Solutions to Exercise 2

1.

$$E(\tilde{\theta}_1) = E(\frac{1}{n+1}\sum_{i=1}^{n} X_i) = \frac{1}{n+1}\sum_{i=1}^{n} E(X_i) = \frac{1}{n+1} n\theta \neq \theta$$

$\therefore\ \tilde{\theta}_1$ is a biased estimator for θ.

2. (a)

$$\begin{aligned}
L &= f_x(x;\theta) \\
&= \prod_{i=1}^{n} f_{x_i}(x_i;\theta) \quad x_i \in (0,\infty) \\
&= \prod_{i=1}^{n} \frac{\beta}{\theta} x_i^{\beta-1} \exp(\frac{-x_i^{\beta}}{\theta}) \\
\log L &= n\log\frac{\beta}{\theta} + (\beta-1)\sum_{i=1}^{n}\log x_i - \frac{1}{\theta}\sum_{i=1}^{n} x_i^{\beta} \\
\frac{\partial \log L}{\partial \theta} &= \frac{-n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i^{\beta}
\end{aligned}$$

Set $\dfrac{\partial \log L}{\partial \theta} = 0$,

$$\Rightarrow \frac{-n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i^{\beta} = 0$$

$$\Rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i^{\beta}$$

(b) Let $Y_i = X_i^{\beta} \Rightarrow \frac{dy}{dx} = \beta X^{\beta-1}$

$$\begin{aligned}
\therefore\ f_Y(y;\theta) &= f_X(x;\theta)|\frac{dx}{dy}| \\
&= f_x(\sqrt[\beta]{y};\theta)|\beta x^{\beta-1}|^{-1} \\
&= \frac{\beta}{\theta}(x)^{\beta-1}\exp(\frac{-(\sqrt[\beta]{y})^{\beta}}{\theta}) \cdot |\beta X^{\beta-1}|^{-1} \\
&= \frac{1}{\theta}\exp(\frac{-y}{\theta}), \quad y \in (0,\infty)
\end{aligned}$$

which is the pdf of exponential distribution with parameter $\frac{1}{\theta}$.

$$\therefore\ E(\hat{\theta}) = E(\frac{1}{n}\sum_{i=1}^{n} X_i^{\beta}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i^{\beta}) = \frac{1}{n}E(Y_i) = \frac{1}{n}\cdot n\cdot\theta = \theta$$

$\therefore\ \hat{\theta}$ is unbiased for θ.
$\because\ Y \sim Exponential(\frac{1}{\theta})$

$\therefore\ E(Y)=\theta, \quad Var(Y)=\theta^2$

$$\begin{aligned} Var(\hat{\theta}) &= Var(\frac{1}{n}\sum_{i=1}^{n}X_i^{\beta}) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}Var(X_i^{\beta}) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}Var(Y_i) \\ &= \frac{1}{n^2}\cdot n\cdot\theta^2 \\ &= \frac{\theta^2}{n} \end{aligned}$$

$$\because\ \lim_{n\to\infty}Var(\hat{\theta})=\lim_{n\to\infty}\frac{\theta^2}{n}=0$$

$$\therefore\ \hat{\theta}\text{ is consistent for }\theta.$$

3. (a)

$$M_1'=\frac{1}{n}\sum_{i=1}^{n}X_i=E(\tilde{x}=\frac{1}{\tilde{p}}=\bar{x})$$

$$\therefore\ \tilde{p}=\frac{1}{\bar{x}}$$

(b) Note that $\bar{x}$ is the sample average of how many trials required to get the first success, so intuitively, probability of success should be approximately equal to $\frac{1}{\bar{x}}$.

(c) $\bar{x}=\frac{1}{20}(3+34+7+\ldots+21+15+16)=\frac{252}{20}=12.6$

$\therefore\ \tilde{p}=\frac{1}{\bar{x}}=\frac{1}{12.6}=0.0794$

4. (a)

$$M_1'=\frac{1}{n}\sum_{i=1}^{n}X_i=E(\tilde{x})=\frac{(\tilde{\theta}-1)+(\tilde{\theta}+1)}{2}=\tilde{\theta}=\bar{x}$$

$$\therefore\ \tilde{\theta}=\bar{x}$$

(b) $E(\bar{x})=E(x)=\theta$ therefore $\bar{x}$ is unbiased estimator for θ.

(c) $\tilde{\theta}=\bar{x}=\frac{1}{5}(6.61+\ldots+7.26)=7.382)$

(d) $\frac{1}{2}[\min(X_i)+\max(X_i)]=\frac{1}{2}(6.61+8.36)=7.485$

5.

$$\begin{aligned} E(Y) &= E\Big[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\Big] \\ &= \frac{1}{n}E\Big(\sum_{i=1}^{n}(X_i-\mu)^2\Big) \\ &= \frac{1}{n}\cdot n\ Var(X_i) \\ &= \theta \end{aligned}$$

$\therefore$ Y is unbiased estimator for θ.

6. (a)

$$M_1' = \frac{1}{n}\sum_{i=1}^{n} X_i = \widetilde{E(X)} = \tilde{\lambda}$$

$$\therefore\ \tilde{\lambda} = \bar{X}$$

(b) $\tilde{\lambda} = \bar{X} = \frac{1}{11}(1+0+\ldots+1+1) = \frac{18}{11} = 1.636$

(c) $\bar{X} = 1.636$

$S_{n-1}^2 = \frac{1}{11-1}\sum^{1} 1_{i=1}(X_i - \bar{X})^2 = \frac{1}{10}[(1-1.636)^2 + \ldots + (1-1.636)^2] = 1.6545$

Since the difference between $\bar{X}$ and S_{n-1}^2 is quite small ($\bar{X} \approx S_{n-1}^2$), this information supports the assumption that X ~Poission.

7. X ~Hypergeometric Distribution with n_1 = size of population 1 (orange balls), N = total population (orange and blue balls)= 64, r = sample size = 8

$$M_1' = \frac{1}{30}\sum_{i=1}^{30} X_i = 8(\frac{\tilde{n_1}}{64}) = \frac{\tilde{n_1}}{8}$$

$\therefore \tilde{n_1} = 8\bar{X} = \frac{8}{30}(3+0+\ldots+1+2) = 11.73$,

$\therefore$ we guess the value of n_1 is 12 by the method of moments.

8.

$$\begin{aligned}
L &= f_{\underset{\sim}{X}}(\underset{\sim}{X};\theta) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} \exp\{-\frac{1}{2\theta}(X_i-\mu)^2\} \\
\log L &= \frac{-n}{2}\log 2\pi - \frac{n}{2}\log\theta - \frac{1}{2\theta}\sum_{i=1}^{n}(X_i-\mu)^2 \\
\frac{\partial}{\partial\theta}\log L = 0 &\Rightarrow \frac{-n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n}(X_i-\mu)^2 = 0 \\
&\Rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 \quad \text{which is MLE for } \theta
\end{aligned}$$

9. (a)

$$\begin{aligned}
f(x;\theta) &= \frac{1}{\theta^2}x\exp\{-\frac{x}{\theta}\}, 0 < x < \infty, 0 < \theta < \infty \\
L &= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \prod_{i=1}^{n} \frac{x_i}{\theta^2}\exp\{-\frac{x_i}{\theta}\} \\
&= (\frac{1}{\theta})^{2n}\prod_{i=1}^{n} x_i \exp\{-\sum_{i=1}^{n}\frac{x_i}{\theta}\} \\
\log L &= -2n\log\theta + \sum_{i=1}^{n}\log x_i - \sum\frac{x_i}{\theta} \\
\frac{\partial}{\partial\theta}\log L &= \frac{-2n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i = 0
\end{aligned}$$

$$\Rightarrow \quad \hat{\theta} = \frac{1}{2n}\sum_{i=1}^{n} x_i \quad \text{which is MLE for } \theta$$

(b)

$$\begin{aligned}
f(x;\theta) &= \frac{1}{2\theta^3}x^2\exp\{\frac{-x}{\theta}\}, 0 < x < \infty, 0 < \theta < \infty \\
L &= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \prod_{i=1}^{n} \frac{1}{2\theta^3}x_i^2\exp\{-\frac{x_i}{\theta}\} \\
&= (\frac{1}{\theta^3})^n \prod_{i=1}^{n} x_i \exp\{-\sum_{i=1}^{n}\frac{x_i}{\theta}\} \\
\log L &= -n\log 2 - 3n\log\theta + \sum_{i=1}^{n}\log x_i^2 - \sum\frac{x_i}{\theta} \\
\frac{\partial}{\partial\theta}\log L &= \frac{-3n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i = 0 \\
&\Rightarrow \quad \hat{\theta} = \frac{1}{3n}\sum_{i=1}^{n} x_i \quad \text{which is MLE for } \theta
\end{aligned}$$

(c)

$$\begin{aligned}
f(x;\theta) &= \frac{1}{2}\exp^{-|x-\theta|}, -\infty < x < \infty, -\infty < \theta < \infty \\
L &= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \prod_{i=1}^{n} \frac{1}{2}\exp^{-|x-\theta|} \\
&= \frac{1}{2^n}\exp\left(-\sum_{i=1}^{n}|x_i-\theta|\right) \\
\log L &= -n\log 2 - \sum_{i=1}^{n}|x_i-\theta|
\end{aligned}$$

In order to maximize L, we should maximize $-\sum_{i=1}^{n}|x_i-\theta|$, i.e. we want to minimize $\sum_{i=1}^{n}|x_i-\theta|$.
Since

$$\frac{d|x_i-\theta|}{d\theta} = \begin{cases} 1 & x_i < \theta \\ -1 & x_i > \theta \end{cases},$$

Therefore $\frac{d\log L}{d\theta} = 0$ if there are as equal number of observations <u>above θ</u> and <u>below θ</u>.
So $\hat{\theta}$ should be equal to the <u>median</u> of $(X_1, \ldots, X_n)$..

10. (a)

$$\begin{aligned}
f(x;\theta) &= \frac{1}{\theta}x^{\frac{1-\theta}{\theta}}, \quad 0 < x < 1, 0 < \theta < \infty \\
L &= \prod_{i=1}^{n} f(x_i;\theta)
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{n} \frac{1}{\theta} x_i^{\frac{1-\theta}{\theta}} \\
&= \left(\frac{1}{\theta}\right)^n \prod_{i=1}^{n} x_i^{\frac{1-\theta}{\theta}} \\
\log L &= n \log(\frac{1}{\theta}) + \frac{1-\theta}{\theta} \sum_{i=1}^{n} \log x_i \\
\frac{\partial}{\partial \theta} \log L &= \frac{-n}{\theta} - \frac{1}{\theta^2} \sum_{i=1}^{n} \log x_i \\
\frac{\partial}{\partial \theta} \log L = 0 &\Rightarrow \frac{n}{\theta} = -\frac{1}{\theta^2} \sum_{i=1}^{n} \log x_i \\
&\Rightarrow \hat{\theta} = \frac{-1}{n} \sum_{i=1}^{n} \log x_i \quad \text{which is MLE of } \theta
\end{aligned}
$$

(b) Let $Y = -\log X, dy/dx = -1/X, x = e^{-y}$

$$
\begin{aligned}
f_Y(y;\theta) &= f_x(e^{-y};\theta) \cdot |-1/x|^{-1} \\
&= \frac{1}{\theta}\, e^{-y(\frac{1-\theta}{\theta})} \cdot |\frac{-1}{e^{-y}}|^{-1} \\
&= \frac{1}{\theta}\, e^{-y(\frac{1-\theta}{\theta})} \cdot e^{-y} \\
&= \frac{1}{\theta}\, e^{-y/\theta}
\end{aligned}
$$

$\therefore Y \sim \exp(\frac{1}{\theta})$,

$$
E(\hat{\theta}) = E(\frac{-1}{n} \sum_{i=1}^{n} \log X_i) = E(\frac{1}{n} \sum_{i=1}^{n} Y_i) = \frac{1}{n} \cdot nE(Y_i) = E(Y_i) = \theta
$$

$\therefore$ $\hat{\theta}$ is an unbiased estimator for θ.

(Remark: for part b, same technique as Q.2b, please see back !)

11. Since $X \sim$ exponential $(\frac{1}{\theta})$,

$$
E(\bar{X}) = E\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{1}{n} \cdot (n\theta) = \theta
$$

$\therefore$ $\bar{X}$ is an unbiased estimator for θ.

$$
Var(\bar{X}) = Var\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) = \frac{1}{n^2} \cdot (n\theta^2) = \frac{\theta^2}{n}
$$

12.

$$
E(X_i^2) = Var(X_i) + [E(X_i)]^2 = \theta + 0^2 = \theta
$$

$$
\therefore\ E\left(\frac{1}{n} \sum_{i=1}^{n} X_i^2\right) = \frac{1}{n} \sum_{i=1}^{n} E(X_i^2) = \frac{1}{n} \cdot (n\theta) = \theta
$$

$\therefore$ $\frac{1}{n} \sum_{i=1}^{n} X_i^2$ is an unbiased of estimator for θ.

Since $X_i \sim iid\ N(0, \theta)$,

$$\frac{X_i - 0}{\sqrt{\theta}} \sim iid\ N(0, 1)$$

$$\Rightarrow \frac{X_i^2}{\theta} \sim iid\ \chi^2_{(1)}$$

$$\Rightarrow Var\left(\frac{X_i^2}{\theta}\right) = 2 \times 1 = 2$$

$$\Rightarrow Var(X_i^2) = 2\theta^2$$

$$\therefore\ Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n^2}Var(\sum_{i=1}^{n} X_i^2) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i^2) = \frac{1}{n^2} \cdot nVar(X_i^2) = \frac{1}{n} \cdot 2\theta^2 = \frac{2\theta^2}{n}$$

13. $Var(Y_1) = 2Var(Y_2)$

    Let $Y = k_1Y_1 + k_2Y_2$.

    Since Y is unbiased for θ,

    $\Rightarrow E(Y) = \theta$
    $\Rightarrow E(k_1Y_1 + k_2Y_2) = \theta$
    $\Rightarrow k_1E(Y_1) + k_2E(Y_2) = \theta$
    $\Rightarrow k_1\theta + k_2\theta = \theta, \qquad \because\ E(Y_1) = E(Y_2) = \theta$
    $\Rightarrow k_1 + k_2 = 1$

$$\begin{aligned}
\therefore\ Var(Y) &= Var(k_1Y_1 + k_2Y_2) \\
&= k_1^2Var(Y_1) + k_2^2Var(Y_2) \\
&= 2k_1^2Var(Y_2) + k_2^2Var(Y_2), \quad \text{since } Var(Y_1) = 2Var(Y_2) \\
&= (2k_1^2 + k_2^2)Var(Y_2) \\
&= [2k_1^2 + (1 - k_1)^2]Var(Y_2), \quad \text{since } k_1 + k_2 = 1 \\
&= (3k_1^2 - 2k_1 + 1)Var(Y_2)
\end{aligned}$$

We want to minimize $Var(Y)$ which is equivalent to minimize $g(k_1) = 3k_1^2 - 2k_1 + 1$

$$g'(k_1) = 6k_1 - 2$$

$$g'(k_1) = 0 \Rightarrow k_1 = 1/3$$

$$k_2 = 1 - 1/3 = 2/3$$

$$\therefore Y = \frac{1}{3}Y_1 + \frac{2}{3}Y_2$$

14.

$$X \sim Bin(n; \theta), \quad E(X) = n\theta, \quad Var(X) = n\theta(1 - \theta)$$

$$\begin{aligned}
E\left(n\frac{X}{n}\left(1-\frac{X}{n}\right)\right) &= E\left(X-\frac{X^2}{n}\right)\\
&= E(X)-\frac{1}{n}E(X^2)\\
&= n\theta-\frac{1}{n}[Var(X)+[E(X)]^2]\\
&= n\theta-\frac{1}{n}[n\theta(1-\theta)+(n\theta)^2]\\
&= n\theta-\theta(1-\theta)-n\theta^2\\
&= \theta(n-1+\theta-n\theta)\\
&= (n-1)\theta(1-\theta)\\
&\neq n\theta(1-\theta)=Var(X)
\end{aligned}$$

$\therefore\ n(\frac{X}{n})(1-\frac{X}{n})$ is biased for $Var(X)=n\theta(1-\theta)$

15.

$$E\left(\frac{X_1+2X_2+X_3}{4}\right)=\frac{1}{4}\left[E(X_1)+2E(X_2)+E(X_3)\right]=\frac{1}{4}(\mu+2\mu+\mu)=\mu$$

$\therefore\ \frac{X_1+2X_2+X_3}{4}$ is unbiased for μ.

$$E\left(\frac{X_1+X_2+X_3}{3}\right)=\frac{1}{3}\left[E(X_1)+E(X_2)+E(X_3)\right]=\frac{1}{3}(\mu+\mu+\mu)=\mu$$

$\therefore\ \frac{X_1+X_2+X_3}{3}$ is unbiased for μ.

$$Var\left(\frac{X_1+2X_2+X_3}{4}\right)=\frac{1}{16}\left[Var(X_1)+4Var(X_2)+Var(X_3)\right]=\frac{1}{16}(\sigma^2+4\sigma^2+\sigma^2)=\frac{3}{8}\sigma^2$$

$$Var\left(\frac{X_1+X_2+X_3}{3}\right)=\frac{1}{9}\left[Var(X_1)+Var(X_2)+Var(X_3)\right]=\frac{1}{9}(\sigma^2+\sigma^2+\sigma^2)=\frac{1}{3}\sigma^2$$

Therefore the efficiency of $\frac{X_1+2X_2+X_3}{4}$ relative to $\frac{X_1+X_2+X_3}{3}$ is

$$\frac{Var\left(\frac{X_1+X_2+X_3}{3}\right)}{Var\left(\frac{X_1+2X_2+X_3}{4}\right)}=\frac{\frac{1}{3}\sigma^2}{\frac{3}{8}\sigma^2}=\frac{8}{9}$$

16. (a)

$$Var(\hat{\theta_1})=Var(\frac{X}{n})=\frac{1}{n^2}Var(X)=\frac{1}{n^2}\cdot n\theta(1-\theta)=\frac{\theta(1-\theta)}{n}=\frac{1}{4n}\qquad\text{(since }\theta=\frac{1}{2})$$

$$Var(\hat{\theta_2})=Var(\frac{X+1}{n+2})=Var(\frac{X}{n+2})=\frac{1}{(n+2)^2}\cdot n\theta(1-\theta)=\frac{n\theta(1-\theta)}{(n+2)^2}$$

$$
\begin{aligned}
E\left[(\hat{\theta_2}-\theta)^2\right] &= Var(\hat{\theta_2}-\theta)+E\left[(\hat{\theta_2}-\theta)\right]^2 \\
&= Var(\hat{\theta_2})+E\left[(\hat{\theta_2}-\theta)\right]^2 \qquad (\because\ \theta \text{ is constant}) \\
&= \frac{n\theta(1-\theta)}{(n+2)^2}+(\frac{n\theta+1}{n+2}-\theta)^2 \\
&= \frac{n\theta(1-\theta)}{(n+2)^2}+\frac{(1-2\theta)^2}{(n+2)^2} \\
&= \frac{1}{(n+2)^2}[n\theta-n\theta^2+1-4\theta+4\theta^2] \\
&= \frac{1}{(n+2)^2}\times\frac{n}{4} \qquad (\text{since } \theta=\frac{1}{2})
\end{aligned}
$$

Since $E\left[(\hat{\theta_2}-\theta)^2\right] < Var(\hat{\theta_1})$, we have

$$\frac{n}{4(n+2)^2} < \frac{1}{4n}$$

$$\Rightarrow n^2 < (n+2)^2$$

$$\Rightarrow n^2 < n^2+4n+4$$

$$\Rightarrow 4n+4 > 0$$

$$\Rightarrow n > -1$$

(b)

$$E\left[(\hat{\theta_3}-\theta)^2\right] = \left[(\frac{1}{3}-\frac{1}{2})^2\right] = \frac{1}{36}$$

$$\therefore\ E\left[(\hat{\theta_3}-\theta)^2\right] < Var(\hat{\theta_1})$$

$$\Rightarrow \frac{1}{36} < \frac{1}{4n}$$
$$\Rightarrow n < 9$$

17.

$$M_1' = \frac{1}{n}\sum_{i=1}^{n} X_i = \widetilde{E(X)} = \tilde{\lambda}$$

$$\Rightarrow \tilde{\lambda} = \bar{X}$$

18. The n independent measurements of the radius r of the circle are

$$R_1, R_2, \ldots, R_n \sim iid\ N(r, \sigma^2)$$

If we can find an unbiased estimator of r^2, we can find an unbiased estimator of πr^2 (area). Observed that

$$\begin{aligned}
E\left[\left(\frac{1}{n}\sum_{i=1}^{n} R_i\right)^2\right] &= E(\bar{R}^2) \\
&= Var(\bar{R}) + E\left[(\bar{R})\right]^2 \\
&= \frac{Var(R)}{n} + E\left[(R)\right]^2 \\
&= \frac{\sigma^2}{n} + r^2
\end{aligned}$$

Also, $E\left[\frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R})^2\right] = E(S_{n-1}^2) = \sigma^2$.

$\left(\frac{1}{n}\sum_{i=1}^{n} R_i\right)^2 - \frac{1}{n}(\frac{1}{n-1})\sum_{i=1}^{n}(R_i - \bar{R})^2$ is an unbiased estimator for r^2 since

$$E\left[\left(\frac{1}{n}\sum_{i=1}^{n} R_i\right)^2\right] - \frac{1}{n}(\frac{1}{n-1})E\left[\sum_{i=1}^{n}(R_i - \bar{R})^2\right] = \left(\frac{\sigma^2}{n} + r^2\right) - \frac{1}{n}\cdot\sigma^2 = r^2,$$

$$\therefore\ \pi\left[\left(\frac{1}{n}\sum_{i=1}^{n} R_i\right)^2 - \frac{1}{n}(\frac{1}{n-1})\sum_{i=1}^{n}(R_i - \bar{R})^2\right]$$

is an unbiased estimator for πr^2 (area).

19. X_i's independent $N(\mu, \sigma_i^2)$, $i = 1, 2, \ldots, n$

It is NOT possible to estimate all the parameters from the n observations.

Now assume σ_i^2 are known, $i = 1, \ldots, n$

$$\begin{aligned}
f_{X_i}(x_i; \mu) &= \frac{1}{\sqrt{2\pi}\sigma_i}\exp\left\{-\frac{1}{2\sigma_i^2}(x_i - \mu)^2\right\} \\
L &= f_{\underset{\sim}{X}}(\underset{\sim}{X}; \mu) \\
&= \prod_{i=1}^{n} f_{X_i}(x_i; \mu) \\
&= \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma_i}\exp\left\{-\frac{1}{2\sigma_i^2}(x_i - \mu)^2\right\} \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^n\left(\prod_{i=1}^{n}\frac{1}{\sigma_i}\right)\exp\left\{-\sum_{i=1}^{n}\frac{1}{2\sigma_i^2}(x_i - \mu)^2\right\} \\
\log L &= \frac{-n}{2}\log(2\pi) - \sum_{i=1}^{n}\log\sigma_i - \sum_{i=1}^{n}\frac{1}{2\sigma_i^2}(x_i - \mu)^2 \\
\frac{\partial}{\partial\mu}\log L &= -\sum_{i=1}^{n}\frac{1}{2\sigma_i^2}2(x_i - \mu)(-1) \\
&= \sum_{i=1}^{n}\frac{(x_i - \mu)}{\sigma_i^2}
\end{aligned}$$

Set equal to 0, then

$$\sum_{i=1}^{n} \frac{x_i - \mu}{\sigma_i^2} = 0$$

$$\sum_{i=1}^{n} \frac{x_i}{\sigma_i^2} = \hat{\mu} \sum_{i=1}^{n} \frac{1}{\sigma_i^2}$$

$$\hat{\mu} = \sum_{i=1}^{n} \frac{x_i}{\sigma_i^2} / \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \quad \text{which is MLE for} \mu.$$

And $\sum_{i=1}^{n} \frac{1}{\sigma_i^2}$, this is call weighted sample mean.

20. (a)

$$E(X^2) = Var(X) + [E(X)]^2 = \sigma^2 + 0^2 = \sigma^2$$

$\therefore$ X^2 is an unbiased estimator of σ^2.

(b)

$$\begin{aligned} L &= f_X(x;\theta) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{\frac{-1}{2\sigma^2}x^2\right\} \\ \log L &= \frac{-1}{2}\log(2\pi) - \frac{1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}x^2 \\ \frac{\partial}{\partial\sigma^2}\log L &= -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}x^2 \end{aligned}$$

Set equal to 0, we get

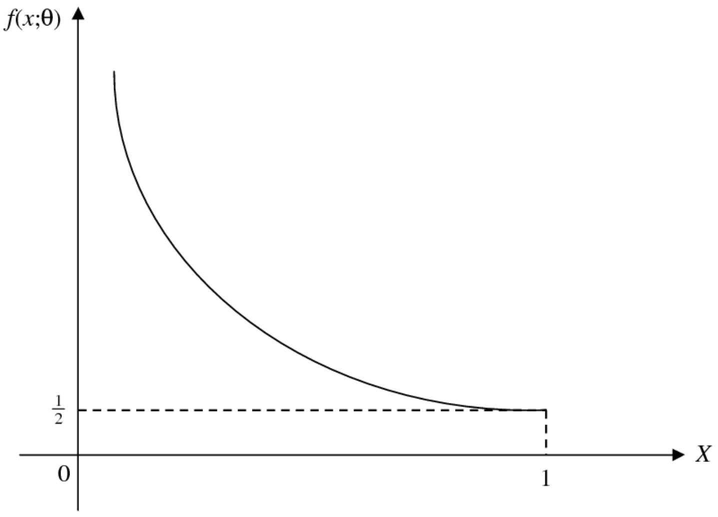
$$\frac{-1}{2\sigma^2} + \frac{1}{2\sigma^4}x^2 = 0$$

$$\Rightarrow \hat{\sigma}^2 = x^2$$

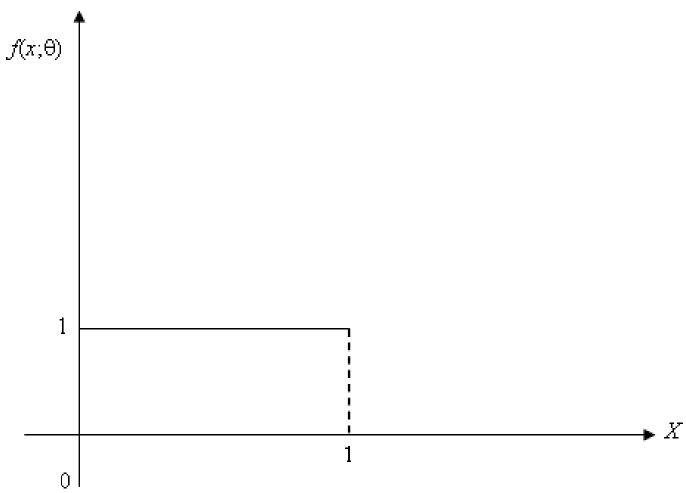
$\therefore$ The MLE for σ is $\hat{\sigma} = \sqrt{x^2} = |X|$.

(c) $M_2' = \widetilde{E(X^2)} = \tilde{\sigma}^2$
Also, $\widetilde{E(X^2)} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 = X^2 \quad (\because\ n = 1)$
$\therefore$ $\hat{\sigma} = \sqrt{X^2} = |X|$.
(Method of moment estimator is equal to MLE in this question.)

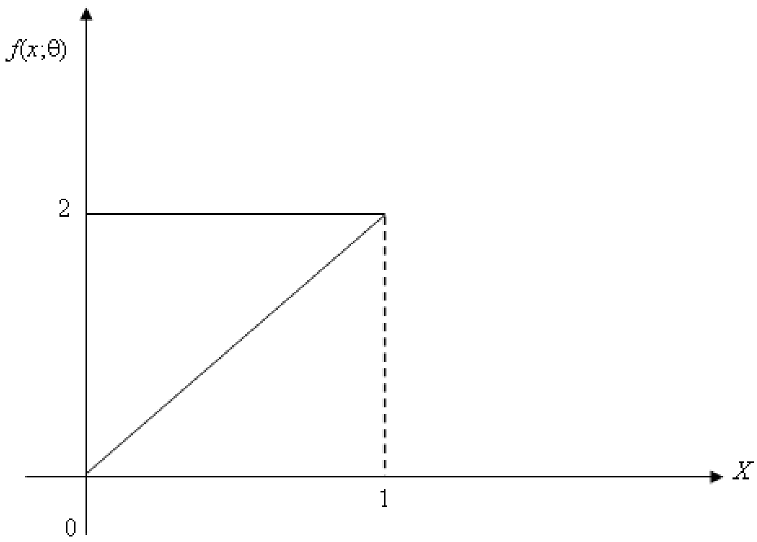
21. (a) (i) $\theta = \frac{1}{2}$,
$f(x;\theta) = \frac{1}{2}X^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$



(ii) $\theta = 1$,
$f(x;\theta) = 1 \cdot x^{1-1} = 1$



(iii) $\theta = 1$,
$f(x;\theta) = 2x^{2-1} = 2x$



(b)

$$\begin{aligned}
L &= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \prod_{i=1}^{n} \theta x_i^{\theta-1} \\
&= \theta^n \prod_{i=1}^{n} x_i^{\theta-1} \\
\log L &= n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i \\
\frac{\partial}{\partial\theta}\log L &= \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i
\end{aligned}$$

Set equal to 0, we get

$$\frac{n}{\theta} + \sum_{i=1}^{n}\log x_i = 0$$

$$\hat{\theta} = \frac{-n}{\sum_{i=1}^{n}\log x_i} = \frac{-n}{\log\prod_{i=1}^{n} x_i}$$

(c) (i)

$$\log\prod_{i=1}^{10} x_i = -18.21 \Rightarrow \hat{\theta} = -\frac{10}{-18.21} = 0.549$$

(ii)

$$\log\prod_{i=1}^{10} x_i = -4.52 \Rightarrow \hat{\theta} = -\frac{10}{-4.52} = 2.212$$

(iii)

$$\log \prod_{i=1}^{10} x_i = -10.43 \Rightarrow \hat{\theta} = -\frac{10}{-10.43} = 0.959$$

22. (a) $f_{Y_1}(y_1;\theta) = 3[1-F_X(y_1)]^2 f_X(y_1;\theta) = 3(1-\frac{y_1}{\theta})^2(\frac{1}{\theta})$,

$$\begin{aligned}
E(Y_1) &= \int_0^\theta y_1 f_{Y_1}(y_1;\theta)dy_1 \\
&= \int_0^\theta y_1 \left(\frac{3}{\theta}\right)\left(1-\frac{y_1}{\theta}\right)^2 dy_1 \\
&= \frac{3}{\theta^3}\int_0^\theta y_1(\theta-y_1)^2 dy_1 \\
&= \frac{3}{\theta^3}\int_0^\theta (y_1^3 - 2\theta y_1^2 + \theta^2 y_1)dy_1 \\
&= \frac{3}{\theta^3}\left[\frac{y_1^4}{4} - \frac{2\theta y_1^3}{3} + \frac{\theta^2 y_1^2}{2}\right]_0^\theta \\
&= \frac{3}{\theta^3}\left[\frac{\theta^4}{4} - \frac{2\theta^4}{3} + \frac{\theta^4}{2}\right] \\
&= \frac{\theta}{4}
\end{aligned}$$

$$\therefore\ E(4Y_1) = 4E(Y_1) = 4\cdot\frac{\theta}{4} = \theta$$

Therefore $4Y_1$ is unbiased for θ.

$$\begin{aligned}
E(Y_1^2) &= \int_0^\theta y_1^2 f_{Y_1}(y_1;\theta)\ dy_1 \\
&= \int_0^\theta y_1^2 \left(\frac{3}{\theta}\right)\left(1-\frac{y_1}{\theta}\right)^2 dy_1 \\
&= \frac{3}{\theta^3}\int_0^\theta y_1^2(\theta-y_1)^2 dy_1 \\
&= \frac{3}{\theta^3}\int_0^\theta (y_1^4 - 2\theta y_1^3 + \theta^2 y_1^2)\ dy_1 \\
&= \frac{3}{\theta^3}\left[\frac{y_1^5}{5} - \frac{2}{4}\theta y_1^4 + \frac{1}{3}\theta^2 y_1^3\right]_0^\theta \\
&= \frac{3}{\theta^3}\left[\frac{\theta^5}{5} - \frac{\theta^5}{2} + \frac{\theta^5}{3}\right] \\
&= \frac{\theta^2}{10}
\end{aligned}$$

$$\begin{aligned}
Var(Y_1) &= E(Y_1^2) - [E(Y_1^2)]^2 \\
&= \frac{\theta^2}{10} - \left(\frac{\theta}{4}\right)^2 \\
&= \frac{3}{80}
\end{aligned}$$

$$Var(4Y_1) = 16Var(Y_1) = 16 \times \frac{3}{80}\theta^2 = \frac{3}{5}\theta^2$$

(b)

$$f_{Y_2}(y_2;\theta) = 3![F_X(y_2)]f(y_2)[1-F_X(y_2)] = 6(\frac{y_2}{\theta})(\frac{1}{\theta})(1-\frac{y_2}{\theta}) = \frac{6}{\theta^3}(\theta y_2 - y_2^2), \quad 0 < y_2 < \theta$$

$$\begin{aligned}
E(Y_2) &= \int_0^\theta y_2 f_{Y_2}(y_2;\theta)dy_2 \\
&= \int_0^\theta y_1\left(\frac{3}{\theta}\right)\left(1-\frac{y_1}{\theta}\right)^2 dy_1 \\
&= \int_0^\theta y_2\frac{6}{\theta^3}(\theta y_2 - y_2^2)dy_2 \\
&= \frac{6}{\theta^3}\int_0^\theta(\theta y_2^2 - y_2^3)dy_2 \\
&= \frac{6}{\theta^3}\left[\frac{1}{3}\theta y_2^3 - \frac{1}{4}y_2^4\right]_0^\theta \\
&= \frac{6}{\theta^3}\left[\frac{\theta^4}{3} - \frac{\theta^4}{4}\right] \\
&= \frac{\theta}{2}
\end{aligned}$$

$$\therefore\ E(2Y_2) = 2E(Y_2) = 2(\frac{\theta}{2}) = \theta$$

Therefore $2Y_2$ is unbiased for θ.

$$\begin{aligned}
E(Y_2^2) &= \int_0^\theta y_2^2 f_{Y_2}(y_2;\theta)\ dy_2 \\
&= \int_0^\theta \frac{6y_2^2}{\theta^3}\left(\theta y_2 - y_2^2\right) dy_2 \\
&= \frac{6}{\theta^3}\int_0^\theta(\theta y_2^3 - y_2^4)dy_2 \\
&= \frac{6}{\theta^3}\left[\frac{1}{4}\theta y_2^4 - \frac{1}{5}y_2^5\right]_0^\theta \\
&= \frac{6}{\theta^3}\left[\frac{\theta^5}{4} - \frac{\theta^5}{5}\right] \\
&= \frac{3}{10}\theta^2 \\
\therefore\ Var(Y_2) &= E(Y_2^2) - [E(Y_2^2)]^2 \\
&= \frac{3}{10}\theta^2 - \left(\frac{1}{2}\theta\right)^2 \\
&= \frac{1}{20}\theta^2
\end{aligned}$$

$$\therefore\ Var(2Y_2) = 4Var(Y_2) = 4(\frac{1}{20}\theta^2) = \frac{1}{5}\theta^2$$

(c)

$$f_{Y_3}(y_3;\theta) = 3[F_X(y_3)]^2 f_X(y_3) = 3(\frac{y_3}{\theta})^2(\frac{1}{\theta}), \quad 0 < y_3 < \theta$$

$$\begin{aligned} E(Y_3) &= \int_0^\theta y_3 \frac{3}{\theta^3} y_3^2 dy_3 \\ &= \frac{3}{\theta^3}\int_0^\theta y_3^3 dy_3 \\ &= \frac{3}{4\theta^3}[y_3^4]_0^\theta \\ &= \frac{3}{4}\theta \end{aligned}$$

$$\therefore\ E(\frac{4}{3}Y_3) = \frac{4}{3}E(Y_3) = \frac{4}{3}(\frac{3}{4})\theta = \theta$$

Therefore $\frac{4}{3}Y_3$ is unbiased for θ.

$$\begin{aligned} E(Y_3^2) &= \int_0^\theta y_3^2 \left(\frac{3}{\theta^3}\right) y_3^2\ dy_3 \\ &= \frac{3}{\theta^3}\int_0^\theta y_3^4\ dy_3 \\ &= \frac{3}{5\theta^3}\left[y_3^5\right]_0^\theta \\ &= \frac{3}{5}\theta^2 \end{aligned}$$

$$\begin{aligned} \therefore\ Var(Y_3) &= E(Y_3^2) - [E(Y_3^2)]^2 \\ &= \frac{3}{5}\theta^2 - \left(\frac{3}{4}\theta\right)^2 \\ &= \frac{3}{80}\theta^2 \end{aligned}$$

$$\therefore\ Var(\frac{4}{3}Y_3) = (\frac{4}{3})^2 Var(Y_3) = \frac{16}{9}(\frac{3}{80}\theta^2) = \frac{1}{15}\theta^2$$

24. (a) $E(\omega\bar{X}_1 + (1-\omega)\bar{X}_2)$
$= \omega E(\bar{X}_1) + (1-\omega)E(\bar{X}_2)$
$= \omega E(X_i) + (1-\omega)E(X_i)$
$= \omega \cdot \mu + (1-\omega)\cdot \mu$
$= \mu$
$\therefore\ \omega\bar{X}_1 + (1-\omega)\bar{X}_2$ is unbiased for μ.
$Var(\omega\bar{X}_1 + (1-\omega)\bar{X}_2)$

$= \omega^2 Var(\bar{X}_1) + (1-\omega)^2 Var(\bar{X}_2)$

$= \omega^2(\frac{\sigma^2}{n_1}) + (1-\omega)^2(\frac{\sigma^2}{n_2})$

Let
$$g(\omega) = \omega^2(\frac{\sigma^2}{n_1}) + (1-\omega)^2(\frac{\sigma^2}{n_2})$$

$$g'(\omega) = 2\omega(\frac{\sigma^2}{n_1}) - 2(1-\omega)(\frac{\sigma^2}{n_2})$$

$$
\begin{aligned}
g'(\omega) = 0 \quad &\Rightarrow \quad 2\omega(\frac{\sigma^2}{n_1}) - 2(1-\omega)(\frac{\sigma^2}{n_2}) = 0 \\
&\Rightarrow \quad \omega(\frac{1}{n_1} + \frac{1}{n_2}) = \frac{1}{n_2} \\
&\Rightarrow \quad \omega = \frac{n_1}{n_1+n_2}
\end{aligned}
$$

(b) When $\omega = \frac{1}{2}$, let $\hat{\mu}_1 = \frac{1}{2}\bar{X}_1 + \frac{1}{2}\bar{X}_2$,

$$
\Rightarrow Var(\hat{\mu}_1) = \frac{1}{4}\cdot\frac{\sigma^2}{n_1} + \frac{1}{4}\cdot\frac{\sigma^2}{n_2} = \frac{\sigma^2}{4}(\frac{1}{n_1} + \frac{1}{n_2})
$$

When $\omega = \frac{n_1}{n_1+n_2}$, let $\hat{\mu}_2 = \frac{n_1}{n_1+n_2}\bar{X}_1 + \frac{n_2}{n_1+n_2}\bar{X}_2$

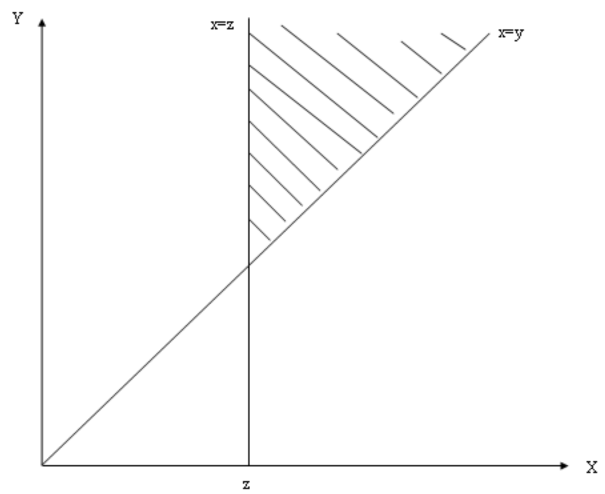
$$
\Rightarrow Var(\hat{\mu}_2) = (\frac{n_1}{n_1+n_2})^2\cdot\frac{\sigma^2}{n_1} + (\frac{n_2}{n_1+n_2})^2\cdot\frac{\sigma^2}{n+2} = \frac{\sigma^2}{(n_1+n_2)^2}(n_1+n_2) = \frac{\sigma^2}{n_1+n_2}
$$

$\therefore$ The efficiency of the estimator with $\omega = \frac{1}{2}$ relative to that with $\omega = \frac{n_1}{n_1+n_2}$

$$
= \frac{Var(\hat{\mu}_2)}{Var(\hat{\mu}_1)} = \frac{\frac{\sigma^2}{n_1+n_2}}{\frac{\sigma^2}{4}(\frac{1}{n_1} + \frac{1}{n_2})} = \frac{4n_1n_2}{(n_1+n_2)^2}
$$

25. In order to find the MLEs of λ and μ, we need to find the joint pdf $f_{Z,W}(z,w;\lambda,\mu)$.
In this question, the pdf technique cannot be used. Why???
So we use cdf technique.

$$
\begin{aligned}
f_{Z,W}(z,w) &= \frac{\partial}{\partial z}P(Z \le z, W = w) \quad (W \text{ is discrete, } Z \text{ is continuous}) \\
&= \frac{\partial}{\partial z}[1 - P(Z > z, W = w)] \\
&= \frac{-\partial}{\partial z}P(Z > z, W = w) \quad w = 0 \ or \ 1 \qquad (1)
\end{aligned}
$$



Now

$$
\begin{aligned}
& P(Z \geq z, W = 1) \\
= \; & P(\min(X, Y) \geq z, W = 1) \\
= \; & P(X \geq z, X = Z) \\
= \; & P(X \geq z, X < Y) \\
= \; & \int_z^\infty \int_x^\infty f_{X,Y}(x, y) \; dy \; dx \\
= \; & \int_z^\infty \int_x^\infty \frac{1}{\lambda} e^{-x/\lambda} \cdot \frac{1}{\mu} e^{-y/\mu} \; dy \; dx \\
= \; & \int_z^\infty \frac{1}{\lambda} e^{-x/\lambda} \cdot \left[-e^{-y/\mu} \right]_x^\infty \; dx \\
= \; & \int_z^\infty \frac{1}{\lambda} \cdot e^{-x/\lambda} \cdot e^{-x/\mu} \; dx \\
= \; & \int_z^\infty \frac{1}{\lambda} e^{-x(\frac{1}{\lambda} + \frac{1}{\mu})} \; dx \\
= \; & -\frac{1}{\lambda} (\frac{1}{\lambda} + \frac{1}{\mu})^{-1} \cdot \left[e^{-x(\frac{1}{\lambda} + \frac{1}{\mu})} \right]_z^\infty \\
= \; & \frac{\lambda \mu}{\lambda(\lambda + \mu)} \cdot e^{-z(\frac{1}{\lambda} + \frac{1}{\mu})} \\
= \; & \frac{\mu}{\lambda + \mu} \cdot e^{-z(\frac{1}{\lambda} + \frac{1}{\mu})}
\end{aligned}
$$

Similarly $P(Z \geq z, W = 0) = \frac{\lambda}{\lambda + \mu} \cdot e^{-z(\frac{1}{\lambda} + \frac{1}{\mu})}$

$$
\begin{aligned}
& \therefore \; P(Z \geq z, W = w) \\
= \; & [P(Z \geq z, W = 1)]^w [P(Z \geq z, W = 0)]^{1-w} \quad (W \sim \text{Bernoulli distribution}) \\
= \; & [\frac{\mu}{\lambda + \mu} \cdot e^{-z(\frac{1}{\lambda} + \frac{1}{\mu})}]^w [\frac{\lambda}{\lambda + \mu} \cdot e^{-z(\frac{1}{\lambda} + \frac{1}{\mu})}]^{1-w} \\
= \; & \frac{1}{\lambda + \mu} \cdot \exp\left\{ \frac{-(\lambda + \mu)}{\lambda \mu} z \right\} \mu^w \lambda^{1-w}
\end{aligned}
$$

$$
\begin{aligned}
f_{Z,W}(z, w) &= \frac{-\partial}{\partial z} P(Z \geq z, W = w) \\
&= \frac{-1}{\lambda + \mu} \cdot \left(\frac{-\lambda + \mu}{\lambda \mu} \right) \exp\left\{ \frac{-\lambda + \mu}{\lambda \mu} z \right\} \mu^w \lambda^{1-w} \\
&= exp\left\{ \frac{-\lambda + \mu}{\lambda \mu} z \right\} \mu^{(w-1)} \lambda^{-w} \\
\therefore \; L &= f_{\underset{\sim}{Z}, \underset{\sim}{W}}(\underset{\sim}{z}, \underset{\sim}{w}; \lambda, \mu) \\
&= \prod_{i=1}^n f_{Z_i, W_i}(z_i, w_i; \lambda, \mu) \\
&= \prod_{i=1}^n exp\left\{ \frac{-\lambda + \mu}{\lambda \mu} z_i \right\} \mu^{(w_i - 1)} \lambda^{-w_i} \\
&= \mu^{\sum_{i=1}^n w_i - n} \lambda^{-\sum_{i=1}^n w_i} \; \exp\left\{ \frac{-\lambda + \mu}{\lambda \mu} z_i \right\} \\
\log L &= (\sum_{i=1}^n w_i - n) \log \mu - (\sum_{i=1}^n w_i) \log \lambda - \frac{\lambda + \mu}{\lambda \mu} \sum_{i=1}^n z_i \\
\frac{\partial}{\partial \lambda} \log L &= \frac{-1}{\lambda} \sum_{i=1}^n w_i + \frac{1}{\lambda^2} \sum_{i=1}^n z_i
\end{aligned}
$$

$$\frac{\partial}{\partial\mu}\log L = \frac{1}{\mu}(\sum_{i=1}^{n} w_i - n) + \frac{1}{\mu^2}\sum_{i=1}^{n} z_i$$

Set to 0 $\Rightarrow$

$$\hat{\lambda} = \frac{\sum_{i=1}^{n} z_i}{\sum_{i=1}^{n} w_i} = \frac{\frac{1}{n}\sum_{i=1}^{n} z_i}{\frac{1}{n}\sum_{i=1}^{n} w_i} = \frac{\bar{Z}}{\bar{W}}$$

$$\hat{\mu} = \frac{\sum_{i=1}^{n} z_i}{(n - \sum_{i=1}^{n} w_i)} = \frac{\frac{1}{n}\sum_{i=1}^{n} z_i}{(1 - \frac{1}{n}\sum_{i=1}^{n} w_i)} = \frac{\bar{Z}}{(1-\bar{W})}$$

26. $f(x|\theta) = \theta x^{\theta-1}$

(a)

$$L = f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta) = \prod_{i=1}^{n} f_{X_i}(x_i|\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n(\prod_{i=1}^{n} x_i)^{\theta-1}$$

$$\log L = n\log\theta + (\theta-1)\log(\prod_{i=1}^{n} x_i) = n\log\theta + (\theta-1)\sum_{i=1}^{n}\log x_i$$

$$\frac{\partial}{\partial\theta}\log L = \frac{n}{\theta} + \sum_{i=1}^{n}\log x_i$$

Set to be 0 $\Rightarrow$

$$\hat{\theta} = \frac{-n}{\sum_{i=1}^{n}\log x_i}$$

which is MLE for θ.
Let $Y = -\log X,\ X = e^{-Y}$,

$$\begin{aligned} f_Y(y) &= f_X(e^{-y}) \cdot \left|\frac{dx}{dy}\right| \\ &= \theta(e^{-y})^{\theta-1} \cdot |-e^{-y}| \\ &= \theta e^{-\theta y} \end{aligned}$$

$\therefore\ Y \sim$ Exponential (θ) and hence

$$W = \sum_{i=1}^{n} Y_i = -\sum_{i=1}^{n}\log x_i \sim Gamma(n,\theta)$$

$$\begin{aligned} E(\frac{1}{W}) &= \int_0^\infty \frac{1}{w}\cdot\frac{w^{n-1}e^{-\theta w}}{\theta^{-n}\Gamma(n)}\ dw \\ &= \frac{\theta}{n-1}\int_0^\infty \frac{w^{n-2}e^{-\theta w}}{\theta^{-(n-1)}\Gamma(n-1)}\ dw \\ &= \frac{\theta}{n-1}\cdot 1 \\ &= \frac{\theta}{n-1} \\ E(\frac{1}{W^2}) &= \int_0^\infty \frac{1}{w^2}\cdot\frac{w^{n-1}e^{-\theta w}}{\theta^{-n}\Gamma(n)}\ dw \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \frac{w^{n-3}e^{-\theta w}}{\theta^{-n}\Gamma(n)}\ dw \\
&= \frac{\theta^2}{(n-1)(n-2)} \int_0^\infty \frac{w^{n-3}e^{-\theta w}}{\theta^{-(n-2)}\Gamma(n-2)}\ dw \\
&= \frac{\theta^2}{(n-1)(n-2)}
\end{aligned}
$$

$$
\begin{aligned}
Var(\hat{\theta}) &= Var(\frac{-n}{\sum_{i=1}^n \log x_i}) \\
&= n^2 Var(\frac{1}{W}) \\
&= n^2 \left[E(\frac{1}{W^2}) - [E(\frac{1}{W})]^2 \right] \\
&= n^2 \left[\frac{\theta^2}{(n-1)(n-2)} - \frac{\theta^2}{(n-1)^2} \right] \\
&= \frac{\theta^2 n^2}{(n-1)^2(n-2)} \\
&= \frac{\frac{\theta^2}{n}}{(1-\frac{1}{n})^2(1-\frac{2}{n})} \\
&\to 0 \quad as \quad n \to \infty
\end{aligned}
$$

(b)

$$
E(X) = \int_0^1 x \cdot \theta x^{\theta-1}\ dx = \int_0^1 \theta x^\theta\ dx = \frac{\theta}{\theta+1}[x^{\theta+1}]_0^1 = \frac{\theta}{\theta+1}
$$

$$
\therefore\ M_1' = \frac{1}{n}\sum_{i=1}^n X_i = \widetilde{E(X)} = \frac{\tilde{\theta}}{\tilde{\theta}+1}
$$

$$
\begin{aligned}
\frac{\tilde{\theta}}{\tilde{\theta}+1} = \bar{X} \quad &\Rightarrow \quad (\tilde{\theta}+1)\bar{X} = \tilde{\theta} \\
&\Rightarrow \quad \tilde{\theta}\bar{X} - \tilde{\theta} = -\bar{X} \\
&= \quad \tilde{\theta} = \frac{\bar{X}}{1-\bar{X}}
\end{aligned}
$$

27. (a) MLE for $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$.

$$
\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix} \right)
$$

$$
f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{(1-\rho^2)}\sigma_X\sigma_Y} \exp\left\{ \frac{-1}{2(1-\rho^2)} \left[(\frac{x-\mu_X}{\sigma_X})^2 + (\frac{y-\mu_Y}{\sigma_Y})^2 - 2\rho(\frac{x-\mu_X}{\sigma_X})(\frac{y-\mu_Y}{\sigma_Y}) \right] \right\}
$$

$$
\begin{aligned}
&L \\
&= f_{\underset{\sim}{X},\underset{\sim}{Y}}(\underset{\sim}{x},\underset{\sim}{y}) \\
&= \prod_{i=1}^n f_{X_i,Y_i}(x_i, y_i; \mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)
\end{aligned}
$$

$$= \prod_{i=1}^{n} \left[4\pi^2(1-\rho^2)\sigma_X^2\sigma_Y^2\right]^{\frac{-1}{2}} \exp\left\{\left[(\frac{x_i-\mu_X}{\sigma_X})^2 + (\frac{y_i-\mu_Y}{\sigma_Y})^2 - 2\rho(\frac{x_i-\mu_X}{\sigma_X})(\frac{y_i-\mu_Y}{\sigma_Y})\right]\right\}$$

$$\begin{aligned} &\log L \\ = \; & \frac{-n}{2}\log(4\pi^2) - \frac{n}{2}\log(1-\rho^2) - \frac{n}{2}\log\sigma_X^2 - \frac{n}{2}\log\sigma_Y^2 \\ & -\frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}(\frac{x_i-\mu_X}{\sigma_X})^2 - \frac{1}{2(1-\rho^2)}\sum_{i=1}^{n}(\frac{y_i-\mu_Y}{\sigma_Y})^2 + \frac{\rho}{1-\rho^2}\sum_{i=1}^{n}(\frac{x_i-\mu_X}{\sigma_X})(\frac{y_i-\mu_Y}{\sigma_Y}) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial\mu_X}\log L &= \frac{1}{(1-\rho^2)}\sum_{i=1}^{n}(\frac{x_i-\mu_X}{\sigma_X}) - \frac{\rho}{1-\rho^2}\sum_{i=1}^{n}(\frac{y_i-\mu_Y}{\sigma_X\sigma_Y}) \\ \frac{\partial}{\partial\mu_Y}\log L &= \frac{1}{(1-\rho^2)}\sum_{i=1}^{n}(\frac{y_i-\mu_Y}{\sigma_Y}) - \frac{\rho}{1-\rho^2}\sum_{i=1}^{n}(\frac{x_i-\mu_X}{\sigma_X\sigma_Y}) \\ \frac{\partial}{\partial\sigma_X^2}\log L &= \frac{-n}{2\sigma_X^2} + \frac{-1}{2(1-\rho^2)}\frac{1}{\sigma_X^4}\cdot\sum_{i=1}^{n}(x_i-\mu_X)^2 - \frac{\rho}{1-\rho^2}\sum_{i=1}^{n}(\frac{x_i-\mu_X}{2(\sigma_X^2)^{3/2}})(\frac{y_i-\mu_Y}{\sigma_Y}) \\ \frac{\partial}{\partial\sigma_Y^2}\log L &= \frac{-n}{2\sigma_Y^2} + \frac{-1}{2(1-\rho^2)}\frac{1}{\sigma_Y^4}\cdot\sum_{i=1}^{n}(y_i-\mu_Y)^2 - \frac{\rho}{1-\rho^2}\sum_{i=1}^{n}(\frac{y_i-\mu_Y}{2(\sigma_Y^2)^{3/2}})(\frac{x_i-\mu_X}{\sigma_X}) \\ \frac{\partial}{\partial\rho}\log L &= \frac{-n}{2}(\frac{1}{1-\rho^2})(-2\rho) \\ &+\frac{(-2\rho)}{2(1-\rho^2)^2}\left[\sum_{i=1}^{n}(\frac{x_i-\mu_X}{\sigma_X})^2 + \sum_{i=1}^{n}(\frac{y_i-\mu_Y}{\sigma_Y})^2 - 2\rho\sum_{i=1}^{n}(\frac{x_i-\mu_X}{\sigma_X})(\frac{y_i-\mu_Y}{\sigma_Y})\right] \\ &+\frac{1}{1-\rho^2}\sum_{i=1}^{n}(\frac{x_i-\mu_X}{\sigma_X})(\frac{y_i-\mu_Y}{\sigma_Y}) \end{aligned}$$

Set all above 5 partial derivative equations equal to 0, we have

$$\frac{1}{\hat\sigma_X}\sum_{i=1}^{n}(x_i-\hat\mu_X) - \frac{\hat\rho}{\hat\sigma_Y}\sum_{i=1}^{n}(y_i-\hat\mu_Y) = 0 \tag{1}$$

$$\frac{1}{\hat\sigma_Y}\sum_{i=1}^{n}(y_i-\hat\mu_Y) - \frac{\hat\rho}{\hat\sigma_X}\sum_{i=1}^{n}(x_i-\hat\mu_X) = 0 \tag{2}$$

$$-n + \frac{1}{(1-\hat\rho^2)\hat\sigma_X^2}\sum_{i=1}^{n}(x_i-\hat\mu_X)^2 - \frac{\hat\rho}{(1-\hat\rho^2)}\sum_{i=1}^{n}(\frac{x_i-\hat\mu_X}{\hat\sigma_X})(\frac{y_i-\hat\mu_Y}{\hat\sigma_Y}) = 0 \tag{3}$$

$$-n + \frac{1}{(1-\hat\rho^2)\hat\sigma_Y^2}\sum_{i=1}^{n}(y_i-\hat\mu_Y)^2 - \frac{\hat\rho}{(1-\hat\rho^2)}\sum_{i=1}^{n}(\frac{x_i-\hat\mu_X}{\hat\sigma_X})(\frac{y_i-\hat\mu_Y}{\hat\sigma_Y}) = 0 \tag{4}$$

$$\begin{aligned} n\hat\rho - \frac{\hat\rho}{(1-\hat\rho^2)}\left[\sum_{i=1}^{n}(\frac{x_i-\hat\mu_X}{\hat\sigma_X})^2 + \sum_{i=1}^{n}(\frac{y_i-\hat\mu_Y}{\hat\sigma_Y})^2 - 2\hat\rho\sum_{i=1}^{n}(\frac{x_i-\hat\mu_X}{\hat\sigma_X})(\frac{y_i-\hat\mu_Y}{\hat\sigma_Y})\right] + \\ \sum_{i=1}^{n}(\frac{x_i-\hat\mu_X}{\hat\sigma_X})(\frac{y_i-\hat\mu_Y}{\hat\sigma_Y}) = 0 \qquad (5) \end{aligned}$$

From (1), (2), since $\begin{vmatrix} 1 & -\hat{\rho} \\ -\hat{\rho} & 1 \end{vmatrix} = 1 - \hat{\rho}^2 \neq 0$,
we get

$$\frac{1}{\hat{\sigma}_X}\sum_{i=1}^{n}(x_i - \hat{\mu}_X) = \frac{1}{\hat{\sigma}_Y}\sum_{i=1}^{n}(y_i - \hat{\mu}_Y) = 0$$

$$\Rightarrow \hat{\mu}_X = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}, \quad \text{and} \quad \hat{\mu}_Y = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y} \qquad (\text{MLE for } \mu_X, \mu_Y)$$

Put back MLEs of μ_X and μ_Y into (3), (4), (5).
From (3), (4), we have

$$\frac{1}{(1-\hat{\rho}^2)\hat{\sigma}_X^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 = n + \frac{\hat{\rho}}{(1-\hat{\rho}^2)}\sum_{i=1}^{n}(\frac{x_i - \bar{x}}{\hat{\sigma}_X})(\frac{y_i - \bar{y}}{\hat{\sigma}_Y}) = \frac{1}{(1-\hat{\rho}^2)\hat{\sigma}_Y^2}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

$$\Rightarrow \frac{1}{\hat{\sigma}_X}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{\hat{\sigma}_Y}\sum_{i=1}^{n}(y_i - \bar{y})^2 \qquad (*)$$

and

$$\begin{aligned} \sum_{i=1}^{n}(\frac{x_i - \bar{x}}{\hat{\sigma}_X})(\frac{y_i - \bar{y}}{\hat{\sigma}_Y}) &= \frac{1-\hat{\rho}^2}{\hat{\rho}}\left[\frac{1}{(1-\hat{\rho}^2)\hat{\sigma}_X^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 - n\right] \qquad (**) \\ &= \frac{1-\hat{\rho}^2}{\hat{\rho}}\left[\frac{1}{(1-\hat{\rho}^2)\hat{\sigma}_Y^2}\sum_{i=1}^{n}(y_i - \bar{y})^2 - n\right] \end{aligned}$$

Put them into (5), we have

$$n\hat{\rho} - \frac{\hat{\rho}}{(1-\hat{\rho}^2)}\left[\sum_{i=1}^{n}(\frac{x_i - \bar{x}}{\hat{\sigma}_X})^2 + \overbrace{\sum_{i=1}^{n}(\frac{x_i - \bar{x}}{\hat{\sigma}_X})^2)} + 2\overbrace{\left[n(1-\hat{\rho}^2) - \frac{1}{\hat{\sigma}_X^2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right]}\right]$$

$$+ \overbrace{\frac{1-\hat{\rho}^2}{\hat{\rho}}\left[\frac{1}{(1-\hat{\rho}^2)\hat{\sigma}_X^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 - n\right]} = 0$$

$$\Rightarrow n\hat{\rho} - \frac{\hat{\rho}}{(1-\hat{\rho}^2)}\left[2n(1-\hat{\rho}^2)\right] + \frac{1}{\hat{\rho}\hat{\sigma}_X^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 - \frac{n(1-\hat{\rho}^2)}{\hat{\rho}} = 0$$

$$\Rightarrow -n\hat{\rho} + \frac{1}{\hat{\sigma}_X^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 - n(1-\hat{\rho}^2) = 0$$

$$\Rightarrow n = \frac{1}{\hat{\sigma}_X^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{\hat{\sigma}_Y^2}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

$$\Rightarrow \hat{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \quad \text{and} \quad \hat{\sigma}_Y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2 \qquad (\text{MLE for } \sigma_X^2, \sigma_Y^2)$$

Put

$$\frac{1}{\hat{\sigma}_X^2}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{1}{\hat{\sigma}_Y^2}\sum_{i=1}^{n}(y_i-\bar{y})^2 \qquad \text{to (5),}$$

$$n\hat{\rho} - \frac{\hat{\rho}}{(1-\hat{\rho}^2)}\left[n+n-2\hat{\rho}\sum_{i=1}^{n}(\frac{x_i-\bar{x}}{\hat{\sigma}_X})(\frac{y_i-\bar{y}}{\hat{\sigma}_Y})\right] + \sum_{i=1}^{n}(\frac{x_i-\hat{\mu}_X}{\hat{\sigma}_X})(\frac{y_i-\hat{\mu}_Y}{\hat{\sigma}_Y}) = 0$$

$$\Rightarrow n\hat{\rho}(1-\hat{\rho}^2) - 2n\hat{\rho} + 2\hat{\rho}^2\sum_{i=1}^{n}(\frac{x_i-\bar{x}}{\hat{\sigma}_X})(\frac{y_i-\bar{y}}{\hat{\sigma}_Y}) + (1-\hat{\rho}^2)\sum_{i=1}^{n}(\frac{x_i-\bar{x}}{\hat{\sigma}_X})(\frac{y_i-\bar{y}}{\hat{\sigma}_Y}) = 0$$

$$\Rightarrow -(n\hat{\rho}+n\hat{\rho}^3) + (1+\hat{\rho}^2)\sum_{i=1}^{n}(\frac{x_i-\bar{x}}{\hat{\sigma}_X})(\frac{y_i-\bar{y}}{\hat{\sigma}_Y}) = 0$$

$$\Rightarrow n\hat{\rho} = \sum_{i=1}^{n}(\frac{x_i-\bar{x}}{\hat{\sigma}_X})(\frac{y_i-\bar{y}}{\hat{\sigma}_Y})$$

$$\Rightarrow \hat{\rho} = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\cdot\frac{1}{n}(\sum_{i=1}^{n}(y_i-\bar{y})^2}}$$

$$\Rightarrow \hat{\rho} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2\cdot\sum_{i=1}^{n}(y_i-\bar{y})^2}} \qquad \text{MLE for } \rho$$

(b) Method of moments estimates

$$E(X)=\mu_X,\ \therefore\ M'_X = \frac{1}{n}\sum_{i=1}^{n}x_i = \widetilde{E(X)} = \tilde{\mu}_X \Rightarrow \tilde{\mu}_X = \frac{1}{n}\sum_{i=1}^{n}x_i = \bar{x}$$

$$E(Y)=\mu_Y,\ \therefore\ M'_Y = \frac{1}{n}\sum_{i=1}^{n}y_i = \widetilde{E(Y)} = \tilde{\mu}_Y \Rightarrow \tilde{\mu}_Y = \frac{1}{n}\sum_{i=1}^{n}y_i = \bar{y}$$

$$E(X^2) = Var(X) + [E(X)]^2 = \sigma_X^2 + \mu_X^2$$

$$\therefore\ M'_{X_2} = \frac{1}{n}\sum_{i=1}^{n}x_i^2 = \widetilde{E(X^2)} = \tilde{\sigma}_X^2 + \tilde{\mu}_X^2$$

$$\tilde{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \bar{x}^2 = \frac{1}{n}[\sum_{i=1}^{n}x_i^2 - n\bar{x}^2]$$

$$\therefore\ \tilde{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

Similarly, $E(Y^2) = \sigma_Y^2 + \mu_Y^2$,

$$\therefore \; M'_{Y^2} = \frac{1}{n}\sum_{i=1}^{n} y_i^2 = \widetilde{E(Y^2)} = \tilde{\sigma}_Y^2 + \tilde{\mu}_Y^2$$

$$\therefore \; \tilde{\sigma}_Y^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

$$\begin{aligned}
E(XY) &= COV(X,Y) + E(X)E(Y) \\
&= \rho\sigma_X\sigma_Y \\
M'_{XY} &= \frac{1}{n}\sum_{i=1}^{n} x_i y_i = \widetilde{E(XY)} \\
&= \widetilde{COV(XY)} + \widetilde{E(X)}\widetilde{E(Y)} \\
&= \tilde{\rho}\tilde{\sigma}_X\tilde{\sigma}_Y + \tilde{\mu}_X\tilde{\mu}_Y \\
&= \tilde{\rho}\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2} \cdot \sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2} + \bar{x}\bar{y} \\
\tilde{\rho} &= \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y}}{\frac{1}{n}\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}} \\
\tilde{\rho} &= \frac{\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}} \\
\tilde{\rho} &= \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y}}{\frac{1}{n}\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}} \\
\tilde{\rho} &= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}}
\end{aligned}$$

$\therefore$ The MLEs and the method of moments estimates of $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ and ρ are all the same.

28.

$$\begin{aligned}
f(X;\theta) &= \theta^X(1-\theta)^{1-X} \\
L &= f_{\underset{\sim}{X}}(\underset{\sim}{x};\theta) \\
&= \prod_{i=1}^{n} f(x_i;\theta) \\
&= \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} \\
&= \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} \\
\log L &= (\sum_{i=1}^{n} x_i)\log\theta + (n - \sum_{i=1}^{n} x_i)\log(1-\theta) \\
\frac{\partial}{\partial\theta}\log L &= \frac{1}{\theta}\sum_{i=1}^{n} x_i - \frac{1}{1-\theta}(n - \sum_{i=1}^{n} x_i) \\
\frac{\partial}{\partial\theta}\log L = 0 &\Rightarrow \frac{1}{\theta}\sum_{i=1}^{n} x_i = \frac{1}{1-\theta}(n - \sum_{i=1}^{n} x_i) \\
&\Rightarrow (1-\theta)\sum_{i=1}^{n} x_i = (n - \sum_{i=1}^{n} x_i)\theta
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \quad & \sum_{i=1}^{n} x_i = n\theta \\
\Rightarrow \quad & \hat{\theta} = \frac{1}{\sum_{i=1}^{n} x_i} \qquad \text{which is MLE for } \theta
\end{aligned}
$$

$$
\begin{aligned}
\log f(X;\theta) &= X\log\theta + (1-X)\log(1-\theta) \\
\frac{\partial}{\partial\theta}\log f(X;\theta) &= \frac{X}{\theta} - \frac{1-X}{1-\theta} \\
\frac{\partial^2}{\partial\theta^2}\log f(X;\theta) &= \frac{-X}{\theta^2} - \frac{1-X}{(1-\theta)^2} \\
E\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right] &= E\left[\frac{-X}{\theta^2} - \frac{1-X}{(1-\theta)^2}\right] \\
&= \frac{-}{\theta^2}E[X] - \frac{1}{(1-\theta)^2}E[1-X] \\
&= \frac{-}{\theta^2}X\theta - \frac{1}{(1-\theta)^2}(1-\theta) \\
&= \frac{-1}{\theta} - \frac{1}{1-\theta} \\
&= -\frac{(1-\theta)+\theta}{\theta(1-\theta)} \\
&= -\frac{1}{\theta(1-\theta)}
\end{aligned}
$$

$$
\therefore \text{The CRLB } = \frac{1}{-nE\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right]} = \frac{\theta(1-\theta)}{n}
$$

Since

$$
\begin{aligned}
Var(\hat{\theta}) &= Var(\frac{1}{n}\sum_{i=1}^{n} X_i) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) \\
&= \frac{1}{n^2}\cdot n(\theta)(1-\theta) \\
&= \frac{\theta(1-\theta)}{n} \\
&= \text{CRLB}
\end{aligned}
$$

$\therefore\ \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a <u>fully efficient</u> estimator (UMVUE) for θ.

29.

$$
\begin{aligned}
f(X;\theta) &= \frac{1}{\theta}\exp(\frac{-X}{\theta}) \\
\log f(X;\theta) &= \log(\frac{1}{\theta}) - \frac{X}{\theta} \\
\frac{\partial^2}{\partial\theta^2}\log f(X;\theta) &= \frac{-1}{\theta} + \frac{X}{\theta^2}
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial^2}{\partial\theta^2}\log f(X;\theta) &= \frac{1}{\theta^2}-\frac{2X}{\theta^3}\\
E\left[\frac{\partial^2}{\partial\theta^2}\log f(X;\theta)\right] &= \frac{1}{\theta^2}-\frac{2}{\theta^3}E(X)\\
&= \frac{1}{\theta^2}-\frac{2}{\theta^3}\cdot\theta\\
&= -\frac{1}{\theta^2}
\end{aligned}$$

$$\therefore \text{The CRLB } = \frac{1}{-nE\left[\frac{\partial^2}{\partial\theta^2}\log Lf(X;\theta)\right]} = \frac{\theta^2}{n}$$

$$\because \; Var(\bar{X}) = Var(\frac{1}{n}\sum_{i=1}^{n}X_i) = \frac{Var(X)}{n} = \frac{\theta^2}{n} = \text{CRLB}$$

$\therefore\;\; \bar{X} = \frac{1}{n}\sum_{i=1}^{n}X_i$ is a <u>fully efficient</u> estimator (UMVUE) for θ.