Math 243: Mid-term examination 2009/2010

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- This exam is 1.5 hours long.
- To get full points for each question, please show your work as much as possible.

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1. Consider a random variable X having a density defined by

$$f_X(x) = \frac{1}{2}e^{-|x|}, \quad \text{for } x \in (-\infty, \infty),$$

and $f_X(x) = 0$ otherwise.

(a) (3 marks) Find the moment generating function of X.

(b) (1 marks) Let $Y = \frac{1}{\sigma}(X - \mu)$, where $\mu \in (-\infty, \infty)$ and $\sigma^2 \in (0, \infty)$. Find the moment generating function of Y.

(c) (2 marks) What are the mean and variance of Y?

2. (6 marks) Suppose that $\{X_1, \ldots, X_n\}$ is a random sample of size n from a uniform distribution $U[0, \theta]$ with a density

$$f_X(x|\,\theta) = \frac{1}{\theta}, \quad \text{if } x \in [0, \theta] \quad \text{and } \theta \in (0, \infty),$$

and $f_X(x|\,\theta) = 0$ otherwise.

Among all order statistics $X_{(1)}, \ldots, X_{(n)}$, find the one with minimum variance.

3. Consider a random sample $\{X_1, \ldots, X_n\}$ from a density

$$f(x|\,\theta) = \theta x^{\theta-1}, \quad 0 < x < 1,$$

and $f(x|\,\theta) = 0$ otherwise, where $\theta > 0$. Let $g(\theta) = 1/\theta$.

(a) (2 marks) Show that the MLE for $g(\theta)$ is $-\frac{1}{n}\sum_{i=1}^{n} \ln X_i$.

(b) (3 marks) Is it unbiased for $g(\theta)$? If yes, prove it; if not, why?

Given that the regularity conditions are satisfied, then

(c) (3 marks) find the C-R inequality for $g(\theta)$.

(d) (2 marks) show that the MLE is the UMVUE for $g(\theta)$.

4. Let $X_1, \ldots, X_n$ be i.i.d. discrete random variables with $P(X_i = 0) = 1 - P(X_i = 1) = \frac{\theta}{2} + \frac{1}{4}$, for $i = 1, \ldots, n$, where $\theta \in [0, 1]$ is an unknown parameter.

(a) (5 marks) Find the maximum likelihood estimator $\hat{\theta}$ for θ.

(b) (3 marks) For $\bar{X} \in [1/4, 3/4]$, show that

$$\sqrt{n}(\hat{\theta} - \theta) \to N(0, \frac{(3-2\theta)(1+2\theta)}{4}),$$

as $n \to \infty$.

5. Consider a random sample $\{X_1, X_2\}$ from density

$$f_X(x|\theta) = \frac{3x^2}{\theta^3} I_{(0<x<\theta)},$$

where $\theta > 0$.

(a) (2 marks) Are $\hat{\theta}_1 = \frac{2}{3}(X_1 + X_2)$ and $\hat{\theta}_2 = \frac{7}{6}\max(X_1, X_2)$ unbiased for θ?

(b) (4 marks) Find the mean squared errors (MSEs) of $\hat{\theta}_1$ and $\hat{\theta}_2$, and compare those estimators.

(c) (4 marks) Prove that in the sense of MSE, $T_{8/7}$ is the best among the estimators in form of $T_c = c\max(X_1, X_2)$.

Please put this paper inside your answer book at the end of the examination.

Solutions to Mid-term examination

1a) $\because f_X(x) = \frac{1}{2}e^{-|x|}$, $\forall x \in (-\infty, \infty)$ and $= 0$ otherwise

$\therefore$
$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} \frac{1}{2} e^{tx} e^{-|x|} dx$$
$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{tx} e^{x} dx + \int_{0}^{\infty} e^{tx} e^{-x} dx\right]$$
$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{(t+1)x} dx + \int_{0}^{\infty} e^{(t-1)x} dx\right]$$
$$= \frac{1}{2}\left[\frac{1}{t+1} e^{(t+1)x}\Big|_{-\infty}^{0} + \frac{1}{t-1} e^{(t-1)x}\Big|_{0}^{\infty}\right] \text{ for } t \neq -1 \text{ and } 1$$

$\because$ $e^{(t+1)x}\Big|_{-\infty}^{0}$ exists when $t+1 > 0$ and
$e^{(t-1)x}\Big|_{0}^{\infty}$ exists when $t-1 < 0$

$$\therefore M_X(t) = \frac{1}{2}\left[\frac{1}{t+1} - \frac{1}{t-1}\right] = \frac{1}{1-t^2} \text{ if } -1 < t < 1$$ #.

b) Let $Y = \frac{1}{\sigma}(X - \mu)$,

i.e. $$M_Y(t) = E(e^{tY}) = E(e^{\frac{t}{\sigma}(X-\mu)}) = e^{-\frac{\mu t}{\sigma}} E(e^{\frac{t}{\sigma}X})$$
$$= e^{\frac{-\mu t}{\sigma}} \frac{1}{1-\left(\frac{t}{\sigma}\right)^2}, \text{ if } \left|\frac{t}{\sigma}\right| < 1$$ #

c). Consider $M_X(t)$,

$$\because \frac{d}{dt}M_X(t) = -(1-t^2)^{-2}(-2t) = \frac{2t}{(1-t^2)^2}$$

and $$\frac{d^2}{dt^2}M_X(t) = 2(1-t^2)^{-2} - 2(2t)(1-t^2)^{-3}(-2t) = \frac{2}{(1-t^2)^2} + \frac{8t^2}{(1-t^2)^3}$$

$$\therefore E(X) = \frac{d}{dt}M_X(t)\Big|_{t=0} = 0 \text{ and } Var(X) = E(X^2) = \frac{d^2}{dt^2}M_X(t)\Big|_{t=0} = 2$$

$$\Rightarrow E(Y) = \frac{1}{\sigma}[E(X) - \mu] = \frac{-\mu}{\sigma} \text{ and } Var(Y) = \frac{1}{\sigma^2}Var(X) = \frac{2}{\sigma^2}$$ #

2). $\because\ X_i \overset{iid}{\sim} U_{[0,\theta]}$

$$\underbrace{\qquad\qquad\qquad}_{i-1}\overset{i^{th}}{\bullet}\underbrace{\qquad\qquad\qquad}_{n-i}$$

↳ For any $i=1,2,\dots,n$,
the pdf of $X_{(i)}$ is

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!\,1!\,(n-i)!}\,[P(X<x)]^{i-1} f_X(x)\,[P(X\geq x)]^{n-i} \quad \text{for } 0<x<\theta$$

$$= \frac{n!}{(i-1)!\,(n-i)!}\left(\frac{x}{\theta}\right)^{i-1}\frac{1}{\theta}\left(1-\frac{x}{\theta}\right)^{n-i}$$

Note that $$1=\int_0^\theta f_{X_{(i)}}(x)\,dx = \frac{n!}{(i-1)!\,(n-i)!}\,\frac{1}{\theta}\int_0^\theta \left(\frac{x}{\theta}\right)^{i-1}\left(1-\frac{x}{\theta}\right)^{n-i}dx$$

$$\Rightarrow \int_0^\theta \left(\frac{x}{\theta}\right)^{i-1}\left(1-\frac{x}{\theta}\right)^{n-i}dx = \frac{(i-1)!\,(n-i)!}{n!}\,\theta \quad \text{——} \quad (*)$$

Thus,
$$E(X_{(i)}) = \int_0^\theta x f_{X_{(i)}}(x)\,dx$$

$$= \frac{n!}{(i-1)!\,(n-i)!}\int_0^\theta \left(\frac{x}{\theta}\right)^{(i+1)-1}\left(1-\frac{x}{\theta}\right)^{(n+1)-(i+1)}$$

$$= \frac{n!}{(i-1)!\,(n-i)!}\,\frac{(i+1-1)!\,(n-i)!}{(n+1)!}\,\theta \quad \text{by } (*)$$

$$= \frac{i}{n+1}\theta$$

Similarly,
$$E(X_{(i)}^2) = \int_0^\theta x^2 f_{X_{(i)}}(x)\,dx = \frac{n!\,\theta}{(i-1)!\,(n-i)!}\int_0^\theta \left(\frac{x}{\theta}\right)^{(i+2)-1}\left(1-\frac{x}{\theta}\right)^{(n+2)-(i+2)}dx$$

$$= \frac{n!\,\theta}{(i-1)!\,(n-i)!}\,\frac{(i+2-1)!\,(n-i)!}{(n+2)!}\,\theta \quad \text{by } (*)$$

$$= \frac{(i+1)\,i}{(n+2)(n+1)}\theta^2$$

i.e.

$$Var(X_{(i)}) = \frac{i(i+1)}{(n+2)(n+1)}\theta^2 - \left(\frac{i}{n+1}\theta\right)^2 = \frac{i(i+1)(n+1) - i^2(n+2)}{(n+2)(n+1)^2}\theta^2$$

$$= \frac{(n+1)i - i^2}{(n+2)(n+1)^2}\theta^2, \text{ for } i=1,2,\dots,n$$

Let $a(i) = (n+1)i - i^2$, for $i=1,2,\dots,n$

$\because$ $a(i) < a(i+1) \Leftrightarrow (n+1)i - i^2 < (n+1)(i+1) - (i+1)^2$

$\Leftrightarrow (n+1)i - i^2 < (n+1)i + (n+1) - i^2 - 2i - 1$

$\Leftrightarrow 2i < n$

and

$a(i) > a(i+1) \Leftrightarrow 2i > n$

$\therefore$ $Var(X_{(i)})$ increases when i is from 1 to $\frac{n}{2}$ (if n is even) or $\frac{n-1}{2}$ (if n is odd) and decreases when i is from $\frac{n}{2}$ (if n is even) or $\frac{n+1}{2}$ (if n is odd)

Also, $Var(X_{(1)}) = \frac{n}{(n+2)(n+1)^2}\theta^2 = Var(X_{(n)})$

Therefore, $X_{(1)}$ and $X_{(n)}$ have minimum variances among n order statistics $\divideontimes$

Remark: $a(n-i+1) = (n+1)(n-i+1) - (n-i+1)^2$

$= (n-i+1)(n+1-n+i-1)$

$= (n+1)i - i^2 = a(i)$

So, $a(\cdot)$ is symmetric

3a) ∵ $f(x|\theta)=\begin{cases}\theta x^{\theta-1}, & \text{if } 0<x<1\\ 0, & \text{otherwise},\end{cases}$

where $\theta>0$

∴ $\mathcal{L}(\theta)=\prod_{i=1}^{n} f(x_i|\theta)$ for a r.s. $\{X_1,\dots,X_n\}$

$= \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$

$\Rightarrow \ell(\theta)=\ln\mathcal{L}(\theta)=n\ln\theta+(\theta-1)\sum_{i=1}^{n}\ln x_i$

$\Rightarrow 0=\frac{d}{d\theta}\ell(\theta)\Big|_{\theta=\hat\theta}=\frac{n}{\hat\theta}+\sum_{i=1}^{n}\ln x_i \Rightarrow \hat\theta=\frac{n}{-\sum_{i=1}^{n}\ln x_i}$, where we have one critical point

∵ $0<x_i<1$

∴ $-\ln x_i>0$ for all $i=1,\dots,n$

$\Rightarrow \hat\theta=\frac{n}{-\sum_{i=1}^{n}\ln X_i}>0$, i.e. $\hat\theta\in\Theta$

∵ $\frac{d^2}{d\theta^2}\ell(\theta)\Big|_{\theta=\hat\theta}=\frac{-n}{\hat\theta^2}<0$

∴ the MLE for θ is $\hat\theta=\frac{n}{-\sum_{i=1}^{n}\ln X_i}$

By the invariance property of MLE, we have a result that the MLE for $g(\theta)=\frac{1}{\theta}$ is $g(\hat\theta)=\frac{1}{\hat\theta}=-\frac{1}{n}\sum_{i=1}^{n}\ln X_i$ #

b). Consider $Y=-\ln X$,

For $y>0$,

$P(Y\le y)=P(-\ln X\le y)=P(X\ge e^{-y})=\int_{e^{-y}}^{1}\theta x^{\theta-1}dx=1-e^{-\theta y}$

∴ $f_Y(y)=\theta e^{-\theta y}$ if $y>0$, which is the density of $Exp(\theta)$

$\Rightarrow Y_i=-\ln X_i\sim Exp(\theta)$ for $i=1,2,\dots,n$

Note that $E(Y_i) = \frac{1}{\theta}$ for $i=1,2,\ldots,n$

$\therefore \quad E(g(\hat{\theta})) = E\left[\frac{1}{n}\sum_{i=1}^{n}(-\ln X_i)\right] = E(Y_1) = \frac{1}{\theta} = g(\theta)$

$\Rightarrow g(\hat{\theta})$ is unbiased #.

Given that the regularity conditions are satisfied,

c). $\because$ the CR lower bound for $g(\theta)$ is

$$\frac{(g'(\theta))^2}{I_{X_1,\ldots,X_n}(\theta)} = \frac{\left(\frac{d}{d\theta}\frac{1}{\theta}\right)^2}{E\left[\frac{d}{d\theta}\ln L(\theta)\right]^2} = \frac{\left(\frac{-1}{\theta^2}\right)^2}{-E\left[\frac{d^2}{d\theta^2}\ln L(\theta)\right]} = \frac{\frac{1}{\theta^4}}{-\left(\frac{-n}{\theta^2}\right)} = \frac{1}{n\theta^2}$$

$\therefore$ for any unbiased estimator $T = T(X_1,\ldots,X_n)$ of $g(\theta) = \frac{1}{\theta}$, we have -

$$Var(T) \geq \frac{1}{n\theta^2}$$

#.

d). From (a), we have that

$$\ln L(\theta) = \ln f_{X_1,\ldots,X_n}(\theta) = n\ln\theta + (\theta-1)\sum_{i=1}^{n}\ln X_i$$

$$\therefore \quad \frac{d}{d\theta}\ln f_{X_1,\ldots,X_n}(\theta) = \frac{n}{\theta} + \sum_{i=1}^{n}\ln X_i = -n\left[\frac{-1}{n}\sum_{i=1}^{n}\ln X_i - \frac{1}{\theta}\right]$$

$$= -n\left[\hat{\theta} - \frac{1}{\theta}\right]$$

i.e. $\hat{\theta} = \frac{-1}{n}\sum_{i=1}^{n}\ln X_i$, the MLE for $g(\theta)$, is the UMVUE of $g(\theta) = \frac{1}{\theta}$ #

4(a). $\because X_i \overset{iid}{\sim} Bin(1, p)$, where $p = P(X_i=1) = 1-\frac{\theta}{2}-\frac{1}{4}$ and $i=1,2,\dots,n$

$\therefore \mathcal{L}(\theta) = \prod_{i=1}^{n} P(X_i = x_i) = \prod_{i=1}^{n}\left[p^{x_i}(1-p)^{1-x_i}\right] = p^{\sum x_i}(1-p)^{n-\sum x_i}$

$\Rightarrow l(\theta) = \ln\mathcal{L}(\theta) = \left(\sum_{i=1}^{n} x_i\right)\ln\left(\frac{3}{4}-\frac{\theta}{2}\right) + \left(n-\sum_{i=1}^{n} x_i\right)\ln\left(\frac{\theta}{2}+\frac{1}{4}\right)$

$0 = \frac{d}{d\theta} l(\theta)\Big|_{\theta=\tilde{\theta}} = \frac{-\frac{1}{2}\sum_{i=1}^{n} x_i}{\frac{3}{4}-\frac{\tilde{\theta}}{2}} + \frac{\frac{1}{2}\left(n-\sum_{i=1}^{n} x_i\right)}{\frac{\tilde{\theta}}{2}+\frac{1}{4}} \Rightarrow \tilde{\theta} = \frac{3}{2} - 2\overline{X}$, where $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

and $\frac{d^2}{d\theta^2} l(\theta)\Big|_{\theta=\tilde{\theta}} = \frac{-\frac{1}{2}\left(\frac{1}{2}\right)\sum_{i=1}^{n} x_i}{\left(\frac{3}{4}-\frac{\tilde{\theta}}{2}\right)^2} - \frac{\frac{1}{2}\left(\frac{1}{2}\right)\left(n-\sum_{i=1}^{n} x_i\right)}{\left(\frac{\tilde{\theta}}{2}+\frac{1}{4}\right)^2} < 0$

Thus,

$\tilde{\theta}$ is the MLE for θ when $\tilde{\theta} \in [0,1]$, i.e. if $\frac{1}{4} \le \overline{X} \le \frac{3}{4}$

$\because$ if $\overline{X} > \frac{3}{4}$, then $\tilde{\theta} < 0$, which implies that $l(\theta) = \ln\mathcal{L}(\theta)$ is a decreasing function of $\theta \in [0,1]$. In other words, $l(\theta)$ achieves its maximum at $\theta = 0$, so the MLE for θ is 0

Similarly, if $\overline{X} < \frac{1}{4}$, then $\tilde{\theta} > 1$, which implies that $l(\theta)$ is an increasing function of $\theta \in [0,1]$. Thus, $l(\theta)$ achieves its maximum at $\theta = 1$, so the MLE for θ is 1

$\therefore$ The MLE $\hat{\theta}$ for θ

$$\text{is } \begin{cases} 0 & \text{if } \overline{X} > \frac{3}{4} \\ \tilde{\theta} = \frac{3}{2} - 2\overline{X} & \text{if } \frac{1}{4} \le \overline{X} \le \frac{3}{4} \\ 1 & \text{if } \overline{X} < \frac{1}{4} \end{cases}$$

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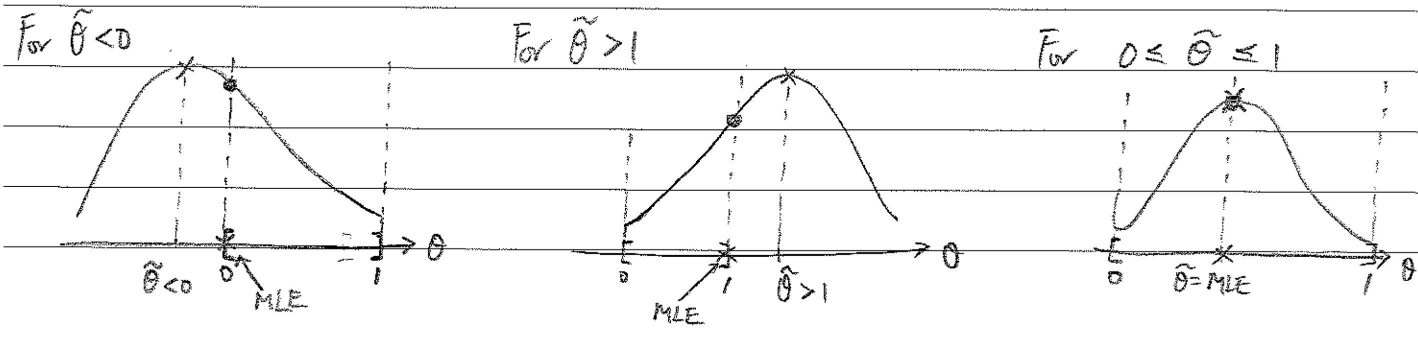

4b). For $\bar{X} \in [\frac{1}{4}, \frac{3}{4}]$, i.e. $\hat{\theta} = \tilde{\theta} = \frac{3}{2} - 2\bar{X}$

$\because$ $X_i \overset{iid}{\sim} Bin(1, p)$ with $E(X_i) = p = \frac{3}{4} - \frac{\theta}{2} < \infty$ and
$Var(X_i) = p(1-p) = (\frac{3}{4} - \frac{\theta}{2})(\frac{\theta}{2} + \frac{1}{4}) < \infty$

$\therefore$ By CLT,

$$\frac{\sqrt{n}(\bar{X} - p)}{\sqrt{p(1-p)}} \longrightarrow N(0,1), \text{ as } n \to \infty$$

(Method 1)

Since $E(\hat{\theta}) = \frac{3}{2} - 2E(\bar{X}) = \frac{3}{2} - 2p = \theta$ and
$Var(\hat{\theta}) = 4Var(\bar{X}) = \frac{4}{n}(\frac{3}{4} - \frac{\theta}{2})(\frac{\theta}{2} + \frac{1}{4}) = \frac{1}{4n}(3-2\theta)(2\theta+1)$,

by the property of normal distribution, we have

$$\hat{\theta} = \frac{3}{2} - 2\bar{X} \longrightarrow N(E(\hat{\theta}), Var(\hat{\theta})) \quad \text{as } n \to \infty$$

i.e. $\hat{\theta} \to N(\theta, \frac{1}{4n}(3-2\theta)(2\theta+1))$

or $\sqrt{n}(\hat{\theta} - \theta) \to N(0, \frac{1}{4}(3-2\theta)(2\theta+1))$ #

(Method 2)

Let $g(t) = \frac{3}{2} - 2t$, $g'(t) = -2$,

by Delta method, we have

$$\sqrt{n}(g(\bar{X}) - g(p)) \longrightarrow N(0, (g'(p)\sqrt{p(1-p)})^2)$$

i.e.

$$\sqrt{n}(\hat{\theta} - \theta) \to N(0, 4(\frac{3}{4} - \frac{\theta}{2})(\frac{\theta}{2} + \frac{1}{4}) = \frac{1}{4}(3-2\theta)(2\theta+1)), \text{ as } n \to \infty$$ #

(Method 3)

$\because$ $I_{X_1, \ldots, X_n}(\theta) = -E(\frac{\partial^2}{\partial \theta^2} \ln L(\theta)) = -E\left[\frac{-\frac{1}{4}n\bar{X}}{(\frac{3}{4} - \frac{\theta}{2})^2} - \frac{\frac{1}{4}(n - n\bar{X})}{(\frac{\theta}{2} + \frac{1}{4})^2}\right]$

$$= \frac{\frac{n}{4}E(\bar{X})}{(\frac{3}{4} - \frac{\theta}{2})^2} + \frac{\frac{n}{4}(1 - E(\bar{X}))}{(\frac{\theta}{2} + \frac{1}{4})^2} = \frac{\frac{n}{4}}{(\frac{3}{4} - \frac{\theta}{2})} + \frac{\frac{n}{4}}{(\frac{\theta}{2} + \frac{1}{4})}$$

∴ By the asymptotic property of the MLE, we have

$$\hat{\theta}-\theta \to N\left(0, \frac{1}{nI_{X_1}(\theta)} = \frac{1}{I_{X_1,\dots,X_n}(\theta)} = \frac{1}{\frac{n}{4}\left[\frac{1}{\frac{3}{4}-\frac{\theta}{2}} + \frac{1}{\frac{\theta}{2}+\frac{1}{4}}\right]}\right.$$

$$= \frac{1}{\frac{n}{4}\left[\frac{1}{(\frac{3}{4}-\frac{\theta}{2})(\frac{\theta}{2}+\frac{1}{4})}\right]}$$

$$\left. = \frac{(3-2\theta)(2\theta+1)}{4n}\right)$$

as $n \to \infty$ #

(a) For $i=1,2$,

$$E(X_i) = \int_0^\theta x f_{X_i}(x|\theta)\,dx = \int_0^\theta \frac{3x^3}{\theta^3}\,dx = \frac{3}{\theta^3}\frac{1}{4}x^4\Big|_0^\theta = \frac{3}{4}\theta$$

∴ $E(\hat{\theta}_1) = \frac{2}{3}(E(X_1)+E(X_2)) = \frac{2}{3}(\frac{3}{4}\theta+\frac{3}{4}\theta) = \theta$, so $\hat{\theta}_1$ is unbiased for θ

Consider $Y = \max(X_1, X_2)$, for $y \in (0,\theta)$,

∵ $P(Y \le y) = P(\max(X_1,X_2) \le y) = [P(X_1 \le y)]^2$,

$$= \left[\int_0^y \frac{3x^2}{\theta^3}\,dx\right]^2 = \left(\frac{y}{\theta}\right)^6$$

∴ $f_Y(y) = 6(\frac{y}{\theta})^5(\frac{1}{\theta})$ if $0<y<\theta$

i.e. $E(Y) = \frac{6}{\theta^6}\int_0^\theta y^6\,dy = \frac{6}{7}\theta \Rightarrow \theta = E(\frac{7}{6}Y) = E(\frac{7}{6}\max(X_1,X_2))$

$$= E(\hat{\theta}_2)$$

so $\hat{\theta}_2$ is also unbiased for θ #

b) For $i=1,2$,

$$E(X_i^2) = \int_0^\theta \frac{3}{\theta^3} x^4 \, dx = \frac{3}{5}\theta^2,$$

$$Var(X_i) = \frac{3}{5}\theta^2 - \left(\frac{3}{4}\theta\right)^2 = \frac{3}{80}\theta^2$$

Thus,

$$MSE(\hat{\theta}_1) = Var(\hat{\theta}_1) \quad (\because \hat{\theta}_1 \text{ is unbiased for } \theta)$$

$$= Var\left(\frac{4}{3}\bar{X}\right), \text{ where } \bar{X} = \frac{1}{2}(X_1 + X_2)$$

$$= \frac{16}{9} \frac{Var(X_1)}{2}$$

$$= \frac{8}{9} \times \frac{3}{80}\theta^2 = \frac{1}{30}\theta^2$$

For $\hat{\theta}_2 = \frac{7}{6}\max(X_1, X_2)$,

$$\because E(Y^2) = \frac{6}{\theta^6}\int_0^\theta y^7 \, dy = \frac{3}{4}\theta^2$$

$$\therefore Var(Y) = \frac{3}{4}\theta^2 - \left(\frac{6}{7}\theta\right)^2 = \frac{3}{196}\theta^2$$

$$\Rightarrow MSE(\hat{\theta}_2) = Var(\hat{\theta}_2) \quad (\because \hat{\theta}_2 \text{ is unbiased for } \theta)$$

$$= Var\left(\frac{7}{6}Y\right)$$

$$= \frac{49}{36} Var(Y)$$

$$= \frac{49}{36} \cdot \frac{3}{196}\theta^2 = \frac{\theta^2}{12(4)} = \frac{\theta^2}{48}$$

Thus, $\hat{\theta}_2$ has a smaller MSE than $\hat{\theta}_1$ #

c). ∵ $T_c = cY$

∴ $Var(T_c) = c^2 Var(Y) = c^2(\frac{5}{196}\theta^2)$
and $E(T_c) = cE(Y) = c(\frac{6}{7}\theta)$

i.e. $MSE(T_c) = Var(T_c) + [E(T_c) - \theta]^2$

$= c^2 Var(Y) + [cE(Y) - \theta]^2$

$= c^2 Var(Y) + [c^2(E(Y))^2 - 2c\theta E(Y) + \theta^2]$

$= c^2[Var(Y) + (E(Y))^2] - 2c\theta E(Y) + \theta^2$

$0 = \frac{d}{dc} MSE(T_c)\Big|_{c=c^*} = 2c^*[Var(Y) + (E(Y))^2] - 2\theta E(Y)$

$\Rightarrow c^* = \frac{\theta E(Y)}{Var(Y) + (E(Y))^2} = \frac{\frac{6}{7}\theta^2}{\frac{5}{196}\theta^2 + \frac{36}{49}\theta^2} = \frac{8}{7}$

and $\frac{d^2}{dc^2} MSE(T_c)\Big|_{c=c^*} = 2[Var(Y) + E(Y)^2] > 0$

Thus,
$T_{c^*} = T_{\frac{8}{7}}$ has the smallest MSE among the estimators of θ in ~~form~~ of $T_c = c\max(X_1, X_2)$, i.e. it is the best #