1. **(8 marks)** Let X_1, X_2 be random variables having the bivariate normal distribution with parameters μ_1, μ_2, σ_1, σ_2, ρ (correlation coefficient between X_1 and X_2), i.e.,

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N_2\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right).$$

Set

$$Y_1 = \frac{X_1 - \mu_1}{\sigma_1} + \frac{X_2 - \mu_2}{\sigma_2}, \; Y_2 = \frac{X_1 - \mu_1}{\sigma_1} - \frac{X_2 - \mu_2}{\sigma_2}.$$

Find the probability density functions of Y_1 and Y_2. Are they independent?

Ans.

$$\begin{aligned} Z_1 &= \frac{X_1 - \mu_1}{\sigma_1} \sim N(0,1) \\ Z_2 &= \frac{X_2 - \mu_1}{\sigma_2} \sim N(0,1) \\ Cov(Z_1, Z_2) &= \rho \end{aligned}$$

Prove that Y_1 and Y_2 are independent, use any method below:

(a) Since

$$\begin{aligned} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} &= \begin{pmatrix} Z_1 + Z_2 \\ Z_1 - Z_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \end{aligned}$$

We consider the terms inside exponential, i.e.,

$$\begin{aligned} & \frac{1}{2} \begin{pmatrix} Z_1 & Z_2 \end{pmatrix}^T \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}^{-1} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \\ = & \frac{1}{2(1-\rho^2)} \begin{pmatrix} Z_1 & Z_2 \end{pmatrix}^T \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \\ = & \frac{1}{8(1-\rho^2)} \begin{pmatrix} Y_1 & Y_2 \end{pmatrix}^T \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^T \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\ = & \frac{1}{8(1-\rho^2)} \begin{pmatrix} Y_1 & Y_2 \end{pmatrix}^T \begin{pmatrix} 2(1-\rho) & 0 \\ 0 & 2(1+\rho) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \end{aligned}$$

$\Rightarrow$ Y_1 & Y_2 are independent.

(b) Y_1, Y_2 are random variables having the bivariate normal distribution as linear combination of a multivariate random vector has a multivariate normal distribution.

$$\begin{aligned} Y_1 &\sim N(0, 2(1+\rho)) \\ Y_2 &\sim N(0, 2(1-\rho)) \end{aligned}$$

$Cov(Y_1, Y_2) = 0 \; \Rightarrow \; Y_1$ & Y_2 are independent.

2. If $X_1, X_2, \ldots, X_n$ are independently and normally distributed with the same mean μ but different variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$. Let $U = \sum_{i=1}^n (X_i/\sigma_i^2)/\sum_{j=1}^n (1/\sigma_j^2)$ and $V = \sum_{i=1}^n (X_i - U)^2/\sigma_i^2$. Note that U and V are independently distributed.

(a) **(4 marks)** Find the distribution of U.

Ans.

$$\begin{aligned}
X_i &\sim N(\mu, \sigma_i^2) \\
\frac{X_i}{\sigma_i^2} &\sim N\left(\frac{\mu}{\sigma_i^2}, \frac{1}{\sigma_i^2}\right) \\
\sum_{i=1}^n \frac{X_i}{\sigma_i^2} &\sim N\left(\mu \sum_{i=1}^n \frac{1}{\sigma_i^2}, \sum_{i=1}^n \frac{1}{\sigma_i^2}\right) \\
U = \frac{\sum_{i=1}^n \frac{X_i}{\sigma_i^2}}{\sum_{j=1}^n \frac{1}{\sigma_j^2}} &\sim N\left(\mu, \frac{1}{\sum_{j=1}^n \frac{1}{\sigma_j^2}}\right)
\end{aligned}$$

(b) **(10 marks)** Find the distribution of V .

Ans.

$$\begin{aligned}
\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma_i}\right)^2 &= \sum_{i=1}^n \left[\frac{(X_i - U) + (U - \mu)}{\sigma_i}\right]^2 \\
&= \sum_{i=1}^n \left(\frac{X_i - U}{\sigma_i}\right)^2 + \sum_{i=1}^n \left(\frac{U - \mu}{\sigma_i}\right)^2
\end{aligned}$$

because the cross-product term is equal to

$$2\sum_{i=1}^n \frac{(X_i - U)(U - \mu)}{\sigma_i^2} = 2(U - \mu)\sum_{i=1}^n \frac{X_i - U}{\sigma_i^2} = 0$$

Then,

$$\underbrace{\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma_i}\right)^2}_{\sim \chi^2(n)} = V + \underbrace{\sum_{i=1}^n \left(\frac{U - \mu}{\sigma_i}\right)^2}_{\sim \chi^2(1)}$$

Since U and V are independent, thus $V \sim \chi^2(n-1)$.

3. Let X_1, ...,X_n be a random sample from a location distribution family

$$f(x;\theta) = \frac{1}{\theta}\exp\left(-\frac{x-\delta}{\theta}\right) I(x \geq \delta) \ .$$

Note that $Y_i = X_i - \delta \sim \exp\left(\frac{1}{\theta}\right)$.

(a) Assume that <u>δ is equal to zero</u>.

i. **(2 marks)** Prove that the moment generating function of X_i is equal to $1/(1-\theta t)$.

<u>Ans.</u>

$$\begin{aligned} \text{m.g.f.} = E(e^{tx}) &= \int_0^\infty e^{tx} \ \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right) dx \\ &= \int_0^\infty \frac{1}{\theta}\exp\left(-\frac{(1-\theta t)x}{\theta}\right) dx \\ &= \frac{1}{\theta} \ \frac{\theta}{1-\theta t} \\ &= \frac{1}{1-\theta t} \end{aligned}$$

ii. **(4 marks)** Find the distribution of $\sum_{i=1}^n X_i$.

<u>Ans.</u>

$$\sum_{i=1}^n X_i \sim \text{Gamma}(n, 1/\theta)$$

iii. **(4 marks)** Find the distribution of $2\sum_{i=1}^n X_i/\theta$.

<u>Ans.</u>

$$\frac{2}{\theta}\sum_{i=1}^n X_i \sim \chi^2(2n)$$

(b) Assume that <u>δ is known</u>.

i. **(4 marks)** Find the method of moments estimator, $\tilde{\theta}$, for θ. Is it unbiased?

<u>Ans.</u>

$$\tilde{\theta} = \frac{1}{n}\sum_{i=1}^n (x_i - \delta)$$

It is unbiased if $E(X_i - \delta) = \theta$.

ii. **(4 marks)** Find the maximum likelihood estimator, $\hat{\theta}$, for θ. Is it unbiased?

Ans.

$$
\begin{aligned}
\text{likelihood} \;\; L(\theta) &= \frac{1}{\theta^n}\text{exp}\left(-\frac{1}{\theta}\sum_{i=1}^{n}(x_i-\delta)\right) \\
\text{loglikelihood} \;\; \text{log}L(\theta) &= -n\text{log}(\theta) - \frac{\sum_{i=1}^{n}(x_i-\delta)}{\theta} \\
\frac{\partial \text{log}L(\theta)}{\partial\theta} &= -\frac{n}{\theta} + \frac{\sum_{i=1}^{n}(x_i-\delta)}{\theta^2} \;=\; 0 \\
\Rightarrow \hat{\theta} &= \frac{1}{n}\sum_{i=1}^{n}(x_i-\delta)
\end{aligned}
$$

It is unbiased if $E(X_i - \delta) = \theta$.

iii. **(2 marks)** Find the mean squared error of $\hat{\theta}$.

Ans.

$$
\begin{aligned}
\text{MSE}(\hat{\theta}) &= \text{Var}(\hat{\theta}) \\
&= \frac{1}{n}\text{Var}(X_i - \delta) \\
&= \frac{\theta^2}{n}
\end{aligned}
$$

iv. **(4 marks)** Let $\tau(\theta) = Pr(X_1 > 1+\delta)$, find its maximum likelihood estimator, $\widehat{\tau(\theta)}$.

Ans.

$$
\begin{aligned}
\tau(\theta) &= Pr(X_1 > 1+\delta) \\
&= \int_{1+\delta}^{\infty}\frac{1}{\theta}\text{exp}\left(-\frac{x-\delta}{\theta}\right)dx \\
&= \int_{1}^{\infty}\frac{1}{\theta}\text{exp}\left(-\frac{y}{\theta}\right)dy \\
&= -\text{exp}\left(-\frac{y}{\theta}\right)\Bigg|_{1}^{\infty} \\
&= \text{exp}\left(-\frac{1}{\theta}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \widehat{\tau(\theta)} &= \text{exp}\left(-\frac{1}{\hat{\theta}}\right) \\
&= \text{exp}\left(-\frac{n}{\sum_{i=1}^{n}(x_i-\delta)}\right)
\end{aligned}
$$

v. **(6 marks)** Find Cramer-Rao lower bound for the variance of unbiased estimators of $\tau(\theta)$.

Ans.

$$
\begin{aligned}
\log f_{X_i}(x_i,\theta) &= -\log(\theta) - \frac{X_i-\delta}{\theta} \\
\frac{\partial}{\partial\theta}\log f_{X_i}(x_i,\theta) &= -\frac{1}{\theta} + \frac{x_i-\delta}{\theta^2} \\
\frac{\partial^2}{\partial\theta^2}\log f_{X_i}(x_i,\theta) &= \frac{1}{\theta^2} - \frac{2(x_i-\delta)}{\theta^3} \\
E\left[\frac{\partial^2}{\partial\theta^2}\log f_{X_i}(x_i,\theta)\right] &= -\frac{1}{\theta^2} \\
\tau(\theta) &= \exp\left(-\frac{1}{\theta}\right) \\
\tau'(\theta) &= \frac{1}{\theta^2}\exp\left(-\frac{1}{\theta}\right) \\
\Rightarrow \text{ C-R lower bound } &= \frac{\left(\frac{1}{\theta^2}\exp\left(-\frac{1}{\theta}\right)\right)^2}{\frac{n}{\theta^2}} \\
&= \frac{\exp\left(-\frac{2}{\theta}\right)}{n\theta^2}
\end{aligned}
$$

vi. **(6 marks)** Find the limiting distribution of $\widehat{\tau(\theta)}$ by Delta method. What phenomenon do you observe?

Ans.
As $n \to \infty$,

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}(x_i-\delta) &\to N\left(\theta,\ \frac{\theta^2}{n}\right) \\
\Rightarrow \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\delta)\ -\ \theta\right) &\to N\left(0,\ \theta^2\right) \\
\exp\left(-\frac{n}{\sum_{i=1}^{n}(x_i-\delta)}\right) &\to N\left(\exp\left(-\frac{1}{\theta}\right),\ \frac{\exp\left(-\frac{2}{\theta}\right)}{n\theta^2}\right) \\
\Rightarrow \sqrt{n}\left(\exp\left(-\frac{n}{\sum_{i=1}^{n}(x_i-\delta)}\right)\ -\ \exp\left(-\frac{1}{\theta}\right)\right) &\to N\left(0,\ \frac{\exp\left(-\frac{2}{\theta}\right)}{\theta^2}\right)
\end{aligned}
$$

As $n \to \infty$, the maximum likelihood estimator of $\tau(\theta)$ is unbiased, normally distributed and fully efficiency, i.e., its variance is equal to C-R lower bound.

(c) Assume that $\underline{\theta \text{ is known}}$.

i. **(12 marks)** Find the maximum likelihood estimator, $\hat{\delta}$, for δ. Is it unbiased? Hence or otherwise, find the unbiased estimator for δ.

Ans.
The maximum likelihood estimator, $\hat{\delta}$, for δ is $X_{(1)}$.

$$\begin{aligned}
F_X(y) &= \int_\delta^y \frac{1}{\theta}\exp\left(-\frac{x-\delta}{\theta}\right) dx \\
&= \int_0^{y-\delta} \frac{1}{\theta}\exp\left(-\frac{u}{\theta}\right) du \\
&= -\exp\left(-\frac{u}{\theta}\right)\Big|_0^{y-\delta} \\
&= 1-\exp\left(-\frac{y-\delta}{\theta}\right) \\
\Rightarrow 1-F_X(y) &= \exp\left(-\frac{y-\delta}{\theta}\right)
\end{aligned}$$

$$\begin{aligned}
f_{X_{(1)}}(y) &= n(1-F_X(y))^{n-1} f_X(y) \\
&= \frac{n}{\theta}\exp\left(-\frac{n(y-\delta)}{\theta}\right) \\
\Rightarrow X_{(1)}-\delta &\sim \exp\left(\frac{n}{\theta}\right) \\
\Rightarrow E(X_{(1)}) &= \frac{n}{\theta}+\delta
\end{aligned}$$

$X_{(1)} - \frac{n}{\theta}$ is unbiased estimator for δ.