MATH243 Statistical Inference

<u>Exercise 4</u> (Hypothesis Testing)

<u>Type I Error, Type II Error & Power</u>

1. A single observation of a random variable having a geometric distribution is used to test the null hypothesis $\theta = \theta_0$ against the alternative hypothesis $\theta = \theta_1 > \theta_0$. If the null hypothesis is rejected if and only if the observed value of the random variable is greater than or equal to the positive integer k, find expressions for the probabilities of type I and type II errors.

2. Let X_1 and X_2 constitute a random sample of size 2 from the population given by

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1} & \text{for } \ 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

   If the critical region $x_1 x_2 \geq \frac{3}{4}$ is used to test the null hypothesis $\theta = 1$ against the alternative hypothesis $\theta = 2$, what is the power of this test at $\theta = 2$?

3. Let $X_1, X_2, ..., X_n$ denote a random sample from a normal distribution $n(\theta, 100)$. Show that $C = \left\{ (x_1, x_2, ..., x_n); c \leq \bar{x} = \sum_1^n x_1/n \right\}$ is a best critical region for testing $H_0 : \theta = 75$ against $H_1 : \theta = 78$. Find n and c so that

$$\Pr[(X_1, X_2, ..., X_n) \in C; H_0] = \Pr[\bar{X} \geq c; H_0] = 0.05$$

   and

$$\Pr[(X_1, X_2, ..., X_n) \in C; H_1] = \Pr[\bar{X} \geq c; H_1] = 0.90, \text{approximately.}$$

4. Let X have the p.d.f. $f(x;\theta) = \theta^x(1-\theta)^{1-x}, x = 0, 1$, zero elsewhere. We test the simple hypothesis $H_0 : \theta = \frac{1}{4}$ against the alternative composite hypothesis $H_1 : \theta < \frac{1}{4}$ by taking a random sample of size 10 and rejecting $H_0 : \theta = \frac{1}{4}$ if and only if the observed values $x_1, x_2, ..., x_{10}$ of the sample items are such that $\sum_1^{10} x_i \leq 1$. Find the power function $Q(\theta), 0 < \theta \leq \frac{1}{4}$, of this test.

5. Let X have a p.d.f. of the form $f(x;\theta) = 1/\theta, 0 < x < \theta$, zero elsewhere. Let $Y_1 < Y_2 < Y_3 < Y_4$ denote the order statistics of a random sample of size 4 from this distribution. Let the observed value of Y_4 be y_4. We reject $H_0 : \theta = 1$ and accept $H_1 : \theta \neq 1$ if either $y_4 \leq \frac{1}{2}$ or $y_4 \geq 1$. Find the power function $Q(\theta), 0 < \theta$, of the test.

6. Consider a normal distribution of the form $n(\theta, 4)$. The simple hypothesis $H_0 : \theta = 0$ is rejected, and the alternative composite hypothesis $H_1 : \theta > 0$ is accepted if and only if the observed mean $\bar{x}$ of a random sample of size 25 is greater than or equal to $\frac{3}{5}$. Find the power function $Q(\theta), 0 \leq \theta$, of this test.

7. Consider the two independent normal distributions $n(\mu_1, 400)$ and $n(\mu_2, 225)$. Let $\theta = \mu_1 - \mu_2$. Let $\bar{x}$ and $\bar{y}$ denote the observed means of two independent random samples, each of size n, from these two distributions. We reject $H_0 : \theta = 0$ and accept $H_1 : \theta > 0$ if and only if $\bar{x} - \bar{y} \geq c$. If $Q(\theta)$ is the power function of this test, find n and c so that $Q(0) = 0.05$ and $Q(10) = 0.90$, approximately.

UMP Test:

8. Suppose that $\underset{\sim}{X} = (X_1, ..., X_n)$ is a random sample from the exponential distribution with parameter θ. Show that there exists a uniformly most powerful test at significance level α of $H_0 : \theta = \theta_0 (\theta_0 \in R^+)$ against the one-sided alternative hypothesis $H_1 : \theta > \theta_0$.

9. Given a random sample of size n from a normal population with $\mu = 0$, use the Neyman-Pearson lemma to construct the most powerful critical region of size α to test the null hypothesis $\sigma = \sigma_0$ against the alternative $\sigma = \sigma_1 > \sigma_0$.

10. Let $X_1, X_2, ..., X_{10}$ be a random sample from a distribution which is $n(\theta_1, \theta_2)$. Find a best test of the simple hypothesis $H_0 : \theta_1 = \theta_1' = 0, \theta_2 = \theta_2' = 1$ against the alternative simple hypothesis $H_1 : \theta_1 = \theta_1'' = 1, \theta_2 = \theta_2'' = 4$.

11. Let $X_1, X_2, ..., X_{25}$ denote a random sample of size 25 from a normal distribution $n(\theta, 100)$. Find a uniformly most powerful critical region of size $\alpha = 0.10$ for testing $H_0 : \theta = 75$ against $H_1 : \theta > 75$.

12. Let $X_1, X_2, ..., X_n$ denote a random sample from a normal distribution $n(\theta, 16)$. Find the sample size n and a uniformly most powerful test of $H_0 : \theta = 25$ against $H_1 : \theta < 25$ with power function $Q(\theta)$ so that approximately $Q(25) = 0.10$ and $Q(23) = 0.90$.

13. Let $X_1, X_2, ..., X_{20}$ be a random sample of size 20 from a Poisson distribution with mean θ. Show that the critical region defined by $\sum_{1}^{20} x_i \geq 5$ is a uniformly most powerful critical region for testing $H_0 : \theta = \frac{1}{10}$ against $H_1 : \theta > \frac{1}{10}$. What is α, the significance level of the test ?

14. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables, each with the Poisson distribution of parameter θ (and therefore of mean θ and variance θ). Find the best size α test of $H_0 : \theta = 1$ against $H_1 : \theta = 1.21$. By using the Central Limit Theorem to approximate the distribution of $\sum_i X_i$, find the smallest value of n required to make $\alpha = 0.05$ and $\beta \leq 0.1$.

15. Let $X_1, \ldots, X_n$ be independent r.v.'s distributed as $N(\mu, \sigma^2)$, where μ is unknown and σ^2 is known. Show that the sample size n can be determined so that when testing the hypothesis $H_o : \mu = 0$ against the alternative $H_a : \mu = 1$, one has predetermined values for α and β. What is the numerical value of n if $\alpha = 0.05$, $\beta = 0.1$ and $\sigma^2 = 1$?

Likelihood Ratio Test

16. Suppose that $(X_1, X_2, ..., X_{100})$ is a random sample from a normal distribution with mean θ and standard deviation 1.8. Find the likelihood ratio test for testing $H_0 : \theta = 2$ against $H_1 : \theta \neq 2$ using a significance level of 0.05.

17. The number of successes in n trials is to be used to test the null hypothesis that the parameter θ of a binomial population equals $\frac{1}{2}$ against the alternative that it does not equal $\frac{1}{2}$.

    (a) Find an expression for the likelihood ratio statistic.

    (b) Use the result of part (a) to show that the critical region of the likelihood ratio test can be written as
    $$x \cdot \ln x + (n - x) \cdot \ln(n - x) \geq K$$
    where x is the observed number of successes.

    (c) Study the graph of $f(x) = x \cdot \ln x + (n - x) \cdot \ln(n - x)$, in particular its minimum and its symmetry, to show that the critical region of this likelihood ratio test can also be written as
    $$\left| x - \frac{n}{2} \right| \geq K$$

where K is a constant which depends on the size of the critical region.

18. A random sample of size n from a normal population with unknown mean and variance is to be used to test the null hypothesis $\mu = \mu_0$ against the alternative $\mu \neq \mu_0$. Show that the values of the likelihood ratio statistic can be written in the form

$$\lambda = \left(1 + \frac{t^2}{n-1}\right)^{-n/2}$$

where $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$. Note that the likelihood ratio test can, thus, be based on the t distribution.

19. Given a random sample of size n from a normal population with unknown mean and variance, find an expression for the likelihood ratio statistic for testing the null hypothesis $\sigma = \sigma_0$ against the alternative hypothesis $\sigma \neq \sigma_0$.

20. Let $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ..., Y_m)$ be random samples from the independent normal distributions $N(\theta_1, \theta_3)$ and $N(\theta_2, \theta_4)$, respectively.

    Show that the likelihood ratio for testing $H_0 : \theta_1 = \theta_2,\ \theta_3 = \theta_4$ against a general alternative is given by

$$\frac{\left[\sum_{i=1}^{n}(x_i - \bar{x})^2 \Big/ n\right]^{n/2} \left[\sum_{j=1}^{m}(y_j - \bar{y})^2 \Big/ m\right]^{m/2}}{\left\{\left[\sum_{i=1}^{n}(x_i - u)^2 + \sum_{j=1}^{m}(y_j - u)^2\right] \Big/ (m+n)\right\}^{(m+n)/2}}$$

    where $u = (n\bar{x} + m\bar{y})/(n+m)$.

21. Let $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ..., Y_n)$ be random samples from the independent normal distributions $N(\theta_1, \theta_3)$ and $N(\theta_2, \theta_4)$, respectively.

    Show that the likelihood ratio test for testing $H_0 : \theta_3 = \theta_4,\ \theta_1$ and θ_2 unspecified, against $H_1 : \theta_3 \neq \theta_4,\ \theta_1$ and θ_2 unspecified, can be based on the random variable

$$F = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 \Big/ (n-1)}{\sum_{j=1}^{m}(y_j - \bar{y})^2 \Big/ (m-1)}$$

    Hence, construct the critical region at significance level α.

22. Suppose that we have two independent random samples: $X_1, \ldots, X_n$ are exponential(θ), with density

$$f(x|\theta) = \theta e^{-\theta x}, \quad x > 0$$

    and $Y_1, \ldots, Y_n$ are exponential (μ).

    (a) Find the form of the likelihood ratio test for testing $H_0 : \theta = \mu$ against $H_1 : \theta \neq \mu$.

    (b) Show that the test in part (a) can be expressed in terms of the statistic

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_i}$$

    (c) Find the distribution of T when H_0 is true.

Goodness of Fit Test

23. The evidence of 300 easts of a certain six-sided die is to be used to examine, at a significance level of 0.10, the null hypothesis that the die is fair. Given the results of the hypothesis that the die is fair. Given the results of the following table, carry out an appropriate test.

Face Number	Observed Frequency
1	47
2	43
3	68
4	48
5	53
6	41

24. The data of the following table were obtained form a random sample of 300 car owners, each of whom was classified both according to age and to the number of accidents he or she had been involved in during the past two years. Using a test at the 5% level of significance, examine the null hypothesis that there is no dependence of accident rate on age in the sample population.

		Number of Accidents		
		0	1	2 or more
Age	Under 22 year	10	21	14
	Between 22 and 32 years	22	43	10
	Over 32 years	81	80	19

25. Show that for a 2×2 contingency table, the realization of the Pearson statistic can be expressed as

$$\frac{n\left(x_{11}x_{22} - x_{12}x_{21}\right)^2}{a_1 a_2 b_1 b_2}$$

where x_{ij} represents the observed frequency in the ij^{th} cell $(i = 1, 2; j = 1, 2)$ and $a_i = x_{i1} + x_{i2}, b_j = x_{1j} + x_{2j}$, also n is the total number of observations.

26. A geneticist suspects that susceptibility to a certain disease is heritable. He takes a random sample of 30 individuals and classifies each according to

(a) whether or not he or she has ever had the disease;

(b) whether or not either or both of his parents have ever had the disease.

His results are as follows:

	Parents	
Observed frequencies	Have had disease	Have not had disease
Individuals HAVE had disease	10	5
Individuals HAVE NOT had disease	3	12

Using the chi-squared test with Yates' correction at significance level 0.01, examine the null hypothesis that susceptibility to the disease is not heritable.

27. Two hundred randomly selected electronic devices of a certain type were tested and the following frequency distribution of their 'lift-times', measured in months, was compiled:

Lift time (months)	Observed freq.
$0 \leq x < 3$	53
$3 \leq x < 6$	42
$6 \leq x < 9$	35
$9 \leq x$	70

Using a test at significance level 0.01, examine the null hypothesis H_0 that the life time distribution is an exponential distribution with mean 12.

28. The data $x_1, \ldots, x_n$ has been observed and it is known that X_i is a sample from a Poisson distribution with an unknown mean λ_i. It is desired to test $H_0 : \lambda_1 = \ldots = \lambda_n$ against a general alternative hypothesis that the λ_i are arbitrary. Derive the approximate large sample likelihood ratio test. What would you conclude for data (3,4,1,6,5)? Take $\alpha = 0.05$.