Math 202 Exercises on Landau's Big-Oh and Little-Oh Notations

301. For $c \in \mathbb{R}$, a *neighborhood* of c is any interval of the form $(c-\varepsilon, c+\varepsilon)$, where $\varepsilon > 0$. For $c = +\infty$, a *neighborhood* of c is any interval of the form $(a, +\infty)$, where $a \in \mathbb{R}$. For $c = -\infty$, a *neighborhood* of c is any interval of the form $(-\infty, a)$, where $a \in \mathbb{R}$.

    (a) Prove that if N_1 and N_2 are neighborhoods of c, then $N_1 \cap N_2$ is also a neighborhood of c.

    (b) Let I be an interval of positive length that contains c or has c as an endpoint. Let $f(x)$ and $g(x)$ be functions on I. In some books, the phrase "$f(x) = O\big(g(x)\big)$ *as* $x \to c$" is defined as there exist a neighborhood N of c and $A \in \mathbb{R}$ such that for every $x \in I \cap N$, we have $|f(x)| \le A|g(x)|$.

    In that case, prove that $O\big(g(x)\big) \pm O\big(g(x)\big) = O\big(g(x)\big)$ and $o\big(g(x)\big) \pm O\big(g(x)\big) = O\big(g(x)\big)$ as $x \to c$.

302. (a) Prove that $o\big(g(x)\big) \pm o\big(g(x)\big) = o\big(g(x)\big)$ as $x \to c$.

    (b) Prove that $o\big(g(x)\big) \pm O^*\big(g(x)\big) = O^*\big(g(x)\big)$ as $x \to c$.

    (c) Prove that $o\big(g_1(x)\big)o\big(g_2(x)\big) = o\big(g_1(x)g_2(x)\big)$ as $x \to c$.

    (d) Prove that $o\big(g_1(x)\big)O^*\big(g_2(x)\big) = o\big(g_1(x)g_2(x)\big)$ as $x \to c$.

303. In this exercise, $f(x) = O\big(g(x)\big)$ refers to the meaning stated in lecture and not that of exercise 301 above.

    (a) Prove that $O\big(g_1(x)\big)O\big(g_2(x)\big) = O\big(g_1(x)g_2(x)\big)$.

    (b) Prove that for every $p > 0$, $O\big(g(x)\big)^p = O\big(g(x)^p\big)$.

    (c) Prove that for every $r \in \mathbb{R}$, $O^*\big(g(x)\big)^r = O^*\big(g(x)^r\big)$ as $x \to c$ whenever the r-th power can be defined on both sides.

    (d) Prove that $o\big(g_1(x)\big)O\big(g_2(x)\big) = o\big(g_1(x)g_2(x)\big)$ as $x \to c$.

304. (a) Prove that for every $\varepsilon > 0$, $\ln x = o\big(\frac{1}{x^\epsilon}\big)$ as $x \to 0$ on $I = (0, +\infty)$.

    (b) Prove that for every $\varepsilon > 0$, $\ln x = o(x^\epsilon)$ as $x \to +\infty$ on $I = (0, +\infty)$.

    (c) Prove that $\sqrt{x + \sqrt{x + \sqrt{x}}} \sim \sqrt[8]{x}$ as $x \to 0$ on $I = (0, +\infty)$.

    (d) Prove that $\sqrt{x + \sqrt{x + \sqrt{x}}} \sim \sqrt{x}$ as $x \to +\infty$ on $I = (0, +\infty)$.

305. Find real numbers a, b and c such that as $x \to 0$, $\ln\big(\frac{\sin x}{x}\big) = ax^2 + bx^4 + cx^6 + o(x^6)$ on $I = (-\pi, \pi)$.

306. Find $\displaystyle\lim_{x \to +\infty} \Big((x^3 - x^2 + \frac{x}{2})e^{1/x} - \sqrt{x^6 - 1}\Big)$.

307. Find real numbers a and b such that as $x \to 0$, $x = (a + b\cos x)\sin x + O^*(x^5)$.

308. Prove that for every $k \in \mathbb{R}$, $\displaystyle\sum_{n=1}^{\infty} \sin(\pi\sqrt{n^2 + k^2})$ converges.

309. For which $p \in \mathbb{R}$, will $\displaystyle\sum_{n=3}^{\infty} \big(\ln\sec\frac{\pi}{n}\big)^p$ converges? (*Hint*: $\tan x \sim x$ and $\ln(1+x) \sim x$ as $x \to 0$.)

310. Determine with proof if $\displaystyle\sum_{n=1}^{\infty} \big(\cos\frac{1}{n}\big)^{n^3}$ converges or not. (*Hint*: $(\cos a)^b = e^{b\ln\cos a}$.)

301 (a) For $c \in \mathbb{R}$, $(c-\varepsilon, c+\varepsilon) \cap (c-\varepsilon', c+\varepsilon') = (c-\min(\varepsilon,\varepsilon'), c+\min(\varepsilon,\varepsilon'))$ is a neighborhood of c.

For $c=+\infty$, $(a,+\infty) \cap (a', +\infty) = (\max(a,a'), +\infty)$ is a neighborhood of $+\infty$.

For $c=-\infty$, $(-\infty, a) \cap (-\infty, a') = (-\infty, \min(a,a'))$ is a neighborhood of $-\infty$.

(b) Let $f_1(x) = O(g(x))$, $f_2(x) = O(g(x))$, $f_3(x) = o(g(x))$ as $x \to c$. Then $\exists$ neighborhoods N_1, N_2 of c and $A_1, A_2 \in \mathbb{R}$ such that $\forall x \in I \cap N_1$, $|f_1(x)| \le A_1 |g(x)|$; $\forall x \in I \cap N_2$, $|f_2(x)| \le A_2 |g(x)|$.

$\lim_{x\to c} \frac{f_3(x)}{g(x)} = 0 \Rightarrow \exists$ neighborhood N_3 of c such that $\forall x \in (I \cap N_3) \setminus \{c\}$, $\left|\frac{f_3(x)}{g(x)}\right| < 1$.

Then $\forall x \in I \cap (N_1 \cap N_2)$, $|f_1(x) \pm f_2(x)| \le |f_1(x)| + |f_2(x)| \le (A_1 + A_2)|g(x)|$.

So $O(g(x)) \pm O(g(x)) = O(g(x))$ as $x \to c$.

Also $\forall x \in I \cap (N_3 \cap N_1)$,

$$|f_3(x) \pm f_1(x)| \le |f_3(x)| + |f_1(x)| \le \begin{cases} (1+A_1)|g(x)| & \text{if } c \notin I \cap (N_3 \cap N_1) \\ \left(\max\left(1, \frac{|f_3(c)|}{|g(c)|}\right) + A_1\right)|g(x)| & \text{if } c \in I \cap (N_3 \cap N_1) \end{cases}$$

So $o(g(x)) \pm O(g(x)) = O(g(x))$ as $x \to c$.

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302 (a) Let $f_1(x) = o(g(x))$, $f_2(x) = o(g(x))$ as $x \to c$. Then $\lim_{x\to c} \frac{f_1(x)}{g(x)} = 0$, $\lim_{x\to c} \frac{f_2(x)}{g(x)} = 0$.

So $\lim_{x\to c} \frac{f_1(x) \pm f_2(x)}{g(x)} = 0 \Leftrightarrow o(g(x)) \pm o(g(x)) = o(g(x))$ as $x \to c$.

(b) Let $f_1(x) = o(g(x))$, $f_2(x) = O^*(g(x))$ as $x \to c$. Then $\lim_{x\to c} \frac{f_1(x)}{g(x)} = 0$, $\lim_{x\to c} \frac{f_2(x)}{g(x)} = k \ne 0$.

So $\lim_{x\to c} \frac{f_1(x) \pm f_2(x)}{g(x)} = \pm k \ne 0 \Leftrightarrow o(g(x)) \pm O^*(g(x)) = O^*(g(x))$ as $x \to c$.

(c) Let $f_1(x) = o(g_1(x))$, $f_2(x) = o(g_2(x))$ as $x \to c$. Then $\lim_{x\to c} \frac{f_1(x)}{g_1(x)} = 0$, $\lim_{x\to c} \frac{f_2(x)}{g_2(x)} = 0$.

So $\lim_{x\to c} \frac{f_1(x) f_2(x)}{g_1(x) g_2(x)} = 0 \Leftrightarrow o(g_1(x)) o(g_2(x)) = o(g_1(x) g_2(x))$ as $x \to c$.

(d) Let $f_1(x) = o(g_1(x))$, $f_2(x) = O^*(g_2(x))$ as $x \to c$. Then $\lim_{x\to c} \frac{f_1(x)}{g_1(x)} = 0$, $\lim_{x\to c} \frac{f_2(x)}{g_2(x)} = k \ne 0$.

So $\lim_{x\to c} \frac{f_1(x) f_2(x)}{g_1(x) g_2(x)} = 0 \Leftrightarrow o(g_1(x)) O^*(g_2(x)) = o(g_1(x) g_2(x))$ as $x \to c$.

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303 (a) Let $f_1(x) = O(g_1(x))$, $f_2(x) = O(g_2(x))$. Then $\exists A_1, A_2 \in \mathbb{R}$ such that $\forall x \in I$, we have $|f_1(x)| \le A_1 |g_1(x)|$, $|f_2(x)| \le A_2 |g_2(x)| \Rightarrow |f_1(x) f_2(x)| \le A_1 A_2 |g_1(x) g_2(x)|$.

So $O(g_1(x)) O(g_2(x)) = O(g_1(x) g_2(x))$.

(b) Let $f(x) = O(g(x))$. Then $\exists A \in \mathbb{R}$ such that $\forall x \in I$, $|f(x)| \le A|g(x)|$.

So $|f(x)^p| \le A^p |g(x)^p| \Leftrightarrow O(g(x))^p = O(g(x)^p)$.

(c) Let $f(x) = O^*(g(x))$. Then $\lim_{x\to c} \frac{f(x)}{g(x)} = k \Rightarrow \lim_{x\to c} \frac{f(x)^r}{g(x)^r} = k^r \ne 0 \Leftrightarrow O^*(g(x))^r = O^*(g(x)^r)$.

(d) Let $f_1(x)=o(g_1(x))$ as $x\to c$ and $f_2(x)=O(g_2(x))$. Then $\lim_{x\to c}\frac{f_1(x)}{g_1(x)}=0$ and $\exists A\in\mathbb{R}$ such that, $\forall x\in I$, $|f_2(x)|\le A|g_2(x)|$. So $\left|\frac{f_1(x)f_2(x)}{g_1(x)g_2(x)}\right|\le A\left|\frac{f_1(x)}{g_1(x)}\right|\to 0$ as $x\to c$. Therefore, $\lim_{x\to c}\frac{f_1(x)f_2(x)}{g_1(x)g_2(x)}=0 \Leftrightarrow o(g_1(x))O(g_2(x))=o(g_1(x)g_2(x))$.

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304 (a) $$\lim_{x\to 0}\frac{\ln x}{\frac{1}{x^\varepsilon}}=\lim_{x\to 0}\frac{\frac{1}{\varepsilon}(\varepsilon\ln x)}{\frac{1}{x^\varepsilon}}=\frac{1}{\varepsilon}\lim_{x\to 0}\frac{\ln x^\varepsilon}{\frac{1}{x^\varepsilon}}\underset{y=x^\varepsilon}{\uparrow}=\frac{1}{\varepsilon}\lim_{y\to 0}\frac{\ln y}{\underset{\to+\infty}{\frac{1}{y}}}$$
$$=\frac{1}{\varepsilon}\lim_{y\to 0}\frac{1/y}{-1/y^2}=\frac{1}{\varepsilon}\lim_{y\to 0}(-y)=0. \quad\therefore\quad \ln x=o\left(\frac{1}{x^\varepsilon}\right)\text{ as } x\to 0$$

(b) $$\lim_{x\to+\infty}\frac{\ln x}{x^\varepsilon}\underset{y=x^\varepsilon}{\uparrow}=\frac{1}{\varepsilon}\lim_{y\to+\infty}\frac{\ln y}{\underset{\to+\infty}{y}}=\frac{1}{\varepsilon}\lim_{y\to+\infty}\frac{1/y}{1}=0.\quad\therefore\ \ln x=o(x^\varepsilon)\text{ as } x\to+\infty.$$

(c) $$\lim_{x\to 0}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt[8]{x}}=\lim_{x\to 0}\sqrt{\frac{x}{x^{1/4}}+\frac{\sqrt{x+\sqrt{x}}}{x^{1/4}}}=\lim_{x\to 0}\sqrt{x^{3/4}+\sqrt{\frac{x}{x^{1/2}}+\frac{\sqrt{x}}{x^{1/2}}}}$$
$$=\lim_{x\to 0}\sqrt{x^{3/4}+\sqrt{\sqrt{x}+1}}=\sqrt{0+\sqrt{\sqrt{0}+1}}=1.\quad\therefore\ \sqrt{x+\sqrt{x+\sqrt{x}}}\sim\sqrt[8]{x}\text{ as } x\to 0.$$

(d) $$\lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x}}=\lim_{x\to+\infty}\sqrt{\frac{x}{x}+\frac{\sqrt{x+\sqrt{x}}}{x}}=\lim_{x\to+\infty}\sqrt{1+\sqrt{\frac{x}{x^2}+\frac{\sqrt{x}}{x^2}}}=\lim_{x\to+\infty}\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{x^{3/2}}}}=1.$$
$\therefore\ \sqrt{x+\sqrt{x+\sqrt{x}}}\sim\sqrt{x}$ as $x\to+\infty$.

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305 $$\ln\frac{\sin x}{x}=\ln\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+o(x^7)}{x}=\ln\left(1+\left(-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+o(x^6)\right)\right)$$
$$=\left(-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+o(x^6)\right)-\frac{1}{2}\left(-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+o(x^6)\right)^2+\frac{1}{3}\left(-\frac{x^2}{6}+\frac{x^4}{120}-\frac{x^6}{5040}+o(x^6)\right)^3+o(x^6)$$
$$=-\frac{x^2}{6}+\left(\frac{1}{120}-\frac{1}{72}\right)x^4+\left(-\frac{1}{5040}+\frac{1}{720}+\frac{1}{3\cdot 6^3}\right)x^6+o(x^6)$$
$$=-\frac{x^2}{6}-\frac{x^4}{180}-\frac{x^6}{2835}+o(x^6)\quad\text{as } x\to 0.$$

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306 As $x\to+\infty$, $$\left(x^3-x^2+\frac{x}{2}\right)e^{1/x}-\sqrt{x^6-1}=\left(x^3-x^2+\frac{x}{2}\right)\left(1+\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}+o\left(\frac{1}{x^3}\right)\right)-x^3\left(1-\frac{1}{x^6}\right)^{1/2}$$
$$=x^3+\frac{1}{6}+o(1)-x^3\left(1-\frac{1}{2x^6}-o\left(\frac{1}{x^6}\right)\right)=\frac{1}{6}+o(1).$$ So limit is $\frac{1}{6}$.

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307 $$x-(a+b\cos x)\sin x=x-a\sin x-\frac{b}{2}\sin 2x$$
$$=x-a\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+O^*(x^7)\right)-\frac{b}{2}\left(2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}+O^*(x^7)\right)$$
$$=(1-a-b)x+\left(\frac{a}{6}+\frac{2b}{3}\right)x^3-\left(\frac{a}{120}+\frac{2b}{15}\right)x^5+O^*(x^7)$$
$$\left.\begin{array}{l}1-a-b=0\\ \frac{a}{6}+\frac{2b}{3}=0\end{array}\right\}\Rightarrow a=\frac{4}{3},\ b=-\frac{1}{3}\Rightarrow\frac{a}{120}+\frac{2b}{15}\ne 0\quad\therefore\ x=\left(\frac{4}{3}-\frac{1}{3}\cos x\right)\sin x+O^*(x^5).$$

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$$\sin\left(\pi\sqrt{n^2+k^2}\right)=\sin\left(n\pi\sqrt{1+\left(\frac{k}{n}\right)^2}\right)=\sin\left(n\pi\left(1+\frac{1}{2}\frac{k^2}{n^2}+O^*\left(\frac{1}{n^4}\right)\right)\right)$$

$$=\sin\left(n\pi+\frac{k^2\pi}{2n}+O^*\left(\frac{1}{n^3}\right)\right)=(-1)^n\sin\left(\frac{k^2\pi}{2n}+O^*\left(\frac{1}{n^3}\right)\right)=(-1)^n\frac{k^2\pi}{2n}+O^*\left(\frac{1}{n^3}\right)$$

$\sin(n\pi+\alpha)=(-1)^n\sin\alpha$ $\qquad$ $\sin x = x+O(x^3)$

$\sum_{n=1}^{\infty}(-1)^n\frac{k^2\pi}{2n}+\sum_{n=1}^{\infty}O^*\left(\frac{1}{n^3}\right)$ converges by alternating series and p-test.

$\therefore \sum_{n=1}^{\infty}\sin\left(\pi\sqrt{n^2+k^2}\right)$ converges

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$$\left(\ln\sec\frac{\pi}{n}\right)^p=\left(\ln\left(1+\tan^2\frac{\pi}{n}\right)^{\frac{1}{2}}\right)^p=\frac{1}{2^p}\left(\ln\left(1+\tan^2\frac{\pi}{n}\right)\right)^p\sim\frac{1}{2^p}\tan^{2p}\frac{\pi}{n}\sim\frac{\pi^{2p}}{2^pn^{2p}}$$

as $x\to 0$, $\tan x\sim x$ $\qquad$ $\ln(1+x)\sim x$ as $x\to 0$

$\sum_{n=3}^{\infty}\frac{1}{n^{2p}}$ Converges $\Leftrightarrow 2p>1 \Leftrightarrow p>\frac{1}{2}$. So $\sum_{n=3}^{\infty}\left(\ln\sec\frac{\pi}{n}\right)^p$ converges $\Leftrightarrow p>\frac{1}{2}$.

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310 As $n\to\infty$,

$$\left(\cos\frac{1}{n}\right)^{n^3}=\left(\cos\frac{1}{n}\right)^{n^3}=e^{n^3\ln\cos\frac{1}{n}}=e^{n^3\ln\left(1-\frac{1}{2n^2}+O^*\left(\frac{1}{n^4}\right)\right)}=e^{n^3\left(-\frac{1}{2n^2}+O^*\left(\frac{1}{n^4}\right)\right)}$$

$$=e^{-\frac{n}{2}+O^*\left(\frac{1}{n}\right)}\sim e^{-\frac{n}{2}}$$ since $\lim_{n\to\infty}O^*\left(\frac{1}{n}\right)=\lim_{n\to\infty}\frac{O^*\left(\frac{1}{n}\right)}{\frac{1}{n}\cdot n}=\lim_{n\to\infty}\frac{k}{n}=0\Rightarrow\lim_{n\to\infty}e^{O^*\left(\frac{1}{n}\right)}=1$.

$\sum_{n=1}^{\infty}e^{-\frac{n}{2}}=\sum_{n=1}^{\infty}\left(\underbrace{\frac{1}{e^{1/2}}}_{<1}\right)^n$ Converges by geometric series test. $\therefore \sum_{n=1}^{\infty}\left(\cos\frac{1}{n}\right)^{n^3}$ converges.

<u>Math 2031 Exercises on Stolz' Theorem</u>

311. Let $b > a > 0$ and $d > 0$. Let $x_n = \dfrac{a(a+d)(a+2d)\cdots(a+nd)}{b(b+d)(b+2d)\cdots(b+nd)}$. Show $\lim_{n\to\infty} x_n$ exists, then find its value. (*Hint*: for value of limit, consider $(nx_n)/n$.)

312. Let $\lim_{n\to\infty} a_n = a$. Find $\lim_{n\to\infty}(a_n + \frac{1}{2}a_{n-1} + \frac{1}{4}a_{n-2} + \cdots + \frac{1}{2^n}a_0)$.

313. Let $x_n = \sum_{k=1}^{n} \ln\binom{n}{k}$, where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. Find $\lim_{n\to\infty} \dfrac{x_n}{n^2}$.

314. Let $\{a_n\}$ be a sequence satisfying $\lim_{n\to\infty}(a_{n+1} - \frac{a_n}{2}) = 0$. Use Stolz' theorem to prove that $\lim_{n\to\infty} a_n = 0$. (*Hint*: $2^n a_n/2^n$.)

315. Let $\{x_n\}$ be a sequence satisfying $\lim_{n\to\infty}(x_n - x_{n-2}) = 0$. Use Stolz' theorem to prove that $\lim_{n\to\infty} \dfrac{x_n - x_{n-1}}{n} = 0$. (*Hint*: Intertwine sequence.)

316. Let $\{x_n\}$ be a sequence and let $y_1 = 0$ and $y_n = x_{n-1} + 2x_n$ for $n = 2, 3, 4, \ldots$. If $\{y_n\}$ converges to y, then use Stolz' theorem to prove that $\{x_n\}$ also converges. (*Hint*: Intertwine and $4^n x_{2n}/4^n$.)

317. If $a_n > 0$ and $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = a$, then prove that $\lim_{n\to\infty} \sqrt[n]{a_n} = a$.

318. (a) Prove that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n = \gamma$ exists. This is called *<u>Euler's constant</u>*. (*Hint*: $\int_k^{k+1} (\frac{1}{k} - \frac{1}{x})dx$.)

    (b) Find $\lim_{n\to\infty} n\Big(\sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma\Big)$.

319. Let $\sum_{n=1}^{\infty} a_n$ converge and $\{p_n\}$ be strictly increasing with limit $+\infty$.

    (a) Prove that $\lim_{n\to\infty} \dfrac{p_1a_1 + p_2a_2 + \cdots + p_na_n}{p_n} = 0$. (*Hint*: $A_n = a_1 + \cdots + a_n$.)

    (b) Prove that if $\lim_{n\to\infty} na_n$ exists, then it is 0.

320. (a) Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Prove that $\lim_{n\to\infty} \dfrac{x_1y_n + x_2y_{n-1} + \cdots + x_ny_1}{n} = xy$. (*Hint*: Note $\dfrac{\sum_{i=1}^n x_iy_{n-i+1}}{n} = \dfrac{\sum_{i=1}^n (x_i - x)y_{n-i+1}}{n} + x\dfrac{\sum_{i=1}^n y_i}{n}$.)

    (b) Let $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$. Let $c_n = a_1b_n + a_2b_{n-1} + \cdots + a_nb_1$. (The sequence $\{c_n\}$ is called the *<u>convolution</u>* of $\{a_n\}, \{b_n\}$. The series $\sum_{n=1}^{\infty} c_n$ is called the *<u>Cauchy product</u>* of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.) If $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} c_n = ab$.

(311) We have $0 < \frac{x_n}{x_{n-1}} = \frac{a+nd}{b+nd} < 1 \Rightarrow 0 < x_n < x_{n-1} \Rightarrow \lim_{n\to\infty} x_n = x \in \mathbb{R}$.

Then $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n x_n}{n} \overset{\text{Stolz}}{=} \lim_{n\to\infty}\left((n+1)x_{n+1} - n x_n\right) = \lim_{n\to\infty}\left((n+1)x_n \frac{a+(n+1)d}{b+(n+1)d} - n x_n\right)$

$= \lim_{n\to\infty} x_n \left(\frac{(n+1)(a-b+d)+b}{b+(n+1)d}\right) = x\,\frac{a-b+d}{d}$. Since $\frac{a-b+d}{d} \neq 1$, $x = 0$.

(312) $\lim_{n\to\infty}\left(a_n + \frac{1}{2}a_{n-1} + \dots + \frac{1}{2^n}a_0\right) = \lim_{n\to\infty} \frac{a_0 + 2a_1 + \dots + 2^n a_n}{2^n} \underset{2^n \nearrow +\infty}{\overset{\text{Stolz}}{=}} \lim_{n\to\infty} \frac{2^{n+1}a_{n+1}}{2^{n+1}-2^n} = \lim_{n\to\infty} 2a_{n+1} = 2a$.

(313) $\lim_{n\to\infty} \frac{x_n}{n^2} \underset{n^2 \nearrow +\infty}{\overset{\text{Stolz}}{=}} \lim_{n\to\infty} \frac{x_{n+1}-x_n}{2n+1} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n+1} \ln\binom{n+1}{k} - \sum_{k=1}^{n} \ln\binom{n}{k}}{2n+1} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n}\left[\ln\binom{n+1}{k} - \ln\binom{n}{k}\right]}{2n+1}$

$= \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \ln\binom{n+1}{k}/\binom{n}{k}}{2n+1} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \ln(n+1) - \ln(n+1-k)}{2n+1} = \lim_{n\to\infty} \frac{n\ln(n+1) - \sum_{k=1}^{n} \ln k}{2n+1 \nearrow +\infty}$

$\underset{2n+1 \nearrow +\infty}{\overset{\text{Stolz}}{=}} \lim_{n\to\infty} \frac{(n+1)\ln(n+2) - n\ln(n+1) - \ln(n+1)}{2} = \frac{1}{2}\lim_{n\to\infty} \ln\left(1+\frac{1}{n+1}\right)^{n+1} = \frac{1}{2}\ln e = \frac{1}{2}$.

(314) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^n a_n}{2^n} \overset{\text{Stolz}}{=} \lim_{n\to\infty} \frac{2^{n+1}a_{n+1} - 2^n a_n}{2^{n+1}-2^n} = \lim_{n\to\infty} 2\left[a_{n+1} - \frac{a_n}{2}\right] = 0$.

(315) Let $y_n = \frac{x_n - x_{n-1}}{n}$, $\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} \frac{x_{2n} - x_{2n-1}}{2n} = \lim_{n\to\infty} \frac{(x_{2n+2} - x_{2n+1}) - (x_{2n} - x_{2n-1})}{2}$

$= \frac{1}{2}\lim_{n\to\infty}\left((x_{2n+2} - x_{2n}) - (x_{2n+1} - x_{2n-1})\right) = 0$.

$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{x_{2n-1} - x_{2n-2}}{2n-1} = \lim_{n\to\infty} \frac{(x_{2n+1} - x_{2n}) - (x_{2n-1} - x_{2n-2})}{2} = \frac{1}{2}\lim_{n\to\infty}\left((x_{2n+1} - x_{2n-1}) - (x_{2n} - x_{2n-2})\right) = 0$

By intertwining sequence theorem, $\lim_{n\to\infty} y_n = 0$.

(316) $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} \frac{4^n x_{2n}}{4^n} = \lim_{n\to\infty} \frac{4^{n+1}x_{2n+2} - 4^n x_{2n}}{4^{n+1} - 4^n = 3\cdot 4^n} = \frac{1}{3}\lim_{n\to\infty}(4x_{2n+2} - x_{2n})$

$= \frac{1}{3}\lim_{n\to\infty}(4x_{2n+2} - 2x_{2n+1} + 2x_{2n+1} - x_{2n}) = \frac{1}{3}\lim_{n\to\infty}(2y_{2n+2} - y_{2n+1}) = \frac{2y-y}{3} = \frac{y}{3}$.

$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{4^n x_{2n-1}}{4^n} = \lim_{n\to\infty} \frac{4^{n+1}x_{2n+1} - 4^n x_{2n-1}}{3\cdot 4^n} = \frac{1}{3}\lim_{n\to\infty}(4x_{2n+1} - x_{2n-1}) = \frac{1}{3}(2y_{2n+1} - y_{2n}) = \frac{y}{3}$.

By the intertwining sequence theorem, $\lim_{n\to\infty} x_n = y/3$.

(317) $\lim_{n\to\infty} \frac{\ln a_n}{n} \underset{n \nearrow +\infty}{\overset{\text{Stolz}}{=}} \lim_{n\to\infty}(\ln a_{n+1} - \ln a_n) = \lim_{n\to\infty} \ln\frac{a_{n+1}}{a_n} = \ln a \Rightarrow \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{\frac{\ln a_n}{n}} = a$.

(318) (a) $\forall x \in [k, k+1]$, $\frac{1}{k+1} \le \frac{1}{x} \le \frac{1}{k} \Rightarrow 0 \le \sum_{k=1}^{n} \int_{k}^{k+1} \underbrace{\left(\frac{1}{k} - \frac{1}{x}\right)}_{\ge 0} dx \le \sum_{k=1}^{\infty} \int_{k}^{k+1} \left(\frac{1}{k} - \frac{1}{k+1}\right) dx = 1$

$\Rightarrow \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \int_{1}^{n+1} \frac{1}{x} dx \right) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) \right) = \gamma$ exists. Adding this to $\lim_{n\to\infty} (\ln(n+1) - \ln n) = \lim_{n\to\infty} \ln \frac{n+1}{n} = \ln 1 = 0$, we get $\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = \gamma$.

(b) $\lim_{n\to\infty} n \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma \right) = \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{1}{k} - \ln n - \gamma}{1/n} \underset{0/0 \text{ form}}{\overset{\text{Stolz}}{=}} \lim_{n\to\infty} \frac{\frac{1}{n+1} - \ln(n+1) + \ln n}{1/(n+1) - 1/n}$

$= \lim_{n\to\infty} n(n+1)\left[\ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} \right] = \lim_{n\to\infty} n(n+1) \left[\frac{1}{n} - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) - \frac{1}{n+1} \right]$

$= \lim_{n\to\infty} \left(1 - \frac{n+1}{2n} + \frac{n+1}{n} o(1) \right) = \frac{1}{2}$

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(319) (a) Let $A_n = \sum_{k=1}^{n} a_k$. We have $a_1 = A_1$, $a_n = A_n - A_{n-1}$ for $n > 1$,

$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n = p_1 A_1 + p_2 (A_2 - A_1) + \cdots + p_n (A_n - A_{n-1}) = (p_1 - p_2) A_1 + \cdots + (p_{n-1} - p_n) A_{n-1} + p_n A_n$

So $\lim_{n\to\infty} \frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_n} = \lim_{n\to\infty} \left(\frac{(p_1 - p_2) A_1 + \cdots + (p_{n-1} - p_n) A_{n-1}}{p_n} + A_n \right)$

$\overset{\text{Stolz}}{=} \lim_{n\to\infty} \left(\frac{(p_n - p_{n+1}) A_n}{p_{n+1} - p_n} + A_n \right) = \lim_{n\to\infty} (-A_n + A_n) = 0.$

(b) If $\lim_{n\to\infty} n a_n = x$, then by (a) and Stolz, $0 \overset{p_n = n}{=} \lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + n a_n}{n} = \lim_{n\to\infty} (n+1) a_{n+1} = x$

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(320) (a) Since $\lim_{n\to\infty} y_n = y$ exists, $\{y_n\}$ is bounded, say $|y_n| \le M$ for all n. Now

$\left| \frac{\sum_{i=1}^{n} (x_i - x) y_{n-i+1}}{n} \right| \le \frac{\sum_{i=1}^{n} |x_i - x| M}{n}$ and $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} |x_i - x| M}{n} \overset{\text{Stolz}}{=} \lim_{n\to\infty} |x_{n+1} - x| M = 0$.

Also, $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} y_i}{n} \overset{\text{Stolz}}{=} \lim_{n\to\infty} y_{n+1} = y$. So $\frac{\sum_{i=1}^{n} x_i y_{n-i+1}}{n} = \frac{\sum_{i=1}^{n} (x_i - x) y_{n-i+1}}{n} + x \frac{\sum_{i=1}^{n} y_i}{n} \to 0 + xy = xy$.

(b) Let $x_n = a_1 + a_2 + \cdots + a_n$, $y_1 = 0$, $y_n = b_1 + \cdots + b_{n-1}$ for $n > 1$, then $x_n \to a$, $y_n \to b$.

Let $z_n = x_1 y_n + \cdots + x_n y_1$, then

$z_{n+1} - z_n = x_1 (y_{n+1} - y_n) + x_2 (y_n - y_{n-1}) + \cdots + x_n (y_2 - y_1) + x_{n+1} y_1$

$= a_1 b_n + (a_1 + a_2) b_{n-1} + \cdots + (a_1 + a_2 + \cdots + a_n) b_1$

$= (a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1) + (a_1 b_{n-1} + \cdots + a_{n-1} b_1) + \cdots + a_1 b_1$

$= c_n + c_{n-1} + \cdots + c_1$.

By (a), $ab = \lim_{n\to\infty} \frac{z_n}{n} \underset{n \uparrow +\infty}{\overset{\text{Stolz}}{=}} \lim_{n\to\infty} (z_{n+1} - z_n) = \sum_{n=1}^{\infty} c_n$.

Math 202(Introduction to Real Analysis) **Spring 2010-2011**

Homework #4 – Due Tuesday, May 3, 2010 at 10:30am

Be sure to write your name (as shown on your student ID card) and your tutorial session number on the homework! Show work. Answers are worth very little. **Make a copy of your homework and submit the original.**

1. (a) Let $c \in (a,b)$ and n be a positive integer. Let $f : (a,b) \to \mathbb{R}$ be $n-1$ times differentiable on (a,b) and n times differentiable at c. Using l'Hopital's rule, prove that as $x \to c$, $f(x) = f(c) + \dfrac{f'(c)}{1!}(x-c) + \cdots + \dfrac{f^{(n)}(c)}{n!}(x-c)^n + o((x-c)^n)$.
   Remark. The term $o((x-c)^n)$ in the above expansion is called the Peano remainder.

   (b) Let $0 < x_1 < \pi/2$ and $x_{n+1} = \sin x_n$ for $n \geq 1$. Prove that $\{x_n\}$ is decreasing and $\lim\limits_{n\to\infty} x_n = 0$. Use Stolz' theorem to do $\lim\limits_{n\to\infty} \dfrac{n}{1/x_n^2}$, then show $x_n^2 \sim 3/n$ as $n \to \infty$.

2. Let $g : (a,b] \to \mathbb{R}$ be bounded and $S_g = \{x \in (a,b] : g \text{ discontinuous at } x\}$ be of measure 0. Let $f : [a,b] \to \mathbb{R}$ be a function such that $f(x) = g(x)$ for all $x \in (a,b]$.

   (a) Prove that f is Riemann integrable on $[a,b]$.

   (b) Prove that the value of $\displaystyle\int_a^b f(x)\,dx$ does not depend on the value of $f(a)$.

   (c) Prove that the improper integral $\displaystyle\int_a^b g(x)\,dx$ is equal to $\displaystyle\int_a^b f(x)\,dx$.

   Remarks. Thus, we may treat $\displaystyle\int_0^1 \frac{\sin x}{x}\,dx, \int_0^1 x^x\,dx, \ldots$ as proper integrals.

3. (a) Prove the improper integral $\displaystyle\int_0^\infty \sin(x^2)\,dx$ converges. (*Hint*: Substitute $t = x^2$.)

   (b) Show that the improper integral $\displaystyle\int_0^\infty \frac{du}{1+u^4}$ converges to $\dfrac{\pi}{2\sqrt{2}}$. (*Hint*: Consider $\displaystyle\int_0^\infty = \int_0^1 + \int_1^\infty$ and substitute $t = \dfrac{1}{u}$ in $\displaystyle\int_1^\infty$. Then $y = u - \dfrac{1}{u}$ for $\displaystyle\int_0^1 \frac{1+u^2}{1+u^4}du$.)

   (c) Show that $\displaystyle\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ formally by writing the integral as a double integral, then interchanging the order of integration. (*Hint*: From the example $\displaystyle\int_0^\infty e^{-u^2}du = \frac{\sqrt{\pi}}{2}$ done in class, get $\dfrac{1}{\sqrt{t}} = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^\infty e^{-tu^2}du$. Consider part (a).)

4. Use the integral criterion to show every increasing function $f : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$.

Solutions to Homework #4 (Math 202, Spring 2005-2006)

① (a) Let $g(x)=f(x)-f(c)-\frac{f'(c)}{1!}(x-c)-\cdots-\frac{f^{(n)}(c)}{n!}(x-c)^n$ and $h(x)=(x-c)^n$. Then

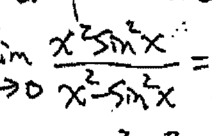
$$\frac{g'(x)}{h'(x)}=\frac{f'(x)-f'(c)-\frac{f''(c)}{1!}(x-c)-\cdots-\frac{f^{(n)}(c)}{(n-1)!}(x-c)^{n-1}}{n(x-c)^{n-1}},\ \ldots,\ \frac{g^{(n-1)}(x)}{h^{(n-1)}(x)}=\frac{f^{(n-1)}(x)-f^{(n-1)}(c)-\frac{f^{(n)}(c)}{1!}(x-c)}{n(n-1)\cdots 2(x-c)}$$

are all of $\frac{0}{0}$ form as $x\to c$. However,

$$\lim_{x\to c}\frac{g^{(n-1)}(x)}{h^{(n-1)}(x)}=\lim_{x\to c}\frac{1}{n!}\left(\frac{f^{(n-1)}(x)-f^{(n-1)}(c)}{x-c}-f^{(n)}(c)\right)\overset{\checkmark}{=}0.$$

since we are given $f^{(n)}(c)$ exists.

By l'Hopital's rule, $\lim_{x\to c}\frac{g(x)}{h(x)}=0$, which is equivalent to the required condition by the little-oh notation.

(b) 

For $\theta\in(0,\pi)$, $0\le 2\sin\frac{\theta}{2}<\theta$. So for $x\in(0,\frac{\pi}{2})$, $0<\sin x<x$.
Hence $0<x_{n+1}=\sin x_n<x_n$. By the monotone sequence theorem, $x=\lim_{n\to\infty}x_n$ exists and $x\in[0,\frac{\pi}{2})$. Then $x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\sin x_n=\sin x$. So $x=0$.

Next, we apply Stolz' theorem to compute

$$\lim_{n\to\infty}nx_n^2=\lim_{n\to\infty}\frac{n\to\infty}{1/x_n^2\to\infty}=\lim_{n\to\infty}\frac{(n+1)-n}{1/x_{n+1}^2-1/x_n^2}=\lim_{n\to\infty}\frac{x_n^2x_{n+1}^2}{x_n^2-x_{n+1}^2}=\lim_{n\to\infty}\frac{x_n^2\sin^2x_n}{x_n^2-\sin^2x_n}$$ and

to compute the last limit, we apply the sequential limit theorem to the fact

$$\lim_{x\to0}\frac{x^2\sin^2x}{x^2-\sin^2x}=\lim_{x\to0}\frac{x^2(x-\frac{1}{6}x^3+o(x^3))^2}{x^2-(x-\frac{1}{6}x^3+o(x^3))^2}=\lim_{x\to0}\frac{x^4+o(x^4)}{\frac{1}{3}x^4+o(x^4)}=3$$

$$\left(\text{or }\lim_{x\to0}\frac{x^2\sin^2x}{x^2-\sin^2x}=\lim_{x\to0}\left(\frac{x^3}{x-\sin x}\right)\left(\frac{x}{x+\sin x}\right)\frac{\sin^2x}{x^2}=\lim_{x\to0}\left(\frac{3x^2}{1-\cos x}\right)\left(\frac{1}{1+\cos x}\right)1=\lim_{x\to0}\frac{6x}{\sin x}\frac{1}{2}\cdot1=3\right).$$

So $\lim_{n\to\infty}nx_n^2=3\iff\lim_{n\to\infty}\frac{x_n^2}{3/n}=1\iff x_n^2\sim\frac{3}{n}$ as $n\to\infty$.

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② (a) (Proof of f is Riemann integrable on $[a,b]$)

Since g is bounded on $(a,b]$, say $|g(x)|\le K$ for all $x\in(a,b]$; so for every such f, we have $|f(x)|\le\max(K,|f(a)|)$ for all $x\in[a,b]$. So f is bounded on $[a,b]$. Next $S_f\subseteq S_g\cup\{a\}$ and S_g is of measure 0 imply S_f is of measure 0. By Lebesgue's theorem, f is Riemann integrable on $[a,b]$.

(b) (Proof of $\int_a^b f(x)\,dx$ does not depend on the value of $f(a)$).

Solution 1

Let f_1 and f_2 be two such f. Then $h(x)=f_1(x)-f_2(x)=\begin{cases}0 & \text{if } x\in(a,b]\\ f_1(a)-f_2(a) & \text{if } x=a.\end{cases}$

Switching f_1 and f_2 if necessary, we may assume $f_1(a)\ge f_2(a)$ so that $h(x)\ge0$ on $[a,b]$.
For a partition $P_{x_1}=\{a,x_1,b\}$, $U(h,P_{x_1})=(f_1(a)-f_2(a))(x_1-a)$. Now we have

$$0\le\int_a^b f_1(x)\,dx-\int_a^b f_2(x)\,dx=\int_a^b h(x)\,dx=\inf\{U(h,P):P\text{ partition of }[a,b]\}\le\inf\{U(h,P_{x_1}):a<x_1<b\}=0.$$

$\therefore\ \int_a^b f_1(x)\,dx=\int_a^b f_2(x)\,dx.$

Solution 2

Before the fundamental theorem of Calculus, we showed $F(t)=\int_a^t f(x)\,dx$ is uniformly continuous on $[a,b]$. So $\int_t^b f(x)dx=\int_a^b f(x)dx-\int_a^t f(x)dx$ is continuous at a. By the uniqueness of limit, for all such f, $\int_a^b f(x)dx$ equals $\lim\limits_{t\to a^+}\int_t^b f(x)dx$, which is unique.

(c) (Proof of $\int_a^b g(x)dx=\int_a^b f(x)dx$).

Again, since $F(t)=\int_a^t f(x)\,dx$ is uniformly continuous on $[a,b]$, so $\int_t^b f(x)dx=\int_a^b f(x)dx-\int_a^t f(x)dx$ is continuous on $[a,b]$. Now $S_g\cap[t,b]$ is of measure 0 implies g is locally integrable on $(a,b]$. Then $\int_a^b g(x)dx=\lim\limits_{t\to a^+}\int_t^b g(x)dx=\lim\limits_{t\to a^+}\int_t^b f(x)dx=\int_a^b f(x)dx$.

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③ (a) Since $\sin(x^2)$ is continuous on $[0,1]$, $\int_0^1\sin(x^2)\,dx<\infty$. Next,

$$\int_1^c\sin(x^2)dx\underset{t=x^2}{=}\int_1^{c^2}\frac{\sin t}{2\sqrt t}dt=\frac{\cos t}{2\sqrt t}\Big|_1^{c^2}+\int_1^{c^2}\frac{\cos t}{4t^{3/2}}dt.$$ Since $\left|\frac{\cos t}{4t^{3/2}}\right|\le\frac{1}{4t^{3/2}}$ on $[1,+\infty)$

and $\int_1^\infty\frac{1}{t^{3/2}}dt<\infty$ by p-test, by absolute convergence test and comparison test, we get $\int_1^\infty\frac{\cos t}{4t^{3/2}}dt<\infty$. So $\int_1^\infty\sin(x^2)dx=\frac{-\cos 1}{2}+\int_1^\infty\frac{\cos t}{4t^{3/2}}dt<\infty$ and $\int_0^\infty\sin(x^2)dx<\infty$.

(b)
$$\int_1^\infty\frac{du}{1+u^4}\underset{t=1/u}{=}\int_1^0\frac{-1/t^2\,dt}{1+1/t^4}=\int_0^1\frac{t^2dt}{t^4+1}.$$ So $$\int_0^\infty\frac{du}{1+u^4}=\int_0^1\frac{du}{1+u^4}+\int_1^\infty\frac{du}{1+u^4}=\int_0^1\frac{1+u^2}{1+u^4}du$$

$$=\int_0^1\frac{\frac{1}{u^2}+1}{\frac{1}{u^2}+u^2}du\underset{y=u-\frac1u}{=}\int_{-\infty}^0\frac{dy}{y^2+2}=\frac{1}{\sqrt2}\operatorname{Arctan}\frac{y}{\sqrt2}\Big|_{-\infty}^0=0-\left(\frac{1}{\sqrt2}\left(-\frac{\pi}{2}\right)\right)=\frac{\pi}{2\sqrt2}.$$

Alternatively, we can also compute $u^4+1=\underbrace{(u^4+2u^2+1)}_{=(u^2+1)^2}-2u^2=(u^2+\sqrt2u+1)(u^2-\sqrt2u-1)$

$$\int_0^\infty\frac{du}{1+u^4}\underset{\text{as before}}{=}\int_0^1\frac{1+u^2}{1+u^4}du=\frac12\int_0^1\frac{du}{u^2+\sqrt2u+1}+\frac12\int_0^1\frac{du}{u^2-2\sqrt u+1}$$

$$=\frac12\int_0^1\frac{du}{(u+\frac{1}{\sqrt2})^2+\frac12}+\frac12\int_0^1\frac{du}{(u-\frac{1}{\sqrt2})^2+\frac12}$$

$$=\frac12\left(\sqrt2\operatorname{Arctan}\sqrt2\left(u+\frac{1}{\sqrt2}\right)\Big|_0^1\right)+\frac12\left(\sqrt2\operatorname{Arctan}\sqrt2\left(u-\frac{1}{\sqrt2}\right)\Big|_0^1\right)$$

$$=\frac{1}{\sqrt2}\left(\operatorname{Arctan}(\sqrt2+1)-\frac{\pi}{4}+\operatorname{Arctan}(\sqrt2-1)+\frac{\pi}{4}\right)$$

$$=\frac{1}{\sqrt2}\frac{\pi}{2}$$ since $\sqrt2+1=\frac{1}{\sqrt2-1}$ implies $\operatorname{Arctan}(\sqrt2+1)+\operatorname{Arctan}(\sqrt2-1)=\frac{\pi}{2}$

(c) By an example in lecture, $\int_0^\infty e^{-u^2}du = \frac{\sqrt{\pi}}{2}$. Then $\int_0^\infty e^{-tu^2}du \underset{w=\sqrt{t}u}{=} \int_0^\infty e^{-w^2}\frac{dw}{\sqrt{t}} = \frac{1}{\sqrt{t}}\frac{\sqrt{\pi}}{2}$

So $\frac{1}{\sqrt{t}} = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-tu^2}du$. Now

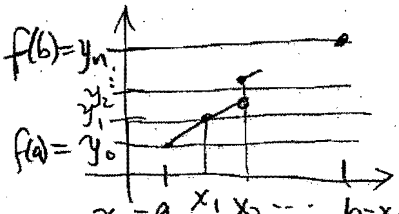
$$\int_0^\infty \sin(x^2)dx \underset{t=x^2}{=} \int_0^\infty \frac{\sin t}{2\sqrt{t}}dt = \int_0^\infty \frac{\sin t}{2}\left(\frac{2}{\sqrt{\pi}}\int_0^\infty e^{-tu^2}du\right)dt = \frac{1}{\sqrt{\pi}}\int_0^\infty\int_0^\infty e^{-tu^2}\sin t\,du\,dt$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty \underbrace{\int_0^\infty e^{-tu^2}\sin t\,dt}\,du = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{1}{1+u^4}du = \frac{1}{\sqrt{\pi}}\frac{\pi}{2\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Set $c=u^2$, $\int_0^\infty e^{-ct}\sin t\,dt = -e^{-ct}\cos t\Big|_0^\infty - \int_0^\infty ce^{-ct}\cos t\,dt = 1 - c\int_0^\infty e^{-ct}\cos t\,dt$

$$= 1 - c\left(e^{-ct}\sin t\Big|_0^\infty + \int_0^\infty ce^{-ct}\sin t\,dt\right) = 1 - c^2\int_0^\infty e^{-ct}\sin t\,dt \Rightarrow \int_0^\infty e^{-ct}\sin t\,dt = \frac{1}{1+c^2}$$

④ Solution 1 $\forall \varepsilon > 0$, let $\{y_0 = f(a) \le y_1 \le \cdots \le y_n = f(b)\}$ be a partition of $[f(a), f(b)]$ such that $\forall i = 0, 1, \dots, n-1$, $y_{i+1} - y_i < \frac{\varepsilon}{b-a}$. For these i's,



let $x_i = \inf\{x \in [a,b] : y_i \le f(x)\}$ and $x_n = b$. Then $P = \{x_0 = a \le x_1 \le \cdots \le x_n = b\}$ is a partition of $[a,b]$. Note on $[x_i, x_{i+1}]$, $y_i \le f(x) \le y_{i+1}$. Now

$$U(f,P) - L(f,P) \le \sum_{i=0}^{n-1} y_{i+1}(x_{i+1}-x_i) - \sum_{i=0}^{n-1} y_i(x_{i+1}-x_i) = \sum_{i=0}^{n-1}(y_{i+1}-y_i)(x_{i+1}-x_i) < \frac{\varepsilon}{(b-a)}\underbrace{\sum_{i=0}^{n-1}(x_{i+1}-x_i)}_{=b-a} = \varepsilon$$

By integral criterion, f is Riemann integrable on $[a,b]$.

Solution 2 $\forall \varepsilon > 0$, by Archimedian principle, there exists $n \in \mathbb{N}$ such that $n > (f(b)-f(a))\frac{b-a}{\varepsilon}$. Let $P = \{x_0 = a \le x_1 \le \cdots \le x_n = b\}$ be the partition of $[a,b]$ such that $x_i = a + i\left(\frac{b-a}{n}\right)$ for $i = 0, 1, \dots, n$. Then $[x_0, x_1], \dots, [x_{n-1}, x_n]$ all have lengths $\frac{b-a}{n}$. Now $U(f,P) - L(f,P) = \sum_{i=0}^{n-1}(M_i - m_i)\frac{b-a}{n} \le (f(b)-f(a))\frac{b-a}{n} < \varepsilon$ where $M_i = \sup\{f(x) : x \in [x_i, x_{i+1}]\}$ and $m_i = \inf\{f(x) : x \in [x_i, x_{i+1}]\}$. By integral criterion, f is Riemann integrable on $[a,b]$.

$M_i \le f(x_{i+1}) \le m_i$ since f is increasing $\Rightarrow \sum_{i=0}^{n-1}(M_i - m_i) \le \sum_{i=0}^{n-1}(m_{i+1} - m_i) = \overset{\le f(b)}{m_n} - \overset{= f(a)}{m_0} \le f(b) - f(a)$