MATH3332 Quiz 2

1. *(30 pts)* Consider the polynomial kernel $K(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^T \boldsymbol{y})^2$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$. Find an explicit feature map $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ satisfying $\langle \phi(\boldsymbol{x}), \phi(\boldsymbol{y}) \rangle = K(\boldsymbol{x}, \boldsymbol{y})$, where the inner product the standard inner product in $\mathbb{R}^3$.

2. Let V be the space of all continuously differentiable real valued functions on $[a, b]$.

   (a) *(20 pts)* Define $\langle f, g \rangle = \int_a^b f(t)g(t)dt + \int_a^b f'(t)g'(t)dt$, show $\langle \cdot, \cdot \rangle$ is an inner product. *(hint: check the conditions in the definition of inner product.)*

   (b) *(20 pts)* Define that $\|f\| = \int_a^b |f(t)|dt + \int_a^b |f'(t)|dt$. Prove that this defines a norm on V.

3. *(30 pts)* Recall we say a function $K(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is symmetric positive semi-definite (SPSD) if

   - $K(\boldsymbol{x}, \boldsymbol{y}) = K(\boldsymbol{y}, \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
   - For any m and vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m \in \mathbb{R}^n$, the matrix

   $$\begin{bmatrix} K(\boldsymbol{y}_1, \boldsymbol{y}_1) & \ldots & K(\boldsymbol{y}_1, \boldsymbol{y}_m) \\ \vdots & \ddots & \\ K(\boldsymbol{y}_m, \boldsymbol{y}_1) & & K(\boldsymbol{y}_m, \boldsymbol{y}_m) \end{bmatrix}$$

   is symmetric positive semi-definite.

   Now let $K_1, \ldots, K_d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be SPSD functions, show $\sum_{i=1}^d K_i$ is also a SPSD function.