(1) (5%, 1% each) Consider two random variables $X(\mu)$ and $Y(\mu)$ with the joint density

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\}$$

with $|\rho| < 1$. Determine whether each of the following statements is TRUE or FALSE. No proof or explanation is necessary.

Sol: Clearly, $X(\mu)$ and $Y(\mu)$ are jointly Gaussian with zero mean, unit variance, and correlation coefficient $\rho = E\{X(\mu)Y(\mu)\}$. Here, ρ is also the covariance between $X(\mu)$ and $Y(\mu)$.

(a) If $E\{X(\mu)Y(\mu)\} = 0$, then $X(\mu)$ is independent of $Y(\mu)$.

Sol: TRUE. With $\rho = E\{X(\mu)Y(\mu)\} = 0$, $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. Thus, $X(\mu)$ and $Y(\mu)$ are mutually independent.

(b) It is true that $\text{Var}\{X(\mu) + Y(\mu)\} = 2$.

Sol: FALSE. Here is a counterexample. With $\rho \neq 0$,

$$\begin{aligned} & \text{Var}\{X(\mu) + Y(\mu)\} \\ = \; & E\{X^2(\mu)\} + 2E\{X(\mu)Y(\mu)\} + E\{Y^2(\mu)\} \\ = \; & 2 + 2\rho \end{aligned}$$

can not be 2.

(c) It is true that $E\{X^2(\mu)Y^2(\mu)\} = 1 + 2\rho^2$.

Sol: TRUE. Because $X(\mu)$ and $Y(\mu)$ are jointly Gaussian random variables with mean zero,

$$\begin{aligned} E\{X^2(\mu)Y^2(\mu)\} &= E\{X^2(\mu)\}E\{Y^2(\mu)\} + 2E^2\{X(\mu)Y(\mu)\} \\ &= 1 + 2\rho^2. \end{aligned}$$

(d) If $X(\mu)$ is independent of $Y(\mu)$, then $\rho = 0$.

Sol: TRUE. Because $X(\mu)$ and $Y(\mu)$ are mutually independent, they are uncorrelated with zero correlation coefficient. Thus, $\rho = 0$.

(e) It is true that $\text{Var}\{X(\mu)|Y(\mu)\} = 1 - \rho^2$.

Sol: TRUE. Recall that we have learned in class that if $X(\mu)$ and $Y(\mu)$ are two jointly Gaussian random variables, with mean $m_X = 0$ and $m_Y = 0$, variance $\sigma_X^2 = 1$ and $\sigma_Y^2 = 1$, and covariance $C_{XY} = C_{YX} = \rho$, then the conditional density of $X(\mu)$, given $Y(\mu)$, is also Gaussian with conditional variance

$$\text{Var}\{X(\mu)|Y(\mu)\} = \sigma_X^2 - \rho^2/\sigma_Y^2 = 1 - \rho^2.$$

(2) (3%, 1% each) Let $\{X_n(\mu); n = 0, 1, 2, ...\}$ be a Markov chain. Which of the following are Markov chains? No proof or explanation is necessary.

(a) $\{X_{m+r}(\mu); r = 0, 1, ...\}$ with m a positive integer.

Sol: It is a Markov chain. Because $\{X_n(\mu); n = 0, 1, 2, ...\}$ is a Markov chain, we have $\Pr\{X_{m+r}(\mu)|X_{m+r-1}(\mu) = x_{m+r-1}, X_{m+r-2}(\mu) = x_{m+r-2}, ..., X_m(\mu) = x_m\} = \Pr\{X_{m+r}(\mu)|X_{m+r-1}(\mu) = x_{m+r-1}\}$. As a result, $\{X_{m+r}(\mu); r = 0, 1, ...\}$ is a Markov chain.

(b) $\{X_{2m}(\mu); m = 0, 1, ...\}$.

Sol: It is a Markov chain. Let $\Gamma \triangleq \{X_{2m}(\mu); m = 0, 1, ..., r-1\}$. Then,

$$\begin{aligned}
\Pr\{X_{2r}(\mu)|\Gamma\} &= \frac{\Pr\{X_{2r}(\mu), \Gamma\}}{\Pr\{\Gamma\}} \\
&= \sum_k \frac{\Pr\{X_{2r}(\mu), X_{2r-1}(\mu) = k, \Gamma\}}{\Pr\{\Gamma\}} \\
&= \sum_k \frac{\Pr\{X_{2r}(\mu)|X_{2r-1}(\mu) = k, \Gamma\} \Pr\{X_{2r-1}(\mu) = k, \Gamma\}}{\Pr\{\Gamma\}} \\
&= \sum_k \frac{\Pr\{X_{2r}(\mu)|X_{2r-1}(\mu) = k\} \Pr\{X_{2r-1}(\mu) = k|\Gamma\} \Pr\{\Gamma\}}{\Pr\{\Gamma\}} \\
&= \sum_k \Pr\{X_{2r}(\mu)|X_{2r-1}(\mu) = k\} \Pr\{X_{2r-1}(\mu) = k|X_{2r-2}(\mu) = x_{2r-2}\} \\
&= \Pr\{X_{2r}(\mu)|X_{2r-2}(\mu) = x_{2r-2}\}.
\end{aligned}$$

As a result, $\{X_{2m}(\mu); m = 0, 1, ...\}$ is a Markov chain.

(c) The sequence of pairs $\{(X_m(\mu), X_{m+1}(\mu)); m = 0, 1, ...\}$.

Sol: It is a Markov chain. Let $Y_m(\mu) \triangleq (X_m(\mu), X_{m+1}(\mu))$ for $m = 0, 1, ...$ Then, we have

$$\begin{aligned}
\Pr\ & \{Y_{m+1}(\mu) = (a, b)|Y_m(\mu) = (c, d), ..., Y_0(\mu) = (e, f)\} \\
= \ & \Pr\{X_{m+2}(\mu) = b|X_{m+1}(\mu) = a\} \times \mathbf{1}_{a=d} \\
= \ & \Pr\{Y_{m+1}(\mu) = (a, b)|Y_m(\mu) = (c, d)\}
\end{aligned}$$

where $\mathbf{1}_{a=d}$ is the indicator function. As a result, the sequence of pairs $\{(X_m(\mu), X_{m+1}(\mu)); m = 0, 1, ...\}$ is a Markov chain.

(3) (2%, 1% each) Determine whether each of the following statements is TRUE or FALSE. No proof or explanation is necessary.

(a) The statistic of a Gaussian process can be completely characterized by its mean function and autocorrelation function.

Sol: TRUE. Knowing mean function and autocorrelation function suffices to describe the first two order statistics and thus the statistic of a Gaussian process.

(b) The statistic of a continuous-time Markov process can be completely characterized by its transition probability density.

Sol: FALSE. Knowing the first-order density and the transition probability density suffices to describe the statistic of a Markov process. Therefore, knowing the transition probability density only is not sufficient.

(4) (8%) Consider the hard limiter

$$g(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}.$$

Let $X(\mu)$ be a continuous random variable and $Y(\mu)$ be another discrete random variable defined from $X(\mu)$ through $Y(\mu) = g(X(\mu))$.

(a) (4%) Express the probability distribution function (i.e., $F_Y(y)$) and the probability density function of $Y(\mu)$ (i.e., $f_Y(y)$) in terms of the probability distribution function of $X(\mu)$ (i.e., $F_X(x)$). Your answer may be given with the aid of the unit step function $u(y)$ defined by $u(y) = 1$ if $y \geq 0$ and $u(y) = 0$ otherwise, and the special relationship $\frac{d}{dy}u(y) = \delta(y)$ with $\delta(y)$ being the Dirac delta.

Sol: By definition,

$$\begin{aligned} F_Y(y) &= \Pr\{Y(\mu) \leq y\} = \begin{cases} 1, & y \geq 1 \\ 0, & y < -1 \\ \Pr\{X(\mu) < 0\}, & -1 \leq y < 1 \end{cases} \\ &= \begin{cases} 1, & y \geq 1 \\ 0, & y < -1 \\ F_X(0), & -1 \leq y < 1 \end{cases} \\ &= F_X(0) \cdot [u(y+1) - u(y-1)] + u(y-1) \\ &= F_X(0) \cdot u(y+1) + (1 - F_X(0))u(y-1) \end{aligned}$$

where $u(y)$ is the unit step function defined by $u(y) = 1$ if $y \geq 0$ and $u(y) = 0$ otherwise. Now, by using the relationship of special functions, $du(y)/dy = \delta(y)$ with $\delta(y)$ being the Dirac delta function, i.e., $\delta(y) = \int_{-\infty}^{y} u(x)dx$, we can represent $f_Y(y) = dF_Y(y)/dy$

$$f_Y(y) = F_X(0) \cdot \delta(y+1) + (1 - F_X(0)) \cdot \delta(y-1).$$

(b) (4%) Let $X(\mu)$ be a Gaussian random variable with zero mean and unit variance. That is, the probability density function of $X(\mu)$ is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x^2\}.$$

Find the mean and variance of $Y(\mu)$.

Sol: Because

$$F_X(0) = \int_{-\infty}^{0} f_X(x)dx = \int_{0}^{\infty} f_X(x)dx = 1/2$$

we have

$$f_Y(y) = \frac{1}{2}[\delta(y+1) + \delta(y-1)].$$

Thus, $E\{Y(\mu)\} = 0$ and $\mathrm{Var}\{Y(\mu)\} = E\{Y^2(\mu)\} = 1$.

(5) (5%) Let $S_X(\omega)$ and $R_X(\tau)$ be the power spectrum and autocorrelation of a complex-valued wide-sense stationary random process $X(\mu, t)$. That is, $S_X(\omega)$ and $R_X(\tau)$ are a Fourier-transformable pair. Also, define $S_T(\mu, \omega)$ by

$$S_T(\mu,\omega) = \frac{1}{2T}\left|\int_{-T}^{T} X(\mu,t)\exp\{-j\omega t\}dt\right|^2$$

with $j = \sqrt{-1}$. Prove that

$$\lim_{T\to\infty} E\{S_T(\mu,\omega)\} = S_X(\omega).$$

Hints:

- The Fourier transform of a window function $U_T(t) = \begin{cases} 1 & ,|t| \le 2T \\ 0 & ,|t| > 2T \end{cases}$ is $\mathcal{F}\{U_T(t)\}$ $= 2\frac{\sin(2T\omega)}{\omega}$. Also, $\mathcal{F}\{(1-\frac{|t|}{2T})U_T(t)\} = \frac{2\sin^2(T\omega)}{T\omega^2}$.
- $\lim_{T\to\infty}\frac{2\sin^2(T\omega)}{T\omega^2} = 2\pi\delta(\omega)$, with $\delta(\omega)$ being a Dirac delta function.

Sol: By definition, $\lim_{T\to\infty} E\{S_T(\mu,\omega)\}$ can be expressed as

$$\begin{aligned}
\lim_{T\to\infty} E\{S_T(\mu,\omega)\} &= \lim_{T\to\infty} E\{\frac{1}{2T}\left|\int_{-T}^{T} X(\mu,t)\exp\{-j\omega t\}dt\right|^2\} \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_X(t_1-t_2)\exp\{-j\omega(t_1-t_2)\}dt_1dt_2 \\
&= \lim_{T\to\infty}\int_{-2T}^{2T}(1-\frac{|\tau|}{2T})R_X(\tau)\exp\{-j\omega\tau\}d\tau \\
&= \lim_{T\to\infty}\mathcal{F}\{R_X(t)\cdot(1-\frac{|t|}{2T})U_T(t)\} \\
&= \lim_{T\to\infty}\frac{1}{2\pi}\mathcal{F}\{R_X(t)\} * \mathcal{F}\{(1-\frac{|t|}{2T})U_T(t)\} \\
&= \frac{1}{2\pi}S_X(\omega) * \lim_{T\to\infty}\mathcal{F}\{(1-\frac{|t|}{2T})U_T(t)\} \\
&= \frac{1}{2\pi}S_X(\omega) * 2\pi\delta(\omega) \\
&= S_X(\omega). \qquad\qquad \text{Q.E.D.}
\end{aligned}$$

(6) (4%) Define the random process $X(\mu, \omega)$ by

$$X(\mu,\omega) = \int_{-T}^{T}[f(t) + n(\mu,t)]\exp\{-j\omega t\}dt$$

i.e., a Fourier transform of the random process $f(t) + n(\mu, t)$ over the time interval $(-T, T)$ with $T > 0$. Here, $f(t)$ is deterministic with Fourier transform $F(\omega)$, i.e., $F(\omega) = \int_{-\infty}^{\infty} f(t)\exp\{-j\omega t\}dt$. $n(\mu, t)$ is a wide-sense stationary random noise with mean zero and power spectrum $S_n(\omega) = 1$. It is known that $\mathrm{Var}\{X(\mu, \omega)\} = \alpha T$. Find α.

Sol: By definition, $E\{X(\mu,\omega)\}$ is derived as

$$\begin{aligned}
E\{X(\mu,\omega)\} &= E\{\int_{-T}^{T}[f(t)+n(\mu,t)]\exp\{-j\omega t\}dt\} \\
&= \int_{-T}^{T}[f(t)+E\{n(\mu,t)\}]\exp\{-j\omega t\}dt \\
&= \int_{-T}^{T} f(t)\exp\{-j\omega t\}dt.
\end{aligned}$$

Next, $\mathrm{Var}\{X(\mu,\omega)\}$ is given by

$$\begin{aligned}
&\mathrm{Var}\{X(\mu,\omega)\} \\
=\ & E\{|X(\mu,\omega)-E\{X(\mu,\omega)\}|^2\} \\
=\ & E\{|\int_{-T}^{T}[f(t)+n(\mu,t)]\exp\{-j\omega t\}dt \\
& -\int_{-T}^{T} f(t)\exp\{-j\omega t\}dt|^2\} \\
=\ & E\{|\int_{-T}^{T} n(\mu,t)]\exp\{-j\omega t\}dt|^2\} \\
=\ & \int_{-T}^{T}\int_{-T}^{T} E\{n(\mu,t_1)n^*(\mu,t_2)\}\exp\{-j\omega(t_1-t_2)\}dt_1dt_2 \\
=\ & \int_{-T}^{T}\int_{-T}^{T} \delta(t_1-t_2)\exp\{-j\omega(t_1-t_2)\}dt_1dt_2 \\
=\ & 2T.
\end{aligned}$$

Thus, $\alpha = 2$.

(7) (7%) Let $Z(\mu,t)$ be the random signal obtained by switching between two values 0 and 1 according to the events in a counting process $N(\mu,t)$, $t \geq 0$. Let

$$\Pr\{N(\mu,t)=k\} = \frac{1}{1+t}\left(\frac{t}{1+t}\right)^k,\ k=0,1,...$$

with $N(\mu,0)=0$ by default. Suppose that $N(\mu,t)$ has stationary increments and that $\Pr\{Z(\mu,0)=0\} = \Pr\{Z(\mu,0)=1\} = \frac{1}{2}$. Answer the following.

(a) (3%) Find $\Pr\{Z(\mu,t)=n\}$ for $n \in \{0,1\}$.

Sol: Now,

$$\begin{aligned}
& \Pr\{\text{There are even numbers of arrivals in } [0,t]\} \\
=\ & \sum_{k=0}^{\infty}\frac{1}{1+t}\left(\frac{t}{1+t}\right)^{2k} \\
=\ & \frac{1}{1+t}\frac{1}{1-(\frac{t}{1+t})^2} \\
\left(\ \sum_{k=0}^{\infty}x^k =\right.\ & \left.\frac{1}{1-x} \text{ for } |x|<1\right) \\
=\ & \frac{1+t}{1+2t}
\end{aligned}$$

$$\begin{aligned} & \Pr\{\text{There are odd numbers of arrivals in } [0,t]\} \\ = & \ 1 - \Pr\{\text{There are even numbers of arrivals in } [0,t]\} \\ = & \ \frac{t}{1+2t}. \end{aligned}$$

Thus, for $t \geq 0$,

$$\begin{aligned} & \Pr\{Z(\mu,t)=0\} \\ = & \ \Pr\{Z(\mu,t)=0|Z(\mu,0)=0\}\Pr\{Z(\mu,0)=0\} + \\ & \ \Pr\{Z(\mu,t)=0|Z(\mu,0)=1\}\Pr\{Z(\mu,0)=1\} \\ = & \ \frac{1}{2}(\frac{1+t}{1+2t}) + \frac{1}{2}\frac{t}{1+2t} \\ = & \ \frac{1}{2} \end{aligned}$$

$$\Pr\{Z(\mu,t)=1\} = 1 - \Pr\{Z(\mu,t)=0\} = \frac{1}{2}.$$

(b) (2%) Find $E\{Z(\mu,t)\}$ for $t \geq 0$.

Sol: $Z(\mu,t)$ has mean

$$E\{Z(\mu,t)\} = \Pr\{Z(\mu,t)=1\} = \frac{1}{2}$$

(c) (2%) Find $E\{Z(\mu,t_1)Z(\mu,t_2)\}$ for $t_1,t_2 \geq 0$.

Sol: For $0 < t_1 \leq t_2$,

$$\begin{aligned} & E\{Z(\mu,t_1)Z(\mu,t_2)\} \\ = & \ \Pr\{Z(\mu,t_1)=1, Z(\mu,t_2)=1\} \\ = & \ \Pr\{Z(\mu,t_2)=1|Z(\mu,t_1)=1\}\Pr\{Z(\mu,t_1)=1\} \\ = & \ \Pr\{Z(\mu,t_2-t_1)=1|Z(\mu,0)=1\}\Pr\{Z(\mu,t_1)=1\} \\ & \ \text{(because } N(\mu,t) \text{ has stationary increment.)} \\ = & \ \frac{1}{2}(\frac{1+t_2-t_1}{1+2(t_2-t_1)}). \end{aligned}$$

Similarly, for $0 < t_2 \leq t_1$,

$$E\{Z(\mu,t_1)Z(\mu,t_2)\} = \frac{1}{2}(\frac{1+t_1-t_2}{1+2(t_1-t_2)}).$$

Thus, we have

$$E\{Z(\mu,t_1)Z(\mu,t_2)\} = \frac{1+|t_2-t_1|}{2+4|t_2-t_1|}$$

for $t_1, t_2 \geq 0$.

(8) (8%, 4% each) A shop has N machines in operation initially at time $t=0$ and one technician to repair them. A machine remains in the working state for an exponentially distributed time with mean $1/\beta$ and independently of the others. The technician works on one machine at a time, and it takes him an exponentially distributed time with mean $1/\alpha$ to repair each machine. Let $X(\mu,t)$ be the number of working machines at time t. It is known that $X(\mu,t)$ is a continuous-time homogeneous Markov chain. Answer the following:

(a) Let $W_n(\mu)$ be the time till the next breakdown of machine n, and $T(\mu)$ be the time till the next breakdown of any machine. Find the conditional density of $T(\mu)$ given that $X(\mu,t)=k$.

Sol: Now, we can express

$$T(\mu)=\min\{W_1(\mu),W_2(\mu),...,W_k(\mu)\}$$

when $X(\mu,t)=k$ and find its conditional distribution as

$$\begin{aligned}
\Pr\{T(\mu)>t|X(\mu,t)=k\} &= \Pr\{\min\{W_1(\mu),W_2(\mu),...,W_k(\mu)\}>t\}\\
&= \Pr\{W_1(\mu)>t,W_2(\mu)>t,...,W_k(\mu)>t\}\\
&= \Pr\{W_1(\mu)>t\}\Pr\{W_2(\mu)>t\}\cdots\Pr\{W_k(\mu)>t\}\\
&\quad (W_n(\mu)\text{'s are independent.})\\
&= \exp\{-k\beta t\}.\\
&\quad (W_n(\mu)\text{'s are identically distributed with rate } \beta.)
\end{aligned}$$

Thus, if $X(\mu,t)=k$, then the time until the next machine breakdown is an exponentially distributed random variable with mean $1/(k\beta)$.

(b) Find the steady-state state probabilities p_i's for $X(\mu,t)$.

Sol: Let $\gamma_{i,j}$ be the transition rate (of probability flow) at which $X(\mu,t)$ enters state j from state i. Then, we have

$$\begin{aligned}
\gamma_{i,i+1} &= \alpha \text{ for } i=0,1,...,N-1\\
\gamma_{i,i-1} &= i\beta \text{ for } i=1,2,...,N\\
\gamma_{i,j} &= 0 \text{ otherwise.}
\end{aligned}$$

Using these transition rates, the global balance equations when $X(\mu,t)$ settles into steady state are given by

$$\begin{aligned}
\alpha p_0 &= \beta p_1\\
(\alpha+j\beta)p_j &= \alpha p_{j-1}+(j+1)\beta p_{j+1} \text{ for } j=1,2,...,N-1\\
\alpha p_{N-1} &= N\beta p_N.
\end{aligned}$$

This set of global balance equations can be solved by first finding

$$p_j=\frac{\alpha}{j\beta}p_{j-1}=\frac{(\alpha/\beta)^j}{j!}p_0 \text{ for } j=1,2,...,N$$

and then deriving

$$p_0=\frac{1}{\sum_{j=0}^{N}\frac{(\alpha/\beta)^j}{j!}}$$

by using the identity $\sum_{j=0}^{N}p_j=1$. Thus,

$$p_j=\frac{\frac{(\alpha/\beta)^j}{j!}}{\sum_{j=0}^{N}\frac{(\alpha/\beta)^j}{j!}} \text{ for } j=0,1,...,N.$$

(9) (6%) Let us consider a cell in a cellular phone system with the following system model: 1) There are K channels available in the cell. 2) The interarrival times between initiating calls are independent and identically distributed (i.i.d.) and exponentially distributed with rate λ. 3) The (service) times that serviced calls occupy an assigned channel are i.i.d. and exponentially distributed with rate β. 4) Service times and interarrival times are mutually independent. Let us define the number of serviced calls as the state and model the single cell system as a mean-ergodic $M/M/K/K$ queuing system. Assume that $\lambda < \beta$ and that the single cell system settles into steady state. Answer the following subquestions:

(a) (4%) Find the steady-state state probability of the event that there are k ($k \geq 0$) calls being serviced in terms of λ and β.

Sol: Let p_k denotes the steady-state state probability of the event that there are k calls being serviced. The global balance equations are given by

$$\begin{aligned}
&\lambda p_0 = \beta p_1 \\
&\lambda p_{j-1} + (j+1)\beta p_{j+1} = (\lambda + j\beta) p_j \quad j = 1, 2, ..., K-1 \\
&\lambda p_{K-1} = K\beta p_K
\end{aligned}$$

which have the solutions

$$p_j = \frac{\alpha^j}{j!} \left\{ \sum_{l=0}^{K} \frac{\alpha^l}{l!} \right\}^{-1} \quad \text{for } j = 0, 1, ..., K$$

with $\alpha = \frac{\lambda}{\beta}$.

(b) (2%) Find the blocking probability that an initiating call finds no channel available and is rejected.

Sol: Because an initiating call is blocked when it finds no channel available, the blocking probability is given by

$$p_{Block} = p_K = \frac{\alpha^K}{K!} \left\{ \sum_{l=0}^{K} \frac{\alpha^l}{l!} \right\}^{-1}.$$