Fall 02/03

1.a $E(Y) = \frac{0+\theta}{2} = \frac{\theta}{2}$, $var(Y) = \frac{(\theta-0)^2}{12} = \frac{\theta^2}{12}$

b. $P(Z \le z) = \prod_{i=1}^{n} P(Y_i \le z) = P(Y_i \le z)^n = \left(\frac{z}{\theta}\right)^n$

$f(z) = \frac{d}{dy} P(Z \le z) = \frac{n z^{n-1}}{\theta^n}$, $z \in (0, \theta)$

c. $E(Z) = \int_0^\theta z \cdot \frac{n z^{n-1}}{\theta^n} dz = \frac{n}{\theta^n} \int_0^\theta z^n dz = \frac{n}{\theta^n} \frac{z^{n+1}}{n+1}\Big|_0^\theta = \frac{n\theta}{n+1}$

$E(Z^2) = \int_0^\theta z^2 \frac{n z^{n-1}}{\theta^n} dz = \frac{n}{\theta^n} \int_0^\theta z^{n+1} dy = \frac{n}{\theta^n} \cdot \frac{z^{n+2}}{n+2}\Big|_0^\theta = \frac{n\theta^2}{n+2}$

$var(Z) = E(Z^2) - (EZ)^2 = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2 = \frac{n\theta^2}{(n+1)^2(n+2)}$

d. $E(\bar{Y}) = E(Y) = \frac{\theta}{2}$, $var(\bar{Y}) = \frac{var(Y)}{n} = \frac{\theta^2}{12n}$

e. From (c), $E(Z) = \frac{n\theta}{n+1} \Rightarrow E\left(\frac{n+1}{n} Z\right) = \theta \Rightarrow \tilde{\theta}_1 = \left(\frac{n+1}{n}\right) Z$

From (d), $E(\bar{Y}) = \frac{\theta}{2} \Rightarrow E(2\bar{Y}) = \theta \Rightarrow \tilde{\theta}_2 = 2\bar{Y}$

f. $var(\tilde{\theta}_1) = var\left(\frac{n+1}{n} Z\right) = \left(\frac{n+1}{n}\right)^2 var Z = \left(\frac{n+1}{n}\right)^2 \cdot \frac{n\theta^2}{(n+1)^2(n+2)} = \frac{\theta^2}{n(n+2)}$

$var(\tilde{\theta}_2) = var(2\bar{Y}) = 4\, var\,\bar{Y} = 4 \cdot \frac{\theta^2}{12n} = \frac{\theta^2}{3n}$

$var(2\bar{Y}) = \frac{\theta^2}{3n} \geq \frac{\theta^2}{n(n+2)} = var\left(\frac{n+1}{n} Z\right)$ when $n \geq 1$,

So $\frac{n+1}{n} Z$ is more efficient than $2\bar{Y}$.

2.

a. $f_{X_i}(x) = \frac{\theta^x e^{-\theta}}{x!}$

$L(\theta) = \prod_{i=1}^{n} f_{X_i}(x_i) = \frac{\theta^{\Sigma x_i} e^{-n\theta}}{\prod_{i=1}^{n} x_i!}$

$\log L(\theta) = \Sigma x_i \log\theta - n\theta - \log \prod_{i=1}^{n} x_i!$

$\frac{\partial}{\partial\theta} \log L(\theta) = \frac{\Sigma x_i}{\theta} - n = 0$

$\Rightarrow \hat{\theta} = \frac{\Sigma x_i}{n} = \bar{x}$

$\therefore \hat{\theta}^2 = \bar{x}^2$

b. $E(a\bar{x} + b\bar{x}^2) = a E\bar{x} + b E\bar{x}^2 = a\theta + b[var\,\bar{x} + (E\bar{x})^2] = a\theta + b\left(\frac{\theta}{n} + \theta^2\right)$

$= \left(a + \frac{b}{n}\right)\theta + b\theta^2$

$\Rightarrow a + \frac{b}{n} = 0$ and $b = 1 \Rightarrow a = -\frac{1}{n}$ and $b = 1$

$$var\left(-\frac{\bar{X}}{n}+\bar{X}^2\right) = var\left(\frac{\bar{X}}{n}\right) + var(\bar{X}^2) - 2cov\left(\frac{\bar{X}}{n}, \bar{X}^2\right)$$

$$Var(\bar{X}^2) = E\bar{X}^4 - (E\bar{X}^2)^2$$

$$m_X(t) = \exp\{\theta(e^t-1)\}$$

$$m_X'(t) = \theta e^t e^{\theta(e^t-1)}$$

$$m_X''(t) = \theta e^t e^{\theta(e^t-1)}[1+\theta e^t]$$

$$m_X^{(3)}(t) = \theta e^t e^{\theta(e^t-1)}[1+3\theta e^t+\theta^2 e^{2t}]$$

$$m_X^{(4)}(t) = \theta e^t e^{\theta(e^t-1)}[1+7\theta e^t+6\theta^2 e^{2t}+\theta^3 e^{3t}]$$

$E\bar{X} = m_X'(t)|_{t=0} = \theta$, $EX^3 = m_X'''(t)|_{t=0} = \theta(1+3\theta+\theta^2)$

$E\bar{X}^2 = m_X''(t)|_{t=0} = \theta(1+\theta)$, $EX^4 = m_X^{(4)}(t)|_{t=0} = \theta(1+7\theta+6\theta^2+\theta^3)$

$$E\bar{X}^4 = E\left(\frac{(\sum X_i)^4}{n^4}\right) = \frac{1}{n^4}E\Big(\sum X_i^4 + 3\sum_{i=1}^n\sum_{j\neq i} X_i^2X_j^2 + 4\sum_{i=1}^n\sum_{j\neq i}X_i^3X_j + 6\sum_{i=1}^n\sum_{j\neq i}\sum_{k\neq i,j}X_i^2X_jX_k$$
$$+ \sum_{i=1}^n\sum_{j\neq i}\sum_{k\neq i,j}\sum_{l\neq i,j,k}X_iX_jX_kX_l\Big)$$

$$= \frac{1}{n^4}\{n[\theta(1+7\theta+6\theta^2+\theta^3)] + 3n(n-1)(\theta(\theta+1))^2 + 4n(n-1)[\theta(1+3\theta+\theta^2)]\theta$$
$$+ 6n(n-1)(n-2)[\theta(1+\theta)]\cdot\theta\cdot\theta + n(n-1)(n-2)(n-3)\theta^4\}$$

$$= \frac{\theta}{n^3} + \frac{7\theta^2}{n^2} + \frac{6\theta^3}{n} + \theta^4$$

$$E\bar{X}^3 = E\left(\frac{(\sum X_i)^3}{n^3}\right) = \frac{1}{n^3}E((\sum X_i)^3) = \frac{1}{n^3}E\Big(\sum X_i^3 + 3\sum_{i=1}^n\sum_{j\neq i}X_i^2X_j + \sum_{i=1}^n\sum_{j\neq i}\sum_{k\neq i,j}X_iX_jX_k\Big)$$
$$= \frac{1}{n^3}\{n[\theta(1+3\theta+\theta^2)] + 3n(n-1)\theta^2(\theta+1) + n(n-1)(n-2)\theta^3\}$$
$$= \frac{1}{n^3}[n^3\theta^3 + 3n^2\theta^2 + n\theta] = \frac{\theta}{n^2} + \frac{3\theta^2}{n} + \theta^3$$

$$E\bar{X}^2 = var\,\bar{x} - (E\bar{X})^2 = \frac{\theta}{n} + \theta^2$$

$$Var(\bar{X}^2) = E\bar{X}^4 - (E\bar{X}^2)^2 = \frac{\theta}{n^3} + \frac{7\theta^2}{n^2} + \frac{6\theta^3}{n} + \theta^4 - \left(\frac{\theta}{n}+\theta^2\right)^2 = \frac{\theta}{n^3} + \frac{6\theta^2}{n^2} + \frac{4\theta^3}{n}$$

$$var\left(\frac{\bar{X}}{n}\right) = \left(\frac{1}{n^2}\right)\left(\frac{\theta}{n}\right) = \frac{\theta}{n^3}$$

$$cov\left(\frac{\bar{X}}{n}, \bar{X}^2\right) = E\left(\frac{\bar{X}^3}{n}\right) - E(\bar{X}^2)E\left(\frac{\bar{X}}{n}\right) = \frac{1}{n}\left(\frac{\theta}{n^2} + \frac{3\theta^2}{n} + \theta^3\right) - \left(\frac{\theta}{n}+\theta^2\right)\left(\frac{\theta}{n}\right)$$
$$= \frac{\theta}{n^3} + \frac{2\theta^2}{n^2}$$

$$\therefore var\left(-\frac{\bar{X}}{n}+\bar{X}^2\right) = \frac{\theta}{n^3} + \left(\frac{\theta}{n^3} + \frac{6\theta^2}{n^2} + \frac{4\theta^3}{n}\right) - 2\left(\frac{\theta}{n^3} + \frac{2\theta^2}{n^2}\right)$$
$$= \frac{2\theta^2}{n^2} + \frac{4\theta^3}{n}$$

$$E\left(\frac{\partial^2}{\partial\theta^2}\log f_X(x_i,\theta)\right) = E\left(\frac{\partial}{\partial\theta}\left(-1+\frac{x_i}{\theta}\right)\right) = E\left(-\frac{x_i}{\theta^2}\right) = -\frac{1}{\theta^2}(\theta) = -\frac{1}{\theta}$$

$$\therefore \text{CRLB for } \theta^2 = -\frac{(2\theta)^2}{n(-\frac{1}{\theta})} = \frac{4\theta^3}{n}$$

The CRLB is not attained since

$$\sum \frac{\partial}{\partial\theta}\log f(x_i;\theta) = \sum\left(-1+\frac{x_i}{\theta}\right) = \frac{n}{\theta}(\bar{x}-\lambda) ,$$

So only the UMVUE of λ can achieve the CRLB.

3.

a. $X \sim Bin(n;\theta)$

$$f(x;\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x} = \exp\{n\log(1-\theta)+\log\binom{n}{x}+[\log\tfrac{\theta}{1-\theta}]x\}$$

$\Rightarrow \sum_{i=1}^{m} X_i$ is a complete minimal suff. stat.

$$S = \sum_{i=1}^{m} X_i \sim Bin(mn,\theta)$$

$$E(g(S)) = \theta^k$$

$$\Rightarrow \sum_{s=0}^{mn} g(s)\binom{mn}{s}\theta^s(1-\theta)^{mn-s} = \theta^k$$

$$\sum_{s=0}^{mn} g(s)\binom{mn}{s}\theta^{s-k}(1-\theta)^{mn-s} = 1$$

$$\sum_{s=0}^{mn}\binom{mn-k}{s-k}\theta^{s-k}(1-\theta)^{mn-k-(s-k)} \cdot \binom{mn}{s}\Big/\binom{mn-k}{s-k}\cdot g(s) = 1$$

$$\Rightarrow g(s) = \binom{mn-k}{s-k}\Big/\binom{mn}{s} I_{\{k,k+1,\cdots,mn\}}(s)$$

$\therefore$ UMVUE of θ^k is $\binom{mn-k}{\sum X_i-k}\Big/\binom{mn}{\sum X_i} I_{\{k,k+1,\cdots,mn\}}(\sum X_i)$

b. $X \sim \exp(\lambda)$

$$f(x;\lambda) = \lambda e^{-\lambda x} = \exp\{\log\lambda - \lambda x\}$$

$\Rightarrow \sum_{i=1}^{n} X_i$ is a complete minimal suff. stat.

$$S = \sum_{i=1}^{n} X_i \sim Gamma(n,\lambda)$$

$$E(g(S)) = \lambda^r$$

$$\Rightarrow \int g(s)\cdot\frac{s^{n-1}e^{-\lambda s}}{\Gamma(n)\lambda^{-n}}\,ds = \lambda^r$$

$$\int g(s)\cdot\frac{s^{n-1}e^{-\lambda s}}{\Gamma(n)\lambda^{-(n-r)}}\,ds = 1$$

$$\int \frac{s^{(n-r)-1}e^{-\lambda s}}{\Gamma(n-r)\lambda^{-(n-r)}}\cdot\frac{s^r\,\Gamma(n-r)}{\Gamma(n)}\cdot g(s) = 1$$

$$\Rightarrow g(s) = \frac{\Gamma(n)}{s^r\,\Gamma(n-r)}$$

$\therefore$ UMVUE of λ^r is $\dfrac{\Gamma(n)}{(\sum X_i)^r\,\Gamma(n-r)}$.