Solution of Homework 3

506 Sketch

$$|b_n-b_m| = \left|\sqrt{\frac{a_n}{a_n+3}+5}-\sqrt{\frac{a_m}{a_m+3}+5}\right| \le \sqrt{\left|\left(\frac{a_n}{a_n+3}+5\right)-\left(\frac{a_m}{a_m+3}+5\right)\right|}$$

$$= \sqrt{\left|\frac{a_n}{a_n+3}-\frac{a_m}{a_m+3}\right|} = \sqrt{\left|\frac{3a_n-3a_m}{\underbrace{(a_n+3)}_{>0}\underbrace{(a_m+3)}_{>0}}\right|} \le \sqrt{\frac{3|a_n-a_m|}{3\times 3}} = \sqrt{\frac{|a_n-a_m|}{3}} < \varepsilon$$

$\Leftrightarrow |a_n-a_m| < 3\varepsilon^2$.

Solution $\forall \varepsilon>0$, since $\{a_n\}$ is Cauchy, $\exists K\in\mathbb{N}$ such that $m,n\ge K \Rightarrow |a_n-a_m|<3\varepsilon^2$ $\xRightarrow{\text{by sketch above}} |b_n-b_m|<\varepsilon$

707 Solution If $f(a)=f(b)$, then $1-f(a)=1-f(b)$, $1-a^9=f(1-f(a))=f(1-f(b))=1-b^9$. So $a^9=b^9$. Taking 9^{th} root, $a=b$. Hence, f is injective. Since f is differentiable (hence continuous), by the continuous injection theorem, f is strictly monotone. Since $f'(1)<0$, f is strictly decreasing.

Let $g(x)=f(x)-x^{2013}$. Then $g(1)=f(1)-1^{2013}<0$. Since f is strictly decreasing, $0<1 \Rightarrow f(0)>f(1)=0$ $\Rightarrow g(0)=f(0)-0^{2013}=f(0)>0$. By the intermediate value theorem, $\exists r\in\mathbb{R}, g(r)=0$. $\therefore f(r)=r^{2013}$.

803 Solution By Taylor's theorem, $\exists\,\theta_0$ between 0 and x and $\exists\,\theta_1$ between $-x$ and 0 such that

$$f(x)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2}(x-0)^2+\frac{f'''(0)}{6}(x-0)^3+\frac{f^{(4)}(\theta_0)}{24}(x-0)^4$$

and

$$f(-x)=f(0)+f'(0)(-x-0)+\frac{f''(0)}{2}(-x-0)^2+\frac{f'''(0)}{6}(-x-0)^3+\frac{f^{(4)}(\theta_1)}{24}(-x-0)^4.$$

Adding these, we get

$$f(x)+f(-x)=2f(0)+f''(0)x^2+\left(f^{(4)}(\theta_0)+f^{(4)}(\theta_1)\right)\frac{x^4}{24}.$$

Solving for $f''(0)$, we get

$$f''(0)=\frac{f(x)-2f(0)+f(-x)}{x^2}-\frac{\left(f^{(4)}(\theta_0)+f^{(4)}(\theta_1)\right)x^2}{24}.$$

$$\therefore \left|f''(0)-\frac{f(x)-2f(0)+f(-x)}{x^2}\right| \le \frac{|f^{(4)}(\theta_0)|+|f^{(4)}(\theta_1)|}{24}x^2$$

$$\le \frac{1+1}{24}x^2=\frac{x^2}{12}.$$

901 (b)

Solution $\forall\varepsilon>0$, since S is of measure 0, $\exists (a_1,b_1),(a_2,b_2),\dots$ such that $S\subseteq\bigcup_{i=1}^{\infty}(a_i,b_i)$ and $\sum_{i=1}^{\infty}|a_i-b_i|<\varepsilon/2$, then $T\subseteq\bigcup_{i=1}^{\infty}(2a_i,2b_i)$ and $\sum_{i=1}^{\infty}|2a_i-2b_i|<\varepsilon$. $\therefore T$ is of measure 0.

Since f is Riemann integrable, by Lebesgue's Theorem, S_f is of measure 0. Observe that if $f(x)$ is continuous at w, then $g(x)=f(\frac{x}{2})$ is continuous at $2w$. Taking contrapositive, if g is discontinuous at $2w$, then f is discontinuous at w. (This means if $w\in S_g$, then $2w\in S_f$) So we have $S_g\subseteq\{2w : w\in S_f\}$. By first part, we have $\{2w : w\in S_f\}$ is measure 0. Then S_g is of measure 0. By Lebesgue theorem, g is integrable.