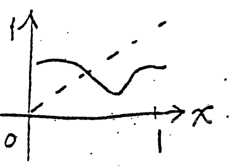
(61) For every r, $f(r)$ is the maximum or minimum of $f(x)$ on some interval (a,b) containing r. Then $a<r<b$. By density of rational numbers, there are $c,d\in\mathbb{Q}$ such that $a<c<r$ and $r<d<b$. Let $S=\{(q_0,q_1): q_0,q_1\in\mathbb{Q} \text{ and } q_0<q_1\}$, then $S\subseteq \mathbb{Q}\times\mathbb{Q}$ and so S is countable.

$$f(\mathbb{R})=\{f(r): r\in\mathbb{R}\}\subseteq\{\max_{c<x<d} f(x) : (c,d)\in S\}\cup\{\min_{c<x<d} f(x) : (c,d)\in S\}$$

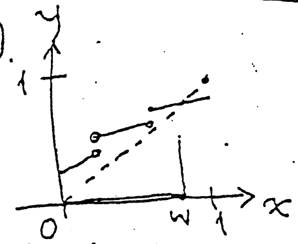
$$=\underbrace{\bigcup_{(c,d)\in S}}_{\text{countable}}\underbrace{\{\max_{c<x<d} f(x),\ \min_{c<x<d} f(x)\}}_{\text{finite}}$$ which is countable by the countable union theorem

So $f(\mathbb{R})$ is countable. By the intermediate value theorem, f is constant.

(62) Suppose such function g exists. We first show g is injective. (If $g(a)=g(b)$, then $-a^9=g(g(a))=g(g(b))=-b^9 \Rightarrow a=b$.) Since g is continuous and injective, by the continuous injection theorem, g is strictly increasing or strictly decreasing. If g is strictly increasing, then $x<y\Rightarrow g(x)<g(y)\Rightarrow g(g(x))<g(g(y))$. If g is strictly decreasing, then $x<y \Rightarrow g(x)>g(y)\Rightarrow g(g(x))<g(g(y))$. So in both cases, $g(g(x))$ is strictly increasing, which cannot equal to the decreasing function $-x^9$, a contradiction. So no such g exists.

(63) 

Let $g(x)=f(x)-x$, then g is continuous on $[0,1]$ because f is continuous on $[0,1]$. Since $f(0),f(1)\in[0,1]$, so $g(0)=f(0)-0\geq 0$ and $g(1)=f(1)-1\leq 0$. By the intermediate value theorem, there is at least one w between 0 and 1 such that $g(w)=0$. Then $f(w)=w$.

(64) 

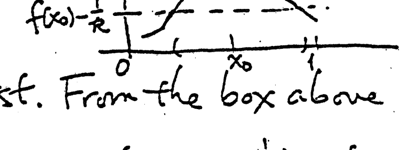
Let $S=\{t\in[0,1]: t<f(t)\}$. Since $0\in S$ and S is bounded above by 1, $\sup S=w\in[0,1]$. By the supremum limit theorem, there is a sequence $t_n\in S$ converging to w. By the monotone function theorem, $w=\lim_{n\to\infty} t_n\leq \lim_{n\to\infty} f(t_n)\overset{\text{seq. limit thm}}{=} f(w-)\overset{\text{monotone function theorem}}{\leq} f(w)$. In particular, $w\neq 1$. So $w<1$. Let $\{s_n\}$ be a strictly decreasing sequence in $[0,1]$ converging to w. Since $s_n>w$, $s_n\notin S$ and so $w=\lim_{n\to\infty} s_n \underset{s_n\notin S}{\geq} \lim_{n\to\infty} f(s_n) \underset{\text{seq. limit theorem}}{=} f(w+) \underset{\text{monotone function theorem}}{\geq} f(w)$. Therefore, $w=f(w)$.

(65) f is injective because $f(a)=f(b)\Rightarrow 0=|f(a)-f(b)|\geq|a-b|\Rightarrow a=b$. Next, since f is continuous and injective, f is strictly monotone by the continuous injection theorem. To show f is surjective, let $w\in\mathbb{R}$ and $M=|w-f(0)|$. The given inequality implies $|f(M)-f(0)|\geq|M-0|=M=|w-f(0)|$ and $|f(0)-f(-M)|\geq|0-(-M)|=M=|w-f(0)|$. Since f is strictly monotone, $f(0)$ is between $f(-M)$ and $f(M)$. The inequalities above imply w is closer to $f(0)$ than $f(M)$ and $f(-M)$. So w is between $f(-M)$ and $f(M)$. The intermediate value theorem implies $w=f(x)$ for some x between $-M$ and M. So f is surjective. Therefore, f is bijective.

(66) Since $M = \sup_{x\in[0,1]} f(x)$, $\left(\int_0^1 f(x)^n dx\right)^{\frac{1}{n}} \le \left(\int_0^1 M^n dx\right)^{\frac{1}{n}} = M$ for all $n\in\mathbb{N}$. By the extreme value theorem, $M = f(x_0)$ for some $x_0\in[0,1]$. For every $k\in\mathbb{N}$, we consider $g(x) = f(x) - \left(f(x_0) - \frac{1}{k}\right)$ on $[0,1]$. Since g is continuous and $g(x_0) = \frac{1}{k} > 0$, by the sign preserving property, there is $\delta>0$ such that $g(x)>0$ ($\Leftrightarrow f(x) > M - \frac{1}{k}$) on the interval $(x_0-\delta, x_0+\delta)\cap[0,1]$. Let a, b be the endpoints of the interval with $a<b$. Since $f(x)>0$, $\left(\int_a^b (M-\frac{1}{k})^n dx\right)^{\frac{1}{n}} < \left(\int_a^b f(x)^n dx\right)^{\frac{1}{n}} \le \left(\int_0^1 f(x)^n dx\right)^{\frac{1}{n}}$.

So $\boxed{(M-\frac{1}{k})(b-a)^{\frac{1}{n}} \le \left(\int_0^1 f(x)^n dx\right)^{\frac{1}{n}} \le M}$. Since $\lim_{n\to\infty} (b-a)^{\frac{1}{n}} = 1$, we have

$M - \frac{1}{k} \le \lim_{n\to\infty} \left(\int_0^1 f(x)^n dx\right)^{\frac{1}{n}} \le M$ for every $k\in\mathbb{N}$. As $k\to\infty$, we get by sandwich theorem that $\lim_{n\to\infty} \left(\int_0^1 f(x)^n dx\right)^{\frac{1}{n}} = M$.



Comments: In fact, the limit must exist. From the box above we have

$\left|\left(\int_0^1 f(x)^n dx\right)^{\frac{1}{n}} - M\right| \le M - (M-\frac{1}{k})(b-a)^{\frac{1}{n}} = (M-\frac{1}{k})\left(1-(b-a)^{\frac{1}{n}}\right) + \frac{1}{k}$.

For every $\varepsilon>0$, by the Archimedean Principle, there is $k\in\mathbb{N}$ such that $\frac{1}{k} < \frac{\varepsilon}{2}$ and $\frac{1}{k} < M$. With one such k, since $\lim_{n\to\infty}(b-a)^{\frac{1}{n}} = 1$, there is $K\in\mathbb{N}$ such that

$n\ge K \Rightarrow \left|(b-a)^{\frac{1}{n}} - 1\right| < \frac{\varepsilon}{2(M-\frac{1}{k})}$. Then

$$n\ge K \Rightarrow \left|\left(\int_0^1 f(x)^n dx\right)^{\frac{1}{n}} - M\right| \le M - (M-\frac{1}{k})(b-a)^{\frac{1}{n}} = (M-\frac{1}{k})(1-(b-a)^{\frac{1}{n}}) + \frac{1}{k} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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(67) Since $f(0) = 0 = 0^2$, so $f(x) = x^2$ for all $x\in\mathbb{R}$. Then for every $x_0\in\mathbb{R}$,

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{x^2-x_0^2}{x-x_0} = \lim_{x\to x_0} x+x_0 = 2x_0.$$

Remarks We have $f(x) = 2x = \begin{cases} 2x & \text{if } x\ne 0 \\ 0 & \text{if } x=0\end{cases} \ne \begin{cases} \frac{d}{dx}(x^2) & \text{if } x\ne 0 \\ \frac{d}{dx}(x) & \text{if } x=0.\end{cases}$ This is to illustrate that if $f(x) = \begin{cases} h_0(x) & \text{if } x\in S \\ h_1(x) & \text{if } x\notin S\end{cases}$, then in general, $f'(x) \ne \begin{cases} h_0'(x) & \text{if } x\in S \\ h_1'(x) & \text{if } x\notin S.\end{cases}$

For $g(x) = |\cos x|$, let $r(x) = |x|$ and $s(x) = \cos x$, then $r'(x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \\ \text{not exist} & \text{if } x=0\end{cases}$ and $s'(x) = -\sin x$. By chain rule, if $\cos x > 0$ ($\Leftrightarrow x\in\bigcup_{n\in\mathbb{Z}}\left((2n-\frac{1}{2})\pi, (2n+\frac{1}{2})\pi\right)$), then $g'(x) = (r\circ s)'(x) = r'(s(x))\cdot s'(x) = -\sin x$; if $\cos x<0$ ($\Leftrightarrow x\in\bigcup_{n\in\mathbb{Z}}\left((2n+\frac{1}{2})\pi, (2n+\frac{3}{2})\pi\right)$), then $g'(x) = (r\circ s)'(x) = r'(s(x))\cdot s'(x) = \sin x$. If $\cos x = 0$ ($\Leftrightarrow x = (2n\pm\frac{1}{2})\pi$, $n\in\mathbb{Z}$), then $\lim_{t\to x^+}\frac{|\cos t|-|\cos x|}{t-x} = \left|\lim_{t\to x^+}\frac{\cos t}{t-x}\right| = 1$, but $\lim_{t\to x^-}\frac{|\cos t|-|\cos x|}{t-x} = -\left|\lim_{t\to x^+}\frac{\cos t}{t-x}\right| = -1$, so $g'(x)$ doesn't exist.

Remarks Even $r'(0)$ doesn't exist, $(s\circ r)(x) = \cos|x| = \cos x$ has derivative $-\sin x$ everywhere!

(68) $$\frac{f(b_n)-f(a_n)}{b_n-a_n} - f'(c) = \left(\frac{f(b_n)-f(c)}{b_n-a_n} + \frac{f(c)-f(a_n)}{b_n-a_n}\right) - f'(c)\left(\frac{b_n-c}{b_n-a_n} + \frac{c-a_n}{b_n-a_n}\right)$$
$$= \frac{f(b_n)-f(c)}{b_n-c}\,\frac{b_n-c}{b_n-a_n} + \frac{f(c)-f(a_n)}{c-a_n}\,\frac{c-a_n}{b_n-a_n} - f'(c)\frac{b_n-c}{b_n-a_n} - f'(c)\frac{c-a_n}{b_n-a_n}$$
$$= \left(\frac{f(b_n)-f(c)}{b_n-c} - f'(c)\right)\underbrace{\frac{b_n-c}{b_n-a_n}}_{\le 1} + \left(\frac{f(c)-f(a_n)}{c-a_n} - f'(c)\right)\underbrace{\frac{c-a_n}{b_n-a_n}}_{\le 1}.$$
So $$\left|\frac{f(b_n)-f(a_n)}{b_n-a_n} - f'(c)\right| \le \underbrace{\left|\frac{f(b_n)-f(c)}{b_n-c} - f'(c)\right|}_{\to 0 \text{ as } n\to\infty}\cdot 1 + \underbrace{\left|\frac{f(c)-f(a_n)}{c-a_n} - f'(c)\right|}_{\to 0 \text{ as } n\to\infty}\cdot 1 \to 0$$
$\therefore \lim_{n\to\infty} \frac{f(b_n)-f(a_n)}{b_n-a_n} = f'(c)$.

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(69) $$f(x) = \begin{cases} x^3 & \text{if } x\ge 0 \\ -x^3 & \text{if } x<0 \end{cases} \Rightarrow f'(x) = \begin{cases} 3x^2 & \text{if } x>0 \\ -3x^2 & \text{if } x<0 \\ 0 & \text{if } x=0 \end{cases} \Rightarrow f''(x) = \begin{cases} 6x & \text{if } x>0 \\ -6x & \text{if } x<0 \\ 0 & \text{if } x=0 \end{cases} = 6|x| \text{ is continuous}$$
$|f'(0)| = \left|\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}\right| = \lim |x|^2 = 0$ $\quad |f''(0)| = \left|\lim_{x\to 0}\frac{f'(x)-f'(0)}{x-0}\right| = \lim_{x\to 0} 3|x| = 0 \Rightarrow f\in C^2(\mathbb{R})$.
$f'''(0) = \lim_{x\to 0}\frac{f''(x)-f''(0)}{x-0} = 6\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

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(70) $|f'(b)| = \left|\lim_{x\to b}\frac{f(x)-f(b)}{x-b}\right| = \lim_{x\to b}\left|\frac{f(x)-f(b)}{x-b}\right| \le \lim_{x\to b}|x-b| = 0$ for every $b\in\mathbb{R}$. So $f'\equiv 0$.
Therefore, f is a constant function. The same is true if 2 is replaced by $n>1$ because $\left|\frac{f(x)-f(b)}{x-b}\right| \le |x-b|^{n-1} \to 0$ as $x\to b$. However if 2 is replaced by 1, then it is not true as can be seen by taking $f(x)=x$, then $|f(a)-f(b)| = |a-b|$ and f is not constant.

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(71) Since f has roots at ± 1 with multiplicities n, so $f(\pm 1) = f'(\pm 1) = \cdots = f^{(n-1)}(\pm 1) = 0$.
Since $f(-1) = f(1) = 0$, by Rolle's theorem, there is $x_0\in(-1,1)$ such that $f'(x_0)=0$.
Then f' has at least three distinct roots $-1, x_0, 1$. By Rolle's theorem, f'' will have at least four distinct roots. Repeating this until the $(n-1)^{st}$ derivative, we see that $f^{(n-1)}$ will have at least $n+1$ distinct roots. So by Rolle's theorem, $f^{(n)}$ will have at least n distinct roots. Since $\deg f^{(n)} = n$, $f^{(n)}$ has exactly n distinct roots.

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(72) Let $g(x) = e^{-x}f(x)$, then $g'(x) = -e^{-x}f(x) + e^{-x}f'(x) = e^{-x}(f'(x)-f(x)) \le 0$.
So $g(x)$ is decreasing on $[0,\infty)$. Then $g(x)\le g(0) = f(0) = 0$ for $x\in[0,\infty)$.
So $f(x) = e^{x}g(x) \le 0$ for $x\in[0,\infty)$.

(73) We first show $x_n = f(\frac{1}{n})$ is a Cauchy sequence. For every $\varepsilon > 0$, let $K \in \mathbb{N}$ such that $K > \frac{2}{\varepsilon}$ (by Archimedian principle). Then $m, n \geq K \Rightarrow |x_m - x_n| = |f(\frac{1}{m}) - f(\frac{1}{n})| \underset{\text{mean-value theorem}}{=} |f'(x_0)||\frac{1}{m} - \frac{1}{n}|$ $\leq 2|\frac{1}{m} - \frac{1}{n}| \leq 2(\frac{1}{K} - 0) = \frac{2}{K} < \varepsilon$. $\Rightarrow 0 < \frac{1}{m}, \frac{1}{n} \leq \frac{1}{K}$

Next, to show $\lim_{x \to 0^+} f(x)$ exists, it is enough to show $\lim_{n\to\infty} f(t_n)$ exists for every $t_n \to 0$ in $(0, +\infty)$ by the remark following the sequential limit theorem. For every $t_n \to 0$ in $(0, +\infty)$, $\{t_n\}$ is a Cauchy sequence by Cauchy's theorem. We will show $\lim_{n\to\infty} f(t_n)$ exists by showing $\{f(t_n)\}$ is a Cauchy sequence. For every $\varepsilon > 0$, since $\{t_n\}$ is Cauchy, $\exists K_1 \in \mathbb{N}$ such that $m, n \geq K_1 \Rightarrow |t_m - t_n| < \frac{\varepsilon}{2} \Rightarrow |f(t_m) - f(t_n)| = |f'(y_0)(t_m - t_n)|$ $\leq 2|t_m - t_n| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$. So $\{f(t_n)\}$ is Cauchy, $\lim_{n\to\infty} f(t_n)$ exists by Cauchy's theorem.

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(74) For $0 < x < \frac{\pi}{2}$, consider the function $f: [0, x] \to \mathbb{R}$ defined by $f(t) = \ln(\cos t)$. Now f is continuous on $[0, x]$ and differentiable on $(0, x)$. By mean-value theorem, $|\ln(\cos x)| = |f(x) - f(0)| = |f'(t_0)(x - 0)| = |(-\tan t_0)x|$ for some t_0 on $(0, x)$. Now $\tan$ is strictly increasing on $(0, \frac{\pi}{2})$, $\tan t_0 < \tan x$. $\therefore |\ln(\cos x)| \leq (\tan t_0)x < x \tan x$.

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(75) Let $|f|$ has maximum value M on $[0, \frac{1}{2}]$. Since $|f|$ is continuous on $[0, \frac{1}{2}]$, so by extreme value theorem, $M = |f(w)|$ for some $w \in [0, \frac{1}{2}]$. By mean value theorem, there is $x_0 \in (0, w)$ such that $f(w) - f(0) = f'(x_0)(w - 0)$. Then $M = |f(w) - f(0)| \leq |f'(x_0)||w|$ $\leq |f(x_0)| \frac{1}{2} \leq \frac{M}{2}$. Since $0 \leq M \leq \frac{M}{2}$, we get $M = 0$. Then $f(x) = 0$ for all $x \in [0, \frac{1}{2}]$. Similarly, replacing $[0, \frac{1}{2}]$ by $[\frac{1}{2}, 1]$ and using $f(\frac{1}{2}) = 0$ instead of $f(0) = 0$, the argument above shows $f(x) = 0$ for all $x \in [\frac{1}{2}, 1]$. Keep on going, we will get $f(x) = 0$ for all $x \geq 0$.

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(76) Since $\lim_{h\to 0} f(x_0+h) + f(x_0-h) - 2f(x_0) = 0$ and $\lim_{h\to 0} h^2 = 0$, we consider using l'Hopital's rule

$$\lim_{h\to 0} \frac{f'(x_0+h) - f'(x_0-h)}{2h} = \frac{1}{2} \lim_{h\to 0} \left(\frac{f'(x_0+h) - f'(x_0)}{h} + \frac{f'(x_0-h) - f'(x_0)}{-h} \right) = \frac{1}{2}(f''(x_0) + f''(x_0)) = f''(x_0).$$

By l'Hopital's rule, $\lim_{h\to 0} \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2} = f''(x_0)$.

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(77) Since $\frac{d^4}{d\theta^4} \cos\theta = \cos\theta$, by Taylor's theorem, there is $\theta_0 \in (0, \theta)$ such that

$\cos\theta = 1 + 0(\theta - 0) - \frac{1}{2!}(\theta - 0)^2 + \frac{0}{3!}(\theta - 0)^3 + \frac{\cos\theta_0}{4!}(\theta - 0)^4$. Since $0 \leq \theta_0 \leq \theta \leq \frac{\pi}{2}$, so $0 \leq \cos\theta_0 \leq 1$.

Therefore, $1 - \frac{\theta^2}{2} \leq \cos\theta \leq 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$.