MATH 2031 Introduction to Real Analysis

November 19, 2012

Tutorial Note 8

Limits of sequence

(I) **Definition:**
A sequence $x_1, x_2, \cdots$ (or written as $\{x_n\}_{n\in\mathbb{N}}$) converges to a number x (or has limit x) iff

$$\forall\varepsilon > 0,\ \exists K \in \mathbb{N} \text{ such that } n \geq K \Rightarrow |x_n - x| < \varepsilon$$

(II) **Monotone Sequence Theorem**
If $\{x_n\}$ is increasing and bounded above, then $\lim_{n\to\infty} x_n = sup\{x_1, x_2, x_3 \cdots\}$;
Similarly, If $\{x_n\}$ is decreasing and bounded below, then $\lim_{n\to\infty} x_n = inf\{x_1, x_2, x_3 \cdots\}$.

(III) **Intertwining Sequence Theorem**
If $\lim_{m\to\infty} x_{2m-1} = x$ and $\lim_{m\to\infty} x_{2m} = x$, then $\lim_{n\to\infty} x_n = x$

(IV) **Nested Interval Theorem**
If $\forall n \in \mathbb{N}$, $I_n = [a_n, b_n]$ and $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, then $\bigcap_{n=1}^{\infty} I_n = [a, b]$, where $a = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n = b$.
If $\lim_{n\to\infty}(b_n - a_n) = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{x\}$ for some $x \in \mathbb{R}$.

Problem 1 Prove that $\lim_{n\to\infty}\left(\frac{2}{n+1} - \frac{1}{n^3} + \frac{5\sqrt{n}}{\sqrt{n}+1}\right) = 5$ by checking the definition of limit.

Scratch work:

$$\text{"Want: Find a suitable } K \in \mathbb{N} \text{ such that}$$
$$\forall\varepsilon > 0,\ n \geq K \Rightarrow \left|\frac{2}{n+1} - \frac{1}{n^3} + \frac{5\sqrt{n}}{\sqrt{n}+1} - 5\right| < \varepsilon\text{"}$$

We may apply triangular inequality and the "Max Trick". (Beware that sometimes the estimate of the triangular inequality would be "too large"!)
Considering $\left|\frac{2}{n+1} - \frac{1}{n^3} + \frac{5\sqrt{n}}{\sqrt{n}+1} - 5\right| \leq \left|\frac{2}{n+1}\right| + \left|\frac{1}{n^3}\right| + \left|\frac{5\sqrt{n}}{\sqrt{n}+1} - 5\right|$ and requiring each term to be less than $\frac{\varepsilon}{3}$, we can get what we want.

$$\left|\frac{2}{n+1}\right| < \frac{\varepsilon}{3}, \quad \left|\frac{1}{n^3}\right| < \frac{\varepsilon}{3}, \quad \left|\frac{5\sqrt{n}}{\sqrt{n}+1} - 5\right| < \frac{\varepsilon}{3}\ ,$$

which is equivalent to the following

$$\frac{6}{\varepsilon} - 1 < n, \quad \sqrt[3]{\frac{3}{\varepsilon}} < n, \quad \left(\frac{15}{\varepsilon} - 1\right)^2 < n.$$

Solution:
$\forall \varepsilon > 0$, by Archimedean's principle, $\exists K \in \mathbb{N}$ such that $K > max\{\frac{6}{\varepsilon} - 1, \sqrt[3]{\frac{3}{\varepsilon}}, (\frac{15}{\varepsilon} - 1)^2\}$. Then for any $n \geq K$, we have the inequalities:

$$\frac{6}{\varepsilon} - 1 < n, \quad \sqrt[3]{\frac{3}{\varepsilon}} < n, \quad \left(\frac{15}{\varepsilon} - 1\right)^2 < n$$

which are equivalent to

$$\left|\frac{2}{n+1}\right| < \frac{\varepsilon}{3}, \quad \left|\frac{1}{n^3}\right| < \frac{\varepsilon}{3}, \quad \left|\frac{5\sqrt{n}}{\sqrt{n}+1} - 5\right| < \frac{\varepsilon}{3}.$$

Then $\left|\frac{2}{n+1} - \frac{1}{n^3} + \frac{5\sqrt{n}}{\sqrt{n}+1} - 5\right| \leq \left|\frac{2}{n+1}\right| + \left|\frac{1}{n^3}\right| + \left|\frac{5\sqrt{n}}{\sqrt{n}+1} - 5\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$

Then we get $\lim\limits_{n\to\infty} \left(\frac{2}{n+1} - \frac{1}{n^3} + \frac{5\sqrt{n}}{\sqrt{n}+1}\right) = 5$ by definition of limit.

Problem 2 (Adapted from Apostal) If $0 < x_1 < 1$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for all $n \geq 1$, show that $\{x_n\}$ converges and find its limit.

Scratch:
In this kind of recurrence relation problems, we usually need to try a few terms to get a "sense" of whether it is a monotonic sequence or a intertwining sequence.
Since $x_2 = 1 - \sqrt{1 - x_1} \iff 1 - x_1 = (1 - x_2)^2 \iff x_1 = 2x_2 - x_2^2 \iff x_1 - x_2 = x_2(1 - x_2)$, $\{x_n\}$ seems to be decreasing.

Solution:
Claim $\forall n \in \mathbb{N}$, $0 < x_n < 1$:
For $n = 1$, $0 < x_1 < 1$.
Assume $0 < x_k < 1$.
For $n = k + 1$, $x_{k+1} = 1 - \sqrt{1 - x_k}$. Since $0 < x_k < 1$, we have $0 < \sqrt{1 - x_k} < 1$ and thus $0 < x_{k+1} = 1 - \sqrt{1 - x_k} < 1$.
By mathematical induction, $0 < x_n < 1$ $\forall n \in \mathbb{N}$. So $\{x_n\}$ is bounded from below by 0.

Then for any $n \geq 1$, $x_{n+1} = 1 - \sqrt{1 - x_n} \iff 1 - x_n = (1 - x_{n+1})^2 \iff x_n = 2x_{n+1} - x_{n+1}^2 \iff x_n - x_{n+1} = x_{n+1}(1 - x_{n+1}) > 0$.
So $\{x_n\}$ is decreasing.

Thus by monotonic sequence theorem, we see that $\{x_n\}$ converges, say to $\lim\limits_{n\to\infty} x_n = x$

From the relation $x_{n+1} = 1 - \sqrt{1 - x_n}$ for all $n \geq 1$ and $\lim\limits_{n\to\infty} x_n = x = \lim\limits_{n\to\infty} x_{n+1}$,
we get $x = 1 - \sqrt{1 - x} \iff \sqrt{1 - x}(1 - \sqrt{1 - x}) = 0$.
i.e. $\sqrt{1 - x} = 0$ or $1 - \sqrt{1 - x} = 0$, and both of them give $0 = x = \lim\limits_{n\to\infty} x_n$.

Problem 3 (Adapted from Rudin) Fix $\gamma > 1$. Take $x_1 > \sqrt{\gamma}$, and define

$$x_{n+1} = \frac{\gamma + x_n}{1 + x_n} = x_n + \frac{\gamma - x_n^2}{1 + x_n}.$$

Show that $\{x_n\}$ converges and find its limit.

Scratch:
Observation:
$\gamma > 1$, $x_1 > \sqrt{\gamma} > 1$ and $x_{n+1} = \dfrac{\gamma + x_n}{1 + x_n} > 1$.

$$\begin{aligned}
x_2 - \sqrt{\gamma} &= \frac{\gamma + x_1}{1 + x_1} - \sqrt{\gamma} \\
&= \frac{\gamma + x_1 - \sqrt{\gamma}(1 + x_1)}{1 + x_1} \\
&= \frac{\gamma + x_1 - \sqrt{\gamma} - \sqrt{\gamma}x_1}{1 + x_1} \\
&= \frac{-\sqrt{\gamma}(1 - \sqrt{\gamma}) + x_1(1 - \sqrt{\gamma})}{1 + x_1} \\
&= \frac{(x_1 - \sqrt{\gamma})(1 - \sqrt{\gamma})}{1 + x_1} \\
&< 0 \qquad \text{as } 1 + x_1 > 0,\ x_1 > \sqrt{\gamma} \text{ and } 1 < \sqrt{\gamma}
\end{aligned}$$

i.e $1 < x_2 < \sqrt{\gamma} < x_1$.

Also, $x_3 = x_2 + \dfrac{\gamma - x_2^2}{1 + x_2}$ and $x_2 < \sqrt{\gamma}$, $x_3 > x_2$ and

$$\begin{aligned}
x_1 - x_3 &= x_1 - \frac{\gamma + x_2}{1 + x_2} \\
&= x_1 - \frac{\gamma + \left(\frac{\gamma + x_1}{1 + x_1}\right)}{1 + \left(\frac{\gamma + x_1}{1 + x_1}\right)} \\
&= \frac{2(x_1^2 - \gamma)}{(1 + \gamma) + 2x_1} \\
&> 0
\end{aligned}$$

By similar argument as above we can get $x_2 < x_4 < \sqrt{\gamma} < x_3 < x_1$.
We suspect that $\{x_n\}$ is intertwining. So we are going to show that $\forall n \in \mathbb{N}$,

$$x_{2n} < x_{2n+2} < \sqrt{\gamma} < x_{2n+3} < x_{2n+1}.$$

Solution:
This statement can be proved by mathematical induction on $n \in \mathbb{N}$ and by the following equality:

$$x_{k+2} - x_k = \frac{2(\gamma - x_k^2)}{(1 + \gamma) + 2x_k}$$

We then get a collection of nested intervals $\{I_n = [x_{2n}, x_{2n+1}] |\ n \in \mathbb{N}\}$ and that $\{x_{2n}\}$ is increasing and bounded above by $\sqrt{\gamma}$; $\{x_{2n+1}\}$ is decreasing and bounded below by $\sqrt{\gamma}$.

Thus $\lim_{n\to\infty} x_{2n}$ and $\lim_{n\to\infty} x_{2n+1}$ exist, say $\lim_{n\to\infty} x_{2n} = a$ and $\lim_{n\to\infty} x_{2n+1} = b$.

Follow from the above identity with $k = 2n$, $x_{2n+2} - x_{2n} = \dfrac{2(\gamma - x_{2n}^2)}{(1+\gamma) + 2x_{2n}}$, we get $0 = a - a = \dfrac{2(\gamma - a^2)}{(1+\gamma) + 2a}$. i.e $a^2 = \gamma$ as for all $n \in \mathbb{N}$, $x_n > 1$, thus $a = \sqrt{\gamma}$.

Consider

$$\begin{aligned} b - a &= \lim_{n\to\infty} (x_{2n+1} - x_{2n}) \\ &= \lim_{n\to\infty} \left(\frac{\gamma - x_{2n}^2}{1 + x_{2n}} \right) \\ &= \frac{\gamma - a^2}{1 + a} \\ &= 0 \end{aligned}$$

By Nested Interval Theorem, $\lim_{n\to\infty} x_{2n} = a = \sqrt{\gamma} = b = \lim_{n\to\infty} x_{2n+1}$.
Then by Intertwining Theorem, $\lim_{m\to\infty} x_m = \sqrt{\gamma}$.