MATH 2031 Introduction to Real Analysis

April 9, 2013

Tutorial Note 17

Riemann Integral (Con't Proper Integral)

(I) **Definition:**

(i) A set $S \subseteq \mathbb{R}$ is of measure zero iff $\left(\begin{array}{c} \forall \varepsilon > 0,\ \exists \text{ intervals } (a_1, b_1), (a_2, b_2) \cdots \text{ such that} \\ S \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \text{ and } \sum_{k=1}^{\infty} (b_k - a_k) < \varepsilon \end{array} \right)$.

(ii) A property is said to hold almost everywhere (a.e.) iff the property holds except on a set of measure zero.

Remarks:

(i) Countable sets are of measure zero, but uncountable sets may or may not be of measure zero. (The set of irrational numbers in $[a, b]$ for $a < b$ has measure $b - a$, while the Cantor set is uncountable but of measure zero)

(ii) A countable union of measure zero sets is also of measure zero.

(iii) Subsets of a measure zero set are again of measure zero.

(iv) The limit of a sequence of Riemann integrable functions on $[a, b]$ may not be a Riemann integrable function.

(II) **Lebesgue's Theorem**
Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

$$f \text{ is integrable on } [a, b] \iff f \text{ is continuous a.e. on } [a, b]$$

(III) **Definition:**
For an integrable function $f(x)$ on $[a, b]$ and $c \in [a, b]$,
the function $F(x) = \displaystyle\int_c^x f(t)dt$ is called an anti-derivative (or primitive function) of f.

(IV) **Theorem**
If f is an integrable function on $[a, b]$ and $c \in [a, b]$, then $F(x) = \displaystyle\int_c^x f(t)dt$ is uniformly continuous on $[a, b]$.

(V) **Fundamental theorem of Calculus**

(i) If f is integrable on $[a, b]$, continuous at $x_0 \in [a, b]$ and $F(x) = \displaystyle\int_c^x f(t)dt$, where $c \in [a, b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

(ii) If G is differentiable on $[a, b]$ and G' is integrable on $[a, b]$, then $\displaystyle\int_a^b G'(x)dx = G(b) - G(a) = G\Big|_a^b$.
(G' may not be continuous.)

(VI) **Properties:**

(i) For $c \in [a,b]$, f is integrable on $[a,b] \iff f$ is integrable on $[a,c]$ and on $[c,b]$.

(ii) (i) If $f, g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$, then $f \pm g$, fg, are also integrable on $[a,b]$.

(ii) If $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ and g is bounded and continuous on $f([a,b])$, then $g \circ f$ is also integrable on $[a,b]$.

Remark:
Even if both $f : [a,b] \to [c,b]$ is integrable on $[a,b]$ and $g : [c,b] \to \mathbb{R}$ is integrable on $[c,d]$, $g \circ f$ may not be integrable on $[a,b]$.

(iii) Let f, g be integrable on $[a,b]$

(i) $\displaystyle\int_a^b (f(x) \pm g(x))\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

$\forall c \in \mathbb{R}, \quad \displaystyle\int_a^b cf(x)dx = c\int_a^b f(x)dx$

(ii) If $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$. Also, $\displaystyle\left|\int_a^b f(x)dx\right| = \int_a^b |f(x)|\, dx$

(iii) $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, for all $c \in [a,b]$

(iv) **Integration by parts**
If f, g are differentiable on $[a,b]$ and f', g' are integrable on $[a,b]$, then

$$\int_a^b f(x)g'(x)dx = fg\Big|_a^b - \int_a^b f'(x)g(x)dx$$

(v) **Change of variable Formula**
If $\phi : [a,b] \to \mathbb{R}$ is differentiable, ϕ' is integrable on $[a,b]$ and f continuous on $\phi([a,b])$, then

$$\int_{\phi(a)}^{\phi(b)} f(t)dt = \int_a^b f(\phi(x))\phi'(x)dx$$

Problem 1 Show that the following sets are of measure zero:

(i) $A = \left\{x \middle| \sin x = \dfrac{1}{n}, n \in \mathbb{N}\right\}$

(ii) $B = \{x | f_1(x)f_2(x)\cdots f_n(x) = 0\}$, where $\{f_k\}_{k=1}^n$ is a collection of real-valued functions such that each of the sets $\{x | f_k(x) = 0\}$ is of measure zero

(iii) $C = \{x | g_1(x) + g_2(x) + \cdots + g_n(x) = 0\}$, where $\{g_k\}_{k=1}^n$ is a collection of non-negative real-valued functions such that each of the sets $\{x | g_k(x) = 0\}$ is of measure zero

Solution:

(i) Notice that

$$A = \left\{x \middle| \sin x = \frac{1}{n}, n \in \mathbb{N}\right\} = \bigcup_{n\in\mathbb{N}} \left\{x \middle| \sin x = \frac{1}{n}\right\} = \bigcup_{n\in\mathbb{N}} \underbrace{\left\{x \middle| 2k\pi + \arcsin\frac{1}{n}, k \in \mathbb{Z}\right\}}_{\text{countable}}$$

Since A is a countable union of countable sets, A is countable. Thus A is of measure zero.

(ii) Since we have $f_1(x)f_2(x)\cdots f_n(x) = 0$ iff one of the $f_k(x) = 0$, we have

$$B = \{x | f_1(x)f_2(x)\cdots f_n(x) = 0\} \subseteq \bigcup_{k=1}^n \{x | f_k(x) = 0\}$$

Since the latter set is a countable union of measure zero sets, it is also of measure zero.
Now B is a subset of a measure zero set, so B is also of measure zero.

(iii) Since all g_k are non-negative functions, we have

$$g_1(x) + g_2(x) + \cdots + g_n(x) = 0 \qquad \Rightarrow \qquad g_k(x) = 0, \quad \forall k \in \{1, 2, \cdots, n\}$$

From this we get,

$$C = \{x | g_1(x) + g_2(x) + \cdots + g_n(x) = 0\} = \bigcap_{k=1}^{n} \{x | g_k(x) = 0\}$$

The latter set is a subset of $\{x|g_k(x) = 0\}$ for every $k \in \{1, 2, \cdots, n\}$, which is of measure zero. Therefore C is of measure zero.

Problem 2 Let $f, g : [a, b] \to \mathbb{R}$ be a monotone function, show that $f(x)\pm g(x)$, $\max\{f(x), g(x)\}$ and $\min\{f(x), g(x)\}$ are all integrable on $[a, b]$.

Solution:
Recall that Monotone Function Theorem asserts that a monotone function on $[a, b]$ has at most countably many discontinuous points.

Let S_f, S_g be the sets of discontinuous points of f and g on $[a, b]$ respectively. Then by the theorem, S_f and S_g are both most countable, thus they are both of measure zero.

Then the set of discontinuous points of $f(x) \pm g(x)$ on $[a, b]$, $S_{f(x)\pm g(x)} \subseteq S_f \cup S_g$.
As $S_{f(x)\pm g(x)}$ is a subset of a measure zero set, it is also of measure zero.
Therefore, $f(x) \pm g(x)$ is continuous a.e. and by Lebesgue theorem, $f(x) \pm g(x)$ is integrable.

The cases of $\max\{f(x), g(x)\}$ and $\min\{f(x), g(x)\}$ are very similar as above, once you have checked that

$$\max\{f(x), g(x)\} = \frac{1}{2}\big[f(x) + g(x) + |f(x) - g(x)|\big] \text{ and } \min\{f(x), g(x)\} = \frac{1}{2}\big[(f(x) + g(x)) - |f(x) - g(x)|\big]$$

and then apply similar argument as above.
These 2 cases are left as exercises.

Problem 3 Let g be an integrable function on $[a, b]$ for $a, b \in \mathbb{R}$, define $f : [a, b] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is a prime number} \\ g(x) & \text{otherwise} \end{cases}$$

Prove that f is integrable on $[a, b]$.

Solution:
Similar as before, we will check on the set S_f of discontinuous points of f on $[a, b]$, and conclude by Lebesgue Theorem.

Since g is integrable on $[a, b]$, the set S_g of discontinuous points of g on $[a, b]$ is of measure zero.
Also, the set of all prime numbers on $[a, b]$, denoted as $\mathcal{P}_{[a,b]}$, is finite and thus again of measure zero.

Then we get $S_f \subseteq S_g \cup \mathcal{P}_{[a,b]}$. Since S_f is a subset of a union of 2 measure zero sets, S_f is also of measure zero.

Now f is continuous a.e. so it is integrable on $[a, b]$ by Lebesgue Theorem.

Remark:
S_f does not necessarily equal to $S_g \cup \mathcal{P}_{[a,b]}$. Even if f is piecewisely defined, f can still be continuous at those "junctions".