Solution for MATH243 08 Fall final

Q1: (a) $L(\theta) = \prod_{i=1}^{n} f(x_i, \mu) = \prod_{i=1}^{n} \frac{e^{-\mu}\mu^{x_i}}{x_i!} = e^{-n\mu}\mu^{\sum x_i} / \prod_{i=1}^{n} x_i!$

$l(\theta) = \log L(\theta) = -n\mu + \sum_{i=1}^{n} x_i \log\mu - \log(\prod_{i=1}^{n} x_i!)$

$\Rightarrow \frac{\partial l(\theta)}{\partial \theta} = -n + \sum x_i / \mu = 0 \quad \Rightarrow \hat{\mu} = \bar{x}$

by the invariant property of MLE, $\bar{x}^2$ is MLE for μ^2

(b) $E\bar{x}^2 = E\frac{(\sum x_i)^2}{n^2} = \frac{1}{n^2} n\mu(n\mu+1)$
$= \mu^2 + \frac{\mu}{n} \neq \mu^2$

Therefore $\bar{x}^2$ is biased for μ^2

Since $\sum x_i \sim Poi(n\mu)$
$\Rightarrow E(\sum x_i) = n\mu$
$\Rightarrow Var(\sum x_i) = n\mu$
$\Rightarrow E(\sum x_i)^2 = Var(\sum x_i) + E(\sum x_i)^2$
$= n\mu + (n\mu)^2$
$= n\mu(n\mu+1)$

(c) $\sum x_i \sim Poi(n\mu)$, $E(\sum x_i) = n\mu$, $E(\sum x_i)^2 = n\mu(n\mu+1)$
$E(\bar{x}) = \mu \quad E(\bar{x}^2) = \mu^2 + \frac{\mu}{n}$

Thus the MSE is $E(\bar{x}^2 - \mu^2)^2 = E(\bar{x}^4 - 2\bar{x}^2\mu^2 + \mu^4)$
$= E\bar{x}^4 - 2\mu^2 E\bar{x}^2 + \mu^4$

Since $E(\bar{x}^4) = E(\sum x_i)^4 / n^4 \Rightarrow E(\sum x_i)^4 = (n\mu)^4 + 6(n\mu)^3 + 7(n\mu)^2 + (n\mu)$

$\Rightarrow E\bar{x}^4 = \mu^4 + \frac{6}{n}\mu^3 + \frac{7}{n^2}\mu^2 + \frac{\mu}{n^3}$

$\Rightarrow$ MSE $E(\bar{x}^2 - \mu^2)^2 = \mu^4 + \frac{6}{n}\mu^3 + \frac{7}{n^2}\mu^2 + \frac{\mu}{n^3} - 2\mu^2(\mu^2 + \frac{\mu}{n}) + \mu^4$
$= \frac{4\mu^3}{n} + \frac{8\mu^3}{n^2} + \frac{\mu}{n^3}$

(d) $f(x_i, \mu) = \frac{e^{-\mu}\mu^{x_i}}{x_i!}$

$\log f(x_i, \mu) = -\mu + x_i \log\mu - \log x_i!$

$\frac{\partial}{\partial\mu}\log f(x_i, \mu) = -1 + \frac{x_i}{\mu}$

$\frac{\partial^2}{\partial\mu^2}\log f(x_i, \mu) = -\frac{x_i}{\mu^2}$

$-E(\frac{\partial^2}{\partial\mu^2}\log f(x_i, \mu)) = \frac{\mu}{\mu^2} = \frac{1}{\mu}$

then the C.R.L.B for μ^2 is $\frac{(2\mu)^2}{n \cdot \frac{1}{\mu}} = \frac{4\mu^3}{n}$

(e). Since $\sum_{i=1}^{n} x_i$ is Complete and sufficient then since
$E\bar{x}^2 = \mu^2 + \frac{\mu}{n}$ in part (b)
then consider $E(\bar{x}^2 - \frac{\bar{x}}{n}) = E\bar{x}^2 - E\bar{x}/n = \mu^2 + \frac{\mu}{n} - \frac{\mu}{n} = \mu^2$, and $\bar{x}^2 - \frac{\bar{x}}{n}$ is function of C-S statistics, then $\bar{x}^2 - \frac{\bar{x}}{n}$ is UMVUE for μ^2.

Q2: (a). $X_1, X_2, \cdots, X_n \sim \text{Bernoulli}(\theta) \Rightarrow EX_1 = EX_2 = \theta$

then $E(X_1 - X_2) = EX_1 - EX_2 = 0$ but $p(X_1 - X_2 = 0) \neq 1$

that means $p(X_1 = X_2) \neq 1$

Thus $X_1 - X_2$ is not a complete statistics.

(b) Since Bernolli(θ) belongs to exponential family

$f_\theta(x) = \theta^{x_i}(1-\theta)^{1-x_i}$

$\Rightarrow f_\theta(x) = \exp\{x_i \log\theta + (1-x_i)\log(1-\theta)\}$

$= \exp\{x_i(\log\theta + \log(1-\theta)) - (1-x_i)\}$

$\Rightarrow \Sigma X_i$ is Complete and sufficient statistics.

(c) Since $X_1 \sim \text{Bernoulli}(\theta) \Rightarrow EX_1 = 0 \cdot P(X_1=0) + 1 \cdot P(X_1=1) = \theta$

which means X_1 is an unbiased estimator for θ

(d). Consider $\eta(x) = E(X_1 | \sum_{i=1}^n X_i)$

Since $\Sigma X_i \sim \text{Bin}(n,\theta)$, then

$E(X_1 | \Sigma X_i) = Pr(X_1 = 1 | \Sigma X_i = x) = \dfrac{Pr(X_1=1, \Sigma X_i = x)}{Pr(\Sigma X_i = x)}$

$= \dfrac{Pr(X_1=1, \sum_{i=2}^n X_i = x-1)}{Pr(\sum_{i=1}^n X_i = x)} = \dfrac{Pr(X_1=1)\, Pr(\sum_{i=2}^n X_i = x-1)}{Pr(\sum_{i=1}^n X_i = x)} = \dfrac{\cancel{\theta} \cdot \binom{n-1}{x-1} \cancel{\theta^{x-1}} \cancel{(1-\theta)^{n-x}}}{\binom{n}{x} \cancel{\theta^x} \cancel{(1-\theta)^{n-x}}}$

$= \dfrac{(n-1)!}{(x-1)!(n-x)!} \cdot \dfrac{(n-x)!\,x!}{n!} = \dfrac{x}{n}$

then ~~ΣX_i~~ $E(X_1 | \sum_{i=1}^n X_i) = \dfrac{\Sigma X_i}{n}$ is an improved estimator by Rao-Blackwell Theorem.

(e) Since ΣX_i is C-S statistics and $\Sigma X_i \sim \text{Bin}(n,\theta)$, then

$\sum_{s=0}^n h(s) C_n^s \theta^s (1-\theta)^{n-s} = \theta^m$

i.e. $\sum_{s=0}^n h(s) C_n^s \theta^{s-m} (1-p)^{n-s} = 1$

$\Rightarrow \sum_{s=m}^n C_n^s \theta^{s-m} (1-\theta)^{n-s} h(s) = \sum_{k=0}^{n-m} C_n^{k+m} \theta^k (1-\theta)^{n-m-k} h(k+m) = 1$

observe that $\sum_{k=0}^{n-m} C_{n-m}^k \theta^k (1-\theta)^{n-m-k} = 1$, therefore

$h(s) = \dfrac{C_{n-m}^{s-m}}{C_n^s}, \quad m \le s \le n.$

and $h(s) = 0$, when $s < m$

Combine the above results, we know the UMVUE for p^m is

$$\eta(S) = \begin{cases} 0, & S < m \\ \frac{C_{n-m}^{S-m}}{C_n^S}, & m \le S \le n \end{cases}, \quad S = \sum_{i=1}^n X_i.$$

(f) $$p(X_1 + \cdots + X_m = k) = \begin{cases} C_m^k (1-\theta)^{m-k} \theta^k, & 0 \le k \le m \\ 0, & k > m \end{cases}$$

$$\Rightarrow C_m^k (1-\theta)^{m-k} \theta^k = C_m^k \sum_{i=0}^{m-k} C_{m-k}^i (-p)^i p^k$$
$$= C_m^k \sum_{i=0}^{m-k} C_{m-k}^i (-1)^i \theta^{i+k}$$

From part (a), we known $C_{n-(i+k)}^{S-(i+k)} / C_n^S$ is UMVUE for θ^{i+k}

Therefore $$\eta(S) = \begin{cases} C_m^k \sum_{i=0}^{m-k} C_{m-k}^i (-1)^i \cdot C_{n-(i+k)}^{S-(i+k)} / C_n^S \\ 0, & k > m \end{cases}$$

and $S = \sum_{i=1}^n X_i$ is a complete suff. statistics, then $\eta(S)$ is UMVUE for $p(X_1 + \cdots + X_m = k)$

3. (a). $L(\theta) = \exp\{n\theta - \sum_{i=1}^n X_i\} I(X_{(1)} \ge \theta)$, by factorization theorem, then $X_{(1)}$ is sufficient statistics.

(b) $$F_{X_{(1)}}(x) = p(X_{(1)} \le x) = 1 - p(X_{(1)} - \theta \ge x - \theta) = 1 - \prod_{i=1}^n p(X_i - \theta > x - \theta)$$
$$= 1 - \exp\{-n(x-\theta)\}$$

$f_{X_{(1)}}(x) = n\exp\{-n(x-\theta)\}$

then let $g(\cdot)$ be any function, if $Eg(X_{(1)}) = 0$ for any θ, then we must have

$$\int_\theta^\infty u(x)\, n \exp\{-n(x-\theta)\} dx = 0$$
$$\Rightarrow \int_\theta^\infty u(x) \exp\{-nx\} dx = 0$$

taking derivative of left hand side with respect to θ,

$$-u(\theta) \exp\{-n\theta\} = 0 \quad \text{for all } \theta$$
$$\Rightarrow p(u(\theta) = 0) = 1$$

Then $X_{(1)}$ is complete statistics

~~(c)~~. By part (b), the distn of $X_{(1)}$ is exponential with rate n, with a location parameter θ; That is $X_{(1)} - \theta$ has exponential distn with rate n.

(c) By part (b), $EX_{(1)} = E\{X_{(1)} - \theta\} + \theta = \frac{1}{n} + \theta$

and ~~$EX_{(1)}^2 = Var(X_{(1)}) + (\frac{1}{n}+\theta)^2 = \frac{1}{n^2} + (\frac{1}{n}+\theta)^2$~~

$Var(X_{(1)}) = Var(X_{(1)} - \theta) = \frac{1}{n^2}$

(d). Since $EX_{(1)} = \frac{1}{n} + \theta \Rightarrow EX_{(1)} - \frac{1}{n} = \theta$

Since $X_{(1)}$ is complete and sufficient statistics, then $X_{(1)} - \frac{1}{n}$ is UMVUE for θ

(e). $EX_{(1)}^2 = Var(X_{(1)}) + (EX_{(1)})^2 = \frac{1}{n^2} + (\frac{1}{n}+\theta)^2 = \theta^2 + \frac{2\theta}{n} + \frac{2}{n^2}$

then $E\left[X_{(1)}^2 - \frac{2X_{(1)}}{n}\right] = \theta^2 + \frac{2\theta}{n} + \frac{2}{n^2} - \frac{2}{n}(\frac{1}{n}+\theta)$

$= \theta^2 + \cancel{\frac{2\theta}{n}} + \cancel{\frac{2}{n^2}} - \cancel{\frac{2}{n^2}} - \cancel{\frac{2\theta}{n}}$

$= \theta^2$

then $X_{(1)}^2 - \frac{2X_{(1)}}{n}$ is the UMVUE of θ^2

4. (a) $\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu = \mu_1 (> \mu_0) \end{cases}$

By N-P theorem, $C_1 = \{\underset{\sim}{x} : \frac{f_{\underset{\sim}{x}}(\underset{\sim}{x}: \mu=\mu_0)}{f_{\underset{\sim}{x}}(\underset{\sim}{x}: \mu=\mu_1)} \le k\}$

$\Rightarrow \frac{f_{\underset{\sim}{x}}(\underset{\sim}{x}: \mu=\mu_0)}{f_{\underset{\sim}{x}}(\underset{\sim}{x}: \mu=\mu_1)} = \frac{\prod_{i=1}^n f_{x_i}(x_i: \mu=\mu_0)}{\prod_{i=1}^n f_{x_i}(x_i: \mu=\mu_1)} = \prod_{i=1}^n \frac{\mu_0^{x_i} e^{-\mu_0}}{x_i!} \Big/ \prod_{i=1}^n \frac{\mu_1^{x_i} e^{-\mu_1}}{x_i!}$

$= \frac{\mu_0^{\Sigma x_i} e^{-n\mu_0}}{\cancel{\prod_{i=1}^n x_i!}} \Big/ \frac{\mu_1^{\Sigma x_i} e^{-n\mu_1}}{\cancel{\prod_{i=1}^n x_i!}} = (\frac{\mu_0}{\mu_1})^{\Sigma x_i} e^{-n(\mu_0-\mu_1)} = (\frac{\mu_0}{\mu_1})^{\Sigma x_i} e^{n(\mu_1-\mu_0)} \le k$

$\overset{(\mu_0<\mu_1)}{\Longleftrightarrow} \sum_{i=1}^n x_i \ge k$ ##

$\sum_{i=1}^n X_i \sim Poisson(n\mu)$, By CLT, $\sum_{i=1}^n X_i \overset{\cdot}{\sim} N(n\theta, n\theta)$

when $\mu_0 = 1$, $\mu_1 = 1.21$

$\Rightarrow \begin{cases} H_0: \mu = 1 \\ H_1: \mu = 1.21 \end{cases}$ $\begin{matrix} \Rightarrow P(\sum_{i=1}^n X_i \ge k \mid \mu=1) = 0.05 \\ \Rightarrow P(\sum_{i=1}^n X_i < k \mid \mu=1.21) \le 0.1 \end{matrix}$

$\Rightarrow \begin{cases} P(Z \ge \frac{k-0.5-n\mu}{\sqrt{n\mu}} \mid \mu=1) = 0.05 \\ P(Z < \frac{k-0.5-n\mu}{\sqrt{n\mu}} \mid \mu=1.21) \le 0.1 \end{cases}$

$\Rightarrow \begin{cases} P(Z \ge \frac{k-0.5-n}{\sqrt{n}}) = 0.05 \\ P(Z < \frac{k-0.5-1.21n}{\sqrt{1.21n}}) \le 0.1 \end{cases}$

$\Rightarrow \begin{cases} \frac{k-n-0.5}{\sqrt{n}} = 1.645 \\ \frac{k-1.21n-0.5}{\sqrt{1.21n}} \le -1.28 \end{cases}$ $\Rightarrow \begin{cases} k = n + 1.645\sqrt{n} + 0.5 \\ k \le -1.28\sqrt{1.21n} + 1.21n + 0.5 \end{cases}$

$\Rightarrow \quad n+1.645\sqrt{n}+0.5 \le 1.28\sqrt{1.21n}+1.21n+0.5$

$\Rightarrow \quad 0.21n \ge 3.053\sqrt{n}$

$\Rightarrow \quad \sqrt{n} \ge 3.053/0.21 \quad \Rightarrow \quad n \ge 211.4$

$\therefore$ Smallest value of n is 212.

(b).(i) $\begin{cases} H_0: \theta=\theta_0 \\ H_1: \theta>\theta_0 \end{cases}$

$f_X(x;\theta)=\frac{1}{\sqrt{2\pi\theta}}\exp\{-\frac{1}{2\theta}(x-\mu)^2\}=\exp\{-\frac{1}{2}\log(2\pi\theta)-\frac{1}{2\theta}(x-\mu)^2\}$

$\therefore$ $f_X(x;\theta)$ belongs to the exponential family of p.d.f where $c(\theta)=-\frac{1}{2\theta}$ and $d(x)=(x-\mu)^2$

Since $c(\theta)=-\frac{1}{2\theta}$ is increasing, the critical region of the ump test is in the form

$C_1=\{\underset{\sim}{x}:\sum_{i=1}^{n}d(x_i)\ge k\}=\{\underset{\sim}{x}:\sum_{i=1}^{n}(x_i-\mu)^2\ge k\}$

Under $H_0: \theta=\theta_0$, $\frac{1}{\theta_0}\sum_{i=1}^{n}(X_i-\mu)^2\sim\chi^2_{(n)}$

$$\begin{aligned}\alpha &= p(\underset{\sim}{X}\in C_1 \mid H_0)\\ &= p(\textstyle\sum_{i=1}^{n}(X_i-\mu)^2\ge k \mid \theta=\theta_0)\\ &= p(\frac{1}{\theta_0}\textstyle\sum_{i=1}^{n}(X_i-\mu)^2\ge\frac{1}{\theta_0}k \mid \theta=\theta_0)\\ &= p(\chi^2_n\ge\frac{1}{\theta_0}k)\end{aligned}$$

$\Rightarrow \frac{1}{\theta_0}k=\chi^2_n(\alpha) \quad \Rightarrow k=\theta_0\chi^2_n(\alpha)$

$\therefore$ the ump test at level of significant α is to reject H_0 when $\sum_{i=1}^{n}(x_i-\mu)^2\ge\theta_0\chi^2_n(\alpha)$

(ii). When $\theta_0=4$, $\alpha=0.05$ and $n=25$, the power at $\theta=12$ is

$$\begin{aligned}&p(\underset{\sim}{X}\in C_1 \mid \theta=12)\\ &= p(\textstyle\sum_{i=1}^{25}(X_i-\mu)^2\ge(4)\chi^2_{25}(0.05) \mid \theta=12)\\ &= p(\frac{1}{12}\textstyle\sum_{i=1}^{25}(X_i-\mu)^2\ge\frac{1}{3}\chi^2_{25}(0.05) \mid \theta=12)\\ &= p(\chi^2_{25}\ge\frac{1}{3}\chi^2_{25}(0.05))=p(\chi^2_{25}\ge\frac{1}{3}(37.652))\\ &= p(\chi^2_{25}\ge 12.551)\\ &> p(\chi^2_{25}\ge 13.12)\\ &= 0.975\end{aligned}$$

25. (a) Since $\theta_{01}, \theta_{02}, \cdots, \theta_{0m}$ are known specified values, H_0 is a simple hypothesis.

The likelihood function is

$$L(\theta_1, \cdots, \theta_m, x) = \text{constant} \times \prod_{i=1}^{m} \theta_i^{x_i}$$

The numerator of the likelihood ratio

$$\sup\{L(\theta, x) : \theta \in \Theta_0\} = \text{constant} \times \prod_{i=1}^{m} \theta_{0i}^{x_i}$$

The denominator of the likelihood ratio involves finding the MLE for θ

$$\sup\{L(\theta, x) : \theta \in \Theta\} = \text{constant} \times \prod_{i=1}^{n} \left(\frac{x_i}{n}\right)^{x_i}$$

(since $\theta_1 + \cdots + \theta_m = 1$

$$\log L(\theta) = \sum_{i=1}^{n} x_i \log \theta_i = \sum_{i=1}^{m-1} x_i \log \theta_i + x_m \log\left(1 - \sum_{i=1}^{m-1} \theta_i\right)$$

$$\frac{\partial}{\partial \theta} \log L(\theta) = \frac{x_i}{\theta_i} - \frac{x_m}{1-\sum \theta_i}, \quad i = 1, \cdots, m-1$$

$$\Rightarrow \hat{\theta}_i = \frac{x_i}{n}, \quad i, \cdots, n \quad)$$

Then $\lambda(x) = \prod_{i=1}^{m} \left(\frac{n\theta_{0i}}{x_i}\right)^{x_i}$

By the likelihood ratio test, H_0 is rejected for $\lambda(x) \leq k$ or equivalently $-2\log \lambda(x) \geq k'$

i.e $C_1 = \{x : -2\log \lambda(x) \geq k'\}$

$$= \{x : 2\sum_{i=1}^{m} x_i \log \frac{x_i}{n\theta_{0i}} \geq \chi^2_{(m-1, \alpha)}\} \text{ for large } n$$

(b) (i) $m=4$, $n=3839$. $x_1 = 1997$, $x_2 = 906$, $x_3 = 904$, $x_4 = 32$, $\alpha = 0.05$

Now H_0: $\theta_1 = \frac{9}{16}$, $\theta_2 = \frac{3}{16}$, $\theta_3 = \frac{3}{16}$, $\theta_4 = \frac{1}{16}$

By the ~~result~~ result of part (a),

$$2\sum_{i=1}^{4} x_i \log \frac{x_i}{n\theta_{0i}} = 2\left(1997 \times \log \frac{1997}{3839 \times \frac{9}{16}} + 906 \times \log \frac{906}{3839 \times \frac{3}{16}} + 904 \times \log \frac{904}{3839 \times \frac{3}{16}} + 32 \times \log \frac{32}{3839 \times \frac{1}{16}}\right)$$

$$= 387.51 \quad > \quad \chi^2_{3, 0.05} = 7.81$$

Therefore, we reject H_0.

(ii) Since here $n \cdot \theta_{0i} \geq 3839 \times \frac{1}{16} \approx 240$, then we can use Pearson's goodness of fit test.

$$G = \sum_{i=1}^{m} \frac{(x_i - n\theta_{0i})^2}{n\theta_{0i}} = \frac{(1997 - 3839 \times \frac{9}{16})^2}{3839 \times \frac{9}{16}} + \frac{(906 - 3839 \times \frac{3}{16})^2}{3839 \times \frac{3}{16}}$$

$$+ \frac{(904 - 3839 \times \frac{3}{16})^2}{3839 \times \frac{3}{16}} + \frac{(32 - 3839 \times \frac{1}{16})^2}{3839 \times \frac{1}{16}} = 287.714$$

Since $Q = 287.714 > \chi^2_3(0.05) = 7.81$

$\Rightarrow H_0$ is ~~not~~ ~~very~~ rejected.

We got the same conclusion as part (i) ##

26. (a) Since $Y_{1i} \sim N(\gamma_{10} + \gamma_{11}(X_{1i} - \bar{X}_1), \sigma^2)$

$Y_{2i} \sim N(\gamma_{20} + \gamma_{21}(X_{2i} - \bar{X}_2), \sigma^2)$

under the alternative hypothesis $\gamma_{11} \neq \gamma_{21}$

Then $L(\theta) = L(\gamma_{10}, \gamma_{11}, \gamma_{20}, \gamma_{21}, \sigma^2)$

$$= (2\pi\sigma^2)^{-n} \exp\left(- \frac{\sum(Y_{1i} - \gamma_{10} - \gamma_{11}(X_{1i} - \bar{X}_1))^2 + \sum(Y_{2i} - \gamma_{20} - \gamma_{21}(X_{2i} - \bar{X}_2))^2}{2\sigma^2} \right)$$

$$\log L(\theta) = l(\theta) = -n\log(2\pi\sigma^2) - \frac{\sum_{i=1}^n (Y_{1i} - \gamma_{10} - \gamma_{11}(X_{1i} - \bar{X}_1))^2}{2\sigma^2} - \frac{\sum_{i=1}^n (Y_{2i} - \gamma_{20} - \gamma_{21}(X_{2i} - \bar{X}_2))^2}{2\sigma^2}$$

$$\Rightarrow \begin{cases} \frac{\partial l(\theta)}{\partial \gamma_{10}} = -\frac{2}{2\sigma^2} \sum_{i=1}^n (Y_{1i} - \gamma_{10} - \gamma_{11}(X_{1i} - \bar{X}_1))(-1) = 0 \\ \frac{\partial l(\theta)}{\partial \gamma_{11}} = +\frac{2}{2\sigma^2} \sum_{i=1}^n (Y_{1i} - \gamma_{10} - \gamma_{11}(X_{1i} - \bar{X}_1))(X_{1i} - \bar{X}_1) = 0 \\ \frac{\partial l(\theta)}{\partial \gamma_{20}} = -\frac{2}{2\sigma^2} \sum_{i=1}^n (Y_{2i} - \gamma_{20} - \gamma_{21}(X_{2i} - \bar{X}_2))(-1) = 0 \\ \frac{\partial l(\theta)}{\partial \gamma_{21}} = +\frac{2}{2\sigma^2} \sum_{i=1}^n (Y_{2i} - \gamma_{20} - \gamma_{21}(X_{2i} - \bar{X}_2))(X_{2i} - \bar{X}_2) = 0 \\ \frac{\partial l(\theta)}{\partial \sigma^2} = -n\frac{2\pi}{2\pi\sigma^2} + \frac{\sum_{i=1}^n (Y_{1i} - \gamma_{10} - \gamma_{11}(X_{1i} - \bar{X}_1))^2}{2\sigma^4} + \frac{\sum_{i=1}^n (Y_{2i} - \gamma_{20} - \gamma_{21}(X_{2i} - \bar{X}_2))^2}{2\sigma^4} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \hat{\gamma}_{10} = \bar{Y}_1 - \hat{\gamma}_{11} \frac{1}{n}\sum_{i=1}^n (X_{1i} - \bar{X}_1) = \bar{Y}_1 \\ \hat{\gamma}_{11} = \frac{S_{x_1y_1}}{S_{x_1x_1}} \\ \hat{\gamma}_{20} = \bar{Y}_2 \\ \hat{\gamma}_{21} = \frac{S_{x_2y_2}}{S_{x_2x_2}} \\ \hat{\sigma}_0^2 = \frac{(S_{y_1y_1} + \hat{\gamma}_{11}^2 S_{x_1x_1} - 2\hat{\gamma}_{11} S_{x_1y_1}) + (S_{y_2y_2} + \hat{\gamma}_{21}^2 S_{x_2x_2} - 2\hat{\gamma}_{21} S_{x_2y_2})}{2n} \end{cases}$$

(b) Since under $H_0: \gamma_{11} = \gamma_{21} = \gamma$

Then $L(\theta) = L(\gamma_{10}, \gamma_{11}, \gamma, \sigma^2)$

$$= (2\pi\sigma^2)^{-n} \exp\left\{ - \frac{\sum(Y_{1i} - \gamma_{10} - \gamma(X_{1i} - \bar{X}_1))^2 + \sum(Y_{2i} - \gamma_{20} - \gamma(X_{2i} - \bar{X}_2))^2}{2\sigma^2} \right\}$$

$$\log L(\theta) = l(\theta) = -n\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\left\{ \sum_{i=1}^n (Y_{1i} - \gamma_{10} - \gamma(X_{1i} - \bar{X}_1))^2 \right\} - \frac{1}{2\sigma^2}\left\{ \sum_{i=1}^n (Y_{2i} - \gamma_{20} - \gamma(X_{2i} - \bar{X}_2))^2 \right\}$$

$$\begin{cases} \dfrac{\partial l(\theta)}{\partial \gamma_{10}} = -\dfrac{2}{2\sigma^2}\sum_{i=1}^{n}(Y_{1i}-\gamma_{10}-\gamma(X_{1i}-\bar{X}_1))(-1) = 0 \\ \dfrac{\partial l(\theta)}{\partial \gamma_{20}} = -\dfrac{2}{2\sigma^2}\sum_{i=1}^{n}(Y_{2i}-\gamma_{20}-\gamma(X_{2i}-\bar{X}_2))(-1) = 0 \\ \dfrac{\partial l(\theta)}{\partial \gamma} = +\dfrac{2}{2\sigma^2}\sum_{i=1}^{n}(Y_{1i}-\gamma_{10}-\gamma(X_{1i}-\bar{X}_1))(X_{1i}-\bar{X}_1) \\ \qquad + \dfrac{2}{2\sigma^2}\sum_{i=1}^{n}(Y_{2i}-\gamma_{20}-\gamma(X_{2i}-\bar{X}_2))(X_{2i}-\bar{X}_2) = 0 \\ \dfrac{\partial l(\theta)}{\partial \sigma^2} = -n\dfrac{2\pi}{2\pi\sigma^2} + \dfrac{\sum_{i=1}^{n}(Y_{1i}-\gamma_{10}-\gamma(X_{1i}-\bar{X}_1))^2}{2\sigma^4} + \dfrac{\sum_{i=1}^{n}(Y_{2i}-\gamma_{20}-\gamma(X_{2i}-\bar{X}_2))^2}{2\sigma^4} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \hat{\gamma}_{10} = \bar{Y}_1, \quad \hat{\gamma}_{20} = \bar{Y}_2 \\ \hat{\gamma} = \dfrac{S_{x_1y_1}+S_{x_2y_2}}{S_{x_1x_1}+S_{x_2x_2}} \\ \hat{\sigma}_1^2 = \left[(S_{y_1y_1}+S_{y_2y_2}) - \dfrac{(S_{x_1y_1}+S_{x_2y_2})^2}{S_{x_1x_1}+S_{x_2x_2}}\right] / 2n \end{cases}$$

(c) By part (a) & part (b), $\begin{cases} H_0: \gamma_{11}=\gamma_{21}=\gamma \\ H_1: \gamma_{11}\neq\gamma_{21} \end{cases}$

under H_0: $L_0(\theta) = (2\pi\hat{\sigma}_0^2)^{-n}\exp\{-n\}$

not under H_0: $L_1(\theta) = (2\pi\hat{\sigma}_1^2)^{-n}\exp\{-n\}$

$$\Lambda(X) = \frac{L_0(\theta)}{L_1(\theta)} = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)^{-n} \#$$

$$\hat{\sigma}_0^2 = \frac{1}{2n}\left[\left(S_{y_1y_1}+\frac{S_{x_1y_1}^2}{S_{x_1x_1}^2}S_{x_1x_1} - 2\frac{S_{x_1y_1}}{S_{x_1x_1}}S_{x_1y_1}\right) + \left(S_{y_2y_2}+\frac{S_{x_2y_2}^2}{S_{x_2x_2}^2}S_{x_2x_2} - 2\frac{S_{x_2y_2}}{S_{x_2x_2}}S_{x_2y_2}\right)\right]$$

$$= \frac{1}{2n}\left[\frac{S_{y_1y_1}S_{x_1x_1}-S_{x_1y_1}^2}{S_{x_1x_1}} + \frac{S_{y_2y_2}S_{x_2x_2}-S_{x_2y_2}^2}{S_{x_2x_2}}\right]$$

$$= \frac{1}{2n}\frac{S_{x_1x_1}S_{x_2x_2}}{S_{x_1x_1}+S_{x_2x_2}}\left[(\right.$$

$$\Rightarrow \hat{\sigma}_0^2 = \frac{1}{2n}\left[S_{y_1y_1}-\hat{\beta}_1^2S_{x_1x_1}+S_{y_2y_2}-\hat{\beta}_2^2S_{x_2x_2}\right] \triangleq \frac{1}{2n}\text{Res S.S}\,|H_1.$$

$$\Rightarrow \hat{\sigma}_1^2 = \frac{1}{2n}\left[(S_{y_1y_1}+S_{y_2y_2}) - \frac{(S_{x_1y_1}+S_{x_2y_2})^2}{S_{x_1x_1}+S_{x_2x_2}}\right]$$

$$= \frac{1}{2n}\left[S_{y_1y_1}+S_{y_2y_2}-\hat{\gamma}^2(S_{x_1x_1}+S_{x_2x_2})\right] \quad \left(\text{By } \hat{\gamma} = \frac{\hat{\gamma}_{11}S_{x_1x_1}+\hat{\gamma}_{21}S_{x_2x_2}}{S_{x_1x_1}+S_{x_2x_2}}\right)$$

$$\triangleq \frac{1}{2n}\text{Res S.S}\,|H_0$$

$\Rightarrow$

Then $\text{Res S.S}|H_1 - \text{Res S.S}|H_0$

$$= \hat{\gamma}_{11}^2S_{x_1x_1}+\hat{\gamma}_{21}^2S_{x_2x_2} - \frac{\hat{\gamma}_{11}^2S_{x_1x_1}^2+2\hat{\gamma}_{11}\hat{\gamma}_{21}S_{x_1x_1}S_{x_2x_2}+\hat{\gamma}_{21}^2S_{x_2x_2}^2}{S_{x_1x_1}+S_{x_2x_2}}$$

$$= \frac{S_{x_1x_1}S_{x_2x_2}}{S_{x_1x_1}+S_{x_2x_2}}\left[\hat{\gamma}_{11}-\hat{\gamma}_{21}\right]^2$$

$$\left(\text{By } \hat{\gamma}^2(S_{x_1x_1}+S_{x_2x_2}) = \frac{\hat{\gamma}_{11}^2S_{x_1x_1}^2+2\hat{\gamma}_{11}\hat{\gamma}_{21}S_{x_1x_1}S_{x_2x_2}+\hat{\gamma}_{21}^2S_{x_2x_2}^2}{S_{x_1x_1}+S_{x_2x_2}}\right)$$

(d) Here $\lambda(X) = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)^{-n}$

$-2\log\lambda(X) = 2n\log\left(\frac{Res\,S.S\,|H_0}{Res\,S.S\,|H_1}\right) = 10 \times \log\frac{356.645913}{339.82327444} \overset{2*}{=} 0.4832$

$< \chi_1^2(0.05) = 3.84$

Then we don't reject ~~the null~~ H_0

(e) Since the test statistic

$T = \frac{(\hat{\gamma}_1 - \hat{\gamma}_2)^2}{\left(\frac{1}{S_{x_1x_1}} + \frac{1}{S_{x_2x_2}}\right)\frac{Res\,S.S\,|H_1}{2n-4}} \sim F_{1,2n-4}$

$\hat{\gamma}_1 = 0.7435222929$, $\hat{\gamma}_2 = 1.194464159$, $\hat{\gamma} = 0.928695652$

then we get $T \approx \cancel{0.297}\ 0.792 < F_{1,6}(0.05) = 5.99$

Then we don't reject H_0