MATH 243 Statistical Inference

Final Examination - Fall 98/99

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1. (a) Two hundred randomly selected electronic devices of a certain type were tested and the following frequency distribution of their 'lift-times', measured in months, was compiled:

Lift time (months)	Observed freq.
$0 \le x < 3$	53
$3 \le x < 6$	42
$6 \le x < 9$	35
$9 \le x$	70

Using a test at significance level 0.01, examine the null hypothesis H_0 that the life time distribution is an exponential distribution with mean 12.

1. (b) The data of the following table were obtained from a random sample of 300 car owners, each of whom was classified both according to age and to the number of accidents he or she had been involved in during the past two years. Using a test at the 5% level of significance, examine the null hypothesis that there is no dependence of accident rate on age in the sample population.

		Number of Accidents		
		0	1	2 or more
	Under 22 year	10	21	14
Age	Between 22 and 32 years	22	43	10
	Over 32 years	81	80	19

2. (a) If X is the number of successes in n binomial trials, $\hat{\theta}_1 = X/n$, and $\hat{\theta}_2 = (X+1)/(n+2)$, for what values of θ is $E\left[\left(\hat{\theta}_2 - \theta\right)^2\right]$ less than $E\left[\left(\hat{\theta}_1 - \theta\right)^2\right]$? Do you consider $\hat{\theta}_1$ as a better estimator than $\hat{\theta}_2$?

2. (b) Let $Y_1 < Y_2 < Y_3$ be the order statistics of a random sample of size 3 from the uniform distribution having p.d.f. $f(x;\theta) = 1/\theta,\ 0 < x < \theta,\ 0 < \theta < \infty$, zero elsewhere. Show that $4Y_1$ and $\frac{4}{3}Y_3$ are unbiased statistics for θ. Find the variance of each of these unbiased statistics.

3. Let $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ..., Y_n)$ be random samples from the independent normal distributions $N(\theta_1, \theta_3)$ and $N(\theta_2, \theta_4)$, respectively.

(a) Show that the likelihood ratio for testing $H_0 : \theta_1 = \theta_2,\ \theta_3 = \theta_4$ against a general alternative is given by

$$\frac{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2 \Big/ n\right]^{n/2} \left[\sum_{j=1}^{m}(y_j - \bar{y})^2 \Big/ m\right]^{m/2}}{\left\{\left[\sum_{i=1}^{n}(x_i - u)^2 + \sum_{j=1}^{m}(y_j - u)^2\right] \Big/ (m+n)\right\}^{(m+n)/2}}$$

where $u = (n\bar{x} + m\bar{y})/(n+m)$.

3. (b) Show that the likelihood ratio test for testing $H_0 : \theta_3 = \theta_4,\ \theta_1$ and θ_2 unspecified, against $H_1 : \theta_3 \neq \theta_4,\ \theta_1$ and θ_2 unspecified, can be based on the random variable

$$F = \frac{\sum\limits_{i=1}^{n} (x_i - \bar{x})^2 \Big/ (n-1)}{\sum\limits_{j=1}^{m} (y_j - \bar{y})^2 \Big/ (m-1)}$$

Hence, construct the critical region at significance level α.

4. (a) Let a random sample of size n be taken from a distribution of the discrete type with p.d.f. $f(x;\theta) = 1/\theta,\ x = 1, 2, ..., \theta$, zero elsewhere, where θ is an unknown positive integer.

   (i) Show that the largest item, say Y, of the sample is a complete sufficient statistic for θ.

   (ii) Prove that

   $$[Y^{n+1} - (Y-1)^{n+1}]/[Y^n - (Y-1)^n]$$

   is the best statistic for θ.

4. (b) Let $(X_1, ..., X_n)$ be a sample from $P_0(\lambda)$, find the UMVUE of $\tau(\lambda) = \dfrac{\lambda^k e^{-\lambda}}{k!}$ for any $k = 0, 1, 2, ...$.

5. Suppose that $(X_1, ..., X_n)\,, n > 2$, is a random sample from an exponential distribution with p.d.f. of the form

$$f(x;\lambda) = \lambda e^{-\lambda x}, \qquad x \in [0, \infty),$$

where $\lambda \in (0, \infty)$

(a) Show that the Cramer-Rao lower bound for estimating λ is λ^2/n.

(b) Show further that the statistic $1/\bar{X}$ is biased for λ, but that

$$T = \frac{n-1}{\sum\limits_{k=1}^{n} X_k}$$

is an unbiased estimator for λ.

(c) Find $Var[T]$ and compare it with the Cramer-Rao lower bound.