3. (a) $C_1 = \{ \underset{\sim}{x} : \sum_{i=1}^{n} X_i \geq k \}$

$\sum_{i=1}^{n} X_i \sim N(n\mu, n)$

$P_r(\sum_{i=1}^{n} X_i \geq k) = \alpha \Rightarrow P_r(Z \geq \frac{k - n\mu_0}{\sqrt{n}}) = \alpha$

$\Rightarrow \frac{k - n\mu_0}{\sqrt{n}} = z_\alpha$

$\Rightarrow k = \sqrt{n} z_\alpha + n\mu_0$

$\Rightarrow C_1 = \{ \underset{\sim}{x} : \sum_{i=1}^{n} X_i \geq n\mu_0 + \sqrt{n} z_\alpha \}$

OR $C_1 = \{ \underset{\sim}{x} : \bar{x} \geq \mu_0 + \frac{z_\alpha}{\sqrt{n}} \}$

$P_r(\sum_{i=1}^{n} X_i \geq n\mu_0 + \sqrt{n} z_\alpha \mid \mu) = P_r(Z \geq \frac{n\mu_0 + \sqrt{n} z_\alpha - n\mu}{\sqrt{n}})$

$= P_r(Z \geq \sqrt{n}\underbrace{(\mu_0 - \mu)}_{\text{+ve under } H_0} + z_\alpha)$

$\leq \alpha$

(b) $C_1 = \{ \underset{\sim}{x} : \sum_{i=1}^{n} X_i^2 \geq k \}$

$X_i \sim N(0, \sigma^2) \Rightarrow \frac{X_i}{\sigma} \sim N(0,1)$

$\Rightarrow \frac{X_i^2}{\sigma^2} \sim \chi^2_{(1)}$

$\Rightarrow \frac{\sum_{i=1}^{n} X_i^2}{\sigma^2} \sim \chi^2_{(n)}$

$P_r(\sum_{i=1}^{n} X_i^2 \geq k) = \alpha \Rightarrow P_r(\chi^2_{(n)} \geq \frac{k}{\sigma_0^2}) = \alpha$

$\Rightarrow k = \sigma_0^2 \chi^2_\alpha(n)$

$\Rightarrow C_1 = \{ \underset{\sim}{x} : \sum_{i=1}^{n} X_i^2 \geq \sigma_0^2 \chi^2_\alpha(n) \}$

$P_r(\sum_{i=1}^{n} X_i^2 \geq \sigma_0^2 \chi^2_\alpha(n) \mid \sigma^2 \} = P_r\{ \chi^2_{(n)} \geq \frac{\sigma_0^2 \chi^2_\alpha(n)}{\sigma^2})$

1 under H_0

$\leq \alpha$

(c) $k = \chi^2_{0.05}(n)$

$P_r(\chi^2_{(n)} \geq \frac{\chi^2_{0.05}(n)}{2.5}) = 0.95 \Rightarrow \frac{\chi^2_{0.05}(n)}{2.5} = \chi^2_{0.95}(n) \Rightarrow \frac{\chi^2_{0.05}(n)}{\chi^2_{0.95}(n)} = 2.5$

$\Rightarrow n \geq 27$

4. (a). Under H_0 : $\hat{\mu}_0 = \dfrac{\sum_{i=1}^n y_{1i} + \sum_{i=1}^n y_{2i} + \sum_{i=1}^n y_{3i}}{3n} = \dfrac{\bar{y}_1 + \bar{y}_2 + \bar{y}_3}{3}$

$$\hat{\sigma}_0^2 = \frac{\sum_{i=1}^n (y_{1i} - \hat{\mu}_0)^2 + \sum_{i=1}^n (y_{2i} - \hat{\mu}_0)^2 + \sum_{i=1}^n (y_{3i} - \hat{\mu}_0)^2}{3n}$$

Under H_1 : $\hat{\mu}_1 = \bar{y}_1$, $\hat{\mu}_2 = \bar{y}_2$, $\hat{\mu}_3 = \bar{y}_3$

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n (y_{1i} - \bar{y}_1)^2 + \sum_{i=1}^n (y_{2i} - \bar{y}_2)^2 + \sum_{i=1}^n (y_{3i} - \bar{y}_3)^2}{3n}$$

$$\lambda(\underset{\sim}{y_1}, \underset{\sim}{y_2}, \underset{\sim}{y_3}) = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}^2}\right)^{-3n/2}$$

$$= \left(\frac{\sum_{i=1}^n (y_{1i} - \hat{\mu}_0)^2 + \sum_{i=1}^n (y_{2i} - \hat{\mu}_0)^2 + \sum_{i=1}^n (y_{3i} - \hat{\mu}_0)^2}{\sum_{i=1}^n (y_{1i} - \bar{y}_1)^2 + \sum_{i=1}^n (y_{2i} - \bar{y}_2)^2 + \sum_{i=1}^n (y_{3i} - \bar{y}_3)^2}\right)^{-3n/2}$$

(b) $$-2 \log \lambda(\underset{\sim}{y_1}, \underset{\sim}{y_2}, \underset{\sim}{y_3}) = 3n \log\left(\frac{\sum_{i=1}^n (y_{1i} - \hat{\mu}_0)^2 + \sum_{i=1}^n (y_{2i} - \hat{\mu}_0)^2 + \sum_{i=1}^n (y_{3i} - \hat{\mu}_0)^2}{\sum_{i=1}^n (y_{1i} - \bar{y}_1)^2 + \sum_{i=1}^n (y_{2i} - \bar{y}_2)^2 + \sum_{i=1}^n (y_{3i} - \bar{y}_3)^2}\right)$$

$$\sim \chi^2_{(2)}$$

$$\Rightarrow 3 * 6 * \log \frac{106074.2778}{94616.8333} = 2.05748 < 5.991 \text{ (can't reject } H_0\text{)}$$

(c) $\sum_{i=1}^n (y_{1i} - \hat{\mu}_0)^2 + \sum_{i=1}^n (y_{2i} - \hat{\mu}_0)^2 + \sum_{i=1}^n (y_{3i} - \hat{\mu}_0)^2$

$= \sum_{i=1}^n (y_{1i} - \bar{y}_1)^2 + n(\bar{y}_1 - \hat{\mu}_0)^2 + \sum_{i=1}^n (y_{2i} - \bar{y}_2)^2 + n(\bar{y}_2 - \hat{\mu}_0)^2 + \sum_{i=1}^n (y_{3i} - \bar{y}_3)^2 + n(\bar{y}_3 - \hat{\mu}_0)^2$

$= n[(\bar{y}_1 - \hat{\mu}_0)^2 + (\bar{y}_2 - \hat{\mu}_0)^2 + (\bar{y}_3 - \hat{\mu}_0)^2] + \sum_{i=1}^n (y_{1i} - \bar{y}_1)^2 + \sum_{i=1}^n (y_{2i} - \bar{y}_2)^2 + \sum_{i=1}^n (y_{3i} - \bar{y}_3)^2$

$\Rightarrow \dfrac{n[(\bar{y}_1 - \hat{\mu}_0)^2 + (\bar{y}_2 - \hat{\mu}_0)^2 + (\bar{y}_3 - \hat{\mu}_0)^2]}{\sigma^2} \sim \chi^2_{(2)}$

$$\lambda(\underset{\sim}{y_1}, \underset{\sim}{y_2}, \underset{\sim}{y_3}) = \left(1 + \frac{n[(\bar{y}_1 - \hat{\mu}_0)^2 + (\bar{y}_2 - \hat{\mu}_0)^2 + (\bar{y}_3 - \hat{\mu}_0)^2]}{\sum_{i=1}^n (y_{1i} - \bar{y}_1)^2 + \sum_{i=1}^n (y_{2i} - \bar{y}_2)^2 + \sum_{i=1}^n (y_{3i} - \bar{y}_3)^2}\right)^{-3n/2}$$

$C_1 = \{(\underset{\sim}{y_1}, \underset{\sim}{y_2}, \underset{\sim}{y_3}) : \lambda(\underset{\sim}{y_1}, \underset{\sim}{y_2}, \underset{\sim}{y_3}) \le k\}$

$$\Rightarrow C_1 = \left\{(\underset{\sim}{y_1}, \underset{\sim}{y_2}, \underset{\sim}{y_3}) : \frac{n[(\bar{y}_1 - \hat{\mu}_0)^2 + (\bar{y}_2 - \hat{\mu}_0)^2 + (\bar{y}_3 - \hat{\mu}_0)^2]}{\sum_{i=1}^n (y_{1i} - \bar{y}_1)^2 + \sum_{i=1}^n (y_{2i} - \bar{y}_2)^2 + \sum_{i=1}^n (y_{3i} - \bar{y}_3)^2} \ge k_1\right\}$$

$$\Rightarrow C_1 = \left\{(\underset{\sim}{y_1}, \underset{\sim}{y_2}, \underset{\sim}{y_3}) : \frac{n[(\bar{y}_1 - \hat{\mu}_0)^2 + (\bar{y}_2 - \hat{\mu}_0)^2 + (\bar{y}_3 - \hat{\mu}_0)^2]/2}{[\sum_{i=1}^n (y_{1i} - \bar{y}_1)^2 + \sum_{i=1}^n (y_{2i} - \bar{y}_2)^2 + \sum_{i=1}^n (y_{3i} - \bar{y}_3)^2]/(3n-3)} \ge F_{(2, 3n-3, \alpha)}\right\}$$

$$\downarrow$$

$$\frac{11457.4444/2}{94616.8333/15} = \frac{5728.7222}{6307.7889} = 0.91$$

(can't reject H_0)