MATH2033 Mathematical Analysis
Suggested Solution of Problem Set 6

Problem 1

Prove the following limits using the definition of limits (ε-δ definition)

(a) $\lim\limits_{x\to 1}\frac{x}{x+1}=\frac{1}{2}$.

(b) $\lim\limits_{x\to c}x^3=c^3$, where $c\in\mathbb{R}$.

(c) $\lim\limits_{x\to 0}x\sin\frac{1}{x}=0$ and $\lim\limits_{x\to\frac{\pi}{2}}x\cos x=0$.

☺Solution

(a) For any $\varepsilon>0$, we take $\delta=\min(1,2\varepsilon)$. Then for any $0<|x-1|<\delta$, we have

$$\left|\frac{x}{x+1}-\frac{1}{2}\right|=\left|\frac{x-1}{2(x+1)}\right|\ \underset{\substack{|x-1|<1\\0<x<2\\\Rightarrow 1<x+1<3}}{\le}\ \frac{1}{2(1)}|x-1|\ \underset{|x-1|<2\varepsilon}{<}\ \varepsilon$$

So $\lim\limits_{x\to 1}\frac{x}{x+1}=\frac{1}{2}$ by definition

(b) Recall the triangle inequality $|a+b|\le|a|+|b|$. By taking $a=x-y$ and $b=y$, we have

$$|x|\le|x-y|+|y|\Rightarrow|x-y|\ge|x|-|y|.$$

For any $\varepsilon>0$, we take $\delta=\min\left(1,\frac{\varepsilon}{3|c|^2+3|c|+1}\right)$ Then for any $0<|x-c|<\delta$, we have

$$\begin{aligned}|x^3-c^3|&=|x-c||x^2+cx+c^2|\\&\le|x-c|(|x|^2+|c||x|+|c|^2)\le^{(*)}|x-c|((|c|+1)^2+|c|(|c|+1)\\&\quad+|c|^2)\le|x-c|(3|c|^2+3|c|+1)<\varepsilon\end{aligned}$$

(*Note: The inequality follows from the fact that $|x|-|c|\le|x-c|\le 1\Rightarrow|x|\le|c|+1$).

Therefore, we conclude that $\lim\limits_{x\to c}x^3=c^3$.

(c) Prove of $\lim\limits_{x\to 0}x\sin\frac{1}{x}=0$

For any $\varepsilon>0$, we take $\delta=\varepsilon$. Then for any $|x-0|<\varepsilon$, we have

$$\left|x\sin\frac{1}{x}-0\right|\le|x|<\varepsilon.$$

So $\lim\limits_{x\to 0}x\sin\frac{1}{x}=0$ by definition.

Prove of $\lim\limits_{x\to\frac{\pi}{2}}x\cos x=0$

For any $\varepsilon>0$, we take $\delta=^{(*)}\min(\frac{\pi}{4},\frac{\pi}{2}-\cos^{-1}\frac{4\varepsilon}{3\pi},\cos^{-1}\left(-\frac{4\varepsilon}{3\pi}\right)-\frac{\pi}{2})$. (*Note: We take $\cos^{-1}x\in[0,\pi]$ for $x\in[-1,1]$.

Then for any $0 < \left|x - \frac{\pi}{2}\right| < \delta$,we have

$$-\frac{4\varepsilon}{3\pi} = \cos\left(\cos^{-1}\left(-\frac{4\varepsilon}{3\pi}\right)\right) < \cos\left(\frac{\pi}{2} + \delta\right) < \cos x < \cos\left(\frac{\pi}{2} - \delta\right)$$

$$< \cos\left(\cos^{-1}\frac{4\varepsilon}{3\pi}\right) = \frac{4\varepsilon}{3\pi}$$

$$\Rightarrow |\cos x| < \frac{4\varepsilon}{3\pi}$$

Hence, we deduce that

$$|x\cos x - 0| \overset{\substack{\left|x-\frac{\pi}{2}\right|<\frac{\pi}{4} \\ \Rightarrow \frac{\pi}{4}<x<\frac{3\pi}{4}}}{<} \frac{3\pi}{4}|\cos x| < \frac{3\pi}{4}\left(\frac{4\varepsilon}{3\pi}\right) = \varepsilon.$$

So we conclude that $\lim_{x\to\frac{\pi}{2}} x\cos x = 0$.

Problem 2

Prove the following limits using the definition of limits

(a) $\lim_{x\to\infty} \cos\frac{1}{x} = 1$ (☺Hint: Recall that $\frac{1}{x} \to 0$ when $x \to \infty$, so $\frac{1}{x} < \frac{\pi}{2}$ when x is large).

(b) $\lim_{x\to-\infty} e^x = 0$

(c) $\lim_{x\to\infty} e^x = \infty$

☺Solution

(a) For any $\varepsilon > 0$, we take $K = \max\left(\frac{2}{\pi}, \frac{1}{\cos^{-1}(1-\varepsilon)}\right)$. Then for any $x > K$, we have

$$\cos\frac{1}{x} > \cos\left(\frac{1}{\frac{1}{\cos^{-1}(1-\varepsilon)}}\right) = 1 - \varepsilon.$$

(*Note that $0 < \frac{1}{x} < \frac{\pi}{2}$ for $x > K$ and y is decreasing over $y \in \left[0, \frac{\pi}{2}\right]$.)

Together with the fact that $\cos\frac{1}{x} \le 1 < 1 + \varepsilon$, we have

$$1 - \varepsilon < \cos\frac{1}{x} < 1 + \varepsilon \Rightarrow -\varepsilon < \cos\frac{1}{x} - 1 < \varepsilon \Rightarrow \left|\cos\frac{1}{x} - 1\right| < \varepsilon.$$

So we get $\lim_{x\to\infty} \cos\frac{1}{x} = 1$.

(b) For any $\varepsilon > 0$, we pick $K = \ln\varepsilon$. Then for any $x < K$, we have

$$|e^x - 0| = e^x < e^{\ln\varepsilon} = \varepsilon.$$

So $\lim_{x\to-\infty} e^x = 0$ by definition.

(c) For any $M > 0$, we take $K = \ln M$. Then for any $x > K$, we have

$$e^x > e^{\ln M} = M.$$

So $\lim_{x\to-\infty} e^x = \infty$ by definition.

Problem 3

We let $[x]$ denotes the greatest integer less than or equal to x.

(a) We let c be an integer. Determine if the limits $\lim_{x\to c}[x]$ exists.

(😊Hint: Try an example when $c = 3$)

(b) We let d be a non-integer. Determine if the limits $\lim_{x\to d}[x]$ exists

😊Solution

(a) We note that

$$\lim_{x\to c^-}[x] = \lim_{x\to c^-}(c-1) = c-1 \quad and \quad \lim_{x\to c^+}[x] = \lim_{x\to c^+} c = c$$

Since $\lim_{x\to c^-}[x] \neq \lim_{x\to c^+}[x]$, so $\lim_{x\to c}[x]$ does not exists when c is integer.

(b) If d is not integer, then

$$\lim_{x\to d^-}[x] = [d] \quad and \quad \lim_{x\to d^+}[x] \overset{[d]<d<x<[d]+1}{=} [d] = [d].$$

Since $\lim_{x\to d^-}[x] = \lim_{x\to d^+}[x]$, so $\lim_{x\to d}[x] = [d]$ exists.

Problem 4

We let $f: \mathbb{R} \to \mathbb{R}$ be a function which $\lim_{x\to 0} f(x) = L \in \mathbb{R}$. Let $a > 0$ be a positive number and define $g: \mathbb{R} \to \mathbb{R}$ as $g(x) = f(ax)$.

(a) Show that $\lim_{x\to 0} g(x) = L$ using the definition of limits.

(b) Redo **(a)** using the sequential limits theorem.

😊Solution

(a) For any $\varepsilon > 0$,

- Since $\lim_{x\to 0} f(x) = L$, then there exists $\delta_1 > 0$ such that

$$|f(x) - L| < \varepsilon \quad for \;\; 0 < |x - 0| < \delta_1.$$

- We take $\delta = \frac{\delta_1}{a} > 0$, then for any $0 < |x - 0| < \delta \Rightarrow 0 < |ax| < \delta_1$, we have

$$|g(x) - L| = |f(ax) - L| < \varepsilon.$$

Thus, $\lim_{x\to 0} g(x) = L$ by definition.

(b) For any convergent sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = 0$.

- Since $\lim_{x\to 0} f(x) = L$, then $\lim_{n\to\infty} f(y_n) = L$ for any sequence $\{y_n\}$ with $\lim_{n\to\infty} y_n = 0$.
- As $\lim_{n\to\infty} ax_n = 0$, we take $y_n = ax_n$ and deduce that

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(ax_n) = \lim_{n\to\infty} f(y_n) = L.$$

Problem 5

(a) We let $f: \mathbb{R} \to \mathbb{R}$ be a function which $\lim_{x\to x_0} f(x) = L$. Show that there exists $\delta > 0$ and $M > 0$ such that $|f(x)| < M$ for all $|x - x_0| < \delta$.

(😊Hint: You can consider the definition of limits)

(b) We let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be two functions which $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Using the definition of limits (ε-δ definition), prove that $\lim_{x\to a} f(x)g(x) = LM$.

(☺Hint: Write $f(x)g(x) - LM = f(x)g(x) - f(x)M + f(x)M - LM$. Also, the result in (a) is also useful.)

☺Solution

Using triangle inequality, we deduce that for any $x, y \in \mathbb{R}$

$$|x| \le |(x-y)+y| \le |x-y| + |y| \Rightarrow |x| - |y| \le |x-y|.$$

(a) Since $\lim_{x\to x_0} f(x) = L$, then for $\varepsilon = 1$, there exists $\delta > 0$ such that

$$\underbrace{|f(x) - L|}_{\ge |f(x)|-|L|} < \varepsilon = 1 \Rightarrow |f(x)| \le \underbrace{|L| + 1}_{=M}.$$

(b) As $\lim_{x\to a} f(x) = L$, one can deduce from (a) that there exists $C_1 > 0$, $\delta_1 > 0$ such that

$$|f(x)| \le C_1 \quad for\ \ 0 < |x-a| < \delta_1$$

Using the definition of limits, we deduce that for any $\varepsilon > 0$, there exists $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$|f(x) - L| \le \frac{\varepsilon}{2|M|} \quad for\ \ 0 < |x-a| < \delta_2$$

$$|g(x) - M| \le \frac{\varepsilon}{2C_1} \quad for\ \ 0 < |x-a| < \delta_3$$

We pick $\delta = \min(\delta_1, \delta_2, \delta_3)$, then for any $0 < |x-a| < \delta$, we have

$$\begin{aligned} |\,f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &\le |f(x)||(g(x) - M)| + |M||f(x) - L| \le C_1\left(\frac{\varepsilon}{2C_1}\right) + |M|\left(\frac{\varepsilon}{2|M|}\right) \\ &= \varepsilon. \end{aligned}$$

(*Provided that $M \ne 0$. When $M = 0$, $|\,f(x)g(x) - LM| \le |f(x)||(g(x) - M)| \le C_1\left(\frac{\varepsilon}{2C_1}\right) = \frac{\varepsilon}{2} < \varepsilon$.)

Thus, $\lim_{x\to a} f(x)g(x) = LM$ by definition.

Problem 6

We let $f: \mathbb{R} \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} x & if\ \ x \in \mathbb{Q} \\ 0 & if\ otherwise \end{cases}.$$

(a) Show that $\lim_{x\to 0} f(x)$ exists.

(b) Show that $\lim_{x\to c} f(x)$ does not exist for any $c \ne 0$.

☺Solution

(a) We shall argue that $\lim_{x\to 0} f(x) = 0$ by definition of limit. For any $\varepsilon > 0$, we take $\delta = \varepsilon$. Then for any $0 < |x - 0| < \delta < \varepsilon$, we have

$$|f(x) - 0| = |f(x)| = \begin{cases} |x| & if\ \ x \in \mathbb{Q} \\ 0 & if\ otherwise \end{cases} \le |x| < \varepsilon.$$

Thus we conclude that $\lim_{x\to 0} f(x) = 0$.

(b) From the result established in Example 2 of Lecture Note 6, there exists a sequence of rational number $\{q_n\}$ and a sequence of irrational number $\{r_n\}$ which both sequences converge to c.

Then we deduce that

$$\lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} q_n = c \neq 0 \quad and \quad \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} r_n = 0$$

Since $\lim_{n\to\infty} f(q_n) \neq \lim_{n\to\infty} f(r_n)$, it follows from sequential limit theorem that $\lim_{x\to c} f(x)$ does not exist.

Continuity

Problem 7

We consider a function $f: (0,\infty) \to \mathbb{R}$ defined by $f(x) = \frac{[x]}{x}$, where $[x]$ denotes the greatest integer less than or equal to x.

(a) Determine if $f(x)$ is continuous at $x = 1$.

(b) Determine if $f(x)$ is continuous at $x = 2.5$.

☺Solution

(a) We observe that $f(1) = \frac{1}{1} = 1$. On the other hand, we deduce that

$$\lim_{x\to 1^+} \frac{[x]}{x} = \lim_{x\to 1^+} \frac{1}{x} = 1 \quad and \quad \lim_{x\to 1^-} \frac{[x]}{x} = \lim_{x\to 1^-} \frac{0}{x} = \lim_{x\to 1^-} 0 = 0$$

Since $\lim_{x\to 1^+} \frac{[x]}{x} \neq \lim_{x\to 1^-} \frac{[x]}{x}$, so $\lim_{x\to 1} \frac{[x]}{x}$ does not exist. Thus, f is not continuous at $x = 1$.

(b) We observe that $f(2) = \frac{2}{2.5} = \frac{4}{5}$. On the other hand, we deduce that

$$\lim_{x\to 2.5^+} \frac{[x]}{x} = \lim_{x\to 2.5^+} \frac{2}{x} = \frac{4}{5} \quad and \quad \lim_{x\to 2.5^-} \frac{[x]}{x} = \lim_{x\to 2.5^-} \frac{2}{x} = \frac{4}{5}$$

Then $\lim_{x\to 2.5} \frac{[x]}{x}$ exist. Since $\lim_{x\to 2.5} \frac{[x]}{x} = \frac{4}{5} = f(2)$, we conclude that f is not continuous at $x = 2.5$.

Problem 8

(a) We let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be two continuous functions on $\mathbb{R}$, show that the function $h(x) = \min\big(f(x), g(x)\big)$ is continuous on $\mathbb{R}$.

(b) We let $f_1, f_2, \dots, f_n: \mathbb{R} \to \mathbb{R}$ be n continuous functions on $\mathbb{R}$. Using the result of (a), show that $p(x) = \min\big(f_1(x), f_2(x), \dots, f_n(x)\big)$ is continuous on $\mathbb{R}$.

(☺Hint: You can try mathematical induction)

☺Solution

(a) One can verify that (left as exercise)

$$\min\big(f(x), g(x)\big) = \frac{1}{2}\big(f(x) + g(x)\big) - \frac{|f(x) - g(x)|}{2}.$$

Since $f(x), g(x)$ and $|x|$ are continuous, then

- $f(x)+g(x)$ and $f(x)-g(x)$ are continuous
- $|f(x)-g(x)|$ is continuous (i.e. composition).
- $\min\big(f(x),g(x)\big)=\frac{1}{2}\big(f(x)+g(x)\big)-\frac{|f(x)-g(x)|}{2}$ is continuous.

(b) We shall prove it by induction.

- For $n=1$, $\min\big(f_1(x)\big)=f_1(x)$ is continuous.
- Assume that the statement is true for $n=k$, then for $n=k+1$,
  $\min\big(f_1(x),f_2(x),\dots,f_{k+1}(x)\big)=\min\Big(\min\big(f_1(x),f_2(x),\dots,f_k(x)\big),f_{k+1}(x)\Big)$
  Since both $\min\big(f_1(x),f_2(x),\dots,f_k(x)\big)$ and $f_{k+1}(x)$ are continuous by assumption, so $\min\big(f_1(x),f_2(x),\dots,f_{k+1}(x)\big)$ is also continuous by the result of (a). So the statement is also true for $n=k+1$.
- By induction, $\min\big(f_1(x),f_2(x),\dots,f_n(x)\big)$ is continuous for all $n\in\mathbb{N}$ and the proof is completed.

Problem 8

We consider a function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} x-x^3 & if\ \ x\in\mathbb{Q} \\ 0 & if\ otherwise\end{cases}.$$

(a) Show that the function is continuous at $x=0$ and $x=\pm1$.

(b) Show that the function is not continuous at point $x=x_0$ where $x_0\neq 0,-1,1$.

☺Solution

(a) We first examine the continuity at $x=0$. To do so, we shall argue that $\lim\limits_{x\to0}f(x)=0$ using sequential limits theorem.

We consider any convergent sequence $\{x_n\}$ with $\lim\limits_{n\to\infty}x_n=0$. One can see that

$$0\leq|f(x_n)|=\begin{cases}|x_n-x_n^3| & if\ \ x\in\mathbb{Q}\\ 0 & if\ otherwise\end{cases}\leq|x_n-x_n^3|$$

By taking limits on both sides, we get

$$0\leq\lim_{n\to\infty}|f(x_n)|\leq\lim_{n\to\infty}|x_n-x_n^3|\overset{\lim\limits_{n\to\infty}x_n=0}{=\!=}0.$$

It follows from sandwich theorem that $\lim\limits_{n\to\infty}|f(x_n)|=0$. Hence, we deduce that $\lim\limits_{n\to\infty}f(x_n)=0$ (see note) and $\lim\limits_{x\to0}f(x)=0$ by sequential limits theorem.

Since $\lim\limits_{x\to0}f(x)=f(0)=0$, so $f(x)$ is continuous at $x=0$.

Using similar methods, one can deduce that $f(x)$ is also continuous at $x=\pm1$ (as $\lim\limits_{x\to1}f(x)=f(1)=0$ and $\lim\limits_{x\to-1}f(x)=f(-1)=0$)

(b) Note that $x-x^3=0\Leftrightarrow x(1-x)(1+x)=0\Leftrightarrow x=0,1\ or-1$. Thus for any $x_0\neq0,-1,1$, we have $x_0-x_0^3\neq0$.

- We let $\{r_n\}$ be a sequence of rational number which $\lim\limits_{n\to\infty}r_n=x_0$ (see Example 2 in Lecture Note 6). Then we have $\lim\limits_{n\to\infty}f(r_n)=\lim\limits_{n\to\infty}(r_n-r_n^3)=x_0-x_0^3\neq0.$

- We let $\{s_n\}$ be a sequence of irrational number which $\lim_{n\to\infty} s_n = x_0$ (see Example 2 in Lecture Note 6). Then we have $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} 0 = 0$.

Since $\lim_{n\to\infty} f(r_n) \neq \lim_{n\to\infty} f(s_n)$, it follows from sequential limit theorem that $\lim_{x\to x_0} f(x)$ does not exist and $f(x)$ is not continuous at $x = x_0$.

Problem 9

Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ which f is discontinuous at any point on $\mathbb{R}$ but $|f|$ is continuous on $\mathbb{R}$.

😊Solution

We consider a function f defined by

$$f(x) = \begin{cases} 1 & if \;\; x \in \mathbb{Q} \\ -1 & if \;\; x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}.$$

We first argue that $f(x)$ is discontinuous at any $x_0 \in \mathbb{R}$.

- We let $\{r_n\}$ be a sequence of rational number which $\lim_{n\to\infty} r_n = x_0$ (see Example 2 in Lecture Note 6). Then we have $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 1 = 1$.
- We let $\{s_n\}$ be a sequence of irrational number which $\lim_{n\to\infty} s_n = x_0$ (see Example 2 in Lecture Note 6). Then we have $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} -1 = -1$.

Since $\lim_{n\to\infty} f(r_n) \neq \lim_{n\to\infty} f(s_n)$, it follows from sequential limit theorem that $\lim_{x\to x_0} f(x)$ does not exist and $f(x)$ is not continuous at $x = x_0$.

On the other hand, we have $|f(x)| = 1$ for all $x \in \mathbb{R}$ which is continuous at any $x = x_0$ (since $\lim_{x\to x_0} |f(x)| = \lim_{x\to x_0} 1 = 1 = |f(x_0)|$).

Problem 10

We let $f, g: \mathbb{R} \to \mathbb{R}$ be two continuous functions on $\mathbb{R}$ such that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

😊Solution

Assignment 3 problem. Solution will be posted later.

Problem 11

(a) Show that the equation $x = \cos x$ has a solution in the interval $\left[0, \frac{\pi}{2}\right]$.

(b) Show that the equation $x^4 + 7x^3 - 9 = 0$ has at least two real solutions.

😊Solution

(a) We let $g(x) = x - \cos x$. Note that

- $g(x)$ is continuous on $\left[0, \frac{\pi}{2}\right]$;
- $g(0) = 0 - \cos 0 = -1 < 0$ and
- $g(1) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} > 0$.

It follows from intermediate value theorem that there is $x_0 \in \left(0, \frac{\pi}{2}\right)$ such that $g(x_0) = 0 \Leftrightarrow x_0 = \cos x_0$.

(b) We let $f(x) = x^4 + 7x^3 - 9$

- We consider the interval $[0,1]$.
  Since $f(0) = -9 < 0$ and $f(2) = 63 > 0$. It follows from intermediate value theorem that there exists $c_1 \in (0,1)$ such that $f(c_1) = 0$.
- We consider the interval $[-10,0]$
  Since $f(0) = -9 < 0$ and $f(-10) = 2991 > 0$. It follows from intermediate value theorem that there exists $c_2 \in (-10,0)$ such that $f(c_2) = 0$.

Therefore, we conclude that there are at least two real roots.

Problem 12

We let $L > 0$ be a positive number and let $f: [a, b] \to \mathbb{R}$ be a continuous function. Suppose that for any $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that

$$|f(x_n) - L| < \frac{1}{2^n}.$$

Show that there exists $x^* \in [a, b]$ such that $f(x^*) = L$.

😊Solution

Note that the sequence $\{x_n\}$ is bounded (since $a \le x_n \le b$), it follows from Bolzano-Weierstrass theorem that there exists a convergent subsequence $\{x_{n_k}\}$ with $\lim_{k\to\infty} x_{n_k} = x^*$, where $x^* \in [a, b]$.

On the other hand, we note that

$$\left|f\left(x_{n_k}\right) - L\right| < \frac{1}{2^{n_k}} \quad for \ \ k \in \mathbb{N}.$$

By taking $k \to \infty$ (or $n_k \to \infty$), we have

$$0 \le \lim_{k\to\infty} \left|f\left(x_{n_k}\right) - L\right| \le \lim_{k\to\infty} \frac{1}{2^{n_k}} = 0.$$

It follows from sandwich theorem that $\lim_{k\to\infty} \left|f\left(x_{n_k}\right) - L\right| = 0$

Since $f(x)$ and $|x|$ are continuous, then $g(x) = |f(x) - L|$ is also continuous at $x = x^*$. It follows that

$$\lim_{k\to\infty} \underbrace{\left|f\left(x_{n_k}\right) - L\right|}_{g\left(x_{n_k}\right)} = \lim_{x\to x^*} g(x) = g(x^*) = |f(x^*) - L| = 0 \Rightarrow f(x^*) = L.$$

Problem 13

We let $f: [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Suppose that there exists $c \in (a, b)$ such that $f(c) > f(x)$ for all $x \in [a, b]$, show that $f(x)$ is not injective.

(☺Hint: Draw a figure and get some idea)

😊Solution

We pick $y_0 \in \left(\max\left(f(a), f(b)\right), f(c)\right)$.

- Since $f(a) \leq \max\big(f(a), f(b)\big) < y_0$ and $f(c) > y_0$, it follows from intermediate value theorem (since f is continuous) that there exists $c_1 \in (a, c)$ such that $f(c_1) = y_0$.
- Since $f(b) \leq \max\big(f(a), f(b)\big) < y_0$ and $f(c) > y_0$, it follows from intermediate value theorem (since f is continuous) that there exists $c_2 \in (c, b)$ such that $f(c_2) = y_0$.
- Since (a, c) and (c, b) are disjoint, so $c_1 \neq c_2$. Since $f(c_1) = f(c_2) = y_0$, thus f is not injective.