04/05 MATH 243 Mid-term exam (solution)

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right)$$

$$Y_1 = \frac{X_1-\mu_1}{\sigma_1} + \frac{X_2-\mu_2}{\sigma_2}, \qquad Y_2 = \frac{X_1-\mu_1}{\sigma_1} - \frac{X_2-\mu_2}{\sigma_2}$$

$$\Rightarrow \frac{X_1-\mu_1}{\sigma_1} = \frac{Y_1+Y_2}{2}, \qquad \frac{X_2-\mu_2}{\sigma_2} = \frac{Y_1-Y_2}{2}$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2) \quad |J(x_1,x_2)|^{-1}$$

where $J(x_1,x_2) = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$

First we find the Jacobian:

$$\frac{\partial y_1}{\partial x_1} = \frac{1}{\sigma_1}, \quad \frac{\partial y_1}{\partial x_2} = \frac{1}{\sigma_2}, \quad \frac{\partial y_2}{\partial x_1} = \frac{1}{\sigma_1}, \quad \frac{\partial y_2}{\partial x_2} = -\frac{1}{\sigma_2}$$

$$J = \begin{vmatrix} \frac{1}{\sigma_1} & \frac{1}{\sigma_2} \\ \frac{1}{\sigma_1} & -\frac{1}{\sigma_2} \end{vmatrix} = -\frac{2}{\sigma_1\sigma_2}$$

Then

$$\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}(x_1,x_2) \, |J|^{-1} \\
&= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right] \right\} \left\{ \frac{\sigma_1\sigma_2}{2} \right\} \\
&= \frac{1}{4\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{y_1+y_2}{2}\right)^2 - \frac{2\rho(y_1+y_2)(y_1-y_2)}{4} + \left(\frac{y_1-y_2}{2}\right)^2 \right] \right\} \\
&= \frac{1}{4\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{y_1^2+y_2^2-\rho(y_1^2-y_2^2)}{2} \right] \right\} \\
&= \frac{1}{4\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{4(1-\rho^2)} \left[(1-\rho)y_1^2 + (1+\rho)y_2^2 \right] \right\} \\
&= \left(\frac{1}{\sqrt{2\pi(\sqrt{2(1+\rho)})^2}} \exp\left\{ -\left(\frac{y_1}{\sqrt{2(1+\rho)}}\right)^2 \Big/ 2 \right\} \right) \left(\frac{1}{\sqrt{2\pi(\sqrt{2(1-\rho)})^2}} \exp\left\{ -\left(\frac{y_2}{\sqrt{2(1-\rho)}}\right)^2 \Big/ 2 \right\} \right)
\end{aligned}$$

The pdf of Y_1 and Y_2

$$f_{Y_1}(y_1) = \frac{1}{\sqrt{2\pi(\sqrt{2(1+\rho)})^2}} \exp\left\{ -\left(\frac{y_1}{\sqrt{2(1+\rho)}}\right)^2 \Big/ 2 \right\} \qquad Y_1 \sim N(0, 2(1+\rho))$$

$$f_{Y_2}(y_2) = \frac{1}{\sqrt{2\pi(\sqrt{2(1-\rho)})^2}} \exp\left\{ -\left(\frac{y_2}{\sqrt{2(1-\rho)}}\right)^2 \Big/ 2 \right\} \qquad Y_2 \sim N(0, 2(1-\rho))$$

Therefore, Y_1 and Y_2 are indep.

Here is the solution for $\frac{e^{-\lambda}\lambda^k}{k!}$; $k = 0, 1, 2, \cdots$

a) $$CRLB = -\frac{(\tau'(\lambda))^2}{nE\left[\frac{\partial^2}{\partial \lambda^2} \log f(x,\lambda)\right]}$$

$$f(x,\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\log f(x,\lambda) = -\lambda + x \log \lambda - \log x!$$

$$\frac{\partial}{\partial \lambda} \log f(x,\lambda) = -1 + \frac{x}{\lambda}$$

$$\frac{\partial^2}{\partial \lambda^2} \log f(x,\lambda) = -\frac{x}{\lambda^2}$$

$$E\left(\frac{\partial^2}{\partial \lambda^2} \log f(x,\lambda)\right) = -\frac{1}{\lambda^2} E(x) = -\frac{1}{\lambda}$$

$$\tau(\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \cdots \qquad 2\lambda e^{-\lambda} + e^{-\lambda}\lambda^2$$

$$\tau'(\lambda) = \begin{cases} e^{-\lambda}, & k = 0 \\ -\frac{e^{-\lambda}\lambda^k}{k!} + \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}, & k = 1, 2, \cdots \end{cases}$$

$$\therefore \; CRLB = \begin{cases} \frac{\lambda e^{-2\lambda}}{n}, & k = 0 \\ \frac{\lambda}{n}\left(\frac{e^{-\lambda}\lambda^k}{k!} - \frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}\right)^2 & k = 1, 2, \cdots \end{cases}$$

b) $$E\left(\frac{1}{n}\sum_{i=1}^{n} I_{\{k\}}(X_i)\right) = \frac{1}{n}\sum_{i=1}^{n} E\, I_{\{k\}}(X_i)$$
$$= E\, I_{\{k\}}(X)$$
$$= 1 \times P(X=k) + 0 \times P(X \neq k)$$
$$= \frac{e^{-\lambda}\lambda^k}{k!}$$

which is unbiased

c) $$Var\left(\frac{1}{n}\sum_{i=1}^{n} I_{\{k\}}(X_i)\right)$$
$$= \frac{1}{n^2}\sum_{i=1}^{n} Var\left(I_{\{k\}}(X_i)\right)$$
$$= \frac{1}{n} Var\left(I_{\{k\}}(X)\right) \qquad I_{\{k\}}(X) \sim Bi\left(1, \frac{e^{-\lambda}\lambda^k}{k!}\right)$$
$$= \frac{1}{n}\frac{e^{-\lambda}\lambda^k}{k!}\left(1 - \frac{e^{-\lambda}\lambda^k}{k!}\right)$$

$$\frac{2e^{-\lambda}\lambda^2}{n}\left(1 - \frac{e^{-\lambda}\lambda^2}{2}\right)$$

a) $f(x,\lambda) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

$= \exp\{-\lambda + x\log\lambda - \log x!\}$

Then, $a(\lambda) = -\lambda$

$b(x) = -\log x!$

$c(\lambda) = \log\lambda$

$d(x) = x$

$D = \{0, 1, 2, \cdots\}$

$A = (0, \infty)$

$a'(\lambda) = -1$, $c'(\lambda) = \frac{1}{\lambda} \neq 0$

$\therefore$ $f(x,\lambda)$ belongs to exponential family. $\Sigma d(x_i) = \Sigma x_i$ is a complete minimal sufficient statistic.

(Method 2)

$$\sum_{s=0}^{\infty} h(s)\,\frac{e^{-n\lambda}(n\lambda)^{s}}{s!} = \frac{e^{-\lambda}\lambda^{k}}{k!}$$

$$\Rightarrow \sum_{s=0}^{\infty} h(s)\,\frac{e^{-(n-1)\lambda}\, n^{s}\,\lambda^{s-k}\,k!}{s!} = 1$$

$$\Rightarrow \sum_{s=0}^{\infty} h(s)\,\frac{n^{s}\,k!\,(s-k)!}{(n-1)^{s-k}\,s!}\;\frac{e^{-(n-1)\lambda}\,[(n-1)\lambda]^{s-k}}{(s-k)!} = 1$$

$$\Rightarrow h(s) = \frac{(n-1)^{s-k}}{n^{s}}\,\frac{s!}{k!\,(s-k)!}$$

is the UMVUE of $\dfrac{e^{-\lambda}\lambda^{k}}{k!}$.

3.a) $E(X) = \frac{1}{2}(\theta - a + \theta + b) = \frac{1}{2}(2\theta - a + b) = \theta - \frac{1}{2}(a-b)$

$\therefore \frac{1}{n}\sum_{i=1}^{n} X_i = \widehat{EX} = \hat{\theta} - \frac{1}{2}(a-b)$

$\therefore \hat{\theta} = \bar{X} + \frac{1}{2}(a-b)$

$$\begin{aligned} E(\hat{\theta}) &= E(\bar{X} + \tfrac{1}{2}(a-b)) \\ &= EX_i + \tfrac{1}{2}(a-b) \\ &= \theta - \tfrac{1}{2}(a-b) + \tfrac{1}{2}(a-b) \\ &= \theta \end{aligned}$$

which is unbiased

b) $$\begin{aligned} Var(\hat{\theta}) &= Var(\bar{X} + \tfrac{1}{2}(a-b)) \\ &= \tfrac{1}{n} Var X_i \\ &= \tfrac{1}{n} \frac{(\theta + b - (\theta - a))^2}{12} \\ &= \frac{(a+b)^2}{12n} \end{aligned}$$

c) The likelihood function is

$$L(\theta, x) = \begin{cases} \frac{1}{b+a} & \text{if } y_n - b \le \theta \le y_1 + a \\ 0 & \text{otherwise} \end{cases}$$

Since $\forall i$, $\theta - a \le X_i \le \theta + b$

$\Rightarrow \theta - a \le X_{(1)} \le \cdots \le X_{(n)} \le \theta + b$

$\Rightarrow \theta - a \le y_1 \le \cdots \le y_n \le \theta + b$

$\Rightarrow y_n - b \le \theta \le y_1 + a$

The max. cannot be found by differentiation

But it is clear that the max value is $\frac{1}{b+a}$, $\forall \theta \in [y_n - b, y_1 + a]$

i.e for every statistic $t(x_1, \cdots, x_n)$ such that

$y_n - b \le t(x_1, \cdots, x_n) \le y_1 + a$

are the max. likelihood estimator.

d) let $U = \frac{1}{2}(Y_n + Y_1)$, $V = \frac{1}{2}(Y_n - Y_1)$, $\theta - a \le y_1 \le y_n \le \theta + b$

then $Y_1 = U - V$, $Y_n = U + V$, $\theta - a \le u - v \le u + v \le \theta + b$

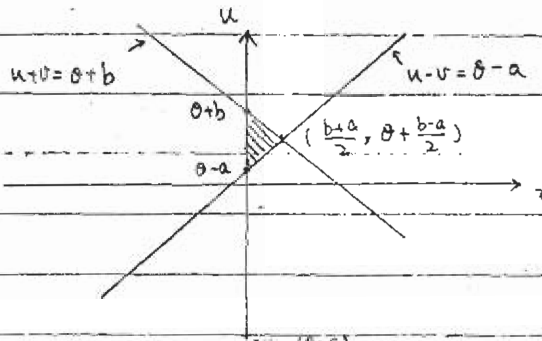
First we find the Jacobian:

$\frac{\partial u}{\partial y_1} = \frac{1}{2}$, $\frac{\partial u}{\partial y_n} = \frac{1}{2}$, $\frac{\partial v}{\partial y_1} = -\frac{1}{2}$, $\frac{\partial v}{\partial y_n} = \frac{1}{2}$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Then

$$f_{U,V}(u,v) = f_{Y_1,Y_2}(y_1,y_2)\,|J|^{-1}$$
$$= 2^{n-1} n(n-1)(v)^{n-2} / (b+a)^n, \quad \theta - a \le u - v \le u + v \le \theta + b$$



$$f_U(u) = \int_0^{u-(\theta-a)} f_{U,V}(u,v)\,dv, \quad u \in [\theta - a, \theta + \tfrac{b-a}{2}]$$

$$= \int_0^{(\theta+b)-u} f_{U,V}(u,v)\,dv, \quad u \in [\theta + \tfrac{b-a}{2}, \theta + b]$$

Thus,
$$f_U(u) = \begin{cases} n 2^{n-1} (u - \theta + a)^{n-1} / (b+a)^n, & u \in [\theta - a, \theta + \frac{b-a}{2}] \\ n 2^{n-1} (\theta + b - u)^{n-1} / (b+a)^n, & u \in [\theta + \frac{b-a}{2}, \theta + b] \end{cases}$$

$$E(Y) = \frac{a-b}{2} + \frac{n 2^{n-1}}{(b+a)^n} \int_{\theta-a}^{\theta+\frac{b-a}{2}} u(u-\theta+a)^{n-1}\,du + \frac{n 2^{n-1}}{(b+a)^n} \int_{\theta+\frac{b-a}{2}}^{\theta+b} u(\theta+b-u)^{n-1}\,du$$

$$= \frac{a-b}{2} + \frac{n 2^{n-1}}{(b+a)^n} \left[\frac{(u-\theta+a)^{n+1}}{n+1} + \frac{(\theta-a)(u-\theta+a)^n}{n} \right] \Bigg|_{\theta-a}^{\theta+\frac{b-a}{2}} + \frac{n 2^{n-1}}{(b+a)^n} \left[\frac{(\theta+b-u)^{n+1}}{n+1} - \frac{(\theta+b)(\theta+b-a)^n}{n} \right] \Bigg|_{\theta+\frac{b-a}{2}}^{\theta+b}$$

$$= \frac{a-b}{2} + \frac{n(b+a)}{4(n+1)} + \frac{\theta-a}{2} - \frac{n(b+a)}{4(n+1)} + \frac{\theta+b}{2}$$

$$= \theta$$

which is unbiased

e)

$\hat{Var}(Y) = E((Y-\theta)^2) = E(\frac{Y_1+Y_n}{2}-\theta)^2 + 2(\frac{a-b}{2})(\frac{Y_1+Y_n}{2}-\theta) + E(\frac{a-b}{2})^2$

$= E(W^2) - (\frac{b-a}{2})^2$, where $W = \frac{Y_1+Y_n}{2} - \theta = u - \theta$

Then, $f_W(w) = \begin{cases} n 2^{n-1}(w+a)^{n-1}/(b+a)^n, & w \in [-a, \frac{b-a}{2}] \\ n 2^{n-1}(b-w)^{n-1}/(b+a)^n & w \in [\frac{b-a}{2}, b] \end{cases}$

$\Rightarrow$ ~~Var(Y)~~ =

$\therefore Var(Y) = \int_{-a}^{\frac{b-a}{2}} w^2 n 2^{n-1}(w+a)^{n-1}/(b+a)^n \, dw + \int_{\frac{b-a}{2}}^{b} w^2 n 2^{n-1}(b-w)^{n-1}/(b+a)^n \, dw - (\frac{b-a}{2})^2$

$= \int_{-a}^{\frac{b-a}{2}} w^2 2^{n-1}/(b+a)^n \, d(w+a)^n - \int_{\frac{b-a}{2}}^{b} w^2 2^{n-1}/(b+a)^n \, d(b-w)^n - (\frac{b-a}{2})^2$

$= \frac{2^{n-1}w^2(w+a)^n}{(b+a)^n}\Big|_{-a}^{\frac{b-a}{2}} - \int_{-a}^{\frac{b-a}{2}} \frac{2^n w(w+a)^n}{(b+a)^n} dw - \frac{2^{n-1}w^2(b-w)^n}{(b+a)^n}\Big|_{\frac{b-a}{2}}^{b} + \int_{\frac{b-a}{2}}^{b} \frac{2^n w(b-w)^n}{(b+a)^n} dw - (\frac{b-a}{2})^2$

$= \frac{1}{2}(\frac{b-a}{2})^2 - \frac{2^n}{(b+a)^n}\left[\frac{(w+a)^{n+2}}{n+2} - \frac{a(w+a)^{n+1}}{n+1}\right]\Big|_{-a}^{\frac{b-a}{2}} + \frac{1}{2}(\frac{b-a}{2})^2 + \frac{2^n}{(b+a)^n}\left[\frac{(b-w)^{n+2}}{n+2} - \frac{b(b-w)^{n+1}}{n+1}\right]\Big|_{\frac{b-a}{2}}^{b} - (\frac{b-a}{2})^2$

$= (\frac{b-a}{2})^2 - \frac{(b+a)^2}{4(n+2)} + \frac{a(b+a)}{2(n+1)} - \frac{(b+a)^2}{4(n+2)} + \frac{b(b+a)}{2(n+1)} - (\frac{b-a}{2})^2$

$= (\frac{b-a}{2})^2 + \frac{(a+b)^2}{2(n+1)} - \frac{(b+a)^2}{2(n+2)} - (\frac{b-a}{2})^2$

$= \frac{(b+a)^2}{2(n+1)(n+2)}$

Since Y is unbiased for θ and $\lim_{n\to\infty} Y = \lim_{n\to\infty} \frac{(b+a)^2}{2(n+1)(n+2)} = 0$.

$\therefore$ Y is consistent for θ.

and $\because Var(\hat{\theta}) = \frac{(a+b)^2}{12} > \frac{(a+b)^2}{2(n+1)(n+2)} = Var(Y)$ for $n > 2$

$\therefore Var(Y)$ is less then $Var(\hat{\theta})$

$\Rightarrow Y$ is more efficient //