Math 2033 (Homework 3) Solutions

① Given $x_n \neq -1$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} x_n = 0$, then show that $\lim_{n\to\infty} \frac{x_n}{1+x_n} = 0$ by checking the definition of limit.

Solution For every $\varepsilon > 0$, since $\lim_{n\to\infty} x_n = 0$, $\exists K_1 \in \mathbb{N}$ such that

15 marks $n \geq K_1 \Rightarrow |x_n - 0| < \frac{1}{2} \Leftrightarrow x_n \in (-\frac{1}{2}, \frac{1}{2}) \Leftrightarrow 1 + x_n \in (\frac{1}{2}, \frac{3}{2})$ $\Leftrightarrow \frac{1}{1+x_n} \in (\frac{2}{3}, 2)$ and $\exists K_2 \in \mathbb{N}$ such that $n \geq K_2 \Rightarrow |x_n - 0| < \frac{\varepsilon}{2}$.

5 marks Let $K = \max(K_1, K_2)$. Then $n \geq K \Rightarrow n \geq K_1$ and $n \geq K_2 \Rightarrow$

5 marks $\left|\frac{x_n}{1+x_n} - 0\right| = \frac{|x_n|}{1+x_n} \leq 2|x_n| < \varepsilon$.

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② Let $a_1 = 9$ and $a_{n+1} = \frac{\sqrt{a_n} + 2a_n}{3}$ for $n = 1, 2, 3, \ldots$. Prove that $a_1, a_2, a_3, \ldots$ converges and find its limit.

Solution (Observe that $a_1 = 9$, $a_2 = \frac{\sqrt{9} + 2\times 9}{3} = 7$, $a_3 = \frac{\sqrt{7} + 2\times 7}{3} < \frac{17}{3}$. Suspect $a_1, a_2, a_3, \ldots$ decreasing) We claim $a_n > a_{n+1} > 1$.

10 marks For $n = 1$, $a_1 = 9 > a_2 = 7 > 1$. Suppose $a_n > a_{n+1} > 1$. Then $\sqrt{a_n} > \sqrt{a_{n+1}} > 1$. So $\frac{\sqrt{a_n} + 2a_n}{3} > \frac{\sqrt{a_{n+1}} + 2a_{n+1}}{3} > \frac{\sqrt{1} + 2}{3}$, i.e. $a_{n+1} > a_{n+2} > 1$.

10 marks By the monotone sequence theorem, $a_1, a_2, a_3, \ldots$ converges to some a. Then $a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{\sqrt{a_n} + 2a_n}{3} = \frac{\sqrt{a} + 2a}{3} \Rightarrow a = \sqrt{a}$ $\Rightarrow a = 0$ or 1.

5 marks Since $a_n > 1$, $a \geq 1$. Therefore the answer is 1.

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③ Let $w_1, w_2, w_3, \ldots$ be a sequence such that for $k = 1, 2, 3, \ldots$, we have $|w_{k+1} - w_k| < \frac{1}{2^k}$. Then prove that $w_1, w_2, w_3, \ldots$ is a Cauchy sequence.

Solution For every $\varepsilon > 0$, by Archimedean Principle, there is integer $K > \frac{1}{\varepsilon}$.

15 marks Then $m > n \geq K \Rightarrow |w_m - w_n| = |(w_m - w_{m-1}) + (w_{m-1} - w_{m-2}) + \cdots + (w_{n+1} - w_n)|$ $\leq |w_m - w_{m-1}| + |w_{m-1} - w_{m-2}| + \cdots + |w_{n+1} - w_n|$ $< \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^n}$

5 marks $< \sum_{j=n}^{\infty} \frac{1}{2^j} = \frac{1}{2^{n-1}} \leq \frac{1}{2^{K-1}} \leq \frac{1}{K} < \varepsilon$. The case $m < n$ is similar.

5 marks The case $m = n$ leads to $|w_m - w_n| = 0 < \varepsilon$. Therefore, $w_1, w_2, w_3, \ldots$ is a Cauchy sequence.

④ Show that for every $t \in \mathbb{R}$, there is a strictly increasing sequence of irrational numbers $t_1, t_2, t_3, \ldots$ converging to t.

<u>Solution</u> Let $t \in \mathbb{R}$. By the density of irrational numbers,

8 marks there is $t_1 \in \mathbb{R} \setminus \mathbb{Q}$ such that $t-1 < t_1 < t$. Suppose $t_n < t$

9 marks has been chosen, then we use the density of irrational numbers to choose $t_{n+1} \in \mathbb{R} \setminus \mathbb{Q}$ such that $\max(t_n, t-\frac{1}{n+1}) < t_{n+1} < t$.

8 marks Then $t_n < t_{n+1}$ and $t - \frac{1}{n} < t_n < t$ imply $\lim_{n\to\infty} t_n = t$ by the Sandwich theorem.