1. (a) First compute the statistics $\bar{x} = 3$, $\bar{y} = (6+c)/5$. Since the regression line passes through $(\bar{x}, \bar{y})$, we have $\bar{y} = 1.65$ indicating $c = 2.25$.
   Then $\sum(y_i - \bar{y})^2 = \sum y_i^2 - n\bar{y}^2 = 3.075$.

   (b) A 98% confidence interval for β_0 is given by $\hat{\beta}_0 \mp t_{0.01,23}\hat{\sigma}\sqrt{1/25+0}$, whose length is

$$2 \cdot t_{0.01,23}\hat{\sigma}/5 = 2 \times 2.499867 \times 10/5 \approx 9.9995.$$

   (c) We have $Res.SS = TSS - Reg.SS = 425 - 225 = 200$, and then $\hat{\sigma}^2 = Res.SS/8 = 25$.
   Since $Reg.SS = \hat{\beta}_1^2 S_{x,x}$, we have $|\hat{\beta}_1| = \sqrt{225/400} = 3/4$.
   The absolute test statistic is given by

$$|t| = \frac{|\hat{\beta}_1|}{\hat{\sigma}\sqrt{c_{11}}} = \frac{3/4}{5/20} = 3.$$

   Then $|t| > t_{0.025,8} = 2.306004$. Hence at the significant level of $\alpha = 0.05$, we reject H_0.

2. (a) We compute

$$\begin{aligned} RSS &= Y^\top Y - \hat{\beta}^\top X^\top Y \\ &= 1324 - (1.237088, -1.007608, 2.089488) \cdot (138, 962, 1002)^\top = 28.934. \end{aligned}$$

   The unbiased estimate of σ^2 is thus given by $\hat{\sigma}^2 = RSS/(n-3) = 1.702$.

   (b) The test statistic is given by

$$t = \frac{\hat{\beta}_2 - 2}{\hat{\sigma} c_{22}} = \frac{2.089488 - 2}{\sqrt{1.702} \times \sqrt{0.014734}} \approx 0.5651$$

   Compute the critical value $t_{0.025,17} = 2.109816 > |t|$. Hence at the significant level of $\alpha = 0.05$, we do not reject H_0.

Assume that $\boldsymbol{\beta_0 = 2}$.

   (c) **(2 marks)** Write down $\underset{\sim}{X}^T\underset{\sim}{X}$, $(\underset{\sim}{X}^T\underset{\sim}{X})^{-1}$ and $\underset{\sim}{X}^T\underset{\sim}{Y}$ in terms of values of summary statistics.
   $\underset{\sim}{X}^T\underset{\sim}{X} = \begin{pmatrix} \sum_{i=1}^n x_{i1}^2 & \sum_{i=1}^n x_{i1}x_{i2} \\ \sum_{i=1}^n x_{i1}x_{i2} & \sum_{i=1}^n x_{i2}^2 \end{pmatrix} = \begin{pmatrix} 1012 & 875 \\ 875 & 834 \end{pmatrix}$, then
   $(\underset{\sim}{X}^T\underset{\sim}{X})^{-1} = \begin{pmatrix} 0.010640 & -0.011163 \\ -0.011163 & 0.012911 \end{pmatrix}$.
   $\underset{\sim}{X}^T\underset{\sim}{Y} = \begin{pmatrix} \sum_{i=1}^n x_{i1}y_i - 2\sum_{i=1}^n x_{i1} \\ \sum_{i=1}^n x_{i2}y_i - 2\sum_{i=1}^n x_{i2} \end{pmatrix} = \begin{pmatrix} 714 \\ 774 \end{pmatrix}$.

   (d) $\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = (\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\underset{\sim}{X}^T\underset{\sim}{Y} = \begin{pmatrix} -1.043262 \\ 2.022607 \end{pmatrix}$.
   Thus, the fitted line is given by

$$\hat{y} = 2 - 1.043262x_1 + 2.022607x_2.$$

(e) According to the formula on page 9 of chapter 1, we have

$$\begin{aligned} RSS &= \underset{\sim}{Y}^\top \underset{\sim}{Y} - \hat{\beta}^\top \underset{\sim}{X}^\top \underset{\sim}{Y} = 852 - 820.6087 = 31.3913 \\ \hat{\sigma}^2 &= \frac{RSS}{n-p'} = \frac{31.3913}{20-2} = 1.743961. \end{aligned}$$

3. (a)

$$\begin{aligned} \hat{\beta}_1 &= \frac{S_{xy}}{S_{xx}} = \frac{\sum_{i=1}^{3}(i-2)y_i}{\sum_{i=1}^{3}(i-2)^2} = \frac{1}{2}(y_3 - y_1) \\ \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} = \frac{1}{3}\sum_{i=1}^{3} y_i - \sum_{i=1}^{3}(i-2)y_i = \frac{4}{3}y_1 + \frac{1}{3}y_2 - \frac{2}{3}y_3 \end{aligned}$$

(b)

$$\begin{aligned} Var(\hat{\beta}_1) &= \frac{\sum_{i=1}^{3}(i-2)^2 i\sigma^2}{\left(\sum_{i=1}^{3}(i-2)^2\right)^2} = \frac{\sigma^2 + 3\sigma^2}{4} = \sigma^2 \\ Var(\hat{\beta}_0) &= Var(\frac{y_1+y_2+y_3}{3} - \sum_{i=1}^{3}(i-2)y_i) \\ &= Var(\sum_{i=1}^{3}(\frac{7}{3} - i)e_i) = \sum_{i=1}^{3}(\frac{7}{3} - i)^2 i\sigma^2 \\ &= \frac{16}{9}\sigma^2 + \frac{1}{9}2\sigma^2 + \frac{4}{9}3\sigma^2 = \frac{10}{3}\sigma^2 \end{aligned}$$

(c) Write $\hat{\boldsymbol{e}} = (\boldsymbol{I} - \boldsymbol{H})(\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{e}) = (\boldsymbol{I} - \boldsymbol{H})\boldsymbol{e}$
where

$$\boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \quad \boldsymbol{X}^T\boldsymbol{X} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix} \quad (\boldsymbol{X}^T\boldsymbol{X})^{-1} = \begin{pmatrix} \frac{7}{3} & -1 \\ -1 & \frac{1}{2} \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{H} = \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$

$$\Rightarrow \quad \boldsymbol{I} - \boldsymbol{H} = \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\begin{aligned} E(\hat{\boldsymbol{e}}) &= (\boldsymbol{I} - \boldsymbol{H})E(\boldsymbol{e}) = 0 \\ Var(\hat{\boldsymbol{e}}) &= \sigma^2(\boldsymbol{I} - \boldsymbol{H})\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}(\boldsymbol{I} - \boldsymbol{H}) = \frac{\sigma^2}{3}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} \end{aligned}$$