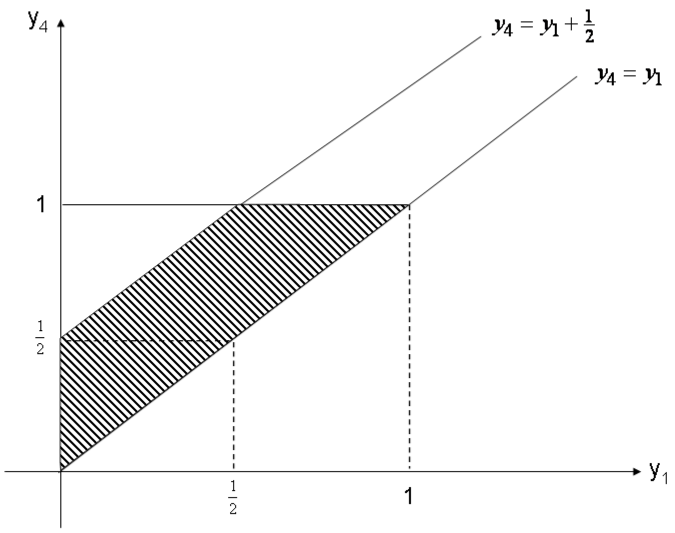
Solutions to Exercise 1

44. Consider $Y_1 \le Y_2 \le Y_3 \le Y_4$ be the order statistics of the random sample with increasing order, so

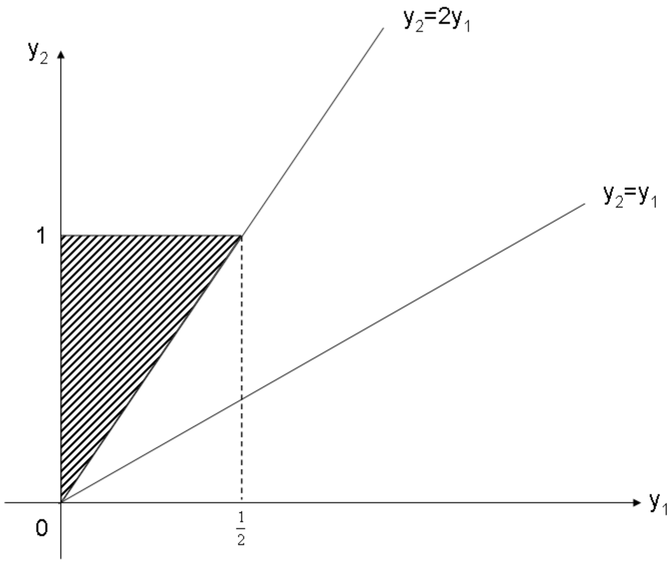
$$\begin{aligned}
f_{Y_1,Y_4}(y_1,y_4) &= \frac{4!}{(1-1)!(4-1-1)!(4-4)!}\cdot \\
&\quad \Big[F(y_1)\Big]^{1-1}\Big[F(y_4)-F(y_1)\Big]^{4-1-1}\Big[1-F(y_4)\Big]^{4-4} f(y_1)f(y_4) \\
&= 12(y_4-y_1)^2, \qquad 0<y_1<y_4<1
\end{aligned}$$

$$\begin{aligned}
P\left(Y_4-Y_1<\frac{1}{2}\right) &= \int_0^{\frac{1}{2}}\int_{y_1}^{\frac{1}{2}+y_1} 12(y_4-y_1)^2 dy_4\ dy_1 + \int_{\frac{1}{2}}^{1}\int_{y_1}^{1} 12(y_4-y_1)^2 dy_4\ dy_1 \\
&= \int_0^{\frac{1}{2}} 4(y_4-y_1)^3\Big|_{y_1}^{\frac{1}{2}+y_1}\ dy_1 + \int_{\frac{1}{2}}^{1} 4(y_4-y_1)^3\Big|_{y_1}^{1}\ dy_1 \\
&= 4\int_0^{\frac{1}{2}}\left(\frac{1}{2}\right)^3 dy_1 + 4\int_{\frac{1}{2}}^{1}(1-y_1)^3 dy_1 \\
&= 4\left(\frac{1}{8}y_1\Big|_0^{\frac{1}{2}}\right) + \left[-(1-y_1)^4\Big|_{\frac{1}{2}}^{1}\right] \\
&= \frac{5}{16}
\end{aligned}$$



45.

$$\begin{aligned}
P(Y_4 \geq 3) &= P(\max\{X_1, X_2, X_3, X_4\} \geq 3) \\
&= 1 - P(\max\{X_1, X_2, X_3, X_4\} < 3) \\
&= 1 - P(X_1 < 3, X_2 < 3, X_3 < 3, X_4 < 4) \\
&= 1 - \prod_{i=1}^{4} P(X_i < 3) \qquad \text{(by independent)} \\
&= 1 - \prod_{i=1}^{4} P(X < 3) \qquad \text{(by identically distributed)} \\
\text{Now } P(X < 3) &= P(X \leq 3) \qquad (X \text{ is a continuous r.v.}) \\
&= \int_0^3 f(x)dx \\
&= \int_0^3 e^{-x}dx \\
&= \Big[-e^{-x}\Big]_0^3 \\
&= 1 - e^{-3} \\
\therefore\ P(Y_4 \geq 3) &= 1 - (1 - e^{-3})^4 = 0.1848
\end{aligned}$$



46. Let $X_1, X_2, \ldots, X_5$ be a random sample of size 5 from a distribution having pdf $f(x)$

$$
\begin{aligned}
P(\min\{X_1, X_2, \ldots, X_5\} \le y_1) &= 1 - P(\min\{X_1, X_2, \ldots, X_5\} > y_1) \\
&= 1 - P(X_1 > y_1, X_2 > y_1, \ldots, X_5 > y_1) \\
&= 1 - \prod_{i=1}^{5} P(X_i > y_1) = 1 - \Big[P(X > y_1)\Big]^5 \qquad \text{(by iid)} \\
&= 1 - \Big[1 - P(X \le y_1))\Big]^5 = 1 - \left(1 - \frac{y_1}{6}\right)^5 \\
P(\min\{X_1, X_2, \ldots, X_5\} = y_1) &= P(\min\{X_1, X_2, \ldots, X_5\} \le y_1) - P(\min\{X_1, X_2, \ldots, X_5\} < y_1) \\
&= \Big[1 - (1 - \frac{y_1}{6})^5\Big] - P(\min\{X_1, X_2, \ldots, X_5\} \le y_1 - 1) \\
&= \Big[1 - (1 - \frac{y_1}{6})^5\Big] - \Big[1 - (1 - \frac{y_1 - 1}{6})^5\Big] \\
&= \left(\frac{7 - y_1}{6}\right)^5 - \left(\frac{6 - y_1}{6}\right)^5, \qquad y_1 = 1, 2, \ldots, 6
\end{aligned}
$$

which is the p.d.f. of the smallest item of a random sample of size 5.

47. Let X_1, X_2 be a random sample of size 2 from a distribution having pdf

$$
f(x) = \begin{cases} 2(1 - x), & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}
$$

Let $Y_1 \le Y_2$ be the corresponding order statistics.

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1, y_2) &= 2! f(y_1) f(y_2) \\
&= 2[2(1 - y_1)][2(1 - y_2)] \qquad\qquad \text{for } y_1 \le y_2 \\
&= 8(1 - y_1)(1 - y_2) \\
P(Y_2 \ge 2Y_1) &= \int_0^1 \int_0^{\frac{1}{2}y_2} f_{Y_1,Y_2}(y_1, y_2)\, dy_1 dy_2 \\
&= \int_0^1 \int_0^{\frac{1}{2}y_2} 8(1 - y_1)(1 - y_2)\, dy_1 dy_2 \\
&= -4 \int_0^1 \Big[(1 - y_1)^2 (1 - y_2)\Big]_0^{\frac{1}{2}y_2} dy_2 \\
&= -4 \int_0^1 \Big[(1 - \frac{1}{2}y_2)^2 (1 - y_2) - (1 - y_2)\Big] dy_2 \\
&= -4 \int_0^1 \left(1 - y_2 + \frac{1}{4}y_2^2 - 1\right)(1 - y_2)\, dy_2 \\
&= -4 \int_0^1 \left(-y_2 + \frac{1}{4}y_2^2 + y_2^2 - \frac{1}{4}y_2^3\right) dy_2 \\
&= -4 \int_0^1 \left(-y_2 + \frac{5}{4}y_2^2 - \frac{1}{4}y_2^3\right) dy_2 \\
&= \int_0^1 (4y_2 - 5y_2^2 + y_2^3)\, dy_2 \\
&= \left[2y_2^2 - \frac{5}{3}y_2^3 + \frac{y_2^4}{4}\right]_0^1 \\
&= 2 - \frac{5}{3} + \frac{1}{4} = \frac{7}{12}
\end{aligned}
$$

You can also write the integral as:

$$
P(Y_2 \ge 2Y_1) = \int_0^{\frac{1}{2}} \int_{2y_1}^1 f_{Y_1,Y_2}(y_1, y_2)\, dy_2 dy_1 = \ldots = \frac{7}{12}
$$

48.

$$\begin{cases} Z_1 &= Y_1/Y_2 \\ Z_2 &= Y_2/Y_3 \\ Z_3 &= Y_3 \end{cases} \Rightarrow \begin{cases} Y_1 &= Z_1Z_2Z_3 \\ Y_2 &= Z_2Z_3 \\ Y_3 &= Z_3 \end{cases}$$

$$\therefore\ f_{Z_1,Z_2,Z_3}(z_1,z_2,z_3) = f_{Y_1,Y_2,Y_3}(z_1z_2z_3, z_2z_3, z_3)\ |J| \qquad \text{for } 0 < z_1z_2z_3 \le z_2z_3 \le z_3 < 1$$

where

$$|J| = \begin{vmatrix} \frac{\partial y_1}{\partial z_1} & \frac{\partial y_1}{\partial z_2} & \frac{\partial y_1}{\partial z_3} \\ \frac{\partial y_2}{\partial z_1} & \frac{\partial y_2}{\partial z_2} & \frac{\partial y_2}{\partial z_3} \\ \frac{\partial y_3}{\partial z_1} & \frac{\partial y_3}{\partial z_2} & \frac{\partial y_3}{\partial z_3} \end{vmatrix} = \begin{vmatrix} z_2z_3 & z_1z_3 & z_1z_2 \\ 0 & z_3 & z_2 \\ 0 & 0 & 1 \end{vmatrix} = z_2z_3^2$$

Note that the domain $0 < z_1z_2z_3 \le z_2z_3 \le z_3 < 1$ is equivalent to $\begin{cases} 0 < z_1 \le 1 \\ 0 < z_2 \le 1 \\ 0 < z_3 \le 1 \end{cases}$

$$\begin{aligned} f_{Z_1,Z_2,Z_3}(z_1,z_2,z_3) &= \Big[3!f(y_1)f(y_2)f(y_3)\Big]\cdot(z_2z_3^2) \\ &= 6\times 2y_1\times 2y_2\times 2y_3\times z_2z_3^2 \\ &= 6\times 2(z_1z_2z_3)\times 2(z_2z_3)\times 2(z_3)\times z_2z_3^2 \\ &= 48z_1z_2^3z_3^5 \end{aligned}$$

$$\begin{aligned} f_{Z_1,Z_2}(z_1,z_2) &= \int_0^1 48z_1z_2^3z_3^5\ dz_3 \\ &= \Big[8z_1z_2^3z_3^6\Big]_0^1 \\ &= 8z_1z_2^3 \qquad \text{for } \begin{cases} 0 < z_1 \le 1 \\ 0 < z_2 \le 1 \end{cases} \end{aligned}$$

$$\begin{aligned} f_{Z_1}(z_1) &= \int_0^1 8z_1z_2^3\ dz_2 \\ &= \Big[2z_1z_2^4\Big]_0^1 \\ &= 2z_1, \qquad 0 < z_1 \le 1 \end{aligned}$$

$$\begin{aligned} f_{Z_2}(z_2) &= \int_0^1 8z_1z_2^3\ dz_1 \\ &= \Big[4z_1^2z_2^3\Big]_0^1 \\ &= 4z_2^3, \qquad 0 < z_2 \le 1 \end{aligned}$$

$$\begin{aligned} f_{Z_3}(z_3) &= f_{Y_3}(y_3) \\ &= \frac{3!}{(3-1)!(3-3)!}[F(z_3)]^{3-1}[1-F(z_3)]^{3-3}f(z_3) \\ &= 3(z_3^2)^2(2z_3) \\ &= 6z_3^5 \qquad 0 < z_3 \le 1 \end{aligned}$$

$$
\begin{aligned}
f_{Z_1}(z_1)f_{Z_2}(z_2)f_{Z_3}(z_3) &= (2z_1)(4z_2^3)(6z_3^5) \\
&= 48z_1z_2^3z_3^5 \\
&= f_{Z_1,Z_2,Z_3}(z_1,z_2,z_3) \qquad \text{for } \left\{ \begin{array}{ccccc} 0 & < & z_1 & \leq & 1 \\ 0 & < & z_2 & \leq & 1 \\ 0 & < & z_3 & \leq & 1 \end{array} \right.
\end{aligned}
$$

$\therefore\ Z_1, Z_2, Z_3$ are mutually independent.

49.

$$
\begin{aligned}
f_{Y_1,Y_3}(y_1,y_3) &= \frac{3!}{(1-1)!(3-1-1)!(3-3)!}[F(y_1)]^{1-1}[F(y_3)-F(y_1)]^{3-1-1}[1-F(y_3)]^{3-3}f(y_1)f(y_3) \\
&= 6(y_3-y_1) \qquad (\because F(y_1)=y_1, F(y_3)=y_3)
\end{aligned}
$$

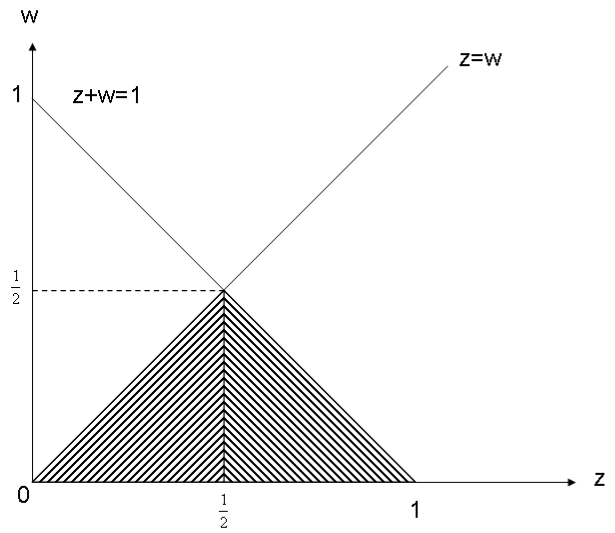
In order to find pdf of $Z = \frac{1}{2}(Y_1+Y_3)$, we let

$$
W = \frac{1}{2}(Y_3 - Y_1) \Rightarrow \left\{ \begin{array}{ccc} Y_1 & = & Z - W \\ Y_3 & = & Z + W \end{array} \right.
$$

$$
\therefore\ f_{Z,W}(z,w) = f_{Y_1,Y_3}(z-w, z+w)\cdot |J| \qquad \text{for } 0 < z-w \leq z+w \leq 1
$$

$$
\text{where } |J| = \left| \begin{array}{cc} \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \\ \frac{\partial y_2}{\partial z} & \frac{\partial y_2}{\partial w} \end{array} \right| = \left| \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right| = 2
$$

$$
\begin{aligned}
f_{Z,W}(z,w) &= 6[(z+w)-(z-w)]\cdot|2| \\
&= 24w \qquad \text{for } 0 < z-w \leq z+w \leq 1
\end{aligned}
$$



$$
\begin{aligned}
f_Z(z) &= \begin{cases} \displaystyle\int_0^z 24w\ dw & 0 < z < \frac{1}{2} \\ \displaystyle\int_0^{1-z} 24w\ dw & \frac{1}{2} < z < 1 \end{cases} \\
&= \begin{cases} \left[12w^2\right]_0^z & 0 < z < \frac{1}{2} \\ \left[12w^2\right]_0^{1-z} & \frac{1}{2} < z < 1 \end{cases} \\
&= \begin{cases} 12z^2 & 0 < z < \frac{1}{2} \\ 12(1-z)^2 & \frac{1}{2} < z < 1 \end{cases}
\end{aligned}
$$

50.

$$
\begin{aligned}
f_{Y_1}(y_1) &= 2(1-F(y_1))f(y_1) \qquad \text{where} \\
F(y_1) &= \int_{-\infty}^{y_1} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \\
f(y_1) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \qquad -\infty < y < \infty \\
\therefore\ E(Y_1) &= \int_{-\infty}^{\infty} y_1 f_{Y_1}(y_1)\ dy_1 \\
&= \int_{-\infty}^{\infty} y_1 \cdot 2(1-F(y_1))f(y_1)\ dy_1 \\
&= 2\int_{-\infty}^{\infty} y_1 f(y_1)\ dy_1 - \int_{-\infty}^{\infty} 2y_1 \int_{-\infty}^{y_1} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{y_1^2}{2\sigma^2}\right\}\ dy_1 \\
&= 0 - \int_{-\infty}^{\infty} \frac{1}{\pi\sigma^2} \int_{-\infty}^{y_1} y_1 \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \exp\left\{-\frac{y_1^2}{2\sigma^2}\right\}\ dx\ dy_1 \\
&= -\frac{1}{\pi\sigma^2} \int_{-\infty}^{\infty} \int_{x}^{\infty} y_1 \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \exp\left\{-\frac{y_1^2}{2\sigma^2}\right\}\ dy_1\ dx \\
&= -\frac{1}{\pi} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \left[\exp\left\{-\frac{y_1^2}{2\sigma^2}\right\}\right]_x^{\infty} dx \\
&= -\frac{1}{\pi} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2}{\sigma^2}\right\} dx \\
&= -\frac{1}{\pi} \cdot \sqrt{2\pi} \cdot \left(\frac{\sigma}{\sqrt{2}}\right) \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \cdot (\frac{\sigma}{\sqrt{2}})} \exp\left\{-\frac{x^2}{2(\frac{\sigma}{\sqrt{2}})^2}\right\} dx \\
&= -\frac{\sigma}{\sqrt{\pi}}
\end{aligned}
$$

51. $X_1, X_2, X_3 \sim N(6, 4)$ i.i.d

$$
\begin{aligned}
P(\max_i\{X_i\} > 8) &= 1 - P(\max_i\{X_i\} \le 8) \\
&= 1 - P(X_1 \le 8, X_2 \le 8, X_3 \le 8) \\
&= 1 - \prod_{i=1}^{3} P(X_i \le 8) \qquad\qquad (\because X_i \text{ are i.i.d}) \\
P(X_i \le 8) &= P\left(Z \le \frac{8-6}{\sqrt{4}}\right) \\
&= P(Z \le 1) = 0.8413
\end{aligned}
$$

$$
\begin{aligned}
P(\max_i\{X_i\} > 8) &= 1 - (0.8413)^3 \\
&= 0.4045
\end{aligned}
$$

52.

$$
f(x) = \frac{x+1}{2}, \qquad\qquad -1 < x < 1
$$

$$
\begin{aligned}
P(X > 0) &= \int_0^1 \frac{x+1}{2} dx \\
&= \frac{1}{2}\left[\frac{x^2}{2} + x\right]_0^1 \\
&= \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{3}{4}
\end{aligned}
$$

$$
P(\text{exactly four items exceed zero}) = \binom{5}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) = 0.3955
$$

53. Let X and Y be the horizontal and vertical distances of the landing pt of the arrow from the center.

$$
X \sim N(0,1), \quad Y \sim N(0,1)
$$

$$
\Rightarrow Z = X^2 + Y^2 \sim \chi^2(2), \qquad f(z) = \frac{e^{-z/2}}{2}, \qquad z > 0
$$

(i)

$$
\begin{aligned}
P(X^2 + Y^2 > 4^2) &= P(Z > 16) \\
&= 1 - \int_0^{16} \frac{e^{-z/2}}{2} dz \\
&= 1 - [1 - e^{-8}] \\
&= e^{-8} \\
&= 0.0003355
\end{aligned}
$$

(ii)

$$
\begin{aligned}
P(3^2 < X^2 + Y^2 < 4^2) &= P(9 < Z < 16) \\
&= \int_9^{16} \frac{e^{-z/2}}{2} dz \\
&= e^{-9/2} - e^{-8} \\
&= 0.01077
\end{aligned}
$$

(iii)

$$\begin{aligned} P(X^2+Y^2<1) &= P(Z<1) \\ &= \int_0^1 \frac{e^{-z/2}}{2} dz \\ &= 1-e^{-1/2} \\ &= 0.3935 \end{aligned}$$

P(an arrow shot by the archer will land in the second and third ring)
= 1- 0.0003355 - 0.01077 - 0.3935 = 0.5954
So,

$$\begin{aligned} P(\text{Robin qualifies}) &= (0.3935)^4 + \binom{4}{3}(0.3935)^3(0.5954) + \binom{4}{3}(0.3935)^3(0.01077) \\ &\quad + \binom{4}{3}(0.3935)^3(0.0003355) + \binom{4}{2}(0.3935)^2(0.5954)^2 \\ &= 0.5011 \end{aligned}$$

54.

$$\begin{aligned} X_1+X_2 &\sim N(1.6+1.3, 1+1.2+2(0.7)) \\ &= N(2.9, 3.6) \\ P(|X_1+X_2|<1) &= P\left(\frac{-1-2.9}{\sqrt{3.6}} < Z < \frac{1-2.9}{\sqrt{3.6}}\right) \\ &= P(-2.055 < Z < -1.001) \\ &= 0.1385 \end{aligned}$$