(1) (10%) Consider the deterministic system $\mathbf{T}_1$ with real-valued input process $X(\mu, t)$ and real-valued output process $Y(\mu, t)$ being related by

$$Y(\mu, t) = \mathbf{T}_1[X(\mu, t)] = \sum_{k=0}^{K} h^k X(\mu, t-k)$$

where h is a deterministic real-valued factor with $0 < h < 1$. Also, feed $Y(\mu, t)$ into the other deterministic system $\mathbf{T}_2$ with real-valued output process $Z(\mu, t)$ which is related to $Y(\mu, t)$ by

$$Z(\mu, t) = \mathbf{T}_2[Y(\mu, t)] = Y(\mu, -t).$$

Answer the following questions:

(a) (2%) Prove that the system $\mathbf{T}_1$ is linear and time-invariant. Also, find the impulse response of the system in terms of the Dirac delta function $\delta(t)$.

Sol: First, $\mathbf{T}_1$ is linear because

$$\begin{aligned}
\mathbf{T}_1[ax(t) + by(t)] &= \sum_{k=0}^{K} h^k [ax(t-k) + by(t-k)] \\
&= a\sum_{k=0}^{K} h^k x(t-k) + b\sum_{k=0}^{K} h^k y(t-k) \\
&= a\mathbf{T}_1[x(t)] + b\mathbf{T}_1[y(t)].
\end{aligned}$$

Second, $\mathbf{T}$ is time-invariant because if $\mathbf{T}_1[x(t)] = z(t)$, then

$$\begin{aligned}
\mathbf{T}_1[x(t-c)] &= \sum_{k=0}^{K} h^k x(t-k-c) \\
&= z(t-c).
\end{aligned}$$

Last, the impulse response is derived as

$$\begin{aligned}
h(t) &= \mathbf{T}_1[\delta(t)] \\
&= \sum_{k=0}^{K} h^k \delta(t-k)
\end{aligned}$$

where $\delta(t)$ denotes the Dirac delta function.

(b) (1%) Let $X(\mu, t)$ and $Y(\mu, t)$ are both wide-sense stationary random processes with means η_X and η_Y. It is known that $\eta_Y = \alpha(h)\eta_X$ with $\alpha(h)$ a function of h. Determine $\alpha(h)$ in a closed-form expression.

Sol: Because η_Y is derived as

$$\begin{aligned}\eta_Y &= E\{Y(\mu,t)\} \\ &= \sum_{k=0}^{K} h^k E\{X(\mu,t-k)\} \\ &= \sum_{k=0}^{K} h^k \eta_X \\ &= \frac{1-h^{K+1}}{1-h}\eta_X,\end{aligned}$$

we have $\alpha(h) = \frac{1-h^{K+1}}{1-h}$.

(c) (2%) If $X(\mu,t)$ is a Gaussian random process with mean $\eta_X(t) = 0$ and autocorrelation $R_X(t_1,t_2) = \delta(t_1-t_2)$, find the second-order density of $Y(\mu,t)$, i.e., the joint probability density function of random variables $Y(\mu,t_1)$ and $Y(\mu,t_2)$ for any two distinct time points t_1 and t_2.

Sol: Because $X(\mu,t)$ is a stationary Gaussian random process and $\mathbf{T}_1$ is linear and time-invariant, $Y(\mu,t)$ is a stationary Gaussian random process. Hence, $Y(\mu,t_1)$ and $Y(\mu,t_2)$ are jointly Gaussian and their joint probability density function is determined by their mean, variance, and covariance which are derived as

$$E\{Y(\mu,t_1)\} = 0 = E\{Y(\mu,t_2)\}$$

$$\begin{aligned}Var\{Y(\mu,t_1)\} &= Var\{Y(\mu,t_2)\} \\ &= E\left\{\left[\sum_{k=0}^{K} h^k X(\mu,t-k)\right]^2\right\} \\ &= \sum_{k_1=0}^{K}\sum_{k_2=0}^{K} h^{k_1+k_2}\delta_{k_1,k_2} \\ &= \sum_{k=0}^{K} h^{2k} \\ &= \frac{1-h^{2(K+1)}}{1-h^2} \\ &= \sigma^2\end{aligned}$$

$$\begin{aligned}Cov\{Y(\mu,t_1),Y(\mu,t_2)\} &= E\left\{\sum_{k_1=0}^{K} h^{k_1}X(\mu,t_1-k_1)\sum_{k_2=0}^{K} h^{k_2}X(\mu,t_2-k_2)\right\} \\ &= \sum_{k_1=0}^{K}\sum_{k_2=0}^{K} h^{k_1+k_2}\delta(t_1-t_2-k_1+k_2).\end{aligned}$$

Therefore, the second-order density of $Y(\mu,t)$ is given by

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi\sigma^2\sqrt{1-\gamma^2}} \exp\{\frac{-1}{2(1-\gamma^2)}[\frac{y_1^2}{\sigma^2} - 2\gamma\frac{y_1 y_2}{\sigma^2} + \frac{y_2^2}{\sigma^2}]\}$$

with $\gamma \triangleq Cov\{Y(\mu,t_1),Y(\mu,t_2)\}/\sigma^2$.

(d) (2%) If $X(\mu,t)$ is a wide-sense stationary random process with mean $\eta_X(t)=0$ and autocorrelation $R_X(t_1,t_2)=\delta(t_1-t_2)$, find the power spectrum of $Y(\mu,t)$.

Sol: Because the autocorrelation function of $Y(\mu,t)$ is derived as

$$\begin{aligned} R_Y(t_1,t_2) &= E\{Y(\mu,t_1)Y(\mu,t_2)\} \\ &= \sum_{k_1=0}^{K}\sum_{k_2=0}^{K} h^{k_1+k_2}\delta(\tau-k_1+k_2) \\ &= R_Y(\tau) \end{aligned}$$

with $\tau \triangleq t_1 - t_2$, the power spectrum of $Y(\mu,t)$ is obtained as

$$\begin{aligned} S_Y(f) &= \mathcal{F}\{R_Y(\tau)\} \\ &= \sum_{k_1=0}^{K}\sum_{k_2=0}^{K} h^{k_1+k_2}\mathcal{F}\{\delta(\tau-k_1+k_2)\} \\ &= \sum_{k_1=0}^{K}\sum_{k_2=0}^{K} h^{k_1+k_2}e^{-jk_1\omega}e^{jk_2\omega} \\ &= \sum_{k_1=0}^{K} h^{k_1}e^{-jk_1\omega}\sum_{k_2=0}^{K} h^{k_2}e^{jk_2\omega} \\ &= \left|\sum_{k=0}^{K} h^k e^{-jk\omega}\right|^2 \\ &= \left|\frac{1-h^{K+1}e^{-j\omega(K+1)}}{1-he^{-j\omega}}\right|^2 \end{aligned}$$

where $\mathcal{F}\{\cdot\}$ is the Fourier transform operator.

(e) (3%) If $X(\mu,t)$ is a wide-sense stationary random process with mean $\eta_X(t)=0$ and autocorrelation $R_X(t_1,t_2)=\delta(t_1-t_2)$, find the autocorrelation of $Z(\mu,t)$. Since $Z(\mu,t)$ is wide-sense stationary as well, find the power spectrum of $Z(\mu,t)$.

Sol: The autocorrelation function of $Z(\mu,t)$ is given by

$$\begin{aligned} R_Z(t_1,t_2) &= E\{Z(\mu,t_1)Z(\mu,t_2)\} \\ &= E\{Y(\mu,-t_1)Y(\mu,-t_2)\} \\ &= R_Y(t_2-t_1) \\ &= \sum_{k_1=0}^{K}\sum_{k_2=0}^{K} h^{k_1+k_2}\delta(t_2-t_1-k_1+k_2) \end{aligned}$$

where the autocorrelation of $Y(\mu, t)$ comes from (d). Next, since $Z(\mu, t)$ has zero mean and is wide-sense stationary, it has the power spectrum

$$\begin{aligned}
S_Z(f) &= \mathcal{F}\{R_Z(\tau)\} = \mathcal{F}\{R_Y(-\tau)\} \\
&= \sum_{k_1=0}^{K}\sum_{k_2=0}^{K} h^{k_1+k_2}\mathcal{F}\{\delta(-\tau - k_1 + k_2)\} \\
&= \sum_{k_1=0}^{K}\sum_{k_2=0}^{K} h^{k_1+k_2}e^{jk_1\omega}e^{-jk_2\omega} \\
&= \left|\sum_{k=0}^{K} h^k e^{-jk\omega}\right|^2 \\
&= S_Y(f).
\end{aligned}$$

(2) (3%) Consider the random process $X(\mu, t)$ for $|t| < 1$ which has mean zero, i.e., $\eta_X(t) = 0$, and autocorrelation $R_X(t, s) = \cos(\pi(t - s)) + 1 - 2\sin^2(\pi(t - s))$ for $|t| < 1$ and $|s| < 1$. Find the Karhunen-Loève expansion of $X(\mu, t) + X(\mu, -t)$ in the interval $(-1, 1)$.

Sol: Define $Y(\mu, t) \triangleq X(\mu, t) + X(\mu, -t)$ for $|t| < 1$. Obviously, $Y(\mu, t)$ is also wide-sense stationary with mean zero and autocorrelation

$$\begin{aligned}
& R_Y(t, s) = E\{Y(\mu, t)Y(\mu, s)\} \\
= \;& R_X(t, s) + R_X(-t, s) + R_X(t, -s) + R_X(-t, -s) \\
= \;& 2R_X(t, s) + 2R_X(-t, s)
\end{aligned}$$

since $R_X(t, s) = R_X(-t, -s)$ and $R_X(-t, s) = R_X(t, -s)$. Now, $R_X(t, s)$ and $R_X(-t, s)$ can be rewritten as $R_X(t, s) = \cos(\pi(t - s)) + \cos(2\pi(t - s))$ and $R_X(-t, s) = \cos(\pi(t + s)) + \cos(2\pi(t + s))$. Therefore, $R_Y(t, s)$ can be rewritten as

$$\begin{aligned}
R_Y(t, s) &= 2[\cos(\pi(t - s)) + \cos(\pi(t + s))] \\
&\quad +2[\cos(2\pi(t - s)) + \cos(2\pi(t + s))] \\
&= 4\cos(\pi t)\cos(\pi s) + 4\cos(2\pi t)\cos(2\pi s).
\end{aligned}$$

By Mercer's theorem, we have

$$\begin{aligned}
R_Y(t, s) &= \sum_{k=1}^{\infty} \rho_k\phi_k(t)\phi_k^*(s) \qquad (4) \\
&= 4\cos(\pi t)\cos(\pi s) + 4\cos(2\pi t)\cos(2\pi s) \qquad (5)
\end{aligned}$$

where ρ_k's and $\phi_k(t)$'s are eigenvalues and eigenfunctions of $R_Y(t, s)$, respectively. From (4) and (5), it is straightforward to observe that

$$\begin{cases} \rho_1 = 4 \text{ and } \phi_1(t) = \cos(\pi t) \\ \rho_2 = 4 \text{ and } \phi_2(t) = \cos(2\pi t) \end{cases}$$

and $\rho_k = 0$ and $\phi_k(t) = 0$ otherwise. Therefore, the Karhunen-Loève expansion of $Y(\mu, t)$ in the interval $(-1, 1)$ is given by

$$Y(\mu, t) = \sum_{k=1}^{2} b_k(\mu)\phi_k(t)$$

where $b_k(\mu) \triangleq \int_{-1}^{1} Y(\mu, t)\phi_k(t)dt$ with $E\{b_k^2(\mu)\} = 4$.

(3) (2%, 1% each) Determine whether each of the following functions can be the power spectrum of a real-valued wide-sense stationary random process? Explain your answer. (Any correct answer without explanation will result in zero point.)

(a) $S_1(\omega) = \ln\{1 + \frac{1}{|\omega|}\}$

(b) $S_2(\omega) = \exp\{\omega^3 + \omega^4\}$

Sol: The power spectrum $S_X(\omega)$ of a real-valued wide-sense stationary random process $X(\mu, t)$ has to satisfy two conditions: (i) $S_X(\omega) \geq 0$ for all ω and (ii) $S_X(\omega) = S_X(-\omega)$ for all ω (even function). Based on these conditions, $S_2(\omega)$ can not be a power spectrum and $S_1(\omega)$ can be a power spectrum. The reasons are give below.

(a) $S_1(\omega)$ is nonnegative and even.

(b) $S_2(\omega)$ is not even.

(4) (3%) Denote $\widehat{W}(\mu, t)$ as the Hilbert transform of the wide-sense stationary real-valued Gaussian random process $W(\mu, t)$ which has mean zero and autocorrelation $R_W(\tau)$. Describe the joint statistic of random processes $\widehat{W}(\mu, t)$ and $W(\mu, t)$. A complete description of joint statistic is required and needs to be explained.

Sol: Because Hilbert transform is an (ideal) LTI system with impulse response $h(t) = \frac{1}{\pi t}$, $W(\mu, t)$ and

$$\widehat{W}(\mu, t) = \int_{-\infty}^{\infty} W(\mu, t - x)\frac{1}{\pi x}dx$$

are jointly Gaussian random processes. To describe the joint statistic completely, the mean functions, autocorrelation functions, and covariance function are required. First, because $W(\mu, t)$ has mean zero, $\widehat{W}(\mu, t)$ has mean zero as well. Second, because Hilbert transform is an all-pass system, we obtain

$$S_{\widehat{W}}(\omega) = S_W(\omega)$$

and thus

$$R_{\widehat{W}}(\tau) = R_W(\tau).$$

Thus, $\widehat{W}(\mu,t)$ and $W(\mu,t)$ have identical autocorrelation. Finally, the covariance of $W(\mu,t)$ and $\widehat{W}(\mu,t)$ is given by

$$\begin{aligned}
\mathrm{Cov}\{W(\mu,t),\widehat{W}(\mu,s)\} &= E\{W(\mu,t)\widehat{W}(\mu,s)\} \\
&= E\{W(\mu,t)\int_{-\infty}^{\infty} W(\mu,s-x)\frac{1}{\pi x}dx\} \\
&= \int_{-\infty}^{\infty} E\{W(\mu,t)W(\mu,s-x)\}\frac{1}{\pi x}dx \\
&= \int_{-\infty}^{\infty} R_W(s-t-x)\frac{1}{\pi x}dx \\
&= \widehat{R}_W(s-t)
\end{aligned}$$

where $\widehat{R}_W(\tau)$ denotes the Hilbert transform of $R_W(\tau)$.

(5) (6%, 2% each) If packets enter a network router with two input ports, namely Port A and Port B, according to a Poisson process with a rate of λ packets per minute, and if each packet enters Port A independently of the others, with probability p $(0 < p < 1)$, then find the probabilities of the following events (in terms of λ and p):

(a) *Event A:* No packet enters the router for two minutes.

Sol: Without loss of generality, we consider the first two minutes. Because the number of packets entering the router is a Poisson process $N(\mu,t)$ with rate λ, we have

$$\Pr\{N(\mu,2)=0\} = e^{-2\lambda}.$$

(b) *Event B:* In a given N minutes, there are K packets entering the router and L of them entering Port A in every minute, where N is a positive integer and K and L are nonnegative integers with $0 \le L \le K$.

Sol: Without loss of generality, we consider the first N minutes. Because the number of packets entering Port A is a Poisson process $N_A(\mu,t)$ with rate λp and the number of packets entering Port B is a Poisson process $N_B(\mu,t)$ with rate $\lambda(1-p)$, we have

$$\begin{aligned}
&\Pr\{N_A(\mu,k+1)-N_A(\mu,k)=L, N_B(\mu,k+1)-N_B(\mu,k)=K-L \\
&\qquad \text{for } k=0,1,...,N-1\} \\
=\ & [\Pr\{N_A(\mu,k+1)-N_A(\mu,k)=L, N_B(\mu,k+1)-N_B(\mu,k)=K-L\}]^N \\
=\ & \left\{[e^{-\lambda p}\frac{(\lambda p)^L}{L!}][e^{-\lambda(1-p)}\frac{[\lambda(1-p)]^{K-L}}{(K-L)!}]\right\}^N \\
=\ & \left\{e^{-\lambda}\frac{\lambda^K(p)^L(1-p)^{K-L}}{L!(K-L)!}\right\}^N.
\end{aligned}$$

(c) *Event C:* There is only one packet entering Port A in the first minute, provided that only N packets enter the router in the first N minutes, where N is a positive integer with $N \geq 2$. (Your answer must not contain the parameter λ.)

Sol: Because the number of packets entering Port A is a Poisson process $N_A(\mu, t)$ with rate λp and the number of packets entering Port B is a Poisson process $N_B(\mu, t)$ with rate $\lambda(1-p)$, we have

$$\begin{aligned}
& \Pr\{N_A(\mu,1) - N_A(\mu,0) = 1 | N(\mu,N) - N(\mu,0) = N\} \\
= \; & \frac{\Pr\{N_A(\mu,1) - N_A(\mu,0) = 1, N(\mu,N) - N(\mu,0) = N\}}{\Pr\{N(\mu,N) - N(\mu,0) = N\}} \\
= \; & \frac{[e^{-\lambda p}\frac{\lambda p}{1!}]\sum_{k=0}^{N-1}[e^{-\lambda(1-p)}\frac{[\lambda(1-p)]^k}{k!}][e^{-\lambda(N-1)}\frac{[\lambda(N-1)]^{N-1-k}}{(N-1-k)!}]}{e^{-\lambda N}\frac{(\lambda N)^N}{N!}} \\
= \; & \frac{p}{N^{N-1}}\sum_{k=0}^{N-1}\binom{N-1}{k}(1-p)^k(N-1)^{N-1-k} \\
= \; & p(1-\frac{p}{N})^{N-1}.
\end{aligned}$$

(6) (3%) Consider the random process

$$X(\mu,t) = A(\mu)\cos(\omega_c t + \phi(\mu)) + n(\mu,t)$$

where ω_c is a deterministic radian frequency. Here, $n(\mu,t)$ is a stationary Gaussian random process with mean zero and autocorrelation $R_n(\tau) = \delta(\tau)$. $A(\mu)$ is a Rayleigh random variable with probability density function $f_A(a) = ae^{-a^2/2}u(a)$ and $u(a)$ a unit step function. $\phi(\mu)$ is a uniform random variable in $[0, 2\pi)$. Moreover, $A(\mu)$, $\phi(\mu)$, and $n(\mu,t)$ are mutually independent. Is $X(\mu,t)$ a wide-sense stationary process? Is $X(\mu,t)$ a Gaussian process? Prove your answer.

Sol: $X(\mu,t)$ is a strict-sense stationary Gaussian process. Let us show it below.

(i) Show that $X(\mu,t)$ is a Gaussian process: Let $X_i(\mu) \triangleq X(\mu,t_i) = A(\mu)\cos(\omega_c t_i + \phi(\mu)) + n(\mu,t_i)$. Now, for any positive integer N, we have

$$\begin{aligned}
& \Phi_{X_1,X_2,\ldots,X_N}(\omega_1,\omega_2,\ldots,\omega_N) \\
= \; & E\{\exp\{j\sum_{i=1}^{N}\omega_i X_i(\mu)\}\} \\
= \; & E\{\exp\{j\sum_{i=1}^{N}\omega_i[A(\mu)\cos(\omega_c t_i + \phi(\mu)) + n(\mu,t_i)]\}\} \\
= \; & E\{\exp\{j\sum_{i=1}^{N}\omega_i A(\mu)\cos(\omega_c t_i + \phi(\mu))\}\}E\{\exp\{j\sum_{i=1}^{N}\omega_i n(\mu,t_i)\}\} \\
& \text{(independence)}
\end{aligned}$$

$$= E\{\exp\{j\sum_{i=1}^{N}\omega_i A(\mu)\cos(\omega_c t_i+\phi(\mu))\}\}\exp\{\frac{-1}{2}\sum_{i=1}^{N}\sum_{l=1}^{N}\omega_i\omega_l C_{il}\}$$

$(n(\mu,t)$ is Gaussian$)$

with $C_{il} \triangleq E\{n(\mu,t_i)n(\mu,t_l)\} = C_{|i-l|} = R_n(t_i-t_l)$ and $C_0=\sigma_n^2$. Also, we can express

$$E\{\exp\{j\sum_{i=1}^{N}\omega_i A(\mu)\cos(\omega_c t_i+\phi(\mu))\}\} = E\{\exp\{j[\alpha U(\mu)+\beta V(\mu)]\}$$

where $\alpha \triangleq \sum_{i=1}^{N}\omega_i\cos(\omega_c t_i)$, $\beta \triangleq -\sum_{i=1}^{N}\omega_i\sin(\omega_c t_i)$, $U(\mu)\triangleq A(\mu)\cos(\phi(\mu))$ and $V(\mu)\triangleq A(\mu)\sin(\phi(\mu))$. Because

$$\begin{aligned} f_{U,V}(u,v) &= \frac{1}{a}f_{A,\phi}(a,\phi) = \frac{1}{a}f_A(a)f_\phi(\phi) \\ &= \frac{1}{\sqrt{2\pi}}\exp\{-\frac{u^2}{2}\}\frac{1}{\sqrt{2\pi}}\exp\{-\frac{v^2}{2}\}, \end{aligned}$$

$U(\mu)$ and $V(\mu)$ are two independent and identically distributed Gaussian random variables with zero mean and unit variance. Thus,

$$\begin{aligned} E\{\exp\{j[\alpha U(\mu)+\beta V(\mu)]\} &= E\{\exp\{j\alpha U(\mu)\}\}E\{\exp\{j\beta V(\mu)\}\} \\ &= \exp\{-\frac{1}{2}(\alpha^2+\beta^2)\} \\ &= \exp\{-\frac{1}{2}[(\sum_{i=1}^{N}\omega_i\cos(\omega_c t_i))^2+(\sum_{i=1}^{N}\omega_i\sin(\omega_c t_i))^2]\} \\ &= \exp\{-\frac{1}{2}\sum_{i=1}^{N}\sum_{l=1}^{N}\omega_i\omega_l\cos(\omega_c(t_i-t_l))\}. \end{aligned}$$

It thus follows that

$$\begin{aligned} &\Phi_{X_1,X_2,\ldots,X_N}(\omega_1,\omega_2,\ldots,\omega_N) \\ &= \exp\{-\frac{1}{2}\sum_{i=1}^{N}\sum_{l=1}^{N}\omega_i\omega_l\cos(\omega_c(t_i-t_l))\}\exp\{\frac{-1}{2}\sum_{i=1}^{N}\sum_{l=1}^{N}\omega_i\omega_l C_{il}\} \\ &= \exp\{\frac{-1}{2}\sum_{i=1}^{N}\sum_{l=1}^{N}\omega_i\omega_l[\cos(\omega_c(t_i-t_l))+R_n(t_i-t_l)]\} \end{aligned}$$

which shows that $X_1(\mu), X_2(\mu), \ldots, X_N(\mu)$ are jointly Gaussian random variables. This proves that $X(\mu,t)$ is a Gaussian random process.

(ii) Because $E\{X(\mu,t)\}=0$ and

$$\begin{aligned} &R_X(t_1,t_2) \\ &= E\{A^2(\mu)\}E\{\cos(\omega_c t_1+\phi(\mu))\cos(\omega_c t_2+\phi(\mu))\} \\ &\quad +R_n(t_1-t_2) \\ &= \cos(\omega_c(t_1-t_2))+R_n(t_1-t_2), \end{aligned}$$

$X(\mu,t)$ is a stationary Gaussian random process.

(7) (3%) Let $X_1(\mu), X_2(\mu), ..., X_N(\mu)$ be independent Poisson random variables with parameters λ_1, λ_2,..., and λ_N, respectively. Prove that the sum random variable $Y(\mu) = \sum_{n=1}^N X_n(\mu)$ is a Poisson random variable with parameter $\sum_{n=1}^N \lambda_n$.

(Hint: Note that the Poisson random variable with parameter λ_n has the probability mass $\Pr\{X_n(\mu) = k\} = \exp\{-\lambda_n\}\lambda_n^k/k!$ for $k = 0, 1, ...$ You may give your proof in terms of moment generating function.)

Sol: We first find the moment generating function of $X_n(\mu)$, i.e., $E\{\exp\{sX_n(\mu)\}\}$ for permissible complex s, as follows.

$$\begin{aligned}
E\{\exp\{sX_n(\mu)\}\} &= \sum_{k=0}^{\infty} \exp\{-\lambda_n\} \frac{(\lambda_n)^k}{k!} [\exp\{s\}]^k \\
&= \exp\{-\lambda_n\} \sum_{k=0}^{\infty} \frac{(\lambda_n \exp\{s\})^k}{k!} \\
&= \exp\{-\lambda_n\} \exp\{\lambda_n \exp\{s\}\} \\
&= \exp\{\lambda_n[\exp\{s\} - 1]\}.
\end{aligned}$$

Next, the moment generating function of $Y(\mu)$ is

$$\begin{aligned}
E\{\exp\{sY(\mu)\}\} &= E\{\exp\{s \sum_{n=1}^{N} X_n(\mu)\}\} \\
&= \prod_{n=1}^{N} E\{\exp\{sX_n(\mu)\}\} \\
&\quad (X_n(\mu)\text{'s are independent}) \\
&= \prod_{n=1}^{N} \exp\{\lambda_n[\exp\{s\} - 1]\} \\
&= \exp\{(\sum_{n=1}^{N} \lambda_n)[\exp\{s\} - 1]\}
\end{aligned}$$

which shows that $Y(\mu) = \sum_{n=1}^N X_n(\mu)$ is a Poisson random variable with parameter $\sum_{n=1}^N \lambda_n$.