1. (a). ① $\|x\|_1 = \sum_{i=1}^{n} |x_i| \geq 0$

and $\|x\|_1 = 0 \iff x_i = 0,\ i = 1, \cdots, n.$

$\iff x = 0.$

② $\forall a \in \mathbb{R}.$

$$\|ax\|_1 = \sum_{i=1}^{n} |ax_i| = |a| \cdot \sum_{i=1}^{n} |x_i|$$

$$= |a| \, \|x\|_1$$

③ $\forall x, y \in \mathbb{R}^n$

$$\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + |y_i|$$

$$= \|x\|_1 + \|y\|_1.$$

(b) ① $\|A\|_{2\to 2} = \max_{\|x\|_2=1} \|Ax\|_2 \geq 0$

$\|A\|_{2\to 2} = 0 \iff \max_{\|x\|_2=1} \|Ax\|_2 = 0$

$\iff A = 0$

② $\forall a \in \mathbb{R}$

$$\|aA\|_{2\to 2} = \max_{\|x\|_2=1} \|aAx\|_2$$

$$= |a| \cdot \max_{\|x\|_2=1} \|Ax\|_2$$

$$= |a| \cdot \|A\|_{2\to 2}.$$

③ $\forall A, B \in \mathbb{R}^{m\times n}.$

$$\|A+B\|_{2\to 2} = \max_{\|x\|_2=1} \|(A+B)x\|_2$$

$$\leq \max_{\|x\|_2=1} \|Ax\|_2 + \|Bx\|_2$$

$$\leq \max_{\|x\|_2=1} \|Ax\|_2 + \max_{\|x\|_2=1} \|Bx\|_2$$

$$= \|A\|_{2\to 2} + \|B\|_{2\to 2}.$$

2. (a) $\|x-y\|+\|y\| \geq \|x\|$, $\|y-x\|+\|x\| \geq \|y\|$ $\Rightarrow$ $\big|\|x\|-\|y\|\big| \leq \|x-y\|$.

(b) $\big|\|x^{(k)}\|-\|x\|\big| \leq \|x^{(k)}-x\| \xrightarrow{k\to+\infty} 0$

$\Rightarrow \lim\limits_{k\to+\infty} \|x^{(k)}\|-\|x\| = 0$

$\Rightarrow \lim\limits_{k\to+\infty} \|x^{(k)}\| = \|x\|$.

(c). $\|x-y\| \leq \|x-x^{(k)}\| + \|x^{(k)}-y\| \xrightarrow{k\to+\infty} 0$

$\Rightarrow \|x-y\|=0$

$\Rightarrow x=y$.

3. (a) $f(b) := (a_1-b)^2 + \cdots + (a_m-b)^2$

Set $f'(b)=0 \Rightarrow b=\frac{1}{m}(a_1+\cdots+a_m)$

$f''(b) \geq 0$ (b is the minimizer)

(b). Since $|x|$ is not differentiable, consider the subgradient instead. $a_{(1)} \geq a_{(2)} \geq \cdots \geq a_{(m)}$.

$g(b) := |a_1-b| + \cdots + |a_m-b|$

$$\partial g(b) = \begin{cases} \left[\sum\limits_{a_i\neq b} \operatorname{sgn}(b-a_i) - \sum\limits_{a_i=b} 1,\ \sum\limits_{a_i\neq b}\operatorname{sgn}(b-a_i)+\sum\limits_{a_i=b} 1\right], & b\in\{a_1,\cdots,a_m\}, \\ \sum\limits_{i=1}^{m} \operatorname{sgn}(b-a_i) & b\notin\{a_1,\cdots,a_m\}. \end{cases}$$

For b to minimize $g(b)$, $0\in\partial g(b)$

If m is odd. $\Rightarrow$ $0\in\partial g(a_{(\frac{m+1}{2})})$

If m is even $\Rightarrow$ $0\in\partial g(a_{(\frac{m}{2})})$, $0\in\partial g(a_{(\frac{m}{2}+1)})$

$\forall a\in[a_{(\frac{m}{2})}, a_{(\frac{m}{2}+1)}]$, $0\in\partial g(a)$.

4. (a) k-means:

$$z_j = \frac{1}{|G_j|}\left(\sum_{x \in G_j} x\right)$$

x has nonnegative entries

$\Downarrow$

z_j has nonnegative entries

k-median:

z_j = median of G_j $\Rightarrow$ z_j has nonnegative entries

(b) k-means:

$$\sum_{l=1}^{n} x_{il} = 1$$

$$\sum_{l=1}^{n} z_{jl} = \frac{1}{|G_j|} \sum_{l=1}^{n} \sum_{i \in G_j} x_{il} = \frac{|G_j|}{|G_j|} = 1.$$

k-medians:

eg: $(0.7, 0.2, 0.1)$
$(0.2 \quad 0.1 \quad 0.7)$
$(0.1 \quad 0.7 \quad 0.2)$

in the same group.

$z = (0.2 \quad 0.2 \quad 0.2)$.

(c) k-means: average

k-medians: majority.