29 Oct

$H_0: \mu_{ij} = \mu$

Can't reject H_0. STOP the analysis

Reject H_0

method — 2 levels

variety — 3 levels

Regression model (Model I)

$$y_k = \beta_0 + \alpha_1 * m_{1,k} + \beta_1 * v_{1,k} + \beta_2 * v_{2,k} + \gamma_{11} * m_{1,k} * v_{1k} + \gamma_{12} * m_{1,k} * v_{2,k} + e_k \qquad k = 1, \ldots, N$$

$\underset{n\times 6}{X}$

$\Rightarrow$ ANOVA model

$y_{ijk} = \mu_{ij} + e_{ijk}$

$i = 1, \ldots, a$

$j = 1, \ldots, b$

$k = 1, \ldots, n_{ij}$

method	Variety	$E(y_k)$	μ
1	1	$\beta_0 + \alpha_1 + \beta_1 + \cancel{\gamma_{11}}$	μ_{11}
	2	$\beta_0 + \alpha_1 + \beta_2 + \cancel{\gamma_{12}}$	μ_{12}
	3	$\beta_0 + \alpha_1$	μ_{13}
2	1	$\beta_0 + \beta_1$	μ_{21}
	2	$\beta_0 + \beta_2$	μ_{22}
	3	β_0	μ_{23}

α_1 α_1 α_1

$H_0: \gamma_{ij} = 0$ for $i = 1, \ldots, a$
$j = 1, \ldots, b$

Example : $H_0: \gamma_{11} = \gamma_{12} = 0$

$\Rightarrow H_0: \mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}$

$\Rightarrow H_0: \mu_{11} - \mu_{21} = \mu_{12} - \mu_{22}$

$\mu_{11} - \mu_{21} = \mu_{13} - \mu_{23}$

$\Rightarrow H_0: \mu_{11} - \mu_{12} - \mu_{21} + \mu_{22} = 0$

$\mu_{11} - \mu_{13} - \mu_{21} + \mu_{23} = 0$

$\Rightarrow H_0: \underset{\sim}{c}\, \underset{\sim}{\beta} = \underset{\sim}{d}$

where $\underset{2\times 6}{\underset{\sim}{c}} = \begin{pmatrix} 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \end{pmatrix}$ $\quad \underset{\sim}{\beta} = \begin{pmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{21} \\ \mu_{22} \\ \mu_{23} \end{pmatrix}$ $\quad \underset{\sim}{d} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow F = \frac{(\underset{\sim}{c}\hat{\underset{\sim}{\beta}})^T (\underset{\sim}{c} \boxed{(\underset{\sim}{X}^T \underset{\sim}{X})^{-1}} \underset{\sim}{c}^T)^{-1} \underset{\sim}{c}\hat{\underset{\sim}{\beta}} / r}{\hat{\sigma}^2} \quad r = 2$$

$$\begin{pmatrix} \frac{1}{n_{11}} & & & & & \\ & \frac{1}{n_{12}} & & & \underset{\sim}{0} & \\ & & \frac{1}{n_{13}} & & & \\ & & & \frac{1}{n_{21}} & & \\ & \underset{\sim}{0} & & & \frac{1}{n_{22}} & \\ & & & & & \frac{1}{n_{23}} \end{pmatrix}$$

$$\underset{\sim}{c}(\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \underset{\sim}{c}^T = \begin{pmatrix} \frac{1}{n_{11}} + \frac{1}{n_{12}} + \frac{1}{n_{21}} + \frac{1}{n_{22}} & \frac{1}{n_{11}} + \frac{1}{n_{21}} \\ \frac{1}{n_{11}} + \frac{1}{n_{21}} & \frac{1}{n_{11}} + \frac{1}{n_{13}} + \frac{1}{n_{21}} + \frac{1}{n_{23}} \end{pmatrix}$$

$$\hat{\underset{\sim}{\beta}}^T = (\bar{y}_{11\cdot} \quad \bar{y}_{12\cdot} \quad \bar{y}_{13\cdot} \quad \bar{y}_{21\cdot} \quad \bar{y}_{22\cdot} \quad \bar{y}_{23\cdot})$$

$$\underset{\sim}{c}\hat{\underset{\sim}{\beta}} = (\bar{y}_{11\cdot} - \bar{y}_{12\cdot} - \bar{y}_{21\cdot} + \bar{y}_{22\cdot} \quad \bar{y}_{11\cdot} - \bar{y}_{13\cdot} - \bar{y}_{21\cdot} + \bar{y}_{23\cdot})$$

$$\Rightarrow F = \frac{45.823889 / 2}{19.397222} = 1.18 < F_{0.05, 2, \underset{= 30}{\boxed{36 - 2*3}}}$$

Can't reject $H_0 \Rightarrow \gamma_{11} = \gamma_{12} = 0$

Interaction terms are not significant

Model I: $y_k = \underline{\beta_0} + \underline{\alpha_1} * M_{1,k} + \underline{\beta_1} * V_{1,k} + \underline{\beta_2} * V_{2,k} + e_k \qquad k = 1, \_\_\_, N$

4 unknown parameters

$$N = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij}$$

$\Rightarrow$ For ANOVA model $\quad y_{ijk} = \boxed{\mu_{ij}} + e_{ijk}$

$\mu_{11}, \mu_{12}, \mu_{13}, \mu_{21}, \mu_{22}, \mu_{23} = 6$ unknown parameters

Can't use this model. This model assumes $\gamma_{11} \neq 0$ & $\gamma_{12} \neq 0$ (with interaction terms)

Main effect — method, variety

- method effect

H_0: method effect is insignificant

$\Rightarrow H_0: \mu_{1\cdot} = \mu_{2\cdot}$ ← pop. mean for method 2

pop. mean for method 1

Model

$y_k = \beta_0 + \alpha_1 * M_{1k} + \beta_1 * V_{1k} + \beta_2 * V_{2k} + e_k$

$$\Rightarrow H_0: \frac{\mu_{11}+\mu_{12}+\mu_{13}}{3} = \frac{\mu_{21}+\mu_{22}+\mu_{23}}{3}$$

$$\Rightarrow H_0: \beta_0+\alpha_1+\beta_1+\beta_0+\alpha_1+\beta_2+\beta_0+\alpha_1$$
$$= \beta_0+\beta_1+\beta_0+\beta_2+\beta_0$$

$\Rightarrow H_0: \alpha_1 = 0$ ← $\hat{\alpha}_1$, s.e. of $\hat{\alpha}_1$

t

$\underset{\sim}{c}\,\underset{\sim}{\beta} = \underset{\sim}{d}$

$$\Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_1 \\ \beta_1 \\ \beta_2 \end{pmatrix} = 0$$

Similarly H_0: variety effect is insignificant

$\Rightarrow H_0: \mu_{\cdot 1} = \mu_{\cdot 2} = \mu_{\cdot 3}$

$$\Rightarrow H_0: \frac{\mu_{11}+\mu_{21}}{2} = \frac{\mu_{12}+\mu_{22}}{2} = \frac{\mu_{13}+\mu_{23}}{2}$$

$$\Rightarrow H_0: \beta_0+\alpha_1+\beta_1+\beta_0+\beta_1 = \beta_0+\alpha_1+\beta_2+\beta_0+\beta_2$$
$$= \beta_0+\alpha_1+\beta_0$$

$\Rightarrow H_0: 2\beta_1 = 2\beta_2 = 0$

$\Rightarrow H_0: \beta_1 = \beta_2 = 0$ ← $\underset{\sim}{c}\,\underset{\sim}{\beta} = \underset{\sim}{d}$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_1 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- estimate σ^2

ⓐ

$$\text{Res S.S.} = \underset{\sim}{Y}^T \underset{\sim}{Y} - \underset{\sim}{\hat\beta}^T \underset{\sim}{X}^T \underset{\sim}{Y}$$

$$= \underset{\sim}{Y}^T \underset{\sim}{Y} - \underset{\sim}{Y}^T \underset{\sim}{X} \underbrace{\boxed{(\underset{\sim}{X}^T \underset{\sim}{X})^{-1}}}_{4\times 4} \underset{\sim}{X}^T \underset{\sim}{Y}$$

$$\underset{n\times 4}{\underset{\sim}{X}} = \begin{array}{cccc|cc} \beta_0 & \alpha_1 & \beta_1 & \beta_2 & \text{method} & \text{variety} \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 1 & 0 & 0 & 0 & 2 & 3 \end{array}$$

ⓑ Total S.S. = Res S.S. + $\underline{\text{Reg S.S.}}$

Model without interactions

$y_k = \beta_0 + \alpha_1 * M_{1k} + \beta_1 * V_{1k} + \beta_2 * V_{2k} + e_k$

Reg S.S. ← method, variety

$\underline{\text{Model with interaction terms.}}$

$y_k = \beta_0 + \alpha_1 * M_{1k} + \beta_1 * V_{1k} + \beta_2 * V_{2k} + \gamma_{11} * M_{1k} * V_{1k} + \gamma_{12} * M_{1k} * V_{2k} + e_k$

Reg S.S. ← method, variety, interaction

$\Rightarrow$ Res S.S. for the model without interactions

= Res S.S. for the model with interactions

+ SS due to interactions

Total SS = Reg S.S. + Res S.S. | with interaction

= R(method, variety, interaction | β_0)

+ Res S.S. | with interaction

Example in p.11

Res S.S. | without interaction

$= (581.916667 + 45.823889)$

$= 627.74$

d.f. $= 30 + 2 = 32$ d.f. $= 30$

$\hat{\sigma}^2_{\text{no int}} = 19.6168$ $\hat{\sigma}^2_{\text{int}} = 19.397222$

MATH 3423

~~MATH 3423~~

$\frac{\text{Res S.S.}|\text{no int}}{\sigma^2} \sim \chi^2_{32}$ $\frac{\text{Res S.S.}|\text{int}}{\sigma^2} \sim \chi^2_{30}$

~~Expectation = 32~~

$E\left(\frac{\text{Res S.S.}|\text{no int}}{\sigma^2}\right) = \cancel{32}\ d.f.$

~~$\text{Var}\left(\frac{\hat{\sigma}^2_{\text{no int}}}{32}\right)$~~

✓ $\text{Var}(\hat{\sigma}^2_{\text{no int}}) = \frac{2\sigma^4}{32}$

$\text{Var}(\hat{\sigma}^2_{\text{int}}) = \frac{2\sigma^4}{30}$

$$\boxed{\begin{aligned} &\frac{SSE}{\sigma^2} \sim \chi^2_{df} \\ \Rightarrow\ &E\left(\frac{SSE}{\sigma^2}\right) = df \\ &\text{Var}\left(\frac{SSE}{\sigma^2}\right) = 2 * df \\ \Rightarrow\ &\text{Var}(SSE) = 2 * df\,\sigma^4 \\ \Rightarrow\ &\text{Var}\left(\frac{SSE}{df}\right) = \frac{2 * df * \sigma^4}{df^2} \\ &= \frac{2\sigma^4}{df} \end{aligned}}$$

Use $\hat{\sigma}^2_{\text{no int}}$ to est. σ^2

for getting a more precise estimate

provided that the interaction terms are insignificant

Otherwise, $\hat{\sigma}^2_{\text{no int}}$ is a biased est.

①

$$\underset{n\times 4}{\underset{\sim}{X}} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

method	variety
1	1
1	2
1	3
2	1
2	2
2	3

$\frac{18}{36} = \frac{1}{2}$ $\qquad$ $\frac{12}{36} = \frac{1}{3}$

Centred model

$$y_k = \beta_0' + \alpha_1(m_{1k} - \overline{m_1}) + \beta_1(v_{1k} - \bar{v}_1) + \beta_2(v_{2k} - \bar{v}_2) + e_k \qquad k = 1, \ldots, N$$

$\frac{1}{3}$

$$\underset{n\times 4}{\underset{\sim}{X}} = \begin{pmatrix} 1 & \frac12 & \frac23 & -\frac13 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac12 & -\frac13 & \frac23 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \frac12 & -\frac13 & -\frac13 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & -\frac12 & \frac23 & -\frac13 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & -\frac12 & -\frac13 & \frac23 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & -\frac12 & -\frac13 & -\frac13 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$1 - \frac12 = \frac12 \qquad 1 - \frac13 = \frac23$

$0 - \frac12 = -\frac12 \qquad 0 - \frac13 = -\frac13$

$$a_{11} = \left(\tfrac12\right)^2 * 18 + \left(-\tfrac12\right)^2 * 18 = 9$$

$$a_{12} = \tfrac12 * \tfrac23 * 6 + \tfrac12 * \left(-\tfrac13\right) * 12 + \left(-\tfrac12\right) * \tfrac23 * 6 + \left(-\tfrac12\right) * \left(-\tfrac13\right) * 12 = 0$$

$$\underset{4\times 4}{\underset{\sim}{X}^T \underset{\sim}{X}} = \begin{pmatrix} 36 & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & a_{13} \\ 0 & a_{12} & a_{22} & a_{23} \\ & a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$a_{13} = 0$

$$a_{22} = \left(\tfrac23\right)^2 * 12 + \left(-\tfrac13\right)^2 * 24 = 8$$

$a_{33} = 8$

$$a_{23} = \left(\tfrac23\right)\left(-\tfrac13\right) * 12 + \left(-\tfrac13\right) * \tfrac23 * 12 + \left(-\tfrac13\right) * \left(-\tfrac13\right) * 12 = -4$$

$$\underset{\sim}{X}^T \underset{\sim}{X} = \begin{pmatrix} 36 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 8 & \cancel{0} \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

$n_{ij} = n \; \forall i, j$

$-4\,!$

$$Var(\hat{\underset{\sim}{\beta}}) = \sigma^2 (\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \qquad \hat{\underset{\sim}{\beta}} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\alpha}_1 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix}$$

$$\hat{\underset{\sim}{\beta}} = (\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \boxed{\underset{\sim}{X}^T \underset{\sim}{Y}}$$

$$\begin{pmatrix} \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} y_{ijk} \\ \frac{1}{2}\sum_{j=1}^{b}\sum_{k=1}^{n_{1j}} y_{1jk} - \frac{1}{2}\sum_{j=1}^{b}\sum_{k=1}^{n_{2j}} y_{2jk} \\ \frac{2}{3}\sum_{i=1}^{a}\sum_{k=1}^{n_{i1}} y_{i1k} - \frac{1}{3}\sum_{i=1}^{a}\sum_{k=1}^{n_{i2}} y_{i2k} - \frac{1}{3}\sum_{i=1}^{a}\sum_{j=1}^{n_{i3}} y_{i3k} \\ -\frac{1}{3}\sum_{i=1}^{a}\sum_{k=1}^{n_{i1}} y_{i1k} + \frac{2}{3}\sum_{i=1}^{a}\sum_{k=1}^{n_{i2}} y_{i2k} - \frac{1}{3}\sum_{i=1}^{a}\sum_{j=1}^{n_{i3}} y_{i3k} \end{pmatrix}$$

$\hat{\alpha}_1 =$

$$\underset{\sim}{X}^T \underset{\sim}{Y} = \begin{pmatrix} 696.8 \\ \frac{1}{2}(428.6 - 268.2) \\ \frac{2}{3} * 242.7 - \frac{1}{3} * 244.8 - \frac{1}{3} * 209.3 \\ -\frac{1}{3} * 242.7 + \frac{2}{3} * 244.8 - \frac{1}{3} * 209.3 \end{pmatrix}$$

Example

Method	Variety 1	Variety 2	Variety 3	Sum	CSS
1	22.3	19.8	20		
	25.8	28.3	17		
	22.8	26.8	24		
	28.3	27.3	22.5		
	21.3	26.8	28		
	18.3	26.8	22.5		
Sum	138.8	155.8	134	$\boxed{428.6} \leftarrow \sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} y_{ijk}$	
Corrected S.S.	61.333333	47.333333	68.833333	221.237778	
2	16.4	24.5	11.8		
	14.4	16	14.3		
	21.4	11	21.3		
	19.9	7.5	6.3		
	10.4	14.5	7.8		
	21.4	15.5	13.8		
Sum	103.9	89	75.3	$\boxed{268.2} \leftarrow \sum_{j=1}^{b}\sum_{k=1}^{n_{2j}} y_{2jk}$	
Corrected S.S.	97.208333	163.833333	143.375	472.62	
	$\sum_{i=1}^{a}\sum_{k=1}^{n_{i1}} y_{i1k}$	$\sum_{i=1}^{a}\sum_{k=1}^{n_{i2}} y_{i2k}$	$\sum_{i=1}^{a}\sum_{k=1}^{n_{i3}} y_{i3k}$		
Sum	$\boxed{242.7}$	$\boxed{244.8}$	$\boxed{209.3}$	$\boxed{696.8} \leftarrow \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}} y_{ijk}$	
Corrected S.S.	260.0425	583.02	499.349167	1408.53	

Source of Variation of	Sum of Squares	Degrees of freedom	Mean Square	Computed f
Method	714.671111	1	714.671111	36.84
Variety	66.117222	2	33.058611	1.71
Interaction	45.823889	2	22.911944	1.18
Error	581.916667	30	19.397222	
Total	1408.528889	35		

Test "interaction" effect is equivalent to test $H_0 : \mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}$.

As the interaction terms are not significant, we re-construct the ANOVA table.