Stochastic Processes and Applications, Fall 2016
Homework Three (5%)

1. (1%) You are told the theorem,

   "If $\underline{X}(\mu)$ and $\underline{Y}(\mu)$ are two real-valued jointly Gaussian random vectors, n and m dimensional, respectively, with mean vectors $\underline{m}_X$ and $\underline{m}_Y$, covariance matrices Λ_X and Λ_Y, and cross covariance matrix C_{XY}, then the conditional density of the random vector $\underline{X}(\mu)$, given $\underline{Y}(\mu)$, is also Gaussian with mean

$$\underline{m}_{X|Y} = E\{\underline{X}(\mu)|\underline{Y}(\mu)\} = \underline{m}_X + C_{XY}\Lambda_Y^{-1}(\underline{Y}(\mu) - \underline{m}_Y)$$

   and conditional covariance matrix given by

$$\Lambda_{X|Y} = \Lambda_X - C_{XY}\Lambda_Y^{-1}C_{YX}$$

   ."

   Also you are given a new real-valued l-dimensional random vector $\underline{Z}(\mu)$, defined by

$$\underline{Z}(\mu) = A\underline{X}(\mu) + B\underline{Y}(\mu)$$

   where A is a real-valued l X n deterministic matrix and B is a real-valued l X m deterministic matrix. Suppose you are not aware of the theorem "the linear transformations of Gaussian random vectors produce Gaussian random vectors," prove that $\underline{Z}(\mu)$ is a Gaussian random vector and derive its mean vector $\underline{m}_Z$ and covariance matrix Λ_Z. (Hint: investigating if the characteristic function of $\underline{Z}(\mu)$ is that of a Gaussian random vector.)

Sol: By definition,

$$\begin{aligned}
& E\{\exp\{j\underline{\omega}^T\underline{Z}(\mu)\}\} = E\{E\{\exp\{j\underline{\omega}^T\underline{Z}(\mu)\}|\underline{Y}(\mu)\}\} \\
= \ & E\{E\{\exp\{j\underline{\omega}^T A\underline{X}(\mu)\}|\underline{Y}(\mu)\} \cdot \exp\{j\underline{\omega}^T B\underline{Y}(\mu)\}\} \\
= \ & E\{\exp\{j\underline{\omega}^T A[\underline{m}_X + C_{XY}\Lambda_Y^{-1}(\underline{Y}(\mu) - \underline{m}_Y)]\} \cdot \exp\{j\underline{\omega}^T B\underline{Y}(\mu)\}\} \\
& \cdot \exp\{-\frac{1}{2}\underline{\omega}^T A[\Lambda_X - C_{XY}\Lambda_Y^{-1}C_{YX}]A^T\underline{\omega}\} \\
= \ & E\{\exp\{j\underline{\omega}^T[B + AC_{XY}\Lambda_Y^{-1}]\underline{Y}(\mu)\}\} \\
& \cdot \exp\{j[\underline{\omega}^T A\underline{m}_X - \underline{\omega}^T AC_{XY}\Lambda_Y^{-1}\underline{m}_Y]\} \\
& \cdot \exp\{-\frac{1}{2}\underline{\omega}^T A[\Lambda_X - C_{XY}\Lambda_Y^{-1}C_{YX}]A^T\underline{\omega}\} \\
= \ & \exp\{j\underline{\omega}^T[B + AC_{XY}\Lambda_Y^{-1}]\underline{m}_Y\} \\
& \cdot \exp\{-\frac{1}{2}\underline{\omega}^T[B + AC_{XY}\Lambda_Y^{-1}]\Lambda_Y[B + AC_{XY}\Lambda_Y^{-1}]^T\underline{\omega}\} \\
& \cdot \exp\{j[\underline{\omega}^T A\underline{m}_X - \underline{\omega}^T AC_{XY}\Lambda_Y^{-1}\underline{m}_Y]\} \\
& \cdot \exp\{-\frac{1}{2}\underline{\omega}^T A[\Lambda_X - C_{XY}\Lambda_Y^{-1}C_{YX}]A^T\underline{\omega}\} \\
= \ & \exp\{j\underline{\omega}^T[B\underline{m}_Y + A\underline{m}_X]\} \cdot \exp\{-\frac{1}{2}\Phi\}
\end{aligned}$$

where Φ is given by

$$\begin{aligned}\Phi &= \underline{\omega}^T B\Lambda_Y B^T\underline{\omega} + \underline{\omega}^T AC_{XY}\Lambda_Y^{-1}\Lambda_Y B^T\underline{\omega} + \underline{\omega}^T B\Lambda_Y(\Lambda_Y^{-1})^T C_{XY}^T A^T\underline{\omega} \\ &+\underline{\omega}^T AC_{XY}\Lambda_Y^{-1}\Lambda_Y(\Lambda_Y^{-1})^T C_{XY}^T A^T\underline{\omega} \\ &+\underline{\omega}^T A\Lambda_X A^T\underline{\omega} - \underline{\omega}^T AC_{XY}\Lambda_Y^{-1}C_{YX}A^T\underline{\omega}.\end{aligned}$$

Using $C_{XY} = C_{YX}^T$, $\Lambda_Y\Lambda_Y^{-1} = I$, and $\Lambda_Y^{-1} = (\Lambda_Y^{-1})^T$, Φ simplifies to

$$\begin{aligned}\Phi &= \underline{\omega}^T B\Lambda_Y B^T\underline{\omega} + \underline{\omega}^T AC_{XY}B^T\underline{\omega} + \underline{\omega}^T BC_{XY}^T A^T\underline{\omega} \\ &+\underline{\omega}^T A\Lambda_X A^T\underline{\omega}.\end{aligned}$$

Since $\underline{\omega}^T AC_{XY}B^T\underline{\omega} = \underline{\omega}^T BC_{XY}^T A^T\underline{\omega}$ is a scalar, we further have

$$\begin{aligned}\Phi &= \underline{\omega}^T B\Lambda_Y B^T\underline{\omega} + 2\underline{\omega}^T AC_{XY}B^T\underline{\omega} + \underline{\omega}^T A\Lambda_X A^T\underline{\omega} \\ &= \underline{\omega}^T[B\Lambda_Y B^T + 2AC_{XY}B^T + A\Lambda_X A^T]\underline{\omega}.\end{aligned}$$

Thus, we show that the characteristic function of $\underline{Z}(\mu)$

$$\begin{aligned}E\{\exp\{j\underline{\omega}^T\underline{Z}(\mu)\}\} &= \exp\{j\underline{\omega}^T[B\underline{m}_Y + A\underline{m}_X]\} \\ &\cdot\exp\{-\frac{1}{2}\underline{\omega}^T[B\Lambda_Y B^T + 2AC_{XY}B^T + A\Lambda_X A^T]\underline{\omega}\}\end{aligned}$$

is of the form of the characteristic function of a Gaussian random vector. This proves that $\underline{Z}(\mu)$ is a Gaussian random vector with mean $B\underline{m}_Y + A\underline{m}_X$ and covariance $B\Lambda_Y B^T + 2AC_{XY}B^T + A\Lambda_X A^T$.

(2) (1%, 0.5% each) Consider the stationary Gaussian random process $X(\mu, t)$ with mean zero and autocorrelation $R_X(\tau) = 1 - |\tau|$ if $|\tau| < 1$ and $R_X(\tau) = 0$ otherwise. Now, form two random variables $A(\mu) = X(\mu, t+1)$ and $B(\mu) = X(\mu, t-1)$. Derive the following:

(a) $E\{(A(\mu) + B(\mu))^{2n+1}\}$ for any odd positive integer n.

(b) $\Pr\{A(\mu) < B(\mu)\}$.

Sol: Because $A(\mu)$ and $B(\mu)$ are both defined from Gaussian $X(\mu, t)$, they are jointly Gaussian. Also, because $A(\mu)$ and $B(\mu)$ are uncorrelated, i.e., $E\{A(\mu)B(\mu)\} = R_X(2) = 0 = E\{A(\mu)\}E\{B(\mu)\}$, they are mutually independent. Further, by definition, $A(\mu)$ and $B(\mu)$ are identically Gaussian distributed with zero mean and unit variance. With this statistic, we derive the following.

(a) Because $A(\mu)$ and $B(\mu)$ are independent and identically distributed Gaussian with mean zero and unit variance, $A(\mu) + B(\mu)$ is also Gaussian and has mean zero and variance two. Thus, its odd moment $E\{(A(\mu) + B(\mu))^{2n+1}\}$ is zero.

(b) Because $A(\mu)$ and $B(\mu)$ are independent and identically distributed Gaussian with mean zero and unit variance, $A(\mu) - B(\mu)$ is also Gaussian and has mean zero and variance two. Thus,

$$\Pr\{A(\mu) < B(\mu)\} = \Pr\{A(\mu) - B(\mu) < 0\} = \frac{1}{2}.$$

(3) (1%) Consider the wide-sense stationary random process $X(\mu,t)$ with zero mean and autocorrelation $R_X(\tau)=\cos(2\pi\tau)$. Find the Karhunen-Loève expansion of $X(\mu,t)$ in the interval $(0,1)$.

Sol: In order to find the Karhunen-Loève expansion of $X(\mu,t)$ in the interval $(0,T)$, we should find the eigenfunctions of $R_X(t,s)$. By Mercer's theorem, we have

$$\begin{aligned}
R_X(t,s) &= \sum_{k=1}^{\infty}\rho_k\phi_k(t)\phi_k^*(s) && \text{((a))}\\
&= \cos(2\pi(t-s)) && \text{((b))}\\
&= \frac{1}{2}\times\sqrt{2}\cos(2\pi t)\times\sqrt{2}\cos(2\pi s)\\
&\quad+\frac{1}{2}\times\sqrt{2}\sin(2\pi t)\times\sqrt{2}\sin(2\pi s) && \text{((c))}
\end{aligned}$$

where ρ_k's and $\phi_k(t)$'s are eigenvalues and eigenfunctions of $R_X(t,s)$, respectively. From (c), it is straightforward to observe that

$$\begin{cases}\rho_1=\frac{1}{2}\text{ and }\phi_1(t)=\sqrt{2}\cos(\frac{2\pi t}{T})\\ \rho_2=\frac{1}{2}\text{ and }\phi_2(t)=\sqrt{2}\sin(\frac{2\pi t}{T})\end{cases}.$$

Therefore, the Karhunen-Loève expansion of $X(\mu,t)$ in the interval $(0,1)$ is given by

$$X(\mu,t)=\sqrt{2}a(\mu)\cos(2\pi t)+\sqrt{2}b(\mu)\sin(2\pi t)$$

where $a(\mu)\triangleq\sqrt{2}\int_0^1 X(\mu,t)\cos(2\pi t)dt$ and $b(\mu)\triangleq\sqrt{2}\int_0^1 X(\mu,t)\sin(2\pi t)dt$ with $E\{a^2(\mu)\}=E\{b^2(\mu)\}=\frac{1}{2}$.

(4) (1%) Denote $\widehat{W}(\mu,t)$ as the Hilbert transform of the real-valued wide-sense stationary Gaussian random process $W(\mu,t)$ which has mean zero and autocorrelation $R_W(\tau)$. Express $E\{(\widehat{W}(\mu,t_1)+W(\mu,t_2))^2\}$ in terms of $R_W(t_1-t_2)$ and $\widehat{R}_W(t_1-t_2)$.

Sol: Because Hilbert transform is an (ideal) LTI system with impulse response $h(t)=\frac{1}{\pi t}$, we have

$$\begin{aligned}
&E\{(\widehat{W}(\mu,t_1)+W(\mu,t_2))^2\}\\
=\;&E\{\widehat{W}(\mu,t_1)\widehat{W}(\mu,t_1)\}+E\{W(\mu,t_2)W(\mu,t_2)\}+2E\{\widehat{W}(\mu,t_1)W(\mu,t_2)\}\\
=\;&2R_W(\tau)+2E\{\int_{-\infty}^{\infty}W(\mu,t_1-x)W(\mu,t_2)\frac{1}{\pi x}dx\}\\
&\text{(because WSS } W(\mu,t)\text{ and }\widehat{W}(\mu,t)\text{ have the same autocorrelation)}\\
=\;&2R_W(\tau)+2\int_{-\infty}^{\infty}R_W(\tau-x)\frac{1}{\pi x}dx\\
=\;&2R_W(\tau)+2\widehat{R}_W(\tau)
\end{aligned}$$

with $\tau=t_1-t_2$.

(5) (1%) Consider a linear and time invariant system with continuous real input $X(\mu, t)$, continuous real output $Y(\mu, t)$, continuous real impulse response $h(t)$, and system function $H(\omega)$. Let $X(\mu, t)$ and $Y(\mu, t)$ are both wide-sense stationary random processes with means η_X and η_Y, respectively, autocorrelations $R_X(\tau)$ and $R_Y(\tau)$, respectively, and power spectrums $S_X(\omega)$ and $S_Y(\omega)$, respectively. Also, let $h(t) = 1$ if $|t| < 1$ and $h(t) = 0$ otherwise.

(a) It is known that $\eta_Y = \alpha\eta_X$ with α a constant. Determine α.

Sol: By definition, we have $H(\omega) = \mathcal{F}\{h(t)\} = \int_{-1}^{1} e^{-j\omega t}dt = \frac{e^{-j\omega t}}{-j\omega}|_{-1}^{1} = \frac{2\sin\omega}{\omega}$. Further, since $\eta_Y = H(0)\eta_X$, $\alpha = H(0) = 2$.

(b) Express $R_Y(\tau)$ in terms of $R_X(\tau)$. Derivation is required.

Sol: Now,

$$
\begin{aligned}
R_Y(\tau) &= E\{Y(\mu, t+\tau)Y(\mu, t)\} \\
&= E\{\int_{-\infty}^{\infty} X(\mu, t+\tau-\theta)h(\theta)d\theta \int_{-\infty}^{\infty} X(\mu, t-\lambda)h(\lambda)d\lambda\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau-\theta+\lambda)h(\theta)h(\lambda)d\theta d\lambda \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau-\tau')h(\tau'+\lambda)h(\lambda)d\tau' d\lambda \\
&= \int_{-\infty}^{\infty} R_X(\tau-\tau')d\tau' \int_{-\infty}^{\infty} h(\tau'+\lambda)h(\lambda)d\lambda \\
&= R_X(\tau) * \rho(t)
\end{aligned}
$$

where $\rho(t) = \int_{-\infty}^{\infty} h(t+\lambda)h(\lambda)d\lambda = \int_{-1}^{1} h(t+\lambda)d\lambda = \int_{t-1}^{t+1} h(\beta)d\beta = \begin{cases} 0 & |t| > 2 \\ 2-|t| & |t| \le 2 \end{cases}$.

(c) Express $S_Y(\omega)$ in terms of $S_X(\omega)$. Derivation is required.

Sol: Now,

$$
\begin{aligned}
S_Y(\omega) &= \mathcal{F}\{R_Y(\tau)\} \\
&= \mathcal{F}\{R_X(\tau) * \rho(t)\} \\
&= S_X(\omega)\mathcal{F}\{\rho(t)\} \\
&= \frac{4S_X(\omega)\sin^2\omega}{\omega^2}.
\end{aligned}
$$