

Chapter 6 Limits $\epsilon = \text{epsilon}$ $\delta = \text{delta}$

We say a sequence x_1, x_2, x_3, \dots is in S iff every term x_1, x_2, x_3, \dots is an element of the set S .

A sequence x_1, x_2, x_3, \dots in \mathbb{R} is bounded above iff the set $\{x_1, x_2, x_3, \dots\}$ is bounded above in \mathbb{R} . Similarly, one can define sequence bounded below or bounded in \mathbb{R} .

Notations $\forall x, y \in \mathbb{R}$, let $d(x, y) = |x - y|$. This is the distance between x and y .

$$\forall \epsilon > 0, c \in \mathbb{R}, |x - c| < \epsilon \Leftrightarrow -\epsilon < x - c < \epsilon$$

$$\Leftrightarrow c - \epsilon < x < c + \epsilon \Leftrightarrow x \in (c - \epsilon, c + \epsilon)$$

\uparrow ϵ -neighborhood of c .

Intuitive meaning of limit of sequences

A sequence x_1, x_2, x_3, \dots in \mathbb{R} has $c \in \mathbb{R}$ as limit means " x_n may be as close to c as desired when n is sufficiently large" or more loosely "as n tends to ∞ , $d(x_n, c) = |x_n - c|$ goes to 0."

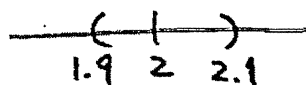
Warning The words "close", "large", "tends", "goes to" are not precise as they involve personal judgements.

Example Let $x_n = \frac{2n^2 - 1}{n^2 + 1}$. We may think its limit is 2.

For every $\epsilon > 0$, Consider the open interval $(2 - \epsilon, 2 + \epsilon)$. If the limit is 2, then we should be able to see $x_n, x_{n+1}, x_{n+2}, \dots$ in $(2 - \epsilon, 2 + \epsilon)$ eventually!

$x_n = \frac{2n^2-1}{n^2+1}$ has 2 as limit should mean that, $\varepsilon > 0$
 for every interval $(2-\varepsilon, 2+\varepsilon)$,
 the sequence x_1, x_2, x_3, \dots
 will get into the interval
 and stay in the interval when n is sufficiently large.

Checking For $\varepsilon = 0.1$, how large should n be so
 x_n will be in $(2-\varepsilon, 2+\varepsilon) = (1.9, 2.1)$?

 Note $x_n \in (2-\varepsilon, 2+\varepsilon)$

$$\Leftrightarrow 2-\varepsilon < x_n < 2+\varepsilon$$

$$\Leftrightarrow -\varepsilon < x_n - 2 < \varepsilon$$

$$\Leftrightarrow |x_n - 2| < \varepsilon$$

$$|x_n - 2| = \left| \frac{2n^2-1}{n^2+1} - 2 \right| = \left| \frac{2n^2-1-2(n^2+1)}{n^2+1} \right| = \frac{3}{n^2+1} < \varepsilon$$

$$\Leftrightarrow \frac{3}{\varepsilon} < n^2+1 \Leftrightarrow \frac{3}{\varepsilon} - 1 < n^2 \Leftrightarrow n > \sqrt{\frac{3}{\varepsilon} - 1}$$

For $\varepsilon = 0.1$, $n > \sqrt{\frac{3}{0.1} - 1} = \sqrt{29} \approx 5.38$, so $n \geq 6 = K$
 is enough.

For $\varepsilon = 0.01$, $n > \sqrt{\frac{3}{0.01} - 1} = \sqrt{299} \approx 17.3$, so $n \geq 18 = K$
 is enough.

For $\varepsilon = 4$, $n^2 > \frac{3}{4} - 1 = -\frac{1}{4}$, so $n \geq 1 = K$ is
 enough.

So $\forall \varepsilon > 0$, let $K = \left\lceil \max\left(\frac{3}{\varepsilon} - 1, 1\right) \right\rceil$, then

$$n \geq K \Rightarrow x_n \in (2-\varepsilon, 2+\varepsilon)$$

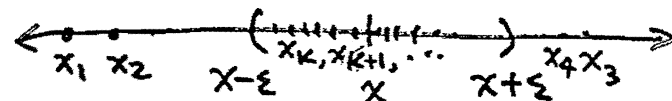
$$\therefore x_K, x_{K+1}, x_{K+2}, \dots \in (2-\varepsilon, 2+\varepsilon)$$

Note for different ε , the value of K will be different.
 We say K depends on ε in such situation.

Definition A sequence x_1, x_2, x_3, \dots converges
 to a number x (or has limit x) iff

$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ (depends on ε) such that

$$x_K, x_{K+1}, x_{K+2}, \dots \in (x-\varepsilon, x+\varepsilon)$$



equivalently,

$$\forall \varepsilon > 0, \exists K \in \mathbb{N} \text{ such that } n \geq K \Rightarrow |x_n - x| < \varepsilon$$

⤴ This version is easier to do computations.

Remarks ① For simple sequences, given ε , it may
 be easy to compute K exactly. However, for
 complicated sequences, all we need to do is to
 show such K exists.

② If we are given x_1, x_2, \dots has limit x , then we
 may set any positive ε and there is a K for
 us to use.

If we are asked to prove x_1, x_2, \dots has limit x ,
 then for every positive ε , we have to find a K
 or show such a K exists as in the definition.

Examples ① $v_n = c$. Prove $\{v_n\}$ converges to c .
 \uparrow sequence v_1, v_2, v_3, \dots

Solution $\forall \varepsilon > 0$, $\overbrace{c-\varepsilon \quad c \quad c+\varepsilon}^{\text{let } K=1, \text{ then}}$
 $n \geq K \Rightarrow |v_n - c| = |c - c| = 0 < \varepsilon$.

② $w_n = c - \frac{1}{n}$. Prove $\{w_n\}$ converges to c .

Solution $\forall \varepsilon > 0$, $\overbrace{c-\varepsilon \quad c \quad c+\varepsilon}^{\text{why is such } n?}$
 $w_n = c - \frac{1}{n}$

(Scratch works: $|w_n - c| = \frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$.)

By Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\varepsilon}$. Then $n \geq K \Rightarrow |w_n - c| = \frac{1}{n} \leq \frac{1}{K} < \varepsilon$.

③ $x_n = \frac{n}{(\cos n) - n}$. Prove $\{x_n\}$ converges to -1 .

Solution $\forall \varepsilon > 0$, $\overbrace{x_1 \quad c-\varepsilon \quad c \quad c+\varepsilon}^{\text{let } K=1, \text{ then}}$
 $x_1 = \frac{n}{(\cos n) - n}$

(Scratch works: $|x_n - (-1)| = \left| \frac{\cos n}{(\cos n) - n} \right|$ \uparrow difficult to solve for n
 $\overbrace{-1 \quad 0 \quad n}^{\text{let } K=1, \text{ then}}$ $n-1 < |\cos n - n| \Rightarrow \frac{1}{n-1} < \varepsilon \Leftrightarrow n > 1 + \frac{1}{\varepsilon}$.)

By Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > 1 + \frac{1}{\varepsilon}$. Then $n \geq K \Rightarrow |x_n - (-1)| = \left| \frac{\cos n}{(\cos n) - n} \right| \leq \frac{1}{n-1} < \varepsilon$.

④ $y_n = (-1)^n$. Prove $\{y_n\}$ does not converge.

Solution Assume $\{y_n\}$ converges to y .
 $\overbrace{-1 \quad 1}^{\text{For } \varepsilon=1, \exists K \text{ such that } n \geq K \Rightarrow (-1)^n - y < \varepsilon}$
 y_1, y_3, \dots y_2, y_4, \dots
 $n \text{ odd} \Rightarrow y \in (-1-\varepsilon, -1+\varepsilon) = (-1.1, -0.9)$
 $n \text{ even} \Rightarrow y \in (-\varepsilon, \varepsilon) = (0.9, 1.1)$. No y satisfies both.

⑤ $z_n = n^{1/n}$. Prove $\{z_n\}$ converges to 1.

Scratch. (Hard to solve $|z_n - 1| = n^{1/n} - 1 < \varepsilon$.)
 (Let $u_n = |z_n - 1| = n^{1/n} - 1 \geq 0$. Then $n^{1/n} = 1 + u_n$,
 $n = (1 + u_n)^n = 1 + nu_n + \frac{n(n-1)}{2}u_n^2 + \dots \geq \frac{n(n-1)}{2}u_n^2$.
 Solving for u_n , we get $u_n \leq \sqrt{\frac{2}{n-1}} < \varepsilon \Leftrightarrow n > 1 + \frac{2}{\varepsilon^2}$.)
Solution $\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > 1 + \frac{2}{\varepsilon^2}$. Then

$$n \geq K \Rightarrow |z_n - 1| \leq \sqrt{\frac{2}{n-1}} \leq \sqrt{\frac{2}{K-1}} < \varepsilon$$

Uniqueness of Limit If $\{x_n\}$ converges to x and y , then $x = y$. (So we may introduce the notation $\lim_{n \rightarrow \infty} x_n = x$.)

Given: ① $\{x_n\}$ converges to x ($\forall \varepsilon_1 > 0 \exists K_1 \in \mathbb{N}$ such that $n \geq K_1 \Rightarrow |x_n - x| < \varepsilon_1$)

② $\{x_n\}$ converges to y ($\forall \varepsilon_2 > 0 \exists K_2 \in \mathbb{N}$ such that $n \geq K_2 \Rightarrow |x_n - y| < \varepsilon_2$)

To Prove: $x = y$ ($\Leftrightarrow \forall \varepsilon > 0, |x - y| < \varepsilon$ Infinitesimal Principle)

Proof. $\forall \varepsilon > 0$, let $\varepsilon_1 = \frac{\varepsilon}{2} > 0$ and $\varepsilon_2 = \frac{\varepsilon}{2} > 0$. Then
 $\exists K_1 \in \mathbb{N}$ such that $n \geq K_1 \Rightarrow |x_n - x| < \varepsilon_1 = \frac{\varepsilon}{2}$
 $\exists K_2 \in \mathbb{N}$ such that $n \geq K_2 \Rightarrow |x_n - y| < \varepsilon_2 = \frac{\varepsilon}{2}$.
 Let $n = \max(K_1, K_2)$. Then $n \geq K_1$ and $n \geq K_2$.

So $|x - y| = |x - x_n + x_n - y| \leq |x - x_n| + |x_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
 \therefore by infinitesimal principle, $x = y$ by triangle inequality

Example

⑥ Prove $\left\{ \frac{2n}{n+5} + \frac{n^8}{n^8+n^5+1} \right\}$ converges to 3 by checking the definition of limit.

Scratch ① When n is large, $\frac{2n}{n+5} \approx 2$, $\frac{n^8}{n^8+n^5+1} \approx 1$.

② $\forall \varepsilon > 0$, $\left| \frac{2n}{n+5} + \frac{n^8}{n^8+n^5+1} - 3 \right| < \varepsilon$ is hard to solve for n

$$= \left| \frac{2n}{n+5} - 2 + \frac{n^8}{n^8+n^5+1} - 1 \right|$$

$$\leq \left| \frac{2n}{n+5} - 2 \right| + \left| \frac{n^8}{n^8+n^5+1} - 1 \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

is easier to solve

③ $\left| \frac{2n}{n+5} - 2 \right| = \left| \frac{-10}{n+5} \right| = \frac{10}{n+5} < \frac{\varepsilon}{2}$ if $n > \frac{20}{\varepsilon} - 5$

$$\left| \frac{n^8}{n^8+n^5+1} - 1 \right| = \left| \frac{-n^5-1}{n^8+n^5+1} \right| = \frac{n^5+1}{n^8+n^5+1} < \frac{2n^5}{n^8} = \frac{2}{n^3} < \frac{\varepsilon}{2}$$

Solution $\forall \varepsilon > 0$, by Archimedean Principle, if $n > \sqrt[3]{4/\varepsilon}$.

$\exists K_1 \in \mathbb{N}$ such that $K_1 > \frac{20}{\varepsilon} - 5$ and

$\exists K_2 \in \mathbb{N}$ such that $K_2 > \sqrt[3]{4/\varepsilon}$.

Let $K = \max(K_1, K_2)$. Then

$$n \geq K \Rightarrow n \geq K_1 > \frac{20}{\varepsilon} - 5 \text{ and } n \geq K_2 > \sqrt[3]{4/\varepsilon}$$

$$\Rightarrow \left| \frac{2n}{n+5} + \frac{n^8}{n^8+n^5+1} - 3 \right| \leq \left| \frac{2n}{n+5} - 2 \right| + \left| \frac{n^8}{n^8+n^5+1} - 1 \right|$$

$$= \frac{10}{n+5} + \frac{n^5+1}{n^8+n^5+1} < \frac{10}{n+5} + \frac{2}{n^3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Boundedness Theorem If $\{x_n\}$ converges, then the set $\{x_1, x_2, x_3, \dots\}$ is bounded (above and below).

Given: $\{x_n\}$ converges to some $x \in \mathbb{R}$ ($\forall \varepsilon > 0 \exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |x_n - x| < \varepsilon$)

To Prove: $\{x_1, x_2, x_3, \dots\}$ is bounded ($\Leftrightarrow \exists M \in \mathbb{R} \forall x_n, |x_n| \leq M$)

Proof. Let $x = \lim_{n \rightarrow \infty} x_n$. For $\varepsilon = 1$, $\exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |x_n - x| < 1 \Rightarrow |x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$.

Let $M = \max(|x_1|, |x_2|, \dots, |x_{K-1}|, 1 + |x|)$. Then

$$\forall n \in \mathbb{N}, n \geq K \Rightarrow |x_n| < 1 + |x| \leq M$$

$$n < K \Rightarrow x_n = x_1 \text{ or } x_2 \text{ or } \dots = x_{K-1} \Rightarrow |x_n| \leq M.$$

Remarks The converse of the boundedness theorem is false. $x_n = (-1)^n$ $\{x_1, x_2, x_3, \dots\} = \{-1, 1\}$ is bounded but $\{x_n\}$ does not converge by example ④.

Remarks The following are equivalent:

- ① $\{x_n\}$ converges to x ($\forall \varepsilon > 0 \exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |x_n - x| < \varepsilon$)
- ② $\{x_n - x\}$ converges to 0 ($\forall \varepsilon > 0 \exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |(x_n - x) - 0| < \varepsilon$)
- ③ $\{|x_n - x|\}$ converges to 0 ($\forall \varepsilon > 0 \exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |x_n - x| - 0| < \varepsilon$.)

Computation Formulas for Limits

Given: ① $\lim_{n \rightarrow \infty} x_n = x$ ($\forall \varepsilon_1 > 0, \exists K_1 \in \mathbb{N}, n \geq K_1 \Rightarrow |x_n - x| < \varepsilon_1$)

② $\lim_{n \rightarrow \infty} y_n = y$ ($\forall \varepsilon_2 > 0, \exists K_2 \in \mathbb{N}, n \geq K_2 \Rightarrow |y_n - y| < \varepsilon_2$)

To prove: $\lim_{n \rightarrow \infty} (x_n + y_n) = x + y$

($\forall \varepsilon > 0, \exists K \in \mathbb{N}, n \geq K \Rightarrow |(x_n + y_n) - (x + y)| < \varepsilon$)

Idea: $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)|$
 $\leq \underbrace{|x_n - x|}_{< \varepsilon/2} + \underbrace{|y_n - y|}_{< \varepsilon/2}$

Proof. $\forall \varepsilon > 0$, let $\varepsilon_1 = \varepsilon/2 > 0$ and $\varepsilon_2 = \varepsilon/2 > 0$.

By ①, $\exists K_1, n \geq K_1 \Rightarrow |x_n - x| < \varepsilon_1 = \varepsilon/2$.

By ②, $\exists K_2, n \geq K_2 \Rightarrow |y_n - y| < \varepsilon_2 = \varepsilon/2$.

Max trick Let $K = \max(K_1, K_2) \in \mathbb{N}$.

$n \geq K \Rightarrow \begin{cases} n \geq K_1 \\ \text{and} \\ n \geq K_2 \end{cases} \Rightarrow \begin{cases} |x_n - x| < \varepsilon/2 \\ \text{and} \\ |y_n - y| < \varepsilon/2 \end{cases}$

$\Rightarrow |(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)|$ (*)

$\leq |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Given: ① and ② above

To Prove: $\lim_{n \rightarrow \infty} (x_n - y_n) = x - y$

($\forall \varepsilon > 0, \exists K \in \mathbb{N}, n \geq K \Rightarrow |(x_n - y_n) - (x - y)| < \varepsilon$)

Proof. Just change the 3 + signs in (*) to - signs above.

Lemma

If (a) $\{a_n\}$ is bounded ($\exists M > 0$ such that $\forall n, |a_n| \leq M$) and (b) $\lim_{n \rightarrow \infty} b_n = 0$ ($\forall \varepsilon_1 > 0, \exists K_1 \in \mathbb{N}, n \geq K_1 \Rightarrow |b_n - 0| < \varepsilon_1$) then $\lim_{n \rightarrow \infty} a_n b_n = 0$ ($\forall \varepsilon > 0, \exists K \in \mathbb{N}, n \geq K \Rightarrow |a_n b_n - 0| < \varepsilon$)

Idea $|a_n b_n - 0| = |a_n b_n| \leq M |b_n| = M |b_n - 0| < M \varepsilon_1 = \varepsilon$

should choose $\varepsilon_1 = \frac{\varepsilon}{M}$.

Proof. $\forall \varepsilon > 0$, let $\varepsilon_1 = \frac{\varepsilon}{M}$, where M is as in (a).

By (b), $\exists K = K_1 \in \mathbb{N}, n \geq K \Rightarrow |b_n - 0| < \varepsilon_1 = \frac{\varepsilon}{M}$

$\Rightarrow |a_n b_n - 0| = |a_n b_n| \leq M |b_n| = M |b_n - 0| < M \varepsilon_1 = \varepsilon$.

Given: ① $\lim_{n \rightarrow \infty} x_n = x$ ($\Leftrightarrow \lim_{n \rightarrow \infty} (x_n - x) = 0$), ② $\lim_{n \rightarrow \infty} y_n = y$ ($\Leftrightarrow \lim_{n \rightarrow \infty} (y_n - y) = 0$)

To Prove: $\lim_{n \rightarrow \infty} x_n y_n = xy$ ($\Leftrightarrow \lim_{n \rightarrow \infty} (x_n y_n - xy) = 0$)
by earlier remark,

Proof. $x_n y_n - xy = x_n y_n - x_n y + x_n y - xy$
 $= x_n (y_n - y) + y (x_n - x)$ } (Δ)

Since $\{x_n\}$ converges, $\{x_n\}$ is bounded by boundedness theorem.
Constant sequence $\{y\}$ is bounded

$\therefore \lim_{n \rightarrow \infty} (x_n y_n - xy) = \lim_{n \rightarrow \infty} (x_n (y_n - y) + y (x_n - x))$ by (Δ)

$= \lim_{n \rightarrow \infty} x_n (y_n - y) + \lim_{n \rightarrow \infty} y (x_n - x)$ by $\lim(a_n + b_n) = \lim a_n + \lim b_n$

$= 0 + 0$ by lemma

$= 0$

Given: ① $\lim_{n \rightarrow \infty} x_n = x$, ② $\forall n \in \mathbb{N}, y_n \neq 0$
and ③ $\lim_{n \rightarrow \infty} y_n = y \neq 0$ ($\forall \varepsilon_1 > 0, \exists K_1 \in \mathbb{N}, n \geq K_1 \Rightarrow |y_n - y| < \varepsilon_1$)

To Prove: $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y}$.

Proof (Step 1) We will show $\lim_{n \rightarrow \infty} \frac{1}{y_n} = \frac{1}{y}$ first.
($\forall \varepsilon > 0, \exists K \in \mathbb{N}, n \geq K \Rightarrow |\frac{1}{y_n} - \frac{1}{y}| < \varepsilon$.)

Since $\frac{1}{2}|y| > 0$, by ③, $\exists K_0 \in \mathbb{N}, n \geq K_0 \Rightarrow |y_n - y| < \frac{1}{2}|y|$
 $\Rightarrow |y| = |y_n - (y_n - y)| \leq |y_n| + |y_n - y| < |y_n| + \frac{1}{2}|y|$
 $\Rightarrow \frac{1}{2}|y| < |y_n|$
 $\Rightarrow \frac{1}{|y_n|} < \frac{1}{\frac{1}{2}|y|}$.

$\forall \varepsilon > 0$, let $\varepsilon_1 = \frac{1}{2}|y|^2 \varepsilon > 0$. By ③, $\exists K_1 \in \mathbb{N}$ such that
 $n \geq K_1 \Rightarrow |y_n - y| < \varepsilon_1 = \frac{1}{2}|y|^2 \varepsilon$.

Max trick Let $K = \max(K_0, K_1)$. Then

$n \geq K \Rightarrow \begin{cases} n \geq K_0 \\ \text{and} \\ n \geq K_1 \end{cases} \Rightarrow \begin{cases} \frac{1}{|y_n|} \leq \frac{1}{\frac{1}{2}|y|} \\ |y_n - y| < \frac{1}{2}|y|^2 \varepsilon \end{cases}$
 $\Rightarrow |\frac{1}{y_n} - \frac{1}{y}| = |\frac{y - y_n}{y_n y}| = \frac{|y_n - y|}{|y_n||y|} < \frac{\frac{1}{2}|y|^2 \varepsilon}{\frac{1}{2}|y||y|} = \varepsilon$.

(Step 2) $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} x_n \frac{1}{y_n} = x \frac{1}{y} = \frac{x}{y}$.

by $\lim(a_n b_n) = (\lim a_n)(\lim b_n)$
and step 1.

Recall $|a - b| < r \Leftrightarrow a \in (b - r, b + r)$.

Sandwich Theorem (or Squeeze Limit Theorem)

If ① $\forall n \in \mathbb{N}, x_n \leq w_n \leq y_n$

and ② $\lim_{n \rightarrow \infty} x_n = z = \lim_{n \rightarrow \infty} y_n$

($\forall \varepsilon > 0 \exists K_1 \in \mathbb{N}, n \geq K_1 \Rightarrow |x_n - z| < \varepsilon$)

($\forall \varepsilon > 0 \exists K_2 \in \mathbb{N}, n \geq K_2 \Rightarrow |y_n - z| < \varepsilon$)

then $\lim_{n \rightarrow \infty} w_n = z$ ($\forall \varepsilon > 0 \exists K \in \mathbb{N}, n \geq K \Rightarrow |w_n - z| < \varepsilon$)

Proof. $\forall \varepsilon > 0$, let $K = \max(K_1, K_2)$, where K_1, K_2 are as in ②. Then

$n \geq K \Rightarrow \begin{cases} n \geq K_1 \\ \text{and} \\ n \geq K_2 \end{cases} \Rightarrow \begin{cases} |x_n - z| < \varepsilon \\ \text{and} \\ |y_n - z| < \varepsilon \end{cases} \Leftrightarrow \begin{cases} x_n \in (z - \varepsilon, z + \varepsilon) \\ y_n \in (z - \varepsilon, z + \varepsilon) \end{cases}$

by ① $\Rightarrow w_n \in (z - \varepsilon, z + \varepsilon) \Leftrightarrow |w_n - z| < \varepsilon$.

Example Let $w_n = \frac{[10^n \sqrt{2}]}{10^n} \in \mathbb{Q}$ for all $n \in \mathbb{N}$.

(Note $w_1 = 1.4$, $w_2 = 1.41$, $w_3 = 1.414$, $w_4 = 1.4142$, ...)

Then $10^n \sqrt{2} - 1 < [10^n \sqrt{2}] \leq 10^n \sqrt{2}$ and so

$$\frac{10^n \sqrt{2} - 1}{10^n} < \frac{[10^n \sqrt{2}]}{10^n} = w_n \leq \sqrt{2}.$$

Since $\lim_{n \rightarrow \infty} \frac{10^n \sqrt{2} - 1}{10^n} = \sqrt{2}$, by sandwich theorem, $\lim_{n \rightarrow \infty} w_n = \sqrt{2}$.

Remark We may replace $\sqrt{2}$ by any real number.
Every real number is the limit of a sequence in \mathbb{Q} .

Limit Inequality

If ① $\forall n \in \mathbb{N}, a_n \geq 0$
and ② $\lim_{n \rightarrow \infty} a_n = a$ ($\forall \varepsilon > 0 \exists K \in \mathbb{N}, n \geq K \Rightarrow |a_n - a| < \varepsilon$)
then $a \geq 0$.

Proof. Assume $a < 0$. Then let $\varepsilon = |a| = -a > 0$.

By ②, $\exists K \in \mathbb{N}, n \geq K \Rightarrow |a_n - a| < \varepsilon = -a$
 $\Rightarrow a_n - a < -a$
 $\Rightarrow a_n < 0$, contradiction to ①.

Remarks ① If $\forall n \in \mathbb{N}, x_n \leq y_n$ and $\lim_{n \rightarrow \infty} x_n = x, \lim_{n \rightarrow \infty} y_n = y$,
then $(a_n = y_n - x_n \geq 0, \lim_{n \rightarrow \infty} a_n = y - x \geq 0) \Rightarrow x \leq y$.

② If $\forall n \in \mathbb{N}, a \leq x_n \leq b$ and $\lim_{n \rightarrow \infty} x_n = x$, then
 $(a = \lim_{n \rightarrow \infty} a \leq \lim_{n \rightarrow \infty} x_n = x, x = \lim_{n \rightarrow \infty} x_n \leq \lim_{n \rightarrow \infty} b = b) \Rightarrow a \leq x \leq b$.

Equivalently, if $\forall n \in \mathbb{N}, x_n \in [a, b]$ and $\lim_{n \rightarrow \infty} x_n = x$, then
 $x \in [a, b]$. This is not true for open intervals !!!

$\frac{1}{n} > 0, \lim_{n \rightarrow \infty} \frac{1}{n} = 0, \frac{1}{n} \in (0, +\infty), \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \notin (0, +\infty)$.

Supremum Limit Theorem

Let c be an upper bound of a nonempty set S . Then
 $(\exists w_n \in S \text{ such that } \lim_{n \rightarrow \infty} w_n = c) \Leftrightarrow c = \sup S$.

Proof. (\Rightarrow) $\exists w_n \in S, \lim_{n \rightarrow \infty} w_n = c$. Since $w_n \in S, w_n \leq \sup S \leq c$.
 Taking limit, $c \leq \sup S \leq c \Rightarrow c = \sup S$. c is an upper bound.

(\Leftarrow) $c = \sup S$. By supremum property, $\forall n \in \mathbb{N}, \exists w_n \in S$ such that
 $c - \frac{1}{n} = \sup S - \frac{1}{n} < w_n \leq \sup S = c$. Sandwich $\Rightarrow \lim_{n \rightarrow \infty} w_n = c$.

Infimum Limit Theorem

Let c be a lower bound of a nonempty set S . Then
 $(\exists w_n \in S \text{ such that } \lim_{n \rightarrow \infty} w_n = c) \Leftrightarrow c = \inf S$.

Proof is similar to the proof of supremum limit theorem.

Examples ① Let $S = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$.

$0 \leq \frac{1}{n} \forall n \in \mathbb{N} \Rightarrow 0$ is a lower bound of S .
 $w_n = \frac{1}{n} \in S, \lim_{n \rightarrow \infty} w_n = 0 \Rightarrow \inf S = 0$.

② Let $S = \{x\pi + \frac{1}{y} : x \in \mathbb{Q} \cap (0, 1], y \in [1, 2]\}$.

$\forall x \in \mathbb{Q} \cap (0, 1], y \in [1, 2], x\pi + \frac{1}{y} > 0\pi + \frac{1}{2} = \frac{1}{2}$
 $\Rightarrow \frac{1}{2}$ is a lower bound of S .
 $w_n = \frac{1}{n}\pi + \frac{1}{2} \in S, \lim_{n \rightarrow \infty} w_n = \frac{1}{2} \Rightarrow \inf S = \frac{1}{2}$.

③ Let A and B be bounded sets in \mathbb{R} .

Let $A - 2B = \{a - 2b : a \in A, b \in B\}$.

Prove $\sup(A - 2B) = \sup A - 2 \inf B$.

Solution. Since A bounded, $\sup A$ exists in \mathbb{R} . Since B bounded, $\inf B$ exists in \mathbb{R} . $\forall a \in A, b \in B$, we have
 $a \leq \sup A, \inf B \leq b \Rightarrow a - 2b \leq \sup A - 2 \inf B$.
 $\therefore c = \sup A - 2 \inf B$ is an upper bound of $A - 2B$.

By supremum limit theorem, $\exists a_n \in A, \lim_{n \rightarrow \infty} a_n = \sup A$.

By infimum limit theorem, $\exists b_n \in B, \lim_{n \rightarrow \infty} b_n = \inf B$.

Then $a_n - 2b_n \in A - 2B$ and $\lim_{n \rightarrow \infty} (a_n - 2b_n) = \sup A - 2 \inf B$.
 \therefore by supremum limit theorem, $\sup(A - 2B) = \sup A - 2 \inf B$.

Question: How can we show a sequence has a limit if it is given by a recurrent relation? For example,

$$x_1 = 2 \text{ and } x_{n+1} = \sqrt{3x_n - 2} \text{ for } n=1, 2, 3, \dots$$

Definition Let $\{x_n\}$ be a sequence of numbers.

$x_{n_1}, x_{n_2}, x_{n_3}, \dots$ is a subsequence of $\{x_n\}$ iff $n_1 < n_2 < n_3 < \dots$ and $n_j \in \mathbb{N} \forall j=1, 2, 3, \dots$

Examples For sequence x_1, x_2, x_3, \dots , if we set $n_j = j^2$, then we get $x_1, x_4, x_9, x_{16}, \dots$, which is a subsequence because $1 < 4 < 9 < 16 < \dots$

If we set $n_j = 2j+1$, then we get $x_3, x_5, x_7, x_9, \dots$, which is a subsequence because $3 < 5 < 7 < 9 < \dots$

Remarks $n_1 < n_2 < n_3 < \dots$ and $n_j \in \mathbb{N} \forall j=1, 2, 3, \dots$
 $\Rightarrow n_j \geq j \forall j=1, 2, 3, \dots$

We can prove this by mathematical induction. For $j=1$, $n_1 \in \mathbb{N} \Rightarrow n_1 \geq 1$. If $n_j \geq j$, then $n_{j+1} > n_j \geq j$ and $n_{j+1} \in \mathbb{N} \Rightarrow n_{j+1} \geq j+1$.

Subsequence Theorem If $\lim_{n \rightarrow \infty} x_n = x$, then for every subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \dots$, we have $\lim_{j \rightarrow \infty} x_{n_j} = x$.

Proof. $\forall \epsilon > 0$, since $\lim_{n \rightarrow \infty} x_n = x$, $\exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |x_n - x| < \epsilon$. Then

$$j \geq K \Rightarrow n_j \geq j \geq K \Rightarrow |x_{n_j} - x| < \epsilon.$$

Terminologies Let $\{x_n\}$ be a sequence of real numbers.

$\{x_n\}$ is increasing iff $x_1 \leq x_2 \leq x_3 \leq \dots$

$\{x_n\}$ is decreasing iff $x_1 \geq x_2 \geq x_3 \geq \dots$

$\{x_n\}$ is strictly increasing iff $x_1 < x_2 < x_3 < \dots$

$\{x_n\}$ is strictly decreasing iff $x_1 > x_2 > x_3 > \dots$

$\{x_n\}$ is monotone iff $\{x_n\}$ is increasing or decreasing.

$\{x_n\}$ is strictly monotone iff $\{x_n\}$ is strictly increasing or strictly decreasing.

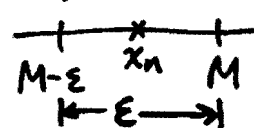
Monotone Sequence Theorem If $\{x_n\}$ is increasing and bounded above, then $\lim_{n \rightarrow \infty} x_n = \sup \{x_1, x_2, x_3, \dots\}$. (Similarly, if $\{x_n\}$ is decreasing and bounded below, then $\lim_{n \rightarrow \infty} x_n = \inf \{x_1, x_2, x_3, \dots\}$.)

Proof. Since $\{x_n\}$ is bounded above, $M = \sup \{x_1, x_2, x_3, \dots\}$ exists. $\forall \epsilon > 0$, by the supremum property, $\exists x_K$ such that $M - \epsilon < x_K \leq M$. Then $x_K \in (M - \epsilon, M]$. So

$$n \geq K \Rightarrow x_K \leq x_n \leq \sup \{x_1, x_2, x_3, \dots\} = M$$

$$\Rightarrow x_n \in (M - \epsilon, M]$$

$$\Rightarrow |x_n - M| < \epsilon$$



The decreasing case is similar.

Examples ① Let $0 < c < 1$ and $x_n = c^{1/n}$ for $n=1, 2, 3, \dots$.

Then $x_n < 1 \forall n$. Also,

$$c^{n+1} < c^n \Rightarrow x_n = c^{1/n} = (c^{n+1})^{1/(n+1)} < (c^n)^{1/(n+1)} = c^{n/(n+1)} = c^{1/n} = x_{n+1}$$

By the monotone sequence theorem, $\{x_n\}$ has a limit x .

Now $x_{2n} = (c^{1/(2n)})^2 = c^{1/n} = x_n$. Taking limits on both sides, by subsequence theorem, $x^2 = x$. So $x=0$ or 1 .

Since $0 < c = x_1 \leq x = \sup\{x_1, x_2, x_3, \dots\}$, we get $x=1$.

(Similarly, if $c \geq 1$, then $x_n = c^{1/n}$ will decrease to the limit 1 .)

② Does $\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}$ represent a real number?

Here, the question is if $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}$ for $n=1, 2, 3, \dots$ converges to a real number.

Scratch Work $x_1 = \sqrt{2} < x_2 = \sqrt{2 + \sqrt{2}} < x_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$
We suspect $\{x_n\}$ is strictly increasing.

Assume $\lim_{n \rightarrow \infty} x_n = x$. Then $x^2 = \lim_{n \rightarrow \infty} x_{n+1}^2 = \lim_{n \rightarrow \infty} (2 + x_n) = 2 + x$
 $\Rightarrow x^2 - x - 2 = (x-2)(x+1) = 0 \Rightarrow x=2$ or -1 reject.

Solution. We will show $x_n < x_{n+1} < 2 \forall n \in \mathbb{N}$ by math induction. For $n=1$, $x_1 = \sqrt{2} < x_2 = \sqrt{2 + \sqrt{2}} < 2$.

If $x_n < x_{n+1} < 2$, then $x_{n+2} = \sqrt{2 + x_{n+1}} < \sqrt{2 + 2} = 2$
 $\Rightarrow x_{n+1} = \sqrt{x_{n+2}} < \sqrt{x_{n+1} + 2} = x_{n+2} < \sqrt{2 + 2} = \sqrt{4} = 2$.

So monotone sequence theorem $\Rightarrow \lim_{n \rightarrow \infty} x_n = x$ exists.

As in scratch work, Since $\sqrt{2} = x_1 \leq x$, we set $x=2$.

Note If $\lim_{n \rightarrow \infty} a_n = x$ and $\lim_{n \rightarrow \infty} b_n = x$, then we expect

$a_1, b_1, a_2, b_2, a_3, b_3, \dots$ converges to x .

Intertwining Sequence Theorem

If ① $\lim_{m \rightarrow \infty} x_{2m-1} = x$ ($\forall \epsilon > 0 \exists K_1 \in \mathbb{N}, m \geq K_1 \Rightarrow |x_{2m-1} - x| < \epsilon$)

and ② $\lim_{m \rightarrow \infty} x_{2m} = x$ ($\forall \epsilon > 0 \exists K_2 \in \mathbb{N}, m \geq K_2 \Rightarrow |x_{2m} - x| < \epsilon$)

then $\lim_{n \rightarrow \infty} x_n = x$ ($\forall \epsilon > 0 \exists K \in \mathbb{N}, n \geq K \Rightarrow |x_n - x| < \epsilon$).

Proof. $\forall \epsilon > 0$, let K_1, K_2 be as in conditions ①, ②.

Let $K = \max(2K_1 - 1, 2K_2)$. Then

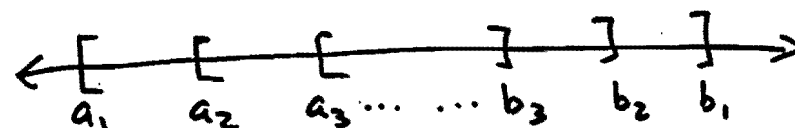
$$n \geq K \Rightarrow \begin{cases} n \geq 2K_1 - 1 \\ \text{and} \\ n \geq 2K_2 \end{cases} \Rightarrow \begin{cases} n \text{ odd} \Rightarrow n = 2m - 1 \text{ with } m \geq K_1 \\ n \text{ even} \Rightarrow n = 2m \text{ with } m \geq K_2 \end{cases}$$

$$\Rightarrow \begin{cases} n \text{ odd} \Rightarrow |x_n - x| = |x_{2m-1} - x| < \epsilon \\ n \text{ even} \Rightarrow |x_n - x| = |x_{2m} - x| < \epsilon \end{cases}$$

Nested Interval Theorem $\swarrow a_n, b_n \in \mathbb{R}$

If $\forall n \in \mathbb{N}, I_n = [a_n, b_n]$ and $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$,
then $\bigcap_{n=1}^{\infty} I_n = [a, b]$, where $a = \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n = b$.

If $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{x\}$ for some $x \in \mathbb{R}$.



Proof. $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ implies $\{a_n\}$ is increasing and bounded above by b_1 and $\{b_n\}$ is decreasing and bounded below by a_1 . By monotone sequence theorem,

$$\lim_{n \rightarrow \infty} a_n = \sup \{a_1, a_2, a_3, \dots\} = a, \quad \lim_{n \rightarrow \infty} b_n = \inf \{b_1, b_2, b_3, \dots\} = b.$$

Since $a_n \leq b_n \quad \forall n$, taking limit on both sides, get $a \leq b$.

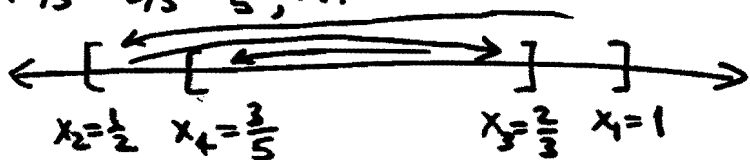
$$\text{Hence } x \in \bigcap_{n=1}^{\infty} [a_n, b_n] \Leftrightarrow \forall n \in \mathbb{N}, a_n \leq x \leq b_n \\ \Leftrightarrow a \leq x \leq b \Leftrightarrow x \in [a, b].$$

If $0 = \lim_{n \rightarrow \infty} (b_n - a_n) = b - a$, then $a = b$, $\bigcap_{n=1}^{\infty} I_n = \{a\}$.

Example. Does $\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$ represent a real number?

Here the question is if $x_1 = 1$ and $x_{n+1} = \frac{1}{1+x_n}$ for $n=1, 2, 3, \dots$ converges to a real number.

Scratch Work $x_1 = 1, x_2 = \frac{1}{1+1} = \frac{1}{2}, x_3 = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}, x_4 = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}, \dots$



Solution (Step 1: Form I_n and show $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$)

Define $I_n = [x_{2n}, x_{2n-1}]$ for $n=1, 2, 3, \dots$

We will show $I_n \supseteq I_{n+1}$ by math induction for $n \in \mathbb{N}$.

$$I_n \supseteq I_{n+1} \Leftrightarrow x_{2n} \leq x_{2n+2} \leq x_{2n+1} \leq x_{2n-1} \quad \forall n \in \mathbb{N}.$$

$$\text{For } n=1, x_2 = \frac{1}{2} \leq x_4 = \frac{3}{5} \leq x_3 = \frac{2}{3} \leq x_1 = 1.$$

If $x_{2n} \leq x_{2n+2} \leq x_{2n+1} \leq x_{2n-1}$, then case n is true

$$1 + x_{2n} \leq 1 + x_{2n+2} \leq 1 + x_{2n+1} \leq 1 + x_{2n-1}$$

$$\Rightarrow \frac{1}{1+x_{2n}} = x_{2n+1} \geq \frac{1}{1+x_{2n+2}} = x_{2n+3} \geq \frac{1}{1+x_{2n+1}} = x_{2n+2} \geq \frac{1}{1+x_{2n-1}} = x_{2n}$$

$$\Rightarrow 1 + x_{2n+1} \geq 1 + x_{2n+3} \geq 1 + x_{2n+2} \geq 1 + x_{2n}$$

$$\Rightarrow \frac{1}{1+x_{2n+1}} = x_{2n+2} \leq \frac{1}{1+x_{2n+3}} = x_{2n+4} \leq \frac{1}{1+x_{2n+2}} = x_{2n+3} \leq \frac{1}{1+x_{2n}} = x_{2n+1}$$

case $n+1$ is true. So $\bigcap_{n=1}^{\infty} I_n = [a, b]$, where $\lim_{n \rightarrow \infty} x_{2n} = a$ and $\lim_{n \rightarrow \infty} x_{2n-1} = b$.

(Step 2: Show $\lim_{n \rightarrow \infty} |x_{2n} - x_{2n-1}| = 0$ and compute limit.)

$$|x_{m+1} - x_m| = \left| \frac{1}{1+x_m} - \frac{1}{1+x_{m-1}} \right| = \frac{|x_m - x_{m-1}|}{(1+x_m)(1+x_{m-1})} \\ \leq \frac{|x_m - x_{m-1}|}{(1+\frac{1}{2})(1+\frac{1}{2})} = \frac{4}{9} |x_m - x_{m-1}|$$

$$\text{So } |x_{2n} - x_{2n-1}| \leq \left(\frac{4}{9}\right)^{n-1} |x_{2n-1} - x_{2n-2}| \leq \left(\frac{4}{9}\right)^{n-2} |x_{2n-2} - x_{2n-3}| \\ \leq \dots \leq \left(\frac{4}{9}\right)^{2n-2} |x_2 - x_1| = \left(\frac{4}{9}\right)^{2n-2} \frac{1}{2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By Sandwich theorem, $\lim_{n \rightarrow \infty} |x_{2n} - x_{2n-1}| = 0$. By the nested interval theorem, $\lim_{n \rightarrow \infty} x_{2n} = x = \lim_{n \rightarrow \infty} x_{2n-1}$. By the intertwining sequence theorem, $\lim_{n \rightarrow \infty} x_n = x$. Taking limit of $x_{n+1} = \frac{1}{1+x_n}$, we get $x = \frac{1}{1+x} \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$. Since $\frac{-1-\sqrt{5}}{2} \notin I_1$, so $x = \frac{-1+\sqrt{5}}{2}$.

Alternative way to do step 2

From end of step 1, we have $\lim_{n \rightarrow \infty} x_{2n} = a, \lim_{n \rightarrow \infty} x_{2n-1} = b$.

Now $x_{2n} = \frac{1}{1+x_{2n-1}} \Rightarrow a = \frac{1}{1+b}$ by taking limit.

Also $x_{2n+1} = \frac{1}{1+x_{2n}} \Rightarrow b = \frac{1}{1+a}$ by taking limit.

$$\left. \begin{array}{l} a = \frac{1}{1+b} \\ b = \frac{1}{1+a} \end{array} \right\} \Rightarrow a(1+b) = 1 = b(1+a) \Rightarrow a+ab = b+ab \Rightarrow a=b$$

$$\left. \begin{array}{l} \text{Then } \lim_{n \rightarrow \infty} x_n = a \text{ and } a = \frac{1}{1+a} \\ x_n \in I_1 \Rightarrow a \in I_1 \end{array} \right\} \Rightarrow a = \frac{-1+\sqrt{5}}{2} \text{ as } \frac{-1-\sqrt{5}}{2} \notin I_1.$$

Question How can we prove a sequence converges without identifying the limit?

In the 19th century, Cauchy introduced the following

Definition $\{x_n\}$ is a Cauchy sequence iff $\forall \varepsilon > 0$
 $\exists K \in \mathbb{N}$ such that $n, m \geq K \Rightarrow |x_n - x_m| < \varepsilon$.

Remarks This means the terms are as close as desired when the indices are sufficiently large.

Example Let $x_n = \frac{1}{n^2}$. Show $\{x_n\}$ is Cauchy.

Scratch Work Say $m \geq n$, $|x_n - x_m| = \frac{1}{n^2} - \frac{1}{m^2} < \frac{1}{n^2} < \varepsilon$
 $n > \frac{1}{\sqrt{\varepsilon}}$ is enough.

Solution. $\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\sqrt{\varepsilon}}$. Then

$$n, m \geq K \Rightarrow |x_n - x_m| = \left| \frac{1}{n^2} - \frac{1}{m^2} \right| < \frac{1}{K^2} < \varepsilon.$$

Cauchy's Theorem $\{x_n\}$ converges $\Leftrightarrow \{x_n\}$ is Cauchy.

Proof (\Rightarrow) Given: $\forall \varepsilon_0 > 0 \exists K_0 \in \mathbb{N}, n \geq K_0 \Rightarrow |x_n - x| < \varepsilon_0$.

To prove: $\forall \varepsilon > 0, \exists K \in \mathbb{N}, m, n \geq K \Rightarrow |x_m - x_n| < \varepsilon$.

Idea: $|x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x - x_n|$
 $< \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\forall \varepsilon > 0$, let $\varepsilon_0 = \varepsilon/2$. We are given that $\exists K_0 \in \mathbb{N}$,

$n \geq K_0 \Rightarrow |x_n - x| < \varepsilon/2$. Set $K = K_0$. Then

$$m, n \geq K \Rightarrow |x_m - x_n| \leq |x_m - x| + |x - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

(\Leftarrow) We are given $\{x_n\}$ is Cauchy. We are to prove $\{x_n\}$ converges to some x . We will do this in 3 steps.

Step 1 $\{x_n\}$ is Cauchy $\Rightarrow \{x_1, x_2, x_3, \dots\}$ is bounded

Step 2 (Bolzano-Weierstrass Theorem)

$\{x_1, x_2, x_3, \dots\}$ is bounded $\Rightarrow \exists$ subsequence $\{x_{n_k}\}$ which converges.

Step 3 $\{x_n\}$ is Cauchy and a subsequence $\{x_{n_k}\}$ converges to x $\Rightarrow \{x_n\}$ converges to x .

For step 1, we modify the proof of the boundedness theorem.

For $\varepsilon = 1$, $\exists K \in \mathbb{N}, n, m \geq K \Rightarrow |x_n - x_m| < \varepsilon = 1$.

So the case $m = K$ means $n \geq K \Rightarrow |x_n - x_K| < 1$

$$\Rightarrow |x_n| = |x_n - x_K + x_K| \leq |x_n - x_K| + |x_K| < 1 + |x_K|.$$

Let $M = \max\{|x_1|, |x_2|, \dots, |x_{K-1}|, 1 + |x_K|\}$.

Then $\forall n \in \mathbb{N}, n = 1, 2, \dots, K-1 \Rightarrow |x_n| \leq M$

$$n \geq K \Rightarrow |x_n| < 1 + |x_K| \leq M.$$

$\therefore \{x_1, x_2, x_3, \dots\}$ is bounded.

For step 2

Let $a_1 = \inf \{x_n\}$, $b_1 = \sup \{x_n\}$ and $I_1 = [a_1, b_1]$.

Let m_1 be the midpoint of I_1 .

If $[a_1, m_1]$ contains infinitely many terms of $\{x_n\}$, then let $a_2 = a_1$, $b_2 = m_1$, and $I_2 = [a_2, b_2]$. Otherwise, $[m_1, b_1]$ contains infinitely many terms of $\{x_n\}$, then let $a_2 = m_1$, $b_2 = b_1$, and $I_2 = [a_2, b_2]$. Let m_2 be the midpoint of I_2 . Keep repeating, we get $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ and since I_{n+1} is either the left or the right half of I_n , we have $\lim_{n \rightarrow \infty} \underbrace{(b_n - a_n)}_{\text{length of } I_n} = \lim_{n \rightarrow \infty} \frac{b_1 - a_1}{2^{n-1}} = 0$. $\therefore \bigcap_{n=1}^{\infty} I_n = \{x\}$.

Take $n_1 = 1$, then $x_{n_1} = x_1 \in I_1$. Since I_2 has infinitely many terms, $\exists x_{n_2} \in I_2$ with $n_2 > n_1$. Keep repeating, we get $x_{n_k} \in I_k$ and $n_1 < n_2 < n_3 < \dots$. So $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. \downarrow as $k \rightarrow \infty$
Now $x_{n_k}, x \in I_k \Rightarrow |x_{n_k} - x| \leq \underbrace{b_k - a_k}_{\text{length of } I_k} \rightarrow 0$
 $\therefore \{x_{n_k}\}$ converges to x .

For step 3 $\forall \varepsilon > 0$, $\{x_n\}$ Cauchy $\Rightarrow \exists K_1 \in \mathbb{N}$
 $m, n \geq K_1 \Rightarrow |x_n - x_m| < \varepsilon/2$
 $\{x_{n_j}\}$ converges to $x \Rightarrow \exists K_2 \in \mathbb{N}$, $j \geq K_2 \Rightarrow |x_{n_j} - x| < \varepsilon/2$.
Let $K = \max(K_1, K_2)$. Then
 $n \geq K \Rightarrow \begin{cases} n_K \geq K \geq K_1 \Rightarrow |x_n - x_{n_K}| < \varepsilon/2 \\ K \geq K_2 \Rightarrow |x_{n_K} - x| < \varepsilon/2 \end{cases}$
 $\Rightarrow |x_n - x| = |x_n - x_{n_K} + x_{n_K} - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Example Let $x_1 = \sin 1$ and $x_k = x_{k-1} + \frac{\sin k}{k^2}$. Prove $\{x_n\}$ converges.

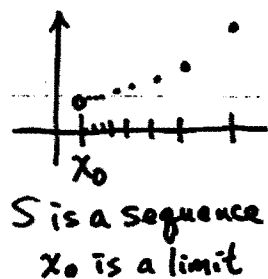
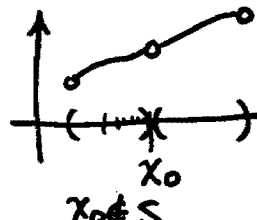
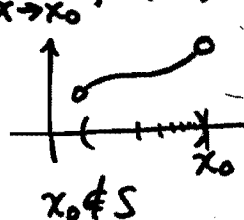
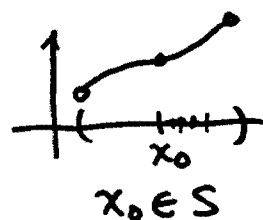
Solution (Scratch work) Check Cauchy Condition

$$\begin{aligned} m > n &\Rightarrow |x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - \dots - x_n| \\ &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n| \\ &= \left| \frac{\sin m}{m^2} \right| + \left| \frac{\sin(m-1)}{(m-1)^2} \right| + \dots + \left| \frac{\sin(n+1)}{(n+1)^2} \right| \\ &\leq \frac{1}{m^2} + \frac{1}{(m-1)^2} + \dots + \frac{1}{(n+1)^2} \\ &< \frac{1}{m(m-1)} + \frac{1}{(m-1)(m-2)} + \dots + \frac{1}{(n+1)n} \\ &= \left(\frac{1}{n} - \frac{1}{n+1} \right) + \dots + \left(\frac{1}{m-2} - \frac{1}{m-1} \right) + \left(\frac{1}{m-1} - \frac{1}{m} \right) \\ &= \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \varepsilon \leftarrow n > \frac{1}{\varepsilon} \text{ is enough} \end{aligned}$$

$\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\varepsilon}$. Then $n, m \geq K \Rightarrow |x_m - x_n| < \left| \frac{1}{n} - \frac{1}{m} \right| < \frac{1}{K} < \varepsilon$.
 $\therefore \{x_n\}$ is a Cauchy sequence. $\therefore \{x_n\}$ converges.

Limit of Functions

Question Let S be an interval (more generally a set). Let $f: S \rightarrow \mathbb{R}$ be a function. At which number x_0 can we consider $\lim_{x \rightarrow x_0} f(x)$?



What do these cases have in common about x_0 and S ?

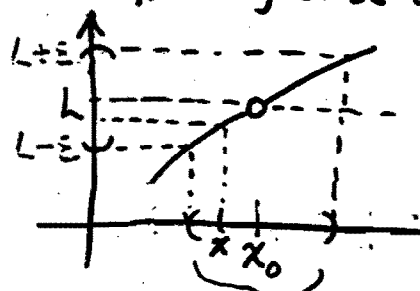
Definition x_0 is an accumulation point (or limit point or cluster point) of S iff $\exists w_n \in S$ such that $\lim_{n \rightarrow \infty} w_n = x_0$, $w_n \neq x_0$

Remarks Accumulation points may or may not be in S .

Notation We write $w_n \rightarrow x_0$ in $S - \{x_0\}$ to mean $w_n \in S$, $w_n \neq x_0$ and $\lim_{n \rightarrow \infty} w_n = x_0$.

Convention When discussing $\lim_{x \rightarrow x_0} f(x)$, we will assume x_0 is an accumulation point of the domain of f .

Let $f: S \rightarrow \mathbb{R}$, $\lim_{x \rightarrow x_0} f(x) = L$ roughly means for any desired distance $\varepsilon > 0$, when $x \in S$, $x \neq x_0$ is sufficiently close to x_0 , we can obtain



$d(f(x), L) < \varepsilon$
distance between $f(x)$ and L

Precise Sufficiently close to x_0

Definition $\lim_{x \rightarrow x_0} f(x) = L$ iff $\forall \varepsilon > 0 \exists \delta > 0$

$\forall x \in S, x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\} \Rightarrow |f(x) - L| < \varepsilon$

Equivalently, $\forall \varepsilon > 0 \exists \delta > 0 \forall x \in S$,
 $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

↗ This is easier to do computations.

Examples ① Let $f(x) = \frac{x^3 - 3x^2}{x - 3} = x^2 \frac{(x - 3)}{x - 3}$. (check $\lim_{x \rightarrow 3} f(x) = 9$)

Scratch Work $x \neq 3 \Rightarrow f(x) = x^2$ $|f(x) - 9| = |x^2 - 9| < \varepsilon$
 $\Leftrightarrow x^2 \in (9 - \varepsilon, 9 + \varepsilon) \Leftrightarrow x \in (\sqrt{9 - \varepsilon}, \sqrt{9 + \varepsilon})$ for $\varepsilon < 9$
 $\leftarrow \delta_1 \quad \leftarrow \delta_2 \rightarrow$
 $\left(\frac{1}{\sqrt{9 - \varepsilon}} \quad 3 \quad \sqrt{9 + \varepsilon} \right)$ Let $\delta = \min(\delta_1, \delta_2) = \min(3 - \sqrt{9 - \varepsilon}, \sqrt{9 + \varepsilon} - 3)$
Then $0 < |x - 3| < \delta \Rightarrow x \in (\sqrt{9 - \varepsilon}, \sqrt{9 + \varepsilon})$
 $\Rightarrow |f(x) - 9| = |x^2 - 9| < \varepsilon$

② Let $g: [0, \infty) \rightarrow \mathbb{R}$ be defined by $g(x) = \sqrt{x}$.

Check: $\lim_{x \rightarrow 0} g(x) = 0$ and $\lim_{x \rightarrow 4} g(x) = 2$.

Solution. (Scratch Work) $|g(x) - 0| = \sqrt{x} < \varepsilon$ $x < \varepsilon^2$ is enough
 $|x - 0|$

$\forall \varepsilon > 0$, let $\delta = \varepsilon^2$, then $\forall x \in [0, \infty)$, $0 < |x - 0| = x < \delta = \varepsilon^2$
 $\Rightarrow |g(x) - 0| = \sqrt{x} < \varepsilon$.

(Scratch Work) $|g(x) - 2| = |\sqrt{x} - 2| = \frac{|x - 4|}{\sqrt{x} + 2} \leq \frac{|x - 4|}{2} < \varepsilon$
 $|x - 4| < 2\varepsilon$ is enough.

$\forall \varepsilon > 0$, let $\delta = 2\varepsilon$, then $\forall x \in [0, \infty)$,
 $0 < |x - 4| < \delta = 2\varepsilon \Rightarrow |g(x) - 2| \leq \frac{|x - 4|}{2} < \varepsilon$.

③ Let $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be defined by $f(x) = \frac{1}{5x}$.

Check: $\lim_{x \rightarrow 2} f(x) = \frac{1}{10}$.

Solution (Scratch Work) x is close to 2
 $|f(x) - \frac{1}{10}| = \left| \frac{1}{5x} - \frac{1}{10} \right| = \frac{|x - 2|}{10x} \leq \frac{|x - 2|}{10} < \varepsilon$
 $|x - 2| < 10\varepsilon$ are enough. if $x \geq 1 \leftarrow \delta < 1$
 min trick and $x \geq 1$

$\forall \varepsilon > 0$, let $\delta = \min(1, 10\varepsilon)$, then $\forall x \in \mathbb{R} \setminus \{0\}$,
 $0 < |x - 2| < \delta \Rightarrow \begin{cases} |x - 2| < 1 \\ \text{and} \\ |x - 2| < 10\varepsilon \end{cases} \Rightarrow \begin{cases} x \in (1, 3) \leftarrow x \geq 1 \\ \text{and} \\ |x - 2| < 10\varepsilon \end{cases}$

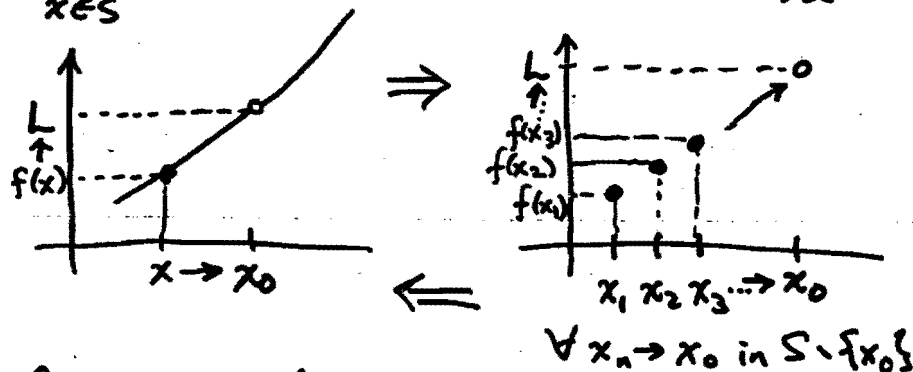
$\Rightarrow |f(x) - \frac{1}{10}| = \frac{|x - 2|}{10x} \leq \frac{|x - 2|}{10} < \varepsilon$.

Recall " $x_n \rightarrow x_0$ in $S \setminus \{x_0\}$ " means $\forall n \in \mathbb{N}, x_n \in S$
 $x_n \neq x_0, \lim_{n \rightarrow \infty} x_n = x_0$

Sequential Limit Theorem (S.L.T.)

Let $f: S \rightarrow \mathbb{R}$ be a function and x_0 be an accumulation point of S . Then

$$\lim_{x \rightarrow x_0} f(x) = L \Leftrightarrow \forall x_n \rightarrow x_0 \text{ in } S \setminus \{x_0\}, \lim_{n \rightarrow \infty} f(x_n) = L.$$



Proof (\Rightarrow)
 Given: $\lim_{x \rightarrow x_0} f(x) = L$ ($\forall \varepsilon > 0 \exists \delta > 0, \forall x \in S$
 $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$)

$x_n \rightarrow x_0$ in $S \setminus \{x_0\}$ ($\forall \delta > 0 \exists K \in \mathbb{N}$
 $n \geq K \Rightarrow |x_n - x_0| < \delta$)

So $\forall \varepsilon > 0 \exists K \in \mathbb{N}$
 $n \geq K \Rightarrow 0 < |x_n - x_0| < \delta \Rightarrow |f(x_n) - L| < \varepsilon$
 $\therefore \lim_{n \rightarrow \infty} f(x_n) = L$

(\Leftarrow) Assume $\lim_{x \rightarrow x_0} f(x) \neq L$.

$\sim (\forall \varepsilon > 0 \exists \delta > 0, \forall x \in S, 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon)$
 $= \exists \varepsilon > 0 \forall \delta > 0 \exists x \in S, 0 < |x - x_0| < \delta \text{ and } |f(x) - L| \geq \varepsilon.$

for $\delta = 1 \exists x_1 \in S, 0 < |x_1 - x_0| < 1$ and $|f(x_1) - L| \geq \varepsilon$

for $\delta = \frac{1}{2} \exists x_2 \in S, 0 < |x_2 - x_0| < \frac{1}{2}$ and $|f(x_2) - L| \geq \varepsilon$

\vdots
 for $\delta = \frac{1}{n} \exists x_n \in S, 0 < |x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon$
 \vdots

$\therefore x_n \in S, 0 < |x_n - x_0| < \frac{1}{n} \Rightarrow x_n \neq x_0$ and $\lim_{n \rightarrow \infty} x_n = x_0$

$\therefore x_n \rightarrow x_0$ in $S \setminus \{x_0\}$. Then $\lim_{n \rightarrow \infty} f(x_n) = L$.

$$|f(x_n) - L| \geq \varepsilon \Rightarrow 0 = |L - L| = \lim_{n \rightarrow \infty} |f(x_n) - L| \geq \varepsilon$$

Contradicting $\varepsilon > 0$.

Application ① $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \xrightarrow{\text{S.L.T.}} \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$
 $x = \frac{1}{n} \rightarrow 0$

\Leftarrow need not just $x_n = \frac{1}{n}$, need all $(x_n \rightarrow 0$ to
 have $\lim_{n \rightarrow \infty} (1+x_n)^{1/x_n} = e$ $x_n \neq 0$

② If $\lim_{x \rightarrow x_0} f(x) = L_1, \lim_{x \rightarrow x_0} g(x) = L_2$ then prove
 $\lim_{x \rightarrow x_0} (f(x) + g(x)) = L_1 + L_2$
 $x \in S \quad (*) \quad x \in S \quad (**)$

Solution 1 $\forall x_n \rightarrow x_0$ in $S \setminus \{x_0\}$,

by S.L.T., $(*) \Rightarrow \lim_{n \rightarrow \infty} f(x_n) = L_1$

$(**) \Rightarrow \lim_{n \rightarrow \infty} g(x_n) = L_2$.

By computation formula, $\lim_{n \rightarrow \infty} (f(x_n) + g(x_n)) = L_1 + L_2$

By S.L.T., $\lim_{x \rightarrow x_0} (f(x) + g(x)) = L_1 + L_2$.

Solution 2

$$\lim_{x \rightarrow x_0} f(x) = L_1 \quad (\forall \varepsilon_1 > 0 \exists \delta_1 > 0 \forall x \in S \quad 0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon_1)$$

$$\lim_{x \rightarrow x_0} g(x) = L_2 \quad (\forall \varepsilon_2 > 0 \exists \delta_2 > 0 \forall x \in S \quad 0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - L_2| < \varepsilon_2)$$

$\forall \varepsilon > 0$, let $\varepsilon_1 = \frac{\varepsilon}{2}$ and $\varepsilon_2 = \frac{\varepsilon}{2}$. From above, get $\delta_1, \delta_2 > 0$

Set $\delta = \min(\delta_1, \delta_2)$. Then $\forall x \in S$,

$$0 < |x - x_0| < \delta \Rightarrow 0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{2}$$

$$0 < |x - x_0| < \delta \Rightarrow 0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - L_2| < \frac{\varepsilon}{2}$$

$$\Rightarrow |(f(x) + g(x)) - (L_1 + L_2)| = |f(x) - L_1 + g(x) - L_2|$$

$$\leq |f(x) - L_1| + |g(x) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\therefore \lim_{x \rightarrow x_0} (f(x) + g(x)) = L_1 + L_2$$

Similarly

If $\lim_{x \rightarrow x_0} f(x) = L_1, \lim_{x \rightarrow x_0} g(x) = L_2$, then

$$\lim_{x \rightarrow x_0} (f(x) - g(x)) = L_1 - L_2$$

$$\lim_{x \rightarrow x_0} f(x)g(x) = L_1 L_2$$

$$\lim_{x \rightarrow x_0} f(x)/g(x) = L_1/L_2$$

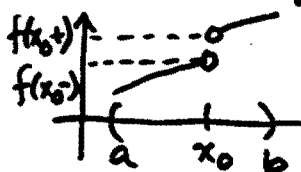
(provided $g(x) \neq 0$ and $L_2 \neq 0$)

If $f(x) \leq g(x) \leq h(x)$ for all $x \in S$, $\lim_{x \rightarrow x_0} f(x) = L = \lim_{x \rightarrow x_0} h(x)$, then $\lim_{x \rightarrow x_0} g(x) = L$.

If $f(x) \geq 0$ for all $x \in S$ and $\lim_{x \rightarrow x_0} f(x) = L$, then $L \geq 0$.

One-sided Limits

Definitions For $f: (a, b) \rightarrow \mathbb{R}$ and $x_0 \in (a, b)$,



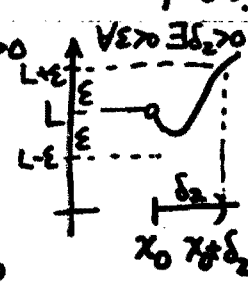
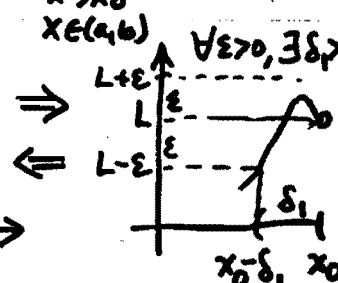
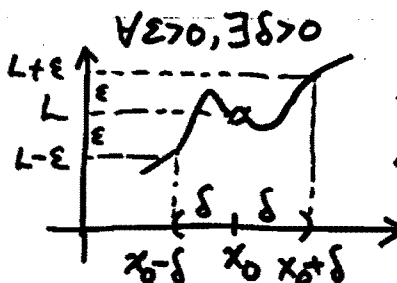
left hand limit of f at x_0

$$f(x_0-) = \lim_{x \rightarrow x_0-} f(x) = \lim_{x \rightarrow x_0} f(x) \quad x \in (a, x_0)$$

right hand limit of f at x_0

$$f(x_0+) = \lim_{x \rightarrow x_0+} f(x) = \lim_{x \rightarrow x_0} f(x) \quad x \in (x_0, b)$$

Theorem For $x_0 \in (a, b)$, $\lim_{x \rightarrow x_0} f(x) = L \Leftrightarrow f(x_0-) = L = f(x_0+) = L$



Proof

$$\lim_{x \rightarrow x_0} f(x) = L$$

$$x \rightarrow x_0$$

$$x \in (a, b)$$

$$\forall \varepsilon > 0 \exists \delta > 0$$

$$\forall x \in (a, b),$$

$$0 < |x - x_0| < \delta$$

$$\Rightarrow |f(x) - L| < \varepsilon$$

\Rightarrow
 let $\delta_1 = \delta$
 $\delta_2 = \delta$

$$f(x_0-) = L$$

$$\lim_{x \rightarrow x_0} f(x)$$

$$x \rightarrow x_0$$

$$x \in (a, x_0)$$

$$\forall \varepsilon > 0 \exists \delta_1 > 0$$

$$\forall x \in (a, x_0)$$

$$0 < |x - x_0| < \delta_1$$

$$\Rightarrow |f(x) - L| < \varepsilon$$

$$f(x_0+) = L$$

$$\lim_{x \rightarrow x_0} f(x)$$

$$x \rightarrow x_0$$

$$x \in (x_0, b)$$

$$\forall \varepsilon > 0 \exists \delta_2 > 0$$

$$\forall x \in (x_0, b)$$

$$0 < |x - x_0| < \delta_2$$

$$\Rightarrow |f(x) - L| < \varepsilon$$

\Leftarrow
let $\delta = \min(\delta_1, \delta_2)$

Definitions Let $f: S \rightarrow \mathbb{R}$ be a function.

- ① f is increasing on S iff $\forall x, y \in S, x < y \Rightarrow f(x) \leq f(y)$.
- ② f is decreasing on S iff $\forall x, y \in S, x < y \Rightarrow f(x) \geq f(y)$.
- ③ f is strictly increasing on S iff $\forall x, y \in S, x < y \Rightarrow f(x) < f(y)$.
- ④ f is strictly decreasing on S iff $\forall x, y \in S, x < y \Rightarrow f(x) > f(y)$.
- ⑤ f is monotone on S iff f is increasing or decreasing on S .
- ⑥ f is strictly monotone on S iff f is strictly increasing or strictly decreasing on S .
- ⑦ f is bounded above on S iff $\{f(x): x \in S\}$ is bounded above.
- ⑧ f is bounded below on S iff $\{f(x): x \in S\}$ is bounded below.
- ⑨ f is bounded on S iff f is bounded above and below.

Monotone Function Theorem

- ① If f is increasing on (a, b) , then $\forall x_0 \in (a, b)$,
 $f(x_0^-) = \sup \{f(x): a < x < x_0\} \Rightarrow f(x_0^-) \leq f(x_0) \leq f(x_0^+)$
 and $f(x_0^+) = \inf \{f(x): x_0 < x < b\}$
 If f is bounded below, then $f(a^+) = \inf \{f(x): a < x < b\}$.
 If f is bounded above, then $f(b^-) = \sup \{f(x): a < x < b\}$.
- ② f has countably many discontinuous points on (a, b)
 $J = \{x_0: x_0 \in (a, b), f(x_0^-) \neq f(x_0^+)\}$ is countable.

Remarks Similarly, the theorem is true for decreasing functions and all other kinds of intervals.

Proof. ① If $a < x < x_0 < b$, then $f(x) \leq f(x_0)$ as f is increasing. So $\{f(x): a < x < x_0\}$ is bounded above by $f(x_0)$. Hence, $M = \sup \{f(x): a < x < x_0\}$ exists by completeness axiom.

To show $f(x_0^-) = M$, $\forall \varepsilon > 0$, by the supremum property, $\exists c \in (a, x_0)$ such that $M - \varepsilon < f(c) \leq M$. Let $\delta = x_0 - c$.

Then $\forall x \in (a, x_0)$, $\frac{M - \varepsilon}{M - f(c)} < \frac{x - c}{x_0 - c} < 1$
 $0 < |x - x_0| < \delta \Rightarrow x \in (x_0 - \delta, x_0) = (c, x_0)$
 $\Rightarrow c < x < x_0 \Rightarrow f(c) \leq f(x) \leq M$
 $\Rightarrow |f(x) - M| = M - f(x) \leq M - f(c) < \varepsilon$

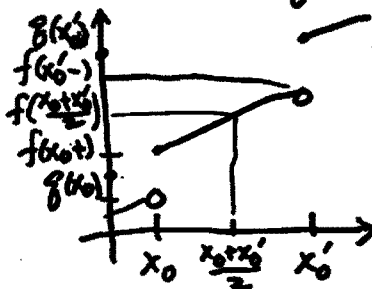
$\therefore f(x_0^-) = \lim_{\substack{x \rightarrow x_0 \\ x \in (a, x_0)}} f(x) = M = \sup \{f(x): a < x < x_0\} \leq f(x_0)$.

The other parts of ① are similarly proved.

- ② f is discontinuous at $x_0 \in (a, b) \Leftrightarrow f(x_0^-) < f(x_0^+)$.
 $\Rightarrow \exists g(x_0) \in \mathbb{Q}$ such that $f(x_0^-) < g(x_0) < f(x_0^+)$
 by density of \mathbb{Q} .

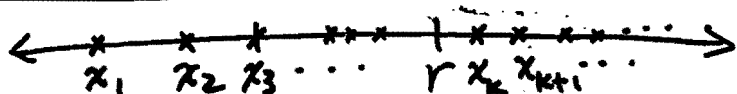
The function $g: J = \{x_0: f(x_0^-) < f(x_0^+)\} \rightarrow \mathbb{Q}$ is injective because $\forall x_0, x_0' \in J$,

$x_0 < x_0' \Rightarrow g(x_0) < f(x_0^+) \leq f\left(\frac{x_0 + x_0'}{2}\right) \leq f(x_0'^-) < g(x_0')$

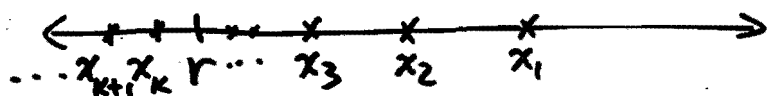


By injection theorem, J is countable.

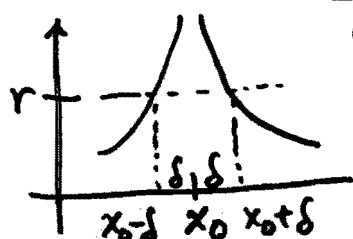
Infinite Limits



Definitions $\{x_n\}$ diverges to $+\infty$ (or $\lim_{n \rightarrow \infty} x_n = +\infty$)
iff $\forall r \in \mathbb{R}, \exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow x_n > r$.



$\{x_n\}$ diverges to $-\infty$ (or $\lim_{n \rightarrow \infty} x_n = -\infty$) iff $\lim_{n \rightarrow \infty} -x_n = +\infty$,
 $\forall r \in \mathbb{R}, \exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow x_n < r$.



Let $f: S \rightarrow \mathbb{R}$ and x_0 be an accumulation point of S .

f diverges to $+\infty$ as x tends to x_0

(or $\lim_{x \rightarrow x_0, x \in S} f(x) = +\infty$) iff

$\forall r \in \mathbb{R} \exists \delta > 0$ such that $\forall x \in S$
 $x \neq x_0$ and $x \in (x_0 - \delta, x_0 + \delta) \Rightarrow f(x) > r$.

$$0 < |x - x_0| < \delta$$

f diverges to $-\infty$ as x tends to x_0 (or $\lim_{x \rightarrow x_0, x \in S} f(x) = -\infty$)

iff $\lim_{x \rightarrow x_0, x \in S} -f(x) = +\infty$,

$\forall r \in \mathbb{R} \exists \delta > 0$ such that $\forall x \in S$

$$0 < |x - x_0| < \delta \Rightarrow f(x) < r.$$

Limit at Infinity

Recall $\lim_{n \rightarrow \infty} a_n = L$ iff $\forall \epsilon > 0 \exists K \in \mathbb{N} n \geq K \Rightarrow |x_n - L| < \epsilon$.

Let $f: S \rightarrow \mathbb{R}$ be a function and $+\infty, -\infty$ are accumulation points of S (that is, \exists sequences in S with $+\infty, -\infty$ as limits). $L \in \mathbb{R}$.

Definitions $\lim_{x \rightarrow +\infty, x \in S} f(x) = L$ iff $\forall \epsilon > 0 \exists K \in \mathbb{R}$
 $x \geq K \Rightarrow |f(x) - L| < \epsilon$.

$\lim_{x \rightarrow -\infty, x \in S} f(x) = L$ iff $\lim_{x \rightarrow +\infty, x \in S} f(-x) = L$ iff $\forall \epsilon > 0 \exists K \in \mathbb{R}$
 $x \leq K \Rightarrow |f(x) - L| < \epsilon$.

Recall $\lim_{n \rightarrow \infty} a_n = +\infty$ iff $\forall r \in \mathbb{R}, \exists K \in \mathbb{N}$
 $n \geq K \Rightarrow a_n > r$.

Definitions $\lim_{x \rightarrow +\infty, x \in S} f(x) = +\infty$ iff $\forall r \in \mathbb{R}, \exists K \in \mathbb{R}$
 $x \geq K \Rightarrow f(x) > r$.

$\lim_{x \rightarrow +\infty, x \in S} f(x) = -\infty$ iff $\lim_{x \rightarrow +\infty, x \in S} -f(x) = +\infty$ iff $\forall r \in \mathbb{R}, \exists K \in \mathbb{R}$
 $x \geq K \Rightarrow f(x) < r$.

$\lim_{x \rightarrow -\infty, x \in S} f(x) = +\infty$ iff $\lim_{x \rightarrow +\infty, x \in S} f(-x) = +\infty$ iff $\forall r \in \mathbb{R}, \exists K \in \mathbb{R}$
 $x \leq K \Rightarrow f(x) > r$.

$\lim_{x \rightarrow -\infty, x \in S} f(x) = -\infty$ iff $\lim_{x \rightarrow +\infty, x \in S} -f(-x) = +\infty$ iff $\forall r \in \mathbb{R}, \exists K \in \mathbb{R}$
 $x \leq K \Rightarrow f(x) < r$.

Chapter 7 Continuity

Definition A function $f: S \rightarrow \mathbb{R}$ is continuous at $x_0 \in S$ iff $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ (more precisely, $\forall \varepsilon > 0 \exists \delta > 0$ such that $\forall x \in S, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$)

For $E \subseteq S$, we say f is continuous on E iff f is continuous at every element of E . Also, we say f is continuous iff f is continuous on the domain S .

Sequential Continuity Theorem (S.C.T.) Drop $x_n \neq x_0$ requirement.

$f: S \rightarrow \mathbb{R}$ is continuous at $x_0 \in S \iff \forall x_n \rightarrow x_0 \text{ in } S, \lim_{n \rightarrow \infty} f(x_n) = f(x_0) = f(\lim_{n \rightarrow \infty} x_n)$

Proof. Just modify the proof of the sequential limit theorem by replacing ① L by $f(x_0)$

② $0 < |x - x_0| < \delta$ by $|x - x_0| < \delta$

③ $x_n \rightarrow x_0$ in $S \setminus \{x_0\}$ by $x_n \rightarrow x_0$ in S (delete $x_n \neq x_0$ requirement)

Examples ① Since $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$, so $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(\theta) = \begin{cases} \frac{\sin \theta}{\theta} & \text{if } \theta \neq 0 \\ 1 & \text{if } \theta = 0 \end{cases}$ is continuous at $x_0 = 0$.

Also $\lim_{n \rightarrow \infty} n \sin \frac{1}{n} = \lim_{n \rightarrow \infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{n \rightarrow \infty} f(\frac{1}{n}) = f(0) = 1$ by S.C.T.

② $\exists f: \mathbb{R} \rightarrow \mathbb{R}$ discontinuous (not continuous) at every $x \in \mathbb{R}$. Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$.

$\forall x_0 \in \mathbb{R}, n \in \mathbb{N}$, by density of \mathbb{Q} and density of $\mathbb{R} \setminus \mathbb{Q}$, $\exists r_n \in \mathbb{Q}, s_n \in \mathbb{R} \setminus \mathbb{Q}$, both $r_n, s_n \in (x_0, x_0 + \frac{1}{n})$.

So $\lim_{n \rightarrow \infty} r_n = x_0 = \lim_{n \rightarrow \infty} s_n$ by Sandwich theorem.

Then $\lim_{n \rightarrow \infty} f(r_n) = \lim_{n \rightarrow \infty} 1 = 1, \lim_{n \rightarrow \infty} f(s_n) = \lim_{n \rightarrow \infty} 0 = 0$, so

$\lim_{x \rightarrow x_0} f(x)$ cannot exist by S.C.T. $\therefore f$ is discontinuous at x_0 .

Theorem If $f, g: S \rightarrow \mathbb{R}$ are continuous at $x_0 \in S$, then $f \pm g, fg, f/g$ (provided $g(x_0) \neq 0$) are continuous at x_0 .

Proof f, g continuous at $x_0 \in S \iff \lim_{x \rightarrow x_0} f(x) = f(x_0), \lim_{x \rightarrow x_0} g(x) = g(x_0)$

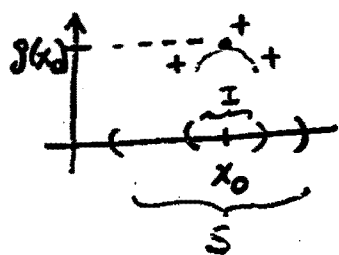
by definition of continuity at x_0 $\Rightarrow \lim_{x \rightarrow x_0} (f \pm g)(x) = (f \pm g)(x_0)$
 see applications of S.L.T. $\lim_{x \rightarrow x_0} (fg)(x) = (fg)(x_0)$
 $\lim_{x \rightarrow x_0} \left(\frac{f}{g}\right)(x) = \left(\frac{f}{g}\right)(x_0)$
 $\iff f \pm g, fg, \frac{f}{g}$ is continuous at x_0

Theorem If $f: S \rightarrow \mathbb{R}$ is continuous at x_0 , $f(S) \subseteq S'$ and $g: S' \rightarrow \mathbb{R}$ is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0 .

Proof. By S.C.T., we need to show $\forall x_n \rightarrow x_0$ in S , $\lim_{n \rightarrow \infty} (g \circ f)(x_n) = (g \circ f)(x_0)$. By S.C.T., since f is continuous at x_0 , $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$. So $f(x_n) \rightarrow f(x_0)$ in S' . Since g is continuous at $f(x_0)$, by S.C.T., $\lim_{n \rightarrow \infty} (g \circ f)(x_n) = \lim_{n \rightarrow \infty} g(f(x_n)) = g(f(x_0)) = (g \circ f)(x_0)$.

Below, S will denote an interval of positive length.

Sign Preserving Property



If $g: S \rightarrow \mathbb{R}$ is continuous and $g(x_0) > 0$, then \exists an interval $I = (x_0 - \delta, x_0 + \delta)$ with $\delta > 0$ such that $g(x) > 0$ for all $x \in S \cap I$.

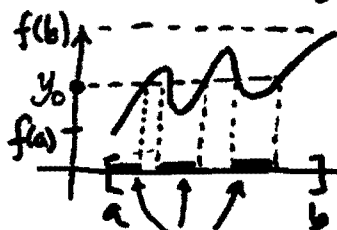
(Similarly for the case $g(x_0) < 0$.)

Proof Let $\varepsilon = g(x_0) > 0$. Note $(g(x_0) - \varepsilon, g(x_0) + \varepsilon) = (0, 2g(x_0))$. Since g is continuous at x_0 , $\exists \delta > 0$ such that $\forall x \in S$, $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon$.

$$\underbrace{x \in S \cap (x_0 - \delta, x_0 + \delta)}_{= I} \Rightarrow \underbrace{g(x) \in (g(x_0) - \varepsilon, g(x_0) + \varepsilon)}_{= (0, 2g(x_0))} \Rightarrow g(x) > 0$$

Intermediate Value Theorem

If $f: [a, b] \rightarrow \mathbb{R}$ is continuous and y_0 is between $f(a)$ and $f(b)$, then $\exists x_0 \in [a, b]$ such that $f(x_0) = y_0$.



Proof Case 1: $y_0 = f(a)$. Take $x_0 = a$.

Case 2: $y_0 = f(b)$. Take $x_0 = b$.

Case 3: $f(a) < y_0 < f(b)$.

$S = \{x \in [a, b] : f(x) \leq y_0\}$
(Case $f(a) > y_0 > f(b)$ is similar.) Let $S = \{x \in [a, b] : f(x) \leq y_0\}$. $S \neq \emptyset$ as $a \in S$. S is bounded above by b . By completeness axiom, $x_0 = \sup S$ exists.

By supremum limit theorem, $\exists x_n \in S$ such that $\lim_{n \rightarrow \infty} x_n = x_0$. $x_n \in [a, b] \Rightarrow a \leq x_n \leq b \Rightarrow a \leq x_0 \leq b \Rightarrow x_0 \in [a, b]$. By S.C.T., $f(x_0) = \lim_{n \rightarrow \infty} f(x_n) \leq y_0$.

Assume $f(x_0) < y_0$. Then $x_0 \neq b$ since $y_0 < f(b)$.

Define $g(x) = y_0 - f(x)$ on $[a, b]$. Then $g(x_0) = y_0 - f(x_0) > 0$.

By sign preserving property, there is interval $I = (x_0 - \delta, x_0 + \delta)$ such that $\forall x \in [a, b] \cap I$, $g(x) > 0$.

Now $x_0 < b \Rightarrow \exists x_1 \in (x_0, b] \cap (x_0, x_0 + \delta) \subseteq [a, b] \cap I$
 $x_0 < x_0 + \delta$

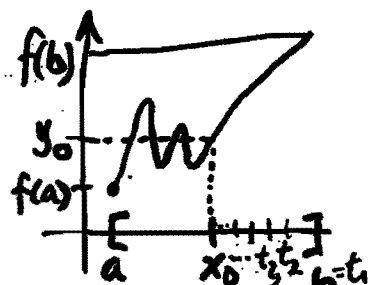
$$\Rightarrow g(x_1) = y_0 - f(x_1) > 0$$

$$\Rightarrow f(x_1) < y_0, \text{ but } x_1 > x_0, x_1 \in [a, b]$$

$\therefore f(x_0) = y_0$. \nwarrow contradict $x_0 = \sup S$.

Alternative Ending (Avoiding sign preserving property)

As in the previous proof, we get $f(x_0) \leq y_0$.



Since $y_0 < f(b)$, we get $x_0 \neq b$.

Let $t_n = x_0 + \frac{1}{n}(b - x_0) \in (x_0, b]$.

$\lim_{n \rightarrow \infty} t_n = x_0$ and $t_n > x_0 = \sup S$

So $t_n \notin S \Rightarrow f(t_n) > y_0$.

By S.C.T., $f(x_0) = \lim_{n \rightarrow \infty} f(t_n) \geq y_0 \therefore f(x_0) = y_0$.

Exercise

Let $f: [0, 1] \rightarrow [0, 1]$ be an increasing function (perhaps discontinuous). Suppose $0 < f(0)$ and $f(1) < 1$. Show that f has at least one fixed point. (A fixed point of f is an element r in the domain of f such that $f(r) = r$.)

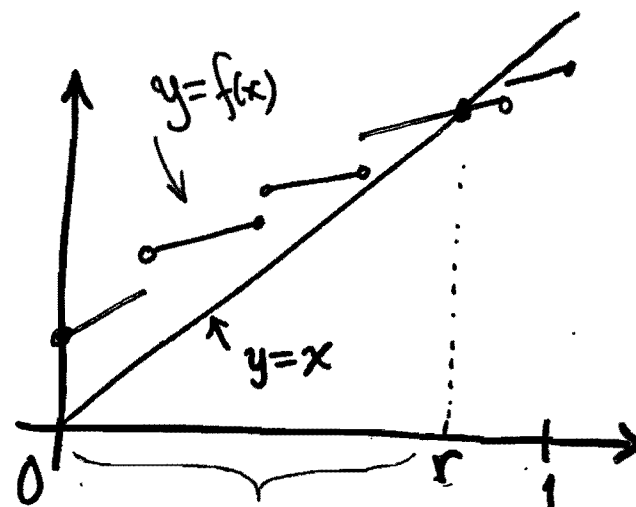
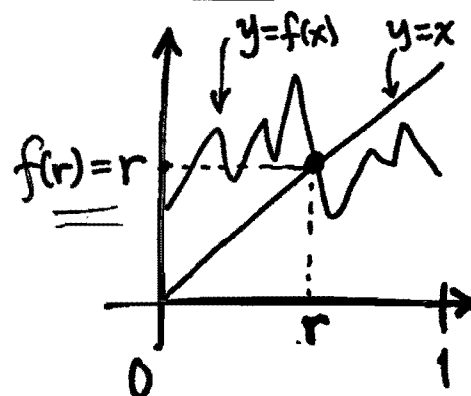
Hint: Sketch the graph of f and consider

$$S = \{t \in [0, 1] : t \leq f(t)\}.$$

Does it have a supremum?

Use monotone function theorem and S.L.T.

Fixed Point



$$S = \{t \in [0, 1] : t \leq f(t)\}$$

Examples ① The equation $x^5 + 3x + \sin x = \cos x + 10$ has a solution.

Let $f(x) = x^5 + 3x + \sin x - \cos x - 10$. Then f is continuous.

$$f(0) = -11 \text{ and } 26 = 2^5 + 3 \cdot 2 - 1 - 1 - 10 \leq f(2)$$

So 0 is between $f(0)$ and $f(2)$. By intermediate value theorem, $\exists x_0 \in [0, 2]$ such that $f(x_0) = 0$. Then

$$x_0^5 + 3x_0 + \sin x_0 = \cos x_0 + 10. \leftarrow x_0 \text{ is a solution of equation.}$$

② Every odd degree polynomial with real coefficients has at least one real root.

Let $P(x) = x^n + a_1x^{n-1} + \dots + a_n$ with n odd.

Let $x_0 = 1 + |a_1| + \dots + |a_n| \geq 1$. Then

$$P(x_0) = x_0^n + a_1x_0^{n-1} + \dots + a_n \Rightarrow x_0^n - P(x_0) = -a_1x_0^{n-1} - \dots - a_n$$

$$P(-x_0) = -x_0^n + a_1x_0^{n-1} - \dots + a_n \Rightarrow x_0^n + P(-x_0) = a_1x_0^{n-1} - \dots + a_n$$

$$\Rightarrow \begin{cases} x_0^n - P(x_0) \\ x_0^n + P(-x_0) \end{cases} \leq |a_1x_0^{n-1}| + \dots + |a_n|$$

$$\leq |a_1|x_0^{n-1} + \dots + |a_n|x_0^{n-1}$$

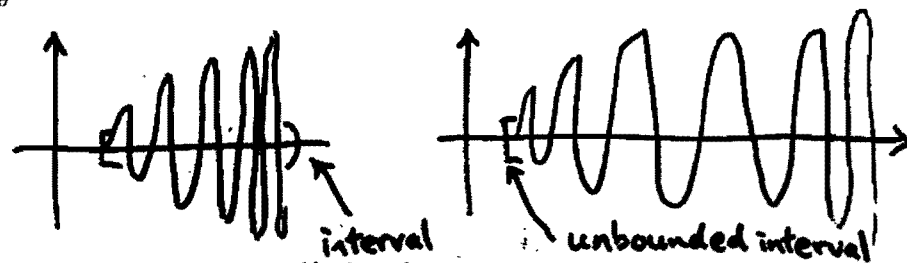
$$= (|a_1| + \dots + |a_n|)x_0^{n-1}$$

$$< x_0^n = x_0 - 1$$

$$\Rightarrow P(x_0) > 0 \text{ and } P(-x_0) < 0$$

$$\Rightarrow 0 \text{ is between } P(-x_0) \text{ and } P(x_0)$$

$$\Rightarrow P \text{ has a real root between } -x_0 \text{ and } x_0.$$



Examples of continuous function with no maximum nor minimum values.

Extreme Value Theorem Let $a, b \in \mathbb{R}$ with $a \leq b$.

If $f: [a, b] \rightarrow \mathbb{R}$ is continuous, then $\exists x_0, w_0 \in [a, b]$ such that $f(w_0) \leq f(x) \leq f(x_0) \quad \forall x \in [a, b]$.

So range of $f = \{f(x) : x \in [a, b]\} = f([a, b])$ is the interval $[f(w_0), f(x_0)]$. In particular, f is bounded on $[a, b]$. $f(x_0) = \sup \{f(x) : x \in [a, b]\} = \max_{x \in [a, b]} f(x)$ and $f(w_0) = \inf \{f(x) : x \in [a, b]\} = \min_{x \in [a, b]} f(x)$.

Proof. Assume $f([a, b])$ is not bounded above. Then every $n \in \mathbb{N}$ is not an upper bound. So $\exists z_n \in [a, b]$ with $f(z_n) > n$. By Bolzano-Weierstrass theorem, $\{z_n\}$ has a subsequence $\{z_{n_j}\}$ converging to some $z_0 \in [a, b]$. Since f is continuous at z_0 , $\lim_{n \rightarrow \infty} f(z_{n_j}) = f(z_0)$ by S.C.T. By boundedness theorem, $\{f(z_{n_j})\}$ is bounded. However, $f(z_{n_j}) > n_j \geq j \Rightarrow \{f(z_{n_j})\}$ is unbounded, a contradiction.

$\therefore f([a, b])$ is bounded above and $M = \sup f([a, b])$ exists.

By supremum limit theorem, $\exists x_n \in [a, b]$ such that $M = \lim_{n \rightarrow \infty} f(x_n)$. By Bolzano-Weierstrass theorem, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ converging to some $x_0 \in [a, b]$. By S.C.T., $f(x_0) = f(\lim_{i \rightarrow \infty} x_{n_i}) = \lim_{i \rightarrow \infty} f(x_{n_i}) = M$.
Similarly, $\exists w_0 \in [a, b]$ with $f(w_0) = \inf f([a, b])$.

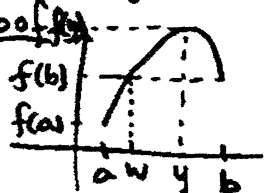
Application. Let $f: [a, b]$ be continuous. Then

$$-\infty < \int_a^b f(x) dx < +\infty \text{ because } \exists x_0, w_0 \in [a, b] \text{ such that } f(w_0) \leq f(x) \leq f(x_0) \Rightarrow \int_a^b f(w_0) dx \leq \int_a^b f(x) dx \leq \int_a^b f(x_0) dx \Rightarrow -\infty < f(w_0)(b-a) \leq \int_a^b f(x) dx < f(x_0)(b-a) < +\infty.$$

Continuous Injection Theorem

If f is continuous and injective on $[a, b]$, then f is strictly monotone on $[a, b]$ and $f([a, b]) = [f(a), f(b)]$ or $[f(b), f(a)]$. (The theorem is true for any other nonempty interval.)

Proof.



Since f is injective, either $f(a) < f(b)$ or $f(a) > f(b)$. Suppose $f(a) < f(b)$.

$\forall y \in (a, b)$, $f(y) > f(b)$ is false

for otherwise, by intermediate value theorem,

$\exists w \in (a, y)$ with $f(w) = f(b)$, contradicting injectivity of f .

Similarly, $f(y) < f(a)$ is false. So $a < y < b \Rightarrow f(a) < f(y) < f(b)$.

Then similarly, $a < x < y \leq b \Rightarrow f(a) \leq f(x) < f(y) \leq f(b)$.

$\therefore f$ is strictly increasing on $[a, b]$ and $f([a, b]) = [f(a), f(b)]$.

The case $f(a) > f(b)$ is similar.

Application The Continuous injection theorem will be used to prove the following theorem, which will be used to prove the $dx/dy = 1/(dy/dx)$ rule for differentiation.

Continuous Inverse Theorem

If f is continuous and injective on $[a, b]$, then

$f^{-1}: f([a, b]) \rightarrow [a, b]$ is continuous and surjective.

(The theorem is true for any other nonempty interval.)

Proof. f^{-1} is surjective because $c \in [a, b] \Rightarrow f(c) \in f([a, b])$ and $f^{-1}(f(c)) = c$.

By Continuous injection theorem, f is strictly monotone. Say strictly increasing. Then f^{-1} is also strictly increasing.

Assume f^{-1} is discontinuous at some $y_0 = f(x_0) \in f([a, b])$. Then either $a \leq f^{-1}(y_0-) < f^{-1}(y_0) = x_0 \leq b$ or $a \leq x_0 = f^{-1}(y_0) < f^{-1}(y_0+) \leq b$. by Monotone function theorem.

This implies either the interval $(f^{-1}(y_0-), f^{-1}(y_0))$ or the interval $(f^{-1}(y_0), f^{-1}(y_0+))$ is not in the range of f^{-1} . This contradicts f^{-1} is surjective.

$\therefore f^{-1}$ is continuous.

