MATH202 Introduction to Analysis (2007 Fall-2008 Spring)

Tutorial Note #4

Note: Please read the note "Note on Countability" which is available at tutorial homepage http://ihome.ust.hk/~mathlcm

More Examples in Countability

Example 1 (Warm up)

Is the set $S = \{a\sqrt{2} + b\sqrt{3} + c\sqrt{5} : a \in \mathbf{N}, b \in \mathbf{Z}, c \in \mathbf{Q}\}$ countable?

IDEA: First, we see all the variables lies in the sets which are countable, so we can prove it using countable union theorem.

Solution:

$$S = \{a\sqrt{2} + b\sqrt{3} + c\sqrt{5} : a \in \mathbf{N}, b \in \mathbf{Z}, c \in \mathbf{Q}\}$$

$$= \bigcup_{a\in\mathbf{N}} \{a\sqrt{2} + b\sqrt{3} + c\sqrt{5} : b \in \mathbf{Z}, c \in \mathbf{Q}\} \quad \text{where a is fixed}$$

$$= \bigcup_{a\in\mathbf{N}} \bigcup_{b\in\mathbf{Z}} \{a\sqrt{2} + b\sqrt{3} + c\sqrt{5} : c \in \mathbf{Q}\} \quad \text{where a and b are fixed}$$

$$= \bigcup_{a\in\mathbf{N}} \bigcup_{b\in\mathbf{Z}} \bigcup_{c\in\mathbf{Q}} \{a\sqrt{2} + b\sqrt{3} + c\sqrt{5}\}$$

Since $\{a\sqrt{2} + b\sqrt{3} + c\sqrt{5}\}$ has only 1 elements and therefore countable, then apply the countable union theorem. We see

$$\rightarrow S = \bigcup_{a\in\mathbf{N}} \bigcup_{b\in\mathbf{Z}} \bigcup_{c\in\mathbf{Q}} \{a\sqrt{2} + b\sqrt{3} + c\sqrt{5}\} \text{ is also countable}$$

Example 2

a) Let $-1 \le y \le 1$, show that the set $A = \{x \in \mathbf{R} : \sin x = y\}$ is countable

b) Show that $B = \{x \in \mathbf{R} : 6\sin^2 x - 5\sin x + 1 = 0\}$ is countable

Solution:

a) Note that

$\sin x = y \rightarrow x = n\pi + (-1)^n \sin^{-1} y$ (for $n \in \mathbf{Z}$)

Therefore A can be rewritten as

$A = \{x \in \mathbf{R} : \sin x = y\} = \{n\pi + (-1)^n \sin^{-1} y : n \in \mathbf{Z})$

Define a function $f: A \rightarrow \mathbf{Z}$ which $f(n\pi + (-1)^n \sin^{-1} y) = n$

We can see f is bijective

Since $\mathbf{Z}$ is countable and f is bijection, therefore A is countable

b) $B = \{x \in \mathbf{R}: 6\sin x - 5\sin x + 1 = 0\}$

$= \{x \in \mathbf{R}: (2\sin x - 1)(3\sin x - 1) = 0\}$

$= \{x \in \mathbf{R}: \sin x = \frac{1}{2} \text{ or } \sin x = \frac{1}{3}\}$

$= \{x \in \mathbf{R}: \sin x = \frac{1}{2}\} \cup \{x \in \mathbf{R}: \sin x = \frac{1}{2}\}$

Since $\{x \in \mathbf{R}: \sin x = \frac{1}{2}\}$ and $\{x \in \mathbf{R}: \sin x = \frac{1}{3}\}$ are countable by a)

Therefore B is countable (by countable union theorem)

☺Exercise 1

Show that the set $\{x \in \mathbf{R}: \tan^4 x - 4 = 0\}$ is countable

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Example 3

Determine whether the set $S = T \cap U$, where $T = \mathbf{R} \backslash \mathbf{Q}$ and $U = \mathbf{R} \backslash \{\sqrt{m} + \sqrt{n}: m, n \in \mathbf{N}\}$ is countable or not.

Hint: Consider $\mathbf{R} \backslash (T \cap U)$

Solution:

(Step 1) Using the hint, we first consider the set $\mathbf{R} \backslash (T \cap U)$

$\mathbf{R} \backslash (T \cap U) = (\mathbf{R} \backslash T) \cup (\mathbf{R} \backslash U)$ (Note: $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$)

$= \mathbf{R} \backslash (\mathbf{R} \backslash \mathbf{Q}) \cup \mathbf{R} \backslash (\mathbf{R} \backslash \{\sqrt{m} + \sqrt{n}: m, n \in \mathbf{N}\})$

$= \mathbf{Q} \cup \{\sqrt{m} + \sqrt{n}: m, n \in \mathbf{N}\}$

$\mathbf{Q}$ is countable, and

For $\{\sqrt{m} + \sqrt{n}: m, n \in \mathbf{N}\}$

Consider a map $f: \mathbf{N} \times \mathbf{N} \to \{\sqrt{m} + \sqrt{n}: m, n \in \mathbf{N}\}$

which $f(m, n) = \sqrt{m} + \sqrt{n}$. It is surjective

Since $\mathbf{N} \times \mathbf{N}$ is countable and by surjection theorem**,** $\{\sqrt{m} + \sqrt{n}: m, n \in \mathbf{N}\}$ is also countable.

So $\mathbf{R}\backslash(\mathrm{T} \cap \mathrm{U})$ is also countable by countable union theorem.

(Step 2) Until now, we do not know whether $\mathrm{T} \cap \mathrm{U}$ is countable or not. So let us assume it is countable first.

Then note that $\underbrace{[\mathbf{R}\backslash(\mathrm{T} \cap \mathrm{U})]}_{\text{(countable)}} \cup \underbrace{(\mathrm{T} \cap \mathrm{U})}_{\text{(countable)}} = \mathbf{R}$ (since $\mathrm{T} \cap \mathrm{U} \subseteq \mathbf{R}$)

L.H.S. is countable by countable union theorem which implies $\mathbf{R}$ is countable. But we know $\mathbf{R}$ is uncountable. So it leads to contradiction.

Therefore S is uncountable

Example 4 Let S is the set of all non-constant polynomials with coefficients in G, where $i = \sqrt{-1}$ and $\mathrm{G} = \{a + bi: a, b \in \mathbf{Z})$

Solution:

(Step 0) We first rewrite the S into mathematical form:

$\mathrm{S} = \{a_n x^n + a_{n-1}x^{n-1} + \cdots. + a_1 + a_0 : a_0, a_1, \ldots a_n \in \mathrm{G} \text{ and } a_n \neq 0 \text{ and } n = 1,2,\ldots\}$

(Step 1) Since the set is too big for us, let decompose it into smaller sets by fixing the degree of the set

$$\mathrm{S} = \bigcup_{n=1}^{\infty} \{a_n x^n + a_{n-1}x^{n-1} + \cdots. + a_1 + a_0 : \ a_0, a_1, \ldots a_n \in \mathrm{G} \text{and } a_n \neq 0\}$$

(Step 2: Since all variables are in set G, we need to know whether G is countable or not)

Consider a map $f: \mathbf{Z} \times \mathbf{Z} \to \mathrm{G}$, which $f(a, b) = a + bi$. Clearly f is bijective

Since $\mathbf{Z} \times \mathbf{Z}$ is countable, by bijection theorem, G is countable

(Step 3: Now G is countable, we can further decompose S into following form)

$$\mathrm{S} = \bigcup_{n=1}^{\infty} \{a_n x^n + a_{n-1}x^{n-1} + \cdots. + a_1 + a_0 : \ a_0, a_1, \ldots a_n \in \mathrm{G} \text{and } a_n \neq 0\}$$

$$= \bigcup_{n=1}^{\infty} \bigcup_{a_n \in Z\backslash\{0\}} \bigcup_{a_{n-1} \in Z} \bigcup_{a_{n-2} \in Z} \ldots.. \bigcup_{a_0 \in Z} \{a_n x^n + a_{n-1}x^{n-1} + \cdots. + a_1 + a_0\}$$

The rightmost set has only 1 element and therefore countable

So S is countable by applying countable union theorem for $n + 1$ times.

Note: I will NOT post the solutions of these exercises on the web, you should try to work on them. You may submit the solution to me so that I can give some comments to your work.

☺Exercise 2 (Practice Exercise #89f)

Determine the set

$$S = Q(\sqrt{2}) = \{\frac{a + b\sqrt{2}}{c + d\sqrt{2}}: a, b, c, d \in \mathbf{Q}, c + d\sqrt{2} \neq 0\}$$

Is countable or not.

☺Exercise 3 (Practice Exercise #89g)

Determine the set

$$S = \{x^2 + y^2 + z^2: x \in A \cap B, y \in \mathbf{Q} \cap A, z \in B \cap \mathbf{Q}\}$$

Is countable or not.

where A is non-empty **countable** subset of R and B is an **uncountable** subset of R.

(Hint: You should first determine whether $A \cap B, \mathbf{Q} \cap A, B \cap \mathbf{Q}$ are countable or not)

☺Exercise 4 (Practice Exercise #89h)

Determine the set

$$S = \{x - y: x, y \in A\}$$

Is countable or not.

Where A is a uncountable subset of $\mathbf{R}.$

☺Exercise 5 (Practice Exercise #89m)

Determine the set

$$S = R \backslash \{a + b\sqrt{2} - c\sqrt{3}: a, b, c \in T\}$$

Is countable or not.

Where $T = \{r\pi: r \in \mathbf{Q}\}$

(Hint: First check whether$\{a + b\sqrt{2} - c\sqrt{3}: a, b, c \in T\}$ is countable or not.