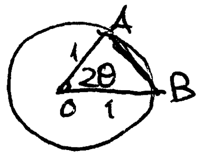
Some useful inequalities

① $|\sin\theta| \le |\theta|$ for all $\theta \in \mathbb{R}$

Reason If $|\theta| \ge 1$, then $|\sin\theta| \le 1 \le |\theta|$. If $|\theta| < 1$, then



draw $\triangle AOB$ with $OA = OB = 1$ and $\angle AOB = 2\theta$.

$AB = 2\sin\theta \le \overset{\frown}{AB} = 2\theta$.

② $\forall a, b \in \mathbb{R}$, $|\sin a - \sin b| \le |a-b|$ and $|\cos a - \cos b| \le |a-b|$.

Reason $|\sin a - \sin b| = \left|2\sin\frac{a-b}{2}\cos\frac{a+b}{2}\right| \le 2\left|\sin\frac{a-b}{2}\right| \le 2\left|\frac{a-b}{2}\right| = |a-b|$.

$|\cos a - \cos b| = |\sin(\frac{\pi}{2}-a) - \sin(\frac{\pi}{2}-b)| \le |(\frac{\pi}{2}-a)-(\frac{\pi}{2}-b)| = |a-b|$.

③ If $a, b > 0$ and $x \in (0,1]$, then $|a^x - b^x| \le |a-b|^x$.

(In particular, $|\sqrt[n]{a} - \sqrt[n]{b}| \le \sqrt[n]{|a-b|}$ for $n = 2, 3, 4, \ldots$).

Reason. In case $a > b$, let $c = \frac{b}{a}$, then $0 < c < 1$. So $c^{1-x} \le 1$

$\Rightarrow c \le c^x \Rightarrow \underbrace{c - c^x \le 0}_{(*)}$. Also $0 < 1-c < 1$. So $\underbrace{1-c \le (1-c)^x}_{(**)}$.

Adding $(*)$ and $(**)$, $1 - c^x \le (1-c)^x$. Multiplying by a^x, we get

$\underbrace{a^x - b^x}_{=|a^x-b^x|} \le \underbrace{(a-b)^x}_{|a-b|^x}$, the case $b > a$ is similar and case $a = b$ is obvious.

④ $\forall x \ge 0$, $\ln(1+x) \le x$

Reason $\ln(1+x) = \int_1^{1+x} \frac{1}{t}\,dt \le \int_1^{1+x} 1\,dt = x$.