Solutions to Exercise 3

1.

$$
\begin{aligned}
f(X;\theta) &= \theta^X(1-\theta)^{1-X} \\
L &= f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta) \\
&= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} \\
&= \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} \\
\log L &= (\sum_{i=1}^{n} x_i)\log\theta + (n-\sum_{i=1}^{n} x_i)\log(1-\theta) \\
\frac{\partial}{\partial\theta}\log L &= \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}(n-\sum_{i=1}^{n} x_i) \\
\frac{\partial}{\partial\theta}\log L = 0 \quad &\Rightarrow \quad \frac{1}{\theta}\sum_{i=1}^{n} x_i = \frac{1}{1-\theta}(n-\sum_{i=1}^{n} x_i) \\
&\Rightarrow \quad (1-\theta)\sum_{i=1}^{n} x_i = (n-\sum_{i=1}^{n} x_i)\theta \\
&\Rightarrow \quad \sum_{i=1}^{n} x_i = n\theta \\
&\Rightarrow \quad \hat{\theta} = \frac{1}{\sum_{i=1}^{n} x_i} \qquad \text{which is MLE for } \theta
\end{aligned}
$$

$$
\begin{aligned}
\log f(X;\theta) &= X\log\theta + (1-X)\log(1-\theta) \\
\frac{\partial}{\partial\theta}\log f(X;\theta) &= \frac{X}{\theta} - \frac{1-X}{1-\theta} \\
\frac{\partial^2}{\partial\theta^2}\log f(X;\theta) &= \frac{-X}{\theta^2} - \frac{1-X}{(1-\theta)^2} \\
E\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right] &= E\left[\frac{-X}{\theta^2} - \frac{1-X}{(1-\theta)^2}\right] \\
&= \frac{-}{\theta^2}E[X] - \frac{1}{(1-\theta)^2}E[1-X] \\
&= \frac{-}{\theta^2}X\theta - \frac{1}{(1-\theta)^2}(1-\theta) \\
&= \frac{-1}{\theta} - \frac{1}{1-\theta} \\
&= -\frac{(1-\theta)+\theta}{\theta(1-\theta)} \\
&= -\frac{1}{\theta(1-\theta)}
\end{aligned}
$$

$$
\therefore \text{ The CRLB } = \frac{1}{-nE\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right]} = \frac{\theta(1-\theta)}{n}
$$

Since

$$\begin{aligned} Var(\hat{\theta}) &= Var(\frac{1}{n}\sum_{i=1}^{n} X_i) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) \\ &= \frac{1}{n^2}\cdot n(\theta)(1-\theta) \\ &= \frac{\theta(1-\theta)}{n} \\ &= \text{CRLB} \end{aligned}$$

$\therefore\ \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a <u>fully efficient</u> estimator (UMVUE) for θ.

2.

$$\begin{aligned} f(X;\theta) &= \frac{1}{\theta}\exp(\frac{-X}{\theta}) \\ \log f(X;\theta) &= \log(\frac{1}{\theta}) - \frac{X}{\theta} \\ \frac{\partial^2}{\partial\theta^2}\log f(X;\theta) &= \frac{-1}{\theta} + \frac{X}{\theta^2} \\ \frac{\partial^2}{\partial\theta^2}\log f(X;\theta) &= \frac{1}{\theta^2} - \frac{2X}{\theta^3} \\ E\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right] &= \frac{1}{\theta^2} - \frac{2}{\theta^3}E(X) \\ &= \frac{1}{\theta^2} - \frac{2}{\theta^3}\cdot\theta \\ &= -\frac{1}{\theta^2} \end{aligned}$$

$$\therefore \text{The CRLB } = \frac{1}{-nE\left[\frac{\partial^2}{\partial\theta^2}\log Lf(X;\theta)\right]} = \frac{\theta^2}{n}$$

$$\because\ Var(\bar{X}) = Var(\frac{1}{n}\sum_{i=1}^{n} X_i) = \frac{Var(X)}{n} = \frac{\theta^2}{n} = \text{CRLB}$$

$\therefore\ \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a <u>fully efficient</u> estimator (UMVUE) for θ.

3. (a)

$$\begin{aligned} X &\sim Poisson(\theta) \\ f_X(x;\theta) &= \theta^x e^{-\theta}/x! \\ &= \exp\{x\log\theta - \log(x!) - \theta\} \\ &= \exp\{-\theta - \log(x!) + x\log\theta\} \end{aligned}$$

$$\therefore\ a(\theta) = -\theta,\ b(X) = \log(X!),\ c(\theta) = \log\theta,\ d(X) = X$$

$\therefore\ Poisson(\theta)$ belongs to exponential family.

(b)

$$\begin{aligned}
X &\sim Bin(n,\theta) \quad (n \text{ is known}) \\
f_X(x;\theta) &= \begin{pmatrix} n \\ x \end{pmatrix} \theta^n (1-\theta)^{n-x} \\
&= \exp\left\{\log\begin{pmatrix} n \\ x \end{pmatrix} + x\log\theta + (n-x)\log(1-\theta)\right\} \\
&= \exp\left\{n\log(1-\theta) + \log\begin{pmatrix} n \\ x \end{pmatrix} + x\log(\frac{\theta}{1-\theta})\right\}
\end{aligned}$$

$$\therefore\ a(\theta) = n\log(1-\theta),\ b(X) = \log\begin{pmatrix} n \\ X \end{pmatrix},\ c(\theta) = \log(\frac{\theta}{1-\theta}),\ d(X) = X$$

$\therefore$ $Bin(n,\theta)$ belongs to exponential family.

(c) Note: X here means the trial number on which the rth success occurs.

$$\begin{aligned}
X &\sim Neg.Bin.(r,\theta) \quad (r \text{ is known}) \\
f_X(x;\theta) &= \begin{pmatrix} x-1 \\ r-1 \end{pmatrix} \theta^r (1-\theta)^{x-r} \\
&= \exp\left\{\log\begin{pmatrix} x-1 \\ r-1 \end{pmatrix} + r\log\theta + (x-r)\log(1-\theta)\right\} \\
&= \exp\left\{r\log(\frac{\theta}{1-\theta}) + \log\begin{pmatrix} x-1 \\ r-1 \end{pmatrix} + +x\log(1-\theta)\right\}
\end{aligned}$$

$$\therefore\ a(\theta) = r\log(\frac{\theta}{1-\theta}),\ b(X) = \log\begin{pmatrix} X-1 \\ r-1 \end{pmatrix},\ c(\theta) = \log(1-\theta),\ d(X) = X$$

$\therefore$ $Neg.Bin.(r,\theta)$ belongs to exponential family.

4. (a)

$$\begin{aligned}
X &\sim gamma(k,\theta) \quad (k > 0 \text{ is known}) \\
f_X(X;\theta) &= \frac{x^{k-1}e^{-x\theta}}{\Gamma(k)\theta^{-k}} \\
&= \exp\{(k-1)\log x - x\theta - \log[\Gamma(k)] + k\log\theta\} \\
&= \exp\{k\log\theta - \log[\Gamma(k)] + (k-1)\log x - x\theta\}
\end{aligned}$$

$$\therefore\ a(\theta) = k\log\theta,\ b(X) = -\log[\Gamma(k)] + (k-1)\log X,\ c(\theta) = -\theta,\ d(X) = X$$

$\therefore$ $Gamma(k,\theta), k > 0$ belongs to exponential family.

(b)

$$\begin{aligned}
X &\sim N(\theta,1) \\
f_X(x;\theta) &= \frac{1}{\sqrt{2\pi}\cdot 1}\exp\left\{\frac{-1}{2(1)}(x-\theta)^2\right\} \\
&= \exp\left\{\frac{-1}{2}\log(2\pi) - \frac{1}{2}(x-\theta)^2\right\} \\
&= \exp\left\{\frac{-1}{2}\log(2\pi) - \frac{1}{2}x^2 + \theta x - \frac{1}{2}\theta^2\right\} \\
&= \exp\left\{-\frac{1}{2}\theta^2 - \frac{1}{2}(x^2 + \log 2\pi) + \theta x\right\}
\end{aligned}$$

$$\therefore\ a(\theta) = -\frac{1}{2}\theta^2,\ b(X) = -\frac{1}{2}(X^2 + \log 2\pi),\ c(\theta) = \theta,\ d(X) = X$$

$\therefore$ $N(\theta, 1)$ belongs to exponential family.

(c)

$$\begin{aligned} X &\sim N(0, \theta) \\ f_X(x;\theta) &= \frac{1}{\sqrt{2\pi\theta}} \exp\left\{\frac{-1}{2\theta}(x-0)^2\right\} \\ &= \exp\left\{\frac{-1}{2}\log(2\pi\theta) - \frac{1}{2\theta}x^2\right\} \end{aligned}$$

$$\therefore\ a(\theta) = -\frac{1}{2}\log(2\pi\theta),\ b(X) = 0,\ c(\theta) = -\frac{1}{2\theta},\ d(X) = X^2$$

$\therefore$ $N(0, \theta)$ belongs to exponential family.

5.

$$\begin{aligned} f_X(x;\theta) &= (1-\theta)^x\theta \\ &= \exp\{x\log(1-\theta) + \log\theta\} \\ &= \exp\{\log\theta + x\log(1-\theta)\} \end{aligned}$$

$$\therefore\ a(\theta) = \log\theta,\ b(X) = 0,\ c(\theta) = \log(1-\theta),\ d(X) = X$$

$\therefore$ *Geometric*(θ) belongs to exponential family, and $\sum_{i=1}^n d(X_i) = \sum_{i=1}^n X_i$ is a sufficient statistic for θ.

6.

$$\begin{aligned} f_X(x;\theta) &= \frac{1}{\theta}\exp(-\frac{x}{\theta}), \quad 0 < x < \infty, 0 < \theta < \infty \\ &= \exp\left\{-\log\theta - \frac{x}{\theta}\right\} \end{aligned}$$

$$\therefore\ a(\theta) = -\log\theta,\ b(X) = 0,\ c(\theta) = \frac{-1}{\theta},\ d(X) = X$$

$\therefore$ $Exp(\frac{1}{\theta}) = Gamma(1, \frac{1}{\theta})$ belongs to exponential family, and $\sum_{i=1}^n d(X_i) = \sum_{i=1}^n X_i$ is a sufficient statistic for θ.

7.

$$\begin{aligned} f_X(x;\theta) &= \frac{1}{\sqrt{2\pi\theta}} \exp\left\{-\frac{1}{2\theta}(x-0)^2\right\} \\ &= \exp\left\{\frac{-1}{2}\log(2\pi\theta) - \frac{1}{2\theta}x^2\right\} \\ &= \exp\{a(\theta) + b(x) + c(\theta)d(x)\} \end{aligned}$$

$$\therefore\ a(\theta) = -\frac{1}{2}\log(2\pi\theta),\ b(X) = 0,\ c(\theta) = -\frac{1}{2\theta},\ d(X) = X^2$$

$\therefore$ $N(0, \theta)$ belongs to exponential family, and $\sum_{i=1}^n d(X_i) = \sum_{i=1}^n X_i^2$ is a sufficient statistic for θ.

8.

$$
\begin{aligned}
f_{X_i}(x_i;\theta) &= \frac{1}{\theta} I_{[0,\theta]}(x_i) \\
f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta) &= \prod_{i=1}^{n} f_{X_i}(x_i;\theta) \\
&= \prod_{i=1}^{n} \frac{1}{\theta} I_{[0,\theta]}(x_i) \\
&= (\frac{1}{\theta})^n \prod_{i=1}^{n} I_{[0,\theta]}(x_i) \\
&= (\frac{1}{\theta})^n I_{[0,Y_n]}(y_1)\ I_{[Y_1,\theta]}(y_n) \cdot 1
\end{aligned}
$$

$$
\because\ g = (\frac{1}{\theta})^n I_{[0,Y_n]}(y_1)\ I_{[Y_1,\theta]}(y_n), \quad h = 1
$$

$\therefore$ By factorization theorem, $Y_n = \max(X_1, \ldots, X_n)$ is sufficient for θ. (refer to example 4 in Ch-2 P.14).

9. By Q.5(a), $d(X) = X$, $\sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ

10. (a) $X_1, \ldots, X_n \sim$ *iid Bernoulli*$(\theta) \Rightarrow E(X_k) = \theta, k = 1, 2, \ldots, n$
$\therefore$ X_k is an unbiased estimator for θ.
In order to show that X_k is NOT sufficient statistics for θ, we can just find another statistic $T = t(X_1, \ldots, X_n)$ such that $P(T|X_k = x_k)$ depends on θ.
Now, let $T = X_i, i \neq k$.

$$
\begin{aligned}
\therefore\quad & P(X_i = x_i | X_k = x_k) \\
= \quad & \frac{P(X_i = x_i, X_k = x_k)}{P(X_k = x_k)} \\
= \quad & \frac{P(X_i = x_i)P(X_k = x_k)}{P(X_k = x_k)} \\
= \quad & P(X_i = x_i) \\
= \quad & \theta^{x_i}(1-\theta)^{1-x_i} \qquad \text{which depends on } \theta
\end{aligned}
$$

$$
\therefore\ X_k \text{ is NOT sufficient statistic for } \theta
$$

(b)

$$
\begin{aligned}
f_X(x;\theta) &= \theta^x(1-\theta)^{1-x} \\
&= \exp\{x\log\theta + (1-x)\log(1-\theta)\} \\
&= \exp\left\{\log(1-\theta) + x\log(\frac{\theta}{1-\theta})\right\}
\end{aligned}
$$

$$
\therefore\ a(\theta) = \log(1-\theta),\ b(X) = 0,\ c(\theta) = \log(\frac{\theta}{1-\theta}),\ d(X) = X
$$

$\therefore$ *Bernoulli*(θ) belongs to exponential family, and $\sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

(c) Now, we have X_1 is unbiased for θ and $\sum_{i=1}^n X_i$ is sufficient statistic for θ,
$\therefore$ by Rao-Blackwell theorem, $T = E(X_1|\sum_{i=1}^n X_i)$ is an improved unbiased estimator for θ.

In order to calculate T, we need the distribution of $X_1|\sum_{i=1}^n X_i = r$

$$\begin{aligned}
\Rightarrow \quad & P(X_1 = 1|\sum_{i=1}^n X_i = r) \\
= \quad & \frac{P(X_1 = 1, \sum_{i=1}^n X_i = r)}{P(\sum_{i=1}^n X_i = r)} \\
= \quad & \frac{P(X_1 = 1, \sum_{i=2}^n X_i = r-1)}{P(\sum_{i=1}^n X_i = r)} \\
= \quad & \frac{P(X_1 = 1)\cdot P(\sum_{i=2}^n X_i = r-1)}{P(\sum_{i=1}^n X_i = r)} \qquad \because X_1, \sum_{i=2}^n X_i \text{ are independent} \\
= \quad & \frac{\theta \binom{n-1}{r-1} \theta^{r-1}(1-\theta)^{(n-1)-(r-1)}}{\binom{n}{r} \theta^{r-1}(1-\theta)^{n-r}} \\
= \quad & \frac{\binom{n-1}{r-1}}{\binom{n}{r}} \\
= \quad & \frac{(n-1)!}{(r-1)!(n-r)!} \times \frac{r!(n-r)!}{n!} \\
= \quad & \frac{r}{n}
\end{aligned}$$

$$\therefore\ P(X_1 = 0|\sum_{i=1}^n X_i = r) = 1 - \frac{r}{n}$$

$$\therefore\ E(X_1|\sum_{i=1}^n X_i = r) = 1\cdot P(X_1 = 1|\sum_{i=1}^n X_i = r) + 0\cdot P(X_1 = 0|\sum_{i=1}^n X_i = r) = \frac{r}{n}$$

$$T = E(X_1|\sum_{i=1}^n X_i) = \frac{1}{n}\sum_{i=1}^n X_i = \bar{X} \text{ is an improved unbiased estimator for } \theta.$$

The alternative method to get an improved estimate is as follows. Let $r = \sum_{i=1}^n x_i$, which is sufficient and complete statistic.

$$\begin{aligned}
\sum_{i=1}^n h(r)\binom{n}{r}\theta^r(1-\theta)^{n-r} &= \theta \\
\sum_{i=1}^n h(r)\binom{n}{r}\theta^{r-1}(1-\theta)^{n-r} &= 1 \\
\sum_{i=1}^n h(r)\frac{n}{r}\binom{n-1}{r-1}\theta^{r-1}(1-\theta)^{(n-1)-(r-1)} &= 1 \\
\Rightarrow \quad h(r)\frac{n}{r} &= 1 \\
\Rightarrow \quad h(r) &= \frac{r}{n}
\end{aligned}$$

11. (a) $E(X_1) = \theta$ and $E(X_2) = \theta$
$\therefore$ X_1 and X_2 are unbiased estimators for θ.

$$\begin{aligned} f_{X_2|X_1}(x_2|x_1) &= \frac{f_{X_2,X_1}(x_2,x_1)}{f_{X_1}(x_1)} \\ &= \frac{f_{X_2}(x_2)f_{X_1}(x_1)}{f_{X_1}(x_1)} \\ &= f_{X_2}(x_2) \\ &= \frac{1}{\theta}\exp\{-\frac{1}{\theta}x_2\} \quad \text{which depends on } \theta \end{aligned}$$

$\therefore$ X_1 is not sufficient statistic for θ.

(Care that X_1, X_2 are now continuous !!!)

Similarly, $f_{X_2|X_1}(x_2|x_1) = f_{X_1}(x_1) = \frac{1}{\theta}\exp\{-\frac{1}{\theta}x_1\}$ which depends on θ,
$\therefore$ X_2 is also not sufficient statistic for θ.

(b)

$$\begin{aligned} f_X(x;\theta) &= \frac{1}{\theta}\exp\{-\frac{x}{\theta}\}, \quad x \geq 0 \\ &= \exp\{-\log\theta - \frac{x}{\theta}\} \end{aligned}$$

$$a(\theta) = -\log\theta, b(X) = 0, c(\theta) = -\frac{1}{\theta}, d(X) = X$$

$\therefore$ Exponential distribution with mean θ belongs to exponential family.
$\therefore \sum_{i=1}^{2} d(X_i) = \sum_{i=1}^{2} X_i = X_1 + X_2$ is sufficient for θ.

(c)

$$\begin{aligned} f_{X_1|X_1+X_2}(x_1|r) &= \frac{f_{X_1,X_1+X_2}(x_1,r)}{f_{X_1+X_2}(r)}, \quad 0 \leq x_1 \leq r \\ &= \frac{f_{X_1,X_2}(x_1, r-x_1)}{f_{X_1+X_2}(r)} \\ & \quad (\because\ X_1 \sim exp(\theta), X_2 \sim exp(\theta), X_1+X_2 \sim gamma(2,\theta)) \\ &= \frac{\frac{1}{\theta}\exp(-\frac{1}{\theta}x_1)\cdot\frac{1}{\theta}\exp(-\frac{1}{\theta}(r-x_1))}{\frac{1}{\Gamma(2)}\theta^{-2}\cdot r^{2-1}\exp(-\frac{r}{\theta})} \\ &= \frac{\exp(-\frac{1}{\theta}r)}{r\exp(-\frac{r}{\theta})} \qquad \because\ \Gamma(2) = 1 \\ &= \frac{1}{r} \end{aligned}$$

$$\therefore\ E(X_1|X_1+X_2=r) = \int_0^r x_1 f_{X_1|X_1+X_2}(x_1|r)\, dx_1 = \int_0^r x_1\cdot\frac{1}{r}\, dx_1 = \left[\frac{x_1^2}{2r}\right]_0^r = \frac{r}{2}$$

$\therefore$ By Rao-Blackwell theorem, the desired estimator is $\frac{r}{2} = \frac{1}{2}(X_1 + X_2)$.

12. (a)

$$\begin{matrix} Y_1 = X_1 + X_2 \\ Y_2 = X_2 \end{matrix} \Rightarrow \begin{matrix} X_1 = Y_1 - Y_2 \\ X_2 = Y_2 \end{matrix}$$

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}(x_1,x_2)\begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} \\
&= f_{X_1}(x_1)f_{X_2}(x_2)\cdot \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \\
&= \frac{1}{\theta}\exp(-\frac{x_1}{\theta})\cdot\frac{1}{\theta}\exp(-\frac{x_2}{\theta}) \\
&= \frac{1}{\theta}\exp(-\frac{y_1-y_2}{\theta})\cdot\frac{1}{\theta}\exp(-\frac{y_2}{\theta}) \\
&= \frac{1}{\theta^2}\exp(-\frac{y_1}{\theta}), \qquad 0<y_2<y_1<\infty.
\end{aligned}
$$

(b)

$$
\because\ Y_2 = X_2, \quad \therefore\ f_{Y_2}(y_2) = f_{X_2}(x_2) = \exp(-\frac{x_2}{\theta})
$$

$$
\begin{aligned}
E(Y_2) &= \int_0^\infty y_2\frac{1}{\theta}\exp\left(-\frac{y_2}{\theta}\right)\,dy_2 \\
&= \left[-y_2\exp\left(\frac{-y_2}{\theta}\right)\right]_0^\infty + \int_0^\infty \exp\left(\frac{-y_2}{\theta}\right)\,dy_2, \qquad \text{by parts} \\
&= 0+\left[-\theta\exp\left(-\frac{y_2}{\theta}\right)\right]_0^\infty \\
&= \theta
\end{aligned}
$$

$\therefore$ Y_2 is an unbiased estimator for θ.

$$
\begin{aligned}
E(Y_2^2) &= \int_0^\infty y_2^2\frac{1}{\theta}\exp\left(-\frac{y_2}{\theta}\right)\,dy_2 \\
&= \left[-y_2^2\exp\left(-\frac{y_2}{\theta}\right)\right]_0^\infty + \int_0^\infty 2y_2\exp\left(-\frac{y_2}{\theta}\right)\,dy_2, \qquad \text{by parts} \\
&= 0+2\theta\int_0^\infty y_2\frac{1}{\theta}\exp\left(-\frac{y_2}{\theta}\right)\,dy_2 \\
&= 2\theta\cdot E(Y_2) \\
&= 2\theta\cdot\theta \\
&= 2\theta^2 \\
\therefore\ Var(Y_2) &= E(Y_2^2)-[E(Y_2)]^2 \\
&= 2\theta^2-\theta^2 \\
&= \theta^2
\end{aligned}
$$

In order to apply Rao-Blackwell theorem, we need to find $E(Y_2/Y_1)$ because Y_2 is unbiased for θ (by part b) and $Y_1 = X_1+X_2 = \sum_{i=1}^2 X_i$ is sufficient for θ (see Q.3 part b).

$$
\begin{aligned}
f_{Y_2|Y_1}(y_2|y_1) &= \frac{f_{Y_2,Y_1}(y_2,y_1)}{f_{Y_1}(y_1)} \\
&= \frac{f_{Y_2,Y_1-Y_2}(y_2,y_1-y_2)}{f_{Y_1}(y_1)} \\
&= \frac{f_{X_2,X_1}(x_2,x_1)}{f_{X_1+X_2}(x_1+x_2)} \\
&= \frac{\frac{1}{\theta}\exp(-\frac{x_2}{\theta})\cdot\frac{1}{\theta}\exp(-\frac{x_1}{\theta})}{\frac{1}{\Gamma(2)}\theta^{-2}(x_1+x_2)^{2-1}\exp(-\frac{(x_1+x_2)}{\theta})}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{x_1 + x_2}, \qquad (\Gamma(2) = 1) \\
&= \frac{1}{y_1}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \; E(Y_2|Y_1 = y_1) &= \int_0^{y_1} y_2 \cdot f_{Y_2|Y_1}(y_2|y_1)\; dy_2 \\
&= \int_0^{y_1} y_2 y_2 \cdot \frac{1}{y_1}\; dy_2 \\
&= \left[\frac{y_2^2}{2y_1}\right]_0^{y_1} \\
&= \frac{y_1}{2}
\end{aligned}
$$

$\therefore$ By Rao-Blackwell theorem, an improved unbiased estimator is $\frac{1}{2}Y_1 = \frac{1}{2}(X_1 + X_2) = \bar{X}$.

The alternative method to get an improved estimate is as follows. Y_1 is sufficient and complete statistic and $Y_1 \sim gamma(2, \frac{1}{\theta})$.

$$
\begin{aligned}
\Rightarrow \; h(y_1)\frac{\Gamma(3)}{\Gamma(2)y_1} &= 1 \\
\Rightarrow \; h(y_1) &= \frac{\Gamma(2)y_1}{\Gamma(3)} \\
&= \frac{y_1}{2} \\
&= \frac{X_1 + X_2}{2}
\end{aligned}
$$

$$
\begin{aligned}
f_X(x;\theta) &= \frac{1}{\theta}\exp(-\frac{x}{\theta}) \\
\log f_X(x;\theta) &= -\log\theta - \frac{x}{\theta} \\
\frac{\partial}{\partial\theta}\log f_X(x;\theta) &= -\frac{1}{\theta} + \frac{x}{\theta^2} \\
\frac{\partial^2}{\partial\theta^2}\log f_X(x;\theta) &= \frac{1}{\theta^2} - \frac{2x}{\theta^3} \\
E\left[\frac{\partial^2}{\partial\theta^2}\log f_X(x;\theta)\right] &= E\left[\frac{1}{\theta^2} - \frac{2x}{\theta^3}\right] \\
&= \frac{1}{\theta^2} - \frac{2}{\theta^3}E(X) \\
&= \frac{1}{\theta^2} - \frac{2\theta}{\theta^3} \\
&= -\frac{1}{\theta^2}
\end{aligned}
$$

$$
\therefore \text{ The CRLB for } \theta = \frac{-1}{n \cdot E[\frac{\partial^2}{\partial\theta^2}\log f_X(x;\theta)]} = \frac{-1}{2 \cdot (\frac{-1}{\theta^2})} = \frac{\theta^2}{2}
$$

Also, variance of improved estimator

$$
= Var(\bar{X}) = \frac{Var(X)}{n} = \frac{\theta^2}{n} \left(\because \; X \sim \exp(\frac{1}{\theta})\right) = \text{CRLB for } \theta
$$

$\therefore$ The variance of improved estimator $\bar{X}$ attains the CRLB.

13.

$$\begin{aligned} f_X(x;\theta) &= \frac{1}{6\theta^4}x^3\exp(-\frac{x}{\theta}), \qquad 0<x<\infty,\ 0<\theta<\infty \\ &= \exp\Big\{-\log(6\theta^4)+3\log x-\frac{x}{\theta}\Big\} \end{aligned}$$

$$\therefore\ a(\theta)=-\log(6\theta^4), b(X)=3\log X, c(\theta)=-\frac{1}{\theta}, d(X)=X$$

$$\therefore\ f_X(x;\theta) \text{ belongs to the exponential family.}$$

The complete sufficient statistic for θ is $Y_1=\sum_{i=1}^n d(X_i)=\sum_{i=1}^n X_i$.

$$\begin{aligned} E(X) &= \int_0^\infty x f_X(x;\theta)\ dx \\ &= \int_0^\infty \frac{x}{6\theta^4}x^3\exp(-\frac{x}{\theta})\ dx \\ &= \frac{1}{6\theta^4}\int_0^\infty x^4\exp(-\frac{x}{\theta})\ dx, \qquad \text{by parts} \\ &= \left[\frac{1}{6\theta^3}(-x^4)\right]_0^\infty + \frac{1}{6\theta^3}\int_0^\infty 4x^3\exp(-\frac{x}{\theta})\ dx \\ &= 0+\frac{4}{6\theta^3}\int_0^\infty x^3\exp(-\frac{x}{\theta})\ dx, \qquad \text{by parts} \\ &= \left[\frac{4}{6\theta^3}(-x^3)\right]_0^\infty + \frac{4}{6\theta^2}\int_0^\infty 3x^2\exp(-\frac{x}{\theta})\ dx, \qquad \text{by parts} \\ &= 0+\left[\frac{2}{\theta}(-x^2)\exp(-\frac{x}{\theta})\right]_0^\infty + \frac{2}{\theta}\int_0^\infty 2x\exp(-\frac{x}{\theta})\ dx, \qquad \text{by parts} \\ &= 0+0+\left[4(-x)\exp(-\frac{x}{\theta})\right]_0^\infty + 4\int_0^\infty \exp(-\frac{x}{\theta})\ dx \\ &= 0+0+0+\left[-4\theta\exp(-\frac{x}{\theta})\right]_0^\infty \\ &= 4\theta \\ \therefore\ E(\frac{1}{4n}\sum_{i=1}^n X_i) &= \frac{1}{4n}\sum_{i=1}^n E(X_i) \\ &= \frac{1}{4n}n(4\theta) \\ &= \theta \end{aligned}$$

$$\Rightarrow\ \frac{1}{4n}\sum_{i=1}^n X_i \text{ is unbiased for } \theta$$

Since $\frac{1}{4n}\sum_{i=1}^n X_i=\frac{1}{4n}Y_1$ is a function of Y_1 (complete sufficient statistic), it is the UMVUE for θ,

$$\therefore\ \varphi(Y_1)=\frac{1}{4n}Y_1=\frac{1}{4n}\sum_{i=1}^n X_i$$

Also, $\varphi(Y_1)=\frac{1}{4n}Y_1$ is itself a complete sufficient statistic for θ since Y_1 is a complete sufficient statistic.

14. $X \sim \exp(\theta)$.

Given $Y = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

$\Rightarrow\ Y = \sum_{i=1}^{n} X_i \sim gamma(n, \theta)$

$\Rightarrow\ f_Y(y) = \frac{\theta^n}{\Gamma(n)} y_{n-1} e^{-\theta y}$

$f_X(x;\theta) = \theta e^{-\theta x} = \exp(\log\theta - \theta x)$

$$\therefore\ a(\theta) = \log\theta, b(X) = 0, c(\theta) = -\theta, d(X) = X$$

$$\therefore\ f_X(x;\theta) \text{ belongs to the exponential family.}$$

$$\text{and } \sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i \text{ is complete and sufficient.}$$

$$\begin{aligned}
\therefore\ E(\frac{n-1}{Y}) &= (n-1)\int_0^\infty \frac{1}{y} f_Y(y)\ dy \\
&= (n-1)\int_0^\infty \frac{1}{y}\frac{\theta^n}{\Gamma(n)} y^{n-1} e^{-\theta y}\ dy \\
&= \frac{(n-1)\theta}{\Gamma(n)}\int_0^\infty \theta^{n-1} y^{n-2} e^{-\theta y}\ dy \\
&= \frac{(n-1)\theta}{\Gamma(n)}\Gamma(n-1)\int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)} y^{n-2} e^{-\theta y}\ dy \\
&= (n-1)\theta \cdot \frac{1}{n-1} \cdot 1 \\
&= \theta
\end{aligned}$$

$$\because\ \Gamma(n) = (n-1)\Gamma(n-1) \quad \text{and} \quad \frac{\theta^{n-1}}{\Gamma(n-1)} y^{n-2} e^{-\theta y} \text{ is pdf of gamma}(n-1, \frac{1}{\theta})$$

Since $\frac{n-1}{Y}$ is function of complete sufficient statistic, $\frac{n-1}{Y}$ is UMVUE for θ.

15.

$$\begin{aligned}
f_X(x;\theta) &= \theta^x (1-\theta)^{1-x} \\
&= \exp\{x\log\theta + (1-x)\log\theta\} \\
&= \exp\left\{\log(1-\theta) + x\log\frac{\theta}{1-\theta}\right\}
\end{aligned}$$

$$\therefore\ a(\theta) = \log(1-\theta), b(X) = 0, c(\theta) = \log(\frac{\theta}{1-\theta}), d(X) = X,$$

$\therefore$ Bin$(1,\theta)$ belongs to the exponential family and $\sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i$ is complete and sufficient for θ.

Since we usually to use $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ to estimate true mean $= \theta$, we try to check $E(n\bar{X}(1-\bar{X}))$ whether it is equal to $n\theta(1-\theta)$.

Now

$$\begin{aligned}
E(n\bar{X}(1-\bar{X})) &= E(\sum_{i=1}^{n} X_i(1-\bar{X})) \\
&= \frac{1}{n}E(\sum_{i=1}^{n} X_i(n - \sum_{i=1}^{n} X_i))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}E(Y(n-Y)) \quad \text{where } Y=\sum_{i=1}^{n} X_i \\
&= \frac{1}{n}[E(nY)-E(Y^2)] \\
&= \frac{1}{n}[nE(Y)-Var(Y)-[E(Y)]^2] \\
&= \frac{1}{n}[n\cdot n\theta - n\theta(1-\theta)-n^2\theta^2] \\
&= \frac{1}{n}[n^2\theta+n\theta^2-n\theta-n^2\theta^2] \\
&= \theta(n-1)-\theta^2(n-1) \\
&= (n-1)\theta(1-\theta) \\
\therefore\ E[\frac{n^2}{n-1}\bar{X}(1-\bar{X})] &= n\theta(1-\theta)
\end{aligned}
$$

Since $\frac{n^2}{n-1}\bar{X}(1-\bar{X})$ is function of complete sufficient statistic $\sum_{i=1}^{n} X_i$, it is the UMVUE for $n\theta(1-\theta)$.

16. By Q.6(c), $\sum_{i=1}^{n} d(X_i)=\sum_{i=1}^{n} X_i^2$ is complete sufficient statistic for θ.

Now

$$
\begin{aligned}
&\because\ X_i \sim N(0,\theta) \\
\Rightarrow\ &\frac{X_i-0}{\sqrt{\theta}} \sim N(0,1) \\
\Rightarrow\ &\frac{X_i^2}{\theta} \sim \chi^2(1) \\
\Rightarrow\ &\frac{1}{\theta}\sum_{i=1}^{n} X_i^2 \sim \chi^2(n) \\
\Rightarrow\ &\frac{1}{\theta}Y \sim \chi^2(n) \qquad \text{where } Y=\sum_{i=1}^{n} X_i^2
\end{aligned}
$$

$$\therefore\ E(\frac{1}{\theta}Y)=n \text{ and } Var(\frac{1}{\theta}Y)=2n$$

$$\Rightarrow E(Y)=n\theta \text{ and } Var(Y)=2n\theta^2$$

$$\therefore\ E(Y^2)=Var(Y)+[E(Y)]^2=2n\theta^2+n^2\theta^2=(2n+n^2)\theta^2$$

$$\therefore\ E\left[\frac{1}{n^2+2n}Y^2\right]=\theta^2$$

$\therefore\ \frac{1}{n^2+2n}Y^2$ is unbiased for θ^2 and since it is function of $\sum_{i=1}^{n} X_i^2=Y$, $\therefore$ it is UMVUE for θ^2.

17.

$$f_X(x;\theta)=\theta e^{-\theta x}=\exp(\log\theta-\theta x)$$

$$\Rightarrow a(\theta) = \log\theta, b(X) = 0, c(\theta) = -\theta, d(X) = X$$

$\therefore$ $f(x;\theta)$ belongs to exponential family and $\sum_{i=1}^n d(X_i) = \sum_{i=1}^n X_i$ is a complete sufficient statistic for θ.

Let $Y = \sum_{i=1}^n X_i \sim \text{gamma}(n,\theta)$

$$\begin{aligned}
&\Rightarrow \quad E(t(Y)) = \theta \\
&\Rightarrow \quad \int_0^\infty t(y)\frac{y^{n-1}e^{-\theta y}}{\theta^{-n}\Gamma(n)}\,dy = \theta \\
&\Rightarrow \quad \int_0^\infty t(y)\frac{y^{n-1}e^{-\theta y}}{\theta^{-(n-1)}\Gamma(n)}\,dy = 1 \\
&\Rightarrow \quad \int_0^\infty t(y)\cdot y\frac{\Gamma(n-1)}{\Gamma(n)}\cdot\frac{y^{(n-1)-1}e^{-\theta y}}{\theta^{-(n-1)}\Gamma(n-1)}\,dy = 1 \\
&\Rightarrow \quad t(y)y\frac{\Gamma(n-1)}{\Gamma(n)} = 1 \\
&\Rightarrow \quad t(y) = \frac{\Gamma(n)}{\Gamma(n-1)}\cdot\frac{1}{y} = \frac{n-1}{y}
\end{aligned}$$

$\therefore$ $t(Y) = \frac{n-1}{\sum_{i=1}^n X_i}$ is UMVUE for θ.

18. By Q.5(a),$\sum_{i=1}^n X_i$ is complete and sufficient statistic for λ.

Observe that $\tau(\lambda) = \frac{\lambda^k e^{-\lambda}}{k!} = P(X_1 = k)$.

$\therefore$ An unbiased estimator of $\tau(\lambda)$ is $I_{(k)(X_1)},\ k = 0,1,2,\dots$

Since $E[I_{(k)}(X_1)] = P(X_1 = k) = \frac{\lambda^k e^{-\lambda}}{k!} = \tau(\lambda)$,

$\therefore$ By Rao-Blackwell theorem, $E[I_{(k)}(X_1)|\sum_{i=1}^n X_i]$ is the UMVUE for $\tau(\lambda)$.

Note that $\sum_{i=1}^n X_i \sim$Poisson $(n\lambda)$ and $\sum_{i=2}^n X_i \sim$ Poisson$((n-1)\lambda)$

$$\begin{aligned}
\text{For } n = 1, \quad & E[I_{(k)}(X_1)|X_1] \\
=\ & I_{(k)}(X_1) \\
\text{For } n > 1, \quad & E[I_{(k)}(X_1)|\sum_{i=1}^n X_1 = s] \\
=\ & P(X_1 = k|\sum_{i=1}^n X_i = s) \\
=\ & \frac{P(X_1 = k, \sum_{i=1}^n X_i = s)}{P(\sum_{i=1}^n X_i = s)} \\
=\ & \frac{P(X_1 = k)P(\sum_{i=2}^n X_i = s-k)}{P(\sum_{i=1}^n X_i = s)} \\
=\ & \begin{cases} 0 & \text{for } s < k \\ \dfrac{\dfrac{\lambda^k e^{-\lambda}}{k!}\cdot\dfrac{[(n-1)\lambda]^{s-k}e^{-(n-1)\lambda}}{(s-k)!}}{\dfrac{(n\lambda)^s e^{-n\lambda}}{s!}} & \text{for } s \geq k \end{cases} \\
=\ & \begin{cases} 0 & \text{for } s < k \\ \begin{pmatrix} s \\ k \end{pmatrix}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{s-k} & \text{for } s \geq k \end{cases}
\end{aligned}$$

$$E\left[I_{(k)}(X_1)|\sum_{i=1}^n X_1\right] = \left(\begin{array}{c}\sum_{i=1}^n X_i\\ k\end{array}\right)\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{\sum_{i=1}^n X_i - k} I_{\{k,k+1,\ldots\}}(\sum_{i=1}^n X_i)$$

Method 2:

$$\begin{aligned}
& S = \sum_{i=1}^n X_i \sim Po(n,\lambda) \\
& \sum_{s=0}^{\infty} h(s)\frac{(n\lambda)^s e^{-\lambda n}}{s!} = \tau(\lambda) = \frac{\lambda^k e^{-\lambda}}{k!} \\
\Rightarrow\ & \sum_{s=0}^{\infty} \frac{h(s) n^s \lambda^{s-k} e^{-(n-1)\lambda} k!}{s!} = 1 \\
\Rightarrow\ & \sum_{s=0}^{\infty} \frac{h(s)n^s}{(n-1)^{s-k}}\left(\begin{array}{c}s\\ k\end{array}\right)\frac{((n-1)\lambda)^{s-k}e^{-(n-1)\lambda}}{(s-k)!} = 1 \\
\Rightarrow\ & \left(\begin{array}{c}s\\ k\end{array}\right)\frac{h(s)n^s}{(n-1)^{s-k}} = 1 \\
\Rightarrow\ & h(s) = \left(\begin{array}{c}s\\ k\end{array}\right)\frac{(n-1)^{s-k}}{n^s} = \ldots
\end{aligned}$$

20.

$$\begin{aligned}
f(X;\theta) &= \frac{X^{\alpha-1}e^{-X/\theta}}{\theta^\alpha\Gamma(\alpha)} \\
&= \exp\left\{(\alpha-1)\log X - \frac{X}{\theta} - \alpha\log\theta - \log\Gamma(\alpha)\right\} \\
&= \exp\left\{-\alpha\log\theta - \log\Gamma(\alpha) + (\alpha-1)\log X - \frac{X}{\theta}\right\}
\end{aligned}$$

$$\therefore\ a(\theta) = -\alpha\log\theta, b(X) = -\log\Gamma(\alpha) + (\alpha-1)\log X, c(\theta) = -\frac{1}{\theta}, d(X) = X$$

$\therefore\ f(X;\theta)$ belongs to exponential family and $\sum_{i=1}^n X_i$ is complete sufficient statistic for θ.

$$\because\ E\left[\frac{1}{n\alpha}\sum_{j=1}^n X_j\right] = \frac{1}{n\alpha}\sum_{j=1}^n E(X_j) = \frac{1}{n\alpha}\cdot n(\alpha\theta) = \theta \quad (X_j \sim gamma(\alpha, \frac{1}{\theta}))$$

and $\frac{1}{n\alpha}\sum_{j=1}^n X_j$ is function of complete sufficient statistic,

$$\therefore\ U(X_1,\ldots,X_n) = \frac{1}{n\alpha}\sum_{j=1}^n X_j \text{ is UMVUE for } \theta$$

$$\begin{aligned}
\log f(X;\theta) &= -\alpha\log\theta - \log\Gamma(\alpha) + (\alpha-1)\log X - \frac{X}{\theta} \\
\frac{\partial}{\partial\theta}\log f(X;\theta) &= \frac{-\alpha}{\theta} + \frac{X}{\theta^2} \\
\frac{\partial^2}{\partial\theta^2}\log f(X;\theta) &= \frac{\alpha}{\theta^2} - \frac{2X}{\theta^3}
\end{aligned}$$

$$\begin{aligned}
E\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right] &= E\left[\frac{\alpha}{\theta^2} - \frac{2X}{\theta^3}\right] \\
&= \frac{\alpha}{\theta^2} - \frac{2\alpha\theta}{\theta^3} \\
&= \frac{-\alpha}{\theta^2} \\
\therefore\ \text{CRLB for } \theta &= \frac{-1}{n\cdot E\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right]} \\
&= \frac{-1}{n\cdot(\frac{-\alpha}{\theta^2})} \\
&= \frac{\theta^2}{n\alpha} \\
Var(U(X_1,\ldots,X_n)) &= Var(\frac{1}{n\alpha}\sum_{j=1}^{n} X_j) \\
&= \frac{1}{n^2\alpha^2}\sum_{j=1}^{n} Var(X_j) \\
&\quad (\because\ X_j \sim gamma(\alpha, \frac{1}{\theta}), Var(X_j) = \alpha\theta^2) \\
&= \frac{1}{n^2\alpha^2}\cdot n(\alpha\theta^2) \\
&= \frac{\theta^2}{n\alpha} \\
&= \text{CRLB for } \theta
\end{aligned}$$

21. $X \sim \exp(\theta)$

$f(x;\theta) = \theta e^{-\theta x} = \exp(\log\theta - \theta x)$

$$a(\theta) = \log\theta, b(X) = 0, c(\theta) = -\theta, d(X) = X$$

$\therefore$ $f(x;\theta)$ belongs to exponential family and $\sum_{i=1}^{n} X_i$ is complete sufficient statistic for θ.
Let

$$\begin{aligned}
& S = \sum_{i=1}^{n} X_i \sim gamma(n,\theta) \\
\Rightarrow\ & E[h(S)] = R(x,\theta) \\
\Rightarrow\ & \int_0^\infty h(s)\frac{s^{n-1}e^{-\theta s}}{\Gamma(n)\theta^{-n}}\ ds = R(x,\theta) = P(X > x) = e^{-\theta x} \\
\Rightarrow\ & \int_0^\infty \frac{h(s)s^{n-1}}{(s-x)^{n-1}}\cdot\frac{(s-x)^{n-1}}{\Gamma(n)\theta^{-n}}e^{-\theta(s-x)}\ ds = 1 \\
\Rightarrow\ & \frac{h(s)s^{n-1}}{(s-x)^{n-1}} = 1 = I_{(x,\infty)}(s) \\
& \text{Since } X \text{ must be greater than } x,\ \therefore\ s = \sum_{i=1}^{n} X_i > x \\
\Rightarrow\ & h(s) = \frac{(s-x)^{n-1}}{s^{n-1}}I_{(x,\infty)}(s) = (1-\frac{x}{s})^{n-1}I_{(x,\infty)}(s)
\end{aligned}$$

$\therefore$ $(1-\frac{x}{\sum_{i=1}^{n} X_i})^{n-1}I_{(x,\infty)}(\sum_{i=1}^{n} X_i)$ is UMVUE for $R(x;\theta)$.

22.

$$
\begin{aligned}
f_{Y_3}(y_3) &= \frac{5!}{2!2!}[F_X(y_3)]^2 f_X(y_3)[1-F_X(y_3)]^2 \\
&= 30\left(\frac{y_3}{\theta}\right)^2\left(\frac{1}{\theta}\right)\left(1-\frac{y_3}{\theta}\right)^2 \\
&= \frac{30}{\theta^5}y_3^2(\theta-y_3)^2, \quad 0<y_3<\theta
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ E(Y_3) &= \int_0^\theta y_3 f_{Y_3}(y_3)\ dy_3 \\
&= \int_0^\theta y_3\frac{30}{\theta^5}y_3^2(\theta-y_3)^2\ dy_3 \\
&= \int_0^\theta y_3\frac{30}{\theta^5}(\theta^2y_3^2-2\theta y_3^3+y_3^4)\ dy_3 \\
&= \int_0^\theta \frac{30}{\theta^5}(\theta^2y_3^3-2\theta y_3^4+y_3^5)\ dy_3 \\
&= \frac{30}{\theta^5}\left[\theta^2\frac{y_3^4}{4}-2\theta\frac{y_3^5}{5}+\frac{y_3^6}{6}\right]_0^\theta \\
&= 30\theta\left(\frac{1}{4}-\frac{2}{5}+\frac{1}{6}\right) \\
&= 30\theta\left(\frac{15-24+10}{60}\right) \\
&= \frac{\theta}{2} \\
\therefore\ E(2Y_3) &= 2E(Y_3)=2(\frac{\theta}{2})=\theta
\end{aligned}
$$

$$\therefore\ 2Y_3 \text{ is an unbiased statistic for } \theta.$$

Joint pdf of Y_3 and Y_5,

$$
\begin{aligned}
f_{Y_3,Y_5}(y_3,y_5) &= \frac{5!}{2!1!0!}[F_X(y_3)]^2 f_X(y_3)[F_X(y_5)-F_X(y_3)]f_X(Y_5) \\
&= 60(\frac{y_3}{\theta})^2(\frac{1}{\theta})(\frac{y_5}{\theta}-\frac{y_3}{\theta})\frac{1}{\theta} \\
&= \frac{60}{\theta^5}(y_3^2y_5-y_3^3), \quad 0<y_3\le y_5<\theta \\
f_{Y_5}(y_5) &= \frac{5!}{4!0!}[F_X(y_5)]^4 f_X(y_5) \\
&= 5(\frac{y_5}{\theta})^4(\frac{1}{\theta}) \\
&= \frac{5y_5^4}{\theta^5}, \quad 0<y_5<\theta \\
\therefore\ f_{Y_3|Y_5}(y_3|y_5) &= \frac{{}_{Y_3,Y_5}(y_3,y_5)}{f_{Y_5}(y_5)} \\
&= \frac{\frac{60}{\theta^5}(y_3^2y_5-y_3^3)}{\frac{5y_5^4}{\theta^5}} \\
&= \frac{12y_3^2(y_5-y_3)}{y_5^4} \\
\therefore\ E(Y_3|Y_5=y_5) &= \int_0^{y_5} y_3 f_{Y_3|Y_5}(y_3|y_5)\ dy_3
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{y_5} y_3 \frac{12y_3^2(y_5-y_3)}{y_5^4}\ dy_3 \\
&= \frac{12}{y_5^4}\int_0^{y_5} y_3^3(y_5-y_3)\ dy_3 \\
&= \frac{12}{y_5^4}\int_0^{y_5} (y_3^3y_5-y_3^4)\ dy_3 \\
&= \frac{12}{y_5^4}[\frac{1}{4}y_3^4y_5-\frac{1}{5}y_3^5]_0^{y_5} \\
&= 12(\frac{1}{4}y_5-\frac{1}{5}y_5) \\
&= \frac{3}{5}y_5
\end{aligned}
$$

$$
\begin{aligned}
\varphi(y_5) &= E(2Y_3|Y_5=y_5) = 2E(Y_3|Y_5=y_5) \\
&= \frac{6}{5}y_5
\end{aligned}
$$

$$
\begin{aligned}
E(Y_3^2) &= \int_0^{\theta} y_3^2(\frac{30}{\theta^5})(\theta^2y_3^3-2\theta y_3^3+y_3^4)\ dy_3 \\
&= \frac{30}{\theta^5}\int_0^{\theta}(\theta^2y_3^4-2\theta y_3^5+y_3^6)\ dy_3 \\
&= \frac{30}{\theta^5}[\frac{1}{5}\theta^2y_3^5-\frac{2}{6}\theta y_3^6+\frac{1}{7}y_3^7]_0^{\theta} \\
&= 30\ \theta^2(\frac{1}{5}-\frac{1}{3}+\frac{1}{7}) \\
&= \frac{30\ \theta}{105} \\
&= \frac{2}{7}\theta^2 \\
\therefore\ Var(2Y_3) &= E[(2Y_3)^2]-[E(2Y_3)]^2 \\
&= 4E(Y_3^2)-\theta^2 \\
&= \frac{8}{7}\theta^2-\theta^2 = \frac{1}{7}\theta^2
\end{aligned}
$$

Note that $\varphi(Y_5)$ is unbiased for θ since

$$
\begin{aligned}
E[\varphi(Y_5)] &= E[E(2Y_3|Y_5)] = E(2Y_3) = \theta \quad (\because\ E(g(X)) = E[E(g(X)|Y)]) \\
E\left([\varphi(Y_5)]^2\right) &= \int_0^{\theta}(\frac{6}{5}y_5)^2\cdot f_{Y_5}(y_5)\ dy_5 \\
&= \int_0^{\theta}(\frac{6}{5})^2y_5^2\cdot\frac{5y_5^4}{\theta^5}\ dy_5 \\
&= \frac{36}{5\theta^5}\int_0^{\theta}y_5^6\ dy_5 \\
&= \frac{36}{5\theta^5}[\frac{y_5^7}{7}]_0^{\theta} \\
&= \frac{36}{35}\theta^2 \\
Var[\varphi(Y_5)] &= E[(\varphi(Y_5))]-[E(\varphi(Y_5))]^2 \\
&= \frac{36}{35}\theta^2-\theta^2 = \frac{\theta^2}{35}
\end{aligned}
$$

$$Var[\varphi(Y_5)] = \frac{\theta^2}{35} < \frac{1}{7}\theta^2 = Var(2Y_3)$$

23. (a)

$$\begin{aligned}
f_Y(y) &= \int_0^y f_{X,Y}(x,y)\ dx, \quad 0 < x < y < \infty \\
&= \int_0^y \frac{2}{\theta^2} \exp\left\{-\frac{(x+y)}{\theta}\right\}\ dx \\
&= \frac{2}{\theta^2} \exp\left\{\frac{-y}{\theta}\right\} \int_0^y e^{-x/\theta}\ dx \\
&= \frac{2}{\theta^2} \exp\left\{\frac{-y}{\theta}\right\} \left[-\theta e^{-x/\theta}\right]_0^y \\
&= \frac{2}{\theta^2} \exp\left\{\frac{-y}{\theta}\right\} [-\theta e^{-y/\theta} + \theta] \\
&= \frac{2}{\theta^2} \exp\left\{-\frac{y}{\theta}\right\} (1 - \exp\{-\frac{y}{\theta}\}), \quad 0 < y < \infty
\end{aligned}$$

$$\begin{aligned}
\therefore\ E(Y) &= \int_0^\infty y f_Y(y)\ dy \\
&= \int_0^\infty y \cdot \frac{2}{\theta}\ \exp\left(-\frac{y}{\theta}\right)\left(1 - \exp(-\frac{y}{\theta})\right)\ dy \\
&= \int_0^\infty \frac{2y}{\theta} \exp\left(\frac{-y}{\theta}\right)\ dy - \int_0^\infty \frac{2y}{\theta} \exp\left(\frac{-2y}{\theta}\right)\ dy \\
&= \left[-2ye^{-y/\theta}\right]_0^\infty + \int_0^\infty 2e^{-y/\theta}\ dy + \left[ye^{-2y/\theta}\right]_0^\infty - \int_0^\infty e^{-2y/\theta}\ dy \\
&= \left[-2\theta e^{-y/\theta}\right]_0^\infty + \left[\frac{\theta}{2} e^{-2y/\theta}\right]_0^\infty \\
&= 2\theta - \theta/2 \\
&= 3\theta/2
\end{aligned}$$

$$\begin{aligned}
E(Y^2) &= \int_0^\infty y^2 f_Y(y)\ dy \\
&= \int_0^\infty y^2 \cdot \frac{2}{\theta}\ \exp\left(\frac{-y}{\theta}\right)\left(1 - \exp\frac{-y}{\theta}\right)\ dy \\
&= \int_0^\infty \frac{2y^2}{\theta} \exp\left(\frac{-y}{\theta}\right)\ dy - \int_0^\infty \frac{2y^2}{\theta} \exp\left(\frac{-2y}{\theta}\right)\ dy \\
&= \left[-2y^2 e^{-y/\theta}\right]_0^\infty + \int_0^\infty 4ye^{-y/\theta}\ dy + \left[y^2 e^{-2y/\theta}\right]_0^\infty - \int_0^\infty 2ye^{-2y/\theta}\ dy \\
&= \left[-4\theta ye^{-y/\theta}\right]_0^\infty + \int_0^\infty 4\theta e^{-y/\theta}\ dy + \left[\theta ye^{-2y/\theta}\right]_0^\infty - \int_0^\infty \theta e^{-2y/\theta}\ dy \\
&= \left[-4\theta^2 e^{-y/\theta}\right]_0^\infty + \left[\frac{\theta^2}{2} e^{-2y/\theta}\right]_0^\infty \\
&= 4\theta^2 - \frac{\theta^2}{2} \\
&= \frac{7\theta^2}{2} \\
\therefore\ Var(Y) &= E(Y)^2 - [E(Y)]^2 \\
&= \frac{7\theta^2}{2} - (\frac{3\theta}{2})^2 = \frac{5\theta^2}{4}
\end{aligned}$$

(b)

$$\begin{aligned}
f_X(x) &= \int_x^\infty f_{X,Y}(x,y)\ dy \\
&= \int_x^\infty \frac{2}{\theta^2}\exp\left\{-\frac{(x+y)}{\theta}\right\}\ dy \\
&= \frac{2}{\theta^2}\exp\left\{\frac{-x}{\theta}\right\}\int_x^\infty e^{-y/\theta}\ dy \\
&= \frac{2}{\theta^2}\exp\left\{\frac{-x}{\theta}\right\}[-\theta e^{-y/\theta}]_x^\infty \\
&= \frac{2}{\theta}\exp\left\{\frac{-2x}{\theta}\right\}, \quad 0<x<\infty \\
\therefore f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{\frac{2}{\theta^2}\exp\left\{-\frac{(x+y)}{\theta}\right\}}{\frac{2}{\theta}\exp\left\{\frac{-2x}{\theta}\right\}} \\
&= \frac{1}{\theta}\exp\left\{\frac{x-y}{\theta}\right\}, \quad 0<x<y<\infty
\end{aligned}$$

$$\begin{aligned}
\therefore\ E(Y|X=x) &= \int_x^\infty y\cdot f_{Y|X}(y|x)\ dy \\
&= \int_x^\infty y\frac{1}{\theta}\exp\left\{\frac{x-y}{\theta}\right\}\ dy \\
&= \frac{1}{\theta}\exp(\frac{x}{\theta})\int_x^\infty ye^{-y/\theta}\ dy, \quad \text{by parts} \\
&= \exp(\frac{x}{\theta})\left\{\left[-ye^{-y/\theta}\right]_x^\infty + \int_x^\infty e^{-y/\theta}\ dy\right\} \\
&= x+\exp(\frac{x}{\theta})\left[-\theta e^{-y/\theta}\right]_x^\infty \\
&= x+\theta
\end{aligned}$$

$$\begin{aligned}
f_X(x) &= \frac{2}{\theta}\exp\{-\frac{2x}{\theta}\} \\
\therefore\ X &\sim \exp(\frac{2}{\theta}) \\
\therefore\ Var(X+\theta) &= Var(X) \\
&= (\frac{\theta}{2})^2 \\
&= \frac{\theta^2}{4} \\
&< \frac{5\theta^2}{4} \\
&= Var(Y)
\end{aligned}$$

25. (a)

$$\begin{aligned}
f(x;p) &= p^x(1-p)^{1-x} \\
L(p) &= f_{\underset{\sim}{X}}(\underset{\sim}{x};p) \\
&= \prod_{i=1}^n f_{X_i}(x_i;p)
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} \\
&= p^{\sum_{i=1}^{n} x_i}(1-p)^{n-\sum_{i=1}^{n} x_i} \\
\log L(p) &= \left(\sum_{i=1}^{n} x_i\right)\log p + \left(n - \sum_{i=1}^{n} x_i\right)\log(1-p) \\
\frac{\partial}{\partial p}\log L(p) &= \left(\sum_{i=1}^{n} x_i\right)\frac{1}{p} - \left(n - \sum_{i=1}^{n} x_i\right)\frac{1}{1-p} \\
\text{Setting equal to 0} &\Rightarrow \hat{p} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}
\end{aligned}
$$

$\therefore$ By invariant property of MLE, MLE for $\theta = (1-\hat{p})^2 = (1-\bar{x})^2$

(b)

$$
\begin{aligned}
& X_1 + X_2 \sim Bin(2,p) \\
\therefore\ E(\hat{\theta}) &= 1 \cdot P(X_1 + X_2 = 0) + 0 \cdot P(X_1 + X_2 \neq 0) \\
&= \binom{2}{0} p^0 (1-p)^{2-0} \\
&= (1-p)^2 \\
&= \theta
\end{aligned}
$$

$\therefore\ \hat{\theta}$ is an unbiased estimator of θ.

(c)

$$
\begin{aligned}
f(x;p) &= p^x(1-p)^{1-x} \\
&= \exp\{x\log p + (1-x)\log(1-p)\} \\
&= \exp\left\{\log(1-p) + x\log\left(\frac{p}{1-p}\right)\right\}
\end{aligned}
$$

$$
\therefore\ a(p) = \log(1-p), b(X) = 0, c(p) = \left(\frac{p}{1-p}\right), d(X) = X
$$

$\therefore\ f(X;p)$ belongs to exponential family and $\sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for p.
Let $S = \sum_{i=1}^{n} X_i \sim Bin(n;p)$
By Rao-Blackwell theorem, UMVUE for $\theta = E(\hat{\theta}|S=s)$, $\hat{\theta}$ is unbiased, S is sufficient.

$$
\begin{aligned}
\Rightarrow E(\hat{\theta}|S=s) &= 1 \cdot P(X_1 + X_2 = 0|S=s) + 0 \cdot P(X_1 + X_2 \neq 0|S=s) \\
&= \frac{P(X_1 + X_2 = 0, \sum_{i=1}^{n} X_i = s)}{P(\sum_{i=1}^{n} X_i = s)} \\
&= \frac{P(X_1 + X_2 = 0, \sum_{i=3}^{n} X_i = s)}{P(\sum_{i=1}^{n} X_i = s)} \\
&= \frac{P(X_1 + X_2 = 0) \cdot P(\sum_{i=3}^{n} X_i = s)}{P(\sum_{i=1}^{n} X_i = s)} \\
&= \frac{\binom{2}{0} p^0(1-p)^{2-0} \binom{n-2}{s} p^s(1-p)^{n-2-s}}{\binom{n}{s} p^s(1-p)^{n-s}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(n-2)!}{s!(n-2-s)!} \cdot \frac{s!(n-s)!}{n!} \\
&= \frac{(n-s)(n-s-1)}{n(n-1)}
\end{aligned}
$$

$\therefore$ UMVUE for $\theta = (1-p)^2$ is $\frac{1}{n(n-1)}(n - \sum_{i=1}^{n} X_i)(n - \sum_{i=1}^{n} X_i - 1)$

Method 2

$$
\begin{aligned}
& \sum_{s=0}^{\infty} h(s) \binom{n}{s} p^s (1-p)^{n-s} = \theta = (1-p)^2 \\
\Rightarrow \quad & \sum_{s=0}^{\infty} h(s) \binom{n}{s} p^s (1-p)^{n-s-2} = 1 \\
\Rightarrow \quad & \sum_{s=0}^{\infty} \frac{h(s) n(n-1)}{(n-s-1)(n-s)} \binom{n-2}{s} p^s (1-p)^{n-s-2} = 1 \\
\Rightarrow \quad & h(s) = \frac{(n-s-1)(n-s)}{n(n-1)} = \ldots
\end{aligned}
$$

26.

$$
f_X(x) = \frac{1}{2\theta - \theta} = \frac{1}{\theta}
$$

$$
F_X(x) = \int_{\theta}^{x} \frac{1}{\theta}\, dt = \left[\frac{t}{\theta}\right]_{\theta}^{x} = \frac{x}{\theta} - 1 = \frac{1}{\theta}(x - \theta)
$$

$$
\begin{aligned}
f_{Y_1}(y_1) &= \frac{n!}{(1-1)!(n-1)!}[F_X(y_1)]^{1-1}[1 - F_X(y_1)]^{n-1} f_X(y_1) \\
&= n\left(1 - \frac{1}{\theta}(y_1 - \theta)\right)^{n-1} \frac{1}{\theta} \\
&= \frac{n}{\theta}\left[2 - \frac{y_1}{\theta}\right]^{n-1}, \qquad \theta < y_1 < 2\theta
\end{aligned}
$$

$$
\begin{aligned}
f_{Y_n}(y_n) &= \frac{n!}{(n-1)!(n-n)!}[F_X(y_n)]^{n-1}[1 - F_X(y_n)]^{n-n} f_X(y_n) \\
&= n\left(1 - \frac{1}{\theta}(y_n - \theta)\right)^{n-1} \frac{1}{\theta} \\
&= \frac{n}{\theta}\left[\frac{y_n}{\theta} - 1\right]^{n-1}, \qquad \theta < y_n < 2\theta
\end{aligned}
$$

$$
\begin{aligned}
f_{Y_1,Y_n}(y_1, y_n) &= \frac{n!}{(1-1)!(n-1-1)!(n-n)!}[F_X(y_1)]^{1-1}[F_X(y_n) - F_X(y_1)]^{n-1-1} \\
&\quad \cdot [1 - F_X(y_n)]^{n-n} f_X(y_1) f_X(y_n) \\
&= n(n-1)\left(\frac{y_n - \theta}{\theta} - \frac{y_1 - \theta}{\theta}\right)^{n-2} \frac{1}{\theta} \cdot \frac{1}{\theta} \\
&= n(n-1)\left(\frac{1}{\theta^n}\right)(y_n - y_1)^{n-2}, \qquad \theta < y_1 \le y_n < 2\theta
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ E(Y_1) &= \int_\theta^{2\theta} y_1 f_{Y_1}(y_1)\ dy_1 \\
&= \frac{n}{\theta}\int_\theta^{2\theta} y_1\left(2-\frac{y_1}{\theta}\right)^{n-1}\ dy_1 \\
&= \frac{n}{\theta^n}\int_\theta^{2\theta} y_1(2\theta-y_1)^{n-1}\ dy_1 \\
&= \frac{n}{\theta^n}\int_\theta^{0} -(2\theta-z)z^{n-1}\ dz, \qquad\qquad \text{let } z=2\theta-y_1, dz=-dy_1 \\
&= \frac{n}{\theta^n}\int_0^{\theta} (-z^n+2\theta z^{n-1})\ dz \\
&= \frac{n}{\theta^n}\left[\frac{-1}{n+1}\ z^{n+1}+\frac{2\theta}{n}\ z^n\right]_0^\theta \\
&= \frac{n}{\theta^n}\left[\frac{-\theta^{n+1}}{n+1}+\frac{2\theta^{n+1}}{n}\right] \\
&= \frac{-n\theta}{n+1}+2\theta \\
&= \frac{n+2}{n+1}\ \theta
\end{aligned}
$$

$$
\begin{aligned}
E(Y_n) &= \int_\theta^{2\theta} y_n f_{Y_n}(y_n)\ dy_n \\
&= \frac{n}{\theta}\int_\theta^{2\theta} y_n\left(\frac{y_n}{\theta}-1\right)^{n-1}\ dy_n \\
&= \frac{n}{\theta^n}\int_\theta^{2\theta} y_n(y_n-\theta)^{n-1}\ dy_n \\
&= \frac{n}{\theta^n}\int_0^{\theta} (z+\theta)z^{n-1}\ dz, \qquad\qquad \text{let } z=y_n-\theta, dz=dy_n \\
&= \frac{n}{\theta^n}\int_0^{\theta} (z^n+\theta z^{n-1})\ dz \\
&= \frac{n}{\theta^n}\left[\frac{1}{n+1}\ z^{n+1}+\frac{\theta}{n}\ z^n\right]_0^\theta \\
&= \frac{n}{\theta^n}\left[\frac{\theta^{n+1}}{n+1}+\frac{\theta^{n+1}}{n}\right] \\
&= \frac{n\theta}{n+1}+\theta \\
&= \frac{2n+1}{n+1}\ \theta
\end{aligned}
$$

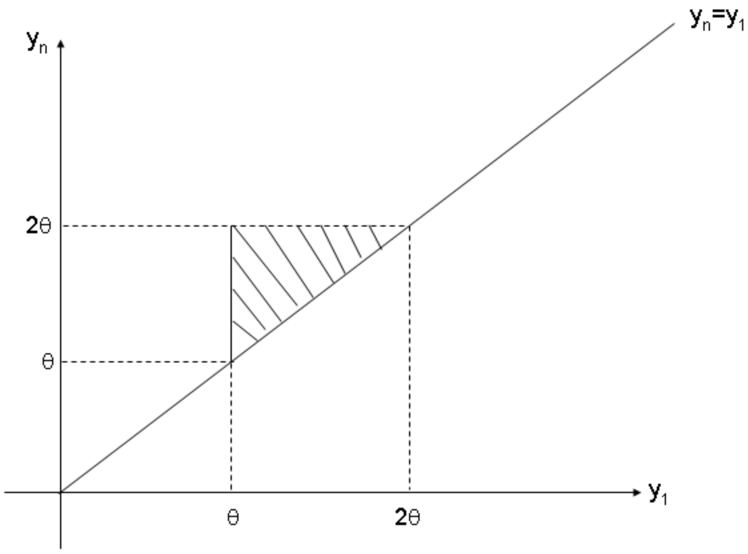
$$
\begin{aligned}
E(U_1) &= E\left(\frac{n+1}{2n+1}\ Y_n\right) \\
&= \frac{n+1}{2n+1}\cdot E(Y_n) \\
&= \frac{n+1}{2n+1}\cdot\frac{2n+1}{n+1}\ \theta = \theta \\
E(U_2) &= E\left(\frac{n+1}{5n+4}\ (2Y_n+Y_1)\right) \\
&= \frac{n+1}{5n+4}\ [2E(Y_n)+E(Y_1)]
\end{aligned}
$$

$$= \quad \frac{n+1}{5n+4}\left[\frac{4n+2}{n+1}\theta + \frac{n+2}{n+1}\theta\right] = \theta$$

$\therefore$ U_1 and U_2 are unbiased estimators for θ.

$$\begin{aligned}
E(Y_1^2) &= \int_\theta^{2\theta} y_1^2 f_{Y_1}(y_1)\ dy_1 \\
&= \frac{n}{\theta}\int_\theta^{2\theta} y_1^2\left(2-\frac{y_1}{\theta}\right)^{n-1}\ dy_1 \\
&= \frac{n}{\theta^n}\int_\theta^{2\theta} y_1^2(2\theta-y_1)^{n-1}\ dy_1 \\
&= \frac{n}{\theta^n}\int_\theta^{0} -(2\theta-z)^2 z^{n-1}\ dz, \qquad \text{let } z = 2\theta - y_1, dz = -dy_1 \\
&= \frac{n}{\theta^n}\int_0^{\theta} (z^{n+1} - 4\theta z^n + 4\theta^2 z^{n-1})\ dz \\
&= \frac{n}{\theta^n}\left[\frac{1}{n+2}\ z^{n+2} - \frac{4\theta}{n+1}\ z^{n+1} + \frac{4\theta^2}{n}\ z^n\right]_0^\theta \\
&= \frac{n}{\theta^n}\left[\frac{\theta^{n+2}}{n+2} - \frac{4\theta^{n+2}}{n+1} + \frac{4\theta^{n+2}}{n}\right] \\
&= \left[\frac{n}{n+2} - \frac{4n}{n+1} + 4\right]\theta^2
\end{aligned}$$

$$\begin{aligned}
E(Y_n) &= \int_\theta^{2\theta} y_n^2 f_{Y_n}(y_n)\ dy_n \\
&= \frac{n}{\theta}\int_\theta^{2\theta} y_n^2\left(\frac{y_n}{\theta}-1\right)^{n-1}\ dy_n \\
&= \frac{n}{\theta^n}\int_\theta^{2\theta} y_n^2(y_n-\theta)^{n-1}\ dy_n \\
&= \frac{n}{\theta^n}\int_0^{\theta} (z+\theta)^2 z^{n-1}\ dz, \qquad \text{let } z = y_n - \theta, dz = dy_n \\
&= \frac{n}{\theta^n}\int_0^{\theta} (z^{n+1} + 2\theta z^n + \theta^2 z^{n-1})\ dz \\
&= \frac{n}{\theta^n}\left[\frac{1}{n+2}\ z^{n+2} + \frac{2\theta}{n+1}\ z^{n+1} + \frac{\theta^2}{n}\ z^n\right]_0^\theta \\
&= \left[\frac{n}{n+2} + \frac{2n}{n+1} + 1\right]\theta^2
\end{aligned}$$



$$
\begin{aligned}
E(Y_1Y_n) &= \int_\theta^{2\theta}\int_\theta^{y_n} y_1 y_n f_{Y_1,Y_n}(y_1,y_n)\ dy_1\ dy_n \\
&= \frac{n(n-1)}{\theta^n}\int_\theta^{2\theta}\int_\theta^{y_n} y_1 y_n (y_n-y_1)^{n-2}\ dy_1\ dy_n \\
&= \frac{n(n-1)}{\theta^n}\int_\theta^{2\theta} y_n \int_\theta^{y_n} y_1 (y_n-y_1)^{n-2}\ dy_1\ dy_n \\
&= \frac{n(n-1)}{\theta^n}\int_\theta^{2\theta} y_n \int_{y_n-\theta}^{0} -(y_n-z)z^{n-2}\ dz\ dy_n, \qquad \text{let } z=y_n-y_1, dz=-dy_1 \\
&= \frac{n(n-1)}{\theta^n}\int_\theta^{2\theta} y_n \int_0^{y_n-\theta} (y_n\ z^{n-2}-z^{n-1})\ dz\ dy_n \\
&= \frac{n(n-1)}{\theta^n}\int_\theta^{2\theta} y_n \left[\frac{y_n}{n-1}z^{n-1}-\frac{1}{n}z^n\right]_0^{y_n-\theta}\ dy_n \\
&= \frac{n(n-1)}{\theta^n}\int_\theta^{2\theta} y_n \left[\frac{1}{n-1}y_n\ (y_n-\theta)^{n-1}-\frac{1}{n}(y_n-\theta)^n\right]\ dy_n \\
&= \frac{n(n-1)}{\theta^n}\left\{\int_\theta^{2\theta}\frac{1}{n-1}\ y_n^2\ (y_n-\theta)^{n-1}\ dy_n-\int_\theta^{2\theta}\frac{1}{n}\ y_n\ (y_n-\theta)^n\ dy_n\right\} \\
&= \frac{n(n-1)}{\theta^n}\left\{\frac{1}{n-1}\int_0^\theta (z+\theta)^2\ z^{n-1}\ dz-\frac{1}{n}\int_0^\theta (z+\theta)\ z^n\ dz\right\}, \\
&\qquad (\text{let } z=y_n-\theta, dz=dy_n) \\
&= \frac{n(n-1)}{\theta^n}\left\{\frac{1}{n-1}\left[\frac{z^{n+2}}{n+2}+\frac{2\theta z^{n+1}}{n+1}+\frac{\theta^2 z^n}{n}\right]_0^\theta-\frac{1}{n}\left[\frac{1}{n+2}z^{n+2}+\frac{\theta}{n+1}z^{n+1}\right]_0^\theta\right\} \\
&= \frac{n(n-1)}{\theta^n}\left\{\frac{1}{n-1}\left(\frac{\theta^{n+2}}{n+2}+\frac{2\theta^{n+2}}{n+1}+\frac{\theta^{n+2}}{n}\right)-\frac{1}{n}\left(\frac{\theta^{n+2}}{n+2}+\frac{\theta^{n+2}}{n+1}\right)\right\} \\
&= n\left(\frac{1}{n+2}+\frac{2}{n+1}+\frac{1}{n}\right)\theta^2-(n-1)\left(\frac{1}{n+2}+\frac{1}{n+1}\right)\theta^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left(2 + \frac{1}{n+2}\right)\theta^2 \\
&= \frac{2n+5}{n+2}\theta^2
\end{aligned}
$$

$$
\begin{aligned}
Var(Y_1) &= E(Y_1^2) - [E(Y_1)]^2 \\
&= \left(\frac{n}{n+2} - \frac{4n}{n+1} + 4\right)\theta^2 + \left[\left(\frac{n+2}{n+1}\right)\ \theta\right]^2 \\
&= \left(\frac{n}{n+2} - \frac{4n}{n+1} + 4 - \left(1 + \frac{1}{n+1}\right)^2\right)\theta^2 \\
&= \left(\frac{n}{n+2} - \frac{4n}{n+1} + 4 - 1 - \frac{2}{n+1} - \frac{1}{(n+1)^2}\right)\theta^2 \\
&= \left(\frac{n}{n+2} - \frac{4n}{n+1} + 3 - \frac{1}{(n+1)^2}\right)\theta^2 \\
&= \frac{n\theta^2}{(n+1)^2(n+2)}
\end{aligned}
$$

$$
\begin{aligned}
Var(Y_n) &= E(Y_n^2) - [E(Y_n)]^2 \\
&= \left(\frac{n}{n+2} + \frac{2n}{n+1} + 1\right)\theta^2 - \left(\frac{2n+1}{n+1}\ \theta\right)^2 \\
&= \left(\frac{n}{n+2} + \frac{2n}{n+1} + 1 - \left(2 - \frac{1}{n+1}\right)^2\right)\theta^2 \\
&= \left(\frac{n}{n+2} + \frac{2n}{n+1} + 1 - 4 + \frac{4}{n+1} - \frac{1}{(n+1)^2}\right)\theta^2 \\
&= \left(\frac{n}{n+2} + \frac{2n+4}{n+1} - 3 - \frac{1}{(n+1)^2}\right)\theta^2 \\
&= \frac{n\theta^2}{(n+1)^2(n+2)}
\end{aligned}
$$

$$
\begin{aligned}
Cov(Y_1, Y_n) &= E(Y_1Y_n) - E(Y_1)E(Y_n) \\
&= \left(\frac{2n+5}{n+2}\right)\theta^2 - \left[\left(\frac{n+2}{n+1}\right)\ \theta\left(\frac{2n+1}{n+1}\right)\ \theta\right] \\
&= \frac{\theta^2}{(n+2)(n+1)^2}[(2n+5)(n+1)^2 - (n+2)(n+2)(2n+1)] \\
&= \frac{\theta^2}{(n+2)(n+1)^2}
\end{aligned}
$$

$$
\begin{aligned}
Var(U_1) &= Var\left(\frac{n+1}{2n+1}\ Y_n\right) \\
&= \left(\frac{n+1}{2n+1}\right)^2 Var(Y_n) \\
&= \left(\frac{n+1}{2n+1}\right)^2 \cdot \frac{n\theta^2}{(n+1)^2(n+2)} \\
&= \frac{n\theta^2}{(2n+1)^2(n+2)}
\end{aligned}
$$

$$
Var(U_2) = Var[\left(\frac{n+1}{5n+4}\right)(2Y_n + Y_1)]
$$

$$
\begin{aligned}
&= \left(\frac{n+1}{5n+4}\right)^2 Var(2Y_n + Y_1) \\
&= \left(\frac{n+1}{5n+4}\right)^2 [Var(Y_1) + 4Cov(Y_1, Y_n) + 4Var(Y_n)] \\
&= \left(\frac{n+1}{5n+4}\right)^2 \left[\frac{n\theta^2}{(n+1)^2(n+2)} + \frac{4\theta^2}{(n+2)(n+1)^2} + \frac{4n\theta^2}{(n+1)^2(n+2)}\right] \\
&= \frac{\theta^2}{(5n+4)^2}\left(\frac{5n+4}{n+2}\right) \\
&= \frac{\theta^2}{(5n+4)(n+2)}
\end{aligned}
$$

$$
\begin{aligned}
Var(U_1) - Var(U_2) &= \frac{\theta^2}{n+2}\left[\frac{n}{(2n+1)^2} - \frac{1}{5n+4}\right] \\
&= \frac{\theta^2}{n+2} \cdot \frac{n^2-1}{(2n+1)^2(5n+4)} \\
&> 0 \qquad (\because\ n^2 - 1 > 0 \text{ for } n > 1)
\end{aligned}
$$

$\therefore$ U_2 is better than U_1 for estimating θ.

27. $f(x, \theta) = \exp\{-\frac{1}{2}\log(2\pi) - \frac{1}{2}\theta^2 - \frac{1}{2}x^2 + \theta x\}$, so $\sum X_i$ is C-S for θ
Note that $\bar{X} \sim N(\theta, \frac{1}{n})$,
so $E(\bar{X}^2) = Var(\bar{X}) + (E(\bar{X}))^2 = \frac{1}{n} + \theta^2 \Rightarrow E(\bar{X}^2 - \frac{1}{n}) = \theta^2$
$\therefore$ $\bar{X}^2 - \frac{1}{n}$ is the UMVUE for θ^2.

The mgf of $\bar{X}$ is $m_{\bar{X}}(t) = \exp\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\}$

$$
\begin{aligned}
m'_{\bar{X}}(t) &= \left(\theta + \frac{t}{n}\right)\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} \\
m''_{\bar{X}}(t) &= \left(\theta + \frac{t}{n}\right)^2 \exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} + \frac{1}{n}\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} \\
m'''_{\bar{X}}(t) &= \left(\theta + \frac{t}{n}\right)^3 \exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} + \frac{3}{n}\left(\theta + \frac{t}{n}\right)\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} \\
m^{(4)}_{\bar{X}}(t) &= \left(\theta + \frac{t}{n}\right)^4 \exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} + 3\left(\theta + \frac{t}{n}\right)^2 (\frac{1}{n})\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} \\
&\quad + \frac{3}{n}\left(\theta + \frac{t}{n}\right)^2 \exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} + \frac{3}{n}\left(\frac{1}{n}\right)\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
E(\bar{X}^4) &= m^{(4)}_{\bar{X}}(0) \\
&= \theta^4 + 3\theta^2\left(\frac{1}{n}\right) + \left(\frac{3}{n}\right)\theta^2 + \left(\frac{3}{n}\right)\left(\frac{1}{n}\right) \\
&= \theta^4 + \frac{6}{n}\,\theta^2 + \frac{3}{n^2} \\
Var(\bar{X}^2 - \frac{1}{n}) &= Var(\bar{X}^2) \\
&= E\bar{X}^4 - (E\bar{X}^2)^2 \\
&= \theta^4 + \frac{6}{n}\,\theta^2 + \frac{3}{n^2} - (\theta^2 + \frac{1}{n})^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{4}{n}\right)\theta^2 + \frac{2}{n^2} \\
\log f(x;\theta) &= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\theta^2 - \frac{1}{2}x^2 + \theta x \\
\frac{\partial}{\partial\theta}\log f(x;\theta) &= -\theta + x \\
\frac{\partial^2}{\partial\theta^2}\log f(x;\theta) &= -1 \\
E\left[\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)\right] &= -1
\end{aligned}
$$

$$\therefore \text{ CRLB for } \theta^2 = \frac{(2\theta)^2}{n} = \frac{4\theta^2}{n}$$

$$Var(\bar{X}^2 - \frac{1}{n}) = \frac{4\theta^2}{n} + \frac{2}{n^2} > \frac{4\theta^2}{n} = \text{CRLB for } \theta^2$$

28. Recall that $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ and $\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \bar{X})^2 \sim \chi^2_{(n-1)}$ are independent.
Note:

$$\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \mu)^2 \sim \chi^2_{(n)}$$

$$\frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \bar{X})^2 \sim \chi^2_{(n-1)}$$

Let $S = \sum_{i=1}^n (X_i - \bar{X})^2$ and $Y = \frac{S}{\sigma^2} \sim \chi^2_{(n-1)}$

$$
\begin{aligned}
\therefore\ f_Y(y) &= \frac{y^{\frac{n-1}{2}-1}\cdot e^{-y/2}}{2^{\frac{n-1}{2}}\cdot\Gamma(\frac{n-1}{2})} \\
\therefore\ E\left(\frac{1}{\sqrt{S}}\right) &= E\left(\frac{1}{\sigma\sqrt{Y}}\right) \\
&= \int_0^\infty \frac{1}{\sigma\sqrt{y}}\ f_Y(y)\ dy \\
&= \int_0^\infty \frac{1}{\sigma\sqrt{y}}\cdot\frac{y^{\frac{n-1}{2}-1}\cdot e^{-y/2}}{2^{\frac{n-1}{2}}\cdot\Gamma(\frac{n-1}{2})}\ dy \\
&= \int_0^\infty \frac{1}{\sigma}\cdot\frac{y^{\frac{n-2}{2}-1}\cdot e^{-y/2}}{2^{\frac{n-1}{2}}\cdot\Gamma(\frac{n-1}{2})}\ dy \\
&= \frac{2^{-1/2}\cdot\Gamma(\frac{n-2}{2})}{\sigma\cdot\Gamma(\frac{n-1}{2})}\int_0^\infty \frac{y^{\frac{n-2}{2}-1}\cdot e^{-y/2}}{2^{\frac{n-2}{2}}\cdot\Gamma(\frac{n-2}{2})}\ dy \\
&= \frac{1}{\sigma}\frac{\Gamma(\frac{n-2}{2})}{\sqrt{2}\cdot\Gamma(\frac{n-1}{2})} \quad \text{pdf of } \chi^2_{(n-2)}
\end{aligned}
$$

$$
\begin{aligned}
& \therefore\ E\left[\left(\frac{\bar{X}}{\sqrt{S}}\right)\cdot\frac{\sqrt{2}\cdot\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\right] \\
= \;& E(\bar{X})\cdot E\left[\frac{1}{\sqrt{S}}\cdot\frac{\sqrt{2}\cdot\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\right] \quad \because\ \bar{X} \text{ and } S \text{ are independent} \\
= \;& \mu\cdot\frac{1}{\sigma} \\
= \;& \frac{\mu}{\sigma}
\end{aligned}
$$

$$
\therefore\ \frac{\sqrt{2}\cdot\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\cdot\left(\frac{\bar{X}}{\sqrt{S}}\right) = \frac{\sqrt{2}\cdot\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\cdot\left(\frac{\bar{X}}{\sqrt{\sum_{i=1}^{n}(X_i-\bar{X})^2}}\right)
$$

is the UMVUE for $\frac{\mu}{\sigma}$.

29.

$$
\begin{aligned}
E(U) &= E[c\bar{X}+(1-c)\bar{Y}] \\
&= cE(\bar{X})+(1-c)E(\bar{Y}) \\
&= cE(X)+(1-c)E(Y) \\
&= c\theta+(1-c)\theta \\
&= \theta \\
& \therefore\ U \text{ is an unbiased estimator of } \theta
\end{aligned}
$$

$$
\begin{aligned}
Var(U) &= Var(c\bar{X}+(1-c)\bar{Y}) \\
&= c^2Var(\bar{X})+(1-c)^2Var(\bar{Y}) \quad (X, Y \text{ ar independent}) \\
&= c^2\left(\frac{\sigma_1^2}{m}\right)+(1-c)^2\left(\frac{\sigma_2^2}{n}\right) \\
\text{Let } g(c) &= c^2\left(\frac{\sigma_1^2}{m}\right)+(1-c)^2\left(\frac{\sigma_2^2}{n}\right) \\
&= c^2\left(\frac{\sigma_1^2}{m}\right)+(1-2c+c^2)\left(\frac{\sigma_2^2}{n}\right) \\
g'(c) &= 2c\left(\frac{\sigma_1^2}{m}\right)+(2c-2)\left(\frac{\sigma_2^2}{n}\right) \\
g'(c)=0 &\Rightarrow 2c\left(\frac{\sigma_1^2}{m}\right)+(2c-2)\left(\frac{\sigma_2^2}{n}\right)=0 \\
&\Rightarrow \left(\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}\right)c=\frac{\sigma_2^2}{n} \\
&\Rightarrow c=\frac{\frac{\sigma_2^2}{n}}{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}
\end{aligned}
$$