Stochastic Processes and Applications, Fall 2016
Homework Five (5%)

(1) (1%) Let $X_1(\mu), X_2(\mu), ..., X_N(\mu)$ be independent Poisson random variables with parameters λ_1, λ_2,..., and λ_N, respectively. Prove that the sum random variable $Y(\mu) = \sum_{n=1}^{N} X_n(\mu)$ is a Poisson random variable with parameter $\sum_{n=1}^{N} \lambda_n$.

(Hint: Note that the Poisson random variable with parameter λ_n has the probability mass $\Pr\{X_n(\mu) = k\} = \exp\{-\lambda_n\}\lambda_n^k/k!$ for $k = 0, 1, ...$You may give your proof in terms of moment generating function.)

Sol: We first find the moment generating function of $X_n(\mu)$, i.e., $E\{\exp\{sX_n(\mu)\}\}$ for permissible complex s, as follows.

$$\begin{aligned}
E\{\exp\{sX_n(\mu)\}\} &= \sum_{k=0}^{\infty} \exp\{-\lambda_n\} \frac{(\lambda_n)^k}{k!} [\exp\{s\}]^k \\
&= \exp\{-\lambda_n\} \sum_{k=0}^{\infty} \frac{(\lambda_n \exp\{s\})^k}{k!} \\
&= \exp\{-\lambda_n\} \exp\{\lambda_n \exp\{s\}\} \\
&= \exp\{\lambda_n [\exp\{s\} - 1]\}.
\end{aligned}$$

Next, the moment generating function of $Y(\mu)$ is

$$\begin{aligned}
E\{\exp\{sY(\mu)\}\} &= E\{\exp\{s \sum_{n=1}^{N} X_n(\mu)\}\} \\
&= \prod_{n=1}^{N} E\{\exp\{sX_n(\mu)\}\} \\
&\quad (X_n(\mu)\text{'s are independent}) \\
&= \prod_{n=1}^{N} \exp\{\lambda_n [\exp\{s\} - 1]\} \\
&= \exp\{(\sum_{n=1}^{N} \lambda_n)[\exp\{s\} - 1]\}
\end{aligned}$$

which shows that $Y(\mu) = \sum_{n=1}^{N} X_n(\mu)$ is a Poisson random variable with parameter $\sum_{n=1}^{N} \lambda_n$.

(2) (1.5%, 0.5% each) Suppose that a network router handles data packets that arrive according to a Poisson process $N(\mu, t)$ with rate λ packets per minute. Answer the following:

(a) What is the probability that no packet arrives for the first 15 and last 15 seconds of a minute.

(b) Find the probability that there are j arrivals in the first $t - d$ minutes when there are k arrivals in a particular period of t minutes, with $j \in \{0, 1, ..., k\}$, k a nonnegative integer, and $0 < d < t$.

(c) Let $I(\mu)$ and $J(\mu)$ be the numbers of data packets arriving in the first minute and the last minute in a specific hour. Find the probability that $I(\mu) + J(\mu) = 1$.

Sol: (a) The probability is given by

$$
\begin{aligned}
& \Pr\{N(\mu,\frac{1}{4}) - N(\mu,0) = 0, N(\mu,1) - N(\mu,\frac{3}{4}) = 0\} \\
= & \Pr\{N(\mu,\frac{1}{4}) - N(\mu,0) = 0\} \Pr\{N(\mu,1) - N(\mu,\frac{3}{4}) = 0\} \\
= & \exp\{-\frac{\lambda}{2}\}.
\end{aligned}
$$

(Poisson process has independent and stationary increments)

(b) The probability is given by

$$
\begin{aligned}
& \Pr\{N(\mu,t-d) = j | N(\mu,t) = k\} \\
= & \frac{\Pr\{N(\mu,t-d) = j, N(\mu,t) = k\}}{\Pr\{N(\mu,t) = k\}} \\
= & \frac{\Pr\{N(\mu,t-d) = j\} \Pr\{N(\mu,t) - N(\mu,t-d) = k-j\}}{\Pr\{N(\mu,t) = k\}} \\
= & \frac{\frac{\lambda^j (t-d)^j}{j!} e^{-\lambda(t-d)} \frac{\lambda^{k-j} d^{k-j}}{(k-j)!} e^{-\lambda d}}{\frac{\lambda^k t^k}{k!} e^{-\lambda t}} \\
= & \binom{k}{j} (1 - \frac{d}{t})^j (\frac{d}{t})^{k-j}
\end{aligned}
$$

which is binomial distributed.

(c) First, find the probability

$$
\begin{aligned}
\Pr\{I(\mu) + J(\mu) = 1\} &= \Pr\{I(\mu) = 0, J(\mu) = 1\} + \Pr\{I(\mu) = 1, J(\mu) = 0\} \\
&= \Pr\{N(\mu,1) - N(\mu,0) = 0, N(\mu,60) - N(\mu,59) = 1\} + \\
&\quad \Pr\{N(\mu,1) - N(\mu,0) = 1, N(\mu,60) - N(\mu,59) = 0\} \\
&= 2 \Pr\{N(\mu,1) = 1\} \Pr\{N(\mu,1) = 0\} \\
&\quad (N(\mu,t) \text{ has independent and stationary increments} \\
&\quad \text{with } N(\mu,0) = 0) \\
&= 2\lambda t e^{-2\lambda t}.
\end{aligned}
$$

(3) (1%) Suppose that the time required to service a customer in a ticket booth is an exponential random variable $T(\mu)$ with parameter β, i.e., $f_T(t) = \beta \exp\{-\beta t\}$ for $t \geq 0$ and $f_T(t) = 0$ otherwise. If customers arrive at the booth according to a Poisson process $N(\mu,t)$ with rate λ and can wait in a queue until they are serviced, find the probability that there are k customer arrivals (with $k = 0, 1, ...$) during one customer's service time.

Sol: The probability is given by

$$
\begin{aligned}
\Pr\{N(\mu,t) = k\} &= E\{E\{\Pr\{N(\mu,t) = k\} | T(\mu)\}\} \\
&= E\{\frac{\lambda^k}{k!} T(\mu)^k e^{-\lambda T(\mu)}\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\lambda^k}{k!}\int_0^\infty t^k \beta \exp\{-(\beta+\lambda)t\}dt \\
&= \frac{\lambda^k}{k!}\frac{\beta}{(\beta+\lambda)^{k+1}}\int_0^\infty x^k \exp\{-x\}dx \\
\text{(using } x &= (\beta+\lambda)t) \\
&= \frac{\lambda^k}{k!}\frac{\beta}{(\beta+\lambda)^{k+1}}\Gamma(k+1) \\
\text{(using the definition } \Gamma(k+1) &= \int_0^\infty x^k \exp\{-x\}dx) \\
&= \frac{\beta\lambda^k}{(\beta+\lambda)^{k+1}}. \\
(\Gamma(k+1) &= k! \text{ for a nonnegative integer } k)
\end{aligned}
$$

(4) (1.5%, 0.5% each) Let $Z(\mu, t)$ be the random signal obtained by switching between two values 0 and 1 according to the events in a counting process $N(\mu, t)$, $t \geq 0$. Let

$$\Pr\{N(\mu,t)=k\} = \frac{1}{1+\lambda t}\left(\frac{\lambda t}{1+\lambda t}\right)^k,\ k=0,1,...$$

with $N(\mu, 0) = 0$ by default and $\lambda > 0$. Assume that $\Pr\{Z(\mu, 0) = 0\} = p$ with $0 < p < 1$. Answer the following.

(a) Find $\Pr\{Z(\mu, t) = n\}$ for $n \in \{0, 1\}$.

(b) Suppose that $N(\mu, t)$ has stationary increments. Find $E\{Z(\mu, t_1)Z(\mu, t_2)\}$ for $0 < t_1 \leq t_2$.

(c) Find the condition under which $Z(\mu, t)$ is wide-sense stationary. Also, find the mean and autocorrelation for such a wide-sense stationary $Z(\mu, t)$.

Sol: (a) Now,

$$
\begin{aligned}
& \Pr\{\text{There are even numbers of arrivals in } [0,t]\} \\
=\ & \sum_{k=0}^{\infty} \frac{1}{1+\lambda t}\left(\frac{\lambda t}{1+\lambda t}\right)^{2k} \\
=\ & \frac{1}{1+\lambda t}\frac{1}{1-(\frac{\lambda t}{1+\lambda t})^2} \\
(\ \textstyle\sum_{k=0}^{\infty} x^k =\ & \frac{1}{1-x} \text{ for } |x|<1) \\
=\ & \frac{1+\lambda t}{1+2\lambda t}
\end{aligned}
$$

$$
\begin{aligned}
& \Pr\{\text{There are odd numbers of arrivals in } [0,t]\} \\
=\ & \sum_{k=0}^{\infty} \frac{1}{1+\lambda t}\left(\frac{\lambda t}{1+\lambda t}\right)^{2k+1} \\
=\ & \frac{1}{1+\lambda t}\frac{\frac{\lambda t}{1+\lambda t}}{1-(\frac{\lambda t}{1+\lambda t})^2} \\
=\ & \frac{\lambda t}{1+2\lambda t}.
\end{aligned}
$$

Thus, for $t \geq 0$,

$$\begin{aligned}
& \Pr\{Z(\mu,t)=0\} \\
= \; & \Pr\{Z(\mu,t)=0|Z(\mu,0)=0\}\Pr\{Z(\mu,0)=0\} + \\
& \Pr\{Z(\mu,t)=0|Z(\mu,0)=1\}\Pr\{Z(\mu,0)=1\} \\
= \; & p(\frac{1+\lambda t}{1+2\lambda t}) + (1-p)\frac{\lambda t}{1+2\lambda t}
\end{aligned}$$

$$\begin{aligned}
& \Pr\{Z(\mu,t)=1\} \\
= \; & 1-\Pr\{Z(\mu,t)=0\} \\
= \; & (1-p)(\frac{1+\lambda t}{1+2\lambda t}) + p\frac{\lambda t}{1+2\lambda t}.
\end{aligned}$$

(b) Next, for $0 < t_1 \leq t_2$,

$$\begin{aligned}
& E\{Z(\mu,t_1)Z(\mu,t_2)\} \\
= \; & \Pr\{Z(\mu,t_1)=1, Z(\mu,t_2)=1\} \\
= \; & \Pr\{Z(\mu,t_2)=1|Z(\mu,t_1)=1\}\Pr\{Z(\mu,t_1)=1\} \\
= \; & \Pr\{Z(\mu,t_2-t_1)=1|Z(\mu,0)=1\}\Pr\{Z(\mu,t_1)=1\} \\
& \text{(because } N(\mu,t) \text{ has stationary increment.)} \\
= \; & \frac{1+\lambda(t_2-t_1)}{1+2\lambda(t_2-t_1)} \times [(1-p)(\frac{1+\lambda t_1}{1+2\lambda t_1}) + p\frac{\lambda t_1}{1+2\lambda t_1}].
\end{aligned}$$

(c) $Z(\mu,t)$ has mean

$$\begin{aligned}
E\{Z(\mu,t)\} &= \Pr\{E\{Z(\mu,t)=1\} \\
&= (1-p)(\frac{1+\lambda t}{1+2\lambda t}) + p\frac{\lambda t}{1+2\lambda t}
\end{aligned}$$

and autocorrelation

$$\begin{aligned}
& E\{Z(\mu,t_1)Z(\mu,t_2)\} \\
= \; & \begin{cases} \frac{1+\lambda(t_2-t_1)}{1+2\lambda(t_2-t_1)} \times [(1-p)(\frac{1+\lambda t_1}{1+2\lambda t_1}) + p\frac{\lambda t_1}{1+2\lambda t_1}], & 0 < t_1 \leq t_2 \\ \frac{1+\lambda(t_1-t_2)}{1+2\lambda(t_1-t_2)} \times [(1-p)(\frac{1+\lambda t_2}{1+2\lambda t_2}) + p\frac{\lambda t_2}{1+2\lambda t_2}] & 0 < t_2 < t_1 \end{cases}.
\end{aligned}$$

The mean function is a constant and the autocorrelation function depends on time difference only when $p = 1/2$. In the latter case,

$$\begin{aligned}
E\{Z(\mu,t)\} &= \frac{1}{2} \\
E\{Z(\mu,t_1)Z(\mu,t_2)\} &= \frac{1+\lambda|t_2-t_1|}{1+2\lambda|t_2-t_1|}.
\end{aligned}$$