1. Consider the following limit:

$$\lim_{n\to+\infty} \frac{a_0 n^k + a_1 n^{k-1} + \dots + a_{k-1} n + a_k}{b_0 n^l + b_1 n^{l-1} + \dots + b_{l-1} n + b_l},$$

where k, l are positive integers. $a_0 \neq 0$. $b_0 \neq 0$.

$$\lim_{x\to 1} \frac{4x^3}{x^2+1} \ ?$$

$$\lim_{n\to\infty} \frac{3n^5+n}{4n^6+1} = 0.$$

$$\frac{a_0 n^k + a_1 n^{k-1} + \dots + a_{k-1} n + a_k}{b_0 n^l + b_1 n^{l-1} + \dots + b_{l-1} n + b_l} = \frac{a_0 + a_1 \frac{1}{n} + \dots + a_{k-1} \frac{1}{n^{k-1}} + a_k \frac{1}{n^k}}{b_0 + b_1 \frac{1}{n} + \dots + b_{l-1} \frac{1}{n^{l-1}} + b_l \frac{1}{n^l}} \cdot n^{k-l}.$$

$$\lim_{n\to+\infty} \frac{a_0 + a_1 \frac{1}{n} + \dots + a_{k-1} \frac{1}{n^{k-1}} + a_k \frac{1}{n^k}}{b_0 + b_1 \frac{1}{n} + \dots + b_{l-1} \frac{1}{n^{l-1}} + b_l \frac{1}{n^l}} = \frac{a_0}{b_0} \neq 0.$$

$$\lim_{n\to+\infty} n^{k-l} = \begin{cases} 0 & k<l \\ 1 & k=l \\ +\infty & k>l \end{cases}$$

Then,
$$\lim_{n\to+\infty} \frac{a_0 n^k + a_1 n^{k-1} + \dots + a_{k-1} n + a_k}{b_0 n^l + b_1 n^{l-1} + \dots + b_{l-1} n + b_l} = \begin{cases} 0 & k<l \\ \frac{a_0}{b_0} & k=l \\ +\infty & k>l. \end{cases}$$

Convergence of sequence: ① Bounded. increasing / de..

2. Fibonacci Sequence. (Rabbit Sequence).

$$a_1 = 1, \quad a_2 = 1, \quad a_3 = 2, \quad a_4 = 3,$$

$$a_{n+1} = a_n + a_{n-1}. \quad n = 2, 3, 4, \dots$$

Set $b_n = \frac{a_{n+1}}{a_n}$ and b_n reveals the growth of $\{a_n\}$.

$$\lim_{n\to+\infty} b_n = ?$$

$$b_n = \frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n} = 1 + \frac{1}{b_{n-1}}$$

When $b_n > \frac{\sqrt{5}+1}{2}$, $b_{n+1} < \frac{\sqrt{5}+1}{2}$.

When $b_n < \frac{\sqrt{5}+1}{2}$, $b_{n+1} > \frac{\sqrt{5}+1}{2}$.

$\Rightarrow$ $\{b_n\}$ is not a monotonic sequence.

By induction we find: $b_{2k-1} \in \left(0, \frac{\sqrt{5}+1}{2}\right)$

$b_{2k} \in \left(\frac{\sqrt{5}+1}{2}, +\infty\right)$. $k=1,2,3,\dots$

$$b_{2k+2} - b_{2k} = 1 + \frac{1}{1+\frac{1}{b_{2k}}} - b_{2k} = \frac{\left(\frac{\sqrt{5}+1}{2} - b_{2k}\right)\left(\frac{\sqrt{5}-1}{2} + b_{2k}\right)}{1+b_{2k}} < 0.$$

$$b_{2k+1} - b_{2k-1} = 1 + \frac{1}{1+\frac{1}{b_{2k-1}}} - b_{2k-1} = \frac{\left(\frac{\sqrt{5}+1}{2} - b_{2k-1}\right)\left(\frac{\sqrt{5}-1}{2} + b_{2k-1}\right)}{1+b_{2k-1}} > 0.$$

$\Rightarrow \{b_{2k}\}$ is a bounded monotonically decreasing sequence.

$\{b_{2k+1}\}$ is a bounded monotonically increasing sequence.

Denote $\lim\limits_{n\to+\infty} b_{2n} = a$, $\lim\limits_{n\to+\infty} b_{2n+1} = b$.

$b_{2k+1} = 1 + \frac{1}{b_{2k}}$. $\Rightarrow$ $b = 1 + \frac{1}{a}$ $\Rightarrow$ $ab = a+1$

$b_{2k} = 1 + \frac{1}{b_{2k-1}}$ $\Rightarrow$ $a = 1 + \frac{1}{b}$. $\Rightarrow$ $ab = b+1$

$\Rightarrow a=b$. $\Rightarrow$ $a=b=\frac{1+\sqrt{5}}{2}$.

3. Prove: Sequence $\{(1+\frac{1}{n})^n\}$ is increasing and $\{(1+\frac{1}{n})^{n+1}\}$ is decreasing. They converge to the same limit.

Proof: Use this inequality: $(a_1 a_2 \cdots a_n)^{\frac{1}{n}} \le \frac{a_1+a_2+\dots+a_n}{n}$ $(a_k \ge 0, k=1,2,\dots,n)$.

$x_n = (1+\frac{1}{n})^n$. $y_n = (1+\frac{1}{n})^{n+1}$.

$\left((1+\frac{1}{n})^n \cdot 1\right)^{\frac{1}{n+1}} \le \frac{n(1+\frac{1}{n})+1}{n+1}$

$n+1$,

$x_n = (1+\frac{1}{n})^n \cdot 1 \le \left[\frac{n(1+\frac{1}{n})+1}{n+1}\right]^{n+1} = x_{n+1}$

$\frac{1}{y_n} = \left(\frac{n}{n+1}\right)^{n+1} \cdot 1 \le \left[\frac{\left(\frac{n}{n+1}\right)\cdot(n+1)+1}{n+2}\right]^{n+2} = \frac{1}{y_{n+1}}$.

$y_n \ge y_{n+1}$. $\Rightarrow \{x_n\}$ is increasing $\{y_n\}$ is decreasing.

$2 = x_1 \le x_n < y_n \le y_1 = 4.$ $\Rightarrow$ $\{x_n\}, \{y_n\}$ are bounded.

$\Rightarrow$ They converge.

Because $y_n = \underbrace{(1+\frac{1}{n})}_{\downarrow\ 1.} x_n$. $\Rightarrow$ $\lim_{n\to+\infty} y_n = \lim_{n\to+\infty} x_n$

$e := \lim_{n\to+\infty}(1+\frac{1}{n})^n = \lim_{n\to+\infty}(1+\frac{1}{n})^{n+1}$.

4. $p>0$, $a_n \triangleq 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p}$

$= a_n.$ $= 1 + \frac{1}{2^p} + \frac{1}{3^p}$ $> f(1) + f(2) + \dots + f(n)$

$\to +\infty$, $+\infty$, $f(x) = \frac{1}{x^p}$

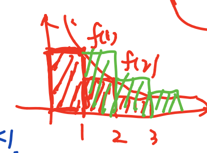
If $p>1$, then $\{a_n\}$ converges. $f(1)+f(2)+\dots + f(n)$ Integral,

If $p\le 1$. then $\{a_n\}$ goes to $+\infty$. $\int_0^{+\infty} \frac{1}{x^p}\,dx$, $p>1 \to$ finite; $p \le 1$ infinite

$\parallel$ $a_n \to +\infty$

Proof: $\{a_n\}$ is increasing. ($a_n < a_{n+1}$).

Consider its bound.



If $p>1$. Denote. $r = \frac{1}{2^{p-1}}$. $0<r<1$.

$$\frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{1}{2^{p-1}} = r$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p}\cdot 4 = \frac{1}{4^{p-1}} = r^2$$

...

$$\frac{1}{2^{kp}} + \frac{1}{(2^k+1)^p} + \dots + \frac{1}{(2^{k+1}-1)^p} < \frac{2^k}{2^{kp}} = r^k.$$

$$a_n \le a_{2^n-1} < 1 + r + r^2 + \dots + r^{n-1} < \frac{1}{1-r}$$

$\Rightarrow$ $\{a_n\}$ converge.

If $p \le 1$.

$$\frac{1}{2^p} \ge \frac{1}{2}$$

$$\frac{1}{3^p} + \frac{1}{4^p} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

$$\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}+\frac{1}{8^p}>\frac{4}{8}=\frac{1}{2}$$

$\cdots$

$$a_{2^n}\geq 1+\frac{n}{2}.$$

$$\lim_{n\to+\infty} a_n = +\infty.$$

5. Denote $b_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln n$. Prove $\{b_n\}$ converges.

Proof:

From Problem3, we know

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}.$$

$$\Rightarrow \frac{1}{n+1} < \ln\frac{n+1}{n} < \frac{1}{n}.$$

$$b_{n+1}-b_n = \frac{1}{n+1}-\ln^{n+1}+\ln^n = \frac{1}{n+1}-\ln\frac{n+1}{n} < 0.$$

$$\begin{aligned} b_n &= 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\ln^n \\ &> \ln\frac{2}{1}+\ln\frac{3}{2}+\ldots+\ln\frac{n+1}{n}-\ln^n \\ &= \ln^{n+1}-\ln^n > 0. \end{aligned}$$

$\Rightarrow$ $\{b_n\}$ has a lower bound. $\{b_n\}$ is decreasing.

$\Rightarrow$ $\{b_n\}$ converges.

Denote $\gamma = \lim_{n\to+\infty} b_n$: Euler constant.