Fall 02/03

1 a $E(Y) = \frac{0+\theta}{2} = \frac{\theta}{2}$, $var(Y) = \frac{(\theta-0)^2}{12} = \frac{\theta^2}{12}$

b. $P(Z \le z) = \prod_{i=1}^{n} P(Y_i \le z) = P(Y_i \le z)^n = \left(\frac{z}{\theta}\right)^n$

$f(z) = \frac{d}{dy} P(Z \le z) = \frac{n z^{n-1}}{\theta^n}$, $z \in (0,\theta)$

c. $E(Z) = \int_0^\theta z \cdot \frac{n z^{n-1}}{\theta^n} dz = \frac{n}{\theta^n} \int_0^\theta z^n dz = \frac{n}{\theta^n} \left.\frac{z^{n+1}}{n+1}\right|_0^\theta = \frac{n\theta}{n+1}$

$E(Z^2) = \int_0^\theta z^2 \frac{n z^{n-1}}{\theta^n} dz = \frac{n}{\theta^n} \int_0^\theta z^{n+1} dy = \frac{n}{\theta^n} \cdot \left.\frac{z^{n+2}}{n+2}\right|_0^\theta = \frac{n\theta^2}{n+2}$

$var(Z) = E(Z^2) - (EZ)^2 = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2 = \frac{n\theta^2}{(n+1)^2(n+2)}$

d. $E(\bar{Y}) = E(Y) = \frac{\theta}{2}$, $var(\bar{Y}) = \frac{var(Y)}{n} = \frac{\theta^2}{12n}$

e. From (c), $E(Z) = \frac{n\theta}{n+1} \Rightarrow E\left(\frac{n+1}{n} Z\right) = \theta \Rightarrow \tilde{\theta}_1 = \left(\frac{n+1}{n}\right) Z$

From (d), $E(\bar{Y}) = \frac{\theta}{2} \Rightarrow E(2\bar{Y}) = \theta \Rightarrow \tilde{\theta}_2 = 2\bar{Y}$

f. $var(\tilde{\theta}_1) = var\left(\frac{n+1}{n} Z\right) = \left(\frac{n+1}{n}\right)^2 var Z = \left(\frac{n+1}{n}\right)^2 \cdot \frac{n\theta^2}{(n+1)^2(n+2)} = \frac{\theta^2}{n(n+2)}$

$var(\tilde{\theta}_2) = var(2\bar{Y}) = 4\, var\,\bar{Y} = 4 \cdot \frac{\theta^2}{12n} = \frac{\theta^2}{3n}$

$var(2\bar{Y}) = \frac{\theta^2}{3n} \ge \frac{\theta^2}{n(n+2)} = var\left(\frac{n+1}{n} Z\right)$ when $n \ge 1$,

So $\frac{n+1}{n} Z$ is more efficient than $2\bar{Y}$.

2.

a $f_{X_i}(x) = \frac{\theta^x e^{-\theta}}{x!}$

$L(\theta) = \prod_{i=1}^{n} f_{X_i}(x_i) = \frac{\theta^{\sum X_i} e^{-n\theta}}{\prod_{i=1}^{n} X_i!}$

$\log L(\theta) = \sum X_i \log\theta - n\theta - \log \prod_{i=1}^{n} X_i!$

$\frac{\partial}{\partial\theta} \log L(\theta) = \frac{\sum X_i}{\theta} - n = 0$

$\Rightarrow \hat{\theta} = \frac{\sum X_i}{n} = \bar{X}$

$\therefore \hat{\theta}^2 = \bar{X}^2$

b. $E(a\bar{X} + b\bar{X}^2) = a E\bar{X} + b E\bar{X}^2 = a\theta + b[var\bar{X} + (E\bar{X})^2] = a\theta + b\left(\frac{\theta}{n} + \theta^2\right)$

$= \left(a + \frac{b}{n}\right)\theta + b\theta^2$

$\Rightarrow a + \frac{b}{n} = 0$ and $b = 1 \Rightarrow a = -\frac{1}{n}$ and $b = 1$

c. $var(-\frac{X}{n}+\bar{X}^2) = var(\frac{X}{n}) + var(X^2) - 2cov(\frac{X}{n}, X^2)$

$Var(\bar{X}^2) = E\bar{X}^4 - (E\bar{X}^2)^2$

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$m_X(t) = \exp\{\theta(e^t-1)\}$

$m_X'(t) = \theta e^t e^{\theta(e^t-1)}$

$m_X''(t) = \theta e^t e^{\theta(e^t-1)}[1+\theta e^t]$

$m_X^{(3)}(t) = \theta e^t e^{\theta(e^t-1)}[1+3\theta e^t + \theta^2 e^{2t}]$

$m_X^{(4)}(t) = \theta e^t e^{\theta(e^t-1)}[1+7\theta e^t + 6\theta^2 e^{2t} + \theta^3 e^{3t}]$

$E\bar{X} = m_X'(t)|_{t=0} = \theta$, $EX^3 = m_X'''(t)|_{t=0} = \theta(1+3\theta+\theta^2)$

$E\bar{X}^2 = m_X''(t)|_{t=0} = \theta(1+\theta)$, $EX^4 = m_X^{(4)}(t)|_{t=0} = \theta(1+7\theta+6\theta^2+\theta^3)$

$E\bar{X}^4 = E\left(\frac{(\Sigma X_i)^4}{n^4}\right) = \frac{1}{n^4}E\Big(\Sigma X_i^4 + 3\sum_{i=1}^{n}\sum_{j\neq i} X_i^2X_j^2 + 4\sum_{i=1}^{n}\sum_{j\neq i} X_i^3X_j + 6\sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i,j} X_i^2X_jX_k$
$+ \sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i,j}\sum_{\ell\neq i,j,k} X_iX_jX_kX_\ell\Big)$

$= \frac{1}{n^4}\{n[\theta(1+7\theta+6\theta^2+\theta^3)] + 3n(n-1)(\theta(\theta+1))^2 + 4n(n-1)[\theta(1+3\theta+\theta^2)]\theta$
$+ 6n(n-1)(n-2)[\theta(1+\theta)]\cdot\theta\cdot\theta + n(n-1)(n-2)(n-3)\theta^4\}$

$= \frac{\theta}{n^3} + \frac{7\theta^2}{n^2} + \frac{6\theta^3}{n} + \theta^4$

$E\bar{X}^3 = E\left(\frac{(\Sigma X_i)^3}{n^3}\right) = \frac{1}{n^3}E((\Sigma X_i)^3) = \frac{1}{n^3}E\Big(\Sigma X_i^3 + 3\sum_{i=1}^{n}\sum_{j\neq i}X_i^2X_j + \sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i,j}X_iX_jX_k\Big)$

$= \frac{1}{n^3}\{n[\theta(1+3\theta+\theta^2)] + 3n(n-1)\theta^2(\theta+1) + n(n-1)(n-2)\theta^3\}$

$= \frac{1}{n^3}[n^3\theta^3 + 3n^2\theta^2 + n\theta] = \frac{\theta}{n^2} + \frac{3\theta^2}{n} + \theta^3$

$E\bar{X}^2 = var\,\bar{X} - (E\bar{X})^2 = \frac{\theta}{n} + \theta^2$

$Var(\bar{X}^2) = E\bar{X}^4 - (E\bar{X}^2)^2 = \frac{\theta}{n^3} + \frac{7\theta^2}{n^2} + \frac{6\theta^3}{n} + \theta^4 - \left(\frac{\theta}{n}+\theta^2\right)^2 = \frac{\theta}{n^3} + \frac{6\theta^2}{n^2} + \frac{4\theta^3}{n}$

$var\left(\frac{\bar{X}}{n}\right) = \left(\frac{1}{n^2}\right)\left(\frac{\theta}{n}\right) = \frac{\theta}{n^3}$

$cov\left(\frac{\bar{X}}{n}, \bar{X}^2\right) = E\left(\frac{\bar{X}^3}{n}\right) - E(\bar{X}^2)E\left(\frac{\bar{X}}{n}\right) = \frac{1}{n}\left(\frac{\theta}{n^2} + \frac{3\theta^2}{n} + \theta^3\right) - \left(\frac{\theta}{n}+\theta^2\right)\left(\frac{\theta}{n}\right)$

$= \frac{\theta}{n^3} + \frac{2\theta^2}{n^2}$

$\therefore var\left(-\frac{\bar{X}}{n} + \bar{X}^2\right) = \frac{\theta}{n^3} + \left(\frac{\theta}{n^3} + \frac{6\theta^2}{n^2} + \frac{4\theta^3}{n}\right) - 2\left(\frac{\theta}{n^3} + \frac{2\theta^2}{n^2}\right)$

$= \frac{2\theta^2}{n^2} + \frac{4\theta^3}{n}$

d. $E\left(\frac{\partial^2}{\partial\theta^2}\log f_X(x_i,\theta)\right) = E\left(\frac{\partial}{\partial\theta}\left(-1+\frac{x_i}{\theta}\right)\right) = E\left(-\frac{x_i}{\theta^2}\right) = -\frac{1}{\theta^2}(\theta) = -\frac{1}{\theta}$

$\therefore$ CRLB for $\theta^2 = -\dfrac{(2\theta)^2}{n(-\frac{1}{\theta})} = \dfrac{4\theta^3}{n}$

The CRLB is not attained since

$\sum \frac{\partial}{\partial\theta}\log f(x_i;\theta) = \sum\left(-1+\frac{x_i}{\theta}\right) = \frac{n}{\theta}(\bar{x}-\lambda)$,

So only the UMVUE of λ can achieve the CRLB.

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a. $X \sim Bin(n;\theta)$

$f(x;\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x} = \exp\{n\log(1-\theta)+\log\binom{n}{x}+[\log\frac{\theta}{1-\theta}]x\}$

$\Rightarrow \sum_{i=1}^{m} X_i$ is a complete minimal suff. stat.

$S = \sum_{i=1}^{m} X_i \sim Bin(mn,\theta)$

$E(g(S)) = \theta^k$

$\Rightarrow \sum_{s=0}^{mn} g(s)\binom{mn}{s}\theta^s(1-\theta)^{mn-s} = \theta^k$

$\sum_{s=0}^{mn} g(s)\binom{mn}{s}\theta^{s-k}(1-\theta)^{mn-s} = 1$

$\sum_{s=0}^{mn}\binom{mn-k}{s-k}\theta^{s-k}(1-\theta)^{mn-k-(s-k)} \cdot \binom{mn}{s}\Big/\binom{mn-k}{s-k}\cdot g(s) = 1$

$\Rightarrow g(s) = \binom{mn-k}{s-k}\Big/\binom{mn}{s}\, I_{\{k,k+1,\dots,mn\}}(s)$

$\therefore$ UMVUE of θ^k is $\binom{mn-k}{\sum X_i-k}\Big/\binom{mn}{\sum X_i} I_{\{k,k+1,\dots,mn\}}(\sum X_i)$

b. $X \sim \exp(\lambda)$

$f(x;\lambda) = \lambda e^{-\lambda x} = \exp\{\log\lambda - \lambda x\}$

$\Rightarrow \sum_{i=1}^{n} X_i$ is a complete minimal suff. stat.

$S = \sum_{i=1}^{n} X_i \sim Gamma(n,\lambda)$

$E(g(S)) = \lambda^r$

$\Rightarrow \int g(s)\cdot\dfrac{s^{n-1}e^{-\lambda s}}{\Gamma(n)\lambda^{-n}}\,ds = \lambda^r$

$\int g(s)\cdot\dfrac{s^{n-1}e^{-\lambda s}}{\Gamma(n)\lambda^{-(n-r)}}\,ds = 1$

$\int \dfrac{s^{(n-r)-1}e^{-\lambda s}}{\Gamma(n-r)\lambda^{-(n-r)}}\cdot\dfrac{s^r\Gamma(n-r)}{\Gamma(n)}\cdot g(s) = 1$

$\Rightarrow g(s) = \dfrac{\Gamma(n)}{s^r\,\Gamma(n-r)}$

$\therefore$ UMVUE of λ^r is $\dfrac{\Gamma(n)}{(\sum X_i)^r\,\Gamma(n-r)}$.

4 $f(x) = \lambda e^{-\lambda x} = \exp\{\log\lambda - \lambda x\}$,

so $d(x) = x$, $c(\lambda) = -\lambda$ which is a decreasing function.

$\Rightarrow C_1 = \{\underset{\sim}{x} : \sum_{i=1}^{n} x_i \le k\}$

Since $X_i \sim \exp(\lambda)$, so $\Sigma X_i \sim \text{Gamma}(n, \lambda)$, $2\lambda \Sigma X_i \sim \chi^2_{2n}$

$\Rightarrow$ the UMP test is $C_1 : \{\underset{\sim}{x} : \sum_{i=1}^{n} x_i \le \chi^2_{2n}(\alpha)\}$

$\begin{cases} H_0 : \lambda = \lambda_0 \\ H_a : \lambda > \lambda_0 \end{cases}$, now $C_1 : \{\underset{\sim}{x} : \max_i (x_i) < c\}$

$$P(\max_i (x_i) < c) = P(x_i < c)^n = [\int_0^c \lambda e^{-\lambda x}]^n = [1 - e^{-\lambda c}]^n$$

$$P(\max_i (x_i) < c \mid H_0) = \alpha \Rightarrow [1 - e^{-\lambda_0 c}]^n = \alpha \Rightarrow e^{-\lambda_0 c} = 1 - \alpha^{1/n}$$

$$\Rightarrow e^{-c} = (1 - \alpha^{1/n})^{1/\lambda_0}$$

$\therefore$ Power function of this test $= P(\text{reject } H_0 \mid H_1) = [1 - e^{-\lambda_1 c}]^n$

$$= \{1 - [(1-\alpha^{1/n})^{1/\lambda_0}]^{\lambda_1}\}^n$$

$$= [1 - (1-\alpha^{1/n})^{\lambda_1/\lambda_0}]^n$$

So, $[1 - (1 - 0.05^{1/n})^2]^n \geq 0.8$

$\Rightarrow n = 38$ (try and error)

Yes, since the previous test is the UMP test.

(a). $\begin{cases} H_0: \lambda_1 = \cdots = \lambda_n \\ H_1: \lambda_i\text{'s are not all equal} \end{cases}$, $f_{X_i}(x_i;\theta) = \dfrac{\lambda_i^{x_i} e^{-\lambda_i}}{x_i!}$

$$L(\lambda) = f_{\underline{X}}(\underline{x};\lambda_1,\dots,\lambda_n) = \prod_{i=1}^{n} f_{X_i}(x_i;\lambda_i) = \frac{\prod_{i=1}^{n}(\lambda_i^{x_i})\, e^{-\sum \lambda_i}}{\prod_{i=1}^{n} x_i!} \quad \text{—— } (**)$$

Numerator: $\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda$

$$L(\lambda) = \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!} \quad \text{—— } (*)$$

$$\log L(\lambda) = \sum_{i=1}^{n} x_i \log\lambda - n\lambda - \log \prod_{i=1}^{n} x_i!$$

$$\frac{\partial \log L(\lambda)}{\partial\lambda} = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n$$

Set to 0 $\Rightarrow$ $\hat\lambda = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$

$\therefore \sup\{L(\lambda): \lambda\in\Theta_0\} = (\bar{x})^{n\bar{x}} e^{-n\bar{x}} \Big/ \prod_{i=1}^{n} x_i!$ (substitute $\bar{x}$ into $(*)$ for λ)

Denominator:

$$\log L(\lambda) = \sum_{i=1}^{n}(x_i \log\lambda_i) - \sum_{i=1}^{n}\lambda_i - \log\prod_{i=1}^{n} x_i!$$

$$\frac{\partial \log L(\lambda)}{\partial\lambda_i} = \frac{x_i}{\lambda_i} - 1 \quad \text{for } i = 1, 2, \dots, n$$

$$\frac{\partial \log L(\lambda)}{\partial\lambda_i} = 0 \Rightarrow \begin{cases} \frac{x_1}{\lambda_1} - 1 = 0 \\ \vdots \\ \frac{x_n}{\lambda_n} - 1 = 0 \end{cases} \Rightarrow \begin{cases} \hat\lambda_1 = x_1 \\ \vdots \\ \hat\lambda_n = x_n \end{cases}$$

$\therefore \sup\{L(\lambda): \lambda\in\Theta\} = \prod_{i=1}^{n} x_i^{x_i} e^{-\sum_{i=1}^{n} x_i} \Big/ \prod_{i=1}^{n} x_i!$ (substitute $\hat\lambda_i = x_i$ into $(**)$)

$$\therefore \lambda(\underline{x}) = \frac{(\bar{x})^{n\bar{x}} e^{-n\bar{x}}}{\prod_{i=1}^{n} x_i^{x_i}\, e^{-\sum x_i}} = \frac{(\bar{x})^{n\bar{x}}}{\prod_{i=1}^{n} x_i^{x_i}}$$

For large n, $-2\log\lambda(\underline{x}) \doteq \chi^2_{(r)}$ where $r = (n-1) - 0 = n-1$

$\therefore$ Approximate test is to reject H_0 if $-2\log\lambda(\underline{x}) \geq \chi^2_{(n-1)}(\alpha)$

$\Downarrow$

i.e. $-2\left(n\bar{x}\log\bar{x} - \sum_{i=1}^{n} x_i \log x_i\right) \geq \chi^2_{(n-1)}(\alpha)$

$$\begin{aligned} &\sum x_i \log x_i - n\bar{x}\log\bar{x} \\ &= \sum(\bar{x}+u_i)\log(\bar{x}+u_i) - n\bar{x}\log\bar{x} \\ &= \sum(\bar{x}+u_i)\left(\log\bar{x} + \frac{u_i}{\bar{x}}\right) - n\bar{x}\log\bar{x} \\ &= \log\bar{x}\sum u_i + \sum \bar{x}\frac{u_i}{\bar{x}} + \sum u_i\frac{u_i}{\bar{x}} \\ &= \frac{\sum u_i^2}{\bar{x}} = \frac{\sum(x_i-\bar{x})^2}{\bar{x}} \end{aligned}$$

5(b) $\begin{cases} H_0: \theta = 1 \\ H_1: \theta = 1.21 \end{cases}$

By N-P theorem, $C_1 = \left\{ \underset{\sim}{x} : \dfrac{f_{\underset{\sim}{X}}(\underset{\sim}{x}; \theta=1)}{f_{\underset{\sim}{X}}(\underset{\sim}{x}; \theta=1.21)} \le K \right\}$

Now, $\dfrac{f_{\underset{\sim}{X}}(\underset{\sim}{x}; \theta=1)}{f_{\underset{\sim}{X}}(\underset{\sim}{x}; \theta=1.21)} = \dfrac{\prod_{i=1}^{n} f_{X_i}(x_i; \theta=1)}{\prod_{i=1}^{n} f_{X_i}(x_i; \theta=1.21)} = \dfrac{\prod_{i=1}^{n} \dfrac{1^{x_i} e^{-1}}{x_i!}}{\prod_{i=1}^{n} \dfrac{1.21^{x_i} e^{-1.21}}{x_i!}} = \dfrac{1^{\sum_{i=1}^{n} x_i} e^{-n}}{1.21^{\sum_{i=1}^{n} x_i} e^{-1.21n}} \le K$

$\Rightarrow \quad (1.21)^{-\sum_{i=1}^{n} x_i} e^{0.21n} \le K$

$\Rightarrow \quad \left(-\sum_{i=1}^{n} x_i\right)\left[\log(1.21)\right](0.21)n \le K_1$

$\Rightarrow \quad \sum_{i=1}^{n} x_i \ge K_2$

$\therefore$ $C_1 = \{ \underset{\sim}{x} : \sum_{i=1}^{n} x_i \ge K \}$ is the UMP test. //

$\sum_{i=1}^{n} X_i \sim \text{Poisson}(n\theta)$,

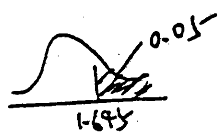
By CLT, $\sum_{i=1}^{n} X_i \approx N(n\theta, n\theta)$.

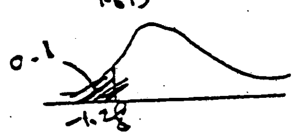
$\begin{cases} \alpha = 0.05 \Rightarrow & P\left(\sum_{i=1}^{n} X_i \ge K \mid \theta=1\right) = 0.05 \\ \beta \le 0.1 \Rightarrow & P\left(\sum_{i=1}^{n} X_i < K \mid \theta=1.21\right) \le 0.1 \end{cases}$

$\Rightarrow \begin{cases} P\left(Z \ge \dfrac{K-0.5-n\theta}{\sqrt{n\theta}} \,\middle|\, \theta=1\right) = 0.05 \\ P\left(Z < \dfrac{K-0.5-n\theta}{\sqrt{n\theta}} \,\middle|\, \theta=1.21\right) \le 0.1 \end{cases}$ (do correction !)

$\Rightarrow \begin{cases} P\left(Z \ge \dfrac{K-0.5-n}{\sqrt{n}}\right) = 0.05 \\ P\left(Z < \dfrac{K-0.5-1.21n}{\sqrt{1.21n}}\right) \le 0.1 \end{cases}$

$\Rightarrow \begin{cases} \dfrac{K-n-0.5}{\sqrt{n}} = 1.645 \\ \dfrac{K-1.21n-0.5}{\sqrt{1.21n}} \le -1.28 \end{cases}$





$\Rightarrow \begin{cases} K = n + 1.645\sqrt{n} + 0.5 \\ K \le -1.28\sqrt{1.21n} + 1.21n + 0.5 \end{cases}$

$\Rightarrow \quad \therefore \quad n + 1.645\sqrt{n} + 0.5 \le -1.28\sqrt{1.21n} + 1.21n + 0.5$

$\Rightarrow \quad 0.21n \ge 3.053\sqrt{n}$

$\Rightarrow \quad \sqrt{n} \ge 3.053/0.21 \quad \Rightarrow \quad n \ge 211.4$

$\therefore$ Smallest value of n is 212 //

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