MATH2033 Mathematical Analysis
Problem Set 1

Problem 1

Write down the negation of the following statements:

(a) x is divisible by 3 or 4.

(b) If x and y are positive, then $x + y > 0$.

(c) There exists a differentiable function $f(x)$ such that $\frac{df}{dx} + 2x = 0$ for all $x \in \mathbb{R}$.

(d) For any $\varepsilon > 0$, there exists a positive integer K such that $|x_n - L| < \varepsilon$ for all $n \geq K$
(*In this problem, $\{x_1, x_2, x_3, \dots\}$ denotes a sequence of real number)

Problem 2

(a) We let $\{x_1, x_2, x_3, \dots\}$ be a sequence of real numbers defined by $x_1 = 2$ and $x_{n+1} = 2x_n + 1$. Is it true that x_n is prime number for all positive integers n. Explain your answer. (😊Hint: Calculate x_2, x_3, x_4, x_5, x_6)

(b) We let n be a positive integer.

(i) If n^2 is multiple of 4, is it true that n is multiple of 4? Explain your answer.

(ii) If n^2 is multiple of 3, is it true that n is multiple of 3? Explain your answer.

(c) We let $f(x)$ be a function. Prove or disprove the following statement

(d) "If $f(0) = 0$, then $f'(0) = 0$."

Problem 3

We let $f(x)$ be a function.

Determine if each of the following statements is correct or not.

(a) Suppose that $f(x) > 0$ for all $x \in (1,4)$ (i.e. $1 < x < 4$), then $f(2)f(3) > 0$.

(b) Suppose that $f(x) > 0$ for some $x \in (1,4)$, then $f(2)f(3) > 0$.

Problem 4

Prove that $\sqrt[3]{3}$ is an irrational number.

Problem 5

Prove that there does *not* exist integers a and b such that $21a + 30b = 1$.

Problem 6

We let a and b be two real numbers. Prove that if $a, b > 0$, then $\frac{2}{a} + \frac{2}{b} \neq \frac{4}{a+b}$.

Problem 7

We let x be a non-zero rational number and y be an irrational number, show that $x + y$ and xy are both irrational.

Problem 8 (Harder)
We let x, y, z be three positive integers satisfying $x^2 + y^2 = z^2$. Show that if x and y are relatively prime (i.e. H.C.F. of x and y is 1), then one of them is odd and another one is even.

Problem 9
We let $f(x)$ be a function satisfying $f(ax + by) = af(x) + bf(y)$ for all real numbers a, b, x, y. Show that $f(z_1) = 0$ and $f(z_2) = 0$ if and only if $f(z_1 + z_2) = 0$ and $f(z_1 - z_2) = 0$.

Problem 10
Prove that a positive integer n is divisible by 9 if and only if the sum of digits of n is divisible by 9.
(😊Hint: We write $n = d_r d_{r-1} \dots d_1 d_0$ in decimal representation, where each d_i represents a digit of n. Then n can be expressed as

$$n = d_r \times 10^r + d_{r-1} \times 10^{r-1} + \cdots + d_1 \times 10 + d_0.)$$