Stochastic Processes and Applications, Fall 2015
Homework Five (5%)

1. (1.5%) Let $X(\mu,t)$ and $Y(\mu,t)$ be independent Poisson random processes with rates λ_X and λ_Y, respectively. Answer the following:

(A) (0.5%) Find the characteristic functions of $X(\mu,t)$ for a fixed t, i.e., $E\{\exp\{j\omega X(\mu,t)\}\}$.

Sol: Now,

$$\begin{aligned}
E\{\exp\{j\omega X(\mu,t)\}\} &= \sum_{n=0}^{\infty}\exp\{-\lambda_X t\}\frac{(\lambda_X t)^n}{n!}[\exp\{j\omega\}]^n \\
&= \exp\{-\lambda_X t\}\sum_{n=0}^{\infty}\frac{(\lambda_X t\exp\{j\omega\})^n}{n!} \\
&= \exp\{-\lambda_X t\}\exp\{\lambda_X t\exp\{j\omega\}\} \\
&= \exp\{\lambda_X t[\exp\{j\omega\}-1]\}
\end{aligned}$$

(B) (1%) Prove that $X(\mu,t)+Y(\mu,t)$ is also a Poisson process with rate $\lambda_X+\lambda_Y$.

Sol: Let $N(\mu,t)=X(\mu,t)+Y(\mu,t)$. We are going to show that $N(\mu,t)$ is a Poisson process with rate $\lambda_X+\lambda_Y$. First, $N(\mu,t)$ has stationary increment because

$$\begin{aligned}
& \Pr\{N(\mu,t+s)-N(\mu,t)=K\} \\
= \; & \Pr\{[X(\mu,t+s)+Y(\mu,t+s)]-[X(\mu,t)-Y(\mu,t)]=K\} \\
= \; & \sum_{k=0}^{K}\Pr\{X(\mu,t+s)-X(\mu,t)=k, Y(\mu,t+s)-Y(\mu,t)=K-k\} \\
= \; & \sum_{k=0}^{K}\Pr\{X(\mu,t+s)-X(\mu,t)=k\}\Pr\{Y(\mu,t+s)-Y(\mu,t)=K-k\} \\
= \; & \sum_{k=0}^{K}\Pr\{X(\mu,s)-X(\mu,0)=k\}\Pr\{Y(\mu,s)-Y(\mu,0)=K-k\} \\
= \; & \sum_{k=0}^{K}\Pr\{X(\mu,s)-X(\mu,0)=k, Y(\mu,s)-Y(\mu,0)=K-k\} \\
= \; & \Pr\{[X(\mu,s)+Y(\mu,s)]-[X(\mu,0)-Y(\mu,0)]=K\} \\
= \; & \Pr\{N(\mu,s)-N(\mu,0)=k\} \quad \forall t,s\geq 0, \forall K\in\mathcal{N}.
\end{aligned}$$

Second, because $X(\mu,t)$ and $Y(\mu,t)$ are independent and $\{X(\mu,t_j)-X(\mu,s_j)\}_{j=0}^{n}$ and $\{Y(\mu,t_j)-Y(\mu,s_j)\}_{j=0}^{n}$ are independent random variables, $\{N(\mu,t_j)-N(\mu,s_j)\}_{j=0}^{n}$ are also independent increments and thus $N(\mu,t)$ has independent increment, where we let $s_1\leq t_1\leq s_2\leq t_2\leq \ldots \leq s_n\leq t_n$. Third, it is shown that

$$\begin{aligned}
\Pr\{N(\mu,t)=K\} &= \Pr\{X(\mu,t)+Y(\mu,t)=K\} \\
&= \sum_{k=0}^{K}\Pr\{X(\mu,t)=k\}\Pr\{Y(\mu,t)=K-k\} \\
&= \sum_{k=0}^{K}[e^{-\lambda_X}\frac{\lambda_X^k}{k!}][e^{-\lambda_Y}\frac{\lambda_Y^{K-k}}{(K-k)!}]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{-\lambda_X+\lambda_Y}}{K!}\sum_{k=0}^{K}\frac{K!}{k!(K-k)!}\lambda_X^k\lambda_Y^{K-k} \\
&= e^{-\lambda_X+\lambda_Y}\frac{(\lambda_X+\lambda_Y)^K}{K!}.
\end{aligned}
$$

Because $N(\mu,t)$ has stationary and independent increments and $\Pr\{N(\mu,t) = K\} = e^{-\lambda_X+\lambda_Y}\frac{(\lambda_X+\lambda_Y)^K}{K!}$, $N(\mu,t)$ is a Poisson process with rate $\lambda_X+\lambda_Y$.

2. (1%) Let $\{X(\mu,t); t \geq 0\}$ be the random telegraph signal of rate λ, and $\{Y(\mu,t); t \geq 0\}$ be a continuous-time process derived from $X(\mu,t)$ as follows:

   a. $Y(\mu,t)$ takes values from the set $\{0,1\}$.

   b. $\Pr\{X(\mu,0) = +1\} = \Pr\{X(\mu,0) = -1\} = \dfrac{1}{2}$.

   c. Each time $X(\mu,t)$ changes polarity, $Y(\mu,t)$ switches value between 0 and 1.

   d. If $X(\mu,t)$ does not change polarity, $Y(\mu,t)$ does not switch value, either.

   (A) (0.5%) Find $E\{Y(\mu,t)\}$.

   Sol: Because $\{X(\mu,t); t \geq 0\}$ is a random telegraph signal with $\Pr\{X(\mu,0) = +1\} = \Pr\{X(\mu,0) = -1\} = \frac{1}{2}$,

   $$\Pr\{X(\mu,t) = +1\} = \Pr\{X(\mu,t) = -1\} = \frac{1}{2}$$

   for $t \geq 0$. Because $Y(\mu,t)$ is obtained by switching between the values 0 and 1 according to the polarity change in $X(\mu,t)$,

   $$\Pr\{Y(\mu,t) = +1\} = \frac{1}{2} = \Pr\{Y(\mu,t) = 0\}.$$

   Thus, $E\{Y(\mu,t)\} = \frac{1}{2}$.

   (B) (0.5%) Find the autocorrelation function of $Y(\mu,t)$.

   Sol: For $t_2 \geq t_1$, the autocorrelation function of $Y(\mu,t)$ is

   $$
   \begin{aligned}
   R_Y(t_1,t_2) &= E\{Y(\mu,t_1)Y(\mu,t_2)\} \\
   &= \Pr\{Y(\mu,t_1) = 1, Y(\mu,t_2) = 1\} \\
   &= \Pr\{Y(\mu,t_2) = 1 | Y(\mu,t_1) = 1\}\Pr\{Y(\mu,t_1) = 1\} \\
   &= \frac{1}{2}\Pr\{\text{There are even numbers of polarity changes} \\
   &\quad \text{in } X(\mu,t) \text{ from } t_1 \text{ to } t_2\} \\
   &= \frac{1}{2}\times\frac{1}{2}[1+\exp\{-2\lambda(t_2-t_1)\}].
   \end{aligned}
   $$

   Thus, the autocorrelation function of $Y(\mu,t)$ is given by $R_Y(t_1,t_2) = \frac{1}{4}[1+\exp\{-2\lambda|t_2-t_1|\}]$.

3. (0.5%) Packets arrive at a network router according to a Poisson process of rate λ packets per second. Find the probability that in a one-second period K packets arrive in the first half second and no packet arrives in the last half second, where K is a nonnegative integer.

Sol: Let $N(\mu, t), t \geq 0$, be the Poisson process of rate λ packets per second. Now, we want to find

$$\begin{aligned}
& \Pr\{N(\mu, \frac{1}{2}) - N(\mu, 0) = K, N(\mu, 1) - N(\mu, \frac{1}{2}) = 0\} \\
= \; & \Pr\{N(\mu, \frac{1}{2}) - N(\mu, 0) = K\} \Pr\{N(\mu, 1) - N(\mu, \frac{1}{2}) = 0\} \\
& \text{(because of independent increments)} \\
= \; & \Pr\{N(\mu, \frac{1}{2}) = K\} \Pr\{N(\mu, \frac{1}{2}) = 0\} \\
& \text{(because of stationary increments and } N(\mu, 0) \text{ is zero by default)} \\
= \; & \frac{(\frac{1}{2}\lambda)^K}{K!} \exp\{-\lambda\}.
\end{aligned}$$

4. (2%) Messages arrive at a customer from two telephone lines according to independent Poisson processes of rates λ_1 and λ_2 messages per second, respectively.

(A) (0.5%) Find the probability that a message arrives first on line one.

Sol: Let $T_n(\mu)$ be the time till the first arrival on line n. Thus,

$$\begin{aligned}
\Pr\{T_1(\mu) < T_2(\mu)\} &= \int_0^\infty \Pr\{x < T_2(\mu) | T_1(\mu) = x\} f_{T_1}(x) dx \\
&= \int_0^\infty \exp\{-\lambda_2 x\} \lambda_1 \exp\{-\lambda_1 x\} dx \\
&= \lambda_1 \int_0^\infty \exp\{-(\lambda_1 + \lambda_2) x\} dx \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\end{aligned}$$

(B) (0.5%) Find the probability density function for the time until a message arrives on either line.

Sol: Now, $T(\mu) = \min\{T_1(\mu), T_2(\mu)\}$ is the time until a message arrives on either line. Because $T_1(\mu)$ and $T_2(\mu)$ are independent,

$$\begin{aligned}
\Pr\{T(\mu) > x\} &= \Pr\{\min\{T_1(\mu), T_2(\mu)\} > x\} \\
&= \Pr\{T_1(\mu) > x, T_2(\mu) > x\} \\
&= \Pr\{T_1(\mu) > x\} \Pr\{T_2(\mu) > x\} \\
&= \exp\{-\lambda_1 x\} \exp\{-\lambda_2 x\} \\
&= \exp\{-(\lambda_1 + \lambda_2) x\}.
\end{aligned}$$

Thus, the probability density function of $T(\mu)$ is

$$f_T(x) = (\lambda_1 + \lambda_2) \exp\{-(\lambda_1 + \lambda_2) x\}, \; x \geq 0.$$

(C) (0.5%) Find the probability $\Pr\{N(\mu, t) = n\}$ for any nonnegative integer n where $N(\mu, t)$ is the total number of messages on both lines that arrive in an interval of length t.

Sol: Now, because the arrival patterns from both lines are independent Poisson processes, $N(\mu, t)$ is Poisson process of rate $\lambda = \lambda_1 + \lambda_2$. Thus,

$$\Pr\{N(\mu, t) = n\} = \exp\{-\lambda t\}\frac{(\lambda t)^n}{n!}.$$

(D) (0.5%) Find the probability that there is no message on both lines for the first second given that there is one message in a two-second period.

Sol: The probability is given by

$$\begin{aligned}
& \Pr\{N(\mu, \frac{1}{2}) = 0 | N(\mu, 1) = 1\} \\
= \; & \frac{\Pr\{N(\mu, \frac{1}{2}) = 0, N(\mu, 1) = 1\}}{\Pr\{N(\mu, 1) = 1\}} \\
= \; & \frac{\Pr\{N(\mu, \frac{1}{2}) = 0, N(\mu, 1) - N(\mu, \frac{1}{2}) = 1\}}{\Pr\{N(\mu, 1) = 1\}} \\
= \; & \frac{\Pr\{N(\mu, \frac{1}{2}) = 0\} \Pr\{N(\mu, 1) - N(\mu, \frac{1}{2}) = 1\}}{\Pr\{N(\mu, 1) = 1\}} \\
& \text{(because of independent increments)} \\
= \; & \frac{\Pr\{N(\mu, \frac{1}{2}) = 0\} \Pr\{N(\mu, \frac{1}{2}) = 1\}}{\Pr\{N(\mu, 1) = 1\}} \\
& \text{(because of stationary increments)} \\
= \; & \frac{\exp\{-\frac{\lambda}{2}\} \exp\{-\frac{\lambda}{2}\}(\frac{\lambda}{2})}{\exp\{-\lambda\}\lambda} \\
= \; & \frac{1}{2}.
\end{aligned}$$