Stochastic Processes and Applications, Fall 2016
Homework Four (5%)

(1) (1%) Consider the random process

$$X(\mu, t) = A(\mu)\cos(\omega_c t + \phi(\mu)) + n(\mu, t)$$

where $n(\mu, t)$ is a strict-sense stationary Gaussian random process with mean zero and autocorrelation $R_n(\tau) = \delta(\tau)$, $A(\mu)$ is a Rayleigh random variable with probability density function $f_A(a) = ae^{-a^2/2}u(a)$ and $u(a)$ a unit step function, $\phi(\mu)$ is a uniform random variable in $[0, 2\pi)$, and $A(\mu)$, $\phi(\mu)$ and $n(\mu, t)$ are mutually independent. Prove that $X(\mu, t)$ is a strict-sense stationary Gaussian process.

Sol: For a given time t_i, define $X_i(\mu) \triangleq X(\mu, t_i) = A(\mu)\cos(\omega_c t_i + \phi(\mu)) + n(\mu, t_i)$. Now, for any positive integer N, we have

$$\begin{aligned}
& \Phi_{X_1, X_2, \ldots, X_N}(\omega_1, \omega_2, \ldots, \omega_N) \\
= \;& E\{\exp\{j\sum_{i=1}^{N}\omega_i X_i(\mu)\}\} \\
= \;& E\{\exp\{j\sum_{i=1}^{N}\omega_i[A(\mu)\cos(\omega_c t_i + \phi(\mu)) + n(\mu, t_i)]\}\} \\
= \;& E\{\exp\{j\sum_{i=1}^{N}\omega_i A(\mu)\cos(\omega_c t_i + \phi(\mu))\}\}E\{\exp\{j\sum_{i=1}^{N}\omega_i n(\mu, t_i)\}\} \\
& (A(\mu),\ \phi(\mu) \text{ and } n(\mu, t) \text{ are mutually independent}) \\
= \;& E\{\exp\{j\sum_{i=1}^{N}\omega_i A(\mu)\cos(\omega_c t_i + \phi(\mu))\}\}\exp\{\frac{-1}{2}\sum_{i=1}^{N}\sum_{l=1}^{N}\omega_i\omega_l C_{il}\} \\
& (n(\mu, t) \text{ is a Gaussian process with mean zero})
\end{aligned}$$

with $C_{il} \triangleq E\{n(\mu, t_i)n(\mu, t_l)\} = C_{|i-l|} = R_n(t_i - t_l)$. Also, we can express

$$E\{\exp\{j\sum_{i=1}^{N}\omega_i A(\mu)\cos(\omega_c t_i + \phi(\mu))\}\} = E\{\exp\{j[\alpha U(\mu) + \beta V(\mu)]\}$$

where $\alpha \triangleq \sum_{i=1}^{N}\omega_i\cos(\omega_c t_i)$, $\beta \triangleq -\sum_{i=1}^{N}\omega_i\sin(\omega_c t_i)$, $U(\mu) \triangleq A(\mu)\cos(\phi(\mu))$ and $V(\mu) \triangleq A(\mu)\sin(\phi(\mu))$. Because

$$\begin{aligned}
f_{U,V}(u, v) &= \frac{1}{a}f_{A,\phi}(a, \phi) = \frac{1}{a}f_A(a)f_\phi(\phi) \\
&= \frac{1}{\sqrt{2\pi}}\exp\{-\frac{u^2}{2}\} \times \frac{1}{\sqrt{2\pi}}\exp\{-\frac{v^2}{2}\}
\end{aligned}$$

$U(\mu)$ and $V(\mu)$ are independent and identically distributed Gaussian random variables with zero mean and unit variance. Thus,

$$E\{\exp\{j[\alpha U(\mu) + \beta V(\mu)]\} = E\{\exp\{j\alpha U(\mu)\}\}E\{\exp\{j\beta V(\mu)\}\}$$

$$
\begin{aligned}
&= \exp\{-\frac{1}{2}(\alpha^2+\beta^2)\} \\
&= \exp\{-\frac{1}{2}[(\sum_{i=1}^{N}\omega_i\cos(\omega_c t_i))^2+(\sum_{i=1}^{N}\omega_i\sin(\omega_c t_i))^2]\} \\
&= \exp\{-\frac{1}{2}\sum_{i=1}^{N}\sum_{l=1}^{N}\omega_i\omega_l\cos(\omega_c(t_i-t_l))\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\Phi_{X_1,X_2,...,X_N}(\omega_1,\omega_2,...,\omega_N) \\
=\ &\exp\{-\frac{1}{2}\sum_{i=1}^{N}\sum_{l=1}^{N}\omega_i\omega_l\cos(\omega_c(t_i-t_l))\}\exp\{\frac{-1}{2}\sum_{i=1}^{N}\sum_{l=1}^{N}\omega_i\omega_l C_{il}\} \\
=\ &\exp\{\frac{-1}{2}\sum_{i=1}^{N}\sum_{l=1}^{N}\omega_i\omega_l[\cos(\omega_c(t_i-t_l))+R_n(t_i-t_l)]\}
\end{aligned}
$$

which shows that $X_1(\mu), X_2(\mu), ..., X_N(\mu)$ are jointly Gaussian random variables. This proves that $X(\mu,t)$ is a Gaussian random process. Also, because the N-th order characteristic function $\Phi_{X_1,X_2,...,X_N}(\omega_1,\omega_2,...,\omega_N)$ is invariant to a common time shift, e.g., $t_i \to t_i+\tau$ for $i=1,2,...,N$, $X(\mu,t)$ is a strict-sense stationary Gaussian random process.

(2) (1%) Let $n_+(\mu,t)$, $\widehat{n}(\mu,t)$ and $\widetilde{n}(\mu,t)$ be the pre-envelope, the Hilbert transform, and the complex envelope of wide-sense stationary real-valued random process $n(\mu,t)$, respectively, related by

$$
n_+(\mu,t)=n(\mu,t)+j\widehat{n}(\mu,t)=\widetilde{n}(\mu,t)\exp\{j2\pi f_c t\}.
$$

Determine whether each of the following statements is true or false. Prove the statement if it is true and explain the reason if it is false.

(a) *Statement A:* $n(\mu,t)$ and $\widehat{n}(\mu,t)$ are jointly wide-sense stationary real-valued random processes.

(b) *Statement B:* $\widehat{n}(\mu,t)$ is a wide-sense stationary complex-valued random process.

(c) *Statement C:* $\widetilde{n}(\mu,t)$ is a wide-sense stationary complex-valued random process.

Sol: *Statement A* and *Statement B* are true, but *Statement C* is false. The arguments are given below. Note first that

$$
\widehat{n}(\mu,t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{n(\mu,x)}{t-x}dx=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{n(\mu,t-x)}{x}dx.
$$

(a) We first prove that $n(\mu,t)$ and $\widehat{n}(\mu,t)$ are jointly wide-sense stationary (WSS) real-valued random processes when $n(\mu,t)$ is WSS. Now, if $E\{n(\mu,t)\}=C$, then

$$
E\{\widehat{n}(\mu,t)\}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{C}{x}dx=\text{a constant}
$$

Also,

$$
\begin{aligned}
R_{\widehat{n}}(t_1,t_2) &= E\{\widehat{n}(\mu,t_1)\widehat{n}(\mu,t_2)\} \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{E\{n(\mu,t_1-x)n(\mu,t_2-y)\}}{xy}dxdy \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{R_n(t_1-x,t_2-y)}{xy}dxdy \\
&= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{R_n(t_1-t_2-x+y)}{xy}dxdy \\
&= \text{a function of } t_1-t_2.
\end{aligned}
$$

Thus, $\widehat{n}(\mu,t)$ is WSS. Further, because

$$
\begin{aligned}
E\{n(\mu,t_1)\widehat{n}(\mu,t_2)\} &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{E\{n(\mu,t_1)n(\mu,t_2-x)\}}{x}dx \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R_n(t_1,t_2-x)}{x}dx \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{R_n(t_1-t_2+x)}{x}dx \\
&= \text{a function of } t_1-t_2.
\end{aligned}
$$

and so is $E\{\widehat{n}(\mu,t_1)n(\mu,t_2)\}$, $n(\mu,t)$ and $\widehat{n}(\mu,t)$ are jointly WSS.

(b) Because the real part $n(\mu,t)$ and the imaginary part $\widehat{n}(\mu,t)$ of a complex-valued random process $n_+(\mu,t)=n(\mu,t)+j\widehat{n}(\mu,t)$ are jointly WSS, $n_+(\mu,t)$ is WSS.

(3) Note that $\widetilde{n}(\mu,t)=n_+(\mu,t)\exp\{-j2\pi f_c t\}$. Now, $E\{\widetilde{n}(\mu,t)\}=E\{n_+(\mu,t)\}\exp\{-j2\pi f_c t\}$ depends on time t if $E\{n_+(\mu,t)\}$ is nonzero. Thus, $\widetilde{n}(\mu,t)$ may not be WSS.

(3) (1%) Define the random process $h(\mu,t)=[1+\cos(2\pi f_1 t+\phi(\mu))]\cos(2\pi f_2 t+\theta(\mu))$ for $f_1<f_2$, where $\phi(\mu)$ and $\theta(\mu)$ are independent and identically distributed random variables which are uniform distributed in $[0,2\pi)$. Derive the mean and autocorrelation of the Hilbert transform $\widehat{h}(\mu,t)$, i.e., $E\{\widehat{h}(\mu,t)\}$ and $E\{\widehat{h}(\mu,t_1)\widehat{h}(\mu,t_2)\}$.

Sol: Since $f_1<f_2$, we can use Bedrosian's theorem to obtain

$$
\begin{aligned}
\widehat{h}(\mu,t) &= [1+\cos(2\pi f_1 t+\phi(\mu))]\cdot H.T.\{\cos(2\pi f_2 t+\theta(\mu))\} \\
&= [1+\cos(2\pi f_1 t+\phi(\mu))]\sin(2\pi f_2 t+\theta(\mu)).
\end{aligned}
$$

Thus, the mean $E\{\widehat{h}(\mu,t)\}$ is given by

$$
\begin{aligned}
E\{\widehat{h}(\mu,t)\} &= [1+E\{\cos(2\pi f_1 t+\phi(\mu))\}]E\{\sin(2\pi f_2 t+\theta(\mu))\} \\
&\quad (\phi(\mu) \text{ and } \theta(\mu) \text{ are independent}) \\
&= 0. \\
(E\{\sin(2\pi f_2 t+\theta(\mu))\} &= \frac{1}{2\pi}\int_0^{2\pi}\sin(2\pi f_2 t+\theta)d\theta = 0)
\end{aligned}
$$

The autocorrelation $E\{\widehat{h}(\mu,t_1)\widehat{h}(\mu,t_2)\}$ is given by

$$\begin{aligned}
& E\{\widehat{h}(\mu,t_1)\widehat{h}(\mu,t_2)\} \\
= \; & E\{[1+\cos(2\pi f_1 t_1+\phi(\mu))][1+\cos(2\pi f_1 t_2+\phi(\mu))]\} \\
& \times E\{\sin(2\pi f_2 t_1+\theta(\mu))\sin(2\pi f_2 t_2+\theta(\mu))\} \\
& (\phi(\mu) \text{ and } \theta(\mu) \text{ are independent}) \\
= \; & [1+E\{\cos(2\pi f_1 t_1+\phi(\mu))\cos(2\pi f_1 t_1+\phi(\mu))\}] \\
& \times E\{\sin(2\pi f_2 t_1+\theta(\mu))\sin(2\pi f_2 t_2+\theta(\mu))\} \\
= \; & \frac{1}{2}[1+\frac{1}{2}\cos(2\pi f_1(t_1-t_2))]\cos(2\pi f_2(t_1-t_2)) \\
(E\{\cos(2\pi f_1 t+\phi(\mu))\} = \; & \frac{1}{2\pi}\int_0^{2\pi}\cos(2\pi f_2 t+\phi)d\phi = 0)
\end{aligned}$$

where we have used the results

$$\begin{aligned}
& E\{\cos(2\pi f_1 t_1+\phi(\mu))\cos(2\pi f_1 t_2+\phi(\mu))\} \\
= \; & \frac{1}{2}\{E\{\cos(2\pi f_1(t_1+t_2)+2\phi(\mu))\}+\cos(2\pi f_1(t_1-t_2))\} \\
= \; & \frac{1}{2}\cos(2\pi f_1(t_1-t_2))
\end{aligned}$$

and

$$\begin{aligned}
& E\{\sin(2\pi f_2 t_1+\theta(\mu))\sin(2\pi f_2 t_2+\theta(\mu))\} \\
= \; & \frac{1}{2}\{\cos(2\pi f_2(t_1-t_2))-E\{\cos(2\pi f_2(t_1+t_2)+2\theta(\mu))\}\} \\
= \; & \frac{1}{2}\cos(2\pi f_2(t_1-t_2)).
\end{aligned}$$

(4) (1%) Consider the real-valued linear and time-invariant system with system function $H(\omega)$, jointly wide-sense stationary input $X(\mu,t)$ and output $Y(\mu,t)$. Show that if

$$R_{XX}(\tau)=R_{YY}(\tau) \text{ and } R_{XY}(-\tau)=-R_{XY}(\tau)$$

then $H(\omega)=jB(\omega)$ where $B(\omega)$ is a function taking value in $\{-1,+1\}$. Also, find $B(\omega)$ if $Y(\mu,t)=\widehat{X}(\mu,t)$.

Sol: Taking Fourier transform of $R_{XX}(\tau)=R_{YY}(\tau)$ and $R_{XY}(-\tau)=-R_{XY}(\tau)$ gives

$$S_{XX}(\omega)=S_{YY}(\omega) \text{ and } S_{XY}(-\omega)=-S_{XY}(\omega). \tag{1}$$

Because $S_{YY}(\omega)=S_{XX}(\omega)|H(\omega)|^2$ and $S_{XY}(\omega)=S_{XX}(\omega)H^*(\omega)$, we have from (1) that

$$|H(\omega)|^2=1 \text{ and } H(-\omega)=-H(\omega).$$

Because the system impulse response is real-valued, $H(\omega)$ is Hermitian symmetric, i.e., $H^*(\omega)=H(-\omega)$ which yields $H^*(\omega)=-H(\omega)$. This, in conjunction with $|H(\omega)|^2=1$, shows that $H(\omega)$ is pure imaginary and has the form $H(\omega)=jB(\omega)$ where $B(\omega)$ is a function taking value in $\{-1,+1\}$.

Further, if $Y(\mu,t)=\widehat{X}(\mu,t)$, then the system is a Hilbert transform with $H(\omega)=-j\,\mathrm{sgn}\,\omega$ and thus $B(\omega)=-\mathrm{sgn}\,\omega$.

(5) (1%) Let $S_X(\omega)$ and $R_X(\tau)$ be the power spectrum and autocorrelation of a complex-valued wide-sense stationary random process $X(\mu, t)$. That is, $S_X(\omega)$ and $R_X(\tau)$ are a Fourier-transformable pair. Also, define $S_T(\mu, \omega)$ by

$$S_T(\mu, \omega) = \frac{1}{2T} \left| \int_{-T}^{T} X(\mu, t) \exp\{-j\omega t\} dt \right|^2$$

with $j = \sqrt{-1}$. Prove that

$$\lim_{T\to\infty} E\{S_T(\mu, \omega)\} = S_X(\omega).$$

Hints:

- The Fourier transform of a window function $U_T(t) = \begin{cases} 1 & , |t| \le 2T \\ 0 & , |t| > 2T \end{cases}$ is $\mathcal{F}\{U_T(t)\}$ $= 2\frac{\sin(2T\omega)}{\omega}$. Also, $\mathcal{F}\{(1 - \frac{|t|}{2T})U_T(t)\} = \frac{2\sin^2(T\omega)}{T\omega^2}$.
- $\lim_{T\to\infty} \frac{2\sin^2(T\omega)}{T\omega^2} = 2\pi\delta(\omega)$, with $\delta(\omega)$ being a Dirac delta function.

Sol: By definition, $\lim_{T\to\infty} E\{S_T(\mu, \omega)\}$ can be expressed as

$$\begin{aligned}
\lim_{T\to\infty} E\{S_T(\mu, \omega)\} &= \lim_{T\to\infty} E\{\frac{1}{2T} \left| \int_{-T}^{T} X(\mu, t) \exp\{-j\omega t\} dt \right|^2\} \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{T} R_X(t_1 - t_2) \exp\{-j\omega(t_1 - t_2)\} dt_1 dt_2 \\
&= \lim_{T\to\infty} \int_{-2T}^{2T} (1 - \frac{|\tau|}{2T}) R_X(\tau) \exp\{-j\omega\tau\} d\tau \\
&= \lim_{T\to\infty} \mathcal{F}\{R_X(t) \cdot (1 - \frac{|t|}{2T}) U_T(t)\} \\
&= \lim_{T\to\infty} \frac{1}{2\pi} \mathcal{F}\{R_X(t)\} * \mathcal{F}\{(1 - \frac{|t|}{2T}) U_T(t)\} \\
&= \frac{1}{2\pi} S_X(\omega) * \lim_{T\to\infty} \mathcal{F}\{(1 - \frac{|t|}{2T}) U_T(t)\} \\
&= \frac{1}{2\pi} S_X(\omega) * 2\pi\delta(\omega) \\
&= S_X(\omega).
\end{aligned}$$