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1. Kernel function $K(x,y)$ satisfies $k(x,y) = \langle \phi(x), \phi(y) \rangle$

when $K(x,y) = (x^T y)^2$

let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$K(x,y) = (x_1 y_1 + x_2 y_2)^2$$

$$K(x,y) = x_1^2 y_1^2 + 2 x_1 y_1 x_2 y_2 + x_2^2 y_2^2$$

So, we can observe that $K(x,y)$ is inner product from following vectors:-

$$K(x,y) = \langle (x_1^2, \sqrt{2} x_1 x_2, x_2^2), (y_1^2, \sqrt{2} y_1 y_2, y_2^2) \rangle$$

So, $\phi(x) = \begin{pmatrix} x_1^2 \\ \sqrt{2} x_1 x_2 \\ x_2^2 \end{pmatrix}$ and it is similar for $\phi(y)$

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2.a). $\langle f,g\rangle = \int_a^b f(t)g(t)\,dt + \int_a^b f'(t)\,g'(t)\,dt$

Let $k \in V$,

$\langle k,k\rangle = \int_a^b k(t)k(t)\,dt + \int_a^b k'(t)\,k'(t)\,dt$

$\langle k,k\rangle = \int_a^b k(t)^2\,dt + \int_a^b [k'(t)]^2\,dt$

$\langle k,k\rangle \geq 0$

and $\langle k,k\rangle = 0 \iff k(t)^2 = 0$ and $k'(t)^2 = 0$

$k(t) = 0$ and $k'(t) = 0$.

$\langle \alpha f_1 + \beta f_2, g\rangle = \int_a^b [\alpha f_1(t) + \beta f_2(t)]\,g(t)\,dt + \int_a^b [\alpha f_1'(t) + \beta f_2'(t)]\,g'(t)\,dt$

$= \alpha\int_a^b f_1(t)g(t)\,dt + \beta\int_a^b f_2(t)\,g(t)\,dt +$
$\alpha\int_a^b f_1'(t)\,g'(t)\,dt + \beta\int_a^b f_2'(t)\,g'(t)\,dt$

$= \alpha\langle f_1,g\rangle + \beta\langle f_2,g\rangle.$

$\langle g,f\rangle = \int_a^b g(t)f(t)\,dt + \int_a^b g'(t)\,f'(t)\,dt$

$= \int_a^b f(t)g(t)\,dt + \int_a^b f'(t)g'(t)\,dt$

$= \langle f,g\rangle$

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2b). $\|f\| = \int_a^b |f(t)|dt + \int_a^b |f'(t)|dt \geq 0$

① since $|\cdot| \geq 0$. and $\int_a^b |\cdot| d\cdot \geq 0$. $\forall a.b \in \mathbb{R}$

$\|f\| = 0 \iff f(t) = 0$ and $f'(t) = 0$

②

$\|\alpha f\| = \int_a^b |\alpha f(t)|dt + \int_a^b |\alpha f'(t)|dt$

$= |\alpha| \int_a^b |f(t)|dt + |\alpha| \int_a^b |f'(t)|dt$

$= |\alpha| \left(\int_a^b |f(t)|dt + \int_a^b |f'(t)|dt \right)$

$= |\alpha| \|f\|$

③ $\|f+g\| = \int_a^b |f(t)+g(t)|dt + \int_a^b |f'(t)+g'(t)|dt$

$= \int_a^b |f(t)|dt + \int_a^b |g(t)|dt + \int_a^b |f'(t)|dt + \int_a^b |g'(t)|dt$

$= \|f\| + \|g\|$

$\therefore \|f\|$ is a norm on V.

$\leq \int_a^b |f(t)| + |g(t)|dt + \int_a^b |f'(t)| + |g'(t)| dt$

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Q3. 1. Showing $\sum_{i=1}^{d} k_i(x,y) = \sum_{i=1}^{d} k_i(y,x)$

① Let $\sum_{i=1}^{d} k_i(x,y) = (k_1 + k_2 + \cdots + k_d)(x,y)$

$= k_1(x,y) + k_2(x,y) + \cdots + k_d(x,y)$

$= k_1(y,x) + k_2(y,x) + \cdots + k_d(y,x)$

$= (k_1 + k_2 + \cdots + k_d)(y,x)$

$= \sum_{i=1}^{d} k_i(y,x)$

② in ① condition, we have

$\sum_{i=1}^{d} k_i(y_1, y_1) = \sum_{i=1}^{d} k_i(y_1, y_1)$

$\vdots$

$\sum_{i=1}^{d} k_i(y_1, y_m) = \sum_{i=1}^{d} k_i(y_m, y_1)$

$\vdots$

$\sum_{i=1}^{d} k_i(y_m, y_m) = \sum_{i=1}^{d} k_i(y_m, y_m)$

So ~~the~~ we proved that the matrix

$$M = \begin{bmatrix} \sum_{i=1}^{d} k_i(y_1, y_1) & \cdots & \sum_{i=1}^{d} k_i(y_1, y_m) \\ \vdots & & \\ \sum_{i=1}^{d} k_i(y_m, y_1) & \cdots & \sum_{i=1}^{d} k_i(y_m, y_m) \end{bmatrix}$$

is SPSD,

since $C^T K C \geq 0$ and $C^T M C \geq 0$ also.

~~(M is just linear combination of K).~~

$\because C^T K_1 C \geq 0, C^T K_2 C \geq 0 \cdots C^T K_m C \geq 0.$

$\therefore \sum_{i=1}^{d} k_i(x, y)$ is a SPSD function.