1. $X_1, \ldots, X_n$ is a random variable having the bernoulli distribution with the parameter θ.

   (a) **(2 marks)** Let $S = \sum_{i=1}^{n} X_i$, find the distribution of S.

   (b) **(2 marks)** Find the maximum likelihood estimate of $\theta(1-\theta)$.

   (c) **(2 marks)** Is it unbiased estimator? No mark will be given if the answer is "Yes" or "No".

   (d) **(2 marks)** Find the Cramer-Rao Lower Bound for the variance of an unbiased estimator for $\theta(1-\theta)$.

   (e) **(2 marks)** Does the variance of any unbiased estimator for $\theta(1-\theta)$ achieve this bound? Why? Explain in details.

   (f) **(3 marks)** Find the limiting distribution of the maximum likelihood estimate for $\theta(1-\theta)$ by Central Limit Theorem and Delta method. What phenomenon do you observe?

2. $X_1, X_2, ..., X_n$ are observations of a random sample of size n from the geometric distribution with probability distribution $f(x, \theta) = \theta(1-\theta)^x$ for $x = 0, 1, \ldots$.

   (a) **(2 marks)** Find the distribution of $\sum\limits_{i=1}^{n} X_i$.

   (b) **(2 marks)** Find the estimator from the method of moment.

   (c) **(2 marks)** Find the estimator from the method of maximum likelihood.

   (d) **(2 marks)** Find the maximum likelihood estimator for $E(X)$.

   (e) **(2 marks)** Is the maximum likelihood estimator for $E(X)$ unbiased? If yes, find its variance. If no, find its mean squared error.

   (f) **(2 marks)** Is the variance of any unbiased estimator for $E(X)$ equal to the Cramer-Rao Lower Bound? No need to find the Cramer-Rao Lower Bound.

3. Let $X_1, \ldots, X_n$ are independently uniformly distributed on $(\theta, 2\theta)$.

   (a) **(2 marks)** Find the estimators from the method of moment and the method of maximum likelihood.

   (b) **(2 marks)** Find the expectation and variance of the estimator from the method of moments.

   (c) **(4 marks)** Find the expectation and variance of the estimator from the method of maximum likelihood.

(d) **(2 marks)** Hence or otherwise, construct two unbiased estimators of θ based on the two estimators in part (a).

(e) **(2 marks)** Compare the variances of the two unbiased estimates in (d) and comment briefly.

4. If $X_1, X_2, \ldots, X_n$ are independently and normally distributed with the same mean μ but different **known** variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$.

(a) **(3 marks)** Find the estimator from the method of moment. Prove that it is an unbiased estimator for μ. Find its variance.

(b) **(3 marks)** Find the estimator from the method of maximum likelihood. Prove that it is an unbiased estimator for μ. Find its variance.

(c) **(2 marks)** Which of the estimators from part (a) or part (b) is more efficient? Explain in details

(d) **(4 marks)** Let

$$W = \frac{\sum_{i=1}^{n} \frac{X_i}{\sigma_i^2}}{\sum_{j=1}^{n} \frac{1}{\sigma_j^2}} .$$

Find its distribution. Hence or otherwise, construct the $(1-\alpha)100\%$ confidence interval for μ.

(e) **(2 marks)** Find the distribution of $X_i - W$.

(f) **(3 marks)** Are W and $X_i - W$ independent? Explain in details.

(g) **(1 mark)** Find the distribution of

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma_i} \right)^2 .$$

(h) **(1 mark)** Find the distribution of

$$\sum_{i=1}^{n} \frac{1}{\sigma_i^2} (W - \mu)^2 .$$

(i) **(4 marks)** Hence or otherwise, find the distribution of

$$\sum_{i=1}^{n} \frac{(X_i - W)^2}{\sigma_i^2} .$$

5. **(Bonus)** Consider a random sample $\{X_1, X_2\}$ from density

$$f_X(x|\theta) = \frac{3x^2}{\theta^3} I_{(0<x<\theta)},$$

where $\theta > 0$.

(a) **(2 marks)** Are $\hat{\theta}_1 = \frac{2}{3}(X_1 + X_2)$ and $\hat{\theta}_2 = \frac{7}{6}\max(X_1, X_2)$ unbiased for θ?

(b) **(4 marks)** Find the mean squared errors (MSEs) of $\hat{\theta}_1$ and $\hat{\theta}_2$, and compare those estimators.

(c) **(4 marks)** Prove that in the sense of MSE, $T_{8/7}$ is the best estimator of θ among the estimators in form of $T_c = c\max(X_1, X_2)$.