(1) (10%) Consider the deterministic system $\mathbf{T}_1$ with real-valued input process $X(\mu, t)$ and real-valued output process $Y(\mu, t)$ being related by

$$Y(\mu, t) = \mathbf{T}_1[X(\mu, t)] = \sum_{k=0}^{K} h^k X(\mu, t-k)$$

where h is a deterministic real-valued factor with $0 < h < 1$. Also, feed $Y(\mu, t)$ into the other deterministic system $\mathbf{T}_2$ with real-valued output process $Z(\mu, t)$ which is related to $Y(\mu, t)$ by

$$Z(\mu, t) = \mathbf{T}_2[Y(\mu, t)] = Y(\mu, -t).$$

Answer the following questions:

(a) (2%) Prove that the system $\mathbf{T}_1$ is linear and time-invariant. Also, find the impulse response of the system in terms of the Dirac delta function $\delta(t)$.

(b) (1%) Let $X(\mu, t)$ and $Y(\mu, t)$ are both wide-sense stationary random processes with means η_X and η_Y. It is known that $\eta_Y = \alpha(h)\eta_X$ with $\alpha(h)$ a function of h. Determine $\alpha(h)$ in a closed-form expression.

(c) (2%) If $X(\mu, t)$ is a Gaussian random process with mean $\eta_X(t) = 0$ and autocorrelation $R_X(t_1, t_2) = \delta(t_1 - t_2)$, find the second-order density of $Y(\mu, t)$, i.e., the joint probability density function of random variables $Y(\mu, t_1)$ and $Y(\mu, t_2)$ for any two distinct time points t_1 and t_2.

(d) (2%) If $X(\mu, t)$ is a wide-sense stationary random process with mean $\eta_X(t) = 0$ and autocorrelation $R_X(t_1, t_2) = \delta(t_1 - t_2)$, find the power spectrum of $Y(\mu, t)$.

(e) (3%) If $X(\mu, t)$ is a wide-sense stationary random process with mean $\eta_X(t) = 0$ and autocorrelation $R_X(t_1, t_2) = \delta(t_1 - t_2)$, find the autocorrelation of $Z(\mu, t)$. Since $Z(\mu, t)$ is wide-sense stationary as well, find the power spectrum of $Z(\mu, t)$.

(2) (3%) Consider the random process $X(\mu, t)$ for $|t| < 1$ which has mean zero, i.e., $\eta_X(t) = 0$, and autocorrelation $R_X(t, s) = \cos(\pi(t - s)) + 1 - 2\sin^2(\pi(t - s))$ for $|t| < 1$ and $|s| < 1$. Find the Karhunen-Loève expansion of $X(\mu, t) + X(\mu, -t)$ in the interval $(-1, 1)$.

(3) (2%, 1% each) Determine whether each of the following functions can be the power spectrum of a real-valued wide-sense stationary random process? Explain your answer. (Any correct answer without explanation will result in zero point.)

(a) $S_1(\omega) = \ln\{1 + \frac{1}{|\omega|}\}$

(b) $S_2(\omega) = \exp\{\omega^3 + \omega^4\}$

(4) (3%) Denote $\widehat{W}(\mu, t)$ as the Hilbert transform of the wide-sense stationary real-valued Gaussian random process $W(\mu, t)$ which has mean zero and autocorrelation $R_W(\tau)$. Describe the joint statistic of random processes $\widehat{W}(\mu, t)$ and $W(\mu, t)$. A complete description of joint statistic is required and needs to be explained.

(5) (6%, 2% each) If packets enter a network router with two input ports, namely Port A and Port B, according to a Poisson process with a rate of λ packets per minute, and if each packet enters Port A independently of the others, with probability p $(0 < p < 1)$, then find the probabilities of the following events (in terms of λ and p):

(a) *Event A:* No packet enters the router for two minutes.

(b) *Event B:* In a given N minutes, there are K packets entering the router and L of them entering Port A in every minute, where N is a positive integer and K and L are nonnegative integers with $0 \leq L \leq K$.

(c) *Event C:* There is only one packet entering Port A in the first minute, provided that only N packets enter the router in the first N minutes, where N is a positive integer with $N \geq 2$. (Your answer must not contain the parameter λ.)

(6) (3%) Consider the random process

$$X(\mu, t) = A(\mu)\cos(\omega_c t + \phi(\mu)) + n(\mu, t)$$

where ω_c is a deterministic radian frequency. Here, $n(\mu, t)$ is a stationary Gaussian random process with mean zero and autocorrelation $R_n(\tau) = \delta(\tau)$. $A(\mu)$ is a Rayleigh random variable with probability density function $f_A(a) = ae^{-a^2/2}u(a)$ and $u(a)$ a unit step function. $\phi(\mu)$ is a uniform random variable in $[0, 2\pi)$. Moreover, $A(\mu)$, $\phi(\mu)$, and $n(\mu, t)$ are mutually independent. Is $X(\mu, t)$ a wide-sense stationary process? Is $X(\mu, t)$ a Gaussian process? Prove your answer.

(7) (3%) Let $X_1(\mu), X_2(\mu), ..., X_N(\mu)$ be independent Poisson random variables with parameters λ_1, λ_2,..., and λ_N, respectively. Prove that the sum random variable $Y(\mu) = \sum_{n=1}^{N} X_n(\mu)$ is a Poisson random variable with parameter $\sum_{n=1}^{N} \lambda_n$.

(Hint: Note that the Poisson random variable with parameter λ_n has the probability mass $\Pr\{X_n(\mu) = k\} = \exp\{-\lambda_n\}\lambda_n^k/k!$ for $k = 0, 1, ...$ You may give your proof in terms of moment generating function.)