Problem Set 4.

Definition of Supremum and Infimum.

A : set

$M^* = \sup A$ if and only if

$\begin{cases} ① \ M^* \text{ is upper bound of } A \\ ② \ \forall \varepsilon > 0, \ \exists x \in A, \ \text{s.t. } x > \sup A - \varepsilon. \end{cases}$

$m^* = \inf A$ if and only if

① m^* is lower bound of A

② $\forall \varepsilon > 0$, $\exists x \in A$, s.t. $x < \inf A + \varepsilon$.

Density of rational numbers and irrational numbers.

Property 4: Density of rantional number.

Property 5: Density of irrational number.

Property 6: Nested Interval Theorem.

Problem 1:

Find the supremum and infimum, if exists as a number, for the following sets:

(a) $A = \{e^{-x} \mid x \in (0,1) \cap \mathbb{Q}\}$ ✓

bound. (b) $B = \{\cos \frac{1}{n} \mid n \in \mathbb{N}\}$ ✓

(c) $C = \{1 - \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$ ✓

$x \to 0,$
$x \to 1,$

Solution:

(a). $e^{-1} < e^{-x} < 1$, $x \in (0,1) \cap \mathbb{Q}$

Claim: supremum of A is $\underline{1}$ $= M^*$

$a_n = \frac{1}{n}$, $n = 1, 2, 3, \ldots$, $a_n \in (0,1) \cap \mathbb{Q}$

$\left(\lim_{n\to\infty} e^{-a_n} = 1. \right)$

$\rightarrow$ ① $e^{-x} < 1$. $x \in (0,1) \cap \mathbb{Q}$ ✓

$\rightarrow$ ② For any $\varepsilon > 0$, $\exists N$ s.t.

$e^{-\frac{1}{N}} > 1 - \varepsilon$ $\qquad$ $e^{-a_N} > 1 - \varepsilon$.

$-\frac{1}{N} > \log(1-\varepsilon)$,

$\frac{1}{N} < -\log(1-\varepsilon)$

$N > -\frac{1}{\log(1-\varepsilon)}$

$\Rightarrow \sup A = 1$.

Claim: infimum of A is e^{-1}

$b_n = 1 - \frac{1}{n}$ $\qquad$ $b_n \in (0,1) \cap \mathbb{Q}$

$b_n \to 1$.

$\left(\lim_{n\to\infty} e^{-b_n} = e^{-1} \right)$

$\rightarrow$ ① $e^{-x} > e^{-1}$ $\quad$ $x \in (0,1) \cap \mathbb{Q}$

$\rightarrow$ ② For any $\varepsilon > 0$, $\exists N$ s.t.

$e^{-(1-\frac{1}{N})} < e^{-1} + \varepsilon$ $\qquad$ $e^{-b_N} < e^{-1} + \varepsilon$

$\Leftrightarrow$

$\Rightarrow \inf A = e^{-1}$.

$e^{\frac{1}{N}} < 1 + \varepsilon e$

$\frac{1}{N} < \log(1 + \varepsilon e)$

$N > \frac{1}{\log(1+\varepsilon e)}$

(b). $B = \{ \cos\frac{1}{n} \mid n \in \mathbb{N} \}$

$\frac{1}{n} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ $\quad [0, \frac{\pi}{2}]$,

$\cos x$ is decreasing over $[0, \frac{\pi}{2}]$.

$\cos\frac{1}{n}$ is increasing against n.

$1 > \cos\frac{1}{n} \geq \cos 1$

$\inf B = \cos 1$ when $n = 1$,

Claim: $\sup B = 1$.

$\rightarrow$ ① $\cos B \leq 1$

$\rightarrow$ ② $\forall \varepsilon > 0$, $\exists N$ s.t. $\cos\frac{1}{N} > 1 - \varepsilon$.

$\frac{1}{N} < \arccos(1-\varepsilon)$.

$\rightarrow, N > \frac{1}{\arccos(1-\varepsilon)}$

$\Rightarrow \sup B = 1$.

(C). $C = \{1 - \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$

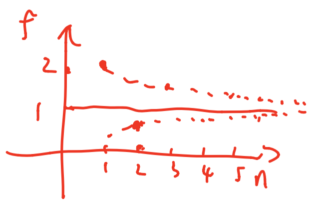
$n = 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6,$

$\quad 2, \quad \frac{1}{2}, \quad \frac{4}{3}, \quad \frac{3}{4}, \quad \frac{6}{5}, \quad \frac{5}{6},$

n is odd. $n = 2k+1$, $k = 0, 1, 2, \ldots$

$$\underbrace{1 - \frac{(-1)^n}{n}}_{\downarrow} = 1 - \frac{-1}{2k+1} = 1 + \frac{1}{2k+1}$$

is decreasing against n.



n is even $n = 2k$, $k = 1, 2, \ldots$

$$\underbrace{1 - \frac{(-1)^n}{n}}_{\downarrow} = 1 - \frac{1}{2k}$$

is increasing against n.

$\inf C = \frac{1}{2}$

$\sup C = 2$.

Problem 2: Find the supremum and infimum, if exists as a number, for the following sets:

(a) $D = \{\frac{1}{n} - \frac{1}{m} \mid m \in \mathbb{N}, n \in \mathbb{N}\}$

(b) $E = \{a + b \mid a \in (0,1) \cap \mathbb{Q}, b \in [1,2) \setminus \mathbb{Q}\}$

(a). $\frac{1}{n}$. $0 < \frac{1}{n} \leq 1$

$1 \leftarrow \frac{1}{n} - \frac{1}{m}$?

$\frac{1}{m}$. $0 < \frac{1}{m} \leq 1$

$n = 1$

$-1 < \frac{1}{n} - \frac{1}{m} < 1$.

$m \to +\infty$,

$\frac{1}{m} \to 0$.

Claim: $\sup_{m,n} \frac{1}{n} - \frac{1}{m} = 1$.

① $\frac{1}{n} - \frac{1}{m} \leq 1$.

② take $n=1$.

$\forall \varepsilon > 0$. $\exists M > 0$, s.t. $1-\frac{1}{M} \geq 1-\varepsilon$.

$\frac{1}{n}-\frac{1}{M} \geq 1-\varepsilon$.

$\Rightarrow \sup_{m,n} \frac{1}{n}-\frac{1}{m}=1$

Claim $\inf_{m,n} \frac{1}{n}-\frac{1}{m}=-1$.

① $\frac{1}{n}-\frac{1}{m} \geq -1$

② take $m=1$.

$\forall \varepsilon > 0$. $\exists N > 0$. s.t. $\frac{1}{N}-1 \leq -1+\varepsilon$.

$\Rightarrow \inf_{m,n} \frac{1}{n}-\frac{1}{m}=-1$.

(b). $E=\{a+b \mid a \in (0,1) \cap \mathbb{Q},\ b \in (1,2) \setminus \mathbb{Q}\}$

$0 < a < 1$

$1 < b < 2$

$1 < a+b < 3$.

Claim : $\sup_{a,b} a+b=3$.

Claim: $\inf_{a,b} a+b=1$

Problem 3: We let S be a bounded subset in $\mathbb{R}$ and let $S_0 \subseteq S$ be a subset of S.

relationship subsets.
$S_0 \subseteq S$.
sup sup
inf inf.

(a) Show that the supremum and infimum of S_0 exist and satisfy $\inf S_0 \geq \inf S$ and $\sup S_0 \leq \sup S$.

(b) Suppose that $S_0 \subset S$ (i.e. S_0 is proper subset of S), is it always true that $\inf S_0 > \inf S$ and $\sup S_0 < \sup S$? Explain your answer.

Proof: (a) S is bounded.

$\exists M_1, M_2$, s.t. $\forall x \in S$,

$M_1 \le x \le M_2.$

S_0 is a subset of S

$\Rightarrow \forall x \in S_0 \subseteq S.$

$M_1 \le x \le M_2$

$\Rightarrow$ S_0 is bounded

$\Rightarrow$ supremum and infimum of S_0 exist.

$S_0 \subseteq S,$

elementwise $\forall x \in S_0, \; x \in S,$

$\forall x \in S_0 \subseteq S$

$\inf S \le x \le \sup S.$

$\Rightarrow \inf S \le \inf S_0.$

$\sup S_0 \le \sup S.$

(b). Not !!! absolute value small. 10. -10.

$S = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{10, -10\}$

" = "

$\sup S = 10 \quad \inf S = -10.$

$S_0 = \{\frac{1}{n} \mid n \in \mathbb{N}, n \text{ is odd}\} \cup \{10, -10\}$

$\sup S_0 = 10, \quad \inf S = -10.$

$S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ $\quad \{\frac{1}{n} \mid n \in \mathbb{N}, n \text{ is even}\},$

$\sup S = 1, \quad \inf S = 0.$

$S_0 = \{\frac{1}{n} \mid n \in \mathbb{N}, n \text{ is odd}\}$

$\sup S = 1. \quad \inf S = 0.$

Problem 4: Prove the following statements using Archimedean property;

(a) We let $I_n = [0, \frac{1}{n}]$ for every $n \in \mathbb{N}$. If $x > 0$, prove that $x \notin \bigcap_{n=1}^{+\infty} I_n$.

Nested interval theorem. $[a,b] = \bigcap_{n=1}^{+\infty} I_n$

$\subset I_n.$

$I_n = [a_n, b_n].$ $\quad \{a\}.$

? $\quad$? $I_n = [a_n, b_n).$

(b) We let $I_n' = (0, \frac{1}{n})$ for every $n \in \mathbb{N}$, prove that $\bigcap_{n=1}^{+\infty} I_n = \phi.$

?. $I_n = [a_n, b_n]$

?. $I_n = (a_n, b_n)$.

Add some other conditions.

(c) We let $K_n = [n, +\infty)$ for every $n \in \mathbb{N}$. prove that

$$\bigcap_{n=1}^{+\infty} K_n = \phi.$$

Proof: Archimedean Principle:

For any $x \in \mathbb{R}$, there exists a positive integer $n \in \mathbb{N}$ such that $n > x$.

(a) For any $x > 0$, select any $x > 0$.

there exists a positive integer n s.t.

$$n > \frac{1}{x}.$$

$$\Rightarrow \quad x > \frac{1}{n}.$$

$$\Rightarrow \quad x \notin [0, \tfrac{1}{n}] = I_n$$

$$\Rightarrow x \notin \bigcap_{n=1}^{+\infty} I_n$$

(b) $0 \notin \bigcap_{n=1}^{+\infty} I_n$

From (a), we know that for any $x > 0$.

$$x \notin \bigcap_{n=1}^{+\infty} I_n.$$

$$\Rightarrow \bigcap_{n=1}^{+\infty} I_n = \phi$$

(c) By contradiction, if $x \in \bigcap_{n=1}^{+\infty} K_n$

By Archimedean Principle,

$\exists\, N > 0$ s.t. $N > x$.

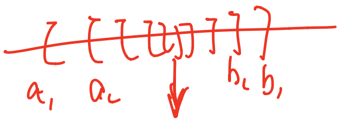
Then $x \notin K_N$.

$$\Rightarrow x \notin \bigcap_{n=1}^{+\infty} K_n.$$

Problem 7:

Nested interval Theorem:

We let $\{I_n = [a_n, b_n] \mid n \in \mathbb{N}\}$ be a set of closed intervals such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Then $\bigcap_{n=1}^{+\infty} I_n = [a, b]$, where $a = \sup\{a_n \mid n \in \mathbb{N}\}$, $b = \inf\{b_n \mid n \in \mathbb{N}\}$



Suppose that $\inf\{b_n - a_n \mid n \in \mathbb{N}\} = 0$, prove that $\bigcap_{n=1}^{+\infty} I_n$ contains a single element.

$b_n - a_n$,

$\downarrow$

0,

Proof: Prove $a = b$.

$\Leftrightarrow$ For any $\varepsilon > 0$, $\exists N$ s.t. $b_N - a_N < \varepsilon$

$0 \leq b - a < \varepsilon$

$\forall \varepsilon > 0$.

Because $[a, b] \subset I_N = [a_N, b_N]$

$0 \leq b - a \leq b_N - a_N < \varepsilon$

$\Rightarrow$ $0 \leq b - a < \varepsilon$ holds for any ε

$\Rightarrow$ $a = b$.

$\Rightarrow$ $\bigcap_{n=1}^{+\infty} I_n = \{a\}$