Classwork 1: solution

1. Let $U_i,\ i = 1, 2, \ldots, n$ are i.i.d sample from Uniform[0,1], and $X_n = (\prod_{i=1}^{n} U_i)^{-\frac{1}{n}}$, show that

$$\sqrt{n}(X_n - e) \to N(0, e^2).$$

Solutions:
First, X_n can be written as $X_n = exp(-\frac{1}{n}\sum_{i=1}^{n} log(U_i))$, let $Y = -log(U)$, it is not difficulty to find the p.d.f of Y is $f_Y(y) = exp(-y),\quad y \geq 0$, which is the density of standard exponential distribution, i.e., $Y \sim exp(1)$. So the expectation and variance of Y is $E(Y) = Var(Y) = 1$. From the independence of U_i, we know Y_i is also independent.
Denote $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, according to CLT, we can get $\sqrt{n}(\bar{Y} - 1) \sim N(0, 1)$. Actually,

$$X_n = exp(-\frac{1}{n}\sum_{i=1}^{n} log(U_i)) = exp(\bar{Y}) = g(\bar{Y}),$$

where $g(y) = exp(y)$. Note that $exp(1)$ exists and is **NOT** 0, thus by delta method, we have

$$\sqrt{n}(g(\bar{Y}) - g(1)) \to N(0, [g'(1)1]^2),$$

that is

$$\sqrt{n}(X_n - e) \to N(0, e^2).$$

2. Let $X_1 \sim N(4, 2), X_2 \sim N(-2, 3)$ and $Cov(X_1, X_2) = -1$. Set $Y_1 = 2X_1 - X_2$ and $Y_2 = X_1 + X2$.
(a) Does the random vector of $(X_1, X_2)^T$ have a bivariate normal distribution? No mark will be given if the answer of "Yes" or "No" is given.
(b) Are Y_1 and Y_2 independent? If yes, explain; if no, find the joint p.d.f.
(c) Find the probability density functions of Y_1 and Y_2 respectively.

Solutions:
(a) Yes, the expectation of $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ is $\mu = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$, and the variance-covariance matrix is $\Sigma = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$.
Let $L = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, then $LL^T = \begin{pmatrix} a^2 & ab \\ ab & b^2 + c^2 \end{pmatrix}$. We may take $a = \sqrt{2}, b = -\frac{1}{\sqrt{2}}$ and $c = \sqrt{\frac{5}{2}}$, then $LL^T = \Sigma$.
Therefore, there exist $p = 2$ independent standard normal variables, i.e., $Z_1, Z_2 \sim N(0, 1)$, a p-vector of $\mu = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and a $p \times p$ matrix $L = \begin{pmatrix} \sqrt{2} & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{\frac{5}{2}} \end{pmatrix} = Chol(\Sigma)$ such that $\mathbf{X} = \mu + Chol(\Sigma)\mathbf{Z}$, so the random vector $(X_1, X_2)^T$ has a bivariate normal distribution.
(b) Y_1 and Y_2 are independent. $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ is a linear function of $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, so Y also follows bivariate normal distribution and

$$Cov(Y_1, Y_2) = Cov(2X_1 - X_2, X_1 + X_2) = 2Var(X_1) - var(X_2) + Cov(X_1, X_2) = 0.$$

Thus, Y_1 and Y_2 are independent.
(c)Since

$$E(Y_1) = 2E(X_1) - E(X_2) = 10,$$
$$E(Y_2) = E(X_1) + E(X_2) = 2,$$
$$Var(Y_1) = 4Var(X_1) + Var(X_2) - 4Cov(X_1, X_2) = 15,$$
$$Var(Y_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2) = 3,$$

therefore, $(Y_1, Y_2)^T \sim N_2(\mu, \Sigma)$, with $\mu = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$, and the variance-covariance matrix is $\Sigma = \begin{pmatrix} 15 & 0 \\ 0 & 3 \end{pmatrix}$.
Thus, we have $Y_1 \sim N(10, 15)$ and $Y_2 \sim N(2, 3)$.