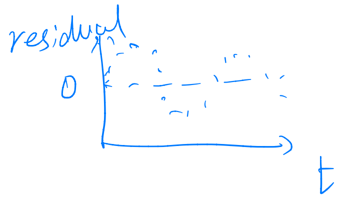
Tutorial Notes 8 of MATH3424

1 Summary of course material

1. Autocorrelation:

   When the observations have a **natural sequential order**, the correlation is referred to as **autocorrelation**.

2. Runs test: n_1 residuals positive and n_2 residuals negative

   Run test statistic= $\frac{\#\text{ of runs } -\mu}{\sigma}$ with

   $$\mu = \frac{2n_1n_2}{n_1+n_2}+1, \quad \sigma^2 = \frac{2n_1n_2(2n_1n_2-n_1-n_2)}{(n_1+n_2)^2(n_1+n_2-1)}$$

   Under H_0: the residuals are uncorrelated random, this test statistic (approximately) follows a distribution $N(0,1)$.

   second-order

   $\cdots\ \epsilon_{-2}\ \epsilon_{-1}\ \epsilon_0\ \epsilon_1 \cdots$

3. Durbin-Watson Statistic

   $\epsilon_t = \rho_1\epsilon_{t-1} + \rho_2\epsilon_{t-2} + \omega_t$

   (a) Assumption: successive errors are correlated, i.e., $\epsilon_t = \rho\epsilon_{t-1}+\omega_t$, $|\rho| \le 1$, with $\omega_t \overset{i.i.d}{\sim} N(0,\sigma_\omega^2)$. This is the first-order autocorrelation.

   $|\rho|<1$

   $\epsilon_t = \rho\epsilon_{t-1} + \omega_t$
   $= \rho(\rho\epsilon_{t-2}+\omega_{t-1})+\omega_t$
   $= \rho^2\epsilon_{t-2} + \rho\omega_{t-1} + \omega_t$
   $= \rho^s\epsilon_{t-s} + \sum_{i=0}^{s-1}\rho^i\omega_{t-i}$
   $= \sum_{i=0}^{\infty}\rho^i\omega_{t-i}$

   $Var(\epsilon_t) = \sum_{i=0}^{\infty}\rho^{2i}Var(\omega_{t-i})$
   $= \sigma_\omega^2\sum_{i=0}^{\infty}\rho^{2i}$
   $= \sigma_\omega^2\cdot\frac{1}{1-\rho^2}$

   (b) The Durbin-Watson statistic:

   $$d = \frac{\sum_{t=2}^{n}(e_t-e_{t-1})^2}{\sum_{t=1}^{n}e_t^2}$$

   where e_i is the i-th ordinary least squares (OLS) residual. The statistics d is used for testing the null hypothesis $H_0: \rho = 0$ against an alternative $H_1: \rho > 0$.

   i. $d < d_L$, reject H_0
   ii. $d > d_U$, do not reject H_0
   iii. $d_L < d < d_U$, the test is inconclusive.

   (c) An estimate of ρ is given by

   $$\hat{\rho} = \frac{\sum_{t=2}^{n}e_te_{t-1}}{\sum_{t=1}^{n}e_t^2}, \quad d \approx 2(1-\hat{\rho})$$

4. Remedial measures for autocorrelation

   (a) Addition of predictor variables

   (b) Transformation (Cochrane-Orcutt Procedure)

   $$y_t^* = \beta_0^* + \beta_1^*x_t^* + \omega_t$$

where

$$
\begin{aligned}
y_t^* &= y_t - \rho y_{t-1} \\
x_t^* &= x_t - \rho x_{t-1} \\
\beta_0^* &= \beta_0(1 - \rho) \\
\beta_1^* &= \beta_1
\end{aligned}
$$

2 Questions

2.1

For each of the following tests concerning the autocorrelation parameter ρ in the following regression model with first-order autocorrelation:

$$Y_t = \beta_0 + \beta_1 X_{t1} + \beta_2 X_{t2} + \beta_3 X_{t3} + \epsilon_t, \quad \epsilon_t = \rho\epsilon_{t-1} + \omega_t$$

State the appropriate decision rule based on the Durbin-Watson test statistic for a sample of size 38:

(1) $H_0 : \rho = 0, \quad H_a : \rho > 0, \quad \alpha = .01$

(2) $H_0 : \rho = 0, \quad H_a : \rho < 0, \quad \alpha = .05$

(3) $H_0 : \rho = 0, \quad H_a : \rho \neq 0, \quad \alpha = .02$

Define $d = \dfrac{\sum_{t=2}^{38} (e_t - e_{t-1})^2}{\sum_{t=1}^{38} e_t^2}$

$d_L(n, p, \alpha)$, $d_u(n, p, \alpha)$

(1) If $d < d_L(38, 3, 0.01)$, then we reject H_0.

If $d > d_u(38, 3, 0.01)$, then we do not reject H_0

Otherwise, no conclusion.

(2) If $d' = 4 - d < d_L(38, 3, 0.05)$, then we reject H_0.

If $d' > d_u(38, 3, 0.05)$, ... do not reject H_0.

(3) Conduct two test

$$\begin{cases} ① \ H_0^{(1)} : \rho = 0 & H_a^{(1)} : \rho > 0, \quad \alpha = 0.01 \\ ② \ H_0^{(2)} : \rho = 0 & H_a^{(2)} : \rho < 0, \quad \alpha = 0.01 \end{cases}$$

If at least one of ① and ② is rejected, then H_0 is rejected. otherwise we do not reject H_0.

2.2

A staff analyst for a manufacturer of microcomputer components has compiled monthly data for the past 16 months on the value of industry production of processing unit that use the. e components (X, in million dollars) and the value of the firm's components used (Y, in thousand dollars). The analyst believes that a simple linear regression relation is appropriate but anticipates positive autocorrelation. The data follow:

t:	1	2	3	...	14	15	16
X_t:	2.052	2.026	2.002	...	2.080	2.102	2.150
Y_t:	102.9	101.5	100.8	...	104.8	105.0	107.2

1. Fit a simple linear regression model by ordinary least squares and obtain the residuals. Also obtain s.e.$(\hat{\beta}_0)$ and s.e.$(\hat{\beta}_1)$.

```
Call:
lm(formula = Y ~ X, data = df)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.91277 -0.67136  0.09514  0.53886  1.80259 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   -7.739      7.175  -1.079    0.299    
X             53.953      3.520  15.329 3.82e-10 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.9543 on 14 degrees of freedom
Multiple R-squared:  0.9438,    Adjusted R-squared:  0.9398 
F-statistic:   235 on 1 and 14 DF,  p-value: 3.818e-10
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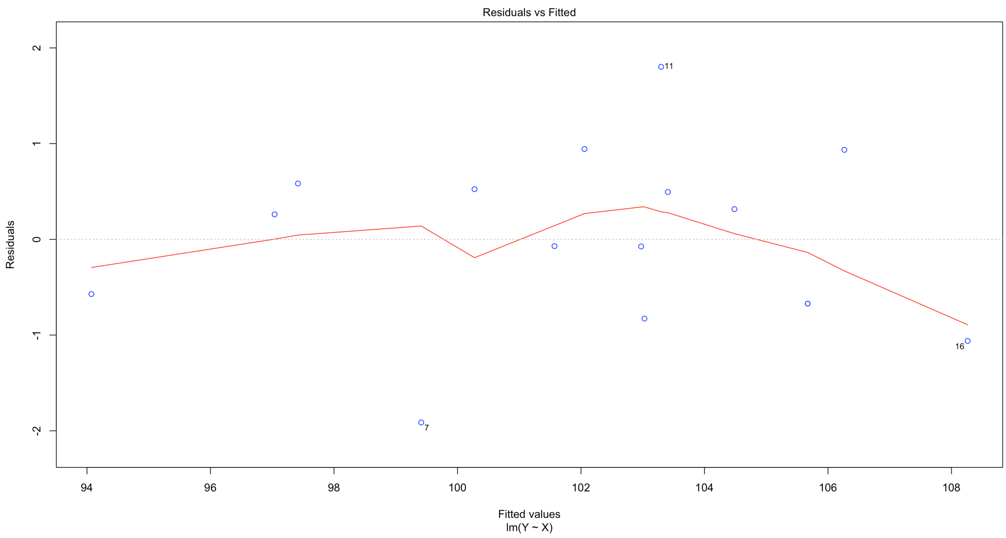
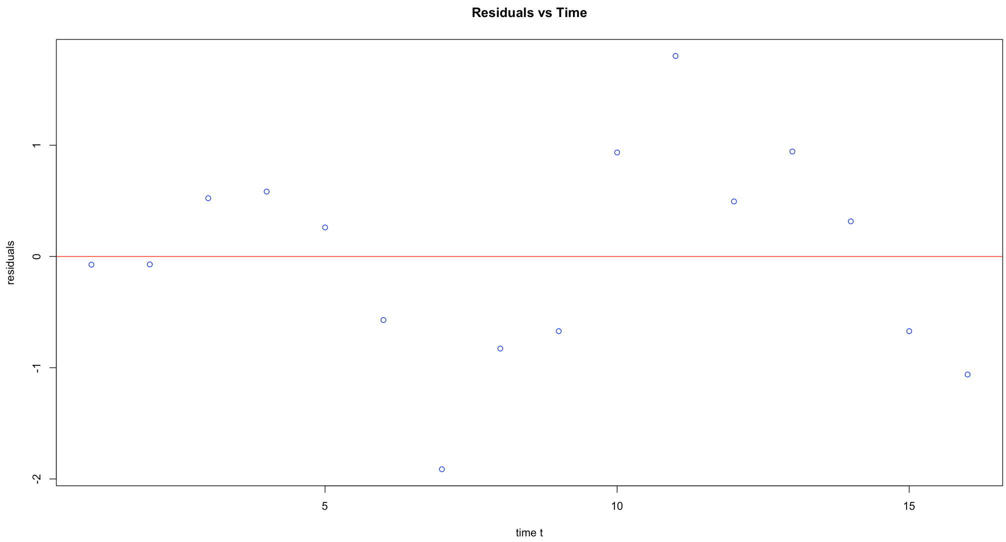
s.e.$(\hat{\beta}_0) = 7.175$, s.e$(\hat{\beta}_1) = 3.520$

2. Plot the residuals against time and explain whether you find any evidence of positive autocorrelation.





3. Conduct a formal test for positive autocorrelation using $\alpha = .05$. State the alternatives, decision rule, and conclusion. Is the residual analysis in part (b) in accord with the test result?

	d_L	d_U
$p=1$, $n=16$	1.10	1.37

$H_0: \rho = 0 \qquad H_a: \rho > 0$

$$d = \frac{\sum_{2}^{16} (e_t - e_{t-1})^2}{\sum_{1}^{16} e_t^2} = 0.8566 < d_L$$

So we reject H_0, there is positive autocorrelation.

4. The analyst has decided to employ regression model with first-order autocorrelation and use the Cochrane-Orcutt procedure to fit the model.

 (a) Obtain a point estimate of the autocorrelation parameter. How well does the approximate relationship $d \approx 2(1-\hat{\rho})$ hold here between this point estimate and the Durbin-Watson test statistic?

$$\hat{\rho} = \frac{\sum_{t=2}^{n} e_t e_{t-1}}{\sum_{t=1}^{n} e_t^2} = 0.5273$$

$$2(1-\hat{\rho}) = 0.94534$$

 (b) Use one iteration to obtain the estimates $\hat{\beta}_0^*$ and $\hat{\beta}_1^*$ of the regression coefficients β_0^* and β_1^* in transformed model ((7.4) in lecture slides) and state the estimated regression function. Also obtain s.e.$(\hat{\beta}_0^*)$ and s.e.$(\hat{\beta}_1^*)$.

```
Call:
lm(formula = Ystar ~ Xstar)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.51142 -0.43478 -0.05777  0.41365  1.42613 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   -1.073      4.121   -0.26    0.799    
Xstar         51.244      4.261   12.03 2.04e-08 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.8135 on 13 degrees of freedom
Multiple R-squared:  0.9175,	Adjusted R-squared:  0.9112 
F-statistic: 144.6 on 1 and 13 DF,  p-value: 2.036e-08
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$\hat{\beta}_0^* = -1.073$, $\hat{\beta}_1^* = 51.244$ $\hat{y}_t^* = -1.073 + 51.244 x_t^*$

s.e.$(\hat{\beta}_0^*) = 4.121$ s.e.$(\hat{\beta}_1^*) = 4.261$

(c) Test whether any positive autocorrelation remains after the first iteration using $\alpha = .05$. State the alternatives, decision rule, and conclusion.

$$y_t^* = \beta_0^* + \beta_1^* x_t^* + \underline{\omega_t}$$

test $H_0: \rho^* = 0$, $H_a: \rho^* > 0$, $\omega_t = \rho^* \omega_{t-1} + \xi_t$

		d_L	d_U
$p=1$,	$n=15$	1.08	1.36

$$d = 1.4149 > d_U$$

So we do not reject H_0.

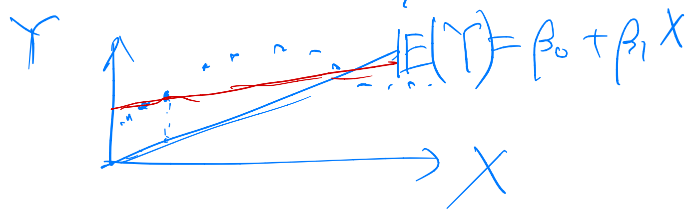
(d) Restate the estimated regression function obtained in part (b) in terms of the original variables. Also obtain s.e.$(\hat{\beta}_0)$ and s.e.$(\hat{\beta}_1)$. Compare the estimated regression coefficients obtained with tbe Cochrane-Orcutt procedure and their estimated standard deviations with those obtained with ordinary least squares in part 1.

since $\hat{\beta}_0 = \dfrac{\hat{\beta}_0^*}{1-\hat{\rho}}$, $\hat{\beta}_1 = \hat{\beta}_1^*$

$$s.e.(\hat{\beta}_0) = \frac{s.e.(\hat{\beta}_0^*)}{1-\hat{\rho}} = 8.7185, \quad s.e.(\hat{\beta}_1) = s.e.(\hat{\beta}_1^*) = 4.216$$

In part 1: $s.e.(\hat{\beta}_0) = 7.175$ $s.e.(\hat{\beta}_1) = 3.520$

$$\hat{\sigma}^2 = MSE = \frac{\sum (y_i - \hat{y}_i)^2}{n-2}$$



(e) On the basis of the results in parts 4(c) and 4(d), does the Cochrane-Orcutt procedure appear to have been effective here?

$s.e.(\hat{\beta}_0) \propto \hat{\sigma}$

$s.e.(\hat{\beta}_1) \propto \hat{\sigma}$

Yes.