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Q1: $\mathcal{F}(\mathbb{R}, \mathbb{R})$

(1) $f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x)$ Associativity. $f + (g+h) = (f+g) + h.$

(2) $f(x) + g(x) = g(x) + f(x).$ Commutativity.

(3) 0-element: $0(x) : \mathbb{R} \to \mathbb{R}$
$x \mapsto 0$

$$f(x) + 0(x) = 0(x) + f(x) = f(x).$$

$$0(x) + f(x) = 0 + f(x) = f(x).$$

(4) $-f(x) + f(x) = 0$

(5) $\forall f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$, $1_{\mathbb{R}} \in \mathbb{R}.$

$$1_{\mathbb{R}} \cdot f(x) = f(x)$$

(6) $\forall f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$

$$(\alpha\beta) \cdot f(x) = \alpha \cdot (\beta f(x))$$

(7) $\forall f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$

$$(\alpha + \beta) f(x) = \alpha f(x) + \beta f(x).$$

(8) $\forall f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ and $\alpha \in \mathbb{R}$

$$\alpha (f(x) + g(x)) = \alpha f(x) + \alpha g(x).$$

Rk.: The element in $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a function.

$$f \in \mathcal{F}(\mathbb{R}, \mathbb{R}).$$

$$f = g \iff f(x) = g(x), \ \forall x \in \mathbb{R}.$$

Q2:

(5) $1_{\mathbb{R}} \cdot u = u.$

$$1_{\mathbb{R}} \cdot u = (1_{\mathbb{R}} \cdot u_1, 0) = (u_1, 0) \neq (u_1, u_2) = u.$$

R.k.: $V = \mathbb{R}^2$ $\quad u+v = (u_1+v_1, u_2+v_2)$ $\quad k\cdot(u_1, u_2) = (ku_1, ku_2)$

$$(V, +, \cdot) \quad \mathbb{R} \times V \xrightarrow{\cdot} V$$

$$k, v \quad\quad k\cdot v$$

Q3:

$V = \mathbb{R}^2$

$u+v = (u_1 \cdot v_1, u_2 \cdot v_2)$

$ku = (ku_1, ku_2+1)$ $\quad ku = (ku_1, ku_2)$

(1) $u + (v+w) = (u+v)+w = (u_1 v_1 w_1, u_2 v_2 w_2)$. ✓

(2) $u+v = v+u = (u_1 v_1, u_2 v_2)$.

(3) $(1,1)$: zero element wrt. "+" rule defined above.

$$(1,1) + u = (1\cdot u_1, 1\cdot u_2) = u$$

(4) $u \in V$, $u = (1, 0)$. find $-u = (x, y)$

$$u + (-u) = (1\cdot x, 0\cdot y) \overset{\text{Want}}{=} (1, 1).$$

$$0 \cdot y = 1. \text{ no sol.!}$$

If V were a vector space,

it should have the zero element $(1,1)$

Since:

assume the zero element is (x, y),

then $(x, y) + (u_1, u_2) = (u_1, u_2)$

LHS $= (x \cdot u_1, y \cdot u_2)$

RHS $= (u_1, u_2)$

$$\begin{cases} x \cdot u_1 = u_1 \\ y \cdot u_2 = u_2 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$$

Q4:

need to check:

$$\left.\begin{array}{l} \|x\| - \|y\| \le \|x-y\| \\ \|y\| - \|x\| \le \|x-y\| \end{array}\right\} \Rightarrow \big|\|x\| - \|y\|\big| \le \|x-y\|$$

$$\|x-y\| + \|y\| \ge \|x-y+y\| = \|x\|$$

$$\|y-x\| + \|x\| \ge \|y-x+x\| = \|y\|$$

Q5:

- $x = (x_1, \dots, x_n)$

$\|x\|_1 = |x_1| + \dots + |x_n|$

$\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2}$

Want to show

$$\underbrace{|x_1| + \dots + |x_n| \le \sqrt{n}\sqrt{x_1^2 + \dots + x_n^2}}$$

$$\Uparrow \text{ C-S}$$

$$(x_1^2 + \dots + x_n^2)(1 + \dots + 1) \ge (|x_1| + \dots + |x_n|)^2$$

- $\|x\|_\infty = \max_{1\le i\le n} |x_i|$

Want to show:

$$\|x\|_2 = \sqrt{x_1^2 + \dots + x_n^2} \le \sqrt{n \cdot \max_{1\le i\le n} |x_i|^2}$$

$$= \sqrt{n \cdot \|x\|_\infty^2}$$

$$= \sqrt{n} \cdot \|x\|_\infty$$

- $y_k = \dfrac{|x_k|}{\|x\|_q}$

$y_k \le 1 \Rightarrow y_k^p \ge y_k^q$

$\Rightarrow \|y\|_p \ge 1$

$\Rightarrow \|x\|_p \ge \|x\|_q.$

$$\|y\|_p^p = |y_1|^p + \dots + |y_n|^p$$

$$\ge |y_1|^q + \dots + |y_n|^q$$

$$= \frac{|x_1|^q + \dots + |x_n|^q}{\|x\|_q^q}$$

$$= \frac{\|x\|_q^q}{\|x\|_q^q}$$

$$= 1.$$