Q1.

(a)
$$\partial g_1(x) = \begin{cases} \{-1\} & x<0 \\ [-1,0] & x=0 \\ \{2x\} & x>0 \end{cases}$$

(b).
$$\partial g_2(x) = \begin{cases} \left[\frac{x_1}{\sqrt{x_1^2+x_2^2}}, \frac{x_2}{\sqrt{x_1^2+x_2^2}}, \frac{x_3}{\sqrt{x_3^2+x_4^2}}, \frac{x_4}{\sqrt{x_3^2+x_4^2}}\right]^T, & x_1^2+x_2^2 \neq 0, \quad x_3^2+x_4^2 \neq 0. \\ \left\{\left[a, b, \frac{x_3}{\sqrt{x_3^2+x_4^2}}, \frac{x_4}{\sqrt{x_3^2+x_4^2}}\right]^T : a^2+b^2 \le 1\right\} & x_1^2+x_2^2 = 0, \quad x_3^2+x_4^2 \neq 0. \\ \left\{\left[\frac{x_1}{\sqrt{x_1^2+x_2^2}}, \frac{x_2}{\sqrt{x_1^2+x_2^2}}, c, d\right]^T : c^2+d^2 \le 1\right\} & x_1^2+x_2^2 \neq 0, \quad x_3^2+x_4^2 = 0. \\ \left\{[a, b, c, d]^T : a^2+b^2 \le 1, \ c^2+d^2 \le 1\right\}. & x_1^2+x_2^2 = 0, \quad x_3^2+x_4^2 = 0. \end{cases}$$

(c).
$$\partial g_3(x) = \begin{cases} \{(1,0)^T\} & x_1 > x_2 \\ \{(0,1)^T\} & x_1 < x_2 \\ \{(u_1,u_2)^T : u_1 \ge 0, u_2 \ge 0, u_1+u_2 = 1\}. & x_1 = x_2 \end{cases}$$

(d)
$$\partial_1 g_4(x) = 2(|x_1|+|x_2|) \cdot \partial|x_1| = \begin{cases} 2(|x_1|+|x_2|) & x_1 > 0 \\ [-2|x_2|, 2|x_2|] & x_1 = 0 \\ -2(|x_1|+|x_2|) & x_1 < 0. \end{cases}$$

$$\partial_2 g_4(x) = 2(|x_1|+|x_2|) \cdot \partial|x_2| = \begin{cases} 2(|x_1|+|x_2|) & x_2 > 0 \\ [-2|x_1|, 2|x_1|] & x_2 = 0 \\ -2(|x_1|+|x_2|) & x_2 < 0. \end{cases}$$

$$\partial g_4(x) = \begin{bmatrix} \partial_1 g_4(x) \\ \partial_2 g_4(x) \end{bmatrix}.$$

Q2. denote $F(x) = \frac{1}{2}\|x - x^{(k)}\|_2^2 + \alpha_k g(x)$.

Suppose x_0, y_0 are two distinct minimizers of F.

claim: $F(\frac{x_0+y_0}{2}) < \frac{1}{2}(F(x_0) + F(y_0))$.

proof of the claim:

$$g(\frac{x+y}{2}) \le \frac{1}{2}(g(x) + g(y)) \quad \text{convexity.} \quad (*)$$

$$\|\frac{x+y}{2} - x^{(k)}\|_2^2 = \frac{\|x\|_2^2 + \|y\|_2^2}{4} + \frac{\langle x, y\rangle}{2} + \|x^{(k)}\|_2^2 - \langle x+y, x^{(k)}\rangle$$

$$\frac{1}{2}(\|x - x^{(k)}\|_2^2 + \|y - x^{(k)}\|_2^2) = \frac{1}{2}(\|x\|_2^2 + \|y\|_2^2) - \langle x+y, x^{(k)}\rangle + \|x^{(k)}\|_2^2$$

Since $\langle x, y\rangle \le \frac{1}{2}(\|x\|_2^2 + \|y\|_2^2)$ $(\frac{1}{2}\|x - y\|_2^2 \ge 0)$.

with "=" iff $x = y$

$$\Rightarrow \frac{1}{2}\|\frac{x+y}{2} - x^{(k)}\|_2^2 < \frac{1}{2}(\frac{1}{2}\|x - x^{(k)}\|_2^2 + \frac{1}{2}\|y - x^{(k)}\|_2^2) \quad (**)$$

Together with $(*)$ $(**)$, we proved the claim.

Q3: Let $f(x) = \frac{1}{2}\|Ax-b\|_2^2$

$g(x) = \sum_{i=1}^{n} r_i(x)$.

$$\nabla f(x) = A^T(Ax-b).$$

$$\partial r(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \\ [0,1] & t=0 \end{cases}$$

Now let $y = \text{prox}_{\alpha_k \lambda}\, g(x)$

$$= \underset{y}{\text{argmin}}\ \frac{1}{2}\|y-x\|_2^2 + \alpha_k \lambda g(y).$$

$$\Leftrightarrow \quad x_i \in y_i + \alpha_k \lambda \partial r(y_i) \quad \forall\ 1 \le i \le n.$$

$$\Leftrightarrow \quad x_i \in \begin{cases} y_i & y_i<0 \\ y_i+\alpha_k\lambda & y_i>0 \\ [0,\alpha_k\lambda] & y_i=0 \end{cases} \quad \Rightarrow \quad y_i = T'(x_i) = \begin{cases} x_i & x_i<0 \\ 0 & x_i \in [0,\alpha_k\lambda] \\ x_i-\alpha_k\lambda & x_i>\alpha_k\lambda. \end{cases}$$

$\Rightarrow$ FBS gives:

$$x^{(k+1)} = \text{prox}_{\alpha_k\lambda}\, g\left(x^{(k)} - \alpha_k \nabla f(x^{(k)})\right)$$

$$\Leftrightarrow \quad x^{(k+1)} = \underbrace{T'}_{\text{elementwise}}\left(x^{(k)} - \alpha_k A^T(Ax^{(k)}-b)\right) \quad \text{where} \quad T'(x) = \begin{cases} x & x<0 \\ 0 & 0 \le x \le \alpha_k\lambda \\ x-\alpha_k\lambda & x>\alpha_k\lambda \end{cases}$$