1. **(9 marks)** Let $U_1, ..., U_n$ be a random sample from the $U(0,\theta)$, where θ is the unknown parameter.

   (a) **(1 mark)** Find the moment estimator of θ. Is it unbiased? Hence or otherwise, find an unbiased estimator for θ.

   (b) **(3 marks)** Find the maximum likelihood estimator of θ. Is it unbiased? Hence or otherwise, find an unbiased estimator for θ.

   (c) **(3 marks)** Find the variance of unbiased estimators from (a) and (b). Which unbiased estimator for θ is more efficient?

   (d) **(2 marks)** Suppose a random sample with sample size six is drawn. The values are as follow.

   0.3, 1.2, 1.8, 2.4, 4.1, 5.5

   Calculate the moment estimator and the maximum likelihood estimator. Hence or otherwise, state one problem of moment estimator **other than efficiency**

a. $E(U) = \frac{\theta}{2}$. $\hat{\theta} = 2 \cdot \bar{U}$

$\because$ $E(\hat{\theta}) = 2E(\bar{U}) = 2 \cdot \frac{\theta}{2} = \theta$

$\therefore$ it is unbiased, unbiased estimator $= \hat{\theta} = 2\bar{U}$

b. $L(\theta) = \prod_{i=1}^{n} I(0 < U_i < \theta) \cdot \frac{1}{\theta} = I(0 < U_{(n)} < \theta) \cdot \frac{1}{\theta^n}$

Hence it is maximized when θ is as small as possible and $\theta > U_{(n)}$, Hence. $\hat{\theta} = U_{(n)}$

$F_{U_{(n)}}(y) = P(U_{(n)} \le y) = (P(U_1 \le y))^n = \left(\frac{y}{\theta}\right)^n$

$\therefore$ $f_{U_{(n)}}(y) = \frac{n y^{n-1}}{\theta^n}$

$\therefore$ $E(U_{(n)}) = \int_0^\theta \frac{n y^n}{\theta^n} dy = \frac{n}{n+1}\theta \Rightarrow E\left(\frac{n+1}{n} U_{(n)}\right) = \theta$

$\therefore$ $\frac{n+1}{n} U_{(n)}$ is an unbiased estimator of θ

c. $Var(2(\bar{U})) = \frac{4}{n} Var(U) = \frac{4}{n} \cdot \frac{\theta^2}{12} = \frac{\theta^2}{3n}$

$$E(U_{(n)}^2) = \int_0^\theta \frac{ny^{n+1}}{\theta^n} dy = \frac{n}{n+2} \theta^2$$

$$\therefore Var(U_{(n)}) = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 = \theta^2 \left(\frac{n((n+1)^2 - n(n+2))}{(n+2)(n+1)^2} \right)$$

$$= \theta^2 \left(\frac{n}{(n+2)(n+1)^2} \right)$$

$$\therefore Var\left(\frac{n+1}{n} U_{(n)}\right) = \frac{(n+1)^2}{n^2} \cdot \frac{n}{n+2(n+1)^2} \cdot \theta^2 = \frac{\theta^2}{n(n+2)}$$

$$\because \frac{\theta^2}{n(n+2)} \leq \frac{\theta^2}{3n} \quad \text{for } n \geq 1$$

Hence. $\frac{n+1}{n} U_{(n)}$ is more efficient.

e. moment estimator = 5.1

MLE = 5.5

problem: moment estimator may not valid, since from the data, $\theta \geq 5.5$

d. $L(\theta) = I(0 < U_{(n)} < \theta) \cdot \frac{1}{\theta^n} I(\theta > 1)$

Hence it is maximized when $\theta > U_{(n)}$ and $\theta > 1 \Rightarrow \theta > \max(U_{(n)}, 1)$

i.e. MLE $= \max(U_{(n)}, 1)$

f. $f_{Y_n}(y) = \frac{4y^3}{\theta^4}$ for $0 < y < 1$

$$0.95 = P(c\theta < Y_n < \theta) = \int_{c\theta}^{\theta} \frac{4y^3}{\theta^4} dy = \frac{1}{\theta^4}(\theta^4 - c^4\theta^4) = 1 - c^4$$

$\therefore c = \sqrt[4]{0.05}$ or 0.4729

Hence. $P\left(Y_n < \theta < \frac{Y_n}{\sqrt[4]{0.05}}\right) = 0.95$

$\Rightarrow$ 95% C.I. of θ is $Y_n, \frac{Y_n}{\sqrt[4]{0.05}}$.

2. **(9 marks)** Let $X_1, ..., X_n$ be a random sample from $Gamma(T, \theta)$, where θ is the unknown parameter and T is a **known** positive integer. The probability density function is as follow.

$$f(x) = \frac{\theta^T x^{T-1} e^{-\theta x}}{(T-1)!}$$

(a) **(4 marks)** Find the maximum likelihood estimators for θ and $\frac{1}{\theta}$. Are they unbiased? Hence or otherwise, find unbiased estimators for θ and $\frac{1}{\theta}$.

(b) **(3 marks)** Find the Cramer-Rao lower bound for the variance of unbiased estimators of θ and $\frac{1}{\theta}$.

(c) **(2 marks)** Do the variances of unbiased estimators for θ and $\frac{1}{\theta}$ achieve their corresponding Cramer-Rao lower bound? Why? Explain in details.

2a. $L(\theta) = \prod_{i=1}^{n} \frac{\theta^T x_i^{T-1} e^{-\theta x_i}}{(T-1)!} = \frac{\theta^{nT} \prod_{i=1}^{n} x_i^{T-1} e^{-\theta \sum_{i=1}^{n} x_i}}{((T-1)!)^n}$

$\therefore \log L(\theta) = nT \log\theta + \sum_{i=1}^{n} (T-1)\log x_i - \theta \sum_{i=1}^{n} x_i - n\log(T-1)!$

$\frac{\partial}{\partial\theta} \log L(\theta) = \frac{nT}{\theta} - \sum_{i=1}^{n} x_i = 0$

$\Rightarrow \quad \theta = \frac{nT}{\sum_{i=1}^{n} x_i}$

$\therefore$ MLE of $\theta = \hat{\theta} = \frac{T}{\bar{X}}$

By invariant property, MLE of $\frac{1}{\theta} = \frac{1}{\hat{\theta}} = \frac{\bar{X}}{T}$

$\because E\left(\frac{1}{\hat{\theta}}\right) = E\left(\frac{\bar{X}}{T}\right) = \frac{1}{T} E(X_1) = \frac{1}{T} \cdot \frac{T}{\theta} = \frac{1}{\theta}$

$\therefore \frac{1}{\hat{\theta}}$ is unbiased estimator of $\frac{1}{\theta}$

$E\left(\frac{T}{\bar{X}}\right) = E\left(\frac{nT}{\sum_{i=1}^{n} X_i}\right) \quad \because \sum_{i=1}^{n} X_i \sim Gamma(nT, \theta)$

$= \int_0^\infty \frac{nT}{y} \frac{\theta^{nT} y^{nT-1} e^{-\theta y}}{(nT-1)!} dy$

$= nT \int_0^\infty \frac{\theta^{nT} y^{nT-2} e^{-\theta y}}{(nT-1)!} dy$

$$= \frac{nT\theta}{nT-1} \int_0^\infty \underbrace{\frac{\theta^{nT-1} y^{nT-2} e^{-\theta y}}{(nT-2)!}}_{\text{density of Gamma}(nT-1,\ \theta)} dy$$

$$= \frac{nT\theta}{nT-1}$$

Hence. $E\left(\frac{T}{X}\right) = \frac{nT\theta}{nT-1} \Rightarrow E\left(\frac{nT-1}{nT} \cdot \frac{nT}{\sum_{i=1}^n X_i}\right) = \theta$

$\therefore \frac{nT-1}{\sum_{i=1}^n X_i}$ is an unbiased estimator of θ.

b. $\frac{\partial}{\partial\theta} \log L(\theta) = \frac{nT}{\theta} - \sum_{i=1}^n X_i$

$\frac{\partial^2}{\partial\theta^2} \log L(\theta) = -\frac{nT}{\theta^2}$

$\therefore$ CRLB of $\theta = \frac{1}{-\left(-\frac{nT}{\theta^2}\right)} = \frac{\theta^2}{nT}$

CRLB of $\frac{1}{\theta} = \frac{\left(-\frac{1}{\theta^2}\right)^2}{-\left(-\frac{nT}{\theta^2}\right)} = \frac{1}{nT\theta^2}$

c. $\because \frac{\partial}{\partial\theta} \log L(\theta) = -nT\left(\frac{\sum_{i=1}^n X_i}{nT} - \frac{1}{\theta}\right)$

$\therefore \frac{\sum_{i=1}^n X_i}{nT}$ can achieve the CRLB.

and $\frac{\partial}{\partial\theta} \log L(\theta) = \left(\frac{nT}{\theta} - \sum_{i=1}^n X_i\right) = \frac{nT}{\theta^2}\left(\theta - \frac{\sum_{i=1}^n X_i}{nT} \cdot \theta^2\right)$

which can not express as $\frac{\partial}{\partial\theta} \log L(\theta) = A(n,\theta)(T(X_1,\dots,X_n) - \theta)$

$\therefore$ the unbiased estimator of θ can not achieve the CRLB.

3. **(9 marks)** Let $X_1, ..., X_n$ be a random sample from $N(\mu, \sigma^2)$, where μ and σ^2 is the unknown parameters. (Define the notations you use carefully.)

   (a) **(3 marks)** Find the maximum likelihood estimators for μ and σ^2. Find the expectation and variance of the estimators.

   (b) **(4 marks)** Find the maximum likelihood estimator for $\frac{\mu}{\sigma}$. Find the expectation of the estimator. Is it unbiased? Hence or otherwise, find the unbiased estimator.

   (c) **(2 marks)** Find the variance of the unbiased estimator in part (b).

a. $$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}.$$

$$\log L(\theta) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial \log L(\theta)}{\partial \mu} = +\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0 \Rightarrow \mu = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}.$$

$$\frac{\partial \log L(\theta)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i-\mu)^2 \Rightarrow \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2.$$

$$\therefore \hat{\mu} = \bar{X}, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 \equiv S_n^2$$

$$E(\bar{X}) = \mu, \quad Var(\bar{X}) = \frac{\sigma^2}{n}.$$

By ~~theorem~~. $$\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\therefore E(\hat{\sigma}^2) = E\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right) = \frac{\sigma^2}{n}\cdot E\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2}\right)$$

$$= \frac{\sigma^2}{n}\cdot n-1 = \frac{n-1}{n}\sigma^2$$

$$Var(\hat{\sigma}^2) = Var\,\frac{\sigma^2}{n}\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sigma^2}\right) = \frac{\sigma^4}{n^2}\cdot 2(n-1) = \frac{2(n-1)\sigma^4}{n^2}$$

b. By invariant property, MLE of $\frac{\mu}{\sigma} = \frac{\hat{\mu}}{\hat{\sigma}} = \frac{\bar{X}}{S_n}$

$$E\left(\frac{\bar{X}}{S_n}\right) = E(\bar{X}) \cdot E\left(\frac{1}{S_n}\right) = \mu \cdot E\left(\frac{1}{S_n}\right) \quad \because \bar{X} \perp S_n \text{ By theorem}$$

$$\Rightarrow E\left(\frac{\bar{X}}{S_n}\right) = \mu \cdot E\left(\frac{1}{S_n}\right)$$

$$\because E\left(\sqrt{\frac{\sigma^2}{nS_n^2}}\right) = \int_0^\infty \frac{1}{\sqrt{x}} \cdot \frac{x^{\frac{n-1}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)} \, dx$$

$$= \int_0^\infty \frac{x^{\frac{n-2}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)} \, dx$$

$$= \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \cdot \frac{1}{\sqrt{2}} \cdot \int_0^\infty \underbrace{\frac{x^{\frac{n-2}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{n-2}{2}} \cdot \Gamma\left(\frac{n-2}{2}\right)}}_{\text{density of } \chi^2_{n-2}} \, dx$$

$$\Rightarrow \frac{\sigma}{\sqrt{n}} E\left(\frac{1}{S_n}\right) = \frac{1}{\sqrt{2}} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \Rightarrow E\left(\frac{1}{S_n}\right) = \sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \cdot \frac{1}{\sigma}$$

$$\therefore E\left(\frac{\bar{X}}{S_n}\right) = \sqrt{\frac{n}{2}} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \cdot \frac{\mu}{\sigma}$$

$\Rightarrow \sqrt{\frac{2}{n}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} \cdot \frac{\bar{X}}{S_n}$ is an unbiased estimator of $\frac{\mu}{\sigma}$

$\because E\left(\frac{\bar{X}^2}{S_n^2}\right) = E(\bar{X}^2)\, E\left(\frac{1}{S_n^2}\right)$ By independent

$E(\bar{X}^2) = Var(\bar{X}) + (E(\bar{X}))^2 = \frac{\sigma^2}{n} + \mu^2.$

$E\left(\frac{\sigma^2}{nS_n^2}\right) = \int_0^\infty \frac{1}{x} \frac{x^{\frac{n-1}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}\,dx$

$= \frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-1}{2})} \cdot \frac{1}{2} \underbrace{\int_0^\infty \frac{x^{\frac{n-3}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{n-3}{2}}\Gamma(\frac{n-3}{2})}\,dx}_{\text{density of } \chi^2_{n-3}}$

$\Rightarrow E\left(\frac{1}{S_n^2}\right) = \frac{n}{2} \frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-1}{2})} \cdot \frac{1}{\sigma^2}$

$\therefore Var\left(\sqrt{\frac{2}{n}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})} \frac{\bar{X}}{S_n}\right) = E\left(\frac{2}{n}\left(\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\right)^2 \frac{\bar{X}^2}{S_n^2}\right) - \frac{\mu^2}{\sigma^2}$

$= \frac{2}{n}\left(\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\right)^2 \cdot \frac{\sigma^2 + n\mu^2}{n} \cdot \frac{n}{2} \frac{\Gamma(\frac{n-3}{2})}{\Gamma(\frac{n-1}{2})} \cdot \frac{1}{\sigma^2} - \frac{\mu^2}{\sigma^2}$

$= \frac{\Gamma(\frac{n-1}{2})\,\Gamma(\frac{n-3}{2})}{(\Gamma(\frac{n-2}{2}))^2} \cdot \left(\frac{1}{n} + \frac{\mu^2}{\sigma^2}\right) - \frac{\mu^2}{\sigma^2}.$