Assignment 1: Solution

1. (a) **Assume that $\boldsymbol{\beta_0 = 2}$**

   i. The model is $y_i' = \beta_1 x_{i1} + \beta_2 x_{i2} + e_i$ where $y_i' = y_i - 2$.

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X^T X})^{-1}\boldsymbol{X^T Y} = \begin{pmatrix} 860 & -1025 \\ -1025 & 1228 \end{pmatrix}^{-1} \begin{pmatrix} 1309 \\ -1552 \end{pmatrix} = \begin{pmatrix} 0.225115 & 0.187901 \\ 0.187901 & 0.157654 \end{pmatrix} \begin{pmatrix} 1309 \\ -1552 \end{pmatrix}$$
$$= \begin{pmatrix} 3.053183 \\ 1.283401 \end{pmatrix}$$

   Hence, the fitted line is $\hat{y} = 2 + 3.053183x_1 + 1.283401x_2$.

   ii. ResS.S $= \boldsymbol{Y^T Y} - \hat{\boldsymbol{\beta}}^{\boldsymbol{T}} \boldsymbol{X^T Y} = 2142 - 2004.778195 = 137.221805$
   $\hat{\sigma}^2 = \frac{137.221805}{20-2} = 7.623434$

   (b) **Assume that $\boldsymbol{2\beta_1 = \beta_2}$.**

   i. The model is $y_i = \beta_0 + \beta_1 x_i' + e_i$ where $x_i' = x_{i1} + 2x_{i2}$.
   $\hat{\beta}_1 = \frac{S_{x'y}}{S_{x'x'}} = \frac{S_{x_1y}+2S_{x_2y}}{S_{x_1x_1}+4S_{x_1x_2}+4S_{x_2x_2}} = \frac{-357.2}{423.8} = -0.842850$
   $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}' = 4.441485$
   Hence, the fitted line is $\hat{y} = 4.441485 - 0.842850(x_1 + 2x_2)$.

   ii. ResS.S $= S_{yy} - \hat{\beta}_1 S_{x'y} = 485.8 - (-0.842850)(-357.2) = 184.73398$
   $\hat{\sigma}^2 = \frac{184.73398}{20-2} = 10.262999$

   iii. $(\boldsymbol{X^T X})^{-1} = \begin{pmatrix} 20 & -158 \\ -158 & 1672 \end{pmatrix}^{-1} = \begin{pmatrix} 0.197263 & 0.018641 \\ 0.018641 & 0.002360 \end{pmatrix}$
   s.e of $\hat{\beta}_1 = \sqrt{10.262999 * 0.002360} = 0.155630$
   $t_{obs} = |\frac{\hat{\beta}_1 - 2}{0.155630}| = 18.266722 > t_{18,0.025} = 2.101$
   Hence, we reject H_0.

   (c) **Assume that $\boldsymbol{\beta_0, \beta_1}$ and $\boldsymbol{\beta_2}$ are unknown.**

   i. Consider Centered Model $y_i = \beta_0' + \beta_1 x_{i1}' + \beta_2 x_{i2}' + e_i$ where $x_{ij}' = x_{ij} - \bar{x}_j, \quad j = 1, 2.$

$$\hat{\boldsymbol{\beta}}_{\boldsymbol{c}} = (\boldsymbol{X_c^T X_c})^{-1}\boldsymbol{X_c^T Y} = \begin{pmatrix} 0.05 & 0 & 0 \\ 0 & 0.227525 & 0.187453 \\ 0 & 0.187453 & 0.157737 \end{pmatrix} \begin{pmatrix} 222 \\ 271.6 \\ -314.4 \end{pmatrix}$$
$$= \begin{pmatrix} 11.1 \\ 2.860567 \\ 1.319722 \end{pmatrix}$$

   $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x_1} - \hat{\beta}_2 \bar{x_2} = 3.768878$
   Hence, the fitted line is $\hat{y} = 3.768878 + 2.860567x_1 + 1.319722x_2$.

   ii. ResS.S $= S_{yy} - \hat{\beta}_1 S_{x_1y} - \hat{\beta}_2 S_{x_2y} = 123.790600$
   $\hat{\sigma}^2 = \frac{123.790600}{20-3} = 7.2818$

   iii. $H_0 : 2\beta_1 = \beta_2 \Leftrightarrow 2\beta_1 - \beta_2 = 0$
   Recall in centered model,
   $(\boldsymbol{X_c^T X_c})^{-1} = \begin{pmatrix} 0.05 & 0 & 0 \\ 0 & 0.227525 & 0.187453 \\ 0 & 0.187453 & 0.157737 \end{pmatrix}$
   Hence,
   $Var(2\hat{\beta}_1 - \hat{\beta}_2) = 4Var(\hat{\beta}_1) - 4Cov(\hat{\beta}_1, \hat{\beta}_2) + Var(\hat{\beta}_2) = \hat{\sigma}^2 \{4(0.227525) - 4(0.187453) + 0.157737\} = 2.315794$
   $t_{obs} = |\frac{2\hat{\beta}_1 - \hat{\beta}_2}{\sqrt{2.315794}}| = 2.89229 > t_{17,0.025} = 2.110$
   Hence, we reject H_0.

2.

$$\sum_{i=1}^{n}(y_i-\hat{y}_i)^2 = \sum_{i=1}^{n_1}\Big(y_i-\hat{\beta_0}^{(1)*}-\hat{\beta}_1(x_i-\bar{x}_1)\Big)^2+\sum_{i=n_1+1}^{n_1+n_2}\Big(y_i-\hat{\beta_0}^{(2)*}-\hat{\beta}_1(x_i-\bar{x}_2)\Big)^2$$

$$\begin{cases} (1) & \dfrac{\partial}{\partial\hat{\beta}_0^{(1)*}}\displaystyle\sum_{i=1}^{n}(y_i-\hat{y}_i)^2 = 2\sum_{i=1}^{n_1}\Big(y_i-\hat{\beta_0}^{(1)*}-\hat{\beta}_1(x_i-\bar{x}_1)\Big)(-1) \\ (2) & \dfrac{\partial}{\partial\hat{\beta}_0^{(2)*}}\displaystyle\sum_{i=1}^{n}(y_i-\hat{y}_i)^2 = 2\sum_{i=n_1+1}^{n_1+n_2}\Big(y_i-\hat{\beta_0}^{(2)*}-\hat{\beta}_1(x_i-\bar{x}_2)\Big)(-1) \\ (3) & \dfrac{\partial}{\partial\hat{\beta}_1}\displaystyle\sum_{i=1}^{n}(y_i-\hat{y}_i)^2 = 2\sum_{i=1}^{n_1}\Big(y_i-\hat{\beta_0}^{(1)*}-\hat{\beta}_1(x_i-\bar{x}_1)\Big)\Big(-(x_i-\bar{x}_1)\Big) \\ & \qquad +2\displaystyle\sum_{i=n_1+1}^{n_1+n_2}\Big(y_i-\hat{\beta_0}^{(2)*}-\hat{\beta}_1(x_i-\bar{x}_2)\Big)\Big(-(x_i-\bar{x}_2)\Big) \end{cases}$$

Set them equal to 0.

$$(1) \qquad \sum_{i=1}^{n_1}\Big(y_i-\hat{\beta_0}^{(1)*}-\hat{\beta}_1(x_i-\bar{x}_1)\Big)=0 \Rightarrow$$

$$\begin{aligned} \hat{\beta_0}^{(1)*} &= \frac{1}{n_1}\sum_{i=1}^{n_1}\Big(y_i-\hat{\beta}_1(x_i-\bar{x}_1)\Big) \\ &= \bar{y_1} \end{aligned}$$

$$(2) \qquad \sum_{i=n_1+1}^{n_1+n_2}\Big(y_i-\hat{\beta_0}^{(2)*}-\hat{\beta}_1(x_i-\bar{x}_2)\Big)=0 \Rightarrow$$

$$\begin{aligned} \hat{\beta_0}^{(2)*} &= \frac{1}{n_2}\sum_{i=n_1+1}^{n_1+n_2}\Big(y_i-\hat{\beta}_1(x_i-\bar{x}_2)\Big) \\ &= \bar{y_2} \end{aligned}$$

$$\begin{aligned} (3) \quad & \sum_{i=1}^{n_1}\Big(y_i-\hat{\beta_0}^{(1)*}-\hat{\beta}_1(x_i-\bar{x}_1)\Big)\Big(-(x_i-\bar{x}_1)\Big) \\ &+\sum_{i=n_1+1}^{n_1+n_2}\Big(y_i-\hat{\beta_0}^{(2)*}-\hat{\beta}_1(x_i-\bar{x}_2)\Big)\Big(-(x_i-\bar{x}_2)\Big) = 0 \qquad \Rightarrow \\ \hat{\beta}_1\left[\sum_{i=1}^{n_1}(x_i-\bar{x}_1)^2+\sum_{i=n_1+1}^{n_1+n_2}(x_i-\bar{x}_2)^2\right] &= \sum_{i=1}^{n_1}(y_i-\hat{\beta_0}^{(1)*})(x_i-\bar{x}_1) \\ &+ \sum_{i=n_1+1}^{n_1+n_2}(y_i-\hat{\beta_0}^{(2)*})(x_i-\bar{x}_2) \qquad (*) \end{aligned}$$

Sub $\hat{\beta_0}^{(1)*}$ and $\hat{\beta_0}^{(2)*}$ into $(*)$

$$\begin{aligned} \hat{\beta}_1\left[\sum_{i=1}^{n_1}(x_i-\bar{x}_1)^2+\sum_{i=n_1+1}^{n_1+n_2}(x_i-\bar{x}_2)^2\right] &= \sum_{i=1}^{n_1}(y_i-\bar{y_1})(x_i-\bar{x}_1)+\sum_{i=n_1+1}^{n_1+n_2}(y_i-\bar{y_2})(x_i-\bar{x}_2) \\ &= \sum_{i=1}^{n_1}(x_i-\bar{x}_1)y_i+\sum_{i=n_1+1}^{n_1+n_2}(x_i-\bar{x}_2)y_i \\ \Rightarrow \qquad \hat{\beta}_1 &= \frac{\displaystyle\sum_{i=1}^{n_1}(x_i-\bar{x}_1)y_i+\sum_{i=n_1+1}^{n_1+n_2}(x_i-\bar{x}_2)y_i}{\displaystyle\sum_{i=1}^{n_1}(x_i-\bar{x}_1)^2+\sum_{i=n_1+1}^{n_1+n_2}(x_i-\bar{x}_2)^2} \end{aligned}$$

Or, write

$$
\underset{\sim}{Y} = \begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ y_{n_1} \\ \hline y_{n_1+1} \\ \vdots \\ \vdots \\ y_{n_1+n_2} \end{pmatrix}, \quad \underset{\sim}{X} = \begin{pmatrix} 1 & 0 & x_1 - \bar{x}_1 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 1 & 0 & x_{n_1} - \bar{x}_1 \\ \hline 0 & 1 & x_{n_1+1} - \bar{x}_2 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 0 & 1 & x_{n_1+n_2} - \bar{x}_2 \end{pmatrix}, \quad \underset{\sim}{\beta} = \begin{pmatrix} \beta_0^{(1)*} \\ \beta_0^{(2)*} \\ \beta_1 \end{pmatrix}
$$

Then,

$$
\underset{\sim}{X}^T \underset{\sim}{X} = \begin{pmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & \sum_{i=1}^{n_1} (x_i - \bar{x}_1)^2 + \sum_{i=n_1+1}^{n_1+n_2} (x_i - \bar{x}_2)^2 \end{pmatrix}, \quad \underset{\sim}{X}^T \underset{\sim}{Y} = \begin{pmatrix} \sum_{i=1}^{n_1} y_i \\ \sum_{i=n_1+1}^{n_1+n_2} y_i \\ \sum_{i=1}^{n_1} (x_i - \bar{x}_1) y_i + \sum_{i=n_1+1}^{n_1+n_2} (x_i - \bar{x}_2) y_i \end{pmatrix}
$$

$\Rightarrow$ get $\hat{\beta}_0^{(1)*}$, $\hat{\beta}_0^{(2)*}$ and $\hat{\beta}_1$.

3. According to the definition of least square method, and by the computational procedure of page 2 in Chapter 1 lecture note, least square estimation of slope is

$$
\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}},
$$

where

$$
\begin{aligned}
S_{xy} &= \sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x}) \\
&= \sum_{i=1}^{n} [\beta_0 + \beta_1 x_i + \epsilon_i - (\beta_0 + \beta_1 \bar{x} + \bar{\epsilon})](x_i - \bar{x}) \\
&= \beta_1 \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x}) + \sum_{i=1}^{n} \epsilon_i (x_i - \bar{x}) - \bar{\epsilon} \sum_{i=1}^{n} (x_i - \bar{x}) \\
&= \beta_1 \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x}) + \sum_{i=1}^{n} \epsilon_i (x_i - \bar{x}).
\end{aligned}
$$

Note that $S_{xx} = \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})$ and define $d_i = \frac{x_i - \bar{x}}{S_{xx}}$, then we can obtain the least squares slope is given by

$$
\hat{\beta}_1 = \beta_1 + \sum_{i=1}^{n} d_i \epsilon_i, \quad where \;\; d_i = \frac{x_i - \bar{x}}{S_{xx}}.
$$

According to conclusion of page 2 in chapter 1, least square estimation of β_0 is

$$
\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}.
$$

Note that $\bar{y} = \beta_0 + \beta_1 \bar{x} + \bar{\epsilon}$ and $\hat{\beta}_1 = \beta_1 + \sum_{i=1}^{n} d_i \epsilon_i$, thus, we have

$$
\begin{aligned}
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\
&= \beta_0 + \beta_1 \bar{x} + \bar{\epsilon} - (\beta_1 + \sum_{i=1}^{n} d_i \epsilon_i) \bar{x} \\
&= \beta_0 + \bar{\epsilon} - \bar{x} \sum_{i=1}^{n} d_i \epsilon_i.
\end{aligned}
$$

4. $y_i = \beta_0 + \beta_1 x_{i1} + e_i \quad \& \quad y_i \sim N(\beta_0, \beta_1 x_{i1}, \sigma^2) \quad \text{for } i = 1, ..., n$
   $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1}$. Then,

$$\begin{aligned}
Var(\hat{y}_i) &= Var(\hat{\beta}_0) + Var(\hat{\beta}_1)x_{i1}^2 + 2Cov(\hat{\beta}_0, \hat{\beta}_1)x_{i1} \\
&= \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}_1^2}{S_{x_1x_1}} + \frac{x_{i1}^2}{S_{x_1x_1}} - \frac{2\bar{x}_1 x_{i1}}{S_{x_1x_1}} \right) \\
&= \sigma^2 \left(\frac{1}{n} + \frac{(x_{i1} - \bar{x}_1)^2}{S_{x_1x_1}} \right) \\
\Rightarrow \hat{y}_i &\sim N\left(\beta_0 + \beta_1 x_{i1}, \sigma^2 \left(\frac{1}{n} + \frac{(x_{i1} - \bar{x}_1)^2}{S_{x_1x_1}} \right) \right)
\end{aligned}$$

$$\begin{aligned}
Cov(y_i,\ \hat{y}_i) &= Cov(\beta_0 + \beta_1 x_{i1} + e_i,\ \bar{y} + \hat{\beta}_1(x_{i1} - \bar{x}_1)) \\
&= Cov\left(e_i,\ \frac{1}{n}\sum_{i=1}^{n} y_i + \frac{(x_{i1} - \bar{x}_1)}{S_{x_1x_1}} \sum_{i=1}^{n} (x_{i1} - \bar{x}_1) y_i \right) \\
&= \sigma^2 \left(\frac{1}{n} + \frac{(x_{i1} - \bar{x}_1)^2}{S_{x_1x_1}} \right) \\
Var(y_i - \hat{y}_i) &= Var(y_i) + Var(\hat{y}_i) - 2Cov(y_i,\ \hat{y}_i) \\
&= \sigma^2 \left(1 - \frac{1}{n} - \frac{(x_{i1} - \bar{x}_1)^2}{S_{x_1x_1}} \right) \\
\Rightarrow y_i - \hat{y}_i &\sim N\left(0, \sigma^2 \left(1 - \frac{1}{n} - \frac{(x_{i1} - \bar{x}_1)^2}{S_{x_1x_1}} \right) \right)
\end{aligned}$$

5. First regress y on x_1 and x_2 using centered model.

$$y_i = \beta_0' + \beta_1 x_{i1}' + \beta_2 x_{i2}' + e_i$$

$$X^T X = \begin{pmatrix} \text{n} & 0 & 0 \\ 0 & S_{x_1x_1} & S_{x_1x_2} \\ 0 & S_{x_1x_2} & S_{x_2x_2} \end{pmatrix} \quad (X^T X)^{-1} = \begin{pmatrix} 1/\text{n} & 0 & 0 \\ 0 & S_{x_2x_2}/c & -S_{x_1x_2}/c \\ 0 & -S_{x_1x_2}/c & S_{x_1x_1}/c \end{pmatrix}$$

$$X^T Y = \begin{pmatrix} \sum_i y_i \\ S_{x_1y} \\ S_{x_2y} \end{pmatrix}$$
$$\text{where } c = S_{x_1x_1} S_{x_2x_2} - S_{x_1x_2}^2$$

Therefore, we have

$$\hat{\beta}_1 = \frac{S_{x_2x_2} S_{x_1y} - S_{x_1x_2} S_{x_2y}}{S_{x_1x_1} S_{x_2x_2} - S_{x_1x_2}^2}$$
$$\hat{\beta}_2 = \frac{S_{x_1x_1} S_{x_2y} - S_{x_1x_2} S_{x_1y}}{S_{x_1x_1} S_{x_2x_2} - S_{x_1x_2}^2}$$

Then regress x_2 on $\hat{y}$, and similarly we have

$$\hat{\gamma}_1 = \frac{S_{\hat{y}\hat{y}} S_{x_1x_2} - S_{x_1\hat{y}} S_{x_2\hat{y}}}{S_{x_1x_1} S_{\hat{y}\hat{y}} - S_{x_1\hat{y}}^2} \tag{1}$$
$$\hat{\gamma}_2 = \frac{S_{\hat{y}\hat{y}} S_{x_1x_2} - S_{x_1\hat{y}} S_{x_2\hat{y}}}{S_{x_1x_1} S_{\hat{y}\hat{y}} - S_{x_1\hat{y}}^2} \tag{2}$$

Next, try to express $S_{x_1\hat{y}}$, $S_{x_2\hat{y}}$ and $S_{\hat{y}\hat{y}}$ by $S_{x_1x_1}$, $S_{x_1x_2}$, and $S_{x_2x_2}$.

$$\begin{aligned}
S_{x_1\hat{y}} &= \sum_i (x_{i1} - \bar{x_1})(\hat{y}_i - \bar{\hat{y}}) \\
&= \sum_i (x_{i1} - \bar{x_1})(\hat{\beta}_1(x_{i1} - \bar{x}_1) + \hat{\beta}_2(x_{i2} - \bar{x}_2)) \\
&= \hat{\beta}_1 S_{x_1x_1} + \hat{\beta}_2 S_{x_1x_2} \\
S_{x_2\hat{y}} &= \hat{\beta}_1 S_{x_1x_2} + \hat{\beta}_2 S_{x_2x_2} \\
S_{\hat{y}\hat{y}} &= \hat{\beta}_1^2 S_{x_1x_1} + 2\hat{\beta}_1\hat{\beta}_2 S_{x_1x_2} + \hat{\beta}_2^2 S_{x_2x_2}
\end{aligned}$$

Then substitute above three into (1) and (2), the followings are achieved

$$\begin{aligned}
\hat{\gamma}_1 &= -\frac{\hat{\beta}_1}{\hat{\beta}_2} = -\frac{S_{x_2x_2}S_{x_1y} - S_{x_1x_2}S_{x_2y}}{S_{x_1x_1}S_{x_2y} - S_{x_1x_2}S_{x_1y}} \\
\hat{\gamma}_2 &= \frac{1}{\hat{\beta}_2} = \frac{S_{x_1x_1}S_{x_2x_2} - S_{x_1x_2}^2}{S_{x_1x_1}S_{x_2y} - S_{x_1x_2}S_{x_1y}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{\epsilon}_i &= \hat{x}_{i2} - x_{i2} \\
&= \hat{\gamma}_0 + \hat{\gamma}_1 x_{i1} + \hat{\gamma}_2 \hat{y}_i - x_{i2} \\
&= \bar{x}_2 - \hat{\gamma}_1 \bar{x}_1 - \hat{\gamma}_2 \bar{\hat{y}} + \hat{\gamma}_1 x_{i1} + \hat{\gamma}_2 \hat{y}_i - x_{i2} \\
&= \bar{x}_2 + \hat{\gamma}_1 (x_{i1} - \bar{x}_1) + \hat{\gamma}_2 (\hat{y}_i - \bar{\hat{y}}) - x_{i2} \\
&= \bar{x}_2 - \frac{\hat{\beta}_1}{\hat{\beta}_2}(x_{i1} - \bar{x}_1) + \frac{1}{\hat{\beta}_2}(\hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} - \hat{\beta}_0 - \hat{\beta}_1 \bar{x}_1 - \hat{\beta}_2 \bar{x}_2) - x_{i2} \\
&= \bar{x}_2 + \frac{1}{\hat{\beta}_2}(\hat{\beta}_2(x_{i2} - \bar{x}_2)) - x_{i2} = 0
\end{aligned}$$