Solutions to Exercise 1

1. (a)
$$P(X < 6.0171) = P\left(Z < \frac{6.0171 - 6.05}{\sqrt{0.0004}}\right) = P(Z < -1.645) = 0.05$$

(b) Let N be the two of boxes that are less than 6.1071, then $N \sim Bin(9, 0.05)$
$$\begin{aligned} P(N \leq 2) &= P(N = 0) + P(N = 1) + P(N = 2) \\ &= \binom{9}{0} 0.05^0 (1 - 0.05)^{9-0} + \binom{9}{1} 0.05^1 (1 - 0.05)^{9-1} + \binom{9}{2} 0.05^0 (1 - 0.05)^{9-2} \\ &= 0.9916 \end{aligned}$$

(c) $\bar{X} \sim N(6.05, \frac{0.0004}{9})$
Therefore:
$$\begin{aligned} P(\bar{X} \leq 6.035) &= P\left(Z \leq \frac{6.035 - 6.05}{\sqrt{\frac{0.0004}{6}}}\right) \\ &= P(Z \leq -2.25) \\ &= 1 - 0.9878 = 0.01 \end{aligned}$$

2. X_1 and X_2 are independent. Therefore:
$$X_1 - X_2 \sim N(47.88 - 43.04, 2.19 + 14.89) = N(4.84, 17.08)$$

3. Let $X_i \sim N(1.18, 0.07^2), \qquad i = 1, 2, 3 \qquad Y \sim N(3.22, 0.09^2)$
Assume X_i and Y are independent.
$$\begin{aligned} E(X_1 + X_2 + X_3 - Y) &= 3 \times 1.18 - 3.22 \qquad \because X_i \text{ are i.i.d.} \\ &= 0.32 \\ Var(X_1 + X_2 + X_3 - Y) &= Var(X_1 + X_2 + X_3) + Var(Y) \\ &= Var(X_1) + Var(X_2) + Var(X_3) + Var(Y) \\ &= 3 \times 0.007^2 + 0.009^2 = 0.0228 \end{aligned}$$

$$\begin{aligned} \therefore \qquad X_1 + X_2 + X_3 - Y &\sim N(0.32, 0.0228) \\ P(X_1 + X_2 + X_3 - Y > 0) &= P\left(Z > \frac{0 - 0.32}{\sqrt{0.0228}}\right) \\ &= P(Z > -2.12) = 0.983 \end{aligned}$$

4. Assume X, Y are independent.
$$\begin{aligned} X - Y &\sim N(529 - 474, 5732 + 6368) = N(55, 12100) \\ P(X > Y) &= P(X - Y > 0) \\ &= P\left(Z > \frac{0 - 55}{\sqrt{12100}}\right) \\ &= P(Z > -0.5) = 0.6915 \end{aligned}$$

5. $\bar{X} \simeq N(40, \frac{8}{32}) = N(40, \frac{1}{4})$ by C.L.T.
$$\begin{aligned} P(39.75 \leq \bar{X} \leq 41.25) &\approx P\left(\frac{39.75 - 40}{\sqrt{0.25}} \leq Z \leq \frac{41.25 - 40}{\sqrt{0.25}}\right) \\ &= P(-0.5 \leq Z \leq 2.5) \\ &= P(Z \leq 2.5) - P(Z \leq -0.5) \\ &= 0.9938 - (1 - 0.6915) = 0.6853 \end{aligned}$$

6. (a)

$$E(\bar{X}) = E\left(\frac{1}{30}\sum_{i=1}^{30} X_i\right) = \frac{1}{30}\sum_{i=1}^{30} E(X_i) = \frac{1}{30} \times 30 \times 24.43 = 24.43$$

(b)

$$Var(\bar{X}) = Var\left(\frac{1}{30}\sum_{i=1}^{30} X_i\right) = \frac{1}{30^2}\sum_{i=1}^{30} Var(X_i) = \frac{1}{30^2} \times 30 \times 2.2 = \frac{2.2}{30}$$

(c) By C.L.T. $\quad \bar{X} \simeq N(24.43, \frac{2.2}{30})$

$$\begin{aligned} P(24.17 \leq \bar{X} \leq 24.82) &\approx P\left(\frac{24.17 - 24.43}{\sqrt{\frac{2.2}{30}}} \leq Z \leq \frac{24.82 - 24.43}{\sqrt{\frac{2.2}{30}}}\right) \\ &= P(-0.96 \leq Z \leq 1.44) \\ &= P(Z \leq 1.44) - P(Z \leq -0.96) \\ &= 0.9251 - (1 - 0.8315) = 0.7566 \end{aligned}$$

7. $X \sim Bin(48, 0.75) \simeq N(48 \times 0.75, 48 \times 0.75 \times (1 - 0.75)) = N(36, 9)$

$$\begin{aligned} P(35 \leq X \leq 40) &\approx P\left(\frac{34.5 - 36}{\sqrt{9}} \leq Z \leq \frac{40.5 - 36}{\sqrt{9}}\right) \\ &= P(-0.5 \leq Z \leq 1.5) \\ &= P(Z \leq 1.5) - P(Z \leq -0.5) \\ &= 0.9332 - (1 - 0.6915) = 0.6247 \end{aligned}$$

8. $X \sim Bin(100, 0.9) \simeq N(100 \times 0.9, 100 \times 0.9 \times (1 - 0.9)) = N(90, 9)$

$$\begin{aligned} P(89 \leq X \leq 94) &\approx P\left(\frac{88.5 - 90}{\sqrt{9}} \leq Z \leq \frac{94.5 - 90}{\sqrt{9}}\right) \\ &= P(-0.5 \leq Z \leq 1.5) \\ &= P(Z \leq 1.5) - P(Z \leq -0.5) \\ &= 0.9332 - (1 - 0.6915) = 0.6247 \end{aligned}$$

9. (a) $X \sim N(21.37, 0.16)$

$$P(X < 20.857) = P\left(Z < \frac{20.857 - 21.37}{\sqrt{0.16}}\right) = P(Z < -1.2825) = 1 - 0.8997 = 0.1$$

(b) By C.L.T. $\quad Y \sim Bin(100, 1) \simeq N(100 \times 0.1, 100 \times 0.1 \times (1 - 0.1)) = N(10, 9)$

$$P(Y \leq 5) \approx P(Z \leq \frac{5.5 - 10}{\sqrt{9}}) = P(Z \leq -1.5) = 1 - 0.9332 = 0.0668$$

(c) $\bar{X} \sim N(21.37, \frac{0.16}{100})$

$$\begin{aligned} P(21.31 \leq \bar{X} \leq 21.39) &\approx P\left(\frac{21.31 - 21.37}{\sqrt{\frac{0.16}{100}}} \leq Z \leq \frac{21.39 - 21.37}{\sqrt{\frac{0.16}{100}}}\right) \\ &= P(-1.5 \leq Z \leq 0.5) \\ &= P(Z \leq 0.5) - P(Z \leq -1.5) \\ &= 0.6915 - (1 - 0.9332) = 0.6247 \end{aligned}$$

10. $X \sim Po(4829) \simeq N(4829, 4829)$ by C.L.T.

$$\begin{aligned} P(4776 \leq X \leq 4857) &\approx P\left(\frac{4775.5 - 4829}{\sqrt{4829}} \leq Z \leq \frac{4857.5 - 4829}{\sqrt{4829}}\right) \\ &= P(-0.77 \leq Z \leq 0.41) \\ &= P(Z \leq 0.41) - P(Z \leq -0.77) \\ &= 0.6591 - (1 - 0.7794) = 0.4385 \end{aligned}$$

11. $Y \sim Bi(1000, \frac{18}{38}) \simeq N(1000 \times \frac{18}{38}, 1000 \times \frac{18}{38}(1 - \frac{18}{38})) = N(473.68, 249.31)$ by C.L.T.

$$\begin{aligned} P(Y > 500) &\approx P\left(Z \geq \frac{500.5 - 473.68}{\sqrt{249.31}}\right) \\ &= P(Z \geq 1.70) \\ &= 1 - 0.9554 = 0.0446 \end{aligned}$$

12. $Y_i \sim N(1, 9)$, $\quad i.i.d. \quad i = 1, 2, \ldots, 25$
X_i and Y_i are independent. Hence:

$$\bar{X} \sim N\left(0, \frac{16}{25}\right), \quad \bar{Y} \sim N\left(1, \frac{9}{25}\right), \quad \bar{X} - \bar{Y} \sim N\left(0 - 1, \frac{16}{25} + \frac{9}{25}\right) = N(-1, 1)$$

$$\begin{aligned} P(\bar{X} > \bar{Y}) &= P(\bar{X} - \bar{Y} > 0) \\ &= P\left(Z > \frac{0 - (-1)}{\sqrt{1}}\right) \\ &= P(Z > 1) \\ &= 1 - 0.8413 = 0.1587 \end{aligned}$$

13. $Y \sim Bi(72, \frac{1}{3}) \simeq N(72 \times \frac{1}{3}, 72, \frac{1}{3} \times (1 - \frac{1}{3})) = N(24, 16)$ by C.L.T.

$$\begin{aligned} P(22 \leq X \leq 28) &\approx P\left(\frac{21.5 - 24}{\sqrt{16}} \leq Z \leq \frac{28.5 - 24}{\sqrt{16}}\right) \\ &= P(-0.625 \leq Z \leq 1.125) \\ &= P(Z \leq 1.125) - P(Z \leq -0.625) \\ &\approx P(Z \leq 1.13) - P(Z \leq -0.63) \\ &= 0.8708 - (1 - 0.7357) = 0.6065 \end{aligned}$$

14. $Y \sim Bi(400, \frac{1}{5}) \simeq N(400 \times \frac{1}{5}, 400 \times \frac{1}{5} \times (1 - \frac{1}{5})) = N(80, 64)$ by C.L.T.

$$\begin{aligned} P\left(\frac{Y}{400} > 0.25\right) &= P(Y > 100) \\ &\approx P\left(Z \geq \frac{100.5 - 80}{\sqrt{64}}\right) \\ &= P(Z \geq 2.56) \\ &= 1 - 0.9948 = 0.0052 \end{aligned}$$

15. $Y \sim Bi(100, \frac{1}{2}) \simeq N(100 \times \frac{1}{2}, 100 \times \frac{1}{2} \times (1 - \frac{1}{2})) = N(50, 25)$ by C.L.T.

$$\begin{aligned} P(Y = 50) &\approx P\left(\frac{49.5 - 50}{\sqrt{25}} \leq Z \leq \frac{50.5 - 50}{\sqrt{25}}\right) \\ &= P(-0.1 \leq Z \leq 0.1) \\ &= 2 \times P(0 \leq Z \leq 0.1) \\ &= 2 \times (0.5398 - 0.5) = 0.0796 \end{aligned}$$

17. X_i are i.i.d. with p.d.f. $f(x) = 1 - \frac{x}{2}, \qquad 0 \le x \le 2.$

(a)

$$\mu = E(X_i) = \int_0^2 xf(x)dx = \int_0^2 x(1-\frac{x}{2})dx = \left[\frac{x^2}{2} - \frac{x^3}{6}\right]_0^2 = 2 - 8/6 = 2/3$$

$$E(X_i^2) = \int_0^2 x^2 f(x)dx = \int_0^2 x^2(1-\frac{x}{2})dx = \left[\frac{x^3}{3} - \frac{x^4}{8}\right]_0^2 = 8/3 - 2 = 2/3$$

$$\therefore\ \sigma^2 = Var(X_i) = E(X_i^2) - E(X_i)^2 = 2/3 - (2/3)^2 = 2/9$$

(b) By C.L.T.

$$\bar{X} \simeq N\left(\mu, \frac{\sigma^2}{n}\right) = N\left(\frac{2}{3}, \frac{2}{9} \times \frac{1}{18}\right) = N\left(\frac{2}{3}, \frac{1}{81}\right)$$

Hence

$$\begin{aligned} P(2/3 \le \bar{X} \le 5/6) &\approx P\left(\frac{2/3 - 2/3}{\sqrt{1/81}} \le Z \le \frac{5/6 - 2/3}{\sqrt{1/81}}\right) \\ &= P(0 \le Z \le 9/6) \\ &= 0.4332 \end{aligned}$$

19. $X_1, X_2, \ldots, X_{100} \sim \chi^2(50), \qquad E(X_i) = 50, \qquad Var(X_i) = 2 \times 50 = 100.$
By C.L.T., $\bar{X} \simeq N(50, 100/100) = N(50, 1)$

$$P(49 < \bar{X} < 51) \approx P\left(\frac{49-50}{\sqrt{1}} \le Z \le \frac{51-50}{\sqrt{1}}\right) = P(-1 \le Z \le 1) = 0.6826$$

20. $X_1, X_2, \ldots, X_{100} \sim Gamma(2,4), \qquad \alpha = 2, \beta = 4$
By table, $E(X_i) = \alpha(\frac{1}{\lambda}) = \alpha\beta = 2 \times 4 = 8, \qquad (\frac{1}{\lambda} = \beta)$
$Var(X_i) = \alpha(\frac{1}{\lambda^2}) = \alpha\beta^2 = 2 \times 4^2 = 32$

By C.L.T., $\bar{X} \simeq N(8, \frac{32}{128}) = N(8, 1/4)$

$$\begin{aligned} P(7 < \bar{X} < 9) &\approx P\left(\frac{7-8}{\sqrt{1/4}} \le Z \le \frac{9-8}{\sqrt{1/4}}\right) \\ &= P(-2 \le Z \le 2) \\ &= 0.9544 \end{aligned}$$

21. $X_1, \ldots, X_{15}$ are i.i.d. with pdf $f(x) = 3x^2, \qquad 0 < x < 1.$

$$E(X_i) = \int_0^1 xf(x)dx = \int_0^1 x(3x^2)dx = \int_0^1 3x^3 dx = \left[\frac{3}{4}x^4\right]_0^1 = \frac{3}{4}$$

$$E(X_i^2) = \int_0^1 x^2 f(x)dx = \int_0^1 x^2(3x^2)dx = \int_0^1 3x^4 dx = \left[\frac{3}{5}x^5\right]_0^1 = \frac{3}{5}$$

$$\therefore\ Var(X_i) = E(X_i^2) - E(X_i)^2 = 3/5 - (3/4)^2 = 3/80$$

By C.L.T., $\bar{X} \simeq N(3/4, \frac{3}{80} \times \frac{1}{15}) = N(3/4, 1/400)$

$$\begin{aligned} P(3/5 < \bar{X} < 4/5) &\approx P\left(\frac{3/5 - 3/4}{\sqrt{1/400}} \le Z \le \frac{4/5 - 3/4}{\sqrt{1/400}}\right) \\ &= P(-3 \le Z \le 1) \\ &= 0.84 \end{aligned}$$

22. Note that Y is discrete.

$$E(X_i) = \sum_{x=1}^{6} x f(x) = \sum_{x=1}^{6} \frac{x}{6} = \frac{1}{6}(1+2+\ldots+6) = 3.5$$

$$E(X_i^2) = \sum_{x=1}^{6} x^2 f(x) = \sum_{x=1}^{6} \frac{x^2}{6} = \frac{1}{6}(1^2+2^2+\ldots+6^2) = 91/6$$

$$\therefore\ Var(X_i) = E(X_i^2) - E(X_i)^2 = 91/6 - (7/2)^2 = 35/12$$

By C.L.T., $Y = \sum_{i=1}^{12} X_i \simeq N(12 \times 3.5, 12 \times \frac{35}{12}) = N(42, 35)$

$$\begin{aligned} P(36 < Y < 48) &\approx P\left(\frac{35.5-42}{\sqrt{35}} \le Z \le \frac{48.5-42}{\sqrt{35}}\right) \\ &= P(-1.1 \le Z \le 1.1) \\ &= 0.7286 \end{aligned}$$

23.

$$f(x) = \frac{1}{x^2}, \qquad 1 < x < \infty$$

$$f(X<3) = \int_1^3 f(x)dx = \int_1^3 \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^3 = -\frac{1}{3}+1 = \frac{2}{3}$$

By C.L.T., let

$$Y \sim Bi(72, P(X<3)) = Bi(72, 2/3) \simeq N(72 \times \frac{2}{3}, 72 \times \frac{2}{3} \times \frac{1}{3}) = N(48, 16)$$

$$P(Y > 50) \approx P\left(Z \ge \frac{50.5-48}{\sqrt{16}}\right) = P(Z \ge 0.625) = 0.2660$$

24. Let $X_i \sim Uniform(-\frac{1}{2}, \frac{1}{2}) \quad i.i.d.$
$f(x) = 1, \qquad x \in (-\frac{1}{2}, \frac{1}{2})$

$$E(X_i) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x f(x) dx = \left[\frac{1}{2}x^2\right]_{-\frac{1}{2}}^{\frac{1}{2}} = 0$$

$$E(X_i^2) = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 f(x) dx = \left[\frac{1}{3}x^3\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{3}(\frac{1}{4}) = \frac{1}{12}$$

$$\therefore\ Var(X_i) = E(X_i^2) - E(X_i)^2 = \frac{1}{12}$$

By C.L.T., $\sum_{i=1}^{48} X_i \simeq N(48 \times 0, 48 \times \frac{1}{12}) = N(0, 4)$

$$\begin{aligned} P\left(-2 \le \sum_{i=1}^{48} X_i \le 2\right) &\approx P\left(\frac{-2-0}{\sqrt{4}} \le Z \le \frac{2-0}{\sqrt{4}}\right) \\ &= P(-1 \le Z \le 1) \\ &= 0.6826 \end{aligned}$$

25. 90% C.I. for μ

$$\left[\bar{X} \pm Z_{(\frac{0.1}{2})} \frac{S}{\sqrt{n}}\right] = \left[49.2 \pm 1.645 \times \frac{25}{\sqrt{36}}\right] = [47.83, 50.57]$$

26. (a) n is now large enough, so we can use normal distribution.
$\therefore$ 99% C.I. for μ

$$\left[\bar{X} \pm Z_{(\frac{0.1}{2})}\frac{S}{\sqrt{n}}\right] = \left[680 \pm 2.576 \times \frac{35}{\sqrt{42}}\right] = [666.1, 693.9]$$

(b) $\chi^2(\frac{\alpha}{2},\ n-1) = \chi^2(0.025,\ 41) = 59.3417$
$\chi^2(1-\frac{\alpha}{2},\ n-1) = \chi^2(0.975,\ 41) = 24.4331$

$\therefore$ 99% C.I. for σ^2

$$\left[\frac{(n-1)S^2}{\chi^2(\frac{\alpha}{2}, n-1)}, \frac{(n-1)S^2}{\chi^2(1-\frac{\alpha}{2}, n-1)}\right] = \left[\frac{41 \times 35^2}{59.3417}, \frac{41 \times 35^2}{24.4331}\right] = [846.37, 2055.61]$$

$\therefore$ 99% C.I. for σ

$$[\sqrt{846.37}, \sqrt{2055.61}] = [29.1, 45.3]$$

27. Since n is large enough, we can use normal table.
$\therefore$ 90% C.I. for $\mu_1 - \mu_2$

$$\left[\bar{X} - \bar{Y} \pm Z_{(\frac{\alpha}{2})}\sqrt{\frac{S_X^2}{n_1} + \frac{S_Y^2}{n_2}}\right] = \left[984 - 1121 \pm 1.645\sqrt{\frac{8742}{45} + \frac{9411}{52}}\right] = [-168.9, -105.1]$$

Since 0 is not inside the interval, we conclude that the time until failure is larger for the 2th type of light bulb.

28. (a) Since population standard deviation is known, we use normal distribution.
95% C.I. for μ

$$\left[\bar{X} \pm Z_{(\frac{\alpha}{2})}\frac{\sigma}{\sqrt{n}}\right] = \left[2.4 \pm 1.96 \times \frac{0.2}{\sqrt{22}}\right] = [2.316, 2.484]$$

(b) We are t-distribution since population standard deviation is unknown and $n < 30$, $(t_{(0.025,21)} = 2.08)$
Assumption: population follows normal distribution. 95% C.I. for μ

$$\left[\bar{X} \pm t_{(\frac{\alpha}{2}, n-1)}\frac{\sigma}{\sqrt{n}}\right] = \left[2.4 \pm 2.08 \times \frac{0.2}{\sqrt{22}}\right] = [2.311, 2.489]$$

29.

$$t_{(\frac{\alpha}{2}, n-1)} = t_{(0.005,13)} = 3.012$$

99% C.I. for μ

$$\left[\bar{X} \pm t_{(\frac{\alpha}{2}, n-1)}\frac{\sigma}{\sqrt{n}}\right] = \left[32.132 \pm 3.012 \times \frac{2596}{\sqrt{14}}\right] = [30.042, 34.222]$$

30. (a) Estimator:

$$S^2 = \frac{(n_1-1)S_X^2 + (n_2-1)S_Y^2}{n_1+n_2-2} = \frac{(13-1)82.6 + (11-1)112.6}{13+11-2} = 96.2364$$

Since $t_{(0.025,22)} = 2.074$, 99% C.I. for $\mu_1 - \mu_2$

$$\left[\bar{X} - \bar{Y} \pm t_{(\frac{\alpha}{2}, n_1+n_2-2)}S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right] = \left[74.5 - 71.8 \pm 2.074\sqrt{96.2364}\sqrt{\frac{1}{13} + \frac{1}{11}}\right] = [-5.6, 11.0]$$

(b)

$$F(\frac{\alpha}{2};r_1,r_2) = F(0.05;12,10) = 2.91, \text{ and } F(\frac{\alpha}{2};r_2,r_1) = F(0.05;10,12) = 2.76$$
$$\text{where } r_1 = n_1 - 1, \; r_2 = n_2 - 1$$

90% C.I. for $\frac{\sigma_1^2}{\sigma_2^2}$

$$\left[\frac{1}{F(\frac{\alpha}{2};r_2,r_1)}\frac{S_x^2}{S_y^2}, F(\frac{\alpha}{2};r_1,r_2)\frac{S_x^2}{S_y^2}\right] = \left[\frac{1}{2.75}\times\frac{82.6}{112.6}, 2.91\times\frac{82.6}{112.6}\right] = [0.2668, 2.1347]$$

31. 99% C.I. for P

$$\left[\frac{y}{n} \pm Z_{(\frac{\alpha}{2})}\sqrt{\frac{\frac{y}{n}\times(1-\frac{y}{n})}{n}}\right] = \left[\frac{32}{200} \pm 2.576\sqrt{\frac{\frac{32}{200}\times(1-\frac{32}{200})}{200}}\right] = [0.09, 0.23]$$

32.

$$\begin{aligned}
90\% \text{ C.I. for } P_1 - P_2 &= \left[\frac{y_1}{n_1} - \frac{y_2}{n_2} \pm Z_{(\frac{\alpha}{2})}\sqrt{\frac{\frac{y_1}{n_1} - (1-\frac{y_1}{n_1})}{n_1} + \frac{\frac{y_2}{n_2} - (1-\frac{y_2}{n_2})}{n_2}}\right] \\
&= \left[\frac{62}{100} - \frac{74}{100} \pm 1.645\sqrt{\frac{\frac{62}{100}\times\frac{38}{100}}{100} + \frac{\frac{74}{100}\times\frac{26}{100}}{100}}\right] \\
&= [-0.2276, -0.0124]
\end{aligned}$$

33.

$$\chi^2(\frac{\alpha}{2}, n-1) = \chi^2(0.05, 20) = 31.41$$
$$\chi^2(1-\frac{\alpha}{2}, n-1) = \chi^2(0.95, 20) = 10.851$$

90% C.I. for σ^2

$$\left[\frac{(n-1)S^2}{\chi^2(\frac{\alpha}{2}, n-1)}, \frac{(n-1)S^2}{\chi^2(1-\frac{\alpha}{2}, n-1)}\right] = [\frac{20\times 562.8}{31.41}, \frac{20\times 562.8}{10.851}] = [358.4, 1037.3]$$

34. Let the c.d.f. of Y be $F(y)$

X_1 and X_2 are i.i.d. and $A = \{(x_1^2, x_2^2) : x_1^2 + x_2^2 \le y\}$

$$\begin{aligned}
F(y) &= P(X_1 + X_2 \le y) \\
&= \int\int_A \frac{1}{2\pi}e^{-\frac{1}{2}(x_1^2+x_2^2)}dx_1dx_2 \\
&= \int_0^{\sqrt{y}}\int_0^{2\pi}\frac{1}{2\pi}e^{-\frac{r^2}{2}}rd\theta dr, \qquad \text{let } x_1 = r\cos\theta, x_2 = r\sin\theta \\
&= \int_0^{\sqrt{y}}\left[\frac{1}{2\pi}e^{-\frac{r^2}{2}}r\theta\right]_0^{2\pi}dr \\
&= \int_0^{\sqrt{y}} e^{-\frac{r^2}{2}}rdr \\
&= \int_0^{\frac{y}{2}} e^{-u}du, \qquad \text{let } u = \frac{r^2}{2}, du = rdr \\
&= \left[-e^{-u}\right]_0^{\frac{y}{2}} = 1 - e^{-\frac{y}{2}}
\end{aligned}$$

$$\therefore\; f(y) = F'(y) = \frac{1}{2}e^{-y/2}, \qquad 0 < y < \infty \quad \text{which is pdf of } \chi^2(2)$$

we obtain: $Y \sim \chi^2(2)$

35.

$$
\begin{array}{l}
X_1 \sim Po(\mu_1), \qquad Y = X_1 + X_2 \sim Po(\mu) \\
\text{m.g.f. of } Y = E(e^{tY}) = E(e^{t(X_1+X_2)}) = E(e^{tX_1})E(e^{tX_2})
\end{array}
$$

We can check from the table that:

$$E(e^{t(X_1+X_2)}) = e^{\mu(e^t-1)} \text{ and } E(e^{tX_1}) = e^{\mu_1(e^t-1)}$$

Then we can obtain

$$E(e^{tX_2}) = e^{(\mu-\mu_1)(e^t-1)} \qquad \text{which is m.g.f. of Po}(\mu - \mu_1)$$

$$\therefore\ X_2 \sim Po(\mu - \mu_1)$$

36.

$$
\begin{aligned}
f_Y(y) &= P(Y = y) \\
&= P(X^3 = y) \\
&= P(X = y^{1/3}) \\
&= \begin{cases} \left(\frac{1}{2}\right)^{y^{1/3}} & y = 1^3, 2^3, 3^3, \ldots \\ 0 & \text{elsewhere} \end{cases}
\end{aligned}
$$

37.

$$
\begin{array}{l}
\frac{dy}{dx} = 3x^2 = 3(y^{1/3})^2 \\
f_Y(y) = f_X(x)\left|\frac{dy}{dx}\right| = \frac{1}{9}(y^{1/3})^2\left|\frac{1}{3}y^{-2/3}\right| = \frac{1}{27} \qquad\qquad 0 < y < 27
\end{array}
$$

38.

$$
\begin{aligned}
& Y = X^2, \frac{dy}{dx} = 2x = x(y^{1/2}) \\
f_Y(y) &= f_X(x)\left|\frac{dy}{dx}\right| \\
&= f_X(y^{1/2})\left|\frac{1}{2}y^{-1/2}\right| \\
&= 2y^{1/2} \cdot e^{-(y^{1/2})^2} \times \left|\frac{1}{2}y^{-1/2}\right| = e^{-y} \qquad 0 < y < \infty
\end{aligned}
$$

39. $F \sim F(r_1, r_2)$

Let

$$
\begin{aligned}
Y &= \frac{1}{1 + (r_1/r_2)F} \\
&\Rightarrow\ 1 + (r_1/r_2)F = \frac{1}{Y} \\
&\Rightarrow\ F = \frac{r_2}{r_1}\left(\frac{1-Y}{Y}\right)
\end{aligned}
$$

$$\therefore\ \frac{df}{dy} = \frac{r_2}{r_1}\left(\frac{-1}{y^2}\right) = -\frac{r_2}{r_1}\left(\frac{1}{y^2}\right)$$

$$
\begin{aligned}
f_Y(y) &= f_F(f)\left|\frac{df}{dy}\right| \\
&= f_F\left(\frac{r_2}{r_1}\left(\frac{1-y}{y}\right)\right)\left|\frac{df}{dy}\right| \\
&= \frac{\Gamma\left(\dfrac{r_1+r_2}{2}\right)}{\Gamma\left(\dfrac{r_1}{2}\right)\Gamma\left(\dfrac{r_2}{2}\right)}\left(\frac{r_2}{r_1}\right)^{\frac{r_1}{2}}\frac{\left[\dfrac{r_2}{r_1}\left(\dfrac{1-y}{y}\right)\right]^{\frac{r_1-2}{2}}}{\left[1+\dfrac{r_1}{r_2}\times\dfrac{r_2}{r_1}\left(\dfrac{1-y}{y}\right)\right]^{\frac{r_1+r_2}{2}}}\left(\frac{r_2}{r_1}\right)\left(\frac{1}{y^2}\right) \\
&= \frac{\Gamma\left(\dfrac{r_1+r_2}{2}\right)}{\Gamma\left(\dfrac{r_1}{2}\right)\Gamma\left(\dfrac{r_2}{2}\right)}\frac{\left(\dfrac{1-y}{y}\right)^{\frac{r_1-2}{2}}}{\left(1+\dfrac{1-y}{y}\right)^{\frac{r_1+r_2}{2}}}\left(\frac{1}{y^2}\right) \\
&= \frac{\Gamma\left(\dfrac{r_1+r_2}{2}\right)}{\Gamma\left(\dfrac{r_1}{2}\right)\Gamma\left(\dfrac{r_2}{2}\right)}(1-y)^{\frac{r_1}{2}-1}y^{-\frac{r_1}{2}+1+\frac{r_1+r_2}{2}-2} \\
&= \frac{\Gamma\left(\dfrac{r_1+r_2}{2}\right)}{\Gamma\left(\dfrac{r_1}{2}\right)\Gamma\left(\dfrac{r_2}{2}\right)}y^{\frac{r_2}{2}-1}(1-y)^{\frac{r_1}{2}-1} \\
&= \frac{1}{B\left(\dfrac{r_2}{2},\dfrac{r_1}{2}\right)}\;y^{\frac{r_2}{2}-1}(1-y)^{\frac{r_1}{2}-1} \qquad \text{where}\left(\frac{1}{B\left(\dfrac{r_2}{2},\dfrac{r_1}{2}\right)}=\frac{\Gamma\left(\dfrac{r_1+r_2}{2}\right)}{\Gamma\left(\dfrac{r_1}{2}\right)\Gamma\left(\dfrac{r_2}{2}\right)}\right) \\
& \therefore\ Y\sim B\left(\frac{r_2}{2},\frac{r_1}{2}\right)
\end{aligned}
$$

40.

$$
\begin{aligned}
F_Y(y) &= P(Y\le y)=P(X^2\le y) \\
&= P(-\sqrt{y}\le X\le\sqrt{y}) \\
&= F_X(\sqrt{y})-F_X(-\sqrt{y}) \\
&= \frac{1}{2}\sqrt{y}-\frac{1}{2}(-\sqrt{y}), \qquad 0<y<1
\end{aligned}
$$

We obtain

$$
f_Y(y)=F_Y'(y)=\frac{1}{2}y^{-1/2} \qquad 0<y<1
$$

41.

$$
\begin{cases} Y_1 = X_1-X_2 \\ Y_2 = X_1+X_2 \end{cases} \Rightarrow \begin{cases} X_1 = \frac{1}{2}(Y_1+Y_2) \\ X_2 = \frac{1}{2}(Y_2-Y_1) \end{cases}
$$

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= \left(\frac{2}{3}\right)^{x_1+x_2}\left(\frac{1}{3}\right)^{2-x_1-x_2} \\
&= \begin{cases} \left(\dfrac{2}{3}\right)^{y_2}\left(\dfrac{1}{3}\right)^{2-y_2} & (y_1,y_2)=(0,0),(-1,1),(1,1),(0,2) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

(Note that X_1, X_2 and Y_1, Y_2 are discrete)

If discrete, $f_{Y_1,\ldots,Y_n} = \sum_A f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)$, the summation is over those

$$(x_1,\ldots,x_n),(y_1,\ldots,y_k) = (g_1,(x_1,\ldots,x_n),\ldots,g_k(x_1,\ldots,x_n))$$

42.

$$\left\{\begin{array}{lcl} Y_1 & = & X_1+X_2 \\ Y_2 & = & X_1-X_2 \end{array}\right. \Rightarrow \left\{\begin{array}{lcl} X_1 & = & \frac{1}{2}(Y_1+Y_2) \\ X_2 & = & \frac{1}{2}(Y_1-Y_2) \end{array}\right.$$

$$\therefore\ f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(x_1,x_2)|J|, \quad 0<y_1<\infty, 0<y_2<\infty$$

where

$$|J| = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left|-\frac{1}{2}\right| = \frac{1}{2}$$

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}\left(\frac{1}{2}(y_1+y_2),\frac{1}{2}(y_1-y_2)\right)\cdot\left(\frac{1}{2}\right), \qquad \because X_1,X_2\sim N(\mu,\sigma^2) \\ &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{1}{2}(y_1+y_2)-\mu\right]^2\right\}\cdot\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{1}{2}(y_1-y_2)-\mu\right]^2\right\}\cdot\left(\frac{1}{2}\right) \\ &= \frac{1}{4\pi\sigma^2}\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{1}{4}(y_1^2+2y_1y_2+y_2^2)-(y_1+y_2)\mu+\mu^2\right.\right. \\ &\quad \left.\left.+\frac{1}{4}(y_1^2-2y_1y_2+y_2^2)-(y_1-y_2)\mu+\mu^2\right]\right\} \\ &= \frac{1}{4\pi\sigma^2}\exp\left\{-\frac{1}{2\sigma^2}\left[\frac{1}{2}y_1^2-2y_1\mu+2\mu^2+\frac{1}{2}y_2^2\right]\right\} \\ &= \frac{1}{4\pi\sigma^2}\exp\left\{-\frac{1}{4\sigma^2}(y_1-2\mu)^2\right\}\exp\left\{-\frac{1}{4\sigma^2}y_2^2\right\} \\ &= H(y_1)\cdot K(y_2) \end{aligned}$$

$\therefore\ Y_1$ and Y_2 are independent.

Remark: If we can show $f_{Y_1,Y_2}(y_1,y_2) = H(y_1)\cdot K(y_2)$ for some functions H and K, then Y_1,Y_2 are mutually independent.

It is not necessary that $H(y_1)$ is pdf of Y_1 and $K(y_2)$ is pdf of Y_2.

43.

$$\left\{\begin{array}{lcl} Y_1 & = & X_1^2+X_2^2 \\ Y_2 & = & X_2 \end{array}\right. \Rightarrow \left\{\begin{array}{lcl} X_1^2 & = & Y_1-Y_2^2 \\ X_2 & = & Y_2 \end{array}\right.$$

Now, $X_2\sim N(0,1)$, $X_1^2\sim\chi^2(1)$ and since X_1 and X_2 are independent, X_1^2 and X_2 are also independent.

$$\begin{aligned} f_{X_1^2,X_2}(x_1^2,x_2) &= f_{X_1^2}(x_1^2)f_{X_2}(x_2) \\ &= \frac{(x_1^2)^{\frac{1}{2}-1}e^{-\frac{x_1^2}{2}}}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})}\times\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}x_2^2\right\} \\ &= \frac{1}{2\pi}x_1^{-1}\exp\left\{-\frac{1}{2}(x_1^2+x_2^2)\right\}, \qquad \because (\Gamma(\frac{1}{2})=\pi) \end{aligned}$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1^2,X_2}(y_1-y_2^2,y_2)\ |J|$$

$$\text{where} \qquad |J| = \begin{vmatrix} \dfrac{\partial x_1^2}{\partial y_1} & \dfrac{\partial x_1^2}{\partial y_2} \\ \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & -2y_2 \\ \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore\ f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi}(y_1-y_2^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}y_1\right\}, \qquad y_1-y_2^2 \geq 0, \qquad (-\sqrt{y_1} \leq y_2 \leq \sqrt{y_1})$$

$$\begin{aligned}
f_{Y_1}(y_1) &= \int_{-\sqrt{y_1}}^{\sqrt{y_1}} f_{Y_1,Y_2}(y_1,y_2)dy_2 \\
&= \int_{-\sqrt{y_1}}^{\sqrt{y_1}} \frac{1}{2\pi\sqrt{y_1-y_2^2}} \cdot \exp\left\{-\frac{1}{2}y_1\right\}\ dy_2 \\
&= \frac{1}{2\pi}\exp\left\{-\frac{1}{2}y_1\right\} \int_{-\sqrt{y_1}}^{\sqrt{y_1}} \frac{1}{\sqrt{y_1-y_2^2}}\ dy_2 \\
&\quad (\text{let } y_2 = \sqrt{y_1}\sin\theta, dy_2 = \sqrt{y_1}\cos\theta d\theta) \\
&= \frac{1}{2\pi}\exp\left\{-\frac{1}{2}y_1\right\} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{y_1}\cos\theta}{\sqrt{y_1-y_1\sin^2\theta}}\ d\theta \\
&= \frac{1}{2\pi}\exp\left\{-\frac{1}{2}y_1\right\} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{y_1}\cos\theta}{\sqrt{y_1}\cos\theta}\ d\theta \\
&= \frac{1}{2\pi}\exp\left\{-\frac{1}{2}y_1\right\} \Big[\theta\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= \frac{1}{2\pi}\exp\left\{-\frac{1}{2}y_1\right\} \cdot \pi \\
&= \frac{1}{2}\exp\left\{-\frac{1}{2}y_1\right\} \qquad\qquad 0 < y_1 < \infty
\end{aligned}$$