The Hong Kong University of Science & Technology

MATH3423 - Statistical Inference
Final Examination - Fall 2014/2015

Answer ALL Questions | **Date: 12 December 2014**

Full marks: 80 + 10 for Bonus | **Time Allowed: 3 hours**

- DO NOT open the exam paper until instructed to do so.
- It is a closed-book examination.
- Five questions are included in this paper.
- Give detailed explanation how to obtain the final answer. NO mark will be given if only the final answer is written down.
- Unless specified, numerical answers should be EITHER exact OR corrected to 6 decimal places.
- You may write on the both sides of the examination booklet.
- Cheating is a serious offense. Students caught cheating are subject to a zero score as well as additional penalties.

Name : ______________________

Student Number : ______________

Signature : ____________________

For marking use only:

Question No.	Marks	Out of
1		20
2		20
3		20
4		20
5		10

1. Let $X_1, ..., X_n$ be a random sample from the Bernoulli(θ), where θ is the unknown parameter.

   (a) (**2 marks**) Find the complete and sufficient statistic for θ. Find its distribution.

   (b) (**3 marks**) Find the UMVUE for θ^2.

   (c) (**3 marks**) Find the CRLB for θ^2. Is the variance of the UMVUE for θ^2 equal to its CRLB? Explain in details.

   (d) (**2 marks**) Find the limiting distribution of the maximum likelihood estimator for θ^2 as $n \to \infty$ by Delta method. What phenomenon do you observe?

   (e) (**5 marks**) Find the UMVUE of $P(X_1 + X_2 + X_3 = 1)$.

   (f) (**5 marks**) Find the maximum likelihood estimator for the variance of $\sum X_i$, i.e., $n\theta(1-\theta)$. Is it unbiased? Hence or otherwise, find the UMVUE for the variance of of $\sum X_i$.

2. Let $X_1, ..., X_n$ be a r.s. from the continuous uniform distribution in the interval $(\theta, 2\theta)$, $\theta \in (0, \infty)$.

   Hint:

$$f(y_1, y_n) = n(n-1)(y_n - y_1)^{n-2}/\theta^n \qquad \theta \leq y_1 \leq y_n \leq 2\theta$$

   and

$$Cov(Y_1, Y_n) = \frac{\theta^2}{(n+1)^2(n+2)} .$$

   (a) (**2 marks**) Find the method of moments estimator, $\tilde{\theta}$, for θ. Is it unbiased? Hence or otherwise, find an unbiased estimator of θ as a function of $\tilde{\theta}$. What is its corresponding variance?

   (b) (**3 marks**) Find $E(Y_1)$, where $Y_1 = \min(X_1, \ldots, X_n)$. Hence or otherwise, find an unbiased estimator of θ as a function of Y_1.

   (c) (**3 marks**) Find $E(Y_n)$, where $Y_n = \max(X_1, \ldots, X_n)$. Hence or otherwise, find an unbiased estimator of θ as a function of Y_n.

   (d) (**9 marks**) Define the unbiased estimators of θ in parts (b) and (c) as U_a and U_b, respectively. Find a constant k so that the unbiased estimator, $kU_a + (1-k)U_b$, has the smallest variance. What is the variance of this unbiased estimator?

   (e) (**3 marks**) Does the UMVUE for θ exist? If yes, find it; if no, explain in details.

3. Individuals were classified according to gender and according to whether or not they were color-blind as follows:

	Male	Female
Normal	x_{11}	x_{12}
Color-blind	x_{21}	x_{22}

Let $X = (X_{11}, X_{12}, X_{21}, X_{22}) \sim$ multinomial $(n, P_{11}, P_{12}, P_{21}, P_{22})$.

(a) Test the hypothesis $H_0 : P_{11} = \frac{p}{2}, P_{12} = \frac{p^2}{2} + pq, P_{21} = \frac{q}{2}, P_{22} = \frac{q^2}{2}$, where $q = 1 - p$, against $H_1 : (P_{11}, P_{12}, P_{21}, P_{22})$ takes any other value in $[0, 1]^4$ at the level of significance α.

i. (**4 marks**) Find the likelihood ratio statistic and then derive the approximate large sample likelihood ratio test.

ii. (**2 marks**) Write down the Pearson's goodness of fit test statistic and state the critical region for this test.

(b) Suppose $x_{11} = 442, x_{12} = 514, x_{21} = 38, x_{22} = 6$. Perform the following tests at $\alpha = 0.05$. State clearly the hypothesis statements, value of test statistic, critical value and your conclusion for each test.

i. (**6 marks**) Test whether the null hypothesis $H_0 : P_{11} = \frac{p}{2}, P_{12} = \frac{p^2}{2} + pq, P_{21} = \frac{q}{2}, P_{22} = \frac{q^2}{2}$ is true by the two tests derived above.

ii. (**4 marks**) Test the hypothesis that color blindness is independent of gender. No need to make the Yates's Correction.

iii. (**4 marks**) Test whether the probabilities of color-blind individuals for male and female are equal by z test. No need to make the continuity correction.

4. If $X_1, X_2, \ldots, X_n$ are independently and normally distributed with the same unknown mean μ but different known variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$.

   (a) **(4 marks)** Find the maximum likelihood estimator of μ. Hence, find its distribution.

   (b) **(6 marks)** Construct the UMP test for testing $H_0 : \mu \leq \mu_0$ against $H_1 : \mu > \mu_0$ at a significance level of α.

   (c) **(2 marks)** Based on the test in part (b), calculate the power of test at $\mu_1 = 1$, where $\mu_1 \in \Theta_1$, when $\alpha = 0.05$, $\mu_0 = 0$, $n = 10$, $\sigma_1^2 = \ldots = \sigma_5^2 = 1$ and $\sigma_6^2 = \ldots = \sigma_{10}^2 = 2$. Round the value to two decimal places before finding the probability.

   (d) Assuming that $\mu = 0$ and all σ_j^2, for $j = 1, \ldots, n$, are equal to σ^2 but unknown, consider another hypothesis testing problem with $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$ at the level of significance α.

      i. (**4 marks**) Find the expression of the likelihood ratio statistic.

      ii. (**4 marks**) Hence, derive the exact likelihood ratio test at the significance level of α.

5. (**Bonus : 10 marks**) Consider a random sample of a fixed size n, $\{X_1, \ldots, X_n\}$, from a p.m.f. given by

$$p_{-1} = P(X_i = -1) = \frac{1-\theta}{2}, \quad p_0 = P(X_i = 0) = \frac{1}{2}, \quad p_1 = P(X_i = 1) = \frac{\theta}{2},$$

where $0 \leq \theta \leq 1$. Let $n_{-1} = \sum_{i=1}^n I_{\{X_i=-1\}}$, $n_0 = \sum_{i=1}^n I_{\{X_i=0\}}$, and $n_1 = \sum_{i=1}^n I_{\{X_i=1\}}$. Given that $(n_{-1}, n_0, n_1) \sim$ multinomial(n, p_{-1}, p_0, p_1).

Find the maximum likelihood estimator, $\hat{\theta}$, for θ. Find $E(\hat{\theta})$. Hence or otherwise, find an unbiased estimator for θ

********** **END** **********