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$p = 1$

$$\Rightarrow \qquad \begin{cases} \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}_1 \\ \\ \hat{\beta}_1 = \dfrac{\sum_{i=1}^{n} (x_{i1} - \bar{x}_1)(y_i - \bar{y})}{\sum_{i=1}^{n} (x_{i1} - \bar{x}_1)^2} = \dfrac{S_{x_1 y}}{S_{x_1 x_1}} \end{cases}$$

Properties of $\hat{e}_i$

$$\sum_{i=1}^{n} \hat{e}_i = 0$$

$$\sum_{i=1}^{n} x_{ij} \hat{e}_i = 0 \quad \text{for} \quad j = 1, \ldots p$$

$$\Rightarrow \qquad \underset{\sim}{X}^T \underset{\sim}{e} = \underset{\sim}{0}$$

Estimates of $\boldsymbol{\beta}$

Find $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_p$ such that $\sum_{i=1}^{n} \hat{e}_i^2$ is minimized

$$\Rightarrow \qquad \hat{\underset{\sim}{\beta}} = (\underset{\sim}{X}^T \underset{\sim}{X})^{-1} \underset{\sim}{X}^T \underset{\sim}{Y}$$

Centered model

Original model: $y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip} + e_i$

Centered model:

$$\begin{aligned} y_i &= \beta_0' + \beta_1'(x_{i1} - \bar{x}_1) + \ldots + \beta_p'(x_{ip} - \bar{x}_p) + e_i \\ &= \beta_0' + \beta_1' x_{i1} + \ldots + \beta_p' x_{ip} - \beta_1 \bar{x}_1 - \ldots - \beta_p \bar{x}_p + e_i \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \beta_0 &= \beta_0' - \beta_1' \bar{x}_1 - \ldots - \beta_p' \bar{x}_p \\ \beta_j &= \beta_j' \quad \text{for} \quad j = 1, \ldots, p \end{aligned}$$

$\Rightarrow$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}_1 - \ldots - \hat{\beta}_p \bar{x}_p$$

$$\begin{pmatrix} \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_p \end{pmatrix} = (\underset{\sim}{X}_c{}^T \underset{\sim}{X}_c)^{-1} \underset{\sim}{X}_c{}^T \underset{\sim}{Y}$$

where

$$\underset{\sim}{X}_c^T \underset{\sim}{X}_c = \begin{pmatrix} S_{x_1,x_1} & S_{x_1,x_2} & \dots & S_{x_1,x_p} \\ \vdots & \vdots & \vdots & \vdots \\ S_{x_1,x_p} & S_{x_2,x_p} & \dots & S_{x_p,x_p} \end{pmatrix}$$

$$\underset{\sim}{X}_c^T \underset{\sim}{Y} = \begin{pmatrix} S_{x_1,y} \\ \vdots \\ S_{x_p,y} \end{pmatrix}$$

and

$$\begin{aligned} S_{u,v} &= \sum_{i=1}^{n} (u_i - \bar{u})(v_i - \bar{v}) \\ &= \sum_{i=1}^{n} (u_i - \bar{u}) v_i \qquad \text{since } \sum_{i=1}^{n} (u_i - \bar{u}) = 0 \\ &= \sum_{i=1}^{n} u_i v_i - n \bar{u} \bar{v} \end{aligned}$$