Solution to Presentation Exercises

502 Since $\{x_n\}$ is Cauchy, it is bounded. So $\exists M$ such that $\forall n \in \mathbb{N}$, $|x_n| < M$.
Since $\{x_n\}$ is Cauchy, $\forall \varepsilon > 0$, $\exists K_1 \in \mathbb{N}$ such that $m, n \geq K_1 \Rightarrow |x_m - x_n| < \frac{\varepsilon}{3}$. Also $\exists K_2 \in \mathbb{N}$ such that $m, n \geq K_2 \Rightarrow |x_m - x_n| < \frac{\varepsilon}{6M}$. Let $K = \max\{K_1, K_2\}$.
Then $m, n \geq K \Rightarrow m, n \geq K_1$ and $m, n \geq K_2$, $m+1, n+1 \geq K_1$

$$\Rightarrow |y_m - y_n| \leq |x_{n+1} - x_{m+1}| + |x_n^2 - x_m^2| + |\cos x_n - \cos x_m|$$
$$\leq \varepsilon/3 + |x_n + x_m||x_n - x_m| + |x_n - x_m|$$
$$< \varepsilon/3 + 2M \cdot \varepsilon/6M + \varepsilon/3 = \varepsilon.$$

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504 Observe that

$$\boxed{\begin{aligned} |B_m - B_n| &= |B_m - \sqrt{A_m} + \sqrt{A_m} - \sqrt{A_n} + \sqrt{A_n} - B_n| \\ &\leq |B_m - \sqrt{A_m}| + |\sqrt{A_m} - \sqrt{A_n}| + |\sqrt{A_n} - B_n| \\ &\leq |\sqrt{A_{m+2011}} - \sqrt{A_m}| + |\sqrt{A_m} - \sqrt{A_n}| + |\sqrt{A_n} - \sqrt{A_{n+2011}}| \\ &\leq \sqrt{|A_{m+2011} - A_m|} + \sqrt{|A_m - A_n|} + \sqrt{|A_n - A_{n+2011}|} \end{aligned}}$$

$\forall \varepsilon > 0$, since $\{A_n\}$ is Cauchy, $\exists K \in \mathbb{N}$ such that $m, n \geq K \Rightarrow |A_m - A_n| < (\frac{\varepsilon}{3})^2$. Then
$m, n \geq K \Rightarrow m, n, m+2011, n+2011 \geq K$

$$\Rightarrow \sqrt{|A_{m+2011} - A_m|} < \sqrt{(\tfrac{\varepsilon}{3})^2} = \varepsilon/3$$
$$\sqrt{|A_m - A_n|} < \sqrt{(\tfrac{\varepsilon}{3})^2} = \varepsilon/3$$
$$\sqrt{|A_n - A_{n+2011}|} < \sqrt{(\varepsilon/3)^2} = \varepsilon/3$$
$$\Rightarrow |B_m - B_n| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

by observation above

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505 <u>Solution 1</u> Observe that

$$|a_n \sin a_n - a_m \sin a_m| = |a_n \sin a_n - a_m \sin a_n + a_m \sin a_n - a_m \sin a_m|$$
$$\leq |(a_n - a_m)\sin a_n| + |a_m(\sin a_n - \sin a_m)|$$
$$\leq |a_n - a_m| + |a_m||a_n - a_m|.$$

Since $\{a_n\}$ is Cauchy, it is bounded, say $|a_n| \leq M$ for all n.
$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ such that $m, n \geq K \Rightarrow |a_n - a_m| < \frac{\varepsilon}{1+M}$.
Then $|a_n \sin a_n - a_m \sin a_m| \leq (1 + |a_m|)|a_n - a_m| < (1+M)\frac{\varepsilon}{1+M} = \varepsilon$.

505

Solution 2

Let $f(x) = x\sin x$. Then $f'(x) = \sin x + x\cos x$. So $|f'(x)| \le 1 + |x|$. Since $\{a_n\}$ is Cauchy, it is bounded, Say $|a_n| \le M$ for all n.

$\forall \varepsilon > 0, \exists K \in \mathbb{N}$ such that $m, n \ge K \Rightarrow |a_n - a_m| < \frac{\varepsilon}{1+M}$.

Then $|a_n \sin a_n - a_m \sin a_m| = |f(a_n) - f(a_m)|$.

$\overset{\text{Mean Value theorem}}{=} |f'(x)||a_n - a_m| < (1+M)\frac{\varepsilon}{1+M} = \varepsilon$

for some x between a_n, a_m
$\Rightarrow x \in [-M, M]$

601

$\forall \varepsilon > 0$, let $\delta = 4\varepsilon^2 > 0$. $\forall x \in (0, \infty)$

$$0 < |x-1| < \delta \Rightarrow \left|\frac{1}{\sqrt{x}+1} - \frac{1}{2}\right| = \left|\frac{1-\sqrt{x}}{2(\sqrt{x}+1)}\right|$$

Use $|\sqrt{a}-\sqrt{b}| \le \sqrt{|a-b|}$; $\sqrt{x} > 0$

$$\le \frac{\sqrt{|1-x|}}{2} < \frac{\sqrt{\delta}}{2} = \varepsilon$$

Since $\delta = 4\varepsilon^2$

Variations

multiply by $\frac{1+\sqrt{x}}{1+\sqrt{x}}$

$$\left|\frac{1-\sqrt{x}}{2(\sqrt{x}+1)}\right| = \left|\frac{1-x}{2(\sqrt{x}+1)^2}\right| < \frac{\delta}{2} = \varepsilon \qquad \delta = 2\varepsilon$$

$$\left|\frac{1-\sqrt{x}}{2(\sqrt{x}+1)}\right| = \frac{|1-x|}{2(\sqrt{x}+1)^2} < \frac{\delta}{2(\sqrt{\frac{1}{2}}+1)^2} \le \varepsilon \qquad \delta = \min(\delta_1, \delta_2)$$

$\delta_1 = \frac{1}{2}$ $\quad |x-1| < \delta_1 = \frac{1}{2} \Rightarrow x \in (\frac{1}{2}, \frac{3}{2})$

$\delta_2 = 2(\sqrt{\frac{1}{2}}+1)^2\varepsilon$

$$\left|\frac{1-\sqrt{x}}{\underbrace{2(\sqrt{x}+1)}_{>2>1}}\right| < \frac{|1-\sqrt{x}|}{1} \le |1-\sqrt{x}|\underbrace{|1+\sqrt{x}|}_{>1} = |1-x| < \delta = \varepsilon$$

605

Solution 1

Observe that $|\sqrt{a}-\sqrt{b}| \le \sqrt{|a-b|}$

$$\left|\sqrt{\frac{1}{2+\sqrt{x}}} - \underbrace{\frac{1}{2}}_{=\sqrt{1/4}}\right| \le \sqrt{\left|\frac{1}{2+\sqrt{x}} - \frac{1}{4}\right|} = \sqrt{\left|\frac{2-\overset{\sqrt{4}}{\sqrt{x}}}{4(2+\underbrace{\sqrt{x}}_{\ge 0})}\right|} \le \frac{\sqrt[4]{|4-x|}}{\sqrt{8}} < \varepsilon$$

$\Leftrightarrow |4-x| < (\sqrt{8}\varepsilon)^4 = 64\varepsilon^4$

$\forall \varepsilon > 0$, let $\delta = 64\varepsilon^4$. Then
$\forall x \in [0, \infty), 0 < |x-4| < \delta \Rightarrow \left|\sqrt{\frac{1}{2+\sqrt{x}}} - \frac{1}{2}\right| < \varepsilon$.

(605) Solution 2 Observe that $\sqrt{a}-\sqrt{b}=\dfrac{a-b}{\sqrt{a}+\sqrt{b}}$ and so

$$\left|\sqrt{\frac{1}{2+\sqrt{x}}}-\frac{1}{2}\right|=\frac{\left|\frac{1}{2+\sqrt{x}}-\frac{1}{4}\right|}{\underbrace{\sqrt{\frac{1}{2+\sqrt{x}}}+\frac{1}{2}}_{\geq 0}}\leq 2\left|\frac{2-\sqrt{x}}{4(2+\sqrt{x})}\right|\leq\frac{1}{4}\frac{|4-x|}{\underbrace{2+\sqrt{x}}_{\geq 0}}\leq\frac{|4-x|}{8}<\varepsilon$$

$$\Leftrightarrow |4-x|<8\varepsilon$$

$\forall\varepsilon>0$, let $\delta=8\varepsilon$. Then

$\forall x\in[0,\infty)$, $0<|x-4|<\delta\Rightarrow\left|\sqrt{\frac{1}{2+\sqrt{x}}}-\frac{1}{2}\right|<\varepsilon$.

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(606) Sketch

$$|f(x)-1|=\left|\frac{x}{1+2x}+\frac{2}{2+\sqrt{x}}-1\right|=\left|\left(\frac{x}{1+2x}-\frac{1}{3}\right)+\left(\frac{2}{2+\sqrt{x}}-\frac{2}{3}\right)\right|$$

$$\leq\left|\frac{x}{1+2x}-\frac{1}{3}\right|+\left|\frac{2}{2+\sqrt{x}}-\frac{2}{3}\right|=\left|\frac{x-1}{3(1+2x)}\right|+\left|\frac{2-2\sqrt{x}}{3(2+\sqrt{x})}\right|$$

$$\leq\frac{|x-1|}{3}+\frac{2\sqrt{|1-x|}}{3\times 2}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

if $|x-1|<\frac{3\varepsilon}{2}$ and $\sqrt{|x-1|}<\frac{3\varepsilon}{2}$ $\Leftrightarrow |x-1|<\frac{9\varepsilon^2}{4}$

Solution $\forall\varepsilon>0$, let $\delta=\min\left\{\frac{3\varepsilon}{2},\frac{9\varepsilon^2}{4}\right\}>0$, then

$\left.\begin{array}{l}x\in[0,+\infty)\\ 0<|x-1|<\delta\end{array}\right\}\Rightarrow |x-1|<3\varepsilon/2$ and $|x-1|<9\varepsilon^2/4$

$\Rightarrow |f(x)-1|<\varepsilon$

by sketch above.

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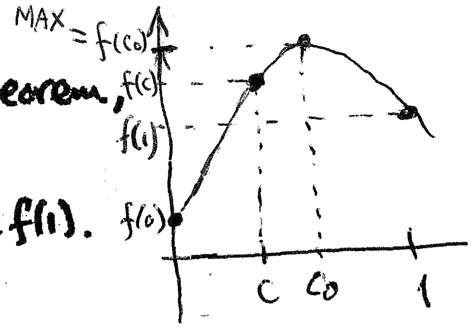
(704) Assume such f exists. Let $g(x)=f(x)+f(x+1)$.
Then g is continuous and the range of g is a subset of $\mathbb{R}\setminus\mathbb{Q}$. From the intermediate value theorem, the range of g is an interval. Hence, the range of g must be a single point.
$\therefore$ g is a constant function.
Similarly, $h(x)=f(x)-f(x+1)$ is a constant function. Then $f(x)=\frac{g(x)+h(x)}{2}$ is a constant function, which cannot satisfy the condition
$f(x)\in\mathbb{Q}\Leftrightarrow f(x+1)\in\mathbb{R}\setminus\mathbb{Q}$, Contradiction.

802 On $[0,1]$, by extreme value theorem, $\exists\, c_0 \in [0,1]$ such that

$$f(c_0) = \max\{f(t) : t \in [0,1]\} \geq f(c) > f(0) \text{ and } f(1).$$



So $c_0 \in (0,1)$. Hence, $f'(c_0) = 0$.

By mean value theorem, $\exists\, \theta_0, \theta_1 \in (0,1)$ such that

$$f'(0) = f'(0) - f'(c_0) = f''(\theta_0)(0 - c_0)$$

$$f'(1) = f'(1) - f'(c_0) = f''(\theta_1)(1 - c_0).$$

So
$$|f'(0)| + |f'(1)| \leq |f''(\theta_0)| c_0 + |f''(\theta_1)|(1-c_0)$$
$$\leq 2010\, c_0 + 2010\,(1 - c_0)$$
$$= 2010.$$

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804 On $[0,1]$, by extreme value theorem, $\exists\, c_0 \in [0,1]$ such that

$$f(c_0) = \max\{f(t) : t \in [0,1]\} = 2 > 0 = f(0) \text{ and } f(1).$$

So $c_0 \in (0,1)$. Hence $f'(c_0) = 0$.

By Taylor's Theorem, $\exists\, \theta_0, \theta_1 \in (0,1)$ such that

$$\underbrace{f(0)}_{=0} = \underbrace{f(c_0)}_{=2} + \underbrace{f'(c_0)}_{=0}(0 - c_0) + \frac{f''(\theta_0)}{2}(0 - c_0)^2 \quad 0 < \theta_0 < c$$

$$\underbrace{f(1)}_{=0} = \underbrace{f(c_0)}_{=2} + \underbrace{f'(c_0)}_{=0}(1 - c_0) + \frac{f''(\theta_1)}{2}(1 - c_0)^2 \quad c < \theta_1 < 1$$

Solving for $f''(\theta_0), f''(\theta_1)$, we get

$$f''(\theta_0) = \frac{-4}{c_0^2} \quad \text{and} \quad f''(\theta_1) = \frac{-4}{(1-c_0)^2}$$

Now
$$-\frac{4}{c_0^2} \leq -16 \iff \frac{1}{4} \geq c_0^2 \iff 0 < c_0 \leq \frac{1}{2}$$

$$-\frac{4}{(1-c_0)^2} \leq -16 \iff \frac{1}{4} \geq (1-c_0)^2 \iff \frac{1}{2} \leq c_0 < 1$$

So if $c_0 \in (0, \frac{1}{2}]$, take $\theta = \theta_0$;
if $c_0 \in [\frac{1}{2}, 1)$, take $\theta = \theta_1$.

(805) By Taylor's theorem, $\exists\, a, b \in \mathbb{R}$ such that $f(0) = f(1) + f'(1)(0-1) + \frac{f''(1)}{2}(0-1)^2 + \frac{f'''(a)}{6}(0-1)^3$

and $f(2) = f(1) + f'(1)(2-1) + \frac{f''(1)}{2}(2-1)^2 + \frac{f'''(b)}{6}(2-1)^3$.

Adding these and cancelling $f(0)+f(2) = 2f(1)$,

$$0 = 0 + 0 + f''(1) - \frac{f'''(a)}{6} + \frac{f'''(b)}{6}.$$

Rearranging terms, we get $f'''(a) - f'''(b) = 6f''(1)$.

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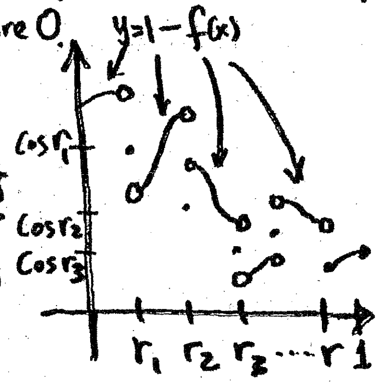
(902) (b) Since $\{r_n\}$ is strictly increasing and bounded above by 1, $\lim_{n\to\infty} r_n = r$ exists.

Notation
For function $h: [a,b] \to \mathbb{R}$, denote
$S_h = \left\{x \;\middle|\; \begin{array}{l} x \in [a,b] \\ h \text{ is discontinuous at } x \end{array}\right\}$

Next, f is continuous at $x \Leftrightarrow 1-f$ is continuous at x.
So $S_f = S_{1-f}$, which is of measure 0.

For g, we have

$$S_g \subseteq \underbrace{\underset{\substack{\| \\ S_f \\ \text{measure } 0}}{S_{1-f}} \cup \underbrace{\{r_1, r_2, r_3, \ldots\} \cup \{r\}}_{\substack{\text{Countable} \\ \Rightarrow \text{measure } 0}}}_{\text{measure } 0}$$



$\therefore S_g$ is of measure 0.

$\therefore g$ is Riemann integrable on $[0,1]$.

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(903) If $f(x)$ is continuous at $x = x_0 \in [0,1)$, then $|f(x)-1|$ is continuous at $x = x_0$. Taking contrapositive, if $|f(x)-1|$ is discontinuous at $x = x_0$, then $f(x)$ is discontinuous at $x = x_0$. This means $x_0 \in S_{|f-1|} \Rightarrow x_0 \in S_f$. Hence, $S_{|f-1|} \subseteq S_f$.

Since f is integrable, by Lebesgue's theorem, S_f is of measure 0.

$S_{|f-1|} \subseteq S_f \Rightarrow S_{|f-1|}$ is of measure 0 by example done in class.

If $f(x)$ is continuous at $x = x_1 \in [0,1]$, then $h(x) = f(x-1)$ is continuous at $x = x_1 + 1 \in [1,2]$. Using contrapositive, we see $S_h \cap \underbrace{[1,2]}_{\text{domain of } h} \subseteq \{1+x : x \in S_f\}$.

Claim: $T = \{1+x : x \in S_f\}$ is of measure 0.

Reason: S_f is of measure 0 $\Rightarrow \forall \varepsilon > 0\ \exists\, (a_1,b_1), (a_2,b_2), (a_3,b_3), \ldots$ such that $S_f \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$ and $\sum_{n=1}^{\infty} |a_n - b_n| < \varepsilon$

$\Rightarrow T \subseteq \bigcup_{n=1}^{\infty} (1+a_n, 1+b_n)$ and $\sum_{n=1}^{\infty} |(1+a_n) - (1+b_n)| < \varepsilon$

Finally, $S_F \subseteq (S_{|f-1|} \cap [0,1)) \cup \{1\} \cup (S_h \cap [1,2]) \subseteq S_{|f-1|} \cup \{1\} \cup T$

The union of $S_{|f-1|}, \{1\}, T$ is of measure 0 by example done in class. measure 0

$\therefore S_F$ is of measure 0. $\therefore F$ is integrable on $[0,2]$.