01-02

1. $X_1, X_2, \ldots, X_m$ and $Y_1, \ldots, Y_n$ are two independent random sample with the same mean θ and known variance σ_1^2 and σ_2^2 respectn

For every $c \in [0,1]$, $U = c\bar{X} + (1-c)\bar{Y}$

$E(U) = E(c\bar{X} + (1-c)\bar{Y})$

$\quad = cE(\bar{X}) + (1-c)E(\bar{Y}) = c\theta + (1-c)\theta = \theta$

$\therefore U$ is an unbiased estimator of θ.

$Var(U) = Var(c\bar{X} + (1-c)\bar{Y})$

$\quad = c^2 Var(\bar{X}) + (1-c)^2 Var(\bar{Y})$

$\quad = c^2 \frac{\sigma_1^2}{m} + (1-c)^2 \frac{\sigma_2^2}{n}$

Let $g(c) = c^2 \frac{\sigma_1^2}{m} + (1-c)^2 \frac{\sigma_2^2}{n} = \left(\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}\right)c^2 - 2c\frac{\sigma_2^2}{n} + \frac{\sigma_2^2}{n}$

$g'(c) = 2c\frac{\sigma_1^2}{m} - 2(1-c)\frac{\sigma_2^2}{n}$

$g'(c) = 0 \Rightarrow 2c\frac{\sigma_1^2}{m} - 2(1-c)\frac{\sigma_2^2}{n} = 0$

$c\left(\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}\right) = \frac{\sigma_2^2}{n}$

$c = \frac{\sigma_2^2}{n}\left(\frac{mn}{n\sigma_1^2 + m\sigma_2^2}\right)$

Since $g(c)$ is quadratic in c and opening upward, there is only one critical point and it is the minimum point of $g(c)$.

Thus, $c = m\sigma_2^2/(n\sigma_1^2 + m\sigma_2^2)$ minimize the $Var(U)$.

2. $X_1, \dots, X_n$ iid r.v's from $U(\alpha, \beta)$.

(i) $L(\alpha,\beta) = f_{X_1,\dots,X_n}(x_1,\dots,x_n;\alpha,\beta)$

$$= \prod_{i=1}^{n} f_{X_i}(x_i;\alpha,\beta)$$

$$= \prod_{i=1}^{n} \frac{1}{\beta-\alpha} I\{\alpha < x_i < \beta\}$$

$$= \left(\frac{1}{\beta-\alpha}\right)^n \prod_{i=1}^{n} I\{\alpha < x_i < \beta\}$$

$$= \left(\frac{1}{\beta-\alpha}\right)^n I\{\alpha < x_{(1)} \le x_{(n)} < \beta\}$$

which achieves the maximum when $\hat{\alpha} = X_{(1)}$ and $\hat{\beta} = X_{(n)}$

(ii) $P(\hat{\alpha} \le y) = 1 - P(\hat{\alpha} > y) = 1 - P(\min\{X_1,\dots,X_n\} > y)$ $\quad \alpha < y < \beta$

$= 1 - \prod_{i=1}^{n} P(X_i > y) = 1 - \left[1 - \frac{y-\alpha}{\beta-\alpha}\right]^n = 1 - \left(\frac{\beta-y}{\beta-\alpha}\right)^n \Rightarrow f_{\hat{\alpha}}(y) = n\left(\frac{\beta-y}{\beta-\alpha}\right)^{n-1}\frac{1}{\beta-\alpha}$

$P(\hat{\beta} \le y) = P(\max\{X_1,\dots,X_n\} < y) = \prod_{i=1}^{n} P(X_i \le y)$

$$= \left(\frac{y-\alpha}{\beta-\alpha}\right)^n$$

$f_{\hat{\beta}}(y) = n\left(\frac{y-\alpha}{\beta-\alpha}\right)^{n-1}\frac{1}{\beta-\alpha}$

$f_{\hat{\alpha},\hat{\beta}}(y_1, y_n)$

$= \frac{n!}{0!(n-2)!0!}[P(X \le y_1)]^0[P(X \le y_n) - P(X \le y_1)]^{n-2}[1 - P(X \le y_n)]^0 f(y_1) f(y_n)$

$= n(n-1)\left(\frac{1}{\beta-\alpha}\right)^{n-2}[(y_n-\alpha) - (y_1-\alpha)]^{n-2}\frac{1}{\beta-\alpha}\frac{1}{\beta-\alpha}$

$= n(n-1)(y_n - y_1)^{n-2}/(\beta-\alpha)^n$

3. $X_1, \ldots, X_n$ are independent r.v.'s from $U(\theta - a, \theta + b)$, where $a, b > 0$ are known and $\theta \in R$.

$E(X) = \frac{1}{2}(\theta - a + \theta + b) = \frac{1}{2}(2\theta - a + b) = \theta - \frac{1}{2}(a - b)$

$\therefore \frac{1}{n}\sum_{i=1}^{n} X_i = \widetilde{E(X)} = \tilde{\theta} - \frac{1}{2}(a-b)$

$\therefore \quad \hat{\theta} = \bar{X} + \frac{1}{2}(a-b)$

$Var(\tilde{\theta}) = Var(\bar{X} + \frac{1}{2}(a-b)) = Var(\bar{X}) = (\frac{1}{n})\frac{[(\theta+b)-(\theta-a)]^2}{12}$

$= \frac{(b+a)^2}{12n}$

4. (i) The asymptotic distribution of $\bar{X}_n$ is $N(\mu_1, \sigma^2/n)$.

The asymptotic distribution of $\bar{Y}_n$ is $N(\mu_2, \sigma^2/n)$.

Since $\bar{X}_n$ and $\bar{Y}_n$ are independent, $Var(\bar{X}_n - \bar{Y}_n) = Var(\bar{X}_n) + Var(\bar{Y}_n) = \frac{\sigma^2}{n} + \frac{\sigma^2}{n}$

$$= \frac{2}{n}\sigma^2$$

Thus, the asymptotic distribution of $\bar{X}_n - \bar{Y}_n$ is $N(\mu_1 - \mu_2, \frac{2}{n}\sigma^2)$

(ii) $X \sim Bin(1000, 0.03)$

$$P\left(\frac{X}{1000} \le 5\%\right) = P(X \le 50)$$

$$= P(X \le 50.5) = P\left(\frac{X - 30}{\sqrt{1000(0.03)(1-0.03)}} \le \frac{50.5 - 30}{\sqrt{1000(0.03)(1-0.03)}}\right)$$

$$\approx P(Z \le 20.5/\sqrt{29.1})$$

$$= P(Z \le 3.8) = 1$$

5. Since X_1, X_2 and X_3 are independent r.v.'s distributed as $N(0,1)$, the joint distribution of X_1, X_2 and X_3 is multivariate normal distribution.

Also, the joint distribution of Y_1, Y_2 and Y_3 is multivariate normal distribution where $Y_1 = -\frac{1}{\sqrt{2}}X_1 + \frac{1}{\sqrt{2}}X_2$, $Y_2 = -\frac{1}{\sqrt{3}}X_1 - \frac{1}{\sqrt{3}}X_2 + \frac{1}{\sqrt{3}}X_3$,

$Y_3 = \frac{1}{\sqrt{6}}X_1 + \frac{1}{\sqrt{6}}X_2 + \frac{2}{\sqrt{6}}X_3$.

$Cov(Y_1, Y_2) = Cov(-\frac{1}{\sqrt{2}}X_1 + \frac{1}{\sqrt{2}}X_2, -\frac{1}{\sqrt{3}}X_1 - \frac{1}{\sqrt{2}}X_2 + \frac{1}{\sqrt{3}}X_3)$

$= \frac{1}{\sqrt{6}}Var(X_1) - \frac{1}{\sqrt{6}}Var(X_2) = \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{6}} = 0$

$Cov(Y_1, Y_3) = Cov(-\frac{1}{\sqrt{2}}X_1 + \frac{1}{\sqrt{2}}X_2, \frac{1}{\sqrt{6}}X_1 + \frac{1}{\sqrt{6}}X_2 + \frac{2}{\sqrt{6}}X_3)$

$= -\frac{1}{\sqrt{12}}Var(X_1) + \frac{1}{\sqrt{12}}Var(X_2) = -\frac{1}{\sqrt{12}} + \frac{1}{\sqrt{12}} = 0$

$Cov(Y_2, Y_3) = Cov(-\frac{1}{\sqrt{3}}X_1 - \frac{1}{\sqrt{3}}X_2 + \frac{1}{\sqrt{3}}X_3, \frac{1}{\sqrt{6}}X_1 + \frac{1}{\sqrt{6}}X_2 + \frac{2}{\sqrt{6}}X_3)$

$= -\frac{1}{\sqrt{18}}Var(X_1) - \frac{1}{\sqrt{18}}Var(X_2) + \frac{2}{\sqrt{18}}Var(X_3) = -\frac{1}{\sqrt{18}} - \frac{1}{\sqrt{18}} + \frac{2}{\sqrt{18}} = 0$

Thus, the r.v's Y_1, Y_2 and Y_3 are independent.

$E(Y_1) = E(-\frac{1}{\sqrt{2}}X_1 + \frac{1}{\sqrt{2}}X_2) = 0$

$E(Y_2) = E(-\frac{1}{\sqrt{3}}X_1 - \frac{1}{\sqrt{3}}X_2 + \frac{1}{\sqrt{3}}X_3) = 0$

$E(Y_3) = E(\frac{1}{\sqrt{6}}X_1 + \frac{1}{\sqrt{6}}X_2 + \frac{2}{\sqrt{6}}X_3) = 0$

$Var(Y_1) = Var(-\frac{1}{\sqrt{2}}X_1 + \frac{1}{\sqrt{2}}X_2) = \frac{1}{2}Var(X_1) + \frac{1}{2}Var(X_2) = 1$

$Var(Y_2) = Var(-\frac{1}{\sqrt{3}}X_1 - \frac{1}{\sqrt{3}}X_2 + \frac{1}{\sqrt{3}}X_3) = \frac{1}{3}Var(X_1) + \frac{1}{3}Var(X_2) + \frac{1}{3}Var(X_3) = 1$

$Var(Y_3) = Var(\frac{1}{\sqrt{6}}X_1 + \frac{1}{\sqrt{6}}X_2 + \frac{2}{\sqrt{6}}X_3) = \frac{1}{6}Var(X_1) + \frac{1}{6}Var(X_2) + \frac{4}{6}Var(X_3) = 1$

$\therefore$ the joint p.d.f. of $\underset{\sim}{Y} = (Y_1, Y_2, Y_3)^T$ is

$f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = \frac{1}{(2\pi)^{\frac{3}{2}}|\Sigma|^{\frac{1}{2}}} \exp\{-\frac{1}{2}(\underset{\sim}{y} - \underset{\sim}{\mu})^T \Sigma^{-1} (\underset{\sim}{y} - \underset{\sim}{\mu})\}$

where $\underset{\sim}{\mu} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \frac{1}{(2\pi)^{3/2}} \exp\{-\frac{1}{2}\underset{\sim}{y}^T \underset{\sim}{y}\}$