MATH2033 Mathematical Analysis (2021 Spring)
Midterm Examination

Time allowed: 90 minutes (7:30p.m.-9:00p.m.)

Instructions: Answer ALL problems. Full details must be clearly shown to receive full credits. Please submit your work via the submission system in canvas before 9:15p.m.. Late submission will not be accepted. Your submission must be

- **100% handwritten (typed solution will not be accepted)**
- **In a single pdf. files (other file format will not be accepted)**
- **With your full name (as shown in your student ID card) and student ID number on the first page of your submission.**

Problem 1 (30 marks)

(a) **(i)** State the definition of limits of sequence (i.e. $\lim_{n\to\infty} x_n = L \in \mathbb{R}$)

(ii) State the definition of Cauchy sequence.

(b) Using the definition of limits, prove that $\lim_{n\to\infty} \frac{1}{n^4-4n+10} = 0$.

(c) We let $\{x_n\}$ be a sequence of real number such that $\lim_{n\to\infty} x_n = a$, where a is a positive real number. Using the definition of limits, show that $\lim_{n\to\infty} \sqrt[3]{x_n + a} = \sqrt[3]{2a}$.

Problem 2 (20 marks)

We consider a sequence of real number $\{x_n\}$ defined by

$$x_1 = 2 \quad and \quad x_{n+1} = 1 + \frac{x_n^2}{1+x_n^2} \quad for\, n \in \mathbb{N}.$$

(a) Show that the sequence $\{x_n\}$ is monotone.

(b) Hence, show that the sequence $\{x_n\}$ converges and find its limits.

Problem 3 (22 marks)

Recall that the cubic root of 2 (denoted by $\sqrt[3]{2}$) is defined as a real number x satisfying

$$x^3 = 2.$$

In this problem, you are asked to prove the existence of the cubic root $\sqrt[3]{2}$. To do so, we consider the set $S = \{r \in \mathbb{Q} |\, r > 0 \ and \ r^3 < 2\}$.

(a) Prove that $x = \sup S$ exists.

(b) Show that the supremum x satisfy $x^3 = 2$.

Problem 4 (28 marks)

(a) **(6 marks)** We let $[x]$ denotes the greatest integer less than or equal to x. For example, $[7.2] = 7$, $[7.9] = 7$, $[7] = 7$. We consider the set

$$T = \left\{\frac{[x]^2}{y} \,|\, x \in \mathbb{R}\backslash\mathbb{Q} \ and \ y \in \mathbb{Z}\backslash\{0\}\right\}.$$

Determine if the set T is countable.

(b) We let m be a real number. We consider a set S which is the collection of all sequences of integers $\{x_n\}$ that converges to m. That is,

$$S = \left\{\{x_n\} | x_n \in \mathbb{Z} \ for\ all\, n \in \mathbb{N} \ and \ \lim_{n\to\infty} x_n = m\right\}.$$

(i) **(10 marks)** If m is *not* integer, show that S must be an empty set.

(ii) **(12 marks)** If m is an integer, show that S is countable.

(😊Hint: If the sequence $\{x_n\}$ converges, what will happen to x_n when n is large?)

End of paper