Math 2033 (Mathematical Analysis) Spring 2015

Final Examination – (Duration: 150 minutes)

Directions: This is a closed book exam. **For every problem of this exam, detailed written works supported by correct reasons must be shown legibly to receive credits**. Answers alone are worth very little. Calculators are not allowed.

Notations: $\mathbb{R}$ is the set of all real numbers. $\mathbb{Q}$ is the set of all rational numbers.

Problems

1. (10 marks) Prove that there exist infinitely many positive real numbers r such that the equation $2^x r^3 = \pi^y$ does not have any solution with $x, y \in \mathbb{Q}$.

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2. (15 marks) Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence of real numbers in $[1, +\infty)$. For every positive integer n, let $y_n = x_{2n} - \dfrac{x_n}{x_n + 1}$. Prove that $y_1, y_2, y_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

   (Do not use Cauchy's theorem that said a sequence converges if and only if it is a Cauchy sequence, otherwise you will get zero mark.)

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3. (20 marks) Let $f : \mathbb{R} \to (0, +\infty)$ be a function such that $\lim_{x\to 1} f(x) = 1$. Prove that $\lim_{x\to 1} \sin\left(\dfrac{5\pi}{4}\sqrt[3]{2f(x)+6}\right) = 1$ by checking the ε-δ definition of limit of function.

   (Do not use any computation formula for limits, sandwich theorem or l'Hopital's rule, otherwise, you will get zero mark.)

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4. (20 marks) Let $f : [0,1] \to [0,1]$ be continuous and injective with $f(0) < f(1)$. Determine how many solution(s) the equation $\dfrac{1 - f(x)}{1 + f(x)} = \dfrac{x^2}{2 - x^2}$ has and prove your answer is correct.

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5. (20 marks) Let $f : \mathbb{R} \to \mathbb{R}$ be n-time differentiable for $n = 1, 2, 3$. If $f(2) = 4, f(1) = 2$ and $f''(0) = 1$, then prove that there exist $a, b, c \in [0, 2]$ such that
$$8f'''(a) - 3f''(b) + 6f'(c) = 0.$$

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6. (25 marks) Let $f : [0,1] \to [0,1]$ be an increasing function. Define $g : [0,1] \to [0,1]$ by
$$g(x) = \begin{cases} f(2x) & \text{if } x \in [0, 1/2) \\ 1 - f(2x-1) & \text{if } x \in [1/2, 1] \end{cases}.$$
   Prove that g is Riemann integrable on $[0,1]$ by checking the integral criterion.

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