Lecture 21, 25-04-2019

Chapter 8 Riemann integral

Focus on proper integral.

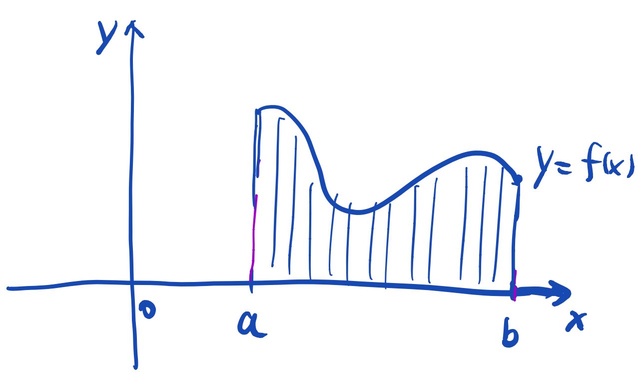
$f: [a,b] \to \mathbb{R}$, $a < b$, $a, b \neq \infty$.

f is bounded, say $\exists K > 0$ s.t $|f(x)| \leq K$, $\forall x \in [a,b]$.

f may be discontinuous on $[a,b]$.

We may further assume that $f \geq 0$.

We want to compute the "area" under the graph of f



Three principals for defining "area":

① The area of a rectangle with two sides of length L_1 and L_2 is $L_1 \cdot L_2$.

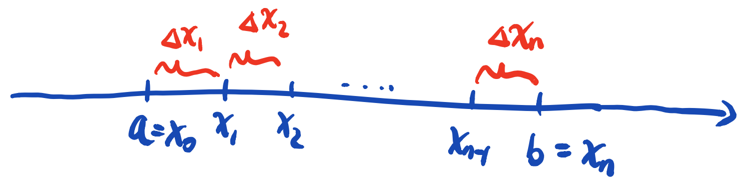
② If $A \subseteq B$, then

Area of $A \leq$ Area of B.

③ If $A \cap B = \phi$, then

Area of $A \cup B$ = Area of A + Area of B

Partion of $[a,b]$:



Def : $P = \{ x_0, x_1, \cdots x_n \}$ is a partition of $[a,b]$

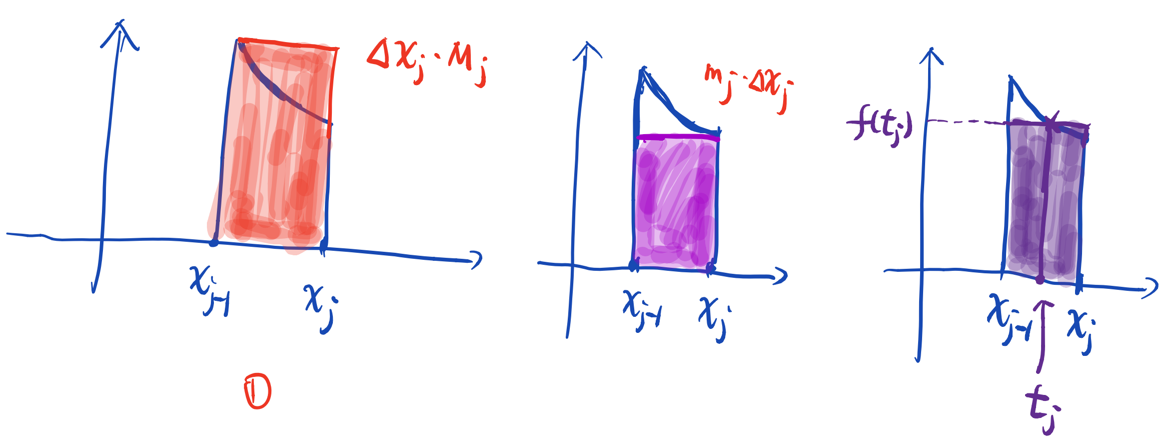
iff $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

Let $\Delta x_j = x_j - x_{j-1}$, $1 \le j \le n$.

$\|P\| = \max\{ \Delta x_1, \cdots \Delta x_n \}$ is called the mesh size of P.

The partion P divides the whole area into small pieces.

Three ways to approximate the area of each small piece



More precisely,

On each interval $[x_{j-1}, x_j]$, we define

$$m_j = \inf \{ f(x) : x \in [x_{j-1}, x_j] \},$$

$$M_j = \sup \{ f(x) : x \in [x_{j-1}, x_j] \}$$

1842-1917

$L(f, P) = \sum_{j=1}^{n} m_j \Delta x_j$: the lower Darboux sum w.r.t P

$U(f, P) = \sum_{j=1}^{n} M_j \Delta x_j$: the upper Darboux sum w.r.t P.

One may also take an arbitary $t_j \in [x_{j-1}, x_j]$ for $1 \leq j \leq n$

1826-1866

and define $S = \sum_{j=1}^{n} f(t_j) \Delta x_j$: Riemann sum w.r.t P.

It's clear that $L(f, P) \leq S \leq U(f, P)$

The true area A, if exists, should satisfy

$$L(f, P) \leq A \leq U(f, P) \quad \text{for all } P.$$

There are two approaches to define the area A, or the integral $\int_a^b f(t)\,dt$:

$$\begin{cases} \text{Darboux integral} \\ \text{Riemann integral} \end{cases}$$

Later, we will see that the two are actually equivalent, i.e

Darboux integral = Riemann integral

$\downarrow$

easier to understand

and easier to use.

Darboux Integral

Darboux's definition:

Let $\underline{\int_a^b} f(x)dx = \sup \{ L(f,P) : P \text{ is a partition of } [a,b] \}$

called the lower integral of f on $[a,b]$

$\overline{\int_a^b} f(x)dx = \inf \{ U(f,P) : P \text{ is a partition of } [a,b] \}$

called the upper integral of f on $[a,b]$

Def: f is Darboux Integrable iff $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$

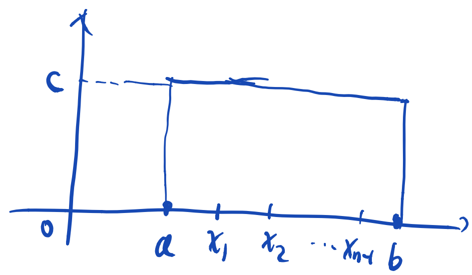
In that case, we write $\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$

If $b \leq a$, we define $\int_a^b f(x)dx = -\int_b^a f(x)dx$

In particular, $\int_a^a f(x)dx = 0$.

Examples

Example 1. $f(x) \equiv c$ on $[a,b]$ is integrable.



Solution: $\forall$ partition $P = \{x_0, \cdots x_n\}$. on any $[x_{j-1}, x_j]$,

$$m_j = M_j = c \qquad m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$$

$$\Rightarrow L(f,P) = \sum m_j \Delta x_j = \sum c \Delta x_j = C(b-a)$$

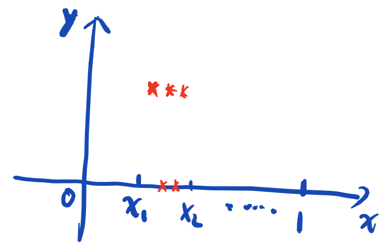
$$U(f,P) = \sum M_j \Delta x_j = \sum c \Delta x_j = c(b-a)$$

$$\Rightarrow \underline{\int_a^b} f(x)dx = \sup\{L(f,P) : P \text{ is a partion of } [a,b]\} = c(b-a)$$

$$\overline{\int_a^b} f(x)dx = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\} = C(b-a) = \underline{\int_a^b} f(x)dx$$

$\Rightarrow f(x) \equiv c$ is integrable

Example 2. $f: [0,1] \to \mathbb{R}$ defined by



$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \bar{\in} \mathbb{Q} \end{cases}$$

f is NOT integrable on $[0,1]$. The reason is below:

$\forall$ partion $P = \{x_0, \cdots x_n\}$ on $[0,1]$

On any $[x_{j-1}, x_j]$, $m_j = 0$ by the density of irrational numbers, $M_j = 1$ by the density of rational numbers

$m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$

$M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\}$

$$\Rightarrow L(f, P) = \sum m_j \Delta x_j = 0$$

$$U(f, P) = \sum M_j \Delta x_j = \sum \Delta x_j = 1$$

$$\Rightarrow \underline{\int_a^b} f(x)\,dx = \sup\{L(f,P) : P \text{ is a partition}\} = 0$$

$$\overline{\int_a^b} f(x)\,dx = \inf\{U(f,P) : P \text{ is a partition}\} = 1 \neq \underline{\int_a^b} f(x)\,dx$$

Properties of Partition

Def ①: For partitions P, P', we say P' is a refinement of P (or P' is finer than P) iff $P \subseteq P'$

②: For partition P_1, P_2, we call $P_1 \cup P_2$ the common refinement of P_1 and P_2.

Example: $[a, b] = [0, 1]$ $\quad P = \{0, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \cdots \frac{10}{10}\}$

$$P' = \{0, \frac{1}{20}, \frac{2}{20}, \frac{3}{20}, \cdots \frac{20}{20}\}$$

then $P \subset P'$, P' is a refinement of P.

Let $P_1 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $P_2 = \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}\}$

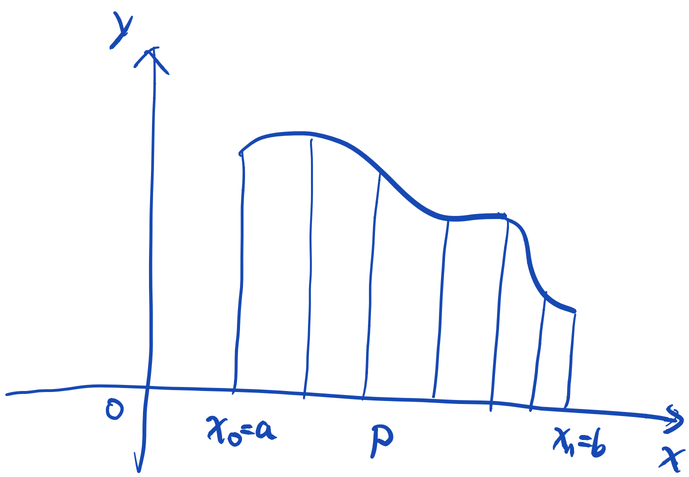
then $P_1 \cup P_2 = \{0, \frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{3}{5}, \frac{2}{3}, \frac{4}{5}, 1\}$

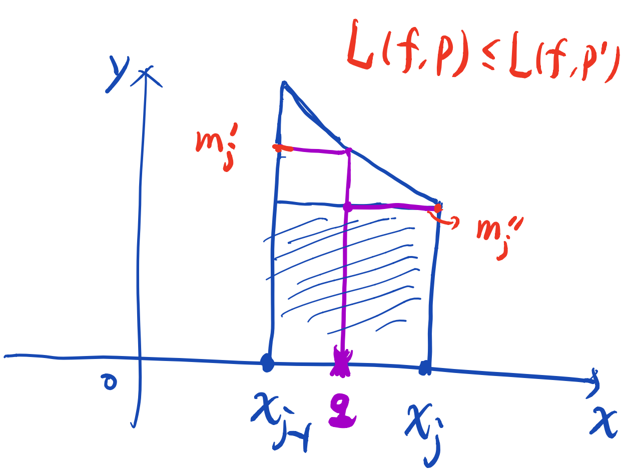
is the common refinement of P_1 and P_2

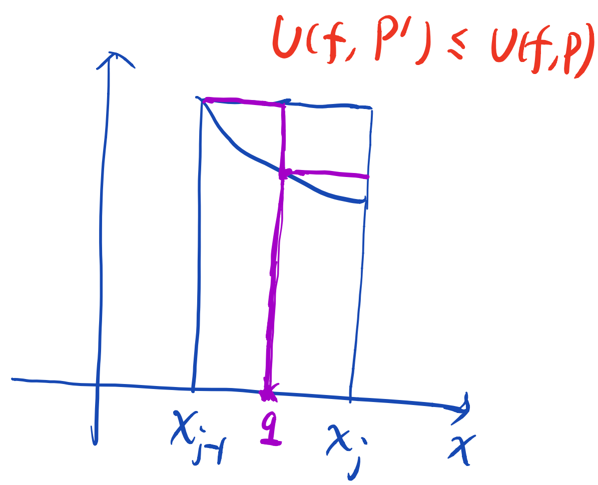
Refinement Theorem (finer partions gives better bound of area)

THM: If $P \subset P'$, then

$$L(f,P) \le L(f,P') \le U(f,P') \le U(f,P) \quad \cdots (*)$$







proof: Let $P = \{x_0 = a, x_1, x_2, \cdots x_n = b\}$,

$P' = P \cup Q$ where $Q \cap P = \phi$ and

$Q = \{q_1, \cdots q_m\}$ for some $m \in \mathbb{N}$.

Let $P_1 = P \cup \{q_1\}$, we show that $(\star)$ hold with P' being replaced by P_1. Indeed, since $q_1 \in [a,b]$, we may assume that $x_{j-1} < q_1 < x_j$ for some j.

then $P_1 = \{x_0, x_1, \cdots x_{j-1}, q_1, x_j, x_{j+1}, \cdots x_n\}$.

Let $m_j' = \inf\{f(x) : x \in [x_{j-1}, q_1]\}$

$m_j'' = \inf\{f(x) : x \in [q_1, x_j]\}$.

Since $[x_{j-1}, q_1]$, $[q_1, x_j]$ are subset of $[x_{j-1}, x_j]$

$m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\} \le m_j'$

$m_j \le m_j''$.

$$\Rightarrow \quad L(f,P) = \sum_{k=1}^{n} m_k \Delta x_k = \sum_{k=1}^{j-1} m_k \Delta x_k + m_j \Delta x_j + \sum_{k=j+1}^{n} m_k \Delta x_k$$

$$L(f,P_1) = \sum_{k=1}^{j-1} m_k \Delta x_k + \boxed{m_j'(q_1 - x_{j-1}) + m_j''(x_j - q_1)} + \sum_{k=j+1}^{n} m_k \Delta x_k$$

$$\geq L(f,P)$$

Continuing this argument,

$$L(f,P_1) \leq L(f,P_2) \leq \cdots \leq L(f,P_m) = L(f,P')$$

where $P_2 = P_1 \cup \{q_2\}$, $\cdots$ $P_k = P_{k-1} \cup \{q_k\}$ for $1 \leq k \leq m$.

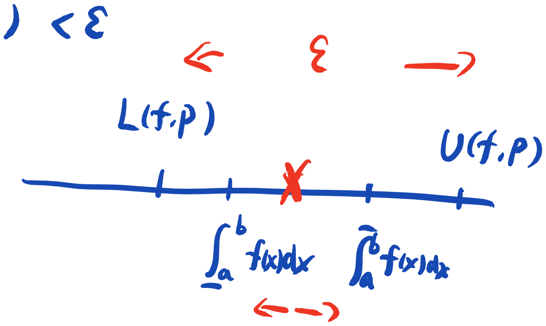
Therefore $L(f,P) \leq L(f,P')$.

Similarly, we can show that $U(f,P') \leq U(f,P)$

Integral criterion

THM: Let f be bounded on $[a,b]$. f is Darboux integrable on $[a,b]$ $\Leftrightarrow$ $\forall \varepsilon>0$, $\exists$ partition P of $[a,b]$ s.t

$$U(f,P)-L(f,P)<\varepsilon$$



Pf:

$\Leftarrow$ $\forall \varepsilon>0$,

$$\underline{\int_a^b} f(x)\,dx \geq L(f,P)$$

$$\overline{\int_a^b} f(x)\,dx \leq U(f,P)$$

$$\Rightarrow \quad 0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq U(f,P)-L(f,P) < \varepsilon$$

By infenitesimal principal, $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx = 0$

$\Rightarrow$ f is D-integrable

$(\Rightarrow)$ f is D-integrable $\Rightarrow$ $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)\,dx$.

$\forall \varepsilon > 0$, by Supremum property, $\exists P_1$ s.t

$$\underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} < L(f, P_1) \leq \underline{\int_a^b} f(x)\,dx$$

by Infimum property, $\exists P_2$ s.t

$$\overline{\int_a^b} f(x)\,dx \leq U(f, P_2) < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2}$$

Take $P = P_1 \cup P_2$. Then the refinement thm $\Rightarrow$

$$L(f, P_1) \leq L(f, P) \leq U(f, P) \leq U(f, P_2)$$

$$\begin{aligned}
\Rightarrow \quad U(f,P) - L(f,P) &\leq U(f, P_2) - L(f, P_1) \\
&< \overline{\int_a^b} f(x) + \frac{\varepsilon}{2} - \left(\underline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} \right) \\
&= \overline{\int_a^b} f(x) - \underline{\int_a^b} f(x) + \varepsilon \\
&= \varepsilon
\end{aligned}$$