(1) (6%) Cars arrive at a bridge entrance according to a Poisson process of rate $\lambda= 15$ cars per minute.

(a) (2%) Find the probability that in a given 4 minute period there are 3 car arrivals during the first minute and 2 car arrivals in the last minute.

Sol: Let $N(\mu, t)$ be the Poisson process. Now,

$$\begin{aligned}
& \Pr\{N(\mu,1)=3, N(\mu,4)-N(\mu,3)=2\} \\
= \ & \Pr\{N(\mu,1)=3\}\Pr\{N(\mu,4)-N(\mu,3)=2\} \\
& \text{(because of independent increments)} \\
= \ & \Pr\{N(\mu,1)=3\}\Pr\{N(\mu,1)=2\} \\
& \text{(because of stationary increments)} \\
= \ & \left(e^{-15}\frac{15^3}{3!}\right)\left(e^{-15}\frac{15^2}{2!}\right) \\
= \ & e^{-30}\frac{15^5}{12}.
\end{aligned}$$

(b) (4%) Find the mean and variance of the time of the tenth car arrival, given that the time of the fifth car arrival is T minutes.

Sol: Let $S_n(\mu)$ be the nth arrival time and $T_n(\mu)$ be the nth interarrival time. Thus, $S_n(\mu) = \sum_{k=1}^{n} T_k(\mu)$. Now,

$$E\{S_{10}(\mu)|S_5(\mu)=T\} = E\{T + \sum_{k=6}^{10} T_k(\mu)\} = T + \frac{5}{\lambda} = T + \frac{1}{3}$$

$$Var\{S_{10}(\mu)|S_5(\mu)=T\} = Var\{T + \sum_{k=6}^{10} T_k(\mu)\} = \frac{5}{\lambda^2} = \frac{1}{45}.$$

(2) (4%) Consider a linear and time-invariant system with impulse response $h(t)$, input process $X(\mu, t)$, and output process $Y(\mu, t)$. Show that if $h(t) = 0$ outside the time interval $(0, T)$ and $X(\mu, t)$ is a white noise with mean zero, then $R_Y(t_1, t_2) = 0$ for $|t_1 - t_2| > T$.

Sol: Since the considered system is linear and time-invariant, $Y(\mu, t)$ is given by $Y(\mu, t) = \int_0^T X(\mu, t-\tau)h(\tau)d\tau$. Therefore, when $|t_1 - t_2| > T$, $R_Y(t_1, t_2)$ is derived as

$$\begin{aligned}
R_Y(t_1,t_2) &= E\{Y(\mu,t_1)Y(\mu,t_2)\} \\
&= E\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\mu,t_1-\tau_1)h(\tau_1)X(\mu,t_2-\tau_2)h(\tau_2)d\tau_2 d\tau_1\}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha\delta(t_1-\tau_1-t_2+\tau_2)h(\tau_1)h(\tau_2)d\tau_2 d\tau_1 \\
&= \alpha\int_{-\infty}^{\infty} h(\tau_1)h(\tau_1+t_2-t_1)d\tau_1 \\
&= \alpha\int_{0}^{T} h(\tau_1)h(\tau_1+t_2-t_1)d\tau_1 \\
&= 0
\end{aligned}
$$

where the last equality stems from the fact that $|t_1-t_2|>T$ and thus $h(\tau_1)$ and $h(\tau_1+t_2-t_1)$ can not be nonzero at the same time.

(3) (4%) Consider the wide-sense stationary real-valued random process $X(\mu,t)$ which is defined in the time interval $(-1,1)$, and has mean $E\{X(\mu,t)\}=m$ and autocorrelation $R_X(\tau)=m^2+\cos(\pi\tau)$, where m is a real constant. Find the Karhunen-Loéve expansion of $X(\mu,t)$ in the interval $(-1,1)$.

Sol: Let $C_X(\tau)$ be the autocovariance of $X(\mu,t)$. Then, $C_X(\tau)=R_X(\tau)-m^2=\cos(\pi\tau)$. Also, let $\phi_n(t)$ and λ_n be the nth eigenfunction and eigenvalue, respectively, of $C_X(\tau)$ for $n=0,1,...$ Then, $\{\phi_n(t)\}$ and $\{\lambda_n\}$ are the solution to

$$
\begin{aligned}
& \int_{-1}^{1} C_X(t-\tau)\phi_n(\tau)d\tau = \lambda_n\phi_n(t) \\
\Rightarrow\ & \int_{-1}^{1} \cos(\pi(t-\tau))\phi_n(\tau)d\tau = \lambda_n\phi_n(t) \\
\Rightarrow\ & \lambda_n\phi_n(t) = \cos(\pi t)\int_{-1}^{1}\cos(\pi\tau)\phi_n(\tau)d\tau \\
& +\sin(\pi t)\int_{-1}^{1}\sin(\pi\tau)\phi_n(\tau)d\tau \qquad (1)
\end{aligned}
$$

for $-1<t<1$. From (1), it is straightforward to observe that $X(\mu,t)-m$ in the interval $(-1,1)$ can be expanded by

$$
\begin{cases}
\lambda_0 = 0 \text{ and } \phi_0(t)=1/\sqrt{2}, & -1<t<1 \\
\lambda_1 = 1 \text{ and } \phi_1(t)=\cos(\pi t), & -1<t<1 \\
\lambda_2 = 1 \text{ and } \phi_2(t)=\sin(\pi t), & -1<t<1
\end{cases}
$$

and

$$
\begin{cases}
\lambda_{2k+1} = 0 \text{ and } \phi_{2k+1}(t)=\cos((k+1)\pi t), & -1<t<1 \\
\lambda_{2k+2} = 0 \text{ and } \phi_{2k+2}(t)=\sin((k+1)\pi t), & -1<t<1
\end{cases}
$$

for a positive integer k. Therefore, the Karhunen-Loéve expansion of $X(\mu,t)-m$ in the interval $(-1,1)$ is given by

$$
X(\mu,t)-m = a(\mu)\phi_1(t)+b(\mu)\phi_2(t)
$$

where $a(\mu) \triangleq \int_{-1}^{1} X(\mu,t)\phi_1(t)dt$ and $b(\mu) \triangleq \int_{-1}^{1} X(\mu,t)\phi_2(t)dt$ with $E\{a(\mu)\} = E\{b(\mu)\} = 0$, $E\{a^2(\mu)\} = 1$, $E\{b^2(\mu)\} = 1$, and $Cov\{a(\mu), b(\mu)\} = 0$. Thus, the Karhunen-Loéve expansion of $X(\mu,t)$ in the interval $(-1,1)$ is given by

$$X(\mu,t) = \sqrt{2}m\phi_0(t) + a(\mu)\phi_1(t) + b(\mu)\phi_2(t).$$

(4) (6%, 2% each) Let $X(\mu,t) \triangleq \cos(2\pi f_1 t + \theta(\mu))$ and $Y(\mu,t) \triangleq \cos(2\pi f_2 t + \phi(\mu))$ where $f_1 > f_2 > 0$, $\theta(\mu)$ and $\phi(\mu)$ are statistically independent and identically distributed random variables with the identical uniform density over $[0, 2\pi)$. Also, let $\widehat{X}(\mu,t)$ and $\widehat{Y}(\mu,t)$ be the Hilbert transforms of $X(\mu,t)$ and $Y(\mu,t)$, respectively. Answer the following sub-questions.

(a) Derive the power spectral densities of $\widehat{X}(\mu,t)$ and $\widehat{Y}(\mu,t)$.

Sol: Since $\widehat{X}(\mu,t)$ and $\widehat{Y}(\mu,t)$ can be created by passing $X(\mu,t)$ and $Y(\mu,t)$ through a linear and time-invariant system with frequency response $H(f) = -j\,\mathrm{sgn}(f)$, respectively, the power spectral densities of $\widehat{X}(\mu,t)$ and $\widehat{Y}(\mu,t)$ are given by

$$S_{\widehat{X}}(f) = S_X(f)|H(f)|^2 = S_X(f)|-j\,\mathrm{sgn}(f)|^2 = S_X(f) \text{ and } S_{\widehat{Y}}(f) = S_Y(f).$$

Moreover, because the autocorrelation functions $R_X(\tau)$ and $R_Y(\tau)$ are given by

$$\begin{aligned} R_X(\tau) &= E\{\cos(2\pi f_1(t+\tau) + \theta(\mu))\cos(2\pi f_1 t + \theta(\mu))\} \\ &= \frac{1}{2}\cos(2\pi f_1 \tau) \end{aligned}$$

and

$$R_Y(\tau) = \frac{1}{2}\cos(2\pi f_2 \tau),$$

$S_X(f)$ and $S_Y(f)$ are obtained as

$$\begin{aligned} S_X(f) &= \mathcal{F}\{R_X(\tau)\} \\ &= \frac{\delta(f-f_1) + \delta(f+f_1)}{4} \end{aligned}$$

and

$$\begin{aligned} S_Y(f) &= \mathcal{F}\{R_Y(\tau)\} \\ &= \frac{\delta(f-f_2) + \delta(f+f_2)}{4}. \end{aligned}$$

Therefore, we get $S_{\widehat{X}}(f) = \frac{\delta(f-f_1)+\delta(f+f_1)}{4}$ and $S_{\widehat{Y}}(f) = \frac{\delta(f-f_2)+\delta(f+f_2)}{4}$.

(b) Determine whether $\widehat{X}(\mu,t)$ and $Y(\mu,t)$ for a fixed t are orthogonal, i.e., whether $E\{\widehat{X}(\mu,t)Y(\mu,t)\}$ is zero or not.

Sol: Since $\theta(\mu)$ and $\phi(\mu)$ are statistically independent, we have

$$\begin{aligned} E\{\widehat{X}(\mu,t)Y(\mu,t)\} &= E\{\widehat{X}(\mu,t)\}E\{Y(\mu,t)\} \\ &= 0 \end{aligned}$$

where the last equality is because

$$\begin{aligned} E\{Y(\mu,t)\} &= E\{\cos(2\pi f_2 t+\phi(\mu))\} \\ &= \int_0^{2\pi} \cos(2\pi f_2 t+\phi)d\phi \\ &= 0. \end{aligned}$$

Thus, they are orthogonal.

(c) Find the Hilbert transform of the product process $\widehat{X}(\mu,t)\widehat{Y}(\mu,t)$.

Sol: Since $\widehat{X}(\mu,t)$ is highpass and $\widehat{Y}(\mu,t)$ is lowpass, we know from Bedrosian's Theorem that

$$\begin{aligned} H.T.[\widehat{X}(\mu,t)\widehat{Y}(\mu,t)] &= H.T.[\widehat{X}(\mu,t)]\widehat{Y}(\mu,t) \\ &= -X(\mu,t)\widehat{Y}(\mu,t) \\ &= -\cos(2\pi f_1 t+\theta(\mu))\sin(2\pi f_2 t+\phi(\mu)). \end{aligned}$$

(5) (4%) Let $n(\mu,t)$ be a narrowband stationary random process with mean zero and power spectral density $S_n(f)$ given by

$$S_n(f) = \frac{1}{2(2\pi\sigma^2)^{1/2}}\left[\exp\left\{-\frac{(f-f_c)^2}{2\sigma^2}\right\}+\exp\left\{-\frac{(f+f_c)^2}{2\sigma^2}\right\}\right]$$

with $\sigma^2 \ll f_c$. Suppose that we wish to represent $n(\mu,t)$ in the form of the quadrature representation

$$n(\mu,t) = n_c(\mu,t)\cos(2\pi f_c t) - n_s(\mu,t)\sin(2\pi f_c t).$$

Evaluate the autocorrelations of $n_c(\mu,t)$ and $n_s(\mu,t)$.

(Note that you can decompose $S_n(f)$ as $S_n(f) = S_L(f+f_c)+S_L(f-f_c)$ with the low-poss spectrum $S_L(f)$ under the narrowband assumption $\sigma^2 \ll f_c$. Also, the following formula may be useful: $\int_0^\infty \exp\left\{-\frac{f^2}{2\sigma^2}\right\}\cos(2\pi f\tau)df = (\pi\sigma^2/2)^{1/2}\exp\{-2(\pi\sigma\tau)^2\}$.)

Sol: Now, we can express $S_n(f) = S_L(f+f_c)+S_L(f-f_c)$ where

$$S_L(f) = \frac{1}{2(2\pi\sigma^2)^{1/2}}\exp\left\{-\frac{f^2}{2\sigma^2}\right\}.$$

Therefore, both $n_c(\mu,t)$ and $n_s(\mu,t)$ have mean zero and identical autocorrelation

$$\begin{aligned}
R_{n_c}(\tau) &= R_{n_s}(\tau) = 2\int_{-\infty}^{\infty} S_L(f)\exp\{j2\pi f\tau\}df \\
&= 2\int_{-\infty}^{\infty} S_L(f)\cos(2\pi f\tau)df \\
&= 4\int_{0}^{\infty} S_L(f)\cos(2\pi f\tau)df \\
&= \frac{2}{(2\pi\sigma^2)^{1/2}}\int_{0}^{\infty} \exp\left\{-\frac{f^2}{2\sigma^2}\right\}\cos(2\pi f\tau)df \\
&= \exp\left(-2(\pi\sigma\tau)^2\right)
\end{aligned}$$

where we have used the formula

$$\int_{0}^{\infty} \exp\left\{-\frac{f^2}{2\sigma^2}\right\}\cos(2\pi f\tau)df = (\pi\sigma^2/2)^{1/2}\exp\{-2(\pi\sigma\tau)^2\}.$$

(6) (6%) Consider a linear and time-invariant system with input $X(\mu,t)$ and output $Y(\mu,t)$, which are related by

$$Y(\mu,t) = X(\mu,t) + X(\mu,t-1).$$

Also, let $X(\mu,t)$ be a real-valued Gaussian random process with mean function $\eta_X(t)$ and autocorrelation $R_X(t_1,t_2)$.

(a) (4%) Express the mean function $\eta_Y(t)$ and the autocorrelation $R_Y(t_1,t_2)$ of $Y(\mu,t)$ in terms of $\eta_X(t)$ and $R_X(t_1,t_2)$.

Sol: The mean function of $Y(\mu,t)$ is given by $\eta_Y(t) = \eta_X(t) + \eta_X(t-1)$. The autocorrelation $R_Y(t_1,t_2)$ of $Y(\mu,t)$ is given by $R_Y(t_1,t_2) = E\{Y(\mu,t_1)Y(\mu,t_2)\}$ $= R_X(t_1,t_2) + R_X(t_1-1,t_2) + R_X(t_1,t_2-1) + R_X(t_1-1,t_2-1)$.

(b) (2%) Can $Y(\mu,t)$ be a stationary Gaussian random process? Prove your answer.

Sol: Because the system is linear and $X(\mu,t)$ is Gaussian, $Y(\mu,t)$ is Gaussian. Now, if we further let $X(\mu,t)$ be stationary, then $Y(\mu,t)$ is also stationary. This is because from (a) $\eta_Y(t)$ is a constant and $R_Y(t_1,t_2)$ depends only on t_1-t_2 if $X(\mu,t)$ is stationary.

(7) (6%, 2% each) Let $n_+(\mu,t)$, $\widehat{n}(\mu,t)$ and $\widetilde{n}(\mu,t)$ be the pre-envelope, the Hilbert transform, and the complex envelope of wide-sense stationary real-valued random process $n(\mu,t)$, respectively, related by

$$n_+(\mu,t) = n(\mu,t) + j\widehat{n}(\mu,t) = \widetilde{n}(\mu,t)\exp\{j2\pi f_c t\}.$$

Determine whether each of the following statements is true or false. Prove the statement if it is true and explain the reason if it is false.

(a) *Statement A:* $n(\mu,t)$ and $\widehat{n}(\mu,t)$ are jointly wide-sense stationary real-valued random processes.

(b) *Statement B:* $n_+(\mu,t)$ is a wide-sense stationary complex-valued random process.

(c) *Statement C:* $\widetilde{n}(\mu,t)$ is a wide-sense stationary complex-valued random process.

Sol: *Statement A* and *Statement B* are true, but *Statement C* is false. The arguments are given below. Note first that

$$\widehat{n}(\mu,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{n(\mu,x)}{t-x}dx = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{n(\mu,t-x)}{x}dx.$$

(a) We first prove that $n(\mu,t)$ and $\widehat{n}(\mu,t)$ are jointly wide-sense stationary (WSS) real-valued random processes when $n(\mu,t)$ is WSS. Now, if $E\{n(\mu,t)\} = C$, then

$$E\{\widehat{n}(\mu,t)\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{C}{x}dx = \text{a constant}$$

Also,

$$\begin{aligned}
R_{\widehat{n}}(t_1,t_2) &= E\{\widehat{n}(\mu,t_1)\widehat{n}(\mu,t_2)\} \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{E\{n(\mu,t_1-x)n(\mu,t_2-y)\}}{xy}dxdy \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{R_n(t_1-x,t_2-y)}{xy}dxdy \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{R_n(t_1-t_2-x+y)}{xy}dxdy \\
&= \text{a function of } t_1-t_2.
\end{aligned}$$

Thus, $\widehat{n}(\mu,t)$ is WSS. Further, because

$$\begin{aligned}
E\{n(\mu,t_1)\widehat{n}(\mu,t_2)\} &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{E\{n(\mu,t_1)n(\mu,t_2-x)\}}{x}dx \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R_n(t_1,t_2-x)}{x}dx \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R_n(t_1-t_2+x)}{x}dx \\
&= \text{a function of } t_1-t_2.
\end{aligned}$$

and so is $E\{\widehat{n}(\mu,t_1)n(\mu,t_2)\}$, $n(\mu,t)$ and $\widehat{n}(\mu,t)$ are jointly WSS.

(b) Because the real part $n(\mu,t)$ and the imaginary part $\widehat{n}(\mu,t)$ of a complex-valued random process $n_+(\mu,t) = n(\mu,t) + j\widehat{n}(\mu,t)$ are jointly WSS, $n_+(\mu,t)$ is WSS.

(c) Note that $\widetilde{n}(\mu,t) = n_+(\mu,t)\exp\{-j2\pi f_c t\}$. Now, $E\{\widetilde{n}(\mu,t)\} = E\{n_+(\mu,t)\}\exp\{-j2\pi f_c t\}$ depends on time t if $E\{n_+(\mu,t)\}$ is nonzero. Thus, $\widetilde{n}(\mu,t)$ may not be WSS.