MATH3332 Tutorial Sheet 1

Question 1
Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ with addition and scalar multiplication as follows:

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (kf)(x) &= kf(x)\end{aligned}$$

for all $k \in \mathbb{R}$ and $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$. Show that $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a vector space.

Question 2
Let $V = \mathbb{R}^2$, and let $u, v \in V$ such that $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Define the addition and scalar multiplication for all $u, v \in V$ and $k \in \mathbb{R}$ as follows:

$$\begin{aligned}u+v &= (u_1+v_1, u_2+v_2)\\ ku &= (ku_1, 0)\end{aligned}$$

Determine whether or not this set under these operations is a vector space.

Question 3
Let $V = \mathbb{R}^2$, and let $u, v \in V$ such that $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Define the addition and scalar multiplication for all $u, v \in V$ and $k \in \mathbb{R}$ as follows:

$$\begin{gathered}u+v = (u_1v_1, u_2v_2)\\ ku = (ku_1, ku_2+1)\end{gathered}$$

Determine whether or not this set under these operations is a vector space.

Question 4
Let V be a normed vector space and let $x, y \in V$. Show that

$$\big|\|x\| - \|y\|\big| \leq \|x-y\|$$

Question 5
Let $x \in \mathbb{R}^n$ be an arbitrary vector. Recall the *p-norm* of x is defined to be:

$$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$$

Prove the following results:

- $\|x\|_1 \leq \sqrt{n}\|x\|_2$
- $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$
- $\|x\|_p \geq \|x\|_q$ whenever $1 \leq p \leq q < +\infty$