(1) (10%, 1% each) Determine whether each of the following statements is **true** or **false**. No proof or explanation is necessary.

- *Statement A:* If events $\mathcal{A}$ and $\mathcal{B}$ are statistically independent and if events $\mathcal{B}$ and $\mathcal{C}$ are statistically independent, then events $\mathcal{A}$ and $\mathcal{C}$ are statistically independent.

Sol: False. Let event $\mathcal{C}$ be the complement of event $\mathcal{A}$. If events $\mathcal{A}$ and $\mathcal{B}$ are statistically independent, events $\mathcal{B}$ and $\mathcal{C}$ are statistically independent as well. However, events $\mathcal{A}$ and $\mathcal{C}$ are NOT statistically independent.

- *Statement B:* If two events $\mathcal{A}$ and $\mathcal{B}$ are statistically independent, then $\mathcal{A}$ can not be a proper set of $\mathcal{B}$.

Sol: False. Let $\mathcal{B}$ be the universe $\mathcal{U}$ and $\mathcal{A}$ be a proper set of $\mathcal{U}$. Then, $\mathcal{A}$ is a proper set of $\mathcal{B}$ and

$$P(\mathcal{A} \cap \mathcal{B}) = P(\mathcal{A}) = P(\mathcal{A})P(\mathcal{U}) = P(\mathcal{A})P(\mathcal{B}).$$

Therefore, events $\mathcal{A}$ and $\mathcal{B}$ are statistically independent.

- *Statement C:* If two events are mutually exclusive, then they are statistically independent.

Sol: False. Let events $\mathcal{A}$ and $\mathcal{B}$ be mutually exclusive with $P(\mathcal{A}) \neq 0$ and $P(\mathcal{B}) \neq 0$. Then, we have

$$P(\mathcal{A} \cap \mathcal{B}) = 0 \neq P(\mathcal{A})P(\mathcal{B})$$

which indicates that events $\mathcal{A}$ and $\mathcal{B}$ are NOT statistically independent.

- *Statement D:* If two events $\mathcal{A}$ and $\mathcal{B}$ are statistically independent, then the probability of their union is equal to the sum of marginal probabilities, i.e., $P(\mathcal{A} \cup \mathcal{B}) = P(\mathcal{A}) + P(\mathcal{B})$.

Sol: False. Let $\mathcal{B}$ be the universe $\mathcal{U}$ and $P(\mathcal{A}) > 0$. Then, $P(\mathcal{A} \cup \mathcal{B}) = 1 < P(\mathcal{A}) + P(\mathcal{B})$ and thus $P(\mathcal{A} \cup \mathcal{B}) \neq P(\mathcal{A}) + P(\mathcal{B})$.

- *Statement E:* If $P(\mathcal{A}|\mathcal{B}) = P(\mathcal{A})$ with $P(\mathcal{B}) \neq 0$, then events $\mathcal{A}$ and $\mathcal{B}$ are not statistically independent.

Sol: False. If $P(\mathcal{A}|\mathcal{B}) = P(\mathcal{A})$, then $P(\mathcal{A} \cap \mathcal{B}) = P(\mathcal{A}|\mathcal{B})P(\mathcal{B}) = P(\mathcal{A})P(\mathcal{B})$ and thus events $\mathcal{A}$ and $\mathcal{B}$ are statistically independent.

- *Statement F:* If a wide-sense stationary random process $X(\mu, t)$ has mean zero, then the power spectrum of $X(\mu, t)$ must be zero on $f = 0$, i.e., $S_X(0) = 0$.

Sol: False. Let $X(\mu, t) \triangleq c(\mu) + n(\mu, t)$ where $c(\mu)$ is a random variable with zero mean and $E\{c^2(\mu)\} = C$ with a positive finite C and $n(\mu, t)$ is a WSS random process with zero mean and autocorrelation function $R_n(\tau) = \delta(\tau)$. Further, we assume that $c(\mu)$ and $n(\mu, t)$ are mutually independent. Then, the mean of $X(\mu, t)$ is zero and the autocorrelation function of $X(\mu, t)$ is given by

$$\begin{aligned} R_X(\tau) &= E\{X(\mu, t+\tau)X(\mu, t)\} \\ &= E\{(c(\mu) + n(\mu, t+\tau))(c(\mu) + n(\mu, t))\} \\ &= E\{c^2(\mu)\} + E\{n(\mu, t+\tau)n(\mu, t))\} \\ &= C + \delta(\tau). \end{aligned}$$

Therefore, $X(\mu, t)$ is a zero-meaned WSS random process with nonzero spectrum on $f = 0$.

- *Statement G:* The function $R(\tau) = \sin(\pi\tau)\exp\{-|\tau|\}$ can be the autocorrelation function of a wide-sense stationary random process.

Sol: False. Because $R(0) = 0 \le R(1/2)$, $R(\tau) = \sin(\pi\tau)\exp\{-|\tau|\}$ can NOT be the autocorrelation function of a wide-sense stationary random process.

- *Statement H:* If $Z(\mu) \triangleq \sum_{n=1}^{N} l_n X_n(\mu)$ is a Gaussian random variable for any set of real-valued constants $\{l_n\}_{n=1}^{N}$ with l_n's constrained to be not all zeros simultaneously, then every $X_n(\mu)$ is marginally Gaussian distributed.

Sol: True. Let $l_k = 1$ and $l_m = 0$ for $m \neq k$. Then, $Z(\mu) = \sum_{n=1}^{N} l_n X_n(\mu) = X_k(\mu)$ is Gaussian distributed.

- *Statement I:* The Gaussian process $X(\mu, t)$ with mean function $\eta_X(t) = 0$ and autocorrelation function $R_X(t_1, t_2) = \delta(t_1 - t_2)$ (with $\delta(\tau)$ a Dirac delta function) possesses the strict-sense stationary property.

Sol: True. For Gaussian random processes, WSS implies SSS.

- *Statement J:* For any event $\mathcal{A}$ and any continuous random variable $X(\mu)$,

$$P(\mathcal{A}) = P(\mathcal{A}|X(\mu) \le x)F_X(x) + P(\mathcal{A}|X(\mu) > x)(1 - F_X(x)).$$

Sol: True. Define event $\mathcal{B} \triangleq \{\mu | X(\mu) \le x\}$ and denote $\mathcal{B}^c$ as its complement. We have $P(A) = P(\mathcal{A} \cap \mathcal{B}) + P(\mathcal{A} \cap \mathcal{B}^c) = P(\mathcal{A}|\mathcal{B})P(\mathcal{B}) + P(\mathcal{A}|\mathcal{B}^c)P(\mathcal{B}^c)$ $= P(\mathcal{A}|X(\mu) \le x)F_X(x) + P(\mathcal{A}|X(\mu) > x)(1 - F_X(x))$.

(2) (4%; 2% each) Let $X_1(\mu)$, $X_2(\mu)$, $X_3(\mu)$ be independent and identically distributed (i.i.d.) continuous random variables with a common continuous probability density function $f_X(x)$ and a common cumulative probability distribution function $F_X(x)$. Now, form $Y_1(\mu)$, $Y_2(\mu)$, $Y_3(\mu)$ in a way that for a given outcome μ

$$\begin{aligned} \text{if } X_{k_1}(\mu) &\le X_{k_2}(\mu) \le X_{k_3}(\mu), \text{ then} \\ Y_1(\mu) &= X_{k_1}(\mu),\ Y_2(\mu) = X_{k_2}(\mu), \text{ and } Y_3(\mu) = X_{k_3}(\mu) \end{aligned}$$

where $X_{k_1}(\mu), X_{k_2}(\mu), X_{k_3}(\mu)$ are the three numbers $X_1(\mu), X_2(\mu), X_3(\mu)$ arranged in a nondecreasing order.

(a) Find the probability density function of $Y_2(\mu)$ (i.e., $f_{Y_2}(y)$) in terms of $f_X(\cdot)$ and $F_X(\cdot)$.

Sol: Denote $X(\mu)$ as a random variable with probability density function $f_X(x)$ and cumulative probability distribution function $F_X(x)$. For a differential dy,

$$\begin{aligned}
& \Pr\{y \leq Y_2(\mu) < y + dy\} \\
= \; & f_{Y_2}(y)dy \\
= \; & \Pr\{\text{one of } X_1(\mu), X_2(\mu), X_3(\mu) \text{ are smaller than } y, \text{ another} \\
& \text{is not smaller than } y + dy, \text{ and} \\
& \text{the rest is in } [y, y + dy)\} \\
= \; & \binom{3}{1}\binom{2}{1} \Pr\{X(\mu) < y\} \Pr\{X(\mu) \geq y + dy\} \Pr\{y \leq X(\mu) < y + dy\} \\
= \; & 6F_X(y)(1 - F_X(y + dy)) \cdot f_X(y)dy.
\end{aligned}$$

As $dy \to 0$, we thus have

$$f_{Y_2}(y) = 6F_X(y)(1 - F_X(y))f_X(y).$$

(b) Find the joint probability density function of $Y_1(\mu)$ and $Y_3(\mu)$ (i.e., $f_{Y_1,Y_3}(w, z)$) in terms of $f_X(\cdot)$ and $F_X(\cdot)$.

Sol: For $z > w$ and differentials dw and dz,

$$\begin{aligned}
& \Pr\{w \leq Y_1(\mu) < w + dw, z \leq Y_3(\mu) < z + dz\} = f_{Y_1,Y_3}(w, z)dwdz \\
= \; & \Pr\{\text{one of } X_1(\mu), X_2(\mu), X_3(\mu) \text{ is in } [w + dw, z), \\
& \text{another is in } [z, z + dz), \text{ and the rest is in } [w, w + dw)\} \\
= \; & \binom{3}{1}\binom{2}{1} \Pr\{w + dw \leq X(\mu) < z\} \Pr\{z \leq X(\mu) < z + dz\} \\
& \Pr\{w \leq X(\mu) < w + dw\} \\
= \; & 6\left(F_X(z) - F_X(w + dw)\right) \cdot f_X(z)dz \cdot f_X(w)dw.
\end{aligned}$$

As $dz \to 0$ and $dw \to 0$, we thus have

$$f_{Y_1,Y_3}(w, z) = 6\left(F_X(z) - F_X(w)\right) f_X(z)f_X(w)$$

which holds for $z > w$ and $f_{Y_1,Y_N}(w, z) = 0$ otherwise.

3) (6%; 2% each) Let $\{X_n(\mu); n = 1, 2, ...\}$ be a random sequence consisting of i.i.d. binary-valued random variables, each taking value $+1$ or -1 with equal probability, i.e., $\Pr\{X_n(\mu) = +1\} = \Pr\{X_n(\mu) = -1\} = \frac{1}{2}$. Define a new random variable

$$Z_n(\mu) \triangleq \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k(\mu).$$

(a) Find the characteristic function $\Phi_{Z_n}(\omega)$ of $Z_n(\mu)$. If we put the answer in the form of

$$\Phi_{Z_n}(\omega) \triangleq \exp\{ng(\omega, n)\}$$

what is $g(\omega, n)$? Give your answer in terms of cos function. (Note: $\cos(x) = \frac{1}{2}\{\exp\{jx\} + \exp\{-jx\}\}$).

Sol: By definition,

$$\begin{aligned} g(\omega, n) &= \frac{1}{n}\ln[\Phi_{Z_n}(\omega)] = \frac{1}{n}\ln[\prod_{k=1}^{n}\Phi_{X_k}(\frac{\omega}{\sqrt{n}})] \\ &= \frac{1}{n}\sum_{k=1}^{n}\ln[\Phi_{X_k}(\frac{\omega}{\sqrt{n}})]. \end{aligned}$$

Because $\Phi_{X_k}(\frac{\omega}{\sqrt{n}}) = \frac{1}{2}\exp\{j\frac{\omega}{\sqrt{n}}\} + \frac{1}{2}\exp\{-j\frac{\omega}{\sqrt{n}}\} = \cos(\frac{\omega}{\sqrt{n}})$, we have

$$g(\omega, n) = \ln[\cos(\frac{\omega}{\sqrt{n}})].$$

(b) Evaluate the limit $\lim_{n\to\infty} ng(\omega, n)$.

Sol: By Central Limit Theorem,

$$\lim_{n\to\infty}\Phi_{Z_n}(\omega) = \exp\{-\frac{1}{2}\omega^2\} = \exp\{\lim_{n\to\infty} ng(\omega, n)\}.$$

Thus, $\lim_{n\to\infty} ng(\omega, n) = -\frac{1}{2}\omega^2$. Another method is also given here:

$$\lim_{n\to\infty} ng(\omega, n) = \lim_{n\to\infty}\frac{\ln[\cos(\frac{\omega}{\sqrt{n}})]}{1/n}.$$

By L$'$Hopital's rule,

$$\begin{aligned} \lim_{n\to\infty} ng(\omega, n) &= \lim_{n\to\infty}\frac{\frac{-\sin(\frac{\omega}{\sqrt{n}})}{\cos(\frac{\omega}{\sqrt{n}})}n^{-3/2}(-\frac{\omega}{2})}{-1/n^2} \\ &= (-\frac{\omega}{2})\cdot\lim_{n\to\infty}\frac{\tan(\frac{\omega}{\sqrt{n}})}{n^{-1/2}}. \end{aligned}$$

Applying L$'$Hopital's rule again, we have

$$\lim_{n\to\infty} ng(\omega, n) = -\frac{1}{2}\omega^2.$$

(c) Write down the Chernoff bound

$$\Pr\{Z_n(\mu) \geq \epsilon\} \leq \min_{\lambda\geq 0} E\{\exp\{\lambda(Z_n(\mu) - \epsilon)\}\} \triangleq \min_{\lambda\geq 0}\exp\{f(\lambda, n, \epsilon)\}$$

for some positive constant ϵ. Identify exactly the exponent function $f(\lambda, n, \epsilon)$ in the bound in terms of cosh function. (Note: $\cosh(x) = \frac{1}{2}\{\exp\{x\} + \exp\{-x\}\}$).

Sol: Now,

$$\begin{aligned}
\Pr\{Z_n(\mu) \geq \epsilon\} &\leq \min_{\lambda \geq 0} E\{\exp\{\lambda(Z_n(\mu) - \epsilon)\}\} \\
&= \min_{\lambda \geq 0} \exp\{-\lambda\epsilon\} E\{\exp\{\lambda Z_n(\mu)\}\} \\
&= \min_{\lambda \geq 0} \exp\{-\lambda\epsilon\} \Phi_{Z_n}(-j\lambda) \\
&= \min_{\lambda \geq 0} \exp\{-\lambda\epsilon\} \exp\{n \ln[\cos(-\frac{j\lambda}{\sqrt{n}})]\} \\
&= \min_{\lambda \geq 0} \exp\{-\lambda\epsilon + n \ln[\cosh(\frac{\lambda}{\sqrt{n}})]\}
\end{aligned}$$

since $\cosh(x) = \frac{1}{2}\{\exp\{x\} + \exp\{-x\}\} = \cos(-jx)$. Thus,

$$f(\lambda, n, \epsilon) = -\lambda\epsilon + n \ln[\cosh(\frac{\lambda}{\sqrt{n}})].$$

(4) (2%) Assume that two random variables $X(\mu)$ and $Y(\mu)$ are observed by a receiver. The receiver makes decision based on the metric $Z(\mu) \triangleq X(\mu)Y(\mu)$. An error occurs if $Z(\mu) > Q^{-1}(\alpha)$ and the decision is correct otherwise, where Q^{-1} is the inverse function of $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\{\frac{-1}{2}y^2\}dy$ and α is a positive design value with $0 < \alpha < 1$. It is also known that $X(\mu)$ and $Y(\mu)$ are mutually independent, that $X(\mu)$ is a Gaussian random variable with zero mean and unit variance, and that $Y(\mu)$ is a binary-valued random variable with the common probability mass function $\Pr\{Y(\mu) = y\} = \frac{1}{2}$ if $y = +1$ or $y = -1$, and $\Pr\{Y(\mu) = y\} = 0$ otherwise. Now, determine the probability of incorrect decision for the receiver, i.e., $\Pr\{Z(\mu) > Q^{-1}(\alpha)\}$. Your answer should be expressed in terms of α.

Sol: Given $Y(\mu)$, $Z(\mu)$ is conditionally Gaussian distributed with zero mean and unit variance. This conditional density has nothing to do with $Y(\mu)$. Thus, $Z(\mu)$ is a Gaussian random variable with zero mean and unit variance. Therefore, the probability of incorrect decision is given by $\Pr\{Z(\mu) > Q^{-1}(\alpha)\}$, which is derived as

$$\begin{aligned}
\Pr\{Z(\mu) &> Q^{-1}(\alpha)\} = Q(Q^{-1}(\alpha)) \\
&= \alpha.
\end{aligned}$$

(5) (4%; 2% each) Consider a linear and time invariant system with continuous real input $X(\mu, t)$, continuous real output $Y(\mu, t)$, continuous real impulse response $h(t)$, and system function $H(\omega)$. Let $X(\mu, t)$ and $Y(\mu, t)$ be both wide-sense stationary random processes with means η_X and η_Y, respectively, and power spectrums $S_X(\omega)$ and $S_Y(\omega)$, respectively. Also, let $h(t) = 1$ if $|t| < 1/2$ and $h(t) = 0$ otherwise.

(a) It is known that $\eta_Y = \alpha\eta_X$ with α a constant. Determine α.

Sol: Because $\eta_Y(t) = \eta_X(t) * h(t) = \int_{-\infty}^{\infty} \eta_X(t-x)h(x)dx = \eta_X \int_{-\infty}^{\infty} h(x)dx$, $\alpha = \int_{-\infty}^{\infty} h(x)dx = 1$.

(b) Express $S_Y(\omega)$ in terms of $S_X(\omega)$. Derivation is required.

Sol: Because $H(\omega) = \mathcal{F}\{h(t)\} = \sin(\omega/2)/(\omega/2)$,

$$S_Y(\omega) = S_X(\omega)\frac{\sin^2(\omega/2)}{(\omega/2)^2}.$$

(6) (2%) Let $A(\mu)$ and $B(\mu)$ be independent and identically distributed Gaussian random variables with mean zero and unit variance. Also, denote p as the probability that the random process $X(\mu, t) = A(\mu) - B(\mu)t$ crosses the t axis in the time interval $(0, T)$. Show that $\pi p = \arctan(T)$.

Sol: $X(\mu, t)$ crosses the t axis in the time interval $(0, T)$ if and only if $\exists\, t_1 \in (0, T)$ such that $A(\mu) - B(\mu)t_1 = 0$. Hence, we have $p = \Pr\{0 < \frac{A(\mu)}{B(\mu)} < T\}$. Further, because $A(\mu)$ and $B(\mu)$ are independent and identically distributed Gaussian random variables with mean zero and unit variance, $C(\mu) \triangleq \frac{A(\mu)}{B(\mu)}$ is a Cauchy random variable with probability distribution function $F_C(c) = \frac{1}{2} + \frac{1}{\pi}\arctan(c)$. As a result, p is given by

$$\begin{aligned} p &= \Pr\{0 < \frac{A(\mu)}{B(\mu)} < T\} \\ &= F_C(T) - F_C(0) \\ &= \frac{1}{\pi}\arctan(T) \end{aligned}$$

and thus $\pi p = \arctan(T)$.

(7) (2%) Consider the real-valued Gaussian random process $X(\mu, t)$ which have mean zero, i.e., $\eta_X(t) = E\{X(\mu, t)\} = 0$ and autocorrelation $R_X(t_1, t_2) = \delta(t_1 - t_2)$ with $\delta(t)$ being the Dirac delta function. Also define K new random processes $Y_1(\mu, t), Y_2(\mu, t), ..., Y_K(\mu, t)$ by the outputs of the K linear time-invariant systems with real-valued impulse responses $h_1(t), h_2(t), ..., h_K(t)$, respectively, and common input $X(\mu, t)$. Prove that $\sum_{k=1}^{K} Y_k(\mu, t)$ is strict-sense stationary.

Sol: Now, $Y_k(\mu, t)$ can be represented as $Y_k(\mu, t) = X(\mu, t) * h_k(t)$ with $*$ being the convolution operator. Thus, we can represent sum process $Z(\mu, t) \triangleq \sum_{k=1}^{K} Y_k(\mu, t)$ by

$$\begin{aligned} Z(\mu, t) &= \sum_{k=1}^{K} X(\mu, t) * h_k(t) \\ &= X(\mu, t) * \sum_{k=1}^{K} h_k(t). \end{aligned}$$

Thus, the mean function and the autocorrelation function of $Z(\mu, t)$ are given by

$$\begin{aligned} \eta_Z(t) &\triangleq E\{Z(\mu, t)\} = \eta_X(t) * \sum_{k=1}^{K} h_k(t) = 0 \\ R_Z(t_1, t_2) &\triangleq E\{Z(\mu, t_1)Z(\mu, t_2)\} \end{aligned}$$

$$
\begin{aligned}
&= R_X(t_1,t_2) * \sum_{k_1=1}^{K} h_{k_1}(t_1) * \sum_{k_2=1}^{K} h_{k_2}(t_2) \\
&= \delta(t_1-t_2) * \sum_{k_1=1}^{K} h_{k_1}(t_1) * \sum_{k_2=1}^{K} h_{k_2}(t_2) \\
&= \sum_{k_1=1}^{K} h_{k_1}(t_1-t_2) * \sum_{k_2=1}^{K} h_{k_2}(t_2) \\
&= \sum_{k_1=1}^{K}\sum_{k_2=1}^{K} h_{k_1}(t_1-t_2) * h_{k_2}(t_2)
\end{aligned}
$$

where $h_{k_1}(t_1-t_2) * h_{k_2}(t_2)$ can be expressed as

$$
\begin{aligned}
h_{k_1}(t_1-t_2) * h_{k_2}(t_2) &= \int_{-\infty}^{\infty} h_{k_1}(t_1-\tau)h_{k_2}(t_2-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h_{k_1}(x)h_{k_2}(x+t_2-t_1)dx \\
(x &= t_1-\tau)
\end{aligned}
$$

and is a function of t_1-t_2. Since $\eta_Z(t)$ is a constant and $R_Z(t_1,t_2)$ is a function of t_1-t_2 only, $Z(\mu,t)$ is wide-sense stationary. Further, $Z(\mu,t)$ is a Gaussian process because it is the result of a linear transform of Gaussian process $X(\mu,t)$. Thus, $Z(\mu,t)$ is strict-sense stationary since a wide-sense stationary Gaussian process is strict-sense stationary.