

The Hong Kong University of Science & Technology
MATH3423 - Statistical Inference
Midterm Examination - Fall 2015/2016

Answer ALL Questions

Date: 28 October 2015

Full marks: 60

Time Allowed: 2.5 hours

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- DO NOT open the exam paper until instructed to do so.
 - It is a closed-book examination.
 - Five questions are included in this paper.
 - To get full points for each question, please show your work as much as possible.
 - Cheating is a serious offense. Students caught cheating are subject to a zero score as well as additional penalties.

Name : _____

Student Number : _____

Signature : _____

For marking use only:

| Question No. | Marks | Out of |
|--------------|-------|--------|
| 1 | | 13 |
| 2 | | 12 |
| 3 | | 12 |
| 4 | | 23 |
| 5 | | 10 |

1. X_1, \dots, X_n is a random variable having the bernoulli distribution with the parameter θ .
 - (a) **(2 mark)** Let $S = \sum_{i=1}^n X_i$, find the distribution of S .
 - (b) **(2 marks)** Find the maximum likelihood estimate of $\theta(1 - \theta)$.
 - (c) **(2 marks)** Is it unbiased estimator? No mark will be given if the answer is “Yes” or “No”.
 - (d) **(2 marks)** Find the Cramer-Rao Lower Bound for the variance of an unbiased estimator for $\theta(1 - \theta)$.
 - (e) **(2 mark)** Does the variance of any unbiased estimator for $\theta(1 - \theta)$ achieve this bound? Why? Explain in details.
 - (f) **(3 marks)** Find the limiting distribution of the maximum likelihood estimate for $\theta(1 - \theta)$ by Central Limit Theorem and Delta method. What phenomenon do you observe?

Solutions:

- (a) Since $X_i \stackrel{\text{i.i.d.}}{\sim} \text{Bernoulli}(\theta)$, we have

$$\begin{aligned} M_{X_i}(t) &= (1 - \theta) + \theta e^t \\ M_S(t) &= M_{\sum_{i=1}^n X_i}(t) = \Pi_{i=1}^n M_{X_i}(t) = ((1 - \theta) + \theta e^t)^n \end{aligned}$$

Therefore, $S = \sum_{i=1}^n X_i \sim \text{Bin}(n, \theta)$.

- (b)

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}) &= \theta^{\sum_{i=1}^n x_i} (1 - \theta)^{n - \sum_{i=1}^n x_i} \\ l(\theta) &= \log f_{\mathbf{X}}(\mathbf{x}) = \left(\sum_{i=1}^n x_i \right) \log \theta + \left(n - \sum_{i=1}^n x_i \right) \log(1 - \theta) \\ l'(\theta) &= \frac{\sum_{i=1}^n x_i}{\theta} + \frac{n - \sum_{i=1}^n x_i}{\theta - 1} \end{aligned}$$

Taking $l'(\theta) = 0$, we have $\theta = \bar{x}$ and $l''(\theta) < 0$ at $\theta = \bar{x}$. Therefore, we get the MLE for θ , which is $\hat{\theta} = \bar{X}$. Then by the invariance property of MLE, we get the MLE of $\theta(1 - \theta)$ is $\hat{\theta}(1 - \hat{\theta}) = \bar{X}(1 - \bar{X})$.

- (c) By part (a), we have

$$\begin{aligned} E(S) &= n\theta \\ \text{Var}(S) &= n\theta(1 - \theta) \\ E(S^2) &= n\theta(1 - \theta + n\theta) \end{aligned}$$

Since $\hat{\theta} = \bar{X} = \frac{1}{n}S$, we have

$$\begin{aligned} E(\hat{\theta}) &= \frac{1}{n}E(S) = \theta \\ E(\hat{\theta}^2) &= \frac{1}{n^2}E(S^2) = \frac{\theta(1 - \theta + n\theta)}{n} \\ E(\hat{\theta}(1 - \hat{\theta})) &= \theta - \frac{\theta(1 - \theta + n\theta)}{n} = \frac{n - 1}{n}\theta(1 - \theta) \neq \theta(1 - \theta) \end{aligned}$$

Therefore, it is not unbiased.

- (d) By part(c), take $T = \frac{n}{n-1}\bar{X}(1 - \bar{X})$ and $E(T) = \theta(1 - \theta)$, so T is an unbiased estimator of $g(\theta) = \theta(1 - \theta)$.

$$\begin{aligned}
 \log f_{X_i}(x_i; \theta) &= x_i \log \theta + (1 - x_i) \log(1 - \theta) \\
 \frac{\partial^2}{\partial \theta^2} \log f_{X_i}(x_i; \theta) &= -\frac{x_i}{\theta^2} - \frac{1 - x_i}{(1 - \theta)^2} \\
 E\left(\frac{\partial^2}{\partial \theta^2} \log f_{X_i}(x_i; \theta)\right) &= -\frac{1}{\theta} - \frac{1}{1 - \theta} = -\frac{1}{\theta(1 - \theta)} \\
 g'(\theta) &= 1 - 2\theta \\
 \text{CR Lower Bound} &= -\frac{g'(\theta)^2}{nE\left(\frac{\partial^2}{\partial \theta^2} \log f_{X_i}(x_i; \theta)\right)} \\
 &= \frac{(1 - 2\theta)^2}{\frac{n}{\theta(1 - \theta)}}
 \end{aligned}$$

(e)

$$\frac{\partial}{\partial \theta} \log f_{\mathbf{X}}(\mathbf{x}; \theta) = \frac{\sum_{i=1}^n x_i}{\theta} - \frac{n - \sum_{i=1}^n x_i}{1 - \theta} = \frac{\sum_{i=1}^n x_i - n\theta}{\theta(1 - \theta)} = \frac{n}{\theta(1 - \theta)} \left(\frac{\sum_{i=1}^n x_i}{n} - \theta \right)$$

There is no function $A(n, \theta)$ s.t. $\frac{\partial}{\partial \theta} \log f_{\mathbf{X}}(\mathbf{x}; \theta) = A(n, \theta)[h(\mathbf{x}) - g(\theta)]$, where $g(\theta) = \theta(1 - \theta)$ and $h(\mathbf{x})$ is an unbiased estimator of $g(\theta)$. Therefore, no unbiased estimator of $g(\theta)$ achieve the CR lower bound.

- (f) By C.L.T., we have $\hat{\theta} \rightarrow N\left(\theta, \frac{\theta(1-\theta)}{n}\right)$. Then by Delta Method, we have $g(\hat{\theta}) \rightarrow N\left(g(\theta), \frac{g'(\theta)^2 \theta(1-\theta)}{n}\right) = N\left(\theta(1 - \theta), \frac{(1-2\theta)^2 \theta(1-\theta)}{n}\right)$. So the limiting variance of MLE can achieve the CR lower bound.

2. X_1, X_2, \dots, X_n are observations of a random sample of size n from the geometric distribution with probability distribution $f(x, \theta) = \theta(1 - \theta)^x$

- (a) **(2 marks)** Find the distribution of $\sum_{i=1}^n X_i$.
- (b) **(2 mark)** Find the estimator from the method of moment.
- (c) **(2 mark)** Find the estimator from the method of maximum likelihood.
- (d) **(2 marks)** Find the maximum likelihood estimator for $E(X)$.
- (e) **(2 marks)** Is the maximum likelihood estimator for $E(X)$ unbiased? If yes, find its variance. If no, find its mean squared error.
- (f) **(2 marks)** Is the variance of any unbiased estimator for $E(X)$ equal to the Cramer-Rao Lower Bound? No need to find the Cramer-Rao Lower Bound.

Solutions:

(a) Since $X_i \stackrel{\text{i.i.d.}}{\sim} \text{Geometric}(\theta)$, we have

$$\begin{aligned} M_{X_i}(t) &= \frac{\theta}{1 - (1 - \theta)e^t} \\ M_S(t) &= M_{\sum_{i=1}^n X_i}(t) = \Pi_{i=1}^n M_{X_i}(t) = \left(\frac{\theta}{1 - (1 - \theta)e^t} \right)^n \end{aligned}$$

Therefore, $S = \sum_{i=1}^n X_i \sim \text{NB}(n, \theta)$.

(b) For MME:

$$\begin{aligned} M'_1 &= \widetilde{E(X)} \\ \frac{1}{n} \sum_{i=1}^n x_i &= \frac{\widetilde{1 - \theta}}{\theta} \\ \hat{\theta} &= \frac{1}{1 + \bar{X}} \end{aligned}$$

(c)

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}) &= \theta^n (1 - \theta)^{\sum_{i=1}^n x_i} \\ l(\theta) &= \log f_{\mathbf{X}}(\mathbf{x}) = n \log \theta + \left(\sum_{i=1}^n x_i \right) \log(1 - \theta) \\ l'(\theta) &= \frac{n}{\theta} + \frac{\sum_{i=1}^n x_i}{\theta - 1} \end{aligned}$$

Taking $l'(\theta) = 0$, we have $\theta = \frac{1}{1 + \bar{x}}$ and $l''(\theta) < 0$ at $\theta = \frac{1}{1 + \bar{x}}$. Therefore, we get the MLE for θ , which is $\hat{\theta} = \frac{1}{1 + \bar{X}}$.

(d) $EX = \frac{1 - \theta}{\theta}$. Therefore, by the invariant property of MLE, we have the MLE of EX is $\frac{1 - \hat{\theta}}{\hat{\theta}} = \bar{X}$

(e)

$$E(\bar{X}) = \frac{1}{n} \sum_{i=1}^n E(X_i) = \frac{1 - \theta}{\theta}$$

Therefore, the MLE of EX is unbiased.

$$\text{Var}(\bar{X}) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i) = \frac{1 - \theta}{n\theta^2}$$

(f)

$$\begin{aligned} \frac{\partial}{\partial \theta} \log f_{\mathbf{X}}(\mathbf{x}; \theta) &= \frac{n}{\theta} - \frac{\sum_{i=1}^n x_i}{1 - \theta} = \frac{n(1 - \theta) - \sum_{i=1}^n x_i \theta}{\theta(1 - \theta)} \\ &= \frac{-n}{1 - \theta} \left(\bar{x} - \frac{1 - \theta}{\theta} \right) \end{aligned}$$

Therefore, the variance of the unbiased estimator \bar{X} is equal to the CR lower bound.

3. Let X_1, \dots, X_n are independently uniformly distributed on $(\theta, 2\theta)$.

- (a) **(2 marks)** Find the estimators from the method of moment and the method of maximum likelihood.
- (b) **(2 mark)** Find the expectation and variance of the estimator from the method of moments.
- (c) **(4 marks)** Find the expectation and variance of the estimator from the method of maximum likelihood.
- (d) **(2 mark)** Hence or otherwise, construct two unbiased estimators of θ based on the two estimators in part (a).
- (e) **(2 marks)** Compare the variances of the two unbiased estimates in (d) and comment briefly.

Solutions:

- (a) For MME:

$$\begin{aligned} M'_1 &= \widetilde{E(X)} \\ \frac{1}{n} \sum_{i=1}^n x_i &= \frac{\theta + 2\theta}{2} \\ \tilde{\theta} &= \frac{2}{3} \bar{X} \end{aligned}$$

For MLE:

$$\begin{aligned} f_{X_i}(x_i) &= \frac{1}{\theta} I_{(\theta \leq x_i \leq 2\theta)} \\ f_{\mathbf{X}}(\mathbf{x}) &= \frac{1}{\theta^n} I_{(\theta \leq x_{(1)} \leq x_{(n)} \leq 2\theta)} = \frac{1}{\theta^n} I_{(\frac{x_{(n)}}{2} \leq \theta \leq x_{(1)})} \end{aligned}$$

Therefore, from the graph of $f_{\mathbf{X}}(\mathbf{x})$, we can see that MLE of θ is $\hat{\theta} = \frac{X_{(n)}}{2}$.

- (b)

$$\begin{aligned} E(\bar{X}) &= \frac{1}{n} \sum_{i=1}^n E(X_i) = \frac{3}{2}\theta \\ Var(\bar{X}) &= \frac{1}{n^2} \sum_{i=1}^n Var(X_i) = \frac{1}{n^2} \sum_{i=1}^n \frac{\theta^2}{12} = \frac{\theta^2}{12n} \\ E(\tilde{\theta}) &= \frac{2}{3} E(\bar{X}) = \theta \\ Var(\tilde{\theta}) &= \frac{4}{9} Var(\bar{X}) = \frac{\theta^2}{27n} \end{aligned}$$

(c)

$$\begin{aligned}f_{X_i}(t) &= \frac{1}{\theta} I_{(\theta \leq t \leq 2\theta)} \\F_{X_i}(t) &= \frac{t - \theta}{\theta} I_{(\theta \leq t \leq 2\theta)} + I_{(2\theta \leq t)} \\f_{X_{(n)}}(t) &= n(F_{X_i}(t))^{n-1} f_{X_i}(t) \\&= n \frac{(t - \theta)^{n-1}}{\theta^n} I_{(\theta \leq t \leq 2\theta)} \\E(X_{(n)}) &= \int_{\theta}^{2\theta} \frac{nt(t - \theta)^{n-1}}{\theta^n} dt \\&= \int_0^{\theta} \frac{n(y + \theta)y^{n-1}}{\theta^n} dy \quad y = t - \theta \\&= \frac{2n + 1}{n + 1} \theta \\E(\hat{\theta}) &= \frac{2n + 1}{2n + 2} \theta \\E(X_{(n)}^2) &= \int_{\theta}^{2\theta} \frac{nt^2(t - \theta)^{n-1}}{\theta^n} dt \\&= \int_0^{\theta} \frac{n(y + \theta)^2 y^{n-1}}{\theta^n} dy \quad y = t - \theta \\&= \frac{4n^2 + 8n + 2}{(n + 1)(n + 2)} \theta^2 \\Var(X_{(n)}) &= E(X_{(n)}^2) - E(X_{(n)})^2 = \frac{n}{(n + 1)^2(n + 2)} \theta^2 \\Var(\hat{\theta}) &= \frac{1}{4} Var(X_{(n)}) = \frac{n}{4(n + 1)^2(n + 2)} \theta^2\end{aligned}$$

(d) By part(b), we have $\tilde{\theta} = \frac{2}{3} \bar{X}$ is an unbiased estimator of θ .

By part(c), take $S = \frac{n+1}{2n+1} X_{(n)}$ and $E(S) = \theta$, so S is an unbiased estimator of θ .

(e)

$$\begin{aligned}Var(\tilde{\theta}) &= \frac{\theta^2}{27n} \\Var(S) &= \frac{(n + 1)^2}{(2n + 1)^2} Var(X_{(n)}) = \frac{n}{(2n + 1)^2(n + 2)} \theta^2\end{aligned}$$

Therefore, when n is large, $Var(S)$ is much smaller than $Var(\tilde{\theta})$, and since they are both unbiased, S is better than $\tilde{\theta}$.

4. If X_1, X_2, \dots, X_n are independently and normally distributed with the same mean μ but different **known** variances $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$.

(a) **(3 marks)** Find the estimator from the method of moment. Prove that it is an unbiased estimator for μ . Find its variance.

- (b) **(3 marks)** Find the estimator from the method of maximum likelihood. Prove that it is an unbiased estimator for μ . Find its variance.
- (c) **(2 mark)** Which of the estimators from part (a) or part (b) is more efficient? Explain in details
- (d) **(4 marks)** Let $W = \sum_{i=1}^n (X_i/\sigma_i^2) / \sum_{j=1}^n (1/\sigma_j^2)$. Find its distribution. Hence or otherwise, construct the $(1 - \alpha)100\%$ confidence interval for μ .
- (e) **(2 marks)** Find the distribution of $X_i - W$.
- (f) **(3 marks)** Are W and $X_i - W$ independent? Explain in details.
- (g) **(1 mark)** Find the distribution of

$$\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma_i} \right)^2 .$$

- (h) **(1 mark)** Find the distribution of

$$\sum_{i=1}^n \frac{1}{\sigma_i^2} (W - \mu)^2 .$$

- (i) **(4 marks)** Hence or otherwise, find the distribution of

$$\sum_{i=1}^n (X_i - W)^2 / \sigma_i^2 .$$

Solutions:

- (a) For MME:

$$\begin{aligned} M'_1 &= \widetilde{E(X)} \\ \frac{1}{n} \sum_{i=1}^n x_i &= \tilde{\mu} \\ \tilde{\mu} &= \bar{X} \\ E(\bar{X}) &= \frac{1}{n} \sum_{i=1}^n E(X_i) = \mu \\ Var(\bar{X}) &= \frac{1}{n^2} \sum_{i=1}^n Var(X_i) = \frac{1}{n^2} \sum_{i=1}^n \sigma_i^2 \end{aligned}$$

(b)

$$\begin{aligned}
f_{\mathbf{X}}(\mathbf{x}) &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma_i^2}} \\
l(\mu) &= \log f_{\mathbf{X}}(\mathbf{x}) = -\frac{1}{2} \sum_{i=1}^n \log(2\pi\sigma_i^2) - \sum_{i=1}^n \frac{(x_i-\mu)^2}{2\sigma_i^2} \\
l'(\mu) &= \sum_{i=1}^n \frac{(x_i-\mu)}{\sigma_i^2}
\end{aligned}$$

Taking $l'(\mu) = 0$, we have $\mu = \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} \sum_{i=1}^n \frac{x_i}{\sigma_i^2}$ and $l''(\mu) < 0$ at $\mu = \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} \sum_{i=1}^n \frac{x_i}{\sigma_i^2}$. Therefore, we get the MLE for μ , which is $\hat{\mu} = \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} \sum_{i=1}^n \frac{X_i}{\sigma_i^2}$.

$$\begin{aligned}
E(\hat{\mu}) &= \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} \sum_{i=1}^n E\left(\frac{X_i}{\sigma_i^2}\right) = \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} \sum_{i=1}^n \frac{\mu}{\sigma_i^2} = \mu \\
Var(\hat{\mu}) &= \frac{1}{\left(\sum_{i=1}^n \frac{1}{\sigma_i^2}\right)^2} \sum_{i=1}^n \frac{Var(X_i)}{(\sigma_i^2)^2} = \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{Var(\bar{X})}{Var(\hat{\mu})} &= \frac{\frac{1}{n^2} \sum_{i=1}^n \sigma_i^2}{\frac{1}{\sum_{j=1}^n \frac{1}{\sigma_j^2}}} = \frac{1}{n^2} \left(\sum_{i=1}^n \sigma_i^2 \right) \left(\sum_{j=1}^n \frac{1}{\sigma_j^2} \right) \\
&= \frac{1}{n^2} \left(\sum_{i=j} \frac{\sigma_i^2}{\sigma_j^2} + \sum_{i \neq j} \frac{\sigma_i^2}{\sigma_j^2} \right) = \frac{1}{n^2} \left(n + \sum_{i \neq j} \frac{\sigma_i^2}{\sigma_j^2} \right) \\
&= \frac{1}{n^2} \left(n + \sum_{i > j} \left(\frac{\sigma_i^2}{\sigma_j^2} + \frac{\sigma_j^2}{\sigma_i^2} \right) \right) \\
&\geq \frac{1}{n^2} \left(n + \sum_{i > j} 2 \right) = 1
\end{aligned}$$

Therefore $Var(\bar{X}) \geq Var(\hat{\mu})$, which implies $\hat{\mu}$ is more efficient.

(d)

$$\begin{aligned}
X_i &\sim N(\mu, \sigma_i^2) \\
\frac{X_i}{\sigma_i^2} &\sim N\left(\frac{\mu}{\sigma_i^2}, \frac{1}{\sigma_i^2}\right) \\
\sum_{i=1}^n \frac{X_i}{\sigma_i^2} &\sim N\left(\mu \sum_{i=1}^n \frac{1}{\sigma_i^2}, \sum_{i=1}^n \frac{1}{\sigma_i^2}\right) \\
W = \frac{\sum_{i=1}^n \frac{X_i}{\sigma_i^2}}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} &\sim N\left(\mu, \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}\right)
\end{aligned}$$

Therefore, a $(1 - \alpha)100\%$ C.I. for μ is $W \pm z_{\alpha/2} \sqrt{\frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}}$

(e) Take $m = Var(W) = \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}$, which is a constant.

$$\begin{aligned}
E(X_i - W) &= E(X_i) - EW = 0 \\
Cov(X_i, W) &= Cov(X_i, \frac{m}{\sigma_i^2} X_i) = \frac{m}{\sigma_i^2} \sigma_i^2 = m \\
Var(X_i - W) &= Var(X_i) + Var(W) - 2Cov(X_i, W) \\
&= \sigma_i^2 + m - 2Cov(X_i, \frac{m}{\sigma_i^2} X_i) \\
&= \sigma_i^2 + m - 2 \frac{m}{\sigma_i^2} \sigma_i^2 \\
&= \sigma_i^2 - m \\
&= \sigma_i^2 - \frac{1}{\sum_{j=1}^n \frac{1}{\sigma_j^2}}
\end{aligned}$$

Therefore, $X_i - W \sim N(0, \sigma_i^2 - \frac{1}{\sum_{j=1}^n \frac{1}{\sigma_j^2}})$

(f)

$$Cov(W, X_i - W) = Cov(W, X_i) - Var(W) = m - m = 0$$

,where m is the same as the one defined in part(e). Then if W and $X_i - W$ follow bivariate normal distribution, then W and $X_i - W$ are independent.

The covariance matrix between W and $X_i - W$ is $\Sigma = \begin{pmatrix} m & 0 \\ 0 & \sigma_i^2 - m \end{pmatrix}$. Let $L = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$.

Then LL^T is $\begin{pmatrix} a^2 & ab \\ ab & b^2 + c^2 \end{pmatrix}$. We may take $a = \sqrt{m}$, $b=0$ and $c = \sqrt{\sigma_i^2 - m}$, and then $LL^T = \Sigma$. Therefore, there exist 2 independent standard normal variables, i.e., $Z_1, Z_2 \sim N(0,1)$, a 2-vector of μ and a 2×2 matrix $L = \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{\sigma_i^2 - m} \end{pmatrix} = \text{Chol}(\Sigma)$ such that $\mathbf{X} = \mu + \text{Chol}(\Sigma)\mathbf{Z}$, so the random vector of $(W, X_i - W)^T$ has a bivariate normal distribution.

Or since $\begin{pmatrix} W \\ X_i - W \end{pmatrix}$ is a linear transformation of $\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$, so the random vector $\begin{pmatrix} W \\ X_i - W \end{pmatrix}$

follows a bivariate normal distribution.

Therefore, together with $Cov(W, X_i - W) = 0$, W and $X_i - W$ are independent.

(g) since $X_i \stackrel{\text{i.i.d.}}{\sim} N(\mu, \sigma_i^2)$, we have

$$\frac{X_i - \mu}{\sigma_i} \sim N(0, 1) \quad (1)$$

$$\left(\frac{X_i - \mu}{\sigma_i}\right)^2 \sim \chi_1^2 \quad (2)$$

$$\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma_i}\right)^2 \sim \chi_n^2 \quad (3)$$

The (3) comes from (2) and the independence of X_i , $i=1,2,\dots,n$.

(h) From part(d), we have $W \sim N(\mu, \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}})$, so

$$\begin{aligned}\frac{W - \mu}{\sqrt{\frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}}} &\sim N(0, 1) \\ \left(\frac{W - \mu}{\sqrt{\frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}}}}\right)^2 &\sim \chi_1^2 \\ \sum_{i=1}^n \frac{1}{\sigma_i^2} (W - \mu)^2 &\sim \chi_1^2\end{aligned}$$

(i)

$$\begin{aligned}\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma_i}\right)^2 &= \sum_{i=1}^n \left(\frac{(X_i - W) + (W - \mu)}{\sigma_i}\right)^2 \\ &= \sum_{i=1}^n \left(\frac{X_i - W}{\sigma_i}\right)^2 + \sum_{i=1}^n \left(\frac{W - \mu}{\sigma_i}\right)^2\end{aligned}$$

because the cross-product term is equal to

$$2 \sum_{i=1}^n \frac{(X_i - W)(W - \mu)}{\sigma_i^2} = 2(W - \mu) \sum_{i=1}^n \frac{X_i - W}{\sigma_i^2} = 0$$

Then,

$$\begin{aligned}\sum_{i=1}^n \left(\frac{X_i - \mu}{\sigma_i}\right)^2 &= \sum_{i=1}^n \left(\frac{X_i - W}{\sigma_i}\right)^2 + \sum_{i=1}^n \left(\frac{W - \mu}{\sigma_i}\right)^2 \\ &\sim \chi_n^2 \qquad \qquad \qquad \sim \chi_1^2\end{aligned}$$

Since W and $X_i - W$ are independent, thus $\sum_{i=1}^n \left(\frac{X_i - W}{\sigma_i}\right)^2 \sim \chi_{n-1}^2$. The detailed proof can be carried out using investigating on the MGFs.

5. **(Bonus)** Consider a random sample $\{X_1, X_2\}$ from density

$$f_X(x|\theta) = \frac{3x^2}{\theta^3} I_{(0 < x < \theta)},$$

where $\theta > 0$.

(a) **(2 marks)** Are $\hat{\theta}_1 = \frac{2}{3}(X_1 + X_2)$ and $\hat{\theta}_2 = \frac{7}{6} \max(X_1, X_2)$ unbiased for θ ?

(b) **(4 marks)** Find the mean squared errors (MSEs) of $\hat{\theta}_1$ and $\hat{\theta}_2$, and compare those estimators.

(c) **(4 marks)** Prove that in the sense of MSE, $T_{8/7}$ is the best estimator of θ among the estimators in form of $T_c = c \max(X_1, X_2)$.

Solutions:

(a) For $\hat{\theta}_1$,

$$\begin{aligned} EX &= \int_0^\theta x \frac{3x^2}{\theta^3} dx = \frac{3}{4}\theta \\ EX^2 &= \int_0^\theta x^2 \frac{3x^2}{\theta^3} dx = \frac{3}{5}\theta^2 \\ Var(X) &= EX^2 - (EX)^2 = \frac{3}{80}\theta^2 \\ E\hat{\theta}_1 &= \frac{2}{3}(EX_1 + EX_2) = \theta \end{aligned}$$

Therefore, $\hat{\theta}_1$ is unbiased.

For $\hat{\theta}_2$, take $T = \max(X_1, X_2)$.

$$\begin{aligned} f_T(t) &= 2F_X(t)f_X(t)\frac{6t^5}{\theta^6}I_{(0 \leq t \leq \theta)} \\ ET &= \int_0^\theta t \frac{6t^5}{\theta^6} dt = \frac{6}{7}\theta \\ E\hat{\theta}_2 &= \theta \end{aligned}$$

Therefore, $\hat{\theta}_2$ is unbiased.

(b) Since both $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased, MSEs of $\hat{\theta}_1$ and $\hat{\theta}_2$ are equal to their variance respectively.

$$\begin{aligned} Var(\hat{\theta}_1) &= \frac{4}{9}(Var(X_1) + Var(X_2)) = \frac{1}{30}\theta^2 \\ E(T^2) &= \int_0^\theta t^2 \frac{6t^5}{\theta^6} dt = \frac{3}{4}\theta^2 \\ Var(T) &= ET^2 - (ET)^2 = \frac{3}{196}\theta^2 \\ Var(\hat{\theta}_2) &= \frac{1}{48}\theta^2 \end{aligned}$$

(c)

$$\begin{aligned} T_c &= c\max(X_1, X_2) = cT \\ E(T_c) &= \frac{6c}{7}\theta \\ Var(T_c) &= \frac{3c^2}{196}\theta^2 \\ MSE \text{ of } T_c &= Var(T_c) + (ET_c - \theta)^2 \\ &= \frac{3c^2}{196}\theta^2 + \left(\frac{6c}{7}\theta - \theta\right)^2 \\ &= \left(\frac{3}{4}c^2 - \frac{12}{7}c + 1\right)\theta^2 \end{aligned}$$

Therefore, MSE of T_c achieves its minimum at $c = -\frac{-\frac{12}{7}}{\frac{3}{4}} = \frac{8}{7}$, so $T_{\frac{8}{7}}$ is the best estimator of θ among T_c in the sense of MSE.