1. (a) $\langle 0, x\rangle = \langle x - x, x\rangle = \langle x, x\rangle - \langle x, x\rangle = 0$

$\langle x, 0\rangle = \langle 0, x\rangle = 0$

(b). "$\Rightarrow$"

take $\alpha = \beta = 1$.

take $\beta = 0$.

"$\Leftarrow$"

$$\langle \alpha x_1 + \beta x_2, y\rangle = \langle \alpha x_1, y\rangle + \langle \beta x_2, y\rangle$$
$$= \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle$$

2. (a) use parallelogram law, we have:

$$\|x+y+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - \|x+z-y\|^2$$
$$= 2\|y+z\|^2 + 2\|x\|^2 - \|y+z-x\|^2$$
$$\|z+x-y\|^2 + \|z-(x-y)\|^2 = 2\|z\|^2 + 2\|x-y\|^2$$

$$f(x+y, z) = \frac{1}{2}\left(\|x+y+z\|^2 - \|x+y\|^2 - \|z\|^2\right)$$
$$= \frac{1}{2}\Big[\frac{1}{2}\left(2\|x+z\|^2 + 2\|y\|^2 - \|x+z-y\|^2\right)$$
$$+ \frac{1}{2}\left(2\|y+z\|^2 + 2\|x\|^2 - \|y+z-x\|^2\right)$$
$$- \|x+y\|^2 - \|z\|^2\Big]$$
$$= \frac{1}{2}\Big[\left(\|x+z\|^2 + \|y+z\|^2 + \|x\|^2 + \|y\|^2\right) - \|z\|^2 - \|x-y\|^2 - \|x+y\|^2 - \|z\|^2\Big]$$
$$= \frac{1}{2}\Big[\left(\|x+z\|^2 + \|y+z\|^2 + \|x\|^2 + \|y\|^2\right) - \|z\|^2 - 2\|x\|^2 - 2\|y\|^2 - \|z\|^2\Big]$$

$$f(x, z) + f(y, z) = \frac{1}{2}\left(\|x+z\|^2 - \|x\|^2 - \|z\|^2\right) + \frac{1}{2}\left(\|y+z\|^2 - \|y\|^2 - \|z\|^2\right)$$

$$\Rightarrow \quad f(x+y, z) = f(x, z) + f(y, z)$$

(b).
$$f(-x, y) = \frac{1}{2}\left(\|-x+y\|^2 - \|-x\|^2 - \|y\|^2\right)$$
$$= \frac{1}{2}\left(2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$$
$$= \frac{1}{2}\left(-\|x+y\|^2 + \|x\|^2 + \|y\|^2\right)$$
$$= -f(x, y).$$

(c). $f(x,y)^2 = \left[\frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2)\right]^2$

$\le \left[\frac{1}{2}\left((\|x\|+\|y\|)^2 - \|x\|^2 - \|y\|^2\right)\right]^2$

$= \left(\frac{1}{2}\, 2\|x\|\cdot\|y\|\right)^2$

$= f(x,x)\cdot f(y,y).$

(d). • $f(x,y) = f(y,x)$ trivial

• $f(x,x) \geq 0$, $f(x,x) = 0 \Leftrightarrow x = 0$.

• $f(x+y, z) = f(x,z) + f(y,z)$ from 2(a)

• It remains to show $\forall\, \alpha \in \mathbb{R}$, $f(\alpha x, y) = \alpha f(x,y)$.

First show this is true for $\alpha \in \mathbb{Z}$

$$f(\alpha x, y) = \alpha f(x,y) \quad \text{use 2(a) and induction.}$$

Then show this is true for $\alpha \in \mathbb{Q}$,

$\alpha = \frac{p}{q}$. $p, q \neq 0 \in \mathbb{Z}$.

$q f(\alpha x, y) = q f(\frac{p}{q} x, y)$

$= pq f(\frac{x}{q}, y)$

$= p f(x,y)$

$\Rightarrow f(\alpha x, y) = \frac{p}{q} f(x,y) = \alpha f(x,y)$.

Now we show this is true for all $\alpha \in \mathbb{R}$. For $\forall\, r \in \mathbb{Q}$

$|f(\alpha x, y) - \alpha f(x,y)| = |f(\alpha x, y) - f(r x, y) + r f(x,y) - \alpha f(x,y)|$

$\le |f((\alpha - r)x, y)| + |r - \alpha|\, |f(x,y)|$

$\le \left(f((\alpha-r)x, (\alpha-r)x)\cdot f(y,y)\right)^{\frac{1}{2}} + |r-\alpha| \left(f(x,x)\, f(y,y)\right)^{\frac{1}{2}}$

$= 2|\alpha - r|\cdot\left(f(x,x)\cdot f(y,y)\right)^{\frac{1}{2}}.$

$\lim_{r\to\alpha} |f(\alpha x, y) - \alpha f(x,y)| \le \lim_{r\to\alpha} 2|\alpha-r|\cdot\left(f(x,x)\cdot f(y,y)\right)^{\frac{1}{2}} = 0$

$\Rightarrow |f(\alpha x, y) - \alpha f(x,y)| = 0$

3.

$$\underset{x \in X}{\operatorname{argmin}} \|y - x\| = \underset{x \in X}{\operatorname{argmin}} \|y - x\|^2$$

$$= \underset{x \in X}{\operatorname{argmin}} \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle$$

$$= \underset{x \in X}{\operatorname{argmin}}\ 2 - 2\langle x, y \rangle$$

$$= \underset{x \in X}{\operatorname{argmax}} \langle x, y \rangle.$$

$$= \underset{x \in X}{\operatorname{argmin}} \arccos \langle x, y \rangle.$$

4.

$$\phi(x) = \begin{bmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2}x_1x_2 \end{bmatrix}.$$