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國立臺灣大學期中學期考試答案卷
National Taiwan University Midterm/Final Examination Answer Sheet

課程編號 Course no. EE5481

科目 Course title Stochastic Process 學院 College ______ 學系 Department ______ 組 Division ______ 年級 Year ______

考試日期 Date 2022 年 Y 1 月 M 14 日 D 學號 Student ID no. T09023(3 姓名 Name 梁尚瑋

從此處開始寫起。試卷用紙務須節用。非經主試認可不得續用其他紙張作答。

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Q2a). Pr{there are even numbers of arrivals in $[0,t]$}

$$= \sum_{k=0}^{\infty} \frac{1}{1+e^{-\lambda t}}\left(\frac{e^{-\lambda t}}{1+e^{-\lambda t}}\right)^{2k}$$

$$= \frac{1}{1+e^{-\lambda t}} \frac{1}{1-\left(\frac{e^{-\lambda t}}{1+e^{-\lambda t}}\right)^2} \quad \left(\because \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ for } |x|<1\right)$$

$$= \frac{1}{1+e^{-\lambda t}} \frac{1}{1-\frac{e^{-2\lambda t}}{(1+e^{-\lambda t})^2}} = \frac{1}{1+e^{-\lambda t}} \frac{1}{\frac{(1+e^{-\lambda t})^2 - e^{-2\lambda t}}{(1+e^{-\lambda t})^2}}$$

$$= \frac{1}{1+e^{-\lambda t}} \frac{(1+e^{-\lambda t})^2}{1+2e^{-\lambda t}+e^{-2\lambda t}-e^{-2\lambda t}}$$

$$= \frac{1+e^{-\lambda t}}{1+2e^{-\lambda t}}$$

Pr{there are odd numbers of arrivals in $[0,t]$}

$$= 1 - \text{above probability} = 1 - \frac{1+e^{-\lambda t}}{1+2e^{-\lambda t}} = \frac{1+2e^{-\lambda t}-1-e^{-\lambda t}}{1+2e^{-\lambda t}} = \frac{e^{-\lambda t}}{1+2e^{-\lambda t}}$$

So, for $t \geq 0$, $Pr\{X(\mu,t)=0\} = Pr\{X(\mu,t)=0 \mid X(\mu,0)=0\} Pr\{X(\mu,0)=0\} +$

$Pr\{X(\mu,t)=0 \mid X(\mu,0)=1\} Pr\{X(\mu,0)=1\} = \frac{1}{2}\left(\frac{1+e^{-\lambda t}}{1+2e^{-\lambda t}}\right) + \frac{1}{2}\left(\frac{e^{-\lambda t}}{1+2e^{-\lambda t}}\right)$

$= \frac{1}{2}$, and $Pr\{X(\mu,t)=1\} = 1 - Pr\{X(\mu,t)=0\} = \frac{1}{2}$.

b). Find $E\{X(\mu,t)\}$:

$E\{X(\mu,t)\} = Pr\{X(\mu,t)=1\} = \frac{1}{2}$.

c). Find $\lim_{t\to\infty} E\{X(\mu,t)X(\mu,0)\}$.

$E\{X(\mu,t)X(\mu,0)\} = Pr\{X(\mu,t)=1, X(\mu,0)=1\}$

$= Pr\{X(\mu,t)=1 \mid X(\mu,0)=1\} Pr\{X(\mu,0)=1\}$

$= \frac{1}{2}\left(\frac{1+t}{1+2t}\right)$

$\lim_{t\to\infty} E\{X(\mu,t)X(\mu,0)\} = \frac{1}{2}$

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Q4 a) $\Phi_{N_I}(w;t) = \sum_{i=0}^{\infty} e^{jwi} \frac{\lambda_I^i}{i!} e^{-\lambda_I} = e^{-\lambda_I} \sum_{i=0}^{\infty} \frac{(e^{jw}\lambda_I)^i}{i!} = e^{-\lambda_I} e^{\lambda_I e^{jw}} = \exp\{\lambda_I (e^{jw}-1)\}$

b) By the theorem introduced in class, If $N_1(\mu,t)_{t\geq 0}$ is poisson process with rate λ_1, and $N_2(\mu,t)_{t\geq 0}$ is poisson process with rate λ_2, and N_1 is independent with N_2, then $N(\mu,t) = N_1(\mu,t) + N_2(\mu,t)_{t\geq 0}$ is a poisson process with $\lambda = \lambda_1 + \lambda_2$.

So ~~, the characteristic function of A is: $\Phi_A(w;t) = \sum_{i=0}^{\infty} e^{jwi} \frac{(\lambda_I)^i}{i!} e^{-\lambda_I}$~~
~~$= e^{-\lambda_A} \sum_{i=0}^{\infty} \frac{(e^{jw}\lambda_A)^i}{i!} = e^{-\lambda_A} e^{\lambda_A e^{jw}} = \exp\{\lambda_A (e^{jw}-1)\}$. with $\lambda_A = \lambda_I + \lambda_{II}$.~~

c). B is not a poisson process. $P(B(\mu,t) = -1) \geq P(N_I(\mu,t) = 0, N_{II}(\mu,t) = 1)$.

$= P(N_I(\mu,t)=0)\, P(N_{II}(\mu,t)=1)$

$= e^{-\lambda_I t} \cdot e^{-\lambda_{II} t} \cdot t \cdot \lambda_{II} > 0$. So it is not a poisson process

~~$\Phi_B(w;t) = \sum_{i=0}^{\infty} e^{jwi} \left(\frac{\lambda_I^i}{i!} e^{-\lambda_I} - \frac{\lambda_{II}^i}{i!} e^{-\lambda_{II}}\right) =$~~

$\Phi_A(w;t) = \sum_{i=0}^{\infty} e^{jwi} \left(\frac{\lambda_I^i}{i!} e^{\lambda_I} + \frac{\lambda_{II}^i}{i!} e^{\lambda_{II}}\right) = \sum_{i=0}^{\infty} e^{jwi}\left(\frac{\lambda_I}{i!} e^{\lambda_I}\right) + \sum_{i=0}^{\infty} e^{jwi}\left(\frac{\lambda_{II}}{i!} e^{\lambda_{II}}\right)$

$= \exp\{\lambda_I (e^{jw}-1)\} + \exp\{\lambda_{II}(e^{jw}-1)\}$ (from (a))

$= \exp\{e^{jw}-1\}(e^{\lambda_I} + e^{\lambda_{II}})$

$\Phi_B(w;t) = \sum_{i=0}^{\infty} e^{jwi}\left(\frac{\lambda_{II}^i}{i!} e^{\lambda_{II}} - \frac{\lambda_I^i}{i!} e^{\lambda_I}\right) = \sum_{i=0}^{\infty} e^{jwi}\left(\frac{\lambda_{II}}{i!} e^{\lambda_{II}}\right) - \sum_{i=0}^{\infty} e^{jwi}\left(\frac{\lambda_I}{i!} e^{\lambda_I}\right)$

$= \exp\{\lambda_{II}(e^{jw}-1)\} - \exp\{\lambda_I (e^{jw}-1)\}$

$= \exp\{e^{jw}-1\}(e^{\lambda_{II}} - e^{\lambda_I})$

Q1(a). First, evaluate $E\{Y(\mu,t)\}$, $E(A(\mu)) = \int_{-\infty}^{\infty} a^2 \exp\{\frac{-a^2}{2}\} u(a)\, da = \lim_{d\to\infty}\lim_{c\to-\infty} [-\exp\{-a^2/2\} u(a)]_c^d$

$= 0$., $E(\cos(2\pi f t + \phi(\mu)) = 0$, ($\because \phi(\mu)$ is uniformly distributed in $(0, 2\pi)$), $E(N_c(\mu,t)) = E(N_s(\mu,t)) = 0$, so $E(Y(\mu,t)) = 0$.

$R_X(t_1,t_2) = E\{A(\mu)^2\} E\{\cos(2\pi f t_1 + \phi(\mu)) \cos(2\pi f t_2 + \phi(\mu))\} + R_{N_c}(t_1-t_2) - R_{N_s}(t_1-t_2)$

$= \cos(2\pi f(t_1-t_2)) + R_{N_c}(t_1-t_2) - R_{N_s}(t_1-t_2)$

it is only depend on τ, so it is wide sense stationary.

b). let $Y_i(\mu) \triangleq Y(\mu,t_i) = A(\mu)\cos(2\pi f t_i + \phi(\mu)) + N_c(\mu,t_i)\cos(2\pi f t_i) - N_s(\mu,t_i)\sin(2\pi f t_i)$

for $N \in \mathbb{N}$, ~~$\Phi_{Y_1,Y_2,\ldots,Y_N}(w_1,w_2,\ldots,w_N) = E\{\exp\{j\sum_{i=1}^{N} w_i Y_i(\mu)\}\} = E\{\exp\{j\sum_{i=1}^{N} w_i [A(\mu)\cos(2\pi f t_i + \phi(\mu)) +$~~
~~$N_c(\mu,t_i)\cos(2\pi f t_i) - N_s(\mu,t_i)\sin(2\pi f t_i)]\}\} = E\{\exp\{j\sum_{i=1}^{N} w_i A(\mu)\cos(2\pi f t_i + \phi(\mu))\}\}$~~ $E\{\exp\{j\sum_{i=1}^{N} w_i$

$N_c(\mu,t_i)\cos(2\pi f t_i)\}\}$ $E\{\exp\{j\sum_{i=1}^{N} w_i(-N_s(\mu,t_i)\sin(2\pi f t_i)\}\}$

~~Q15 cont'd).~~ $\cancel{E\{\exp\{j\sum_{i=1}^{N} w_i A(\mu)\cos(2\pi f t_i + \phi(\mu))\}\} \cdot \exp\{\frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N} w_i w_k C_{ik}\} \cdot \exp\{\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} w_j w_k C_{jk}\}}$

Q16. let $X(\mu,t) = A(\mu)\sin(2\pi f t + \phi(\mu)) + N_c(\mu,t)\sin(2\pi f t) - N_s(\mu,t)\cdot\cos(2\pi f t)$

then by Jacobian, X and Y are independent and identically distributed Gaussian random variables which have mean $=0$ and variance $=1$, for a fixed t. Thus, the first order density of $Y(\mu,t)$ is of Gaussian density with mean $=0$ and variance $=1$, i.e. $f_Y(y) = \frac{1}{\sqrt{2\pi}}\exp\{-\frac{y^2}{2}\}$

c). $E\{Y^3(\mu,t)\}$ = From (b), $Y(\mu,t)$ is Gaussian distributed with mean $=0$ and variance $=1$ for a fixed t. Thus $E\{Y^3(\mu,t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{y^3\exp\{-\frac{y^2}{2}\}}_{\text{this is odd function}} dy = 0$

d). $Y(\mu,t) = A(\mu)\cos(2\pi f t + \phi(\mu)) + N_c(\mu,t)\cos(2\pi f t) - N_s(\mu,t)\sin(2\pi f t)$

$Y(\mu, t+\frac{1}{4f}) = A(\mu)\cos(2\pi f(t+\frac{1}{4f}) + \phi(\mu)) + N_c(\mu,t)\cos(2\pi f(t+\frac{1}{4f})) - N_s(\mu,t)\sin(2\pi f(t+\frac{1}{4f}))$

~~By~~

$= -A(\mu)\sin(2\pi f t + \phi(\mu)) - N_c(\mu,t)\sin(2\pi f t) + N_s(\mu,t)\cos(2\pi f t)$

By Jacobian, $Y(\mu,t)$ and $Y(\mu, t+\frac{1}{4f})$ are i.i.d. Gaussian random variables which have 0 mean and unit variance

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~~Q3. $\Pr\{Z(\mu)>Q^{-1}(\alpha)\} = 1-\Pr\{Z(\mu)\le Q^{-1}(\alpha)\}$~~

~~$= 1-\Pr\{X(\mu)Y(\mu)\le Q^{-1}(\alpha)\}$~~

~~$1-\int_{2}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{1}{2}y^2\}dy$~~

~~$= Q.A$~~

Q4d). $\Pr\{A(\mu,2t)=n+k,\ B(\mu,2t)=n-k \mid N_I(\mu,t)=k\}$

$= \Pr\{N_{II}(\mu,2t)=n \mid N_I(\mu,t)=k\}$

$= \Pr\{N_{II}(\mu,2t)=n\}$ (independent).

$= \cancel{e^{-\lambda_{II}}\frac{(\lambda_{II})^{2t}}{(2t)!}}\quad e^{-\lambda_{II}\cdot 2t}\frac{(\lambda\cdot 2t)^n}{n!}$

Q3. $Z(\mu)\triangleq X(\mu)Y(\mu)$

$f_X(x)=\frac{1}{\sqrt{2\pi}}\exp\{-\frac{x^2}{2}\}$

$f_Y(y)=\begin{cases}\frac{1}{2} & \text{if } y=+1\\ \frac{1}{2} & \text{if } y=-1\end{cases}$

$f_{XY}(x,y)=\begin{cases}\frac{1}{2\sqrt{2\pi}}\exp\{-\frac{x^2}{2}\} & \text{if } y=+1\\ \frac{1}{2\sqrt{2\pi}}\exp\{-\frac{x^2}{2}\} & \text{if } y=-1\end{cases}$

~~第四頁~~

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考試日期 Date ____ 年 Y ____ 月 M ____ 日 D 學號 Student ID no. T1·7·23·3 姓名 Name 梁高淺

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Q5.a). We can express $T(\mu) = \min\{W_1(\mu), W_2(\mu), \dots, W_k(\mu)\}$ when $X(\mu,t) = k$.

$$\Pr\{T(\mu) > t \mid X(\mu,t) = k\} = \Pr\{\min\{W_1(\mu), W_2(\mu), \dots, W_k(\mu)\} > t\}$$
$$= \Pr\{W_1(\mu) > t, W_2(\mu) > t, \dots, W_k(\mu) > t\}$$
$$= \Pr\{W_1(\mu) > t\}\Pr\{W_2(\mu) > t\} \dots \Pr\{W_k(\mu) > t\}$$
$$= \exp(-k\beta t)$$

$\therefore$ density : $\begin{cases} \frac{1}{k\beta} e^{-k\beta t} & , t \geq 0 \\ 0 & , t < 0. \end{cases}$

b). Let r_{ij} be transition rate at which $X(\mu,t)$ enters state j from state i

$r_{i,i+1} = \alpha$ for $i = 0, 1, \dots, N-1$ || $r_{i,i-1} = i\beta$ for $i = 1, 2, \dots N$ || $r_{ij} = 0$ otherwise.

By the global balance equation, $\alpha p_0 = \beta p_1$

$$(\alpha + j\beta) p_j = \alpha p_{j-1} + (j+1)\beta p_{j+1} \text{ for } j = 1, 2, \dots, N-1$$

$$\alpha p_{N-1} = N\beta p_N$$

$\Rightarrow p_j = \frac{\alpha}{j\beta} p_{j-1} = \frac{(\alpha/\beta)^j}{j!} p_0$

$\Rightarrow p_0 = \frac{1}{\sum_{j=0}^{N} \frac{(\alpha/\beta)^j}{j!}}$

from the identity that $\sum_{j=0}^{N} p_j = 1$, we have :

$$p_j = \frac{\frac{(\alpha/\beta)^j}{j!}}{\sum_{j=0}^{N} \frac{(\alpha/\beta)^j}{j!}} \quad \text{for } j = 0, 1, \dots, N$$

c). Pr{technician can take a break} = Pr{all machine remains working}

$$= (1/\beta)^N$$

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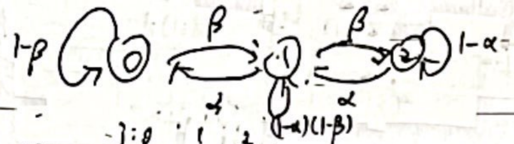
Q7 a). False (b) True (c) True (d) True (e) True

Q6 a). P(working part fails) = α, P(fail part repaired) = β.



$$\mathbb{P} = \begin{array}{c|ccc} i \backslash j & 0 & 1 & 2 \\ \hline 0 & & & \\ 1 & & & \\ 2 & & & \end{array} \quad \begin{bmatrix} 1-\beta & \beta & 0 \\ \alpha & (1-\alpha)(1-\beta) & \beta \\ 0 & \alpha & 1-\alpha \end{bmatrix}$$

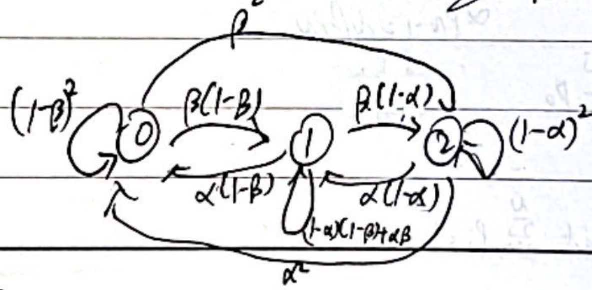
b). $\cancel{(1-\beta)p_0 = \beta p_1}$ $\cancel{\beta p_0 = \alpha p_1}$

$$\begin{cases} \pi_0 = (1-\beta)\pi_0 + \alpha\pi_1 \\ \pi_1 = \beta\pi_0 + (1-\alpha)(1-\beta)\pi_1 + \alpha\pi_2 \\ \pi_2 = \beta\pi_1 + (1-\alpha)\pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \beta\pi_0 = \alpha\pi_1 \\ (1-\alpha)(1-\beta)\pi_1 = \beta\pi_0 + \alpha\pi_2 \\ \alpha\pi_2 = \beta\pi_1, \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

Using the hint $\pi_1 = \frac{2\alpha\beta}{(\alpha+\beta)^2}$,

~~from $\beta\pi_0 = \alpha\pi_1$, $\pi_0 = \frac{2\alpha^2}{(\alpha+\beta)^2}$. from $\alpha\pi_2 = \beta\pi_1$, $\pi_2 = \frac{2\beta^2}{(\alpha+\beta)^2}$~~



b).
$$\begin{cases} \pi_0 = (1-\beta)^2\pi_0 + \alpha(1-\beta)\pi_1 + \alpha^2\pi_2 \\ \pi_1 = \beta(1-\beta)\pi_0 + ((1-\alpha)(1-\beta)+\alpha\beta)\pi_1 + \alpha(1-\alpha)\pi_2 \\ \pi_2 = \beta^2\pi_0 + \beta(1-\alpha)\pi_1 + (1-\alpha)^2\pi_2 \\ \cancel{\pi_1 + \pi_2 + \pi_3 = 1} \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

c).
$$\mathbb{P}_\theta = [p_{ij}] = \begin{array}{c|ccc} i \backslash j & 0 & 1 & 2 \\ \hline 0 & & & \\ 1 & & & \\ 2 & & & \end{array} \quad \begin{bmatrix} (1-\beta)^2 & \beta(1-\beta) & \beta^2 \\ \alpha(1-\beta) & (1-\alpha)(1-\beta)+\alpha\beta & \beta(1-\alpha) \\ \alpha^2 & \alpha(1-\alpha) & (1-\alpha)^2 \end{bmatrix}$$

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Q6 b contd). Solving the below equation:

$$\begin{cases} \pi_0 = (1-\beta)^2 \pi_0 + \alpha(1-\beta)\pi_1 + \alpha^2 \pi_2 \\ \pi_1 = \beta(1-\beta)\pi_0 + \left((1-\alpha)(1-\beta)+\alpha\beta\right)\pi_1 + \alpha(1-\alpha)\pi_2 \\ \pi_2 = \beta^2 \pi_0 + \beta(1-\alpha)\pi_1 + (1-\alpha)^2 \pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases}$$

Using the hint of $\pi_1 = \frac{2\alpha\beta}{(\alpha+\beta)^2}$, we have:

$$\pi_0 = (1-\beta)^2 \pi_0 + \alpha(1-\beta)\left(\frac{2\alpha\beta}{(\alpha+\beta)^2}\right) + \alpha^2 \pi_2$$

$$\pi_2 = \beta^2 \pi_0 + \beta(1-\alpha)\left(\frac{2\alpha\beta}{(\alpha+\beta)^2}\right) + (1-\alpha)^2 \pi_2$$

$$\pi_0 + \pi_2 = 1 - \frac{2\alpha\beta}{(\alpha+\beta)^2} = \frac{\alpha^2+\beta^2}{(\alpha+\beta)^2}$$

Solving the above we have $\pi_0 = \frac{\alpha^2}{(\alpha+\beta)^2}$, $\pi_2 = \frac{\beta^2}{(\alpha+\beta)^2}$.

So, $\pi_k = \lim_{n\to\infty} Pr\{X_n(u) = k\}$ for $k \in \{0,1,2\}$,

$$\pi_0 = \frac{\alpha^2}{(\alpha+\beta)^2}, \quad \pi_1 = \frac{2\alpha\beta}{(\alpha+\beta)^2}, \quad \pi_2 = \frac{\beta^2}{(\alpha+\beta)^2}.$$