MATH2033 Mathematical Analysis (2021 Spring)
Problem Set 4

Problem 1

Find the supremum and infimum, if exists as a number, for the following sets

(a) $A = \{e^{-x} | x \in (0,1) \cap \mathbb{Q}\}$.

(b) $B = \left\{\cos\frac{1}{n} | n \in \mathbb{N}\right\}$ (😊Hint: The function $\cos x$ is decreasing over $\left[0, \frac{\pi}{2}\right]$)

(c) $C = \left\{1 - \frac{(-1)^n}{n} | n \in \mathbb{N}\right\}$

Problem 2

Find the supremum and infimum, if exists as a number, for the following sets

(a) (A bit harder) $D = \left\{\frac{1}{n} - \frac{1}{m} | m \in \mathbb{N}, n \in \mathbb{N}\right\}$

(b) (A bit harder) $E = \{a + b | a \in (0,1) \cap \mathbb{Q},\ b \in (1,2) \backslash \mathbb{Q}\}$.

Problem 3

We let S be a bounded subset in $\mathbb{R}$ and let $S_0 \subseteq S$ be a subset of S_0

(a) Show that the supremum and infimum of S_0 exist and satisfy $\inf S_0 \geq \inf S$ and $\sup S_0 \leq \sup S$.

(b) Suppose that $S_0 \subset S$ (i.e. S_0 is proper subset of S), is it always true that $\inf S_0 > \inf S$ and $\sup S_0 < \sup S$? Explain your answer.

Problem 4

Prove the following statements using Archimedean property.

(a) We let $I_n = \left[0, \frac{1}{n}\right]$ for every $n \in \mathbb{N}$. If $x > 0$, prove that $x \notin \bigcap_{n=1}^{\infty} I_n$.

(b) We let $J_n = \left(0, \frac{1}{n}\right)$ for every $n \in \mathbb{N}$, prove that $\bigcap_{n=1}^{\infty} J_n = \phi$.

(c) We let $K_n = [n, \infty)$ for every $n \in \mathbb{N}$, prove that $\bigcap_{n=1}^{\infty} K_n = \phi$

Problem 5

We let X be a non-empty set. We let $f, g: X \to \mathbb{R}$ be two functions which the ranges $f(X)$ and $g(X)$ are both bounded.

(a) Show that $\sup\{f(x) + g(x) | x \in X\} \leq \sup\{f(x) | x \in X\} + \sup\{g(x) | x \in X\}$. Provide an example which the strict inequality holds.

(b) Show that $\inf\{f(x) + g(x) | x \in X\} \geq \inf\{f(x) | x \in X\} + \inf\{g(x) | x \in X\}$. Provide an example which the strict inequality holds.

Problem 6

We let X and Y be two non-empty sets. We let $h: X \times Y \to \mathbb{R}$ be a function which $h(X \times Y)$ is bounded (*Note: Here, $h = h(x, y)$ is a function of two variables where $x \in X$ and $y \in Y$.)

We define two functions $f: X \to \mathbb{R}$ and $g: Y \to \mathbb{R}$ to be

$$f(x) = \sup\{h(x,y)|y \in Y\} \quad and \quad g(y) = \sup\{h(x,y)|x \in X\}.$$

(a) Suppose that $h(x,y) = 2x + y$ and $X = Y = [0,1]$, compute $f(x)$ and $g(y)$.

(b) (Independent of (a)) Prove that

$$\sup\{g(y)|y \in Y\} \leq \inf\{f(x)|x \in X\}.$$

(c) (Independent of (a)) Prove that

$$\sup\{h(x,y)|x \in X, y \in Y\} = \sup\{f(x)|x \in X\} = \sup\{g(y)|y \in Y\}.$$

(*Note: The above equation is known as *principle of the iterated supremum*. The principle suggests that the supremum of a function $h(x,y)$ can be found through the following two steps procedure:

- For each $x \in X$, we first find the supremum ("maximum") of $h(x,y)$ over all possible of y and call this maximum be $f(x)$.
- Given $f(x)$ obtained, we find the final supremum by finding the supremum of $f(x)$ over all possible values of X.

Problem 7 (A bit harder)

We consider the nested interval theorem (see Theorem 6 of Lecture Note 4) as follows:

Nested Interval Theorem
We let $\{I_n = [a_n, b_n] | n \in \mathbb{N}\}$ be a set of closed intervals such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Then $\bigcap_{n=1}^{\infty} I_n = [a,b]$, where $a = \sup\{a_n | n \in \mathbb{N}\}$ and $b = \inf\{b_n | n \in \mathbb{N}\}$.

Suppose that $\inf\{b_n - a_n | n \in \mathbb{N}\} = 0$, prove that $\bigcap_{n=1}^{\infty} I_n$ contains a single element.

(😊Hint: It suffices to argue that $a = b$. This can be done by first showing $0 \leq b - a < \varepsilon$ for any $\varepsilon > 0$ and conclude that $a = b$ using infinitesimal property.)

Problem 8

(a) Using mathematical induction, prove that

$$\cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\cos\left(\frac{n+1}{2}\theta\right)\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

for all positive integer n. Here, $\theta \neq k\pi$ for any $k \in \mathbb{Z}$.

(b) We let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers defined by

$$a_0 = \sqrt{2}, \qquad a_n = \sqrt{2 + a_{n-1}} \quad for \quad n = 1,2, \ldots$$

Using mathematical induction, prove that

$$a_n = 2\cos\frac{\pi}{2^{n+2}}$$

for all $n = 0,1,2, \ldots$.

(c) We let $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Using mathematical induction, prove that

$$A^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}$$

for any positive integer n.

Problem 9

Using mathematical induction, prove that

(a) $(1+x)^n \geq 1+nx$ for any positive integer n, where $x \geq -1$ is real number.

(b) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1}-1\right)$ for all positive integer n.

Problem 10

We let $P(n)$ be a statement which depends on the positive integer n. The second principle of mathematical induction states that $P(n)$ is true for all positive integer n if all of the following conditions hold:

- $P(1)$ and $P(2)$ are true
- If $P(k)$ and $P(k+1)$ are true for some integer k, then $P(k+2)$ is also true.

(a) Prove the principle using well-ordering principle.

(b) Using the second principle of mathematical induction, prove the following statement: We let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers defined by

$$a_1 = 1, \qquad a_2 = 7, \qquad a_{n+2} - 4a_{n+1} + 3a_n = 0 \quad for \;\; n = 1,2,3,\ldots$$

Then $a_n = 3^n - 2$ for all $n \in \mathbb{N}$.