Math 2033 (Homework 2) L1 Fall 2021

Problems (Due Oct. 11 at 11:59 pm)

① Prove that there is a bijection from $[0,1]$ to $(0,1]$.

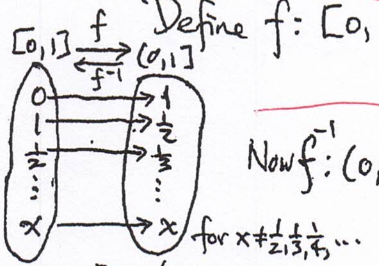
② Determine if the set A of all intersection points in $\mathbb{R}^2$ of the family of lines $\{y=mx : m\in\mathbb{Z}\}$ with the family of circles $\{x^2+y^2=r^2 : r\in\mathbb{Q}\}$ is countable or uncountable. Here A is the set of all points in $\mathbb{R}^2$ that are on at least one of the lines $y=mx$ $(m\in\mathbb{Z})$ and at least one of the circles $x^2+y^2=r^2$ $(r\in\mathbb{Q})$.

③ Prove that there exist infinitely many positive real numbers r such that the equation $2^x+3^y+5^z=r$ has no solution $(x,y,z)\in\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$.

④ Let T be a nonempty subset of the interval $(0,1)$. If every finite subset $\{x_1,x_2,\dots,x_n\}$ of T (with no two of $x_1,x_2,\dots,x_n$ equal) has the property that $x_1^2+x_2^2+\dots+x_n^2<1$, then prove that T is a countable set.

Solution of Math 2033 Homework # 2

① Let $X=\{0,1,\frac12,\frac13,\cdots\}$. Then $[0,1]\setminus X=(0,1)\setminus\{\frac12,\frac13,\frac14,\cdots\}=(0,1]\setminus X$. 5



Define $f:[0,1]\to(0,1]$ by $f(x)=\begin{cases} x & \text{if } x\in(0,1)\setminus\{\frac12,\frac13,\frac14,\cdots\}\\ 1 & \text{if } x=0\\ \frac{1}{n+1} & \text{if } x=\frac1n,\ n=1,2,3,\cdots\end{cases}$ 5

Now $f^{-1}:(0,1]\to[0,1]$ given by $f^{-1}(x)=\begin{cases} x & \text{if } x\in(0,1)\setminus\{\frac12,\frac13,\frac14,\cdots\}\\ 0 & \text{if } x=1\\ \frac{1}{n-1} & \text{if } x=\frac1n,\ n=2,3,4,\cdots\end{cases}$ 5

is the inverse of f. $\therefore f$ is a bijection.

② For every $(m,r)\in\mathbb{Z}\times\mathbb{Q}$, $\begin{cases} y=mx\\ x^2+y^2=r^2\end{cases} \Rightarrow \begin{cases} x^2+(mx)^2=r^2\\ y=mx\end{cases} \Rightarrow \begin{cases} x^2=\frac{r^2}{1+m^2}\\ y=mx\end{cases}$ 10

has at most 2 solutions since $x^2=r^2/(1+m^2)$ has at most 2 solutions and at most 1 y for each solution of x.

Let $S_{(m,r)}$ be the solutions of the system $\begin{cases} y=mx\\ x^2+y^2=r^2\end{cases}$ for $(m,r)\in\mathbb{Z}\times\mathbb{Q}$. 5

Then $S=\bigcup_{\underbrace{(m,r)\in\mathbb{Z}\times\mathbb{Q}}_{\text{countable}}}\underbrace{S_{(m,r)}}_{\text{finite, hence countable}}$ is countable by countable union theorem. 10

③ For every $(x,y,z)\in\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$, let $S_{(x,y,z)}=\{2^x+3^y+5^z\}$. ← 1 element set. 5

then $S=\{2^x+3^y+5^z : (x,y,z)\in\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}\}=\bigcup_{\underbrace{(x,y,z)\in\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}}_{\text{countable}}}\underbrace{S_{(x,y,z)}}_{\text{finite, hence countable}}$

is countable by countable union theorem. 10

By example 6 (on transparencies page 21) $(0,+\infty)\setminus S$ is uncountable. 5

Hence $(0,+\infty)\setminus S$ is not a finite set (finite sets are countable).

So $(0,+\infty)\setminus S$ is infinite. $\therefore$ there are infinitely many $r\in(0,\infty)$ and r is not in S. 5

④ Note $(0,1)=\bigcup_{k=1}^{\infty}[\frac{1}{k+1},\frac1k)$ 5

So every element of T ($\subseteq(0,1)$) is in at least one of the interval $[\frac{1}{k+1},\frac1k)$. 5

If $x_1,x_2,\cdots,x_n$ are in T and $[\frac{1}{k+1},\frac1k)$, then $1>x_1^2+\cdots+x_n^2\geq\underbrace{\frac{1}{(k+1)^2}+\cdots+\frac{1}{(k+1)^2}}_{n\text{ times}}$. 10

From this we get $1>\frac{n}{(k+1)^2}$. So $n<(k+1)^2$.

Let $T_k=T\cap[\frac{1}{k+1},\frac1k)$, then T_k has less than $(k+1)^2$ elements. 10

So T_k is a finite set. $\therefore T=\bigcup_{k=1}^{\infty}\underbrace{T_k}_{\text{finite, hence countable}}$ is countable by the countable union theorem. 5