MATH3332 Quiz 1

1. Determine whether or not the set under the operations is a vector space. Give justification for your answer.

   (a) *(20 pts)* Let $V = \mathbb{R}^2$, and let $u, v \in V$ such that $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Define the addition and scalar multiplication for all $u, v \in V$ and $k \in \mathbb{R}$ as follows:

   $$\begin{aligned} u + v &= (u_1 + v_1, u_2 + v_2) \\ ku &= (ku_1, 0) \end{aligned}$$

   (b) *(20 pts)* Let $V = \mathbb{R}^2$, and let $u, v \in V$ such that $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Define the addition and scalar multiplication for all $u, v \in V$ and $k \in \mathbb{R}$ as follows:

   $$\begin{aligned} u + v &= (u_1 v_1, u_2 v_2) \\ ku &= (ku_1, ku_2 + 1) \end{aligned}$$

2. Let $(V, \|\cdot\|)$ be a normed vector space.

   (a) *(10 pts)* Prove that, for all $\boldsymbol{x}, \boldsymbol{y} \in V$,

   $$|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x} - \boldsymbol{y}\|.$$

   (b) *(10 pts)* Let $\{\boldsymbol{x}^{(k)}\}_{k \in \mathbb{N}}$ be a convergent sequence in V with limit $\boldsymbol{x} \in V$. Prove that

   $$\lim_{k \to \infty} \|\boldsymbol{x}^{(k)}\| = \|\boldsymbol{x}\|.$$

   (*Hint: Use part (a).*)

   (c) *(10 pts)* Let $\{\boldsymbol{x}^{(k)}\}_{k \in \mathbb{N}}$ be a sequence in V and $\boldsymbol{x}, \boldsymbol{y} \in V$. Prove that, if

   $$\boldsymbol{x}^{(k)} \to \boldsymbol{x}, \quad \text{and} \quad \boldsymbol{x}^{(k)} \to \boldsymbol{y},$$

   then $\boldsymbol{x} = \boldsymbol{y}$.

3. Let $x \in \mathbb{R}^n$ be an arbitrary vector. Recall the *p-norm* of x is defined to be:

   $$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$$

   Prove the following results:

   (a) *(10 pts)* $\|x\|_1 \leq \sqrt{n}\|x\|_2$

   (b) *(10 pts)* $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$

   (c) *(10 pts)* $\|x\|_p \geq \|x\|_q$ whenever $1 \leq p \leq q < +\infty$