Additional problems

(1). $(1+\frac{1}{n})^n < (1+\frac{1}{n})^n(1+\frac{1}{n}) = (1+\frac{1}{n})^{n+1}$

so we can find that $(1+\frac{1}{n})^{n+1} \downarrow$

$(1+\frac{1}{n})^n \uparrow$

we $n\to\infty$ the limit is same. we define it as Eular number e.

(2). we first prove $\frac{1}{x+1} < \ln(1+\frac{1}{x}) < \frac{1}{x}$

we have $\frac{1}{x+1} < \frac{1}{t} < \frac{1}{x}$ when $x>0$ $x<t<x+1$

so $\int_{x}^{x+1} \frac{1}{x+1}\,dt < \int_{x}^{x+1} \frac{1}{t}\,dt < \int_{x}^{x+1} \frac{1}{x}\,dt$

$\Rightarrow \frac{1}{x+1} < \ln(1+\frac{1}{x}) < \frac{1}{x}$ for $x>0$

Let $x_n = 1+\frac{1}{2}+\dots+\frac{1}{n} - \ln n$

$x_{n+1} - x_n = \frac{1}{n+1} - \ln(1+\frac{1}{n}) < 0$ so $x_n \downarrow$.

$\ln n = \ln\left(\frac{n}{n-1}\cdot\frac{n-1}{n-2}\cdot\ \cdots\ \frac{2}{1}\right)$

$= \sum_{k=1}^{n-1} \cancel{\ln\left(\frac{k}{k+1}\right)} \ln\left(\frac{k+1}{k}\right)$

$x_n = \sum_{k=1}^{n-1}\left[\frac{1}{k} - \ln(1+\frac{1}{k})\right] + \frac{1}{n}$ $\quad \frac{1}{k} > \ln(1+\frac{1}{k})$

so $x_n > \frac{1}{n} > 0$ $\quad x_n \downarrow$ 0

$\{x_n\}$ monotone and bounded limit exits

define it as Eular constant $\lim_{n\to\infty}(1+\frac{1}{2}+\dots+\frac{1}{n} - \ln n) = \gamma \approx 0.57721566$

γ and e are irrational number.