Definitions

A set S is <u>countably infinite</u> $\Leftrightarrow \exists$ bijection $f: \mathbb{N} \to S$.

A set S is <u>countable</u> $\Leftrightarrow$ S is finite or countably infinite.

A set S is <u>uncountable</u> $\Leftrightarrow$ S is not countable.

Basic Examples of Countable Sets

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{N}\times\mathbb{N}, \mathbb{Q}\times\mathbb{Q}, \ldots$

Basic Examples of Uncountable Sets

$\mathbb{R}$, intervals with more than 1 elements, $\mathbb{R}\setminus\mathbb{Q}$

$\mathbb{C}$, $P(\mathbb{N})$, $\{0,1\}\times\{0,1\}\times\{0,1\}\times\cdots$, $\underset{\text{uncountable}}{A} \setminus \underset{\text{countable}}{B}$

Countable Union Theorem

① $A_1, A_2, A_3, \cdots$ are countable $\Rightarrow \bigcup_{n=1}^{\infty} A_n$ is countable.

② S countable and $\forall s \in S$, A_s is countable

$\Rightarrow \bigcup_{s\in S} A_s$ is countable.

countable

Countable Subset Theorem

Let $A \subseteq B$. If B is countable, then A is countable.

If A is uncountable, then B is uncountable.

Product Theorem

If $A_1, A_2, \cdots, A_n$ are countable, $n \in \mathbb{N}$, then $A_1\times A_2\times\cdots\times A_n$ is countable.

Bijection Theorem

If $\exists$ bijection $f: A \to B$, then either both A and B are countable or both A and B are uncountable.

Countability

① Can you decompose the given set into a group of subsets?

If so, try to use countable union theorem

② Is it a subset of some countable set? If so, then it's countable.

Is it containing a set that is uncountable? If so, then it is uncountable.

③ Is there a bijection from the set to a known example that is countable (or that is uncountable)?

① (2002 L3 Midterm)

Let S be the set of all ordered pairs (p, C), where $p=(x,y)\in\mathbb{Q}\times\mathbb{Q}$ and

C is the circle with center p, radius $|xy|+1$

Determine if S is countable or not.

Solution. Let T be the set of all such circle C.

Note for each $(x,y)\in\mathbb{Q}\times\mathbb{Q}$, we can let $C_{x,y,|xy|+1}$ denote the circle with center at $p=(x,y)$ and radius $|xy|+1$.

$$T=\{C_{x,y,|xy|+1} : (x,y)\in\mathbb{Q}\times\mathbb{Q}\}$$

$$=\bigcup_{\underbrace{(x,y)\in\mathbb{Q}\times\mathbb{Q}}_{\text{countable}}}\underbrace{\{C_{x,y,|xy|+1}\}}_{\text{1 circle, finite}\Rightarrow\text{countable}}$$

$\therefore T$ is countable by the countable union theorem.

$$S=\{(p,C): p=(x,y)\in\mathbb{Q}\times\mathbb{Q},\ C=C_{x,y,|xy|+1}\}$$

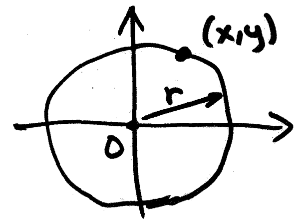
$$\subseteq\underbrace{\underbrace{(\mathbb{Q}\times\mathbb{Q})}_{\text{countable}}\times\underbrace{T}_{\text{countable}}}_{\text{countable by product theorem.}}$$

$\therefore S$ is countable by countable subset theorem.

② P countable subset of $\mathbb{R}^2$. (2003 Final)

Prove $\exists$ circle C with center at $(0,0)$, radius >0 and every point of C is not in P.

Solution. $P=\{(x_1,y_1),(x_2,y_2),\ldots\}$



C contains a point (x,y) of P

$\Longleftrightarrow r=\sqrt{x^2+y^2}$.

Let $S=\{\sqrt{x^2+y^2} : (x,y)\in P\}$.

$$=\bigcup_{\underbrace{(x,y)\in P}_{\text{countable}}}\underbrace{\{\sqrt{x^2+y^2}\}}_{\text{1 element}\Rightarrow\text{countable}}$$

By countable union theorem, S is countable.

Then $(0,+\infty)\setminus S$ is uncountable.

Take $r\in(0,+\infty)\setminus S$. We get a desired circle center at origin and radius r.

③ (2004 L2 Midterm)

Let S be the set of all intersection points $(x,y)\in\mathbb{R}^2$ of the graphs of the equations

$$x^2+my^2=1 \quad\text{and}\quad mx^2+y^2=1,$$

where $m\in\mathbb{Z}\setminus\{-1,1\}$.

Determine if S is countable or uncountable.

Solution For each $m\in\mathbb{Z}\setminus\{-1,1\}$, let

$$S_m=\{(x,y): x,y\in\mathbb{R},\ x^2+my^2=1 \text{ and } mx^2+y^2=1\}.$$

$$\left.\begin{array}{l} x^2+my^2=1 \\ mx^2+y^2=1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x^2+my^2=1 \\ y^2=1-mx^2 \end{array}\right\} \Rightarrow x^2+m(1-mx^2)=1$$

$$\Rightarrow (1-m^2)x^2+m-1=0.$$

at most 2 such x's

For such x, at most 2 y's.

$\therefore$ S_m has at most 4 elements.

$$S=\bigcup_{\underbrace{m\in\mathbb{Z}\setminus\{-1,1\}}_{\text{countable}}} S_m \text{ is countable by countable union theorem.}$$

S_m: countable

countable since $\mathbb{Z}\setminus\{-1,1\}\subseteq\mathbb{Z}$.

④ 2007 Midterm

Let S be a nonempty countable subset of the interval $(0,+\infty)$. Prove that there exists a positive real number which is not the area of any triangle whose three sides have lengths in S.

Solution. Let Area(a,b,c) denote the area of a triangle with sides having lengths a,b,c.

$$T=\{(a,b,c): a,b,c\in S \text{ and } a,b,c \text{ are lengths of sides of a triangle}\}$$

$$\subseteq\{(a,b,c): a,b,c\in S\}=\underbrace{S\times S\times S}_{\text{countable by prod. thm}}$$

S countable $\Rightarrow$ countable by prod. thm

By countable subset theorem,

T is countable.

$$A=\{\text{Area}(a,b,c):(a,b,c)\in T\}$$

$$=\bigcup_{(a,b,c)\in T}\underbrace{\{\text{Area}(a,b,c)\}}_{\substack{1 \text{ number} \\ \Rightarrow \text{countable}}} \text{ is countable by countable union theorem.}$$

T: countable

$\mathbb{R}^+=(0,+\infty)$ is uncountable since $\mathbb{R}^+\supseteq\underbrace{(0,1)}_{\text{uncountable}}$

$\therefore$ $\mathbb{R}^+\setminus A$ is uncountable

$\therefore$ $\mathbb{R}^+\setminus A\neq\emptyset$. So there exists a positive real number which is not the area of any triangle whose three sides have lengths in S.

⑤ 2008 Midterm

Prove that there exists a positive real number c which does not equal to any number of the form $2^{a+b\sqrt{2}}$, where $a,b \in \mathbb{Q}$.

Solution. $S = \{ 2^{a+b\sqrt{2}} : a,b \in \mathbb{Q} \}$

$$= \underbrace{\bigcup_{(a,b)\in \mathbb{Q}\times\mathbb{Q}}}_{\text{Countable}} \underbrace{\{2^{a+b\sqrt{2}}\}}_{\text{1 number} \Rightarrow \text{Countable}}$$

By countable union theorem, S is countable.

Also, $\mathbb{R}^+ = (0, +\infty)$ is uncountable since $\mathbb{R}^+ \supseteq \underbrace{(0,1)}_{\text{uncountable}}$

$\therefore$ $\mathbb{R}^+ \setminus S$ is uncountable

$\therefore$ $\mathbb{R}^+ \setminus S$ is nonempty

$\therefore$ $\exists c \in \mathbb{R}^+$ and $c \notin S$,

that is c is not of the form $2^{a+b\sqrt{2}}$, where $a,b \in \mathbb{Q}$.

⑥ 2009 Fall Midterm

Let S be the set of all points (x,y) in the coordinate plane that satisfy the equations

$$x^2+y^2=a^2 \text{ and } y = x^2 - x^3 + b$$

for some $a,b \in \mathbb{Q}$ with $a \neq b$. Determine (with proof) if S is countable or not.

Solution ① Let $T = \{(a,b) : a,b \in \mathbb{Q}, a \neq b\}$.

Then $T \subseteq \mathbb{Q}\times\mathbb{Q}$. Since $\mathbb{Q}\times\mathbb{Q}$ is countable by product theorem, so T is countable by countable subset theorem.

② For $a,b \in \mathbb{Q}$ with $a \neq b$ ($\Leftrightarrow (a,b) \in T$),

$$\left.\begin{array}{l} x^2+y^2=a^2 \\ y = x^2-x^3+b \end{array}\right\} \Rightarrow \left.\begin{array}{l} x^2+(x^2-x^3+b)^2 = a^2 \\ y = x^2-x^3+b \end{array}\right\} \Rightarrow \begin{array}{l}\text{at most 4 } x\text{'s} \\ \text{One } y \text{ for each } x.\end{array}$$

Let $S_{(a,b)} = \{(x,y) : x^2+y^2=a^2 \text{ and } y = x^2-x^3+b\}$.

Then $S_{(a,b)}$ has at most 4 elements, hence $S_{(a,b)}$ is countable.

$$S = \{(x,y) : x^2+y^2=a^2 \text{ and } y=x^2-x^3+b \text{ for some } (a,b)\in T\}$$

$$= \bigcup_{(a,b)\in \underbrace{T}_{\text{countable by ①}}} \underbrace{S_{(a,b)}}_{\text{countable by ②}}$$

is countable by countable union theorem.