Solutions to Exercise 4

1.
$$\begin{cases} H_0 : \theta = \theta_0 \\ H_1 : \theta = \theta_1 \ (> \theta_0) \end{cases}$$

H_0 is rejected if $x \geq k$. $\quad X \sim Geometric(\theta),\ P(X = x) = (1-\theta)^{x-1}\theta.$

$$\begin{aligned} P(\text{making a type I error}) &= P(\text{reject } H_0 | H_0 \text{ is true}) \\ &= P(X \geq k | \theta = \theta_0) \\ &= \sum_{x=k}^{\infty} (1-\theta_0)^{x-1}\theta_0 \\ &= \frac{\theta_0(1-\theta_0)^{k-1}}{1-(1-\theta_0)} \\ &= (1-\theta_0)^{k-1} \end{aligned}$$

$$\begin{aligned} P(\text{making a type II error}) &= P(\text{not reject } H_0 | H_0 \text{ is false}) \\ &= P(X < k | \theta = \theta_1) \\ &= 1 - P(X \geq k | \theta = \theta_1) \\ &= 1 - \sum_{x=k}^{\infty} (1-\theta_1)^{x-1}\theta_1 \\ &= 1 - \frac{\theta_1(1-\theta_1)^{k-1}}{1-(1-\theta_1)} \\ &= 1 - (1-\theta_1)^{k-1} \end{aligned}$$

2. X_1 and X_2 is a random sample of size 2 from the population given by

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$\begin{cases} H_0: & \theta = 1 \\ H_1: & \theta = 2 \end{cases}$$

$$\begin{aligned} c_1 &= \left\{ \underset{\sim}{x} : x_1 x_2 \geq \frac{3}{4} \right\} \\ f(x_1, x_2; \theta) &= (\theta x_1^{\theta-1}) \cdot (\theta x_2^{\theta-1}) \\ &= \theta^2 x_1^{\theta-1} x_2^{\theta-1} \qquad \text{for } 0 < x_1 < 1 \text{ and } 0 < x_2 < 1 \end{aligned}$$

$$\begin{aligned}
\text{Power of this test at } (\theta = 2) &= P(\underset{\sim}{X} \in c_1 | \theta = 2) \\
&= P(X_1 X_2 \geq \frac{3}{4} | \theta = 2) \\
&= \iint_{\{x_1 x_2 \geq \frac{3}{4}\}} f(x_1, x_2; 2)\ dx_1\ dx_2 \\
&= \int_{\frac{3}{4}}^{1} \int_{\frac{3}{4x_2}}^{1} 4x_1 x_2\ dx_1\ dx_2 \\
&= \int_{\frac{3}{4}}^{1} \Big[2x_2 x_1^2\Big]_{\frac{3}{4x_2}}^{1}\ dx_2 \\
&= \int_{\frac{3}{4}}^{1} 2x_2 \left((1)^2 - \left(\frac{3}{4x_2}\right)^2\right)\ dx_2 \\
&= \int_{\frac{3}{4}}^{1} \left(2x_2 - \frac{9}{8x_2}\right)\ dx_2 \\
&= \left[x_2^2 - \frac{9}{8}\ln x_2\right]_{\frac{3}{4}}^{1} \\
&= \left[\left(1^2 - \frac{9}{8}\ln(1)\right) - \left((\frac{3}{4})^2 - \frac{9}{8}\ln(\frac{3}{4})\right)\right] \\
&= 0.1139
\end{aligned}$$

3. $X_1, X_2, \ldots, X_n$ is a random sample from $N(\theta, 100)$.

$$\begin{cases} H_0: & \theta = 75 \\ H_1: & \theta = 78 \end{cases}$$

$$c = \left\{(x_1, x_2, \ldots, x_n) : \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \geq c\right\}$$

In order to show c is a best critical region for the above test (simple v.s. simple), we may use the Neyman-Pearson theorem.

$$\begin{aligned}
f_{\underset{\sim}{X}}(\underset{\sim}{x}; \theta) &= (2\pi(100))^{-n/2} \exp\left\{-\frac{1}{2(100)}\sum_{i=1}^{n}(x_i - \theta)^2\right\} \\
c_1 &= \left\{\underset{\sim}{x} : \frac{f_{\underset{\sim}{X}}(\underset{\sim}{x}\,; \theta = 75)}{f_{\underset{\sim}{X}}(\underset{\sim}{x}\,; \theta = 78)} \leq k\right\}
\end{aligned}$$

By Neyman-Pearson theorem,

$$\begin{aligned}
\frac{f_{\underset{\sim}{X}}(\underset{\sim}{x}\,;\theta=75)}{f_{\underset{\sim}{X}}(\underset{\sim}{x}\,;\theta=78)} &= \frac{(2\pi(100))^{-n/2}\exp\left\{-\frac{1}{200}\sum_{i=1}^{n}(x_i-75)^2\right\}}{(2\pi(100))^{-n/2}\exp\left\{-\frac{1}{200}\sum_{i=1}^{n}(x_i-78)^2\right\}} \\
&= \exp\left\{-\frac{1}{200}\sum_{i=1}^{n}\left[(x_i-75)^2-(x_i-78)^2\right]\right\} \\
&= \exp\left\{-\frac{1}{200}\sum_{i=1}^{n}\left(x_i^2-150x_i+75^2-x_i^2+156x_i-78^2\right)\right\} \\
&= \exp\left\{-\frac{1}{200}\sum_{i=1}^{n}(6x_i-459)\right\}\le k \\
&\Rightarrow -\frac{6}{200}\sum_{i=1}^{n}x_i+\frac{459n}{200}\le k' \\
&\Rightarrow -\frac{6}{200}\sum_{i=1}^{n}x_i\le k'' \\
&\Rightarrow \sum_{i=1}^{n}x_i\ge c
\end{aligned}$$

$\therefore$ C is a best critical region for the above test.

$$\begin{aligned}
P[(X_1,X_2,\ldots,X_n)\in c\,;H_0] &= P(\bar{X}\ge c\,;H_0)=0.05 \\
&\Rightarrow P\left(Z\ge\frac{c-75}{\sqrt{100/n}}\right)=0.05 \\
&\Rightarrow \frac{c-75}{\sqrt{100/n}}=1.645 \\
&\Rightarrow c-75=\frac{16.45}{\sqrt{n}} \qquad (1) \\
P[(X_1,X_2,\ldots,X_n)\in c\,;H_1] &= P(\bar{X}\ge c\,;H_1)=0.90 \\
&\Rightarrow P\left(Z\ge\frac{c-78}{\sqrt{100/n}}\right)=0.90 \\
&\Rightarrow \frac{c-78}{\sqrt{100/n}}=-1.28 \\
&\Rightarrow c-78=\frac{-12.8}{\sqrt{n}} \qquad (2) \\
(1)-(2) &\Rightarrow (c-75)-(c-78)=\frac{16.45}{\sqrt{n}}-\frac{-12.8}{\sqrt{n}} \\
3\sqrt{n} &= 29.25 \\
n &= 95.0625, \qquad \therefore \text{take } n=96 \\
\therefore\ c-75 &= \frac{16.45}{96} \\
c &= 76.68
\end{aligned}$$

4. $X_1,\ldots,X_{10}$ is a random sample from the p.d.f $f(x;\theta)=\theta^x(1-\theta)^{1-x}, \quad x=0,1$

$$\begin{cases} H_0: & \theta=\frac{1}{4} \\ H_1: & \theta<\frac{1}{4} \end{cases}$$

H_0 is rejected if the observed values $\sum_{i=1}^{10}x_i\le 1$

$$
\begin{aligned}
\text{For } 0<\theta\le\frac{1}{4}, \qquad & \text{Power function of the test } = K(\theta) = P(\text{reject } H_0|\theta) \\
&= P\left(\sum_{i=1}^{10} x_i \le 1|\theta\right) \\
&= \binom{10}{0}\theta^0(1-\theta)^{10-0} + \binom{10}{1}\theta^1(1-\theta)^{10-1} \quad (Note \sum_{i=1}^{10} x_i \sim Bin(10,\theta)) \\
&= (1-\theta)^{10} + 10\theta(1-\theta)^9 \\
&= (1-\theta)^9(10\theta + (1-\theta)) \\
&= (1-\theta)^9(9\theta+1)
\end{aligned}
$$

5. Let $Y_1 < Y_2 < Y_3 < Y_4$ denote the order statistics of a random sample of size 4 from the distribution with p.d.f. $f(x;\theta) = \frac{1}{\theta}, 0 < x < \theta$. Note that $F(x;\theta) = \frac{x}{\theta}, 0 < x < \theta$.

$$
\begin{cases} H_0: & \theta = 1 \\ H_1: & \theta \neq 1 \end{cases}
$$

H_0 is rejected if either the observed value Y_4, $y_4 \le \frac{1}{2}$ or $y_4 \ge 1$.

$$
\begin{aligned}
\text{For } \theta>0, \qquad & \text{Power function of the test } = K(\theta) = P(\text{reject } H_0|\theta) \\
&= P(Y_4 \le \frac{1}{2} \text{ or } Y_4 \ge 1|\theta) \\
&= P(Y_4 \le \frac{1}{2}|\theta) + P(Y_4 \ge 1|\theta) \\
&= \left[\frac{1}{\theta}\min\{\frac{1}{2},\theta\}\right]^4 + 1 - P(Y_4 < 1|\theta) \quad (\text{remind that } P(Y_4 \le y) = [P(X \le y)]^4) \\
&= \frac{1}{\theta^4}[\min\{\frac{1}{2},\theta\}]^4 + 1 - [\frac{1}{\theta}\min\{1,\theta\}]^4 \\
&= 1 + \frac{1}{\theta^4}\left[(\min\{\frac{1}{2},\theta\})^4 - (\min\{1,\theta\})^4\right] \ldots
\end{aligned}
$$

Or

Critical region $= \{\underset{\sim}{x} : y_4 \le \frac{1}{2} \text{ or } y_4 \ge 1\}$

Note that $f_{Y_4}(y_4) = 4(\frac{y_4}{\theta})^3(\frac{1}{\theta}) = \frac{4}{\theta^4}y_4^3, \qquad 0 < y_4 < \theta$

For $\theta > 0$, Power function of the test $= K(\theta) = P(\text{reject } H_0|\theta)$

$$\begin{aligned}
&= P\left(Y_4 \leq \frac{1}{2} \text{ or } Y_4 \geq 1|\theta\right) \\
&= P\left(Y_4 \leq \frac{1}{2}|\theta\right) + P(Y_4 \geq 1|\theta) \\
&= \begin{cases} P(Y_4 \leq \frac{1}{2}|\theta) + P(Y_4 \geq 1|\theta) & \text{for } \theta \geq 1 \\ P(Y_4 \leq \frac{1}{2}|\theta) & \text{for } \frac{1}{2} < \theta < 1 \\ 1 & \text{for } 0 \leq \theta \leq \frac{1}{2} \end{cases} \\
&= \begin{cases} 1 - P(\frac{1}{2} < Y_4 < 1|\theta) & \text{for } \theta \geq 1 \\ \int_0^{\frac{1}{2}} \frac{4}{\theta^4}\, y_4^3\, dy_4 & \text{for } \frac{1}{2} < \theta < 1 \\ 1 & \text{for } 0 \leq \theta \leq \frac{1}{2} \end{cases} \\
&= \begin{cases} 1 - \int_{\frac{1}{2}}^1 \frac{4}{\theta^4}\, y_4^3\, dy_4 & \text{for } \theta \geq 1 \\ \left[\frac{1}{\theta^4}\, y_4^4\right]_0^{\frac{1}{2}} & \text{for } \frac{1}{2} < \theta < 1 \\ 1 & \text{for } 0 \leq \theta \leq \frac{1}{2} \end{cases} \\
&= \begin{cases} 1 - \left[\left(\frac{1}{\theta^4}\right)\, y_4^4\right]_{\frac{1}{2}}^1 & \text{for } \theta \geq 1 \\ \frac{1}{\theta^4}\left(\frac{1}{2}\right)^4 & \text{for } \frac{1}{2} < \theta < 1 \\ 1 & \text{for } 0 \leq \theta \leq \frac{1}{2} \end{cases} \\
&= \begin{cases} 1 - \frac{1}{\theta^4}\left[1^4 - \left(\frac{1}{2}\right)^4\right] = 1 - \frac{15}{16\theta^4} & \text{for } \theta \geq 1 \\ \frac{1}{16\theta^4} & \text{for } \frac{1}{2} < \theta < 1 \\ 1 & \text{for } 0 \leq \theta \leq \frac{1}{2} \end{cases}
\end{aligned}$$

6. $X_1, X_2, \ldots, X_{25}$ is a random sample of size 25 from $N(\theta, 4)$

$$\begin{cases} H_0: & \theta = 0 \\ H_1: & \theta > 0 \end{cases}$$

H_0 is rejected if either the observed mean $\bar{x} \geq \frac{3}{5}$.

For $\theta \geq 0$, Power function of the test $= K(\theta) = P(\text{reject } H_0|\theta)$

$$\begin{aligned}
&= P\left(\bar{X} \geq \frac{3}{5}|\theta\right) \\
&= P\left(Z \geq \frac{\frac{3}{5} - \theta}{\sqrt{4/25}}\right) \\
&= P\left(Z \geq \frac{3 - 5\theta}{2}\right) \\
&= 1 - \Phi\left(\frac{3 - 5\theta}{2}\right) \qquad \text{where } \Phi(\cdot) \text{is the c.d.f. of } N(0,1)
\end{aligned}$$

7. $X_1, X_2, \ldots, X_n$ is a random sample of size n from $N(\mu_1, 400)$

$Y_1, Y_2, \ldots, Y_n$ is a random sample of size n from $N(\mu_2, 225)$

$$\begin{cases} H_0: & \theta = 0 \\ H_1: & \theta > 0 \end{cases} \qquad \text{where } \theta = \mu_1 - \mu_2$$

H_0 is rejected if the observed values $\bar{x} - \bar{y} \geq c$.

$$
\begin{aligned}
\text{Power function of the test } K(\theta) &= P(\text{reject } H_0|\theta) \\
&= P(\bar{X}-\bar{Y} \geq c|\theta) \\
K(0) = 0.05 &\Rightarrow P(\bar{X}-\bar{Y} \geq c|\theta = 0) = 0.05 \\
&\Rightarrow P\left(Z \geq \frac{c-0}{\sqrt{\frac{400}{n}+\frac{225}{n}}}\right) = 0.05 \\
&\Rightarrow \frac{c}{25/\sqrt{n}} = 1.645 \\
&\Rightarrow c = \frac{41.125}{\sqrt{n}} \qquad (1) \\
K(10) = 0.90 &\Rightarrow P(\bar{X}-\bar{Y} \geq c|\theta = 10) = 0.90 \\
&\Rightarrow P\left(Z \geq \frac{c-10}{\sqrt{\frac{400}{n}+\frac{225}{n}}}\right) = 0.90 \\
&\Rightarrow \frac{c-10}{25/\sqrt{n}} = -1.28 \\
&\Rightarrow c-10 = -\frac{32}{\sqrt{n}} \\
&\Rightarrow c = -\frac{32}{\sqrt{n}} + 10 \qquad (2) \\
\text{From (1), (2)} \qquad \frac{41.125}{\sqrt{n}} &= -\frac{32}{\sqrt{n}} + 10 \\
n &= 53.47, \qquad \therefore \text{take } n = 54 \\
c &= \frac{41.125}{\sqrt{54}} = 5.596
\end{aligned}
$$

8. $X_1, X_2, \ldots, X_n$ is a random sample from the exponential distribution with parameter θ.

$$
f_X(x;\theta) = \theta e^{-\theta x}
$$

$$
\begin{cases} H_0: & \theta = \theta_0 \qquad (\theta_0 \in \Re^+) \\ H_1: & \theta > \theta_0 \end{cases}
$$

In order to show there exists a uniformly most powerful test at significance α of $H_0 : \theta = \theta_0$ against the one-sided alternative hypothesis $\theta > \theta_0$, we may check that the p.d.f. $f_X(x,\theta)$ can be written in the exponential form $\exp\{a(\theta) + b(x) + c(\theta)d(x)\}$ and check that $c(\theta)$ should be either increasing or decreasing on Θ.

$$
f_X(x;\theta) = \theta e^{-\theta x} = \exp\{\log\theta - \theta x\} \qquad \text{where } a(\theta) = \log\theta, b(x) = 0, c(\theta) = -\theta, d(x) = x
$$

Now, $c(\theta) = -\theta$ is a decreasing function on Θ.

$\therefore$ the uniformly most powerful test exists and $c_1 = \{\underset{\sim}{x} : \sum_{i=1}^n d(x_i) = \sum_{i=1}^n x_i \leq k\}$ where k is determined by α. In fact, $\sum_{i=1}^n X_i \sim Gamma(n,\theta)$ and $2\theta\sum_{i=1}^n X_i \sim \chi^2(n)$.

9. $X_1, X_2, \ldots, X_n$ is a random sample of size n from $N(0,\sigma^2)$

$$
\begin{cases} H_0: & \sigma = \sigma_0 \\ H_1: & \sigma = \sigma_1 (> \sigma_0) \end{cases}
$$

By Neyman-Pearson theorem,

$$f_{\underset{\sim}{X}}(\underset{\sim}{x};\sigma) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-0)^2\right\} = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}x_i^2\right\}$$

$$c_1 = \left\{\underset{\sim}{x} : \frac{f_{\underset{\sim}{X}}(\underset{\sim}{x}\,;\sigma_0)}{f_{\underset{\sim}{X}}(\underset{\sim}{x}\,;\sigma_1)} \le k\right\}$$

$$\begin{aligned}
& \frac{f_{\underset{\sim}{X}}(\underset{\sim}{x}\,;\sigma_0)}{f_{\underset{\sim}{X}}(\underset{\sim}{x}\,;\sigma_1)} \\
= \; & \frac{\left(\frac{1}{\sqrt{2\pi}\sigma_0}\right)^n \exp\left\{-\frac{1}{2\sigma_0^2}\sum_{i=1}^n x_i^2\right\}}{\left(\frac{1}{\sqrt{2\pi}\sigma_1}\right)^n \exp\left\{-\frac{1}{2\sigma_1^2}\sum_{i=1}^n x_i^2\right\}} \\
= \; & \left(\frac{\sigma_1}{\sigma_0}\right)^n \exp\left\{\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\cdot\frac{1}{2}\sum_{i=1}^{n}x_i^2\right\} \le k \\
\Rightarrow \; & \exp\left\{\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\cdot\frac{1}{2}\sum_{i=1}^{n}x_i^2\right\} \le k' \\
\Rightarrow \; & \frac{1}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum_{i=1}^{n}x_i^2 \le k'' \\
\Rightarrow \; & \sum_{i=1}^{n}x_i^2 \ge K \qquad\qquad (\because\ \sigma_1 > \sigma_0 \Rightarrow \frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2} < 0) \\
\therefore\ c_1 \; = \; & \left\{\underset{\sim}{x} : \sum_{i=1}^{n}x_i^2 \ge K\right\}
\end{aligned}$$

$$\begin{aligned}
& P(\underset{\sim}{X} \in c_1 | H_0 \text{ is true}) = \alpha \\
\Rightarrow \; & P\left(\sum_{i=1}^{n}x_i^2 \ge K | \sigma=\sigma_0\right) = \alpha \\
\Rightarrow \; & P\left(\frac{1}{\sigma_0^2}\sum_{i=1}^{n}x_i^2 \ge \frac{1}{\sigma_0^2}K | \sigma=\sigma_0\right) = \alpha \\
\Rightarrow \; & P\left(\chi^2_{(n)} \ge \frac{1}{\sigma_0^2}K\right) = \alpha \\
\Rightarrow \; & \frac{1}{\sigma_0^2}K = \chi^2_{(n,\alpha)} \\
\Rightarrow \; & K = \sigma_0^2\chi^2_{(n,\alpha)}
\end{aligned}$$

$\therefore$ the most powerful critical region of size α to test $H_0 : \sigma = \sigma_0$ v.s. $H_1 : \sigma = \sigma_1 (> \sigma_0)$ is $c_1 = \{\underset{\sim}{x} : \sum_{i=1}^n x_i^2 \ge \sigma_0^2\ \chi^2_{(n,\alpha)}\}$

10. $X_1, \ldots, X_{10}$ is a random sample of size n from $N(\theta_1, \theta_2)$

$$\begin{cases} H_0 : & \theta_1 = \theta_1' = 0, \quad \theta_2 = \theta_2' = 1 \\ H_1 : & \theta_1 = \theta_1'' = 1, \quad \theta_2 = \theta_2'' = 4 \end{cases}$$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta_1,\theta_2) = (2\pi\theta_2)^{-10/2}\exp\left\{-\frac{1}{2\theta_2}\sum_{i=1}^{10}(x_i-\theta_1)^2\right\}$$

$$c_1 = \left\{ \underset{\sim}{x} : \frac{f_{\underset{\sim}{X}}(\underset{\sim}{x}\,; \theta_1', \theta_2')}{f_{\underset{\sim}{X}}(\underset{\sim}{x}\,; \theta_1'', \theta_2'')} \leq k \right\}$$

By Neyman-Pearson theorem,

$$\begin{aligned}
\frac{f_{\underset{\sim}{X}}(\underset{\sim}{x}\,; \theta_1', \theta_2')}{f_{\underset{\sim}{X}}(\underset{\sim}{x}\,; \theta_1'', \theta_2'')} &= \frac{(2\pi(1))^{-5} \exp\left\{-\frac{1}{2(1)} \sum_{i=1}^{10} (x_i - 0)^2\right\}}{(2\pi(4))^{-5} \exp\left\{-\frac{1}{2(4)} \sum_{i=1}^{10} (x_i - 1)^2\right\}} \\
&= (4)^5 \exp\left\{-\frac{1}{2}\sum_{i=1}^{10} x_i^2\right\} \exp\left\{\frac{1}{8}\sum_{i=1}^{10}(x_i - 1)^2\right\} \leq k \\
&\Rightarrow -\frac{1}{2}\sum_{i=1}^{10} x_i^2 + \frac{1}{8}\sum_{i=1}^{10}(x_i - 1)^2 \leq k' \\
&\Rightarrow -\frac{1}{8}\sum_{i=1}^{10}(4x_i^2 - x_i^2 + 2x_i - 1) \leq k' \\
&\Rightarrow -\frac{1}{8}\sum_{i=1}^{10}(3x_i^2 + 2x_i - 1) \leq k' \\
&\Rightarrow -\frac{1}{8}\sum_{i=1}^{10}\left[3\left(x_i + \frac{1}{3}\right)^2 - 3\left(\frac{1}{3}\right)^2 - 1\right] \leq k' \\
&\Rightarrow \sum_{i=1}^{10}\left(x_i + \frac{1}{3}\right)^2 \geq K
\end{aligned}$$

$\therefore$ a best test is to reject H_0 if $\sum_{i=1}^{10}\left(x_i + \frac{1}{3}\right)^2 \geq K$

11. $X_1, \ldots, X_{25}$ is a random sample of size 25 from $N(\theta, 100)$

$$\begin{cases} H_0: & \theta = 75 \\ H_1: & \theta > 75 \end{cases}$$

In order to find a uniformly most powerful critical region, we may check that the p.d.f. can be written as the exponential form $\exp\{a(\theta) + b(x) + c(\theta)d(x)\}$ and also $c(\theta)$ needs to be either increasing or decreasing on Θ.

$$\begin{aligned}
f(x;\theta) &= \frac{1}{\sqrt{2\pi 100}} \exp\left\{-\frac{1}{200}(x-\theta)^2\right\} \\
&= \exp\left\{-\frac{1}{2}\log(200\pi) - \frac{1}{200}(x^2 - 2x\theta + \theta^2)\right\} \\
&= \exp\left\{-\frac{1}{2}\log(200\pi) - \frac{1}{200}\theta^2 - \frac{1}{200}x^2 + \frac{\theta}{100}x\right\}
\end{aligned}$$

$\therefore$ $c(\theta) = \frac{\theta}{100}$ is an increasing function on $\Theta = [75, \infty)$

$\therefore$ a uniformly most powerful critical region for testing $H_0 : \theta = 75$ against $H_1 : \theta > 75$ is

$$c_1 = \left\{ \underset{\sim}{x} : \sum_{i=1}^{25} d(x_i) \geq k \right\} = \left\{ \underset{\sim}{x} : \sum_{i=1}^{25} x_i \geq k \right\}$$

$$
\begin{aligned}
& P(\text{reject } H_0 | H_0 \text{ is true}) = 0.01 \qquad\qquad (\text{size } \alpha = 0.1) \\
\Rightarrow\quad & P\left(\sum_{i=1}^{25} x_i \geq k | \theta = 75\right) = 0.1 \\
\Rightarrow\quad & P\left(Z \geq \frac{k - 25(75)}{\sqrt{(25)(100)}}\right) = 0.1 \\
\Rightarrow\quad & \frac{k - 1875}{50} = 1.28 \\
\Rightarrow\quad & k = 1939
\end{aligned}
$$

$\therefore$ the uniformly most powerful critical region of size 0.01 for the test is $c_1 = \left\{ \underset{\sim}{x} : \sum_{i=1}^{n} x_i \geq 1939 \right\}$.

Alternatively, we may try to extend the result by the Neyman-Pearson theorem discussed in the classes.

12. $X_1, \ldots, X_n$ is a random sample of size n from $N(\theta, 16)$

$$
\begin{cases} H_0: & \theta = 25 \\ H_1: & \theta < 25 \end{cases}
$$

Similar to Q.11,

$$
\begin{aligned}
f(x;\theta) &= \frac{1}{\sqrt{2\pi(16)}} \exp\left\{ -\frac{1}{2(16)} (x - \theta)^2 \right\} \\
&= \exp\left\{ -\frac{1}{2}\log(32\pi) - \frac{1}{32}(x^2 - 2x\theta + \theta^2) \right\} \\
&= \exp\left\{ -\frac{1}{2}\log(32\pi) - \frac{1}{32}\theta^2 - \frac{1}{32}x^2 + \frac{\theta}{16}x \right\}
\end{aligned}
$$

$\therefore\ c(\theta) = \frac{\theta}{16}$ is an increasing function on $\Theta = (-\infty, 25]$

$\therefore$ a uniformly most powerful critical region for testing $H_0 : \theta = 25$ against $H_1 : \theta < 25$ is

$$
c_1 = \left\{ \underset{\sim}{x} : \sum_{i=1}^{n} d(x_i) \leq k_1 \right\} = \left\{ \underset{\sim}{x} : \sum_{i=1}^{n} x_i \leq k_1 \right\}
$$

$$
\begin{aligned}
\text{Power function of the test} &= K(\theta) = P(\underset{\sim}{x} \in c_1|\theta) \\
&= P\left(\sum_{i=1}^{n} x_i \le k_1|\theta\right) \\
K(25) = 0.10 &\Rightarrow P\left(\sum_{i=1}^{n} x_i \le k_1|\theta = 25\right) = 0.10 \\
&\Rightarrow P\left(Z \le \frac{k_1 - 25n}{\sqrt{n(16)}}\right) = 0.10 \\
&\Rightarrow \frac{k_1 - 25n}{4\sqrt{n}} = -1.28 \\
&\Rightarrow k_1 = 25n - 5.12\sqrt{n} \qquad (1) \\
K(23) = 0.90 &\Rightarrow P\left(\sum_{i=1}^{n} x_i \le k_1|\theta = 23\right) = 0.90 \\
&\Rightarrow P\left(Z \le \frac{k_1 - 23n}{\sqrt{n(16)}}\right) = 0.90 \\
&\Rightarrow \frac{k_1 - 23n}{4\sqrt{n}} = 1.28 \\
&\Rightarrow k_1 = 23n + 5.12\sqrt{n} \qquad (2)
\end{aligned}
$$

From (1) and (2),

$$
\begin{aligned}
25n - 5.12\sqrt{n} &= 23n + 5.12\sqrt{n} \\
\Rightarrow 2n &= 10.24\sqrt{n} \\
\Rightarrow n &= 26.21 \\
\therefore\ n &= 27 \\
\therefore\ k_1 &= 25(27) - 5.12\sqrt{27} = 648.40
\end{aligned}
$$

$\therefore$ the uniformly most powerful test is to reject H_0 if

$$
\underset{\sim}{X} \in c_1 = \left\{ \underset{\sim}{x} : \sum_{i=1}^{n} x_i \le 648.40 \right\}.
$$

13. $X_1, \ldots, X_{20}$ is a random sample of size 20 from Poisson(θ).

$$
\begin{cases} H_0: & \theta = \frac{1}{10} \\ H_1: & \theta > \frac{1}{10} \end{cases}
$$

$C_1 = \{\underset{\sim}{x} : \sum_{i=1}^{20} x_i \ge 5\}$

In order to show C_1 is the uniformly most powerful critical region for the above simple against one-sided composite test, we may check whether the critical region is equivalent to the critical region which can be obtained by the exponential form of the p.d.f.

$$
f(x;\theta) = \frac{\theta^x e^{-\theta}}{x!} = \exp\{-\theta - \log(x!) + x\log\theta\}
$$

$\therefore$ $c(\theta) = \log\theta$ is an increasing function on $\Theta = [\frac{1}{10}, \infty)$

$\therefore$ a uniformly most powerful critical region for testing $H_0 : \theta = \frac{1}{10}$ against $H_1 : \theta > \frac{1}{10}$ is

$$
C_1' = \left\{ \underset{\sim}{x} : \sum_{i=1}^{20} d(x_i) \ge k \right\} = \left\{ \underset{\sim}{x} : \sum_{i=1}^{20} x_i \ge k \right\}
$$

Hence, C_1 is the uniformly most powerful critical region for the test.

The significance level of the test:

$$\begin{aligned}
\alpha &= P(\underset{\sim}{X} \in C_1 | H_0 \text{ is true}) \\
&= P\left(\sum_{i=1}^{20} X_i \geq 5 | \theta = \frac{1}{10}\right) \\
&= 1 - P\left(\sum_{i=1}^{20} X_i < 5 | \theta = \frac{1}{10}\right) \\
&= 1 - P\left(\sum_{i=1}^{20} X_i \leq 4 | \theta = \frac{1}{10}\right), \qquad \text{Note: } \sum_{i=1}^{20} X_i \sim Poi(20\ \theta) \\
&= 1 - \left[\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} + \frac{2^3 e^{-2}}{3!} + \frac{2^4 e^{-2}}{4!}\right] \\
&= 1 - (0.1353 + 0.2707 + 0.2707 + 0.1804 + 0.0902) \\
&= 0.0527
\end{aligned}$$

14. $X_1, \ldots, X_n$ are iid random variables, each with the Poisson distribution of parameter θ.

$$f_X(x;\theta) = \frac{\theta^x e^{-\theta}}{x!}$$

$$\begin{cases} H_0: & \theta = 1 \\ H_1: & \theta = 1.21 \end{cases}$$

By Neyman-Pearson theorem, the most powerful test is to reject H_0 when

$$\begin{aligned}
& \frac{f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta=1)}{f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta=1.21)} \leq k \\
\Rightarrow\ & \frac{\prod_{i=1}^{n} f_{X_i}(x_i;\theta=1)}{\prod_{i=1}^{n} f_{X_i}(x_i;\theta=1.21)} \leq k \\
\Rightarrow\ & \frac{\frac{(1)^{\sum_{i=1}^n x_i} e^{-1}}{\prod_{i=1}^n x_i!}}{\frac{(1.21)^{\sum_{i=1}^n x_i} e^{-1.21}}{\prod_{i=1}^n x_i!}} \leq k \\
\Rightarrow\ & (1.21)^{-\sum_{i=1}^n x_i} e^{0.21} \leq k \\
\Rightarrow\ & \left(-\sum_{i=1}^{n} x_i\right) \log(1.21)(0.21) \leq k \\
\Rightarrow\ & \sum_{i=1}^{n} x_i \geq k
\end{aligned}$$

$\therefore$ the critical region of the most powerful test of $H_0: \theta = 1$ against $H_1: \theta = 1.21$ is

$$C_1 = \{\underset{\sim}{x} : \sum_{i=1}^{n} x_i \geq k\}$$

Note that $\sum_{i=1}^{n} X_i \sim \text{Poi}(n\theta)$.

$$\alpha \geq P\left(\sum_{i=1}^{n} X_i \geq k|\theta = 1\right) \geq P(Y \geq k) \text{ where } Y \sim Poi(n)$$

$\Rightarrow k = q(n;\alpha)$ where $q(n;\alpha)$ is the smallest integer which satisfies $P(Y \geq q(n;\alpha)) \leq \alpha$.
$\therefore$ the best size α test of $H_0 : \theta = 1$ against $H_1 : \theta = 1.21$ is to reject H_0 when $\sum_{i=1}^{n} X_i \geq q(n;\alpha)$.

By Central Limit Theorem, $\sum_{i=1}^{n} X_i \approx N(n\theta, n\theta)$
Now, $\alpha = 0.05$ and $\beta \leq 0.1$

$$\left\{\begin{array}{rcl} P\left(\sum_{i=1}^{n} X_i \geq k|\theta = 1\right) & = & 0.05 \\ P\left(\sum_{i=1}^{n} X_i < k|\theta = 1.21\right) & \leq & 0.1 \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{rcl} P\left(Z \geq \dfrac{k-n-0.5}{\sqrt{n}}\right) & = & 0.05 \\ P\left(Z < \dfrac{k-1.21n-0.5}{\sqrt{1.21n}}\right) & \leq & 0.1 \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{rcl} \dfrac{k-n-0.5}{\sqrt{n}} & = & 1.645 \\ \dfrac{k-1.21n-0.5}{\sqrt{1.21n}} & \leq & -1.28 \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{rcll} k-n-0.5 & = & 1.645\sqrt{n} & \Rightarrow k = n + 1.645\sqrt{n} + 0.5 \qquad (1) \\ k-1.21n-0.5 & \leq & -1.28\sqrt{1.21n} & \qquad (2) \end{array}\right.$$

Sub (1) into (2),

$$\begin{array}{rcl} n + 1.645\sqrt{n} + 0.5 - 1.21n - 0.5 & \leq & -1.408\sqrt{n} \\ \Rightarrow 0.21n & \geq & 3.053\sqrt{n} \\ \Rightarrow \sqrt{n} & \geq & \dfrac{3.053}{0.21} \\ \Rightarrow n & \geq & 211.4 \end{array}$$

$\therefore$ the smallest value of n required to make $\alpha = 0.05$ and $\beta \leq 0.1$ is 212.

15. $X_1, \ldots, X_n$ are independent r.v.'s distributed as $N(\mu, \sigma)$ where μ is unknown and σ is known.

$$\left\{\begin{array}{ll} H_0 : & \mu = 0 \\ H_1 : & \mu = 1 \end{array}\right.$$

The critical region of the test of above hypothesis is:

$$C_1 = \left\{ \underset{\sim}{x} : \frac{f_{\underset{\sim}{X}}(\underset{\sim}{x}, \mu = 0)}{f_{\underset{\sim}{X}}(\underset{\sim}{x}, \mu = 1)} \leq k \right\} \quad \text{(by Neyman-Pearson theorem)}$$

$$
\begin{aligned}
\frac{f_{\underset{\sim}{X}}(\underset{\sim}{x}, \mu = 0)}{f_{\underset{\sim}{X}}(\underset{\sim}{x}, \mu = 1)} &= \exp\left\{-\frac{1}{2\sigma}\left[\sum_{i=1}^{n}(x_i - 0)^2 - \sum_{i=1}^{n}(x_i - 1)^2\right]\right\} \\
&= \exp\left\{\frac{n}{2\sigma}\left[1 - 2\bar{x}\right]\right\} \le k \\
\Rightarrow \quad \frac{n}{2\sigma}[1 - 2\bar{x}] &\le \log k \\
\Rightarrow \quad \bar{x} &\ge \frac{1}{2} - \frac{\sigma}{n}\log k = k \\
\Rightarrow \quad C_1 &= \{\underset{\sim}{x} : \bar{x} \ge k\}
\end{aligned}
$$

Under H_0, $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N(0, \frac{\sigma}{n})$ and under $H_1, \bar{X} \sim N(1, \frac{\sigma}{n})$,

$$
\begin{aligned}
& \alpha = P(\underset{\sim}{X} \in C_1 | H_0) \\
\Rightarrow \quad & \alpha = P(\bar{X} \ge k | H_0) = P\left(Z \ge \frac{k}{\sqrt{\sigma/n}}\right) \\
\Rightarrow \quad & \frac{k}{\sqrt{\sigma/n}} = z_\alpha \\
& \beta = P(\underset{\sim}{X} \notin C_1 | H_1) \\
\Rightarrow \quad & \beta = P(\bar{X} < k | H_1) = P\left(Z < \frac{k-1}{\sqrt{\sigma/n}}\right) \\
\Rightarrow \quad & \beta = P\left(Z < z_\alpha - \frac{1}{\sqrt{\sigma/n}}\right) \\
\Rightarrow \quad & z_\beta = -z_\alpha + \frac{1}{\sqrt{\sigma/n}} \\
\Rightarrow \quad & \sqrt{\frac{\sigma}{n}} = \frac{1}{z_\alpha + z_\beta} \\
\Rightarrow \quad & n = (z_\alpha + z_\beta)^2 \sigma^2
\end{aligned}
$$

$\therefore$ the sample size n can be determined.

If $\alpha = 0.05$, $\beta = 0.1$ and $\sigma = 1$

$$
n \ge (1.645 + 1.28)^2(1) = 8.5264 \quad \Rightarrow \quad \therefore \; n = 9
$$

16. $X_1, X_2, \ldots, X_{100}$ is a random sample from $N(\theta, 1.8^2)$

$$
\begin{cases} H_0: & \theta = 2 \\ H_1: & \theta \ne 2 \end{cases}
$$

$$
\Theta = \{2\}, \Theta_1 = \Re \setminus \{2\}, \Theta = \Re
$$

$$
\begin{aligned}
L(\theta, \underset{\sim}{x}) &= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}(1.8)} \exp\left\{-\frac{1}{2(1.8)^2}(x_i-\theta)^2\right\} \\
&= (6.48\pi)^{-\frac{n}{2}} \exp\left\{-\frac{1}{6.48}\sum_{i=1}^{n}(x_i-\theta)^2\right\} \\
\log L(\theta, \underset{\sim}{x}) &= -\frac{n}{2}\log(6.48\pi) - \frac{1}{6.48}\sum_{i=1}^{n}(x_i-\theta)^2 \\
\frac{\partial}{\partial\theta}\log L &= -\frac{2}{6.48}\sum_{i=1}^{n}(x_i-\theta)(-1) \\
&= \frac{1}{3.24}\sum_{i=1}^{n}(x_i-\theta) \\
\frac{\partial}{\partial\theta}\log L = 0 &\Rightarrow \frac{1}{3.24}\sum_{i=1}^{n}(x_i-\theta) = 0 \\
&\Rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}(\text{M.L.E.})
\end{aligned}
$$

The Likelihood ratio is:

$$
\begin{aligned}
\lambda(\underset{\sim}{x}) &= \frac{\sup\{L(\theta,\underset{\sim}{x}) : \theta \in \Theta_0\}}{\sup\{L(\theta,\underset{\sim}{x}) : \theta \in \Theta\}} \\
&= \frac{L(2,\underset{\sim}{x})}{L(\hat{\theta},\underset{\sim}{x})} \\
&= \frac{(6.48\pi)^{-\frac{n}{2}} \exp\left\{-\frac{1}{6.48}\sum_{i=1}^{n}(x_i-2)^2\right\}}{(6.48\pi)^{-\frac{n}{2}} \exp\left\{-\frac{1}{6.48}\sum_{i=1}^{n}(x_i-\bar{x})^2\right\}} \\
&= \exp\left\{-\frac{1}{6.48}\left[\sum_{i=1}^{n}(x_i-2)^2 - \sum_{i=1}^{n}(x_i-\bar{x})^2\right]\right\}
\end{aligned}
$$

17. $X \sim Bin(n,\theta)$

$$
\begin{cases}
H_0: & \theta = \frac{1}{2} \\
H_1: & \theta \neq \frac{1}{2}
\end{cases}
$$

(a) An expression for the likelihood ratio statistic is:

$$
\lambda(x) = \frac{\sup\{L(\theta,x) : \theta \in \Theta_0\}}{\sup\{L(\theta,x) : \theta \in \Theta\}} \qquad \text{where } \Theta_0 = \{\frac{1}{2}\}, \Theta = \left\{\theta : \theta \in (0,1)\right\}
$$

The likelihood function is:

$$
\begin{aligned}
L(\theta, x) &= f(x;\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x} \\
\log L(\theta, x) &= \log \binom{n}{x} + x\log\theta + (n-x)\log(1-\theta) \\
\frac{\partial}{\partial\theta}\log L(\theta, x) &= \frac{x}{\theta} - \frac{n-x}{1-\theta} = 0 \\
\Rightarrow \frac{x}{\theta} &= \frac{n-x}{1-\theta} \\
\Rightarrow x - \theta x &= n\theta - \theta x \\
\Rightarrow \hat{\theta} &= \frac{1}{n}X \qquad\qquad \text{(M.L.E.)} \\
\therefore\ \lambda(x) &= \frac{\binom{n}{x} (\frac{1}{2})^x (1-\frac{1}{2})^{n-x}}{\binom{n}{x} (\frac{x}{n})^x\ (1-\frac{x}{n})^{n-x}} \\
&= \left(\frac{n}{2x}\right)^x \left(\frac{1}{2}\right)^{n-x} \left(\frac{n}{n-x}\right)^{n-x} \\
&= \left(\frac{n}{2}\right)^n \left(\frac{1}{x}\right)^x \left(\frac{1}{n-x}\right)^{n-x}
\end{aligned}
$$

(b) H_0 is rejected if $\lambda(x) \leq k$

$$
\begin{aligned}
& \lambda(x) \leq k \\
\Rightarrow\ & \left(\frac{n}{2}\right)^n \left(\frac{1}{x}\right)^x \left(\frac{1}{n-x}\right)^{n-x} \leq k \\
\Rightarrow\ & x^x\ (n-x)^{n-x} \geq k' \\
\Rightarrow\ & x\log x + (n-x)\log(n-x) \geq k
\end{aligned}
$$

i.e. the critical region of the likelihood ratio test can be written as

$$x\log x + (n-x)\log(n-x) \geq k$$

(c)

$$
\begin{aligned}
f(x) &= x\log x + (n-x)\log(n-x) \\
f'(x) &= \log x + x\cdot\frac{1}{x} - \log(n-x) - (n-x)\left(\frac{1}{n-x}\right) \\
&= \log x + 1 - \log(n-x) - 1 \\
&= \log x - \log(n-x) \\
\text{Set } f'(x) &= 0 \\
\Rightarrow \log x - \log(n-x) &= 0 \\
\Rightarrow \qquad \log x &= \log(n-x) \\
\therefore \qquad x &= n-x \\
\therefore \qquad x &= \frac{n}{2} \\
f''(x) &= \frac{1}{x} + \frac{1}{n-x} \\
f''\left(\frac{n}{2}\right) &= \frac{2}{n} + \frac{1}{n-\frac{n}{2}} \\
&= \frac{n}{4} > 0
\end{aligned}
$$

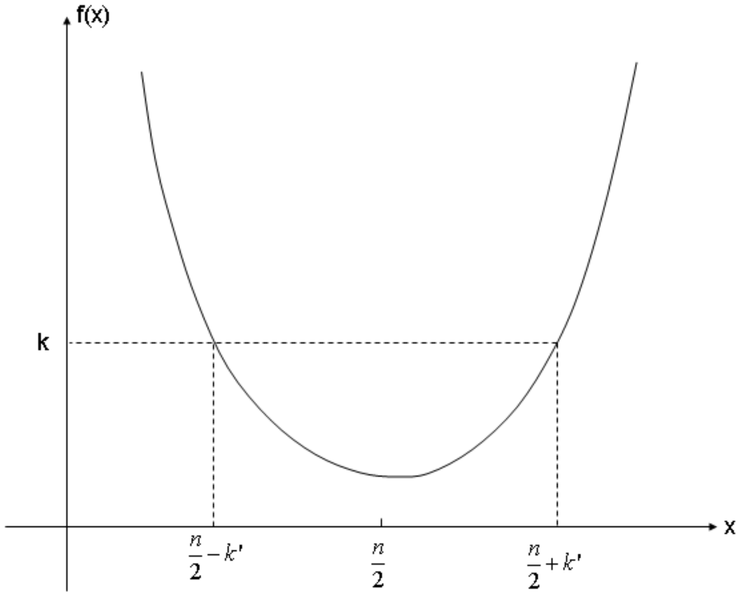
$$\therefore \; f(x) \text{ attains a minimum at } x = \frac{n}{2}$$

Now,

$$\begin{aligned} f(n-x) &= (n-x)\log(n-x) + (n-(n-x))\log(n-(n-x)) \\ &= (n-x)\log(n-x) + x\log x \\ &= f(x) \end{aligned}$$

$$\therefore \; f(x) \text{ is symmetric about } \frac{n}{2}$$

Since $f(x)$ attains the minimum at $x = \frac{n}{2}$ and also symmetric about $x = \frac{n}{2}$, the critical region of this likelihood ratio test $f(x) \geq k$ can also be written as $|x - \frac{n}{2}| \geq k'$ where k' is a constant which depends on the size of the critical region.



18. $X_1, \ldots, X_n$ is a random sample of size n from $N(\mu, \sigma^2)$ where σ^2 is unknown.

$$\begin{cases} H_0: & \mu = \mu_0 \\ H_1: & \mu \neq \mu_0 \end{cases}$$

$$\begin{aligned} \Theta_0 &= \{(\mu, \sigma^2) : \mu = \mu_0, \; \sigma^2 \in \Re^+\} \\ \Theta &= \{(\mu, \sigma^2) : \mu \in \Re, \; \sigma^2 \in \Re^+\} \end{aligned}$$

$$L(\mu, \sigma^2, \underset{\sim}{x}) = f_{\underset{\sim}{X}}(\underset{\sim}{x}; \mu, \sigma^2) = (2\pi\sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right\}$$

The likelihood ratio

$$\lambda(x) = \frac{\sup\{L(\mu, \sigma^2, \underset{\sim}{x}) : (\mu, \sigma^2) \in \Theta_0\}}{\sup\{L(\mu, \sigma^2, \underset{\sim}{x})(\mu, \sigma^2) \in \Theta\}}$$

In order to find the numerator of $\lambda(\underset{\sim}{x})$, we need to find the M.L.E. for μ and σ^2 subject to

$(\mu, \sigma^2) \in \Theta_0$.

$$
\begin{aligned}
\log L_0 &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_0)^2 \\
&= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_0)^2 \\
\frac{\partial}{\partial\sigma^2}\log L_0 &= -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i - \mu_0)^2 \\
\frac{\partial}{\partial\sigma^2}\log L_0 &= 0 \\
\Rightarrow \qquad \hat{\sigma_0^2} &= \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2 \\
\therefore\ \sup\{L(\mu, \sigma^2, \underset{\sim}{x}) : (\mu, \sigma^2) \in \Theta_0\} &= (2\pi\hat{\sigma_0^2})^{-n/2}\exp\left\{-\frac{1}{2\hat{\sigma_0^2}}\sum_{i=1}^{n}(x_i - \mu_0)^2\right\} \\
&= \left(\frac{2\pi}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2\right)^{-n/2} e^{-n/2}
\end{aligned}
$$

Similarly, in order to find the denominator of $\lambda(x)$, we need to find the M.L.E. for μ and σ^2 subject to $(\mu, \sigma^2) \in \Theta$.

$$
\begin{aligned}
\log L &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2 \\
&= -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2
\end{aligned}
$$

$$
\begin{cases}
\dfrac{\partial \log L}{\partial \mu} = -\dfrac{1}{2\sigma^2}(2)\displaystyle\sum_{i=1}^{n}(x_i - \mu)(-1) = \dfrac{1}{\sigma^2}\displaystyle\sum_{i=1}^{n}(x_i - \mu) \\
\dfrac{\partial \log L}{\partial \sigma^2} = -\dfrac{n}{2\sigma^2} + \dfrac{1}{2(\sigma^2)^2}\displaystyle\sum_{i=1}^{n}(x_i - \mu)^2
\end{cases}
$$

$$
\Rightarrow \begin{cases}
\dfrac{\partial \log L}{\partial \mu} = 0 \\
\dfrac{\partial \log L}{\partial \sigma^2} = 0
\end{cases}
$$

$$
\Rightarrow \begin{cases}
\dfrac{1}{\sigma^2}\displaystyle\sum_{i=1}^{n}(x_i - \mu) = 0 \\
-\dfrac{n}{2\sigma^2} + \dfrac{1}{2(\sigma^2)^2}\displaystyle\sum_{i=1}^{n}(x_i - \mu)^2 = 0
\end{cases}
$$

$$
\Rightarrow \begin{cases}
\hat{\mu} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}X_i \\
\hat{\sigma^2} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(X_i - \bar{X})^2 \qquad \text{(M.L.E.)}
\end{cases}
$$

(Remark: the above is a system of equations not just two independent equations.)

$$
\begin{aligned}
\therefore \quad \sup\{L(\mu,\sigma^2,\underset{\sim}{x})(\mu,\sigma^2)\in\Theta\} &= (2\pi\hat{\sigma^2})^{-n/2}\exp\left\{-\frac{1}{2\hat{\sigma^2}}\sum_{i=1}^{n}(x_i-\hat{\mu})^2\right\} \\
&= \left(\frac{2\pi}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{-n/2} e^{-n/2} \\
\therefore \qquad \lambda(\underset{\sim}{x}) &= \left(\frac{\sum_{i=1}^{n}(x_i-\mu_0)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{-n/2} \\
&= \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2+n(\bar{x}-\mu_0)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{-n/2} \\
&= \left(1+\frac{n(\bar{x}-\mu_0)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)^{-n/2} \\
&= \left(1+\frac{n(\bar{x}-\mu_0)^2}{(n-1)S^2}\right)^{-n/2} \qquad \text{where } S^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 \\
&= \left(1+\frac{t^2}{n-1}\right)^{-n/2} \qquad \text{where } t=\frac{\bar{x}-\mu_0}{S/\sqrt{n}}\sim t_{n-1}
\end{aligned}
$$

19. $X_1,\ldots,X_n$ is a random sample of size n from $N(\mu,\sigma^2)$ where μ and σ^2 are unknown.

$$
\begin{cases} H_0: & \sigma=\sigma_0 \\ H_1: & \sigma\neq\sigma_0 \end{cases}
$$

$$
\begin{aligned}
\Theta_0 &= \{(\mu,\sigma):\mu\in\Re,\ \sigma=\sigma_0\} \\
\Theta &= \{(\mu,\sigma):\mu\in\Re,\ \sigma\in\Re^+\}
\end{aligned}
$$

$$
L(\mu,\sigma,\underset{\sim}{x})=(2\pi\sigma^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}
$$

The likelihood ratio

$$
\lambda(x)=\frac{\sup\{L(\mu,\sigma,\underset{\sim}{x}):(\mu,\sigma)\in\Theta_0\}}{\sup\{L(\mu,\sigma,\underset{\sim}{x})(\mu,\sigma)\in\Theta\}}
$$

The find the numerator of $\lambda(\underset{\sim}{x})$, we need $\hat{\mu_0}$.

$$
\begin{aligned}
\log L_0 &= -\frac{n}{2}\log(2\pi)-\frac{n}{2}\log\sigma_0^2-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i-\mu)^2 \\
\frac{\partial\log L_0}{\partial\mu} &= -\frac{1}{2\sigma_0^2}(2)\sum_{i=1}^{n}(x_i-\mu)(-1) \\
&= \frac{1}{\sigma_0^2}\sum_{i=1}^{n}(x_i-\mu) \\
\frac{\partial\log L_0}{\partial\mu}=0 \quad &\Rightarrow \quad \frac{1}{\sigma_0^2}\sum_{i=1}^{n}(x_i-\mu)=0 \\
&\Rightarrow \quad \hat{\mu}_0=\frac{1}{n}\sum_{i=1}^{n}X_i
\end{aligned}
$$

To find the denominator of $\lambda(x)$, we need $\hat{\mu}$ and $\hat{\sigma}$.

(Note: we may use the invariant property of M.L.E. to find $\hat{\sigma}^2$ instead of $\hat{\sigma}$ since the calculation is simpler.)

$$\left\{\begin{array}{rcl}\dfrac{\partial \log L}{\partial \mu} &=& 0\\ \dfrac{\partial \log L}{\partial \sigma^2} &=& 0\end{array}\right. \Rightarrow \left\{\begin{array}{rcl}\hat{\mu} &=& \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i\\ \hat{\sigma}^2 &=& \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(X_i-\bar{X})^2\end{array}\right.$$

$$\lambda(\underset{\sim}{x}) = \frac{(2\pi\sigma_0^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma_0^2}\sum_{i=1}^n(x_i-\hat{\mu}_0)^2\right\}}{(2\pi\hat{\sigma}^2)^{-n/2}\exp\left\{-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^n(x_i-\hat{\mu})^2\right\}} = \frac{\left(\sigma_0^2\right)^{-n/2}\exp\left\{-\dfrac{1}{2\sigma_0^2}\displaystyle\sum_{i=1}^{n}(x_i-\bar{x})^2\right\}}{\left(\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{-n/2}\exp\left\{-\dfrac{n}{2}\right\}}$$

20. Let $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_n)$ be random samples from the independent normal distribution $N(\theta_1, \theta_3)$ and $N(\theta_2, \theta_4)$ respectively.

$$\left\{\begin{array}{ll} H_0: & \theta_1=\theta_2, \quad \theta_3=\theta_4\\ H_1: & \text{otherwise}\end{array}\right.$$

$$\begin{aligned} L(\theta_1,\theta_2,\theta_3,\theta_4,\underset{\sim}{x},\underset{\sim}{y}) &= f(\underset{\sim}{x},\underset{\sim}{y};\theta_1,\theta_2,\theta_3,\theta_4)\\ &= (2\pi\theta_3^2)^{-n/2}\exp\left\{-\frac{1}{2\theta_3^2}\sum_{i=1}^{n}(x_i-\theta_1)^2\right\}(2\pi\theta_4^2)^{-m/2}\exp\left\{-\frac{1}{2\theta_4^2}\sum_{j=1}^{m}(y_j-\theta_2)^2\right\}\end{aligned}$$

The likelihood ratio:

$$\lambda(\underset{\sim}{x},\underset{\sim}{y}) = \frac{\sup\{L(\theta_1,\theta_2,\theta_3,\theta_4,\underset{\sim}{x},\underset{\sim}{y}) : (\theta_1,\theta_2,\theta_3,\theta_4)\in\Theta_0\}}{\sup\{L(\theta_1,\theta_2,\theta_3,\theta_4,\underset{\sim}{x},\underset{\sim}{y}) : (\theta_1,\theta_2,\theta_3,\theta_4)\in\Theta\}}$$

$$\text{where } \Theta_0 = \{(\theta_1,\theta_2,\theta_3,\theta_4) : \theta_1=\theta_2=\mu,\ \theta_3=\theta_4=\sigma^2,\ \mu\in\Re,\ \sigma^2\in\Re^+\}$$
$$\text{and } \Theta = \{(\theta_1,\theta_2,\theta_3,\theta_4) : \theta_1\in\Re,\ \theta_2\in\Re,\ \theta_3\in\Re^+,\ \theta_4\in\Re^+\}$$

Denominator:

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\theta_3 - \frac{1}{2\theta_3}\sum_{i=1}^{n}(x_i-\theta_1)^2 - \frac{m}{2}\log(2\pi) - \frac{m}{2}\log\theta_4 - \frac{1}{2\theta_4}\sum_{j=1}^{m}(y_j-\theta_2)^2$$

$$\begin{cases} \dfrac{\partial \log L}{\partial \theta_1} &= -\dfrac{1}{\theta_3}\sum_{i=1}^{n}(x_i-\theta_1)(-1) &= \dfrac{1}{\theta_3}\sum_{i=1}^{n}(x_i-\theta_1) \\ \dfrac{\partial \log L}{\partial \theta_2} &= -\dfrac{1}{\theta_4}\sum_{j=1}^{m}(y_j-\theta_2)(-1) &= \dfrac{1}{\theta_4}\sum_{j=1}^{m}(y_j-\theta_2) \\ \dfrac{\partial \log L}{\partial \theta_3} &= -\dfrac{n}{2\theta_3}+\dfrac{1}{2\theta_3^2}\sum_{i=1}^{n}(x_i-\theta_1)^2 \\ \dfrac{\partial \log L}{\partial \theta_4} &= -\dfrac{m}{2\theta_4}+\dfrac{1}{2\theta_4^2}\sum_{j=1}^{m}(y_j-\theta_2)^2 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial \log L}{\partial \theta_1} &= 0 \\ \dfrac{\partial \log L}{\partial \theta_2} &= 0 \\ \dfrac{\partial \log L}{\partial \theta_3} &= 0 \\ \dfrac{\partial \log L}{\partial \theta_4} &= 0 \end{cases}$$

$$\Rightarrow \begin{cases} \hat{\theta}_1 &= \dfrac{1}{n}\sum_{i=1}^{n}X_i &= \bar{X} \\ \hat{\theta}_2 &= \dfrac{1}{m}\sum_{j=1}^{m}Y_j &= \bar{Y} \\ \hat{\theta}_3 &= \dfrac{1}{n}\sum_{i=1}^{n}(X_i-\hat{\theta}_1)^2 &= \dfrac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 \\ \hat{\theta}_4 &= \dfrac{1}{m}\sum_{j=1}^{n}(Y_j-\hat{\theta}_2)^2 &= \dfrac{1}{m}\sum_{j=1}^{m}(Y_j-\bar{Y})^2 \end{cases}$$

$\therefore$ The denominator is

$$\left[(2\pi)\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{-n/2}\exp\left\{-\frac{n}{2}\right\}\left[(2\pi)\frac{1}{m}\sum_{j=1}^{m}(y_j-\bar{y})^2\right]^{-m/2}\exp\left\{-\frac{m}{2}\right\}$$

Numerator: $\theta_1=\theta_2=\mu, \theta_3=\theta_4=\sigma^2$

$$\begin{aligned} L_0 &= (2\pi\sigma^2)^{-n/2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}(2\pi\sigma^2)^{-m/2}\exp\left\{-\frac{1}{2\sigma^2}\sum_{j=1}^{m}(y_j-\mu)^2\right\} \\ &= (2\pi\sigma^2)^{-\frac{1}{2}(m+n)}\exp\left\{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\mu)^2+\sum_{j=1}^{m}(y_j-\mu)^2\right]\right\} \\ \log L_0 &= -\frac{1}{2}(m+n)\log(2\pi)-\frac{1}{2}(m+n)\log\sigma^2-\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i-\mu)^2+\sum_{j=1}^{m}(y_j-\mu)^2\right] \end{aligned}$$

$$\begin{cases} \dfrac{\partial \log L_0}{\partial \mu} &= -\dfrac{1}{\sigma^2}\left[\displaystyle\sum_{i=1}^{n}(x_i-\mu)(-1)+\sum_{j=1}^{m}(y_j-\mu)(-1)\right] \\ \dfrac{\partial \log L_0}{\partial \sigma^2} &= -\dfrac{1}{2\sigma^2}(m+n)+\dfrac{1}{2\sigma^4}\left[\displaystyle\sum_{i=1}^{n}(x_i-\mu)^2+\sum_{j=1}^{m}(y_j-\mu)^2\right] \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial \log L_0}{\partial \mu} &= 0 \\ \dfrac{\partial \log L_0}{\partial \sigma^2} &= 0 \end{cases}$$

$$\Rightarrow \begin{cases} \displaystyle\sum_{i=1}^{n} x_i - n\mu + \sum_{j=1}^{m} y_j - m\mu = 0 \\ \hat{\sigma}^2 = \dfrac{1}{m+n}\left[\displaystyle\sum_{i=1}^{n}(X_i-\hat{\mu})^2+\sum_{j=1}^{m}(Y_j-\hat{\mu})^2\right] \end{cases}$$

$$\Rightarrow \begin{cases} \hat{\mu} &= \dfrac{1}{m+n}\left(\displaystyle\sum_{i=1}^{n} X_i + \sum_{j=1}^{m} Y_j\right) = \dfrac{1}{m+n}(n\bar{X}+m\bar{Y}) = u \\ \hat{\sigma}^2 &= \dfrac{1}{m+n}\left[\displaystyle\sum_{i=1}^{n}(X_i-u)^2+\sum_{j=1}^{m}(Y_j-u)^2\right] \qquad \text{where } u = \dfrac{(n\bar{X}+m\bar{Y})}{n+m} \end{cases}$$

$\therefore$ The numerator is

$$\left\{(2\pi)\left(\frac{1}{m+n}\right)\left[\sum_{i=1}^{n}(x_i-u)^2+\sum_{j=1}^{m}(y_j-u)^2\right]\right\}^{-\frac{1}{2}(m+n)} \exp\left\{-\frac{m+n}{2}\right\}$$

$$\therefore \qquad \lambda(\underset{\sim}{x},\underset{\sim}{y}) = \frac{\left\{(2\pi)\left(\frac{1}{m+n}\right)\left[\sum_{i=1}^{n}(x_i-u)^2+\sum_{j=1}^{m}(y_j-u)^2\right]\right\}^{-\frac{1}{2}(m+n)} \exp\left\{-\frac{m+n}{2}\right\}}{\left[(2\pi)\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{-n/2} \exp\left\{-\frac{n}{2}\right\}\cdot\left[(2\pi)\frac{1}{m}\sum_{j=1}^{m}(y_j-\bar{y})^2\right]^{-m/2} \exp\left\{-\frac{m}{2}\right\}}$$

$$= \frac{\left[\sum_{i=1}^{n}(x_i-\bar{x})^2/n\right]^{n/2}\left[\sum_{j=1}^{m}(y_j-\bar{y})^2/m\right]^{m/2}}{\left\{\left[\sum_{i=1}^{n}(x_i-u)^2+\sum_{j=1}^{m}(y_j-u)^2\right]/(m+n)\right\}^{\frac{(m+n)}{2}}} \qquad \text{where } u = \frac{(n\bar{x}+m\bar{y})}{n+m}$$

21.

$$\begin{cases} H_0: & \theta_3 = \theta_4, \quad \theta_1 \text{ and } \theta_2 \text{ unspecified} \\ H_1: & \theta_3 \neq \theta_4, \quad \theta_1 \text{ and } \theta_2 \text{ unspecified} \end{cases}$$

$$\begin{aligned} L(\theta_1,\theta_2,\theta_3,\theta_4,\underset{\sim}{x},\underset{\sim}{y}) &= f(\underset{\sim}{x},\underset{\sim}{y};\theta_1,\theta_2,\theta_3,\theta_4) \\ &= (2\pi\theta_3)^{-n/2}\exp\left\{-\frac{1}{2\theta_3}\sum_{i=1}^{n}(x_i-\theta_1)^2\right\}(2\pi\theta_4)^{-m/2}\exp\left\{-\frac{1}{2\theta_4}\sum_{j=1}^{m}(y_j-\theta_2)^2\right\} \end{aligned}$$

The likelihood ratio

$$\lambda(\underset{\sim}{x}, \underset{\sim}{y}) = \frac{\sup\{L(\theta_1, \theta_2, \theta_3, \theta_4, \underset{\sim}{x}, \underset{\sim}{y}) : (\theta_1, \theta_2, \theta_3, \theta_4) \in \Theta_0\}}{\sup\{L(\theta_1, \theta_2, \theta_3, \theta_4, \underset{\sim}{x}, \underset{\sim}{y}) : (\theta_1, \theta_2, \theta_3, \theta_4) \in \Theta\}}$$

$$\text{where } \Theta_0 = \{(\theta_1, \theta_2, \theta_3, \theta_4) : \theta_3 = \theta_4 = \sigma^2, \theta_1 \in \Re,\ \theta_2 \in \Re,\ \sigma^2 \in \Re^+\}$$

$$\text{and } \Theta = \{(\theta_1, \theta_2, \theta_3, \theta_4) : \theta_1 \in \Re,\ \theta_2 \in \Re,\ \theta_3 \in \Re^+,\ \theta_4 \in \Re^+\}$$

The denominator is same as the denominator of Q20.

Numerator: $\theta_3 = \theta_4 = \sigma^2$

$$\begin{aligned} L_0 &= (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \theta_1)^2\right\} (2\pi\sigma^2)^{-m/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{j=1}^{m}(y_j - \theta_2)^2\right\} \\ \log L_0 &= -\frac{1}{2}(m+n)\log(2\pi) - \frac{1}{2}(m+n)\log\sigma^2 - \frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i - \theta_1)^2 + \sum_{j=1}^{m}(y_j - \theta_2)^2\right] \end{aligned}$$

$$\left\{\begin{aligned} \frac{\partial \log L_0}{\partial \theta_1} &= -\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \theta_1)(-1) \\ \frac{\partial \log L_0}{\partial \theta_2} &= -\frac{1}{\sigma^2}\sum_{j=1}^{m}(y_j - \theta_2)(-1) \\ \frac{\partial \log L_0}{\partial \sigma^2} &= -\frac{n+m}{2\sigma^2} + \frac{1}{2\sigma^4}\left[\sum_{i=1}^{n}(x_i - \theta_1)^2 + \sum_{j=1}^{m}(y_j - \theta_2)^2\right] \end{aligned}\right.$$

$$\Rightarrow \left\{\begin{aligned} \frac{\partial \log L_0}{\partial \theta_1} &= 0 \\ \frac{\partial \log L_0}{\partial \theta_2} &= 0 \\ \frac{\partial \log L_0}{\partial \sigma^2} &= 0 \end{aligned}\right.$$

$$\Rightarrow \left\{\begin{aligned} \hat{\theta_1} &= \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} \\ \hat{\theta_2} &= \frac{1}{m}\sum_{j=1}^{m} Y_j = \bar{Y} \\ \hat{\sigma^2} &= \frac{1}{m+n}\left[\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{j=1}^{m}(Y_j - \bar{Y})^2\right] \end{aligned}\right.$$

$\therefore$ The numerator is

$$\left\{(2\pi)\left(\frac{1}{m+n}\right)\left[\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{j=1}^{m}(y_j - \bar{y})^2\right]\right\}^{-\frac{(m+n)}{2}} \exp\left\{-\frac{m+n}{2}\right\}$$

$$\therefore \qquad \lambda(\underset{\sim}{x}, \underset{\sim}{y}) = \frac{\left\{(2\pi)\left(\frac{1}{m+n}\right)\left[\sum_{i=1}^{n}(x_i-\bar{x})^2+\sum_{j=1}^{m}(y_j-\bar{y})^2\right]\right\}^{-\frac{1}{2}(m+n)} \exp\left\{-\frac{m+n}{2}\right\}}{\left[(2\pi)\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{-n/2}\exp\left\{-\frac{n}{2}\right\}\cdot\left[(2\pi)\frac{1}{m}\sum_{j=1}^{m}(y_j-\bar{y})^2\right]^{-m/2}\exp\left\{-\frac{m}{2}\right\}}$$

$$= \frac{\left\{\frac{1}{m+n}\left[\sum_{i=1}^{n}(x_i-\bar{x})^2+\sum_{j=1}^{m}(y_j-\bar{y})^2\right]\right\}^{-\frac{(m+n)}{2}}}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{-n/2}\left[\frac{1}{m}\sum_{j=1}^{m}(y_j-\bar{y})^2\right]^{-m/2}}$$

H_0 is rejected when $\lambda(\underset{\sim}{x}, \underset{\sim}{y}) \leq k$

$$\frac{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{n/2}\left[\frac{1}{m}\sum_{j=1}^{m}(y_j-\bar{y})^2\right]^{m/2}}{\left\{\frac{1}{m+n}\left[\sum_{i=1}^{n}(x_i-\bar{x})^2+\sum_{j=1}^{m}(y_j-\bar{y})^2\right]\right\}^{\frac{(m+n)}{2}}} \leq k$$

$$\text{iff} \quad \frac{\left[\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{n/2}\left[\sum_{j=1}^{m}(y_j-\bar{y})^2\right]^{m/2}}{\left[\sum_{i=1}^{n}(x_i-\bar{x})^2+\sum_{j=1}^{m}(y_j-\bar{y})^2\right]^{\frac{(m+n)}{2}}} \leq k'$$

$$\text{iff} \quad \frac{\left[\sum_{i=1}^{n}(x_i-\bar{x})^2/\sum_{j=1}^{m}(y_j-\bar{y})^2\right]^{n/2}}{\left[1+\sum_{i=1}^{n}(x_i-\bar{x})^2/\sum_{j=1}^{m}(y_j-\bar{y})^2\right]^{\frac{(m+n)}{2}}} \leq k' \qquad (*)$$

$$\text{iff} \quad \sum_{i=1}^{n}(x_i-\bar{x})^2/\sum_{j=1}^{m}(y_j-\bar{y})^2 \leq k_1 \text{ or } \sum_{i=1}^{n}(x_i-\bar{x})^2/\sum_{j=1}^{m}(y_j-\bar{y})^2 \geq k_2$$

$$\text{iff} \quad F \leq K_1 \text{ or } F \geq K_2 \qquad \text{where } F = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2/(n-1)}{\sum_{j=1}^{m}(y_j-\bar{y})^2/(m-1)}$$

(A closer look on (*): Consider

$$g(z) = \frac{z^{n/2}}{(1+z)^{(m+n)/2}} = \frac{1}{(1+z)^{m/2}(1+\frac{1}{z})^{n/2}} \qquad \text{for } z > 0$$

$\therefore$ when z is small, $(1+\frac{1}{z})^{n/2}$ is large and $(1+z)^{m/2}$ close to 1.

But when z is large, $(1+z)^{m/2}$ is large and $(1+\frac{1}{z})^{n/2}$ close to 1.

Thus, in both cases, $g(z)$ will be large.)

Note that $F \sim F$-distribution with d.f. (n-1) and (m-1).

The critical region at significance level α is:

$$C_1 = \{(\underset{\sim}{x}, \underset{\sim}{y}) : F \le F_{1-\frac{\alpha}{2}}(n-1, m-1) \text{ or } F \ge F_{\frac{\alpha}{2}}(n-1, m-1)\}$$

22. $X_1, \ldots, X_n$ is a random sample of size n from exponential(θ).

$Y_1, \ldots, Y_n$ is another random sample of size n from exponential(μ).

$$\begin{aligned} f(x|\theta) &= \theta e^{-\theta x}, \quad x > 0 \\ f(y|\mu) &= \mu e^{-\mu y}, \quad y > 0 \end{aligned}$$

(a)

$$\begin{cases} H_0: & \theta = \mu \\ H_1: & \theta \ne \mu \end{cases}$$

$$\Theta_0 = \{(\theta, \mu) : \theta = \mu,\ \theta > 0,\ \mu > 0\} \qquad\qquad \Theta = \Re^+ \times \Re^+ = \Re^{+^2}$$

$$\begin{aligned} L(\theta, \mu) &= f_{\underset{\sim}{X}, \underset{\sim}{Y}}(\underset{\sim}{x}, \underset{\sim}{y}; \theta, \mu) \\ &= \left[\prod_{i=1}^n f_{X_i}(x_i; \theta)\right] \left[\prod_{i=1}^n f_{Y_i}(y_i; \mu)\right] \\ &= \theta^n e^{-\theta \sum_{i=1}^n x_i} \cdot \mu^n e^{-\mu \sum_{i=1}^n y_i} \end{aligned}$$

The likelihood ratio:

$$\lambda(\underset{\sim}{x}, \underset{\sim}{y}) = \frac{\sup\{L(\theta, \mu) : (\theta, \mu) \in \Theta_0\}}{\sup\{L(\theta, \mu) : (\theta, \mu) \in \Theta\}}$$

Numerator: $(\theta, \mu) \in \Theta_0, \theta = \mu = \lambda$ where $\lambda > 0$

$$\begin{aligned} L(\theta, \mu) &= L(\lambda, \lambda) \\ &= \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \cdot \lambda^n e^{-\lambda \sum_{i=1}^n y_i} \\ &= \lambda^n e^{-\lambda(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i)} \\ \log L(\lambda, \lambda) &= 2n \log \lambda - \lambda \left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right) \\ \frac{\partial}{\partial \lambda} \log L(\lambda, \lambda) &= \frac{2n}{\lambda} - \left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right) \\ \frac{\partial \log L(\lambda, \lambda)}{\partial \lambda}\Big|_{\lambda = \hat{\lambda}} = 0 &\Rightarrow \hat{\lambda} = (2n)\left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right)^{-1} \\ \sup\{L(\theta, \mu) : (\theta, \mu) \in \Theta_0\} &= \left[(2n)\left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right)^{-1}\right]^{2n} \\ &\quad \cdot \exp\left\{-(2n)\left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right)^{-1} \left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right)\right\} \\ &= \left[(2n)\left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right)^{-1}\right]^{2n} \exp\{-2n\} \end{aligned}$$

Denominator: $(\theta, \mu) \in \Theta$

$$
\begin{aligned}
\log L(\theta,\mu) &= n\log\theta - \theta\sum_{i=1}^n x_i + n\log\mu - \mu\sum_{i=1}^n y_i \\
\frac{\partial}{\partial\theta}\log L(\theta,\mu) &= \frac{n}{\theta} - \sum_{i=1}^n x_i \\
\frac{\partial}{\partial\mu}\log L(\theta,\mu) &= \frac{n}{\mu} - \sum_{i=1}^n y_i
\end{aligned}
$$

$$
\begin{aligned}
& \left\{\begin{array}{rcl}
\frac{\partial}{\partial\theta}\log L(\theta,\mu)|_{(\theta,\mu)=(\hat\theta,\hat\mu)} &=& 0 \\
\frac{\partial}{\partial\mu}\log L(\theta,\mu)|_{(\theta,\mu)=(\hat\theta,\hat\mu)} &=& 0
\end{array}\right. \\
\Rightarrow & \left\{\begin{array}{rcl}
\frac{n}{\theta} - \sum_{i=1}^n x_i &=& 0 \\
\frac{n}{\mu} - \sum_{i=1}^n y_i &=& 0
\end{array}\right. \\
\Rightarrow & \left\{\begin{array}{rcl}
\hat\theta &=& n\left(\sum_{i=1}^n x_i\right)^{-1} \\
\hat\mu &=& n\left(\sum_{i=1}^n y_i\right)^{-1}
\end{array}\right.
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad & \sup\{L(\theta,\mu) : (\theta,\mu)\in\Theta\} \\
&= \left[n\left(\sum_{i=1}^n x_i\right)^{-1}\right]^n \exp\left\{-n\left(\sum_{i=1}^n x_i\right)^{-1}\left(\sum_{i=1}^n x_i\right)\right\} \\
&\quad \cdot \left[n\left(\sum_{i=1}^n y_i\right)^{-1}\right]^n \exp\left\{-n\left(\sum_{i=1}^n y_i\right)^{-1}\left(\sum_{i=1}^n y_i\right)\right\} \\
&= n^{2n}\left(\sum_{i=1}^n x_i\right)^{-n}\left(\sum_{i=1}^n y_i\right)^{-n}\exp\{-2n\}
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \lambda(\underset{\sim}{x},\underset{\sim}{y}) &= \frac{\left[(2n)\left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right)^{-1}\right]^{2n}\exp\{-2n\}}{n^{2n}\left(\sum_{i=1}^n x_i\right)^{-n}\left(\sum_{i=1}^n y_i\right)^{-n}\exp\{-2n\}} \\
&= \frac{2^{2n}\left(\sum_{i=1}^n x_i\right)^{n}\left(\sum_{i=1}^n y_i\right)^{n}}{\left(\sum_{i=1}^n x_i + \sum_{i=1}^n y_i\right)^{2n}}
\end{aligned}
$$

$\therefore$ The critical region of the likelihood ratio test is:

$$C_1 = \left\{ (\underset{\sim}{x}, \underset{\sim}{y}) : \frac{2^{2n}\left(\sum_{i=1}^{n} x_i\right)^n \left(\sum_{i=1}^{n} y_i\right)^n}{\left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i\right)^{2n}} \leq k \right\}$$

(b)

$$\begin{aligned}
& \frac{2^{2n}\left(\sum_{i=1}^{n} x_i\right)^n \left(\sum_{i=1}^{n} y_i\right)^n}{\left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i\right)^{2n}} \leq k \\
\Leftrightarrow \quad & \left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i + \sum_{i=1}^n y_i}\right)^n \left(\frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n x_i + \sum_{i=1}^n y_i}\right)^n \leq k' \\
\Leftrightarrow \quad & \left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i + \sum_{i=1}^n y_i}\right) \left(1 - \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i + \sum_{i=1}^n y_i}\right) \leq k'' \\
\Leftrightarrow \quad & T \leq K_1 \quad \text{or} \quad T \geq K_2 \quad \text{where } T = \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n x_i + \sum_{i=1}^n y_i}
\end{aligned}$$

$\therefore$ The critical region in part (a) could be expressed as

$$C_2 = \{(\underset{\sim}{x}, \underset{\sim}{y}) : t \leq K_1 \quad \text{or} \quad t \geq K_2\}$$

(c) When H_0 is true, then $\theta = \mu = \lambda$ where $\lambda > 0$.
Thus, $2\lambda \sum_{i=1}^n x_i \sim \chi^2_{2n}$ since

$$\begin{aligned}
m_{2\lambda \sum_{i=1}^n x_i}(s) &= E\left(e^{(s)2\lambda \sum_{i=1}^n X_i}\right) \\
&= \prod_{i=1}^{n} E\left(e^{(2\lambda s)X_i}\right) \\
&= \left(\frac{\lambda}{\lambda - 2\lambda s}\right)^n \\
&= (1-2s)^{-n} = (1-2s)^{-2n/2}
\end{aligned}$$

which is the moment generating function of χ^2_{2n}

$$\begin{aligned}
\text{Similarly,} \qquad 2\lambda \sum_{i=1}^{n} Y_i &\sim \chi^2_{2n} \\
\frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i} &= \frac{2\lambda \sum_{i=1}^n Y_i/2n}{2\lambda \sum_{i=1}^n X_i/2n} \sim F_{(2n,2n)} \\
T &= \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i} \\
&= \frac{1}{1 + \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}}
\end{aligned}$$

have a distribution $(1+F)^{-1}$ under H_0 where $F \sim F_{(2n,2n)}$

23. $X_1, X_2, \ldots, X_6 \sim$ Multinomial $(300, \theta_1, \theta_2, \ldots, \theta_6)$

$$\begin{cases} H_0: & \theta_1 = \frac{1}{6}, \theta_2 = \frac{1}{6}, \theta_3 = \frac{1}{6}, \ldots, \theta_6 = \frac{1}{6}, \\ H_1: & \text{otherwise} \end{cases} \qquad \sum_{i=1}^{6} \theta_i = 1$$

The likelihood function is $L(\theta_1, \theta_2, \ldots, \theta_6, \underset{\sim}{x}) = \text{constant} \cdot \prod_{i=1}^{6} \theta_i^{x_i}$

The numerator of the likelihood ratio is $\sup\{L(\theta, \underset{\sim}{x}) : \theta \in \Theta_0\} = \text{constant} \cdot \prod_{i=1}^{6} \left(\frac{1}{6}\right)^{x_i}$

Note that the M.L.E. for θ_i is $\frac{x_i}{n} = \frac{x_i}{300}$

The denominator of the likelihood ratio is $\sup\{L(\theta, \underset{\sim}{x}) : \theta \in \Theta\} = \text{constant} \cdot \prod_{i=1}^{6} \left(\frac{x_i}{300}\right)^{x_i}$

$$\therefore \ \lambda(\underset{\sim}{x}) = \frac{\text{constant} \ \cdot \prod_{i=1}^{6} \left(\frac{1}{6}\right)^{x_i}}{\text{constant} \ \cdot \prod_{i=1}^{6} \left(\frac{x_i}{300}\right)^{x_i}} = \prod_{i=1}^{6} \left(\frac{300}{6x_i}\right)^{x_i}$$

By the likelihood ratio test, we reject H_0 if $\lambda(\underset{\sim}{x}) \le k$ or equivalently $-2\log\lambda(\underset{\sim}{x}) \ge k'$.
Since $n = 300$ which is large, $-2\log\lambda(\underset{\sim}{x}) \approx \chi_r^2$ where $r = (6-1) - 0 = 5$

$$\therefore \ C_1 = \{-2\log\lambda(\underset{\sim}{x}) \ge \chi^2_{5,0.01}\}$$

$$\begin{aligned} \text{Now,} \ -2\log\lambda(\underset{\sim}{x}) &= -2\log\prod_{i=1}^{6}\left(\frac{300}{6x_i}\right)^{x_i} \\ &= 2\sum_{i=1}^{6} x_i \log\left(\frac{x_i}{50}\right) \\ &= 2[-2.908 - 6.485 + 20.909 - 1.959 + 3.088 - 8.136] \\ &= 9.018 \\ \chi^2_{5,0.01} &= 9.24 > 9.018 = -2\log\lambda(\underset{\sim}{x}) \\ & \quad \therefore \ \text{we do not reject } H_0 \end{aligned}$$

Alternatively, $n = 300$ which is large enough, we may use the Pearson's goodness of fit test.

Now, the test statistic

$$\begin{aligned} G &= \sum_{i=1}^{6} \frac{\left(x_i - 300(\frac{1}{6})\right)^2}{300(\frac{1}{6})} \\ &= 0.18 + 0.98 + 6.48 + 0.08 + 0.18 + 1.62 \\ &= 9.52 > 9.24 = \chi^2_{5,0.01} \\ & \quad \therefore \ H_0 \text{ is rejected.} \end{aligned}$$

(Note: It is normally used the same large sample distribution as $-2\log\lambda(\underset{\sim}{x})$ under H_0)

24.

$$\begin{cases} H_0: & \text{accident rate is independent of age in the sample population} \\ H_1: & \text{otherwise} \end{cases}$$

$$H_0 \text{ is rejected if } \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\left(x_{ij} - \frac{a_i b_j}{300}\right)^2}{\frac{a_i b_j}{300}} \geq \chi^2_{(3-1)(3-1),0.05}$$

where a_i is the i^{th} row total and b_j is the j^{th} column total.

Now

$$\begin{aligned} a_1 &= 10 + 21 + 14 = 45 \\ a_2 &= 22 + 43 + 10 = 75 \\ a_3 &= 81 + 80 + 19 = 180 \\ b_1 &= 10 + 22 + 81 = 113 \\ b_2 &= 21 + 43 + 80 = 144 \\ b_3 &= 14 + 10 + 19 = 43 \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{The test statistic is} &= 2.850 + 0.017 + 8.838 + 1.383 \\ &\quad +1.361 + 0.052 + 2.570 + 0.474 + 1.792 \\ &= 19.337 \end{aligned}$$

$$\begin{aligned} \chi^2_{(3-1)(3-1),0.05} &= \chi^2_{4,0.05} = 9.49 \\ \therefore \quad \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\left(x_{ij} - \frac{a_i b_j}{300}\right)^2}{\frac{a_i b_j}{300}} &= 19.337 > 9.49 = \chi^2_{(3-1)(3-1),0.05} \\ &\quad \therefore\ H_0 \text{ is rejected.} \end{aligned}$$

25. For a 2×2 contingency table, the realization of the Pearson statistic is

$$
\begin{aligned}
& n\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{x_{ij}^2}{a_i b_j}-1\right) \\
= & n\left(\frac{x_{11}^2}{a_1b_1}+\frac{x_{12}^2}{a_1b_2}+\frac{x_{21}^2}{a_2b_1}+\frac{x_{22}^2}{a_2b_2}-1\right) \\
= & \frac{n(a_2b_2x_{11}^2+a_2b_1x_{12}^2+a_1b_2x_{21}^2+a_1b_1x_{22}^2-a_1a_2b_1b_2)}{a_1a_2b_1b_2} \\
= & \frac{n}{a_1a_2b_1b_2}\left[a_2b_2x_{11}^2+a_2b_1x_{12}^2+a_1b_2x_{21}^2+a_1b_1(x_{22}^2-a_2b_2)\right] \\
= & \frac{n}{a_1a_2b_1b_2}\left[a_2b_2x_{11}^2+a_2b_1x_{12}^2+a_1b_2x_{21}^2+a_1b_1\Big(x_{22}^2-(x_{21}+x_{22})(x_{12}+x_{22})\Big)\right] \\
= & \frac{n}{a_1a_2b_1b_2}\left[a_2b_2x_{11}^2+a_2b_1x_{12}^2+a_1b_2x_{21}^2+a_1b_1\Big(x_{22}^2-(x_{21}x_{12}+x_{21}x_{22}+x_{22}x_{12}+x_{22}^2)\Big)\right] \\
= & \frac{n}{a_1a_2b_1b_2}\left[a_2b_2x_{11}^2+a_2b_1x_{12}^2+a_1b_2x_{21}^2-a_1b_1(x_{21}x_{12}+x_{21}x_{22}+x_{22}x_{12})\right] \\
= & \frac{n}{a_1a_2b_1b_2}\left[a_2b_2x_{11}^2+a_2b_1x_{12}^2+a_1b_2x_{21}^2-a_1b_1b_2x_{21}-a_1b_1x_{22}x_{12}\right] \\
= & \frac{n}{a_1a_2b_1b_2}\left[a_2b_2x_{11}^2+a_2b_1x_{12}^2+a_1b_2\Big(x_{21}^2-(x_{11}+x_{21})x_{21}\Big)-a_1b_1x_{22}x_{12}\right] \\
= & \frac{n}{a_1a_2b_1b_2}\left[a_2b_2x_{11}^2+a_2b_1x_{12}^2-a_1b_2x_{11}x_{21}-a_1b_1x_{22}x_{12}\right] \\
= & \frac{n}{a_1a_2b_1b_2}\left[b_2x_{11}(a_2x_{11}-a_1x_{21})+b_1x_{12}(a_2x_{12}-a_1x_{22})\right] \\
= & \frac{n}{a_1a_2b_1b_2}\left[b_2x_{11}\Big((x_{21}+x_{22})x_{11}-(x_{11}+x_{12})x_{21}\Big)+b_1x_{12}\Big((x_{21}+x_{22})x_{12}-(x_{11}+x_{12})x_{22}\Big)\right] \\
= & \frac{n}{a_1a_2b_1b_2}\left[b_2x_{11}(x_{11}x_{22}-x_{12}x_{21})+b_1x_{12}(x_{21}x_{12}-x_{11}x_{22})\right] \\
= & \frac{n}{a_1a_2b_1b_2}\left[(x_{11}x_{22}-x_{12}x_{21})\Big((x_{12}+x_{22})x_{11}-(x_{11}+x_{21})\Big)x_{12}\right] \\
= & \frac{n}{a_1a_2b_1b_2}\left[(x_{11}x_{22}-x_{12}x_{21})(x_{11}x_{22}-x_{12}x_{21})\right] \\
= & \frac{n(x_{11}x_{22}-x_{12}x_{21})^2}{a_1a_2b_1b_2}
\end{aligned}
$$

26.

$$
\begin{cases} H_0: & \text{the certain disease is not heritble} \\ H_1: & \text{otherwise} \end{cases}
$$

With Yates' correction, the corrected Pearson statistic is

$$
\begin{aligned}
& \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{(|x_{ij}-\frac{a_ib_j}{30}|-0.5)^2}{\frac{a_ib_j}{30}} \qquad \text{where } a_i \text{ is the } i^{th} \text{ row total and } b_j \text{ is the } j^{th} \text{ row total} \\
= & \frac{(|10-\frac{(15)(13)}{30}|-0.5)^2}{\frac{(15)(13)}{30}}+\frac{(|5-\frac{(15)(17)}{30}|-0.5)^2}{\frac{(15)(17)}{30}}+\frac{(|3-\frac{(15)(13)}{30}|-0.5)^2}{\frac{(15)(13)}{30}}+\frac{(|12-\frac{(15)(17)}{30}|-0.5)^2}{\frac{(15)(17)}{30}} \\
= & 1.385+1.059+1.385+1.059=4.888
\end{aligned}
$$

$$
\chi^2_{(2-1)(2-1),0.01}=\chi^2_{1,0.01}=6.63
$$

$\therefore$ the corrected Pearson statistic $= 4.88 < 6.63 = \chi^2_{(2-1)(2-1),0.01}$

$\therefore$ H_0 is not rejected at significant level 0.01

27.

$$\begin{cases} H_0: & \text{the life time distribution is an exponential distribution with mean 12} \\ H_1: & \text{otherwise} \end{cases}$$

$[a_i, b_i)$	$[0,3)$	$[3,6)$	$[6,9)$	$[9,\infty)$
$P(a_i \le X \le b_i)$	0.2212	0.1723	0.1342	0.4723
expected freq.	44.24	34.46	26.84	94.46

Consider

$$\begin{cases} H_0: & \theta_1 = 0.2112, \theta_2 = 0.1723, \theta_3 = 0.1342, \theta_4 = 0.4723 \\ H_1: & (\theta_1, \theta_2, \theta_3, \theta_4) \text{ takes any value other than } (0.2212, 0.1723, 0.1342, 0.4723) \end{cases}$$

The Pearson's statistic is:

$$\begin{aligned} \sum_{i=1}^{4} \frac{(n_i - 200\theta_{0i})^2}{200\theta_{0i}} &= \frac{(53-44.24)^2}{44.24} + \frac{(42-33.46)^2}{34.46} + \frac{(35-26.84)^2}{26.84} + \frac{(70-94.46)^2}{94.46} \\ &= 1.735 + 1.650 + 2.481 + 6.334 = 12.20 \end{aligned}$$

$$\chi^2_{4-1-0,0.01} = \chi^2_{3,0.01} = 11.34$$

$\therefore$ H_0 is rejected at significant level 0.01

28. The data $x_1, \ldots, x_n$ has been observed and it is known that x_i is a sample from a Poisson distribution with an unknown mean λ_i.

$$\begin{cases} H_0: & \lambda_1 = \ldots = \lambda_n \\ H_1: & \lambda_i's \text{ are not all equal} \end{cases}$$

$$\begin{aligned} \Theta_0 &= \{(\lambda, \lambda, \ldots, \lambda) : \lambda > 0\} \\ \Theta &= \Re^+ \times \Re^+ \times \ldots \times \Re^+ = \Re^{+n} \\ f_{X_i}(x_i; \lambda_i) &= \frac{\lambda_i^{x_i} e^{-\lambda_i}}{x_i!} \end{aligned}$$

$$L(\underset{\sim}{\lambda}) = f_{\underset{\sim}{X}}(\underset{\sim}{x}; \lambda_1, \ldots, \lambda_n) = \prod_{i=1}^{n} f_{X_i}(x_i; \lambda_i) = \frac{\prod_{i=1}^{n} (\lambda_i^{x_i}) e^{-\sum_{i=1}^{n} \lambda_i}}{\prod_{i=1}^{n} x_i!}$$

The likelihood ratio:

$$\lambda(\underset{\sim}{x}) = \frac{\sup\{L(\underset{\sim}{\lambda}) : \underset{\sim}{\lambda} \in \Theta_0\}}{\sup\{L(\underset{\sim}{\lambda}) : \underset{\sim}{\lambda} \in \Theta\}}$$

Numerator: $\underset{\sim}{\lambda} \in \Theta_0, \lambda_1 = \lambda_2 = \ldots = \lambda_n = \lambda$

$$\begin{aligned} L(\underset{\sim}{\lambda}) &= (\lambda)^{\sum_{i=1}^{n} x_i} e^{-n\lambda} / \prod_{i=1}^{n} x_i! \\ \log L(\underset{\sim}{\lambda}) &= \sum_{i=1}^{n} x_i \log \lambda - n\lambda - \log \prod_{i=1}^{n} x_i! \\ \frac{\partial \log L(\underset{\sim}{\lambda})}{\partial \lambda} &= \frac{\sum_{i=1}^{n} x_i}{\lambda} - n \\ \frac{\partial \log L(\underset{\sim}{\lambda})}{\partial \lambda}\Big|_{\lambda=\hat{\lambda}} = 0 \quad &\Rightarrow \quad \hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i = \bar{x} \\ \sup\{L(\underset{\sim}{\lambda}) : \underset{\sim}{\lambda} \in \Theta_0\} &= (\bar{x})^{n\bar{x}} e^{-n\bar{x}} / \prod_{i=1}^{n} x_i! \end{aligned}$$

Denominator: $\underset{\sim}{\lambda} \in \Theta$

$$\begin{aligned}
\log L(\underset{\sim}{\lambda}) &= \sum_{i=1}^{n} x_i \log \lambda_i - \sum_{i=1}^{n} \lambda_i - \log \prod_{i=1}^{n} x_i! \\
\frac{\partial \log L(\underset{\sim}{\lambda})}{\partial \lambda_i} &= \frac{x_i}{\lambda_i} - 1 \qquad \text{for } i = 1, 2, \ldots, n \\
\frac{\partial \log L(\underset{\sim}{\lambda})}{\partial \lambda_i}\Big|_{\underset{\sim}{\lambda}=\hat{\underset{\sim}{\lambda}}} = 0 &\Rightarrow \left\{ \begin{array}{rcl} \frac{x_1}{\lambda_1} - 1 &=& 0 \\ &\vdots& \\ \frac{x_n}{\lambda_n} - 1 &=& 0 \end{array} \right. \\
&\Rightarrow \left\{ \begin{array}{rcl} \hat{\lambda}_1 &=& x_1 \\ &\vdots& \\ \hat{\lambda}_n &=& x_n \end{array} \right. \\
\therefore \qquad \sup\{L(\underset{\sim}{\lambda}) : \underset{\sim}{\lambda} \in \Theta\} &= \frac{\prod_{i=1}^{n} x_i^{x_i} e^{-\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \\
\therefore \qquad \lambda(\underset{\sim}{x}) &= \frac{(\bar{x})^{n\bar{x}} e^{-n\bar{x}}}{\prod_{i=1}^{n} x_i^{x_i} e^{-\sum_{i=1}^{n} x_i}} \\
&= \frac{(\bar{x})^{n\bar{x}}}{\prod_{i=1}^{n} x_i^{x_i}}
\end{aligned}$$

For large n, $-2\log\lambda(\underset{\sim}{x}) \approx \chi^2_r$ where r = number of free parameters in Θ - number of free parameters in $\Theta_0 = n - 1$.

$\therefore$ The approximate large sample likelihood ratio test is to reject H_0 when $-2\log\lambda(\underset{\sim}{x}) \geq \chi^2_{n-1}(\alpha)$

$$\therefore \ -2\left(n\bar{x}\log\bar{x} - \sum_{i=1}^{n} x_i \log x_i\right) \geq \chi^2_{n-1}(\alpha)$$

For data (3,4,1,6,5), n=5

$$\bar{x} = (3+4+1+6+5) = \frac{19}{5} = 3.8$$

$$\begin{aligned}
-2(5(3.8)\log(3.8) - 3\log 3 - 4\log 4 - 1\log 1 - 6\log 6 - 5\log 5) &= 4.55 \\
&= 9.488 = \chi^2_{5-1}(0.05)
\end{aligned}$$

$$\therefore \ H_0 \text{ is not rejected at significance level } \alpha = 0.05$$