Final 2004/2005

1 a) $f(x,p) = p^x (1-p)^{1-x}$, $x \in \{0,1\}$

$= \exp\{\log(1-p) + x\log(\frac{p}{1-p})\}$

$a(p) = \log(1-p)$., $b(x) = 0$, $c(p) = \log(\frac{p}{1-p})$. $d(x) = x$.

belongs to the exponential family

$\Rightarrow$ $X_1 + X_2$ is sufficient statistics for p and it is minimal

b) $E(X_1X_2) \overset{iid}{=} E(X_1)E(X_2) = p \cdot p = p^2$

So X_1X_2 is an unbiased estimator of p^2

c) $X_1X_2 = 0$ or 1 and $X_1X_2 = 1$ if and only if $X_1 = 1, X_2 = 1$

i.e $X_1X_2 = \begin{cases} 1 & , \text{ if } X_1 + X_2 = 2 \\ 0 & , \text{ if } X_1 + X_2 = 0 \text{ or } 1 \end{cases}$ (*)

$E(X_1X_2 | X_1 + X_2) = Pr(X_1X_2 = 1 | X_1X_2) = \begin{cases} 1 & , \text{ if } X_1 + X_2 = 2 \\ 0 & , \text{ if } X_1 + X_2 = 0 \text{ or } 1 \end{cases}$

$\therefore E(X_1X_2 | X_1 + X_2) = X_1X_2$.

So X_1X_2 is already the best unbiased estimator of p^2.

d) $L(p) = p^{x_1}(1-p)^{1-x_1} \cdot p^{x_2}(1-p)^{1-x_2} = p^{x_1+x_2}(1-p)^{2-x_1-x_2}$

$\log L(p) = (x_1 + x_2)\log p + (2 - x_1 - x_2)\log(1-p)$

$\frac{\partial}{\partial p}\log L(p) = \frac{x_1 + x_2}{p} - \frac{2 - x_1 - x_2}{1-p} = \frac{2(\frac{x_1+x_2}{2}) - p}{p(1-p)} = 0$

$\Rightarrow \hat{p} = \frac{X_1 + X_2}{2}$ is the M.L.E ~~for~~ of p .

$\Rightarrow \hat{p^2} = \frac{(X_1 + X_2)^2}{4}$ is the M.L.E of p^2

e). For X_1X_2 : by (*) in c). $Pr(X_1X_2 = 0) = 1 - p^2$

$Pr(X_1X_2 = 1) = p^2$

$E(X_1X_2 - p^2)^2 = (0 - p^2)^2(1 - p^2) + (1 - p^2)^2 \cdot p^2 = \frac{3}{16}$.

For $\frac{(X_1+X_2)^2}{4}$: $\begin{cases} Pr(\frac{(X_1+X_2)^2}{4} = 0) = (1-p)^2 \\ Pr(\frac{(X_1+X_2)^2}{4} = 1/4) = 2p(1-p) \\ Pr(\frac{(X_1+X_2)^2}{4} = 1) = p^2 \end{cases}$

~~MSE~~ the estimator from d) is better

$E(\frac{(X_1+X_2)^2}{4} - p^2) = (0 - p^2)^2(1-p)^2 + (1/4 - p^2)^2 2p(1-p) + (1 - p^2)^2 p^2 = \frac{5}{32}$

2. $f_X(x) = \frac{1}{2\theta-\theta} = \frac{1}{\theta}$, $F_X(x) = \int_\theta^x \frac{1}{\theta}\,dt = \left[\frac{t}{\theta}\right]_\theta^x = \frac{x}{\theta}-1 = \frac{1}{\theta}(x-\theta)$

$$f_{Y_1}(y_1) = \frac{n!}{(1-1)!(n-1)!}\left[F_X(y_1)\right]^{1-1}\left[1-F_X(y_1)\right]^{n-1} f_X(y_1)$$

$$= n\left(1-\frac{1}{\theta}(y_1-\theta)\right)^{n-1}\frac{1}{\theta} = \underline{\frac{n}{\theta}\left[2-\frac{y_1}{\theta}\right]^{n-1}}, \quad \theta<y_1<2\theta$$

$$f_{Y_n}(y_n) = \frac{n!}{(n-1)!(n-n)!}\left[F_X(y_n)\right]^{n-1}\left[1-F_X(y_n)\right]^{n-n} f_X(y_n)$$

$$= n\left[\frac{1}{\theta}(y_n-\theta)\right]^{n-1}\frac{1}{\theta} = \underline{\frac{n}{\theta}\left(\frac{y_n}{\theta}-1\right)^{n-1}}, \quad \theta<y_n<2\theta$$

$$f_{Y_1,Y_n}(y_1,y_n) = \frac{n!}{(1-1)!(n-1-1)!(n-n)!}\left[F_X(y_1)\right]^{1-1}\left[F_X(y_n)-F_X(y_1)\right]^{n-1-1}\cdot\left[1-F_X(y_n)\right]^{n-n} f_X(y_1)f_X(y_n)$$

$$= n(n-1)\left(\frac{y_n-\theta}{\theta}-\frac{y_1-\theta}{\theta}\right)^{n-2}\frac{1}{\theta}\cdot\frac{1}{\theta}$$

$$= \underline{n(n-1)\frac{1}{\theta^n}(y_n-y_1)^{n-2}}, \quad \theta<y_1\le y_n<2\theta.$$

$$\therefore E(Y_1) = \int_\theta^{2\theta} y_1 f_{Y_1}(y_1)\,dy_1 = \frac{n}{\theta}\int_\theta^{2\theta} y_1\left(2-\frac{y_1}{\theta}\right)^{n-1}dy_1$$

$$= \frac{n}{\theta^n}\int_\theta^{2\theta} y_1(2\theta-y_1)^{n-1}\,dy_1$$

$$= \frac{n}{\theta^n}\int_\theta^0 -(2\theta-z)z^{n-1}\,dz \qquad \text{let } z = 2\theta-y_1,\ dz = -dy_1$$

$$= \frac{n}{\theta^n}\int_0^\theta\left(-z^n+2\theta z^{n-1}\right)dz$$

$$= \frac{n}{\theta^n}\left[\frac{-1}{n+1}z^{n+1}+\frac{2\theta}{n}z^n\right]_0^\theta$$

$$= \frac{n}{\theta^n}\left[\frac{-\theta^{n+1}}{n+1}+\frac{2\theta^{n+1}}{n}\right]$$

$$= \frac{-n\theta}{n+1}+2\theta$$

$$= \frac{n+2}{n+1}\theta$$

$$E(Y_n) = \int_\theta^{2\theta} y_n f_{Y_n}(y_n)\,dy_n = \frac{n}{\theta}\int_\theta^{2\theta} y_n\left(\frac{y_n}{\theta}-1\right)^{n-1}dy_n$$

$$= \frac{n}{\theta^n}\int_\theta^{2\theta} y_n(y_n-\theta)^{n-1}\,dy_n$$

$$= \frac{n}{\theta^n}\int_0^\theta (z+\theta)z^{n-1}\,dz \qquad \text{let } z = y_n-\theta,\ dz = dy_n$$

$$= \frac{n}{\theta^n}\int_0^\theta\left(z^n+\theta z^{n-1}\right)dz$$

$$= \frac{n}{\theta^n}\left[\frac{1}{n+1}z^{n+1}+\frac{\theta}{n}z^n\right]_n^\theta$$

$$= \frac{n}{\theta^n}\left(\frac{\theta^{n+1}}{n+1}+\frac{\theta^{n+1}}{n}\right)$$

$$= \frac{n\theta}{n+1}+\theta$$

$$= \frac{2n+1}{n+1}\theta$$

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$$\begin{aligned}
Cov(Y_1, Y_n) &= E(Y_1, Y_n) - E(Y_1)\,E(Y_n) \\
&= \left(\frac{2n+5}{n+2}\right)\theta^2 - \left[\left(\frac{n+2}{n+1}\theta\right)\left(\frac{2n+1}{n+1}\theta\right)\right] \\
&= \frac{\theta^2}{(n+2)(n+1)^2}\left[(2n+5)(n+1)^2 - (n+2)(n+2)(2n+1)\right] \\
&= \frac{\theta^2}{(n+2)(n+1)^2}
\end{aligned}$$

$$\begin{aligned}
\therefore Var(U_1) &= Var\left(\frac{n+1}{2n+1} Y_n\right) \\
&= \left(\frac{n+1}{2n+1}\right)^2 Var(Y_n) \\
&= \frac{(n+1)^2}{(2n+1)^2} \cdot \frac{n\theta^2}{(n+1)^2(n+2)} \\
&= \frac{n\theta^2}{(2n+1)^2(n+2)}
\end{aligned}$$

$$\begin{aligned}
Var(U_2) &= Var\left[\left(\frac{n+1}{5n+4}\right)(2Y_n + Y_1)\right] \\
&= \frac{(n+1)^2}{(5n+4)^2}\left[Var(Y_1 + 2Y_n)\right] \\
&= \left(\frac{n+1}{5n+4}\right)^2\left[Var(Y_1) + 4\,cov(Y_1, Y_n) + 4\,Var(Y_n)\right] \\
&= \left(\frac{n+1}{5n+4}\right)^2\left[\frac{n\theta^2}{(n+1)^2(n+2)} + \frac{4\theta^2}{(n+2)(n+1)^2} + \frac{4n\theta^2}{(n+1)^2(n+2)}\right] \\
&= \frac{\theta^2}{(5n+4)^2}\left(\frac{5n+4}{n+2}\right) \\
&= \frac{\theta^2}{(5n+4)(n+2)}
\end{aligned}$$

$$\begin{aligned}
\therefore Var(U_1) - Var(U_2) &= \frac{\theta^2}{n+2}\left[\frac{n}{(2n+1)^2} - \frac{1}{5n+4}\right] \\
&= \frac{\theta^2}{n+2} \cdot \frac{n^2-1}{(2n+1)^2(5n+4)} \\
&> 0 \qquad (\because n^2-1 > 0 \text{ for } n > 1)
\end{aligned}$$

$\therefore$ U_2 is better than U_1 for estimating θ. //

$$U_1 = \frac{n+1}{n+2} Y_1$$

$$U_n = \frac{n+1}{2n+1} Y_n$$

$$E(aU_1 + bU_n) = aE(U_1) + bE(U_2) = \theta$$

$$\Rightarrow a + b = 1$$

$$\Rightarrow a = 1 - b$$

$$a^2 Var(U_1) + b^2 Var(U_n) + 2ab \, cov(U_1, U_n)$$

$$= a^2 Var(U_1) + (1-a)^2 Var(U_n) + 2a(1-a) \, cov(U_1, U_n)$$

$$Var(U_1) = \frac{(n+1)^2}{(n+2)^2} Var(Y_1)$$

$$= \frac{(n+1)^2}{(n+2)^2} * \frac{n\theta^2}{(n+1)^2(n+2)}$$

$$Var(U_n) = \frac{(n+1)^2}{(2n+1)^2} * \frac{n\theta^2}{(n+1)^2(n+2)}$$

$$cov(U_1, U_n) = \frac{(n+1)^2}{(n+2)(2n+1)} * \frac{\theta^2}{(n+2)(n+1)^2}$$

$$= \cancel{\left[a^2 \frac{(n+1)^2}{(n+2)^2} + (1-a)^2\right.}$$

$$= \left\{ \frac{a^2 n}{(n+2)^2} + \frac{(1-a)^2 n}{(2n+1)^2} + \frac{2a(1-a)}{(n+2)(2n+1)} \right\} * \frac{(n+1)^2 \theta^2}{(n+2)(n+1)^2}$$

$$\frac{\cancel{2}an}{(n+2)^2} - \frac{\cancel{2}(1-a)n}{(2n+1)^2} + \frac{\cancel{2}(1-2a)}{(n+2)(2n+1)} = 0$$

$$\Rightarrow an(2n+1)^2 - (1-a)n(n+2)^2 + (1-2a)(n+2)(2n+1) = 0$$

$$\Rightarrow an(4n^2+2n+1) - (1-a)n(n^2+4n+4) + (1-2a)(2n^2+5n+2) = 0$$

$$\Rightarrow a\left[n(2n+1)^2 + n(n+2)^2 - 2(n+2)(2n+1)\right] - \left[n(n+2)^2 - (n+2)(2n+1)\right] = 0.$$

$$\Rightarrow a^* = \frac{n+2}{5n+4}$$

$$b = \frac{2(n+1)}{5n+4}$$