MATH 2033 HW-6 Due Oct 18.

1. Prove that the irrational numbers are dense in real numbers, that is, for any real numbers a and b that $a < b$, there exists an irrational number z such that $a < z < b$.

2. Find a bounded sequence of real numbers that is not convergent. You need to prove that the sequence you provide is not convergent.

3. Let $(a_k)_{k\in\mathbb{N}}$ be a convergent sequence. Show that any of its subsequences converges to the same limit.

4. Let $(a_k)_{k\in\mathbb{N}}$ be a bounded sequence of real numbers. Denote $a^* = \limsup_{k\to\infty} a_k$. Notice that a^* is a real number. Prove that there exists a subsequence of $(a_k)_{k\in\mathbb{N}}$ that converges to the real number a^*.

5. Let $(a_k)_{k\in\mathbb{N}}$ be a bounded sequence of real numbers. Let b be a real number such that $b > \limsup_{k\to\infty} a_k$. Prove that there exists $N \in \mathbb{N}$ such that

$$b > a_k \quad \text{for all } k > N.$$