MATH2033 Mathematical Analysis (2021 Spring)
Assignment 1

Submission deadline of Assignment 1: **11:59p.m. of 4th Mar, 2020 (Thurs)**

Instruction: Please complete all required problems. Full details (including description of methods used and explanation, key formula and theorem used and final answer) must be shown **clearly** to receive full credits. Marks can be deducted for incomplete solution or unclear solution.

Please submit your completed work via the submission system in canvas before the deadline. Late assignment will not be accepted.

Your submission must (1) be hand-written (typed assignment will not be accepted), (2) in a single pdf. file (other file formats will not be accepted) and (3) contain your full name and student ID on the first page of the assignment.

Problem 1

Write down the opposite statement (negation) for each of the following statements

- **(a)** I will watch a movie and have a dinner outside if tomorrow is sunny or not rainy.
- **(b)** $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|x-y| < \delta$, then $|f(x)-f(y)| < \varepsilon$.
- **(c)** $\forall x \in S$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|x-y| < \delta$, then $|f(x)-f(y)| < \varepsilon$.
- **(d)** $\forall \varepsilon > 0$, $\exists N > 0$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $m, n \geq N$ and $x \in \mathbb{R}$.
- **(e)** $\forall x \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists N > 0$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $m, n \geq N$.

Problem 2

We let $f: A \to B$ be a function. For any subset $Y \subseteq B$, we define the *inverse image* of Y under f (denoted by $f^{-1}(Y)$) as the collection of elements in the domain A that maps to elements in $f(Y)$. That is,

$$f^{-1}(Y) = \{x \in A | f(x) \in Y\}.$$

Prove the following statements

- **(a)** $U \subseteq f^{-1}(f(U))$ for any subset $U \subseteq A$. Give an example which $U \subset f^{-1}(f(U))$
- **(b)** $f(f^{-1}(V)) \subseteq V$ for any subset $V \subseteq B$. Give an example which $f(f^{-1}(V)) \subset V$
- **(c)** $f(\bigcup_{\alpha \in I} X_\alpha) = \bigcup_{\alpha \in I} f(X_\alpha)$ and $f^{-1}(\bigcup_{\alpha \in I} Y_\alpha) = \bigcup_{\alpha \in I} f^{-1}(Y_\alpha)$.
  Here, X_α is subset of A and Y_α is subset of B for all $\alpha \in I$
- **(d)** $f(\bigcap_{\alpha \in I} X_\alpha) \subseteq \bigcap_{\alpha \in I} f(X_\alpha)$ and $f^{-1}(\bigcap_{\alpha \in I} Y_\alpha) = \bigcap_{\alpha \in I} f^{-1}(Y_\alpha)$.

(*Note: In (c) and (d), I is called index set. For example,

$$\bigcup_{\alpha \in I} X_\alpha = \{x | x \in X_\alpha \ \ for\ some\ i \in \alpha\} \quad and \quad \bigcap_{\alpha \in I} X_\alpha = \{x | x \in X_\alpha \ \ for\ all\ i \in \alpha\}$$

(*Note 2: Here, $A \subset B$ means that A is proper subset of B in the sense that $A \subseteq B$ but $A \neq B$)

Problem 3

We let $f: X \to Y$ be a function, prove that f is injective if and only if $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq X$.

(☺Hint: To prove " $\Leftarrow$ " (i.e. $f(A \cap B) = f(A) \cap f(B)$ implies f is injective) part, you can consider "proof by contradiction" and derive a contradiction by considering suitable choices of A and B).

Problem 4

We let $f_1(x), f_2(x), f_3(x), \ldots$ be functions (where $f_k: \mathbb{R} \to \mathbb{R}$ for all $k \in \mathbb{N}$). It is given that

$$A_k = \{x \in \mathbb{R} | f_k(x) = 0\}$$

is countable for any $k \in \mathbb{N}$.

(a

Proposition 16 *Let A_n for $n \in \mathbb{N}$ be countable sets. Then $\bigcup_{n=1}^{\infty} A_n$ is a countable set.*

Proof[1] If some of the A_n are empty then we can just leave them out. If there

(b

are only finitely many non-empty sets left then the result follows by the above remark. Otherwise assume A_n are already the non-empty ones and let

$$\begin{aligned} A_0 &= \{a_{00}, a_{01}, a_{02}, a_{03}, \ldots\} \\ A_1 &= \{a_{10}, a_{11}, a_{12}, a_{13}, \ldots\} \\ A_2 &= \{a_{20}, a_{21}, a_{22}, a_{23}, \ldots\} \\ &\vdots \end{aligned}$$

(*Not

Then following the diagonals again

$$\bigcup_{n=1}^{\infty} A_n = \{a_{00}, a_{01}, a_{10}, a_{02}, a_{11}, a_{20}, \ldots\}. \qquad \blacksquare$$

Note that the above proof requires us to *choose* a list for each A_n, simultaneously.

Problem 5

Prove that the power set $\mathcal{P}(\mathbb{N})$, which is a collection of all subsets (including empty set) of $\mathbb{N}$, is uncountable. Here, $\mathbb{N}$ is the set of positive integers (natural numbers).

(*Note: Mathematically, we can express the power set $\mathcal{P}(\mathbb{N})$ as

$$\mathcal{P}(\mathbb{N}) = \{A | A \subseteq \mathbb{N}\}.)$$

End of Assignment 1