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Hypothesis testing.

$H_0 : \underset{\sim}{\beta_1} = \underset{\sim}{0}$

vs $H_1 : \underset{\sim}{\beta_1} \neq \underset{\sim}{0}$

e.g. $p = 4$, $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$

$$\underset{\sim}{\beta_1} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \qquad \underset{\sim}{\beta_2} = \begin{pmatrix} \beta_0 \\ \beta_3 \\ \beta_4 \end{pmatrix}$$

$H_0 : \beta_1 = \beta_2 = 0$

ⓐ likelihood - ratio test

$$= -2 \log \left\{ \frac{\sup_{\hat{\underset{\sim}{\beta}} \in \Theta_0} L(\underset{\sim}{\beta})}{\sup_{\beta \in \Theta} L(\underset{\sim}{\beta})} \right\}$$

$$= -2 \log \left\{ \frac{L(\hat{\underset{\sim}{\beta_2}})}{L(\hat{\underset{\sim}{\beta}})} \right\}$$

$$= -2 \log \left\{ \frac{L(\hat{\underset{\sim}{\beta_2}}) / L(\hat{\underset{\sim}{\beta}})}{L(\hat{\underset{\sim}{\beta}}) / L(\hat{\underset{\sim}{\beta}})} \right\}$$

grouped data

$$P_i = \frac{r_i}{n_i}$$

$$= -2 \log \{ L(\hat{\underset{\sim}{\beta_2}}) / L(\underset{\sim}{\beta}) \} - (-2 \log \{ L(\hat{\underset{\sim}{\beta}}) / L(\underset{\sim}{\beta}) \})$$

$$\lambda(\underset{\sim}{\beta_1} | \underset{\sim}{\beta_2}) = \lambda(\underset{\sim}{\beta_2}) - \lambda(\underset{\sim}{\beta})$$

more indep. variables

$\Rightarrow$ smaller deviance

$$\dot{\sim} \chi^2_r$$

r = # of free para. under Θ − # of free para under Θ_0

e.g. $\underset{\sim}{\beta_1} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ $H_0 : \underset{\sim}{\beta_1} = \underset{\sim}{0}$

$\Rightarrow r = 5 - 3 = 2$

linear regression

$$\cancel{\log}\ \text{likelihood} = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{\sum_{i=1}^{n}(y_i-\mu_i)^2}{2\sigma^2}\right\}$$

$$\text{log-likelihood} = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{\sum_{i=1}^{n}(y_i-\mu_i)^2}{2\sigma^2}$$

linear regression $\quad \mu_i = \underset{\sim}{x}_i^T \underset{\sim}{\beta}$

$$\log L(\hat{\underset{\sim}{\beta}}) = -\frac{n}{2}\log(2\pi\hat{\sigma}^2) - \frac{\sum_{i=1}^{n}(y_i-\underset{\sim}{x}_i^T\hat{\underset{\sim}{\beta}})^2}{2\hat{\sigma}^2}$$

log-likelihood for the saturated model

$$= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\hat{\mu}_i)^2 - \frac{n}{2}\log(2\pi\sigma^2)$$

$$\frac{\partial}{\partial\hat{\mu}_i}\log L(\hat{\mu}_i) = 0 \Rightarrow \hat{\mu}_i = y_i$$

$$\Rightarrow \text{log-likelihood} = \log L(\hat{\underset{\sim}{\mu}}) = -\frac{n}{2}\log(2\pi\sigma^2)$$

$$\Rightarrow \text{Deviance} = -2\log\left\{\frac{\sup_{\theta\in\Theta_0} L(\theta)}{\boxed{\sup_{\theta\in\Theta} L(\theta)}}\right\}$$

saturate model

$$= \frac{1}{\sigma^2}\boxed{\sum_{i=1}^{n}(y_i-\underset{\sim}{x}_i^T\hat{\underset{\sim}{\beta}})^2}$$

Res S.S.

(b) Score test

$$\underset{\sim}{U}^T(\underset{\sim}{\beta}_0)\, \underset{\sim}{I}^{-1}(\underset{\sim}{\beta}_0)\, U(\underset{\sim}{\beta}_0)$$

$1 \times p'$

Fisher Information matrix

1st derivative of log-likelihood

(c) Wald test

$H_0 : \beta_j = 0$

$$\hat{\beta}_j \mathrel{\dot{\sim}} N(\beta_j, c^{jj})$$

$\underset{\sim}{C} = \underset{\sim}{I}$

c^{jj} - $\quad$ th element in $\underset{\sim}{C}^{-1}$

$$\frac{\hat{\beta}_j - \beta_j}{\sqrt{c^{jj}}} \mathrel{\dot{\sim}} N(0, 1)$$

$$\left(\frac{\hat{\beta}_j - \beta_j}{\sqrt{c^{jj}}}\right)^2 \mathrel{\dot{\sim}} \chi^2_{(1)}$$

Section 4 of Chapter 1

$$\hat{\underset{\sim}{\beta}} \mathrel{\dot{\sim}} N(\underset{\sim}{\beta}, \underset{\sim}{C}^{-1}) \qquad \hat{\underset{\sim}{\beta}} \sim N(\underset{\sim}{\beta}, (\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\sigma^2)$$

$H_0 : \underset{\sim}{\beta}_1 = \underset{\sim}{0}$

$H_0 : \underset{\sim}{C}\,\underset{\sim}{\beta} = \underset{\sim}{d}$

$H_0 : \underset{\sim}{L}\,\underset{\sim}{\beta} = \underset{\sim}{d}$

$$F = \frac{(\underset{\sim}{C}\hat{\underset{\sim}{\beta}} - \underset{\sim}{d})^T (\underset{\sim}{C}(\underset{\sim}{X}^T\underset{\sim}{X})^{-1}\underset{\sim}{C}^T)^{-1} (\underset{\sim}{C}\hat{\underset{\sim}{\beta}} - \underset{\sim}{d})}{r\hat{\sigma}^2}$$

e.g. $H_0 : \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = 0 \Rightarrow H_0 : \begin{matrix} \beta_1 = 0 \\ \beta_2 = 0 \end{matrix}$

$$\Rightarrow H_0 : \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = 0$$

$$\text{test stat.} = (\underset{\sim}{L}\hat{\underset{\sim}{\beta}} - \underset{\sim}{d})^T (\underset{\sim}{L}\,\underset{\sim}{C}^{-1}\underset{\sim}{L}^T)^{-1} (\underset{\sim}{L}\hat{\underset{\sim}{\beta}} - \underset{\sim}{d})$$

$$\xrightarrow{D} \chi^2_r$$

r = Rank of $\underset{\sim}{L}$ $\quad$ e.g. $r = 2$

Example 1

for one unit increase in logload

(e) H_0: odds ratio = 1

$\Rightarrow H_0: \exp(\beta_1) = 1$

$\Rightarrow H_0: \beta_1 = 0$

Wald test $= \dfrac{\hat{\beta}_1 - 0}{\sqrt{c^{11}}} = \dfrac{1.14 - 0}{\sqrt{0.007968}} =$

(f) H_0: Prob | logload $= 4$ = ~~0.6~~ ~~0.~~ 0.6

$\Rightarrow H_0: \dfrac{\exp(\beta_0 + \beta_1 * 4)}{1 + \exp(\beta_0 + \beta_1 * 4)} = 0.6$

$\Rightarrow H_0: \beta_0 + \beta_1 * 4 = \log\left(\dfrac{0.6}{1-0.6}\right) = 0.4054$

$H_0: \underbrace{\begin{pmatrix} 0 & 4 \end{pmatrix}}_{L} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = 0.4054$

rank of $L = 1$

Wald test $= \left(\dfrac{(\hat{\beta}_0 + \hat{\beta}_1 * 4) - (0.4054)}{\sqrt{\boxed{Var(\hat{\beta}_0 + 4\hat{\beta}_1)}}}\right)^2 =$

$Var(\hat{\beta}_0) + 16\, Var(\hat{\beta}_1) + 8\, cov(\hat{\beta}_0, \hat{\beta}_1)$

$= 0.002848$

Critical value $= \chi^2_{\alpha, 1}$

Example 2 – two independent variables

Deviance

Deviance and Pearson Goodness-of-Fit Statistics				
Criterion	**Value**	**DF**	**Value/DF**	**Pr > ChiSq**
Deviance	29.7723	35	0.8506	0.7184
Pearson	39.0106	35	1.1146	0.2942

Parameter estimates with confidence interval

Analysis of Maximum Likelihood Estimates					
Parameter	**DF**	**Estimate**	**Standard Error**	**Wald Chi-Square**	**Pr > ChiSq**
Intercept	1	-9.5293	~~3.2331~~	~~8.6873~~	~~0.0032~~
volume	1	3.8820	~~1.4286~~	~~7.3844~~	~~0.0066~~
rate	1	2.6490	0.9142	~~8.3966~~	~~0.0038~~

given

Covariance matrix

$Var(\hat{\beta}_0)$ $Var(\hat{\beta}_1)$ $Var(\hat{\beta}_2)$

Estimated Covariance Matrix			
Parameter	**Intercept**	**volume**	**rate**
Intercept	10.45283	-4.32469	-2.729
volume	-4.32469	2.040791	0.99885
rate	-2.729	0.99885	0.835746

$\log_e\left(\frac{\hat{P}}{1-\hat{P}}\right) = -9.5293 + 3.882 * \text{volume} + 2.649 * \text{rat}$

a) Write down the fitted line.
b) Find the 95% confidence interval of unknown parameters.
c) Estimate the odds ratio for one unit increase in volume with its 95% confidence interval.
d) Estimate the odds ratio for one unit increase in rate with its 95% confidence interval.
e) Estimate the odds ratio for one unit increase in volume & one unit increase in rate with its 95% confidence interval.
f) Estimate the probability of success when volume=1 & rate=1 with its 95% confidence interval.

(b) $\hat{\beta}_0 \pm 1.96 * \cancel{3.2331} \sqrt{10.45283} = (-15.866, -3.1925)$

$\hat{\beta}_1 \pm 1.96 * \sqrt{2.040791} = (1.0821, 6.6819)$

$\hat{\beta}_2 \pm 1.96 * \sqrt{0.835746} = (0.8573, 4.4408)$

c) odds ratio $= \dfrac{\frac{p}{1-p}\Big|_{\text{when volume} = a+1}}{\frac{p}{1-p}\Big|_{\text{when volume} = a}}$

$$= \frac{\exp(\beta_0 + \beta_1 * (a+1) + \beta_2 \text{ rate})}{\exp(\beta_0 + \beta_1 * a + \beta_2 \text{rate})}$$

$$= \exp(\beta_1)$$

$\widehat{\text{odds ratio}} = \exp(\hat{\beta}_1) = \exp(3.8820) = \boxed{48.522} > 1$

95% C.I. of β_1 = (exp(1.0821), exp(6.6819))

= (2.931, 797.865)

~~If x~~ If volume increases, does prob. increase?

odds ratio of case A vs case B > 1

⇒ odds of case A > odds of case B

⇒ prob at case A > prob at case B

d) odds ratio for one unit increase in rate

$$= \frac{\frac{p}{1-p}\Big|_{\text{rate} = a+1}}{\frac{p}{1-p}\Big|_{\text{rate} = a}}$$

$$= \frac{\exp(\beta_0 + \beta_1 * \text{volume} + \beta_2 (a+1))}{\exp(\beta_0 + \beta_1 * \text{volume} + \beta_2 (a))}$$

$$= \exp(\beta_2)$$

$\widehat{\text{odds ratio}} = \exp(\hat{\beta}_2) = \exp(2.649) = \boxed{14.14} > 1$

95% C.I. of $\exp(\beta_2)$ = (exp(0.8573), exp(4.4408))

95% C.I. of β_2

= (2.357, 84.844)

Which one is more important factor to affect the Prob. ?

6

$\widehat{\text{odds ratio}}$ of ~~vol~~ volume = 48.522

$\widehat{\text{odds ratio}}$ of rate = 14.14

	est	s.e.	Wald chi-square	p-value
volume	3.8820	1.4286	$(3.882/1.4286)^2 = 7.3844$	0.0066
rate	2.6490	0.9142	$(2.6490/0.9142)^2 = 8.3966$	0.0038

$] < 0.05$

both are important factor to affect the Prob.

(e) Odds ratio for one unit ~~[illegible]~~ increase in volume & rate

$$\text{Odds ratio} = \frac{\frac{P}{1-P}\Big|_{\text{volume}=a+1,\ \text{rate}=b+1}}{\frac{P}{1-P}\Big|_{\text{volume}=a,\ \text{rate}=b}}$$

$$= \frac{\exp(\beta_0 + \beta_1(a+1) + \beta_2(b+1))}{\exp(\beta_0 + \beta_1(a) + \beta_2(b))}$$

$$= \exp(\beta_1 + \beta_2)$$

odds ratio for one unit increase in rate

$$= \underline{\exp(\beta_1)} * \exp(\beta_2)$$

odds rate for one unit increase in volume

$$\widehat{\text{odds ratio}} = \exp(\hat{\beta}_1 + \hat{\beta}_2) = 686.1$$

95% C.I. for $\exp(\beta_1 + \beta_2) = (\hat{\beta}_1 + \hat{\beta}_2) \pm 1.96\ \underline{\text{s.e. of } (\hat{\beta}_1 + \hat{\beta}_2)}$

$$\sqrt{Var(\hat{\beta}_1 + \hat{\beta}_2)} \quad 0.835746$$

$$= \sqrt{Var(\hat{\beta}_1) + Var(\hat{\beta}_2) + 2\,cov(\hat{\beta}_1, \hat{\beta}_2)}$$

2.040791

0.99885

$= (\exp(2.2039), \exp(10.8582))$

(f) Prob at volume = 1 & rate = 1

$$= \frac{\exp(\beta_0 + \beta_1 + \beta_2)}{1 + \exp(\beta_0 + \beta_1 + \beta_2)}$$

~~PE~~ Estimate $= \dfrac{\exp(\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2)}{1 + \exp(\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2)} = 0.0475$

95% C.I. of $\beta_0 + \beta_1 + \beta_2 = (\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2) \pm 1.96 \sqrt{\boxed{Var(\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2)}}$

$Var(\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2)$

$= Var(\hat{\beta}_0) + \underset{-4.32469}{Var(\hat{\beta}_1)} + \underset{-2.7729}{Var(\hat{\beta}_2)}$

$+ \underset{0.99885}{\underline{2\,Cov(\hat{\beta}_0, \hat{\beta}_1)}} + \underline{2\,Cov(\hat{\beta}_0, \hat{\beta}_2)}$

$+ \underline{2\,Cov(\hat{\beta}_1, \hat{\beta}_2)}$

$= 1.2197$

$\doteq (\ W_e\ ,\ W_u\ )$

95% C.I. of Prob at volume = 1 & rate = 1

$$= \left(\frac{\exp(W_e)}{1 + \exp(W_e)} , \frac{\exp(W_u)}{1 + \exp(W_u)} \right)$$

$H_0 : Prob = 0.4$

$\Rightarrow H_0 : \dfrac{\exp(\beta_0 + \beta_1 + \beta_2)}{1 + \exp(\beta_0 + \beta_1 + \beta_2)} = 0.4$

$H_0 : L\underset{\sim}{\beta} = \underset{\sim}{d}$

$\Rightarrow H_0 : \boxed{(1\ \ 1\ \ 1)} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = -0.4054$

rank = 1

$\Rightarrow H_0 : (\beta_0 + \beta_1 + \beta_2) = \ln\left(\dfrac{0.4}{1-0.4}\right) = \ln\left(\dfrac{2}{3}\right) = -0.4054$

test stat. $= \left(\dfrac{(\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2) - \ln(\frac{2}{3})}{\sqrt{Var(\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2)}} \right)^2 =$

Critical ~~val~~ value $= \chi^2_{\alpha, r}$ $r = 1$

Example in case-control study (grouped data)

Data

categorial variable with two levels 0 & 1

reference level.

Obs	exposure	r	n
1	0	2	8
2	1	11	15

```
/* exposure 1='High Cholesterol Diet'
            0='Low Cholesterol  Diet';
   response 1='Yes'
            0='No' */
```

$H_0: P_i = \frac{\exp(\beta_0+\beta_1 X_i)}{1+\exp(\beta_0+\beta_1 X_i)}$

$i = 0, 1$

Deviance

H $\rightarrow$ P_0 & P_1 # of unknown paramtrs $= 2$

H_0 $\rightarrow$ β_0, β_1 # of unknown paramtrs $= 2$

Why?

Deviance and Pearson Goodness-of-Fit Statistics				
Criterion	Value	DF	Value/DF	Pr > ChiSq
Deviance	0.0000	0	.	.
Pearson	0.0000	0	.	.

$d.f. = s-(p+1) = 0$

2   1 (exposure)

two levels of exposures

OR two observations

Parameter estimates by logistic regression

Analysis of Maximum Likelihood Estimates						
Parameter		DF	Estimate	Standard Error	Wald Chi-Square	Pr > ChiSq
Intercept		1	-1.0986	0.8165	1.8104	~~0.1785~~
exposure	1	1	2.1102	1.0038	4.4195	~~0.0355~~

Covariance matrix

Estimated Covariance Matrix		
Parameter	Intercept	exposure1
Intercept	0.666666	-0.66667
exposure1	-0.66667	1.007576

Can be obtained from the relationship of (P_0, P_1) & (β_0, β_1)

P_0, P_1 ← prob when exposure = 1

prob when exposure = 0

β_0, β_1 in logistic regression model