Solutions to Presentation Exercises

(22). For $r\in\mathbb{Q}$, $D_r=\{x\in\mathbb{R} \mid \underbrace{x^5+x+2}_{\text{degree 5 polynomial}}=r\}$ has at most 5 elements, so D_r is countable. Now $D=\bigcup_{r\in\mathbb{Q}} D_r$, $\mathbb{Q}$ is countable and each D_r is countable for $r\in\mathbb{Q}$, so by the countable union theorem, D is countable.

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(23) Let $\mathbb{Q}^+$ be the positive rational numbers. Since $\mathbb{Q}^+\subseteq\mathbb{Q}$ and $\mathbb{Q}$ is countable, by the countable subset theorem, $\mathbb{Q}^+$ is countable. Now the function $f:\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}^+\to E$ defined by letting $f(x,y,r)$ be the circle centered at (x,y) and radius r is a bijection. Since $\mathbb{Q}$ and $\mathbb{Q}^+$ are countable, by the product theorem, $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}^+$ is countable. By the bijection theorem, E is countable.

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(24) Suppose $x^4+ax-5=0$ has a rational root r. (If $r=0$, then $r^4+ar-5\neq 0$.) We get $r\neq 0$ and $r^4+ar-5=0 \Rightarrow a=\frac{5-r^4}{r}\in\mathbb{Q}$. So $F\subseteq\mathbb{Q}$. Therefore, F is countable.

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(25) Since X is nonempty, let $a_0\in X$. Consider the subset $G'=\{a_0^3+b^3: b\in Y\}$ of G. The function $f:Y\to G'$ defined by $f(b)=a_0^3+b^3$ is a bijection (From $w=a_0^3+b^3 \Leftrightarrow b=\sqrt[3]{w-a_0^3}$, we see $g:G'\to Y$ defined by $g(w)=\sqrt[3]{w-a_0^3}$ is the inverse of f.) Since Y is uncountable, so G' is uncountable. Since $G'\subseteq G$, so G is also uncountable.

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(89)(a) For a fixed $m\in\mathbb{Z}$, the curves $y=\pi x$ and $y=x^3+x+m$ intersect in at most 3 points (because $\pi x=x^3+x+m \Rightarrow x^3+(1-\pi)x+m=0$.) Now $S=\bigcup_{\underset{\text{countable}}{m\in\mathbb{Z}}}\underbrace{\{(x,y): y=\pi x, y=x^3+x+m\}}_{\text{at most 3 points hence countable}}$ is countable by the countable union theorem.

(b) For a fixed $m\in\mathbb{Z}$, the curves $y=x^3+x+1$ and $y=mx$ intersects in at most 3 points (because $mx=x^3+x+1 \Rightarrow x^3+(1-m)x+1=0$). Now $S=\bigcup_{\underset{\text{countable}}{m\in\mathbb{Z}}}\underbrace{\{(x,y): y=x^3+x+1, y=mx\}}_{\text{at most 3 points hence countable}}$ is countable by the countable union theorem.

(d) Taking $b=0$, we see that $S\supseteq M$. Since M is uncountable, so S is uncountable by the countable subset theorem.

(e) Note if $x=|a|$, then $a=x$ or $-x$. So

$$S=\{a+b: |a|\in M, b\in\mathbb{Q}\}=\{x+b: x\in M, b\in\mathbb{Q}\}\cup\{-x+b: x\in M, b\in\mathbb{Q}\}$$
$$=\bigcup_{\underset{\text{countable}}{(x,b)\in M\times\mathbb{Q}}}\underbrace{\{x+b,-x+b\}}_{\text{2 elements, countable}}$$

is countable by the countable union theorem.

(89) (f) The set $S_0 = \{a+b\sqrt{2} : a, b \in \mathbb{Q}\} = \bigcup_{\substack{(a,b) \in \underbrace{\mathbb{Q}\times\mathbb{Q}}_{\text{Countable}}}} \underbrace{\{a+b\sqrt{2}\}}_{\substack{1 \text{ element} \\ \text{finite, countable}}}$ is countable.

The set $\{c+d\sqrt{2} : c, d \in \mathbb{Q},\ c+d\sqrt{2} \neq 0\} = S_0 \setminus \{0\}$ is also countable by Countable Subset theorem.

$\therefore S = \mathbb{Q}(\sqrt{2}) = \{\frac{x}{y} : x \in S_0, y \in S_0 \setminus \{0\}\} = \bigcup_{(x,y) \in \underbrace{S_0 \times (S_0 \setminus \{0\})}_{\text{countable}}} \underbrace{\{\frac{x}{y}\}}_{\substack{1 \text{ element} \\ \text{finite, countable}}}$ is countable

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(j) Since A is countable, $\mathbb{R} \setminus A$ must be uncountable. Taking $y=0$, we have $S \supseteq \mathbb{R} \setminus A$. By the Countable subset theorem, S is uncountable.

(k) Since A is countable, $\mathbb{R} - A$ must be uncountable. Let $a \in A$, then S contains the subset $S_a = \{(a,y) : y \in \mathbb{R} \setminus A\}$. The function $f : \mathbb{R} \setminus A \to S_a$ defined by $f(y) = (a,y)$ is a bijection with $f^{-1}(a,y) = y$. Since $\mathbb{R} \setminus A$ is uncountable, so S_a is uncountable. Then S is uncountable by the Countable subset theorem.

(l) $S = \bigcup_{x \in \mathbb{Z}} S_x$, where $S_x = \{x+y\sqrt{2} : y \in A\}$. The function $f : A \to S_x$ defined by $f(y) = x+y\sqrt{2}$ is a bijection with $f^{-1}(x+y\sqrt{2}) = y$. Since A is countable, each S_x is countable, then $S = \bigcup_{x \in \mathbb{Z}} S_x$ is countable by the countable union theorem.

(m) Since $f : \mathbb{Q} \to T$ defined by $f(r) = r\pi$ is a bijection with $f^{-1}(t) = \frac{t}{\pi}$, so T is countable.
The set $U = \{a+b\sqrt{2} - c\sqrt{3} : a, b, c \in T\} = \bigcup_{\substack{(a,b,c) \in \underbrace{T\times T\times T}_{\substack{\text{Countable} \\ \text{by product theorem}}}}} \underbrace{\{a+b\sqrt{2}-c\sqrt{3}\}}_{\substack{1 \text{ element set} \\ \Rightarrow \text{Countable}}}$ is countable by the countable union theorem.
Then $S = \mathbb{R} \setminus U$ is uncountable.