1. Determine whether each of the following functions of vectors in $\mathbb{R}^n$ is linear. If it is a linear function, give its inner product representation, i.e., an vector $\boldsymbol{a} \in \mathbb{R}^n$ for which $f(x) = \langle \boldsymbol{a}, \boldsymbol{x} \rangle$ for all $\boldsymbol{x} \in \mathbb{R}^n$. If it is not linear, give specific $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ for which superposition fails, i.e., $f(\alpha\boldsymbol{x} + \beta\boldsymbol{x}) \neq \alpha f(\boldsymbol{x}) + \beta f(\boldsymbol{y})$.

   (a) The spread of values of the vector, defined as $f(\boldsymbol{x}) = \max_k x_k - \min_k x_k$.

   (b) The difference of the last element and the first, $f(\boldsymbol{x}) = x_n - x_1$.

   (c) The median of a vector, where we will assume $n = 2k + 1$ is odd. The median of the vector $\boldsymbol{x}$ is defined as the $(k + 1)$-st largest number among the entries of $\boldsymbol{x}$. For example, the median of $(7.1, 3.2, 1.5)$ is 1.5.

   (d) Vector extrapolation, defined as $x_n + (x_n - x_{n-1})$, for $n \geq 2$. (This is a simple prediction of what x_{n+1} would be, based on a straight line drawn through x_n and x_{n-1}.)

2. Let V be a Hilbert space. Let S_1 and S_2 be two hyperplanes in V defined by

$$S_1 = \{\boldsymbol{x} \in V \mid \langle \boldsymbol{a}_1, \boldsymbol{x} \rangle = b_1\}, \quad S_2 = \{\boldsymbol{x} \in V \mid \langle \boldsymbol{a}_2, \boldsymbol{x} \rangle = b_2\}.$$

   Let $\boldsymbol{y} \in V$ be given. We consider the projection of $\boldsymbol{y}$ onto $S_1 \cap S_2$, i.e., the solution of

$$\min_{x \in S_1 \cap S_2} \|\boldsymbol{x} - \boldsymbol{y}\|. \tag{1}$$

   (a) Prove that $S_1 \cap S_2$ is a plane, i.e., if $\boldsymbol{x}, \boldsymbol{z} \in S_1 \cap S_2$, then $(1 + t)\boldsymbol{z} - t\boldsymbol{x} \in S_1 \cap S_2$ for any $t \in \mathbb{R}$.

   (b) Prove that $\boldsymbol{z}$ is a solution of (1) if and only if $\boldsymbol{z} \in S_1 \cap S_2$ and

$$\langle \boldsymbol{z} - \boldsymbol{y}, \boldsymbol{z} - \boldsymbol{x} \rangle = 0, \quad \forall \boldsymbol{x} \in S_1 \cap S_2. \tag{2}$$

   (c) Find an explicit solution of (1).

   (d) Prove the solution found in part (c) is unique.

3. Let $\{(\boldsymbol{x}_i, y_i)\}_{i-1}^N$ be given with $\boldsymbol{x}_i \in \mathbb{R}^n$ and $y_i \in \mathbb{R}$. Assume $N < n$, and $\boldsymbol{x}_i$, $i = 1, 2, \ldots, N$, are linearly independent. Consider the ridge regression

$$\min_{\boldsymbol{a} \in \mathbb{R}^n} \sum_{i-1}^{N} \left(\langle \boldsymbol{a}, \boldsymbol{x}_i \rangle - y_i\right)^2 + \lambda \|\boldsymbol{a}\|_2^2,$$

   where $\lambda \in \mathbb{R}$ is a regularization parameter, and we set the bias $b = 0$ for simplicity.

   (a) Prove that the solution must be in the form of $\boldsymbol{a} = \sum_{i-1}^{N} c_i \boldsymbol{x}_i$ for some $\boldsymbol{c} = [c_1, c_2, \ldots, c_N]^T \in \mathbb{R}^N$. (*Hint: Similar to the proof of the representer theorem.*)

(b) Re-express the minimization in terms of $\boldsymbol{c} \in \mathbb{R}^N$, which has fewer unknowns than the original formulation.