MATH202 Introduction to Analysis (2007 Fall-2008 Spring)

Tutorial Note #3

Proof on Sets

HOW TO PROVE THINGS?

Last tutorial, we have go through some operations and discuss some simple proofs about comparing sets. This tutorial, we will go through the proof of theorem about general sets. The proof actually relies on definitions ONLY.

Recall

$\boxed{\mathbf{A \subseteq B \leftrightarrow \text{for any } x \in A, \text{we have } x \in B}}$

$\boxed{\mathbf{A = B \leftrightarrow A \subseteq B \text{ and } B \subseteq A}}$

We first start with some simple proof:

Example 1 Suppose $\mathrm{A \subseteq B}$, show that $\mathrm{A \cup C \subseteq B \cup C}$

Proof:

We need to show for any $\boldsymbol{x \in A \cup C \rightarrow x \in B \cup C}$

For any $\boldsymbol{x \in A \cup C}$

$\boldsymbol{\rightarrow x \in A \text{ or } x \in C}$ ***(***$\boldsymbol{x \in A \cup C \rightarrow x \in A \text{ or } x \in C}$***)***

$\boldsymbol{\rightarrow x \in B \text{ or } x \in C}$ ***(It is because*** $\boldsymbol{A \subseteq B}$***, i.e. for any*** $\boldsymbol{x \in A \rightarrow x \in B}$***)***

$\boldsymbol{\rightarrow x \in B \cup C}$ ($\boldsymbol{x \in B \cup C \leftrightarrow x \in B \text{ or } x \in C}$)

So $\boldsymbol{A \cup C \subseteq B \cup C}$***.***

Example 2 Show if $\mathrm{A \subseteq B}$, then $\mathrm{A \cap B = A}$

Proof:

We need to show $\mathrm{A \cap B \subseteq A}$ and $\mathrm{A \subseteq A \cap B}$

(Step 1: $\mathrm{A \cap B \subseteq A}$)

For any $\mathrm{x \in A \cap B}$

$\rightarrow \mathrm{x \in A \text{ and } x \in B}$ ($\mathrm{x \in A \cap B \leftrightarrow x \in A}$ and $\mathrm{x \in B}$)

$\rightarrow$ $\mathrm{x \in A}$

$\rightarrow \mathrm{A \cap B \subseteq A}$

(Step 2: $\mathrm{A \subseteq A \cap B}$)

For any $\mathrm{x \in A}$

We also have $\mathrm{x} \in \mathrm{B}$ (since $\mathrm{A} \subseteq \mathrm{B}$, ***i.e. for any*** $\boldsymbol{x} \in \boldsymbol{A} \rightarrow \boldsymbol{x} \in \boldsymbol{B}$)

$\rightarrow \mathrm{x} \in \mathrm{A}$ and $\mathrm{x} \in \mathrm{B}$

$\rightarrow \mathrm{x} \in \mathrm{A} \cap \mathrm{B}$

$\rightarrow \mathrm{A} \subseteq \mathrm{A} \cap \mathrm{B}$

Combine two steps, we have $\mathrm{A} \cap \mathrm{B} = \mathrm{A}$

☺Exercise

Show that if $\mathrm{A} \subseteq \mathrm{B}$, then $\mathrm{A} \cup \mathrm{B} = \mathrm{B}$

Besides proving if-then statement, we need to show some identities of sets. We start with some simple examples:

Example 3 Show that $(\mathrm{X} \backslash \mathrm{Y}) \backslash \mathrm{Z} = (\mathrm{X} \backslash \mathrm{Z}) \backslash \mathrm{Y}$

(Step 1: $(\boldsymbol{X} \backslash \boldsymbol{Y}) \backslash \boldsymbol{Z} \subseteq (\boldsymbol{X} \backslash \boldsymbol{Z}) \backslash \boldsymbol{Y}$***)***

For any $\boldsymbol{a} \in (\boldsymbol{X} \backslash \boldsymbol{Y}) \setminus \boldsymbol{Z}$

$\rightarrow \boldsymbol{a} \in \boldsymbol{X} \backslash \boldsymbol{Y}$ ***and*** $\boldsymbol{a} \notin \boldsymbol{Z}$ ***(We treat*** $\boldsymbol{X} \backslash \boldsymbol{Y}$ ***is a set)***

$\rightarrow (\boldsymbol{a} \in \boldsymbol{X} \ \boldsymbol{and} \ \boldsymbol{a} \notin \boldsymbol{Y}) \ \boldsymbol{and} \ \boldsymbol{a} \notin \boldsymbol{Z}$

$\rightarrow (\boldsymbol{a} \in \boldsymbol{X} \ \boldsymbol{and} \ \boldsymbol{a} \notin \boldsymbol{Z}) \ \boldsymbol{and} \ \boldsymbol{a} \notin \boldsymbol{Y}$ ***(Change order)***

$\rightarrow \boldsymbol{a} \in \boldsymbol{X} \backslash \boldsymbol{Z}$ ***and*** $\boldsymbol{a} \notin \boldsymbol{Y}$

$\rightarrow \boldsymbol{a} \in (\boldsymbol{X} \backslash \boldsymbol{Z}) \backslash \boldsymbol{Y}$

So $(\boldsymbol{X} \backslash \boldsymbol{Y}) \backslash \boldsymbol{Z} \subseteq (\boldsymbol{X} \backslash \boldsymbol{Z}) \backslash \boldsymbol{Y}$

(Step 2: $(\boldsymbol{X} \backslash \boldsymbol{Z}) \backslash \boldsymbol{Y} \subseteq (\boldsymbol{X} \backslash \boldsymbol{Y}) \backslash \boldsymbol{Z}$***)*** ***(Try to fill in the detail by yourself)***

So $(\boldsymbol{X}\backslash\boldsymbol{Z})\backslash\boldsymbol{Y} \subseteq (\boldsymbol{X}\backslash\boldsymbol{Y})\backslash\boldsymbol{Z}$

Therefore $(\boldsymbol{X}\backslash\boldsymbol{Y})\backslash\boldsymbol{Z} = (\boldsymbol{X}\backslash\boldsymbol{Z})\backslash\boldsymbol{Y}$

Example 4
Show $\mathrm{X}\backslash(\mathrm{Y} \cap \mathrm{Z}) = (\mathrm{X}\backslash\mathrm{Y}) \cup (\mathrm{X}\backslash\mathrm{Z})$

Proof:

(Step 1: $\boldsymbol{A} \subseteq \boldsymbol{B}$***)***

For any $\boldsymbol{x} \in \boldsymbol{X}\backslash(\boldsymbol{Y} \cap \boldsymbol{Z})$

$\rightarrow \boldsymbol{x} \in \boldsymbol{X}$ ***and*** $\boldsymbol{x} \notin \boldsymbol{Y} \cap \boldsymbol{Z}$

$\rightarrow \boldsymbol{x} \in \boldsymbol{X}\ \boldsymbol{and}\ (\boldsymbol{x} \notin \boldsymbol{Y}\ \boldsymbol{or}\ \boldsymbol{x} \notin \boldsymbol{Z})$ ***(Recall in Logic: ~(x and y) = ~x or ~y)***

$\rightarrow (\boldsymbol{x} \in \boldsymbol{X}\ \boldsymbol{or}\ \boldsymbol{x} \notin \boldsymbol{Y})\boldsymbol{and}\ (\boldsymbol{x} \in \boldsymbol{X}\ \boldsymbol{or}\ \boldsymbol{x} \notin \boldsymbol{Z})$

$\rightarrow \boldsymbol{x} \in \boldsymbol{X}\backslash\boldsymbol{Y}$ ***or*** $\boldsymbol{x} \in \boldsymbol{X}\backslash\boldsymbol{Z}$

$\rightarrow \boldsymbol{x} \in (\boldsymbol{X}\backslash\boldsymbol{Y}) \cup (\boldsymbol{X}\backslash\boldsymbol{Z})$

$\rightarrow \boldsymbol{X}\backslash(\boldsymbol{Y} \cap \boldsymbol{Z}) \subseteq (\boldsymbol{X}\backslash\boldsymbol{Y}) \cup (\boldsymbol{X}\backslash\boldsymbol{Z})$

(Step 2: $\boldsymbol{A} \supseteq \boldsymbol{B}$***)***

By reversing the step in Step 1, we can show $\boldsymbol{X}\backslash(\boldsymbol{Y} \cap \boldsymbol{Z}) \supseteq (\boldsymbol{X}\backslash\boldsymbol{Y}) \cup (\boldsymbol{X}\backslash\boldsymbol{Z})$

So $\boldsymbol{X}\backslash(\boldsymbol{Y} \cap \boldsymbol{Z}) = (\boldsymbol{X}\backslash\boldsymbol{Y}) \cup (\boldsymbol{X}\backslash\boldsymbol{Z})$***.***

☺Exercise

Show $\mathrm{X}\backslash(\mathrm{Y} \cup \mathrm{Z}) = (\mathrm{X}\backslash\mathrm{Y}) \cap (\mathrm{X}\backslash\mathrm{Z})$

(Step 1: $\mathrm{X}\backslash(\mathrm{Y} \cup \mathrm{Z}) \subseteq (\mathrm{X}\backslash\mathrm{Y}) \cap (\mathrm{X}\backslash\mathrm{Z})$)

(Step 2: $\mathrm{X}\backslash(\mathrm{Y} \cup \mathrm{Z}) \supseteq (\mathrm{X}\backslash\mathrm{Y}) \cap (\mathrm{X}\backslash\mathrm{Z})$)

Given a set formula, how can we say whether it is true or not? If the formula is simple, we can first check ***by drawing diagrams***.

Case i) If your answer is TRUE, then prove it using technique shown above
Case ii) If your answer is FALSE, then provide a counter-example

Example 5

a) Is it always true that $(A \cup B) \cap C = A \cup (B \cap C)$

b) Is it always true that if $A \cup B = A \cup C$, then $B = C$

c) Is it always true that $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$

Solution:

a) FALSE

Let $\boldsymbol{A = \{1, 2\}}$***,*** $\boldsymbol{B = \{2, 3\}}$ ***and*** $\boldsymbol{C = \{1\}}$***, then***

$(\boldsymbol{A} \cup \boldsymbol{B}) \cap \boldsymbol{C} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\} \cap \{\mathbf{1}\} = \{\mathbf{1}\}$

$\boldsymbol{but\ A} \cup (\boldsymbol{B} \cap \boldsymbol{C}) = \{\mathbf{1}, \mathbf{2}\} \cup \boldsymbol{\phi} = \{\mathbf{1}, \mathbf{2}\}$

b) FALSE

Let $\boldsymbol{A = \{1, 2, 3\}}$ $\boldsymbol{B = \{1, 2\}}$ ***and*** $\boldsymbol{C = \{2, 3\}}$

It is easy to see $\boldsymbol{A} \cup \boldsymbol{B} = \boldsymbol{A} \cup \boldsymbol{C} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$

But $\boldsymbol{B \neq C}$

c) TRUE

The proof is in Example 6.

Remark: When constructing counter-example, it is an good idea to use some simple sets (e.g. empty set or single element set) which are easy to handle.

☺Exercise

Determine whether the following statements are true or not. For the false statement, provide a counter-example.

a) $(A \cup B) \cap C = A \cup (B \cap C)$ for **<u>all sets A, B and C</u>**

b) $(A \cup B) \cap C \neq A \cup (B \cap C)$ for **<u>all sets of A, B and C</u>**

(This problem is suggested by one of our classmates.)

c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Part II: About Functions

Recall:

Let $f: A \to B$ be a map/function from A to B, we say

f is **injective** (or one to one) if $f(x_1) = f(x_2) \to x_1 = x_2$

f is **surjective** (or onto) if for any $y \in B$, there exists $x \in A$, such that $f(x) = y$

f is **bijective** (or 1-1 correspond) if f is injective and surjective

(Note: To prove f is NOT injective or surjective, by logic, all we need to do is to find a counter-example)

Example 6

Let $f: R \to R$ be a function, determine which of the following functions are injective, surjective or bijective.

a) $f(x) = x^2$

b) $f(x) = x^3$

Solution:

a) ***(Injective?)*** $f(x)$ ***is not injective because*** $f(2) = f(-2) = 4$ ***but*** $2 \neq -2$

(Surjective?) $f(x)$ ***is not surjective because pick*** $y = -1$***, we can not find*** $x \in R$ ***such that*** $x^2 = -1$

(Bijective?) We can see it is not bijective

b) ***(Injective?)*** $f(x_1) = f(x_2) \to x_1^3 = x_2^3 \to x_1 = x_2$***, so f is injective***

(Surjective?) For any $y \in R$***, pick*** $x = \sqrt[3]{y}$***, then we have*** $f(x) = \left(\sqrt[3]{y}\right)^3 = y$

So f is surjective

(Bijective?) It is bijective

Example 6

For $a < b$, (a, b) will denote the open interval $\{x: x \in R, a < x < b\}$. Find a bijective function $f: (0,1) \to (a, b)$

Since we see that (0,1) is similar to (a,b) (just difference in size), therefore we can define the map in this way:

$$f(t) = a + t(b - a) \; for \; t \in (0, 1)$$

One can show this map is bijective (left as exercise)

Example 7

Let S be the set of all positive real numbers x such that $0 < x < 1$ and x has a decimal representation consisted of finitely many digits (i.e. $x = 0.a_1a_2a_3 \dots a_k$ where a_i is digit (0-9)). Find a bijective function $g: N \to S$

Solution:

The trouble here is the number of decimal places of x is finite and can be 1,2,3,4,…..

Therefore, to find the map, we can do it in this way:

(Step 1: Consider all $x \in S$ which only one decimal place)

Let

$$\mathbf{1 \to 0.1}$$
$$\mathbf{2 \to 0.2}$$
$$\dots ..$$
$$\mathbf{9 \to 0.9}$$

(Step 2: Consider $x \in S$ with only two demical places)

(Start with last digit with 1 until 9)

$$\mathbf{10 \to 0.01}$$
$$\mathbf{11 \to 0.11}$$
$$\mathbf{12 \to 0.21}$$
$$\dots \dots$$
$$\mathbf{21 \to 0.12}$$
$$\dots \dots$$
$$\mathbf{99 \to 0.99}$$

We construct in this way, from the pattern (you may work out more), we see that the map is actually the following:

$$\mathbf{f(\overline{a_1a_2a_3 \dots .a_k}) = 0.a_ka_{k-1}a_{k-2} \dots .a_2a_1}$$

Once again, you can check the map is bijective.

☺Exercise

1. Show that the maps in Example 6 and 7 are bijective

2. Find a bijective map f which maps $(0,3)$ to $(8,20)$
   Find a bijective map g which maps $(3,4)$ to $(9,11)$

3. Define function f and g: $R \to R$ by

$$f(x) = \begin{cases} 0 \text{ if } x > 0 \\ 1 \text{ if } x \leq 0 \end{cases} \text{ and } g(x) = 1 - 2x$$

   a) For each f and g, determine whether it is injective or surjective.
   b) Furthermore, compute $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x)$