MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #16

Differentiability and its property

Basic Definition:

A function $f: I \to \mathbf{R}$ is said to be **differentiable** at x_0 if and only if the limit

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

exists. (where I is an interval of positive length)

A function is said to be **differentiable** iff it is differentiable on every $x_0 \in I$

Example 1

Show that the function $f(x) = x^{\frac{1}{3}}$ is differentiable on $\mathbf{R}\backslash\{0\}$

Solution:

It is just a matter of computation. Note that for $x_0 \neq 0$

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{x^{\frac{1}{3}}-x_0^{\frac{1}{3}}}{x-x_0} = \lim_{x\to x_0} \frac{x^{\frac{1}{3}}-x_0^{\frac{1}{3}}}{\left(x^{\frac{1}{3}}-x_0^{\frac{1}{3}}\right)\left(x^{\frac{2}{3}}+x^{\frac{1}{3}}x_0^{\frac{1}{3}}+x_0^{\frac{2}{3}}\right)}$$

$$= \lim_{x\to x_0} \frac{1}{x^{\frac{2}{3}}+x^{\frac{1}{3}}x_0^{\frac{1}{3}}+x_0^{\frac{2}{3}}} = \frac{1}{x_0^{\frac{2}{3}}+x_0^{\frac{2}{3}}+x_0^{\frac{2}{3}}} = \frac{1}{3x_0^{\frac{2}{3}}}$$

So $f(x)$ is differentiable on $R\backslash\{0\}$ with $f'(x_0) = 1/3x_0^{\frac{2}{3}}$

Example 2

Show that the function $f(x) = |x|$ is not differentiable at $x = 0$

Solution:

Note that $f(0) = 0$

$$f'(0^-) = \lim_{x\to 0^-} \frac{f(x)-0}{x-0} = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

$$f'(0^+) = \lim_{x\to 0^+} \frac{f(x)-0}{x-0} = \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1$$

We see $f(0^-) \neq f(0^+)$, therefore $\lim_{x\to 0} \frac{f(x)-0}{x-0}$ does not exist.

Hence $f(x)$ is not differentiable at $x = 0$

Example 3 (Important!)

Show that the function $f(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ -x & \text{if } x \in \mathbf{R}\backslash\mathbf{Q} \end{cases}$ is not differentiable at 0

IDEA: To check the function is not differentiable $\rightarrow$ check the limit does not exist $\rightarrow$ use sequential limit theorem

Solution:

For any $x_0 \in \mathbf{R}$, there exists rational $\{r_n\}$ and irrational $\{q_n\}$ sequences such that $r_n \rightarrow 0$ and $q_n \rightarrow 0$.

Consider

$$\lim_{n\to\infty} \frac{f(r_n) - f(0)}{r_n - 0} = \lim_{n\to\infty} \frac{r_n - 0}{r_n - 0} = \lim_{n\to\infty} 1 = 1$$

$$\lim_{n\to\infty} \frac{f(q_n) - f(0)}{q_n - 0} = \lim_{n\to\infty} \frac{-q_n - 0}{q_n - 0} = \lim_{n\to\infty} -1 = -1$$

Hence by sequential limit theorem, $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist

Example 4 (Practice Exercise #70)

Let $f: \mathbf{R} \rightarrow \mathbf{R}$ satisfy $|f(a) - f(b)| < |a - b|^2$ for any $a, b \in \mathbf{R}$. Show that $f(x)$ is a constant function.

Solution:

Consider

$$|f'(x)| = \lim_{h\to 0} \left|\frac{f(x+h) - f(x)}{h}\right| < \lim_{h\to 0} \frac{|(x+h) - x|^2}{|h|} = \lim_{h\to 0} |h| = 0$$

So we have $|f'(x)| = 0 \rightarrow f'(x) = 0$

Hence $f(x)$ is constant.

(*Note: We have used the fact that Using the fact that $f'(x) = 0 \leftrightarrow f(x) = c$. We will prove this result in later example)

Theorem: (Inverse Function Theorem)

Suppose f is continuous and injective on (a, b) and if $f(x)$ is differentiable and $f'(x_0) \neq 0$ for some $x_0 \in (a, b)$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

One application of this theorem is to compute the derivative of some inverse functions namely $\ln x$ (the inverse of e^x), $\sin^{-1} x$ (the inverse of $\sin x$), $x^{\frac{1}{n}}$ (the inverse of x^n) etc.

Example 5

Use the fact that $\frac{d}{dx}e^x = e^x$. Find the derivative of $g(x) = \ln x$ for $x \in (0, \infty)$

Solution:

It is easy to see e^x is continuous, differentiable and $\frac{d}{dx}e^x = e^x \neq 0$.

To show e^x is injective, note that e^x is increasing function, so $e^a = e^b \to a = b$.
Hence by inverse function theorem, $\ln x$ is differentiable on $(0, \infty)$ and

$$g'(x) = \frac{d}{dx}\ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

Example 6

Find the derivative of $h(x) = \sin^{-1} x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Solution:

For $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, it is clear that $\sin x$ is continuous, differentiable and injective.

Note that $\frac{d}{dx}\sin x = \cos x$, by inverse function theorem,

$$h'(x) = \frac{d}{dx}\sin^{-1} x = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1 - (\sin(\sin^{-1} x))^2}} = \frac{1}{\sqrt{1 - x^2}}$$

There are two important theorems in differentiation

Theorem: (Rolle's Theorem)
Let $f: [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and satisfy $f(a) = f(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$

Theorem: (Mean Value Theorem) (or MVT in short)
Let $f: [a, b] \to \mathbf{R}$ be continuousnon $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

In previous tutorial, we have seen an application of mean value theorem in Cauchy Sequence (See Tutorial Note #14). In the following examples, we will show some more applications of them.

Example 7

Suppose $f: \mathbf{R} \to \mathbf{R}$ is differentiable and $f'(x) = 0$ for all $x \in \mathbf{R}$. Show that $f(x)$ is constant function.

Solution:

We prove by contradiction, suppose f is not constant, then there exists $x_1, x_2 \in \mathbf{R}$ such that $f(x_1) \neq f(x_2)$. Since f is differentiable, by mean value theorem

$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$ for some $c \in (x_1, x_2)$

Since $f'(x) = 0$, then $\frac{f(x_2)-f(x_1)}{x_2-x_1} = 0 \to f(x_2) = f(x_1)$ which leads to contradiction.

Example 8

Let $f: [1,3] \to R$ be differentiable on $[1,3]$ and suppose $f'(x) = 4 - [f(x)]^2$. Is $f(3) - f(1) = 10$ possible?

Solution:

Using mean value theorem, we have

$\frac{f(3)-f(1)}{3-1} = f'(c)$ for some $c \in (1,3)$

$\to f(3) - f(1) = 2f'(c) = 2(4 - [f(c)]^2) < 2(4) = 8$

Hence $f(3) - f(1) = 10$ is impossible.

Example 9

Let f be defined in $[0,1]$ and has a bounded derivative in $[0,1]$. Show that the sequence $\{a_n\}$ converges where $a_n = f\left(\frac{1}{n}\right)$

IDEA: Since we do not have any idea about the limit of sequence a_n, hence to prove a_n converges, we will show a_n is Cauchy.

For any $\varepsilon > 0$, we need to find K such that for $m, n > K$, we have $|a_m - a_n| < \varepsilon$

Note that

$$|a_m - a_n| = \left|f\left(\frac{1}{n}\right) - f\left(\frac{1}{m}\right)\right| = f'(c)\left|\frac{1}{n} - \frac{1}{m}\right| < M\left|\frac{1}{n} - \frac{1}{m}\right| \leq M\left(\left|\frac{1}{n}\right| + \left|\frac{1}{m}\right|\right)$$

$$< M\left(\frac{\varepsilon}{2M} + \frac{\varepsilon}{2M}\right) = \varepsilon$$

We require both $\frac{1}{n} < \frac{\varepsilon}{2M} \to n > \frac{2M}{\varepsilon}$ (and $m > 2M/\varepsilon$)

Solution:

For any $\varepsilon > 0$, by Archimedean Property, there exists K such that $K > \frac{2M}{\varepsilon}$

Then for $m, n > K$, from the above argument , we get $|a_m - a_n| < \varepsilon$.
Hence $\{a_n\}$ is Cachy and therefore converges (by Cauchy Theorem)

Example 10 (Numeral Estimation)
Show that $2 < 65^{\frac{1}{6}} < 2 + \frac{1}{192}$

Solution:

It is clear that $65^{\frac{1}{6}} > 64^{\frac{1}{6}} = 2$

Let $f(x) = x^{\frac{1}{6}}$ and $f'(x) = \frac{1}{6}x^{-\frac{5}{6}}$, applying mean value theorem, we have

$$\frac{f(65) - f(64)}{65 - 64} = f'(c) \quad \text{for some } c \in (64,65)$$

$$\rightarrow 65^{\frac{1}{6}} - 64^{\frac{1}{6}} = \frac{1}{6}c^{-\frac{5}{6}} < \frac{1}{6}(64)^{-\left(\frac{5}{6}\right)} = \frac{1}{192}$$

$$\rightarrow 65^{\frac{1}{6}} < 2 + \frac{1}{192}$$

In the following, there are some suggested exercises, you should try to do them in order to understand the material. If you have any questions about them, you are welcome to find me during office hours. You are also welcome to submit your work to me and I can give some comments to your work.

☺Exercise 1

Use the definition to find the derivative of the functions

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 2x & \text{if } x = 0 \end{cases} \quad \text{and} \quad g(x) = |\cos x|$$

☺Exercise 2

Let $f(x) = \begin{cases} 3x + 2 & \text{if } x \in \mathbf{Q} \\ x^2 - 3x + 5 & \text{if } x \in \mathbf{R}\backslash\mathbf{Q} \end{cases}$, is f differentiable at $x = 3$

☺Exercise 3

Let a_n and b_n be 2 sequences ($a_n \neq b_n$) such that $a_n \rightarrow c$ and $b_n \rightarrow c$, suppose $f'(c)$ exists. Show that $\lim_{n\rightarrow\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n} = f'(c)$.

☺Exercise 4

Compute the derivative of $\tan^{-1} x$, $\cos^{-1} x$, $x^{\frac{1}{n}}$ (for $n \in \mathbf{N}$)

☺Exercise 5

Let $f(x)$ and $g(x)$ be differentiable function such that $f'(x) = g'(x)$ for all $x \in \mathbf{R}$. Show that $f(x) = g(x) + C$ for some constant C (Use Example 7 to help you)

☺Exercise 6

Let $f: [0, \infty) \rightarrow \mathbf{R}$ be continuous and $f(0) = 0$. If $|f'(x)| < |f(x)|$ for every $x > 0$, show that $f(x) = 0$ for every $x \in [0, \infty)$

(Hint: Consider the interval $\left[0, \left(\frac{1}{2}\right)\right]$, since $f(x)$ is continuous, then it must have maximum (say M) in $\left[0, \frac{1}{2}\right]$ by extreme value theorem, show that $M = 0$ and conclude $f\left(\frac{1}{2}\right) = 0$. Repeat the process in $[\frac{1}{2}, 1]$ and so on.)

☺Exercise 7

For $0 < x < \frac{\pi}{2}$, prove that $|\ln\cos x| < x\tan x$

☺Exercise 8

Let f satisfy all the hypothesis of the mean value theorem on $[a, b]$. Show that if $f'(x) > 0$, then $f(x)$ is increasing.

(Hint: You need to show $a > b \rightarrow f(a) > f(b)$)

☺Exercise 9

Give an example of a function that is differentiable on $(0,1)$ and $f(0) = f(1)$ but does not satisfy the Rolle's Theorem.

(Hint: To construct such function, note that f is differentiable on $(0,1)$ only imply f is continuous on **(0,1) (NOT [0,1]),** compare with the conditions in Rolle's Theorem)