# Tutorial for 10-27 and 10-30.

1. C-R inequality. $\begin{cases} \theta \\ g(\theta) \end{cases}$

- Under the regularity conditions, the variance of an UNBIASED estimator $T(\boldsymbol{X}) = T(X_1, \dots, X_n)$ for $\theta$, based on a set of random variables $\boldsymbol{X} = \{X_1, \dots, X_n\}$ from their joint pdf $f_{X_1,\dots,X_n}(\cdot\,|\theta)$ satisfies the following inequality:

$$Var(T(\boldsymbol{X})) \geq \frac{1}{I_{X_1,\dots,X_n}(\theta)} = \frac{1}{E\left[\frac{\partial}{\partial\theta}\ln f_{X_1,\dots,X_n}(X_1,\dots,X_n|\theta)\right]^2}.$$

This inequality is well-known as ***the C-R inequality*** for $\theta$, and its lower bound is often called ***the CR lower bound*** (or *CRLB*) for $\theta$. It means that no any unbiased estimator for $\theta$ based on a set of random variables $\boldsymbol{X} = \{X_1, \dots, X_n\}$ can have a variance smaller than CRLB for $\theta$.

by Lemma 3

$$Var(T(\boldsymbol{X})) \geq \frac{1}{-E\left[\frac{\partial^2}{\partial\theta^2}\ln f_{X_1,\dots,X_n}(X_1,\dots,X_n|\theta)\right]}.$$

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If a rs $\{X_1, \dots, X_n\}$ of size $n$ is considered, then we would have

$$Var(T(\boldsymbol{X})) \geq \frac{1}{nI_{X_1}(\theta)} = \frac{1}{nE\left[\frac{\partial}{\partial\theta}\ln f_{X_1}(X_1|\theta)\right]^2}.$$

or

$$Var(T(\boldsymbol{X})) \geq \frac{1}{-nE\left[\frac{\partial^2}{\partial\theta^2}\ln f_{X_1}(X_1|\theta)\right]}.$$

- $g(\theta)$

More general

Often we want to estimate a function of $\theta$, $g(\theta)$, instead of $\theta$. If $T(\boldsymbol{X}) = T(X_1, \dots, X_n)$ is an UNBIASED estimator for $g(\theta)$, then the CR inequality for $g(\theta)$ is, if the regularity conditions hold,

$$Var(T(\boldsymbol{X})) \geq \frac{\left[\frac{d}{d\theta}g(\theta)\right]^2}{I_{X_1,\dots,X_n}(\theta)} = \frac{\left[\frac{d}{d\theta}g(\theta)\right]^2}{E\left[\frac{\partial}{\partial\theta}\ln f_{X_1,\dots,X_n}(X_1,\dots,X_n|\theta)\right]^2}.$$

An unbiased estimator whose variance can achieve the CRLB for $g(\theta)$ is the UMVUE for $g(\theta)$. Indeed, the result in Section 2.5 is a special case of this section with $g(\theta) = \theta$.

**Theorem 1**: Under the regularity conditions, the CR equality holds if and only if

$$\frac{\partial}{\partial\theta}\ln f_{X_1,\dots,X_n}(X_1,\dots,X_n|\theta) = \overset{①}{A(\theta,n)}[\overset{②}{T'(X_1,\dots,X_n) - h(\theta)}],$$

where $A(\theta,n) \neq 0$. Then, $T'(X_1,\dots,X_n)$ is an UMVUE for $h(\theta)$.

Note that the above condition is ***unique up to an Euclidean transformation*** of $T'$ and $h(\theta)$. In other words, if $T'(X_1,\dots,X_n)$ is an UMVUE for $h(\theta)$ because they satisfies the above condition, then $aT'(X_1,\dots,X_n) + b$ is an UMVUE for $ah(\theta) + b$, where $a \neq 0$.

① $A(\theta, n)$: $\begin{cases} \text{depends on } \theta \text{ or } n \\ \text{not related to } X_i \text{ (sample)} \end{cases}$

$$\begin{cases} Var(T(X)) > CRLB & \text{sub-efficient} \\ Var(T(X)) < CRLB & \text{super-efficient} \\ Var(T(X)) = CRLB & \text{efficient} \end{cases}$$

(FYI)

An example of super-efficiency:

Let $X_1, X_2, \dots, X_n \sim U(0,\theta)$. show that

1. assumption in CR theorem doesn't hold → regulatority condition
2. Information inequality doesn't apply to UMVUE of $\theta$

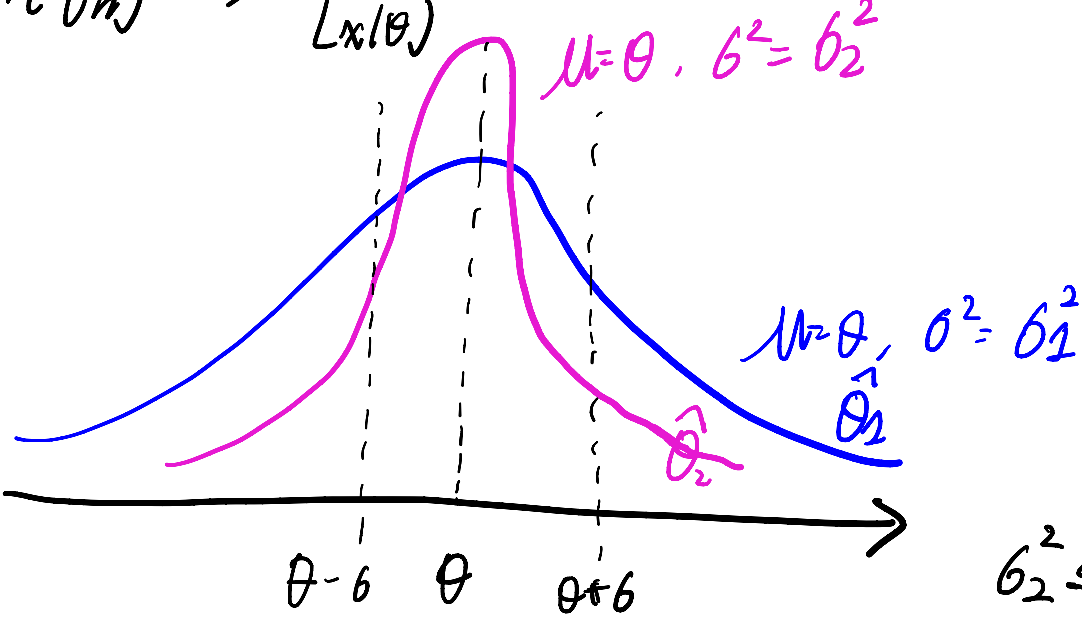
$Var(T(X))$ v.s. CRLB

**Theorem 2**: Consider a random sample $\{X_1, \dots, X_n\}$ of size $n$ from a parametric distribution with a pdf $f_X(\cdot\,|\theta)$ or a pmf $p_X(\cdot\,|\theta)$. Then, under the regularity and other conditions,

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{d} N\left(0, \frac{1}{I_X(\theta)}\right).$$

In other words, the asymptotic variance of $\hat{\theta}_n$ is $\frac{1}{nI_X(\theta)}$, the CRLB for $\theta$.

$$Var(\hat{\theta}_n) \rightarrow \frac{1}{I_X(\theta)}$$




$\sigma_2^2 \le \sigma_1^2$

$$\sigma_1^2 < \sigma_2^2 \Rightarrow Var(\hat{\theta}_1) \ge Var(\hat{\theta}_2)$$

$$\Rightarrow \frac{1}{I_X(\hat{\theta}_1)} \ge \frac{1}{I_X(\hat{\theta}_2)}$$

$$\Rightarrow I_X(\hat{\theta}_1) \le I_X(\hat{\theta}_2)$$

$\hat{\theta}_2$ is more informative $\Rightarrow$ closer to $\theta$

less error (such as MLE)