Assignment 1: Solution

Q1:a

$$\begin{aligned}
X &\sim Bin(n,p) \\
E(X) &= \sum_{x=0}^{n} x\binom{n}{x}p^x(1-p)^{n-x} \\
&= \sum_{x=1}^{n} n\binom{n-1}{x-1}p^x(1-p)^{n-x} \\
&= np\sum_{x=1}^{n} \binom{n-1}{x-1}p^{x-1}(1-p)^{(n-1)-(x-1)} \\
&= np
\end{aligned}$$

The 2nd equality holds because $x\binom{n}{x}p^x(1-p)^{n-x}$ is equal to 0 when x = 0

$$\begin{aligned}
E(X^2) &= \sum_{x=0}^{n} x^2\binom{n}{x}p^x(1-p)^{n-x} \\
&= np\sum_{x=1}^{n} x\binom{n-1}{x-1}p^{x-1}(1-p)^{(n-1)-(x-1)} \\
&\overset{x-1=j}{=} np\sum_{j=0}^{n-1}(j+1)\binom{n-1}{j}p^j(1-p)^{(n-1)-j} \\
&= np(\sum_{j=0}^{n-1} j\binom{n-1}{j}p^j(1-p)^{(n-1)-j} + \sum_{j=0}^{n-1}\binom{n-1}{j}p^j(1-p)^{(n-1)-j}) \\
&= np((n-1)p+1) \\
&= n^2p^2 - np^2 + np \\
Var(X) &= EX^2 - (EX)^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p)
\end{aligned}$$

Q1:b

$$\begin{aligned}
X &\sim Poisson(\lambda) \qquad p(x) = \frac{\lambda^x e^{-\lambda}}{x!} \\
E(X) &= \sum_{x=0}^{\infty} x p(x) \\
&= \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!} \\
&= \lambda \cdot 1 \\
&= \lambda
\end{aligned}$$

$$\begin{aligned}
E(X(X-1)) &= \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x e^{-\lambda}}{x!} \\
&= \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2} e^{-\lambda}}{(x-2)!} \\
&= \lambda^2 \cdot 1 \\
&= \lambda^2
\end{aligned}$$

$$\begin{aligned}
Var(X) &= E(X(X-1)) + EX - (EX)^2 \\
&= \lambda^2 + \lambda - \lambda^2 = \lambda
\end{aligned}$$

Q1:c

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\dfrac{(x-\mu)^2}{2\sigma^2}} \\
EX &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} x e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&\overset{x-\mu=y}{=} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} (y+\mu) e^{-\dfrac{y^2}{2\sigma^2}}\, dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} y e^{-\dfrac{y^2}{2\sigma^2}}\, dy + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \mu e^{-\dfrac{y^2}{2\sigma^2}}\, dy \\
&= 0 + \mu \cdot 1 = \mu
\end{aligned}
$$

The last but two equality holds since $ye^{-\dfrac{y^2}{2\sigma^2}}$ is an odd function and its integration from $-\infty$ to ∞ is equal to 0.
For Var(X), first we notice that:

$$
\begin{aligned}
\int_0^{\infty} z^2 e^{-\dfrac{1}{2}z^2}\, dz &= \int_0^{\infty} z e^{-\dfrac{1}{2}z^2}\, d\frac{1}{2}z^2 = \int_0^{\infty} z\, d(-e^{-\dfrac{1}{2}z^2}) \\
&= z e^{-\dfrac{1}{2}z^2} |_0^{\infty} + \int_0^{\infty} e^{-\dfrac{1}{2}z^2}\, dz \\
&= 0 + \sqrt{\frac{\pi}{2}}
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
Var(X) &= E(X-\mu)^2 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} (x-\mu)^2 e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&\overset{\frac{x-\mu}{\sigma}=z}{=} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z^2 \sigma^2 e^{-\dfrac{z^2}{2}}\, dz \\
&= 2\frac{1}{\sqrt{2\pi}}\sigma^2 \int_0^{\infty} z^2 e^{-\dfrac{z^2}{2}}\, dz \\
&= 2\frac{1}{\sqrt{2\pi}}\sigma^2 \sqrt{\frac{\pi}{2}} = \sigma^2
\end{aligned}
$$

Q2:a For t$\geq$ 0,

$$
\begin{aligned}
F_x(t) = P(X \leq t) &= P(-logU \leq t) \\
&= P(logU \geq t) \\
&= P(U \geq e^{-t}) \\
&= 1 - e^{-t} \\
f_x(t) &= \frac{d}{dt} F_x(t) = e^{-t}
\end{aligned}
$$

Otherwise

$$
F_x(t) = 0 \qquad f_x(t) = 0
$$

36.

$$\begin{aligned} f_Y(y) &= P(Y = y) \\ &= P(X^3 = y) \\ &= P(X = y^{1/3}) \\ &= \begin{cases} \left(\frac{1}{2}\right)^{y^{1/3}} & y = 1^3, 2^3, 3^3, \ldots \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

37.

$$\frac{dy}{dx} = 3x^2 = 3(y^{1/3})^2$$

$$f_Y(y) = f_X(x)\left|\frac{dy}{dx}\right| = \frac{1}{9}(y^{1/3})^2\left|\frac{1}{3}y^{-2/3}\right| = \frac{1}{27} \qquad 0 < y < 27$$

38.

$$Y = X^2, \frac{dy}{dx} = 2x = x(y^{1/2})$$

$$\begin{aligned} f_Y(y) &= f_X(x)\left|\frac{dy}{dx}\right| \\ &= f_X(y^{1/2})\left|\frac{1}{2}y^{-1/2}\right| \\ &= 2y^{1/2} \cdot e^{-(y^{1/2})^2} \times \left|\frac{1}{2}y^{-1/2}\right| = e^{-y} \qquad 0 < y < \infty \end{aligned}$$

3. Let the joint pdf of X and Y be

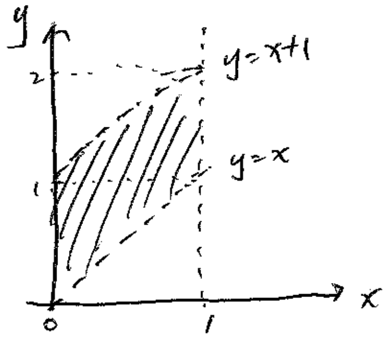
$$f_{X,Y}(x,y) = \begin{cases} 1, & \text{for } 0 < x < 1, \quad x < y < x+1; \\ 0, & \text{otherwise.} \end{cases}$$

(a) [5 *marks*] Find the marginal pdfs of X and Y.

(b) [5 *marks*] What are the variances of X and Y?

(c) [5 *marks*] Determine the correlation coefficient ρ_{XY} of X and Y.

[Solution] Consider the following picture of the domain for the joint pdf



(a) The marginal pdf of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = \begin{cases} \int_x^{x+1} 1\, dy = 1, & \text{for } 0 < x < 1, \quad 1 \\ 0 & \text{otherwise.} \quad 1 \end{cases}$$

The marginal pdf of Y is then given by

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \begin{cases} \int_0^y 1\, dx = y, & \text{for } 0 < y \le 1 \quad 1 \\ \int_{y-1}^1 1\, dx = 2-y, & \text{for } 1 < y < 2 \quad 1 \\ 0, & \text{otherwise.} \quad 1 \end{cases}$$

(b) Based on f_X, we have

$$E(X) = \int_0^1 x\, dx = \frac{1}{2}, \quad 0.5$$
$$E(X^2) = \int_0^1 x^2\, dx = \frac{1}{3}. \quad 0.5$$

Thus, $Var(X) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$. 1

Based on f_Y, we have

$$E(Y) = \int_0^1 y^2 dy + \int_1^2 (2y - y^2)dy = 1\,, \quad 1$$
$$E(Y^2) = \int_0^1 y^3 dy + \int_1^2 (2y^2 - y^3)dy = \frac{7}{6}. \quad 1$$

So, $Var(Y) = \frac{7}{6} - 1^2 = \frac{1}{6}$. 1

P.3

(c) Note that

$$E(XY) = \int_0^1 \int_x^{x+1} xy f_{X,Y}(x,y)\, dydx = \int_0^1 \left[x\left(\int_x^{x+1} y\, dy\right)\right] dx = \int_0^1 \frac{x}{2}\left[(x+1)^2 - x^2\right] dx = \frac{7}{12}. \quad 3$$

Thus, $Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{12}$, 1 and then

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{\frac{1}{12}}{\sqrt{\frac{1}{12}}\sqrt{\frac{1}{6}}} = \frac{1}{\sqrt{2}} \quad (\text{or } \frac{\sqrt{2}}{2}). \quad 1$$

P.4

5. (a) [5 *marks*] Consider a standard normal random variable Z, i.e. $Z \sim N(0,1)$. Show that

$$Z^2 \sim Gamma(\tfrac{1}{2}, \tfrac{1}{2}).$$

(b) [5 *marks*] **Use the result in (a)** to determine the distribution of $\frac{(X_1-X_2)^2}{2}$ if X_1 and X_2 are independent random variables from $N(0,1)$.

In the following two parts, consider n independent random variables $X_1, \ldots, X_n$ from $N(0,1)$.

(c) [5 *marks*] Let $\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$. What is the distribution of $\bar{X}$?

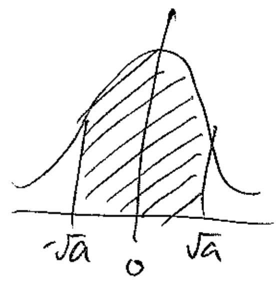
(d) [10 *marks*] Let $W = \sum_{i=1}^{n}(X_i - \bar{X})^2$. Given that W and $\bar{X}$ are independent, what is the distribution of W. [Hint: use $W = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$]

[Solution] (a) Consider the cdf of Z^2:

1 For $a \le 0$,

$$P(Z^2 \le a) = 0$$

2 For $a > 0$,



$$P(Z^2 \le a) = P(-\sqrt{a} \le Z \le \sqrt{a})$$

$$= 2\Phi(\sqrt{a}) - 1 \quad \text{(by the symmetry of } N(0,1)\text{)}$$

Thus,

$$f_{Z^2}(a) = \frac{d}{da}P(Z^2 \le a) = \begin{cases} 2\Phi'(\sqrt{a})\frac{d}{da}\sqrt{a} = 2f_Z(\sqrt{a})\frac{1}{2\sqrt{a}} \\ \quad = \frac{1}{\sqrt{a}}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\sqrt{a})^2} \\ \quad = \frac{1}{\Gamma(\frac{1}{2})}(\frac{1}{2})^{\frac{1}{2}} a^{\frac{1}{2}-1} e^{-\frac{1}{2}a}, & \text{for } a > 0 \\ 0, & \text{otherwise} \end{cases}$$

the pdf of Gamma$(\frac{1}{2},\frac{1}{2})$

$\Rightarrow Z^2 \sim Gamma(\frac{1}{2},\frac{1}{2})$ #

~~Alternative approach is based on the mgf technique~~

can also use the mgf technique to prove this result

P. 6

~~(b) Consider~~

(b) If X_1 and X_2 are indept. r.v.s from $N(0,1)$

then

$$-X_2 \sim N(0,1) \quad \text{and} \quad \underline{X_1 \perp -X_2}$$

Thus,

$$\boxed{X_1 - X_2 \sim N(0, 2)} \quad (2) \qquad (\because Var(X_1 - X_2) = Var(X_1) + Var(X_2) = 1 + 1 = 2)$$

$$\Rightarrow \boxed{\frac{X_1 - X_2}{\sqrt{2}} \sim N(0,1)} \quad (2)$$

Then by (a), $\dfrac{(X_1 - X_2)^2}{2} \sim Gamma(\frac{1}{2}, \frac{1}{2})$ (1) #

Consider n indept. $N(0,1)$ r.v.s. $X_1, \ldots, X_n$

(c).

$$M_{\overline{X}}(t) = M_{\frac{1}{n}\sum_{j=1}^{n} X_j}(t) = M_{\sum_{j=1}^{n} X_j}\left(\frac{t}{n}\right) \quad (1)$$

$$= M_{X_1}\left(\frac{t}{n}\right) \cdots M_{X_n}\left(\frac{t}{n}\right) \quad (1) \quad (\because \text{indept of } X_1 \ldots X_n)$$

$$= \left[M_{X_1}\left(\frac{t}{n}\right)\right]^n \quad (1) \quad \because X_1 \ldots X_n \text{ have the same dist. } N(0,1)$$

$$= \left[e^{\frac{1}{2}\left(\frac{t}{n}\right)^2}\right]^n \quad (1)$$

$$= e^{\frac{1}{2}\left(\frac{1}{n}\right)t^2} = e^{0t + \frac{1}{2}\left(\frac{1}{n}\right)t^2}$$

$\uparrow$ the mgf of $N(0, \frac{1}{n})$

By the uniqueness of mgf,

$$\overline{X} \sim N\left(0, \frac{1}{n}\right) \quad (1) \; \#$$

(C) Alternatively,

$$\sum_{j=1}^{n} X_j \sim N(0, n) \quad ④$$

thus $\bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j \sim N(0, \frac{n}{n^2} = \frac{1}{n})$ # ①

(d) Let $W = \sum_{i=1}^{n}(X_i - \bar{X})^2$, and $W \perp \bar{X}$

First, note that

$$W = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2,$$

from (c), $\bar{X} \sim N(0, \frac{1}{n}) \Rightarrow \sqrt{n}\bar{X} \sim N(0,1)$

$\Rightarrow (\sqrt{n}\bar{X})^2 = n\bar{X}^2 \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$

by (a) ②

Similarly, $\because X_i \sim N(0,1)$

$\therefore \underline{X_i^2 \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})}$ by (a) ①

$\because X_1, \ldots, X_n$ are indept of each other

$\Rightarrow X_1^2, \ldots, X_n^2$ are also indept of each other

and all of them follows $\text{Gamma}(\frac{1}{2}, \frac{1}{2})$

By the result of the sum of indept Gamma-distributed r.vs,

we have $\sum_{i=1}^{n} X_i^2 \sim \text{Gamma}(\frac{n}{2}, \frac{1}{2})$ ②

Next, consider the mgf of $\sum_{i=1}^{n} X_i^2 = W + n\bar{X}^2$

Note that $W \perp \bar{X} \Rightarrow W \perp n\bar{X}^2$ ③

$\therefore M_{\sum_{i=1}^{n} X_i^2}(t) = M_{W+n\bar{X}^2}(t) = M_W(t)\, M_{n\bar{X}^2}(t)$ (P.8)

$$L.H.S. = \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{\frac{n}{2}} \quad \text{for } t<\frac{1}{2}$$

$$R.H.S. = M_W(t) \cdot \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{\frac{1}{2}} \quad \text{for } t<\frac{1}{2}$$

$$\Rightarrow \quad M_W(t) = \left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{\frac{n-1}{2}} \quad \text{for } t<\frac{1}{2} \qquad 1$$

By the uniqueness of mgf,
we can conclude that

$$W \sim \text{Gamma}\left(\frac{n-1}{2}, \frac{1}{2}\right) \qquad 1$$

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Math2421 Final Exam (Spring, 2012)

"Solution"

1. [20 Marks]

Consider a circle of radius R, and suppose that a point within the circle is randomly chosen in such a manner that all region within the circle of equal area are equally likely to contain the point. In other words, the point is uniformly distributed within the circle. If we let the center of the circle denote the origin and define X and Y to be the coordinates of the point chosen, then since (X, Y) is equally likely to be near each point in the circle, it follows that the joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} c, & \text{if } x^2+y^2 \le R^2; \\ 0, & \text{otherwise,} \end{cases}$$

for some value of c.

(a) Determine c.

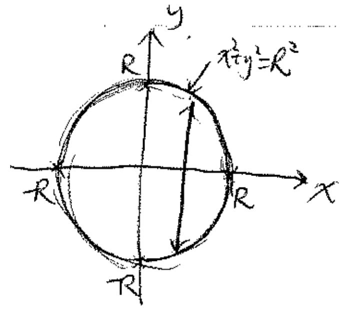
(b) Find the marginal pdfs of X alone and Y alone.

(c) Compute the probability that D, the distance from the origin to the point selected, is less than or equal to a.

(d) Find $E(D)$.

a) $\because \; 1 = \iint_{x^2+y^2\le R^2} c\,dxdy = c\iint_{x^2+y^2\le R^2} dxdy = c\pi R^2 \Rightarrow c = \frac{1}{\pi R^2}$ ✱

b). The marginal pdf of X alone is given by



$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \begin{cases} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2} dy, & \text{if } x\in(-R,R) \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2-x^2}, & \text{if } x\in(-R,R) \\ 0, & \text{otherwise} \end{cases}$$ ✱

~~Similarly,~~ By symmetry,

$$f_Y(y) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2-y^2}, & \text{if } y\in(-R,R) \\ 0, & \text{otherwise} \end{cases}$$ ✱

c). Note that $\because D = \sqrt{X^2+Y^2}$ is a continuous r.v. and $D \in [0,R]$

So, if $a \le 0$, $P(D\le a) = 0$ ✱

if $a > R$, $P(D \le a) = 1$ ✱

if $a \in [0,R]$,

$P(D\le a) = P(X^2+Y^2 \le a^2) = \iint_{x^2+y^2\le a^2} f_{X,Y}(x,y)\,dxdy$

$= \frac{1}{\pi R^2}(\pi a^2) = \frac{a^2}{R^2}$ ✱

d). According to the result in part (c), we know that the pdf of D is

$$f_D(a) = \frac{d}{da}F_D(a) = \frac{d}{da}P(D \le a) = \begin{cases} \frac{2a}{R^2}, & \text{if } a \in [0, R] \\ 0, & \text{otherwise} \end{cases}$$

Thus,

$$E(D) = \int_{-\infty}^{\infty} a f_D(a)\, da = \int_0^R a\left(\frac{2a}{R^2}\right) da = \frac{2}{R^2}\left[\frac{1}{3}a^3\Big|_0^R\right] = \frac{2}{3}R$$

3. Consider independent random variables $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1$ and 2.

   (a) Follow the four steps below to show that

$$\sum_{i=1}^{2} X_i \sim N(\sum_{i=1}^{2} \mu_i, \sum_{i=1}^{2} \sigma_i^2).$$

   Define $W_i = X_i - \mu_i$, for $i = 1$ and 2.

   i. [4 Marks] What are the distributions of W_1 and W_2?

   ii. [5 Marks] Show that W_1 and W_2 are independent.

   iii. [10 Marks] Prove that the distribution of $W_1 + W_2$ is $N(0, \sigma_1^2 + \sigma_2^2)$ without using the mgf technique.

   iv. [3 Marks] Show that $\sum_{i=1}^{2} X_i \sim N(\sum_{i=1}^{2} \mu_i, \sum_{i=1}^{2} \sigma_i^2)$.

   (b) [3 Marks] Use the result in part (a) to show that

$$\sum_{i=1}^{2} c_i X_i \sim N(\sum_{i=1}^{2} c_i \mu_i, \sum_{i=1}^{2} c_i^2 \sigma_i^2).$$

a i). For $i=1$ and 2,

$$W_i = X_i - \mu_i = (\mu_i + \sigma_i Z_i) - \mu_i \text{, where } Z_i \sim N(0,1)$$
$$= \sigma_i Z_i$$

no explanation (lose 1 pt)

Thus,

$$W_i \sim N(0, \sigma_i^2) \text{ ※}$$

ii). Note that the possible values of W_i, $i=1,2$, are any real numbers.
For any $s, t \in \mathbb{R}$,

$$f_{W_1|W_2}(s|t) = \frac{d}{ds} P(W_1 \le s \mid W_2 = t)$$
$$= \frac{d}{ds} P(X_1 \le s+\mu_1 \mid X_2 = t+\mu_2)$$
$$\cancel{= \frac{d}{ds} P(X_1 \le s+\mu_1)} \quad \cancel{\because X_1 \text{ and } X_2 \text{ are independent}}$$

or

$$\frac{d}{ds} P(X_1 \le s+\mu_1) = \frac{d}{ds} P(W_1 \le s) = f_{W_1}(s)$$

$$= f_{X_1|X_2}(s+\mu_1 \mid t+\mu_2)$$
$$= f_{X_1}(s+\mu_1) \quad (\because X_1 \text{ and } X_2 \text{ are independent})$$
$$= \frac{d}{ds} P(X_1 \le s+\mu_1)$$
$$= \frac{d}{ds} P(W_1 \le s) = f_{W_1}(s)$$

i.e. W_1 and W_2 are also independent ※

$$f_{W_1+W_2}(a) = \int_{-\infty}^{\infty} f_{W_1}(a-w_2)\, f_{W_2}(w_2)\, dw_2, \text{ where } f_{W_i}(w_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{1}{2}\frac{w_i^2}{\sigma_i^2}}, \text{ for } i=1,2$$

$$\Rightarrow \boxed{f_{W_1+W_2}(a) = \frac{1}{2\pi}\frac{1}{\sigma_1\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\frac{(a-w_2)^2}{\sigma_1^2}+\frac{w_2^2}{\sigma_2^2}\right]} dw_2}$$

Note that

$$\boxed{\frac{(a-w_2)^2}{\sigma_1^2}+\frac{w_2^2}{\sigma_2^2} = \left[\sqrt{\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}}\, w_2 - \frac{\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}\frac{a}{\sigma_1}\right]^2 + \frac{a^2}{\sigma_1^2+\sigma_2^2}}$$

$$\Rightarrow f_{W_1+W_2}(a) = \frac{1}{2\pi}\frac{1}{\sigma_1\sigma_2} e^{-\frac{1}{2}\frac{a^2}{\sigma_1^2+\sigma_2^2}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\sqrt{\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}}\, w_2 - \frac{a\sigma_2}{\sigma_1\sqrt{\sigma_1^2+\sigma_2^2}}\right]^2} dw_2}$$

$$= \left(\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\right)^{-\frac{1}{2}}\sqrt{2\pi}$$

$$\text{(5)} = \sqrt{\frac{2\pi(\sigma_1^2\sigma_2^2)}{\sigma_1^2+\sigma_2^2}}$$

Thus,

$$f_{W_1+W_2}(a) = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} e^{-\frac{1}{2}\frac{a^2}{\sigma_1^2+\sigma_2^2}} \leftarrow \text{ a pdf of } N(0, \sigma_1^2+\sigma_2^2)$$

#

iv) $\because W_1+W_2 \sim N(0, \sigma_1^2+\sigma_2^2)$
$\quad \& \ W_i = X_i - \mu_i, \ i=1,2$
$\therefore X_1+X_2 - (\mu_1+\mu_2) \sim N(0, \sigma_1^2+\sigma_2^2)$
$\Rightarrow X_1+X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

OR

$\because W_1+W_2 \sim N(0, \sigma_1^2+\sigma_2^2)$
$\therefore W_1+W_2 = \sqrt{\sigma_1^2+\sigma_2^2}\, Z$
$\& \ W_i = X_i - \mu_i, \ i=1,2$
i.e. $\boxed{X_1+X_2 = W_1+W_2+\mu_1+\mu_2 = (\mu_1+\mu_2)+\sqrt{\sigma_1^2+\sigma_2^2}\, Z}$
$\sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$ #

b) Let $Y_i = c_i X_i$

$\because$ $X_i \sim N(\mu_i, \sigma_i^2)$

$\therefore$ $Y_i \sim N(c_i\mu_i, c_i^2\sigma_i^2)$

(because $Y_i = c_i X_i = c_i(\sigma_i + \mu_i Z) = c_i\sigma_i + c_i\mu_i Z$)

Let $\mu_i' = c_i\mu_i$, $\sigma_i' = c_i\sigma_i$ i.e. $Y_i \sim N(\mu_i', \sigma_i'^2)$

Then by (a), we know that

$$\underline{\sum_{i=1}^{2} c_i X_i = \sum_{i=1}^{2} Y_i \sim N(\underbrace{\sum_{i=1}^{2}\mu_i'}_{\sum_{i=1}^{2} c_i\mu_i}, \underbrace{\sum_{i=1}^{2}\sigma_i'^2}_{\sum_{i=1}^{2} c_i^2\sigma_i^2}\ )}$$

$(\because X_1 \perp X_2 \Rightarrow Y_1 \perp Y_2)$