2010 Math 202 Spring Midterm

Problem 1 Let $f:[1,3]\to\mathbb{R}$ be defined by $f(x)=\dfrac{1}{\sqrt[4]{x^2+6x}}$.
Prove $\lim_{x\to 2} f(x)=\frac{1}{2}$ by checking definition of limit.

Scratch Work

$$\left|f(x)-\tfrac{1}{2}\right| = \left|\sqrt[4]{\frac{1}{x^2+6x}}-\sqrt[4]{\frac{1}{16}}\right| \le \sqrt[4]{\left|\frac{1}{x^2+6x}-\frac{1}{16}\right|}$$

$$=\sqrt[4]{\frac{|x^2+6x-16|}{16(x^2+6x)}} = \sqrt[4]{\frac{|x+8||x-2|}{16(x^2+6x)}} \underset{1\le x\le 3}{\le} \sqrt[4]{\frac{11|x-2|}{112}}$$

Solution $\forall \varepsilon>0$, let $\delta=\frac{112}{11}\varepsilon^4$, then

$$\forall x\in[1,3],\ 0<|x-2|<\delta \Rightarrow \left|f(x)-\tfrac{1}{2}\right| \le \sqrt[4]{\frac{11|x-2|}{112}} < \sqrt[4]{\frac{11\delta}{112}}=\varepsilon.$$

Variation

$$\left|f(x)-\tfrac{1}{2}\right| = \left|\frac{1}{\sqrt[4]{x^2+6x}}-\frac{1}{2}\right| = \left|\frac{2-\sqrt[4]{x^2+6x}}{2\sqrt[4]{x^2+6x}}\right| \times \left|\frac{2+\sqrt[4]{x^2+6x}}{2+\sqrt[4]{x^2+6x}}\right|$$

$$=\frac{|4-\sqrt{x^2+6x}|}{2\sqrt[4]{x^2+6x}\,(2+\sqrt[4]{x^2+6x})} \times \left|\frac{4+\sqrt{x^2+6x}}{4+\sqrt{x^2+6x}}\right|$$

$$=\frac{|16-(x^2+6x)|}{2\sqrt[4]{x^2+6x}\,(2+\sqrt[4]{x^2+6x})(4+\sqrt{x^2+6x})} \underset{x\in[1,3]}{\le} \frac{|x+8||x-2|}{2\cdot 1(2)(4)}$$

$$\le \frac{11}{16}|x-2| < \varepsilon \text{ if } |x-2|<\frac{16}{11}\varepsilon.$$

Problem 2 Let $a_1, a_2, a_3, \dots$ be a Cauchy sequence of real numbers. Let $b_n=\sin^2(a_n+a_{2n})$. Prove that $b_1, b_2, b_3, \dots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

Scratch Work

$$|b_n-b_m| = |\sin^2(a_n+a_{2n})-\sin^2(a_m+a_{2m})|$$

$$=|\sin(a_n+a_{2n})+\sin(a_m+a_{2m})|\,|\sin(a_n+a_{2n})-\sin(a_m+a_{2m})|$$

$$\le 2|(a_n+a_{2n})-(a_m+a_{2m})|$$

$$\le 2(|a_n-a_m|+|a_{2n}-a_{2m}|)$$

Solution $\forall\varepsilon>0$, since $\{a_n\}$ is Cauchy, $\exists K\in\mathbb{N}$ such that $n,m\ge K \Rightarrow |a_n-a_m|<\varepsilon/4$.
Then $n,m\ge K \Rightarrow n,m,2n,2m\ge K$

$$\Rightarrow |b_n-b_m| \le 2(|a_n-a_m|+|a_{2n}-a_{2m}|) < 2\left(\frac{\varepsilon}{4}+\frac{\varepsilon}{4}\right)=\varepsilon.$$

Variation Let $f(x)=\sin^2 x$, then $f'(x)=2\sin x\cos x$
By mean-value theorem, $|f(c)-f(d)|=|f'(\theta)(c-d)| \le 2|c-d|$

$$|b_n-b_m| = |f(a_n+a_{2n})-f(a_m+a_{2m})| \le 2|(a_n+a_{2n})-(a_m+a_{2m})| \le 2\left(|a_n-a_m|+|a_{2n}-a_{2m}|\right)$$

Problem 3 Prove that there does not exist any continuous function $f:\mathbb{R}\to\mathbb{R}$ such that
$f(x)$ is rational $\Leftrightarrow$ $f(x+1)$ is irrational.

Solution Assume such f exists. Let $g(x)=f(x)+f(x+1)$. Then g is continuous and the range of g is a subset of $\mathbb{R}\setminus\mathbb{Q}$. From the intermediate value theorem, the range of g is an interval. Hence, the range of g must be a single point.
$\therefore$ g is a constant function.
Similarly, $h(x)=f(x)-f(x+1)$ is a constant function. Then $f(x)=\dfrac{g(x)+h(x)}{2}$ is a constant function, which cannot satisfy the condition
$f(x)\in\mathbb{Q} \Leftrightarrow f(x+1)\in\mathbb{R}\setminus\mathbb{Q}$, Contradiction.

Problem 4 Let $f:\mathbb{R}\to\mathbb{R}$ be twice differentiable and for all $x\in[0,1]$, $|f''(x)|\le 2010$.
If there exists $c\in(0,1)$ such that $f(c)>f(0)$ and $f(c)>f(1)$, then prove that

$$|f'(0)|+|f'(1)|\le 2010.$$

Solution On $[0,1]$, by extreme value theorem, $\exists\, c_0\in[0,1]$ such that

$$f(c_0)=\max\{f(t): t\in[0,1]\}\ge f(c)> f(0) \text{ and } f(1).$$

So $c_0\in(0,1)$. Hence, $f'(c_0)=0$.
By mean value theorem, $\exists\, \theta_0,\theta_1\in(0,1)$ such that

$$f'(0)=f'(0)-f'(c_0)=f''(\theta_0)(0-c_0)$$
$$f'(1)=f'(1)-f'(c_0)=f''(\theta_1)(1-c_0).$$

So
$$\begin{aligned}|f'(0)|+|f'(1)| &\le |f''(\theta_0)|c_0+|f''(\theta_1)|(1-c_0)\\ &\le 2010\,c_0+2010(1-c_0)\\ &=2010.\end{aligned}$$