Riemann considered $\lim_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j$ as the integral of $f(x)$ on $[a,b]$.

However, this limit is <u>not</u> a limit of a sequence nor a limit of a function. Since the t_j's may be arbitrary, it is not clear how the Riemann sums are approximating the integral.

Observe that $\forall j=1,2,\dots,n$, $-K \le m_j \le f(t_j) \le M_j \le K$.

So $\underbrace{\sum_{j=1}^{n} -K\Delta x_j}_{} \le \underbrace{\sum_{j=1}^{n} m_j \Delta x_j}_{} \le \underbrace{\sum_{j=1}^{n} f(t_j)\Delta x_j}_{} \le \underbrace{\sum_{j=1}^{n} M_j \Delta x_j}_{} \le \underbrace{\sum_{j=1}^{n} K\Delta x_j}_{}$

i.e. $-K(b-a) \le L(f,P) \le S \le U(f,P) \le K(b-a)$

In particular, $U(f,P)$ and $L(f,P)$ are bounded above and below. Also, $\forall$ partition P,

$$L(f,P) \le U(f,P).$$

How about if we have 2 partitions P_1 and P_2?

First, note $P_1 \cup P_2$ is also a partition.

<u>Definitions</u>

① For partitions P, P', we say P' is a <u>refinement</u> of P (or P' is <u>finer</u> than P) iff $P \subseteq P'$.

② For partitions P_1, P_2, we say $P_1 \cup P_2$ is the <u>common refinement</u> of P_1 and P_2.

<u>Refinement Theorem</u> If $P \subseteq P'$, then

$$L(f,P) \le L(f,P') \le U(f,P') \le U(f,P).$$

<u>Proof.</u> It suffices to consider the case $P' = P \cup \{w\}$, say $P = \{\underset{a}{\overset{\shortparallel}{x_0}}, x_1, \dots, x_{j-1}, x_j, \dots, \underset{b}{\overset{\shortparallel}{x_n}}\}$ and $x_{j-1} < w < x_j$.

Since $[x_{j-1}, w], [w, x_j] \subseteq [x_{j-1}, x_j]$, so

$m' = \inf\{f(x) : x \in [x_{j-1}, w]\} \ge m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$

and $m'' = \inf\{f(x) : x \in [w, x_j]\} \ge m_j$. Then

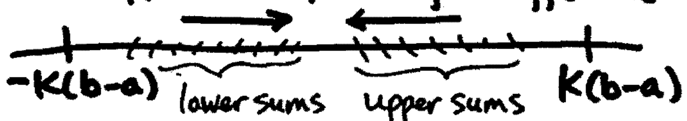
$L(f,P) = \dots + m_j \underbrace{\Delta x_j}_{=(w-x_{j-1})+(x_j-w)} + \dots \le \dots + m'(w-x_{j-1}) + m''(x_j - w) + \dots = L(f,P')$.

Similarly, $U(f,P') \le U(f,P)$.

For partitions P_1, P_2, since $P_1, P_2 \subseteq P_1 \cup P_2$,

$$L(f,P_1) \le L(f,P_1\cup P_2) \le U(f,P_1 \cup P_2) \le U(f,P_2).$$

So lower sums $\le$ upper sums, even for different partitions!



$(L)\int_a^b f(x)\,dx = \sup\{L(f,P) : P \text{ partition of } [a,b]\} = \underline{\int_a^b} f(x)\,dx$

is the <u>lower integral</u> of $f(x)$ on $[a,b]$.

$(U)\int_a^b f(x)\,dx = \inf\{U(f,P) : P \text{ partition of } [a,b]\} = \overline{\int_a^b} f(x)\,dx$

is the <u>upper integral</u> of $f(x)$ on $[a,b]$.

Definitions

$f(x)$ is (Riemann) integrable on $[a,b]$ iff

$$(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx.$$

In that case, we write $\int_a^b f(x)\,dx$ for this number.

If $b \le a$, define $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

In particular, $\int_a^a f(x)\,dx = 0$.

Questions Are there integrable functions? Are there non-integrable functions?

Examples ① $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ is nonintegrable on $[a,b]$ where $a<b$.

On any $[x_{j-1}, x_j]$, $m_j = 0$ by density of irrationals and $M_j = 1$ by density of rationals. So for all partition P of $[a,b]$,

$$L(f,P) = \sum_{j=1}^{n} m_j \Delta x_j = 0, \quad U(f,P) = \sum_{j=1}^{n} M_j \Delta x_j = \sum_{j=1}^{n} \Delta x_j = b-a.$$

$$(L)\int_a^b f(x)\,dx = \sup_P L(f,P) = 0, \quad (U)\int_a^b f(x)\,dx = \inf_P U(f,P) = b-a$$

different

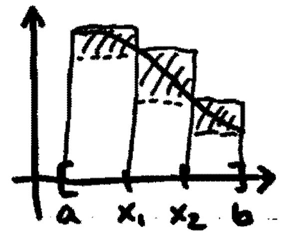
② $f(x) = c \ (\forall x \in [a,b])$ is integrable on $[a,b]$.

On any $[x_{j-1}, x_j]$, $m_j = c = M_j$. So $\forall$ partition P,

$$L(f,P) = \sum m_j \Delta x_j = \underbrace{\sum c \Delta x_j}_{=c(b-a)} = \sum M_j \Delta x_j = U(f,P).$$

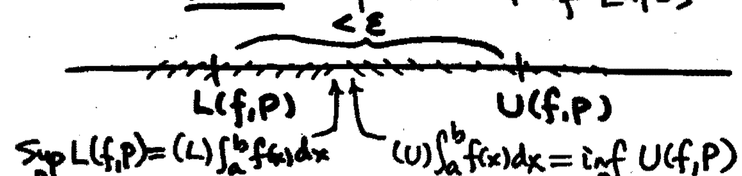
$$\therefore (L)\int_a^b f(x)\,dx = c(b-a) = (U)\int_a^b f(x)\,dx.$$

Continuous functions on $[a,b]$ are integrable. For that we need



Theorem (Integral Criterion)
Let $f(x)$ be bounded on $[a,b]$.
$f(x)$ is Riemann integrable on $[a,b]$
$\Leftrightarrow \forall \varepsilon > 0 \ \exists$ partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

Proof ($\Leftarrow$) $\forall \varepsilon > 0$ $\exists$ partition P of $[a,b]$

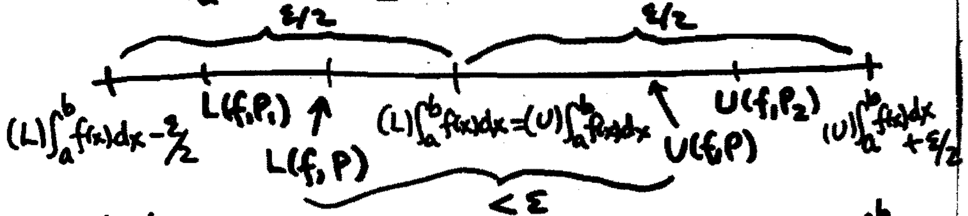


$$\varepsilon > U(f,P) - L(f,P) \ge (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx \ge 0.$$

$\therefore (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx = 0$ by infinitesimal principle.

($\Rightarrow$) $\forall \varepsilon > 0$, by supremum property, $\exists P_1$ such that

$$(L)\int_a^b f(x)\,dx - \frac{\varepsilon}{2} < L(f,P_1) \le (L)\int_a^b f(x)\,dx$$



Similarly, $\exists P_2$ such that $(U)\int_a^b f(x)\,dx \le U(f,P_2) \le (U)\int_a^b f(x)\,dx + \varepsilon/2$.

Let $P = P_1 \cup P_2$, then

$$U(f,P) - L(f,P) < \left((U)\int_a^b f(x)\,dx + \frac{\varepsilon}{2}\right) - \left((L)\int_a^b f(x)\,dx - \frac{\varepsilon}{2}\right) = \varepsilon.$$

Recall $f: S \to \mathbb{R}$ is continuous at $t \in S$ means $\forall \varepsilon > 0, \exists \delta > 0$ (δ depends on ε and t) such that $\forall x \in S \quad |x-t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon$.

Definition $f: S \to \mathbb{R}$ is <u>uniformly continuous</u> iff $\forall \varepsilon > 0, \exists \delta > 0$ (δ depends only on ε) such that $\forall x, t \in S, \ |x-t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon$.

Remark For any set S, $f: S \to \mathbb{R}$ uniformly continuous $\Rightarrow$ f is continuous (at every $t \in S$) because the δ in the definition can be used for every $t \in S$.

For closed and bounded intervals, the <u>converse</u> is true.

Uniform Continuity theorem

If $f: [a,b] \to \mathbb{R}$ is continuous, then it is uniformly continuous.

Proof. Assume f is not uniformly continuous. Then $\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x, t \in [a,b]$, $|x-t| < \delta$ and $|f(x) - f(t)| \geq \varepsilon$.

$\delta = 1 \quad \exists x_1, t_1 \in [a,b] \quad |x_1 - t_1| < \delta = 1$ and $|f(x_1) - f(t_1)| \geq \varepsilon$

$\delta = \frac{1}{2} \quad x_2, t_2 \quad |x_2 - t_2| < \delta = \frac{1}{2} \quad |f(x_2) - f(t_2)| \geq \varepsilon$

$\vdots$

$\delta = \frac{1}{n} \quad x_n, t_n \quad |x_n - t_n| < \delta = \frac{1}{n} \quad |f(x_n) - f(t_n)| \geq \varepsilon$.

By Bolzano-Weierstrass theorem, $\exists x_{n_j} \to w \in [a,b]$. Then

$|t_{n_j} - w| \leq |t_{n_j} - x_{n_j}| + |x_{n_j} - w| < \frac{1}{n_j} + |x_{n_j} - w| \to 0$ as $j \to \infty$.

So $t_{n_j} \to w$. Then f continuous implies

$$0 = |f(w) - f(w)| = \lim_{j \to \infty} |f(x_{n_j}) - f(t_{n_j})| \geq \varepsilon$$

contradicts $\varepsilon > 0$.

Theorem If $f: [a,b] \to \mathbb{R}$ is continuous, then it is integrable.

Proof. We check the integral criterion. $\forall \varepsilon > 0$, since f is uniformly continuous, $\exists \delta > 0 \ \forall x, t \in [a,b]$,

$$|x-t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon/(b-a).$$

Let P be a partition of $[a,b]$ such that $\max_{1 \leq j \leq n} |x_j - x_{j-1}| < \delta$

$a \quad \underbrace{\text{|||||||||||}}_{\text{all} < \delta} \quad b$

On $[x_{j-1}, x_j]$, by extreme value theorem, $\exists w_j, u_j \in [x_{j-1}, x_j]$ such that

$f(w_j) = \sup\{f(x) : x \in [x_{j-1}, x_j]\} = M_j$ and $f(u_j) = m_j$.

use f continuous here also

Then

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} (M_j - m_j) \Delta x_j = \sum_{j=1}^{n} (f(w_j) - f(u_j)) \Delta x_j$$

$$< \sum_{j=1}^{n} \frac{\varepsilon}{b-a} \Delta x_j = \frac{\varepsilon}{b-a} \sum_{j=1}^{n} \Delta x_j = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

$\therefore$ by integral criterion, f is integrable on $[a,b]$.

Remarks (Exercise) If $f: [a,b] \to \mathbb{R}$ is continuous except at finitely many $c_1, c_2, \ldots, c_n \in [a,b]$, then f is integrable on $[a,b], [a,c_1], \ldots, [c_i, c_{i+1}], \ldots, [c_n, b]$ and

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_{n-1}}^{c_n} f(x)dx + \int_{c_n}^{b} f(x)dx.$$

Questions How bad can an integrable function be discontinuous? Which functions are integrable?

Answer f integrable on $[a,b] \Leftrightarrow S_f = \{x \in [a,b] : f \text{ discontinuous at } x\}$ is a zero-length set

Questions What is a zero-length set? Which sets are zero-length?

Definitions ① A set $S \subseteq \mathbb{R}$ is of measure 0 (or has zero-length or is a null set) iff $\forall \varepsilon > 0$, $\exists$ intervals $(a_1,b_1),(a_2,b_2),(a_3,b_3),\cdots$ such that $S \subseteq \bigcup_{n=1}^{\infty}(a_n,b_n)$ and $\sum_{n=1}^{\infty}|a_n-b_n| < \varepsilon$.

② A property is said to hold almost everywhere (a.e.) (or almost surely (a.s.)) iff the property holds except on a set of measure 0.

Lebesgue's Theorem (1902)

Let $f:[a,b] \to \mathbb{R}$ be a bounded function.

f is integrable on $[a,b] \Leftrightarrow f$ is continuous a.e. on $[a,b]$

(that means f is continuous on $[a,b]$ except on a set of measure 0).

Remarks So all we need to check is that

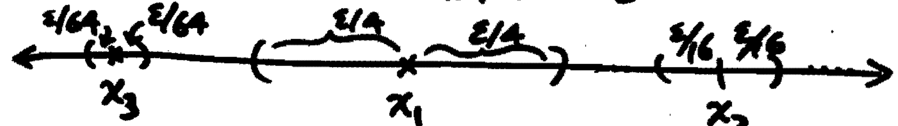
$$S_f = \{x \in [a,b] : f \text{ is discontinuous at } x\}$$

is of measure 0.

Examples ① Empty set $\emptyset$ is of measure 0 because $\emptyset \subseteq \bigcup_{n=1}^{\infty}(0,0)$ and $\sum_{n=1}^{\infty}|0-0| = 0 < \varepsilon$. So Lebesgue's theorem implies every continuous function on $[a,b]$ is integrable.

② A countable set $\{x_1,x_2,\cdots\}$ is of measure 0 because $\{x_1,x_2,\cdots\} \subseteq \bigcup_{n=1}^{\infty}(x_n-\frac{\varepsilon}{4^n}, x_n+\frac{\varepsilon}{4^n})$ and

$$\sum_{n=1}^{\infty}\left|(x_n-\tfrac{\varepsilon}{4^n})-(x_n+\tfrac{\varepsilon}{4^n})\right| = \sum_{n=1}^{\infty}\frac{2\varepsilon}{4^n} = \frac{2\varepsilon}{3} < \varepsilon.$$



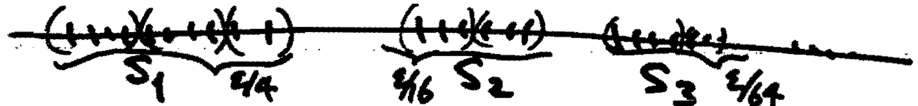
Since montone functions have countably many jumps by the monotone function theorem, so Lebesgue's theorem implies monotone functions are integrable on $[a,b]$.

③ Uncountable sets may or may not be of measure 0. The Cantor set is uncountable (by exercise 29) and is of measure 0. At stage n, there are 2^n subintervals of $[0,1]$, each of length $\frac{1}{3^n}$. So $\lim_{n\to\infty} 2^n(\frac{1}{3^n}) = 0$.

For $a<b$, $[a,b]$ is uncountable, but its length $b-a>0$. Since $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ is discontinuous everywhere on $[a,b]$, $S_f = [a,b]$, which is not of measure 0, so Lebesgue's theorem implies f is not integrable on $[a,b]$.

④ A countable union of sets of measure 0 is of measure 0.

To see this, let $S_1, S_2, S_3, \cdots$ be sets of measure 0 and $S = \bigcup_{n=1}^{\infty} S_n$. Use the idea in example 2 !



$\forall \varepsilon > 0$, since S_n is of measure 0 and $\frac{\varepsilon}{4^n} > 0$, by the definition of measure 0, $\exists$ open intervals $(a_{n,1}, b_{n,1})$, $(a_{n,2}, b_{n,2}), (a_{n,3}, b_{n,3}), \cdots$ such that

$$S_n \subseteq \bigcup_{i=1}^{\infty} (a_{n,i}, b_{n,i}) \text{ and } \sum_{i=1}^{\infty} |a_{n,i} - b_{n,i}| \le \frac{\varepsilon}{4^n}.$$

Then $S \subseteq \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} (a_{n,i}, b_{n,i})$ and $\sum_{n=1}^{\infty}\sum_{i=1}^{\infty} |a_{n,i} - b_{n,i}| \le \sum_{n=1}^{\infty} \frac{\varepsilon}{4^n} = \varepsilon/3 < \varepsilon$.

$\therefore$ S is of measure 0.

⑤ If S is of measure 0 and $S' \subseteq S$, then S' is of measure 0.

To see this, $\forall \varepsilon > 0$, since S is of measure 0, $\exists$ open interval $(a_1, b_1), (a_2, b_2), (a_3, b_3), \cdots$ such that

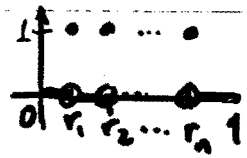
$$S \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \text{ and } \sum_{n=1}^{\infty} |a_n - b_n| < \varepsilon$$

Since $S' \subseteq S$, $S' \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$ and $\sum_{n=1}^{\infty} |a_n - b_n| < \varepsilon$.

$\therefore S'$ is of measure 0.

⑥ The limit of a sequence of Riemann integrable functions on $[a,b]$ may not be Riemann integrable on $[a,b]$.

To see this, note $\mathbb{Q} \cap [0,1]$ is countable. So we can arrange its elements as $r_1, r_2, r_3, \cdots$ without repetition and without omission. Define



$$f_n(x) = \begin{cases} 1 & \text{if } x = r_1, r_2, \cdots \text{ or } r_n \\ 0 & \text{otherwise} \end{cases}$$

$f_n(x)$ is discontinuous only at $r_1, r_2, \cdots, r_n$

$S_{f_n} = \{r_1, r_2, \cdots, r_n\}$ is countable, hence S_{f_n} is of measure 0. By Lebesgue's theorem, f_n is Riemann integrable on $[0,1]$. Next,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = r_1, r_2, r_3, \cdots \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \notin \mathbb{Q} \cap [0,1], \end{cases}$$

which is not Riemann integrable on $[0,1]$.

We will postpone the proof of Lebesgue's Theorem. Here we will use it to prove some basic facts.

Theorem For $c \in [a,b]$, f is integrable on $[a,b] \Longleftrightarrow$ f is integrable on $[a,c]$ and on $[c,b]$.

Proof. Note f bounded on $[a,b] \Longleftrightarrow f$ bounded on $[a,c], [c,b]$.

Let S, S_1, S_2 be the sets of discontinuous points of f on $[a,b], [a,c], [c,b]$, respectively. Note $S_1, S_2 \subseteq S$.

($\Rightarrow$) f integrable on $[a,b]$ $\underset{\text{by Lebesgue}}{\Longleftrightarrow}$ S is of measure 0 $\underset{\text{by example 5}}{\Longrightarrow}$ S_1, S_2 are of measure 0 $\underset{\text{by Lebesgue}}{\Longleftrightarrow}$ f is integrable on $[a,c], [c,b]$

($\Leftarrow$) Note $S \subseteq S_1 \cup S_2$. Since S_1 and S_2 are of measure 0, by examples 4 and 5, S is of measure 0. $\therefore f$ is integrable on $[a,b]$

Theorem If $f,g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$, then $f+g$, $f-g$, fg are also integrable on $[a,b]$.

Proof. f,g integrable on $[a,b]$ $\Rightarrow$ f,g bounded on $[a,b]$ $\Rightarrow$ $f+g, f-g, fg$ are bounded on $[a,b]$.

Next, note that if f,g are continuous at x, then $f+g$ is also continuous at x. Taking contrapositive, if $f+g$ is discontinuous at x, then f or g is discontinuous at x.

So $x\in S_{f+g}\Rightarrow x\in S_f\cup S_g$. $\therefore S_{f+g}\subseteq S_f\cup S_g$.

f,g integrable on $[a,b]$ $\Leftrightarrow$ S_f, S_g are of measure 0 $\Rightarrow$ S_{f+g} is of measure 0 $\Leftrightarrow$ $f+g$ is integrable on $[a,b]$.

Similarly, $f-g$, fg are integrable on $[a,b]$.

Theorem If $f:[a,b]\to\mathbb{R}$ is integrable on $[a,b]$ and g is bounded and continuous on $f([a,b])$, then $g\circ f$ is integrable on $[a,b]$. (In particular, taking $g(x)=|x|, x^2, e^x, \cos x, \ldots$ respectively, we see

f integrable on $[a,b]\Rightarrow |f|, f^2, e^f, \cos f$ integrable on $[a,b]$.)

Proof. Note g bounded on $f([a,b])$ implies $g\circ f$ is bounded on $[a,b]$.

Note g is continuous on $f([a,b])$. So if f is continuous at $x\in[a,b]$, then $g\circ f$ is continuous at $x\in[a,b]$.

Taking contrapositive, we see that $S_{g\circ f}\subseteq S_f$.

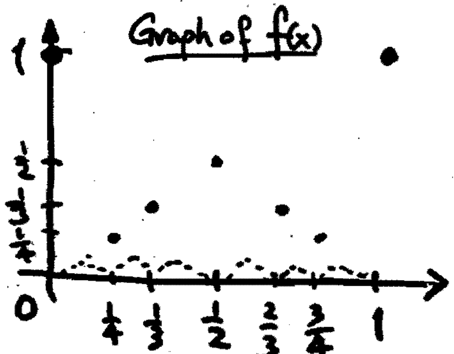
f integrable on $[a,b]$ $\Leftrightarrow$ S_f is of measure 0 $\Rightarrow$ $S_{g\circ f}$ is of measure 0 $\Leftrightarrow$ $g\circ f$ is integrable on $[a,b]$.

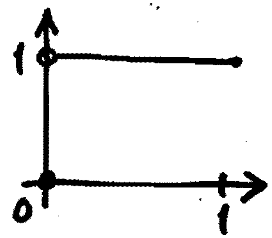
Remarks Even if $f:[a,b]\to[c,d]$ is integrable on $[a,b]$ and $g:[c,d]\to\mathbb{R}$ is integrable on $[c,d]$, $g\circ f$ may <u>not</u> be integrable on $[a,b]$. Here is an example.

Define $f:[0,1]\to[0,1]$ by
$$f(x)=\begin{cases}\frac{1}{n} & \text{if } x=\frac{m}{n}\in(0,1] \\ & m,n \text{ positive integers with no common prime factor} \\ 0 & \text{if } x\in[0,1]\setminus\mathbb{Q} \\ 1 & \text{if } x=0\end{cases}$$

and define $g:[0,1]\to[0,1]$ by
$$g(x)=\begin{cases}0 & \text{if } x=0\\ 1 & \text{if } x\in(0,1].\end{cases}$$

Graph of $f(x)$





Graph of $g(x)$

Exercise: $S_f=[0,1]\cap\mathbb{Q}$ $\quad S_g=\{0\}$

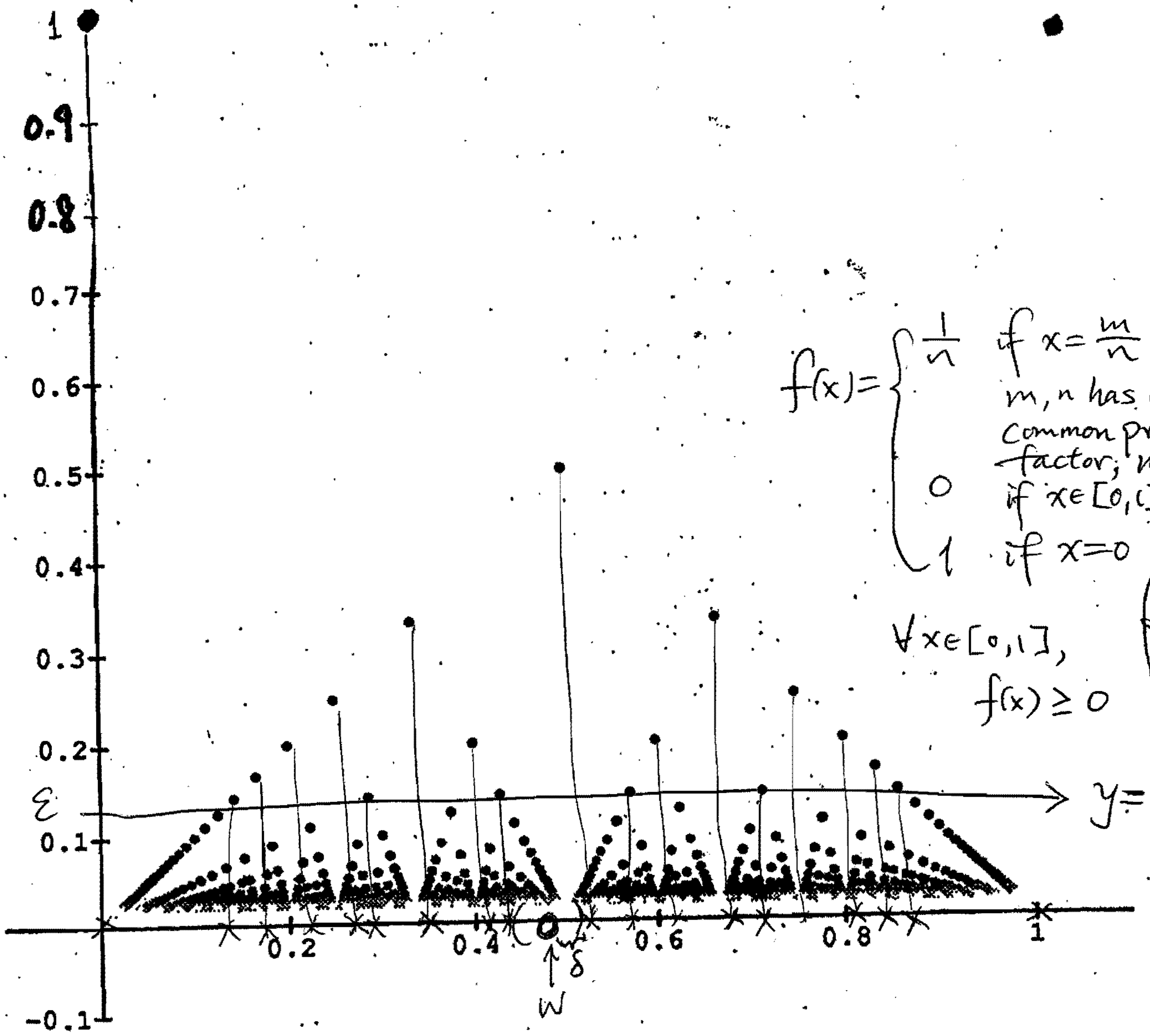
Countable $\Rightarrow$ measure 0

By Lebesgue's theorem, f,g are integrable on $[0,1]$.

However,
$$(g\circ f)(x)=\begin{cases}1 & \text{if } x=\frac{m}{n} \text{ or } 0\in[0,1]\cap\mathbb{Q}\\ 0 & \text{if } x\in[0,1]\setminus\mathbb{Q}\end{cases}$$

is <u>not</u> integrable on $[0,1]$.

Graph of $f(x)$



$$f(x)=\begin{cases} \frac{1}{n} & \text{if } x=\frac{m}{n} \text{ and } m,n \text{ has no common prime factor; } m,n\geq 1 \\ 0 & \text{if } x\in[0,1]\setminus\mathbb{Q} \\ 1 & \text{if } x=0 \end{cases}$$

(page 88)

$\forall x\in[0,1]$, $f(x)\geq 0$

<u>Exercise</u>: $S_f=[0,1]\cap\mathbb{Q}$ (i.e. f discontinuous at <u>only</u> <u>rational</u> numbers on $[0,1]$)

<u>Solution</u> $\boxed{\text{We will show for every } w\in[0,1],\ \lim_{x\to w} f(x)=0.}$

By definition, we have to show $\forall\varepsilon>0,\ \exists\delta>0$ such that $0<|x-w|<\delta$ and $x\in[0,1]$ imply $\underbrace{|f(x)-0|}_{=f(x)}<\varepsilon$.

For every $\varepsilon>0$, we have $\frac{1}{N+1}<\varepsilon\leq\frac{1}{N}$, where $N=\left[\frac{1}{\varepsilon}\right]$ (greatest integer $\leq 1/\varepsilon$).

Let $S=\{0,1,\frac{1}{2},\frac{1}{3},\frac{2}{3},\cdots,\frac{1}{N},\frac{2}{N},\cdots,\frac{N-1}{N}\}$. Let $\delta=\min\{|x-w|: x\in S, x\neq w\}$

<u>Then S finite $\Rightarrow\delta>0$. Now $0<|x-w|<\delta\Rightarrow x\notin S\Rightarrow f(x)\neq 1,\frac{1}{2},\cdots,\frac{1}{N}$</u>

$\Rightarrow |f(x)-0|=f(x)\leq\frac{1}{N+1}<\varepsilon$. $\therefore$ $\underbrace{\lim_{x\to w} f(x)=f(w)}_{f \text{ continuous at } w}\Leftrightarrow f(w)=0\Leftrightarrow w\in[0,1]\setminus\mathbb{Q}$.

Simple Properties of Riemann Integrals

Let f and g be integrable on $[a,b]$.

① $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

$\forall c \in \mathbb{R}$, $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

Proof. Recall f integrable means

$$\int_a^b f(x)dx = \sup\{L(f,P) : P \text{ partition of } [a,b]\}$$
$$= \inf\{U(f,P) : P \text{ partition of } [a,b]\}.$$

So by supremum property and infimum property, $\forall \varepsilon > 0$, $\exists P_1, P_2, P_3, P_4$ such that

$$\int_a^b f(x)dx - \tfrac{\varepsilon}{2} < L(f,P_1) \le \int_a^b f(x)dx$$
$$\int_a^b g(x)dx - \tfrac{\varepsilon}{2} < L(g,P_2) \le \int_a^b g(x)dx$$
$$\int_a^b f(x)dx \le U(f,P_3) < \int_a^b f(x)dx + \tfrac{\varepsilon}{2}$$
$$\int_a^b g(x)dx \le U(g,P_4) < \int_a^b g(x)dx + \tfrac{\varepsilon}{2}.$$

Then for the common refinement $P = P_1 \cup P_2 \cup P_3 \cup P_4$,

$$\int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon < L(f,P) + L(g,P)$$
$$\le L(f+g,P) \le \int_a^b (f(x)+g(x))dx \le U(f+g,P)$$
$$< U(f,P) + U(g,P) < \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

$U(f+g,P) < U(f,P) + U(g,P)$ is similar to $L(f,P) + L(g,P) \le L(f+g,P)$

$L(f,P) + L(g,P) \le L(f+g,P)$

Proof. $L(f,P) = \sum_{i=1}^n m_i \Delta x_i$, $L(g,P) = \sum_{i=1}^n n_i \Delta x_i$, $L(f+g,P) = \sum_{i=1}^n k_i \Delta x_i$

where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$, $n_i = \inf\{g(x) : x \in [x_{i-1}, x_i]\}$, $k_i = \inf\underbrace{\{f(x)+g(x) : x \in [x_{i-1}, x_i]\}}_{\text{Call this set } T}$

$m_i + n_i \le f(x) + g(x)$ for all $x \in [x_{i-1}, x_i]$

$\Rightarrow m_i + n_i$ is a lower bound of $T \Rightarrow m_i + n_i \le \inf T = k_i$ (greatest lower bound)

$\Rightarrow \sum m_i \Delta x_i + \sum n_i \Delta x_i \le \sum k_i \Delta x_i$.

Next $\int_a^b -f(x)dx = \inf\{U(-f,P) : P \text{ partition of } [a,b]\}$

$-\sup S = \inf(-S)$, $\sup S$; $\underbrace{\quad}_{-S}$ 0 $\underbrace{\quad}_{S}$

$$= \inf\{-L(f,P) : P \text{ partition of } [a,b]\}$$
$$= -\sup\{L(f,P) : P \text{ partition of } [a,b]\}$$
$$= -\int_a^b f(x)dx.$$

So $\int_a^b (f(x) - g(x))dx = \int_a^b (f(x) + (-g(x)))dx$

$$= \int_a^b f(x)dx + \int_a^b -g(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

For $\int_a^b cf(x)dx = c\int_a^b f(x)dx$,

Case $c = 0$: $\int_a^b 0 f(x)dx = \int_a^b 0\, dx = 0 = 0 \cdot \int_a^b f(x)dx$.

Case $c > 0$: $\int_a^b cf(x)dx = \sup\{L(cf,P) : P \text{ partition of } [a,b]\}$

$$= \sup\{c\,L(f,P) : \text{〃}\}$$
$$= c\sup\{L(f,P) : \text{〃}\}$$
$$= c\int_a^b f(x)dx.$$

Case $c < 0$: $\int_a^b cf(x)dx = \int_a^b -(-cf(x))dx$

$$= -\int_a^b -cf(x)dx$$

$-c > 0 \rightarrow$ $= -(-c)\int_a^b f(x)dx$

$$= c\int_a^b f(x)dx.$$

② If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$.
Also, $\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|dx$.

Proof. We have $g-f \ge 0$ on $[a,b]$, which implies $L(g-f,P) \ge 0$ $\forall$ partition P of $[a,b]$. So

$\int_a^b (g(x)-f(x))dx = \sup\{L(g-f,P) : P \text{ partition of } [a,b]\} \ge 0$

$\therefore$ $\int_a^b g(x)dx - \int_a^b f(x) \ge 0$, i.e. $\int_a^b f(x)dx \le \int_a^b g(x)dx$.

Next $-|f| \le f \le |f|$ on $[a,b]$. So
$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$, which is the same as $\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|dx$.

---

③ $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for $c \in [a,b]$.

Proof. $\forall$ partition P of $[a,b]$, let $P' = P \cup \{c\}$, $P_1 = P' \cap [a,c]$ and $P_2 = P' \cap [c,b]$. Then $P \subseteq P'$, P_1 is a partition of $[a,c]$ and P_2 is a partition of $[c,b]$.

Let $A = \{L(f,P) : P \text{ partition of } [a,b]\}$
and $B = \{L(f,P') : P \text{ partition of } [a,b] \text{ and } P' = P \cup \{c\}\}$

P' is also partition of $[a,b] \Rightarrow B \subseteq A \Rightarrow \sup B \le \sup A$

$P \subseteq P' \underset{\text{refinement theorem}}{\Rightarrow} L(f,P) \le L(f,P') \Rightarrow \sup A \le \sup B$ $\quad\therefore \sup A = \sup B$.

$\int_a^b f(x)\,dx = \sup A = \sup B$
$= \sup\{ L(f,P_1) + L(f,P_2) : \begin{matrix} P_1 \text{ partition of } [a,c] \\ P_2 \text{ partition of } [c,b] \end{matrix} \}$
$= \sup\{L(f,P_1) : P_1 \text{ partition of } [a,c]\} + \sup\{L(f,P_2) : P_2 \text{ partition of } [c,b]\}$
$= \int_a^c f(x)dx + \int_c^b f(x)dx$.

Definition For an integrable function $f(x)$ on $[a,b]$ and $c \in [a,b]$, the function $F(x) = \int_c^x f(t)dt$ is called an antiderivative (or a primitive function) of f.

Example For $x \in [-1,1]$, define $f(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0. \end{cases}$

f is discontinuous at 0.
$S_f = \{0\} \Rightarrow f$ is integrable on $[-1,1]$.

$F(x) = \int_0^x f(t)dt = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases} = |x|$

is continuous on $[-1,1]$, but not differentiable at 0.
(hence uniformly continuous on $[-1,1]$ by the uniform continuity theorem)

Theorem If f is integrable on $[a,b]$ and $c \in [a,b]$, then $F(x) = \int_c^x f(t)dt$ is uniformly continuous on $[a,b]$.

Proof. f integrable $\Rightarrow f$ bounded $\Rightarrow \exists K>0$ $|f(x)| \le K$ on $[a,b]$
$\forall \varepsilon > 0$, let $\delta = \varepsilon/K$, then
$|x-w| < \delta \Rightarrow |F(x)-F(w)| = \left| \int_w^x f(t)dt \right| \le K|x-w| < K\delta = \varepsilon$.

Fundamental Theorem of Calculus Let $c, x_0 \in [a,b]$.

(1) If f is integrable on $[a,b]$, continuous at x_0 and $F(x) = \int_c^x f(t)\,dt$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$. $\left(\frac{d}{dx}\int_c^x f(t)\,dt\right)(x_0) = f(x_0)$

Proof f cont. at $x_0 \iff \forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x \in [a,b],\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

then
$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| = \left|\frac{\int_{x_0}^x f(t)\,dt}{x-x_0} - \frac{\int_{x_0}^x f(x_0)\,dt}{x-x_0}\right|$$
$$\le \frac{1}{|x-x_0|}\left|\int_{x_0}^x (f(t)-f(x_0))\,dt\right| < \frac{1}{|x-x_0|}\varepsilon|x-x_0| = \varepsilon.$$

By definition of limit, $\lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0)$.

$\therefore F'(x_0) = f(x_0)$.

(2) If G is differentiable on $[a,b]$ and G' integrable on $[a,b]$, then $\int_a^b G'(x)\,dx = G(b) - G(a)$.

(Note G' may not be continuous.) $\int_a^b \frac{d}{dx}G(x)\,dx = G\Big|_a^b$

Proof. $\forall \varepsilon > 0$, by integral criterion, $\exists$ partition P of $[a,b]$ such that $U(G',P) - L(G',P) < \varepsilon$.

$a = x_0 \quad x_1 \quad x_2 \quad \cdots \quad b = x_n$

By mean value theorem, $\exists\, t_j \in [x_{j-1}, x_j]$ such that $G(x_j) - G(x_{j-1}) = G'(t_j)(x_j - x_{j-1})$. Then
$$L(G',P) \le \sum_{j=1}^n G'(t_j)(x_j - x_{j-1}) = G(b) - G(a) \le U(G',P).$$

$L(G',P) \quad \int_a^b G'(x)\,dx \quad G(b)-G(a) \quad U(G',P)$

$\therefore \left|\int_a^b G'(x)\,dx - (G(b)-G(a))\right| < \varepsilon$. Let $\varepsilon \to 0$.

Integration by Parts

If f, g are differentiable on $[a,b]$ and f', g' are integrable on $[a,b]$, then $\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$.

Proof. $\int_a^b (fg)'(x)\,dx = f(b)g(b) - f(a)g(a)$
$$\| $$
$$\int_a^b (f(x)g'(x) + f'(x)g(x))\,dx$$

Subtracting $\int_a^b f'(x)g(x)\,dx$ from both sides, we get formula.

Change of Variable Formula

If $\phi: [a,b] \to \mathbb{R}$ is differentiable, ϕ' integrable on $[a,b]$ and f continuous on $\phi([a,b])$, then
$$\int_{\phi(a)}^{\phi(b)} f(t)\,dt = \int_a^b f(\phi(x))\,\phi'(x)\,dx.$$

Proof. Let $g(x) = \int_{\phi(a)}^{u=\phi(x)} f(t)\,dt$. By part (1) of the Fundamental Theorem of Calculus and Chain Rule, $g'(x) = \frac{dg}{du}\frac{du}{dx} = f(\phi(x))\,\phi'(x)$, which is integrable on $[a,b]$.

So
$$\int_a^b f(\phi(x))\,\phi'(x)\,dx = \int_a^b g'(x)\,dx$$
$$= g(b) - g(a)$$
$$= \int_{\phi(a)}^{\phi(b)} f(t)\,dt.$$

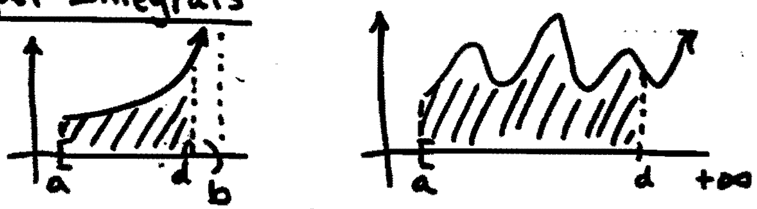
Improper Setting: f is an unbounded function or f is defined on an interval that is not closed or not bounded

Definition Let I be an interval. A function $f: I \to \mathbb{R}$ is locally integrable iff f is integrable on every closed and bounded subintervals of I. We denote this by $f \in L_{loc}(I)$.

Example. If f is continuous on an interval I, then f is locally integrable because f is continuous on every closed and bounded subinterval of I, hence integrable there.

Improper Integrals

Case 1



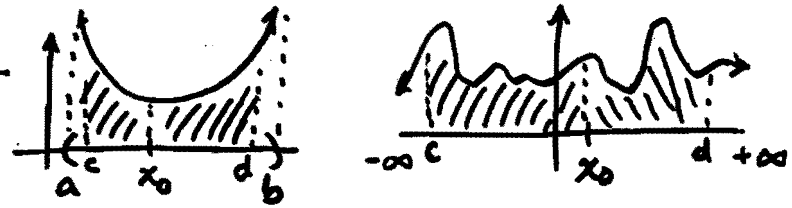
Let $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$, $I = [a,b)$, $f \in L_{loc}(I)$. The improper integral of f on $[a,b)$ is

$$\int_a^b f(x)\,dx = \lim_{d \to b^-} \int_a^d f(x)\,dx \text{ provided the limit exists in } \mathbb{R}.$$

In this case, we say f is improper integrable on $[a,b)$.

The case $I = (a,b]$ with $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R}$ is similarly defined: $\int_a^b f(x)\,dx = \lim_{d \to a^+} \int_d^b f(x)\,dx$.

Case 2



Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$, $I = (a,b)$, $f \in L_{loc}(I)$. The improper integral of f on (a,b) is $\;x_0 \in I$

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^{x_0} f(x)\,dx + \lim_{d \to b^-} \int_{x_0}^d f(x)\,dx$$

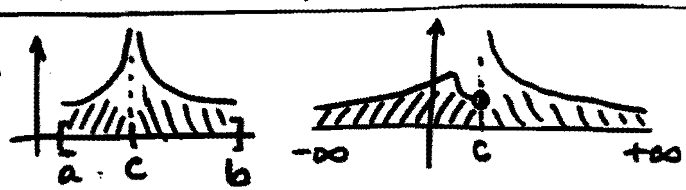
provided the limits exist in $\mathbb{R}$. In this case, we say f is improper integrable on (a,b).

Remark The answer does not depend on x_0. For another x_0', the first term

$$\lim_{c \to a^+} \int_c^{x_0'} f(x)\,dx = \lim_{c \to a^+} \int_c^{x_0} f(x)\,dx + \underbrace{\int_{x_0}^{x_0'} f(x)\,dx}_{\text{number because } f \text{ integrable on } [x_0, x_0']}$$

So left side is a number iff the right side is a number. The second term is similar.

Case 3



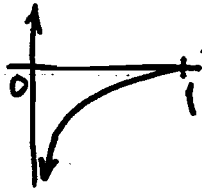
Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$, I be an interval with endpoints a, b, $I_0 = I \cap (-\infty, c)$, $I_1 = I \cap (c, +\infty)$ for $c \in (a,b)$. $f \in L_{loc}(I_0)$, $f \in L_{loc}(I_1)$. The improper integral of f on I is

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

provided both integrals are numbers. In that case, f is improper integrable on I.

In each case, if the improper integral is a number, then we say the improper integral <u>converges</u>, otherwise we say it <u>diverges</u>.

<u>Examples</u> ① Consider $\int_0^1 \ln x\, dx$.



$\ln x$ continuous on $(0,1] \Rightarrow \ln x \in L_{loc}((0,1])$

$$\int_0^1 \ln x\, dx = \lim_{c\to 0^+} \int_c^1 \ln x\, dx \quad \text{integration by parts}$$

$$= \lim_{c\to 0^+} (x\ln x - x)\Big|_c^1$$

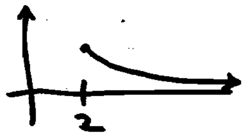
$$= \lim_{c\to 0^+} (-1 - \underbrace{c\ln c}_{} + c) = -1$$

$\ln x$ improper integrable on $(0,1]$.

$\therefore \int_0^1 \ln x\, dx$ converges to -1.

$(\ln c)/(\frac{1}{c}) \to 0$ by l'Hopital's rule

② Consider $\int_2^{+\infty} \frac{1}{x^2} dx$.



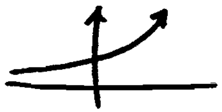
$\frac{1}{x^2}$ Continuous on $[2,+\infty) \Rightarrow \frac{1}{x^2} \in L_{loc}([2,+\infty))$

$$\int_2^{+\infty} \frac{1}{x^2} dx = \lim_{d\to+\infty} \int_2^d \frac{1}{x^2} dx$$

$$= \lim_{d\to+\infty} -\frac{1}{x}\Big|_2^d = \lim_{d\to+\infty}\left(-\frac{1}{d}+\frac{1}{2}\right) = \frac{1}{2}$$

$\therefore \int_2^{+\infty} \frac{1}{x^2} dx$ converges to $\frac{1}{2}$.

③ Consider $\int_{-\infty}^{+\infty} e^x dx$. $\quad e^x \in L_{loc}((-\infty,+\infty))$.



Take $x_0 = 0$. $\int_0^{+\infty} e^x dx = \lim_{d\to+\infty} \int_0^d e^x dx = \lim_{d\to+\infty} e^x\Big|_0^d$

$= \lim_{d\to+\infty} (e^d - 1) = +\infty$, not a number.

$\therefore e^x$ is not improper integrable on $(-\infty,+\infty)$.

$\int_{-\infty}^{\infty} e^x dx$ diverges.

<u>Question</u> What if the improper integral cannot be computed?

<u>P-test</u> For $0<a<\infty$, $\int_a^{+\infty} \frac{1}{x^p} dx < \infty \Leftrightarrow p>1$.

Also, $\int_0^a \frac{1}{x^p} dx < \infty \Leftrightarrow p<1$.

<u>Comparison Test</u> Suppose $0 \le f(x) \le g(x)$ on interval I and $f, g \in L_{loc}(I)$. If g is improper integrable on I, then f is improper integrable on I. (Taking contrapositive, if f is not improper integrable on I, then g is not improper integrable on I.)

<u>Limit Comparison Test</u> Suppose $f(x), g(x) > 0$ on $(a,b]$ and $f, g \in L_{loc}((a,b])$.

If $\lim_{x\to a^+} \frac{g(x)}{f(x)}$ is a positive number L, then either (both $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converges) or (both diverges).

If $\lim_{x\to a^+} \frac{g(x)}{f(x)} = 0$, then $\left(\int_a^b f(x)dx \text{ converges} \Rightarrow \int_a^b g(x)dx \text{ converges}\right)$.

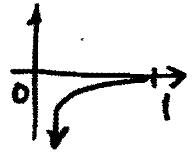
If $\lim_{x\to a^+} \frac{g(x)}{f(x)} = +\infty$, then $\left(\int_a^b f(x)dx \text{ diverges} \Rightarrow \int_a^b g(x)dx \text{ diverges}\right)$.

In the case $[a,b)$, we take $\lim_{x\to b^-} \frac{g(x)}{f(x)}$. Results are similar.

<u>Absolute Convergence Test</u> Let $f \in L_{loc}(I)$.

If $|f|$ is improper integrable on I, then f is improper integrable on I.

Examples ④ Consider $\int_0^1 \frac{\ln x}{1+x^2}dx$. $\frac{\ln x}{1+x^2} \in L_{loc}((0,1])$.



On $(0,1]$, $\left|\frac{\ln x}{1+x^2}\right| = \frac{|\ln x|}{1+x^2} \le |\ln x| = -\ln x$

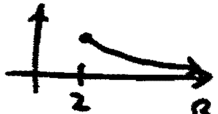
By example ①, $\int_0^1 |\ln x|dx = -\int_0^1 \ln x dx = 1$.

By comparison test, $\int_0^1 \left|\frac{\ln x}{1+x^2}\right| dx$ converges.

By absolute convergence test, $\int_0^1 \frac{\ln x}{1+x^2}dx$ converges.

---

⑤ Consider $\int_2^{+\infty} \frac{dx}{\sqrt{x^2-1}}dx$.

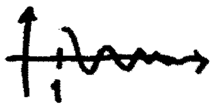


On $[2,+\infty)$, $0 < \frac{1}{x} < \frac{1}{\sqrt{x^2-1}}$. $\int_2^{+\infty} \frac{1}{x}dx = \infty$ by p-test

By comparison test, $\int_2^{+\infty} \frac{dx}{\sqrt{x^2-1}}$ diverges.

---

⑥ Consider $\int_1^{+\infty} \frac{\sin x}{x}dx$.



$$\int_1^c \frac{\sin x}{x}dx = \left.\frac{-\cos x}{x}\right|_1^c - \int_1^c \frac{\cos x}{x^2}dx$$
$$= \frac{-\cos c}{c} + \cos 1 - \int_1^c \frac{\cos x}{x^2}dx$$

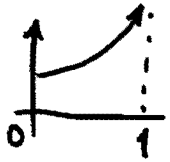
Since $|\cos c| \le 1$, $\lim_{c\to+\infty} \frac{-\cos c}{c} = 0$.

On $[1,+\infty)$, $\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$ and $\int_1^{+\infty} \frac{1}{x^2}dx < \infty$ by p-test.

By comparison test, $\int_1^{+\infty} \left|\frac{\cos x}{x^2}\right| dx < \infty$. By absolute convergence test, $\int_1^{+\infty} \frac{\cos x}{x^2}dx$ converges. $\therefore$ $\int_1^{+\infty} \frac{\sin x}{x}dx$ converges.

---

⑦ Consider $\int_0^1 \frac{dx}{1-x^3}$.



$\frac{1}{1-x^3} = \frac{1}{1-x}\left(\frac{1}{1+x+x^2}\right)$, $\lim_{x\to1^-} \frac{1}{1+x+x^2} = \frac{1}{3}$

So $\lim_{x\to1^-} \frac{\frac{1}{1-x^3}}{\frac{1}{3(1-x)}} = 1$.

$\int_0^1 \frac{1}{3(1-x)}dx = \lim_{d\to1^-} \int_0^d \frac{1}{3(1-x)}dx = \lim_{d\to1^-} -\frac{1}{3}\ln(1-d) = +\infty$.

By limit comparison test, $\int_0^1 \frac{1}{1-x^3}dx$ diverges.

---

⑧ Consider $\int_0^5 \frac{dx}{\sqrt[3]{7x+2x^4}}$.

$\frac{1}{\sqrt[3]{7x+2x^4}} = \frac{1}{\sqrt[3]{x}}\left(\frac{1}{\sqrt[3]{7+2x^3}}\right)$, $\lim_{x\to0^+} \frac{1}{\sqrt[3]{7+2x^3}} = \frac{1}{\sqrt[3]{7}}$

So $\lim_{x\to0^+} \left(\frac{1}{\sqrt[3]{7x+2x^4}}\right) / \frac{1}{\sqrt[3]{7x}} = 1$.

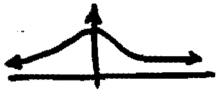
$\int_0^5 \frac{1}{\sqrt[3]{7x}}dx = \frac{1}{\sqrt[3]{7}} \int_0^5 \frac{1}{x^{1/3}}dx < \infty$ by p-test.

By limit comparison test, $\int_0^5 \frac{dx}{\sqrt[3]{7x+2x^4}}$ converges.

Cauchy Principal Value of Integrals

Definition. Let $f\in L_{loc}(\mathbb{R})$. The principal value of $\int_{-\infty}^{\infty} f(x)\,dx$ is $\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{c\to+\infty}\int_{-c}^{c} f(x)\,dx$.

Examples ① Consider $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$ and $\text{P.V.}\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$.



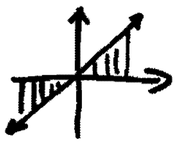
$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \int_{-\infty}^{0}\frac{1}{1+x^2}dx + \int_{0}^{+\infty}\frac{1}{1+x^2}dx$$

$$= \lim_{c\to-\infty}\int_{c}^{0}\frac{1}{1+x^2}dx + \lim_{d\to+\infty}\int_{0}^{d}\frac{1}{1+x^2}dx = \lim_{c\to-\infty}(-\arctan c) + \lim_{d\to+\infty}(\arctan d)$$

$$= \frac{\pi}{2}+\frac{\pi}{2} = \pi.$$

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \lim_{c\to+\infty}\int_{-c}^{c}\frac{1}{1+x^2}dx = \lim_{c\to+\infty}(2\arctan c) = 2\cdot\frac{\pi}{2} = \pi.$$

② Consider $\int_{-\infty}^{\infty} x\,dx$ and $\text{P.V.}\int_{-\infty}^{\infty} x\,dx$.



$$\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{+\infty} x\,dx = \lim_{c\to-\infty}\left(-\frac{c^2}{2}\right) + \lim_{d\to+\infty}\left(\frac{d^2}{2}\right)$$

$$= -\infty + \infty, \text{ not exist.}$$

$$\text{P.V.}\int_{-\infty}^{\infty} x\,dx = \lim_{c\to+\infty}\int_{-c}^{c} x\,dx = \lim_{c\to+\infty}\frac{x^2}{2}\Big|_{-c}^{c} = \lim_{c\to+\infty} 0 = 0.$$

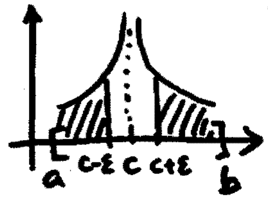
So $\int_{-\infty}^{\infty} f(x)\,dx$ and $\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx$ may be different.

Theorem If the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ exists in $\mathbb{R}$, then $\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx$ exists and equals the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$. The converse is false by example ②.

Proof. If $\int_{-\infty}^{\infty} f(x)\,dx$ exists, then $\lim_{d\to-\infty}\int_{d}^{0} f(x)\,dx$ and $\lim_{c\to+\infty}\int_{0}^{c} f(x)\,dx$ both exist as numbers. So

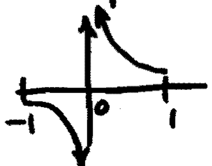
$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{c\to+\infty}\int_{-c}^{c} f(x)\,dx = \lim_{c\to+\infty}\left(\int_{-c}^{0} f(x)\,dx + \int_{0}^{c} f(x)\,dx\right)$$

$$= \lim_{d\to-\infty}\int_{d}^{0} f(x)\,dx + \lim_{c\to+\infty}\int_{0}^{c} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$



Definition Let I be an interval with endpoints a and b, let $c\in(a,b)$, $I_0 = I\cap(-\infty,c)$ and $I_1 = I\cap(c,+\infty)$. Let $f\in L_{loc}(I_0)$ and $f\in L_{loc}(I_1)$. Define the principal value of $\int_a^b f(x)\,dx$ as $\text{P.V.}\int_a^b f(x)\,dx = \lim_{\varepsilon\to0^+}\left(\int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx\right)$.

Example Consider $\int_{-1}^{1}\frac{1}{x}dx$ and $\text{P.V.}\int_{-1}^{1}\frac{1}{x}dx$.



$$\int_0^1\frac{1}{x}dx = \lim_{c\to0^+}\int_c^1\frac{1}{x}dx = \lim_{c\to0^+}(-\ln c) = +\infty \text{ not a number.}$$

So $\int_{-1}^{1}\frac{1}{x}dx$ diverges.

$$\text{P.V.}\int_{-1}^{1}\frac{1}{x}dx = \lim_{\varepsilon\to0^+}\left(\int_{-1}^{-\varepsilon}\frac{1}{x}dx + \int_{\varepsilon}^{1}\frac{1}{x}dx\right) = \lim_{\varepsilon\to0^+}(\ln|-\varepsilon| - \ln|\varepsilon|) = 0.$$

Remarks ① $\int_{-1}^{1}\frac{1}{x}dx = \ln|x|\Big|_{-1}^{1} = \ln 1 - \ln 1 = 0$ is incorrect as the fundamental theorem of calculus requires $f(x)=\ln|x|$ differentiable on the whole interval $[-1,1]$ and $f'(x)=\frac{1}{x}$ bounded integrable!

② There is a theorem "$\int_a^b f(x)\,dx$ converges $\Rightarrow \text{P.V.}\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$." The proof is similar to the one above.

Proof of p-test Since $\int_a^1 \frac{1}{x^p}dx < \infty$, so

$$\int_a^\infty \frac{1}{x^p}dx < \infty \Leftrightarrow \int_1^\infty \frac{1}{x^p}dx < \infty \overset{\text{integral test}}{\Leftrightarrow} \sum_{n=1}^\infty \frac{1}{n^p} < \infty \Leftrightarrow p > 1.$$

$$\int_0^a \frac{1}{x^p}dx < \infty \Leftrightarrow \int_0^1 \frac{1}{x^p}dx < \infty$$

$$\Leftrightarrow \int_1^\infty \frac{1}{y^{2-p}}dy < \infty \qquad \int_c^1 \frac{1}{x^p}dx = \int_1^{1/c} \frac{1}{y^{2-p}}dy, \quad y = 1/x, \quad dy = -1/x^2 dx$$

$$\Leftrightarrow 2-p > 1 \Leftrightarrow p < 1.$$

Proof of Comparison Test For the case $I = [a,b)$, if

$$\left.\begin{array}{l} 0 \le f \le g \text{ on } [a,b) \\ g \text{ improper integrable on } [a,b) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \int_a^d f(x)dx \text{ is increasing when } d \nearrow b \\ \int_a^d f(x)dx \le \int_a^b g(x)dx < \infty \end{array}\right.$$

$$\overset{\text{Monotone function theorem}}{\Longrightarrow} \int_a^b f(x)dx = \lim_{d\to b^-} \int_a^d f(x)dx < \infty.$$

$\therefore f$ is improper integrable on $[a,b)$.

The cases $(a,b]$ and (a,b) are similar.

Proof of Limit Comparison Test On $(a,b]$, $f(x), g(x) > 0$.

Case $\lim_{x\to a^+} \frac{g(x)}{f(x)} = L$ positive number. For $\varepsilon = \frac{L}{2} > 0$, $\exists \delta > 0$ such that

$\forall x \in (a, a+\delta) \Rightarrow \frac{L}{2} = L - \varepsilon < \frac{g(x)}{f(x)} < L + \varepsilon = \frac{3L}{2}$. Then

$$\frac{L}{2}\int_a^{a+\delta} f(x)dx \le \int_a^{a+\delta} g(x)dx \le \frac{3L}{2}\int_a^{a+\delta} f(x)dx. \leftarrow \text{by comparison test.}$$

So $\int_a^{a+\delta} f(x)dx < \infty \Leftrightarrow \int_a^{a+\delta} g(x)dx < \infty$.

Since $f, g \in L_{loc}((a,b])$ and $[a+\delta, b] \subseteq (a,b]$, so

$\int_{a+\delta}^b f(x)dx < \infty$, $\int_{a+\delta}^b g(x) < \infty$. Therefore,

$$\int_a^b f(x)dx < \infty \Leftrightarrow \int_a^b g(x)dx < \infty.$$

Case $\lim_{x\to a^+} \frac{g(x)}{f(x)} = 0$ For $\varepsilon = 1 > 0$, $\exists \delta' > 0$ such that

$\forall x \in (a, a+\delta') \Rightarrow 0 < \frac{g(x)}{f(x)} < 1 \Rightarrow 0 < g(x) < f(x)$. Then

$$0 \le \int_a^{a+\delta'} g(x)dx \le \int_a^{a+\delta'} f(x)dx. \leftarrow \text{by comparison test}$$

So $\int_a^b f(x)dx < \infty \Rightarrow \int_a^{a+\delta'} f(x)dx < \infty \Rightarrow \int_a^{a+\delta'} g(x)dx < \infty$

$\Rightarrow \int_a^b g(x)dx < \infty$.

Case $\lim_{x\to a^+} \frac{g(x)}{f(x)} = +\infty$ For $r = 1$, $\exists \delta'' > 0$ such that

$\forall x \in (a, a+\delta'') \Rightarrow \frac{g(x)}{f(x)} > 1 \Rightarrow g(x) > f(x)$. Then

$$\int_a^{a+\delta''} g(x)dx \ge \int_a^{a+\delta''} f(x)dx. \leftarrow \text{by comparison test}$$

So $\int_a^b f(x)dx = +\infty \Rightarrow \int_a^b g(x)dx = +\infty$.

Proof of Absolute Convergence Test

$-|f| \le f \le |f|$ on I $\Rightarrow 0 \le f + |f| \le 2|f|$ on I

$|f|$ improper integrable on I $\Rightarrow f + |f|$ improper integrable on I by comparison test

$\Rightarrow f = (f+|f|) - |f|$ improper integrable on I.