Question 1

part 1

Note that : $\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$

$\Rightarrow E(\frac{nS_n^2}{\sigma^2}) = n-1$; $E(\frac{(n-1)S_{n-1}^2}{\sigma^2}) = n-1$

$Var(\frac{nS_n^2}{\sigma^2}) = 2(n-1)$; $E(\frac{(n-1)S_{n-1}^2}{\sigma^2}) = 2(n-1)$

$\Rightarrow E(S_n^2) = \frac{(n-1)\sigma^2}{n}$; $E(S_{n-1}^2) = \sigma^2$

$Var(S_n^2) = \frac{2(n-1)\sigma^4}{n^2}$; $Var(S_{n-1}^2) = \frac{2\sigma^4}{n-1}$

$$
\begin{aligned}
MSE(S_n^2) &= Var(S_n^2) - (E(S_n^2) - \sigma^2)^2 \\
&= \frac{2(n-1)\sigma^4}{n^2} - \left(\frac{(n-1)\sigma^2}{n} - \sigma^2\right)^2 \\
&= \frac{2(n-1)\sigma^4}{n^2} - \left(\frac{\sigma^2}{n}\right)^2 \\
&= \frac{2n\sigma^4 - 2\sigma^4 - \sigma^4}{n^2} \\
&= \frac{2n\sigma^4 - \sigma^4}{n^2} = \frac{(2n-1)\sigma^4}{n^2} \quad //
\end{aligned}
$$

$$
\begin{aligned}
MSE(S_{n-1}^2) &= Var(S_{n-1}^2) - (E(S_{n-1}^2) - \sigma^2)^2 \\
&= \frac{2\sigma^4}{n-1} + 0 \\
&= \frac{2\sigma^4}{n-1} \quad //
\end{aligned}
$$

$MSE(S^2) = \frac{2(n-1)\sigma^4}{n^2} < \frac{2n}{n^2}\sigma^4 = \frac{2}{n}\sigma^4 < \frac{2\sigma^4}{n-1} = MSE(S_{n-1}^2)$

$\therefore$ S^2 has smaller MSE than S_{n-1}^2

part2

$$k\sum_{i=1}^{n}(X_i-\bar{X})^2 = k(n-1)S_{n-1}^2$$

$$E(k(n-1)S_{n-1}^2) = k(n-1)\sigma^2$$

$$Var(k(n-1)S_{n-1}^2) = \frac{2k^2(n-1)^2\sigma^4}{n-1}$$ //

$$\begin{aligned} MSE(k(n-1)S_{n-1}^2) &= Var[k(n-1)S_{n-1}^2] + [E(k(n-1)S_{n-1}^2)-\sigma^2]^2 \\ &= \frac{2k^2(n-1)^2\sigma^4}{n-1} + [(k(n-1)-1)\sigma^2]^2 \\ &= \frac{2k^2(n-1)^2\sigma^4}{n-1} + (k(n-1)-1)^2\sigma^4 \\ &= \left[\frac{2(k(n-1))^2}{n-1} + (k(n-1))^2 - 2k(n-1)+1\right]\sigma^4 \\ &= \left[\frac{n+1}{n-1}(k(n-1))^2 - 2k(n-1)+1\right]\sigma^4 \end{aligned}$$ //

Let $f[k(n-1)] = \frac{n+1}{n-1}(k(n-1))^2 - 2k(n-1)+1$

when $k(n-1) = \frac{n-1}{n+1} \Rightarrow f[k(n-1)]$ achieve its minimal value

$$\begin{aligned} MSE\left(\frac{n-1}{n+1}S_{n-1}^2\right) &= \left[\frac{n+1}{n-1}\left(\frac{n-1}{n+1}\right)^2 - 2\left(\frac{n-1}{n+1}\right)+1\right]\sigma^4 \\ &= \left[\frac{n-1}{n+1} - 2\left(\frac{n-1}{n+1}\right) + \frac{n+1}{n+1}\right]\sigma^4 \\ &= \frac{-n+1+n+1}{n+1}\sigma^4 \\ &= \frac{2}{n+1}\sigma^4 \leqslant \frac{2n-1}{n^2}\sigma^4 = MSE(S^2) \end{aligned}$$

$\Rightarrow$ when $k(n-1) = \frac{n-1}{n+1} \Leftrightarrow k = \frac{1}{n+1}$

$k\sum_{i=1}^{n}(X_i-\bar{X})^2$ has an minimal MSE.

Question 2

$\boxed{Q2}$

- Since $X \sim U[0;\theta]$ (X and X_i are i.i.d $\forall i=1,..,n$),
then from the solution of Question 3: $\mathbb{E}X = \frac{0+\theta}{2} = \frac{\theta}{2}$.
Since $\theta = 2\mathbb{E}X$, then MME of θ is $2\overline{X}$.
We also proved in Question 3 that $L(\theta) = \begin{cases} \left(\frac{1}{\theta}\right)^n : 0 \le x_i \le \theta \quad \forall i=1,..,n \\ 0 \text{ : otherwise} \end{cases}$

and $L(\theta)$ attains its maximum when θ is minimum.
The MLE of θ is a value of θ for which $0 \le x_i \le \theta \quad \forall i=1,..,n$
(since $\left(\frac{1}{\theta}\right)^n > 0$). Then MLE is $\hat{\theta} = \max(x_1,..,x_n) = X_{(n)}$.

- The MSE of MLE for θ is:

$$\mathbb{E}\left[\left(X_{(n)} - \theta\right)^2\right] = \int_{-\infty}^{+\infty} (x-\theta)^2 f_{X_{(n)}}(x)\,dx.$$

By Theorem 7 (in lecture notes): $f_{X_{(n)}}(x) = \frac{n!}{(n-1)!\,(n-n)!} f_X(x) \cdot [F_X(x)]^{n-1} \cdot 1$

$$= n \cdot \frac{1}{\theta} \cdot \left(\frac{x}{\theta}\right)^{n-1} \quad \text{if } x \in [0;\theta]$$

Otherwise if $x \notin [0;\theta] \Rightarrow f_{X_{(n)}}(x) = 0$.

Then $\mathbb{E}\left(\left[X_{(n)} - \theta\right)^2\right] = \int_0^\theta (x-\theta)^2 n \cdot \frac{x^{n-1}}{\theta^n}\,dx$

$$= \frac{n}{\theta^n} \int_0^\theta (x-\theta)^2 x^{n-1}\,dx$$

$$= \frac{1}{\theta^n} \int_0^\theta (x-\theta)^2\, d(x^n) \qquad \Bigg\| \text{ Let } u = (x-\theta)^2, \; v = x^n$$

integration by parts $\rightarrow$ $$= \frac{1}{\theta^n} \cdot \left(\left[(x-\theta)^2 x^n\right]_0^\theta - \int_0^\theta x^n\, d(x-\theta)^2 \right)$$

$$= \frac{1}{\theta^n} \left(- \int_0^\theta x^n \cdot 2(x-\theta)\,dx \right)$$

$$= -\frac{2}{\theta^n} \left(\left[\frac{x^{n+2}}{n+2}\right]_0^\theta - \left[\frac{x^{n+1}}{n+1}\theta\right]_0^\theta \right)$$

$$= -\frac{2}{\theta^n} \left(\frac{\theta^{n+2}}{n+2} - \frac{\theta^{n+2}}{n+1} \right)$$

$$= 2\left(\frac{\theta^2}{n+1} - \frac{\theta^2}{n+2} \right) = \frac{2\theta^2}{(n+1)(n+2)}$$

• The MSE of MME for θ is:

$$\mathbb{E}\left[(2\overline{X}-\theta)^2\right] = \cancel{\int_{-\infty}^{+\infty}} \mathbb{E}\left[4\overline{X}^2 - 4\overline{X}\theta + \theta^2\right]$$

$$= 4\mathbb{E}(\overline{X}^2) - 4\theta\mathbb{E}(\overline{X}) + \theta^2$$

$$= 4\left(\operatorname{Var}\overline{X} + (\mathbb{E}(\overline{X}))^2\right) - 4\theta\,\mathbb{E}(\overline{X}) + \theta^2$$

$$= 4\left(\frac{\operatorname{Var} X_n}{n} + (\mathbb{E}(X_n))^2\right) - 4\theta\,\mathbb{E}(X_n) + \theta^2$$

We derived earlier that $\mathbb{E}X_n = \frac{0+\theta}{2} = \frac{\theta}{2}$ $\left(\mathbb{E}X_i = \frac{\theta}{2}\ \forall i=1,..,n\right)$

We also proved in Question 3 that $\operatorname{Var} X = \frac{(b-a)^2}{12}$ if $X \sim U[a;b]$.

Since $X_n \sim U[0;\theta]$, then $\operatorname{Var} X_n = \frac{\theta^2}{12}$.

Therefore, $\mathbb{E}\left[(2\overline{X}-\theta)^2\right] = 4\left(\frac{\theta^2}{12n} + \frac{\theta^2}{4}\right) - 4\theta\cdot\frac{\theta}{2} + \theta^2$

$$= \frac{\theta^2}{3n} + \theta^2 - 2\theta^2 + \theta^2$$

$$= \frac{\theta^2}{3n}$$

• Now compare $\frac{2\theta^2}{(n+1)(n+2)}$ and $\frac{\theta^2}{3n}$.

$$\frac{\theta^2}{3n} - \frac{2\theta^2}{(n+1)(n+2)} = \theta^2\cdot\frac{(n+1)(n+2) - 2\cdot 3n}{3n(n+1)(n+2)} = \theta^2\cdot\frac{n^2-3n+2}{3n(n+1)(n+2)}$$

$$= \theta^2\cdot\frac{(n-1)(n-2)}{3n(n+1)(n+2)} > 0 \quad (\text{since } \theta>0,\ n>2)$$

Then $\frac{\theta^2}{3n} > \frac{2\theta^2}{(n+1)(n+2)}$, i.e. MSE ~~for~~ of MME is larger than that of MLE.

Therefore, in the sense of MSE, MLE is better than MME.

Question 3

(a)

$\boxed{Q3}$

(a). Let $a = \mu - \sqrt{3}\delta$, $b = \mu + \sqrt{3}\delta$.

Since $X \sim U[a, b]$ (X and X_i are i.i.d $\forall i = 1, .., n$), then $f_X(x) = \begin{cases} \frac{1}{b-a} : a \le x \le b \\ 0 : \text{otherwise} \end{cases}$

$$\text{Then } \mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{a} 0x\,dx + \int_a^b \frac{x}{b-a}\,dx + \int_b^{+\infty} 0x\,dx$$

$$= \left[\frac{x^2}{2(b-a)}\right]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

$$\text{which means } \mathbb{E}X = \frac{(\mu - \sqrt{3}\delta) + (\mu + \sqrt{3}\delta)}{2} = \mu.$$

$$\bullet\ \mathbb{E}X^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_{-\infty}^{a} 0x^2\,dx + \int_a^b \frac{x^2}{b-a}\,dx + \int_b^{+\infty} 0x\,dx$$

$$= \left[\frac{x^3}{3(b-a)}\right]_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + ab + b^2}{3}$$

$$\text{Var}\,X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{{}^{4/}a^2 + ab + b^2}{3} - \frac{{}^{3/}(b+a)^2}{4} = \frac{a^2 + b^2 - 2ab}{12} = \frac{(b-a)^2}{12}$$

$$\text{Which means } \text{Var}\,X = \frac{((\mu + \sqrt{3}\delta) - (\mu - \sqrt{3}\delta))^2}{12} = \frac{(2\sqrt{3}\delta)^2}{12} = \frac{12\delta^2}{12} = \delta^2$$

(b)

version 1

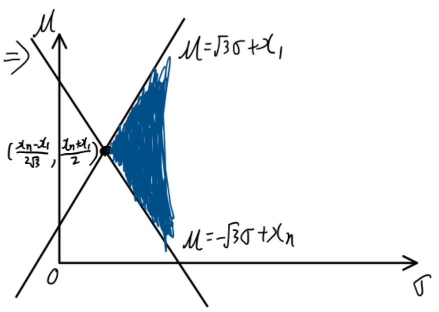
$$(b)\quad f(x \mid \mu, \sigma) = \frac{1}{2\sqrt{3}\sigma} I_{\{\mu - \sqrt{3}\sigma \le x \le \mu + \sqrt{3}\sigma\}}$$

$$L(\mu, \sigma) = \prod_{i=1}^{n} f(x_i \mid \mu, \sigma) I_{\{\mu - \sqrt{3}\sigma \le x_1, \dots x_n \le \mu + \sqrt{3}\sigma\}}$$

$$= \prod_{i=1}^{n} \frac{1}{2\sqrt{3}\sigma} I_{\{\mu - \sqrt{3}\sigma \le x_1 \le x_n \le \mu + \sqrt{3}\sigma\}}$$

$$= \left(\frac{1}{2\sqrt{3}\sigma}\right)^n I_{\{\mu - \sqrt{3}\sigma \le x_1\}} I_{\{\mu + \sqrt{3}\sigma \ge x_n\}}$$

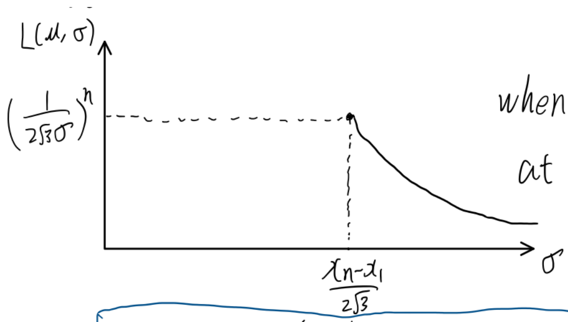
$$\left.\begin{array}{l} \mu \le \sqrt{3}\sigma + x_1 \\ \mu \ge -\sqrt{3}\sigma + x_n \end{array}\right\} \Rightarrow$$



The blue zone shown above is $I_{blue} = I_{\{\mu - \sqrt{3}\sigma \le x_1\}} I_{\{\mu + \sqrt{3}\sigma \ge x_n\}}$ $(\sigma \in (0, +\infty))$

$$\ell(\mu, \sigma) = \log L(\mu, \sigma) I_{blue} = -n(\log 2\sqrt{3} + \log \sigma) I_{blue}$$

$$\frac{\partial \ell(\mu, \sigma)}{\partial \sigma} = -\frac{n}{\sigma} I_{\{\sigma \ge \frac{x_n - x_1}{2\sqrt{3}}\}} < 0$$



when $\sigma = \frac{x_n - x_1}{2\sqrt{3}}$, $L(\mu,\sigma)$ get its maximum, at the same time, $\mu = \frac{x_n + x_1}{2}$

$$\therefore \boxed{\begin{aligned} &MLE(\mu) = \frac{x_n + x_1}{2} \\ &MLE(\sigma) = \frac{x_n - x_1}{2\sqrt{3}} \qquad (x_1 \le x_2 \le \dots \le x_n) \end{aligned}}$$

version 2

(b) $L = \prod_{i=1}^{n} \frac{1}{2\sqrt{3}\sigma} = \left(\frac{1}{2\sqrt{3}\sigma}\right)^n \implies \log L = \log\left(\frac{1}{2\sqrt{3}\sigma}\right)^n = -n\log(2\sqrt{3}\sigma)$

To maximize $\log L$, $\hat{\sigma}$ should be as small as possible

we have $\begin{cases} \mu - \sqrt{3}\sigma \le \min\{X_1, \dots, X_n\} \\ \mu + \sqrt{3}\sigma \ge \max\{X_1, \dots, X_n\} \end{cases}$

then $\sqrt{3}\sigma \ge \mu - \min\{X_1, \dots, X_n\}$ and $\sqrt{3}\sigma \ge \max\{X_1, \dots, X_n\} - \mu$

To minimize $\hat{\sigma}$, $\hat{\mu} = \frac{\min\{X_1, \dots, X_n\} + \max\{X_1, \dots, X_n\}}{2}$

$\sqrt{3}\hat{\sigma} = \max\{X_1, \dots, X_n\} - \hat{\mu}$

$$\begin{cases} \hat{\mu} = \frac{\min\{X_1, \dots, X_n\} + \max\{X_1, \dots, X_n\}}{2} \\ \hat{\sigma} = \frac{\max\{X_1, \dots, X_n\} - \min\{X_1, \dots, X_n\}}{2\sqrt{3}} \end{cases}$$

With $\{X_1 \dots X_n\}$ as order statistics,

$$\begin{cases} \hat{\mu} = \frac{X_1 + X_n}{2} \\ \hat{\sigma} = \frac{X_n - X_1}{2\sqrt{3}} \end{cases} \text{ as MLEs}$$

version 3

3、 $f(x) = \begin{cases} 0, \text{ if } x \notin [\mu-\sqrt{3}\sigma, \mu+\sqrt{3}\sigma] \\ \frac{1}{2\sqrt{3}\sigma}, \text{ if } x \in [\mu-\sqrt{3}\sigma, \mu+\sqrt{3}\sigma] \end{cases}$

For MME, we have:

$\hat{\mu}_1 = \frac{1}{n}(X_1+X_2+\cdots+X_n) = \bar{X} = \mu$ --- ①

$\hat{\mu}_2 = \frac{1}{n}(X_1^2+X_2^2+\cdots+X_n^2) = \frac{1}{3}[(\mu-\sqrt{3}\sigma)^2+(\mu+\sqrt{3}\sigma)^2+(\mu-\sqrt{3}\sigma)(\mu+\sqrt{3}\sigma)] = \mu^2+\sigma^2$ --- ②

$② - ①^2 = \frac{1}{n}(X_1^2+\cdots+X_n^2) + [\frac{1}{n}(X_1+\cdots+X_n)]^2 = \frac{1}{n}[(X_1-\bar{X})^2+\cdots+(X_n-\bar{X})^2]$

$= S_n^2 = \sigma^2 \Rightarrow \sigma = S_n$, where $S_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2}$

So we get: $\mu(MME) = \bar{X}$ and $\sigma(MME) = S_n$

For MLE, we have: $L(\mu,\sigma) = \prod_{i=1}^{n} f(X_i | \mu,\sigma)$

$= \frac{1}{(2\sqrt{3}\sigma)^n} 1(\mu-\sqrt{3}\sigma \le X_1, X_2, \cdots, X_n \le \mu+\sqrt{3}\sigma)$

$= \frac{1}{(2\sqrt{3}\sigma)^n} 1(\mu-\sqrt{3}\sigma \le X_{(1)} \le X_{(n)} \le \mu+\sqrt{3}\sigma)$

Since $L(\mu,\sigma)$ is decreasing with respect to σ, $L(\mu,\sigma)$ will be maximized when σ is minimized. We have the constraints:

$\begin{cases} \mu-\sqrt{3}\sigma \le X_{(1)} \\ \mu+\sqrt{3}\sigma \ge X_{(n)} \end{cases} \Rightarrow \sigma \ge \frac{X_{(n)}-X_{(1)}}{2\sqrt{3}}$. So $\sigma(MLE) = \frac{X_{(n)}-X_{(1)}}{2\sqrt{3}}$

Also, when $\sigma = \frac{X_{(n)}-X_{(1)}}{2\sqrt{3}}$, we have:

$\begin{cases} \mu \le X_{(1)}+\sqrt{3}\sigma \\ \mu \ge X_{(n)}-\sqrt{3}\sigma \end{cases} \Rightarrow \frac{1}{2}(X_{(1)}+X_{(n)}) \le \mu \le \frac{1}{2}(X_{(1)}+X_{(n)}) \Rightarrow \mu(MLE) = \frac{X_{(1)}+X_{(n)}}{2}$

Question 4

4. a) $l(\theta|X_1,\dots X_n\in(0,1)) = \ln\prod_{i=1}^{n} f(x|\theta) = \ln\prod_{i=1}^{n}\theta x^{\theta-1}$

$= \sum_{i=1}^{n}[\ln\theta + (\theta-1)\ln X_i] = n\ln\theta + (\theta-1)\sum_{i=1}^{n}\ln X_i$

$l'(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n}\ln X_i \qquad l''(\theta) = \frac{-n}{\theta^2}.$

$l'(\theta)=0 \Rightarrow \frac{n}{\theta} + \sum_{i=1}^{n}\ln X_i = 0 \Rightarrow \theta = \frac{-n}{\sum_{i=1}^{n}\ln X_i}, \quad l''(\theta)\Big|_{\theta=\frac{-n}{\sum_{i=1}^{n}\ln X_i}} < 0.$

Therefore, the MLE of θ: $\hat{\theta} = \frac{-n}{\sum_{i=1}^{n}\ln X_i}$.

MLE of $g(\theta)=\frac{1}{\theta}$: $g(\hat{\theta}) = -\frac{\sum_{i=1}^{n}\ln X_i}{n}$

b) $\mathbb{E}[g(\hat{\theta})] = \mathbb{E}\left[-\frac{\sum_{i=1}^{n}\ln X_i}{n}\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[-\ln X_i]$

Since the cdf of $-\ln X_i$: $P(-\ln X_i \le x) = \begin{cases} 0 & \text{if } x<0 \\ P(X_1 \ge e^{-x}) = 1-P(X_1<e^{-x}) = 1-\int_0^{e^{-x}}\theta t^{\theta-1}dt = 1-e^{-x\theta}; & \text{if } x\ge 0 \end{cases}$

$-\ln X_1 \sim \exp(\theta)$. $\mathbb{E}[-\ln X_i] = \frac{1}{\theta}$.

$\Rightarrow \mathbb{E}[g(\hat{\theta})] = \frac{1}{n}\cdot\frac{n}{\theta} = \frac{1}{\theta} = g(\theta)$. Therefore MLE is an unbiased estimator

c) $Var(T(X)) \ge \frac{[\frac{d}{d\theta}g(\theta)]^2}{I_{X_1,\dots X_n}(\theta)} = \frac{[\frac{d}{d\theta}g(\theta)]^2}{E[\frac{\partial}{\partial\theta}\ln f_{X_1,\dots X_n}(X_1,\dots X_n|\theta)]^2} = \frac{g'(\theta)^2}{-\mathbb{E}[\frac{\partial^2}{\partial\theta^2}\ln f_{X_1,\dots X_n}(X_1,\dots X_n|\theta)]}$

$= \frac{(-\frac{1}{\theta^2})^2}{-\mathbb{E}[-\frac{n}{\theta^2}]} = \frac{(-\frac{1}{\theta^2})^2}{\frac{n}{\theta^2}} = \frac{1}{n\theta^2}$. Therefore, the C-R inequality is $Var(T(X)) \ge \frac{1}{n\theta^2}$

d) $Var(g(\hat{\theta})) = Var\left(-\frac{\sum_{i=1}^{n}\ln X_i}{n}\right) = \frac{1}{n^2}\sum_{i=1}^{n}Var(-\ln X_i) = \frac{1}{n^2}\left(n\cdot\frac{1}{\theta^2}\right) = \frac{1}{n\theta^2}$.

Therefore, MLE found in (a) is the UMVUE of $g(\theta)$.