Chapter 4. Series

Definitions. A *series* is the summation of a countable set of numbers in a specific order. If there are finitely many numbers, then the series is a *finite series*, otherwise it is an *infinite series*. The numbers are called *terms*. The sum of the first n terms is called the *n-th partial sum of the series*.

An infinite series is of the form $\underbrace{a_1}_{1^{\text{st}} \text{ term}} + \underbrace{a_2}_{2^{\text{nd}} \text{ term}} + \underbrace{a_3}_{3^{\text{rd}} \text{ term}} + \ldots$ or we may write it as $\sum_{k=1}^{\infty} a_k$.
The first partial sum is $S_1 = a_1$. The second partial sum is $S_2 = a_1 + a_2$. The n^{th} partial sum is $S_n = a_1 + a_2 + \ldots + a_n$.

Series are used frequently in science and engineering to solve problems or approximate solutions. (*E.g.* trigonometric or logarithm tables were computed using series in the old days.)

Examples.(1) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots = ?$ ($S_n = 1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n} = 2-\frac{1}{2^n}$, $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots = \lim_{n\to\infty}\left(2-\frac{1}{2^n}\right) = 2$.)
We say the series *converges* to 2, which is called the *sum* of the series.

(2) $1+1+1+1+1+1+\ldots = \infty$ ($S_n = \underbrace{1+1+\ldots+1}_{n} = n$, $\lim_{n\to\infty} S_n = \infty$.) We say the series *diverges* (to ∞).

(3) $1-1+1-1+1-1+1-1+\ldots$. ($S_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$, $\lim_{n\to\infty} S_n$ doesn't exist.) We say the series *diverges*.

Definitions. A series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots$ *converges* to a number S iff $\lim_{n\to\infty}(a_1 + a_2 + \ldots + a_n) = \lim_{n\to\infty} S_n = S$.
In that case, we may write $\sum_{k=1}^{\infty} a_k = S$ and say S is the *sum* of the series. A series *diverges* to ∞ iff the partial sum S_n tends to infinity as n tends to infinity. A series *diverges* iff it does not converge to any number.

Remarks. (1) For every series $\sum_{k=1}^{\infty} a_k$, there is a sequence (of partial sums) $\{S_n\}$. Conversely, if the partial sum sequence $\{S_n\}$ is given, we can find the terms a_n as follows: $a_1 = S_1$, $a_2 = S_2 - S_1$, ..., $a_k = S_k - S_{k-1}$ for $k > 1$. Then $a_1 + \ldots + a_n = S_1 + (S_2 - S_1) + \ldots + (S_n - S_{n-1}) = S_n$. So $\{S_n\}$ is the partial sum sequence of $\sum_{k=1}^{\infty} a_k$. Conceptually, series and sequences are equivalent. So to study series, we can use facts about sequences.
(2) Let N be a positive integer. $\sum_{k=1}^{\infty} a_k$ converges to A if and only if $\sum_{k=N}^{\infty} a_k$ converges to $B = A - (a_1 + \cdots + a_{N-1})$ because

$$B = \lim_{n\to\infty}(a_N + \cdots + a_n) = \lim_{n\to\infty}(a_1 + a_2 + \cdots + a_n) - (a_1 + \cdots + a_{N-1}) = A - (a_1 + \cdots + a_{N-1}).$$

So to see if a series converges, we may ignore finitely many terms.

Theorem. *If* $\sum_{k=1}^{\infty} a_k$ *converges to* A *and* $\sum_{k=1}^{\infty} b_k$ *converges to* B, *then*

$$\sum_{k=1}^{\infty}(a_k + b_k) = A + B = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k, \quad \sum_{k=1}^{\infty}(a_k - b_k) = A - B = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k, \quad \sum_{k=1}^{\infty} c a_k = cA = c\sum_{k=1}^{\infty} a_k$$

for any constant c.

For simple series such as geometric or telescoping series, we can find their sums.

Theorem (Geometric Series Test). *We have*

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty}(1+r+r^2+\ldots+r^n) = \lim_{n\to\infty}\frac{1-r^{n+1}}{1-r} = \begin{cases} \dfrac{1}{1-r} & \textit{if } |r|<1 \\ \text{doesn't exist} & \textit{otherwise} \end{cases}.$$

Example. $0.999\cdots = \frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots = \frac{9}{10}(\frac{1}{1-\frac{1}{10}}) = 1 = 1.000\cdots$. So, this shows that the number 1 has two decimal representations!

Theorem (Telescoping Series Test). *We have* $\sum_{k=1}^{\infty}(b_k - b_{k+1}) = \lim_{n\to\infty}\big((b_1-b_2)+(b_2-b_3)+\cdots+(b_n-b_{n+1})\big)$ $= \lim_{n\to\infty}(b_1-b_{n+1}) = b_1 - \lim_{n\to\infty} b_{n+1}$ *converges if and only if* $\lim_{n\to\infty} b_n$ *is a number.*

Examples. (1) $\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \sum_{k=1}^{\infty}(\frac{1}{k}-\frac{1}{k+1}) = \big(1-\frac{1}{2}\big)+\big(\frac{1}{2}-\frac{1}{3}\big)+\big(\frac{1}{3}-\frac{1}{4}\big)+\cdots = 1-\lim_{n\to\infty}\frac{1}{n+1} = 1.$

(2) $\sum_{k=1}^{\infty}(5^{1/k}-5^{1/(k+1)}) = (5-\sqrt{5})+(\sqrt{5}-\sqrt[3]{5})+\cdots = 5-\lim_{k\to\infty}5^{1/(k+1)} = 5-5^0 = 4.$

If a series is not geometric or telescoping, we can only determine if it converges or diverges. This can be done most of the time by applying some standard tests. If the series converges, it may be extremely difficult to find the sum!

Theorem (Term Test). *If* $\sum_{k=1}^{\infty} a_k$ *converges, then* $\lim_{k\to\infty} a_k = 0$. *(If* $\lim_{k\to\infty} a_k \neq 0$, *then the series* $\sum_{k=1}^{\infty} a_k$ *diverges.) If* $\lim_{k\to\infty} a_k = 0$, *the series* $\sum_{k=1}^{\infty} a_k$ *may or may not converge.*

(**Reason.** Suppose $\sum_{k=1}^{\infty} a_k$ converges to S. Then $\lim_{n\to\infty} S_n = S$ and $\lim_{k\to\infty} a_k = \lim_{k\to\infty}(S_k - S_{k-1}) = S - S = 0$.)

Term test is only good for series that are suspected to be divergent!

Examples. (1) $1+1+1+1+\ldots$. Here $a_k = 1$ for all k, so $\lim_{k\to\infty} a_k = 1$. Series diverges.

(2) $\sum_{k=1}^{\infty}\cos(\frac{1}{k}) = \cos 1 + \cos\frac{1}{2} + \cos\frac{1}{3} + \ldots$ diverges because $\lim_{k\to\infty}\cos(\frac{1}{k}) = \cos 0 = 1 \neq 0$.

(3) $\sum_{k=1}^{\infty}\cos k = \cos 1 + \cos 2 + \cos 3 + \ldots$ diverges because $\lim_{k\to\infty}\cos k \neq 0$. (Otherwise, $\lim_{k\to\infty}\cos k = 0$. Then $\lim_{k\to\infty}|\sin k| = \lim_{k\to\infty}\sqrt{1-\cos^2 k} = 1$ and $0 = \lim_{k\to\infty}|\cos(k+1)| = \lim_{k\to\infty}|\cos k\cos 1 - \sin k \sin 1| = \sin 1 \neq 0$, a contradiction.)

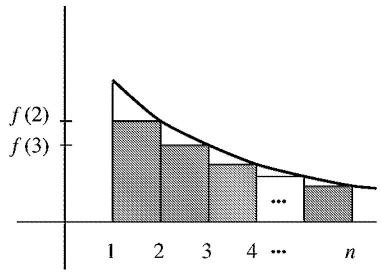
(4) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$. Here $a_k = (-\frac{1}{2})^{k-1}$ for all k, so $\lim_{k\to\infty} a_k = 0$. (Term test doesn't apply!) Series converges by the geometric series test.

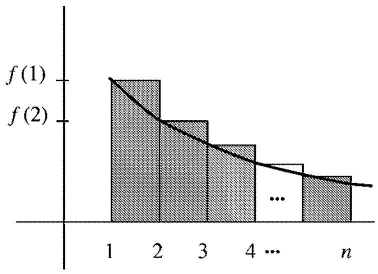
(5) $1+\underbrace{\frac{1}{2}+\frac{1}{2}}_{\text{2 times}}+\underbrace{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}_{\text{4 times}}+\underbrace{\frac{1}{8}+\ldots+\frac{1}{8}}_{\text{8 times}}+\ldots$. We have $\lim_{k\to\infty} a_k = 0$. (Term test doesn't apply.) Series diverges to ∞ because $S_1 \leq S_2 \leq S_3 \leq \cdots$ and $S_{2^n-1} = n$ has limit ∞.

For a *nonnegative* series $\sum_{k=1}^{\infty} a_k$ (i.e. $a_k \geq 0$ for every k), we have $S_1 \leq S_2 \leq S_3 \leq \ldots$ and $\lim_{n\to\infty} S_n$ must exist as a number or equal to $+\infty$. So either $\sum_{k=1}^{\infty} a_k$ converges to a number or $\sum_{k=1}^{\infty} a_k$ diverges to $+\infty$. (In short, either $\sum_{k=1}^{\infty} a_k = S$ or $\sum_{k=1}^{\infty} a_k = +\infty$.) For nonnegative series, we have the following tests.

Theorem (Integral Test). *Let $f : [1, +\infty) \to \mathbb{R}$ decrease to 0 as $x \to +\infty$. Then $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(x)\,dx < \infty$. (Note in general, $\sum_{k=1}^{\infty} f(k) \neq \int_1^{\infty} f(x)\,dx$.)*

(**Reason.** This follows from $f(2) + f(3) + \ldots + f(n) + \cdots \leq \int_1^{\infty} f(x)dx \leq f(1) + f(2) + \ldots + f(n-1) + \cdots$ as shown in the figures below.)





Examples. (1) Consider the convergence or divergence of $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$.

As $x \nearrow \infty$, $1 + x^2 \nearrow \infty$, so $\frac{1}{1+x^2} \searrow 0$. Now $\int_1^{\infty} \frac{1}{1+x^2}dx = \arctan x\Big|_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} < \infty$. So $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$ converges.

(2) Consider the convergence or divergence of $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ and $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$.

As $x \nearrow \infty$, $x \ln x$ and $x(\ln x)^2 \nearrow \infty$, so their reciprocals decrease to 0. Now $\int_2^{\infty} \frac{dx}{x \ln x} = \ln(\ln x)\Big|_2^{\infty} = \infty$. So $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ diverges. Next $\int_2^{\infty} \frac{dx}{x(\ln x)^2} = -\frac{1}{\ln x}\Big|_2^{\infty} = \frac{1}{\ln 2} < \infty$. So $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges.

Theorem (p-test). *For a real number p, $\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$ converges if and only if $p > 1$.*

(**Reason.** For $p \leq 0$, the terms are at least 1, so the series diverges by term test. For $p > 0$, $f(x) = \frac{1}{x^p}$ decreases to 0 as $x \to +\infty$. Since $\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{x^{-p+1}}{-p+1}\Big|_1^{\infty} = \frac{1}{p-1}$ if $p > 1$, $\int_1^{\infty} \frac{1}{x^p}\,dx = (\ln x)|_1^{\infty} = \infty$ if $p = 1$ and $\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{x^{-p+1}}{-p+1}\Big|_1^{\infty} = \infty$ if $p < 1$, the integral test gives the conclusion.)

Remarks. For even positive integer p, the value of $\zeta(p)$ was computed by Euler back in 1736. He got

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \ldots, \quad \zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \quad \ldots,$$

where $B_0 = 1$ and $(k+1)B_k = -\sum_{m=0}^{k-1} \binom{k+1}{m} B_m$ for $k \geq 1$. The values of $\zeta(3), \zeta(5), \ldots$ are unknown. Only in the 1980's, R. Apery was able to show $\zeta(3)$ was irrational.

Theorem (Comparison Test). *Given $v_k \geq u_k \geq 0$ for every k. If $\sum_{k=1}^{\infty} v_k$ converges, then $\sum_{k=1}^{\infty} u_k$ converges. If $\sum_{k=1}^{\infty} u_k$ diverges, then $\sum_{k=1}^{\infty} v_k$ diverges.*

(**Reason.** $v_k \geq u_k \geq 0 \Rightarrow \sum_{k=1}^{\infty} v_k \geq \sum_{k=1}^{\infty} u_k \geq 0$. If $\sum_{k=1}^{\infty} v_k$ is a number, then $\sum_{k=1}^{\infty} u_k$ is a number. If $\sum_{k=1}^{\infty} u_k = +\infty$, then $\sum_{k=1}^{\infty} v_k = +\infty$.)

Theorem (Limit Comparison Test). *Given $u_k, v_k > 0$ for every k. If $\lim_{k\to\infty} \frac{v_k}{u_k}$ is a positive number L, then either (both $\sum_{k=1}^{\infty} u_k$ and $\sum_{k=1}^{\infty} v_k$ converge) or (both diverge to $+\infty$). If $\lim_{k\to\infty} \frac{v_k}{u_k} = 0$, then $\sum_{k=1}^{\infty} u_k$ converges $\Rightarrow \sum_{k=1}^{\infty} v_k$ converges. If $\lim_{k\to\infty} \frac{v_k}{u_k} = \infty$, then $\sum_{k=1}^{\infty} u_k$ diverges $\Rightarrow \sum_{k=1}^{\infty} v_k$ diverges.*

(**Sketch of Reason.** For k large, $\frac{v_k}{u_k} \approx L$. For $L > 0$, $\sum v_k \approx \sum L u_k = L \sum u_k$. If one series converges, then the other also converges. If one diverges (to $+\infty$), so does the other. For $L = 0$, $v_k < u_k$ eventually. For $L = \infty$, $v_k > u_k$ eventually. So the last two statements follow from the comparison test.)

Examples. Consider the convergence or divergence of the following series:

(1) $\sum_{k=1}^{\infty} \frac{1}{k^2}\cos\left(\frac{1}{k}\right)$ (2) $\sum_{k=2}^{\infty} \frac{3^k}{k^2-1}$ (3) $\sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{k^2+5k}$ (4) $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$.

Solutions. (1) Since $0 < \frac{1}{k^2}\cos\left(\frac{1}{k}\right) < \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p-test, $\sum_{k=1}^{\infty} \frac{1}{k^2}\cos\left(\frac{1}{k}\right)$ converges.

(2) Since $0 < \left(\frac{3}{2}\right)^k \leq \frac{3^k}{k^2-1}$ for $k \geq 2$ and $\sum_{k=2}^{\infty} \left(\frac{3}{2}\right)^k$ diverges by the geometric series test, $\sum_{k=2}^{\infty} \frac{3^k}{k^2-1}$ diverges.

(3) When k is large, $\frac{\sqrt{k+1}}{k^2+5k} \approx \frac{\sqrt{k}}{k^2} = \frac{1}{k^{3/2}}$. We compute $\lim_{k\to\infty} \frac{\frac{\sqrt{k+1}}{k^2+5k}}{\frac{\sqrt{k}}{k^2}} = \lim_{k\to\infty} \sqrt{\frac{k+1}{k}}\frac{k^2}{k^2+5k} = 1$. Since $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ converges by p-test, $\sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{k^2+5k}$ converges by the limit comparison test.

(4) When k is large, $\frac{1}{k}$ is close to 0, so $\sin\left(\frac{1}{k}\right)$ is close to $\frac{1}{k}$ because $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ (i.e. $\sin\theta \approx \theta$ as $\theta \to 0$). We compute $\lim_{k\to\infty} \frac{\sin(\frac{1}{k})}{\frac{1}{k}} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by p-test, $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)$ diverges by the limit comparison test.

For series with *alternate* positive and negative terms, we have the following test.

Theorem (Alternating Series Test). *If c_k decreases to 0 as $k \to \infty$ (i.e. $c_1 \geq c_2 \geq c_3 \geq \ldots \geq 0$ and $\lim_{k\to\infty} c_k = 0$), then $\sum_{k=1}^{\infty} (-1)^{k+1} c_k = c_1 - c_2 + c_3 - c_4 + c_5 - \ldots$ converges.*

(**Reason.** Since $c_1 \geq c_2 \geq c_3 \geq \ldots \geq 0$, we have $0 \leq S_2 \leq S_4 \leq S_6 \leq \ldots \leq S_5 \leq S_3 \leq S_1$. Since $\lim_{n\to\infty} |S_n - S_{n-1}| = \lim_{n\to\infty} c_n = 0$, the distances between the partial sums decrease to 0 and so $\lim_{n\to\infty} S_n$ must exist.)

Examples. Both $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$ and $\sum_{k=1}^{\infty} e^{-k} \cos k\pi$ converge by the alternating series test because as $k \nearrow \infty$, $k \ln k \nearrow \infty$ and $e^k \nearrow \infty$, so $1/(k \ln k) \searrow 0$ and $e^{-k} \searrow 0$ and $\cos k\pi = (-1)^k$.

For series with *arbitrary* positive or negative term, we have the following tests.

Theorem (Absolute Convergence Test). *If* $\sum_{k=1}^{\infty} |a_k|$ *converges, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

(**Reason.** From $-|a_k| \leq a_k \leq |a_k|$, we get $0 \leq a_k + |a_k| \leq 2|a_k|$. Since $\sum_{k=1}^{\infty} 2|a_k|$ converges, so by the comparison test, $\sum_{k=1}^{\infty}(a_k + |a_k|)$ converges. Then $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(a_k + |a_k|) - \sum_{k=1}^{\infty} |a_k|$ converges.)

Definition. We say $\sum_{k=1}^{\infty} a_k$ *converges absolutely* iff $\sum_{k=1}^{\infty} |a_k|$ converges. We say $\sum_{k=1}^{\infty} a_k$ *converges conditionally* iff $\sum_{k=1}^{\infty} a_k$ converges, but $\sum_{k=1}^{\infty} |a_k|$ *diverges.*

Examples. Determine if the following series converge absolutely or conditionally

(a) $\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$ (b) $\sum_{k=1}^{\infty} \frac{\cos k\pi}{1+k}$.

Solutions. (a) $\sum_{k=1}^{\infty} \left|\frac{\cos k}{k^3}\right| \leq \sum_{k=1}^{\infty} \frac{1}{k^3}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges by p-test, it follows that $\sum_{k=1}^{\infty} \left|\frac{\cos k}{k^3}\right|$ converges by the comparison test. So $\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$ converges absolutely by the absolute convergence test.

(b) $\sum_{k=1}^{\infty} \left|\frac{\cos k\pi}{1+k}\right| = \sum_{k=1}^{\infty} \frac{1}{1+k}$ because $\cos k\pi = (-1)^k$. $\int_1^{\infty} \frac{dx}{1+x} = \ln(1+x)\Big|_1^{\infty} = \infty \Rightarrow \sum_{k=1}^{\infty} \frac{1}{1+k}$ diverges. However, $\frac{1}{1+k}$ decreases to 0 as $k \to +\infty$. So by the alternating series test, $\sum_{k=1}^{\infty} \frac{\cos k\pi}{1+k} = \sum_{k=1}^{\infty}(-1)^k \frac{1}{1+k}$ converges. Therefore $\sum_{k=1}^{\infty} \frac{\cos k\pi}{1+k}$ converges conditionally.

Theorem (Ratio Test). *If* $a_k \neq 0$ *for every* k *and* $\lim_{k\to\infty} |a_{k+1}/a_k|$ *exists, then*

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| \begin{cases} < 1 & \Rightarrow \sum_{k=1}^{\infty} a_k \text{ converges absolutely} \\ = 1 & \Rightarrow \sum_{k=1}^{\infty} a_k \text{ may converge } \left(e.g. \sum_{k=1}^{\infty} \frac{1}{k^2}\right) \text{ or diverge } \left(e.g. \sum_{k=1}^{\infty} \frac{1}{k}\right). \\ > 1 & \Rightarrow \sum_{k=1}^{\infty} a_k \text{ diverges} \end{cases}$$

(**Sketch of reason.** Let $r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$, then for k large, $\left|\frac{a_{k+1}}{a_k}\right|, \left|\frac{a_{k+2}}{a_{k+1}}\right|, \ldots, \left|\frac{a_{k+n}}{a_{k+n-1}}\right| \approx r$, so $|a_{k+n}| \approx |a_k|r^n$ and $|a_k| + |a_{k+1}| + |a_{k+2}| + \ldots \approx |a_k|(1 + r + r^2 + r^3 + \ldots)$ which converges if $r < 1$ by the geometric series test and $\sum a_{k+n} \approx \sum \pm a_k r^n$ diverges if $r > 1$ by the term test.)

Theorem (Root Test). *If* $\lim_{k\to\infty}\sqrt[k]{|a_k|}$ *exists, then*

$$\lim_{k\to\infty}\sqrt[k]{|a_k|}\begin{cases} < 1 & \Rightarrow \sum_{k=1}^{\infty} a_k \textit{ converges absolutely} \\ = 1 & \Rightarrow \sum_{k=1}^{\infty} a_k \textit{ may converge } \left(e.g. \sum_{k=1}^{\infty}\frac{1}{k^2}\right) \textit{ or diverge } \left(e.g. \sum_{k=1}^{\infty}\frac{1}{k}\right). \\ > 1 & \Rightarrow \sum_{k=1}^{\infty} a_k \textit{ diverges} \end{cases}$$

(**Sketch of reason.** Let $r = \lim_{k\to\infty}\sqrt[k]{|a_k|}$, then for k large, $\sqrt[k]{|a_k|} \approx r$. So $|a_k| \approx r^k$, $\sum |a_k| \approx \sum r^k$.)

Examples. Consider the convergence or divergence of the following series:

(1) $\displaystyle\sum_{k=1}^{\infty}\frac{1}{3^k - 2^k}$ (2) $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{k^k}$.

Solutions. (1) Since $\displaystyle\lim_{k\to\infty}\frac{\frac{1}{3^{k+1}-2^{k+1}}}{\frac{1}{3^k-2^k}} = \lim_{k\to\infty}\frac{3^k-2^k}{3^{k+1}-2^{k+1}} = \lim_{k\to\infty}\frac{\frac{1}{3}-(\frac{2}{3})^k\frac{1}{3}}{1-(\frac{2}{3})^{k+1}} = \frac{1}{3} < 1$, by the ratio test, $\displaystyle\sum_{k=1}^{\infty}\frac{1}{3^k-2^k}$ converges. Alternatively, since $\displaystyle\lim_{k\to\infty}\sqrt[k]{\frac{1}{3^k-2^k}} = \lim_{k\to\infty}\frac{1}{\sqrt[k]{3^k-2^k}} = \lim_{k\to\infty}\frac{1}{3\sqrt[k]{1-(\frac{2}{3})^k}} = \frac{1}{3} < 1$, by the root test, $\displaystyle\sum_{k=1}^{\infty}\frac{1}{3^k-2^k}$ converges.

(2) Since $\displaystyle\lim_{k\to\infty}\frac{(k+1)!}{(k+1)^{k+1}}\frac{k^k}{k!} = \lim_{k\to\infty}\frac{1}{(1+\frac{1}{k})^k} = \frac{1}{e} < 1$, by the ratio test, $\displaystyle\sum_{k=1}^{\infty}\frac{k!}{k^k}$ converges.

Remarks. You may have observed that in example (1), the limit you got for applying the root test was the same as the limit you got for applying the ratio test. This was not an accident!

Theorem. *If* $a_k > 0$ *for all* k *and* $\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = r \in \mathbb{R}$, *then* $\lim_{k\to\infty}\sqrt[k]{a_k} = r$. *(This implies that the root test can be applied to more series than the ratio test.)*

Examples. (1) Let $a_k = k$, then $\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \lim_{k\to\infty}\frac{k+1}{k} = 1$. So, $\lim_{k\to\infty}\sqrt[k]{k} = 1$.

(2) Let $a_k = \frac{k!}{k^k}$, then $\lim_{k\to\infty}\frac{a_{k+1}}{a_k} = \frac{1}{e}$ as above. So $\lim_{k\to\infty}\sqrt[k]{a_k} = \lim_{k\to\infty}\frac{\sqrt[k]{k!}}{k} = \frac{1}{e}$, i.e. when k is large, $k! \approx \left(\frac{k}{e}\right)^k$, which is a simple version of what is called *Stirling's formula.* It is useful for estimating $n!$ when n is large. For example, since $\log_{10}\frac{100}{e} \approx 1.566$, so $\frac{100}{e} \approx 10^{1.566}$, then we get $100! \approx 10^{156.6}$, which has about 157 digits.

Theorem (Summation by Parts). *Let* $S_j = \sum_{k=1}^{j} a_k = a_1 + a_2 + \ldots + a_j$ *and* $\Delta b_k = \frac{b_{k+1}-b_k}{(k+1)-k} = b_{k+1} - b_k$, *then*

$$\sum_{k=1}^{n} a_k b_k = S_n b_n - \sum_{k=1}^{n-1} S_k \Delta b_k.$$

(**Reason.** Note $a_1 = S_1$ and $a_k = S_k - S_{k-1}$ for $k > 1$. So,

$$\sum_{k=1}^{n} a_k b_k = S_1 b_1 + (S_2 - S_1) b_2 + \ldots + (S_n - S_{n-1}) b_n$$
$$= S_n b_n - S_1(b_2 - b_1) - \ldots - S_{n-1}(b_n - b_{n-1}).)$$

Example. Show that $\sum_{k=1}^{\infty} \frac{\sin k}{k}$ converges.

Let $a_k = \sin k$ and $b_k = \frac{1}{k}$. Using the identity $\sin m \sin \frac{1}{2} = \frac{1}{2}\left(\cos(m - \frac{1}{2}) - \cos(m + \frac{1}{2})\right)$, we have

$$S_k = \sin 1 + \sin 2 + \cdots + \sin k = \frac{\cos \frac{1}{2} - \cos(k + \frac{1}{2})}{2 \sin \frac{1}{2}}.$$

This implies $|S_k| \le \frac{1}{\sin(1/2)}$ for every k. Applying summation by parts and noting that $\lim_{n\to\infty} \frac{S_n}{n} = 0$, we get

$$\sum_{k=1}^{\infty} \frac{\sin k}{k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin k}{k} = \lim_{n\to\infty} \left(\frac{S_n}{n} - \sum_{k=1}^{n-1} S_k \left(\frac{1}{k+1} - \frac{1}{k} \right) \right) = \sum_{k=1}^{\infty} S_k \left(\frac{1}{k} - \frac{1}{k+1} \right).$$

Now $\sum_{k=1}^{\infty} \left| S_k \left(\frac{1}{k} - \frac{1}{k+1} \right) \right| \le \frac{1}{\sin(1/2)} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \frac{1}{\sin(1/2)}$ by the telescoping series test. So by the absolute convergence test, $\sum_{k=1}^{\infty} \frac{\sin k}{k} = \sum_{k=1}^{\infty} S_k \left(\frac{1}{k} - \frac{1}{k-1} \right)$ converges.

Inserting Parentheses and Rearrangements of Series.

Definition. We say $\sum_{k=1}^{\infty} b_k$ is obtained from $\sum_{k=1}^{\infty} a_k$ by *inserting parentheses* iff there is a strictly increasing function $p : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ such that $p(0) = 0$, $b_1 = a_1 + \cdots + a_{p(1)}$, $b_2 = a_{p(1)+1} + \cdots + a_{p(2)}$, $b_3 = a_{p(2)+1} + \cdots + a_{p(3)}, \ldots$. (Note b_n is the sum of $k_n = p(n) - p(n-1)$ terms.)

Grouping Theorem. *Let $\sum_{k=1}^{\infty} b_k$ be obtained from $\sum_{k=1}^{\infty} a_k$ by inserting parentheses. If $\sum_{k=1}^{\infty} a_k$ converges to s, then $\sum_{k=1}^{\infty} b_k$ will converge to s. Next, if $\lim_{n\to\infty} a_n = 0$, k_n is bounded and $\sum_{k=1}^{\infty} b_k$ converges to s, then $\sum_{k=1}^{\infty} a_k$ will converge to s.*

(**Reason.** Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$. For the first part, $\sum_{k=1}^{\infty} b_k = \lim_{n\to\infty} t_n = \lim_{n\to\infty} \sum_{k=1}^{p(n)} a_k = s$. For the second part, let $p(n) - p(n-1)$ be bounded by M. For a positive integer j, let $p(i) \le j < p(i+1)$. For $r = 1, 2, \ldots, M$, define $c_{r,j} = \begin{cases} a_{p(i)+r} & \text{if } p(i) + r \le j \\ 0 & \text{if } p(i) + r > j \end{cases}$. Then $\sum_{k=1}^{\infty} a_k = \lim_{j\to\infty} s_j = \lim_{i\to\infty} t_i + \lim_{j\to\infty} (c_{1,j} + \cdots + c_{M,j}) = s + 0 + \cdots + 0 = s.)$

Examples. (1) Since $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ converges to 1, so by the theorem,

$$\frac{1}{2} + \left(\frac{1}{4} + \frac{1}{8} \right) + \left(\frac{1}{16} + \frac{1}{32} + \frac{1}{64} \right) + \left(\frac{1}{128} + \frac{1}{256} + \frac{1}{512} + \frac{1}{1024} \right) + \cdots = 1.$$

(2) $(1-1)+(1-1)+\cdots$ converges to 0, but $1-1+1-1+\cdots$ diverges by term test. So $\lim_{n\to\infty} a_n = 0$ is important. Also, $(1-1)+\left(\frac{1}{2}+\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}\right)+\cdots$ converges to 0. However, the series without parentheses diverges (as $S_{n^2}=1$ and $S_{n^2+n}=0$) even though the terms have limit 0. So k_n bounded is important.

(3) Since $\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots=\sum_{j=1}^{\infty}\left(\frac{1}{2j-1}-\frac{1}{2j}\right)=\sum_{j=1}^{\infty}\frac{1}{2j(2j-1)}$ converges (by the limit comparison test with $\sum_{j=1}^{\infty}\frac{1}{j^2}$), so by the theorem, $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$ converges to the same sum.

Definition. $\sum_{k=1}^{\infty} b_k$ is a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ iff there is a bijection $\sigma:\mathbb{N}\to\mathbb{N}$ such that $b_k=a_{\sigma(k)}$.

Example. Given $\ln 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$ (which converges conditionally). Consider the rearrangement $\underbrace{1+\frac{1}{3}}_{2+}\underbrace{-\frac{1}{2}}_{1-}+\underbrace{\frac{1}{5}+\frac{1}{7}}_{2+}\underbrace{-\frac{1}{4}}_{1-}+\underbrace{\frac{1}{9}+\frac{1}{11}}_{2+}\underbrace{-\frac{1}{6}}_{1-}+\ldots$. Observe that

$$\begin{array}{rl} & (1-\tfrac{1}{2})+(\tfrac{1}{3}-\tfrac{1}{4})+(\tfrac{1}{5}-\tfrac{1}{6})+(\tfrac{1}{7}-\tfrac{1}{8})+\ldots=\ln 2\\ + & \quad\;\; \tfrac{1}{2} \qquad\quad -\tfrac{1}{4} \qquad\quad +\tfrac{1}{6} \qquad\quad -\tfrac{1}{8} \quad \ldots=\tfrac{1}{2}\ln 2\\ \hline & 1 \quad +(\tfrac{1}{3}-\tfrac{1}{2})+ \quad \tfrac{1}{5} \quad +(\tfrac{1}{7}-\tfrac{1}{4})+\ldots=\tfrac{3}{2}\ln 2.\end{array}$$

Riemann's Rearrangement Theorem. *Let $a_k\in\mathbb{R}$ and $\sum_{k=1}^{\infty} a_k$ converge conditionally. For any $x\in\mathbb{R}$ or $x=\pm\infty$, there is a rearrangement $\sum_{k=1}^{\infty} a_{\sigma(k)}$ of $\sum_{k=1}^{\infty} a_k$ such that $\sum_{k=1}^{\infty} a_{\sigma(k)}=x$.*

(**Sketch of reason.** Let $p_k=\begin{cases} a_k & \text{if } a_k\ge 0\\ 0 & \text{if } a_k<0\end{cases}$ and $q_k=\begin{cases} 0 & \text{if } a_k\ge 0\\ |a_k| & \text{if } a_k<0\end{cases}$. Then $a_k=p_k-q_k$ and $|a_k|=p_k+q_k$. Now both $\sum_{k=1}^{\infty} p_k$, $\sum_{k=1}^{\infty} q_k$ must diverge to $+\infty$. (If both converges, then their sum $\sum_{k=1}^{\infty}|a_k|$ will be finite, a contradiction. If one converges and the other diverges to $+\infty$, then $\sum_{k=1}^{\infty} a_k=\sum_{k=1}^{\infty} p_k-\sum_{k=1}^{\infty} q_k$ will diverges to $\pm\infty$, a contradiction also.) Let u_n, v_n be sequences of real numbers having limits x and $u_n<v_n$, $u_n<v_{n+1}$, $v_1>0$. Now let $P_1, P_2, \ldots$ be the nonnegative terms of $\sum_{k=1}^{\infty} a_k$ in the order they occur and $Q_1, Q_2, \ldots$ be the absolute value of the negative terms in the order they occur. Since $\sum_{k=1}^{\infty} P_k$, $\sum_{k=1}^{\infty} Q_k$ differ from $\sum_{k=1}^{\infty} p_k$, $\sum_{k=1}^{\infty} q_k$ only by zero terms, they also diverges to $+\infty$. Let m_1, k_1 be the smallest integers such that $P_1+\cdots+P_{m_1}>v_1$ and $P_1+\cdots+P_{m_1}-Q_1-\cdots-Q_{k_1}<u_1$. Let m_2, k_2 be the smallest integers such that $P_1+\cdots+P_{m_1}-Q_1-\cdots-Q_{k_1}+P_{m_1+1}+\cdots+P_{m_2}>v_2$ and $P_1+\cdots+P_{m_1}-Q_1-\cdots-Q_{k_1}+P_{m_1+1}+\cdots+P_{m_2}-Q_{k_1+1}-\cdots-Q_{k_2}<u_2$ and continue this way. This is possible since the sums of P_k and Q_k are $+\infty$. Now if s_n, t_n are the partial sums of this series $P_1+\cdots+P_{m_1}-Q_1-\cdots-Q_{k_1}+\cdots$ whose last terms are P_{m_n}, Q_{k_n}, respectively, then $|s_n-v_n|\le P_{m_n}$ and $|t_n-u_n|\le Q_{k_n}$ by the choices of m_n, k_n. Since P_n, Q_n have limit 0, so s_n, t_n must have limit x. As all other partial sums are squeezed by s_n and t_n, the series we constructed must have limit x.)

Dirichlet's Rearrangement Theorem. *If $a_k\in\mathbb{R}$ and $\sum_{k=1}^{\infty} a_k$ converges absolutely, then every rearrangement $\sum_{k=1}^{\infty} a_{\sigma(k)}$ converges to the same sum as $\sum_{k=1}^{\infty} a_k$.*

(**Reason.** Define p_k, q_k as in the last proof. Since $p_k, q_k \le |a_k|$, $\sum_{k=1}^{\infty} p_k$, $\sum_{k=1}^{\infty} q_k$ converge, say to p and q, respectively. Since $a_{\sigma(k)} = p_{\sigma(k)} - q_{\sigma(k)}$, we may view $\sum_{k=1}^{\infty} p_{\sigma(k)}$ as a rearrangement of the nonnegative terms of $\sum_{k=1}^{\infty} a_k$ and inserting zeros where $a_{\sigma(k)} < 0$. For any positive integer m, the partial sum $s_m = \sum_{k=1}^{m} p_{\sigma(k)} \le \sum_{k=1}^{\infty} p_k = p$. Since $p_k \ge 0$, the partial sum s_m is also increasing, hence $\sum_{k=1}^{\infty} p_{\sigma(k)}$ converges. Now, for every positive integer n, $\sum_{k=1}^{n} p_k \le \sum_{k=1}^{\infty} p_{\sigma(k)} \le p$. As $n \to \infty$, we get $\sum_{k=1}^{\infty} p_{\sigma(k)} = p$. Similarly, $\sum_{k=1}^{\infty} q_{\sigma(k)} = q$. Then $\sum_{k=1}^{\infty} a_{\sigma(k)} = p - q = \sum_{k=1}^{\infty} a_k$.)

Example. $\sum_{k=1}^{\infty}(-\frac{1}{2})^k = -\frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \ldots$ converges (absolutely) to $\dfrac{-\frac{1}{2}}{1-(-\frac{1}{2})} = -\frac{1}{3}$.

$$-\frac{1}{2} + \frac{1}{2^2} + \underbrace{\frac{1}{2^4} - \frac{1}{2^3}}_{\text{2 terms}} + \underbrace{\frac{1}{2^8} - \frac{1}{2^7} + \frac{1}{2^6} - \frac{1}{2^5}}_{\text{4 terms}} + \underbrace{\frac{1}{2^{16}} - \frac{1}{2^{15}} + \frac{1}{2^{14}} - \frac{1}{2^{13}} + \frac{1}{2^{12}} - \frac{1}{2^{11}} + \frac{1}{2^{10}} - \frac{1}{2^9}}_{\text{8 terms}} + \ldots$$

is a rearrangement of $\sum_{k=1}^{\infty}(-\frac{1}{2})^k$, so it also converges to $-\frac{1}{3}$.

Remarks. As a consequence of the rearrangement theorem, the sum of a nonnegative series is the same no matter how the terms are rearranged.

Complex Series

Complex numbers $S_1, S_2, S_3, \ldots$ with $S_n = u_n + i v_n$ are said to have limit $\lim_{n\to\infty} S_n = u + iv$ iff $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. A *complex series* is a series where the terms are complex numbers. The definitions of convergent, absolutely convergent and conditional convergent are the same. The remarks and the basic properties following the definitions of convergent and divergent series are also true for complex series.

The geometric series test, telescoping series test, term test, absolute convergence test, ratio test and root test are also true for complex series. For $z_k = x_k + i y_k$, we have $\sum_{k=1}^{\infty} z_k$ converges to $z = x + iy$ if and only if $\sum_{k=1}^{\infty} x_k$ converges to x and $\sum_{k=1}^{\infty} y_k$ converges to y. So complex series can be reduced to real series for study if necessary.

Examples. (1) Note $\lim_{n\to\infty} i^n \ne 0$ (otherwise $0 = \lim_{n\to\infty} |i^n| = \lim_{n\to\infty} 1$ is a contradiction). So $\sum_{k=1}^{\infty} i^k$ diverges by term test.

(2) If $|z| \le 1$, then $\left|\frac{z^k}{k^2}\right| \le \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p-test implies $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ converges absolutely. However, if $|z| > 1$, then $\lim_{k\to\infty}\left|\frac{z^{k+1}}{(k+1)^2}\frac{k^2}{z^k}\right| = \lim_{k\to\infty} \frac{k^2}{(k+1)^2}|z| = |z| > 1$ implies $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ diverges by the ratio test.