

Defn. $f: [a, b] \rightarrow \mathbb{R}$ is Riemann integrable on $[a, b]$ if $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$

(1) $f(x) = |x-1|$ on $[0, 2]$

Let partition of $[0, 2]$:
 $P = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$
 $x_i = \frac{i}{n}, x_{i-1} = \frac{i-1}{n}, x_0 = 0, x_n = 2$
 $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$
 $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$
 $M_i = \sup_{x \in [x_{i-1}, x_i]} |x-1| = \frac{n-i}{n}$
 $m_i = \inf_{x \in [x_{i-1}, x_i]} |x-1| = \frac{i-1}{n}$
 $M_i - m_i = |x_i - x_{i-1}| = \frac{1}{n}$
 $\sum_{i=1}^n M_i - m_i = \sum_{i=1}^n \frac{1}{n} = 1$
 $\int_0^2 |x-1| dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n M_i - m_i \Delta x$
 $= \lim_{n \rightarrow \infty} \frac{1}{n} (M_1 - m_1 + M_2 - m_2 + \dots + M_n - m_n)$
 $\leq M_1 (x_1 - x_0) + M_2 (x_2 - x_1) + \dots + M_n (x_n - x_{n-1})$
 $\leq (x_1 - 0) + (x_2 - x_1) + \dots + (x_n - x_{n-1})$
 $\leq (x_n - 0) + (x_1 - x_0) + (x_2 - x_1) + \dots + (x_{n-1} - x_{n-2})$
 $\leq (x_n - 0) + (x_1 - x_0) + (x_2 - x_1) + \dots + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1})$
 $\leq \sum_{i=1}^n (x_i - x_{i-1}) = 1$
 $\int_0^2 |x-1| dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n m_i - M_i \Delta x$
 $= \lim_{n \rightarrow \infty} \frac{1}{n} (m_1 - M_1 + m_2 - M_2 + \dots + m_n - M_n)$
 $\geq m_1 (x_1 - x_0) + m_2 (x_2 - x_1) + m_3 (x_3 - x_2) + \dots + m_n (x_n - x_{n-1})$
 $\geq (x_1 - x_0) + (x_2 - x_1) + \dots + (x_n - x_{n-1})$
 $\geq (x_n - x_0) + (x_1 - x_0) + (x_2 - x_1) + \dots + (x_{n-1} - x_{n-2}) + (x_n - x_{n-1})$
 $\geq \sum_{i=1}^n (x_i - x_{i-1}) = 1$

(2) $f(x) = \frac{x}{x+1}$ on $[0, 1]$

$\int_0^1 \frac{x}{x+1} dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$

$\Delta x = \frac{1}{n}, x_i = \frac{i}{n}, x_{i-1} = \frac{i-1}{n}, x_0 = 0, x_n = 1$
 $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = \frac{i}{i+1}$
 $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = \frac{i-1}{i+1}$
 $M_i - m_i = \frac{1}{(i+1)^2} = \frac{1}{n^2}$
 $\sum_{i=1}^n M_i - m_i = \sum_{i=1}^n \frac{1}{n^2} = \frac{n}{n+1}$
 $\int_0^1 \frac{x}{x+1} dx = \lim_{n \rightarrow \infty} \frac{n}{n+1}$
 $= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}}$
 $= 1$

(3) $f(x) = x^2$ on $[0, 1]$

$\int_0^1 x^2 dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$

$\Delta x = \frac{1}{n}, x_i = \frac{i}{n}, x_{i-1} = \frac{i-1}{n}, x_0 = 0, x_n = 1$
 $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = x_i^2 = \left(\frac{i}{n}\right)^2$
 $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = x_{i-1}^2 = \left(\frac{i-1}{n}\right)^2$
 $M_i - m_i = \left(\frac{i}{n}\right)^2 - \left(\frac{i-1}{n}\right)^2 = \frac{2i-1}{n^2}$
 $\sum_{i=1}^n M_i - m_i = \sum_{i=1}^n \frac{2i-1}{n^2} = \frac{n^2 + n}{2n^2} = \frac{1}{2} + \frac{1}{n}$
 $\int_0^1 x^2 dx = \lim_{n \rightarrow \infty} \frac{1}{2} + \frac{1}{n}$
 $= \frac{1}{2}$

(4) $f(x) = x$ on $[0, 1]$

$\int_0^1 x dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$

$\Delta x = \frac{1}{n}, x_i = \frac{i}{n}, x_{i-1} = \frac{i-1}{n}, x_0 = 0, x_n = 1$
 $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = x_i = \frac{i}{n}$
 $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = x_{i-1} = \frac{i-1}{n}$
 $M_i - m_i = \frac{i}{n} - \frac{i-1}{n} = \frac{1}{n}$
 $\sum_{i=1}^n M_i - m_i = \sum_{i=1}^n \frac{1}{n} = 1$
 $\int_0^1 x dx = \lim_{n \rightarrow \infty} 1$
 $= 1$

credits. Marks can be deducted for incomplete solution or unclear solution.

Please submit your completed work via the submission system in canvas before the deadline. Late assignment will not be accepted.

Your submission must (1) be hand-written (typed assignment will not be accepted), (2) in a single pdf. file (other file formats will not be accepted) and (3) contain your full name and student ID on the first page of the assignment.

Problem 1

- (a) Using the definition of integrability or integral criterion, prove that $f(x) = |x-1|$ is integrable on $[0, 2]$.
- (b) Using the definition of integrability or integral criterion, prove that the function $f: [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not integrable on $[0, 1]$.

Problem 2

We let f, g, h be three bounded functions on $[a, b]$ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in [a, b]$. Suppose that f, h are integrable on $[a, b]$ and $\int_a^b f(x) dx = \int_a^b h(x) dx$.

(a) Show that g is integrable on $[a, b]$.

(b) Show that $\int_a^b g(x) dx = \int_a^b f(x) dx$.

Problem 3

- (a) We let $f, g: [a, b] \rightarrow \mathbb{R}$ be two bounded Riemann integrable functions on $[a, b]$, show that the function $h(x) = \min(f(x), g(x))$ is also Riemann integrable on $[a, b]$.

- (b) We let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function on $[a, b]$.

Suppose that f^2 is Riemann integrable, is it true that f is Riemann integrable? Explain your answer.

Suppose that f^3 is Riemann integrable, is it true that f is Riemann integrable? Explain your answer.

If your answer is yes, please give a proof. If your answer is no, please give a counter-example.)

$S(p, g, p) \leq S(g, p) \leq S(h, p)$
We get,
 $\int_a^b S(p, g) \leq \int_a^b S(g, p) \leq \int_a^b S(h, p)$
 $\Rightarrow \int_a^b g \leq \int_a^b S(g, p) \leq \int_a^b h$
 $\Rightarrow \int_a^b g = \int_a^b h$
 $\Rightarrow \int_a^b g = \int_a^b h \leq \int_a^b S(h, p)$
 $\Rightarrow \int_a^b g = \int_a^b h \Rightarrow \int_a^b g = \int_a^b S(h, p)$

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$\exists f(x), g(x)$ both Riemann integrable on $[a, b]$
 $\exists P(x) = \frac{1}{2} [f(x) + g(x)]$ and $\exists P(x) - g(x)$ both Riemann integrable
 $\min\{P, g\} = \frac{1}{2} [f(x) - g(x)] - \frac{1}{2} [f(x) - g(x)]$
Difference of two Riemann integrable functions
Riemann integrable
Hence $\min\{P, g\}$ Riemann integrable

(1) Not true
Let $P(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$
 $P(x)$ not Riemann integrable
but $P^2(x) = 1 \forall x \in \mathbb{R}$; constant function which
is Riemann integrable

(2) True
Let $P(x)$ is Riemann integrable and $\Phi(x)$ continuous
then $\Phi(P(x))$ Riemann integrable

Using this theorem, let $g(x) = x^2$ is continuous
 $g(x) = x^2$ is Riemann integrable given
By the previous theorem, $\Phi(g(x)) = \Phi(x^2)$
 $= (\Phi(x))^2$
 $= f(x)$
 $\Phi(g(x))$ Riemann integrable (Prove)
Hence Φ