MATH 2033 HW4

1(a) Check f(x) differentiable at $x=0$,

$$\lim_{h\to 0} \frac{h^n \sin\frac{1}{h}}{h} = f'(0) = 0$$

At $n=1$, $\lim_{h\to 0} \frac{h\sin\frac{1}{h}}{h} = \lim_{h\to 0} \sin\frac{1}{h} = \infty$ unbounded

At $n=2$, $\lim_{h\to 0} \frac{h^2\sin\frac{1}{h}}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0 = f'(0)$

Hence, f(x) is differentiable at $x=0$ for $n>1$

(b) Take derivative of f(x), we have

$$f'(x) = \begin{cases} nx^{n-1}\sin\frac{1}{x} + x^n\cos\frac{1}{x}\left(-\frac{1}{x^2}\right) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

Then,

$$\lim_{h\to 0} nh^{n-1}\sin\frac{1}{h} - h^{n-2}\cos\frac{1}{h} = f'(0) = 0$$

With same idea in 1a), ~~[illegible]~~

At $n=2$, $\lim_{h\to 0} nh\sin\frac{1}{h} - \cos\frac{1}{h} = \infty$, unbounded

Hence, f(x) is continuous differentiable at $x=0$ for $n>2$.

2a) $f'(x_0) = \lim_{x \to x_0^+} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0^-} \frac{f(x_0) - f(x_0-h)}{h} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$

We have, $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0-h)}{2h} = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0) - f(x_0-h) + f(x_0)}{2h}$

$= \frac{1}{2} \lim_{h \to 0} \left(\frac{f(x_0+h) - f(x_0)}{h} + \frac{f(x_0) - f(x_0-h)}{h} \right) = \frac{1}{2} \lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} + \frac{1}{2} \lim_{h \to 0^-} \frac{f(x_0) - f(x_0-h)}{h}$

$= \frac{1}{2} \left(f'(x_0) + f'(x_0) \right) = f'(x_0)$

2b) No, it is not necessary.

Let $f(x) = |x|$ and $x_0 = 0$,

$$\lim_{h \to 0} \frac{f(x_0+h) - f(x_0-h)}{2h} = \lim_{h \to 0} \frac{f(h)-f(-h)}{2h} = \lim_{h \to 0} \frac{|h| - |-h|}{2h} = 0$$

However, $\lim_{h \to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = 1$

$\lim_{h \to 0^-} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = -1$, $f(x)$ is not differentiable at $x = x_0$

3) let $(a,b) \subset [0,1]$, then $f(x)$ differentiable on (a,b)

By Mean-value thm, $f'(x) = \dfrac{f(b)-f(a)}{b-a}$

Since $b-a \neq 0$, $f'(x)(b-a) = f(b) - f(a)$

$\Rightarrow |f'(x)||b-a| = |f(b) - f(a)| < M \cdot |b-a|$

We know $\{\frac{1}{n}\}$ is a cauchy's sequence, let $\forall \varepsilon > 0$, take $k = [\frac{2}{\varepsilon}] + 1$

$|\frac{1}{m} - \frac{1}{n}| \leq |\frac{1}{m}| + |\frac{1}{n}| < \frac{1}{k} + \frac{1}{k} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, $\forall m,n \geq k$

With $a_n = f(\frac{1}{n})$, $|a_m - a_n| = |f(\frac{1}{m}) - f(\frac{1}{n})| < M|\frac{1}{m} - \frac{1}{n}|$

$\Rightarrow |a_m - a_n| < M|\frac{1}{m} - \frac{1}{n}| < M \cdot \varepsilon < M \cdot \frac{\varepsilon}{M} = \varepsilon$, for $\forall m,n \geq k$.

Hence $\{a_n\}$ is cauchy's sequence and $\{a_n\}$ converges.

4) As $f(x)$ is differentiable and $n+1$ point $\in [a,b]$ intersect with $y=0$

By local extrema thm, $\exists\, x_0$, $f'(x_0)=0$

And we have $n+1$ distinct root to obtain n local extrema $\in (a,b)$

Therefore, we can concluded that $f^{(n)}(c)=0$, $c \in (a,b)$ exist

5) let $f(x) = 1-x+\frac{x^2}{2}$, $g(x) = e^{-x}$, $h(x) = 1-x$

$f(x)-g(x) = 1-x+\frac{x^2}{2}-e^{-x}$, $\frac{d}{dx}(f(x)-g(x)) = -1+x+e^{-x} > 0, x>0$

$f(x)-g(x)$ is continuous for $x \geq 0$

Then, $f(x)-g(x)$ is strictly increasing function $x \in [0,\infty)$

$f(0)-g(0)=0$, $f(x)-g(x)>0$, where $x>0$

$\Rightarrow f(x) > g(x)$, $1-x^2+\frac{x^2}{2} > e^{-x}$

$g(x)-h(x) = e^{-x}-1+x$, $g(x)-h(x)$ is continuous for $x \geq 0$

$\frac{d}{dx}(g(x)-h(x)) = -e^{-x}+1 > 0$, for $x>0$.

Then $g(x)-h(x)$ is strictly increasing function $x \in [0,\infty)$

$g(0)-h(0) = e^0-1+0 = 0$, $g(x)-h(x)>0$, where $x>0$

$\Rightarrow g(x) > h(x)$, $e^{-x} > 1-x$

Hence. $1-x+\frac{x^2}{2} > e^{-x} > 1-x$

b) Using Taylor thm, $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0+t(x-x_0))(x-x_0)^2$

where $x, t \in [0,1]$

$$f(0) = 0 = f(x_0) + f'(x_0)(-x_0) + \frac{1}{2} f''(x_0+t(-x_0))(-x_0)^2$$

$$f(1) = 0 = f(x_0) + f'(x_0)(1-x_0) + \frac{1}{2} f''(x_0+t(1-x_0))(1-x_0)^2$$

$$f(1) - f(0) = f'(x_0) + \frac{1}{2} f''(x_0+t(1-x_0))(1-x_0)^2 - \frac{1}{2} f''(x_0+t(-x_0))(x_0)^2 = 0$$

$$\Rightarrow |f'(x_0)| \leq \frac{1}{2}\left[\left|f''(x_0+t(-x_0))(x_0)^2\right| + \left|f''(x_0+t(1-x_0))(1-x_0)^2\right|\right]$$

Take $x_0 = \frac{1}{2}$,

$$|f'(\tfrac{1}{2})| \leq \frac{1}{2}\left|f''(\tfrac{1-t}{2})(\tfrac{1}{4})\right| + \frac{1}{2}\left|f''(\tfrac{1+t}{2})(\tfrac{1}{4})\right| = \frac{1}{8}\left(\left|f''(\tfrac{1-t}{2})\right| + \left|f''(\tfrac{1+t}{2})\right|\right)$$

Since $\frac{1-t}{2}$ and $\frac{1+t}{2} \in [0,1]$, $|f''(\frac{1-t}{2})| \leq A$, $|f''(\frac{1+t}{2})| \leq A$

$$|f'(\tfrac{1}{2})| \leq \frac{1}{8} \cdot 2A = \frac{A}{4}.$$