quiz1

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(a) Checking condition 5: $1_{\mathbb{R}} \cdot u = u$ if V is vector space.

$$1_{\mathbb{R}} \cdot u = (1_{\mathbb{R}} \cdot u_1, 0) = (u_1, 0) \neq (u_1, u_2) = u.$$

∴ Rule 5 is violated and V is not a vector space.

b). Checking condition 1:

True, $u+(v+w) = (u+v)+w = (u_1v_1w_1, u_2v_2w_2)$

checking condition 2:

True, $u+v = v+u = (u_1v_1 + u_2v_2)$

checking condition 3:

Let (x,y) be the zero element,

$$(x,y)+(u_1,u_2) = (u_1,u_2)$$

$$(xu_1, yu_2) = (u_1,u_2)$$

$$\therefore x=1,\ y=1.$$

$$(1,1)+u = (u_1,u_2) = u.$$

Checking condition 4:

$u \in V$, let $u=(1,0)$,

find $-u = (x,y)$.

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$= (1\cdot x, 0\cdot y)$

and it should equals to $(1,1)$.

Then we have: $0\cdot y = 1$

which is no solution,

so V is not a vector space.

2.a). ① checking $\|x\| - \|y\| \leq \|x-y\|$,

by triangle inequality,

$\|x-y\| + \|y\| \geqslant \|x-y+y\| = \|x\|$

$\|x-y\| \geqslant \|x\| - \|y\|$

$\|x\| - \|y\| \leq \|x-y\|$

② checking $\|y\| - \|x\| \leq \|x-y\|$,

by triangle inequality,

$\|y-x\| + \|x\| \geqslant \|y-x+x\| = \|y\|$

$\|y\| - \|x\| \leq \|y-x\|$

$\|y\| - \|x\| \leq \|x-y\|$

$\therefore \quad \big|\|x\| - \|y\|\big| \leq \|x-y\|$

2 b). $\because \{x^{(k)}\}_{k\in\mathbb{N}}$ is a convergent sequence,

$\therefore \lim\limits_{k\to+\infty} \|x^{(k)} - x\| = 0.$

Using the result of 2(a).

$$0 \le \left| \|x^{(k)}\| - \|x\| \right| \le \|x^{(k)} - x\|$$

Taking limit on both sides:

$$0 \le \lim_{k\to+\infty} \left| \|x^{(k)}\| - \|x\| \right| \le \lim_{k\to+\infty} \|x^{(k)} - x\|$$

$$0 \le \lim_{k\to+\infty} \left| \|x^{(k)}\| - \|x\| \right| \le 0$$

By sandwitch rules,

$$\lim_{k\to+\infty} \left| \|x^{(k)}\| - \|x\| \right| = 0$$

$$\left| \lim_{k\to+\infty} \|x^{(k)}\| - \|x\| \right| = 0$$

$$\lim_{k\to+\infty} \|x^{(k)}\| = \|x\|$$

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2(c). ~~lim~~ if $x^{(k)} \to x$ and $x^{(k)} \to y$,

$$\|x^{(k)} - x\| \le \|x^{(k)} - x^{(k-1)}\| + \|x^{(k-1)} - x\|$$
$$\le \|x^{(k)} - x^{(k-1)}\| + \|x^{(k-1)} - x^{(k-2)}\| + \|x^{(k-2)} - x\|$$
$$\le \sum_{i=1}^{k} \|x^{i} - x^{i-1}\|$$

and ~~let $\epsilon = y-x$~~

2d). Suppose $x \neq y$,

let $\epsilon = \frac{|y-x|}{10}$.

There exist N_1 s.t. if $n > N_1$ then

$|x^{N_1} - x| < \epsilon$

There exist N_2 s.t. if $n > N_2$ then

$|x^{N_2} - y| < \epsilon$

let $N = \max(N_1, N_2)$, if $n > N$ then

$|x^{N} - x| < \epsilon$ and $|x^{N} - y| < \epsilon$

But then by triangle inequality,

$|y-x| \le |x^{N} - y| + |x - x^{N}| < \frac{2}{10}|y-x|$

This is impossible, so $x \neq y$ is false and $x = y$.

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3(a). $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$

By cauchy schwarz inequality,

$|\vec{x}\cdot\vec{y}| \le \|\vec{x}\|\,\|\vec{y}\|$

$|\vec{x}\cdot\vec{y}| = \|\vec{x}\|\,\|\vec{y}\|$ if $\vec{x} = c\vec{y}$.

$(\|x\|_1)^2 = (|x_1| + \cdots + |x_n|)^2 = (|x_1| + \cdots + |x_n|)(|x_1| + \cdots + |x_n|)$

$\le (|x_1|^2 + \cdots + |x_n|^2)(1 + \cdots + 1)$

$= n\cdot\|x\|_2^2$

b). $\|x\|_\infty = \max_{1\le i\le n} |x_i|$

$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2} \le \sqrt{n\cdot\max_{1\le i\le n} |x_i|^2}$

$= \sqrt{n\,\|x\|_\infty^2}$

$= \sqrt{n}\cdot\|x\|_\infty$

Then, $\|x\|_1^2 \le n\|x\|_2^2$

$\|x\|_1 \le \sqrt{n}\,\|x\|_2$ //

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3(c). ~~let~~ Want to prove $\|x\|_p \geq \|x\|_q$ whenever $1 \leq p \leq q < +\infty$.

Let $y_k = \frac{|x_k|}{\|x\|_q} \Rightarrow$ since $\|x\|_q \geq |x_k|$

$y_k \leq 1 \Rightarrow y_k^p \geq y_k^q$

since $\|y\|_p^p = |y_1|^p + \cdots + |y_k|^p$

$\geq |y_1|^q + \cdots + |y_n|^q$

$= \frac{|x_1|^q + \cdots + |x_n|^q}{\|x\|_q^q}$

$= \frac{\|x\|_q^q}{\|x\|_q^q}$

$= 1$

$\Rightarrow \|y\|_p \geq 1$

$\Rightarrow |y_1|^p + \cdots + |y_n|^p \geq 1$

Using def. of $y_k = \frac{|x_k|}{\|x\|_q}$,

$\Rightarrow \frac{|x_1|^p}{\|x\|_q^p} + \cdots + \frac{|x_n|^p}{\|x\|_q^p} \geq 1$

$\Rightarrow \frac{\|x\|_p^p}{\|x\|_q^p} \geq 1$

$\Rightarrow \frac{\|x\|_p}{\|x\|_q} \geq 1$

$\Rightarrow \|x\|_p \geq \|x\|_q$