MATH 3332 Data Analytic Tools
Homework 2

Due date: 19 October, 6pm, Monday

1. Let V be a vector space, and $\langle\cdot,\cdot\rangle$ be an inner product on V. Use the definition of inner product to prove the following.

   (a) Prove that $\langle \mathbf{0}, \boldsymbol{x}\rangle = \langle \boldsymbol{x}, \mathbf{0}\rangle = 0$ for any $\boldsymbol{x} \in V$. Here $\mathbf{0}$ is the zero vector in V.

   (b) Prove that the second condition

   $$\langle \alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2, \boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle + \beta\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle, \qquad \forall\ \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y} \in V\ , \alpha, \beta \in \mathbb{R}$$

   is equivalent to

   $$\langle \boldsymbol{x}_1 + \boldsymbol{x}_2, \boldsymbol{y}\rangle = \langle \boldsymbol{x}_1, \boldsymbol{y}\rangle + \langle \boldsymbol{x}_2, \boldsymbol{y}\rangle \quad \text{and} \quad \langle \alpha\boldsymbol{x}, \boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x}, \boldsymbol{y}\rangle, \qquad \forall\ \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}, \boldsymbol{y} \in V,\ \alpha \in \mathbb{R}.$$

2. Let V be a vector space with a norm $\|\cdot\|$ that satisfies the parallelogram identity

   $$\|\boldsymbol{x} + \boldsymbol{y}\|^2 + \|\boldsymbol{x} - \boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2, \qquad \forall\ \boldsymbol{x}, \boldsymbol{y} \in V.$$

   Note that we don't have an inner product on V so far. For any $\boldsymbol{x}, \boldsymbol{y} \in V$, define

   $$f(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{2}\left(\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2\right)$$

   Obviously, for all $\boldsymbol{x}, \boldsymbol{y} \in V$, we have $f(\boldsymbol{x}, \boldsymbol{x}) \geq 0$ and $f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x})$. Also, $f(\boldsymbol{x}, \boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$.

   (a) Prove $f(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z}) = f(\boldsymbol{x}, \boldsymbol{z}) + f(\boldsymbol{y}, \boldsymbol{z})$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$.

   (b) Prove $f(-\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{x}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$.

   (c) Prove $(f(\boldsymbol{x}, \boldsymbol{y}))^2 \leq f(\boldsymbol{x}, \boldsymbol{x})f(\boldsymbol{y}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$.

   (d) (*Bonus question!*) Prove that $f(\boldsymbol{x}, \boldsymbol{y})$ is an inner product on V whose induced norm is $\|\cdot\|$. (*Hint: From Q1(b) and Q2(a), it suffices to prove $f(\alpha\boldsymbol{x}, \boldsymbol{y}) = \alpha f(\boldsymbol{x}.\boldsymbol{y})$ for all $\alpha \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in V$. You first prove this identity for rational α, and then use limit and Q1(c) to show it for any real α.*)

   This question proved that the parallelogram identity is also a sufficient condition for a norm to be induced by an inner product. Combined with the parallelogram law on inner product spaces, we see that the parallelogram identity is a necessary and sufficient condition for a norm to be an induced by an inner product.

3. Consider an inner product space V with the induced norm. Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset V$ be a set of vectors in V with $\|\boldsymbol{x}_i\| = 1$ for all i. Given a vector $\boldsymbol{y} \in V$ with $\|\boldsymbol{y}\| = 1$, show that the following two things are the same:

   - finding the vector in X that has the smallest distance to $\boldsymbol{y}$ (i.e., solving $\min_{\boldsymbol{x} \in X} \|\boldsymbol{x} - \boldsymbol{y}\|$)

- finding the vector in X that has the smallest angle to $\boldsymbol{y}$ (i.e., solving $\min_{\boldsymbol{x} \in X} \arccos\langle \boldsymbol{x}, \boldsymbol{y} \rangle$)

4. Consider the polynomial kernel $K(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^T \boldsymbol{y})^2$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$. Find an explicit feature map $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ satisfying $\langle \phi(\boldsymbol{x}), \phi(\boldsymbol{y}) \rangle = K(\boldsymbol{x}, \boldsymbol{y})$, where the inner product the standard inner product in $\mathbb{R}^3$.

5. (*You don't need to do anything for this question.*) A good Matlab code and demonstration of kernel K-means can be found at
`http://www.dcs.gla.ac.uk/~srogers/firstcourseml/matlab/chapter6/kernelkmeans.html`
Read the code. Run the code in Matlab, if possible, to see how kernel K-means works for nonlinear data.