F-test

8 October

$$F = \frac{\boxed{\text{Increase in Reg S.S.}}/r}{\hat{\sigma}^2} = F = \frac{\boxed{(C\hat{\beta} - d)^T (C(X^TX)^{-1}C^T)^{-1} (C\hat{\beta} - d)}/r}{\hat{\sigma}^2}$$

r: d.f. of Reg S.S. | full − d.f. of Reg S.S. | reduced

r: rank of C

Can be proved

Res S.S. | reduced

$\boxed{C(X^TX)^{-1}C^T}$

(a) $H_0: \beta_3 = 0$

$\Rightarrow$ F-test

$\tilde{\beta}_R = \boxed{(X_R^T X_R)^{-1}} X_R^T Y$

$\underset{n\times 3}{X_R} \Rightarrow \underset{3\times 3}{(X_R^T X_R)^{-1}}$

$H_0: \underset{1\times 4}{(0 \ 0 \ 0 \ 1)} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = 0$

$C(X^TX)^{-1}C^T \ {}_{1\times 1}$

~~(b) $H_0: \beta_1 = \beta_2 = 0$~~

$\Rightarrow$ t-test

$$t = \frac{\hat{\beta}_3 - \beta_3 \text{ under } H_0}{\text{s.e. of } \hat{\beta}_3}$$

(b) $H_0: \beta_1 = \beta_2 = 0$

$\Rightarrow H_0: \beta_1 = 0, \ \beta_2 = 0$

− F-test

$\underset{n\times 2}{X_R} \Rightarrow X_R^T X_R \ {}_{2\times 2}$

β_0, β_3

$$\begin{pmatrix} 1 & x_{13} \\ \vdots & \vdots \\ 1 & x_{n3} \end{pmatrix}$$

$C(X^TX)^{-1}C^T \ {}_{2\times 2}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(c) $H_0: \beta_1 = \beta_2$

$\Rightarrow H_0: \beta_1 - \beta_2 = 0$

− t-test

pt. est

value under H_0

$$t = \frac{(\hat{\beta}_1 - \hat{\beta}_2) - 0}{\text{s.e. of } (\hat{\beta}_1 - \hat{\beta}_2)}$$

$\sqrt{\widehat{Var}(\hat{\beta}_1) + \widehat{Var}(\hat{\beta}_2) - 2\widehat{Cov}(\hat{\beta}_1, \hat{\beta}_2)}$

① - F-test

Reduced model : $y_i = \beta_0 + \beta_1 x_{i1} + \beta_1 x_{i2} + \beta_3 x_{i3} + e_i$

$$\Rightarrow y_i = \beta_0 + \beta_1 \underbrace{(x_{i1} + x_{i2})}_{} + \beta_3 x_{i3} + e_i$$

$$\begin{array}{cccc} & \beta_0 & \beta_1 \; x_{i1}' & \beta_3 \end{array}$$

$$\underset{\sim}{X}_R = \begin{pmatrix} 1 & x_{11}+x_{12} & x_{13} \\ \vdots & \vdots & \vdots \\ 1 & x_{n1}+x_{n2} & x_{n3} \end{pmatrix}_{n\times 3}$$

$$\Rightarrow \underset{\sim}{X}_R^T \underset{\sim}{X}_R \quad {}_{3\times 3}$$

$$\text{Res S.S.} \mid \text{reduced model} = S_{yy} - \beta_1 \boxed{S_{x_1' y}} - \beta_3 S_{x_3 y}$$

$$\sum_{i=1}^{n} (x_{i1}' - \bar{x}_1')\, y_i$$

$$= \sum_{i=1}^{n} \left((x_{i1} + x_{i2}) - (\bar{x}_1 + \bar{x}_2)\right) y_i$$

$$= \sum_{i=1}^{n} \left[(x_{i1} - \bar{x}_1) + (x_{i2} - \bar{x}_2)\right] y_i$$

$$= S_{x_1 y} + S_{x_2 y}$$

② $\cancel{\underset{\sim}{C} (\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \underset{\sim}{C}^T}$ $\quad H_0 : \underset{\sim}{C} \underset{\sim}{\beta} = \underset{\sim}{d}$

$$\Rightarrow H_0 : \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{C} (\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \underset{\sim}{C}^T \quad {}_{1\times 1}$$

① – Sequential Sum of squares – Type I S.S.

② – Partial Sum of squares – Type III S.S.

① ~~SS~~ Sequential S.S.

$$R(\beta_1, \cdots, \beta_p \mid \beta_0)\left[\begin{array}{l} \beta_0 \\ +\beta_1 \quad \Big] R(\beta_1 \mid \beta_0) \\ \qquad\qquad + \\ +\beta_2 \quad \Big] R(\beta_2 \mid \beta_1, \beta_0) \\ \quad \vdots \\ \qquad\qquad + \\ +\beta_{p-1} \quad \Big] R(\beta_p \mid \beta_{p-1}, \beta_{p-2}, \cdots, \beta_2, \beta_1, \beta_0) \\ +\beta_p \end{array}\right.$$

$y_i = \beta_0 + e_i$

$y_i = \beta_0 + \beta_1 X_{i1} + e_i$

$y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + e_i$

$$\Rightarrow R(\beta_1, \cdots, \beta_p \mid \beta_0) = R(\beta_1 \mid \beta_0) + R(\beta_2 \mid \beta_1, \beta_0) + \cdots + R(\beta_p \mid \beta_{p-1}, \beta_{p-2}, \cdots, \beta_2, \beta_1, \beta_0)$$

② Partial S.S. Model : $y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip} + e_i$

$$R(\beta_1 \mid \underbrace{\beta_2, \beta_3, \cdots, \beta_{p-1}, \beta_p}_{\text{all other indep. variables}}, \beta_0)$$

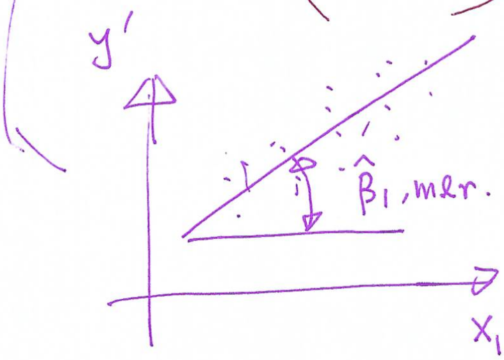
Section 7

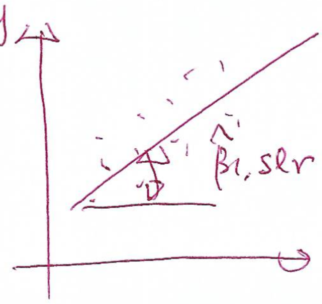
m.l.r.

e.g. Model : $y_i = \beta_0 + \boxed{\beta_1} X_{i1} + \beta_2 \boxed{X_{i2}} + \cancel{\beta_3 X_{i3}} + e_i$

partial relationship between y & X_1

s.l.r. $y_i = \beta_0 + \boxed{\beta_1} X_{i1} + e_i$





X_1 all other indep. variables than X_1

added variable plot

y' is the residual of y on $\boxed{X_2}$

X_1' is the residual of X_1 on $\boxed{X_2}$

① Important factors, i.e. selection of best model

② prediction on ~~a~~ new observation.

Section 5 Prediction

$\left\langle \begin{array}{l} E(y) \text{ at } \underset{\sim}{x}_0 \leftarrow \text{new observation} \\ y \text{ at } \underset{\sim}{x}_0 \\ \uparrow \text{individual value of } y \end{array} \right.$

$p=1$

Model: $y_i = \beta_0 + \beta_1 x_{i1} + e_i$

New observation: x_{01}

- $E(y)$ at $x_1 = x_{01}$

$= \mu_{y|x_{01}}$

$= \beta_0 + \beta_1 x_{01} + \cancel{E(e_0)}$ 0

- pt est. $\hat{\mu}_{y|x_{01}} = \hat{\beta}_0 + \hat{\beta}_1 x_{01}$ — linear combination of normal r.v.

$$Var(\hat{\mu}_{y|x_0}) = Var(\hat{\beta}_0 + \hat{\beta}_1 x_{01})$$

$$= \underline{Var(\hat{\beta}_0)} + x_{01}^2 \underline{Var(\hat{\beta}_1)} + 2 x_{01} \underline{cov(\hat{\beta}_0, \hat{\beta}_1)}$$

Section 3

$$= \sigma^2\left(\frac{1}{n} + \frac{\bar{x}_1^2}{S_{x_1x_1}}\right) + x_{01}^2 \frac{\sigma^2}{S_{x_1x_1}} - 2x_{01}\frac{\sigma^2 \bar{x}_1}{S_{x_1x_1}}$$

$$= \sigma^2\left(\frac{1}{n} + \frac{(x_{01}-\bar{x}_1)^2}{S_{x_1x_1}}\right)$$

$$\text{s.e. of } \hat{\mu}_{y|x_0} = \hat{\sigma}\sqrt{\frac{1}{n} + \frac{(x_{01}-\bar{x}_1)^2}{S_{x_1x_1}}}$$

$$\hat{\mu}_{y|x_{01}} \sim N\left(\beta_0 + \beta_1 x_{01},\ \sigma^2\left(\frac{1}{n} + \frac{(x_{01}-\bar{x}_1)^2}{S_{x_1x_1}}\right)\right)$$

$$\Rightarrow \frac{\hat{\mu}_{y|x_{01}} - (\overbrace{\boxed{\beta_0 + \beta_1 x_{01}}}^{\mu_{y|x_{01}}})}{\cancel{\sigma}\sqrt{\frac{1}{n} + \frac{(x_{01}-\bar{x}_1)^2}{S_{x_1x_1}}}} \sim N(0,1)$$

$\sim t(n-2)$

$$\sqrt{\underbrace{\boxed{\frac{Res\ S.S.}{\cancel{\sigma^2}} \Big/ (n-2)}}_{=\tilde{\sigma}^2}} \sim \chi^2(n-\cancel{p})\ 2$$

$$\Rightarrow \frac{\hat{\mu}_{y|x_{01}} - \mu_{y|x_{01}}}{\hat{\sigma}\sqrt{\frac{1}{n} + \frac{(x_{01}-\bar{x}_1)^2}{S_{x_1x_1}}}} \sim t(n-2)$$

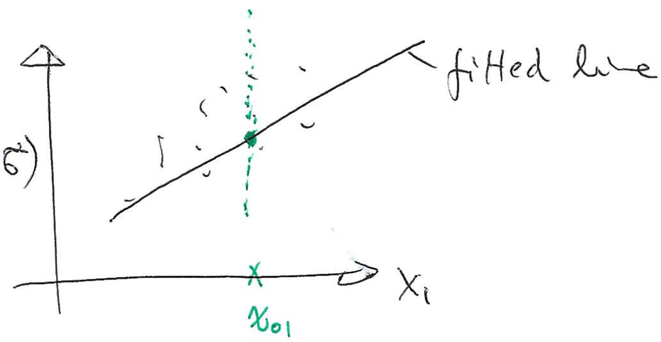
$100(1-\alpha)\%$ C.I. of $\mu_{y|x_{01}}$

$$(\hat{\beta}_0 + \hat{\beta}_1 x_{01}) \pm t_{\alpha/2}(n-2)\,\hat{\sigma}\sqrt{\frac{1}{n} + \frac{(x_{01}-\bar{x}_1)^2}{S_{x_1x_1}}}$$

$H_0 = \mu_{y|x_{01}} = 2$

$$t = \frac{(\hat{\beta}_0 + \hat{\beta}_1 x_{01}) - 2}{\hat{\sigma}\sqrt{\frac{1}{n} + \frac{(x_{01}-\bar{x}_1)^2}{S_{x_1x_1}}}}$$

e.g. in p.33



Individual prediction

y at x_{01}

Model : $y_i = \beta_0 + \beta_1 x_{i1} + e_i$

$\Rightarrow$ $y_0 = \beta_0 + \beta_1 x_{01} + \boxed{e_0}$ ← unobservable

pt. est. $\hat{y}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_{01} + \boxed{\hat{e}_0}$ → take it equal to zero

$E(\hat{e}_0) = 0$

$\Rightarrow \hat{y}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_{01}$

↑ same as $\boxed{\widehat{E(y)} \text{ at } x_{01}}$ → unbiased est. of $E(y)$ at x_{01}

Is $\hat{y}_0$ unbiased for y_0?

$$E(\hat{y}_0) = E(\hat{\beta}_0 + \hat{\beta}_1 x_{01})$$
$$= \beta_0 + \beta_1 x_{01}$$

$\Rightarrow \hat{y}_0$ is biased est. for y_0

Consider $\hat{y}_0 - y_0 = (\hat{\beta}_0 + \hat{\beta}_1 x_{01}) - (\beta_0 + \beta_1 x_{01} + e_0)$

~~$E(\hat{y}_0$~~ Is $\hat{y}_0 - y_0$ unbiased for 0 ?

$$E(\hat{y}_0 - y_0) = (\underbrace{E(\hat{\beta}_0)}_{\beta_0} + \underbrace{E(\hat{\beta}_1)}_{\beta_1} x_{01}) - (\beta_0 + \beta_1 x_{01} + \underbrace{E(e_0)}_{0}))$$

$$= 0$$

$$Var(\hat{y}_0 - y_0) = Var[(\hat{\beta}_0 + \hat{\beta}_1 x_{01}) - (\beta_0 + \beta_1 x_{01} + e_0)]$$

$\boxed{\hat{\beta}_0} + \boxed{\hat{\beta}_1} x_{01}$ $\qquad$ $\beta_0 + \beta_1 x_{01} + \boxed{e_0}$

$$= Var(\hat{\beta}_0 + \hat{\beta}_1 x_{01} - e_0)$$

$$\cancel{= Var(\hat{\beta}_0) + Var(\hat{\beta}_1 x_{01}) + Var(e_0)}$$

$$= Var(\hat{\beta}_0 + \hat{\beta}_1 \cdot x_{01}) + Var(e_0)$$

new observation

$$-2\underbrace{cov(\hat{\beta}_0 + \hat{\beta}_1 x_{01}, e_0)}_{0}$$

$\boxed{\begin{array}{l} \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}_1 \\ \hat{\beta}_1 = \dfrac{S_{x_1 y}}{S_{x_1 x_1}} \end{array}}$

independent (all observations are indep.)

based on original obs.

$$= \sigma^2\left(\frac{1}{n} + \frac{(x_{01} - \bar{x}_1)^2}{S_{x_1 x_1}}\right) + \sigma^2$$

$$= \sigma^2\left(1 + \frac{1}{n} + \frac{(x_{01} - \bar{x}_1)^2}{S_{x_1 x_1}}\right)$$

$$\hat{y}_0 - y_0 \sim N\left(0, \sigma^2\left(1 + \frac{1}{n} + \frac{(x_{01} - \bar{x}_1)^2}{S_{x_1 x_1}}\right)\right)$$ prediction interval

$(1-\alpha)100\%$ C.I. of individual value of y at $x_1 = x_{01}$

$$\Rightarrow (\hat{\beta}_0 + \hat{\beta}_1 x_{01}) \pm t_{\alpha/2, n-2}\, \hat{\sigma} \sqrt{\boxed{1} + \frac{1}{n} + \frac{(x_{01} - \bar{x}_1)^2}{S_{x_1 x_1}}}$$ wider

$Var(e_0)$

example in p.34

For any p New observation $\underset{\sim}{x}_0^T = (1, x_{01}, \cdots, x_{op})$

a row in $\underset{\sim}{X}$

design matrix

Mean prediction

$$\mu_{y|\underset{\sim}{x}_0} = \beta_0 + \beta_1 x_{01} + \cdots + \beta_p x_{op}$$

pt. est. $\hat{\mu}_{y|\underset{\sim}{x}_0} = \hat{\beta}_0 + \hat{\beta}_1 x_{01} + \cdots + \hat{\beta}_p x_{op}$

$$= (1, x_{01}, \cdots, x_{op}) \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_p \end{pmatrix}$$

$$= \underset{\sim}{x}_0^T \boxed{\hat{\underset{\sim}{\beta}}} \sim MN(\underset{\sim}{\beta}, (\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\sigma^2)$$

By theorem 3.2 in p.14

$$E(\hat{\mu}_{y|\underset{\sim}{x}_0}) = E(\underset{\sim}{x}_0^T \hat{\underset{\sim}{\beta}})$$

$$= \underset{\sim}{x}_0^T E(\hat{\underset{\sim}{\beta}})$$

$$= \underset{\sim}{x}_0^T \underset{\sim}{\beta} = \mu_{y|\underset{\sim}{x}_0} \leftarrow \text{unbiased est.}$$

$$Var(\hat{\mu}_{y|\underset{\sim}{x}_0}) = Var(\underset{\sim}{x}_0^T \hat{\underset{\sim}{\beta}})$$

$$= \underset{\sim}{x}_0^T \boxed{Var(\hat{\underset{\sim}{\beta}})} \underset{\sim}{x}_0 \quad \sim (\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\sigma^2$$

$$= \sigma^2 \underset{\sim}{x}_0^T (\underset{\sim}{X}^T\underset{\sim}{X})^{-1} \underset{\sim}{x}_0$$

$$\Rightarrow \hat{\mu}_{y|\underset{\sim}{x}_0} \sim MN(\underbrace{\underset{\sim}{x}_0^T \underset{\sim}{\beta}}_{\mu_{y|\underset{\sim}{x}_0}}, \cancel{\sigma^2} \; \sigma^2 \underset{\sim}{x}_0^T (\underset{\sim}{X}^T\underset{\sim}{X})^{-1} \underset{\sim}{x}_0)$$

$$\Rightarrow \frac{\dfrac{\hat{\mu}_{y|\underset{\sim}{x}_0} - \mu_{y|\underset{\sim}{x}_0}}{\sigma\sqrt{\underset{\sim}{x}_0^T (\underset{\sim}{X}^T\underset{\sim}{X})^{-1} \underset{\sim}{x}_0}} \sim N(0,1)}{\sqrt{\dfrac{Res.S.S.}{\sigma^2} / (n-p')}} \sim t(n-p')$$

$$\frac{Res.S.S.}{\sigma^2} \sim \chi^2(n-p')$$

$(1-\alpha)100\%$ C.I. for mean prediction at $\underset{\sim}{x} = \underset{\sim}{x}_0$

$$\hat{\mu}_{y|\underset{\sim}{x}_0} \pm t_{\alpha/2,\, n-p'}\, \hat{\sigma} \sqrt{\underset{\sim}{x}_0^T (\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \underset{\sim}{x}_0}$$

Individual prediction

$(1-\alpha)100\%$ prediction interval at $\underset{\sim}{x} = \underset{\sim}{x}_0$

$$\hat{\mu}_{y|\underset{\sim}{x}_0} \pm t_{\alpha/2,\, n-p'}\, \hat{\sigma} \sqrt{1 + \underset{\sim}{x}_0^T (\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \underset{\sim}{x}_0}$$

Var(e_0)

Example in p.7 $\quad \mu_{y|\underset{\sim}{x}_0}$ at $x_1 = 3,\ x_2 = 8,\ x_3 = 9$

$$\hat{\mu}_{y|x_0} = \underset{\sim}{x}_0^T \hat{\underset{\sim}{\beta}}$$

$$= (1, 3, 8, 9) \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{pmatrix}$$

$$= 24.2232$$

p.7