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Properties of estimators

1. $\underset{\sim}{\hat{\beta}}$

For $p = 1$

By the Theorem 3.1 in page 11,

$$\begin{aligned}\hat{\beta}_0 &= \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{(x_{i1}-\bar{x}_1)\bar{x}_1}{S_{x_1x_1}}\right)y_i \\ \hat{\beta}_1 &= \sum_{i=1}^{n}\left(\frac{x_{i1}-\bar{x}_1}{S_{x_1x_1}}\right)y_i\end{aligned}$$

Then

$$\begin{aligned}Var(\hat{\beta}_0) &= \left(\frac{1}{n} + \frac{\bar{x}_1^2}{S_{x_1x_1}}\right)\sigma^2 \\ &= \frac{\sigma^2\sum_{i=1}^{n}x_{i1}^2}{nS_{x_1x_1}} \\ Var(\hat{\beta}_1) &= \frac{\sigma^2}{S_{x_1x_1}} \\ Cov(\hat{\beta}_0, \hat{\beta}_1) &= -\frac{\sigma^2\sum_{i=1}^{n}x_{i1}}{nS_{x_1x_1}}\end{aligned}$$

$$\begin{pmatrix}\hat{\beta}_0 \\ \hat{\beta}_1\end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix}\beta_0 \\ \beta_1\end{pmatrix}, \begin{pmatrix}\sigma^2(\frac{1}{n} + \frac{\bar{x}_1^2}{S_{x_1x_1}}) & -\frac{\bar{x}_1\sigma^2}{S_{x_1x_1}} \\ -\frac{\bar{x}_1\sigma^2}{S_{x_1x_1}} & \frac{\sigma^2}{S_{x_1x_1}}\end{pmatrix}\right)$$

For any p

By the Theorem 3.2 in page 14,

$$\underset{\sim}{\hat{\beta}} \sim \mathrm{N}(\underset{\sim}{\beta}, \sigma^2(\underset{\sim}{X}^T\underset{\sim}{X})^{-1})$$

2. $\hat{e}_i$

For $p = 1$

$$\hat{e}_i = \sum_{j=1}^{n}\left(\delta_{ij} - (c_j + d_j x_{i1})\right)y_j$$

where $c_j = \frac{1}{n} - \frac{(x_{j1}-\bar{x}_1)\bar{x}_1}{S_{x_1x_1}}$, $d_j = \frac{x_{j1}-\bar{x}_1}{S_{x_1x_1}}$ and $\delta_{ij} = \begin{cases}1 & \text{if } i = j \\ 0 & \text{if } i \neq j\end{cases}$

For any p

$$\underset{\sim}{\hat{e}} = (\underset{\sim}{I} - \underset{\sim}{H})\,\underset{\sim}{Y}$$

3. **Residual Sum of Squares**

For p=1

$$\begin{aligned} \mathrm{E}(S_{yy}) &= (n-1)\sigma^2 + \beta_1^2 S x_1 x_1 \quad \text{and} \\ \mathrm{E}(\hat{\beta}_1^2) &= \mathrm{Var}(\hat{\beta}_1) + (\mathrm{E}(\hat{\beta}_1))^2 \\ \mathrm{E}(RSS) &= (n-2)\sigma^2 \\ \Rightarrow \quad \hat{\sigma}^2 &= \frac{RSS}{n-2} \end{aligned}$$

For any p

By the Theorem 3.3 in page 16,

$$\mathrm{E}(\mathrm{RSS}) = (n-p')\sigma^2.$$

Thus,

$$\hat{\sigma}^2 \text{ (unbiased estimator)} = \frac{RSS}{n-p'}$$