Math 243 Midterm

1. $X_1, \ldots, X_n \overset{iid}{\sim} N(0,1)$

Define $\overline{X_k} = \frac{1}{k}\sum_{i=1}^{k} X_i$ and $\overline{X_{n-k}} = \frac{1}{n-k}\sum_{i=k+1}^{n} X_i$

(a) Observe that $\overline{X_k}$ and $\overline{X_{n-k}}$ are independent. Also,

$\overline{X_k} \sim N(0, \frac{1}{k})$ and $\overline{X_{n-k}} \sim N(0, \frac{1}{n-k})$.

$$Var\left(\tfrac{1}{2}(\overline{X_k} + \overline{X_{n-k}})\right)$$
$$= \tfrac{1}{4} Var(\overline{X_k}) + \tfrac{1}{4} Var(\overline{X_{n-k}})$$
$$= \tfrac{1}{4}\left(\tfrac{1}{k} + \tfrac{1}{n-k}\right) = \left(\tfrac{1}{4}\right)\frac{n-k+k}{(k)(n-k)} = \frac{n}{4k(n-k)}$$
$$\therefore \tfrac{1}{2}(\overline{X_k} + \overline{X_{n-k}}) \sim N\left(0, \frac{n}{4k(n-k)}\right)$$

(b) $\sqrt{k}\,\overline{X_k} \sim N(0,1)$ and $\sqrt{n-k}\,\overline{X_{n-k}} \sim N(0,1)$

Also, $\sqrt{k}\,\overline{X_k}$ and $\sqrt{n-k}\,\overline{X_{n-k}}$ are independent.

Thus, $(\sqrt{k}\,\overline{X_k})^2 + (\sqrt{n-k}\,\overline{X_{n-k}})^2 = k\overline{X_k}^2 + (n-k)^2\overline{X_{n-k}}^2 \sim \chi^2_{(2)}$

(c) $X_1^2 \sim \chi^2_{(1)}$

$X_2^2 \sim \chi^2_{(2)}$

Since X_1^2 and X_2^2 are independent,

$\frac{X_1^2}{X_2^2} \sim F_{(1,1)}$

2. X_1, X_2, X_3 is a r.s. with p.d.f. $f(x) = 2x$, $0 < x < 1$, zero otherwise.

let m_0 be the median of the distribution.

$$P(\min\{X_1, X_2, X_3\} > m_0)$$
$$= P(X_1 > m_0, X_2 > m_0, X_3 > m_0)$$
$$= \prod_{i=1}^{3} P(X_i > m_0) = \prod_{i=1}^{3}(1 - P(X_i \le m_0))$$
$$= (1 - F_X(m_0))^3$$
$$= (1 - \tfrac{1}{2})^3 \quad \text{since } X_i \text{ is a continuous r.v.}$$
$$= \tfrac{1}{8}$$

3. $X \sim \text{Bernoulli}(\theta)$

(a) $E(t_1(X)) = E(X)$

$= \theta$

$\therefore t_1(X)$ is unbiased for θ.

$E(t_2(X)) = E(\frac{1}{2})$

$= \frac{1}{2} \neq \theta$ in general

$\therefore t_2(X)$ is not unbiased for θ.

(b) M.S.E. of $t_1(X)$

$= E(t_1(X) - \theta)^2 = E(X-\theta)^2$

$= \theta(1-\theta)$

M.S.E. of $t_2(X)$

$= E(t_2(X) - \theta)^2 = E(\frac{1}{2}-\theta)^2$

$= (\theta - \frac{1}{2})^2$

Let $g(\theta) =$ M.S.E. of $t_1(X)$ − M.S.E. of $t_2(X)$

$= \theta(1-\theta) - (\theta - \frac{1}{2})^2$

$= (\theta - \theta^2) - (\theta^2 - \theta + \frac{1}{4})$

$= -2\theta^2 + 2\theta - \frac{1}{4}$

$g(\theta) = 0$

$\Rightarrow -2\theta^2 + 2\theta - \frac{1}{4} = 0$

$\Rightarrow \theta = \dfrac{-2 \pm \sqrt{2^2 - 4(-2)(-\frac{1}{4})}}{2(-2)}$

$= \frac{1}{2} \pm \frac{\sqrt{2}}{4}$

Observe that $g(\theta)$ is opening downward, so

$g(\theta) \geq 0$ iff $\frac{1}{2} - \frac{\sqrt{2}}{4} \leq \theta \leq \frac{1}{2} + \frac{\sqrt{2}}{4}$.

Thus, M.S.E. of $t_1(X) <$ M.S.E. of $t_2(X)$

when $0 < \theta < \frac{1}{2} - \frac{\sqrt{2}}{4}$ and $\frac{1}{2} + \frac{\sqrt{2}}{4} < \theta < 1$

4. $X_1, \ldots, X_n$ is a random sample from the geometric density

$$f(x;\theta) = \theta(1-\theta)^x I_{\{0,1,\ldots\}}(x)$$

where $0<\theta<1$

(a)
$$E(X) = \sum_{x=0}^{\infty} x\theta(1-\theta)^x$$
$$= \sum_{x=1}^{\infty} x\theta(1-\theta)^x$$
$$= (1-\theta)\sum_{x=1}^{\infty} x\theta(1-\theta)^{x-1}$$
$$= \frac{(1-\theta)}{\theta}$$

$$\therefore \widetilde{E(X)} = \bar{X}$$
$$\Rightarrow \frac{(1-\tilde{\theta})}{\tilde{\theta}} = \bar{X}$$
$$\therefore \tilde{\theta} = \frac{1}{\bar{X}+1}$$

(b)
$$L(\theta) = f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta) = \prod_{i=1}^{n} f_{X_i}(x_i;\theta)$$
$$= \prod_{i=1}^{n} \theta(1-\theta)^{x_i} I_{\{0,1,\ldots\}}(x_i)$$
$$\log L(\theta) = n\log\theta + \left(\sum_{i=1}^{n} x_i\right)\log(1-\theta) + \sum_{i=1}^{n}\log I_{\{0,1,\ldots\}}(x_i)$$
$$\frac{d}{d\theta}\log L(\theta) = \frac{n}{\theta} - \frac{\sum_{i=1}^{n} x_i}{1-\theta}$$
$$\frac{d}{d\theta}\log L(\theta)\Big|_{\theta=\hat{\theta}} = 0$$
$$\Rightarrow \frac{n}{\hat{\theta}} - \frac{\sum_{i=1}^{n} X_i}{1-\hat{\theta}} = 0$$
$$\Rightarrow \frac{n}{\hat{\theta}} = \frac{\sum_{i=1}^{n} X_i}{1-\hat{\theta}}$$
$$\therefore \hat{\theta} = \frac{1}{\bar{X}+1}$$

(c)
$$\mu = E(X)$$
$$= \frac{1-\theta}{\theta} = \frac{1}{\theta} - 1 \quad \text{by (a)}$$
$$\Rightarrow \theta = \frac{1}{1+\mu}$$
$$\therefore \log L(\mu) = \log f_{\underset{\sim}{X}}(\underset{\sim}{x};\mu)$$
$$= n\log\left(\frac{1}{1+\mu}\right) + \left(\sum_{i=1}^{n} x_i\right)\log\left(1-\frac{1}{1+\mu}\right) + \sum_{i=1}^{n}\log I_{\{0,1,\ldots\}}(x_i)$$
$$= -n\log(1+\mu) + \left(\sum_{i=1}^{n} x_i\right)\log(\mu) - \left(\sum_{i=1}^{n} x_i\right)\log(1+\mu) + \sum_{i=1}^{n}\log I_{\{0,1,\ldots\}}(x_i)$$
$$= -(n + \sum_{i=1}^{n} x_i)\log(1+\mu) + \left(\sum_{i=1}^{n} x_i\right)\log(\mu) + \sum_{i=1}^{n}\log I_{\{0,1,\ldots\}}(x_i)$$

4 (c) (Cont.)

$$\frac{d}{d\mu}\log L(\mu)$$
$$= -\frac{n+\sum_{i=1}^{n}x_i}{1+\mu}+\frac{\sum_{i=1}^{n}x_i}{\mu}$$

$$\frac{d}{d\mu}\log L(\mu)\Big|_{\mu=\hat{\mu}}=0 \Rightarrow \frac{n+\sum_{i=1}^{n}X_i}{1+\hat{\mu}}=\frac{\sum_{i=1}^{n}X_i}{\hat{\mu}}$$

$$\Rightarrow \left(n+\sum_{i=1}^{n}X_i\right)\hat{\mu}=\sum_{i=1}^{n}X_i+\hat{\mu}\sum_{i=1}^{n}X_i$$
$$n\hat{\mu}=\sum_{i=1}^{n}X_i$$
$$\therefore \hat{\mu}=\frac{1}{n}\sum_{i=1}^{n}X_i$$

(d) $$\frac{d}{d\theta}\log f_X(x;\theta)$$
$$=\frac{d}{d\theta}[\log\theta + x\log(1-\theta)]$$
$$=\frac{1}{\theta}-\frac{x}{1-\theta}$$
$$\frac{d^2}{d\theta^2}\log f_X(x;\theta)=-\frac{1}{\theta^2}-\frac{x}{(1-\theta)^2}$$

$$\therefore E\left(\frac{d^2}{d\theta^2}\log f_X(x;\theta)\right)=E\left(-\frac{1}{\theta^2}-\frac{X}{(1-\theta)^2}\right)=\frac{-1}{\theta^2}-\frac{(1-\theta)}{\theta(1-\theta)^2}=\frac{-1}{\theta^2}-\frac{1}{\theta(1-\theta)}$$

Now, $\frac{d}{d\theta}(1-\theta)=-1$

$\therefore$ C.R. lower bound for the variance of unbiased estimators of $(1-\theta)$ is

$$\frac{(-1)^2}{-n\left[\frac{-1}{\theta^2}-\frac{1}{\theta(1-\theta)}\right]}$$
$$=\frac{1}{n\left(\frac{1}{\theta^2}+\frac{1}{\theta(1-\theta)}\right)}$$
$$=\frac{\theta^2(1-\theta)}{n(1-\theta+\theta)}$$
$$=\frac{\theta^2(1-\theta)}{n}$$