Math 2033 (Mathematical Analysis) **Spring 2016**

Homework # 1 – Due Monday, March 14, 2016 at 3:00pm

Be sure to write your name (as shown on your student ID card) and your tutorial session number on the homework! Show work. Answers are worth very little. **Make a copy of your homework and submit the original.**

1. Show there is a bijection from $[0,1]$ to $(0,1]$. (*Hint*: Consider omiting $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots$ from the domain and codomain first.)

2. Determine if the set A of all intersection points in $\mathbb{R}^2$ of the family of lines $\{y = mx : m \in \mathbb{Z}\}$ with the family of circles $\{x^2 + y^2 = r^2 : r \in \mathbb{Q}\}$ is countable or uncountable. Here A is the set of all points in $\mathbb{R}^2$ that are on at least one of the lines $y = mx$ $(m \in \mathbb{Z})$ and at least one of the circles $x^2 + y^2 = r^2$ $(r \in \mathbb{Q})$.

3. Prove that there exist infinitely many positive real numbers r such that the equation $2^x + 3^y + 5^z = r$ has no solution $(x, y, z) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$.

   (*Hint*: Is the set $S = \{2^x + 3^y + 5^z : (x, y, z) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}\}$ countable?)

4. Let T be a nonempty subset of the interval $(0, 1)$. If every finite subset $\{x_1, x_2, \ldots, x_n\}$ of T (with no two of $x_1, x_2, \ldots, x_n$ equal) has the property that $x_1^2 + x_2^2 + \cdots + x_n^2 < 1$, then prove that T is a countable set.

   (*Hint:* For every $k \in \mathbb{N}$, how many elements of T can be in $\left[\frac{1}{k+1}, \frac{1}{k}\right)$? Do you use the Archimedean principle anywhere?)

5. Let D be a nonempty subset of $\mathbb{R}$ with $\inf D = 1$ and $\sup D = 5$. Determine (with proof) the supremum of the set

$$E = \left\{ x(y + \sqrt{2}) - \frac{1}{x} \ : \ x \in D, \quad y \in [0, \sqrt{2}) \cap \mathbb{Q} \right\}.$$

6. Let A, B, C be nonempty subsets of $\mathbb{R}$ such that $A \subseteq B \subseteq C$. Suppose C is bounded above in $\mathbb{R}$. If $\sup A = w = \sup C$, then prove that $\sup B = w$.