MATH2033 Mathematical Analysis

Lecture Note 2

Sets and functions

Set Notation

A *set* A is a collection of distinct objects (they can be numbers, letters or anything you like). An object inside the set is called an *element* of the set A.

Some examples of sets

$$\overbrace{A}^{set} = \overbrace{\{1,3,5,7,9\}}^{elements}$$ (set of all odd numbers between 1 and 10)

$B = \{1,2,3,4,5 \ldots\ldots\}$ (set of all positive integers)

$C = \{0, +3, -3, +6, -6, +9, -9, \ldots\}$ (set of multiple of 3)

$D = \{all\ real\ numbers\} = \mathbb{R}$ (set of real numbers)

Mathematically, we write

- $p \in B$ if the element p is in the set B ("∈" means "belongs to") and
- $p \notin B$ if the element p is NOT in the set B.

For example : If $E = \{2,3,4,5\}$, then $3 \in E$ and $\sqrt{6} \notin E$.

General description of sets

In general, we describe the set by mentioning the common properties that the objects in the set have. In particular

$$E = \{x \mid x \text{ has certain properties}\}.$$

Example 1

$$A = \left\{ x \mid x \text{ is prime and } \underbrace{0 < x \leq 10}_{\substack{x \text{ lies between} \\ 0 \text{ and } 10}} \right\} = \{2,3,5,7\}.$$

$$B = \{x \mid x > 0 \text{ and } x \text{ is multiple of } 3\} = \{3,6,9,12, \dots.\}$$

$$\begin{aligned} C &= \{x \mid x^2 \leq 100 \text{ and } x \text{ is negative integer}\} \\ &= \{x: -10 \leq x \leq 10 \text{ and } x \text{ is negative integer}\} \\ &= \{-10, -9, -8, -7, -6, -5, -4, -3, -2, -1\}. \end{aligned}$$

$$D = \{f(x) \mid f(0) = 1 \text{ and } f(x) \text{ is continuous}\}$$

Common sets in Mathematics (or in this course)

$\phi =$ empty set (the set containing nothing)

$\mathbb{N} = \{1,2,3,4,5,..\}$ (the set of all positive integers)

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ (the set of all integers)

$\mathbb{Q} = \left\{\frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\right\}$, (the set of rational numbers)

$[a, b] = \{x | a \le x \le b\}$, $[a, b) = \{x | \ a \le x < b\}$, $(a, \infty) = \{x | \ x > a\}$, (intervals)

$\mathbb{R} =$ the set of real numbers

$\mathbb{C} = \{a + bi | \ a \in \mathbb{R}, b \in \mathbb{R}\}$ the set of all complex numbers

Note: In mathematics, we usually write

"$x \in \mathbb{R}$" to represent "x is real", "$x \in \mathbb{N}$" to represent "x is positive integer",

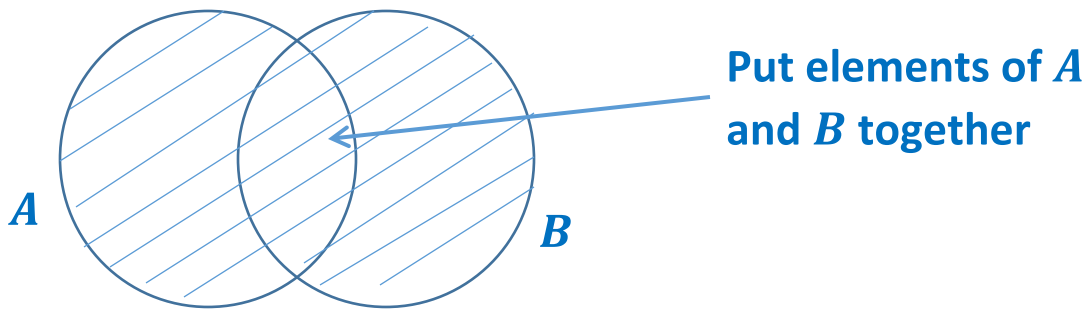
"$x \in [a, b]$" to represent "$a \le x \le b$" or "x lies between a and b".

Set operations

1. Unions of sets

We let A, B be two sets. The *union* of two sets, denoted by $A \cup B$, is defined as

$$A \cup B = \{x | x \in A \ \ or \ \ x \in B\}.$$



Example 2

If $A = \{1,3\}$ and $B = \{2,4,5\}$, then $A \cup B = \{1,2,3,4,5\}$.

If $A = \{1,2,3,4\}$, $B = \{3,4,5,6\}$, then $A \cup B = \{1,2,3,3,4,4,5,6\} = \{1,2,3,4,5,6\}$.

Remarks

- In set notation, repeated elements (say 3,4 in the last example) count only once.
- In general, we can define the union of n sets $(A_1, A_2, \dots, A_n)$ as

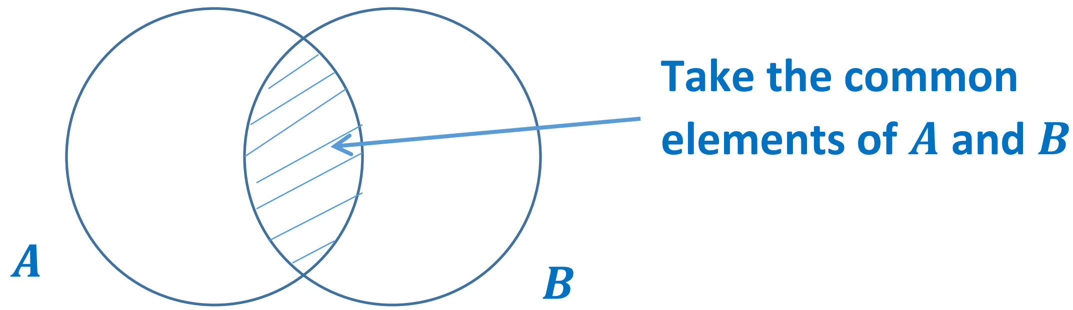
$$\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cup \dots \cup A_n = \{x | x \in A_1 \ \ or \ \ x \in A_2 \ \ or \ \ \dots \ or \ \ x \in A_n\}$$

$$= \{x | x \in A_i \ \ for \ some \ i\}$$

2. Intersection of sets

We let A, B be two sets. The *intersection* of two sets, denoted by $A \cap B$, is defined as

$$A \cap B = \{x | x \in A \;\; and \;\; x \in B\}.$$



Example 3

If $A = \{1,3\}$ and $B = \{2,3,4\}$, then $A \cap B = \{3\}$.

If $A = \{2,6,8\}$ and $B = \{\sqrt{3}, \sqrt{7}\}$, then $A \cap B = \phi$ (i.e. there is no common elements between two sets).

In general, we can define the intersection of n sets to be

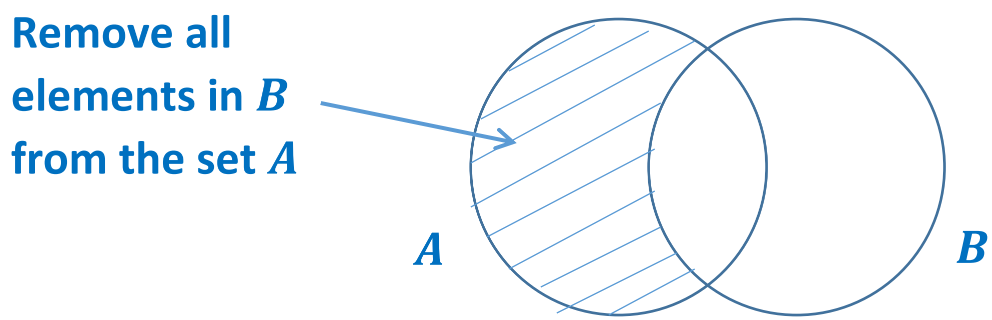
$$\bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap \cdots \cap A_n = \{x | x \in A_1 \;\; and \;\; x \in A_2 \;\; and \;\; \ldots \;\; and \;\; x \in A_n\}$$

$$= \{x | x \in A_i \;\; for \; all \; i\}$$

3. Complement of sets

We let A, B be two sets. The *complement* of B in A, denoted by $A\backslash B$, is defined as

$$A\backslash B = \{x|x \in A \ \ and \ \ x \notin B\}$$



Example 4

If $A = \{2,3,4,5,6\}$ and $B = \{1,2,3,4\}$, then $A\backslash B = \{5,6\}$ (since the elements "2,3,4" are in B.

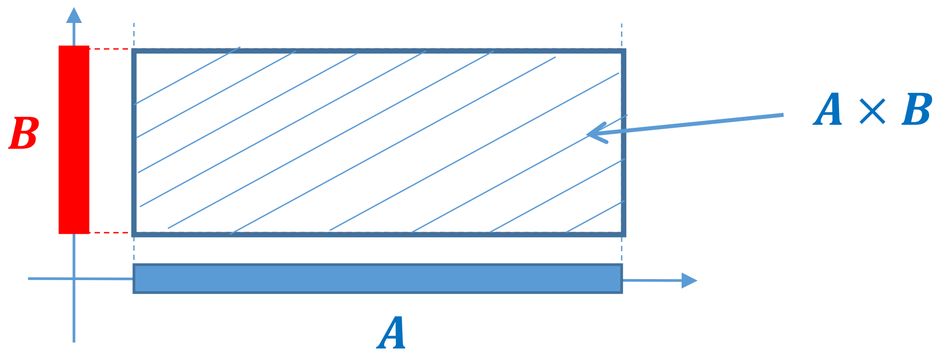
If $A = \{1,3,5\}$ and $B = \{2,4,6\}$, then $A\backslash B = \{1,3,5\}$ (since no elements In A are in B)

If $A = \{1,2,3,4\}$ and $B = \{1,2,3,4,5,6\}$, then $A\backslash B = \phi$ (since every element in A is in B also)

4. Cartesian product

The Cartesian product of A and B, denoted by $A \times B$, is defined as

$$A \times B = \{(x, y) | \; x \in A \; \; and \; \; y \in B\}.$$



Example 5

If $A = B = \mathbb{R}$, then $A \times B = \underbrace{\{(x, y) | \; x \in \mathbb{R} \; \; and \; \; y \in \mathbb{R}\}}_{denoted \; by \; \mathbb{R}^2}$ represents the 2D-plane.

Remark

In general, we define the Cartesian product of n sets as

$$A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \dots, x_n) | x_i \in A_i \; \; for \; i = 1,2, \dots, n\}.$$

Application of set operations

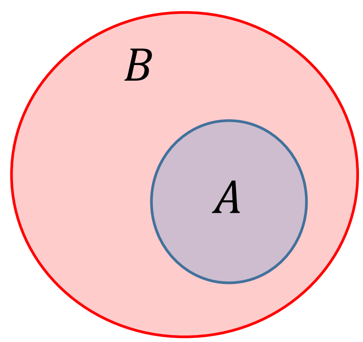
In practice, these set operations allow us to express a complex set in terms of simple sets so that we can analyze the set easily. For example:

- Irrational number is a number (e.g. $\sqrt{2}, e$) which *cannot* be expressed in the form of $\frac{m}{n}$, where m, n are integers. Then the set of irrational number can be expressed as $\mathbb{R}\backslash\mathbb{Q}$ (i.e. remove all rational numbers from the set of real numbers).

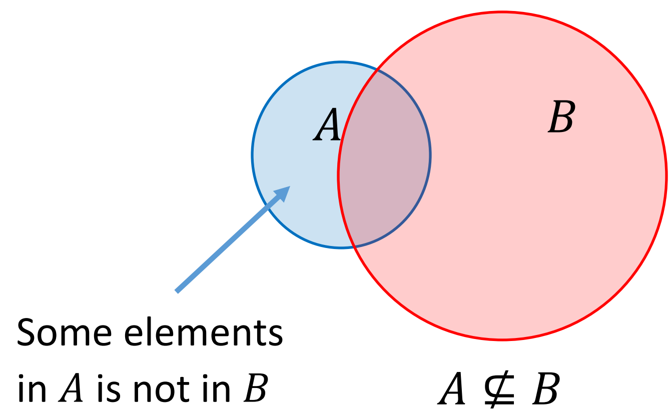
- We consider the set
$$A = \{x \in \mathbb{R} | \ x^5 + ax - a^2 = 0 \ \ for\ some\ a \in \mathbb{N}\}.$$
Then the set can be expressed as
$$A = \bigcup_{n=1}^{\infty} \underbrace{\{x \in \mathbb{R} | \ x^5 + nx - n^2 = 0\}}_{denoted\ by\ A_n}.$$
To study the set A, one can study the set A_n by finding the roots of the equation $x^5 + nx - n^2 = 0$.

Comparison between sets

- **(Equality of sets)** We say two sets A, B are equal (we write $A = B$) only when two sets contain the same elements. For example:
  - ✓ If $A = \{1,2,3\}$ and $B = \{1,2,3\}$, then $A = B$.
  - ✓ If $A = \{1,3,4\}$ and $B = \{1,2,3\}$, then $A \neq B$.
- **(Subset)** Given two sets A and B, we say A is *subset* of B (denoted by $A \subseteq B$) if <u>every elements in A is also an element in B</u>. For example:
  - ✓ If $A = \{1,3\}$ and $B = \{0,1,3,4\}$, then $A \subseteq B$.
  - ✓ If $A = \{2,4\}$ and $B = \{0,1,3,4\}$, then $A \nsubseteq B$. (since $2 \in A$ but $2 \notin B$)



$A \subseteq B$



$A \nsubseteq B$

According to the definition, we observe that $A = B \Leftrightarrow A \subseteq B$ AND $B \subseteq A$.

Example 6

(a) We let A, B and C be three non-empty subsets such that $B \subset C$, show that $A \cup B \subseteq A \cup C$.

(b) Is it always true that $A \cup B \subset A \cup C$? Explain your answer.

Solution

(a) We shall prove it using the definition. For any $x \in A \cup B$, then we have

$$x \in A \cup B \Rightarrow x \in A \ \ or \ \ x \in B$$

$$\overset{B \subset C}{\Rightarrow} \ x \in A \ \ or \ \ x \in C \Rightarrow x \in A \cup C.$$

The element x is also in $A \cup C$, so we deduce that $A \cup B \subseteq A \cup C$.

(b) The statement is incorrect in general. Mathematically, one can disprove the statement by *providing a suitable counter-example*.

We pick $A = \{1, 2, 3,4\}$, $B = \{1\}$ and $C = \{1,2\}$.

One can see that A, B, C are non-empty, $B \subset C$ but

$$A \cup B = \{1, 2, 3,4\} = A \cup C.$$

Example 7 (Proving set identity)

We let Ω be a non-empty subset and $A_1, A_2, A_3, \ldots$ be subsets of Ω. Show that

$$\Omega \backslash \left(\bigcap_{n=1}^{\infty} A_n \right) = \bigcup_{n=1}^{\infty} (\Omega \backslash A_n).$$

😊Solution

Step 1: Prove $\boldsymbol{\Omega \backslash (\bigcap_{n=1}^{\infty} A_n) \subseteq \bigcup_{n=1}^{\infty} (\Omega \backslash A_n)}$.

$$x \in \Omega \backslash \left(\bigcap_{n=1}^{\infty} A_n \right) \Rightarrow x \in \Omega \ \ and \ \ x \notin \bigcap_{n=1}^{\infty} A_n$$

$$\Rightarrow x \in \Omega \ \ and \ \ x \notin A_n \ \ for\ some\ n$$

$$\Rightarrow x \in \Omega \backslash A_n \ \ for\ some\ n \Rightarrow x \in \bigcup_{n=1}^{\infty} (\Omega \backslash A_n)$$

So we get $\Omega \backslash (\bigcap_{n=1}^{\infty} A_n) \subseteq \bigcup_{n=1}^{\infty} (\Omega \backslash A_n)$.

Step 2: Prove $\boldsymbol{\Omega}\backslash(\bigcap_{n=1}^{\infty} \boldsymbol{A_n}) \supseteq \bigcup_{n=1}^{\infty}(\boldsymbol{\Omega}\backslash \boldsymbol{A_n})$.

$$x \in \bigcup_{n=1}^{\infty}(\Omega\backslash A_n) \Rightarrow x \in \Omega\backslash A_n \ \ for\ some\ n$$

$$\Rightarrow x \in \Omega \ \ and \ \ x \notin A_n \ \ for \ \ some\ n$$

$$\Rightarrow x \in \Omega \ \ and \ \ x \notin \bigcap_{n=1}^{\infty} A_n \Rightarrow x \in \Omega\backslash\left(\bigcap_{n=1}^{\infty} A_n\right)$$

So we deduce that $\Omega\backslash(\bigcap_{n=1}^{\infty} A_n) \supseteq \bigcup_{n=1}^{\infty}(\Omega\backslash A_n)$ and the result follows.

Example 8 (Self-reading)

We let X, Y and Z be three sets. Prove that

$$(X\backslash Y)\backslash Z = (X\backslash Z)\backslash Y.$$

😊Solution

Step 1: Prove that $(\boldsymbol{X}\backslash\boldsymbol{Y})\backslash\boldsymbol{Z} \subseteq (\boldsymbol{X}\backslash\boldsymbol{Z})\backslash\boldsymbol{Y}$

$x \in (X\backslash Y)\backslash Z$

$\Rightarrow x \in X\backslash Y \ \ and \ \ x \notin Z$

$\Rightarrow (x \in X \ \ and \ \ x \notin Y) \ \ and \ \ x \notin Z$

$\Rightarrow (x \in X \ \ and \ \ x \notin Z) \ \ and \ \ x \notin Y$

$\Rightarrow x \in X\backslash Z \ \ and \ \ x \notin Y \Rightarrow x \in (X\backslash Z)\backslash Y$

So we have $(X\backslash Y)\backslash Z \subseteq (X\backslash Z)\backslash Y$

Step 2: Prove that $(\boldsymbol{X}\backslash\boldsymbol{Z})\backslash\boldsymbol{Y} \subseteq (\boldsymbol{X}\backslash\boldsymbol{Y})\backslash\boldsymbol{Z}$

One can establish the proof by interchanging the role of Y and Z in the proof of Step 1. So we get $(X\backslash Y)\backslash Z \subseteq (X\backslash Z)\backslash Y$.

Therefore we conclude that $(X\backslash Y)\backslash Z = (X\backslash Z)\backslash Y$

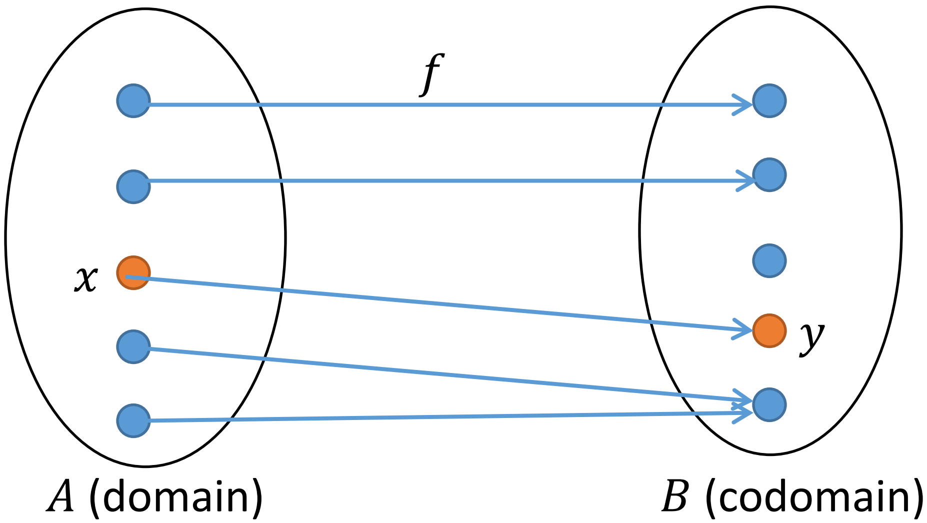
Functions

A function $f(x)$ from set A to set B, denoted by

$$f: \underbrace{A}_{domain} \rightarrow \underbrace{B}_{codomain},$$

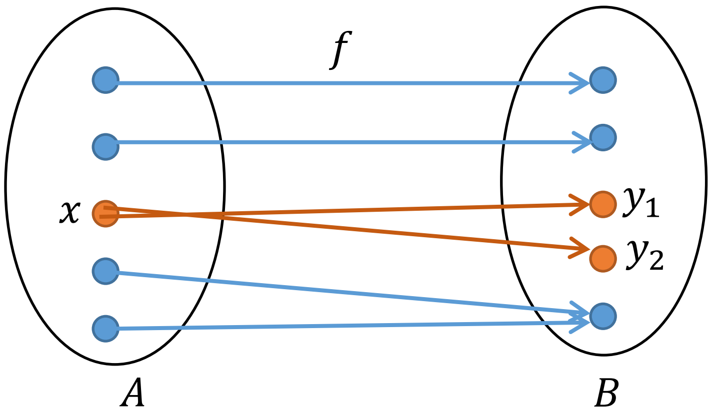
assigns (maps) each element of A to exactly one element of B.

Mathematically, we write $f\,(\underbrace{x}_{\in A}) = \underbrace{y}_{\in B}$.



According to the definition, one would expect that a function $f: A \to B$ is well-defined in the sense that $x = y$ implies $f(x) = f(y)$.

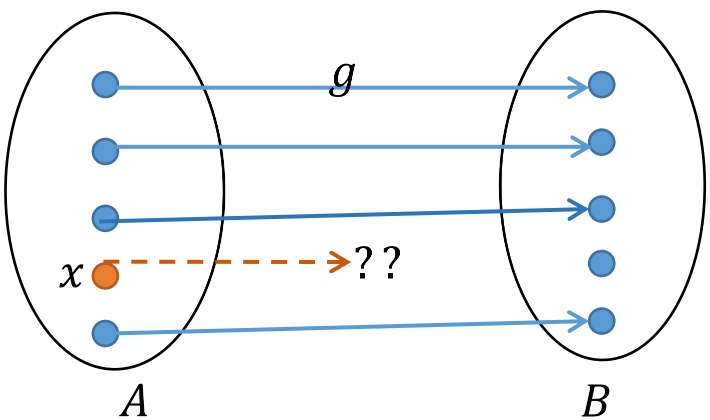
The following shows some examples of not well-defined functions



f is not function since $f(x)$ has two possible values y_1 or y_2.

Example:

$f: [0, \infty) \to \mathbb{R}, \quad f(x^2) = x$



f is not function since $f(x)$ is not defined for some x.

Example:

$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \log x$

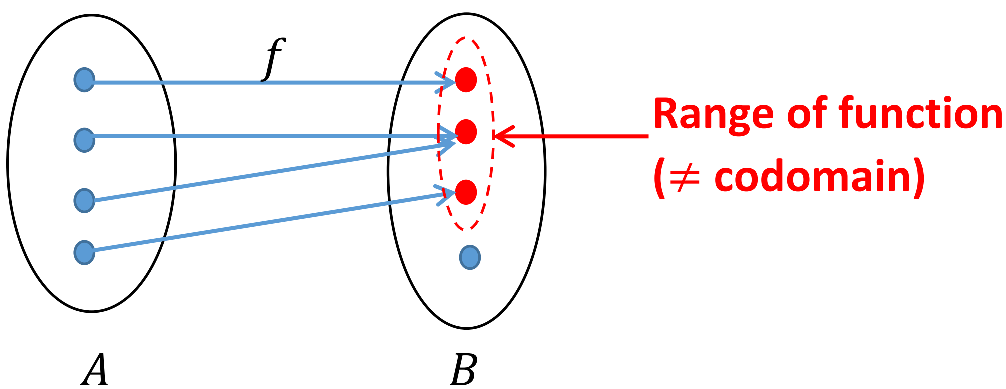
$f(-2) = \log(-2)$ is not defined!

Some terminologies

- The *domain* of a function (denoted by "$dom\ f$") is the collection of numbers that can be "put" (in the sense that the value of $f(x)$ is defined) in the function.
- The *codomain* of a function (denoted by "$codom\ f$") is the set which all possible outputs of the function $f(x)$ lie in this set.
- The *range* of a function (denoted by $f(A)$) is the collection of all possible outputs of the function. Mathematically, we write

$$f(A) = \{y \in B |\ y = f(x)\ \ for\ some\ x \in A\}.$$

(In general, the range of $f(x)$ does not necessarily cover the whole codomain.)



Example 9

Find the largest possible domain for each of the following functions.

$(a)\ \ f_1(x) = x^2 - 2x - 3$ $\qquad (b)\ \ f_2(x) = \dfrac{1}{x^2 - 2x - 3}$

$(c)\ \ f_3(x) = \dfrac{x^2 - 1}{x - 1}$ $\qquad (d)\ \ f_4(x) = \sqrt{4 - x^2}$

Solution

(a) Since $x^2 - 2x - 3$ for every $x \in \mathbb{R}$, thus the domain of $f_1 = \mathbb{R}$.

(b) Note that $f_2(x)$ is not defined when $x^2 - 2x - 3 = 0$.

$$x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x = 3 \ \ or \ \ x = -1.$$

Thus the domain of $f_2 = \mathbb{R}\backslash\{1,3\}$.

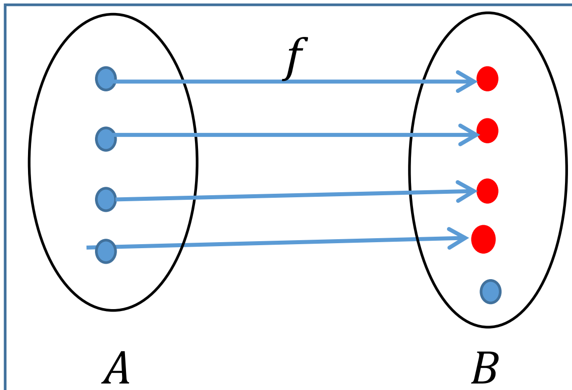
(c) Note that $f_3(x)$ is not defined when $x - 1 = 0$, i.e. $x = 1$. Thus the domain of $f_3 = \mathbb{R}\backslash\{1\})$.

(d) Note that $f_4(x)$ is defined only when $4 - x^2 \geq 0$, i.e. $-2 \leq x \leq 2$. Thus the domain of $f_4 = [-2, 2]$.

Injective function, surjective function, bijective function

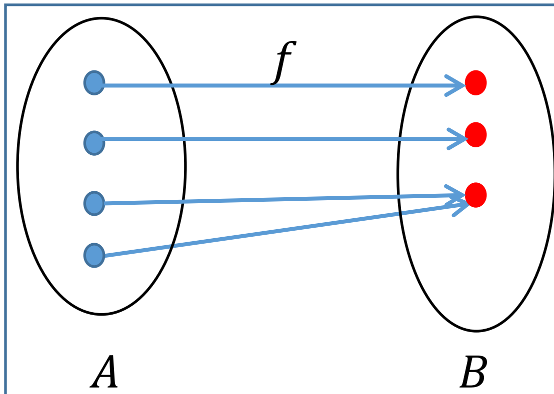
We let $f: A \to B$ be a function. We say

- f is injective if and only if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
- f is surjective if and only if for any $y \in B$, there exists $x \in A$ (one or multiple) such that $f(x) = y$.
- f is bijective if and only if f is both injective and surjective.



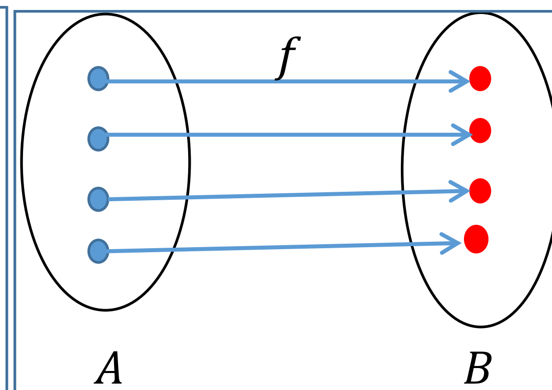
Injective function

Each element in A is mapped to an unique element in B.



Surjective function

The range of f covers the entire codomain B.



Bijective function

There is one-to-one correspondence between element in A and element in B

Example 10

We consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4$.

(a) Determine if the function f is injective and surjective.

(b) How about the case if codomain of the function is changed to $[0, \infty)$?

😊Solution

(a) We first argue that f is *not* injective. To see this, we pick $x_1 = 2$ and $x_2 = -2$. One can verify that $f(x_1) = f(x_2) = 16$.
On the other hand, f is *not* surjective. Since $x^4 \geq 0$ for all $x \in \mathbb{R}$, so it is impossible to find a real number x such that $x^4 = -1$.

(b) By the same reason as in (a), we deduce that f is not injective.
On the other hand, for any $y \in [0, \infty)$, we take $x = y^{\frac{1}{4}}$. One can show that $f(x) = \left(y^{\frac{1}{4}}\right)^4$. So f is *surjective*.

Remark of Example 10

In general, the injectivity and surjectivity of a function also depend on the choices of domain and codomain of the function.

Example 11

We let $f: A \to B$ and $g: B^* \to C$ be two functions such that $f(A) \subseteq B^*$. We define the composition of g by f, denoted by $g \circ f$, as a function

$$(g \circ f)(x) = g\big(f(x)\big).$$

Prove that

(a) If both f and g are injective, determine if $g \circ f$ is also injective.
(b) If both f and g are surjective, determine if $g \circ f$ is also surjective.

😊Solution

(a) Since f, g are injective, we have (1) $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ and (2) $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$.
For any z_1, z_2 such that $(g \circ f)(z_1) = (g \circ f)(z_2)$, we have

$$g\big(f(z_1)\big) = g\big(f(z_2)\big) \quad \overset{g \text{ is injective}}{\Rightarrow} \quad f(z_1) = f(z_2) \quad \overset{f \text{ is injective}}{\Rightarrow} \quad z_1 = z_2.$$

So $g \circ f$ is injective.

(b) It appears that $g \circ f$ may not necessarily to be surjective. To see this, we let the functions f and g to be

- $f: \mathbb{R} \to [0, \infty)$, $f(x) = x^2$;
- $g: \mathbb{R} \to (0, \infty)$, $g(x) = e^x$

One can show that f, g are both surjective (left as exercise), $f(\mathbb{R}) = [0, \infty) \subseteq \mathbb{R} = dom(g)$ and

$$(g \circ f)(x) = g\big(f(x)\big) = g(x^2) = e^{x^2}.$$

We consider $z = e^{-1} \in (0, \infty)$, one can see that

$$(g \circ f)(x) = -1 \Rightarrow e^{x^2} = e^{-1} \Rightarrow x^2 = -1.$$

One cannot find a real number x such that $x^2 = -1$ since $x^2 \geq 0$ for all $x \in \mathbb{R}$. So $(g \circ f)(x)$ may not be surjective.

Additional remark of (b)

Suppose that $B^* = B$, one can show that $(g \circ f)(x)$ is surjective as follows:

For any $z \in C$, there exists $y_0 \in B$ such that $g(y_0) = z$ since g is surjective.

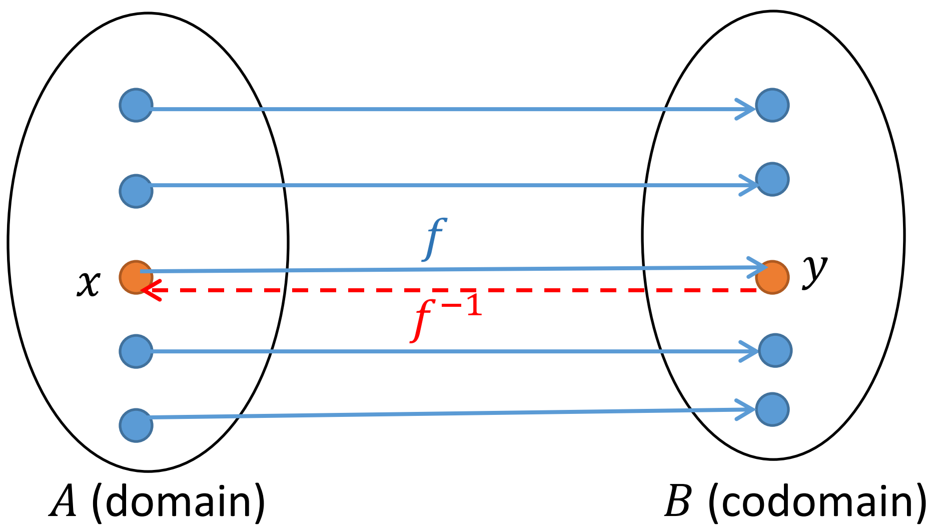
As $y_0 \in B = codom\ f$ and f is surjective, there exists $x_0 \in A$ such that $f(x_0) = y_0$.

We pick $x = x_0$. Then $(g \circ f)(x) = g\big(f(x_0)\big) = g(y_0) = z$ and $(g \circ f)(x)$ is surjective.

Question: Why do we need the concept of injective, surjective and bijective?

Scenario 1: Inverse function

Recall that a function $f: A \to B$ takes an element x in domain A and assigns it to another element $y = f(x)$ in the codomain B.



Question:

Given the value of $y \in B$, can we find value of x such that

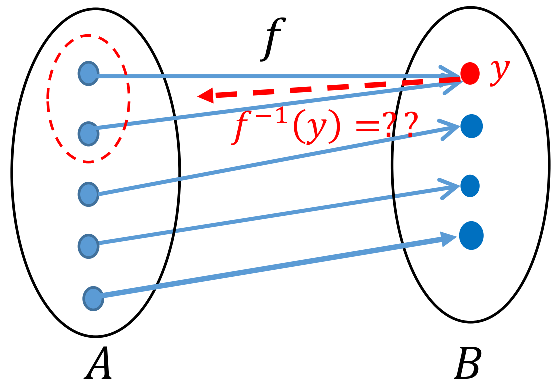
$$f(x) = y?$$

The inverse function of $f(x)$, denoted by f^{-1}, as a function that satisfies

$$x = f^{-1}(y) \Leftrightarrow y = f(x).$$

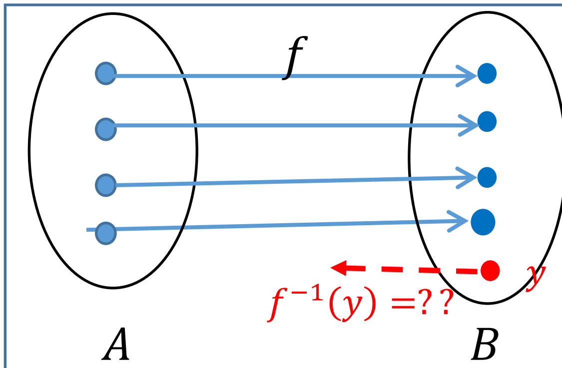
However, the inverse function f^{-1} may not be necessarily to be well-defined for any function f. For example,

Case 1: The function is not injective.



For the elements y, we have more than 1 $x \in A$ satisfying $f(x) = y$.

Case 2: The function is not surjective.

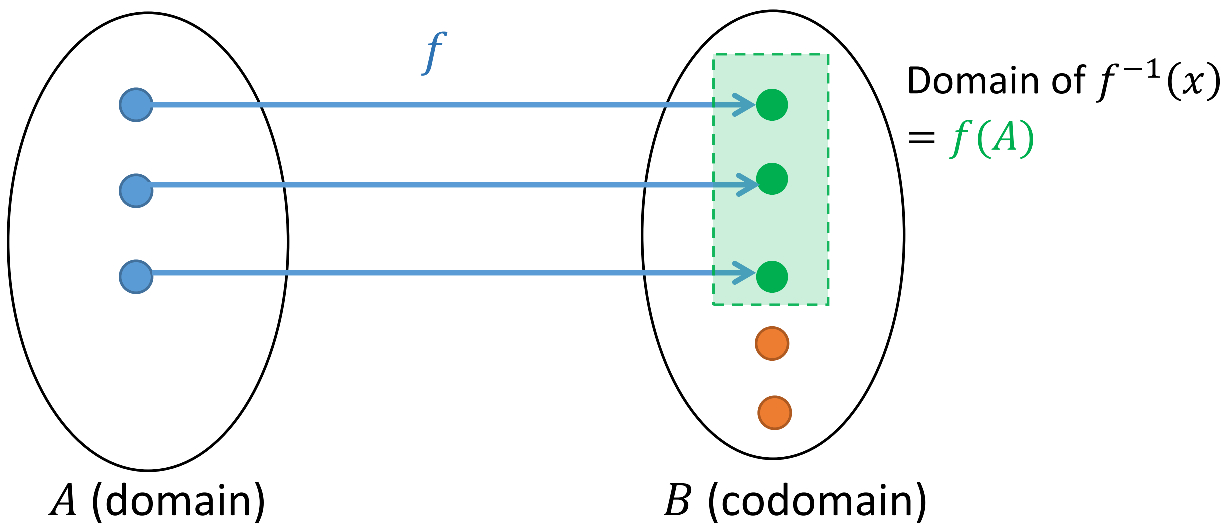


For elements which is not in the range of f, one cannot find x such that $f(x) = y$.

In order that the inverse function f^{-1} is well-defined in the sense that it is a function, we require that f should be bijective (i.e. injective and surjective).

In some cases when the function f is not bijective, it is still possible to define the inverse function for f, provided that f is injective.

To do so, one needs to modify the *domain of* f^{-1} as the range of f (i.e. $f(A)$) instead of the entire codomain B.



For example, we take $f: \mathbb{R} \to \mathbb{R}$ to be $f(x) = e^x$. One can show that e^x is injective. Then the domain of the inverse function $f^{-1}(x) = \ln x$ is $f(\mathbb{R}) = (0, \infty)$.

Some examples of common inverse functions used

$f(x)$	Inverse of $f(x)$
$f_1: \mathbb{R} \to [0, \infty)$, $f_1(x) = e^x$	$f_1^{-1}(x) = \ln x$ Domain = $(0, \infty)$
$f_2: [0, \infty) \to [0, \infty)$ $f_2(x) = x^2$	$f_2^{-1}(x) = \sqrt{x}$ Domain = $[0, \infty)$
$f_3: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$, $f_3(x) = \sin x$	$f_3^{-1}(x) = \sin^{-1} x\ (= \arcsin x)$ Domain = $[-1,1]$
$f_4: [0, \pi] \to [-1,1)$, $f_4(x) = \cos x$	$f_4^{-1}(x) = \cos^{-1} x\ (= \arccos x)$ Domain = $[-1,1]$
$f_5(x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$, $f_5(x) = \tan x$	$f_5^{-1}(x) = \tan^{-1} x\ (= \arctan x)$ Domain =$(-\infty, \infty)$

Example 12

We let $f:[a,b]\to\mathbb{R}$ be a function which f is continuous and strictly increasing (i.e. $f(x_1)<f(x_2)$ for any $a\le x_1<x_2\le b$)

Show that the inverse function of f exists and find the domain of f^{-1}.

😊Solution (A bit informal)

Our goal is to argue that f is injective. We consider $f(x_1)=f(x_2)$.

Suppose that $x_1\neq x_2$. Since f is strictly increasing, it follows that either

$$f(x_1)<f(x_2)\ (if\ x_1<x_2)\quad or\quad f(x_1)>f(x_2)\quad (if\ x_1>x_2).$$

So we always have $f(x_1)\neq f(x_2)$ and it leads to contradiction.

Hence, it follows that $x_1=x_2$ and f is injective. Then the inverse of f exists.

By plotting the graph of f, we see that f increases from $f(a)$ to $f(b)$. So the domain of f^{-1} is $f([a,b])=[f(a),f(b)]$.

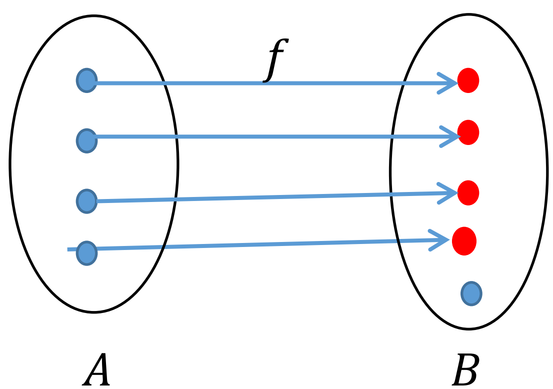
Remark: The formal proof of second part requires intermediate value theorem which will be discussed later (continuity).

Scenario 2: Comparing the "size" between sets

In some cases (e.g. examining the countability of the sets), one may want to compare the number of elements between sets. This can be done by examining the mapping between sets.

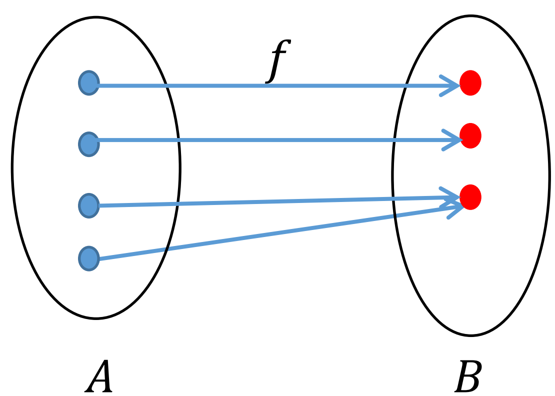
More precisely, we let A and B be two finite sets and $f: A \to B$ be a function. We consider the following 3 cases.

<u>Case 1: If f is injective</u>



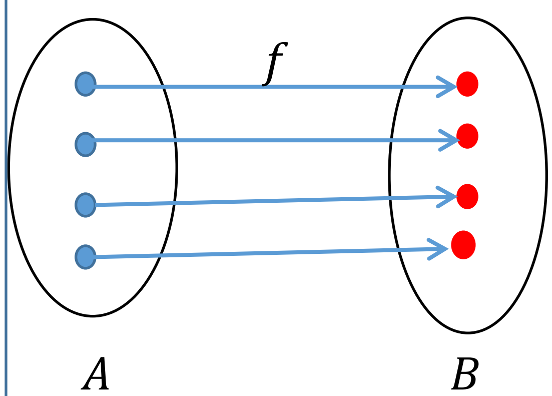
The number of elements contained in B is likely to be more than that in A.

<u>Case 2: If f is surjective</u>



The number of elements contained in A is likely to be more than that in B.

<u>Case 3: If f is bijective</u>



The number of elements contained in A is same as that in B.

Equivalent relation

Roughly speaking, equivalent relation is a tool to divide a set into several disjoint subsets based on certain criterion chosen.

Formal definition of equivalent relation

We let S be a set. The relation R is defined as a collection of pairs $(a, b) \in S \times S$ which two elements in each pair satisfy certain property:

$$R = \{(a, b) \in S \times S |\ (a, b)\ \ has\ certain\ property\}.$$

Mathematically, we write $a \sim b$ if $(a, b) \in \mathbb{R}$.

The relation R (or $\sim$) is said to be *equivalent relation* if it has the following three properties:

- *(Reflexive property)* For any $x \in S$, we have $x \sim x$ (or $(x, x) \in R$)
- *(Symmetric property)* If $x \sim y$ (or $(x, y) \in R$), then $y \sim x$ (or $(y, x) \in R$)
- *(Transitive property)* If $x \sim y$ and $y \sim z$, then $x \sim z$.

Example 13

We let $S = \mathbb{N}$. We define a relation R which $x \sim y$ if and only if x, y have same remainder when they are divided by 3 (Mathematically, it can be shown that this condition is equivalent to $x - y$ is divisible by 3). Show that $\sim$ is an equivalent relation.

☺Solution

It suffices to check those three conditions:

- (Reflexivity property) For any $x \in S$, we have $x - x = 0$ which is divisible by 3, so $x \sim x$.
- (Symmetric property) For any $x, y \in S$ which $x \sim y$, we get $x - y = 3m$ for some integer m. It is clear that $y - x = -3m = 3(-m)$ is also divisible by 3. Thus $y \sim x$.
- (Transitive property) For any $x, y, z \in S$ which $x \sim y$ and $y \sim z$, we have $x - y = 3p$ and $y - z = 3q$ for some integers p and q. Then $x - z = (x - y) + (y - z) = 3\underbrace{(p - q)}_{integer}$ is divisible by 3 and hence $x \sim z$.

Hence, we conclude that $\sim$ is an equivalent relation.

Example 14

(a) We let $\mathbb{R}$ be the set of real number and define a relation $\sim$ which $x \sim y$ if and only if $x \geq y$. Determine if $\sim$ defines an equivalent relation.

(b) We let $\mathbb{R}\backslash\{0\}$ be the set of non-zero real number and define a relation $\sim$ which $x \sim y$ if and only if $xy < 0$ (i.e. x, y have opposite sign). Determine if $\sim$ defines an equivalent relation.

☺Solution

(a) One can show that both reflexivity and transitivity hold for this relation $\sim$ (left as exercise). However, the symmetric property does *not* hold. To see this, we take $x = 3$ and $y = 2$. We have $x \sim y$ (since $x \geq y$) but $y \nsim x$ (since $y < x$). So $\sim$ is not equivalent relation.

(b) One can show that both *reflexivity* and *transitivity* do not hold.

- Since $x^2 > 0$ for all $x \neq 0$, so $x \nsim x$ and reflexivity fails.
- We pick $x = 3$, $y = -1$ and $z = 2$. Then $x \sim y$ (since $xy = -3 < 0$) and $y \sim z$ (since $yz = -2 < 0$) but $x \nsim z$ (since $xz = 6$), so the transitivity fails.

So $\sim$ is not equivalent relation.

Application of equivalent relation -- Equivalent class

We let S be a set and $\sim$ be an equivalent relation associated with this set S. One can use the equivalent relation and divide the set S into several groups which the elements which are equivalent (in the sense that $x \sim y$) are put in a same group.

As an example, we revisit Example 12.

- We take $x = 1 \in \mathbb{N}$ and seek other elements y which are $y \sim x$. One can observe that the elements $4,7,10, \dots$ are equivalent to $x = 1$ since $x - y$ is divisible by 3. By putting these elements together, we got the subset $[1] = \{1,4,7, \dots.\}$.
- We take $x = 2 \in \mathbb{N}$. Using similar method, one can deduce that $2,5,8, \dots$ are equivalent to $x = 2$. Thus we obtain the second subset $[2] = \{2,5,8, \dots\}$
- Lastly, we take $x = 3 \in \mathbb{N}$. Using similar method, one can deduce that $3,6,9, \dots$ are equivalent to $x = 3$. Thus we obtain the second subset $[3] = \{3,6,9, \dots\}$

We observe that the equivalent relation can divide the set into several subsets (i.e. $S = [1] \cup [2] \cup [3]$ such that the elements in each subset have "common properties" (e.g. the elements in $[1]$ contains all elements which have remainder 1 when it is divided by 3).

The subsets generated from the equivalent relation are called *equivalent class*.

Definition (Equivalent class)

We let S be a set and $\sim$ be an equivalent relation on the set S. For any $x \in S$, the equivalent class containing x, denoted by $[x]$, is defined as

$$[x] = \{y \in S |\, y \sim x\}.$$

Some remarks

- According to the definition, the equivalent class $[x]$ contains all elements that are equivalent to class.
- Due to the transitivity property of equivalent relation, it implies that any pair of elements in the same equivalent class are equivalent. That is, $y \sim z$ for all $y, z \in [x]$.
- For every element $x \in S$, one can generate an equivalent class $[x]$. Some of these equivalent classes may duplicate in the sense that they contain exactly the same elements.
  - ✓ We consider $1 \in \mathbb{N}$ and $4 \in \mathbb{N}$ (which $1 \sim 4$) in the above numerical example, we see that $[1] = [4] = \{1,4,7,10, \dots.\}$.
  - ✓ If two equivalent classes are distinct, we observe that they are disjoint in the sense that they do not have the common elements (e.g. $[2]$, $[3]$ in the above example).

Theorem (Properties of equivalent class)

We let S be a set and $\sim$ be the equivalent relation on the set S. We let $[x]$ and $[y]$ be two equivalent classes containing $x \in S$ and $y \in S$ respectively.

(1) If $x \sim y$, then $[x] = [y]$

(2) If $x \nsim y$, then $[x] \cap [y] = \phi$

Proof of the theorem

To prove (1), we shall prove that $[x] \subseteq [y]$ and $[y] \subseteq [x]$.

For any $z \in [x]$, we have $x \sim z$. Since $x \sim y$, it follows from transitivity property that $y \sim z$ and $z \in [y]$. Thus $[x] \subseteq [y]$.

By the similar argument (interchanging the role of x and y in earlier), we get $[y] \subseteq [x]$. Thus, we conclude that $[x] = [y]$.

To prove (2), we assume that $[x] \cap [y] \neq \phi$ and let $z \in [x] \cap [y]$. Since $x \sim z$ and $y \sim z$, it follows from transitivity property that $x \sim y$. It contradicts the fact that $x \nsim y$. So we deduce that $[x] \cap [y] \neq \phi$.

Further examples of equivalent classes

Example 15

We consider the set of non-zero real number $\mathbb{R}\backslash\{0\}$ and define a relation $\sim$ on $\mathbb{R}\backslash\{0\}$ as

$$x \sim y \Leftrightarrow xy > 0.$$

- **(a)** Prove that $\sim$ is an equivalent relation.
- **(b)** Describe the equivalent classes generated from this equivalent relation.

😊Solution

- **(a)** **(Reflexive)** Since $x^2 > 0$ for any non-zero real x, thus $x \sim x$.
  (Symmetric) For any non-zero real numbers x, y such that $x \sim y$, we have $xy > 0$, it follows that $yx = xy > 0$ and $y \sim x$.
  (Transitive) For any non-zero x, y, z such that $x \sim y$ and $y \sim z$, we have $xy > 0$ and $yz > 0$, then $(xy)(yz) = xz(y^2) > 0$. Since $y^2 > 0$, it must be that $xz > 0$. Therefore, $x \sim z$.
  So $\sim$ is equivalent relation.
- **(b)** We take $x_1 > 0$. In order that $x_1 \sim y$ (i.e. $xy > 0$), then y must be positive. Thus, the equivalent class $[x_1]$ contains all positive real number.
  On the other hand, we take $x_2 < 0$. Using similar argument, we see that $y < 0$ in order that $x_2 \sim y$. This the equivalent class $[x_2]$ contains all negative real number.
  Therefore, we conclude that there are two equivalent classes and each equivalent class contains elements with same sign (positive or negative)

Example 16

We let $A = \{a_1, a_2, a_3, \dots, a_n\}$ be a finite set and let $\mathcal{P}(A)$ be the collection of all non-empty subsets of A. That is,

$$\mathcal{P}(A) = \{B \mid B \neq \phi \ \ and \ \ B \subseteq A\}.$$

We let R be a relation on $\mathcal{P}(A)$which $C \sim D$ if there is a bijective function f from C to D.

(a) Show that R is an equivalent relation on $\mathcal{P}(A)$.

(b) Describe the equivalent classes generated from this equivalent relation.

☺Solution

(a) **(Reflexivity)** For any $C = \{a_{n_1}, \dots, a_{n_k}\} \in \mathcal{P}(A)$, then one can construct a bijection map $f: C \to C$ by $f(a_{n_i}) = a_{n_i}$. So $C \sim C$.

(Symmetry) Suppose that $C \sim D$, then there exists a bijective function $f: C \to D$. Then there exists inverse function $f^{-1}: \underbrace{D}_{f(C)} \to C$.

One can show that f^{-1} is bijective:

- (Injective) $f^{-1}(y_1) = f^{-1}(y_2) \Rightarrow f\left(f^{-1}(y_1)\right) = f\left(f^{-1}(y_2)\right) \Rightarrow y_1 = y_2$.

- (Surjective) For any $y \in C$, we take $x = f(y) \in D$, then
$$f^{-1}(x) = f^{-1}\big(f(y)\big) = y.$$

Thus there is a bijective function from D to C, so $D \sim C$.

(Transitive) For any sets $C, D, E \in \mathcal{P}(A)$ which $C \sim D$ and $D \sim E$, there exists bijection functions $f: C \to D$ and exists another bijection function $g: D \to E$. To construct a bijective function from C to E, we consider *composite function* $(g \circ f)(x) = g\big(f(x)\big)$ which is a mapping from C to E.

Using the result in Example 11, we can show that $g\big(f(x)\big)$ is injective and surjective (and hence bijective). So we conclude that $C \sim E$.

Therefore, $\sim$ is the equivalent relation.

(b) We let $C \in \mathcal{P}(A)$ which has k $(1 \leq k \leq n)$ elements. Suppose that $C \sim D$ for some $D \in \mathcal{P}(A)$ and there is bijective function $f: C \to D$, then C and D must have same number of elements. Thus, the equivalence class $[C]$ should contain all subsets with k elements.

Hence, there are n equivalent classes $[C_1], [C_2], \dots, [C_n]$ where $[C_k]$ contains all subsets with k elements.