MATH 2031 Introduction to Real Analysis

October 11, 2012

Tutorial Note 5

Infinite series

(I) Definition of infinite series $\sum_{k=1}^{\infty} a_k$ and partial sum $S_n = \sum_{k=1}^{n} a_k$,
converges ($\lim_{n\to\infty} S_n \in \mathbb{R}$) and divergence ($\lim_{n\to\infty} S_n = \infty$ or $\lim_{n\to\infty} S_n$ doesn't exist)

(II) List of tests for infinite series (Summarized in the transparencies P.37)

Problem 1 Determine if the following series converges or diverges.

1. $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \left(\frac{5}{8}\right)^k$
2. $\sum_{k=1}^{\infty} \frac{ln(e+k)}{k}$

Solution:

1. Since $0 \le \frac{1}{k(k+1)} \left(\frac{5}{8}\right)^k \le \frac{1}{2} \left(\frac{5}{8}\right)^k$ and $\frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{5}{8}\right)^k$ converges by geometric series test, $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \left(\frac{5}{8}\right)^k$ converges by comparison test.
2. Since $\ln(x)$ is an increasing function as its derivative$= \frac{1}{x}$ is always positive for $x > 0$, so $\ln(e+k) \ge \ln(e) = 1$, and we get $\frac{ln(e+k)}{k} \ge \frac{1}{k}$. From p-test, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges for $p \le 1$, so $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

   Thus, $\sum_{k=1}^{\infty} \frac{ln(e+k)}{k}$ diverges by comparison test.

Problem 2 Show that $\sum_{k=1}^{\infty} e^{(\frac{1}{k})^2}$ converges.

Solution:
Since $\lim_{k\to\infty} \frac{e^{(\frac{1}{k})^2}}{(\frac{1}{k})^2} = \lim_{x\to 0} \frac{e^{x^2}}{x^2} = \lim_{x\to 0} \frac{2xe^{x^2}}{2x} = \lim_{x\to 0} e^{x^2} = e^0 = 1$ and $e^{(\frac{1}{k})^2}$ and $\frac{1}{k^2}$ are positive for any $k \ge 1$.
Also $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p-test, so $\sum_{k=1}^{\infty} e^{(\frac{1}{k})^2}$ converges by limit comparison test.

Problem 3 Show that $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ converges.

Solution:
Since the series involves $(-1)^{k+1}$, we may consider the alternate series test (alt. series test).
As $\frac{1}{\sqrt{k}}$ is decreasing and $\lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$, so by alt. series test $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ converges.

Remark:

From the notes, we know that $\sum_{k=1}^{\infty} a_k$ converges absolutely $\Rightarrow \sum_{k=1}^{\infty} a_k$ converges.

i.e. $\sum_{k=1}^{\infty} |a_k|$ converges $\Rightarrow \sum_{k=1}^{\infty} a_k$ converges.

However, the converse" $\sum_{k=1}^{\infty} a_k$ converges $\Rightarrow \sum_{k=1}^{\infty} |a_k|$ converges" may not generally true.

Consider the series above, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ converges, but

$\sum_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{\sqrt{k}}\right| = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$ diverges by p-test.

Problem 4 Show that $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2(k^3+1)}$ converges.

Solution: As $\sin(x)$ is oscillating but bounded by 1, we may consider the absolute convergence test.

$\left|\frac{\sin(k)}{k^2(k^3+1)}\right| = \frac{1}{k^2(k^3+1)} \leq \frac{1}{k^2(k^3)} = \frac{1}{k^5}$.

As $\sum_{k=1}^{\infty} \frac{1}{k^5}$ converges by p-test and by comparison test, $\sum_{k=1}^{\infty} \left|\frac{\sin(k)}{k^2(k^3+1)}\right|$ converges.

Thus, by absolute convergence test, $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2(k^3+1)}$ converges.

Problem 5 Let $r > 0$, find all values of r such that the series $\sum_{k=1}^{\infty} \frac{r^k}{k^2+k^3}$ converges.

Solution:

Let $a_k = \frac{r^k}{k^2+k^3}$ consider

$$\begin{aligned}
\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty} \left|\frac{\frac{r^{(k+1)}}{(k+1)^2+(k+1)^3}}{\frac{r^k}{k^2+k^3}}\right| \\
&= \lim_{k\to\infty} \frac{r(k^2+k^3)}{(k+1)^2+(k+1)^3} \\
&= \lim_{k\to\infty} \frac{r(k^2+k^3)}{(k+1)^2+(k+1)^3}\left(\frac{\frac{1}{k^3}}{\frac{1}{k^3}}\right) \\
&= \lim_{k\to\infty} \frac{r(\frac{1}{k}+1)}{(\frac{1}{k})(1+\frac{1}{k})^2+(1+\frac{1}{k})^3} \\
&= r
\end{aligned}$$

By ratio test, $\sum_{k=1}^{\infty} \frac{r^k}{k^2+k^3} \begin{cases} \text{converges} & \text{if } r < 1 \\ \text{may or may not converge} & \text{if } r = 1 \\ \text{diverges} & \text{if } r > 1 \end{cases}$.

For $r = 1$, we get $\frac{1}{k^2+k^3} = \frac{1}{k^2(1+k)} \leq \frac{1}{k(1+k)} = \frac{1}{k} - \frac{1}{1+k}$

By telescoping series test, $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{1+k}\right) = 1 - \lim_{k\to\infty} \frac{1}{1+k} = 1$, which converges.

(or $\frac{1}{k^2+k^3} = \frac{1}{k^2(1+k)} \leq \frac{1}{k^2}$ and use p-test.)

Thus, by comparison test, $\sum_{k=1}^{\infty} \frac{1}{k^2+k^3}$ converges.

Therefore, for $0 < r \leq 1$, $\sum_{k=1}^{\infty} \frac{r^k}{k^2+k^3}$ converges.