Extra Exercises on Limit of Sequences

Recall on page 48 of the transparencies, we have the *definition of limit of sequences* as follows:

A sequence $x_1, x_2, x_3, \ldots$ converges to a number x (or has limit x) if and only if for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $n \geq K$ implies $|x_n - x| < \varepsilon$.

On page 50 of the transparencies, there is example 6, which shows how to do a limit problem by checking the definition of limit. For the exercises below, apply similar reasoning as in example 6 to give a proof of the limit in the exercise by checking the definition of limit.

Exercise A. Prove that $\lim\limits_{n\to\infty}\Big(\dfrac{2n^2-1}{4n^2}+\dfrac{3n}{2n+1}\Big) = 2$ by checking the definition of limit of sequences.

Exercise B. Prove that $\lim\limits_{n\to\infty}\Big(\dfrac{n^4}{3n^4-2}-\dfrac{1-2n}{3n}\Big) = 1$ by checking the definition of limit of sequences.

Next we present variations of these exercises. First we recall on page 50 of the transparencies, there is the *boundedness theorem*, which said if a sequence $x_1, x_2, x_3, \ldots$ has a limit, then $\{x_1, x_2, x_3, \ldots\}$ is bounded (above and below). This means there exists a positive number M such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Example 1. Let $a_n > 0$ and $\lim\limits_{n\to\infty} a_n = 2$, then prove $\lim\limits_{n\to\infty} \dfrac{a_n}{3a_n+5} = \dfrac{2}{11}$ by checking the definition of limit of sequences.

Sketch work. Since $a_n > 0$, so $3a_n + 5 > 5$. Then

$$\Big|\frac{a_n}{3a_n+5}-\frac{2}{11}\Big| = \Big|\frac{11a_n-(6a_n+10)}{11(3a_n+5)}\Big| = \frac{5|a_n-2|}{11(3a_n+5)} \leq \frac{5|a_n-2|}{11\times 5} = \frac{|a_n-2|}{11} < \varepsilon \quad \text{if} \quad |a_n-2| < 11\varepsilon.$$

Solution. For every $\varepsilon > 0$, since $\lim\limits_{n\to\infty} a_n = 2$, by the definition of limit of sequences, there exists $K \in \mathbb{N}$ such that $n \geq K$ implies $|a_n - 2| < 11\varepsilon$. Then

$$\Big|\frac{a_n}{3a_n+5}-\frac{2}{11}\Big| = \Big|\frac{11a_n-(6a_n+10)}{11(3a_n+5)}\Big| = \frac{5|a_n-2|}{11(3a_n+5)} \leq \frac{5|a_n-2|}{11\times 5} = \frac{|a_n-2|}{11} < \varepsilon.$$

Example 2. Let $\lim\limits_{n\to\infty} x_n = 3$. Prove that $\lim\limits_{n\to\infty}\Big(\dfrac{x_n}{\sqrt{n}}+\dfrac{9n}{n+1}\Big) = 9$.

Sketch work. First $\dfrac{x_n}{\sqrt{n}} \to 0$ and $\dfrac{9n}{n+1} \to 9$. By boundedness theorem, $\exists M > 0$ such that $\forall n \in \mathbb{N}$, $|x_n| \leq M$. So $\Big|\dfrac{x_n}{\sqrt{n}} - 0\Big| \leq \dfrac{M}{\sqrt{n}} < \dfrac{\varepsilon}{2}$ if $n > \big(\dfrac{2M}{\varepsilon}\big)^2$. Next, $\Big|\dfrac{9n}{n+1} - 9\Big| = \dfrac{9}{n+1} < \dfrac{9}{n} < \dfrac{\varepsilon}{2}$ if $n > \dfrac{18}{\varepsilon}$.

Solution. By boundedness theorem, $x_n \to 3$ implies $\exists M > 0$ such that $\forall n \in \mathbb{N}$, $|x_n| \leq M$. For every $\varepsilon > 0$, by the Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \max\big\{\big(\dfrac{2M}{\varepsilon}\big)^2, \dfrac{18}{\varepsilon}\big\}$. Then $n \geq K$ implies

$$\Big|\frac{x_n}{\sqrt{n}}+\frac{9n}{n+1}-9\Big| \leq \Big|\frac{x_n}{\sqrt{n}}-0\Big| + \Big|\frac{9n}{n+1}-9\Big| \leq \frac{M}{\sqrt{n}}+\frac{9}{n+1} \leq \frac{M}{\sqrt{n}}+\frac{9}{n} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Exercise C. Let $b_n > 0$ and $\lim\limits_{n\to\infty} b_n = 1$. Prove that $\lim\limits_{n\to\infty}\Big(\dfrac{b_n}{1+b_n^2}+\dfrac{3n}{n+4}\Big) = \dfrac{7}{2}$ by checking the definition of limit of sequences. (Hint: Find the limit of each term. Use triangle inequality as in example 2 above.)

Exercise D. Let $c_n > 0$. Prove that if $\{c_n\}$ converges to 2, then $\lim\limits_{n\to\infty}\Big(\dfrac{1}{n+c_n}+\dfrac{c_n}{c_n+2}\Big) = \dfrac{1}{2}$ by checking the definition of limit of sequences. (Hint: Find the limit of each term. Use triangle inequality as in example 2 above.)