17 Sept

Likelihood $(\hat\beta_0, \hat\beta_1, \tilde\sigma^2)$

$$= \left(\frac{1}{2\pi}\right)^{n/2} (\tilde\sigma^2)^{-n/2} \exp\left\{ - \frac{\sum_{i=1}^n (y_i - (\hat\beta_0 + \hat\beta_1 x_{i1}))^2}{2\tilde\sigma^2} \right\} \qquad \tilde\sigma^2 = \frac{\sum_{i=1}^n (y_i - \hat y_i)^2}{n}$$

$$= \left(\frac{1}{2\pi}\right)^{n/2} \left(\boxed{\tilde\sigma^2}\right)^{-n/2} \exp\left\{-\frac{n}{2}\right\}$$

Likelihood $\uparrow$ $\quad \tilde\sigma^2 \downarrow$

$\Rightarrow$ We look for the model with smallest Res S.S.

Section 3 Properties of $\hat\beta$ & Res S.S.

① Properties of $\hat\beta$

$p=1$ p.11 of chapter 1

Theorem 3.1 $\quad y_i \sim N(\mu_i, \sigma^2) \qquad \mu_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_p x_{ip}$

$$\operatorname{cov}(y_i, y_j) = 0$$

$$\operatorname{cov}\left(\sum_{i=1}^n c_i y_i, \sum_{j=1}^n d_j y_j\right) = \sum_{i=1}^n c_i d_i \underbrace{\operatorname{Var}(y_i)}_{\sigma^2}$$

$$\hat\beta_0 = \bar y - \hat\beta_1 \bar x_1 \qquad \hat\beta_1 = \frac{S_{x_1 y}}{S_{x_1 x_1}}$$

$$S_{x_1y} = \sum_{i=1}^n (x_{i1} - \bar x_1) y_i - \bar y \underbrace{\boxed{\sum_{i=1}^n (x_{i1} - \bar x_1)}}_{=0}$$

$$= \frac{\sum_{i=1}^n y_i}{n} - \frac{\sum_{i=1}^n (x_{i1} - \bar x_1)(y_i - \bar y)}{\sum_{i=1}^n (x_{i1} - \bar x_1)^2} \bar x_1$$

$$= \frac{\sum_{i=1}^n y_i}{n} - \frac{\bar x_1}{\sum_{i=1}^n (x_{i1} - \bar x_1)^2} \sum_{i=1}^n (x_{i1} - \bar x_1) \boxed{y_i}$$

$$= \sum_{i=1}^n \underbrace{\left(\frac{1}{n} - \frac{(x_{i1} - \bar x_1)\bar x_1}{\sum_{i=1}^n (x_{i1} - \bar x_1)^2} \right)}_{c_i} \boxed{y_i} \quad \sim N$$

$$\sim N(\beta_0 \quad , \quad)$$

Model : $y_i = \beta_0 + \beta_1 x_{i1} + e_i$

$$E(y_i) = \beta_0 + \beta_1 x_{i1}$$

$$Var(y_i) = \sigma^2$$

$$E(\hat{\beta}_0) = \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{(x_{i1} - \bar{x}_1)\bar{x}_1}{S_{x_1x_1}} \right) E(y_i)$$

$$= \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{(x_{i1} - \bar{x}_1)\bar{x}_1}{S_{x_1x_1}} \right) (\beta_0 + \beta_1 x_{i1})$$

$$= \beta_0 \underbrace{\sum_{i=1}^{n} \left(\frac{1}{n} - \frac{(x_{i1} - \bar{x}_1)\bar{x}_1}{S_{x_1x_1}} \right)}_{1 - \underbrace{\frac{\sum_{i=1}^{n}(x_{i1}-\bar{x}_1)\bar{x}_1}{S_{x_1x_1}}}_{=0}} + \beta_1 \underbrace{\sum_{i=1}^{n} \left(\frac{1}{n} - \frac{(x_{i1} - \bar{x}_1)\bar{x}_1}{S_{x_1x_1}} \right) x_{i1}}_{\sum_{i=1}^{n}\frac{x_{i1}}{n} - \underbrace{\frac{\sum_{i=1}^{n}(x_{i1}-\bar{x}_1)\bar{x}_1 x_{i1}}{S_{x_1x_1}}}_{= \bar{x}_1 \frac{\sum_{i=1}^{n}(x_{i1}-\bar{x}_1)(x_{i1}-\bar{x}_1)}{S_{x_1x_1}}}}$$

$= \beta_0$ — unbiased estimator

$$\boxed{\begin{aligned} &Var(aX + bY) \\ &= a^2 Var(X) + b^2 Var(Y) \\ &\quad + 2ab\, cov(X, Y) \end{aligned}}$$

$$Var(\hat{\beta}_0) = Var\left(\sum_{i=1}^{n} c_i y_i \right)$$

$$= \sum_{i=1}^{n} c_i^2 \boxed{Var(y_i)} = \sigma^2$$

$$= \sigma^2 \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{(x_{i1} - \bar{x}_1)\bar{x}_1}{S_{x_1x_1}} \right)^2$$

$$= \sigma^2 \left[\sum_{i=1}^{n} \left(\frac{1}{n^2} - \frac{2(x_{i1} - \bar{x}_1)\bar{x}_1}{S_{x_1x_1}} + \frac{(x_{i1} - \bar{x}_1)^2 \bar{x}_1^2}{S_{x_1x_1}^2} \right) \right]$$

$$\sum_{i=1}^{n} \frac{1}{n^2} \rightarrow \frac{1}{n} \qquad \sum_{i=1}^{n} \frac{2(x_{i1}-\bar{x}_1)\bar{x}_1}{S_{x_1x_1}} \rightarrow 0 \qquad \frac{\sum_{i=1}^{n}(x_{i1}-\bar{x}_1)^2\, \bar{x}_1^2}{S_{x_1x_1}^2},\ \sum_{i=1}^{n}(x_{i1}-\bar{x}_1)^2 \rightarrow S_{x_1x_1} \ \rightarrow \frac{\bar{x}_1^2}{S_{x_1x_1}}$$

$$= \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}_1^2}{S_{x_1x_1}} \right)$$

$$\Rightarrow \hat{\beta}_0 \sim N\left(\beta_0, \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}_1^2}{S_{x_1x_1}} \right) \right)$$

$$\hat\beta_1 = \frac{S_{x_1 y}}{S_{x_1 x_1}}$$

$$= \frac{\sum_{i=1}^{n} (x_{i1} - \bar{x}_1)\, y_i}{S_{x_1 x_1}}$$

$$= \sum_{i=1}^{n} \underbrace{\frac{x_{i1} - \bar{x}_1}{S_{x_1 x_1}}}_{d_i} \underbrace{\boxed{y_i}}_{\sim N}$$

$$\hat\beta_1 \sim N(\quad , \quad)$$

$E(y_i)$

$$E(\hat\beta_1) = \sum_{i=1}^{n} \frac{x_{i1} - \bar{x}_1}{S_{x_1 x_1}} (\beta_0 + \beta_1 x_{i1})$$

$$= \beta_0 \underbrace{\boxed{\sum_{i=1}^{n} \frac{(x_{i1} - \bar{x}_1)}{S_{x_1 x_1}}}}_{=\,0} + \beta_1 \underbrace{\boxed{\sum_{i=1}^{n} \frac{(x_{i1} - \bar{x}_1)(x_{i1} - \bar{x}_1)}{S_{x_1 x_1}}}}_{=\,1}$$

$$= \beta_1 \quad \text{— unbiased estimator}$$

$$Var(\hat\beta_1) = \sum_{i=1}^{n} \left(\frac{x_{i1} - \bar{x}_1}{S_{x_1 x_1}} \right)^2 \boxed{Var(y_i)} \leftarrow \sigma^2$$

$$= \sigma^2 \frac{\sum_{i=1}^{n} (x_{i1} - \bar{x}_1)^2}{S_{x_1 x_1}^2} \quad S_{x_1 x_1}$$

$$= \frac{\sigma^2}{S_{x_1 x_1}}$$

$$\Rightarrow \hat\beta_1 \sim N\left(\beta_1 , \frac{\sigma^2}{S_{x_1 x_1}} \right)$$

$$Cov(\hat\beta_0, \hat\beta_1) = Cov\left(\sum_{i=1}^{n} c_i y_i , \sum_{j=1}^{n} d_j y_j \right)$$

$$c_i = \left(\frac{1}{n} - \frac{(x_{i1} - \bar{x}_1)\bar{x}_1}{S_{x_1 x_1}} \right) \qquad d_j = \frac{x_{i1} - \bar{x}_1}{S_{x_1 x_1}}$$

$$= \sum_{i=1}^{n} c_i d_i \boxed{Var(y_i)} = \sigma^2$$

$$= \sigma^2 \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{(x_{i1}-\bar{x}_1)\bar{x}_1}{S_{x_1x_1}} \right) \left(\frac{x_i - \bar{x}_1}{S_{x_1x_1}} \right)$$

$$= \sigma^2 \left(\underbrace{\sum_{i=1}^{n} \frac{x_{i1}-\bar{x}_1}{n\, S_{x_1x_1}}}_{=\,0} - \underbrace{\frac{\overbrace{\boxed{\sum_{i=1}^{n} (x_{i1}-\bar{x}_1)^2}}^{=\,S_{x_1x_1}}\,\bar{x}_1}{S_{x_1x_1}^2}}_{=\,\frac{\bar{x}_1}{S_{x_1x_1}}} \right)$$

$$= -\frac{\bar{x}_1}{S_{x_1x_1}}\sigma^2$$

$$\Rightarrow \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}, \underbrace{\begin{pmatrix} \overset{Var(\hat{\beta}_0)}{\boxed{\sigma^2\left(\frac{1}{n}+\frac{\bar{x}_1^2}{S_{x_1x_1}}\right)}} & \overset{cov(\hat{\beta}_0,\hat{\beta}_1)}{\boxed{-\frac{\bar{x}_1}{S_{x_1x_1}}\sigma^2}} \\ \underset{cov(\hat{\beta}_1,\hat{\beta}_0)}{\boxed{-\frac{\bar{x}_1}{S_{x_1x_1}}\sigma^2}} & \underset{Var(\hat{\beta}_1)}{\boxed{\frac{\sigma^2}{S_{x_1x_1}}}} \end{pmatrix}_{2\times 2}}_{\Sigma \text{ variance \& covariance matrix}} \right)$$

$p > 1$ $\quad \hat{\beta} = (X^T X)^{-1} X^T \boxed{Y}$ $\sim MN(X\beta, \sigma^2 I)$

Theorem 3, 2 (p.14 of Chapter 1)

$$E(\underset{\sim}{c}\,\underset{\sim}{z} + \underset{\sim}{d}) = \underset{\sim}{c}\, E(\underset{\sim}{z}) + \underset{\sim}{d}$$

$$Var(\underset{\sim}{c}\,\underset{\sim}{z} + \underset{\sim}{d}) = \underset{\sim}{c}\, Var(\underset{\sim}{z})\, \underset{\sim}{c}^T$$

$$Cov(\underset{\sim}{c}\,\underset{\sim}{z}, \underset{\sim}{d}\,\underset{\sim}{z}) = \underset{\sim}{c}\, Var(\underset{\sim}{z})\, \underset{\sim}{d}^T$$

$E(cz+d) = cE(z) + d$

$Var(cz+d) = c^2 Var(z)$

$Cov(cz, dz) = cd\, Var(z)$

$$\underset{p'\times 1}{\hat{\beta}} = \underbrace{\boxed{\underset{p'\times p'}{(X^T X)^{-1}}\ \underset{p'\times 1}{X^T}}}_{\underset{p'\times 1}{\underset{\sim}{C}}}\ \underset{n\times 1}{Y}$$

$$E(\hat{\beta}) = (X^T X)^{-1} X^T \boxed{E(Y)} = X\beta$$

$$= (X^T X)^{-1} X^T X \beta = \beta$$

$$Var(\hat{\beta}) = \underbrace{(X^T X)^{-1} X^T}_{C} \underbrace{\boxed{Var(Y)}}_{= \sigma^2 I} \underbrace{((X^T X)^{-1} X^T)^T}_{C^T}$$

$$= \sigma^2 (X^T X)^{-1} X^T \underbrace{((X^T X)^{-1} X^T)^T}_{X (X^T X)^{-1}}$$

$$= (X^T X)^{-1} \sigma^2$$

$$\Rightarrow \hat{\beta} \sim MN(\beta, \sigma^2 (X^T X)^{-1})$$

e.g. $p = 1$ $\underset{n\times 2}{X} = \begin{pmatrix} 1 & x_{11} \\ \vdots & \vdots \\ 1 & x_{n1} \end{pmatrix}$ $\quad X^T X = \begin{pmatrix} n & \sum_{i=1}^n x_{i1} \\ \sum_{i=1}^n x_{i1} & \sum_{i=1}^n x_{i1}^2 \end{pmatrix}$

$$(X^T X)^{-1} = \frac{1}{\boxed{n \sum_{i=1}^n x_{i1}^2 - (\sum_{i=1}^n x_{i1})^2}} \begin{pmatrix} \sum_{i=1}^n x_{i1}^2 & -\sum_{i=1}^n x_{i1} \\ -\sum_{i=1}^n x_{i1} & n \end{pmatrix}$$

$$n \sum_{i=1}^n x_{i1}^2 - (\sum_{i=1}^n x_{i1})^2 = n\left(\sum_{i=1}^n x_{i1}^2 - \frac{1}{n}(n\bar{x}_1)^2\right) = n\left(\sum_{i=1}^n x_{i1}^2 - n\bar{x}_1^2\right) = n S_{x_1 x_1}$$

$$\Rightarrow Var(\hat{\beta}_0) = \frac{\sum_{i=1}^n x_{i1}^2}{n S_{x_1 x_1}} \sigma^2$$

$$\sum_{i=1}^n x_{i1}^2 - n\bar{x}_1^2 = S_{x_1 x_1}$$

$$\Rightarrow \sum_{i=1}^n x_{i1}^2 = S_{x_1 x_1} + n\bar{x}_1^2$$

$$Var(\hat{\beta}_1) = \frac{n}{n S_{x_1 x_1}} \sigma^2 = \frac{\sigma^2}{S_{x_1 x_1}}$$

$$Cov(\hat{\beta}_0, \hat{\beta}_1) = -\frac{\sum_{i=1}^n x_{i1}}{n S_{x_1 x_1}} \sigma^2$$

e.g. $\underline{p=1}$ β_0 is known

$$\underset{n\times 1}{\underset{\sim}{X}} = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix} \qquad \underset{\sim}{X}^T\underset{\sim}{X} = \sum_{i=1}^{n} x_{i1}^2$$

$$(\underset{\sim}{X}^T\underset{\sim}{X})^{-1} = \frac{1}{\sum_{i=1}^{n} x_{i1}^2}$$

$$Var(\hat{\beta}_1) = \sigma^2 / \sum_{i=1}^{n} x_{i1}^2$$

2 Properties of $\hat{e}_i$

$\underline{p=1}$

$$\hat{e}_i = y_i - \hat{y}_i$$

$$= y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_{i1})$$

$$= y_i - \left(\sum_{j=1}^{n} c_j y_j + \sum_{j=1}^{n} d_j x_{i1} y_j\right)$$

$$= y_i - \sum_{j=1}^{n} (c_j + d_j x_{i1}) y_j$$

$$= \sum_{j=1}^{n} (\delta_{ij} - (c_j + d_j x_{i1})) \boxed{y_j} \qquad \boxed{\delta_{ij}} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$\sim N$ kronecker delta

$$\hat{e}_i \sim N(\underset{E(\hat{e}_i)}{\uparrow} \quad , \quad \underset{Var(\hat{e}_i)}{\uparrow})$$

$\underline{p>1}$

$$\underset{\sim}{\hat{e}} = \underset{\sim}{Y} - \underset{\sim}{\hat{Y}}$$

$$= \underset{\sim}{Y} - \underset{\sim}{X}\underset{\sim}{\hat{\beta}}$$

$$= \underset{\sim}{Y} - \underset{\sim}{X}(\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\underset{\sim}{X}^T\underset{\sim}{Y}$$

$$= (\underset{\sim}{I} - \underset{\sim}{X}(\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\underset{\sim}{X}^T)\underset{\sim}{Y}$$

$$= (\underset{\sim}{I} - \underset{\sim}{H})\boxed{\underset{\sim}{Y}} \sim MN(\underset{\sim}{X}\underset{\sim}{\beta}, \sigma^2 \underset{\sim}{I})$$

$$\Rightarrow \underset{\sim}{\hat{e}} \sim MN(\quad , \quad)$$

③ Properties of Res S.S.

$p=1$ $\quad$ Res S.S. $= \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$

(β_0 is unknown) $\quad = S_{yy} - \hat{\beta}_1 \boxed{S_{x_1 y}}$ $\quad \frac{S_{x_1 y}}{S_{x_1 x_1}} = \hat{\beta}_1$

$= S_{yy} - \hat{\beta}_1^2 S_{x_1 x_1}$

$$E(\text{Res S.S.}) = \boxed{E(S_{yy})} - S_{x_1 x_1}\boxed{E(\hat{\beta}_1^2)}$$

$E(S_{yy}) \neq (n-1)\sigma^2$

$E(\hat{\beta}_1^2) = Var(\hat{\beta}_1) + (E(\hat{\beta}_1))^2 = \frac{\sigma^2}{S_{x_1 x_1}} + \beta_1^2$

$E(y_i)$ = diff means $= \beta_0 + \beta_1 x_{i1}$

$$E(S_{yy}) = E\left(\sum_{i=1}^{n}(y_i - \bar{y})^2\right)$$

$$= E\left(\sum_{i=1}^{n} y_i^2 - n\bar{y}^2\right)$$

$$= \boxed{E\left(\sum_{i=1}^{n} y_i^2\right)} - n\boxed{(E(\bar{y}^2))}$$

$Var(y_i) = \sigma^2$

$E(y_i) = \beta_0 + \beta_1 x_{i1}$

$E\left(\sum_{i=1}^{n} y_i^2\right) = \sum_{i=1}^{n} \underbrace{E(y_i^2)}_{Var(y_i) + (E(y_i))^2} \overset{?}{\neq} n E(y_i^2)$

$E(\bar{y}^2) = Var(\bar{y}) + (E(\bar{y}))^2$, $\quad Var(\bar{y}) = \sigma^2/n$, $\quad (E(\bar{y}))^2 = (\beta_0 + \beta_1 \bar{x}_1)^2$

$$= \sum_{i=1}^{n}\left(\sigma^2 + (\beta_0 + \beta_1 x_{i1})^2\right)$$

$$E(S_{yy}) = n\sigma^2 + \sum_{i=1}^{n}(\beta_0 + \beta_1 x_{i1})^2 - n\left(\frac{\sigma^2}{n} + (\beta_0 + \beta_1 \bar{x}_1)^2\right)$$

$$= (n-1)\sigma^2 + \beta_1^2 S_{x_1 x_1}$$

$$\sum_{i=1}^{n}(\beta_0 + \beta_1 x_{i1})^2 - n(\beta_0 + \beta_1 \bar{x}_1)^2$$

$$= \sum_{i=1}^{n}(\beta_0^2 + 2\beta_0\beta_1 x_{i1} + \beta_1^2 x_{i1}^2) - n(\beta_0^2 + 2\beta_0\beta_1 \bar{x}_1 + \beta_1^2 \bar{x}_1^2)$$

$$= \beta_1^2\left[\sum_{i=1}^{n} x_{i1}^2 - n\bar{x}_1^2\right] = \beta_1^2 S_{x_1 x_1}$$

$$E(\text{Res S.S.}) = (n-1)\sigma^2 + \beta_1^2 S_{x_1 x_1} - S_{x_1 x_1}\left(\sigma^2/S_{x_1 x_1} + \beta_1^2\right)$$

$= (n-2)\sigma^2$

$\Rightarrow$ unbiased est. of $\sigma^2 = \dfrac{\text{Res S.S.}}{n-2}$

$p \geq 1$ $\quad \hat{\sigma}^2 = \dfrac{\text{Res S.S.}}{\boxed{n} - \boxed{p'}}$

n = # of obs. $\quad p'$ = # of unknown para. in the model

① Prove $E(\text{Res S.S.}) = (n-p')\sigma^2$

② dist. of Res S.S. ?

Theorem 3.3 ~~in~~ (p.16 of ~~Theor~~ Chapter 1)

$E(\underset{\sim}{Y}) = \underset{\sim}{\mu}$, $\quad cov(\underset{\sim}{Y}) = \underset{\sim}{\Sigma}$

quadratic form of $\underset{\sim}{Y}$

Then $E(\boxed{\underset{\sim}{Y}^T \underset{\sim}{A} \underset{\sim}{Y}}) = trace(\underset{\sim}{A}\underset{\sim}{\Sigma}) + \underset{\sim}{\mu}^T \underset{\sim}{A} \underset{\sim}{\mu}$

Model : $\underset{\sim}{Y} = \underset{\sim}{X}\underset{\sim}{\beta} + \underset{\sim}{e}$ $\quad E(\underset{\sim}{Y}) = \underset{\sim}{X}\underset{\sim}{\beta}$

$Var(\underset{\sim}{Y}) = \sigma^2 \underset{\sim}{I}$

$$\text{Res S.S.} = \underset{\sim}{Y}^T\underset{\sim}{Y} - \hat{\underset{\sim}{\beta}}^T \underset{\sim}{X}^T \underset{\sim}{Y}$$

$$= \underset{\sim}{Y}^T\underset{\sim}{Y} - \underbrace{\underset{\sim}{Y}^T\underset{\sim}{X}(\underset{\sim}{X}^T\underset{\sim}{X})^{-1}}_{\hat{\underset{\sim}{\beta}}^T}\underset{\sim}{X}^T\underset{\sim}{Y} \qquad \hat{\underset{\sim}{\beta}} = (\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\underset{\sim}{X}^T\underset{\sim}{Y}$$

$$= \underset{\sim}{Y}^T(\underset{\sim}{I} - \underbrace{\underset{\sim}{X}(\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\underset{\sim}{X}^T}_{\underset{\sim}{H}})\underset{\sim}{Y}$$

$$= \underset{\sim}{Y}^T(\underset{\sim}{I} - \underset{\sim}{H})\underset{\sim}{Y}$$

$$E(\text{Res S.S.}) = E(\underset{1\times n}{\underset{\sim}{Y}^T} \underbrace{\overset{n\times n}{(\underset{\sim}{I} - \underset{\sim}{H})}}_{\underset{\sim}{A}} \underset{n\times 1}{\underset{\sim}{Y}})$$

$$\text{trace}(\underset{\sim}{A} \underset{\sim}{\Sigma}) = \text{trace}\left((\underset{\sim}{I} - \underset{\sim}{H}) \sigma^2 \underset{\sim}{I}\right)$$

$$= \sigma^2 \, \text{trace}(\underset{\sim}{I} - \underset{\sim}{H})$$

$$= \sigma^2 \{ \text{trace}(\underset{n\times n}{\underset{\sim}{I}}) - \text{trace}(\underset{\sim}{X} (\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \underset{\sim}{X}^T) \}$$

$$= \sigma^2 \{ n - \text{trace}(\underbrace{(\underset{\sim}{X}^T \underset{\sim}{X})^{-1} (\underset{p'\times n}{\underset{\sim}{X}^T} \underset{n\times p'}{\underset{\sim}{X}})}_{I_{p'\times p'}}) \}$$

\# of indep. variables

$$p' = 1 + p$$

$$= \sigma^2 \{ n - p' \}$$

$$\underset{\sim}{\mu}^T \underset{\sim}{A} \underset{\sim}{\mu} = (\underset{\sim}{X} \underset{\sim}{\beta})^T (\underset{\sim}{I} - \underset{\sim}{X} (\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \underset{\sim}{X}^T) (\underset{\sim}{X} \underset{\sim}{\beta})$$

$$\mu = E(Y) = \underset{\sim}{X} \underset{\sim}{\beta}$$

$$= \underset{\sim}{\beta}^T \underset{\sim}{X}^T (\underset{\sim}{I} - \underset{\sim}{X} (\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \underset{\sim}{X}^T) \underset{\sim}{X} \underset{\sim}{\beta}$$

$$= \underset{\sim}{\beta}^T (\underset{\sim}{X}^T \underset{\sim}{X} - \underset{\sim}{X}^T \underset{\sim}{X} \underbrace{(\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \underset{\sim}{X}^T \underset{\sim}{X}}_{\underset{\sim}{I}}) \underset{\sim}{\beta}$$

$$= 0$$

$$\Rightarrow E(\text{Res S.S.}) = \sigma^2 (n - p') + 0$$

$$\Rightarrow \hat{\sigma}^2 = \frac{\text{Res S.S.}}{n - p'} \text{ is unbiased for } \sigma^2$$