Solutions to Presentation Exercises

(195) <u>Sketch work</u> $\left|\frac{y_n}{x_n+1}-\frac{y_m}{x_m+1}\right|=\frac{|y_nx_m+y_n-y_mx_n-y_m|}{\underbrace{(x_n+1)}_{>1}\underbrace{(x_m+1)}_{>1}}\underset{x_n,x_m>0}{\leq}\frac{|y_nx_m-y_mx_n|+|y_n-y_m|}{1\cdot 1}$

(*) $\leq |y_nx_m-y_nx_n+y_nx_n-y_mx_n|+|y_n-y_m|\leq |y_n||x_m-x_n|+|y_n-y_m||x_n|+|y_n-y_m|$

<u>Solution</u> $\{x_n\},\{y_n\}$ Cauchy $\Rightarrow \{x_n\},\{y_n\}$ bounded $\Rightarrow \exists M_1,M_2>0$ such that $\forall n\in\mathbb{N}$, $|x_n|<M_1$ and $|y_n|<M_2$. For every $\varepsilon>0$, since $\{x_n\},\{y_n\}$ are Cauchy, $\exists K_1,K_2,K_3\in\mathbb{N}$ such that

$n,m\geq K_1\Rightarrow |x_m-x_n|<\frac{\varepsilon}{3M_2}$

$n,m\geq K_2\Rightarrow |y_n-y_m|<\frac{\varepsilon}{3M_1}$

$n,m\geq K_3\Rightarrow |y_n-y_m|<\frac{\varepsilon}{3}$.

Let $K=\max\{K_1,K_2,K_3\}$. Then from (*) above,

$m,n\geq K\Rightarrow \left|\frac{y_n}{x_n+1}-\frac{y_m}{x_m+1}\right|<M_2\frac{\varepsilon}{3M_2}+M_1\frac{\varepsilon}{3M_1}+\frac{\varepsilon}{3}=\varepsilon.$

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(207) (<u>Sketch</u> $\left|\frac{3x}{x^2+2}-1\right|=\frac{|x^2-3x+2|}{x^2+2}\leq\frac{|x-2||x-1|}{2}<\frac{2|x-1|}{2}=|x-1|<\varepsilon$

$|x-1|<1\Rightarrow x\in(0,2)\Rightarrow x-2\in(-2,0)$.)

<u>Solution</u>

$\forall\varepsilon>0$, let $\delta=\min(1,\varepsilon)$. Then

$0<|x-1|<\delta\Rightarrow \begin{matrix}|x-1|<1\\ \text{and}\\ |x-1|<\varepsilon\end{matrix}\Rightarrow \begin{matrix}x\in(0,2)\\ \text{and}\\ |x-1|<\varepsilon\end{matrix}\overset{x-2\in(-2,0)}{\Rightarrow}\left|\frac{3x}{x^2+2}-1\right|=\frac{|x^2-3x+2|}{x^2+2}\leq\frac{|x-2||x-1|}{2}<\frac{2}{2}|x-1|$

$=|x-1|<\varepsilon.$

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(208) (<u>Sketch</u> $|b_m-b_n|\leq|b_n-a_n|+|a_n-a_m|+|a_m-b_m|\leq\frac{1}{n}+|a_n-a_m|+\frac{1}{m}$.)

$\forall\varepsilon>0$, since $\{a_n\}$ is Cauchy, $\exists K_1\in\mathbb{N}$ such that $m,n\geq K_1\Rightarrow|a_n-a_m|<\frac{\varepsilon}{3}$.

Next let $K_2>\frac{3}{\varepsilon}$, then $m,n\geq K_2\Rightarrow\frac{1}{m},\frac{1}{n}\leq\frac{1}{K_2}<\frac{\varepsilon}{3}$.

Let $K=\max(K_1,K_2)$. Then $m,n\geq K\Rightarrow m,n\geq K_1$ and $m,n\geq K_2$

$\Rightarrow|b_m-b_n|\leq|b_n-a_n|+|a_n-a_m|+|a_m-b_m|\leq\frac{1}{n}+|a_n-a_m|+\frac{1}{m}<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$

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(214) <u>Notes</u> $\left|\sqrt{x_n^2+y_n^2}-\sqrt{x_m^2+y_m^2}\right|\leq\sqrt{|(x_n^2+y_n^2)-(x_m^2+y_m^2)|}\leq\sqrt{\underbrace{|x_n^2-x_m^2|}_{=|x_n+x_m||x_n-x_m|}+\underbrace{|y_n^2-y_m^2|}_{=|y_n+y_m||y_n-y_m|}}$

Since Cauchy sequences are bounded, $\exists M_1,M_2>0$ such that $|x_n|\leq M_1$ and $|y_n|\leq M_2$ for all n. For every $\varepsilon>0$, $\exists K_1,K_2\in\mathbb{N}$ such that

$m,n\geq K_1\Rightarrow|x_n-x_m|<\frac{\varepsilon^2}{4M_1}$ and $m,n\geq K_2\Rightarrow|y_n-y_m|<\frac{\varepsilon^2}{4M_2}$.

Let $K=\max\{K_1,K_2\}$. Then $n,m\geq K\Rightarrow n,m\geq K_1$ and $n,m\geq K_2$

$\Rightarrow\left|\sqrt{x_n^2+y_n^2}-\sqrt{x_m^2+y_m^2}\right|\leq\sqrt{|x_n^2-x_m^2|+|y_n^2-y_m^2|}\underset{\text{by notes above}}{<}\sqrt{2M_1\frac{\varepsilon^2}{4M_1}+2M_2\frac{\varepsilon^2}{4M_2}}=\varepsilon.$

(243) Sketch $|f(x)-\frac{1}{2}| = \left|\sqrt[4]{\frac{1}{x^2+6x}} - \sqrt[4]{\frac{1}{16}}\right| \le \sqrt[4]{\left|\frac{1}{x^2+6x} - \frac{1}{16}\right|} = \sqrt[4]{\frac{|x^2+6x-16|}{16(x^2+6x)}} = \sqrt[4]{\frac{|x+8||x-2|}{16(x^2+6x)}} \le \sqrt[4]{\frac{11|x-2|}{112}}$

$x\in[1,3] \Rightarrow x+8\in[9,11]$
$x^2+6x\in[7,27]$

Solution $\forall \varepsilon>0$, let $\delta = \frac{112}{11}\varepsilon^4 > 0$, then
$\forall x\in[1,3]$, $0<|x-2|<\delta \Rightarrow |f(x)-\frac{1}{2}| \overset{\text{by sketch work}}{\le} \sqrt[4]{\frac{11|x-2|}{112}} < \sqrt[4]{\frac{11\delta}{112}} = \varepsilon$.

(259) Sketch $|B_m-B_n| = |B_m-\sqrt{A_m}+\sqrt{A_m}-\sqrt{A_n}+\sqrt{A_n}-B_n| \le |B_m-\sqrt{A_m}|+|\sqrt{A_m}-\sqrt{A_n}|+|\sqrt{A_n}-B_n|$
$\le |\sqrt{A_{m+2011}}-\sqrt{A_m}|+|\sqrt{A_m}-\sqrt{A_n}|+|\sqrt{A_n}-\sqrt{A_{n+2011}}| \le \sqrt{|A_{m+2011}-A_m|}+\sqrt{|A_m-A_n|}+\sqrt{|A_n-A_{n+2011}|}$
because $\sqrt{A_{m+2011}} \le B_m \le \sqrt{A_m}$

Solution $\forall\varepsilon>0$, since $\{A_n\}$ is Cauchy, $\exists K\in\mathbb{N}$ such that $m,n\ge K \Rightarrow |A_m-A_n| < (\frac{\varepsilon}{3})^2$
Then $m,n\ge K \Rightarrow m, n, m+2011, n+2011 \ge K \Rightarrow \sqrt{|A_{m+2011}-A_m|} < \sqrt{(\frac{\varepsilon}{3})^2} = \varepsilon/3$ and
$\sqrt{|A_m-A_n|} < \sqrt{(\frac{\varepsilon}{3})^2} = \varepsilon/3$ and $\sqrt{|A_n-A_{n+2011}|} < \sqrt{(\frac{\varepsilon}{3})^2} = \varepsilon/3 \Rightarrow |B_m-B_n| \underset{\text{by sketch above}}{<} \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon$.

(274) Sketch $\left|\sqrt{\frac{1}{2+\sqrt{x}}} - \underbrace{\frac{1}{2}}_{=\sqrt{\frac{1}{4}}}\right| \le \sqrt{\left|\frac{1}{2+\sqrt{x}}-\frac{1}{4}\right|} = \sqrt{\left|\frac{2-\sqrt{x}}{\underbrace{4(2+\sqrt{x})}_{\ge 0}}\right|} \le \frac{\sqrt[4]{|4-x|}}{\sqrt{8}} < \varepsilon \iff |4-x| < (\sqrt{8}\varepsilon)^4$

$2=\sqrt{4}$

Solution $\forall\varepsilon>0$, let $\delta = (\sqrt{8}\varepsilon)^4$, then $\forall x\in[0,\infty)$, $0<|x-4|<\delta \Rightarrow \left|\sqrt{\frac{1}{2+\sqrt{x}}}-\frac{1}{2}\right| < \varepsilon$.
by sketch above

(276) Let $g(x) = f(x)-x^2$ for $x\in[0,2]$, Then g is continuous on $[0,2]$ since $f(x)$ and x^2 are continuous on $[0,2]$. We have $g(2) = f(2)-2^2 = 0-4<0$. Next, for $x\ne 1$,
$f(x)-2 = \frac{f(x)-2}{\sqrt{x}-1}(\sqrt{x}-1)$. Then $f(1)-2 = \lim_{x\to1}(f(x)-2) = \lim_{x\to1}\frac{f(x)-2}{\sqrt{x}-1}\lim_{x\to1}(\sqrt{x}-1) = 1\times 0 = 0$.
So $f(1)=2$. Then $g(1) = f(1)^2-1^2 = 2-1>0$. By the intermediate value theorem, $\exists x\in[1,2]$ such that $g(x)=0$, hence $f(x)=x^2$.

(279) Sketch $|\sin(a_n^2+\sqrt{a_n}) - \sin(a_m^2+\sqrt{a_m})| \le |a_n^2+\sqrt{a_n}-a_m^2-\sqrt{a_m}| \le |a_n^2-a_m^2|+|\sqrt{a_n}-\sqrt{a_m}|$
$\le |a_n+a_m||a_n-a_m| + \sqrt{|a_n-a_m|} \le (|a_n|+|a_m|)|a_n-a_m| + \sqrt{|a_n-a_m|}$

Solution Since $\{a_n\}$ is Cauchy, it is bounded, So $\exists M>0, \forall n, |a_n|<M$. $\forall\varepsilon>0$, choose $K_1, K_2\in\mathbb{N}$ so that $n,m\ge K_1 \Rightarrow |a_n-a_m| < \frac{\varepsilon}{4M}$ and $n,m\ge K_2 \Rightarrow |a_n-a_m| < \frac{\varepsilon^2}{4}$.
Let $K=\max(K_1,K_2)$, then $n,m\ge K \Rightarrow n,m\ge K_1$ and $n,m\ge K_2 \Rightarrow \underbrace{|c_n-c_m|}_{= |\sin(a_n^2+\sqrt{a_n})-\sin(a_m^2+\sqrt{a_m})|}$
$< 2M\frac{\varepsilon}{4M} + \sqrt{\frac{\varepsilon^2}{4}} = \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$.

(289) Sketch $|f(x)-1| = \left|\frac{x}{1+2x} + \frac{2}{2+\sqrt{x}} - 1\right| = \left|\left(\frac{x}{1+2x} - \frac{1}{3}\right) + \left(\frac{2}{2+\sqrt{x}} - \frac{2}{3}\right)\right|$

$x \to 1 \Rightarrow \frac{x}{1+2x} \to \frac{1}{3}, \ \frac{2}{2+\sqrt{x}} \to \frac{2}{3}$

$$\le \left|\frac{x}{1+2x} - \frac{1}{3}\right| + \left|\frac{2}{2+\sqrt{x}} - \frac{2}{3}\right| = \left|\frac{x-1}{3\underbrace{(1+2x)}_{\ge 0}}\right| + \left|\frac{2-2\sqrt{x}}{3\underbrace{(2+\sqrt{x})}_{\ge 0}}\right|$$

$$\le \frac{|x-1|}{3} + \frac{\cancel{2}|1-\sqrt{x}|}{3 \times \cancel{2}}$$

$$\le \frac{|x-1|}{3} + \frac{\sqrt{|1-x|}}{3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

if $|x-1| < \frac{3\varepsilon}{2}$ and $\sqrt{|x-1|} < \frac{3\varepsilon}{2}$

$\Leftrightarrow |x-1| < \left(\frac{3\varepsilon}{2}\right)^2$

Solution. $\forall \varepsilon > 0$, let $\delta = \min\left\{\frac{3\varepsilon}{2}, \left(\frac{3\varepsilon}{2}\right)^2\right\} > 0$, then

$\forall x \in [0, +\infty)$, $0 < |x-1| < \delta \Rightarrow |x-1| < \frac{3\varepsilon}{2}$ and $|x-1| < \left(\frac{3\varepsilon}{2}\right)^2 \xRightarrow{\text{by sketch above}} |f(x)-1| < \varepsilon$.

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(314) Since $f: [0,1] \to [0,1]$ is continuous injective, by the continuous injection theorem f is strictly monotone. Since $f(0) < f(1)$, f is strictly increasing. Cross-multiplying $\frac{1-f(x)}{1+f(x)} = \frac{x^2}{2-x^2}$ and simplifying, we get the equation $f(x) = 1-x^2$. Function $g(x) = 1-x^2$ is strictly decreasing and continuous on $[0,1]$. So $h(x) = f(x) - (1-x^2)$ is strictly increasing and continuous. Using $0 \le f(0) < f(1) \le 1$, we have $h(0) = f(0) - 1 < 0$ and $h(1) = f(1) > 0$. By the intermediate value theorem, $h(x) = 0$ for some $x \in [0,1]$. Since h is strictly increasing, there is exactly 1 solution.
