MATH 243 Statistical Inference

Final Examination - Fall 1999/2000

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1. Let $X_1, \ldots, X_n$ be i.i.d. with p.d.f.

$$f(x|\theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta, \quad \theta > 0.$$

   Estimate θ using both the method of moments and maximum likelihood. Which one should be preferred and why?

2. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size $n > 2$ from a distribution with p.d.f. $f(x;\theta) = \theta e^{-\theta x}, 0 < x < \infty$, zero elsewhere, and $\theta > 0$. Find the best estimator for θ.

3. Suppose $X_1, \ldots, X_n$ are independent random variable with distribution $B(1,p)$.

   (a) Find the maximum likelihood estimator of $\theta = (1-p)^2$.

   (b) Show that $\hat{\theta}$ is an unbiased estimator of θ, where $\hat{\theta} = 1$ if $X_1 + X_2 = 0$ and $\hat{\theta} = 0$ otherwise.

   (c) Find the best estimator for θ.

4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables, each with the Poisson distribution of parameter θ (and therefore of mean θ and variance θ). Find the best size α test of $H_0 : \theta = 1$ against $H_1 : \theta = 1.21$. By using the Central Limit Theorem to approximate the distribution of $\sum_i X_i$, find the smallest value of n required to make $\alpha = 0.05$ and $\beta \le 0.1$.

5. The data $x_1, \ldots, x_n$ has been observed and it is known that X_i is a sample from a Poisson distribution with an unknown mean λ_i. It is desired to test $H_0 : \lambda_1 = \ldots = \lambda_n$ against a general alternative hypothesis that the λ_i are arbitrary. Derive the approximate large sample likelihood ratio test. What would you conclude for data (3,4,1,6,5)? Take $\alpha = 0.05$.

6. Suppose that we have two independent random samples: $X_1, \ldots, X_n$ are exponential(θ), with density

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0$$

   and $Y_1, \ldots, Y_n$ are exponential (μ).

   (a) Find the form of the likelihood ratio test for testing $H_0 : \theta = \mu$ against $H_1 : \theta \neq \mu$.

   (b) Show that the test in part (a) can be expressed in terms of the statistic

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_i}$$

   (c) Find the distribution of T when H_0 is true. (Bonus: 5 marks)