HW4

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HW4(2)

MATH 3332 Data Analytic Tools
Homework 4

Due date: 16 November, 6pm, Monday

1. Find the gradient of the following functions $f : V \mapsto \mathbb{R}$, where V is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.
   (a) $f(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}\|$ for a given $\boldsymbol{a} \in V$, and $\boldsymbol{x} \neq \boldsymbol{a}$.
   (b) $f(\boldsymbol{x}) = \|2\boldsymbol{x} - \boldsymbol{a}\|^2$ for a given $\boldsymbol{a} \in V$.
   (c) $f(\boldsymbol{x}) = \frac{1}{\|\boldsymbol{x}\|}$, where $\boldsymbol{x} \neq \boldsymbol{0}$.
   (d) $f(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2 + \lambda\|\boldsymbol{B}\boldsymbol{x}\|^2$, where $\lambda > 0$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{B} \in \mathbb{R}^{p \times n}$.
   (e) $V = \mathbb{R}^n$ and $f(\boldsymbol{x}) = \sum_{i=1}^n \sqrt{x_i^2 + c}$ for some $c > 0$.
   (f) Let $f : \mathbb{R}^m \to \mathbb{R}$ be differentiable everywhere. Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ be a matrix. Let $g : \mathbb{R}^m \to \mathbb{R}$ be defined by
   $$g(\boldsymbol{x}) = f(\boldsymbol{A}\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^n.$$
   Express $\nabla g(\boldsymbol{x})$ in terms of $\boldsymbol{A}$ and gradient of f.
2. Determine whether or not the following functions are convex.
   (a) $f(x) = e^x - 1$ on $\mathbb{R}$.
   (b) $f(\boldsymbol{x}) = \sum_{i=1}^n \sqrt{|x_i|}$ on $\mathbb{R}^n$.
3. Let f_1 and f_2 be two convex functions from $\mathbb{R}^n$ to $\mathbb{R}$.
   (a) Let
   $$f(\boldsymbol{x}) = f_1(\boldsymbol{x}) + f_2(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$
   Prove that f is convex.
   (b) Let
   $$g(\boldsymbol{x}) = f_1(x) - f_2(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$
   Give an example of f_1 and f_2 such that g is not convex.
   (c) Let
   $$h(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}, \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$
   Prove that h is convex.
   (d) Let
   $$k(\boldsymbol{x}) = \min\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}, \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$
   Give an example of f_1 and f_2 such that k is not convex.

1. Find the gradient of the following functions $f : V \mapsto \mathbb{R}$, where V is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.
   (a) $f(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}\|$ for a given $\boldsymbol{a} \in V$, and $\boldsymbol{x} \neq \boldsymbol{a}$.

For $x \in V - \{a\}$, we have

$$f(x) = \|x - a\| = (\|x-a\|^2)^{\frac{1}{2}}$$

Let $g(t) = \sqrt{t}$ and $h(x) = \|x-a\|^2$, let $g(t) = \sqrt{t}$ and $h(x) = \|x-a\|^2$,

$$f(x) = g(h(x))$$

$$\nabla g(h(x)) = g'(h(x))\,\nabla h(x)$$

$$g'(t) = \frac{1}{2}t^{-\frac{1}{2}}$$

$$\nabla h(x) = \nabla\|x-a\|^2 = 2(x-a)$$

$$\therefore\ \nabla f(x) = \frac{1}{2}(\|x-a\|^2)^{-\frac{1}{2}} \cdot 2(x-a)$$

$$= \frac{x-a}{\|x-a\|}$$

(b) $f(\boldsymbol{x}) = \|2\boldsymbol{x} - \boldsymbol{a}\|^2$ for a given $\boldsymbol{a} \in V$.

b). Fix $x_0 \in V$,

$$f(x) - f(x_0) = \|2x-a\|^2 - \|2x_0 - a\|^2$$

$$= \|2x - 2x_0 + 2x_0 - a\|^2 - \|2x_0 - a\|^2$$

$$= \|2x - 2x_0\|^2 + 2\langle 2x - 2x_0, 2x_0 - a\rangle$$

$$= 4\|x - x_0\|^2 + \langle x - x_0, 4(2x_0 - a)\rangle$$

$$\therefore\ f(x) - f(x_0) - \langle x - x_0, 4(2x_0 - a)\rangle$$

$$= 4\|x - x_0\|^2$$

or equivalently,

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0) - \langle x - x_0, 4(x_0 - a)\rangle|}{\|x - x_0\|}$$

$$= \lim_{x \to \infty} 4\|x - x_0\|$$

$$= 0$$

$\therefore \nabla f(x_0) = 4(2x_0 - a)$

Since $x_0 \in V$ is arbitrary.

$$\nabla f(x) = 4(2x - a) \quad \forall x \in V.$$

(c) $f(\boldsymbol{x}) = \frac{1}{\|\boldsymbol{x}\|}$, where $\boldsymbol{x} \neq \mathbf{0}$.

c). Fix $x \in V - \{0\}$,

$$f(x) = \frac{1}{\|x\|} = (\|x\|^2)^{-\frac{1}{2}}$$

let $g(t) = \frac{1}{\sqrt{t}}$ and $h(x) = \|x\|^2$

$$f(x) = g(h(x))$$

$$\nabla g(h(x)) = g'(h(x)) \nabla h(x)$$

$$g'(t) = -\frac{1}{2} t^{-\frac{3}{2}}$$

$$\nabla h(x) = \nabla \|x\|^2 = 2x$$

$$\therefore \nabla f(x) = -\frac{1}{2}(\|x\|^2)^{-\frac{3}{2}} \cdot 2x$$

$$= -\frac{x}{\|x\|^3}$$

(d) $f(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|^2 + \lambda\|\boldsymbol{B}\boldsymbol{x}\|^2$, where $\lambda > 0$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{B} \in \mathbb{R}^{p \times n}$.

$$\|Ax-b\|^2 = (Ax-b)^T(Ax-b)$$
$$= x^TA^TAx-(Ax)^Tb - b^TAx + b^Tb$$
$$= x^TA^TAx - 2b^TAx + b^Tb$$
$$= x^TA^TAx - 2(A^Tb)^Tx + b^Tb$$

Let $\varphi(x) = c^Tx$ where $c = A^Tb$

$$\varphi(x) = \sum_{j=1}^{n} c_j x_j$$

$$\frac{\partial \varphi}{\partial x_k} = \sum_{j=1}^{n} c_j \frac{\partial x_j}{\partial x_k} = \sum_{j=1}^{n} c_j \delta_{jk} = c_k$$

$$\Rightarrow \nabla\varphi(x) = c = A^Tb.$$

Let $\phi(x) = x^TBx$ where $B = A^TA$

$$\phi(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} x_i x_j$$

$$\frac{\partial \phi}{\partial x_k} = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} \frac{\partial (x_i x_j)}{\partial x_k}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}(\delta_{ik}x_j + x_i\delta_{jk})$$

$$= \sum_{j=1}^{n} b_{kj} x_j + \sum_{i=1}^{n} b_{ik} x_i$$

$$= (Bx)_k + \sum_{i=1}^{n} (B^T)_{ki} x_i$$

$$= (Bx)_k + (B^T x)_k$$

$$\therefore \nabla \phi(x) = (B + B^T) x \quad -(*)$$

$$\therefore \nabla(\|Ax - b\|_2^2) = (A^T A + (A^T A)^T) x - 2A^T b$$

and $(A^T A)^T = A^T (A^T)^T = A^T A$

$$\Rightarrow \nabla(\|Ax - b\|_2^2) = 2A^T A x - 2A^T b$$

Now solving $\nabla(\|Bx\|^2)$,

$$\|Bx\|^2 = (Bx)^T (Bx)$$

$$\|Bx\|^2 = x^T B^T B x = x^T (B^T B)^T x$$

From $(*)$, we know that $\nabla(\|Bx\|^2)$

$$= (B^T B + (B^T B)^T) x$$

$$= (B^T B + B^T B) x$$

$$= (B^T B + B^T B)x$$
$$= 2(B^T B)x$$

$\therefore \nabla(f(x))$

$$= \frac{1}{2}(2A^T A x - 2A^T b) + 2\lambda(B^T B)x$$
$$= A^T A x - A^T b + 2\lambda(B^T B)x$$

e). (e) $V = \mathbb{R}^n$ and $f(x) = \sum_{i=1}^{n} \sqrt{x_i^2 + c}$ for some $c > 0$.

$$f(x) = \sum_{i=1}^{n} \sqrt{x_i^2 + c} = \sum_{i=1}^{n} \left(\sqrt{x_i^2 + c} - \sqrt{x_{0,i}^2 + c}\right) + \sum_{i=1}^{n} \sqrt{x_{0,i}^2 + c}$$

for some $x_0 = (x_{0,1}, x_{0,2}, \dots, x_{0,n}) \in V$

$$f(x) = \sum_{i=1}^{n} \frac{(x_i^2 + c) - (x_{0,i}^2 + c)}{\sqrt{x_i^2 + c} + \sqrt{x_{0,i}^2 + c}} + f(x_0)$$

$$= \sum_{i=1}^{n} (x_i - x_{0,i}) \frac{(x_i + x_{0,i})}{\sqrt{x_i^2 + c} + \sqrt{x_{0,i}^2 + c}} + f(x_0)$$

$$\sum_{i=1} (x_i - x_{0,i}) \frac{\qquad}{\sqrt{x_i^2+c}+\sqrt{x_{0,i}^2+c}} + f(x_0)$$

$$= \sum_{i=1}^{n} (x_i - x_{0,i}) \frac{x_{0,i}}{\sqrt{x_{0,i}^2+c}} + \sum_{i=1}^{n} (x_i - x_{0,i}) \left[\frac{x_{0,i}}{\sqrt{x_{0,i}^2+c}} - \frac{(x_i + x_{0,i})}{\sqrt{x_i^2+c}+\sqrt{x_{0,i}^2+c}}\right] + f(x_0)$$

For ■ term, let $K = \left(\frac{x_{0,1}}{\sqrt{x_{0,1}^2+c}}, \frac{x_{0,2}}{\sqrt{x_{0,2}^2+c}}, \dots, \frac{x_{0,n}}{\sqrt{x_{0,n}^2+c}}\right)^T$

$$\therefore \sum_{i=1}^{n} (x_i - x_{0,i}) \frac{x_{0,i}}{\sqrt{x_{0,i}^2+c}} = \langle x - x_0, K \rangle$$

For ■ term, $x_0 \to x_{0,i}$ as $x \to x_0$,

$$\frac{x_{0,i}}{\sqrt{x_{0,i}^2+c}} - \frac{(x_i + x_{0,i})}{\sqrt{x_i^2+c}+\sqrt{x_{0,i}^2+c}} \to 0$$

$$(x_i - x_{0,i}) \left[\frac{x_{0,i}}{\sqrt{x_{0,i}^2+c}} - \frac{x_i + x_{0,i}}{\sqrt{x_i^2+c}+\sqrt{x_{0,i}^2+c}}\right] = o(x_i - x_{0,i})$$

$$\sum_{i=1}^{n} (x_i - x_{0,i}) \left[\frac{x_{0,i}}{\sqrt{x_{0,i}^2+c}} - \frac{x_i + x_{0,i}}{\sqrt{x_i^2+c}+\sqrt{x_{0,i}^2+c}}\right] = o(\| x_i - x_{0,i} \|)$$

$$\therefore f(x) - f(x_0) - \langle x - x_0, K \rangle = o(\| x - x_0 \|)$$

$$\Rightarrow \nabla(f(x_0)) = K = \left(\frac{x_{0,1}}{\sqrt{x_{0,1}^2 + c}}, \frac{x_{0,2}}{\sqrt{x_{0,2}^2 + c}}, \dots, \frac{x_{0,n}}{\sqrt{x_{0,n}^2 + c}}\right)^T$$

$$\nabla f(x) = \left(\frac{x_1}{\sqrt{x_1^2 + c}}, \frac{x_2}{\sqrt{x_2^2 + c}}, \dots, \frac{x_n}{\sqrt{x_n^2 + c}}\right)^T$$

f). Haven't done :(

2. Determine whether or not the following functions are convex.

(a) $f(x) = e^x - 1$ on $\mathbb{R}$.

(b) $f(\boldsymbol{x}) = \sum_{i=1}^{n} \sqrt{|x_i|}$ on $\mathbb{R}^n$.

2a). $f(x) = e^x - 1$

$\forall x, y \in \mathbb{R}, \alpha \in [0,1],$

if $x = y$,

$f(\alpha x + (1-\alpha) y) = f(\alpha x + (1-\alpha) x) = f(x) =$

$\alpha f(x) + (1-\alpha) f(x) = \alpha f(x) + (1-\alpha) f(y)$

if $x \neq y$,

$f(\alpha x + (1-\alpha) y) - \alpha f(x) - (1-\alpha) f(y)$

$= e^{\alpha x + (1-\alpha) y} - 1 - \alpha(e^x - 1) - (1-\alpha)(e^y - 1)$

$= e^{\alpha x + (1-\alpha) y} - \alpha e^x - (1-\alpha) e^y$

$= e^{\alpha(x-y)+y} - \alpha(e^x - e^y) - e^y$

let $g(\alpha) = e^{\alpha(x-y)+y} - \alpha(e^x - e^y) - e^y$

$g'(\alpha) = (x-y) e^{\alpha(x-y)+y} - (e^x - e^y)$

$g''(\alpha) = (x-y)^2 e^{\alpha(x-y)+y} > 0 \quad \forall \alpha \in [0,1]$

$$g''(\alpha) = (x-y)^2 e^{\alpha(x-y)+y} > 0 \quad \forall \alpha \in [0,1]$$

$\therefore g$ is minimum at boundary, 0 or 1.

$$g(0) = 0$$
$$g(1) = 0$$

$$\therefore g(\alpha) \leq 0 \quad \forall \alpha \in [0,1]$$

$$\Rightarrow f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) y$$

$\Rightarrow f(x)$ is convex

b). $f(x) = \sum_{i=1}^{n} \sqrt{|x_i|}$

let $x = (1,0,0,\cdots,0)^T$, $y = (16,0,0,\cdots,0)^T \in \mathbb{R}^n$,

$$\alpha = \frac{7}{15} \in (0,1)$$

$$\alpha x + (1-\alpha) y = \frac{7}{15}(1,0,0,\cdots 0)^T + \frac{8}{15}(16,0,0,\cdots 0)^T$$

$$= (9,0,0,\cdots 0)^T$$

$$f(\alpha x+(1-\alpha)y) = f((9,0,0,\cdots,0)^T)$$

$$= \sqrt{9}+\sqrt{0}+\sqrt{0}+\cdots\sqrt{0} = 3$$

$$f(x) = f((1,0,\cdots,0)^T) \qquad f(y) = f((16,0,0,\cdots,0)^T)$$

$$= 1 \qquad\qquad = \sqrt{16}+\sqrt{0}+\sqrt{0}+\cdots+\sqrt{0}$$

$$= 4$$

$$\alpha f(x)+(1-\alpha)f(y) = \frac{7}{15}(1)+\frac{8}{15}(4) = \frac{39}{15} < 3$$

$$\therefore f(\alpha x+(1-\alpha)y) > \alpha f(x)+(1-\alpha)f(y)$$

when $x=(1,0,\cdots,0)^T$, $y=(16,0,0,\cdots,0)^T$, $\alpha=\frac{7}{15}$.

$\Rightarrow f(x)$ is not convex.

3. Let f_1 and f_2 be two convex functions from $\mathbb{R}^n$ to $\mathbb{R}$.

(a) Let
$$f(\boldsymbol{x}) = f_1(\boldsymbol{x}) + f_2(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$
Prove that f is convex.

(b) Let
$$g(\boldsymbol{x}) = f_1(\boldsymbol{x}) - f_2(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$
Give an example of f_1 and f_2 such that g is not convex.

(c) Let
$$h(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}, \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$
Prove that h is convex.

(d) Let
$$k(\boldsymbol{x}) = \min\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}, \quad \forall\, \boldsymbol{x} \in \mathbb{R}^n.$$
Give an example of f_1 and f_2 such that k is not convex.

a). $f(x) = f_1(x) + f_2(x)$

Given f_1 and f_2 are convex

$\Rightarrow$ for any $x, y \in \mathbb{R}^n$, $\alpha \in [0,1]$,

$f_1(\alpha x + (1-\alpha)y) \leq \alpha f_1(x) + (1-\alpha) f_1(y)$

$f_2(\alpha x + (1-\alpha)y) \leq \alpha f_2(x) + (1-\alpha) f_2(y)$

$\therefore f(\alpha x + (1-\alpha)y) \leq [\alpha f_1(x) + (1-\alpha) f_1(y)] + [\alpha f_2(x) + (1-\alpha) f_2(y)]$

$= \alpha(f_1(x) + f_2(x)) + (1-\alpha)(f_1(y) + f_2(y))$

$= \alpha f(x) + (1-\alpha) f(y)$

$= \alpha f(x) + (1-\alpha) f(y)$

$\Rightarrow f$ is convex function.

b). let $f_1(x) = (x-2)^2 + 1$,

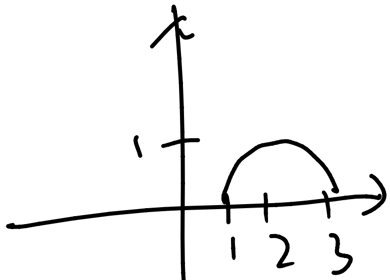
$f_2(x) = 2(x-2)^2$

$g(x) = f_1(x) - f_2(x)$

$g(x) = (x-2)^2 + 1 - 2(x-2)^2$

$g(x) = -(x-2)^2 + 1$

and we know that the graph of $g(x)$ is



$\Rightarrow g(x)$ is concave.

when $f_1(x) = (x-2)^2 + 1$,

$f_2(x) = 2(x-2)^2$

c). $\forall x, y \in \mathbb{R}^n$, $\alpha \in [0,1]$,

$$h(\alpha x + (1-\alpha) y) = \max\{f_1(\alpha x + (1-\alpha) y), f_2(\alpha x + (1-\alpha) y)\}$$

Given $f_1(x)$ and $f_2(x)$ are convex,

$$f_1(\alpha x + (1-\alpha) y) \leq \alpha f_1(x) + (1-\alpha) f_1(y)$$

$$f_2(\alpha x + (1-\alpha) y) \leq \alpha f_2(x) + (1-\alpha) f_2(y)$$

$$h(\alpha x + (1-\alpha) y) \leq \max\{\alpha f_1(x) + (1-\alpha) f_1(y), \alpha f_2(x) + (1-\alpha) f_2(y)\}$$

$$\leq \max\{\alpha f_1(x), \alpha f_2(x)\} + \max\{(1-\alpha) f_1(y), (1-\alpha) f_2(y)\}$$

$$\leq \alpha \max\{f_1(x), f_2(x)\} + (1-\alpha) \max\{f_1(y), f_2(y)\}$$

$$= \alpha h(x) + (1-\alpha) h(y)$$

$\Rightarrow h$ is convex

$\Rightarrow h$ is convex

d). let $f_1(x) = x_1$, $f_2(x) = 0$, $\forall x \in \mathbb{R}^n$

$$f_1(\alpha x + (1-\alpha)y) = \alpha x_1 + (1-\alpha)y_1 = \alpha f_1(x) + (1-\alpha)f_1(y)$$

$$f_2(\alpha x + (1-\alpha)y) = 0 = \alpha f_2(x) + (1-\alpha)f_2(y)$$

$\Rightarrow f_1, f_2$ are convex.

let $x = (-1, 0, 0, \cdots, 0)^T$, $y = (1, 0, 0, \cdots, 0)^T \in \mathbb{R}^n$,

$\alpha = 0.5 \in [0,1]$,

$$K(\alpha x + (1-\alpha)y) = K(\tfrac{1}{2}(-1, 0, 0, \cdots, 0)^T + \tfrac{1}{2}(1, 0, 0, \cdots, 0)^T)$$

$$= K((0, 0, \cdots, 0)^T)$$

$$= \min\{f_1(0, 0, \cdots, 0)^T, f_2(0, 0, \cdots, 0)^T\}$$

$$= \min\{0, 0\} = 0$$

$$K(x) = K((-1, 0, 0, \cdots, 0)^T)$$

$$= \min\{f_1(-1, 0, \cdots 0)^T, f_2(-1, 0, 0, \cdots, 0)^T\}$$

$$= \min\{-1, 0\} = -1$$

$$K(y) = K((1, 0, 0, \cdots, 0)^T)$$

$$K(y) = K((1,0,0,\cdots,0))$$

$$= \min\{f_1(1,0,0,\cdots,0)^T, f_2(1,0,0\ldots,0)^T\}$$

$$= \min\{1,0\} = 0$$

$$\alpha K(x) + (1-\alpha)K(y) = \frac{1}{2}(-1) + \frac{1}{2}(0) = -\frac{1}{2} < 0$$

$\therefore$ $K(\alpha x + (1-\alpha)y) > \alpha K(x) + (1-\alpha)K(y) \Rightarrow$

K is not convex.