Tutorial for 2020-10-20 and 2020-10-23

1. How can we evaluate an estimator.

- Mean Square Error (MSE) —— Closeness

(i) Definition: $MSE = E[\hat{\theta}(X) - \theta]^2$

(ii) the smaller the MSE is. the better the estimator is.

$$E(\hat{\theta}_1(X) - \theta)^2 \leq E(\hat{\theta}_2(X) - \theta)^2$$

$\Longrightarrow \hat{\theta}_1(X)$ is better than $\hat{\theta}_1(X)$

(iii) Unbiasedness

$E(\hat{\theta}) = \theta$ for all $\theta \in \Theta$ $\Rightarrow$ $\hat{\theta}$ is unbiased for θ

otherwise it is biased.

$\Downarrow$

$$E(\hat{\theta}(X) - \theta)^2 = Var(\hat{\theta}(X)) + \underline{bias^2} \quad ①$$

$\to$ $bias = E(\hat{\theta}(X)) - \theta$

IF $\hat{\theta}(X)$ is unbiased =

$bias = E(\hat{\theta}(X)) - \theta = 0$

$① = Var(\hat{\theta}(X)) = MSE$ $\to$ It means the MSE is smallest under the condition that $\hat{\theta}$ is unbiased.

$\Downarrow$

Uniform minimum Variance unbiased estimator (UMVUE)

2. Some Concepts.

- **Asymptotic Unbiased.**

Definition (Asymptotic Unbiasedness): A sequence of estimator, $\{\hat{\theta}_n : n = 1,2,\ldots,\}$, based on a r.s. of size n is said to be ***asymptotically unbiased*** if $\lim_{n\to\infty} E(\hat{\theta}_n) = \theta$, for all $\theta \in \Theta$.

$$\begin{cases} \lim_{n\to\infty} E(\hat{\theta}_n) - \theta = 0 & \text{Asymptotic Unbiased} \\ E(\hat{\theta}) - \theta = 0 & \text{(unbiased)} \end{cases}$$

- **Consistency.**

Definition (Consistency): A sequence of estimator, $\{\hat{\theta}_n : n = 1,2,\ldots,\}$, based on a r.s. of size n is said to be ***consistency*** if, for any $\epsilon > 0$, $\lim_{n\to\infty} P(|\hat{\theta}_n - \theta| \le \epsilon) = 1$, for all $\theta \in \Theta$.

Consistency $\rightleftarrows$ asymptotic Unbiased ? ? ?

$$\begin{cases} ① \text{ asymptotic unbiased} \\ ② \ Var(\hat{\theta}_n) \to 0 \text{ as } n \to \infty \end{cases} \Rightarrow \text{Consistency}$$

- **Asymptotic Normality**

Definition (Asymptotic Normality): A sequence of estimator, $\{\hat{\theta}_n : n = 1,2,\ldots,\}$, based on a r.s. of size n is said to be ***asymptotically normal*** if $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N(0, \sigma_\theta^2)$.

$\begin{cases} \text{MME} \\ \text{MLE} \end{cases}$ are consistent, asymptotically unbiased and asymptotically normal.

3. UMVUE

- U: Uniform
- MV: minimum Variance
- U: unbiased
- E: estimator

- Lemma 1: UMVUE is unique.

- How to find UMVUE ???
  1. Cramér - Rao inequity (The C-R inequality)
  2. Complete and Sufficient Statistic

① C-R inequality

- Fisher Information.

Definition 1. *Define the Fisher information of random variables* $\{X_1, \ldots, X_n\}$ *by*

$$I_{X_1,\ldots,X_n}(\theta) = E_{X_1,\ldots,X_n}[\frac{\partial}{\partial \theta} \ln L(\theta)]^2, \tag{1}$$

where $E_{X_1,\ldots,X_n}$ *is the usual expectation with respect to a joint pdf or joint pmf of* X_1, X_2, $\ldots$ *and* X_n, $L(\theta) = f_{X_1,\ldots,X_n}(x_1,\ldots,x_n|\theta)$ *for continuous cases, and* $L(\theta) = P(X_1 = x_1, \ldots, X_n = x_n|\theta)$ *for discrete cases.*

$$I_X(\theta) = E_X[\frac{\partial}{\partial \theta} \ln L(\theta)]^2$$
$$= E_X[l'(\theta)]^2$$

Lemma 2. *Suppose that* X *is a random variable from a density* $f_X(\cdot|\theta)$. *Under the regularity conditions,*

$$E_X[\frac{\partial}{\partial \theta} \ln f_X(X|\theta)] = 0, \quad \textit{for all } \theta \in \Theta.$$

Corollary 1. *By Lemma 2, we have* $I_X(\theta) = Var\left(\frac{\partial}{\partial \theta} \ln f_X(X|\theta)\right)$.

Lemma 3. *Under the regularity conditions,*

$$E_X[\frac{\partial}{\partial\theta}\ln f_X(X|\theta)]^2 = -E_X[\frac{\partial^2}{\partial\theta^2}\ln f_X(X|\theta)]. \tag{2}$$

Lemma 4. *If X and Y are independent and have densities $f_X(\cdot)$ and $f_Y(\cdot)$ satisfying the regularity conditions, respectively, then*

$$I_{X,Y}(\theta) = I_X(\theta) + I_Y(\theta).$$

For a rs $\{X_1, \dots, X_n\}$ of size n from a distribution with a density function $f(\cdot\,|\theta)$, by Lemma 4, we can see that the Fisher information about θ contained in the random sample is

$$I_{X_1,\dots,X_n}(\theta) = I_{X_1}(\theta) + \cdots + I_{X_n}(\theta) = nI_{X_1}(\theta).$$

Lemma 5. Under the regularity conditions, for any statistic $T(\boldsymbol{X})$ for θ, we have

$$I_{T(\boldsymbol{X})}(\theta) \leq I_{\boldsymbol{X}}(\theta),$$

where $I_{T(\boldsymbol{X})}(\theta) = E_{X_1,\dots X_n}\left[\frac{d}{d\theta}\ln f_T(T(\boldsymbol{X})|\theta)\right]^2$ and $f_T(\cdot\,|\theta)$ is the density function of $T(\boldsymbol{X})$.

Fisher Information Calculation

1. To find Fisher Information for the PDF

$$f_\theta(x) = \frac{3\theta^3}{(x+\theta)^4} \quad x>0,\ \theta>0$$

2. Let $X_1, X_2, \ldots, X_n \sim U(0, \theta)$, to find the F.I for θ