MATH3423 - Statistical Inference

Assignment 1 (Deadline: 5pm of 9 September)

1. Find E(X) and Var(X) for the following distributions

   (a) $X \sim \text{Bi}(n, P)$;

   (b) $X \sim \text{Poisson}(\lambda)$;

   (c) $X \sim N(\mu, \sigma^2)$.

2. (a) Lt $U \sim \text{Uniform}(0,1)$. Find the p.d.f. of $X = -\log(U)$.

   (b) Let X have the p.d.f. $f(x) = (\frac{1}{2})^x$, $x = 1, 2, 3, ...$, zero elsewhere. Find the p.d.f. of $Y = X^3$.

   (c) Let X have the p.d.f. $f(x) = x^2/9$, $0 < x < 3$, zero elsewhere. Find the p.d.f. of $Y = X^3$.

   (d) If the p.d.f. of X is $f(x) = 2xe^{-x^2}$, $0 < x < \infty$, zero elsewhere, determine the p.d.f. of $Y = X^2$.

3. (Q3 in final exam of MATH2421 - Spring 2011)

   Let the joint pdf of X and Y be

   $$f_{X,Y}(x,y) = \begin{cases} 1 & \text{for} 0 < x < 1, x < y < x+1; \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the marginal pdfs of X and Y.

   (b) What are the variances of X and Y?

   (c) Determine the correlation coefficient ρ_{XY} of X and Y.

4. (Q5 in final exam of MATH2421 - Spring 2011)

   (a) Consider a standard normal random variable Z, i.e., $Z \sim N(0,1)$. Show that

   $$Z^2 \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$$

   (b) Use the result in (a) to determine the distribution of $\frac{(X_1 - X_2)^2}{2}$ if X_1 and X_2 are independent random variables from $N(0,1)$.

   In the following two parts, consider n independent random variables $X_1, \ldots, X_n$ from $N(0,1)$.

   (c) Let $\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$. What is the distribution of $\bar{X}$?

   (d) Let $W = \sum_{i=1}^{n}(X_i - \bar{X})^2$. Given that W and $\bar{X}$ are independent, what is the distribution of W.
   Hint: Use $W = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$.

5. (Q1 in final exam of MATH2421 - Spring 2012)

   Consider a circle of radius R, and suppose that a point within the circle is randomly chosen in such a manner that all region within the circle of equal area are equally likely to contain the point. In other words, the point is uniformly distributed within the circle. If we let the center of the circle denote the origin and define X and Y to be the coordinates of the point chosen, then since (X, Y) is equally likely to be near each point in the circle, it follows that the joint pdf of X and Y is given by

$$f_{X,Y}(x,y) \;=\; \begin{cases} c & \text{if} x^2+y^2 \le R^2; \\ 0 & \text{otherwise} \end{cases}$$

for some value of c.

(a) Determine c.

(b) Find the marginal pdfs of X alone and Y alone.

(c) Compute the probability that D, the distance from the origin to the point selected, is less than or equal to a.

(d) Find $\mathrm{E}(D)$.

6. (Q3 in final exam of MATH2421 - Spring 2012)

Consider independent random variables $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1$ and 2.

(a) Follow the four steps below to show that

$$\sum_{i=1}^{2} X_i \;\sim\; N\left(\sum_{i=1}^{2} \mu_i, \sum_{i=1}^{2} \sigma_i^2\right)$$

Define $W_i = X_i - \mu_i$ for $i = 1$ and 2.

i. What are the distributions of W_1 and W_2?

ii. Show that W_1 and W_2 are independent.

iii. Prove that the distribution of $W_1 + W_2$ is $N(0, \sigma_1^2 + \sigma_2^2)$.

iv. Show that $\sum_{i=1}^{2} X_i \;\sim\; N\left(\sum_{i=1}^{2} \mu_i, \sum_{i=1}^{2} \sigma_i^2\right)$.

(b) Use the result in part (a) to show that

$$\sum_{i=1}^{2} c_i X_i \;\sim\; N\left(\sum_{i=1}^{2} c_i \mu_i, \sum_{i=1}^{2} c_i^2 \sigma_i^2\right)$$