03/04 final exam

1a. $Y_1 \sim N(4-\theta, \sigma^2)$, $Y_2 \sim N(2+2\theta, \sigma^2)$

$$L(y_1, y_2 ; \theta, \sigma^2) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[(y_1+\theta-4)^2 + (y_2-2\theta-2)^2\right]\right\}$$

$$\log L = \log C - \frac{1}{2\sigma^2}\left[(y_1+\theta-4)^2 + (y_2-2\theta-2)^2\right]$$

$$\frac{\partial}{\partial\theta}\log L = -\frac{1}{2\sigma^2}\left[2(y_1+\theta-4) \mp (2)(2)(y_2-2\theta-2)\right] = 0$$

$$\Rightarrow \quad 2y_1 - 4y_2 + 10\theta = 0$$

$$\Rightarrow \quad \hat{\theta} = \frac{-y_1 + 2y_2}{5}$$

b. $E(\hat{\theta}) = E\left(-\frac{y_1}{5} + \frac{2y_2}{5}\right) = -\frac{1}{5}(4-\theta) + \frac{2}{5}(2+2\theta) = \theta$

$\Rightarrow \hat{\theta}$ is an unbiased estimator for θ.

c. $\mathrm{var}(\hat{\theta}) = \frac{1}{25}\sigma^2 + \frac{4}{25}\sigma^2 = \frac{1}{5}\sigma^2$

d. $E(W) = E(Y_1 + Y_2 - 6) = 4-\theta + (2+2\theta) - 6 = \theta$,

So, W is unbiased.

$\mathrm{var}(Y_1 + Y_2 - 6) = \sigma^2 + \sigma^2 = 2\sigma^2$

So, $\hat{\theta}$ is more efficient since $\frac{\sigma^2}{5} = \mathrm{var}(\hat{\theta}) < \mathrm{var}(W) = 2\sigma^2$.

2. a) $X \sim gamma(2, \theta)$

$$L(x, \theta) = \prod_{i=1}^{n} \theta^2 x_i e^{-\theta x_i} = \theta^{2n} e^{-\theta \sum_{i=1}^{n} x_i} \cdot \prod_{i=1}^{n} x_i$$

$$\log L(x, \theta) = 2n \log\theta - \theta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log x_i$$

$$\Rightarrow \frac{\partial \log L(x,\theta)}{\partial \theta} = \frac{2n}{\theta} - \sum_{i=1}^{n} x_i \Rightarrow \hat{\theta} = 2n / \sum_{i=1}^{n} x_i$$

$E(X) = \frac{2}{\theta}$, $Var(X) = \frac{2}{\theta^2}$. MLE for Variance is $\frac{2}{\hat{\theta}^2} = \frac{(\sum x_i)^2}{2n^2}$

b). $\frac{\partial^2 \log L(x,\theta)}{\partial \theta^2} = -\frac{2n}{\theta^2}$ $\therefore$ CRLB for $\frac{1}{\theta} = -\frac{(-\frac{1}{\theta^2})^2}{-\frac{2n}{\theta^2}} = \frac{1}{2n\theta^2}$

$f(x, \theta) = \theta^2 x e^{-\theta x} = \exp(2\log\theta + \log x + (-\theta x))$.

$a(\theta) = 2\log\theta$, $b(x) = \log x$. $c(\theta) = -\theta$ $d(x) = x$.

$\Rightarrow \sum_{i=1}^{n} x_i$ is complete and sufficient.

and $E(\bar{X}) = \frac{2}{\theta}$ $\Rightarrow \frac{\bar{X}}{2}$ is UMVUE for $\frac{1}{\theta}$

$$Var(\frac{\bar{X}}{2}) = \frac{1}{4} Var(\bar{X}) = \frac{1}{4} \cdot \frac{1}{n^2} \cdot Var(\sum X_i) = \frac{1}{4} \cdot \frac{1}{n^2} \cdot n \cdot \frac{2}{\theta^2} = \frac{1}{2n\theta^2}$$

$\Rightarrow Var(\frac{\bar{X}}{2}) = CRLB$.

c.) $S = \sum_{i=1}^{n} x_i \sim gamma(2n, \theta)$

$$E(\frac{1}{S}) = \int_0^{\infty} \frac{1}{S} \frac{\theta^{2n} S^{2n-1} e^{-\theta S}}{\Gamma(2n)} dS$$

$$= \int_0^{\infty} \frac{\theta^{2n}}{\Gamma(2n)} \cdot S^{2n-2} e^{-\theta S} dS$$

$$= \int_0^{\infty} \frac{\Gamma(2n-1)}{\theta^{2n-1}} \cdot \frac{\theta^{2n}}{\Gamma(2n)} \cdot \frac{\theta^{2n-1}}{\Gamma(2n-1)} \cdot S^{2n-2} \cdot e^{-\theta S} dS$$

$$= \frac{\Gamma(2n-1)\theta^{2n}}{\Gamma(2n)\theta^{2n-1}} = \theta / 2n-1$$

$\Rightarrow$ UMVUE for $\theta = 2n-1 / \sum_{i=1}^{n} x_i$.

$$CRLB = -\frac{1}{-2n/\theta^2} = \frac{\theta^2}{2n}$$

$\therefore$ UMVUE for θ > CRLB.

3a. $\hat{\theta} = X_{(n)} = Y_n$

b. $P(Y_n < y) = [P(X_i < y)]^n = [\int_0^y \frac{2x}{\theta^2} dx]^n = \left[\frac{1}{\theta^2} x^2 \Big|_0^y\right]^n = \left(\frac{y}{\theta}\right)^{2n}$

$f(y) = \frac{d}{dy} P(Y_n < y) = \frac{2n y^{2n-1}}{\theta^{2n}}$

$E(X_{(n)}) = E(Y_n) = \int_0^\theta y \cdot \frac{2n y^{2n-1}}{\theta^{2n}} dy$

$= \int_0^\theta \frac{2n y^{2n}}{\theta^{2n}} dy = \frac{2n}{\theta^{2n}} \cdot \frac{y^{2n+1}}{2n+1} \Big|_0^\theta$

$= \frac{2n}{2n+1} \theta$

$\Rightarrow E\left(\frac{2n+1}{2n} X_{(n)}\right) = \theta$

c. $E(X_{(n)}^2) = E(Y_n^2) = \int_0^\theta y^2 \cdot \frac{2n y^{2n-1}}{\theta^{2n}} dy$

$= \int_0^\theta \frac{2n y^{2n+1}}{\theta^{2n}} dy = \frac{2n}{\theta^{2n}} \cdot \frac{y^{2n+2}}{2n+2} \Big|_0^\theta$

$= \frac{2n}{2n+2} \theta^2$

$\Rightarrow E\left(\frac{2n+2}{2n} X_{(n)}^2\right) = \theta^2$.

$E\left(\frac{n+1}{n} X_{(n)}^2\right) = \theta^2$

4. a) $X_1, \cdots, X_n$ is a random sample from $exp(\theta)$

$$\begin{cases} H_0: \theta = \theta_0 \\ H_1: \theta = \theta_a \ (\theta_a < \theta_0) \end{cases} \qquad f_{\underset{\sim}{x}}(\underset{\sim}{x}, \theta) = \theta^n e^{-\theta \sum_{i=1}^n x_i}$$

By N-P theorem $C_1 = \{ \underset{\sim}{x} : \frac{f_{\underset{\sim}{x}}(\underset{\sim}{x}, \theta_0)}{f_{\underset{\sim}{x}}(\underset{\sim}{x}, \theta_a)} \le k \}$

$$\frac{f_{\underset{\sim}{x}}(\underset{\sim}{x}, \theta_0)}{f_{\underset{\sim}{x}}(\underset{\sim}{x}, \theta_a)} \le k \Rightarrow \frac{\theta_0^n e^{-\theta_0 \sum_{i=1}^n x_i}}{\theta_a^n e^{-\theta_0 \sum_{i=1}^n x_i}} \le k$$

$$\Rightarrow n \log \theta_0 - \theta_0 \sum_{i=1}^n x_i - n \log \theta_a + \theta_a \sum_{i=1}^n x_i \le \log k$$

$$\Rightarrow (\theta_a - \theta_0) \sum_{i=1}^n x_i \le k' \Rightarrow \sum_{i=1}^n x_i \ge k''$$

the critical region $C_1 = \{ \underset{\sim}{x} : \sum_{i=1}^n x_i \ge k'' \}$

b) Yes. Because C_1 above doesn't contain θ_a.

c) Yes. suppose it is at significant level α.

$$\text{in} \begin{cases} H_0: \theta = \theta_0 \\ H_a: \theta \ge \theta_a \end{cases} \qquad X \sim exp(\theta) \Rightarrow \sum_{i=1}^n x_i \sim gamma(n, \theta) \Rightarrow 2\theta \sum_{i=1}^n x_i \sim \chi^2_{2n}$$

then $C_1 = \{ \underset{\sim}{x} : \sum_{i=1}^n x_i \ge k \} \Rightarrow 2\theta_0 k = \chi^2_{2n,\alpha} \Rightarrow k = \frac{\chi^2_{2n,\alpha}}{2\theta_0}$

$$\text{~~then~~ } Q(\theta) = Pr\left(\sum_{i=1}^n x_i \ge \frac{\chi^2_{2n,\alpha}}{2\theta_0}\right) = Pr\left(2\theta \sum_{i=1}^n x_i \ge \frac{\theta}{\theta_0} \chi^2_{2n,\alpha}\right)$$

$$= Pr\left(Y \ge \frac{\theta}{\theta_0} \chi^2_{2n,\alpha}\right), \quad Y \sim \chi^2_{2n}$$

It is obvious that when $\theta = \theta_0$ $Q(\theta)$ is max. $(\theta \ge \theta_0)$

$$\Rightarrow C_1 \text{ is UMP test for } \begin{cases} H_0: \theta \le \theta_0 \\ H_1: \theta > \theta_0 \end{cases}$$

d). $n = 10$, $\alpha = 0.05$. $\begin{cases} H_0: \theta = 5 \\ H_a: \theta = 2 \end{cases} \Rightarrow \theta_0 = 5$

$$k = \frac{\chi^2_{2n,\alpha}}{2\theta_0} = \frac{\chi^2_{20,0.05}}{10} = 3.141$$

$$\Rightarrow C_1 = \{ \underset{\sim}{x} : \sum_{i=1}^{10} x_i > 3.141 \}$$

Q5.
(a) joint p.d.f. $= \prod_{i=1}^{m} \frac{e^{-\theta_1}\theta_1^{x_i}}{x_i!} \prod_{j=1}^{n} \frac{e^{-\theta_2}\theta_2^{y_j}}{y_j!}$

$$= e^{-(m\theta_1+n\theta_2)} \prod_{i=1}^{m} \frac{\theta_1^{x_i}}{x_i!} \prod_{j=1}^{n} \frac{\theta_2^{y_j}}{y_j!}$$

log likelihood $= -(m\theta_1+n\theta_2) + \sum_{i=1}^{m} x_i \log\theta_1 - \sum_{i=1}^{m} \log x_i! + \sum_{j=1}^{n} y_j \log\theta_2 - \sum_{j=1}^{n} \log y_j!$

Under $H_0: \theta_1 = \theta_2 = \theta_0$

$$\frac{\partial \log \text{likelihood}}{\partial\theta} = -(m+n) + \frac{1}{\theta_0}\sum_{i=1}^{m} x_i + \frac{1}{\theta_0}\sum_{j=1}^{n} y_j = 0 \Rightarrow \hat\theta_0 = \frac{\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j}{m+n} = \frac{m\bar x + n\bar y}{m+n}$$

Under Θ,

$$\hat\theta_1 = \frac{\sum_{i=1}^{m} x_i}{m}, \quad \hat\theta_2 = \frac{\sum_{j=1}^{n} y_j}{n}$$

$$\lambda(\underset{\sim}{x}) = \frac{e^{-(m\hat\theta_0+n\hat\theta_0)} \prod_{i=1}^{m} \frac{\hat\theta_0^{x_i}}{x_i!} \prod_{j=1}^{n} \frac{\hat\theta_0^{y_j}}{y_j!}}{e^{-(m\hat\theta_1+n\hat\theta_2)} \prod_{i=1}^{m} \frac{\hat\theta_1^{x_i}}{x_i!} \prod_{j=1}^{n} \frac{\hat\theta_2^{y_j}}{y_j!}}$$

$$= \frac{\hat\theta_0^{\left(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j\right)}}{\hat\theta_1^{\sum_{i=1}^{m} x_i}\ \hat\theta_2^{\sum_{j=1}^{n} y_j}}$$

$$= \frac{\left(\frac{m\bar x+n\bar y}{m+n}\right)^{(m\bar x+n\bar y)}}{\bar x^{m\bar x}\ \bar y^{n\bar y}}$$

$$= \left(\frac{m\bar x+n\bar y}{(m+n)\bar x}\right)^{m\bar x} \left(\frac{m\bar x+n\bar y}{(m+n)\bar y}\right)^{n\bar y}$$

(b) $r = 2-1 = 1$,

$$-2\log\lambda(\underset{\sim}{x}) = -2\{m\bar x\log(m\bar x+n\bar y) - m\bar x\log((m+n)\bar x) + n\bar y\log(m\bar x+n\bar y) - n\bar y\log((m+n)\bar y)\}$$

$$= -2\{(m\bar x+n\bar y)\log(m\bar x+n\bar y) - (m\bar x+n\bar y)\log((m+n)\bar x)\}.$$

$$\Rightarrow C = \{x : -2\log\lambda(x) \geq \chi_1^2\}.$$

6: $X_1, \cdots, X_n$ is random sample from $U(0,\theta)$

$$\begin{cases} H_0: \theta = 1.6 \\ H_a: \theta = 2 \end{cases}$$

a) i) $n=1$ $Q(\theta) = Pr(X > k \mid \theta)$

$Q(2) = Pr(X > k \mid 2) = 0.9 = 1-\beta$

$\Rightarrow \int_k^2 \frac{1}{2} dx = 0.9 \Rightarrow 1 - \frac{k}{2} = 0.9 \Rightarrow k = 0.2$

ii) $\alpha = Q(1.6) = Pr(X > 0.2 \mid 1.6) = \int_{0.2}^{1.6} \frac{1}{1.6} dx = 0.875$

b) i) $n=1$.

$\alpha = Q(1.6) = Pr(X > k \mid 1.6) = 0.05$

$\Rightarrow \int_k^{1.6} \frac{1}{1.6} dx = 0.05 \Rightarrow k = 1.52$

ii) $Q(2) = Pr(X > 1.52 \mid 2) = \int_{1.52}^{2} \frac{1}{2} dx = 0.24$.

c) $n = 8$.

$C_1 = \{ \underset{\sim}{x} : Y_n \geq k \}$ $\quad f_Y(y) = ny^{n-1}/\theta^n$.

$Q(\theta) = P_r(Y \geq k)$

$= 1 - \int_0^k \frac{ny^{n-1}}{\theta^n} dy$

$= 1 - \frac{n}{\theta^n} \int_0^k y^{n-1} dy$

$= 1 - k^n/\theta^n$, $(n=8)$

$Q(1.6) = 1 - k^8/\theta^8 = 0.05 \Rightarrow k = 1.5898$.

$\Rightarrow C_1 = \{ \underset{\sim}{x} : Y \geq 1.5898 \}$.