MATH2033 Mathematical Analysis (2021 Spring)
Suggested Solution of Final Examination

Problem 1 (18 marks)

We consider a function $f:[-1,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} x^m & if\ \ x=\pm 1,\pm\frac{1}{2},\pm\frac{1}{3},\pm\frac{1}{4},\dots\dots \\ 0 & if\ otherwise\end{cases}\quad ,$$

where m is a positive integer.

(a) (9 marks) Find the value(s) of m such that $f(x)$ is continuous at $x=0$.

(b) (9 marks) Find the value(s) of m such that $f(x)$ is differentiable at $x=0$.

☺Solution

(a) We shall argue that $\lim\limits_{x\to 0} f(x)=f(0)=0$ for any $m\ge 1$.

Since $f(x)=0$ or x^m, it follows that $|f(x)|\le|x^m|=|x|^m$ and

$$-|x|^m\le -|f(x)|\le f(x)\le |f(x)|\le |x|^m.$$

Note that $\lim\limits_{x\to 0}|x|^m=0$, it follows from sandwich theorem that

$$\lim_{x\to 0} f(x)=0=f(0).$$

So $f(x)$ is continuous at $x=0$ for any $m\ge 1$.

(b) For any positive integer $m\ge 2$, we note that for $x\ne 0$

$$|x|^{m-1}\le -\left|\frac{f(x)}{x}\right|\le\frac{f(x)-f(0)}{x-0}=\frac{f(x)}{x}\le\left|\frac{f(x)}{x}\right|\overset{|f(x)|\le|x|^m}{\le}|x|^{m-1}.$$

Note that $\lim\limits_{x\to 0}|x|^{m-1}=0$ for $m2$, it follows from sandwich theorem that

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=0.$$

So $f(x)$ is differentiable at $x=0$ for $m\ge 2$.

For $m=1$, we consider two sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_n=\frac{1}{n}$ and $y_n=\frac{1}{n\sqrt{2}}$.

Then we have $f(x_n)=x_n$ and $f(y_n)=0$ so that

$$\lim_{n\to\infty}\frac{f(x_n)-f(0)}{x_n-0}=\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n}}=\lim_{n\to\infty}1=1.$$

$$\lim_{n\to\infty}\frac{f(y_n)-f(0)}{y_n-0}=\lim_{n\to\infty}\frac{0}{y_n}=\lim_{n\to\infty}0=0.$$

Since $\lim\limits_{n\to\infty}\frac{f(x_n)-f(0)}{x_n-0}\ne\lim\limits_{n\to\infty}\frac{f(y_n)-f(0)}{y_n-0}$, then the limits $\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exist for $m=1$. So $f(x)$ is not differentiable at $x=0$ for $m=1$.

Problem 2 (10 marks)

We let $f: [0,2] \to \mathbb{R}$ be a continuous function. Show that there exists $c \in [0,1]$ such that

$$f(c+1) - f(c) = \frac{1}{2}\big(f(2) - f(0)\big).$$

☺Solution

We consider a function $g: [0,1] \to \mathbb{R}$ defined by

$$g(x) = f(x+1) - f(x).$$

Since $f(x)$ is continuous on $[0,2]$, it follows that $g(x)$ is also continuous on $[0,1]$. Then the statement is equivalent to

$$g(c) = \underbrace{\frac{1}{2}\big(g(1) + g(0)\big)}_{denoted\ by\ K} \quad for\ \ some\ c \in [0,1].$$

If $g(0) = K$ or $g(1) = K$, then the above equation holds for $c = 0$ (if $g(0) = K$) or $c = 1$ (if $g(1) = K$).

If $g(0) \neq K$ and $g(1) \neq K$, since $K = \underbrace{\frac{g(0)+g(1)}{2}}_{\substack{midpoint\ of \\ g(0)\ and\ g(1)}} \in \big(g(0), g(1)\big)$, it follows from intermediate value theorem that there exists $c \in (0,1)$ such that

$$g(c) = K = \frac{1}{2}\big(g(1) + g(0)\big).$$

Problem 3 (18 marks)

(a) **(8 marks)** We let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathbb{R}$ such that $|f'(x)| \leq C$ for all $x \in \mathbb{R}$, where C is a positive constant. We let $\{x_n\}$ be a Cauchy sequence. Show that the sequence $\{y_n\}$ defined by $y_n = f(x_n)$ is also a Cauchy sequence.

(b) **(10 marks)** We let $f: (a,b) \to \mathbb{R}$ be 4-times differentiable function on (a,b) such that $\left|f^{(4)}(x)\right| \leq M$ for all $x \in (a,b)$. Show that

$$\left|\frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} - f''(x_0)\right| \leq \frac{M}{12}h^2$$

for any x_0 and h satisfying $a < x_0 - h < x_0 < x_0 + h < b$.

☺Solution

(a) For any $\varepsilon > 0$, we note that

- Since $\{x_n\}$ is Cauchy sequence, then there exists $K \in \mathbb{N}$ such that for $m, n \geq K$,
$$|x_m - x_n| < \frac{\varepsilon}{C}.$$
- For any $x_m \neq x_n$, we apply mean value theorem on $f(x)$ over the interval $[x_m, x_n]$ and deduce that there exists $c \in (x_m, x_n)$ such that
$$\frac{f(x_m) - f(x_n)}{x_m - x_n} = f'(c) \Rightarrow f(x_m) - f(x_n) = f'(c)(x_m - x_n).$$

Then it follows that for any $m, n \geq K$

$$|y_m - y_n| = |f(x_m) - f(x_n)| = |f'(c)(x_m - x_n)| \leq C|x_m - x_n| < C\left(\frac{\varepsilon}{C}\right) = \varepsilon.$$

(*Note: Although the above inequality requires that $x_m \neq x_n$, it can be see that the inequality also holds for $x_m = x_n$ since $|y_m - y_n| = 0 < \varepsilon$ in this case.

So it follows that $\{y_n\}$ is Cauchy sequence by definition.

(b) For any $x \in (a,b)$ and $x_0 \in (a,b)$, we apply Taylor theorem on $f(x)$ and deduce that there exists $c_x \in (x_0, x)$ such that

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \frac{f^{(4)}(c_x)}{4!}(x-x_0)^4.$$

By taking $x = x_0 + h$ and $x = x_0 - h$, we have

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f^{(3)}(x_0)}{3!}h^3 + \frac{f^{(4)}(c_1)}{4!}h^4 \ldots (*)$$

$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f^{(3)}(x_0)}{3!}h^3 + \frac{f^{(4)}(c_2)}{4!}h^4 \ldots (**)$$

where $c_1 \in (x_0.x_0+h)$ and $c_2 \in (x_0-h, x_0)$

Then it follow that

$$\left|\frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} - f''(x_0)\right|$$

$$= \left|\frac{f''(x_0)h^2 + \frac{f^{(4)}(c_1)}{4!}h^4 + \frac{f^{(4)}(c_2)}{4!}h^4}{h^2} - f''(x_0)\right|$$

$$= \left|\frac{f^{(4)}(c_1)}{24}h^2 + \frac{f^{(4)}(c_2)}{24}h^2\right| \overset{f^{(4)}(x)\le M}{\widehat{\le}} \frac{M}{24}h^4 + \frac{M}{24}h^4 = \frac{M}{12}h^2.$$

Problem 4 (18 marks)

(a) **(8 marks)** We let $f:(a,b) \to \mathbb{R}$ be n-times differentiable function and suppose that $f^{(n)}(x) > 0$ for all $x \in (a,b)$. Show that $f(x) = 0$ has at most n solutions in the interval (a,b).

(b) **(10 marks)** We consider the equation $4x^2 - 8x + 5 = 2^x$.

(i) Show that the equation has at least one solution over $(0,1)$.

(ii) Show that the equation has exactly two solutions over $(0,2)$.

☺Solution

(a) Suppose that $f(x) = 0$ has at least $(n+1)$ solutions over (a,b). We let $x_1 < x_2 < \cdots < x_{n+1}$ be some solutions (may not all) of $f(x) = 0$.

For any $k = 1,2,\ldots,n$, we apply Rolle's theorem on $f(x)$ over $[x_k, x_{k+1}]$ and deduce that there exists $c_k \in (x_k, x_{k+1})$ such that

$$f'(c_k) = 0.$$

So $f'(x) = 0$ has at least n solutions.

By applying Rolle's theorem on $f'(x)$ over $[c_k, c_{k+1}]$ for $k = 1,2,\ldots,n-1$, we deduce that there exists $d_k \in (x_k, x_{k+1})$ such that

$$f''(d_k) = (f')'(d_k) = 0.$$

Then $f''(x) = 0$ has at least $n-1$ solutions.

By repeating this process, we deduce that $f^{(3)}(x) = 0$ has at least $n-2$ solutions, $f^{(4)}(x) = 0$ has at least $n-3$ solutions and so on.
Finally, we deduce that $f^{(n)}(x) = 0$ has at least 1 solution over (a, b) which contradicts to the assumption that $f^{(n)}(x) > 0$ for all $x \in (a, b)$.
Therefore we conclude that $f(x) = 0$ has at most n solutions.

(b) **(i)** We let $f(x) = 4x^2 - 8x + 5 - 2^x$ which is continuous over $\mathbb{R}$. Note that

- ✓ $f(0) = 0 - 0 + 5 - 1 = 4 > 0$ and
- ✓ $f(1) = 4 - 8 + 5 - 2 = -1 < 0$

It follows from intermediate value theorem that there exists $c \in (0,1)$ such that $f(c) = 0$ so that $f(x) = 0$ (equivalent to $4x^2 - 8x + 5 = 2^x$) has at least one solution over $(0,1)$.

(ii) On the other hand, we see that $f(2) = 16 - 16 + 5 - 4 = 1 > 0$. It follows from intermediate value theorem that there exists $d \in (1,2)$ (and $d \neq c$) such that $f(d) = 0$. So $f(x) = 0$ has at least two solutions over $(0,2)$.
Furthermore, we note that

$$f''(x) = 8 - 2^x (\ln 2)^2 > 8 - 2^2 \underbrace{(\ln 2)^2}_{<1} > 0 \quad for \;\; x \in (0,2).$$

It follows from the result of (a) that the equation $f(x) = 0$ has at most two solutions over $(0,2)$. Combining the earlier result, we deduce that the equation $f(x) = 0$ has exactly two solutions.

Problem 5 (20 marks)

(a) **(10 marks)** We let $[a, b]$ (where $a < b$) be an closed interval. For any closed interval $[c, d] \subseteq [a, b]$ (where $a < c < d < b$), we define a function $g: [a, b] \to \mathbb{R}$ as

$$g(x) = \begin{cases} 1 & if \;\; x \in [c, d] \\ 0 & if \;\; otherwise \end{cases}'$$

Using integral criterion or the definition of integrability, determine if $g(x)$ is integrable.

(b) **(10 marks)** We let $f: [a, b] \to \mathbb{R}$ be a bounded Riemann integrable function and let $g: [a, b] \to \mathbb{R}$ be another bounded function such that the set $\{x \in [a, b] | f(x) \neq g(x)\} = \{x_1, x_2, \dots, x_n\}$ where $a < x_1 < x_2 < \cdots < x_n < b$.

(i) Show that $g(x)$ is integrable. (☺Hint: Consider the function $h(x) = g(x) - f(x)$)

(ii) Show that

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

☺Solution

(a) For any $\varepsilon > 0$, we consider the following partition

$$\mathcal{P} = \left\{ \underbrace{a}_{x_0}, \underbrace{c-\delta}_{x_1}, \underbrace{c+\delta}_{x_2}, \underbrace{d-\delta}_{x_3}, \underbrace{d+\delta}_{x_4}, \underbrace{b}_{x_5} \right\}$$

where $\delta > 0$ is some positive constant (the value will be determined later).
Under this partition, we have

$$U(\mathcal{P},g)-L(\mathcal{P},g)=\sum_{k=1}^{n}(M_k-m_k)(x_k-x_{k-1})$$
$$=(0-0)(x_1-x_0)+(1-0)(x_2-x_1)+(1-1)(x_3-x_2)+(1-0)(x_4-x_3)$$
$$+(0-0)(x_5-x_4)$$
$$=2\delta+2\delta=4\delta.$$
By taking $\delta<\frac{\varepsilon}{4}$, we deduce that
$$U(\mathcal{P},g)-L(\mathcal{P},g)<4\left(\frac{\varepsilon}{4}\right)=\varepsilon.$$
So $g(x)$ integrable on $[a,b]$ by integral criterion.

(b) **(i)** We consider the function $h(x)=g(x)-f(x)$. Note that
$$h(x)=\begin{cases} g(x_k)-f(x_k) & if\ \ x=x_1,x_2,\dots,x_n \\ 0 & if\ otherwise\end{cases}.$$
One can see that $h(x)$ is not continuous at $x=x_k$ if $g(x_k)-f(x_k)\neq 0$. Thus, the number of discontinuity points of $h(x)$ is *at most* n and therefore finite. So it follows that $h(x)$ is integrable over $[a,b]$.

Therefore, we conclude that the function
$$g(x)=\underbrace{f(x)}_{integrable}+\underbrace{\big(g(x)-f(x)\big)}_{integrable}.$$
is integrable over $[a,b]$.

(ii) We shall prove that $\int_a^b h(x)dx=0$. To facilitate the analysis, we let
$$M=\sup\{h(x_1),h(x_2),\dots,h(x_n)\}\quad and\quad m=\inf\{h(x_1),h(x_2),\dots,h(x_n)\}.$$
For any $\varepsilon>0$, we consider the partition
$$\mathcal{P}=\{a,x_1-\delta,x_1+\delta,x_2-\delta,x_2+\delta,\dots.,x_n-\delta,x_n+\delta,b\},$$
where $\delta=\frac{\varepsilon}{2nM}$.

Then we have
$$\int_a^b h(x)dx\le U(\mathcal{P},h)\le\sum_{k=1}^{n}M\big(x_k+\delta-(x_k-\delta)\big)=2nM\delta\overset{\delta=\frac{\varepsilon}{2nM}}{\cong}\varepsilon.$$
By taking $\varepsilon\to 0^+$, we have $\int_a^b h(x)dx\le 0$.

On the other hand,
$$\int_a^b h(x)dx\ge L(\mathcal{P},h)\ge\sum_{k=1}^{n}m\big(x_k+\delta-(x_k-\delta)\big)=2nm\delta=\frac{m}{M}\varepsilon.$$
By taking $\varepsilon\to 0^+$, we have $\int_a^b h(x)dx\ge 0$.

So it follows that $\int_a^b h(x)dx=0$.

Hence, it follows from property of integral that
$$\int_a^b g(x)dx=\int_a^b f(x)dx+\int_a^b\underbrace{\big(g(x)-f(x)\big)}_{h(x)}dx=\int_a^b f(x)dx.$$

Problem 6 (16 marks)

We let $f: \mathbb{R} \to \mathbb{R}$ be a function.

(a) **(12 marks)** We let L be a real number. Show that $\lim_{x\to+\infty} f(x) = L$ if and only if $\lim_{n\to\infty} f(x_n) = L$ for any sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = +\infty$.

(b) **(4 marks)** Does the limits $\lim_{x\to\infty} \frac{\sin x}{2+\cos x}$ converge to a real number? Explain your answer.

☺Solution

(a) ("⇒" part)

We let $\{x_n\}$ be a sequence which $\lim_{n\to\infty} x_n = +\infty$. We shall argue that $\lim_{n\to\infty} f(x_n) = L$ using the definition of limits.

For any $\varepsilon > 0$,

- ✓ Since $\lim_{x\to+\infty} f(x) = L$, then there exists $M > 0$ such that $|f(x) - L| < \varepsilon$ when $x > M$.
- ✓ Since $\lim_{n\to\infty} x_n = +\infty$, then there exists $K \in \mathbb{N}$ such that $x_n > M$ for $n \geq K$.

For this integer K, it follows that when $n \geq K$,

$$|f(x_n) - L| \overset{\substack{x_n > M \\ for\ n \geq K}}{<} \varepsilon.$$

So $\lim_{n\to\infty} f(x_n) = L$ using the definition of limits.

("⇐" part)

Suppose that $\lim_{x\to+\infty} f(x) \neq L$, then there exists $\varepsilon_0 > 0$ such that for any $M > 0$, there exists x_0 such that $x_0 > M$ and $|f(x_0) - L| \geq \varepsilon_0$.

By taking $M = n$ (where $n \in \mathbb{N}$), we deduce that there exists x_n satisfying

$$x_n > M = n \quad and \quad |f(x_n) - L| \geq \varepsilon_0 \ldots (*)$$

By repeating the process for all positive integer n, we obtain a sequence $\{x_n\}$ such that each x_n satisfies the inequalities $(*)$.

Note that $\lim_{n\to\infty} x_n = +\infty$, it follows that $\lim_{n\to\infty} f(x_n) = L$, then for $\varepsilon = \varepsilon_0$, there exists $K \in \mathbb{N}$ such that

$$|f(x_n) - L| < \varepsilon = \varepsilon_0 \quad for\ \ n \geq K.$$

This contradicts to the inequality $(*)$ since the inequality $(*)$ is supposed to be valid for all positive integer n. Hence, we conclude that $\lim_{x\to+\infty} f(x) = L$.

(b) We consider two sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_n = 2n\pi \quad and \quad y_n = 2n\pi + \frac{\pi}{2}.$$

We observe that $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} y_n = +\infty$. On the other hand, we deduce that

$$\lim_{n\to\infty} \frac{\sin x_n}{2 + \cos x_n} = \lim_{n\to\infty} \frac{0}{2+1} = 0$$

$$\lim_{n\to\infty} \frac{\sin y_n}{2 + \cos y_n} = \lim_{n\to\infty} \frac{1}{2+0} = \frac{1}{2}.$$

Since $\lim_{n\to\infty} \frac{\sin x_n}{2+\cos x_n} \neq \lim_{n\to\infty} \frac{\sin y_n}{2+\cos y_n}$, so the limits $\lim_{x\to\infty} \frac{\sin x}{2+\cos x}$ does not exist by the result of **(a)**.