Tutorial for 10-06 and 10-09

1. Point Estimation:

- idea: a $T(x) \to$ a unknown parameter $g(\theta)$
  - $x = (x_1, x_2, \ldots, x_n)^T \xleftarrow{\text{realization}} X = (X_1, X_2, \ldots, X_n)^T$
    (Or $\{X_i, i = 1, 2, \ldots, n\}$)
  - a population with pdf $f(\cdot|\theta)$ or pmf $p(\cdot|\theta)$
- "Point": opposite to Interval.
  $T(X)$ ← Capital X (not $T(x)$) is a random variable indeed.
- Method: (i) method of moments estimation (MME)
  (ii) maximun likelihood ... ... (MLE)

2. method of moments estimation (MME)

Basic idea of the method of moments estimation:
If the function can be specified, then we replace the population moments by their corresponding sample moments. The function of these sample moments is called ***the method of moments estimator*** (MME) for the parameter of interest.

Definition (MME): Suppose that there are k unknown parameters $\theta_1, \ldots, \theta_k$. If we can rewrite them in them of the first k or more moments, i.e.

$$\begin{cases} \theta_1 = g_1(\mu_1', \mu_2', \ldots, \mu_k', \ldots) \\ \theta_2 = g_2(\mu_1', \mu_2', \ldots, \mu_k', \ldots) \\ \vdots \\ \theta_k = g_k(\mu_1', \mu_2', \ldots, \mu_k', \ldots) \end{cases},$$

then, the method of moments estimator (MME), denoted by $(\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_k)$, of $(\theta_1, \theta_2, \ldots, \theta_k)$ is

$$\begin{cases} \tilde{\theta}_1 = g_1(\bar{X}, \overline{X^2}, \ldots, \overline{X^k}, \ldots) \\ \tilde{\theta}_2 = g_2(\bar{X}, \overline{X^2}, \ldots, \overline{X^k}, \ldots) \\ \vdots \\ \tilde{\theta}_k = g_k(\bar{X}, \overline{X^2}, \ldots, \overline{X^k}, \ldots) \end{cases}.$$

$\left\{ \begin{array}{l} \text{(i) not unique} \\ \text{(ii) biased} \end{array} \right. \Rightarrow$ RULE: use fewer and lower moments
$k \downarrow$.

Theorem (Multivariate Delta Method): Let $\{\boldsymbol{X}_n \in R^k : n = 1,2,\dots\}$ be a sequence of random vectors that for a constant vector $\boldsymbol{a} \in R^k$,

$$\sqrt{n}(\boldsymbol{X}_n - \boldsymbol{a}) \xrightarrow{d} \boldsymbol{Y}, \qquad as\ n \to \infty,$$

where $\boldsymbol{Y}$ is a random vector in R^k.

gradient

If a function $h: R^k \to R$ has a derivative $\nabla h(\boldsymbol{a}) \neq \boldsymbol{0}$, then we have

$$\sqrt{n}\big(h(\boldsymbol{X}_n) - h(\boldsymbol{a})\big) \xrightarrow{d} \nabla h(\boldsymbol{a})\boldsymbol{Y}, \qquad as\ n \to \infty,$$

where $\nabla h = \left(\frac{\partial h(t_1,t_2,\dots,t_k)}{\partial t_1}, \frac{\partial h(t_1,t_2,\dots,t_k)}{\partial t_2}, \dots, \frac{\partial h(t_1,t_2,\dots,t_k)}{\partial t_k}\right)$. → Partial derivatives

Rewrite to:

We usually know Y's distribution.

$$X_n \xrightarrow{d} \frac{Y}{\sqrt{n}} + a \quad \Rightarrow \quad h(X_n) \xrightarrow{d} \frac{\nabla h(a) \cdot Y}{\sqrt{n}} + h(a)$$

Examples:

$X_1, X_2, \dots, X_n$ iid $N(\mu, \sigma^2)$ ($\bar{X}$ is the mean, S^2 is the sample variance)

Find the limiting d.f.'s of ① $(\bar{X})^2$, ② $1/\bar{X}$, ③ S^2, ④ S, ⑤ $S/\bar{X}$.

Solution:

① $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ $\qquad (\bar{X})^2$? $\to g(x) = x^2$ kernal.

According univariate Delta method:

$$\sqrt{n}\big(g(Y_n) - g(a)\big) \xrightarrow{d} N(0, [g'(a)b]^2), \qquad as\ n \to \infty.$$

Delute method:

- Univariate:

$X \xrightarrow{d} Y$

$\Rightarrow g(X) \xrightarrow{d} g$

$g'(x) = 2x$

$$g(\bar{x}) \sim N\left(\mu^2, \frac{4\mu^2 \cdot \sigma^2}{n}\right)$$

③ S^2 Suppose $Y_i := X_i - \mu \Rightarrow Y_i \sim N(0, \sigma^2)$

Take $g(x, y) = y - x^2$.

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \frac{1}{n}\sum_{i=1}^{n}Y_i^2 - (\bar{Y})^2 = \overline{Y^2} - (\bar{Y})^2$$

$$= g(\bar{Y}, \overline{Y^2})$$

$$\tilde{\mu} = E(\bar{Y}, \overline{Y^2}) = (0, \sigma^2)$$

$$\begin{cases} E(\bar{Y}) = 0 \\ E(\overline{Y^2}) = Var(\bar{Y}) - [E(\bar{Y})]^2 = \sigma^2 \end{cases}$$

$$\nabla g(\hat{u}) = \left(\frac{\partial g}{\partial \bar{Y}}, \frac{\partial g}{\partial \overline{Y^2}}\right) = (-2\underset{\downarrow}{\overset{0}{x}}, \underset{\downarrow}{\overset{1}{1}})^T = (0, 1)^T$$

$$\Sigma = \begin{pmatrix} Var(Y) & cor(Y, Y^2) \\ Cov(Y, Y^2) & Var(Y^2) \end{pmatrix} = \begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_4 & \mu^4 - \sigma^4 \end{pmatrix}$$

→ central moment

$$g(\tilde{u}) = \sigma^2.$$

$$[\nabla g(\tilde{u})]^T \Sigma \nabla g(\tilde{u}) = \mu_4 - \sigma^4$$

$$\Rightarrow S^2 \xrightarrow{d} N\left(\sigma^2, \frac{\mu_4 - \sigma^4}{n}\right)$$

From Lecture notes:

Theorem 5 (Delta Method): Let $\{Y_n: n = 1,2,\ldots\}$ be a sequence of random variables that for a constant a and a positive constant b,

$$\sqrt{n}(Y_n - a) \xrightarrow{d} N(0, b^2), \qquad as\ n \to \infty.$$

$Y_n \xrightarrow{d} N(a, \frac{b^2}{n})$

Then, for any given function g that $g'(a)$ exists and $g'(a) \neq 0$, we have

$$\sqrt{n}(g(Y_n) - g(a)) \xrightarrow{d} N(0, [g'(a)b]^2), \qquad as\ n \to \infty.$$

$g(Y_n) \xrightarrow{d} N(g(a), \frac{[g'(a)b]^2}{n})$

$\Downarrow$

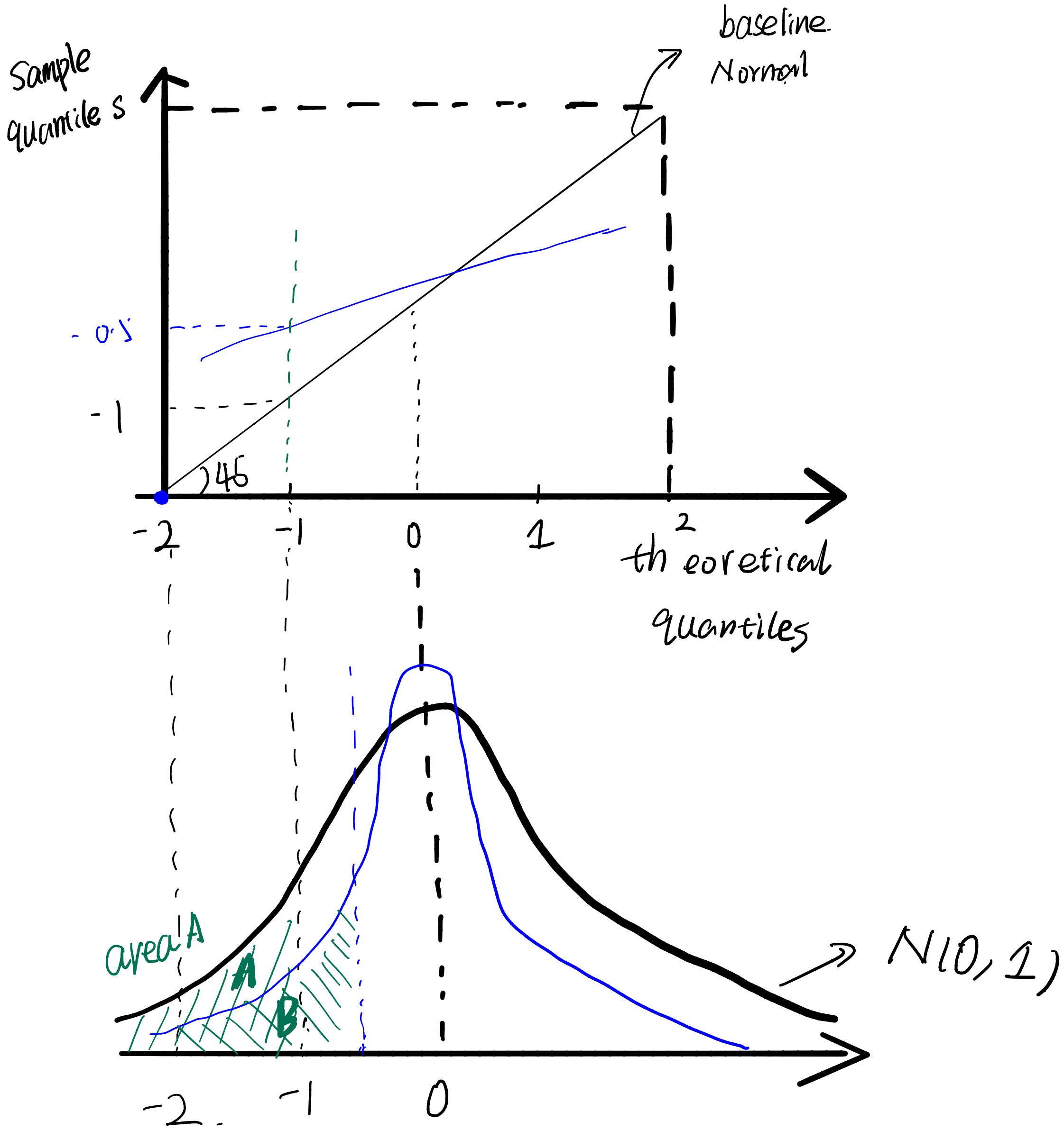
$\frac{\sqrt{n}(g(Y_n) - g(a))}{g'(a)b} \xrightarrow{d} N(0,1)$

The univariate case

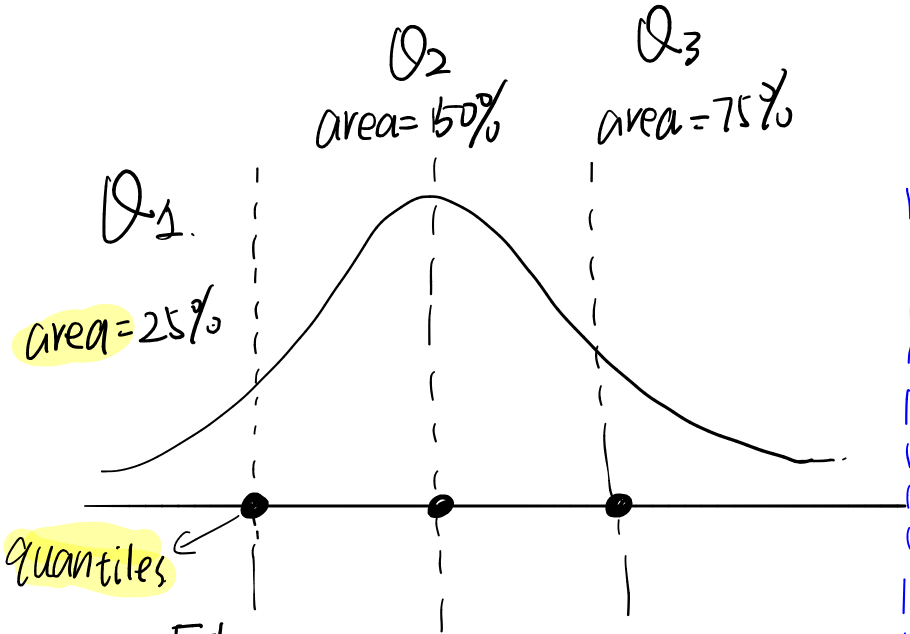
THEOREM **6.4** *Let $X_1, \ldots, X_n$ be i.i.d. with $\mu = EX_1$ and $\sigma^2 = Var(X_1) \in (0,\infty)$, and $g'(\mu) \neq 0$. Then,*

$$g(\bar{X}) \to_d N\left(g(\mu), n^{-1}[g'(\mu)]^2\sigma^2\right), \quad \Longleftrightarrow \quad \frac{\sqrt{n}(g(\bar{X}) - g(\mu))}{g'(\mu)\sigma} \to_d N(0,1). \qquad (5.1)$$

QQ plot



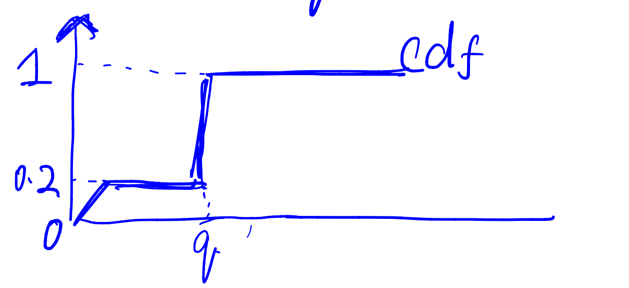
$A = B$



$q = F^{-1}(\text{area})$

quartile $\neq$ fn

☆ quantile can be not unique



$0.2 - 0.8 = q$