Solutions to Practice Exercises

① $\sim((x>0 \text{ and } x<1) \text{ or } x=-1) = \sim(x>0 \text{ and } x<1) \text{ and } x\neq -1$

$= (x\leq 0 \text{ or } x\geq 1) \text{ and } x\neq -1$

② $\sim(x>0 \text{ and } (x<1 \text{ or } x=-1)) = x\leq 0 \text{ or } \sim(x<1 \text{ or } x=-1)$

$= x\leq 0 \text{ or } (x\geq 1 \text{ and } x\neq -1)$

$= x\leq 0 \text{ or } x\geq 1$

③ $\sim(\forall \triangle ABC, \angle A+\angle B+\angle C=180^\circ) = \exists \triangle ABC$ such that $\angle A+\angle B+\angle C\neq 180^\circ$

(There is a triangle ABC such that $\angle A+\angle B+\angle C\neq 180^\circ$.)

④ $\sim(\exists$ man such that man does not have wife$) = \forall$ man, man has a wife

(Every man has a wife.)

⑤ $\sim(\forall x.\ \exists y$ such that $x+y=0) = \exists x\ \forall y,\ x+y\neq 0$

(There is an x such that for every y, $x+y\neq 0$.)

⑥ $\sim(\exists\alpha\ \forall\beta\ \exists\gamma$ such that $|\alpha-\beta|<\gamma) = \forall\alpha\ \exists\beta\ \forall\gamma,\ |\alpha-\beta|\geq\gamma$.

⑦ $\sim($If $(x>0)$ and $(y>0)$, then $x+y>0) = (x>0)$ and $(y>0)$ and $(x+y\leq 0)$

⑧ (a) If $\angle B\neq\angle C$ in $\triangle ABC$, then $AB\neq AC$ in $\triangle ABC$.

(b) If a function is not continuous, then it is not differentiable.

(c) If $\lim_{x\to 0}(f(x)+g(x))\neq a+b$, then $\lim_{x\to 0} f(x)\neq a$ or $\lim_{x\to 0} g(x)\neq b$.

(d) If $x\neq \frac{-b+\sqrt{b^2-4c}}{2}$ and $x\neq\frac{-b-\sqrt{b^2-4c}}{2}$, then $x^2+bx+c\neq 0$.

⑨ (a) $(\{x,y,z\}\cup\{w,z\})\setminus\{u,v,w\} = \{w,x,y,z\}\setminus\{u,v,w\} = \{x,y,z\}$.

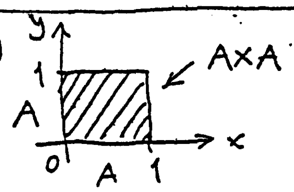
(b) $\{1,2\}\times\{3,4\}\times\{5\} = \{(1,3,5),(1,4,5),(2,3,5),(2,4,5)\}$.

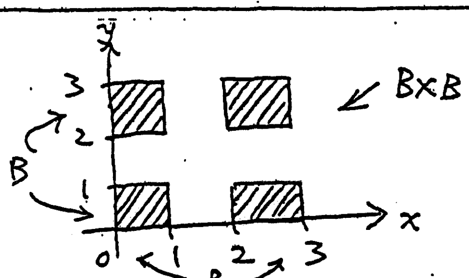
(c) $\mathbb{Z}\cap[0,10]\cap\{n^2+1 : n\in\mathbb{N}\} = \{0,1,2,\cdots,10\}\cap\{2,5,10,\cdots\} = \{2,5,10\}$.

(d) $\{n\in\mathbb{N} : 5<n<9\}\setminus\{2m : m\in\mathbb{N}\} = \{6,7,8\}\setminus\{2,4,6,8,10,\cdots\} = \{7\}$.

(e) $([0,2]\setminus[1,3])\cup([1,3]\setminus[0,2]) = [0,1)\cup(2,3]$.

⑩ (i)





(ii) $A=B$ (Reason: For every $a\in A$, $b\in B$, we have $(a,b)\in A\times B = B\times A$. By the definition of Cartesian product, this means $a\in B$, $b\in A$. So $A\subseteq B$ and $B\subseteq A$.)

(11) (a) If $x \in A \cup B$, then $x \in A$ or $x \in B$, which implies $x \in A$ or $x \in C$ (because $B \subseteq C$ and $x \in B$ will yield $x \in C$). So $x \in A \cup C$.
So every element of $A \cup B$ is also an element of $A \cup C$. Therefore, $A \cup B \subseteq A \cup C$.

(b) If $x \in (X \setminus Y) \setminus Z$, then $x \in X \setminus Y$ and $x \notin Z$. So $x \in X$ and $x \notin Y$ and $x \notin Z$. Then $x \in X$ and $x \notin Z$ and $x \notin Y$. Hence, $x \in X \setminus Z$ and $x \notin Y$. Therefore, $x \in (X \setminus Z) \setminus Y$. We get $(X \setminus Y) \setminus Z \subseteq (X \setminus Z) \setminus Y$.

Interchanging Y, Z everywhere in the last paragraph, we also get $(X \setminus Z) \setminus Y \subseteq (X \setminus Y) \setminus Z$.
Therefore, $(X \setminus Y) \setminus Z = (X \setminus Z) \setminus Y$.

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(12) (i) False. For example, $A = \mathbb{R} \setminus \mathbb{Q}$, $B = \mathbb{Q} = C$, then $(A \cup B) \cap C = \mathbb{R} \cap \mathbb{Q} = \mathbb{Q}$, $A \cup (B \cap C) = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}$

(ii) False. For example, $A = \mathbb{R} = B$, $C = \mathbb{Q}$, then $A \cup B = \mathbb{R} = A \cup C$, but $B \neq C$.

(iii) True. (Reason: For every $x \in A \setminus (B \cup C)$, we have $x \in A$ and $x \notin B \cup C$. Now $\boxed{x \notin B \cup C = \sim(x \in B \cup C) = \sim((x \in B) \text{ or } (x \in C)) = x \notin B \text{ and } x \notin C}$. So $x \in A \setminus B$ and $x \in A \setminus C$. We get $x \in (A \setminus B) \cap (A \setminus C)$. $\therefore$ $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$.
Next we reverse steps. For every $x \in (A \setminus B) \cap (A \setminus C)$, we have $x \in A \setminus B$ and $x \in A \setminus C$. So $x \in A$ and $x \notin B$ and $x \notin C$. By the box above, we get $x \in A$ and $x \notin B \cup C$. So $x \in A \setminus (B \cup C)$. $\therefore$ $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$.)

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(13) (i) For every $x \in A \cup C$, we have $x \in A$ or $x \in C$. If $x \in A$, then $A \subseteq B$ implies $x \in B$. If $x \in C$, then $C \subseteq D$ implies $x \in D$. So $x \in B$ or $x \in D$, which implies $x \in B \cup D$.

(ii) False. For example, let $A = \{0\}$, $C = \{1\}$, $B = \{0, 1\} = D$, then $A \cup C = \{0, 1\} = B \cup D$.

(iii) Yes. (Reason: Since $(\frac{1}{n}, 2)_{\mathbb{Q}} \subseteq [\frac{1}{n}, 2)_{\mathbb{Q}}$ for each n, so as in part (i),

$$\bigcup_{n=1}^{\infty} (\tfrac{1}{n}, 2)_{\mathbb{Q}} = (1,2)_{\mathbb{Q}} \cup (\tfrac{1}{2}, 2)_{\mathbb{Q}} \cup (\tfrac{1}{3}, 2)_{\mathbb{Q}} \cup \cdots \subseteq [1,2)_{\mathbb{Q}} \cup [\tfrac{1}{2}, 2)_{\mathbb{Q}} \cup [\tfrac{1}{3}, 2)_{\mathbb{Q}} \cup \cdots = \bigcup_{n=1}^{\infty} [\tfrac{1}{n}, 2)_{\mathbb{Q}}.$$

For the reverse inclusion, since $[\frac{1}{n}, 2)_{\mathbb{Q}} \subseteq (\frac{1}{n+1}, 2)_{\mathbb{Q}}$ for each n, we have

$$\bigcup_{n=1}^{\infty} [\tfrac{1}{n}, 2)_{\mathbb{Q}} = [1,2)_{\mathbb{Q}} \cup [\tfrac{1}{2}, 2)_{\mathbb{Q}} \cup [\tfrac{1}{3}, 2)_{\mathbb{Q}} \cup \cdots \subseteq (\tfrac{1}{2}, 2)_{\mathbb{Q}} \cup (\tfrac{1}{3}, 2)_{\mathbb{Q}} \cup (\tfrac{1}{4}, 2)_{\mathbb{Q}} \cup \cdots = \bigcup_{n=1}^{\infty} (\tfrac{1}{n}, 2)_{\mathbb{Q}}$$

because $(\frac{1}{2}, 2)_{\mathbb{Q}} = (1,2)_{\mathbb{Q}} \cup (\frac{1}{2}, 2)_{\mathbb{Q}}$.

Actually, $\bigcup_{n=1}^{\infty} [\frac{1}{n}, 2)_{\mathbb{Q}} = (0,2)_{\mathbb{Q}} = \bigcup_{n=1}^{\infty} (\frac{1}{n}, 2)_{\mathbb{Q}}$ but this is less rigorous.

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(14) f is not injective because $f(1) = 0 = f(2)$. f is not surjective because $f(\mathbb{R}) = \{0, 1\} \neq \mathbb{R}$. g is injective because $g(x) = g(y) \Leftrightarrow 1 - 2x = 1 - 2y$ implies $x = y$.
g is surjective because for every $y \in \mathbb{R}$, $y = g(\frac{1-y}{2})$ and so $g(\mathbb{R}) = \mathbb{R}$.
$f \circ g : \mathbb{R} \to \mathbb{R}$ is given by $(f \circ g)(x) = f(g(x)) = f(1-2x) = \begin{cases} 0 & \text{if } 1-2x > 0 \\ 1 & \text{if } 1-2x \leq 0 \end{cases}$
$= \begin{cases} 0 & \text{if } \frac{1}{2} > x \\ 1 & \text{if } \frac{1}{2} \leq x \end{cases}$. $g \circ f : \mathbb{R} \to \mathbb{R}$ is given by $(g \circ f)(x) = g(f(x)) = \begin{cases} 1 = g(0) & \text{if } x > 0 \\ -1 = g(1) & \text{if } x \leq 0 \end{cases}$.

(15) (i) To show f is injective, let $f(x)=f(y)$. Then $x=(g\circ f)(x)=g(f(x))=g(f(y))$ $=(g\circ f)(y)=y$. Next we will show f is surjective. For every $b\in B$, since $b=(f\circ g)(b)=f(g(b))$, we see that $b\in f(A)$. $\therefore$ $f(A)=B$.

(ii) To show $h\circ f$ is injective, let $(h\circ f)(x)=(h\circ f)(y)$. Then $h(f(x))=h(f(y))$. Since h is injective, we get $f(x)=f(y)$. Since f is injective, we get $x=y$. Next we will show $h\circ f$ is surjective. For every $c\in C$, since h is surjective, $C=h(B)$, which implies $c=h(b)$ for some $b\in B$. Since f is surjective, $B=f(A)$, which implies $b=f(a)$ for some $a\in A$. Then $c=h(b)=h(f(a))$ $=(h\circ f)(a)\in(h\circ f)(A)$. $\therefore$ $(h\circ f)(A)=C$.

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(16) For the 'at most once' case, to show f is injective, let $f(x_0)=f(y_0)$. Using the choice $b=f(x_0)$, we see that the line $y=b$ intersects the graph of f at the point $(x_0, f(x_0))$ and at the point $(y_0, f(y_0))$. Since the intersection is at most one point, we have $(x_0, f(x_0))=(y_0, f(y_0))$, which implies $x_0=y_0$.

For the 'at least once' case, we can conclude f is surjective. (Reason: For every $b\in B$, the line $y=b$ intersects the graph of f at least once. This implies there is a point (a,b) on the graph of f. Then $b=f(a)\in f(A)$. $\therefore$ $f(A)=B$.)

(Comments: Combining the two cases, we see that if for every $b\in B$, the horizontal line $y=b$ intersects the graph of f exactly once, then f is a bijection. This "horizontal line test" is useful to check bijections by inspecting the graphs.)

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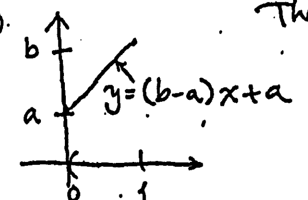
(17)



The function $f:(0,1)\to(a,b)$ defined by $f(x)=(b-a)x+a$ is a bijection. (This is clear from the graph. As x varies from a to b, $f(x)$ takes each of the values between a and b exactly once.) Since $(0,1)$ is uncountable, by the bijection theorem we see that (a,b) is uncountable. Since $(a,b)\subseteq[a,b]$, by the countable subset theorem, $[a,b]$ is uncountable.

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(18) Let $S=\{(0,y): y\in\mathbb{R}\setminus\mathbb{Q}\}$. The function $f:\mathbb{R}\setminus\mathbb{Q}\to S$ with $f(y)=(0,y)$ is a bijection as $f^{-1}:S\to\mathbb{R}\setminus\mathbb{Q}$ with $f^{-1}(0,y)=y$ is its inverse. Since $\mathbb{R}\setminus\mathbb{Q}$ is uncountable, by the bijection theorem, S is uncountable. Since $S\subseteq\mathbb{Q}\times(\mathbb{R}\setminus\mathbb{Q})$, by the countable subset theorem, $\mathbb{Q}\times(\mathbb{R}\setminus\mathbb{Q})$ is uncountable.

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(19) For $n,m\in\mathbb{Z}$, $\frac{1}{2^n}+\frac{1}{3^m}\in\mathbb{Q}$. So $A\subseteq\mathbb{Q}$. Since $\mathbb{Q}$ is countable, by the countable subset theorem, A is countable.
