19 Nov

Example 3

categorical variable

exposure	r	n
0	2	8
1	11	15

prob. when exposure = 1 P_1

" " " = 0 P_0

Fit a logistic regression on exposure $\Rightarrow \beta_0, \beta_1$ $p' = 2$

$$\underset{\text{data}}{\frac{11}{15}} = \hat{P}_1 \underset{\text{model}}{=} \frac{\exp(\hat\beta_0 + \hat\beta_1)}{1+\exp(\hat\beta_0+\hat\beta_1)} \qquad \underset{\text{data}}{\frac{2}{8}} = \hat{P}_0 \underset{\text{model}}{=} \frac{\exp(\hat\beta_0)}{1+\exp(\hat\beta_0)}$$

① Odds ratio of exposure

Data $\Rightarrow$ $$\frac{\frac{P}{1-P}\Big|\text{exposure}=1}{\frac{P}{1-P}\Big|\text{exposure}=0} \underset{\text{data}}{=} \frac{\frac{11}{\cancel{15}}\Big/\frac{4}{\cancel{15}}}{\frac{2}{\cancel{8}}\Big/\frac{6}{\cancel{8}}} = \frac{11\times 6}{2\times 4} = 8.25$$

Model $\Rightarrow$ $$\frac{\exp(\hat\beta_0+\hat\beta_1)}{\exp(\hat\beta_0)} = \exp(\hat\beta_1) \qquad \hat\beta_1 = \ln\left(\frac{a*d}{b*c}\right)$$

$\Rightarrow \exp(\hat\beta_1) = 8.25$

$\Rightarrow \hat\beta_1 = \ln(8.25) = 2.1102$

② $$\frac{\exp(\hat\beta_0)}{1+\exp(\hat\beta_0)} \underset{\text{model}}{=} \hat{P}_0 \Rightarrow \exp(\hat\beta_0) = \frac{\hat{P}_0 \cancel{|\exp=0}}{1-\hat{P}_0\cancel{|\exp=0}} \underset{\text{data}}{=} \frac{2/8}{6/8} = \frac{1}{3}$$

exp	r	n−r	n
1	11 a	4 b	15
0	2 c	6 d	8

$\Rightarrow \hat\beta_0 = \ln\left(\frac{2}{6}\right) = -1.0986$

$\hat\beta_0 = \ln\left(\frac{c}{d}\right)$

(a) $\ln\left(\frac{\hat{P}}{1-\hat{P}}\right) = -1.0986 + 2.1102\, I\{\text{exposure}=1\}$

$$\mathrm{Var}(\hat{\beta}_1) = \mathrm{Var}\Big(\underbrace{\ln\left(\frac{a\,d}{b\,c}\right)}_{\mathrm{Var}(f(a,b,c,d))}\Big)$$

$$\mathrm{Var}(f(a,b,c,d)) \approx \mathrm{Var}\Big(\overbrace{f(\mu_a, \mu_b, \mu_c, \mu_d)}^{0} +$$

$$\frac{\partial f}{\partial a}\Big|_{\substack{a=\mu_a, b=\mu_b, c=\mu_c, d=\mu_d \\ \cancel{\mu_a, \mu_b}}} (a-\mu_a)$$

$$+ \frac{\partial f}{\partial b}\Big|_{a=\mu_a, b=\mu_b, c=\mu_c, d=\mu_d} (b-\mu_b)$$

$$+ \frac{\partial f}{\partial c}\Big|_{a=\mu_a, b=\mu_b, c=\mu_c, d=\mu_d} (c-\mu_c)$$

$$+ \frac{\partial f}{\partial d}\Big|_{a=\mu_a, b=\mu_b, c=\mu_c, d=\mu_d} (d-\mu_d)\Big)$$

$$+ \cdots$$

$$\mathrm{Var}\left(\frac{\partial f}{\partial a}(a-\mu_a)\right) = \left(\frac{\partial f}{\partial a}\right)^2\Big|_{a=\mu_a, \ldots, d=\mu_d} \mathrm{Var}(a-\mu_a)$$

$$f = \ln\left(\frac{a\,d}{b\,c}\right) \Rightarrow \frac{\partial f}{\partial a} = \frac{b\,c}{a\,d} * \frac{d}{bc}$$

$$= \frac{1}{a}\Big|_{a=\mu_a, \ldots, d=\mu_d}$$

$$= \frac{1}{\mu_a}$$

Assume count data $\sim Po(\mu)$

i.e. $a \sim Po(\mu_a)$

$E(a) = \mu_a$

$\mathrm{Var}(a) = \mu_a$

$\Rightarrow \mathrm{Var}(a-\mu_a) = \mu_a$

$$\Rightarrow \mathrm{Var}\left(\frac{\partial f}{\partial a}(a-\mu_a)\right) = \left(\frac{1}{\mu_a}\right)^2 * \mu_a = \frac{1}{\mu_a}$$

$$\mathrm{Var}\left(\frac{\partial f}{\partial b}(b-\mu_b)\right) = \frac{1}{\mu_b}$$

$$\mathrm{Var}\left(\frac{\partial f}{\partial c}(c-\mu_c)\right) = \frac{1}{\mu_c}$$

$$\mathrm{Var}\left(\frac{\partial f}{\partial d}(d-\mu_d)\right) = \frac{1}{\mu_d}$$

$$\frac{\partial f}{\partial b} = \frac{\partial}{\partial b}\ln\left(\frac{a\,d}{b\,c}\right)$$

$$= \frac{b\,c}{a\,d}\left(-\frac{a\,d}{b^2 c}\right)$$

$$= -\frac{1}{b}\Big|_{a=\mu_a, \ldots, d=\mu_b}$$

$$\mathrm{Var}\left(\frac{\partial f}{\partial b}(b-\mu_b)\right) = \left(-\frac{1}{\mu_b}\right)^2 * \mu_b$$

$$= \frac{1}{\mu_b}$$

$$\Rightarrow \widehat{Var}(\hat{\beta}_1) = \widehat{Var}\left(\ln\left(\frac{ad}{bc}\right)\right) \approx \frac{1}{\hat{\mu}_a} + \frac{1}{\hat{\mu}_b} + \frac{1}{\hat{\mu}_c} + \frac{1}{\hat{\mu}_d}$$

$$\approx \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

e.g. $\widehat{Var}(\hat{\beta}_1) \approx \frac{1}{11} + \frac{1}{4} + \frac{1}{2} + \frac{1}{6}$

$$= 1.007576$$

$$\widehat{Var}(\hat{\beta}_0) = Var\left(\underbrace{\ln\left(\frac{c}{d}\right)}\right)$$

$$f(c,d) \approx f(\mu_c, \mu_d) + \frac{\partial f}{\partial c}\Big| (c-\mu_c) + \frac{\partial f}{\partial d}\Big| (d-\mu_d)$$

$$\approx \frac{1}{\mu_c} + \frac{1}{\mu_d} \Rightarrow \widehat{Var}(\hat{\beta}_0) \approx \frac{1}{c} + \frac{1}{d}$$

e.g. $\widehat{Var}(\hat{\beta}_0) \approx \frac{1}{2} + \frac{1}{6} = 0.6667$

$$cov(\hat{\beta}_0, \hat{\beta}_1) = cov\left(\underbrace{\ln\left(\frac{c}{d}\right)}, \underbrace{\ln\left(\frac{ad}{bc}\right)}\right)$$

$$f(c,d) = f_0(\mu_c, \mu_d) + \underline{\frac{\partial f_0}{\partial c}(c-\mu_c)} + \underline{\frac{\partial f_0}{\partial d}(d-\mu_d)}$$

$$f_1(\mu_a, \mu_b, \mu_c, \mu_d) + \frac{\partial f_1}{\partial a}(a-\mu_a) + \frac{\partial f_1}{\partial b}(b-\mu_b) + \underline{\frac{\partial f_1}{\partial c}(c-\mu_c)} + \underline{\frac{\partial f_1}{\partial d}(d-\mu_d)}$$

$cov(\hat{\beta}_0, \hat{\beta}_1)$

$$= cov\left(\underbrace{\frac{\partial f_0}{\partial c}\Big|}_{=\frac{1}{c}\big|_{c=\mu_c}}(c-\mu_c) + \underbrace{\frac{\partial f_0}{\partial d}\Big|}_{=-\frac{1}{d}\big|_{d=\mu_d}}(d-\mu_d),\ \underbrace{\frac{\partial f_1}{\partial c}\Big|}_{=-\frac{1}{c}\big|_{c=\mu_c}}(c-\mu_c) + \underbrace{\frac{\partial f_1}{\partial d}\Big|}_{=\frac{1}{d}\big|_{d=\mu_d}}(d-\mu_d)\right)$$

$$= cov\left(\frac{\partial f_0}{\partial c}\Big|(c-\mu_c), \frac{\partial f_1}{\partial c}\Big|(c-\mu_c)\right) + cov\left(\frac{\partial f_0}{\partial d}\Big|(d-\mu_d), \frac{\partial f_1}{\partial d}\Big|(d-\mu_d)\right)$$

$$= -\frac{1}{\mu_c} - \frac{1}{\mu_d}$$

$\Rightarrow \text{cov}(\hat{\beta}_0, \hat{\beta}_1) \approx -\left(\frac{1}{c} + \frac{1}{d}\right)$

e.g. $\text{cov}(\hat{\beta}_0, \hat{\beta}_1) \approx -\left(\frac{1}{2} + \frac{1}{6}\right) = -0.6667$

(b) 95% C.I. of unknown parameters

β_0 : $\hat{\beta}_0 \pm 1.96 *$ s.e. of $\hat{\beta}_0$ $= -1.0986 \pm 1.96\sqrt{0.6667}$

β_1 : $\hat{\beta}_1 \pm 1.96 *$ s.e. of $\hat{\beta}_1$ $= 2.1102 \pm 1.96\sqrt{1.007576}$

$= (\hat{\beta}_{1l}, \hat{\beta}_{1u})$

(c) $\widehat{\text{odds ratio}} = \exp(\hat{\beta}_1) = \exp(2.1102)$

95% C.I. of odds ratio $= (\exp(\hat{\beta}_{1l}), \exp(\hat{\beta}_{1u}))$

(d) Estimate the prob. of getting a disease for patients with $\underbrace{\text{high cholestrol diet}}_{\text{exposure}=1}$ and its 95% C.I.

$$P = \frac{\exp(\beta_0 + \beta_1)}{1+\exp(\beta_0+\beta_1)}$$

pt. est. $\hat{P} = \frac{\exp(\hat{\beta}_0 + \hat{\beta}_1)}{1+\exp(\hat{\beta}_0+\hat{\beta}_1)} = \frac{\exp(-1.0986+2.1102)}{1+\exp(-1.0986+2.1102)}$ = Ans.

From data $\hat{P}_1 = \frac{11}{15} = 0.7333$

95% C.I. of $\beta_0 + \beta_1$

pt. est. of $\beta_0 + \beta_1 = \hat{\beta}_0 + \hat{\beta}_1 = -1.0986 + 2.1102 = 1.0116$

s.e. of $\beta_0 + \beta_1 = \sqrt{Var(\hat{\beta}_0) + Var(\hat{\beta}_1) + 2\,\text{cov}(\hat{\beta}_0, \hat{\beta}_1)} = 0.5839$

$\Rightarrow$ 95% C.I. of $\beta_0 + \beta_1 = (W_l, W_u)$

95% C.I. of $\hat{P}_1 = \left(\frac{\exp(W_l)}{1+\exp(W_l)}, \frac{\exp(W_u)}{1+\exp(W_u)}\right)$

(e) $P_0 = \frac{\exp(\beta_0)}{1+\exp(\beta_0)}$

Example 4 (2015 Final)

categorical variable with 3 levels = A, B, $\boxed{P}$ ⇒ 2 dummy

Consider a study of the analgesic effect of treatments on elderly patients with neuralgia. Patient is assigned to take one type of treatments (either A or B) or a placebo (P) randomly. The response variable is whether the patient reported pain or not. Researchers recorded gender of the patients, age and the duration of complaint before the treatment began.

Pr(no pain)

Consider a logistic model for the probability of no pain on TREATMENT and age. Note that the categorical variable TREATMENT has three levels and "P" is chosen as reference group. The table below shows the summary of the maximum likelihood estimates and their variance and covariance matrix.

$Var(\hat\beta_0)$ $\quad$ $Cov(\hat\beta_0, \hat\beta_3)$ $\quad$ $Var(\hat\beta_3)$

Parameter	Estimate	Covariance Matrix			
		Intercept	TREATMENT=A	TREATMENT=B	Age
Intercept	16.5564	35.27268	1.625089	3.061703	-0.51924
TREATMENT=A	2.6825	1.625089	0.762709	0.515533	-0.02899
TREATMENT=B	3.2551	3.061703	0.515533	1.001707	-0.05034
Age	-0.2581	-0.51924	-0.02899	-0.05034	0.007715

Based on the above table, answer the following questions.

1. Write down the fitted model. $\ln\left(\frac{p}{1-p}\right) = 16.5564 + 2.6825\, I\{treatment = A\} + 3.2551\, I\{treatment = B\} - 0.2581 * age$ $\quad$ case A $\quad$ case B

2. Estimate the odds ratio of no pain for a patient taking treatment A verse a patient taking treatment B at the same age. Test whether the odds ratio is equal to 1 at $\alpha = 0.1$. State clearly the test statistic, critical value and your decision. odds for case A $= \exp(\beta_0 + \beta_1 + \beta_3 * age)$
odds for case B $= \exp(\beta_0 + \beta_2 + \beta_3 * age)$ ⇒ odds ratio $= \exp(\beta_1 - \beta_2)$

$H_0: \exp(\beta_1 - \beta_2) = 1$ ⇒ $\beta_1 - \beta_2 = 0$ ⇒ $\left(\frac{\hat\beta_1 - \hat\beta_2}{s.e.\ of\ (\hat\beta_1 - \hat\beta_2)}\right)^2 \sim \chi^2_{\alpha, 1}$

3. Estimate the odds ratio of no pain (with 90% confidence interval) for one unit increase in age for a patient taking treatment A. $\exp(\beta_3)$

4. Estimate odds and then the probability of no pain (with 95% confidence interval) for the following two cases: (1) a 65 year-old patient taking placebo P; (2) a 75 year-old patient taking treatment B.

5. Are the probabilities of no pain for the two cases in (d) equal? State clearly the test statistic, critical value and your decision. Set $\alpha = 0.1$

6. From the data set, it is noted that 30 (out of 40 patients taking treatment either A or B) and 5 (out of 20 patients taking placebo) showed no pain. Estimate the odds ratio of taking treatment for the model of probability of no pain on treatment.

4 ↕

7. Using the data set in the previous part to estimate the odds ratio of taking treatment for the model of probability of no pain on treatment using the method of weighted least squares.

4. (i) odds $= \exp(\boxed{\hat\beta_0 + \hat\beta_3 * 65})$ $\quad$ s.e. of $\hat\beta_0 + 65\hat\beta_3 = \left(Var(\hat\beta_0) + 65^2 Var(\hat\beta_3) + 130\, Cov(\hat\beta_0, \hat\beta_3)\right)^{1/2}$

(ii) odds $= \exp(\boxed{\hat\beta_0 + \hat\beta_2 + \hat\beta_3 * 75})$ $\quad$ s.e. of $\hat\beta_0 + \hat\beta_2 + 75\hat\beta_3$

$= \left(Var(\hat\beta_0) + Var(\hat\beta_2) + 75^2 Var(\hat\beta_3) + 2\, Cov(\hat\beta_0, \hat\beta_2) + 150\, Cov(\hat\beta_0, \hat\beta_3) + 150(\hat\beta_0, \hat\beta_3)\right)^{1/2}$

5. Prob(i) = Prob(ii)

$$\text{Prob}(i) = \frac{\exp(\beta_0 + 65 * \beta_3)}{1+\exp(\beta_0+65*\beta_3)}$$

$\Rightarrow H_0: \beta_0 + 65\beta_3 = \beta_0 + \beta_2 + 75 * \beta_3$

$$\text{Prob}(ii) = \frac{\exp(\beta_0 + \beta_2 + \beta_3 * 75)}{1+\exp(\beta_0+\beta_2+\beta_3*75)}$$

$\Rightarrow H_0: \beta_2 + 10\beta_3 = 0$

$$\text{Wald Chi-square} = \left(\frac{\hat{\beta}_2 + 10\hat{\beta}_3 - 0}{\boxed{s.e.\ of\ (\hat{\beta}_2 + 10\hat{\beta}_3)}}\right)^2$$

$$\left(Var(\hat{\beta}_2) + 100\, Var(\hat{\beta}_3) + 20\, cov(\hat{\beta}_2, \hat{\beta}_3)\right)^{1/2}$$

6.

treatment	r	n
A & B	30	40
P	5	20

Fit a model of Pr(no pain) on treatment.

	r	n-r	n
A & B	30 a	10 b	40
P	5 c	15 d	20

$$\widehat{\text{Odds ratio}} = \frac{a * d}{b * c}$$

$$= \frac{30 * 15}{10 * 5} \leftarrow \exp(\hat{\beta}_1)$$

$\ln\left(\frac{a\,d}{bc}\right)$

~~Var(odds ratio)~~ ① 95% C.I. of $\beta_1 = \hat{\beta}_1 \pm 1.96\, s.e.$

$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)^{1/2}$

② 95% C.I. of $\exp(\hat{\beta}_1)$

R^2

liner regression $\quad R^2 = \dfrac{\text{Reg S.S.}}{\text{total S.S.}} = 1 - \dfrac{\text{Res S.S.}}{\boxed{\text{Total S.S.}}}$

Res S.S. when all β's equal to zero

Model becomes:

$y_i = \beta_0 + e_i$

$y_i \sim N(\underset{\sim}{x}_i^T \underset{\sim}{\beta}, \sigma^2)$

likelihood function $(\underset{\sim}{\beta}, \sigma^2)$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2}} \exp\left\{ - \frac{\sum_{i=1}^{n}(y_i - \underset{\sim}{x}_i^T \underset{\sim}{\beta})^2}{2\sigma^2} \right\}$$

likelihood function $(\hat{\underset{\sim}{\beta}}, \hat{\sigma}^2)$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(y_i - \underset{\sim}{x}_i^T \hat{\underset{\sim}{\beta}})^2}{n} \leftarrow \text{Res S.S.}$$

$$= (2\pi)^{-\frac{n}{2}} (\hat{\sigma}^2)^{-\frac{n}{2}} \exp\left(-\frac{n}{2}\right)$$

$$= (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{n}{2}\right) \left(\frac{\text{Res S.S.}}{n}\right)^{-\frac{n}{2}} \Rightarrow \text{Res S.S.} = \text{constant} * \left(L(\hat{\underset{\sim}{\beta}}, \hat{\sigma}^2)\right)^{-2/n}$$

$$R^2 = 1 - \frac{\text{Res S.S.}}{\text{Res S.S.} \mid \text{model with } \beta_0}$$

Logistic regression

Cox & Snell

$$R^2 = 1 - \left(\frac{L_M}{L_0}\right)^{-2/n}$$

L_M: $L(\hat{\underset{\sim}{\beta}}, \hat{\sigma}^2)$

L_0: model with β_0

$$= 1 - \left(\frac{L_0}{L_M}\right)^{2/n}$$

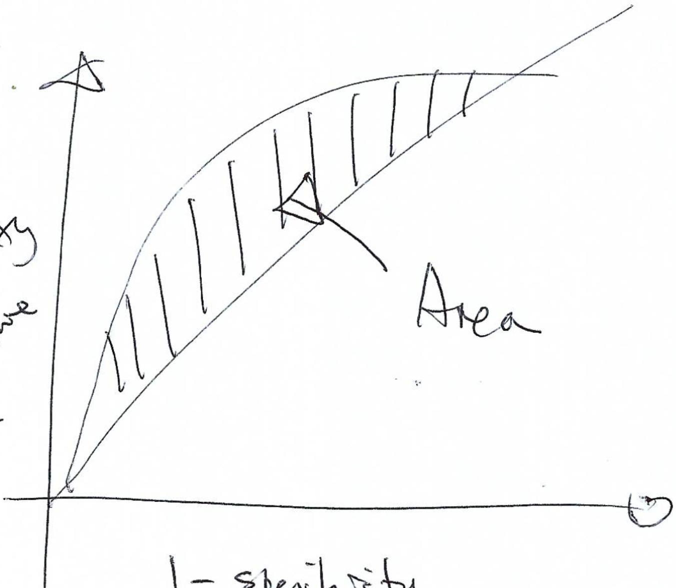
~~$R^2 = 1$~~

$$R^2_{max} = 1 - L_0^{2/n}$$

$$\Rightarrow \text{max-rescaled } R^2 = \frac{1 - \left(\frac{L_0}{L_M}\right)^{2/n}}{1 - L_0^{2/n}}$$

to make sure ~~R^2~~ the max. value of $R^2 = 1$

ROC



sensitivity
= true positive rate

~~= # tp~~

$$= \frac{\text{\# of true positive}}{\text{total \# of positive}}$$

1 - specificity

= false positive rate

$$= \frac{\text{\# of false positive}}{\text{total \# of negative}}$$