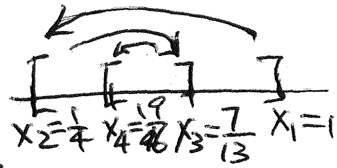
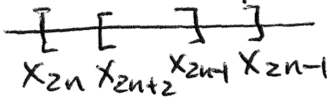
Solutions to Presentation Exercises

92(c) Sketch: $x_1 = 1$, $x_2 = \frac{1}{4}$, $x_3 = \frac{7}{13}$, $x_4 = \frac{19}{46}$



Let $I_n = [x_{2n}, x_{2n-1}]$. We want to show $I_n \supseteq I_{n+1}$ first.

Claim: $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$



We will prove this by math induction.

For $n=1$, we have $x_2 = \frac{1}{4} \le x_4 = \frac{19}{46} \le x_3 = \frac{7}{13} \le x_1 = 1$

Suppose Case n: $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$. We have $x_{k+1} = \frac{2-x_k}{3+x_k} = \frac{5}{3+x_k} - 1$.

So $3+x_{2n} \le 3+x_{2n+2} \le 3+x_{2n+1} \le 3+x_{2n-1}$,

then $\frac{5}{3+x_{2n}} \ge \frac{5}{3+x_{2n+2}} \ge \frac{5}{3+x_{2n+1}} \ge \frac{5}{3+x_{2n-1}}$,

then $\underbrace{\frac{5}{3+x_{2n}} - 1}_{=x_{2n+1}} \ge \underbrace{\frac{5}{3+x_{2n+2}} - 1}_{=x_{2n+3}} \ge \underbrace{\frac{5}{3+x_{2n+1}} - 1}_{=x_{2n+2}} \ge \underbrace{\frac{5}{3+x_{2n-1}} - 1}_{x_{2n}}$,

then $3+x_{2n+1} \ge 3+x_{2n+3} \ge 3+x_{2n+2} \ge 3+x_{2n}$

then $\frac{5}{3+x_{2n+1}} \le \frac{5}{3+x_{2n+3}} \le \frac{5}{3+x_{2n+2}} \le \frac{5}{3+x_{2n}}$

then $\underbrace{\frac{5}{3+x_{2n+1}} - 1}_{=x_{2n+2}} \le \underbrace{\frac{5}{3+x_{2n+3}} - 1}_{=x_{2n+4}} \le \underbrace{\frac{5}{3+x_{2n+2}} - 1}_{=x_{2n+3}} \le \underbrace{\frac{5}{3+x_{2n}} - 1}_{x_{2n+1}}$

We get Case $n+1$: $x_{2n+2} \le x_{2n+4} \le x_{2n+3} \le x_{2n+1}$.

Then x_{2n} is increasing and bounded above by $x_1 = 1$ and x_{2n+1} is decreasing and bounded below by $x_2 = \frac{1}{4}$.

By monotone sequence theorem, $\lim_{n\to\infty} x_{2n} = a$, $\lim_{n\to\infty} x_{2n+1} = b$ for some $a, b \in \mathbb{R}$. Now

$$b = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} \frac{2-x_{2n}}{3+x_{2n}} = \frac{2-a}{3+a} \Rightarrow 3b+ab = 2-a$$

and
$$a = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} \frac{2-x_{2n-1}}{3+x_{2n-1}} = \frac{2-b}{3+b} \Rightarrow 3a+ab = 2-b$$

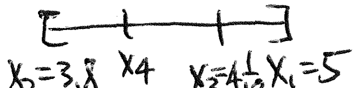
Subtracting, we get $3(b-a) = (b-a)$
$\Rightarrow b-a=0 \Rightarrow a=b$.

By intertwining sequence theorem, x_n converges to a. Then

$$a = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{2-x_n}{3+x_n} = \frac{2-a}{3+a} \Rightarrow a^2+4a-2=0 \Rightarrow a = -2 \pm \sqrt{6}.$$

Since $I_1 = [\frac{1}{4}, 1]$ contains all x_n, $a = \lim_{n\to\infty} x_n = -2+\sqrt{6}$ as $-2-\sqrt{6} \notin I_1$.

92(h) Sketch: $x_1=5$, $x_2=3\frac{4}{5}=3.8$, $x_3=4\frac{1}{19}$, $x_4=3+\frac{4}{4\frac{1}{19}}>3+\frac{4}{5}=x_2$



Let $I_n=[x_{2n},x_{2n-1}]$. We want to show $I_n \supseteq I_{n+1}$ first.

Claim: $x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}$ (We will prove by math induction).

For $n=1$, we have $x_2=3\frac{4}{5}\le x_4=3\frac{4}{4\frac{1}{19}}\le x_3=4\frac{1}{19}\le x_1=5$.

Suppose case n: $x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}$. We have $x_{k+1}=3+\frac{4}{x_k}$.

So $\frac{4}{x_{2n}}\ge\frac{4}{x_{2n+2}}\ge\frac{4}{x_{2n+1}}\ge\frac{4}{x_{2n-1}}$, then $\underbrace{3+\frac{4}{x_{2n}}}_{=x_{2n+1}}\ge\underbrace{3+\frac{4}{x_{2n+2}}}_{=x_{2n+3}}\ge\underbrace{3+\frac{4}{x_{2n+1}}}_{=x_{2n+2}}\ge\underbrace{3+\frac{4}{x_{2n-1}}}_{=x_{2n}}$,

then $\frac{4}{x_{2n+1}}\le\frac{4}{x_{2n+3}}\le\frac{4}{x_{2n+2}}\le\frac{4}{x_{2n}}$, then $\underbrace{3+\frac{4}{x_{2n+1}}}_{=x_{2n+2}}\le\underbrace{3+\frac{4}{x_{2n+3}}}_{=x_{2n+4}}\le\underbrace{3+\frac{4}{x_{2n+2}}}_{=x_{2n+3}}\le\underbrace{3+\frac{4}{x_{2n}}}_{=x_{2n+1}}$

We get case $n+1$: $x_{2n+2}\le x_{2n+4}\le x_{2n+3}\le x_{2n+1}$.

Then x_{2n} is increasing and bounded above by $x_1=5$ and x_{2n+1} is decreasing and bounded below by $x_2=3\frac{4}{5}$.

By monotone sequence theorem, $\lim_{n\to\infty}x_{2n}=a$, $\lim_{n\to\infty}x_{2n+1}=b$ for some $a,b\in\mathbb{R}$.

Now $b=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}\left(3+\frac{4}{x_{2n}}\right)=3+\frac{4}{a}\Rightarrow ab=3a+4$

and $a=\lim_{n\to\infty}x_{2n}=\lim_{n\to\infty}\left(3+\frac{4}{x_{2n-1}}\right)=3+\frac{4}{b}\Rightarrow ab=3b+4$

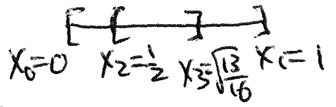
$\Rightarrow a=b$

By intertwining sequence theorem, x_n converges to a. Then

$a=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\left(3+\frac{4}{x_n}\right)=3+\frac{4}{a}\Rightarrow a^2-3a-4=0\Rightarrow a=-1$ or 4.

Since $I_1=[3.8,5]$ contains all x_n, $a=\lim_{n\to\infty}x_n=4$ as $-1\notin I_1$.

(96) <u>Sketch</u>: $x_0=0$, $x_1=1$, $x_2=\sqrt{\frac{1}{4}+0}=\frac{1}{2}$, $x_3=\sqrt{\frac{1}{16}+\frac{3}{4}}=\sqrt{\frac{13}{16}}$



Let $I_n=[x_{2n},x_{2n+1}]$ for $n=0,1,2,\ldots$ $\quad I_1=[0,1]$

We want to prove $I_n\supseteq I_{n+1}$ first. $\quad \boxed{x_{n+1}=\sqrt{\frac{1}{4}x_n^2+\frac{3}{4}x_{n-1}^2}}$

<u>Claim</u>: $x_{2n}\le x_{2n+2}\le x_{2n+3}\le x_{2n+1}$

For case $n=0$, we have $x_0=0\le x_2=\frac{1}{2}\le x_3=\sqrt{\frac{13}{16}}\le x_1=1$.

Suppose case n: $x_{2n}\le x_{2n+2}\le x_{2n+3}\le x_{2n+1}$. Then

$$x_{2n+2}=\sqrt{\tfrac{1}{4}x_{2n+2}^2+\tfrac{3}{4}x_{2n+2}^2}\underset{x_{2n+2}\le x_{2n+3}}{\le}\underbrace{\sqrt{\tfrac{1}{4}x_{2n+3}^2+\tfrac{3}{4}x_{2n+2}^2}}_{=x_{2n+4}}\underset{x_{2n+2}\le x_{2n+3}}{\le}\sqrt{\tfrac{1}{4}x_{2n+3}^2+\tfrac{3}{4}x_{2n+3}^2}=x_{2n+3}$$

and
$$x_{2n+4}=\sqrt{\tfrac{1}{4}x_{2n+4}^2+\tfrac{3}{4}x_{2n+4}^2}\underset{x_{2n+4}\le x_{2n+3}}{\le}\underbrace{\sqrt{\tfrac{1}{4}x_{2n+4}^2+\tfrac{3}{4}x_{2n+3}^2}}_{=x_{2n+5}}\underset{x_{2n+4}\le x_{2n+3}}{\le}\sqrt{\tfrac{1}{4}x_{2n+3}^2+\tfrac{3}{4}x_{2n+3}^2}=x_{2n+3}.$$

Then $x_{2n+2}\le x_{2n+4}\le x_{2n+5}\le x_{2n+3}$.

By nested interval theorem, $\lim_{n\to\infty}x_{2n}=a$, $\lim_{n\to\infty}x_{2n+1}=b$ for some $a,b\in\mathbb{R}$.

Then $b=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}\sqrt{\frac{1}{4}x_{2n}^2+\frac{3}{4}x_{2n-1}^2}=\sqrt{\frac{1}{4}a^2+\frac{3}{4}b^2}\Rightarrow b^2=\frac{1}{4}a^2+\frac{3}{4}b^2\Rightarrow a=b.$

So $\lim_{n\to\infty}x_n=a$. To find a, we write $x_{n+1}^2=\frac{1}{4}x_n^2+\frac{3}{4}x_{n-1}^2$ for $n\ge 1$.

$$\left.\begin{aligned}
x_2^2&=\tfrac{1}{4}x_1^2+\tfrac{3}{4}x_0^2\\
x_3^2&=\tfrac{1}{4}x_2^2+\tfrac{3}{4}x_1^2\\
&\ \ \vdots\\
x_k^2&=\tfrac{1}{4}x_{k-1}^2+\tfrac{3}{4}x_{k-2}^2\\
x_{k+1}^2&=\tfrac{1}{4}x_k^2+\tfrac{3}{4}x_{k-1}^2
\end{aligned}\right\}$$

Summing these, we get

$$x_2^2+\cdots+x_{k+1}^2=\tfrac{3}{4}x_0^2+(x_1^2+\cdots+x_{k-1}^2)+\tfrac{1}{4}x_k^2.$$

Cancelling, we get $x_{k+1}^2+\frac{3}{4}x_k^2=x_1^2+\frac{3}{4}x_0^2=1$

Taking limit as $k\to\infty$, we get $x^2+\frac{3}{4}x^2=1$.

This gives $x=\pm\sqrt{\frac{4}{7}}$. Since $-\sqrt{\frac{4}{7}}\notin I_1$, $x=\sqrt{\frac{4}{7}}$.

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(107) For $\varepsilon>0$, since $\{x_n\}$ and $\{y_n\}$ converge to A, by definition, there exist $K_1,K_2\in\mathbb{N}$ such that $n\ge K_1\Rightarrow |x_n-A|<\varepsilon$ and $n\ge K_2\Rightarrow |y_n-A|<\varepsilon$. Let $K=\max(K_1,K_2)$. Then

$n\ge K\Rightarrow n\ge K_1$ and $n\ge K_2\Rightarrow |x_n-A|<\varepsilon$ and $|y_n-A|<\varepsilon$

$\Rightarrow |z_n-A|<\varepsilon$ since $z_n=\max\{x_n,y_n\}$
$=x_n$ or y_n

Therefore, $\{z_n\}$ converges to A.

Ⓐ Sketch work: $\frac{2n^2-1}{4n^2} \to \frac{1}{2}$, $\frac{3n}{2n+1} \to \frac{3}{2}$

$$\left|\frac{2n^2-1}{4n^2} + \frac{3n}{2n+1} - 2\right| = \left|\left(\frac{2n^2-1}{4n^2} - \frac{1}{2}\right) + \left(\frac{3n}{2n+1} - \frac{3}{2}\right)\right| \le \left|\frac{2n^2-1}{4n^2} - \frac{1}{2}\right| + \left|\frac{3n}{2n+1} - \frac{3}{2}\right|$$

$$= \frac{1}{4n^2} + \frac{3}{2(2n+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$ if $\frac{1}{4n^2} < \frac{\varepsilon}{2}$ ($\Leftrightarrow n > \sqrt{\frac{1}{2\varepsilon}}$) and $\frac{3}{2(2n+1)} < \frac{\varepsilon}{2}$ ($\Leftrightarrow n > \frac{1}{2}(\frac{3}{\varepsilon} - 1)$)

Solution $\forall \varepsilon > 0$, $\exists$ $K \in \mathbb{N}$ and $K > \max\{\sqrt{\frac{1}{2\varepsilon}}, \frac{1}{2}(\frac{3}{\varepsilon}-1)\}$ by Archimedean principle.
Then $n \ge K \Rightarrow \left|\frac{2n^2-1}{4n^2} + \frac{3n}{2n+1} - 2\right| < \varepsilon$ as in the box above.

Ⓑ Sketch work: $\frac{n^4}{3n^4-2} \to \frac{1}{3}$, $\frac{1-2n}{3n} \to -\frac{2}{3}$

$$\left|\frac{n^4}{3n^4-2} - \frac{1-2n}{3n} - 1\right| = \left|\left(\frac{n^4}{3n^4-2} - \frac{1}{3}\right) - \left(\frac{1-2n}{3n} - \left(-\frac{2}{3}\right)\right)\right| \le \left|\frac{n^4}{3n^4-2} - \frac{1}{3}\right| + \left|\frac{1-2n}{3n} + \frac{2}{3}\right|$$

$$= \frac{2}{3(3n^4-2)} + \frac{1}{3n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$ if $\frac{2}{3(3n^4-2)} < \frac{\varepsilon}{2}$ ($\Leftrightarrow n > \sqrt[4]{\frac{1}{3}(\frac{4}{3\varepsilon}+2)}$) and $\frac{1}{3n} < \frac{\varepsilon}{2}$ ($\Leftrightarrow n > \frac{2}{3\varepsilon}$)

Solution $\forall \varepsilon > 0, \exists$ $K \in \mathbb{N}$ and $K > \max\{\sqrt[4]{\frac{1}{3}(\frac{4}{3\varepsilon}+2)}, \frac{2}{3\varepsilon}\}$ by Archimedean principle.
Then $n \ge K \Rightarrow \left|\frac{n^4}{3n^4-2} - \frac{1-2n}{3n} - 1\right| < \varepsilon$ as in the box above.

Ⓒ Sketch work: $\frac{b_n}{1+b_n^2} \to \frac{1}{2}$, $\frac{3n}{n+4} \to 3$

$$\left|\frac{b_n}{1+b_n^2} + \frac{3n}{n+4} - \frac{7}{2}\right| = \left|\left(\frac{b_n}{1+b_n^2} - \frac{1}{2}\right) + \left(\frac{3n}{n+4} - 3\right)\right| \le \left|\frac{b_n}{1+b_n^2} - \frac{1}{2}\right| + \left|\frac{3n}{n+4} - 3\right|$$

$$= \frac{|-b_n^2+2b_n-1|}{2(1+b_n^2)} + \frac{12}{n+4} \le \frac{|b_n-1|^2}{2} + \frac{12}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$ if $|b_n - 1| < \sqrt{\varepsilon}$ and $n > \frac{24}{\varepsilon}$

Solution $\forall \varepsilon > 0$, Since $b_n \to 1$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |b_n - 1| < \sqrt{\varepsilon}$.
By Archimedean principle, $\exists$ $K \in \mathbb{N}$ such that $K > \max\{K_1, \frac{24}{\varepsilon}\}$. Then
$n \ge K \Rightarrow \begin{matrix} n \ge K_1 \\ n > 24/\varepsilon \end{matrix} \Rightarrow \left|\frac{b_n}{1+b_n^2} + \frac{3n}{n+4} - \frac{7}{2}\right| < \varepsilon$ as in the box above.

Ⓓ Sketch work: $\frac{1}{n+c_n} \to 0$, $\frac{c_n}{c_n+2} \to \frac{1}{2}$

$$\left|\frac{1}{n+c_n} + \frac{c_n}{c_n+2} - \frac{1}{2}\right| = \left|\left(\frac{1}{n+c_n} - 0\right) + \left(\frac{c_n}{c_n+2} - \frac{1}{2}\right)\right| \le \frac{1}{n+c_n} + \left|\frac{c_n}{c_n+2} - \frac{1}{2}\right|$$

$$= \frac{1}{n+c_n} + \frac{|c_n-2|}{2(c_n+2)} < \frac{1}{n} + \frac{|c_n-2|}{4} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$ if $\frac{1}{n} < \frac{\varepsilon}{2}$ ($\Leftrightarrow n > \frac{2}{\varepsilon}$) and $|c_n - 2| < 2\varepsilon$

Solution $\forall \varepsilon > 0$, Since $c_n \to 2$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |c_n - 2| < 2\varepsilon$.
By Archimedean Principle, $\exists$ $K \in \mathbb{N}$ such that $K > \max\{K_1, \frac{2}{\varepsilon}\}$. Then
$n \ge K \Rightarrow \begin{matrix} n \ge K_1 \\ n > 2/\varepsilon \end{matrix} \Rightarrow \begin{matrix} |c_n - 2| < \varepsilon \\ n > 2/\varepsilon \end{matrix} \Rightarrow \left|\frac{1}{n+c_n} + \frac{c_n}{c_n+2} - \frac{1}{2}\right| < \varepsilon$ as in the box above.