The Hong Kong University of Science & Technology

MATH243 – Statistical Inference

Final Examination – Fall 07/08

Answer ALL Questions **Date: 14 December 2007**

Time allowed: 3 hours

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1. Consider on systems with failure times $X_1,\cdots,X_n$ assumed to be independent and identically distribution, exp (θ), distributions, i.e., $f(x_i)=\frac{1}{\theta}e^{-x_i/\theta}$, $x_i>0$.

(a) **(1 mark)** Find the complete and sufficient statistic for θ.
(b) **(1 mark)** Find the maximum likelihood estimate of θ.
(c) **(1 mark)** Is the m.l.e. in (b) UMVUE? Why?
(d) **(2 marks)** Find the maximum likelihood estimate of the variance of the m.l.e. in (b).
(e) **(1 mark)** Find the maximum likelihood estimate of the probability $\Pr(X_1\ge 1)$ that one system with last at least a month.
(f) **(2 marks)** Find the Cramér-Rao Lower bound for variances of unbiased estimators of the probability $\Pr(X_1\ge 1)$.
(g) **(2 marks)** Find the UMVUE of the probability $\Pr(X_1\ge 1)$.
(h) **(2 marks)** For the following sets of 10 observations from this distribution: 0.0256, 0.3051, 0.0278, 0.8971, 0.0739, 0.3191, 0.7379, 0.3671, 0.9763, 0.0102, calculate the value of the method of moments estimate for $1/\theta$.

2. Let $X_1,\ldots,X_n$ be a sample from Uniform (θ_1, θ_2) where θ_1 and θ_2 are unknown. Note that

$$f_{Y_\alpha,Y_\beta}(x,y)=\frac{n!}{(\alpha-1)!(\beta-\alpha-1)!(n-\beta)!}[F(x)]^{\alpha-1}[F(y)-F(x)]^{\beta-\alpha-1}[1-F(y)]^{n-\beta}f(x)f(y).$$

(a) **(1 mark)** Find the MLE of θ_1 and θ_2.
(b) **(2 marks)** Find the distributions of $X_{\max}$ and $X_{\min}$, respectively.
(c) **(2 marks)** Find $E(X_{\max})$ and $E(X_{\min})$.
(d) **(2 marks)** Find the sufficient statistics for θ_1 and θ_2. Prove that they are complete.
(e) **(1 mark)** Hence or otherwise, find the UMVUE of $\frac{\theta_1+\theta_2}{2}$.

3. A sample of independent data, (X_1,Y_1), ..., (X_n,Y_n) may be expressed as $Y_i=\alpha+\beta X_i+\varepsilon_i$, where α and β are unknown parameters, X_i is fixed and $\varepsilon_i\sim N(0,\sigma^2)$ with known σ^2

(a) **(2 marks)** Find the distribution of Y_i.
(b) **(2 marks)** Find the maximum likelihood estimators for unknown parameters, α, β.
(c) **(4 marks)** Prove that the maximum likelihood estimators for α and β are unbiased.
(d) **(2 marks)** Find the complete and sufficient statistics for α and β.
(e) **(1 mark)** Are the maximum likelihood estimators for α and β the best estimators? Explain in details.
(f) **(3 marks)** Find the variance of the maximum likelihood estimator for β. What is its distribution?
(g) **(3 marks)** Hence or otherwise, construct a 95% confidence interval of the maximum likelihood estimator for β.

4. Three independent random samples, X_{ij} for $i = 1, 2, 3$, and $j = 1, ..., n$, are normally distributed with mean μ_i and variance σ^2. Let $\overline{X}_i = \sum_{j=1}^{n} X_{ij}/n$ and $S_i^2 = \sum_{j=1}^{n} (X_{ij} - \overline{X}_i)^2/(n-1)$ for $i = 1, 2, 3$. Consider the problem of testing $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_0$ versus H_1: at least two are not equal.

(a) **(2 marks)** Find the maximum likelihood estimators of μ_0 and σ^2 under Θ_0.

(b) **(2 marks)** Find the maximum likelihood estimators of μ_1, μ_2, μ_3 and σ^2 under Θ.

(c) **(3 marks)** Find the likelihood ratio and then derive the approximate large sample likelihood ratio test.

(d) **(2 marks)** Find the distribution of $\sum_{i=1}^{3}\sum_{j=1}^{n} (X_{ij} - \overline{X}_i)^2/\sigma^2$. Explain in details. Marks will be reduced without detailed explanation.

(e) **(4 marks)** Find the distribution of $n\sum_{i=1}^{3}(\overline{X}_i - \overline{\overline{X}})^2/\sigma^2$, where $\overline{\overline{X}} = \frac{1}{3n}\sum_{i=1}^{3}\sum_{j=1}^{n} X_{ij}$. Write down the steps in details. Explain in details. Marks will be reduced without detailed explanation.

(f) **(3 marks)** Are $\sum_{i=1}^{3}\sum_{j=1}^{n} (X_{ij} - \overline{X}_i)^2$ and $\sum_{i=1}^{3}(\overline{X}_i - \overline{\overline{X}})^2$ independent? Why? Then, find the distribution of $3n(n-1)\sum_{i=1}^{3}(\overline{X}_i - \overline{\overline{X}})^2 \Big/ 2\sum_{i=1}^{3}\sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)^2$. Explain in details. Marks will be reduced without detailed explanation.

(g) **(2 marks)** Construct the exact likelihood ratio test.

(h) **(5 marks)** Three catalysts are used in a chemical process. The following are yield data from the process:

Catalyst		
1	**2**	**3**
77.5	81.5	78.1
82.0	82.3	80.2
80.6	81.4	81.5
84.9	79.5	83.0
81.0	83.0	82.1

Use the tests derived in (c) & (g) to test $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_0$ at α =0.05. Are the conclusions different? Write down the test statistic and critical value for each test clearly.

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