Math 3423 Tutorial

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About me ☺

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- 2 tutorial sessions:
  - 18:00 – 18:50 on Tuesday
  - 19:00 – 19:50 on Fridau

Tutorial (For 2020-09-15 and 2020-09-18)

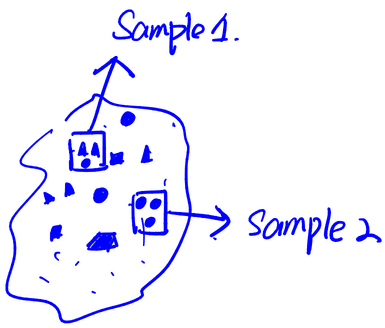
1. what is statistical inference =
{ ① To known unknown things by Sampling
  ② parametric method :

- Target : To find good <u>estimators</u> to study population

- <u>population</u>

population (Distribution, cdf. pdf) $\xrightarrow{\text{know}}$ USE IT.

population (Distribution, cdf. pdf) $\xrightarrow{\text{don't know}}$ SAmpling



SAmpling
$\Downarrow$
Introduce Uncertainty.

Random Variable (r.V.) $\Leftarrow$ (Don't have a certain solution/answer)

- <u>estimator</u> : $T(\cdot)$

To estimate the uncertainty

{ Parametric method $= \theta \Leftrightarrow F_\theta \Rightarrow F$ (where θ is a parameter)
{ non-parametric ...... = don't assume a distribution

No distribution assumption {
- Kendall τ (Tau) : Compare two Variables to get their Correlation
- Statistical boostrap method : estimate the accuracy distribution of a statistic
- Anderson - Darling Test : To test whether a R.V. follow a certain distribution.

Hence,

iid $X_1, X_2, \dots X_n$ statistic $T(X_1, X_2, \dots, X_n)$

$$\begin{array}{c} \text{data} \rightarrow \text{estimators} \rightarrow \text{population } (X) \\ \uparrow \text{- - - - - Sampling - - - - -} \end{array}$$

e.g. If population $X \sim N(\mu, \sigma^2)$

$\mu, \sigma^2 \longrightarrow$ Normal

$\therefore T(X_1, X_2, \dots, X_n)$ estimate μ and σ^2.

2. 3 important issues:

① point estimation:

Using 1 value to make an estimation.

e.g. $X \sim N(\underline{\underline{\mu}}, \sigma^2)$ and σ^2 is known.

$X_1, X_2, \dots X_n$ (from X) $\rightarrow$ $\bar{X}$ mean.

② Interval estimation:

Using an interval, instead of 1 value to contain the actual value

e.g. $\bar{X} \pm Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$ for μ (σ is known)

$100(1-\alpha)\%$ chance that this interval contain μ.

Question:

There is $100(1-\alpha)\%$ chance that μ falls into the interval $[\bar{X} - Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}, \bar{X} - Z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}]$

③ Hypothesis test:

e.g. $\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu > \mu_0 \end{cases}$ one-way HT

$\begin{cases} H_0: \mu = \mu_0 \\ H_1: \mu \neq \mu_0 \end{cases}$ Two-way HT

3. Summary.

If we have a population $X \sim N(\mu, \sigma^2)$. And we have n copies of X: $x_1, x_2, \ldots, x_N$ (iid)

Target	μ	σ		
μ		✓	• $\bar{X} = \sum_i^N X_i / N$ • C.I. $[\bar{x}_N - z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x}_n - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}]$	Z-distribut
μ		✗	• $\bar{X} = \sum_i^N X_i / N$ • C.I $[\bar{x}_N - t_{\frac{\alpha}{2}}^{(n-1)} \cdot \frac{S_{n-1}}{\sqrt{n}}, \bar{x}_N + t_{\frac{\alpha}{2}}^{(n-1)} \frac{S_{n-1}}{\sqrt{n}}]$	t-distribu
σ	✓		• $S_{n-1}^2 \to \sigma^2$ • C.I. $\left[\frac{(n-1) S_{n-1}^2}{\chi^2_{\frac{\alpha}{2}}(n-1)}, \frac{(n-1) S_{n-1}^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)}\right]$	Chi-dist
σ	✗			

- Some useful distribution.

① multivariate normal distribution.

$$f_X(\boldsymbol{x}) = (2\pi)^{-\frac{p}{2}}|\Sigma|^{-\frac{1}{2}}e^{-\underbrace{\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}_{\Delta^2}},$$

qudratic form

denoted by $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \Sigma)$.

☆ $\Delta^2 = -\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)$

$= -\frac{1}{2}X^T\Sigma^{-1}X + X^T\Sigma^{-1}\mu + \text{Constant}.$

Gaussian distirbution $\Leftrightarrow$ Constant

② Chi-square distribution.

- $X_1 \sim N(0,1) \Rightarrow X_1^2 \sim \chi^2(1)$

$\Rightarrow \sum_{i=1}^{N} X_i^2 \sim \chi^2(N)$

- $Y_i \sim \chi^2(r_i) \Rightarrow \sum_{i=1}^{k} Y_i \sim \chi^2(\sum_{i=1}^{k} r_i)$

③ t-distribution.

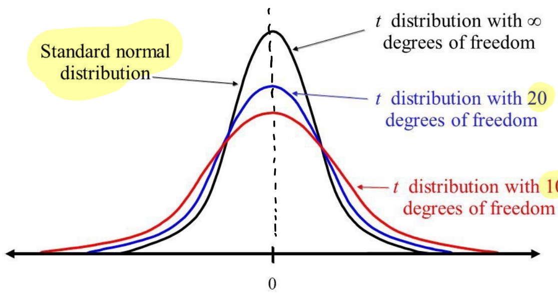
Definition (t distribution): If $Z \sim N(0,1)$, $U \sim \chi^2(r)$, and they are independent, then the distribution of

$$T := \frac{Z}{\sqrt{U/r}} = \frac{\text{standard Normal}}{\text{Chi-square}}$$

is called a t distribution with r degrees of freedom, denoted by $t(r)$.

t Distribution

The t-distribution is used when n is small and σ is unknown.



Note that the pdf of $T \sim t(r)$ can be shown to be

$$f_T(t) = \frac{\Gamma[(r+1)/2]}{\sqrt{\pi r}\,\Gamma(r/2)}(1+t^2/r)^{-(r+1)/2},$$

Standard Normal:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

when $r \to \infty$

$f_T(t) = \frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\cdot\Gamma(\frac{r}{2})} \cdot (1+\frac{t^2}{r})^{-\frac{(r+1)}{2}}$

$= \frac{1}{\sqrt{\pi r}} \cdot (1+\frac{t^2}{r})^{-\frac{1}{2}} \cdot (1+\frac{t^2}{r})^{-\frac{r}{2}}$

$= (1+\frac{t^2}{r})^{\frac{r}{t^2}\cdot\frac{t^2}{r}\cdot\frac{1}{2}} \quad (1+\frac{t^2}{r})^{\frac{r}{t^2}\cdot\frac{t^2}{r}\cdot -\frac{r}{2}}$

$\exp(-\frac{t^2}{2r}) \cdot \exp(-\frac{t^2}{2})$

$\Gamma(\frac{r}{2}+\frac{1}{2}) \quad \Gamma(\frac{r}{2})$

$(\frac{r}{2}-\frac{1}{2})(\frac{r}{2}-\frac{3}{2})$

$(\frac{r}{2}+\frac{1}{2})(\frac{r}{2}-\frac{1}{2}) \quad)(\frac{r}{2}-1)$

$(\frac{r}{2}$

④ F distribution

Definition (F distribution): If $X \sim \chi^2(r_1)$, $Y \sim \chi^2(r_2)$, and they are independent, then the distribution of

$$F(r_1, r_2) := \frac{X/r_1}{Y/r_2} = \frac{\text{Chi-square / df}}{\text{Chi-square / df}}$$

is called a F distribution with r_1 and r_2 degrees of freedom, denoted by $F(r_1, r_2)$.

Note that the pdf of $F \sim F(r_1, r_2)$ can be shown to be

$$f_F(w) = \frac{\Gamma[(r_1+r_2)/2]}{\Gamma(r_1/2)\,\Gamma(r_2/2)} \left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} w^{\frac{r_1}{2}-1}(1 + r_1 w/r_2)^{-(r_1+r_2)/2},$$

Appendix