Math 2033 (Mathematical Analysis) Spring 2015

Midterm

Directions: This is a closed book exam. Every student must show work in every problem with *full* details legibly to receive marks. Calculators are not permitted in this midterm. *Answers alone are worth very little!*

Notations: $\mathbb{R}$ denotes the set of all real numbers. $\mathbb{Q}$ denotes the set of all rational numbers. The variable n in the problems below takes on positive integer values $1, 2, 3, \ldots$.

1. (12 marks) Let A be a nonempty bounded subset of $\mathbb{R}$ such that $\inf A = 1$ and $\sup A = 2$. Let
$$B = \Big\{\sqrt{y}\cos x : x \in \Big(0, \frac{\pi}{3}\Big] \cap \mathbb{Q},\ y \in A\Big\}.$$
Prove that B is bounded. Determine (with proof) the infimum and supremum of B.

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2. (a) (10 marks) Prove that the sequence $\{w_n\}$ converges, where
$$w_1 = 6 \quad \text{and} \quad \text{for } n = 1, 2, 3, \ldots,\ w_{n+1} = 6 - \frac{9}{w_n}$$
and find its limit. Show all details.

   (b) (14 marks) Prove that the sequence $\{x_n\}$ converges, where
$$x_1 = 60 \quad \text{and} \quad \text{for } n = 1, 2, 3, \ldots,\ x_{n+1} = 8 + \frac{120}{x_n}$$
and find its limit. Show all details.

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3. (24 marks) Let $y_1, y_2, y_3, \ldots$ and $z_1, z_2, z_3, \ldots$ be sequences of real numbers such that both converge to 4. Prove that
$$\lim_{n\to\infty}\Big(\frac{9}{z_n^2 + 2} + \frac{5}{y_n - 2}\Big) = 3$$
by checking the definition of limit of a sequence only.

   (Do not use computation formulas, sandwich theorem or l'Hopital's rule! Otherwise, you will get zero mark for this problem.)

–End of Paper–