Brief Descriptions of Facts

Completeness Axiom In $\mathbb{R}$,
every set that is bounded above has a supremum;
every set that is bounded below has an infimum.

Supremum Property Let S be a set that is bounded above. Then $\forall \varepsilon > 0$, $\exists x \in S$ such that

$$\sup S - \varepsilon < x \le \sup S.$$

Supremum Limit Theorem Let S be bounded above and c is an upper bound of S. Then

$$c = \sup S \iff \exists x_n \in S \text{ with } \lim_{n\to\infty} x_n = c.$$

Intermediate Value Theorem Let f be continuous on $[a,b]$ and w is between $f(a)$ and $f(b)$. Then

$\exists\, t \in [a,b]$ such that

$$f(t) = w.$$

Monotone Function Theorem Let f be monotone on (a,b). Then ① $\forall x_0 \in (a,b)$, $f(x_0-) = \lim_{x\to x_0^-} f(x)$, $f(x_0+) = \lim_{x\to x_0+} f(x)$ both exist

② f has countably many discontinuities on (a,b).

Continuous Injection Theorem, Continuous Inverse Theorem

f continuous and injective on $[a,b]$ $\Rightarrow$ f is strictly monotone
$\Rightarrow$ f^{-1} is continuous on $f([a,b])$

Monotone Sequence Theorem

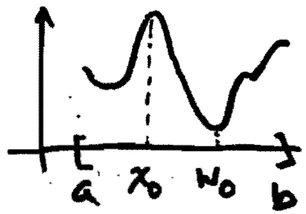
$$x_1 \le x_2 \le x_3 \le \cdots \le M \Rightarrow \lim_{n\to\infty} x_n = \sup\{x_1, x_2, x_3, \ldots\}$$

$$x_1 \ge x_2 \ge x_3 \ge \cdots \ge m \Rightarrow \lim_{n\to\infty} x_n = \inf\{x_1, x_2, x_3, \ldots\}$$

Bolzano-Weierstrass Theorem

If $x_1, x_2, x_3, \ldots \in [a,b]$, then $\exists$ subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ having a limit in $[a,b]$.
(indices $n_1 < n_2 < n_3 < \cdots$

Extreme Value Theorem Let f be continuous on $[a,b]$. Then $\exists\, x_0, w_0 \in [a,b]$ such that



$$f(x_0) = \sup\{f(x) : x \in [a,b]\} = \text{maximum of } f(x) \text{ on } [a,b]$$

$$f(w_0) = \inf\{f(x) : x \in [a,b]\} = \text{minimum of } f(x) \text{ on } [a,b].$$

Question How can we prove a sequence converges without identifying the limit?

In the 19th century, Cauchy introduced the following

Definition $\{x_n\}$ is a Cauchy sequence iff $\forall \varepsilon > 0$ $\exists K \in \mathbb{N}$ such that $n, m \geq K \Rightarrow |x_n - x_m| < \varepsilon$.

Remarks This means the terms are as close as desired when the indices are sufficiently large.

Example Let $x_n = \frac{1}{n^2}$. Show $\{x_n\}$ is Cauchy.

Scratch Work Say $m \geq n$, $|x_n - x_m| = \frac{1}{n^2} - \frac{1}{m^2} < \frac{1}{n^2} < \varepsilon$

$n > \frac{1}{\sqrt{\varepsilon}}$ is enough.

Solution. $\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\sqrt{\varepsilon}}$. Then

$$n, m \geq K \Rightarrow |x_n - x_m| = \left|\frac{1}{n^2} - \frac{1}{m^2}\right| < \frac{1}{K^2} < \varepsilon.$$

Topics to be Covered

0. Differentiation
1. Big-Oh and Small-Oh Notations
   Stolz' Theorem (L'Hopital's Rule for sequences)
2. Riemann Integration and Improper Integrals
3. Preview of
   Sequence and Series of Functions
   - Limit Superior and Limit Inferior
   - Pointwise and Uniform Convergence
4. Introduction to Metric Space Theory
   - Open, Closed, Compact, Connected Sets

   OR

4'. Introduction to Fourier Series

Chapter 8 Differentiation

Definitions Let S be an interval of positive length. A function $f: S \to \mathbb{R}$ is differentiable at $x_0 \in S$ iff $f'(x_0) = \lim\limits_{\substack{x \to x_0 \\ x \in S}} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists in $\mathbb{R}$. Also, f is differentiable iff f is differentiable at every element of S.

Theorem If $f: S \to \mathbb{R}$ is differentiable at $x_0 \in S$, then it is continuous at x_0.

Proof. Since $f(x) = \dfrac{f(x)-f(x_0)}{x-x_0}(x-x_0) + f(x_0)$, so $\lim\limits_{x \to x_0} f(x) = f'(x_0) \cdot 0 + f(x_0) = f(x_0)$.

Theorem (Differentiation Formulas)
If $f, g: S \to \mathbb{R}$ is differentiable at x_0, then $f+g$, $f-g$, fg, f/g (when $g(x_0) \neq 0$) are differentiable at x_0.
In fact, $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Proof. $\dfrac{(f \pm g)(x) - (f \pm g)(x_0)}{x - x_0} = \dfrac{f(x)-f(x_0)}{x-x_0} \pm \dfrac{g(x)-g(x_0)}{x-x_0}$.

Take limit as $x \to x_0$ on both sides, $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$.

$$\frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0}$$

$$= \frac{f(x)-f(x_0)}{x-x_0} g(x) + f(x_0) \frac{g(x)-g(x_0)}{x-x_0}.$$

Take limit as $x \to x_0$, $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

$$\frac{(f/g)(x) - (f/g)(x_0)}{x - x_0} = \frac{1}{x-x_0}\left[\frac{f(x)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)}\right]$$

$$= \frac{1}{g(x)g(x_0)}\left[\frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{x - x_0}\right]$$

$$= \frac{1}{g(x)g(x_0)}\left[\frac{f(x)-f(x_0)}{x-x_0} g(x_0) - f(x_0) \frac{g(x)-g(x_0)}{x-x_0}\right].$$

Take limit as $x \to x_0$, $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$.

Theorem (Chain Rule)
If $f: S \to \mathbb{R}$ is differentiable at x_0, $f(S) \subseteq S'$ and $g: S' \to \mathbb{R}$ is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0)) f'(x_0)$.

Proof. Define $h: S' \to \mathbb{R}$ by $h(y) = \begin{cases} \frac{g(y)-g(f(x_0))}{y-f(x_0)} & \text{if } y \neq f(x_0) \\ g'(f(x_0)) & \text{if } y = f(x_0) \end{cases}$

Then h is continuous at $f(x_0)$ because

$$\lim_{y\to f(x_0)} h(y) = \lim_{y\to f(x_0)} \frac{g(y)-g(f(x_0))}{y-f(x_0)} = g'(f(x_0)) = h(f(x_0)).$$

Now $g(y) - g(f(x_0)) \overset{(*)}{=} h(y)(y-f(x_0))$ if $y \neq f(x_0)$ and also if $y = f(x_0)$. So it is true for all $y \in S'$.

$$\underbrace{\lim_{x\to x_0} \frac{(g\circ f)(x)-(g\circ f)(x_0)}{x-x_0}}_{(g\circ f)'(x_0)} = \lim_{x\to x_0} \frac{h(f(x))(f(x)-f(x_0))}{x-x_0} \quad \text{by } (*)$$
$$= h(f(x_0))f'(x_0) = g'(f(x_0))f'(x_0).$$

Remarks f differentiable at x_0 <u>does not imply</u> f' is continuous at x_0. Here is an example.

Let $f: \mathbb{R}\to\mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

As $x \to 0$, $|f(x)| \le |x^2 \sin\frac{1}{x}| \le x^2 \to 0 \Rightarrow \lim_{x\to 0} f(x) = 0$ by sandwich theorem. So f is continuous.

For $x \neq 0$, $f'(x) = (x^2 \sin\frac{1}{x})' = 2x\sin\frac{1}{x} + x^2\cos(\frac{1}{x})(-\frac{1}{x^2})$
$= 2x\sin\frac{1}{x} - \cos(\frac{1}{x})$.

For $x = 0$, $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x\sin\frac{1}{x} = 0.$

So f is differentiable. as $x\to 0$, $|x \sin\frac{1}{x}| \le |x| \to 0$

Finally, $\lim_{x\to 0} f'(x) = \lim_{x\to 0}(2x\sin\frac{1}{x} - \cos\frac{1}{x})$ doesn't exist ($\neq f'(0)$).

$\therefore$ f' is not continuous at 0 and hence f'' doesn't exist at 0.

Exercise $g(x) = \begin{cases} x^2 \sin\frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is differentiable, but $g'(x)$ is not continuous at 0 and $g'(x)$ is unbounded on every open interval containing 0.

Example If $h(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ x & \text{if } x = 0, \end{cases}$ is $h'(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$?

The answer is no! $h(x) = 0$ for all x. So $h'(x) = 0$ for all x. In particular, $h'(0) = 0 \neq 1$.

Notations Let S be an interval of positive length.

$C^0(S) = C(S)$ is the set of all continuous functions on S.

$\forall n \in \mathbb{N}$, $C^n(S)$ is the set of all functions $f: S \to \mathbb{R}$ such that the n-th derivative $f^{(n)}$ is continuous on S.

$C^\infty(S)$ is the set of all functions having n^{th} derivatives for all $n \in \mathbb{N}$. Functions in $C^1(S)$ are said to be <u>continuously differentiable</u> on S.

Inverse Function Theorem If f is continuous and injective on (a,b) and $f'(x_0) \neq 0$ for some $x_0 \in (a,b)$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = 1/f'(x_0).$$

If $y = f(x)$, then $x = f^{-1}(y)$ and so $\left.\frac{dx}{dy}\right|_{\text{at } y_0} = 1\Big/\left.\frac{dy}{dx}\right|_{\text{at } x_0}$.

Proof. Define $g(x)=\begin{cases}\dfrac{x-x_0}{f(x)-f(x_0)} & \text{if } x\neq x_0\\ 1/f'(x_0) & \text{if } x=x_0\end{cases}$. then g is continuous at x_0 because $\lim\limits_{x\to x_0} g(x)=\lim\limits_{x\to x_0}\dfrac{x-x_0}{f(x)-f(x_0)}=\dfrac{1}{f'(x_0)}=g(x_0)$.

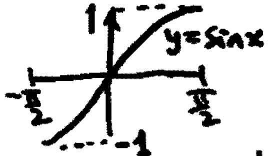
Since f is continuous and injective on (a,b), by the continuous inverse theorem, f^{-1} is continuous.

So $\lim\limits_{y\to y_0} f^{-1}(y)=f^{-1}(y_0)=x_0$. For $y\neq y_0$, $g(f^{-1}(y))=\dfrac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}$.

$\therefore\ (f^{-1})'(y_0)=\lim\limits_{y\to y_0}\dfrac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}=\lim\limits_{y\to y_0} g(f^{-1}(y))=g(f^{-1}(y_0))$

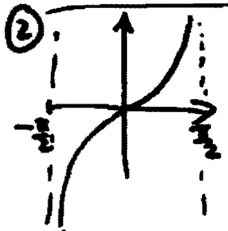
$=g(x_0)=1/f'(x_0)$.

Examples ① If $y=f(x)=\sin x$ on $(-\frac{\pi}{2},\frac{\pi}{2})$, then f is differentiable and injective on $(-\frac{\pi}{2},\frac{\pi}{2})$.



We have $x=f^{-1}(y)=\sin^{-1}y=\text{Arcsin}\,y$ for $y\in(-1,1)$ and

$\frac{d}{dy}(\text{Arcsin}\,y)=\frac{d}{dy}(\sin^{-1}y)=\frac{dx}{dy}=1/\frac{dy}{dx}=1/\cos x=1/\sqrt{1-y^2}$

② 

$y=f(x)=\tan x$ on $(-\frac{\pi}{2},\frac{\pi}{2})$ is differentiable and injective on $(-\frac{\pi}{2},\frac{\pi}{2})$.

$x=f^{-1}(y)=\tan^{-1}y=\text{Arctan}\,y$ for $y\in(-\infty,\infty)$ and

$\frac{d}{dy}(\text{Arctan}\,y)=\frac{d}{dy}(\tan^{-1}y)=\frac{dx}{dy}=1/\frac{dy}{dx}=1/\sec^2 x=1/(1+\tan^2 x)=1/(1+y^2)$.

Local Extremum Theorem

Let $f:(a,b)\to\mathbb{R}$ be differentiable. If $f(x_0)=\min\limits_{x\in(a,b)} f(x)$ or $f(x_0)=\max\limits_{x\in(a,b)} f(x)$, then $f'(x_0)=0$.

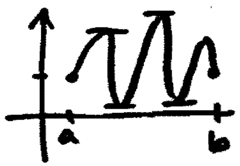
Proof. If $f(x_0)=\min\limits_{x\in(a,b)} f(x)$, then

$$0\le \lim_{x\to x_0^+}\frac{\overbrace{f(x)-f(x_0)}^{\ge 0}}{\underbrace{x-x_0}_{>0}}=f'(x_0)=\lim_{x\to x_0^-}\frac{\overbrace{f(x)-f(x_0)}^{\ge 0}}{\underbrace{x-x_0}_{<0}}\le 0 .$$

$\therefore f'(x_0)=0$. The case $f(x_0)=\max\limits_{x\in(a,b)} f(x)$ is similar.

Remark The theorem is false in general for closed interval, for example, $f(x)=x$ on $[-1,1]$. $f(1)=\max\limits_{x\in[-1,1]} f(x)$, but $f'(1)=1\neq 0$.

Rolle's Theorem Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$, then there is (at least one) $z_0\in(a,b)$ such that $f'(z_0)=0$.

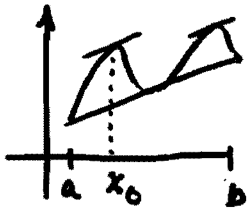


Proof. If f is a constant function, then $f'(x)=0$ for any $x\in(a,b)$. Otherwise, by the extreme value theorem, $\exists\, x_0, w_0\in[a,b]$ such that

$$f(x_0)=\max_{x\in[a,b]} f(x) > \min_{x\in[a,b]} f(x)=f(w_0).$$

Then either $f(x_0)\neq f(a)$ or $f(w_0)\neq f(a)$.

Then x_0 or $w_0\in(a,b)$. By last theorem, $f'(x_0)=0$ or $f'(w_0)=0$.



Mean-Value Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, x_0\in(a,b)$ such that

$$f(b)-f(a)=f'(x_0)(b-a).$$

Proof. Define $F(x)=f(x)-\Big(\underbrace{\frac{f(b)-f(a)}{b-a}(x-a)+f(a)}_{\text{linear function through }(a,f(a)),\,(b,f(b))}\Big)$.

Then $F(a)=0=F(b)$.

By Rolle's theorem, $\exists\, x_0\in(a,b)$ such that $F'(x_0)=0$.

Since $F'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$, we get $f'(x_0)=\frac{f(b)-f(a)}{b-a}$.

Examples. ① $\forall a,b\in\mathbb{R}$, prove $|\sin b-\sin a|\le|b-a|$.

Solution. The case $a=b$ is clear. If $a<b$, then by mean-value theorem, for $f(x)=\sin x$, $\exists\, x_0\in(a,b)$ such that $|\sin b-\sin a|=|\cos(x_0)\,(b-a)|\le|b-a|$. The case $b<a$ is similar.

② Prove $(1+x)^\alpha\ge 1+\alpha x$ for $x>-1$ and $\alpha\ge 1$.

Solution. Let $f(x)=(1+x)^\alpha-1-\alpha x$. Then $f(0)=0$.

Case 1: $x>0$ $\quad(1+x)^\alpha-1-\alpha x=f(x)-f(0)=f'(x_0)(x-0)$

$\exists\, x_0\in(0,x)$ such that $\quad=\alpha\big(\underbrace{(1+x_0)^{\alpha-1}-1}_{\ge 0}\big)\underbrace{x}_{>0}\ge 0$

Case 2: $-1<x<0$

$\exists\, x_0\in(x,0)$ such that $\quad(1+x)^\alpha-1-\alpha x=f(x)-f(0)=f'(x_0)(x-0)$

$\quad=\alpha\big(\underbrace{(1+x_0)^{\alpha-1}-1}_{\le 0}\big)\underbrace{x}_{<0}\ge 0$

③ Prove that $\ln x \le x-1$ for $x>0$.

Solution. Let $f(x) = \ln x - x + 1$, then $f(1)=0$.

If $x>1$, then $\exists\, x_0 \in (1,x)$ such that

$$\ln x - x + 1 = f(x) = f(x) - f(1) = f'(x_0)(x-1) = \underbrace{\left(\frac{1}{x_0}-1\right)}_{<0}\underbrace{(x-1)}_{>0} < 0.$$

The case $0<x<1$ is similar.

④ Approximate $\sqrt{16.1}$.

Let $f(x)=\sqrt{x}$. Then $f(16.1)-f(16) = f'(c)(16.1-16)$ for some $c\in(16,16.1)$. Now $c \approx 16$. So

$$f(16.1)-f(16) \approx f'(16)(16.1-16) = \frac{1}{2\sqrt{16}}(0.1) = 0.0125$$

$\therefore \sqrt{16.1} - 4 \approx 0.0125$, $\sqrt{16.1} \approx 4.0125$.

Theorem (for Curve Tracing)

$$\text{If } \begin{Bmatrix} f' \ge 0 \\ f' > 0 \\ f' \le 0 \\ f' < 0 \\ f' \ne 0 \\ f' \equiv 0 \end{Bmatrix} \text{ everywhere on } (a,b), \text{ then } f \text{ is } \begin{Bmatrix} \text{increasing} \\ \text{strictly increasing} \\ \text{decreasing} \\ \text{strictly decreasing} \\ \text{injective} \\ \text{constant} \end{Bmatrix} \text{ respectively}$$

Proof. If $x,y\in(a,b)$, $x<y$, then by mean value theorem, $\exists\, x_0 \in (x,y)$ such that

$$f(y)-f(x) = f'(x_0)\underbrace{(y-x)}_{>0} \begin{Bmatrix} \ge 0 \\ >0 \\ \le 0 \\ <0 \\ \ne 0 \\ =0 \end{Bmatrix}. \therefore \begin{Bmatrix} f(x)\le f(y) \\ f(x)<f(y) \\ f(x)\ge f(y) \\ f(x)>f(y) \\ f(x)\ne f(y) \\ f(x)=f(y) \end{Bmatrix}.$$

Remarks For differentiable function f,

$$\text{if } f \text{ is } \begin{Bmatrix} \text{strictly increasing} \\ \text{strictly decreasing} \\ \text{injective} \end{Bmatrix}, \text{ then } \begin{Bmatrix} f'>0 \\ f'<0 \\ f'\ne 0 \end{Bmatrix} \text{ everywhere}$$

may be false!

Examples ① $f(x)=x^3$ is strictly increasing and injective, but $f'(0)=0$. ② $f(x)=-x^3$ is strictly decreasing, but $f'(0)=0$.

For differentiable function $f:(a,b)\to\mathbb{R}$,

$$\text{if } f \text{ is } \begin{Bmatrix} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{Bmatrix}, \text{ then } \begin{Bmatrix} f'\ge 0 \\ f'\le 0 \\ f'=0 \end{Bmatrix} \text{ everywhere on } (a,b)$$

is true.

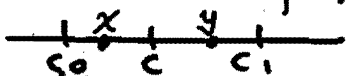
Proof. For $x, x_0 \in (a,b)$,

$$f \text{ is } \begin{Bmatrix} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{Bmatrix} \Rightarrow \frac{f(x)-f(x_0)}{x-x_0} \begin{Bmatrix} \ge 0 \\ \le 0 \\ =0 \end{Bmatrix}$$

$$\Rightarrow f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \begin{Bmatrix} \ge 0 \\ \le 0 \\ =0 \end{Bmatrix}.$$

Local Tracing Theorem

If $f:[a,b]\to\mathbb{R}$ is continuous and $f'(c)>0$ for some $c\in[a,b]$, then $\exists\, c_0, c_1\in\mathbb{R}$ such that $c_0<c<c_1$ and $f(x)<f(c)<f(y)$ $\forall\, x,y\in[a,b]$ and $c_0<x<c$, $c<y<c_1$.



A similar result for the case $f'(c)<0$ is true and the inequality becomes $f(x)>f(c)>f(y)$.

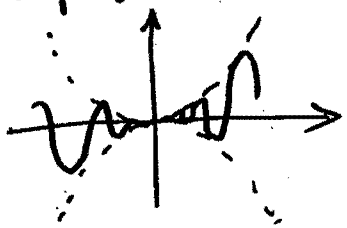
Proof. Let $f'(c)>0$. Assume there is no such c_0. Then $\forall\, n=1,2,3,\cdots$, $\exists\, x_n\in[a,b]$ and $c-\frac{1}{n}<x_n<c$ satisfying $f(x_n)\ge f(c)$. This will lead to

$$f'(c)=\lim_{n\to\infty}\frac{\overbrace{f(x_n)-f(c)}^{\ge 0}}{\underbrace{x_n-c}_{<0}}\le 0,\ \text{contradiction.}$$

The other parts are similar.

Remarks If we only know $f'(c)\ge 0$, we do not have a similar result. For example, let

$$f(x)=\begin{cases}x^2\sin\frac{1}{x} & \text{if } x\ne 0\\ 0 & \text{if } x=0.\end{cases}$$



We have $f'(0)=0$, but on every open interval $(c_0,0)$ or $(0,c_1)$, $f(x)$ takes both positive and negative values.

Generalized Mean-Value Theorem

If f,g are continuous on $[a,b]$ and are differentiable on (a,b), then $\exists\, x_0\in(a,b)$ such that

$$g'(x_0)\big(f(b)-f(a)\big)=f'(x_0)\big(g(b)-g(a)\big).\quad (*)$$

Proof. Define $F(x)=g(x)\big(f(b)-f(a)\big)-f(x)\big(g(b)-g(a)\big)$. Then $F(a)=g(a)f(b)-f(a)g(b)=F(b)$. By Rolle's Theorem, $\exists\, x_0\in(a,b)$ such that $F'(x_0)=0$. So we get $(*)$.

Remark If $g(b)\ne g(a)$, then $(*)$ can be put as

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(x_0)}{g'(x_0)}.$$

($\frac{0}{0}$ form of L'Hôpital's Rule)

1. Let f,g be differentiable on (a,b)
2. $g(x),\ g'(x)\ne 0$ $\forall x\in(a,b)$
3. $\lim_{x\to a^+}f(x)=0=\lim_{x\to a^+}g(x)$
4. $\lim_{x\to a^+}\frac{f'(x)}{g'(x)}=L$, where $L\in\mathbb{R}$ or $L=-\infty$ or $L=+\infty$.

Then $\lim_{x\to a^+}\frac{f(x)}{g(x)}=\lim_{x\to a^+}\frac{f'(x)}{g'(x)}$.

$\boxed{\text{The case } x\to b^- \text{ is similar.}}$

Proof. Define $f(a)=0$ and $g(a)=0$. $\forall x\in(a,b)$, f,g are continuous on $[a,x]$ and differentiable on (a,x). By generalized mean value theorem, $\exists\, x_0\in(a,x)$ such that

$$\frac{f(x)}{g(x)}=\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(x_0)}{g'(x_0)}.$$

As $x\to a^+$, $x_0\to a^+$. $\therefore \frac{f(x)}{g(x)}=\frac{f'(x_0)}{g'(x_0)}\to L$.

($\frac{*}{\infty}$ form of L'Hôpital's Rule)

① Let f, g be differentiable on (a,b)

② $g(x),\ g'(x) \neq 0 \quad \forall x \in (a,b)$

③ $\lim_{x\to a^+} g(x) = \infty$ ← No need $\lim_{x\to a^+} f(x)$ exists!

④ $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, where $L \in \mathbb{R}$ or $L=-\infty$ or $L=+\infty$

Then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$.

The case $x \to b^-$ is similar.

Proof. We do the case $L \in \mathbb{R}$ first. By ④, $\exists$ interval $I=(a, a+\delta_0)$ such that $t \in I \Rightarrow \left|\frac{f'(t)}{g'(t)} - L\right| < \varepsilon/2$.
Let $y \in I$. $\forall x \in I$, by generalized mean-value theorem, $\exists t \in I$ such that $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)}$.

Multiply by $\frac{g(x)-g(y)}{g(x)}$, add $\frac{f(y)}{g(x)}$, then subtract $\frac{f'(t)}{g'(t)}$ on both sides. Get $\frac{f(x)}{g(x)} - \frac{f'(t)}{g'(t)} = -\frac{g(y)}{g(x)}\frac{f'(t)}{g'(t)} + \frac{f(y)}{g(x)}$.

So $\left|\frac{f(x)}{g(x)} - \frac{f'(t)}{g'(t)}\right| \le \left|\frac{g(y)}{g(x)}\right|\left(|L|+\frac{\varepsilon}{2}\right) + \left|\frac{f(y)}{g(x)}\right|$. ← consider $x \to a^+$

By ③, the right side has limit 0. So $\exists$ interval $J=(a, a+\delta_1)$ so that $\forall x \in J$, the right side is at most $\varepsilon/2$.
Then $\forall x \in I \cap J = (a, a+\min\{\delta_0, \delta_1\})$

$$\left|\frac{f(x)}{g(x)} - L\right| \le \left|\frac{f(x)}{g(x)} - \frac{f'(t)}{g'(t)}\right| + \left|\frac{f'(t)}{g'(t)} - L\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The cases $L = \pm\infty$ follow by making simple modifications.

Examples ① Let $f(x) = x^2 \sin\frac{1}{x}$ and $g(x) = \sin x$ on $(0, \frac{\pi}{2})$.

Since $\underbrace{-x^2}_{\to 0} \le x^2 \sin\frac{1}{x} \le \underbrace{x^2}_{\to 0}$, so $\lim_{x\to 0^+} f(x) = 0$. $\lim_{x\to 0^+} g(x) = 0$.

$\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{2x\sin\frac{1}{x} - \cos\frac{1}{x}}{\underbrace{\cos x}_{\to 1}}$ doesn't exist as $\lim_{x\to 0^+} \cos\frac{1}{x}$ does not exist.

$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{x}{\sin x}\left(x \sin\frac{1}{x}\right) = 1 \cdot 0 = 0 \neq \lim_{x\to 0^+} \frac{f'(x)}{g'(x)}$.

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② $\forall r \in \mathbb{R}$, $\lim_{x\to +\infty} \frac{x^r}{e^x} = 0$. (To see this, choose $n > |r|$. Then $x^r \le x^n$ on $[1,\infty)$. So $0 \le \frac{x^r}{e^x} \le \frac{x^n}{e^x}$ on $[1,\infty)$. Since $\frac{d^n}{dx^n} x^n = n!$ and $\lim_{x\to+\infty} \frac{n!}{e^x} = 0$, applying L'Hopital's rule n-times, we see $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$. $\therefore \lim_{x\to+\infty} \frac{x^r}{e^x} = 0$.)

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③ Let $f: (a, +\infty) \to \mathbb{R}$ be differentiable. Then

$$\lim_{x\to+\infty} (f'(x) + f(x)) = 0 \Rightarrow \lim_{x\to+\infty} f(x) = 0 = \lim_{x\to+\infty} f'(x).$$

(To see this, we apply ($\frac{*}{\infty}$)-form of L'Hôpital's rule as follow:

$$\lim_{x\to\infty} f(x) = \lim_{x\to+\infty} \frac{f(x)e^x}{\underbrace{e^x}_{\to\infty}} = \lim_{x\to\infty} \frac{f'(x)e^x + f(x)e^x}{e^x} = \lim_{x\to+\infty} f'(x) + f(x) = 0$$

and $\lim_{x\to+\infty} f'(x) = \lim_{x\to+\infty}\left(\left(f'(x)+f(x)\right) - f(x)\right) = 0 - 0 = 0$.)

Remarks In O.D.E., if $\lim_{x\to+\infty} g(x) = 0$, then every solution $y=f(x)$ of $\frac{dy}{dx} + y = g(x)$ satisfies $\lim_{x\to\infty} f(x) = 0$ by the reason above.

④ Let $f(x)=2x+\sin x$ and $g(x)=2x-\sin x$ on $(-\infty,+\infty)$.
As $x\to+\infty$, $f(x), g(x)\to+\infty$.

$\lim\limits_{x\to+\infty}\frac{f'(x)}{g'(x)} = \lim\limits_{x\to+\infty}\frac{2+\cos x}{2-\cos x}$ doesn't exist

↳ $\begin{cases} x=2n\pi & \text{limit}=3 \\ x=(2n+1)\pi & \text{limit}=1/3 \end{cases}$

but $\lim\limits_{x\to+\infty}\frac{f(x)}{g(x)} = \lim\limits_{x\to+\infty}\frac{2x+\sin x}{2x-\sin x} = \lim\limits_{x\to+\infty}\frac{2+\frac{\sin x}{x}}{2-\frac{\sin x}{x}} = \frac{2}{2}=1$

Taylor's Theorem Let $f:(a,b)\to\mathbb{R}$ be n-times differentiable. $\forall x,c\in(a,b)$, $\exists x_0$ between x and c such that

$$f(x)=\boxed{f(c)+\frac{f'(c)}{1!}(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n-1)}(c)}{(n-1)!}(x-c)^{n-1}} + \underbrace{\frac{f^{(n)}(x_0)}{n!}(x-c)^n}_{R_n(x)}.$$

(n^{th} Taylor expansion of f about c) $R_n(x)$ Lagrange form of the remainder.

Proof. Let I be the closed interval with x and c as endpoints. For $t\in I$, define $\underline{g(t)=(n-1)!\sum_{k=0}^{n-1}\frac{f^{(k)}(t)}{k!}(x-t)^k}$, where $f^{(0)}=f$ and define $p(t)=-\frac{(x-t)^n}{n}$. We have

$$g'(t)=f^{(n)}(t)(x-t)^{n-1} \quad \text{and} \quad p'(t)=(x-t)^{n-1}.$$

By generalized mean value theorem, $\exists x_0$ between x and c such that

$$\underbrace{g'(x_0)}_{f^{(n)}(x_0)(x-x_0)^{n-1}}\underbrace{(p(x)-p(c))}_{(x-c)^n/n} = \underbrace{p'(x_0)}_{(x-x_0)^{n-1}}(\underbrace{g(x)}_{(n-1)!f(x)}-g(c))$$

$$\Rightarrow f(x)=\frac{g(c)}{(n-1)!}+\frac{f^{(n)}(x_0)}{n!}(x-c)^n=\sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k+\frac{f^{(n)}(x_0)}{n!}(x-c)^n.$$

Taylor Expansions of Common Functions at $c=0$

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+R_{n+1}(x)=\sum_{k=0}^{n}\frac{x^k}{k!}+R_{n+1}(x)$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+R_{2n+2}(x)$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+R_{2n+3}(x)$$

$$(1+x)^a=1+\sum_{k=1}^{n}\underbrace{\frac{a(a-1)\cdots(a-k+1)}{k!}}_{=\binom{a}{k}=C_a^k}x^k+R_{n+1}(x)$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+\frac{(-1)^n x^n}{n}+R_{n+1}(x)$$

$$\operatorname{Arctan} x=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+R_{2n+3}(x)$$

$$\operatorname{Arcsin} x=x+\sum_{k=1}^{n}\underbrace{\frac{1\cdot3\cdot5\cdots(2k-1)}{2\cdot4\cdot6\cdots(2k)}}_{=\frac{(2k-1)!!}{(2k)!!}}\frac{x^{2k+1}}{2k+1}+R_{2n+3}(x)$$

Notation:

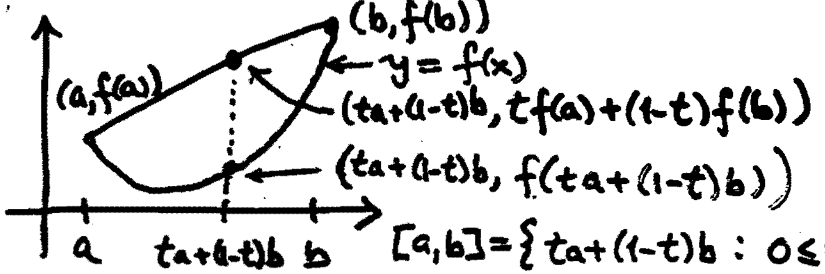
$$m!!=\begin{cases}1\cdot3\cdot5\cdots m & \text{if } m \text{ is odd}\\ 2\cdot4\cdot6\cdots m & \text{if } m \text{ is even}\end{cases}$$

n^{th} Taylor expansion is also called n^{th} Taylor polynomial.

If we let $n\to\infty$, the n^{th} Taylor expansion goes to $\sum_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k$. This is called the Taylor series of $f(x)$ about c.

$f^{(0)}(c)=f(c)$ $(x-c)^0=1$ even if $x=c$.

Appendix 1: Convex and Concave Functions



$[a,b]=\{ta+(1-t)b : 0\le t\le 1\}$

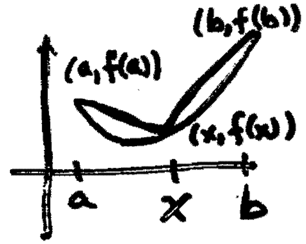
Definitions ① Let I be an interval and $f: I\to\mathbb{R}$. We say f is a convex function on I iff
$\forall a,b\in I,\ 0\le t\le 1,\ f(ta+(1-t)b)\le tf(a)+(1-t)f(b)$.

② f is a concave function on I iff
$\forall a,b\in I,\ 0\le t\le 1,\ f(ta+(1-t)b)\ge tf(a)+(1-t)f(b)$

Remarks ① A function is convex on I iff every chord joining $(a,f(a))$ and $(b,f(b))$ with $a,b\in I$ is always above or on the curve $y=f(x)$. A function is concave on I iff every chord is below or on the curve.

② f is strictly convex iff $f(ta+(1-t)b)<tf(a)+(1-t)f(b)$ for $0<t<1$
Similarly for strictly concave.
Strictly convex functions are the ones whose chords are above the curve (except the endpoints are on the curve, of course). Similarly for strictly concave functions.

③ f is convex $\Leftrightarrow -f$ is concave.
f is strictly convex $\Leftrightarrow -f$ is strictly concave.



Theorem f is convex on I iff the slope of the chords always increase in the sense that
$\forall a,x,b\in I$,
$$a<x<b \Rightarrow \frac{f(x)-f(a)}{x-a}\le\frac{f(b)-f(x)}{b-x}.$$

Proof. Note $x=ta+(1-t)b$ for some $t\in[0,1]$ $\Leftrightarrow 0\le t=\frac{b-x}{b-a}\le 1$.
$$\frac{f(x)-f(a)}{x-a}\le\frac{f(b)-f(x)}{b-x}\Leftrightarrow f(x)\le\frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)$$
$$\Leftrightarrow f(ta+(1-t)b)\le tf(a)+(1-t)f(b).$$

Theorem For f differentiable on I, f is convex on I $\Leftrightarrow f'$ is increasing on I ($\Leftrightarrow f''\ge 0$ on I for f twice differentiable on I). ← from curve tracing theorem.

Proof ($\Rightarrow$) $\forall a,b\in I$ with $a<b$, by last theorem,
$$f'(a)=\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}\le\lim_{x\to a^+}\frac{f(b)-f(x)}{b-x}=\frac{f(b)-f(a)}{b-a}$$
$$=\lim_{x\to b^-}\frac{f(x)-f(a)}{x-a}\le\lim_{x\to b^-}\frac{f(b)-f(x)}{b-x}=f'(b).$$
($\Leftarrow$) $\forall a,x,b\in I$ with $a<x<b$, by the mean-value theorem, $\exists r,s$ such that $a<r<x<s<b$ and
$$\frac{f(x)-f(a)}{x-a}=f'(r)\le f'(s)=\frac{f(b)-f(x)}{b-x}.$$
By last theorem, f is convex on I.

Theorem If f is convex on (a,b), then f is continuous on (a,b).

Proof. $\forall x_0 \in (a,b)$, consider $u, v, w \in (a,b)$ such that $u < x_0 < v < w$. Then

$$\frac{f(x_0)-f(u)}{x_0-u} \le \frac{f(v)-f(x_0)}{v-x_0} \le \frac{f(w)-f(v)}{w-v}.$$

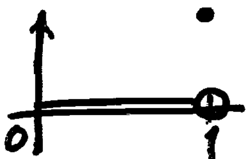
Solving for $f(v)$, we get

$$\frac{f(x_0)-f(u)}{x_0-u}(v-x_0)+f(x_0) \le f(v) \le \frac{f(w)-f(v)}{w-v}(v-x_0)+f(x_0).$$

Take limit as $v \to x_0^+$, we get $f(x_0) \le f(x_0+) \le f(x_0)$.
So $f(x_0+) = f(x_0)$. Similarly, $f(x_0-) = f(x_0)$ by taking $u < v < x_0 < w$. Therefore, f is continuous on (a,b).

Remark and Example The above theorem may not be true for $[a,b]$. For example,

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$



is convex on $[0,1]$ by checking the definition or checking the slope of chords is increasing. However, f is not continuous at 1.

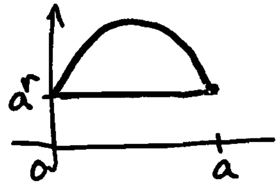
Example Prove that if $a, b \ge 0$ and $0 < r < 1$, then $|a^r - b^r| \le |a-b|^r$.

In particular, $|\sqrt[n]{a} - \sqrt[n]{b}| \le \sqrt[n]{|a-b|}$ $\quad (*)$
for $n = 2, 3, 4, \ldots$.

Solution. We may assume $a \ge b$, otherwise interchange them.
Define $f : [0,a] \to \mathbb{R}$ by $f(x) = x^r + (a-x)^r$.
Since $r - 1 < 0$, so

$$f''(x) = r(r-1)(x^{r-2} + (a-x)^{r-2}) \le 0.$$

So f is concave on $[0,a]$.
Since $f(0) = a^r = f(a)$, we get



$$f(x) = x^r + (a-x)^r \ge a^r \quad \forall x \in [0,a].$$

If $b \in [0,a]$ (i.e. $0 \le b \le a$), then

$$f(b) = b^r + (a-b)^r \ge a^r \Rightarrow |a^r - b^r| = a^r - b^r \le (a-b)^r = |a-b|^r.$$

Remark $(*)$ is the case $r = \frac{1}{n}$ for $n = 2, 3, 4, \ldots$
$(*)$ is useful in some exercises.