DEPARTMENT OF MATHEMATICS

# Math2033 TA note 7

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## 1 LIMIT

**Example 1.** *Let* $\{a_n\}$ *be a sequence satisfying* $\lim\limits_{n\to\infty}(a_{n+1}-\frac{a_n}{2})=0$. *Prove that* $\lim\limits_{n\to\infty} a_n = 0$.

*Solution:* Because

$$\lim_{n\to\infty}(a_{n+1}-\frac{a_n}{2})=0.$$

Then for $\epsilon>0, \exists K_1\in\mathbb{N}, \forall n\geq K_1, |a_{n+1}-\frac{a_n}{2}|<\frac{\epsilon}{4}$. Then

$$\forall n\geq K_1, |a_{n+1}|<\frac{|a_n|}{2}+\frac{\epsilon}{4}.$$

Then we prove by mathematical induction principle that

$$\forall n\geq K_1, |a_n|\leq\frac{|a_{K_1}|}{2^{n-K_1}}+\frac{\epsilon}{2}.$$

Firstly, $|a_{K_1+1}|\leq\frac{|a_{K_1}|}{2}+\frac{\epsilon}{2}$. If when $n=k$, $|a_k|\leq\frac{|a_{K_1}|}{2^{k-K_1}}+\frac{\epsilon}{2}$. Then for $n=k+1$,

$$\begin{aligned}|a_{k+1}| &\leq \frac{|a_k|}{2}+\frac{\epsilon}{4}\\ &\leq \frac{\frac{|a_{K_1}|}{2^{k-K}}+\frac{\epsilon}{2}}{2}+\frac{\epsilon}{4}\\ &= \frac{|a_{K_1}|}{2^{k+1-K_1}}+\frac{\epsilon}{2}.\end{aligned}$$

So by mathematical induction principle, the statement is true that

$$\forall n\geq K_1, |a_n|\leq\frac{|a_{K_1}|}{2^{n-K_1}}+\frac{\epsilon}{2}.$$

There exist $K_2$ such that $\forall n \geq K_2, \frac{|a_{K_1}|}{2^{n-K_1}} < \frac{\epsilon}{2}$. So for $\epsilon > 0, \exists K = \max\{K_1, K_2\}, \forall n \geq K$,

$$|a_n| \leq \frac{|a_{K_1}|}{2^{n-K_1}} + \frac{\epsilon}{2} < \epsilon.$$

That is $\lim\limits_{n\to\infty} a_n = 0$.

**Example 2.** *If $\{x_n\}$ is a sequence such that $|x_{k+1} - x_k| < \frac{1}{2^k}$ for $k = 1, 2, 3, \ldots$, then show that $\{x_n\}$ is a Cauchy sequence.*

*Solution:* For every $\epsilon > 0$, there is a integer $K = 2 - [\frac{\ln \epsilon}{\ln 2}]$. Then $n \geq K, p \in \mathbb{N}$ implies

$$\begin{aligned}
|x_{n+p} - x_n| &= |(x_{n+p} - x_{n+p-1}) + (x_{n+p-1} - x_{n+p-2}) + \cdots + (x_{n+1} - x_n)| \\
&\leq |x_{n+p} - x_{n+p-1}| + |x_{n+p-1} - x_{n+p-2}| + \cdots + |x_{n+1} - x_n| \\
&< \frac{1}{2^{n+p-1}} + \frac{1}{2^{n+p-2}} + \cdots + \frac{1}{2^n} \\
&< \sum_{j=n}^{\infty} \frac{1}{2^j} = \frac{1}{2^{n-1}} \\
&\leq \frac{1}{2^{K-1}} < \epsilon
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence.

**Example 3.** *We can find a divergent sequence $\{a_n\}$ such that for any $p \in \mathbb{N}$, $\lim\limits_{n\to\infty}(a_{n+p} - a_n) = 0$.*

*Solution:* Take

$$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

It is divergent. However, for any $p \in \mathbb{N}$,

$$a_{n+p} - a_n = \frac{1}{n+1} + \cdots + \frac{1}{n+p} \leq \frac{p}{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

**Remark 4.** *The Cauchy sequence, $p$ is independent of $n$ while the previous one is dependent of $n$.*

**Definition 5.** *$f(x)$ converges to $L$ as $x$ tends to $x_0$ in $S$ iff for every $\epsilon > 0$, there exists a $\delta > 0$ such that $x \in S$ and $0 < |x - x_0| < \delta$ imply $|f(x) - L| < \epsilon$.*

**Example 6.** *(1) Let $f : (1,3) \to \mathbb{R}$ be defined by $f(x) = x^2 + \frac{1}{x}$. Prove that $\lim_{x\to 2} f(x) = \frac{9}{2}$ by checking the definition.*

*(2) Let $f : (1,4) \to \mathbb{R}$ be defined by $f(x) = |x^2 - 9|$. Prove that $\lim_{x\to 2} f(x) = 5$ by checking the definition.*

*Solution:* (1) For every $\epsilon > 0$, take $\delta = 2\epsilon/11 > 0$. If $0 < |x - 2| < \delta$ and $x \in (1,3)$, then

$$\begin{aligned}
\left|f(x) - \frac{9}{2}\right| &= \left|x^2 + \frac{1}{x} - \frac{9}{2}\right| = \left|(x^2 - 4) + \left(\frac{1}{x} - \frac{1}{2}\right)\right| \leq |x^2 - 4| + \left|\frac{1}{x} - \frac{1}{2}\right| \\
&= |x+2||x-2| + \frac{|x-2|}{2|x|} \leq 5|x-2| + \frac{1}{2}|x-2| = \frac{11}{2}|x-2| < \frac{11}{2}\delta = \epsilon.
\end{aligned}$$

(2) For every $\epsilon > 0$, take $\delta = \min\{1, \epsilon/5\} > 0$. If $0 < |x-2| < \delta$ and $x \in (1,4)$, then $x \in (1,3)$ and $f(x) = |x^2 - 9| = 9 - x^2$. Therefore

$$|f(x) - 5| = |9 - x^2 - 5| = |4 - x^2| = |2 - x||2 + x| < 5|x-2| < 5\delta \leq \epsilon.$$