MATH3332 Quiz 3

1. Let V be a Hilbert space. Let S_1 and S_2 be two hyperplanes in V defined by (here we assume $\boldsymbol{a}_1, \boldsymbol{a}_2$ are linear independent)

$$S_1 = \{\boldsymbol{x} \in V \mid \langle \boldsymbol{a}_1, \boldsymbol{x} \rangle = b_1\}, \quad S_2 = \{\boldsymbol{x} \in V \mid \langle \boldsymbol{a}_2, \boldsymbol{x} \rangle = b_2\}$$

Let $\boldsymbol{y} \in V$ be given. We consider the projection of $\boldsymbol{y}$ onto $S_1 \cap S_2$, i.e., the solution of

$$\min_{\boldsymbol{x} \in S_1 \cap S_2} \|\boldsymbol{x} - \boldsymbol{y}\|$$

   (a) *(10 pts)*Prove that $S_1 \cap S_2$ is a plane, i.e., if $\boldsymbol{x}, \boldsymbol{z} \in S_1 \cap S_2$, then $(1+t)\boldsymbol{z} - t\boldsymbol{x} \in S_1 \cap S_2$ for any $t \in \mathbb{R}$

   (b) *(10 pts)*Prove that z is a solution of (1) if and only if $z \in S_1 \cap S_2$ and

$$\langle \boldsymbol{z} - \boldsymbol{y}, \boldsymbol{z} - \boldsymbol{x} \rangle = 0, \quad \forall \boldsymbol{x} \in S_1 \cap S_2$$

   (c) *(15 pts)*Find an explicit solution of (1) .

   (d) *(15 pts)*Prove the solution found in part (c) is unique.

2. *(20 pts)*Let $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^N$ be given with $\boldsymbol{x}_i \in \mathbb{R}^n$ and $y_i \in \mathbb{R}$. Assume $N < n$ and $\boldsymbol{x}_i$ are linearly independent. Give the closed form solution to the ridge regression

$$\min_{\boldsymbol{a} \in \mathbb{R}^n} \sum_{i=1}^{N} (\langle \boldsymbol{a}, \boldsymbol{x}_i \rangle - y_i)^2 + \lambda \|\boldsymbol{a}\|_2^2$$

In other words, suppose we write $\boldsymbol{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N]^T$ and $\boldsymbol{y} = [y_1, \ldots, y_N]^T$, represent $\boldsymbol{a}$ using the matrix $\boldsymbol{X}$ and vector $\boldsymbol{y}$ and $\lambda \boldsymbol{I}$.

3. *(This question might be challenging, write out your ideas and partial scores will be given.)*

   (a) *(15 pts)* Show that there exist a Hilbert space H and a transformation $\Phi : \mathbb{R}^n \to H$ such that

$$\langle \Phi(u), \Phi(v) \rangle = 2\langle u, v \rangle^2 + 5\langle u, v \rangle^3 \quad \text{for all } u, v \in \mathbb{R}^n$$

   (*Hint: Consider* $H = \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n}$)

   (b) *(15 pts)* More generally, consider a polynomial $f : \mathbb{R} \to \mathbb{R}$ with non-negative coefficients, and construct H and Φ such that

$$\langle \Phi(u), \Phi(v) \rangle = f(\langle u, v \rangle) \quad \text{for all } u, v \in \mathbb{R}^n$$