MATH2033 Mathematical Analysis (2021 Spring)
Suggested Solution of Problem Set 3

Problem 1

Prove that the following sets are countably infinite using the definition.

(a) $A = \{n \in \mathbb{N} |\ n \text{ is not multiple of } 5\}$.

(b) $B = \{n \in \mathbb{Z} |\ n \text{ is odd number}\}$.

😊Solution

(a) One can construct a mapping $f: \mathbb{N} \to A$ as follows:

$$f(4k+1) = 5k+1, \quad f(4k+2) = 5k+2, \quad f(4k+3) = 5k+3,$$
$$f(4k+4) = 5k+4 \ \ for\ \ all\ k = 0,1,2,\dots$$

One can prove that f is bijective as follows:

- (injective) For any $f(x_1) = f(x_2)$, we have $f(x_1) = f(x_2) = 5k + r$ for some $k = 0,1,2,\dots$ and $r = 1,2,3,4$. We let $x_1 = 4m + p$, then $f(x_1) = 5m + p = 5k + r \Rightarrow 5(m-k) = p - r$.
  Since L.H.S. is divisible by 5, it follows $p - r$ is also multiple of 5. But $-3 \leq p - r \leq 3$. So $p - r = 0 \Rightarrow p = r$. This implies that $5(m-k) = 0 \Rightarrow m = k$
  Thus $x_1 = 4k + r$. Similarly, $x_2 = 4k + r$. So $x_1 = x_2$ and f is injective.
- (Surjective) For any $y = 5k + r \in A$, we pick $x = 4k + r$, then $f(x) = f(4k + r) = 5k + r$. So f is surjective.

Since f is bijection, then A is countable by definition.

(b) For any $n \in \mathbb{Z}$, we can write $n = 2m + 1$ for some $m \in \mathbb{Z}$. One can construct a mapping $g: \mathbb{N} \to B$ as

$$g(1) = 2(0) + 1, \quad g(2n) = 2n + 1, \quad g(2n+1) = 2(-n) + 1 = -2n + 1.$$

One can prove that g is bijective (using similar method as in (a)). So B is countable by definition.

Problem 2

We let $A_1, A_2, A_3, \dots$ be subsets of $\mathbb{R}$. Suppose that the set

$$S = A_1 \times A_2 \times A_3 \times \dots$$

is countable. Prove that there are only finitely many sets that have more than one elements.

😊Solution

We shall prove it by contradiction. Assuming that there are infinitely many sets that have more than one elements, we let these sets be $A_{n_1}, A_{n_2}, A_{n_3}, \dots$, where $n_1 < n_2 < n_3 < \cdots$.
Next, we define a mapping $f: (A_1 \times A_2 \times A_3 \times \dots) \to \left(A_{n_1} \times A_{n_2} \times A_{n_3} \times \dots\right)$ to be

$$f\big((x_1, x_2, x_3, \cdots)\big) = \left(x_{n_1}, x_{n_2}, x_{n_3}, \dots\right).$$

One can show that the mapping f is surjective. That is, for any $\vec{a} = (a_1, a_2, a_3, \dots) \in \left(A_{n_1} \times A_{n_2} \times A_{n_3} \times \dots\right)$, we take $\vec{x} = \left(x_1, x_2, \dots, \underbrace{x_{n_1}}_{a_1}, \dots, \underbrace{x_{n_2}}_{a_2}, \dots, \underbrace{x_{n_3}}_{a_3}, \dots\right) \in A_1 \times A_2 \times A_3 \times \dots$. Then we have $f(\vec{x}) = \vec{a}$.

Since each of $A_{n_k}s$ has at least two elements, one can mimic the proof in Example 4 of lecture note 3 (i.e. $\{0,1\} \times \{0,1\} \times \ldots$ is uncountable) that $A_{n_1} \times A_{n_2} \times A_{n_3} \times \ldots$ is uncountable. It follows from surjection theorem that $S = A_1 \times A_2 \times A_3 \times \ldots$ is uncountable and there is contradiction.

Problem 3

We let $f_1, f_2, f_3, \ldots : \mathbb{R} \to [0, \infty)$ be a collection of functions from $\mathbb{R}$ to $[0, \infty)$. For any positive integer n, we define

$$A_n = \{x \in \mathbb{R}: f_n(x) = 0\}.$$

We consider a set defined by

$$S = \left\{ x \in \mathbb{R} \middle| \sum_{n=1}^{\infty} f_n(x) = 0 \right\}$$

(a) Show that if A_n is countable for some $n \in \mathbb{N}$, then S is countable.

(b) Suppose A_n is uncountable for all $n \in \mathbb{N}$,

(i) Is it true that S is always uncountable?

(☺Note: If your answer is yes, give a proof. If your answer is no, give a counter-example.)

(ii) Is it true that S is always countable?

(☺Note: If your answer is yes, give a proof. If your answer is no, give a counter-example.)

☺Solution

(a) Since $f_n(x) \geq 0$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$ (as the codomain is $[0, \infty)$), then one can show that

$$\sum_{n=1}^{\infty} f_n(x) = 0 \Leftrightarrow f_n(x) = 0 \ \ for\ all\ n \in \mathbb{N}.$$

(*Proof:* "$\Leftarrow$" part is clear. For "$\Rightarrow$" part, suppose that $f_k(x) = 0$ for some k$\in \mathbb{N}$, it follows that $\sum_{n=1}^{\infty} f_n(x) \geq f_k(x) > 0$ and this leads to contradiction.)

Then it follows that

$$S = \bigcap_{n=1}^{\infty} A_n.$$

Since A_m is countable for some $m \in \mathbb{N}$, it follows from countable subset theorem that the set $S = \bigcap_{n=1}^{\infty} A_n \subseteq A_m$ is also countable.

(b) (i) The answer is no. To see this, we let $f_1(x) = \begin{cases} 1 & if\ \ x \geq 0 \\ 0 & if\ \ x < 0 \end{cases}$, $f_2(x) = \begin{cases} 0 & if\ \ x \geq 0 \\ 1 & if\ \ x < 0 \end{cases}$ and $f_3(x) = f_4(x) = \cdots = 0$ for all $x \in \mathbb{R}$. One can show that

- $A_1 = (-\infty, 0)$ and $A_2 = [0, \infty)$ are uncountable and $A_3 = A_4 = \cdots = \mathbb{R}$ are also uncountable.
- Since $A_1 \cap A_2 = \phi$, then $S = \bigcap_{n=1}^{\infty} A_n = \phi$ which is countable.

(ii) The answer is no. To see this, we let $f_1(x) = f_2(x) = f_3(x) = \cdots = 0$ for all $x \in \mathbb{R}$. One can see that

- $A_1 = A_2 = \cdots = \mathbb{R}$ are uncountable and
- $S = \bigcap_{n=1}^{\infty} A_n = \mathbb{R}$ which is also uncountable.

Problem 4

Determine if the set C defined by

$$C = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x^2 + y^2 = r^2\}$$

is countable.

(😊Hint: Recall that $x^2 + y^2 = r^2$ represents the equation of circle with radius r in $\mathbb{R}^2$-plane)

😊Solution

Recall that each point (x, y) on the circle $x^2 + y^2 = r^2$ can be expressed as $(x, y) = (r\cos\theta, r\sin\theta)$ where $\theta \in [0,2\pi)$.

Based on this fact, we construct a mapping $f: C \to [0,2\pi)$ as

$$f(x, y) = f(r\cos\theta, r\sin\theta) = \theta.$$

One can show that f is bijective since

- $f(x_1, y_1) = f(x_2, y_2) \Rightarrow x_1 = x_2 = r\cos\theta$ and $y_1 = y_2 = r\sin\theta$. So $(x_1, y_1) = (x_2, y_2)$ and f is injective.
- For any $\theta \in [0,2\pi)$, we pick $(x, y) = (r\cos\theta, r\sin\theta) \in C$, then $f(x, y) = \theta$. So f is surjective.

Since $[0,2\pi) \supseteq (0,2\pi)$ and $(0,2\pi)$ is uncountable (recall that any open interval (a, b) is uncountable. Then $[0,2\pi)$ is also uncountable by countable subset theorem.

Hence, C is uncountable by bijection theorem.

Problem 5

(a) Determine if the set D defined by

$$D = \{x \in \mathbb{R}: \tan^{10} x - 3\tan^3 x + 1 = 0\}$$

is countable.

(b) Determine if the set E defined by

$$E = \left\{x \in \mathbb{R} | a\cos 2x + b\cos x + c = 0 \quad for\ some\ a, b, c, \in \mathbb{Q}\backslash\{0\}\right\}$$

is countable.

😊Solution

(a) We define the set A as

$$A = \{y \in \mathbb{R} | y^{10} - 3y^3 + 1 = 0\}.$$

Note that $y^{10} - 3y^3 + 1 = 0$ has at most 10 real roots. So A is finite and therefore countable.

On the other hand, the set D can be expressed as

$$D=\{x\in\mathbb{R}:\tan^{10}x-3\tan^3 x+1=0\}=\bigcup_{y\in A}\{x\in\mathbb{R}|\tan x=y\}$$
$$=\bigcup_{y\in A}\{n\pi+\tan^{-1}y\,|n\in\mathbb{Z}\}=\bigcup_{y\in A}\bigcup_{n\in\mathbb{Z}}\{n\pi+\tan^{-1}y\}$$

Since $\{n\pi+\tan^{-1}y\}$ has 1 element and therefore countable, then $\bigcup_{n\in\mathbb{Z}}\{n\pi+\tan^{-1}y\}$ is countable by countable union theorem (as $\mathbb{Z}$ is countable). Hence, $D=\bigcup_{y\in A}(\bigcup_{n\in\mathbb{Z}}\{n\pi+\tan^{-1}y\})$ is also countable by countable union theorem (as A is countable.)

(b) We write $\cos 2x=2\cos^2 x-1$, then the equation can be rewritten as
$a\cos 2x+b\cos x+c=0\Rightarrow a(2\cos^2 x-1)+b\cos x+c=0$
$\Rightarrow 2a\cos^2 x+b\cos x+c-a=0.$

We define the set B as
$$B=\{y\in\mathbb{R}|2ay^2+by+c-a=0\}.$$
Note that the equation has at most 2 real roots. So B is finite and therefore countable.

On the other hand, the set D can be expressed as
$$E=\{x\in\mathbb{R}:2a\cos^2 x+b\cos x+c-a=0\}=\bigcup_{y\in B}\{x\in\mathbb{R}|\cos x=y\}$$
$$=\bigcup_{y\in B}\{2n\pi\pm\cos^{-1}y\,|n\in\mathbb{Z}\}=\bigcup_{y\in B}\bigcup_{n\in\mathbb{Z}}\{2n\pi\pm\cos^{-1}y\}$$

Since $\{2n\pi\pm\cos^{-1}y\}$ has 2 elements and therefore countable, then $\bigcup_{n\in\mathbb{Z}}\{2n\pi\pm\cos^{-1}y\}$ is countable by countable union theorem (as $\mathbb{Z}$ is countable). Hence, $D=\bigcup_{y\in B}(\bigcup_{n\in\mathbb{Z}}\{2n\pi\pm\cos^{-1}y\})$ is also countable by countable union theorem (as A is countable.)

Problem 6

We let $f:A\to B$ be a function, where A,B are non-empty set.

(a) If A is countable, determine if $f(A)$ is countable.

(b) We consider the case when A is uncountable

(i) If f is injective, show that $f(A)$ is also uncountable by mimicking the proof of the injection theorem.

(ii) Is $f(A)$ always uncountable if the function f is not injective? Explain your answer.

😊Solution

(a) Since A is countable, we can write $A=\{a_1,a_2,a_3,\ldots\}$. It follows that
$$f(A)=\{f(a_1),f(a_2),f(a_3),\ldots\}.$$
By removing the repeated elements, we get
$$f(A)=\{f(a_{n_1}),f(a_{n_2}),f(a_{n_3}),\ldots\}.$$
Then we can construct the mapping $g:\mathbb{N}\to f(A)$, as
$$g(k)=f(a_{n_k})\quad for\ \ k=1,2,3,\ldots.$$

One can show that g is bijective so that $f(A)$ is countable by definition.

(b) (i) Suppose that $f(A)$ is countable, we write $f(A) = \{b_1, b_2, b_3, \dots\}$. By deleting the repeated elements, then we have

$$f(A) = \{b_{n_1}, b_{n_2}, b_{n_3}, \dots\}.$$

Since f is injective, there exists an inverse mapping $f^{-1}: f(A) \to A$ which $a_{n_k} = f^{-1}(b_{n_k})$ so the set A can be expressed as

$$A = \{f^{-1}(b_{n_1}), f^{-1}(b_{n_2}), \dots\}$$

Thus one can construct a bijection $g: \mathbb{N} \to A$ as

$$g(k) = f^{-1}(b_{n_k}).$$

So A is countable and this leads to contradiction.

(ii) The answer is negative. To see this, we pick $b \in B$ and define a mapping $f: A \to B$ by

$$f(a) = b \quad for \ \ all\ a \in A.$$

One can see that f is not injective since there are at least two different a_1, a_2 (as A is uncountable) which $f(a_1) = f(a_2)$. On the other hand, $f(A) = \{b\}$ contains 1 element which is countable.

Problem 7

We let $A, B \subseteq \mathbb{R}$ be two uncountable sets.

(a) Is it always true that $A \backslash B$ is uncountable?

(b) Is it always true that $A \backslash B$ is countable?

(😊Note: If your answer is yes, give a proof. If your answer is no, give a counter-example.)

😊Solution

(a) The answer is **No**. To see this, we pick $A = [a, b]$ and $B = (a, b)$ with $a < b$. Note that both A and B are uncountable, but $A \backslash B = \{a, b\}$ which is countable (as it is a finite set)

(b) The answer is **No**. We pick $A = (0,1)$ and $B = (2,3)$. Note that both A, B are uncountable but $A \backslash B = (0,1)$ is also uncountable.

Problem 8 (Harder)

We let A be set of all functions from the set $\{0,1\}$ to the set of positive integers $\mathbb{N}$. That is,

$$A = \{f | f: \{0,1\} \to \mathbb{N}\}.$$

Show that A is countable.

😊Solution

Note that each element f in the set A can be expressed as a pair $(f(0), f(1)) \in \mathbb{N} \times \mathbb{N}$ (since the domain of f is $\{0,1\}$).

Then we can construct a mapping $g: A \to \mathbb{N} \times \mathbb{N}$ as

$$g(f) = (f(0), f(1)).$$

One can show that g is bijective. That is,

- If $g(f_1) = g(f_2) = (x, y)$, we must have $f_1(0) = f_2(0) = x$ and $f_1(1) = f_2(1) = y$. So it follows that $f_1 = f_2$ and g in injective.

- For any $(x, y) \in \mathbb{N} \times \mathbb{N}$, we choose $f: \{0,1\} \to \mathbb{N}$ such that $f(0) = x$ and $f(1) = y$. We see that $f \in A$ and $g(f) = \big(f(0), f(1)\big) = (x, y)$. So f is surjective.

As $\mathbb{N}$ is countable, so $\mathbb{N} \times \mathbb{N}$ is also countable. Thus, A is countable by bijection theorem.

Problem 9

Show that for any open interval (a, b), there are infinitely many irrational numbers that lie in this interval.

Solution

Note that the set of irrational number over the interval (a, b) can be expressed as $(a, b)\setminus\mathbb{Q}$. Since (a, b) is uncountable and $\mathbb{Q}$ is countable, it follows that $(a, b)\setminus\mathbb{Q}$ is uncountable and must be infinite (since finite set is always countable).