MATH3332 Quiz 4

The deadline for SUBMISSION is 8PM.

1. Find the gradient of the following functions $f : \mathbb{R}^n \to \mathbb{R}$, where $\mathbb{R}^n$ is equipped with the standard inner product and norm ($\langle \cdot, \cdot \rangle$ and $\| \cdot \|_2$).

   You can use the result from the assignment directly.

   (a) *(15pts)* $f(\boldsymbol{x}) = \frac{1}{\|2\boldsymbol{x}-\boldsymbol{a}\|_2}$ for all $\boldsymbol{x} \neq \frac{\boldsymbol{a}}{2}$.

   (b) *(15pts)* Let $g : \mathbb{R}^m \to \mathbb{R}$ be the function such that $g(\boldsymbol{y}) = \sum_{i=1}^m \sqrt{y_i^2 + c}$ for some constant $c > 0$ for all $\boldsymbol{y} \in \mathbb{R}^m$. Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ be a matrix. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined

   $$f(\boldsymbol{x}) = g(\boldsymbol{A}\boldsymbol{x}), \forall \boldsymbol{x} \in \mathbb{R}^n$$

   Calculate $\nabla f(\boldsymbol{x})$.

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined such that $f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{x}$, where $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Define the *conjugate* of f to be:

   $$f^*(\boldsymbol{y}) = \sup_{\boldsymbol{x}\in\mathbb{R}^n} [\boldsymbol{y}^T\boldsymbol{x} - f(\boldsymbol{x})]$$

   (a) *(20pts)* Give the explicit expression for $f^*(\boldsymbol{y})$.

   (b) *(20pts)* Calculate the conjugate of $f^*(\boldsymbol{y})$, i.e., $(f^*)^*(\boldsymbol{z})$.

3. *(30pts)* Define the *convex envelope* of a function $g : D \subset \mathbb{R}^n \to \mathbb{R}$ to be the largest convex function $h : D \subset \mathbb{R}^n \to \mathbb{R}$ such that $h(\boldsymbol{x}) \leq g(\boldsymbol{x}), \forall \boldsymbol{x} \in D$.

   Consider

   $$g(\boldsymbol{x}) = \|\boldsymbol{x}\|_0 = |\{i \in \{1, 2, \dots, n\} : \boldsymbol{x}_i \neq 0\}|$$

   on the range $D = [-1, 1]^n$. Here $|A|$ is the cardinality of the set A.

   The convex envelope $h(\boldsymbol{z})$ of a function g can be computed using the following: $h = (g^*)^*$, where the function g^* is the conjugate defined in question 2, i.e.,

   $$g^*(\boldsymbol{y}) = \sup_{\boldsymbol{x}\in D}[\boldsymbol{y}^T\boldsymbol{x} - g(\boldsymbol{x})], \forall \boldsymbol{y} \in \mathbb{R}^n$$

   and

   $$h(\boldsymbol{z}) = (g^*)^*(\boldsymbol{z}) = \sup_{\boldsymbol{y}\in\mathbb{R}^n} [\boldsymbol{z}^T\boldsymbol{y} - g^*(\boldsymbol{y})], \forall \boldsymbol{z} \in D$$

   Show that the convex envelope of $g(\boldsymbol{x})$ is $h(\boldsymbol{z}) = \|\boldsymbol{z}\|_1 = \sum_{i=1}^n |\boldsymbol{z}_i|$ by finishing the following proof:

   Proof. First calculate $g^*(\boldsymbol{y})$. From the definition, we know that $\forall \boldsymbol{y} \in \mathbb{R}^n$:

   $$\begin{aligned} g^*(\boldsymbol{y}) &= \sup_{\boldsymbol{x}\in D}[\boldsymbol{y}^T\boldsymbol{x} - g(\boldsymbol{x})] \\ &= \sup_{\boldsymbol{x}\in D}[\sum_{i=1}^n x_i y_i - \|\boldsymbol{x}\|_0] \\ &= \sup_{\boldsymbol{x}\in D}[(x_i y_i - I\{x_i \neq 0\})] \\ &= \sum_{i=1}^n \sup_{x_i\in[-1,1]} (x_i y_i - I\{x_i \neq 0\}) \end{aligned}$$

Here the function $I\{x_i \neq 0\} = 0$ if $x_i = 0$ and $I\{x_i \neq 0\} = 1$ if $x_i \neq 0$.

Now consider each summand $\sup_{x_i \in [-1,1]}(x_i y_i - I\{x_i \neq 0\})$ for different choice of y_i and you can get the expression for $g^*(\boldsymbol{y}), \forall \boldsymbol{y} \in \mathbb{R}^n$. And $(g^*)^*(\boldsymbol{z})$ can be computed in a similar way (i.e., consider it as the sum of n components).

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