The Hong Kong University of Science & Technology

MATH243 – Statistical Inference

Final Examination – Fall 03/04

Answer <u>ALL</u> Questions | **Date: 11 December 2003, Thursday**

Time allowed: 3 Hours

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1. Let Y_1 and Y_2 be independent random variables. Consider the following model:

$$Y_1 = 4 - \theta + \varepsilon_1$$
$$Y_2 = 2 + 2\theta + \varepsilon_2$$

where θ is a constant, $\varepsilon_1 \sim$ Normal $(0,\ \sigma^2)$ and $\varepsilon_2 \sim$ Normal $(0,\ \sigma^2)$.

(a) **(3 marks)** Find the maximum likelihood estimate, $\hat{\theta}$, for θ.

(b) **(2 marks)** Find $E(\hat{\theta})$. Is $\hat{\theta}$ an unbiased estimator for θ?

(c) **(2 marks)** Find $Var(\hat{\theta})$

(d) **(3 marks)** Let $W = Y_1 + Y_2 - 6$. Is W unbiased? Which estimator is more efficient for θ? Why?

2. Let $X_1, X_2, ..., X_n$ be a random sample from the following distribution:

$$f_X(x;\theta) = \begin{cases} \theta^2 x e^{-\theta x} & x > 0,\ \theta > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) **(2 marks)** Find the maximum likelihood estimator for θ. Hence, find the maximum likelihood estimator for the <u>variance</u> of the distribution.

(b) **(4 marks)** Find the UMVUE for $\frac{1}{\theta}$ and its variance. Find the Cramer Rao lower bound for unbiased estimators of $\frac{1}{\theta}$. Is the variance of the UMVUE for $\frac{1}{\theta}$ equal to Cramer Rao lower bound?

(c) **(4 marks)** Find the UMVUE for θ. Is the variance of the UMVUE for θ equal to the Cramer Rao lower bound for unbiased estimators of θ? Explain! No calculation of CRLB is needed.

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3. Let $X_1, X_2, ..., X_n$ be a random sample from the following distribution:

$$f_X(x;\theta) = \begin{cases} \dfrac{2x}{\theta^2} & 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

(a) **(2 marks)** Find the MLE for θ.

(b) **(4 marks)** Show that the MLE is a biased estimator for θ and find a multiple of it that is an unbiased estimator for θ.

(c) **(4 marks)** Find an unbiased estimator for θ^2.

4. Let $X_1, X_2, ..., X_n$ be a random sample of size n from probability density function given by

$$f_X(x \mid \theta) = \begin{cases} \theta e^{-\theta x} & x > 0, \ \theta > 0 \\ 0 & \text{elsewhere} \end{cases}$$

(a) **(3 marks)** Find the rejection region for the most powerful test of: $H_0 : \theta = \theta_0$ versus $H_a : \theta = \theta_a$ where $\theta_a < \theta_0$.

(b) **(2 marks)** Is the test in (a) the uniformly most powerful test for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$? Explain!

(c) **(2 marks)** Is the test in (a) the uniformly most power test for testing $H_0 : \theta \geq \theta_0$ versus $H_a : \theta < \theta_0$? Explain!

(d) **(3 marks)** For a random sample of size n = 10, use the test in (a) to test $H_0 : \theta = 5$ versus $H_a : \theta = 2$. Find the rejection region for $\alpha = 0.05$.

5. Let $Y_1, Y_2, ..., Y_n$ denote a random sample from a population having a Poisson distribution with mean θ_1. Let $X_1, X_2, ..., X_m$ denote a random sample from a population having a Poisson distribution with mean θ_2. We are interested in testing the hypothesis:

$$H_0 : \theta_1 = \theta_2 \qquad \text{versus} \qquad H_a : \theta_1 \neq \theta_2$$

(a) **(7 marks)** Find the likelihood ratio test for testing these hypotheses and simplify as much as possible.

(b) **(3 marks)** What are the degrees-of-freedom for the chi-square approximation to the test in part (a)? Give the form of the test.

6. Assume that $X_1, X_2, ..., X_n$ is a random sample from the $U\ (0,\ \theta)$ distribution, and that you want to test: $H_0 : \theta = 1.6$ against $H_a : \theta = 2$.

(a) **(3 marks)** Suppose that $n = 1$ (i.e., you have <u>one</u> observation), and consider the rejection region:

$$\textit{reject}\ H_0 \quad \text{if} \quad X > k.$$

(i) Find the test (i.e., the value of k) that has power $1 - \beta = 0.9$.

(ii) What is the significance level (α) of the test in (i)

(b) **(3 marks)** Suppose, again, that $n = 1$ (i.e., you have <u>one</u> observation), and again consider the rejection region:

$$\textit{reject}\ H_0 \quad \text{if} \quad X > k.$$

(i) Find the test (i.e., the value of k) that has significance level (α) 0.05.

(ii) What is the power of the test in (i)

(c) **(4 marks + 4 marks (Bonus))**

Suppose now that $n = 8$ and that the observations are:

1.08, 0.55, 0.12, 0.98, 1.75, 1.34, 0.78, 0.3

Find the most powerful test for $\alpha = 0.05$? What is your conclusion? Explain!

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