MATH 2031 Introduction to Real Analysis

October 8, 2012

Tutorial Note 4

Countability

(I) **Definition:**
A set S is **countably infinite** iff $\exists$ bijection $f : \mathbb{N} \to S$ (i.e. $Card\ S = \aleph_0$)

(II) **Definition:**
A set S is **countable** iff S is finite or S is countably infinite.

(III) **Definition:**
A set S is **uncountable** iff S is not countable.(So a set is uncountable implies it must be infinite).

(IV) Bijection Theorem:
Let $g : S \to T$ be a bijection. S is countable $\Leftrightarrow$ T is countable.

(V) Countable Subset Theorem:
Let $A \subseteq B$. If B is countable, then A is countable.(Contrapositive: If A is uncountable, then B is uncountable.)

(VI) Countable Union Theorem:
If $\forall n \in \mathbb{N}$, A_n is countable, then $\bigcup\limits_{n\in\mathbb{N}} A_n$ is countable. In general, we can take union over a countable set, instead of $\mathbb{N}$.(i.e. countable union of countable sets is countable).

(VII) Product Theorem:
For $n \in \mathbb{N}$, if $A_1, A_2, \cdots A_n$ are countable, then $A_1 \times A_2 \times \cdots \times A_n$ is countable.(i.e. finite product of countable sets is countable).

(VIII) Injection Theorem:
Let $f : A \to B$ be an injection. If B is countable, then A is countable.(contrapositive: if A is uncountable, then B is uncountable.)

(IX) Surjection Theorem:
Let $g : A \to B$ be an surjection. If A is countable, then B is countable.(contrapositive: if B is uncountable, then A is uncountable.)

Problem 1 Determine if the following sets are countable or not

(i) the integers $\mathbb{Z}$

(ii) the rational number $\mathbb{Q}$

(iii) the irrational number $\mathbb{R} \setminus \mathbb{Q}$

Solution:

(i) It's countable since there is a bijection $f : \mathbb{N} \to \mathbb{Z}$ given as follow,

$$f(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

(ii) It's countable.

Since $\mathbb{Q} = \{\dfrac{m}{n} : m, n \in \mathbb{Z} \text{ and } n > 0\} = \bigcup_{m\in\mathbb{Z}} \{\dfrac{m}{n} : m, n \in \mathbb{Z} \text{ and } n > 0\}$

It's a countable union of a countable set, so $\mathbb{Q}$ is countable.

(iii) Recall that $\mathbb{R}$ is uncountable and by above part $\mathbb{Q}$ is a countable set. Assume the irrational number $\mathbb{R} \setminus \mathbb{Q}$ is countable, then $\mathbb{R} = \mathbb{Q} \cup \mathbb{R} \setminus \mathbb{Q}$ which is a union of 2 countable sets. Contradiction.

Hence, $\mathbb{R} \setminus \mathbb{Q}$ is uncountable.

Problem 2 Determine the following set is countable or not

$A = \{(0.a_1a_2 \cdots a_{100}) : a_i \in \mathbb{N}\}$

Solution:

We can consider the following bijection $g : A \to \times_{n=1}^{100}\mathbb{N}$ given by $g((0.a_1a_2 \cdots a_{100})) = (a_1, a_2, \cdots a_{100})$, with the inverse $g^{-1} : \times_{n=1}^{100}\mathbb{N}\ toA$ given by $g^{-1}((a_1, a_2, \cdots , a_{100})) = (0.a_1a_2 \cdots a_{100})$.

As $\mathbb{N}$ is countable, then $\times_{n=1}^{100}\mathbb{N}$ is a finite product of countable set, which is countable. Then A is bijective to $\times_{n=1}^{100}\mathbb{N}$, so it's also countable.

Problem 3 Let $\{x \in \mathbb{R} : x^5 + 8x^3 - 7x + 10 \in \mathbb{N}\}$

Solution:

$\{x \in \mathbb{R} : x^5 + 8x^3 - 7x + 10 \in \mathbb{N}\} = \bigcup_{n\in\mathbb{N}} \{x \in \mathbb{R} : x^5 + 8x^3 - 7x + 10 = n\}$.

Noted that $x^5 + 8x^3 - 7x + 10 = n$ has at most 5 elements as it's a polynomials over $\mathbb{R}$ of degree 5. Thus the set is in fact a countable union of finite set. Therefore, it's countable.

Problem 4 Determine the following sets are countable or not

(i) $A = \{a^4 + e^b : a \in \mathbb{Z} \text{ and } b \in \mathbb{R} \setminus \mathbb{Q}\}$

(ii) $B = \{\sin(|x|) + x : x \in \mathbb{R}\}$

(iii) $C = \{(x^y, \pi z^3) : x, y \in \mathbb{Z}, z \in \mathbb{R} \setminus \mathbb{Q}\}$

(iv) $D = \{(x^2, [y]) : x \in \mathbb{Q} \text{ and } y \in \mathbb{R}\}$ where, $[y]$ is the greatest integer less then y

Solution:

(i) We can see that $A = \{a^4 + e^b : a \in \mathbb{Z} \text{ and } b \in \mathbb{R} \setminus \mathbb{Q}\} = \bigcup_{a\in\mathbb{Z}} \{a^4 + e^b : b \in \mathbb{R} \setminus \mathbb{Q}\} \supseteq \{e^b : b \in \mathbb{R} \setminus \mathbb{Q}\}$

Consider the set $\{e^b : b \in \mathbb{R} \setminus \mathbb{Q}\}$, as e^x is a strictly increasing function, we can define a function $f : \mathbb{R} \setminus \mathbb{Q} \to \{e^b : b \in \mathbb{R} \setminus \mathbb{Q}\}$ given by $f(b) = e^b$ which is injective. (Since $f(b) = f(b') \Rightarrow e^b = e^{b'} \Rightarrow b = b'$).

By injection theorem, as $\mathbb{R} \setminus \mathbb{Q}$ is uncountable, $\{e^b : b \in \mathbb{R} \setminus \mathbb{Q}\}$ is uncountable, then by countable subset theorem, $A \supseteq \{e^b : b \in \mathbb{R} \setminus \mathbb{Q}\}$, so A is also uncountable.

(ii) As $B = \{\sin(|x|) + x : x \in \mathbb{R}\} = \{\sin(x) + x : x \in [0, \infty)\} \cup \{-\sin(x) + x : x \in (-\infty, 0)\}$.

Consider the subset $\{\sin(x) + x : x \in [0, \frac{\pi}{2})\}$ of $\{\sin(x) + x : x \in [0, \infty)\}$, then $\sin(x) + x$ is a strictly increasing function on $[0, \frac{\pi}{2})$ as its derivative is positive, so we can define a function $f : [0, \frac{\pi}{2}) \to \{\sin(x) + x : x \in [0, \frac{\pi}{2})\}$ given by $f(x) = \sin(x) + x$. then by injection theorem, $[0, \frac{\pi}{2})$ is uncountable implies $\{\sin(x) + x : x \in [0, \frac{\pi}{2})\}$ is also uncountable.

As $B \supseteq \{\sin(x) + x : x \in [0, \infty)\} \supseteq \{\sin(x) + x : x \in [0, \frac{\pi}{2})\}$, we get B is uncountable by countable subset theorem.

(iii) Notice that the projection to the second coordinate $P : C = \{(x^y, \pi z^3) : x, y \in \mathbb{Z}, z \in \mathbb{R} \setminus \mathbb{Q}\} \to \{\pi z^3 : z \in \mathbb{R} \setminus \mathbb{Q}\}$ given by $P((a, b)) = b$ is a surjection.

Also, we could see that there is a bijection $f : \{\pi z^3 : z \in \mathbb{R} \setminus \mathbb{Q}\} \to \mathbb{R} \setminus \mathbb{Q}$ given by $f(a) = \sqrt[3]{\dfrac{a}{\pi}}$, with the inverse $f^{-1} : \mathbb{R}\setminus\mathbb{Q} \to \{\pi z^3 : z \in \mathbb{R}\setminus\mathbb{Q}\}$ given by $f^{-1}(z) = \pi z^3$. In particular, $f : \{\pi z^3 : z \in \mathbb{R}\setminus\mathbb{Q}\} \to \mathbb{R}\setminus\mathbb{Q}$ is surjective.

Since the composition of surjections is again surjection, $f \circ P : C \to \mathbb{R} \setminus \mathbb{Q}$ is a surjection. As $\mathbb{R} \setminus \mathbb{Q}$ is an uncountable set, C is also uncountable.

(iv) A priori, the set D has a variable y coming from an uncountable set $\mathbb{R}$. Usually we would guess it's also uncountable. However, here we should beware that such $[y]$ is an integer.

Then $D \subseteq \underbrace{\mathbb{Q} \times \mathbb{Z}}_{\text{countable}}$, so D is countable.