Math 2033 Tutorial Examples

$\mathbb{R}$ is the set of all real numbers
$\mathbb{Q}$ is the set of all rational numbers.

Countable Set Exercises

$\mathbb{N}$ is the set $\{1,2,3,\dots\}$

① Show that the set F of all finite subsets of $\mathbb{N}$ is countable.

② Show that the set $(X \setminus Y) \cup (Y \setminus X)$, where X is a countable set and Y is an uncountable set, is an uncountable set.

$\mathbb{Z}$ is the set of all integers

③ Determine if the set $S = \{x + y\sqrt{2} : x \in \mathbb{Z},\ y \in A\}$, where A is nonempty countable subset of $\mathbb{R}$, is countable.

④ Determine if the set $S = T \cap U$, where $T = \mathbb{R} \setminus \mathbb{Q}$ and $U = \mathbb{R} \setminus \{\sqrt{m} + \sqrt{n} : m, n \in \mathbb{N}\}$ is countable or not.

⑤ Determine if the set W is countable, where W is the set of all intersection points (x,y) of the line $y = \pi x$ with the graphs of all equations $y = x^3 + x + m$, where $m \in \mathbb{Z}$.

⑥ Let P be a countable set of points in $\mathbb{R}^2$. Prove that there exists a circle C with the origin as center and positive radius such that every point of the circle C is not in P. (Note points inside the circle do not belong to the circle.)

Math 2033 Tutorial Example Solutions

① Show that the set F of all finite subsets of $\mathbb{N}$ is countable.

Solution For $k=0,1,2,\cdots$, let S_k be the set of all subsets of $\mathbb{N}$ having exactly k elements. Then $S_0=\{\emptyset\}$ has one element and so S_0 is countable.

For $k\in\mathbb{N}$, the function $f_k : S_k \to \underbrace{\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}}_{k\ \mathbb{N}'s}$ defined by $f(\underbrace{\{n_1,n_2,\cdots,n_k\}}_{\text{in increasing order}}) = (n_1,n_2,\cdots,n_k)$ is an injective function.

Since $\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}$ is countable by the product theorem, we can use the bijection theorem to conclude that S is countable. Then

$$F = S_0 \cup \left(\bigcup_{k=1}^{\infty} S_k\right) \text{ is countable}$$

by the countable union theorem.

② Show that the set $(X \setminus Y) \cup (Y \setminus X)$, where X is a countable set and Y is an uncountable set, is an uncountable set.

<u>Solution</u> We will show $Y \setminus X$ is uncountable first. Suppose $Y \setminus X$ is countable. Since X is countable and $X \cap Y \subseteq X$, we get $X \cap Y$ countable by the countable subset theorem. Then

$$Y = (Y \setminus X) \cup (X \cap Y) \text{ is countable}$$

by the countable union theorem, a contradiction. So $Y \setminus X$ is uncountable. Since

$$Y \setminus X \subseteq (X \setminus Y) \cup (Y \setminus X),$$

$(X \setminus Y) \cup (Y \setminus X)$ is uncountable by the countable subset theorem.

③ Determine if the set $S = \{x + y\sqrt{2} : x \in \mathbb{Z}, y \in A\}$, where A is nonempty countable subset of $\mathbb{R}$, is countable.

Solution Let $S = \bigcup_{x \in \mathbb{Z}} S_x$, where $S_x = \{x + y\sqrt{2} : y \in A\}$.

The function $f : A \to S_x$ defined by $f(y) = x + y\sqrt{2}$ is a bijection (because $f^{-1}(x + y\sqrt{2}) = y$ is the inverse of f).

Since A is countable, each S_x is countable, then $S = \bigcup_{x \in \mathbb{Z}} S_x$ is countable by the countable union theorem.

④ Determine if the set $S = T \cap U$, where $T = \mathbb{R} \setminus \mathbb{Q}$ and $U = \mathbb{R} \setminus \{\sqrt{m} + \sqrt{n} : m, n \in \mathbb{N}\}$ is countable or not.

<u>Solution</u> We have $\{\sqrt{m}+\sqrt{n} : m, n \in \mathbb{N}\} = \bigcup_{(m,n) \in \mathbb{N}\times\mathbb{N}} \{\sqrt{m}+\sqrt{n}\}$ is countable because $\mathbb{N}\times\mathbb{N}$ is countable by product theorem. Since $\mathbb{R} \setminus (T \cap U) = (\mathbb{R} \setminus T) \cup (\mathbb{R} \setminus U) = \mathbb{Q} \cup \{\sqrt{m}+\sqrt{n} : m, n \in \mathbb{N}\}$ is countable, so

$$S = T \cap U = \mathbb{R} \setminus (\mathbb{R} \setminus (T \cap U)) \text{ is uncountable.}$$

(5) Determine if the set W is countable, where W is the set of all intersection points (x,y) of the line $y=\pi x$ with the graphs of all equations $y=x^3+x+m$, where $m \in \mathbb{Z}$.

<u>Solution</u> For a fixed $m \in \mathbb{Z}$, the curves $y=\pi x$ and $y=x^3+x+m$ intersect in at most 3 points (because $\pi x = x^3+x+m \Rightarrow x^3+(1-\pi)x+m=0$). Now

$$S=\underbrace{\bigcup_{m\in\mathbb{Z}}}_{\text{countable set}} \underbrace{\{(x,y): y=\pi x,\ y=x^3+x+m\}}_{\text{at most 3 points, hence countable.}}$$

is countable by the countable union theorem.

⑥ Let P be a countable set of points in $\mathbb{R}^2$. Prove that there exists a circle C with the origin as center and positive radius such that every point of the circle C is not in P. (Note points inside the circle do not belong to the circle.)

Solution The set

$$S = \{\sqrt{x^2+y^2} : (x,y) \in P\} = \underbrace{\bigcup_{(x,y)\in P}}_{\text{Countable}} \underbrace{\{\sqrt{x^2+y^2}\}}_{1 \text{ element}}$$

is countable by the countable union theorem.

Then $\underbrace{(0,\infty)}_{\text{uncountable}} \setminus S$ is uncountable; in particular, nonempty.

Let $r \in (0,\infty) \setminus S$. The circle C with the origin as center and radius $r > 0$ contains no point in P as every point (x,y) in P has distance $\sqrt{x^2+y^2} \neq r$ from the origin.