1. (26 points) .

Let $f \in C([a,b]), a < b$.

5 1.1 Write down the definition of continuity of f at $x_0 \in [a,b]$ using ϵ-δ language.

5 1.2 State the extreme value theorem for f.

6 1.3 Assume that $f(a) < e^a, f(b) > e^b$, show that $\exists \xi \in [a,b]$ s.t $f(\xi) = e^\xi$.

10 1.4 Define $M : [a,b] \to \mathbb{R}$ by

$$M(x) = \sup\{f(t) : a \le t \le x\}.$$

4 6

Show that $M(x)$ is increasing and continuous.

Solution:

1.1 f is continuous at $x_0 \in [a,b]$ iff $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall |x - x_0| < \delta$, $|f(x) - f(x_0)| < \epsilon$.

1.2 If $f : [a,b] \to \mathbb{R}$ is continuous, then there exist $x_0, x_1 \in [a,b]$ such that

$$f(x_0) = \sup_{x\in[a,b]} f(x) = \max_{x\in[a,b]} f(x),$$

$$f(x_1) = \inf_{x\in[a,b]} f(x) = \min_{x\in[a,b]} f(x).$$

1.3 Let $g(x) = f(x) - e^x$, then $g(a) = f(a) - e^a < 0$ and $g(b) = f(b) - e^b > 0$. Since f and e^x are continuous, g is continuous. By intermediate value theorem, $\exists \xi \in [a,b]$ such that $g(\xi) = 0$. So $f(\xi) = e^\xi$.

1.4 $\forall x, y \in [a,b]$ with $x \le y$, we have

$$M(x) = \sup_{t\in[a,x]} f(t) \le \sup_{t\in[a,y]} f(t) = M(y).$$

Therefore, M is increasing.

Since $M(x)$ is increasing, the left handed limit $M(x^-)$ and the right handed $M(x^+)$ limit exist and

$$M(x^{-1}) \le M(x) \le M(x^+).$$

If $M(x) > f(x)$, that is x attains the maximum in (a,x), then

$$M(x^-) = M(x) = M(x^+).$$

If $M(x) = f(x)$, then by the continuity of $f(x)$, we have

$$M(x^-) = M(x) = M(x^+).$$

Therefore, $M(x)$ is continuous.