Stochastic Processes and Applications, Fall 2015
Homework Six (5%)

1. (1%) Let $X_1(\mu), X_2(\mu), ...$ be a random sequence where $X_n(\mu)$'s are independent and identically distributed and take the values $+1$ and -1 with $\Pr\{X_n(\mu) = 1\} = \Pr\{X_n(\mu) = -1\} = 1/2$. Define another sequence $Y_1(\mu), Y_2(\mu), ...$ by $Y_m(\mu) = \sum\limits_{n=1}^{m} X_n(\mu)$. Show that the sequence $Y_1(\mu), Y_2(\mu), ...$ is a Markov process. Also. derive its state probability $p_\beta(m) = \Pr\{Y_m(\mu) = \beta\}$ and one-step transition probability $p_{\alpha\beta}(m) = \Pr\{Y_m(\mu) = \beta | Y_{m-1}(\mu) = \alpha\}$.

   Sol: Now,

$$\begin{aligned}
& \Pr\{Y_m(\mu) = y_m | Y_{m-1}(\mu) = y_{m-1}, ..., Y_1(\mu) = y_1\} \\
= & \Pr\{Y_m(\mu) = y_{m-1} + X_m(\mu) | Y_{m-1}(\mu) = y_{m-1}\} \\
& (X_n(\mu)\text{'s are independent}).
\end{aligned}$$

   Thus, the sequence is Markovian. Because $X_n(\mu)$'s are independent and identically distributed and take the values $+1$ and -1 with $\Pr\{X_n(\mu) = 1\} = \Pr\{X_n(\mu) = -1\} = 1/2$, $Y_m(\mu) = \sum\limits_{n=1}^{m} X_n(\mu)$ has the moment generating function

$$\begin{aligned}
\Phi_m(s) &= \exp\{sY_m(\mu)\} \\
&= \exp\{s \sum_{n=1}^{m} X_n(\mu)\} \\
&= \prod_{n=1}^{m} \exp\{sX_n(\mu)\} \\
&\quad (X_n(\mu)\text{'s are independent}) \\
&= (\frac{1}{2})^m (\exp\{s\} + \exp\{-s\})^m \\
&\quad (X_n(\mu)\text{'s are identically distributed}) \\
&= \sum_{n=0}^{m} (\frac{1}{2})^m \binom{m}{n} \exp\{(m-2n)s\} \\
&= \sum_{n=0}^{m} \Pr\{Y_m(\mu) = m-2n\} \exp\{(m-2n)s\}.
\end{aligned}$$

   Thus, the state probability $p_\beta(m)$ is given by

$$\begin{aligned}
p_\beta(m) &= \Pr\{Y_m(\mu) = \beta\} \\
&= \begin{cases} (\frac{1}{2})^m \binom{m}{n}, & \text{if } \beta = m-2n \text{ and } n \in \{0, 1, ..., m\} \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}$$

   where β takes value in the set $\{-m, -m+2, ..., 0\}$. Also, the one-step transition probability $p_{\alpha\beta}(m) = \Pr\{Y_m(\mu) = \beta | Y_{m-1}(\mu) = \alpha\}$ is given by

$$\begin{aligned}
p_{\alpha\beta}(m) &= \Pr\{Y_m(\mu) = \beta | Y_{m-1}(\mu) = \alpha\} \\
&= \Pr\{X_m(\mu) = \beta - \alpha\} \\
&= \begin{cases} \frac{1}{2}, & \begin{array}{l} \text{if } \beta \in \{\alpha-1, \alpha+1\} \\ \text{and } \alpha = m-1-2n \\ \text{with } n \in \{0, 1, ..., m-1\} \end{array} \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}$$

2. (1%) A certain part of a machine can be in two states: working or undergoing repair. A working part fails during the course of a day with probability α. A part undergoing repair is put into working order during the course of a day with probability β. Let $X_n(\mu)$ be the state of the part. We model that $X_n(\mu)$, $n = 0, 1, 2, ...$ is a two-state homogeneous Markov chain.

(A) (0.5%) Find the m-step state transition probability matrix for any positive integer m.

Sol: Denote $X_n(\mu) = 0$ if the part is working and $X_n(\mu) = 1$ otherwise. Now, the one-step state transition probability matrix $\mathbf{P}$ is given by

$$\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}.$$

Now, the eigenvalues of $\mathbf{P}$ can be obtained from the equation

$$|\mathbf{P} - \lambda \mathbf{I}| = (1-\beta-\lambda)(1-\alpha-\lambda) - \alpha\beta = 0$$

and given by $\lambda_1 = 1$ and $\lambda_2 = 1-\alpha-\beta$. The corresponding eigenvectors can be obtained from the equation

$$\mathbf{P}\mathbf{e}_i = \lambda_i \mathbf{e}_i$$

and given by $\mathbf{e}_1 = \begin{bmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} \frac{-\alpha}{\sqrt{\alpha^2+\beta^2}} \\ \frac{\beta}{\sqrt{\alpha^2+\beta^2}} \end{bmatrix}$. The eigenmatrix $\mathbf{E}$ is thus given by

$$\mathbf{E} = \begin{bmatrix} \sqrt{\frac{1}{2}} & \frac{-\alpha}{\sqrt{\alpha^2+\beta^2}} \\ \sqrt{\frac{1}{2}} & \frac{\beta}{\sqrt{\alpha^2+\beta^2}} \end{bmatrix}$$

which has the inverse

$$\mathbf{E}^{-1} = \frac{\sqrt{2\alpha^2+2\beta^2}}{\alpha+\beta} \begin{bmatrix} \frac{\beta}{\sqrt{\alpha^2+\beta^2}} & \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{bmatrix}.$$

Thus, $\mathbf{P}$ can be decomposed as

$$\mathbf{P} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1}$$

where $\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{bmatrix}$. Therefore, the m-step state transition probability matrix is

$$\begin{aligned}
\mathbf{P}^m &= (\mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1})(\mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1})...(\mathbf{E}\mathbf{\Lambda}\mathbf{E}^{-1}) \\
&= \mathbf{E}\mathbf{\Lambda}^m\mathbf{E}^{-1} \\
&= \frac{\sqrt{2\alpha^2+2\beta^2}}{\alpha+\beta} \begin{bmatrix} \sqrt{\frac{1}{2}} & \frac{-\alpha}{\sqrt{\alpha^2+\beta^2}} \\ \sqrt{\frac{1}{2}} & \frac{\beta}{\sqrt{\alpha^2+\beta^2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^m \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} \frac{\beta}{\sqrt{\alpha^2+\beta^2}} & \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{bmatrix}
\end{aligned}$$

$$= \frac{\sqrt{2\alpha^2+2\beta^2}}{\alpha+\beta}\begin{bmatrix} \sqrt{\frac{1}{2}} & \frac{-\alpha(1-\alpha-\beta)^m}{\sqrt{\alpha^2+\beta^2}} \\ \sqrt{\frac{1}{2}} & \frac{\beta(1-\alpha-\beta)^m}{\sqrt{\alpha^2+\beta^2}} \end{bmatrix}\begin{bmatrix} \frac{\beta}{\sqrt{\alpha^2+\beta^2}} & \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} \\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \end{bmatrix}$$

$$= \frac{1}{\alpha+\beta}\begin{bmatrix} \beta+\alpha(1-\alpha-\beta)^m & \alpha-\alpha(1-\alpha-\beta)^m \\ \beta-\beta(1-\alpha-\beta)^m & \alpha+\beta(1-\alpha-\beta)^m \end{bmatrix}.$$

(B) (0.5%) Assume that $0 < \alpha+\beta < 2$. Find the steady-state state probability for each of the two states.

Sol: Because $0 < \alpha+\beta < 2$, the steady-state state probability vector is given by

$$\begin{aligned} \mathbf{R}_{ss} &= \mathbf{R}(0)\cdot\lim_{m\to\infty}\mathbf{P}^m \\ &= \mathbf{R}(0)\cdot\lim_{m\to\infty}\frac{1}{\alpha+\beta}\begin{bmatrix} \beta+\alpha(1-\alpha-\beta)^m & \alpha-\alpha(1-\alpha-\beta)^m \\ \beta-\beta(1-\alpha-\beta)^m & \alpha+\beta(1-\alpha-\beta)^m \end{bmatrix} \\ &= \frac{1}{\alpha+\beta}\mathbf{R}(0)\begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} \\ &\quad \text{(since } \lim_{m\to\infty}(1-\alpha-\beta)^m \text{ is zero)} \\ &= [\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}] \end{aligned}$$

where $\mathbf{R}(0) = [p, 1-p]$ is the initial state probability vector with p being the initial probability of working state.

3. (1%) A critical part of a machine has an exponentially distributed lifetime with parameter a. Suppose that n spare parts are initially in stock. Also, let $N(\mu, t)$ be the number of spare parts left at time t. Answer the following subquestions:

(A) (0.5%) Find $p_{ij}(t) = \Pr\{N(\mu, t+s) = j | N(\mu, s) = i\}$.

Sol: Let $\widetilde{N}(\mu, t)$ be the number of failures at time t. Because the lifetime is exponentially distributed, $\widetilde{N}(\mu, t)$ is a Poisson random process with rate a. As a result, we have

$$\begin{aligned} p_{ij}(t) &= \Pr\{N(\mu, t+s) = j | N(\mu, s) = i\} \\ &= \Pr\{\widetilde{N}(\mu, t+s) = n-j | \widetilde{N}(\mu, s) = n-i\} \\ &= \begin{cases} \Pr\{\widetilde{N}(\mu, t) \geq i-j\} & n \geq i \geq j = 0 \\ \Pr\{\widetilde{N}(\mu, t) = i-j\} & n \geq i \geq j > 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \sum_{k=i}^{\infty} e^{-at}\frac{(at)^k}{k!} & n \geq i \geq j = 0 \\ e^{-at}\frac{(at)^{i-j}}{(i-j)!} & n \geq i \geq j > 0 \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

(B) (0.5%) Find $p_i(t) = \Pr\{N(\mu, t) = i\}$.

Sol: From (A), we have

$$\begin{aligned} p_i(t) &= \Pr\{N(\mu, t) = i\} \\ &= \begin{cases} \sum_{k=n}^{\infty} e^{-at}\frac{(at)^k}{k!} & i = 0 \\ e^{-at}\frac{(at)^{n-i}}{(n-i)!} & n \geq i > 0 \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

4. (1%) Suppose that two types of customers arrive at queueing system with one server and an infinite queue space according to independent Poisson process of rate $\lambda/2$. Both types of customers require exponentially distributed service times of rate μ. Type 1 customers are always accepted into the system, but type 2 customers are turned away when the total number of customers in the system exceeds K with K a positive integer.

(A) (0.5%) Describe the transition state diagram for $N(\mu, t)$, the total number of customers in the system.

Sol: Denote the number of customers in the system N as the state index in the transition state diagram. Note that the arrival rate into the queueing system is λ if $N < K$ and $\frac{\lambda}{2}$ otherwise. Now, the diagram consists of an infinite number of states, denoted by $N \in \{0, 1, 2, ...\}$, and has the transition rates $\gamma_{n,m}$'s specified by

$$
\begin{aligned}
\gamma_{n,n-1} &= \mu \text{ if } n \text{ is a positive integer} \\
\gamma_{n,n+1} &= \lambda \text{ if } n \text{ is a nonnegative integer and smaller than } K \\
\gamma_{n,n+1} &= \frac{\lambda}{2} \text{ if } n \text{ is a nonnegative integer and not smaller than } K \\
\gamma_{n,m} &= 0 \text{ otherwise.}
\end{aligned}
$$

(B) (0.5%) Does the system has steady state? If yes, find the steady-state state probability for $N(\mu, t)$.

Sol: Use Global balance equations for $M/M/1$ queue to solve the steady-state state probability $p_n = \Pr\{N(\mu, \infty) = n\}$. First, for $0 \le n \le K - 1$,

$$
p_n = \frac{\lambda}{\mu} p_{n-1} = ... = (\frac{\lambda}{\mu})^n p_0.
$$

Next, for $n \ge K$,

$$
\begin{aligned}
p_n &= \frac{\lambda}{2\mu} p_{n-1} = ... = (\frac{\lambda}{2\mu})^{n-K} p_K \\
&= (\frac{\lambda}{2\mu})^{n-K} (\frac{\lambda}{\mu})^K p_0.
\end{aligned}
$$

Because $\sum_{n=0}^{\infty} p_n = 1$, we have

$$
\begin{aligned}
1 &= \sum_{n=0}^{K-1} p_n + \sum_{n=K}^{\infty} p_n \\
&= p_0 \left\{ \sum_{n=0}^{K-1} (\frac{\lambda}{\mu})^n + (\frac{\lambda}{\mu})^K \sum_{n=K}^{\infty} (\frac{\lambda}{2\mu})^{n-K} \right\} \\
&= p_0 \left\{ \frac{1 - (\frac{\lambda}{\mu})^K}{1 - \frac{\lambda}{\mu}} + \frac{(\frac{\lambda}{\mu})^K}{1 - \frac{\lambda}{2\mu}} \right\}.
\end{aligned}
$$

Thus, if $\left\{\frac{1-(\frac{\lambda}{\mu})^K}{1-\frac{\lambda}{\mu}} + \frac{(\frac{\lambda}{\mu})^K}{1-\frac{\lambda}{2\mu}}\right\} > 1$, the system has steady state and the steady-state state probabilities are given by

$$p_n = \begin{cases} (\frac{\lambda}{\mu})^n \left\{\frac{1-(\frac{\lambda}{\mu})^K}{1-\frac{\lambda}{\mu}} + \frac{(\frac{\lambda}{\mu})^K}{1-\frac{\lambda}{2\mu}}\right\}^{-1}, & 0 \le n \le K-1 \\ (\frac{\lambda}{2\mu})^{n-K} (\frac{\lambda}{\mu})^K \left\{\frac{1-(\frac{\lambda}{\mu})^K}{1-\frac{\lambda}{\mu}} + \frac{(\frac{\lambda}{\mu})^K}{1-\frac{\lambda}{2\mu}}\right\}^{-1}, & n \ge K \end{cases}.$$

Otherwise, the system does not have steady state.

5. (1%) Consider an $M/M/1/2$ queueing system in which each customer accepted into the system brings in a profit of α dollars and in which each customer rejection results in a loss of β dollars with $\beta < \alpha$. Denote λ and μ as the arrival rate and the service rate, respectively. Find the condition on λ and μ under which the system breaks even when the system settles into steady state. You may describe the condition in terms of $a\rho^2 + b\rho + c = 0$ with $\rho = \lambda/\mu$ and coefficients a, b, c being functions of α and β.

Sol: Define the number of customers in the system as the state. The transition state diagram for the system consists of three states, namely 0, 1, and 2, with the transition rates $\gamma_{0,1} = \gamma_{1,2} = \lambda$, $\gamma_{1,0} = \gamma_{2,1} = \mu$, and $\gamma_{n,m} = 0$ otherwise. By global balance equations, the steady-state state probabilities are given by $p_n = \frac{\rho^n}{1+\rho+\rho^2}$ for $n = 0, 1, 2$, with $\rho = \lambda/\mu$. The blocking probability that an arriving customer finds no space in the system and is rejected is $p_b = p_2 = \frac{\rho^2}{1+\rho+\rho^2}$. Thus, when the system settles into steady state, the net profit is $\alpha(1 - p_b) - \beta p_b$ and the system breaks even when

$$\begin{aligned} & \alpha(1-p_b) = \beta p_b \\ \Leftrightarrow\ & p_b = \frac{\alpha}{\alpha+\beta} \\ \Leftrightarrow\ & \frac{\rho^2}{1+\rho+\rho^2} = \frac{\alpha}{\alpha+\beta} \\ \Leftrightarrow\ & \beta\rho^2 - \alpha\rho - \alpha = 0 \end{aligned}$$

which is the condition on λ and μ, under which the system breaks even.