The Hong Kong University of Science & Technology

MATH243 – Statistical Inference

Final Examination – Fall 02/03

Answer ALL questions **Date: 12 December 2002 (Thu)**

All Equal Marks **Time allowed: 3 Hours**

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1. (a) Find the mean and variance of the rectangular (or uniform) distribution on $(0,\theta)$, where $\theta > 0$.

   (b) If $Y_1,...,Y_n$ are independently rectangularly distributed on $(0,\theta)$, obtain the probability density function of $Z = \max(Y_1,...,Y_n)$.

   (c) Find the expectation and variance of Z.

   (d) If $Y_1,...,Y_n$ are as in (b) and $\overline{Y} = \sum Y_j / n$, find the expectation and variance of $\overline{Y}$.

   (e) If θ is an unknown parameter, construct two unbiased estimators of θ, one based on Z and one based on $\overline{Y}$.

   (f) Compare the variances of the two unbiased estimates in (e) and comment briefly.

2. Suppose that $X_1,...,X_n$ is a random sample from Poisson distribution with mean θ.

   (a) Find the maximum likelihood estimate of θ^2.

   (b) Obtain an unbiased estimate of θ^2 of the form $a\overline{X} + b\overline{X}^2$ where a and b are constants and $\overline{X} = \sum X_i / n$.

   (c) Find $Var(a\overline{X} + b\overline{X}^2)$.

   Hints: (1) $\dfrac{d^r m(t)}{dt^r}\Big|_{t=0} = E(x^r)$ where $m(t)$ is moment generating function of x.

   (2) $$\left(\sum_{i=1}^{n} x_i\right)^3 = \sum_{i=1}^{n} x_i^3 + 3\sum_{i=1}^{n}\sum_{j\neq i} x_i^2 x_j + \sum_{i=1}^{4}\sum_{j\neq i}\sum_{k\neq i,j} x_i x_j x_k$$

   (3) $$\left(\sum_{i=1}^{n} x_i\right)^4 = \sum_{i=1}^{n} x_i^4 + 3\sum_{i=1}^{n}\sum_{j\neq i} x_i^2 x_j^2 + 4\sum_{i=1}^{n}\sum_{j\neq i} x_i^3 x_j + 6\sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i,j} x_i^2 x_j x_k$$
   $$+ \sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i,j}\sum_{l\neq i,j,k} x_i x_j x_l x_k$$

   (d) What is the Crámer Rao lower bound for the variance of an unbiased estimator of θ^2? Is the Crámer Rao lower bound attained? Why?

3. (a) The observations $X_1, ..., X_m$ are independent and each has a Binomial distribution with index n and parameter θ. Find the minimum variance unbiased estimator of θ^k, where k is an integer $(1 \le k \le mn)$.

   (b) A random sample, $X_1, ..., X_n$, is taken from the exponential distribution with parameter λ. Find the minimum variance unbiased estimator of $\lambda^r (0 < r < n)$ where r is an integer.

4. The observations $X_1, ..., X_n$ are independent and have an exponential distribution with unknown parameter λ. Find a uniformly most powerful test of the null hypothesis $H_0 : \lambda = \lambda_0$ against the alternative $H_A : \lambda > \lambda_0$.

   In a particular situation measurement of each individual observation is costly although comparing observations is not. For this reason a different test is suggested in which H_0 is rejected if the observed value of $T = \max_i(X_i)$ is less than a constant c which is chosen at significance level $\alpha (0 < \alpha < 1)$. Find the power function of this test. Hence, determine the minimum sample size n required to obtain power at least 0.8 against the alternative $\lambda_1 = 0.002$ when $\lambda_0 = 0.001$ and $\alpha = 0.05$.

   Is the sample size larger than that required in the previous test? Why? NO calculation is needed.

5. (a) $X_1, ..., X_n$ are independent Poisson random variables with $E(X_i) = \mu_i$ (i = 1, …, n). Derive the approximate large sample likelihood ratio test for the null hypothesis that $\mu_1 = \cdots = \mu_n$ against the alternative that the $\mu_i s$ are arbitrary.

   By writing $X_i = \overline{X} + U_i$, where $\overline{X} = \sum X_i / n$, and assuming that the $U_i s$ are small compared with $\overline{X}$, show that the test in (a) approximately compares $s^2 = \sum (X_i - \overline{X})^2 / (n-1)$ with $\overline{X}$. Comment.

   Hint: Take Log(1 + a) $\approx a$ if a is small.

   (b) Let X_1, …, X_n be i.i.d. random variables, each with the Poison distribution of parameter μ. The best test of $H_0 : \mu = \mu_0$ against $H_1 : \mu = \mu_1 (> \mu_0)$ is $C_1 = \left\{ \underset{\sim}{x} : \sum_{i=1}^{n} x_i \ge k \right\}$. By using the central limit theorem to approximate the distribution of $\sum_{i=1}^{n} x_i$, find the smallest value of n required to obtain power at least 0.9 against the alternative $\mu_1 = 1.21$ when $\mu_0 = 1$ and $\alpha = 0.05$.

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