# MATH3423 Statistical Inference - Final Examination (Fall 11/12)

Time allowed: Two hours

There are four questions in this paper, each with 15 marks.

Answer all questions.

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Name: __________________

Student ID: ________________

1. Let $X_1, ..., X_n$ be a random sample from a Poisson distribution with mean parameter $\theta$.

   (a) (**2 marks**) Find the complete and sufficient statistic for $\theta$. What is its distribution?

   (b) (**3 marks**) Find the UMVUE of $\theta^3$. Is its variance equal to CRLB of all unbiased estimators for $\theta^3$? Explain.

   (c) (**5 marks**) Find the UMVUE of $P(X_1 = 0)$. Find the CRLB of all unbiased estimators for $P(X_1 = 0)$.

   (d) (**3 marks**) Find the UMVUE of $P(X_1 = 1)$.

   (e) (**2 marks**) Hence or otherwise, find the UMVUE of $P(X_1 > 1)$.

2. Let $X_1, ..., X_n$ be a random sample from the exponential distribution with parameter $\theta$, where the probability density function is

$$f(x) = \theta e^{-\theta x}$$

   (a) (**2 marks**) Find the complete and sufficient statistic for $\theta$. What is its distribution?

   (b) (**3 marks**) Find the UMVUE of $\theta$.

   (c) (**5 marks**) Find the MLE and UMVUE of $P(X_1 > a)$.

   (d) (**3 marks**) Find the CRLB of all unbiased estimators for $P(X_1 > a)$. Is the variance of the UMVUE in (c) equal to CRLB of all unbiased estimators for $P(X_1 > a)$? Explain.

   (e) (**2 marks**) What is the UMVUE of $P(X_1 > b | X_1 > a)$ where $b > a$?

3. Let $X_1, ..., X_n$ be a random sample from the $N(\mu, \sigma^2)$. **Assume $\mu = 0$ for parts (a) and (b).**

   (a) (**3 marks**) Find the UMP test for $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 < \sigma_0^2$ at significant level $\alpha$.

   (b) (**4 marks**) For the UMP test in part (a), find the power function $Q(\sigma^2)$. Express it in terms of $P(\chi_a^2 \leq b)$ for some constants of $a$ and $b$. Hence or otherwise, show that the UMP test in part (a) is also the UMP test for $H_0 : \sigma^2 \geq \sigma_0^2$ versus $H_1 : \sigma^2 < \sigma_0^2$ at the level of significance $\alpha$.

   (c) Consider another hypothesis testing problem with $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$ at the level of significance $\alpha$. **(Remark: $\mu \neq 0$ in this part.)**

      i. (**3 marks**) Find the expression of the likelihood ratio statistic.

      ii. (**5 marks**) Show that the critical region of this likelihood ratio test can be written as
      $$\{T(X_1, X_2, \ldots, X_n) : T(X_1, X_2, \ldots, X_n) \leq K_1 \quad \text{or} \quad T(X_1, X_2, \ldots, X_n) \geq K_2\}$$
      where $T(X_1, X_2, \ldots, X_n)$ is a function of data and $K_1$ and $K_2$ are constants which depend on the size of the critical region. Then, find the constants $K_1$ and $K_2$ by setting $P(T(X_1, X_2, \ldots, X_n) \leq K_1 | H_0) = P(T(X_1, X_2, \ldots, X_n) \geq K_2 | H_0)$.

4. (a) Let $Y_1, ..., Y_n$ be a random sample from the $Bin(1, \theta)$. Consider the hypothesis $H_0 : \theta = 0.5$ versus $H_1 : \theta \neq 0.5$ at the level of significance $\alpha$.

      i. (**1 mark**) Find an expression of the likelihood ratio statistic.

      ii. (**2 marks**) Show that the critical region of this likelihood ratio test can be written as $\{T(X_1, X_2, \ldots, X_n) : |T(X_1, X_2, \ldots, X_n) - 0.5| > C\}$.

      iii. (**3 marks**) Using central limit theorem, find the constant $C$ and state the critical region for this test.

   (b) Let $X = (X_1, X_2)$, where $X_i$ denoted as the number of occurrences, have multinomial distribution with parameters $n, p_1, p_2$. Consider the hypothesis $H_0 : p_1 = p_2$ versus $H_1 : p_1 \neq p_2$ at the level of significance $\alpha$.

      i. (**4 marks**) Find the likelihood ratio statistic and then derive the approximate large sample likelihood ratio test.

      ii. (**2 marks**) Write down the Pearson's goodness of fit test statistic and state the critical region for this test.

   (c) (**3 marks**) Suppose there are 55 males and 45 females in a sample. Using the three tests derived above to test the hypothesis that the proportions of males and females are equal at $\alpha = 0.05$.

   (d) (**Bonus: 4 marks**) Show that the tests from part (a) and part (b)(ii) are equivalent.

********** **END** **********