The Hong Kong University of Science & Technology

MATH3423 - Statistical Inference
Midterm Examination - Fall 2014/2015

Answer ALL Questions **Date: 23 October 2014**

Full marks: 70 **Time Allowed: 80 minutes**

- DO NOT open the exam paper until instructed to do so.
- It is a closed-book examination.
- Three questions are included in this paper.
- Give detailed explanation how to obtain the final answer. NO mark will be given if only the final answer is written down.
- Cheating is a serious offense. Students caught cheating are subject to a zero score as well as additional penalties.

Name : ______________________

Student Number : ______________

Signature : ____________________

For marking use only:

Question No.	Marks	Out of
1		8
2		14
3(a)		10
3(b)		26
3(c)		12

1. **(8 marks)** Let X_1, X_2 be random variables having the bivariate normal distribution with parameters μ_1, μ_2, σ_1, σ_2, ρ (correlation coefficient between X_1 and X_2), i.e.,

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N_2\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right).$$

Set

$$Y_1 = \frac{X_1 - \mu_1}{\sigma_1} + \frac{X_2 - \mu_2}{\sigma_2}, \; Y_2 = \frac{X_1 - \mu_1}{\sigma_1} - \frac{X_2 - \mu_2}{\sigma_2}.$$

Find the probability density functions of Y_1 and Y_2. Are they independent?

2. If $X_1, X_2, \ldots, X_n$ are independently and normally distributed with the same mean μ but different variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$. Let $U = \sum_{i=1}^n (X_i/\sigma_i^2) / \sum_{j=1}^n (1/\sigma_j^2)$ and $V = \sum_{i=1}^n (X_i - U)^2/\sigma_i^2$. Note that U and V are independently distributed.

   (a) **(4 marks)** Find the distribution of U.

   (b) **(10 marks)** Find the distribution of V .

3. Let X_1, ...,X_n be a random sample from a location distribution family

$$f(x;\theta) = \frac{1}{\theta}\exp\left(-\frac{x-\delta}{\theta}\right) I(x \geq \delta) .$$

Note that $Y_i = X_i - \delta \sim \exp\left(\frac{1}{\theta}\right)$.

   (a) Assume that δ is equal to zero.

      i. **(2 marks)** Prove that the moment generating function of X_i is equal to $1/(1-\theta t)$.

      ii. **(4 marks)** Find the distribution of $\sum_{i=1}^n X_i$.

      iii. **(4 marks)** Find the distribution of $2\sum_{i=1}^n X_i/\theta$.

   (b) Assume that δ is known.

      i. **(4 marks)** Find the method of moments estimator, $\tilde{\theta}$, for θ. Is it unbiased?

      ii. **(4 marks)** Find the maximum likelihood estimator, $\hat{\theta}$, for θ. Is it unbiased?

      iii. **(2 marks)** Find the mean squared error of $\hat{\theta}$.

      iv. **(4 marks)** Let $\tau(\theta) = Pr(X_1 > 1 + \delta)$, find its maximum likelihood estimator, $\widehat{\tau(\theta)}$.

      v. **(6 marks)** Find Cramer-Rao lower bound for the variance of unbiased estimators of $\tau(\theta)$.

      vi. **(6 marks)** Find the limiting distribution of $\widehat{\tau(\theta)}$ by Delta method. What phenomenon do you observe?

   (c) Assume that θ is known.

      i. **(12 marks)** Find the maximum likelihood estimator, $\hat{\delta}$, for δ. Is it unbiased? Hence or otherwise, find the unbiased estimator for δ.