Stochastic Processes and Applications, Fall 2015
Homework Four (5%)

1. (1%) If a complex-valued random process $Z(\mu,t) = X(\mu,t) + jY(\mu,t)$ is wide-sense stationary (with $j = \sqrt{-1}$, $X(\mu,t) \triangleq \mathrm{Re}\{Z(\mu,t)\}$ and $Y(\mu,t) \triangleq \mathrm{Im}\{Z(\mu,t)\}$), determine whether each of the following statements is true or false. Prove or explain your answer.

(A) (0.25%) $Z^*(\mu,t)$ is wide-sense stationary.

Sol: True. First, if $E\{Z(\mu,t)\} = \eta_Z$ is independent of t, then $E\{Z^*(\mu,t)\} = \eta_Z^*$ is also independent of t. Further, if $E\{Z(\mu,t+\tau)Z^*(\mu,t)\} = R_Z(\tau)$ depends on τ only, then $E\{Z^*(\mu,t+\tau)Z(\mu,t)\} = E\{Z(\mu,t)Z^*(\mu,t+\tau)\} = R_Z(-\tau)$ depends on τ only as well. Hence, $Z^*(\mu,t)$ is wide-sense stationary.

(B) (0.25%) $Z(\mu,t)$ and $Z^*(\mu,t)$ are jointly wide-sense stationary.

Sol: False. Consider the following counterexample. Let $Z(\mu,t) = A(\mu)e^{j\omega t}$ where $A(\mu)$ is a real-valued random variable with mean zero and variance $\sigma_A^2 > 0$ and ω is a constant. Because $E\{Z(\mu,t)\} = 0$ and

$$\begin{aligned}
R_Z(t+\tau,t) &= E\{Z(\mu,t+\tau)Z^*(\mu,t)\} \\
&= E\{A(\mu)e^{j\omega(t+\tau)}A(\mu)e^{-j\omega t}\} \\
&= E\{A^2(\mu)\}e^{j\omega\tau} \\
&= \sigma_A^2 e^{j\omega\tau}
\end{aligned}$$

depends on τ only, $Z(\mu,t)$ is wide-sense stationary. However, because

$$\begin{aligned}
R_{ZZ^*}(t+\tau,t) &= E\{Z(\mu,t+\tau)Z(\mu,t)\} \\
&= E\{A(\mu)e^{j\omega(t+\tau)}A(\mu)e^{j\omega t}\} \\
&= E\{A^2(\mu)\}e^{j\omega(2t+\tau)} \\
&= \sigma_A^2 e^{j\omega(2t+\tau)}
\end{aligned}$$

depends on both t and τ, $Z(\mu,t)$ and $Z^*(\mu,t)$ are **NOT** jointly wide-sense stationary.

(C) (0.25%) $X(\mu,t)$ is wide-sense stationary.

Sol: False. Consider the following counterexample. Let $Z(\mu,t) = A(\mu)e^{j\omega t}$ where $A(\mu)$ is a real-valued random variable with mean zero and variance $\sigma_A^2 > 0$ and ω is a constant. Note that it is shown in (B) that $Z(\mu,t)$ is wide-sense stationary. Further, because $X(\mu,t) = \mathrm{Re}\{Z(\mu,t)\} = A(\mu)\cos(\omega t)$, we have

$$\begin{aligned}
R_X(t+\tau,t) &= E\{X(\mu,t+\tau)X(\mu,t)\} \\
&= E\{A(\mu)\cos(\omega(t+\tau))A(\mu)\cos(\omega t)\} \\
&= E\{A^2(\mu)\}\frac{\cos(\omega(2t+\tau)) + \cos(\omega\tau)}{2} \\
&= \frac{\sigma_A^2}{2}\left[\cos(\omega(2t+\tau)) + \cos(\omega\tau)\right].
\end{aligned}$$

Therefore, $R_X(t+\tau,t)$ depends on both t and τ, and thus $X(\mu,t)$ is **NOT** wide-sense stationary.

(D) (0.25%) $X(\mu,t)$ and $Y(\mu,t)$ are jointly wide-sense stationary.

Sol: False. Consider the following counterexample. Let $Z(\mu,t) = A(\mu)e^{j\omega t}$ where $A(\mu)$ is a real-valued random variable with mean zero and variance $\sigma_A^2 > 0$ and ω is a constant. From (C), we know that $X(\mu,t)$ is not wide-sense stationary and thus $X(\mu,t)$ and $Y(\mu,t)$ are **NOT** jointly wide-sense stationary.

2. (1%) Consider the real-valued bandpass stationary Gaussian noise process $n(\mu,t)$ which has zero mean and the power spectral density $S_n(f)$, given by

$$S_n(f) = \begin{cases} 1 - \big|f - f_c\big|/B, & \big|f - f_c\big| \le B \\ 1 - \big|f + f_c\big|/B, & \big|f + f_c\big| \le B \\ 0, & \text{otherwise} \end{cases}$$

with $f_c \gg B$. Let $n_+(\mu,t)$, $\widehat{n}(\mu,t)$ and $\widetilde{n}(\mu,t)$ be the pre-envelope, the Hilbert transform, and the complex envelope of $n(\mu,t)$, respectively, related by

$$n_+(\mu,t) = n(\mu,t) + j\widehat{n}(\mu,t) = \widetilde{n}(\mu,t)\exp\{j2\pi f_c t\}.$$

Determine whether each of the following statements is true or false. Prove or explain your answer.

(A) (0.25%) $n(\mu,t)$, $n_+(\mu,t)$, $\widehat{n}(\mu,t)$ and $\widetilde{n}(\mu,t)$ are jointly Gaussian processes.

Sol: True. Because $\widehat{n}(\mu,t)$ is a linear transform of the Gaussian process $n(\mu,t)$, $n(\mu,t)$ and $\widehat{n}(\mu,t)$ are jointly Gaussian. Further, because $n_+(\mu,t)$ and $\widetilde{n}(\mu,t)$ are linear combinations of $n(\mu,t)$ and $\widehat{n}(\mu,t)$, $n(\mu,t)$, $n_+(\mu,t)$, $\widehat{n}(\mu,t)$ and $\widetilde{n}(\mu,t)$ are jointly Gaussian.

(B) (0.25%) $E\{n_+(\mu,t)\widehat{n}(\mu,t)\widetilde{n}(\mu,t)\} = 0$.

Sol: True. Because $n(\mu,t)$ and $\widehat{n}(\mu,t)$ are jointly Gaussian and $R_{\widehat{n}n}(0) = 0$ (from the lecture note), $n(\mu,t)$ and $\widehat{n}(\mu,t)$ are independent Gaussian random variables with zero mean for a fixed t. As a result, we have

$$\begin{aligned} E\{n_+(\mu,t)\widehat{n}(\mu,t)\widetilde{n}(\mu,t)\} &= E\{[n(\mu,t) + j\widehat{n}(\mu,t)]^2\widehat{n}(\mu,t)\}\exp\{-j2\pi f_c t\} \\ &= E\{n^2(\mu,t)\widehat{n}(\mu,t) + 2jn(\mu,t)\widehat{n}^2(\mu,t) - \widehat{n}^3(\mu,t)\}\exp\{-j2\pi f_c t\} \\ &= 0 \end{aligned}$$

where the last equation is due to the fact that

$$\begin{aligned} E\{n^2(\mu,t)\widehat{n}(\mu,t)\} &= E\{n^2(\mu,t)\}\underbrace{E\{\widehat{n}(\mu,t)\}}_{=0} \\ &= 0, \end{aligned}$$

$$\begin{aligned} E\{n(\mu,t)\widehat{n}^2(\mu,t)\} &= \underbrace{E\{n(\mu,t)\}}_{=0}E\{\widehat{n}^2(\mu,t)\} \\ &= 0, \end{aligned}$$

and

$$E\{\widehat{n}^3(\mu,t)\} = 0.$$

This can also be shown alternatively. First, because $n(\mu, t)$ has zero mean, $n_+(\mu, t)$, $\widehat{n}(\mu, t)$ and $\widetilde{n}(\mu, t)$ have zero mean (from lecture note). Second, because $n_+(\mu, t)$, $\widehat{n}(\mu, t)$ and $\widetilde{n}(\mu, t)$ are jointly Gaussian processes (from (a)) with zero mean, the expectation of their product is zero (from the lemma that the expectation of any odd-numbered jointly Gaussian random variables with mean zero is zero.).

(C) (0.25%) If $\widetilde{n}(\mu, t) = n_c(\mu, t) + j n_s(\mu, t)$, then $n_c(\mu, t)$ and $n_s(\mu, t)$ are independent Gaussian random processes which have the identical first-order density.

Sol: True. Because $n_c(\mu, t)$ and $n_s(\mu, t)$ are linear combinations of jointly Gaussian random processes $n(\mu, t)$ and $\widehat{n}(\mu, t)$ with zero mean, $n_c(\mu, t)$ and $n_s(\mu, t)$ are jointly Gaussian with zero mean. Because the local symmetry of $S_n(f)$ holds, $R_{n_c n_s}(\tau) = 0$ and thus $n_c(\mu, t)$ and $n_s(\mu, t)$ are independent. In addition, because $R_{n_s}(0) = R_{n_c}(0)$, $n_c(\mu, t)$ and $n_s(\mu, t)$ has identical variance for a fixed t. There facts imply that $n_c(\mu, t)$ and $n_s(\mu, t)$ are independent Gaussian random processes which have the identical first-order density.

(D) (0.25%) For a fixed t, $n(\mu, t)$ and $\widehat{n}(\mu, t)$ are independent and identically distributed Gaussian random variables.

Sol: True. Because $\widehat{n}(\mu, t)$ is a linear transform of the Gaussian process $n(\mu, t)$ with zero mean, $n(\mu, t)$ and $\widehat{n}(\mu, t)$ are jointly Gaussian with zero mean. Further, because $R_{\widehat{n}n}(0) = 0$ and $R_{\widehat{n}}(\tau) = R_n(\tau)$, $n(\mu, t)$ and $\widehat{n}(\mu, t)$ are independent and identically distributed Gaussian random variables for a fixed t.

3. (1%) Let $X(\mu, t) \triangleq \cos(2\pi f_1 t + \theta(\mu))$ and $Y(\mu, t) \triangleq \cos(2\pi f_2 t + \phi(\mu))$ where $f_1 > f_2 > 0$, $\theta(\mu)$ and $\phi(\mu)$ are statistically independent and identically distributed random variables with the identical uniform density over $[0, 2\pi)$. Also, let $\widehat{X}(\mu, t)$ and $\widehat{Y}(\mu, t)$ be the Hilbert transforms of $X(\mu, t)$ and $Y(\mu, t)$, respectively. Answer the following sub-questions.

(A) (0.5%) Derive the power spectral densities of $\widehat{X}(\mu, t)$ and $\widehat{Y}(\mu, t)$.

Sol: Since $\widehat{X}(\mu, t)$ and $\widehat{Y}(\mu, t)$ can be created by passing $X(\mu, t)$ and $Y(\mu, t)$ through an linear and time-invariant system with frequency response $H(f) = -j\,\mathrm{sgn}(f)$, respectively, the power spectral densities of $\widehat{X}(\mu, t)$ and $\widehat{Y}(\mu, t)$ is given by

$$S_{\widehat{X}}(f) = S_X(f)|H(f)|^2 = S_X(f)|-j\,\mathrm{sgn}(f)|^2 = S_X(f) \text{ and } S_{\widehat{Y}}(f) = S_Y(f).$$

Moreover, because the autocorrelation functions $R_X(\tau)$ and $R_Y(\tau)$ are given by

$$\begin{aligned} R_X(\tau) &= E\{\cos(2\pi f_1(t+\tau) + \theta(\mu))\cos(2\pi f_1 t + \theta(\mu))\} \\ &= \frac{1}{2}\cos(2\pi f_1 \tau) \end{aligned}$$

and

$$R_Y(\tau) = \frac{1}{2}\cos(2\pi f_2 \tau),$$

$S_X(f)$ and $S_Y(f)$ are obtained as

$$\begin{aligned} S_X(f) &= \mathcal{F}\{R_X(\tau)\} \\ &= \frac{\delta(f - f_1) + \delta(f + f_1)}{4} \end{aligned}$$

and

$$\begin{aligned} S_Y(f) &= \mathcal{F}\{R_Y(\tau)\} \\ &= \frac{\delta(f-f_2)+\delta(f+f_2)}{4}. \end{aligned}$$

Therefore, we get $S_{\widehat{X}}(f) = \frac{\delta(f-f_1)+\delta(f+f_1)}{4}$ and $S_{\widehat{Y}}(f) = \frac{\delta(f-f_2)+\delta(f+f_2)}{4}$.

(B) (0.5%) Determine whether $\widehat{X}(\mu,t)$ and $Y(\mu,t)$ for a fixed t are orthogonal, i.e., $E\{\widehat{X}(\mu,t)Y(\mu,t)\} = E\{\widehat{X}(\mu,t)\}E\{Y(\mu,t)\}$.

Sol: Since $\theta(\mu)$ and $\phi(\mu)$ are statistically independent, we have

$$\begin{aligned} E\{\widehat{X}(\mu,t)Y(\mu,t)\} &= E\{\widehat{X}(\mu,t)\}E\{Y(\mu,t)\} \\ &= 0 \end{aligned}$$

where the last equality is because

$$\begin{aligned} E\{Y(\mu,t)\} &= E\{\cos(2\pi f_2 t + \phi(\mu))\} \\ &= \int_0^{2\pi} \cos(2\pi f_2 t + \phi) d\phi \\ &= 0. \end{aligned}$$

Thus, they are orthogonal.

4. (1%) Let $x(t) = \mathrm{Re}\{\tilde{x}(t)\exp\{j2\pi f_c t\}\}$ be a narrowband bandpass signal, centered around the frequency f_c and with $\tilde{x}(t)$ being the corresponding complex envelope. Let $h(t) = \mathrm{Re}\{\tilde{h}(t)\exp\{j2\pi(2f_c)t\}\}$ be the impulse response of a linear and time-invariant (LTI) bandpass system, centered around $2f_c$. The bandwidth of $h(t)$ is much smaller than $2f_c$. Let us feed $y(t) \triangleq x^2(t)$ into the LTI system and represent the output by $z(t)$. Also let $z(t) = Re\{\tilde{z}(t)\exp\{j2\pi(2f_c)t\}\}$ with $\tilde{z}(t)$ being the corresponding complex envelope. Express $\tilde{z}(t)$ in terms of $\tilde{x}(t)$ and $\tilde{h}(t)$.

Sol: The bandpass system output is given by

$$\begin{aligned} z(t) &= x^2(t) * h(t) \\ &= \int_{-\infty}^{\infty} x^2(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} \mathrm{Re}^2\{\widetilde{x}(\tau)\exp\{j2\pi f_c\tau\}\}\,\mathrm{Re}\{\widetilde{h}(t-\tau)\exp\{j4\pi f_c(t-\tau)\}\}d\tau \\ &= \frac{1}{4}\int_{-\infty}^{\infty} [\widetilde{x}^2(\tau)\exp\{j4\pi f_c\tau\} + \widetilde{x}^{*2}(\tau)\exp\{-j4\pi f_c\tau\} + 2\widetilde{x}(\tau)\widetilde{x}^*(\tau)] \\ &\quad \cdot \mathrm{Re}\{\widetilde{h}(t-\tau)\exp\{j4\pi f_c(t-\tau)\}\}d\tau \qquad (+) \\ &= \frac{1}{4}\int_{-\infty}^{\infty} [\widetilde{x}^2(\tau)\exp\{j4\pi f_c\tau\} + \widetilde{x}^{*2}(\tau)\exp\{-j4\pi f_c\tau\}] \\ &\quad \cdot \mathrm{Re}\{\widetilde{h}(t-\tau)\exp\{j4\pi f_c(t-\tau)\}\}d\tau \end{aligned}$$

since the last term in the bracket in (+) is filtered out by the bandpass system. We can further have that

$$
\begin{aligned}
z(t) &= \frac{1}{8}\int_{-\infty}^{\infty}[\widetilde{x}^2(\tau)\exp\{j4\pi f_c\tau\}+\widetilde{x}^{*2}(\tau)\exp\{-j4\pi f_c\tau\}] \\
&\quad\cdot[\widetilde{h}(t-\tau)\exp\{j4\pi f_c(t-\tau)\}+\widetilde{h}^*(t-\tau)\exp\{-j4\pi f_c(t-\tau)\}]d\tau \\
&= \frac{1}{8}\int_{-\infty}^{\infty}[\widetilde{x}^2(\tau)\widetilde{h}(t-\tau)\exp\{j4\pi f_c t\}+\widetilde{x}^{*2}(\tau)\widetilde{h}^*(t-\tau)\exp\{-j4\pi f_c t\}]d\tau \\
&\quad+\frac{1}{8}\int_{-\infty}^{\infty}[\widetilde{x}^2(\tau)\widetilde{h}^*(t-\tau)\exp\{j4\pi f_c(2\tau-t)\} \\
&\quad+\widetilde{x}^{*2}(\tau)\widetilde{h}(t-\tau)\exp\{-j4\pi f_c(2\tau-t)\}]d\tau \qquad (*) \\
&= \frac{1}{4}\mathrm{Re}\{\int_{-\infty}^{\infty}\widetilde{x}^2(\tau)\widetilde{h}(t-\tau)d\tau\cdot\exp\{j4\pi f_c t\}\}
\end{aligned}
$$

because the second integral in $(*)$ is integrated to zero when the bandwidths of $\tilde{x}(t)$ and $\widetilde{h}(t)$ are much smaller than $2f_c$. Thus,

$$\widetilde{z}(t)=\frac{1}{2}\widetilde{x}^2(t)*\frac{1}{2}\widetilde{h}(t).$$

5. (1%) Consider the random process

$$Y(\mu,t)=A(\mu)\cos(2\pi ft+\phi(\mu))$$

where $f>0$ is a constant; $A(\mu)$ is a real-valued random variable with the probability density function $f_A(a)=a\exp\{-\frac{a^2}{2}\}u(a)$, where $u(a)=1$ if $a\geq 0$ and $u(a)=0$ if $a<0$; and $\phi(\mu)$ is a real-valued random variable, which is uniformly distributed in $[0,2\pi)$. Furthermore, $A(\mu)$ and $\phi(\mu)$ are mutually independent. Find

(A) (0.2%) $E\{Y(\mu,t)\}$.

Sol: $E\{Y(\mu,t)\}=E\{A(\mu)\}E\{\cos(2\pi ft+\phi(\mu))\}=0$ since $\phi(\mu)$ is uniform in $[0,2\pi)$.

(B) (0.2%) $Var\{Y(\mu,t)\}$.

Sol: $Var\{Y(\mu,t)\}=E\{Y^2(\mu,t)\}=\frac{1}{2}E\{A^2(\mu)\}=\frac{1}{2}\int_0^\infty a^3\exp\{-\frac{a^2}{2}\}da=1$.

(C) (0.2%) The probability density function of $Y(\mu,t)$..

Sol: Let $X(\mu,t)=A(\mu)\sin(2\pi ft+\phi(\mu))$. By Jacobian, $X(\mu,t)$ and $Y(\mu,t)$ are independent and identically distributed Gaussian random variables which have zero mean and unit variance, for a fixed t. Thus, the first-order density of $Y(\mu,t)$ is of a Gaussian density with zero mean and unit variance.

(D) (0.2%) $E\{Y^3(\mu,t)\}$.

Sol: From (C), $Y(\mu,t)$ is Gaussian distributed with zero mean and unit variance for a fixed t. Thus, $E\{Y^3(\mu,t)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty y^3\exp\{-\frac{y^2}{2}\}dy=0$ since the integrand is an odd function.

(E) (0.2%) The joint probability density function of $Y(\mu,0)$ and $Y(\mu,\frac{1}{4f})$.

Sol: $Y(\mu,0)=A(\mu)\cos(\phi(\mu))$ and $Y(\mu,\frac{1}{4f})=-A(\mu)\sin(\phi(\mu))$. By Jacobian, $Y(\mu,0)$ and $Y(\mu,\frac{1}{4f})$ are independent and identically distributed Gaussian random variables which have zero mean and unit variance.