Q1.

(a) No. $x = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\alpha = 1$ $\beta = -1$

$f(\alpha x + \beta y) = 2$

$\alpha f(x) + \beta f(y) = 0$

(b) $a = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^n$.

(c) No. $x = (1, 0, 0)^T$

$y = (1, 2, 0)^T$

$f(x+y) = 2$

$f(x) + f(y) = 0 + 1 = 1$

(d) $a = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 2 \end{bmatrix}$

Q2.

(a) $\langle a_1, (1+t)z - tx \rangle = (1+t)b_1 - tb_1 = b_1$

$\langle a_2, (1+t)z - tx \rangle = (1+t)b_2 - tb_1 = b_2$.

$\Rightarrow (1+t)z - tx \in S_1 \cap S_2$.

(b) $\|z - y\|^2 \leq \|(1+t)z - tx - y\|^2$ ($\forall t, \forall x$).

$= t^2\|z - x\|^2 + \|z - y\|^2 + 2t\langle z - x, z - y \rangle$

$\Rightarrow t^2\|z - x\|^2 + 2t\langle z - x, z - y \rangle \geq 0 \quad \forall t, \forall x$.

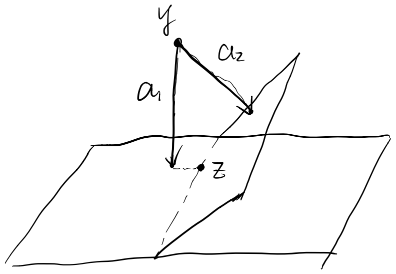
$\Rightarrow \langle z - x, z - y \rangle = 0$

OTOH, $\|x - y\|^2 = \|x - z + z - y\|^2$

$= \|x - z\|^2 + \|z - y\|^2$

$\geq \|z - y\|^2$

(c)



z should be of the form

$$z = y + c_1 a_1 + c_2 a_2.$$

use
$$\begin{cases} \langle a_1, z\rangle = b_1 \\ \langle a_2, z\rangle = b_2 \\ \langle z-x, z-y\rangle = 0 \quad \forall x \end{cases}$$

to solve for c_1, c_2

and we get

$$z = y - \frac{\langle a_1, a_2\rangle(\langle a_2, y\rangle - b_2) - \|a_2\|^2(\langle a_1, y\rangle - b_1)}{\langle a_1, a_2\rangle^2 - \|a_1\|^2\|a_2\|^2} a_1 - \frac{-\langle a_1, a_2\rangle(\langle a_1, y\rangle - b_1) + \|a_1\|^2(\langle a_2, y\rangle - b_2)}{\langle a_1, a_2\rangle^2 - \|a_1\|^2\|a_2\|^2} a_2$$

(d) suppose there are two solution $z_1 \neq z_2$.

then $\forall x \in S_1 \cap S_2$,
$$\langle z_1 - x, z_1 - y\rangle = 0 \quad ①$$
$$\langle z_2 - x, z_2 - y\rangle = 0 \quad ②$$

Take $x = z_2$ in ① and $x = z_1$ in ②

and we get

$$\langle z_1 - z_2, z_1 - y\rangle = 0$$
$$\langle z_2 - z_1, z_2 - y\rangle = 0.$$

$$\Rightarrow \langle z_1 - z_2, z_1 - z_2\rangle = 0$$

$$\Rightarrow z_1 = z_2$$

Q3.

(a)

Pf: For any $a \in \mathbb{R}^n$, we can write

$$a = \sum_{i=1}^{N} c_i x_i + a_s \quad \text{with } \langle a_s, x_i\rangle = 0 \quad i = 1, \cdots, N$$

$$\frac{1}{2}\sum_{i=1}^{N}(\langle a, x_i\rangle - y_i)^2 + \lambda\|a\|^2$$
$$= \frac{1}{2}\sum_{i=1}^{N}\left(\left\langle\sum_{j=1}^{N} c_j x_j + a_s, x_i\right\rangle - y_i\right)^2 + \lambda\left\|a_s + \sum_{i=1}^{N} c_i x_i\right\|^2$$
$$= \frac{1}{2}\sum_{i=1}^{N}\left(\sum_{j=1}^{N} c_j\langle x_j, x_i\rangle - y_i\right)^2 + \lambda\|a_s\|^2 + \lambda\left\|\sum_{i=1}^{N} c_i x_i\right\|^2$$
$$= \frac{1}{2}\sum_{i=1}^{N}\left(\left\langle\sum_{j=1}^{N} c_j x_j, x_i\right\rangle - y_i\right)^2 + \lambda\left\|\sum_{j} c_j x_j\right\|^2 + \underbrace{\lambda\|a_s\|^2}_{\geq 0}$$

$$\Rightarrow a_s = 0$$

(6) $\iff \min_{c \in \mathbb{R}^N} \underbrace{\frac{1}{2} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} c_j \langle x_j, x_i \rangle - y_i \right)^2 + \lambda \left\| \sum_{j=1}^{N} c_j x_j \right\|^2}$

$$\|$$

$$\frac{1}{2} \| X^T X c - y \|^2 + \lambda c^T X^T X c$$

$X = [x_1, \dots, x_N]$