Exercise 1.

For a function $f : \mathbb{R} \to \mathbb{R}$, we say f has a *local (or relative) maximum* at x_0 (if there exists an open interval (a, b) containing x_0 such that $f(x) \le f(x_0)$ for every $x \in (a, b)$). Similarly, we say f has a *local (or relative) minimum* at x_1 (if there exists an open interval (c, d) containing x_1 such that $f(x) \ge f(x_1)$ for every $x \in (c, d)$.) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and has a local maximum or a local minimum at every real number, show that f is a constant function.

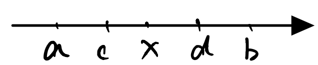
Proof:

Idea: Show that $f(\mathbb{R})$ is both countable and uncountable.

For each $x \in \mathbb{R}$, $\exists$ neighborhood (a, b) s.t. $a < x < b$, and x is the global maximum or minimum on (a, b).

From the density of $\mathbb{Q}$ in $\mathbb{R}$, we know $\exists (c, d) \in \mathbb{Q} \times \mathbb{Q}$ s.t.

$a < c < x < d < b$.



Then $f(x) = \max\limits_{y \in [c,d]} f(y)$ or $\min\limits_{y \in [c,d]} f(y)$.

Hence, we know

$$f(\mathbb{R}) = \bigcup_{x \in \mathbb{R}} f(x)$$

$$\subseteq \bigcup_{(c,d) \in \mathbb{Q} \times \mathbb{Q}} \left\{ \max_{y \in [c,d]} f(y),\ \min_{y \in [c,d]} f(y) \right\}$$

which is countable by the countable union thm.

$\square$

Exercise 2.

If $f(x) = x^3$, then $f(f(x)) = x^9$. Is there a continuous function $g : [-1, 1] \to [-1, 1]$ such that $g(g(x)) = -x^9$ for all $x \in [-1, 1]$? (*Hint*: If such a function g exists, then it is injective.)

Proof:

If so. g is injective (when $g(x_1) = g(x_2) \Rightarrow g(g(x_1)) = g(g(x_2)) \Rightarrow -x_1^9 = -x_2^9 \Rightarrow x_1 = x_2$). Since injective continuous function should be monotone (otherwise, if $x_1 < x_2 < x_3 \Rightarrow g(x_1) \le g(x_2)$ $g(x_3) \ge g(x_2)$. from the intermediate value thm. we can deduce a contradiction). no matter g is increasing or decreasing $g \circ g$ will be increasing (why?). which contradicts to the decreasing nature of $x \mapsto -x^9$.

$\square$

Exercise 3.

Find the derivatives of the functions $f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ x & \text{if } x = 0 \end{cases}$ and $g(x) = |\cos x|$.

Proof:

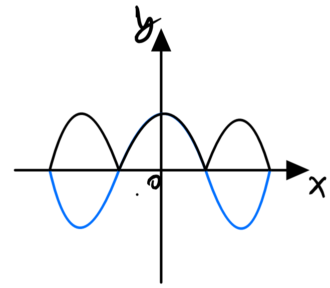
f: when $x \neq 0$. $f'(x) = (x^2)' = 2x$.

when $x = 0$. $\lim\limits_{h \to 0^+} \frac{f(h)-f(0)}{h} = \lim\limits_{h \to 0^+} \frac{h^2}{h} = 0$

$\lim\limits_{h \to 0^-} \frac{f(h)-f(0)}{h} = \lim\limits_{h \to 0^-} h = 0$.

$\therefore f'(0) = 0$

$f'(x) = 2x$



g: when $x \in \left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right)$. $(k \in \mathbb{Z})$.

$g'(x) = (\cos x)' = -\sin x$

when $x \in \left(\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi\right)$ $(k \in \mathbb{Z})$

$g'(x) = (-\cos x)' = \sin x$.

when $x = \frac{\pi}{2} + 2k\pi$. $(k \in \mathbb{Z})$

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$$\begin{cases} \lim\limits_{h \to 0^+} \frac{g(x+h)-g(x)}{h} = \lim\limits_{h \to 0^+} \frac{-\cos(x+h)}{h} = \lim\limits_{h \to 0^+} \frac{\sin(x+h)}{1} = 1 \\ \lim\limits_{h \to 0^-} \frac{g(x+h)-g(x)}{h} = \lim\limits_{h \to 0^-} \frac{\cos(x+h)}{h} = \lim\limits_{h \to 0^-} \frac{-\sin(x+h)}{1} = -1 \end{cases}$$

$\Rightarrow$ non-differentiable at $x = \frac{\pi}{2} + 2k\pi$

when $x = \frac{3}{2}\pi + 2k\pi$ $(k \in \mathbb{Z})$

Similar to above. Exercise to you. we can show that g is not differentiable at $x = \frac{3}{2}\pi + 2k\pi$.

$\square$

Exercise 4.

(a) Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$.
(b) Find all strictly increasing functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Proof:

(a)

idea: Choose some special values. $\Rightarrow$ "generators"

$f(0) = f(0) + f(0) \Rightarrow f(0) = 0. \Rightarrow f(x) + f(-x) = 0 \Rightarrow f(-x) = -f(x)$

Set $f(1) = c \Rightarrow f(1) = f(m \cdot \frac{1}{m}) = m f(\frac{1}{m}) \Rightarrow f(\frac{1}{m}) = \frac{1}{m} f(1)$

$\therefore f(q) = f(\frac{n}{m}) = n f(\frac{1}{m}) = \frac{n}{m} c = cq \Rightarrow$ linear.

(b) From (a). f must follow the form $f(q) = qc$ on $\mathbb{Q}$.
Since f strictly increases, $c > 0$.

From the density of $\mathbb{Q}$ in $\mathbb{R}$. for each $x \in \mathbb{R}$. we can construct two sequences $\{y_n\}$. $\{z_n\} \subset \mathbb{Q}$ running to x increasingly. and decreasing. respectively. Then due to the monotonicity of f. we know

$$c \cdot y_n = f(y_n) < f(x) < f(z_n) = c \cdot z_n$$

From the squeeze thm. $\underbrace{\lim_{n\to\infty} c y_n}_{= cx} \le f(x) \le \underbrace{\lim_{n\to\infty} c \cdot z_n}_{= cx}$

$\therefore f(x) = cx.$

$\square$

Exercise 5

Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable at c and $I_n=[a_n,b_n]$ be such that $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$ and $\bigcap_{n=1}^{\infty}[a_n,b_n]=\{c\}$.

Prove that if $a_n<b_n$ for all $n\in\mathbb{N}$, then $f'(c)=\lim_{n\to\infty}\frac{f(b_n)-f(a_n)}{b_n-a_n}$.

$$\frac{f(b_n)-f(a_n)}{b_n-a_n}=\frac{f(b_n)-f(c)+f(c)-f(a_n)}{b_n-c+c-a_n}$$

$$=\frac{b_n-c}{b_n-a_n}\,\frac{f(b_n)-f(c)}{b_n-c}+\frac{c-a_n}{b_n-a_n}\,\frac{f(c)-f(a_n)}{c-a_n}$$

$$\therefore \left|\frac{f(b_n)-f(a_n)}{b_n-a_n}-f'(c)\right|=\left|\frac{b_n-c}{b_n-a_n}\left(\frac{f(b_n)-f(c)}{b_n-c}-f'(c)\right)+\cdots\right|$$

$$\le \varepsilon\left(\frac{b_n-c}{b_n-an}+\frac{c-a_n}{b_n-a_n}\right)=\varepsilon$$

Proof: We need to show that $\forall\varepsilon>0,\ \exists N$ s.t.

$$\left|\frac{f(b_n)-f(a_n)}{b_n-a_n}-f'(c)\right|\le\varepsilon,\quad \forall n\geq N.$$

By the above decomposition, we have (noticing $\underbrace{\frac{b_n-c}{b_n-a_n}}_{>0}+\underbrace{\frac{c-a_n}{b_n-a_n}}_{>0}=1$).

$$f'(c)-\frac{f(b_n)-f(a_n)}{b_n-a_n}$$

$$=\underbrace{\frac{b_n-c}{b_n-a_n}}_{\in[0,1]}\underbrace{\left(f'(c)-\frac{f(b_n)-f(c)}{b_n-c}\right)}_{①}+\underbrace{\frac{c-a_n}{b_n-a_n}}_{\in[0,1]}\underbrace{\left(f'(c)-\frac{f(c)-f(a_n)}{c-a_n}\right)}_{②}$$

By $\lim_{n\to\infty}[a_n,b_n]=c$ and $a_n<b_n$. We have $\lim_{n\to\infty}b_n=\lim_{n\to\infty}a_n=c$.

thus $\forall\varepsilon>0,\ \exists N$ s.t. $\begin{cases}|①|<\varepsilon/2\\ |②|<\varepsilon/2\end{cases}$, $\forall n\geq N$ by the definition of derivative.

As a result. $\forall n \geq N$, we have

$$\left| f'(c) - \frac{f(b_n) - f(a_n)}{b_n - a_n} \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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