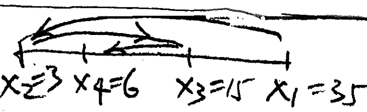
Solution of 2016 Midterm (Math 2033)

① We have $A \subseteq [3,6]$ and $x \in [5, \sqrt{80})$. So $3 \le y \le 6$ and $5 \le x < \sqrt{80}$. Then $9 \le y^2 \le 36$ and $9 \le x^2-16 < 64$ ($\Leftrightarrow 3 \le \sqrt{x^2-16} < 8$). So $12 \le y^2 + \sqrt{x^2-16} < 44$. Hence B is bounded above by 44 and below by 12.

By supremum limit theorem, $\exists\, y_n \in A$ such that $\lim_{n\to\infty} y_n = 6$.
By infimum limit theorem, $\exists\, y_n' \in A$ such that $\lim_{n\to\infty} y_n' = 3$.
Next let $x_n = \sqrt{80} - \frac{1}{n}$ and $x_n' = 5 + \frac{1}{n\sqrt{2}}$, then $x_n, x_n' \in [5, \sqrt{80}) \setminus \mathbb{Q}$ and $\lim_{n\to\infty} x_n = \sqrt{80}$, $\lim_{n\to\infty} x_n' = 5$. Then $y_n^2 + \sqrt{x_n^2-16} \in B$ and $\lim_{n\to\infty} y_n^2 + \sqrt{x_n^2-16} = 44$.
Also $y_n'^2 + \sqrt{x_n'^2 - 16} \in B$ and $\lim_{n\to\infty} y_n'^2 + \sqrt{x_n'^2-16} = 12$. $\therefore \sup B = 44$ and $\inf B = 12$.

---

② Sketch: $x_1 = 35$, $x_2 = 3$, $x_3 = 15$, $x_4 = 6$



$x = \frac{120}{5+x} \Leftrightarrow x^2 + 5x - 120 = 0 \Leftrightarrow x = \frac{-5 \pm \sqrt{25+480}}{2} = \frac{-5\pm\sqrt{505}}{2}$ (reject $-$)

Solution Let $I_n = [x_{2n}, x_{2n-1}]$ for $n = 1,2,3,\ldots$

Claim: $I_n \supseteq I_{n+1}$ ($\Leftrightarrow x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$) for $n=1,2,3,\ldots$

Case $n=1$, $x_2 = 3 \le x_4 = 6 \le x_3 = 15 \le x_1 = 35$

Suppose case n is true. Then $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$. We have

$5 + x_{2n} \le 5 + x_{2n+2} \le 5 + x_{2n+1} \le 5 + x_{2n-1} \Rightarrow \frac{1}{5+x_{2n}} \ge \frac{1}{5+x_{2n+2}} \ge \frac{1}{5+x_{2n+1}} \ge \frac{1}{5+x_{2n-1}}$

$\Rightarrow x_{2n+1} = \frac{120}{5+x_{2n}} \ge x_{2n+3} = \frac{120}{5+x_{2n+2}} \ge x_{2n+2} = \frac{120}{5+x_{2n+1}} \ge x_{2n} = \frac{1}{5+x_{2n-1}}$

$\Rightarrow 5 + x_{2n+1} \ge 5 + x_{2n+3} \ge 5 + x_{2n+2} \ge 5 + x_{2n} \Rightarrow \frac{1}{5+x_{2n+1}} \le \frac{1}{5+x_{2n+3}} \le \frac{1}{5+x_{2n+2}} \le \frac{1}{5+x_{2n}}$

$\Rightarrow x_{2n+2} = \frac{120}{5+x_{2n+1}} \le x_{2n+4} = \frac{120}{5+x_{2n+3}} \le x_{2n+3} = \frac{120}{5+x_{2n+2}} \le x_{2n+1} = \frac{120}{5+x_{2n}}$.

This completes M.I. So the claim is true.

By nested interval theorem, $\lim_{n\to\infty} x_{2n} = a$ and $\lim_{n\to\infty} x_{2n-1} = b$. Now $\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} \frac{120}{5+x_{2n}}$

$\Rightarrow b = \frac{120}{5+a}$ and $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} \frac{120}{5+x_{2n-1}} \Rightarrow a = \frac{120}{5+b}$. We have $5b + ab = 120 = 5a + ab$,

this implies $a = b$. Then $a = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{120}{5+x_n} = \frac{120}{5+a} \Rightarrow a^2 + 5a - 120 = 0$

$\Rightarrow a = \frac{-5\pm\sqrt{25+480}}{2} = \frac{-5\pm\sqrt{505}}{2}$. Since $x_n \ge 3$, $a = \lim_{n\to\infty} x_n$ is $\frac{-5+\sqrt{505}}{2}$.

③ $S=\{r\sqrt{2} : r\in\mathbb{Q}\} = \underbrace{\bigcup_{r\in\mathbb{Q}}}_{\text{countable}} \underbrace{\{r\sqrt{2}\}}_{1\text{ element} \Rightarrow \text{countable}}$ is countable by countable union theorem.

Let $W=\{2^x\sin y - 2^y\cos x : x,y\in S\}$. Then $W=\underbrace{\bigcup_{(x,y)\in S\times S}}_{\text{countable}} \underbrace{\{2^x\sin y - 2^y\cos x\}}_{1\text{ element} \Rightarrow \text{countable}}$ is countable. Then $\underbrace{\mathbb{R}}_{\text{uncountable}}\setminus \underbrace{W}_{\text{countable}}$ is uncountable. So $\mathbb{R}\setminus W$ is an infinite set.

Therefore, there exist infinitely many real numbers c such that $2^x\sin y - 2^y\cos x = c$ has no solution with $x,y\in S$.

---

④ Sketch $\frac{4n\sqrt{n}-3}{n\sqrt{n}+5}\to 1$, $\frac{6n^2-2n+3}{3n^2-1}\to 2$

$$\begin{cases} \left|\frac{4n\sqrt{n}-3}{n\sqrt{n}+5}-4\right| = \frac{|-23|}{n\sqrt{n}+5} \leq \frac{23}{n\sqrt{n}} = \frac{23}{n^{3/2}} < \frac{\varepsilon}{2} \Leftrightarrow \left(\frac{46}{\varepsilon}\right)^{2/3} < n \\ \left|\frac{6n^2-n+3}{3n^2-1}-2\right| = \frac{|-n+5|}{3n^2-1} \leq \frac{n+5}{3n^2-n^2} \leq \frac{6n}{2n^2} = \frac{3}{n} < \frac{\varepsilon}{2} \Leftrightarrow \frac{6}{\varepsilon} < n \end{cases}$$

Solution For every $\varepsilon>0$, by Archimedean principle, $\exists K\in\mathbb{N}$ such that $K>\max\{(\frac{46}{\varepsilon})^{2/3}, \frac{6}{\varepsilon}\}$ (or take $K=\lceil \max\{(\frac{46}{\varepsilon})^{2/3}, \frac{6}{\varepsilon}\}\rceil$)

Then $n\geq K \Rightarrow \left|\frac{4n\sqrt{n}-3}{n\sqrt{n}+5}+\frac{6n^2-n+3}{3n^2-1}-6\right| \leq \left|\frac{4n\sqrt{n}-3}{n\sqrt{n}+5}-4\right| + \left|\frac{6n^2-n+3}{3n^2-1}-2\right|$

$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$