1. $X_1 \ldots X_n \overset{iid}{\sim} Poi(\theta)$

a. $f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!} = \frac{e^{-n\theta}\theta^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} = e^{-\theta n - \log \prod_{i=1}^n x_i! + (\sum_{i=1}^n x_i)\log\theta}$

Hence it is in exp family and $d(\underline{X}) = \sum_{i=1}^n X_i$ is C-S stat.

$\sum_{i=1}^n X_i \sim Poi(n\theta)$

b. $E\left(h\left(\sum_{i=1}^n X_i\right)\right) = \theta^3$

$\Rightarrow \sum_{x=0}^{\infty} h(x) \cdot \frac{e^{-n\theta}(n\theta)^x}{x!} = \theta^3$

$\Rightarrow \sum_{x=0}^{\infty} h(x) \cdot \frac{n^x \cdot e^{-n\theta}\theta^{x-3}}{x!} = 1$

put $h(0) = h(1) = h(2) = 0$

$\Rightarrow \sum_{x=3}^{\infty} h(x) \cdot \frac{n^3}{x(x-1)(x-2)} \cdot \underbrace{\frac{e^{-n\theta}(n\theta)^{x-3}}{(x-3)!}}_{p.m.f\ of\ Poi(n\theta)} = 1$

Hence $h(x) = \frac{x(x-1)(x-2)}{n^3}$

$\because \sum_{i=1}^n X_i$ is C-S stat.

$\therefore h\left(\sum_{i=1}^n X_i\right) = \frac{\left(\sum_{i=1}^n X_i\right)\left(\sum_{i=1}^n X_i - 1\right)\left(\sum_{i=1}^n X_i - 2\right)}{n^3}$ is UMVUE of θ^3

$\because \log f_{\underline{X}}(\underline{x}) = -\theta n - \log \prod_{i=1}^n x_i! + \left(\sum_{i=1}^n x_i\right)(\log\theta)$

$\therefore \frac{\partial}{\partial\theta}\log f_{\underline{X}}(\underline{x},\theta) = -n + \frac{\sum_{i=1}^n X_i}{\theta} = -\frac{n}{\theta}\left(\frac{\sum_{i=1}^n X_i}{n} - \theta\right)$

$= -\frac{n}{\theta^3}\left(\frac{\theta^2\sum_{i=1}^n X_i}{n} - \theta^3\right)$

Hence the CRLB can not be achieved.

c. $P(X_1=0) = e^{-\theta}$

$E\left(h\left(\sum_{i=1}^{n} X_i\right)\right) = e^{-\theta}$

$\Rightarrow \sum_{x=0}^{\infty} h(x) \cdot \frac{e^{-n\theta}(n\theta)^x}{x!} = e^{-\theta}$

$\Rightarrow \sum_{x=0}^{\infty} h(x) \cdot \frac{e^{-(n-1)\theta}(n\theta)^x}{x!} = 1$

$\Rightarrow \sum_{x=0}^{\infty} h(x) \left(\frac{n}{n-1}\right)^x \underbrace{\frac{e^{-(n-1)\theta}((n-1)\theta)^x}{x!}}_{\text{p.m.f of Poi}((n-1)\theta)} = 1$

$\therefore h(x) = \left(\frac{n-1}{n}\right)^x$

$\because \sum_{i=1}^{n} X_i$ is C-S stat

$\therefore h\left(\sum_{i=1}^{n} X_i\right) = \left(\frac{n-1}{n}\right)^{\sum_{i=1}^{n} X_i}$ is UMVUE of $e^{-\theta}$

d. $P(X_1=1) = \theta e^{-\theta}$

$E\left(h\left(\sum_{i=1}^{n} X_i\right)\right) = \theta e^{-\theta}$

$\Rightarrow \sum_{x=0}^{\infty} h(x) \cdot \frac{e^{-(n-1)\theta} n^x \theta^{x-1}}{x!} = 1$

$\Rightarrow \sum_{x=0}^{\infty} h(x) \cdot \frac{n}{x} \cdot \frac{n^{x-1}}{(n-1)^{x-1}} \cdot \underbrace{\frac{e^{-(n-1)\theta}((n-1)\theta)^{x-1}}{(x-1)!}}_{\text{p.m.f of Poi}((n-1)\theta)} = 1$

set $h(x) = 0$

$\Rightarrow h(x) = \frac{x}{n} \cdot \left(\frac{n-1}{n}\right)^{x-1}$

$\because \sum_{i=1}^{n} X_i$ is C-S stat

$\therefore h\left(\sum_{i=1}^{n} X_i\right) = \left(\frac{\sum_{i=1}^{n} X_i}{n}\right)\left(\frac{n-1}{n}\right)^{\left(\sum_{i=1}^{n} X_i\right)-1}$ is UMVUE of $\theta e^{-\theta}$

e. $\because$ $P(X_1 > 1) = 1 - P(X_1 = 0) - P(X_1 = 1)$

$= 1 - e^{-\theta} - \theta e^{-\theta}$

$\therefore E\left(1 - \left(\frac{n-1}{n}\right)^{\sum_{i=1}^{n} X_i} - \left(\frac{\sum_{i=1}^{n} X_i}{n}\right)\left(\frac{n-1}{n}\right)^{\left(\sum_{i=1}^{n} X_i\right)-1}\right) = 1 - e^{-\theta} - \theta e^{-\theta}$

from part c. and d.

$\because \sum_{i=1}^{n} X_i$ is CS stat.

$\therefore 1 - \left(\frac{n-1}{n}\right)^{\sum_{i=1}^{n} X_i} - \left(\frac{\sum_{i=1}^{n} X_i}{n}\right)\left(\frac{n-1}{n}\right)^{\left(\sum_{i=1}^{n} X_i\right)-1}$ is UMVUE of $P(X_1 > 1)$

c. $\because \frac{\partial^2}{\partial \theta^2} \log f_{\underline{X}}(\underline{x}, \theta) = -\frac{\sum_{i=1}^{n} x_i}{\theta^2}$

$\Rightarrow E\left(\frac{\partial^2}{\partial \theta^2} \log f_{\underline{X}}(\underline{x}, \theta)\right) = -\frac{n\theta}{\theta^2} = -\frac{n}{\theta}$

$\therefore$ CRLB of $e^{-\theta} = \frac{(-e^{-\theta})^2}{-\left(-\frac{n}{\theta}\right)} = \frac{\theta e^{-2\theta}}{n}$

2a. $X_1 \ldots, X_n \overset{iid}{\sim} \exp(\theta)$

a. $\because$ $f_{\underline{x}}(\underline{x}) = \prod_{i=1}^{n} \theta e^{-\theta x_i} = \theta^n e^{-\theta \sum_{i=1}^{n} x_i} = e^{n \log\theta - \theta \sum_{i=1}^{n} x_i}$

Hence it is in exp family and $J(\underline{X}) = \sum_{i=1}^{n} X_i$ is C-S stat.

$\sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \theta)$

b. $E\left(h\left(\sum_{i=1}^{n} X_i\right)\right) = \theta$

$\Rightarrow \int_0^\infty h(x) \cdot \frac{\theta^n x^{n-1} e^{-\theta x}}{(n-1)!} \cdot \frac{1}{\theta} \, dx = 1$

$\Rightarrow \int_0^\infty h(x) \frac{x}{n-1} \cdot \underbrace{\frac{\theta^{n-1} \cdot x^{n-2} \cdot e^{-\theta x}}{(n-2)!}}_{\text{p.d.f of } \Gamma(n-1, \theta)} \, dx = 1$

$\therefore h(x) = \frac{n-1}{x}$

$\because \sum_{i=1}^{n} X_i$ is C-S stat.

$\therefore h\left(\sum_{i=1}^{n} X_i\right) = \frac{n-1}{\sum_{i=1}^{n} X_i}$ is UMVUE of θ

c. $P(X_1 > a) = e^{-a\theta}$

$\because L(\theta) = \theta^n e^{-\theta \sum_{i=1}^{n} x_i}$

$\therefore \log L(\theta) = n \log\theta - \theta \sum_{i=1}^{n} x_i$

$\frac{\partial}{\partial\theta} \log L(\theta) = 0 \Rightarrow \frac{n}{\theta} = \sum_{i=1}^{n} x_i \Rightarrow \hat{\theta} = \frac{n}{\sum_{i=1}^{n} X_i} = \frac{1}{\bar{X}}$

$\therefore$ MLE of $e^{-a\theta} = e^{-\frac{a}{\bar{X}}}$ by invariant property.

$$E\left(h\left(\sum_{i=1}^{n} X_i\right)\right) = e^{-a\theta}$$

$$\Rightarrow \int_0^\infty h(x) \frac{\theta^n x^{n-1} e^{-\theta(x-a)}}{(n-1)!} dx = 1$$

set $h(x) = 0$ for $x < a$.

$$\Rightarrow \int_a^\infty h(x) \cdot \left(\frac{x}{x-a}\right)^{n-1} \cdot \underbrace{\frac{\theta^n (x-a)^{n-1} e^{-\theta(x-a)}}{(n-1)!}}_{\text{p.d.f of } \Gamma(n-1,\theta)} dx = 1$$

$$\therefore h(x) = \left(\frac{x-a}{x}\right)^{n-1} \text{ for } x \geq a.$$

$\because \sum_{i=1}^{n} X_i$ is C-S stat

$\therefore h\left(\sum_{i=1}^{n} X_i\right) = \left(1 - \frac{a}{\sum_{i=1}^{n} X_i}\right)^{n-1} I\left(\sum_{i=1}^{n} X_i \geq a\right)$ is UMVUE of $e^{-a\theta}$

d. $\frac{\partial}{\partial\theta} \log f_{\underline{X}}(\underline{x}, \theta) = \frac{n}{\theta} - \sum_{i=1}^{n} X_i$

$$\frac{\partial^2}{\partial\theta^2} \log f_{\underline{X}}(\underline{x}, \theta) = -\frac{n}{\theta^2}$$

$$\therefore \text{CRLB of } e^{-a\theta} = \frac{\left(-a e^{-a\theta}\right)^2}{-\left(-\frac{n}{\theta^2}\right)} = \frac{\theta^2 a^2 e^{-2a\theta}}{n}$$

$$\because \frac{\partial}{\partial\theta} \log f_{\underline{X}}(\underline{x}, \theta) = -\frac{n}{\theta} \cdot e^{a\theta} \left(\underbrace{\frac{\theta e^{-a\theta}}{n}}_{\text{involve } \theta} \sum_{i=1}^{n} X_i - e^{-a\theta}\right)$$

$\therefore$ the CRLB can not be achived

e. $P(X_1 > b \mid X_1 > a) = \frac{P(X_1 > b, X_1 > a)}{P(X_1 > a)} = \frac{P(X_1 > b)}{P(X_1 > a)}$

$$= \frac{e^{-b\theta}}{e^{-a\theta}} = e^{-(b-a)\theta}$$

$\therefore$ UMVUE of $e^{-(b-a)\theta}$ is $\left(1 - \frac{b-a}{\sum_{i=1}^{n} X_i}\right)^{n-1} I\left(\sum_{i=1}^{n} X_i \geq b-a\right)$.

3. $X_1 \ldots X_n \overset{iid}{\sim} N(0, \sigma^2)$

a. $\begin{cases} H_0: \sigma^2 = \sigma_0^2 \\ H_1: \sigma^2 < \sigma_0^2 \end{cases}$

$\because \quad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} = e^{-\frac{1}{2}\log 2\pi\sigma^2 - \frac{x^2}{2\sigma^2}}$

$\because \quad -\frac{1}{2\sigma^2}$ is increasing function of σ^2

$\therefore \quad \{ \sum_{i=1}^n X_i^2 \le k \}$ is CR for UMP test

for $\begin{cases} H_0: \sigma^2 = \sigma_0^2 \\ H_1: \sigma^2 < \sigma_0^2. \end{cases}$

$\alpha = P\left(\sum_{i=1}^n X_i^2 \le k \mid \sigma^2 = \sigma_0^2 \right) = P\left(\sum_{i=1}^n \left(\frac{X_i}{\sigma_0}\right)^2 \le \frac{k}{\sigma_0^2} \mid \sigma^2 = \sigma_0^2 \right)$

$= P\left(\chi_n^2 \le \frac{k}{\sigma_0^2} \mid \sigma^2 = \sigma_0^2 \right) \Rightarrow k = \sigma_0^2 \chi^2_{n,\alpha}$

Hence the UMP test is $\{ \sum_{i=1}^n X_i^2 \le \sigma_0^2 \chi^2_{n,\alpha} \}$

b. $Q(\sigma^2) = P\left(\sum_{i=1}^n X_i^2 \le \sigma_0^2 \chi^2_{n,\alpha} \mid \sigma^2 = \sigma^2 \right)$

$= P\left(\sum_{i=1}^n \left(\frac{X_i}{\sigma}\right)^2 \le \frac{\sigma_0^2}{\sigma^2} \chi^2_{n,\alpha} \mid \sigma^2 = \sigma^2 \right)$

$= P\left(\chi_n^2 \le \frac{\sigma_0^2}{\sigma^2} \chi^2_{n,\alpha} \right)$

Since for $\sigma_1^2 > \sigma_0^2$.

$Q(\sigma_1^2) = P\left(\chi_n^2 \le \frac{\sigma_0^2}{\sigma_1^2} \chi^2_{n,\alpha} \right) \le \underbrace{P\left(\chi_n^2 \le \frac{\sigma_0^2}{\sigma_0^2} \chi^2_{n,\alpha} \right)}_{= \alpha}$

$\Rightarrow \sup_{\sigma^2 > \sigma_0^2} Q(\sigma^2) = \alpha$

$\Rightarrow$ the UMP test of $\begin{cases} H_0: \sigma^2 \ge \sigma_0^2 \\ H_1: \sigma^2 < \sigma_0^2 \end{cases}$ is the same for the part a.

c. $X_1 \dots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$$\begin{cases} H_0: \sigma^2 = \sigma_0^2 \\ H_1: \sigma^2 \neq \sigma_0^2 \end{cases}$$

i) $L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

under H_0: $L(\mu, \sigma_0^2) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i-\mu)^2}$

$$\log L(\mu, \sigma_0^2) = -\frac{n}{2}\log 2\pi\sigma_0^2 - \frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial}{\partial \mu}\log L(\mu, \sigma_0^2) = 0 \Rightarrow \hat{\mu} = \bar{X}$$

under H_a: $L(\mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$

$$\log L(\mu, \sigma^2) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\begin{cases} \frac{\partial}{\partial \mu}\log L(\mu, \sigma^2) = 0 \\ \frac{\partial}{\partial \sigma^2}\log L(\mu, \sigma^2) = 0 \end{cases} \Rightarrow \begin{array}{l} \hat{\mu} = \bar{X} \\ \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 = S_n^2 \end{array}$$

$$\therefore \lambda(\underline{x}) = \frac{\sup_{\mu, \sigma_0^2} L(\mu; \sigma_0^2)}{\sup_{\mu, \sigma^2} L(\mu, \sigma^2)} = \frac{\left(\frac{1}{2\pi\sigma_0^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(X_i-\bar{X})^2}}{\left(\frac{1}{2\pi\hat{\sigma}^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^{n}(X_i-\bar{X})^2}}$$

$$= \left(\frac{S_n^2}{\sigma_0^2}\right)^{\frac{n}{2}} \cdot e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(X_i-\bar{X})^2 + \frac{n}{2}}$$

$$= \left(\frac{1}{n}\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma_0^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(X_i-\bar{X})^2 + \frac{n}{2}}$$

$\therefore \lambda(x) \le k \iff \left(\frac{1}{n}\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma_0^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(X_i-\bar{X})^2+\frac{n}{2}} \le k$

$\iff \left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma_0^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2}\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma_0^2}} \le k.$

Consider $g(x) = x^{\frac{n}{2}} e^{-\frac{x}{2}}$.

$g'(x) = \frac{n}{2} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} - \frac{1}{2} x^{\frac{n}{2}} e^{-\frac{x}{2}}$

$= e^{-\frac{x}{2}} x^{\frac{n}{2}-1}\left(\frac{n}{2}-\frac{1}{2}x\right) = 0 \Rightarrow x = n.$

$g'(x) > 0$ when $x < n$

$g'(x) < 0$ when $x > n$.

$\therefore g(x)$ attain maximum at $x = n$ and decreasing when $x > n$, increasing when $x < n$

$\therefore g(x) \le k \iff x \le k_1$ or $x \ge k_2$ for some k_1, k_2

$\therefore \lambda(x) \le k \iff \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma_0^2} \le k_1$ or $\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma_0^2} \ge k_2$

$\because$ under H_0, $\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma_0^2} \sim \chi^2_{n-1}$

$\therefore P\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma_0^2} \le k_1\right) = P\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma_0^2} \ge k_2\right) = \frac{\alpha}{2}$

$\Rightarrow k_1 = \chi^2_{n-1, 1-\frac{\alpha}{2}}$ $\quad k_2 = \chi^2_{n-1, \frac{\alpha}{2}}$

4a $Y_1 \dots Y_n \overset{iid}{\sim} Bin(1, \theta)$

$$\begin{cases} H_0: \theta = 0.5 \\ H_1: \theta \neq 0.5 \end{cases}$$

i $L(\theta) = \prod_{i=1}^{n} \theta^{y_i} (1-\theta)^{1-y_i}$

$= \theta^{\sum_{i=1}^{n} y_i} (1-\theta)^{n - \sum_{i=1}^{n} y_i}$

under H_0: $L(0.5) = 0.5^n$.

under H_a: $\log L(\theta) = \left(\sum_{i=1}^{n} y_i\right) \log\theta + \left(n - \sum_{i=1}^{n} y_i\right)\log(1-\theta)$

$\frac{\partial}{\partial\theta} \log L(\theta) = \frac{\sum_{i=1}^{n} y_i}{\theta} + \frac{n - \sum_{i=1}^{n} y_i}{1-\theta} = 0$

$\Rightarrow \hat{\theta} = \frac{\sum_{i=1}^{n} y_i}{n} = \bar{Y}$

$\therefore \lambda(y) = \frac{L(0.5)}{\sup_\theta L(\theta)} = \frac{0.5^n}{\bar{Y}^{n\bar{Y}} (1-\bar{Y})^{n-n\bar{Y}}}$

ii $\lambda(y) \leq k \Rightarrow \bar{Y}^{n\bar{Y}} (1-\bar{Y})^{n-n\bar{Y}} \geq k.$

$\Rightarrow n\bar{Y} \log \bar{Y} + n(1-\bar{Y}) \log(1-\bar{Y}) \geq k.$

$\Rightarrow \bar{Y} \log \bar{Y} + (1-\bar{Y}) \log(1-\bar{Y}) \geq k$

let $g(x) = x \log x + (1-x) \log(1-x)$

$g'(x) = 1 + \log x - \log(1-x) - 1 = 0$

$\Rightarrow x = 0.5$

$g'(x) < 0$ when $x < 0.5$

$g'(x) > 0$ when $x > 0.5$

Hence $x = 0.5$ is minimum point and $g(x)$ is decreasing when $x < 0.5$, when $x > 0.5$ $g(x)$ is increasing also, $g(1-x) = (1-x)\log(1-x) + x\log x = g(x)$

$\therefore$ $g(x)$ is symmetric at $x = 0.5$

$\therefore$ $g(x) \geq k \iff |x - 0.5| \geq k'$

Hence $\lambda(\underline{y}) \leq k \iff |\bar{Y} - 0.5| \geq k$

(iii) By CLT $Y_1 \ldots Y_n \overset{iid}{\sim} Bin(1, \theta)$

$$\therefore \frac{\bar{Y} - \theta}{\sqrt{\frac{\theta(1-\theta)}{n}}} \longrightarrow N(0,1)$$

under H_0: $\theta = 0.5$

$$\therefore \frac{\bar{Y} - 0.5}{\sqrt{\frac{0.25}{n}}} \longrightarrow N(0,1)$$

Hence $p(|\bar{Y} - 0.5| \geq k \mid \theta = 0.5)$

$$= p\left(\frac{|\bar{Y} - 0.5|}{\sqrt{\frac{0.25}{n}}} \geq \frac{k}{\sqrt{\frac{0.25}{n}}} \,\Big|\, \theta = 0.5\right) = \alpha$$

By CLT $\Rightarrow$ $p\left(|Z| \geq \frac{k}{\sqrt{\frac{0.25}{n}}} \,\Big|\, \theta = 0.5\right) = \alpha$ $\quad Z \sim N(0,1)$

$$\Rightarrow \frac{k}{\sqrt{\frac{0.25}{n}}} = Z_{\frac{\alpha}{2}} \Rightarrow k = \frac{0.5\, Z_{\frac{\alpha}{2}}}{\sqrt{n}}$$

$\therefore$ reject H_0 when $\left\{ \bar{Y} : |\bar{Y} - 0.5| \geq \frac{0.5 \cdot Z_{\frac{\alpha}{2}}}{\sqrt{n}} \right\}$

b.(i) $(X_1, X_2) \sim \text{mult}(n, p_1, p_2)$ $(p_1 + p_2 = 1)$

$(x_1 + x_2 = n)$

$L(p_1, p_2) = \cdots p_1^{x_1} p_2^{x_2}$

under $H_0: p_1 = p_2 \Rightarrow p_1 = 0.5 = p_2$

$\therefore L(0.5, 0.5) = 0.5^{x_1} \cdot 0.5^{x_2}$

under $H_a: \hat{p}_1 = \frac{x_1}{n}, \hat{p}_2 = \frac{x_2}{n}$

$\therefore \lambda(\underline{x}) = \frac{0.5^{x_1} \cdot 0.5^{x_2}}{\left(\frac{x_1}{n}\right)^{x_1} \left(\frac{x_2}{n}\right)^{x_2}} = \left(\frac{0.5n}{x_1}\right)^{x_1} \left(\frac{0.5n}{x_2}\right)^{x_2}$

for large sample likelihood ratio test.

reject H_0 when $\{\underline{x} : -2\log \lambda(\underline{x}) \geq \chi^2_{1,\alpha}\}$, where

$-2\log\lambda(\underline{x}) = -2\left(x_1 \cdot \log\frac{0.5n}{x_1} + x_2 \log\frac{0.5n}{x_2}\right)$

(ii) $G = \frac{(x_1 - 0.5n)^2}{0.5n} + \frac{(x_2 - 0.5n)^2}{0.5n}$

c. for part a: assume $y_i = \begin{cases} 1 & \text{if male} \\ 0 & \text{if female} \end{cases}$

Hence $\bar{Y} = \frac{55}{n} = 0.55$.

$|\bar{Y} - 0.5| = 0.05 < \sqrt{\frac{0.25}{100}}\ 1.96 = 0.098$

do not reject H_0 under 5% significant level

for part b.

$-2\left(55 \cdot \log\frac{0.5 \cdot 100}{55} + 45 \log\frac{0.5 \cdot 100}{45}\right) = 1.0017$

$\chi^2_{1,0.05} = 3.841 > -2\log\lambda(\tilde{x})$

Hence do not reject H_0 under 5% level of significant

for Pearson's test.

$$G = \frac{(55-50)^2}{50} + \frac{(45-50)^2}{50} = 1$$

reject H_0 when $G > \chi^2_{1,0.05} = 3.841$

Hence do not reject H_0 under 5% level of significant.

d part. a: $P\left(\frac{|\bar{Y}-0.5|}{\sqrt{\frac{0.25}{n}}} \geq K \mid \theta = 0.5 \right) = \alpha$.

By CLT, $\frac{\bar{Y}-0.5}{\sqrt{\frac{0.25}{n}}} \sim N(0,1) \Rightarrow \frac{(\bar{Y}-0.5)^2}{\frac{0.25}{n}} \sim \chi^2_1$

$\therefore P\left(\frac{(\bar{Y}-0.5)^2}{\frac{0.25}{n}} \geq K \right) = \alpha \Rightarrow K = \chi^2_{\alpha,1}$

and. Hence reject H_0 when $\frac{(\bar{Y}-0.5)^2}{\frac{0.25}{n}} \geq \chi^2_{\alpha,1}$

$$G = \frac{(x_1-0.5n)^2}{0.5n} + \frac{(x_2-0.5n)^2}{0.5n} = \frac{x_1^2 + x_2^2 - n(x_1+x_2) + 0.5^2n^2 \times 2}{0.5n}$$

$$= \frac{x_1^2 + n^2 - 2nx_1 + x_1^2 - n^2 + 0.5n^2}{0.5n} = \frac{2x_1^2 - 2nx_1 + 0.5n^2}{0.5n}$$

$$= \frac{x_1^2 - nx_1 + 0.25n^2}{0.25n} = \frac{(x_1-0.5n)^2}{0.25n} = \frac{\left(\frac{x_1}{n}-0.5\right)^2}{\frac{0.25}{n}}$$

$$= \frac{(\bar{Y}-0.5)^2}{\frac{0.25}{n}}$$

and also reject H_0 when $G = \frac{(\bar{Y}-0.5)^2}{\frac{0.25}{n}} > \chi^2_{\alpha,1}$

$\therefore$ this two test are the same asymptotically.