

Solutions to Exercise 2

1.

$$E(\tilde{\theta}_1) = E\left(\frac{1}{n+1} \sum_{i=1}^n X_i\right) = \frac{1}{n+1} \sum_{i=1}^n E(X_i) = \frac{1}{n+1} n\theta \neq \theta$$

$\therefore \tilde{\theta}_1$ is a biased estimator for θ .

2. (a)

$$\begin{aligned} L &= f_x(x; \theta) \\ &= \prod_{i=1}^n f_{x_i}(x_i; \theta) \quad x_i \in (0, \infty) \\ &= \prod_{i=1}^n \frac{\beta}{\theta} x_i^{\beta-1} \exp\left(-\frac{x_i^\beta}{\theta}\right) \\ \log L &= n \log \frac{\beta}{\theta} + (\beta - 1) \sum_{i=1}^n \log x_i - \frac{1}{\theta} \sum_{i=1}^n x_i^\beta \\ \frac{\partial \log L}{\partial \theta} &= \frac{-n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n x_i^\beta \end{aligned}$$

$$\text{Set } \frac{\partial \log L}{\partial \theta} = 0,$$

$$\Rightarrow \frac{-n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n x_i^\beta = 0$$

$$\Rightarrow \hat{\theta} = \frac{1}{n} \sum_{i=1}^n x_i^\beta$$

$$(b) \text{ Let } Y_i = X_i^\beta \Rightarrow \frac{dy}{dx} = \beta X^{\beta-1}$$

$$\begin{aligned} \therefore f_Y(y; \theta) &= f_X(x; \theta) \left| \frac{dx}{dy} \right| \\ &= f_x(\sqrt[\beta]{y}; \theta) |\beta x^{\beta-1}|^{-1} \\ &= \frac{\beta}{\theta} (x)^{\beta-1} \exp\left(-\frac{(\sqrt[\beta]{y})^\beta}{\theta}\right) \cdot |\beta X^{\beta-1}|^{-1} \\ &= \frac{1}{\theta} \exp\left(-\frac{y}{\theta}\right), \quad y \in (0, \infty) \end{aligned}$$

which is the pdf of exponential distribution with parameter $\frac{1}{\theta}$.

$$\therefore E(\hat{\theta}) = E\left(\frac{1}{n} \sum_{i=1}^n X_i^\beta\right) = \frac{1}{n} \sum_{i=1}^n E(X_i^\beta) = \frac{1}{n} E(Y_i) = \frac{1}{n} \cdot n \cdot \theta = \theta$$

$\therefore \hat{\theta}$ is unbiased for θ .

$\therefore Y \sim \text{Exponential}(\frac{1}{\theta})$

$$\therefore E(Y) = \theta, \quad \text{Var}(Y) = \theta^2$$

$$\begin{aligned} \text{Var}(\hat{\theta}) &= \text{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i^\beta\right) \\ &= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i^\beta) \\ &= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(Y_i) \\ &= \frac{1}{n^2} \cdot n \cdot \theta^2 \\ &= \frac{\theta^2}{n} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} \text{Var}(\hat{\theta}) = \lim_{n \rightarrow \infty} \frac{\theta^2}{n} = 0$$

$\therefore \hat{\theta}$ is consistent for θ .

3. (a)

$$\begin{aligned} M'_1 &= \frac{1}{n} \sum_{i=1}^n X_i = E(\tilde{x} = \frac{1}{\tilde{p}} = \bar{x}) \\ \therefore \tilde{p} &= \frac{1}{\bar{x}} \end{aligned}$$

(b) Note that \bar{x} is the sample average of how many trials required to get the first success, so intuitively, probability of success should be approximately equal to $\frac{1}{\bar{x}}$.

$$(c) \bar{x} = \frac{1}{20}(3 + 34 + 7 + \dots + 21 + 15 + 16) = \frac{252}{20} = 12.6$$

$$\therefore \tilde{p} = \frac{1}{\bar{x}} = \frac{1}{12.6} = 0.0794$$

4. (a)

$$\begin{aligned} M'_1 &= \frac{1}{n} \sum_{i=1}^n X_i = E(\tilde{x}) = \frac{(\tilde{\theta} - 1) + (\tilde{\theta} + 1)}{2} = \tilde{\theta} = \bar{x} \\ \therefore \tilde{\theta} &= \bar{x} \end{aligned}$$

(b) $E(\bar{x}) = E(x) = \theta$ therefore \bar{x} is unbiased estimator for θ .

$$(c) \tilde{\theta} = \bar{x} = \frac{1}{5}(6.61 + \dots + 7.26) = 7.382$$

$$(d) \frac{1}{2}[\min(X_i) + \max(X_i)] = \frac{1}{2}(6.61 + 8.36) = 7.485$$

5.

$$\begin{aligned} E(Y) &= E\left[\frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2\right] \\ &= \frac{1}{n} E\left(\sum_{i=1}^n (X_i - \mu)^2\right) \\ &= \frac{1}{n} \cdot n \cdot \text{Var}(X_i) \\ &= \theta \end{aligned}$$

$\therefore Y$ is unbiased estimator for θ .

6. (a)

$$M'_1 = \frac{1}{n} \sum_{i=1}^n X_i = \widetilde{E(X)} = \tilde{\lambda}$$

$$\therefore \tilde{\lambda} = \bar{X}$$

$$(b) \tilde{\lambda} = \bar{X} = \frac{1}{11}(1 + 0 + \dots + 1 + 1) = \frac{18}{11} = 1.636$$

$$(c) \bar{X} = 1.636$$

$$S_{n-1}^2 = \frac{1}{11-1} \sum_{i=1}^1 1_{i=1} (X_i - \bar{X})^2 = \frac{1}{10} [(1 - 1.636)^2 + \dots + (1 - 1.636)^2] = 1.6545$$

Since the difference between \bar{X} and S_{n-1}^2 is quite small ($\bar{X} \approx S_{n-1}^2$), this information supports the assumption that $X \sim \text{Poisson}$.

7. $X \sim \text{Hypergeometric Distribution}$ with $n_1 =$ size of population 1 (orange balls), $N =$ total population (orange and blue balls) = 64, $r =$ sample size = 8

$$M'_1 = \frac{1}{30} \sum_{i=1}^{30} X_i = 8 \left(\frac{\tilde{n}_1}{64} \right) = \frac{\tilde{n}_1}{8}$$

$$\therefore \tilde{n}_1 = 8\bar{X} = \frac{8}{30}(3 + 0 + \dots + 1 + 2) = 11.73,$$

\therefore we guess the value of n_1 is 12 by the method of moments.

8.

$$L = f_{\tilde{X}}(\tilde{X}; \theta)$$

$$= \prod_{i=1}^n \frac{1}{\sqrt{2\pi\theta}} \exp\left\{-\frac{1}{2\theta}(X_i - \mu)^2\right\}$$

$$\log L = -\frac{n}{2} \log 2\pi - \frac{n}{2} \log \theta - \frac{1}{2\theta} \sum_{i=1}^n (X_i - \mu)^2$$

$$\frac{\partial}{\partial \theta} \log L = 0 \Rightarrow -\frac{n}{2\theta} + \frac{1}{2\theta^2} \sum_{i=1}^n (X_i - \mu)^2 = 0$$

$$\Rightarrow \hat{\theta} = \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2 \quad \text{which is MLE for } \theta$$

9. (a)

$$f(x; \theta) = \frac{1}{\theta^2} x \exp\left\{-\frac{x}{\theta}\right\}, 0 < x < \infty, 0 < \theta < \infty$$

$$L = \prod_{i=1}^n f(x_i; \theta)$$

$$= \prod_{i=1}^n \frac{x_i}{\theta^2} \exp\left\{-\frac{x_i}{\theta}\right\}$$

$$= \left(\frac{1}{\theta}\right)^{2n} \prod_{i=1}^n x_i \exp\left\{-\sum_{i=1}^n \frac{x_i}{\theta}\right\}$$

$$\log L = -2n \log \theta + \sum_{i=1}^n \log x_i - \sum \frac{x_i}{\theta}$$

$$\frac{\partial}{\partial \theta} \log L = -\frac{2n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n x_i = 0$$

$$\Rightarrow \hat{\theta} = \frac{1}{2n} \sum_{i=1}^n x_i \quad \text{which is MLE for } \theta$$

(b)

$$\begin{aligned} f(x; \theta) &= \frac{1}{2\theta^3} x^2 \exp\left\{-\frac{x}{\theta}\right\}, 0 < x < \infty, 0 < \theta < \infty \\ L &= \prod_{i=1}^n f(x_i; \theta) \\ &= \prod_{i=1}^n \frac{1}{2\theta^3} x_i^2 \exp\left\{-\frac{x_i}{\theta}\right\} \\ &= \left(\frac{1}{\theta^3}\right)^n \prod_{i=1}^n x_i \exp\left\{-\sum_{i=1}^n \frac{x_i}{\theta}\right\} \\ \log L &= -n \log 2 - 3n \log \theta + \sum_{i=1}^n \log x_i^2 - \sum_{i=1}^n \frac{x_i}{\theta} \\ \frac{\partial}{\partial \theta} \log L &= \frac{-3n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n x_i = 0 \\ \Rightarrow \hat{\theta} &= \frac{1}{3n} \sum_{i=1}^n x_i \quad \text{which is MLE for } \theta \end{aligned}$$

(c)

$$\begin{aligned} f(x; \theta) &= \frac{1}{2} \exp^{-|x-\theta|}, -\infty < x < \infty, -\infty < \theta < \infty \\ L &= \prod_{i=1}^n f(x_i; \theta) \\ &= \prod_{i=1}^n \frac{1}{2} \exp^{-|x_i-\theta|} \\ &= \frac{1}{2^n} \exp\left(-\sum_{i=1}^n |x_i - \theta|\right) \\ \log L &= -n \log 2 - \sum_{i=1}^n |x_i - \theta| \end{aligned}$$

In order to maximize L, we should maximize $-\sum_{i=1}^n |x_i - \theta|$, i.e. we want to minimize $\sum_{i=1}^n |x_i - \theta|$.

Since

$$\frac{d|x_i - \theta|}{d\theta} = \begin{cases} 1 & x_i < \theta \\ -1 & x_i > \theta \end{cases},$$

Therefore $\frac{d \log L}{d\theta} = 0$ if there are as equal number of observations above θ and below θ .

So $\hat{\theta}$ should be equal to the median of (X_1, \dots, X_n) .

10. (a)

$$\begin{aligned} f(x; \theta) &= \frac{1}{\theta} x^{\frac{1-\theta}{\theta}}, \quad 0 < x < 1, 0 < \theta < \infty \\ L &= \prod_{i=1}^n f(x_i; \theta) \end{aligned}$$

$$\begin{aligned}
&= \prod_{i=1}^n \frac{1}{\theta} x_i^{\frac{1-\theta}{\theta}} \\
&= \left(\frac{1}{\theta}\right)^n \prod_{i=1}^n x_i^{\frac{1-\theta}{\theta}} \\
\log L &= n \log\left(\frac{1}{\theta}\right) + \frac{1-\theta}{\theta} \sum_{i=1}^n \log x_i \\
\frac{\partial}{\partial \theta} \log L &= \frac{-n}{\theta} - \frac{1}{\theta^2} \sum_{i=1}^n \log x_i \\
\frac{\partial}{\partial \theta} \log L = 0 &\Rightarrow \frac{n}{\theta} = -\frac{1}{\theta^2} \sum_{i=1}^n \log x_i \\
&\Rightarrow \hat{\theta} = \frac{-1}{n} \sum_{i=1}^n \log x_i \quad \text{which is MLE of } \theta
\end{aligned}$$

(b) Let $Y = -\log X, dy/dx = -1/X, x = e^{-y}$

$$\begin{aligned}
f_Y(y; \theta) &= f_X(e^{-y}; \theta) \cdot |-1/x|^{-1} \\
&= \frac{1}{\theta} e^{-y(\frac{1-\theta}{\theta})} \cdot \left|\frac{-1}{e^{-y}}\right|^{-1} \\
&= \frac{1}{\theta} e^{-y(\frac{1-\theta}{\theta})} \cdot e^{-y} \\
&= \frac{1}{\theta} e^{-y/\theta}
\end{aligned}$$

$\therefore Y \sim \exp(\frac{1}{\theta}),$

$$E(\hat{\theta}) = E\left(\frac{-1}{n} \sum_{i=1}^n \log X_i\right) = E\left(\frac{1}{n} \sum_{i=1}^n Y_i\right) = \frac{1}{n} \cdot nE(Y_i) = E(Y_i) = \theta$$

$\therefore \hat{\theta}$ is an unbiased estimator for θ .

(Remark: for part b, same technique as Q.2b, please see back !)

11. Since $X \sim \text{exponential}(\frac{1}{\theta})$,

$$E(\bar{X}) = E\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n} \sum_{i=1}^n E(X_i) = \frac{1}{n} \cdot (n\theta) = \theta$$

$\therefore \bar{X}$ is an unbiased estimator for θ .

$$Var(\bar{X}) = Var\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n^2} \sum_{i=1}^n Var(X_i) = \frac{1}{n^2} \cdot (n\theta^2) = \frac{\theta^2}{n}$$

12.

$$E(X_i^2) = Var(X_i) + [E(X_i)]^2 = \theta + 0^2 = \theta$$

$$\therefore E\left(\frac{1}{n} \sum_{i=1}^n X_i^2\right) = \frac{1}{n} \sum_{i=1}^n E(X_i^2) = \frac{1}{n} \cdot (n\theta) = \theta$$

$\therefore \frac{1}{n} \sum_{i=1}^n X_i^2$ is an unbiased estimator for θ .

Since $X_i \sim iid N(0, \theta)$,

$$\frac{X_i - 0}{\sqrt{\theta}} \sim iid N(0, 1)$$

$$\Rightarrow \frac{X_i^2}{\theta} \sim iid \chi_{(1)}^2$$

$$\Rightarrow Var\left(\frac{X_i^2}{\theta}\right) = 2 \times 1 = 2$$

$$\Rightarrow Var(X_i^2) = 2\theta^2$$

$$\therefore Var\left(\frac{1}{n} \sum_{i=1}^n X_i^2\right) = \frac{1}{n^2} Var\left(\sum_{i=1}^n X_i^2\right) = \frac{1}{n^2} \sum_{i=1}^n Var(X_i^2) = \frac{1}{n^2} \cdot n Var(X_i^2) = \frac{1}{n} \cdot 2\theta^2 = \frac{2\theta^2}{n}$$

13. $Var(Y_1) = 2Var(Y_2)$

Let $Y = k_1 Y_1 + k_2 Y_2$.

Since Y is unbiased for θ ,

$$\Rightarrow E(Y) = \theta$$

$$\Rightarrow E(k_1 Y_1 + k_2 Y_2) = \theta$$

$$\Rightarrow k_1 E(Y_1) + k_2 E(Y_2) = \theta$$

$$\Rightarrow k_1 \theta + k_2 \theta = \theta, \quad \because E(Y_1) = E(Y_2) = \theta$$

$$\Rightarrow k_1 + k_2 = 1$$

$$\begin{aligned} \therefore Var(Y) &= Var(k_1 Y_1 + k_2 Y_2) \\ &= k_1^2 Var(Y_1) + k_2^2 Var(Y_2) \\ &= 2k_1^2 Var(Y_2) + k_2^2 Var(Y_2), \quad \text{since } Var(Y_1) = 2Var(Y_2) \\ &= (2k_1^2 + k_2^2) Var(Y_2) \\ &= [2k_1^2 + (1 - k_1)^2] Var(Y_2), \quad \text{since } k_1 + k_2 = 1 \\ &= (3k_1^2 - 2k_1 + 1) Var(Y_2) \end{aligned}$$

We want to minimize $Var(Y)$ which is equivalent to minimize $g(k_1) = 3k_1^2 - 2k_1 + 1$

$$g'(k_1) = 6k_1 - 2$$

$$g'(k_1) = 0 \Rightarrow k_1 = 1/3$$

$$k_2 = 1 - 1/3 = 2/3$$

$$\therefore Y = \frac{1}{3} Y_1 + \frac{2}{3} Y_2$$

14.

$$X \sim Bin(n; \theta), \quad E(X) = n\theta, \quad Var(X) = n\theta(1 - \theta)$$

$$\begin{aligned}
E\left(n\frac{X}{n}\left(1 - \frac{X}{n}\right)\right) &= E\left(X - \frac{X^2}{n}\right) \\
&= E(X) - \frac{1}{n}E(X^2) \\
&= n\theta - \frac{1}{n}[Var(X) + [E(X)]^2] \\
&= n\theta - \frac{1}{n}[n\theta(1 - \theta) + (n\theta)^2] \\
&= n\theta - \theta(1 - \theta) - n\theta^2 \\
&= \theta(n - 1 + \theta - n\theta) \\
&= (n - 1)\theta(1 - \theta) \\
&\neq n\theta(1 - \theta) = Var(X)
\end{aligned}$$

$\therefore n(\frac{X}{n})(1 - \frac{X}{n})$ is biased for $Var(X) = n\theta(1 - \theta)$

15.

$$E\left(\frac{X_1 + 2X_2 + X_3}{4}\right) = \frac{1}{4}[E(X_1) + 2E(X_2) + E(X_3)] = \frac{1}{4}(\mu + 2\mu + \mu) = \mu$$

$\therefore \frac{X_1 + 2X_2 + X_3}{4}$ is unbiased for μ .

$$E\left(\frac{X_1 + X_2 + X_3}{3}\right) = \frac{1}{3}[E(X_1) + E(X_2) + E(X_3)] = \frac{1}{3}(\mu + \mu + \mu) = \mu$$

$\therefore \frac{X_1 + X_2 + X_3}{3}$ is unbiased for μ .

$$Var\left(\frac{X_1 + 2X_2 + X_3}{4}\right) = \frac{1}{16}[Var(X_1) + 4Var(X_2) + Var(X_3)] = \frac{1}{16}(\sigma^2 + 4\sigma^2 + \sigma^2) = \frac{3}{8}\sigma^2$$

$$Var\left(\frac{X_1 + X_2 + X_3}{3}\right) = \frac{1}{9}[Var(X_1) + Var(X_2) + Var(X_3)] = \frac{1}{9}(\sigma^2 + \sigma^2 + \sigma^2) = \frac{1}{3}\sigma^2$$

Therefore the efficiency of $\frac{X_1 + 2X_2 + X_3}{4}$ relative to $\frac{X_1 + X_2 + X_3}{3}$ is

$$\frac{Var\left(\frac{X_1 + X_2 + X_3}{3}\right)}{Var\left(\frac{X_1 + 2X_2 + X_3}{4}\right)} = \frac{\frac{1}{3}\sigma^2}{\frac{3}{8}\sigma^2} = \frac{8}{9}$$

16. (a)

$$Var(\hat{\theta}_1) = Var\left(\frac{X}{n}\right) = \frac{1}{n^2}Var(X) = \frac{1}{n^2} \cdot n\theta(1 - \theta) = \frac{\theta(1 - \theta)}{n} = \frac{1}{4n} \quad (\text{since } \theta = \frac{1}{2})$$

$$Var(\hat{\theta}_2) = Var\left(\frac{X + 1}{n + 2}\right) = Var\left(\frac{X}{n + 2}\right) = \frac{1}{(n + 2)^2} \cdot n\theta(1 - \theta) = \frac{n\theta(1 - \theta)}{(n + 2)^2}$$

$$\begin{aligned}
E \left[(\hat{\theta}_2 - \theta)^2 \right] &= \text{Var}(\hat{\theta}_2 - \theta) + E \left[(\hat{\theta}_2 - \theta) \right]^2 \\
&= \text{Var}(\hat{\theta}_2) + E \left[(\hat{\theta}_2 - \theta) \right]^2 \quad (\because \theta \text{ is constant}) \\
&= \frac{n\theta(1-\theta)}{(n+2)^2} + \left(\frac{n\theta+1}{n+2} - \theta \right)^2 \\
&= \frac{n\theta(1-\theta)}{(n+2)^2} + \frac{(1-2\theta)^2}{(n+2)^2} \\
&= \frac{1}{(n+2)^2} [n\theta - n\theta^2 + 1 - 4\theta + 4\theta^2] \\
&= \frac{1}{(n+2)^2} \times \frac{n}{4} \quad (\text{since } \theta = \frac{1}{2})
\end{aligned}$$

Since $E \left[(\hat{\theta}_2 - \theta)^2 \right] < \text{Var}(\hat{\theta}_1)$, we have

$$\frac{n}{4(n+2)^2} < \frac{1}{4n}$$

$$\Rightarrow n^2 < (n+2)^2$$

$$\Rightarrow n^2 < n^2 + 4n + 4$$

$$\Rightarrow 4n + 4 > 0$$

$$\Rightarrow n > -1$$

(b)

$$E \left[(\hat{\theta}_3 - \theta)^2 \right] = \left[\left(\frac{1}{3} - \frac{1}{2} \right)^2 \right] = \frac{1}{36}$$

$$\therefore E \left[(\hat{\theta}_3 - \theta)^2 \right] < \text{Var}(\hat{\theta}_1)$$

$$\Rightarrow \frac{1}{36} < \frac{1}{4n}$$

$$\Rightarrow n < 9$$

17.

$$M'_1 = \frac{1}{n} \sum_{i=1}^n X_i = \widetilde{E(X)} = \tilde{\lambda}$$

$$\Rightarrow \tilde{\lambda} = \bar{X}$$

18. The n independent measurements of the radius r of the circle are

$$R_1, R_2, \dots, R_n \sim iid N(r, \sigma^2)$$

If we can find an unbiased estimator of r^2 , we can find an unbiased estimator of πr^2 (area).

Observed that

$$\begin{aligned} E \left[\left(\frac{1}{n} \sum_{i=1}^n R_i \right)^2 \right] &= E(\bar{R}^2) \\ &= Var(\bar{R}) + E[(\bar{R})]^2 \\ &= \frac{Var(R)}{n} + E[(R)]^2 \\ &= \frac{\sigma^2}{n} + r^2 \end{aligned}$$

Also, $E \left[\frac{1}{n-1} \sum_{i=1}^n (R_i - \bar{R})^2 \right] = E(S_{n-1}^2) = \sigma^2$.

$\left(\frac{1}{n} \sum_{i=1}^n R_i \right)^2 - \frac{1}{n} \left(\frac{1}{n-1} \right) \sum_{i=1}^n (R_i - \bar{R})^2$ is an unbiased estimator for r^2 since

$$E \left[\left(\frac{1}{n} \sum_{i=1}^n R_i \right)^2 \right] - \frac{1}{n} \left(\frac{1}{n-1} \right) E \left[\sum_{i=1}^n (R_i - \bar{R})^2 \right] = \left(\frac{\sigma^2}{n} + r^2 \right) - \frac{1}{n} \cdot \sigma^2 = r^2,$$

$$\therefore \pi \left[\left(\frac{1}{n} \sum_{i=1}^n R_i \right)^2 - \frac{1}{n} \left(\frac{1}{n-1} \right) \sum_{i=1}^n (R_i - \bar{R})^2 \right]$$

is an unbiased estimator for πr^2 (area).

19. X_i 's independent $N(\mu, \sigma_i^2)$, $i = 1, 2, \dots, n$

It is NOT possible to estimate all the parameters from the n observations.

Now assume σ_i^2 are known, $i = 1, \dots, n$

$$\begin{aligned} f_{X_i}(x_i; \mu) &= \frac{1}{\sqrt{2\pi}\sigma_i} \exp \left\{ -\frac{1}{2\sigma_i^2} (x_i - \mu)^2 \right\} \\ L &= f_{\tilde{X}}(\tilde{X}; \mu) \\ &= \prod_{i=1}^n f_{X_i}(x_i; \mu) \\ &= \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma_i} \exp \left\{ -\frac{1}{2\sigma_i^2} (x_i - \mu)^2 \right\} \\ &= \left(\frac{1}{\sqrt{2\pi}} \right)^n \left(\prod_{i=1}^n \frac{1}{\sigma_i} \right) \exp \left\{ -\sum_{i=1}^n \frac{1}{2\sigma_i^2} (x_i - \mu)^2 \right\} \\ \log L &= \frac{-n}{2} \log(2\pi) - \sum_{i=1}^n \log \sigma_i - \sum_{i=1}^n \frac{1}{2\sigma_i^2} (x_i - \mu)^2 \\ \frac{\partial}{\partial \mu} \log L &= -\sum_{i=1}^n \frac{1}{2\sigma_i^2} 2(x_i - \mu)(-1) \\ &= \sum_{i=1}^n \frac{(x_i - \mu)}{\sigma_i^2} \end{aligned}$$

Set equal to 0, then

$$\sum_{i=1}^n \frac{x_i - \mu}{\sigma_i^2} = 0$$

$$\sum_{i=1}^n \frac{x_i}{\sigma_i^2} = \hat{\mu} \sum_{i=1}^n \frac{1}{\sigma_i^2}$$

$$\hat{\mu} = \sum_{i=1}^n \frac{x_i}{\sigma_i^2} / \sum_{i=1}^n \frac{1}{\sigma_i^2} \quad \text{which is MLE for } \mu.$$

And $\sum_{i=1}^n \frac{1}{\sigma_i^2}$, this is call weighted sample mean.

20. (a)

$$E(X^2) = Var(X) + [E(X)]^2 = \sigma^2 + 0^2 = \sigma^2$$

$\therefore X^2$ is an unbiased estimator of σ^2 .

(b)

$$\begin{aligned} L &= f_X(x; \theta) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left\{ \frac{-1}{2\sigma^2} x^2 \right\} \\ \log L &= \frac{-1}{2} \log(2\pi) - \frac{1}{2} \log \sigma^2 - \frac{1}{2\sigma^2} x^2 \\ \frac{\partial}{\partial \sigma^2} \log L &= -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4} x^2 \end{aligned}$$

Set equal to 0, we get

$$\begin{aligned} \frac{-1}{2\sigma^2} + \frac{1}{2\sigma^4} x^2 &= 0 \\ \Rightarrow \hat{\sigma}^2 &= x^2 \end{aligned}$$

\therefore The MLE for σ is $\hat{\sigma} = \sqrt{x^2} = |X|$.

(c) $M'_2 = \widetilde{E(X^2)} = \tilde{\sigma}^2$

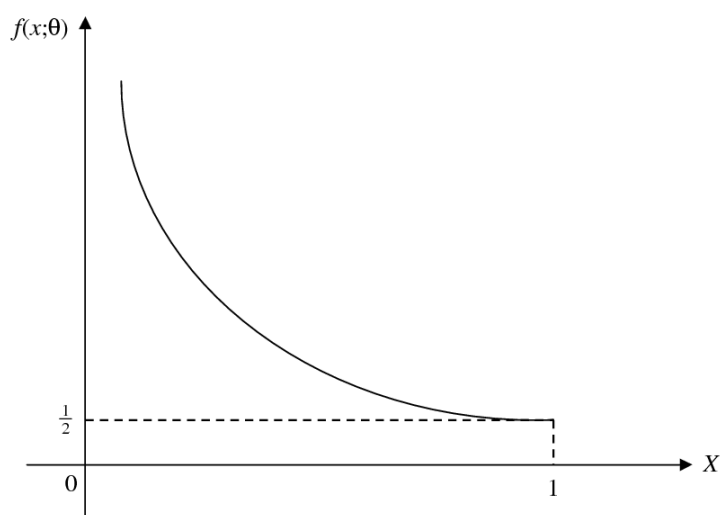
Also, $\widetilde{E(X^2)} = \frac{1}{n} \sum_{i=1}^n X_i^2 = X^2 \quad (\because n=1)$

$\therefore \hat{\sigma} = \sqrt{X^2} = |X|$.

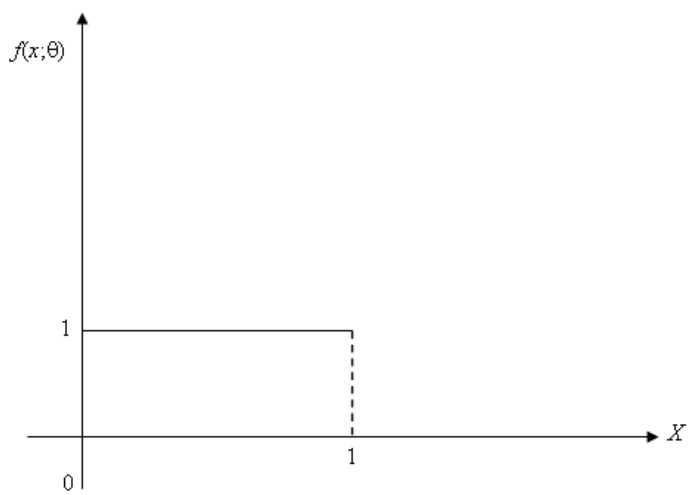
(Method of moment estimator is equal to MLE in this question.)

21. (a) (i) $\theta = \frac{1}{2}$,

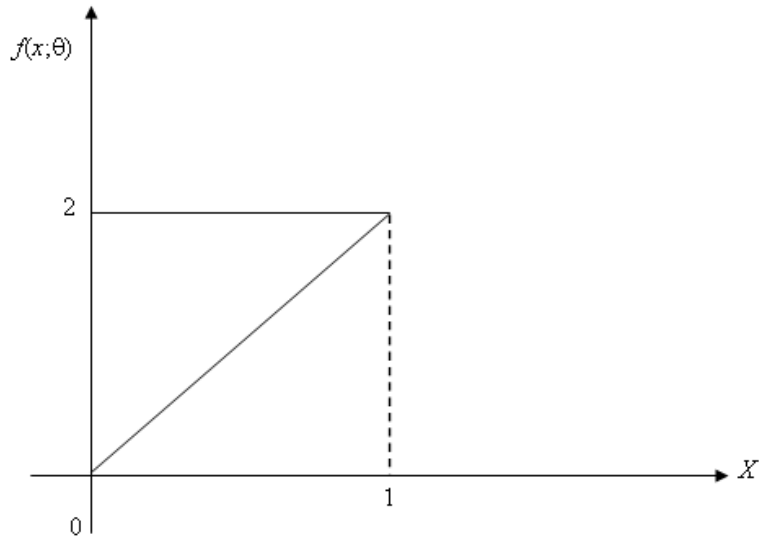
$$f(x; \theta) = \frac{1}{2} X^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$



(ii) $\theta = 1$,
 $f(x; \theta) = 1 \cdot x^{1-1} = 1$



(iii) $\theta = 1$,
 $f(x; \theta) = 2x^{2-1} = 2x$



(b)

$$\begin{aligned}
 L &= \prod_{i=1}^n f(x_i; \theta) \\
 &= \prod_{i=1}^n \theta x_i^{\theta-1} \\
 &= \theta^n \prod_{i=1}^n x_i^{\theta-1}
 \end{aligned}$$

$$\log L = n \log \theta + (\theta - 1) \sum_{i=1}^n \log x_i$$

$$\frac{\partial}{\partial \theta} \log L = \frac{n}{\theta} + \sum_{i=1}^n \log x_i$$

Set equal to 0, we get

$$\frac{n}{\theta} + \sum_{i=1}^n \log x_i = 0$$

$$\hat{\theta} = \frac{-n}{\sum_{i=1}^n \log x_i} = \frac{-n}{\log \prod_{i=1}^n x_i}$$

(c) (i)

$$\log \prod_{i=1}^{10} x_i = -18.21 \Rightarrow \hat{\theta} = -\frac{10}{-18.21} = 0.549$$

(ii)

$$\log \prod_{i=1}^{10} x_i = -4.52 \Rightarrow \hat{\theta} = -\frac{10}{-4.52} = 2.212$$

(iii)

$$\log \prod_{i=1}^{10} x_i = -10.43 \Rightarrow \hat{\theta} = -\frac{10}{-10.43} = 0.959$$

22. (a) $f_{Y_1}(y_1; \theta) = 3[1 - F_X(y_1)]^2 f_X(y_1; \theta) = 3(1 - \frac{y_1}{\theta})^2 (\frac{1}{\theta})$,

$$\begin{aligned} E(Y_1) &= \int_0^\theta y_1 f_{Y_1}(y_1; \theta) dy_1 \\ &= \int_0^\theta y_1 \left(\frac{3}{\theta}\right) \left(1 - \frac{y_1}{\theta}\right)^2 dy_1 \\ &= \frac{3}{\theta^3} \int_0^\theta y_1 (\theta - y_1)^2 dy_1 \\ &= \frac{3}{\theta^3} \int_0^\theta (y_1^3 - 2\theta y_1^2 + \theta^2 y_1) dy_1 \\ &= \frac{3}{\theta^3} \left[\frac{y_1^4}{4} - \frac{2\theta y_1^3}{3} + \frac{\theta^2 y_1^2}{2} \right]_0^\theta \\ &= \frac{3}{\theta^3} \left[\frac{\theta^4}{4} - \frac{2\theta^4}{3} + \frac{\theta^4}{2} \right] \\ &= \frac{\theta}{4} \end{aligned}$$

$$\therefore E(4Y_1) = 4E(Y_1) = 4 \cdot \frac{\theta}{4} = \theta$$

Therefore $4Y_1$ is unbiased for θ .

$$\begin{aligned} E(Y_1^2) &= \int_0^\theta y_1^2 f_{Y_1}(y_1; \theta) dy_1 \\ &= \int_0^\theta y_1^2 \left(\frac{3}{\theta}\right) \left(1 - \frac{y_1}{\theta}\right)^2 dy_1 \\ &= \frac{3}{\theta^3} \int_0^\theta y_1^2 (\theta - y_1)^2 dy_1 \\ &= \frac{3}{\theta^3} \int_0^\theta (y_1^4 - 2\theta y_1^3 + \theta^2 y_1^2) dy_1 \\ &= \frac{3}{\theta^3} \left[\frac{y_1^5}{5} - \frac{2}{4}\theta y_1^4 + \frac{1}{3}\theta^2 y_1^3 \right]_0^\theta \\ &= \frac{3}{\theta^3} \left[\frac{\theta^5}{5} - \frac{\theta^5}{2} + \frac{\theta^5}{3} \right] \\ &= \frac{\theta^2}{10} \end{aligned}$$

$$\begin{aligned} Var(Y_1) &= E(Y_1^2) - [E(Y_1)]^2 \\ &= \frac{\theta^2}{10} - \left(\frac{\theta}{4}\right)^2 \\ &= \frac{3}{80} \end{aligned}$$

$$Var(4Y_1) = 16Var(Y_1) = 16 \times \frac{3}{80}\theta^2 = \frac{3}{5}\theta^2$$

(b)

$$f_{Y_2}(y_2; \theta) = 3![F_X(y_2)]f(y_2)[1 - F_X(y_2)] = 6\left(\frac{y_2}{\theta}\right)\left(\frac{1}{\theta}\right)\left(1 - \frac{y_2}{\theta}\right) = \frac{6}{\theta^3}(\theta y_2 - y_2^2), \quad 0 < y_2 < \theta$$

$$\begin{aligned} E(Y_2) &= \int_0^\theta y_2 f_{Y_2}(y_2; \theta) dy_2 \\ &= \int_0^\theta y_2 \left(\frac{3}{\theta}\right) \left(1 - \frac{y_2}{\theta}\right)^2 dy_2 \\ &= \int_0^\theta y_2 \frac{6}{\theta^3} (\theta y_2 - y_2^2) dy_2 \\ &= \frac{6}{\theta^3} \int_0^\theta (\theta y_2^2 - y_2^3) dy_2 \\ &= \frac{6}{\theta^3} \left[\frac{1}{3} \theta y_2^3 - \frac{1}{4} y_2^4 \right]_0^\theta \\ &= \frac{6}{\theta^3} \left[\frac{\theta^4}{3} - \frac{\theta^4}{4} \right] \\ &= \frac{\theta}{2} \end{aligned}$$

$$\therefore E(2Y_2) = 2E(Y_2) = 2\left(\frac{\theta}{2}\right) = \theta$$

Therefore $2Y_2$ is unbiased for θ .

$$\begin{aligned} E(Y_2^2) &= \int_0^\theta y_2^2 f_{Y_2}(y_2; \theta) dy_2 \\ &= \int_0^\theta \frac{6y_2^2}{\theta^3} (\theta y_2 - y_2^2) dy_2 \\ &= \frac{6}{\theta^3} \int_0^\theta (\theta y_2^3 - y_2^4) dy_2 \\ &= \frac{6}{\theta^3} \left[\frac{1}{4} \theta y_2^4 - \frac{1}{5} y_2^5 \right]_0^\theta \\ &= \frac{6}{\theta^3} \left[\frac{\theta^5}{4} - \frac{\theta^5}{5} \right] \\ &= \frac{3}{10} \theta^2 \\ \therefore \text{Var}(Y_2) &= E(Y_2^2) - [E(Y_2)]^2 \\ &= \frac{3}{10} \theta^2 - \left(\frac{1}{2} \theta\right)^2 \\ &= \frac{1}{20} \theta^2 \end{aligned}$$

$$\therefore \text{Var}(2Y_2) = 4\text{Var}(Y_2) = 4\left(\frac{1}{20} \theta^2\right) = \frac{1}{5} \theta^2$$

(c)

$$f_{Y_3}(y_3; \theta) = 3[F_X(y_3)]^2 f_X(y_3) = 3\left(\frac{y_3}{\theta}\right)^2 \left(\frac{1}{\theta}\right), \quad 0 < y_3 < \theta$$

$$\begin{aligned}
E(Y_3) &= \int_0^\theta y_3 \frac{3}{\theta^3} y_3^2 dy_3 \\
&= \frac{3}{\theta^3} \int_0^\theta y_3^3 dy_3 \\
&= \frac{3}{4\theta^3} [y_3^4]_0^\theta \\
&= \frac{3}{4} \theta
\end{aligned}$$

$$\therefore E\left(\frac{4}{3}Y_3\right) = \frac{4}{3}E(Y_3) = \frac{4}{3}\left(\frac{3}{4}\right)\theta = \theta$$

Therefore $\frac{4}{3}Y_3$ is unbiased for θ .

$$\begin{aligned}
E(Y_3^2) &= \int_0^\theta y_3^2 \left(\frac{3}{\theta^3}\right) y_3^2 dy_3 \\
&= \frac{3}{\theta^3} \int_0^\theta y_3^4 dy_3 \\
&= \frac{3}{5\theta^3} [y_3^5]_0^\theta \\
&= \frac{3}{5} \theta^2
\end{aligned}$$

$$\begin{aligned}
\therefore \text{Var}(Y_3) &= E(Y_3^2) - [E(Y_3)]^2 \\
&= \frac{3}{5} \theta^2 - \left(\frac{3}{4} \theta\right)^2 \\
&= \frac{3}{80} \theta^2
\end{aligned}$$

$$\therefore \text{Var}\left(\frac{4}{3}Y_3\right) = \left(\frac{4}{3}\right)^2 \text{Var}(Y_3) = \frac{16}{9} \left(\frac{3}{80} \theta^2\right) = \frac{1}{15} \theta^2$$

23. (a) $E(\omega \bar{X}_1 + (1 - \omega) \bar{X}_2), \quad 0 \leq \omega \leq 1$

$$\begin{aligned}
&= \omega E(\bar{X}_1) + (1 - \omega) E(\bar{X}_2) \\
&= \omega E(X_i) + (1 - \omega) E(X_i) \\
&= \omega \cdot \mu + (1 - \omega) \cdot \mu \\
&= \mu \\
&\therefore \omega \bar{X}_1 + (1 - \omega) \bar{X}_2 \text{ is unbiased for } \mu
\end{aligned}$$

(b) $\text{Var}(\omega \bar{X}_1 + (1 - \omega) \bar{X}_2)$

$$\begin{aligned}
&= \omega^2 \text{Var}(\bar{X}_1) + (1 - \omega)^2 \text{Var}(\bar{X}_2) \\
&= \omega^2 \frac{\text{Var}(X_1)}{n} + (1 - \omega)^2 \frac{\text{Var}(X_2)}{n} \\
&= \omega^2 \left(\frac{\sigma_1^2}{n}\right) + (1 - \omega)^2 \left(\frac{\sigma_2^2}{n}\right)
\end{aligned}$$

Let

$$g(\omega) = \omega^2 \left(\frac{\sigma_1^2}{n}\right) + (1 - \omega)^2 \left(\frac{\sigma_2^2}{n}\right)$$

$$g'(\omega) = 2\omega\left(\frac{\sigma_1^2}{n}\right) - 2(1-\omega)\left(\frac{\sigma_2^2}{n}\right)$$

$$\begin{aligned} g'(\omega) = 0 &\Rightarrow 2\omega\left(\frac{\sigma_1^2}{n}\right) - 2(1-\omega)\left(\frac{\sigma_2^2}{n}\right) = 0 \\ &\Rightarrow \omega\sigma_1^2 = (1-\omega)\sigma_2^2 \\ &\Rightarrow \omega(\sigma_1^2 + \sigma_2^2) = \sigma_2^2 \\ &\Rightarrow \omega = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \end{aligned}$$

(c) When $\omega = \frac{1}{2}$, let $\hat{\mu}_1 = \frac{1}{2}\bar{X}_1 + \frac{1}{2}\bar{X}_2$,

$$\Rightarrow \text{Var}(\hat{\mu}_1) = \left(\frac{1}{2}\right)^2 \cdot \frac{\text{Var}(X)}{n} + \left(\frac{1}{2}\right)^2 \cdot \frac{\text{Var}(X)}{n} = \frac{1}{4} \frac{\sigma_1^2}{n} + \frac{1}{4} \frac{\sigma_2^2}{n} = \frac{1}{4n}(\sigma_1^2 + \sigma_2^2)$$

When $\omega = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$, let $\hat{\mu}_2 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\bar{X}_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\bar{X}_2$

$$\Rightarrow \text{Var}(\hat{\mu}_2) = \left(\frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)^2 \frac{\sigma_1^2}{n} + \left(\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\right)^2 \frac{\sigma_2^2}{n} = \frac{\sigma_1^2 \sigma_2^2}{n(\sigma_1^2 + \sigma_2^2)}$$

\therefore The efficiency of the estimator of part (a) with $\omega = \frac{1}{2}$ relative to $\omega = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}$

$$= \frac{\text{Var}(\hat{\mu}_2)}{\text{Var}(\hat{\mu}_1)} = \frac{\frac{(\sigma_1^2 \sigma_2^2)}{n(\sigma_1^2 + \sigma_2^2)}}{\frac{1}{4n}(\sigma_1^2 + \sigma_2^2)} = \frac{4\sigma_1^2 \sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2}$$

24. (a) $E(\omega\bar{X}_1 + (1-\omega)\bar{X}_2)$
 $= \omega E(\bar{X}_1) + (1-\omega)E(\bar{X}_2)$
 $= \omega E(X_i) + (1-\omega)E(X_i)$
 $= \omega \cdot \mu + (1-\omega) \cdot \mu$
 $= \mu$
 $\therefore \omega\bar{X}_1 + (1-\omega)\bar{X}_2$ is unbiased for μ .
 $\text{Var}(\omega\bar{X}_1 + (1-\omega)\bar{X}_2)$
 $= \omega^2 \text{Var}(\bar{X}_1) + (1-\omega)^2 \text{Var}(\bar{X}_2)$
 $= \omega^2 \left(\frac{\sigma^2}{n_1}\right) + (1-\omega)^2 \left(\frac{\sigma^2}{n_2}\right)$

Let

$$g(\omega) = \omega^2 \left(\frac{\sigma^2}{n_1}\right) + (1-\omega)^2 \left(\frac{\sigma^2}{n_2}\right)$$

$$g'(\omega) = 2\omega \left(\frac{\sigma^2}{n_1}\right) - 2(1-\omega) \left(\frac{\sigma^2}{n_2}\right)$$

$$\begin{aligned} g'(\omega) = 0 &\Rightarrow 2\omega \left(\frac{\sigma^2}{n_1}\right) - 2(1-\omega) \left(\frac{\sigma^2}{n_2}\right) = 0 \\ &\Rightarrow \omega \left(\frac{1}{n_1} + \frac{1}{n_2}\right) = \frac{1}{n_2} \\ &\Rightarrow \omega = \frac{n_1}{n_1 + n_2} \end{aligned}$$

(b) When $\omega = \frac{1}{2}$, let $\hat{\mu}_1 = \frac{1}{2}\bar{X}_1 + \frac{1}{2}\bar{X}_2$,

$$\Rightarrow \text{Var}(\hat{\mu}_1) = \frac{1}{4} \cdot \frac{\sigma^2}{n_1} + \frac{1}{4} \cdot \frac{\sigma^2}{n_2} = \frac{\sigma^2}{4} \left(\frac{1}{n_1} + \frac{1}{n_2} \right)$$

When $\omega = \frac{n_1}{n_1+n_2}$, let $\hat{\mu}_2 = \frac{n_1}{n_1+n_2}\bar{X}_1 + \frac{n_2}{n_1+n_2}\bar{X}_2$

$$\Rightarrow \text{Var}(\hat{\mu}_2) = \left(\frac{n_1}{n_1+n_2} \right)^2 \cdot \frac{\sigma^2}{n_1} + \left(\frac{n_2}{n_1+n_2} \right)^2 \cdot \frac{\sigma^2}{n_2} = \frac{\sigma^2}{(n_1+n_2)^2} (n_1+n_2) = \frac{\sigma^2}{n_1+n_2}$$

\therefore The efficiency of the estimator with $\omega = \frac{1}{2}$ relative to that with $\omega = \frac{n_1}{n_1+n_2}$

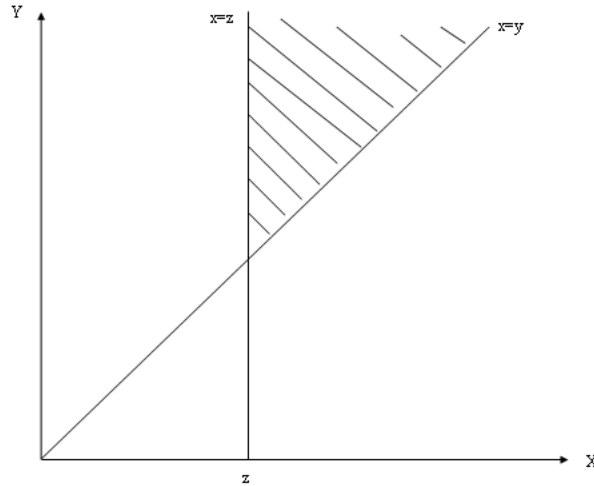
$$= \frac{\text{Var}(\hat{\mu}_2)}{\text{Var}(\hat{\mu}_1)} = \frac{\frac{\sigma^2}{n_1+n_2}}{\frac{\sigma^2}{4} \left(\frac{1}{n_1} + \frac{1}{n_2} \right)} = \frac{4n_1n_2}{(n_1+n_2)^2}$$

25. In order to find the MLEs of λ and μ , we need to find the joint pdf $f_{Z,W}(z, w; \lambda, \mu)$.

In this question, the pdf technique cannot be used. Why???

So we use cdf technique.

$$\begin{aligned} f_{Z,W}(z, w) &= \frac{\partial}{\partial z} P(Z \leq z, W = w) \quad (W \text{ is discrete, } Z \text{ is continuous}) \\ &= \frac{\partial}{\partial z} [1 - P(Z > z, W = w)] \\ &= \frac{-\partial}{\partial z} P(Z > z, W = w) \quad w = 0 \text{ or } 1 \end{aligned} \quad (1)$$



Now

$$\begin{aligned} &P(Z \geq z, W = 1) \\ &= P(\min(X, Y) \geq z, W = 1) \\ &= P(X \geq z, X = Z) \end{aligned}$$

$$\begin{aligned}
&= P(X \geq z, X < Y) \\
&= \int_z^\infty \int_x^\infty f_{X,Y}(x,y) dy dx \\
&= \int_z^\infty \int_x^\infty \frac{1}{\lambda} e^{-x/\lambda} \cdot \frac{1}{\mu} e^{-y/\mu} dy dx \\
&= \int_z^\infty \frac{1}{\lambda} e^{-x/\lambda} \cdot \left[-e^{-y/\mu} \right]_x^\infty dx \\
&= \int_z^\infty \frac{1}{\lambda} \cdot e^{-x/\lambda} \cdot e^{-x/\mu} dx \\
&= \int_z^\infty \frac{1}{\lambda} e^{-x(\frac{1}{\lambda} + \frac{1}{\mu})} dx \\
&= -\frac{1}{\lambda} \left(\frac{1}{\lambda} + \frac{1}{\mu} \right)^{-1} \cdot \left[e^{-x(\frac{1}{\lambda} + \frac{1}{\mu})} \right]_z^\infty \\
&= \frac{\lambda\mu}{\lambda(\lambda + \mu)} \cdot e^{-z(\frac{1}{\lambda} + \frac{1}{\mu})} \\
&= \frac{\mu}{\lambda + \mu} \cdot e^{-z(\frac{1}{\lambda} + \frac{1}{\mu})}
\end{aligned}$$

Similarly $P(Z \geq z, W = 0) = \frac{\lambda}{\lambda + \mu} \cdot e^{-z(\frac{1}{\lambda} + \frac{1}{\mu})}$

$$\begin{aligned}
&\therefore P(Z \geq z, W = w) \\
&= [P(Z \geq z, W = 1)]^w [P(Z \geq z, W = 0)]^{1-w} \quad (W \sim \text{Bernoulli distribution}) \\
&= \left[\frac{\mu}{\lambda + \mu} \cdot e^{-z(\frac{1}{\lambda} + \frac{1}{\mu})} \right]^w \left[\frac{\lambda}{\lambda + \mu} \cdot e^{-z(\frac{1}{\lambda} + \frac{1}{\mu})} \right]^{1-w} \\
&= \frac{1}{\lambda + \mu} \cdot \exp \left\{ \frac{-(\lambda + \mu)}{\lambda\mu} z \right\} \mu^w \lambda^{1-w} \\
f_{Z,W}(z, w) &= \frac{-\partial}{\partial z} P(Z \geq z, W = w) \\
&= \frac{-1}{\lambda + \mu} \cdot \left(\frac{-\lambda + \mu}{\lambda\mu} \right) \exp \left\{ \frac{-\lambda + \mu}{\lambda\mu} z \right\} \mu^w \lambda^{1-w} \\
&= \exp \left\{ \frac{-\lambda + \mu}{\lambda\mu} z \right\} \mu^{(w-1)} \lambda^{-w} \\
\therefore L &= f_{\tilde{Z}, \tilde{W}}(\tilde{z}, \tilde{w}; \lambda, \mu) \\
&= \prod_{i=1}^n f_{Z_i, W_i}(z_i, w_i; \lambda, \mu) \\
&= \prod_{i=1}^n \exp \left\{ \frac{-\lambda + \mu}{\lambda\mu} z_i \right\} \mu^{(w_i-1)} \lambda^{-w_i} \\
&= \mu^{\sum_{i=1}^n w_i - n} \lambda^{-\sum_{i=1}^n w_i} \exp \left\{ \frac{-\lambda + \mu}{\lambda\mu} \sum_{i=1}^n z_i \right\} \\
\log L &= \left(\sum_{i=1}^n w_i - n \right) \log \mu - \left(\sum_{i=1}^n w_i \right) \log \lambda - \frac{\lambda + \mu}{\lambda\mu} \sum_{i=1}^n z_i \\
\frac{\partial}{\partial \lambda} \log L &= \frac{-1}{\lambda} \sum_{i=1}^n w_i + \frac{1}{\lambda^2} \sum_{i=1}^n z_i \\
\frac{\partial}{\partial \mu} \log L &= \frac{1}{\mu} \left(\sum_{i=1}^n w_i - n \right) + \frac{1}{\mu^2} \sum_{i=1}^n z_i
\end{aligned}$$

Set to 0 \Rightarrow

$$\hat{\lambda} = \frac{\sum_{i=1}^n z_i}{\sum_{i=1}^n w_i} = \frac{\frac{1}{n} \sum_{i=1}^n z_i}{\frac{1}{n} \sum_{i=1}^n w_i} = \frac{\bar{Z}}{\bar{W}}$$

$$\hat{\mu} = \frac{\sum_{i=1}^n z_i}{(n - \sum_{i=1}^n w_i)} = \frac{\frac{1}{n} \sum_{i=1}^n z_i}{(1 - \frac{1}{n} \sum_{i=1}^n w_i)} = \frac{\bar{Z}}{(1 - \bar{W})}$$

26. $f(x|\theta) = \theta x^{\theta-1}$

(a)

$$L = f_{\tilde{X}}(\tilde{x}; \theta) = \prod_{i=1}^n f_{X_i}(x_i|\theta) = \prod_{i=1}^n \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^n x_i \right)^{\theta-1}$$

$$\log L = n \log \theta + (\theta - 1) \log \left(\prod_{i=1}^n x_i \right) = n \log \theta + (\theta - 1) \sum_{i=1}^n \log x_i$$

$$\frac{\partial}{\partial \theta} \log L = \frac{n}{\theta} + \sum_{i=1}^n \log x_i$$

Set to be 0 \Rightarrow

$$\hat{\theta} = \frac{-n}{\sum_{i=1}^n \log x_i}$$

which is MLE for θ .

Let $Y = -\log X$, $X = e^{-Y}$,

$$\begin{aligned} f_Y(y) &= f_X(e^{-y}) \cdot \left| \frac{dy}{dx} \right| \\ &= \theta (e^{-y})^{\theta-1} \cdot |-e^{-y}| \\ &= \theta e^{-\theta y} \end{aligned}$$

$\therefore Y \sim \text{Exponential}(\theta)$ and hence

$$W = \sum_{i=1}^n Y_i = - \sum_{i=1}^n \log x_i \sim \text{Gamma}(n, \theta)$$

$$\begin{aligned} E\left(\frac{1}{W}\right) &= \int_0^\infty \frac{1}{w} \cdot \frac{w^{n-1} e^{-\theta w}}{\theta^{-n} \Gamma(n)} dw \\ &= \frac{\theta}{n-1} \int_0^\infty \frac{w^{n-2} e^{-\theta w}}{\theta^{-(n-1)} \Gamma(n-1)} dw \\ &= \frac{\theta}{n-1} \cdot 1 \\ &= \frac{\theta}{n-1} \\ E\left(\frac{1}{W^2}\right) &= \int_0^\infty \frac{1}{w^2} \cdot \frac{w^{n-1} e^{-\theta w}}{\theta^{-n} \Gamma(n)} dw \\ &= \int_0^\infty \frac{w^{n-3} e^{-\theta w}}{\theta^{-n} \Gamma(n)} dw \\ &= \frac{\theta^2}{(n-1)(n-2)} \int_0^\infty \frac{w^{n-3} e^{-\theta w}}{\theta^{-(n-2)} \Gamma(n-2)} dw \end{aligned}$$

$$\begin{aligned}
&= \frac{\theta^2}{(n-1)(n-2)} \\
\text{Var}(\hat{\theta}) &= \text{Var}\left(\frac{-n}{\sum_{i=1}^n \log x_i}\right) \\
&= n^2 \text{Var}\left(\frac{1}{W}\right) \\
&= n^2 \left[E\left(\frac{1}{W^2}\right) + [E\left(\frac{1}{W}\right)]^2 \right] \\
&= n^2 \left[\frac{\theta^2}{(n-1)(n-2)} - \frac{\theta^2}{(n-1)^2} \right] \\
&= \frac{\theta^2 n^2}{(n-1)^2(n-2)} \\
&= \frac{\frac{\theta^2}{n}}{(1 - \frac{1}{n})^2(1 - \frac{2}{n})} \\
&\rightarrow 0 \quad \text{as } n \rightarrow \infty
\end{aligned}$$

(b)

$$E(X) = \int_0^1 x \cdot \theta x^{\theta-1} dx = \int_0^1 \theta x^\theta dx = \frac{\theta}{\theta+1} [x^{\theta+1}]_0^1 = \frac{\theta}{\theta+1}$$

$$\therefore M'_1 = \frac{1}{n} \sum_{i=1}^n X_i = \widetilde{E(X)} = \frac{\tilde{\theta}}{\tilde{\theta}+1}$$

$$\begin{aligned}
\frac{\tilde{\theta}}{\tilde{\theta}+1} = \bar{X} &\Rightarrow (\tilde{\theta}+1)\bar{X} = \tilde{\theta} \\
&\Rightarrow \tilde{\theta}\bar{X} - \tilde{\theta} = -\bar{X} \\
&= \tilde{\theta} = \frac{\bar{X}}{1-\bar{X}}
\end{aligned}$$

27. (a) MLE for $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$.

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2 \left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} \right)$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{(1-\rho^2)\sigma_X\sigma_Y}} \exp \left\{ \frac{-1}{2(1-\rho^2)} \left[\left(\frac{x-\mu_X}{\sigma_X} \right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y} \right)^2 - 2\rho \left(\frac{x-\mu_X}{\sigma_X} \right) \left(\frac{y-\mu_Y}{\sigma_Y} \right) \right] \right\}$$

$$\begin{aligned}
&L \\
&= f_{\tilde{X}, \tilde{Y}}(\tilde{x}, \tilde{y}) \\
&= \prod_{i=1}^n f_{X_i, Y_i}(x_i, y_i; \mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho) \\
&= \prod_{i=1}^n [4\pi^2(1-\rho^2)\sigma_X^2\sigma_Y^2]^{-\frac{1}{2}} \exp \left\{ \left[\left(\frac{x_i - \mu_X}{\sigma_X} \right)^2 + \left(\frac{y_i - \mu_Y}{\sigma_Y} \right)^2 - 2\rho \left(\frac{x_i - \mu_X}{\sigma_X} \right) \left(\frac{y_i - \mu_Y}{\sigma_Y} \right) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
&\log L \\
&= \frac{-n}{2} \log(4\pi^2) - \frac{n}{2} \log(1-\rho^2) - \frac{n}{2} \log \sigma_X^2 - \frac{n}{2} \log \sigma_Y^2 \\
&\quad - \frac{1}{2(1-\rho^2)} \sum_{i=1}^n \left(\frac{x_i - \mu_X}{\sigma_X} \right)^2 - \frac{1}{2(1-\rho^2)} \sum_{i=1}^n \left(\frac{y_i - \mu_Y}{\sigma_Y} \right)^2 + \frac{\rho}{1-\rho^2} \sum_{i=1}^n \left(\frac{x_i - \mu_X}{\sigma_X} \right) \left(\frac{y_i - \mu_Y}{\sigma_Y} \right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial \mu_X} \log L &= \frac{1}{(1-\rho^2)} \sum_{i=1}^n \left(\frac{x_i - \mu_X}{\sigma_X} \right) - \frac{\rho}{1-\rho^2} \sum_{i=1}^n \left(\frac{y_i - \mu_Y}{\sigma_X \sigma_Y} \right) \\
\frac{\partial}{\partial \mu_Y} \log L &= \frac{1}{(1-\rho^2)} \sum_{i=1}^n \left(\frac{y_i - \mu_Y}{\sigma_Y} \right) - \frac{\rho}{1-\rho^2} \sum_{i=1}^n \left(\frac{x_i - \mu_X}{\sigma_X \sigma_Y} \right) \\
\frac{\partial}{\partial \sigma_X^2} \log L &= \frac{-n}{2\sigma_X^2} + \frac{-1}{2(1-\rho^2)} \frac{1}{\sigma_X^4} \cdot \sum_{i=1}^n (x_i - \mu_X)^2 - \frac{\rho}{1-\rho^2} \sum_{i=1}^n \left(\frac{x_i - \mu_X}{2(\sigma_X^2)^{3/2}} \right) \left(\frac{y_i - \mu_Y}{\sigma_Y} \right) \\
\frac{\partial}{\partial \sigma_Y^2} \log L &= \frac{-n}{2\sigma_Y^2} + \frac{-1}{2(1-\rho^2)} \frac{1}{\sigma_Y^4} \cdot \sum_{i=1}^n (y_i - \mu_Y)^2 - \frac{\rho}{1-\rho^2} \sum_{i=1}^n \left(\frac{y_i - \mu_Y}{2(\sigma_Y^2)^{3/2}} \right) \left(\frac{x_i - \mu_X}{\sigma_X} \right) \\
\frac{\partial}{\partial \rho} \log L &= \frac{-n}{2} \left(\frac{1}{1-\rho^2} \right) (-2\rho) \\
&\quad + \frac{(-2\rho)}{2(1-\rho^2)^2} \left[\sum_{i=1}^n \left(\frac{x_i - \mu_X}{\sigma_X} \right)^2 + \sum_{i=1}^n \left(\frac{y_i - \mu_Y}{\sigma_Y} \right)^2 - 2\rho \sum_{i=1}^n \left(\frac{x_i - \mu_X}{\sigma_X} \right) \left(\frac{y_i - \mu_Y}{\sigma_Y} \right) \right] \\
&\quad + \frac{1}{1-\rho^2} \sum_{i=1}^n \left(\frac{x_i - \mu_X}{\sigma_X} \right) \left(\frac{y_i - \mu_Y}{\sigma_Y} \right)
\end{aligned}$$

Set all above 5 partial derivative equations equal to 0, we have

$$\frac{1}{\hat{\sigma}_X} \sum_{i=1}^n (x_i - \hat{\mu}_X) - \frac{\hat{\rho}}{\hat{\sigma}_Y} \sum_{i=1}^n (y_i - \hat{\mu}_Y) = 0 \quad (1)$$

$$\frac{1}{\hat{\sigma}_Y} \sum_{i=1}^n (y_i - \hat{\mu}_Y) - \frac{\hat{\rho}}{\hat{\sigma}_X} \sum_{i=1}^n (x_i - \hat{\mu}_X) = 0 \quad (2)$$

$$-n + \frac{1}{(1-\hat{\rho}^2)\hat{\sigma}_X^2} \sum_{i=1}^n (x_i - \hat{\mu}_X)^2 - \frac{\hat{\rho}}{(1-\hat{\rho}^2)} \sum_{i=1}^n \left(\frac{x_i - \hat{\mu}_X}{\hat{\sigma}_X} \right) \left(\frac{y_i - \hat{\mu}_Y}{\hat{\sigma}_Y} \right) = 0 \quad (3)$$

$$-n + \frac{1}{(1-\hat{\rho}^2)\hat{\sigma}_Y^2} \sum_{i=1}^n (y_i - \hat{\mu}_Y)^2 - \frac{\hat{\rho}}{(1-\hat{\rho}^2)} \sum_{i=1}^n \left(\frac{x_i - \hat{\mu}_X}{\hat{\sigma}_X} \right) \left(\frac{y_i - \hat{\mu}_Y}{\hat{\sigma}_Y} \right) = 0 \quad (4)$$

$$\begin{aligned}
n\hat{\rho} - \frac{\hat{\rho}}{(1-\hat{\rho}^2)} \left[\sum_{i=1}^n \left(\frac{x_i - \hat{\mu}_X}{\hat{\sigma}_X} \right)^2 + \sum_{i=1}^n \left(\frac{y_i - \hat{\mu}_Y}{\hat{\sigma}_Y} \right)^2 - 2\hat{\rho} \sum_{i=1}^n \left(\frac{x_i - \hat{\mu}_X}{\hat{\sigma}_X} \right) \left(\frac{y_i - \hat{\mu}_Y}{\hat{\sigma}_Y} \right) \right] + \\
\sum_{i=1}^n \left(\frac{x_i - \hat{\mu}_X}{\hat{\sigma}_X} \right) \left(\frac{y_i - \hat{\mu}_Y}{\hat{\sigma}_Y} \right) = 0 \quad (5)
\end{aligned}$$

From (1), (2), since $\begin{vmatrix} 1 & -\hat{\rho} \\ -\hat{\rho} & 1 \end{vmatrix} = 1 - \hat{\rho}^2 \neq 0$,

we get

$$\frac{1}{\hat{\sigma}_X} \sum_{i=1}^n (x_i - \hat{\mu}_X) = \frac{1}{\hat{\sigma}_Y} \sum_{i=1}^n (y_i - \hat{\mu}_Y) = 0$$

$$\Rightarrow \hat{\mu}_X = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}, \quad \text{and} \quad \hat{\mu}_Y = \frac{1}{n} \sum_{i=1}^n y_i = \bar{y} \quad (\text{MLE for } \mu_X, \mu_Y)$$

Put back MLEs of μ_X and μ_Y into (3), (4), (5).

From (3), (4), we have

$$\begin{aligned} \frac{1}{(1-\hat{\rho}^2)\hat{\sigma}_X^2} \sum_{i=1}^n (x_i - \bar{x})^2 &= n + \frac{\hat{\rho}}{(1-\hat{\rho}^2)} \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\hat{\sigma}_X} \right) \left(\frac{y_i - \bar{y}}{\hat{\sigma}_Y} \right) = \frac{1}{(1-\hat{\rho}^2)\hat{\sigma}_Y^2} \sum_{i=1}^n (y_i - \bar{y})^2 \\ \Rightarrow \frac{1}{\hat{\sigma}_X} \sum_{i=1}^n (x_i - \bar{x})^2 &= \frac{1}{\hat{\sigma}_Y} \sum_{i=1}^n (y_i - \bar{y})^2 \quad (*) \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\hat{\sigma}_X} \right) \left(\frac{y_i - \bar{y}}{\hat{\sigma}_Y} \right) &= \frac{1-\hat{\rho}^2}{\hat{\rho}} \left[\frac{1}{(1-\hat{\rho}^2)\hat{\sigma}_X^2} \sum_{i=1}^n (x_i - \bar{x})^2 - n \right] \\ &= \frac{1-\hat{\rho}^2}{\hat{\rho}} \left[\frac{1}{(1-\hat{\rho}^2)\hat{\sigma}_Y^2} \sum_{i=1}^n (y_i - \bar{y})^2 - n \right] \quad (**) \end{aligned}$$

Put them into (5), we have

$$\begin{aligned} n\hat{\rho} - \frac{\hat{\rho}}{(1-\hat{\rho}^2)} &\left[\sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\hat{\sigma}_X} \right)^2 + \overbrace{\sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\hat{\sigma}_X} \right)^2} + \overbrace{2 \left[n(1-\hat{\rho}^2) - \frac{1}{\hat{\sigma}_X^2} \sum_{i=1}^n (x_i - \bar{x})^2 \right]} \right] \\ &+ \frac{1-\hat{\rho}^2}{\hat{\rho}} \left[\frac{1}{(1-\hat{\rho}^2)\hat{\sigma}_X^2} \sum_{i=1}^n (x_i - \bar{x})^2 - n \right] = 0 \\ \Rightarrow n\hat{\rho} - \frac{\hat{\rho}}{(1-\hat{\rho}^2)} [2n(1-\hat{\rho}^2)] &+ \frac{1}{\hat{\rho}\hat{\sigma}_X^2} \sum_{i=1}^n (x_i - \bar{x})^2 - \frac{n(1-\hat{\rho}^2)}{\hat{\rho}} = 0 \\ \Rightarrow -n\hat{\rho} + \frac{1}{\hat{\sigma}_X^2} \sum_{i=1}^n (x_i - \bar{x})^2 - n(1-\hat{\rho}^2) &= 0 \\ \Rightarrow n = \frac{1}{\hat{\sigma}_X^2} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{\hat{\sigma}_Y^2} \sum_{i=1}^n (y_i - \bar{y})^2 \\ \Rightarrow \hat{\sigma}_X^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \quad \text{and} \quad \hat{\sigma}_Y^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2 \quad (\text{MLE for } \sigma_X^2, \sigma_Y^2) \end{aligned}$$

Put

$$\frac{1}{\hat{\sigma}_X^2} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{\hat{\sigma}_Y^2} \sum_{i=1}^n (y_i - \bar{y})^2 \quad \text{to (5),}$$

$$n\hat{\rho} - \frac{\hat{\rho}}{(1-\hat{\rho}^2)} \left[n + n - 2\hat{\rho} \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\hat{\sigma}_X} \right) \left(\frac{y_i - \bar{y}}{\hat{\sigma}_Y} \right) \right] + \sum_{i=1}^n \left(\frac{x_i - \hat{\mu}_X}{\hat{\sigma}_X} \right) \left(\frac{y_i - \hat{\mu}_Y}{\hat{\sigma}_Y} \right) = 0$$

$$\Rightarrow n\hat{\rho}(1 - \hat{\rho}^2) - 2n\hat{\rho} + 2\hat{\rho}^2 \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\hat{\sigma}_X}\right) \left(\frac{y_i - \bar{y}}{\hat{\sigma}_Y}\right) + (1 - \hat{\rho}^2) \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\hat{\sigma}_X}\right) \left(\frac{y_i - \bar{y}}{\hat{\sigma}_Y}\right) = 0$$

$$\Rightarrow -(n\hat{\rho} + n\hat{\rho}^3) + (1 + \hat{\rho}^2) \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\hat{\sigma}_X}\right) \left(\frac{y_i - \bar{y}}{\hat{\sigma}_Y}\right) = 0$$

$$\Rightarrow n\hat{\rho} = \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{\hat{\sigma}_X}\right) \left(\frac{y_i - \bar{y}}{\hat{\sigma}_Y}\right)$$

$$\Rightarrow \hat{\rho} = \frac{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 \cdot \frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2}}$$

$$\Rightarrow \hat{\rho} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \cdot \sum_{i=1}^n (y_i - \bar{y})^2}} \quad \text{MLE for } \rho$$

(b) Method of moments estimates

$$E(X) = \mu_X, \therefore M'_X = \frac{1}{n} \sum_{i=1}^n x_i = \widetilde{E(X)} = \tilde{\mu}_X \Rightarrow \tilde{\mu}_X = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

$$E(Y) = \mu_Y, \therefore M'_Y = \frac{1}{n} \sum_{i=1}^n y_i = \widetilde{E(Y)} = \tilde{\mu}_Y \Rightarrow \tilde{\mu}_Y = \frac{1}{n} \sum_{i=1}^n y_i = \bar{y}$$

$$E(X^2) = \text{Var}(X) + [E(X)]^2 = \sigma_X^2 + \mu_X^2$$

$$\therefore M'_{X^2} = \frac{1}{n} \sum_{i=1}^n x_i^2 = \widetilde{E(X^2)} = \tilde{\sigma}_X^2 + \tilde{\mu}_X^2$$

$$\tilde{\sigma}_X^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - \bar{x}^2 = \frac{1}{n} [\sum_{i=1}^n x_i^2 - n\bar{x}^2]$$

$$\therefore \tilde{\sigma}_X^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

Similarly, $E(Y^2) = \sigma_Y^2 + \mu_Y^2$,

$$\therefore M'_{Y^2} = \frac{1}{n} \sum_{i=1}^n y_i^2 = \widetilde{E(Y^2)} = \tilde{\sigma}_Y^2 + \tilde{\mu}_Y^2$$

$$\therefore \tilde{\sigma}_Y^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2$$

$$\begin{aligned}
E(XY) &= COV(X, Y) + E(X)E(Y) \\
&= \rho\sigma_X\sigma_Y \\
M'_{XY} &= \frac{1}{n} \sum_{i=1}^n x_i y_i = \widetilde{E(XY)} \\
&= \widetilde{COV(XY)} + \widetilde{E(X)}\widetilde{E(Y)} \\
&= \tilde{\rho}\tilde{\sigma}_X\tilde{\sigma}_Y + \tilde{\mu}_X\tilde{\mu}_Y \\
&= \tilde{\rho} \sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \cdot \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2} + \bar{x}\bar{y} \\
\tilde{\rho} &= \frac{\frac{1}{n} \sum_{i=1}^n x_i y_i - \bar{x}\bar{y}}{\frac{1}{n} \sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}} \\
\tilde{\rho} &= \frac{\sum_{i=1}^n x_i y_i - n\bar{x}\bar{y}}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}} \\
\tilde{\rho} &= \frac{\frac{1}{n} \sum_{i=1}^n x_i y_i - \bar{x}\bar{y}}{\frac{1}{n} \sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}} \\
\tilde{\rho} &= \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}}
\end{aligned}$$

\therefore The MLEs and the method of moments estimates of $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ and ρ are all the same.