MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #5

Series

In Chapter 4, we have learnt many different kinds of tests to check whether a given infinite series converge or not. In this note, we will revise all these tests and see how these tests are applied to the problems.

Part 1: Test for Special Series (for non-negative series)

(Geometric Series Test) $$S = \sum_{n=0}^{\infty} ar^n$$ S converges when $r < 1$ and diverges when $r \geq 1$
(Telescoping Series Test) $$S = \sum_{n=1}^{\infty}(b_n - b_{n+1}) = b_1 - \lim_{n\to\infty} b_{n+1}$$ S converges if $\lim_{n\to\infty} b_{n+1} = L$, diverge if $\lim_{n\to\infty} b_{n+1} = \pm\infty$ or does not exist.
(P-series Test) $$S = \sum_{k=1}^{\infty} \frac{1}{k^p}$$ S converges if $p > 1$, diverge if $p \leq 1$

Part 2: Some useful tests (FOR NON-NEGATIVE SERIES ONLY)

In many cases, the series may be complicated. To determine whether the series converges or not, we need some more powerful tests.

(Term test) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges *(Original Theorem: If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$)*

(Caution: $\lim_{n\to\infty} a_n = 0$ does not imply $\sum_{n=1}^{\infty} a_n$ converges, one counter-example is $1 + \frac{1}{2} + \frac{1}{3} + \cdots.. = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges by p-test (even though $\lim_{k\to\infty} \frac{1}{k} = 0$)

For the case when $\lim_{n\to\infty} a_n = 0$, the term test can't tell you whether the series converges or diverges. The following test suggests under some conditions, we can check convergence of series by checking some integral

(Integral Test) If $f:[1,+\infty)\to\mathbf{R}$ is a decreasing function and $\lim_{x\to\infty} f(x)=0$ Then $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(k) < \infty$ $\sum_{k=1}^{\infty} f(k)$ diverges if and only if $\int_1^{\infty} f(k) = \infty$

Important Note: To use the integral test, you need to check $\mathbf{f'(x)\leq 0}$, for $\mathbf{1\leq x<\infty}$ and $\mathbf{\lim_{x\to\infty} f(x)=0}$.

A little bit investigation to the proof of integral test. We can have a modified version of integral test

(Modified Integral Test) Let n be a positive integer If $f:[n,+\infty)\to\mathbf{R}$ is a decreasing function and $\lim_{x\to\infty} f(x)=0$ Then $\sum_{k=n}^{\infty} f(k)$ converges if and only if $\int_n^{\infty} f(k) < \infty$ $\sum_{k=n}^{\infty} f(k)$ diverges if and only if $\int_n^{\infty} f(k) = \infty$

Example 1 Determine whether the series $$\sum_{k=1}^{\infty} ke^{-k} \text{ and } \sum_{k=1}^{\infty} k^5e^{-k}$$ converges or not.

Solution:

(For $\sum_{k=1}^{\infty} ke^{-k}$)

Let $f(x)=xe^{-x}$

(Step 1: Check!!)

$$f'(x)=e^{-x}-xe^{-x}=(1-x)e^{-x}\leq 0 \text{ for } 1\leq x<\infty$$

$$\lim_{x\to\infty} xe^{-x}=\lim_{x\to\infty}\frac{x}{e^x}=\lim_{x\to\infty}\frac{1}{e^x}=0$$

So we can apply integral test

(Step 2: Use Integral Test)

Note $\int_1^{\infty} xe^{-x}\,dx=-\int_1^{\infty} x\,de^{-x}=-xe^{-x}|_1^{\infty}+\int_1^{\infty} e^{-x}\,dx$

$= e^{-1} - e^{-x}|_1^\infty = e^{-1} + e^{-1} = 2e^{-1} < \infty$

Therefore by integral test, $\sum_{k=1}^{\infty} ke^{-k}$ converges.

(For $\sum_{k=1}^{\infty} k^5 e^{-k}$)

Let $f(x) = x^5 e^{-x}$

(Step 1: Check!!)

$$\lim_{x\to\infty} x^5 e^{-x} = \lim_{x\to\infty} \frac{x^5}{e^x} = \lim_{x\to\infty} \frac{5x^4}{e^x} = \lim_{x\to\infty} \frac{(5\times 4)x^3}{e^x} = \cdots = \lim_{x\to\infty} \frac{5!}{e^x} = 0$$

But $f'(x) = 5x^4e^{-x} - x^5e^{-x} = x^4e^{-x}(5-x) \le 0$ **ONLY WHEN $x \ge 5$**

So we can't apply the integral test directly

(Step 2: Split the series)

$$\sum_{k=1}^{\infty} k^5 e^{-k} = \underbrace{\sum_{k=1}^{4} k^5 e^{-k}}_{\text{Finite}} + \underbrace{\sum_{k=5}^{\infty} k^5 e^{-k}}_{\text{Apply integral test}}$$

(Step 3: Apply modified integral test)

We apply integration test for 5 times to the integral, we get

$$\int_5^\infty x^5 e^{-x}\,dx = 5^5e^{-5} + 5\times 5^4 e^{-5} + \cdots. + 5!\,e^{-5} < \infty$$

Therefore $\sum_{k=5}^{\infty} k^5 e^{-k}$ converges, and so $\sum_{k=1}^{\infty} k^5 e^{-k}$ also converges.

Example 2:

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{\tan\left(\frac{1}{k}\right)}{k^2}$$

Converges or not

Solution:

Let $f(x) = \frac{\tan\left(\frac{1}{x}\right)}{x^2}$

(Step 1: Check!!)

$$f'(x) = -\frac{\sec^2\left(\frac{1}{x}\right)}{x^4} - \frac{2\tan\left(\frac{1}{x}\right)}{x^3} < 0$$

$$\lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{x^2} = 0$$

(Step 2: Apply Integral test)

$$\int_1^\infty \frac{\tan\left(\frac{1}{x}\right)}{x^2}dx = -\int_1^\infty \tan\left(\frac{1}{x}\right)d\left(\frac{1}{x}\right) = \ln\left(\cos\left(\frac{1}{x}\right)\right)\Big|_1^\infty = \ln 1 - \ln(\cos 1) < \infty$$

(Note: $\int \tan x dx = -\ln(\cos x) + c$)

Therefore the series $\sum_{k=1}^\infty \frac{\tan\left(\frac{1}{k}\right)}{k^2}$ converges

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☺Exercise 1

Check whether the series $\sum_1^\infty \frac{\ln k}{k}$ converges

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Another useful test is the comparison test.

(Comparison Test) Given $v_k \geq u_k \geq 0$ for every k. If $\sum_{k=1}^\infty v_k$ converges, then $\sum_{k=1}^\infty u_k$ converges If $\sum_{k=1}^\infty u_k$ diverges, then $\sum_{k=1}^\infty v_k$ diverges

To check the convergence of a given sequence by comparison test, we need to use some series which is known to be convergence and divergence.

Example 3 (Practice Exercise #30g) Check whether the series $$\sum_{k=1}^\infty \left(\frac{k+2}{k+1}\right)\left(\frac{2}{3}\right)^k$$ Converges or not

Solution:

$$\sum_{k=1}^\infty \left(\frac{k+2}{k+1}\right)\left(\frac{2}{3}\right)^k = \sum_{k=1}^\infty \left(1+\frac{1}{k+1}\right)\left(\frac{2}{3}\right)^k \leq \sum_{k=1}^\infty \left(1+\frac{1}{2}\right)\left(\frac{2}{3}\right)^k = \left(\frac{3}{2}\right)\sum_{k=1}^\infty \left(\frac{2}{3}\right)^k$$

Note $\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k$ converges by geometric series test. $\left(\frac{3}{2}\right)\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k$ also converges.

Therefore by comparison test, $\sum_{k=1}^{\infty}\left(\frac{k+2}{k+1}\right)\left(\frac{2}{3}\right)^k$ also converges

Example 4 Check whether the series $$\sum_{k=1}^{\infty}\frac{\ln(2007+k)}{k}$$ Converges or not

Solution:

Note that $\ln(2007+k) \geq \ln(2007+1) = \ln 2008 > 1$

$$\text{Therefore,} \sum_{k=1}^{\infty}\frac{\ln(2007+k)}{k} > \sum_{k=1}^{\infty}\frac{1}{k}$$

Since $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges by p-test, therefore $\sum_{k=1}^{\infty}\frac{\ln(2007+k)}{k}$ diverges by comparison test.

☺Exercise 2

Check whether the series

$$\sum_{k=1}^{\infty}\frac{k}{\sqrt[3]{k+1}\sqrt[3]{k+2}\sqrt[3]{k+3}\ldots\sqrt[3]{k+2007}}$$

Converges or not

(Hint: $\sqrt[3]{k+n} \geq \sqrt[3]{??}$ for $n = 1,2,3,\ldots$ Guess what ?? is)

In some cases, comparison may not efficient since we need to find some inequality. This process can be difficult in many series (especially when you meet the function like $e^x, \sin x, \cos x, \ldots\ldots$ etc) So to deal with this situation, we need to have a more powerful test.

(Limit Comparison Test)

Given $u_k, v_k > 0$

If $\lim_{k\to\infty} \frac{v_k}{u_k} = L$ where L is a number,

Then $\sum_{k=1}^{\infty} u_k$ converges (diverges) $\leftrightarrow$ $\sum_{k=1}^{\infty} v_k$ converges (diverges)

(The case $\lim_{k\to\infty} \frac{v_k}{u_k} = 0$ and $\lim_{k\to\infty} \frac{v_k}{u_k} = \infty$)

To check whether a given series $\sum_{k=1}^{\infty} u_k$ converges or not using limit comparison test, we need to find out v_k.

The following facts are useful in finding our v_k

(1) $\sin\left(\frac{1}{k}\right) \approx \frac{1}{k}$, $\tan\left(\frac{1}{k}\right) \approx \frac{1}{k}$ when k is large

(2) $a_n k^n + a_{n-1} k^{n-1} + \cdots . + a_1 k + a_0 \approx a_n k^n$ (for $a_n \neq 0$)

(3) $(1 + \frac{x}{k}) \approx e^x$ where x is a fixed number

Example 5

Check whether the series

$$\sum_{k=1}^{\infty} \frac{k^5 - 3k + 3}{k^7 + 3k^2 - 4k + 1}$$

Converges or not

Solution:

Note that for k is large

$$(\text{our } u_k)\,\frac{k^5 - 3k + 3}{k^7 + 3k^2 - 4k + 1} \approx \frac{k^5}{k^7} = \frac{1}{k^2}\ (\text{our } v_k)$$

(Step 1: Find $\lim_{k\to\infty} \frac{u_k}{v_k}$)

$$\lim_{k\to\infty} \frac{u_k}{v_k} = \lim_{k\to\infty} \frac{\left(\frac{k^5 - 3k + 3}{k^7 + 3k^2 - 4k + 1}\right)}{\frac{1}{k^2}} = 1$$

(Step 2: Use limit comparison test)

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p-test ($p = 2$)

Therefore by limit comparison test, $\sum_{k=1}^{\infty} \frac{k^5 - 3k + 3}{k^7 + 3k^2 - 4k + 1}$ converges

Example 6 (Difficult Example)
Check whether the series

$$\sum_{k=1}^{\infty} e^{-\left(\sin\left(\frac{1}{k}\right)\right)^{-2}}$$

converges or not.

(Note: It is a good application of limit comparison test)

Solution:

Note that $\sin\left(\frac{1}{k}\right) \approx \frac{1}{k}$ when k is large

Therefore $e^{-\left(\sin\left(\frac{1}{k}\right)\right)^{-2}} \approx e^{-\left(\frac{1}{k}\right)^{-2}} = e^{-k^2}$ (our v_k)

(Step 1: Find $\lim_{k\to\infty} \frac{u_k}{v_k}$)

By some computation, we get $\lim_{k\to\infty} \frac{e^{-\left(\sin\left(\frac{1}{k}\right)\right)^{-2}}}{e^{-k^2}} = 1$

(Step 2: Use limit comparison test)
Note that

$$\sum_{k=1}^{\infty} e^{-k^2} = e^{-1} + e^{-4} + e^{-9} + \cdots \ldots < e^{-1} + e^{-2} + e^{-3} + e^{-4} \ldots \ldots = \sum_{k=1}^{\infty} e^{-k}$$

$\sum_{k=1}^{\infty} e^{-k}$ converges by geometric series test.
By comparison test $\sum_{k=1}^{\infty} e^{-k^2}$ converges.

By limit comparison test, $\sum_{k=1}^{\infty} e^{-\left(\sin\left(\frac{1}{k}\right)\right)^{-2}}$ also converges

Remark: In some cases, we need to use several test to test the convergence of series. It looks complicated but will be easy after several practices. In next tutorial, we will go over more tests which help us to tackle more difficult series.

The following are additional exercises for you. I strongly advise all of you to do some of them. You may submit your solution to me and I can give some comment to you.

☺Exercise 3

Check whether the series

$$\sum_{\mathrm{k}=1}^{\infty} \frac{\mathrm{k}}{(\mathrm{k}+1)!}$$

Converges or not. (Hint: $\mathrm{k}=(\mathrm{k}+1)-1$)

☺Exercise 4

Check whether the series

$$\sum_{\mathrm{k}=1}^{\infty} \frac{\mathrm{k}^3+5}{\mathrm{k}^7-\mathrm{k}^2+1}$$

Converges or not

☺Exercise 5

Check whether the series

$$\sum_{\mathrm{k}=1}^{\infty} \mathrm{k}e^{-\mathrm{k}^2}$$

Converges or not (Hint: Use Integral Test)

☺Exercise 6

Check whether the series

$$\sum_{\mathrm{k}=1}^{\infty} \frac{1}{\sqrt{\mathrm{k}}} \sin\left(\frac{1}{\sqrt{k}}\right)$$

Converges or not (Hint: When $\mathrm{k} \to \infty$, $\sqrt{\mathrm{k}} \to \infty$)

☺Exercise 7

Check whether the series

$$\sum_{\mathrm{k}=1}^{\infty} \frac{\tan^{-1} k}{k^2+1}$$

Converges or not (Hint: Note that $\frac{\mathrm{d}}{\mathrm{dx}} \tan^{-1} x = \frac{1}{1+x^2}$ and apply integral test)