(1) (10%, 1% each) Determine whether each of the following statements is **true** or **false**. No proof or explanation is necessary.

- *Statement A:* The function $R(\tau) = |\tau| \exp\{-|\tau|\}$ can be the autocorrelation function of a wide-sense stationary random process.

Sol: False. This is because $R(0) \leq R(\tau) = |\tau| \exp\{-|\tau|\}$ violates the property that $R(0) \geq |R(\tau)|$ holds for a wide-sense stationary random process.

- *Statement B:* $A(\omega) = \exp\{\omega + 2\omega^2 - \omega^4\}$ can be the power spectrum of a real-valued wide-sense stationary random process.

Sol: False. The power spectrum $S_X(\omega)$ of a real-valued wide-sense stationary random process $X(\mu, t)$ has to satisfy two conditions (i) $S_X(\omega) \geq 0$ for all ω and (ii) $S_X(\omega) = S_X(-\omega)$ for all ω (even function). Based on these conditions, $A(\omega)$ can not be a power spectrum because it is not even.

- *Statement C:* The output process $Y(\mu, t)$ from a deterministic linear and time-invariant system with real-valued impulse response $h(t)$ and real-valued input process $X(\mu, t)$ is wide-sense stationary if $X(\mu, t)$ is wide-sense stationary.

Sol: True. This is because $E\{Y(\mu, t)\} = \eta_X \int_{-\infty}^{\infty} h(\tau) d\tau$ is a constant and $E\{Y(\mu, t+\tau)Y(\mu, t)\} = R_X(\tau) * h(\tau) * h(-\tau)$ depends only on time difference, where $*$ denotes the convolution operator.

- *Statement D:* The Hilbert transform $\widehat{W}(\mu, t)$ of the real-valued Gaussian random process $W(\mu, t)$ is a real-valued Gaussian random process.

Sol: True. This is because Hilbert transform is an LTI system and linearly transforming a Gaussian process yields a Gaussian process.

- *Statement E:* If $X(\mu, t)$ and $Y(\mu, t)$ are independent Poisson random processes, then $X(\mu, t) + Y(\mu, t)$ is also a Poisson process.

Sol: True. The sum of independent Poisson processes forms a Poisson process.

- *Statement F:* Consider a linear and time-invariant system with input process $X(\mu, t)$, output process $Y(\mu, t)$ and impulse response $h(t)$. If $X(\mu, t)$ is wide sense stationary, then $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$ where $S_X(\omega)$ and $S_Y(\omega)$ are power spectrums of $X(\mu, t)$ and $Y(\mu, t)$, respectively, and $H(\omega)$ is the Fourier transform of $h(t)$.

Sol: True. Since $X(\mu, t)$ is wide sense stationary, so is the LTI system output process $Y(\mu, t)$. Therefore, both processes have well defined power spectrums that are related by $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$.

- *Statement G:* Let $X_1(\mu), X_2(\mu), ..., X_n(\mu)$ be jointly Gaussian random variables with zero mean. If $X_i(\mu)$ and $X_j(\mu)$ are orthogonal for any $i \neq j$, then $X_1(\mu), X_2(\mu), ..., X_n(\mu)$ are mutually independent.

Sol: True. It is because independence in pairs does imply mutual independence for jointly Gaussian random variables.

- *Statement H:* Let $X(\mu)$ and $Y(\mu)$ be independent and identically distributed real-valued Gaussian random variables which have zero mean and unit variance. Define two new random variables $W(\mu)$ and $V(\mu)$ by $W(\mu) = X(\mu) + Y(\mu)$ and $V(\mu) = X(\mu) + aY(\mu)$ where a is a real constant. There exists a positive value for a that makes $W(\mu)$ and $V(\mu)$ independent.

Sol: False. First, $W(\mu)$ and $V(\mu)$ are jointly Gaussian with zero means since they are linear transforms of zero-mean Gaussian random variables $X(\mu)$ and $Y(\mu)$. Second, the covariance of $W(\mu)$ and $V(\mu)$ is $E\{W(\mu)V(\mu)\} = E\{X^2(\mu)\} + aE\{Y^2(\mu)\} = 1 + a$. Thus, only when $a = -1$, $W(\mu)$ and $V(\mu)$ are uncorrelated and thus independent. When a is positive-valued, $W(\mu)$ and $V(\mu)$ are correlated and thus dependent on each other.

- *Statement I:* If a complex-valued random process $Z(\mu, t) = X(\mu, t) + jY(\mu, t)$ is wide-sense stationary (with $j = \sqrt{-1}$, $X(\mu, t) \triangleq \text{Re}\{Z(\mu, t)\}$ and $Y(\mu, t) \triangleq \text{Im}\{Z(\mu, t)\}$), then $Z^*(\mu, t)$ is wide-sense stationary.

Sol: True. First, if $E\{Z(\mu, t)\} = \eta_Z$ is independent of t, then $E\{Z^*(\mu, t)\} = \eta_Z^*$ is also independent of t. Further, if $E\{Z(\mu, t+\tau)Z^*(\mu, t)\} = R_Z(\tau)$ depends on τ only, then $E\{Z^*(\mu, t+\tau)Z(\mu, t)\} = E\{Z(\mu, t)Z^*(\mu, t+\tau)\} = R_Z(-\tau)$ depends on τ only as well. Hence, $Z^*(\mu, t)$ is wide-sense stationary.

- *Statement J:* If $X(\mu, t)$ and $Y(\mu, t)$ are independent Poisson processes, then $X(\mu, t) - Y(\mu, t)$ is a Poisson process.

Sol: False. This is because $X(\mu, t) - Y(\mu, t)$ is NOT a counting process.

(2) (5%) Consider the real-valued bandpass stationary Gaussian noise process $n(\mu, t)$ which has zero mean and the power spectral density $S_n(f)$, given by

$$S_n(f) = \begin{cases} \frac{1}{4B}, & |f - f_c| \leq B \\ \frac{1}{4B}, & |f + f_c| \leq B \\ 0, & \text{otherwise} \end{cases}$$

with $f_c \gg B > 0$. Let $\widehat{n}(\mu, t)$, $n_c(\mu, t)$ and $n_s(\mu, t)$ be the Hilbert transform, the inphase component, and the quadrature-phase component of $n(\mu, t)$, respectively, related by

$$\begin{aligned} \widehat{n}(\mu, t) &= \int_{-\infty}^{\infty} \frac{n(\mu, t - x)}{\pi x} dx \\ n(\mu, t) &= n_c(\mu, t)\cos(2\pi f_c t) - n_s(\mu, t)\sin(2\pi f_c t). \end{aligned}$$

Also, let $\widetilde{n}(\mu, t) = n_c(\mu, t) + jn_s(\mu, t)$ where $n_c(\mu, t)$ and $n_s(\mu, t)$ are real-valued inphase and quadrature-phase components of of $n(\mu, t)$, respectively. Answer the following subquestions.

(A) (2%) You are told that $n(\mu, t)$ and $\widehat{n}(\mu, t)$ are jointly wide-sense stationary Gaussian processes which have zero mean. Derive the cross-correlation $R_{n,\widehat{n}}(\tau) \triangleq E\{n(\mu, t+\tau)\widehat{n}(\mu, t)\}$. A closed-form expression is required.

Sol: By definition,

$$\begin{aligned}
R_{n,\widehat{n}}(\tau) &= E\{n(\mu, t+\tau)\widehat{n}(\mu, t)\} \\
&= \int_{-\infty}^{\infty} \frac{1}{\pi x} E\{n(\mu, t+\tau)n(\mu, t-x)\}dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\pi x} R_n(\tau + x)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\pi x} R_n^*(\tau + x)dx \\
&\quad (R_n(\tau + x) \text{ is real-valued}) \\
&= \int_{-\infty}^{\infty} (-j\,\mathrm{sgn}(f))(S_n(f)\exp\{j2\pi f\tau\})^* df \\
&\quad (\text{Parseval's relation}) \\
&= \frac{-j}{4B}\int_{|f-f_c|\le B} \exp\{-j2\pi f\tau\}df \\
&\quad + \frac{j}{4B}\int_{|f+f_c|\le B} \exp\{-j2\pi f\tau\}df \\
&= \frac{-j\exp\{-j2\pi f_c\tau\}}{4B}\int_{|y|\le B} \exp\{-j2\pi y\tau\}dy \\
&\quad + \frac{j\exp\{j2\pi f_c\tau\}}{4B}\int_{|y|\le B} \exp\{-j2\pi y\tau\}dy \\
&= \frac{-\sin(2\pi f_c\tau)}{2B}\int_{|y|\le B} \exp\{-j2\pi y\tau\}dy \\
&= -\sin(2\pi f_c\tau)\,\mathrm{sinc}(2B\tau)
\end{aligned}$$

where $\mathrm{sinc}\,x \triangleq \frac{\sin \pi x}{\pi x}$.

(B) (1%) Find all possible values of z for which $n(\mu, t+z)$ and $\widehat{n}(\mu, t)$ are independent for any fixed time point t.

Sol: Because $n(\mu, t)$ and $\widehat{n}(\mu, t)$ are jointly wide-sense stationary Gaussian processes with mean zero, $n(\mu, t+z)$ and $\widehat{n}(\mu, t)$ are independent for any fixed time point t if and only if $R_{n,\widehat{n}}(z) = 0$ (i.e., $n(\mu, t+z)$ and $\widehat{n}(\mu, t)$ are uncorrelated). From (A), we have that

$$R_{n,\widehat{n}}(z) = -\sin(2\pi f_c z)\,\mathrm{sinc}(2Bz)$$

and thus $R_{n,\widehat{n}}(z) = 0$ when $z = \frac{n}{2B}$ for any nonzero integer n or $z = \frac{m}{2f_c}$ for any integer m. Thus, $n(\mu, t+z)$ and $\widehat{n}(\mu, t)$ are independent for any fixed time point t, when z is given by

$$\begin{aligned}
z &= \frac{n}{2B} \text{ for any nonzero integer } n \\
\text{or } z &= \frac{m}{2f_c} \text{ for any integer } m.
\end{aligned}$$

(C) (1%) Find all possible values of z for which $n_c(\mu, t+z)$ and $n_s(\mu, t)$ are independent for any fixed time point t.

Sol: Because $n_c(\mu, t)$ and $n_s(\mu, t)$ are linear combinations of jointly Gaussian random processes $n(\mu, t)$ and $\widehat{n}(\mu, t)$ with zero mean, $n_c(\mu, t)$ and $n_s(\mu, t)$ are jointly Gaussian with zero mean. Because the local symmetry of $S_n(f)$ holds, $R_{n_c,n_s}(\tau) = 0$ and thus $n_c(\mu, t)$ and $n_s(\mu, t)$ are independent. Thus, z can be any real value to make that $n_c(\mu, t+z)$ and $n_s(\mu, t)$ are independent for any fixed time point t.

(D) (1%) Determine whether $n(\mu, t)$ and $\widehat{n}(\mu, t)$ for a fixed t are independent and identically distributed Gaussian random variables.

Sol: True. Because $\widehat{n}(\mu, t)$ is a linear transform of the Gaussian process $n(\mu, t)$ with zero mean, $n(\mu, t)$ and $\widehat{n}(\mu, t)$ are jointly Gaussian with zero mean. Further, because $n(\mu, t)$ and $\widehat{n}(\mu, t)$ have the same autocorrelation with $R_{n,\widehat{n}}(0) = 0$, $n(\mu, t)$ and $\widehat{n}(\mu, t)$ are independent and identically distributed Gaussian random variables for a fixed t.

(3) (4%) Let two real-valued $X(\mu, t)$ and $Y(\mu, t)$ be jointly wide-sense stationary. Show that if $E\{|X(\mu, a) - Y(\mu, a)|^2\} = 0$ for some constant a, then $R_X(\tau) = R_Y(\tau) = R_{XY}(\tau)$ for all values of τ. (Hint: For any two real-valued random variables $A(\mu)$ and $B(\mu)$, $E^2\{A(\mu)B(\mu)\} \leq E\{A^2(\mu)\}E\{B^2(\mu)\}$ from Schwartz's inequality.)

Sol: Define $A(\mu) = X(\mu, t+\tau)$ and $B(\mu) = X(\mu, t) - Y(\mu, t)$. Now,

$$
\begin{aligned}
E\{A(\mu)B(\mu)\} &= R_X(\tau) - R_{XY}(\tau) \\
E\{A^2(\mu)\} &= R_X(0) \\
E\{B^2(\mu)\} &= E\{X^2(\mu, t)\} + E\{Y^2(\mu, t)\} - 2E\{X(\mu, t)Y(\mu, t)\} \\
&= R_X(0) + R_Y(0) - 2R_{XY}(0).
\end{aligned}
$$

Since $E\{|X(\mu, a) - Y(\mu, a)|^2\} = R_X(0) + R_Y(0) - 2R_{XY}(0) = 0$, we have $E\{B^2(\mu)\} = 0$ and, from Schwartz's inequality, i.e., $E^2\{A(\mu)B(\mu)\} \leq E\{A^2(\mu)\}E\{B^2(\mu)\}$, $R_X(\tau) = R_{XY}(\tau)$ for all τ. Similarly, if we let $A(\mu) = Y(\mu, t)$ and $B(\mu) = Y(\mu, t+\tau) - X(\mu, t+\tau)$, then we can also come out with $R_Y(\tau) = R_{XY}(\tau)$ for all τ. Thus, $R_X(\tau) = R_Y(\tau) = R_{XY}(\tau)$ for all τ.

(4) (4%) Consider the wide-sense stationary real-valued Gaussian random process $X(\mu, t)$ with mean zero and autocorrelation $R_X(\tau)$ and power spectrum $S_X(\omega)$. Define a new wide-sense stationary random process $Y(\mu, t) = X^2(\mu, t)$ with mean η_Y and power spectrum $S_Y(\omega)$. Express η_Y and $S_Y(\omega)$ in terms of $R_X(\tau)$ and $S_X(\omega)$.

Sol: By definition, we have

$$
\begin{aligned}
\eta_Y &= E\{Y(\mu, t)\} \\
&= E\{X^2(\mu, t)\} \\
&= R_X(0)
\end{aligned}
$$

and

$$\begin{aligned}
R_Y(\tau) &= E\{Y(\mu,t+\tau)Y(\mu,t)\} \\
&= E\{X^2(\mu,t+\tau)X^2(\mu,t)\} \\
&= E\{X^2(\mu,t+\tau)\}E\{X^2(\mu,t)\} + 2\left[E\{X(\mu,t+\tau)X(\mu,t)\}\right]^2 \\
&= R_X^2(0) + 2R_X^2(\tau) \qquad (1)
\end{aligned}$$

where (1) is because $X(\mu,t)$ is zero-meaned stationary Gaussian random process and

$$\begin{aligned}
E\{\prod_{i=1}^{4} X_i(\mu)\} &= E\{X_1(\mu)X_2(\mu)\}E\{X_3(\mu)X_4(\mu)\} \\
&\quad +E\{X_1(\mu)X_3(\mu)\}E\{X_2(\mu)X_4(\mu)\} \\
&\quad +E\{X_1(\mu)X_4(\mu)\}E\{X_2(\mu)X_3(\mu)\}
\end{aligned}$$

holds for jointly Gaussian random variables $X_1(\mu)$, $X_2(\mu)$, $X_3(\mu)$, and $X_4(\mu)$. Using (1), $S_Y(\omega)$ is derived as

$$\begin{aligned}
S_Y(\omega) &= \mathcal{F}\{R_Y(\tau)\} \\
&= \mathcal{F}\{R_X^2(0) + 2R_X^2(\tau)\} \\
&= 2\pi R_X^2(0)\delta(\omega) + \frac{1}{\pi}S_X(\omega) * S_X(\omega)
\end{aligned}$$

where $\mathcal{F}$ and $*$ denote the Fourier transform and convolution operators, respectively.

(5) (4%) Consider the wide-sense stationary random process $X(\mu,t)$ with zero mean and autocorrelation $R_X(\tau) = 4\cos^2(\pi\tau) - 2$. Find the Karhunen-Loève expansion of $X(\mu,t)$ in the interval $(0,1)$.

Sol: In order to find the Karhunen-Loève expansion of $X(\mu,t)$ in the interval $(0,1)$, we should find the eigenfunctions of $R_X(t,s)$. Here, $R_X(t,s) = R_X(t-s) = 4\cos^2(\pi(t-s)) - 2 = 2\cos(2\pi(t-s))$.By Mercer's theorem, we have

$$\begin{aligned}
R_X(t,s) &= \sum_{k=1}^{\infty} \rho_k \phi_k(t)\phi_k^*(s) \qquad \text{(5-1)} \\
&= 2\cos(2\pi(t-s)) \qquad \text{(5-2)} \\
&= \sqrt{2}\cos(2\pi t)\sqrt{2}\cos(2\pi s) \\
&\quad +\sqrt{2}\sin(2\pi t)\sqrt{2}\sin(2\pi s) \qquad \text{(5-3)}
\end{aligned}$$

where ρ_k's and $\phi_k(t)$'s are eigenvalues and eigenfunctions of $R_X(t,s)$, respectively. From (5-3), it is straightforward to observe that

$$\begin{cases} \rho_1 = 1 \text{ and } \phi_1(t) = \sqrt{2}\cos(2\pi t) \\ \rho_2 = 1 \text{ and } \phi_2(t) = \sqrt{2}\sin(2\pi t) \end{cases}.$$

Therefore, the Karhunen-Loève expansion of $X(\mu,t)$ in the interval $(0,1)$ is given by

$$X(\mu,t) = a(\mu)\sqrt{2}\cos(2\pi t) + b(\mu)\sqrt{2}\sin(2\pi t)$$

where $a(\mu) \triangleq \sqrt{2}\int_0^1 X(\mu,t)\cos(2\pi t)dt$ and $b(\mu) \triangleq \sqrt{2}\int_0^1 X(\mu,t)\sin(2\pi t)dt$ with $E\{a^2(\mu)\} = E\{b^2(\mu)\} = 1$.

(6) (4%, 2% each) Let $\{N(\mu,t); t \geq 0\}$ be a continuous-time Poisson process of rate λ. Suppose that each time an arrival occurs, a coin is flipped and the outcome (heads or tails) is recorded. Let $\{N_h(\mu,t); t \geq 0\}$ and $\{N_t(\mu,t); t \geq 0\}$ denote the numbers of heads and tails recorded up to time t, respectively. Assume that p is the probability of heads recorded, with $0 < p < 1$.

(a) Find the autocorrelation function of $N(\mu,t)$.

Sol: When $t_1 \leq t_2$, $R_N(t_1,t_2)$ is derived as

$$\begin{aligned} R_N(t_1,t_2) &= E\{N(\mu,t_1)N(\mu,t_2)\} \\ &= E\{N(\mu,t_1)\left[N(\mu,t_2) - N(\mu,t_1)\right]\} + E\{N^2(\mu,t_1)\} \\ &= E\{N(\mu,t_1)\}E\{N(\mu,t_2) - N(\mu,t_1)\} + E\{N^2(\mu,t_1)\} \\ &= \lambda t_1 \lambda (t_2 - t_1) + \lambda t_1 + \lambda^2 t_1^2 \qquad (2) \\ &= \lambda t_1 + \lambda^2 t_1 t_2 \end{aligned}$$

where (2) stems from the fact that when t is given, $N(\mu,t)$ is a Poisson random variable with $\Pr\{N(\mu,t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ and thus

$$E\{N(\mu,t)\} = \lambda t \text{ and } R_N(t,t) = E\{N^2(\mu,t)\} = \lambda t + \lambda^2 t^2.$$

Similarly, we have $R_N(t_1,t_2) = \lambda t_2 + \lambda^2 t_1 t_2$ for $t_1 \geq t_2$. Therefore, $R_N(t_1,t_2)$ is given by

$$R_N(t_1,t_2) = \begin{cases} \lambda t_1 + \lambda^2 t_1 t_2 & t_1 \leq t_2 \\ \lambda t_2 + \lambda^2 t_1 t_2 & t_1 \geq t_2 \end{cases}.$$

(b) Find $\Pr\{N_h(\mu,t) = j, N_t(\mu,t) = k | N(\mu,t) = k+j\}$ for all nonnegative integers k, j.

Sol: By definition, we have

$$\begin{aligned} & \Pr\{N_h(\mu,t) = j, N_t(\mu,t) = k | N(\mu,t) = k+j\} \\ = & \Pr\{\text{Heads occurs } j \text{ times and tails occurs } k \text{ times} | \text{Flipping a coin } k+j \text{ times}\} \\ = & \tbinom{k+j}{j} p^j (1-p)^k. \end{aligned}$$

(7) (4%, 2% each) Cars arrive at a bridge entrance according to a Poisson process of rate λ= 15 cars per minute.

(a) Find the probability that in a given 4 minute period there are 3 car arrivals during the first minute and 2 car arrivals in the last minute.

(b) Find the mean and variance of the time of the tenth car arrival, given that the time of the fifth car arrival is T minutes.

Sol: Let $N(\mu,t)$ be the Poisson process.

(a) Now,

$$
\begin{aligned}
& \Pr\{N(\mu,1)=3, N(\mu,4)-N(\mu,3)=2\} \\
= \; & \Pr\{N(\mu,1)=3\}\Pr\{N(\mu,4)-N(\mu,3)=2\} \\
& \text{(because of independent increments)} \\
= \; & \Pr\{N(\mu,1)=3\}\Pr\{N(\mu,1)=2\} \\
& \text{(because of stationary increments)} \\
= \; & \left(e^{-15}\frac{15^3}{3!}\right)\left(e^{-15}\frac{15^2}{2!}\right) \\
= \; & e^{-30}\frac{15^5}{12}.
\end{aligned}
$$

(b) Let $S_n(\mu)$ be the nth arrival time and $T_n(\mu)$ be the nth interarrival time. Thus, $S_n(\mu) = \sum_{k=1}^{n} T_k(\mu)$. Now,

$$
E\{S_{10}(\mu)|S_5(\mu)=T\} = E\{T + \sum_{k=6}^{10} T_k(\mu)\} = T + \frac{5}{\lambda} = T + \frac{1}{3}
$$

$$
Var\{S_{10}(\mu)|S_5(\mu)=T\} = Var\{T + \sum_{k=6}^{10} T_k(\mu)\} = \frac{5}{\lambda^2} = \frac{1}{45}.
$$