2012 Fall Midterm

① Let S be the set of all points $(x,y)\in\mathbb{R}^2$ that satisfy the system of equations

$$x+y=mx^2-x^3 \text{ and } mx+y^4=x^6-7mx^3+2$$

for some $m\in\mathbb{Q}$. Determine (with proof) if S is countable or not.

Solution If $m\in\mathbb{Q}$, then

$$\left.\begin{array}{l} x+y=mx^2-x^3 \\ mx+y^4=x^6-7mx^3+2 \end{array}\right\} \Leftrightarrow \begin{array}{l} y=mx^2-x^3-x \\ mx+(mx^2-\overset{③}{x^3}-x)^{④}=x^6-7mx^3+2 \end{array}$$

$\uparrow$ degree 12

So there are at most 12 x's and 1 y for each x.

$$\therefore S=\{(x,y): (x,y)\in\mathbb{R}^2, m\in\mathbb{Q}, x+y=mx^2-x^3, mx+y^4=x^6-7mx^3+2\}$$

$$=\bigcup_{\underbrace{m\in\mathbb{Q}}_{\text{countable}}} \underbrace{\{(x,y): (x,y)\in\mathbb{R}^2, x+y=mx^2-x^3, mx+y^4=x^6-7mx^3+2\}}_{\text{at most 12 } (x,y)\text{'s} \Rightarrow \text{finite} \Rightarrow \text{countable}}$$

is countable by the countable union theorem.

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② Determine (with proof) all positive real numbers b such that the series $\sum_{k=1}^{\infty}\frac{2^{k+3}}{\sqrt{k}(\sqrt{b}+1)^k}$ converges.

Be sure to prove you have gotten all such b.

Solution By ratio test,

$$\lim_{k\to\infty}\frac{2^{k+4}}{\sqrt{k+1}(\sqrt{b}+1)^{k+1}}\frac{\sqrt{k}(\sqrt{b}+1)^k}{2^{k+3}}=\lim_{k\to\infty}\frac{2}{\sqrt{b}+1}\sqrt{\frac{k}{k+1}}=\frac{2}{\sqrt{b}+1}$$

If $\frac{2}{\sqrt{b}+1}<1$ ($\Leftrightarrow b>1$), then the series converges.

If $\frac{2}{\sqrt{b}+1}>1$ ($\Leftrightarrow b<1$), then the series diverges.

If $\frac{2}{\sqrt{b}+1}=1$ ($\Leftrightarrow b=1$), the series $\sum_{k=1}^{\infty}\frac{8}{k^{1/2}}$ diverges by p-test as $1/2<1$.

Remark We can also do it by root test

$$\lim_{k\to\infty}\sqrt[k]{\frac{2^{k+3}}{\sqrt{k}(\sqrt{b}+1)^k}}=\lim_{k\to\infty}\frac{2^{1+3/k}}{k^{1/2k}(\sqrt{b}+1)}=\frac{2}{\sqrt{b}+1}.$$

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③ Let S be a nonempty countable subset of $\mathbb{R}$. Prove that there exists a positive real number r such that the equation $5^x+7^y=\sqrt{r}$ does not have any solution with $x,y\in S$.

Solution $5^x+7^y=\sqrt{r} \Leftrightarrow (5^x+7^y)^2=r$.

Let $T=\{(5^x+7^y)^2 : x,y\in S\}$. Then

$$T=\bigcup_{(x,y)\in \underbrace{S\times S}_{\text{countable}}}\underbrace{\{(5^x+7^y)^2\}}_{1\text{ element}\Rightarrow\text{finite}\Rightarrow\text{countable}} \text{ is countable.}$$

$\therefore (0,\infty)\setminus T$ is uncountable

$\therefore \exists\, r\in(0,\infty)$ and $r\notin T$

$\therefore$ there exists $r>0$ and $r\neq(5^x+7^y)^2$ with $x,y\in S$.

$\Leftrightarrow 5^x+7^y\neq\sqrt{r}$