MATH 3332 Data Analytic Tools

Quiz 1

September 25, 2019

Please follow the exam rules. No cheating will be tolerated!

1. Consider a vector

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ -4 \\ -1 \\ 2 \end{pmatrix}. \tag{1}$$

Compute 1) $\|\boldsymbol{a}\|_1$, 2)$\|\boldsymbol{a}\|_2$, 3) $\|\boldsymbol{a}\|_\infty$.

1) $\|a\|_1 = 3+4+1+2 = 10$

2) $\|a\|_2 = \sqrt{3^2+(-4)^2+(-1)^2+2^2} = \sqrt{30}$

3) $\|a\|_\infty = \max\{3, 4, 1, 2\} = 4$

2. Let $\boldsymbol{x} \in \mathbb{R}^n$. Show:

1) $\|\mathbf{x}\|_\infty \leqslant \|\boldsymbol{x}\|_2 \leqslant \sqrt{n}\|\mathbf{x}\|_\infty$.

2) $\|\boldsymbol{x}\|_\infty \leqslant \|\boldsymbol{x}\|_1 \leqslant n\|\boldsymbol{x}\|_\infty$.

proof: 1) $\|x\|_\infty^2 = \left(\max_i |x_i|\right)^2 = \max_i |x_i|^2 \leq \sum_{i=1}^n |x_i|^2 = \|x\|_2^2 \Rightarrow \|x\|_\infty \leq \|x\|_2$.

$\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^n \max_j |x_j|^2\right)^{1/2} = \left(n \cdot \|x\|_\infty^2\right)^{1/2} = \sqrt{n}\|x\|_\infty$.

$\therefore$ $\|\mathbf{x}\|_\infty \leqslant \|\boldsymbol{x}\|_2 \leqslant \sqrt{n}\|\mathbf{x}\|_\infty$.

2) $\|x\|_\infty = \max_i |x_i| \leq \sum_{i=1}^n |x_i| = \|x\|_1$

$\|x\|_1 = \sum_{i=1}^n |x_i| \leq n \cdot \left(\max_i |x_i|\right) = n\|x\|_\infty$

$\therefore$ $\|\boldsymbol{x}\|_\infty \leqslant \|\boldsymbol{x}\|_1 \leqslant n\|\boldsymbol{x}\|_\infty$. #

3. Given $\boldsymbol{x} \in \mathbb{R}^n$ and $p \geqslant 1$, $p \in \mathbb{R}$.

   1) What is the formula of the vector p-norm of $\boldsymbol{x}$?

   2) Prove that

$$\|\boldsymbol{x}\|_\infty = \max_{i=1}^{n} |x_i|, \tag{2}$$

   where x_i is the i-th element of vector $\boldsymbol{x}$, and $|\cdot|$ means the absolute value.

   3) Show that $\|\boldsymbol{x}\|_\infty$ is indeed a norm.

1) $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

2) Method 1

Let $M = \max_{i=1}^{n} |x_i|$, then $0 \leq \frac{|x_i|}{M} \leq 1$, for $i = 1, \cdots, n$.

$$\|x\|_\infty = \lim_{p\to\infty} \|x\|_p = \lim_{p\to\infty} \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

$$= \lim_{p\to\infty} \left(M^p \sum_{i=1}^{n} \left(\frac{|x_i|}{M}\right)^p\right)^{1/p}$$

$$\leq \lim_{p\to\infty} M n^{\frac{1}{p}}$$

$$= M \lim_{p\to\infty} n^{\frac{1}{p}} = M.$$

Method 2.

Let $M = \max_{i=1}^{n} |x_i|$, then $0 \leq \frac{|x_i|}{M} \leq 1$, for $i = 1, \cdots, n$.

$$M^p \leq \|x\|_p^p = \sum_{i=1}^{n} |x_i|^p \leq n M^p$$

$$\therefore \quad M \leq \|x\|_p \leq n^{\frac{1}{p}} M.$$

$$\therefore \quad \lim_{p\to\infty} M \leq \lim_{p\to\infty} \|x\|_p \leq \lim_{p\to\infty} n^{\frac{1}{p}} M.$$

i.e. $\quad M \leq \|x\|_\infty \leq M$

$\therefore \|x\|_\infty = M$ by Sandwich Theorem #