MATH202 Introduction to Analysis (2007 Fall & 2008 Spring)

Tutorial Note #8

Real Number (Part 2)

Note: For the theorems covered in Chapter 5, please refer to P.1 of tutorial note #7

Part 1: Finding Supremum and infinmum of a given set (Review)

> Example 1
>
> Find the infinmum and supremum of the set
>
> $$S = \left\{\frac{1}{p} + \frac{2}{q} + \frac{3}{r} : p, q, r \in \mathbf{N}\right\}$$

(Step 1: Find the upper bound and lower bound)

Note that $\frac{1}{p} + \frac{2}{q} + \frac{3}{r} \leq \frac{1}{1} + \frac{2}{1} + \frac{3}{1} = 6$ (Since $p, q, r \geq 1$)

Which gives the upper bound is 6.

Since $\frac{1}{p} + \frac{2}{q} + \frac{3}{r} > 0$, so one possible lower bound is 0

(Step 2)

Since the upper bound 6 can be achieved by an elements in S $(\text{Put } p = q = r = 1, \text{we have} \frac{1}{p} + \frac{2}{q} + \frac{3}{r} = 6)$. Hence the supreme is 6.

Next, note that 0 is lower bound of S,

(to show $\inf S = 0$, here we use infimum limit theorem)

Pick $w_k = \frac{1}{k} + \frac{2}{k} + \frac{3}{k} \in S$ (Just simply set $p = q = r = k$)

Then by simple computation gives $\lim_{k\to\infty} w_k = 0$

Therefore by infimum limit theorem, we conclude $\inf S = 0$

Difficult Situations

The set involves set operations such as intersection, union or complement.

> Example 2
>
> Find the infimum and supremum of the set
>
> $$S = \left\{1 + \frac{2}{n} : n \in \mathbf{N}\right\} \cup \left\{\frac{4 - 2q}{q} : q \in \mathbf{Q} \cap (1, \infty)\right\}$$

(Step 1) IDEA: We first find out the upper bound and lower bound of each set and combine them

	Upper Bound	Lower Bound
$A = \left\{1 + \frac{2}{n} : n \in \mathbf{N}\right\}$	3 (when $n = 1$)	1 (when $n \to \infty$)
$B = \left\{\frac{4 - 2q}{q} : q \in \mathbf{Q} \cap (1, \infty)\right\}$	2 (when $q = 1$)	-2 (when $q \to \infty$)

For $S = A \cup B$,

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                  -2                                   1          2              3
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We can see all elements in $A \cup B$ lies between -2 and 3. Therefore the lower bound and upper bound is -2 and 3 respectively.

(Step 2)

To show $\inf S = -2$

From the graph, we see this lower bound is achieved by elements in B. Since every elements in B must be in $A \cup B = S$, it is suffice to find $\{w_n\} \in B$ (and it should be in S) such that $\lim_{n\to\infty} w_n = -2$

Pick $q = 1 + n$ and set $w_n = \frac{4-2(1+n)}{1+n} = \frac{2-2n}{1+n} \in B$ (and $w_n \in A \cup B$)

And $\lim_{n\to\infty} w_n = \lim_{n\to\infty} \frac{2-2n}{1+n} = \lim_{n\to\infty} \frac{\frac{2}{n}-2}{\frac{1}{n}+1} = -2$

Therefore by infimum limit theorem, we conclude $\inf S = -2$

Similarly to show $\sup S = 3$, we see that the upper bound is achieved by elements in A. Therefore we just need to concentrate on A

Since $3 = 1 + \frac{2}{1} \in A$, the upper bound is in A and therefore also in $A \cup B$, hence

$\sup S = 3$

Example 3

Find the supremum and infimum of the set

$$S = \left\{1 + \frac{2}{n} : n \in \mathbf{N}\right\} \cap \left\{\frac{4 - 2q}{q} : q \in \mathbf{Q} \cap (1, \infty))\right\}$$

(Step 1) From Example 2, we have

	Upper Bound	Lower Bound
$A = \left\{1 + \frac{2}{n} : n \in \mathbf{N}\right\}$	3 (when $n = 1$)	1 (when $n \to \infty$)
$B = \left\{\frac{4 - 2q}{q} : q \in \mathbf{Q} \cap (1, \infty)\right\}$	2 (when $q = 1$)	-2 (when $q \to \infty$)

For $S = A \cap B$,

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------------------(------------------------------------<---------------)------------->
                 -2                                   1               2              3
```

We can see all elements in $A \cap B$ lies between 1 and 2. Therefore the lower bound and upper bound is 1 and 2 respectively.

(Step 2)

If we use infimum limit theorem and supremum limit theorem, it will be difficult for us to find w_n since we need to ensure ALL w_n's are in both sets. Of course, you can try to do it. But here I will use proof by contradiction.

(Show $\inf S = 1$, note this lower bound is achieved by A)

Suppose $\inf S > 1$,

Since $\inf A = 1$ **(☺Left as exercise for you, you need to prove it in your midterm)**

Then there exists an element $x \in A$, such that $\inf S > x > 1$.

(Let $x = 1 + \frac{2}{n}$, since 2 is upper bound, therefore $1 + \frac{2}{n} \leq 2 \rightarrow n \geq 2$)

Next, we will show $x \in B$ (Then it will imply $x \in A \cap B$, which will give the contradiction that $\inf S$ is being lower bound)

IDEA: We need to find q such that $\boldsymbol{x = 1 + \frac{2}{n} = \frac{4-2q}{q}}$

Set $1 + \frac{2}{n} = \frac{4-2q}{q} \rightarrow \mathbf{q} = \frac{\mathbf{4n-2}}{\mathbf{3n}} > 1$ **for** $\mathbf{n \geq 2}$ (so $q \in \mathbf{Q} \cap (1, \infty)$)

Hence $x = 1 + \frac{2}{n} = \frac{4-2q}{q} \in B$, so $x \in A \cap B$, it leads to contradiction.

Therefore $\inf S = 1$

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☺Exercise 1

a) In Example 3, show that $\inf A = 1$ (where $A = \left\{1 + \frac{2}{n} : n \in \mathbf{N}\right\}$)

b) In Example 3, show that $\sup S = \frac{5}{3}$ (which completes the solution)

(Comment: It is quite difficult for you)

Example 4 (Example 2 of Tutorial Note #7)

Find the infimum and supremum of the set

$$S = \left\{x - \frac{1}{n} : x \in [0,1] \cap \mathbf{Q}, n \in \mathbf{N}\right\} \backslash [-1, \frac{1}{2})$$

(Step 1: Finding upper bound and lower bound)

For the set $S = \left\{x - \frac{1}{n} : x \in [0,1) \cap \mathbf{Q}, n \in \mathbf{N}\right\} \backslash [-1, \frac{1}{2})$

Since $x - \frac{1}{n} < 1 - 0 = 1$, then 1 is the upper bound of S

Therefore $x - \frac{1}{n} > 0 - 1 = -1$,

For S

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-1 1/2 1

From the graph, we see the upper bound and lower bound are 1 and 1/2 respectively.

(Step 2)

We may use limit theorem to show $\sup S = 1$ and $\inf S = 1/2$, however it is quite difficult for some students to construct the sequence. (Please refer to Example 2 of Tutorial Note 7 for detail) Here I present another alternative.

(Show $\inf S = 1/2$)

Since $1/2 \in S$, (we can pick $x = \frac{3}{4}$ and $n = 4$, then $x - \frac{1}{n} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$)

Therefore, $\inf S = 1/2$

(Show $\sup S = 1$)

Suppose $\sup S < 1$, by density of rational number, there exists a rational number p/q such that $\sup S < \frac{p}{q} < 1$, since $\frac{p}{q}$ is positive, then p and q are both positive integers*.

(Show $\frac{p}{q} \in S$).Note that $\frac{p}{q} = \frac{p+1}{q} - \frac{1}{q}$, by letting $\boldsymbol{x = \frac{p+1}{q} \geq 0}$ ***and*** $\mathbf{x \leq 1}$ ***(why?)***

and $\boldsymbol{n = q}$, then $\frac{p}{q} = x - \frac{1}{n} \in S$. Contradict to the fact that $\sup S$ is the upper bound.

Therefore $\sup S = 1$

(*Remark: since the lower bound of S is $1/2$, therefore it is OK for us to assume p/q is positive.)

Unknown Set

Example 5 Let A and B be subsets of non-negative real number with $\sup A = 7$, $\sup B = 2$, a) Find the supreme of $S = \{x^2 + 3y^3 : x \in A, y \in B\}$ b) Will the supreme in a) remain the same if the word "non-negative" is omitted?

a) Note that $0 \le x \le 7$ and $0 \le y \le 2$

Then $x^2 + 3y^3 \le 7^2 + 3(2)^3 = 73$, the upper bound is 73

To show $\sup S = 73$, Since we do not what exactly A and B are, so we can't construct our w_n directly. Instead, we can use the converse of supermum limit theorem to obtain such sequence.

Note that $\sup A = 7$

There exist $\{a_1, a_2, \ldots\ldots a_n, \ldots.\} \in S$ such that $\lim_{n\to\infty} a_n = 7$

Similarly for $\sup B = 2$

There exist $\{b_1, b_2, \ldots\ldots b_n, \ldots.\} \in S$ such that $\lim_{n\to\infty} b_n = 2$

Now take our $w_n = a_n^2 + 3b_n^3 \in S$ and $\lim_{n\to\infty} w_n = 7^2 + 3(2^3) = 73$

Therefore by supermum limit theorem, we get $\sup S = 73$

b) If the word "non-negative" is omitted, $\sup S$ may not be 73.

Example: Pick $A = (-10,7)$ and $B = (0,2)$, it is clear that $\sup A = 7, \sup B = 2$

But $x^2 + 3y^3 \le (-10)^2 + 3(2)^3 = 124$ and $\sup S = 124 \neq 73$

(Note: That's why the word non-negative is important in this problem)

Example 6 Let A be a non-empty subset of **R** such that $\inf B = -1$ and $\sup B = 1$, find the supremum and infimum (if any) of the set $S = \{b^4 - 5b + 20 : b \in B\}$

(Step 1: Find the upper bound and lower bound)

Let $f(b) = b^4 - 5b + 20$

Then $f'(b) = 4b^3 - 5 < 0$ for $-1 \le b \le 1$

So f is decreasing, therefore $f(1) \le f(b) \le f(-1) \to 16 \le f(b) \le 26$

Therefore the upper bound and lower bound of S are 26 and 16 respectively.

(Step 2)

We first claim $\sup S = 24$

First since $\inf B = -1$, therefore by infimum limit theorem(another direction), there exists $\{b_n\} \in B$ such that $\lim_{n\to\infty} b_n = -1$
Pick $w_n = b_n^4 - 5b_n + 20 \in S$
Then $\lim_{n\to\infty} w_n = (-1)^4 - 5(-1) + 20 = 26$
Hence by supreme limit theorem, $\sup S = 26$

Since $\sup B = 1$, therefore by supreme limit theorem(another direction), there exists $\{b_n\} \in B$ such that $\lim_{n\to\infty} b_n = 1$
Pick $w_n = b_n^4 - 5b_n + 20 \in S$
Then $\lim_{n\to\infty} w_n = (1)^4 - 5(1) + 20 = 16$
Hence by infimum limit theorem, $\inf S = 16$

(You can also refer to Tutorial Note #7 for more exercises)

☺Exercise 2

Find the supremum and infimum of the sets

$$S = \left\{\frac{2}{n} - x^2 : n \in \mathbf{N}, x \in \mathbf{Q}\right\}, T = \{3x^2 - \frac{1}{y} : x \in (0,2) \cap Q, y \in (1,5]\}$$

$$U = \{\sqrt{2}x + \sqrt{3}y :\ x, y \in [1,4]\backslash\mathbf{Q}\}$$

(Hint: in set U, $[1,4]\backslash Q$ means the set of all irrational (non-rational) number between 1 and 4)

☺Exercise 3

Find the supermum and infimum of the sets

$$S = \bigcup_{n=1}^{5}\left[\frac{1}{n}, 2 - \frac{1}{n}\right], T = \{x^2 + y^2 : x, y \in (-2,1) \cap \mathbf{Q}\} \cap (2,7]$$

☺Exercise 4

Suppose $A_n \subseteq (-\infty, 2)$ and $\sup A_n = x_n$ for $n = 1,2,3,\ldots,10$. Prove that

$$\sup\left(\bigcup_{k=1}^{10} x_k\right) = \max(x_1, x_2, \ldots, x_{10})$$

(Hint: With loss of generality, you may assume $x_1 \le x_2 \le \cdots . \le x_{10}$

☺Exercise 5

Let $S = \{\pi - x^2 : x \in (-2,2) \cap Q\}$ and $T = \left\{\frac{x-y+z}{2x+y+z} : x, y, z \in (2,9) \cap \mathbf{Q}\right\}$

Find the supreme and infimum (if any) of $S \cap T$

(Hint: Show that all elements in S are irrational and show all elements in T are rational)