(88) (a) S is a <u>countably infinite</u> set iff there exists a bijection $f: \mathbb{N} \to S$.

(b) S is a <u>countable</u> set iff S is a finite set or a countably infinite set.

(c) A series $\sum_{k=1}^{\infty} a_k$ <u>converges</u> to a number S iff $\lim_{n\to\infty}(a_1+a_2+\cdots+a_n) = S$.

(d) A nonempty subset S of $\mathbb{R}$ is <u>bounded above</u> iff there exists some $M \in \mathbb{R}$ such that $x \le M$ for all $x \in S$.

(e) $\tilde{M}$ is the <u>supremum</u> of a subset S of $\mathbb{R}$ that is bounded above iff $\tilde{M}$ is an upper bound of S and $\tilde{M} \le M$ for all upper bounds M of S.

(f) A sequence $\{x_n\}$ <u>converges</u> to a number x iff for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $n \ge K$ implies $|x_n - x| < \varepsilon$.

(g) A sequence $\{x_n\}$ is a <u>Cauchy</u> sequence iff for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $m, n \ge K$ implies $|x_m - x_n| < \varepsilon$.

(h) x is an <u>accumulation point</u> of a set S iff there exists a sequence $\{x_n\}$ in S such that $x_n \ne x$ for all n and $\lim_{n\to\infty} x_n = x$.

(i) $f: S \to \mathbb{R}$ has a <u>limit</u> L at x_0 iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $x \in S$ and $0 < |x - x_0| < \delta$ imply $|f(x) - L| < \varepsilon$.

(j) $f: S \to \mathbb{R}$ is <u>continuous</u> at $x_0 \in S$ iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $x \in S$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$.

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(89) (a) For a fixed $m \in \mathbb{Z}$, the curves $y = \pi x$ and $y = x^3 + x + m$ intersect in at most 3 points (because $\pi x = x^3 + x + m \Rightarrow x^3 + (1-\pi)x + m = 0$.) Now $S = \underbrace{\bigcup_{m\in\mathbb{Z}}}_{\text{countable}} \underbrace{\{(x,y): y = \pi x, y = x^3 + x + m\}}_{\text{at most 3 points hence countable}}$ is countable by the countable union theorem.

(b) For a fixed $m \in \mathbb{Z}$, the curves $y = x^3 + x + 1$ and $y = mx$ intersects in at most 3 points (because $mx = x^3 + x + 1 \Rightarrow x^3 + (1-m)x + 1 = 0$). Now $S = \underbrace{\bigcup_{m\in\mathbb{Z}}}_{\text{countable}} \underbrace{\{(x,y): y = x^3 + x + 1, y = mx\}}_{\text{at most 3 points hence countable}}$ is countable by the countable union theorem.

(c) For a fixed $m \in \mathbb{N}$, the curves $x^2 + y^2 = 1$ and $xy = \frac{1}{m}$ intersect in at most 4 points (because $x^2 + (\frac{1}{mx})^2 = 1 \Rightarrow x^4 - x^2 + \frac{1}{m^2} = 0$.) Now $S = \underbrace{\bigcup_{m\in\mathbb{N}}}_{\text{countable}} \underbrace{\{(x,y): x^2 + y^2 = 1, xy = \frac{1}{m}\}}_{\text{at most 4 points hence countable}}$ is countable by the countable union theorem.

(d) Taking $b = 0$, we see that $S \supseteq M$. Since M is uncountable, so S is uncountable by the countable subset theorem.

(e) Note if $x = |a|$, then $a = x$ or $-x$. So

$$S = \{a + b : |a| \in M, b \in \mathbb{Q}\} = \{x + b : x \in M, b \in \mathbb{Q}\} \cup \{-x + b : x \in M, b \in \mathbb{Q}\}$$

$$= \underbrace{\bigcup_{(x,b)\in M\times\mathbb{Q}}}_{\text{countable}} \underbrace{\{x + b, -x + b\}}_{\text{2 elements, countable}}$$

is countable by the countable union theorem.

(89)(f) The set $S_0=\{a+b\sqrt{2}: a,b\in\mathbb{Q}\}=\bigcup_{\underbrace{(a,b)\in\mathbb{Q}\times\mathbb{Q}}_{\text{Countable}}}\underbrace{\{a+b\sqrt{2}\}}_{\substack{1\text{ element}\\ \text{finite, countable}}}$ is countable.

The set $\{c+d\sqrt{2}: c,d\in\mathbb{Q},\ c+d\sqrt{2}\neq 0\}=S_0\setminus\{0\}$ is also countable by Countable Subset theorem.

$\therefore S=\mathbb{Q}(\sqrt{2})=\{\frac{x}{y}: x\in S_0,\ y\in S_0\setminus\{0\}\}=\bigcup_{(x,y)\in\underbrace{S_0\times(S_0\setminus\{0\})}_{\text{countable}}}\underbrace{\{\frac{x}{y}\}}_{\substack{1\text{ element}\\ \text{finite, countable}}}$ is countable

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(g) Since $A\cap B\subseteq A$, $\mathbb{Q}\cap A\subseteq\mathbb{Q}$, $B\cap\mathbb{Q}\subseteq\mathbb{Q}$ and A, $\mathbb{Q}$ are countable, so by the countable subset theorem, $A\cap B$, $\mathbb{Q}\cap A$, $B\cap\mathbb{Q}$ are countable. For $x\in A\cap B$, $y\in\mathbb{Q}\cap A$ and $z\in B\cap\mathbb{Q}$, let $S_{x,y,z}=\{x^2+y^2+z^2\}$. Then $S_{x,y,z}$ is a one element set. So $S_{x,y,z}$ is countable.

Finally, $S=\bigcup_{\underbrace{(x,y,z)\in(A\cap B)\times(\mathbb{Q}\cap A)\times(B\cap\mathbb{Q})}_{\text{countable by product theorem}}}\underbrace{S_{x,y,z}}_{\text{Countable}}$ is countable by the countable union theorem

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(h) Let $y_0\in A$ and $T=\{x-y_0: x\in A\}$. Then $T\subseteq S$. Now $f:A\to T$ defined by $f(x)=x-y_0$ is a bijection (with $f^{-1}(t)=t+y_0$). By bijection theorem, A uncountable $\Rightarrow$ T uncountable. Finally, since $T\subseteq S$, S must also be uncountable by contrapositive of countable subset theorem

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(i) Solution 1 For $x\in A$, let $S_x=\{x^2+y^2: y\in A\}=\bigcup_{y\in A}\underbrace{\{x^2+y^2\}}_{\substack{1\text{ element}\\ \text{so countable}}}$, then S_x is countable by countable union theorem. Then $S=\bigcup_{x\in A}S_x$ is countable by countable union theorem.

Solution 2 A countable $\underset{\text{product theorem}}{\Rightarrow}$ $A\times A$ countable $\underset{\text{Countable union theorem}}{\Rightarrow}$ $S=\bigcup_{(x,y)\in A\times A}\{x^2+y^2\}$ is countable.

Solution 3 The function $f: A\times A\to S$ defined by $f(x,y)=x^2+y^2$ is surjective. Since A is countable, $A\times A$ is countable by product theorem. Then S is countable by the surjection theorem.

(j) Since A is countable, $\mathbb{R}\setminus A$ must be uncountable. Taking $y=0$, we have $S\supseteq\mathbb{R}\setminus A$. By the countable subset theorem, S is uncountable.

(k) Since A is countable, $\mathbb{R}-A$ must be uncountable. Let $a\in A$, then S contains the subset $S_a=\{(a,y): y\in\mathbb{R}\setminus A\}$. The function $f:\mathbb{R}\setminus A\to S_a$ defined by $f(y)=(a,y)$ is a bijection (with $f^{-1}(a,y)=y$). Since $\mathbb{R}\setminus A$ is uncountable, so S_a is uncountable. Then S is uncountable by the countable subset theorem.

(l) $S=\bigcup_{x\in\mathbb{Z}}S_x$, where $S_x=\{x+y\sqrt{2}: y\in A\}$. The function $f: A\to S_x$ defined by $f(y)=x+y\sqrt{2}$ is a bijection (with $f^{-1}(x+y\sqrt{2})=y$). Since A is countable, each S_x is countable, then $S=\bigcup_{x\in\mathbb{Z}}S_x$ is countable by the countable union theorem.

89 (m) Since $f: \mathbb{Q} \to T$ defined by $f(r) = r\pi$ is a bijection with $f^{-1}(t) = \frac{t}{\pi}$, so T is countable. The set $\mathcal{U} = \{a + b\sqrt{2} - c\sqrt{3} : a, b, c \in T\} = \bigcup_{\underbrace{(a,b,c) \in T \times T \times T}_{\text{Countable by product theorem}}} \underbrace{\{a + b\sqrt{2} - c\sqrt{3}\}}_{\text{1 element set} \Rightarrow \text{Countable}}$ is Countable by the countable union theorem.
Then $S = \mathbb{R} \setminus \mathcal{U}$ is uncountable.

(n) $\{\sqrt{m} + \sqrt{n} : m, n \in \mathbb{N}\} = \bigcup_{\underbrace{(m,n) \in \mathbb{N} \times \mathbb{N}}_{\text{Countable by product theorem}}} \underbrace{\{\sqrt{m} + \sqrt{n}\}}_{\text{one element} \Rightarrow \text{Countable}}$ is countable.

Since $\mathbb{R} \setminus (T \cap \mathcal{U}) = (\mathbb{R} \setminus T) \cup (\mathbb{R} \setminus \mathcal{U}) = \mathbb{Q} \cup \{\sqrt{m} + \sqrt{n} : m, n \in \mathbb{N}\}$ is countable, so $S = T \cap \mathcal{U} = \mathbb{R} \setminus (\mathbb{R} \setminus (T \cap \mathcal{U}))$ is uncountable.

(o) Consider the subset of S of squares having the unit circle at the origin as circumcircle. This subset is uncountable because for every $\alpha \in [0, \frac{\pi}{2})$, there is a unique square having $(\cos\alpha, \sin\alpha)$ as a vertex and $[0, \frac{\pi}{2})$ is uncountable. So S is uncountable.

(p). $G = \bigcup_{\underbrace{(a,b) \in \mathbb{Z} \times \mathbb{Z}}_{\text{Countable}}} \underbrace{\{a + bi\}}_{\text{1 element} \Rightarrow \text{Countable}}$ is countable by countable union theorem. Let S_n be the degree n polynomials. So $S_n = \bigcup_{\underbrace{(a_0, a_1, \dots, a_n) \in G \times G \times \cdots \times G \setminus \{0\}}_{\text{Countable}}} \{a_0 + a_1 x + \cdots + a_n x^n\}$ (1 element $\Rightarrow$ countable) is countable. Then $S = \bigcup_{n \in \mathbb{N}} S_n$ is countable by countable union theorem.

90 (a) Alternating Series Test $\sum_{k=1}^{\infty} \frac{\cos k\pi}{k^2 + 2^k} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2 + 2^k}$. As $k \uparrow \infty$, $k^2 + 2^k \uparrow \infty$ and $\frac{1}{k^2 + 2^k} \downarrow 0$. So $\sum_{k=1}^{\infty} \frac{\cos k\pi}{k^2 + 2^k}$ Converges.

Comparison Test Since $\frac{e^{\sqrt{k}}}{\sqrt{k}} > \frac{1}{\sqrt{k}} = \frac{1}{k^{1/2}}$ and $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ diverges by p-test, so $\sum_{k=1}^{\infty} \frac{e^{\sqrt{k}}}{\sqrt{k}}$ diverges.

(b) Ratio test $\lim_{k\to\infty} \frac{(2(k+1))!}{3^{k+1}(k+1)^4} \frac{3^k k^4}{(2k)!} = \lim_{k\to\infty} \frac{(2k+2)(2k+1)}{3} \left(\frac{k}{k+1}\right)^4 = \infty \Rightarrow \sum_{k=1}^{\infty} \frac{(2k)!}{3^k k^4}$ diverges.

Absolute Convergence Test and Comparison Test $\left|\frac{(\cos k)(\sin 2k)}{2^k}\right| \le \left(\frac{1}{2}\right)^k$ and $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ Converges
$\Rightarrow \sum_{k=1}^{\infty} \left|\frac{(\cos k)(\cos 2k)}{2^k}\right|$ Converges $\Rightarrow \sum_{k=1}^{\infty} \frac{(\cos k)(\sin 2k)}{2^k}$ Converges.

(c) Term test $\lim_{k\to\infty} \frac{1}{2}\left(\cos\frac{1}{k} + \sin\frac{1}{k}\right) = \frac{1}{2}(1 + 0) = \frac{1}{2} \ne 0 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{2}\left(\cos\frac{1}{k} + \sin\frac{1}{k}\right)$ diverges.

Limit Comparison Test $\lim_{k\to\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 0 \Rightarrow \lim_{k\to\infty} \frac{\sin\left(\frac{1}{k} - \frac{1}{k+1}\right)}{\frac{1}{k} - \frac{1}{k+1}} \overset{\theta = \frac{1}{k} - \frac{1}{k+1}}{=} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.
Since $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \lim_{k\to\infty} \frac{1}{k+1} = 1$ (by telescoping series test), $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k} - \frac{1}{k+1}\right)$ converges

(90) (d) Since $0 \le \frac{2^k+3^k}{1^k+4^k} \le \frac{3^k+3^k}{4^k} = 2\left(\frac{3}{4}\right)^k$ and $\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^k$ converges by geometric series test, so $\sum_{k=1}^{\infty} \frac{2^k+3^k}{1^k+4^k}$ converges by comparison test.

Since $\lim_{k\to\infty} \cos\left(\sin\frac{1}{k}\right) = \cos(\sin 0) = \cos 0 = 1 \ne 0$, $\sum_{k=1}^{\infty} \cos\left(\sin\frac{1}{k}\right)$ diverges by term test.

(e) $\lim_{k\to\infty} \frac{2^{1/k}+3^{1/k}}{1^{1/k}+4^{1/k}} = \frac{2^0+3^0}{1^0+4^0} = 1 \ne 0 \Rightarrow \sum_{k=1}^{\infty} \frac{2^{1/k}+3^{1/k}}{1^{1/k}+4^{1/k}}$ diverges by term test.

Since $\cos k\pi = (-1)^k$ and $k \nearrow \infty \Rightarrow \frac{1}{k\pi} \searrow 0 \Rightarrow \sin\frac{1}{k\pi} \searrow 0$, by the alternating series test, $\sum_{k=1}^{\infty} (\cos k\pi)(\sin\frac{1}{k\pi})$ converges

sin is increasing on $[0, \frac{1}{\pi}]$.

(f) Since $\lim_{k\to\infty} \frac{(k+1)!^2}{((k+1)^2)!} \Big/ \frac{(k!)^2}{(k^2)!} = \lim_{k\to\infty} \frac{(k+1)!^2 (k^2)!}{(k!)^2((k+1)^2)!} = \lim_{k\to\infty} \frac{1}{(k^2+1)(k^2+2)\cdots(k^2+2k)} = 0 < 1$,

by ratio test, $\sum_{k=1}^{\infty} \frac{(k!)^2}{(k^2)!}$ converges.

Note $0 < (\cos\frac{1}{k})(\sin\frac{1}{k})(\tan\frac{1}{k}) \le (\sin\frac{1}{k})(\tan\frac{1}{k})$. Since $\lim_{k\to\infty} \frac{(\sin\frac{1}{k})(\tan\frac{1}{k})}{\frac{1}{k^2}} \underset{\theta=\frac{1}{k}}{=} \lim_{\theta\to 0} \frac{\sin\theta}{\theta}\frac{\tan\theta}{\theta} = 1$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p-test, so $\sum_{k=1}^{\infty} (\sin\frac{1}{k})(\tan\frac{1}{k})$ converges by limit comparison test. Therefore $\sum_{k=1}^{\infty} (\cos\frac{1}{k})(\sin\frac{1}{k})(\tan\frac{1}{k})$ converges by comparison test.

(g) $\left|\frac{2^k\cos k}{(k-1)!}\right| \le \frac{2^k}{(k-1)!}$. Now $\lim_{k\to\infty} \frac{2^{k+1}}{k!} \Big/ \frac{2^k}{(k-1)!} = \lim_{k\to\infty} \frac{2}{k} = 0 < 1$. So by the ratio test, $\sum_{k=2}^{\infty} \frac{2^k}{(k-1)!}$ converge. By the comparison test, $\sum_{k=2}^{\infty} \left|\frac{2^k\cos k}{(k-1)!}\right|$ converges. By the absolute convergence test, $\sum_{k=2}^{\infty} \frac{2^k\cos k}{(k-1)!}$ converges.

$\lim_{k\to\infty} \frac{\sin(\frac{1}{k})}{\ln k} \Big/ \frac{\frac{1}{k}}{\ln k} = \lim_{k\to\infty} \frac{\sin(\frac{1}{k})}{1/k} \underset{\theta=\frac{1}{k}}{=} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. Now as $k\to\infty$, $k\ln k$ increases to ∞, $\frac{1}{k\ln k}$ decreases to 0. Since $\int_2^{\infty} \frac{1}{x\ln x}dx = \ln(\ln x)\Big|_2^{\infty} = \infty$, by the integral test, $\sum_{k=2}^{\infty} \frac{1}{k\ln k}$ diverges. So by the limit comparison test, $\sum_{k=2}^{\infty} \frac{\sin(\frac{1}{k})}{\ln k}$ diverges.

(h) (When k is large, $\frac{k^\pi+\cos k\pi}{3+k^4} \sim \frac{k^\pi}{k^4}$.) $\lim_{k\to\infty} \frac{\frac{k^\pi+\cos k\pi}{3+k^4}}{\frac{k^\pi}{k^4}} = \lim_{k\to\infty} \frac{1+\frac{\cos k\pi}{k^\pi}}{\frac{3}{k^4}+1} = 1$. Since $\sum_{k=1}^{\infty} \frac{k^\pi}{k^4} = \sum_{k=1}^{\infty} \frac{1}{k^{4-\pi}}$ diverges by p-test (because $4-\pi \le 1$), so $\sum_{k=1}^{\infty} \frac{k^\pi+\cos k\pi}{3+k^4}$ diverges.

Next $\sum_{k=1}^{\infty} \frac{k^\pi\cos k\pi}{3k^4} = \sum_{k=1}^{\infty} \frac{(-1)^k}{3k^{4-\pi}}$ and $a_k = \frac{1}{3k^{4-\pi}}$ decreases to 0, so by the alternating series test, $\sum_{k=1}^{\infty} \frac{k^\pi\cos k\pi}{3k^4}$ converges.

(90)(i) $\lim_{k\to\infty} \frac{(2k+2)!}{(k+2)!\,k!} \frac{(k+1)!\,(k-1)!}{(2k)!} = \lim_{k\to\infty} \frac{(2k+2)(2k+1)}{(k+2)(k)} = 4 > 1 \overset{\text{ratio test}}{\Longrightarrow} \sum_{k=2}^{\infty} \frac{(2k)!}{(k+1)!(k-1)!}$ diverges

$\lim_{k\to\infty} k \cos(\frac{1}{k^2}) = \infty \cdot \cos 0 = \infty \cdot 1 = \infty \neq 0 \overset{\text{term test}}{\Longrightarrow} \sum_{k=1}^{\infty} k \cos(\frac{1}{k^2})$ diverges.

(j) $\lim_{k\to\infty} \frac{(3(k+1))!}{(k+1)!\,(2(k+1))!} \Big/ \frac{3k!}{k!(2k)!} = \lim_{k\to\infty} \frac{(3k+3)(3k+2)(3k+1)}{(k+1)(2k+2)(2k+1)} = \frac{27}{4} > 1 \Rightarrow \sum_{k=1}^{\infty} \frac{(3k)!}{k!(2k)!}$ diverges by ratio test.

$0 \le \frac{\cos(1/k)}{k^2-1} \le \frac{1}{k^2-1} < \frac{2}{k^2}$ for $k \ge 2$. Since $\sum_{k=2}^{\infty} \frac{2}{k^2} = 2\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges by p-test,

so $\sum_{k=2}^{\infty} \frac{\cos(1/k)}{k^2-1}$ converges by comparison test.

(k) Ratio Test $\lim_{k\to\infty} \frac{(k+1)!}{(2(k+1)-1)!} \Big/ \frac{k!}{(2k-1)!} = \lim_{k\to\infty} \frac{k+1}{(2k+1)2k} = 0 < 1 \Rightarrow \sum_{k=1}^{\infty} \frac{k!}{(2k-1)!}$ converges.

Alternating Series Test $\sum_{k=1}^{\infty} \frac{\cos k\pi}{\sqrt{k}+1} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k}+1}$, the sequence $\{\frac{1}{\sqrt{k}+1}\}$ decrease to 0 because $k > k' \Rightarrow \sqrt{k} > \sqrt{k'} \Rightarrow \frac{1}{\sqrt{k}+1} < \frac{1}{\sqrt{k'}+1}$ and $\lim_{k\to\infty} \frac{1}{\sqrt{k}+1} = 0$.

So $\sum_{k=1}^{\infty} \frac{\cos k\pi}{\sqrt{k}+1}$ converges.

(l) Ratio Test $\lim_{k\to\infty} \frac{2^{k+1}(k+1)^2}{(k+1)!} \Big/ \frac{2^k k^2}{k!} = \lim_{k\to\infty} \frac{2(k+1)}{k^2} = 0 < 1 \Rightarrow \sum_{k=1}^{\infty} \frac{2^k k^2}{k!}$ converges.

Limit Comparison Test $\lim_{k\to\infty} \frac{\frac{1}{\sqrt{k}} \sin(\frac{1}{\sqrt{k}})}{\frac{1}{\sqrt{k}}\frac{1}{\sqrt{k}}} = \lim_{k\to\infty} \frac{\sin\frac{1}{\sqrt{k}}}{\frac{1}{\sqrt{k}}} = \lim_{\substack{\theta\to 0 \\ \theta = \frac{1}{\sqrt{k}}}} \frac{\sin\theta}{\theta} = 1$. Since $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges by p-test, so $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \sin(\frac{1}{\sqrt{k}})$ diverges.

(m) Since $\frac{1}{k}$ decreases to 0 as $k \nearrow \infty$, by alternating series test, $\sum_{k=1}^{\infty} \frac{1}{k \cos k\pi} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges.

Since $0 \le \frac{k^2 \sin(1/k)}{(2k+1)!} \le \frac{k^2}{(2k+1)!}$ and $\lim_{k\to\infty} \frac{(k+1)^2}{(2k+3)!} \Big/ \frac{k^2}{(2k+1)!} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^2 \frac{1}{(2k+3)(2k+2)} = 0$, by ratio test, $\sum_{k=1}^{\infty} \frac{k^2}{(2k+1)!}$ converges. By comparison test, $\sum_{k=1}^{\infty} \frac{k^2 \sin(1/k)}{(2k+1)!}$ converges.

(n) By root test, $\lim_{k\to\infty} \cos(1+\frac{1}{k}) = \cos 1 < 1 \Rightarrow \sum_{k=1}^{\infty} \cos^k(1+\frac{1}{k})$ converges.

By term test, $\lim_{k\to\infty} \frac{\cos(\sin(\frac{1}{k}))}{\sin(\cos(\frac{1}{k}))} = \frac{1}{\sin 1} \neq 0 \Rightarrow \sum_{k=1}^{\infty} \frac{\cos(\sin(1/k))}{\sin(\cos(1/k))}$ diverges.

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(91)(a) For $m, n \in \mathbb{N}$, $0 < \frac{1}{m} + \frac{1}{n}$ and $\frac{1}{1} + \frac{1}{1} = \frac{2}{1} \notin S$. So $S \subseteq (0, 1+\frac{1}{2}]$. Then S has lower bound 0 and upper bound $\frac{3}{2}$. Let $x_k = \frac{1}{k} + \frac{1}{k+1}$, then $x_k \in S$. (Note $\frac{2}{k+1} < x_k < \frac{2}{k}$.) Since $\lim_{k\to\infty} x_k = 0 + 0 = 0$, by the infimum limit theorem, $\inf S = 0$. Next, every upper bound $M \ge \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \in S$. So $\sup S = \frac{3}{2}$.

(91) (b) For $x,y\in[\frac{1}{2},1)$, $1=\frac{1}{2}+\frac{1}{2}\le x+y<1+1=2$. So $S\subseteq[1,2)$. Then S has lower bound 1 and upper bound 2. Take $x=y=\frac{1}{2}+\frac{1}{2\sqrt{2}k}\in[\frac{1}{2},1)$. Then $x_k=x+y\in S$. (Note x_k is irrational, so $x_k\neq 2-\frac{1}{n}$ for all $n\in\mathbb{N}$.) Since $\lim_{k\to\infty}x_k=\frac{1}{2}+\frac{1}{2}=1$, by the infimum limit theorem, $\inf S=1$. Next, take $x=y=1-\frac{1}{\sqrt{2}k}$. Then $w_k=x+y\in S$ and $\lim_{k\to\infty}w_k=1+1=2$. By the supremum limit theorem, $\sup S=2$.

(c) For $x\in[0,1]\cap\mathbb{Q}$, $n\in\mathbb{N}$, $-1=0-\frac{1}{1}\le x-\frac{1}{n}<1-0=1$. So $S\subseteq[\frac{1}{2},1)$. Then $\frac{1}{2}$ is a lower bound of S and 1 is an upper bound of S. Now every lower bound $m\le\frac{1}{2}=1-\frac{1}{2}\in S$, so $\inf S=\frac{1}{2}$. Also let $x_n=1-\frac{1}{n+1}\in S$, then $\lim_{n\to\infty}x_n=1$. By supremum limit theorem, $\sup S=1$.

(d) (When $x\to\pi$, $\frac{x-\pi}{x+\pi}\to 0$ and when $x\to\infty$, $\frac{x-\pi}{x+\pi}\to 1$.) We will show that $\inf S=0$ and $\sup S=1$. For $x\in(\mathbb{R}\setminus\mathbb{Q})\cap[\pi,\infty)$, $0\le\frac{x-\pi}{x+\pi}\Leftrightarrow\pi\le x$, which is true. So 0 is a lower bound of S. Also $0=\frac{\pi-\pi}{\pi+\pi}\in S$. So every lower bound $t\le\frac{\pi-\pi}{\pi+\pi}=0$. $\therefore\ \inf S=0$.

For $x\in(\mathbb{R}\setminus\mathbb{Q})\cap[\pi,\infty)$, $\frac{x-\pi}{x+\pi}\le 1\Leftrightarrow x-\pi\le x+\pi$, which is true. So 1 is an upper bound of S. Now $w_n=\frac{n\pi-\pi}{n\pi+\pi}\in S$ for every $n\in\mathbb{N}$. Since $\lim_{n\to\infty}w_n=1$, so by the supremum limit theorem, $\sup S=1$.

(e) (When $x\to 0$, $\frac{x-\pi}{x+\pi}\to -1$ and when $x\to\infty$, $\frac{x-\pi}{x+\pi}\to 1$.) We will show that $\inf S=-1$ and $\sup S=1$. For $x\in\mathbb{Q}\cap[0,\infty)$, $-1\le\frac{x-\pi}{x+\pi}\Leftrightarrow -x-\pi\le x-\pi$ $\Leftrightarrow 0\le x$, which is true. So -1 is a lower bound of S. Also $-1=\frac{0-\pi}{0+\pi}\in S$. So every lower bound $t\le\frac{0-\pi}{0+\pi}=-1$. $\therefore\ \inf S=-1$.

For $x\in\mathbb{Q}\cap[0,\infty)$, $\frac{x-\pi}{x+\pi}\le 1\Leftrightarrow x-\pi\le x+\pi$, which is true. So 1 is an upper bound of S. Now $w_n=\frac{n-\pi}{n+\pi}\in S$ for every $n\in\mathbb{N}$. Since $\lim_{n\to\infty}w_n=1$, so by the supremum limit theorem, $\sup S=1$.

(f) For $x\in\mathbb{Q}\cap[0,1]$, $y\in(-1,0]$, $-1=0^3+(-1)^3<x^3+y^3\le 1^3+0^3=1$. So -1 is a lower bound of S and 1 is an upper bound of S. Note $1=1^3+0^3\in S$. So for every upper bound M of S, $M\ge 1$. Therefore, $\sup S=1$.

Next for every $n\in\mathbb{N}$, $w_n=0^3+(-\frac{n}{n+1})^3\in S$ and $\lim_{n\to\infty}w_n=-1$. So by the infimum limit theorem, $\inf S=-1$.

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(9) (g) Since $0 < \frac{\sqrt{2}}{m+n} + \frac{1}{k\sqrt{2}} \le \frac{\sqrt{2}}{1+1} + \frac{1}{1\sqrt{2}} = \frac{\sqrt{2}}{2} + \frac{1}{\sqrt{2}} = \sqrt{2}$, S is bounded below by 0 and above by $\sqrt{2}$. Now every upper bound $M \ge \sqrt{2} \in S$, so $\sup S = \sqrt{2}$. Next considering $a_n = \frac{\sqrt{2}}{n+n} + \frac{1}{n\sqrt{2}} \in S$, we have $\lim_{n\to\infty} a_n = 0$, which is a lower bound. So by the infimum limit theorem, $\inf S = 0$.

(h) $S = [0, \frac{1}{2}) \cup [\frac{2}{3}, \frac{3}{4}) \cup [\frac{4}{5}, \frac{5}{6}) \cup \cdots$. Since $0 \le 1 - \frac{1}{2k-1}$ and $1 - \frac{1}{2k} < 1$ for $k = 1, 2, 3, \cdots$, so $0 \le x < 1$ for all $x \in S$. So S is bounded below by 0 and above by 1. Since every lower bound $m \le 0 \in S$, so $\inf S = 0$. Next since $1 - \frac{1}{2k-1} \in S$ and $\lim_{k\to\infty}(1 - \frac{1}{2k-1}) = 1$, so by the supremum limit theorem, $\sup S = 1$.

(i) For $x, y \in (0,1] \cap \mathbb{Q}$, $0 \le \sqrt{x} + y^2 \le \sqrt{1} + 1^2 = 2$. So 0 is a lower bound and 2 is an upper bound. Now let $w_n = \sqrt{\frac{1}{n}} + (\frac{1}{n})^2$ for $n = 1, 2, 3, \cdots$, then $w_n \in S$ and $\lim_{n\to\infty} w_n = 0$. So by infimum limit theorem, $\inf S = 0$. Next, $2 = \sqrt{1} + 1^2 \in S$ and so every upper bound of S is greater than or equal to 2. Therefore, $\sup S = 2$.

(j) Since $0 \le \frac{1}{n} + x \le 2$ for $x \in [0,1] \cap \mathbb{Q}$, $n = 1, 2, 3, \cdots$, the set S is bounded below by 0 and bounded above by 2. We will show $\inf S = 0$ and $\sup S = 2$.
Since 0 is a lower bound, $0 \le \inf S$. For $n = 1, 2, 3, \cdots$, $\frac{1}{n} = \frac{1}{n} + 0 \in S$ and so $\inf S \le \frac{1}{n}$. Then $\inf S \le \lim_{n\to\infty} \frac{1}{n} = 0$. So $\inf S = 0$.
Since 2 is an upper bound, $\sup S \le 2$. If m is an upper bound of S, then $m \ge \frac{1}{1} + 1 = 2 \in S$. So $\sup S = 2$.

(k) Since $0 \le x + y \le 2$ for $x \in [0,1] \cap \mathbb{Q}$, $y \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$, S is bounded below by 0 and bounded above by 2. We will show $\inf S = 0$ and $\sup S = 2$.
Let $w_n = \frac{1}{n} + \frac{1}{n\sqrt{2}}$, then $w_n \in S$ and $\lim_{n\to\infty} w_n = 0$. So by infimum limit theorem, $\inf S = 0$.
Let $v_n = \frac{n}{n+1} + \frac{1}{n+1\sqrt{2}}$, then $v_n \in S$ and $\lim v_n = 2$. So by supremum limit theorem, $\sup S = 2$.

(l) Note $x(x+1) \le 0 \Leftrightarrow x \in [-1, 0]$. So $S = [-1, 0] \cap (\mathbb{R} \setminus \mathbb{Q})$. Hence S is bounded below by -1 and above by 0. We will show $\inf S = -1$ and $\sup S = 0$.
Let $w_n = -\frac{1}{n+1\sqrt{2}}$, then $w_n \in S$ and $\lim_{n\to\infty} w_n = -1$. So by infimum limit theorem, $\inf S = -1$.
Let $v_n = -\frac{1}{n\sqrt{2}}$, then $v_n \in S$ and $\lim_{n\to\infty} v_n = 0$. So by supremum limit theorem, $\sup S = 0$.

(m) $\forall \frac{k}{n!} \in S$, $0 < \frac{k}{n!} < \sqrt{2}$. So S has lower bound 0 and upper bound $\sqrt{2}$. Will show $\inf S = 0$, $\sup S = \sqrt{2}$.
If $\sup S < \sqrt{2}$, then by density of rational, there is $\frac{m}{n} \in \mathbb{Q}$ such that $\sup S < \frac{m}{n} < \sqrt{2}$. However, $\frac{m}{n} = \frac{m(n-1)!}{n!} \in S$, contradicting $\sup S$ is an upper bound of S. $\therefore \sup S = \sqrt{2}$.
If $\inf S > 0$, then by density of rational, there is $\frac{p}{q} \in \mathbb{Q}$ such that $0 < \frac{p}{q} < \inf S$. However, $\frac{p}{q} = \frac{p(q-1)!}{q!} \in S$, contradicting $\inf S$ is a lower bound of S. $\therefore \inf S = 0$.

91 (n) Note $S=\bigcup_{n=1}^{10}([\frac{1}{n\sqrt{2}}, 2-\frac{1}{n}]\setminus\mathbb{Q}) = [\frac{1}{10\sqrt{2}}, 1.9]\setminus\mathbb{Q}$. So S is bounded below by $\frac{1}{10\sqrt{2}}$ and above by 1.9. We will show $\inf S = \frac{1}{10\sqrt{2}}$ and $\sup S = 1.9$.

Since $\frac{1}{10\sqrt{2}} \in S$, every lower bound $m \le \frac{1}{10\sqrt{2}}$, so $\inf S = \frac{1}{10\sqrt{2}}$.

Next, let $w_n = 1.9 - \frac{1}{n\sqrt{2}}$, then $\frac{1}{10\sqrt{2}} < 1 < 1.9 - \frac{1}{\sqrt{2}} \le w_n < 1.9$. So $w_n \in S$. Since $\lim_{n\to\infty} w_n = 1.9$, by the supremum limit theorem, $\sup S = 1.9$.

(o) $0 \le x^2+y^3+z^4 \le 1+1+1=3$ for $x\in(-1,0)\setminus\mathbb{Q}$, $y\in(0,1)\cap\mathbb{Q}$, $z\in(-1,1)$. So 0 is a lower bound and 3 is an upper bound of S. Since $(-\frac{1}{n\sqrt{2}})^2+(\frac{1}{n+1})^3+(\frac{1}{n+1})^4$ is in S and has limit 0, so $\inf S=0$. Since $(-1+\frac{1}{n\sqrt{2}})^2+(1-\frac{1}{n+1})^3+((1-\frac{1}{n+1})^4$ is in S and has limit 3, so $\sup S=3$.

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92 (a) (Note $x_1=1<x_2=\frac{1}{2}+\sqrt{1}=\frac{3}{2}<x_3=\frac{3}{4}+\sqrt{\frac{3}{2}}=\frac{3+2\sqrt{6}}{4}$. Also $x=\frac{x}{2}+\sqrt{x} \Rightarrow x=0$ or 4.)

We will show $x_n \le x_{n+1} \le 4$ by induction. For $n=1$, $1\le\frac{3}{2}\le 4$. Next suppose $x_n\le x_{n+1}\le 4$. Then $\frac{x_n}{2}\le\frac{x_{n+1}}{2}\le 2$ and $\sqrt{x_n}\le\sqrt{x_{n+1}}\le\sqrt{4} \Rightarrow x_{n+1}=\frac{x_n}{2}+\sqrt{x_n}\le x_{n+2}=\frac{x_{n+1}}{2}+\sqrt{x_{n+1}} \le 2+\sqrt{4}=4$. Therefore, $\{x_n\}$ is increasing and bounded above. By the monotone sequence theorem, $\lim_{n\to\infty}x_n = x$ exists. Then $x=\frac{x}{2}+\sqrt{x} \Rightarrow x=0$ or 4. Since $x_1>1$, $\lim_{n\to\infty}x_n = x = 4$.

(b) (Note $x_1=1<x_2=2<x_3=\sqrt{2}+\sqrt{1}=\sqrt{2}+1$, so we suspect $\{x_n\}$ is increasing.)

We will show $x_n\le x_{n+1}$ for all $n\in\mathbb{N}$ by induction. The cases $n=1,2$ are true as shown above. Assume the cases $n<k$ are true. For the case $n=k$, we have $x_k\le x_{k+1} \Leftrightarrow \sqrt{x_{k-1}}+\sqrt{x_{k-2}}\le\sqrt{x_k}+\sqrt{x_{k-1}} \Leftrightarrow x_{k-2}\le x_k$, which is true by cases $n=k-2$ $(x_{k-2}\le x_{k-1})$ and $n=k-1$ $(x_{k-1}\le x_k)$. So $\{x_n\}$ is increasing.

Next we will show $x_n\le 4$ for all $n\in\mathbb{N}$. For $n=1,2$, this is clear. Assume the cases $n<k$ are true, then $x_k=\sqrt{x_{k-1}}+\sqrt{x_{k-2}}\le\sqrt{4}+\sqrt{4}=4$. So by induction, $x_n\le 4$ for all $n\in\mathbb{N}$. By the monotone sequence theorem, $\{x_n\}$ converges. Let $x=\lim_{n\to\infty}x_n$, then $x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}(\sqrt{x_n}+\sqrt{x_{n-1}})=2\sqrt{x} \Rightarrow x=0$ or 4. Since $1=x_1\le x$, $x=4$.

(c) $x_2=\frac{1}{4}<x_4=\frac{19}{46}<x_3=\frac{7}{13}<x_1=1$. Assume $x_{2n}<x_{2n+2}<x_{2n+1}<x_{2n-1}$. Now $x_{k+1}=\frac{2-x_k}{3+x_k}=\frac{5}{3+x_k}-1$. So $x_{2n+1}=\frac{5}{3+x_{2n}}-1>x_{2n+3}=\frac{5}{3+x_{2n+2}}-1>x_{2n+2}=\frac{5}{3+x_{2n+1}}-1>x_{2n}=\frac{5}{3+x_{2n-1}}-1$. Repeating this once more, we get $x_{2n+2}=\frac{5}{3+x_{2n+1}}-1<x_{2n+4}=\frac{5}{3+x_{2n+3}}-1<x_{2n+3}=\frac{5}{3+x_{2n+2}}-1<x_{2n+1}=\frac{5}{3+x_{2n}}-1$. Therefore, $x_{2k}<x_{2k+2}<x_{2k+1}<x_{2k-1}$ for all k by mathematical induction.

(c) (Cont.) Now $|x_m - x_{m-1}| = \left|\frac{2-x_{m-1}}{3+x_{m-1}} - \frac{2-x_{m-2}}{3+x_{m-2}}\right| = \frac{5|x_{m-1}-x_{m-2}|}{(3+x_{m-1})(3+x_{m-2})} \le \frac{5|x_{m-1}-x_{m-2}|}{(3\frac{1}{4})(3\frac{1}{4})} = \frac{80}{169}|x_{m-1}-x_{m-2}|$
$\le \cdots \le \left(\frac{80}{169}\right)^{m-2}|x_2-x_1|$. Since $\lim\limits_{m\to\infty}\left(\frac{80}{169}\right)^{m-2}|x_2-x_1|=0$, $\lim\limits_{m\to\infty}|x_m-x_{m-1}|=0$ and $\lim\limits_{k\to\infty}|x_{2k}-x_{2k-1}|=0$. By the nested interval theorem and intertwining sequence theorem, $\{x_n\}$ converges to some x. Now $x=\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}\frac{2-x_n}{3+x_n}=\frac{2-x}{3+x}$. Solving, we find $x=-2\pm\sqrt{6}$. Since $x_n \ge x_2=\frac{1}{4}$, $x=-2+\sqrt{6}$.

Alternatively, after we showed $x_{2k}<x_{2k+2}<x_{2k+1}<x_{2k-1}$ for all k, we can argue as follow. Since $\{x_{2n}\}$ is increasing and bounded above by x_1, $\{x_{2n}\}$ must converge to some a by the monotone sequence theorem. Also $\{x_{2n+1}\}$ is decreasing and bounded below by x_2, so $\{x_{2n+1}\}$ must converge to some b by the monotone sequence theorem. Then

$$\left.\begin{aligned} b&=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}\frac{2-x_{2n}}{3+x_{2n}}=\frac{2-a}{3+a}\\ a&=\lim_{n\to\infty}x_{2n}=\lim_{n\to\infty}\frac{2-x_{2n-1}}{3+x_{2n-1}}=\frac{2-b}{3+b}\end{aligned}\right\}\Rightarrow \begin{aligned}3b+ab&=2-a\\ 3a+ab&=2-b\end{aligned}$$

$$\overset{\text{subtracting}}{\Longrightarrow} 3(b-a)=b-a \Rightarrow b=a.$$

By the intertwining sequence theorem, $\{x_n\}$ converges. Then the limit of $\{x_n\}$ is found as above.

(d) $x_3=\frac{1}{2}$ $x_2=\frac{3}{4}$ $x_1=\frac{15}{16}$

$x=1-\sqrt{1-x} \Leftrightarrow \sqrt{1-x}=1-x \Leftrightarrow 1-x=(1-x)^2 \Leftrightarrow (1-x)x=0 \Leftrightarrow x=0$ or 1

We will prove $0<x_{n+1}<x_n$ for $n=1,2,\cdots$ by induction. For $n=1$, $0<x_2=\frac{3}{4}<x_1=\frac{15}{16}$. Assume $0<x_{n+1}<x_n$. Then $1>1-x_{n+1}>1-x_n \Rightarrow 1>\sqrt{1-x_{n+1}}>\sqrt{1-x_n}$
$\Rightarrow 0<1-\sqrt{1-x_{n+1}}=x_{n+2}<1-\sqrt{1-x_n}=x_{n+1}$,
completing the induction. Therefore $\{x_n\}$ is decreasing and bounded below. By the monotone sequence theorem, $\{x_n\}$ converges to some limit x. Then $x=\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}(1-\sqrt{1-x_n})=(1-\sqrt{1-x})$. So $x=0$ or 1. Since $x_n<1$ and $\{x_n\}$ is decreasing, $x=0$.
Now $\lim\limits_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim\limits_{n\to\infty}\frac{1-\sqrt{1-x_n}}{x_n}\frac{(1+\sqrt{1-x_n})}{(1+\sqrt{1-x_n})}=\lim\limits_{n\to\infty}\frac{1-(1-x_n)}{x_n(1+\sqrt{1-x_n})}=\lim\limits_{n\to\infty}\frac{1}{1+\sqrt{1-x_n}}=\frac{1}{2}$.

(e) $x_2=\frac{3}{2}$ $x_4=\frac{8}{5}$ $x_3=\frac{5}{3}$ $x_1=2$

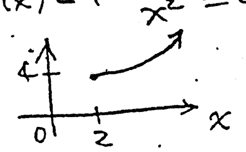
Let $I_n=[x_{2n},x_{2n-1}]$. We will show $x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}$. The case $n=1$ is shown on the left. Suppose the case $n=k$ is true, i.e. $x_{2k}\le x_{2k+2}\le x_{2k+1}\le x_{2k-1}$. Since $x_n=\frac{a_{n+1}}{a_n}=\frac{a_n+a_{n-1}}{a_n}=1+\frac{1}{x_{n-1}}$, so

$\underbrace{1+\frac{1}{x_{2k}}}_{x_{2k+1}}\ge\underbrace{1+\frac{1}{x_{2k+2}}}_{x_{2k+3}}\ge\underbrace{1+\frac{1}{x_{2k+1}}}_{x_{2k+2}}\ge\underbrace{1+\frac{1}{x_{2k-1}}}_{x_{2k}}$ and $\underbrace{1+\frac{1}{x_{2k+1}}}_{x_{2k+2}}\le\underbrace{1+\frac{1}{x_{2k+3}}}_{x_{2k+4}}\le\underbrace{1+\frac{1}{x_{2k+2}}}_{x_{2k+3}}\le\underbrace{1+\frac{1}{x_{2k}}}_{x_{2k+1}} \Rightarrow \begin{cases}n=k+1\\ \text{is true}\end{cases}$

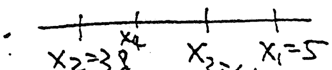
This implies $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$. Next we will show $\lim\limits_{n\to\infty}(x_{2n-1}-x_{2n})=0$. Note that
$|x_m-x_{m+1}|=\left|\left(1+\frac{1}{x_{m-1}}\right)-\left(1+\frac{1}{x_m}\right)\right|=\frac{|x_{m-1}-x_m|}{x_{m-1}x_m}\le\frac{4}{9}|x_{m-1}-x_m| \Rightarrow |x_{2n-1}-x_{2n}|\le\left(\frac{4}{9}\right)^{2n-2}\left(\frac{1}{2}\right)$
Since $\lim\limits_{n\to\infty}\left(\frac{4}{9}\right)^{2n-2}\left(\frac{1}{2}\right)=0$, by the squeeze limit theorem, $\lim\limits_{n\to\infty}(x_{2n-1}-x_{2n})=0$. By the nested interval theorem and the intertwining sequence theorem, $\{x_n\}$ converges, say to x.
Then $x=\lim\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}\left(1+\frac{1}{x_{n-1}}\right)=1+\frac{1}{x} \Rightarrow x=\frac{1\pm\sqrt{5}}{2}$. Since $x_n>0$, so $x=\frac{1+\sqrt{5}}{2}$

§2 (f) ($x=1-\frac{1}{4x} \Rightarrow 4x^2=4x-1 \Rightarrow 0=4x^2-4x+1=(2x-1)^2 \Rightarrow x=\frac{1}{2}$.)

We will show $x_n \geq x_{n+1} \geq \frac{1}{2}$ for $n=1,2,3,\cdots$ by induction. We have $x_1=1 \geq x_2=\frac{3}{4} \geq \frac{1}{2}$.
Assume $x_n \geq x_{n+1} \geq \frac{1}{2}$. Then $\frac{1}{4x_n} \leq \frac{1}{4x_{n+1}} \leq \frac{1}{2}$ and $x_{n+1}=1-\frac{1}{4x_n} \geq x_{n+2}=1-\frac{1}{4x_{n+1}} \geq 1-\frac{1}{4(\frac{1}{2})}=\frac{1}{2}$,
completing the induction. So $\{x_n\}$ is decreasing and bounded below by $\frac{1}{2}$.
By monotone sequence theorem, $\{x_n\}$ converges to some x. Then $x=\lim\limits_{n\to\infty} x_{n+1}$
$=\lim\limits_{n\to\infty}\left(1-\frac{1}{4x_n}\right)=1-\frac{1}{4x}$. So $x=1-\frac{1}{4x}$. As above, $x=\frac{1}{2}$.

(g) Since $f'(x)=1-\frac{4}{x^2} \geq 0$ for $x \geq 2$ and $\lim\limits_{x\to\infty}(x+\frac{4}{x})=\infty$, $f(x)$ is increasing
to ∞. 

$x_1=4$, $x_2=\frac{5}{2}=2.5$, $x_3=\frac{1}{2}(2.5+1.6)=2.05$. We suspect $\{x_n\}$ is decreasing.
(If $\{x_n\}$ converges to x, then $x=\lim\limits_{n\to\infty} x_{n+1}=\lim\limits_{n\to\infty}\frac{1}{2}(x_n+\frac{4}{x_n})=\frac{1}{2}(x+\frac{4}{x})$, which
implies $x=\pm 2$. Since $x_n>0$ by induction, $x=2$.)
We will show $2 \leq x_{n+1} \leq x_n$ for $n=1,2,\cdots$. (This implies $\{x_n\}$ is decreasing
and bounded below by 2. By the monotone sequence theorem, we get $\{x_n\}$ converges.)
For $n=1$, $2 \leq x_2=2.5 \leq x_1=4$. Suppose $2 \leq x_{n+1} \leq x_n$. Then since $f(x)=x+\frac{4}{x}$
is increasing for $x \geq 2$, we get $2=\frac{1}{2}f(2) \leq x_{n+2}=\frac{1}{2}f(x_{n+1}) \leq x_{n+1}=\frac{1}{2}f(x_n)$,
completing the induction.

(h) $x_1=5$, $x_2=3\frac{4}{5}=3.8$, $x_3=4\frac{1}{19}$, $x_4=3+\frac{4}{x_3}>3+\frac{4}{5}=x_2$. 

Define $I_n=[x_{2n}, x_{2n-1}]$, we will show $x_{2n} \leq x_{2n+2} \leq x_{2n+1} \leq x_{2n-1}$, i.e. $I_{n+1} \subseteq I_n$.
The case $n=1$ is done above. Suppose the case n is true. Then $\frac{4}{x_{2n}} \geq \frac{4}{x_{2n+2}} \geq \frac{4}{x_{2n+1}} \geq \frac{4}{x_{2n-1}}$
$\Rightarrow x_{2n+1} \geq x_{2n+3} \geq x_{2n+2} \geq x_{2n} \Rightarrow \frac{4}{x_{2n+1}} \leq \frac{4}{x_{2n+3}} \leq \frac{4}{x_{2n+2}} \leq \frac{4}{x_{2n}} \Rightarrow x_{2n+2} \leq x_{2n+4} \leq x_{2n+3} \leq x_{2n+1}$,
completing the induction.
*Next observe that $|x_{m+1}-x_m|=\left|\frac{4}{x_m}-\frac{4}{x_{m-1}}\right|=\frac{4|x_m-x_{m-1}|}{x_m x_{m-1}} \leq \frac{4}{(3.8)^2}|x_m-x_{m-1}|$. So
$|x_{2n-1}-x_{2n}| \leq \frac{4}{(3.8)^2}|x_{2n-2}-x_{2n-1}| \leq \cdots \leq \left(\frac{4}{(3.8)^2}\right)^{2n-3}|x_2-x_1|$. Since $\frac{4}{(3.8)^2}<1$, $\lim\limits_{n\to\infty}\left(\frac{4}{(3.8)^2}\right)^{2n-3}|x_2-x_1|=0$
and $\lim\limits_{n\to\infty}|x_{2n-1}-x_{2n}|=0$. Hence $\bigcap\limits_{n=1}^{\infty} I_n=\{x\}$ and $\lim\limits_{n\to\infty} x_{2n}=x=\lim\limits_{n\to\infty} x_{2n-1}$. So by the intertwining
sequence theorem, $\{x_n\}$ converges to x.
Taking limit of $x_{n+1}=3+\frac{4}{x_n}$, we get $x=3+\frac{4}{x} \Rightarrow x^2-3x-4=0 \Rightarrow x=-1$ or 4.
Since $x \in I_1=[3.8, 5]$, so $x=4$.

* Alternatively, the 2nd paragraph can be replaced by the following argument. Let $x_{2n}\to a$, $x_{2n-1}\to b$,
then $\left(x_{2n+1}=3+\frac{4}{x_{2n}} \Rightarrow b=3+\frac{4}{a}\right)$ and $\left(x_{2n}=3+\frac{4}{x_{2n-1}} \Rightarrow a=3+\frac{4}{b}\right)$. So $(b-3)a=4=(a-3)b$,
$ba-3a=ab-3b \Rightarrow a=b$. Therefore, $\{x_n\}$ converges to $x=a=b$.

(12)(i) $x_1=2,\ x_2=\frac{3}{2}=1.5,\ x_3=\frac{4}{3}=1.33\cdots$. We suspect $\{x_n\}$ is decreasing.
(If $\{x_n\}$ converges to x, then $x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}(2-\frac{1}{x_n})=2-\frac{1}{x}$, which implies $x=1$ by algebra.)
We will show $1\le x_{n+1}\le x_n$ for $n=1,2,\cdots$; (this implies $\{x_n\}$ is decreasing and bounded below by 1. By the monotone sequence theorem, we get $\{x_n\}$ converges.)
For $n=1$, we have $1\le x_2=1.5\le x_1=2$. Suppose $1\le x_{n+1}\le x_n$, then $\frac{1}{1}\ge\frac{1}{x_{n+1}}\ge\frac{1}{x_n}$ and $1=2-\frac{1}{1}\le x_{n+2}=2-\frac{1}{x_{n+1}}\le x_{n+1}=2-\frac{1}{x_n}$, completing M. I.

(j) $(x_1=0<x_2=\frac{0^2+4}{5}=\frac{4}{5}<x_3=\frac{(\frac{4}{5})^2+4}{5}$ $\quad x=\frac{x^2+4}{5}\iff x^2-5x+4=(x-1)(x-4)=0,)$ $\quad x=1\quad x=4$
We will show $x_n\le x_{n+1}\le 1$ by math induction. For $n=1$, $x_1=0\le x_2=\frac{4}{5}\le 1$.
Suppose $x_n\le x_{n+1}\le 1$. Then $x_n^2+4\le x_{n+1}^2+4\le 1^2+4$. Dividing by 5, we get $x_{n+1}\le x_{n+2}\le 1$ completing the induction. This shows $\{x_n\}$ is increasing and bounded above. By monotone sequence theorem, $\{x_n\}$ converges to some $x\in\mathbb{R}$. Now
$x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\frac{x_n^2+4}{5}=\frac{x^2+4}{5}\Rightarrow x^2-5x+4=0\Rightarrow x=1$ or $\cancel{4}$ $\leftarrow$ rejected as $x_n\le 1$

(k) (Note $x_1=1>x_2=\sqrt{1}-\frac{1}{4}=\frac{3}{4}>x_3=\sqrt{\frac{3}{4}}-\frac{1}{4}=\frac{2\sqrt{3}-1}{4}$ and $x=\sqrt{x}-\frac{1}{4}\Rightarrow x=\frac{1}{4}$.)
We will show $x_n\ge x_{n+1}\ge\frac{1}{4}$ by induction. For $n=1$, $1\ge\frac{3}{4}\ge\frac{1}{4}$. Suppose $x_n\ge x_{n+1}\ge\frac{1}{4}$. Then $x_{n+1}=\sqrt{x_n}-\frac{1}{4}\ge x_{n+2}=\sqrt{x_{n+1}}-\frac{1}{4}\ge\frac{1}{4}=\sqrt{\frac{1}{4}}-\frac{1}{4}$. Therefore, $\{x_n\}$ is decreasing and bounded below. By the monotone sequence theorem, $\lim_{n\to\infty}x_n=x$ exists. Then $x=\sqrt{x}-\frac{1}{4}\Rightarrow x=\frac{1}{4}$. So $\lim_{n\to\infty}x_n=\frac{1}{4}$.

(l) (Note $x_1=3>x_2=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}>x_3=\sqrt{1-\frac{2}{2+\sqrt{3}}}=\sqrt{\frac{3}{2+\sqrt{3}}}$ and $x=\sqrt{1-\frac{1}{x+1}}=\sqrt{\frac{x}{x+1}}\Rightarrow$ $x=0$ or $x(x+1)-1=0$ $\Rightarrow x=0$ or $\frac{-1\pm\sqrt{5}}{2}$.) We will show $x_n\ge x_{n+1}\ge\frac{-1+\sqrt{5}}{2}$. For $n=1$, $3>\frac{\sqrt{3}}{2}\approx\frac{1.73}{2}>\frac{-1+\sqrt{5}}{2}\approx\frac{1.2}{2}$.
Suppose $x_n\ge x_{n+1}\ge\frac{-1+\sqrt{5}}{2}$. Then $x_n+1\ge x_{n+1}+1\ge\frac{1+\sqrt{5}}{2}\Rightarrow\frac{1}{x_n+1}\le\frac{1}{x_{n+1}+1}\le\frac{2}{1+\sqrt{5}}\frac{(1-\sqrt{5})}{(1-\sqrt{5})}=\frac{\sqrt{5}-1}{2}$
and so $\sqrt{1-\frac{1}{x_n+1}}\ge\sqrt{1-\frac{1}{x_{n+1}+1}}\ge\sqrt{1-(\frac{\sqrt{5}-1}{2})}=\sqrt{\frac{3-\sqrt{5}}{2}}=\frac{-1+\sqrt{5}}{2}$ (as $(\frac{-1+\sqrt{5}}{2})^2=\frac{6-2\sqrt{5}}{4}=\frac{3-\sqrt{5}}{2}$)
So $x_{n+1}\ge x_{n+2}\ge\frac{-1+\sqrt{5}}{2}$. Therefore, $\{x_n\}$ is decreasing and bounded below. By the monotone sequence theorem, $\lim_{n\to\infty}x_n=x$ exists. Then $x=\sqrt{1-\frac{1}{x+1}}=\sqrt{\frac{x}{x+1}}\Rightarrow$
$x=0$ or $\frac{-1+\sqrt{5}}{2}$. Since $x_n\ge\frac{-1+\sqrt{5}}{2}>0>\frac{-1-\sqrt{5}}{2}$, $\lim_{n\to\infty}x_n=x=\frac{-1+\sqrt{5}}{2}$.

(m) We claim that $0<x_n<1$ for $n=1,2,3,\cdots$. The case $n=1$ is given. Suppose $0<x_n<1$, then $0<x_{n+1}=\frac{x_n^3+6}{7}<\frac{1+6}{7}=1$, completing the induction. Next, $x_{n+1}-x_n=\frac{x_n^3+6}{7}-x_n=\frac{x_n^3-7x_n+6}{7}=\frac{(x_n-1)(x_n-2)(x_n+3)}{7}>0$ implies $\{x_n\}$ is increasing. Since it is bounded above by 1, $\{x_n\}$ converges to some x by monotone sequence theorem. We have $7x=\lim_{n\to\infty}7x_{n+1}=\lim_{n\to\infty}x_n^3+6=x^3+6\Rightarrow x^3-7x+6=0\Rightarrow x=1,2$ or -3.
Since $0<x_n<1$. $x=1$.

(92)(n) ($x=\sqrt{3x-2} \Rightarrow x^2-3x+2=0 \Rightarrow x=1$ or 2.) If $x_1=1$ and $x_n=1$, then $x_{n+1}=\sqrt{3\cdot 1-2}=1$ and so $\lim_{n\to\infty} x_n=1$ in that case. If $x_1\in(1,2]$, then we claim $1<x_n\le x_{n+1}\le 2$. For $x_n>1$ $x_n\le x_{n+1} \Leftrightarrow x_n^2\le 3x_n-2 \Leftrightarrow x_n^2-3x_n+2=(x_n-1)(x_n-2)\le 0 \Leftrightarrow 1\le x_n\le 2$. Since $1<x_1\le 2$, so if $1<x_n\le 2$, then $1<x_n\le x_{n+1}=\sqrt{3x_n-2}\le\sqrt{3\cdot 2-2}=2$, completing induction. So $\lim_{n\to\infty}x_n$ exists in this case. It is a root of $x=\sqrt{3x-2}$ in $(1,2]$. So $\lim_{n\to\infty}x_n=2$ in this case. If $x_1\in(2,\infty)$, then we claim $x_n\ge x_{n+1}\ge 2$. For $x_n>2$, $x_{n+1}\le x_n \Leftrightarrow 3x_n-2\le x_n^2$ $\Leftrightarrow x_n^2-3x_n+2=(x_n-1)(x_n-2)\ge 0 \Leftrightarrow x_n\ge 2$. Since $x_1>2$, so if $x_n>2$, then $x_n\ge x_{n+1}=\sqrt{3x_n-2}\ge\sqrt{3\cdot 2-2}=2$, completing induction. So $\lim_{n\to\infty}x_n$ exists in this case. It is a root of $x=\sqrt{3x-2}$ in $[2,\infty)$. $\therefore \lim_{n\to\infty}x_n=2$ in this case.

(o) ($x_1\le x_2$. If $x_2=0$, then $x_3=\frac{1}{3}$, so suspect $\{x_n\}$ is increasing. The equation $x=\frac{1}{3}(1+x+x^3)$ has $x=1$ as a root. So $x=\frac{1}{3}(1+x+x^3)\Leftrightarrow x^3-2x+1=0$ can be solved by factoring $x-1$. The roots are 1, $\frac{-1+\sqrt5}{2}$, $\frac{-1-\sqrt5}{2}$. Note $a_1\le a_2<\frac{-1+\sqrt5}{2}$.)

Claim: $x_n\le x_{n+1}\le\frac{-1+\sqrt5}{2}$. (Note $\frac{-1-\sqrt5}{2}<\frac{-1+\sqrt5}{2}<1$.)

Case $n=1$ is true as $x_1\le x_2\le\frac{-1+\sqrt5}{2}$. Case $n=2$ is true because $x_2\le\frac{1}{2}\Leftrightarrow$ $x_2\le\frac{1}{3}(1+x_2)=x_3\le\frac{1}{3}\left(1+\frac{-1+\sqrt5}{2}\right)=\frac{1}{3}\left(\frac{1+\sqrt5}{2}\right)<\frac{-1+\sqrt5}{2}$. Assume cases $n-1$ and n, we have $x_{n-1}\le x_n\le\frac{-1+\sqrt5}{2}$ and $x_n\le x_{n+1}\le\frac{-1+\sqrt5}{2}$. So

$$x_{n+1}=\frac{1}{3}(1+x_n+x_{n-1}^3)\le\frac{1}{3}(1+x_{n+1}+x_n^3)=x_{n+2}\le\frac{1}{3}\left(1+\frac{-1+\sqrt5}{2}+\left(\frac{-1+\sqrt5}{2}\right)^3\right)=\frac{-1+\sqrt5}{2}.$$

Completing induction.

By monotone sequence theorem, $\lim_{n\to\infty}x_n$ exists. It is a root of $x=\frac{1}{3}(1+x+x^3)$ in $\left[0,\frac{-1+\sqrt5}{2}\right]$, so $\lim_{n\to\infty}x_n=\frac{-1+\sqrt5}{2}$.

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(93) From $x_2=a_1-a_2\le x_4=a_1-a_2+a_3-a_4\le x_3=a_1-a_2+a_3\le x_1=a_1$, we define $I_n=[x_{2n},x_{2n-1}]$. We claim $I_1\supseteq I_2\supseteq I_3\supseteq I_4\supseteq\cdots$. For this, we have to check $I_n=[x_{2n},x_{2n-1}]\supseteq I_{n+1}=[x_{2n+2},x_{2n+1}]$. (Since $\{a_n\}$ is decreasing, $x_{2n}\le x_{2n+2}=x_{2n}+a_{2n+1}-a_{2n+2}\le x_{2n+1}=x_{2n}+a_{2n+1}=x_{2n-1}-a_{2n}+a_{2n+1}\le x_{2n-1}$.) Finally since $\lim_{n\to\infty}|x_{2n}-x_{2n-1}|=\lim_{n\to\infty}a_{2n}=0$, we have $\bigcap_{n=1}^{\infty}I_n=\{x\}$, $\lim_{n\to\infty}x_{2n}=x=\lim_{n\to\infty}x_{2n-1}$. So $\lim_{n\to\infty}x_n=x$.

<u>Alternative Solution</u> Applying summation by parts, we get $x_n=S_na_n-\sum_{k=1}^{n-1}S_k(a_{k+1}-a_k)$, where $S_j=\sum_{k=1}^{j}(-1)^{k+1}=\begin{cases}0 & \text{if } j \text{ is even}\\ 1 & \text{if } j \text{ is odd}\end{cases}$. Since $\{a_n\}$ is a decreasing sequence with limit 0 and $0\le S_n\le 1$, we have $\lim_{n\to\infty}S_na_n=0$. Also, $-S_k(a_{k+1}-a_k)\ge 0$ so that $y_n=-\sum_{k=1}^{n-1}S_k(a_{k+1}-a_k)$ is increasing. Since $y_n\le\sum_{k=1}^{n-1}1(a_k-a_{k+1})=a_1-a_n\le a_1$, by monotone sequence theorem, $\lim_{n\to\infty}y_n$ exists. Then $\lim_{n\to\infty}x_n=\lim_{n\to\infty}S_na_n+\lim_{n\to\infty}y_n$ exists.

(94) Observe $a_1=a\le a_2=\frac{a+b}{2}=\sqrt{\frac{a^2+2ab+b^2}{4}}\le b_2=\sqrt{\frac{2a^2+2b^2}{4}}\le b_1=b$. We will try to show $a_n\le a_{n+1}\le b_{n+1}\le b_n$ by mathematical induction. Case $n=1$ is done above. Suppose case $n=k$ is true. For case $n=k+1$, since $a_{k+1}\le b_{k+1}$,

$$a_{k+1}\le a_{k+2}=\frac{a_{k+1}+b_{k+1}}{2}=\sqrt{\frac{a_{k+1}^2+2a_{k+1}b_{k+1}+b_{k+1}^2}{4}}\le b_{k+2}=\sqrt{\frac{2a_{k+1}^2+2b_{k+1}^2}{4}}\le b_{k+1}.$$

So $\{a_n\}$ is increasing and bounded above by $b_1=b$, hence converges to some A. Also $\{b_n\}$ is decreasing and bounded below by $a_1=a$, hence converges to some B. Since $a_{n+1}=\frac{a_n+b_n}{2}$, so $A=\lim_{n\to\infty} a_{n+1}=\lim_{n\to\infty}\frac{a_n+b_n}{2}=\frac{A+B}{2}\Rightarrow A=B$.

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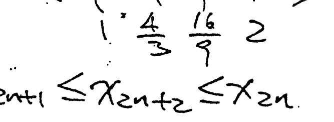
(95) (i) If $a\le b$ and $0<t<1$, then $a=ta+(1-t)a\le ta+(1-t)b\le tb+(1-t)b=b$.

(ii) (Note $x_1=1$, $x_2=2$, $x_3=\frac{1}{3}2+\frac{2}{3}1=\frac{4}{3}$, $x_4=\frac{1}{3}\frac{4}{3}+\frac{2}{3}2=\frac{16}{9}$



)

Let $I_n=[x_{2n-1},x_{2n}]$, then we will show $I_n\supseteq I_{n+1}$, i.e. $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$ for all $n\in\mathbb{N}$. Now $x_{2n-1}\le x_{2n+1}=\frac{1}{3}x_{2n}+\frac{2}{3}x_{2n-1}=\frac{2}{3}x_{2n-1}+\frac{1}{3}x_{2n}\le x_{2n}$ by part (i). Also $x_{2n+1}\le x_{2n+2}=\frac{1}{3}x_{2n+1}+\frac{2}{3}x_{2n}\le x_{2n}$ by part (i) again. So we get $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$ for every $n\in\mathbb{N}$. Note $1\le x_{2n-1}\le x_{2n}\le 2$. By the monotone sequence theorem, $\{x_{2n-1}\}$ converges to a and $\{x_{2n}\}$ converges to b for some $a,b\in\mathbb{R}$. Since $x_{2n+1}=\frac{1}{3}x_{2n}+\frac{2}{3}x_{2n-1}$, let $n\to\infty$, we get $a=\frac{1}{3}b+\frac{2}{3}a$ $\Rightarrow a=b$. By the intertwining sequence theorem, $\{x_n\}$ converges.

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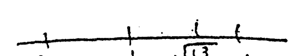
(96) ($x_0=0$, $x_1=1$, $x_2=\sqrt{\frac{1}{4}1+\frac{3}{4}0}=\frac{1}{2}$, $x_3=\sqrt{\frac{1}{4}\frac{1}{4}+\frac{3}{4}1}=\sqrt{\frac{13}{16}}$



)

If $x_n\le x_{n-1}$, then $x_n=\sqrt{\frac{1}{4}x_n^2+\frac{3}{4}x_n^2}\le x_{n+1}=\sqrt{\frac{1}{4}x_n^2+\frac{3}{4}x_{n-1}^2}\le x_{n-1}=\sqrt{\frac{1}{4}x_{n-1}^2+\frac{3}{4}x_{n-1}^2}$

If $x_{n-1}<x_n$, then $x_{n-1}=\sqrt{\frac{1}{4}x_{n-1}^2+\frac{3}{4}x_{n-1}^2}\le x_{n+1}=\sqrt{\frac{1}{4}x_n^2+\frac{3}{4}x_{n-1}^2}\le x_n=\sqrt{\frac{1}{4}x_n^2+\frac{3}{4}x_n^2}$.

So x_{n+1} is always between x_{n-1} and x_n. Define $I_n=[x_{2n},x_{2n+1}]$ for $n=0,1,2,\cdots$ Then $x_{2n}\le x_{2n+2}\le x_{2n+3}\le x_{2n+1}$ for $n=0,1,2,\cdots$. So $[0,1]=I_0\supseteq I_1\supseteq I_2\supseteq\cdots$.

By nested interval theorem, $\lim_{n\to\infty}x_{2n}=a$ and $\lim_{n\to\infty}x_{2n+1}=b$ exist. Taking limit of $x_{2n+1}=\sqrt{\frac{1}{4}x_{2n}^2+\frac{3}{4}x_{2n-1}^2}$, we get $b=\sqrt{\frac{1}{4}a^2+\frac{3}{4}b^2}\Rightarrow a=b$. By intertwining sequence theorem, x_n converges to some limit x.

To find x, write

$$\begin{aligned}
\cancel{x_2^2}&=\tfrac{1}{4}x_1^2+\tfrac{3}{4}x_0^2\\
\cancel{x_3^2}&=\tfrac{1}{4}\cancel{x_2^2}+\tfrac{3}{4}x_1^2\\
&\ \vdots\\
\tfrac{3}{4}x_k^2\quad \cancel{x_k^2}&=\tfrac{1}{4}\cancel{x_{k-1}^2}+\tfrac{3}{4}\cancel{x_{k-2}^2}\\
x_{k+1}^2&=\tfrac{1}{4}\cancel{x_k^2}+\tfrac{3}{4}\cancel{x_{k-1}^2}
\end{aligned}$$

Adding these equations and cancelling common terms on both sides, we get $x_{k+1}^2+\frac{3}{4}x_k^2=x_1^2+\frac{3}{4}x_0^2=1$. Taking limit, we get $\frac{7}{4}x^2=1$. So $x=\sqrt{\frac{4}{7}}$.

(96) $x_{n+1}=\sqrt{\frac{1}{4}x_n^2+\frac{3}{4}x_{n-1}^2} \Rightarrow x_{n+1}^2=\frac{1}{4}x_n^2+\frac{3}{4}x_{n-1}^2 \Rightarrow 4x_{n+1}^2=x_n^2+3x_{n-1}^2$

$\Rightarrow 4x_{n+1}^2-4x_n^2=-3x_n^2+3x_{n-1}^2 \Rightarrow x_{n+1}^2-x_n^2=-\frac{3}{4}(x_n^2-x_{n-1}^2)=\cdots=\left(-\frac{3}{4}\right)^n\underbrace{(x_1^2-x_0^2)}_{=1}$

$\Rightarrow x_{n+1}^2=\underbrace{\sum_{k=0}^{n}(x_{k+1}^2-x_k^2)}_{\text{telescoping series}}=\sum_{k=0}^{n}\left(-\frac{3}{4}\right)^k \Rightarrow x_n=\sqrt{\sum_{k=0}^{n-1}\left(-\frac{3}{4}\right)^k} \Rightarrow \lim_{n\to\infty}x_n=\sqrt{\sum_{k=0}^{\infty}\left(-\frac{3}{4}\right)^k}=\sqrt{\frac{1}{1-\left(-\frac{3}{4}\right)}}=\sqrt{\frac{4}{7}}.$

Geometric series

(97) Assume S is unbounded. Then for every $n \in \mathbb{N}$, there is $x_n \in S$ outside $[-n,n]$, i.e. $|x_n| > n$. We are given that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Then $\{x_{n_k}\}$ is bounded. Since $|x_{n_k}| > n_k \geq k$ can be arbitrarily large, $\{x_{n_k}\}$ cannot be bounded, a contradiction. Therefore S is bounded.

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(98) We have $x \in A, y \in A \Rightarrow x^2+y^2 \leq (\sup A)^2 + (\sup A)^2 = 2(\sup A)^2$. So $2(\sup A)^2$ is an upper bound for B.
By supremum limit theorem, there is a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = \sup A$. Then $\{x_n^2 + x_n^2\}$ is a sequence in B and $\lim_{n\to\infty}(x_n^2+x_n^2) = 2(\sup A)^2$. So by the Supremum limit theorem, $\sup B = 2(\sup A)^2$.

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(99) For $x \in \bigcup_{n=1}^{10} A_n$, $x \in A_n$ for some $n \Rightarrow x \leq x_n = \sup A_n \leq \max(x_1, \cdots, x_{10})$.
So $\max(x_1,\cdots,x_{10})$ is an upper bound of $\bigcup_{n=1}^{10} A_n$. Let $x_i = \max(x_1,\cdots,x_{10})$, then since $x_i = \sup A_i$, there is $\{a_n\}$ in A_i such that $\lim_{n\to\infty} a_n = x_i$. Since $\{a_n\} \in \bigcup_{n=1}^{10} A_n$, so $x_i = \sup(\bigcup_{i=1}^{10} A_i)$. $\therefore \sup(\bigcup_{i=1}^{10} A_i) = \max(x_1,\cdots,x_{10})$.

Alternative Solution

As in first solution, $x_i = \max(x_1,\cdots,x_{10})$ is an upper bound of $\bigcup_{n=1}^{10} A_n$.
For any upper bound M of $\bigcup_{n=1}^{10} A_n$, $M \geq x$ for every $x \in \bigcup_{n=1}^{10} A_n$. Since $A_i \subseteq \bigcup_{n=1}^{10} A_n$, $M \geq x$ for every $x \in A_i$. So M is an upper bound of A_i, too. Then $M \geq x_i$.
So $x_i = \max(x_1,\cdots,x_{10})$ is the least upper bound of $\bigcup_{n=1}^{10} A_n$.

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(100) Since $f(x,y) \in [0,1]$, all inf and sup expressions exist by completeness axiom.
For every $x_0 \in \mathbb{R}$, $h(y) = \inf\{f(x,y) : x \in \mathbb{R}\} \leq f(x_0,y) \leq g(x_0) = \sup\{f(x_0,y) : y \in \mathbb{R}\}$.
So $g(x_0)$ is an upper bound of $\{h(y) : y \in \mathbb{R}\}$. Then $\sup\{h(y) : y \in \mathbb{R}\} \leq g(x_0)$.
So $\sup\{h(y) : y \in \mathbb{R}\}$ is a lower bound of $\{g(x_0) : x_0 \in \mathbb{R}\}$. Therefore, $\sup\{h(y) : y \in \mathbb{R}\} \leq \inf\{g(x_0) : x_0 \in \mathbb{R}\}$.

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(101) Let $x \in \mathbb{R}$. By the density of irrational numbers, there is $x_1 \in \mathbb{R} \setminus \mathbb{Q}$ such that $x-1 < x_1 < x$. Suppose $x_n < x$ has been chosen, then we use density of irrational numbers to choose $x_{n+1} \in \mathbb{R} \setminus \mathbb{Q}$ such that $\max(x_n, x - \frac{1}{n+1}) < x_{n+1} < x$. Then $x_n < x_{n+1}$ and $x - \frac{1}{k} < x_k < x$ implies $\lim_{k\to\infty} x_k = x$ by the squeeze limit theorem.

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(102) (Note $\frac{1}{n^2} < \frac{\varepsilon}{2} \Leftrightarrow \sqrt{\frac{2}{\varepsilon}} < n$ and $\frac{\sqrt{2}}{n^3} < \frac{\varepsilon}{2} \Leftrightarrow \sqrt[3]{\frac{2\sqrt{2}}{\varepsilon}} < n$.) For every $\varepsilon > 0$, by the Archimedean principle, there exists $K \in \mathbb{N}$ such that $K > \max\left(\sqrt{\frac{2}{\varepsilon}}, \sqrt[3]{\frac{2\sqrt{2}}{\varepsilon}}\right)$.
Then $n \geq K \Rightarrow \left|\left(\frac{1}{n^2} - \frac{\sqrt{2}}{n^3}\right) - 0\right| \leq \frac{1}{n^2} + \frac{\sqrt{2}}{n^3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So $\lim_{n\to\infty}\left(\frac{1}{n^2} - \frac{\sqrt{2}}{n^3}\right) = 0$ by definition.

(103) (Note $\frac{2}{n+1}<\frac{\varepsilon}{2} \iff \frac{4}{\varepsilon}-1<n$ and $\frac{1}{n^2}<\frac{\varepsilon}{2} \iff \sqrt{\frac{2}{\varepsilon}}<n$.) For every $\varepsilon>0$, by the Archimedean principle, there is $K\in\mathbb{N}$ such that $K>\max\left(\frac{4}{\varepsilon}-1, \sqrt{\frac{2}{\varepsilon}}\right)$.
Then $n\geq K \Rightarrow \left|\left(\frac{2}{n+1}-\frac{1}{n^2}\right)-0\right| \leq \frac{2}{n+1}+\frac{1}{n^2} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. So $\lim\limits_{n\to\infty}\left(\frac{2}{n+1}-\frac{1}{n^2}\right)=0$ by definition.

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(104) For every $\varepsilon>0$, since $\lim\limits_{n\to\infty} x_n=0$, there is $K_1\in\mathbb{N}$ such that $n\geq K_1 \Rightarrow |x_n-0|<\frac{\varepsilon}{2}$. By the Archimedean principle, there is $K_2\in\mathbb{N}$ such that $K_2>\frac{2}{\varepsilon}$. Let $K=\max(K_1,K_2)$ then $n\geq K \Rightarrow \left|\left(x_n+\frac{1}{n}\right)-0\right| \leq |x_n-0|+\frac{1}{n} < \frac{\varepsilon}{2}+\frac{1}{K_2} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. Therefore, $\lim\limits_{n\to\infty}\left(x_n+\frac{1}{n}\right)=0$ by definition.

$n\geq K_1$ $\quad$ $n\geq K_2 \Rightarrow \frac{1}{n}\leq\frac{1}{K_2}<\frac{\varepsilon}{2}$

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(105) Since $\lim\limits_{n\to\infty} x_n=\frac{1}{2}$, so for $\varepsilon_0=\frac{1}{3}$, there is $K_1\in\mathbb{N}$ such that $n\geq K_1 \Rightarrow |x_n-\frac{1}{2}|<\varepsilon_0=\frac{1}{3}$ $\Rightarrow -\frac{1}{3}<x_n-\frac{1}{2}<\frac{1}{3} \Rightarrow \frac{1}{6}<x_n<\frac{5}{6} \Rightarrow |x_n^n-0|<\left(\frac{5}{6}\right)^n$. So for every $\varepsilon>0$, let $K=\max\left(K_1, \left\lceil\frac{\ln 1/\varepsilon}{\ln 6/5}\right\rceil\right)$, then $n\geq K \Rightarrow |x_n^n-0|<\left(\frac{5}{6}\right)^n\leq\varepsilon$.

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(106) Since $\lim\limits_{n\to\infty} x_n=8$, so for $\varepsilon_0=8$, there is $K_1\in\mathbb{N}$ such that $n\geq K_1 \Rightarrow |x_n-8|<\varepsilon_0=8$ $\Rightarrow -8<x_n-8 \Rightarrow x_n>0$. For $\varepsilon>0$, there is $K_2\in\mathbb{N}$ such that $n\geq K_2 \Rightarrow |x_n-8|<4\varepsilon$. Let $K=\max(K_1,K_2)$, then $n\geq K \Rightarrow n\geq K_1$ and $n\geq K_2$. Since $x_n>0$ for $n\geq K$ and $|x_n-8|=|\sqrt[3]{x_n}-2|\,|(\sqrt[3]{x_n})^2+2\sqrt[3]{x_n}+4|>|\sqrt[3]{x_n}-2|\,4$, so $|\sqrt[3]{x_n}-2|<\frac{1}{4}|x_n-8|<\frac{1}{4}4\varepsilon=\varepsilon$.

Alternative Solution: Claim: $|\sqrt[3]{x}-\sqrt[3]{y}|\leq\sqrt[3]{|x-y|}$ for $x,y\geq 0$. Let $u=\max(x,y)$ and $v=\min(x,y)$, then we have to show $\sqrt[3]{u}-\sqrt[3]{v}\leq\sqrt[3]{u-v}$ ($\iff \sqrt[3]{u}\leq\sqrt[3]{v}+\sqrt[3]{u-v} \iff u\leq v+3v^{2/3}(u-v)^{1/3}+3v^{1/3}(u-v)^{2/3}+(u-v)=u+3v^{2/3}(u-v)^{1/3}+3v^{1/3}(u-v)^{2/3}$) which is true. For the problem, since $\lim\limits_{n\to\infty}x_n=8$, so for $\varepsilon_0=8$, there is $K_1\in\mathbb{N}$ such that $n\geq K_1 \Rightarrow |x_n-8|<\varepsilon_0=8 \Rightarrow -8<x_n-8 \Rightarrow x_n>0$. For $\varepsilon>0$, there is $K_2\in\mathbb{N}$ such that $|x_n-8|<\varepsilon^3$. Then for $n\geq K=\max(K_1,K_2)$, $|\sqrt[3]{x_n}-2|\leq\sqrt[3]{|x_n-8|}<\sqrt[3]{\varepsilon^3}=\varepsilon$.

---

(107) Let $\varepsilon>0$. Since $\{x_n\}$ and $\{y_n\}$ converge to A, so by definition, there are $K_1, K_2\in\mathbb{N}$ such that $n\geq K_1$ implies $|x_n-A|<\varepsilon$, and $n\geq K_2$ implies $|y_n-A|<\varepsilon$. Let $K=\max(K_1,K_2)$, then $n\geq K \Rightarrow n\geq K_1$ and $n\geq K_2 \Rightarrow |x_n-A|<\varepsilon$ and $|y_n-A|<\varepsilon$ $\Rightarrow |z_n-A|<\varepsilon$ (because $z_n=x_n$ or y_n.)

---

(108) For every $\varepsilon>0$, by Archimedean principle, there is integer $K>\frac{1}{\varepsilon}$.
Then $\underset{m>n}{m,n\geq K} \Rightarrow$

$$\begin{aligned}|x_m-x_n| &= |(x_m-x_{m-1})+(x_{m-1}-x_{m-2})+\cdots+(x_{n+1}-x_n)| \\ &\leq |x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n| \\ &< \frac{1}{2^{m-1}}+\frac{1}{2^{m-2}}+\cdots+\frac{1}{2^n} < \sum_{j=n}^{\infty}\frac{1}{2^j}=\frac{1}{2^{n-1}}\leq\frac{1}{2^{K-1}}\leq\frac{1}{K}<\varepsilon\end{aligned}$$

The case $m<n$ is similar. The case $m=n$ leads to $|x_m-x_n|=0<\varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence.

$K\leq 2^{K-1}$ can be proved by mathematical induction.

(109) (a) $f(x)$ converges to L (or has limit L) as x tends to x_0 in S iff for every $\varepsilon>0$, there exists a $\delta>0$ such that $x\in S$ and $0<|x-x_0|<\delta$ imply $|f(x)-L|<\varepsilon$.

(b) For every $\varepsilon>0$, take $\delta=\frac{2}{11}\varepsilon>0$. If $0<|x-2|<\delta$ and $x\in(1,3)$, then

$$|f(x)-\tfrac{9}{2}|=|(x^2+\tfrac{1}{x})-\tfrac{9}{2}|=|(x^2-4)+(\tfrac{1}{x}-\tfrac{1}{2})|\le|x^2-4|+|\tfrac{1}{x}-\tfrac{1}{2}|=|x+2||x-2|+\frac{|x-2|}{2|x|}$$
$$\le 5|x-2|+\tfrac{1}{2}|x-2|=\tfrac{11}{2}|x-2|<\tfrac{11}{2}\delta=\varepsilon.$$

(c) Solution 1 For every $\varepsilon>0$, take $\delta=\frac{\varepsilon}{6}>0$. If $0<|x-2|<\delta$ and $x\in(1,4)$, then

$$|f(x)-5|=||x^2-9|-|-5||\underset{\uparrow}{\le}|x^2-4|=|x-2||x+2|\le 6|x-2|<6\delta=\varepsilon.$$

by exercise 40, $||a|-|b||\le|a-b|$ $a=x^2-9$, $b=-5$

Solution 2 (Note that for $x\in[1,3]$, $x^2-9\le 0 \Rightarrow f(x)=9-x^2$.)

For every $\varepsilon>0$, take $\delta=\min(1,\frac{\varepsilon}{5})>0$. If $0<|x-2|<\delta$, then $|x-2|<1 \Rightarrow$ $x\in(1,3)\Rightarrow |f(x)-5|=|(9-x^2)-5|=|4-x^2|=|2-x||2+x|\le 5|x-2|<5\delta\le\varepsilon$.

---

(110) Since $\max(a,b)+\min(a,b)=a+b$ and $\max(a,b)-\min(a,b)=|a-b|$, so adding the two equation and dividing by 2, we get $\max(a,b)=\frac{a+b+|a-b|}{2}$.

Let S_f, S_g, S_h be the set of jumps of f,g,h, respectively. If f,g are continuous at x, then $h=\frac{f+g+|f-g|}{2}$ will also be continuous at x. Taking contrapositive, if $x\in S_h$, then $x\in S_f\cup S_g$. So $S_h\subseteq S_f\cup S_g$. By the monotone function theorem, S_f, S_g are countable. By the countable union theorem, $S_f\cup S_g$ is countable. By the countable subset theorem, S_h is countable.

---

(111) Define $f(x)=\begin{cases}\sin\pi x & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$. For every $m\in\mathbb{Z}$, $|f(x)|\le|\sin\pi x|\to 0$ as $x\to m$, So $\lim_{x\to m}f(x)=0=f(m)$. So f is continuous at every $m\in\mathbb{Z}$. For $x_0\notin\mathbb{Z}$, let $r_n\in\mathbb{Q}$ and $s_n\notin\mathbb{Q}$ such that $\lim_{n\to\infty}r_n=x_0=\lim_{n\to\infty}s_n$. Then $\lim_{n\to\infty}f(r_n)=\sin\pi x_0\ne 0=\lim_{n\to\infty}f(s_n)$. So f is not continuous at x_0 by the sequential continuity theorem.

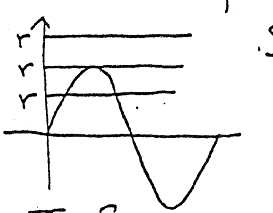
---

(112) (a) If $f:[a,b]\to\mathbb{R}$ is continuous and y_0 is between $f(a)$ and $f(b)$, then there is (at least one) $x_0\in[a,b]$ such that $f(x_0)=y_0$.

(b) Define $g:[0,1]\to\mathbb{R}$ by $g(x)=f(x)-f(x+1)$. Note $g(0)=f(0)-f(1)$ and $g(1)$ $=f(1)-f(2)=f(1)-f(0)=-g(0)$. So $g(1)$ and $g(0)$ are of opposite sign. Since g is continuous on $[0,1]$, by the intermediate value theorem, $\exists c\in[0,1]$ such that $0=g(c)=f(c)-f(c+1)$. Then $f(c)=f(c+1)$.

(c) Observe that $|t|^r+|2t|^r+|3t|^r=|4t|^r+|5t|^r$ for every $t\in\mathbb{R}$ is equivalent to $1+2^r+3^r=4^r+5^r$. We will show this equation has a solution. Let $f(r)=1+2^r+3^r-4^r-5^r$, which is continuous. Since $f(0)=1$, $f(1)=-3$, by the intermediate value theorem, there is $r\in(0,1)$ such that $f(r)=0$. For this r, let $g(t)=|t|^r$, then $g(t)+g(2t)+g(3t)=g(4t)+g(5t)$. $\forall t\in\mathbb{R}$

(113) (a) For a fixed rational r, $\{x : \sin x = r\} = \bigcup_{k \text{ even integer}} \{x : \sin x = r,\ x \in [k\pi, k\pi + 2\pi)\}$



Since $\sin x = r$ on $[k\pi, k\pi+2\pi)$ has at most 2 solutions,

$\{x : \sin x = r\} = \underbrace{\bigcup_{k \text{ even integer}}}_{\text{countable}} \underbrace{\{x : \sin x = r,\ x \in [k\pi, k\pi + 2\pi)\}}_{\text{countable}}$ is countable.

So $T = \{x : \sin x \in \mathbb{Q}\} = \underbrace{\bigcup_{r \in \mathbb{Q}}}_{\text{countable}} \underbrace{\{x : \sin x = r\}}_{\text{countable}}$ is countable.

(b) For every $x \in [0,1]$, $\sin f(x) \in \mathbb{Q}$ implies $f(x) \in T$. So $f([0,1]) = \{f(x) : x \in [0,1]\} \subseteq T$. By (a), T is countable, so $f([0,1])$ is countable.

Assume f is not a constant function, then $f([0,1])$ contains an interval (of positive length) by the intermediate value theorem. Then $f([0,1])$ is uncountable, a contradiction. Therefore, f is a constant function.

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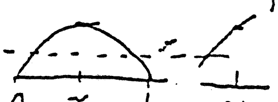
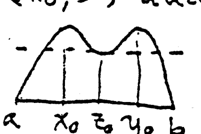
(114) Suppose such a function exists. Let a, b be the solutions of $f(x) = 0$ with $a < b$.



Case 1 $\max_{x \in [a,b]} f(x) = f(x_0) \neq 0$. Let y_0 be the other solution of $f(x) = f(x_0)$. If $y_0 \notin [a,b]$, then by the intermediate value theorem, there will be 3 solutions of $f(x) = \frac{1}{2} f(x_0)$, one on (a, x_0), one on (x_0, b) and one between y_0 and the closer endpoint of $[a,b]$ to y_0.



If $y_0 \in [a,b]$, then let $f(z_0) = \min_{x \in [x_0, y_0]} f(x)$ with $z_0 \in [x_0, y_0]$. Let $w = \max\{f(z_0), 0\}$, then by the intermediate value theorem, there are at least 3 solutions of $f(z) = w$, one on (a, x_0), one on (x_0, y_0), one on (y_0, b).

Thus, whether $y_0 \notin [a,b]$ or $y_0 \in [a,b]$ will lead to a contradiction.

Case 2 $\min_{x \in [a,b]} f(x) \neq 0$. This case is similar to case 1. (Turn figures upside down.)

Case 3 $\max_{x \in [a,b]} f(x) = 0 = \min_{x \in [a,b]} f(x)$. Then $f(x) \equiv 0$ on $[a,b]$, a contradiction.

---

(115) (a) $f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$. $-\frac{x^4}{|x|} \le f(x) \le \frac{x^4}{|x|} \Rightarrow \lim_{x \to 0} f(x) = 0 = f(0)$ (Sandwich Theorem)

So f is continuous at 0.

(b) $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} [f(x - x_0) + f(x_0)] = \lim_{x \to x_0} f(x - x_0) + f(x_0) = \lim_{y \to 0} f(y) + f(x_0) = 0 + f(x_0) = f(x_0)$

So f is continuous at every $x_0 \in \mathbb{R}$.

(c) $f(x) \equiv 0$ satisfies $f(x+y) = f(x) + f(y)$ and $|f(x)| \le x^4/|x|$ for $x \neq 0$.

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(116) Since $\{g(x) : x \in [1,2]\} = [3,4]$, so there are $x_0, x_1 \in [1,2]$ such that $g(x_0) = 3$ and $g(x_1) = 4$. On the closed interval I with endpoints x_0 and x_1, since $f : I \to [3,4]$, $(f-g)(x_0) = f(x_0) - 3 \ge 0$ and $(f-g)(x_1) = f(x_1) - 4 \le 0$, $f - g$ is continuous on I, so by intermediate value theorem, there is $c \in I \subseteq [1,2]$ such that $(f-g)(c) = 0$. So $f(c) = g(c)$.

(117) (a) Observe that $|x_{k+1}-x_k| = |f(x_k)-f(x_{k-1})| \le \frac{1}{2}|x_k - x_{k-1}|$. Repeating this, we get $|x_{k+1}-x_k| \le \frac{1}{2}|x_k-x_{k-1}| \le (\frac{1}{2})^2 |x_{k-1}-x_{k-2}| \le \cdots \le (\frac{1}{2})^{k-1}|x_2-x_1|$. So for $m>n$, we have

$$|x_m - x_n| = |(x_m-x_{m-1})+(x_{m-1}-x_{m-2})+\cdots+(x_{n+1}-x_n)|$$
$$\le |x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n|$$
$$\le \left((\tfrac{1}{2})^{m-2}+(\tfrac{1}{2})^{m-3}+\cdots+(\tfrac{1}{2})^{n-1}\right)|x_2-x_1| \le (\tfrac{1}{2})^{n-2}|x_2-x_1|.$$

$$\sum_{k=n-1}^{\infty}\left(\tfrac{1}{2}\right)^k = \left(\tfrac{1}{2}\right)^{n-2}$$

If $x_1=x_2$, then $x_m=x_n$ for all m,n and $\{x_n\}$ is a constant sequence. Hence $\{x_n\}$ converges and is a Cauchy sequence. If $x_1 \ne x_2$, then for every $\varepsilon>0$, by the Archimedean principle, there is $K\in\mathbb{N}$ such that $K > 2 - \log_2 \frac{\varepsilon}{|x_2-x_1|}$, which implies $(\frac{1}{2})^{K-2}|x_2-x_1| < \varepsilon$. So $m,n\ge K \Rightarrow |x_m-x_n| \le (\frac{1}{2})^{K-2}|x_2-x_1| < \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence.

(b) Let $w\in\mathbb{R}$. Define $\{x_n\}$ as in (a). Then $\{x_n\}$ is a Cauchy sequence by (a). By Cauchy's theorem, $\{x_n\}$ converges to some $x\in\mathbb{R}$. We have

$$x \underset{\text{Subsequence theorem}}{=} \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) \underset{\text{Sequential continuity theorem}}{=} f(\lim_{n\to\infty} x_n) = f(x).$$

---

(118) Define $f(x) = \begin{cases} (x-1)^2 \sin\frac{1}{x-1} & \text{if } x\in(0,1)\cup(1,2) \\ 0 & \text{if } x=1 \end{cases}$. For $x\in(0,1)\cup(1,2)$, by product rule, $f'(x) = 2(x-1)\sin\frac{1}{x-1} - \cos\frac{1}{x-1}$. For $x=1$, $f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1}(x-1)\sin\frac{1}{x-1}$ $= 0$ as $|(x-1)\sin\frac{1}{x-1}| \le |x-1| \to 0$ as $x\to 1$. So f is differentiable on $(0,2)$. However, $\lim_{x\to1} f'(x) = -\lim_{x\to 1}\cos\frac{1}{x-1}$ doesn't exist. So $f'(x)$ is not continuous at 1.

---

(119) We have $\left|\frac{f(a)-f(b)}{a-b}\right| \le \frac{\sin^2|a-b|}{|a-b|}$ for $a\ne b$, a,b in $(0,\pi)$. Since $\lim_{a\to b}\frac{\sin^2|a-b|}{|a-b|}$ $= \lim_{\theta\to0}\frac{\sin^2\theta}{\theta} = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to0}\sin\theta = 1\cdot 0 = 0$, we have $f'(b) = \lim_{a\to b}\frac{f(a)-f(b)}{a-b} = 0$ for every b. Therefore, f is a constant function.

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(120) (a) Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $x_0\in(a,b)$ such that $f(b)-f(a) = f'(x_0)(b-a)$.

(b) By the mean value theorem, $|\sin b - \sin a| = |(\cos x_0)(b-a)| \le 1|b-a|$. If there is a K such that $|f(b)-f(a)| \le K|b-a|$ for every $a,b\in\mathbb{R}$, then $|f'(a)| = \lim_{b\to a}\left|\frac{f(b)-f(a)}{b-a}\right| \le K$ for every $a\in\mathbb{R}$. Since $f'(0)=\cos 0 = 1$, so $K\ge 1$. Therefore, the smallest K is 1.

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(121) Since $\lim_{x\to0}\frac{f'(x)}{1} = \lim_{x\to 0} f'(x)$ exists, by l'Hopital's rule, we have $\lim_{x\to0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to0}\frac{f'(x)}{1}$ $= \lim_{x\to0} f'(x)$ exists in $\mathbb{R}$. Therefore, $f'(0) \underset{\text{by definition of } f'(0)}{=} \lim_{x\to0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to0} f'(x)$, i.e. f' is continuous at 0.

(122) By the mean value theorem, $|\sin 5b - \sin 5a| = |(5\cos 5x_0)(b-a)| \le 5|b-a|$. So for every $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{5} > 0$. With this δ, we have for every $a, b \in \mathbb{R}$, $|b-a| < \delta \Rightarrow |\sin 5b - \sin 5a| \le 5|b-a| < 5\delta = \varepsilon$.

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(123) For every $\varepsilon > 0$, since f is uniformly continuous, so $\exists\, \delta > 0$ such that $\forall x, y \in \mathbb{R}$ $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon^2$. Then $|x-y| < \delta \Rightarrow |\sqrt{f(x)} - \sqrt{f(y)}| \le \sqrt{|f(x)-f(y)|} < \varepsilon$, (where we used $|\sqrt{a} - \sqrt{b}| \le \sqrt{|a-b|}$ ~~as in homework 2, #236(b)~~). Therefore, $\sqrt{f(x)}$ is also uniformly continuous.

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(124) (b) Solution 1. (using Lebesgue's Theorem) $\Rightarrow f, g$ bounded on $[0,2]$ $\Rightarrow h$ bounded on $[0,2]$.

Since f, g are Riemann integrable on $[0,2]$, so $S_f = \{x \in [0,2] : f \text{ is discontinuous at } x\}$ and $S_g = \{x \in [0,2] : g \text{ is discontinuous at } x\}$ are of measure 0.

Now for $x \in [0,1)$, h is discontinuous at x if and only if f is discontinuous at x. Also for $x \in (1,2]$, h is discontinuous at x if and only if g is discontinuous at x. (These are because $h = f$ on $[0,1)$ and $h = g$ on $(1,2]$.) So

$$S_h = \{x \in [0,2] : h \text{ is discontinuous at } x\} \subseteq (S_f \cap [0,1)) \cup (S_g \cap (1,2]) \cup \{1\}$$
$$\subseteq S_f \cup S_g \cup \{1\}.$$

Since $S_f, S_g, \{1\}$ are of measure 0, we have $S_f \cup S_g \cup \{1\}$ is of measure 0. Then S_h is also of measure 0. Therefore, h is Riemann integrable by Lebesgue's Theorem.

Solution 2 (using integral criterion)

Since f and g are integrable on $[0,2]$, they are bounded on $[0,2]$. So there are $m, M \in \mathbb{R}$ such that $m \le f(x), g(x) \le M$ for all $x \in [0,2]$.

If $m = M$, then $h(x)$ is a constant function, hence h is integrable on $[0,2]$.

If $m < M$, then for every $\varepsilon > 0$, let $d \in (0,1)$ such that $1 - d < \frac{\varepsilon}{3(M-m)}$.

By 82(ii), f is integrable on $[0,d]$ and g is integrable on $[1,2]$.

By integral criterion, there are partitions P_1 of $[0,d]$ and P_2 of $[1,2]$ such that $U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{3}$ and $U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{3}$.

Now $P = P_1 \cup P_2$ is a partition of $[0,2]$ and

$$U(h,P) - L(h,P) \le U(f,P_1) - L(f,P_1) + (M-m)(1-d) + U(g,P_2) - L(g,P_2)$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

(125) Note $\max(f,g)+\min(f,g)=f+g$ and $\max(f,g)-\min(f,g)=|f-g|$. Subtracting, then dividing by 2, we have $h=\min(f,g)=\frac{f+g-|f-g|}{2}$. If f,g are integrable, then $f+g$, $f-g$ are integrable. Since $|x|$ is continuous, so $|f-g|$ is also integrable. Therefore, $h=\frac{f+g-|f-g|}{2}$ is integrable.

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(126) Since $\mathbb{Q}\cap[0,1]$ is countable, let $r_1,r_2,r_3,\dots$ be a listing of the elements of $\mathbb{Q}\cap[0,1]$ without repetition nor omission. Define $f_n(x)=\begin{cases}1 & \text{if } x=r_1 \text{ or } r_2 \text{ or} \dots \text{or } r_n\\ 0 & : \text{otherwise}\end{cases}$. Then on $[0,1]$, f_n is discontinuous exactly at $r_1,r_2,\dots,r_n$. Since $\{r_1,r_2,\dots,r_n\}$ is countable, hence of measure 0, f_n is Riemann integrable by Lebesgue's theorem.

Now $f(x)=\lim_{n\to\infty} f_n(x)=\begin{cases}1 & \text{if } x=r_i \text{ for } i=1,2,3,\dots\\ 0 & \text{otherwise}\end{cases}=\begin{cases}1 & \text{if } x\in\mathbb{Q}\cap[0,1]\\ 0 & \text{otherwise}\end{cases}$.

On $[0,1]$, f is discontinuous everywhere as is shown in class. Since $[0,1]$ is not of measure 0, f is not Riemann integrable by Lebesgue's theorem.

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(127) (a) Since $\left|\frac{\cos 3x}{1+x^2}\right|=\frac{|\cos 3x|}{1+x^2}\le\frac{1}{1+x^2}$ and $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\lim_{c\to-\infty}\int_c^0\frac{1}{1+x^2}dx+\lim_{d\to+\infty}\int_0^d\frac{1}{1+x^2}dx$

$=\lim_{c\to-\infty}(-\operatorname{Arctan} c)+\lim_{d\to+\infty}(\operatorname{Arctan} d)=\frac{\pi}{2}+\frac{\pi}{2}=\pi$, so $\frac{1}{1+x^2}$ is improper integrable on $(-\infty,\infty)$.

By comparison test, $\left|\frac{\cos 3x}{1+x^2}\right|$ is improper integrable on $(-\infty,\infty)$. By the absolute convergence test, $\frac{\cos 3x}{1+x^2}$ is improper integrable on $(-\infty,\infty)$. So $\int_{-\infty}^{\infty}\frac{\cos 3x}{1+x^2}dx$ exists.

(b) Since $\int_{-\infty}^{\infty}\frac{\cos 3x}{1+x^2}dx$ is improper integrable on $(-\infty,\infty)$, so P.V. $\int_{-\infty}^{\infty}\frac{\cos 3x}{1+x^2}dx$ exists.

(c) $\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}dx=\lim_{c\to0^-}\int_{-1}^{c}\frac{1}{\sqrt[3]{x}}dx+\lim_{d\to0^+}\int_d^1\frac{1}{\sqrt[3]{x}}dx=\lim_{c\to0^-}\left(\frac{3}{2}x^{2/3}\right)\Big|_{-1}^{c}+\lim_{d\to0^+}\left(\frac{3}{2}x^{2/3}\right)\Big|_d^1=-\frac{3}{2}+\frac{3}{2}=0$

(d) Since $\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}dx$ exists as an improper integral, so PV $\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}dx$ also exist.

Alternatively, PV $\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}dx=\lim_{\varepsilon\to0^+}\left(\int_{-1}^{-\varepsilon}\frac{1}{\sqrt[3]{x}}dx+\int_{\varepsilon}^{1}\frac{1}{\sqrt[3]{x}}dx\right)=\lim_{\varepsilon\to0^+}0=0$

$1/\sqrt[3]{x}$ is an odd function.

(e) $\int_0^{\infty}\sin x\,dx=\lim_{c\to+\infty}\int_0^c\sin x\,dx=\lim_{c\to+\infty}(-\cos x)\Big|_0^c=\lim_{c\to+\infty}(-\cos c+1)$ doesn't exist.

So $\int_{-\infty}^{\infty}\sin x\,dx$ doesn't exist.

(f) PV $\int_{-\infty}^{\infty}\sin x\,dx=\lim_{c\to+\infty}\int_{-c}^{c}\sin x\,dx=\lim_{c\to+\infty}(-\cos x)\Big|_{-c}^{c}=\lim_{c\to+\infty}0=0$.

Solutions to Math 202 Past Exam Problems (Part I)

(128) Since $0+2 \le w+z \le 1+3$, we have $7 = 2^2+3 \le f(w+z) \le 4^2+3 = 19$. So 7 is a lower bound of S and 19 is an upper bound of S.

Let $w_n = 0$ and $z_n = 2+\frac{1}{\sqrt{2}n}$, then $f(w_n+z_n) \in S$ and $f(w_n+z_n) = (w_n+z_n)^2+3$ converges to 7. $\therefore$ $\inf S = 7$ by infimum limit theorem. Next let $w_n = 1$ and $z_n = 3-\frac{1}{\sqrt{2}n}$, then $f(w_n+z_n) \in S$ and $f(w_n+z_n) = (w_n+z_n)^2+3$ converges to 19. $\therefore$ $\sup S = 19$ by supremum limit theorem.

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(129) $\lim_{k\to\infty} \frac{2^{k+1}\sqrt{k+1}}{(2k+2)!}\,\frac{(2k)!}{2^k\sqrt{k}} = \lim_{k\to\infty} \frac{1}{(2k+1)\sqrt{k(k+1)}} = 0 < 1 \Rightarrow \sum_{k=1}^{\infty} \frac{2^k\sqrt{k}}{(2k)!}$ converges by ratio test.

$|(\cos k)(\sin\frac{1}{k^2})| \le \sin(\frac{1}{k^2})$; $\lim_{k\to\infty} \frac{\sin(\frac{1}{k^2})}{\frac{1}{k^2}} \underset{\theta=\frac{1}{k^2}}{=} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by p-test ($p=2>1$)

$\Rightarrow \sum_{k=1}^{\infty} \sin(\frac{1}{k^2})$ converges by limit comparison test $\Rightarrow \sum_{k=1}^{\infty}(\cos k)(\sin\frac{1}{k^2})$ converges by the comparison test and the absolute convergence test.

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(130) Let $W = \{(a,b,c,d) : a,b,c,d \in \mathbb{Q},\ (a,b) \ne (c,d)\}$. Then $W \subseteq \underbrace{\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}}_{\text{countable by product theorem}}$ implies W is countable. For $(a,b,c,d) \in W$, let $L_{(a,b,c,d)}$ be the line through (a,b) and (c,d).

Then $S = \bigcup_{(a,b,c,d)\in W} \underbrace{\{L_{(a,b,c,d)}\}}_{1 \text{ element} \Rightarrow \text{countable}}$ is countable by countable union theorem.

Let $V = \{(L,L') : L, L' \in S,\ L \ne L'\}$. Then $V \subseteq \underbrace{S\times S}_{\text{countable by product theorem}}$ implies V is countable.

Now $T = \bigcup_{(L,L')\in V} \underbrace{(L\cap L')}_{0 \text{ or } 1 \text{ element} \Rightarrow \text{countable}}$ is countable by countable union theorem.

Alternatively, each L in S has equation of the form $ax+by=c$ for some $a,b,c \in \mathbb{Q}$. The intersection point of two such lines (if they intersect) is in $\mathbb{Q}\times\mathbb{Q}$ (by algebra or Cramer's rule). So $T \subseteq \mathbb{Q}\times\mathbb{Q}$. Hence T is countable.

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(131) For every $\varepsilon > 0$, since $\lim_{n\to\infty} x_n = 0$, $\exists K_1$ such that $n \ge K_1 \Rightarrow |x_n - 0| < \frac{1}{2} \Leftrightarrow x_n \in (-\frac{1}{2}, \frac{1}{2})$ $\Leftrightarrow 1 + x_n \in (\frac{1}{2}, \frac{3}{2}) \Leftrightarrow \frac{1}{1+x_n} \in (\frac{2}{3}, 2)$ and $\exists K_2$ such that $n \ge K_2 \Rightarrow |x_n - 0| < \frac{\varepsilon}{2}$. Let $K = \max(K_1, K_2)$. Then $n \ge K \Rightarrow n \ge K_1$ and $n \ge K_2 \Rightarrow \left|\frac{x_n}{1+x_n} - 0\right| = \frac{|x_n|}{1+x_n} \le 2|x_n| < \varepsilon$.

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(132) For every $\varepsilon > 0$, since $\lim_{n\to\infty} x_n = w$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |x_n - w| < \varepsilon$.

In particular, for $n = 2K_1 \ge K_1$, we have $x_{2K_1} < w$ and $|x_{2K_1} - w| = w - x_{2K_1} < \varepsilon$.

Let $\underline{K = 2K_1 - 1}$. Then for $n \ge K = 2K_1 - 1$, we have $K_1 \le \frac{n+1}{2}$, which implies $x_{2K_1} \in \{x_{2k} : k \in \mathbb{N},\ k \le \frac{n+1}{2}\}$. So $x_{2K_1} \le y_n < w \Rightarrow \underbrace{|y_n - w|}_{= w - y_n} \le \underbrace{|x_{2K_1} - w|}_{= w - x_{2K_1}} < \varepsilon$.

finite set ↑ ↑ bounded above by w

$\Rightarrow y_n < w$

(133) $w \in (\frac{\pi}{4}, \frac{\pi}{3}) \setminus \mathbb{Q} \Rightarrow \frac{\sqrt{2}}{2} \le f(w) \le \frac{\sqrt{3}}{2}$. For $n \in \mathbb{N}$, $\frac{\sqrt{2}}{2} - 1 \le f(w) - \frac{1}{n} \le \frac{\sqrt{3}}{2}$. So $\frac{\sqrt{2}}{2} - 1$ is a lower bound of S and $\frac{\sqrt{3}}{2}$ is an upper bound of S.

Let $w_k = \frac{\pi}{4} + \frac{\pi}{k+12}$ and $n = 1$, then $f(w_k) - \frac{1}{1} \in S$ and $f(w_k) - \frac{1}{1} \to \frac{\sqrt{2}}{2} - 1$ $\therefore \frac{\sqrt{2}}{2} - 1 = \inf S$ by infimum limit theorem. Next let $w_k = \frac{\pi}{3} - \frac{\pi}{k+12}$ and $n = k$, then $f(w_k) - \frac{1}{k} \in S$ and $f(w_k) - \frac{1}{k} \to \frac{\sqrt{3}}{2} - 0 = \frac{\sqrt{3}}{2}$. $\therefore \frac{\sqrt{3}}{2} = \sup S$ by supremum limit theorem.

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(134) $\lim_{k\to\infty} \frac{(2k+3)^5}{(k+1)!} \frac{k!}{(2k+1)^5} = \lim_{k\to\infty} \left(\frac{2k+3}{2k+1}\right)^5 \frac{1}{k+1} = 0 < 1 \Rightarrow \sum_{k=1}^{\infty} \frac{(2k+1)^5}{k!}$ converges by ratio test.

$\left|\frac{\cos k}{k^4+k+1}\right| \le \frac{1}{k^4+k+1} \le \frac{1}{k^4}$ and $\sum_{k=1}^{\infty} \frac{1}{k^4}$ converges by p-test $\underset{p=4>1}{}$ $\Rightarrow \sum_{k=1}^{\infty} \left|\frac{\cos k}{k^4+k+1}\right|$ converges by comparison test $\Rightarrow \sum_{k=1}^{\infty} \frac{\cos k}{k^4+k+1}$ converge by absolute convergence test.

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(135) In the case $x = 0$, S will contain all circles passing through $(1,1)$ and $(0,0)$. For every point (other than $(\frac{1}{2}, \frac{1}{2})$) on the perpendicular bisector of the segment joining $(0,0)$ to $(1,1)$, it can be used as a center for a unique such circle as it is equal distance from $(0,0)$ and $(1,1)$. As the perpendicular bisector minus $(\frac{1}{2}, \frac{1}{2})$ contains uncountably many points, S will contain uncountably many circles. So S is uncountable.

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(136) For every $\varepsilon > 0$, since $\lim_{n\to\infty} x_n = 0$, $\exists K_1$ such that $n \ge K_1 \Rightarrow |x_n - 0| < \frac{1}{2} \Leftrightarrow x_n \in (-\frac{1}{2}, \frac{1}{2})$ $\Leftrightarrow x_n - 1 \in (-\frac{3}{2}, -\frac{1}{2}) \Leftrightarrow \left|\frac{1}{x_n - 1}\right| = \frac{1}{1 - x_n} \in (\frac{2}{3}, 2)$ and $\exists K_2$ such that $n \ge K_2 \Rightarrow |x_n - 0| < \frac{\varepsilon}{2}$.

Let $K = \max(K_1, K_2)$. Then $n \ge K \Rightarrow n \ge K_1$ and $n \ge K_2 \Rightarrow \left|\frac{x_n}{x_n - 1} - 0\right| = \frac{|x_n|}{1 - x_n} \le 2|x_n| < \varepsilon$.

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(137) For every $\varepsilon > 0$, since $\lim_{n\to\infty} x_n = w$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |x_n - w| < \varepsilon$. In particular, for $n = 2K_1 \ge K_1$, we have $x_{2K_1} > w$ and $|x_{2K_1} - w| = x_{2K_1} - w < \varepsilon$.

Let $\underline{\underline{K = 2K_1 - 1}}$. Then for $n \ge K = 2K_1 - 1$, we have $K_1 \le \frac{n+1}{2}$, which implies $x_{2K_1} \in \{x_{2k} : k \in \mathbb{N}, k \le \frac{n+1}{2}\}$. So $x_{2K_1} \ge y_n > w \Rightarrow \underset{y_n - w}{|y_n - w|} \le \underset{x_{2K_1} - w}{|x_{2K_1} - w|} < \varepsilon$

finite set; bounded below by w $\Rightarrow y_n > w$

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(138) For $n\in\mathbb{N}$, $1+(-1)^n=0$ or 2. For $w\in[1,\sqrt{2})\setminus\mathbb{Q}$, $1\le w\le\sqrt{2}$. So we have $1=0^2+1^2\le f(1+(-1)^n,w)\le 2^2+(\sqrt{2})^2=6$. So 1 is a lower bound of S and 6 is an upper bound of S.

Let $n=1$, $w_k=1+\frac{\sqrt{2}-1}{k+1}$, then $f(1+(-1)^n,w_k)=f(0,w_k)\in S$ and $f(0,w_k)=w_k^2\to 1$. $\therefore 1=\inf S$ by infimum limit theorem. Next, let $n=2$, $w_k=\sqrt{2}-\frac{1}{k+2}$, then $f(1+(-1)^n,w_k)=f(2,w_k)\in S$ and $f(2,w_k)=2^2+w_k^2\to 2^2+(\sqrt{2})^2=6$. $\therefore 6=\sup S$ by supremum limit theorem.

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(139) $\lim_{k\to\infty}\frac{2^{k+1}}{(k+1)^3(3k+3)!}\cdot\frac{k^3(3k)!}{2^k}=\lim_{k\to\infty}2\left(\frac{k}{k+1}\right)^3\frac{1}{(3k+3)(3k+2)(3k+1)}=0<1\Rightarrow\sum_{k=1}^{\infty}\frac{2^k}{k!(3k)!}$ converges by ratio test.

$\left|\left(\frac{1}{e}+\frac{1}{k}\right)^k\sin k\right|\le\left(\frac{1}{e}+\frac{1}{k}\right)^k$ and $\lim_{k\to\infty}\sqrt[k]{\left(\frac{1}{e}+\frac{1}{k}\right)^k}=\lim_{k\to\infty}\left(\frac{1}{e}+\frac{1}{k}\right)=\frac{1}{e}<1\Rightarrow\sum_{k=1}^{\infty}\left(\frac{1}{e}+\frac{1}{k}\right)^k$ converges by root test $\Rightarrow\sum_{k=1}^{\infty}\left|\left(\frac{1}{e}+\frac{1}{k}\right)^k\sin k\right|$ converges by comparison test $\Rightarrow\sum_{k=1}^{\infty}\left(\frac{1}{e}+\frac{1}{k}\right)^k\sin k$ converges by absolute convergence test.

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(140) $x_1=1<x_2=\frac{\sqrt{1^2+4}}{2}=\frac{\sqrt{5}}{2}<x_3=\frac{\sqrt{\frac{5}{4}+2\sqrt{5}}}{2}$ $\qquad x=\frac{\sqrt{x^2+4x}}{2}\Rightarrow 4x^2=x^2+4x\Rightarrow x=0,\frac{4}{3}$.

$\frac{5}{4}+2\sqrt{5}>1+2\sqrt{4}=5$ $\qquad 5=(\sqrt{5})^2<\left(\frac{8}{3}\right)^2=\frac{64}{9}$ rejected

$\therefore$ We will show $x_n\le x_{n+1}\le\frac{4}{3}$ by math induction. For $n=1$, $x_1=1<x_2=\frac{\sqrt{5}}{2}<\frac{4}{3}$.

Suppose $x_n\le x_{n+1}\le\frac{4}{3}$. Then $x_n^2+4x_n\le x_{n+1}^2+4x_{n+1}\le\left(\frac{4}{3}\right)^2+4\left(\frac{4}{3}\right)$. Taking square root and dividing by 2, we get $x_{n+1}\le x_{n+2}\le\frac{4}{3}$ completing the induction. This shows $\{x_n\}$ is increasing and bounded above. By monotone sequence theorem, $\{x_n\}$ converges to some $x\in\mathbb{R}$. Now $x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\frac{\sqrt{x_n^2+4x_n}}{2}=\frac{\sqrt{x^2+4x}}{2}\Rightarrow 4x^2=x^2+4x\Rightarrow x=\frac{4}{3}$ or 0

rejected as $x_1>0$

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(141) Let T be the set of all circles on the coordinate plane with center $(x,y)\in\mathbb{Q}\times\mathbb{Q}$ and radius $r\in\mathbb{Q}^+$. Then $T=\bigcup_{(x,y,r)\in\underbrace{\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}^+}_{\text{countable}}}\{C_{(x,y,r)}\}$ is countable.

$\mathbb{Q}^+ = \mathbb{Q}\cap(0,\infty)$ $\qquad C_{(x,y,r)}$: one circle with center (x,y), radius r

Now $S\subseteq\underbrace{(\mathbb{Q}\times\mathbb{Q})\times T}_{\text{countable by product theorem}}$ implies S is countable.

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(142) For every $\varepsilon>0$, since $\lim_{n\to\infty}x_n=1$, $\exists K_1$ such that $n\ge K_1\Rightarrow|x_n-1|<1\Leftrightarrow x_n\in(0,2)$ and $\exists K_2$ such that $n\ge K_2\Rightarrow|x_n-1|<\frac{\varepsilon}{3}$. Let $K=\max(K_1,K_2)$. Then $n\ge K$ $\Rightarrow n\ge K_1$ and $n\ge K_2\Rightarrow|x_n^2-1|=|x_n-1|(x_n+1)\le 3|x_n-1|<\varepsilon$. $\qquad x_n+1\in(1,3)$

(143) (a) Since $\{x_n\}$ is bounded, $\exists M$ such that $|x_n| \le M \iff -M \le x_n \le M$ for all n. Then $-M-1 \in S$ and S is bounded above by M.

(b) By supremum limit theorem, $\exists\, w_j \in S$ and $w_j \to s = \sup S$.
$\forall j \in \mathbb{N}$; $s+\frac{1}{j} > \sup S \Rightarrow s+\frac{1}{j} \notin S \iff$ there are only finitely many $n \in \mathbb{N}$ such that $s+\frac{1}{j} < x_n$.
Since $w_j \in S$, there are infinitely many $n \in \mathbb{N}$ such that $w_j < x_n$ and of these x_n's, only finitely many satisfy $s+\frac{1}{j} < x_n$. So there are infinitely many $x_n \in (w_j, s+\frac{1}{j}]$ for every j. Now pick $x_{n_1} \in (w_1, s+1]$. Once x_{n_j} is picked in $(w_j, s+\frac{1}{j}]$, since there are infinitely many x_n's in $(w_{j+1}, s+\frac{1}{j+1}]$ we can pick a $x_{n_{j+1}}$ with $n_{j+1} > n_j$ in $(w_{j+1}, s+\frac{1}{j+1}]$. So $\{x_{n_j}\}$ is a subsequence of x_n. Now $w_j \to s$, $s+\frac{1}{j} \to s$ and $w_j < x_{n_j} < s+\frac{1}{j}$ imply $x_{n_j} \to s$ by sandwich theorem.

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(144) For positive b, since $\lim_{k\to\infty} \sqrt[k]{(b+\frac{1}{k})^k} = \lim_{k\to\infty}(b+\frac{1}{k}) = b$, by root test, the series will converge if $b<1$ and will diverge if $b>1$. When $b=1$, $(b+\frac{1}{k})^k \ge 1^k = 1$ so that $\lim_{k\to\infty}(b+\frac{1}{k})^k \ne 0$. By term test, the series will diverge if $b=1$.
The answer is $0<b<1$.

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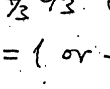
(145) For $a \in \mathbb{R}$, $S_a = \{\theta : \theta \in \mathbb{R}, \sin\theta = a\} = \bigcup_{\substack{n\in\mathbb{Z} \\ \uparrow \\ \text{countable}}} \underbrace{\{\theta : \theta \in [2n\pi, 2(n+1)\pi), \sin\theta = a\}}_{\text{at most 2 elements hence countable}}$
is countable by union theorem.



So $S = \bigcup_{a\in A} \underset{\nwarrow \text{countable}}{S_a}$ is countable by union theorem.
Next $T = \mathbb{R} \setminus S$ is uncountable since $\mathbb{R}$ is uncountable and S is countable.

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(146) ($x_1 = 6$, $x_2 = \frac{40}{15} = \frac{8}{3}$, $x_3 = \dfrac{\frac{64}{9}+4}{\frac{16}{3}+3} = \dfrac{64+36}{48+27} = \dfrac{100}{75} = \dfrac{4}{3}$ $\quad \frac{4}{3}\ \ \frac{8}{3}\ \ 6$ Suspect $\{x_n\}$ is decreasing.
$x = \dfrac{x^2+4}{2x+3} \Rightarrow 2x^2+3x = x^2+4 \Rightarrow x^2+3x-4=0 \Rightarrow x = 1$ or -4.)
We claim $x_n \ge 1$ and $x_n \ge x_{n+1}$ for $n=1,2,\cdots$. Note $x_1 = 6 \ge 1$ and suppose $x_n \ge 1$, then
$x_{n+1} = \dfrac{x_n^2+4}{2x_n+3} \ge 1 \overset{x_n \ge 1}{\iff} x_n^2+4 \ge 2x_n+3 \iff x_n^2-2x_n+1 = (x_n-1)^2 \ge 0$, which is true.
So by induction $x_n \ge 1$ for $n=1,2,\cdots$. Next, for $n=1,2,\cdots$
$$x_n - x_{n+1} = x_n - \frac{x_n^2+4}{2x_n+3} = \frac{2x_n^2+3x_n-(x_n^2+4)}{2x_n+3} = \frac{x_n^2+3x_n-4}{2x_n+3} = \frac{(x_n+4)(x_n-1)}{2x_n+3} \overset{x_n\ge 1}{\ge} 0$$
So $x_n \ge x_{n+1}$. By monotone sequence theorem, since $\{x_n\}$ is decreasing and bounded below by 1, we see $\{x_n\}$ converges to some $x \in \mathbb{R}$. Then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n^2+4}{2x_n+3} = \frac{x^2+4}{2x+3}$ implies $x = 1$ or -4. Since $x_n \ge 1$, we have $x = 1$.

(147) Solution 1

Let $y_n = \frac{x_n}{x_n+1}$, then by algebra, $x_n = \frac{y_n}{1-y_n}$ and $x_n - 1 = \frac{2y_n-1}{1-y_n} = \frac{2(y_n-\frac{1}{2})}{1-y_n}$.

Note if $|y_n - \frac{1}{2}| < \frac{1}{3}$, then $y_n \in (\frac{1}{6}, \frac{5}{6})$ so that $1-y_n \in (\frac{1}{6}, \frac{5}{6})$, $|x_n - 1| \le \frac{2|y_n-\frac{1}{2}|}{1/6}$.

For every $\varepsilon > 0$, since $\lim_{n\to\infty} y_n = \frac{1}{2}$, there is K_1 such that $n \ge K_1 \Rightarrow |y_n - \frac{1}{2}| < \frac{1}{3}$ and there is K_2 such that $n \ge K_2 \Rightarrow |y_n - \frac{1}{2}| < \frac{\varepsilon}{12}$. Let $K = \max(K_1, K_2)$, We have $n \ge K \Rightarrow n \ge K_1$ and $n \ge K_2 \Rightarrow |x_n - 1| \le \frac{2|y_n - \frac{1}{2}|}{1/6} = 12|y_n - \frac{1}{2}| < \varepsilon$.

Solution 2 Since $\lim_{n\to\infty} \frac{x_n}{x_n+1} = \frac{1}{2}$, $\exists K_1 \in \mathbb{N}$ such that $|\frac{x_n}{x_n+1} - \frac{1}{2}| < \frac{1}{4}$. Then

$\frac{x_n}{x_n+1} - \frac{1}{2} < \frac{1}{4} \Rightarrow \frac{x_n}{x_n+1} < \frac{3}{4} \Rightarrow \frac{1}{x_n+1} > 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow 0 < x_n + 1 < 4$. Next,

$\exists K_2 \in \mathbb{N}$ such that $\left|\frac{x_n}{x_n+1} - \frac{1}{2}\right| = \left|\frac{x_n - 1}{2(x_n+1)}\right| < \frac{\varepsilon}{8}$. Let $K = \max(K_1, K_2)$.

Then $n \ge K \Rightarrow n \ge K_1$ and $n \ge K_2 \Rightarrow |x_n - 1| = \left|\frac{x_n-1}{2(x_n+1)}\right| \cdot 2|x_n+1| < \frac{\varepsilon}{8} \cdot 2 \cdot 4 = \varepsilon$.

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(148) Let $V = \{\cos c : c \in S\}$ Since $S \subseteq [0, \frac{\pi}{2}]$, $U, V \subseteq [0,1]$, $x = \sup V$ and $y = \inf U$ exist in $\mathbb{R}$.
By Supremum limit theorem, $\exists \cos c_n \in V$ (with $c_n \in S$) such that $\lim_{n\to\infty} \cos c_n = x$. For $a, b \in S$,
$\cos^2 a + \cos^2 b \le 2x^2$ and $\lim_{n\to\infty} (\cos^2 c_n + \cos^2 c_n) = 2x^2$ imply $2x^2 = \sup T = \frac{1}{2}$. Hence $x = \frac{1}{2}$.
For $c \in S$, $\sin c = \sqrt{1-\cos^2 c} \ge \sqrt{1 - (\frac{1}{2})^2} = \frac{\sqrt{3}}{2}$ Since $\lim_{n\to\infty} \sin c_n = \lim_{n\to\infty} \sqrt{1-\cos^2 c_n} = \frac{\sqrt{3}}{2}$ We have $\inf U = \frac{\sqrt{3}}{2}$.

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(149) For $m \in \mathbb{Z} \setminus \{-1, 1\}$, let $S_m = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}, x^2 + my^2 = 1 \text{ and } mx^2 + y^2 = 1\}$.
Then S_m has at most 4 elements because $y^2 = 1 - mx^2 \Rightarrow \underbrace{x^2 + m(1-mx^2) = 1}_{(1-m^2)x^2 + m = 1}$

$(1-m^2)x^2 + m = 1$ has at most 2 solutions for x.

So $S = \underbrace{\bigcup_{m \in \mathbb{Z} \setminus \{-1,1\}}}_{\text{Countable}} \overbrace{S_m}^{\text{Countable}}$ is countable.

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(150) ($x_1 = 9$, $x_2 = \frac{\sqrt{9} + 2\cdot 9}{3} = 7$, $x_3 = \frac{\sqrt{7} + 2\cdot 7}{3} < \frac{3 + 2\cdot 7}{3} = 5\frac{2}{3}$. Suspect $\{x_n\}$ decreasing
$x = \frac{\sqrt{x} + 2x}{3} \Rightarrow x = \sqrt{x} \Rightarrow x = 0$ or 1.)
We claim: $x_n > x_{n+1} > 1$. For $n=1$, $x_1 = 9 > x_2 = 7 > 1$. Suppose $x_n > x_{n+1} > 1$
Then $\sqrt{x_n} > \sqrt{x_{n+1}} > 1$. So $\frac{\sqrt{x_n} + 2x_n}{3} > \frac{\sqrt{x_{n+1}} + 2x_{n+1}}{3} > \frac{\sqrt{1} + 2}{3}$, i.e. $x_{n+1} > x_{n+2} > 1$.
By the monotone sequence theorem, $\{x_n\}$ converges to some x. Then
$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{\sqrt{x_n} + 2x_n}{3} = \frac{\sqrt{x} + 2x}{3} \Rightarrow x = \sqrt{x} \Rightarrow x = 0$ or 1. Since
$x_n > 1$, $x \ge 1$. Therefore the answer is 1.

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(151) $\lim_{k\to\infty} \sqrt[k]{\frac{(b+a_k)^k}{k}} = \lim_{k\to\infty} \frac{b+a_k}{\sqrt[k]{k}} = b$ (because $\sum a_k$ converges $\Rightarrow \lim_{k\to\infty} a_k = 0$ and $\lim_{k\to\infty} \sqrt[k]{k} = 1$.)
So if $0 < b < 1$, then $\sum_{k=1}^{\infty} \frac{(b+a_k)^k}{k}$ converges. If $b > 1$, then $\sum_{k=1}^{\infty} \frac{(b+a_k)^k}{k}$ diverges
If $b = 1$, then $\frac{(b+a_k)^k}{k} = \frac{(1+a_k)^k}{k} > \frac{1}{k}$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by p-test, $p=1$, so
$\sum_{k=1}^{\infty} \frac{(b+a_k)^k}{k}$ diverges.

(152) $\lim_{k\to\infty} x_{2k} = 0.5 \Rightarrow$ for $\varepsilon_0 = 0.2$, $\exists K_0 \in \mathbb{N}$, $k \ge K_0 \Rightarrow |x_{2k} - 0.5| < \varepsilon_0 \Rightarrow x_{2k} \in (0.3, 0.7)$

$\lim_{k\to\infty} x_{2k+1} = 0.6 \Rightarrow$ for $\varepsilon_1 = 0.1$, $\exists K_1 \in \mathbb{N}$, $k \ge K_1 \Rightarrow |x_{2k+1} - 0.6| < \varepsilon_1 \Rightarrow x_{2k+1} \in (0.4, 0.7)$.

$\forall \varepsilon > 0$, $(0.7)^n \le \varepsilon \Leftrightarrow n \ge \left\lceil \frac{\ln \varepsilon}{\ln 0.7} \right\rceil$. Let $K = \max\left(2K_0, 2K_1+1, \left\lceil \frac{\ln \varepsilon}{\ln 0.7} \right\rceil\right)$.

Then $n \ge K \Rightarrow n \ge 2K_0$ and $n \ge 2K_1 + 1$ and $n \ge \left\lceil \frac{\ln \varepsilon}{\ln 0.7} \right\rceil$

<u>Case n is even</u> $n = 2k \ge 2K_0 \Rightarrow k \ge K_0 \Rightarrow |x_n^n - 0| = x_{2k}^{2k} < (0.7)^n \le \varepsilon$

<u>Case n is odd</u> $n = 2k+1 \ge 2K_1 + 1 \Rightarrow k \ge K_1 \Rightarrow |x_n^n - 0| = x_{2k+1}^{2k+1} < (0.7)^n \le \varepsilon$.

So $n \ge K \Rightarrow |x_n^n - 0| < \varepsilon$.

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(153) $\forall t \in I$, $A_t \subseteq A \Rightarrow x_t = \sup A_t \le \sup A \Rightarrow \sup\{x_t : t \in I\} \le \sup A$. Conversely, by supremum limit theorem, $\exists\, c_n \in A$ such that $\lim_{n\to\infty} c_n = \sup A$. $\forall n$, $c_n \in A = \bigcup_{t\in I} A_t$ $\Rightarrow c_n \in A_t$ for some $t \in I \Rightarrow c_n \le \sup A_t = x_t \le \sup\{x_t : t \in I\} \Rightarrow \sup A = \lim_{n\to\infty} c_n \le \sup\{x_t : t \in I\}$.

$\therefore \sup A = \sup\{x_t : t \in I\}$.

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(154) Since $\left|\frac{\cos 3x}{\sqrt{x}}\right| \le \frac{1}{\sqrt{x}}$ and $\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1 = 2$, so by the comparison test, $\int_0^1 \left|\frac{\cos 3x}{\sqrt{x}}\right| dx$ exists. By the absolute convergence theorem, $\int_0^1 \frac{\cos 3x}{\sqrt{x}}$ exists.

---

(155) Assume such continuous function f exists. If $f(a) = f(b)$, then $-a = f(f(a)) = f(f(b))$ $= -b \Rightarrow a = b$. So f is injective. By the continuous injection theorem, f is strictly monotone on $\mathbb{R}$. If f is strictly increasing, then $x < y \Rightarrow f(x) < f(y) \Rightarrow f(f(x)) < f(f(y))$. If f is strictly decreasing, then $x < y \Rightarrow f(x) > f(y) \Rightarrow f(f(x)) < f(f(y))$. In both cases, we have $f(f(x))$ is strictly increasing. However, $f(f(x)) = -x$ is not strictly increasing, a contradiction. Therefore no such continuous function f exists.

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(156) Since $\{x_n\}$ is Cauchy, for every $\varepsilon > 0$, there exists K such that $m, n \ge K$ implies $|x_m - x_n| < \frac{\varepsilon}{5}$. Then $|\sin 5x_m - \sin 5x_n| \underset{\substack{\uparrow \\ \text{mean-value theorem}}}{=} |5\cos t\,(x_m - x_n)| \le 5|x_m - x_n| < \varepsilon$.

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(157) Since f, g are Riemann integrable on $[0,2]$, $\max(f,g) = \frac{f+g+|f-g|}{2}$, $\min(f,g) = \frac{f+g-|f-g|}{2}$ are also integrable by theorem and remark in notes. By Lebesgue's theorem,

$S_{max} = \{x \in [0,2] : \max(f,g) \text{ discontinuous at } x\}$
$S_{min} = \{x \in [0,2] : \min(f,g) \text{ discontinuous at } x\}$ are both of measure 0.

Then $S_{max} \cap [0,1)$ and $S_{min} \cap (1,2]$ are both of measure 0. Since

$$S_h = \{x \in [0,2] : h \text{ is discontinuous at } x\} \subseteq \underbrace{(S_{max} \cap [0,1)) \cup (S_{min} \cap (1,2]) \cup \{1\}}_{\text{of measure } 0},$$

we see S_h is of measure 0. By Lebesgue's theorem, h is Riemann integrable on $[0,2]$.

Remarks

f, g Riemann integrable on $[0,2] \Rightarrow f, g$ bounded on $[0,2] \Rightarrow h$ bounded on $[0,2]$.

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(158) $\sum_{k=1}^{\infty} \sin^k(1+\frac{1}{k})$ Root Test $\lim_{k\to\infty} \sqrt[k]{|\sin^k(1+\frac{1}{k})|} = \lim \sin(1+\frac{1}{k}) = \sin 1 < 1 \Rightarrow$ Series converges.

Comparison Test Since $0 \le \sin^k(1+\frac{1}{k}) \le \sin^k(1\frac{1}{2})$ and $0 < \sin(1\frac{1}{2}) < 1$, so $\sum_{k=1}^{\infty} \sin^k(1\frac{1}{2})$ converges by geometric series test. Hence $\sum_{k=1}^{\infty} \sin^k(1+\frac{1}{k})$ converges by comparison test.

$\sum_{k=1}^{\infty} \frac{1-\cos(1/k)}{1/k^2}$ Term Test $\lim_{k\to\infty} \frac{1-\cos(1/k)}{1/k^2} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \lim \frac{\sin\theta}{2\theta} = \frac{1}{2} \ne 0 \Rightarrow$ Series diverges

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(159) $\inf A = 0$ and $\sup A = 1$ imply $A \subseteq [0,1]$. On $[0,1]$, $\frac{d}{dx}(x^3-4x+1) = 3x^2-4 = 3(x^2-\frac{4}{3}) < 0$.
So $f(x) = x^3-4x+1$ is strictly decreasing on $[0,1]$. So for every $a \in A$,
$f(1) = -2 \le f(a) = a^3-4a+1 \le f(0) = 1$. By infimum limit theorem and supremum limit theorem, there are sequences $\{r_n\}, \{s_n\}$ such that $\lim_{n\to\infty} r_n = 0$, $\lim_{n\to\infty} s_n = 1$ due to $\inf A = 0$ and $\sup A = 1$. Then $\{f(r_n)\}, \{f(s_n)\}$ are sequences in S such that $\lim_{n\to\infty} f(r_n) = 1$ and $\lim_{n\to\infty} f(s_n) = -2$. By infimum limit theorem and supremum limit theorem, $\inf S = -2$ and $\sup S = 1$.

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(160) The set $S = \{\sqrt{x^2+y^2} : (x,y) \in P\} = \bigcup_{(x,y)\in P \leftarrow \text{countable}} \{\sqrt{x^2+y^2}\}$ is countable by countable union theorem. ($\leftarrow$ 1 element, hence countable)

Then $(0,\infty) \setminus S$ is uncountable; in particular, nonempty.

Let $r \in \underbrace{(0,\infty)}_{\text{uncountable}} \setminus S$. The circle C with the origin as center and radius $r > 0$ contains no point in P as every point (x,y) in P has distance $\sqrt{x^2+y^2} \ne r$ from origin.

(161) (a) Let $A_r=\{f(t): 0<|t-w|<r\}$. Note $0<r_1\le r_2 \Rightarrow A_{r_1}\subseteq A_{r_2}$, which implies

$u\le \underset{m(r_2)}{\underset{\|}{\inf A_{r_2}}} \le \underset{m(r_1)}{\underset{\|}{\inf A_{r_1}}} \le \underset{M(r_1)}{\underset{\|}{\sup A_{r_1}}} \le \underset{M(r_2)}{\underset{\|}{\sup A_{r_2}}} \le v$. Hence $M(r)$ is increasing and $m(r)$ is decreasing on $(0,\infty)$. By the monotone function theorem*, $\lim_{r\to 0+} m(r)=m(0+)$ and $\lim_{r\to 0+} M(r)=M(0+)$ exist.

*Remark For $r\le 0$, we may define $M(r)=u$ and $m(r)=v$ so that $M(r)$ is increasing and $m(r)$ is decreasing on $(-\infty,\infty)$.

(b) If $\lim_{r\to 0+} m(r)=L=\lim_{r\to 0+} M(r)$, then for $x\ne w$, let $r=2|x-w|$ to get $0<|x-w|<r$ and $m(r)\le f(x)\le M(r)$. Now $x\to w \Rightarrow r\to 0^+ \xRightarrow{\text{Sandwich theorem}} f(x)\to L$. $f(x)\in(L-\frac{\varepsilon}{2},L+\frac{\varepsilon}{2})$

Conversely, if $\lim_{x\to w} f(x)=L$, then $\forall\varepsilon>0\ \exists\delta>0$ such that $0<|x-w|<\delta\Rightarrow|f(x)-L|<\frac{\varepsilon}{2}$.

This means $A_\delta\subseteq(L-\frac{\varepsilon}{2},L+\frac{\varepsilon}{2})$. So $0<r<\delta\Rightarrow$ $\qquad \Rightarrow x\in A_\delta=(w-\delta,w+\delta)$

$L-\varepsilon<L-\frac{\varepsilon}{2}\le m(\delta)\le m(r)\le M(r)\le M(\delta)\le L+\frac{\varepsilon}{2}<L-\varepsilon$,

i.e. $0<|r-0|<\delta\Rightarrow|m(r)-L|<\varepsilon$ and $|M(r)-L|<\varepsilon$ $\therefore\ \lim_{r\to 0+} m(r)=L=\lim_{r\to 0+} M(r)$.

One student presented the following alternative proof of the converse part:
By (a) and sequential limit theorem, $\lim_{r\to 0+} M(r)=\lim_{n\to\infty} M(\frac{1}{n})$. By the Supremum Property, $\exists t_n$ such that $0<|t_n-w|<\frac{1}{n}$ and $M(\frac{1}{n})-\frac{1}{n}<f(t_n)\le M(\frac{1}{n})$. Since $0<|t_n-w|<\frac{1}{n}$, by Sandwich theorem, $\lim_{n\to\infty} t_n=w$. By the sequential limit theorem, $\lim_{x\to w} f(x)=L\Rightarrow\lim_{n\to\infty} f(t_n)=L$. Since $f(t_n)\le M(\frac{1}{n})\le f(t_n)+\frac{1}{n}$, by Sandwich theorem, $\lim_{n\to\infty} M(\frac{1}{n})=L$. So $\lim_{r\to 0+} M(r)=L$. Similarly, $\lim_{r\to 0+} m(r)=L$.

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(162) Solution 1 $\forall\varepsilon>0$, Since $f:\mathbb{R}\to\mathbb{R}$ is uniformly continuous, $\exists\delta>0$ such that $|x-t|<\delta\Rightarrow|f(x)-f(t)|<\varepsilon$. Now $|x-t|<\delta\Rightarrow$

$$\left|\frac{1}{1+f(x)^2}-\frac{1}{1+f(t)^2}\right|=\left|\frac{f(x)^2-f(t)^2}{(1+f(x)^2)(1+f(t)^2)}\right|\le\frac{|f(x)|+|f(t)|}{(1+f(x)^2)(1+f(t)^2)}|f(x)-f(t)|$$

$$=\left(\frac{|f(x)|}{1+f(x)^2}\frac{1}{1+f(t)^2}+\frac{1}{1+f(x)^2}\frac{|f(t)|}{1+f(t)^2}\right)|f(x)-f(t)|<\left(\frac{1}{2}\cdot1+1\cdot\frac{1}{2}\right)\varepsilon=\varepsilon.$$

Note $\frac{r}{1+r^2}\le\frac{1}{2}\Leftrightarrow 0\le \underset{(1-r)^2}{\underset{\|}{1-2r+r^2}}$ is true.

Remarks If we let $a=|f(x)|$, $b=|f(t)|$, then the key step is to show $\frac{a+b}{(1+a^2)(1+b^2)}\le 1$. Multiplying by denominator and transferring terms, this is equivalent to showing $0\le(1+a^2)(1+b^2)-(a+b)=1+a^2+b^2+a^2b^2-a-b=\frac{1}{2}+(a^2-a+\frac{1}{4})+(b^2-b+\frac{1}{4})+a^2b^2$
$=\frac{1}{2}+(a-\frac{1}{2})^2+(b-\frac{1}{2})^2+a^2b^2$, which is clear.

Solution 2 Let $h(x)=\frac{1}{1+x^2}$, then $|h'(x)|=\left|\frac{2x}{(1+x^2)^2}\right|\le\frac{1+x^2}{(1+x^2)^2}=\frac{1}{1+x^2}\le 1$ $\forall x\in\mathbb{R}$.
By mean-value theorem, $\forall a,b\in\mathbb{R}$, $|h(a)-h(b)|=|h'(c)(a-b)|\le|a-b|$.
$\forall\varepsilon>0$, Since $f:\mathbb{R}\to\mathbb{R}$ is uniformly continuous, $\exists\delta>0$ such that $|x-t|<\delta\Rightarrow$ $|f(x)-f(t)|<\varepsilon$. Then $\left|\frac{1}{1+f(x)^2}-\frac{1}{1-f(t)^2}\right|=|h(f(x))-h(f(t))|\le|f(x)-f(t)|<\varepsilon$.
Therefore, $g(x)=\frac{1}{1+f(x)^2}$ is also uniformly continuous.

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(163) Since f is Riemann integrable, for every $\varepsilon>0$, $\exists$ partition $P_1=\{0=x_0<x_1<\cdots<x_n=1\}$ Such that $U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{3}$. Choose $r\in(0,x_1)$ and $r<\frac{\varepsilon}{6}$. Also choose $s\in(x_{n-1},1)$ and $1-s<\frac{\varepsilon}{6}$. Let $P=P_1\cup\{r,s\}$, then $U(f,P)-L(f,P)<\frac{\varepsilon}{3}$ by the refinement theorem. Note $-1\le g(x)\le 1$ $\forall x\in[0,1]$. Then

$$U(g,P)-L(g,P)\le\Big(\max_{x\in[0,r]}g(x)-\min_{x\in[0,r]}g(x)\Big)r+\big(U(f,P)-L(f,P)\big)+\Big(\max_{x\in[s,1]}g(x)-\min_{x\in[s,1]}g(x)\Big)(1-s)$$
$$<2\cdot\frac{\varepsilon}{6}+\frac{\varepsilon}{3}+2\cdot\frac{\varepsilon}{6}=\varepsilon.$$

By the Integral Criterion, g is Riemann integrable.

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(164) Since $\lim_{x\to0^+}\frac{(\sin x)/x^{3/2}}{1/x^{1/2}}=\lim_{x\to0^+}\frac{\sin x}{x}=1$, so $\int_0^1\frac{\sin x}{x^{3/2}}dx<\infty\iff\int_0^1\frac{1}{x^{1/2}}dx<\infty$ by the limit Comparison test. Now $\int_0^1\frac{1}{x^{1/2}}dx=2x^{1/2}\Big|_0^1=2$, so $\int_0^1\frac{\sin x}{x^{3/2}}dx<\infty$.
For $\int_1^\infty\frac{\sin x}{x^{3/2}}dx$, we have $\left|\frac{\sin x}{x^{3/2}}\right|\le\frac{1}{x^{3/2}}$ and $\int_1^\infty\frac{1}{x^{3/2}}dx<\infty$ by p-test ($p=3/2>1$), So by the Comparison test, $\int_1^\infty\left|\frac{\sin x}{x^{3/2}}\right|dx<\infty$. By the absolute convergence test, $\int_1^\infty\frac{\sin x}{x^{3/2}}dx<\infty$; therefore, $\int_0^\infty\frac{\sin x}{x^{3/2}}dx$ converges.

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(165) Let $S_f=\{x: f \text{ is discontinuous at } x\}$ and similarly for S_g and S_{fg}. By the monotone function theorem, S_f and S_g are countable sets. If f and g are continuous at x, then fg is continuous at x. Taking contrapositive, if fg is discontinuous at x, then f is discontinuous at x or g is discontinuous at x. So $S_{fg}\subseteq S_f\cup S_g$.
Since S_f, S_g countable $\underset{\uparrow}{\Longrightarrow}$ $S_f\cup S_g$ countable $\underset{\uparrow}{\Longrightarrow}$ S_{fg} countable, we are done.
Countable union theorem   Countable subset theorem

(166) We have shown in class that $h(x)=\begin{cases}1 & \text{if } x\in[0,1]\cap\mathbb{Q}\\ 0 & \text{if } x\in[0,1]\setminus\mathbb{Q}\end{cases}$ is not Riemann integrable on $[0,1]$. So it suffices to define $g(x)=\begin{cases}1 & \text{if } x=\frac{1}{n},\ n=1,2,3,\dots\\ 0 & \text{if } x\in[0,1]\setminus\{1,\frac12,\frac13,\dots\}\end{cases}$

Then $g\circ f(x)=\begin{cases}g(\frac{1}{n})=1 & \text{if } x\in[0,1]\cap\mathbb{Q}\\ g(0)=0 & \text{if } x\in[0,1]\setminus\mathbb{Q}\end{cases} = h(x)$ is not Riemann integrable on $[0,1]$.

Now g is discontinuous only on $S_g=\{1,\frac12,\frac13,\dots\}$, which is countable, hence of measure 0. So the bounded function g is Riemann integrable on $[0,1]$ by Lebesgue Theorem.

Another solution, define $g(x)=\begin{cases}0 & \text{if } x=0\\ 1 & \text{if } 0<x\le 1\end{cases}$. $S_g=\{0\}\Rightarrow$ g is Riemann integrable on $[0,1]$, $g\circ f(x)=h(x)$ is not Riemann integrable on $[0,1]$

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(167) By mean value theorem, $|\sin a-\sin b|=|(\cos c)(a-b)|\le|a-b|$ for all $a,b\in\mathbb{R}$. So $|f(x)-f(y)|\le|x^2-y^2|$. For every $\varepsilon>0$, by Archimedian principle, there exist $K>\frac{1}{\sqrt{\varepsilon}}$. If $m,n\ge K$, then

$|x_m-x_n|=|f(\frac1m)-f(\frac1n)|\le\left|\frac{1}{m^2}-\frac{1}{n^2}\right|\le\frac{1}{K^2}<\varepsilon$. $\therefore\ \{x_n\}$ is a Cauchy Sequence.

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(168) $\forall x\in(2,3]\setminus\mathbb{Q}$ and $n=1,2,3,\dots$, $\frac1x+\frac{1}{n\sqrt2}<\frac12+\frac{1}{\sqrt2}$. So $\frac12+\frac{1}{\sqrt2}$ is an upper bound of S. Let $x_n=\frac{1}{2+\frac{1}{n\sqrt2}}+\frac{1}{\sqrt2}$, then $x_n\in S$ and $\lim_{n\to\infty}x_n=\frac12+\frac{1}{\sqrt2}$. By the Supremum limit theorem, $\sup S=\frac12+\frac{1}{\sqrt2}$.

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(169) (a) $f:S\to\mathbb{R}$ is continuous at $x_0\in S$ iff $\lim_{x\to x_0}f(x)=f(x_0)$, i.e. $\forall\varepsilon>0\ \exists\delta>0$ such that $\forall x\in S$, $|x-x_0|<\delta\Rightarrow|f(x)-f(x_0)|<\varepsilon$.

(b) $f(2)=1$. $|f(x)-1|=\left|\frac{2+3x}{x^2+4}-1\right|=\frac{|x^2-3x+2|}{x^2+4}=\frac{|x-2||x-1|}{x^2+4}$. If $x\in(1,3)$, then $x-1\in(0,2)$. For every $\varepsilon>0$, let $\delta=\min(1,2\varepsilon)>0$. $\forall x\in S$, $|x-2|<\delta\Rightarrow|x-2|<1$, $|x-2|<2\varepsilon$ $\Rightarrow x\in(1,3)$, $|x-2|<2\varepsilon\Rightarrow|f(x)-1|=\frac{|x-2||x-1|}{x^2+4}\le\frac{|x-2|2}{4}<\varepsilon$.

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(170) By the mean value theorem, if $a,b>0$, then $|e^{-a}-e^{-b}|=|(-e^{-c})(a-b)|\le e^{-0}|a-b|$.
c between a and b
$\forall\varepsilon>0$, since $\{x_n\}$ is Cauchy, $\exists K$ such that $m,n\ge K\Rightarrow|x_m-x_n|<\varepsilon$. Then $|e^{-x_m}-e^{-x_n}|\le|x_m-x_n|<\varepsilon$.

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(171) (a) Since $S\setminus\{0,1\}\subseteq S$, $S\setminus\{0,1\}$ is of measure 0. For every $\varepsilon>0$, $\exists$ intervals (a_1,b_1), (a_2,b_2), (a_3,b_3), ... such that $S\setminus\{0,1\}\subseteq\bigcup_{n=1}^{\infty}(a_n,b_n)$ and $\sum_{n=1}^{\infty}|a_n-b_n|<\frac{\varepsilon}{2}$. Let $(c_n,d_n)=(a_n,b_n)\cap(0,1)$, then $S\setminus\{0,1\}\subseteq\bigcup_{n=1}^{\infty}(c_n,d_n)$ and $\sum_{n=1}^{\infty}|c_n-d_n|\le\sum_{n=1}^{\infty}|a_n-b_n|<\frac{\varepsilon}{2}$. Then $T\setminus\{0,1\}\subseteq\bigcup_{n=1}^{\infty}(c_n^2,d_n^2)$ and $\sum_{n=1}^{\infty}|c_n^2-d_n^2|=\sum_{n=1}^{\infty}|c_n+d_n||c_n-d_n|\le 2\sum_{n=1}^{\infty}|c_n-d_n|<\varepsilon$. So $T\setminus\{0,1\}$ is of measure 0. Since $T\subseteq\underbrace{(T\setminus\{0,1\})\cup\{0,1\}}_{\text{both are of measure }0}$, T is of measure 0.

(b) f and h are given to be bounded. $g(t)=\sqrt{t}$ is continuous at x^2. So if f is continuous at $g(x^2)=\sqrt{x^2}=x$, then $h=f\circ g$ is continuous at x^2. Taking contrapositive, if h is discontinuous at x^2, then f is discontinuous at x. So $S_h=\{x^2\in[0,1]: h \text{ is discontinuous at } x^2\}\subseteq\{x^2: x\in S_f\}$.

$\therefore$ f integrable $\overset{\text{by Leb. Thm}}{\Longleftrightarrow}$ S_f is of measure 0 $\overset{\text{by (a)}}{\Longrightarrow}$ $\{x^2: x\in S_f\}$ is of measure 0 $\Rightarrow$ S_h is of measure 0 $\overset{\text{by Leb. Thm}}{\Longleftrightarrow}$ h integrable.

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(172) (a) $\displaystyle\lim_{k\to\infty}\frac{3^{k+1}}{(2(k+1))!\,(k+1)!}\cdot\frac{(2k)!\,k!}{3^k}=\lim_{k\to\infty}\frac{3}{(2k+2)(2k+1)(k+1)}=0<1\Rightarrow\sum_{k=1}^{\infty}\frac{3^k}{(2k)!\,k!}$ converges by ratio test

(b) By term test, $\sum_{k=1}^{\infty}|a_k|$ converges $\Rightarrow\lim_{k\to\infty}a_k=0$. Now $\displaystyle\lim_{k\to\infty}\frac{\left|\frac{a_k}{1+a_k}\right|}{|a_k|}=\lim_{k\to\infty}\frac{1}{|1+a_k|}=1$ and $\sum_{k=1}^{\infty}|a_k|$ converges $\Rightarrow\sum_{k=1}^{\infty}\left|\frac{a_k}{1+a_k}\right|$ converges $\Rightarrow\sum_{k=1}^{\infty}\frac{a_k}{1+a_k}$ converges.

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(173) $x_1=4>x_2=4-\frac{4}{4}=3>x_3=4-\frac{4}{3}=8/3=2\frac{2}{3}$ $\quad x=4-\frac{4}{x}\Rightarrow x^2-4x+4=0\Rightarrow x=2$.

Claim: $x_n>x_{n+1}>2$. For $n=1$, $x_1=4>x_2=3>x_3=2\frac{2}{3}$ is true. Assume case n is true. Then $x_n>x_{n+1}>2\Rightarrow\frac{4}{x_n}<\frac{4}{x_{n+1}}<\frac{4}{2}\Rightarrow 4-\frac{4}{x_n}>4-\frac{4}{x_{n+1}}>4-\frac{4}{2}$ $\Rightarrow x_{n+1}>x_{n+2}>2$. $\therefore$ the claim is true for all n. By the monotone sequence theorem, $\lim_{n\to\infty}x_n=x$ exists. Then $x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}(4-\frac{4}{x_n})=4-\frac{4}{x}\Rightarrow x=2$.

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(174) We have $0-2\sqrt{2}\le x-y\sqrt{2}\le 1-(-2)\sqrt{2}=1+2\sqrt{2}$. Since $\sup A=1$ and $\inf A=0$, by the supremum limit theorem and the infimum limit theorem, $\exists\, a_n\in A$ such that $\lim_{n\to\infty}a_n=1$ and $\exists\, a_n'\in A$ such that $\lim_{n\to\infty}a_n'=0$. Then $a_n'-(2-\frac{1}{n})\sqrt{2}\in S$ and $\lim_{n\to\infty}(a_n'-(2-\frac{1}{n})\sqrt{2})=0-2\sqrt{2}=-2\sqrt{2}$ and $a_n-(-2)\sqrt{2}\in S$ and $\lim_{n\to\infty}(a_n-(-2)\sqrt{2})=1+2\sqrt{2}$. So $\inf S=-2\sqrt{2}$ (by infimum limit theorem) and $\sup S=1+2\sqrt{2}$ by the supremum limit theorem.

(175) Since $\lim_{n\to\infty} b_n = 3$, for $\varepsilon_0 = 1$, $\exists K_0 \in \mathbb{N}$ such that $n \geq K_0 \Rightarrow |b_n - 3| < 1$ $\Rightarrow b_n \in (2,4) \Rightarrow b_n + 2 \in (4,6)$. Now for $\varepsilon > 0$, $\exists K_1 \in \mathbb{N}$ such that $n \geq K_1 \Rightarrow |a_n - 2| < 2\varepsilon$ and $\exists K_2 \in \mathbb{N}$ such that $n \geq K_2 \Rightarrow |b_n - 3| < 2\varepsilon$. Let $K = \max(K_0, K_1, K_2)$. Then $n \geq K \Rightarrow n \geq K_1, K_2, K_3 \Rightarrow$

$$\left|\frac{a_n+3}{b_n+2} - 1\right| = \left|\frac{a_n+1-b_n}{b_n+2}\right| = \left|\frac{(a_n-2)-(b_n-3)}{b_n+2}\right| \leq \frac{|a_n-2|+|b_n-3|}{4} < \frac{2\varepsilon+2\varepsilon}{4} = \varepsilon.$$

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(176) (a) $\lim_{k\to\infty} \frac{(3(k+1))!}{(k+1)!\,2^{k+1}} \frac{k!\,2^k}{(3k)!} = \lim_{k\to\infty} \frac{\cancel{(3k+3)}^{3}(3k+2)(3k+1)}{\cancel{(k+1)}\,2} = \infty > 1 \Rightarrow \sum_{k=1}^{\infty} \frac{(3k)!}{k!\,2^k}$ diverges, by ratio test

(b) By term test, $\sum_{k=1}^{\infty} |a_k|$ converges $\Rightarrow \lim_{k\to\infty} a_k = 0$. Now $\lim_{k\to\infty} \frac{|a_k/\cos a_k|}{|a_k|} = \lim_{k\to\infty} \frac{1}{|\cos a_k|} = 1$ and $\sum_{k=1}^{\infty} |a_k|$ converges $\Rightarrow \sum_{k=1}^{\infty} |a_k/\cos a_k|$ converges $\Rightarrow \sum_{k=1}^{\infty} a_k/\cos a_k$ converges.

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(177) $x_1 = 1 < x_2 = \frac{1^2+15}{8} = 2 < x_3 = \frac{2^2+15}{8} = 2\frac{3}{8}$ $\quad x = \frac{x^2+15}{8} \Rightarrow x^2 - 8x + 15 = 0 \Rightarrow x = 3$ or 5

Claim: $x_n < x_{n+1} < 3$. For $n=1$, $x_1 = 1 < x_2 = 2 < x_3 = 2\frac{3}{8}$. Assume case n is true. Then $x_n < x_{n+1} < 3 \Rightarrow \frac{x_n^2+15}{8} < \frac{x_{n+1}^2+15}{8} < \frac{3^2+15}{8} \Leftrightarrow x_{n+1} < x_{n+2} < 3$.

$\therefore$ the claim is true for all n. By monotone sequence theorem, $\lim_{n\to\infty} x_n = x$ exists. Then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n^2+15}{8} = \frac{x^2+15}{8} \Rightarrow x = 3$ or 5.

Since $x_n < 3$, $x = 3$.

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(178) Note $\frac{1}{2} \leq y < 1 \Rightarrow 2 \geq \frac{1}{y} > 1$ and $-\frac{1}{2} \geq -y > -1 \Rightarrow 2 - \frac{1}{2} \geq \frac{1}{y} - y \geq 1 - 1 = 0$

For $x \in A$, $9 = 6\left(2 - \frac{1}{2}\right) \geq x\left(\frac{1}{y} - y\right) = \frac{x}{y} - xy > 2\cdot 0 = 0$. By the Supremum limit theorem, $\exists x_n \in A$ such that $\lim_{n\to\infty} x_n = 6$, then $\frac{x_n}{1/2} - x_n\left(\frac{1}{2}\right)$ has limit $6\left(2 - \frac{1}{2}\right) = 9$ and $\frac{x_n}{1-\frac{1}{n+2}} - x_n\left(1 - \frac{1}{n+2}\right)$ has limit $\frac{6}{1} - 6 = 0$. By the Supremum limit theorem, $\sup S = 9$. By the Infimum limit theorem, $\inf S = 0$.

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(179) Since $\lim_{n\to\infty} a_n = 1$, for $\varepsilon_0 = 1$, $\exists K_0 \in \mathbb{N}$ such that $n \geq K_0 \Rightarrow |a_n - 1| < 1$ $\Rightarrow a_n \in (0,2)$. Let $K = \max\left(K_0, \left[\frac{6}{\varepsilon} + 2\right]\right)$. Then $n \geq K \Rightarrow$ $n \geq K_0$ and $n \geq \frac{6}{\varepsilon} + 2 \Rightarrow \left|\frac{a_n^2+n}{n-a_n} - 1\right| = \left|\frac{a_n^2+a_n}{n-a_n}\right| < \frac{2^2+2}{n-2} = \frac{6}{n-2} \leq \varepsilon.$

(180) (a) $f(x)$ converges to L as x tends to x_0 iff for every $\varepsilon>0$, there exists $\delta>0$ such that for every $x\in S$, $0<|x-x_0|<\delta$ implies $|f(x)-L|<\varepsilon$.

(b) Solution 1 $\forall \varepsilon>0$, set $\delta=\frac{\varepsilon}{\sqrt{2}}>0$

$\forall x\in(0.5,+\infty)$

Used $|\sqrt{a}-\sqrt{b}|\le\sqrt{|a-b|}$; $x>0$

$$0<|x-1|<\delta \Rightarrow \left|\sqrt{x+\tfrac{1}{x}}-\sqrt{2}\right| \le \sqrt{\left|x+\tfrac{1}{x}-2\right|} = \sqrt{\left|\frac{x^2-2x+1}{x}\right|} = \sqrt{\frac{(x-1)^2}{x}}$$

$$= \frac{|x-1|}{\sqrt{x}} < \sqrt{2}\,|x-1| < \varepsilon$$

$x>0.5$; need $\delta=\frac{\varepsilon}{\sqrt{2}}$

Solution 2 $\forall\varepsilon>0$, set $\delta=\sqrt{\frac{\varepsilon}{\sqrt{2}}}$

$\forall x\in(0.5,+\infty)$

$|\sqrt{a}-\sqrt{b}|=\frac{|a-b|}{\sqrt{a}+\sqrt{b}}$; $\sqrt{x+\frac{1}{x}}\ge 0$; $x>0.5$

$$0<|x-1|<\delta \Rightarrow \left|\sqrt{x+\tfrac{1}{x}}-\sqrt{2}\right| = \frac{|x+\frac{1}{x}-2|}{\sqrt{x+\frac{1}{x}}+\sqrt{2}} \le \frac{(x-1)^2/x}{\sqrt{2}} < \frac{2}{\sqrt{2}}(x-1)^2 < \varepsilon$$

need $\delta=\sqrt{\varepsilon/\sqrt{2}}$

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(181) Solution 1 Let $b_n=na_n$. Then $b_{n+1}=(n+1)a_{n+1}=na_n+\frac{\cos n}{(n+1)^2}=b_n+\frac{\cos n}{(n+1)^2}$

To show $\lim\limits_{n\to\infty} na_n=\lim\limits_{n\to\infty} b_n$ exists, by Cauchy's theorem, it is the same as showing $\{b_n\}$ is a Cauchy sequence.

$\forall\varepsilon>0$, take $K\in\mathbb{N}$ such that $K>\frac{1}{\varepsilon}$.

$m,n\ge K \Rightarrow |b_m-b_n| = |(b_m-b_{m-1})+(b_{m-1}-b_{m-2})+\cdots+(b_{n+1}-b_n)|$

Say $m\ge n$

$$\le \left|\frac{\cos(m-1)}{m^2}+\frac{\cos(m-2)}{(m-1)^2}+\cdots+\frac{\cos n}{(n+1)^2}\right|$$

$$\le \frac{1}{m^2}+\frac{1}{(m-1)^2}+\cdots+\frac{1}{(n+1)^2}$$

$$< \frac{1}{m(m-1)}+\frac{1}{(m-1)(m-2)}+\cdots+\frac{1}{(n+1)n}$$

$$= \left(\frac{1}{m-1}-\frac{1}{m}\right)+\left(\frac{1}{m-2}-\frac{1}{m-1}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= \frac{1}{n}-\frac{1}{m} < \frac{1}{n} \le \frac{1}{K} < \varepsilon$$

need $K>1/\varepsilon$

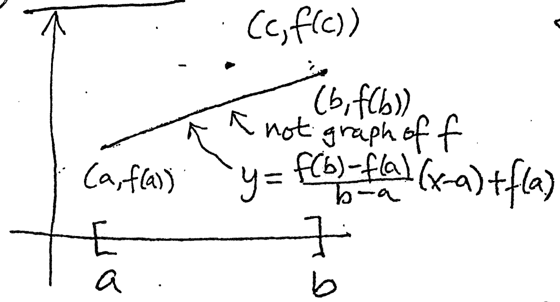
Solution 2 Let $b_n=na_n$. Then $b_{n+1}=(n+1)a_{n+1}=na_n+\frac{\cos n}{(n+1)^2}=b_n+\frac{\cos n}{(n+1)^2}$.

Then $b_{n+1}-b_n=\frac{\cos n}{(n+1)^2}$. So $b_n=b_1+\sum\limits_{k=2}^{n}(b_k-b_{k-1})=1+\sum\limits_{k=2}^{n}\frac{\cos(k-1)}{k^2}$. Hence,

$\lim\limits_{n\to\infty} b_n$ exists $\Leftrightarrow \lim\limits_{n\to\infty}\left(1+\sum\limits_{k=2}^{n}\frac{\cos(k-1)}{k^2}\right)=1+\sum\limits_{k=2}^{\infty}\frac{\cos(k-1)}{k^2}$ converges.

Since $\left|\frac{\cos(k-1)}{k^2}\right|\le\frac{1}{k^2}$ and $\sum\limits_{k=2}^{\infty}\frac{1}{k^2}$ converges by p-test ($p=2>1$), by comparison test and absolute convergence test, $\sum\limits_{k=2}^{\infty}\frac{\cos(k-1)}{k^2}$ converges. So $\lim\limits_{n\to\infty} b_n$ exists.

(182) Solution 1



Since the graph of f is not a line segment, there exists $c \in (a,b)$ such that $(c, f(c))$ is either above or below the line segment joining $(a, f(a))$ and $(b, f(b))$. So either $f(c) > \frac{f(b)-f(a)}{b-a}(c-a)+f(a)$ or $f(c) < \frac{f(b)-f(a)}{b-a}(c-a)+f(a)$.

In the former case, solving for $\frac{f(b)-f(a)}{b-a}$, we get $\frac{f(c)-f(a)}{c-a} > \frac{f(b)-f(a)}{b-a}$.

By mean-value theorem, $\exists\, x_2 \in (a,c) \subseteq (a,b)$ such that $f'(x_2) = \frac{f(c)-f(a)}{c-a}$.

Then $\frac{f(b)-f(a)}{b-a} < f'(x_2)$. Next, we should consider $\frac{f(c)-f(b)}{c-b}$ in view of the expression $\frac{f(c)-f(a)}{c-a}$. We have

$$f(c)-f(b) > \frac{f(b)-f(a)}{b-a}(c-a)+f(a)-f(b) = (f(b)-f(a))\Big(\underbrace{\frac{c-a}{b-a}-1}_{=\frac{c-b}{b-a}}\Big).$$

Since $c-b<0$, we get $\frac{f(c)-f(b)}{c-b} < \frac{f(b)-f(a)}{b-a}$. By the mean-value theorem, $\exists\, x_1 \in (b,c) \subseteq (a,b)$ such that $f'(x_1) = \frac{f(c)-f(b)}{c-b}$.

Therefore, $f'(x_1) < \frac{f(b)-f(a)}{b-a} < f'(x_2)$. The latter case is similar.

Solution 2 Let $L(x) = \frac{f(b)-f(a)}{b-a}(x-a)+f(a)$. Assume there is no $x \in (a,b)$ such that $f'(x) < \frac{f(b)-f(a)}{b-a}$. Then $f'(x) \geq \frac{f(b)-f(a)}{b-a}$ for all $x \in (a,b)$. So $(f(x)-L(x))' = f'(x) - \frac{f(b)-f(a)}{b-a} \geq 0$ for all $x \in (a,b)$. This implies $f(x)-L(x)$ is increasing.

Since $f(a)-L(a) = 0 = f(b)-L(b)$, so $f(x)-L(x)=0$ $\forall x \in [a,b]$. Then $f(x) = L(x)$, contradicting the graph of f is not a line segment.

$\therefore \exists\, x_1 \in (a,b)$ such that $f'(x_1) < \frac{f(b)-f(a)}{b-a}$. Similarly, $\exists\, x_2 \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} < f'(x_2)$.

(183) Solution 1

Since f, g are continuous on $[0,1]$, $h(x)=g(x)-f(x)$ is also continuous on $[0,1]$.
By the extreme value theorem, $\exists\, u, v \in [0,1]$ such that $h(u)=\min_{x\in[0,1]} h(x)$ and $h(v)=\max_{x\in[0,1]} h(x)$. So $\forall x \in [0,1]$, $h(u) \le h(x) \le h(v)$. Then

$$h(u) \le h(x_n) = g(x_n)-f(x_n) = f(x_{n+1})-f(x_n) \le h(v).$$

(Now $f(x_{n+1})-f(x_n)$ suggests mean-value theorem or telescoping series. Since f is not known to be differentiable, we consider telescoping series.)
We have $h(x_1)+\cdots+h(x_n)=f(x_{n+1})-f(x_1)$.
Since $h(u)\le h(x_1), \cdots, h(x_n) \le h(v)$, so $h(u) \le c_n = \frac{h(x_1)+\cdots+h(x_n)}{n} \le h(v)$.
By the intermediate value theorem, $\exists\, w_n \in [u,v] \subseteq [0,1]$ such that $h(w_n)=c_n$.
Now $|c_n| = \left|\frac{f(x_{n+1})-f(x_1)}{n}\right| \le \frac{2\max_{x\in[0,1]} f(x)}{n} \to 0$ as $n\to\infty$. By the Bolzano-Weierstrass theorem, $\exists\, w_{n_i} \to w \in [0,1]$. Then $h(w)=\lim_{i\to\infty} h(w_{n_i}) = \lim_{i\to\infty} c_{n_i} = 0$. So $g(w)=f(w)$.

Solution 2 Define $h: [0,1] \to \mathbb{R}$ by $h(x)=g(x)-f(x)$. Assume $h(x) \ne 0$ for all $x\in[0,1]$. By the intermediate value theorem, either $h(x)>0$ for all $x\in[0,1]$ or $h(x)<0$ for all $x\in[0,1]$.
In the former case, $h(x_n)=g(x_n)-f(x_n)=f(x_{n+1})-f(x_n)>0$.
Then $f(x_{n+1})>f(x_n)$. So $\{f(x_n)\}$ is strictly increasing. Since $f(x_n) \le \max_{x\in[0,1]} f(x)$, $\{f(x_n)\}$ is bounded above.
So $f(x_n) \to c \in [0,1]$. Then $g(x_n)=f(x_{n+1}) \to c \in [0,1]$ by subsequence theorem. So $h(x_n)=g(x_n)-f(x_n) \to c-c=0$.
By Bolzano-Weierstrass theorem, $\exists\, x_{n_i} \to x \in [0,1]$.
Then $h(x)=\lim_{i\to\infty} h(x_{n_i}) = 0$, a contradiction.
The latter case is similar.
Therefore, there must exist some $x\in[0,1]$ such that $h(x)=0$, then $g(x)=f(x)$.