Solutions of Presentation Exercises

(108) For every $\varepsilon>0$, by Archimedean principle, there is integer $K>\frac{1}{\varepsilon}$.

Then $m,n\geq K$, $m>n$ $\Rightarrow$ $|x_m-x_n| = |(x_m-x_{m-1})+(x_{m-1}-x_{m-2})+\cdots+(x_{n+1}-x_n)|$

$\leq |x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n|$

$< \frac{1}{2^{m-1}}+\frac{1}{2^{m-2}}+\cdots+\frac{1}{2^n} < \sum_{j=n}^{\infty}\frac{1}{2^j} = \frac{1}{2^{n-1}} \leq \frac{1}{2^{K-1}} \leq \frac{1}{K} < \varepsilon$

The case $m<n$ is similar. The case $m=n$ leads to $|x_m-x_n|=0<\varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence.

$K\leq 2^{K-1}$ can be proved by mathematical induction.

(109) (b) Sketch $x\to 2$, $x^2\to 4$, $\frac{1}{x}\to\frac{1}{2}$. $x\in(1,3)\Rightarrow x+2\in(3,5)$, $\frac{1}{2|x|}\in(\frac{1}{6},\frac{1}{2})$

$\Rightarrow |x^2+\frac{1}{x}-\frac{9}{2}| = |x^2-4+\frac{1}{x}-\frac{1}{2}| \leq |x^2-4|+|\frac{1}{x}-\frac{1}{2}| = |x+2||x-2|+\frac{|x-2|}{2|x|} < 5|x-2|+\frac{1}{2}|x-1| = \frac{11}{2}|x-1|$

Solution $\forall\varepsilon>0$, take $\delta=\frac{2}{11}\varepsilon>0$. Then $|x-1|<\delta \Rightarrow |x-1|<\frac{2}{11}\varepsilon \Rightarrow$

$|f(x)-\frac{9}{2}| = |x^2+x-\frac{9}{2}| < \frac{11}{2}|x-1| < \varepsilon$

(c) Claim: $||a|-|b|| \leq |a-b|$.

Proof. If $|a|\geq|b|$, then $||a|-|b|| = |a|-|b| \leq |a-b|$ via triangle inequality & $|a| = |(a-b)+b| \leq |a-b|+|b|$

If $|b|\geq|a|$, then $||a|-|b|| = ||b|-|a|| \leq |b-a| = |a-b|$. from the $|a|\geq|b|$ case.

Sketch $x\to 2$, $f(x)=|x^2-9|\to|-5|=5$ $x\in(1,4)\Rightarrow x+2\in(3,6)$ $|x-2|<\frac{\varepsilon}{6}$

$\Rightarrow |f(x)-5| = ||x^2-9|-|-5|| \leq |x^2-9-(-5)| = |x^2-4| = |x+2||x-2| \leq 6|x-2| < \varepsilon$

Solution $\forall\varepsilon>0$, let $\delta=\frac{\varepsilon}{6}>0$, then $0<|x-2|<\delta$ and $x\in(1,4) \Rightarrow 0<|x-2|<\frac{\varepsilon}{6}$ and $x\in(1,4)$

$\Rightarrow |f(x)-5|<\varepsilon$

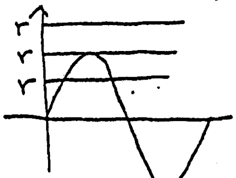
Intermediate Value Theorem

(112) (a) If $f:[a,b]\to\mathbb{R}$ is continuous and y_0 is between $f(a)$ and $f(b)$, then there is (at least one) $x_0\in[a,b]$ such that $f(x_0)=y_0$.

(b) Define $g:[0,1]\to\mathbb{R}$ by $g(x)=f(x)-f(x+1)$. Note $g(0)=f(0)-f(1)$ and $g(1)=f(1)-f(2)=f(1)-f(0)=-g(0)$. So $g(1)$ and $g(0)$ are of opposite sign. Since g is continuous on $[0,1]$, by the intermediate value theorem, $\exists c\in[0,1]$ such that $0=g(c)=f(c)-f(c+1)$. Then $f(c)=f(c+1)$.

(c) Observe that $|t|^r+|2t|^r+|3t|^r=|4t|^r+|5t|^r$ for every $t\in\mathbb{R}$ is equivalent to $1+2^r+3^r=4^r+5^r$. We will show this equation has a solution. Let $f(r)=1+2^r+3^r-4^r-5^r$, which is continuous. Since $f(0)=1$, $f(1)=-3$, by the intermediate value theorem, there is $r\in(0,1)$ such that $f(r)=0$. For this r, let $g(t)=|t|^r$, then $g(t)+g(2t)+g(3t)=g(4t)+g(5t)$ for all $r\in\mathbb{R}$.

(113) (b) For a fixed rational r, $\{x : \sin x = r\} = \bigcup_{k \text{ even integer}} \{x : \sin x = r, x \in [k\pi, k\pi + 2\pi)\}$



Since $\sin x = r$ on $[k\pi, k\pi+2\pi)$ has at most 2 solutions,

$$\{x : \sin x = r\} = \underbrace{\bigcup_{k \text{ even integer}}}_{\text{countable}} \underbrace{\{x : \sin x = r,\ x \in [k\pi, k\pi+2\pi)\}}_{\text{countable}} \text{ is countable}$$

So $T = \{x : \sin x \in \mathbb{Q}\} = \underbrace{\bigcup_{r \in \mathbb{Q}}}_{\text{Countable}} \underbrace{\{x : \sin x = r\}}_{\text{Countable}}$ is countable.

For every $x \in [0,1]$, $\sin f(x) \in \mathbb{Q}$ implies $f(x) \in T$. So $f([0,1]) = \{f(x) : x \in [0,1]\} \subseteq T$. By (a), T is countable, so $f([0,1])$ is countable.
Assume f is not a constant function, then $f([0,1])$ contains an interval (of positive length) by the intermediate value theorem. Then $f([0,1])$ is uncountable, a contradiction. Therefore, f is a constant function.

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(156) Since $\{x_n\}$ is Cauchy, for every $\varepsilon > 0$, there exists K such that $m, n \geq K$ implies $|x_m - x_n| < \frac{\varepsilon}{5}$. Then $|\sin 5x_m - \sin 5x_n| \underset{\text{by useful inequalities}}{\leq} |5x_m - 5x_n| = 5|x_m - x_n| < \varepsilon$.

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(165) Let $S_f = \{x : f \text{ is discontinuous at } x\}$ and similarly for S_g and S_{fg}. By the monotone function theorem, S_f and S_g are countable sets. If f and g are continuous at x, then fg is continuous at x. Taking contrapositive, if fg is discontinuous at x, then f is discontinuous at x or g is discontinuous at x. So $S_{fg} \subseteq S_f \cup S_g$. Since S_f, S_g countable $\underset{\text{Countable union theorem}}{\Longrightarrow}$ $S_f \cup S_g$ countable $\underset{\text{Countable subset theorem}}{\Longrightarrow}$ S_{fg} countable, we are done.

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(167) By the useful inequalities, $|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.
So $|f(x) - f(y)| \leq |\sin(x^2) - \sin(y^2)| \leq |x^2 - y^2|$. For every $\varepsilon > 0$, by Archimedean Principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\sqrt{\varepsilon}}$. Then $m, n \geq K$ implies $\frac{1}{m^2}, \frac{1}{n^2} \in (0, \frac{1}{K^2}]$
$\Rightarrow |x_m - x_n| = |f(\frac{1}{m}) - f(\frac{1}{n})| \leq |\frac{1}{m^2} - \frac{1}{n^2}| \leq \frac{1}{K^2} - 0 = \frac{1}{K^2} < \varepsilon$. $\therefore \{x_n\}$ is a Cauchy seq.

(180) (a) $f(x)$ converges to L as x tends to x_0 iff for every $\varepsilon>0$, there exists $\delta>0$ such that for every $x\in S$, $0<|x-x_0|<\delta$ implies $|f(x)-L|<\varepsilon$.

(b) Solution 1 $\forall\varepsilon>0$, set $\delta=\frac{\varepsilon}{\sqrt{2}}>0$

$\forall x\in(0.5,+\infty)$

Used $|\sqrt{a}-\sqrt{b}|\le\sqrt{|a-b|}$; $x>0$

$$0<|x-1|<\delta \Rightarrow \left|\sqrt{x+\tfrac{1}{x}}-\sqrt{2}\right| \le \sqrt{\left|x+\tfrac{1}{x}-2\right|} = \sqrt{\left|\frac{x^2-2x+1}{x}\right|} = \sqrt{\frac{(x-1)^2}{x}}$$

$$= \frac{|x-1|}{\sqrt{x}} < \sqrt{2}\,|x-1| < \varepsilon$$

$x>0.5$; need $\delta=\frac{\varepsilon}{\sqrt{2}}$

Solution 2 $\forall\varepsilon>0$, set $\delta=\sqrt{\frac{\varepsilon}{\sqrt{2}}}$

$\forall x\in(0.5,+\infty)$

$|\sqrt{a}-\sqrt{b}|=\frac{|a-b|}{\sqrt{a}+\sqrt{b}}$; $\sqrt{x+\frac{1}{x}}\ge 0$; $x>0.5$

$$0<|x-1|<\delta \Rightarrow \left|\sqrt{x+\tfrac{1}{x}}-\sqrt{2}\right| = \frac{\left|x+\frac{1}{x}-2\right|}{\sqrt{x+\frac{1}{x}}+\sqrt{2}} \le \frac{(x-1)^2/x}{\sqrt{2}} < \frac{2}{\sqrt{2}}(x-1)^2 < \varepsilon$$

need $\delta=\sqrt{\varepsilon/\sqrt{2}}$
