The Hong Kong University of Science & Technology

MATH243 – Statistical Inference

Final Examination – Fall 04/05

Answer ALL questions **Date: 17 Dec 2004 (Fri)**

Time allowed: 2 Hours

---

1. Suppose X_1, X_2 ~ i.i.d. Bernoulli (p), i.e., they are independent and identically distributed and

$$X_1 = \begin{cases} 1 & \text{with probability } p, \\ 0 & \text{with probability } 1 - p. \end{cases}$$

(a) **(1 mark)** Find the minimal sufficient statistic for p.

(b) **(1 mark)** Show that $X_1 X_2$ is an unbiased estimator of p^2.

(c) **(3 marks)** Find $E(X_1 X_2 \mid X_1 + X_2)$. Is it the best unbiased estimator of p^2 ?

(d) **(5 marks)** Find the maximum likelihood estimator of p. Hence or otherwise, find the maximum likelihood estimator of p^2. Is it unbiased?

(e) **(5 marks)** Consider the two estimators that you found in (c) and (d). Which one has a smaller mean squared error when $p = \frac{1}{2}$?

2. Let $X_1, ..., X_n$ be i.i.d. v.v.'s from the $U(\theta, 2\theta)$, $\theta \in \Omega = (0, \infty)$, distribution.

(a) **(3 marks)** Find $E(Y_1)$. Hence or otherwise, find an unbiased estimator of θ as a function of Y_1.

(b) **(3 marks)** Find $E(Y_n)$. Hence or otherwise, find an unbiased estimator of θ as a function of Y_n.

(c) **(9 marks)** Define the unbiased estimators of θ in parts (a) and (b) as U_a and U_b, respectively. Find a constant k so that the unbiased estimator, $kU_a + (1-k)U_b$, has the smallest variance.

Hint: $f_{Y_1, Y_n(y_1, y_n)} = n(n-1)(y_n - y_1)^{n-2} / \theta^n$ for $\theta \le y_1 \le y_n \le 2\theta$ and $Cov(Y_1, Y_n) = \dfrac{\theta^2}{(n+1)^2(n+2)}$.

3. (a) **(5 marks)** Consider a family of normal distribution $N(\mu, 1)$ with unknown mean μ as a parameter. Construct the UMP test with level of significance α for $H_0 : \mu = \mu_0$ and $H_1 : \mu > \mu_1$. Hence, construct the UMP test with level of significance α for $H_0 : \mu \le \mu_0$ and $H_1 : \mu > \mu_1$. Write down all steps in details.

(b) **(5 marks)** Consider a family of normal distributions $N(0, \sigma^2)$ with variance σ^2 as unknown parameter. Construct the UMP test with level of significance α for $H_0 : \sigma^2 = \sigma_0^2$ and $H_1 : \sigma^2 > \sigma_0^2$. Hence, construct the UMP test with level of significance α for $H_0 : \sigma^2 \le \sigma_0^2$ and $H_1 : \sigma^2 > \sigma_0^2$. Write down all steps in details.

(c) **(5 marks)** Use the result in part (b), calculate the sample size required to achieve at least 95% power at level of significance 5% for testing $H_0 : \sigma^2 = 1$ and $H_1 : \sigma^2 = 2.5$.

Note: Use z_α to define the upper percentage point corresponding to the upper tail α of a standard normal distribution and use $\chi^2_\alpha(n)$ to define the upper percentage point corresponding to the upper tail α of a chi-square distribution with *n* degrees of freedom.

4. Let $(y_{11}, y_{12}, \cdots, y_{1n})$, $(y_{21}, y_{22}, \cdots, y_{2n})$, and $(y_{31}, y_{32}, \cdots, y_{3n})$ be random samples from the independent normal distribution $N(\mu_1, \sigma^2)$, $N(\mu_2, \sigma^2)$, and $N(\mu_3, \sigma^2)$, respectively.

(a) **(8 marks)** Construct the likelihood ratio test for testing $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_0$ against H_1: they are not all equal.

Hint: (i) Find the m.l.e. of μ_0 and σ^2 under H_0

(ii) Find the m.l.e. of μ_1, μ_2, μ_3 and σ^2 under H_1

(b) **(7 marks)** Derive the approximate large sample likelihood ratio test. Specify the hypothesis to be tested, test statistic and your conclusion for the data set below at α = 0.05.

	Group		
	1	2	3
	551	595	639
	457	580	615
	430	508	511
	731	583	573
	499	633	648
	632	517	677
Total	3320	3416	3663
Mean	553.33	569.33	610.53

(c) **(Bonus: 4 marks)** Construct the exact likelihood ratio test with level of significance α. Do you get the same conclusion on the data set as part (b)?

Hint: Write down the likelihood ratio test in the form of $(1 + \lambda(y_1, y_2, y_3))^{-3n/2}$.

****** **END** ******