Math 243 Final 99/2000

1. $X_1, \dots, X_n \overset{iid}{\sim} f(x|\theta) = \frac{1}{\theta}$, $0 \le x \le \theta$, $\theta > 0$ $(U[0,\theta])$

$E(X) = \frac{1}{2}(0+\theta) = \frac{\theta}{2}$

Method of moments:

$$\frac{\tilde{\theta}}{2} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\therefore \tilde{\theta} = \frac{2}{n}\sum_{i=1}^{n} X_i = 2\bar{X}$$

Maximum likelihood:

$$L(\theta) = f_{\underline{X}}(\underline{x};\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{1}{\theta} I_{[0,\theta]}(x_i)$$

$$= \frac{1}{\theta^n} I_{[0,y_n]}(y_1) I_{[y_1,\theta]}(y_n)$$ where y_i is the i-th order statistic.

Observe that $\frac{1}{\theta}$ is a decreasing function on θ when $\theta > 0$.
But θ could not be smaller than y_n.
Thus, the maximum likelihood estimator of θ is $Y_n = \max\{X_1, \dots, X_n\}$.

$$P(Y_n \le y) = P(X_1 \le y, X_2 \le y, \dots, X_n \le y) \qquad 0 \le y \le \theta$$

$$= \prod_{i=1}^{n} P(X_i \le y)$$

$$= \left(\frac{y}{\theta}\right)^n$$

$$\therefore f_{Y_n}(y) = \frac{n y^{n-1}}{\theta^n}$$

$E(\tilde{\theta}) = E(2\bar{X}) = 2E(\bar{X}) = 2\left(\frac{\theta}{2}\right) = \theta$ ($\tilde{\theta}$ is unbiased for θ)

$$E(\hat{\theta}) = E(Y_n) = \int_0^\theta y f_{Y_n}(y)\,dy$$

$$= \int_0^\theta y \frac{n y^{n-1}}{\theta^n}\,dy$$

$$= \left(\frac{n}{n+1}\right)\frac{y^{n+1}}{\theta^n}\Big|_0^\theta = \frac{n\theta}{n+1}$$ ($\hat{\theta}$ is not unbiased for θ)

$$\text{M.S.E. of } \tilde{\theta} = E(\tilde{\theta}-\theta)^2 = Var(\tilde{\theta})$$

$$= Var(2\bar{X}) = 4\,Var(\bar{X}) = \frac{4}{n}Var(X) = \frac{4}{n}\frac{(\theta-0)^2}{12} = \frac{\theta^2}{3n}$$

$$\text{M.S.E. of } \hat{\theta} = E(\hat{\theta}-\theta)^2 = E(Y_n-\theta)^2 = \int_0^\theta (y-\theta)^2 f_{Y_n}(y)\,dy$$

$$= \int_0^\theta (y^2 - 2\theta y + \theta^2)\frac{n y^{n-1}}{\theta^n}\,dy$$

$$= \left(\frac{n}{n+2}\right)\frac{y^{n+2}}{\theta^n}\Big|_0^\theta - \left(\frac{2n}{n+1}\right)\frac{y^{n+1}}{\theta^{n-1}}\Big|_0^\theta + \frac{y^n}{\theta^{n+2}}\Big|_0^\theta$$

$$= \left(\frac{n}{n+2}\right)\theta^2 - \left(\frac{2n}{n+1}\right)\theta^2 + \theta^2 = (n+2)^{-1}(n+1)^{-1}[n^2+n-2n^2-4n+n^2+3n+2]\theta^2$$

$$= 2(n+2)^{-1}(n+1)^{-1}\theta^2 < \frac{\theta^2}{3n} \quad \text{for } n \ge 1.$$

1. (cont.)
Therefore, if it is preferred to have an unbiased estimator, $\tilde{\theta} = 2\bar{X}$ should be chosen.
However, if it is preferred to have an estimator having smaller mean square error, $\hat{\theta} = Y_n = \max\{X_1, \dots, X_n\}$ should be chosen.

2. $X_1, \dots, X_n$ is a random sample of size $n>2$ from a distribution with p.d.f. $f(x;\theta)=\theta e^{-\theta x}$, $0<x<\infty$, zero elsewhere, and $\theta>0$.

$$f(x;\theta)=\theta e^{-\theta x}$$
$$=\exp\{\log\theta-\theta x\}$$
$$=\exp\{a(\theta)+b(x)+c(\theta)d(x)\}$$

where $a(\theta)=\log\theta$, $b(x)=0$, $c(\theta)=-\theta$, $d(x)=x$,
$A=(0,\infty)$ and $D=\mathbb{R}^+$

$\therefore$ $f(x;\theta)$ is a p.d.f. which belongs to exponential family.

$\therefore$ $\sum_{i=1}^n d(X_i)=\sum_{i=1}^n X_i$ is a complete and sufficient statistic for θ.

$m_X(t)=E(e^{tX})=\left(\frac{\theta}{\theta-t}\right)$ (moment generating function of X)

Let $Y=\sum_{i=1}^n X_i$

$$m_Y(t)=E(e^{tY})=E(e^{t\sum_{i=1}^n X_i})=\prod_{i=1}^n E(e^{tX_i})$$
$$=\left(\frac{\theta}{\theta-t}\right)^n$$ which is the moment generating function of Gamma(n,θ).

$$E(t(Y))=\theta$$
$$\Rightarrow \int_0^\infty t(y)\frac{y^{n-1}e^{-\theta y}}{\theta^{-n}\Gamma(n)}dy=\theta$$
$$\Rightarrow \int_0^\infty t(y)\frac{y^{n-1}e^{-\theta y}}{\theta^{-(n-1)}\Gamma(n)}dy=1$$
$$\Rightarrow \int_0^\infty t(y)y\frac{\Gamma(n-1)}{\Gamma(n)}\frac{y^{(n-1)-1}e^{-\theta y}}{\theta^{-(n-1)}\Gamma(n-1)}dy=1$$
$$\Rightarrow t(y)y\frac{\Gamma(n-1)}{\Gamma(n)}=1$$
$$\Rightarrow t(y)=\left(\frac{(n-1)!}{(n-2)!}\right)y^{-1}$$
$$\therefore\quad t(Y)=\left(\frac{(n-1)!}{(n-2)!}\right)Y^{-1}$$

$\therefore$ the best estimator for θ is $\overset{n-1}{\cancel{\left(\frac{n-1}{n-2}\right)}}\left(\sum_{i=1}^n X_i\right)^{-1}$

3. $X_1, \ldots, X_n$ are independent random variable with distribution $B(1, p)$.
$f(x;p) = p^x(1-p)^{1-x}$, $x = 0, 1$.

(a) $L(p) = f_{\underset{\sim}{X}}(\underset{\sim}{x};p) = \prod_{i=1}^{n} f_{X_i}(x_i;p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i}$

$\log L(p) = \sum_{i=1}^n x_i \log p + (n - \sum_{i=1}^n x_i)\log(1-p)$

$\frac{\partial \log L(p)}{\partial p} = (\sum_{i=1}^n x_i)(\frac{1}{p}) - (n - \sum_{i=1}^n x_i)(\frac{1}{1-p})$

$\left.\frac{\partial \log L(p)}{\partial p}\right|_{p=\hat{p}} = 0 \Rightarrow \hat{p} = \frac{1}{n}\sum_{i=1}^n x_i = \overline{X}$

$\therefore$ the maximum likelihood estimator of $\theta = (1-p)^2$ is
$\hat{\theta} = (1-\hat{p})^2 = (1-\overline{X})^2$

(b) $\hat{\theta} = \begin{cases} 1 & \text{if } X_1 + X_2 = 0 \\ 0 & \text{otherwise} \end{cases}$

Note that $X_1 + X_2 \sim Bin(2, p)$
$\therefore E(\hat{\theta}) = 1 \cdot P(X_1 + X_2 = 0) + 0 \cdot P(X_1 + X_2 \neq 0)$
$= \binom{2}{0} p^0 (1-p)^2 = (1-p)^2 = \theta$
$\therefore \hat{\theta}$ is an unbiased estimator of θ.

(c) $f(x;p) = p^x(1-p)^{1-x}$
$= \exp\{x \log p + (1-x)\log(1-p)\}$
$= \exp\{\log(1-p) + x \log(\frac{p}{1-p})\}$
$= \exp\{a(p) + b(x) + c(p)d(x)\}$

where $a(p) = \log(1-p)$, $b(x) = 0$, $c(p) = \log\frac{p}{1-p}$, $d(x) = x$
$A = (0, 1)$, $D = \{0, 1\}$

$\therefore f(x;p)$ is a p.d.f. which belongs to exponential family.
$\therefore \sum_{i=1}^n d(X_i) = \sum_{i=1}^n X_i$ a complete and sufficient statistic for p.

Let $S = \sum_{i=1}^n X_i$. Note that $S \sim Bin(n, p)$.

$E(\hat{\theta} | S = s) = 1 \cdot P(X_1 + X_2 = 0 | S = s) + 0 \cdot P(X_1 + X_2 \neq 0 | S = s)$

$= \frac{P(X_1 + X_2 = 0, \sum_{i=1}^n X_i = s)}{P(\sum_{i=1}^n X_i = s)}$

$= \frac{P(X_1 + X_2 = 0, \sum_{i=3}^n X_i = s)}{P(\sum_{i=1}^n X_i = s)} = \frac{P(X_1 + X_2 = 0)P(\sum_{i=3}^n X_i = s)}{P(\sum_{i=1}^n X_i = s)}$

3 (c) (cont.)

Again, $\sum_{i=3}^{n} X_i \sim Bin(n-2, p)$.

$$\therefore E(\hat{\theta} \mid S=s) = \frac{\binom{2}{0} p^0 (1-p)^2 \binom{n-2}{s} p^s (1-p)^{n-2-s}}{\binom{n}{s} p^s (1-p)^{n-s}}$$

$$= \frac{\frac{(n-2)!}{s!(n-2-s)!}}{\frac{n!}{s!(n-s)!}} = \frac{(n-s)(n-s-1)}{n(n-1)}$$

$\therefore$ the best estimator of $\theta = (1-p)^2$ is $\frac{1}{n(n-1)}\left(n - \sum_{i=1}^{n} X_i\right)\left(n - \sum_{i=1}^{n} X_i - 1\right)$.

4. $X_1, X_2, \ldots, X_n$ are i.i.d. random variables, each with the Poisson distribution of parameter θ. $f_X(x;\theta) = \frac{\theta^x e^{-\theta}}{x!}$

$$\begin{cases} H_0: \theta = 1 \\ H_1: \theta = 1.21 \end{cases}$$

By Neyman-Pearson theorem, the most powerful test is to reject H_0 when

$$\frac{f_X(x;\theta=1)}{f_X(x;\theta=1.21)} \le k$$

$$\Leftrightarrow \frac{\prod_{i=1}^n f_{X_i}(x_i;\theta=1)}{\prod_{i=1}^n f_{X_i}(x_i;\theta=1.21)} \le k \Leftrightarrow \frac{\dfrac{(1)^{\sum_{i=1}^n x_i} e^{-1}}{\prod_{i=1}^n x_i!}}{\dfrac{(1.21)^{\sum_{i=1}^n x_i} e^{-1.21}}{\prod_{i=1}^n x_i!}} \le k$$

$$\Leftrightarrow (1.21)^{-\sum_{i=1}^n x_i} e^{0.21} \le k$$

$$\Leftrightarrow \left(-\sum_{i=1}^n x_i\right) \log(1.21)(0.21) \le k$$

$$\Leftrightarrow \sum_{i=1}^n x_i \ge K$$

$\therefore C_1 = \{x : \sum_{i=1}^n x_i \ge K\}$ is the critical region of the most powerful test of $H_0: \theta = 1$ against $H_1: \theta = 1.21$.

Note that $\sum_{i=1}^n X_i \sim Poi(n\theta)$.

$$\alpha \ge P\left(\sum_{i=1}^n X_i \ge K \mid \theta = 1\right)$$

$$\ge P(Y \ge K) \quad \text{where } Y \sim Poi(n)$$

$\Rightarrow K = \xi(n;\alpha)$ where $\xi(n;\alpha)$ is the smallest integer which satisfies $P(Y \ge \xi(n;\alpha)) \le \alpha$.

$\therefore$ the best size α test of $H_0: \theta = 1$ against $H_1: \theta = 1.21$ is to reject H_0 when $\sum_{i=1}^n X_i \ge \xi(n;\alpha)$.

By Central limit theorem, $\sum_{i=1}^n X_i \mathrel{\dot\sim} N(n\theta, n\theta)$.

Now, $\alpha = 0.05$ and $\beta \le 0.1$.

$$\begin{cases} P\left(\sum_{i=1}^n X_i \ge K \mid \theta = 1\right) = 0.05 \\ P\left(\sum_{i=1}^n X_i < K \mid \theta = 1.21\right) \le 0.1 \end{cases}$$

$$\Rightarrow \begin{cases} P\left(Z \ge \frac{K - n - 0.5}{\sqrt{n}}\right) = 0.05 \\ P\left(Z < \frac{K - 1.21n - 0.5}{\sqrt{1.21n}}\right) \le 0.1 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{K - n - 0.5}{\sqrt{n}} = 1.645 \\ \frac{K - 1.21n - 0.5}{\sqrt{1.21n}} \le -1.28 \end{cases}$$

4. (cont.)

$$\Rightarrow \begin{cases} K - n - 0.5 = 1.645\sqrt{n} \Rightarrow K = n + 1.645\sqrt{n} + 0.5 & -① \\ K - 1.21n - 0.5 \le -1.28(1.1\sqrt{n}) & -② \end{cases}$$

sub ① into ②,

$$n + 1.645\sqrt{n} + 0.5 - 1.21n - 0.5 \le -1.408\sqrt{n}$$

$$\Rightarrow 0.21n \ge 3.053\sqrt{n}$$

$$\Rightarrow \quad \sqrt{n} \ge \frac{3.053}{0.21}$$

$$\Rightarrow \quad n \ge 211.4$$

$\therefore$ the smallest value of n required to make $\alpha = 0.05$ and $\beta \le 0.1$ is 212.

5. The data $x_1, \dots, x_n$ has been observed and it is known that x_i is a sample from a Poisson distribution with an unknown mean λ_i.

$$\begin{cases} H_0: \lambda_1 = \cdots = \lambda_n \\ H_1: \lambda_i\text{'s are not all equal} \end{cases}$$

$$\Theta_0 = \{(\lambda, \lambda, \dots, \lambda) : \lambda > 0\}$$

$$\Theta = \mathbb{R}^+ \times \mathbb{R}^+ \times \cdots \times \mathbb{R}^+ = \mathbb{R}^{+^n}$$

$$f_{X_i}(x_i ; \lambda_i) = \frac{\lambda_i^{x_i} e^{-\lambda_i}}{x_i!}$$

$$L(\underset{\sim}{\lambda}) = f_{\underset{\sim}{X}}(\underset{\sim}{x} ; \lambda_1, \dots, \lambda_n) = \prod_{i=1}^n f_{X_i}(x_i ; \lambda_i) = \frac{\prod_{i=1}^n (\lambda_i^{x_i}) e^{-\sum_{i=1}^n \lambda_i}}{\prod_{i=1}^n x_i!}$$

the likelihood ratio:

$$\lambda(\underset{\sim}{x}) = \frac{\sup\{L(\underset{\sim}{\lambda}) : \underset{\sim}{\lambda} \in \Theta_0\}}{\sup\{L(\underset{\sim}{\lambda}) : \underset{\sim}{\lambda} \in \Theta\}}$$

Numerator: $\underset{\sim}{\lambda} \in \Theta_0$, $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda$,

$$L(\underset{\sim}{\lambda}) = (\lambda)^{\sum_{i=1}^n x_i} e^{-n\lambda} / \prod_{i=1}^n x_i!$$

$$\log L(\underset{\sim}{\lambda}) = \sum_{i=1}^n x_i \log \lambda - n\lambda - \log \prod_{i=1}^n x_i!$$

$$\frac{\partial \log L(\underset{\sim}{\lambda})}{\partial \lambda} = \frac{\sum_{i=1}^n x_i}{\lambda} - n$$

$$\left.\frac{\partial \log L(\underset{\sim}{\lambda})}{\partial \lambda}\right|_{\lambda=\hat{\lambda}} = 0 \Rightarrow \hat{\lambda} = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}$$

$$\therefore \sup\{L(\underset{\sim}{\lambda}) : \underset{\sim}{\lambda} \in \Theta_0\} = (\bar{x})^{n\bar{x}} e^{-n\bar{x}} / \prod_{i=1}^n x_i!$$

Denominator: $\underset{\sim}{\lambda} \in \Theta$.

$$\log L(\underset{\sim}{\lambda}) = \sum_{i=1}^n x_i \log \lambda_i - \sum_{i=1}^n \lambda_i - \log \prod_{i=1}^n x_i!$$

$$\frac{\partial \log L(\underset{\sim}{\lambda})}{\partial \lambda_i} = \frac{x_i}{\lambda_i} - 1 \quad \text{for } i = 1, 2, \dots, n$$

$$\left.\frac{\partial \log L(\underset{\sim}{\lambda})}{\partial \lambda_i}\right|_{\underset{\sim}{\lambda}=\hat{\underset{\sim}{\lambda}}} = 0 \Rightarrow \begin{cases} \frac{x_1}{\hat{\lambda}_1} - 1 = 0 \\ \vdots \\ \frac{x_n}{\hat{\lambda}_n} - 1 = 0 \end{cases} \Rightarrow \begin{cases} \hat{\lambda}_1 = x_1 \\ \vdots \\ \hat{\lambda}_n = x_n \end{cases}$$

$$\therefore \sup\{L(\underset{\sim}{\lambda}) : \underset{\sim}{\lambda} \in \Theta\} = \prod_{i=1}^n x_i^{x_i} e^{-\sum_{i=1}^n x_i} / \prod_{i=1}^n x_i!$$

$$\therefore \lambda(\underset{\sim}{x}) = \frac{(\bar{x})^{n\bar{x}} e^{-n\bar{x}}}{\prod_{i=1}^n x_i^{x_i} e^{-\sum_{i=1}^n x_i}} = \frac{(\bar{x})^{n\bar{x}}}{\prod_{i=1}^n x_i^{x_i}}$$

For large n, $-2\log \lambda(\underset{\sim}{x}) \mathrel{\dot\sim} \chi^2_r$

where r = no. of free parameters in Θ − no. of free parameters in Θ_0 = $n-1$.

5. (cont.)

$\therefore$ the approximate large sample likelihood ratio test is to reject H_0 when $-2\log\lambda(\underset{\sim}{x}) \geq \chi^2_{n-1}(\alpha)$

i.e. $-2(n\bar{x}\log\bar{x} - \sum_{i=1}^{n} x_i \log x_i) \geq \chi^2_{n-1}(\alpha)$

For data $(3, 4, 1, 6, 5)$, $n=5$,

$\bar{x} = \frac{1}{5}(3+4+1+6+5) = \frac{19}{5} = 3.8$.

$-2(5(3.8)\log(3.8) - 3\log 3 - 4\log 4 - (1)\log(1) - 6\log 6 - 5\log 5)$

$= 4.55 < 9.488 = \chi^2_{5-1}(0.05)$

$\therefore$ H_0 is not rejected at significance level $\alpha = 0.05$.

5. $X_1, \dots, X_n$ is a random sample of size n from exponential (θ).
$Y_1, \dots, Y_n$ is another random sample of size n from exponential (μ).

$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0$

$f(y|\mu) = \mu e^{-\mu y}, \quad y > 0$

(a) $\begin{cases} H_0: \theta = \mu \\ H_1: \theta \neq \mu \end{cases}$

$\Theta_0 = \{(\theta, \mu) : \theta = \mu, \ \theta > 0, \ \mu > 0\}$

$\Theta = \mathbb{R}^+ \times \mathbb{R}^+ = \mathbb{R}^{+2}$

$L(\theta, \mu) = f_{\underset{\sim}{X}, \underset{\sim}{Y}}(\underset{\sim}{x}, \underset{\sim}{y}; \theta, \mu) = \left[\prod_{i=1}^n f_{X_i}(x_i; \theta)\right]\left[\prod_{i=1}^n f_{Y_i}(y_i; \mu)\right]$

$= \theta^n e^{-\theta \sum_{i=1}^n x_i} \mu^n e^{-\mu \sum_{i=1}^n y_i}$

the likelihood ratio:

$\lambda(\underset{\sim}{x}, \underset{\sim}{y}) = \dfrac{\sup\{L(\theta, \mu) : (\theta, \mu) \in \Theta_0\}}{\sup\{L(\theta, \mu) : (\theta, \mu) \in \Theta\}}$

Numerator: $(\theta, \mu) \in \Theta_0$, $\theta = \mu = \lambda$ where $\lambda > 0$

$L(\theta, \mu) = L(\lambda, \lambda)$

$= \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \lambda^n e^{-\lambda \sum_{i=1}^n y_i} = \lambda^{2n} e^{-\lambda(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i)}$

$\log L(\lambda, \lambda) = 2n \log \lambda - \lambda(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i)$

$\dfrac{\partial \log L(\lambda, \lambda)}{\partial \lambda} = \dfrac{2n}{\lambda} - (\sum_{i=1}^n x_i + \sum_{i=1}^n y_i)$

$\dfrac{\partial \log L(\lambda, \lambda)}{\partial \lambda}\Big|_{\lambda = \hat{\lambda}} = 0 \Rightarrow \hat{\lambda} = (2n)(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i)^{-1}$

$\therefore \sup\{L(\theta, \mu) : (\theta, \mu) \in \Theta_0\}$

$= [(2n)(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i)^{-1}]^{2n} \exp\{-(2n)(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i)^{-1}(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i)\}$

$= [(2n)(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i)^{-1}]^{2n} \exp\{-2n\}$

Denominator: $(\theta, \mu) \in \Theta$

$\log L(\theta, \mu) = n \log \theta - \theta \sum_{i=1}^n x_i + n \log \mu - \mu \sum_{i=1}^n y_i$

$\dfrac{\partial \log L(\theta, \mu)}{\partial \theta} = \dfrac{n}{\theta} - \sum_{i=1}^n x_i$

$\dfrac{\partial \log L(\theta, \mu)}{\partial \mu} = \dfrac{n}{\mu} - \sum_{i=1}^n y_i$

6 (a) (cont.)

$$\begin{cases} \frac{\partial \log L(\theta,\mu)}{\partial \theta}\Big|_{(\theta,\mu)=(\hat\theta,\hat\mu)} = 0 \\ \frac{\partial \log L(\theta,\mu)}{\partial \mu}\Big|_{(\theta,\mu)=(\hat\theta,\hat\mu)} = 0 \end{cases} \Rightarrow \begin{cases} \frac{n}{\hat\theta} - \sum_{i=1}^n x_i = 0 \\ \frac{n}{\hat\mu} - \sum_{i=1}^n y_i = 0 \end{cases} \Rightarrow \begin{cases} \hat\theta = n\left(\sum_{i=1}^n x_i\right)^{-1} \\ \hat\mu = n\left(\sum_{i=1}^n y_i\right)^{-1} \end{cases}$$

$\therefore \sup\{L(\theta,\mu) : (\theta,\mu) \in \Theta\}$

$$= \left[n\left(\sum_{i=1}^n x_i\right)^{-1}\right]^n \exp\left\{-n\left(\sum_{i=1}^n x_i\right)^{-1}\left(\sum_{i=1}^n x_i\right)\right\} \left[n\left(\sum_{i=1}^n y_i\right)^{-1}\right]^n \exp\left\{-n\left(\sum_{i=1}^n y_i\right)^{-1}\left(\sum_{i=1}^n y_i\right)\right\}$$

$$= n^{2n}\left(\sum_{i=1}^n x_i\right)^{-n}\left(\sum_{i=1}^n y_i\right)^{-n} \exp\{-2n\}$$

$$\therefore \lambda(\underset{\sim}{x},\underset{\sim}{y}) = \frac{\left[(2n)\left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right)^{-1}\right]^{2n}\exp\{-2n\}}{n^{2n}\left(\sum_{i=1}^n x_i\right)^{-n}\left(\sum_{i=1}^n y_i\right)^{-n}\exp\{-2n\}}$$

$$= \frac{2^{2n}\left(\sum_{i=1}^n x_i\right)^n\left(\sum_{i=1}^n y_i\right)^n}{\left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right)^{2n}}$$

$\therefore$ the critical region of the likelihood ratio test is:

$$C_1 = \left\{(\underset{\sim}{x},\underset{\sim}{y}) : \frac{2^{2n}\left(\sum_{i=1}^n x_i\right)^n\left(\sum_{i=1}^n y_i\right)^n}{\left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right)^{2n}} \le k\right\}$$

(b)

$$\frac{2^{2n}\left(\sum_{i=1}^n x_i\right)^n\left(\sum_{i=1}^n y_i\right)^n}{\left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right)^{2n}} \le k$$

$$\Leftrightarrow \left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i + \sum_{i=1}^n y_i}\right)^n \left(\frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n x_i + \sum_{i=1}^n y_i}\right)^n \le k'$$

$$\Leftrightarrow \left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i + \sum_{i=1}^n y_i}\right)\left(1 - \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i + \sum_{i=1}^n y_i}\right) \le k''$$

$\Leftrightarrow \quad T \le K_1$ or $T \ge K_2$

where $T = \dfrac{\sum_{i=1}^n X_i}{\sum_{i=1}^n X_i + \sum_{i=1}^n Y_i}$

$\therefore$ the critical region in part (a) could be expressed as

$C_2 = \{(\underset{\sim}{x},\underset{\sim}{y}) : t \le K_1 \text{ or } t \ge K_2\}$

(c) When H_0 is true, then $\theta = \mu = \lambda$ where $\lambda > 0$.

Thus, $2\lambda\sum_{i=1}^n X_i \sim \chi^2(2n)$ since

$$m_{2\lambda\sum_{i=1}^n X_i}(s) = E\left(e^{(s)2\lambda\sum_{i=1}^n X_i}\right) = \prod_{i=1}^n E\left(e^{(2\lambda s)X_i}\right) = \left(\frac{\lambda}{\lambda - 2\lambda s}\right)^n = (1-2s)^{-n} = (1-2s)^{-\frac{2n}{2}}$$

which is the moment generating function of $\chi^2(2n)$

Similarly, $2\lambda\sum_{i=1}^n Y_i \sim \chi^2(2n)$

$$\therefore \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i} = \frac{2\lambda\sum_{i=1}^n Y_i/2n}{2\lambda\sum_{i=1}^n X_i/2n} \sim F_{(2n,2n)}$$

6 (c) (cont.)

$\therefore$ $T = \dfrac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i + \sum_{i=1}^{n} Y_i} = \dfrac{1}{1 + \sum_{i=1}^{n} Y_i / \sum_{i=1}^{n} X_i}$ have a distribution

same as $(1+F)^{-1}$ under H_0 where $F \sim F(2n, 2n)$.