(1) (4%, 2% each) Let $X_n(\mu)$'s, $n = 1, ..., N$, be independent and identically distributed (iid) Gaussian random variables with zero mean and unit variance. Also, let $Y_n(\mu)$'s, $n = 1, ..., N$, be iid binary random variables with the common probability density function $f_Y(y) = \frac{1}{2}$ if $y = +1$ or $y = -1$, and $f_Y(y) = 0$ otherwise. In addition, $X_n(\mu)$'s, $Y_n(\mu)$'s, $n = 1, ..., N$, are mutually independent. Now, define a new random variable $Z(\mu) \triangleq \sum_{n=1}^{N} X_n(\mu)Y_n(\mu)$. Answer the following questions:

(a) Derive the probability density function of the random variable $Z(\mu)$.

Sol: Given $Y_n(\mu)$'s, $Z(\mu)$ is conditionally Gaussian distributed with zero mean and variance N. This conditional density has nothing to do with $Y_n(\mu)$'s. Thus, $Z(\mu)$ is a Gaussian random variable with zero mean and variance N.

(b) Find the probability $\Pr\{Z(\mu) > 0\}$.

Sol: Since $Z(\mu)$ is a Gaussian random variable with zero mean, the probability $\Pr\{Z(\mu) > 0\}$ is given by $Q(0) = 1/2$.

(2) (3%; 1% each) Let $X(\mu)$ and $Y(\mu)$ be two real-valued random variables with probability density functions

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(x - m_X)^2\}$$
$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(y - m_Y)^2\}.$$

Determine whether each of the following statements is true or not. Explain your answer. (Any correct answer without explanation will result in zero point.)

(a) If $E\{X(\mu)Y(\mu)\} = E\{X(\mu)\}E\{Y(\mu)\}$, then $X(\mu)$ and $Y(\mu)$ are jointly Gaussian.

(b) $X(\mu) + Y(\mu)$ is a Gaussian random variable.

(c) If conditioned on $Y(\mu)$ the random variable $X(\mu)$ is Gaussian distributed, then $X(\mu)$ and $Y(\mu)$ are jointly Gaussian.

Sol: All are false.

(a) False. This is because "uncorrelated marginally Gaussian" does NOT imply "jointly Gaussian."

(b) False. This is because $X(\mu)$ and $Y(\mu)$ may NOT be jointly Gaussian.

(c) False. This is because

$$\begin{aligned} f_{X,Y}(x,y) &= f_{X|Y}(x|y)f_Y(y) \\ &= \frac{1}{\sqrt{2\pi\sigma_{X|Y}^2}} \exp\{-\frac{1}{2\sigma_{X|Y}^2}(x - m_{X|Y}(y))^2\} \\ &\quad \cdot \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\{-\frac{1}{2\sigma_Y^2}(y - m_Y)^2\} \end{aligned}$$

may not have a quadratic exponent in x and y.

(3) (6%, 2% each) Consider the experiment of rolling two fair dices independently. Define two random variables $X(\mu)$ and $Y(\mu)$ as the values of both dices that face upward after a single trial. Also, define the random variables $Z(\mu) = X(\mu) + Y(\mu)$ and $W(\mu) = X(\mu)Y(\mu)$.

(a) Determine the probability $\Pr\{Z(\mu) = n\}$ for all integer n.

Sol: Now,

$$\Pr\{Z(\mu) = n\} = \begin{cases} 1/36, & n = 2, 12 \\ 2/36, & n = 3, 11 \\ 3/36, & n = 4, 10 \\ 4/36, & n = 5, 9 \\ 5/36, & n = 6, 8 \\ 6/36, & n = 7 \\ 0, & \text{otherwise} \end{cases}.$$

(b) Determine the conditional probability $\Pr\{Z(\mu) = n \mid X(\mu) = m\}$ for all integers n and m.

Sol: Next,

$$\Pr\{Z(\mu) = n|X(\mu) = m\} = \begin{cases} 1/6, & (n, m) \in A \\ 0, & \text{otherwise} \end{cases}.$$

where $A = \{(m + k, m)|m \in \{1, 2, 3, 4, 5, 6\}, k \in \{1, 2, 3, 4, 5, 6\}\}$.

(c) Determine the variance $\mathrm{Var}\{W(\mu)\}$.

Sol: Since $X(\mu)$ and $Y(\mu)$ are independent, we have

$$\begin{aligned} E\{W(\mu)\} &= E\{X(\mu)Y(\mu)\} \\ &= E\{X(\mu)\}E\{Y(\mu)\} \\ &= 49/4. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Var}\{W(\mu)\} &= E\{W^2(\mu)\} - E^2\{W(\mu)\} \\ &= E\{X^2(\mu)\}E\{Y^2(\mu)\} - (49/4)^2 \\ &= (91/6)^2 - (49/4)^2 \\ &= 11515/144. \end{aligned}$$

(4) (4%) Define x_u as the u-percentile of the continuous-typed random variable $X(\mu)$ (i.e., $F_X(x_u) = u$ with F_X being the distribution function of $X(\mu)$). Prove that $x_{1-u} = -x_u$ if the density of $X(\mu)$ is an even function and if $F_X(x)$ increases monotonically with its argument x.

Sol: Since the density of $X(\mu)$ is an even function, we have $\int_{-\infty}^{-x} f_X(t)dt = \int_x^{\infty} f_X(t)dt$. Therefore, $F_X(-x) = 1 - F_X(x)$ and $F_X(-x_u) = 1 - F_X(x_u) = 1 - u = F_X(x_{1-u})$. If $F_X(x)$ is strictly increasing, then $F_X(x)$ is one-to-one and thus $-x_u = x_{1-u}$. This completes the proof. Noteworthily, when $F_X(x)$ is not strictly increasing, $-x_u = x_{1-u}$ is not true in general.

(5) (2%; 1% each) Determine whether each of the following functions can be the autocorrelation function of a real-valued wide-sense stationary random process. Explain your answer. (Any correct answer without explanation will result in zero point.)

(a) $R_1(\tau) = \exp\{-|\tau|\}$

(b) $R_2(\tau) = \frac{1-|\tau|+|\tau|^2}{1+|\tau|}$.

Sol: $R_X(\tau)$ of a real-valued WSS $X(\mu, t)$ has to satisfy three conditions: (i) $R_X(0) \geq |R_X(\tau)| \geq 0$, for all τ, (ii) $R_X(\tau) = R_X(-\tau)$, for all τ, and (iii) $R_X(\tau)$ is nonnegative definite. $R_1(\tau)$ can be the autocorrelation function of a real-valued wide-sense stationary random process, but $R_2(\tau)$ can not, because

(a) $R_1(\tau)$ satisfies all three conditions,

(b) $R_2(0) = 1 < R_2(\tau)$ for $\tau = 3$.

(6) (3%) Prove that for any real-valued random variables $X(\mu)$ and $Y(\mu)$

$$|E\{X(\mu)Y(\mu)\}|^2 \leq E\{X^2(\mu)\}E\{Y^2(\mu)\}.$$

Sol: Consider the second moment of the random variable $X(\mu) + aY(\mu)$ with a being a real number. By definition, we have

$$\begin{aligned} E\{(X(\mu) + aY(\mu))^2\} &= E\{X^2(\mu)\} + 2aE\{X(\mu)Y(\mu)\} + a^2E\{Y^2(\mu)\} \\ &\geq 0 \quad \text{for any real number } a. \end{aligned}$$

Because the above expression is a quadratic form of a and is nonnegative for any real number a, the discriminant $4|E\{X(\mu)Y(\mu)\}|^2 - 4E\{X^2(\mu)\}E\{Y^2(\mu)\}$ must not be larger than zero. Therefore, we have

$$4|E\{X(\mu)Y(\mu)\}|^2 - 4E\{X^2(\mu)\}E\{Y^2(\mu)\} \leq 0$$

which implies

$$|E\{X(\mu)Y(\mu)\}|^2 \leq E\{X^2(\mu)\}E\{Y^2(\mu)\}.$$

(7) (4%, 2% each) Consider wide-sense stationary random processes $X(\mu, t)$ and $Y(\mu, t)$ which are related by $Y(\mu, t) = \sum_{n=1}^{2N}(-1)^n X(\mu, t+n)$. Express (a) $R_Y(\tau)$ in terms of $R_X(\tau)$, and (b) $S_Y(\omega)$ in terms of $S_X(\omega)$. Make expressions as neat as possible.

Sol: (a) The autocorrelation $R_Y(t_1,t_2)$ is given by

$$\begin{aligned}
R_Y(t_1,t_2) &= E\{Y(\mu,t_1)Y^*(\mu,t_2)\} \\
&= \sum_{n,m=1}^{2N} E\{(-1)^n X(\mu,t_1+n)(-1)^m X^*(\mu,t_2+m)\} \\
&= \sum_{n,m=1}^{2N} (-1)^{n+m} E\{X(\mu,t_1+n)X^*(\mu,t_2+m)\} \\
&= \sum_{n,m=1}^{2N} (-1)^{n+m} R_X(\tau+n-m) \\
&= R_Y(\tau)
\end{aligned}$$

where $\tau = t_1 - t_2$. (b) The spectrum $S_Y(\omega)$ is given by

$$\begin{aligned}
S_Y(\omega) &= \mathcal{F}\{R_Y(\tau)\} \\
&= \mathcal{F}\{\sum_{n,m=1}^{2N} (-1)^{n+m} R_X(\tau+n-m)\} \\
&= \sum_{n,m=1}^{2N} (-1)^{n+m} \mathcal{F}\{R_X(\tau+n-m)\} \\
&= S_X(\omega) \sum_{n,m=1}^{2N} (-1)^{n+m} e^{j\omega(n-m)} \\
&= S_X(\omega) |\sum_{n=1}^{2N} (-1)^n e^{j\omega n}|^2 \\
&= S_X(\omega) |\frac{1-e^{j2N\omega}}{1+e^{j\omega}}|^2.
\end{aligned}$$

(8) (4%) Consider a linear and time-invariant system with impulse response $h(t)$, input process $X(\mu,t)$, and output process $Y(\mu,t)$. Show that if $h(t)=0$ outside the time interval $(-T,T)$ and $X(\mu,t)$ is a zero-mean white noise, then $R_Y(t_1,t_2)=0$ for $|t_1-t_2|>2T$.

Sol: Since the considered system is linear and time-invariant, $Y(\mu,t)$ is given by $Y(\mu,t) = \int_0^T X(\mu,t-\tau)h(\tau)d\tau$. Therefore, when $|t_1-t_2|>2T$, $R_Y(t_1,t_2)$ is derived as

$$\begin{aligned}
R_Y(t_1,t_2) &= E\{Y(\mu,t_1)Y^*(\mu,t_2)\} \\
&= E\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\mu,t_1-\tau_1)h(\tau_1)X^*(\mu,t_2-\tau_2)h^*(\tau_2)d\tau_2 d\tau_1\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha\delta(t_1-\tau_1-t_2+\tau_2)h(\tau_1)h^*(\tau_2)d\tau_2 d\tau_1 \\
&= \alpha\int_{-\infty}^{\infty} h(\tau_1)h^*(\tau_1+t_2-t_1)d\tau_1
\end{aligned}$$

$$
\begin{aligned}
&= \alpha \int_{-T}^{T} h(\tau_1) h^*(\tau_1 + t_2 - t_1) d\tau_1 \\
&= 0
\end{aligned}
$$

where the last equality stems from the fact that $|t_1 - t_2| > 2T$ and thus $h(\tau_1)$ and $h^*(\tau_1 + t_2 - t_1)$ can not be nonzero at the same time.