Stochastic Processes and Applications, Fall 2015
Homework Two (5%)

1. (1%) In the fair-coin experiment, Pr{heads shows} = Pr{tail shows} = 1/2. Suppose that a random process $X(\mu, t)$ is defined accordingly to be $X(\mu, t) = \sin(\pi t)$ if μ is the outcome "heads shows" and $X(\mu, t) = 2t$ if μ is the outcome "tail shows". Find the mean function $\eta_X(t)$ and the autocorrelation function $R_X(t_1, t_2)$ of $X(\mu, t)$.

Sol: By definition, $\eta_X(t) = E\{X(\mu, t)\} = \frac{1}{2}\sin(\pi t) + t$ and

$$\begin{aligned} R_X(t_1, t_2) &= E\{X(\mu, t_1)X(\mu, t_2)\} \\ &= \frac{1}{2}\sin(\pi t_1)\sin(\pi t_2) + 2t_1t_2. \end{aligned}$$

2. (1%) Consider the binary phase-shift keyed (BPSK) modulation signal

$$X(\mu, t) = \sum_{n=-\infty}^{\infty} D_n(\mu)\cos(2\pi f_c t)p(t - nT)$$

where f_c is the center frequency in hertz, $p(t)$ is the real-valued rectangular pulse waveform defined by $p(t) = 1$ for $0 \le t < T$ and $p(t) = 0$ otherwise, T is the symbol time with f_cT being an integer multiple of 2π, and $D_n(\mu)$ is the real-valued random data symbol which is assumed to take value in the binary set $\{-1, 1\}$ with probability $\Pr\{D_n(\mu) = -1\} = \Pr\{D_n(\mu) = 1\} = 1/2$. It is also assumed that $D_n(\mu)$'s are independent. Derive the mean function and the average power of $X(\mu, t)$.

Sol: Note that the data sequence ..., $D_{n-1}(\mu), D_n(\mu)$, ... has mean $E\{D_n(\mu)\} = 0$ and correlation $E\{D_n(\mu)D_m(\mu)\} = 0$ if $m \neq n$ and $E\{D_n(\mu)D_m(\mu)\} = 1$ if $m = n$. The mean and the average power of $X(\mu, t)$ are derived as

$$\begin{aligned} E\{X(\mu, t)\} &= E\{\sum_{n=-\infty}^{\infty} D_n(\mu)\cos(2\pi f_c t)p(t - nT)\} \\ &= \sum_{n=-\infty}^{\infty} E\{D_n(\mu)\}\cos(2\pi f_c t)p(t - nT) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} E\{X^2(\mu, t)\} &= E\{\sum_{n=-\infty}^{\infty} D_n(\mu)\cos(2\pi f_c t)p(t - nT) \\ &\quad \times \sum_{m=-\infty}^{\infty} D_m(\mu)\cos(2\pi f_c t)p(t - mT)\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E\{D_n(\mu)D_m(\mu)\} \cos^2(2\pi f_c t) p(t-nT)p(t-mT) \\
&= \sum_{n=-\infty}^{\infty} p(t-nT)p(t-nT)\cos^2(2\pi f_c t) \\
&= \cos^2(2\pi f_c t).
\end{aligned}
$$

3. (1%) Consider the real-valued Gaussian random process $X(\mu,t)$ which have mean zero, i.e., $\eta_X(t) = E\{X(\mu,t)\} = 0$ and autocorrelation $R_X(t_1,t_2) = \delta(t_1 - t_2)$ with $\delta(t)$ being the Dirac delta function. Also define K new random processes $Y_1(\mu,t), Y_2(\mu,t), ..., Y_K(\mu,t)$ by the outputs of the K linear time-invariant systems with real-valued impulse responses $h_1(t), h_2(t), ..., h_K(t)$, respectively, and common input $X(\mu,t)$. Derive the mean function and the autocorrelation function of the sum process $\sum_{k=1}^{K} Y_k(\mu,t)$ in terms of $h_1(t), h_2(t), ..., h_K(t)$. Also, determine whether $\sum_{k=1}^{K} Y_k(\mu,t)$ is stationary in any sense.

Sol: Now, $Y_k(\mu,t)$ can be represented as $Y_k(\mu,t) = X(\mu,t) * h_k(t)$ with $*$ being the convolution operator. Thus, we can represent sum process $Z(\mu,t) \triangleq \sum_{k=1}^{K} Y_k(\mu,t)$ by

$$
\begin{aligned}
Z(\mu,t) &= \sum_{k=1}^{K} X(\mu,t) * h_k(t) \\
&= X(\mu,t) * \sum_{k=1}^{K} h_k(t).
\end{aligned}
$$

Thus, the mean function and the autocorrelation function of $Z(\mu,t)$ are given by

$$
\begin{aligned}
\eta_Z(t) &\triangleq E\{Z(\mu,t)\} = \eta_X(t) * \sum_{k=1}^{K} h_k(t) = 0 \\
R_Z(t_1,t_2) &\triangleq E\{Z(\mu,t_1)Z(\mu,t_2)\} \\
&= R_X(t_1,t_2) * \sum_{k_1=1}^{K} h_{k_1}(t_1) * \sum_{k_2=1}^{K} h_{k_2}(t_2) \\
&= \delta(t_1 - t_2) * \sum_{k_1=1}^{K} h_{k_1}(t_1) * \sum_{k_2=1}^{K} h_{k_2}(t_2) \\
&= \sum_{k_1=1}^{K} h_{k_1}(t_1 - t_2) * \sum_{k_2=1}^{K} h_{k_2}(t_2) \\
&= \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} h_{k_1}(t_1 - t_2) * h_{k_2}(t_2)
\end{aligned}
$$

where $h_{k_1}(t_1 - t_2) * h_{k_2}(t_2)$ can be expressed as

$$\begin{aligned} h_{k_1}(t_1 - t_2) * h_{k_2}(t_2) &= \int_{-\infty}^{\infty} h_{k_1}(t_1 - \tau) h_{k_2}(t_2 - \tau) d\tau \\ &= \int_{-\infty}^{\infty} h_{k_1}(x) h_{k_2}(x + t_2 - t_1) dx \\ (x &= t_1 - \tau) \end{aligned}$$

and is a function of $t_1 - t_2$. Since $\eta_Z(t)$ is a constant and $R_Z(t_1, t_2)$ is a function of $t_1 - t_2$ only, $Z(\mu, t)$ is wide-sense stationary. Further, $Z(\mu, t)$ is a Gaussian process because it is the result of a linear transform of Gaussian process $X(\mu, t)$. Thus, $Z(\mu, t)$ is strict-sense stationary since a wide-sense stationary Gaussian process is strict-sense stationary.

4. (1%) Consider a memoryless hard limiter system with input $X(\mu, t)$ and output $Y(\mu, t)$ related by

$$Y(\mu, t) = \begin{cases} +1, & \text{if } X(\mu, t) \geq 0 \\ -1, & \text{if } X(\mu, t) < 0 \end{cases}.$$

Also, let $X(\mu, t)$ be a stationary white Gaussian random process with mean zero and autocorrelation $\delta(\tau)$. Find the mean function and the autocorrelation function of $Y(\mu, t)$.

Sol: Note that $Y(\mu, t)$ is binary-valued and has the probability mass

$$\begin{aligned} \Pr\{Y(\mu, t) &= +1\} = \Pr\{X(\mu, t) \geq 0\} = \frac{1}{2} \\ \Pr\{Y(\mu, t) &= -1\} = \Pr\{X(\mu, t) < 0\} = \frac{1}{2} \end{aligned}$$

because $X(\mu, t)$ has mean zero and thus an even probability density function for every t. By definition,

$$\begin{aligned} E\{Y(\mu, t)\} &= \Pr\{Y(\mu, t) = +1\} - \Pr\{Y(\mu, t) = -1\} = 0. \\ E\{Y(\mu, t_1) Y(\mu, t_2)\} &= \Pr\{X(\mu, t_1) X(\mu, t_2) \geq 0\} - \Pr\{X(\mu, t_1) X(\mu, t_2) < 0\} \\ &= 2 \Pr\{X(\mu, t_1) X(\mu, t_2) \geq 0\} - 1. \end{aligned}$$

For $t_1 \neq t_2$, $\Pr\{X(\mu, t_1) X(\mu, t_2) \geq 0\}$ is obtained as

$$\begin{aligned} & \Pr\{X(\mu, t_1) X(\mu, t_2) \geq 0\} \\ = \; & \Pr\{X(\mu, t_1) \geq 0, X(\mu, t_2) \geq 0\} + \Pr\{X(\mu, t_1) < 0, X(\mu, t_2) < 0\} \\ = \; & \Pr\{X(\mu, t_1) \geq 0\} \Pr\{X(\mu, t_2) \geq 0\} + \Pr\{X(\mu, t_1) < 0\} \Pr\{X(\mu, t_2) < 0\} \\ & \text{(since } X(\mu, t_1) \text{ and } X(\mu, t_2) \text{ are independent.)} \\ = \; & (\frac{1}{2})^2 + (\frac{1}{2})^2 \\ = \; & \frac{1}{2} \end{aligned}$$

and thus $E\{Y(\mu,t_1)Y(\mu,t_2)\}=0$. For $t_1 \neq t_2$, $E\{Y(\mu,t_1)Y(\mu,t_2)\}$ is obtained as

$$E\{Y(\mu,t_1)Y(\mu,t_2)\}=1.$$

5. (1%) Let $X(\mu,t)$ be a real-valued wide-sense stationary random process with mean zero, autocorrelation $R_X(\tau)$, and power spectrum $S_X(\omega)$. Also, define the new process $Y(\mu,t)=\sum_{k=1}^{N} k\cdot X(\mu,t+k)$ for a positive integer N. Express the autocorrelation and the power spectrum of $Y(\mu,t)$ in terms of $R_X(\tau)$ and $S_X(\omega)$, respectively.

Sol: Obviously, $Y(\mu,t)$ has mean zero. By definition, the autocorrelation function of $Y(\mu,t)$ is given by

$$\begin{aligned}
R_Y(t_1,t_2) &= E\{Y(\mu,t_1)Y(\mu,t_2)\} \\
&= E\{\sum_{k_1,k_2=1}^{N} k_1k_2\cdot X(\mu,t_1+k_1)X(\mu,t_2+k_2)\} \\
&= \sum_{k_1,k_2=1}^{N} k_1k_2\cdot R_X(\tau+k_1-k_2) \\
&= R_Y(\tau)
\end{aligned}$$

with $\tau=t_1-t_2$, which is a function of time difference. Thus, $Y(\mu,t)$ is wide-sense stationary. The power spectrum of $Y(\mu,t)$ is obtained as

$$\begin{aligned}
S_Y(\omega) &= \mathcal{F}\{R_Y(\tau)\} \\
&= \sum_{k_1,k_2=1}^{N} k_1k_2\cdot S_X(\omega)e^{-j\omega(k_2-k_1)} \\
&= S_X(\omega)|\sum_{k=1}^{N} e^{-j\omega k}|^2.
\end{aligned}$$