03/04 final exam

1a. $Y_1 \sim N(4-\theta, \sigma^2)$, $Y_2 \sim N(2+2\theta, \sigma^2)$

$$L(y_1, y_2 ; \theta, \sigma^2) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[(y_1+\theta-4)^2 + (y_2-2\theta-2)^2\right]\right\}$$

$$\log L = \log C - \frac{1}{2\sigma^2}\left[(y_1+\theta-4)^2 + (y_2-2\theta-2)^2\right]$$

$$\frac{\partial}{\partial\theta}\log L = -\frac{1}{2\sigma^2}\left[2(y_1+\theta-4) \mp (2)(2)(y_2-2\theta-2)\right] = 0$$

$$\Rightarrow \quad 2y_1 - 4y_2 + 10\theta = 0$$

$$\Rightarrow \quad \hat{\theta} = \frac{-y_1 + 2y_2}{5}$$

b. $E(\hat{\theta}) = E\left(-\frac{y_1}{5} + \frac{2y_2}{5}\right) = -\frac{1}{5}(4-\theta) + \frac{2}{5}(2+2\theta) = \theta$

$\Rightarrow \hat{\theta}$ is an unbiased estimator for θ.

c. $\text{Var}(\hat{\theta}) = \frac{1}{25}\sigma^2 + \frac{4}{25}\sigma^2 = \frac{1}{5}\sigma^2$

d. $E(W) = E(Y_1 + Y_2 - 6) = 4-\theta + (2+2\theta) - 6 = \theta$,

So, W is unbiased.

$\text{var}(Y_1 + Y_2 - 6) = \sigma^2 + \sigma^2 = 2\sigma^2$

So, $\hat{\theta}$ is more efficient since $\frac{\sigma^2}{5} = \text{var}(\hat{\theta}) < \text{var}(W) = 2\sigma^2$.

2 a) $X \sim gamma(2, \theta)$

$$L(x, \theta) = \prod_{i=1}^{n} \theta^2 x_i e^{-\theta x_i} = \theta^{2n} e^{-\theta \sum_{i=1}^{n} x_i} \cdot \prod_{i=1}^{n} x_i$$

$$\log L(x, \theta) = 2n \log\theta - \theta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log x_i$$

$$\Rightarrow \frac{\partial \log L(x,\theta)}{\partial \theta} = \frac{2n}{\theta} - \sum_{i=1}^{n} x_i \Rightarrow \hat{\theta} = 2n / \sum_{i=1}^{n} x_i$$

$E(X) = \frac{2}{\theta}$, $Var(X) = \frac{2}{\theta^2}$. MLE for Variance is $\frac{2}{\hat{\theta}^2} = \frac{(\sum x_i)^2}{2n^2}$

b). $\frac{\partial^2 (\log L(x,\theta))}{\partial \theta^2} = -\frac{2n}{\theta^2}$ $\therefore$ CRLB for $\frac{1}{\theta} = -\frac{(-\frac{1}{\theta^2})^2}{-\frac{2n}{\theta^2}} = \frac{1}{2n\theta^2}$

$$f(x, \theta) = \theta^2 x e^{-\theta x} = \exp(2\log\theta + \log x + (-\theta x)).$$

$a(\theta) = 2\log\theta$, $b(x) = \log x$. $c(\theta) = -\theta$ $d(x) = x$.

$\Rightarrow \sum_{i=1}^{n} x_i$ is complete and sufficient.

and $E(\bar{x}) = \frac{2}{\theta}$ $\Rightarrow$ $\frac{\bar{x}}{2}$ is UMVUE for $\frac{1}{\theta}$

$$Var(\frac{\bar{x}}{2}) = \frac{1}{4} Var(\bar{x}) = \frac{1}{4} \cdot \frac{1}{n^2} \cdot Var(\sum x_i) = \frac{1}{4} \cdot \frac{1}{n^2} \cdot n \cdot \frac{2}{\theta^2} = \frac{1}{2n\theta^2}$$

$\Rightarrow Var(\frac{\bar{x}}{2}) = CRLB$.

c.) $S = \sum_{i=1}^{n} x_i \sim gamma(2n, \theta)$

$$E(\frac{1}{S}) = \int_0^{\infty} \frac{1}{s} \frac{\theta^{2n} s^{2n-1} e^{-\theta s}}{\Gamma(2n)} ds$$

$$= \int_0^{\infty} \frac{\theta^{2n}}{\Gamma(2n)} \cdot s^{2n-2} e^{-\theta s} ds$$

$$= \int_0^{\infty} \frac{\Gamma(2n-1)}{\theta^{2n-1}} \cdot \frac{\theta^{2n}}{\Gamma(2n)} \cdot \frac{\theta^{2n-1}}{\Gamma(2n-1)} \cdot s^{2n-2} \cdot e^{-\theta s} ds$$

$$= \frac{\Gamma(2n-1)\theta^{2n}}{\Gamma(2n)\theta^{2n-1}} = \theta / 2n-1$$

$\Rightarrow$ UMVUE for $\theta = 2n-1 / \sum_{i=1}^{n} x_i$.

$$CRLB = -\frac{1}{-2n/\theta^2} = \frac{\theta^2}{2n}$$

$\therefore$ UMVUE for $\theta > CRLB$.

3a. $\hat{\theta} = X_{(n)} = Y_n$

b. $P(Y_n < y) = [P(X_i < y)]^n = [\int_0^y \frac{2x}{\theta^2} dx]^n = \left[\frac{1}{\theta^2} x^2 \Big|_0^y\right]^n = \left(\frac{y}{\theta}\right)^{2n}$

$f(y) = \frac{d}{dy} P(Y_n < y) = \frac{2n y^{2n-1}}{\theta^{2n}}$

$E(X_{(n)}) = E(Y_n) = \int_0^\theta y \cdot \frac{2n y^{2n-1}}{\theta^{2n}} dy$

$= \int_0^\theta \frac{2n y^{2n}}{\theta^{2n}} dy = \frac{2n}{\theta^{2n}} \cdot \frac{y^{2n+1}}{2n+1} \Big|_0^\theta$

$= \frac{2n}{2n+1} \theta$

$\Rightarrow E\left(\frac{2n+1}{2n} X_{(n)}\right) = \theta$

c. $E(X_{(n)}^2) = E(Y_n^2) = \int_0^\theta y^2 \cdot \frac{2n y^{2n-1}}{\theta^{2n}} dy$

$= \int_0^\theta \frac{2n y^{2n+1}}{\theta^{2n}} dy = \frac{2n}{\theta^{2n}} \cdot \frac{y^{2n+2}}{2n+2} \Big|_0^\theta$

$= \frac{2n}{2n+2} \theta^2$

$\Rightarrow E\left(\frac{2n+2}{2n} X_{(n)}^2\right) = \theta^2$.

$E\left(\frac{n+1}{n} X_{(n)}^2\right) = \theta^2$