78

Final 1/02

1\. (a) $L(\mu_1, \mu_2, \sigma^2) = f_{\underset{\sim}{X}, \underset{\sim}{Y}}(\underset{\sim}{x}, \underset{\sim}{y}) = \prod_{i=1}^{m} f_{X_i}(x_i) \prod_{j=1}^{n} f_{Y_j}(y_j)$

$= (2\pi\sigma^2)^{-\frac{m}{2}} \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{m}(x_i-\mu_1)^2\} (2\pi\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_j-\mu_2)^2\}$

$\log L(\mu_1, \mu_2, \sigma^2) = -\frac{m}{2}\log(2\pi\sigma^2) - \frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{m}(x_i-\mu_1)^2 - \frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_j-\mu_2)^2$

$\frac{\partial}{\partial\mu_1}\log L = -\frac{1}{\sigma^2}\sum_{i=1}^{m}(x_i-\mu_1)(-1)$

$\frac{\partial}{\partial\mu_2}\log L = -\frac{1}{\sigma^2}\sum_{j=1}^{n}(y_j-\mu_2)(-1)$

$\frac{\partial}{\partial\sigma^2}\log L = -\frac{m}{2\sigma^2} - \frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{m}(x_i-\mu_1)^2 + \frac{1}{2\sigma^4}\sum_{j=1}^{n}(y_j-\mu_2)^2$

$\frac{\partial}{\partial\mu_1}\log L(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}^2) = 0, \quad \frac{\partial}{\partial\mu_2}\log L(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}^2) = 0, \quad \frac{\partial}{\partial\sigma^2}\log L(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}^2) = 0$

$$\Rightarrow \begin{cases} \frac{1}{\hat{\sigma}^2}\sum_{i=1}^{m}(x_i-\hat{\mu}_1) = 0 \\ \frac{1}{\hat{\sigma}^2}\sum_{j=1}^{n}(y_j-\hat{\mu}_2) = 0 \\ -\frac{m}{2\hat{\sigma}^2} - \frac{n}{2\hat{\sigma}^2} + \frac{1}{2\hat{\sigma}^4}\sum_{i=1}^{m}(x_i-\hat{\mu}_1)^2 + \frac{1}{2\hat{\sigma}^4}\sum_{j=1}^{n}(y_j-\hat{\mu}_2)^2 = 0 \end{cases}$$

$\Rightarrow \hat{\mu}_1 = \frac{1}{m}\sum_{i=1}^{m}X_i = \overline{X}, \quad \hat{\mu}_2 = \frac{1}{n}\sum_{j=1}^{n}Y_j = \overline{Y}, \quad \hat{\sigma}^2 = \frac{1}{m+n}\left[\sum_{i=1}^{m}(X_i-\overline{X})^2 + \sum_{j=1}^{n}(Y_j-\overline{Y})^2\right]$

(b) $(n-1)S_{n-1}^2/\sigma^2 \sim \chi^2_{(n-1)}$

$E[(S_{n-1}^2 - \sigma^2)^2] = Var(S_{n-1}^2 - \sigma^2) + [E(S_{n-1}^2 - \sigma^2)]^2 = Var(S_{n-1}^2)$

$= \frac{\sigma^4}{(n-1)^2}Var(\chi^2_{(n-1)}) = \frac{2\sigma^4}{n-1}$

$E[(\tilde{\sigma}^2 - \sigma^2)^2] = Var(\tilde{\sigma}^2 - \sigma^2) + [E(\tilde{\sigma}^2 - \sigma^2)]^2 = Var(\tilde{\sigma}^2) + (\frac{n-1}{n+1} - 1)^2\sigma^4$

$= \frac{\sigma^4}{(n+1)^2}Var(\chi^2_{(n-1)}) + \frac{4\sigma^4}{(n+1)^2} = \frac{\sigma^4}{(n+1)^2}(2(n-1)+4) = \frac{2\sigma^4}{n+1}$

2. $f_{X_i}(x_i) = \frac{1}{\theta} I_{(0,\theta)}(x_i)$

$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} \frac{1}{\theta} I_{(0,\theta)}(x_i) = \frac{1}{\theta^n} \prod_{i=1}^{n} I_{(0,\theta)}(x_i) = \frac{1}{\theta^n} I_{(0,\max\{x_i\})}(\min\{x_i\}) I_{(\min\{x_i\},\theta)}(\max\{x_i\})$

$\therefore$ $\max\{X_i\} = X_{(n)}$ is the sufficient statistic for θ.

$P(X_{(n)} \le x) = P(\max\{X_i\} \le x) = P(X_1 \le x, \dots, X_n \le x) = \prod_{i=1}^{n} P(X_i \le x) = \left(\frac{x}{\theta}\right)^n$ for $x \in (0,\theta)$

$\therefore$ $f_{X_{(n)}}(x) = \left(\frac{1}{\theta^n}\right) n x^{n-1}$ for $x \in (0,\theta)$

$E(X) = \frac{\theta}{2}$, $Var(X) = \frac{\theta^2}{12}$

$E(g_1(X_{(n)})) = \frac{\theta}{2}$

$\Rightarrow \int_0^\theta g_1(x) \left(\frac{1}{\theta^n}\right) n x^{n-1} dx = \frac{\theta}{2}$

$\Rightarrow \int_0^\theta 2 g_1(x) \left(\frac{1}{\theta^{n+1}}\right) n x^{n-1} dx = 1$

$\Rightarrow 2 g_1(x) n x^{n-1} = (n+1) x^{n+1-1} \Rightarrow g_1(x) = \frac{n+1}{2n} x$

$\therefore$ an unbiased estimator of the $E(X)$ is $\frac{n+1}{2n} X_{(n)}$

$E(g_2(X_{(n)})) = \frac{\theta^2}{12}$

$\Rightarrow \int_0^\theta g_2(x) \left(\frac{1}{\theta^n}\right) n x^{n-1} dx = \frac{\theta^2}{12}$

$\Rightarrow \int_0^\theta 12 g_2(x) \left(\frac{1}{\theta^{n+2}}\right) n x^{n-1} dx = 1$

$\Rightarrow 12 g_2(x) n x^{n-1} = (n+2) x^{n+2-1} \Rightarrow g_2(x) = \frac{n+2}{12n} x^2$

$\therefore$ an unbiased estimator of the $Var(X)$ is $\frac{n+2}{12n} X_{(n)}$

3. $f_X(x;\mu,\sigma^2)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\}=\exp\{-\frac{1}{2}\log(2\pi)-\frac{1}{2}\log\sigma^2-\frac{1}{2\sigma^2}(x-\mu)^2\}$

$=\exp\{-\frac{1}{2}\log(2\pi)-\frac{1}{2}\log\sigma^2-\frac{\mu^2}{2\sigma^2}+\frac{\mu}{\sigma^2}x-\frac{1}{2\sigma^2}x^2\}$

$a(\mu,\sigma^2)=-\frac{1}{2}\log(2\pi)-\frac{1}{2}\log\sigma^2-\frac{\mu^2}{2\sigma^2}$, $b(x)=0$, $c_1(\mu,\sigma^2)=\frac{\mu}{\sigma^2}$, $c_2(\mu,\sigma^2)=-\frac{1}{2\sigma^2}$,

$d_1(x)=x$, $d_2(x)=x^2$.

$\therefore$ $f_X(x;\mu,\sigma^2)$ belongs to the pdf of exponential family and $(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ is a complete minimal sufficient statistic for (μ,σ^2)

Thus, $(\sum_{i=1}^n X_i, \sum_{i=1}^n (X_i-\bar{X})^2)$ is also a complete minimal sufficient statistic for (μ,σ^2). Also, $\sum_{i=1}^n X_i$ and $\sum_{i=1}^n (X_i-\bar{X})^2$ are independent.

$\bar{X}\sim N(\mu,\frac{\sigma^2}{n})$ and $\frac{1}{\sigma^2}\sum_{i=1}^n (X_i-\bar{X})^2\sim\chi^2_{(n-1)}$

Let $S=\sum_{i=1}^n (X_i-\bar{X})^2$ and $Y=\frac{1}{\sigma^2}S$

$\therefore f_Y(y)=y^{\frac{n-1}{2}-1}e^{-y/2}/[2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})]$

$\therefore E(1/\sqrt{S})=E(\frac{1}{\sigma\sqrt{Y}})=\int_0^\infty \frac{1}{\sigma\sqrt{y}}\,y^{\frac{n-1}{2}-1}e^{-y/2}/[2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})]\,dy$

$=\frac{1}{\sigma}\int_0^\infty \frac{y^{\frac{n-2}{2}-1}e^{-y/2}}{2^{\frac{n-1}{2}-1}\Gamma(\frac{n-1}{2})}dy=\frac{2^{-\frac{1}{2}}\Gamma(\frac{n-2}{2})}{\sigma\,\Gamma(\frac{n-1}{2})}\int_0^\infty \frac{y^{\frac{n-2}{2}-1}e^{-y/2}}{2^{\frac{n-2}{2}}\Gamma(\frac{n-2}{2})}dy$

$=\frac{1}{\sigma}\frac{\Gamma(\frac{n-2}{2})}{\sqrt{2}\,\Gamma(\frac{n-1}{2})}$

$\therefore E\left[\left(\frac{\bar{X}}{\sqrt{S}}\right)\left(\frac{\sqrt{2}\,\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\right)\right]=E(\bar{X})E\left[\frac{\sqrt{2}\,\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\left(\frac{1}{\sqrt{S}}\right)\right]=\frac{\mu}{\sigma}$

$\therefore \frac{\sqrt{2}\,\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\left(\frac{\bar{X}}{\sqrt{S}}\right)$ is the UMVUE of $\frac{\mu}{\sigma}$.

4. (i) $f_X(x) = \theta^x(1-\theta)^{1-x} \qquad x \in \{0,1\}$

$\log f_X(x) = x\log\theta + (1-x)\log(1-\theta)$

$\frac{\partial}{\partial\theta}\log f_X(x) = \frac{x}{\theta} - \frac{1-x}{1-\theta}$

$\frac{\partial^2}{\partial\theta^2}\log f_X(x) = -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}$

$E\left[\frac{\partial^2}{\partial\theta^2}\log f_X(X)\right] = E\left[-\frac{1}{\theta^2}X - \frac{1}{(1-\theta)^2}(1-X)\right] = -\frac{1}{\theta} - \frac{1}{1-\theta} = -\frac{1}{\theta(1-\theta)}$

$\frac{d}{d\theta}[\theta(1-\theta)] = 1-2\theta$

$\therefore$ the Cramer-Rao lower bound for the variance of unbiased estimators of $\theta(1-\theta)$ is $\dfrac{\left\{\frac{d}{d\theta}[\theta(1-\theta)]\right\}^2}{-nE\left[\frac{\partial^2}{\partial\theta^2}\log f_X(X)\right]^2} = \dfrac{(1-2\theta)^2}{\frac{n}{\theta(1-\theta)}} = \frac{1}{n}\theta(1-\theta)(1-2\theta)^2$

(ii) $f_X(x) = \exp\{x\log\theta + (1-x)\log(1-\theta)\} = \exp\{\log(1-\theta) + x\log(\frac{\theta}{1-\theta})\}$

$a(\theta) = \log(1-\theta), \quad b(x) = 0, \quad c(\theta) = \log\frac{\theta}{1-\theta}, \quad d(x) = x$

$A = (0,1), \quad D = \{0,1\}$

$\therefore f_X(x)$ belongs to exponential family

and $\sum_{i=1}^{n} X_i$ is complete and sufficient statistic for θ.

Guess $\bar{X}(1-\bar{X})$ where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$E[\bar{X}(1-\bar{X})] = E(\bar{X}) - E(\bar{X}^2) = \theta - [Var(\bar{X}) + [E(\bar{X})]^2]$

$= \theta - \frac{\theta(1-\theta)}{n} - \theta^2 = (\frac{n-1}{n})\theta(1-\theta)$

$\therefore$ the UMVUE for $\theta(1-\theta)$ is $\frac{n}{n-1}\bar{X}(1-\bar{X})$.

5. (i) $\begin{cases} H_0: \theta_1 = \theta_2 = \theta \\ H_1: \theta_1 \neq \theta_2 \end{cases}$

$$\lambda(\underset{\sim}{x}, \underset{\sim}{y}) = \frac{\sup\{L(\theta_1, \theta_2): \theta_1 = \theta_2 = \theta \in (0,\infty)\}}{\sup\{L(\theta_1, \theta_2): \theta_1 \in (0,\infty),\ \theta_2 \in (0,\infty)\}}$$

$$L(\theta_1, \theta_2) = f_{\underset{\sim}{X}, \underset{\sim}{Y}}(\underset{\sim}{x}, \underset{\sim}{y}) = \prod_{i=1}^{m} f_{X_i}(x_i) \prod_{j=1}^{n} f_{Y_j}(y_j)$$

$$= \prod_{i=1}^{m} \frac{1}{\theta_1} \exp\{-\frac{x_i}{\theta_1}\} \prod_{j=1}^{n} \frac{1}{\theta_2} \exp\{-\frac{y_j}{\theta_2}\}$$

$$= \left(\frac{1}{\theta_1^m}\right)\left(\frac{1}{\theta_2^n}\right) \exp\{-\frac{1}{\theta_1}\sum_{i=1}^{m} x_i - \frac{1}{\theta_2}\sum_{j=1}^{n} y_j\}$$

Numerator: $L(\theta, \theta) = \left(\frac{1}{\theta}\right)^{m+n} \exp\{-\frac{1}{\theta}(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j)\}$

$$\log L(\theta, \theta) = -(m+n)\log\theta - \frac{1}{\theta}(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j)$$

$$\frac{\partial}{\partial\theta}\log L(\theta,\theta) = -\frac{m+n}{\theta} + \frac{1}{\theta^2}(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j)$$

$$\frac{\partial}{\partial\theta}\log L(\hat\theta, \hat\theta) = 0 \Rightarrow \hat\theta = \frac{1}{m+n}(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j)$$

Denominator: $\log L(\theta_1, \theta_2) = -m\log\theta_1 - n\log\theta_2 - \frac{1}{\theta_1}\sum_{i=1}^{m} x_i - \frac{1}{\theta_2}\sum_{j=1}^{n} y_j$

$$\frac{\partial}{\partial\theta_1}\log L(\theta_1, \theta_2) = -\frac{m}{\theta_1} + \frac{1}{\theta_1^2}\sum_{i=1}^{m} x_i$$

$$\frac{\partial}{\partial\theta_2}\log L(\theta_1, \theta_2) = -\frac{n}{\theta_2} + \frac{1}{\theta_2^2}\sum_{j=1}^{n} y_j$$

$\frac{\partial}{\partial\theta_1}\log L(\hat\theta_1, \hat\theta_2) = 0$ and $\frac{\partial}{\partial\theta_2}\log L(\hat\theta_1, \hat\theta_2) = 0$

$\Rightarrow \hat\theta_1 = \frac{1}{m}\sum_{i=1}^{m} x_i$ and $\hat\theta_2 = \frac{1}{n}\sum_{j=1}^{n} y_j$

$$\lambda(\underset{\sim}{x}, \underset{\sim}{y}) = \frac{\left[\frac{1}{m+n}(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j)\right]^{-m-n} \exp\{-[\frac{1}{m+n}(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j)]^{-1}(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j)\}}{\left[\frac{1}{m}\sum_{i=1}^{m} x_i\right]^{-m}\left[\frac{1}{n}\sum_{j=1}^{n} y_j\right]^{-n} \exp\{-[\frac{1}{m}\sum_{i=1}^{m} x_i]^{-1}\sum_{i=1}^{m} x_i - [\frac{1}{n}\sum_{j=1}^{n} y_j]^{-1}\sum_{j=1}^{n} y_j\}}$$

$$= \frac{(m+n)^{m+n}(\sum_{i=1}^{m} x_i)^m(\sum_{j=1}^{n} y_j)^n}{m^m n^n(\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j)^{m+n}}$$

the likelihood ratio test rejects H_0 if $(\underset{\sim}{X}, \underset{\sim}{Y}) \in C_1$ where

$C_1 = \{(\underset{\sim}{x}, \underset{\sim}{y}): \lambda(\underset{\sim}{x}, \underset{\sim}{y}) \le k\}$

(ii) $\lambda(\underset{\sim}{x}, \underset{\sim}{y}) \le k$

$\Leftrightarrow (\sum_{i=1}^{m} x_i)^m (\sum_{j=1}^{n} y_j)^n / (\sum_{i=1}^{m} x_i + \sum_{j=1}^{n} y_j)^{m+n} \le k_1$

$\Leftrightarrow [\sum_{i=1}^{m} x_i / \sum_{j=1}^{n} y_j]^m / [(\sum_{i=1}^{m} x_i / \sum_{j=1}^{n} y_j) + 1]^{m+n} \le k_1$

$\Leftrightarrow \sum_{i=1}^{m} x_i / \sum_{j=1}^{n} y_j \le k_2$ or $\sum_{i=1}^{m} x_i / \sum_{j=1}^{n} y_j \ge k_3$

Under H_0, $\frac{2}{\theta}\sum_{i=1}^{m} X_i \sim \chi^2_{(2m)}$ and $\frac{2}{\theta}\sum_{j=1}^{n} Y_j \sim \chi^2_{(2n)}$, $\therefore \sum_{i=1}^{m} X_i / \sum_{j=1}^{n} Y_j \sim F_{(2m, 2n)}$

$\therefore C_1 = \{(\underset{\sim}{x}, \underset{\sim}{y}): \sum_{i=1}^{m} x_i / \sum_{j=1}^{n} y_j \le F_{(2m,2n)}(1-\frac{\alpha}{2})$ or $\sum_{i=1}^{m} x_i / \sum_{j=1}^{n} y_j \ge F_{(2m,2n)}(\frac{\alpha}{2})\}$

power of the test $= P((\underset{\sim}{X}, \underset{\sim}{Y}) \in C_1 \mid H_1)$ $\quad(\theta_2 \sum_{i=1}^{m} X_i / \theta_1 \sum_{j=1}^{n} Y_j \overset{\text{under } H_1}{\sim} F_{(2m,2n)})$

$= P(F_{(2m,2n)} \le \frac{\theta_2}{\theta_1} F_{(2m,2n)}(1-\frac{\alpha}{2})) + P(F_{(2m,2n)} \ge \frac{\theta_2}{\theta_1} F_{(2m,2n)}(\frac{\alpha}{2}))$

6. $\begin{cases} H_0: \text{Grades} \sim N(75, 9^2) \\ H_1: \text{otherwise} \end{cases}$

$n = 3 + 12 + 10 + 4 + 1 = 30$

$(a, b]$	$(-\infty, 49]$	$(49, 59]$	$(59, 74]$	$(74, 89]$	$(89, \infty)$
$P(a < X \le b)$	≈ 0	≈ 0	0.3707	0.6293	≈ 0
expected frequency	0	0	11.121	18.879	0

To obtain expected frequency ≥ 5, we adopt the partition:

$(-\infty, 74]$, $(74, \infty)$

Now, $\sum_{i=1}^{2} \frac{(n_i - n\theta_{0i})^2}{n\theta_{0i}} = \frac{(15 - 11.121)^2}{11.121} + \frac{(15 - 18.879)^2}{18.879} = 2.150 < \chi^2_{(1, 0.05)} = 3.841$

$\therefore$ H_0 cannot be rejected at $\alpha = 0.05$.

6. $\begin{cases} H_0: \text{Grades} \sim N(75, 9^2) \\ H_1: \text{otherwise} \end{cases}$

$n = 3 + 12 + 10 + 4 + 1 = 30$

$(a, b]$	$(-\infty, 49]$	$(49, 59]$	$(59, 74]$	$(74, 89]$	$(89, \infty)$
$P(a < X \le b) = P\left(\frac{a-75}{9} < Z \le \frac{b-75}{9}\right)$	0.0019	0.0356	0.4187	0.4844	0.0594
expected frequency	0.057	1.068	12.561	14.532	1.782

To obtain expected frequency ≥ 5, we adopt the partition:
$(-\infty, 74]$ and $(74, \infty)$

Now, $G = \frac{(15-13.686)^2}{13.686} + \frac{(15-16.314)^2}{16.314} = 0.23 < \chi^2_{(1, 0.05)} = 3.841$

$\therefore H_0$ is not rejected at $\alpha = 0.05$.

7. (i) $\begin{cases} H_0: \theta = \theta_0 \\ H_1: \theta < \theta_0 \end{cases}$

$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} = \exp\{-\log\theta - \frac{x}{\theta}\}$

$a(\theta) = -\log\theta$, $b(x) = 0$, $c(\theta) = -\frac{1}{\theta}$, $d(x) = x$

$A = (0, \infty)$, $D = (0, \infty)$

$\therefore$ $f(x;\theta)$ belongs to the p.d.f. of exponential family.

$\because$ $c(\theta) = -\frac{1}{\theta}$ is increasing

$\therefore$ the UMP test rejects H_0 when $\sum_{i=1}^{n} d(X_i) \le k$ (i.e. $\sum_{i=1}^{n} X_i \le k$)

$\because$ $\frac{2}{\theta}\sum_{i=1}^{n} X_i \sim \chi^2_{(2n)}$

$\alpha = P(\sum_{i=1}^{n} X_i \le k \mid \theta = \theta_0) = P(\chi^2_{(2n)} \le \frac{2}{\theta_0} k)$

$\therefore$ $\frac{2k}{\theta_0} = \chi^2_{(2n)}(1-\alpha) \Rightarrow k = \frac{\theta_0}{2}\chi^2_{(2n)}(1-\alpha)$

$\therefore$ at level of significance α, the UMP test for testing the hypothesis $H_0: \theta = \theta_0$ v.s. $H_1: \theta < \theta_0$ rejects H_0 when

$\sum_{i=1}^{n} X_i \le \frac{\theta_0}{2}\chi^2_{(2n)}(1-\alpha)$

Now, $\sup_{\theta: \theta \ge \theta_0} P(\sum_{i=1}^{n} X_i \le \frac{\theta_0}{2}\chi^2_{(2n)}(1-\alpha) \mid \theta) = \sup_{\theta: \theta \ge \theta_0} P(\chi^2_{(2n)} \le \frac{\theta_0}{\theta}\chi^2_{(2n)}(1-\alpha))$

$= P(\chi^2_{(2n)} \le \frac{\theta_0}{\theta_0}\chi^2_{(2n)}(1-\alpha)) = P(\chi^2_{(2n)} \le \chi^2_{(2n)}(1-\alpha)) = \alpha$

$\therefore$ the critical region $\{\underline{x} : \sum_{i=1}^{n} x_i \le \frac{\theta_0}{2}\chi^2_{(2n)}(1-\alpha)\}$ is also the critical region for the UMP test for testing the hypothesis $H_0: \theta \ge \theta_0$ v.s. $H_1: \theta < \theta_0$ at level of significance α.

(ii) $\begin{cases} Q(\theta_0) = Q(1000) = 0.05 \\ Q(\theta_1) = Q(250) \ge 0.95 \end{cases}$

where $Q(\theta) = P(\sum_{i=1}^{n} X_i \le \frac{1000}{2}\chi^2_{(2n)}(1-0.05) \mid \theta)$

$\therefore$ $Q(\theta_1) \ge 0.95 \Rightarrow P(\sum_{i=1}^{n} X_i \le 500\chi^2_{(2n)}(1-0.05) \mid \theta = 500) \ge 0.95$

$\Rightarrow P(\chi^2_{(2n)} \le 4\chi^2_{(2n)}(0.95)) \ge 0.95$

$\Rightarrow 4\chi^2_{(2n)}(0.95) \ge \chi^2_{(2n)}(1-0.95) = \chi^2_{(2n)}(0.05)$

$\Rightarrow \chi^2_{(2n)}(0.95) / \chi^2_{(2n)}(0.05) \ge \frac{1}{4} = 0.25$

$\Rightarrow$ $2n = 13$

$\Rightarrow$ $n = 6.5$

$\therefore$ take $n = 7$.

8. $X \sim Bin(n, \theta)$, $\theta \in (0,1)$, $x = \{0, 1, 2, \dots, n\}$

(i) $f_X(x;\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} = \exp\{\log\binom{n}{x} + x\log\theta + (n-x)\log(1-\theta)\}$
$= \exp\{ n\log(1-\theta) + \log\binom{n}{x} + x\log(\frac{\theta}{1-\theta})\}$.

$a(\theta) = n\log(1-\theta)$, $b(x) = \log\binom{n}{x}$, $c(\theta) = \log\frac{\theta}{1-\theta}$, $d(x) = x$

$\therefore f_X(x;\theta)$ belongs to the p.d.f. of exponential family.

$\because$ $c(\theta) = \log(\frac{\theta}{1-\theta})$ is increasing,

$\therefore$ the UMP test for testing the hypothesis $H_0: \theta = \theta_0$ v.s. $H_1: \theta > \theta_0$ rejects H_0 when $X \geq k$.

$\alpha \geq P(X \geq k \mid \theta = \theta_0) = P(Bin(n, \theta_0) \geq k) \Rightarrow k = k(\alpha, n, \theta_0)$

$\therefore$ the UMP test for testing the hypothesis $H_0: \theta = \theta_0$ v.s. $H_1: \theta > \theta_0$ rejects H_0 when $X \geq k(\alpha, n, \theta_0)$ at level of significance α.

Now, $\sup_{\theta:\theta\leq\theta_0} P(X \geq k(\alpha, n, \theta_0) \mid \theta) = \sup_{\theta:\theta\leq\theta_0} P(Bin(n,\theta) \geq k(\alpha, n, \theta_0))$
$= P(Bin(n, \theta_0) \geq k(\alpha, n, \theta_0)) \leq \alpha$

$\therefore$ the critical region $\{X : X \geq k(\alpha, n, \theta_0)\}$ is also the critical region for the UMP test for testing the hypothesis $H_0: \theta \leq \theta_0$ v.s. $H_1: \theta > \theta_0$ at level of significance α.

(ii) For $n = 10$, $\theta_0 = 0.25$, $\alpha = 0.05$,

$0.05 \geq P(Bin(10, 0.25) \geq k(0.05, 10, 0.25))$

$\Rightarrow \sum_{l=k(0.05,10,0.25)}^{10} P(X = l) \leq 0.05$

$\Rightarrow \sum_{l=k(0.05,10,0.25)}^{10} \binom{10}{l} (0.25)^l (1-0.25)^{10-l} \leq 0.05$

$\Rightarrow \sum_{l=k(0.05,10,0.25)}^{10} \binom{10}{l} (0.25)^{10} (3)^{10-l} \leq 0.05 \Rightarrow k(0.05, 10, 0.25) = 6$

$\therefore$ the test in (i) becomes rejecting H_0 when $X \geq 6$.

(iii) $Q(\theta_1) = P(X \geq 6 \mid \theta = \theta_1)$

θ_1	0.375	0.5	0.625	0.75	0.875
$Q(\theta_1)$	0.1275	0.3770	0.6943	0.9219	0.9955

(iv) $\begin{cases} Q(\theta_0) = Q(0.125) = 0.1 \\ Q(\theta_1) = Q(0.25) \geq 0.9 \end{cases}$

where $Q(\theta) = P(X \geq k(0.1, n, 0.125) \mid \theta)$

$\approx P\left(Z \geq \frac{k(0.1, n, 0.125) - 0.5 - n\theta}{\sqrt{n\theta(1-\theta)}}\right)$ where $Z \sim N(0,1)$

(iv) $\therefore \begin{cases} \dfrac{k(0.1, n, 0.125) - 0.5 - n(0.125)}{\sqrt{n(0.125)(0.875)}} = 1.28 \\ \dfrac{k(0.1, n, 0.125) - 0.5 - n(0.25)}{\sqrt{n(0.25)(0.75)}} \leq -1.28 \end{cases}$

$\Rightarrow \begin{cases} k(0.1, n, 0.125) - 0.5 = 0.4233\sqrt{n} + 0.125n \\ k(0.1, n, 0.125) - 0.5 \leq -0.5543\sqrt{n} + 0.25n \end{cases}$

$\Rightarrow \quad 0.4233\sqrt{n} + 0.125n \leq -0.5543\sqrt{n} + 0.25n$

$\Rightarrow 0.125n \geq 0.9776\sqrt{n}$

$\Rightarrow \quad \sqrt{n} \geq 7.8208 \Rightarrow n \geq 61.2$

$\therefore$ take $n = 62$.