Stochastic Processes and Applications, Fall 2015
Homework Three (5%)

1. (2.5%, 0.5% each) Consider the deterministic system $\mathbf{T}$ with input process $X(\mu,t)$ and output process $Y(\mu,t)$ being related by

$$Y(\mu,t)=\mathbf{T}[X(\mu,t)]=\frac{1}{T}\int_t^{t+1}(\alpha-t)X(\mu,\alpha)d\alpha.$$

Answer the following questions:

(a) Show that the system $\mathbf{T}$ is linear and time-invariant. Also, find the impulse response of the system.

Sol: First, $\mathbf{T}$ is linear because

$$\begin{aligned}\mathbf{T}[ax(t)+by(t)] &= \frac{1}{T}\int_t^{t+1}(\alpha-t)\left[ax(\alpha)+by(\alpha)\right]d\alpha \\ &= \frac{a}{T}\int_t^{t+1}(\alpha-t)x(\alpha)d\alpha \\ &\quad +\frac{b}{T}\int_t^{t+1}(\alpha-t)y(\alpha)d\alpha \\ &= a\mathbf{T}[x(t)]+b\mathbf{T}[y(t)].\end{aligned}$$

Second, $\mathbf{T}$ is time-invariant because if $\mathbf{T}[x(t)]=z(t)$, then

$$\begin{aligned}\mathbf{T}[x(t-c)] &= \frac{1}{T}\int_t^{t+1}(\alpha-t)x(\alpha-c)d\alpha \\ (\text{let } \theta\triangleq\alpha-c) &= \frac{1}{T}\int_{t-c}^{t-c+1}(\theta+c-t)x(\theta)d\theta \\ &= z(t-c).\end{aligned}$$

Last, the impulse response is derived as

$$\begin{aligned}h(t) &= \mathbf{T}[\delta(t)] \\ &= \frac{1}{T}\int_t^{t+1}(\alpha-t)\delta(\alpha)d\alpha \\ &= \begin{cases}\frac{-t}{T} & \text{if } -1<t<0 \\ 0 & \text{otherwise}\end{cases}.\end{aligned}$$

(b) Let $X(\mu,t)$ and $Y(\mu,t)$ are both wide-sense stationary random processes with means η_X and η_Y. It is known that $\eta_Y=\alpha\eta_X$ with α a constant. Determine α.

Sol: Because the system $\mathbf{T}$ is linear and time-invariant, we have

$$\begin{aligned}\alpha &= \mathcal{F}\{h(t)\}|_{\omega=0} \\ &= \int_{-1}^{0}\frac{-t}{T}dt \\ &= \frac{-1}{2T}t^2|_{-1}^{0} \\ &= \frac{1}{2T}\end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform operator.

(c) If $X(\mu,t)$ has mean $\eta_X(t)=0$ and autocorrelation $R_X(t_1,t_2)=\delta(t_1-t_2)$ with $\delta(\tau)$ being a Dirac delta function (i.e., $\int_{-\infty}^{\infty} g(x)\delta(x-t)dx = g(t)$ for any well-defined function $g(t)$), find the mean function $\eta_Y(t)$ and the autocorrelation function $R_Y(t_1,t_2)$ of $Y(\mu,t)$.

Sol: The mean function and the autocorrelation function of $Y(\mu,t)$ are derived as

$$\begin{aligned}
\eta_Y(t) &= E\{Y(\mu,t)\} \\
&= E\{\frac{1}{T}\int_t^{t+1}(\alpha-t)X(\mu,\alpha)d\alpha\} \\
&= \frac{1}{T}\int_t^{t+1}(\alpha-t)E\{X(\mu,\alpha)\}d\alpha \\
&= 0 \qquad (1)
\end{aligned}$$

and

$$\begin{aligned}
R_Y(t_1,t_2) &= E\{Y(\mu,t_1)Y(\mu,t_2)\} \\
&= \frac{1}{T^2}\int_{t_2}^{t_2+1}\int_{t_1}^{t_1+1}(\alpha_1-t_1)(\alpha_2-t_2)R_X(\alpha_1,\alpha_2)d\alpha_1 d\alpha_2 \\
&= \frac{1}{T^2}\int_{t_2}^{t_2+1}\int_{t_1}^{t_1+1}(\alpha_1-t_1)(\alpha_2-t_2)\delta(\alpha_1-\alpha_2)d\alpha_1 d\alpha_2 \\
&= \begin{cases} \frac{1}{T^2}\int_{t_1}^{t_2+1}(\alpha_2-t_1)(\alpha_2-t_2)d\alpha_2, & 0\le t_1-t_2<1 \\ \frac{1}{T^2}\int_{t_2}^{t_1+1}(\alpha_2-t_1)(\alpha_2-t_2)d\alpha_2, & -1<t_1-t_2<0 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{T^2}\int_0^{1-\tau}(x+\tau)x dx, & 0\le\tau<1 \\ \frac{1}{T^2}\int_0^{1+\tau}x(x-\tau)dx, & -1<\tau<0 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{T^2}\int_0^{1-|\tau|}x(x+|\tau|)dx, & |\tau|<1 \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \frac{1}{T^2}(\frac{1}{3}+\frac{1}{6}|\tau|)(1-|\tau|)^2, & |\tau|<1 \\ 0, & \text{otherwise} \end{cases} \qquad (2) \\
&= R_Y(\tau)
\end{aligned}$$

where $\tau \triangleq t_1-t_2$.

(d) If $X(\mu,t)$ is a Gaussian random process with mean $\eta_X(t)=0$ and autocorrelation $R_X(t_1,t_2)=\delta(t_1-t_2)$, find the second-order density of $Y(\mu,t)$, i.e., the joint probability density function of random variables $Y(\mu,t_1)$ and $Y(\mu,t_2)$ for any two distinct time points t_1 and t_2.

Sol: Because $X(\mu,t)$ is a Gaussian random process and $\mathbf{T}$ is linear and time-invariant, $Y(\mu,t)$ is a Gaussian random process. Hence, $Y(\mu,t_1)$ and $Y(\mu,t_2)$ are jointly Gaussian with mean for any two distinct time points t_1 and t_2. Further, because the mean and autocorrelation functions of $Y(\mu,t)$ are already derived in (1) and (2), respectively, the mean, variance, and covariance of $Y(\mu,t_1)$ and $Y(\mu,t_2)$ are given by

$$E\{Y(\mu,t_1)\} = 0$$

$$
\begin{aligned}
E\{Y(\mu, t_2)\} &= 0 \\
Var\{Y(\mu, t_1)\} &= Var\{Y(\mu, t_2)\} = R_Y(0) = \frac{1}{3T^2} = \sigma^2 \\
Cov\{Y(\mu, t_1), Y(\mu, t_2)\} &= R_Y(t_1 - t_2).
\end{aligned}
$$

Therefore, the second-order density of $Y(\mu, t)$ is given by

$$
f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\gamma^2}} \exp\{\frac{-1}{2(1-\gamma^2)}[\frac{x^2}{\sigma^2} - 2\gamma\frac{xy}{\sigma^2} + \frac{y^2}{\sigma^2}]\}
$$

where $\gamma \triangleq R_Y(t_1 - t_2)/\sigma^2$.

(e) If $X(\mu, t)$ is a Gaussian random process with mean $\eta_X(t) = 0$ and autocorrelation $R_X(t_1, t_2) = \delta(t_1 - t_2)$, find the power spectrum of $Y(\mu, t)$.

Sol: The power spectrum of $Y(\mu, t)$ is given by

$$
\begin{aligned}
S_Y(f) &= \mathcal{F}\{R_Y(\tau)\} \\
&= \frac{1}{T^2}\int_{-1}^{1}(\frac{1}{3} + \frac{1}{6}|\tau|)(1-|\tau|)^2 e^{-j\omega\tau}d\tau \\
&= \frac{1}{T^2}\int_{-1}^{1}\left[\frac{1}{3} - \frac{1}{2}|\tau| + \frac{1}{6}|\tau|^3\right] e^{-j\omega\tau}d\tau \\
&= \frac{1}{T^2\omega^4}[e^{j\omega}(j\omega - 1) - e^{-j\omega}(j\omega + 1) + \omega^2 + 2] \qquad (3) \\
&= \frac{1}{T^2\omega^4}[-2\omega\sin\omega - 2\cos\omega + \omega^2 + 2]
\end{aligned}
$$

where $\mathcal{F}\{\cdot\}$ is the Fourier transform operator. Note that (3) stems from the following equalities

$$
\int_{-1}^{1} e^{-j\omega\tau}d\tau = \frac{1}{j\omega}\left[e^{j\omega} - e^{-j\omega}\right]
$$

$$
\int_{-1}^{1} |\tau|e^{-j\omega\tau}d\tau = \frac{1}{(j\omega)^2}\left[2 - (j\omega+1)e^{-j\omega} + (j\omega - 1)e^{j\omega}\right]
$$

$$
\begin{aligned}
\int_{-1}^{1} |\tau|^3 e^{-j\omega\tau}d\tau &= \frac{1}{(j\omega)^4}[((j\omega)^3 - 3(j\omega)^2 + 6(j\omega) - 6)e^{j\omega} \\
&\quad -((j\omega)^3 + 3(j\omega)^2 + 6(j\omega) + 6)e^{-j\omega} + 12].
\end{aligned}
$$

2. (1%) Consider the quadrature representation of the zero-mean, real-valued random process

$$
n(\mu, t) = n_c(\mu, t)\cos(2\pi f_c t) - n_s(\mu, t)\sin(2\pi f_c t)
$$

where $f_c > 0$. Prove that if $n(\mu, t)$ is wide-sense stationary, then $n_c(\mu, t)$ and $n_s(\mu, t)$ are jointly wide-sense stationary with

$$
R_{n_c}(\tau) = R_{n_s}(\tau) = R_n(\tau)\cos(2\pi f_c\tau) + \widehat{R}_n(\tau)\sin(2\pi f_c\tau)
$$

and

$$
R_{n_c n_s}(\tau) = -R_{n_s n_c}(\tau) = R_n(\tau)\sin(2\pi f_c\tau) - \widehat{R}_n(\tau)\cos(2\pi f_c\tau).
$$

(Note that $\widehat{R}_n(\tau)$ represents the Hilbert transform of $R_n(\tau)$.)

Sol: First note that $n_c(\mu,t)$ and $n_s(\mu,t)$ can be expressed in terms of $n(\mu,t)$ and $\widehat{n}(\mu,t)$ as

$$\begin{aligned} n_c(\mu,t) &= n(\mu,t)\cos(2\pi f_c t) + \widehat{n}(\mu,t)\sin(2\pi f_c t) \\ n_s(\mu,t) &= \widehat{n}(\mu,t)\cos(2\pi f_c t) - n(\mu,t)\sin(2\pi f_c t). \end{aligned}$$

Because $n(\mu,t)$ is wide-sense stationary, $n(\mu,t)$ and $\widehat{n}(\mu,t)$ are jointly wide-sense stationary. From the class note, we also have that $R_n(\tau) = \widehat{R}_n(\tau)$ and that $R_{\widehat{n}n}(\tau) = -R_{n\widehat{n}}(\tau) = \widehat{R}_n(\tau)$. Now, from the above equations, both $n_c(\mu,t)$ and $n_s(\mu,t)$ have mean zero since both $n(\mu,t)$ and $\widehat{n}(\mu,t)$ have mean zero. In what follows, we show that $R_{n_c}(\tau)$, $R_{n_s}(\tau)$, $R_{n_cn_s}(\tau)$ and $R_{n_sn_c}(\tau)$ depend only on time difference.

$$\begin{aligned} R_{n_c}(\tau) &= E\{n_c(\mu,t+\tau)n_c(\mu,t)\} \\ &= R_n(\tau)\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) + R_{n\widehat{n}}(\tau)\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t) \\ &\quad + R_{\widehat{n}n}(\tau)\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) + R_{\widehat{n}}(\tau)\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t) \\ &= R_n(\tau)[\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) + \sin(2\pi f_c(t+\tau))\sin(2\pi f_c t)] \\ &\quad + \widehat{R}_n(\tau)[\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) - \cos(2\pi f_c(t+\tau))\sin(2\pi f_c t)] \\ &= R_n(\tau)\cos(2\pi f_c\tau) + \widehat{R}_n(\tau)\sin(2\pi f_c\tau) \end{aligned}$$

$$\begin{aligned} R_{n_s}(\tau) &= E\{n_s(\mu,t+\tau)n_s(\mu,t)\} \\ &= R_{\widehat{n}}(\tau)\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) - R_{\widehat{n}n}(\tau)\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t) \\ &\quad - R_{n\widehat{n}}(\tau)\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) + R_n(\tau)\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t) \\ &= R_n(\tau)[\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) + \sin(2\pi f_c(t+\tau))\sin(2\pi f_c t)] \\ &\quad - \widehat{R}_n(\tau)[\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t) - \sin(2\pi f_c(t+\tau))\cos(2\pi f_c t)] \\ &= R_n(\tau)\cos(2\pi f_c\tau) + \widehat{R}_n(\tau)\sin(2\pi f_c\tau) \end{aligned}$$

$$\begin{aligned} R_{n_cn_s}(\tau) &= E\{n_c(\mu,t+\tau)n_s(\mu,t)\} \\ &= -R_n(\tau)\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t) + R_{n\widehat{n}}(\tau)\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) \\ &\quad - R_{\widehat{n}n}(\tau)\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t) + R_{\widehat{n}}(\tau)\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) \\ &= R_n(\tau)[\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) - \cos(2\pi f_c(t+\tau))\sin(2\pi f_c t)] \\ &\quad - \widehat{R}_n(\tau)[\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t) + \cos(2\pi f_c(t+\tau))\cos(2\pi f_c t)] \\ &= R_n(\tau)\sin(2\pi f_c\tau) - \widehat{R}_n(\tau)\cos(2\pi f_c\tau) \end{aligned}$$

and

$$\begin{aligned} R_{n_sn_c}(\tau) &= E\{n_s(\mu,t+\tau)n_c(\mu,t)\} \\ &= -R_n(\tau)\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) - R_{n\widehat{n}}(\tau)\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t) \\ &\quad + R_{\widehat{n}n}(\tau)\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) + R_{\widehat{n}}(\tau)\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t) \\ &= R_n(\tau)[\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t) - \sin(2\pi f_c(t+\tau))\cos(2\pi f_c t)] \\ &\quad + \widehat{R}_n(\tau)[\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) + \sin(2\pi f_c(t+\tau))\sin(2\pi f_c t)] \\ &= -R_n(\tau)\sin(2\pi f_c\tau) + \widehat{R}_n(\tau)\cos(2\pi f_c\tau). \end{aligned}$$

Since $n_c(\mu, t)$ and $n_s(\mu, t)$ are zero-meaned and their autocorrelation and cross-correlation functions depend on the time difference τ only, $n_c(\mu, t)$ and $n_s(\mu, t)$ are jointly wide-sense stationary.

3. (0.5%) Let $X(\mu, t)$ be a real-valued random process with zero mean and autocorrelation

$$R_X(t_1, t_2) = \delta(t_1 - t_2)$$

where $\delta(t)$ is an Dirac delta function. Find a Karhunen Loéve expansion of $X(\mu, t)$ for $0 < t < T$ (with $T > 0$).

Sol: In order to find the Karhunen-Loève expansion of $X(\mu, t)$ in the interval $(0, T)$, we should find the eigenfunctions of $R_X(\tau)$. The eigenvalues ρ_k's and eigenfunctions $\phi_k(t)$'s of $R_X(\tau)$ have to satisfy the following equation:

$$\int_0^T R_X(t, s)\phi_k(s)ds = \int_0^T \delta(t - s)\phi_k(s)ds = \phi_k(t) = \rho_k\phi_k(t) \quad t \in (0, T).$$

This means that and any orthonormal function set $\{\phi_k(t), k = 1, 2, ...\}$ can be used to expand $X(\mu, t)$, with the same eigenvalue $\rho_k = 1$ for all k. Therefore, the Karhunen-Loève expansion of $X(\mu, t)$ in the interval $(0, T)$ is given by

$$X(\mu, t) = \sum_{n=1}^{\infty} x_n(\mu)\phi_n(t)$$

where $\{\phi_k(t), k = 1, 2, ...\}$ is an orthonormal function set and $x_n(\mu)$'s are given by

$$x_n(\mu) = \int_0^T X(\mu, t)\phi_n(t)dt \text{ with } E\{x_n^2(\mu)\} = 1.$$

4. (1%) Let $X(\mu, t)$ be a Gaussian random process with mean $\eta_X(t) = 0$ and autocorrelation function $R_X(t_1, t_2) = \frac{1}{2}(|t_1| + |t_2| - |t_1 - t_2|)$. Also, define two random variables $Y(\mu) = X(\mu, \alpha)$ and $Z(\mu) = X(\mu, -\alpha)$ with α being a real number and nonzero.

(a) (0.5%) Find the joint probability density function of $Y(\mu)$ and $Z(\mu)$.

Sol: Because $X(\mu, t)$ is Gaussian, $Y(\mu)$ and $Z(\mu)$ are jointly Gaussian. Further, the statistics of $Y(\mu)$ and $Z(\mu)$ are derived as

$$E\{Y(\mu)\} = E\{Z(\mu)\} = 0$$

$$Var\{Y(\mu)\} = R_X(\alpha, \alpha) = |\alpha|$$

$$Var\{Z(\mu)\} = R_X(-\alpha, -\alpha) = |\alpha|$$

and

$$Cov\{Y(\mu), Z(\mu)\} = R_X(\alpha, -\alpha) = \frac{1}{2}(|\alpha| + |\alpha| - 2|\alpha|) = 0.$$

Therefore, the joint probability density function of $Y(\mu)$ and $Z(\mu)$ is given by

$$f_{Y,Z}(y, z) = \frac{1}{2\pi|\alpha|} \exp\{\frac{-(y^2 + z^2)}{2|\alpha|}\}.$$

(b) (0.5%) Find $E\{Y^n(\mu)Z^{n+1}(\mu)\}$ for any positive integer n.

Sol: Recall that

$$E\{X_1(\mu)X_2(\mu)\cdots X_n(\mu)\} = \begin{cases} 0, & n \text{ is odd} \\ \sum_{all\ pairs}^{\frac{n!}{(\frac{n}{2})!2^{n/2}}} \left[\prod_{i_1 \neq i_2}^{\frac{n}{2}} E\{X_{i_1}(\mu)X_{i_2}(\mu)\} \right], & n \text{ is even} \end{cases}.$$

holds for zero-mean jointly Gaussian random variables $X_1(\mu), X_2(\mu), ..., X_n(\mu)$. Because $2n+1$ is odd, we have

$$E\{Y^n(\mu)Z^{n+1}(\mu)\} = 0.$$

for any positive integer n.