MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #17

Differentiability (Part 2)

Generalized Mean Value Theorem: (OR Cauchy Mean Value Theorem)
If $f(x)$ and $g(x)$ are both continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$\big(g(b)-g(a)\big)f'(c) = \big(f(b)-f(a)\big)g'(c)$$

If $g(a) \neq g(b)$, we can rewrite the formula as

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Proof: (Since it will be useful in later examples)
Define

$$F(x) = g(x)[f(b)-f(a)] - f(x)[g(b)-g(a)]$$

Note that $\mathbf{F(a)} = \mathbf{F(b)} = g(a)f(b) - f(a)g(b)$, then by Rolle's Theorem,
$\exists c \in (a,b)$ such that $F'(c) = 0$,
$\rightarrow \big(g(b)-g(a)\big)f'(c) = \big(f(b)-f(a)\big)g'(c)$

Example 1
By Consider Suitable $f(x)$ and $g(x)$, show that for $x > 0$

$$\frac{\tan^{-1} x}{e^x - 1} < 1$$

<u>Solution:</u>

Pick $f(x) = \tan^{-1} x \rightarrow f'(x) = \frac{1}{1+x^2}$ and $g(x) = e^x \rightarrow g'(x) = e^x$

Using Generalized Mean Value Theorem on $[0,x]$, we get

$$\frac{f(x)-f(0)}{g(x)-g(0)} = \frac{f'(c)}{g'(c)} \qquad \text{where } c \in (0,x)$$

$$\rightarrow \frac{\tan^{-1} x}{e^x - 1} = \frac{1}{e^c(1+c^2)} < \frac{1}{e^0(1+0)} = 1$$

Example 2
a) Suppose $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) (where $b > a > 0$). Show that there exists $c \in (a,b)$ such that

$$f(b) - f(a) = \ln\left(\frac{b}{a}\right) c f'(c)$$

b) Show that the sequence $b_n = n(b^{\frac{1}{n}} - 1)$ converges to $\ln b$ (for $b > 1$)

(Use of L'Hopital's Rule is NOT allowed)

Solution:

(IDEA: Note that $f(b)-f(a)=\ln\left(\frac{b}{a}\right)cf'(c)$

$\rightarrow f(b)-f(a)=(\ln b-\ln a)cf'(c)\rightarrow \dfrac{f(b)-f(a)}{\ln b-\ln a}=cf'(c)$

a) Put $g(x)=\ln x$ and apply generalized mean value theorem, we get

$$\frac{f(b)-f(a)}{\ln b-\ln a}=\frac{f'(c)}{\frac{1}{c}}\rightarrow\frac{f(b)-f(a)}{\ln b-\ln a}=cf'(c)\rightarrow f(b)-f(a)=\ln\left(\frac{b}{a}\right)cf'(c)$$

b) (IDEA: Since the limit is given, we try to prove by definition directly)

(Note that

$|b_n-\ln b|=\left|n\left(b^{\frac{1}{n}}-1\right)-\ln b\right|$

$=\left|n\left(b^{\frac{1}{n}}-1\right)-n\left(1^{\frac{1}{n}}-1\right)-\ln b\right|$ since $n\left(1^{\frac{1}{n}}-1\right)=0$

$\boxed{\text{Pick } f(x)=n\left(x^{\frac{1}{n}}-1\right)\rightarrow f'(x)=n\left(\frac{1}{n}x^{\frac{1}{n}-1}\right)=x^{\frac{1}{n}-1}\text{, apply a)}}$

$=\left|c\left(c^{\frac{1}{n}-1}\right)\ln b-\ln b\right|$ $c\in(1,b)$

$=\left|c^{\frac{1}{n}}\ln b-\ln b\right|=\ln b\left|c^{\frac{1}{n}}-1\right|$

$<(\ln b)(b^{\frac{1}{n}}-1)<\varepsilon$

We require $\ln b\left(b^{\frac{1}{n}}-1\right)<\varepsilon\rightarrow n>\frac{\ln b}{\ln\left(\frac{\varepsilon}{\ln b}+1\right)}$

Solution:

By Archimedean Property, there exists K such that $K>\frac{\ln b}{\ln\left(\frac{\varepsilon}{\ln b}+1\right)}$

Then for $n>K$, we get $|b_n-\ln b|<\varepsilon$

Hence $\{b_n\}$ converges and $\lim_{n\rightarrow\infty}b_n=\ln b$

Next, we will show you an example which Cauchy Sequence cannot be applied directly.

Example 3

Let $f(x)$ and $g(x)$ are 2 functions defined on $[a,+\infty)$ such that both are continuous on $[a,+\infty)$ and differentiable on $(a,+\infty)$, $g(x)\neq 0$. Suppose $\lim_{x\rightarrow\infty}f(x)=\lim_{x\rightarrow\infty}g(x)=0$, show that there exists $x_0\in(a,\infty)$ such that

$$\frac{f(a)}{g(a)}=\frac{f'(x_0)}{g'(x_0)}$$

(IDEA: The conclusion looks like the Generalized Mean Value Theorem

$$\frac{f(a)-f(\infty)}{g(a)-g(\infty)}=\frac{f'(x_0)}{g'(x_0)}$$

But there are 2 differences:

1. $f(\infty)$ and $g(\infty)$ are undefined, even though $\lim_{x\to\infty} f(x)=\lim_{x\to\infty} g(x)=0$
2. The Theorem can only be applied in $[a,b]$ BUT NOT $[a,\infty)$

There are two possible methods, the first one is to review the proof of Generalized Mean Value Theorem and the second one is try to construct 2 new functions and apply the theorem to new functions.

<u>Solution: (Method 1)</u>

Consider the function

$F(x)=-g(x)f(a)+f(x)g(a)$

$F(a)=-g(a)f(a)+f(a)g(a)=0$

$$\lim_{x\to\infty} F(x)=\lim_{x\to\infty} -g(x)f(a)+f(x)g(a)=0+0=0$$

Case i) If $F(x)=0$, then $F'(x)=0 \to \frac{f(a)}{g(a)}=\frac{f'(x)}{g'(x)}$

Case ii) If $F(x)\neq 0$,

Pick c such that $F(c)\neq 0$, assume $F(c)>0$ (The case when $F(c)<0$ is similar, left as exercise)

Since $\lim_{x\to\infty} F(x)=0$

For $\varepsilon=\frac{F(c)}{2}$, there exists $M\in\mathbf{R}$ such that $x\geq M\to |F(x)|<\frac{F(c)}{2}$

Then the maximum should occurs in (a,M)

Using Extreme Value Theorem on $[a,M]$, there exists $x_0\in(a,M)$ such that $F(x)\leq F(x_0)$

By local extreme theorem, $F'(x_0)=0\to -g'(x_0)f(a)+f'(x_0)g(a)=0$

$$\to \frac{f(a)}{g(a)}=\frac{f'(x_0)}{g'(x_0)}$$

☺Exercise 1

From Example 2, try to work out the case $F(c)<0$

Solution: (Method 2)

Construct two functions $\mathrm{F(x)}$ and $\mathrm{G(x)}$ defined on $[0, e^{-a}]$ to be

$$F(x) = \begin{cases} f(-\ln x) = f\left(\ln\left(\frac{1}{x}\right)\right) & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases} \quad \text{and}$$

$$G(x) = \begin{cases} g(-\ln x) = g\left(\ln\left(\frac{1}{x}\right)\right) & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$$

(We define $F(0) = G(0) = 0$ because the fact that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$.) You can check $F(x)$ and $G(x)$ are continuous on $[0, e^{-a}]$ and differentiable on $(0, e^{-a})$.

Applying Generalized Mean Value Theorem, we have

$$\frac{F(e^{-a}) - F(0)}{G(e^{-a}) - G(0)} = \frac{F'(c)}{G'(c)} \quad \text{for some } c \in (0, e^{-a})$$

$$\to \frac{f(a)}{g(a)} = \frac{-\frac{1}{c}f'(-\ln c)}{-\frac{1}{c}g(-\ln c)} = \frac{f'(-\ln c)}{g'(-\ln c)} = \frac{f'(x_0)}{g'(x_0)} \quad \text{where } x_0 = -\ln c \in (a, \infty)$$

<u>Theorem: L'Hopital's Rule</u>

Suppose $f, g: (a, b) \to \mathbf{R}$ are differentiable functions on (a, b), with $g'(x) \neq 0$ for all $x \in (a, b)$. Suppose that either

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \left(\frac{0}{0} \text{ case}\right)$$

$$\text{or} \quad \lim_{x\to a} g(x) = \pm\infty \quad \left(\frac{*}{\infty} \text{ case}\right)$$

If $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$

where $L \in \mathbf{R}$ or $L = \pm\infty$

☹Caution: "$\lim_{x\to a} \frac{f'(x)}{g'(x)}$ does not exist" DOES NOT imply "$\lim_{x\to a} \frac{f(x)}{g(x)} = L$ does not exist"

Theorem: (Taylor Theorem)

Let $f: (a, b) \to R$ be n-time differentiable on (a, b), then for any $x, c \in (a, b)$, there exists $x_0 \in (a, b)$ such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots.. + \frac{f^{(n-1)}(c)}{(n-1)!}(x - c)^{n-1} + \frac{f^{(n)}(x_0)}{n!}(x - c)^n$$

Using Taylor Theorem, one can express some differentiable functions into series. Some Examples are shown (The derivation is left as exercise)

Example 4

(1) $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \ldots .. = \sum_{k=0}^{\infty}(-1)^k x^k$

(2) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .. = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

(3) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .. = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$

(4) $\cos = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .. = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$

(5) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots .. = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{k}$

(6) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots .. = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1}$

Solution:

(1) Let $f(x) = \frac{1}{1+x}$, by computing a first few derivatives, we get

$$f'(x) = -\frac{1}{(1+x)^2}, f''(x) = \frac{2}{(1+x)^3}, f^{(3)}(x) = \frac{3!}{(1+x)^4} \ldots f^{(n)}(x) = \frac{(-1)^n n!}{(1+x)^{n+1}}$$

Therefore by Taylor Theorem (at $c = 0$), we have

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots .. + \frac{f^{(n)}(0)}{(n)!}x^n + \cdots ..$$

$$\rightarrow \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \ldots .. = \sum_{k=0}^{\infty}(-1)^k x^k$$

(2)-(6) are left as exercises

Example 5

Let $f: I \rightarrow \mathbf{R}$ be $(n+1)$-times differentiable on I (where I is any interval). If $f^{(n+1)}(x) = 0$ for all $x \in I$, then on the interval I, f is a polynomial with degree at most n

Solution:

Applying Taylor Theorem up to x^{n+1} terms (around any $c \in I$, we get

$$f(x) = f(c) + f'(c)(x-c) + \cdots . + \frac{f^{(n)}(c)(x-c)^n}{n!} + \frac{f^{(n+1)}(x_0)(x-c)^{n+1}}{(n+1)!}$$

Since $f^{(n+1)}(x_0) = 0$, then

$$f(x) = f(c) + f'(c)(x-c) + \cdots . + \frac{f^{(n)}(c)(x-c)^n}{n!}$$

which is a polynomial with degree at most n.

Try to work on the following exercises to understand the material, you are welcome to give your solution to me for comments.

☺Exercise 2

Show that $\dfrac{\ln(1+x^2)}{e^{x^2}} < 1 \quad for\ all\ x \in \mathbf{R}$

☺Exercise 3

Show that the sequence $a_n = n\tan^{-1}\left(\frac{1}{n}\right)$ converges to 1

(Hint: Use Mean Value Theorem and consider $f(x) = n\tan^{-1}\left(\frac{x}{n}\right)$)

☺Exercise 4

If $f(x)$ and $g(x)$ satisfy the conditions of Generalized Mean Value Theorem in $[a, b]$ and $a < b$. Prove that there exists $c \in (a, b)$ such that

$$\frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}$$

(Hint: The above equality can be rewritten as

$$f(c)g'(c) + f'(c)g(c) - f'(c)g(b) - f(a)g'(c) = 0$$

$$\rightarrow \frac{d}{dx}\big(f(x)g(x) - f(x)g(b) - f(a)g(c)\big) = 0$$

Apply the mean value theorem to appropriate $F(x)$ (you need to construct it))

☺Exercise 5

Let C be a curve in a plane with parameterization $\mathbf{r}(t) = (x(t), y(t))$, $t \in [0,1]$ and has derivatives at all $t \in (0,1)$. Show that if $x(0) \neq x(1)$, then there exists $t_0 \in (0,1)$ such that the tangent to C at $(x(t_0), y(t_0))$ is parallel to the line segment joining $((x(0), y(0))$ and $((x(1), y(1))$

(Hint: Draw the graph)

☺Exercise 6

Let $f: I \rightarrow \mathbf{R}$ and assume that $f^{(n)}(x) = 0$ for all $x \in I$ and $f^{(k)}(x_0) = 0$ for $0 \leq k \leq n - 1$ and spme $x_0 \in I$. Show that f is a constant function.

(Hint: Apply Taylor Theorem around $c =$???)

☺Exercise 7

Derive the formula $(2) - (6)$ in Example 4

(Hint: For (5), note that $\frac{d}{dx}\ln(1 + x) = \frac{1}{1+x}$, expand the R.H.S.

For (6), the method is similar to (5))