Welcome to Math 2033 (Math. Analysis)

Main Items in the Syllabus

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Office Hour : Mondays 11:00 – 11:50
or by appointment

Prerequisite : Math 1014 or 1018 or 1020 or 1024

Website for Lecture Notes or Transparencies

https://www.math.ust.hk/~makyli/UG.html

Just scroll down to the Math 2033 part.

Grading System:

https://grading.math.ust.hk/checkgrade/

Grade Scheme :

Tutorial Presentations

Homeworks	5%	Total: 100%
Presentations	10%	
Midterm	30%	
Final Exam	55%	

At the end of the course, student must get at least 40% out of the total 100% of the grade scheme above to pass the course.

Homework : Make a copy, then submit the original. Homework solutions must be legible!

Tutorial Presentations:

1 to 3 students per group

0 for member who is absent

1 for member who tries with little success

2 for member who gets half way but not complete

3 for member who solves and presents completely.

TA will choose one presenter for each problem and one member to answer questions for that problem.

Every member (who is not absent) may get different mark.

What is Analysis?

Algebra	Equations
Geometry	Figures
Number Theory	Integers, Rational Numbers
Analysis	Limit, Continuity, Differentiation, Integration, ...

Number Theory vs Analysis

$$\frac{2^x - x^2}{4x^4+1} = 987654321$$

Number Theory: any integer solution?

Analysis: any real solution?

③

$$f(x) = \frac{2^x - x^2}{4x^4+1}$$ is continuous on $\mathbb{R}$

$f(0) = 1$

$2^{10} = 1024 > 10^3$

$$f(100) = \frac{2^{100} - 100^2}{4 \cdot 100^4 + 1} > \frac{10^{30} - 10^4}{10 \cdot 10^8} > 10^{20}$$

> 987654321

There is a real solution in $[0, 100]$.
We can find the solution to as many decimal place as we like by bisection method: Test $x_0 = 50$. Then
$x_1 = 25$, $x_2 = 12.5$, ...
or $x_1 = 75$, ... $\quad x = \lim_{n \to \infty} x_n$

$$2^x - x^2 = 987654321(4x^4+1)$$

If x is integer, then
x cannot be negative (otherwise $0 < 2^x < 1$)
$x \neq 0 \quad 1 \neq 987654321$
$x \neq 1 \quad 1 \neq 987654321(5)$

If $x \geq 2$, then $2^x = 4a$

$$x^2 = \begin{cases} (2b)^2 = 4b^2 & \text{if } x \text{ is even} \\ (2b+1)^2 = 4b^2+4b+1 & \text{if } x \text{ is odd} \end{cases}$$

$$2^x - x^2 = \begin{cases} 4a - 4b^2 = 4c \\ 4a - (4b^2+4b+1) = 4c - 1 \end{cases}$$

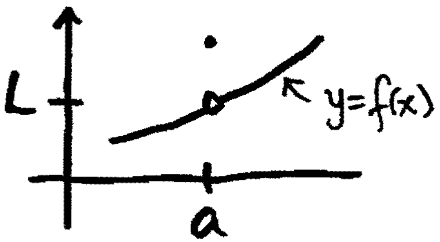
$$987654321(4x^4+1)$$
$$= (4d+1)(4x^4+1) = 4e+1$$

$4c - 1 \neq 4e + 1$ no solution

Analysis solves equations by using limit concepts.

Number theory solves equations by studying forms of numbers, not by approximations.

<u>Limit</u> $\lim_{x\to a} f(x) = L$



"As x gets close to a, $f(x)$ gets close to L."

"Close" is a feeling, no way to judge!

$$\left.\begin{array}{l} f_1(x) \to L_1 \\ f_2(x) \to L_2 \\ \vdots \\ f_{1000}(x) \to L_{1000} \end{array}\right\} \begin{array}{l} \text{Is} \\ f_1(x)+f_2(x)+\cdots+f_{1000}(x) \\ \text{"close" to} \\ L_1+L_2+\cdots+L_{1000}\ ? \end{array}$$

Although we learned this as a fact in calculus, one can challenge this fact as follow:

If $f_1(x), f_2(x), \cdots, f_{1000}(x)$ are 49.9 and
$L_1, L_2, \cdots, L_{1000}$ are 50,
then $f_i(x)$ may be considered close to L_i,
but $f_1(x)+\cdots+f_{1000}(x)$ is 49900 and
$L_1+\cdots+L_{1000}$ is 50000,
which are 100 units apart, not that close.

If $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$,

then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$

"L'Hopital's Rule"

Is this correct?

Let $f(x) = x^2 \sin\frac{1}{x}$ and $g(x) = \sin x$

$$\left|x^2 \sin\tfrac{1}{x}\right| \le |x|^2 \to 0 \quad \text{as } x \to 0$$

$$\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x).$$

Apply formula above:

$$\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin x} = \lim_{x\to 0} \frac{\overbrace{2x\sin\frac{1}{x}}^{\to 0} - \overbrace{\cos\frac{1}{x}}^{\text{"oscillate"}}}{\underbrace{\cos x}_{\to 1}}$$

limit doesn't exist!

How about

$$\frac{x^2\sin\frac{1}{x}}{\sin x} = \underbrace{\left(\frac{x}{\sin x}\right)}_{\downarrow \atop 1}\underbrace{\left(x\sin\frac{1}{x}\right)}_{\downarrow \atop 0} \to 0 \quad \text{as } x\to 0$$

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x} = 0 \ ?$$ This is correct!

When is $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$?

$\displaystyle\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ is not enough.

What additional conditions do we need?

Why those conditions are enough?

Need to do proofs!!!

The correct l'Hopital's rule requires the additional condition $\displaystyle\lim_{x\to a}\frac{f'(x)}{g'(x)}$ = number or $+\infty$ or $-\infty$. This is not stated in some secondary school textbooks. So don't just accept what books or teachers tell you. Look at a proof to decide.

Question: Let f be a continuous function on $\mathbb{R}$. Must f be differentiable at every x in $\mathbb{R}$?

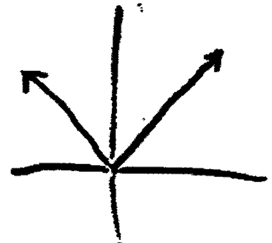
Answer: No, $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

is continuous on $\mathbb{R}$. However,

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h-0}{h} = 1$$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{-h-0}{h} = -1$$

So $f'(0)$ doesn't exist.



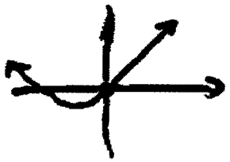
Question: What is the derivative of

$$f(x) = \begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x=0 \\ x^2+x & \text{if } x<0 \end{cases} \quad ?$$

Is it

$$f'(x) = \begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ 2x+1 & \text{if } x<0 \end{cases} \quad ?$$

No, $f'(0)=1$. Why?



$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h-0}{h} = 1$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{(h^2+h)-0}{h}$$

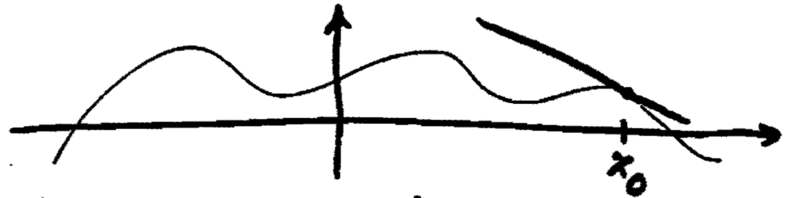
$$= \lim_{h\to 0} (h+1)$$

$$= 1.$$

$\therefore f'(0)=1.$

Question: Which one is true?

① Every continuous function $f:\mathbb{R}\to\mathbb{R}$ is differentiable at some x_0



② There exists a continuous function $f:\mathbb{R}\to\mathbb{R}$, which is not differentiable at every x_0.

② is true. So we cannot always differentiate a continuous function!

Why is ② true? Need to do proofs!!!

Logic = Rules for Reasoning

Chapter 1

Notations $\sim$ (or $\neg$) not, the opposite of

quantifiers
- $\forall$ for all, for any, for every
- $\exists$ there is (at least one), there exists, there are (some)

p, q variables of statements or phrases

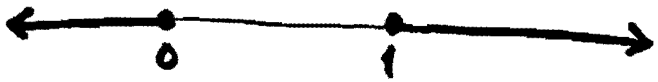
Negation = Taking opposite

① $\sim(\sim p) = p$

② $\sim(p \text{ and } q) = (\sim p) \text{ or } (\sim q)$

Example $\underbrace{x>0}_{p}$ and $\underbrace{x<1}_{q}$

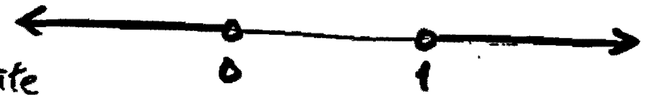
Opposite



$\underbrace{x \le 0}_{\sim p}$ or $\underbrace{x \ge 1}_{\sim q}$.

③ $\sim(p \text{ or } q) = (\sim p) \text{ and } (\sim q)$

Example $\underbrace{x<0}_{p}$ or $\underbrace{x>1}_{q}$



Opposite

$\underbrace{x \ge 0}_{\sim p}$ and $\underbrace{x \le 1}_{\sim q}$

④ $\sim(\forall x\ \exists y \cdots\ p) = \exists x\ \forall y \cdots \sim p$

Example 1 For every $x \ge 0$, x has a square root

True $\forall x \ge 0$ (x has a square root)

Opposite There exists $x \ge 0$ such that

False x does not have square root.

$\exists x \ge 0 \sim$(x has a square root).

Example 2 For every $x \geq 0$, there is $y \geq 0$ such that $y^2 = x$.

True

$$\forall x \geq 0 \quad \exists y \geq 0 \quad y^2 = x$$

Opposite There exists $x \geq 0$ such that for every $y \geq 0$, $y^2 \neq x$.

False

$$\exists x \geq 0 \quad \forall y \geq 0 \quad \sim(y^2 = x).$$

Conditional Statements (If-then statements)

$$\left.\begin{array}{l}\text{If } p, \text{ then } q\\ p \text{ implies } q\\ p \text{ only if } q\\ p \text{ is sufficient for } q\\ q \text{ is necessary for } p\end{array}\right\} \quad p \Rightarrow q$$

⑤ $\sim(p \Rightarrow q) = p \text{ and } (\sim q)$

Example If $x \geq 0$, then $|x| = x$

True

$$(x \geq 0) \Longrightarrow (|x| = x)$$

Opposite $x \geq 0$ and $|x| \neq x$

False $(x \geq 0)$ and $\sim(|x| = x)$

Remark

$$\begin{aligned} p \Rightarrow q &= \sim(\sim(p \Rightarrow q)) && \text{by ①}\\ &= \sim(p \text{ and } (\sim q)) && \text{by ⑤}\\ &= (\sim p) \text{ or } \sim(\sim q) && \text{by ②}\\ &= (\sim p) \text{ or } q && \text{by ①}\end{aligned}$$

Terminologies

For the statement "If p, then q" ($p \Rightarrow q$), its converse is "If q, then p" ($q \Rightarrow p$), its contrapositive is "If $\sim q$, then $\sim p$" $(\sim q) \Rightarrow (\sim p)$

Examples 1

Statement If $x=-3$, then $x^2=9$ (True)

Converse If $x^2=9$, then $x=-3$ (False, as x maybe 3)

Contrapositive If $x^2 \neq 9$, then $x \neq -3$ (True)

Example 2

Statement $(x=-3) \Rightarrow (2x=-6)$ (True)

Converse $(2x=-6) \Rightarrow (x=-3)$ (True)

Contrapositive $(2x \neq -6) \Rightarrow (x \neq -3)$ (True)

Example 3

Statement If $|x|=3$, then $x=-3$ (False, as x may be 3)

Converse If $x=-3$, then $|x|=3$ (True)

Contrapositive If $x \neq -3$, then $|x| \neq 3$ (False, as x may be 3)

Remarks ① Contrapositive = statement

$$\begin{aligned}(\sim q) \Rightarrow (\sim p) &= \sim(\sim q) \text{ or } (\sim p) && \text{by earlier remark} \\ &= q \text{ or } (\sim p) && \text{by ①} \\ &= (\sim p) \text{ or } q \\ &= p \Rightarrow q && \text{by earlier remark}\end{aligned}$$

② "If p, then q" and "If q, then p" are true

We will say "p if and only if q" or "p is necessary and sufficient for q".

Abbreviation if and only if = iff

③ $\forall \alpha \forall \beta = \forall \beta \forall \alpha$

$\exists \alpha \exists \beta = \exists \beta \exists \alpha$

$\forall \alpha \exists \beta \neq \exists \beta \forall \alpha$ → EXAMPLE

Every student is assigned a number

« $\forall$ student $\exists$ number (student is assigned number)

$\exists$ number $\forall$ student (Student is assigned number)

« There is a number such that every student is assigned the number.

EXERCISES Negate each of the following

1. If $\triangle ABC$ is a right triangle, then $a^2+b^2=c^2$.

2. $\forall \varepsilon>0\ \exists \delta>0$ such that

$$0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon$$

3. No news is good news. ← Ambiguous statement
More than one interpretations

Solutions

1. $\triangle ABC$ is a right triangle and $a^2+b^2\neq c^2$

2. $\exists \varepsilon>0\ \forall \delta>0$, we have

$$0<|x-x_0|<\delta \text{ and } |f(x)-L|\geq \varepsilon.$$

3. Interpretation I All news are bad news

Opposite: There exists a good news.

Interpretation II If no news is received, then it is good news.

Opposite: No news is received and it is not good news.

When ambiguous statement is presented, ask for the intended interpretation, then negate that interpretation.

Chapter 2 Sets ← Language to communicate math efficiently and precisely

A set is a collection of math "objects"
usually numbers, functions, ordered-pairs, ...

The objects in the set are the elements of the set.

We write $x \in S$ iff x is an element of set S.

$x \notin S$ iff x is not an element of set S.

Example Let $\mathbb{Z}$ be the set of all integers.
Then $-54 \in \mathbb{Z}$ and $\sqrt{2} \notin \mathbb{Z}$.

A set is finite iff it has finitely many elements.
A set is infinite iff it has infinitely many elements.
The empty set is the set having no element and is denoted by $\emptyset$.

Common Sets in Math

- $\mathbb{N}$ the set of all positive integers (natural number)
- $\mathbb{Z}$ the set of all integers ← $\mathbb{Z}$ is for "Zahlen" (German for "Number")
- $\mathbb{Q}$ the set of all rational numbers
- $\mathbb{R}$ the set of all real numbers
- $\mathbb{C}$ the set of all complex numbers

Set Descriptions

① List elements enclosed in braces

$S = \{1,2,3\}$, $\mathbb{N} = \{1,2,3,4,\cdots\}$, $\emptyset = \{\ \}$,

$\mathbb{Z} = \{\cdots,-4,-3,-2,-1,0,1,2,3,4,\cdots\}$

② Write the form of the elements, followed by a colon, followed by descriptions of variables inside braces.

$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$

$\mathbb{R} = \{x : x \text{ is a real number}\}$

$\mathbb{C} = \{x+iy : x \in \mathbb{R}, y \in \mathbb{R}, i = \sqrt{-1}\}$

$[a,b] = \{x : x \in \mathbb{R} \text{ and } a \leq x \leq b\}$

ℓ_m ← the line with equation $y = mx$

$= \{(x,y) : x, y \in \mathbb{R} \text{ and } y = mx\}$

$= \{(x,mx) : x \in \mathbb{R}\}$

Let A and B be sets.

A is a <u>subset</u> of B (or B <u>contains</u> A) iff every element of A is also an element of B. In this case, we write $A \subseteq B$.
In particular, $\emptyset \subseteq S$ for every set S.

We say $A=B$ iff $A \subseteq B$ and $B \subseteq A$

A is a <u>proper subset</u> of B iff $A \subseteq B$ and $A \neq B$. In this case, we write $A \subset B$.

<u>Example</u>. Let $A=\{1,2\}$, $B=\{1,2,3\}$ and $C=\{1,1,2,3\}$. Then $A \subset B = C$.

<u>Remarks</u> Repeated elements count only once.
B and C are 3 element sets.
A has 2 elements
$\{4,4,4,4,\ldots\}$ has 1 element only.
← a finite set!

If $X \subseteq Y$, then the number of elements of X is less than or equal to the number of elements of Y.

Let S be a set. The <u>power set</u> of S is the set of all subsets of S. It is denoted by $P(S)$ or 2^S.

<u>Examples</u>

$S=\emptyset$ Then $\emptyset \subseteq S$ $\quad P(S)=\{\emptyset\}$

$S=\{x\}$ Then $\emptyset, \{x\} \subseteq S$ $\quad P(S)=\{\emptyset,\{x\}\}$

$S=\{x,y\}$ Then $\emptyset, \{x\}, \{y\}, \{x,y\} \subseteq S$

$P(S)=\{\emptyset, \{x\}, \{y\}, \{x,y\}\}$

If S has n elements, then $P(S)$ has 2^n elements.

<u>Set Operations</u> Let $A, B, C, D, \ldots$ be sets.
Their <u>union</u> is

$A \cup B \cup C \cup D \cup \cdots = \{x : x$ is an element in at <u>least one</u> of the sets $A, B, C, D, \ldots\}$

<u>Examples</u> $\{p,q\} \cup \{r\} = \{p,q,r\}$

$\{x,y,z\} \cup \{v,w,x,y\} = \{v,w,x,y,z\}$

$\mathbb{R} \cup \mathbb{Q} = \mathbb{R} = \mathbb{Q} \cup \mathbb{R}$, $\mathbb{N} \cup \mathbb{Z} \cup \mathbb{Q} = \mathbb{Q}$

$S \cup \emptyset = S = \emptyset \cup S$ for every set S.

The <u>intersection</u> of $A, B, C, D, \cdots$ is
$A \cap B \cap C \cap D \cap \cdots = \{x : x \text{ is an element in every one of the sets } A, B, C, D, \cdots\}$

<u>Examples</u> $\{p, q\} \cap \{r\} = \emptyset$

$\{x, y, z\} \cap \{v, w, x, y, z\} \cap \{u, v, w, x\} = \{x\}$

$\mathbb{R} \cap \mathbb{Q} \cap [0,1] = \{x : x \in \mathbb{Q} \text{ and } 0 \leq x \leq 1\}$

$S \cap \emptyset = \emptyset = \emptyset \cap S$ for every set S.

The <u>Cartesian product</u> of $A, B, C, D, \cdots$ is
$A \times B \times C \times D \times \cdots = \{(a, b, c, d, \ldots) : a \in A, b \in B, c \in C, d \in D, \cdots\}$

<u>Examples</u> $\mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\} = \mathbb{R}^2$

$\mathbb{N} \times \mathbb{Z} \times \{0, 1\} = \{(x, y, z) : x \in \mathbb{N}, y \in \mathbb{Z}, z = 0 \text{ or } 1\}$

$S \times \emptyset = \emptyset = \emptyset \times S$ for every set S.

If $A \neq B$, then $A \times B \neq B \times A$.

The <u>complement</u> of B in A is
$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

<u>Examples</u> $\mathbb{R} \setminus \mathbb{Q}$ is the set of all irrational numbers.

$\{x, y, z\} \setminus \{w, x\} = \{y, z\}$

$\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) = \{(u, v) : u \text{ rational}, v \text{ irrational}\}$

$S \setminus \emptyset = S$, $\emptyset \setminus S = \emptyset$ for every set S.

The sets $A, B, C, D, \cdots$ are <u>disjoint</u> iff their intersection is the empty set.

The sets $A, B, C, D, \cdots$ are <u>mutually disjoint</u> iff the intersection of <u>every two of the sets</u> is the empty set

<u>Example</u> Let $A = \{x, y\}$, $B = \{y, z\}$, $C = \{z, x\}$

Then A, B, C are disjoint because

$$A \cap B \cap C = \emptyset,$$

but A, B, C are <u>not</u> mutually disjoint because

$A \cap B = \{y\} \neq \emptyset$ for instance.

<u>Remark</u> Mutual disjoint $\Rightarrow$ disjoint

<u>Some Notations</u> n is a positive integer

$$S_1 \cup S_2 \cup \cdots \cup S_n = \bigcup_{k=1}^{n} S_k$$

$$S_1 \cap S_2 \cap \cdots \cap S_n = \bigcap_{k=1}^{n} S_k$$

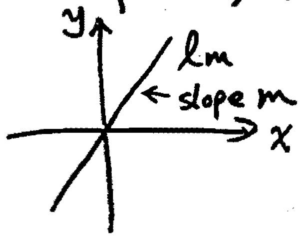
$$S_1 \times S_2 \times \cdots \times S_n = \mathop{\times}_{k=1}^{n} S_k$$

$$S_1 \cup S_2 \cup S_3 \cup \cdots = \bigcup_{k=1}^{\infty} S_k \quad \text{better notation } \left(\bigcup_{k \in \mathbb{N}} S_k \right)$$

not too good, no S_∞ set!

(Similarly for intersection and Cartesian product)

<u>Example</u> If $m \in \mathbb{R}$, let ℓ_m be the line $y = mx$ in the plane, then ℓ_m is not a vertical line



$$\bigcup_{m \in \mathbb{R}} \ell_m = \mathbb{R}^2 \setminus \{(0,y) : y \neq 0\}$$

$$\bigcap_{m \in \mathbb{R}} \ell_m = \{(0,0)\}$$

<u>Examples on Proof Problems</u>

If ① $A \subseteq B$
and ② $C \subseteq D$,
} These are given conditions.

then prove $A \cap C \subseteq B \cap D$.

<u>Strategy</u> According to def. of $\subseteq$, we have to check "every $x \in A \cap C$ is also in $B \cap D$."

<u>Proof</u>:

$x \in A \cap C \iff x \in A$ and $x \in C$ (by def. of $\cap$)

$\Rightarrow x \in B$ and $x \in D$ (by ①, ②, def. of $\subseteq$)

$\iff x \in B \cap D$ (by def. of $\cap$)

$\therefore A \cap C \subseteq B \cap D$ (by def. of $\subseteq$).

<u>Remarks</u> Since this is proved, you may use <u>it</u> (If $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$) to prove any other statement.

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Another Example

Prove $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.

Strategy: To get $=$, check $\subseteq$ and $\supseteq$.

Proof of $\subseteq$

$x \in (A \cup B) \setminus C \Leftrightarrow x \in A \cup B$ and $x \notin C$ (by def of $\setminus$)

$\Leftrightarrow \left\{ \begin{matrix} x \in A \\ \text{or} \\ x \in B \end{matrix} \right\}$ and $x \notin C$ (by def of $\cup$)

$$\left. \begin{matrix} \text{Case 1} & x \in A \text{ and } x \notin C \\ \text{or} & \\ \text{Case 2} & x \in B \text{ and } x \notin C \end{matrix} \right\} \Leftrightarrow \begin{matrix} x \in A \setminus C \\ \text{or} \\ x \in B \setminus C \end{matrix}$$

(by def of $\setminus$)

$\Leftrightarrow x \in (A \setminus C) \cup (B \setminus C)$ (by def of $\cup$)

$\therefore (A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$ (by def of $\subseteq$).

Proof of $\supseteq$

Reverse Steps!

$x \in (A \setminus C) \cup (B \setminus C)$

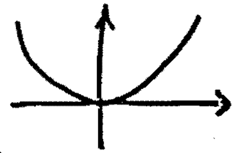
$\Leftrightarrow \begin{matrix} x \in A \setminus C \\ \text{or} \\ x \in B \setminus C \end{matrix}$ (by def. of $\cup$)

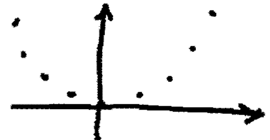
$\Leftrightarrow$ Case 1 or Case 2 above (by def of $\setminus$)

...

You fill in the rest of the details!!!

$f(x) = x^2 \quad x \in \mathbb{R}$ $\qquad$ $g(n) = n^2, \; n \in \mathbb{Z}$





Different functions

Definitions

A function (or map or mapping) f from a set A to a set B (denoted by $f: A \to B$) is a method of assigning to every $a \in A$ exactly one $b \in B$. This b (denoted by $f(a)$) is the value of f at a.

A function must be well-defined in the sense that if $a = a'$, then $f(a) = f(a')$.

A is the domain of f. $\quad A = \operatorname{dom} f$

B is the codomain of f. $\quad B = \operatorname{codom} f$

$f(A) = \{ y : y \in B \text{ and } y = f(x) \text{ for some } x \in A \}$ is the range (or image) of f.

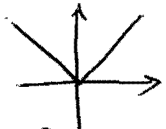
We may say f is a function on A
or f is a B-valued function

The set $\{ (x, f(x)) : x \in A \}$ is the graph of f.

Two functions are equal iff their graphs are the same.

Examples

① The absolute value function on $\mathbb{R}$ is $h:\mathbb{R}\to\mathbb{R}$ given by $h(x)=\begin{cases} x & \text{if } x\ge 0 \\ -x & \text{if } x<0. \end{cases}$



There may be more than one parts in the formula of the function.

In this example, the codomain can be any set containing $[0,\infty)$. The function will be the same because the graph is the same.

② The following attempt to define a function is not well-defined. Let $x_n=(-1)^n$ for all $n\in\mathbb{N}$. Define $f:\{x_1,x_2,\cdots\}\to\mathbb{R}$ by $f(x_n)=n$. It is not well-defined because

$$x_1=-1=x_3, \text{ but } f(x_1)=1\neq 3=f(x_3).$$

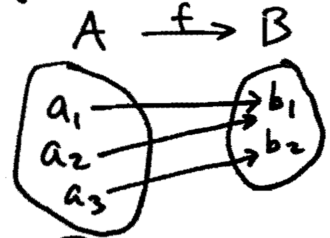
Types of Functions — Definitions

① The identity function on a set S is $I_S:S\to S$ given by $I_S(x)=x$ for all $x\in S$.

② Let $f:A\to B$, $g:B'\to C$ be functions and $f(A)\subseteq B'$. The composition of g by f is $g\circ f:A\to C$ given by $(g\circ f)(x)=g(f(x))$.

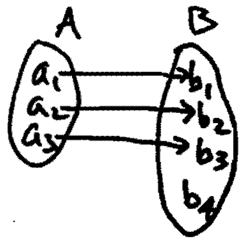
③ Let $f:A\to B$ be a function and $C\subseteq A$. The restriction of f to C is $f|_C:C\to B$ given by $f|_C(x)=f(x)$ for all $x\in C$.

④ $f:A\to B$ is surjective (or onto) iff $f(A)=B$.



The values of f in B may repeat, but no element of B will be omitted as a value. So A has at least as many elements as B.

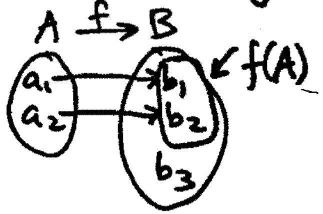
⑤ $f:A\to B$ is injective (or one-to-one) iff $f(x)=f(y)$ implies $x=y$.

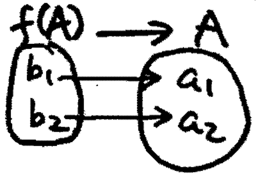


The values of f in B do not repeat, but some elements of B may be omitted as a value.

So B has at least as many elements as A.

⑥ For an injective function $f: A \to B$, the <u>inverse function of f</u> is $f^{-1}: f(A) \to A$ given by $f^{-1}(y) = x \iff f(x) = y$.





⑦ $f: A \to B$ is <u>bijective</u> (or <u>a one-to-one correspondence</u>) iff f is injective and surjective.

A and B have the same number of elements.

<u>Exercises</u>

(a) $f: A \to B$ is a bijection iff $\exists\, g: B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$.

(b) If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f: A \to C$ is a bijection.

(c) Let $A, B \subseteq \mathbb{R}$ and $f: A \to B$ be a function. If $\forall b \in B$, the <u>horizontal line $y = b$</u> <u>intersects</u> the <u>graph</u> of f <u>exactly once</u>, then f is a bijection.

<u>Equivalence Relation</u>

<u>Definition</u> A <u>relation</u> on a set S is any subset of $S \times S$.

A relation R on a set S is an <u>equivalence relation</u> iff ① $\forall x \in S$, $(x,x) \in R$ (Reflexive Prop.)

② $(x,y) \in R \Rightarrow (y,x) \in R$ (Symmetric Property)

③ $(x,y), (y,z) \in R \Rightarrow (x,z) \in R$ (Transitive Prop.)

<u>Notations</u> Write $x \sim y$ iff $(x,y) \in R$.

$\forall x \in S$, write $\underline{[x]} = \{y : x \sim y\}$

called "the equivalence class containing x."

<u>Facts</u> ① $\Rightarrow \forall x \in S$, $x \sim x \Rightarrow x \in [x]$

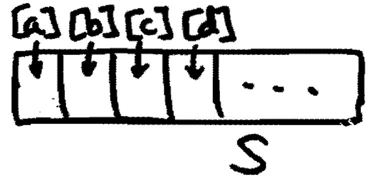
$\Rightarrow \bigcup_{x \in S} [x] = S$.

② $x \sim y \Rightarrow [x] = [y]$ because

$z \in [x] \iff x \sim z \iff y \sim z \iff z \in [y]$

③ $x \not\sim y \Rightarrow [x] \cap [y] = \phi$ (otherwise $z \in [x] \cap [y]$

$\Rightarrow [x] = [z] = [y] \Rightarrow x \sim y$ by ②).



Equivalence relation on S = Partition of S

Examples

$R=\{(T_1,T_2): T_1 \text{ is similar to } T_2\}$

① (Geometry) S = the set of all triangles.

$(T_1,T_2)\in R \Leftrightarrow T_1 \sim T_2 \iff T_1$ is similar to T_2. R is an equivalence relation.

$[T]$ = the set of all triangles similar to T.

$R=\{(m,n): m-n \text{ is even}\}$

② (Arithmetic) $S = \mathbb{Z}$. $m \sim n \iff m-n$ is even

$[0]$ = the set of all $m \in \mathbb{Z}$ such that $m-0$ is even
$= \{\cdots, -4, -2, 0, 2, 4, \cdots\}$

$[1]$ = the set of all $m \in \mathbb{Z}$ such that $m-1$ is even
$= \{\cdots, -3, -1, 1, 3, \cdots\}$

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① and ② are examples of equivalence relations.

③ Let $S=\{0,1\}$ and $R=\{(1,1)\}$. Then R satisfies symmetric and transitive properties, but not reflexive property because $0 \in S$ and $(0,0) \notin R$. $\therefore$ R is not an equivalence relation on S.

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④ For sets S_1 and S_2, define $R=\{(S_1,S_2): \exists \text{ bijection } f: S_1 \to S_2\}$

$S_1 \sim S_2 \iff \exists$ bijection $f: S_1 \to S_2$

This is an equivalence relation.

$S_1 \sim S_2$ ← say S_1 and S_2 have same cardinality

$[S] = \text{Card } S = |S|$ cardinal number of S

Notations Card $\{1,2,\cdots,n\} = n$ for positive integer n

Card $\{1,2,3,\ldots\}$ = Card $\mathbb{N} = \aleph_0$ ← aleph-naught

Card $\mathbb{R} = c$ ← Cardinality of the continuum

Chapter 3 Countability

← a property that distinguishes some infinite sets.

Definitions ① A set S is countably infinite iff $\exists$ bijection $f: \mathbb{N} \to S$ (i.e. Card $S = \aleph_0$).

② A set S is countable iff S is finite or countably infinite. Uncountable = not countable.

Observations

$\exists$ bijection $f: \mathbb{N} \to S \Rightarrow S = \{f(1), f(2), f(3), \ldots\}$

a listing of elements of S with no repetition nor omission

$S = \{s_1, s_2, s_3, \ldots\} \Rightarrow f: \mathbb{N} \to S$ defined by $f(n) = s_n$ is a bijection.

Bijection Theorem Let $g: S \to T$ be a bijection.

S is countable $\Leftrightarrow$ T is countable.

Proof. For finite sets, it is clear as card S = card T.

For infinite sets,

$$\begin{array}{ccc} S \text{ countable} & \Leftrightarrow & \exists \text{ bijection } f: \mathbb{N} \to S \\ \overset{\shortparallel}{\text{countably infinite}} & f = g^{-1} \circ h \ \Uparrow & \Downarrow \ h = g \circ f \\ T \text{ countable} & \Leftrightarrow & \exists \text{ bijection } h: \mathbb{N} \to T \end{array}$$

Remarks Let $g: S \to T$ be a bijection.

S is uncountable $\Leftrightarrow$ T is uncountable.

This is the contrapositive of the bijection theorem.

Basic Examples

① $\mathbb{N}$ is countably infinite as $\underset{\text{identity function}}{I_{\mathbb{N}}}: \mathbb{N} \to \mathbb{N}$ is a bijection

② $\mathbb{Z}$ is countably infinite because

$$\begin{array}{l} \mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \dots\} \\ f \downarrow \quad \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \cdots \\ \mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\} \end{array}$$

$$f(n) = \begin{cases} n/2 & n \text{ even} \\ -\left(\frac{n-1}{2}\right) & n \text{ odd} \end{cases}, \quad f^{-1}(m) = \begin{cases} 2m & m > 0 \\ 1-2m & m \le 0 \end{cases}$$

is a bijection

③ $\mathbb{N} \times \mathbb{N} = \{(m,n) : m, n \in \mathbb{N}\}$ is countably infinite.

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & \cdots \\ (2,1) & (2,2) & (2,3) & \cdots \\ (3,1) & (3,2) & (3,3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Define $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by

$f(1) = (1,1)$
$f(2) = (2,1)$
$f(3) = (1,2)$
$f(4) = (3,1)$
$f(5) = (2,2)$
$f(6) = (1,3)$
$\vdots$

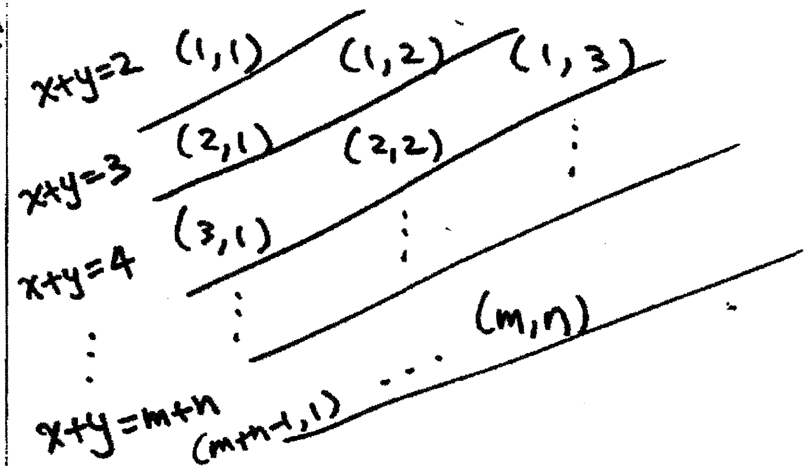
Diagonal Counting Scheme

f injective because no ordered pair is repeated.

f surjective because $\forall (m,n) \in \mathbb{N} \times \mathbb{N}$

$$(m,n) = f\left(\sum_{k=0}^{m+n-2} k + n\right) = f\left(\frac{(m+n-2)(m+n-1)}{2} + n\right)$$

(m,n) is the n^{th} element on the $m+n-1^{th}$ diagonal.



④ Open interval $(0,1)=\{x : x\in\mathbb{R} \text{ and } 0<x<1\}$ is <u>uncountable</u>. $\mathbb{R}$ is uncountable.

Assume $(0,1)$ is countably infinite. Then $\exists$ bijection $f:\mathbb{N}\to(0,1)$. So

$$\begin{aligned} f(1)&=0.\underline{a_{11}}\,a_{12}\,a_{13}\,a_{14}\cdots \\ f(2)&=0.a_{21}\,\underline{a_{22}}\,a_{23}\,a_{24}\cdots \\ f(3)&=0.a_{31}\,a_{32}\,\underline{a_{33}}\,a_{34}\cdots \\ f(4)&=0.a_{41}\,a_{42}\,a_{43}\,\underline{a_{44}}\cdots \\ &\vdots \end{aligned}$$

f <u>surjective</u> $\Rightarrow$ every $x\in(0,1)$ is equal to some $f(n)$

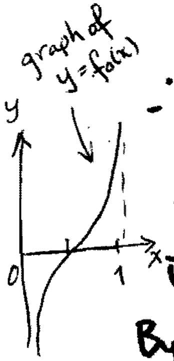
Consider $x=0.b_1b_2b_3\cdots$, where $\forall n=1,2,3,\cdots$,

$$b_n=\begin{cases}2 & \text{if } a_{nn}=1\\ 1 & \text{if } a_{nn}\neq 1\end{cases}\neq a_{nn}.$$

Then $0<x<1$. However $x\neq f(n)$ because $b_n\neq a_{nn}$ for all $n=1,2,3,\dots$ Contradicting the surjectivity of f.

b_n: n^{th} digit of x; a_{nn}: n^{th} digit of $f(n)$

$\therefore$ $(0,1)$ is uncountable.



$f_0:(0,1)\to\mathbb{R}$ given by $f_0(x)=\tan\pi(x-\frac{1}{2})$ is a bijection with $f_0^{-1}(x)=\frac{1}{2}+\frac{\operatorname{Arctan}x}{\pi}$

By bijection theorem, $\mathbb{R}$ is uncountable.

<u>Countable Subset Theorem</u> Let $A\subseteq B$. If B is countable, then A is countable. (Taking contrapositive, if A is uncountable, then B is uncountable.)

<u>Reason</u> B countable $\Rightarrow B=\{b_1,b_2,b_3,\dots\}$ ← no repetition, no omission

From the listing of B, we strike out the elements that are <u>not</u> in A. Then we get a listing of A. Since the listing of B has no repetition and $A\subseteq B$, the listing of A has no repetition nor omission.

<u>Countable Union Theorem</u>

If $\forall n\in\mathbb{N}$, A_n is countable, then $\bigcup_{n\in\mathbb{N}} A_n$ is countable.

(In general, if S is countable, say $f:\mathbb{N}\to S$ is a bijection, and $\forall s\in S$, A_s is countable, then $\bigcup_{s\in S} A_s=\bigcup_{n\in\mathbb{N}} A_{f(n)}$ is countable.)

<u>Reason</u>

$\forall n\in\mathbb{N}$, A_n countable $\Rightarrow$
$$\begin{aligned} A_1&=\{a_{11},a_{12},a_{13},\dots\},\\ A_2&=\{a_{21},a_{22},a_{23},\dots\},\\ A_3&=\{a_{31},a_{32},a_{33},\dots\},\dots\end{aligned}$$

diagonal counting scheme

$$\Rightarrow \bigcup_{n\in\mathbb{N}} A_n=\{a_{11},a_{21},a_{12},a_{31},a_{22},a_{13},\dots\}$$

eliminate repetition along the way, skip blanks

Product Theorem For $n \in \mathbb{N}$, if $A_1, \cdots, A_n$ countable, then $A_1 \times \cdots \times A_n$ is countable.

Reason $n=1$ is clear.

$n=2$ $\quad A_1 = \{x_1, x_2, x_3, \cdots\}$, $A_2 = \{y_1, y_2, y_3, \cdots\}$

$$\Rightarrow A_1 \times A_2 = \{ \begin{array}{l} (x_1,y_1),(x_1,y_2),(x_1,y_3),\cdots \\ (x_2,y_1),(x_2,y_2),(x_2,y_3),\cdots \\ (x_3,y_1),(x_3,y_2),(x_3,y_3),\cdots \\ \vdots \quad\quad \vdots \quad\quad \vdots \end{array}$$

diagonal counting scheme

$$= \{(x_1,y_1),(x_2,y_1),(x_1,y_2),(x_3,y_1),(x_2,y_2),\cdots\}$$

skip blanks if A_1 or A_2 is finite.

$n>2$ $\quad$ Assume case $n-1$ is true. Then

$$A_1 \times \cdots \times A_n = \underbrace{\underbrace{(A_1 \times \cdots \times A_{n-1})}_{\text{Case } n-1 \text{ is true}} \times A_n}_{\text{Case 2 is true}}.$$

Product theorem does <u>not</u> hold for <u>infinitely</u> many countable sets. See example 10.

Examples 5 $\quad \mathbb{Q} = \bigcup_{n \in \mathbb{N}} S_n$, where

$$S_1 = \{\cdots, \tfrac{-2}{1}, \tfrac{-1}{1}, \tfrac{0}{1}, \tfrac{1}{1}, \tfrac{2}{1}, \cdots\}$$
$$S_2 = \{\cdots, \tfrac{-2}{2}, \tfrac{-1}{2}, \tfrac{0}{2}, \tfrac{1}{2}, \tfrac{2}{2}, \cdots\}$$
$$S_3 = \{\cdots, \tfrac{-2}{3}, \tfrac{-1}{3}, \tfrac{0}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \cdots\}$$
$$\vdots$$
$$S_n = \{\cdots, \tfrac{-2}{n}, \tfrac{-1}{n}, \tfrac{0}{n}, \tfrac{1}{n}, \tfrac{2}{n}, \cdots\}$$
$$\vdots$$

$\forall n$, $f_n : \mathbb{Z} \to S_n$, $f_n(m) = \frac{m}{n}$, is a bijection (with $f_n^{-1}(\frac{m}{n}) = m$)

Since $\mathbb{Z}$ is countable, by bijection theorem, S_n is countable. By countable union theorem, $\mathbb{Q}$ is countable.

6 If A is uncountable and B is countable then $A \setminus B$ is uncountable.

(The case $A = \mathbb{R}$, $B = \mathbb{Q}$ $\Rightarrow$ $\mathbb{R} \setminus \mathbb{Q}$ is uncountable)

Reason Assume $A \setminus B$ is countable, then

$$A \cap B \subseteq B \text{ and } B \text{ countable} \underset{\text{countable subset theorem}}{\Longrightarrow} A \cap B \text{ countable}$$

$$\Rightarrow \underbrace{(A \cap B) \cup (A \setminus B)}_{\text{Countable}} = A \text{ countable} \leftarrow \text{contradiction.}$$

by countable union theorem

⑦ Since $\mathbb{R} \subseteq \mathbb{C}$ and $\mathbb{R}$ is uncountable, by the countable subset theorem, $\mathbb{C}$ is uncountable.

$$\underbrace{\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}}_{\text{Countable}} \subseteq \underbrace{\mathbb{R} \subseteq \mathbb{C}}_{\text{uncountable}}$$

$\mathbb{R} \setminus \mathbb{Q} \subseteq$

---

⑧ Determine if $A = \{r\sqrt{m} : m \in \mathbb{N},\ r \in (0,1)\}$ (open interval)

$$B = \{r\sqrt{m} : m \in \mathbb{N},\ r \in (0,1) \cap \mathbb{Q}\}$$

Countable or not.

Solution Taking $m = 1$, we see

$\underbrace{(0,1)}_{\text{uncountable}} = \{r\sqrt{1} : r \in (0,1)\} \subseteq A$

$\therefore$ by countable subset theorem A is uncountable.

$\forall m \in \mathbb{N}$, let $B_m = \{r\sqrt{m} : r \in (0,1) \cap \mathbb{Q}\}$.

Then $(0,1) \cap \mathbb{Q} \subseteq \underset{\text{countable}}{\mathbb{Q}} \Rightarrow (0,1) \cap \mathbb{Q}$ countable by countable subset theorem

$B_m = \bigcup_{r \in \underbrace{(0,1)\cap\mathbb{Q}}_{\text{Countable}}} \underset{\text{finite}}{\{r\sqrt{m}\}}$ is countable by Countable union theorem

$B = \bigcup_{m \in \mathbb{N}} \underbrace{B_m}_{\text{countable}}$ is countable by countable union theorem

⑨ Show that the set L of all lines with equation $y = mx + b$, where $m, b \in \mathbb{Q}$, is Countable.

Solution Define $f : \mathbb{Q} \times \mathbb{Q} \to L$ by letting $f(m,b)$ be the line with equation $y = mx + b$. This is a bijection with f^{-1} sending the line with equation $y = mx + b$ back to (m, b).

Since $\mathbb{Q} \times \mathbb{Q}$ (countable) is countable by product theorem, L is countable by bijection theorem.

---

⑩ Let $A_1 = A_2 = A_3 = \cdots = \{0,1\}$, then $A_1 \times A_2 \times A_3 \times \cdots$ is uncountable.

Solution Assume $A_1 \times A_2 \times A_3 \times \cdots$ is countable.

Then $\exists$ bijection $f : \mathbb{N} \to A_1 \times A_2 \times A_3 \times \cdots$

$f(1) = (\underline{a_{11}}, a_{12}, a_{13}, \cdots)$ All $a_{ij} = 0$ or 1.

$f(2) = (a_{21}, \underline{a_{22}}, a_{23}, \cdots)$

$f(3) = (a_{31}, a_{32}, \underline{a_{33}}, \cdots)$

$\vdots$

Let $b = (b_1, b_2, b_3, \dots)$, where $b_i = \begin{cases} 1 & \text{if } a_{ii} = 0 \\ 0 & \text{if } a_{ii} = 1 \end{cases}$

Then $b \in A_1 \times A_2 \times A_3 \times \cdots$

$\forall i$, $b_i \neq a_{ii} \Rightarrow b \neq f(i) \Rightarrow f$ not surjective

contradiction.

(11) Show $P(\mathbb{N})$ is uncountable.

$\uparrow$ the set of all subsets of $\mathbb{N}$

<u>Solution</u> Define $g: P(\mathbb{N}) \to A_1 \times A_2 \times A_3 \times \cdots$

(all $A_i = \{0,1\}$)

by

$$g(S) = (a_1, a_2, a_3, \ldots), \text{ where}$$

$$a_m = \begin{cases} 1 & \text{if } m \in S \\ 0 & \text{if } m \notin S. \end{cases}$$

For example,

$g(\{1,3,5,7,\ldots\}) = (1,0,1,0,1,0,1,\ldots)$

$g(\emptyset) = (0,0,0,\ldots)$, $g(\mathbb{N}) = (1,1,1,\ldots)$

Note $g^{-1}((a_1,a_2,a_3,\ldots)) = \{m : a_m = 1\}$

$\therefore$ g is a bijection.

Since $A_1 \times A_2 \times A_3 \times \cdots$ is uncountable by example 10,

$P(\mathbb{N})$ is uncountable by bijection theorem.

(12) Show that the set S of all polynomials with integer coefficients is countable.

<u>Solution</u>. Let $S_0 = \mathbb{Z}$. Let

$$S_n = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 : a_n, a_{n-1}, \ldots, a_0 \in \mathbb{Z},\ a_n \neq 0\}$$

Define $f: S_n \to (\mathbb{Z}\setminus\{0\}) \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ by

$$f(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0) = (a_n, a_{n-1}, \ldots, a_0).$$

This is a bijection with

$$f^{-1}(a_n, a_{n-1}, \ldots, a_0) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

Since $(\mathbb{Z}\setminus\{0\}) \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ is countable by product theorem, S_n is countable by bijection theorem.

$$\therefore\ S = \underset{\underset{\mathbb{Z}}{\parallel}}{S_0} \cup \Big(\bigcup_{n\in\mathbb{N}} S_n\Big) \text{ is countable by the countable union theorem}$$

countable

(13) Is every real number a root of some nonconstant polynomial with integer coefficients?

<u>Solution</u>. Let S_n be as in example 12. Then

$$S^* = \bigcup_{\underset{\text{countable}}{n \in \mathbb{N}}} \underset{\text{countable}}{S_n} \text{ is } \underline{\text{countable}} \text{ by countable union theorem.}$$

$\forall f \in S^*$, $\exists n \in \mathbb{N}$ such that $f \in S_n$. Then the set R_f of all roots of f is <u>finite</u> because R_f has at most $n = (\text{degree } f)$ elements.

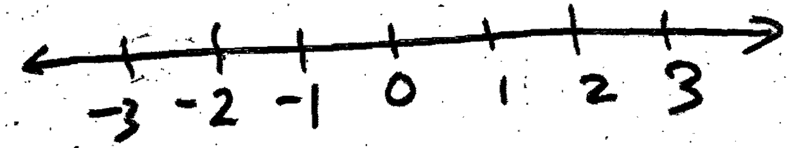
Then $T = \bigcup_{\underset{\text{countable}}{f \in S^*}} \underset{\text{countable}}{R_f}$ is countable by the Countable union theorem.

By example 6,

$$\underset{\text{uncountable}}{\mathbb{R}} \setminus \underset{\text{countable}}{T} = \mathbb{R} \setminus \Big(\bigcup_{f \in S^*} R_f \Big) \text{ is uncountable, hence nonempty.}$$

Therefore, there is a real number not in T. Such a real number is not a root of any nonconstant polynomial with integer coefficients. (Such a number is called <u>transcendental</u>.)

(Stanford Problem)



(1) A submarine at time $=0$ is located at some <u>integer</u> n on $\mathbb{R}$.

(2) Submarine has a constant speed, s, which is also an <u>integer</u>.

(3) At every second, you can fire a missile at some integer.

Is there a strategy to hit the submarine at some time?

Remarks on Examples of Chapter 3

① Every interval I containing at least two numbers is uncountable.

Reason Say $a, b \in I$ with $a<b$. Then $(a,b) \subseteq I$.
Since $f:(0,1) \to (a,b)$ defined by

$f(x) = (b-a)x+a$ has inverse function

$g:(a,b) \to (0,1)$

$g(x) = \dfrac{x-a}{b-a}$,

f is a bijection.

Since $(0,1)$ is uncountable by example 4, (a,b) is uncountable by bijection theorem.

$\therefore$ I is uncountable by countable subset theorem.

---

② We present a second solution to Example 9.

For every $(m,b) \in \mathbb{Q}\times\mathbb{Q}$, let $L_{(m,b)}$ be the set consisted of the line with equation $y=mx+b$.

Then
$$L = \bigcup_{\underbrace{(m,b)\in \mathbb{Q}\times\mathbb{Q}}_{\text{Countable by product theorem}}} \underbrace{L_{(m,b)}}_{\text{one element set}}$$

one element set $\Rightarrow$ finite set $\Rightarrow$ countable set

By countable union theorem, L is countable.

Summary of Countable and uncountable sets

Countable Sets

Finite Sets

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

$\mathbb{N}\times\mathbb{N}$

Countable $\times$ countable

Subsets of Countable Sets, like $\mathbb{Q}\cap[0,1]$

Polynomials with integer coefficients

Uncountable Sets

$(0,1)$, $\mathbb{R}$, intervals with at least 2 numbers

$\mathbb{C}$, $\mathbb{R}\setminus\mathbb{Q}$

$P(\mathbb{N})$

$\{0,1\}\times\{0,1\}\times\{0,1\}\times\cdots$

Uncountable $\times$ Nonempty by surjection theorem

Injection Theorem

Let $f: A \to B$ be injective. If B is countable, then A is countable. (Contrapositive: if A is uncountable, then B is uncountable.)

Reason. Let $h: A \to f(A)$ be given by $h(x) = f(x)$. Then h is injective (since f is injective) and h is surjective (since $h(A) = f(A)$). $\therefore$ h is bijective. Since $f(A) \subseteq B$ and B is countable, we see $f(A)$ is countable (by countable subset theorem). $\therefore$ A is countable (by bijection theorem).

Surjection Theorem

Let $g: A \to B$ be surjective. If A is countable, then B is countable. (Contrapositive: if B is uncountable, then A is uncountable.)

Reason.
$$\underset{g \text{ surjective}}{B =} g(A) = \bigcup_{\underset{\text{countable}}{x \in A}} \underbrace{\{g(x)\}}_{\text{finite}}$$
is countable by the countable union theorem.

Examples (14) Define $f: \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ by $f(x) = (m, n)$, where $x = \frac{m}{n}$ $(m \in \mathbb{Z}, n \in \mathbb{N})$ and the highest common factor (or greatest common divisor) of m, n is 1.

$f(x) = f(x') = (m,n) \Rightarrow x = \frac{m}{n} = x'$

$\therefore$ f is injective. Since $\mathbb{Z} \times \mathbb{N}$ is countable by product theorem, $\mathbb{Q}$ is countable by the injection theorem.

(15) Let A_1 be uncountable and $A_2, \cdots, A_{100}$ be nonempty sets. Then $A_1 \times A_2 \times \cdots \times A_{100}$ is uncountable.

Solution Define $g: A_1 \times A_2 \times \cdots \times A_{100} \to A_1$ by $g(x_1, x_2, \cdots, x_{100}) = x_1$. Since $A_2, \cdots, A_{100} \neq \emptyset$, $\exists\, a_2 \in A_2, \cdots, a_{100} \in A_{100}$. Then $\forall x \in A_1$, $g(x, a_2, \cdots, a_{100}) = x$. So g is surjective. Since A_1 is uncountable, $A_1 \times A_2 \times \cdots \times A_{100}$ is uncountable by surjection theorem.

FAMOUS OPEN MATH PROBLEM

<u>Continuum Hypothesis</u> For every uncountable set S, $\exists$ injective $f : \mathbb{R} \to S$.

<u>Question</u> Is this a true statement ?

1940 Kurt Gödel proved the opposite statement would not lead to any contradiction.

<u>Question</u> Does this mean the opposite statement is true ?

1966 Paul Cohen proved the original statement would not lead to any contradiction.
He won the Fields' medal for this.

<u>Moral</u>: The method of 'proof by contradiction' may not be applied to every statement.