MATH2033 Mathematical Analysis
Problem Set 7

Problem 1

We consider a function $\mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^3 & if \;\; x \in \mathbb{Q} \\ 0 & if \; x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}.$$

(a) Determine if $f(x)$ is differentiable at $x = 0$.
(b) Determine if $f(x)$ is differentiable at $x \neq 0$.
(c) Determine if $f(x)$ is twice differentiable at $x = 0$.

Problem 2

Suppose that $f: \mathbb{R} \to \mathbb{R}$ is differentiable at $x = c$ and $f(c) = 0$. Show that $g(x) = |f(x)|$ is differentiable at $x = c$ if and only if $f'(c) = 0$.

Problem 3 (Harder)

A function $f(x)$ is continuous on (a, b) and has finite derivative $f'(x)$ at every $x \in (a, b) \backslash \{c\}$. Suppose that $\lim_{x \to c} f'(x) = A$, show that f is also differentiable at $x = c$ and $f'(c) = A$.
(😊Hint: Mean value theorem may be useful)

Problem 4

(a) We consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 2x + 1$. Show that the inverse function f^{-1} exists and is differentiable at any $x_0 \in \mathbb{R}$.
(b) We let $g(x) = \tan x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Show that the inverse function $g^{-1}(y) = \tan^{-1} y$ exists and is differentiable at any $y \in \mathbb{R}$. Find $\frac{d}{dy} g^{-1}(y)$.

Problem 5

(a) We let $f(x)$, $g(x)$ be two differentiable functions on $\mathbb{R}$ such that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$, show that $f(x) \leq g(x)$ for all $x \geq 0$.
(b) Show that for any $a > b > 0$, we have $a^{\frac{1}{n}} - b^{\frac{1}{n}} < (a - b)^{\frac{1}{n}}$ for all positive integer $n \geq 2$. (😊Hint: Consider the function $f(x) = x^{\frac{1}{n}} - (x - 1)^{\frac{1}{n}}$ for $x \geq 1$)

Problem 6

It is given that a function $f(x)$ is continuous on $[a, b]$ and is differentiable on (a, b). Suppose that $f(a) = f(b) = 0$, show that for any $\lambda \in \mathbb{R}$, there exists $c \in (a, b)$ such that $f'(c) = \lambda f(c)$. (😊Hint: Apply Rolle's theorem to $g(x)f(x)$, where $g(x)$ is some function depending on λ.)

Problem 7
We let $f(x)$ be a continuous function on $[0,1]$ which $f(0) = 0$ and is differentiable at any $x \in (0,1)$. Prove that if $f'(x)$ is increasing, then a function defined by $g(x) = \frac{f(x)}{x}$ is also increasing.

Problem 8
Suppose that $f(x)$ is differentiable over the interval $(0, \infty)$ and that $\lim_{x \to \infty} f'(x) = 0$. We let $a > 0$ be a positive number and define $g(x) = f(x + a) - f(x)$. Show that $\lim_{x \to \infty} g(x) = 0$.

Problem 9
It is given that a function $f: [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and is differentiable on (a, b). Suppose that $|f'(x)| < 1$ for all $x \in (a, b)$, prove that $f(x) = x$ has at most one solution. (☺Hint: What will happen if there are two or more solutions?)

Problem 10
Show that $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1 + x} \leq 1 + \frac{1}{2}x$ for all $x > 0$.

Problem 11
We let f be a twice differentiable function on (a, b) which $f''(x) \geq 0$ for all $x \in (a, b)$. For any $c \in (a, b)$, show that the graph of $f(x)$ is never below the tangent line to the graph at $\left(c, f(c)\right)$.