Lecture 6

26-02-2019

Review :

① Counting :

∃ injection ∃ injection injection does not ∃.

$$\{1, 2, \cdots m\} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{R}^k \subset \mathbb{C}^k \subset \cdots$$

finite Card $\mathbb{N}$ = Card $\mathbb{N}$ = Card $\mathbb{N}$ < Card $\mathbb{R}$ = Card $\mathbb{R}$ = Card $\mathbb{R}$

Gap : Continuum Hyperthesis

Cardinal number gives the most primitive math property of a set. It is a math terminology of the intuitive concept of "number" of elements in a set. It gives more quantitative information when the set is infinite.

② Real number theory : Four Axioms give the essential properties for a set to be called " real numbers ".

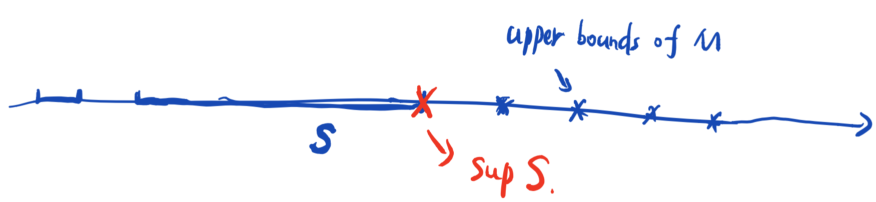
I: Field Axiom : special elements 0, 1

II: Order Axiom : THM : $0 < 1$

III: Well-ordering Axiom : well-ordering for $\mathbb{N}$. Foundation of proof by induction

Def: For a non-empty subset S of $\mathbb{R}$, we say that S is bounded above iff $\exists\, M \in \mathbb{R}$ such that $x \leq M$ for all $x \in S$. Such M is called an upper bound of S.

A supremum (or least upper bound) of S, denoted by $\sup S$, is a upper bound $\tilde{M}$ of S such that $\tilde{M} \leq M$ for all upper bounds M of S.



Remark: ① If M is an upper bound of S, then any $t \geq M$ is also an upper bound of S.

② $\tilde{M}$ may not be in S.

③ If M is an upper bound of S, and $M \in S$. then $M = \sup S$.

Example ① $S = \{ \frac{1}{n} : n \in \mathbb{N} \}$. 2 is an upper bound of S. Actually, every real number ≥ 1 is an upper bound of S. The least upper bound, or $\sup S = 1$.

Example ② $S = \{ x : x \in \mathbb{Q}, \ x < 0 \}$

every rational number ≥ 0 is a upper bound of S.

$\boxed{\sup S = 0}$. Note that $0 \notin S$.

Not trivial, will prove later.

Completeness Axiom (for Supremum)

C-A : Every non-empty subset of $\mathbb{R}$ which is bounded above has a supremum in $\boxed{\mathbb{R}}$.

R.K : ① C-A Does Not Holds for $\mathbb{Q}$. [$\mathbb{Q}$ is not complete in the sense that the supremum of a set of rational numbers may not be a rational number, for instance

$S = \{ x : x \in \mathbb{Q}, x^2 < 2 \}$. Then $S \subseteq \mathbb{Q}$.

But $\sup S = \sqrt{2} \notin \mathbb{Q}$.

② The Four Axioms gives essential properties of real numbers. [or the essential properties of a number system which can be called real number]

③ Starting from 0, 1, one can construct $\mathbb{N}$ and $\mathbb{Z}$, and then $\mathbb{Q}$ using the field axiom. The ordering axiom and well-ordering axiom will yield the natural order in $\mathbb{Q}$. Then the first three axioms generate "a theory of rational numbers".

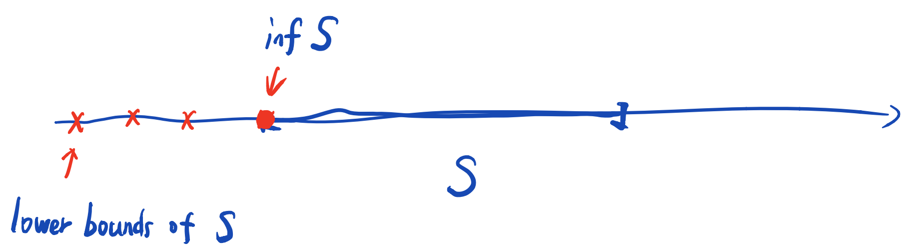
However, $\mathbb{Q}$ is not the right number to do analysis since $\mathbb{Q}$ is not "Complete", or $\mathbb{Q}$ does not form continuum.

④ From $\mathbb{Q}$, one can construct $\mathbb{R}$ using different methods:

Cauchy sequence, Dedekind cut, hyper-real number, ...

All these constructions yield the "same" real number system.

Def : For a non-empty subset S of $\mathbb{R}$, we say that S is bounded below iff $\exists\, m \in \mathbb{R}$ s.t $m \leq x$ for all $x \in S$. Such m is called a lower bound of S. An infimum or greatest lower bound of S (denoted by $\operatorname{Inf} S$ or $\operatorname{glb} S$) is a lower bound $\tilde{m}$ of S such that $\tilde{m} \geq m$ for all lower bounds m of S.



Remark : ① If m is a lower bound of S, then $\forall\, t < m$, t is also a lower bound of S.

② $\operatorname{Inf} S$ may not be in S.

③ If m is a lower bound of S and $m \in S$, then $m = \operatorname{Inf} S$.

The Dual between Sup and Inf

Def: For $\phi \neq B \subset \mathbb{R}$, $c \in \mathbb{R}$, we define

$$-B = \{-x : x \in B\}, \quad cB = \{cx : x \in B\}.$$

THM ①: Let B be a non-empty subset of $\mathbb{R}$, then

B is bounded above $\Rightarrow$ $-B$ is bounded below,

moreover, $\inf(-B)$ exists and we have $\inf(-B) = -\sup B$.

Proof: Step 1. Since B is bounded above, $\exists\, M \in \mathbb{R}$ s.t $b \leq M$, $\forall\, b \in B$.

$\Rightarrow -M \leq -b$, $\forall\, b \in B$ $\Rightarrow -M$ is a lower bound of $-B$.

$\Rightarrow -B$ is bounded below.

Step 2. Since B is bounded above, by C-A, $\sup B$ exists and $\sup B \in \mathbb{R}$.

We show that $-\sup B$ is a lower bound of $-B$.

Indeed, Since $b \leq \sup B$ (by the definition of Sup), $\forall\, b \in B$,

$\Rightarrow -\sup B \leq -b$, $\forall\, b \in B$ $\Rightarrow -\sup B$ is a lower bound of $-B$.

Step 3. We show that $-\sup B$ is the greatest lower bound of $-B$.

$\forall$ lower bound m of $-B$, we have

$m \leq -b, \quad \forall\, b \in B. \quad \Rightarrow \quad b \leq -m, \ \forall\, b \in B$

$\Rightarrow -m$ is an upper bound of B. By the definition of Sup,

$\sup B \leq -m \quad \Rightarrow \quad m \leq -\sup B.$

$\Rightarrow -\sup B$ is indeed the greatest lower bound of $-B$.

$\Rightarrow \inf(-B)$ exists and $\inf(-B) = -\sup B$.

The Dual between Sup and Inf

THM ②: Let B be a non-empty subset of $\mathbb{R}$, then

B is bounded below $\Rightarrow$ $-B$ is bounded above. Moreover

$\inf B$ exists and $\inf B = -\sup(-B)$.

Proof: Exercise (Similar to THM ①).

Remark: we can formulate Completeness Axiom for infimum as follows:

> Every non-empty subset of $\mathbb{R}$ which is bounded below has an infimum in $\mathbb{R}$.

Then THM ① and THM ② shows that:

Completeness axiom for supremum $\Leftrightarrow$ Completeness axiom for Infimum.

Two more properties of Inf and Sup

THM: ③ If $\phi \neq A \subseteq B$, then $\inf B \leq \inf A$

if B is bounded below; $\sup A \leq \sup B$ if

B is bounded above

④. If B is bounded above and $c > 0$, then

$c+B$, cB are bounded above with

$\sup(c+B) = c + \sup B$, $\sup(cB) = c \sup B$.

Proof of ③: If B is bounded below, by THM ②, $\inf B$ exists.

and $\forall\ b \in B$, $b \geq \inf B$. Since $A \subseteq B$,

$a \geq \inf B$, $\forall\ a \in A \subset B$. $\Rightarrow$ $\inf B$ is a lower bound of

A. $\Rightarrow$ A is bounded below. by THM ②, $\inf A$ exists, and

$\inf A \geq \inf B$. Similar arguments hold when B is bounded above.

Proof of ④. Exercise.

Def: Let $\phi \neq S \subset \mathbb{R}$, S is called bounded iff S is bounded above and below.

Remark: S is bounded $\iff$ $\exists M>0$ s.t $|x| \leq M$, $\forall x \in S$.

Exercise: if A and B are bounded, then $A+B$ defined by

$$A+B = \{ a+b : a \in A, b \in B \}$$ is also bounded.

Moreover, we have

$$\inf(A+B) = \inf A + \inf B$$

$$\sup(A+B) = \sup A + \sup B.$$

Theorems derived from the Four Axioms

THM: (Infinetesimal Principal): Let $x, y \in \mathbb{R}$. Then

$$x < y + \varepsilon \text{ for all } \varepsilon > 0 \iff x \le y.$$

Proof: $\Leftarrow$ If $x \le y$, then $\forall\, \varepsilon > 0$, we have

$$x \le \underbrace{y = y+0}_{\text{field axiom}} \underbrace{< y+\varepsilon}_{\text{order axiom}}.$$

$\Rightarrow$ By contradiction, assume $\sim(x \le y)$, then $x > y$ (by ordering axiom,)

$\Rightarrow$ $x - y > y - y = 0$. Let $\varepsilon = x - y$.

then $y + \varepsilon = y + (x-y) = y + x - y = x$. This contradicts

to the assumption $y + \varepsilon > x$. Therefore $x \le y$.

R.K: In the special case, $x = |a-b|$, $y = 0$, we have

$$|a-b| < \varepsilon \text{ for all } \varepsilon > 0 \iff |a-b| \le 0 \iff a = b.$$

Example: $0.\dot{9} = 1$ or $\sum_{n=1}^{\infty} 9 \times 10^{-n} = 1$.

Mathematical Induction Principal

THM: (1) $\forall\ n \in \mathbb{N}$, P_n is a statement that is either true or false.

(2) P_1 is true.

(3) $\forall\ k \in \mathbb{N}$, P_k is true $\Rightarrow$ P_{k+1} is true

Then $\forall n \in \mathbb{N}$, P_n is true.

Proof: By contradiction, assume $\sim(\forall\ n \in \mathbb{N},\ P_n \text{ is true}) =$

$\exists\ n \in \mathbb{N}$ such that P_n is false. Then the

set $S = \{ k : k \in \mathbb{N},\ P_k \text{ is false} \}$ is a nonempty

subset of $\mathbb{N}$. By the well-ordering axiom, S has a minimum $m \in \mathbb{N}$.

Since by (2), $1 \notin S$, we see that $m \geq 2$.

By the definition of S and m (the inf of S), P_m is false

and P_{m-1} is true. This contradicts to (3) that

P_{m-1} is true $\Rightarrow$ P_m is true.

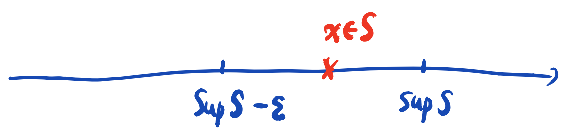
Question: where do we used (1)?

Supremum Property

THM : If S has a supremum in $\mathbb{R}$, then $\forall\ \varepsilon > 0$,

$\exists\ x \in S$ such that

$$\sup S - \varepsilon < x \leq \sup S.$$



Proof : ① By definition of Sup, $x \leq \sup S$, $\forall\ x \in S$.

② Since $\sup S - \varepsilon < \sup S$, by the definition of $\sup S$ again, $\sup S - \varepsilon$ is not an upper bound for S. So $\exists\ x \in S$ such that $\sup S - \varepsilon < x$. By ①, for this x, $x \leq \sup S$.

This proof is completed.