Lecture 4

19-02-2019

Review:

① An Equivalence relation in a set S, is intuitively a "relation" between elements in S which satisfies the reflexive, symmetry and transitive properties. Using math Language, it is a subset $R \subseteq S \times S$ which satisfies the same three properties.

② For the "super-set" which consists of all sets. One can define an equivalence relation in the following way:

$S_1 \sim S_2$ iff $\exists$ bijection $f: S_1 \to S_2$.

This equivalence relation is the foundation of counting.

③ A set S is finite iff

$\exists$ bijection $f: S \to \{1, 2, 3, \cdots m\}$ for some $m \in \mathbb{N}$.

A set S is countably infinite iff

$\exists$ bijection $f: S \to \mathbb{N}$.

A set S is countable if S is finite

or countably infinite.

④. "Counting", intuitively means counting or listing all

the elements in a set S without repetition and omission,

In Math Language, it means finding a bijection

between S and $\{1, 2, 3, \cdots m\}$ for some $m \in \mathbb{N}$ or $\mathbb{N}$.

⑤ Bijection theorem: If $S \sim T$. Then

(S is countable $\Leftrightarrow$ T is countable)

Example 3. $\mathbb{N}\times\mathbb{N} = \{ (m,n) : m, n \in \mathbb{N} \}$ is countably infinite.

[use diagonal counting scheme]

$$\begin{array}{lllll}
(1,1)^{f(1)} & (1,2)^{f(3)} & (1,3)^{f(6)} & (1,4) & \cdots \\
(2,1)\ f(2) & (2,2)_{f(5)} & (2,3) & (2,4) & \cdots \\
(3,1)\ f(4) & (3,2)\ f(8) & (3,3) & (3,4) & \cdots \\
(4,1)^{f(7)} & (4,2) & (4,3) & (4,4) & \cdots \\
(5,1) & \cdots & & &
\end{array}$$

One can check that $f : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ is bijective,

or equivalently, f provides a counting (or listing) of elements in $\mathbb{N}\times\mathbb{N}$ without repetition and omission.

Example 4 : Open interval $(0,1) = \{ x : x \in \mathbb{R},\ 0 < x < 1 \}$

is uncountable.

[More precisely, if "countable" is defined by Def ①. ②,

then $(0,1)$ is not countable]

Proof : By contradiction, assume that $(0,1)$ is countable,

then $\exists$ bijection $f : \mathbb{N} \to (0,1)$. So

$$f(1) = 0.\underline{a_{11}}\, a_{12}\, a_{13}\, a_{14} \cdots = \sum_{n=1}^{\infty} a_{1n} \times 10^{-n}$$

$$f(2) = 0.a_{21}\, \underline{a_{22}}\, a_{23}\, a_{24} \cdots = \sum_{n=1}^{\infty} a_{2n} \times 10^{-n}$$

$$\vdots$$

$$f(m) = 0.a_{m1}\, a_{m2}\, a_{m3}\, a_{m4} \cdots = \sum_{n=1}^{\infty} a_{mn} \times 10^{-n}.$$

Consider $x = 0.b_1 b_2 b_3 \cdots = \sum_{n=1}^{\infty} b_n \times 10^{-n}$

where $b_n = \begin{cases} a_n + 4 & \text{if} \quad a_{nn} \leq 4 \\ a_n - 4 & \text{if} \quad a_{nn} \geq 5 . \end{cases}$

Then $x \in (0, 1)$. However $x \neq f(n)$ for all n.

Since $|a_{nn} - b_n| = 4$,

the n-th digit of x differs from the n-th digit of $f(n)$ by 4.

This contradicts the surjectivity of f.

$\therefore$ $(0,1)$ is not countable.

Remark : $0.19999\cdots = 0.2000\cdots$

Two real numbers may be identical even if their n-th digits differ by 1 or 9.

Countable Subset theorem

THM: Let $A \subseteq B$.

If B is countable, then A is countable.

[Contrapositive: let $A \subseteq B$. if A is not countable, then B is not]

proof: B is countable $\Rightarrow$ $B = \{b_1, b_2, b_3, \dots\}$ where the listing of elements in B has no repetition nor omission. From this listing, we remove elements that are not in A. In doing so, we get a listing of A. Since $A \subseteq B$, the listing has no omission of elements in A. It also has no repetition. As a result, we see that A is countable.

Countable union theorem

THM : $\bigcup_{n\in\mathbb{N}} A_n$ is countable if A_n is countable for $n \geq 1$.

(In general, $\bigcup_{i\in\Lambda} A_i$ is countable if A_i is countable for each $i \in \Lambda$

and Λ is countable)

Proof : A_n is countable for all $n \geq 1$, $\Rightarrow$

$$A_1 = \{ a_{11}, \square, \square, \cdots \}, \cdots$$
$$A_2 = \{ a_{21}, a_{22}, a_{23}, \cdots \}$$
$$\vdots$$
$$A_n = \{ a_{n1}, a_{n2}, a_{n3}, \cdots \}, \cdots$$

Using diagonal counting scheme (skip blanks if the listing of elements in the A_n's is finite), we see that

$\{ a_{11}, a_{21}, a_{12}, a_{31}, \cdots \}$ gives a listing of $\bigcup_{n\in\mathbb{N}} A_n$ with no omission. If we remove repeated elements in the list, then we can get a listing of $\bigcup_{n\in\mathbb{N}} A_n$ with no repetition, nor omission. Hence $\bigcup_{n\in\mathbb{N}} A_n$ is countable.

Product theorem

THM: For $n \in \mathbb{N}$, if $A_1 \cdots A_n$ is countable,

then $A_1 \times \cdots \times A_n$ is countable.

Proof: $n=1$ is clear.

for $n=2$, $A_1 = \{ x_1, x_2 \}$, $A_2 = \{ y_1, y_2, \cdots \}$

$$\Rightarrow A_1 \times A_2 = \left\{ \begin{array}{l} (x_1, y_1), (x_1, y_2), (x_1, y_3), \cdots \\ \vdots \\ (x_n, y_1), (x_n, y_2), (x_n, y_3) \cdots \\ \vdots \end{array} \right\}$$

using diagonal counting scheme (skip blanks in A_1 or A_2 if finite)

$\{ (x_1, y_1), (x_2, y_1), (x_1, y_2), (x_3, y_1), \cdots \}$ is a listing of all elements

in $A_1 \times A_2$ with no repetition nor omission.

Hence $A_1 \times A_2$ is countable.

It following that $A_1 \times A_2 \times A_3 = (A_1 \times A_2) \times A_3$ is countable.

Repeating this argument $\boxed{(n-2) \text{ times,}}$ we see that

$$A_1 \times A_2 \times \cdots \times A_n = (A_1 \times A_2 \times \cdots \times A_{n-1}) \times A_n \text{ is}$$

countable.

R.K: Product theorem does not hold for infinitely many countable sets.

Injection theorem

Let $f: A \to B$ be injective. If B is countable, then A is countable. [Contrapositive: let $f: A \to B$ be injective, if A is not countable, then B is not countable]

Proof: Let $g: A \to f(A)$ be given by

$$g(x) = f(x) \quad \text{for} \quad x \in A.$$

Then g is both injective and surjective, i.e g is a bijection.

Since $f(A) \subseteq B$, and B is countable, $f(A)$ is countable by countable subset theorem.

Therefore A is countable by bijection theorem

Consequence: $\mathbb{R}$ is uncountable since $(0,1) \subset \mathbb{R}$ is uncountable.

Surjection theorem

Let $g: A \to B$ be surjective. If A is countable, then B is countable. [Contrapositive : let $g: A \to B$ be surjective, if B is not countable, then A is not countable].

Proof: $B = f(A) = \{ f(x) : x \in A \}$

$$= \bigcup_{x \in A} \{ f(x) \}$$

is countable by countable union theorem.

Example

Example ⑤ : $\mathbb{Q}$ is countable.

Proof ①: $\mathbb{Q} = \{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \} = \bigcup_{n \in \mathbb{N}} S_n$

where $S_n = \{ \frac{m}{n} : m \in \mathbb{Z} \} = \bigcup_{(m,n) \in \mathbb{Z} \times \mathbb{N}} \{ \frac{m}{n} \}$

It's clear that S_n is countable (since $S_n \sim \mathbb{Z}$)

By countable union theorem, $\mathbb{Q}$ is countable.

Proof ②: Let $S = \{ 2^n \cdot 3^m : n \in \mathbb{N}, m \in \mathbb{N} \}$

Def $g : S \to \mathbb{Q}_+ = \{ \frac{m}{n} : m, n \in \mathbb{N} \}$ $g(2^n \cdot 3^m) = \frac{m}{n}$

then g is surjective. Since $S \subset \mathbb{N}$ and is countable.

by surjective theorem, $\mathbb{Q}_+$ is countable. Since $\mathbb{Q}_+ \sim \mathbb{Q}_- = \{ -q : q \in \mathbb{Q} \}$

$\mathbb{Q}_-$ is countable. $\therefore$ $\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_-$ is countable

Example ⑥. Show that the set S of all polynomials with integer coefficients is countable.

proof: Let $S_n = \{ a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 : a_n, a_{n-1}, \cdots a_0 \in \mathbb{Z}, a_n \neq 0 \}$

For $n=0$, $S_0 = \mathbb{Z}$. (constants are 0th order polynomial)

Then $S = S_0 \cup \bigcup_{n \in \mathbb{N}} S_n$

By countable union theorem, we need only to show that S_n is countable for each $n \in \mathbb{N}$.

Indeed, define $f: S_n \to \mathbb{Z}^{n+1}$ by

$$f(a_n x^n + \cdots a_0) = (a_n, a_{n-1}, \cdots a_0)$$

then f is injective. Since $\mathbb{Z}^{n+1}$ is countable, S_n is countable.

Example ⑦ If A is uncountable, B is countable, then $A \setminus B$ is uncountable.

Proof: By contradition, if $A \setminus B$ is countable, then $A = (A \setminus B) \cup B$. is countable. This contradicts the assumption that A is uncountable. Hence $A \setminus B$ is not countable.

Important case: $\mathbb{R}$ is uncountable, $\mathbb{Q}$ is countable, $\mathbb{R} \setminus \mathbb{Q}$ is uncountable

Remark: $\underbrace{\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}}_{\text{countable}} \subseteq \underbrace{\mathbb{R} \subseteq \mathbb{C}}_{\text{uncountable}}$

Example 8. Show that the set L of all lines with equation

$y = mx + b$, where $m, b \in \mathbb{Q}$ is countable.

proof: Define $f: \mathbb{Q} \times \mathbb{Q} \to L$ by

$f(m, b)$ be the line with equation $y = mx + b$.

then f is surjective. Since $\mathbb{Q} \times \mathbb{Q}$ is countable,

by surjection theorem, L is countable.

Example 9. Let $A_1 = A_2 = A_3 = \cdots = \{0, 1\}$, then

$A_1 \times A_2 \times A_3 \times \cdots$ is uncountable.

[$A_1 \times A_2 \times \cdots = \{(a_1, a_2, a_3, \cdots) : a_n = 0 \text{ or } 1 \text{ for all } n \in \mathbb{N}\}$.]

Proof: Define $f : A_1 \times A_2 \times \cdots \to (0,1)$

$$f(a_1, a_2, \cdots) = a_1 \cdot 2^{-1} + a_2 \cdot 2^{-2} + a_3 \cdot 2^{-3} + \cdots$$
$$= \sum_{n=1}^{\infty} a_n \cdot 2^{-n}$$

then f gives a binary representation (instead of the usual base-10 numeral representation) of real numbers in $(0,1)$

It's clear that f is surjective. By surjection theorem, $A_1 \times A_2 \times \cdots$ is uncountable since $(0,1)$ is uncountable.

Remark: Another proof is similar to the one in example 4.

Example 10. Show $P(\mathbb{N}) = 2^{\mathbb{N}}$: the set of all subset of $\mathbb{N}$ is uncountable.

Proof : There is a one-to-one correspondence between $2^{\mathbb{N}}$ and $A_1 \times A_2 \times \cdots$ where $A_i = \{0,1\}$ for all i.

Define : $g : 2^{\mathbb{N}} \to A_1 \times A_2 \times \cdots$ by

$g(S) = (a_1, a_2, a_3, \cdots)$, where

a_n indicates whether $n \in S$ or not. More precisely

$$a_n = \begin{cases} 1, & \text{if } n \in S \\ 0, & \text{if } n \notin S. \end{cases}$$

For example : $g(\phi) = (0,0,0,\cdots)$, $g\{1,2\} = (1,1,0,0,\cdots)$

One can show that g is a bijection. Hence $2^{\mathbb{N}}$ is uncountable since $A_1 \times A_2 \times \cdots$ is.

Example 11. Is every real number a root of some polynomial with integer coefficients?

Solution: Let S be the set of all polynomials with integer coefficients. [See example 6]. Then S is countable. For each $f \in S$, denote R_f the set of roots of f. Then R_f is finite.

Therefore $T = \bigcup_{f \in S} R_f$ is countable by countable union theorem. Hence $\mathbb{R} \setminus T$ is uncountable.

Note that T is the set of all roots of polynomials with integer coefficients. We see that there are uncountable many real numbers which are not root of polynomials of integer coefficients.

Such numbers are call transcendental, such as π, e.