The Hong Kong University of Science & Technology

MATH243 – Statistical Inference

Final Examination – Fall 2001/2002

Answer ALL questions | **Date: 14 December 2001 (Fri)**

All Equal Marks | **Time allowed: 3 Hours**

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1. (a) Let $X_1, ..., X_m$ and $Y_1, ..., Y_n$ be two independent samples from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$ populations, respectively. Find the maximum likelihood estimators of μ_1, μ_2 and σ^2.

   (b) Calculate the mean squared errors of the estimators S_{n-1}^2 and $\tilde{\sigma}^2 = (n-1)\, S_{n-1}^2 / (n+1)$ for σ^2 when $X_1, ..., X_n$ is a sample from $N(\mu, \sigma^2)$.

2. Let $X_1, ..., X_n$ be independent r.v.s's distributed as $U(0, \theta)$, $\theta \in \Omega = (0, \infty)$. Find unbiased estimators of the mean and variance of the X's depending only on a sufficient statistic for θ.

3. Let $X_1, ..., X_n$ be independent r.v.s's distributed as $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown. Find the UMVU estimator of μ / σ.

4. Let $X_1, ..., X_n$ be a random sample from the Bernoulli distribution, say $P[X = 1] = \theta = 1 - P\,[X = 0]$, $\theta \in (0, 1)$.

   (i) Find the Cramer-Rao lower bound for the variance of unbiased estimators of $\theta(1 - \theta)$.

   (ii) Find the UMVUE of $\theta(1 - \theta)$ if such exists.

5. Let $X_1, ..., X_m$ and $Y_1, ..., Y_n$ be two independent random sample with p.d.f.'s f_1 and f_2, respectively, given below

$$f_1(x;\theta_1) = \frac{1}{\theta_1} \exp\left(-\frac{x}{\theta_1}\right) I_{(0,\infty)}(x), \quad \theta_1 \in \Omega = (0, \infty)$$

$$f_2(y;\theta_2) = \frac{1}{\theta_2} \exp\left(-\frac{y}{\theta_2}\right) I_{(0,\infty)}(y), \quad \theta_2 \in \Omega$$

   (i) Derive the likelihood ratio test for testing $H_0 : \theta_1 = \theta_2$ against $H_1 : \theta_1 \neq \theta_2$.

   (ii) Reduce this test to an F-test at significance level α and indicate how the power of the test can be computed by means of the F tables.

6. Course work grades are often assumed to be normally distributed. In a certain class, suppose that letter grades are given in the following manner: A for grades in (89, 100], B for grades in (74, 89], C for grades in (59, 74], D for grades in (49, 59] and F for grades in [0, 49]. Use the data given below to check the assumption that the data is coming from a N(75, 81) distribution. For this purpose, employ the appropriate χ^2 test and take $\alpha = 0.05$.

A	B	C	D	F
3	12	10	4	1

7. Let $X_1, ..., X_n$ be independent r.v.s's with p.d.f. f given by

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta} I_{(0,\infty)}(x), \quad \theta \in \Omega = (0, \infty)$$

(i) Derive the UMP test for testing the hypothesis $H_0 : \theta \geq \theta_0$ against the alternative $H_1 : \theta < \theta_0$ at level of significance α.

(ii) Determine the minimum sample size n required to obtain power at least 0.95 against the alternative $\theta_1 = 250$ when $\theta_0 = 1{,}000$ and $\alpha = 0.05$.

8. Let X be a r.v. distributed as $B(n, \theta), \theta \in \Omega = (0, 1)$.

(i) Derive the UMP test for testing the hypothesis $H_0 : \theta \leq \theta_0$ against the alternative $H_1 : \theta > \theta_0$ at level of significant α.

(ii) How does the test in (i) become for $n = 10$, $\theta_0 = 0.25$ and $\alpha = 0.05$?

(iii) Compute the power at $\theta_1 = 0.375, 0.625, 0.875$.

(iv) Let now $\theta_0 = 0.125$ and $\alpha = 0.1$ and suppose that we are interested in securing power at least 0.9 against the alternative $\theta_1 = 0.25$. Determine the minimum sample size n required by using the CLT.

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