(1) (4%, 2% each) Let $X_n(\mu)$'s, $n = 1, ..., N$, be independent and identically distributed (iid) Gaussian random variables with zero mean and unit variance. Also, let $Y_n(\mu)$'s, $n = 1, ..., N$, be iid binary random variables with the common probability density function $f_Y(y) = \frac{1}{2}$ if $y = +1$ or $y = -1$, and $f_Y(y) = 0$ otherwise. In addition, $X_n(\mu)$'s, $Y_n(\mu)$'s, $n = 1, ..., N$, are mutually independent. Now, define a new random variable $Z(\mu) \triangleq \sum_{n=1}^{N} X_n(\mu)Y_n(\mu)$. Answer the following questions:

(a) Derive the probability density function of the random variable $Z(\mu)$.

(b) Find the probability $\Pr\{Z(\mu) > 0\}$.

(2) (3%; 1% each) Let $X(\mu)$ and $Y(\mu)$ be two real-valued random variables with probability density functions

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(x - m_X)^2\}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(y - m_Y)^2\}.$$

Determine whether each of the following statements is true or not. Explain your answer. (Any correct answer without explanation will result in zero point.)

(a) If $E\{X(\mu)Y(\mu)\} = E\{X(\mu)\}E\{Y(\mu)\}$, then $X(\mu)$ and $Y(\mu)$ are jointly Gaussian.

(b) $X(\mu) + Y(\mu)$ is a Gaussian random variable.

(c) If conditioned on $Y(\mu)$ the random variable $X(\mu)$ is Gaussian distributed, then $X(\mu)$ and $Y(\mu)$ are jointly Gaussian.

(3) (6%, 2% each) Consider the experiment of rolling two fair dices independently. Define two random variables $X(\mu)$ and $Y(\mu)$ as the values of both dices that face upward after a single trial. Also, define the random variables $Z(\mu) = X(\mu) + Y(\mu)$ and $W(\mu) = X(\mu)Y(\mu)$.

(a) Determine the probability $\Pr\{Z(\mu) = n\}$ for all integer n.

(b) Determine the conditional probability $\Pr\{Z(\mu) = n \mid X(\mu) = m\}$ for all integers n and m.

(c) Determine the variance $\mathrm{Var}\{W(\mu)\}$.

(4) (4%) Define x_u as the u-percentile of the continuous-typed random variable $X(\mu)$ (i.e., $F_X(x_u) = u$ with F_X being the distribution function of $X(\mu)$). Prove that $x_{1-u} = -x_u$ if the density of $X(\mu)$ is an even function and if $F_X(x)$ increases monotonically with its argument x.

(5) (2%; 1% each) Determine whether each of the following functions can be the autocorrelation function of a real-valued wide-sense stationary random process. Explain your answer. (Any correct answer without explanation will result in zero point.)

(a) $R_1(\tau) = \exp\{-|\tau|\}$

(b) $R_2(\tau) = \frac{1-|\tau|+|\tau|^2}{1+|\tau|}$.

(6) (3%) Prove that for any real-valued random variables $X(\mu)$ and $Y(\mu)$

$$|E\{X(\mu)Y(\mu)\}|^2 \leq E\{X^2(\mu)\}E\{Y^2(\mu)\}.$$

(7) (4%, 2% each) Consider wide-sense stationary random processes $X(\mu, t)$ and $Y(\mu, t)$ which are related by $Y(\mu, t) = \sum_{n=1}^{2N}(-1)^n X(\mu, t+n)$. Express (a) $R_Y(\tau)$ in terms of $R_X(\tau)$, and (b) $S_Y(\omega)$ in terms of $S_X(\omega)$. Make expressions as neat as possible.

(8) (4%) Consider a linear and time-invariant system with impulse response $h(t)$, input process $X(\mu, t)$, and output process $Y(\mu, t)$. Show that if $h(t) = 0$ outside the time interval $(-T, T)$ and $X(\mu, t)$ is a zero-mean white noise, then $R_Y(t_1, t_2) = 0$ for $|t_1 - t_2| > 2T$.