Increase in Regression Sum of Squares

Theorem 4.3: When $\underset{\sim}{A}$ and $\underset{\sim}{D}$ are symmetric matrices such that the inverses exist, then

$$\begin{pmatrix} \underset{\sim}{A} & \underset{\sim}{B} \\ \underset{\sim}{B}^T & \underset{\sim}{D} \end{pmatrix}^{-1} = \begin{pmatrix} \underset{\sim}{E}^{-1} & -\underset{\sim}{E}^{-1}\underset{\sim}{F}^T \\ -\underset{\sim}{F}\underset{\sim}{E}^{-1} & \underset{\sim}{D}^{-1} + \underset{\sim}{F}\underset{\sim}{E}^{-1}\underset{\sim}{F}^T \end{pmatrix}$$

where $\underset{\sim}{E} = \underset{\sim}{A} - \underset{\sim}{B}\underset{\sim}{D}^{-1}\underset{\sim}{B}^T$ and $\underset{\sim}{F} = \underset{\sim}{D}^{-1}\underset{\sim}{B}^T$.

Test $H_0 : \beta_1 = \beta_2 = \ldots = \beta_r = 0$. Consider the centered model and write

$$\begin{aligned}\underset{\sim}{Y} &= \begin{pmatrix} \underset{\sim}{j} & \underset{\sim}{X}_r & \underset{\sim}{X}_s \end{pmatrix} \begin{pmatrix} \alpha \\ \underset{\sim}{\beta}_r \\ \underset{\sim}{\beta}_s \end{pmatrix} + \underset{\sim}{e} \\ &= \alpha\underset{\sim}{j} + \underset{\sim}{X}_r\underset{\sim}{\beta}_r + \underset{\sim}{X}_s\underset{\sim}{\beta}_s + \underset{\sim}{e}\end{aligned}$$

$$\begin{pmatrix} \hat{\underset{\sim}{\beta}}_r \\ \hat{\underset{\sim}{\beta}}_s \end{pmatrix} = \begin{pmatrix} \underset{\sim}{X}_r^T\underset{\sim}{X}_r & \underset{\sim}{X}_r^T\underset{\sim}{X}_s \\ \underset{\sim}{X}_s^T\underset{\sim}{X}_r & \underset{\sim}{X}_s^T\underset{\sim}{X}_s \end{pmatrix}^{-1} \begin{pmatrix} \underset{\sim}{X}_r^T\underset{\sim}{Y} \\ \underset{\sim}{X}_s^T\underset{\sim}{Y} \end{pmatrix}$$

$$= \begin{pmatrix} \underset{\sim}{A}_{rr.s}^{-1} & -\underset{\sim}{A}_{rr.s}^{-1}\,\underset{\sim}{X}_r^T\underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1} \\ -(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\,\underset{\sim}{X}_s^T\underset{\sim}{X}_r\underset{\sim}{A}_{rr.s}^{-1} & (\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1} + (\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\,\underset{\sim}{X}_s^T\underset{\sim}{X}_r\underset{\sim}{A}_{rr.s}^{-1}\,\underset{\sim}{X}_r^T\underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1} \end{pmatrix} \begin{pmatrix} \underset{\sim}{X}_r^T\underset{\sim}{Y} \\ \underset{\sim}{X}_s^T\underset{\sim}{Y} \end{pmatrix}$$

where $\underset{\sim}{A}_{rr.s} = \underset{\sim}{X}_r^T\underset{\sim}{X}_r - \underset{\sim}{X}_r^T\underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\underset{\sim}{X}_s^T\underset{\sim}{X}_r$.

Let $\hat{\underset{\sim}{\beta}}_1 = \begin{pmatrix} \hat{\underset{\sim}{\beta}}_r \\ \hat{\underset{\sim}{\beta}}_s \end{pmatrix}$ and $\underset{\sim}{X}_1 = \begin{pmatrix} \underset{\sim}{X}_r & \underset{\sim}{X}_s \end{pmatrix}$.

Then, Reg. S.S. for the p independent variables

$$\begin{aligned} &= \hat{\underset{\sim}{\beta}}_1^T\,\underset{\sim}{X}_1^T\,\underset{\sim}{X}_1\,\hat{\underset{\sim}{\beta}}_1 \\ &= \underset{\sim}{Y}^T\,\underset{\sim}{X}_1\,(\underset{\sim}{X}_1^T\underset{\sim}{X}_1)^{-1}\underset{\sim}{X}_1^T\underset{\sim}{Y} \\ &= \underset{\sim}{Y}^T\underset{\sim}{X}_r\underset{\sim}{A}_{rr.s}^{-1}\,\underset{\sim}{X}_r^T\underset{\sim}{Y} - 2\,\underset{\sim}{Y}^T\underset{\sim}{X}_r\underset{\sim}{A}_{rr.s}^{-1}\,\underset{\sim}{X}_r^T\underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\underset{\sim}{X}_s^T\underset{\sim}{Y} + \underset{\sim}{Y}^T\underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\,\underset{\sim}{X}_s^T\underset{\sim}{Y} + \\ &\quad\ \underset{\sim}{Y}^T\underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\,\underset{\sim}{X}_s^T\underset{\sim}{X}_r\underset{\sim}{A}_{rr.s}^{-1}\,\underset{\sim}{X}_r^T\underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\,\underset{\sim}{X}_s^T\underset{\sim}{Y} \end{aligned}$$

For the model involving only the variables in the complementary set (model under H_0), i.e.,

$$\begin{aligned} \underset{\sim}{Y} &= \alpha\underset{\sim}{j} + \underset{\sim}{X}_s\underset{\sim}{\beta}_s^* + \underset{\sim}{e} \\ \Rightarrow \hat{\underset{\sim}{\beta}}_s^* &= (\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\underset{\sim}{X}_s^T\underset{\sim}{Y} \end{aligned}$$

and the sum of squares due to its fitted regression model (Reg. S.S.) is equal to

$$\underset{\sim}{Y}^T\,\underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\underset{\sim}{X}_s^T\underset{\sim}{Y}$$

Increase in Reg. S.S.

$$\begin{aligned} &= \underset{\sim}{Y}^T\underset{\sim}{X}_r\underset{\sim}{A}_{rr.s}^{-1}\,\underset{\sim}{X}_r^T\underset{\sim}{Y} - 2\,\underset{\sim}{Y}^T\underset{\sim}{X}_r\underset{\sim}{A}_{rr.s}^{-1}\,\underset{\sim}{X}_r^T\underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\underset{\sim}{X}_s^T\underset{\sim}{Y} + \\ &\quad\ \underset{\sim}{Y}^T\underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\,\underset{\sim}{X}_s^T\underset{\sim}{X}_r\underset{\sim}{A}_{rr.s}^{-1}\,\underset{\sim}{X}_r^T\underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\,\underset{\sim}{X}_s^T\underset{\sim}{Y} \\ &= \underset{\sim}{Y}^T\,\underbrace{(\underset{\sim}{X}_r - \underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\underset{\sim}{X}_s^T\underset{\sim}{X}_r)\,\underset{\sim}{A}_{rr.s}^{-1}\,(\underset{\sim}{X}_r^T - \underset{\sim}{X}_r^T\underset{\sim}{X}_s\,(\underset{\sim}{X}_s^T\underset{\sim}{X}_s)^{-1}\underset{\sim}{X}_s^T)}_{\underset{\sim}{A}}\underset{\sim}{Y} \end{aligned}$$

By **Theorem 4.1** , let

$$
\begin{aligned}
\utilde{A} &= (\utilde{X}_r - \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1} \utilde{X}_s^T \utilde{X}_r)\ \utilde{A}_{rr.s}^{-1}\ (\utilde{X}_r^T - \utilde{X}_r^T \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1} \utilde{X}_s^T) \\
\utilde{B} &= \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1} \utilde{X}_s^T
\end{aligned}
$$

and then

$$
\begin{aligned}
\utilde{A}\utilde{B} &= (\utilde{X}_r - \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1} \utilde{X}_s^T \utilde{X}_r)\ \utilde{A}_{rr.s}^{-1}\ (\utilde{X}_r^T - \utilde{X}_r^T \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1} \utilde{X}_s^T) \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1} \utilde{X}_s^T \\
&= \utilde{0}
\end{aligned}
$$

$$
\begin{array}{ccccc}
Reg\, S.S.|_F & = & Reg.\, S.S.|_R & + & Increase\ in\ Reg.S.S. \\
\sim \sigma^2 \chi^2(p, \lambda) & & \sim \sigma^2 \chi^2(p-r, \lambda_1) & &
\end{array}
$$

where $\lambda = \dfrac{1}{\sigma^2} \utilde{\beta}_1^T\ \utilde{X}_1^T\ \utilde{X}_1\ \utilde{\beta}_1$ and $\lambda_1 = \dfrac{1}{\sigma^2} \utilde{\beta}_s^T\ \utilde{X}_s^T\ \utilde{X}_s\ \utilde{\beta}_s$.

By **Theorem 4.2** , $Reg.\, S.S.|_R$ and Increase in $Reg.S.S.$ are independent $\Rightarrow$ $Increase\ in\ Reg.S.S. \sim \sigma^2 \chi^2(r, \lambda_2)$, where $\lambda_2 = \dfrac{1}{\sigma^2} \left(\utilde{\beta}_1^T\ \utilde{X}_1^T\ \utilde{X}_1\ \utilde{\beta}_1\ -\ \utilde{\beta}_s^T\ \utilde{X}_s^T\ \utilde{X}_s\ \utilde{\beta}_s \right)$

Also, Increase in Reg. S.S.

$$
\begin{aligned}
&= \utilde{Y}^T \utilde{X}_r\ \utilde{A}_{rr.s}^{-1}\ \utilde{X}_r^T \utilde{Y}\ -\ 2\, \utilde{Y}^T \utilde{X}_r\ \utilde{A}_{rr.s}^{-1}\ \utilde{X}_r^T \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1} \utilde{X}_s^T \utilde{Y}\ + \\
&\quad \utilde{Y}^T \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1}\ \utilde{X}_s^T \utilde{X}_r\ \utilde{A}_{rr.s}^{-1}\ \utilde{X}_r^T \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1}\ \utilde{X}_s^T \utilde{Y} \\
&= \utilde{Y}^T\ (\utilde{X}_r - \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1} \utilde{X}_s^T \utilde{X}_r)\ \utilde{A}_{rr.s}^{-1}\ (\utilde{X}_r^T - \utilde{X}_r^T \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1} \utilde{X}_s^T) \utilde{Y} \\
&= \underbrace{\utilde{Y}^T\ (\utilde{X}_r - \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1} \utilde{X}_s^T \utilde{X}_r)\ \utilde{A}_{rr.s}^{-1}}_{\hat{\utilde{\beta}}_r^T}\ \utilde{A}_{rr.s}\ \underbrace{\utilde{A}_{rr.s}^{-1}\ (\utilde{X}_r^T - \utilde{X}_r^T \utilde{X}_s\ (\utilde{X}_s^T \utilde{X}_s)^{-1} \utilde{X}_s^T) \utilde{Y}}_{\hat{\utilde{\beta}}_r} \\
&= \hat{\utilde{\beta}}_r^T\ \utilde{A}_{rr.s}\ \hat{\utilde{\beta}}_r \\
&= (\utilde{C} \hat{\utilde{\beta}}_1)^T\ (\utilde{C} (\utilde{X}_1^T \utilde{X}_1)^{-1}\ \utilde{C}^T)^{-1}\ (\utilde{C} \hat{\utilde{\beta}}_1)
\end{aligned}
$$