Assignment 3: Solution

Q1 Since $X_i \overset{\text{i.i.d.}}{\sim} \exp(\theta)$, we have

$$\begin{aligned} M_{2\theta \sum_{i=1}^{n} X_i}(t) &= M_{\sum_{i=1}^{n} X_i}(2\theta t) \qquad since \qquad M_{aX+b}(t) = e^{bt} M_X(at) \\ &= \Pi_{i=1}^{n} M_{X_i}(2\theta t) \qquad since \quad X_i \ are \ i.i.d \\ &= \Pi_{i=1}^{n} \frac{\theta}{\theta - 2\theta t} \\ &= (\frac{1}{1-2\ t})^n \end{aligned}$$

which implies $2\theta \sum_{i=1}^{n} X_i \ \sim \chi^2_{2n}$.

55. (a)

$$\bar{X}_k \sim N\left(0, \frac{1}{k}\right), \qquad \bar{X}_{n-k} \sim N\left(0, \frac{1}{n-k}\right)$$

$$\begin{aligned}
E\left(\frac{1}{2}(\bar{X}_k + \bar{X}_{n-k})\right) &= 0 \\
Var\left(\frac{1}{2}(\bar{X}_k + \bar{X}_{n-k})\right) &= \frac{1}{4}\left(\frac{1}{k} + \frac{1}{n-k}\right) \\
&= \frac{1}{4} \cdot \frac{n}{k(n-k)}
\end{aligned}$$

$$\therefore \qquad \frac{1}{2}(\bar{X}_k + \bar{X}_{n-k}) \sim N\left(0, \frac{n}{4k(n-k)}\right)$$

(b)

$$\begin{aligned}
\sqrt{k}\bar{X}_k \sim N(0,1) &\Rightarrow k\bar{X}_k^2 \sim \chi^2(1) \\
\sqrt{n-k}\bar{X}_{n-k} \sim N(0,1) &\Rightarrow (n-k)\bar{X}_{n-k}^2 \sim \chi^2(1)
\end{aligned}$$

$$\therefore \qquad k\bar{X}_k^2 + (n-k)\bar{X}_{n-k}^2 \sim \chi^2(2)$$

(c)

$$X_1^2 \sim \chi^2(1), \qquad X_2^2 \sim \chi^2(1)$$

$$\begin{aligned}
\frac{X_1^2}{X_2^2} &= \frac{X_1^2/1}{X_2^2/1} \\
&\sim F_{(1,1)}
\end{aligned}$$

56. (a)

$$\bar{X} \sim N(1, \frac{1}{2}), \qquad \bar{Z} \sim N(0, \frac{1}{2}), \qquad \bar{X} + \bar{Z} \sim N(1, 1)$$

(b)

$$\begin{aligned}
X_2 - X_1 \sim N(0,2) &\Rightarrow \frac{1}{\sqrt{2}}(X_2 - X_1) \sim N(0,1) \\
&\Rightarrow \frac{1}{2}(X_2 - X_1)^2 \sim \chi^2(1) \\
Z_2 - Z_1 \sim N(0,2) &\Rightarrow \frac{1}{\sqrt{2}}(Z_2 - Z_1) \sim N(0,1) \\
&\Rightarrow \frac{1}{2}(Z_2 - Z_1)^2 \sim \chi^2(1)
\end{aligned}$$

$$\therefore\ [(X_2 - X_1)^2 + Z_2 - Z_1)^2]/2 \sim \chi^2(2)$$

(c)

$$Z_1 + Z_2 \sim N(0,2) \Rightarrow \frac{1}{\sqrt{2}}(Z_1 + Z_2) \sim N(0,1)$$

Note that

$$\frac{N(0,1)}{\sqrt{\chi^2(r)/r}} \sim t(r)$$

So,

$$\frac{Z_1 + Z_2}{\sqrt{[(X_2 - X_1)^2 + (Z_2 - Z_1)^2]/2}} = \frac{\frac{1}{\sqrt{2}}(Z_1 + Z_2)}{\sqrt{\frac{[(X_2 - X_1)^2 + (Z_2 - Z_1)^2]/2}{2}}} \sim t(2)$$

(d)

$$\begin{aligned}
X_2 + X_1 - 2 \sim N(0,2) &\Rightarrow \frac{1}{2}(X_2 + X_1 - 2)^2 \sim \chi^2(1) \\
X_2 - X_1 \sim N(0,2) &\Rightarrow \frac{1}{2}(X_2 - X_1)^2 \sim \chi^2(1)
\end{aligned}$$

$$\begin{aligned}
\frac{(X_2 + X_1 - 2)^2}{(X_2 - X_1)^2} &= \frac{[\frac{1}{2}(X_2 + X_1 - 2)^2]/1}{[\frac{1}{2}(X_2 - X_1)^2]/1} \\
&\sim F_{(1,1)}
\end{aligned}$$

<u>Q4</u>

(a) Find the p.d.f. of Y_1, $E(Y_1)$ and $Var(Y_1)$, where $Y_1 = \min(X_1, \ldots, X_n)$.

$$\begin{aligned}
f_{Y_1}(y_1) &= \frac{n!}{(1-1)!(n-1)!}[F_X(y_1)]^{1-1}[1-F_X(y_1)]^{n-1}f_X(y_1) \\
&= n\left(1 - \frac{1}{\theta}(y_1 - \theta)\right)^{n-1}\frac{1}{\theta} \\
&= \frac{n}{\theta}\left[2 - \frac{y_1}{\theta}\right]^{n-1} \qquad\qquad \theta < y_1 < 2\theta
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ E(Y_1) &= \int_\theta^{2\theta} y_1 f_{Y_1}(y_1)\ dy_1 \\
&= \frac{n}{\theta}\int_\theta^{2\theta} y_1\left(2 - \frac{y_1}{\theta}\right)^{n-1}\ dy_1 \\
&= \frac{n}{\theta^n}\int_\theta^{2\theta} y_1(2\theta - y_1)^{n-1}\ dy_1 \\
&= \frac{n}{\theta^n}\int_\theta^{0} -(2\theta - z)z^{n-1}\ dz, \qquad\qquad \text{let } z = 2\theta - y_1, dz = -dy_1 \\
&= \frac{n}{\theta^n}\int_0^{\theta}(-z^n + 2\theta z^{n-1})\ dz \\
&= \frac{n}{\theta^n}\left[\frac{-1}{n+1}\ z^{n+1} + \frac{2\theta}{n}\ z^n\right]_0^\theta \\
&= \frac{n}{\theta^n}\left[\frac{-\theta^{n+1}}{n+1} + \frac{2\theta^{n+1}}{n}\right] \\
&= \frac{-n\theta}{n+1} + 2\theta \\
&= \frac{n+2}{n+1}\ \theta
\end{aligned}$$

$$
\begin{aligned}
E(Y_1^2) &= \int_\theta^{2\theta} y_1^2 f_{Y_1}(y_1)\ dy_1 \\
&= \frac{n}{\theta}\int_\theta^{2\theta} y_1^2 \left(2-\frac{y_1}{\theta}\right)^{n-1}\ dy_1 \\
&= \frac{n}{\theta^n}\int_\theta^{2\theta} y_1^2(2\theta-y_1)^{n-1}\ dy_1 \\
&= \frac{n}{\theta^n}\int_\theta^{0} -(2\theta-z)^2 z^{n-1}\ dz, \qquad\qquad \text{let } z=2\theta-y_1, dz=-dy_1 \\
&= \frac{n}{\theta^n}\int_0^{\theta} (z^{n+1}-4\theta z^n+4\theta^2 z^{n-1})\ dz \\
&= \frac{n}{\theta^n}\left[\frac{1}{n+2}\ z^{n+2}-\frac{4\theta}{n+1}\ z^{n+1}+\frac{4\theta^2}{n}\ z^n\right]_0^\theta \\
&= \frac{n}{\theta^n}\left[\frac{\theta^{n+2}}{n+2}-\frac{4\theta^{n+2}}{n+1}+\frac{4\theta^{n+2}}{n}\right] \\
&= \left[\frac{n}{n+2}-\frac{4n}{n+1}+4\right]\theta^2
\end{aligned}
$$

$$
\begin{aligned}
Var(Y_1) &= E(Y_1^2)-[E(Y_1)]^2 \\
&= \left(\frac{n}{n+2}-\frac{4n}{n+1}+4\right)\theta^2+\left[\left(\frac{n+2}{n+1}\right)\ \theta\right]^2 \\
&= \left(\frac{n}{n+2}-\frac{4n}{n+1}+4-\left(1+\frac{1}{n+1}\right)^2\right)\theta^2 \\
&= \left(\frac{n}{n+2}-\frac{4n}{n+1}+4-1-\frac{2}{n+1}-\frac{1}{(n+1)^2}\right)\theta^2 \\
&= \left(\frac{n}{n+2}-\frac{4n}{n+1}+3-\frac{1}{(n+1)^2}\right)\theta^2 \\
&= \frac{n\theta^2}{(n+1)^2(n+2)}
\end{aligned}
$$

(b) Find the p.d.f. of Y_n, $E(Y_n)$ and $Var(Y_n)$, where $Y_n = \max(X_1,\ldots,X_n)$.

$$
\begin{aligned}
f_{Y_n}(y_n) &= \frac{n!}{(n-1)!(n-n)!}[F_X(y_n)]^{n-1}[1-F_X(y_n)]^{n-n}f_X(y_n) \\
&= n\left(1-\frac{1}{\theta}(y_n-\theta)\right)^{n-1}\frac{1}{\theta} \\
&= \frac{n}{\theta}\left[\frac{y_n}{\theta}-1\right]^{n-1} \qquad\qquad \theta<y_n<2\theta
\end{aligned}
$$

$$
\begin{aligned}
E(Y_n) &= \int_{\theta}^{2\theta} y_n f_{Y_n}(y_n)\ dy_n \\
&= \frac{n}{\theta} \int_{\theta}^{2\theta} y_n \left(\frac{y_n}{\theta} - 1 \right)^{n-1}\ dy_n \\
&= \frac{n}{\theta^n} \int_{\theta}^{2\theta} y_n (y_n - \theta)^{n-1}\ dy_n \\
&= \frac{n}{\theta^n} \int_{0}^{\theta} (z+\theta) z^{n-1}\ dz, \qquad\qquad \text{let } z = y_n - \theta, dz = dy_n \\
&= \frac{n}{\theta^n} \int_{0}^{\theta} (z^n + \theta z^{n-1})\ dz \\
&= \frac{n}{\theta^n} \left[\frac{1}{n+1}\ z^{n+1} + \frac{\theta}{n}\ z^n \right]_0^{\theta} \\
&= \frac{n}{\theta^n} \left[\frac{\theta^{n+1}}{n+1} + \frac{\theta^{n+1}}{n} \right] \\
&= \frac{n\theta}{n+1} + \theta \\
&= \frac{2n+1}{n+1}\ \theta
\end{aligned}
$$

$$
\begin{aligned}
E(Y_n^2) &= \int_{\theta}^{2\theta} y_n^2 f_{Y_n}(y_n)\ dy_n \\
&= \frac{n}{\theta} \int_{\theta}^{2\theta} y_n^2 \left(\frac{y_n}{\theta} - 1 \right)^{n-1}\ dy_n \\
&= \frac{n}{\theta^n} \int_{\theta}^{2\theta} y_n^2 (y_n - \theta)^{n-1}\ dy_n \\
&= \frac{n}{\theta^n} \int_{0}^{\theta} (z+\theta)^2 z^{n-1}\ dz, \qquad\qquad \text{let } z = y_n - \theta, dz = dy_n \\
&= \frac{n}{\theta^n} \int_{0}^{\theta} (z^{n+1} + 2\theta z^n + \theta^2 z^{n-1})\ dz \\
&= \frac{n}{\theta^n} \left[\frac{1}{n+2}\ z^{n+2} + \frac{2\theta}{n+1}\ z^{n+1} + \frac{\theta^2}{n}\ z^n \right]_0^{\theta} \\
&= \left[\frac{n}{n+2} + \frac{2n}{n+1} + 1 \right] \theta^2
\end{aligned}
$$

$$
\begin{aligned}
Var(Y_n) &= E(Y_n^2) - [E(Y_n)]^2 \\
&= \left(\frac{n}{n+2} + \frac{2n}{n+1} + 1\right)\theta^2 - \left(\frac{2n+1}{n+1}\ \theta\right)^2 \\
&= \left(\frac{n}{n+2} + \frac{2n}{n+1} + 1 - \left(2 - \frac{1}{n+1}\right)^2\right)\theta^2 \\
&= \left(\frac{n}{n+2} + \frac{2n}{n+1} + 1 - 4 + \frac{4}{n+1} - \frac{1}{(n+1)^2}\right)\theta^2 \\
&= \left(\frac{n}{n+2} + \frac{2n+4}{n+1} - 3 - \frac{1}{(n+1)^2}\right)\theta^2 \\
&= \frac{n\theta^2}{(n+1)^2(n+2)}
\end{aligned}
$$

Solutions to Exercise 2

23. (a) $E(\omega\bar{X}_1 + (1-\omega)\bar{X}_2), \qquad 0 \leq \omega \leq 1$
$= \omega E(\bar{X}_1) + (1-\omega)E(\bar{X}_2)$
$= \omega E(X_i) + (1-\omega)E(X_i)$
$= \omega \cdot \mu + (1-\omega) \cdot \mu$
$= \mu$
$\therefore\ \omega\bar{X}_1 + (1-\omega)\bar{X}_2$ is unbiased for μ

(b) $Var(\omega\bar{X}_1 + (1-\omega)\bar{X}_2)$

$= \omega^2 Var(\bar{X}_1) + (1-\omega)^2 Var(\bar{X}_2)$

$= \omega^2 \frac{Var(X_1)}{n} + (1-\omega)^2 \frac{Var(X_2)}{n}$

$= \omega^2 (\frac{\sigma_1^2}{n}) + (1-\omega)^2 (\frac{\sigma_2^2}{n})$

Let

$$g(\omega) = \omega^2(\frac{\sigma_1^2}{n}) + (1-\omega)^2(\frac{\sigma_2^2}{n})$$

$$g'(\omega) = 2\omega(\frac{\sigma_1^2}{n}) - 2(1-\omega)(\frac{\sigma_2^2}{n})$$

$$\begin{aligned} g'(\omega) = 0 \quad &\Rightarrow \quad 2\omega(\frac{\sigma_1^2}{n}) - 2(1-\omega)(\frac{\sigma_2^2}{n}) = 0 \\ &\Rightarrow \quad \omega\sigma_1^2 = (1-\omega)\sigma_2^2 \\ &\Rightarrow \quad \omega(\sigma_1^2 + \sigma_2^2) = \sigma_2^2 \\ &\Rightarrow \quad \omega = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \end{aligned}$$

(c) When $\omega = \frac{1}{2}$, let $\hat{\mu}_1 = \frac{1}{2}\bar{X}_1 + \frac{1}{2}\bar{X}_2$,

$$\Rightarrow Var(\hat{\mu}_1) = (\frac{1}{2})^2 \cdot \frac{Var(X)}{n} + (\frac{1}{2})^2 \cdot \frac{Var(X)}{n} = \frac{1}{4}\frac{\sigma_1^2}{n} + \frac{1}{4}\frac{\sigma_2^2}{n} = \frac{1}{4n}(\sigma_1^2 + \sigma_2^2)$$

When $\omega = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$, let $\hat{\mu}_2 = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\bar{X}_1 + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\bar{X}_2$

$$\Rightarrow Var(\hat{\mu}_2) = \left(\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2 \frac{\sigma_1^2}{n} + \left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2 \frac{\sigma_2^2}{n} = \frac{\sigma_1^2\sigma_2^2}{n(\sigma_1^2+\sigma_2^2)}$$

$\therefore$ The efficiency of the estimator of part (a) with $\omega = \frac{1}{2}$ relative to $\omega = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$

$$= \frac{Var(\hat{\mu}_2)}{Var(\hat{\mu}_1)} = \frac{\frac{(\sigma_1^2\sigma_2^2)}{n(\sigma_1^2+\sigma_2^2)}}{\frac{1}{4n}(\sigma_1^2+\sigma_2^2)} = \frac{4\sigma_1^2\sigma_2^2}{(\sigma_1^2+\sigma_2^2)^2}$$

(a) For $i=1,2$,

$$E(X_i) = \int_0^\theta x f_{X_i}(x|\theta)\,dx = \int_0^\theta \frac{3x^3}{\theta^3}\,dx = \frac{3}{\theta^3}\frac{1}{4}x^4\Big|_0^\theta = \frac{3}{4}\theta$$

$\therefore$ $E(\hat{\theta}_1) = \frac{2}{3}(E(X_1)+E(X_2)) = \frac{2}{3}(\frac{3}{4}\theta+\frac{3}{4}\theta) = \theta$, so $\hat{\theta}_1$ is unbiased for θ

Consider $Y = \max(X_1, X_2)$, for $y \in (0,\theta)$,

$\therefore$ $P(Y \le y) = P(\max(X_1,X_2) \le y) = [P(X_1 \le y)]^2$,

$$= \left[\int_0^y \frac{3x^2}{\theta^3}\,dx\right]^2 = \left(\frac{y}{\theta}\right)^6$$

$\therefore$ $f_Y(y) = 6(\frac{y}{\theta})^5(\frac{1}{\theta})$ if $0<y<\theta$

i.e. $E(Y) = \frac{6}{\theta^6}\int_0^\theta y^6\,dy = \frac{6}{7}\theta \Rightarrow \theta = E(\frac{7}{6}Y) = E(\frac{7}{6}\max(X_1,X_2))$

$= E(\hat{\theta}_2)$

so $\hat{\theta}_2$ is also unbiased for θ #

b) For $i=1,2$,

$$E(X_i^2)=\int_0^\theta \frac{3}{\theta^3}x^4\,dx=\frac{3}{5}\theta^2, \quad \therefore$$

$$Var(X_i)=\frac{3}{5}\theta^2-\left(\frac{3}{4}\theta\right)^2=\frac{3}{80}\theta^2$$

Thus,

$$MSE(\hat{\theta}_1)=Var(\hat{\theta}_1) \quad (\because \hat{\theta}_1 \text{ is unbiased for } \theta)$$

$$=Var\left(\frac{4}{3}\bar{X}\right), \text{ where } \bar{X}=\frac{1}{2}(X_1+X_2)$$

$$=\frac{16}{9}\,\frac{Var(X_1)}{2}$$

$$=\frac{8}{9}\times\frac{3}{80}\theta^2=\frac{1}{30}\theta^2$$

For $\hat{\theta}_2=\frac{7}{6}\max(X_1,X_2)$,

$$\because E(Y^2)=\frac{6}{\theta^6}\int_0^\theta y^7\,dy=\frac{3}{4}\theta^2$$

$$\therefore \quad Var(Y)=\frac{3}{4}\theta^2-\left(\frac{6}{7}\theta\right)^2=\frac{3}{196}\theta^2$$

$$\Rightarrow MSE(\hat{\theta}_2)=Var(\hat{\theta}_2) \quad (\because \hat{\theta}_2 \text{ is unbiased for } \theta)$$

$$=Var\left(\frac{7}{6}Y\right)$$

$$=\frac{49}{36}Var(Y)$$

$$=\frac{49}{36}\cdot\frac{3}{196}\theta^2=\frac{\theta^2}{12(4)}=\frac{\theta^2}{48}$$

Thus, $\hat{\theta}_2$ has a smaller MSE than $\hat{\theta}_1$ #

c). ∵ $T_c = cY$

∴ $Var(T_c) = c^2 Var(Y) = c^2(\frac{5}{196}\theta^2)$
and $E(T_c) = cE(Y) = c(\frac{6}{7}\theta)$

i.e. $MSE(T_c) = Var(T_c) + [E(T_c) - \theta]^2$

$= c^2 Var(Y) + [cE(Y) - \theta]^2$

$= c^2 Var(Y) + [c^2(E(Y))^2 - 2c\theta E(Y) + \theta^2]$

$= c^2[Var(Y) + (E(Y))^2] - 2c\theta E(Y) + \theta^2$

$0 = \frac{d}{dc} MSE(T_c)\Big|_{c=c^*} = 2c^*[Var(Y) + (E(Y))^2] - 2\theta E(Y)$

$\Rightarrow c^* = \frac{\theta E(Y)}{Var(Y) + (E(Y))^2} = \frac{\frac{6}{7}\theta^2}{\frac{5}{196}\theta^2 + \frac{36}{49}\theta^2} = \frac{8}{7}$

and $\frac{d^2}{dc^2} MSE(T_c)\Big|_{c=c^*} = 2[Var(Y) + E(Y))^2] > 0$

Thus,
$T_{c^*} = T_{\frac{8}{7}}$ has the smallest MSE among the estimators of θ in ~~form~~ of $T_c = c\max(X_1, X_2)$, i.e. it is the best #