MATH 243 Statistical Inference

Midterm Examination - Fall 2000/2001

Answer <u>ALL</u> Questions — **Date: 3 November 2000**
All Equal Marks — **Time Allowed: 2 hours**

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1. Let X_1, X_2, $\ldots$, X_n be a random sample from a Bernoulli distribution with parameter θ.

   (a) i. Show that the method of moments estimator of θ and the maximum likelihood estimator of θ are identical.

   ii. Verify that the estimator found in (a)(i), denoted T_1, is unbiased, and find its variance.

   iii. Find the Cramer Rao lower bound of θ. Does the variance of T_1 achieve the lower bound?

   (b) Suppose that an estimator of the function $\tau = \tau(\theta)$ defined by $\tau(\theta) = \theta^n$ is sought.

   i. Find the probability mass function of the discrete random variable T_2 defined by

   $$T_2 \;=\; \text{Min}(X_1, \ldots, X_n)$$

   and hence show that T_2 is an unbiased estimator of τ.

   ii. Find the variance of T_2.

   iii. Find the maximum likelihood estimator of τ.

2. Let X_1, $\ldots$, X_n be a random sample from the normal density with mean θ and variance $\sigma^2 = 1$. Let T be an estimator of θ.

   (a) Find the maximum likelihood estimator of θ, $\hat{\theta}$.

   (b) Find the mean-square-error of

   $$T_1 \;=\; c\hat{\theta}, \qquad c > 0,$$

   where T_1 is also an estimator of θ. Then, find the values of c, in terms of θ, for which

   $$\text{MSE}(T_1) \;<\; \text{MSE}(\hat{\theta})\,.$$

   What happens to these values of c as $n \to \infty$?
   Hint: $\text{MSE}(T) \;=\; \text{Var}(T) \;+\; [\text{bias}(T)]^2\,.$

3. (a) Let Z_1, Z_2 be a random sample of size 2 from $N(0,1)$ and X_1, X_2 a random sample of size 2 from $N(1,1)$. Suppose the Z_i's are independent of the X_j's. Answer the following:

   i. What is the distribution of $\bar{X} + \bar{Z}$?

   ii. What is the distribution of $[(X_2 - X_1)^2 + (Z_2 - Z_1)^2]/2$?

   iii. What is the distribution of $(Z_1 + Z_2)/\sqrt{[(X_2 - X_1)^2 + (Z_2 - Z_1)^2]/2}$?
   Remark: $(Z_1 + Z_2)$ and $[(X_2 - X_1)^2 + (Z_2 - Z_1)^2]$ are independent.

   iv. What is the distribution of $(X_2 + X_1 - 2)^2/(X_2 - X_1)^2$?
   Remark: $(X_2 + X_1 - 2)$ and $(X_2 - X_1)$ are independent.

   (b) i. What is the probability that the larger of two random observations from any continuous distribution will exceed the median?

   ii. Generalize the above result to samples of size n.