Solutions to Presentation Exercises

(89) (c) For a fixed $m \in \mathbb{N}$, the curves $x^2+y^2=1$ and $xy=\frac{1}{m}$ $\left(\leftrightarrow y=\frac{1}{mx}\right)$ intersect in at most 4 points because $x^2+\left(\frac{1}{mx}\right)^2=1 \Rightarrow x^4-x^2+\frac{1}{m^2}=0$. Then

$$S=\bigcup_{\substack{m\in\mathbb{N}\\ \text{countable}}} \underbrace{\{(x,y): x^2+y^2=1 \text{ and } xy=\tfrac{1}{m}\}}_{\text{at most 4 points, hence countable}}$$ is countable by countable union theorem.

(g) Since $A\cap B \subseteq A$, $\mathbb{Q}\cap A \subseteq \mathbb{Q}$, $B\cap\mathbb{Q}\subseteq\mathbb{Q}$ and A, $\mathbb{Q}$ are countable, so by the countable subset theorem, $A\cap B$, $\mathbb{Q}\cap A$, $B\cap\mathbb{Q}$ are countable. For $x\in A\cap B$, $y\in\mathbb{Q}\cap A$ and $z\in B\cap\mathbb{Q}$, let $S_{x,y,z}=\{x^2+y^2+z^2\}$. Then $S_{x,y,z}$ is a one element set. So $S_{x,y,z}$ is countable.

Finally, $S=\bigcup_{(x,y,z)\in \underbrace{(A\cap B)\times(\mathbb{Q}\cap A)\times(B\cap\mathbb{Q})}_{\text{countable by product theorem}}} \underbrace{S_{x,y,z}}_{\text{countable}}$ is countable by the countable union theorem

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(h) Let $y_0\in A$ and $T=\{x-y_0: x\in A\}$. Then $T\subseteq S$. Now $f: A\to T$ defined by $f(x)=x-y_0$ is bijective with $f^{-1}(t)=t+y_0$. By bijection theorem, A uncountable implies T uncountable. Finally since $T\subseteq S$, S must be uncountable by the contrapositive statement of the countable subset theorem.

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(91) (g) Since $0<\frac{\sqrt{2}}{m+n}+\frac{1}{k\sqrt{2}}\le\frac{\sqrt{2}}{1+1}+\frac{1}{1\sqrt{2}}=\frac{\sqrt{2}}{2}+\frac{1}{\sqrt{2}}=\sqrt{2}$, S is bounded below by 0 and above by $\sqrt{2}$. Now every upper bound $M\ge\sqrt{2}\in S$, so $\sup S=\sqrt{2}$. Next considering $a_n=\frac{\sqrt{2}}{n+n}+\frac{1}{n\sqrt{2}}\in S$, we have $\lim_{n\to\infty}a_n=0$, which is a lower bound. So by the infimum limit theorem, $\inf S=0$.

(h) $S=[0,\frac{1}{2})\cup[\frac{2}{3},\frac{3}{4})\cup[\frac{4}{5},\frac{5}{6})\cup\cdots$. Since $0\le 1-\frac{1}{2k-1}$ and $1-\frac{1}{2k}<1$ for $k=1,2,3,\ldots$, so $0\le x<1$ for all $x\in S$. So S is bounded below by 0 and above by 1. Since every lower bound $m\le 0\in S$, so $\inf S=0$. Next since $1-\frac{1}{2k-1}\in S$ and $\lim_{k\to\infty}\left(1-\frac{1}{2k-1}\right)=1$, so by the supremum limit theorem, $\sup S=1$.

(k) Since $0\le x+y\le 2$ for $x\in[0,1]\cap\mathbb{Q}$, $y\in[0,1]\cap(\mathbb{R}\setminus\mathbb{Q})$, S is bounded below by 0 and bounded above by 2. We will show $\inf S=0$ and $\sup S=2$.

Let $w_n=\frac{1}{n}+\frac{1}{n\sqrt{2}}$, then $w_n\in S$ and $\lim_{n\to\infty}w_n=0$. So by infimum limit theorem, $\inf S=0$.

Let $v_n=\frac{n}{n+1}+\frac{1}{n+1\sqrt{2}}$, then $v_n\in S$ and $\lim v_n=2$. So by supremum limit theorem, $\sup S=2$.

(o) $0\le x^2+y^3+z^4\le 1+1+1=3$ for $x\in(-1,0)\setminus\mathbb{Q}$, $y\in(0,1)\cap\mathbb{Q}$, $z\in(-1,1)$. So 0 is a lower bound and 3 is an upper bound of S. Since $\left(-\frac{1}{n\sqrt{2}}\right)^2+\left(\frac{1}{n+1}\right)^3+\left(\frac{1}{n+1}\right)^4$ is in S and has limit 0, so $\inf S=0$. Since $\left(-1+\frac{1}{n\sqrt{2}}\right)^2+\left(1-\frac{1}{n+1}\right)^3+\left(1-\frac{1}{n+1}\right)^4$ is in S and has limit 3, so $\sup S=3$.

(98) We have $x \in A, y \in A \Rightarrow x^2+y^2 \le (\sup A)^2 + (\sup A)^2 = 2(\sup A)^2$. So $2(\sup A)^2$ is an upper bound for B.

By supremum limit theorem, there is a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = \sup A$. Then $\{x_n^2 + x_n^2\}$ is a sequence in B and $\lim_{n\to\infty}(x_n^2+x_n^2) = 2(\sup A)^2$. So by the Supremum limit theorem, $\sup B = 2(\sup A)^2$.

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(99) For $x \in \bigcup_{n=1}^{10} A_n$, $x \in A_n$ for some $n \Rightarrow x \le x_n = \sup A_n \le \max(x_1, \cdots, x_{10})$.

So $\max(x_1,\cdots,x_{10})$ is an upper bound of $\bigcup_{n=1}^{10} A_n$. Let $x_i = \max(x_1,\cdots,x_{10})$, then since $x_i = \sup A_i$, there is $\{a_n\}$ in A_i such that $\lim_{n\to\infty} a_n = x_i$. Since $\{a_n\} \in \bigcup_{n=1}^{10} A_n$, so $x_i = \sup(\bigcup_{i=1}^{10} A_i)$. $\therefore \sup(\bigcup_{i=1}^{10} A_i) = \max(x_1,\cdots,x_{10})$.

Alternative Solution

As in first solution, $x_i = \max(x_1,\cdots,x_{10})$ is an upper bound of $\bigcup_{n=1}^{10} A_n$. For any upper bound M of $\bigcup_{n=1}^{10} A_n$, $M \ge x$ for all $x \in \bigcup_{n=1}^{10} A_n$. Since $A_i \subseteq \bigcup_{n=1}^{10} A_n$, $M \ge x$ for every $x \in A_i$. So M is an upper bound of A_i, too. Then $M \ge x_i$. So $x_i = \max(x_1,\cdots,x_{10})$ is the least upper bound of $\bigcup_{n=1}^{10} A_n$.

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(141) Let T be the set of all circles on the coordinate plane with center $(x,y) \in \mathbb{Q}\times\mathbb{Q}$ and radius $r \in \mathbb{Q}^+$. Then $T = \underbrace{\bigcup_{(x,y,r)\in \mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}^+}}_{\text{Countable by product theorem}} \underbrace{\{C_{(x,y,r)}\}}_{\text{1 element set} \Rightarrow \text{Countable}}$ where $C_{(x,y,r)}$ is the circle with center (x,y) and radius r is countable by countable union theorem.

Now $\underset{x,y\in\mathbb{Q}\Rightarrow |xy|+1\in\mathbb{Q}^+}{S} \subseteq \underbrace{(\mathbb{Q}\times\mathbb{Q})\times T}_{\text{Countable by product theorem}} \Rightarrow S$ is countable by countable subset theorem.
