MATH 2033 HW-10 Due Nov 22.

1. Suppose that A is a subset of a metric space (X, d). Prove that
(i) $\overline{A} = A \cup \{\text{all accumulation points of } A\}$.
(ii) $\overline{A} = A \cup \partial A$.
(iii) $\partial A = \overline{A} \cap \overline{A^c}$.

2. Prove that a subset Y of a complete metric space (X, d) is also complete with the inherited metric if and only if Y is closed as a subset of X.

3. For the following sequence $(f_n)_{n\in\mathbb{N}}$ of functions, where $f_n : [0, 2\pi] \to \mathbb{R}$ for all $n \in \mathbb{N}$, find all values of $x \in [0, 2\pi]$ such that the sequence $(f_n)_{n\in\mathbb{N}}$ converges, and find the pointwise limit function $f : [0, 2\pi] \to \mathbb{R}$ if it exists.
(i) $f_n(x) = \sin(\frac{x}{n})$;
(ii) $f_n(x) = \sin(nx)$;
(iii) $f_n(x) = \sin^n(x)$.

4. Let $f_n(x) = x^n$ for $n \in \mathbb{N}$.
(i) Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the function $f(x) = 0$ on the interval $(-1, 1)$.
(ii) Show that if we restrict to the domain $[-\frac{1}{2}, \frac{1}{2}]$, the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the function $f(x) = 0$.
(iii) Show that the sequence $(f_n)_{n\in\mathbb{N}}$ does NOT converge uniformly on the interval $(-1, 1)$.

5. Suppose that (X, d) and (X', d') are metric spaces and that $f : X \to X'$ is continuous. For each of the following statements, determine whether or not it is true. If the assertion is true, prove it. If it is not true, give a counterexample.

i. If A is an open subset of X, then $f(A)$ is an open subset of X';

ii. if A is a closed subset of X, then $f(A)$ is a closed subset of X';

iii. if B is a closed subset of X', then $f^{-1}(B)$ is a closed subset of X;