

Homework 1

Monday, 20 September 2021 5:04 AM



Math2033H

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Math 2033 (Homework 1) L1

Fall 2021

Problems (Due September 27 at 11:59 pm)

① Prove that the set $W = \{x \in \mathbb{R} : x^3 - 2x + 5 \in \mathbb{Q}\}$ is countable. (Here \mathbb{R} is the set of all real numbers and \mathbb{Q} is the set of all rational numbers.)

~~②~~ Prove that the set $S = \{b : x^4 + bx - 5 = 0 \text{ has a rational root}\}$ is countable.

③ Determine if the set $B = \{x + \sqrt{2}y : x, y \in \mathbb{N}\}$ is countable or not. (Here $\mathbb{N} = \{1, 2, 3, \dots\}$).

① Prove that the set $W = \{x \in \mathbb{R} : x^3 - 2x + 5 \in \mathbb{Q}\}$ is countable.

$$W = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{Q}, y = x^3 - 2x + 5\}$$

$y = x^3 - 2x + 5$ has at most 3 solutions $\{x_1, x_2, x_3\}$ for a given y .

Since $y \in \mathbb{Q}$ is countable, we can list it as q_1, q_2, q_3, \dots . For a given q_i , it corresponds to $\{x_1, x_2, x_3\}$.
Hence, we can write it as: $(x_1, q_1), (x_2, q_1), (x_3, q_1),$
 $(x_1, q_2), (x_2, q_2), (x_3, q_2),$
 $(x_1, q_3), (x_2, q_3), (x_3, q_3) \dots$

which is still countable.

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Then, we can remove any duplicates and blanks
and non-real values for x_1, x_2, x_3 .

After that we get ω which is countable.

③ $B = \{x + \sqrt{2}y : x, y \in \mathbb{N}\}$.

For every $(x, y) \in \mathbb{N} \times \mathbb{N}$, Define $f: \mathbb{N} \times \mathbb{N} \rightarrow B$ by letting
 $f(x, y) = x + \sqrt{2}y$, (with f^{-1} sending $x + \sqrt{2}y$ back to (x, y)) is a
bijection. Since $\mathbb{N} \times \mathbb{N}$ is countable by the product theorem,
so B is countable by the bijection theorem.

