Math 2033 (Homework 5) L1 FALL 2021

Problems (Due Nov 29 at 11:59 pm)

1. (20) If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $\lim_{x\to 0} f'(x)$ exists, then prove that f' is continuous at 0.

2. (25) If $f : [a,b] \to \mathbb{R}$ is continuous, $f(x) \geq 0$ for all x and $\int_a^b f(x)\,dx = 0$, then prove that $f(x) = 0$ for all $x \in [a,b]$.

3. (25) If $f, g : [0,1] \to \mathbb{R}$ are Riemann integrable, then show that the function $h(x) = \min(f(x), g(x))$ is Riemann integrable on $[0,1]$.

4. (30) Let $f : [0,1] \to [-1,1]$ be Riemann integrable. Using the integral criterion, Prove that

$$g(x) = \begin{cases} f(x) & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 0 \text{ or } 1 \end{cases}$$

is also Riemann integrable on $[0,1]$

① If $f: \mathbb{R} \to \mathbb{R}$ is differentiable and $\lim_{x \to 0} f'(x)$ exists, then prove that f' is continuous at 0.

Solution

(10) Since $\lim_{x \to 0} \frac{f'(x)}{1} = \lim_{x \to 0} f'(x)$ exists, by l'Hopital's rule, we have $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f'(x)}{1} = \lim_{x \to 0} f'(x)$ exists in $\mathbb{R}$.

(10) Therefore, $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} f'(x)$,

by definition of $f'(0)$

i.e. f' is continuous at 0.

② If $f:[a,b]\to\mathbb{R}$ is continuous, $f(x)\geq 0$ for all x and $\int_a^b f(x)\,dx=0$, then prove that $f(x)=0$ for all $x\in[a,b]$.

<u>Solution</u>

(Sol 1) Define $g(t)=\int_a^t f(x)\,dx$. Since f is continuous on $[a,b]$, by the fundamental theorem of calculus, $g'(x)=f(x)\geq 0$

10 for all $x\in[a,b]$. So g is increasing on $[a,b]$. Since

$$0=g(a)\leq g(t)\leq g(b)=\int_a^b f(x)\,dx=0,$$

15 we must have $g(x)=0$ for all $x\in[a,b]$. Then $f(x)=g'(x)=0$ for all $x\in[a,b]$.

(Sol. 2) Assume $f(x_0)>0$ for some $x_0\in[a,b]$. Since f is continuous at x_0, for $\varepsilon=\frac{1}{2}f(x_0)$, there exists $\delta>0$ such that

10 $x\in[a,b]\cap(x_0-\delta,x_0+\delta)$ implies $|f(x)-f(x_0)|<\varepsilon=\frac{f(x_0)}{2}$. Then $-\frac{f(x_0)}{2}<f(x)-f(x_0)$ so that $f(x)>\frac{f(x_0)}{2}>0$.

Now $[a,b]\cap(x_0-\delta,x_0+\delta)$ contains a closed interval

15 $[c,d]$ of positive length. Then

$$0<\int_c^d \frac{f(x_0)}{2}\,dx<\int_c^d f(x)\,dx\leq\int_a^b f(x)\,dx=0,$$

which is a contradiction. So $f(x)=0$ for all $x\in[a,b]$.

3 If $f, g : [0,1] \to \mathbb{R}$ are Riemann integrable, then show that the function $h(x) = \min(f(x), g(x))$ is Riemann integrable on $[0,1]$.

Solution

Note $\max(f,g) + \min(f,g) = f+g$ and $\max(f,g) - \min(f,g) = |f-g|$.
Subtracting, then dividing by 2, we have

12 $h = \min(f,g) = \dfrac{f+g-|f-g|}{2}$.

If f, g are integrable, then $f+g$, $f-g$ are integrable.

13 Since $|x|$ is continuous, so $|f-g|$ is integrable.

Therefore $h = \dfrac{f+g-|f-g|}{2}$ is integrable.

④ Let $f:[0,1]\to[-1,1]$ be Riemann integrable. Using the integral criterion, prove that $g(x)=\begin{cases} f(x) & \text{if } 0<x<1 \\ 0 & \text{if } x=0 \text{ or } 1\end{cases}$ is also Riemann integrable on $[0,1]$.

<u>Solution</u>

Since f is Riemann integrable, for every $\varepsilon>0$, $\exists$ partition $P_1=\{0=x_0<x_1<\cdots<x_n=1\}$ such that $U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{3}$.
15 Choose $r\in(0,x_1)$ and $r<\frac{\varepsilon}{6}$. Also choose $s\in(x_{n-1},1)$ and $1-s<\frac{\varepsilon}{6}$. Let $P=P_1\cup\{r,s\}$, then $U(f,P)-L(f,P)<\frac{\varepsilon}{3}$ by the refinement theorem. Note $-1\le g(x)\le 1$ for all $x\in[0,1]$.

Then

$$U(g,P)-L(g,P)\le\left(\max_{x\in[0,r]}g(x)-\min_{x\in[0,r]}g(x)\right)r+\left(U(f,P)-L(f,P)\right)$$

15

$$+\left(\max_{x\in[s,1]}g(x)-\min_{x\in[s,1]}g(x)\right)(1-s)$$

$$<2\times\frac{\varepsilon}{6}+\frac{\varepsilon}{3}+2\times\frac{\varepsilon}{6}=\varepsilon.$$

By the integral criterion, g is Riemann integrable.