Lecture 22   30-04-2019

Review:

1: Approximate area under the graph of $y=f(x)$ from $x=a$ to $x=b$:

partition $P = \{ x_0 = a, x_1, \cdots x_n = b \}$

1.1 Darboux lower sum : $L(f, P) = \sum_{j=1}^{n} m_j \Delta x_j$

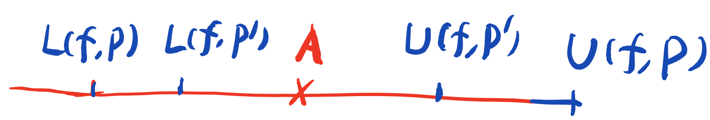
1.2 Darboux Upper sum : $U(f, P) = \sum_{j=1}^{n} M_j \Delta x_j$

1.3 Riemann Sum : $S = \sum_{j=1}^{n} f(t_j) \Delta x_j$

The true area A, if exists, is bounded by

$$L(f, P) \leq A \leq U(f, P)$$

2. Refinement theorem : $P \subseteq P' \Rightarrow$



3. Darboux's definition of integral (or area)

lower integral: $\underline{\int_a^b} f(x)\,dx = \sup \{ L(f,P) : P \text{ is a partition of } [a,b] \}$

upper integral: $\overline{\int_a^b} f(x)\,dx = \inf \{ U(f,P) : P \text{ is a partition of } [a,b] \}$

f is D-integrable iff $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$

[Compare with $\limsup_{n\to\infty} x_n$ (or $\overline{\lim_{n\to\infty}} x_n$), $\liminf_{n\to\infty} x_n$ (or $\underline{\lim_{n\to\infty}} x_n$).

$\lim x_n$ exists iff $\overline{\lim_{n\to\infty}} x_n = \underline{\lim_{n\to\infty}} x_n$]

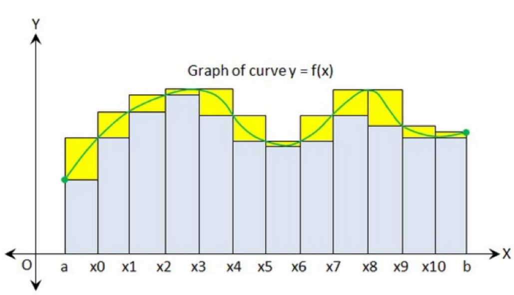
4. Integrable criterion : f is D-integrable iff

$\forall \varepsilon > 0,\ \exists P$ s.t $U(f,P) - L(f,P) < \varepsilon$

(Cauchy criterion for limit) $\forall \varepsilon > 0,\ \exists K \in \mathbb{N}$, s.t

$m,n \geq K \Rightarrow |x_n - x_m| < \varepsilon$

$\Leftrightarrow \quad 0 < \overline{\lim} x_n - \underline{\lim} x_n < \varepsilon$



The area of yellow region is :

$U(f,P) - L(f,P)$

5. Riemann's sum leads to Riemann's definition of Riemann integral. Later we will see Darboux integral = Riemann integral

Question: What functions are integrable?

We shall show that continuous functions are.

Uniform continuity

Recall $f: S \to \mathbb{R}$ is continuous at $y \in S$ iff

$\forall \varepsilon > 0$, $\exists \delta > 0$ (depending on ε and y)

such that $\forall x \in S$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Def: $f: S \to \mathbb{R}$ is uniformly continuous iff

$\forall \varepsilon > 0$, $\exists \delta > 0$ (depending on ε),

such that $\forall x, y \in S$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Remark: $f: S \to \mathbb{R}$ is uniformly continuous

$\Rightarrow$ f is continuous at every point of S, hence

continuous on S. However, the converse may be

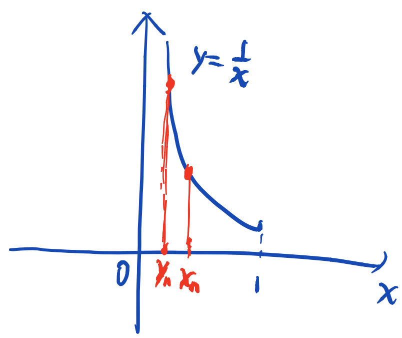
false.

Counter-example

Consider $f:(0,1)\to\mathbb{R}$, $f(x)=\frac{1}{x}$.

f is continuous on $(0,1)$. But not uniformly continuous on $(0,1)$

The reason is below :



Let $x_n=\frac{1}{n}$, $y_n=\frac{1}{2n}$, $n\geq 2$

then $|x_n-y_n|=\frac{1}{2n}$

$$\left|f(x_n)-f(y_n)\right| = \left|\frac{1}{x_n}-\frac{1}{y_n}\right| = 2n \geq 1$$

$\Rightarrow$ $\forall\ \delta>0$, $\exists\ k\in\mathbb{N}$ s.t $\frac{1}{k}<\delta$

$\Rightarrow$ $|x_k-y_k|=\frac{1}{2k}<\delta$, but $|f(x_k)-f(y_k)|\geq 1$.

Uniformly Continuity Theorem

THM: $f \in C[a,b]$, then f is uniformly continuous.

Pf: Step 1. By contradiction, assume f is not uniformly continuous,

i.e $\sim(\forall \varepsilon>0, \exists \delta>0, \forall x,y \in S, |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon)$

$= (\exists \varepsilon_0>0$, s.t $\forall \delta>0, \exists x,y \in S$ with $|x-y|<\delta$ and $|f(x)-f(y)| \geq \varepsilon_0)$

Let $\delta=1$, $\exists x_1, y_1 \in S$, $|x_1-y_1|<\delta=1$ and $|f(x_1)-f(y_1)| \geq \varepsilon_0$

let $\delta=\frac{1}{2}$, $\exists x_2, y_2 \in S$, $|x_2-y_2|<\delta=\frac{1}{2}$ and $|f(x_2)-f(y_2)| \geq \varepsilon_0$

$\vdots$

let $\delta=\frac{1}{n}$, $\exists x_n, y_n \in S$, $|x_n-y_n|<\delta=\frac{1}{n}$ and $|f(x_n)-f(y_n)| \geq \varepsilon_0$

Step 2. Since $\{x_n\} \subseteq [a,b]$, $\boxed{\text{by Bolzano-Weierstrass thm}}$,

$\exists$ subsequence $\{x_{n_j}\}$ s.t $x_{n_j} \to x_0$ in $[a,b]$

Since $y_{n_j} = x_{n_j} + |y_{n_j} - x_{n_j}|$ and $\lim_{j\to\infty} |y_{n_j} - x_{n_j}| = 0$

$y_{n_j} \to x_0$.

Step 3. Since f is continuous at x_0, and $\lim y_{n_j} = \lim x_{n_j} = x_0$

$$\lim_{j\to\infty} |f(x_{n_j}) - f(y_{n_j})| = |f(x_0) - f(x_0)| = 0$$

But $|f(x_{n_j}) - f(y_{n_j})| \geq \varepsilon_0$

by the limit inequality, $\lim_{j\to\infty} |f(x_{n_j}) - f(y_{n_j})| \geq \varepsilon_0$

So $0 \geq \varepsilon_0$ This contradicts to the condition that $\varepsilon_0 > 0$. #.

Remark 1. As a consequence, $y = \frac{1}{x}$ is uniformly continuous on $[a, b]$ for any $a > 0$.

Remark 2. Compare the above proof to the proof of sequential continuity theorem.

Continuity and Integrability

THM: f is continuous on $[a,b]$ $\Rightarrow$ f is Darboux integrable on $[a,b]$.

Pf: Use the integral criterion.

Step 1: $\forall \varepsilon > 0$, since f is continuous on $[a,b]$, hence uniformly continuous on $[a,b]$, therefore $\exists \delta > 0$, s.t $\forall x, y \in [a,b]$

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b-a}.$$

Step 2. Let P be a partition of $[a,b]$ s.t $\|P\| < \delta$

On each $[x_{j-1}, x_j]$, we show that $M_j - m_j < \frac{\varepsilon}{b-a}$.

Note that $\forall x, y$ on $[x_{j-1}, x_j]$, we have

$$|f(x)-f(y)| < \frac{\varepsilon}{b-a} \Rightarrow f(x) < f(y) + \frac{\varepsilon}{b-a} \quad \forall x, y \in [x_{j-1}, x_j]$$

Fix $x \in [x_{j-1}, x_j]$, and let y vary in $[x_{j-1}, x_j]$, we see that

$f(x)$ is a lower bound for the set $\left\{ f(y) + \frac{\varepsilon}{b-a} : y \in [x_{j-1}, x_j] \right\}$

$$\Rightarrow \quad f(x) \le \inf \left\{ f(y) + \frac{\varepsilon}{b-a} : y \in [x_{j-1}, x_j] \right\} = m_j + \frac{\varepsilon}{b-a} \quad \cdots (*)$$

Note that $(\star)$ holds for all $x \in [x_{j-1}, x_j]$ $\Rightarrow$

$m_j + \frac{\varepsilon}{b-a}$ is an upper bound of the set $\{ f(x) : x \in [x_{j-1}, x_j] \}$

$$\Rightarrow \sup \{ f(x) : x \in [x_{j-1}, x_j] \} \leq m_j + \frac{\varepsilon}{b-a}$$

i.e $\quad M_j \leq m_j + \frac{\varepsilon}{b-a}$

$$\Rightarrow \quad M_j - m_j \leq \frac{\varepsilon}{b-a}$$

Step 3. $\quad U(f, P) - L(f, P) = \sum_{j=1}^{n} (M_j - m_j) \Delta x_j$

$$\leq \sum_{j=1}^{n} \frac{\varepsilon}{b-a} \Delta x_j = \frac{\varepsilon}{b-a} \sum_{j=1}^{n} \Delta x_j = \frac{\varepsilon}{b-a} \cdot (b-a)$$

$$= \varepsilon$$

By the integrable criterion, f is integrable.

Riemann's definition of integrability

Recall the Riemann sum $S = \sum_{j=1}^{n} f(t_j)\Delta x_j$, $t_j \in [x_{j-1}, x_j]$

Def: f is Riemann integrable on $[a,b]$ iff

$\lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = I$ in the sense that: $\forall \varepsilon > 0$, $\exists \delta > 0$

s.t $\forall$ partition $P = \{x_0 = a, x_1, \cdots x_n = b\}$ of $[a,b]$

with $\|P\| < \delta$, and $\forall t_j \in [x_{j-1}, x_j]$, $1 \le j \le n$, we

have $\left| \sum_{j=1}^{n} f(t_j)\Delta x_j - I \right| < \varepsilon$

Remark 1: Riemann's definition of integral $\int_a^b f(x)dx$ is a complicated limit when $\|P\| \to 0$. This limit is not a limit of a sequence nor a limit of a function.

Remark 2. The limit, if exists, is unique.

Riemann Integral = Darboux integrable

Darboux's thm : Let f be a bounded function on $[a, b]$, then the following are equivalent :

(1) : f is Darboux integrable and $\int_a^b f(x)dx = I$

(2) f is Riemann integrable and $\lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j) \Delta x_j = I$

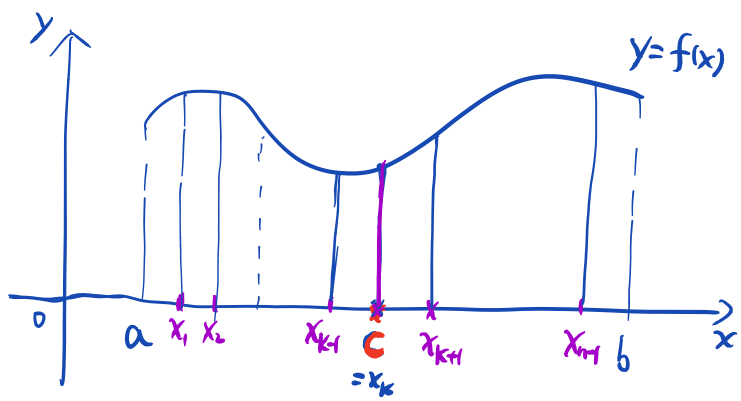
(3) $\forall \varepsilon > 0$, $\exists$ partition $P = \{x_0 = a, x_1, \cdots x_n = b\}$ of $[a, b]$

s.t

$\left| \sum_{j=1}^{n} f(t_j) \Delta x_j - I \right| < \varepsilon$ for all $t_j \in [x_{j-1}, x_j]$

Computational rule of integral

THM: Let $a<c<b$. If f is integrable on $[a,b]$, then f is integrable on $[a,c]$, $[c,b]$. and $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.



proof: Step 1. we show that f is integrable on $[a,c]$ $[c,b]$.

Indeed, f is integrable on $[a,b]$ $\Rightarrow$ $\forall \varepsilon>0$, $\exists$ $P=\{x_0=a, \cdots x_n=b\}$

s.t $U(f,P)-L(f,P) = \sum_{j=1}^{n}(M_j-m_j)\Delta x_j < \varepsilon$ $\cdots(\star)$

WLOG, we may assume that $c\in P$. otherwise we can consider

$P' = P \cup \{c\}$ instead, since by the refinement thm, ($\star$) holds for P'.

Let $c = x_k$. then $P_1 = \{x_0 = a, \dots x_k = c\}$

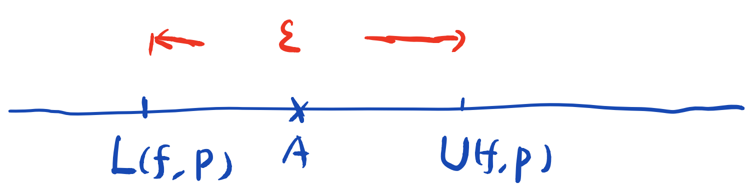
$P_2 = \{x_k = c, \dots x_n = b\}$ are partitions of $[a,c]$ and $[c, b]$ respectively.

$$\Rightarrow \quad U(f, P_1) - L(f, P_1) = \sum_{j=1}^{k} (M_j - m_j)\Delta x_j \le \sum_{j=1}^{n} (M_j - m_j)\Delta x_j = U(f,P) - L(f,P) < \varepsilon$$

$$U(f, P_2) - L(f, P_2) = \sum_{j=k+1}^{n} (M_j - m_j)\Delta x_j \le \sum_{j=1}^{n} (M_j - m_j)\Delta x_j < \varepsilon$$

$\Rightarrow$ f is integrable on $[a,c]$ and $[c,b]$.

Step 2. We show $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$.



Note that $0 \le \int_a^c f\,dx - L(f,P_1) \le U(f,P_1) - L(f,P_1) < \varepsilon$ — ①

$0 \le \int_c^b f\,dx - L(f,P_2) \le U(f,P_2) - L(f,P_2) < \varepsilon$ ②

$0 \le \int_a^b f\,dx - L(f,P) \le U(f,P) - L(f,P) < \varepsilon$

or $-\varepsilon < L(f,P) - \int_a^b f\,dx \le 0$ — — ③

① + ② + ③ $\Rightarrow -\varepsilon < \int_a^c f\,dx + \int_c^b f\,dx - \int_a^b f\,dx + L(f,P) - L(f,P_1) - L(f,P_2) < 2\varepsilon$

$\Rightarrow -\varepsilon < \int_a^c f\,dx + \int_c^b f\,dx - \int_a^b f\,dx < 2\varepsilon$ since $L(f,P) = L(f,P_1) + L(f,P_2)$.

Since $\varepsilon > 0$ is arbitary, by the infinetesimal principal,

$$\int_a^c f\,dx + \int_c^b f\,dx = \int_a^b f\,dx$$