Solutions to Exercise 3

1.

$$
\begin{aligned}
f(X;\theta) &= \theta^X(1-\theta)^{1-X} \\
L &= f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta) \\
&= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} \\
&= \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} \\
\log L &= \left(\sum_{i=1}^{n} x_i\right)\log\theta + \left(n - \sum_{i=1}^{n} x_i\right)\log(1-\theta) \\
\frac{\partial}{\partial\theta}\log L &= \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}\left(n - \sum_{i=1}^{n} x_i\right) \\
\frac{\partial}{\partial\theta}\log L = 0 \quad\Rightarrow\quad & \frac{1}{\theta}\sum_{i=1}^{n} x_i = \frac{1}{1-\theta}\left(n - \sum_{i=1}^{n} x_i\right) \\
\Rightarrow\quad & (1-\theta)\sum_{i=1}^{n} x_i = \left(n - \sum_{i=1}^{n} x_i\right)\theta \\
\Rightarrow\quad & \sum_{i=1}^{n} x_i = n\theta \\
\Rightarrow\quad & \hat{\theta} = \frac{1}{\sum_{i=1}^{n} x_i} \qquad \text{which is MLE for } \theta
\end{aligned}
$$

$$
\begin{aligned}
\log f(X;\theta) &= X\log\theta + (1-X)\log(1-\theta) \\
\frac{\partial}{\partial\theta}\log f(X;\theta) &= \frac{X}{\theta} - \frac{1-X}{1-\theta} \\
\frac{\partial^2}{\partial\theta^2}\log f(X;\theta) &= \frac{-X}{\theta^2} - \frac{1-X}{(1-\theta)^2} \\
E\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right] &= E\left[\frac{-X}{\theta^2} - \frac{1-X}{(1-\theta)^2}\right] \\
&= \frac{-}{\theta^2}E[X] - \frac{1}{(1-\theta)^2}E[1-X] \\
&= \frac{-}{\theta^2}X\theta - \frac{1}{(1-\theta)^2}(1-\theta) \\
&= \frac{-1}{\theta} - \frac{1}{1-\theta} \\
&= -\frac{(1-\theta)+\theta}{\theta(1-\theta)} \\
&= -\frac{1}{\theta(1-\theta)}
\end{aligned}
$$

$$
\therefore \text{ The CRLB} = \frac{1}{-nE\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right]} = \frac{\theta(1-\theta)}{n}
$$

1

Since

$$
\begin{aligned}
Var(\hat{\theta}) &= Var(\frac{1}{n}\sum_{i=1}^{n}X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}Var(X_i) \\
&= \frac{1}{n^2}\cdot n(\theta)(1-\theta) \\
&= \frac{\theta(1-\theta)}{n} \\
&= \text{CRLB}
\end{aligned}
$$

$\therefore \ \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}X_i$ is a <u>fully efficient</u> estimator (UMVUE) for θ.

2.

$$
\begin{aligned}
f(X;\theta) &= \frac{1}{\theta}\exp(\frac{-X}{\theta}) \\
\log f(X;\theta) &= \log(\frac{1}{\theta}) - \frac{X}{\theta} \\
\frac{\partial^2}{\partial\theta^2}\log f(X;\theta) &= \frac{-1}{\theta} + \frac{X}{\theta^2} \\
\frac{\partial^2}{\partial\theta^2}\log f(X;\theta) &= \frac{1}{\theta^2} - \frac{2X}{\theta^3} \\
E\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right] &= \frac{1}{\theta^2} - \frac{2}{\theta^3}E(X) \\
&= \frac{1}{\theta^2} - \frac{2}{\theta^3}\cdot\theta \\
&= -\frac{1}{\theta^2}
\end{aligned}
$$

$$
\therefore \text{The CRLB} = \frac{1}{-nE\left[\frac{\partial^2}{\partial\theta^2}\log Lf(X;\theta)\right]} = \frac{\theta^2}{n}
$$

$$
\because \ Var(\bar{X}) = Var(\frac{1}{n}\sum_{i=1}^{n}X_i) = \frac{Var(X)}{n} = \frac{\theta^2}{n} = \text{CRLB}
$$

$\therefore \ \bar{X} = \frac{1}{n}\sum_{i=1}^{n}X_i$ is a <u>fully efficient</u> estimator (UMVUE) for θ.

3. (a)

$$
\begin{aligned}
X &\sim \ Poisson(\theta) \\
f_X(x;\theta) &= \theta^x e^{-\theta}/x! \\
&= \exp\{x\log\theta - \log(x!) - \theta\} \\
&= \exp\{-\theta - \log(x!) + x\log\theta\}
\end{aligned}
$$

$$
\therefore \ a(\theta) = -\theta, \ b(X) = \log(X!), \ c(\theta) = \log\theta, \ d(X) = X
$$

$\therefore \ Poisson(\theta)$ belongs to exponential family.

(b)

$$X \sim Bin(n, \theta) \quad (n \text{ is known})$$

$$
\begin{aligned}
f_X(x; \theta) &= \binom{n}{x} \theta^n (1 - \theta)^{n-x} \\
&= \exp\left\{ \log\binom{n}{x} + x \log \theta + (n - x) \log(1 - \theta) \right\} \\
&= \exp\left\{ n \log(1 - \theta) + \log\binom{n}{x} + x \log(\frac{\theta}{1 - \theta}) \right\}
\end{aligned}
$$

$$\therefore a(\theta) = n \log(1 - \theta), \ b(X) = \log\binom{n}{X}, \ c(\theta) = \log(\frac{\theta}{1 - \theta}), \ d(X) = X$$

$\therefore Bin(n, \theta)$ belongs to exponential family.

(c) Note: X here means the trial number on which the rth success occurs.

$$X \sim Neg.Bin.(r, \theta) \quad (r \text{ is known})$$

$$
\begin{aligned}
f_X(x; \theta) &= \binom{x - 1}{r - 1} \theta^r (1 - \theta)^{x-r} \\
&= \exp\left\{ \log\binom{x - 1}{r - 1} + r \log \theta + (x - r) \log(1 - \theta) \right\} \\
&= \exp\left\{ r \log(\frac{\theta}{1 - \theta}) + \log\binom{x - 1}{r - 1} + + x \log(1 - \theta) \right\}
\end{aligned}
$$

$$\therefore a(\theta) = r \log(\frac{\theta}{1 - \theta}), \ b(X) = \log\binom{X - 1}{r - 1}, \ c(\theta) = \log(1 - \theta), \ d(X) = X$$

$\therefore Neg.Bin.(r, \theta)$ belongs to exponential family.

4. (a)

$$X \sim gamma(k, \theta) \quad (k > 0 \text{ is known})$$

$$
\begin{aligned}
f_X(X; \theta) &= \frac{x^{k-1} e^{-x\theta}}{\Gamma(k) \theta^{-k}} \\
&= \exp\left\{ (k - 1) \log x - x\theta - \log[\Gamma(k)] + k \log \theta \right\} \\
&= \exp\left\{ k \log \theta - \log[\Gamma(k)] + (k - 1) \log x - x\theta \right\}
\end{aligned}
$$

$$\therefore a(\theta) = k \log \theta, \ b(X) = -\log[\Gamma(k)] + (k - 1) \log X, \ c(\theta) = -\theta, \ d(X) = X$$

$\therefore Gamma(k, \theta), k > 0$ belongs to exponential family.

(b)

$$X \sim N(\theta, 1)$$

$$
\begin{aligned}
f_X(x; \theta) &= \frac{1}{\sqrt{2\pi} \cdot 1} \exp\left\{ \frac{-1}{2(1)} (x - \theta)^2 \right\} \\
&= \exp\left\{ \frac{-1}{2} \log(2\pi) - \frac{1}{2}(x - \theta)^2 \right\} \\
&= \exp\left\{ \frac{-1}{2} \log(2\pi) - \frac{1}{2}x^2 + \theta x - \frac{1}{2}\theta^2 \right\} \\
&= \exp\left\{ -\frac{1}{2}\theta^2 - \frac{1}{2}(x^2 + \log 2\pi) + \theta x \right\}
\end{aligned}
$$

3

$$\therefore \ a(\theta) = -\frac{1}{2}\theta^2, \ b(X) = -\frac{1}{2}(X^2 + \log 2\pi), \ c(\theta) = \theta, \ d(X) = X$$

$\therefore \ N(\theta, 1)$ belongs to exponential family.

(c)

$$X \ \sim \ N(0, \theta)$$
$$f_X(x; \theta) \ = \ \frac{1}{\sqrt{2\pi\theta}} \exp\left\{\frac{-1}{2\theta}(x - 0)^2\right\}$$
$$= \ \exp\left\{\frac{-1}{2}\log(2\pi\theta) - \frac{1}{2\theta}x^2\right\}$$

$$\therefore \ a(\theta) = -\frac{1}{2}\log(2\pi\theta), \ b(X) = 0, \ c(\theta) = -\frac{1}{2\theta}, \ d(X) = X^2$$

$\therefore \ N(0, \theta)$ belongs to exponential family.

5.

$$f_X(x; \theta) \ = \ (1 - \theta)^x \theta$$
$$= \ \exp\left\{x \log(1 - \theta) + \log \theta\right\}$$
$$= \ \exp\left\{\log \theta + x \log(1 - \theta)\right\}$$

$$\therefore \ a(\theta) = \log \theta, \ b(X) = 0, \ c(\theta) = \log(1 - \theta), \ d(X) = X$$

$\therefore \ Geometric(\theta)$ belongs to exponential family, and $\sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

6.

$$f_X(x; \theta) \ = \ \frac{1}{\theta} \exp(-\frac{x}{\theta}), \quad 0 < x < \infty, 0 < \theta < \infty$$
$$= \ \exp\left\{-\log \theta - \frac{x}{\theta}\right\}$$

$$\therefore \ a(\theta) = -\log \theta, \ b(X) = 0, \ c(\theta) = \frac{-1}{\theta}, \ d(X) = X$$

$\therefore \ Exp(\frac{1}{\theta}) = Gamma(1, \frac{1}{\theta})$ belongs to exponential family, and $\sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

7.

$$f_X(x; \theta) \ = \ \frac{1}{\sqrt{2\pi\theta}} \exp\left\{-\frac{1}{2\theta}(x - 0)^2\right\}$$
$$= \ \exp\left\{\frac{-1}{2}\log(2\pi\theta) - \frac{1}{2\theta}x^2\right\}$$
$$= \ \exp\left\{a(\theta) + b(x) + c(\theta)d(x)\right\}$$

$$\therefore \ a(\theta) = -\frac{1}{2}\log(2\pi\theta), \ b(X) = 0, \ c(\theta) = -\frac{1}{2\theta}, \ d(X) = X^2$$

$\therefore \ N(0, \theta)$ belongs to exponential family, and $\sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i^2$ is a sufficient statistic for θ.

8.

$$f_{X_i}(x_i; \theta) = \frac{1}{\theta} I_{[0,\theta]}(x_i)$$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}; \theta) = \prod_{i=1}^{n} f_{X_i}(x_i; \theta)$$

$$= \prod_{i=1}^{n} \frac{1}{\theta} I_{[0,\theta]}(x_i)$$

$$= (\frac{1}{\theta})^n \prod_{i=1}^{n} I_{[0,\theta]}(x_i)$$

$$= (\frac{1}{\theta})^n I_{[0,Y_n]}(y_1) \ I_{[Y_1,\theta]}(y_n) \cdot 1$$

$$\because \ g = (\frac{1}{\theta})^n I_{[0,Y_n]}(y_1) \ I_{[Y_1,\theta]}(y_n), \quad h = 1$$

$\therefore$ By factorization theorem, $Y_n = \max(X_1, \ldots, X_n)$ is sufficient for θ. (refer to example 4 in Ch-2 P.14).

9. By Q.5(a), $d(X) = X$, $\sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ

10. (a) $X_1, \ldots, X_n \sim \textit{iid Bernoulli}(\theta) \Rightarrow E(X_k) = \theta, k = 1, 2, \ldots, n$
   $\therefore \ X_k$ is an unbiased estimator for θ.
   In order to show that X_k is NOT sufficient statistics for θ, we can just find another statistic $T = t(X_1, \ldots, X_n)$ such that $P(T | X_k = x_k)$ depends on θ.
   Now, let $T = X_i, i \neq k$.

$$\therefore \quad P(X_i = x_i | X_k = x_k)$$

$$= \frac{P(X_i = x_i, X_k = x_k)}{P(X_k = x_k)}$$

$$= \frac{P(X_i = x_i)P(X_k = x_k)}{P(X_k = x_k)}$$

$$= P(X_i = x_i)$$

$$= \theta^{x_i}(1 - \theta)^{1-x_i} \qquad \text{which depends on } \theta$$

$$\therefore \ X_k \text{ is NOT sufficient statistic for } \theta$$

(b)

$$f_X(x; \theta) = \theta^x (1 - \theta)^{1-x}$$

$$= \exp\{x \log \theta + (1 - x) \log(1 - \theta)\}$$

$$= \exp\left\{\log(1 - \theta) + x \log(\frac{\theta}{1 - \theta})\right\}$$

$$\therefore \ a(\theta) = \log(1 - \theta), \ b(X) = 0, \ c(\theta) = \log(\frac{\theta}{1 - \theta}), \ d(X) = X$$

$\therefore \ \textit{Bernoulli}(\theta)$ belongs to exponential family, and $\sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

(c) Now, we have X_1 is unbiased for θ and $\sum_{i=1}^{n} X_i$ is sufficient statistic for θ,
$\therefore$ by Rao-Blackwell theorem, $T = E(X_1 | \sum_{i=1}^{n} X_i)$ is an improved unbiased estimator for θ.

In order to calculate T, we need the distribution of $X_1 | \sum_{i=1}^{n} X_i = r$

$$
\begin{aligned}
\Rightarrow \quad & P(X_1 = 1 | \sum_{i=1}^{n} X_i = r) \\
= \quad & \frac{P(X_1 = 1, \sum_{i=1}^{n} X_i = r)}{P(\sum_{i=1}^{n} X_i = r)} \\
= \quad & \frac{P(X_1 = 1, \sum_{i=2}^{n} X_i = r - 1)}{P(\sum_{i=1}^{n} X_i = r)} \\
= \quad & \frac{P(X_1 = 1) \cdot P(\sum_{i=2}^{n} X_i = r - 1)}{P(\sum_{i=1}^{n} X_i = r)} \qquad \because X_1, \sum_{i=2}^{n} X_i \text{ are independent} \\
= \quad & \frac{\theta \dbinom{n-1}{r-1} \theta^{r-1}(1-\theta)^{(n-1)-(r-1)}}{\dbinom{n}{r} \theta^{r-1}(1-\theta)^{n-r}} \\
= \quad & \frac{\dbinom{n-1}{r-1}}{\dbinom{n}{r}} \\
= \quad & \frac{(n-1)!}{(r-1)!(n-r)!} \times \frac{r!(n-r)!}{n!} \\
= \quad & \frac{r}{n}
\end{aligned}
$$

$$\therefore \ P(X_1 = 0 | \sum_{i=1}^{n} X_i = r) = 1 - \frac{r}{n}$$

$$\therefore \ E(X_1 | \sum_{i=1}^{n} X_i = r) = 1 \cdot P(X_1 = 1 | \sum_{i=1}^{n} X_i = r) + 0 \cdot P(X_1 = 0 | \sum_{i=1}^{n} X_i = r) = \frac{r}{n}$$

$$T = E(X_1 | \sum_{i=1}^{n} X_i) = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X} \text{ is an improved unbiased estimator for } \theta.$$

The alternative method to get an improved estimate is as follows. Let $r = \sum_{i=1}^{n} x_i$, which is sufficient and complete statistic.

$$
\begin{aligned}
\sum_{i=1}^{n} h(r) \dbinom{n}{r} \theta^r (1-\theta)^{n-r} &= \theta \\
\sum_{i=1}^{n} h(r) \dbinom{n}{r} \theta^{r-1}(1-\theta)^{n-r} &= 1 \\
\sum_{i=1}^{n} h(r) \frac{n}{r} \dbinom{n-1}{r-1} \theta^{r-1}(1-\theta)^{(n-1)-(r-1)} &= 1 \\
\Rightarrow \quad h(r) \frac{n}{r} &= 1 \\
\Rightarrow \quad h(r) &= \frac{r}{n}
\end{aligned}
$$

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11. (a) $E(X_1) = \theta$ and $E(X_2) = \theta$

$\therefore$ X_1 and X_2 are unbiased estimators for θ.

$$
\begin{aligned}
f_{X_2|X_1}(x_2|x_1) &= \frac{f_{X_2, X_1}(x_2, x_1)}{f_{X_1}(x_1)} \\
&= \frac{f_{X_2}(x_2) f_{X_1}(x_1)}{f_{X_1}(x_1)} \\
&= f_{X_2}(x_2) \\
&= \frac{1}{\theta} \exp\{-\frac{1}{\theta} x_2\} \quad \text{which depends on } \theta
\end{aligned}
$$

$\therefore$ X_1 is not sufficient statistic for θ.

(Care that X_1, X_2 are now continuous !!!)

Similarly, $f_{X_2|X_1}(x_2|x_1) = f_{X_1}(x_1) = \frac{1}{\theta} \exp\{-\frac{1}{\theta} x_1\}$ which depends on θ,
$\therefore$ X_2 is also not sufficient statistic for θ.

(b)

$$
\begin{aligned}
f_X(x; \theta) &= \frac{1}{\theta} \exp\{-\frac{x}{\theta}\}, \quad x \geq 0 \\
&= \exp\{-\log\theta - \frac{x}{\theta}\}
\end{aligned}
$$

$$
a(\theta) = -\log\theta, b(X) = 0, c(\theta) = -\frac{1}{\theta}, d(X) = X
$$

$\therefore$ Exponential distribution with mean θ belongs to exponential family.
$\therefore \sum_{i=1}^{2} d(X_i) = \sum_{i=1}^{2} X_i = X_1 + X_2$ is sufficient for θ.

(c)

$$
\begin{aligned}
f_{X_1|X_1+X_2}(x_1|r) &= \frac{f_{X_1, X_1+X_2}(x_1, r)}{f_{X_1+X_2}(r)}, \quad 0 \leq x_1 \leq r \\
&= \frac{f_{X_1, X_2}(x_1, r - x_1)}{f_{X_1+X_2}(r)} \\
&\quad (\because X_1 \sim exp(\theta), X_2 \sim exp(\theta), X_1 + X_2 \sim gamma(2, \theta)) \\
&= \frac{\frac{1}{\theta} \exp(-\frac{1}{\theta} x_1) \cdot \frac{1}{\theta} \exp(-\frac{1}{\theta}(r - x_1))}{\frac{1}{\Gamma(2)} \theta^{-2} \cdot r^{2-1} \exp(-\frac{r}{\theta})} \\
&= \frac{\exp(-\frac{1}{\theta} r)}{r \exp(-\frac{r}{\theta})} \quad \because \Gamma(2) = 1 \\
&= \frac{1}{r}
\end{aligned}
$$

$$
\therefore E(X_1|X_1 + X_2 = r) = \int_0^r x_1 f_{X_1|X_1+X_2}(x_1|r) \, dx_1 = \int_0^r x_1 \cdot \frac{1}{r} \, dx_1 = \left[\frac{x_1^2}{2r}\right]_0^r = \frac{r}{2}
$$

$\therefore$ By Rao-Blackwell theorem, the desired estimator is $\frac{r}{2} = \frac{1}{2}(X_1 + X_2)$.

12. (a)

$$
\begin{array}{ll}
Y_1 = X_1 + X_2 & \quad X_1 = Y_1 - Y_2 \\
Y_2 = X_2 & \Rightarrow \quad X_2 = Y_2
\end{array}
$$

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2) \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

$$= f_{X_1}(x_1) f_{X_2}(x_2) \cdot \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{\theta} \exp(-\frac{x_1}{\theta}) \cdot \frac{1}{\theta} \exp(-\frac{x_2}{\theta})$$

$$= \frac{1}{\theta} \exp(-\frac{y_1 - y_2}{\theta}) \cdot \frac{1}{\theta} \exp(-\frac{y_2}{\theta})$$

$$= \frac{1}{\theta^2} \exp(-\frac{y_1}{\theta}), \qquad 0 < y_2 < y_1 < \infty.$$

(b)

$$\because Y_2 = X_2, \quad \therefore f_{Y_2}(y_2) = f_{X_2}(x_2) = \exp(-\frac{x_2}{\theta})$$

$$\begin{aligned}
E(Y_2) &= \int_0^\infty y_2 \frac{1}{\theta} \exp\left(-\frac{y_2}{\theta}\right) dy_2 \\
&= \left[-y_2 \exp\left(\frac{-y_2}{\theta}\right)\right]_0^\infty + \int_0^\infty \exp\left(\frac{-y_2}{\theta}\right) dy_2, \qquad \text{by parts} \\
&= 0 + \left[-\theta \exp\left(-\frac{y_2}{\theta}\right)\right]_0^\infty \\
&= \theta
\end{aligned}$$

$\therefore Y_2$ is an unbiased estimator for θ.

$$\begin{aligned}
E(Y_2^2) &= \int_0^\infty y_2^2 \frac{1}{\theta} \exp\left(-\frac{y_2}{\theta}\right) dy_2 \\
&= \left[-y_2^2 \exp\left(-\frac{y_2}{\theta}\right)\right]_0^\infty + \int_0^\infty 2y_2 \exp\left(-\frac{y_2}{\theta}\right) dy_2, \qquad \text{by parts} \\
&= 0 + 2\theta \int_0^\infty y_2 \frac{1}{\theta} \exp\left(-\frac{y_2}{\theta}\right) dy_2 \\
&= 2\theta \cdot E(Y_2) \\
&= 2\theta \cdot \theta \\
&= 2\theta^2 \\
\therefore Var(Y_2) &= E(Y_2^2) - [E(Y_2)]^2 \\
&= 2\theta^2 - \theta^2 \\
&= \theta^2
\end{aligned}$$

In order to apply Rao-Blackwell theorem, we need to find $E(Y_2/Y_1)$ because Y_2 is unbiased for θ (by part b) and $Y_1 = X_1 + X_2 = \sum_{i=1}^2 X_i$ is sufficient for θ (see Q.3 part b).

$$\begin{aligned}
f_{Y_2|Y_1}(y_2|y_1) &= \frac{f_{Y_2,Y_1}(y_2, y_1)}{f_{Y_1}(y_1)} \\
&= \frac{f_{Y_2,Y_1-Y_2}(y_2, y_1 - y_2)}{f_{Y_1}(y_1)} \\
&= \frac{f_{X_2,X_1}(x_2, x_1)}{f_{X_1+X_2}(x_1 + x_2)} \\
&= \frac{\frac{1}{\theta}\exp(-\frac{x_2}{\theta}) \cdot \frac{1}{\theta}\exp(-\frac{x_1}{\theta})}{\frac{1}{\Gamma(2)}\theta^{-2}(x_1 + x_2)^{2-1}\exp(-\frac{(x_1+x_2)}{\theta})}
\end{aligned}$$

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$$= \frac{1}{x_1 + x_2}, \qquad (\Gamma(2) = 1)$$

$$= \frac{1}{y_1}$$

$$\therefore E(Y_2|Y_1 = y_1) = \int_0^{y_1} y_2 \cdot f_{Y_2|Y_1}(y_2|y_1) \, dy_2$$

$$= \int_0^{y_1} y_2 y_2 \cdot \frac{1}{y_1} \, dy_2$$

$$= \left[\frac{y_2^2}{2y_1} \right]_0^{y_1}$$

$$= \frac{y_1}{2}$$

$\therefore$ By Rao-Blackwell theorem, an improved unbiased estimator is $\frac{1}{2}Y_1 = \frac{1}{2}(X_1 + X_2) = \bar{X}$.

The alternative method to get an improved estimate is as follows. Y_1 is sufficient and complete statistic and $Y_1 \sim gamma(2, \frac{1}{\theta})$.

$$\Rightarrow h(y_1) \frac{\Gamma(3)}{\Gamma(2)y_1} = 1$$

$$\Rightarrow h(y_1) = \frac{\Gamma(2)y_1}{\Gamma(3)}$$

$$= \frac{y_1}{2}$$

$$= \frac{X_1 + X_2}{2}$$

$$f_X(x; \theta) = \frac{1}{\theta} \exp(-\frac{x}{\theta})$$

$$\log f_X(x; \theta) = -\log \theta - \frac{x}{\theta}$$

$$\frac{\partial}{\partial \theta} \log f_X(x; \theta) = -\frac{1}{\theta} + \frac{x}{\theta^2}$$

$$\frac{\partial^2}{\partial \theta^2} \log f_X(x; \theta) = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$$

$$E\left[\frac{\partial^2}{\partial \theta^2} \log f_X(x; \theta) \right] = E\left[\frac{1}{\theta^2} - \frac{2x}{\theta^3} \right]$$

$$= \frac{1}{\theta^2} - \frac{2}{\theta^3} E(X)$$

$$= \frac{1}{\theta^2} - \frac{2\theta}{\theta^3}$$

$$= -\frac{1}{\theta^2}$$

$$\therefore \text{ The CRLB for } \theta = \frac{-1}{n \cdot E[\frac{\partial^2}{\partial \theta^2} \log f_X(x; \theta)]} = \frac{-1}{2 \cdot (\frac{-1}{\theta^2})} = \frac{\theta^2}{2}$$

Also, variance of improved estimator

$$= Var(\bar{X}) = \frac{Var(X)}{n} = \frac{\theta^2}{n} \quad \left(\because X \sim \exp(\frac{1}{\theta}) \right) = \text{CRLB for } \theta$$

$\therefore$ The variance of improved estimator $\bar{X}$ attains the CRLB.

13.

$$f_X(x; \theta) = \frac{1}{6\theta^4} x^3 \exp(-\frac{x}{\theta}), \qquad 0 < x < \infty,\ 0 < \theta < \infty$$

$$= \exp\left\{-\log(6\theta^4) + 3\log x - \frac{x}{\theta}\right\}$$

$$\therefore\ a(\theta) = -\log(6\theta^4), b(X) = 3\log X, c(\theta) = -\frac{1}{\theta}, d(X) = X$$

$$\therefore\ f_X(x; \theta) \text{ belongs to the exponential family.}$$

The complete sufficient statistic for θ is $Y_1 = \sum_{i=1}^n d(X_i) = \sum_{i=1}^n X_i$.

$$\begin{aligned}
E(X) &= \int_0^\infty x f_X(x; \theta)\ dx \\
&= \int_0^\infty \frac{x}{6\theta^4} x^3 \exp(-\frac{x}{\theta})\ dx \\
&= \frac{1}{6\theta^4} \int_0^\infty x^4 \exp(-\frac{x}{\theta})\ dx, \qquad \text{by parts} \\
&= \left[\frac{1}{6\theta^3}(-x^4)\right]_0^\infty + \frac{1}{6\theta^3} \int_0^\infty 4x^3 \exp(-\frac{x}{\theta})\ dx \\
&= 0 + \frac{4}{6\theta^3} \int_0^\infty x^3 \exp(-\frac{x}{\theta})\ dx, \qquad \text{by parts} \\
&= \left[\frac{4}{6\theta^3}(-x^3)\right]_0^\infty + \frac{4}{6\theta^2} \int_0^\infty 3x^2 \exp(-\frac{x}{\theta})\ dx, \qquad \text{by parts} \\
&= 0 + \left[\frac{2}{\theta}(-x^2)\exp(-\frac{x}{\theta})\right]_0^\infty + \frac{2}{\theta} \int_0^\infty 2x \exp(-\frac{x}{\theta})\ dx, \qquad \text{by parts} \\
&= 0 + 0 + \left[4(-x)\exp(-\frac{x}{\theta})\right]_0^\infty + 4 \int_0^\infty \exp(-\frac{x}{\theta})\ dx \\
&= 0 + 0 + 0 + \left[-4\theta \exp(-\frac{x}{\theta})\right]_0^\infty \\
&= 4\theta \\
\therefore\ E(\frac{1}{4n}\sum_{i=1}^n X_i) &= \frac{1}{4n}\sum_{i=1}^n E(X_i) \\
&= \frac{1}{4n}n(4\theta) \\
&= \theta
\end{aligned}$$

$$\Rightarrow \frac{1}{4n}\sum_{i=1}^n X_i \text{ is unbiased for } \theta$$

Since $\frac{1}{4n}\sum_{i=1}^n X_i = \frac{1}{4n}Y_1$ is a function of Y_1 (complete sufficient statistic), it is the UMVUE for θ,

$$\therefore\ \varphi(Y_1) = \frac{1}{4n}Y_1 = \frac{1}{4n}\sum_{i=1}^n X_i$$

Also, $\varphi(Y_1) = \frac{1}{4n}Y_1$ is itself a complete sufficient statistic for θ since Y_1 is a complete sufficient statistic.

14. $X \sim \exp(\theta)$.

Given $Y = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ.

$\Rightarrow Y = \sum_{i=1}^{n} X_i \sim gamma(n, \theta)$

$\Rightarrow f_Y(y) = \frac{\theta^n}{\Gamma(n)} y_{n-1} e^{-\theta y}$

$f_X(x; \theta) = \theta e^{-\theta x} = \exp(\log \theta - \theta x)$

$$\therefore \ a(\theta) = \log \theta, b(X) = 0, c(\theta) = -\theta, d(X) = X$$

$$\therefore \ f_X(x; \theta) \text{ belongs to the exponential family.}$$

$$\text{and } \sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i \text{ is complete and sufficient.}$$

$$
\begin{aligned}
\therefore \ E(\frac{n-1}{Y}) &= (n-1) \int_0^\infty \frac{1}{y} f_Y(y) \, dy \\
&= (n-1) \int_0^\infty \frac{1}{y} \frac{\theta^n}{\Gamma(n)} y^{n-1} e^{-\theta y} \, dy \\
&= \frac{(n-1)\theta}{\Gamma(n)} \int_0^\infty \theta^{n-1} y^{n-2} e^{-\theta y} \, dy \\
&= \frac{(n-1)\theta}{\Gamma(n)} \Gamma(n-1) \int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)} y^{n-2} e^{-\theta y} \, dy \\
&= (n-1)\theta \cdot \frac{1}{n-1} \cdot 1 \\
&= \theta
\end{aligned}
$$

$$\because \ \Gamma(n) = (n-1)\Gamma(n-1) \quad \text{and} \quad \frac{\theta^{n-1}}{\Gamma(n-1)} y^{n-2} e^{-\theta y} \text{ is pdf of gamma}(n-1, \frac{1}{\theta})$$

Since $\frac{n-1}{Y}$ is function of complete sufficient statistic, $\frac{n-1}{Y}$ is UMVUE for θ.

15.

$$
\begin{aligned}
f_X(x; \theta) &= \theta^x (1-\theta)^{1-x} \\
&= \exp\{x \log \theta + (1-x) \log \theta\} \\
&= \exp\left\{\log(1-\theta) + x \log \frac{\theta}{1-\theta}\right\}
\end{aligned}
$$

$$\therefore \ a(\theta) = \log(1-\theta), b(X) = 0, c(\theta) = \log(\frac{\theta}{1-\theta}), d(X) = X,$$

$\therefore$ Bin$(1, \theta)$ belongs to the exponential family and $\sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i$ is complete and sufficient for θ.

Since we usually to use $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ to estimate true mean $= \theta$, we try to check $E(n\bar{X}(1-\bar{X}))$ whether it is equal to $n\theta(1-\theta)$.

Now

$$
\begin{aligned}
E(n\bar{X}(1-\bar{X})) &= E(\sum_{i=1}^{n} X_i(1-\bar{X})) \\
&= \frac{1}{n} E(\sum_{i=1}^{n} X_i(n - \sum_{i=1}^{n} X_i))
\end{aligned}
$$

$$= \frac{1}{n}E(Y(n-Y)) \quad \text{where } Y = \sum_{i=1}^{n} X_i$$

$$= \frac{1}{n}[E(nY) - E(Y^2)]$$

$$= \frac{1}{n}[nE(Y) - Var(Y) - [E(Y)]^2]$$

$$= \frac{1}{n}[n \cdot n\theta - n\theta(1-\theta) - n^2\theta^2]$$

$$= \frac{1}{n}[n^2\theta + n\theta^2 - n\theta - n^2\theta^2]$$

$$= \theta(n-1) - \theta^2(n-1)$$

$$= (n-1)\theta(1-\theta)$$

$$\therefore E[\frac{n^2}{n-1}\bar{X}(1-\bar{X})] = n\theta(1-\theta)$$

Since $\frac{n^2}{n-1}\bar{X}(1-\bar{X})$ is function of complete sufficient statistic $\sum_{i=1}^{n} X_i$, it is the UMVUE for $n\theta(1-\theta)$.

16. By Q.6(c), $\sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i^2$ is complete sufficient statistic for θ.

Now

$$\because X_i \sim N(0, \theta)$$

$$\Rightarrow \frac{X_i - 0}{\sqrt{\theta}} \sim N(0, 1)$$

$$\Rightarrow \frac{X_i^2}{\theta} \sim \chi^2(1)$$

$$\Rightarrow \frac{1}{\theta}\sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$$

$$\Rightarrow \frac{1}{\theta}Y \sim \chi^2(n) \qquad \text{where } Y = \sum_{i=1}^{n} X_i^2$$

$$\therefore E(\frac{1}{\theta}Y) = n \text{ and } Var(\frac{1}{\theta}Y) = 2n$$

$$\Rightarrow E(Y) = n\theta \text{ and } Var(Y) = 2n\theta^2$$

$$\therefore E(Y^2) = Var(Y) + [E(Y)]^2 = 2n\theta^2 + n^2\theta^2 = (2n + n^2)\theta^2$$

$$\therefore E\left[\frac{1}{n^2 + 2n}Y^2\right] = \theta^2$$

$\therefore \frac{1}{n^2+2n}Y^2$ is unbiased for θ^2 and since it is function of $\sum_{i=1}^{n} X_i^2 = Y$, $\therefore$ it is UMVUE for θ^2.

17.

$$f_X(x; \theta) = \theta e^{-\theta x} = \exp(\log \theta - \theta x)$$

$$\Rightarrow a(\theta) = \log \theta, b(X) = 0, c(\theta) = -\theta, d(X) = X$$

$\therefore$ $f(x; \theta)$ belongs to exponential family and $\sum_{i=1}^{n} d(X_i) = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ.

Let $Y = \sum_{i=1}^{n} X_i \sim \text{gamma}(n, \theta)$

$$\Rightarrow \quad E(t(Y)) = \theta$$

$$\Rightarrow \quad \int_0^\infty t(y) \frac{y^{n-1} e^{-\theta y}}{\theta^{-n} \Gamma(n)} \, dy = \theta$$

$$\Rightarrow \quad \int_0^\infty t(y) \frac{y^{n-1} e^{-\theta y}}{\theta^{-(n-1)} \Gamma(n)} \, dy = 1$$

$$\Rightarrow \quad \int_0^\infty t(y) \cdot y \frac{\Gamma(n-1)}{\Gamma(n)} \cdot \frac{y^{(n-1)-1} e^{-\theta y}}{\theta^{-(n-1)} \Gamma(n-1)} \, dy = 1$$

$$\Rightarrow \quad t(y) y \frac{\Gamma(n-1)}{\Gamma(n)} = 1$$

$$\Rightarrow \quad t(y) = \frac{\Gamma(n)}{\Gamma(n-1)} \cdot \frac{1}{y} = \frac{n-1}{y}$$

$\therefore$ $t(Y) = \frac{n-1}{\sum_{i=1}^{n} X_i}$ is UMVUE for θ.

18. By Q.5(a), $\sum_{i=1}^{n} X_i$ is complete and sufficient statistic for λ.

Observe that $\tau(\lambda) = \frac{\lambda^k e^{-\lambda}}{k!} = P(X_1 = k)$.

$\therefore$ An unbiased estimator of $\tau(\lambda)$ is $I_{(k)}(X_1)$, $k = 0, 1, 2, \ldots$

Since $E[I_{(k)}(X_1)] = P(X_1 = k) = \frac{\lambda^k e^{-\lambda}}{k!} = \tau(\lambda)$,

$\therefore$ By Rao-Blackwell theorem, $E[I_{(k)}(X_1) | \sum_{i=1}^{n} X_i]$ is the UMVUE for $\tau(\lambda)$.

Note that $\sum_{i=1}^{n} X_i \sim \text{Poisson}(n\lambda)$ and $\sum_{i=2}^{n} X_i \sim \text{Poisson}((n-1)\lambda)$

$$\text{For } n = 1, \quad E[I_{(k)}(X_1) | X_1]$$
$$= I_{(k)}(X_1)$$

$$\text{For } n > 1, \quad E[I_{(k)}(X_1) | \sum_{i=1}^{n} X_1 = s]$$

$$= P(X_1 = k | \sum_{i=1}^{n} X_i = s)$$

$$= \frac{P(X_1 = k, \sum_{i=1}^{n} X_i = s)}{P(\sum_{i=1}^{n} X_i = s)}$$

$$= \frac{P(X_1 = k) P(\sum_{i=2}^{n} X_i = s - k)}{P(\sum_{i=1}^{n} X_i = s)}$$

$$= \begin{cases} 0 & \text{for } s < k \\ \dfrac{\dfrac{\lambda^k e^{-\lambda}}{k!} \cdot \dfrac{[(n-1)\lambda]^{s-k} e^{-(n-1)\lambda}}{(s-k)!}}{\dfrac{(n\lambda)^s e^{-n\lambda}}{s!}} & \text{for } s \geq k \end{cases}$$

$$= \begin{cases} 0 & \text{for } s < k \\ \binom{s}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{s-k} & \text{for } s \geq k \end{cases}$$

$$E\left[I_{(k)}(X_1)|\sum_{i=1}^{n}X_1\right] \;=\; \left(\frac{\sum_{i=1}^{n}X_i}{k}\right)\left(\frac{1}{n}\right)^{k}\left(1-\frac{1}{n}\right)^{\sum_{i=1}^{n}X_i-k}I_{\{k,k+1,\dots\}}\left(\sum_{i=1}^{n}X_i\right)$$

Method 2:

$$S = \sum_{i=1}^{n}X_i \sim Po(n,\lambda)$$

$$\sum_{s=0}^{\infty}h(s)\frac{(n\lambda)^s e^{-\lambda n}}{s!} = \tau(\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\Rightarrow \sum_{s=0}^{\infty}\frac{h(s)n^s\lambda^{s-k}e^{-(n-1)\lambda}k!}{s!} = 1$$

$$\Rightarrow \sum_{s=0}^{\infty}\frac{h(s)n^s}{(n-1)^{s-k}}\left(\begin{array}{c}s\\k\end{array}\right)\frac{((n-1)\lambda)^{s-k}e^{-(n-1)\lambda}}{(s-k)!} = 1$$

$$\Rightarrow \left(\begin{array}{c}s\\k\end{array}\right)\frac{h(s)n^s}{(n-1)^{s-k}} = 1$$

$$\Rightarrow h(s) = \left(\begin{array}{c}s\\k\end{array}\right)\frac{(n-1)^{s-k}}{n^s} = \dots$$

19.

$$\begin{aligned}
f_X(x;\theta) &= \left(\begin{array}{c}k+x-1\\x\end{array}\right)\theta^k(1-\theta)^x\\
&= \exp\left\{\left(\begin{array}{c}k+x-1\\x\end{array}\right)+k\log\theta+x\log(1-\theta)\right\}\\
&= \exp\left\{k\log\theta+\log\left(\begin{array}{c}k+x-1\\x\end{array}\right)+x\log(1-\theta)\right\}
\end{aligned}$$

$$a(\theta) = k\log\theta, b(X) = \log\left(\begin{array}{c}k+X-1\\X\end{array}\right), c(\theta) = \log(1-\theta), d(X) = X$$

$\therefore f_X(x;\theta)$ belongs to exponential family and X is a complete sufficient statistic for θ (Note that only have one observation in this equation).

$$\begin{aligned}
\therefore E(h(X)) &= \tau(\theta) = \frac{1}{\theta}\\
&\Rightarrow \sum_{x=0}^{\infty}h(x)\left(\begin{array}{c}k+x-1\\x\end{array}\right)\theta^k(1-\theta)^x = \frac{1}{\theta}\\
&\Rightarrow \sum_{x=0}^{\infty}h(x)\left(\begin{array}{c}k+x-1\\x\end{array}\right)\theta^{k+1}(1-\theta)^x = 1\\
&\Rightarrow \sum_{x=0}^{\infty}\left(\begin{array}{c}k+x-1\\x\end{array}\right)\left(\begin{array}{c}k+1+x-1\\x\end{array}\right)^{-1}h(x)\left(\begin{array}{c}k+1+x-1\\x\end{array}\right)\theta^{k+1}(1-\theta)^x = 1\\
&\Rightarrow \left(\begin{array}{c}k+x-1\\x\end{array}\right)\left(\begin{array}{c}k+x\\x\end{array}\right)^{-1}h(x) = 1\\
&\Rightarrow \frac{(k+x-1)!}{x!(k-1)!}\times\frac{x!k!}{(k+x)!}h(x) = 1\\
&\Rightarrow h(x) = \frac{k+x}{k} = 1+\frac{x}{k}, \quad \text{is UMVUE for } g(\theta) = \frac{1}{\theta}
\end{aligned}$$

$$
\begin{aligned}
\therefore \quad Var(h(X)) \\
= \quad Var(1 + \frac{X}{k}) \\
= \quad Var(\frac{X}{k}) \\
= \quad \frac{1}{k^2} \frac{k(1-\theta)}{\theta^2} \\
= \quad \frac{1-\theta}{k\theta^2}
\end{aligned}
$$

20.

$$
\begin{aligned}
f(X;\theta) &= \frac{X^{\alpha-1} e^{-X/\theta}}{\theta^\alpha \Gamma(\alpha)} \\
&= \exp\left\{ (\alpha-1)\log X - \frac{X}{\theta} - \alpha \log \theta - \log \Gamma(\alpha) \right\} \\
&= \exp\left\{ -\alpha \log \theta - \log \Gamma(\alpha) + (\alpha-1)\log X - \frac{X}{\theta} \right\}
\end{aligned}
$$

$$
\therefore \; a(\theta) = -\alpha \log \theta, \, b(X) = -\log \Gamma(\alpha) + (\alpha-1)\log X, \, c(\theta) = -\frac{1}{\theta}, \, d(X) = X
$$

$\therefore f(X;\theta)$ belongs to exponential family and $\sum_{i=1}^{n} X_i$ is complete sufficient statistic for θ.

$$
\because \; E\left[\frac{1}{n\alpha} \sum_{j=1}^{n} X_j \right] = \frac{1}{n\alpha} \sum_{j=1}^{n} E(X_j) = \frac{1}{n\alpha} \cdot n(\alpha\theta) = \theta \quad (X_j \sim gamma(\alpha, \frac{1}{\theta}))
$$

and $\dfrac{1}{n\alpha} \sum_{j=1}^{n} X_j$ is function of complete sufficient statistic,

$$
\therefore \; U(X_1, \ldots, X_n) = \frac{1}{n\alpha} \sum_{j=1}^{n} X_j \text{ is UMVUE for } \theta
$$

$$
\begin{aligned}
\log f(X;\theta) &= -\alpha \log \theta - \log \Gamma(\alpha) + (\alpha-1)\log X - \frac{X}{\theta} \\
\frac{\partial}{\partial \theta} \log f(X;\theta) &= \frac{-\alpha}{\theta} + \frac{X}{\theta^2} \\
\frac{\partial^2}{\partial \theta^2} \log f(X;\theta) &= \frac{\alpha}{\theta^2} - \frac{2X}{\theta^3} \\
E\left[\frac{\partial^2}{\partial \theta^2} \log f(X;\theta) \right] &= E\left[\frac{\alpha}{\theta^2} - \frac{2X}{\theta^3} \right] \\
&= \frac{\alpha}{\theta^2} - \frac{2\alpha\theta}{\theta^3} \\
&= \frac{-\alpha}{\theta^2} \\
\therefore \; \text{CRLB for } \theta &= \frac{-1}{n \cdot E\left[\frac{\partial^2}{\partial \theta^2} \log f(X;\theta) \right]} \\
&= \frac{-1}{n \cdot \left(\frac{-\alpha}{\theta^2} \right)}
\end{aligned}
$$

$$= \frac{\theta^2}{n\alpha}$$

$$Var(U(X_1, \ldots, X_n)) = Var(\frac{1}{n\alpha}\sum_{j=1}^{n}X_j)$$

$$= \frac{1}{n^2\alpha^2}\sum_{j=1}^{n}Var(X_j)$$

$$(\because X_j \sim gamma(\alpha, \frac{1}{\theta}), Var(X_j) = \alpha\theta^2)$$

$$= \frac{1}{n^2\alpha^2} \cdot n(\alpha\theta^2)$$

$$= \frac{\theta^2}{n\alpha}$$

$$= \text{CRLB for } \theta$$

21. $X \sim \exp(\theta)$

$f(x;\theta) = \theta e^{-\theta x} = \exp(\log\theta - \theta x)$

$$a(\theta) = \log\theta, b(X) = 0, c(\theta) = -\theta, d(X) = X$$

$\therefore$ $f(x;\theta)$ belongs to exponential family and $\sum_{i=1}^{n}X_i$ is complete sufficient statistic for θ.

Let

$$S = \sum_{i=1}^{n}X_i \sim gamma(n, \theta)$$

$$\Rightarrow E[h(S)] = R(x, \theta)$$

$$\Rightarrow \int_0^\infty h(s)\frac{s^{n-1}e^{-\theta s}}{\Gamma(n)\theta^{-n}} \, ds = R(x,\theta) = P(X > x) = e^{-\theta x}$$

$$\Rightarrow \int_0^\infty \frac{h(s)s^{n-1}}{(s-x)^{n-1}} \cdot \frac{(s-x)^{n-1}}{\Gamma(n)\theta^{-n}}e^{-\theta(s-x)} \, ds = 1$$

$$\Rightarrow \frac{h(s)s^{n-1}}{(s-x)^{n-1}} = 1 = I_{(x,\infty)}(s)$$

$$\text{Since } X \text{ must be greater than } x, \therefore s = \sum_{i=1}^{n}X_i > x$$

$$\Rightarrow h(s) = \frac{(s-x)^{n-1}}{s^{n-1}}I_{(x,\infty)}(s) = (1-\frac{x}{s})^{n-1}I_{(x,\infty)}(s)$$

$\therefore$ $(1 - \frac{x}{\sum_{i=1}^{n}X_i})^{n-1}I_{(x,\infty)}(\sum_{i=1}^{n}X_i)$ is UMVUE for $R(x;\theta)$.

22.

$$f_{Y_3}(y_3) = \frac{5!}{2!2!}[F_X(y_3)]^2 f_X(y_3)[1 - F_X(y_3)]^2$$

$$= 30\left(\frac{y_3}{\theta}\right)^2\left(\frac{1}{\theta}\right)\left(1 - \frac{y_3}{\theta}\right)^2$$

$$= \frac{30}{\theta^5}y_3^2(\theta - y_3)^2, \quad 0 < y_3 < \theta$$

$$\therefore E(Y_3) = \int_0^\theta y_3 f_{Y_3}(y_3) \, dy_3$$

$$= \int_0^\theta y_3\frac{30}{\theta^5}y_3^2(\theta - y_3)^2 \, dy_3$$

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$$= \int_0^\theta y_3 \frac{30}{\theta^5}(\theta^2 y_3^2 - 2\theta y_3^3 + y_3^4) \, dy_3$$

$$= \int_0^\theta \frac{30}{\theta^5}(\theta^2 y_3^3 - 2\theta y_3^4 + y_3^5) \, dy_3$$

$$= \frac{30}{\theta^5} \left[\theta^2 \frac{y_3^4}{4} - 2\theta \frac{y_3^5}{5} + \frac{y_3^6}{6} \right]_0^\theta$$

$$= 30\theta \left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right)$$

$$= 30\theta \left(\frac{15 - 24 + 10}{60} \right)$$

$$= \frac{\theta}{2}$$

$$\therefore \ E(2Y_3) = 2E(Y_3) = 2(\frac{\theta}{2}) = \theta$$

$$\therefore \ 2Y_3 \text{ is an unbiased statistic for } \theta.$$

Joint pdf of Y_3 and Y_5,

$$f_{Y_3,Y_5}(y_3, y_5) = \frac{5!}{2!1!0!}[F_X(y_3)]^2 f_X(y_3)[F_X(y_5) - F_X(y_3)]f_X(Y_5)$$

$$= 60(\frac{y_3}{\theta})^2 (\frac{1}{\theta})(\frac{y_5}{\theta} - \frac{y_3}{\theta})\frac{1}{\theta}$$

$$= \frac{60}{\theta^5}(y_3^2 y_5 - y_3^3), \quad 0 < y_3 \le y_5 < \theta$$

$$f_{Y_5}(y_5) = \frac{5!}{4!0!}[F_X(y_5)]^4 f_X(y_5)$$

$$= 5(\frac{y_5}{\theta})^4 (\frac{1}{\theta})$$

$$= \frac{5y_5^4}{\theta^5}, \quad 0 < y_5 < \theta$$

$$\therefore \ f_{Y_3|Y_5}(y_3|y_5) = \frac{Y_3,Y_5(y_3, y_5)}{f_{Y_5}(y_5)}$$

$$= \frac{\frac{60}{\theta^5}(y_3^2 y_5 - y_3^3)}{\frac{5y_5^4}{\theta^5}}$$

$$= \frac{12y_3^2(y_5 - y_3)}{y_5^4}$$

$$\therefore \ E(Y_3|Y_5 = y_5) = \int_0^{y_5} y_3 f_{Y_3|Y_5}(y_3|y_5) \, dy_3$$

$$= \int_0^{y_5} y_3 \frac{12y_3^2(y_5 - y_3)}{y_5^4} \, dy_3$$

$$= \frac{12}{y_5^4} \int_0^{y_5} y_3^3(y_5 - y_3) \, dy_3$$

$$= \frac{12}{y_5^4} \int_0^{y_5} (y_3^3 y_5 - y_3^4) \, dy_3$$

$$= \frac{12}{y_5^4} [\frac{1}{4} y_3^4 y_5 - \frac{1}{5} y_3^5]_0^{y_5}$$

$$= 12(\frac{1}{4} y_5 - \frac{1}{5} y_5)$$

$$= \frac{3}{5} y_5$$

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$$
\begin{aligned}
\varphi(y_5) &= E(2Y_3|Y_5 = y_5) = 2E(Y_3|Y_5 = y_5) \\
&= \frac{6}{5}y_5
\end{aligned}
$$

$$
\begin{aligned}
E(Y_3^2) &= \int_0^\theta y_3^2(\frac{30}{\theta^5})(\theta^2 y_3^3 - 2\theta y_3^3 + y_3^4)\ dy_3 \\
&= \frac{30}{\theta^5}\int_0^\theta (\theta^2 y_3^4 - 2\theta y_3^5 + y_3^6)\ dy_3 \\
&= \frac{30}{\theta^5}[\frac{1}{5}\theta^2 y_3^5 - \frac{2}{6}\theta y_3^6 + \frac{1}{7}y_3^7]_0^\theta \\
&= 30\ \theta^2(\frac{1}{5} - \frac{1}{3} + \frac{1}{7}) \\
&= \frac{30\ \theta}{105} \\
&= \frac{2}{7}\theta^2 \\
\therefore\ Var(2Y_3) &= E[(2Y_3)^2] - [E(2Y_3)]^2 \\
&= 4E(Y_3^2) - \theta^2 \\
&= \frac{8}{7}\theta^2 - \theta^2 = \frac{1}{7}\theta^2
\end{aligned}
$$

Note that $\varphi(Y_5)$ is unbiased for θ since

$$
\begin{aligned}
E[\varphi(Y_5)] &= E[E(2Y_3|Y_5)] = E(2Y_3) = \theta \quad (\because\ E(g(X)) = E[E(g(X)|Y)]) \\
E\left([\varphi(Y_5)]^2\right) &= \int_0^\theta (\frac{6}{5}y_5)^2 \cdot f_{Y_5}(y_5)\ dy_5 \\
&= \int_0^\theta (\frac{6}{5})^2 y_5^2 \cdot \frac{5y_5^4}{\theta^5}\ dy_5 \\
&= \frac{36}{5\theta^5}\int_0^\theta y_5^6\ dy_5 \\
&= \frac{36}{5\theta^5}[\frac{y_5^7}{7}]_0^\theta \\
&= \frac{36}{35}\theta^2 \\
Var[\varphi(Y_5)] &= E[(\varphi(Y_5))] - [E(\varphi(Y_5))]^2 \\
&= \frac{36}{35}\theta^2 - \theta^2 = \frac{\theta^2}{35} \\
Var[\varphi(Y_5)] &= \frac{\theta^2}{35} < \frac{1}{7}\theta^2 = Var(2Y_3)
\end{aligned}
$$

23. (a)

$$
\begin{aligned}
f_Y(y) &= \int_0^y f_{X,Y}(x,y)\ dx, \quad 0 < x < y < \infty \\
&= \int_0^y \frac{2}{\theta^2}\exp\left\{-\frac{(x+y)}{\theta}\right\}\ dx \\
&= \frac{2}{\theta^2}\exp\left\{\frac{-y}{\theta}\right\}\int_0^y e^{-x/\theta}\ dx \\
&= \frac{2}{\theta^2}\exp\left\{\frac{-y}{\theta}\right\}\left[-\theta e^{-x/\theta}\right]_0^y
\end{aligned}
$$

$$= \frac{2}{\theta^2} \exp\left\{\frac{-y}{\theta}\right\} [-\theta e^{-y/\theta} + \theta]$$

$$= \frac{2}{\theta^2} \exp\left\{-\frac{y}{\theta}\right\} (1 - \exp\{-\frac{y}{\theta}\}), \quad 0 < y < \infty$$

$$\therefore \ E(Y) = \int_0^\infty y f_Y(y) \ dy$$

$$= \int_0^\infty y \cdot \frac{2}{\theta} \ \exp\left(-\frac{y}{\theta}\right) \left(1 - \exp(-\frac{y}{\theta})\right) \ dy$$

$$= \int_0^\infty \frac{2y}{\theta} \exp\left(\frac{-y}{\theta}\right) \ dy - \int_0^\infty \frac{2y}{\theta} \exp\left(\frac{-2y}{\theta}\right) \ dy$$

$$= \left[-2y e^{-y/\theta}\right]_0^\infty + \int_0^\infty 2e^{-y/\theta} \ dy + \left[y e^{-2y/\theta}\right]_0^\infty - \int_0^\infty e^{-2y/\theta} \ dy$$

$$= \left[-2\theta e^{-y/\theta}\right]_0^\infty + \left[\frac{\theta}{2} e^{-2y/\theta}\right]_0^\infty$$

$$= 2\theta - \theta/2$$

$$= 3\theta/2$$

$$E(Y^2) = \int_0^\infty y^2 f_Y(y) \ dy$$

$$= \int_0^\infty y^2 \cdot \frac{2}{\theta} \ \exp\left(\frac{-y}{\theta}\right) \left(1 - \exp\frac{-y}{\theta}\right) \ dy$$

$$= \int_0^\infty \frac{2y^2}{\theta} \exp\left(\frac{-y}{\theta}\right) \ dy - \int_0^\infty \frac{2y^2}{\theta} \exp\left(\frac{-2y}{\theta}\right) \ dy$$

$$= \left[-2y^2 e^{-y/\theta}\right]_0^\infty + \int_0^\infty 4y e^{-y/\theta} \ dy + \left[y^2 e^{-2y/\theta}\right]_0^\infty - \int_0^\infty 2y e^{-2y/\theta} \ dy$$

$$= \left[-4\theta y e^{-y/\theta}\right]_0^\infty + \int_0^\infty 4\theta e^{-y/\theta} \ dy + \left[\theta y e^{-2y/\theta}\right]_0^\infty - \int_0^\infty \theta e^{-2y/\theta} \ dy$$

$$= \left[-4\theta^2 e^{-y/\theta}\right]_0^\infty + \left[\frac{\theta^2}{2} e^{-2y/\theta}\right]_0^\infty$$

$$= 4\theta^2 - \frac{\theta^2}{2}$$

$$= \frac{7\theta^2}{2}$$

$$\therefore \ Var(Y) = E(Y)^2 - [E(Y)]^2$$

$$= \frac{7\theta^2}{2} - (\frac{3\theta}{2})^2 = \frac{5\theta^2}{4}$$

(b)

$$f_X(x) = \int_x^\infty f_{X,Y}(x,y) \ dy$$

$$= \int_x^\infty \frac{2}{\theta^2} \exp\left\{-\frac{(x+y)}{\theta}\right\} \ dy$$

$$= \frac{2}{\theta^2} \exp\left\{\frac{-x}{\theta}\right\} \int_x^\infty e^{-y/\theta} \ dy$$

$$= \frac{2}{\theta^2} \exp\left\{\frac{-x}{\theta}\right\} [-\theta e^{-y/\theta}]_x^\infty$$

$$= \frac{2}{\theta} \exp\left\{\frac{-2x}{\theta}\right\}, \quad 0 < x < \infty$$

$$\therefore f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{\frac{2}{\theta^2} \exp\left\{-\frac{(x+y)}{\theta}\right\}}{\frac{2}{\theta} \exp\left\{\frac{-2x}{\theta}\right\}}$$

$$= \frac{1}{\theta} \exp\left\{\frac{x-y}{\theta}\right\}, \quad 0 < x < y < \infty$$

$$\therefore E(Y|X=x) = \int_x^\infty y \cdot f_{Y|X}(y|x) \, dy$$

$$= \int_x^\infty y \frac{1}{\theta} \exp\left\{\frac{x-y}{\theta}\right\} \, dy$$

$$= \frac{1}{\theta} \exp(\frac{x}{\theta}) \int_x^\infty y e^{-y/\theta} \, dy, \quad \text{by parts}$$

$$= \exp(\frac{x}{\theta}) \left\{ \left[-ye^{-y/\theta}\right]_x^\infty + \int_x^\infty e^{-y/\theta} \, dy \right\}$$

$$= x + \exp(\frac{x}{\theta}) \left[-\theta e^{-y/\theta}\right]_x^\infty$$

$$= x + \theta$$

$$f_X(x) = \frac{2}{\theta} \exp\{-\frac{2x}{\theta}\}$$

$$\therefore X \sim \exp(\frac{2}{\theta})$$

$$\therefore Var(X+\theta) = Var(X)$$

$$= (\frac{\theta}{2})^2$$

$$= \frac{\theta^2}{4}$$

$$< \frac{5\theta^2}{4}$$

$$= Var(Y)$$

24. (a)

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \prod_{i=1}^n \left(\frac{1}{\theta}\right) I_{[1,\theta]}(x_i)$$

$$= \left(\frac{1}{\theta}\right)^n I_{[1,y_n]}(y_1) \cdot I_{[y_1,\theta]}(y_n)$$

$$= \left\{\left(\frac{1}{\theta}\right)^n I_{[y_1,\theta]}(y_n)\right\} \{I_{[1,y_n]}(y_1)\}$$

By factorization theorem, $Y = Y_n = \max(X_1,\ldots,X_n)$ is sufficient for θ.

To check completeness:

$$E(Z(Y)) = \sum_{y=1}^\theta Z(y)P(Y=y)$$

$$= \sum_{y=1}^\theta Z(y) \cdot \left(\frac{1}{\theta}\right)^n (y^n - (y-1)^n) \qquad (*)$$

$$E(Z(Y))=0 \Rightarrow \sum_{y=1}^\theta Z(y) \cdot \left(\frac{1}{\theta}\right)^n (y^n - (y-1)^n) = 0$$

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$$\Rightarrow \sum_{y=1}^{\theta} Z(y) \cdot (y^n - (y-1)^n) = 0, \quad (\because \theta > 0)$$

If $\sum_{y=1}^{\theta} Z(y)(y^n - (y-1)^n) = 0$ for all $\theta \in \{1, 2, \ldots\}$, we need $Z(y) = 0$.

$\therefore$ Y is complete.

$\therefore$ Y is sufficient and complete statistic for θ.

(b)

$$E\left[\frac{Y^{n+1} - (Y-1)^{n+1}}{Y^n - (Y-1)^n}\right]$$

$$= \sum_{y=1}^{\theta} \left(\frac{y^{n+1} - (y-1)^{n+1}}{y^n - (y-1)^n}\right) \cdot P(Y = y)$$

$$= \sum_{y=1}^{\theta} \left(\frac{y^{n+1} - (y-1)^{n+1}}{y^n - (y-1)^n}\right) \cdot \left(\frac{1}{\theta}\right)^n (y^n - (y-1)^n)$$

$$= \sum_{y=1}^{\theta} \left(\frac{1}{\theta}\right)^n (y^{n+1} - (y-1)^{n+1})$$

$$= \left(\frac{1}{\theta}\right)^n [1^{n+1} - (1-1)^{n+1} + 2^{n+1} - 1^{n+1} + 3^{n+1} - 2^{n+1} + \ldots + \theta^{n+1} + (\theta-1)^{n+1}]$$

$$= \left(\frac{1}{\theta}\right)^n \cdot \theta^{n+1}$$

$$= \theta$$

Since $\left(\frac{Y^{n+1} - (Y-1)^{n+1}}{Y^n - (Y-1)^n}\right)$ is function of Y and it is unbiased for θ, it is UMVUE for θ.

$$
\begin{aligned}
(*) \quad P(Y = y) &= P(\max(X_1, \ldots, X_n) = y) \\
&= P(\max(X_1, \ldots, X_n) \le y) - P(\max(X_1, \ldots, X_n) \le y - 1) \\
&= \prod_{i=1}^{n} P(X \le y) - \prod_{i=1}^{n} P(X \le y - 1), \quad \text{by iid of } X \\
&= \prod_{i=1}^{n} \frac{y}{\theta} - \prod_{i=1}^{n} \frac{y-1}{\theta} \\
&= \left(\frac{y}{\theta}\right)^n - \left(\frac{y-1}{\theta}\right)^n
\end{aligned}
$$

25. (a)

$$
\begin{aligned}
f(x; p) &= p^x (1-p)^{1-x} \\
L(p) &= f_{\underset{\sim}{X}}(\underset{\sim}{x}; p) \\
&= \prod_{i=1}^{n} f_{X_i}(x_i; p) \\
&= \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i} \\
&= p^{\sum_{i=1}^{n} x_i} (1-p)^{n - \sum_{i=1}^{n} x_i}
\end{aligned}
$$

$$\log L(p) = \left(\sum_{i=1}^{n} x_i\right)\log p + \left(n - \sum_{i=1}^{n} x_i\right)\log(1-p)$$

$$\frac{\partial}{\partial p}\log L(p) = \left(\sum_{i=1}^{n} x_i\right)\frac{1}{p} - \left(n - \sum_{i=1}^{n} x_i\right)\frac{1}{1-p}$$

Setting equal to $0 \Rightarrow \hat{p} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$

$\therefore$ By invariant property of MLE, MLE for $\theta = (1-\hat{p})^2 = (1-\bar{x})^2$

(b)

$$X_1 + X_2 \sim Bin(2,p)$$

$$\begin{aligned}
\therefore E(\hat{\theta}) &= 1 \cdot P(X_1 + X_2 = 0) + 0 \cdot P(X_1 + X_2 \neq 0)\\
&= \binom{2}{0} p^0 (1-p)^{2-0}\\
&= (1-p)^2\\
&= \theta
\end{aligned}$$

$\therefore$ $\hat{\theta}$ is an unbiased estimator of θ.

(c)

$$\begin{aligned}
f(x;p) &= p^x (1-p)^{1-x}\\
&= \exp\{x\log p + (1-x)\log(1-p)\}\\
&= \exp\left\{\log(1-p) + x\log\left(\frac{p}{1-p}\right)\right\}
\end{aligned}$$

$$\therefore\ a(p) = \log(1-p), b(X) = 0, c(p) = \left(\frac{p}{1-p}\right), d(X) = X$$

$\therefore$ $f(X;p)$ belongs to exponential family and $\sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for p.

Let $S = \sum_{i=1}^{n} X_i \sim Bin(n;p)$

By Rao-Blackwell theorem, UMVUE for $\theta = E(\hat{\theta}|S = s)$, $\hat{\theta}$ is unbiased, S is sufficient.

$$\begin{aligned}
\Rightarrow E(\hat{\theta}|S = s) &= 1 \cdot P(X_1 + X_2 = 0|S = s) + 0 \cdot P(X_1 + X_2 \neq 0|S = s)\\
&= \frac{P(X_1 + X_2 = 0, \sum_{i=1}^{n} X_i = s)}{P(\sum_{i=1}^{n} X_i = s)}\\
&= \frac{P(X_1 + X_2 = 0, \sum_{i=3}^{n} X_i = s)}{P(\sum_{i=1}^{n} X_i = s)}\\
&= \frac{P(X_1 + X_2 = 0) \cdot P(\sum_{i=3}^{n} X_i = s)}{P(\sum_{i=1}^{n} X_i = s)}\\
&= \frac{\binom{2}{0} p^0 (1-p)^{2-0} \binom{n-2}{s} p^s (1-p)^{n-2-s}}{\binom{n}{s} p^s (1-p)^{n-s}}\\
&= \frac{(n-2)!}{s!(n-2-s)!} \cdot \frac{s!(n-s)!}{n!}\\
&= \frac{(n-s)(n-s-1)}{n(n-1)}
\end{aligned}$$

$\therefore$ UMVUE for $\theta = (1-p)^2$ is $\frac{1}{n(n-1)}(n - \sum_{i=1}^{n} X_i)(n - \sum_{i=1}^{n} X_i - 1)$

Method 2

$$\sum_{s=0}^{\infty} h(s) \binom{n}{s} p^s (1-p)^{n-s} = \theta = (1-p)^2$$

$$\Rightarrow \sum_{s=0}^{\infty} h(s) \binom{n}{s} p^s (1-p)^{n-s-2} = 1$$

$$\Rightarrow \sum_{s=0}^{\infty} \frac{h(s)n(n-1)}{(n-s-1)(n-s)} \binom{n-2}{s} p^s (1-p)^{n-s-2} = 1$$

$$\Rightarrow h(s) = \frac{(n-s-1)(n-s)}{n(n-1)} = \ldots$$

26.

$$f_X(x) = \frac{1}{2\theta - \theta} = \frac{1}{\theta}$$

$$F_X(x) = \int_\theta^x \frac{1}{\theta}\, dt = \left[\frac{t}{\theta}\right]_\theta^x = \frac{x}{\theta} - 1 = \frac{1}{\theta}(x - \theta)$$

$$
\begin{aligned}
f_{Y_1}(y_1) &= \frac{n!}{(1-1)!(n-1)!}[F_X(y_1)]^{1-1}[1 - F_X(y_1)]^{n-1} f_X(y_1) \\
&= n\left(1 - \frac{1}{\theta}(y_1 - \theta)\right)^{n-1} \frac{1}{\theta} \\
&= \frac{n}{\theta}\left[2 - \frac{y_1}{\theta}\right]^{n-1}, \qquad \theta < y_1 < 2\theta
\end{aligned}
$$

$$
\begin{aligned}
f_{Y_n}(y_n) &= \frac{n!}{(n-1)!(n-n)!}[F_X(y_n)]^{n-1}[1 - F_X(y_n)]^{n-n} f_X(y_n) \\
&= n\left(1 - \frac{1}{\theta}(y_n - \theta)\right)^{n-1} \frac{1}{\theta} \\
&= \frac{n}{\theta}\left[\frac{y_n}{\theta} - 1\right]^{n-1}, \qquad \theta < y_n < 2\theta
\end{aligned}
$$

$$
\begin{aligned}
f_{Y_1, Y_n}(y_1, y_n) &= \frac{n!}{(1-1)!(n-1-1)!(n-n)!}[F_X(y_1)]^{1-1}[F_X(y_n) - F_X(y_1)]^{n-1-1} \\
&\quad \cdot [1 - F_X(y_n)]^{n-n} f_X(y_1) f_X(y_n) \\
&= n(n-1)\left(\frac{y_n - \theta}{\theta} - \frac{y_1 - \theta}{\theta}\right)^{n-2} \frac{1}{\theta} \cdot \frac{1}{\theta} \\
&= n(n-1)\left(\frac{1}{\theta^n}\right)(y_n - y_1)^{n-2}, \qquad \theta < y_1 \le y_n < 2\theta
\end{aligned}
$$

$$
\begin{aligned}
\therefore E(Y_1) &= \int_\theta^{2\theta} y_1 f_{Y_1}(y_1)\, dy_1 \\
&= \frac{n}{\theta} \int_\theta^{2\theta} y_1 \left(2 - \frac{y_1}{\theta}\right)^{n-1} dy_1 \\
&= \frac{n}{\theta^n} \int_\theta^{2\theta} y_1 (2\theta - y_1)^{n-1} dy_1
\end{aligned}
$$

$$= \frac{n}{\theta^n} \int_\theta^0 -(2\theta - z)z^{n-1}\, dz, \qquad \text{let } z = 2\theta - y_1, dz = -dy_1$$

$$= \frac{n}{\theta^n} \int_0^\theta (-z^n + 2\theta z^{n-1})\, dz$$

$$= \frac{n}{\theta^n} \left[\frac{-1}{n+1} z^{n+1} + \frac{2\theta}{n} z^n \right]_0^\theta$$

$$= \frac{n}{\theta^n} \left[\frac{-\theta^{n+1}}{n+1} + \frac{2\theta^{n+1}}{n} \right]$$

$$= \frac{-n\theta}{n+1} + 2\theta$$

$$= \frac{n+2}{n+1}\theta$$

$$E(Y_n) = \int_\theta^{2\theta} y_n f_{Y_n}(y_n)\, dy_n$$

$$= \frac{n}{\theta} \int_\theta^{2\theta} y_n \left(\frac{y_n}{\theta} - 1 \right)^{n-1} dy_n$$

$$= \frac{n}{\theta^n} \int_\theta^{2\theta} y_n (y_n - \theta)^{n-1}\, dy_n$$

$$= \frac{n}{\theta^n} \int_0^\theta (z + \theta)z^{n-1}\, dz, \qquad \text{let } z = y_n - \theta, dz = dy_n$$

$$= \frac{n}{\theta^n} \int_0^\theta (z^n + \theta z^{n-1})\, dz$$

$$= \frac{n}{\theta^n} \left[\frac{1}{n+1} z^{n+1} + \frac{\theta}{n} z^n \right]_0^\theta$$

$$= \frac{n}{\theta^n} \left[\frac{\theta^{n+1}}{n+1} + \frac{\theta^{n+1}}{n} \right]$$

$$= \frac{n\theta}{n+1} + \theta$$

$$= \frac{2n+1}{n+1}\theta$$

$$E(U_1) = E\left(\frac{n+1}{2n+1} Y_n \right)$$

$$= \frac{n+1}{2n+1} \cdot E(Y_n)$$

$$= \frac{n+1}{2n+1} \cdot \frac{2n+1}{n+1}\theta = \theta$$

$$E(U_2) = E\left(\frac{n+1}{5n+4} (2Y_n + Y_1) \right)$$

$$= \frac{n+1}{5n+4} [2E(Y_n) + E(Y_1)]$$

$$= \frac{n+1}{5n+4} \left[\frac{4n+2}{n+1}\theta + \frac{n+2}{n+1}\theta \right] = \theta$$

$\therefore$ U_1 and U_2 are unbiased estimators for θ.

$$E(Y_1^2) = \int_\theta^{2\theta} y_1^2 f_{Y_1}(y_1)\, dy_1$$

$$= \frac{n}{\theta} \int_{\theta}^{2\theta} y_1^2 \left(2 - \frac{y_1}{\theta}\right)^{n-1} \, dy_1$$

$$= \frac{n}{\theta^n} \int_{\theta}^{2\theta} y_1^2 (2\theta - y_1)^{n-1} \, dy_1$$

$$= \frac{n}{\theta^n} \int_{\theta}^{0} -(2\theta - z)^2 z^{n-1} \, dz, \qquad \text{let } z = 2\theta - y_1, dz = -dy_1$$

$$= \frac{n}{\theta^n} \int_{0}^{\theta} (z^{n+1} - 4\theta z^n + 4\theta^2 z^{n-1}) \, dz$$

$$= \frac{n}{\theta^n} \left[\frac{1}{n+2} z^{n+2} - \frac{4\theta}{n+1} z^{n+1} + \frac{4\theta^2}{n} z^n \right]_0^\theta$$

$$= \frac{n}{\theta^n} \left[\frac{\theta^{n+2}}{n+2} - \frac{4\theta^{n+2}}{n+1} + \frac{4\theta^{n+2}}{n} \right]$$

$$= \left[\frac{n}{n+2} - \frac{4n}{n+1} + 4 \right] \theta^2$$

$$E(Y_n) = \int_{\theta}^{2\theta} y_n^2 f_{Y_n}(y_n) \, dy_n$$

$$= \frac{n}{\theta} \int_{\theta}^{2\theta} y_n^2 \left(\frac{y_n}{\theta} - 1\right)^{n-1} \, dy_n$$

$$= \frac{n}{\theta^n} \int_{\theta}^{2\theta} y_n^2 (y_n - \theta)^{n-1} \, dy_n$$

$$= \frac{n}{\theta^n} \int_{0}^{\theta} (z + \theta)^2 z^{n-1} \, dz, \qquad \text{let } z = y_n - \theta, dz = dy_n$$

$$= \frac{n}{\theta^n} \int_{0}^{\theta} (z^{n+1} + 2\theta z^n + \theta^2 z^{n-1}) \, dz$$

$$= \frac{n}{\theta^n} \left[\frac{1}{n+2} z^{n+2} + \frac{2\theta}{n+1} z^{n+1} + \frac{\theta^2}{n} z^n \right]_0^\theta$$

$$= \left[\frac{n}{n+2} + \frac{2n}{n+1} + 1 \right] \theta^2$$
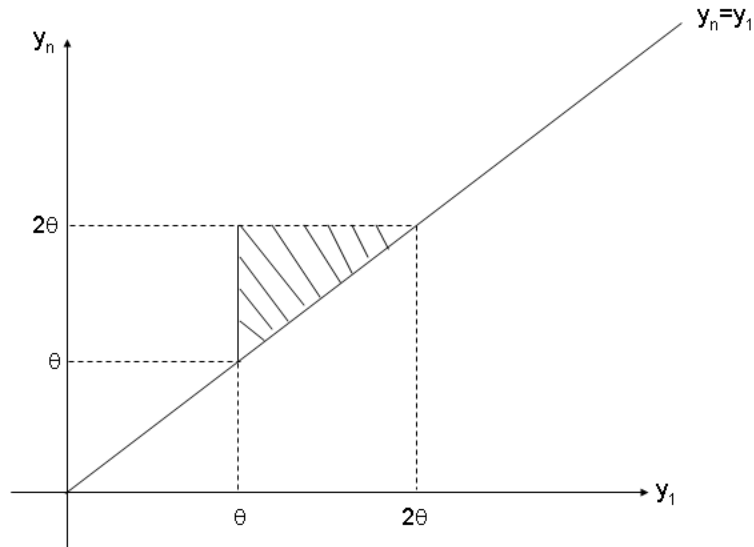
$$
\begin{aligned}
E(Y_1 Y_n) &= \int_\theta^{2\theta} \int_\theta^{y_n} y_1 y_n f_{Y_1,Y_n}(y_1, y_n) \; dy_1 \; dy_n \\
&= \frac{n(n-1)}{\theta^n} \int_\theta^{2\theta} \int_\theta^{y_n} y_1 y_n (y_n - y_1)^{n-2} \; dy_1 \; dy_n \\
&= \frac{n(n-1)}{\theta^n} \int_\theta^{2\theta} y_n \int_\theta^{y_n} y_1 (y_n - y_1)^{n-2} \; dy_1 \; dy_n \\
&= \frac{n(n-1)}{\theta^n} \int_\theta^{2\theta} y_n \int_{y_n-\theta}^0 -(y_n - z) z^{n-2} \; dz \; dy_n, \qquad \text{let } z = y_n - y_1, dz = -dy_1 \\
&= \frac{n(n-1)}{\theta^n} \int_\theta^{2\theta} y_n \int_0^{y_n-\theta} (y_n \, z^{n-2} - z^{n-1}) \; dz \; dy_n \\
&= \frac{n(n-1)}{\theta^n} \int_\theta^{2\theta} y_n \left[\frac{y_n}{n-1} z^{n-1} - \frac{1}{n} z^n \right]_0^{y_n-\theta} dy_n \\
&= \frac{n(n-1)}{\theta^n} \int_\theta^{2\theta} y_n \left[\frac{1}{n-1} y_n \, (y_n - \theta)^{n-1} - \frac{1}{n}(y_n - \theta)^n \right] \; dy_n \\
&= \frac{n(n-1)}{\theta^n} \left\{ \int_\theta^{2\theta} \frac{1}{n-1} y_n^2 \, (y_n - \theta)^{n-1} \; dy_n - \int_\theta^{2\theta} \frac{1}{n} y_n \, (y_n - \theta)^n \; dy_n \right\} \\
&= \frac{n(n-1)}{\theta^n} \left\{ \frac{1}{n-1} \int_0^\theta (z+\theta)^2 \, z^{n-1} \; dz - \frac{1}{n} \int_0^\theta (z+\theta) \, z^n \; dz \right\}, \\
&\qquad \text{(let } z = y_n - \theta, dz = dy_n) \\[2mm]
&= \frac{n(n-1)}{\theta^n} \left\{ \frac{1}{n-1} \left[\frac{z^{n+2}}{n+2} + \frac{2\theta z^{n+1}}{n+1} + \frac{\theta^2 z^n}{n} \right]_0^\theta - \frac{1}{n} \left[\frac{1}{n+2} z^{n+2} + \frac{\theta}{n+1} z^{n+1} \right]_0^\theta \right\} \\
&= \frac{n(n-1)}{\theta^n} \left\{ \frac{1}{n-1} \left(\frac{\theta^{n+2}}{n+2} + \frac{2\theta^{n+2}}{n+1} + \frac{\theta^{n+2}}{n} \right) - \frac{1}{n} \left(\frac{\theta^{n+2}}{n+2} + \frac{\theta^{n+2}}{n+1} \right) \right\} \\
&= n \left(\frac{1}{n+2} + \frac{2}{n+1} + \frac{1}{n} \right) \theta^2 - (n-1) \left(\frac{1}{n+2} + \frac{1}{n+1} \right) \theta^2 \\
&= \left(2 + \frac{1}{n+2} \right) \theta^2 \\
&= \frac{2n+5}{n+2} \theta^2
\end{aligned}
$$

$$
\begin{aligned}
Var(Y_1) &= E(Y_1^2) - [E(Y_1)]^2 \\
&= \left(\frac{n}{n+2} - \frac{4n}{n+1} + 4 \right) \theta^2 + \left[\left(\frac{n+2}{n+1} \right) \theta \right]^2 \\
&= \left(\frac{n}{n+2} - \frac{4n}{n+1} + 4 - \left(1 + \frac{1}{n+1} \right)^2 \right) \theta^2 \\
&= \left(\frac{n}{n+2} - \frac{4n}{n+1} + 4 - 1 - \frac{2}{n+1} - \frac{1}{(n+1)^2} \right) \theta^2 \\
&= \left(\frac{n}{n+2} - \frac{4n}{n+1} + 3 - \frac{1}{(n+1)^2} \right) \theta^2 \\
&= \frac{n\theta^2}{(n+1)^2(n+2)}
\end{aligned}
$$

$$
\begin{aligned}
Var(Y_n) &= E(Y_n^2) - [E(Y_n)]^2 \\
&= \left(\frac{n}{n+2} + \frac{2n}{n+1} + 1 \right) \theta^2 - \left(\frac{2n+1}{n+1} \theta \right)^2
\end{aligned}
$$

$$= \left(\frac{n}{n+2} + \frac{2n}{n+1} + 1 - \left(2 - \frac{1}{n+1} \right)^2 \right) \theta^2$$

$$= \left(\frac{n}{n+2} + \frac{2n}{n+1} + 1 - 4 + \frac{4}{n+1} - \frac{1}{(n+1)^2} \right) \theta^2$$

$$= \left(\frac{n}{n+2} + \frac{2n+4}{n+1} - 3 - \frac{1}{(n+1)^2} \right) \theta^2$$

$$= \frac{n\theta^2}{(n+1)^2(n+2)}$$

$$\begin{aligned}
Cov(Y_1, Y_n) &= E(Y_1 Y_n) - E(Y_1)E(Y_n) \\
&= \left(\frac{2n+5}{n+2} \right) \theta^2 - \left[\left(\frac{n+2}{n+1} \right) \theta \left(\frac{2n+1}{n+1} \right) \theta \right] \\
&= \frac{\theta^2}{(n+2)(n+1)^2} [(2n+5)(n+1)^2 - (n+2)(n+2)(2n+1)] \\
&= \frac{\theta^2}{(n+2)(n+1)^2}
\end{aligned}$$

$$\begin{aligned}
Var(U_1) &= Var\left(\frac{n+1}{2n+1} Y_n \right) \\
&= \left(\frac{n+1}{2n+1} \right)^2 Var(Y_n) \\
&= \left(\frac{n+1}{2n+1} \right)^2 \cdot \frac{n\theta^2}{(n+1)^2(n+2)} \\
&= \frac{n\theta^2}{(2n+1)^2(n+2)}
\end{aligned}$$

$$\begin{aligned}
Var(U_2) &= Var[\left(\frac{n+1}{5n+4} \right) (2Y_n + Y_1)] \\
&= \left(\frac{n+1}{5n+4} \right)^2 Var(2Y_n + Y_1) \\
&= \left(\frac{n+1}{5n+4} \right)^2 [Var(Y_1) + 4Cov(Y_1, Y_n) + 4Var(Y_n)] \\
&= \left(\frac{n+1}{5n+4} \right)^2 \left[\frac{n\theta^2}{(n+1)^2(n+2)} + \frac{4\theta^2}{(n+2)(n+1)^2} + \frac{4n\theta^2}{(n+1)^2(n+2)} \right] \\
&= \frac{\theta^2}{(5n+4)^2} \left(\frac{5n+4}{n+2} \right) \\
&= \frac{\theta^2}{(5n+4)(n+2)}
\end{aligned}$$

$$\begin{aligned}
Var(U_1) - Var(U_2) &= \frac{\theta^2}{n+2} \left[\frac{n}{(2n+1)^2} - \frac{1}{5n+4} \right] \\
&= \frac{\theta^2}{n+2} \cdot \frac{n^2 - 1}{(2n+1)^2(5n+4)} \\
&> 0 \qquad (\because \ n^2 - 1 > 0 \text{ for } n > 1)
\end{aligned}$$

$\therefore \ U_2$ is better than U_1 for estimating θ.

27. $f(x, \theta) = \exp\{-\frac{1}{2} \log(2\pi) - \frac{1}{2}\theta^2 - \frac{1}{2}x^2 + \theta x\}$, so $\sum X_i$ is C-S for θ

Note that $\bar{X} \sim N(\theta, \frac{1}{n})$,

so $E(\bar{X}^2) = Var(\bar{X}) + (E(\bar{X}))^2 = \frac{1}{n} + \theta^2 \Rightarrow E(\bar{X}^2 - \frac{1}{n}) = \theta^2$

$\therefore$ $\bar{X}^2 - \frac{1}{n}$ is the UMVUE for θ^2.

The mgf of $\bar{X}$ is $m_{\bar{X}}(t) = \exp\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\}$

$$
\begin{aligned}
m'_{\bar{X}}(t) &= \left(\theta + \frac{t}{n}\right)\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} \\
m''_{\bar{X}}(t) &= \left(\theta + \frac{t}{n}\right)^2\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} + \frac{1}{n}\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} \\
m'''_{\bar{X}}(t) &= \left(\theta + \frac{t}{n}\right)^3\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} + \frac{3}{n}\left(\theta + \frac{t}{n}\right)\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} \\
m^{(4)}_{\bar{X}}(t) &= \left(\theta + \frac{t}{n}\right)^4\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} + 3\left(\theta + \frac{t}{n}\right)^2(\frac{1}{n})\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} \\
&\quad + \frac{3}{n}\left(\theta + \frac{t}{n}\right)^2\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} + \frac{3}{n}\left(\frac{1}{n}\right)\exp\left\{\theta t + \frac{1}{2}(\frac{1}{n})t^2\right\} \\
E(\bar{X}^4) &= m^{(4)}_{\bar{X}}(0) \\
&= \theta^4 + 3\theta^2\left(\frac{1}{n}\right) + \left(\frac{3}{n}\right)\theta^2 + \left(\frac{3}{n}\right)\left(\frac{1}{n}\right) \\
&= \theta^4 + \frac{6}{n}\theta^2 + \frac{3}{n^2} \\
Var(\bar{X}^2 - \frac{1}{n}) &= Var(\bar{X}^2) \\
&= E\bar{X}^4 - (E\bar{X}^2)^2 \\
&= \theta^4 + \frac{6}{n}\theta^2 + \frac{3}{n^2} - (\theta^2 + \frac{1}{n})^2 \\
&= \left(\frac{4}{n}\right)\theta^2 + \frac{2}{n^2} \\
\log f(x;\theta) &= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\theta^2 - \frac{1}{2}x^2 + \theta x \\
\frac{\partial}{\partial\theta}\log f(x;\theta) &= -\theta + x \\
\frac{\partial^2}{\partial\theta^2}\log f(x;\theta) &= -1 \\
E\left[\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)\right] &= -1
\end{aligned}
$$

$$\therefore \text{ CRLB for } \theta^2 = \frac{(2\theta)^2}{n} = \frac{4\theta^2}{n}$$

$$Var(\bar{X}^2 - \frac{1}{n}) = \frac{4\theta^2}{n} + \frac{2}{n^2} > \frac{4\theta^2}{n} = \text{CRLB for } \theta^2$$

28. Recall that $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ and $\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \bar{X})^2 \sim \chi^2_{(n-1)}$ are independent.

Note:

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2 \sim \chi^2_{(n)}$$

$$\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \sim \chi^2_{(n-1)}$$

Let $S = \sum_{i=1}^{n} (X_i - \bar{X})^2$ and $Y = \frac{S}{\sigma^2} \sim \chi^2_{(n-1)}$

$$\therefore \; f_Y(y) \;=\; \frac{y^{\frac{n-1}{2}-1} \cdot e^{-y/2}}{2^{\frac{n-1}{2}} \cdot \Gamma(\frac{n-1}{2})}$$

$$\therefore \; E\left(\frac{1}{\sqrt{S}}\right) \;=\; E\left(\frac{1}{\sigma\sqrt{Y}}\right)$$

$$=\; \int_0^{\infty} \frac{1}{\sigma\sqrt{y}} \, f_Y(y) \; dy$$

$$=\; \int_0^{\infty} \frac{1}{\sigma\sqrt{y}} \cdot \frac{y^{\frac{n-1}{2}-1} \cdot e^{-y/2}}{2^{\frac{n-1}{2}} \cdot \Gamma(\frac{n-1}{2})} \; dy$$

$$=\; \int_0^{\infty} \frac{1}{\sigma} \cdot \frac{y^{\frac{n-2}{2}-1} \cdot e^{-y/2}}{2^{\frac{n-1}{2}} \cdot \Gamma(\frac{n-1}{2})} \; dy$$

$$=\; \frac{2^{-1/2} \cdot \Gamma(\frac{n-2}{2})}{\sigma \cdot \Gamma(\frac{n-1}{2})} \int_0^{\infty} \frac{y^{\frac{n-2}{2}-1} \cdot e^{-y/2}}{2^{\frac{n-2}{2}} \cdot \Gamma(\frac{n-2}{2})} \; dy$$

$$=\; \frac{1}{\sigma} \frac{\Gamma(\frac{n-2}{2})}{\sqrt{2} \cdot \Gamma(\frac{n-1}{2})} \quad \text{pdf of } \chi^2_{(n-2)}$$

$$\therefore \; E\left[\left(\frac{\bar{X}}{\sqrt{S}}\right) \cdot \frac{\sqrt{2} \cdot \Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\right]$$

$$=\; E(\bar{X}) \cdot E\left[\frac{1}{\sqrt{S}} \cdot \frac{\sqrt{2} \cdot \Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}\right] \quad \because \; \bar{X} \text{ and } S \text{ are independent}$$

$$=\; \mu \cdot \frac{1}{\sigma}$$

$$=\; \frac{\mu}{\sigma}$$

$$\therefore \; \frac{\sqrt{2} \cdot \Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})} \cdot \left(\frac{\bar{X}}{\sqrt{S}}\right) = \frac{\sqrt{2} \cdot \Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})} \cdot \left(\frac{\bar{X}}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2}}\right)$$

is the UMVUE for $\frac{\mu}{\sigma}$.

29.

$$E(U) \;=\; E[c\bar{X} + (1-c)\bar{Y}]$$

$$=\; cE(\bar{X}) + (1-c)E(\bar{Y})$$

$$=\; cE(X) + (1-c)E(Y)$$

$$=\; c\theta + (1-c)\theta$$

$$=\; \theta$$

$$\therefore \; U \text{ is an unbiased estimator of } \theta$$

$$
\begin{aligned}
Var(U) \quad &= \quad Var(c\bar{X} + (1-c)\bar{Y}) \\
&= \quad c^2 Var(\bar{X}) + (1-c)^2 Var(\bar{Y}) \quad (X, Y \text{ ar independent}) \\
&= \quad c^2 \left(\frac{\sigma_1^2}{m} \right) + (1-c)^2 \left(\frac{\sigma_2^2}{n} \right) \\
\text{Let } g(c) \quad &= \quad c^2 \left(\frac{\sigma_1^2}{m} \right) + (1-c)^2 \left(\frac{\sigma_2^2}{n} \right) \\
&= \quad c^2 \left(\frac{\sigma_1^2}{m} \right) + (1 - 2c + c^2) \left(\frac{\sigma_2^2}{n} \right) \\
g'(c) \quad &= \quad 2c \left(\frac{\sigma_1^2}{m} \right) + (2c - 2) \left(\frac{\sigma_2^2}{n} \right) \\
g'(c) = 0 \quad \Rightarrow \quad 2c & \left(\frac{\sigma_1^2}{m} \right) + (2c - 2) \left(\frac{\sigma_2^2}{n} \right) = 0 \\
\Rightarrow \quad & \left(\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n} \right) c = \frac{\sigma_2^2}{n} \\
\Rightarrow \quad c &= \frac{\frac{\sigma_2^2}{n}}{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}
\end{aligned}
$$