# Chapter 6 Limits $\varepsilon$ = epsilon $\delta$ = delta

We say a sequence $x_1, x_2, x_3, \cdots$ is in $S$ iff every term $x_1, x_2, x_3, \cdots$ is an element of the set $S$.

A sequence $x_1, x_2, x_3, \cdots$ in $\mathbb{R}$ is bounded above iff the set $\{x_1, x_2, x_3, \cdots\}$ is bounded above in $\mathbb{R}$. Similarly, one can define sequence bounded below or bounded in $\mathbb{R}$.

Notations $\forall x, y \in \mathbb{R}$, let $d(x,y) = |x-y|$. This is the distance between $x$ and $y$.

$\forall \varepsilon > 0, c \in \mathbb{R}, \ |x-c| < \varepsilon \iff -\varepsilon < x-c < \varepsilon$

$\iff c-\varepsilon < x < c+\varepsilon \iff x \in (c-\varepsilon, c+\varepsilon)$

$\varepsilon$-neighborhood of $c$.

**Intuitive meaning of limit of sequences**

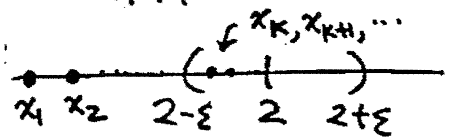
A sequence $x_1, x_2, x_3, \cdots$ in $\mathbb{R}$ has $c \in \mathbb{R}$ as limit means "$x_n$ may be as close to $c$ as desired when $n$ is sufficiently large" or more loosely "as $n$ tends to $\infty$, $d(x_n, c) = |x_n - c|$ goes to $0$."

Warning The words "close", "large", "tends", "goes to" are not precise as they involve personal judgements.

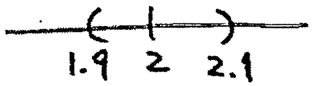
Example Let $x_n = \frac{2n^2-1}{n^2+1}$. We may think its limit is 2.

For every $\varepsilon > 0$, Consider the open interval $(2-\varepsilon, 2+\varepsilon)$. If the limit is 2, then we should be able to see $x_n, x_{n+1}, x_{n+2}, \cdots$ in $(2-\varepsilon, 2+\varepsilon)$ eventually!

$x_n=\dfrac{2n^2-1}{n^2+1}$ has 2 as limit should mean that for <u>every</u> interval $(2-\varepsilon, 2+\varepsilon)$, $\varepsilon>0$, the sequence $x_1, x_2, x_3, \dots$ will <u>get into</u> the interval and <u>stay</u> in the interval when <u>$n$ is sufficiently large</u>.




<u>Checking</u> For $\varepsilon=0.1$, how large should $n$ be so $x_n$ will be in $(2-\varepsilon, 2+\varepsilon)=(1.9, 2.1)$?




Note $x_n \in (2-\varepsilon, 2+\varepsilon)$

$$\iff 2-\varepsilon < x_n < 2+\varepsilon$$
$$\iff -\varepsilon < x_n - 2 < \varepsilon$$
$$\iff |x_n-2| < \varepsilon.$$

$$|x_n-2| = \left|\frac{2n^2-1}{n^2+1}-2\right| = \left|\frac{2n^2-1-2(n^2+1)}{n^2+1}\right| = \frac{3}{n^2+1} < \varepsilon$$

$$\iff \frac{3}{\varepsilon} < n^2+1 \iff \frac{3}{\varepsilon}-1 < n^2 \iff n > \sqrt{\frac{3}{\varepsilon}-1}.$$

For $\varepsilon=0.1$, $n>\sqrt{\frac{3}{0.1}-1}=\sqrt{29}\approx 5.\dots$, so $n\ge 6=K$ is enough.

For $\varepsilon=0.01$; $n>\sqrt{\frac{3}{0.01}-1}=\sqrt{299}\approx 17.\dots$, so $n\ge 18=K$ is enough.

For $\varepsilon=4$, $n^2>\frac{3}{4}-1=-\frac{1}{4}$, so $n\ge 1=K$ is enough.

So $\forall \varepsilon>0$, let $K=\left\lceil \sqrt{\max\left(\frac{3}{\varepsilon}-1, 1\right)}\right\rceil$, then

$n\ge K \Rightarrow x_n\in(2-\varepsilon, 2+\varepsilon)$

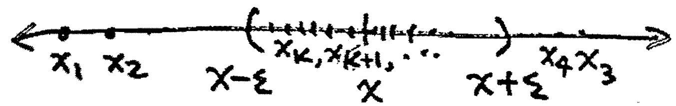
$\therefore x_K, x_{K+1}, x_{K+2}, \dots \in (2-\varepsilon, 2+\varepsilon)$.

Note for different $\varepsilon$, the value of $K$ will be different. We say <u>$K$ depends on $\varepsilon$</u> in such situation.

<u>Definition</u> A sequence $x_1, x_2, x_3, \dots$ <u>converges</u> to a number $x$ (or has <u>limit</u> $x$) iff

$\forall\varepsilon>0$, $\exists K\in\mathbb{N}$ (depends on $\varepsilon$) such that

$$x_K, x_{K+1}, x_{K+2}, \dots \in (x-\varepsilon, x+\varepsilon)$$




equivalently,

$\forall\varepsilon>0$, $\exists K\in\mathbb{N}$ such that $n\ge K \Rightarrow |x_n-x|<\varepsilon$.

This version is easier to do computations.

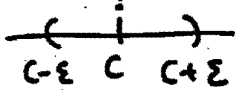
<u>Remarks</u> ① For simple sequences, given $\varepsilon$, it may be easy to compute $K$ exactly. However, for complicated sequences, all we need to do is to show such $K$ <u>exists</u>.

② If we are <u>given</u> $x_1, x_2, \dots$ has limit $x$, then we may set any positive $\varepsilon$ and there is a $K$ for us to use.

If we are asked to <u>prove</u> $x_1, x_2, \dots$ has limit $x$, then for every positive $\varepsilon$, we have to find a $K$ or show such a $K$ exists as in the definition.

**Examples** ① $v_n = c$. Prove $\{v_n\}$ converges to $c$.
↑ sequence $v_1, v_2, v_3, \cdots$

**Solution.** $\forall \varepsilon > 0$,




let $K = 1$, then $n \geq K \Rightarrow |v_n - c| = |c - c| = 0 < \varepsilon$.

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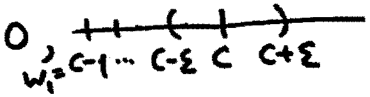
② $w_n = c - \frac{1}{n}$. Prove $\{w_n\}$ converges to $c$.

**Solution** $\forall \varepsilon > 0$,




why is such $n$?

(**Scratch works**: $|w_n - c| = \frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$.)

By Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\varepsilon}$. Then $n \geq K \Rightarrow |w_n - c| = \frac{1}{n} \leq \frac{1}{K} < \varepsilon$.

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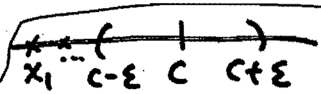
③ $x_n = \frac{n}{(\cos n) - n}$. Prove $\{x_n\}$ converges to $-1$.

**Solution** $\forall \varepsilon > 0$,




(**Scratch works**: $|x_n - (-1)| = \left|\frac{\cos n}{(\cos n) - n}\right|$ $< \varepsilon$ ← difficult to solve for $n$

$\leq \frac{1}{n-1} < \varepsilon \Leftrightarrow n > 1 + \frac{1}{\varepsilon}$.)

By Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > 1 + \frac{1}{\varepsilon}$. Then $n \geq K \Rightarrow |x_n - (-1)| = \left|\frac{\cos n}{(\cos n) - n}\right| \leq \frac{1}{n-1}$
$\leq \frac{1}{K-1} < \varepsilon$.

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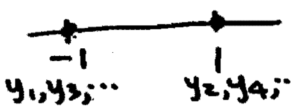
④ $y_n = (-1)^n$. Prove $\{y_n\}$ does not converge.




**Solution** Assume $\{y_n\}$ converges to $y$. For $\varepsilon = 0.1$, $\exists K$ such that $n \geq K \Rightarrow |(-1)^n - y| < \varepsilon$
$n$ odd $\Rightarrow y \in (-1-\varepsilon, -1+\varepsilon) = (-1.1, -0.9)$
$n$ even $\Rightarrow y \in (1-\varepsilon, 1+\varepsilon) = (0.9, 1.1)$. No $y$ satisfies both.

⑤ $z_n = n^{1/n}$. Prove $\{z_n\}$ converges to $1$.

**Scratch.** (Hard to solve $|z_n - 1| = n^{1/n} - 1 < \varepsilon$.)

(Let $u_n = |z_n - 1| = n^{1/n} - 1 \geq 0$. Then $n^{1/n} = 1 + u_n$,

$n = (1+u_n)^n = 1 + n u_n + \frac{n(n-1)}{2} u_n^2 + \cdots \geq \frac{n(n-1)}{2} u_n^2$.

Solving for $u_n$, we get $u_n \leq \sqrt{\frac{2}{n-1}} < \varepsilon \Leftrightarrow n > 1 + \frac{2}{\varepsilon^2}$.)

**Solution** $\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > 1 + \frac{2}{\varepsilon^2}$. Then

$$n \geq K \Rightarrow |z_n - 1| \leq \sqrt{\frac{2}{n-1}} \leq \sqrt{\frac{2}{K-1}} < \varepsilon.$$

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**Uniqueness of Limit** If $\{x_n\}$ converges to $x$ and $y$, then $x = y$. (So we may introduce the notation $\lim_{n\to\infty} x_n = x$.)

**Given**: ① $\{x_n\}$ converges to $x$ ($\forall \varepsilon_1 > 0\ \exists K_1 \in \mathbb{N}$ such that $n \geq K_1 \Rightarrow |x_n - x| < \varepsilon_1$)

② $\{x_n\}$ converges to $y$ ($\forall \varepsilon_2 > 0\ \exists K_2 \in \mathbb{N}$ such that $n \geq K_2 \Rightarrow |x_n - y| < \varepsilon_2$)

**To Prove**: $x = y$ ($\Leftrightarrow \forall \varepsilon > 0$, $|x - y| < \varepsilon$ Infinitesimal Principle)

**Proof.** $\forall \varepsilon > 0$, let $\varepsilon_1 = \frac{\varepsilon}{2} > 0$ and $\varepsilon_2 = \frac{\varepsilon}{2} > 0$. Then

$\exists K_1 \in \mathbb{N}$ such that $n \geq K_1 \Rightarrow |x_n - x| < \varepsilon_1 = \frac{\varepsilon}{2}$

$\exists K_2 \in \mathbb{N}$ such that $n \geq K_2 \Rightarrow |x_n - y| < \varepsilon_2 = \frac{\varepsilon}{2}$.

Let $n = \max(K_1, K_2)$. Then $n \geq K_1$ and $n \geq K_2$.

So $|x - y| = |x - x_n + x_n - y| \underset{\text{by triangle inequality}}{\leq} |x - x_n| + |x_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$\therefore$ by infinitesimal principle, $x = y$.

**Boundedness Theorem** If $\{x_n\}$ converges, then the set $\{x_1, x_2, x_3, \dots\}$ is bounded (above and below).

Given: $\{x_n\}$ converges to some $x \in \mathbb{R}$ ($\forall \varepsilon > 0\ \exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |x_n - x| < \varepsilon$)

To Prove: $\{x_1, x_2, x_3, \cdots\}$ is bounded ($\Leftrightarrow \exists M \in \mathbb{R}$ $\forall x_n,\ |x_n| \leq M$)

Proof. Let $x = \lim_{n\to\infty} x_n$. For $\varepsilon = 1$, $\exists K \in \mathbb{N}$ such that

$$n \geq K \Rightarrow |x_n - x| < 1 \Rightarrow |x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|.$$

Let $M = \max(|x_1|, |x_2|, \dots, |x_{K-1}|, 1 + |x|)$. Then

$\forall n \in \mathbb{N}$, $n \geq K \Rightarrow |x_n| < 1 + |x| \leq M$
$n < K \Rightarrow x_n = x_1$ or $x_2$ or $\dots = x_{K-1} \Rightarrow |x_n| \leq M$.

**Remarks** The converse of the boundedness theorem is false. $x_n = (-1)^n$ $\{x_1, x_2, x_3, \cdots\} = \{-1, 1\}$ is bounded but $\{x_n\}$ does not converge by example ④.

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**Remarks** The following are equivalent:

① $\{x_n\}$ converges to $x$ ($\forall \varepsilon > 0\ \exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |x_n - x| < \varepsilon$)

② $\{x_n - x\}$ converges to $0$ ($\forall \varepsilon > 0\ \exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |(x_n - x) - 0| < \varepsilon$)

③ $\{|x_n - x|\}$ converges to $0$ ($\forall \varepsilon > 0\ \exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow ||x_n - x| - 0| < \varepsilon$.)

## Computation Formulas for Limits

**Given:** ① $\lim_{n\to\infty} x_n = x$ $(\forall \varepsilon_1>0, \exists K_1 \in \mathbb{N}, n\ge K_1 \Rightarrow |x_n - x|<\varepsilon_1)$

② $\lim_{n\to\infty} y_n = y$ $(\forall \varepsilon_2>0, \exists K_2 \in \mathbb{N}, n\ge K_2 \Rightarrow |y_n - y|<\varepsilon_2)$

**To prove:** $\lim_{n\to\infty}(x_n+y_n) = x+y$

$(\forall \varepsilon>0, \exists K \in \mathbb{N}, n\ge K \Rightarrow |(x_n+y_n)-(x+y)|<\varepsilon)$

**Idea:** $|(x_n+y_n)-(x+y)| = |(x_n-x)+(y_n-y)|$

$\le \underbrace{|x_n-x|}_{<\varepsilon/2} + \underbrace{|y_n-y|}_{<\varepsilon/2}$

**Proof.** $\forall \varepsilon>0$, let $\varepsilon_1 = \varepsilon/2 > 0$ and $\varepsilon_2 = \varepsilon/2 > 0$.

By ①, $\exists K_1$, $n \ge K_1 \Rightarrow |x_n - x| < \varepsilon_1 = \varepsilon/2$.

By ②, $\exists K_2$, $n \ge K_2 \Rightarrow |y_n - y| < \varepsilon_2 = \varepsilon/2$.

**Max trick** Let $K = \max(K_1, K_2) \in \mathbb{N}$.

$$n\ge K \Rightarrow \begin{cases} n\ge K_1 \\ \text{and} \\ n \ge K_2 \end{cases} \Rightarrow \begin{cases} |x_n-x|<\varepsilon/2 \\ \text{and} \\ |y_n-y|<\varepsilon/2 \end{cases}$$

$$\Rightarrow |(x_n+y_n)-(x+y)| = |(x_n-x)+(y_n-y)| \qquad (*)$$

$$\le |x_n-x|+|y_n-y| < \varepsilon/2+\varepsilon/2 = \varepsilon.$$

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**Given:** ① and ② above

**To Prove:** $\lim_{n\to\infty}(x_n-y_n) = x-y$

$(\forall \varepsilon>0, \exists K \in \mathbb{N}, n\ge K \Rightarrow |(x_n-y_n)-(x-y)|<\varepsilon)$

**Proof.** Just change the 3 + signs in $(*)$ to − signs above.

## Lemma

If (a) $\{a_n\}$ is bounded ($\exists M>0$ such that $\forall n$, $|a_n|\le M$)
and (b) $\lim_{n\to\infty} b_n = 0$ $(\forall \varepsilon_1>0, \exists K_1 \in \mathbb{N}, n\ge K_1 \Rightarrow |b_n-0|<\varepsilon_1)$
then $\lim_{n\to\infty} a_n b_n = 0$ $(\forall \varepsilon>0, \exists K \in \mathbb{N}, n\ge K \Rightarrow |a_nb_n-0|<\varepsilon)$

**Idea** $|a_nb_n-0| = |a_nb_n| \le M|b_n| = M|b_n-0| < M\varepsilon_1 = \varepsilon$

should choose $\varepsilon_1 = \varepsilon/M$.

**Proof.** $\forall \varepsilon>0$, let $\varepsilon_1 = \varepsilon/M$, where $M$ is as in (a).

By (b), $\exists K = K_1 \in \mathbb{N}$, $n\ge K \Rightarrow |b_n-0|<\varepsilon_1 = \varepsilon/M$

$\Rightarrow |a_nb_n-0| = |a_nb_n| \le M|b_n| = M|b_n-0| < M\varepsilon_1 = \varepsilon$.

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**Given:** ① $\lim_{n\to\infty} x_n = x$ $(\Leftrightarrow \lim_{n\to\infty}(x_n-x)=0)$, ② $\lim_{n\to\infty} y_n = y$ $(\Leftrightarrow \lim_{n\to\infty}(y_n-y)=0)$

**To Prove:** $\lim_{n\to\infty} x_ny_n = xy$ $(\Leftrightarrow \lim_{n\to\infty}(x_ny_n-xy)=0$ by earlier remark)

**Proof.**
$$\left.\begin{aligned} x_ny_n - xy &= x_ny_n - x_ny + x_ny - xy \\ &= x_n(y_n-y) + y(x_n-x) \end{aligned}\right\} (\Delta)$$

Since $\{x_n\}$ converges, $\{x_n\}$ is bounded by boundedness theorem.
Constant sequence $\{y\}$ is bounded

$\therefore \lim_{n\to\infty}(x_ny_n-xy) = \lim_{n\to\infty}(x_n(y_n-y)+y(x_n-x))$ by $(\Delta)$

$= \lim_{n\to\infty} x_n(y_n-y) + \lim_{n\to\infty} y(x_n-x)$ by $\lim(a_n+b_n) = \lim a_n + \lim b_n$

$= 0 + 0$ by lemma

$= 0$

**Given**: ① $\lim_{n\to\infty} x_n = x$, ② $\forall n \in \mathbb{N}$, $y_n \neq 0$

and ③ $\lim_{n\to\infty} y_n = y \neq 0$ $(\forall \varepsilon_1 > 0, \exists K_1 \in \mathbb{N}, n \geq K_1 \Rightarrow |y_n - y| < \varepsilon_1)$

**To Prove**: $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.

**Proof** (Step 1) We will show $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$ first.

$$(\forall \varepsilon > 0, \exists K \in \mathbb{N}, n \geq K \Rightarrow \left|\tfrac{1}{y_n} - \tfrac{1}{y}\right| < \varepsilon)$$

Since $\frac{1}{2}|y| > 0$, by ③, $\exists K_0 \in \mathbb{N}$, $n \geq K_0 \Rightarrow |y_n - y| < \frac{1}{2}|y|$

$\Rightarrow |y| = |y_n - (y_n - y)| \leq |y_n| + |y_n - y| < |y_n| + \frac{1}{2}|y|$

$\Rightarrow \frac{1}{2}|y| < |y_n|$

$\Rightarrow \frac{1}{|y_n|} < \frac{1}{\frac{1}{2}|y|}$.

$\forall \varepsilon > 0$, let $\varepsilon_1 = \frac{1}{2}|y|^2 \varepsilon > 0$. By ③, $\exists K_1 \in \mathbb{N}$ such that $n \geq K_1 \Rightarrow |y_n - y| < \varepsilon_1 = \frac{1}{2}|y|^2 \varepsilon$.

**Max trick** Let $K = \max(K_0, K_1)$. Then

$$n \geq K \Rightarrow \begin{cases} n \geq K_0 \\ \text{and} \\ n \geq K_1 \end{cases} \Rightarrow \begin{cases} \frac{1}{|y_n|} \leq \frac{1}{\frac{1}{2}|y|} \\ |y_n - y| < \frac{1}{2}|y|^2 \varepsilon \end{cases}$$

$$\Rightarrow \left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y_n y}\right| = \frac{|y_n - y|}{|y_n||y|} < \frac{\frac{1}{2}|y|^2 \varepsilon}{\frac{1}{2}|y||y|} = \varepsilon.$$

(Step 2)

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} x_n \frac{1}{y_n} = x \frac{1}{y} = \frac{x}{y}.$$

by $\lim(a_n b_n) = (\lim a_n)(\lim b_n)$ and step 1.

**Recall** $|a - b| < r \Leftrightarrow a \in (b - r, b + r)$.

**Sandwich Theorem** (or **Squeeze Limit Theorem**)

If ① $\forall n \in \mathbb{N}$, $x_n \leq w_n \leq y_n$

and ② $\lim_{n\to\infty} x_n = z = \lim_{n\to\infty} y_n$

$(\forall \varepsilon > 0\ \exists K_1 \in \mathbb{N}, n \geq K_1 \Rightarrow |x_n - z| < \varepsilon$

$\forall \varepsilon > 0\ \exists K_2 \in \mathbb{N}, n \geq K_2 \Rightarrow |y_n - z| < \varepsilon)$

then $\lim_{n\to\infty} w_n = z$ $(\forall \varepsilon > 0\ \exists K \in \mathbb{N}, n \geq K \Rightarrow |w_n - z| < \varepsilon)$

**Proof.** $\forall \varepsilon > 0$, let $K = \max(K_1, K_2)$, where $K_1, K_2$ are as in ②. Then

$$n \geq K \Rightarrow \begin{cases} n \geq K_1 \\ \text{and} \\ n \geq K_2 \end{cases} \Rightarrow \begin{cases} |x_n - z| < \varepsilon \\ \text{and} \\ |y_n - z| < \varepsilon \end{cases} \Leftrightarrow \begin{cases} x_n \in (z - \varepsilon, z + \varepsilon) \\ y_n \in (z - \varepsilon, z + \varepsilon) \end{cases}$$

$$\xRightarrow{\text{by ①}} w_n \in (z - \varepsilon, z + \varepsilon) \Leftrightarrow |w_n - z| < \varepsilon.$$

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**Example** Let $w_n = \frac{[10^n \sqrt{2}]}{10^n} \in \mathbb{Q}$ for all $n \in \mathbb{N}$.

(Note $w_1 = 1.4$, $w_2 = 1.41$, $w_3 = 1.414$, $w_4 = 1.4142, \ldots$)

Then $10^n \sqrt{2} - 1 < [10^n \sqrt{2}] \leq 10^n \sqrt{2}$ and so

$$\frac{10^n \sqrt{2} - 1}{10^n} < \frac{[10^n \sqrt{2}]}{10^n} = w_n \leq \sqrt{2}.$$

Since $\lim_{n\to\infty} \frac{10^n \sqrt{2} - 1}{10^n} = \sqrt{2}$, by sandwich theorem, $\lim_{n\to\infty} w_n = \sqrt{2}$.

**Remark** We may replace $\sqrt{2}$ by any real number. Every real number is the limit of a sequence in $\mathbb{Q}$.

## Limit Inequality

If ① $\forall n\in\mathbb{N}$, $a_n\ge 0$

and ② $\lim_{n\to\infty} a_n = a$ $(\forall\varepsilon>0\ \exists K^{\in\mathbb{N}}, n\ge K \Rightarrow |a_n-a|<\varepsilon)$

then $a\ge 0$.

Proof. Assume $a<0$. Then let $\varepsilon=|a|=-a>0$.

By ②, $\exists K^{\in\mathbb{N}}$, $n\ge K \Rightarrow |a_n-a|<\varepsilon=-a$

$\Rightarrow a_n-a<-a$

$\Rightarrow a_n<0$, contradiction to ①.

Remarks ① If $\forall n\in\mathbb{N}$, $x_n\le y_n$ and $\lim_{n\to\infty}x_n=x$, $\lim_{n\to\infty}y_n=y$, then $(a_n=y_n-x_n\ge 0,\ \lim_{n\to\infty}a_n=y-x\ge 0)$ $x\le y$.

② If $\forall n\in\mathbb{N}$, $a\le x_n\le b$ and $\lim_{n\to\infty}x_n=x$, then $(a=\lim_{n\to\infty}a\le\lim_{n\to\infty}x_n=x,\ x=\lim_{n\to\infty}x_n\le\lim_{n\to\infty}b=b)$ $a\le x\le b$.

Equivalently, if $\forall n\in\mathbb{N}$, $x_n\in[a,b]$ and $\lim_{n\to\infty}x_n=x$, then $x\in[a,b]$. This is not true for open intervals !!!

$\frac{1}{n}>0$, $\lim_{n\to\infty}\frac{1}{n}=0$. $\frac{1}{n}\in(0,+\infty)$, $\lim_{n\to\infty}\frac{1}{n}=0\notin(0,+\infty)$

## Supremum Limit Theorem

Let $c$ be an upper bound of a nonempty set $S$. Then $(\exists w_n\in S$ such that $\lim_{n\to\infty}w_n=c) \Leftrightarrow c=\sup S$.

Proof. $(\Rightarrow)$ $\exists w_n\in S$, $\lim_{n\to\infty}w_n=c$. Since $w_n\in S$, $w_n\le\sup S\le c$. Taking limit, $c\le\sup S\le c$. $\therefore c=\sup S$. $c$ is an upper bound

$(\Leftarrow)$ $c=\sup S$. By supremum property, $\forall n\in\mathbb{N}$, $\exists w_n\in S$ such that $\underbrace{c-\frac{1}{n}=\sup S-\frac{1}{n}}_{\to c}<w_n\le\underbrace{\sup S=c}_{\to c}$. Sandwich $\Rightarrow \lim_{n\to\infty}w_n=c$.

## Infimum Limit Theorem

Let $c$ be a lower bound of a nonempty set $S$. Then $(\exists w_n\in S$ such that $\lim_{n\to\infty}w_n=c) \Leftrightarrow c=\inf S$.

Proof is similar to the proof of supremum limit theorem.

Examples ① Let $S=\{\frac{1}{n}: n\in\mathbb{N}\}=\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\}$.

$0\le\frac{1}{n}\ \forall n\in\mathbb{N} \Rightarrow 0$ is a lower bound of $S$

$w_n=\frac{1}{n}\in S$, $\lim_{n\to\infty}w_n=0$

$\Rightarrow \inf S = 0$.

② Let $S=\{x\pi+\frac{1}{y}: x\in\mathbb{Q}\cap(0,1],\ y\in[1,2]\}$.

$\forall x\in\mathbb{Q}\cap(0,1]$, $y\in[1,2]$, $x\pi+\frac{1}{y}>0\pi+\frac{1}{2}=\frac{1}{2}$

$\Rightarrow \frac{1}{2}$ is a lower bound of $S$

$w_n=\frac{1}{n}\pi+\frac{1}{2}\in S$, $\lim_{n\to\infty}w_n=\frac{1}{2}$

$\Rightarrow \inf S=\frac{1}{2}$.

③ Let $A$ and $B$ be bounded sets in $\mathbb{R}$.

Let $A-2B=\{a-2b: a\in A, b\in B\}$.

Prove $\sup(A-2B)=\sup A-2\inf B$.

Solution. Since $A$ bounded, $\sup A$ exists in $\mathbb{R}$. Since $B$ bounded, $\inf B$ exists in $\mathbb{R}$. $\forall a\in A, b\in B$, we have $a\le\sup A$, $\inf B\le b \Rightarrow a-2b\le\sup A-2\inf B$.

$\therefore$ $c=\sup A-2\inf B$ is an upper bound of $A-2B$.

By supremum limit theorem, $\exists a_n\in A$, $\lim_{n\to\infty}a_n=\sup A$.

By infimum limit theorem, $\exists b_n\in B$, $\lim_{n\to\infty}b_n=\inf B$.

Then $a_n-2b_n\in A-2B$ and $\lim_{n\to\infty}(a_n-2b_n)=\sup A-2\inf B$.

$\therefore$ by supremum limit theorem, $\sup(A-2B)=\sup A-2\inf B$.

**Question**: How can we show a sequence has a limit if it is given by a recurrent relation? For example, $x_1 = 2$ and $x_{n+1} = \sqrt{3x_n - 2}$ for $n = 1, 2, 3, \ldots$

**Definition** Let $\{x_n\}$ be a sequence of numbers. $x_{n_1}, x_{n_2}, x_{n_3}, \cdots$ is a subsequence of $\{x_n\}$ iff $n_1 < n_2 < n_3 < \cdots$ and $n_j \in \mathbb{N}$ $\forall j = 1, 2, 3, \cdots$

**Examples** For sequence $x_1, x_2, x_3, \cdots$, if we set $n_j = j^2$, then we get $x_1, x_4, x_9, x_{16}, \cdots$, which is a subsequence because $1 < 4 < 9 < 16 < \cdots$.

If we set $n_j = 2j+1$, then we get $x_3, x_5, x_7, x_9, \cdots$ which is a subsequence because $3 < 5 < 7 < 9 < \cdots$.

**Remarks** $n_1 < n_2 < n_3 < \cdots$ and $n_j \in \mathbb{N}$ $\forall j = 1, 2, 3, \cdots$
$\Rightarrow n_j \geq j$ $\forall j = 1, 2, 3, \ldots$

We can prove this by mathematical induction. For $j = 1$, $n_1 \in \mathbb{N} \Rightarrow n_1 \geq 1$. If $n_j \geq j$, then $n_{j+1} > n_j \geq j$ and $n_{j+1} \in \mathbb{N} \Rightarrow n_{j+1} \geq j+1$.

**Subsequence Theorem** If $\lim_{n\to\infty} x_n = x$, then for every subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \cdots$, we have $\lim_{j\to\infty} x_{n_j} = x$.

**Proof.** $\forall \varepsilon > 0$, since $\lim_{n\to\infty} x_n = x$, $\exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |x_n - x| < \varepsilon$. Then

$$j \geq K \Rightarrow n_j \geq j \geq K \Rightarrow |x_{n_j} - x| < \varepsilon.$$

**Terminologies** Let $\{x_n\}$ be a sequence of real numbers.

$\{x_n\}$ is increasing iff $x_1 \leq x_2 \leq x_3 \leq \cdots$.

$\{x_n\}$ is decreasing iff $x_1 \geq x_2 \geq x_3 \geq \cdots$.

$\{x_n\}$ is strictly increasing iff $x_1 < x_2 < x_3 < \cdots$.

$\{x_n\}$ is strictly decreasing iff $x_1 > x_2 > x_3 > \cdots$.

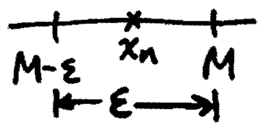
$\{x_n\}$ is monotone iff $\{x_n\}$ is increasing or decreasing.

$\{x_n\}$ is strictly montone iff $\{x_n\}$ is strictly increasing or strictly decreasing.

**Monotone Sequence Theorem** If $\{x_n\}$ is increasing and bounded above, then $\lim_{n\to\infty} x_n = \sup\{x_1, x_2, x_3, \cdots\}$. (Similarly, if $\{x_n\}$ is decreasing and bounded below, then $\lim_{n\to\infty} x_n = \inf\{x_1, x_2, x_3, \cdots\}$.)

**Proof.** Since $\{x_n\}$ is bounded above, $M = \sup\{x_1, x_2, x_3, \cdots\}$ exists. $\forall \varepsilon > 0$, by the supremum property, $\exists x_K$ such that $M - \varepsilon < x_K \leq M$. Then $x_K \in (M - \varepsilon, M]$. So

$$n \geq K \Rightarrow x_K \leq x_n \leq \sup\{x_1, x_2, x_3, \cdots\} = M$$
$$\Rightarrow x_n \in (M - \varepsilon, M]$$
$$\Rightarrow |x_n - M| < \varepsilon$$




The decreasing case is similar.

**Examples** ① Let $0<c<1$ and $x_n = c^{1/n}$ for $n=1,2,3,\cdots$.
Then $x_n<1\ \forall n$. Also,

$$c^{n+1}<c^n \Rightarrow x_n = c^{1/n} = (c^{n+1})^{\frac{1}{n(n+1)}} < (c^n)^{\frac{1}{n(n+1)}} = c^{\frac{1}{n+1}} = x_{n+1}$$

By the monotone sequence theorem, $\{x_n\}$ has a limit $x$.
Now $x_{2n}^2 = (c^{\frac{1}{2n}})^2 = c^{\frac{1}{n}} = x_n$. Taking limits on both sides, by subsequence theorem, $x^2=x$. So $x=0$ or $1$.
Since $0<c=x_1\le x=\sup\{x_1,x_2,x_3,\cdots\}$, we get $x=1$.
(Similarly, if $c\ge 1$, then $x_n=c^{1/n}$ will decrease to the limit $1$.)

② Does $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$ represent a real number?
Here, the question is if $x_1=\sqrt{2}$ and $x_{n+1}=\sqrt{2+x_n}$ for $n=1,2,3,\cdots$ converges to a real numbers.

**Scratch Work** $x_1=\sqrt{2} < x_2=\sqrt{2+\sqrt{2}} < x_3=\sqrt{2+\sqrt{2+\sqrt{2}}}$
We suspect $\{x_n\}$ is strictly increasing.
Assume $\lim\limits_{n\to\infty} x_n = x$. Then $x^2=\lim\limits_{n\to\infty} x_{n+1}^2 = \lim\limits_{n\to\infty}(2+x_n)=2+x$

$$\Rightarrow x^2-x-2=(x-2)(x+1)=0 \Rightarrow x=2 \text{ or } \cancel{-1} \text{ reject.}$$

**Solution.** We will show $x_n<x_{n+1}<2\ \forall n\in\mathbb{N}$ by math induction. For $n=1$, $x_1=\sqrt{2}<x_2=\sqrt{2+\sqrt{2}}<2$.
If $x_n<x_{n+1}<2$, then $x_n+2<x_{n+1}+2<2+2=4$

$$\Rightarrow x_{n+1}=\sqrt{x_n+2}<\sqrt{x_{n+1}+2}=x_{n+2}<\sqrt{2+2}=\sqrt{4}=2.$$

So monotone sequence theorem $\Rightarrow \lim\limits_{n\to\infty} x_n = x$ exists.
As in scratch work, since $\sqrt{2}=x_1\le x$, we get $x=2$.

**Note** If $\lim\limits_{n\to\infty} a_n = x$ and $\lim\limits_{n\to\infty} b_n = x$, then we expect $a_1,b_1,a_2,b_2,a_3,b_3,\cdots$ converges to $x$.

**Intertwining Sequence Theorem**

If ① $\lim\limits_{m\to\infty} x_{2m-1}=x$ $(\forall\varepsilon>0\ \exists K_1\in\mathbb{N}, m\ge K_1\Rightarrow |x_{2m-1}-x|<\varepsilon)$
and ② $\lim\limits_{m\to\infty} x_{2m}=x$ $(\forall\varepsilon>0\ \exists K_2\in\mathbb{N}, m\ge K_2\Rightarrow |x_{2m}-x|<\varepsilon)$
then $\lim\limits_{n\to\infty} x_n=x$ $(\forall\varepsilon>0\ \exists K\in\mathbb{N}, n\ge K\Rightarrow |x_n-x|<\varepsilon)$

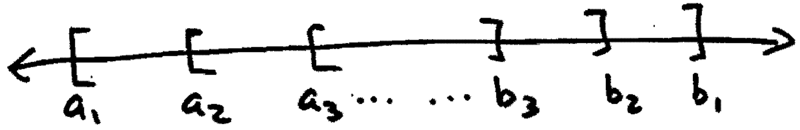
**Proof.** $\forall\varepsilon>0$, let $K_1,K_2$ be as in conditions ①,②.
Let $K=\max(2K_1-1, 2K_2)$. Then

$$n\ge K \Rightarrow \begin{cases} n\ge 2K_1-1 \\ \text{and} \\ n\ge 2K_2\end{cases} \Rightarrow \begin{cases} n \text{ odd} \Rightarrow n=2m-1 \text{ with } m\ge K_1 \\ n \text{ even} \Rightarrow n=2m \text{ with } m\ge K_2\end{cases}$$

$$\Rightarrow \begin{cases} n \text{ odd} \Rightarrow |x_n-x|=|x_{2m-1}-x|<\varepsilon \\ n \text{ even} \Rightarrow |x_n-x|=|x_{2m}-x|<\varepsilon \end{cases}.$$

**Nested Interval Theorem** $a_n,b_n\in\mathbb{R}$

If $\forall n\in\mathbb{N}$, $I_n=[a_n,b_n]$ and $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$, then $\bigcap\limits_{n=1}^{\infty} I_n=[a,b]$, where $a=\lim\limits_{n\to\infty}a_n\le\lim\limits_{n\to\infty}b_n=b$.
If $\lim\limits_{n\to\infty}(b_n-a_n)=0$, then $\bigcap\limits_{n=1}^{\infty} I_n=\{x\}$ for some $x\in\mathbb{R}$.



**Proof.** $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ implies $\{a_n\}$ is increasing and bounded above by $b_1$ and $\{b_n\}$ is decreasing and bounded below by $a_1$. By monotone sequence theorem, $\lim_{n\to\infty} a_n = \sup\{a_1, a_2, a_3, \dots\} = a$, $\lim_{n\to\infty} b_n = \inf\{b_1, b_2, b_3, \dots\} = b$. Since $a_n \le b_n$ $\forall n$, taking limit on both sides, get $a \le b$. Hence $x \in \bigcap_{n=1}^{\infty} [a_n, b_n] \iff \forall n \in \mathbb{N},\ a_n \le x \le b_n$

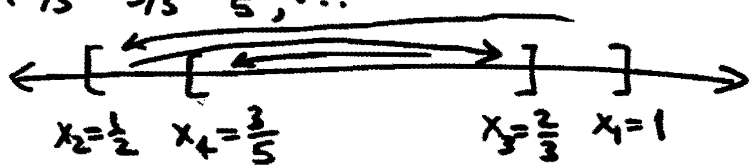
$\iff a \le x \le b \iff x \in [a,b]$.

If $0 = \lim_{n\to\infty}(b_n - a_n) = b - a$, then $a = b$, $\bigcap_{n=1}^{\infty} I_n = \{a\}$.

**Example.** Does $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}$ represent a real number?

Here the question is if $x_1 = 1$ and $x_{n+1} = \frac{1}{1+x_n}$ for $n = 1, 2, 3, \dots$ converges to a real number.

**Scratch Work** $x_1 = 1$, $x_2 = \frac{1}{1+1} = \frac{1}{2}$, $x_3 = \frac{1}{1+\frac{1}{2}} = \frac{1}{3/2} = \frac{2}{3}$, $x_4 = \frac{1}{1+2/3} = \frac{1}{5/3} = \frac{3}{5}, \dots$




**Solution** (Step 1: Form $I_n$ and show $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$)

Define $I_n = [x_{2n}, x_{2n-1}]$ for $n = 1, 2, 3, \dots$.

We will show $I_n \supseteq I_{n+1}$ by math induction for $n \in \mathbb{N}$.

$I_n \supseteq I_{n+1} \iff x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$ $\forall n \in \mathbb{N}$.

For $n=1$, $x_2 = \frac{1}{2} \le x_4 = \frac{3}{5} \le x_3 = \frac{2}{3} \le x_1 = 1$.

If $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$, then case $n$ is true

$1 + x_{2n} \le 1 + x_{2n+2} \le 1 + x_{2n+1} \le 1 + x_{2n-1}$

$\Rightarrow \frac{1}{1+x_{2n}} = x_{2n+1} \ge \frac{1}{1+x_{2n+2}} = x_{2n+3} \ge \frac{1}{1+x_{2n+1}} = x_{2n+2} \ge \frac{1}{1+x_{2n-1}} = x_{2n}$

$\Rightarrow 1 + x_{2n+1} \ge 1 + x_{2n+3} \ge 1 + x_{2n+2} \ge 1 + x_{2n}$

$\Rightarrow \frac{1}{1+x_{2n+1}} = x_{2n+2} \le \frac{1}{1+x_{2n+3}} = x_{2n+4} \le \frac{1}{1+x_{2n+2}} = x_{2n+3} \le \frac{1}{1+x_{2n}} = x_{2n+1}$

case $n+1$ is true. So $\bigcap_{n=1}^{\infty} I_n = [a,b]$, where $\lim_{n\to\infty} x_{2n} = a$ and $\lim_{n\to\infty} x_{2n-1} = b$.

(Step 2: Show $\lim_{n\to\infty} |x_{2n} - x_{2n-1}| = 0$ and compute limit.)

$$|x_{m+1} - x_m| = \left|\frac{1}{1+x_m} - \frac{1}{1+x_{m-1}}\right| = \frac{|x_m - x_{m-1}|}{(1+x_m)(1+x_{m-1})}$$

$$\le \frac{|x_m - x_{m-1}|}{(1+\frac{1}{2})(1+\frac{1}{2})} = \frac{4}{9}|x_m - x_{m-1}|$$

So $|x_{2n} - x_{2n-1}| \le (\frac{4}{9})^1 |x_{2n-1} - x_{2n-2}| \le (\frac{4}{9})^2 |x_{2n-2} - x_{2n-3}|$

$\le \cdots \le (\frac{4}{9})^{2n-2} |x_2 - x_1| = (\frac{4}{9})^{2n-2} \frac{1}{2} \to 0$ as $n \to \infty$.

By Sandwich theorem, $\lim_{n\to\infty} |x_{2n} - x_{2n-1}| = 0$. By the nested interval theorem, $\lim_{n\to\infty} x_{2n} = x = \lim_{n\to\infty} x_{2n-1}$. By the intertwining sequence theorem, $\lim_{n\to\infty} x_n = x$. Taking limit of $x_{n+1} = \frac{1}{1+x_n}$, we get $x = \frac{1}{1+x} \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$. Since $\frac{-1-\sqrt{5}}{2} \notin I_1$, so $x = \frac{-1+\sqrt{5}}{2}$.

Alternative way to do step 2

From end of step 1, we have $\lim_{n\to\infty} x_{2n} = a$, $\lim_{n\to\infty} x_{2n-1} = b$.

Now $x_{2n} = \frac{1}{1+x_{2n-1}} \Rightarrow a = \frac{1}{1+b}$ by taking limit.

Also $x_{2n+1} = \frac{1}{1+x_{2n}} \Rightarrow b = \frac{1}{1+a}$ by taking limit.

$$\left.\begin{aligned} a &= \frac{1}{1+b} \\ b &= \frac{1}{1+a} \end{aligned}\right\} \Rightarrow a(1+b) = 1 = b(1+a) \Rightarrow a+ab = b+ab$$
$$\Rightarrow a = b$$

$$\left.\begin{aligned} &\text{Then } \lim_{n\to\infty} x_n = a \text{ and } a = \frac{1}{1+a} \\ &x_n \in I_1 \Rightarrow a \in I_1 \end{aligned}\right\} \Rightarrow a = \frac{-1+\sqrt{5}}{2}$$
as $\frac{-1-\sqrt{5}}{2} \notin I_1$.

**Question** How can we prove a sequence converges without identifying the limit?

In the $19^{th}$ century, Cauchy introduced the following

**Definition** $\{x_n\}$ is a Cauchy sequence iff $\forall \varepsilon > 0$ $\exists K \in \mathbb{N}$ such that $n, m \geq K \Rightarrow |x_n - x_m| < \varepsilon$.

**Remarks** This means the terms are as close as desired when the indices are sufficiently large.

**Example** Let $x_n = \frac{1}{n^2}$. Show $\{x_n\}$ is Cauchy.

**Scratch Work** Say $m \geq n$, $|x_n - x_m| = \frac{1}{n^2} - \frac{1}{m^2} < \frac{1}{n^2} < \varepsilon$

$n > \frac{1}{\sqrt{\varepsilon}}$ is enough.

**Solution.** $\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\sqrt{\varepsilon}}$. Then

$$n, m \geq K \Rightarrow |x_n - x_m| = \left|\frac{1}{n^2} - \frac{1}{m^2}\right| < \frac{1}{K^2} < \varepsilon.$$

**Cauchy's Theorem** $\{x_n\}$ converges $\Leftrightarrow$ $\{x_n\}$ is Cauchy.

**Proof** ($\Rightarrow$) Given: $\forall \varepsilon_0 > 0$ $\exists K_0 \in \mathbb{N}$, $n \geq K_0 \Rightarrow |x_n - x| < \varepsilon_0$

To prove: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$, $m, n \geq K \Rightarrow |x_m - x_n| < \varepsilon$.

$$\boxed{\textbf{Idea}: |x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x - x_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon}$$

$\forall \varepsilon > 0$, let $\varepsilon_0 = \varepsilon/2$. We are given that $\exists K_0 \in \mathbb{N}$, $n \geq K_0 \Rightarrow |x_n - x| < \varepsilon/2$. Set $K = K_0$. Then

$$m, n \geq K \Rightarrow |x_m - x_n| \leq |x_m - x| + |x - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

($\Leftarrow$) We are given $\{x_n\}$ is Cauchy. We are to prove $\{x_n\}$ converges to some $x$. We will do this in 3 steps.

**Step 1** $\{x_n\}$ is Cauchy $\Rightarrow$ $\{x_1, x_2, x_3, \cdots\}$ is bounded

**Step 2** (Bolzano-Weierstrass Theorem)

$\{x_1, x_2, x_3, \cdots\}$ is bounded $\Rightarrow$ $\exists$ subsequence $\{x_{n_k}\}$ which converges.

**Step 3** $\left.\begin{array}{l}\{x_n\} \text{ is Cauchy} \\ \text{and a subsequence } \{x_{n_k}\} \\ \text{converges to } x\end{array}\right\} \Rightarrow \{x_n\}$ converges to $x$.

**For step 1**, we modify the proof of the boundedness theorem.

For $\varepsilon = 1$, $\exists K \in \mathbb{N}$, $n, m \geq K \Rightarrow |x_n - x_m| < \varepsilon = 1$.

So the case $m = K$ means $n \geq K \Rightarrow |x_n - x_K| < 1$

$$\Rightarrow |x_n| = |x_n - x_K + x_K| \leq |x_n - x_K| + |x_K| < 1 + |x_K|.$$

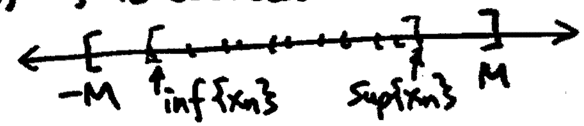
Let $M = \max\{|x_1|, |x_2|, \cdots, |x_{K-1}|, 1 + |x_K|\}$.

Then $\forall n \in \mathbb{N}$, $n = 1, 2, \cdots, K-1 \Rightarrow |x_n| \leq M$

$n \geq K \Rightarrow |x_n| < 1 + |x_K| \leq M$.

$\therefore$ $\{x_1, x_2, x_3, \cdots\}$ is bounded.

**For step 2**




Let $a_1 = \inf\{x_n\}$, $b_1 = \sup\{x_n\}$ and $I_1 = [a_1, b_1]$.

Let $m_1$ be the midpoint of $I_1$.

If $[a_1, m_1]$ contains infinitely many terms of $\{x_n\}$, then let $a_2 = a_1$, $b_2 = m_1$ and $I_2 = [a_2, b_2]$. Otherwise, $[m_1, b_1]$ contains infinitely many terms of $\{x_n\}$, then let $a_2 = m_1$, $b_2 = b_1$ and $I_2 = [a_2, b_2]$. Let $m_2$ be the midpoint of $I_2$. Keep repeating, we get $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ and since $I_{n+1}$ is either the left or the right half of $I_n$, we have $\lim_{n\to\infty} \underbrace{(b_n - a_n)}_{\text{length of } I_n} = \lim_{n\to\infty} \frac{b_1 - a_1}{2^{n-1}} = 0$. $\therefore \bigcap_{n=1}^{\infty} I_n = \{x\}$.

Take $n_1 = 1$, then $x_{n_1} = x_1 \in I_1$. Since $I_2$ has infinitely many terms, $\exists\, x_{n_2} \in I_2$ with $n_2 > n_1$. Keep repeating, we get $x_{n_k} \in I_k$ and $n_1 < n_2 < n_3 < \cdots$. So $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Now $x_{n_k}, x \in I_k \Rightarrow |x_{n_k} - x| \le \underbrace{b_k - a_k}_{\text{length of } I_k} \xrightarrow{\text{as } k\to\infty} 0$

$\therefore \{x_{n_k}\}$ converges to $x$.

For step 3 $\forall \varepsilon > 0$, $\{x_n\}$ Cauchy $\Rightarrow \exists\, K_1 \in \mathbb{N}$
$m, n \ge K_1 \Rightarrow |x_n - x_m| < \varepsilon/2$

$\{x_{n_j}\}$ converges to $x \Rightarrow \exists\, K_2 \in \mathbb{N}$, $j \ge K_2 \Rightarrow |x_{n_j} - x| < \varepsilon/2$.

Let $K = \max(K_1, K_2)$. Then

$$n \ge K \Rightarrow \begin{cases} n_K \ge K \ge K_1 \Rightarrow |x_n - x_{n_K}| < \varepsilon/2 \\ K \ge K_2 \quad \Rightarrow |x_{n_K} - x| < \varepsilon/2 \end{cases}$$

$$\Rightarrow |x_n - x| = |x_n - x_{n_K} + x_{n_K} - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Example Let $x_1 = \sin 1$ and $x_k = x_{k-1} + \frac{\sin k}{k^2}$. Prove $\{x_n\}$ converges.

Solution (Scratch Work) Check Cauchy condition

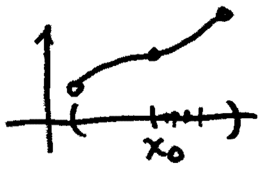
$$\begin{aligned} m > n \Rightarrow |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} \cdots - x_n| \\ &\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &= \left|\frac{\sin m}{m^2}\right| + \left|\frac{\sin(m-1)}{(m-1)^2}\right| + \cdots + \left|\frac{\sin(n+1)}{(n+1)^2}\right| \\ &\le \frac{1}{m^2} + \frac{1}{(m-1)^2} + \cdots + \frac{1}{(n+1)^2} \\ &< \frac{1}{m(m-1)} + \frac{1}{(m-1)(m-2)} + \cdots + \frac{1}{(n+1)n} \\ &= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots + \left(\frac{1}{m-2} - \frac{1}{m-1}\right) + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ &= \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \varepsilon \leftarrow n > \frac{1}{\varepsilon} \text{ is enough} \end{aligned}$$

$\forall \varepsilon > 0$, by Archimedean principle, $\exists\, K \in \mathbb{N}$ such that $K > \frac{1}{\varepsilon}$. Then $n, m \ge K \Rightarrow |x_m - x_n| < \left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{K} < \varepsilon$.

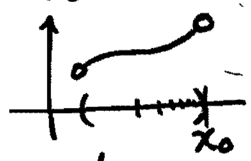
$\therefore \{x_n\}$ is a Cauchy sequence. $\therefore \{x_n\}$ converges.

## Limit of Functions

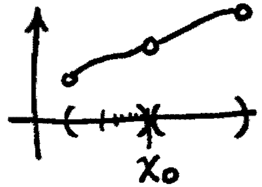
**Question** Let $S$ be an interval (more generally a set). Let $f: S \to \mathbb{R}$ be a function. At which number $x_0$ can we consider $\lim\limits_{x\to x_0} f(x)$ ?



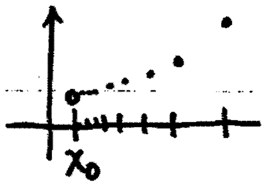

$x_0 \in S$




$x_0 \notin S$




$x_0 \notin S$




$S$ is a sequence
$x_0$ is a limit

What do these cases have in common about $x_0$ and $S$ ?

**Definition** $x_0$ is an accumulation point (or limit point or cluster point) of $S$ iff $\exists\, w_n \in S$, $w_n \neq x_0$ such that $\lim\limits_{n\to\infty} w_n = x_0$.

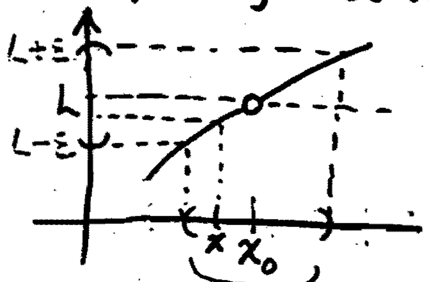
**Remarks** Accumulation points may or may not be in $S$.

**Notation** We write $w_n \to x_0$ in $S \setminus \{x_0\}$ to mean $w_n \in S$, $w_n \neq x_0$ and $\lim\limits_{n\to\infty} w_n = x_0$.

---

**Convention** When discussing $\lim\limits_{x\to x_0} f(x)$, we will assume $x_0$ is an accumulation point of the domain of $f$.

Let $f: S \to \mathbb{R}$, $\lim\limits_{x\to x_0} f(x) = L$ roughly means for any desired distance $\varepsilon > 0$, when $x \in S$, $x \neq x_0$ is sufficiently close to $x_0$, we can obtain




Sufficiently close to $x_0$

$$\underbrace{d(f(x), L)}_{\text{distance between } f(x) \text{ and } L} < \varepsilon .$$

**Precise Definition** $\boxed{\lim\limits_{x\to x_0} f(x) = L}$ iff $\forall \varepsilon > 0\ \exists \delta > 0$

$$\forall x \in S,\ \left.\begin{array}{l} x \in (x_0-\delta, x_0+\delta) \\ x \neq x_0 \end{array}\right\} \Rightarrow |f(x) - L| < \varepsilon .$$

Equivalently, $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in S$,

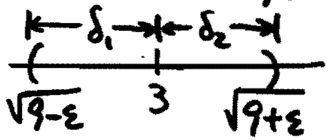
$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon .$$

This is easier to do computations.

**Examples** ① Let $f(x) = \dfrac{x^3 - 3x^2}{x-3} = \dfrac{x^2(x-3)}{x-3}$. Check $\lim\limits_{x\to 3} f(x) = 9$

**Scratch Work** $x \neq 3 \Rightarrow f(x) = x^2$ $\quad |f(x) - 9| = |x^2 - 9| < \varepsilon$

$\Leftrightarrow x^2 \in (9-\varepsilon, 9+\varepsilon) \Leftrightarrow x \in (\sqrt{9-\varepsilon}, \sqrt{9+\varepsilon})$ for $\varepsilon < 9$




Let $\delta = \min(\delta_1, \delta_2) = \min(3 - \sqrt{9-\varepsilon}, \sqrt{9+\varepsilon} - 3)$

Then $0 < |x-3| < \delta \Rightarrow x \in (\sqrt{9-\varepsilon}, \sqrt{9+\varepsilon})$

$\Rightarrow |f(x) - 9| = |x^2 - 9| < \varepsilon$.

② Let $g:[0,\infty)\to\mathbb{R}$ be defined by $g(x)=\sqrt{x}$.

Check: $\lim_{x\to 0} g(x)=0$ and $\lim_{x\to 4} g(x)=2$.

Solution. (Scratch Work) $|g(x)-0|=\sqrt{x}<\varepsilon$ $\quad \underset{|x-0|}{x} < \varepsilon^2$ is enough

$\forall \varepsilon>0$, let $\delta=\varepsilon^2$, then $\forall x\in[0,\infty)$, $0<|x-0|=x<\delta=\varepsilon^2$

$\Rightarrow |g(x)-0|=\sqrt{x}<\varepsilon$.

(Scratch Work) $|g(x)-2|=|\sqrt{x}-2|=\dfrac{|x-4|}{\sqrt{x}+2}\leq\dfrac{|x-4|}{2}<\varepsilon$

$|x-4|<2\varepsilon$ is enough.

$\forall\varepsilon>0$, let $\delta=2\varepsilon$, then $\forall x\in[0,\infty)$,

$0<|x-4|<\delta=2\varepsilon \Rightarrow |g(x)-2|\leq\dfrac{|x-4|}{2}<\varepsilon$.

③ Let $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be defined by $f(x)=\dfrac{1}{5x}$

Check: $\lim_{x\to 2} f(x)=\dfrac{1}{10}$.

Solution (Scratch Work) $x$ is close to 2

$\left|f(x)-\frac{1}{10}\right|=\left|\frac{1}{5x}-\frac{1}{10}\right|=\dfrac{|x-2|}{10x}\leq\dfrac{|x-2|}{10}<\varepsilon$

min trick

$\left.\begin{array}{l}|x-2|<10\varepsilon\\ \text{and } x\geq 1\end{array}\right\}$ are enough. if $x\geq 1$ $\leftarrow \delta<1$

$\forall\varepsilon>0$, let $\delta=\min(1,10\varepsilon)$, then $\forall x\in\mathbb{R}\setminus\{0\}$,

$0<|x-2|<\delta \Rightarrow \left\{\begin{array}{l}|x-2|<1\\ \text{and}\\ |x-2|<10\varepsilon\end{array}\right. \Rightarrow \left\{\begin{array}{l}x\in(1,3) \leftarrow x\geq 1\\ \text{and}\\ |x-2|<10\varepsilon\end{array}\right.$

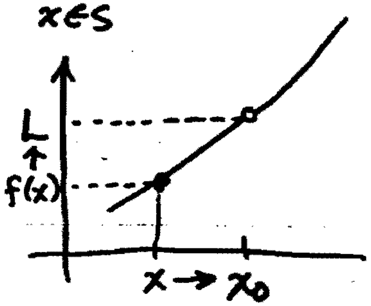
$\Rightarrow \left|f(x)-\frac{1}{10}\right|=\dfrac{|x-2|}{10x}\leq\dfrac{|x-2|}{10}<\varepsilon$.

**Recall** "$x_n \to x_0$ in $S\setminus\{x_0\}$" means $\forall n \in \mathbb{N}$, $x_n \in S$, $x_n \neq x_0$, $\lim\limits_{n\to\infty} x_n = x_0$

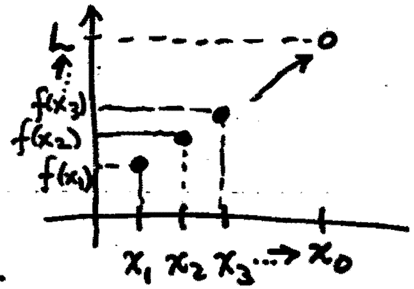
**Sequential Limit Theorem (S.L.T.)**

Let $f: S \to \mathbb{R}$ be a function and $x_0$ be an accumulation point of $S$. Then

$$\lim_{\substack{x\to x_0\\ x\in S}} f(x) = L \iff \forall x_n \to x_0 \text{ in } S\setminus\{x_0\},\ \lim_{n\to\infty} f(x_n) = L.$$




$\Rightarrow$




$\Leftarrow$

$\forall x_n \to x_0$ in $S\setminus\{x_0\}$

**Proof** ($\Rightarrow$)

Given: $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x) = L$ $\left(\forall \varepsilon>0\ \exists \delta>0, \forall x\in S\ \ 0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon\right)$

$x_n \to x_0$ in $S\setminus\{x_0\}$ $\left(\forall \delta>0\ \exists K\in\mathbb{N}\ \ n\geq K \Rightarrow |x_n - x_0|<\delta\right)$

So $\forall \varepsilon>0\ \exists K\in\mathbb{N}$

$n \geq K \Rightarrow \underbrace{0<|x_n-x_0|}_{\text{as } x_n\neq x_0}<\delta \Rightarrow |f(x_n)-L|<\varepsilon$

$\therefore \lim\limits_{n\to\infty} f(x_n) = L$

($\Leftarrow$) Assume $\lim\limits_{x\to x_0} f(x) \neq L$.

$\sim(\forall \varepsilon>0\ \exists\delta>0, \forall x\in S, 0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon)$

$= \exists \varepsilon>0\ \forall\delta>0\ \exists x\in S\ \ 0<|x-x_0|<\delta$ and $|f(x)-L|\geq\varepsilon$.

for $\delta = 1$ $\exists x_1\in S$ $0<|x_1-x_0|<1$ and $|f(x_1)-L|\geq\varepsilon$

for $\delta=\frac{1}{2}$ $\exists x_2\in S$ $0<|x_2-x_0|<\frac{1}{2}$ and $|f(x_2)-L|\geq\varepsilon$

$\vdots$

for $\delta=\frac{1}{n}$ $\exists x_n\in S$ $0<|x_n-x_0|<\frac{1}{n}$ and $|f(x_n)-L|\geq\varepsilon$

$\vdots$

$\therefore x_n\in S$, $0<|x_n-x_0|<\frac{1}{n} \Rightarrow x_n\neq x_0$ and $\lim\limits_{n\to\infty} x_n = x_0$

$\therefore x_n \to x_0$ in $S\setminus\{x_0\}$. Then $\lim\limits_{n\to\infty} f(x_n) = L$.

$|f(x_n)-L|\geq\varepsilon \Rightarrow 0 = |L-L| = \lim\limits_{n\to\infty}|f(x_n)-L| \geq \varepsilon$

Contradicting $\varepsilon>0$.

---

**Application** ① $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = e \overset{\text{SLT}}{\underset{x_n=\frac{1}{n}\to 0}{\Longrightarrow}} \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$\Leftarrow$ need not just $x_n = \frac{1}{n}$, need all $x_n \to 0$, $x_n\neq 0$ to have $\lim\limits_{n\to\infty}(1+x_n)^{1/x_n} = e$

---

② If $\underbrace{\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x) = L_1}_{(*)}$, $\underbrace{\lim\limits_{\substack{x\to x_0\\ x\in S}} g(x) = L_2}_{(**)}$, then prove $\lim\limits_{\substack{x\to x_0\\ x\in S}}(f(x)+g(x)) = L_1+L_2$

**Solution 1** $\forall x_n \to x_0$ in $S\setminus\{x_0\}$,

by S.L.T., $(*) \Rightarrow \lim\limits_{n\to\infty} f(x_n) = L_1$

$(**) \Rightarrow \lim\limits_{n\to\infty} g(x_n) = L_2$.

By computation formula, $\lim\limits_{n\to\infty}(f(x_n)+g(x_n)) = L_1+L_2$

By S.L.T., $\lim\limits_{\substack{x\to x_0\\ x\in S}}(f(x)+g(x)) = L_1+L_2$.

**Solution 2**

$\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x)=L_1$ $\left(\forall \varepsilon_1>0\ \exists \delta_1>0\ \forall x\in S\ \ 0<|x-x_0|<\delta_1 \Rightarrow |f(x)-L_1|<\varepsilon_1\right)$

$\lim\limits_{\substack{x\to x_0\\ x\in S}} g(x)=L_2$ $\left(\forall \varepsilon_2>0\ \exists \delta_2>0\ \forall x\in S\ \ 0<|x-x_0|<\delta_2 \Rightarrow |g(x)-L_2|<\varepsilon_2\right)$

$\forall \varepsilon>0$, let $\varepsilon_1=\frac{\varepsilon}{2}$ and $\varepsilon_2=\frac{\varepsilon}{2}$. From above, get $\delta_1,\delta_2>0$

Set $\delta=\min(\delta_1,\delta_2)$. Then $\forall x\in S$,

$0<|x-x_0|<\delta \Rightarrow$ $0<|x-x_0|<\delta_1 \Rightarrow |f(x)-L_1|<\frac{\varepsilon}{2}$
$\qquad 0<|x-x_0|<\delta_2 \Rightarrow |g(x)-L_2|<\frac{\varepsilon}{2}$

$\Rightarrow |(f(x)+g(x))-(L_1+L_2)| = |f(x)-L_1+g(x)-L_2|$
$\le |f(x)-L_1|+|g(x)-L_2| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$

$\therefore \lim\limits_{\substack{x\to x_0\\ x\in S}} (f(x)+g(x)) = L_1+L_2$

**Similarly**

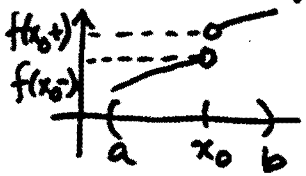
If $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x)=L_1$, $\lim\limits_{\substack{x\to x_0\\ x\in S}} g(x)=L_2$, then
$\lim (f(x)-g(x))=L_1-L_2$
$\lim f(x)g(x)=L_1L_2$
$\lim f(x)/g(x)=L_1/L_2$
(provided $g(x)\neq 0$, $L_2\neq 0$)

If $f(x)\le g(x)\le h(x)$ for all $x\in S$, $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x)=L=\lim\limits_{\substack{x\to x_0\\ x\in S}} h(x)$,
then $\lim\limits_{\substack{x\to x_0\\ x\in S}} g(x)=L$.

If $f(x)\ge 0$ for all $x\in S$ and $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x)=L$, then $L\ge 0$.

**One-sided Limits**

**Definitions** For $f:(a,b)\to\mathbb{R}$ and $x_0\in(a,b)$,



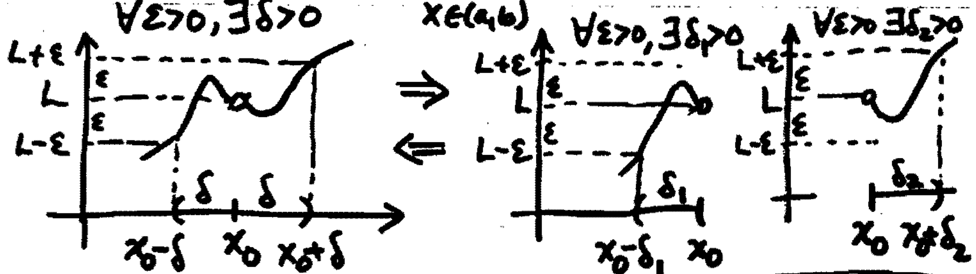

left hand limit of $f$ at $x_0$

$f(x_0-)=\lim\limits_{x\to x_0^-} f(x)=\lim\limits_{\substack{x\to x_0\\ x\in(a,x_0)}} f(x)$

right hand limit of $f$ at $x_0$

$f(x_0+)=\lim\limits_{x\to x_0^+} f(x)=\lim\limits_{\substack{x\to x_0\\ x\in(x_0,b)}} f(x)$.

**Theorem** For $x_0\in(a,b)$, $\lim\limits_{\substack{x\to x_0\\ x\in(a,b)}} f(x)=L \Leftrightarrow f(x_0-)=L=f(x_0+)$




**Proof**

$\lim\limits_{\substack{x\to x_0\\ x\in(a,b)}} f(x)=L$
"$\forall\varepsilon>0\ \exists\delta>0$
$\forall x\in(a,b)$,
$0<|x-x_0|<\delta$
$\Rightarrow |f(x)-L|<\varepsilon$"

$\Longrightarrow$ let $\delta_1=\delta$, $\delta_2=\delta$

$f(x_0-)=L$
$\lim\limits_{\substack{x\to x_0\\ x\in(a,x_0)}} f(x)$
"$\forall\varepsilon>0\ \exists\delta_1>0$
$\forall x\in(a,x_0)$
$0<|x-x_0|<\delta_1$
$\Rightarrow |f(x)-L|<\varepsilon$"

$f(x_0+)=L$
$\lim\limits_{\substack{x\to x_0\\ x\in(x_0,b)}} f(x)$
"$\forall\varepsilon>0\ \exists\delta_2>0$
$\forall x\in(x_0,b)$
$0<|x-x_0|<\delta_2$
$\Rightarrow |f(x)-L|<\varepsilon$

$\Longleftarrow$ let $\delta=\min(\delta_1,\delta_2)$

**Definitions** Let $f: S \to \mathbb{R}$ be a function.

① $f$ is *increasing* on $S$ iff $\forall x,y \in S$, $x<y \Rightarrow f(x) \le f(y)$

② $f$ is *decreasing* on $S$ iff $\forall x,y \in S$, $x<y \Rightarrow f(x) \ge f(y)$

③ $f$ is *strictly increasing* on $S$ iff $\forall x,y \in S$, $x<y \Rightarrow f(x) < f(y)$

④ $f$ is *strictly decreasing* on $S$ iff $\forall x,y \in S$, $x<y \Rightarrow f(x) > f(y)$.

⑤ $f$ is *monotone* on $S$ iff $f$ is increasing or decreasing on $S$.

⑥ $f$ is *strictly monotone* on $S$ iff $f$ is strictly increasing or strictly decreasing on $S$.

⑦ $f$ is *bounded above* on $S$ iff $\{f(x): x \in S\}$ is bounded above.

⑧ $f$ is *bounded below* on $S$ iff $\{f(x): x \in S\}$ is bounded below.

⑨ $f$ is *bounded* on $S$ iff $f$ is bounded above and below.

**Monotone Function Theorem**

(I) If $f$ is increasing on $(a,b)$, then $\forall x_0 \in (a,b)$,

$$\left.\begin{array}{l} f(x_0-) = \sup\{f(x): a<x<x_0\} \\ \text{and } f(x_0+) = \inf\{f(x): x_0<x<b\} \end{array}\right\} \Rightarrow f(x_0-) \le f(x_0) \le f(x_0+)$$

If $f$ is bounded below, then $f(a+) = \inf\{f(x): a<x<b\}$.

If $f$ is bounded above, then $f(b-) = \sup\{f(x): a<x<b\}$.

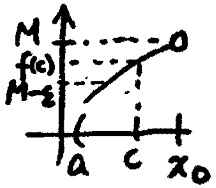
(II) $f$ has countably many discontinuous points on $(a,b)$

$J = \{x_0 : x_0 \in (a,b), f(x_0-) \ne f(x_0+)\}$ is countable.

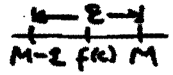
**Remarks** Similarly, the theorem is true for decreasing functions and all other kinds of intervals.

**Proof.** (I) If $a<x<x_0<b$, then $f(x) \le f(x_0)$ as $f$ is increasing. So $\{f(x): a<x<x_0\}$ is bounded above by $f(x_0)$. Hence, $M = \sup\{f(x): a<x<x_0\}$ exists by completeness axiom.

To show $f(x_0-) = M$, $\forall \varepsilon>0$, by the supremum property, $\exists c \in (a,x_0)$ such that $M-\varepsilon < f(c) \le M$. Let $\delta = x_0 - c$.




Then $\forall x \in (a,x_0)$,




$$0<|x-x_0|<\delta \Rightarrow x \in (x_0-\delta, x_0) = (c, x_0)$$
$$\Rightarrow c<x<x_0 \Rightarrow f(c) \le f(x) \le M$$
$$\Rightarrow |f(x)-M| = M-f(x) \le M-f(c) < \varepsilon$$

$$\therefore f(x_0-) = \lim_{\substack{x \to x_0 \\ x \in (a,x_0)}} f(x) = M = \sup\{f(x): a<x<x_0\} \le f(x_0).$$

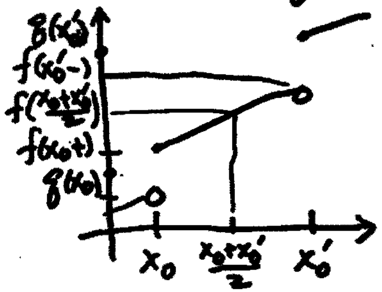
The other parts of (I) are similarly proved.

(II) $f$ is discontinuous at $x_0 \in (a,b) \Leftrightarrow f(x_0-) < f(x_0+)$.

$\Rightarrow \exists g(x_0) \in \mathbb{Q}$ such that $f(x_0-) < g(x_0) < f(x_0+)$ by density of $\mathbb{Q}$.

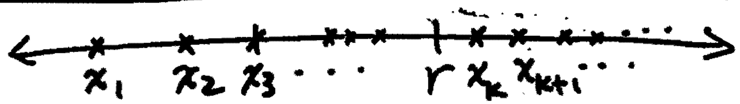
The function $g: J = \{x_0 : f(x_0-) < f(x_0+)\} \to \mathbb{Q}$ is injective because $\forall x_0, x_0' \in J$,

$$x_0 < x_0' \Rightarrow g(x_0) < f(x_0+) \le f\left(\tfrac{x_0+x_0'}{2}\right) \le f(x_0'-) < g(x_0')$$

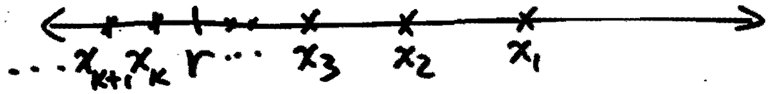



By injection theorem, $J$ is countable.

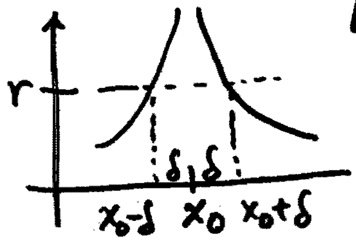
## Infinite Limits




<u>Definitions</u> $\{x_n\}$ <u>diverges to $+\infty$</u> (or $\lim\limits_{n\to\infty} x_n = +\infty$)
iff $\forall r \in \mathbb{R}$, $\exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow x_n > r$.




$\{x_n\}$ <u>diverges to $-\infty$</u> (or $\lim\limits_{n\to\infty} x_n = -\infty$) iff $\lim\limits_{n\to\infty} -x_n = +\infty$,
$\forall r \in \mathbb{R}$, $\exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow x_n < r$.




Let $f : S \to \mathbb{R}$ and $x_0$ be an accumulation point of $S$.
$f$ <u>diverges to $+\infty$ as $x$ tends to $x_0$</u>
(or $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x) = +\infty$) iff

$\forall r \in \mathbb{R}$ $\exists \delta > 0$ such that $\forall x \in S$
$$\underbrace{x \neq x_0 \text{ and } x \in (x_0-\delta, x_0+\delta)}_{0<|x-x_0|<\delta} \Rightarrow f(x) > r.$$

$f$ <u>diverges to $-\infty$ as $x$ tends to $x_0$</u> (or $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x) = -\infty$)
iff $\lim\limits_{\substack{x\to x_0\\ x\in S}} -f(x) = +\infty$,
$\forall r \in \mathbb{R}$ $\exists \delta > 0$ such that $\forall x \in S$
$$0 < |x-x_0| < \delta \Rightarrow f(x) < r.$$

## Limit at Infinity

Recall $\lim\limits_{n\to\infty} a_n = L$ iff $\forall \varepsilon > 0$ $\exists K \in \mathbb{N}$ $n \geq K \Rightarrow |x_n - L| < \varepsilon$.

Let $f : S \to \mathbb{R}$ be a function and $+\infty, -\infty$ are accumulation points of $S$ (that is, $\exists$ sequences in $S$ with $+\infty, -\infty$ as limits). $L \in \mathbb{R}$.

<u>Definitions</u> $\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(x) = L$ iff $\forall \varepsilon > 0$ $\exists K \in \mathbb{R}$
$x \geq K \Rightarrow |f(x) - L| < \varepsilon$.

$\lim\limits_{\substack{x\to-\infty\\ x\in S}} f(x) = L$ iff $\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(-x) = L$ iff $\forall \varepsilon > 0$ $\exists K \in \mathbb{R}$
$x \leq K \Rightarrow |f(x) - L| < \varepsilon$

Recall $\lim\limits_{n\to\infty} a_n = +\infty$ iff $\forall r \in \mathbb{R}$, $\exists K \in \mathbb{N}$
$n \geq K \Rightarrow a_n > r$.

<u>Definitions</u> $\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(x) = +\infty$ iff $\forall r \in \mathbb{R}$, $\exists K \in \mathbb{R}$
$x \geq K \Rightarrow f(x) > r$.

$\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(x) = -\infty$ iff $\lim\limits_{\substack{x\to+\infty\\ x\in S}} -f(x) = +\infty$ iff $\forall r \in \mathbb{R}$, $\exists K \in \mathbb{R}$
$x \geq K \Rightarrow f(x) < r$.

$\lim\limits_{\substack{x\to-\infty\\ x\in S}} f(x) = +\infty$ iff $\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(-x) = +\infty$ iff $\forall r \in \mathbb{R}$, $\exists K \in \mathbb{R}$
$x \leq K \Rightarrow f(x) > r$.

$\lim\limits_{\substack{x\to-\infty\\ x\in S}} f(x) = -\infty$ iff $\lim\limits_{\substack{x\to+\infty\\ x\in S}} -f(-x) = +\infty$ iff $\forall r \in \mathbb{R}$, $\exists K \in \mathbb{R}$
$x \leq K \Rightarrow f(x) < r$.

# Chapter 7 Continuity

**Definition** A function $f: S \to \mathbb{R}$ is continuous at $x_0 \in S$ iff $\lim_{\substack{x \to x_0 \\ x \in S}} f(x) = f(x_0)$ (more precisely, $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x \in S$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$)

For $E \subseteq S$, we say $f$ is continuous on $E$ iff $f$ is continuous at every element of $E$. Also, we say $f$ is continuous iff $f$ is continuous on the domain $S$.

**Sequential Continuity Theorem** (S.C.T.)

$f: S \to \mathbb{R}$ is continuous at $x_0 \in S$ $\Leftrightarrow$ $\forall x_n \to x_0$ in $S$ $\lim_{n\to\infty} f(x_n) = f(x_0) = f(\lim_{n\to\infty} x_n)$

$\boxed{\text{Drop } x_n \neq x_0 \text{ requirement.}}$

Proof. Just modify the proof of the sequential limit theorem by replacing ① $L$ by $f(x_0)$

② $0 < |x - x_0| < \delta$ by $|x - x_0| < \delta$

③ $x_n \to x_0$ in $S \setminus \{x_0\}$ by $x_n \to x_0$ in $S$ (delete $x_n \neq x_0$ requirement)

**Examples** ① Since $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$, so $f: \mathbb{R} \to \mathbb{R}$ defined by $f(\theta) = \begin{cases} \frac{\sin\theta}{\theta} & \text{if } \theta \neq 0 \\ 1 & \text{if } \theta = 0 \end{cases}$ is continuous at $x_0 = 0$.

Also $\lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{n\to\infty} f(\frac{1}{n}) \overset{\text{by S.C.T.}}{=} f(0) = 1$

② $\exists f: \mathbb{R} \to \mathbb{R}$ discontinuous (not continuous) at every $x \in \mathbb{R}$. Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$

$\forall x_0 \in \mathbb{R}, n \in \mathbb{N}$, by density of $\mathbb{Q}$ and density of $\mathbb{R} \setminus \mathbb{Q}$, $\exists r_n \in \mathbb{Q}$, $s_n \in \mathbb{R} \setminus \mathbb{Q}$, both $r_n, s_n \in (x_0, x_0 + \frac{1}{n})$

So $\lim_{n\to\infty} r_n = x_0 = \lim_{n\to\infty} s_n$ by Sandwich theorem.

Then $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 1 = 1$, $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} 0 = 0$, so $\lim_{x \to x_0} f(x)$ cannot exist by S.C.T. $\therefore f$ is discontinuous at $x_0$.

**Theorem** If $f, g: S \to \mathbb{R}$ are continuous at $x_0 \in S$, then $f \pm g$, $fg$, $f/g$ (provided $g(x_0) \neq 0$) are continuous at $x_0$.

**Proof** $f, g$ continuous at $x_0 \in S$ $\Leftrightarrow$ $\lim_{x \to x_0} f(x) = f(x_0)$, $\lim_{x \to x_0} g(x) = g(x_0)$

by definition of continuity at $x_0$

$\Rightarrow \lim_{x \to x_0} (f \pm g)(x) = (f \pm g)(x_0)$

see applications of S.L.T.

$\lim_{x \to x_0} (fg)(x) = (fg)(x_0)$

$\lim_{x \to x_0} (\frac{f}{g})(x) = (\frac{f}{g})(x_0)$

$\Leftrightarrow f \pm g, fg, \frac{f}{g}$ is continuous at $x_0$

**Theorem** If $f: S \to \mathbb{R}$ is continuous at $x_0$, $f(S) \subseteq S'$ and $g: S' \to \mathbb{R}$ is continuous at $f(x_0)$, then $g \circ f$ is continuous at $x_0$.

Proof. By S.C.T., we need to show $\forall x_n \to x_0$ in $S$, $\lim_{n\to\infty} (g\circ f)(x_n) = (g \circ f)(x_0)$. By S.C.T., since $f$ is continuous at $x_0$, $\lim_{n\to\infty} f(x_n) = f(x_0)$. So $f(x_n) \to f(x_0)$ in $S'$. Since $g$ is continuous at $f(x_0)$, by S.C.T.,

$$\lim_{n\to\infty} (g\circ f)(x_n) = \lim_{n\to\infty} g(f(x_n)) = g(f(x_0)) = (g\circ f)(x_0).$$

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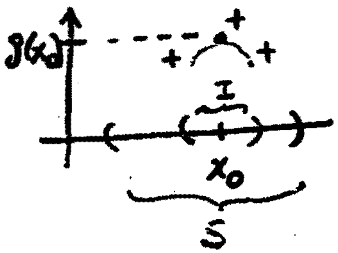
Below, $S$ will denote an interval of positive length.

**Sign Preserving Property**




If $g: S \to \mathbb{R}$ is continuous and $g(x_0) > 0$, then $\exists$ an interval $I = (x_0 - \delta, x_0 + \delta)$ with $\delta > 0$ such that

$g(x) > 0$ for all $x \in S \cap I$.

(Similarly for the case $g(x_0) < 0$.)

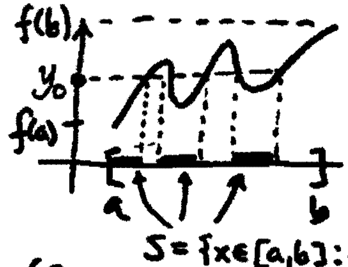
Proof Let $\varepsilon = g(x_0) > 0$. Note $(g(x_0) - \varepsilon, g(x_0)+\varepsilon) = (0, 2g(x_0))$. Since $g$ is continuous at $x_0$, $\exists\, \delta > 0$ such that

$$\underbrace{\forall x \in S, \ |x - x_0| < \delta}_{x \in S \cap \underbrace{(x_0-\delta, x_0+\delta)}_{=I}} \Rightarrow \underbrace{|g(x) - g(x_0)| < \varepsilon}_{g(x) \in (\underbrace{g(x_0)-\varepsilon}_{=0},\, g(x_0)+\varepsilon)}$$

$$\Rightarrow g(x) > 0$$

**Intermediate Value Theorem**

If $f: [a,b] \to \mathbb{R}$ is continuous and $y_0$ is between $f(a)$ and $f(b)$, then $\exists\, x_0 \in [a,b]$ such that $f(x_0) = y_0$.




Proof Case 1: $y_0 = f(a)$. Take $x_0 = a$.

Case 2: $y_0 = f(b)$. Take $x_0 = b$.

Case 3: $f(a) < y_0 < f(b)$.

(Case $f(a) > y_0 > f(b)$ is similar.) Let $S = \{x \in [a,b] : f(x) \le y_0\}$. $S \ne \emptyset$ as $a \in S$. $S$ is bounded above by $b$. By completeness axiom, $x_0 = \sup S$ exists.

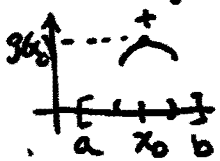
By supremum limit theorem, $\exists\, x_n \in S$ such that $\lim_{n\to\infty} x_n = x_0$.

$x_n \in [a,b] \Rightarrow a \le x_n \le b \Rightarrow a \le x_0 \le b \Rightarrow x_0 \in [a,b]$.

By S.C.T., $f(x_0) = \lim_{n\to\infty} f(x_n) \le y_0$.

Assume $f(x_0) < y_0$. Then $x_0 \ne b$ since $y_0 < f(b)$.

Define $g(x) = y_0 - f(x)$ on $[a,b]$. Then $g(x_0) = y_0 - f(x_0) > 0$.




By sign preserving property, there is interval $I = (x_0 - \delta, x_0 + \delta)$ such that $\forall x \in [a,b] \cap I$, $g(x) > 0$.

Now $x_0 < b \Rightarrow \exists\, x_1 \in (x_0, b] \cap (x_0, x_0+\delta) \subseteq [a,b] \cap I$

$x_0 < x_0 + \delta$

$\Rightarrow g(x_1) = y_0 - f(x_1) > 0$

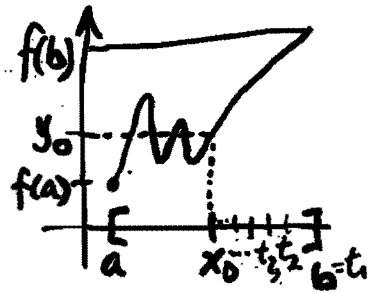
$\Rightarrow f(x_1) < y_0$, but $x_1 > x_0$, $x_1 \in [a,b]$

← contradict $x_0 = \sup S$.

$\therefore f(x_0) = y_0$.

## Alternative Ending (Avoiding sign preserving property)

As in the previous proof, we get $f(x_0) \le y_0$.




Since $y_0 < f(b)$, we get $x_0 \neq b$.

Let $t_n = x_0 + \frac{1}{n}(b - x_0) \in (x_0, b]$.

$\lim_{n\to\infty} t_n = x_0$ and $t_n > x_0 = \sup S$

So $t_n \notin S \Rightarrow f(t_n) > y_0$.

By S.C.T., $f(x_0) = \lim_{n\to\infty} f(t_n) \ge y_0$. $\therefore f(x_0) = y_0$.

## Exercise

Let $f : [0,1] \to [0,1]$ be an increasing function (perhaps discontinuous). Suppose $0 < f(0)$ and $f(1) < 1$. Show that $f$ has at least one fixed point. (A fixed point of $f$ is an element $r$ in the domain of $f$ such that $f(r) = r$.)

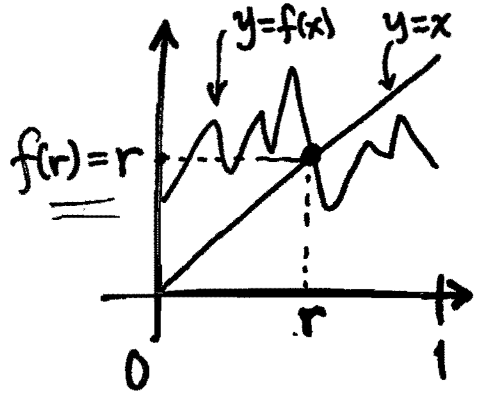
Hint: Sketch the graph of $f$ and consider

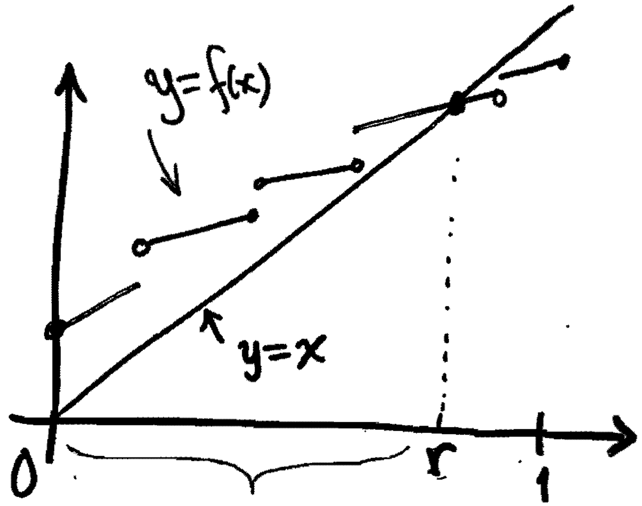
$$S = \{t \in [0,1] : t \le f(t)\}.$$

Does it have a supremum?

Use monotone function theorem and S.L.T.

## Fixed Point




$S = \{t \in [0,1] : t \le f(t)\}$

**Examples** ① The equation $x^5+3x+\sin x=\cos x+10$ has a solution.

Let $f(x)=x^5+3x+\sin x-\cos x-10$. Then $f$ is continuous

$f(0)=-11$ and $26=2^5+3\cdot 2-1-1-10\le f(2)$

So $0$ is between $f(0)$ and $f(2)$. By intermediate value theorem, $\exists x_0\in[0,2]$ such that $f(x_0)=0$. Then

$x_0^5+3x_0+\sin x_0=\cos x_0+10$. ← $x_0$ is a solution of equation.

② Every odd degree polynomial with real coefficients has at least one real root.

Let $P(x)=x^n+a_1x^{n-1}+\cdots+a_n$ with $n$ odd.

Let $x_0=1+|a_1|+\cdots+|a_n|\ge 1$. Then

$P(x_0)=x_0^n+a_1x_0^{n-1}+\cdots+a_n \Rightarrow x_0^n-P(x_0)=-a_1x_0^{n-1}-\cdots-a_n$

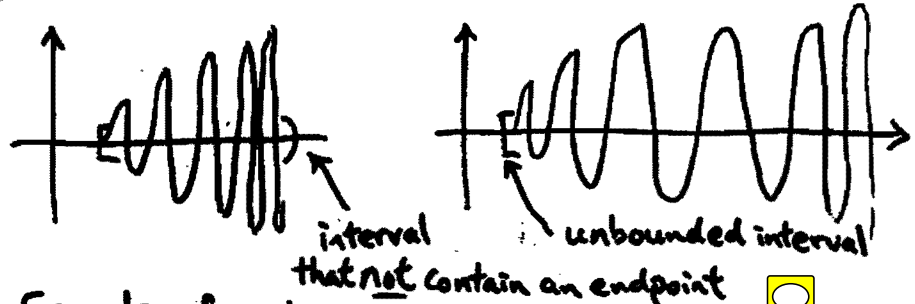
$P(-x_0)=-x_0^n+a_1x_0^{n-1}-\cdots+a_n \Rightarrow x_0^n+P(-x_0)=a_1x_0^{n-1}-\cdots+a_n$

$$\Rightarrow \left.\begin{matrix} x_0^n-P(x_0) \\ x_0^n+P(-x_0)\end{matrix}\right\} \begin{aligned} &\le |a_1x_0|^{n-1}+\cdots+|a_n| \\ &\le |a_1|x_0^{n-1}+\cdots+|a_n|x_0^{n-1} \\ &= (\underbrace{|a_1|+\cdots+|a_n|}_{=x_0-1})\,x_0^{n-1} \\ &< x_0^n \end{aligned}$$

$\Rightarrow$ $P(x_0)>0$ and $P(-x_0)<0$

$\Rightarrow$ $0$ is between $P(-x_0)$ and $P(x_0)$

$\Rightarrow$ $P$ has a real root between $-x_0$ and $x_0$.




Examples of continuous function with no maximum nor minimum values.

**Extreme Value Theorem** Let $a,b\in\mathbb{R}$ with $a\le b$. If $f:[a,b]\to\mathbb{R}$ is continuous, then $\exists x_0,w_0\in[a,b]$ such that $f(w_0)\le f(x)\le f(x_0)$ $\forall x\in[a,b]$.

So range of $f=\{f(x): x\in[a,b]\}=f([a,b])$ is the interval $[f(w_0),f(x_0)]$. In particular, $f$ is bounded on $[a,b]$. $f(x_0)=\sup\{f(x):x\in[a,b]\}=\max\limits_{x\in[a,b]} f(x)$ and $f(w_0)=\inf\{f(x):x\in[a,b]\}=\min\limits_{x\in[a,b]} f(x)$.

Proof. Assume $f([a,b])$ is not bounded above. Then every $n\in\mathbb{N}$ is not an upper bound. So $\exists z_n\in[a,b]$ with $f(z_n)>n$. By Bolzano-Weierstrass theorem, $\{z_n\}$ has a subsequence $\{z_{n_j}\}$ converging to some $z_0\in[a,b]$. Since $f$ is continuous at $z_0$, $\lim\limits_{n\to\infty} f(z_{n_j})=f(z_0)$ by S.C.T. By boundedness theorem, $\{f(z_{n_j})\}$ is bounded. However, $f(z_{n_j})>n_j\ge j \Rightarrow \{f(z_{n_j})\}$ is unbounded, a contradiction.

$\therefore$ $f([a,b])$ is bounded above and $M=\sup f([a,b])$ exists.

By supremum limit theorem, $\exists\, x_n' \in [a,b]$ such that $M = \lim_{n\to\infty} f(x_n)$. By Bolzano-Weierstrass theorem, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ converging to some $x_0 \in [a,b]$. By S.C.T., $f(x_0) = f(\lim_{i\to\infty} x_{n_i}) = \lim_{i\to\infty} f(x_{n_i}) = M$.

Similarly, $\exists\, w_0 \in [a,b]$ with $f(w_0) = \inf f([a,b])$.

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Application. Let $f:[a,b]$ be continuous. Then $-\infty < \int_a^b f(x)\,dx < +\infty$ because $\exists\, x_0, w_0 \in [a,b]$ such that $f(w_0) \le f(x) \le f(x_0) \Rightarrow \int_a^b f(w_0)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(x_0)\,dx \Rightarrow -\infty < f(w_0)(b-a) \le \int_a^b f(x)\,dx < f(x_0)(b-a) < +\infty$

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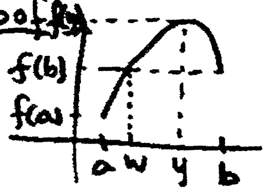
Continuous Injection Theorem

If $f$ is continuous and injective on $[a,b]$, then $f$ is strictly monotone on $[a,b]$ and $f([a,b]) = [f(a), f(b)]$ or $[f(b), f(a)]$. (The theorem is true for any other nonempty interval.)

Proof.




Since $f$ is injective, either $f(a) < f(b)$ or $f(a) > f(b)$. Suppose $f(a) < f(b)$. $\forall y \in (a,b)$, $f(y) > f(b)$ is false for otherwise, by intermediate value theorem, $\exists\, w \in (a,y)$ with $f(w) = f(b)$, contradicting injectivity of $f$. Similarly, $f(y) < f(a)$ is false. So $a < y < b \Rightarrow f(a) < f(y) < f(b)$. Then similarly, $a \le x < y \le b \Rightarrow f(a) \le f(x) < f(y) \le f(b)$. $\therefore f$ is strictly increasing on $[a,b]$ and $f([a,b]) = [f(a), f(b)]$. The case $f(a) > f(b)$ is similar.

Application The continuous injection theorem will be used to prove the following theorem, which will be used to prove the $dx/dy = 1/\frac{dy}{dx}$ rule for differentiation.

Continuous Inverse Theorem

If $f$ is continuous and injective on $[a,b]$, then

$$f^{-1}: f([a,b]) \to [a,b] \text{ is continuous and surjective.}$$

(The theorem is true for any other nonempty interval.)

Proof. $f^{-1}$ is surjective because $c \in [a,b] \Rightarrow f(c) \in f([a,b])$ and $f^{-1}(f(c)) = c$.

By continuous injection theorem, $f$ is strictly monotone, say strictly increasing. Then $f^{-1}$ is also strictly increasing. Assume $f^{-1}$ is discontinuous at some $y_0 = f(x_0) \in f([a,b])$. Then either $a \le f^{-1}(y_0-) < f^{-1}(y_0) = x_0 \le b$

or $a \le x_0 = f^{-1}(y_0) < f^{-1}(y_0+) \le b$. by Monotone function theorem.

This implies either the interval $(f^{-1}(y_0-), f^{-1}(y_0))$ or the interval $(f^{-1}(y_0), f^{-1}(y_0+))$ is not in the range of $f^{-1}$. This contradicts $f^{-1}$ is surjective.

$\therefore f^{-1}$ is continuous.