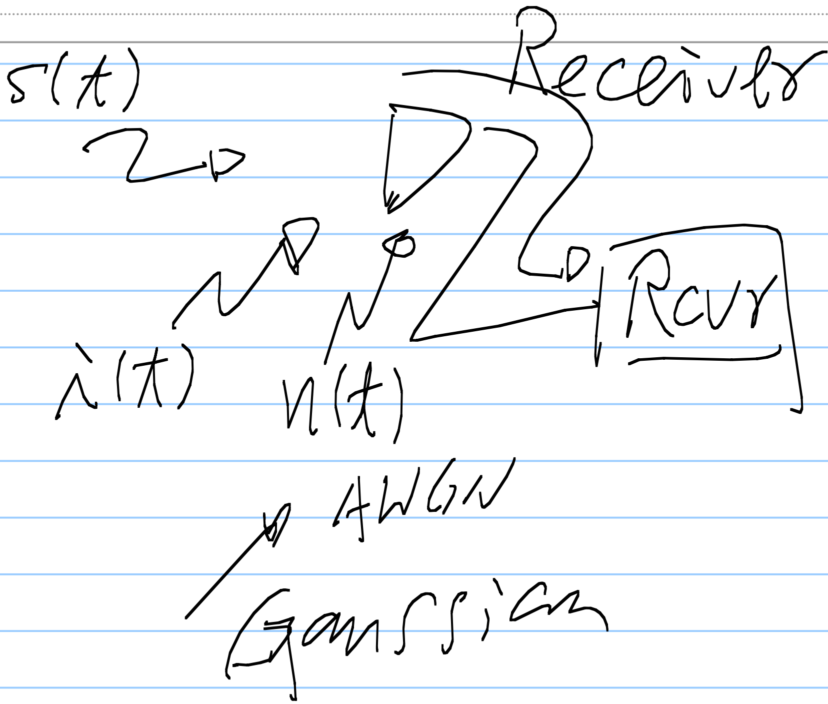
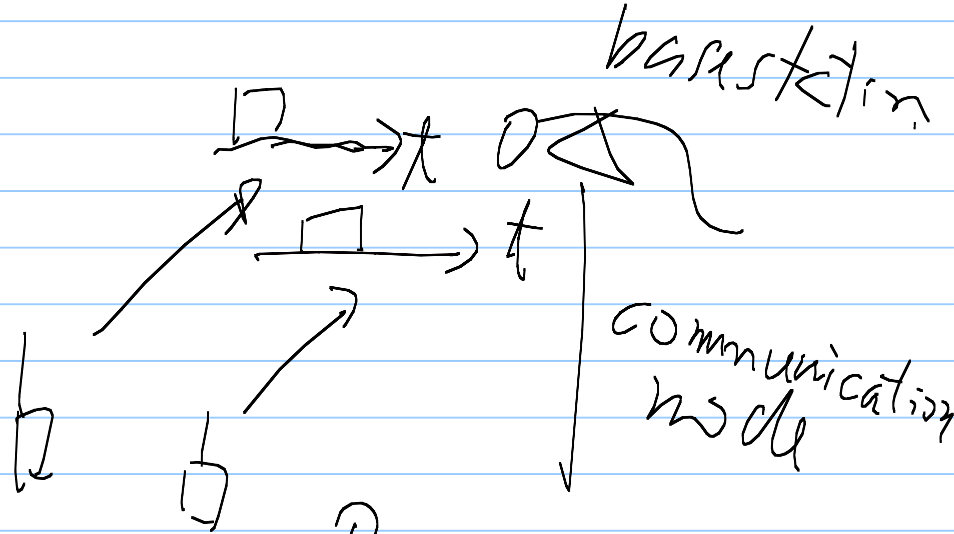
Note 1: Random Process

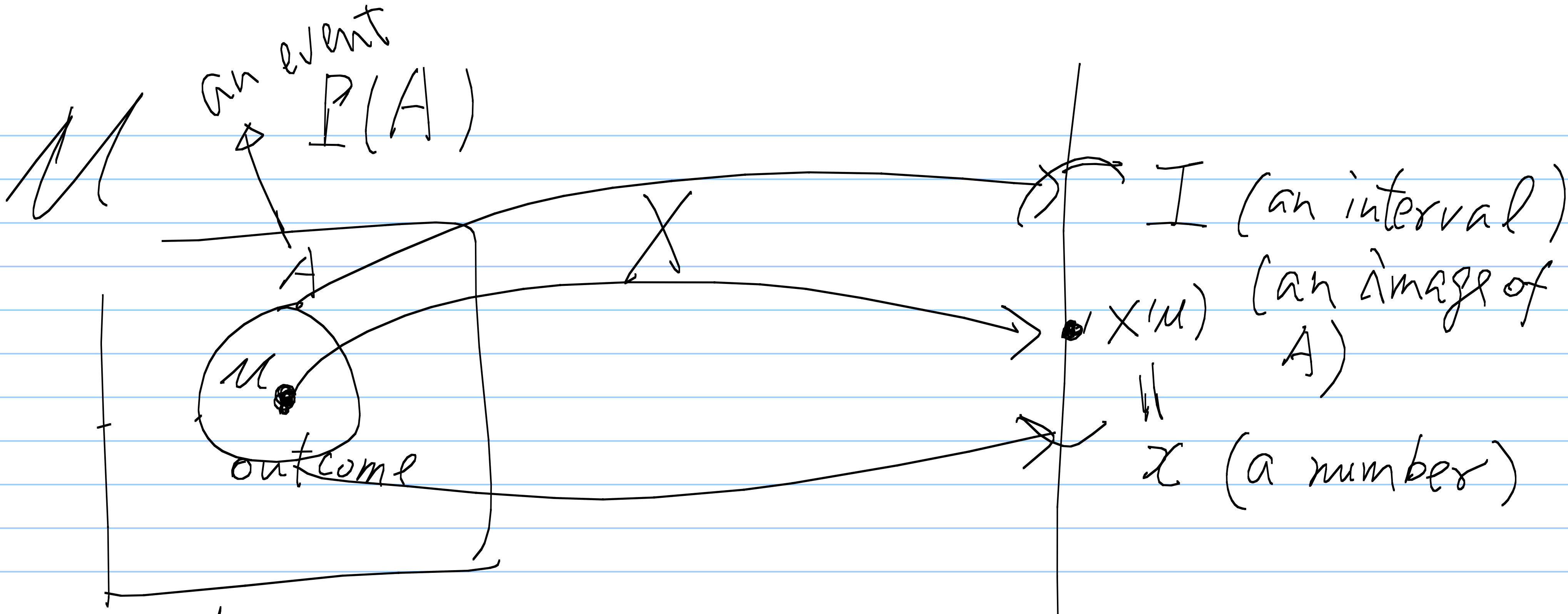
stochastic Process





Poisson

Markov

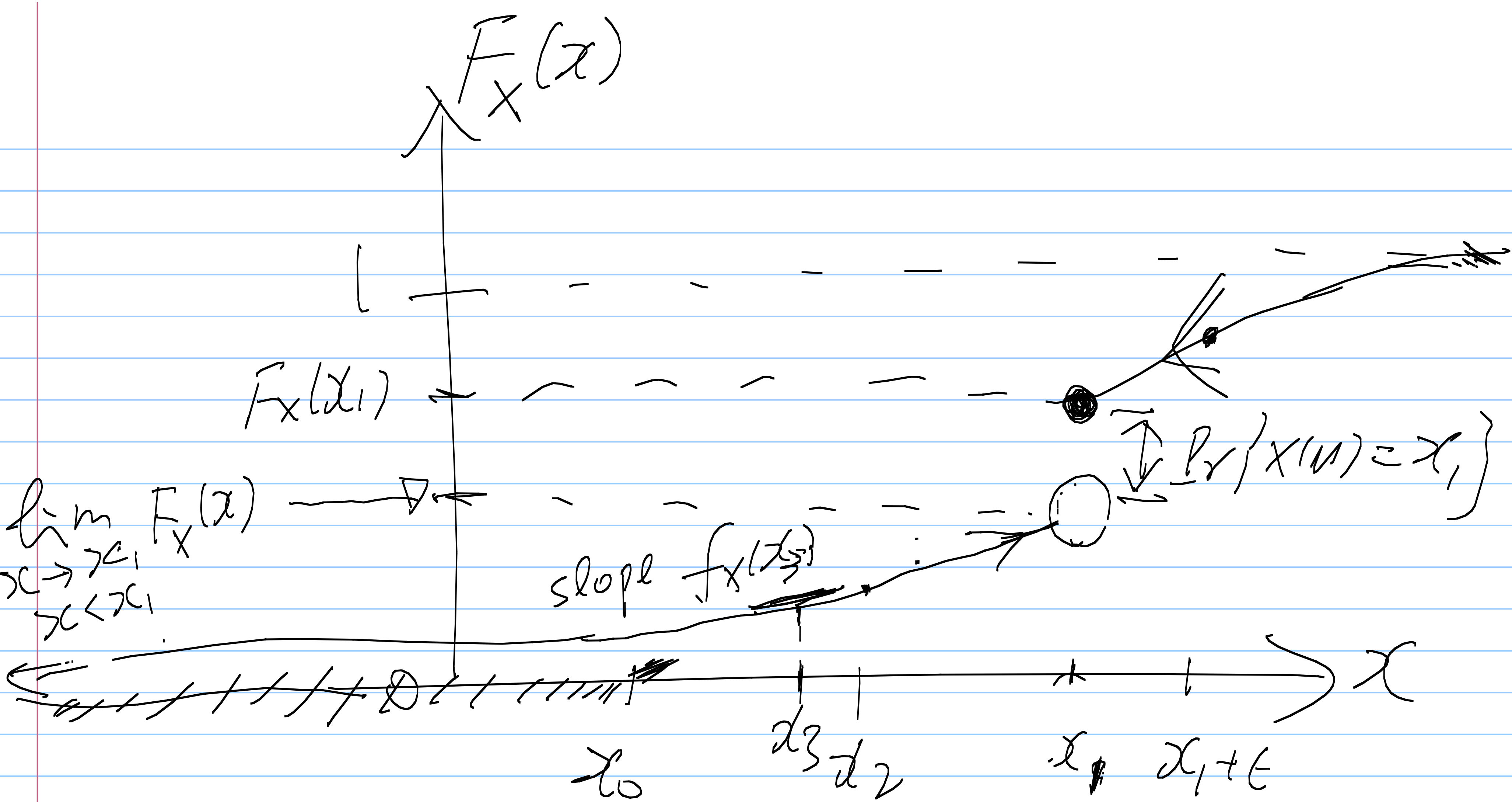


domain

range

$$\{X(u) \in I\} = A \quad \sigma\text{-event}$$

$$P(A) = P_r\{X(u) \in I\}$$



$F_X(x)$
1
$F_X(x_1)$
$\lim_{x \to x_1, x < x_1} F_X(x)$
$\Pr\{X(u) = x_1\}$
slope $f_X(x_3)$
x
0
x_0
x_3
x_2
x_1
$x_1 + \epsilon$

$\Pr\{X(u) = x_2\} = 0$

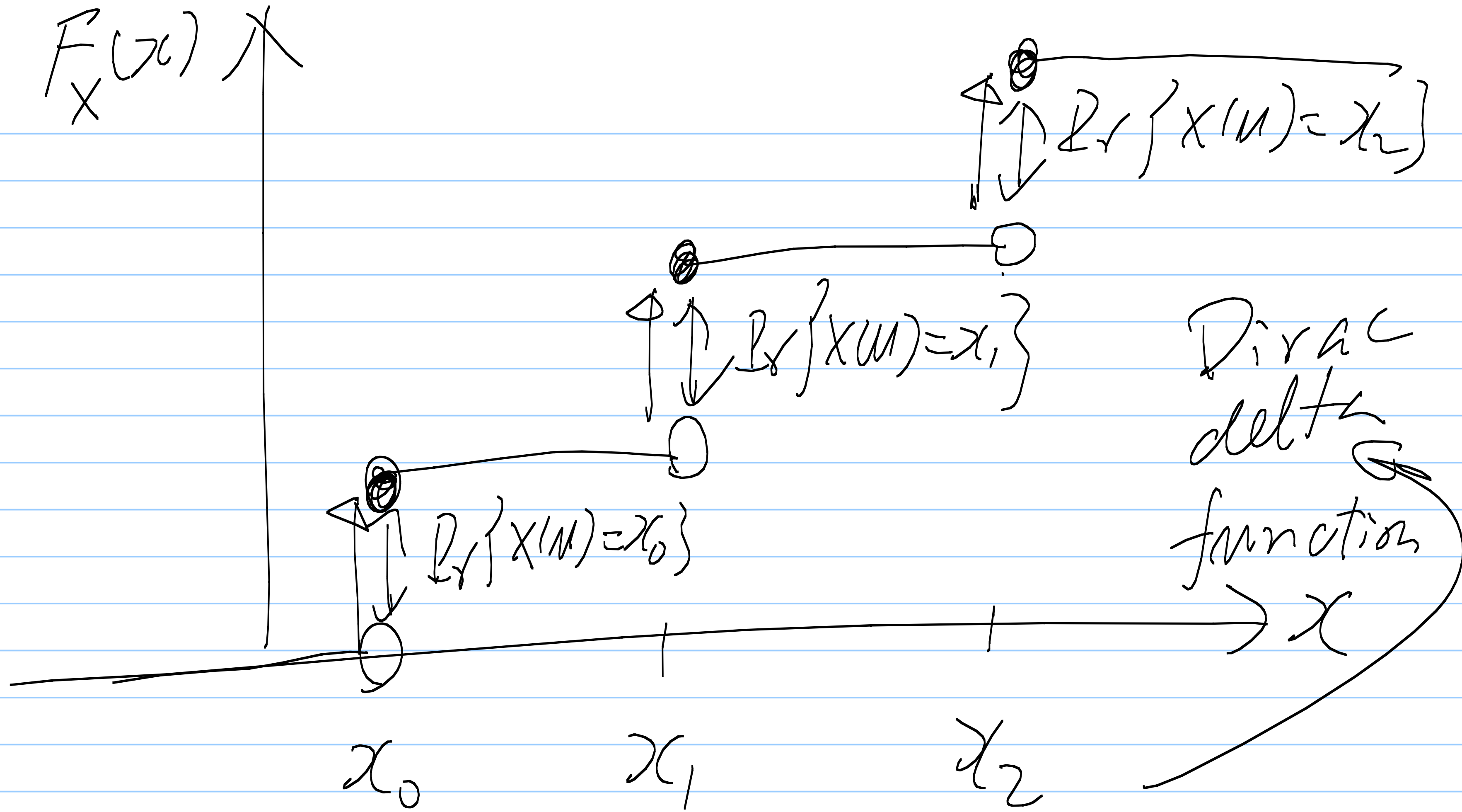
$$\underbrace{\{X(u) = x\}}_{A} = \{X(u) \le x\} - \underbrace{\{X(u) < x\}}_{B}$$

$$A \cap B = \Phi \quad , \quad A \cup B = \{X(u) \le x\}$$

Axiom 3 : $\underbrace{P_r\{X(u) \le x\}}_{F_X(x)} = P_r\{X(u) = x\}$
$$+ \underbrace{P_r\{X(u) < x\}}_{/}$$

$$\lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} F_X(x-\epsilon) = F_X(x^-)$$

$$\Rightarrow \Pr\{X(u) = x\} = F_X(x) - \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} F_X(x-\epsilon)$$



$$\frac{dF_X(x)}{dx} = \sum_{i=0}^{2} P_r\{X(u)=x_i\}\,\delta(x-x_i)$$

$$F_X(x) = \sum_{i=0}^{2} P_r\{X(u) = x_i\} \underbrace{u(x - x_i)}_{\text{unit step function}}$$

$$\frac{du(x)}{dx} = \delta(x)$$

$$\begin{cases} 1 & x \geq x_i \\ 0 & \text{otherwise} \end{cases}$$

Bayes' rule: For two events A and B

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)\,P(A)}{P(B)}$$

$\{X(u) \le x\}$

$P_r\{X(u) \le x\} = F_X(x)$

$P_r\{X(u) \le x \,|\, A\} = F_X(x|A)$

$$P(A \mid \underbrace{\{X(w)=x\}}_{B}) = \frac{P(B \mid A)\,P(A)}{P(B)} = \frac{\epsilon f_X(x \mid A)\,P(A)}{\epsilon f_X(x)}$$

$$\{X(w)=x\} = \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} \{x < X(w) \le x+\epsilon\}$$

$$P(B) = F_X(x+\epsilon) - F_X(x) \xrightarrow{\epsilon \to 0} f_X(x)\,\epsilon$$

$$P(B \mid A) = \frac{F_X(x+\epsilon \mid A) - F_X(x \mid A)}{\epsilon} \cdot \epsilon \xrightarrow{\epsilon \to 0} f_X(\epsilon \mid A)\,\epsilon$$

Ex: For a Gaussian random variable $X(u)$ with density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\eta)^2}{2\sigma^2}}, \quad \forall x$$

Then,

$$E\{X(u)\} = \eta = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$Var\{X(u)\} = \sigma^2 = \int_{-\infty}^{\infty} (x-\eta)^2 f_X(x)\,dx,$$

We say that an rv $X(u)$ can be completely statistically described by either one of the following: its density $f_X(x)$,

its distribution $F_X(x)$,

all its moments $m_1, m_2, \ldots$;

its characteristic $\Phi_X(\omega)$, or

function

its moment generating function $\Phi_X(s)$.

$$E\{X(n)^{2k+1}\} = \int_{-\infty}^{\infty} \underbrace{x^{2k+1}}_{\substack{g(-x)=-g(x) \\ \text{odd function}}} \; \underbrace{e^{-\frac{x^2}{2\sigma_x^2}} \frac{1}{\sqrt{2\pi\sigma_x^2}}}_{\substack{f(x)=f(-x) \\ \text{even function}}} dx$$

$$\underbrace{\text{odd function} \quad \text{even function}}_{\text{odd function}}$$

$h(x) = -h(-x)$

For any function $g(x)$,

$$E\{g(X(u))\} = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

provided with a given density $f_X(x)$.

The Fourier transform $F(\omega)$ exists if its inverse $f(t)$ is absolutely integrable, i.e., $\mathcal{F}^{-1}\{F(\omega)\}$

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty \qquad \underbrace{}_{<\infty}$$

$$\left(\text{pf: } |F(\omega)| = \left| \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt \right| \le \int_{-\infty}^{\infty} |f(t)|\, dt. \right)$$

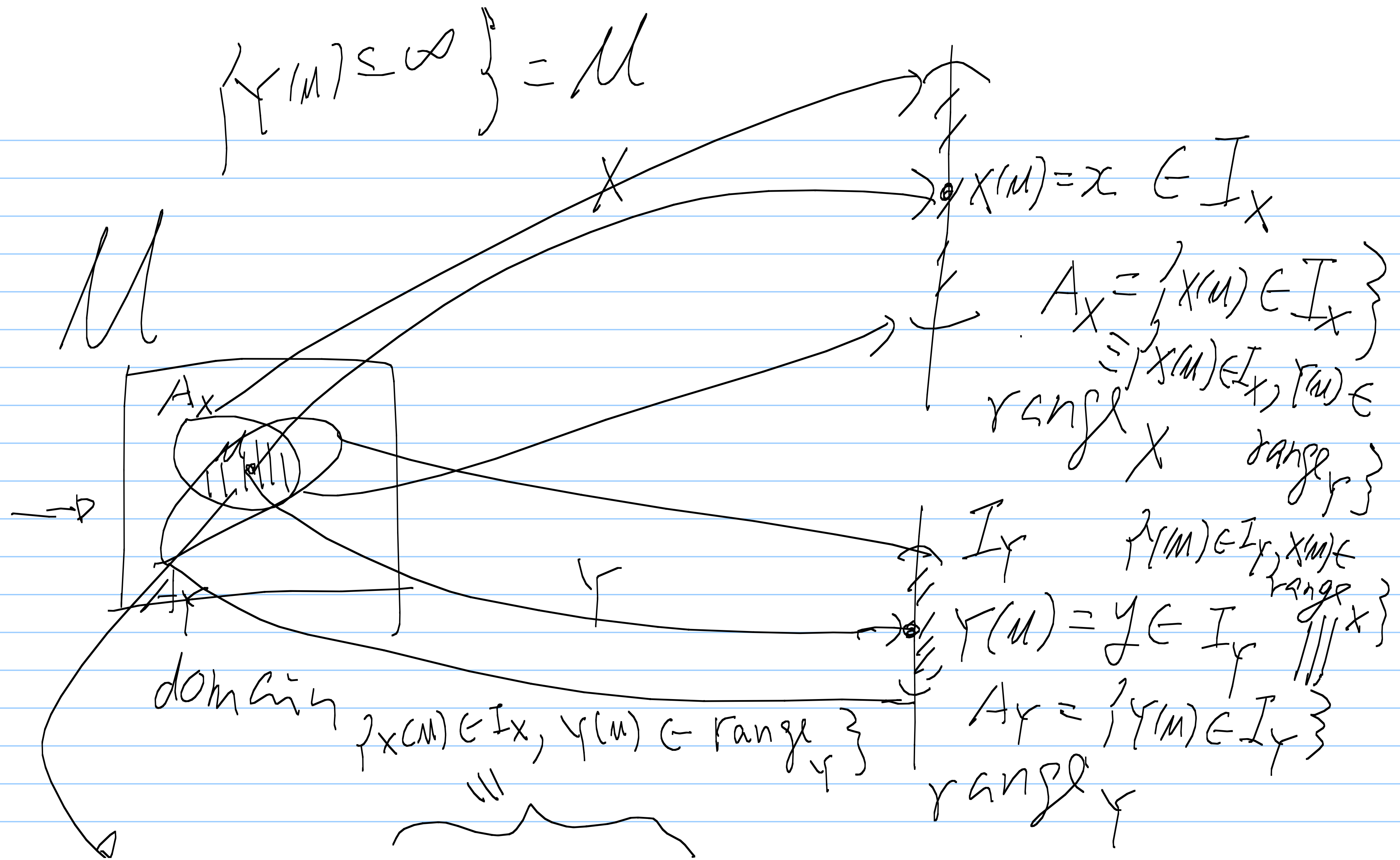
For any random variable $X(u)$,

$$\Phi_X(s) = E\left\{e^{sX(u)}\right\} = \sum_{n=0}^{\infty} \frac{s^n}{n!} \underbrace{E\left\{X(u)^n\right\}}_{m_n}$$

$$\Rightarrow \frac{d^n \overbrace{\Phi_X}^{=E\{e^{sX(u)}\}}(s)}{ds^n} = E\left\{\frac{d^n}{ds^n} e^{sX(u)}\right\} = E\left\{X(u)^n e^{sX(u)}\right\}$$

Both Expectation and differentiation operators are linear.

$$\Rightarrow \frac{d^n \Phi_X(s)}{ds^n}\bigg|_{s=0} = E\{X(w)^n\} = m_n$$



{Y(u) ≤ ∞} = U
X
X(u) = x ∈ I_X
A_X = {X(u) ∈ I_X}
= {X(u) ∈ I_X, Y(u) ∈ range_Y}
range_X
U
A_X
u
A_Y
domain
Y
I_Y
{Y(u) ∈ I_Y, X(u) ∈ range_X}
Y(u) = y ∈ I_Y
A_Y = {Y(u) ∈ I_Y}
range_Y
{X(u) ∈ I_X, Y(u) ∈ range_Y}

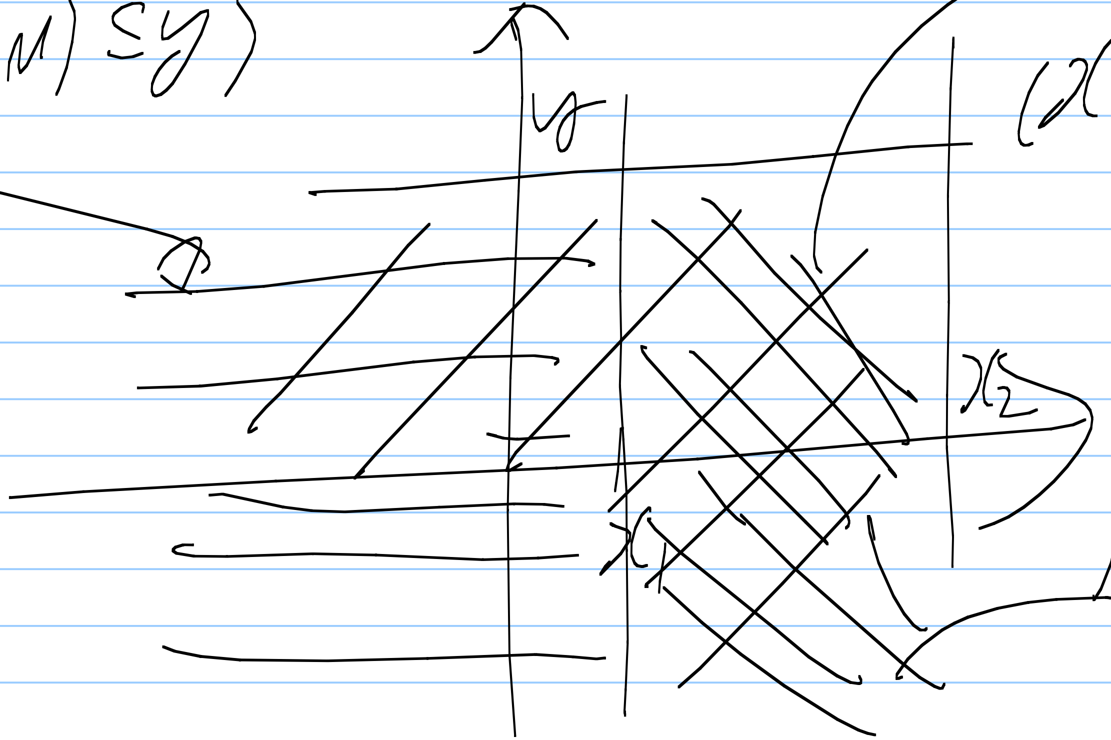
$$\underline{A_X \cap A_Y} = \{X(u) \in I_X\} \cap \{Y(u) \in I_Y\}$$

$$= \{\underbrace{X(u) \in I_X}_{X(u) \le x},\ \underbrace{Y(u) \in I_Y}_{Y(u) \le y}\}$$

$\{X(u) \le x_2,\ Y(u) \le y\}$

(x_2, y)

$\{X(u) \le x_1,\ Y(u) \le y\}$



$\{x_1 < X(u) \le x_2,\ Y(u) \le y\}$

As $dx \to 0$,

$$f_X(x|A)\,\cancel{dx} = \underbrace{\Pr\{x < X(u) \le x + dx \,|\, A\}}$$

$$= \frac{\Pr\{x < X(u) \le x + dx, A\}}{P(A)}$$

$$= \frac{P(A \,|\, x < X(u) \le x + dx)\, \underbrace{\Pr\{x < X(u) \le x + dx\}}}{P(A)}$$

$$= \frac{P(A \,|\, x < X(u) \le x + dx)\, f_X(x)\,\cancel{dx}}{P(A)}$$

$$f_X(x \mid A) = \frac{P(A \mid X(\omega) = x)\, f_X(x)}{P(A)} \quad \#$$