(Solutions of Exercises at the end of Mathematical Induction notes)

① Let $P(n)$ be the statement $1^3+2^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$.

(a) $P(1)$ is $1^3=\frac{1^2(1+1)^2}{4}$, which is $1=1$, hence true.

(b) If $P(n)$ is true, then $1^3+2^3+\cdots+n^3=\frac{n^2(n+1)^2}{4}$. Adding $(n+1)^3$ to both sides, we have

$$1^3+2^3+\cdots+(n+1)^3=\frac{n^2(n+1)^2}{4}+(n+1)^3=(n+1)^2\left[\frac{n^2}{4}+n+1\right]=(n+1)^2\left[\frac{n^2+4n+4}{4}\right]$$
$$=\frac{(n+1)^2(n+2)^2}{4}$$

Then $P(n+1)$ is true. By M.I., we are done.

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② Case $n=1$ $x_2=1-\frac{1}{4x_1}=1-\frac{1}{4\times1}=\frac{3}{4}$. Then $x_1=1>x_2=\frac{3}{4}>\frac{1}{2}$.

Suppose case n is true. Then $x_n>x_{n+1}>\frac{1}{2}$. Taking reciprocal, $\frac{1}{x_n}<\frac{1}{x_{n+1}}<2$. Multiplying by $-\frac{1}{4}$, $-\frac{1}{4x_n}>-\frac{1}{4x_{n+1}}>-\frac{1}{2}$. Adding 1 to all parts, $1-\frac{1}{4x_n}>1-\frac{1}{4x_{n+1}}>1-\frac{1}{2}$, which is $x_{n+1}>x_{n+2}>\frac{1}{2}$. So case $n+1$ is true.

By M.I., we are done.

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③ We will prove $P(n)$: $0<x_{2n}<x_{2n+2}<x_{2n+1}<x_{2n-1}$ for $n=1,2,3,\ldots$

Case $n=1$ $x_2=3+\frac{4}{x_1}=3\frac{4}{5}=\frac{19}{5}=3.8$, $x_3=3+\frac{4}{3.8}=\frac{77}{19}=4\frac{1}{19}$

$x_4=3+\frac{4}{77/19}=3\frac{76}{77}$. So, $0<x_2=3\frac{4}{5}<x_4=3\frac{76}{77}<x_3=4\frac{1}{19}<x_1=5$

Suppose Case n is true. Then $0<x_{2n}<x_{2n+2}<x_{2n+1}<x_{2n-1}$. (We need to show $0<x_{2(n+1)}<x_{2(n+1)+2}<x_{2(n+1)+1}<x_{2(n+1)-1}$.)

From case n, since $0<x_{2n}$, $\frac{4}{x_{2n}}>\frac{4}{x_{2n+2}}>\frac{4}{x_{2n+1}}>\frac{4}{x_{2n-1}}$. Then

$3+\frac{4}{x_{2n}}>3+\frac{4}{x_{2n+2}}>3+\frac{4}{x_{2n+1}}>3+\frac{4}{x_{2n-1}}$. We get $x_{2n+1}>x_{2n+3}>x_{2n+2}>x_{2n}$

Since $0<x_{2n}$, we can repeat these steps once more

$$\frac{4}{x_{2n+1}}<\frac{4}{x_{2n+3}}<\frac{4}{x_{2n+2}}<\frac{4}{x_{2n}},\quad \underset{\displaystyle x_{2n+2}}{\underset{\shortparallel}{3+\frac{4}{x_{2n+1}}}}<\underset{\displaystyle x_{2n+4}}{\underset{\shortparallel}{3+\frac{4}{x_{2n+3}}}}<\underset{\displaystyle x_{2n+3}}{\underset{\shortparallel}{3+\frac{4}{x_{2n+2}}}}<\underset{\displaystyle x_{2n+1}}{\underset{\shortparallel}{3+\frac{4}{x_{2n}}}}$$

$\underbrace{0<x_{2n}<}_{\text{from case } n}$ $x_{2n+2}<x_{2n+4}<x_{2n+3}<x_{2n+1}$ ← Case $n+1$.

By M.I., we're done.