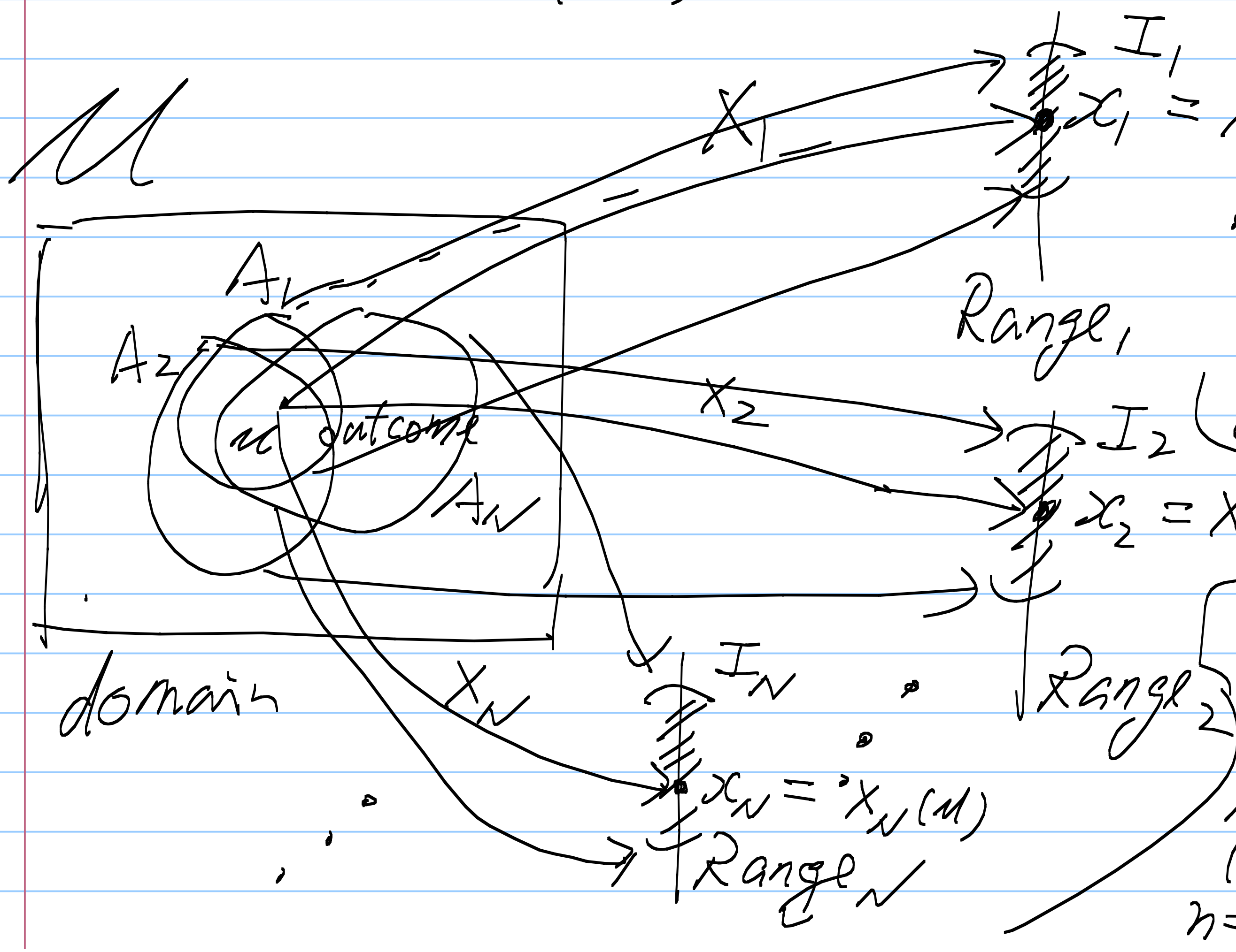
Note 2'

Consider $X_1(u), X_2(u), \cdots, X_N(u)$:



- I_n is the image of A_n, $n = 1, 2, \cdots, N$.
- We can use $\{X_n(u) \in I_n\}$ to denote A_n, $n = 1, 2, \cdots, N$.

Also, $\bigcap_{n=1}^{N} A_n = \bigcap_{n=1}^{N} \{X_n(u) \in I_n\}$

$$\{X_1(u) \in I_1, X_2(u) \in I_2, \cdots, X_N(u) \in I_N\}$$

(Here, $\{X_n(u) \in I_n\} \equiv \{X_1(u) \in \text{Range}_1, \cdots, X_n(u) \in I_n, \cdots, X_N(u) \in \text{Range}_N\}$)

To describe the joint statistic of $X_1(u), X_2(u), \cdots, X_N(u)$, we need to know

$$P_r\{X_1(u) \le x_1, X_2(u) \le x_2, \cdots, X_N(u) \le x_N\}$$
$$= F_{X_1, X_2, \cdots, X_N}(x_1, x_2, \cdots, x_N) \quad \forall x_1, x_2, \cdots, x_N$$

which is the joint cdf of $X_1(u), X_2(u), \cdots, X_N(u)$.

If $F_{X_1, X_2, \cdots, X_N}(x_1, x_2, \cdots, x_N)$ is continuous in all arguments and its first partial derivatives exist, then

$$f_{X_1, \cdots, X_N}(x_1, \cdots, x_N) = \frac{\partial^N}{\partial x_1 \cdots \partial x_N} F_{X_1, \cdots, X_N}(x_1, \cdots, x_N)$$

exists and is called the joint pdf of $X_1(u), X_2(u), \cdots, X_N(u)$.

Defn: $X_1(u), X_2(u), \cdots, X_N(u)$ are called mutually independent if and only if, for <u>any</u> $I_1, I_2, \cdots, I_N$,

$$\Pr\{X_{i_1}(u) \in J_1, X_{i_2}(u) \in J_2, \cdots, X_{i_m}(u) \in J_m\}$$

$$(*) = \prod_{k=1}^{m} \Pr\{X_{i_k}(u) \in J_k\},$$

holds for all $m \in \{2, 3, \cdots, N\}$, all $J_1, J_2, \cdots, J_m$ which are regions belonging to $\{I_1, I_2, \cdots, I_N\}$

with $J_l \neq J_n$ for $l \neq n$, and all $i_1, i_2, \cdots, i_m$
which are values belonging to $\{1, 2, \cdots, N\}$
with $i_l \neq i_n$ for $l \neq n$.

It can be shown that

$X_1(u), X_2(u), \cdots, X_N(u)$ are mutually independent

iff $$F_{X_1, \cdots, X_N}(x_1, \cdots, x_N) = \prod_{n=1}^{N} F_{X_n}(x_n)$$

iff $$f_{X_1, \cdots, X_N}(x_1, \cdots, x_N) = \prod_{n=1}^{N} f_{X_n}(x_n)$$

iff $$\underbrace{\Phi_{X_1, \cdots, X_N}(\omega_1, \cdots, \omega_N)}_{= E\{\exp j \sum_{i=1}^{N} \omega_i X_i(u)\}} = \prod_{n=1}^{N} \Phi_{X_n}(\omega_n)$$

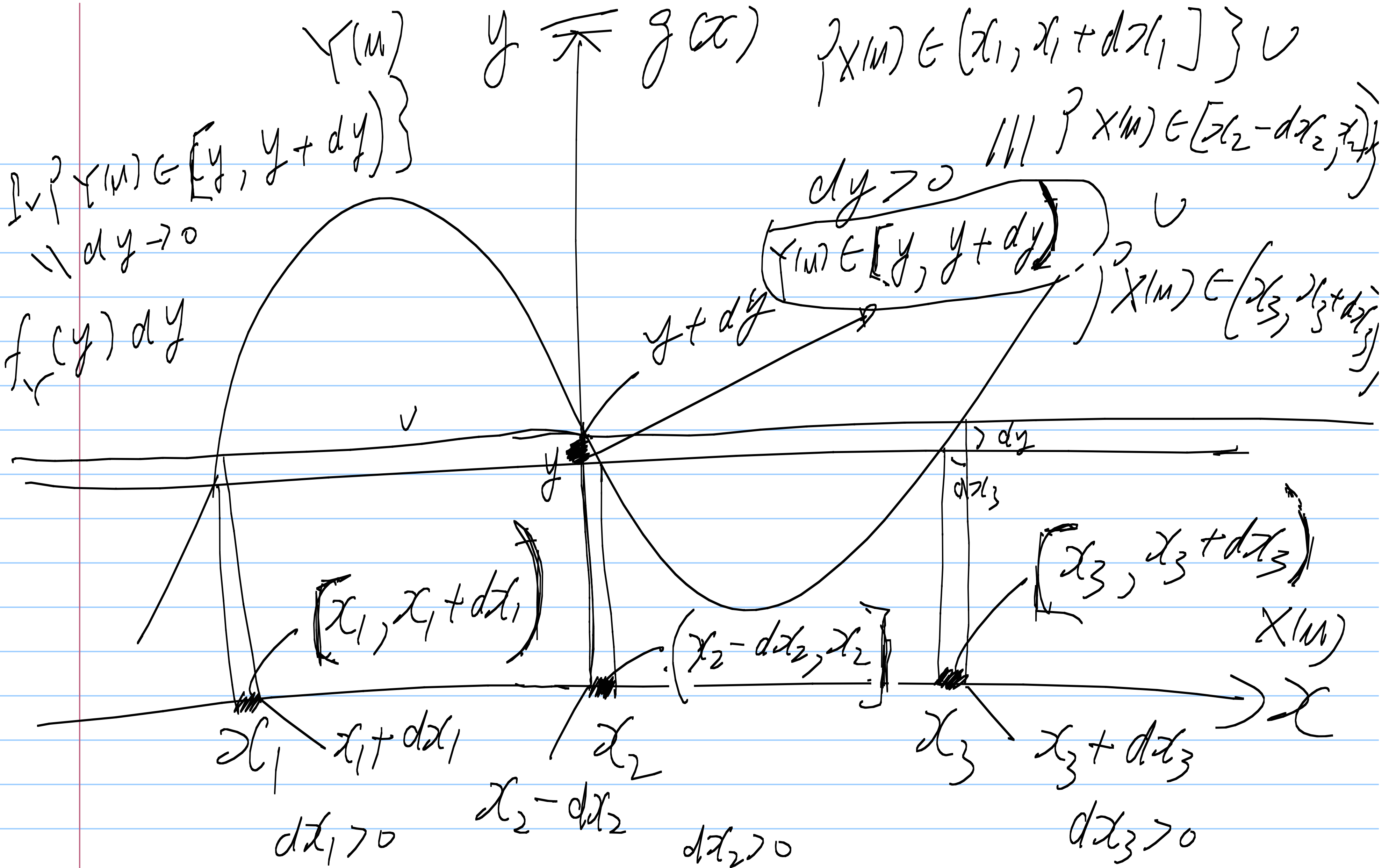
where $\Phi_{X_1,\cdots,X_N}(\omega_1,\cdots,\omega_N) \triangleq E\left\{\exp\left\{j\sum_{n=1}^{N}\omega_n X_n(u)\right\}\right\}$

$\equiv$ joint characteristic function of $X_1(u),\cdots,X_N(u)$.

- For N mutually independent rv's $X_1(u), X_2(u),\cdots,X_N(u)$,

$$E\left\{\prod_{n=1}^{N} g_n(X_n(u))\right\} = \prod_{n=1}^{N} E\left\{g_n(X_n(u))\right\}.$$

$$\underbrace{\int_{-\infty}^{\infty}\cdots\int}_{N\text{-fold}} \prod_{n=1}^{N} g_n(x_n) \underbrace{f_{X_1,\cdots,X_N}(x_1,\cdots,x_N)}_{\prod_{n=1}^{N} f_{X_n}(x_n)} dx_1\cdots dx_N$$



Y(ω) = g(X(ω))
P_Y{Y(ω) ∈ [y, y+dy)}
≈ dy→0
f_Y(y) dy
{X(ω) ∈ (x_1, x_1+dx_1]} ∪
{X(ω) ∈ [x_2−dx_2, x_2)} ∪
{X(ω) ∈ (x_3, x_3+dx_3]}
dy > 0
{Y(ω) ∈ [y, y+dy)}
y + dy
y
dy
dx_3
(x_1, x_1+dx_1]
[x_2−dx_2, x_2)
(x_3, x_3+dx_3]
X(ω)
x
x_1
x_1+dx_1
x_2
x_2−dx_2
x_3
x_3+dx_3
dx_1 > 0
dx_2 > 0
dx_3 > 0

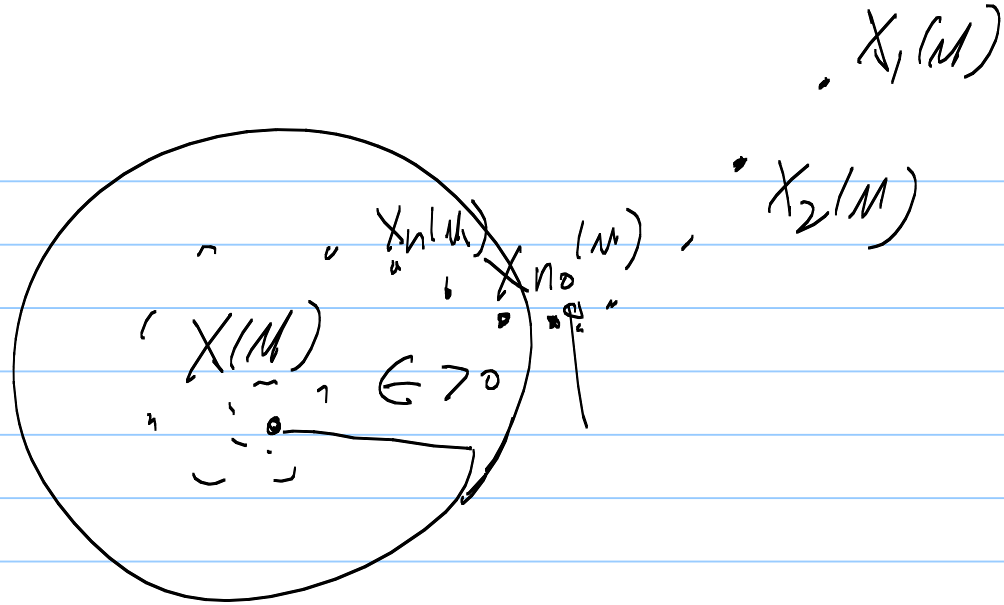
By Jacobian,

$$\Pr\{Y(u) \in [y, y+dy]\} = \Pr\{X(u) \in [x_1, x_1+dx_1]\}$$
$$+ \Pr\{X(u) \in [x_2 - dx_2, x_2]\}$$
$$+ \Pr\{X(u) \in [x_3, x_3+dx_3]\}$$

if $dy \to 0$ $\Bigg\|$

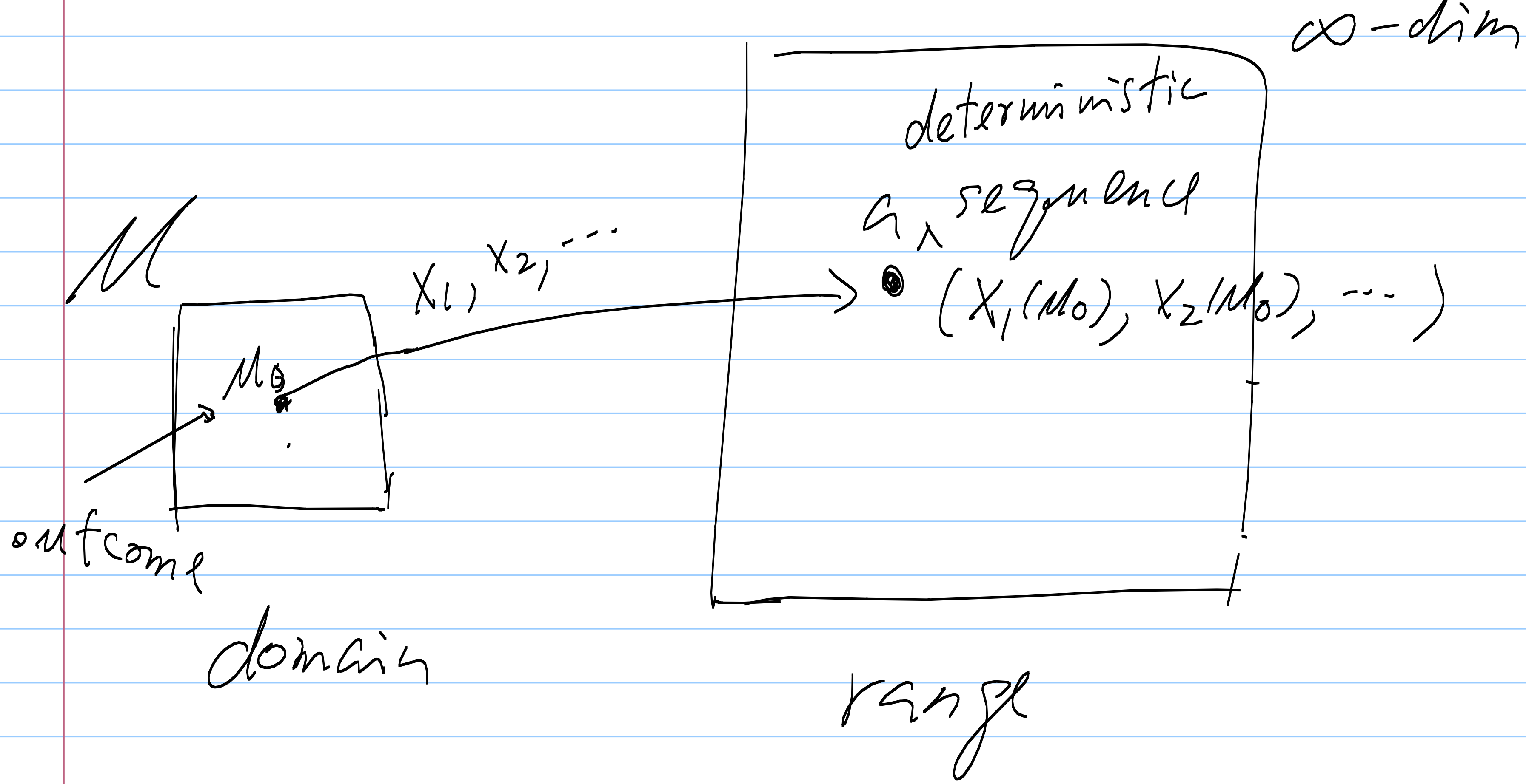
$$f_Y(y)\,dy = f_X(x_1)\,dx_1 + f_X(x_2)\,dx_2 + f_X(x_3)\,dx_3$$

$$f_Y(y) = \sum_{i=1}^{3} f_X(x_i)\left[\underbrace{\frac{dy}{dx_i}}_{\left|\frac{dg(x)}{dx}\right|_{x=x_i}}\right]^{-1}$$

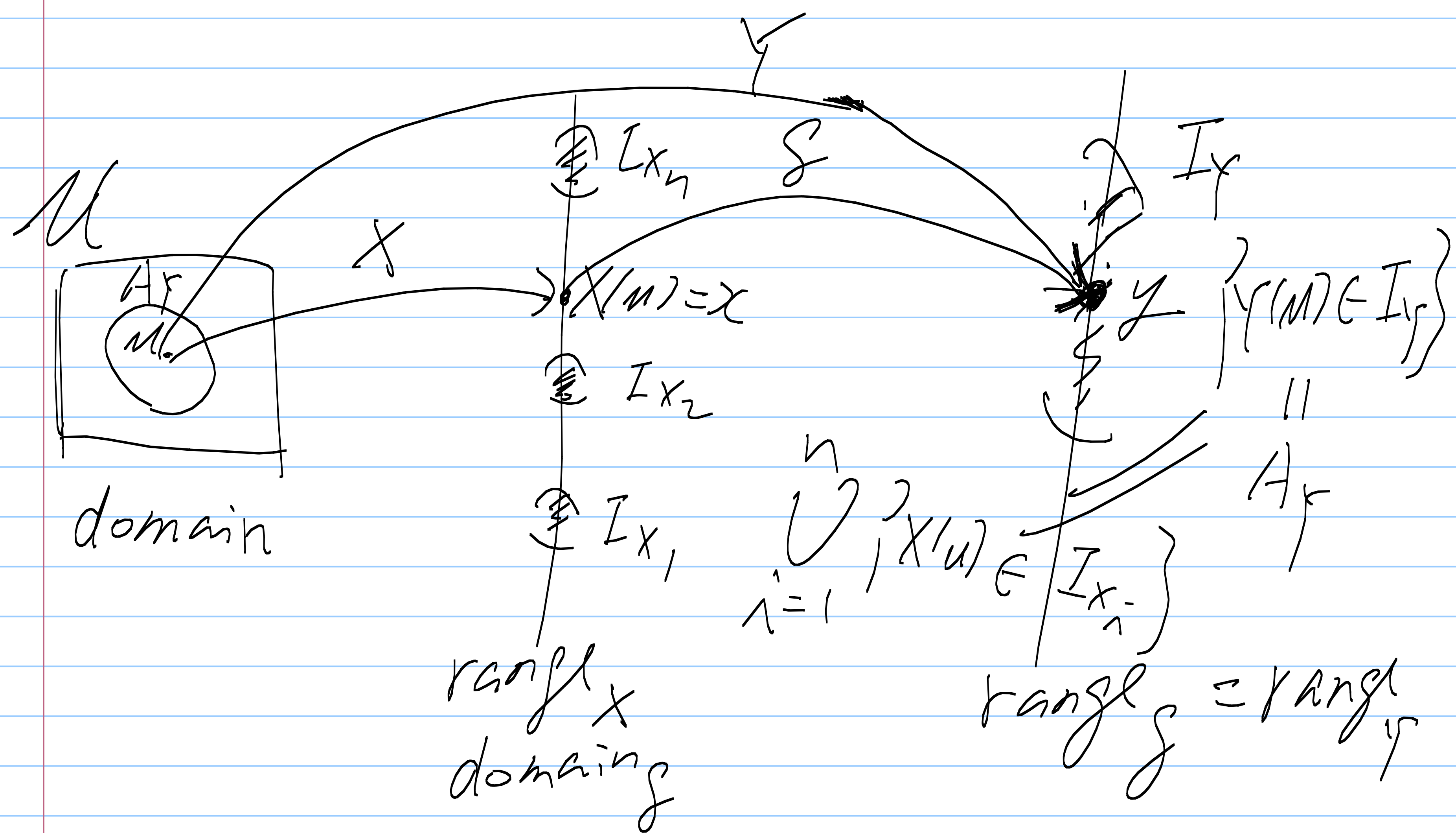


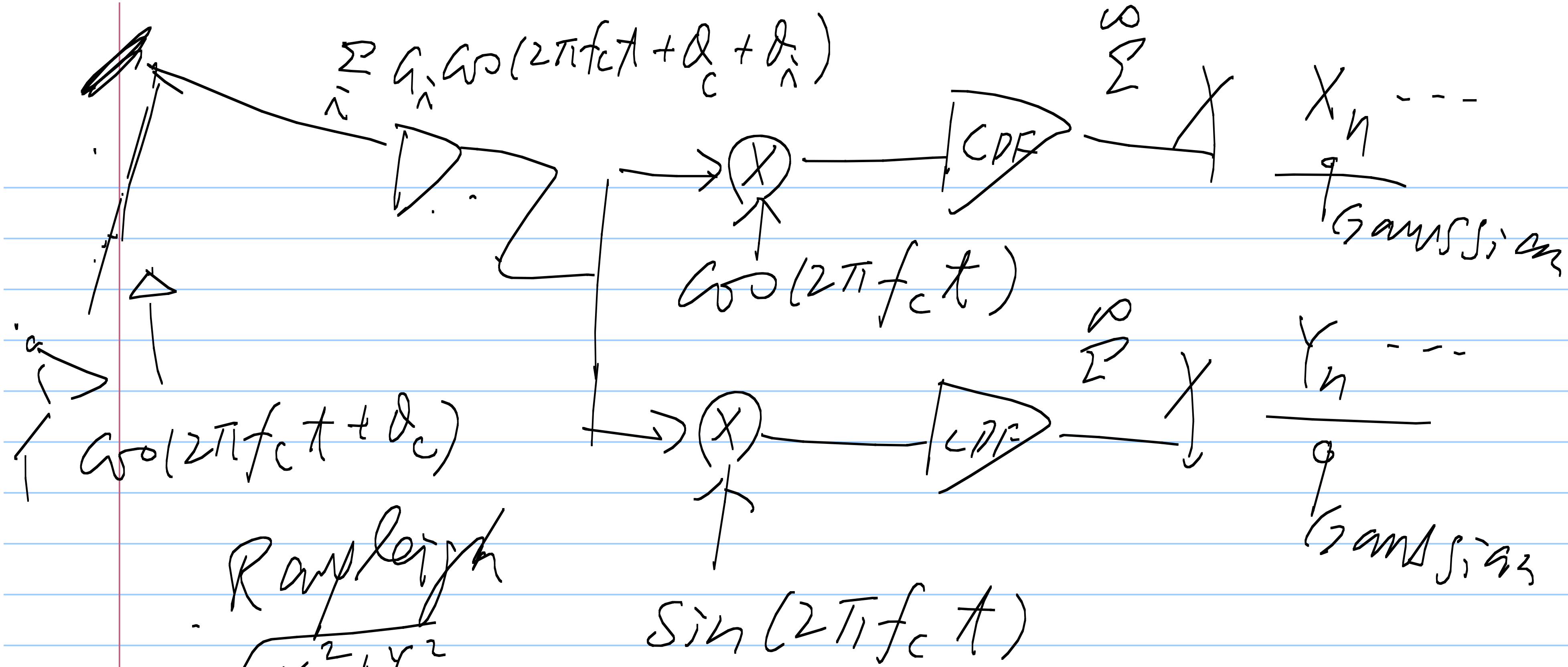
For any $\epsilon > 0$, there exists an n_0 such that $|X_n - X| < \epsilon$ for every $n \geq n_0$.

Consider a random sequence $X_1(\omega), X_2(\omega), \ldots$



Recall: Consider $X(u)$ and $g(X(u)) = Y(u)$





Rayleigh

$\sqrt{X_n^2+Y_n^2}$

$R_n \angle \theta_n = X_n + jY_n$ ~ complex Gaussian

uniform

(X_n, Y_n independent zero mean common variance