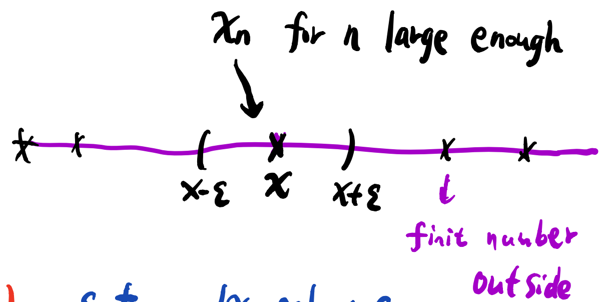
Lecture 9 07-03-2019

Review:

(1) $\lim_{n\to\infty} x_n = x$ iff



① $\forall \varepsilon > 0$, $\exists N$ (depending on ε), s.t $|x_n - x| < \varepsilon$.

for all $n \geq N$

② $\forall \varepsilon > 0$, $\exists N$ (depending on ε), and M (independent of ε),

s.t $|x_n - x| < M\varepsilon$. for all $n \geq N$

③ $\forall \varepsilon > 0$, the number of x_n's such that

$|x_n - x| \geq \varepsilon$ is finite

④ $\lim_{n\to\infty} (x_n - x) = 0$ ⑤ $\lim_{n\to\infty} |x_n - x| = 0$

(2) $\lim_{n\to\infty} x_n$ is unique if $\{x_n\}$ converges.

(3) $\{x_n\}$ converges $\Rightarrow$ $\{x_n : n \geq 1\}$ is bounded

Computation Rule of limit

THM : ① $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ $\Rightarrow$ $\lim_{n\to\infty} x_n + y_n = x + y$

② $c \in \mathbb{R}$, $\lim_{n\to\infty} x_n = x$ $\Rightarrow$ $\lim_{n\to\infty} c\, x_n = c \lim_{n\to\infty} x_n$

Proof of ① : Since $\lim x_n = x$, $\lim y_n = y$

$\forall \varepsilon > 0$, $\exists K_1 \in \mathbb{N}$ s.t $|x_n - x| < \varepsilon/2$ for all $n \geq K_1$.

$\exists K_2 \in \mathbb{N}$ s.t $|y_n - y| < \varepsilon/2$ for all $n \geq K_2$

take $K = \max(K_1, K_2)$, then for $n \geq K$,

we have $|x_n + y_n - x - y| \leq |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

So $\lim (x_n + y_n) = x + y$

② Exercise : two cases : $c = 0$, $c \neq 0$.

Corollary : $\lim_{n\to\infty} (x_n - y_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$

Example: Show that $\left\{ \frac{2n}{n+5} + \frac{n^8}{n^8+1} \right\}$ Converges to 3.

Idea: $x_n = \frac{2n}{n+5} = 2 - \frac{10}{n+5}$

$y_n = \frac{n^8}{n^8+1} = 1 - \frac{1}{n^8+1}$

We need to show that $\lim_{n\to\infty} x_n = 2$ $\lim_{n\to\infty} y_n = 1$

or $\lim_{n\to\infty} \frac{10}{n+5} = 0$. $\lim_{n\to\infty} \frac{1}{n^8+1} = 0$

THM: ③ $\lim_{n\to\infty} x_n = 0$ $\{y_n\}$ is bounded,

$$\Rightarrow \lim_{n\to\infty} x_n y_n = 0.$$

④ $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ $\Rightarrow \lim_{n\to\infty} x_n y_n = xy$

proof of ③. Since $\{y_n\}$ is bounded, $\exists M$ s.t

$|y_n| \leq M \quad \forall n \geq 1.$

Since $\lim_{n\to\infty} x_n = 0$, $\forall \varepsilon > 0$, $\exists K$ s.t $|x_n| < \varepsilon$

for $n \geq K$.

Therefore $|y_n \cdot x_n| < M \cdot \varepsilon$ for all $n \geq K$. So $\lim x_n y_n = 0$.

proof of ④. $x_n y_n - xy = x_n y_n - x_n y + x_n y - xy$

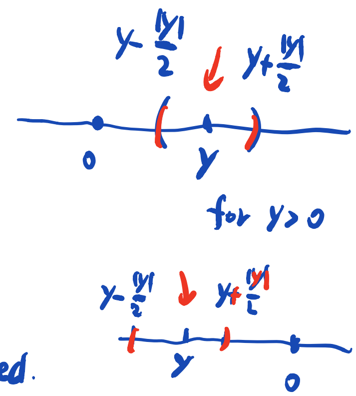
$= x_n(y_n - y) + (x_n - x)y$

Since $\{x_n\}$ converges, $\{x_n : n \geq 1\}$ is bounded, by ③ $\lim x_n(y_n - y) = 0$.

So $\lim(x_n y_n - xy) = \lim x_n(y_n - y) + \lim(x_n - x)y = 0 + 0 = 0$.

THM : ⑤ let $y_n \neq 0$, $\forall\, n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = y$, $y \neq 0$

then $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{y}$.



Proof : $\frac{1}{y_n} - \frac{1}{y} = \frac{y-y_n}{y y_n} = \frac{1}{y} \cdot (y-y_n) \cdot \frac{1}{y_n}$

We first show that $\{ \frac{1}{y_n} \}$ is bounded.

Since $\lim y_n = y$. for $\varepsilon = \frac{|y|}{2}$, $\exists\, K$ s.t

$|y_n - y| < \varepsilon = \frac{|y|}{2}$ for all $n \geq K$.

$\Rightarrow$ $|y_n| > \frac{|y|}{2}$ for all $n \geq K$. [Since $|y_n| > |y| - |y-y_n|$
$> |y| - \frac{|y|}{2}$]

$\Rightarrow$ $\frac{1}{|y_n|} < \frac{2}{|y|}$ for all $n \geq K$.

Set $M = \max \{ \frac{2}{|y|}, \frac{1}{|y_1|}, \frac{1}{|y_2|}, \cdots \frac{1}{|y_K|} \}$

then $|\frac{1}{y_n}| \leq M$ for all $n \geq 1$. $\Rightarrow$ $\{ \frac{1}{y_n} \}$ is bounded

As a result, $\lim (y_n - y) \frac{1}{y_n} = 0$, $\lim y \cdot (y_n - y) \frac{1}{y_n} = 0$

So $\lim (\frac{1}{y_n} - \frac{1}{y}) = 0$

THM : 6 $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$.

$y \neq 0$, $y_n \neq 0$ $\forall n \geq 1$

Then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$

7 $\lim_{n\to\infty} x_n = 0$, $|y_n| \leq |x_n|$

Then $\lim_{n\to\infty} y_n = 0$

Sandwich Theorem (Squeeze limit theorem)

THM: If 1 $\forall n \in \mathbb{N}$, $x_n \le w_n \le y_n$

and 2 $\lim x_n = \lim y_n = z$

then $\lim\limits_{n\to\infty} w_n = z$.

Proof: $w_n = x_n + w_n - x_n$

$x_n \le w_n \le y_n \Rightarrow |w_n - x_n| \le |y_n - x_n|$

Since $\lim y_n = z = \lim x_n$, $\lim (y_n - x_n) = 0$

$\Rightarrow \lim |y_n - x_n| = 0 \Rightarrow \lim |w_n - x_n| = 0$ (by 7)

$\Rightarrow \lim (w_n - x_n) = 0$

$\Rightarrow \lim w_n = \lim x_n + \lim (w_n - x_n) = z + 0 = z$

Example : let $w_n = \dfrac{[10^n\sqrt{2}]}{10^n} \in \mathbb{Q}$ for $n \geq 1$.

Show that $\lim_{n\to\infty} w_n = \sqrt{2}$.

[Note that $\sqrt{2} = 1.4142\cdots$

$$w_1 = \frac{[14.142]}{10} = 1.4 \quad , \quad w_2 = \frac{[141.42]}{100} = 1.41$$

$$w_3 = 1.414.$$

Actually $\sqrt{2} = \sum_{n=0}^{\infty} a_n 10^{-n}$, $\quad w_k = \sum_{n=0}^{n=k} a_n 10^{-n}$]

We have $10^n\sqrt{2} - 1 < [10^n\sqrt{2}] \leq 10^n\sqrt{2}$

$$\Rightarrow \frac{10^n\sqrt{2}-1}{10^n} < \frac{[10^n\sqrt{2}]}{10^n} \leq \frac{10^n\sqrt{2}}{10^n}$$

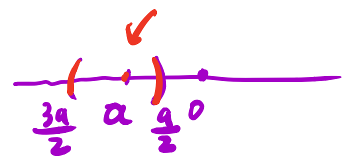
$$\Rightarrow \sqrt{2} - 10^{-n} < w_n \leq \sqrt{2}$$

Since $\lim \sqrt{2} - 10^{-n} = \sqrt{2}$. by sandwich thm, $\lim w_n = \sqrt{2}$.

Limit inequality

THM: (1) If ① $a_n \geq 0, \forall n \geq 1$. ② $\lim\limits_{n\to\infty} a_n = a$.

then $a \geq 0$



(2). If ① $x_n \leq y_n, \forall n \geq 1$. ② $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$

then $x \leq y$.

(3) If ① $a \leq x_n \leq b, \forall n \geq 1$, ② $\lim\limits_{n\to\infty} x_n = x$

then $a \leq x \leq b$. [Exercise]

Proof: (1) By contradiction, assume that $a < 0$. Since $\lim a_n = a$.

for $\varepsilon = -\frac{a}{2}$, $\exists K$, s.t $|a_n - a| < -\frac{a}{2}$ for all $n \geq K$

$\Rightarrow a_n < a - \frac{a}{2} = \frac{a}{2} < 0$, for all $n \geq K$.

Contradicts to $a_n \geq 0 \; \forall n \geq 1$. Therefore $a \geq 0$

(2) Let $w_n = y_n - x_n$, then $w_n \geq 0$. Since $\lim w_n = \lim y_n - \lim x_n = y - x$

So $y - x \geq 0 \Rightarrow x \leq y$

Supremum Limit Theorem

THM: Let c be an upper bound of a non-empty set S. Then

$$\left(\exists\, w_n \in S \text{ s.t } \lim_{n\to\infty} w_n = c \right) \Leftrightarrow c = \sup S.$$

Proof: $\Rightarrow$ Since $w_n \in S$, So $w_n \le \sup S \le c$

Since $\lim_{n\to\infty} w_n = c$, by the limit equality, $c \le \sup S$

But $\sup S \le c \Rightarrow \sup S = c$

$\Leftarrow$ Since $c = \sup S$. by supremum property, $\forall n \in \mathbb{N}$,

$\exists\ w_n \in S$ s.t $c - \frac{1}{n} \le w_n \le c$.

Since $\lim c - \frac{1}{n} = c$. By the sandwich theorem

$\lim w_n = c$

Infimum Limit Theorem

THM: Let c be a lower bound of a nonempty set S. Then

$$\left(\exists\, w_n \in S \text{ s.t } \lim_{n\to\infty} w_n = c \right) \Leftrightarrow c = \inf S.$$

Proof: Similar to Supremum Limit Theorem.

Example: ① $S = \{ \frac{1}{n} : n \in \mathbb{N} \}$

$0 \le \frac{1}{n}, \forall n \in \mathbb{N} \Rightarrow 0$ is a lower bound of S.

$\lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow \inf S = 0$

② $S = \{ x\pi + \frac{1}{y} : x \in \mathbb{Q} \cap [0,1],\ y \in [1,2] \}$

$\forall\ x \in \mathbb{Q} \cap [0,1],\ y \in [1,2],\ x\pi + \frac{1}{y} \ge 0 \cdot \pi + \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \frac{1}{2}$ is a lower bound of S. On the other hand,

$w_n = \frac{1}{n} \cdot \pi + \frac{1}{2} \in S$, $\lim_{n\to\infty} w_n = \frac{1}{2}$ $\therefore \inf S = \frac{1}{2}$.

③ Let A, B be bounded sets in $\mathbb{R}$.

Let $A-2B = \{ a-2b : a \in A, b \in B \}$

Prove that $\sup(A-2B) = \sup A - 2 \inf B$.

Proof: Step 1. Since A, B are bounded, $\sup A$, $\inf B$ exists.

$\forall a \in A, b \in B, \quad a \le \sup A, \quad b \ge \inf B$

Thus $a-2b \le \sup A - 2\inf B$

So $\sup A - 2\inf B$ is an upper bound of $A-2B$.

Step 2. by Supremum / Infimum limit Theorem,

$\exists\ a_n \in A$ s.t $\lim_{n\to\infty} a_n = \sup A$

$\exists\ b_n \in B$ s.t $\lim_{n\to\infty} b_n = \inf B$.

Then $a_n - 2b_n \in A-2B$ and $\lim_{n\to\infty} a_n - 2b_n = \sup A - 2\inf B$

By Supremum Limit theorem, we have $\sup(A-2B) = \sup A - 2\inf B$