1. Recall that for every $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, define

$$\|x\|_\infty = \max_{1 \le k \le n} |x_k|.$$

For every $x, y \in \mathbb{R}^n$, define

$$d_\infty(x, y) = \|x - y\|_\infty.$$

Prove that $(\mathbb{R}^n, d_\infty)$ is a metric space.

2. If $1 \le p < q$, show that the unit ball in $\ell^p(\mathbb{R}^n)$ is contained in the unit ball in $\ell^q(\mathbb{R}^n)$.

3. Consider a point $x \in \mathbb{R}^2$ that lies outside the unit ball in $\ell^1(\mathbb{R}^2)$ and inside the unit ball in $\ell^\infty(\mathbb{R}^2)$. Is there a p between 1 and ∞ such that $\|x\|_p = 1$? Do the same problem in $\mathbb{R}^n$.

4. Prove that any subset of a discrete metric space is both open and closed.

5. Let $1 \le p \le \infty$ be fixed. Find an uncountable number of subsets of $\ell^p(\mathbb{R}^n)$ that are neither open nor closed.