Problem Set 3.

6:50 pm – 7:10 pm

Problem 8

(a) Using mathematical induction, prove that

$$\cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\cos\left(\frac{n+1}{2}\theta\right)\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

→ widely used.

for all positive integer n. Here, $\theta \neq k\pi$ for any $k \in \mathbb{Z}$.

(b) We let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers defined by

$$a_0 = \sqrt{2}, \qquad a_n = \sqrt{2 + a_{n-1}} \quad for \ \ n = 1,2,\ldots$$

Using mathematical induction, prove that

$$a_n = 2\cos\frac{\pi}{2^{n+2}}$$

for all $n = 0,1,2,\ldots$.

Induction.
{ ① Initial case $n=1$.
② $n \to n+1$

(c) We let $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Using mathematical induction, prove that

$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$A^n = \begin{pmatrix} 2^n & 3(2^n-1) \\ 0 & 1 \end{pmatrix}$$

for any positive integer n. A^n.

→ Jordan Decomposition,
Advanced Algebra
Linear

Proof: (a) When $n=1$, Left Hand Side $= \cos\theta$

Right Hand Side $= \cos\theta$.

The Equation holds.

Suppose when $n=k$ the Equation holds:

$$\cos\theta + \cos 2\theta + \ldots + \cos k\theta = \frac{\cos\left(\frac{k+1}{2}\theta\right)\sin\frac{k\theta}{2}}{\sin\frac{\theta}{2}}$$

when $n = k+1$,

Left Hand Side $= \underline{\cos\theta + \cos 2\theta + \ldots + \cos k\theta} + \cos(k+1)\theta$

$$= \frac{1}{\sin\frac{\theta}{2}}\cos\left(\frac{k+1}{2}\theta\right)\sin\frac{k\theta}{2} + \cos(k+1)\theta.$$

$$= \frac{1}{\sin\frac{\theta}{2}}\left(\cos\left(\frac{k+1}{2}\theta\right)\sin\frac{k\theta}{2} + \cos(k+1)\theta\sin\frac{\theta}{2}\right).$$

$\sin(\theta_1 + \theta_2)$
$= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2.$

$\sin\theta_1\cos\theta_2$
$= \frac{1}{2}(\sin(\theta_1+\theta_2) + \sin(\theta_1-\theta_2)).$

$\cos\theta_1\sin\theta_2$
$= \frac{1}{2}(\sin(\theta_1+\theta_2) - \sin(\theta_1-\theta_2)).$

$\cos(k+1)\theta\sin\frac{\theta}{2}$

$= \frac{1}{2}\left(\sin\left(k+1+\frac{1}{2}\right)\theta - \sin\left(k+1-\frac{1}{2}\right)\theta\right)$

$= \frac{1}{2}\left(\sin\left(k+\frac{3}{2}\right)\theta - \sin\left(k+\frac{1}{2}\right)\theta\right).$

$\cos\left(\frac{k+1}{2}\theta\right)\sin\frac{k\theta}{2}$

$= \frac{1}{2}\left(\sin\left(\frac{k+1}{2}+\frac{k}{2}\right)\theta - \sin\left(\frac{k+1}{2}-\frac{k}{2}\right)\theta\right)$

$= \frac{1}{2}\left(\sin\left(k+\frac{1}{2}\right)\theta - \sin\frac{1}{2}\theta\right)$

$\Rightarrow \cos\theta + \dots + \cos(k+1)\theta$

$= \frac{1}{2}\left(\sin\left(k+\frac{3}{2}\right)\theta - \sin\left(k+\frac{1}{2}\right)\theta\right) + \frac{1}{2}\left(\sin\left(k+\frac{1}{2}\right)\theta - \sin\frac{1}{2}\theta\right)$

$= \frac{1}{2}\left(\sin\left(k+\frac{3}{2}\right)\theta - \sin\frac{1}{2}\theta\right)$

$= \frac{1}{2}\cdot 2\cos\left(\frac{k+2}{2}\theta\right)\sin\left(\frac{k+1}{2}\theta\right).$

$\Rightarrow$ The Equation holds for $n=k+1$.

Proof not using induction:

$\sin\frac{\theta}{2}(\cos\theta + \cos 2\theta + \dots + \cos n\theta)$

$= \cos\theta\sin\frac{\theta}{2} + \cos 2\theta\sin\frac{\theta}{2} + \dots + \cos n\theta\sin\frac{\theta}{2}$

$= \frac{1}{2}\left(\cancel{\sin\frac{3}{2}\theta} - \sin\frac{\theta}{2}\right) + \frac{1}{2}\left(\cancel{\sin\frac{5}{2}\theta} - \cancel{\sin\frac{3}{2}\theta}\right) + \dots + \frac{1}{2}\left(\sin\frac{2n+1}{2}\theta - \sin\frac{2n-1}{2}\theta\right)$

$= \frac{1}{2}\left(\sin\frac{2n+1}{2}\theta - \sin\frac{\theta}{2}\right)$

$= \cos\frac{n+1}{2}\theta\sin\frac{n}{2}\theta$

(b). Prove: $a_n = 2\cos\frac{\pi}{2^{n+2}}$

Proof by induction:

When $n=0$, Left Hand Side $=\sqrt{2}$

Right Hand Side $= 2\cdot\cos\frac{\pi}{4} = \sqrt{2}$.

The Equation holds.

Suppose When $n=k$, the Equation holds.

When $n=k+1$,

$a_k = 2\cos\frac{\pi}{2^{k+2}}$

$a_k,\ a_{k+1}$.

$Eq \rightarrow a_{k+1}$

$a_{k+1} = \sqrt{2+a_k}$

$= \sqrt{2+2\cos\frac{\pi}{2^{k+2}}}$

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \longrightarrow$

$= \sqrt{2+2\left(\cos^2\frac{\pi}{2^{k+3}} - \sin^2\frac{\pi}{2^{k+3}}\right)}$

$\sin^2\theta + \cos^2\theta = 1.$

$$= \sqrt{4\cos^2\frac{\pi}{2^{k+3}}}$$

$$= 2\cos\frac{\pi}{2^{k+3}}.$$

The Equation holds when $n=k+1$.

(C). When $n=1$,

The left hand side $= \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$

Right hand side $= \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$

The Equation holds.

Suppose when $n=k$, the Equation holds.

$$A^k = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 2^k & 3(2^k-1) \\ 0 & 1 \end{pmatrix}$$

$$A^{k+1} = A^k \cdot A$$

$$= \begin{pmatrix} 2^k & 3(2^k-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 3(2^{k+1}-1) \\ 0 & 1 \end{pmatrix}$$

The Equation holds for $n=k+1$.

Jordan Decomposition.

A: 2×2. matrix, 100×100. $A^n = ?$

$\exists P$ invertible

$P^{-1}AP = J$ $\quad A = PJP^{-1}$

$A^n = PJ^nP^{-1}$.

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{pmatrix}$$

$$J_k = \begin{pmatrix} \lambda_k & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix} \quad J_k^n = \begin{pmatrix} \lambda_k^n & & \\ & \ddots & \\ & & \lambda_k^n \end{pmatrix}$$

$$\begin{pmatrix} \lambda_k & 1 & & 0 \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix}^n = \begin{pmatrix} \lambda_k & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} + \tilde{J}$$

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \end{pmatrix}$$

Problem 9

Using mathematical induction, prove that

(a) $(1+x)^n \geq 1+nx$ for any positive integer n, where $x \geq -1$ is real number.

(b) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - 1\right)$ for all positive integer n.

$n=1$,

$x+1, =, 1+1.$

Proof: (a) When $n=1$. The inequality holds.

Suppose the inequality holds when $n=k$.

When $n=k+1$,

$$(1+x)^{k+1} = (1+x)^k (1+x)$$
$$\geqslant (1+kx)(1+x)$$
$$= 1+(k+1)x+kx^2$$
$$\geqslant 1+(k+1)x.$$

The inequality holds for $n=k+1$

$\Rightarrow$ The inequality holds for any positive integer.

(b). $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}$

$\underline{\frac{1}{2\sqrt{k}} > \frac{1}{\sqrt{k}+\sqrt{k+1}}}$

$$= 2\left(\frac{1}{2}+\frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{3}}+\dots+\frac{1}{2\sqrt{n}}\right)$$
$$> 2\left(\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\dots+\frac{1}{\sqrt{n}+\sqrt{n+1}}\right)$$
$$= 2\left(\sqrt{2}-1+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}+\dots+\sqrt{n+1}-\sqrt{n}\right)$$
$$= 2(\sqrt{n+1}-1).$$

Proof by induction:

if $1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{k}} > 2(\sqrt{k+1}-1)$.

when $n=k+1$.

$$1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}$$
$$> 2\sqrt{k+1}-2+\frac{1}{\sqrt{k+1}} \qquad \geqslant \underline{2\sqrt{k+2}}\ \underline{-2}.$$
$$= 2\left(\underline{\sqrt{k+1}+\frac{1}{2\sqrt{k+1}}}\right)-\underline{2}$$

$$= 2 \cdot \frac{2k+3}{2\sqrt{k+1}} - 2$$

$$> 2 \cdot \frac{2\sqrt{k+1} \cdot \sqrt{k+2}}{2\sqrt{k+1}} - 2$$

$$= 2\sqrt{k+2} - 2.$$

Problem 10

We let $P(n)$ be a statement which depends on the positive integer n. The second principle of mathematical induction states that $P(n)$ is true for all positive integer n if all of the following conditions hold:

- $P(1)$ and $P(2)$ are true
- If $P(k)$ and $P(k+1)$ are true for some integer k, then $P(k+2)$ is also true.

(a) Prove the principle using well-ordering principle.

(b) Using the second principle of mathematical induction, prove the following statement:
We let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers defined by

$$a_1 = 1, \qquad a_2 = 7, \qquad a_{n+2} - 4a_{n+1} + 3a_n = 0 \quad for \quad n = 1,2,3,\ldots$$

Then $a_n = 3^n - 2$ for all $n \in \mathbb{N}$.

- $P(1)$
- $P(k) \longrightarrow P(k+1)$.

(a). Proof: Prove this by contradiction.

Suppose $P(n)$ is false for some $n \in \mathbb{N}$.

Consider a set:

$$S = \{n \in \mathbb{N}^+ : P(n) \text{ is false}\}.$$

according to Condition 1,

$$1 \bar{\in} S. \quad 2 \bar{\in} S.$$

Since $S \neq \phi$. and $S \subseteq \mathbb{N}$. it follows from well-ordering property that S has the least element and we denote this element by $\inf S = m$.

smallest.

$m \geq 3$.

Note that $P(m-1)$ $P(m-2)$ are true.

According to Condition (2),

$P(m)$ is true.

It leads to contradiction.

(b). We need to verify if $a_{n-1} = 3^{n-1} - 2$ $a_{n-2} = 3^{n-2} - 2$.

then $a_n = 3^n - 2$.

$$\begin{aligned} a_n &= 4a_{n-1} - 3a_{n-2} \\ &= 4 \times (3^{n-1} - 2) - 3 \times (3^{n-2} - 2) \\ &= (4-1) \times 3^{n-1} - 8 + 6 \\ &= 3^n - 2. \end{aligned}$$

$$b_n = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$$

$$\begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} a_{n-2} \\ a_{n-1} \end{pmatrix}$$

$$b_n = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix} b_{n-1} = \underbrace{\begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix}}^{n-2} b_2.$$