MATH 3332 Data Analytic Tools
Homework 6

Due date: 14 December, 6pm, Monday

1. Find the sub-differentials.

   (a) $g_1(x) = \begin{cases} -x & \text{if } x \le 0, \\ x^2 & \text{if } x > 0, \end{cases}$ where $x \in \mathbb{R}$.

   (b) $g_2(\boldsymbol{x}) = \sqrt{x_1^2 + x_2^2} + \sqrt{x_3^2 + x_4^2}$, where $\boldsymbol{x} \in \mathbb{R}^4$.

   (c) $g_3(\boldsymbol{x}) = \|\boldsymbol{x}\|_\infty$, where $\boldsymbol{x} \in \mathbb{R}^2$.

   (d) $g_4(\boldsymbol{x}) = (|x_1| + |x_2|)^2$, where $\boldsymbol{x} \in \mathbb{R}^2$.

2. We consider $\min_{\boldsymbol{x}} g(\boldsymbol{x})$, where $g : \mathbb{R}^n \to \mathbb{R}$ is a convex function. In the backward sub-gradient algorithm (a.k.a. proximal algorithm), we used the iteration: given $\boldsymbol{x}^{(k)}$, generate

$$\boldsymbol{x}^{(k+1)} = \arg\min_{\boldsymbol{x} \in \mathbb{R}^n} \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}^{(k)}\|_2^2 + \alpha_k g(\boldsymbol{x}).$$

   Prove that $\boldsymbol{x}^{(k+1)}$ is uniquely defined, i.e., prove that the solution of the minimization

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}^{(k)}\|_2^2 + \alpha_k g(\boldsymbol{x})$$

   exists and is unique for any $\boldsymbol{x}^{(k)} \in \mathbb{R}^n$ and $\alpha_k > 0$.

3. Consider the following minimization

$$\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2 + \lambda \sum_{i=1}^{n} r(x_i)$$

   where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$, $\lambda > 0$ are given, and $r : \mathbb{R} \to \mathbb{R}$ is defined by

$$r(t) = \begin{cases} 0, & \text{if } t \le 0, \\ t, & \text{if } t \ge 0. \end{cases}$$

   Find a forward-backward splitting algorithm with explicit formulas for solving this minimization problem. (*Hint: Use a forward step for the smooth convex term* $\frac{1}{2}\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2$ *and a backward step for the non-smooth convex term* $\lambda \sum_{i=1}^{n} r(x_i)$*.*)