Stochastic Processes and Applications, Fall 2016
Homework Two (5%)

1. (3%, 0.5% each) Consider the random process

$$Y(\mu, t) = A(\mu)\cos(2\pi f t + \phi(\mu))$$

where $f > 0$ is a constant, $A(\mu)$ is a real-valued random variable with the probability density function $f_A(a) = a\exp\{-\frac{a^2}{2}\}u(a)$, where $u(a) = 1$ if $a \geq 0$ and $u(a) = 0$ if $a < 0$, and $\phi(\mu)$ is a real-valued random variable which is uniformly distributed in $[0, 2\pi)$. Furthermore, $A(\mu)$ and $\phi(\mu)$ are mutually independent. Find

(a) The probability density function of $Y(\mu, t)$.

(b) The joint probability density function of $Y(\mu, 0)$ and $Y(\mu, \frac{1}{4f})$.

(c) The mean function of $Y(\mu, t)$.

(d) The autocorrelation function of $Y(\mu, t)$.

(e) Show that $Y(\mu, t)$ is wide-sense stationary. Also, derive the power spectrum of $Y(\mu, t)$.

(f) Define a new process $Z(\mu, t)$ by $Z(\mu, t) = \sum_{k=0}^{K} z_k Y(\mu, t - k)$ where z_k's are real-valued. Derive the autocorrelation of $Z(\mu, t)$.

Sol: The solution is itemized below.

(a) Let $X(\mu, t) = A(\mu)\sin(2\pi f t + \phi(\mu))$. By Jacobian, $X(\mu, t)$ and $Y(\mu, t)$ are independent and identically distributed Gaussian random variables which have zero mean and unit variance, for a fixed t. Thus, the first-order density of $Y(\mu, t)$ is of a Gaussian density with zero mean and unit variance.

(b) $Y(\mu, 0) = A(\mu)\cos(\phi(\mu))$ and $Y(\mu, \frac{1}{4f}) = -A(\mu)\sin(\phi(\mu))$. By Jacobian, $Y(\mu, 0)$ and $Y(\mu, \frac{1}{4f})$ are independent and identically distributed Gaussian random variables which have zero mean and unit variance.

(c) $m_Y(t) = E\{Y(\mu, t)\} = E\{A(\mu)\}E\{\cos(2\pi f t + \phi(\mu))\} = 0$ since $\phi(\mu)$ is uniform in $[0, 2\pi)$.

(d) $R_Y(\tau) = R_Y(t + \tau, t) = E\{Y(\mu, t + \tau)Y(\mu, t)\} = \frac{1}{2}E\{A^2(\mu)\}\cos(2\pi f\tau) = \cos(2\pi f\tau)$ because $\frac{1}{2}E\{A^2(\mu)\} = \frac{1}{2}\int_0^\infty a^3\exp\{-\frac{a^2}{2}\}da = 1$.

(e) Because $Y(\mu, t)$ has mean zero and autocorrelation $\cos(2\pi f\tau)$ depending only time difference, $Y(\mu, t)$ is wide-sense stationary. Also, the power spectrum of $Y(\mu, t)$ is the Fourier transform of $\cos(2\pi f\tau)$, given by $S_Y(x) = \int_{-\infty}^{\infty}\cos(2\pi f\tau)\exp\{-j2\pi x\tau\}d$ $\frac{1}{2}\delta(x - f) + \frac{1}{2}\delta(x + f)$.

(f) Now, $Z(\mu,t)$ can be expressed in terms of convolution $Z(\mu,t) = Y(\mu,t) * \sum_{k=0}^{K} z_k \delta(t-k) = Y(\mu,t) * h(t)$ with $h(t) = \sum_{k=0}^{K} z_k \delta(t-k)$. Thus, $Z(\mu,t)$ can be regarded as the output to the linear and time-invariant (LTI) system with impulse response $h(t)$ and input $Y(\mu,t)$. Because $Y(\mu,t)$ is wide-sense stationary, so is $Z(\mu,t)$. Note that this LTI system has the deterministic autocorrelation

$$\begin{aligned}
\rho(\tau) &= h(\tau) * h(-\tau) \\
&= \int_{-\infty}^{\infty} h(\tau - x) h(-x) dx \\
&= \sum_{k=0}^{K} \sum_{l=0}^{K} z_l z_k \int_{-\infty}^{\infty} \delta(\tau - x - k)\delta(-x - l) dx \\
&= \sum_{k=0}^{K} \sum_{l=0}^{K} z_l z_k \delta(\tau - k + l) \\
&= \sum_{m=-K}^{K} \delta(\tau - m) \times \sum_{k=\max\{0,m\}}^{\min\{K,K+m\}} z_{k-m} z_k .
\end{aligned}$$

Thus, the autocorrelation of $Z(\mu,t)$ is given by

$$\begin{aligned}
R_Z(\tau) &= R_Y(\tau) * \rho(\tau) \\
&= R_Y(\tau) * \sum_{m=-K}^{K} \delta(\tau - m) \sum_{k=\max\{0,m\}}^{\min\{K,K+m\}} z_{k-m} z_k \\
&= \sum_{m=-K}^{K} R_Y(\tau - m) \sum_{k=\max\{0,m\}}^{\min\{K,K+m\}} z_{k-m} z_k \\
&= \sum_{m=-K}^{K} \sum_{k=\max\{0,m\}}^{\min\{K,K+m\}} z_{k-m} z_k \cos(2\pi f(\tau - m)).
\end{aligned}$$

(2) (0.5%) Let $X(\mu,t)$ be a wide-sense stationary random process with $\eta_X(t) = 0$ and $R_X(\tau) = 0$ for $|\tau| > 1$ and $R_X(\tau) = 1 - |\tau|$ for $|\tau| \le 1$. Is $X(\mu,t)$ mean-ergodic?

Sol: Now,

$$\begin{aligned}
\frac{1}{2T}\int_{-2T}^{2T} (1 - \frac{|\tau|}{2T}) R_X(\tau) d\tau &= \frac{1}{2T}\int_{-1}^{1} (1 - \frac{|\tau|}{2T})(1 - |\tau|) d\tau \\
&< \frac{1}{2T}\int_{-1}^{1} (1 - |\tau|) d\tau \\
&= \frac{1}{4T}
\end{aligned}$$

which shows

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T} (1 - \frac{|\tau|}{2T}) R_X(\tau) d\tau = 0.$$

Thus, $X(\mu, t)$ is mean-ergodic

(3) (1%, 0.5% each) Let $X_1(\mu)$, $X_2(\mu), ..., X_N(\mu)$ be independent and identically distributed (i.i.d.) continuous random variables with a common continuous probability density function (p.d.f.) $f_X(x)$ and cumulative probability distribution function (c.d.f.) $F_X(x)$. Now, form $Y_1(\mu)$, $Y_2(\mu), ...,$ $Y_N(\mu)$ in a way that for a given outcome μ

$$Y_1(\mu) = X_{k_1}(\mu) \leq Y_2(\mu) = X_{k_2}(\mu) \leq ... \leq Y_N(\mu) = X_{k_N}(\mu)$$

where $X_{k_1}(\mu), ..., X_{k_N}(\mu)$ are the N numbers $X_1(\mu), X_2(\mu), ..., X_N(\mu)$ in an increasing order.

(a) Find the p.d.f. of $Y_k(\mu)$ for any $1 \leq k \leq N$.

(b) Find the joint p.d.f. of $Y_N(\mu)$ and $Y_1(\mu)$.

Sol: Solve the subquestions separately.

(a) For a differential dy,

$$\begin{aligned}
& \Pr\{y \leq Y_k(\mu) < y + dy\} = f_{Y_k}(y)dy \\
= \ & \Pr\{(k-1) \text{ of } X_1(\mu), X_2(\mu), ..., X_N(\mu) \text{ are smaller than } y, \\
& (N-k) \text{ of the remainder are not smaller than } y + dy, \text{ and} \\
& \text{the rest is in } [y, y + dy)\} \\
= \ & \binom{N}{k-1}\binom{N-k+1}{N-k} \left(\Pr\{X(\mu) < y\}\right)^{k-1} \left(\Pr\{X(\mu) \geq y\}\right)^{N-k} \\
& \Pr\{y \leq X(\mu) < y + dy\} \\
= \ & \binom{N}{k-1}\binom{N-k+1}{N-k}(F_X(y))^{k-1}(1 - F_X(y + dy))^{N-k} \cdot f_X(y)dy.
\end{aligned}$$

As $dy \to 0$, we thus have

$$f_{Y_k}(y) = \frac{N!}{(k-1)!(N-k)!}(F_X(y))^{k-1}(1 - F_X(y))^{N-k} f_X(y).$$

(b) For $z > w$ and differentials dw and dz,

$$\begin{aligned}
& \Pr\{w \leq Y_1(\mu) < w + dw, z \leq Y_N(\mu) < z + dz\} = f_{Y_1,Y_N}(w, z)dwdz \\
= \ & \Pr\{(N-2) \text{ of } X_1(\mu), X_2(\mu), ..., X_N(\mu) \text{ are in } [w + dw, z), \\
& \text{one of the remainder is in } [z, z + dz), \text{ and} \\
& \text{the rest is in } [w, w + dw)\} \\
= \ & \binom{N}{N-2}\binom{2}{1} \left(\Pr\{w + dw \leq X(\mu) < z\}\right)^{N-2} \Pr\{z \leq X(\mu) < z + dz\} \\
& \Pr\{w \leq X(\mu) < w + dw\} \\
= \ & N(N-1) \left(F_X(z) - F_X(w + dw)\right)^{N-2} \cdot f_X(z)dz \cdot f_X(w)dw.
\end{aligned}$$

As $dz \to 0$ and $dw \to 0$, we thus have

$$f_{Y_1,Y_N}(w,z) = N(N-1)\left(F_X(z) - F_X(w)\right)^{N-2} f_X(z) f_X(w)$$

which holds for $z > w$ and $f_{Y_1,Y_N}(w,z) = 0$ otherwise.

(4) (0.5%) Define $Y_n(\mu) \triangleq u(a - X_n(\mu))$ where $u(x)$ is the unit step function with $u(x) = 1$ if $x \geq 0$ and $u(x) = 0$ otherwise, and $X_n(\mu)$'s are independent and identically distributed with common distribution $F_X(x) = \Pr\{X_n(\mu) \leq x\}$. Is $Y_1(\mu), Y_2(\mu), ...$ mean-ergodic? Also, find $\lim_{n\to\infty} E\{Y_n(\mu)\}$ and $\lim_{n\to\infty} \mathrm{Var}\{Y_n(\mu)\}$.

Sol: Now, the mean and the variance of $Y_n(\mu)$ is given by

$$\begin{aligned} E\{Y_n(\mu)\} &= E\{u(a - X_n(\mu))\} \\ &= \Pr\{a - X_n(\mu) \geq 0\} \\ &= F_X(a) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}\{Y_n(\mu)\} &= E\{u^2(a - X_n(\mu))\} - E^2\{Y_n(\mu)\} \\ &= E\{u(a - X_n(\mu))\} - E^2\{Y_n(\mu)\} \\ &= E\{Y_n(\mu)\} - E^2\{Y_n(\mu)\} \\ &= F_X(a) - F_X^2(a). \end{aligned}$$

Note that $\mathrm{Var}\{Y_n(\mu)\}$ is finite. Because $X_n(\mu)$'s are independent and identically distributed, so are $Y_n(\mu)$'s. Thus,

$$\begin{aligned} \lim_{n\to\infty} E\{Y_n(\mu)\} &= E\{Y_1(\mu)\} = F_X(a) \\ \lim_{n\to\infty} \mathrm{Var}\{Y_n(\mu)\} &= \mathrm{Var}\{Y_1(\mu)\} = F_X(a) - F_X^2(a) \end{aligned}$$

Also, because $Y_n(\mu)$'s are independent and identically distributed, its sample mean sequence $Z_N(\mu)$'s with $Z_N(\mu) = \frac{1}{N}\sum_{n=1}^{N} Y_n(\mu)$ has the common mean

$$E\{Z_N(\mu)\} = E\{Y_n(\mu)\} = F_X(a)$$

and the variance

$$\mathrm{Var}\{Z_N(\mu)\} = \frac{1}{N}\mathrm{Var}\{Y_n(\mu)\} = \frac{F_X(a) - F_X^2(a)}{N}.$$

Because $0 \leq F_X(a) \leq 1$,

$$\lim_{N\to\infty} \mathrm{Var}\{Z_N(\mu)\} = 0.$$

Thus,

$$\begin{aligned} \lim_{N\to\infty} E\{|Z_N(\mu) - F_X(a)|^2\} &= \lim_{N\to\infty} E\{|Z_N(\mu) - E\{Z_N(\mu)\}|^2\} \\ &= \lim_{N\to\infty} \mathrm{Var}\{Z_N(\mu)\} \\ &= 0. \end{aligned}$$

This shows that the sample mean sequence $Z_1(\mu), Z_2(\mu), ...$ converges to $F_X(a)$ in the mean square sense and thus the sequence $Y_1(\mu), Y_2(\mu), ...$ is mean-ergodic.