Solution of Math 2033 (Spring 2015) Midterm

① $\inf A = 1$ and $\sup A = 2 \Rightarrow A \subseteq [1,2]$. For $x \in (0, \frac{\pi}{3}] \cap \mathbb{Q}$, $y \in A$, we have $\sqrt{y} \in [1, \sqrt{2}]$, $\cos x \in [\frac{1}{2}, 1)$. So $\frac{1}{2} \le \sqrt{y}\cos x \le \sqrt{2}$, B is bounded.

$\inf A = 1 \Rightarrow \exists\, y_n \in A$ such that $\lim\limits_{n\to\infty} y_n = 1$ by infimum limit theorem.

$\sup A = 2 \Rightarrow \exists\, y_n' \in A$ such that $\lim\limits_{n\to\infty} y_n' = 2$ by supremum limit theorem.

Let $x_n = \frac{[10^n \frac{\pi}{3}]}{10^n} \in (0, \frac{\pi}{3}] \cap \mathbb{Q}$, then $\lim\limits_{n\to\infty} x_n = \frac{\pi}{3}$

Let $x_n' = \frac{1}{n} \in (0, \frac{\pi}{3}] \cap \mathbb{Q}$, then $\lim\limits_{n\to\infty} x_n' = 0$.

Then $\sqrt{y_n}\cos x_n \in B$ and $\lim\limits_{n\to\infty} \sqrt{y_n}\cos x_n = \frac{1}{2}$. Also, $\sqrt{y_n'}\cos x_n' \in B$, $\lim\limits_{n\to\infty}\sqrt{y_n'}\cos x_n' = \sqrt{2}$

Therefore, $\inf B = \frac{1}{2}$ and $\sup B = \sqrt{2}$.

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② (a) <u>Sketch</u> $w_1 = 6$, $w_2 = 6 - \frac{9}{6} = 4.5$, $w_3 = 6 - \frac{9}{4.5} = 4$

$w_3 \quad w_2 \quad w_1$

$w = 6 - \frac{9}{w} \Rightarrow w^2 - 6w + 9 = 0 \Rightarrow (w-3)^2 = 0 \Rightarrow w = 3$

<u>Claim</u>: $\forall n = 1,2,3,\ldots$, $w_n \ge w_{n+1} \ge 3$.

For $n=1$, $w_1 = 6 \ge w_2 = 4.5 \ge 3$. Suppose $w_n \ge w_{n+1} \ge 3$, then $\frac{9}{w_n} \le \frac{9}{w_{n+1}} \le \frac{9}{3}$.

So $w_{n+1} = 6 - \frac{9}{w_n} \ge w_{n+2} = 6 - \frac{9}{w_{n+1}} \ge 6 - \frac{9}{3} = 3$. By M.I., the claim is true.

Since $\{w_n\}$ is decreasing and bounded below, $\lim\limits_{n\to\infty} w_n = w$ exists.

Then $w = \lim\limits_{n\to\infty} w_{n+1} = \lim\limits_{n\to\infty}(6 - \frac{9}{w_n}) = 6 - \frac{9}{w}$, which implies $\underbrace{w^2 - 6w + 9}_{=(w-3)^2} = 0$.

So $w = 3$.

(b) <u>Sketch</u> $x_1 = 60$, $x_2 = 8 + \frac{120}{60} = 10$, $x_3 = 8 + \frac{120}{10} = 20$, $x_4 = 8 + \frac{120}{20} = 14$

$x_2 = 10 \quad 14 \quad 20 \quad 60 = x_1$

$x = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty}(8 + \frac{120}{x_n}) = 8 + \frac{120}{x} \Rightarrow x^2 - 8x - 120 = 0$

$\Rightarrow \underset{x \in [10,60]}{x} = \frac{8 \pm \sqrt{64+480}}{2} \underset{x>0}{\Rightarrow} x = \frac{8+\sqrt{544}}{2}$

<u>Claim</u>: For $I_n = [x_{2n}, x_{2n-1}]$, we have $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$ for $n = 1,2,3,\ldots$

For $n=1$, $x_2 = 10 \le x_4 = 14 \le x_3 = 20 \le x_1 = 60$. Suppose $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$.

Then $\frac{120}{x_{2n}} \ge \frac{120}{x_{2n+2}} \ge \frac{120}{x_{2n+1}} \ge \frac{120}{x_{2n-1}}$. Also $x_{2n+1} = 8 + \frac{120}{x_{2n}} \ge x_{2n+3} = 8 + \frac{120}{x_{2n+2}} \ge x_{2n+2} = 8 + \frac{120}{x_{2n+1}} \ge x_{2n} = 8 + \frac{120}{x_{2n}}$

Then $\frac{120}{x_{2n+1}} \le \frac{120}{x_{2n+3}} \le \frac{120}{x_{2n+2}} \le \frac{120}{x_{2n}}$. So $x_{2n+2} = 8 + \frac{120}{x_{2n+1}} \le x_{2n+4} = 8 + \frac{120}{x_{2n+3}} \le x_{2n+3} = 8 + \frac{120}{x_{2n+2}} \le x_{2n+1} = 8 + \frac{120}{x_{2n}}$

By M.I., the claim is true.

By nested interval theorem, $\lim\limits_{n\to\infty} x_{2n} = a$ and $\lim\limits_{n\to\infty} x_{2n-1} = b$ exist. Then we have

$a = \lim\limits_{n\to\infty} x_{2n} = \lim\limits_{n\to\infty} 8 + \frac{120}{x_{2n-1}} = 8 + \frac{120}{b}$ and $b = \lim\limits_{n\to\infty} x_{2n+1} = \lim\limits_{n\to\infty}(8 + \frac{120}{x_{2n}}) = 8 + \frac{120}{a}$.

So $8b + 120 = ab = 8a + 120 \Rightarrow a = b$. Then $\lim\limits_{n\to\infty} x_n = x$ exists and $x \in [10, 60]$

$x > 0$, $x = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty}(8 + \frac{120}{x_n}) = 8 + \frac{120}{x} \Rightarrow x^2 - 8x - 120 = 0 \Rightarrow x = \frac{8+\sqrt{544}}{2}$ (as $\frac{8-\sqrt{544}}{2} < 0$).

③ Sketch $\frac{9}{z_n^2+2} \to \frac{9}{18} = \frac{1}{2}$, $\frac{5}{y_n-2} \to \frac{5}{2}$. $\left|\frac{9}{z_n^2+2} - \frac{1}{2}\right| = \frac{|z_n^2-16|}{2(z_n^2+2)} \le \frac{|z_n+4||z_n-4|}{4} \le \frac{9}{4}|z_n-4| < \varepsilon/2$

$|z_n-4|<1 \Rightarrow z_n \in (3,5) \Rightarrow z_n+4 \in (7,9)$ $\quad \Updownarrow \; |z_n-4| < \frac{2}{5}\varepsilon$

$\left|\frac{5}{y_n-2} - \frac{5}{2}\right| = \left|\frac{20-5y_n}{2(y_n-2)}\right| = \frac{5|y_n-4|}{2|y_n-2|} \le \frac{5}{2}|y_n-4| < \frac{\varepsilon}{2} \Leftrightarrow |y_n-4| < \frac{\varepsilon}{5}$.

$|y_n-4|<1 \Rightarrow y_n \in (3,5) \Rightarrow y_n-2 \in (1,3)$

Since $\lim_{n\to\infty} z_n = 4$, $1>0 \Rightarrow \exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |z_n-4|<1 \Rightarrow z_n \in (3,5) \Rightarrow z_n+4 \in (7,9)$

Since $\lim y_n = 4$, $1>0 \Rightarrow \exists K_2 \in \mathbb{N}$ such that $n \ge K_2 \Rightarrow |y_n-4|<1 \Rightarrow y_n \in (3,5) \Rightarrow y_n-2 \in (1,3)$

$\forall \varepsilon>0$, $\frac{2}{9}\varepsilon > 0 \Rightarrow \exists K_3 \in \mathbb{N}$ such that $n \ge K_3 \Rightarrow |z_n-4| < \frac{2}{9}\varepsilon$

$\frac{\varepsilon}{5} > 0 \Rightarrow \exists K_4 \in \mathbb{N}$ such that $n \ge K_4 \Rightarrow |y_n-4| < \frac{\varepsilon}{5}$.

Let $K = \max\{K_1, K_2, K_3, K_4\}$. Then $n \ge K$ implies $n \ge K_1, n \ge K_2, n \ge K_3, n \ge K_4$

$$\Rightarrow \left|\frac{9}{z_n^2+2} + \frac{5}{y_n-2} - 3\right| = \left|\left(\frac{9}{z_n^2+2} - \frac{1}{2}\right) + \left(\frac{5}{y_n-2} - \frac{5}{2}\right)\right| \le \left|\frac{9}{z_n^2+2} - \frac{1}{2}\right| + \left|\frac{5}{y_n-2} - \frac{5}{2}\right|$$

$$= \frac{|z_n^2-16|}{2(z_n^2+2)} + \frac{5|y_n-4|}{2|y_n-2|} \underset{\text{see sketch}}{\le} \frac{|z_n+4||z_n-4|}{4} + \frac{5}{2}|y_n-4| < \frac{9}{4}|z_n-4| + \frac{5}{2}|y_n-4| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
