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Estimation

Maximum Likelihood Estimation

1. Ungrouped data: Response - y_i for $i = 1, \ldots, n$
   Assume $y_i \sim Bernoulli(P_i)$

$$L(\underset{\sim}{\beta}) = \prod_{i=1}^{n_1} \left(\frac{e^{\boldsymbol{x}_i^T \boldsymbol{\beta}}}{1 + e^{\boldsymbol{x}_i^T \boldsymbol{\beta}}} \right) \prod_{i=n_1+1}^{n} \left(\frac{1}{1 + e^{\boldsymbol{x}_i^T \boldsymbol{\beta}}} \right)$$

2. Grouped data: Response - (n_i, r_i) for $i = 1, \ldots, s$
   Assume $r_i \sim Binomial(n_i, P_i)$

$$L(\underset{\sim}{\beta}) = \prod_{i=1}^{s} \left(\frac{e^{\boldsymbol{x}_i^T \boldsymbol{\beta}}}{1 + e^{\boldsymbol{x}_i^T \boldsymbol{\beta}}} \right)^{r_i} \left(\frac{1}{1 + e^{\boldsymbol{x}_i^T \boldsymbol{\beta}}} \right)^{n_i - r_i}$$

The maximum likelihood estimators can be obtained by maximizing the log-likelihood function of $L(\underset{\sim}{\beta})$.

Confidence interval

C.I. of β_j is $\hat{\beta}_j \pm z_{\alpha/2}\sqrt{c^{jj}}$ for $j = 0, 1, \ldots, p$

where the c^{jj} is the $(j+1)^{th}$ diagonal element of $\boldsymbol{C}^{-1}$, where $\boldsymbol{C}^{-1}$ is the estimate of the variance-covariance matrix of the regression coefficients and the $(i,j)^{th}$ element in $\boldsymbol{C}$ is defined as

$$c_{ij} = \frac{-\partial^2 \log L(\hat{\boldsymbol{\beta}})}{\partial \hat{\beta}_i \, \partial \hat{\beta}_j} \quad (i = 0, 1, \ldots, p;\ j = 0, 1, \ldots, p)$$

Odds, Odds ratio & Probability

$$\text{Odds} = \frac{p}{1-p} = \exp\left(\beta_0 + \beta_1 x_1 + \ldots + \beta_p x_p\right)$$

$$\text{Odds Ratio} = \frac{\left.\frac{p}{1-p}\right|_{\text{at A}}}{\left.\frac{p}{1-p}\right|_{\text{at B}}}$$

Parameter	Estimate	Standard Error	Confidence Interval
$\omega = \sum_{i=0}^{p} a_i \beta_i$	$\hat{\omega} = \sum_{i=0}^{p} a_i \hat{\beta}_i$	$s.e.(\hat{\omega})$	$\hat{\omega} \pm 1.96 * s.e.(\hat{\omega}) = (\hat{\omega}_l, \hat{\omega}_u)$
$\exp(\omega)$	$\exp(\hat{\omega})$	$\exp(\hat{\omega}) * s.e.(\hat{\omega})$	$(\exp(\hat{\omega}_l),\ \exp(\hat{\omega}_u))$
$P = \frac{\exp(\omega)}{1+\exp(\omega)}$	$\hat{P} = \frac{\exp(\hat{\omega})}{1+\exp(\hat{\omega})}$	$\frac{\exp(\hat{\omega})}{(1+\exp(\hat{\omega}))^2} * s.e.(\hat{\omega})$	$\left(\frac{\exp(\hat{\omega}_l)}{1+\exp(\hat{\omega}_l)}, \frac{\exp(\hat{\omega}_u)}{1+\exp(\hat{\omega}_u)}\right)$

where $s.e.(\hat{\omega}) = \sqrt{\sum_{i=0}^{p} a_i^2 Var(\hat{\beta}_i) + \sum\sum_{j \neq k} a_j a_k Cov(\hat{\beta}_j, \hat{\beta}_k)}$

Hypothesis testing

1. For testing

$$H_o: P_i = \frac{e^{\boldsymbol{x}_i^T \boldsymbol{\beta}}}{1+e^{\boldsymbol{x}_i^T \boldsymbol{\beta}}},$$

the test statistic is

$$\lambda(\underset{\sim}{\beta}) = -2\log\left[\frac{L(\hat{\underset{\sim}{\beta}})}{L(\hat{\underset{\sim}{P}})}\right]$$

(a) Ungrouped data:

$$\begin{aligned}
L(\underset{\sim}{P}) &= \prod_{i=1}^{n} P_i^{y_i}(1-P_i)^{1-y_i} \\
\text{As} \quad \hat{P}_i &= y_i \text{ if all } \boldsymbol{x}_i \text{ are distinct} \\
\Rightarrow \quad L(\hat{\underset{\sim}{P}}) &= \prod_{i=1}^{n} y_i^{y_i}(1-y_i)^{1-y_i} \\
L(\hat{\underset{\sim}{\beta}}) &= \prod_{i=1}^{n_1}\left(\frac{e^{\boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}}}{1+e^{\boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}}}\right)\prod_{i=n_1+1}^{n}\left(\frac{1}{1+e^{\boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}}}\right) \\
\Rightarrow \quad \lambda(\underset{\sim}{\beta}) &\sim \chi^2_{n-(p+1)}
\end{aligned}$$

(b) Grouped data:

$$\begin{aligned}
L(\underset{\sim}{P}) &= \prod_{i=1}^{s} P_i^{r_i}(1-P_i)^{n_i-r_i} \\
\text{As} \quad \hat{P}_i &= \frac{r_i}{n_i} \\
\Rightarrow \quad L(\hat{\underset{\sim}{P}}) &= \prod_{i=1}^{s}\left(\frac{r_i}{n_i}\right)^{r_i}\left(\frac{n_i-r_i}{n_i}\right)^{n_i-r_i} \\
L(\hat{\underset{\sim}{\beta}}) &= \prod_{i=1}^{s}\left(\frac{e^{\boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}}}{1+e^{\boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}}}\right)^{r_i}\left(\frac{1}{1+e^{\boldsymbol{x}_i^T \hat{\boldsymbol{\beta}}}}\right)^{n_i-r_i} \\
\Rightarrow \quad \lambda(\underset{\sim}{\beta}) &\sim \chi^2_{s-(p+1)}
\end{aligned}$$

The test statistic $\lambda(\underset{\sim}{\beta})$ is called the *deviance* associated with the fitted logistic regression.