Example

⑥ Prove $\left\{\frac{2n}{n+5}+\frac{n^8}{n^8+n^5+1}\right\}$ converges to 3 by checking the definition of limit.

Scratch ① When n is large, $\frac{2n}{n+5}\approx 2$, $\frac{n^8}{n^8+n^5+1}\approx 1$.

② $\forall \varepsilon>0$,

$$\left|\frac{2n}{n+5}+\frac{n^8}{n^8+n^5+1}-3\right| \boxed{<\varepsilon \text{ is hard to solve for } n}$$

$$=\left|\frac{2n}{n+5}-2+\frac{n^8}{n^8+n^5+1}-1\right|$$

$$\le \left|\frac{2n}{n+5}-2\right|+\left|\frac{n^8}{n^8+n^5+1}-1\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \text{ is easier to solve}$$

③ $\left|\frac{2n}{n+5}-2\right|=\left|\frac{-10}{n+5}\right|=\frac{10}{n+5}<\frac{\varepsilon}{2}$ if $n>\frac{20}{\varepsilon}-5$

$\left|\frac{n^8}{n^8+n^5+1}-1\right|=\left|\frac{-n^5-1}{n^8+n^5+1}\right|=\frac{n^5+1}{n^8+n^5+1}<\frac{2n^5}{n^8}=\frac{2}{n^3}<\frac{\varepsilon}{2}$ if $n>\sqrt[3]{4/\varepsilon}$.

Solution $\forall\varepsilon>0$, by Archimedean Principle, $\exists K_1\in\mathbb{N}$ such that $K_1>\frac{20}{\varepsilon}-5$ and $\exists K_2\in\mathbb{N}$ such that $K_2>\sqrt[3]{4/\varepsilon}$.

Let $K=\max(K_1,K_2)$. Then

$n\ge K \Rightarrow n\ge K_1>\frac{20}{\varepsilon}-5$ and $n\ge K_2>\sqrt[3]{4/\varepsilon}$

$$\Rightarrow \left|\frac{2n}{n+5}+\frac{n^8}{n^8+n^5+1}-3\right|\le\left|\frac{2n}{n+5}-2\right|+\left|\frac{n^8}{n^8+n^5+1}-1\right|$$

$$=\frac{10}{n+5}+\frac{n^5+1}{n^8+n^5+1}<\frac{10}{n+5}+\frac{2}{n^3}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Boundedness Theorem If $\{x_n\}$ converges, then the set $\{x_1,x_2,x_3,\dots\}$ is bounded (above and below).

Given: $\{x_n\}$ converges to some $x\in\mathbb{R}$ ($\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $n\ge K\Rightarrow |x_n-x|<\varepsilon$)

To Prove: $\{x_1,x_2,x_3,\dots\}$ is bounded ($\Leftrightarrow \exists M\in\mathbb{R}$ $\forall x_n$, $|x_n|\le M$)

Proof. Let $x=\lim_{n\to\infty}x_n$. For $\varepsilon=1$, $\exists K\in\mathbb{N}$ such that

$$n\ge K\Rightarrow |x_n-x|<1 \Rightarrow |x_n|=|x_n-x+x| \le |x_n-x|+|x| < 1+|x|.$$

Let $M=\max(|x_1|,|x_2|,\dots,|x_{K-1}|,1+|x|)$. Then

$\forall n\in\mathbb{N}$, $n\ge K\Rightarrow |x_n|<1+|x|\le M$

$n<K\Rightarrow x_n=x_1$ or x_2 or $\dots=x_{K-1}\Rightarrow |x_n|\le M$.

Remarks The converse of the boundedness theorem is false. $x_n=(-1)^n$ $\{x_1,x_2,x_3,\dots\}=\{-1,1\}$ is bounded but $\{x_n\}$ does not converge by example ④.

Remarks The following are equivalent:

① $\{x_n\}$ converges to x ($\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $n\ge K\Rightarrow |x_n-x|<\varepsilon$)

② $\{x_n-x\}$ converges to 0 ($\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $n\ge K\Rightarrow |(x_n-x)-0|<\varepsilon$)

③ $\{|x_n-x|\}$ converges to 0 ($\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $n\ge K\Rightarrow \big||x_n-x|-0\big|<\varepsilon$.)