**Stochastic Processes and Applications, Fall 2015**
**Homework One (5%)**

1. (0.5%) Prove that if $\mathcal{A} \cap \mathcal{B} = \Phi$, then $P(\mathcal{A}) \leq P(\overline{\mathcal{B}})$.

   Sol: Because $\mathcal{A} \cap \mathcal{B} = \Phi$, $\mathcal{A}$ is a subset of $\overline{\mathcal{B}}$. Thus, $P(\mathcal{A}) \leq P(\overline{\mathcal{B}})$.

2. (0.5%) Prove that if $P(\mathcal{A}) = P(\mathcal{B}) = 1$, then $P(\mathcal{A} \cap \mathcal{B}) = 1$.

   Sol: Because $P(\mathcal{A}) = P(\mathcal{B}) = 1$, $1 = P(\mathcal{A}) \leq P(\mathcal{A} \cup \mathcal{B})$ and hence

$$1 = P(\mathcal{A} \cup \mathcal{B}) = P(\mathcal{A}) + P(\mathcal{B}) - P(\mathcal{A} \cap \mathcal{B}) = 2 - P(\mathcal{A} \cap \mathcal{B})$$

   which gives $P(\mathcal{A} \cap \mathcal{B}) = 1$.

3. (0.5%) Consider $N$ events $\mathcal{A}_1$,$\mathcal{A}_2$, ..., $\mathcal{A}_N$. Prove the following union-bound inequality

$$P(\bigcup_{n=1}^{N} \mathcal{A}_n) \leq \sum_{n=1}^{N} P(\mathcal{A}_n).$$

   Sol: Using the first and third Axioms of Probability, it can be shown that

$$\begin{aligned} P(\mathcal{A} \cup \mathcal{B}) &= P(\mathcal{A} \cup (\mathcal{B} - \mathcal{A})) \\ &= P(\mathcal{A}) + P(\mathcal{B} - \mathcal{A}) \\ &\leq P(\mathcal{A}) + P(\mathcal{B} - \mathcal{A}) + P(\mathcal{B} \cap \mathcal{A}) \\ &= P(\mathcal{A}) + P(\mathcal{B}) \end{aligned}$$

   for any two events $\mathcal{A}$ and $\mathcal{B}$. Note that the equality holds if and only if $P(\mathcal{B} \cap \mathcal{A}) = 0$. This fact implies that

$$\begin{aligned} P(\bigcup_{n=1}^{N} \mathcal{A}_n) &\leq P(\bigcup_{n=1}^{N-1} \mathcal{A}_n) + P(\mathcal{A}_N) \\ &\leq P(\bigcup_{n=1}^{N-2} \mathcal{A}_n) + P(\mathcal{A}_{N-1}) + P(\mathcal{A}_N) \\ &\leq \ldots \\ &\leq \sum_{n=1}^{N} P(\mathcal{A}_n). \end{aligned}$$

   Note that the equality holds if and only if $P(\bigcap_{n=1}^{N} \mathcal{A}_n) = 0$.

4. (0.5%) For a Gaussian random variable with mean $\eta_X$ and variance $\sigma_X^2$, prove that it has the characteristic function $\Phi_X(\omega) = \exp\{j\omega\eta_X - \frac{1}{2}\sigma_X^2\omega^2\}$.

Sol: By definition, the characteristic function is derived as

$$\begin{aligned}\Phi_X(\omega) &= E\{\exp\{j\omega X(\mu)\}\} \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\{j\omega x\} \exp\{-\frac{(x-\eta_X)^2}{2\sigma_X^2}\} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\{-\frac{(x-\eta_X - j\sigma_X^2\omega)^2 - j2\eta_X\sigma_X^2\omega + \sigma_X^4\omega^2}{2\sigma_X^2}\} dx \\ &= \exp\{j\omega\eta_X - \frac{1}{2}\sigma_X^2\omega^2\} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp(-\frac{(x-\eta_X - j\sigma_X^2\omega)^2}{2\sigma_X^2}) dx}_{=1 \text{ since the integrand is a Gaussian density.}} \\ &= \exp\{j\omega\eta_X - \frac{1}{2}\sigma_X^2\omega^2\}.\end{aligned}$$

5. (0.5%) Show that

$$P(\mathcal{A}) = P(\mathcal{A}|X(\mu) \le x) F_X(x) + P(\mathcal{A}|X(\mu) > x)(1 - F_X(x))$$

for any event $\mathcal{A}$ and any continuous random variable $X(\mu)$.

Sol: Define event $\mathcal{B} \triangleq \{\mu | X(\mu) \le x\}$. We have

$$\begin{aligned}P(\mathcal{A}) &= P(\mathcal{A} \cap \mathcal{B}) + P(\mathcal{A} \cap \mathcal{B}^c) \\ &= P(\mathcal{A}|\mathcal{B})P(\mathcal{B}) + P(\mathcal{A}|\mathcal{B}^c)P(\mathcal{B}^c) \\ &= P(\mathcal{A}|X(\mu) \le x) F_X(x) + P(\mathcal{A}|X(\mu) > x)(1 - F_X(x)).\end{aligned}$$

6. (0.5%) Consider $k$ identical boxes in which each box contains $n$ balls numbered 1 through $n$. One ball is drawn from each box at random. What is the probability that $m$, with $m \in \{1, 2, ..., n\}$, is the smallest number drawn?

Sol: Denote the numbers of balls drawn from box 1,box 2, … ,box $k$ as $X_1(\mu), X_2(\mu), \ldots, X_k(\mu)$. Note that $X_1(\mu), X_2(\mu), \ldots, X_k(\mu)$ are independent and identically distributed random variables. The probability can be expressed as

$$\begin{aligned}& \Pr\{\min\{X_1(\mu), X_2(\mu), \ldots, X_k(\mu)\} = m\} \\ =\ & \Pr\{\min\{X_1(\mu), X_2(\mu), \ldots, X_k(\mu)\} \ge m\} \\ & - \Pr\{\min\{X_1(\mu), X_2(\mu), \ldots, X_k(\mu)\} \ge m+1\} \\ =\ & \prod_{i=1}^{k} \Pr\{X_i(\mu) \ge m\} - \prod_{i=1}^{k} \Pr\{X_i(\mu) \ge m+1\} \\ & \text{(because } X_1(\mu), X_2(\mu), \ldots, X_k(\mu) \text{ are independent.)} \\ =\ & [\Pr\{X_i(\mu) \ge m\}]^k - [\Pr\{X_i(\mu) \ge m+1\}]^k \\ & \text{(because } X_1(\mu), X_2(\mu), \ldots, X_k(\mu) \text{ are identically distributed.)} \\ =\ & \left(\frac{n-m+1}{n}\right)^k - \left(\frac{n-m}{n}\right)^k.\end{aligned}$$

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7. (0.5%) Let $X(\mu)$ and $Y(\mu)$ be independent and identically distributed Gaussian random variables with zero mean zero and unit variance. Find the joint statistic of $\beta X(\mu)+\alpha Y(\mu)$ and $\alpha X(\mu)-\beta Y(\mu)$ where $\alpha$ and $\beta$ are real-valued with $\alpha^2+\beta^2=1$.

Sol: Because $\beta X(\mu)+\alpha Y(\mu)$ and $\alpha X(\mu)-\beta Y(\mu)$ are linear combinations of $X(\mu)$ and $Y(\mu)$, they are jointly Gaussian. Obviously,. Since $X(\mu)$ and $Y(\mu)$ are independent with zero mean and unit variance, both $\beta X(\mu)+\alpha Y(\mu)$ and $\alpha X(\mu)-\beta Y(\mu)$ with $\alpha^2+\beta^2=1$ have zero mean and unit variance. Also,

$$E\{(\beta X(\mu)+\alpha Y(\mu))(\alpha X(\mu)-\beta Y(\mu))\}=\alpha\beta E\{X^2(\mu)\}-\alpha\beta E\{Y^2(\mu)\}=0.$$

Thus, $\beta X(\mu)+\alpha Y(\mu)$ and $\alpha X(\mu)-\beta Y(\mu)$ are uncorrelated and, thus, independent. Thus, $\beta X(\mu)+\alpha Y(\mu)$ and $\alpha X(\mu)-\beta Y(\mu)$ are independent Gaussian random variables with zero mean and unit variance.

8. (0.5%) $X(\mu)$ and $Y(\mu)$ are independent and uniform in the interval $(0,1)$. Find the probability density function of $Z(\mu)=X(\mu)+Y(\mu)$.

Sol: First, $f_Z(z)=0$ for $z\le 0$ or $z\ge 2$. Second,

$$\begin{aligned}F_Z(z) &= \Pr\{X(\mu)+Y(\mu)\le z\}\\ &= \int_0^z\int_0^{z-x} f_Y(y)dyf_X(x)dx\\ &= \int_0^z (z-x)f_X(x)dx\\ &= \int_0^z xdx\\ &= \frac{1}{2}z^2\end{aligned}$$

for $0<z\le 1$, and

$$\begin{aligned}F_Z(z) &= \Pr\{X(\mu)+Y(\mu)\le z\}\\ &= \int_0^1\int_0^{\min\{z-x,1\}} f_Y(y)dyf_X(x)dx\\ &= \int_0^1 \min\{z-x,1\}f_X(x)dx\\ &= \int_0^1 \min\{z-x,1\}dx\\ &= \int_{z-1}^1 (z-x)dx+\int_0^{z-1}dx\\ &= z(2-z)-\frac{1}{2}(1-(z-1)^2)+(z-1)\\ &= \frac{1}{2}z^2-(z-1)^2\end{aligned}$$

for $1<z<2$.

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9. (0.5%) Let $X_n(\mu)$'s, $n = 1, 2, ..., N$, be independent and identically distributed (iid) Gaussian random variables with zero mean, i.e., $E\{X_n(\mu)\} = 0$, and unit variance, i.e., $Var\{X_n(\mu)\} = 1$. Also, let $Y_n(\mu)$'s, $n = 1, 2, ..., N$, be iid binary-valued random variables with the common probability density function $f_Y(y) = \frac{1}{2}$ if $y = +1$ or $y = -1$, and $f_Y(y) = 0$ otherwise. In addition, $X_n(\mu)$'s, $Y_n(\mu)$'s, $n = 1, 2, ..., N$, are mutually independent. Now, define a new random variable $Z(\mu) = \sum_{n=1}^{N} X_n(\mu)Y_n(\mu)$. Find $E\{Z^{2k+1}(\mu)\}\}$ for any nonnegative integer $k$.

Sol: The moment generating function of $Z(\mu)$ is derived as

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\begin{aligned}
\Phi_Z(s) &= E\{\exp\{s\sum_{n=1}^{N} X_n(\mu)Y_n(\mu)\}\} \\
&= \prod_{n=1}^{N} E\{\exp\{sX_n(\mu)Y_n(\mu)\}\} \\
&\quad (X_n(\mu)Y_n(\mu)\text{'s are independent.}) \\
&= \prod_{n=1}^{N} E\{E\{\exp\{sX_n(\mu)Y_n(\mu)\}|Y_n(\mu)\}\} \\
&= \prod_{n=1}^{N} E\{\exp\{\frac{s^2}{2}Y_n^2(\mu)\}\} \\
&\quad (X_n(\mu)\text{ is Gaussian with mean zero and unit variance.}) \\
&= \exp\{\frac{Ns^2}{2}\} \\
&\quad (Y_n(\mu)\text{ is } \pm 1\text{-valued.})
\end{aligned}
$$

This shows that $Z(\mu)$ is Gaussian with mean zero and variance $N$. Thus, because the density of $Z(\mu)$ is an even function, its odd moments, i.e., $E\{Z^{2k+1}(\mu)\}\}$ for any nonnegative integer $k$, are all zero.

10. (0.5%) Consider the real-valued random sequence $X_1(\mu), X_2(\mu), ...$ where $X_n(\mu)$'s are independent and identically distributed positive-valued random variables with common probability density function $f_X(x)$ which is nonzero for nonnegative argument $x$. It is required that $\eta = \int_0^\infty \ln\{x\} f_X(x)dx$ and $\sigma^2 = \int_0^\infty (\ln\{x\} - \eta)^2 f_X(x)dx$ are both finite-valued. Now, define a new real-valued random sequence $Y_1(\mu), Y_2(\mu), ...$ where $Y_N(\mu)$'s are defined by

$$Y_N(\mu) = (\prod_{n=1}^{N} X_n(\mu))^{\frac{1}{N}}.$$

Show that the random sequence $Y_1(\mu), Y_2(\mu), ...$ converges to a fixed value $\alpha$ in the distribution sense. Find $\alpha$.

Sol: Now, define another new random sequence $Z_1(\mu), Z_2(\mu), ...$ where $Z_N(\mu)$'s are defined by

$$Z_N(\mu) = \ln\{Y_N(\mu)\} = \frac{1}{N}\sum_{n=1}^{N}\ln\{X_n(\mu)\}.$$

Now, $Z_N(\mu)$ has mean

$$\begin{aligned}
E\{Z_N(\mu)\} &= E\{\ln\{X_n(\mu)\}\} \\
&= \int_0^\infty \ln\{x\} f_X(x)dx \\
&= \eta
\end{aligned}$$

and variance

$$\begin{aligned}
\mathrm{Var}\{Z_N(\mu)\} &= \frac{1}{N}\mathrm{Var}\{\ln\{X_n(\mu)\}\} \\
&= \frac{1}{N}\int_0^\infty (\ln\{x\} - \eta)^2 f_X(x)dx \\
&= \frac{1}{N}\sigma^2.
\end{aligned}$$

Because $\sigma^2$ is finite valued, $\lim_{N\to\infty}\mathrm{Var}\{Z_N(\mu)\} = \lim_{N\to\infty} E\{(Z_N(\mu) - \eta)^2\} = 0$ and thus the random sequence $Z_1(\mu), Z_2(\mu), ...$ converges to $\eta$ in the mean square sense. Because convergence in the mean square sense implies convergence in the distribution sense, the random sequence $Z_1(\mu), Z_2(\mu), ...$ converges to a fixed value $\eta$ in the distribution sense. Since $Y_N(\mu) = \exp\{Z_N(\mu)\}$, the random sequence $Y_1(\mu), Y_2(\mu), ...$ converges to a fixed value $\exp\{\eta\}$ in the distribution sense. Thus, $\alpha = \exp\{\eta\}$.