The Hong Kong University of Science & Technology

MATH243 – Statistical Inference

Midterm Examination – Fall 02/03

Answer <u>ALL</u> questions — **Date: 1 November 2002 (Fri)**

All Equal Marks — **Time allowed: 2 Hours**

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1. Let X_1, X_2, …, X_n denote a random sample from $B_i(1, p)$.

   (a) Find the maximum likelihood estimator of p.

   (b) Is the estimator in (a) an unbiased estimator of p?

   (c) Find Cramer Rao lower bound for the variance of an unbiased estimator for p.

   (d) Find $Var(\bar{X})$.

   (e) Find $E(\bar{X}(1-\bar{X}))$. Then, find the value of c so that $c\bar{X}(1-\bar{X})$ is an unbiased estimator of $Var(\bar{X})$.

2. Let X_1, X_2, …, X_n be a random sample from a uniform distribution on the interval $[\theta-1/2$, $\theta+1/2]$.

   (a) Find the method of moments estimator for θ.

   (b) Is the estimator in (a) an unbiased estimator for θ?

   (c) Find the variance of the estimator in (a).

   (d) Prove that the mid-range $Y=\frac{1}{2}(Y_1+Y_n)$ where $Y_1=\min(X_i)$ and $Y_n=\max(X_i)$ is an unbiased estimator for θ.

   (e) **OPTIONAL (4 marks)** Find the variance of the estimator Y in (d). Is the variance in (c) greater?

   Hint: $f_{Y_1,Y_n}(y_1,y_n)=n(n-1)(y_n-y_1)^{n-2}$ for $\theta-1/2 \le y_1 \le y_n \le \theta+1/2$

3. Let $X_1, ..., X_n$ be a random sample from $N(\mu,\ \sigma^2)$. Define

$$\overline{X}_k = \frac{1}{k}\sum_{i=1}^{k} X_i\ ,$$

$$\overline{X}_{n-k} = \frac{1}{n-k}\sum_{i=k+1}^{n} X_i\ ,$$

and

$$S_k^2 = \frac{1}{k-1}\sum_{i=1}^{k}(X_i - \overline{X}_k)^2\ ,$$

$$S_{n-k}^2 = \frac{1}{n-k-1}\sum_{i=k+1}^{n}(X_i - \overline{X}_{n-k})^2.$$

Answer the following:

(a) What is the distribution of $\left((k-1)S_k^2 + (n-k-1)S_{n-k}^2\right)/\sigma^2$?

(b) What is the distribution of $\left(\overline{X}_k + \overline{X}_{n-k}\right)/2$?

(c) What is the distribution of $\sigma^{-2}(X_i - \mu)^2$?

(d) What is the distribution of S_k^2 / S_{n-k}^2 ?

(e) What is the distribution of $(\overline{X}_n - \mu)/(S_n / \sqrt{n})$?

If $\mu = 0$ and $\sigma^2 = 1$,

(f) What is the distribution of $k\overline{X}_k^2 + (n-k)\overline{X}_{n-k}^2$?

(g) What is the distribution of X_1^2 / X_2^2 ?

(h) What is the distribution of X_1 / X_n ?

(i) What is the distribution of $(X_2 + X_1)^2 / (X_2 - X_1)^2$?

Note: Write your answer as: $S_n^2 \sim \dfrac{\sigma^2}{n-1}\chi^2(n-1)$.

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