Q1. (a)

$$f_2(x) := \|x-a\|^2 = \|x\|^2 - 2\langle a, x\rangle + \|a\|^2$$

$$\nabla f_2(x) = 2(x-a)$$

$$f_1(t) := \sqrt{t}$$

$$f(x) = f_1 \circ f_2(x)$$

chain rule $\Rightarrow$ $\nabla f(x) = \dfrac{x-a}{\|x-a\|}$

(b). $f(x) = \|2x-a\|^2$

$$= 4\|x\|^2 - 4\langle a, x\rangle + \|a\|^2$$

$$\nabla f(x) = 8x - 4a.$$

(c) $f_1(t) = \frac{1}{t}$, $f_2(x) = \|x\|$

$$\nabla f(x) = \nabla(f_1 \circ f_2)(x)$$

$$= -\frac{1}{\|x\|^2}\frac{x}{\|x\|}$$

$$= -\frac{x}{\|x\|^3}$$

(d) $f(x) = \frac{1}{2}\langle Ax-b, Ax-b\rangle + \lambda x^T B^T B x$

$$\nabla f(x) = A^T(Ax-b) + 2B^TBx.$$

(e) $$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{x_1}{\sqrt{x_1^2+c}} \\ \vdots \\ \frac{x_n}{\sqrt{x_n^2+c}} \end{bmatrix}$$

(f) $f_1(x) = Ax$

$f_2(y) = f(y)$

$g(x) = f_2 \circ f_1(x)$

$$\nabla g(x) = \nabla f_1(x) \cdot \nabla f_2(f_1(x))$$

$$= A^T \nabla f(Ax).$$

Q2: (a) $e^{x-y}-1 \geq x-y$

$\Rightarrow e^{x}-e^{y} \geq e^{y}(x-y)$

$\Rightarrow e^{x}-1 \geq e^{y}-1+e^{y}(x-y) \quad \forall x,y \in \mathbb{R}$

$f(x) \geq f(y)+f'(y)(x-y).$

(b) $x=(0,\cdots,0)$, $y=(1,0,\cdots,0)$

$f(\frac{1}{2}x+\frac{1}{2}y)=\sqrt{\frac{1}{2}}$

$\frac{1}{2}f(x)+\frac{1}{2}f(y)=\frac{1}{2}$

$f(\frac{1}{2}x+\frac{1}{2}y) \not\leq \frac{1}{2}f(x)+\frac{1}{2}f(y)$

not cvx.

Q3: (a) From def.

$f(\alpha x+(1-\alpha)y) = f_1(\alpha x+(1-\alpha)y)+f_2(\alpha x+(1-\alpha)y)$

$\leq \alpha f_1(x)+(1-\alpha)f_1(y)+\alpha f_2(x)+(1-\alpha)f_2(y)$

$= \alpha f(x)+(1-\alpha)f(y). \quad \forall \alpha \in [0,1],\ x,y \in \mathbb{R}^n.$

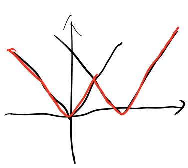
(b). $f_1(x)=0$, $f_2(x)=x_1^2 \quad x \in \mathbb{R}^n.$

(c) $h(\alpha x+(1-\alpha)y) = \max\{f_1(\alpha x+(1-\alpha)y), f_2(\alpha x+(1-\alpha)y)\}.$

$\leq \max\{\alpha f_1(x)+(1-\alpha)f_1(y), \alpha f_2(x)+(1-\alpha)f_2(y)\}.$

$\leq \max\{\alpha f_1(x), \alpha f_2(x)\}+\max\{(1-\alpha)f_1(y),(1-\alpha)f_2(y)\}$

$= \alpha h(x)+(1-\alpha)h(y).$

(d) 

$f_1(x)=|x_1| \quad x \in \mathbb{R}^n$

$f_2(x)=|x_1-1| \quad x \in \mathbb{R}^n.$