1) $\left(\frac{1}{2}\right)^n$, $\left(\frac{1}{2}\right)^n$

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2) $F(2,2)$

3) $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$, $\cancel{\bar{X}}\ X_{n+1} \sim N(\mu, \sigma^2)$

$$\bar{X} - X_{n+1} \sim N\left(0, \sigma^2 \frac{n+1}{n}\right)$$

$$\frac{\bar{X} - X_{n+1}}{\sigma\sqrt{\frac{n+1}{n}}} \sim N(0,1)$$

$$\frac{\cancel{(n-1)}\, n S^2}{\sigma^2} \sim \chi^2_{n-1} \qquad S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$\sqrt{\frac{n}{n+1}}\ \frac{\bar{X} - X_{n+1}}{\sigma} \Big/ \sqrt{\frac{n S^2}{\sigma^2 (n-1)}} \sim t(n-1)$$

$$\Rightarrow C = \sqrt{\frac{n-1}{n+1}}$$

If $n = 8$, $0.8 = P_r(\bar{X} - kS < X_9 < \bar{X} + kS)$

$$= P_r\left(-k < \frac{\bar{X} - X_9}{S} < k\right)$$

$$= P_r\left(-k\sqrt{\frac{7}{9}} < \sqrt{\frac{7}{9}}\,\frac{\bar{X} - X_9}{S} < k\sqrt{\frac{7}{9}}\right)$$

$$\Rightarrow t_{0.1}(7) = 1.415 \Rightarrow k\sqrt{\frac{7}{9}} = 1.415$$

$$\Rightarrow k = \cancel{1.5008}\ 1.6045$$

~~1)~~

~~2) F(2,2)~~

~~3)~~

4) $\cancel{g}\, f(y_4) = \dfrac{4y_4^3}{\theta^4} \qquad 0 < y_4 < \infty$

$$Pr(c\theta < Y_4 < \theta)$$

$$= \int_{c\theta}^{\theta} \frac{4y_4^3}{\theta^4}\, dy_4$$

$$= \frac{y_4^4}{\theta^4}\Bigg|_{c\theta}^{\theta}$$

$$= 1 - c^4 = 0.95$$

$$\Rightarrow c = \sqrt[4]{0.05}$$

$$Pr\left(Y_4 < \theta < \frac{Y_4}{\sqrt[4]{0.05}}\right) = 0.95$$

$$\Rightarrow \left(y_4, \frac{y_4}{\sqrt[4]{0.05}}\right)$$

$$\Rightarrow (y_4, 2.1147\, y_4)$$

5a $E(X) = \frac{1}{n}\sum X_i = \frac{\theta}{2} \Rightarrow \tilde{\theta} = 2\bar{X}$

$f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta) = \prod_{i=1}^{n} \frac{1}{\theta} I\{0 < x_i < \theta\} = \theta^{-n} I\{0 < x_{(1)} \le \cdots \le x_{(n)} < \theta\}$

$\Rightarrow \hat{\theta} = X_{(n)}$

b $E(\tilde{\theta}) = 2E\bar{X} = 2EX = 2(\frac{\theta}{2}) = \theta$

$var(\tilde{\theta}) = 4\, var\, \bar{X} = \frac{4}{n} var X = 4 \cdot \frac{\theta^2}{12n} = \theta^2/3n$

c. $P(X_{(n)} \le y) = \prod_{i=1}^{n} P(X_i \le y) = (\frac{y}{\theta})^n$

$f(y) = ny^{n-1}/\theta^n$

$E(\hat{\theta}) = E(X_{(n)}) = \int_0^\theta y\, n \frac{y^{n-1}}{\theta^n} dy = \frac{n}{\theta^n}\int_0^\theta y^n dy$

$= \frac{n}{\theta^n} \frac{y^{n+1}}{n+1}\Big|_0^\theta = \frac{n}{n+1}\theta$.

$E(\hat{\theta}^2) = \int_0^\theta y^2\, n \frac{y^{n-1}}{\theta^n} dy = \frac{n}{\theta^n}\int_0^\theta y^{n+1} dy = \frac{n}{\theta^n} \frac{y^{n+2}}{n+2}\Big|_0^\theta$

$= \frac{n}{n+2}\theta^2$.

$var(\hat{\theta}) = \frac{n}{n+2}\theta^2 - (\frac{n}{n+1}\theta)^2 = \frac{n\,\theta^2}{(n+1)^2(n+2)}$

d. ~~$2\bar{X}$ and~~ $E(2\bar{x}) = \theta \Rightarrow 2\bar{X}$ is an unbiased est.

$E(\frac{n+1}{n} X_{(n)}) = \theta \Rightarrow \frac{n+1}{n} X_{(n)}$ - - - - - -

e. $var(2\bar{x}) = \theta^2/3n \leftarrow$ by b

$var(\frac{n+1}{n} X_{(n)}) = (\frac{n+1}{n})^2 \cdot \frac{n\,\theta^2}{(n+1)^2(n+2)} = \frac{\theta^2}{n(n+2)}$

When $n \geq 1$, $var(2\bar{x}) \geq var(\frac{n+1}{n} X_{(n)})$,

so $\frac{n+1}{n} X_{(n)}$ is more eff. than $2\bar{X}$.

6a. $f(x_i;\lambda) = \lambda e^{-\lambda x}$

$\log f(x_i;\lambda) = \log \lambda - \lambda x$

$\frac{\partial}{\partial \lambda} \log f(x_i;\lambda) = \frac{1}{\lambda} - x$

$\frac{\partial^2}{\partial \lambda^2} \log f(x_i;\lambda) = -\frac{1}{\lambda^2}$

$E\left[\frac{\partial^2}{\partial \lambda^2} \log f(x_i;\lambda)\right] = -\frac{1}{\lambda^2}$

CRLB for λ is $-\frac{1}{n\left(-\frac{1}{\lambda^2}\right)} = \frac{\lambda^2}{n}$

b. $X_i \sim \exp\left(\frac{1}{\lambda}\right)$

$s = \sum X_i \sim \text{Gamma}\left(n, \frac{1}{\lambda}\right)$

$E\left(\frac{1}{\bar{X}}\right) = E\left(\frac{n}{\sum X_i}\right) = E\left(\frac{n}{s}\right) = n\,E\left(\frac{1}{s}\right)$

$E\left(\frac{1}{s}\right) = \int_0^\infty \frac{1}{s} f_S(s)\,ds = \int_0^\infty \frac{1}{s} \frac{1}{\Gamma(n)} \lambda^n s^{n-1} e^{-\lambda s}\,ds$

$= \frac{\lambda}{\Gamma(n)} \int_0^\infty \lambda^{n-1} s^{n-2} e^{-\lambda s}\,ds$

$= \frac{\lambda}{\Gamma(n)} \Gamma(n-1) \int_0^\infty \frac{1}{\Gamma(n-1)} \lambda^{n-1} s^{(n-1)-1} e^{-\lambda s}\,ds$

$= \frac{\lambda}{n-1}$

$\therefore E\left(\frac{1}{\bar{X}}\right) = n \cdot \frac{\lambda}{n-1} = \frac{n\lambda}{n-1}$

$\therefore E\left(\frac{n-1}{n} \cdot \frac{1}{\bar{X}}\right) = \lambda \Rightarrow \frac{n-1}{n\bar{X}}$ is an unbiased est.

$\cancel{ⓒ.\ var\left(\frac{n-1}{n\bar{x}}\right) = var\,T = ET^2 - (ET)^2}$

c. $var\left(\frac{n-1}{n\bar{x}}\right) = \left(\frac{n-1}{n}\right)^2 var\left(\frac{1}{\bar{x}}\right) = (n-1)^2\, var\left(\frac{1}{s}\right)$

$\cancel{E\left(\frac{1}{\bar{x}}\right)^2 = E\left(\frac{n}{s^2}\right) = n^2 E\left(\frac{1}{s^2}\right)}$

$$E\left(\frac{1}{s^2}\right) = \int_0^\infty \frac{1}{s^2}\,\frac{1}{\Gamma(n)}\,\lambda^n s^{n-1} e^{-\lambda s}\,ds$$

$$= \frac{\lambda^2}{\Gamma(n)} \int_0^\infty \lambda^{n-2} s^{(n-2)-1} e^{-\lambda s}\,ds$$

$$= \frac{\lambda^2}{\Gamma(n)}\,\Gamma(n-2) \int_0^\infty \frac{1}{\Gamma(n-2)}\,\lambda^{n-2} s^{(n-2)-1} e^{-\lambda s}\,ds$$

$$= \frac{\lambda^2}{(n-1)(n-2)}$$

$\cancel{\therefore\ var\left(\frac{n-1}{n\bar{x}}\right)}$ $\cancel{var\left(\frac{1}{\bar{x}}\right) = \frac{\lambda^2}{n(n-1)} - \frac{n^2\lambda^2}{(n-1)^2} = \frac{(n-1)\lambda^2 - n\lambda^2}{n(n-1)^2} =}$

$$var\left(\frac{1}{s}\right) = \frac{\lambda^2}{(n-1)(n-2)} - \frac{\lambda^2}{(n-1)^2} = \frac{\lambda^2}{(n-1)^2(n-2)}$$

$$var\left(\frac{n-1}{n\bar{x}}\right) = \frac{\lambda^2}{n-2}.$$

CRLB for $\lambda = \frac{\lambda^2}{n} < \frac{\lambda^2}{n-2} = var\left(\frac{n-1}{n\bar{x}}\right)$

so $var\left(\frac{n-1}{n\bar{x}}\right) >$ CRLB for λ