5. (28 points) .

Let f be a bounded function on $[0, 1]$

5+5 ← 5.1 Write down the definition of the lower integral of f and the integral criterion.

5.2 Prove the function

8'

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

on $[0, 1]$ is not integrable.

10' 5.3 Let $\{x_n\}$ be a sequence in $[0, 1]$ s.t $\lim_{n\to\infty} x_n = \frac{1}{2}$. Is the function

$$f(x) = \begin{cases} 1, & x \in \{x_n : n \in \mathbb{N}\} \\ 0, & \text{otherwise} \end{cases}$$

integrable? Justify your answer.

5

Solution:

5.1 Lower integral: $\sup\{L(f, P) : P \text{ is a partition of } [a, b]\}$

5

Integral criterion: f is integrable iff $\forall \epsilon > 0$, $\exists P$ such that $U(f, P) - L(f, P) < \epsilon$.

5.2 Let $\epsilon = 1/2$, then for any partition $P = \{x_0 = 1, x_1, \ldots, x_n = 1\}$ of $[0, 1]$, 2'

$$m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\} = 0,$$

3'

$$M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\} = 1.$$

So, we have

$$U(f, P) - L(f, P) = \sum_{j=1}^{n} (M_j - m_j)\Delta x_j = \sum_{j=1}^{n} \Delta x_j = 1 > \frac{1}{2}.$$

3'

Therefore, f is not integrable. 1'

2' 5.3 It is integrable.

$\forall \epsilon > 0$, choose a, b such that $0 < a < \frac{1}{2} < b < 1$ and $b - a < \epsilon/2$. Since $\lim_{n\to\infty} x_n = \frac{1}{2}$, $[0, a] \cup [b, 1]$ only contains finite number of x_n. Suppose they are $\{x_{n_i}\}_{i=1}^{K}$. Then we can choose (a_i, b_i) such that $x_{n_i} \in (a_i, b_i)$ and $b_i - a_i < \frac{\epsilon}{2K}$ for all $i = 1, \ldots, K$. We observe that $f = 0$ on $[0, 1] \setminus ((a, b) \bigcup \cup_{i=1}^{K} (a_i, b_i))$. Consider the partition $P = \{0, 1, a, b\} \bigcup \cup_{i=1}^{K} \{a_i, b_i\}$ of $[0, 1]$, then

$$U(f, P) - L(f, P) \le (b - a) + \sum_{i=1}^{K} (b_i - a_i) < \frac{\epsilon}{2} + K\frac{\epsilon}{2K} = \epsilon.$$

By integral criterion, f is integrable on $[0, 1]$.