Problem Set 5.

Problem 1

Prove the following limits using the definition of limits

(a) $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$

(b) $\lim_{n\to\infty}\sqrt{x_n+y_n}=2$, where $\{x_n\}$ and $\{y_n\}$ are two sequences of positive real number with $\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=2$.

Proof: Recall Definition of limits,

$\lim_{n\to\infty} x_n = x \iff \forall \varepsilon>0.\ \exists K\in\mathbb{N}$ s.t. $|x_n - x|<\varepsilon\ \forall n\geq K.$

(a). $\sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}} < \varepsilon$

$\frac{1}{\sqrt{n}} < 2\varepsilon$

$n > \frac{1}{4\varepsilon^2}$

$\forall \varepsilon>0.$ Choose $K=\left[\frac{1}{4\varepsilon^2}\right]+1$

Then for all $n\geq K$, we have $\left|\sqrt{n+1}-\sqrt{n}\right| < \frac{1}{2\sqrt{n}} < \varepsilon$

Finish the proof. #

(b) $\lim_{n\to\infty} x_n = \lim_{n\to+\infty} y_n = 2.$

$\forall \varepsilon>0.$ Exist $K_1\in\mathbb{N}$ s.t. $\forall n\geq K_1$ $|x_n-2|<\varepsilon$

Exist $K_2\in\mathbb{N}$ s.t. $\forall n>K_2$ $|y_n-2|<\varepsilon$

Choose $K=\max\{K_1,K_2\}$.

$\forall n>K$

$$\left|\sqrt{x_n+y_n}-2\right| = \frac{|x_n+y_n-4|}{\underbrace{\sqrt{x_n+y_n}+2}_{\geq 2}} \leq \frac{|x_n-2|+|y_n-2|}{\sqrt{x_n+y_n}+2} \leq \frac{1}{2}(\varepsilon+\varepsilon) = \varepsilon$$

$\leq |x_n-2|+|y_n-2|.$

Finish the proof.

Problem 2

We let $\{x_n\}$ and $\{y_n\}$ be two sequence of real number with $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}y_n=y$. Suppose that $xy>0$, show that there exists $K\in\mathbb{N}$ such that x_n and y_n have the same sign (either both positive or both negative) when $n\geq K$.

Proof: Choose $\varepsilon_1=\frac{|x|}{2}$ $\exists K_1.$ s.t. $\forall n>K_1$

$|x_n - x| < \varepsilon_1 \Rightarrow \underbrace{x - \frac{|x|}{2}}_{\text{same sign as } x} < x_n < \underbrace{x + \frac{|x|}{2}}_{\text{same sign as } x}$

Choose $\varepsilon_2 = \frac{|y|}{2}$ $\exists K_2$ s.t. $\forall n > K_2$

$|y_n - x| < \varepsilon_2 \Rightarrow \underbrace{y - \frac{|y|}{2}}_{\text{same sign as } y} < y_n < \underbrace{y + \frac{|y|}{2}}_{\text{same sign as } y}$

Take $K = \max\{K_1, K_2\}$.

Then $\forall n > K$.

$x_n y_n > 0$.

Problem 3

(a) Give an example of two divergent sequences $\{x_n\}$, $\{y_n\}$ such that the sequence $\{x_n + y_n\}$ converges.

(b) Give an example of two divergent sequences $\{x_n\}$, $\{y_n\}$ such that the sequence $\{x_n y_n\}$ converges.

$x_n = (-1)^n$

$y_n = (-1)^{n+1}$.

$x_n + y_n = 0$.

$x_n y_n = -1$,

Problem 4

Show that the sequence $\{x_n\}$ defined by $x_n = n^2 - n$ diverges to $+\infty$ using the definition.

Recall how to prove x_n diverges to $+\infty$.

$\forall M > 0$, Choose $N = [\sqrt{M}] + 2$

$\forall n > N$ $x_n = \underline{n^2 - n \geq (n-1)^2} > M$.

Problem 5

We let $\{x_n\}$ be a sequence of positive real number which $\lim_{n\to\infty} x_n = +\infty$. Show that $\lim_{n\to\infty} \frac{1}{x_n} = 0$.

$\lim_{n\to+\infty} x_n = +\infty$.

$$\boxed{\Rightarrow \forall M>0.\ \exists K \in \mathbb{N}.\ \text{s.t.}\ \forall\ n>K\quad x_n > M.}$$

$\forall \varepsilon>0. \quad \exists K \in \mathbb{N} \quad \text{s.t.} \quad \forall n>K \quad x_n > \frac{1}{\varepsilon}$

$$\Rightarrow 0<\frac{1}{x_n}<\varepsilon,$$

$$\Rightarrow \lim_{n\to+\infty} \frac{1}{x_n}=0,$$

Problem 6

Show that the sequence $\{x_n\}$ defined by $x_n = (-1)^n\left(2+\frac{1}{n}\right)$ does not converge.

Proof: $\quad n=2k. \quad \lim\limits_{k\to\infty} x_{2k} = 2$

$n=2k+1 \quad \lim\limits_{k\to+\infty} x_{2k+1} = -2.$