Math 2033 Tutorial Exercises (on Limits of Sequences)

Definitions ① A sequence $x_1, x_2, x_3, \dots \in \mathbb{R}$ has a limit $x \in \mathbb{R}$ if and only if for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ (depends on ε) such that $n \geq K \Rightarrow |x_n - x| < \varepsilon$ (See p. 48 of Lecture Notes). ② $[x]$ denotes the largest integer less than or equal to x.

We will give proofs to the limits of some sequences. Here are facts you can use: $\big||a|-|b|\big| \leq |a-b|$ for all $a, b \in \mathbb{R}$ and $|\sqrt{a}-\sqrt{b}| \leq \sqrt{|a-b|}$ for all $a, b \geq 0$.

Try to do the following exercises:

① Prove $\lim_{n\to\infty} a_n = A$ implies $\lim_{n\to\infty} \frac{a_n + a_{n+1}}{2} = A$ (using the definition of limit of sequences).

② Let $x > 0$ and $a_n = \frac{[x]+[2x]+\dots+[nx]}{n^2}$. Prove $\lim_{n\to\infty} a_n = \frac{x}{2}$.

③ Let $x_n \neq -1$ for all n and $\lim_{n\to\infty} \frac{x_n}{x_n+1} = \frac{1}{2}$. Prove $\lim_{n\to\infty} x_n = 1$.

④ If $\lim_{n\to\infty} a_n = 1$ and all $a_n \neq n$, then prove that $\lim_{n\to\infty} \frac{a_n^2+n}{n-a_n} = 1$.

⑤ Prove that if $\lim_{k\to\infty} x_{2k} = 0.5$ and $\lim_{k\to\infty} x_{2k+1} = 0.6$, then $\lim_{n\to\infty} x_n^n = 0$ by using the definition of limit of sequences.

Math 2033 Solutions of Tutorial Exercises (on limit of Sequences)

(1) Prove $\lim_{n\to\infty} a_n = A$ implies $\lim_{n\to\infty} \frac{a_n + a_{n+1}}{2} = A$ (using the definition of limit of sequences).

Solution For every $\varepsilon > 0$, since $\lim_{n\to\infty} a_n = A$, by the definition of limit of sequence, there exists $K \in \mathbb{N}$ such that $n \geq K \Rightarrow |a_n - A| < \varepsilon$. Then

$$n \geq K \Rightarrow n+1 \geq K \Rightarrow \left|\frac{a_n + a_{n+1}}{2} - A\right| = \left|\frac{a_n - A}{2} + \frac{a_{n+1} - A}{2}\right|$$
$$\leq \left|\frac{a_n - A}{2}\right| + \left|\frac{a_{n+1} - A}{2}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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(2) Let $x > 0$ and $a_n = \dfrac{[x] + [2x] + \cdots + [nx]}{n^2}$. Prove $\lim_{n\to\infty} a_n = \frac{x}{2}$.

Solution We have $y - 1 < [y] \leq y$. So

$$\frac{(x-1) + (2x-1) + \cdots + (nx-1)}{n^2} < a_n \leq \frac{x + 2x + \cdots + nx}{n^2},$$

i.e. $$\frac{\frac{n(n+1)}{2}x - n}{n^2} = \frac{(n+1)x}{2n} - \frac{1}{n} < a_n \leq \frac{\frac{n(n+1)}{2}x}{n^2} = \frac{(n+1)x}{2n}.$$

Since $\lim_{n\to\infty} \left(\frac{(n+1)x}{2n} - \frac{1}{n}\right) = \frac{x}{2} = \lim_{n\to\infty} \frac{(n+1)x}{2n}$, by the Sandwich theorem, $\lim_{n\to\infty} a_n = \frac{x}{2}$.

③ Let $x_n \neq -1$ for all n and $\lim\limits_{n\to\infty} \frac{x_n}{x_n+1} = \frac{1}{2}$. Prove $\lim\limits_{n\to\infty} x_n = 1$.

Solution Since $\lim\limits_{n\to\infty} \frac{x_n}{x_n+1} = \frac{1}{2}$, $\exists K_1 \in \mathbb{N}$ such that $n \geq K_1$ implies $\left|\frac{x_n}{x_n+1} - \frac{1}{2}\right| < \frac{1}{4}$. Then $\frac{x_n}{x_n+1} - \frac{1}{2} < \frac{1}{4} \Rightarrow \frac{x_n}{x_n+1} < \frac{3}{4}$ $\Rightarrow \frac{1}{x_n+1} > 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow 0 < x_n+1 < 4$. Next, $\exists K_2 \in \mathbb{N}$ such that $\left|\frac{x_n}{x_n+1} - \frac{1}{2}\right| = \left|\frac{x_n - 1}{2(x_n+1)}\right| < \frac{\varepsilon}{8}$. Let $K = \max(K_1, K_2)$. Then $n \geq K \Rightarrow n \geq K_1$ and $n \geq K_2 \Rightarrow |x_n - 1| = \left|\frac{x_n-1}{2(x_n+1)}\right| \cdot 2|x_n+1|$ $< \frac{\varepsilon}{8} \cdot 2 \cdot 4 = \varepsilon$.

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④ If $\lim\limits_{n\to\infty} a_n = 1$ and all $a_n \neq n$, then prove that $\lim\limits_{n\to\infty} \frac{a_n^2 + n}{n - a_n} = 1$.

Solution Since $\lim\limits_{n\to\infty} a_n = 1$, for $\varepsilon_0 = 1$, $\exists$ (there exists) $K_0 \in \mathbb{N}$ such that $n \geq K_0 \Rightarrow |a_n - 1| < \varepsilon_0 = 1 \Rightarrow a_n \in (0, 2)$.

Let $K = \max\left(K_0, \left[\frac{6}{\varepsilon} + 2\right]\right)$. Then

$n \geq K \Rightarrow n \geq K_0$ and $n \geq \frac{6}{\varepsilon} + 2$

$$\Rightarrow \left|\frac{a_n^2 + n}{n - a_n} - 1\right| = \left|\frac{a_n^2 + a_n}{n - a_n}\right| < \frac{2^2 + 2}{n - 2} = \frac{6}{n-2} \leq \varepsilon.$$

(5) Prove that if $\lim_{k\to\infty} x_{2k} = 0.5$ and $\lim_{k\to\infty} x_{2k+1} = 0.6$, then $\lim_{n\to\infty} x_n^n = 0$ by using the definition of limit of sequences.

<u>Solution</u> (i) $\lim_{k\to\infty} x_{2k} = 0.5 \Rightarrow$ for $\varepsilon_0 = 0.2$, $\exists K_0 \in \mathbb{N}$, $k \geq K_0$

$\Rightarrow |x_{2k} - 0.5| < \varepsilon_0 \Rightarrow x_{2k} \in (0.3, 0.7)$.

(ii) $\lim_{k\to\infty} x_{2k+1} = 0.6 \Rightarrow$ for $\varepsilon_1 = 0.1$, $\exists K_1 \in \mathbb{N}$, $k \geq K_1$

$\Rightarrow |x_{2k+1} - 0.6| < \varepsilon_1 \Rightarrow x_{2k+1} \in (0.4, 0.7)$.

(iii) For all $\varepsilon > 0$, $(0.7)^n \leq \varepsilon \Leftrightarrow n \geq \left[\frac{\ln \varepsilon}{\ln 0.7}\right]$.

Let $K = \max\left(2K_0, 2K_1 + 1, \left[\frac{\ln \varepsilon}{\ln 0.7}\right]\right)$.

Then $n \geq K \Rightarrow n \geq 2K_0$ and $n \geq 2K_1 + 1$ and $n \geq \left[\frac{\ln \varepsilon}{\ln 0.7}\right]$

(iv) <u>Case n is even</u> $n = 2k \geq 2K_0 \Rightarrow k \geq K_0 \Rightarrow |x_n^n - 0| = x_{2k}^{2k} < (0.7)^n \leq \varepsilon$.

<u>Case n is odd</u> $n = 2k+1 \geq 2K_1 + 1 \Rightarrow k \geq K_1 \Rightarrow |x_n^n - 0| = x_{2k+1}^{2k+1} < (0.7)^n \leq \varepsilon$.

So for $n \geq K$, we get $|x_n^n - 0| < \varepsilon$.