MATH202 Introduction to Analysis (2007 Fall & 2008 Spring)

Tutorial Note #22

Riemann Integration (Part 2)

Recall the criteria of integrability

Theorem: (Integral Criterion)
$f(x)$ is **Riemann-Integrable** if and only if for any $\varepsilon > 0$, there exists a partition P on $[a, b]$ such that $|U(P, f) - L(P, f)| < \varepsilon$

See the tutorial note #19 for the examples.

Here when we apply the integral criteria to show some functions are integrable or not integrable, note that

1) When we prove some functions are integrable, to find the partition
   (Step 1) Sketch the partition first (say uniform cutting)
   (Step 2) Specify the detail of your partition to make $|U(P, f) - L(P, f)| < \varepsilon$

2) When we prove some functions are not integrable, we need to show FOR EVERY PARTITION P, we have $U(P, f) - L(P, f) \geq c$ for some **non-zero number** c.

Even though this criterion is simple but in some situations, when the function is too "complicated", then using the criterion is tedious. Here there is another useful theorem of identifying whether the function is integrable or not.

Lebesgue's Theorem
For a bounded function $f: [a, b] \to \mathbf{R}$, we say f is Riemann integrable on $[a, b]$ if and only if the set of discontinuity points of f $(\text{call it } D_f)$ is of **measure 0** (in order word, $f(x)$ is continuous almost everywhere)

Question: What is measure? What do we meant by measure 0?

In **R**, when we try to measure some sets,

If the set is non-empty interval (say (a, b), $[a, b]$), we can just simply compute the length by $b - a$

-----------(———————————————)--------------------

a b

If the set consists of some disjoint intervals, we can compute the length by adding the lengths of each interval

------(——)-----(——)-----(–)-------(——————)-----------

a b c d e f g h

Total length: $(b - a) + (d - c) + (f - e) + (g - h)$

If the set consists of something that we can't measure the length PHYSICALLY, (for example: 3 distinct points). Then in this case, we have no chance but to approximate their length by measuring the length of some "measurable" things (which is interval for this case)

-------------------(---**x**---)------(---**x**---)----------(---**x**---)-------------------------

$a_1 \quad b_1 \quad a_2 \quad b_2 \quad a_3 \quad b_3$

Hence the length of 3 points $\leq (b_1 - a_1) + (b_2 - a_2) + (b_3 - a_3)$

One can try to make these intervals as small as possible, then the upper bound $(b_1 - a_1) + (b_2 - a_2) + (b_3 - a_3)$ is getting smaller and smaller, finally we can obtain the total length of 3 points should be very close to 0. Then in this case, these 3 points are of **measure 0**.

With this intuition in our mind, here comes a definition of measure 0 set,

> Definition (Measure 0 Set)
>
> A set $S \subseteq \mathbf{R}$ is said to be of measure 0 if and only if for any $\varepsilon > 0$, there is an interval $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n), \ldots.$ such that
>
> 1. $\mathbf{S} \subseteq \bigcup_{\mathbf{n=1}}^{\infty} (\mathbf{a_n}, \mathbf{b_n})$ (which the set is approximated by countable union of intervals)
> 2. $\sum_{\mathbf{n=1}}^{\infty} (\mathbf{b_n} - \mathbf{a_n}) < \varepsilon$ (i.e. the total length of these interval can be very small)

> Example 1
>
> Show that $S = \{x_1, x_2, x_3\}$ is of measure 0

Solution:

For any $\varepsilon > 0$, for each x_i pick a small interval (a_i, b_i) containing x_i with length $< \frac{\varepsilon}{3}$, then we have $S \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$ and $\sum_{n=1}^{3} (b_n - a_n) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

> Example 2
>
> Show that $S = \{x_1, x_2, x_3, x_4, \ldots.\}$ (i.e. any countable set) is of measure 0

☹ **<u>Wrong solution</u>**

For any $\varepsilon > 0$, for each x_i pick a small interval (a_i, b_i) containing x_i with length $< \delta$ (for very small δ, then we have $S \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$ and $\cancel{\sum_{n=1}^{\infty} (b_n - a_n) < \delta + \delta + \delta + \cdots < \varepsilon}$ (WRONG!!!!!!!!!!!).

(**Caution**, for any fix $\delta > 0$, when we add it for infinite many times, it MUST be equal to infinity whenever how small of your δ is.)

☺**<u>Correct Solution: (Use Geometric Series Trick)</u>**

For any $\varepsilon > 0$, for each x_n pick a small interval (a_n, b_n) containing x_i with length $< \frac{\varepsilon}{2^n}$, then $S \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$ and $\sum_{n=1}^{\infty} (b_n - a_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$.

So S is of measure 0.

In summary, we have the following examples of measure 0 sets

1. Empty Set
2. Countable set (Either finite or infinite)
3. **Countable union** of any measure 0 sets (The proof is in the exercise)

   <u>(Caution: The uncountable union of measure 0 set MAY NOT measure 0!)</u>

   <u>Counter-Example:</u>

   A single element $\{\mathbf{x}\}$ is of measure 0.

   But $[\mathbf{0},\mathbf{1}] = \cup_{\mathbf{x}\in[\mathbf{0},\mathbf{1}]} \{\boldsymbol{x}\}$ which is a uncountable union and the length is 1.
4. Cantor Set (Although it is uncountable)

(Remark: Please bear in your mind that <u>SOME uncountable sets may be measure 0</u>. So when you see the set is uncountable, DON'T say it has positive length.)

Example 3

a) Show that the set $\{x: \cos x = \frac{1}{n}, n \in \mathbf{N}\}$ is of measure 0

b) Suppose let $f_1(x), f_2(x), \ldots . f_n(x)$ be n functions such that $\{x: f_n(x) = 0\}$ is of measure 0, show that $\{x: f_1(x)f_2(x) \ldots f_n(x) = 0\}$ is of measure 0.

Solution:

a) First, note that

$$\left\{x: \cos x = \frac{1}{n}, n \in \mathbf{N}\right\} = \cup_{n=1}^{\infty} \left\{x: \cos x = \frac{1}{n}\right\} = \cup_{n=1}^{\infty} \left\{2m\pi + \cos^{-1}\left(\frac{1}{n}\right): m \in \mathbf{N}\right\}$$

Since the set is countable and therefore is of measure 0.

b) Note that $f_1 f_2 \ldots f_n = 0 \rightarrow f_1 = 0 \text{ or } f_2 = 0 \text{ or } \ldots \ldots \text{ or } f_n = 0$

Hence $\{x: f_1(x)f_2(x) \ldots f_n(x) = 0\} = \cup_{k=1}^{n} \{x: f_k(x) = 0\}$, because each of $\{x: f_k(x) = 0\}$ is of measure 0, hence by (3), the original set is also of measure 0

Now let go back to integrability problems.

Example 4:

Which of the following functions are Riemann integrable?

a) $f(x) = 1$

b) $f(x) = \begin{cases} a & \text{if } x = \cdots - 3, -2, -1, 0, 1, 2, 3, \ldots . \\ b & \text{if otherwise} \end{cases}$

c) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{R}\backslash\mathbf{Q} \end{cases}$ in $[0,1]$

d) $f(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ 1 - x & \text{if } x \in \mathbf{R}\backslash\mathbf{Q} \end{cases}$ in $[0,1]$

Solution:

a) Since $f(x)$ is constant and therefore continuous, hence the set of discontinuity is empty and therefore is of measure 0. So $f(x)$ is integrable

b) We can show that $f(x)$ is discontinuous only at $x = ..-2,-1,0,1,2,....$ and continuous at other points. Hence $D_f = \{...-3,-2,1,0,1,2,3..\} = \mathbf{Z}$ and therefore countable. So D_f is of measure 0, hence $f(x)$ is integrable

c) Since $f(x)$ is not continuous at every x_0 in $[0,1]$
Proof: For any $x_0 \in [0,1]$, take a rational sequence $\{r_n\}$ and irrational sequence $\{q_n\}$ such that $r_n \to x_0$ and $q_n \to x_0$,
then $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} 0 = 0$
So by sequential continuity theorem, $f(x)$ is not continuous at x_0.
So $D_f = [0,1]$ which the length is $1 \neq 0$, hence $f(x)$ is not Riemann Integrable.

d) (Technique: When seeing such complicated function, we first find all points where $f(x)$ may be continuous.)
If $f(x)$ is continuous at x_0, then take a rational sequence $\{r_n\}$ and irrational sequence $\{q_n\}$ such that $r_n \to x_0$ and $q_n \to x_0$.
By sequential continuity theorem, we get

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} f(q_n) \to \lim_{n\to\infty} r_n = \lim_{n\to\infty} 1 - q_n \to x_0 = 1 - x_0 \to x_0 = \frac{1}{2}$$

So the only point that the function MAY be continuous is at $x_0 = \frac{1}{2}$. Hence for other points, $f(x)$ MUST be discontinuous. So $D_f \supseteq [0,\frac{1}{2}) \cup (\frac{1}{2},1]$, the R.H.S. has length 1. Therefore the length of $D_f \geq 1 \neq 0$. So $f(x)$ is not integrable on $[0,1]$

Example 5
Let $f(x)$ and $g(x)$ are monotone functions, prove that $f(x) + g(x)$ and $\max\{f(x), g(x)\}$ are Riemann-integrable

Solution:
Since $f(x)$ and $g(x)$ are both monotone and therefore each of them should have countably many discontinuity points (D_f and D_g are countable).

To see whether $f(x) + g(x)$ is integrable, we look at its discontinuity points (D_{f+g}),

Note that $f(x)$ and $g(x)$ are continuous at $x_0 \to f(x) + g(x)$ is continuous at x_0.
By taking contrapositive, we get $f(x) + g(x)$ is discontinuous at x_0, then either $f(x)$ or $g(x)$ is discontinuous at x_0.
Then we have $D_{f+g} \subseteq D_f \cup D_g$

Since D_f and D_g are countable, then $D_f \cup D_g$ is countable, then D_{f+g} is also countable by countable subset theorem. Therefore D_{f+g} is of measure 0. Then $f(x) + g(x)$ is integrable.

For the $\max\{f(x), g(x)\}$, note that $\max\{f(x), g(x)\} = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$, use the fact that if f, g are continuous→ $\frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ is continuous. The argument is similar to the previous one.

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Example 6 (Practice Exercise #124)

Let $f(x)$, $g(x)$ be two integrable functions on [0,2], show that the function defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in [0,1] \\ g(x) & \text{if } x \in (1,2] \end{cases}$$

is Riemann Integrable.

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Solution:

Let D_f and D_g be the discontinuities points of $f(x)$ and $g(x)$ on $[0.2]$ respectively, since f and g are Riemann integrable

Now look at the discontinuous of $h(x)$, the set of discontinuity points

$$\begin{aligned} D_h &\subseteq \{\text{set of discontinuity points on } [0,1)\} \cup \{\text{set of discontinuity points on } (1,2]\} \cup \{1\} \\ &= \{D_f \cap [0,1)\} \cup \{D_g \cap (1,2]\} \cup \{1\} \end{aligned}$$

Note that $D_f \cap [0,1)$ and $D_g \cap (1,2]$ are measure 0 and $\{1\}$ is also measure 0, then the union is also measure 0. So D_h is also of measure 0. Therefore $h(x)$ is Riemann Integrable.

<table><tr><td>Try to work on the following exercises to understand the material, you are welcome to give your solution to me for comments.</td></tr></table>

☺Exercise 1

Which of the following sets are of measure 0?

a) $A = \{x: x^3 - 3x + 2 = 0\}$
b) $B = \{x: f_1(x) + f_2(x) + \cdots + f_n(x) = 0\}$ where f_n's are non-negative functions and $\{x: f_k(x) = 0\}$ is of measure 0 for $k = 1,2,3, \ldots\ldots, n$
c) $C = \{x: \sin x \in \left(\frac{1}{2}, 1\right)\}$
d) $D = \{x: f(x) = 1,2,3, \ldots.\}$ where $f(x)$ is injective function

☺Exercise 2

Determine when the following functions are integrable or not

a) for $x = [-M, M]$ $f(x) = \begin{cases} a & \text{if } x \in \mathbf{Z} \\ b & \text{otherwise} \end{cases}$

b) for $x = [-M, M]$ $f(x) = \begin{cases} a & \text{if } x \in \mathbf{Q} \\ b & \text{otherwise} \end{cases}$ where $a \neq b$

c) for $x \in [0,1]$, $f(x) = \begin{cases} 2^{-n} & \text{if } x \in \left(2^{-(n+1)}, 2^{-n}\right], n \in \mathbf{N} \\ 0 & \text{if } x = 0 \end{cases}$

d) for $x \in [-1,1]$, $f(x) = \begin{cases} 2x & \text{if } x \in \mathbf{Q} \\ x^2 + 8 & \text{otherwise} \end{cases}$

e) for $x \in [-1,1], f(x) = [x]$

☺Exercise 3 (Practice Exercise #125)

If $f, g : [0,2] \to [0,1]$ are Riemann integrable, show that the function $h(x) = \min\{f(x), g(x)\}$ is Riemann integrable on $[0,1]$

☺Exercise 4 (Practice Exercise #157, 2002 Final)

Let $f, g: [0,2] \to \mathbf{R}$ be Riemann integrable. Prove that $h: [0,2] \to \mathbf{R}$ defined by

$$h(x) = \begin{cases} \max\left(f(x), g(x)\right) & \text{if } x \in [0,1] \\ \min\left(f(x), g(x)\right) & \text{if } x \in (1,2] \end{cases}$$

is also Riemann Intergable

☺Exercise 5 (2007 Spring Final)

Let $f_1, f_2: [0,1] \to \mathbf{R}$ be monotone functions, Prove that the function $h: [0,1] \to \mathbf{R}$ defined by $h(x) = \begin{cases} f(x) - g(x) \text{ if } x \in [0, \frac{1}{2}) \\ f(x) + g(x) \text{ if } x \in [\frac{1}{2}, 1] \end{cases}$ is Riemann Integrable on $[0,1]$

☺Exercise 6

Show that if $f(x)$ is Riemann Integrable and $g(x)$ is monotone, then $e^{f(x)g(x)}$ is also Riemann Integrable.

☺Exercise 7

Let $f, g: [a, b] \to \mathbf{R}.$ Assume f is bounded and g is Riemann Integrable on $[a, b]$ and suppose the set $\{x \in [a, b]: f(x) \neq g(x)\}$ has only finitely many points. Show that f is also Riemann Integrable and $\int_a^b f(x)\, dx = \int_a^b g(x) dx.$

(Hint: Note that $f(x) = \left(f(x) - g(x)\right) + g(x)$)

Is the statement remains true if the set $\{x \in [a, b]: f(x) \neq g(x)\}$ is of measure 0 (but may not finitely many points?

☺Exercise 8

Suppose that if $f \geq 0$ and $\int_a^b f(x) dx = 0$, is it true that $f(x) = 0$ for all $x \in [a, b]$

☺Exercise 9

Suppose $|f|$ is integrable on $[0,1]$, is it always true that f is also integrable on [0,1]?

☺Exercise 10 (True or False)

If your answer is YES, give a short proof, if your answer is no, provide a counter example.

a) Measure 0 sets MUST BE countable.

b) A set which has <u>non-zero measure</u> MUST BE uncountable

c) Given a sequence of functions $f_n : [a,b] \to \mathbf{R}$ are Riemann Integrable, then the limit $f(x) = \lim_{n\to\infty} f_n(x)$ (if exist) is also Riemann Integrable.

d) Suppose that $f(x) \geq 0$ and $\int_a^b f(x)dx = 0$, furthermore if $f(x)$ is continuous on $[a,b]$, then $f(x) = 0$ for all $x \in [a,b]$ (Compare your answer with Exercise 8)