Tutorial class 1 Logic & Sets.

(Corresponds to lec 1 + lec 2).

I. Logic

I.1 Statements: denoted by p usually. ← notation: $\forall$, $\exists$.

E.g. ① 1 is a real number.
② 1 is not a real number.
③ $\exists x$ such that $x^2=1$.
④ $\forall x$, $\exists y$ such that $x=2y$.
⑤ $\exists y$, $\forall x$ such that $x=2y$.

I.2 Basic Operation:

For given (mathematical) statements p, q, we have following operations:

Rule: $\sim$($\exists x$ such that x satisfies q).
= $\forall x$, we have x satisfies $\sim q$
$\sim$($\forall x$, we have x satisfies q)
= ($\exists x$ such that x satisfies $\sim q$

① negation : $\sim p$.

② and : $p \wedge q$.

③ or : $p \vee q$.

Example1: $p: x \geq 0$ $q: x<1.$

$\sim p: x<0$ $\sim q: x\geq 1.$

$p \wedge q$: $x\geq 0$ and $x<1$. $p \vee q$: $x \geq 0$ or $x<1$. (i.e. $x \in \mathbb{R}$)

Example2: p: $\exists x>0$, such that $x^2=1$.

q: $\forall x \leq 0$, we have $x^2 \neq 1$.

Finding $\sim p$: define a new statement w: $x^2=1$.

then $p = \exists x>0$ such that x satisfies w.

thus $\sim p = \forall x>0$ we have x satisfies $\sim w$. $x^2 \neq 1$.

$\sim p = \forall x>0$ we have $x^2 \neq 1$.

(1 min) Question: What is $\sim q$?

Answer:

$(\sim p) \wedge q$: $\forall x>0$, we have $x^2 \neq 1$ and $\forall x \leq 0$, we have $x^2\neq 1$.

$\Vert$: $\forall x$, we have $x^2 \neq 1$.

$p \wedge (\sim q)$: $\exists x>0$ such that $x^2=1$ and $(\sim q)$.

$\Vert$ Quesition (1 min)

?: $\exists x$, such that $x^2=1$.

Answer:

I.3 Rules of operations

① $\sim(\sim p) = p$

② $\sim(p \wedge q) = (\sim p) \vee (\sim q)$

③ $\sim(p \vee q) = (\sim p) \wedge (\sim q)$.

Example 3: p: $\exists x > 0,\ x^2 = 1$.

q: $\exists x \leq 0,\ x^2 = 1$.

$p \vee q = \exists x > 0, x^2 = 1$ or $\exists x \leq 0, x^2 = 1$. $\exists x,\ x^2 = 1$.

$\sim(p \vee q)$

approach1: $= \sim(\exists x > 0,\ x^2 = 1)$ and $\sim(\exists x \leq 0,\ x^2 = 1)$

$= \forall x > 0,\ x^2 \neq 1$ and $\forall x \leq 0,\ x^2 \neq 1$

approach2: $= \sim(\exists x,\ x^2 = 1) = \forall x,\ x^2 \neq 1$.

Question: p: $\forall x > 0,\ x^2 = 1$. q: $\forall x \leq 0,\ x^2 = 1$.

$\sim(p \vee q) = ?$ Answer.

I.4 Conditional Structure of Statements.

For statements p, q, we can consider a new statement w, written as

w: If p, then q. Notation: $w: p \Rightarrow q$.

① Conditional structure can be represented using 'negation' and 'or':

$$p \Rightarrow q = (\sim p) \vee q.$$

② As a corollary:

$$\sim (p \Rightarrow q) = \sim ((\sim p) \vee q)$$
$$= p \wedge (\sim q).$$

Converse statement of $p \Rightarrow q$ is defined similar as $q \Rightarrow p$. and by ① we have

$$q \Rightarrow p = (\sim q) \vee p \neq p \wedge (\sim q) = \sim (p \Rightarrow q) \;!!$$

Converse is different with negate!

and we can discuss the converse statement only when the statement has conditional structure!

③ (Contrapositive statement):

$$p \Rightarrow q = (\sim q) \Rightarrow (\sim p).$$

Proof: By def $(\sim q) \Rightarrow (\sim p) = \sim(\sim q) \vee (\sim p) = q \vee (\sim p)$
$= p \Rightarrow q$

Example 4.

$p: x=1.$ $\quad$ $q: x^2=1.$

$\forall x, p \Rightarrow q$: $\forall x$, if $x=1$, then $x^2=1$. ← true

Or equivalently: $= (\sim p) \vee q = (x \neq 1$ or $x^2=1)$

$\forall x, q \Rightarrow p$: $\forall x$, if $x^2=1$, then $x=1$. ← wrong

$= (\sim q) \vee p = (x^2 \neq 1$ or $x=1)$

$\sim(\forall x, p \Rightarrow q)$: $\exists x$, such that $\sim(p \Rightarrow q)$

$= \exists x$ such that $x=1$ and $x^2 \neq 1$. wrong.

Question: Write down $\sim q \Rightarrow \sim p$. Whether it is true or false?

Answer:

II. Set theory

II.1 Basic def of a set.

Def: a set is a collection of objects, and we say the objects in the set are the elements of the set.

Classical paradox: $A = \{x : x \text{ not in set } A\}$.

To avoid such problem, we require that any set A should satisfy:

$\forall x$, the statement $x \in A$ should be either true or false.

$\begin{cases} \text{empty set} \\ \text{finite set} \\ \text{infinite set} \end{cases}$

How to describe a set? $\begin{cases} A = \{x : x \text{ satisfies } p\}. \\ A = \{x_1, \cdots, x_n\}. \\ A = \{1, 2, 3, 4, 5, \cdots\} \end{cases}$

describe the property of elements.

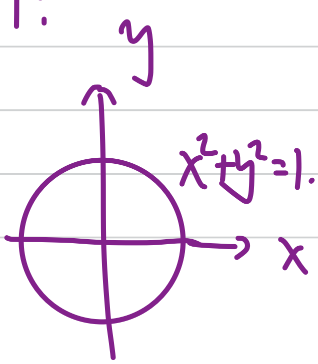
list all elements

Example 0: $\mathbb{R} := \{x : x \text{ is a real number}\}$

$\mathbb{Z} := \{x : x \text{ is a integer}\}$ or $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$

$[a, b] := \{x : \underbrace{x \in \mathbb{R}}_{p} \underbrace{\text{ and }}_{\wedge} \underbrace{a \le x \le b}_{q}\}$

Example 1:



$S := \{(x, y) : x^2 + y^2 = 1,\ x, y \in \mathbb{R}\}.$

II.2 Relation of sets.

For two sets A, B. we say

① A is a subset of B ($A \subseteq B$), if

$\forall x \in A$, we have $x \in B$ holds.

② $A = B$ if $A \subseteq B$ and $B \subseteq A$.

③ A is a proper subset of B ($A \subset B$), if $A \subseteq B$ and $A \neq B$.

Example 2: $A = \mathbb{Z}$, $B = \mathbb{R}$, then $A \subset B$.

$A = \mathbb{Z}$, $B = \mathbb{R}_+ := \{x : x \in \mathbb{R}, x \geq 0\}$, then we have neither $A \subseteq B$ nor $B \subseteq A$.

II.3 Power set

a collection of sets

Def of power set: Let S be a set, then power set of S, denoted by 2^S or $P(S)$, is given by

$$2^S := \{A : A \text{ is a subset of } S\}$$

[30s] Question: $S = \{1\}$. $2^S = \{\phi, 1\}$ or $\{\phi, \{1\}\}$?

II.4 Operations of sets

Consider sets $A_1, A_2, \cdots$

① Union: $\bigcup_{n=1}^{+\infty} A_n := \{x : x \text{ is an element of } A_i \text{ for some } i\}$

② Intersection: $\bigcap_{n=1}^{+\infty} A_n := \{x : x \text{ is an element of } A_i \text{ for all } i\}$

③ Cartesian product:

$$\prod_{n=1}^{+\infty} A_n = A_1 \times \cdots \times A_n \times \cdots = \{(x_1, \cdots, x_n, \cdots) : x_i \in A_i \text{ for all } i\}$$

④ Complement of A_2 in A_1:

$$A_1 \setminus A_2 := \{x : x \in A_1, x \notin A_2\}$$

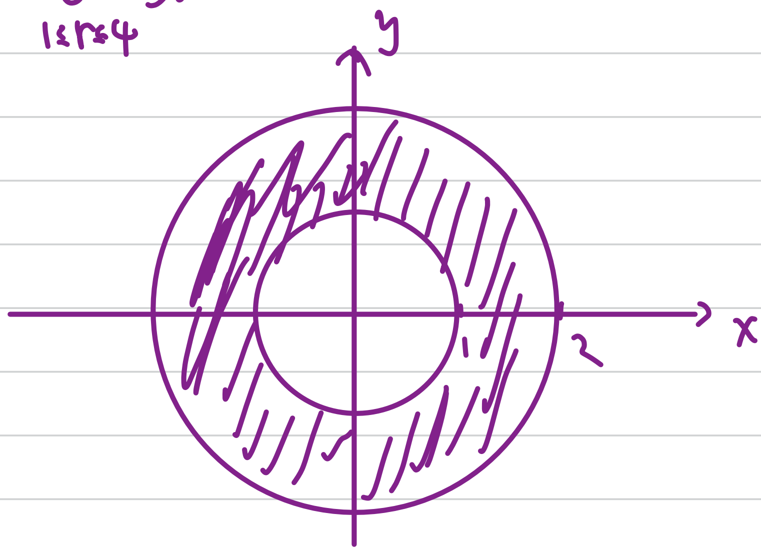
Example 3:

Denote $A := \{n \in \mathbb{Z} : 1 < n < 5\}$ $B := \{2m : m \in \mathbb{Z}, m \geq 0\}$.
then compute $A \cap B$, $A \cup B$, $A \setminus B$, $B \setminus A$:

$A \cap B = \{2, 4\}$, $A \cup B = \{2, 3, 4, 6, 8, 10, 12, \cdots\}$

$A \setminus B = \{3\}$ $B \setminus A = \{0, 6, 8, 10, \cdots\}$

Example 4: denote $S_r := \{(x,y) : x^2+y^2 = r^2\}$, then plot the graph of $\bigcup_{1\le r\le 4} S_r$:



Example 5: Show that for any sets A, B, C, we have

$$(A\setminus B)\setminus C = (A\setminus C)\setminus B.$$

we have

$x \in (A\setminus B)\setminus C$ if and only if

$x \in A\setminus B$ and $x \notin C$ if and only if

$x \in A$ and $x \notin B, x \notin C$ if and only if

$x \in A\setminus C$ and $x \notin B$ if and only if

$x \in (A\setminus C)\setminus B$.

II.5 functions

Def: Given two sets A, B, we say f is a function from A to B, if f is a rule that assign every $a \in A$ exactly to a value $b \in B$, such b is called the value of f at a denoted by $f(a)$.

domain: A

codomain: B

range: $= \{ f(x) : x \in A \}$

graph: $= \{ (x, f(x)), x \in A \}$

Properties:

We say f is

① injective : if $\forall a, a' \in A, a \neq a'$ we have $f(a) \neq f(a')$

② surjective : if $\forall b \in B$, we can find $a \in A$ such that $f(a) = b$.

range $(f) = B$.

③ bijective : if f is both injective and surjective

Example 6:

Consider $A = \mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$

$B = \mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$.

$f(x) = x^2$, then

① whether f is surjective?

② whether f is injective?

③ what about ①, ② when $A = \mathbb{R}$?

Answer:

Example 7:

Suppose A, B are subsets of $\mathbb{R}$ and $f: A \to B$ is a function.

If for every $b \in B$, the horizontal line given by

$\{(x,y): y=b\}$ satisfies it intersect with graph(f) at least once, show that f is surjective:

By the claim, we have $\forall b \in B$, there exists a point

$(x_0, y_0) \in \{(x,y): y=b\} \cap \underbrace{\{(x, f(x)): x \in A\}}_{\text{graph}(f)}$, by

$(x_0, y_0) \in$ graph(f), we have

$$(x_0, y_0) = (x_0, f(x_0)), \text{ thus } y_0 = f(x_0). \quad (*1)$$

On the other hand,

$$(x_0, y_0) \in \{(x,y): y=b\} \text{ implies that } y_0 = b \quad (*2).$$

Combining $(*1)$, $(*2)$, we get $f(x_0) = b$.

By b is aribtrary, we get f is surjective.

Quesition: If "at least once" is replaced by "at most once", show that f is injective.