(41) For $\varepsilon > 0$, since $\lim_{n\to\infty} a_n = A$, by definition of convergence, there is $K \in \mathbb{N}$ such that $n \geq K \Rightarrow |a_n - A| < \varepsilon$. Then

$$n \geq K \Rightarrow n+1 \geq K \Rightarrow \left|\frac{a_n + a_{n+1}}{2} - A\right| = \left|\frac{a_n - A}{2} + \frac{a_{n+1} - A}{2}\right| \leq \left|\frac{a_n - A}{2}\right| + \left|\frac{a_{n+1} - A}{2}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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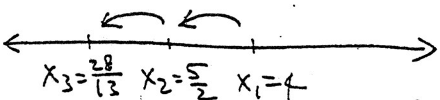
(42) $x_1 = 4,\ x_2 = \frac{5}{2},\ x_3 = \frac{28}{13}$



(We suspect $\{x_n\}$ is decreasing. If $\lim_{n\to\infty} x_n = x$, then $x = \lim_{n\to\infty} x_{n+1} = \lim \frac{4(1+x_n)}{4+x_n} = \frac{4(1+x)}{4+x}$. Solving this, we get $x = \pm 2$. Since $x_n > 0$, $x = 2$.)

We will first show $x_n \geq 2$ for all $n \in \mathbb{N}$ by mathematical induction. For $n=1$, $x_1 = 4 \geq 2$. Assume $x_n \geq 2$, then $2x_n \geq 4 \Rightarrow 4 + 4x_n \geq 8 + 2x_n \Rightarrow x_{n+1} = \frac{4(1+x_n)}{4+x_n} \geq 2$

Next we will show $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$ by mathematical induction. For $n=1$, $x_1 = 4 \geq x_2 = \frac{5}{2}$. Assume $x_n \geq x_{n+1}$. Since $x_{n+1} \geq 2$, so $4x_{n+1} + x_{n+1}^2 \geq 4x_{n+1} + 4$ $\Rightarrow x_{n+1} \geq \frac{4(1+x_{n+1})}{4+x_{n+1}} = x_{n+2}$.

$x_{n+1}^2 \geq 4$

By the monotone sequence theorem, $\{x_n\}$ converges. (In fact, we saw above that $\lim_{n\to\infty} x_n = 2$)

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(43) By AM-GM inequality, $1 + \frac{1}{n+1} = \frac{\overbrace{(1+\frac{1}{n}) + \cdots + (1+\frac{1}{n})}^{n \text{ times}} + 1}{n+1} \geq \sqrt[n+1]{(1+\frac{1}{n})^n \cdot 1}$. Taking $(n+1)$-st power of both sides, we get $\left(1+\frac{1}{n+1}\right)^{n+1} \geq \left(1+\frac{1}{n}\right)^n$. So $\{(1+\frac{1}{n})^n\}$ is increasing. Next, by binomial theorem,

$$\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)^n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 3.$$

---

(44) (a) Since $\{x_n\}$ is bounded, $\exists$ upper bound u, lower bound v for $\{x_n\}$. Then $v \leq x_n \leq u$ for all n. Then $v \leq m_n \leq M_n \leq u$ for all n, i.e. $\{M_n\}$ and $\{m_n\}$ are bounded. Now M_n is an upper bound of $\{x_{n+1}, x_{n+2}, \cdots\}$ and m_n is a lower bound of $\{x_{n+1}, x_{n+2}, \cdots\}$ imply $M_{n+1} \leq M_n$ and $m_n \leq m_{n+1}$. So $\{M_n\}$ is decreasing, $\{m_n\}$ is increasing. By the monotone limit theorem, both $\{M_n\}$ and $\{m_n\}$ converge.

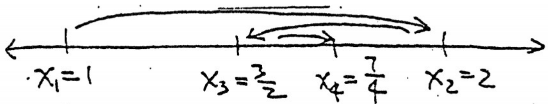
(b) Since $m_n \leq x_n \leq M_n$, so $\lim_{n\to\infty} M_n = x = \lim_{n\to\infty} m_n \Rightarrow \lim_{n\to\infty} x_n = x$ by Sandwich theorem. Conversely, if $\lim_{n\to\infty} x_n = x$, then $\forall \varepsilon > 0\ \exists K$ such that $n \geq K \Rightarrow |x_n - x| < \varepsilon_0 = \varepsilon/2$

$\Rightarrow x_K, x_{K+1}, x_{K+2}, \cdots \in (x - \varepsilon_0, x + \varepsilon_0) \Rightarrow M_K, M_{K+1}, M_{K+2}, \cdots \in [x-\varepsilon_0, x+\varepsilon_0] \subseteq (x-\varepsilon, x+\varepsilon)$

$m_K, m_{K+1}, m_{K+2}, \cdots \in [x-\varepsilon_0, x+\varepsilon_0] \subseteq (x-\varepsilon, x+\varepsilon)$.

So $n \geq K \Rightarrow |M_n - x| < \varepsilon$ and $|m_n - x| < \varepsilon$. $\therefore \lim_{n\to\infty} M_n = x = \lim_{n\to\infty} m_n$.

(45) $x_1=1,\ x_2=2,\ x_3=\frac{3}{2},\ x_4=\frac{7}{4}$



Let $I_n=[x_{2n-1},x_{2n}]$. We will show $I_n \supseteq I_{n+1}$ (i.e. $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$) by mathematical induction. For $n=1$, $x_1=1\le x_3=\frac{3}{2}\le x_4=\frac{7}{4}\le x_2=2$.
Assume $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$. Then $x_{2n+1}\le \frac{x_{2n+1}+x_{2n+2}}{2}(=x_{2n+3})\le x_{2n+2}$ and $x_{2n+3}\le \frac{x_{2n+3}+x_{2n+2}}{2}(=x_{2n+4})\le x_{2n+2}$. So $x_{2n+1}\le x_{2n+3}\le x_{2n+4}\le x_{2n+2}$.

Now $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$ implies $\lim_{n\to\infty}x_{2n-1}=x$ and $\lim_{n\to\infty}x_{2n}=x'$. We will show $x=x'$. (By the intertwining sequence theorem, this will imply $\{x_n\}$ converges.)

Method I. Since x_{k+1} is the midpoint of x_k and x_{k-1}, so $x_{k+1}-x_k=\frac{x_{k-1}-x_k}{2}$.
Then $x_{2n-1}-x_{2n}=\frac{-(x_{2n-2}-x_{2n-1})}{2}=\frac{x_{2n-3}-x_{2n-2}}{2^2}=\cdots=\frac{x_1-x_2}{2^{2n-2}}=\frac{-1}{2^{2n-2}}$. So $\lim_{n\to\infty}|x_{2n-1}-x_{2n}|=0$. By the nested interval theorem, $\bigcap_{n=1}^{\infty} I_n=\{x\}$. So $x=x'$.

Method II. $x=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}\frac{x_{2n}+x_{2n-1}}{2}=\frac{x'+x}{2}\Rightarrow x=x'$.

(Remarks We can find $\lim_{n\to\infty}x_n$ as follow: $x_n=x_1+(x_2-x_1)+(x_3-x_2)+\cdots+(x_n-x_{n-1})$ $=1+1-\frac{1}{2}+\frac{1}{4}-\cdots+(-\frac{1}{2})^{n-1}$. So $\lim_{n\to\infty}x_n=1+(1-\frac{1}{2}+\frac{1}{4}-\cdots)=1+\frac{1}{1-(-\frac{1}{2})}=\frac{5}{3}$.)

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(46) Let $s_n=\sum_{k=2}^{n}|x_k-x_{k-1}|$ and $s=\sum_{k=2}^{\infty}|x_k-x_{k-1}|$. For every $\varepsilon>0$, since $\sum_{k=2}^{\infty}|x_k-x_{k-1}|$ converges $\Leftrightarrow \lim_{n\to\infty}s_n=s$, so $\exists K$ such that $n\ge K\Rightarrow |s_n-s|=\sum_{k=n+1}^{\infty}|x_k-x_{k-1}|<\varepsilon$.
Then for $m,n\ge K$, say $m\ge n$, we have

$$|x_m-x_n|\le|x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n|\le\sum_{k=n+1}^{\infty}|x_k-x_{k-1}|<\varepsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence.

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(47) Claim: $|\sqrt{x}-\sqrt{y}|\le\sqrt{|x-y|}$ for every $x,y\ge 0$.
Proof. Let $u=\max(x,y)$ and $v=\min(x,y)$. Then $|\sqrt{x}-\sqrt{y}|=\sqrt{u}-\sqrt{v}$ and $|x-y|=u-v$.
Now $\sqrt{u}-\sqrt{v}\le\sqrt{u-v}\Leftrightarrow\sqrt{u}\le\sqrt{v}+\sqrt{u-v}\Leftrightarrow u\le\underbrace{v+2\sqrt{v(u-v)}+(u-v)}_{u+2\sqrt{v(u-v)}}$, which is true.

converge $\Rightarrow$ Cauchy

If $a_n\ge0$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}a_n=a$, then for every $\varepsilon>0$, there exists $K\in\mathbb{N}$ such that $m,n\ge K\Rightarrow|a_m-a_n|<\varepsilon^2$. So $m,n\ge K\Rightarrow|\sqrt{a_m}-\sqrt{a_n}|\le\sqrt{|a_m-a_n|}<\sqrt{\varepsilon^2}=\varepsilon$.

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(48) If $x_2=x_1$, then $|x_{n+1}-x_n|\le k|x_n-x_{n-1}|$ implies all $x_n=x_1$. In this case, for every $\varepsilon>0$, take $K=1$ and $m,n\ge K\Rightarrow|x_m-x_n|=0<\varepsilon$. The sequence $\{x_n\}$ is Cauchy.
If $x_2\ne x_1$, then $\forall\varepsilon>0$, let $K>\log_k\frac{(1-k)\varepsilon}{|x_2-x_1|}$ so that $|x_2-x_1|\frac{k^N}{1-k}<\varepsilon$. We have $m,n\ge K$, say $m>n$, implies $|x_m-x_n|\le|x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n|$

$$\le|x_2-x_1|(k^{m-2}+k^{m-3}+\cdots+k^{n-1})$$
$$\le|x_2-x_1|(k^K+k^{K+1}+\cdots)=|x_2-x_1|\frac{k^K}{1-k}<\varepsilon.$$

So the sequence $\{x_n\}$ is Cauchy.

(49) Let $b_n = a_n - A$ and $\beta_n = \frac{b_1+b_2+\cdots+b_n}{n}$, then $\lim_{n\to\infty} \alpha_n = A \Leftrightarrow \lim_{n\to\infty}(\alpha_n - A) = 0 \Leftrightarrow$ $\lim_{n\to\infty} \frac{(a_1-A)+\cdots+(a_n-A)}{n} = 0 \Leftrightarrow \lim_{n\to\infty} \beta_n = 0$, which is to be shown.

Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n - A) = 0$, $\{b_n\}$ is bounded, say $|b_n| \le M$ for all $n \in \mathbb{N}$.

For $\varepsilon > 0$, there is $K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |b_n - 0| < \frac{\varepsilon}{2}$. Let $K = \lceil \max(K_1, \frac{2(K_1-1)M}{\varepsilon}$.

Then $n \ge K \Rightarrow |\beta_n - 0| = \left|\frac{b_1+b_2+\cdots+b_{K_1-1}}{n} + \frac{b_{K_1}+\cdots+b_n}{n}\right| < \frac{(K_1-1)M}{n} + \frac{(n-K_1+1)\frac{\varepsilon}{2}}{n} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$0 < n - K_1 + 1 \le n$

Therefore, $\lim_{n\to\infty} \beta_n = 0$.

To see the converse is false, take $a_n = (-1)^n$. Then $\alpha_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$. So $\lim_{n\to\infty} \alpha_n = 0$ and $\{a_n\}$ doesn't converge.

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(50) Assume $\lim_{n\to\infty} x_n \ne x$. Then $\sim(\forall \varepsilon > 0\ \exists K$, such that $n \ge K \Rightarrow |x_n - x| < \varepsilon)$

$= \exists \varepsilon > 0\ \forall K\ \exists n \ge K$ and $|x_n - x| \ge \varepsilon$. So $\exists \varepsilon > 0$ such that

for $K = 1$, $\exists\, n_1 \ge 1$ and $|x_{n_1} - x| \ge \varepsilon$,

for $K = n_1 + 1$, $\exists\, n_2 \ge n_1 + 1$ and $|x_{n_2} - x| \ge \varepsilon$,

for $K = n_2 + 1$, $\exists\, n_3 \ge n_2 + 1$ and $|x_{n_3} - x| \ge \varepsilon$, ...

Then $n_1 < n_2 < n_3 < \cdots$ and subsequence $\{x_{n_j}\}$ satisfies $|x_{n_j} - x| \ge \varepsilon$ for all j. Since $\{x_{n_j}\}$ is bounded, by Bolzano-Weierstrass theorem, it has a Convergence subsequence $\{x_{n_{j_k}}\}$. Then $\lim_{k\to\infty} x_{n_{j_k}} = x$ and $0 = \lim_{k\to\infty} |x_{n_{j_k}} - x| \ge \varepsilon$ leads to a contradiction. Therefore, $\lim_{n\to\infty} x_n = x$.

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(51) $\forall \varepsilon > 0$, by the Archimedean principle, $\exists m \in \mathbb{N}$ such that $m > \log_2 \frac{1}{\varepsilon}$ ($\Leftrightarrow 2^{-m} < \varepsilon$).

Since f is injective, the set, $T = \{n \in \mathbb{N} : f(n) = 2^{-1}$ or 2^{-2} or $\cdots$ or $2^{-(m-1)}\}$ has at most $m-1$ elements. If the set is empty, then let $K = 1$, otherwise let K be larger than the maximum of T.

Then $n \ge K \Rightarrow n \notin T \Rightarrow |f(n) - 0| = f(n) \le 2^{-m} < \varepsilon$. Therefore, $\lim_{n\to\infty} f(n) = 0$.

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(52) Solution 1 Since $\lim_{n\to\infty}(a_{n+1} - \frac{a_n}{2}) = 0$, so $\forall \varepsilon > 0\ \exists K \in \mathbb{N}$ such that $n \ge K \Rightarrow$ $|a_{n+1} - \frac{a_n}{2}| < \varepsilon_0 = \frac{\varepsilon}{3}$. Let $K' \in \mathbb{N}$ be such that $\frac{|a_K|}{2^{K'}} < \varepsilon_0$. Then

$$m \ge K + K' \Rightarrow |a_m| < \frac{1}{2}|a_{m-1}| + \varepsilon_0 < \frac{1}{2^2}|a_{m-2}| + \frac{\varepsilon_0}{2} + \varepsilon_0 < \cdots$$

$$< \frac{1}{2^{m-K}}|a_K| + \frac{\varepsilon_0}{2^{m-K+1}} + \cdots + \frac{\varepsilon_0}{2} + \varepsilon_0 < \frac{1}{2^{K'}}|a_K| + 2\varepsilon_0 < 3\varepsilon_0 = \varepsilon,$$

$\therefore \lim_{n\to\infty} a_n = 0$.

(52) Solution 2 Let $b_n = a_{n+1} - \frac{1}{2}a_n$. Define $c_1 = c_2 = 0$ and for $k = 1, 2, 3, \ldots$ define $c_{2^k+1} = \cdots = c_{2^{k+1}} = b_k$. Then $b_n \to 0$ implies $c_n \to 0$, which implies (by exercise 49) that we have $\lim\limits_{n\to\infty} \frac{c_1 + c_2 + \cdots + c_n}{n} = 0$.

So, $2\left(\frac{c_1 + c_2 + \cdots + c_{2^{n+1}}}{2^{n+1}}\right) = \frac{2b_1 + 4b_2 + \cdots + 2^n b_n}{2^n} = a_n - \frac{1}{2^n}a_1 \to 0$. $\therefore \lim\limits_{n\to\infty} a_n = 0$.

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(53) For $\varepsilon > 0$, there is $K \in \mathbb{N}$ such that $n \geq K \Rightarrow |x_n - x_{n-2}| < \varepsilon/2$. For $n \geq K$,
$x_n - x_{n-1} = (x_n - x_{n-2}) - (x_{n-1} - x_{n-3}) + (x_{n-2} - x_{n-4}) - \cdots \pm (x_{K+1} - x_{K-1}) \mp (x_K - x_{K-1})$.
So $|x_n - x_{n-1}| \leq (n-K)\frac{\varepsilon}{2} + |x_K - x_{K-1}|$. For $n \geq \frac{2}{\varepsilon}|x_K - x_{K-1}|$, $\left|\frac{x_n - x_{n-1}}{n}\right| \leq \underbrace{\frac{n-K}{n}}_{<1}\frac{\varepsilon}{2} + \frac{|x_K - x_{K-1}|}{n} < 1\cdot\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
$\therefore \lim\limits_{n\to\infty} \frac{x_n - x_{n-1}}{n} = 0$.

---

(54) Let $y_n \to y$ and let $x = y/3$. We will show $\lim\limits_{n\to\infty} x_n = x$. $\forall \varepsilon > 0, \exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |y_n - y| < \frac{\varepsilon}{2}$. Now $\frac{\varepsilon}{2} > |y_n - y| = |x_{n-1} + 2x_n - 3x| = |2(x_n - x) + (x_{n-1} - x)|$ $\geq 2|x_n - x| - |x_{n-1} - x|$. So $|x_n - x| < \frac{\varepsilon}{4} + \frac{1}{2}|x_{n-1} - x|$. Repeat usage of this lead to $|x_{n+m} - x| < \varepsilon/4\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^m}\right) + \frac{1}{2^{m+1}}|x_{n-1} - x| < \frac{\varepsilon}{2} + \frac{1}{2^{m+1}}|x_{n-1} - x|$.
Next choose M large so that $\frac{1}{2^{M+1}}|x_{K-1} - x| < \frac{\varepsilon}{2}$. Then $j \geq K + M$ implies $|x_j - x| < \frac{\varepsilon}{2} + \frac{1}{2^{j-K+1}}|x_{K-1} - x| \leq \frac{\varepsilon}{2} + \frac{1}{2^{M+1}}|x_{K-1} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\therefore \lim\limits_{n\to\infty} x_n = x$.

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(55) (a) If $a_n = 1 - \frac{2}{n(n+1)} = \frac{n^2 + n - 2}{n(n+1)} = \frac{(n-1)(n+2)}{n(n+1)}$, then
$$\prod_{n=2}^{\infty}\left(1 - \frac{2}{n(n+1)}\right) = \lim_{k\to\infty} a_2 a_3 \cdots a_k = \lim_{k\to\infty}\left(\frac{1\cdot\cancel{4}}{\cancel{2}\cdot 3}\,\frac{\cancel{2}\cdot\cancel{5}}{\cancel{3}\cdot\cancel{4}}\cdots\frac{\cancel{(k-1)}(k+2)}{k\cancel{(k+1)}}\right) = \lim_{k\to\infty}\frac{k+2}{3k} = \frac{1}{3}.$$

(b) If $a_n = 1 - \frac{1}{n^2} = \frac{n^2 - 1}{n^2} = \frac{(n-1)(n+1)}{n^2}$, then
$$\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \lim_{k\to\infty} a_2 a_3 \cdots a_k = \lim_{k\to\infty}\left(\frac{1\cdot\cancel{3}}{2^2}\,\frac{\cancel{2}\cdot\cancel{4}}{\cancel{3}^2}\,\frac{\cancel{3}\cdot\cancel{5}}{\cancel{4}^2}\cdots\frac{\cancel{(k-1)}(k+1)}{\cancel{k^2}}\right) = \lim_{k\to\infty}\frac{k+1}{2k} = \frac{1}{2}.$$

(c) Note $\frac{n^3 - 1}{n^3 + 1} = \frac{(n-1)(n^2 + n + 1)}{(n+1)(n^2 - n + 1)}$ and $n^2 + n + 1 = (n+1)^2 - (n+1) + 1$. So
$$\prod_{n=2}^{\infty}\frac{n^3 - 1}{n^3 + 1} = \lim_{k\to\infty}\left(\frac{1\cdot\cancel{7}}{3\cdot\cancel{3}}\,\frac{2\cdot\cancel{13}}{\cancel{4}\cdot\cancel{7}}\,\frac{\cancel{3}\cdot\cancel{21}}{\cancel{5}\cdot\cancel{13}}\cdots\frac{\cancel{(k-1)}(k^2 + k + 1)}{(k+1)\cancel{(k^2 - k + 1)}}\right) = \lim_{k\to\infty}\frac{2(k^2 + k + 1)}{3k(k+1)} = \frac{2}{3}.$$

(d) Note $(1-z)(1+z)(1+z^2)\cdots(1+z^{2^k}) = (1-z^2)(1+z^2)\cdots(1+z^{2^k}) = \cdots$
$= (1 - z^{2^k})(1 + z^{2^k}) = 1 - z^{2^{k+1}}$. So $(1+z)(1+z^2)\cdots(1+z^{2^k}) = \frac{1 - z^{2^{k+1}}}{1 - z}$.
Therefore, $\prod\limits_{n=0}^{\infty}(1 + z^{2^n}) = \lim\limits_{k\to\infty}\frac{1 - z^{2^{k+1}}}{1 - z} = \frac{1}{1 - z}$ as $|z| < 1 \Rightarrow \lim\limits_{k\to\infty} z^{2^{k+1}} = 0$.

(56) Let S be a bounded infinite subset of $\mathbb{R}$. Then we choose $x_1 \in S$. Since S is infinite, there $\exists x_2 \in S$, $x_2 \neq x_1$, $\cdots$, $\exists x_n \in S$, $x_n \neq x_1, \cdots, x_{n-1}$. So the sequence $\{x_n\}$ consists of distinct terms in S. Since S is bounded, $\{x_n\}$ is bounded. By Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergence subsequence, say $\lim_{j\to\infty} x_{n_j} = x_0$. If $x_0 = x_{n_k}$ for some k, then x_0 is the limit of $x_{n_{k+1}}, x_{n_{k+2}}, \cdots$ in $S \setminus \{x_0\}$. So S has x_0 as an accumulation point.

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(57) (Note: $S=(0,+\infty)$, so $x\in S \Rightarrow x+1>1$) For every $\varepsilon>0$, let $\delta = 2\varepsilon$, then for every $x \in S=(0,\infty)$, $0<|x-1|<\delta = 2\varepsilon \Rightarrow \left|\frac{x}{x+1}-\frac{1}{2}\right| = \frac{|x-1|}{2(x+1)} < \frac{2\varepsilon}{2} = \varepsilon$.

---

(58) Suppose $\lim_{x\to x_0} f(x)$ exists at x_0. By density of rational, there is $r_n \in \mathbb{Q}$ such that $x_0 - \frac{1}{n} < r_n < x_0$. By density of irrational, there is $s_n \in \mathbb{R}\setminus\mathbb{Q}$ such that $x_0 - \frac{1}{n} < s_n < x_0$. By squeeze limit theorem, $\lim_{n\to\infty} r_n = x_0 = \lim_{n\to\infty} s_n$. By the sequential limit theorem, $\lim_{x\to x_0} f(x) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 8r_n = 8x_0$ and $\lim_{x\to x_0} f(x) = \lim_{n\to\infty} f(s_n) = \lim_{n\to\infty}(2s_n^2+8) = 2x_0^2+8$. By the uniqueness of limit, $8x_0 = 2x_0^2+8$. So $x_0 = 2$.

Next we show $\lim_{x\to 2} f(x)$ exists. (The limit should be $8\times 2 = 16 = 2\cdot 2^2+8$.)

We have $0 \le |f(x)-16| \le |8x-16| + |(2x^2+8)-16|$ for x rational or irrational.

Since $\lim_{x\to 2}\left(|8x-16|+|(2x^2+8)-16|\right) = 0$, by squeeze limit theorem, $\lim_{x\to 2} f(x) = 16$.

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(59) For $w \in \mathbb{R}$, there is a sequence $\{x_n\}$ of rational numbers converging to w (by practice exercise #39 or last exercise). Since f is continuous at w, by the sequential limit theorem, $f(w) = \lim_{x\to w} f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$.

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(60) (a) ① $f(0+0) = f(0)+f(0) \Rightarrow f(0)=0$. ② $0 = f(x+(-x)) = f(x)+f(-x) \Rightarrow f(-x) = -f(x)$.

③ For $n \in \mathbb{N}$, $f(nx) = nf(x)$ by mathematical induction ($f(1x) = 1f(x)$. If $f(nx)=nf(x)$, then $f((n+1)x) = f(nx+x) = f(nx)+f(x) = nf(x)+f(x) = (n+1)f(x)$.)

④ Taking $x = \frac{1}{n}$ in ③ we get $f(1) = nf(\frac{1}{n}) \Rightarrow f(\frac{1}{n}) = \frac{1}{n}f(1)$.

Taking $x = \frac{1}{k}$ in ③ we get $f(\frac{n}{k}) = nf(\frac{1}{k}) = \frac{n}{k}f(1)$.

⑤ By ②, $f(-\frac{n}{k}) = -f(\frac{n}{k}) = -\frac{n}{k}f(1)$. If $c = f(1)$, then $f(r) \overset{\text{by ①,④,⑤}}{=} cr$ for $r \in \mathbb{Q}$.

Conversely, the function $f(r) = cr$ satisfies $f(x+y) = c(x+y) = cx+cy = f(x)+f(y)$ for any $c \in \mathbb{R}$.

(b) For $w \in \mathbb{R}$, by density of rational numbers, there are $r_n, s_n \in \mathbb{Q}$ such that $w - \frac{1}{n} < r_n < w < s_n < w + \frac{1}{n}$. For f strictly increasing, by part (a), $r_n f(1) = f(r_n) < f(w) < f(s_n) = s_n f(1)$. Taking limit, we get $f(w) = wf(1)$ by squeeze limit theorem. So the functions we are looking for are $f:\mathbb{R}\to\mathbb{R}$ given by $f(w) = cw$, where $c = f(1) > 0$.