Stochastic Processes and Applications, Fall 2016
Homework One (5%)

(1) (0.5%) Define a function of two events $\mathcal{A}$ and $\mathcal{B}$ by

$$g(\mathcal{A};\mathcal{B}) \triangleq \frac{P(\mathcal{A}\cup\mathcal{B})}{P(\mathcal{B})}$$

for all events $\mathcal{A},\mathcal{B}$, with $P(\mathcal{A})$ denoting the probability of an event $\mathcal{A}$. Now, for any given event $\mathcal{B}$, can $g(\mathcal{A};\mathcal{B})$ be a probability measure?

(Hint: A probability measure $h(\mathcal{A})$ has to satisfy the Three Axioms of Probability, namely, (1) $h(\mathcal{A}) \geq 0$ for any event $\mathcal{A}$, (2) $h(\mathcal{U}) = 1$ for the universe space $\mathcal{U}$, and (3) if $\mathcal{A}\cap\mathcal{C}$ is an empty set for events $\mathcal{A}$ and $\mathcal{C}$, then $P(\mathcal{A}\cup\mathcal{C}) = P(\mathcal{A}) + P(\mathcal{C})$.)

Sol: The first and third axioms are satisfied because (1) $g(\mathcal{A};\mathcal{B}) = P(\mathcal{A}\cup\mathcal{B})/P(\mathcal{B}) \geq 0$ for any event $\mathcal{A}$, and (3) if $\mathcal{A}\cap\mathcal{C}$ is an empty set, then

$$\begin{aligned} g(\mathcal{A}\cup\mathcal{C};\mathcal{B}) &= P(\mathcal{A}\cup\mathcal{C}\cup\mathcal{B})/P(\mathcal{B}) = P((\mathcal{A}\cup\mathcal{B})\cup(\mathcal{C}\cup\mathcal{B}))/P(\mathcal{B}) \\ &= P(\mathcal{A}\cup\mathcal{B})/P(\mathcal{B}) + P(\mathcal{C}\cup\mathcal{B})/P(\mathcal{B}) \\ &= g(\mathcal{A};\mathcal{B}) + g(\mathcal{C};\mathcal{B}). \end{aligned}$$

But, the second axiom is not met since (2) $g(\mathcal{U};\mathcal{B}) = P(\mathcal{U}\cup\mathcal{B})/P(\mathcal{B}) = 1/P(\mathcal{B}) \geq 1$ in general. Thus, $g(\mathcal{A};\mathcal{B})$ can not be a probability measure.

(2) (0.5%) If events $\mathcal{A}$ and $\mathcal{B}$ are mutually exclusive and independent *and* if $\mathcal{B}\subset\mathcal{A}$, find $P(\mathcal{B})$.

Sol: Because $P(\mathcal{A}\cap\mathcal{B}) = P(\mathcal{A})P(\mathcal{B})$ and $\mathcal{A}$ and $\mathcal{B}$ are mutually exclusive, we have $P(\mathcal{A})P(\mathcal{B}) = 0$. Further, because $\mathcal{B}\subset\mathcal{A}$, $P(\mathcal{A}\cap\mathcal{B}) = P(\mathcal{B}) = 0$. Thus, $P(\mathcal{B}) = 0$.

(3) (0.5%) Which of the following functions can be the probability density function of a real-valued random variable? Verify your answer. You will be given zero grade if you give a correct answer without explanation.

(a) $f_1(x) = 4 - \frac{1}{2}x$ for $6 \leq x \leq 8$ and $f_1(x) = 0$ elsewhere.

(b) $f_2(x) = \frac{1}{x}$ for $x \geq 2$ and $f_2(x) = 0$ elsewhere.

(c) $f_3(x) = \exp\{-|x|\}$ for $-\infty < x < \infty$.

Sol: A function can be the density function of a real-valued random variable if it is nonnegative and has area one. Now, all three functions are nonnegative. Now, since only $f_1(x)$ has area one and the other two do not, we conclude that only $f_1(x)$ can be the density function of a real-valued random variable.

(4) (1%, 0.5% each) Consider the hard limiter

$$g(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}.$$

Let $X(\mu)$ be a continuous random variable and $Y(\mu)$ be another defined from $X(\mu)$ through $Y(\mu) = g(X(\mu))$.

(a) Express the probability density function of $Y(\mu)$ in terms of the probability distribution function of $X(\mu)$.

(b) Let $X(\mu)$ be a Gaussian random variable with zero mean and unit variance. That is, the probability density function of $X(\mu)$ is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x^2\}.$$

Find the mean and variance of $Y(\mu)$.

Sol: (a) By definition,

$$\begin{aligned} F_Y(y) &= \Pr\{Y(\mu) \leq y\} = \begin{cases} 1, & y \geq 1 \\ 0, & y < -1 \\ \Pr\{X(\mu) < 0\}, & -1 \leq y < 1 \end{cases} \\ &= \begin{cases} 1, & y \geq 1 \\ 0, & y < -1 \\ F_X(0), & -1 \leq y < 1 \end{cases} \\ &= F_X(0) \cdot [u(y+1) - u(y-1)] + u(y-1) \\ &= F_X(0) \cdot u(y+1) + (1 - F_X(0))u(y-1) \end{aligned}$$

where $u(y)$ is the unit step function defined by $u(y) = 1$ if $y \geq 0$ and $u(y) = 0$ otherwise. Now, by using relationship of special functions, $du(y)/dy = \delta(y)$ with $\delta(y)$ the Dirac delta function, i.e., $\delta(y) = \int_{-\infty}^{y} u(x)dx$. Thus,

$$f_Y(y) = F_X(0) \cdot \delta(y+1) + (1 - F_X(0)) \cdot \delta(y-1).$$

(b) Because

$$F_X(0) = \int_{-\infty}^{0} f_X(x)dx = \int_{0}^{\infty} f_X(x)dx = 1/2$$

we have

$$f_Y(y) = \frac{1}{2}[\delta(y+1) + \delta(y-1)].$$

Thus, $E\{Y(\mu)\} = 0$ and $\mathrm{Var}\{Y(\mu)\} = E\{Y^2(\mu)\} = 1$.

(5) (1%) Let $X(\mu)$ be a Gaussian random variable with probability density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x^2\}.$$

Define a new random variable $Y(\mu)$ as follows:

$$\text{If } X = x_0, \text{then } Y = \begin{cases} x_0, & \text{with probability } \frac{1}{2} \\ -x_0, & \text{with probability } \frac{1}{2} \end{cases}.$$

Find the joint probability density function $f_{X,Y}(x,y)$ and the marginal probability density function $f_Y(y)$. What do you observe? Are $X(\mu)$ and $Y(\mu)$ jointly Gaussian? Is $Y(\mu)$ marginally Gaussian?

Sol: Now,

$$f_{X,Y}(x,y) = f_{Y|X}(y|x)f_X(x) = \frac{1}{2}[\delta(y-x)+\delta(y+x)] \cdot \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}x^2\}.$$

With $f_{X,Y}(x,y)$, we have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}y^2\}$$

which means that $Y(\mu)$ is also a Gaussianl random variable. We observe that $X(\mu)$ and $Y(\mu)$ are not jointly Gaussian although they are marginally Gaussian.

(6) (0.5%) Let $X(\mu)$ and $Y(\mu)$ be two independent continuous random variables with probability density functions

$$\begin{aligned} f_X(x) &= \frac{1/\pi}{x^2+1}, \qquad |x| < \infty \\ f_Y(y) &= \exp\{-2|y|\}, \qquad |y| < \infty. \end{aligned}$$

Find $E\{X(\mu)Y(\mu)(X(\mu)+Y(\mu))\}$.

Sol: Because $X(\mu)$ and $Y(\mu)$ are independent, $f_{X,Y}(x,y) = f_Y(y)f_X(x)$. More, since both $f_Y(y)$ and $f_X(x)$ are even functions, $E\{X(\mu)\} = E\{Y(\mu)\} = 0$. Thus, $E\{X(\mu)Y(\mu)(X(\mu)+Y(\mu))\} = E\{X^2(\mu)\}E\{Y(\mu)\} + E\{X(\mu)\}E\{Y^2(\mu)\} = 0$.

(7) (0.5%) Let $X_1(\mu), X_2(\mu), \ldots$ be a random sequence where all random variables $X_n(\mu)$'s are mutually independent and identically distributed with marginal probability density function $f_{X_n}(x) = 1$ for $0 < x < 1$ and $f_{X_n}(x) = 0$ otherwise. Define a new random sequence $Y_1(\mu), Y_2(\mu), \ldots$ in which $Y_n(\mu)$ is defined by

$$Y_n(\mu) = n[1 - \max\{X_1(\mu), X_2(\mu), \ldots, X_n(\mu)\}]$$

with $\max\{x_1, x_2, \ldots, x_n\}$ being the largest value of $x_1, x_2, \ldots, x_n$. Prove that the random sequence $Y_1(\mu), Y_2(\mu), \ldots$ converges in distribution and find the limiting distribution function.

Sol: Now, the complementary distribution function of $Y_n(\mu)$ is given by

$$\begin{aligned}
\Pr\{Y_n(\mu) > y\} &= \Pr\{n[1 - \max\{X_1(\mu), X_2(\mu), ..., X_n(\mu)\}] > y\} \\
&= \Pr\{\max\{X_1(\mu), X_2(\mu), ..., X_n(\mu)\} < 1 - \frac{y}{n}\} \\
&= \Pr\{X_1(\mu) < 1 - \frac{y}{n}, X_2(\mu) < 1 - \frac{y}{n}, ..., X_n(\mu) < 1 - \frac{y}{n}\} \\
&= (\Pr\{X_1(\mu) < 1 - \frac{y}{n}\})^n \\
&\quad \text{(because } X_n(\mu)\text{'s are i.i.d.)} \\
&= \begin{cases} 0, & \text{if } y \geq n \\ (1 - \frac{y}{n})^n, & \text{if } 0 \leq y < n \end{cases} \\
&\quad \text{(because } X_n(\mu) \text{ is uniform in } (0, 1))
\end{aligned}$$

for $y \geq 0$ and $\Pr\{Y_n(\mu) > y\} = 1$ for $y < 0$. Thus, when n approaches to the infinity,

$$\lim_{n\to\infty} \Pr\{Y_n(\mu) > y\} = \begin{cases} 1, & \text{if } y < 0 \\ \lim_{n\to\infty}(1 - \frac{y}{n})^n = \exp\{-y\}, & \text{if } y \geq 0 \end{cases}$$

because for a finite y

$$\begin{aligned}
\lim_{n\to\infty}(1 - \frac{y}{n})^n &= \lim_{n\to\infty} \exp\{n \ln\{1 - \frac{y}{n}\}\} \\
&= \exp\{\lim_{n\to\infty} \frac{\ln\{1 - \frac{y}{n}\}}{1/n}\} \\
&= \exp\{\lim_{n\to\infty} \frac{yn^{-2}/(1 - \frac{y}{n})}{-n^{-2}}\} \\
&= \exp\{\lim_{n\to\infty} y/(\frac{y}{n} - 1)\} \\
&\quad \text{(by L'Hopital rule)} \\
&= \exp\{-y\}.
\end{aligned}$$

Thus, the sequence $Y_1(\mu), Y_2(\mu), ...$ converges in distribution to the exponential distribution function

$$F(y) = \begin{cases} 0, & \text{if } y < 0 \\ 1 - \exp\{-y\}, & \text{if } y \geq 0 \end{cases}.$$

(8) (0.5%) Prove that for any real-valued random variables $X(\mu)$, $Y(\mu)$ and $Z(\mu)$

$$\begin{aligned}
E\{X(\mu)Y(\mu)|Z(\mu)\} &= E\{E\{X(\mu)Y(\mu)|Y(\mu), Z(\mu)\}|Z(\mu)\} \\
&= E\{Y(\mu)E\{X(\mu)|Y(\mu), Z(\mu)\}|Z(\mu)\}.
\end{aligned}$$

Sol: By definition, we have

$$\begin{aligned}
E\{X(\mu)Y(\mu)|Z(\mu)\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y|Z}(x,y|z)dxdy \\
&= \int_{-\infty}^{\infty}[\int_{-\infty}^{\infty} xyf_{X|Y,Z}(x|y,z)dx]f_{Y|,Z}(y|z)dy \\
&= E\{E\{X(\mu)Y(\mu)|Y(\mu),Z(\mu)\}|Z(\mu)\} \\
&= \int_{-\infty}^{\infty} yf_{Y|,Z}(y|z)[\int_{-\infty}^{\infty} xf_{X|Y,Z}(x|y,z)dx]dy \\
&= \int_{-\infty}^{\infty}[yE\{X(\mu)|Y(\mu),Z(\mu)\}]f_{Y|,Z}(y|z)dy \\
&= E\{Y(\mu)E\{X(\mu)|Y(\mu),Z(\mu)\}|Z(\mu)\}.
\end{aligned}$$