# Exercise 1.

## An exercise of mathematical induction:

Using mathematical induction, prove that

$$\cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\cos\left(\frac{n+1}{2}\theta\right)\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

for all positive integer $n$. Here, $\theta \neq k\pi$ for any $k \in \mathbb{Z}$.

**Proof.**

$\boxed{\text{Step 1:}}$ when $n=1$, since

$$\frac{\cos\left(\frac{1+1}{2}\theta\right)\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cos\theta$$

the primitive case is proved.

$\boxed{\text{Step 2:}}$ suppose the claim holds for $n=k$. Then for $n=k+1$, we have:

$$\cos\theta + \cdots + \cos k\theta + \cos(k+1)\theta$$

$$= \frac{\cos\left(\frac{k+1}{2}\theta\right)\cdot\sin\frac{k\theta}{2}}{\sin\frac{\theta}{2}} + \cos(k+1)\theta$$

$$= \frac{\cos\left(\frac{k\theta}{2}+\frac{\theta}{2}\right)\sin\left(\frac{k\theta}{2}\right) + \cos(k+1)\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \frac{1}{\sin\frac{\theta}{2}}\left(\cos\left(\frac{k+2}{2}\theta - \frac{\theta}{2}\right)\sin\frac{k\theta}{2} + \cos((k+1)\theta)\sin\frac{\theta}{2}\right)$$

$$= \frac{1}{\sin\frac{\theta}{2}}\left(\sin\frac{k\theta}{2}\left(\cos\left(\frac{k+2}{2}\theta\right)\cos\frac{\theta}{2}+\sin\left(\frac{k+2}{2}\theta\right)\sin\frac{\theta}{2}\right) + \cos(k+1)\theta\cdot\sin\frac{\theta}{2}\right)$$

$$= \frac{1}{\sin\frac{\theta}{2}}\left(\sin\frac{k\theta}{2}\cdot\cos\left(\frac{k+2}{2}\theta\right)\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\cdot\sin\frac{k\theta}{2}\cdot\sin\frac{k+2}{2}\theta + \sin\frac{\theta}{2}\cdot\cos(k+1)\theta\right)$$

(Recall: $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\cdot\sin\beta$ )

$$= \frac{1}{\sin\frac{\theta}{2}}\left(\sin\frac{k\theta}{2}\cdot\cos\left(\frac{k+2}{2}\theta\right)\cdot\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\cdot\cos\left(\frac{k+2}{2}\theta\right)\cdot\cos\frac{k\theta}{2}\right)$$

$$= \frac{\cos\left(\frac{k+2}{2}\theta\right)\cdot\sin\frac{k+1}{2}\theta}{\sin\frac{\theta}{2}}$$

which is exactly the $k+1$-case. So we've proved the general statement by induction. □

## Exercise 2

① Let $x$ be positive and $a_n = \frac{1}{n^2}\left([x]+[2x]+\cdots+[nx]\right)$. Calculate $\lim a_n$.

Proof: Since $x-1 \le [x] \le x$, we have the inequality

$$a_n = \frac{1}{n^2}([x]+\cdots+[nx])$$

$$\le \frac{1}{n^2}(x+\cdots+nx)$$

$$= \frac{1}{n^2}\left( x + \cdots + nx \right)$$

$$= \frac{1}{n^2} \cdot \frac{n(n+1)}{2} x$$

$$= \frac{x}{2} \cdot \frac{n+1}{n} \xrightarrow{\frac{n+1}{n} \to 1} \frac{x}{2}$$

$$a_n \geq \frac{1}{n^2}\left( (x-1) + \cdots + (nx-1) \right)$$

$$= \frac{1}{n^2}\left( \frac{n(n+1)}{2} x - n \right)$$

$$= \frac{x}{2} \cdot \frac{n+1}{n} - \frac{1}{n} \xrightarrow[\frac{1}{n} \to 0]{\frac{n+1}{n} \to 1} \frac{x}{2}$$

2. Let $x_1 = 4$ and $x_{n+1} = \dfrac{4(1+x_n)}{4+x_n}$ for $n = 1, 2, 3, \ldots$. Plot the first 3 terms on the real line. Then prove the sequence $\{x_n\}$ converges.

**Proof:**

$\boxed{\begin{array}{l} \text{Intuition: } x_n \cdot x_{n+1} + 4x_{n+1} = 4 + 4x_n. \\ \text{if converges, then } x_n, x_{n+1} \to x^*. \\ \Rightarrow (x^*)^2 = 4 \Rightarrow x^* = \pm 2 \\ \text{since } x_n \geq 0. \text{ we guess } x^* = 2. \end{array}}$

→ Just a guess! We cannot use this way to prove the convergence since it's an assumption!

First 3 terms: $x_1 = 4$. $x_2 = \frac{5}{2}$ $x_3 = \frac{28}{13}$

So we try to prove $\{x_n\}$ is decreasing and bounded below.

We claim that $x_n \geq 2$ ($\forall n$). Prove by induction.

1) $x_1 = 4 > 2$ holds

2) Assume $x_n > 2$. then

$$x_{n+1} - 2 = \frac{4(1+x_n) - 2(4+x_n)}{4+x_n} = \frac{\overset{+}{2x_n - 4}}{\underset{+}{4+x_n}} > 0$$

$\Rightarrow x_{n+1} \geq 2$ . So we know $x_n \geq 2$ for all $n$.

Second. we claim that $x_{n+1} \leq x_n$ ($\forall n$). In fact.

$$x_{n+1} - x_n = \frac{4(1+x_n)}{4+x_n} - x_n$$
$$= \frac{\overbrace{4 - x_n^2}^{\leq 0}}{\underbrace{4+x_n}_{>0}} \leq 0$$

Hence. we conclude that $\{x_n\}$ is a decreasing sequence and bounded below , thus converges.

Rmk: use the method in "intuition". we know

$\lim x_n = 2$.

$\square$

Exercise 3.

① Show that for any $c>0$. $\lim\limits_{n\to\infty} \sqrt[n]{c} = 1$.

Proof: ( a rigorous proof only using definition )

When $c<1$ . let $c = \frac{1}{b}$ where $b>1$ . then

$$\lim_{n\to\infty} \sqrt[n]{c} = \frac{1}{\lim\limits_{n\to\infty} \sqrt[n]{b}}$$

So we only need to show the case : $c \geq 1$. When $c=1$ the result is trivial . now suppose $c>1$ .

Since $c>1$, we have $\sqrt[n]{c} \geq 1$ for all $n$. Then for any $\varepsilon>0$, let $N = [\frac{\ln c}{\ln(1+\varepsilon)}]+1$ and if $n \geq N$, we have

$$|\sqrt[n]{c}-1| = \sqrt[n]{c}-1 < c^{\frac{\ln(1+\varepsilon)}{\ln c}} - 1$$

$$= e^{\ln c \cdot \frac{\ln(1+\varepsilon)}{\ln c}} - 1 = \varepsilon$$

So $\lim \sqrt[n]{c} = 1$.

how to find this?

$$|\sqrt[n]{c}-1| = \sqrt[n]{c}-1 < \varepsilon$$

$$\Rightarrow \sqrt[n]{c} < 1+\varepsilon$$

$$\overset{\text{take log}}{\Rightarrow} \frac{1}{n}\ln c < \ln(1+\varepsilon)$$

$$\Rightarrow n > \frac{\ln c}{\ln(1+\varepsilon)}$$

Since every "<" holds without assumption, if we set the final "<" hold, the chain holds as well.

② Show that $\sqrt[n]{n} = 1$

Proof:

(Recall the average inequality

$$(a_1 \cdots a_n)^{\frac{1}{n}} \leq \frac{1}{n}(a_1 + \cdots + a_n) \quad \text{when } a_i \geq 0$$

"=" holds iff $a_1 = a_2 = \cdots = a_n$)

Let $n = \overbrace{1 \cdot 1 \cdot \cdots \cdot 1}^{n-2} \cdot \sqrt{n} \cdot \sqrt{n}$ and apply the average inequality, we have

$$\sqrt[n]{n} \leq \frac{1}{n}(2\sqrt{n} + 1 + \cdots + 1)$$

$$= \frac{1}{n}(2\sqrt{n} + n - 2)$$

$$= \frac{2}{\sqrt{n}} + 1 - \frac{2}{n}$$

and for any given $\varepsilon$. we have $N_1 = [\frac{16}{\varepsilon^2}] + 1$
and $N_2 = [\frac{2}{\varepsilon}] + 1$ s.t.

$$\begin{cases} \left|\frac{2}{\sqrt{n}} - 0\right| < \varepsilon/2 \text{ when } n > N_1 \\ \left|\frac{2}{n} - 0\right| < \varepsilon/2 \text{ when } n > N_2 \end{cases}$$

In conclusion. for each $\varepsilon > 0$. we have $N = \max\{N_1, N_2\}$
such that when $n > N$.

$$\begin{aligned} \left|\sqrt[n]{n} - 1\right| &\overset{\text{since positive}}{=} \sqrt[n]{n} - 1 \\ &\leq \left(\frac{2}{\sqrt{n}} - \frac{2}{n} + 1\right) - 1 \\ &< \frac{2}{\sqrt{n}} + \frac{2}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence. $\lim \sqrt[n]{n} = 1$. $\square$