1, \ldots, b$; $k = 1, \ldots, n_{ij}$

$$\underset{\sim}{\beta} = \begin{pmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{21} \\ \mu_{22} \\ \mu_{23} \end{pmatrix}$$

<u>t - test</u>

$\hat{\mu}_{ij} = \bar{y}_{ij\cdot}$

$H_0 : \mu_{11} = \mu_{21}$ $\quad Var(\hat{\mu}_{ij}) = Var(\bar{y}_{ij\cdot}) = \dfrac{\sigma^2}{n_{ij}}$

$\Rightarrow H_0 : \mu_{11} - \mu_{21} = 0$

$$\Rightarrow t = \frac{\bar{y}_{11\cdot} - \bar{y}_{21\cdot}}{\hat{\sigma}\sqrt{\frac{1}{n_{11\cdot}} + \frac{1}{n_{21\cdot}}}}$$

When $n_{ij} \neq n$ (unbalanced design)

Can't reject $H_0: \gamma_{ij} = 0 \Rightarrow$

① $\hat{\mu}_{1\cdot} = \bar{y}_{1\cdot\cdot}$

Is $\bar{y}_{1\cdot\cdot}$ unbiased est. for $\mu_{1\cdot}$?

Model: $y_{ijk} = \beta_0 + \alpha_i + \beta_j + e_{ijk}$ $\quad a=2,\ b=2$

$\alpha_2 = 0,\ \beta_2 = 0$

$$\mu_{11} = \beta_0 + \alpha_1 + \beta_1$$

$$\mu_{12} = \beta_0 + \alpha_1 + \boxed{\beta_2}$$

$$\mu_{1\cdot} = \frac{\mu_{11} + \mu_{12}}{2} = \frac{\beta_0 + \alpha_1 + \beta_1 + \beta_0 + \alpha_1 + \boxed{\beta_2}}{2}$$

$$= \beta_0 + \alpha_1 + \frac{1}{2}\beta_1 + \frac{1}{2}\boxed{\beta_2} \leftarrow$$

$E(y_{11k}) = \beta_0 + \alpha_1 + \beta_1$

$$\bar{y}_{1\cdot\cdot} = \frac{\sum_{k=1}^{n_{11}} y_{11k} + \sum_{k=1}^{n_{12}} y_{12k}}{n_{11} + n_{12}}$$

$E(y_{12k}) = \beta_0 + \alpha_1 + \boxed{\beta_2}$

$$E(\bar{y}_{1\cdot\cdot}) = \frac{1}{n_{11} + n_{12}} \left[n_{11} \times (\beta_0 + \alpha_1 + \beta_1) + n_{12}(\beta_0 + \alpha_1 + \boxed{\beta_2}) \right]$$

$$= \beta_0 + \alpha_1 + \frac{n_{11}}{n_{11}+n_{12}}\beta_1 + \frac{n_{12}}{n_{11}+n_{12}}\boxed{\beta_2} \leftarrow$$

$\bar{y}_{1\cdot\cdot}$ is biased est. for $\mu_{1\cdot}$ except $n_{11} = n_{12}$

To est. $\hat{\mu}_{1\cdot} = \hat{\beta}_0 + \hat{\alpha}_1 + \frac{1}{2}\hat{\beta}_1 + \frac{1}{2}\boxed{\beta_2}$

② $\hat{\mu}_{1\cdot} - \hat{\mu}_{2\cdot} = \bar{y}_{1\cdot\cdot} - \bar{y}_{2\cdot\cdot} \leftarrow$ biased est.

To est. $\mu_{1\cdot} - \mu_{2\cdot} = \left(\beta_0 + \alpha_1 + \frac{1}{2}\beta_1 + \frac{1}{2}\boxed{\beta_2}\right)$

$$- \left(\beta_0 + \boxed{\alpha_2} + \frac{1}{2}\beta_1 + \frac{1}{2}\boxed{\beta_2}\right)$$

$$= \alpha_1 - \alpha_2$$

est. α_1 & α_2 from Reg. model