Stochastic Processes and Applications, Fall 2016
Homework Six (5%)

(1) (0.5%) Define the discrete-time random process $Y_1(\mu), Y_2(\mu), ...$ by

$$\begin{aligned} Y_n(\mu) &= rY_{n-1}(\mu) + X_n(\mu) \text{ for positive integer } n \\ Y_0(\mu) &= 0 \end{aligned}$$

where $X_n(\mu)$'s are independent and identically distributed random variables. Show that process $Y_1(\mu), Y_2(\mu), ...$ is a Markov process.

Sol: Now,

$$\begin{aligned} f_{Y_n|Y_{n-1},Y_{n-2},...,Y_1}(y_n|y_{n-1}, y_{n-2}, ..., y_1) &= f_{X_n}(y_n - ry_{n-1}) \\ &= f_{Y_n|Y_{n-1}}(y_n|y_{n-1}) \end{aligned}$$

which shows $Y_1(\mu), Y_2(\mu), ...$ is a Markov process.

(2) (1.5%) Given a two-state Markov chain $X_1(\mu), X_2(\mu), ...$ where each $X_n(\mu)$ takes the values 1 and 0 with state probability vector $R(n) = [p_0(n), p_1(n)]$ and one-step state transition probability matrix $\mathbb{P}$ given by

$$\mathbb{P} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Answer the following.

(a) (1%) Find $\lim_{n\to\infty} \mathbb{P}^n$ and $\lim_{n\to\infty} R(n)$.

(b) (0.5%) Find $R(3)$ if $X_1(\mu) = 0$.

Sol: (a) From the definition of $\mathbb{P}$, we define

$$\mathbb{P}^n = \begin{bmatrix} a_n & b_n \\ b_n & a_n \end{bmatrix}$$

for any positive integer n where $a_{n+1} = \frac{2}{3}a_n + \frac{1}{3}b_n$ and $b_{n+1} = \frac{1}{3}a_n + \frac{2}{3}b_n$ since $\mathbb{P}^{n+1} = \mathbb{P}^n\mathbb{P}$. Let us show by induction that

$$1 > a_n > a_{n+1} > b_{n+1} > b_n > 0 \tag{1}$$

for any positive integer n:

(1) (1) is true for $n = 1$ because $a_1 = \frac{2}{3}$, $b_1 = \frac{1}{3}$, $a_2 = \frac{5}{9}$, $b_2 = \frac{4}{9}$ and thus

$$1 > a_1 > a_2 > b_2 > b_1 > 0.$$

(2) Suppose that (1) is true for $n = k$. Now, when $n = k + 1$,

$$\begin{aligned} a_{k+1} &> \frac{2}{3}a_k + \frac{1}{3}a_k = a_k \\ b_{k+1} &> \frac{1}{3}b_k + \frac{2}{3}b_k = b_k \end{aligned}$$

and also

$$a_{k+1} - b_{k+1} = \frac{a_k - b_k}{3} > 0. \qquad (2)$$

Hence, (1) is true for $n = k+1$. This prove (1) for any positive integer n by induction. From (2), it is immediate that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{1}{2}$ because $a_n + b_n = 1$. Thus,

$$\lim_{n\to\infty} \mathbb{P}^n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Also, since $p_0(1) + p_1(1) = 1$, we have

$$\begin{aligned}
\lim_{n\to\infty} R(n+1) &= R(1) \lim_{n\to\infty} \mathbb{P}^n \\
&= [\frac{1}{2}(p_0(1) + p_1(1)), \frac{1}{2}(p_0(1) + p_1(1))] \\
&= [\frac{1}{2}, \frac{1}{2}].
\end{aligned}$$

(b) Next, if $X_1(\mu) = 0$, then

$$\begin{aligned}
R(2) &= [1, 0]\mathbb{P} = [\frac{2}{3}, \frac{1}{3}] \\
R(3) &= [\frac{2}{3}, \frac{1}{3}]\mathbb{P} = [\frac{5}{9}, \frac{4}{9}].
\end{aligned}$$

(3) (1%, 0.5% each) A shop has N machines and one technician to repair them. A machine remains in the working state for an exponentially distributed time with mean $1/\beta$ and independently of the others. The technician works on one machine at a time, and it takes him an exponentially distributed time with mean $1/\alpha$ to repair each machine. Let $X(\mu, t)$ be the number of working machines at time t. Answer the following:

(a) Show that if $X(\mu, t) = k$, then the time until the next machine breakdown is an exponentially distributed random variable with mean $1/(k\beta)$.

(b) Find the steady state state probabilities p_i's for $X(\mu, t)$.

Sol: (a) Let $W_n(\mu)$ be the time till the next breakdown of machine n, and $T(\mu)$ be the time till the next breakdown of any machine. Then, we can express

$$T(\mu) = \min\{W_1(\mu), W_2(\mu), ..., W_k(\mu)\}$$

when $X(\mu, t) = k$ and find its conditional distribution as

$$\begin{aligned}
\Pr\{T(\mu) > t | X(\mu, t) = k\} &= \Pr\{\min\{W_1(\mu), W_2(\mu), ..., W_k(\mu)\} > t\} \\
&= \Pr\{W_1(\mu) > t, W_2(\mu) > t, ..., W_k(\mu) > t\} \\
&= \Pr\{W_1(\mu) > t\} \Pr\{W_2(\mu) > t\} \cdots \Pr\{W_k(\mu) > t\} \\
&\quad (W_n(\mu)\text{'s are independent.}) \\
&= \exp\{-k\beta t\}. \\
&\quad (W_n(\mu)\text{'s are identically distributed with rate } \beta.)
\end{aligned}$$

Thus, if $X(\mu,t)=k$, then the time until the next machine breakdown is an exponentially distributed random variable with mean $1/(k\beta)$.

(b) Let $\gamma_{i,j}$ be the transition rate (of probability flow) at which $X(\mu,t)$ enters state j from state i. Then, we have

$$\begin{aligned}
\gamma_{i,i+1} &= \alpha \text{ for } i=0,1,...,N-1 \\
\gamma_{i,i-1} &= i\beta \text{ for } i=1,2,...,N \\
\gamma_{i,j} &= 0 \text{ otherwise.}
\end{aligned}$$

Using these transition rates, the global balance equations when $X(\mu,t)$ settles into steady state are given by

$$\begin{aligned}
\alpha p_0 &= \beta p_1 \\
(\alpha+j\beta)p_j &= \alpha p_{j-1}+(j+1)\beta p_{j+1} \text{ for } j=1,2,...,N-1 \\
\alpha p_{N-1} &= N\beta p_N.
\end{aligned}$$

This set of global balance equations can be solved by first finding

$$p_j=\frac{\alpha}{j\beta}p_{j-1}=\frac{(\alpha/\beta)^j}{j!}p_0 \text{ for } j=1,2,...,N$$

and then deriving

$$p_0=\frac{1}{\sum_{j=0}^{N}\frac{(\alpha/\beta)^j}{j!}}$$

by using the identity $\sum_{j=0}^{N}p_j=1$. Thus,

$$p_j=\frac{\frac{(\alpha/\beta)^j}{j!}}{\sum_{j=0}^{N}\frac{(\alpha/\beta)^j}{j!}} \text{ for } j=0,1,...,N.$$

(4) (1.5%) N identical balls are distributed in two urns. At time n, a ball is selected at random and it is removed from its present urn and placed in the other urn. Denote $X_n(\mu)$ as the number of balls remaining in urn 1. It is known that $\{X_n(\mu);n=0,1,2,...\}$ an ergodic discrete-time Markov process.

(a) (1%) Find the one-step state transition probabilities, i.e., $p_{ij}=\Pr\{X_{n+1}(\mu)=j|X_n(\mu)=i\}$ for all states i,j, and the stationary probabilities, i.e., π_i for all states i.

(b) (0.5%) Is the Markov process reversible?

Sol: (a) The states of ergodic Markov process $\{X_n(\mu);n=0,1,2,...\}$ are given by

$$X_n(\mu)\in\{0,1,2,...,N\}.$$

Its one-step state transition probabilities are derived as

$$\begin{aligned}
p_{i,j} &= \Pr\{X_{n+1}(\mu)=j|X_n(\mu)=i\} \\
&= \begin{cases} 1-\frac{i}{N} & \text{if } j=i+1 \text{ for } i\in\{0,1,2,...,N-1\} \\ \frac{i}{N} & \text{if } j=i-1 \text{ for } i\in\{1,2,...,N\} \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

The stationary probabilities π_i's can be found by solving

$$\begin{aligned}
\pi_0 &= \pi_1 p_{1,0} = \frac{1}{N}\pi_1 && ((1))\\
\pi_i &= \pi_{i+1}p_{i+1,i} + \pi_{i-1}p_{i-1,i}\\
&= \frac{i+1}{N}\pi_{i+1} + \frac{N-i+1}{N}\pi_{i-1} \text{ for } i \in \{1,2,...,N-1\} && ((2))\\
\pi_N &= \pi_{N-1}\frac{1}{N} && ((3))\\
\sum_{i=0}^{N}\pi_i &= 1. && ((4))
\end{aligned}$$

From (2), we have for $i \in \{1,2,...,N-1\}$ that

$$\pi_{i+1} = \frac{N}{i+1}\pi_i - \frac{N-i+1}{i+1}\pi_{i-1}.$$

If $i = 1$, then $\pi_2 = \frac{N}{2}\pi_1 - \frac{N}{2}\pi_0 = \frac{N-1}{2}\pi_1$.
If $i = 2$, then $\pi_3 = \frac{N}{3}\pi_2 - \frac{N-1}{3}\pi_1 = \frac{N-2}{3}\pi_2$.
If $i = 3$, then $\pi_4 = \frac{N}{4}\pi_3 - \frac{N-2}{4}\pi_2 = \frac{N-3}{4}\pi_3$.
Iteratively, we have $\pi_{i+1} = \frac{N-i}{i+1}\pi_i$ for $i \in \{1,2,...,N-1\}$. Combining (1), we further have

$$\begin{aligned}
\pi_{i+1} &= \frac{N-i}{i+1}\pi_i\\
&= \binom{N}{i+1}\pi_0 \text{ for } i \in \{0,1,...,N-1\}.
\end{aligned}$$

Note that (3) is also satisfied by the above recursion. Using this recursion, we obtain from (4) that

$$\pi_0 = \frac{1}{\sum_{i=0}^{N}\binom{N}{i}} = 2^{-N}$$

and thus

$$\pi_i = 2^{-N}\binom{N}{i} \text{ for } i \in \{0,1,...,N\}.$$

(a) (b) Now, for $|i-j| > 1$

$$\pi_i p_{i,j} = 0 = \pi_j p_{j,i}.$$

For $j = i-1$ and $i \in \{1,2,...,N\}$,

$$\begin{aligned}
\pi_i p_{i,j} &= \pi_i \frac{i}{N} = 2^{-N}\binom{N}{i}\frac{i}{N}\\
&= 2^{-N}\frac{(N-1)!}{(i-1)!(N-i)!}\\
&= 2^{-N}\frac{N!}{(i-1)!(N-i+1)!}\frac{N-i+1}{N}\\
&= \pi_j p_{j,i}.
\end{aligned}$$

Similarly, $\pi_i p_{i,j} == \pi_j p_{j,i}$ for $j = i+1$ and $i \in \{0,1,...,N-1\}$. Thus, the Markov process is reversible.

(5) (0.5%) Consider an $M/M/1$ queueing system in which each customer arrival brings in a profit of $5N$ dollars but in which each unit time of delay costs the system N dollars. Find the range of average arrival rate and average service rate for which the system makes a net profit on average.

Sol: For an $M/M/1$ queueing system, the mean of total delay is given by $\frac{1}{\mu-\lambda}$ where λ and μ are average arrival rate and average service rate, respectively, provided that $\mu > \lambda$. Now, the system can make a net profit only when

$$5N > N\frac{1}{\mu - \lambda}$$

or equivalently

$$\mu > \lambda + \frac{1}{5}.$$