The Hong Kong University of Science & Technology

MATH243 – Statistical Inference

Final Examination – Fall 06/07

Answer ALL Questions | **Date: 13 December 2006 (Wed)**
Time allowed: 3 hours

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1. (a) **(2 marks)** If $X_1, \cdots, X_n$ are independent exp (λ) random variables, i.e.,

$$f(x_i) = \lambda e^{-\lambda x_i} \qquad x_i > 0$$

What is the distribution of $T = 2\lambda \sum_{i=1}^{n} X_i$?

(b) **(2 marks)** If $X_1, \cdots, X_m$ and Y_1 , …, Y_n are independent exp (λ) random variables, what is the distribution of $S = \dfrac{n \sum_{i=1}^{m} X_i}{m \sum_{j=1}^{n} Y_j}$?

(c) Let (*X*, *Y*) have a $N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ distribution.

(i) **(2 marks)** Let (*X*, *Y*) ~ $N(1, 1, 4, 1, \frac{1}{2})$. Find $\Pr(X + 2Y \le 4)$.

(ii) **(2 marks)** Show that *X* + *Y* and *X* – *Y* are independent if and only if $\sigma_1^2 = \sigma_2^2$.

(d) **(4 marks)** Calculate the mean squared error of the estimate $\tilde{\sigma}^2 = \dfrac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n+1}$ for σ^2, where $X_1, \cdots, X_n$ is a sample from $N(\mu, \sigma^2)$.

2. Consider on systems with failure times $X_1, \cdots, X_n$ assumed to be independent and identically distribution, exp (λ), distributions, i.e., $f(x_i) = \lambda e^{-\lambda x_i}$, $x_i > 0$.

(a) **(1 mark)** Find a method of moments estimate of λ.

(b) **(1 mark)** Find the maximum likelihood estimate of λ.

(c) **(2 marks)** Is the estimate in (b) UMVUE? Explain.

(d) **(1 mark)** Find the maximum likelihood estimate of the probability $\Pr(X_1 \ge 1)$ that one system with last at least a month.

(e) **(3 marks)** Find the UMVUE of the probability $\Pr(X_1 \ge 1)$.

(f) **(3 marks)** Find the Cramér-Rao Lower bound for the probability $\Pr(X_1 \ge 1)$.

(g) **(1 mark)** Is the Cramér-Rao Lower bound attainable? Explain. No need to find the variance of UMVUE in (e).

3. (a) Let $X_1,...,X_n$ be a sample from Uniform (θ_1, θ_2) where θ_1 and θ_2 are unknown.

   (i) **(1 mark)** Find the MLE of θ_1 and θ_2.

   (ii) **(1 mark)** Find $E(X_{\max})$ and $E(X_{\min})$.

   (iii) **(4 marks)** Hence or otherwise, find the UMVUE of $\dfrac{\theta_1+\theta_2}{2}$.

   (b) Consider the problem of the choice of estimator of σ^2 based on a random sample of size n from a $N(\mu,\sigma^2)$ distribution. Define $V=\sum_{i=1}^{n}(X_i-\bar{X})^2 \Big/ (n-1)$.

   (i) **(2 marks)** What is the distribution of $(n-1)V/\sigma^2$? Hence, find $E(V)$ and $Var(V)$.

   (ii) **(1 mark)** Find the UMVUE of σ^2.

   (iii) **(3 marks)** Find the UMVUE of $\dfrac{1}{\sigma^2}$ and its variance.

4. (a) Suppose that $X_1,...,X_n$ are independently and identically distributed according to the uniform distribution $U(0,\theta)$. Let the critical function as

$$\delta_c\left(\underset{\sim}{x}\right)=1 \quad \text{if} \quad X_{\max}\ge c$$
$$=0 \quad \text{otherwise}$$

where we reject if $\delta_c\left(\underset{\sim}{x}\right)=1$ and we accept if $\delta_c\left(\underset{\sim}{x}\right)=0$.

   (i) **(2 marks)** Compute the power function of δ_c and show that it is a monotone increasing function of θ.

   (ii) **(3 marks)** Find the value of c in terms of n for testing $H_0:\theta=\frac{1}{2}$ versus $H_1:\theta=\frac{3}{4}$ at α = 0.05. Is it the UMP test for testing $H_0:\theta\le\frac{1}{2}$ versus $H_1:\theta>\frac{1}{2}$ at α = 0.05? Explain your answer in details.

   (iii) **(2 marks)** How large should n be so that δ_c specified in (ii) has power 0.98 for $\theta=\frac{3}{4}$?

   (iv) **(2 marks)** Calculate the p-value if $X_{\max}$= 0.48 in a sample of size n = 20 for testing $H_0:\theta=\frac{1}{2}$ versus $H_1:\theta>\frac{1}{2}$. Is the null hypothesis rejected at α = 0.05?

   (b) **(3 marks)** In 1000 tosses of a coin, 560 heads and 440 tails appear. Construct Pearson goodness-of-fit test to check whether the coin is fair.

5. Let $Y_{11}, Y_{12} \ldots, Y_{1n}$ and $Y_{21}, Y_{22} \ldots, Y_{2n}$ and $Y_{31}, Y_{32} \ldots, Y_{3n}$ be random samples from the independent normal distribution $N(\mu_1, \sigma^2)$, $N(\mu_2, \sigma^2)$ and $N(\mu_3, \sigma^2)$ respectively. Consider the problem of testing $H_0 : \mu_1 = \mu_2 = \mu_0$ versus $H_1 : \mu_1 \neq \mu_2$.

(a) (i) **(2 marks)** Find the likelihood ratio

Hint: (i) Find the m.l.e. of μ_0, μ_3 and σ^2 under Θ_0.

(ii) Find the m.l.e. of μ_1, μ_2, μ_3 and σ^2 under Θ.

(ii) **(2 marks)** Derive the approximate large sample likelihood ratio test.

(iii) **(4 marks)** Construct the exact likelihood ratio test.

(b) Three laboratories are being used to perform chemical analyses.

Laboratory		
A	B	C
62.7	60.7	55.9
64.5	60.3	56.1
63.1	60.9	57.3
59.2	61.4	53.2
60.3	62.3	58.1
Total 309.8	305.6	280.6

(i) **(2 marks)** Use the test obtained in (a) (ii) to test $H_0 : \mu_A = \mu_B$ versus $H_1 : \mu_A \neq \mu_B$ at $\alpha = 0.05$.

(ii) **(2 marks)** Find a 90% confidence interval for σ.

(iii) **(Bonus: 2 marks)** Find the estimate of $\dfrac{\mu_A + \mu_B}{2} - \mu_C$ and then its standard error. Hence or otherwise, construct a 95% confidence interval.

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