HW2

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HW2

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MATH 3332 Data Analytic Tools
Homework 2

Due date: 19 October, 6pm, Monday

1. Let V be a vector space, and $\langle \cdot, \cdot \rangle$ be an inner product on V. Use the definition of inner product to prove the following.

   (a) Prove that $\langle \mathbf{0}, \boldsymbol{x} \rangle = \langle \boldsymbol{x}, \mathbf{0} \rangle = 0$ for any $\boldsymbol{x} \in V$. Here $\mathbf{0}$ is the zero vector in V.

   (b) Prove that the second condition

   $$\langle \alpha \boldsymbol{x}_1 + \beta \boldsymbol{x}_2, \boldsymbol{y} \rangle = \alpha \langle \boldsymbol{x}_1, \boldsymbol{y} \rangle + \beta \langle \boldsymbol{x}_2, \boldsymbol{y} \rangle, \qquad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y} \in V\ , \alpha, \beta \in \mathbb{R}$$

   is equivalent to

   $$\langle \boldsymbol{x}_1 + \boldsymbol{x}_2, \boldsymbol{y} \rangle = \langle \boldsymbol{x}_1, \boldsymbol{y} \rangle + \langle \boldsymbol{x}_2, \boldsymbol{y} \rangle \quad \text{and} \quad \langle \alpha \boldsymbol{x}, \boldsymbol{y} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle, \qquad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}, \boldsymbol{y} \in V,\ \alpha \in \mathbb{R}.$$

2. Let V be a vector space with a norm $\|\cdot\|$ that satisfies the parallelogram identity

   $$\|\boldsymbol{x} + \boldsymbol{y}\|^2 + \|\boldsymbol{x} - \boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2, \qquad \forall\, \boldsymbol{x}, \boldsymbol{y} \in V.$$

   Note that we don't have an inner product on V so far. For any $\boldsymbol{x}, \boldsymbol{y} \in V$, define

   $$f(\boldsymbol{x}, \boldsymbol{y}) := \frac{1}{2}\left(\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2\right)$$

   Obviously, for all $\boldsymbol{x}, \boldsymbol{y} \in V$, we have $f(\boldsymbol{x}, \boldsymbol{x}) \geq 0$ and $f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}, \boldsymbol{x})$. Also, $f(\boldsymbol{x}, \boldsymbol{x}) = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$.

   (a) Prove $f(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z}) = f(\boldsymbol{x}, \boldsymbol{z}) + f(\boldsymbol{y}, \boldsymbol{z})$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$.

   (b) Prove $f(-\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{x}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$.

   (c) Prove $(f(\boldsymbol{x}, \boldsymbol{y}))^2 \leq f(\boldsymbol{x}, \boldsymbol{x}) f(\boldsymbol{y}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$.

   (d) (*Bonus question!*) Prove that $f(\boldsymbol{x}, \boldsymbol{y})$ is an inner product on V whose induced norm is $\|\cdot\|$. (*Hint: From Q1(b) and Q2(a), it suffices to prove $f(\alpha \boldsymbol{x}, \boldsymbol{y}) = \alpha f(\boldsymbol{x}.\boldsymbol{y})$ for all $\alpha \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in V$. You first prove this identity for rational α, and then use limit and Q1(c) to show it for any real α.*)

   This question proved that the parallelogram identity is also a sufficient condition for a norm to be induced by an inner product. Combined with the parallelogram law on inner product spaces, we see that the parallelogram identity is a necessary and sufficient condition for a norm to be an induced by an inner product.

3. Consider an inner product space V with the induced norm. Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset V$ be a set of vectors in V with $\|\boldsymbol{x}_i\| = 1$ for all i. Given a vector $\boldsymbol{y} \in V$ with $\|\boldsymbol{y}\| = 1$, show that the following two things are the same:

   - finding the vector in X that has the smallest distance to $\boldsymbol{y}$ (i.e., solving $\min_{\boldsymbol{x} \in X} \|\boldsymbol{x} - \boldsymbol{y}\|$)

- finding the vector in X that has the smallest angle to $\boldsymbol{y}$ (i.e., solving $\min_{\boldsymbol{x}\in X} \arccos\langle \boldsymbol{x}, \boldsymbol{y}\rangle$)

4. Consider the polynomial kernel $K(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^T\boldsymbol{y})^2$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$. Find an explicit feature map $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ satisfying $\langle \phi(\boldsymbol{x}), \phi(\boldsymbol{y})\rangle = K(\boldsymbol{x}, \boldsymbol{y})$, where the inner product the standard inner product in $\mathbb{R}^3$.

5. (*You don't need to do anything for this question.*) A good Matlab code and demonstration of kernel K-means can be found at
`http://www.dcs.gla.ac.uk/~srogers/firstcourseml/matlab/chapter6/kernelkmeans.html`
Read the code. Run the code in Matlab, if possible, to see how kernel K-means works for nonlinear data.

1. Let V be a vector space, and $\langle \cdot, \cdot \rangle$ be an inner product on V. Use the definition of inner product to prove the following.

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$$\langle \alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2, \boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle + \beta\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle, \qquad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y} \in V\ , \alpha, \beta \in \mathbb{R}$$

   is equivalent to

$$\langle \boldsymbol{x}_1 + \boldsymbol{x}_2, \boldsymbol{y}\rangle = \langle \boldsymbol{x}_1, \boldsymbol{y}\rangle + \langle \boldsymbol{x}_2, \boldsymbol{y}\rangle \quad \text{and} \quad \langle \alpha\boldsymbol{x}, \boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x}, \boldsymbol{y}\rangle, \qquad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}, \boldsymbol{y} \in V,\ \alpha \in \mathbb{R}.$$

1a). let $u \in V$, and $-u \in V$

$\Rightarrow \langle 0, x\rangle = \langle u - u, x\rangle$

$= \langle u, x\rangle + \langle -u, x\rangle$ (By linearity of inner product)

$= \langle u, x\rangle - \langle u, x\rangle$

$= 0$

$\therefore \langle 0, x\rangle = 0$

$\Rightarrow \langle 0, x\rangle = \langle x, 0\rangle = 0$ (commutivity of inner product)

1b). (b) Prove that the second condition

$$\langle \alpha\boldsymbol{x}_1 + \beta\boldsymbol{x}_2, \boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x}_1, \boldsymbol{y}\rangle + \beta\langle \boldsymbol{x}_2, \boldsymbol{y}\rangle, \qquad \forall\, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y} \in V\ , \alpha, \beta \in \mathbb{R}$$

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given $\langle \alpha x_1 + \beta x_2, y\rangle = \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle$ - ①, $\forall x_1, x_2, y \in V, \alpha, \beta \in \mathbb{R}$

putting $\alpha = 1$, $\beta = 1$ in ①, we have

$$\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle \,,,$$

putting $\beta = 0$ in ①, we have

$$\langle \alpha x_1, y\rangle = \alpha\langle x_1, y\rangle$$

Conversely,

given $\begin{cases} \langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle & - ② \\ \langle \alpha x_1, y\rangle = \alpha\langle x_1, y\rangle & - ③ \end{cases}$

$\forall x_1, x_2, y, x \in V, \alpha \in \mathbb{R}$

Let $\alpha, \beta \in \mathbb{R}$, $x_1, x_2 \in V$

$\Rightarrow$, $\beta x_2 \in V$

Let $\alpha, \beta \in \mathbb{R}$, $x_1, \ldots$

$\therefore \alpha x_1, \beta x_2 \in V$

So, replacing x_1 by αx_1 and x_2 by βx_2 in ②

$$\langle \alpha x_1 + \beta x_2, y \rangle = \langle \alpha x_1, y \rangle + \langle \beta x_2, y \rangle$$

Using ③, we have

$$\langle \alpha x_1 + \beta x_2, y \rangle = \alpha \langle x_1, y \rangle + \beta \langle x_2, y \rangle$$

Hence, both statements are equivalent.

Q2.

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(a) Prove $f(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z}) = f(\boldsymbol{x}, \boldsymbol{z}) + f(\boldsymbol{y}, \boldsymbol{z})$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$.

(b) Prove $f(-\boldsymbol{x}, \boldsymbol{y}) = -f(\boldsymbol{x}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$.

(c) Prove $(f(\boldsymbol{x}, \boldsymbol{y}))^2 \leq f(\boldsymbol{x}, \boldsymbol{x}) f(\boldsymbol{y}, \boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in V$.

(d) (*Bonus question!*) Prove that $f(\boldsymbol{x}, \boldsymbol{y})$ is an inner product on V whose induced norm is $\|\cdot\|$. (*Hint: From Q1(b) and Q2(a), it suffices to prove $f(\alpha\boldsymbol{x}, \boldsymbol{y}) = \alpha f(\boldsymbol{x}.\boldsymbol{y})$ for all $\alpha \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in V$. You first prove this identity for rational α, and then use limit and Q1(c) to show it for any real α.*)

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given $f(x,y) = \frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2)$

By parallelogram inequality, we have:

$$f(x,y) = \frac{1}{2}(2\|x\|^2 + 2\|y\|^2 - \|x-y\|^2 - \|x\|^2 - \|y\|^2)$$

$$f(x,y) = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x-y\|^2) \quad -①$$

a). Let $x, y, z \in V$,

$$f(x+y, z) = \frac{1}{2}(\|(x+y)+z\|^2 - \|x+y\|^2 - \|z\|^2)$$

$$f(x,z) = \frac{1}{2}(\|x+z\|^2 - \|x\|^2 - \|z\|^2)$$

$$f(y,z) = \frac{1}{2}(\|y+z\|^2 - \|y\|^2 - \|z\|^2)$$

$$f(y,z) = \frac{1}{2}(\|y\|^2 + \|z\|^2 - \|y-z\|^2) \quad (\text{By } ①)$$

$f(y,z) = \frac{1}{2}(\|y\|^2 + \|z\|^2 - \|y-z\|)$

For $f(x,z) + f(y,z)$,

$f(x,z) + f(y,z) = \frac{1}{2}[\|x+z\|^2 - \|x\|^2 - \|z\|^2 + \|y\|^2 + \|z\|^2 - \|y-z\|^2]$

$= \frac{1}{2}[(\|x+z\|^2 + \|y\|^2) - (\|y-z\|^2 + \|x\|^2)]$

$f(x,z) + f(y,z)$

$f(x,z) + f(y,z) = \frac{1}{2}[\frac{1}{2}(\|(x+z)+y\|^2 + \|(x+z)-y\|^2) - \frac{1}{2}(\|(y-z)+x\|^2 + \|(y-z)-x\|^2)]$

$f(x,z) + f(y,z) = \frac{1}{4}[\|x+y+z\|^2 + \|x+z-y\|^2 - \|x+y-z\|^2 - \|y-z-x\|^2]$

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$f(x,z) + f(y,z) = \frac{1}{4}[\|x+y+z\|^2 - \|x+y-z\|^2]$

$f(x,z) + f(y,z) = \frac{1}{4}[2\|x+y+z\|^2 - 2(\|x+y\|^2 + \|z\|^2)]$

$f(x,z) + f(y,z) = \frac{1}{2}[\|(x+y)+z\|^2 - \|x+y\|^2 - \|z\|^2]$

$f(x,z) + f(y,z) = f(x+y,z)$ //

b). $f(-x,y) = \frac{1}{2}(\|-x+y\|^2 - \|-x\|^2 - \|y\|^2)$

$= \frac{1}{2}(\|y-x\|^2 - \|x\|^2 - \|y\|^2)$

$= \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|y+x\|^2)$ (By parallelogram equality)

$= -\frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2)$

$= -f(x,y)$ //

c). $f(x,x) = \frac{1}{2}(\|2x\|^2 - \|x\|^2 - \|x\|^2)$

$= \frac{1}{2}(4\|x\|^2 - 2\|x\|^2)$

$= \|x\|^2$

$f(y,y) = \frac{1}{2}(\|2y\|^2 - \|y\|^2 - \|y\|^2)$

$= \frac{1}{2}(4\|y\|^2 - 2\|y\|^2)$

$= \|y\|^2$

$< f(x,y)^2$,

For $f(x,y)^2$,

$$f(x,y)^2 = \frac{1}{4}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)^2$$

$$f(x,y)^2 \le \frac{1}{4}\left((\|x\|+\|y\|)^2 - \|x\|^2 - \|y\|^2\right)^2$$

(By triangle inequality)

$$= \frac{1}{4}\left(\|x\|^2 + 2\|x\|\|y\| + \|y\|^2 - \|x\|^2 - \|y\|^2\right)^2$$

$$= \frac{1}{4}\left(4\|x\|^2\|y\|^2\right)$$

$$= \|x\|^2\|y\|^2$$

$$= f(x,x)\,f(y,y)$$

$\therefore\ (f(x,y))^2 \le f(x,x)\,f(y,y)\ .\ \forall\, x,y \in V$ //

d). (d) (*Bonus question!*) Prove that $f(\boldsymbol{x},\boldsymbol{y})$ is an inner product on V whose induced norm is $\|\cdot\|$. (*Hint: From Q1(b) and Q2(a), it suffices to prove $f(\alpha\boldsymbol{x},\boldsymbol{y}) = \alpha f(\boldsymbol{x}.\boldsymbol{y})$ for all $\alpha \in \mathbb{R}$ and $\boldsymbol{x},\boldsymbol{y} \in V$. You first prove this identity for rational α, and then use limit and Q1(c) to show it for any real α.*)

Showing $f(ax,y) = \alpha f(x,y)$. $\forall\, \alpha \in \mathbb{R}$,

from (a), we have

$$f(x+y, z) = f(x,z) + f(y,z) \qquad \forall\, x,y,z \in V$$

Let $n \in \mathbb{N}$, from (a), we have

$$\therefore\ f(n\cdot x, y) = f((n-1)x, y) + f(x,y)$$

$$f(nx\cdot y) = f((n-2)x\cdot y) + 2f(x,y)$$

$$\vdots$$

$$f(nx,y) = nf(x,y), \quad \forall n \in \mathbb{N},\ \forall x,y \in V \quad -(2)$$

Let $m \in \mathbb{N}$,

Then $mf(\frac{1}{m}\cdot x, y) = f((m\cdot\frac{1}{m})x.y)$ (from (1))

$$mf(\tfrac{1}{m}x, y) = f(x,y)$$

$$mf(\tfrac{1}{m}x, y) = f(x,y)$$

$$\Rightarrow f(\tfrac{1}{m}x, y) = \frac{f(x,y)}{m} \quad -(3)$$

from (1) & (3), we have:

$$f(\tfrac{n}{m}x, y) = \tfrac{n}{m}f(x,y) \quad \forall m,n \in \mathbb{N} \ \& \ \forall x,y \in V.$$

from (b), we have

$$f(-\tfrac{n}{m}x, y) = -\tfrac{n}{m}f(x,y)$$

$$\therefore \quad f(qx,y) = q f(x,y) \quad -(4) \quad \forall q \in \mathbb{Q}: \{\text{set of rationals}\}, \ \forall x,y \in V$$

Then, let $\alpha \in \mathbb{R}$, $\because$ $\mathbb{Q}$ is dense in $\mathbb{R}$

$\therefore$ $\exists$ seq^n in $\mathbb{Q}$. say $\{q_n\}_n$, s.t. $\lim_{n\to\infty} q_n \to \alpha$.

Then, for $f(q_n x, y)$,

$$f(q_n x, y) = q_n f(x,y) \quad (\text{from } (4)) \quad \forall n.$$

$$\lim_{n\to\infty} f(q_n x, y) = \lim_{n\to\infty} q_n f(x,y)$$

$\because f: \mathbb{R}^n \to \mathbb{R}$ is a continuous function,

$$\therefore f(\lim_{n\to\infty} q_n x, y) = \alpha f(x,y)$$

$$f(\alpha x, y) = \alpha f(x,y) \ //$$

3. Consider an inner product space V with the induced norm. Let $X = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset V$ be a set of vectors in V with $\|\boldsymbol{x}_i\| = 1$ for all i. Given a vector $\boldsymbol{y} \in V$ with $\|\boldsymbol{y}\| = 1$, show that the following two things are the same:

3.

- finding the vector in X that has the smallest distance to $\boldsymbol{y}$ (i.e., solving $\min_{\boldsymbol{x}\in X} \|\boldsymbol{x} - \boldsymbol{y}\|$)
- finding the vector in X that has the smallest angle to $\boldsymbol{y}$ (i.e., solving $\min_{\boldsymbol{x}\in X} \arccos\langle \boldsymbol{x}, \boldsymbol{y}\rangle$)

Let $X: \{x_1, \ldots, x_N\} \subset V$ and given vector $x \in X$ and $y \in V$ s.t. $\|y\| = 1$, $\|x\| = 1$

$$\|x - y\| = \langle x-y, x-y\rangle^{\frac{1}{2}}$$

($\because$ min $\|x-y\|$ = solving $\min_{x\in X} \|x-y\|^2$)

$$\|x-y\| = \langle x-y, x-y\rangle^{\frac{1}{2}}$$

$$\|x-y\|^2 = \langle x-y, x-y\rangle \quad (\because \text{Solving } \min_{x\in X}\|x-y\| = \text{Solving } \min_{x\in X}\|x-y\|^2)$$

$$\|x-y\|^2 = \langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle$$

$$\|x-y\|^2 = \|x\| - 2\langle x,y\rangle + \|y\|$$

$$\|x-y\|^2 = 2 - 2\langle x,y\rangle$$

$\therefore$ Solving $\min_{x\in X}\|x-y\|^2$ = Solving $\max_{x\in X}\langle x,y\rangle$

Let $A(x,y)$ be the angle of x to y, $x\in X$, $y\in V$.

$\because \|x\| = \|y\| = 1$,

$\therefore \langle x,y\rangle = \cos(A(x,y))$

$$\min_{x\in X}\|x-y\| = \min_{x\in X}\left(\sum_{i=1}^{N}(x_i-y)^2\right)^{\frac{1}{2}}$$

$$= \min_{x\in X}\left[\sum_{i=1}^{N}(x_i^2 - 2x_iy + y^2)\right]^{\frac{1}{2}}$$

$$= \min_{x\in X}\sum_{i=1}^{N}(x_i^2 - 2x_iy + y^2)$$

$$= \min_{x\in X}\sum_{i=1}^{N} 2 - 2x_iy$$

$$= \max_{x\in X}\sum_{i=1}^{N} x_iy$$

$$= \max_{x\in X}\langle x_i, y\rangle$$

$$= \max_{x\in X}\cos(A(x,y))$$

$$= \min_{x\in X} A(x,y)$$

($\because \cos$ is decreasing for $A(x,y)\in[0,\pi]$)

($\because \cos(A(x,y))$ is decreasing for $A(x,y) \in [0,\pi]$)

$= \min_{x \in X} \arccos(A(x,y))$ for $A(x,y) \in [0,\pi]$

$\Rightarrow$ Finding the vector in X that has the smallest distance to y is equivalent to:
finding the vector in X that has the smallest angle to y.

4.

4. Consider the polynomial kernel $K(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}^T \boldsymbol{y})^2$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$. Find an explicit feature map $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ satisfying $\langle \phi(\boldsymbol{x}), \phi(\boldsymbol{y}) \rangle = K(\boldsymbol{x}, \boldsymbol{y})$, where the inner product the standard inner product in $\mathbb{R}^3$.

Kernal function $K(x,y)$ satisfies $k(x,y) = \langle \phi(x), \phi(y) \rangle$.

$K(x,y) = (x^T y)^2$

$K(x,y) = (x_1 y_1 + x_2 y_2)^2$

$K(x,y) = (x_1^2 y_1^2 + 2 x_1 x_2 y_1 y_2 + x_2^2 y_2^2)$

So, it is inner product with two vector:

$(x_1^2, \sqrt{2} x_1 x_2, x_2^2)$ and $(y_1^2, \sqrt{2} y_1 y_2, y_2^2)$

$\therefore \phi(x) = \begin{pmatrix} x_1^2 \\ \sqrt{2} x_1 x_2 \\ x_2^2 \end{pmatrix}$ and similar for y.