MATH243 Statistical Inference

Exercise 2 (**Parameter Estimation I: Unbiasedness, Finding an estimator**)

1. Suppose we have a random sample $(X_1, X_2, ..., X_n)$ from a normal distribution $N(\theta, \sigma^2)$. Show that
$$\tilde{\theta}_1 = \frac{1}{n+1}\sum_{i=1}^{n} X_i$$
is a biased estimator for θ.

2. Consider a random sample $(X_1, ..., X_n)$ taken from a Weibull distribution with parameters $\alpha = \dfrac{1}{\theta}$, and β, where $\beta > 0$ is known. The p.d.f. is
$$f(x;\theta) = \frac{\beta}{\theta}x^{\beta-1} \exp\left(\frac{-x^{\beta}}{\theta}\right), \quad x \in (0, \infty)$$
   (a) Show that the maximum likelihood estimator for θ is given by
$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i^{\beta}$$
   (b) Show that $\hat{\theta}$ is unbiased and consistent for θ.

3. Let $X_1, X_2, ..., X_n$ be a random sample of size n from a geometric distribution for which p is the probability of success.
   (a) Use the method of moments to find a point estimate for p.
   (b) Explain intuitively why your estimate makes good sense.
   (c) Use the following data to give a point estimate of p :

3	34	7	4	19	2	1	19	43	2
22	4	19	11	7	1	2	21	15	16

4. Let $X_1, X_2, ..., X_n$ be a random sample from a uniform distribution on the interval $(\theta - 1, \theta + 1)$.
   (a) Find the method of moments estimator for θ.
   (b) Is your estimator in part (a) an unbiased estimator for θ?
   (c) Given the following $n = 5$ observations of X, give a point estimate of θ : 6.61, 7.70, 6.98, 8.36, 7.26.
   (d) The method of moments estimator actually has greater variance than the estimator $[min(X_i) + max(X_i)]/2$. Compute the value of this estimator for the $n = 5$ observations in (c).

5. Let $X_1, X_2, ..., X_n$ be a random sample of size n from $N(\mu, \sigma^2 = \theta), 0 < \theta < \infty$, where μ is known. Show that $Y = (1/n)\sum_{i=1}^{n}(X_i - \mu)^2$ is an unbiased estimator of θ.

6. During each lecture in a statistics class, let X equal the number of times that Professor Tanis collides with a computer table at the front of the classroom. Assume that the distribution of X is Poisson with mean λ.
   (a) Given n observations of X, find the method of moments estimate of λ.

(b) Give a point estimate of λ using the following 11 observations of X that were collected by Chris:

1 0 1 3 3 0 2 2 4 1 1

(c) Compare the values of $\bar{x}$ and s^2. Does this information support the assumption that X has a Poisson distribution?

7. An urn contains 64 balls of which n_1 are orange and n_2 are blue. A random sample of $r = 8$ balls is selected from the urn without replacement and X is equal to the number of orange balls in the sample. This experiment was repeated 30 times (the 8 balls being returned to the urn before each repetition) yielding the following data:

3	0	0	1	1	1	1	3	1	1	2	0	1	3	1
0	1	0	2	1	1	2	3	2	2	4	3	1	1	2

Using these data, guess the value of n_1 and give a reason for your guess.

8. A random sample $X_1, X_2, ..., X_n$ of size n is taken from $N(\mu, \sigma^2)$, where the variance $\theta = \sigma^2$ is such that $0 < \theta < \infty$ and μ is a known real number. Show that the maximum likelihood estimator for θ is $\hat{\theta} = (1/n)\Sigma(X_i - \mu)^2$.

9. Let $X_1, X_2, ..., X_n$ be a random sample from distributions with the following probability density functions. In each case find the maximum likelihood estimator $\hat{\theta}$.

(a) $f(x;\theta) = (1/\theta^2)xe^{-x/\theta},\ 0 < x < \infty,\ 0 < \theta < \infty.$

(b) $f(x;\theta) = (1/2\theta^3)x^2e^{-x/\theta},\ 0 < x < \infty,\ 0 < \theta < \infty.$

(c) $f(x;\theta) = (1/2)e^{-|x-\theta|},\ -\infty < x < \infty,\ -\infty < \theta < \infty.$

10. Let $f(x;\theta) = (1/\theta)x^{(1-\theta)/\theta},\ 0 < x < 1,\ 0 < \theta < \infty.$

(a) Show that the maximum likelihood estimator of θ is $\hat{\theta} = -(1/n)\sum_{i=1}^{n} \ln X_i$.

(b) Show that $E(\hat{\theta}) = \theta$ and thus $\hat{\theta}$ is an unbiased estimator of θ.

11. Show that the mean $\bar{X}$ of a random sample of size n from a distribution having p.d.f. $f(x;\theta) = (1/\theta)e^{-(x/\theta)},\ 0 < x < \infty,\ 0 < \theta < \infty$, zero elsewhere, is an unbiased statistic for θ and has variance θ^2/n.

12. Let $X_1, X_2, ..., X_n$ denote a random sample from a normal distribution with mean zero and variance $\theta,\ 0 < \theta < \infty$. Show that $\sum_{1}^{n} X_i^2/n$ is an unbiased statistic for θ and has variance $2\theta^2/n$.

13. Let Y_1 and Y_2 be two stochastically independent unbiased statistics for θ. Say the variance of Y_1 is twice the variance of Y_2. Find the constants k_1 and k_2 so that $k_1Y_1 + k_2Y_2$ is an unbiased statistic with smallest possible variance for such a linear combination.

14. If X is a random variable having the binomial distribution with the parameters n and θ, show that $n \cdot \frac{X}{n} \cdot \left(1 - \frac{X}{n}\right)$ is a biased estimator of the variance of X.

15. If X_1, X_2, and X_3 constitute a random sample of size $n = 3$ from a normal population with the mean μ and the variance σ^2, find the efficiency of $\frac{X_1 + 2X_2 + X_3}{4}$ relative to $\frac{X_1 + X_2 + X_3}{3}$.

16. If $\hat{\theta}_1 = \frac{X}{n}$, $\hat{\theta}_2 = \frac{X+1}{n+2}$, and $\hat{\theta}_3 = \frac{1}{3}$ are estimators of the parameter θ of a binomial population and $\theta = \frac{1}{2}$, for what values of n is

(a) the mean square error of $\hat{\theta}_2$ less than the variance of $\hat{\theta}_1$;

(b) the mean square error of $\hat{\theta}_3$ less than the variance of $\hat{\theta}_1$?

17. Given a random sample of size n from a Poisson population, use the method of moments to obtain an estimator for the parameter λ.

18. The radius of a circle is measured with an error of measurement which is distributed $N(0, \sigma^2)$, σ^2 unknown. Given n independent measurements of the radius, find an unbiased estimator of the area of the circle.

19. Observations $X_1, X_2, \ldots X_n$ are drawn from normal populations with the same mean μ but with different variances $\sigma_1^2, \sigma_2^2, \ldots \sigma_n^2$. Is it possible to estimate all the parameters? If we assume that the σ_i^2 are known, what is the maximum likelihood estimator of μ?

20. One observation, X, is taken from a $N(0, \sigma^2)$ population.

    (a) Find an unbiased estimator of σ^2.

    (b) Find the MLE of σ.

    (c) Discuss how the method of moments estimator of σ might be found.

Difficult questions:

21. Let $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $\theta \in \mathbf{\Omega} = \{\theta : 0 < \theta < \infty\}$. Let $X_1, X_2, ..., X_n$ denote a random sample of size n from this distribution.

    (a) Sketch the p.d.f. of X for (i) $\theta = 1/2$, (ii) $\theta = 1$, and (iii) $\theta = 2$.

    (b) Show that $\hat{\theta} = -n/\ln\prod_{i=1}^{n} X_i$ is the maximum likelihood estimator of θ.

    (c) For each of the following three sets of 10 observations, calculate the maximum likelihood estimate:

(i)	0.0256	0.3051	0.0278	0.8971	0.0739
	0.3191	0.7379	0.3671	0.9763	0.0102
(ii)	0.9960	0.3125	0.4374	0.7464	0.8278
	0.9518	0.9924	0.7112	0.2228	0.8609
(iii)	0.4698	0.3675	0.5991	0.9513	0.6049
	0.9917	0.1551	0.0710	0.2110	0.2154

22. Let $Y_1 < Y_2 < Y_3$ be the order statistics of a random sample of size 3 from the uniform distribution having p.d.f. $f(x;\theta) = 1/\theta$, $0 < x < \theta$, $0 < \theta < \infty$, zero elsewhere. Show that $4Y_1, 2Y_2$, and $\frac{4}{3}Y_3$ are all unbiased statistics for θ. Find the variance of each of these unbiased statistics.

23. If $\bar{X}_1$ is the mean of a random sample of size n from a normal population with the mean μ and the variance $\sigma_1^2, \bar{X}_2$ is the mean of a random sample of size n from a normal population with the mean μ and the variance σ_2^2, and the two samples are independent, show that

    (a) $\omega \cdot \bar{X}_1 + (1-\omega) \cdot \bar{X}_2$, where $0 \le \omega \le 1$, is an unbiased estimator of μ;

    (b) the variance of this estimator is a minimum when

$$\omega = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

    (c) Find the efficiency of the estimator of part (a) with $\omega = \frac{1}{2}$ relative to this estimator with

$$\omega = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

24. If $\bar{X}_1$ and $\bar{X}_2$ are the means of independent random samples of size n_1 and n_2 from a normal population with the mean μ and the variance σ^2, show that the variance of the unbiased estimator

$$\omega \cdot \bar{X}_1 + (1-\omega) \cdot \bar{X}_2$$

is a minimum when $\omega = \dfrac{n_1}{n_1+n_2}$. Find the efficiency of the estimator with $\omega = \frac{1}{2}$ relative to the estimator with $\omega = \dfrac{n_1}{n_1+n_2}$.

25. Let X and Y be independent exponential random variables, with

$$f(x|\lambda) = \frac{1}{\lambda} e^{-x/\lambda},\ x > 0, \quad f(y|\mu) = \frac{1}{\mu} e^{-y/\mu},\ y > 0.$$

We observe Z and W with

$$Z = \min(X, Y) \quad \text{and} \quad W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y \end{cases}.$$

Assume that $(Z_i, W_i), i = 1, \ldots, n$, are n iid observations. Find the MLEs of λ and μ.

26. Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x|\theta) = \theta\, x^{\theta-1}, \quad 0 \le x \le 1, \quad 0 < \theta < \infty.$$

(a) Find the MLE of θ, and show that its variance $\to 0$ as $n \to \infty$.

(b) Find the method of moments estimator of θ.

27. For the bivariate normal distribution, show that the MLEs for μ_X, μ_Y, σ_X^2, σ_Y^2 and ρ are the same as the method of moments estimators.