

## Brief Descriptions of Facts

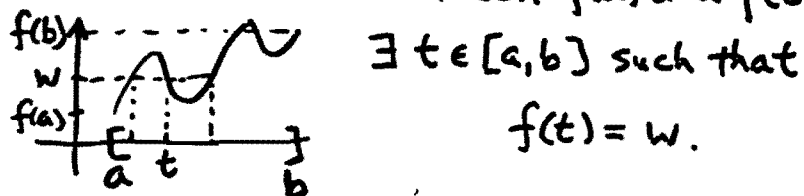
Completeness Axiom In  $\mathbb{R}$ ,  
every set that is bounded above has a supremum;  
every set that is bounded below has an infimum.

Supremum Property Let  $S$  be a set that is bounded above. Then  $\forall \varepsilon > 0, \exists x \in S$  such that  
 $\sup S - \varepsilon < x \leq \sup S$ .

Supremum Limit Theorem Let  $S$  be bounded above and  $c$  is an upper bound of  $S$ . Then

$$c = \sup S \Leftrightarrow \exists x_n \in S \text{ with } \lim_{n \rightarrow \infty} x_n = c.$$

Intermediate Value Theorem Let  $f$  be continuous on  $[a, b]$  and  $w$  is between  $f(a)$  and  $f(b)$ . Then



Monotone Function Theorem Let  $f$  be monotone on  $(a, b)$ .

Then ①  $\forall x_0 \in (a, b), f(x_0^-) = \lim_{x \rightarrow x_0^-} f(x), f(x_0^+) = \lim_{x \rightarrow x_0^+} f(x)$   
both exist

②  $f$  has countably many discontinuities on  $(a, b)$ .

## Continuous Injection Theorem, Continuous Inverse Theorem

$f$  continuous and injective  $\Rightarrow f$  is strictly monotone on  $[a, b]$   
 $\Rightarrow f^{-1}$  is continuous on  $f([a, b])$

## Monotone Sequence Theorem

$$x_1 \leq x_2 \leq x_3 \leq \dots \leq M \Rightarrow \lim_{n \rightarrow \infty} x_n = \sup \{x_1, x_2, x_3, \dots\}$$

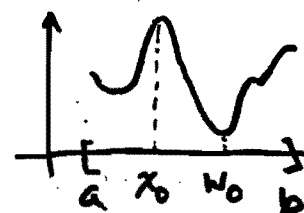
$$x_1 \geq x_2 \geq x_3 \geq \dots \geq m \Rightarrow \lim_{n \rightarrow \infty} x_n = \inf \{x_1, x_2, x_3, \dots\}$$

## Bolzano-Weierstrass Theorem

If  $x_1, x_2, x_3, \dots \in [a, b]$ , then  $\exists$  subsequence  
 $x_{n_1}, x_{n_2}, x_{n_3}, \dots$  having a limit in  $[a, b]$ .

$\hookrightarrow$  indices  $n_1 < n_2 < n_3 < \dots$

Extreme Value Theorem Let  $f$  be continuous on  $[a, b]$ . Then  $\exists x_0, w_0 \in [a, b]$  such that



$$f(x_0) = \sup \{f(x) : x \in [a, b]\}$$

= maximum of  $f(x)$  on  $[a, b]$

$$f(w_0) = \inf \{f(x) : x \in [a, b]\}$$

= minimum of  $f(x)$  on  $[a, b]$ .

Question How can we prove a sequence converges without identifying the limit?

In the 19<sup>th</sup> century, Cauchy introduced the following

Definition  $\{x_n\}$  is a Cauchy sequence iff  $\forall \varepsilon > 0$   
 $\exists K \in \mathbb{N}$  such that  $n, m \geq K \Rightarrow |x_n - x_m| < \varepsilon$ .

Remarks This means the terms are as close as desired when the indices are sufficiently large.

Example Let  $x_n = \frac{1}{n^2}$ . Show  $\{x_n\}$  is Cauchy.

Scratch Work Say  $m \geq n$ ,  $|x_n - x_m| = \frac{1}{n^2} - \frac{1}{m^2} < \frac{1}{n^2} < \varepsilon$   
 $n > \frac{1}{\sqrt{\varepsilon}}$  is enough.

Solution.  $\forall \varepsilon > 0$ , by Archimedean principle,  $\exists K \in \mathbb{N}$  such that  $K > \frac{1}{\sqrt{\varepsilon}}$ . Then

$$n, m \geq K \Rightarrow |x_n - x_m| = \left| \frac{1}{n^2} - \frac{1}{m^2} \right| < \frac{1}{K^2} < \varepsilon.$$

## Topics to be Covered ① Differentiation

- ① Big-Oh and Small-Oh Notations  
 Stolz' Theorem (L'Hopital's Rule for sequences)
- ② Riemann Integration and Improper Integrals
- ③ Preview of  
 Sequence and Series of Functions
  - Limit Superior and Limit Inferior
  - Pointwise and Uniform Convergence
- ④ Introduction to Metric Space Theory
  - Open, Closed, Compact, Connected Sets

OR

- ④' Introduction to Fourier Series

## Chapter 8 Differentiation

Definitions Let  $S$  be an interval of positive length.

A function  $f: S \rightarrow \mathbb{R}$  is differentiable at  $x_0 \in S$

iff  $f'(x_0) = \lim_{\substack{x \rightarrow x_0 \\ x \in S}} \frac{f(x) - f(x_0)}{x - x_0}$  exists in  $\mathbb{R}$ . Also,  $f$  is differentiable iff  $f$  is differentiable at every element of  $S$ .

Theorem If  $f: S \rightarrow \mathbb{R}$  is differentiable at  $x_0 \in S$ , then it is continuous at  $x_0$ .

Proof. Since  $f(x) = \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) + f(x_0)$ ,

so  $\lim_{x \rightarrow x_0} f(x) = f'(x_0) \cdot 0 + f(x_0) = f(x_0)$ .

Theorem (Differentiation Formulas)

If  $f, g: S \rightarrow \mathbb{R}$  is differentiable at  $x_0$ , then  $f+g$ ,  $f-g$ ,  $fg$ ,  $f/g$  (when  $g(x_0) \neq 0$ ) are differentiable at  $x_0$ .

In fact,  $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Proof. 
$$\frac{(f \pm g)(x) - (f \pm g)(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} \pm \frac{g(x) - g(x_0)}{x - x_0}.$$

Take limit as  $x \rightarrow x_0$  on both sides,  $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$ .

$$\begin{aligned} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} &= \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0}. \end{aligned}$$

Take limit as  $x \rightarrow x_0$ ,  $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$ .

$$\begin{aligned} \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0)}{x - x_0} &= \frac{1}{x - x_0} \left[ \frac{f(x)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)} \right] \\ &= \frac{1}{g(x)g(x_0)} \left[ \frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{x - x_0} \right] \\ &= \frac{1}{g(x)g(x_0)} \left[ \frac{f(x) - f(x_0)}{x - x_0} g(x_0) - f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right]. \end{aligned}$$

Take limit as  $x \rightarrow x_0$ ,  $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$ .

Theorem (Chain Rule)

If  $f: S \rightarrow \mathbb{R}$  is differentiable at  $x_0$ ,  $f(S) \subseteq S'$  and  $g: S' \rightarrow \mathbb{R}$  is differentiable at  $f(x_0)$ , then  $g \circ f$  is differentiable at  $x_0$  and  $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$ .

Proof. Define  $h: S' \rightarrow \mathbb{R}$  by  $h(y) = \begin{cases} \frac{g(y) - g(f(x_0))}{y - f(x_0)} & \text{if } y \neq f(x_0) \\ g'(f(x_0)) & \text{if } y = f(x_0) \end{cases}$   
 then  $h$  is continuous at  $f(x_0)$  because

$$\lim_{y \rightarrow f(x_0)} h(y) = \lim_{y \rightarrow f(x_0)} \frac{g(y) - g(f(x_0))}{y - f(x_0)} = g'(f(x_0)) = h(f(x_0)).$$

Now  $g(y) - g(f(x_0)) \stackrel{(*)}{=} h(y)(y - f(x_0))$  if  $y \neq f(x_0)$  and also if  $y = f(x_0)$ . So it is true for all  $y \in S'$ .

$$\lim_{x \rightarrow x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{h(f(x))(f(x) - f(x_0))}{x - x_0} \quad \text{by } (*)$$

$$\underbrace{(g \circ f)'(x_0)} = h(f(x_0)) f'(x_0) = g'(f(x_0)) f'(x_0).$$

Remarks  $f$  differentiable at  $x_0$  does not imply  $f'$  is continuous at  $x_0$ . Here is an example.

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ .

As  $x \rightarrow 0$ ,  $|f(x)| \leq |x^2 \sin \frac{1}{x}| \leq x^2 \rightarrow 0 \Rightarrow \lim_{x \rightarrow 0} f(x) = 0$   
 by sandwich theorem. So  $f$  is continuous.

$$\text{For } x \neq 0, f'(x) = (x^2 \sin \frac{1}{x})' = 2x \sin \frac{1}{x} + x^2 \cos(\frac{1}{x}) (-\frac{1}{x^2})$$

$$= 2x \sin \frac{1}{x} - \cos(\frac{1}{x}).$$

$$\text{For } x = 0, f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} x \sin \frac{1}{x} \nearrow 0.$$

So  $f$  is differentiable. as  $x \rightarrow 0, |x \sin \frac{1}{x}| \leq |x| \rightarrow 0$

Finally,  $\lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} (2x \sin \frac{1}{x} - \cos \frac{1}{x})$  doesn't exist ( $\neq f'(0)$ ).  
 $\therefore f'$  is not continuous at 0 and hence  $f''$  doesn't exist at 0.

Exercise  $g(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$  is differentiable,

but  $g'(x)$  is not continuous at 0 and  $g'(x)$  is unbounded on every open interval containing 0.

Example If  $h(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ x & \text{if } x = 0 \end{cases}$ , is  $h'(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ ?

The answer is no!  $h(x) = 0$  for all  $x$ . So  $h'(x) = 0$  for all  $x$ . In particular,  $h'(0) = 0 \neq 1$ .

Notations Let  $S$  be an interval of positive length.

$C^0(S) = C(S)$  is the set of all continuous functions on  $S$ .

$\forall n \in \mathbb{N}$ ,  $C^n(S)$  is the set of all functions  $f: S \rightarrow \mathbb{R}$

such that the  $n$ -th derivative  $f^{(n)}$  is continuous on  $S$ .

$C^\infty(S)$  is the set of all functions having  $n$ th derivatives for all  $n \in \mathbb{N}$ . Functions in  $C^1(S)$  are said to be continuously differentiable on  $S$ .

Inverse Function Theorem If  $f$  is continuous and injective on  $(a, b)$  and  $f'(x_0) \neq 0$  for some  $x_0 \in (a, b)$ , then  $f^{-1}$  is differentiable at  $y_0 = f(x_0)$  and

$$(f^{-1})'(y_0) = 1/f'(x_0).$$

If  $y = f(x)$ , then  $x = f^{-1}(y)$  and so  $\frac{dx}{dy}$  at  $y_0 = \frac{1}{\frac{dy}{dx} \text{ at } x_0}$ .

Proof. Define  $g(x) = \begin{cases} \frac{x-x_0}{f(x)-f(x_0)} & \text{if } x \neq x_0 \\ 1/f'(x_0) & \text{if } x = x_0 \end{cases}$ . Then  $g$  is

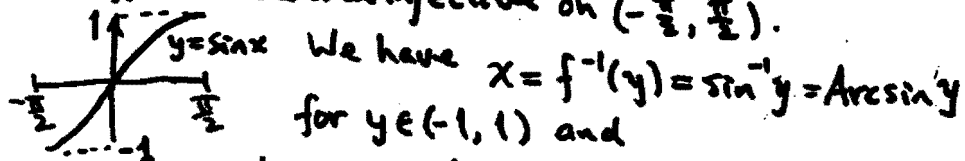
continuous at  $x_0$  because  $\lim_{x \rightarrow x_0} g(x) = \lim_{x \rightarrow x_0} \frac{x-x_0}{f(x)-f(x_0)} = \frac{1}{f'(x_0)} = g(x_0)$ .

Since  $f$  is continuous and injective on  $(a, b)$ , by the continuous inverse theorem,  $f^{-1}$  is continuous.

So  $\lim_{y \rightarrow y_0} f^{-1}(y) = f^{-1}(y_0) = x_0$ . For  $y \neq y_0$ ,  $g(f^{-1}(y)) = \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0}$ .

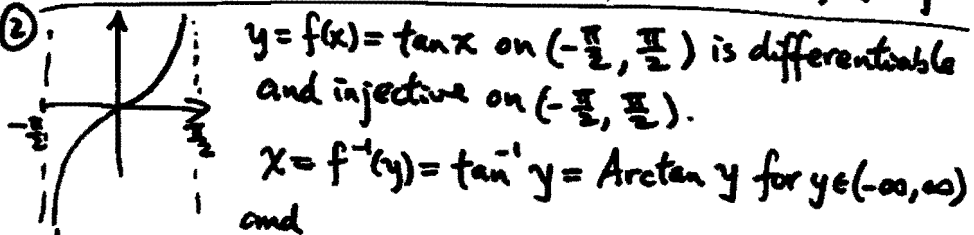
$$\therefore (f^{-1})'(y_0) = \lim_{y \rightarrow y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \rightarrow y_0} g(f^{-1}(y)) = g(f^{-1}(y_0)) = g(x_0) = 1/f'(x_0).$$

Example ① If  $y = f(x) = \sin x$  on  $(-\frac{\pi}{2}, \frac{\pi}{2})$ , then  $f$  is differentiable and injective on  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .



$$\frac{d}{dy}(\text{Arcsin } y) = \frac{d}{dy}(\sin^{-1} y) = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-y^2}}$$

②  $y = f(x) = \tan x$  on  $(-\frac{\pi}{2}, \frac{\pi}{2})$  is differentiable and injective on  $(-\frac{\pi}{2}, \frac{\pi}{2})$ .



$$\frac{d}{dy}(\text{Arctan } y) = \frac{d}{dy}(\tan^{-1} y) = \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\sec^2 x} = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}.$$

## Local Extremum Theorem

Let  $f: (a,b) \rightarrow \mathbb{R}$  be differentiable. If  $f(x_0) = \min_{x \in (a,b)} f(x)$  or  $f(x_0) = \max_{x \in (a,b)} f(x)$ , then  $f'(x_0) = 0$ .

Proof. If  $f(x_0) = \min_{x \in (a,b)} f(x)$ , then

$$0 \leq \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) = \lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

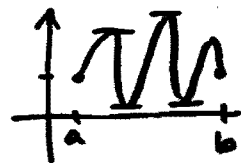
$\begin{matrix} \geq 0 \\ \text{above } f(x_0) \end{matrix}$ 
 $\begin{matrix} \geq 0 \\ \text{below } x - x_0 \end{matrix}$ 
 $\begin{matrix} \leq 0 \\ \text{above } x - x_0 \end{matrix}$

$\therefore f'(x_0) = 0$ . The case  $f(x_0) = \max_{x \in (a,b)} f(x)$  is similar.

Remark The theorem is false in general for closed interval, for example,  $f(x) = x$  on  $[-1, 1]$ .

$f(1) = \max_{x \in [-1, 1]} f(x)$ , but  $f'(1) = 1 \neq 0$ .

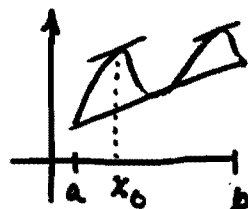
Rolle's Theorem Let  $f$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f(a) = f(b)$ , then there is (at least one)  $z_0 \in (a, b)$  such that  $f'(z_0) = 0$ .



Proof. If  $f$  is a constant function, then  $f'(x) = 0$  for any  $x \in (a, b)$ . Otherwise, by the extreme value theorem,  $\exists x_0, w_0 \in [a, b]$  such that  $f(x_0) = \max_{x \in [a, b]} f(x) > \min_{x \in [a, b]} f(x) = f(w_0)$ .

Then either  $f(x_0) \neq f(a)$  or  $f(w_0) \neq f(b)$ .

Then  $x_0$  or  $w_0 \in (a, b)$ . By last theorem,  $f'(x_0) = 0$  or  $f'(w_0) = 0$ .



## Mean-Value Theorem

If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then  $\exists x_0 \in (a, b)$  such that  $f(b) - f(a) = f'(x_0)(b - a)$ .

Proof. Define  $F(x) = f(x) - \left( \frac{f(b) - f(a)}{b - a} (x - a) + f(a) \right)$ .

Then  $F(a) = 0 = F(b)$ .  $\hookrightarrow$  linear function through  $(a, f(a))$ ,  $(b, f(b))$

By Rolle's Theorem,  $\exists x_0 \in (a, b)$  such that  $F'(x_0) = 0$ .

Since  $F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$ , we get  $f'(x_0) = \frac{f(b) - f(a)}{b - a}$ .

Examples. ①  $\forall a, b \in \mathbb{R}$ , prove  $|\sin b - \sin a| \leq |b - a|$ .

Solution. The case  $a = b$  is clear. If  $a < b$ , then by mean-value theorem, for  $f(x) = \sin x$ ,  $\exists x_0 \in (a, b)$  such that  $|\sin b - \sin a| = |\cos(x_0)(b - a)| \leq |b - a|$ . The case  $b < a$  is similar.

② Prove  $(1+x)^\alpha \geq 1 + \alpha x$  for  $x > -1$  and  $\alpha \geq 1$ .

Solution. Let  $f(x) = (1+x)^\alpha - 1 - \alpha x$ . Then  $f(0) = 0$ .

Case 1:  $x > 0$   $(1+x)^\alpha - 1 - \alpha x = f(x) - f(0) = f'(x_0)(x - 0)$

$$\exists x_0 \in (0, x) \text{ such that } = \underbrace{\alpha(1+x_0)^{\alpha-1}}_{\geq 0} \underbrace{x}_{> 0} \geq 0$$

Case 2:  $-1 < x < 0$

$$\exists x_0 \in (x, 0) \text{ such that } (1+x)^\alpha - 1 - \alpha x = f(x) - f(0) = f'(x_0)(x - 0)$$

$$= \underbrace{\alpha(1+x_0)^{\alpha-1}}_{\leq 0} \underbrace{x}_{< 0} \geq 0$$

③ Prove that  $\ln x \leq x-1$  for  $x > 0$ .

Solution Let  $f(x) = \ln x - x + 1$ , then  $f(1) = 0$ .

If  $x > 1$ , then  $\exists x_0 \in (1, x)$  such that

$$\ln x - x + 1 = f(x) = f(x) - f(1) = f'(x_0)(x-1) \\ = \underbrace{\left(\frac{1}{x_0} - 1\right)}_{< 0} \underbrace{(x-1)}_{> 0} < 0.$$

The case  $0 < x < 1$  is similar.

④ Approximate  $\sqrt{16.1}$ .

Let  $f(x) = \sqrt{x}$ . Then  $f(16.1) - f(16) = f'(c)(16.1 - 16)$  for some  $c \in (16, 16.1)$ . Now  $c \approx 16$ . So

$$f(16.1) - f(16) \approx f'(16)(16.1 - 16) = \frac{1}{2\sqrt{16}}(0.1) = 0.0125.$$

$$\therefore \sqrt{16.1} - 4 \approx 0.0125, \quad \sqrt{16.1} \approx 4.0125.$$

Theorem (for Curve Tracing)

If  $\begin{cases} f' \geq 0 \\ f' > 0 \\ f' \leq 0 \\ f' < 0 \\ f' \neq 0 \\ f' \equiv 0 \end{cases}$  everywhere, then  $f$  is  $\begin{cases} \text{increasing} \\ \text{strictly increasing} \\ \text{decreasing} \\ \text{strictly decreasing} \\ \text{injective} \\ \text{constant} \end{cases}$  on  $(a, b)$  respectively.

Proof. If  $x, y \in (a, b)$ ,  $x < y$ , then by mean value theorem,  $\exists x_0 \in (x, y)$  such that

$$f(y) - f(x) = f'(x_0)(y-x) \begin{cases} \geq 0 \\ > 0 \\ \leq 0 \\ < 0 \\ \neq 0 \\ = 0 \end{cases} \quad \therefore \begin{cases} f(x) \leq f(y) \\ f(x) < f(y) \\ f(x) \geq f(y) \\ f(x) > f(y) \\ f(x) \neq f(y) \\ f(x) = f(y) \end{cases}$$

Remarks For differentiable function  $f$ ,

if  $f$  is  $\begin{cases} \text{strictly increasing} \\ \text{strictly decreasing} \\ \text{injective} \end{cases}$ , then  $\begin{cases} f' > 0 \\ f' < 0 \\ f' \neq 0 \end{cases}$  everywhere may be false!

Examples ①  $f(x) = x^3$  is strictly increasing and injective, but  $f'(0) = 0$ . ②  $f(x) = -x^3$  is strictly decreasing, but  $f'(0) = 0$ .

For differentiable function  $f: (a, b) \rightarrow \mathbb{R}$ ,

if  $f$  is  $\begin{cases} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{cases}$ , then  $\begin{cases} f' \geq 0 \\ f' \leq 0 \\ f' = 0 \end{cases}$  everywhere on  $(a, b)$  is true.

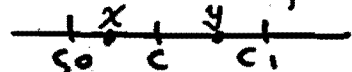
Proof. For  $x, x_0 \in (a, b)$ ,

$$f \text{ is } \begin{cases} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{cases} \Rightarrow \frac{f(x) - f(x_0)}{x - x_0} \begin{cases} \geq 0 \\ \leq 0 \\ = 0 \end{cases} \\ \Rightarrow f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \begin{cases} \geq 0 \\ \leq 0 \\ = 0 \end{cases}$$

## Local Tracing Theorem

If  $f: [a, b] \rightarrow \mathbb{R}$  is continuous and  $f'(c) > 0$  for some  $c \in [a, b]$ , then  $\exists c_0, c_1 \in \mathbb{R}$  such that

$c_0 < c < c_1$  and  $f(x) < f(c) < f(y) \forall x, y \in [a, b]$



and  $c_0 < x < c$   
 $c < y < c_1$ .

A similar result for the case  $f'(c) < 0$  is true and the inequality becomes  $f(x) > f(c) > f(y)$ .

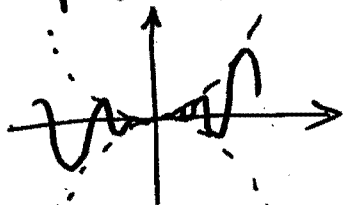
Proof. Let  $f'(c) > 0$ . Assume there is no such  $c_0$ . Then  $\forall n = 1, 2, 3, \dots$ ,  $\exists x_n \in [a, b]$  and  $c - \frac{1}{n} < x_n < c$  satisfying  $f(x_n) \geq f(c)$ . This will lead to

$$f'(c) = \lim_{n \rightarrow \infty} \underbrace{\frac{f(x_n) - f(c)}{x_n - c}}_{\substack{\geq 0 \\ < 0}} \leq 0, \text{ contradiction.}$$

The other parts are similar.

Remarks If we only know  $f'(c) \geq 0$ , we do not have a similar result. For example, let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



We have  $f'(0) = 0$ , but on every open interval  $(c_0, 0)$  or  $(0, c_1)$ ,

$f(x)$  takes both positive and negative values.

## Generalized Mean-Value Theorem

If  $f, g$  are continuous on  $[a, b]$  and are differentiable on  $(a, b)$ , then  $\exists x_0 \in (a, b)$  such that

$$g'(x_0)(f(b) - f(a)) = f'(x_0)(g(b) - g(a)). \quad (*)$$

Proof. Define  $F(x) = g(x)(f(b) - f(a)) - f(x)(g(b) - g(a))$ . Then  $F(a) = g(a)f(b) - f(a)g(b) = F(b)$ . By Rolle's Theorem,  $\exists x_0 \in (a, b)$  such that  $F'(x_0) = 0$ . So we get  $(*)$ .

Remark If  $g(b) \neq g(a)$ , then  $(*)$  can be put as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}.$$

( $\frac{0}{0}$  form of L'Hôpital's Rule)

① Let  $f, g$  be differentiable on  $(a, b)$

②  $g(x), g'(x) \neq 0 \forall x \in (a, b)$

③  $\lim_{x \rightarrow a^+} f(x) = 0 = \lim_{x \rightarrow a^+} g(x)$

④  $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L$ , where  $L \in \mathbb{R}$  or  $L = -\infty$  or  $L = +\infty$ .

Then  $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}$ .

The case  $x \rightarrow b^-$  is similar.

Proof. Define  $f(a) = 0$  and  $g(a) = 0$ .  $\forall x \in (a, b)$ ,  $f, g$  are continuous on  $[a, x]$  and differentiable on  $(a, x)$ . By generalized mean value theorem,  $\exists x_0 \in (a, x)$  such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(x_0)}{g'(x_0)}. \quad \text{As } x \rightarrow a^+, x_0 \rightarrow a^+ \\ \therefore \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)} \rightarrow L.$$



# ( $\frac{\infty}{\infty}$ form of L'Hôpital's Rule)

- ① Let  $f, g$  be differentiable on  $(a, b)$
- ②  $g(x), g'(x) \neq 0 \quad \forall x \in (a, b)$
- ③  $\lim_{x \rightarrow a^+} g(x) = \infty \quad \leftarrow \text{No need } \lim_{x \rightarrow a^+} f(x) \text{ exists!}$
- ④  $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L$ , where  $L \in \mathbb{R}$  or  $L = -\infty$  or  $L = +\infty$ .

Then  $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}$ . The case  $x \rightarrow b^-$  is similar.

Proof. We do the case  $L \in \mathbb{R}$  first. By ④,  $\exists$  interval  $I = (a, a + \delta_0)$  such that  $x \in I \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \frac{\varepsilon}{2}$ .

Let  $y \in I$ .  $\forall x \in I$ , by generalized mean-value theorem,  $\exists t \in I$  such that  $\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)}$ .

Multiply by  $\frac{g(x) - g(y)}{g(x)}$ , add  $\frac{f(y)}{g(x)}$ , then subtract  $\frac{f'(t)}{g'(t)}$  on

both sides. Get  $\frac{f(x)}{g(x)} - \frac{f'(t)}{g'(t)} = -\frac{g(y)}{g(x)} \frac{f'(t)}{g'(t)} + \frac{f(y)}{g(x)}$ .

So  $\left| \frac{f(x)}{g(x)} - \frac{f'(t)}{g'(t)} \right| \leq \left| \frac{g(y)}{g(x)} \right| \left( |L| + \frac{\varepsilon}{2} \right) + \left| \frac{f(y)}{g(x)} \right|$ .  $\leftarrow$  consider  $x \rightarrow a^+$

By ③, the right side has limit 0. So  $\exists$  interval  $J = (a, a + \delta)$  so that  $\forall x \in J$ , the right side is at most  $\frac{\varepsilon}{2}$ .

Then  $\forall x \in I \cap J = (a, a + \min\{\delta_0, \delta\})$

$$\left| \frac{f(x)}{g(x)} - L \right| \leq \left| \frac{f(x)}{g(x)} - \frac{f'(t)}{g'(t)} \right| + \left| \frac{f'(t)}{g'(t)} - L \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The cases  $L = \pm \infty$  follow by making simple modifications.

Examples ① Let  $f(x) = x^2 \sin \frac{1}{x}$  and  $g(x) = \sin x$  on  $(0, \frac{\pi}{2})$ .

Since  $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$ , so  $\lim_{x \rightarrow 0^+} f(x) = 0$ .  $\lim_{x \rightarrow 0^+} g(x) = 0$ .

$\lim_{x \rightarrow 0^+} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow 0^+} \frac{2x \sin \frac{1}{x} - \cos \frac{1}{x}}{\cos x}$  doesn't exist as  $\lim_{x \rightarrow 0^+} \cos \frac{1}{x}$  does not exist.

$\lim_{x \rightarrow 0^+} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0^+} \frac{x}{\sin x} (x \sin \frac{1}{x}) = 1 \cdot 0 = 0 \neq \lim_{x \rightarrow 0^+} \frac{f'(x)}{g'(x)}$ .

②  $\forall r \in \mathbb{R}, \lim_{x \rightarrow +\infty} \frac{x^r}{e^x} = 0$ . (To see this, choose  $n > |r|$ .)

Then  $x^r \leq x^n$  on  $[1, \infty)$ . So  $0 \leq \frac{x^r}{e^x} \leq \frac{x^n}{e^x}$  on  $[1, \infty)$ .

Since  $\frac{d^n}{dx^n} x^n = n!$  and  $\lim_{x \rightarrow +\infty} \frac{n!}{e^x} = 0$ , applying L'Hôpital's rule  $n$ -times, we see  $\lim_{x \rightarrow +\infty} \frac{x^n}{e^x} = 0$ .  $\therefore \lim_{x \rightarrow +\infty} \frac{x^r}{e^x} = 0$ .

③ Let  $f: (a, +\infty) \rightarrow \mathbb{R}$  be differentiable. Then

$$\lim_{x \rightarrow +\infty} (f'(x) + f(x)) = 0 \Rightarrow \lim_{x \rightarrow +\infty} f(x) = 0 = \lim_{x \rightarrow +\infty} f'(x).$$

(To see this, we apply ( $\frac{\infty}{\infty}$ )-form of L'Hôpital's rule as follow:

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{f(x)e^x}{e^x} = \lim_{x \rightarrow +\infty} \frac{f'(x)e^x + f(x)e^x}{e^x} = \lim_{x \rightarrow +\infty} (f'(x) + f(x)) = 0$$

and  $\lim_{x \rightarrow +\infty} f'(x) = \lim_{x \rightarrow +\infty} ((f'(x) + f(x)) - f(x)) = 0 - 0 = 0$ .

Remarks In O.D.E., if  $\lim_{x \rightarrow +\infty} g(x) = 0$ , then every solution  $y = f(x)$  of  $\frac{dy}{dx} + y = g(x)$  satisfies  $\lim_{x \rightarrow +\infty} f(x) = 0$  by the reason above.

④ Let  $f(x) = 2x + \sin x$  and  $g(x) = 2x - \sin x$  on  $(-\infty, +\infty)$ .  
As  $x \rightarrow +\infty$ ,  $f(x), g(x) \rightarrow +\infty$ .

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow +\infty} \frac{2 + \cos x}{2 - \cos x} \text{ doesn't exist}$$

$\hookrightarrow \begin{cases} x = 2n\pi & \text{limit} = 3 \\ x = (2n+1)\pi & \text{limit} = 1/3 \end{cases}$

$$\text{but } \lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow +\infty} \frac{2x + \sin x}{2x - \sin x} = \lim_{x \rightarrow +\infty} \frac{2 + \frac{\sin x}{x}}{2 - \frac{\sin x}{x}} = \frac{2}{2} = 1$$

Taylor's Theorem Let  $f: (a, b) \rightarrow \mathbb{R}$  be  $n$ -times differentiable.  
 $\forall x, c \in (a, b)$ ,  $\exists x_0$  between  $x$  and  $c$  such that

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots + \frac{f^{(n-1)}(c)}{(n-1)!}(x-c)^{n-1} + \frac{f^{(n)}(x_0)}{n!}(x-c)^n$$

( $n^{\text{th}}$  Taylor expansion of  $f$  about  $c$ )  $R_n(x)$  Lagrange form of the remainder

Proof: Let  $I$  be the closed interval with  $x$  and  $c$  as endpoints.  
For  $t \in I$ , define  $g(t) = (n-1)! \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!} (x-t)^k$ , where  $f^{(0)} = f$  and define  $p(t) = -\frac{(x-t)^n}{n}$ . We have

$$g'(t) = f^{(n)}(t)(x-t)^{n-1} \text{ and } p'(t) = (x-t)^{n-1}$$

By generalized mean value theorem,  $\exists x_0$  between  $x$  and  $c$  such that

$$\underbrace{g'(x_0)}_{f^{(n)}(x_0)(x-x_0)^{n-1}} \underbrace{(p(x) - p(c))}_{(x-c)^n/n} = \underbrace{p'(x_0)}_{(x-x_0)^{n-1}} \underbrace{(g(x) - g(c))}_{(n-1)! f(x)}$$

$$\Rightarrow f(x) = \frac{g(c)}{(n-1)!} + \frac{f^{(n)}(x_0)}{n!}(x-c)^n = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n)}(x_0)}{n!}(x-c)^n$$

## Taylor Expansions of Common Functions at $c=0$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + R_{n+1}(x) = \sum_{k=0}^n \frac{x^k}{k!} + R_{n+1}(x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + R_{2n+2}(x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + R_{2n+3}(x)$$

$$(1+x)^a = 1 + \sum_{k=1}^n \underbrace{\frac{a(a-1)\dots(a-k+1)}{k!}}_{= \binom{a}{k} = C_a^k} x^k + R_{n+1}(x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^n x^n}{n} + R_{n+1}(x)$$

$$\text{Arctan } x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + R_{2n+3}(x)$$

$$\text{Arcsin } x = x + \sum_{k=1}^n \underbrace{\frac{1 \cdot 3 \cdot 5 \dots (2k-1)}{2 \cdot 4 \cdot 6 \dots (2k)}}_{= \frac{(2k-1)!!}{(2k)!!}} \frac{x^{2k+1}}{2k+1} + R_{2n+3}(x)$$

Notation:

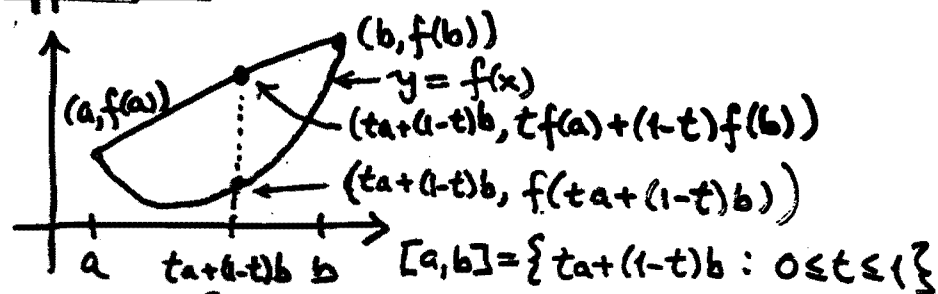
$$m!! = \begin{cases} 1 \cdot 3 \cdot 5 \dots m & \text{if } m \text{ is odd} \\ 2 \cdot 4 \cdot 6 \dots m & \text{if } m \text{ is even} \end{cases}$$

$n^{\text{th}}$  Taylor expansion is also called  $n^{\text{th}}$  Taylor Polynomial

If we let  $n \rightarrow \infty$ , the  $n^{\text{th}}$  Taylor expansion goes to  $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k$ . This is called the Taylor series of  $f(x)$  about  $c$ .

$$f^{(0)}(c) = f(c) \quad (x-c)^0 = 1 \text{ even if } x=c.$$

## Appendix 1: Convex and Concave Functions



**Definitions** ① Let  $I$  be an interval and  $f: I \rightarrow \mathbb{R}$ .

We say  $f$  is a convex function on  $I$  iff

$$\forall a, b \in I, 0 \leq t \leq 1, f(ta + (1-t)b) \leq tf(a) + (1-t)f(b).$$

②  $f$  is a concave function on  $I$  iff

$$\forall a, b \in I, 0 \leq t \leq 1, f(ta + (1-t)b) \geq tf(a) + (1-t)f(b).$$

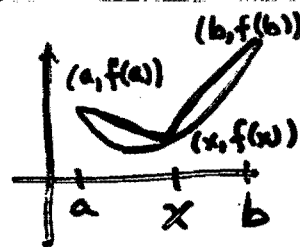
**Remarks** ① A function is convex on  $I$  iff every chord joining  $(a, f(a))$  and  $(b, f(b))$  with  $a, b \in I$  is always above or on the curve  $y = f(x)$ . A function is concave on  $I$  iff every chord is below or on the curve.

②  $f$  is strictly convex iff  $f(ta + (1-t)b) < tf(a) + (1-t)f(b)$  for  $0 < t < 1$ . Similarly for strictly concave.

Strictly convex functions are the ones whose chords are above the curve (except the endpoints are on the curve, of course). Similarly for strictly concave functions.

③  $f$  is convex  $\Leftrightarrow -f$  is concave.

$f$  is strictly convex  $\Leftrightarrow -f$  is strictly concave.



**Theorem**  $f$  is convex on  $I$  iff the slope of the chords always increase in the sense that

$$\forall a, x, b \in I, a < x < b \Rightarrow \frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(x)}{b - x}.$$

**Proof.** Note  $x = ta + (1-t)b \Leftrightarrow 0 \leq t = \frac{b-x}{b-a} \leq 1$  for some  $t \in [0, 1]$ .

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(x)}{b - x} \Leftrightarrow f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

$$\Leftrightarrow f(ta + (1-t)b) \leq tf(a) + (1-t)f(b).$$

**Theorem** For  $f$  differentiable on  $I$ ,  $f$  is convex on  $I \Leftrightarrow f'$  is increasing on  $I$  ( $\Leftrightarrow f'' \geq 0$  on  $I$  for  $f$  twice differentiable on  $I$ ). <sup>from curve tracing theorem.</sup>

**Proof.** ( $\Rightarrow$ )  $\forall a, b \in I$  with  $a < b$ , by last theorem,

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} \leq \lim_{x \rightarrow a^+} \frac{f(b) - f(x)}{b - x} = \frac{f(b) - f(a)}{b - a} \\ &= \lim_{x \rightarrow b^-} \frac{f(x) - f(a)}{x - a} \leq \lim_{x \rightarrow b^-} \frac{f(b) - f(x)}{b - x} = f'(b). \end{aligned}$$

( $\Leftarrow$ )  $\forall a, x, b \in I$  with  $a < x < b$ , by the mean-value theorem,  $\exists r, s$  such that  $a < r < x < s < b$  and

$$\frac{f(x) - f(a)}{x - a} = f'(r) \leq f'(s) = \frac{f(b) - f(x)}{b - x}.$$

By last theorem,  $f$  is convex on  $I$ .

Theorem If  $f$  is convex on  $(a, b)$ , then  $f$  is continuous on  $(a, b)$ .

Proof.  $\forall x_0 \in (a, b)$ , consider  $u, v, w \in (a, b)$  such that  $u < x_0 < v < w$ . Then

$$\frac{f(x_0) - f(u)}{x_0 - u} \leq \frac{f(v) - f(x_0)}{v - x_0} \leq \frac{f(w) - f(v)}{w - v}.$$

Solving for  $f(v)$ , we get

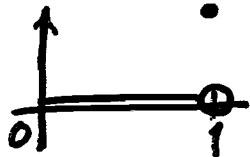
$$\frac{f(x_0) - f(u)}{x_0 - u} (v - x_0) + f(x_0) \leq f(v) \leq \frac{f(w) - f(v)}{w - v} (v - x_0) + f(x_0).$$

Take limit as  $v \rightarrow x_0^+$ , we get  $f(x_0) \leq f(x_0^+) \leq f(x_0)$ .

So  $f(x_0^+) = f(x_0)$ . Similarly,  $f(x_0^-) = f(x_0)$  by taking  $u < v < x_0 < w$ . Therefore,  $f$  is continuous on  $(a, b)$ .

Remark and Example The above theorem may not be true for  $[a, b]$ . For example,

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$



is convex on  $[0, 1]$  by checking the definition or checking the slope of chords is increasing.

However,  $f$  is not continuous at 1.

Example Prove that if  $a, b \geq 0$  and  $0 < r < 1$ , then  $|a^r - b^r| \leq |a - b|^r$ .

In particular,  $|\sqrt[n]{a} - \sqrt[n]{b}| \leq \sqrt[n]{|a - b|}$  (\*)

for  $n = 2, 3, 4, \dots$ .

Solution. We may assume  $a \geq b$ , otherwise interchange them.

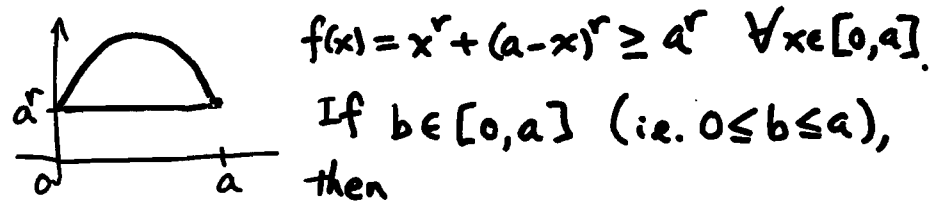
Define  $f: [0, a] \rightarrow \mathbb{R}$  by  $f(x) = x^r + (a - x)^r$ .

Since  $r - 1 < 0$ , so

$$f''(x) = r(r-1)(x^{r-2} + (a-x)^{r-2}) \leq 0.$$

So  $f$  is concave on  $[0, a]$ .

Since  $f(0) = a^r = f(a)$ , we get



$$f(x) = x^r + (a - x)^r \geq a^r \quad \forall x \in [0, a].$$

If  $b \in [0, a]$  (i.e.  $0 \leq b \leq a$ ), then

$$f(b) = b^r + (a - b)^r \geq a^r \Rightarrow |a^r - b^r| = a^r - b^r \leq (a - b)^r = |a - b|^r.$$

Remark (\*) is the case  $r = \frac{1}{n}$  for  $n = 2, 3, 4, \dots$ .

(\*) is useful in some exercises.

## Extra Examples for the Chapter on Differentiation

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① If  $\{x_n\}$  is Cauchy, then prove  $\{\sqrt[5]{x_n}\}$  is Cauchy.

Solution. Since  $\{x_n\}$  is Cauchy, <sup>and  $x_n \geq 0$  for all  $n$</sup>  for every  $\varepsilon > 0$ ,  $\exists K \in \mathbb{N}$  such that  $m, n \geq K \Rightarrow |x_m - x_n| < \varepsilon^5$ .

Then  $|\sqrt[5]{x_m} - \sqrt[5]{x_n}| \leq \sqrt[5]{|x_m - x_n|} < \sqrt[5]{\varepsilon^5} = \varepsilon$ .  
 $\therefore \{\sqrt[5]{x_n}\}$  is Cauchy.

②  $f: \mathbb{R} \rightarrow \mathbb{R}$  is three-times differentiable.

If  $f(-1) = 0$ ,  $f(1) = 1$  and  $f'(0) = 0$ , then prove that  $\sup \{f^{(3)}(x) : -1 < x < 1\} \geq 3$ .

Thoughts  $f^{(3)}$ ,  $f'$ ,  $f$  suggest Taylor's theorem.  $c$  should be  $-1$ ,  $1$  or  $0$ , more likely  $c = 0$ .

Solution By Taylor's theorem, using  $c = 0$ ,

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(\theta)}{6}(x-0)^3 \\ &= f(0) + \frac{f''(0)}{2}x^2 + \frac{f'''(\theta_x)}{6}x^3 \end{aligned}$$

for some  $\theta_x$  between  $x$  and  $0$ . Setting  $x = 1, -1$ ,

$$1 = f(1) = f(0) + \frac{f''(0)}{2} + \frac{f'''(\theta_1)}{6} \text{ for some } \theta_1 \in (0, 1)$$

$$0 = f(-1) = f(0) + \frac{f''(0)}{2} - \frac{f'''(\theta_{-1})}{6} \text{ for some } \theta_{-1} \in (-1, 0).$$

Subtracting yields  $f'''(\theta_1) + f'''(\theta_{-1}) = 6$ .

$$\sup \{f^{(3)}(x) : -1 < x < 1\} \geq \max \{f'''(\theta_1), f'''(\theta_{-1})\} \geq 3.$$

③ Let  $f$  be twice differentiable on  $[0, 2]$ .

$$\forall x \in [0, 2], |f(x)| \leq 1, |f''(x)| \leq 1.$$

Prove that  $\forall x \in [0, 2], |f'(x)| \leq 2$ .

Solution By Taylor's theorem, let  $x \in [0, 2], a \in [0, 2]$

$$f(a) = f(x) + f'(x)(a-x) + \frac{f''(\theta_a)}{2}(a-x)^2$$

for some  $\theta_a$  between  $a$  and  $x$ . Setting  $a = 0, 2$ ,

$$f(0) = f(x) - f'(x)x + \frac{f''(\theta_0)}{2}x^2 \text{ for some } \theta_0 \in (0, x).$$

$$f(2) = f(x) + f'(x)(2-x) + \frac{f''(\theta_2)}{2}(2-x)^2 \text{ for some } \theta_2 \in (x, 2).$$

Subtracting these, we get

$$f(2) - f(0) = 2f'(x) + \frac{f''(\theta_2)}{2}(2-x)^2 - \frac{f''(\theta_0)}{2}x^2.$$

Solving for  $f'(x)$ , we see

$$|f'(x)| = \frac{1}{2} |f(2) - f(0) + \frac{f''(\theta_0)}{2}x^2 - \frac{f''(\theta_2)}{2}(2-x)^2|$$

$$\leq \frac{1}{2} (1 + 1 + \frac{1}{2}x^2 + \frac{1}{2}(2-x)^2)$$

$$= \frac{1}{2} (x^2 - 2x + 4) \quad \left\{ \begin{array}{l} x \in [0, 2] \\ |x-1| \leq 1 \end{array} \right.$$

$$\leq \frac{1}{2} ((x-1)^2 + 3)$$

$$\leq \frac{1}{2} (1 + 3) = 2$$

- ④ Let  $f: [1, 2]$  be continuous and  $f$  be differentiable on  $(1, 2)$ . Prove there exists  $\theta \in (1, 2)$  such that  $f(2) - f(1) = \frac{1}{2} \theta^2 f'(\theta)$ .

(Note mean value theorem only gives  $f(2) - f(1) = f'(\theta_0)(2-1) = f'(\theta_0)$  for some  $\theta_0 \in (1, 2)$ .)

Solution Write

$$\theta^2 f'(\theta) = \frac{f'(\theta)}{1/\theta^2} = \frac{f'(\theta)}{g'(\theta)}, \text{ where } g(\theta) = -\frac{1}{\theta}.$$

By generalized mean value theorem,

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(\theta)}{g'(\theta)} \text{ for some } \theta \in (1, 2).$$

This is just

$$\frac{f(2) - f(1)}{-\frac{1}{2} - (-1)} = \frac{f'(\theta)}{1/\theta^2} \Leftrightarrow f(2) - f(1) = \frac{1}{2} \theta^2 f'(\theta).$$

- ⑤ Let  $0 < a < b$  and let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous with  $f$  differentiable on  $(a, b)$ . Prove  $\exists \theta \in (a, b)$  such that  $\frac{1}{b-a} \left| \frac{f(a)}{a} - \frac{f(b)}{b} \right| = f'(\theta)$ .

Solution. Expanding the left side, we get  $\frac{1}{b-a} \left| \frac{f(a)}{a} - \frac{f(b)}{b} \right| = \frac{bf(a) - af(b)}{b-a} = \frac{f(a)/a - f(b)/b}{1/a - 1/b}$  dividing by  $ab$  in numerator & denominator.

This suggests we consider  $F(x) = f(x)/x$ ,  $G(x) = 1/x$ . By the generalized mean value theorem,

$$\begin{aligned} \frac{f(a)/a - f(b)/b}{1/a - 1/b} &= \frac{F(a) - F(b)}{G(a) - G(b)} = \frac{F'(\theta)}{G'(\theta)} \text{ for some } \theta \in (a, b) \\ &= \frac{\theta f'(\theta) - f(\theta)}{\theta^2} / \left(-\frac{1}{\theta^2}\right) \\ &= f'(\theta). \end{aligned}$$

The result follows.