MATH 2031 Introduction to Real Analysis

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Tutorial Note 13

Differentiation

(I) **Definition:**
Let S be an interval of positive length.
A function $f : S \to \mathbb{R}$ is differentiable at $x_0 \in S$ iff $f'(x_0) = \lim\limits_{\substack{x \to x_0 \\ x \in S}} \frac{f(x) - f(x_0)}{x - x_0}$ exists in $\mathbb{R}$. Also, f is differentiable iff f is differentiable at every element of S.

(II) **Theorem** (use the converse for checking non-differentiability)
If $f : S \to \mathbb{R}$ is differentiable at $x_0 \in S$, then it is continuous at x_0.
(Converse:$f : S \to \mathbb{R}$ is not continuous at $x_0 \in S$ implies it is not differentiable at x_0)

(III) **Differentiation Formulas**

- If $f, g : S \to \mathbb{R}$ are differentiable at x_0, then $f \pm g,\ fg,\ f/g$ (provided that $g(x_0) \neq 0$) are differentiable at x_0. In fact,

$$\begin{aligned} (f \pm g)'(x_0) &= f'(x_0) \pm g'(x_0) \\ (fg)'(x_0) &= f'(x_0)g(x_0) + f(x_0)g'(x_0) \\ \left(\frac{f}{g}\right)'(x_0) &= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)} \end{aligned}$$

- (Chain Rule) If $f : S \to \mathbb{R}$ is differentiable at x_0, $f(S) \subseteq T$ and $g : T \to \mathbb{R}$ is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$

(IV) **Local Extrema Theorem (L.E.T.)**
Let $f : (a, b) \to \mathbb{R}$ be differentiable. If $f(x_0) = \min\limits_{x \in (a,b)} f(x)$ or $f(x_0) = \max\limits_{x \in (a,b)} f(x)$, then $f'(x_0) = 0$.

(V) **Rolle's Theorem**
Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there is (at least one) $w_0 \in (a, b)$ such that $f'(w_0) = 0$.

(VI) **Mean Value Theorem (M.V.T.)**
If f is continuous on $[a, b]$ and differentiable on (a, b), then $\exists x_0 \in (a, b)$ such that

$$f(b) - f(a) = f'(x_0)(b - a)$$

(VII) **Local Tracing Theorem**
If $f : [a, b] \to \mathbb{R}$ is continuous and $f'(c) > 0$ for some $c \in [a, b]$, then $\exists c_0, c_1 \in \mathbb{R}$ such that $c_0 < c < c_1$, and $f(x) < f(c) < f(y)\ \forall x, y \in [a, b]$ and $c_0 < x < c$ and $c < y < c_1$.

Problem 1 Check whether the follow real-valued functions are differentiable or not at 0:

(a) $f(x) = x^2 + bx + c$ where $b, c \in \mathbb{R}$;

(b) $g(x) = |x|$

(c) $h(x) = \begin{cases} x^2 - x & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & \text{for } x \in \mathbb{Q} \end{cases}$

(d) $\widetilde{h}(x) = \begin{cases} x^2 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \\ x^3 + x^2 - 1 & \text{for } x \in \mathbb{Q} \end{cases}$

Solution:

(a) $\dfrac{f(x) - f(0)}{x - 0} = \dfrac{(x^2 + bx + c) - c}{x} = \dfrac{x^2 + bx}{x} = x + b.$

Take limit both side as $x \to 0$, then $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = b$

So. f is differentiable at 0.

(b) We can view g as follows:

$$g(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

We see that

$$g'(0^+) = \lim_{x \to 0^+} \frac{x + 0}{x - 0} = \lim_{x \to 0^+} 1 = 1 \qquad g'(0^-) = \lim_{x \to 0^-} \frac{-x + 0}{x - 0} = \lim_{x \to 0^+} -1 = -1$$

As $g'(0^+) \neq g'(0^-)$, i.e $g'(0)$ does not exist, thus g is not differentiable at 0.

(c) Since we could see that h is defined differently on the rationals and the irrationals, we could use the same trick "Sequential Limit Theorem (S.L.T)". We need to check on sequences of rational numbers and sequences of irrational numbers.

Let r_n be a sequence of rational number that converges to 0 and w_n be a sequence of irrational number that converges to 0, then

$$\lim_{n \to \infty} \frac{h(r_n) - h(0)}{r_n - 0} = \lim_{n \to \infty} 0 = 0, \qquad \lim_{n \to \infty} \frac{h(w_n) - h(0)}{w_n - 0} = \lim_{n \to \infty} \frac{(w_n^2 - w_n) - 0}{w_n} = \lim_{n \to \infty} (w_n - 1) = -1$$

Since the value are different, by Sequential Limit Theorem, $h'(0) = \lim\limits_{x \to 0} \dfrac{h(x) - h(0)}{x - 0}$ does not exist and thus h is not differentiable at 0.

(d) We can do this in 2 ways

Solution 1: Similar as above, try on sequences of rational numbers and sequences of irrational numbers. Let r_n be a sequence of rational numbers that converges to 0 and w_n be a sequence of irrational numbers that converges to 0, then

$$\lim_{n \to \infty} \frac{\widetilde{h}(r_n) - \widetilde{h}(0)}{r_n - 0} = \lim_{n \to \infty} \frac{(r_n^3 - r_n^2 - 1) - (-1)}{r_n - 0} = \lim_{n \to \infty} (r_n^2 - r_n) = 0$$
$$\lim_{n \to \infty} \frac{\widetilde{h}(w_n) - \widetilde{h}(0)}{w_n - 0} = \lim_{n \to \infty} \frac{w_n^2 - (-1)}{w_n} = \lim_{n \to \infty} \left(w_n + \frac{1}{w_m} \right) \text{ does not exist}$$

Since the values are different, by Sequential Limit Theorem, $h'(0) = \lim\limits_{x \to 0} \dfrac{h(x) - h(0)}{x - 0}$ does not exist and thus h is not differentiable at 0.

Solution 2: Check the continuity at 0. (Usually we check the limit at that corresponding point, but this time it is quite tricky. Assume that $\widetilde{h}$ is continuous at x_0, then find what value(s) could x_0 be.)

Assume that $\widetilde{h}$ is continuous at x_0, then $x_0^2 = x_0^3 + x_0^2 - 1$, which gives us the following

$$0 = x_0^3 - 1 = (x_0 - 1)(x_0^2 + x_0 + 1)$$

Thus x_0 can only be 1, i.e $\widetilde{h}$ must not be continuous at 0. Therefore it's not differentiable at 0 either. (Note that the above work has just shown that if $\widetilde{h}$ were continuous at x_0, x_0 could only be 1. However, we don't know whether $\widetilde{h}$ is continuous at 1 or not.)

Problem 2 (Adapted from Apostol) Assume f has a finite derivative in (a, b) (i.e f' exist and $f' < \infty$ for any $x \in (a, b)$) and is continuous on $[a, b]$ with $f(a) = f(b) = 0$.
Prove that for every real λ there is some $c \in (a, b)$ such that $f'(c) = \lambda f(c)$.
Solution:
Idea: Since we are looking for the existence of a certain value c satisfying some differentiability requirement, we should have Rolle's theorem and Mean value theorem in mind. Since we also have $f(a) = f(b) = 0$, we should try Rolle's theorem first.

Since Rolle's theorem implies that the first derivative is zero at some point, we should apply to a particular function that satisfies the criteria and with derivative associated with $f'(c) - \lambda f(c)$. With the help of integrating factor, we have $g(x) = e^{-\lambda x} f(x)$ as our candidate.

Since $g(a) = e^{-\lambda a} f(a) = 0$ and $g(b) = e^{-\lambda b} f(b) = 0$, we have $g(a) = g(b)$.
As $e^{-\lambda x}$ is differentiable over $\mathbb{R}$, and f is differentiable on (a, b) and is continuous on $[a, b]$, so $g(x)$ is differentiable on (a, b) and is continuous on $[a, b]$. Then, by Rolle's theorem, there exists $c \in (a, b)$ such that

$$0 = g'(c) = e^{-\lambda c} f'(c) - \lambda e^{-\lambda c} f(c)$$

As $e^{-\lambda x}$ is always positive, we have $f'(c) - \lambda f(c) = 0$, i.e. $f'(c) = \lambda f(c)$.

Problem 3 (Adapted from Apostol) Assume that f is non-negative and has a finite third derivative f''' in the open interval $(0, 1)$. If $f(x) = 0$ for at least 2 values of x in $(0, 1)$, prove that $f'''(c) = 0$ for some $c \in (0, 1)$.
Solution:
When it involves higher derivatives, we should think of Mean value theorem and Taylor Theorem first.

Since we need the existence of a c such that $f'''(c) = 0$, we need at least 3 distinct points at which the first derivative is zero.

As it is given that $f(x) = 0$ for at least 2 values of x in $(0, 1)$, say, with out loss of generality, $0 < a < b < 1$ and $f(a) = f(b) = 0$, then by mean value theorem, there exists $r \in (a, b) \subset (0, 1)$ such that

$$0 = f(b) - f(a) = f'(r)(b - a).$$

Since $b - a > 0$, we have $f'(r) = 0$.

Then how can we obtain another 2 such points?
In fact, it is given in the condition. As f is non-negative and $f(x) = 0$ at $x = a$ and $x = b$ in $(0, 1)$, thus f must attain the local minimum of in $(0, 1)$.
By Local Extrema Theorem, we have $f'(a) = 0$ and $f'(b) = 0$ at distinct points $a < b$.

Since f is three times differentiable and $f'(a) = 0$, $f'(r) = 0$ and $f'(b) = 0$ for distinct $a, r, b \in (0, 1)$, by Mean value theorem, there exist $s \in (r, b)$ and $t \in (a, r)$ such that

$$0 = f'(b) - f'(r) = f''(s)(b - r)$$
$$0 = f'(r) - f'(a) = f''(t)(r - a)$$

As $a < r < b$, we have $b - r > 0$ and $r - a > 0$, which implies that $f''(s) = 0 = f''(t)$, s and t are distinct since they lie on disjoint intervals.

$$0 = f(b) - f(a) = f'(r)(b - a)$$

Applying the Mean value theorem once more, we see that there exists $c \in (t, s) \subset (0, 1)$ such that

$$0 = f''(s) - f''(t) = f'''(c)(s - t)$$

Since $s - t > 0$, we have $f'''(c) = 0$.

Problem 4 Let $\{a_n\}$ be a Cauchy sequence and $a_n > 0$ for all $n \in \mathbb{N}$. Define $b_n = \sin(a_n)\ln(1+a_n)$, show that $\{b_n\}$ is also a Cauchy sequence.

Scratch:

Idea: Use the inequality obtained from Mean value theorem, for some w between x and y,

$$\begin{aligned}|\sin(x)\ln(1+x) - \sin(y)\ln(1+y)| &= |(\sin w \ln(1+w))'(x-y)| \\ &= \left|\cos w \ln(1+w) + \frac{\sin w}{1+w}\right| |x-y| \\ &\le \left[|\cos w| \ln(1+w) + \frac{|\sin w|}{1+w}\right] |x-y| \\ &\le [\ln(1+w)+1]|x-y|\end{aligned}$$

Then we have

$$\begin{aligned}|b_n - b_m| &= |\sin(a_n)\ln(1+a_n) - \sin(a_m)\ln(1+a_m)| \\ &\le [\ln(1+w)+1]|a_n - a_m|\end{aligned}$$

So requiring that $|a_n - a_m| < \dfrac{\varepsilon}{\ln(1+w)+1}$ is sufficient.

Solution:

Consider $f(x) : [0,\infty) \to \mathbb{R}$ given by $f(x) = \sin(x)\ln(1+x)$, which is clearly continuous on $[0,\infty)$ and differentiable on $(0,\infty)$. Then by mean value theorem, there exists w between $x, y \in (0,\infty)$ such that

$$\begin{aligned}|f(x) - f(y)| &= |(f(w))'(x-y)| \\ &= |(\sin w \ln(1+w))'(x-y)| \\ &= \left|\cos w \ln(1+w) + \frac{\sin w}{1+w}\right| |x-y| \\ &\le \left[|\cos w| \ln(1+w) + \frac{|\sin w|}{1+w}\right] |x-y| \\ &\le \left[\ln(1+w) + \frac{1}{1+w}\right] |x-y| && \text{since } \cos w \text{ and } \sin w \text{ are bounded above by } 1 \\ &\le [\ln(1+w)+1]|x-y| && \text{since } w > 0\end{aligned}$$

Then since $\{a_n\}$ is a Cauchy sequence,
$\forall \varepsilon > 0$, as $w > 0$, $\ln(1+w) > 0$ and $\dfrac{\varepsilon}{\ln(1+w)+1} > 0$,
then there exists $K \in \mathbb{N}$ such that for any $m, n > K$, $|a_n - a_m| < \dfrac{\varepsilon}{\ln(1+w)+1}$

For the same K, for any $m, n > K$,

$$\begin{aligned}|b_n - b_m| &= |\sin(a_n)\ln(1+a_n) - \sin(a_m)\ln(1+a_m)| \\ &\le [\ln(1+w)+1]|a_n - a_m| \\ &< [\ln(1+w)+1]\left[\frac{\varepsilon}{\ln(1+w)+1}\right] = \varepsilon\end{aligned}$$

Thus $\{b_n\}$ is also a Cauchy sequence.