Tutorial Notes 8 of MATH3424

1 Summary of course material

1. Autocorrelation:

   When the observations have a **natural sequential order**, the correlation is referred to as **autocorrelation**.

2. Runs test: n_1 residuals positive and n_2 residuals negative

   Run test statistic= $\frac{\#\text{ of runs } -\mu}{\sigma}$ with

   $$\mu = \frac{2n_1n_2}{n_1+n_2}+1, \quad \sigma^2 = \frac{2n_1n_2(2n_1n_2-n_1-n_2)}{(n_1+n_2)^2(n_1+n_2-1)}$$

   Under H_0: the residuals are uncorrelated random, this test statistic (approximately) follows a distribution $N(0,1)$.

3. Durbin-Watson Statistic

   (a) Assumption: successive errors are correlated, i.e., $\epsilon_t = \rho\epsilon_{t-1} + \omega_t, \quad |\rho| \leq 1$, with $\omega_t \overset{i.i.d}{\sim} N(0, \sigma_\omega^2)$. This is the first-order autocorrelation.

   (b) The Durbin-Watson statistic:

   $$d = \frac{\sum_{t=2}^n (e_t - e_{t-1})^2}{\sum_{t=1}^n e_t^2}$$

   where e_i is the i-th ordinary least squares (OLS) residual. The statistics d is used for testing the null hypothesis $H_0 : \rho = 0$ against an alternative $H_1 : \rho > 0$.

      i. $d < d_L$, reject H_0

      ii. $d > d_U$, do not reject H_0

      iii. $d_L < d < d_U$, the test is inconclusive.

   (c) An estimate of ρ is given by

   $$\hat{\rho} = \frac{\sum_{t=2}^n e_t e_{t-1}}{\sum_{t=1}^n e_t^2}, \quad d \approx 2(1-\hat{\rho})$$

4. Remedial measures for autocorrelation

   (a) Addition of predictor variables

   (b) Transformation (Cochrane-Orcutt Procedure)

   $$y_t^* = \beta_0^* + \beta_1^* x_t^* + \omega_t$$

where

$$
\begin{aligned}
y_t^* &= y_t - \rho y_{t-1} \\
x_t^* &= x_t - \rho x_{t-1} \\
\beta_0^* &= \beta_0(1-\rho) \\
\beta_1^* &= \beta_1
\end{aligned}
$$

2 Questions

2.1

For each of the following tests concerning the autocorrelation parameter ρ in the following regression model with first-order autocorrelation:

$$Y_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \beta_3 X_{t3} + \epsilon_t, \quad \epsilon_t = \rho\epsilon_{t-1} + \omega_t$$

State the appropriate decision rule based on the Durbin-Watson test statistic for a sample of size 38:

(1) $H_0 : \rho = 0, \quad H_a : \rho > 0, \quad \alpha = .01$

(2) $H_0 : \rho = 0, \quad H_a : \rho < 0, \quad \alpha = .05$

(3) $H_0 : \rho = 0, \quad H_a : \rho \neq 0, \quad \alpha = .02$

2.2

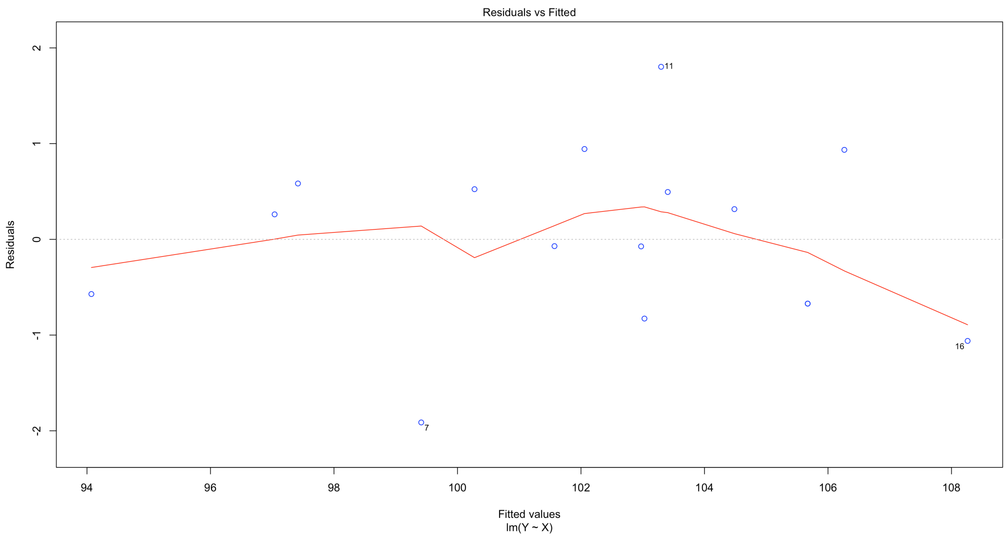
A staff analyst for a manufacturer of microcomputer components has compiled monthly data for the past 16 months on the value of industry production of processing unit that use the. e components (X, in million dollars) and the value of the firm's components used (Y, in thousand dollars). The analyst believes that a simple linear regression relation is appropriate but anticipates positive autocorrelation. The data follow:

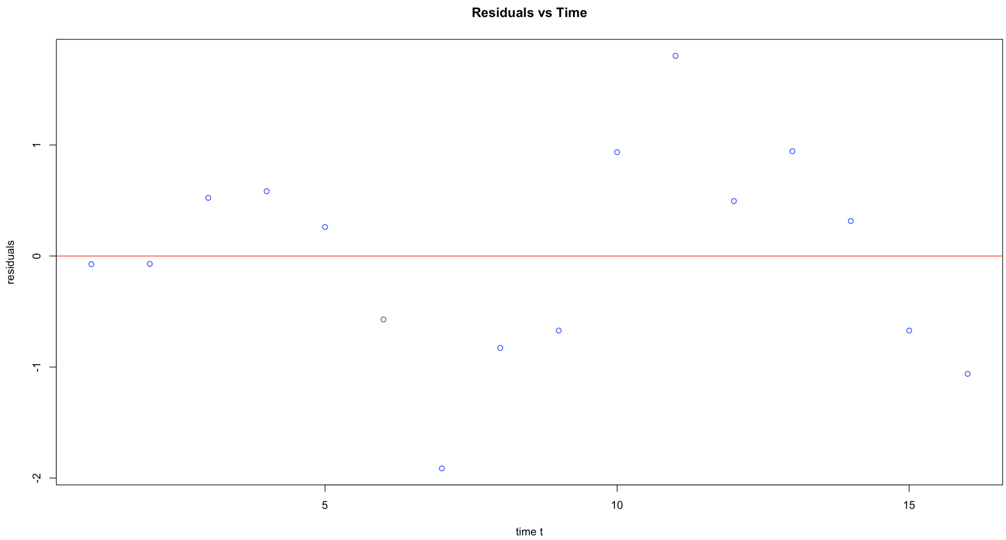
t:	1	2	3	...	14	15	16
X_t:	2.052	2.026	2.002	...	2.080	2.102	2.150
Y_t:	102.9	101.5	100.8	...	104.8	105.0	107.2

1. Fit a simple linear regression model by ordinary least squares and obtain the residuals. Also obtain s.e.($\hat{\beta}_0$) and s.e.($\hat{\beta}_1$).

```
Call:
lm(formula = Y ~ X, data = df)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.91277 -0.67136  0.09514  0.53886  1.80259 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   -7.739      7.175  -1.079    0.299    
X             53.953      3.520  15.329 3.82e-10 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.9543 on 14 degrees of freedom
Multiple R-squared:  0.9438,    Adjusted R-squared:  0.9398 
F-statistic:   235 on 1 and 14 DF,  p-value: 3.818e-10
```

2. Plot the residuals against time and explain whether you find any evidence of positive autocorrelation.





3. Conduct a formal test for positive autocorrelation using $\alpha = .05$. State the alternatives, decision rule, and conclusion. Is the residual analysis in part (b) in accord with the test result?

4. The analyst has decided to employ regression model with first-order autocorrelation and use the Cochrane-Orcutt procedure to fit the model.

   (a) Obtain a point estimate of the autocorrelation parameter. How well does the approximate relationship $d \approx 2(1 - \hat{\rho})$ hold here between this point estimate and the Durbin-Watson test statistic?

   (b) Use one iteration to obtain the estimates $\hat{\beta}_0^*$ and $\hat{\beta}_1^*$ of the regression coefficients β_0^* and β_1^* in transformed model ((7.4) in lecture slides) and state the estimated regression function. Also obtain s.e.$(\hat{\beta}_0^*)$ and s.e.$(\hat{\beta}_1^*)$.

```
Call:
lm(formula = Ystar ~ Xstar)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.51142 -0.43478 -0.05777  0.41365  1.42613 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   -1.073      4.121   -0.26    0.799    
Xstar         51.244      4.261   12.03 2.04e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.8135 on 13 degrees of freedom
Multiple R-squared:  0.9175,    Adjusted R-squared:  0.9112 
F-statistic: 144.6 on 1 and 13 DF,  p-value: 2.036e-08
```

(c) Test whether any positive autocorrelation remains after the first iteration using $\alpha = .05$. State the alternatives, decision rule, and conclusion.

(d) Restate the estimated regression function obtained in part (b) in terms of the original variables. Also obtain s.e.($\hat{\beta}_0$) and s.e.($\hat{\beta}_1$). Compare the estimated regression coefficients obtained with tbe Cochrane-Orcutt procedure and their estimated standard deviations with those obtained with ordinary least squares in part 1.

(e) On the basis of the results in parts 4(c) and 4(d), does the Cochrane-Orcutt procedure appear to have been effective here?