MATH2033 Mathematical Analysis (2021 Spring)
Assignment 1

Submission deadline of Assignment 1: **<u>11:59p.m. of 4th Mar, 2020 (Thurs)</u>**

Instruction: Please complete all required problems. Full details (including description of methods used and explanation, key formula and theorem used and final answer) must be shown **<u>clearly</u>** to receive full credits. Marks can be deducted for incomplete solution or unclear solution.

<u>Please submit your completed work via the submission system in canvas</u> before the deadline. Late assignment will not be accepted.

Your submission must (1) be hand-written (typed assignment will not be accepted), (2) in a single pdf. file (other file formats will not be accepted) and (3) contain your full name and student ID on the first page of the assignment.

Problem 1

Write down the opposite statement (negation) for each of the following statements

- **(a)** I will watch a movie and have a dinner outside if tomorrow is sunny or not rainy.
- **(b)** $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.
- **(c)** $\forall x \in S$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.
- **(d)** $\forall \varepsilon > 0$, $\exists N > 0$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $m, n \geq N$ and $x \in \mathbb{R}$.
- **(e)** $\forall x \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists N > 0$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $m, n \geq N$.

Problem 2

We let $f: A \to B$ be a function. For any subset $Y \subseteq B$, we define the *inverse image* of Y under f (denoted by $f^{-1}(Y)$) as the collection of elements in the domain A that maps to elements in $f(Y)$. That is,

$$f^{-1}(Y) = \{x \in A | f(x) \in Y\}.$$

Prove the following statements

- **(a)** $U \subseteq f^{-1}\big(f(U)\big)$ for any subset $U \subseteq A$. Give an example which $U \subset f^{-1}\big(f(U)\big)$
- **(b)** $f\big(f^{-1}(V)\big) \subseteq V$ for any subset $V \subseteq B$. Give an example which $f\big(f^{-1}(V)\big) \subset V$
- **(c)** $f(\bigcup_{\alpha \in I} X_\alpha) = \bigcup_{\alpha \in I} f(X_\alpha)$ and $f^{-1}(\bigcup_{\alpha \in I} Y_\alpha) = \bigcup_{\alpha \in I} f^{-1}(Y_\alpha)$.
  Here, X_α is subset of A and Y_α is subset of B for all $\alpha \in I$
- **(d)** $f(\bigcap_{\alpha \in I} X_\alpha) \subseteq \bigcap_{\alpha \in I} f(X_\alpha)$ and $f^{-1}(\bigcap_{\alpha \in I} Y_\alpha) = \bigcap_{\alpha \in I} f^{-1}(Y_\alpha)$.

(*Note: In (c) and (d), I is called index set and

$$\bigcup_{\alpha \in I} X_\alpha = \{x | x \in X_\alpha \ \ for\ some\ \alpha \in I\} \quad and \quad \bigcap_{\alpha \in I} X_\alpha = \{x | x \in X_\alpha \ \ for\ all\ \alpha \in I\}$$

(*Note 2: Here, $A \subset B$ means that A is proper subset of B in the sense that $A \subseteq B$ but $A \neq B$)

Problem 3

We let $f: X \rightarrow Y$ be a function, prove that f is injective if and only if $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq X$.

(☺Hint: To prove " $\Leftarrow$ " (i.e. $f(A \cap B) = f(A) \cap f(B)$ implies f is injective) part, you can consider "proof by contradiction" and derive a contradiction by considering suitable choices of A and B).

Problem 4

We let $f_1(x), f_2(x), f_3(x), \ldots$ be functions (where $f_k: \mathbb{R} \rightarrow \mathbb{R}$ for all $k \in \mathbb{N}$). It is given that

$$A_k = \{x \in \mathbb{R} | f_k(x) = 0\}$$

is countable for any $k \in \mathbb{N}$.

(a) Show that for any $n \in \mathbb{N}$, the set

$$S_n = \left\{x \in \mathbb{R} | \prod_{k=1}^{n} f_k(x) = 0\right\}$$

is countable.

(b) Determine if the set

$$S = \left\{x \in \mathbb{R} | \prod_{k=1}^{\infty} f_k(x) = 0\right\}$$

is countable.

(☺Hint: If your answer is yes, please give a mathematical proof. If your answer is no, please give a counter-example (you need to specify the functions $f_k(x)$ in your answer).

(*Note: Here,

$$\prod_{k=1}^{n} f_k(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot \ldots \cdot f_n(x)$$

$$\prod_{k=1}^{\infty} f_k(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot \ldots.$$

Problem 5

Prove that the power set $\mathcal{P}(\mathbb{N})$, which is a collection of all subsets (including empty set) of $\mathbb{N}$, is uncountable. Here, $\mathbb{N}$ is the set of positive integers (natural numbers).

(*Note: Mathematically, we can express the power set $\mathcal{P}(\mathbb{N})$ as

$$\mathcal{P}(\mathbb{N}) = \{A | A \subseteq \mathbb{N}\}.)$$

****End of Assignment 1****