Math 243 Mid-term Exam.

98–99

1. (a) Same as Exercise 1: Q.9

(b) Same as Exercise 1: Q.22

2. $X_1, \dots, X_n$ are iid exponential (λ)

$f_{X_i}(x_i) = \frac{1}{\lambda} e^{-\frac{x_i}{\lambda}}$, $0 \le x_i < \infty$

(a) Let $Y = \min\{X_1, \dots, X_n\}$

In order to find the dist^n of Y, we should find $P(Y \le y)$ and then $f_Y(y)$.

$P(Y \le y) = P(\min\{X_1, \dots, X_n\} \le y)$

(difficult to finish in that direction since we have many cases for $\min\{X_1, \dots, X_n\} \le y$!)

Thus, we consider $P(Y \le y) = 1 - P(Y > y)$

$= 1 - P(\min\{X_1, \dots, X_n\} > y)$

$= 1 - P(X_1 > y, X_2 > y, \dots, X_n > y)$

$= 1 - \prod_{i=1}^{n} P(X_i > y)$

$P(X_i > y) = \int_y^\infty \frac{1}{\lambda} e^{-\frac{x_i}{\lambda}} dx_i = -e^{-\frac{x_i}{\lambda}} \Big|_y^\infty = e^{-y/\lambda}$

$\therefore P(Y \le y) = 1 - \prod_{i=1}^{n} e^{-y/\lambda} = 1 - e^{-\frac{n\lambda}{y}}$

$\therefore f_Y(y) = \frac{d}{dy} P(Y \le y) = \frac{n}{\lambda} e^{-\frac{n}{\lambda} y}$ which is the p.d.f. of exponential$(\frac{\lambda}{n})$

$\therefore E(Y) = \frac{\lambda}{n}$

$\therefore E(nY) = \lambda$

$\therefore nY$ is an unbiased estimator for λ.

(b) Since λ is the mean of X_i, we guess an unbiased estimator for λ is $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Let us check $\overline{X}$ is unbiased for λ or not.

$E(\overline{X}) = E(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}(n\lambda) = \lambda$

$\therefore \overline{X}$ is an unbiased estimator for λ.

$Var(\overline{X}) = Var(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$

$= \frac{1}{n^2}\sum_{i=1}^{n} \lambda^2 = \frac{1}{n^2}(n\lambda^2) = \frac{\lambda^2}{n}$

Now, we want to find the C-R lower bound for estimating λ.

2. (b) (cont.)

$$\log f_{X_i}(x_i;\lambda) = \log \frac{1}{\lambda} e^{-\frac{x_i}{\lambda}} = -\log\lambda - \frac{x_i}{\lambda}$$

$$\frac{\partial}{\partial\lambda}\log f_{X_i}(x_i;\lambda) = -\frac{1}{\lambda} + \frac{x_i}{\lambda^2}$$

$$\frac{\partial^2}{\partial\lambda^2}\log f_{X_i}(x_i;\lambda) = \frac{1}{\lambda^2} - 2\frac{x_i}{\lambda^3}$$

$$E\left[\frac{\partial^2}{\partial\lambda^2}\log f_{X_i}(x_i;\lambda)\right] = E\left(\frac{1}{\lambda^2} - \frac{2X_i}{\lambda^3}\right)$$

$$= \frac{1}{\lambda^2} - \frac{2}{\lambda^3}E(X_i) = \frac{1}{\lambda^2} - \frac{2}{\lambda^3}(\lambda) = -\frac{1}{\lambda^2}$$

$\therefore$ the C-R lower bound for λ is

$$-\frac{1}{nE[\frac{\partial^2}{\partial\lambda^2}\log f_{X_i}(x_i;\lambda)]} = \frac{\lambda^2}{n}$$

$\therefore$ $Var(\bar{X})$ = the C-R lower bound for λ.

$\therefore$ $\bar{X}$ is the minimum variance estimator.

$\therefore$ $\bar{X}$ is more efficient than nY and we consider $\bar{X}$ is a better estimator.

Note: We may also finish it by comparing the $Var(\bar{X})$ and $Var(nY$

3. The random variable $Y_1, \dots, Y_n$ satisfy

$$Y_i = \beta x_i + \varepsilon_i \qquad i = 1, 2, \dots, n$$

where $x_1, \dots, x_n$ are fixed constants, and $\varepsilon_1, \dots, \varepsilon_n$ are iid $N(0, \sigma^2)$, σ^2 unknown.

(a) $\varepsilon_i = Y_i - \beta x_i$

likelihood function:

$$L = f_{\boldsymbol{\varepsilon}}(\underset{\sim}{\varepsilon};\beta,\sigma^2) = \prod_{i=1}^{n} f_{\varepsilon_i}(\varepsilon_i;\beta,\sigma^2) = \prod_{i=1}^{n} f_{\varepsilon_i}(y_i - \beta x_i)$$

$$= \prod_{i=1}^{n}\left[\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)\exp\left\{-\frac{1}{2\sigma^2}(y_i - \beta x_i - 0)^2\right\}\right.$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta x_i)^2\right\}$$

$$\log L = \frac{-n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta x_i)^2$$

$$\frac{\partial}{\partial\beta}\log L = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}2(y_i - \beta x_i)(-x_i)$$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta x_i)x_i$$

$$\frac{\partial}{\partial\beta}\log L = 0$$

$$\Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i - \beta x_i)x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n}x_i y_i - \beta\sum_{i=1}^{n}x_i^2 = 0$$

$\therefore$ max. likelihood estimator of β: $\hat{\beta} = \dfrac{\sum_{i=1}^{n}x_i Y_i}{\sum_{i=1}^{n}x_i^2}$

3 (a) "Alternative"

$Y_i = \beta x_i + \varepsilon_i \quad , \quad i = 1, \dots, n$

Observe that $Y_i \sim N(\beta x_i, \sigma^2)$ and Y_i's are independent.

$\therefore$ the likelihood function is

$$L = f_{Y_1,\dots,Y_n}(y_1,\dots,y_n) = \prod_{i=1}^{n} f_{Y_i}(y_i; \beta, \sigma^2)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right) \exp\left\{-\frac{1}{2\sigma^2}(y_i - \beta x_i)^2\right\}$$ (same likelihood function as before

Now, $$E(\hat{\beta}) = E\left(\frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2}\right)$$

$$= \frac{1}{\sum_{i=1}^{n} x_i^2} E\left(\sum_{i=1}^{n} x_i Y_i\right) = \frac{1}{\sum_{i=1}^{n} x_i^2} \sum_{i=1}^{n} x_i E(Y_i)$$

Now, $E(Y_i) = E(\beta x_i + \varepsilon_i) = \beta x_i + 0 = \beta x_i$

$$\therefore E(\hat{\beta}) = \frac{1}{\sum_{i=1}^{n} x_i^2} \sum_{i=1}^{n} x_i (\beta x_i) = \frac{\beta \sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^2} = \beta$$

$\therefore \hat{\beta}$ is unbiased for β.

(b) $$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i Y_i}{\sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n} x_i(\beta x_i + \varepsilon_i)}{\sum_{i=1}^{n} x_i^2} = \frac{\sum_{i=1}^{n}(\beta x_i^2 + x_i \varepsilon_i)}{\sum_{i=1}^{n} x_i^2} = \frac{\beta \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i \varepsilon_i}{\sum_{i=1}^{n} x_i^2}$$

$$= \beta + \frac{1}{\sum_{i=1}^{n} x_i^2} \sum_{i=1}^{n} x_i \varepsilon_i$$

Since $\varepsilon_i \overset{iid}{\sim} N(0, \sigma^2)$, then $\sum_{i=1}^{n} x_i \varepsilon_i \sim N\left(0, \sum_{i=1}^{n} x_i^2 \sigma^2\right)$

$$\therefore \hat{\beta} \sim N\left(\beta, \left(\frac{1}{\sum_{i=1}^{n} x_i^2}\right)^2 \sum_{i=1}^{n} x_i^2 \sigma^2\right) \sim N\left(\beta, \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}\right)$$

(c) $$E\left(\frac{\sum Y_i}{\sum x_i}\right) = \frac{1}{\sum_{i=1}^{n} x_i} \sum_{i=1}^{n} E(Y_i)$$

$$= \frac{1}{\sum_{i=1}^{n} x_i} \sum_{i=1}^{n} \beta x_i$$ (by part (a))

$$= \beta$$

$\therefore \frac{\sum Y_i}{\sum x_i}$ is an unbiased estimator for β.

(d) $$Var\left(\frac{\sum Y_i}{\sum x_i}\right) = \left(\frac{1}{\sum_{i=1}^{n} x_i}\right)^2 \sum_{i=1}^{n} Var(Y_i)$$

$Var(Y_i) = Var(\beta x_i + \varepsilon_i) = Var(\varepsilon_i) = \sigma^2$

$$\therefore Var\left(\frac{\sum Y_i}{\sum x_i}\right) = \left(\frac{1}{\sum_{i=1}^{n} x_i}\right)^2 n\sigma^2 = \frac{n\sigma^2}{\left(\sum_{i=1}^{n} x_i\right)^2}$$

$$\therefore Var(\hat{\beta}) = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$$ (by part (b), dist$^{\underline{n}}$ of $\hat{\beta}$)

$$\leq \frac{n\sigma^2}{\left(\sum_{i=1}^{n} x_i\right)^2}$$ (since $\sum_{i=1}^{n}(x_i - \bar{x})^2 \geq 0 \Rightarrow \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \geq 0 \Rightarrow \sum_{i=1}^{n} x_i^2 \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}$)