Mid-term (00/01)

1. $X_1, \ldots, X_n$ is a random sample from Bernoulli(θ).

(a) (i) $E(X) = \theta$

$\therefore M_1' = \widetilde{E(X)} = \tilde{\theta}$

$\therefore \tilde{\theta} = M_1' = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$ (method of moments estimator)

$L(\theta; \tilde{x}) = f_{\underset{\sim}{X}}(\underset{\sim}{x}; \theta) = \prod_{i=1}^{n} f_{X_i}(x_i; \theta)$

$= \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i}$

$\log L = \left(\sum_{i=1}^{n} x_i\right)(\log\theta) + \left(n - \sum_{i=1}^{n} x_i\right)\log(1-\theta)$

$\frac{\partial}{\partial\theta}\log L = \left(\sum_{i=1}^{n} x_i\right)\left(\frac{1}{\theta}\right) + \left(n - \sum_{i=1}^{n} x_i\right)\left(\frac{-1}{1-\theta}\right)$

$\frac{\partial}{\partial\theta}\log L\big|_{\theta=\hat{\theta}} = 0 \Rightarrow \left(\sum_{i=1}^{n} x_i\right)\left(\frac{1}{\hat{\theta}}\right) + \left(n - \sum_{i=1}^{n} x_i\right)\left(\frac{-1}{1-\hat{\theta}}\right) = 0$

$\Rightarrow \left(\sum_{i=1}^{n} x_i\right)\left(\frac{1}{\hat{\theta}}\right) = \left(n - \sum_{i=1}^{n} x_i\right)\left(\frac{1}{1-\hat{\theta}}\right)$

$\Rightarrow (1-\hat{\theta})\sum_{i=1}^{n} x_i = \hat{\theta}\left(n - \sum_{i=1}^{n} x_i\right) \Rightarrow \sum_{i=1}^{n} x_i = n\hat{\theta}$

$\Rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}$ (M.L.E.)

$\therefore$ the method of moments estimator of θ and maximum likelihood estimator of θ are identical.

(ii) Denote $T_1 = \overline{X}$

$E(T_1) = E(\overline{X}) = E(X) = \theta$

$\therefore T_1$ is unbiased

$Var(T_1) = Var(\overline{X}) = \frac{1}{n}Var(X) = \frac{\theta(1-\theta)}{n}$

(iii) $f_X(x;\theta) = \theta^x(1-\theta)^{1-x}$

$\log f_X(x;\theta) = x\log\theta + (1-x)\log(1-\theta)$

$\frac{\partial}{\partial\theta}\log f_X(x;\theta) = \frac{x}{\theta} - \frac{1-x}{1-\theta}$

$\frac{\partial^2}{\partial\theta^2}\log f_X(x;\theta) = -x\theta^{-2} + (1-x)(1-\theta)^{-2}(-1) = -x\theta^{-2} - (1-x)(1-\theta)^{-2}$

$E\left[\frac{\partial^2}{\partial\theta^2}\log f_X(x;\theta)\right] = E\left(-X\theta^{-2} - (1-X)(1-\theta)^{-2}\right)$

$= -(\theta)\theta^{-2} - (1-\theta)(1-\theta)^{-2} = -\theta^{-1} - (1-\theta)^{-1}$

$= -\frac{(1-\theta)+\theta}{\theta(1-\theta)} = \frac{-1}{\theta(1-\theta)}$

$\therefore$ the Cramer Rao lower bound of θ is

$-\frac{1}{nE\left[\frac{\partial^2}{\partial\theta^2}\log f_X(x;\theta)\right]} = \frac{\theta(1-\theta)}{n}$

$\therefore$ the variance of T_1 achieve the lower bound.

1. (b) $\tau(\theta)=\theta^n$

(i) $T_2 = \text{Min}\{X_1, \dots, X_n\}$, $\therefore$ T_2 can only take 2 possible values, namely, 0 and

$P(T_2 = 1) = P(\text{Min}\{X_1, \dots, X_n\} = 1)$

$= P(X_1=1, X_2=1, \dots, X_n=1) = \prod_{i=1}^{n} P(X_i = 1) = \prod_{i=1}^{n} \theta = \theta^n$

$P(T_2 = 0) = 1 - P(T_2 = 1) = 1-\theta^n$ $\quad\therefore\quad P(T_2 = t_2) = (\theta^n)^{t_2}(1-\theta^n)^{1-t_2}$

$E(T_2) = (0)P(T_2=0) + (1)P(T_2=1)$

$= (0)(1-\theta^n) + (1)\theta^n = \theta^n$

(ii) $Var(T_2) = E(T_2^2) - [E(T_2)]^2$

$= [(0)^2 P(T_2=0) + (1)^2 P(T_2=1)] - [\theta^n]^2$

$= \theta^n - \theta^{2n} = \theta^n(1-\theta^n)$

(iii) Since the MLE of θ is $\overline{X}$ (from the result of (a)(i)),

the MLE of $\tau(\theta)=\theta^n$ is $\overline{X}^n$ by the invariant property of MLE

2. $X_1, \dots, X_n$ is a random sample from $N(\theta, 1)$.
Let T be an estimator of θ.

(a) $bias(T) = E(T) - \theta$

$$Var(T) = E(T - E(T))^2$$

$$MSE(T) = E(T-\theta)^2$$

$$\begin{aligned}MSE(T) &= E(T-\theta)^2 \\ &= E(T - E(T) + E(T) - \theta)^2 \\ &= E(T-E(T))^2 + 2E[(T-E(T))(E(T)-\theta)] + E(E(T)-\theta)^2 \\ &= Var(T) + 2(E(T)-E(T))(E(T)-\theta) + (E(T)-\theta)^2 \\ &= Var(T) + [bias(T)]^2\end{aligned}$$

(b) $L(\theta; \underset{\sim}{x}) = f_{\underset{\sim}{X}}(\underset{\sim}{x}; \theta) = \prod_{i=1}^n f_{X_i}(x_i; \theta) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi}} \exp\{\frac{-1}{2}(x_i-\theta)^2\}$

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\sum_{i=1}^n (x_i-\theta)^2$$

$$\frac{\partial}{\partial\theta}\log L = -\sum_{i=1}^n (x_i-\theta)(-1) = \sum_{i=1}^n x_i - n\theta$$

$$\frac{\partial}{\partial\theta}\log L|_{\theta=\hat\theta} = 0 \Rightarrow \sum_{i=1}^n x_i - n\hat\theta = 0$$

$$\Rightarrow \hat\theta = \frac{1}{n}\sum_{i=1}^n X_i = \overline{X} \quad (M.L.E.)$$

(c) $Var(T_1) = Var(c\hat\theta) = Var(c\overline{X}) = c^2 Var(\overline{X}) = \frac{c^2}{n} Var(X) = \frac{c^2}{n} \quad (c>0)$

$$\begin{aligned}bias(T_1) &= E(T_1) - \theta = E(c\hat\theta) - \theta = E(c\overline{X}) - \theta = cE(\overline{X}) - \theta \\ &= c\theta - \theta = (c-1)\theta\end{aligned}$$

$\therefore MSE(T_1) = Var(T_1) + [bias(T_1)]^2 = \frac{c^2}{n} + [(c-1)\theta]^2.$

$\therefore MSE(\hat\theta) = \frac{1^2}{n} + [(1-1)\theta]^2 = \frac{1}{n}$

let $g(c) = MSE(T_1) - MSE(\hat\theta)$

$$= \frac{c^2}{n} + (c-1)^2\theta^2 - \frac{1}{n} = \frac{c^2-1}{n} + (c-1)^2\theta^2$$

$$g(c) < 0$$

$$\Rightarrow \frac{c^2-1}{n} + (c-1)^2\theta^2 < 0$$

$$\Rightarrow (\frac{1}{n} + \theta^2)c^2 - 2\theta^2 c + (\theta^2 - \frac{1}{n}) < 0$$

$$\because (\frac{1}{n} + \theta^2) > 0$$

$$\therefore \frac{2\theta^2 - \sqrt{(2\theta^2)^2 - 4(\frac{1}{n}+\theta^2)(\theta^2-\frac{1}{n})}}{2(\frac{1}{n}+\theta^2)} < c < \frac{2\theta^2 + \sqrt{(2\theta^2)^2 - 4(\frac{1}{n}+\theta^2)(\theta^2-\frac{1}{n})}}{2(\frac{1}{n}+\theta^2)}$$

$$\Rightarrow \frac{\theta^2 - \sqrt{\theta^4 - (\theta^4 - \frac{1}{n^2})}}{(\frac{1}{n}+\theta^2)} < c < \frac{\theta^2 + \sqrt{\theta^4 - (\theta^4 - \frac{1}{n^2})}}{(\frac{1}{n}+\theta^2)}$$

$$\Rightarrow \frac{\theta^2 - \frac{1}{n}}{\theta^2 + \frac{1}{n}} < c < 1$$

2 (c) (cont'd)

For the values of c between $\frac{\theta^2 - \frac{1}{n}}{\theta^2 + \frac{1}{n}}$ and 1,

$MSE(T_1) < MSE(\hat{\theta})$.

As $n \to \infty$, $\frac{\theta^2 - \frac{1}{n}}{\theta^2 + \frac{1}{n}} \to 1$.

$\therefore$ as $n \to \infty$, $MSE(T_1) \geq MSE(\hat{\theta})$

3. (a) Z_1, Z_2 is a random sample from $N(0,1)$.

X_1, X_2 is a random sample from $N(1,1)$.

Suppose the Z_i's are independent of the X_j's.

(i) $\bar{X} \sim N(1, \frac{1}{2})$, $\bar{Z} \sim N(0, \frac{1}{2})$

$\therefore \bar{X} \pm \bar{Z} \sim N(1, \frac{1}{2}+\frac{1}{2}) \sim N(1,1)$

(ii) $Z_1 + Z_2 \sim N(0,2)$

$X_2 - X_1 \sim N(0,2) \quad \therefore \frac{1}{2}(X_2 - X_1)^2 \sim \chi^2_{(1)}$

$Z_2 - Z_1 \sim N(0,2) \quad \therefore \frac{1}{2}(Z_2 - Z_1)^2 \sim \chi^2_{(1)}$

$\therefore \frac{1}{2}[(X_2 - X_1)^2 + (Z_2 - Z_1)^2] \sim \chi^2_{(2)}$

$$\therefore \frac{Z_1 + Z_2}{\sqrt{[(X_2-X_1)^2 + (Z_2-Z_1)^2]/2}} = \frac{\frac{1}{\sqrt{2}}(Z_1+Z_2)}{\sqrt{[(\frac{1}{2})((X_2-X_1)^2+(Z_2-Z_1)^2)]/2}} \sim t_2$$

(iii) $\frac{1}{2}[(X_2 - X_1)^2 + (Z_2 - Z_1)^2] \sim \chi^2_{(2)}$

(iv) $X_2 + X_1 - 2 \sim N(0,2) \quad \therefore \frac{1}{2}(X_2 + X_1 - 2)^2 \sim \chi^2_{(1)}$

$X_2 - X_1 \sim N(0,2) \quad \therefore \frac{1}{2}(X_2 - X_1)^2 \sim \chi^2_{(1)}$

$$\therefore \frac{(X_2+X_1-2)^2}{(X_2-X_1)^2} = \frac{[\frac{1}{2}(X_2+X_2-2)^2]}{[\frac{1}{2}(X_2-X_1)^2]} \sim F_{1,1}$$

(b) Let m be the median.

(i) X_1, X_2 is the two random observations from any continuous distribution.

$P(\max\{X_1, X_2\} > m) = 1 - P(\max\{X_1, X_2\} \le m)$

$= 1 - P(X_1 \le m, X_2 \le m) = 1 - \prod_{i=1}^{2} P(X_i \le m) = 1 - \prod_{i=1}^{2}(\frac{1}{2}) = 1 - (\frac{1}{2})^2 = \frac{3}{4}$

(ii) $X_1, \cdots, X_n$ is the n random observations from any continuous distribution.

$P(\max\{X_1, X_2, \cdots, X_n\} > m) = 1 - P(\max\{X_1, X_2, \cdots, X_n\} \le m)$

$= 1 - P(X_1 \le m, X_2 \le m, \cdots, X_n \le m) = 1 - \prod_{i=1}^{n} P(X_i \le m)$

$= 1 - (\frac{1}{2})^n$