Lecture 18

09-04-2019

Review:

① Inverse Function Thm:

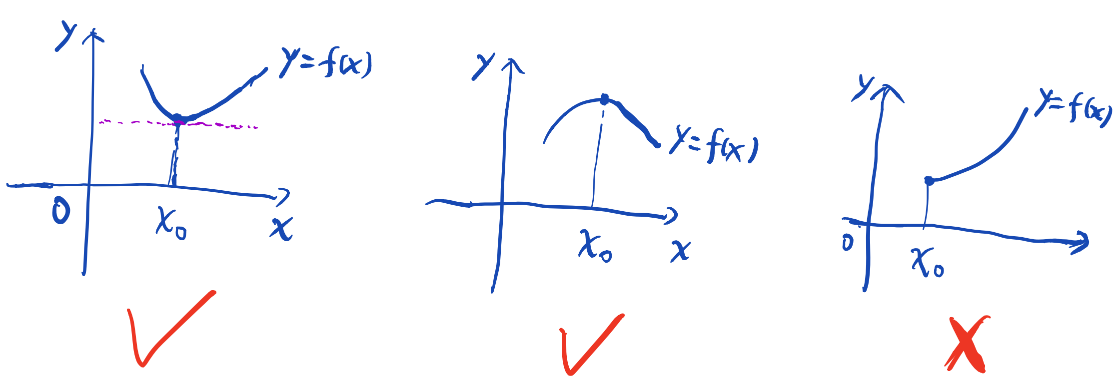
f is continuous + injective + $f'(x_0) \neq 0$

$\Rightarrow$ f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \left.\frac{dx}{dy}\right|_{y=y_0} = \frac{1}{f'(x_0)} = \frac{1}{\left.\frac{dy}{dx}\right|_{x=x_0}}$$

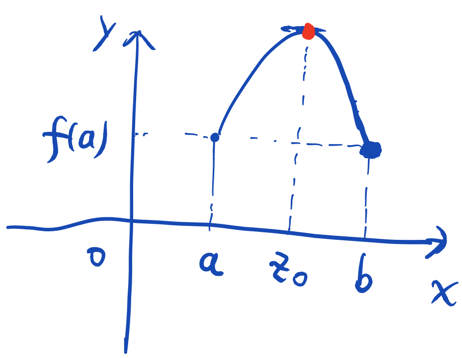
② Local Extremum Thm:

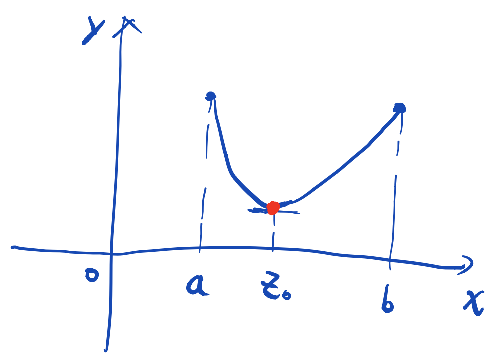
f achieves local extrema at x_0 + x_0 is "inside" the domain of f $\Rightarrow$ $f'(x_0) = 0$

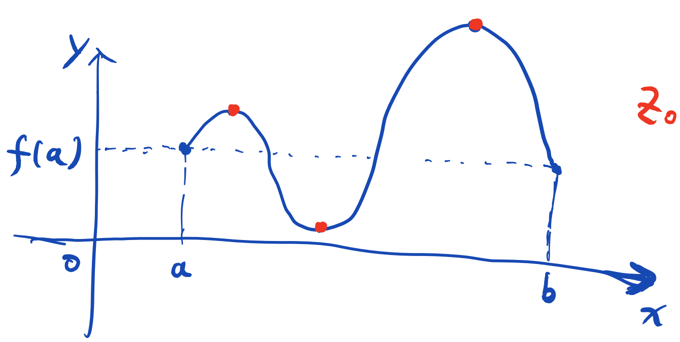


③ Rolle's thm: $f \in C[a,b]$ + differentiable on (a,b) + $f(a) = f(b)$ $\Rightarrow$ $f'(z_0) = 0$ for some $z_0 \in (a,b)$.

Pf: $f(a) = f(b)$ $\Rightarrow$ f achieves at least one extremum inside (a,b). Then apply local extremum thm.







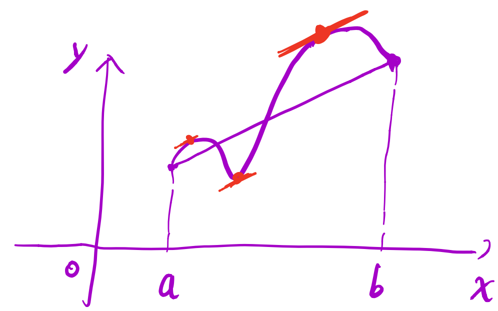
z_0 can be any one of the three local extremum points.

Mean-Value theorem $\Leftarrow\Rightarrow$ Rolle's thm

THM: Let $f \in C[a,b]$ be differentiable on (a,b). then

$\exists\ x_0 \in (a,b)$ s.t $f(b) - f(a) = f'(x_0)\,(b-a)$.

or $f'(x_0) = \dfrac{f(b)-f(a)}{b-a}$



Proof: Define $F(x) = f(x) - \left[\underbrace{\dfrac{f(b)-f(a)}{b-a}(x-a) + f(a)}\right]$

then $F(a) = 0 = F(b)$. By Rolle's thm, $\exists\ x_0 \in (a,b)$

s.t $F'(x_0) = 0 \Rightarrow f'(x_0) - \dfrac{f(b)-f(a)}{b-a} = 0$

$\Rightarrow f'(x_0) = \dfrac{f(b)-f(a)}{b-a}$ or $f(b) - f(a) = f'(x_0)\,(b-a)$.

Remark: If $f(b) = f(a) \Rightarrow f'(x_0) = 0$, i.e Rolle's thm.

Generalized mean-value theorem

THM : Let $f, g \in C[a,b]$ be differentiable on (a,b).

then $\exists\ x_0 \in (a,b)$ s.t

$$g'(x_0)\,(f(b)-f(a)) = f'(x_0)\,(g(b)-g(a)).$$

proof : define $G(x) = g(x)\,(f(b)-f(a)) - f(x)\,(g(b)-g(a))$

then $G'(x) = g'(x)\,(f(b)-f(a)) - f'(x)\,(g(b)-g(a))$

We need to show that $\exists\ x_0 \in (a,b)$ s.t

$$G'(x_0)=0$$

Since $G(a) = g(a)\,(f(b)-f(a)) - f(a)\,(g(b)-g(a)) = g(a)f(b) - f(a)g(b)$

$G(b) = g(b)\,(f(b)-f(a)) - f(b)\,(g(b)-g(a)) = -g(b)f(a) + f(b)g(a)$

$= G(a)$

by Rolle's thm, such x_0 exists.

Remark : ① In the generalized mean-value theorem,

if we take $g(x) = x$, then we

get the mean-value theorem.

②: In the generalize mean-value theorem, we have three cases:

Case 1. $f(b) = f(a)$. $\Rightarrow \exists\ x_0 \in (a,b)$, $f'(x_0) = 0$

Case 2. $g(b) = g(a) \Rightarrow \exists\ x_0 \in (a,b)$, $g'(x_0) = 0$

Case 3. $f(b) \neq f(a)$, $g(b) \neq g(a)$, then $\exists\ x_0 \in (a,b)$ s.t

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}.$$ This implies that

both $f'(x_0) \neq 0$ and $g'(x_0) \neq 0$.

③ We have Local extremum thm $\Rightarrow$ Rolle's thm

$\Leftrightarrow$ $\boxed{\text{mean value thm}}$ $\Leftrightarrow$ generalize mean value thm

Example ①: $\forall a, b \in R$, prove that

$$|\sin b - \sin a| \leq |b-a|.$$

Proof: Case 1. $a=b$, obvious.

Case 2. $a<b$,

define $f(x) = \sin x$, then

$$|\sin b - \sin a| = |f(b) - f(a)| \underset{\uparrow}{=} |f'(z_0)(b-a)| \quad \text{for some } z_0 \in (a,b)$$

mean-value thm

Since $f'(z_0) = \cos z_0$ $\quad |f'(z_0)| \leq 1$

$$\Rightarrow |\sin b - \sin a| = |f'(z_0)| \cdot |b-a| \leq 1 \cdot |b-a| = |b-a|$$

Case 3. $a>b$, similar to the proof of case 2.

Example ② . Show that $(1+x)^\alpha \geq 1+\alpha x$ for $x>-1$

and $\alpha \geq 1$. [Special case : $(1+x)^2 \geq 1+2x$]

Proof : let $f(x) = (1+x)^\alpha - (1+\alpha x)$

We need to show that $f(x) \geq 0$ for all $x>-1$, $\alpha \geq 1$.

Observe that $f(0) = (1+0)^\alpha - (1+\alpha \cdot 0) = 1-1 = 0$

$\Rightarrow f(x) = f(x) - f(0) = f'(z_0)(x-0) = f'(z_0)\, x$ for

some $z_0 \in (0, x)$ or $(x, 0)$ depending on whether

$x>0$ or $x<0$.

Case 1 : $x>0$. $f'(x) = \alpha(1+x)^{\alpha-1} - \alpha = \alpha\left[(1+x)^{\alpha-1} - 1\right]$

Since $1+z_0 > 1$, and $\alpha-1 \geq 0 \Rightarrow (1+z_0)^{\alpha-1} \geq 1$

$\Rightarrow f'(z_0) = \alpha\left[(1+z_0)^{\alpha-1} - 1\right] \geq 0$, $\Rightarrow f(x) = f'(z_0)\, x \geq 0$

Case 2. $-1 < x < 0$, $f'(x) = \alpha\left[(1+x)^{\alpha-1} - 1\right]$

Since $1+z_0 < 1$, and $\alpha-1 \geq 0 \Rightarrow (1+z_0)^{\alpha-1} \leq 1$

$\Rightarrow f'(z_0) = \alpha\left[(1+z_0)^{\alpha-1} - 1\right] \leq 0 \Rightarrow f(x) = f'(z_0)\, x \leq 0$

Example ③. Prove that $\ln x \le x-1$ for $x>0$

Solution : Define $f(x) = \ln x - (x-1) = \ln x - x + 1$

We need to show that $f(x) \le 0$ for all $x>0$.

Observe that $f(1) = \ln 1 - 1 + 1 = 0$

$\Rightarrow$ $f(x) = f(x) - f(1) = f'(z_0)(x-1)$ for some

$z_0 \in (1, x)$ or $(x, 1)$ depending on $x>1$ or $x<1$

Case 1. $x>1$, $f'(x) = \frac{1}{x} - 1 < 0$. $\Rightarrow$ $f'(z_0) < 0$

$\Rightarrow$ $f(x) = f'(z_0)(x-1) < 0$

Case 2. $0<x<1$, $f'(x) = \frac{1}{x} - 1 > 0$ $\Rightarrow$ $f'(z_0) > 0$

$\Rightarrow$ $f(x) = f'(z_0)(x-1) < 0$.

Example 4. Approximate $\sqrt{16.1}$.

The exact value of $\sqrt{16.1} = 4.0124805 \ldots\ldots$

Solution: Define $f(x) = \sqrt{x}$. $\quad f'(x) = \frac{1}{2\sqrt{x}}$

By 1-st order taylor expansion

$$f(16.1) = f(16) + f'(16)(16.1-16) + R(16, 0.1)$$

$$f(x+h) = f(x) + f'(x)h + R(x, h)$$

$\uparrow$ reminder term.

$$\Rightarrow f(16.1) \approx f(16) + f'(16) \cdot 0.1 = \sqrt{16} + \frac{1}{2\times\sqrt{16}} \cdot 0.1 = 4 + \frac{0.1}{8}$$

$$= 4.0125$$

On the other hand, $f(16.1) - f(16) = f'(z_0)\left(16.1-16\right)$ for some $z_0 \in (16, 16.1)$.

But $f'(z_0) = \frac{1}{2\sqrt{z_0}} \approx \frac{1}{2\times\sqrt{16}} = \frac{1}{8}$. $\Rightarrow f(16.1) - f(16) = \frac{1}{8} \times 0.1$

Curve Tracing

THM: If $\left\{\begin{array}{l} f' \geq 0 \\ f' > 0 \\ f' \leq 0 \\ f' < 0 \\ f' \neq 0 \\ f' \equiv 0 \end{array}\right\}$ everywhere on (a,b), then f is $\left\{\begin{array}{l} \nearrow \\ \text{strictly } \nearrow \\ \searrow \\ \text{strictly } \searrow \\ \text{injective} \\ \text{constant} \end{array}\right.$

respectively

proof: $\forall \; a < x < y < b$,

$$f(y) - f(x) = f'(\xi)(y-x) \quad \text{for some } \xi \in (x,y).$$

Remark: If f is differentiable, then $f'>0 \Rightarrow$ strictly $\nearrow$

But strictly $\nearrow$ $\not\Rightarrow$ $f'>0$

Example: $f(x) = x^3$. but $f'(0)=0$

Similarly, strictly $\searrow$ $\not\Rightarrow$ $f'<0$

injective $\not\Rightarrow$ $f'\neq 0$.

However, the following is true.

THM: let $f:(a,b)\to\mathbb{R}$ be differentiable,

if f is $\left\{\begin{matrix}\nearrow \\ \searrow \\ \text{constant}\end{matrix}\right\}$, then $\left\{\begin{matrix} f'\geq 0 \\ f'\leq 0 \\ f'\equiv 0\end{matrix}\right\}$ everywhere on (a,b)

pf: $f'(x) = \lim\limits_{y\to x}\dfrac{f(y)-f(x)}{y-x}$, then apply the limit inequality