MATH 3332 Data Analytic Tools
Homework 4

Due date: 16 November, 6pm, Monday

1. Find the gradient of the following functions $f : V \mapsto \mathbb{R}$, where V is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$.
   (a) $f(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}\|$ for a given $\boldsymbol{a} \in V$, and $\boldsymbol{x} \neq \boldsymbol{a}$.
   (b) $f(\boldsymbol{x}) = \|2\boldsymbol{x} - \boldsymbol{a}\|^2$ for a given $\boldsymbol{a} \in V$.
   (c) $f(\boldsymbol{x}) = \frac{1}{\|\boldsymbol{x}\|}$, where $\boldsymbol{x} \neq \boldsymbol{0}$.
   (d) $f(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2 + \lambda\|\boldsymbol{B}\boldsymbol{x}\|^2$, where $\lambda > 0$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{B} \in \mathbb{R}^{p\times n}$.
   (e) $V = \mathbb{R}^n$ and $f(\boldsymbol{x}) = \sum_{i=1}^n \sqrt{x_i^2 + c}$ for some $c > 0$.
   (f) Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable everywhere. Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ be a matrix. Let $g : \mathbb{R}^m \to \mathbb{R}$ be defined by
   $$g(\boldsymbol{x}) = f(\boldsymbol{A}\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^n.$$
   Express $\nabla g(\boldsymbol{x})$ in terms of $\boldsymbol{A}$ and gradient of f.

2. Determine whether or not the following functions are convex.
   (a) $f(x) = e^x - 1$ on $\mathbb{R}$.
   (b) $f(\boldsymbol{x}) = \sum_{i=1}^n \sqrt{|x_i|}$ on $\mathbb{R}^n$.

3. Let f_1 and f_2 be two convex functions from $\mathbb{R}^n$ to $\mathbb{R}$.
   (a) Let
   $$f(\boldsymbol{x}) = f_1(\boldsymbol{x}) + f_2(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$
   Prove that f is convex.
   (b) Let
   $$g(\boldsymbol{x}) = f_1(x) - f_2(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$
   Give an example of f_1 and f_2 such that g is not convex.
   (c) Let
   $$h(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}, \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$
   Prove that h is convex.
   (d) Let
   $$k(\boldsymbol{x}) = \min\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}, \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$
   Give an example of f_1 and f_2 such that k is not convex.