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Welcome to Math 2033 (Math. Analysis)

Main Items in the Syllabus

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Prerequisite: Math 1014 or 1018 or 1020 or 1024

Website for Lecture Notes or Transparencies

https://www.math.ust.hk/~makyli/UG.html

Just scroll down to the Math 2033 part.

What is Analysis ?

Algebra Equations

Geometry Figures

Number Theory Integers, Rational Numbers

Analysis

Limit, Continuity, Differentiation

Integration, ...

Number Theory vs Analysis

$$\frac{2^x - x^2}{4x^4+1} = 987654321$$

Number Theory : any integer solution ?

Analysis : any real solution ?

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$f(x) = \dfrac{2^x - x^2}{4x^4+1}$ is continuous on $\mathbb{R}$

$f(0) = 1$

$2^{10} = 1024 > 10^3$

$f(100) = \dfrac{2^{100} - 100^2}{4 \cdot 100^4 + 1} > \dfrac{10^{30} - 10^4}{10 \cdot 10^8} > 10^{20}$

> 987654321

There is a real solution in $[0, 100]$.
We can find the solution to as many decimal place as we like by bisection method: Test $x_0 = 50$. Then

$x_1 = 25$, $x_2 = 12.5$, ...

or $x_1 = 75$, ... $\quad x = \lim\limits_{n \to \infty} x_n$

Analysis solves equations by using limit concepts.

$2^x - x^2 = 987654321\,(4x^4+1)$

If x is integer, then

x cannot be negative $\left(\begin{matrix}\text{otherwise} \\ 0 < 2^x < 1\end{matrix}\right)$

$x \neq 0 \qquad 1 \neq 987654321$

$x \neq 1 \qquad 1 \neq 987654321\,(5)$

If $x \geq 2$, then $2^x = 4a$

$x^2 = \begin{cases} (2b)^2 = 4b^2 & \text{if } x \text{ is even} \\ (2b+1)^2 = 4b^2 + 4b + 1 & \text{if } x \text{ is odd} \end{cases}$

$2^x - x^2 = \begin{cases} 4a - 4b^2 = 4c \\ 4a - (4b^2 + 4b + 1) = 4c - 1 \end{cases}$

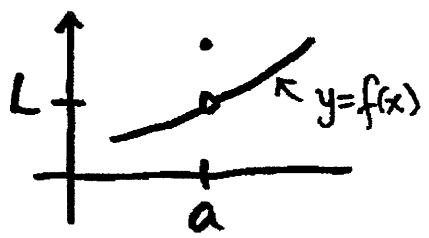
$98765432\underline{\underline{1}}\,(4x^4+1)$

$= (4d+1)(4x^4+1) = 4e+1$

$4c - 1 \neq 4e + 1 \quad$ <u>no solution</u>

Number theory solves equations by studying forms of numbers, not by approximations.

Limit $\lim_{x\to a} f(x) = L$



"As x gets close to a, $f(x)$ gets close to L."

"Close" is a feeling, no way to judge!

$$\left.\begin{array}{l} f_1(x) \to L_1 \\ f_2(x) \to L_2 \\ \vdots \\ f_{1000}(x) \to L_{1000} \end{array}\right\} \begin{array}{l} \text{Is} \\ f_1(x)+f_2(x)+\cdots+f_{1000}(x) \\ \text{"close" to} \\ L_1+L_2+\cdots+L_{1000}\ ? \end{array}$$

Although we learned this as a fact in calculus, one can challenge this fact as follow:

If $f_1(x), f_2(x), \cdots, f_{1000}(x)$ are 49.9 and
$L_1, L_2, \cdots, L_{1000}$ are 50,
then $f_i(x)$ may be considered close to L_i,
but $f_1(x)+\cdots+f_{1000}(x)$ is 49900 and
$L_1+\cdots+L_{1000}$ is 50000,
which are 100 units apart, not that close.

If $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$,

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ "L'Hopital's Rule"

Is this correct?

Let $f(x) = x^2 \sin\frac{1}{x}$ and $g(x) = \sin x$

$|x^2 \sin\frac{1}{x}| \le |x|^2 \to 0$ as $x \to 0$

$\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$.

Apply formula above:

$$\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin x} = \lim_{x\to 0} \frac{\overbrace{2x\sin\frac{1}{x}}^{\to 0} - \overbrace{\cos\frac{1}{x}}^{\text{"oscillate"}}}{\underbrace{\cos x}_{\to 1}}$$

limit doesn't exist!

5

How about

$$\frac{x^2\sin\frac{1}{x}}{\sin x} = \underbrace{\left(\frac{x}{\sin x}\right)}_{\downarrow \atop 1}\underbrace{\left(x\sin\frac{1}{x}\right)}_{\downarrow \atop 0} \to 0 \quad \text{as } x\to 0$$

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x} = 0 \text{ ?}$$ This is correct!

When is $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$?

$\displaystyle\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ is not enough.

What additional conditions do we need?

Why those conditions are enough?

Need to do proofs !!!

The correct l'Hopital's rule requires the additional condition $\displaystyle\lim_{x\to a}\frac{f'(x)}{g'(x)}$ = number or $+\infty$ or $-\infty$. This is not stated in some secondary school textbooks. So don't just accept what books or teachers tell you. Look at a proof to decide.

Question: Let f be a continuous function on $\mathbb{R}$. Must f be differentiable at every x in $\mathbb{R}$?

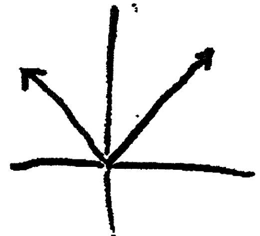
Answer: No. $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

is continuous on $\mathbb{R}$. However,

$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h-0}{h} = 1$$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{-h-0}{h} = -1$$

So $f'(0)$ doesn't exist.



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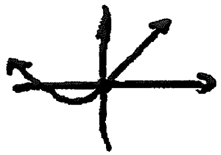
Question: What is the derivative of

$$f(x)=\begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x=0 \\ x^2+x & \text{if } x<0 \end{cases} \quad ?$$

Is it

$$f'(x)=\begin{cases} 1 & \text{if } x>0 \\ 0 & \text{if } x=0 \\ 2x+1 & \text{if } x<0 \end{cases} \quad ?$$

No, $f'(0)=1$. Why?



$$\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{h-0}{h}=1$$

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{(h^2+h)-0}{h}$$

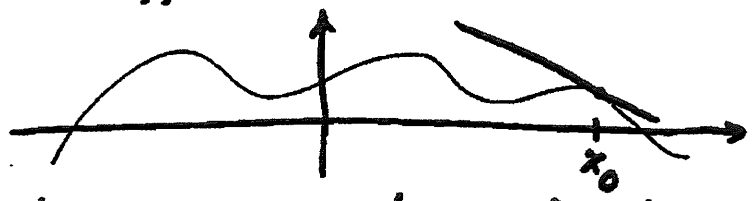
$$=\lim_{h\to 0}(h+1)$$

$$=1.$$

$\therefore f'(0)=1.$

Question: Which one is true?

① Every continuous function $f:\mathbb{R}\to\mathbb{R}$ is differentiable at some x_0



② There exists a continuous function $f:\mathbb{R}\to\mathbb{R}$, which is not differentiable at every x_0.

② is true. So we cannot always differentiate a continuous function!

Why is ② true? Need to do proofs!!!

Logic = Rules for Reasoning

Chapter 1

Notations $\sim$ (or $\neg$) not, the opposite of

quantifiers
- $\forall$ for all, for any, for every
- $\exists$ there is (at least one), there exists, there are (some)

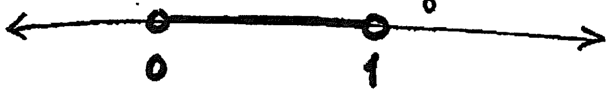
p, q variables of statements or phrases

Negation = Taking opposite

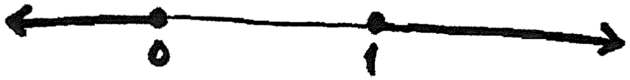
① $\sim(\sim p) = p$

② $\sim(p \text{ and } q) = (\sim p) \text{ or } (\sim q)$

Example $\underbrace{x>0}_{p}$ and $\underbrace{x<1}_{q}$



Opposite

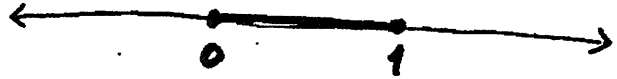


$\underbrace{x \le 0}_{\sim p}$ or $\underbrace{x \ge 1}_{\sim q}$

③ $\sim(p \text{ or } q) = (\sim p) \text{ and } (\sim q)$

Example $\underbrace{x<0}_{p}$ or $\underbrace{x>1}_{q}$

Opposite



$\underbrace{x \ge 0}_{\sim p}$ and $\underbrace{x \le 1}_{\sim q}$

④ $\sim(\forall x\ \exists y \cdots\ p) = \exists x\ \forall y \cdots \sim p$

Example 1 For every $x \ge 0$, x has a square root

True $\forall x \ge 0$ (x has a square root)

Opposite There exists $x \ge 0$ such that x does not have square root.

False $\exists x \ge 0 \sim$(x has a square root).

8

<u>Example 2</u> For every $x \geq 0$, there is $y \geq 0$ such that $y^2 = x$.

True

$$\forall x \geq 0 \ \exists y \geq 0 \ \ y^2 = x$$

<u>Opposite</u> There exists $x \geq 0$ such that for every $y \geq 0$, $y^2 \neq x$.

False

$$\exists x \geq 0 \ \forall y \geq 0 \ \ \sim (y^2 = x).$$

Conditional Statements (If-then statements)

$$\left.\begin{array}{l} \text{If } p \text{, then } q \\ p \text{ implies } q \\ p \text{ only if } q \\ p \text{ is sufficient for } q \\ q \text{ is necessary for } p \end{array}\right\} \ p \Rightarrow q$$

(5) $\sim (p \Rightarrow q) = p$ and $(\sim q)$

<u>Example</u> If $x \geq 0$, then $|x| = x$

True

$$(x \geq 0) \Longrightarrow (|x| = x)$$

<u>Opposite</u> $x \geq 0$ and $|x| \neq x$

False $(x \geq 0)$ and $\sim (|x| = x)$

<u>Remark</u>

$$\begin{aligned} p \Rightarrow q &= \sim (\sim (p \Rightarrow q)) && \text{by (1)} \\ &= \sim (p \text{ and } (\sim q)) && \text{by (5)} \\ &= (\sim p) \text{ or } \sim (\sim q) && \text{by (2)} \\ &= (\sim p) \text{ or } q && \text{by (1)}. \end{aligned}$$

Terminologies

For the statement "If p, then q" ($p \Rightarrow q$), its converse is "If q, then p" ($q \Rightarrow p$), its contrapositive is "If $\sim q$, then $\sim p$" $(\sim q) \Rightarrow (\sim p)$

Examples 1

Statement: If $x=-3$, then $x^2=9$ (True)

Converse: If $x^2=9$, then $x=-3$ (False, as x may be 3)

Contrapositive: If $x^2 \neq 9$, then $x \neq -3$ (True)

Example 2

Statement: $(x=-3) \Rightarrow (2x=-6)$ (True)

Converse: $(2x=-6) \Rightarrow (x=-3)$ (True)

Contrapositive: $(2x \neq -6) \Rightarrow (x \neq -3)$ (True)

Example 3

Statement: If $|x|=3$, then $x=-3$ (False, as x may be 3)

Converse: If $x=-3$, then $|x|=3$ (True)

Contrapositive: If $x \neq -3$, then $|x| \neq 3$ (False, as x may be 3)

Remarks ① Contrapositive = statement

$(\sim q) \Rightarrow (\sim p)$
$= \sim(\sim q)$ or $(\sim p)$ by earlier remark
$= q$ or $(\sim p)$ by ①
$= (\sim p)$ or q
$= p \Rightarrow q$ by earlier remark

② "If p, then q" and "If q, then p" are true

We will say "p if and only if q" or "p is necessary and sufficient for q."

Abbreviation: if and only if = iff

③ $\forall \alpha \forall \beta = \forall \beta \forall \alpha$

$\exists \alpha \exists \beta = \exists \beta \exists \alpha$

$\forall \alpha \exists \beta \neq \exists \beta \forall \alpha$ → EXAMPLE

Every student is assigned a number

(($\forall$ student $\exists$ number (student is assigned number)

$\exists$ number $\forall$ student (Student is assigned number)

((There is a number such that every student is assigned the number.

EXERCISES Negate each of the following

1. If ΔABC is a right triangle, then $a^2+b^2=c^2$.
2. $\forall \varepsilon>0\ \exists \delta>0$ such that
$$0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon$$
3. No news is good news. ↙ Ambiguous statement
More than one interpretations

Solutions

1. ΔABC is a right triangle and $a^2+b^2\neq c^2$
2. $\exists \varepsilon>0\ \forall \delta>0$, we have
$$0<|x-x_0|<\delta \text{ and } |f(x)-L|\geq\varepsilon.$$
3. Interpretation I All news are bad news
Opposite: There exists a good news.
Interpretation II If no news is received, then it is good news.
Opposite: No news is received and it is not good news.

When ambiguous statement is presented, ask for the intended interpretation, then negate that interpretation.

Chapter 2 Sets ← Language to communicate math efficiently and precisely

A **set** is a collection of math "objects"

usually numbers, functions, ordered-pairs, …

The objects in the set are the **elements** of the set.

We write $x \in S$ iff x is an element of set S.

$x \notin S$ iff x is not an element of set S.

Example Let $\mathbb{Z}$ be the set of all integers.

Then $-54 \in \mathbb{Z}$ and $\sqrt{2} \notin \mathbb{Z}$.

A set is **finite** iff it has finitely many elements.

A set is **infinite** iff it has infinitely many elements.

The **empty set** is the set having no element and is denoted by $\emptyset$.

Common Sets in Math natural number

$\mathbb{N}$ the set of all positive integers

$\mathbb{Z}$ the set of all integers ← $\mathbb{Z}$ is for "Zahlen"

German for "Number"

$\mathbb{Q}$ the set of all rational numbers

$\mathbb{R}$ the set of all real numbers

$\mathbb{C}$ the set of all complex numbers

Set Descriptions

① List elements enclosed in braces

$S = \{1, 2, 3\}$, $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$, $\emptyset = \{\ \}$,

$\mathbb{Z} = \{\cdots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \cdots\}$

② Write the form of the elements, followed by a colon, followed by descriptions of variables inside braces.

$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$

$\mathbb{R} = \{x : x \text{ is a real number}\}$

$\mathbb{C} = \{x + iy : x \in \mathbb{R}, y \in \mathbb{R}, i = \sqrt{-1}\}$

$[a, b] = \{x : x \in \mathbb{R} \text{ and } a \leq x \leq b\}$

l_m ← the line with equation $y = mx$

$= \{(x, y) : x, y \in \mathbb{R} \text{ and } y = mx\}$

$= \{(x, mx) : x \in \mathbb{R}\}$

Let A and B be sets.

A is a <u>subset</u> of B (or B <u>contains</u> A) iff every element of A is also an element of B. In this case, we write $A \subseteq B$.

In particular, $\emptyset \subseteq S$ for every set S.

We say $A = B$ iff $A \subseteq B$ and $B \subseteq A$

A is a <u>proper subset</u> of B iff $A \subseteq B$ and $A \neq B$. In this case, we write $A \subset B$.

<u>Example</u>. Let $A = \{1,2\}$, $B = \{1,2,3\}$ and $C = \{1,1,2,3\}$. Then $A \subset B = C$.

<u>Remarks</u> Repeated elements count only once.

B and C are 3 element sets.

A has 2 elements

$\{4,4,4,4,\dots\}$ has 1 element only.
← a finite set!

If $X \subseteq Y$, then the number of elements of X is less than or equal to the number of elements of Y.

Let S be a set. The <u>power set</u> of S is the set of all subsets of S. It is denoted by $P(S)$ or 2^S.

<u>Examples</u>

$S = \emptyset$ Then $\emptyset \subseteq S$ $\quad P(S) = \{\emptyset\}$

$S = \{x\}$ Then $\emptyset, \{x\} \subseteq S$ $\quad P(S) = \{\emptyset, \{x\}\}$

$S = \{x,y\}$ Then $\emptyset, \{x\}, \{y\}, \{x,y\} \subseteq S$

$P(S) = \{\emptyset, \{x\}, \{y\}, \{x,y\}\}$

If S has n elements, then $P(S)$ has 2^n elements.

<u>Set Operations</u> Let $A, B, C, D, \dots$ be sets.

Their <u>union</u> is

$A \cup B \cup C \cup D \cup \dots = \{x : x$ is an element in at <u>least one</u> of the sets $A, B, C, D, \dots\}$

<u>Examples</u> $\{p,q\} \cup \{r\} = \{p,q,r\}$

$\{x,y,z\} \cup \{v,w,x,y\} = \{v,w,x,y,z\}$

$\mathbb{R} \cup \mathbb{Q} = \mathbb{R} = \mathbb{Q} \cup \mathbb{R}$, $\mathbb{N} \cup \mathbb{Z} \cup \mathbb{Q} = \mathbb{Q}$.

$S \cup \emptyset = S = \emptyset \cup S$ for every set S.

The <u>intersection</u> of $A, B, C, D, \dots$ is
$A \cap B \cap C \cap D \cap \cdots = \{x : x \text{ is an element in } \underline{\text{every}} \text{ one of the sets } A, B, C, D, \dots\}$

<u>Examples</u> $\{p, q\} \cap \{r\} = \emptyset$

$\{x, y, z\} \cap \{v, w, x, y, z\} \cap \{u, v, w, x\} = \{x\}$

$\mathbb{R} \cap \mathbb{Q} \cap [0,1] = \{x : x \in \mathbb{Q} \text{ and } 0 \le x \le 1\}$

$S \cap \emptyset = \emptyset = \emptyset \cap S$ for every set S.

The <u>Cartesian product</u> of $A, B, C, D, \dots$ is
$A \times B \times C \times D \times \cdots = \{(a, b, c, d, \dots) : a \in A, b \in B, c \in C, d \in D, \dots\}$

<u>Examples</u> $\mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\} = \mathbb{R}^2$

$\mathbb{N} \times \mathbb{Z} \times \{0, 1\} = \{(x, y, z) : x \in \mathbb{N}, y \in \mathbb{Z}, z = 0 \text{ or } 1\}$

$S \times \emptyset = \emptyset = \emptyset \times S$ for every set S.

If $A \ne B$, then $A \times B \ne B \times A$.

The <u>Complement</u> of B in A is
$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

<u>Examples</u> $\mathbb{R} \setminus \mathbb{Q}$ is the set of all irrational numbers.

$\{x, y, z\} \setminus \{w, x\} = \{y, z\}$

$\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) = \{(u, v) : u \text{ rational}, v \text{ irrational}\}$

$S \setminus \emptyset = S$, $\emptyset \setminus S = \emptyset$ for every set S.

The sets $A, B, C, D, \dots$ are <u>disjoint</u> iff their intersection is the empty set.

The sets $A, B, C, D, \dots$ are <u>mutually disjoint</u> iff the intersection of <u>every two of the sets</u> is the empty set

<u>Example</u> Let $A = \{x, y\}$, $B = \{y, z\}$, $C = \{z, x\}$

Then A, B, C are disjoint because

$$A \cap B \cap C = \emptyset,$$

but A, B, C are <u>not</u> mutually disjoint because $A \cap B = \{y\} \ne \emptyset$ for instance.

<u>Remark</u> Mutual disjoint $\Rightarrow$ disjoint

Some Notations

n is a positive integer

$$S_1 \cup S_2 \cup \cdots \cup S_n = \bigcup_{k=1}^{n} S_k$$

$$S_1 \cap S_2 \cap \cdots \cap S_n = \bigcap_{k=1}^{n} S_k$$

$$S_1 \times S_2 \times \cdots \times S_n = \bigtimes_{k=1}^{n} S_k$$

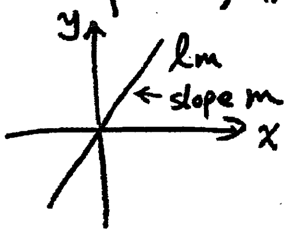
$$S_1 \cup S_2 \cup S_3 \cup \cdots = \bigcup_{k=1}^{\infty} S_k \quad \left(\bigcup_{k \in \mathbb{N}} S_k \right)$$

better notation

not too good, no S_∞ set!

(Similarly for intersection and Cartesian product)

Example If $m \in \mathbb{R}$, let ℓ_m be the line $y = mx$ in the plane, then ℓ_m is not a vertical line



$$\bigcup_{m \in \mathbb{R}} \ell_m = \mathbb{R}^2 \setminus \{(0,y) : y \neq 0\}$$

$$\bigcap_{m \in \mathbb{R}} \ell_m = \{(0,0)\}$$

Examples on Proof Problems

If ① $A \subseteq B$
and ② $C \subseteq D$,

These are given conditions.

then prove $A \cap C \subseteq B \cap D$.

Strategy According to def. of $\subseteq$, we have to check "every $x \in A \cap C$ is also in $B \cap D$."

Proof: $x \in A \cap C \iff x \in A$ and $x \in C$ (by def. of $\cap$)

$\Rightarrow x \in B$ and $x \in D$ (by ①, ②, def. of $\subseteq$)

$\iff x \in B \cap D$ (by def. of $\cap$)

$\therefore A \cap C \subseteq B \cap D$ (by def. of $\subseteq$).

Remarks Since this is proved, you may use it (If $A \subseteq B$ and $C \subseteq D$, then $A \cap C \subseteq B \cap D$) to prove any other statement.

15

Another Example

Prove $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.

Strategy To get $=$, check $\subseteq$ and $\supseteq$.

$\boxed{\text{Proof of } \subseteq}$

$x \in (A \cup B) \setminus C \Leftrightarrow x \in A \cup B$ and $x \notin C$ (by def of $\setminus$)

$\Leftrightarrow \left\{ \begin{matrix} x \in A \\ \text{or} \\ x \in B \end{matrix} \right\}$ and $x \notin C$ (by def of $\cup$)

Case 1 $x \in A$ and $x \notin C$

or

Case 2 $x \in B$ and $x \notin C$

$\Leftrightarrow$ $x \in A \setminus C$ or $x \in B \setminus C$ (by def of $\setminus$)

$\Leftrightarrow x \in (A \setminus C) \cup (B \setminus C)$ (by def of $\cup$)

$\therefore (A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$ (by def of $\subseteq$).

$\boxed{\text{Proof of } \supseteq}$

Reverse Steps!

$x \in (A \setminus C) \cup (B \setminus C)$

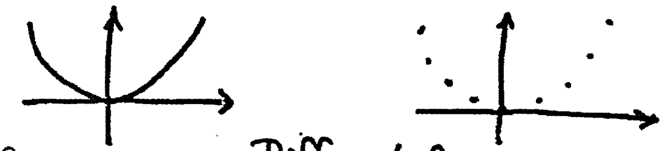
$\Leftrightarrow$ $x \in A \setminus C$ or $x \in B \setminus C$ (by def. of $\cup$)

$\Leftrightarrow$ Case 1 or Case 2 above (by def of $\setminus$)

...

← You fill in the rest of the details!!!!

$f(x) = x^2 \quad x \in \mathbb{R}$ $\qquad g(n) = n^2, \ n \in \mathbb{Z}$



Different functions

Definitions

A **function** (or **map** or **mapping**) f from a set A to a set B (denoted by $f : A \to B$) is a method of assigning to every $a \in A$ **exactly one** $b \in B$. This b (denoted by $f(a)$) is the **value** of f at a.

A function must be **well-defined** in the sense that if $a = a'$, then $f(a) = f(a')$.

A is the **domain** of f. $\quad A = \operatorname{dom} f$

B is the **codomain** of f. $\quad B = \operatorname{codom} f$

$f(A) = \{y : y \in B \text{ and } y = f(x) \text{ for some } x \in A\}$ is the **range** (or **image**) of f.

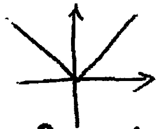
We may say f is a function on A
or f is a B-valued function

The set $\{(x, f(x)) : x \in A\}$ is the **graph** of f.

Two functions are **equal** iff their graphs are the same.

Examples

① The absolute value function on $\mathbb{R}$ is $h: \mathbb{R} \to \mathbb{R}$ given by $h(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$



There may be more than one parts in the formula of the function.

In this example, the codomain can be any set containing $[0, \infty)$. The function will be the same because the graph is the same.

② The following attempt to define a function is not well-defined. Let $x_n = (-1)^n$ for all $n \in \mathbb{N}$. Define $f: \{x_1, x_2, \cdots\} \to \mathbb{R}$ by $f(x_n) = n$. It is not well-defined because

$$x_1 = -1 = x_3, \text{ but } f(x_1) = 1 \neq 3 = f(x_3).$$

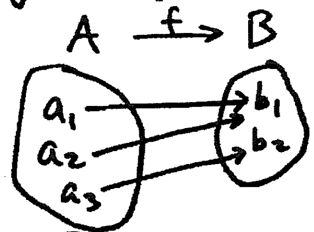
Types of Functions Definitions

① The identity function on a set S is $I_S : S \to S$ given by $I_S(x) = x$ for all $x \in S$.

② Let $f: A \to B$, $g: B' \to C$ be functions and $f(A) \subseteq B'$. The composition of g by f is $g \circ f : A \to C$ given by $(g \circ f)(x) = g(f(x))$.

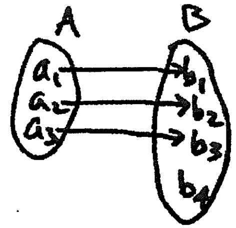
③ Let $f: A \to B$ be a function and $C \subseteq A$. The restriction of f to C is $f|_C : C \to B$ given by $f|_C(x) = f(x)$ for all $x \in C$.

④ $f: A \to B$ is surjective (or onto) iff $f(A) = B$.



The values of f in B may repeat, but no element of B will be omitted as a value. So A has at least as many elements as B.

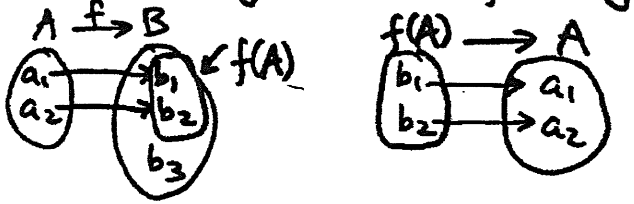
⑤ $f: A \to B$ is injective (or one-to-one) iff $f(x) = f(y)$ implies $x = y$.



The values of f in B do not repeat, but some elements of B may be omitted as a value.

So B has at least as many elements as A.

6. For an injective function $f: A \to B$, the inverse function of f is $f^{-1}: f(A) \to A$ given by $f^{-1}(y) = x \iff f(x) = y$.



7. $f: A \to B$ is bijective (or a one-to-one correspondence) iff f is injective and surjective.
A and B have the same number of elements.

Exercises

(a) $f: A \to B$ is a bijection iff $\exists\, g: B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$.

(b) If $f: A \to B$ and $g: B \to C$ are bijections, then $g \circ f: A \to C$ is a bijection.

(c) Let $A, B \subseteq \mathbb{R}$ and $f: A \to B$ be a function. If $\forall b \in B$, the horizontal line $y = b$ intersects the graph of f exactly once, then f is a bijection.

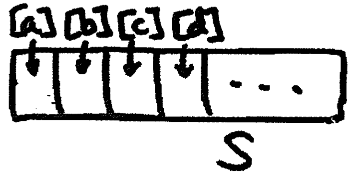
Equivalence Relation

Definition A relation on a set S is any subset of $S \times S$.

A relation R on a set S is an equivalence relation iff
1. $\forall x \in S$, $(x,x) \in R$ (Reflexive Prop.)
2. $(x,y) \in R \Rightarrow (y,x) \in R$ (Symmetric Property)
3. $(x,y), (y,z) \in R \Rightarrow (x,z) \in R$ (Transitive Prop.)

Notations Write $x \sim y$ iff $(x,y) \in R$.
$\forall x \in S$, write $[x] = \{y : x \sim y\}$ ← called "the equivalence class containing x."

Facts
1. $\Rightarrow \forall x \in S$, $x \sim x \Rightarrow x \in [x]$
$\Rightarrow \bigcup_{x \in S} [x] = S$.
2. $x \sim y \Rightarrow [x] = [y]$ because
$z \in [x] \iff x \sim z \iff y \sim z \iff z \in [y]$
3. $x \not\sim y \Rightarrow [x] \cap [y] = \emptyset$ (otherwise $z \in [x] \cap [y]$
$\Rightarrow [x] = [z] = [y] \Rightarrow x \sim y$ by 2).



Equivalence relation on S = Partition of S

Examples

$R=\{(T_1,T_2): T_1 \text{ is similar to } T_2\}$

① (Geometry) S = the set of all triangles.

$(T_1,T_2)\in R \Leftrightarrow T_1 \sim T_2 \iff T_1$ is similar to T_2. R is an equivalence relation.

$[T]$ = the set of all triangles similar to T.

$R=\{(m,n): m-n \text{ is even}\}$

② (Arithmetic) $S=\mathbb{Z}$. $m\sim n \iff m-n$ is even

$[0]$ = the set of all $m\in\mathbb{Z}$ such that $m-0$ is even
$=\{\cdots,-4,-2,0,2,4,\cdots\}$

$[1]$ = the set of all $m\in\mathbb{Z}$ such that $m-1$ is even
$=\{\cdots,-3,-1,1,3,\cdots\}$

---

① and ② are examples of equivalence relations.

③ Let $S=\{0,1\}$ and $R=\{(1,1)\}$. Then R satisfies symmetric and transitive properties, but not reflexive property because $0\in S$ and $(0,0)\notin R$. $\therefore$ R is not an equivalence relation on S.

---

④ For sets S_1 and S_2, define $R=\{(S_1,S_2): \exists \text{ bijection } f:S_1\to S_2\}$

$S_1\sim S_2 \iff \exists$ bijection $f:S_1\to S_2$

This is an equivalence relation.

$S_1\sim S_2$ ← say S_1 and S_2 have same cardinality

$[S]=\text{card } S=|S|$ cardinal number of S

Notations $\text{Card}\{1,2,\cdots,n\}=n$ for positive integer n

$\text{Card}\{1,2,3,\ldots\}=\text{Card }\mathbb{N}=\aleph_0$ ← aleph-naught

$\text{Card }\mathbb{R}=c$ ← Cardinality of the Continuum

Chapter 3 Countability

← a property that distinguishes some infinite sets.

Definitions ① A set S is countably infinite iff $\exists$ bijection $f:\mathbb{N}\to S$ (i.e. $\text{Card } S=\aleph_0$).

② A set S is countable iff S is finite or countably infinite. Uncountable = not countable.

Observations

$\exists$ bijection $f:\mathbb{N}\to S \Rightarrow S=\{f(1),f(2),f(3),\ldots\}$

a listing of elements of S with no repetition nor omission

$S=\{s_1,s_2,s_3,\ldots\} \Rightarrow f:\mathbb{N}\to S$ defined by $f(n)=s_n$ is a bijection.

Bijection Theorem Let $g: S \to T$ be a bijection.

S is countable $\iff$ T is countable.

Proof. For finite sets, it is clear as card S = card T.

For infinite sets,

$$\begin{array}{ccc} S \text{ countable} & \iff & \exists \text{ bijection } f: \mathbb{N} \to S \\ \text{Countably infinite} & f = g^{-1} \circ h \Uparrow & \Downarrow h = g \circ f \\ T \text{ countable} & \iff & \exists \text{ bijection } h: \mathbb{N} \to T \end{array}$$

Remarks Let $g: S \to T$ be a bijection.

S is uncountable $\iff$ T is uncountable.

This is the contrapositive of the bijection theorem.

Basic Examples

① $\mathbb{N}$ is countably infinite as $I_{\mathbb{N}}: \mathbb{N} \to \mathbb{N}$ is a bijection (identity function).

② $\mathbb{Z}$ is countably infinite because

$$\begin{array}{l} \mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \dots\} \\ f \downarrow \quad \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \downarrow \ \cdots \\ \mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\} \end{array}$$

$$f(n) = \begin{cases} n/2 & n \text{ even} \\ -\left(\frac{n-1}{2}\right) & n \text{ odd} \end{cases}, \quad f^{-1}(m) = \begin{cases} 2m & m > 0 \\ 1-2m & m \le 0 \end{cases}$$

is a bijection

③ $\mathbb{N} \times \mathbb{N} = \{(m,n) : m, n \in \mathbb{N}\}$ is countably infinite.

$$\begin{array}{cccc} (1,1) & (1,2) & (1,3) & \cdots \\ (2,1) & (2,2) & (2,3) & \cdots \\ (3,1) & (3,2) & (3,3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Define $f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by

$f(1) = (1,1)$

$f(2) = (2,1)$

$f(3) = (1,2)$

$f(4) = (3,1)$

$f(5) = (2,2)$

$f(6) = (1,3)$

$\vdots$

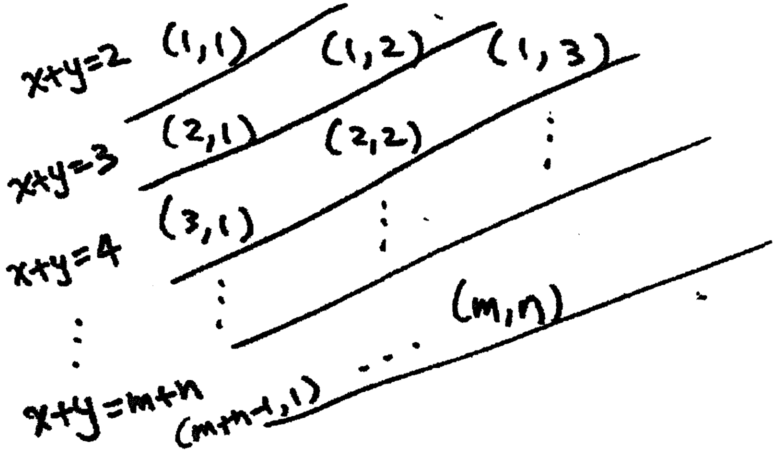
Diagonal Counting Scheme

f injective because no ordered pair is repeated.

f surjective because $\forall (m,n) \in \mathbb{N} \times \mathbb{N}$

$$(m,n) = f\left(\sum_{k=0}^{m+n-2} k + n\right) = f\left(\frac{(m+n-2)(m+n-1)}{2} + n\right)$$

(m,n) is the n^{th} element on the $m+n-1^{th}$ diagonal.



(20)

④ Open interval $(0,1)=\{x : x\in\mathbb{R} \text{ and } 0<x<1\}$ is <u>uncountable</u>. $\mathbb{R}$ is uncountable.

Assume $(0,1)$ is countably infinite. Then $\exists$ bijection $f:\mathbb{N}\to(0,1)$. So

$f(1)=0.\underline{a_{11}}\,a_{12}\,a_{13}\,a_{14}\cdots$
$f(2)=0.a_{21}\,\underline{a_{22}}\,a_{23}\,a_{24}\cdots$
$f(3)=0.a_{31}\,a_{32}\,\underline{a_{33}}\,a_{34}\cdots$
$f(4)=0.a_{41}\,a_{42}\,a_{43}\,\underline{a_{44}}\cdots$
$\vdots \qquad \vdots$

f <u>surjective</u> $\Rightarrow$ every $x\in(0,1)$ is equal to some $f(n)$

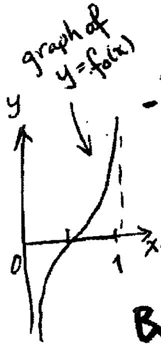
Consider $x=0.b_1b_2b_3\cdots$, where

$\forall n=1,2,3,\cdots,\quad b_n=\begin{cases}2 & \text{if } a_{nn}=1\\ 1 & \text{if } a_{nn}\neq 1\end{cases}\ \neq a_{nn}.$

Then $0<x<1$. However $x\neq f(n)$ because $b_n\neq a_{nn}$ for all $n=1,2,3,\cdots$ Contradicting the surjectivity of f.

b_n: n^{th} digit of x; a_{nn}: n^{th} digit of $f(n)$

$\therefore$ $(0,1)$ is uncountable.



$f_0:(0,1)\to\mathbb{R}$ given by $f_0(x)=\tan\pi(x-\frac{1}{2})$ is a bijection with $f_0^{-1}(x)=\frac{1}{2}+\frac{\operatorname{Arctan} x}{\pi}$

By bijection theorem, $\mathbb{R}$ is uncountable.

<u>Countable Subset Theorem</u> Let $A\subseteq B$.
If B is countable, then A is countable.
(Taking contrapositive, if A is uncountable, then B is uncountable.)

<u>Reason</u> B countable $\Rightarrow B=\{b_1,b_2,b_3,\cdots\}$ ← no repetition, no omission
From the listing of B, we strike out the elements that are <u>not</u> in A. Then we get a listing of A. Since the listing of B has no repetition and $A\subseteq B$, the listing of A has no repetition nor omission.

<u>Countable Union Theorem</u>
If $\forall n\in\mathbb{N}$, A_n is countable, then $\bigcup_{n\in\mathbb{N}} A_n$ is countable.
(In general, if S is countable, say $f:\mathbb{N}\to S$ is a bijection, and $\forall s\in S$, A_s is countable, then $\bigcup_{s\in S} A_s=\bigcup_{n\in\mathbb{N}} A_{f(n)}$ is countable.)

<u>Reason</u>
$\forall n\in\mathbb{N}$, A_n countable $\Rightarrow$
$A_1=\{a_{11},a_{12},a_{13},\cdots\}$,
$A_2=\{a_{21},a_{22},a_{23},\cdots\}$,
$A_3=\{a_{31},a_{32},a_{33},\cdots\}$, $\cdots$

diagonal counting scheme

$\Rightarrow \bigcup_{n\in\mathbb{N}} A_n=\{a_{11},a_{21},a_{12},a_{31},a_{22},a_{13},\cdots\}$

eliminate repetition along the way, skip blanks

21

Product Theorem For $n \in \mathbb{N}$, if $A_1, \cdots, A_n$ countable, then $A_1 \times \cdots \times A_n$ is countable.

Reason $n=1$ is clear.

$n=2$ $\quad A_1 = \{x_1, x_2, x_3, \cdots\}$, $A_2 = \{y_1, y_2, y_3, \cdots\}$

$$\Rightarrow A_1 \times A_2 = \{ \begin{array}{l} (x_1,y_1), (x_1,y_2), (x_1,y_3), \cdots \\ (x_2,y_1), (x_2,y_2), (x_2,y_3), \cdots \\ (x_3,y_1), (x_3,y_2), (x_3,y_3), \cdots \\ \quad\vdots \qquad\quad \vdots \qquad\quad \vdots \end{array}$$

diagonal counting scheme

$= \{(x_1,y_1), (x_2,y_1), (x_1,y_2), (x_3,y_1), (x_2,y_2), \cdots\}$

skip blanks if A_1 or A_2 is finite.

$n>2$ Assume case $n-1$ is true. Then

$$A_1 \times \cdots \times A_n = \underbrace{\underbrace{(A_1 \times \cdots \times A_{n-1})}_{\text{Case } n-1 \text{ is true}} \times A_n}_{\text{Case 2 is true}}.$$

Product theorem does not hold for infinitely many countable sets. See example 10.

Examples 5 $\quad \mathbb{Q} = \bigcup_{n \in \mathbb{N}} S_n$, where

$$S_1 = \{\cdots, \tfrac{-2}{1}, \tfrac{-1}{1}, \tfrac{0}{1}, \tfrac{1}{1}, \tfrac{2}{1}, \cdots\}$$
$$S_2 = \{\cdots, \tfrac{-2}{2}, \tfrac{-1}{2}, \tfrac{0}{2}, \tfrac{1}{2}, \tfrac{2}{2}, \cdots\}$$
$$S_3 = \{\cdots, \tfrac{-2}{3}, \tfrac{-1}{3}, \tfrac{0}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \cdots\}$$
$$\vdots$$
$$S_n = \{\cdots, \tfrac{-2}{n}, \tfrac{-1}{n}, \tfrac{0}{n}, \tfrac{1}{n}, \tfrac{2}{n}, \cdots\}$$
$$\vdots$$

$\forall n$, $f_n : \mathbb{Z} \to S_n$, $f_n(m) = \frac{m}{n}$, is a bijection (with $f_n^{-1}(\frac{m}{n}) = m$)

Since $\mathbb{Z}$ is countable, by bijection theorem, S_n is countable. By countable union theorem, $\mathbb{Q}$ is countable.

6 If A is uncountable and B is countable then $A \setminus B$ is uncountable.

(The case $A = \mathbb{R}$, $B = \mathbb{Q}$ $\Rightarrow$ $\mathbb{R} \setminus \mathbb{Q}$ is uncountable)

Reason Assume $A \setminus B$ is countable, then

$A \cap B \subseteq B$ and B countable $\underset{\text{countable subset theorem}}{\Rightarrow}$ $A \cap B$ countable

$\Rightarrow \underset{\text{Countable}}{(A \cap B) \cup (A \setminus B)} = A$ countable $\leftarrow$ contradiction.

by countable union theorem

(7) Since $\mathbb{R} \subseteq \mathbb{C}$ and $\mathbb{R}$ is uncountable, by the countable subset theorem, $\mathbb{C}$ is uncountable.

$$\underbrace{\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}}_{\text{countable}} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

$\mathbb{R}\setminus\mathbb{Q} \subseteq$ uncountable

(8) Determine if $A=\{r\sqrt{m} : m\in\mathbb{N},\ r\in(0,1)\}$ (open interval)

$B=\{r\sqrt{m} : m\in\mathbb{N},\ r\in(0,1)\cap\mathbb{Q}\}$

Countable or not.

Solution. Taking $m=1$, we see

$\underbrace{(0,1)}_{\text{uncountable}} = \{r\sqrt{1} : r\in(0,1)\} \subseteq A$

$\therefore$ by countable subset theorem A is uncountable.

$\forall m\in\mathbb{N}$, let $B_m=\{r\sqrt{m} : r\in(0,1)\cap\mathbb{Q}\}$.

Then $(0,1)\cap\mathbb{Q} \subseteq \underset{\text{countable}}{\mathbb{Q}} \Rightarrow (0,1)\cap\mathbb{Q}$ countable by countable subset theorem

$B_m = \bigcup_{r\in\underbrace{(0,1)\cap\mathbb{Q}}_{\text{countable}}} \underbrace{\{r\sqrt{m}\}}_{\text{finite}}$ is countable by countable union theorem

$B = \bigcup_{m\in\mathbb{N}} \underbrace{B_m}_{\text{countable}}$ is countable by countable union theorem

(9) Show that the set L of all lines with equation $y=mx+b$, where $m,b\in\mathbb{Q}$, is countable.

Solution. Define $f:\mathbb{Q}\times\mathbb{Q}\to L$ by letting $f(m,b)$ be the line with equation $y=mx+b$. This is a bijection with f^{-1} sending the line with equation $y=mx+b$ back to (m,b).

Since $\underbrace{\mathbb{Q}\times\mathbb{Q}}_{\text{countable}}$ is countable by product theorem, L is countable by bijection theorem.

(10) Let $A_1=A_2=A_3=\cdots=\{0,1\}$, then $A_1\times A_2\times A_3\times\cdots$ is uncountable.

Solution. Assume $A_1\times A_2\times A_3\times\cdots$ is countable. Then $\exists$ bijection $f:\mathbb{N}\to A_1\times A_2\times A_3\times\cdots$

$f(1)=(a_{11}, a_{12}, a_{13}, \cdots)$ All $a_{ij}=0$ or 1.

$f(2)=(a_{21}, a_{22}, a_{23}, \cdots)$

$f(3)=(a_{31}, a_{32}, a_{33}, \cdots)$

$\vdots \qquad \vdots$

Let $b=(b_1,b_2,b_3,\dots)$, where $b_i=\begin{cases}1 & \text{if } a_{ii}=0\\ 0 & \text{if } a_{ii}=1\end{cases}$

Then $b\in A_1\times A_2\times A_3\times\cdots$

$\forall i,\ b_i\neq a_{ii} \Rightarrow b\neq f(i) \Rightarrow f$ not surjective. Contradiction.

(11) Show $P(\mathbb{N})$ is uncountable.
the set of all subsets of $\mathbb{N}$

<u>Solution</u> Define $g : P(\mathbb{N}) \to A_1 \times A_2 \times A_3 \times \cdots$ (all $A_i = \{0,1\}$)

by

$$g(S) = (a_1, a_2, a_3, \ldots), \text{ where } a_m = \begin{cases} 1 & \text{if } m \in S \\ 0 & \text{if } m \notin S. \end{cases}$$

For example,

$$g(\{1,3,5,7,\ldots\}) = (1,0,1,0,1,0,1,\ldots)$$
$$g(\emptyset) = (0,0,0,\ldots), \quad g(\mathbb{N}) = (1,1,1,\ldots)$$

Note $g^{-1}((a_1,a_2,a_3,\ldots)) = \{m : a_m = 1\}$

$\therefore$ g is a bijection.

Since $A_1 \times A_2 \times A_3 \times \cdots$ is uncountable by example 10,

$P(\mathbb{N})$ is uncountable by bijection theorem.

(12) Show that the set S of all polynomials with integer coefficients is countable.

<u>Solution</u>. Let $S_0 = \mathbb{Z}$. Let

$$S_n = \{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 : \begin{array}{c} a_n, a_{n-1}, \ldots, a_0 \in \mathbb{Z} \\ a_n \neq 0 \end{array}\}$$

Define $f : S_n \to (\mathbb{Z} \setminus \{0\}) \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ by

$$f(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0) = (a_n, a_{n-1}, \ldots, a_0).$$

This is a bijection with

$$f^{-1}(a_n, a_{n-1}, \ldots, a_0) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0.$$

Since $(\mathbb{Z} \setminus \{0\}) \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ is countable by product theorem, S_n is countable by bijection theorem.

$$\therefore \ S = \underbrace{S_0}_{\substack{\| \\ \mathbb{Z}}} \cup \Big(\bigcup_{n \in \mathbb{N}} S_n \Big) \text{ is countable by the countable union theorem}$$

countable

(13) Is every real number a root of some nonconstant polynomial with integer coefficients?

<u>Solution</u>. Let S_n be as in example 12. Then

$$S^* = \bigcup_{n\in\mathbb{N}} S_n \text{ is \underline{countable} by countable union theorem.}$$

countable

$\forall f \in S^*$, $\exists n \in \mathbb{N}$ such that $f \in S_n$. Then the set R_f of all roots of f is <u>finite</u> because R_f has at most $n = (\text{degree } f)$ elements.

Then $T = \bigcup_{f\in S^*} R_f$ is countable by the Countable union theorem

countable

By example 6,

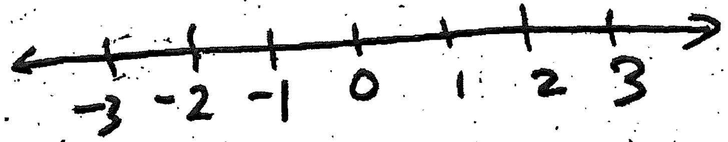
$$\mathbb{R}\setminus T = \mathbb{R}\setminus\Big(\bigcup_{f\in S^*} R_f\Big) \text{ is uncountable,}$$

hence nonempty.

uncountable   countable.

Therefore, there is a real number not in T. Such a real number is not a root of any nonconstant polynomial with integer coefficients.
(Such a number is called <u>transcendental</u>.)

(Stanford Problem)



(1) A submarine at time $= 0$ is located at some <u>integer</u> n on $\mathbb{R}$.

(2) Submarine has a constant speed s, which is also an <u>integer</u>.

(3) At every second, you can fire a missile at some integer.

Is there a strategy to hit the submarine at some time?

Remarks on Examples of Chapter 3

① Every interval I containing at least two numbers is uncountable.

Reason Say $a,b \in I$ with $a<b$. Then $(a,b) \subseteq I$.

Since $f:(0,1) \to (a,b)$ defined by

$f(x) = (b-a)x+a$ has inverse function

$g:(a,b) \to (0,1)$

$g(x) = \dfrac{x-a}{b-a}$,

f is a bijection.

Since $(0,1)$ is uncountable by example 4, (a,b) is uncountable by bijection theorem.

$\therefore$ I is uncountable by countable subset theorem.

---

② We present a second solution to Example 9.

For every $(m,b) \in \mathbb{Q}\times\mathbb{Q}$, let

$L_{(m,b)}$ be the set consisted of the line with equation $y=mx+b$.

Then
$$L = \bigcup_{\substack{(m,b)\in \underbrace{\mathbb{Q}\times\mathbb{Q}}_{\text{Countable by product theorem}}}} \underbrace{L_{(m,b)}}_{\text{one element set}}$$

one element set $\Rightarrow$ finite set $\Rightarrow$ countable set

By countable union theorem, L is countable.

Summary of Countable and uncountable sets

Countable Sets

Finite Sets

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$

$\mathbb{N}\times\mathbb{N}$

Countable $\times$ countable

Subsets of Countable Sets, like $\mathbb{Q}\cap[0,1]$

Polynomials with integer coefficients

Uncountable Sets

$(0,1)$, $\mathbb{R}$, intervals with at least 2 numbers

$\mathbb{C}$, $\mathbb{R}\setminus\mathbb{Q}$

$P(\mathbb{N})$

$\{0,1\}\times\{0,1\}\times\{0,1\}\times\dots$

Uncountable $\times$ Nonempty by surjection theorem

Injection Theorem

Let $f: A \to B$ be injective. If B is countable, then A is countable. (Contrapositive: if A is uncountable, then B is uncountable.)

Reason. Let $h: A \to f(A)$ be given by $h(x) = f(x)$. Then h is injective (since f is injective) and h is surjective (since $h(A) = f(A)$). $\therefore$ h is bijective. Since $f(A) \subseteq B$ and B is countable, we see $f(A)$ is countable ← by countable subset theorem

$\therefore$ A is countable. ← by bijection theorem.

Surjection Theorem

Let $g: A \to B$ be surjective. If A is countable, then B is countable. (Contrapositive: if B is uncountable, then A is uncountable.)

Reason.
$$\underset{g \text{ surjective}}{B =} g(A) = \bigcup_{x \in \underset{\text{countable}}{A}} \underset{\text{finite}}{\{g(x)\}}$$
is countable by the countable union theorem.

Examples (14) Define $f: \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ by $f(x) = (m, n)$, where $x = \frac{m}{n}$ $(m \in \mathbb{Z}, n \in \mathbb{N})$ and the highest common factor of m, n is 1.
or greatest common divisor

$f(x) = f(x') = (m, n) \Rightarrow x = \frac{m}{n} = x'$

$\therefore$ f is injective. Since $\mathbb{Z} \times \mathbb{N}$ is countable by product theorem, $\mathbb{Q}$ is countable by the injection theorem.

---

(15) Let A_1 be uncountable and $A_2, \cdots, A_{100}$ be nonempty sets. Then $A_1 \times A_2 \times \cdots \times A_{100}$ is uncountable.

Solution. Define $g: A_1 \times A_2 \times \cdots \times A_{100} \to A_1$ by $g(x_1, x_2, \cdots, x_{100}) = x_1$. Since $A_2, \cdots, A_{100} \neq \emptyset$, $\exists\, a_2 \in A_2, \cdots, a_{100} \in A_{100}$. Then $\forall x \in A_1$, $g(x, a_2, \cdots, a_{100}) = x$. So g is surjective. Since A_1 is uncountable, $A_1 \times A_2 \times \cdots \times A_{100}$ is uncountable by surjection theorem.

FAMOUS OPEN MATH PROBLEM

Continuum Hypothesis For every uncountable set S, $\exists$ injective $f: \mathbb{R} \to S$.

Question Is this a true statement ?

1940 Kurt Gödel proved the opposite statement would not lead to any contradiction.

Question Does this mean the opposite statement is true ?

1966 Paul Cohen proved the original statement would not lead to any contradiction.
He won the Fields' medal for this.

Moral : The method of 'proof by contradiction' may not be applied to every statement.

Chapter 4 Infinite Series

An infinite series is of the form

$$a_1+a_2+a_3+\cdots \quad \text{or} \quad \sum_{k=1}^{\infty} a_k$$

where $a_1, a_2, a_3, \ldots$ are numbers.

For $n\in\mathbb{N}$, $S_n=\sum_{k=1}^{n} a_k$ is the n^{th} partial sum of the series.

Examples ① $\sum_{k=1}^{\infty}\frac{1}{2^{k-1}} = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots,$

$$S_n=\sum_{k=1}^{n}\frac{1}{2^{k-1}} = 2-\frac{1}{2^n} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{2^{k-1}} = \lim_{n\to\infty} S_n = 2$$

We say the series converges to 2 in this case.

② $\sum_{k=1}^{\infty} 1 = 1+1+1+1+\cdots,\quad S_n=\sum_{k=1}^{n} 1 = n$

We have $\lim_{n\to\infty} S_n=\infty$. We say the series diverges to ∞

③ $\sum_{k=1}^{\infty}(-1)^{k-1} = 1+(-1)+1+(-1)+\cdots;\ S_n=\begin{cases}0 & n \text{ even}\\ 1 & n\text{ odd}\end{cases}$

$\lim_{n\to\infty} S_n$ doesn't exist. We say the series diverges.

Definitions For an infinite series $\sum_{k=1}^{\infty} a_k$,

① it converges to a number S iff $\lim_{n\to\infty} S_n = S$.
  (S is the sum of the series.)

② it diverges to ∞ iff $\lim_{n\to\infty} S_n=\infty$.

③ it diverges iff $\lim_{n\to\infty} S_n$ doesn't exist.

Facts

① For $\sum_{k=1}^{\infty} a_k$ with partial sums S_n, we have

$\boxed{a_1 = S_1}$, $a_2 = (a_1+a_2) - a_1 = S_2 - S_1$, ...

$\underline{k>1} \Rightarrow a_k = (a_1+\cdots+a_k) - (a_1+\cdots+a_{k-1})$

$\Rightarrow \boxed{a_k = S_k - S_{k-1}}$

② For $m \in \mathbb{N}$, $\sum_{k=1}^{\infty} a_k$ converges to A iff $\sum_{k=m}^{\infty} a_k$ converges to

$$B = \lim_{n\to\infty}(a_m+\cdots+a_n) = \lim_{n\to\infty}(S_n-(a_1+\cdots+a_{m-1}))$$

$$= \lim_{n\to\infty} S_n - (a_1+\cdots+a_{m-1}) = A-(a_1+\cdots+a_{m-1}).$$

To check $\sum_{k=1}^{\infty} a_k$ converge, it is enough to check $\sum_{k=m}^{\infty} a_k$ converge for some $m\in\mathbb{N}$.

③ If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, where A, B numbers

then $\sum_{k=1}^{\infty}(a_k+b_k) = A+B = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$

$\sum_{k=1}^{\infty}(a_k-b_k) = A-B = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$

$\forall c\in\mathbb{R}$, $\sum_{k=1}^{\infty}(c\,a_k) = c\sum_{k=1}^{\infty} a_k$.

Geometric Series Test

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty}(1+r+r^2+\cdots+r^n) = \lim_{n\to\infty}\frac{1-r^{n+1}}{1-r}$$

$$= \begin{cases}\frac{1}{1-r} & \text{if } |r|<1\\ \text{doesn't exist} & \text{if } |r|\ge 1.\end{cases}$$

Example $0.999\ldots = \frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots$

$$= \frac{9}{10}\left(1+\frac{1}{10}+\frac{1}{100}+\cdots\right)$$

$$= \frac{9}{10}\,\frac{1}{1-\frac{1}{10}} = 1 = 1.000\cdots$$

Telescoping Series Test

$(b_1-b_2)+(b_2-b_3)+(b_3-b_4)+\cdots$

$$= \sum_{k=1}^{\infty}(b_k-b_{k+1}) = \lim_{n\to\infty}((b_1-b_2)+(b_2-b_3)+\cdots+(b_n-b_{n+1}))$$

$$= \lim_{n\to\infty}(b_1-b_{n+1}) = b_1 - \lim_{n\to\infty} b_{n+1}$$

Examples ① $\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)$ $\quad b_k = \frac{1}{k}$

$$= \left(1-\tfrac12\right)+\left(\tfrac12-\tfrac13\right)+\left(\tfrac13-\tfrac14\right)+\cdots = 1-\lim_{n\to\infty}\frac{1}{n+1} = 1$$

② $\sum_{k=1}^{\infty}\left(5^{1/k}-5^{1/(k+1)}\right) = (5-\sqrt{5})+(\sqrt{5}-\sqrt[3]{5})+\cdots$

$$= 5-\lim_{n\to\infty}5^{1/(n+1)} = 5-5^0 = 5-1 = 4.$$

Term Test If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} a_n = 0$.

(**Contrapositive**: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.)

Reason $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = S \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1})$
$= S - S = 0.$

Examples ① $1+1+1+\cdots = \sum_{k=1}^{\infty} 1$ $\quad a_n = 1,\ \lim_{n\to\infty} a_n = 1 \neq 0$

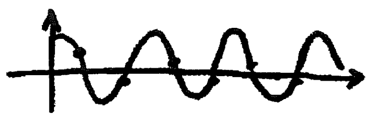
series diverges.

② $\sum_{k=1}^{\infty} \cos\left(\frac{1}{k}\right)$ $\quad a_n = \cos\frac{1}{n},\ \lim_{n\to\infty} a_n = \cos 0 = 1 \neq 0$

series diverges

③ $\sum_{k=1}^{\infty} \cos k$ $\quad a_n = \cos n,\ \lim_{n\to\infty} a_n \neq 0$

series diverges why?



Assume $\lim_{n\to\infty} \cos n = 0$.

Then $\cos 1, \cos 2, \cos 3, \ldots \to 0$

So $\cos 2, \cos 3, \cos 4, \ldots \to 0 \iff \lim_{n\to\infty} \cos(n+1) = 0$

$\lim_{n\to\infty} |\sin n| = \lim_{n\to\infty} \sqrt{1-\cos^2 n} = \sqrt{1-0^2} = 1$

$0 = \lim_{n\to\infty} |\cos(n+1)| = \lim_{n\to\infty} |\underbrace{\cos n}_{0} \cos 1 - \sin n \sin 1|$
$= |\sin 1| \neq 0$

Contradiction.

Question What if $\lim_{n\to\infty} a_n = 0$?

Answer $\sum_{k=1}^{\infty} a_k$ may or may not converge.

Examples ④ $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots = \sum_{k=0}^{\infty}\left(-\frac{1}{2}\right)^k$

$a_n = \left(-\frac{1}{2}\right)^n,\ \lim_{n\to\infty} a_n = 0,$ series converges by geometric series test

$r = -\frac{1}{2},\ |r| = \frac{1}{2} < 1.$

⑤ $1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{2\text{ times}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{4\text{ times}} + \underbrace{\frac{1}{8} + \cdots + \frac{1}{8}}_{8\text{ times}} + \cdots$

$a_1 \geq a_2 \geq a_3 \geq \cdots$ $\quad a_n$ is "decreasing to 0"
$\lim_{n\to\infty} a_n = 0$

$S_1 \leq S_2 \leq S_3 \leq \cdots \quad S_{2^n - 1} = n \quad \lim_{n\to\infty} S_n = \infty$

Series diverges to ∞. S_n is "increasing" to ∞.

Nonnegative Series $\sum_{k=1}^{\infty} a_k$ with $a_k \ge 0 \ \forall k$

$\Rightarrow \forall n,\ S_{n+1} = S_n + a_{n+1} \ge S_n$

$\Rightarrow S_1 \le S_2 \le S_3 \le \cdots \Rightarrow \lim_{n\to\infty} S_n =$ number or $+\infty$

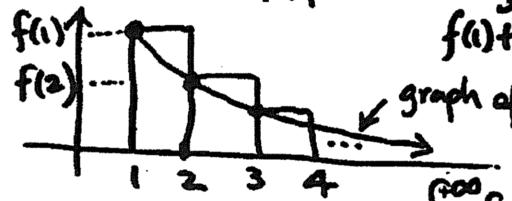
$\Rightarrow \sum_{k=1}^{\infty} a_k$ converges or $\sum_{k=1}^{\infty} a_k$ diverges to $+\infty$.

Integral Test Let $f:[1,\infty)\to\mathbb{R}$ decrease to 0 as $x\to\infty$. Then

$$\sum_{k=1}^{\infty} f(k) \text{ converges} \Leftrightarrow \underbrace{\int_1^{+\infty} f(x)\,dx}_{=\lim_{t\to\infty}\int_1^t f(x)\,dx} < \infty$$

Reason $(\Rightarrow)$ "$\sum_{k=1}^{\infty} f(k)$ converges" means

$f(1)+f(2)+f(3)+f(4)+\cdots < \infty$

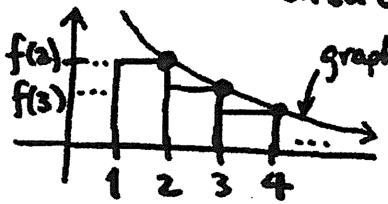


$\int_1^{+\infty} f(x)\,dx \le f(1)+f(2)+\cdots < \infty$

area under graph of f on $[1,\infty)$ — area of rectangles on $[1,\infty)$

$(\Leftarrow)$ "$\int_1^{\infty} f(x)\,dx < \infty$" means

area under graph of f on $[1,\infty) < \infty$



$f(2)+f(3)+\cdots \le \int_1^{+\infty} f(x)\,dx < \infty$

$\Rightarrow \underbrace{f(1)+(f(2)+f(3)+\cdots)}_{=\sum_{k=1}^{\infty} f(k) \text{ converges}} < \infty$

$f(1)\in\mathbb{R}$

Examples (1) Consider $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$. $\quad f(x) = \frac{1}{1+x^2}$

As $x \nearrow \infty$, $1+x^2 \nearrow \infty$, so $\frac{1}{1+x^2} \searrow 0$.

$$\int_1^{\infty} \frac{1}{1+x^2}\,dx = \operatorname{Arctan} x \Big|_1^{\infty} = \operatorname{Arctan}\infty - \operatorname{Arctan} 1 = \frac{\pi}{2} - \frac{\pi}{4} < \infty.$$

$\therefore \sum_{k=1}^{\infty} \frac{1}{1+k^2}$ converges by integral test.

---

(2) Consider $\sum_{k=2}^{\infty} \frac{1}{k\ln k}$ and $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$.

As $x \nearrow \infty$, $\ln x \nearrow \infty$, $x\ln x \nearrow \infty$, $x(\ln x)^2 \nearrow \infty$,

so $\frac{1}{x\ln x} \searrow 0$, $\frac{1}{x(\ln x)^2} \searrow 0$.

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx \underset{u=\ln x}{=} \int_{\ln 2}^{+\infty} \frac{1}{u}\,du = \ln u \Big|_{\ln 2}^{+\infty} = \infty - \ln(\ln 2) = \infty$$

$\therefore \sum_{k=2}^{\infty} \frac{1}{k\ln k}$ diverges to ∞

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx \underset{u=\ln x}{=} \int_{\ln 2}^{+\infty} \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_{\ln 2}^{+\infty} = 0 - \left(-\frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$$

$\therefore \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges.

p-test For $p \in \mathbb{R}$, p constant

$$\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \text{ converges} \iff p > 1.$$

"p-series"

Reason For $p \le 0$, $\frac{1}{k^p} = k^{-p} = k^{|p|} \ge k^0 = 1$

$\Rightarrow \lim_{k\to\infty} \frac{1}{k^p} \ne 0 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges by term test.

For $p > 0$, as $x \nearrow \infty$, $x^p \nearrow \infty$, so $\frac{1}{x^p} \searrow 0$.

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \int_1^{+\infty} x^{-p}\,dx = \begin{cases} \ln x \Big|_1^{\infty} & p = 1 \\ \frac{x^{-p+1}}{-p+1}\Big|_1^{\infty} & 0 < p < 1 \text{ or } p > 1 \end{cases}$$

$$= \begin{cases} +\infty & p = 1 \\ +\infty & 0 < p < 1 \\ \frac{1}{p-1} & p > 1 \end{cases} \qquad \therefore \sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges} \iff p > 1.$$

Known Cases In 1736, Euler showed

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

$$\zeta(4) = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \cdots = \frac{\pi^4}{90}$$

$$\vdots$$

$$\zeta(2n) = r_n \pi^{2n}, \quad r_n \in \mathbb{Q}$$

In 1980, Apery showed

$\zeta(3) = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots$ is <u>irrational</u>.

Comparison Test Given $v_k \ge u_k \ge 0$ $\forall k \in \mathbb{N}$.

$$\sum_{k=1}^{\infty} v_k \text{ converges} \Rightarrow \sum_{k=1}^{\infty} u_k \text{ converges.}$$

(<u>Contrapositive</u>: $\sum_{k=1}^{\infty} u_k$ diverges $\Rightarrow \sum_{k=1}^{\infty} v_k$ ~~converges~~ diverges)

Reason $v_k \ge u_k \ge 0$ $\forall k \Rightarrow \sum_{k=1}^{\infty} v_k \ge \sum_{k=1}^{\infty} u_k \ge 0$

If $\sum_{k=1}^{\infty} v_k$ is a number, then $\sum_{k=1}^{\infty} u_k$ is a number.

If $\sum_{k=1}^{\infty} u_k = +\infty$, then $\sum_{k=1}^{\infty} v_k = +\infty$.

Limit Comparison Test Given $u_k, v_k \ge 0$ $\forall k \in \mathbb{N}$

$\lim_{k\to\infty} \frac{v_k}{u_k} =$ positive number $\Rightarrow$ Both $\sum u_k$, $\sum v_k$ converges or both $\sum u_k$, $\sum v_k$ diverges

$\forall$ large k, $v_k \approx c\, u_k$, $c > 0$

$$\lim_{k\to\infty} \frac{v_k}{u_k} = 0 \Rightarrow \begin{cases} \sum u_k \text{ converges} \Rightarrow \sum v_k \text{ converges} \\ \sum v_k \text{ diverges} \Rightarrow \sum u_k \text{ diverges} \end{cases}$$

$\forall$ large k, $\frac{v_k}{u_k} < 1 \Rightarrow v_k < u_k$

$$\lim_{k\to\infty} \frac{v_k}{u_k} = +\infty \Rightarrow \begin{cases} \sum u_k \text{ diverges} \Rightarrow \sum v_k \text{ diverges} \\ \sum v_k \text{ converges} \Rightarrow \sum u_k \text{ converges.} \end{cases}$$

$\forall$ large k, $\frac{v_k}{u_k} > 1 \Rightarrow v_k > u_k$

Examples (1) Consider $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2}\underbrace{\cos\left(\frac{1}{k}\right)}_{>0 \text{ as } \cos 1, \cos\frac{1}{2}, \dots > 0}$

$$\left.\begin{array}{l} 0 \le \frac{1}{k^2}\cos\left(\frac{1}{k}\right) \le \frac{1}{k^2} \\ \sum_{k=1}^{\infty}\frac{1}{k^2} \text{ converges} \\ \text{p series, } p=2>1 \end{array}\right\} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2}\cos\left(\frac{1}{k}\right) \text{ converges}$$

by comparison test

(2) Consider $\displaystyle\sum_{k=2}^{\infty}\underbrace{\frac{3^k}{k^2-1}}_{>0}$ when k large, $\frac{3^k}{k^2-1}$ is dominated by 3^k

$$0 \le \left(\frac{3}{2}\right)^k < \frac{3^k}{k^2-1} \text{ because } k^2-1 < 2^k \text{ for } k \ge 2.$$

$\displaystyle\sum_{k=1}^{\infty}\left(\frac{3}{2}\right)^k$ diverges, geometric series $r=\frac{3}{2}>1$ $\Rightarrow \displaystyle\sum_{k=1}^{\infty}\frac{3^k}{k^2-1}$ diverges by comparison test

(3) Consider $\displaystyle\sum_{k=1}^{\infty}\underbrace{\frac{\sqrt{k+1}}{k^2+5k}}_{>0}$ when k large, $\frac{\sqrt{k+1}}{k^2+5k} \approx \frac{\sqrt{k}}{k^2} = \frac{1}{k^{3/2}}$

Set $u_k = \frac{\sqrt{k}}{k^2} = \frac{1}{k^{3/2}}$ and $v_k = \frac{\sqrt{k+1}}{k^2+5k}$. $u_k, v_k > 0$.

$$\lim_{k\to\infty}\frac{v_k}{u_k} = \lim_{k\to\infty}\frac{\sqrt{k+1}}{k^2+5k}\cdot\frac{k^2}{\sqrt{k}} = \lim_{k\to\infty}\sqrt{\frac{k+1}{k}}\;\frac{k^2}{k^2+5k} = 1\cdot 1 = 1$$

$\sum u_k = \sum\frac{1}{k^{3/2}}$ converges, p-series $p=3/2>1$ $\Rightarrow \sum v_k = \displaystyle\sum_{k=1}^{\infty}\frac{\sqrt{k+1}}{k^2+5k}$ converges by limit comp. test.

(4) Consider $\displaystyle\sum_{k=1}^{\infty}\underbrace{\sin\left(\frac{1}{k}\right)}_{>0}$ When k large $\sin\left(\frac{1}{k}\right) \approx \frac{1}{k}$ as $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

$\theta = \frac{1}{k}\to 0$, $\sin\theta \approx \theta$

Set $u_k = \frac{1}{k}$, $v_k = \sin\frac{1}{k}$, $u_k, v_k > 0$

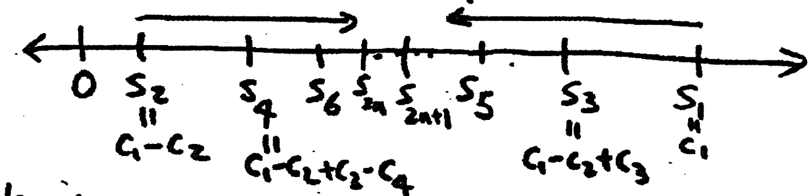
$$\lim_{k\to\infty}\frac{v_k}{u_k} = \lim_{k\to\infty}\frac{\sin(1/k)}{1/k} \underset{\theta=\frac{1}{k}\to 0}{=} \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$\sum u_k = \sum\frac{1}{k}$ diverges, p-series $p=1$ $\Rightarrow \sum v_k = \displaystyle\sum_{k=1}^{\infty}\sin\left(\frac{1}{k}\right)$ diverges by limit comp. test

Alternating Series Test If c_k decreases to 0 as $k\to\infty$ (i.e. $c_1 \ge c_2 \ge c_3 \ge \dots$ and $\lim_{k\to\infty} c_k = 0$), then

$$\sum_{k=1}^{\infty}(-1)^{k+1}c_k = \underbrace{c_1 - c_2 + c_3 - c_4 + c_5 - c_6 + \cdots}_{\text{"alternating series"}} \text{ converges.}$$

Reason For these series, partial sums are as follow



$$\lim_{n\to\infty}|S_{2n} - S_{2n+1}| = \lim_{n\to\infty}(S_{2n+1} - S_{2n}) = \lim_{n\to\infty} c_{2n+1} = 0$$

$\Rightarrow \lim_{n\to\infty} S_n$ is a number $\Rightarrow \displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}c_k$ converges

Examples Consider $\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k}$ and $\sum_{k=1}^{\infty} e^{-k}\underbrace{\cos(k\pi)}_{=(-1)^k}$

For $c_k = \frac{1}{k\ln k}$, as $k\nearrow\infty$, $\ln k\nearrow\infty$, $k\ln k\nearrow\infty$

so $\frac{1}{k\ln k}\searrow 0$. $\therefore \sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k}$ converges by alt. series test.

For $c_k = e^{-k}$, as $k\nearrow\infty$, $-k\searrow-\infty$, $e^{-k}\searrow 0$

$\therefore \sum_{k=1}^{\infty} e^{-k}\cos(k\pi) = \sum_{k=1}^{\infty} (-1)^k e^{-k}$ converges by alt. series test.

Tests for general series $a_k\in\mathbb{R}\ \forall k\in\mathbb{N}$

Absolute Convergence Test $\underbrace{\sum_{k=1}^{\infty} |a_k|}_{\text{converges}} \Rightarrow \underbrace{\sum_{k=1}^{\infty} a_k}_{\text{converges}}$

(Converse is false: $\sum_{k=2}^{\infty}\frac{(-1)^k}{k\ln k}$ converges from above example

$\sum_{k=2}^{\infty}\left|\frac{(-1)^k}{k\ln k}\right| = \sum_{k=2}^{\infty} \frac{1}{k\ln k}$ diverges from integral test page 31, example ②)

Reason for Absolute Convergence Test

$\forall k\in\mathbb{N}$, $-|a_k|\le a_k\le|a_k| \Rightarrow 0\le |a_k|+a_k\le 2|a_k|$

Add $|a_k|$ to all parts

$\underbrace{\sum|a_k| \text{ converges}}_{\text{Given}} \Rightarrow \sum 2|a_k|$ converges

$\Rightarrow \sum(|a_k|+a_k)$ converges by comparison test

$\Rightarrow \underbrace{\sum a_k}_{\text{Converges}} = \sum((|a_k|+a_k)-|a_k|) = \underbrace{\sum(|a_k|+a_k)}_{\text{Converges}} - \underbrace{\sum|a_k|}_{\text{Converges}}$

Definitions $\sum_{k=1}^{\infty} a_k$ converges absolutely iff $\sum_{k=1}^{\infty}|a_k|$ converges

$\sum_{k=1}^{\infty} a_k$ converges conditionally iff $\sum_{k=1}^{\infty}|a_k|$ diverges and $\sum_{k=1}^{\infty} a_k$ converges

Facts to be presented later

Dirichlet proved that for absolute convergent $\sum_{k=1}^{\infty} a_k$,

$\forall$ bijection $f:\mathbb{N}\to\mathbb{N}$, $\sum_{k=1}^{\infty} a_{f(k)} = \sum_{k=1}^{\infty} a_k$.

permutation of terms, same sum

Riemann proved that for condition convergent $\sum_{k=1}^{\infty} a_k$,

$\forall -\infty\le c\le\infty$, $\exists$ bijection $f:\mathbb{N}\to\mathbb{N}$,

$\underbrace{\sum_{k=1}^{\infty} a_{f(k)}}_{\text{permutation of terms}} = c$ ← sum may be arbitrary

Examples Consider $\sum_{k=1}^{\infty}\frac{\cos k}{k^3}$ and $\sum_{k=1}^{\infty} \underbrace{\frac{\cos k\pi}{1+k}}_{=\frac{(-1)^k}{1+k}}$

$$\left.\begin{array}{l} 0\le\left|\frac{\cos k}{k^3}\right|\le\frac{1}{k^3} \\ \sum\frac{1}{k^3} \text{ converges} \\ \text{p-series, } p=3>1 \end{array}\right\} \Rightarrow \sum_{k=1}^{\infty}\left|\frac{\cos k}{k^3}\right| \text{ converges}$$

$\therefore \sum_{k=1}^{\infty}\frac{\cos k}{k^3}$ converges absolutely.

$\sum\left|\frac{\cos k\pi}{1+k}\right| = \sum\frac{1}{1+k}$ As $x\nearrow\infty$, $1+x\nearrow\infty$, so $\frac{1}{1+x}\searrow 0$

$\int_1^{\infty} \frac{1}{1+x}\,dx = \ln(1+x)\Big|_1^{\infty}=\infty \Rightarrow \sum\frac{1}{1+k}$ diverges

$\frac{1}{1+k}\searrow 0 \overset{\text{Alt. series test}}{\Rightarrow} \sum(-1)^k\frac{1}{1+k} = \sum_{k=1}^{\infty}\frac{\cos k\pi}{1+k}$ converges (hence conditionally)

Ratio Test If $\forall k$, $a_k \neq 0$ and $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ exists, then

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| \begin{cases} <1 \Rightarrow \sum a_k \text{ converges absolutely} \\ =1 \Rightarrow \sum a_k \text{ may or may not converge} \\ >1 \Rightarrow \sum a_k \text{ diverges} \end{cases}$$

Reason Let $r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$. Then $\forall k$ large,

$$\left|\frac{a_{k+1}}{a_k}\right|, \left|\frac{a_{k+2}}{a_{k+1}}\right|, \dots, \left|\frac{a_{k+n}}{a_{k+n-1}}\right| \approx r \Rightarrow \left|\frac{a_{k+n}}{a_k}\right| \approx r^n$$

$$\Rightarrow |a_{k+n}| \approx |a_k| r^n$$

$$\Rightarrow |a_k| + |a_{k+1}| + |a_{k+2}| + \cdots \approx |a_k|(1 + r + r^2 + \cdots)$$

So for $r<1$, $|a_k| + |a_{k+1}| + |a_{k+2}| + \cdots \approx \frac{|a_k|}{1-r}$

"hence" $\sum |a_k|$ converges

For $r>1$, $1+r+r^2+\cdots$ diverges, "so" $\lim_{k\to\infty} a_k \neq 0$

"hence" $\sum a_k$ diverges.

Root Test If $\lim_{k\to\infty}\sqrt[k]{|a_k|}$ exists, then

$$\lim_{k\to\infty}\sqrt[k]{|a_k|} \begin{cases} <1 \Rightarrow \sum a_k \text{ converges absolutely} \\ =1 \Rightarrow \sum a_k \text{ may or may not converge} \\ >1 \Rightarrow \sum a_k \text{ diverges.} \end{cases}$$

Reason Let $r = \lim_{k\to\infty}\sqrt[k]{|a_k|}$. Then $\forall k$ large,

$$\sqrt[k]{|a_k|} \approx r \Rightarrow |a_k| \approx r^k \Rightarrow \sum |a_k| \approx \sum r^k.$$

Examples Consider (1) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k - 2^k}$ (2) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$

(1) **Ratio Test**

$$\lim_{k\to\infty} \frac{\frac{1}{3^{k+1}-2^{k+1}}}{\frac{1}{3^k-2^k}} = \lim_{k\to\infty} \frac{3^k-2^k}{3^{k+1}-2^{k+1}} \times \frac{\frac{1}{3^{k+1}}}{\frac{1}{3^{k+1}}} = \lim_{k\to\infty} \frac{\frac{1}{3} - \left(\frac{2}{3}\right)^k \frac{1}{3}}{1 - \left(\frac{2}{3}\right)^{k+1}}$$

$= 1/3 < 1$

$\therefore$ series converges.

Root Test

$$\lim_{k\to\infty} \sqrt[k]{\frac{1}{3^k-2^k}} = \lim_{k\to\infty} \frac{1}{\sqrt[k]{3^k-2^k}} = \lim_{k\to\infty} \frac{1}{3\sqrt[k]{1-\left(\frac{2}{3}\right)^k}} = \frac{1}{3}$$

$\therefore$ series converges.

(2) **Ratio Test**

$$\lim_{k\to\infty} \frac{(k+1)!}{(k+1)^{k+1}} \frac{k^k}{k!} = \lim_{k\to\infty} \frac{k^k}{(k+1)^k} = \lim_{k\to\infty} \frac{1}{\left(1+\frac{1}{k}\right)^k} = \frac{1}{e} < 1.$$

$= \frac{1}{\frac{(k+1)^k}{k^k}}$

$1^\infty \neq 1$

$\therefore$ Series $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$ converges.

<u>Theorem</u> Let $a_k > 0$. If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r \in \mathbb{R}$, then $\lim_{k\to\infty} \sqrt[k]{a_k} = r$. Converse is false.

<u>Examples</u> (1) $a_k = k \Rightarrow \lim_{k\to\infty} \frac{k+1}{k} = 1 \Rightarrow \lim_{k\to\infty} \sqrt[k]{k} = 1$

(2) $a_k = \frac{k!}{k^k}$, $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{e} \Rightarrow \lim_{k\to\infty} \sqrt[k]{\frac{k!}{k^k}} = \frac{1}{e}$

<u>Stirling's Formula</u>

$\forall\, k$ large, $\sqrt[k]{\frac{k!}{k^k}} \approx \frac{1}{e} \Rightarrow \frac{k!}{k^k} \approx \left(\frac{1}{e}\right)^k \Rightarrow k! \approx \left(\frac{k}{e}\right)^k$.

<u>Application</u> Find the number of digits of $100!$ approximately.

$100! \approx \left(\frac{100}{e}\right)^{100}$ $\quad \log_{10} \frac{100}{e} \approx 1.566 \Rightarrow \frac{100}{e} \approx 10^{1.566}$

$\Rightarrow 100! \approx \left(\frac{100}{e}\right)^{100} \approx 10^{156.6}$

$100!$ has about 157 digits.

<u>Summation by Parts</u> Let $S_j = a_1 + a_2 + \cdots + a_j$ and $\Delta b_k = b_{k+1} - b_k$. Then

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\ &= S_1 b_1 + (S_2 - S_1) b_2 + \cdots + (S_n - S_{n-1}) b_n \\ &= S_n b_n - S_1 (b_2 - b_1) - \cdots - S_{n-1}(b_n - b_{n-1}) \\ &= S_n b_n - \sum_{k=1}^{n-1} S_k \Delta b_k .\end{aligned}$$

<u>Example</u> Consider $\sum_{k=1}^{\infty} \frac{\sin k}{k}$. $\quad \frac{\sin k}{k} = \underbrace{(\sin k)}_{a_k} \underbrace{\frac{1}{k}}_{b_k}$

$\sin m \sin\frac{1}{2} = \frac{1}{2}\left(\cos(m - \tfrac{1}{2}) - \cos(m + \tfrac{1}{2})\right)$

$$S_k = \sum_{m=1}^{k} \sin m = \sum_{m=1}^{k} \frac{\cos(m - \frac{1}{2}) - \cos(m + \frac{1}{2})}{2\sin\frac{1}{2}} = \frac{\cos\frac{1}{2} - \cos(k + \frac{1}{2})}{2\sin\frac{1}{2}}$$

$|S_k| \le \frac{1+1}{2\sin\frac{1}{2}} = \frac{1}{\sin\frac{1}{2}} \Rightarrow \lim_{n\to\infty} S_n b_n = \lim_{n\to\infty} \frac{S_n}{n} = 0$

$$\sum_{k=1}^{\infty} \frac{\sin k}{k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin k}{k} = \lim_{n\to\infty} \left(\frac{S_n}{n} - \sum_{k=1}^{n-1} S_k \left(\frac{1}{k+1} - \frac{1}{k}\right)\right) = \sum_{k=1}^{\infty} S_k \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$\sum_{k=1}^{\infty} \left| S_k \left(\frac{1}{k} - \frac{1}{k+1}\right)\right| \le \frac{1}{\sin\frac{1}{2}} \underbrace{\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)}_{\text{example of telescoping series}} = \frac{1}{\sin\frac{1}{2}}$$

$\therefore \sum_{k=1}^{\infty} \frac{\sin k}{k} = \sum_{k=1}^{\infty} S_k \left(\frac{1}{k} - \frac{1}{k+1}\right)$ converges.

Summary

Geometric Series Test for geometric series only $1+a+a^2+\cdots=\sum_{k=1}^{\infty}a^k$

Telescoping Series Test for telescoping series only
$$b_1-\lim_{n\to\infty}b_{n+1}=\sum_{k=1}^{\infty}(b_k-b_{k+1})$$

Term Test
1. Use to show series diverges only
2. May use in the beginning to scan for divergent series

Integral Test for $a_k=f(k)$, $f(x)$ integrable and decreases to 0

p-test
1. Use this for p-series only
2. Use to do Comparison with other series

Comparison Test
Use when you can do inequalities to compare a_k with known examples.

Limit Comparison Test
Use when there are dominated terms in a_k (when k is large) that can be singled out for comparison

Alternating Series Test
for alternating series only with $|a_k|\searrow 0$.

Absolute Convergence Test
for series with positive and negative terms.

Ratio Test for a_k involving $k!$, polynomials in k, k-th power expressions $a_k=(\ldots)^k$

Root Test for k-th power expressions $a_k=(\ldots)^k$

Summation by Parts
for series of the form $\sum a_k b_k$ with $S_n b_n=(a_1+\cdots+a_n)b_n$ having a limit.

$a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9+a_{10}+\cdots$

$$\underbrace{(a_1+a_2)}_{=b_1}+\underbrace{(a_3+a_4+a_5+a_6)}_{=b_2}+\underbrace{(a_7)}_{=b_3}+\underbrace{(a_8+a_9+a_{10})}_{=b_4}+\cdots$$

k_1 terms $\quad k_2$ terms $\quad k_3$ terms $\quad k_4$ terms

$\sum_{k=1}^{\infty} b_k$ is obtained from $\sum_{k=1}^{\infty} a_k$ by inserting parentheses.

Grouping Theorem Let $\sum_{k=1}^{\infty} b_k$ be obtained from $\sum_{k=1}^{\infty} a_k$ by inserting parentheses.

- If $\sum_{k=1}^{\infty} a_k$ converges to S, then $\sum_{k=1}^{\infty} b_k$ converges to S.

The converse is false.

Examples ① $\sum_{k=1}^{\infty}\frac{1}{2^k}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$

$\Rightarrow \frac{1}{2}+\left(\frac{1}{4}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{32}+\frac{1}{64}\right)+\cdots=1$

② $(1-1)+(1-1)+(1-1)+\cdots=0+0+0+\cdots=0$,

but $1-1+1-1+1-1+\cdots$ diverges by term test.

$(1-1)+\left(\frac{1}{2}+\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}\right)+\cdots=0$,

but $1-1+\frac{1}{2}+\frac{1}{2}-\frac{1}{2}-\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}+\cdots$ diverges since $S_{n^2}=1$ and $S_{n^2+n}=0$ so that $\lim_{n\to\infty} S_n$ doesn't exist.

- If $\lim_{n\to\infty} a_n=0$, K_n is bounded, $\sum_{k=1}^{\infty} b_k$ converges to S, then $\sum_{k=1}^{\infty} a_k$ converges to S. $\boxed{\forall n,\ K_n \le \text{Constant}}$

Example ③ $\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots=\sum_{j=1}^{\infty}\left(\frac{1}{2j-1}-\frac{1}{2j}\right)$

$=\sum_{j=1}^{\infty}\frac{1}{2j(2j-1)}$ converges by limit comparison test with $\sum_{j=1}^{\infty}\frac{1}{j^2}$.

Since $\frac{1}{2j-1}, \frac{1}{2j}\to 0$, $\underbrace{\left(\frac{1}{2j-1}-\frac{1}{2j}\right)}_{2\text{ terms}}$ $\forall n,\ K_n=2$

we get $\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

Note $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges by alternating series test. It converges conditionally because $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\sum_{k=1}^{\infty}\frac{1}{k}$ diverges by p-test.

To find the sum of $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$, define $f(x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots$ converge for $x\in[0,1]$ by ratio test

Then $f'(x)=1-x+x^2-x^3+\cdots=\frac{1}{1+x}$ for $x\in[0,1)$

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=f(1)$

$=f(1)-f(0)=\int_0^1 f'(t)\,dt=\int_0^1\frac{1}{1+t}\,dt$

$=\ln(1+t)\Big|_0^1=\ln 2-\ln 1=\underline{\underline{\ln 2}}$

Definition Let $f: \mathbb{N} \to \mathbb{N}$ be a bijection.

$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_{f(k)}$ is a rearrangement of $\sum_{k=1}^{\infty} a_k$.

Example $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

Terms are $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \cdots$

Rearrange terms to $1, \frac{1}{3}, -\frac{1}{2}, \frac{1}{5}, \frac{1}{7}, -\frac{1}{4}, \cdots$

every term appears exactly once.

By Grouping Theorem

$$\left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{3}-\tfrac{1}{4}\right) + \left(\tfrac{1}{5}-\tfrac{1}{6}\right) + \left(\tfrac{1}{7}-\tfrac{1}{8}\right) + \cdots = \ln 2$$

$$+ \quad \tfrac{1}{2} \quad - \tfrac{1}{4} \quad + \tfrac{1}{6} \quad - \tfrac{1}{8} + \cdots = \tfrac{1}{2}\ln 2$$

$$1 + \left(\tfrac{1}{3}-\tfrac{1}{2}\right) + \tfrac{1}{5} + \left(\tfrac{1}{7}-\tfrac{1}{4}\right) + \cdots = \tfrac{3}{2}\ln 2$$

$$\| \quad 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots$$

By Grouping theorem (terms $\to 0$, $k_n \le 2$)

Riemann's Rearrangement theorem

Let $a_k \in \mathbb{R}$ $\forall k$ and $\sum_{k=1}^{\infty} a_k$ converges conditionally.

$\forall x \in \mathbb{R} \cup \{+\infty, -\infty\}$, $\exists$ a rearrangement $\sum_{k=1}^{\infty} b_k$ of $\sum_{k=1}^{\infty} a_k$ such that $\sum_{k=1}^{\infty} b_k = x$.

Dirichlet's Rearrangement Theorem

Let $a_k \in \mathbb{R}$ $\forall k$ and $\sum_{k=1}^{\infty} a_k$ converges absolutely.

$\forall$ rearrangement $\sum_{k=1}^{\infty} b_k$ of $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_k$.

Example $\sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^k = -\frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \cdots = \frac{-\frac{1}{2}}{1-\left(-\frac{1}{2}\right)} = -\frac{1}{3}$

So by Dirichlet's rearrangement theorem,

$$-\frac{1}{2} + \frac{1}{2^2} + \underbrace{\frac{1}{2^4} - \frac{1}{2^3}}_{\text{switched 2 terms}} + \underbrace{\frac{1}{2^8} - \frac{1}{2^7} + \frac{1}{2^6} - \frac{1}{2^5}}_{\text{switched 4 terms}} + \underbrace{\cdots}_{\text{switched } 2^n \text{ terms}} = -\frac{1}{3}$$

Complex Series $z_1+z_2+z_3+\cdots=\sum_{k=1}^{\infty} z_k$, $z_k\in\mathbb{C}$

- $z=a+ib \Rightarrow |z|=\sqrt{a^2+b^2}$
- $S_n=u_n+iv_n$ Definition of Limit

$$\lim_{n\to\infty} S_n = u+iv \Leftrightarrow \lim_{n\to\infty} u_n = u \text{ and } \lim_{n\to\infty} v_n = v$$

- $z_k=x_k+iy_k$ $S_n=z_1+z_2+\cdots+z_n$

$$\sum_{k=1}^{\infty} z_k = \lim_{n\to\infty} S_n = x+iy \Leftrightarrow \sum_{k=1}^{\infty} x_k = x \text{ and } \sum_{k=1}^{\infty} y_k = y$$

- Definitions of absolute convergence and conditional convergence for series are the same.
- Geometric series test, telescoping series test, term test, absolute convergence test, ratio test and root test are true for complex series for the same reasons.

Examples (1) Since $|i|=1$, $\lim_{n\to\infty}|i^n|=\lim_{n\to\infty}1=1\neq 0$

so $\sum_{k=1}^{\infty} i^k$ diverges by term test.

(2) If $|z|\le 1$, then $\left|\frac{z^k}{k^2}\right|\le\frac{1}{k^2}$ and $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by p-test

so $\sum_{k=1}^{\infty}\frac{z^k}{k^2}$ converges absolutely.

If $|z|>1$, then $\lim_{k\to\infty}\left|\frac{z^{k+1}}{(k+1)^2}\frac{k^2}{z^k}\right|=\lim_{k\to\infty}\frac{k^2}{(k+1)^2}|z|=\underbrace{|z|}_{>1}$.

By ratio test, $\sum_{k=1}^{\infty}\frac{z^k}{k^2}$ diverges.

Chapter 5 Real Numbers

The set of all real numbers (denoted by $\mathbb{R}$) satisfies the following axioms:

① Field Axiom
② Order Axiom
③ Well-ordering Axiom
④ Completeness Axiom

An *axiom* is a self-evident statement that is assumed to be foundational in order to obtain more important consequences by deduction.

Field Axiom $\mathbb{R}$ has 2 operations $+$ and $\cdot$ such that $\forall a,b,c\in\mathbb{R}$,

(i) $a+b,\ a\cdot b\in\mathbb{R}$
(ii) $a+b=b+a$, $a\cdot b=b\cdot a$
(iii) $(a+b)+c=a+(b+c)$, $(a\cdot b)\cdot c=a\cdot(b\cdot c)$
(iv) $\exists$ unique elements $0,1\in\mathbb{R}$ with $1\neq 0$ such that $a+0=a$, $a\cdot 1=a$
(v) $\exists\ -a\in\mathbb{R}$ such that $a+(-a)=0$;
if $a\neq 0$, then $\exists\ a^{-1}\in\mathbb{R}$ such that $a\cdot(a^{-1})=1$
(vi) $a\cdot(b+c)=a\cdot b+a\cdot c$.

$-a$ and a^{-1} are unique

Remarks ① From this axiom, we can define

$a-b=a+(-b)$ ← definition of subtraction
$ab=a\cdot b$ ← shorthand notation of multiplication
$\frac{a}{b}=a\cdot(b^{-1})$ for $b\neq 0$ ← definition of division

Define $2=1+1$, $3=2+1$, ...

② $\forall x\in\mathbb{R}$, $x+x\cdot 0=x(1+0)=x\cdot 1=x$. Add $-x$, we get $x\cdot 0=0$.

③ $(-1)(-1)=1$ because $(-1)(-1)=(-1)(-1)+\underbrace{(-1)+1}_{=0}=(-1)(-1+1)+1=(-1)0+1=1$

Order Axiom $\mathbb{R}$ has an (ordering) relation $<$ such that $\forall a,b,c\in\mathbb{R}$,

(i) *exactly one* of the following $a<b$, $a=b$, $b<a$ is true
(ii) if $a<b$ and $b<c$, then $a<c$
(iii) if $a<b$, then $a+c<b+c$
(iv) if $a<b$ and $0<c$, then $ac<bc$.

Remarks We also write $a>b \Leftrightarrow b<a$,
$a\le b \Leftrightarrow a<b$ or $a=b$, $a\ge b\Leftrightarrow b\le a$.
$[a,b]=\{x: x\in\mathbb{R}$ and $a\le x\le b\}$
$(a,b)=\{x: x\in\mathbb{R}$ and $a<x<b\}$
$\max(a_1,\dots,a_n)$ or $\max\{a_1,\dots,a_n\}$ denote the maximum of $a_1,\dots,a_n$ (similarly for $\min(a_1,\dots,a_n)$)
$|x|=\max(x,-x)$ (then $\underbrace{x\le|x| \text{ and } -x\le|x|}_{\Leftrightarrow\ -|x|\le x\le|x|}$)

$|x|\le a \Leftrightarrow x\le a$ and $-x\le a \Leftrightarrow -a\le x\le a$.

Triangle Inequality $\forall x,y\in\mathbb{R}$, $|x+y|\le|x|+|y|$
(Adding $-|x|\le x\le|x|$ and $-|y|\le y\le|y|$, we get $-|x|-|y|\le x+y\le|x|+|y|$. So $|x+y|\le|x|+|y|$.)

$0<1$ Since $1\neq 0$, by (i), $0<1$ or $1<0$.
Assume $1<0$. Then $0=1+(-1)<0+(-1)=-1$.
By (iv), $0=0\cdot(-1)<(-1)\cdot(-1)=1$, contradiction to (i).

CAUTIONS ① $a<b$ and $c<d$ does *not* imply $a-c<b-d$
② $a<b$ does not imply $|a|<|b|$ nor $\frac{a}{c}<\frac{b}{d}$

Well-ordering Axiom $\mathbb{N}=\{1,2,3,\cdots\}$ is well-ordered which means "$\forall$ nonempty $S\subseteq\mathbb{N}$, $\exists\, m\in S$ such that $m\le x$ for all $x\in S$." This m is the least element (or the minimum) of S.

Examples ① $S=$ set of all prime numbers, $m=2$

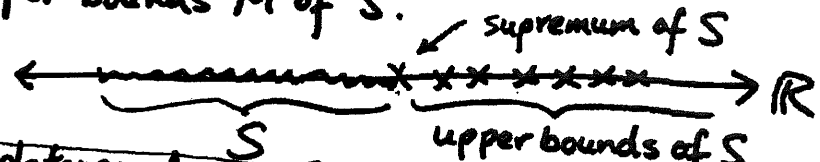
② $S=$ set of all 4-digit positive integers, $m=1000$

③ $S=(\pi,\sqrt{99})\cap\mathbb{N}$, $m=4$

Definitions For a nonempty subset S of $\mathbb{R}$, we say S is bounded above iff $\exists\, M\in\mathbb{R}$ such that $M\ge x$ for all $x\in S$. ← M may not be in S

Such an M is called an upper bound of S.

A supremum or least upper bound of S (denoted by $\sup S$ or $\operatorname{lub} S$) is ① an upper bound $\widetilde{M}$ of S such that ② $\widetilde{M}\le M$ for all upper bounds M of S.



$$\boxed{\text{Completeness Axiom: Every nonempty subset of } \mathbb{R} \text{ which is bounded above has a supremum in } \mathbb{R}.}$$

↑ The supremum may or may not be in the set!!!

Examples ① $S=\{\frac{1}{n}: n\in\mathbb{N}\}=\{1,\frac{1}{2},\frac{1}{3},\cdots\}$

Upper bounds of S: every real number $M\ge 1$

Supremum of S is 1. ← the least number among upper bounds of S

In this case $\sup S=1\in S$.

② $S=\{x: x\in\mathbb{R} \text{ and } x<0\}=(-\infty,0)$

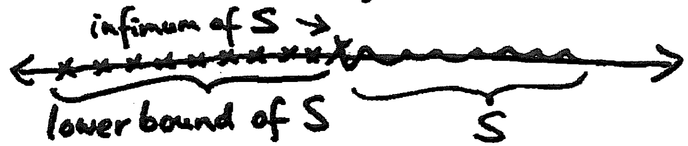
Upper bounds of S: every real number $M\ge 0$

Supremum of S is 0. However, $\sup S=0\notin S$.

Definitions For a nonempty subset S of $\mathbb{R}$, we say S is bounded below iff $\exists\, m\in\mathbb{R}$ such that $m\le x$ for all $x\in S$.

Such an m is called a lower bound of S.

An infimum or greatest lower bound of S (denoted by $\inf S$ or $\operatorname{glb} S$) is ① a lower bound $\widetilde{m}$ of S such that ② $m\le\widetilde{m}$ for all lower bounds m of S.



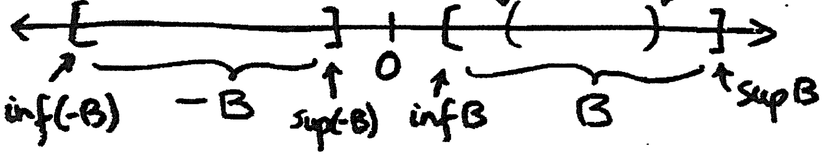
Exercises Let $c\in\mathbb{R}$. Let A,B be nonempty subsets of $\mathbb{R}$. Define

$-B=\{-x: x\in B\}$, $c+B=\{c+x: x\in B\}$,

$cB=\{cx: x\in B\}$,

$A+B=\{x+y: x\in A \text{ and } y\in B\}$.

① B is bounded above $\Leftrightarrow$ $-B$ is bounded below
$\inf(-B) = -\sup B$.
B is bounded below $\Leftrightarrow$ $-B$ is bounded above
$\sup(-B) = -\inf B$.



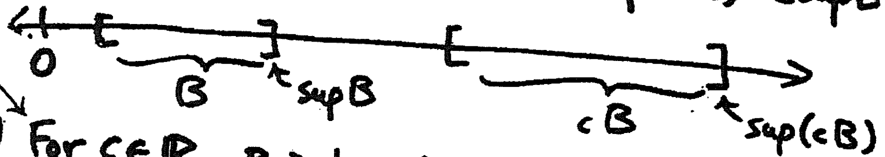
If $\emptyset \neq A \subseteq B$, then $\inf B \leq \inf A$ (when B is bounded below) and $\sup A \leq \sup B$ (when B is bounded above).

Remarks From ① and completeness axiom, we get

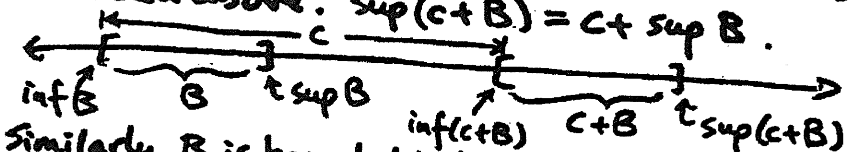
Completeness Axiom for Infimum Every nonempty subset of $\mathbb{R}$ which is bounded below has an infimum in $\mathbb{R}$.

Exercises

② If B is bounded above and $c \geq 0$, then cB is bounded above and $\sup(cB) = c \sup B$.



③ For $c \in \mathbb{R}$, B is bounded above $\Leftrightarrow$ $c+B$ is bounded above. $\sup(c+B) = c + \sup B$.



Similarly, B is bounded below $\Leftrightarrow$ $c+B$ is bounded below. $\inf(c+B) = c + \inf B$.

More generally, if A and B are bounded above and below, then $A+B = \{x+y : x \in A, y \in B\}$ is bounded above and below, $\sup(A+B) = \sup A + \sup B$ and $\inf(A+B) = \inf A + \inf B$. ← Exercises

Definition Let S be a nonempty subset of $\mathbb{R}$. S is <u>bounded</u> iff S is bounded above and below.

Remarks

① S is bounded $\overset{②}{\Rightarrow}$ $\forall x \in S$, $x \leq \sup S$, $\inf S \leq x$

$\overset{③}{\Rightarrow}$ $\forall x \in S$, $|x| \leq \max(\sup S, -\inf S)$ ← $-x \leq -\inf S$

$\overset{④}{\Rightarrow}$ $\exists c \in \mathbb{R}, \forall x \in S$, $|x| \leq c$. ← $-c \leq x \leq c$

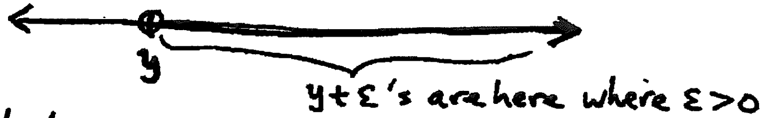
$\therefore$ all 4 statements are equivalent.

Consequences of Axioms

$\alpha, \beta, \gamma, \delta, \varepsilon$ delta epsilon

theorem **(Infinitesimal Principle)** Let $x, y \in \mathbb{R}$.

$(*)$ $x < y + \varepsilon$ for all $\varepsilon > 0 \iff x \le y$



Similarly $y - \varepsilon < x$ for all $\varepsilon > 0 \iff y \le x$.) Order Axiom

Proof. $(\Leftarrow)$ If $x \le y$, then $\forall \varepsilon > 0$, $x \le y = y + 0 < y + \varepsilon$. Field Axiom

$(\Rightarrow)$ If $\forall \varepsilon > 0$, $x < y + \varepsilon$, then assume $x > y$. By order axiom, $\varepsilon_0 = x - y > y - y = 0$. Then $x < y + \varepsilon_0$ by $(*)$. But also $x = y + \varepsilon_0$, contradicting (i) of order axiom. $\therefore x \le y$.

Remarks Letting $x = |a-b|$ and $y = 0$, we have

$$|a-b| < \varepsilon \text{ for all } \varepsilon > 0 \iff |a-b| \le 0 \iff a = b.$$

The principle is often used this way to show expressions are equal.

theorem **(Mathematical Induction Principle)**

(1) $\forall n \in \mathbb{N}$, $A(n)$ is a statement that is either true or false

(2) $A(1)$ is true

(3) $\forall k \in \mathbb{N}$ $A(k)$ true $\Rightarrow A(k+1)$ true

Then $\forall n \in \mathbb{N}$, $A(n)$ is true.

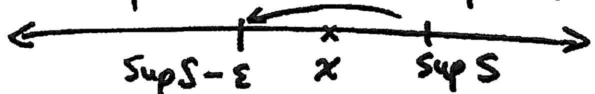
Proof. Assume $\sim(\forall n \in \mathbb{N}, A(n) \text{ is true}) = \exists n \in \mathbb{N}$ such that $A(n)$ is false. Then $S = \{n : A(n) \text{ is false}\}$ is a nonempty subset of $\mathbb{N}$.

By the well-ordering axiom, S has a least element m in S. So $A(m)$ is false and if $A(n)$ is false, then $m \le n$. Taking Contrapositive, if $n < m$, then $A(n)$ is true.

Since $A(1)$ is true, $m \ne 1$. Now $m \in \mathbb{N}$ and $m \ne 1$. $\Rightarrow m \ge 2 \Rightarrow m - 1 \ge 1 \Rightarrow \therefore m - 1 \in \mathbb{N}$. Now $m - 1 < m$. So $A(m-1)$ is true. By (3), we get $A(m)$ is true, contradiction.

theorem **(Supremum Property)** If a set S has a supremum in $\mathbb{R}$ and $\varepsilon > 0$, then $\exists x \in S$ such that

$$\sup S - \varepsilon < x \le \sup S.$$



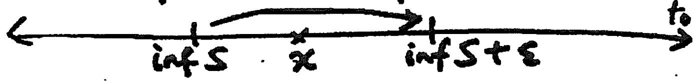
Recall M is an upper bound of S $\iff \forall x \in S$, $x \le M$.

Proof of Supremum Property Since $\sup S - \varepsilon < \sup S$, $\sup S - \varepsilon$ is not an upper bound of S. So $\exists x \in S$ such that $\sup S - \varepsilon < x$. Since $x \in S$, $x \le \sup S$. $\therefore \sup S - \varepsilon < x \le \sup S$.

theorem **(Infimum Property)** If a set S has an infimum in $\mathbb{R}$ and $\varepsilon > 0$, then $\exists x \in S$ such that

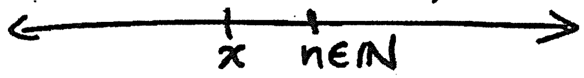
$$\inf S \le x < \inf S + \varepsilon.$$

(Proof is similar to supremum property.)



Theorem

(Archimedean Principle) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > x$.



Proof. Assume $\sim(\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > x)$ $= \exists x \in \mathbb{R}, \forall n \in \mathbb{N}, n \le x$. Then $\mathbb{N}$ is bounded above by x. By the completeness axiom, $\mathbb{N}$ has a supremum in $\mathbb{R}$. By supremum property, $\exists n \in \mathbb{N}$ such that

$\sup \mathbb{N} - 1 < n \le \sup \mathbb{N}$. Then $\sup \mathbb{N} < n+1 \in \mathbb{N}$, a contradiction to $\sup \mathbb{N}$ is an upper bound of $\mathbb{N}$.

Questions How is $\mathbb{Q}$ contained in $\mathbb{R}$? How is $\mathbb{R} \setminus \mathbb{Q}$ contained in $\mathbb{R}$?

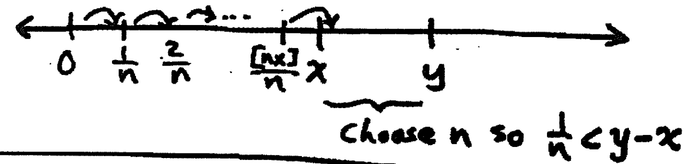
Lemma $\forall x \in \mathbb{R}, \exists$ a least integer (denoted by $\lceil x \rceil$ ← ceiling of x) greater than or equal to x. Similarly, $\exists$ a greatest integer (denoted by $\lfloor x \rfloor$ or $[x]$ ← floor of x) less than or equal to x.

Proof. By Archimedean principle, $\exists n \in \mathbb{N}$ such that $n > |x|$. Then $-n < x < n$. By order axiom, $0 < x+n < 2n$. So $S = \{k : k \in \mathbb{N}, k \ge x+n\}$ is a nonempty subset of $\mathbb{N}$ because $2n \in S$. By the well-ordering axiom, $\exists$ a least positive integer $m \ge x+n$. Then $m-n$ is the least ~~positive~~ integer $\ge x$. So $\lceil x \rceil$ exists. Next, let k be the least ~~positive~~ integer $\ge -x$. Then $-k$ is the greatest ~~positive~~ integer $\le x$. So $\lfloor x \rfloor$ exists.

Theorem

(Density of $\mathbb{Q}$) If $x<y$, then $\exists \frac{m}{n} \in \mathbb{Q}$ such that $x < \frac{m}{n} < y$.

Proof. By Archimedean principle, $\exists n \in \mathbb{N}$ such that $n > \frac{1}{y-x}$. So $ny - nx > 1$. Hence $nx+1 < ny$. Let $m = [nx]+1$, then $m-1 = [nx] \le nx < [nx]+1 = m$. So $nx < m \le nx+1 < ny$. $\therefore x < \frac{m}{n} < y$.

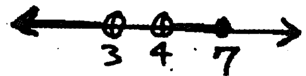


Theorem **(Density of $\mathbb{R} \setminus \mathbb{Q}$)** If $x<y$, then $\exists w \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < w < y$.

Proof. Let $w_0 \in \mathbb{R} \setminus \mathbb{Q}$ (e.g. $w_0 = \sqrt{2}$). By density of $\mathbb{Q}$, $\exists \frac{m}{n} \in \mathbb{Q}$ such that $\frac{x}{|w_0|} < \frac{m}{n} < \frac{y}{|w_0|}$. (If $\frac{m}{n} = 0$, then pick another rational number between 0 and $\frac{y}{|w_0|}$. So we may take $\frac{m}{n} \ne 0$) Let $w = \frac{m}{n}|w_0|$, then $w \in \mathbb{R} \setminus \mathbb{Q}$ and $x < w < y$.

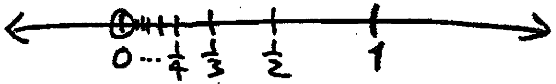
Examples of Supremum and Infimum

① Consider $S = (-\infty, 3) \cup (4, 7]$



S is not bounded below. So S has no infimum. S is bounded above by 7 and every upper bound of S is greater than or equal to 7 because $7 \in S$. So 7 is an upper bound and is the least among upper bounds. $\therefore \sup S = 7$.

② Consider $S=\{\frac{1}{n}: n\in\mathbb{N}\}=\{1,\frac{1}{2},\frac{1}{3},\cdots\}$



$\forall n\in\mathbb{N}$,

$\left.\begin{array}{l}\frac{1}{n}\le 1 \Rightarrow 1 \text{ is an upper bound}\\ 1\in S \Rightarrow \text{every upper bound} \ge 1\end{array}\right\}\Rightarrow \sup S = 1.$

Next we claim $\inf S=0$.

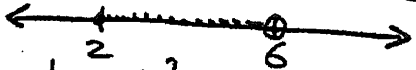
$\forall n\in\mathbb{N}$, $0<\frac{1}{n} \Rightarrow 0$ is a lower bound of S.

(However, $0\notin S$, so we cannot say "every lower bound ≤ 0.") Assume S has a lower bound $t>0$. (To get a contradiction, we will try to get a $\frac{1}{n}\in S$ such that $\frac{1}{n}<t$.) By the Archimedean principle, $\exists n\in\mathbb{N}$ such that $n>\frac{1}{t}$. Then $\frac{1}{n}\in S$ and $\frac{1}{n}<t$, contradicting t is a lower bound of S. So every lower bound $t\le 0$.

$\therefore \inf S=0$.

---

③ Consider $S=[2,6)\cap\mathbb{Q}$



$\left.\begin{array}{l}\forall x\in S,\ 2\le x \Rightarrow 2 \text{ is a lower bound}\\ 2\in S \Rightarrow \text{every lower bound} \le 2\end{array}\right\}\Rightarrow \inf S=2.$

Next we claim $\sup S=6$.

$\forall x\in S$, $x<6 \Rightarrow 6$ is an upper bound of S. Note $6\notin S$.

Assume S has an upper bound $u<6$. Since $2\in S$, $2\le u$. By the density of $\mathbb{Q}$, $\exists r\in\mathbb{Q}$ such that $u<r<6$. Then $r\in[2,6)\cap\mathbb{Q}=S$.

Now $u<r$ contradicts u is an upper bound of S. So every upper bound $u\ge 6$. $\therefore \sup S=6$.

<u>Supremum Limit Theorem</u>

Let c be an <u>upper bound</u> of a nonempty set S. Then

$(\exists w_n\in S \text{ such that } \lim\limits_{n\to\infty} w_n = c) \Leftrightarrow c=\sup S$.

<u>Infimum Limit Theorem</u>

Let c be a <u>lower bound</u> of a nonempty set S. Then

$(\exists w_n\in S \text{ such that } \lim\limits_{n\to\infty} w_n = c) \Leftrightarrow c=\inf S$.

Proofs will be given in the next chapter.

<u>Examples</u> ① Let $S=\{\frac{1}{n}: n\in\mathbb{N}\}=\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots\}$.

$\left.\begin{array}{l}0\le\frac{1}{n}\ \forall n\in\mathbb{N} \Rightarrow 0 \text{ is a lower bound of } S\\ w_n=\frac{1}{n}\in S,\ \lim\limits_{n\to\infty} w_n=0\end{array}\right\}\Rightarrow \inf S = 0.$

---

② Let $S=\{x\pi+\frac{1}{y}: x\in\mathbb{Q}\cap(0,1],\ y\in[1,2]\}$.

$\forall x\in\mathbb{Q}\cap(0,1]$, $y\in[1,2]$, $x\pi+\frac{1}{y}>0\pi+\frac{1}{2}=\frac{1}{2}$

$\left.\begin{array}{l}\Rightarrow \frac{1}{2} \text{ is a lower bound of } S\\ w_n=\frac{1}{n}\pi+\frac{1}{2}\in S,\ \lim\limits_{n\to\infty} w_n=\frac{1}{2}\end{array}\right\}\Rightarrow \inf S=\frac{1}{2}.$

---

③ Let A and B be bounded sets in $\mathbb{R}$.

Let $A-2B=\{a-2b: a\in A,\ b\in B\}$.

Prove $\sup(A-2B)=\sup A-2\inf B$.

<u>Solution.</u> Since A bounded, $\sup A$ exists in $\mathbb{R}$. Since B bounded, $\inf B$ exists in $\mathbb{R}$. $\forall a\in A, b\in B$, we have $a\le\sup A$, $\inf B\le b \Rightarrow a-2b\le\sup A-2\inf B$.

$\therefore$ <u>$c=\sup A-2\inf B$ is an upper bound of $A-2B$.</u>

By supremum limit theorem, $\exists a_n\in A$, $\lim\limits_{n\to\infty} a_n=\sup A$.

By infimum limit theorem, $\exists b_n\in B$, $\lim\limits_{n\to\infty} b_n=\inf B$.

Then $a_n-2b_n\in A-2B$ and $\lim\limits_{n\to\infty}(a_n-2b_n)=\sup A-2\inf B$.

$\therefore$ by supremum limit theorem, $\sup(A-2B)=\sup A-2\inf B$.

Chapter 6 Limits

ε = epsilon $\quad$ δ = delta

We say a sequence $x_1, x_2, x_3, \cdots$ is in S iff every term $x_1, x_2, x_3, \cdots$ is an element of the set S.

A sequence $x_1, x_2, x_3, \cdots$ in $\mathbb{R}$ is bounded above iff the set $\{x_1, x_2, x_3, \cdots\}$ is bounded above in $\mathbb{R}$. Similarly, one can define sequence bounded below or bounded in $\mathbb{R}$.

Notations $\forall x, y \in \mathbb{R}$, let $d(x,y) = |x-y|$. This is the distance between x and y.

$\forall \varepsilon > 0, c \in \mathbb{R}, \; |x-c| < \varepsilon \Leftrightarrow -\varepsilon < x-c < \varepsilon$

$\Leftrightarrow c-\varepsilon < x < c+\varepsilon \Leftrightarrow x \in (c-\varepsilon, c+\varepsilon)$

ε-neighborhood of c.

Intuitive meaning of limit of sequences

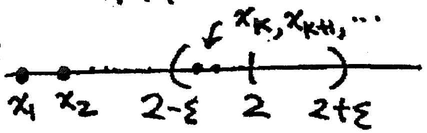
A sequence $x_1, x_2, x_3, \cdots$ in $\mathbb{R}$ has $c \in \mathbb{R}$ as limit means "x_n may be as close to c as desired when n is sufficiently large" or more loosely "as n tends to ∞, $d(x_n, c) = |x_n - c|$ goes to 0."

Warning The words "close", "large", "tends", "goes to" are not precise as they involve personal judgements.

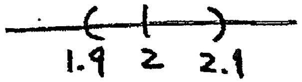
Example Let $x_n = \frac{2n^2-1}{n^2+1}$. We may think its limit is 2.

For every $\varepsilon > 0$, Consider the open interval $(2-\varepsilon, 2+\varepsilon)$. If the limit is 2, then we should be able to see $x_n, x_{n+1}, x_{n+2}, \cdots$ in $(2-\varepsilon, 2+\varepsilon)$ eventually!

$x_n = \frac{2n^2-1}{n^2+1}$ has 2 as limit should mean that for every interval $(2-\varepsilon, 2+\varepsilon)$, $\varepsilon>0$, the sequence $x_1, x_2, x_3, \cdots$ will get into the interval and stay in the interval when n is sufficiently large.



Checking For $\varepsilon = 0.1$, how large should n be so x_n will be in $(2-\varepsilon, 2+\varepsilon) = (1.9, 2.1)$?



Note $x_n \in (2-\varepsilon, 2+\varepsilon)$
$\iff 2-\varepsilon < x_n < 2+\varepsilon$
$\iff -\varepsilon < x_n - 2 < \varepsilon$
$\iff |x_n - 2| < \varepsilon$.

$$|x_n - 2| = \left|\frac{2n^2-1}{n^2+1} - 2\right| = \left|\frac{2n^2-1-2(n^2+1)}{n^2+1}\right| = \frac{3}{n^2+1} < \varepsilon$$

$$\iff \frac{3}{\varepsilon} < n^2+1 \iff \frac{3}{\varepsilon} - 1 < n^2 \iff n > \sqrt{\frac{3}{\varepsilon} - 1}.$$

For $\varepsilon = 0.1$, $n > \sqrt{\frac{3}{0.1} - 1} = \sqrt{29} \approx 5.\cdots$, so $n \geq 6 = K$ is enough.

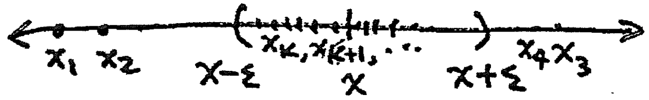
For $\varepsilon = 0.01$; $n > \sqrt{\frac{3}{0.01} - 1} = \sqrt{299} \approx 17.\cdots$, so $n \geq 18 = K$ is enough.

For $\varepsilon = 4$, $n^2 > \frac{3}{4} - 1 = -\frac{1}{4}$, so $n \geq 1 = K$ is enough.

So $\forall \varepsilon > 0$, let $K = \left\lceil \sqrt{\max\left(\frac{3}{\varepsilon} - 1, 1\right)} \right\rceil$, then
$n \geq K \Rightarrow x_n \in (2-\varepsilon, 2+\varepsilon)$
$\therefore x_K, x_{K+1}, x_{K+2}, \cdots \in (2-\varepsilon, 2+\varepsilon)$.

Note for different ε, the value of K will be different. We say K depends on ε in such situation.

Definition A sequence $x_1, x_2, x_3, \cdots$ converges to a number x (or has limit x) iff
$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ (depends on ε) such that
$x_K, x_{K+1}, x_{K+2}, \cdots \in (x-\varepsilon, x+\varepsilon)$



equivalently,
$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |x_n - x| < \varepsilon$
This version is easier to do computations.

Remarks ① For simple sequences, given ε, it may be easy to compute K exactly. However, for complicated sequences, all we need to do is to show such K exists.

② If we are given $x_1, x_2, \cdots$ has limit x, then we may set any positive ε and there is a K for us to use.

If we are asked to prove $x_1, x_2, \cdots$ has limit x, then for every positive ε, we have to find a K or show such a K exists as in the definition.

Examples ① $v_n = c$. Prove $\{v_n\}$ converges to c.
← sequence $v_1, v_2, v_3, \dots$

Solution. $\forall \varepsilon > 0$, $c-\varepsilon \quad c \quad c+\varepsilon$ let $K=1$, then
$n \ge K \Rightarrow |v_n - c| = |c-c| = 0 < \varepsilon$.

② $w_n = c - \frac{1}{n}$. Prove $\{w_n\}$ converges to c.

Solution $\forall \varepsilon > 0$, $w_1 = c-1 \cdots \ c-\varepsilon \quad c \quad c+\varepsilon$ why is such n?

(**Scratch works**: $|w_n - c| = \frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$.)
By Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\varepsilon}$. Then $n \ge K \Rightarrow |w_n - c| = \frac{1}{n} \le \frac{1}{K} < \varepsilon$.

③ $x_n = \frac{n}{(\cos n) - n}$. Prove $\{x_n\}$ converges to -1.

Solution $\forall \varepsilon > 0$, $x_1 \cdots \ c-\varepsilon \quad c \quad c+\varepsilon$

(**Scratch works**: $|x_n - (-1)| = \left|\frac{\cos n}{(\cos n) - n}\right| < \varepsilon$ ← difficult to solve for n

$-1 \quad 0 \quad 1 \quad \cos n \quad n$ $\quad n-1 < |\cos n - n|$

$\le \frac{1}{n-1} < \varepsilon \Leftrightarrow n > 1 + \frac{1}{\varepsilon}$.)

By Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > 1 + \frac{1}{\varepsilon}$. Then $n \ge K \Rightarrow |x_n - (-1)| = \left|\frac{\cos n}{(\cos n) - n}\right| \le \frac{1}{n-1} \le \frac{1}{K-1} < \varepsilon$.

④ $y_n = (-1)^n$. Prove $\{y_n\}$ does not converge.

-1 ($y_1, y_3, \dots$) $\quad 1$ ($y_2, y_4, \dots$)

Solution Assume $\{y_n\}$ converges to y.
For $\varepsilon = 0.1$, $\exists K$ such that $n \ge K \Rightarrow |(-1)^n - y| < \varepsilon$
n odd $\Rightarrow y \in (-1-\varepsilon, -1+\varepsilon) = (-1.1, -0.9)$
n even $\Rightarrow y \in (1-\varepsilon, 1+\varepsilon) = (0.9, 1.1)$. No y satisfies both.

⑤ $z_n = n^{1/n}$. Prove $\{z_n\}$ converges to 1.

Scratch. (Hard to solve $|z_n - 1| = n^{1/n} - 1 < \varepsilon$.)
(Let $u_n = |z_n - 1| = n^{1/n} - 1 \ge 0$. Then $n^{1/n} = 1 + u_n$,
$n = (1+u_n)^n = 1 + n u_n + \frac{n(n-1)}{2} u_n^2 + \cdots \ge \frac{n(n-1)}{2} u_n^2$.
Solving for u_n, we get $u_n \le \sqrt{\frac{2}{n-1}} < \varepsilon \Leftrightarrow n > 1 + \frac{2}{\varepsilon^2}$.)

Solution $\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > 1 + \frac{2}{\varepsilon^2}$. Then
$$n \ge K \Rightarrow |z_n - 1| \le \sqrt{\frac{2}{n-1}} \le \sqrt{\frac{2}{K-1}} < \varepsilon.$$

Uniqueness of Limit If $\{x_n\}$ converges to x and y, then $x = y$. (So we may introduce the notation $\lim_{n\to\infty} x_n = x$.)

Given: ① $\{x_n\}$ converges to x ($\forall \varepsilon_1 > 0\ \exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |x_n - x| < \varepsilon_1$)
② $\{x_n\}$ converges to y ($\forall \varepsilon_2 > 0\ \exists K_2 \in \mathbb{N}$ such that $n \ge K_2 \Rightarrow |x_n - y| < \varepsilon_2$)

To Prove: $x = y$ ($\Leftrightarrow \forall \varepsilon > 0$, $|x - y| < \varepsilon$ Infinitesimal Principle)

Proof. $\forall \varepsilon > 0$, let $\varepsilon_1 = \frac{\varepsilon}{2} > 0$ and $\varepsilon_2 = \frac{\varepsilon}{2} > 0$. Then
$\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |x_n - x| < \varepsilon_1 = \frac{\varepsilon}{2}$
$\exists K_2 \in \mathbb{N}$ such that $n \ge K_2 \Rightarrow |x_n - y| < \varepsilon_2 = \frac{\varepsilon}{2}$.
Let $n = \max(K_1, K_2)$. Then $n \ge K_1$ and $n \ge K_2$.
So $|x - y| = |x - x_n + x_n - y| \le |x - x_n| + |x_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
(the $\le$ step: by triangle inequality)
$\therefore$ by infinitesimal principle, $x = y$.

Example

⑥ Prove $\left\{\frac{2n}{n+5}+\frac{n^8}{n^8+n^5+1}\right\}$ converges to 3 by checking the definition of limit.

Scratch ① When n is large, $\frac{2n}{n+5}\approx 2$, $\frac{n^8}{n^8+n^5+1}\approx 1$.

② $\forall \varepsilon>0$,

$\left|\frac{2n}{n+5}+\frac{n^8}{n^8+n^5+1}-3\right| < \varepsilon$ is hard to solve for n

$=\left|\frac{2n}{n+5}-2+\frac{n^8}{n^8+n^5+1}-1\right|$

$\le\left|\frac{2n}{n+5}-2\right|+\left|\frac{n^8}{n^8+n^5+1}-1\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$ is easier to solve

③ $\left|\frac{2n}{n+5}-2\right|=\left|\frac{-10}{n+5}\right|=\frac{10}{n+5}<\frac{\varepsilon}{2}$ if $n>\frac{20}{\varepsilon}-5$

$\left|\frac{n^8}{n^8+n^5+1}-1\right|=\left|\frac{-n^5-1}{n^8+n^5+1}\right|=\frac{n^5+1}{n^8+n^5+1}<\frac{2n^5}{n^8}=\frac{2}{n^3}<\frac{\varepsilon}{2}$ if $n>\sqrt[3]{4/\varepsilon}$.

Solution $\forall\varepsilon>0$, by Archimedean Principle,

$\exists K_1\in\mathbb{N}$ such that $K_1>\frac{20}{\varepsilon}-5$ and

$\exists K_2\in\mathbb{N}$ such that $K_2>\sqrt[3]{4/\varepsilon}$.

Let $K=\max(K_1,K_2)$. Then

$n\ge K \Rightarrow n\ge K_1>\frac{20}{\varepsilon}-5$ and $n\ge K_2>\sqrt[3]{4/\varepsilon}$

$\Rightarrow \left|\frac{2n}{n+5}+\frac{n^8}{n^8+n^5+1}-3\right|\le\left|\frac{2n}{n+5}-2\right|+\left|\frac{n^8}{n^8+n^5+1}-1\right|$

$=\frac{10}{n+5}+\frac{n^5+1}{n^8+n^5+1}<\frac{10}{n+5}+\frac{2}{n^3}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$

Boundedness Theorem If $\{x_n\}$ converges, then the set $\{x_1,x_2,x_3,\dots\}$ is bounded (above and below).

Given: $\{x_n\}$ converges to some $x\in\mathbb{R}$ ($\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $n\ge K\Rightarrow |x_n-x|<\varepsilon$)

To Prove: $\{x_1,x_2,x_3,\dots\}$ is bounded ($\Leftrightarrow \exists M\in\mathbb{R}$ $\forall x_n$, $|x_n|\le M$)

Proof. Let $x=\lim\limits_{n\to\infty}x_n$. For $\varepsilon=1$, $\exists K\in\mathbb{N}$ such that

$n\ge K\Rightarrow |x_n-x|<1 \Rightarrow |x_n|=|x_n-x+x|$
$\le |x_n-x|+|x|$
$<1+|x|$.

Let $M=\max(|x_1|,|x_2|,\dots,|x_{K-1}|,1+|x|)$. Then

$\forall n\in\mathbb{N}$, $n\ge K\Rightarrow |x_n|<1+|x|\le M$

$n<K\Rightarrow x_n=x_1$ or x_2 or $\dots=x_{K-1}\Rightarrow |x_n|\le M$.

Remarks The converse of the boundedness theorem is false. $x_n=(-1)^n$ $\{x_1,x_2,x_3,\dots\}=\{-1,1\}$ is bounded but $\{x_n\}$ does not converge by example ④.

Remarks The following are equivalent:

① $\{x_n\}$ converges to x ($\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $n\ge K\Rightarrow |x_n-x|<\varepsilon$)

② $\{x_n-x\}$ converges to 0 ($\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $n\ge K\Rightarrow |(x_n-x)-0|<\varepsilon$)

③ $\{|x_n-x|\}$ converges to 0 ($\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $n\ge K\Rightarrow \big||x_n-x|-0\big|<\varepsilon$.)

Computation Formulas for Limits

<u>Given</u>: ① $\lim_{n\to\infty} x_n = x$ ($\forall \varepsilon_1>0, \exists K_1^{\in N}, n\geq K_1 \Rightarrow |x_n - x|<\varepsilon_1$)

② $\lim_{n\to\infty} y_n = y$ ($\forall \varepsilon_2>0, \exists K_2^{\in N}, n\geq K_2 \Rightarrow |y_n - y|<\varepsilon_2$)

<u>To prove</u>: $\lim_{n\to\infty}(x_n+y_n) = x+y$

($\forall \varepsilon>0, \exists K^{\in N}, n\geq K \Rightarrow |(x_n+y_n)-(x+y)|<\varepsilon$)

<u>Idea</u>: $|(x_n+y_n)-(x+y)| = |(x_n-x)+(y_n-y)|$

$\leq \underbrace{|x_n-x|}_{<\varepsilon/2} + \underbrace{|y_n-y|}_{<\varepsilon/2}$

<u>Proof</u>. $\forall \varepsilon>0$, let $\varepsilon_1 = \varepsilon/2>0$ and $\varepsilon_2 = \varepsilon/2 > 0$.

By ①, $\exists K_1$, $n\geq K_1 \Rightarrow |x_n - x| < \varepsilon_1 = \varepsilon/2$.

By ②, $\exists K_2$, $n\geq K_2 \Rightarrow |y_n - y| < \varepsilon_2 = \varepsilon/2$.

<u>Max trick</u> Let $K = \max(K_1, K_2)\in N$.

$$n\geq K \Rightarrow \begin{cases} n\geq K_1 \\ \text{and} \\ n\geq K_2\end{cases} \Rightarrow \begin{cases} |x_n - x| < \varepsilon/2 \\ \text{and} \\ |y_n-y| < \varepsilon/2\end{cases}$$

$$\Rightarrow |(x_n+y_n)-(x+y)| = |(x_n-x)+(y_n-y)| \qquad (*)$$

$$\leq |x_n-x|+|y_n-y| < \varepsilon/2+\varepsilon/2 = \varepsilon.$$

<u>Given</u>: ① and ② above

<u>To Prove</u>: $\lim_{n\to\infty}(x_n-y_n) = x-y$

($\forall\varepsilon>0, \exists K^{\in N}, n\geq K \Rightarrow |(x_n-y_n)-(x-y)|<\varepsilon$)

<u>Proof</u>. Just change the 3 + signs in (*) to − signs above.

Lemma

If (a) $\{a_n\}$ is bounded ($\exists M>0$ such that $\forall n, |a_n|\leq M$)

and (b) $\lim_{n\to\infty} b_n = 0$ ($\forall \varepsilon_1>0, \exists K_1^{\in N}, n\geq K_1 \Rightarrow |b_n - 0|<\varepsilon_1$)

then $\lim_{n\to\infty} a_n b_n = 0$ ($\forall\varepsilon>0, \exists K^{\in N}, n\geq K \Rightarrow |a_nb_n - 0|<\varepsilon$)

<u>Idea</u> $|a_nb_n - 0| = |a_nb_n| \leq M|b_n| = M|b_n-0| < M\varepsilon_1 = \varepsilon$

should choose $\varepsilon_1 = \varepsilon/M$.

<u>Proof</u>. $\forall\varepsilon>0$, let $\varepsilon_1 = \varepsilon/M$, where M is as in (a).

By (b), $\exists K = K_1 \in N$, $n\geq K \Rightarrow |b_n - 0| < \varepsilon_1 = \varepsilon/M$

$\Rightarrow |a_nb_n - 0| = |a_nb_n| \leq M|b_n| = M|b_n - 0| < M\varepsilon_1 = \varepsilon$.

<u>Given</u>: ① $\lim_{n\to\infty} x_n = x$ ($\Leftrightarrow \lim_{n\to\infty}(x_n - x) = 0$), ② $\lim_{n\to\infty} y_n = y$ ($\Leftrightarrow \lim_{n\to\infty}(y_n - y) = 0$)

<u>To Prove</u>: $\lim_{n\to\infty} x_ny_n = xy$ ($\Leftrightarrow \lim_{n\to\infty}(x_ny_n - xy) = 0$ by earlier remark)

<u>Proof</u>.
$$\left.\begin{aligned} x_ny_n - xy &= x_ny_n - x_ny + x_ny - xy \\ &= x_n(y_n-y) + y(x_n-x)\end{aligned}\right\}(\Delta)$$

Since $\{x_n\}$ converges, $\{x_n\}$ is bounded by boundedness theorem.

Constant sequence $\{y\}$ is bounded

$\therefore \lim_{n\to\infty}(x_ny_n - xy) = \lim_{n\to\infty}(x_n(y_n-y)+y(x_n-x))$ by (Δ)

$= \lim_{n\to\infty} x_n(y_n-y) + \lim_{n\to\infty} y(x_n-x)$ by $\lim(a_n+b_n) = \lim a_n + \lim b_n$

$= 0 + 0$ by lemma

$= 0$

Given: ① $\lim_{n\to\infty} x_n = x$, ② $\forall n\in\mathbb{N},\ y_n \neq 0$

and ③ $\lim_{n\to\infty} y_n = y \neq 0$ $(\forall \varepsilon_1>0, \exists K_1^{\in\mathbb{N}}, n\geq K_1 \Rightarrow |y_n - y| < \varepsilon_1)$

To Prove: $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.

Proof (Step 1) We will show $\lim_{n\to\infty}\frac{1}{y_n} = \frac{1}{y}$ first.

$$(\forall\varepsilon>0, \exists K^{\in\mathbb{N}}, n\geq K \Rightarrow |\tfrac{1}{y_n} - \tfrac{1}{y}| < \varepsilon.)$$

Since $\frac{1}{2}|y|>0$, by ③, $\exists K_0^{\in\mathbb{N}}$, $n\geq K_0 \Rightarrow |y_n - y| < \frac{1}{2}|y|$

$\Rightarrow |y| = |y_n - (y_n - y)| \leq |y_n| + |y_n - y| < |y_n| + \frac{1}{2}|y|$

$\Rightarrow \frac{1}{2}|y| < |y_n|$

$\Rightarrow \frac{1}{|y_n|} < \frac{1}{\frac{1}{2}|y|}$.

$\forall\varepsilon>0$, let $\varepsilon_1 = \frac{1}{2}|y|^2\varepsilon > 0$. By ③, $\exists K_1^{\in\mathbb{N}}$ such that $n\geq K_1 \Rightarrow |y_n - y| < \varepsilon_1 = \frac{1}{2}|y|^2\varepsilon$.

Max trick Let $K = \max(K_0, K_1)$. Then

$$n \geq K \Rightarrow \begin{cases} n\geq K_0 \\ \text{and} \\ n \geq K_1\end{cases} \Rightarrow \begin{cases} \frac{1}{|y_n|} \leq \frac{1}{\frac{1}{2}|y|} \\ |y_n - y| < \frac{1}{2}|y|^2\varepsilon \end{cases}$$

$$\Rightarrow \left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y - y_n}{y_n y}\right| = \frac{|y_n - y|}{|y_n||y|} < \frac{\frac{1}{2}|y|^2\varepsilon}{\frac{1}{2}|y||y|} = \varepsilon.$$

(Step 2)

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty} x_n \frac{1}{y_n} = x\frac{1}{y} = \frac{x}{y}.$$

by $\lim(a_n b_n) = (\lim a_n)(\lim b_n)$ and step 1.

Recall $|a-b|<r \Leftrightarrow a\in(b-r, b+r)$.

Sandwich Theorem (or **Squeeze Limit Theorem**)

If ① $\forall n\in\mathbb{N},\ x_n \leq w_n \leq y_n$

and ② $\lim_{n\to\infty} x_n = z = \lim_{n\to\infty} y_n$

$(\forall\varepsilon>0\ \exists K_1^{\in\mathbb{N}}, n\geq K_1 \Rightarrow |x_n - z| < \varepsilon$

$\forall\varepsilon>0\ \exists K_2^{\in\mathbb{N}}, n\geq K_2 \Rightarrow |y_n - z| < \varepsilon)$

then $\lim_{n\to\infty} w_n = z$ $(\forall\varepsilon>0\ \exists K^{\in\mathbb{N}}, n\geq K \Rightarrow |w_n - z| < \varepsilon)$

Proof. $\forall\varepsilon>0$, let $K = \max(K_1, K_2)$, where K_1, K_2 are as in ②. Then

$$n\geq K \Rightarrow \begin{cases} n\geq K_1 \\ \text{and} \\ n\geq K_2\end{cases} \Rightarrow \begin{cases} |x_n - z| < \varepsilon \\ \text{and} \\ |y_n - z| < \varepsilon\end{cases} \Leftrightarrow \begin{cases} x_n \in (z-\varepsilon, z+\varepsilon) \\ y_n \in (z-\varepsilon, z+\varepsilon)\end{cases}$$

$$\xRightarrow{\text{by ①}} w_n \in (z-\varepsilon, z+\varepsilon) \Leftrightarrow |w_n - z| < \varepsilon.$$

Example Let $w_n = \frac{[10^n\sqrt{2}]}{10^n} \in \mathbb{Q}$ for all $n\in\mathbb{N}$.

(Note $w_1 = 1.4$, $w_2 = 1.41$, $w_3 = 1.414$, $w_4 = 1.4142, \ldots$)

Then $10^n\sqrt{2} - 1 < [10^n\sqrt{2}] \leq 10^n\sqrt{2}$ and so

$$\frac{10^n\sqrt{2} - 1}{10^n} < \frac{[10^n\sqrt{2}]}{10^n} = w_n \leq \sqrt{2}.$$

Since $\lim_{n\to\infty}\frac{10^n\sqrt{2}-1}{10^n} = \sqrt{2}$, by sandwich theorem, $\lim_{n\to\infty} w_n = \sqrt{2}$.

Remark We may replace $\sqrt{2}$ by any real number. Every real number is the limit of a sequence in $\mathbb{Q}$.

Limit Inequality

If ① $\forall n\in\mathbb{N}$, $a_n\geq 0$

and ② $\lim\limits_{n\to\infty} a_n = a$ ($\forall\varepsilon>0\ \exists K^{\in\mathbb{N}}, n\geq K \Rightarrow |a_n-a|<\varepsilon$)

then $a\geq 0$.

Proof. Assume $a<0$. Then let $\varepsilon=|a|=-a>0$.

By ②, $\exists K^{\in\mathbb{N}}$, $n\geq K \Rightarrow |a_n-a|<\varepsilon=-a$

$\Rightarrow a_n-a<-a$

$\Rightarrow a_n<0$, contradiction to ①.

Remarks ① If $\forall n\in\mathbb{N}$, $x_n\leq y_n$ and $\lim\limits_{n\to\infty}x_n=x$, $\lim\limits_{n\to\infty}y_n=y$, then ($a_n=y_n-x_n\geq 0$, $\lim\limits_{n\to\infty}a_n=y-x\geq 0$) $x\leq y$.

② If $\forall n\in\mathbb{N}$, $a\leq x_n\leq b$ and $\lim\limits_{n\to\infty}x_n=x$, then ($a=\lim\limits_{n\to\infty}a\leq\lim\limits_{n\to\infty}x_n=x$, $x=\lim\limits_{n\to\infty}x_n\leq\lim\limits_{n\to\infty}b=b$) $a\leq x\leq b$.

Equivalently, if $\forall n\in\mathbb{N}$, $x_n\in[a,b]$ and $\lim\limits_{n\to\infty}x_n=x$, then $x\in[a,b]$. This is not true for open intervals !!!

$\frac{1}{n}>0$, $\lim\limits_{n\to\infty}\frac{1}{n}=0$. $\frac{1}{n}\in(0,+\infty)$, $\lim\limits_{n\to\infty}\frac{1}{n}=0\notin(0,+\infty)$.

Supremum Limit Theorem

Let c be an upper bound of a nonempty set S. Then

($\exists w_n\in S$ such that $\lim\limits_{n\to\infty}w_n=c$) $\Leftrightarrow$ $c=\sup S$.

Proof. ($\Rightarrow$) $\exists w_n\in S$, $\lim\limits_{n\to\infty}w_n=c$. Since $w_n\in S$, $w_n\leq\sup S\leq c$ (c is an upper bound).

Taking limit, $c\leq\sup S\leq c$. $\therefore c=\sup S$.

($\Leftarrow$) $c=\sup S$. By supremum property, $\forall n\in\mathbb{N}$, $\exists w_n\in S$ such that

$\underbrace{c-\frac{1}{n}}_{\to c}=\sup S-\frac{1}{n}<w_n\leq\underbrace{\sup S=c}_{\to c}$. Sandwich $\Rightarrow \lim\limits_{n\to\infty}w_n=c$.

Infimum Limit Theorem

Let c be a lower bound of a nonempty set S. Then

($\exists w_n\in S$ such that $\lim\limits_{n\to\infty}w_n=c$) $\Leftrightarrow$ $c=\inf S$.

Proof is similar to the proof of supremum limit theorem.

Examples ① Let $S=\{\frac{1}{n}:n\in\mathbb{N}\}=\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\}$.

$\left.\begin{array}{l} 0\leq\frac{1}{n}\ \forall n\in\mathbb{N} \Rightarrow 0 \text{ is a lower bound of } S \\ w_n=\frac{1}{n}\in S,\ \lim\limits_{n\to\infty}w_n=0\end{array}\right\}\Rightarrow \inf S = 0$.

---

② Let $S=\{x\pi+\frac{1}{y}: x\in\mathbb{Q}\cap(0,1],\ y\in[1,2]\}$.

$\forall x\in\mathbb{Q}\cap(0,1]$, $y\in[1,2]$, $x\pi+\frac{1}{y}>0\pi+\frac{1}{2}=\frac{1}{2}$

$\left.\begin{array}{l}\Rightarrow\frac{1}{2}\text{ is a lower bound of } S\\ w_n=\frac{1}{n}\pi+\frac{1}{2}\in S,\ \lim\limits_{n\to\infty}w_n=\frac{1}{2}\end{array}\right\}\Rightarrow\inf S=\frac{1}{2}$.

---

③ Let A and B be bounded sets in $\mathbb{R}$.

Let $A-2B=\{a-2b: a\in A, b\in B\}$.

Prove $\sup(A-2B)=\sup A-2\inf B$.

Solution. Since A bounded, $\sup A$ exists in $\mathbb{R}$. Since B bounded, $\inf B$ exists in $\mathbb{R}$. $\forall a\in A, b\in B$, we have $a\leq\sup A$, $\inf B\leq b$ $\Rightarrow a-2b\leq\sup A-2\inf B$.

$\therefore$ $c=\sup A-2\inf B$ is an upper bound of $A-2B$.

By supremum limit theorem, $\exists a_n\in A$, $\lim\limits_{n\to\infty}a_n=\sup A$.

By infimum limit theorem, $\exists b_n\in B$, $\lim\limits_{n\to\infty}b_n=\inf B$.

Then $a_n-2b_n\in A-2B$ and $\lim\limits_{n\to\infty}(a_n-2b_n)=\sup A-2\inf B$.

$\therefore$ by supremum limit theorem, $\sup(A-2B)=\sup A-2\inf B$.

Question: How can we show a sequence has a limit if it is given by a recurrent relation? For example, $x_1 = 2$ and $x_{n+1} = \sqrt{3x_n - 2}$ for $n = 1, 2, 3, \cdots$

Definition Let $\{x_n\}$ be a sequence of numbers. $x_{n_1}, x_{n_2}, x_{n_3}, \cdots$ is a subsequence of $\{x_n\}$ iff $n_1 < n_2 < n_3 < \cdots$ and $n_j \in \mathbb{N}$ $\forall j = 1, 2, 3, \cdots$

Examples For sequence $x_1, x_2, x_3, \cdots$, if we set $n_j = j^2$, then we get $x_1, x_4, x_9, x_{16}, \cdots$, which is a subsequence because $1 < 4 < 9 < 16 < \cdots$.
If we set $n_j = 2j+1$, then we get $x_3, x_5, x_7, x_9, \cdots$ which is a subsequence because $3 < 5 < 7 < 9 < \cdots$.

Remarks $n_1 < n_2 < n_3 < \cdots$ and $n_j \in \mathbb{N}$ $\forall j = 1, 2, 3, \cdots$
$\Rightarrow n_j \geq j$ $\forall j = 1, 2, 3, \cdots$
We can prove this by mathematical induction. For $j = 1$, $n_1 \in \mathbb{N} \Rightarrow n_1 \geq 1$. If $n_j \geq j$, then $n_{j+1} > n_j \geq j$ and $n_{j+1} \in \mathbb{N} \Rightarrow n_{j+1} \geq j+1$.

Subsequence Theorem If $\lim_{n\to\infty} x_n = x$, then for every subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \cdots$, we have $\lim_{j\to\infty} x_{n_j} = x$.

Proof. $\forall \varepsilon > 0$, since $\lim_{n\to\infty} x_n = x$, $\exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |x_n - x| < \varepsilon$. Then

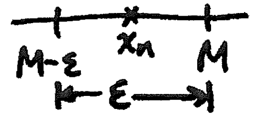
$j \geq K \Rightarrow n_j \geq j \geq K \Rightarrow |x_{n_j} - x| < \varepsilon$.

Terminologies Let $\{x_n\}$ be a sequence of real numbers.
$\{x_n\}$ is increasing iff $x_1 \leq x_2 \leq x_3 \leq \cdots$.
$\{x_n\}$ is decreasing iff $x_1 \geq x_2 \geq x_3 \geq \cdots$.
$\{x_n\}$ is strictly increasing iff $x_1 < x_2 < x_3 < \cdots$
$\{x_n\}$ is strictly decreasing iff $x_1 > x_2 > x_3 > \cdots$.
$\{x_n\}$ is monotone iff $\{x_n\}$ is increasing or decreasing.
$\{x_n\}$ is strictly montone iff $\{x_n\}$ is strictly increasing or strictly decreasing.

Monotone Sequence Theorem If $\{x_n\}$ is increasing and bounded above, then $\lim_{n\to\infty} x_n = \sup\{x_1, x_2, x_3, \cdots\}$. (Similarly, if $\{x_n\}$ is decreasing and bounded below, then $\lim_{n\to\infty} x_n = \inf\{x_1, x_2, x_3, \cdots\}$.)

Proof. Since $\{x_n\}$ is bounded above, $M = \sup\{x_1, x_2, x_3, \cdots\}$ exists. $\forall \varepsilon > 0$, by the supremum property, $\exists x_K$ such that $M - \varepsilon < x_K \leq M$. Then $x_K \in (M-\varepsilon, M]$. So

$n \geq K \Rightarrow x_K \leq x_n \leq \sup\{x_1, x_2, x_3, \cdots\} = M$
$\Rightarrow x_n \in (M-\varepsilon, M]$
$\Rightarrow |x_n - M| < \varepsilon$



The decreasing case is similar.

Examples ① Let $0<c<1$ and $x_n = c^{1/n}$ for $n=1,2,3,\cdots$.
Then $x_n<1\ \forall n$. Also,

$$c^{n+1}<c^n \Rightarrow x_n = c^{1/n} = (c^{n+1})^{\frac{1}{n(n+1)}} < (c^n)^{\frac{1}{n(n+1)}} = c^{\frac{1}{n+1}} = x_{n+1}$$

By the monotone sequence theorem, $\{x_n\}$ has a limit x.
Now $x_{2n}^2 = (c^{\frac{1}{2n}})^2 = c^{\frac{1}{n}} = x_n$. Taking limits on both sides, by subsequence theorem, $x^2=x$. So $x=0$ or 1.
Since $0<c=x_1\le x=\sup\{x_1,x_2,x_3,\cdots\}$, we get $x=1$.
(Similarly, if $c\ge 1$, then $x_n=c^{1/n}$ will decrease to the limit 1.)

② Does $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$ represent a real number?
Here, the question is if $x_1=\sqrt{2}$ and $x_{n+1}=\sqrt{2+x_n}$ for $n=1,2,3,\cdots$ converges to a real numbers.

__Scratch Work__ $x_1=\sqrt{2}<x_2=\sqrt{2+\sqrt{2}}<x_3=\sqrt{2+\sqrt{2+\sqrt{2}}}$
We __suspect__ $\{x_n\}$ is strictly increasing.
Assume $\lim_{n\to\infty} x_n = x$. Then $x^2 = \lim_{n\to\infty} x_{n+1}^2 = \lim_{n\to\infty}(2+x_n) = 2+x$
$\Rightarrow x^2-x-2=(x-2)(x+1)=0 \Rightarrow x=2$ or $\cancel{-1}$ reject.

__Solution__. We will show $x_n<x_{n+1}<2\ \forall n\in\mathbb{N}$ by math induction. For $n=1$, $x_1=\sqrt{2}<x_2=\sqrt{2+\sqrt{2}}<2$.
If $x_n<x_{n+1}<2$, then $x_n+2<x_{n+1}+2<2+2=4$
$\Rightarrow x_{n+1}=\sqrt{x_n+2}<\sqrt{x_{n+1}+2}=x_{n+2}<\sqrt{2+2}=\sqrt{4}=2$.
So monotone sequence theorem $\Rightarrow \lim_{n\to\infty} x_n = x$ exists.
As in scratch work, since $\sqrt{2}=x_1\le x$, we get $x=2$.

__Note__ If $\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} b_n = x$, then we expect $a_1,b_1,a_2,b_2,a_3,b_3,\cdots$ converges to x.

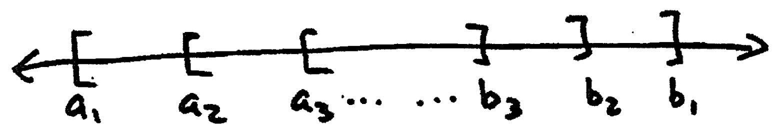
__Intertwining Sequence Theorem__
If ① $\lim_{m\to\infty} x_{2m-1} = x$ ($\forall\varepsilon>0\ \exists K_1\in\mathbb{N}, m\ge K_1 \Rightarrow |x_{2m-1}-x|<\varepsilon$)
and ② $\lim_{m\to\infty} x_{2m} = x$ ($\forall\varepsilon>0\ \exists K_2\in\mathbb{N}, m\ge K_2 \Rightarrow |x_{2m}-x|<\varepsilon$)
then $\lim_{n\to\infty} x_n = x$ ($\forall\varepsilon>0\ \exists K\in\mathbb{N}, n\ge K \Rightarrow |x_n-x|<\varepsilon$).

__Proof__. $\forall\varepsilon>0$, let K_1,K_2 be as in conditions ①,②.
Let $K=\max(2K_1-1,2K_2)$. Then

$$n\ge K \Rightarrow \begin{cases} n\ge 2K_1-1 \\ \text{and} \\ n\ge 2K_2\end{cases} \Rightarrow \begin{cases} n \text{ odd} \Rightarrow n=2m-1 \text{ with } m\ge K_1 \\ n \text{ even} \Rightarrow n=2m \text{ with } m\ge K_2\end{cases}$$

$$\Rightarrow \begin{cases} n \text{ odd} \Rightarrow |x_n-x|=|x_{2m-1}-x|<\varepsilon \\ n \text{ even} \Rightarrow |x_n-x|=|x_{2m}-x|<\varepsilon\end{cases}.$$

__Nested Interval Theorem__ $a_n,b_n\in\mathbb{R}$
If $\forall n\in\mathbb{N}$, $I_n=[a_n,b_n]$ and $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$,
then $\bigcap_{n=1}^{\infty} I_n=[a,b]$, where $a=\lim_{n\to\infty}a_n\le\lim_{n\to\infty}b_n=b$.
If $\lim_{n\to\infty}(b_n-a_n)=0$, then $\bigcap_{n=1}^{\infty} I_n=\{x\}$ for some $x\in\mathbb{R}$.

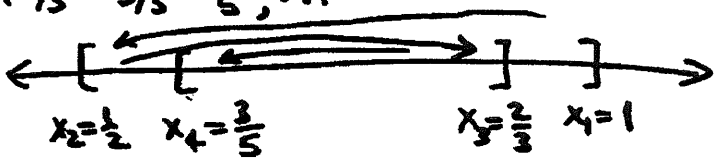


Proof. $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ implies $\{a_n\}$ is increasing and bounded above by b_1 and $\{b_n\}$ is decreasing and bounded below by a_1. By monotone sequence theorem, $\lim_{n\to\infty} a_n = \sup\{a_1,a_2,a_3,\ldots\} = a$, $\lim_{n\to\infty} b_n = \inf\{b_1,b_2,b_3,\ldots\} = b$. Since $a_n \le b_n\ \forall n$, taking limit on both sides, get $a\le b$.
Hence $x \in \bigcap_{n=1}^{\infty} [a_n,b_n] \iff \forall n\in\mathbb{N},\ a_n\le x\le b_n$
$\iff a\le x\le b \iff x\in[a,b]$.
If $0=\lim_{n\to\infty}(b_n-a_n) = b-a$, then $a=b$, $\bigcap_{n=1}^{\infty} I_n = \{a\}$.

Example. Does $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$ represent a real number?

Here the question is if $x_1=1$ and $x_{n+1}=\frac{1}{1+x_n}$ for $n=1,2,3,\ldots$ converges to a real number.

Scratch Work $x_1=1$, $x_2=\frac{1}{1+1}=\frac12$, $x_3=\frac{1}{1+\frac12}=\frac{1}{3/2}=\frac23$,
$x_4=\frac{1}{1+2/3}=\frac{1}{5/3}=\frac35,\ldots$



Solution (Step 1: Form I_n and show $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$)
Define $I_n=[x_{2n},x_{2n-1}]$ for $n=1,2,3,\ldots$
We will show $I_n\supseteq I_{n+1}$ by math induction for $n\in\mathbb{N}$.
$I_n \supseteq I_{n+1} \iff x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}\ \forall n\in\mathbb{N}$.

For $n=1$, $x_2=\frac12 \le x_4=\frac35\le x_3=\frac23\le x_1=1$.
If $x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}$, then case n is true
$1+x_{2n}\le 1+x_{2n+2}\le 1+x_{2n+1}\le 1+x_{2n-1}$
$\Rightarrow \frac{1}{1+x_{2n}} = x_{2n+1} \ge \frac{1}{1+x_{2n+2}} = x_{2n+3} \ge \frac{1}{1+x_{2n+1}} = x_{2n+2} \ge \frac{1}{1+x_{2n-1}} = x_{2n}$
$\Rightarrow 1+x_{2n+1}\ge 1+x_{2n+3}\ge 1+x_{2n+2}\ge 1+x_{2n}$
$\Rightarrow \frac{1}{1+x_{2n+1}} = x_{2n+2}\le \frac{1}{1+x_{2n+3}} = x_{2n+4} \le \frac{1}{1+x_{2n+2}} = x_{2n+3} \le \frac{1}{1+x_{2n}} = x_{2n+1}$
case $n+1$ is true. So $\bigcap_{n=1}^{\infty} I_n=[a,b]$, where $\lim_{n\to\infty} x_{2n}=a$ and $\lim_{n\to\infty} x_{2n-1}=b$.

(Step 2: Show $\lim_{n\to\infty}|x_{2n}-x_{2n-1}|=0$ and compute limit.)

$$|x_{m+1}-x_m| = \left|\frac{1}{1+x_m}-\frac{1}{1+x_{m-1}}\right| = \frac{|x_m-x_{m-1}|}{(1+x_m)(1+x_{m-1})}$$
$$\le \frac{|x_m-x_{m-1}|}{(1+\frac12)(1+\frac12)} = \frac49|x_m-x_{m-1}|$$

So $|x_{2n}-x_{2n-1}| \le (\frac49)^1|x_{2n-1}-x_{2n-2}| \le (\frac49)^2|x_{2n-2}-x_{2n-3}|$
$\le\cdots\le(\frac49)^{2n-2}|x_2-x_1| = (\frac49)^{2n-2}\frac12 \to 0$ as $n\to\infty$.
By Sandwich theorem, $\lim_{n\to\infty}|x_{2n}-x_{2n-1}|=0$. By the nested interval theorem, $\lim_{n\to\infty} x_{2n}=x=\lim_{n\to\infty} x_{2n-1}$. By the intertwining sequence theorem, $\lim_{n\to\infty} x_n=x$. Taking limit of $x_{n+1}=\frac{1}{1+x_n}$, we get $x=\frac{1}{1+x}\Rightarrow x=\frac{-1\pm\sqrt5}{2}$. Since $\frac{-1-\sqrt5}{2}\notin I_1$, so $x=\frac{-1+\sqrt5}{2}$.

Alternative way to do step 2

From end of step 1, we have $\lim_{n\to\infty} x_{2n} = a$, $\lim_{n\to\infty} x_{2n-1} = b$.

Now $x_{2n} = \frac{1}{1+x_{2n-1}} \Rightarrow a = \frac{1}{1+b}$ by taking limit.

Also $x_{2n+1} = \frac{1}{1+x_{2n}} \Rightarrow b = \frac{1}{1+a}$ by taking limit.

$$\left.\begin{aligned} a &= \frac{1}{1+b} \\ b &= \frac{1}{1+a} \end{aligned}\right\} \Rightarrow a(1+b) = 1 = b(1+a) \Rightarrow a+ab = b+ab$$
$$\Rightarrow a = b$$

Then $\left.\begin{aligned} &\lim_{n\to\infty} x_n = a \text{ and } a = \frac{1}{1+a} \\ &x_n \in I_1 \Rightarrow a \in I_1 \end{aligned}\right\} \Rightarrow a = \frac{-1+\sqrt{5}}{2}$

as $\frac{-1-\sqrt{5}}{2} \notin I_1$.

Question How can we prove a sequence converges without identifying the limit?

In the 19^{th} century, Cauchy introduced the following

Definition $\{x_n\}$ is a Cauchy sequence iff $\forall \varepsilon > 0$ $\exists K \in \mathbb{N}$ such that $n, m \geq K \Rightarrow |x_n - x_m| < \varepsilon$.

Remarks This means the terms are as close as desired when the indices are sufficiently large.

Example Let $x_n = \frac{1}{n^2}$. Show $\{x_n\}$ is Cauchy.

Scratch Work Say $m \geq n$, $|x_n - x_m| = \frac{1}{n^2} - \frac{1}{m^2} < \frac{1}{n^2} < \varepsilon$

$n > \frac{1}{\sqrt{\varepsilon}}$ is enough.

Solution. $\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\sqrt{\varepsilon}}$. Then

$$n, m \geq K \Rightarrow |x_n - x_m| = \left|\frac{1}{n^2} - \frac{1}{m^2}\right| < \frac{1}{K^2} < \varepsilon.$$

Cauchy's Theorem $\{x_n\}$ converges $\Leftrightarrow$ $\{x_n\}$ is Cauchy.

Proof ($\Rightarrow$) Given: $\forall \varepsilon_0 > 0$ $\exists K_0^{\in \mathbb{N}}$, $n \geq K_0 \Rightarrow |x_n - x| < \varepsilon_0$

To prove: $\forall \varepsilon > 0$, $\exists K^{\in \mathbb{N}}$, $m, n \geq K \Rightarrow |x_m - x_n| < \varepsilon$.

Idea: $|x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x - x_n|$
$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\forall \varepsilon > 0$, let $\varepsilon_0 = \varepsilon/2$. We are given that $\exists K_0 \in \mathbb{N}$, $n \geq K_0 \Rightarrow |x_n - x| < \varepsilon/2$. Set $K = K_0$. Then

$$m, n \geq K \Rightarrow |x_m - x_n| \leq |x_m - x| + |x - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

($\Leftarrow$) We are given $\{x_n\}$ is Cauchy. We are to prove $\{x_n\}$ converges to some x. We will do this in 3 steps.

Step 1 $\{x_n\}$ is Cauchy $\Rightarrow$ $\{x_1, x_2, x_3, \dots\}$ is bounded

Step 2 (Bolzano-Weierstrass Theorem)
$\{x_1, x_2, x_3, \dots\}$ is bounded $\Rightarrow$ $\exists$ subsequence $\{x_{n_k}\}$ which converges.

Step 3 $\left.\begin{array}{l}\{x_n\} \text{ is Cauchy} \\ \text{and a subsequence } \{x_{n_k}\} \\ \text{converges to } x\end{array}\right\} \Rightarrow \{x_n\}$ converges to x.

For step 1, we modify the proof of the boundedness theorem.

For $\varepsilon = 1$, $\exists K \in \mathbb{N}$, $n, m \geq K \Rightarrow |x_n - x_m| < \varepsilon = 1$.

So the case $m = K$ means $n \geq K \Rightarrow |x_n - x_K| < 1$

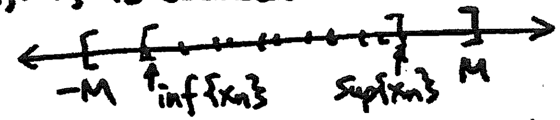
$\Rightarrow |x_n| = |x_n - x_K + x_K| \leq |x_n - x_K| + |x_K| < 1 + |x_K|$.

Let $M = \max\{|x_1|, |x_2|, \dots, |x_{K-1}|, 1 + |x_K|\}$.

Then $\forall n \in \mathbb{N}$, $n = 1, 2, \dots, K-1 \Rightarrow |x_n| \leq M$
$n \geq K \Rightarrow |x_n| < 1 + |x_K| \leq M$.

$\therefore$ $\{x_1, x_2, x_3, \dots\}$ is bounded.

For step 2



Let $a_1 = \inf\{x_n\}$, $b_1 = \sup\{x_n\}$ and $I_1 = [a_1, b_1]$.

Let m_1 be the midpoint of I_1.

If $[a_1, m_1]$ contains infinitely many terms of $\{x_n\}$, then let $a_2 = a_1$, $b_2 = m_1$ and $I_2 = [a_2, b_2]$.
Otherwise, $[m_1, b_1]$ contains infinitely many terms of $\{x_n\}$, then let $a_2 = m_1$, $b_2 = b_1$ and $I_2 = [a_2, b_2]$.
Let m_2 be the midpoint of I_2. Keep repeating, we get $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ and since I_{n+1} is either the left or the right half of I_n, we have
$\lim\limits_{n\to\infty} (\underbrace{b_n - a_n}_{\text{length of } I_n}) = \lim\limits_{n\to\infty} \frac{b_1 - a_1}{2^{n-1}} = 0$. $\therefore \bigcap_{n=1}^{\infty} I_n = \{x\}$.

Take $n_1 = 1$, then $x_{n_1} = x_1 \in I_1$. Since I_2 has infinitely many terms, $\exists\, x_{n_2} \in I_2$ with $n_2 > n_1$. Keep repeating, we get $x_{n_k} \in I_k$ and $n_1 < n_2 < n_3 < \cdots$. So $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.

Now $x_{n_k}, x \in I_k \Rightarrow |x_{n_k} - x| \le \underbrace{b_k - a_k}_{\text{length of } I_k} \xrightarrow{\text{as } k\to\infty} 0$

$\therefore \{x_{n_k}\}$ converges to x.

<u>For step 3</u> $\forall \varepsilon > 0$, $\{x_n\}$ Cauchy $\Rightarrow \exists K_1 \in \mathbb{N}$, $m, n \ge K_1 \Rightarrow |x_n - x_m| < \varepsilon/2$

$\{x_{n_j}\}$ converges to $x \Rightarrow \exists K_2 \in \mathbb{N}$, $j \ge K_2 \Rightarrow |x_{n_j} - x| < \varepsilon/2$.

Let $K = \max(K_1, K_2)$. Then

$$n \ge K \Rightarrow \begin{cases} n_K \ge K \ge K_1 \Rightarrow |x_n - x_{n_K}| < \varepsilon/2 \\ K \ge K_2 \quad \Rightarrow |x_{n_K} - x| < \varepsilon/2 \end{cases}$$
$$\Rightarrow |x_n - x| = |x_n - x_{n_K} + x_{n_K} - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

<u>Example</u> Let $x_1 = \sin 1$ and $x_k = x_{k-1} + \frac{\sin k}{k^2}$. Prove $\{x_n\}$ converges.

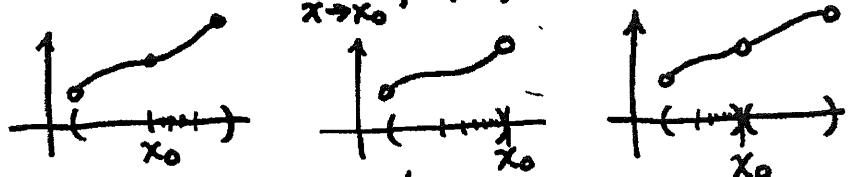
<u>Solution</u> (Scratch Work) Check Cauchy Condition

$$m > n \Rightarrow |x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} \cdots - x_n|$$
$$\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$
$$= \left|\frac{\sin m}{m^2}\right| + \left|\frac{\sin(m-1)}{(m-1)^2}\right| + \cdots + \left|\frac{\sin(n+1)}{(n+1)^2}\right|$$
$$\le \frac{1}{m^2} + \frac{1}{(m-1)^2} + \cdots + \frac{1}{(n+1)^2}$$
$$< \frac{1}{m(m-1)} + \frac{1}{(m-1)(m-2)} + \cdots + \frac{1}{(n+1)n}$$
$$= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots + \left(\frac{1}{m-2} - \frac{1}{m-1}\right) + \left(\frac{1}{m-1} - \frac{1}{m}\right)$$
$$= \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \varepsilon \leftarrow n > \frac{1}{\varepsilon} \text{ is enough}$$

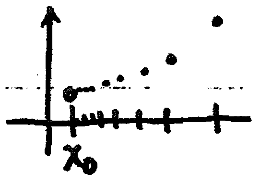
$\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\varepsilon}$. Then $n, m \ge K \Rightarrow |x_m - x_n| < \left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{K} < \varepsilon$.
$\therefore \{x_n\}$ is a Cauchy sequence. $\therefore \{x_n\}$ converges.

Limit of Functions

__Question__ Let S be an interval (more generally a set). Let $f: S \to \mathbb{R}$ be a function. At which number x_0 can we consider $\lim_{x \to x_0} f(x)$?



$x_0 \in S$ $\quad$ $x_0 \notin S$ $\quad$ $x_0 \notin S$



S is a sequence
x_0 is a limit

What do these cases have in common about x_0 and S ?

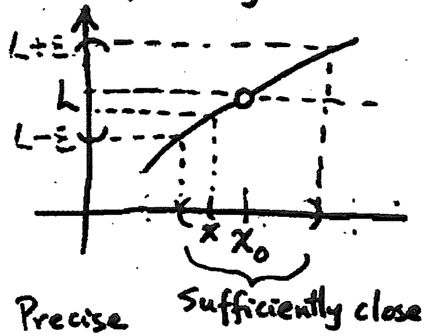
__Definition__ x_0 is an __accumulation point__ (or __limit point__ or __cluster point__) of S iff $\exists\, w_n \in S$, $w_n \neq x_0$ such that $\lim_{n \to \infty} w_n = x_0$.

__Remarks__ Accumulation points __may__ or __may not__ __be__ __in__ __S__.

__Notation__ We write $w_n \to x_0$ in $S \setminus \{x_0\}$ to mean $w_n \in S$, $w_n \neq x_0$ and $\lim_{n \to \infty} w_n = x_0$.

__Convention__ When discussing $\lim_{x \to x_0} f(x)$, we will assume x_0 is an accumulation point of the domain of f.

Let $f: S \to \mathbb{R}$, $\lim_{x \to x_0} f(x) = L$ __roughly__ means for any desired distance $\varepsilon > 0$, when $x \in S$, $x \neq x_0$ is sufficiently close to x_0, we can obtain



$$\underbrace{d(f(x), L)}_{\text{distance between } f(x) \text{ and } L} < \varepsilon .$$

Precise __Definition__ $\boxed{\lim_{x \to x_0} f(x) = L}$ iff $\forall \varepsilon > 0$ $\exists \delta > 0$

$$\forall x \in S, \quad \left.\begin{matrix} x \in (x_0 - \delta, x_0 + \delta) \\ x \neq x_0 \end{matrix}\right\} \Rightarrow |f(x) - L| < \varepsilon .$$

Equivalently, $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall x \in S$,

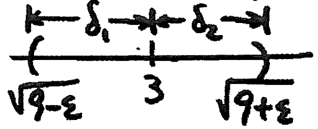
$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon .$$

This is easier to do computations.

__Examples__ ① Let $f(x) = \dfrac{x^3 - 3x^2}{x - 3} = x^2 \dfrac{(x-3)}{x-3}$. __Check__ $\lim_{x \to 3} f(x) = 9$.

__Scratch Work__ $x \neq 3 \Rightarrow f(x) = x^2$ $\quad |f(x) - 9| = |x^2 - 9| < \varepsilon$

$\Leftrightarrow x^2 \in (9 - \varepsilon, 9 + \varepsilon) \Leftrightarrow x \in (\sqrt{9-\varepsilon}, \sqrt{9+\varepsilon})$ for $\varepsilon < 9$



Let $\delta = \min(\delta_1, \delta_2) = \min(3 - \sqrt{9-\varepsilon}, \sqrt{9+\varepsilon} - 3)$

Then $0 < |x - 3| < \delta \Rightarrow x \in (\sqrt{9-\varepsilon}, \sqrt{9+\varepsilon})$

$\Rightarrow |f(x) - 9| = |x^2 - 9| < \varepsilon$.

② Let $g:[0,\infty)\to\mathbb{R}$ be defined by $g(x)=\sqrt{x}$.

<u>Check</u>: $\lim_{x\to 0} g(x)=0$ and $\lim_{x\to 4} g(x)=2$.

<u>Solution</u>. $\boxed{\text{(Scratch Work) } |g(x)-0|=\sqrt{x}<\varepsilon \quad \underset{|x-0|}{\overset{\|}{x}}<\varepsilon^2 \text{ is enough}}$

$\forall \varepsilon>0$, let $\delta=\varepsilon^2$, then $\forall x\in[0,\infty)$, $0<|x-0|=x<\delta=\varepsilon^2$

$\Rightarrow |g(x)-0|=\sqrt{x}<\varepsilon$.

(Scratch Work) $|g(x)-2|=|\sqrt{x}-2|=\dfrac{|x-4|}{\sqrt{x}+2}\le\dfrac{|x-4|}{2}<\varepsilon$

$|x-4|<2\varepsilon$ is enough.

$\forall \varepsilon>0$, let $\delta=2\varepsilon$, then $\forall x\in[0,\infty)$,

$0<|x-4|<\delta=2\varepsilon \Rightarrow |g(x)-2|\le\dfrac{|x-4|}{2}<\varepsilon$.

③ Let $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be defined by $f(x)=\dfrac{1}{5x}$

<u>Check</u>: $\lim_{x\to 2} f(x)=\dfrac{1}{10}$.

<u>Solution</u> (Scratch Work) x is close to 2

$\left|f(x)-\frac{1}{10}\right|=\left|\frac{1}{5x}-\frac{1}{10}\right|=\dfrac{|x-2|}{10x}\le\dfrac{|x-2|}{10}<\varepsilon$

<u>min trick</u>

$\left.\begin{array}{l}|x-2|<10\varepsilon\\ \text{and } x\ge 1\end{array}\right\}$ are enough. if $x\ge 1 \leftarrow \delta<1$

$\overset{\leftarrow 1\rightarrow}{1 \quad 2}$

$\forall \varepsilon>0$, let $\delta=\min(1,10\varepsilon)$, then $\forall x\in\mathbb{R}\setminus\{0\}$,

$0<|x-2|<\delta \Rightarrow \begin{cases}|x-2|<1\\ \text{and}\\ |x-2|<10\varepsilon\end{cases} \Rightarrow \begin{cases}x\in(1,3) \leftarrow x\ge 1\\ \text{and}\\ |x-2|<10\varepsilon\end{cases}$

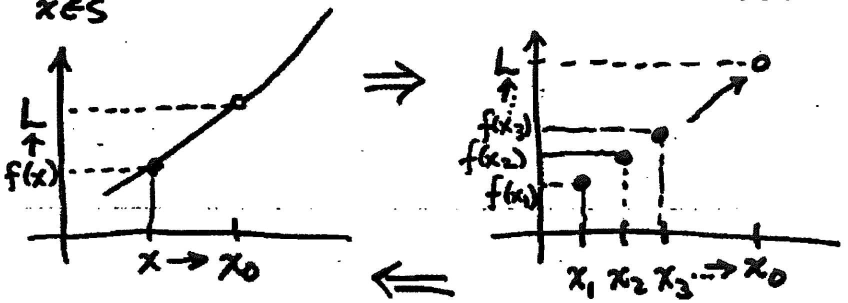
$\Rightarrow \left|f(x)-\frac{1}{10}\right|=\dfrac{|x-2|}{10x}\le\dfrac{|x-2|}{10}<\varepsilon$.

Recall "$x_n \to x_0$ in $S\setminus\{x_0\}$" means $\forall n\in\mathbb{N},\ x_n\in S$, $x_n\neq x_0$, $\lim\limits_{n\to\infty} x_n = x_0$

Sequential Limit Theorem (S.L.T.)

Let $f: S\to\mathbb{R}$ be a function and x_0 be an accumulation point of S. Then

$$\lim_{\substack{x\to x_0\\ x\in S}} f(x) = L \iff \forall x_n\to x_0 \text{ in } S\setminus\{x_0\},\ \lim_{n\to\infty} f(x_n) = L.$$



$\forall x_n\to x_0$ in $S\setminus\{x_0\}$

Proof ($\Rightarrow$)

Given: $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x) = L$ $\left(\forall\varepsilon>0\ \exists\delta>0,\ \forall x\in S\ \ 0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon\right)$

$x_n\to x_0$ in $S\setminus\{x_0\}$ $\left(\forall\delta>0\ \exists K\in\mathbb{N}\ \ n\geq K\Rightarrow |x_n-x_0|<\delta\right)$

So $\forall\varepsilon>0\ \exists K\in\mathbb{N}$

$$n\geq K \Rightarrow \underset{\text{as } x_n\neq x_0}{0<}|x_n-x_0|<\delta \Rightarrow |f(x_n)-L|<\varepsilon$$

$\therefore \lim\limits_{n\to\infty} f(x_n) = L$

($\Leftarrow$) Assume $\lim\limits_{x\to x_0} f(x)\neq L$.

$\sim(\forall\varepsilon>0\ \exists\delta>0,\ \forall x\in S,\ 0<|x-x_0|<\delta\Rightarrow|f(x)-L|<\varepsilon)$

$=\exists\varepsilon>0\ \forall\delta>0\ \exists x\in S\ \ 0<|x-x_0|<\delta$ and $|f(x)-L|\geq\varepsilon$.

for $\delta=1$ $\exists x_1\in S$ $0<|x_1-x_0|<1$ and $|f(x_1)-L|\geq\varepsilon$

for $\delta=\frac12$ $\exists x_2\in S$ $0<|x_2-x_0|<\frac12$ and $|f(x_2)-L|\geq\varepsilon$

$\vdots$

for $\delta=\frac1n$ $\exists x_n\in S$ $0<|x_n-x_0|<\frac1n$ and $|f(x_n)-L|\geq\varepsilon$

$\vdots$

$\therefore x_n\in S,\ 0<|x_n-x_0|<\frac1n \Rightarrow x_n\neq x_0$ and $\lim\limits_{n\to\infty}x_n=x_0$

$\therefore x_n\to x_0$ in $S\setminus\{x_0\}$. Then $\lim\limits_{n\to\infty}f(x_n)=L$.

$|f(x_n)-L|\geq\varepsilon \Rightarrow 0=|L-L|=\lim\limits_{n\to\infty}|f(x_n)-L|\geq\varepsilon$

Contradicting $\varepsilon>0$.

Application ① $\lim\limits_{x\to0}(1+x)^{\frac1x}=e \overset{\text{SLT}}{\underset{x_n=\frac1n\to0}{\Longrightarrow}} \lim\limits_{n\to\infty}\left(1+\frac1n\right)^n=e$

$\Leftarrow$ need not just $x_n=\frac1n$, need all $\underset{x_n\neq0}{x_n\to0}$ to have $\lim\limits_{n\to\infty}(1+x_n)^{1/x_n}=e$

② If $\underbrace{\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x)=L_1}_{(*)}$, $\underbrace{\lim\limits_{\substack{x\to x_0\\ x\in S}} g(x)=L_2}_{(**)}$, then prove $\lim\limits_{\substack{x\to x_0\\ x\in S}}(f(x)+g(x))=L_1+L_2$

Solution 1 $\forall x_n\to x_0$ in $S\setminus\{x_0\}$,

by S.L.T., $(*)\Rightarrow \lim\limits_{n\to\infty}f(x_n)=L_1$

$(**)\Rightarrow\lim\limits_{n\to\infty}g(x_n)=L_2$.

By computation formula, $\lim\limits_{n\to\infty}(f(x_n)+g(x_n))=L_1+L_2$

By S.L.T., $\lim\limits_{\substack{x\to x_0\\ x\in S}}(f(x)+g(x))=L_1+L_2$.

Solution 2

$$\lim_{\substack{x\to x_0\\ x\in S}} f(x)=L_1 \quad (\forall \varepsilon_1>0\ \exists \delta_1>0\ \forall x\in S\ \ 0<|x-x_0|<\delta_1 \Rightarrow |f(x)-L_1|<\varepsilon_1)$$

$$\lim_{\substack{x\to x_0\\ x\in S}} g(x)=L_2 \quad (\forall \varepsilon_2>0\ \exists \delta_2>0\ \forall x\in S\ \ 0<|x-x_0|<\delta_2 \Rightarrow |g(x)-L_2|<\varepsilon_2)$$

$\forall \varepsilon>0$, let $\varepsilon_1=\frac{\varepsilon}{2}$ and $\varepsilon_2=\frac{\varepsilon}{2}$. From above, get $\delta_1, \delta_2>0$.

Set $\delta=\min(\delta_1,\delta_2)$. Then $\forall x\in S$,

$$0<|x-x_0|<\delta \Rightarrow \begin{array}{l} 0<|x-x_0|<\delta_1 \Rightarrow |f(x)-L_1|<\frac{\varepsilon}{2} \\ 0<|x-x_0|<\delta_2 \Rightarrow |g(x)-L_2|<\frac{\varepsilon}{2} \end{array}$$

$$\Rightarrow |(f(x)+g(x))-(L_1+L_2)| = |f(x)-L_1+g(x)-L_2|$$
$$\le |f(x)-L_1|+|g(x)-L_2| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

$$\therefore \lim_{\substack{x\to x_0\\ x\in S}} (f(x)+g(x))=L_1+L_2$$

Similarly

If $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x)=L_1$, $\lim\limits_{\substack{x\to x_0\\ x\in S}} g(x)=L_2$, then

$\lim (f(x)-g(x))=L_1-L_2$

$\lim f(x)g(x)=L_1L_2$

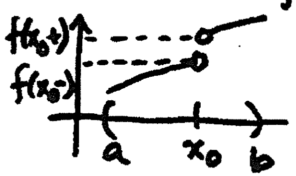
$\lim f(x)/g(x)=L_1/L_2$ (provided $g(x)\neq 0$, $L_2\neq 0$)

If $f(x)\le g(x)\le h(x)$ for all $x\in S$, $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x)=L=\lim\limits_{\substack{x\to x_0\\ x\in S}} h(x)$, then $\lim\limits_{\substack{x\to x_0\\ x\in S}} g(x)=L$.

If $f(x)\ge 0$ for all $x\in S$ and $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x)=L$, then $L\ge 0$.

One-sided Limits

Definitions For $f:(a,b)\to \mathbb{R}$ and $x_0\in(a,b)$,



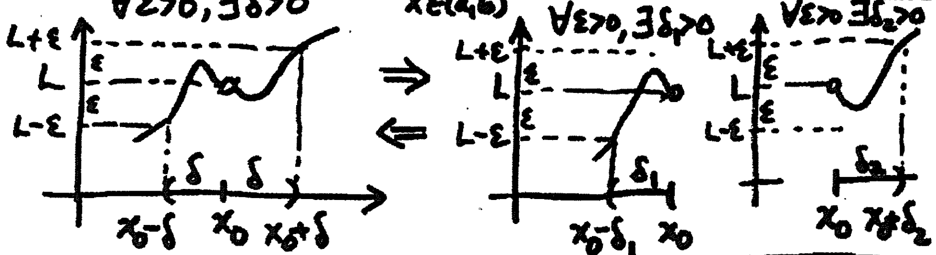
left hand limit of f at x_0

$$f(x_0-)=\lim_{x\to x_0^-} f(x)=\lim_{\substack{x\to x_0\\ x\in(a,x_0)}} f(x)$$

right hand limit of f at x_0

$$f(x_0+)=\lim_{x\to x_0^+} f(x)=\lim_{\substack{x\to x_0\\ x\in(x_0,b)}} f(x).$$

Theorem For $x_0\in(a,b)$, $\lim\limits_{\substack{x\to x_0\\ x\in(a,b)}} f(x)=L \Leftrightarrow f(x_0-)=L=f(x_0+)$



Proof

$\lim\limits_{\substack{x\to x_0\\ x\in(a,b)}} f(x)=L$ "$\forall\varepsilon>0\ \exists\delta>0\ \forall x\in(a,b),\ 0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon$"

$\Longrightarrow$ let $\delta_1=\delta$, $\delta_2=\delta$

$f(x_0-)=L$ $\left(=\lim\limits_{\substack{x\to x_0\\ x\in(a,x_0)}} f(x)\right)$ "$\forall\varepsilon>0\ \exists\delta_1>0\ \forall x\in(a,x_0)\ 0<|x-x_0|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon$"

$f(x_0+)=L$ $\left(=\lim\limits_{\substack{x\to x_0\\ x\in(x_0,b)}} f(x)\right)$ "$\forall\varepsilon>0\ \exists\delta_2>0\ \forall x\in(x_0,b)\ 0<|x-x_0|<\delta_2 \Rightarrow |f(x)-L|<\varepsilon$"

$\Longleftarrow$ let $\delta=\min(\delta_1,\delta_2)$

Definitions Let $f: S \to \mathbb{R}$ be a function.

① f is <u>increasing</u> on S iff $\forall x, y \in S$, $x < y \Rightarrow f(x) \le f(y)$

② f is <u>decreasing</u> on S iff $\forall x, y \in S$, $x < y \Rightarrow f(x) \ge f(y)$

③ f is <u>strictly increasing</u> on S iff $\forall x, y \in S$, $x < y \Rightarrow f(x) < f(y)$

④ f is <u>strictly decreasing</u> on S iff $\forall x, y \in S$, $x < y \Rightarrow f(x) > f(y)$.

⑤ f is <u>monotone</u> on S iff f is increasing or decreasing on S.

⑥ f is <u>strictly monotone</u> on S iff f is strictly increasing or strictly decreasing on S.

⑦ f is <u>bounded above</u> on S iff $\{f(x): x \in S\}$ is bounded above.

⑧ f is <u>bounded below</u> on S iff $\{f(x): x \in S\}$ is bounded below.

⑨ f is <u>bounded</u> on S iff f is bounded above and below.

Monotone Function theorem

(I) If f is increasing on (a,b), then $\forall x_0 \in (a,b)$,

$$\left.\begin{array}{l} f(x_0-) = \sup\{f(x): a < x < x_0\} \\ \text{and } f(x_0+) = \inf\{f(x): x_0 < x < b\} \end{array}\right\} \Rightarrow f(x_0-) \le f(x_0) \le f(x_0+)$$

If f is bounded below, then $f(a+) = \inf\{f(x): a < x < b\}$.

If f is bounded above, then $f(b-) = \sup\{f(x): a < x < b\}$.

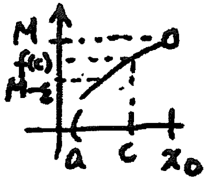
(II) f has countably many discontinuous points on (a,b)

$J = \{x_0 : x_0 \in (a,b), f(x_0-) \ne f(x_0+)\}$ is countable.

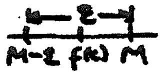
Remarks Similarly, the theorem is true for decreasing functions and all other kinds of intervals.

Proof. (I) If $a < x < x_0 < b$, then $f(x) \le f(x_0)$ as f is increasing. So $\{f(x): a < x < x_0\}$ is bounded above by $f(x_0)$. Hence, $M = \sup\{f(x): a < x < x_0\}$ exists by completeness axiom.

To show $f(x_0-) = M$, $\underline{\forall \varepsilon > 0}$, by the supremum property, $\exists c \in (a, x_0)$ such that $M - \varepsilon < f(c) \le M$. Let $\underline{\delta = x_0 - c}$.



Then $\underline{\forall x \in (a, x_0)}$,



$0 < |x - x_0| < \delta \Rightarrow x \in (x_0 - \delta, x_0) = (c, x_0)$
$\Rightarrow c < x < x_0 \Rightarrow f(c) \le f(x) \le M$
$\Rightarrow |f(x) - M| = M - f(x) \le M - f(c) < \varepsilon$

$\therefore f(x_0-) = \lim\limits_{\substack{x \to x_0 \\ x \in (a, x_0)}} f(x) = M = \sup\{f(x): a < x < x_0\} \le f(x_0)$.

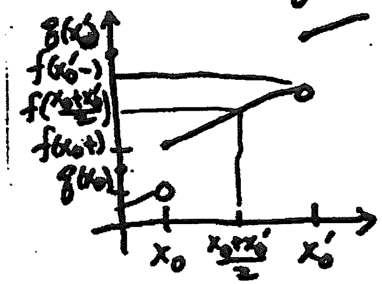
The other parts of (I) are similarly proved.

(II) f is discontinuous at $x_0 \in (a,b) \Leftrightarrow f(x_0-) < f(x_0+)$.

$\Rightarrow \exists\, g(x_0) \in \mathbb{Q}$ such that $f(x_0-) < g(x_0) < f(x_0+)$ by density of $\mathbb{Q}$.

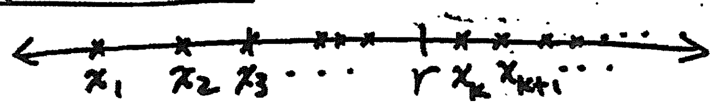
The function $g: J = \{x_0 : f(x_0-) < f(x_0+)\} \to \mathbb{Q}$ is injective because $\forall x_0, x_0' \in J$,

$x_0 < x_0' \Rightarrow g(x_0) < f(x_0+) \le f\left(\frac{x_0 + x_0'}{2}\right) \le f(x_0'-) < g(x_0')$

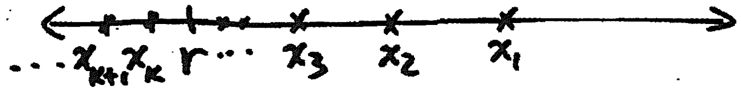


By injection theorem, J is countable.

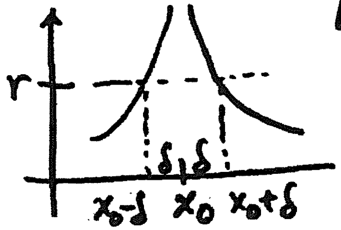
Infinite Limits



Definitions $\{x_n\}$ diverges to $+\infty$ (or $\lim\limits_{n\to\infty} x_n = +\infty$) iff $\forall r\in\mathbb{R}, \exists K\in\mathbb{N}$ such that $n\geq K \Rightarrow x_n > r$.



$\{x_n\}$ diverges to $-\infty$ (or $\lim\limits_{n\to\infty} x_n = -\infty$) iff $\lim\limits_{n\to\infty} -x_n = +\infty$, $\forall r\in\mathbb{R}$, $\exists K\in\mathbb{N}$ such that $n\geq K \Rightarrow x_n < r$.



Let $f: S\to\mathbb{R}$ and x_0 be an accumulation point of S.

f diverges to $+\infty$ as x tends to x_0 (or $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x) = +\infty$) iff

$\forall r\in\mathbb{R}$ $\exists \delta>0$ such that $\forall x\in S$

$$\underbrace{x\neq x_0 \text{ and } x\in(x_0-\delta, x_0+\delta)}_{0<|x-x_0|<\delta} \Rightarrow f(x)>r.$$

f diverges to $-\infty$ as x tends to x_0 (or $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x) = -\infty$) iff $\lim\limits_{\substack{x\to x_0\\ x\in S}} -f(x) = +\infty$,

$\forall r\in\mathbb{R}$ $\exists\delta>0$ such that $\forall x\in S$

$$0<|x-x_0|<\delta \Rightarrow f(x)<r.$$

Limit at Infinity

Recall $\lim\limits_{n\to\infty} a_n = L$ iff $\forall \varepsilon>0$ $\exists K\in\mathbb{N}$ $n\geq K\Rightarrow |x_n - L|<\varepsilon$.

Let $f: S\to\mathbb{R}$ be a function and $+\infty, -\infty$ are accumulation points of S (that is, $\exists$ sequences in S with $+\infty, -\infty$ as limits). $L\in\mathbb{R}$.

Definitions $\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(x) = L$ iff $\forall\varepsilon>0$ $\exists K\in\mathbb{R}$ $x\geq K\Rightarrow |f(x)-L|<\varepsilon$.

$\lim\limits_{\substack{x\to-\infty\\ x\in S}} f(x) = L$ iff $\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(-x) = L$ iff $\forall\varepsilon>0$ $\exists K\in\mathbb{R}$ $x\leq K\Rightarrow|f(x)-L|<\varepsilon$

Recall $\lim\limits_{n\to\infty} a_n = +\infty$ iff $\forall r\in\mathbb{R}, \exists K\in\mathbb{N}$ $n\geq K\Rightarrow a_n>r$.

Definitions $\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(x) = +\infty$ iff $\forall r\in\mathbb{R}, \exists K\in\mathbb{R}$ $x\geq K\Rightarrow f(x)>r$.

$\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(x) = -\infty$ iff $\lim\limits_{\substack{x\to+\infty\\ x\in S}} -f(x) = +\infty$ iff $\forall r\in\mathbb{R}, \exists K\in\mathbb{R}$ $x\geq K\Rightarrow f(x)<r$.

$\lim\limits_{\substack{x\to-\infty\\ x\in S}} f(x) = +\infty$ iff $\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(-x) = +\infty$ iff $\forall r\in\mathbb{R}, \exists K\in\mathbb{R}$ $x\leq K\Rightarrow f(x)>r$.

$\lim\limits_{\substack{x\to-\infty\\ x\in S}} f(x) = -\infty$ iff $\lim\limits_{\substack{x\to+\infty\\ x\in S}} -f(-x) = +\infty$ iff $\forall r\in\mathbb{R}, \exists K\in\mathbb{R}$ $x\leq K\Rightarrow f(x)<r$.

Chapter 7 Continuity

Definition A function $f: S \to \mathbb{R}$ is continuous at $x_0 \in S$ iff $\lim_{\substack{x\to x_0\\ x\in S}} f(x) = f(x_0)$ (more precisely, $\forall \varepsilon>0\ \exists \delta>0$ such that $\forall x\in S$, $|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon$)

For $E \subseteq S$, we say f is continuous on E iff f is continuous at every element of E. Also, we say f is continuous iff f is continuous on the domain S.

Sequential Continuity Theorem (S.C.T.)

$f: S\to\mathbb{R}$ is continuous at $x_0\in S$ $\Leftrightarrow$ $\forall x_n \to x_0$ in S $\lim_{n\to\infty} f(x_n) = f(x_0) = f(\lim_{n\to\infty} x_n)$

Drop $x_n \neq x_0$ requirement.

Proof. Just modify the proof of the sequential limit theorem by replacing ① L by $f(x_0)$

② $0<|x-x_0|<\delta$ by $|x-x_0|<\delta$

③ $x_n\to x_0$ in $S\setminus\{x_0\}$ by $x_n \to x_0$ in S (delete $x_n\neq x_0$ requirement)

Examples ① Since $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$, so $f:\mathbb{R}\to\mathbb{R}$ defined by $f(\theta)=\begin{cases}\frac{\sin\theta}{\theta} & \text{if } \theta\neq 0\\ 1 & \text{if } \theta = 0\end{cases}$ is continuous at $x_0=0$.

Also $\lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{n\to\infty}\frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{n\to\infty} f(\tfrac{1}{n}) \overset{\text{by S.C.T.}}{=} f(0)=1$

② $\exists f:\mathbb{R}\to\mathbb{R}$ discontinuous (not continuous) at every $x\in\mathbb{R}$. Let $f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}.\end{cases}$

$\forall x_0\in\mathbb{R}$, $n\in\mathbb{N}$, by density of $\mathbb{Q}$ and density of $\mathbb{R}\setminus\mathbb{Q}$, $\exists r_n\in\mathbb{Q}$, $s_n\in\mathbb{R}\setminus\mathbb{Q}$, both $r_n, s_n \in (x_0, x_0+\frac{1}{n})$

So $\lim_{n\to\infty} r_n = x_0 = \lim_{n\to\infty} s_n$ by Sandwich theorem.

Then $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 1 = 1$, $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} 0 = 0$, so $\lim_{x\to x_0} f(x)$ cannot exist by S.C.T. $\therefore f$ is discontinuous at x_0.

Theorem If $f, g: S\to\mathbb{R}$ are continuous at $x_0\in S$, then $f\pm g$, fg, f/g (provided $g(x_0)\neq 0$) are continuous at x_0.

Proof f, g continuous at $x_0\in S$ $\Leftrightarrow$ $\lim_{x\to x_0} f(x) = f(x_0)$, $\lim_{x\to x_0} g(x) = g(x_0)$

$\Rightarrow$ $\lim_{x\to x_0}(f\pm g)(x) = (f\pm g)(x_0)$

$\lim_{x\to x_0}(fg)(x) = (fg)(x_0)$

$\lim_{x\to x_0}(\frac{f}{g})(x) = (\frac{f}{g})(x_0)$

See applications of S.C.T.

by definition of Continuity at x_0

$\Leftrightarrow$ $f\pm g$, fg, $\frac{f}{g}$ is continuous at x_0

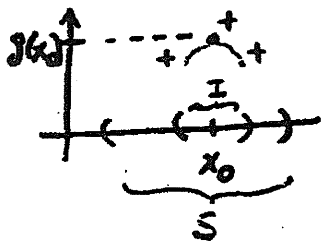
Theorem If $f: S \to \mathbb{R}$ is continuous at x_0, $f(S) \subseteq S'$ and $g: S' \to \mathbb{R}$ is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Proof. By S.C.T., we need to show $\forall x_n \to x_0$ in S, $\lim_{n\to\infty} (g\circ f)(x_n) = (g \circ f)(x_0)$. By S.C.T., since f is continuous at x_0, $\lim_{n\to\infty} f(x_n) = f(x_0)$. So $f(x_n) \to f(x_0)$ in S'. Since g is continuous at $f(x_0)$, by S.C.T.,

$$\lim_{n\to\infty} (g\circ f)(x_n) = \lim_{n\to\infty} g(f(x_n)) = g(f(x_0)) = (g\circ f)(x_0).$$

Below, S will denote an interval of positive length.

Sign Preserving Property



If $g: S \to \mathbb{R}$ is continuous and $g(x_0) > 0$, then $\exists$ an interval $I = (x_0 - \delta, x_0 + \delta)$ with $\delta > 0$ such that

$g(x) > 0$ for all $x \in S \cap I$.

(Similarly for the case $g(x_0) < 0$.)

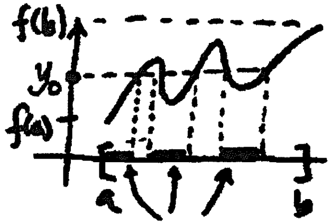
Proof Let $\varepsilon = g(x_0) > 0$. Note $(g(x_0) - \varepsilon, g(x_0) + \varepsilon) = (0, 2g(x_0))$. Since g is continuous at x_0, $\exists \delta > 0$ such that

$$\underbrace{\forall x \in S, \ |x - x_0| < \delta}_{x \in S \cap \underbrace{(x_0-\delta, x_0+\delta)}_{=I}} \Rightarrow \underbrace{|g(x) - g(x_0)| < \varepsilon}_{g(x) \in (\underbrace{g(x_0)-\varepsilon}_{=0},\, g(x_0)+\varepsilon)}$$

$$\Rightarrow g(x) > 0$$

Intermediate Value Theorem

If $f: [a,b] \to \mathbb{R}$ is continuous and y_0 is between $f(a)$ and $f(b)$, then $\exists x_0 \in [a,b]$ such that $f(x_0) = y_0$.



$S = \{x \in [a,b] : f(x) \le y_0\}$

Proof Case 1: $y_0 = f(a)$. Take $x_0 = a$.

Case 2: $y_0 = f(b)$. Take $x_0 = b$.

Case 3: $f(a) < y_0 < f(b)$.

(Case $f(a) > y_0 > f(b)$ is similar.) Let $S = \{x \in [a,b] : f(x) \le y_0\}$.

$S \neq \emptyset$ as $a \in S$. S is bounded above by b. By completeness axiom, $x_0 = \sup S$ exists.

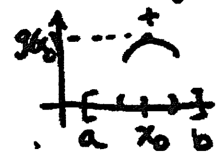
By supremum limit theorem, $\exists x_n \in S$ such that $\lim_{n\to\infty} x_n = x_0$.

$x_n \in [a,b] \Rightarrow a \le x_n \le b \Rightarrow a \le x_0 \le b \Rightarrow x_0 \in [a,b]$.

By S.C.T., $f(x_0) = \lim_{n\to\infty} f(x_n) \le y_0$.

Assume $f(x_0) < y_0$. Then $x_0 \neq b$ since $y_0 < f(b)$.

Define $g(x) = y_0 - f(x)$ on $[a,b]$. Then $g(x_0) = y_0 - f(x_0) > 0$.



By sign preserving property, there is interval $I = (x_0 - \delta, x_0 + \delta)$ such that $\forall x \in [a,b] \cap I$, $g(x) > 0$.

Now $x_0 < b \Rightarrow \exists x_1 \in (x_0, b] \cap (x_0, x_0 + \delta) \subseteq [a,b] \cap I$

$x_0 < x_0 + \delta$

$\Rightarrow g(x_1) = y_0 - f(x_1) > 0$

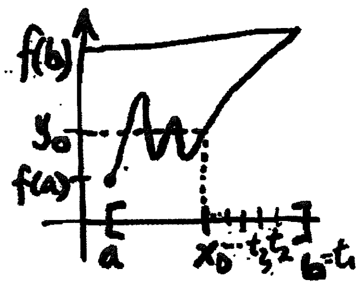
$\Rightarrow f(x_1) < y_0$, but $x_1 > x_0$, $x_1 \in [a,b]$

$\leftarrow$ contradict $x_0 = \sup S$.

$\therefore f(x_0) = y_0$.

Alternative Ending (Avoiding sign preserving property)

As in the previous proof, we get $f(x_0) \leq y_0$.



Since $y_0 < f(b)$, we get $x_0 \neq b$.

Let $t_n = x_0 + \frac{1}{n}(b - x_0) \in (x_0, b]$.

$\lim_{n\to\infty} t_n = x_0$ and $t_n > x_0 = \sup S$

So $t_n \notin S \Rightarrow f(t_n) > y_0$.

By S.C.T., $f(x_0) = \lim_{n\to\infty} f(t_n) \geq y_0$. $\therefore f(x_0) = y_0$.

Exercise

Let $f: [0,1] \to [0,1]$ be an increasing function (perhaps discontinuous). Suppose $0 < f(0)$ and $f(1) < 1$. Show that f has at least one fixed point. (A fixed point of f is an element r in the domain of f such that $f(r) = r$.)

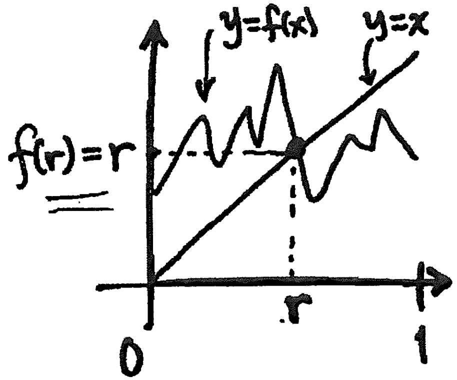
Hint: Sketch the graph of f and consider

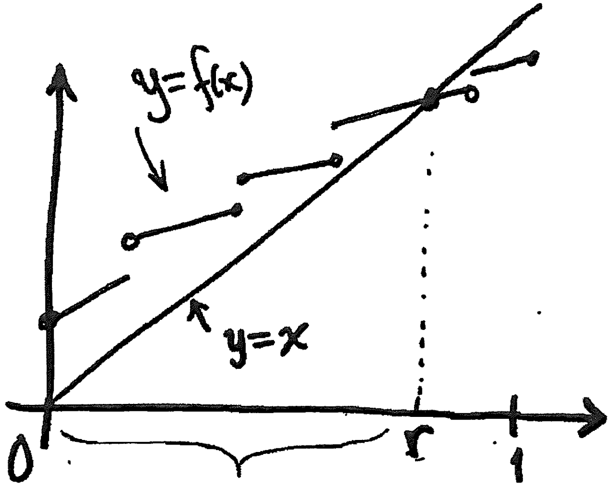
$$S = \{t \in [0,1] : t \leq f(t)\}.$$

Does it have a supremum?

Use monotone function theorem and S.L.T.

Fixed Point





$S = \{t \in [0,1] : t \leq f(t)\}$

Examples ① The equation $x^5+3x+\sin x=\cos x+10$ has a solution.

Let $f(x)=x^5+3x+\sin x-\cos x-10$. Then f is continuous.

$f(0)=-11$ and $26=2^5+3\cdot2-1-1-10\le f(2)$

So 0 is between $f(0)$ and $f(2)$. By intermediate value theorem, $\exists x_0\in[0,2]$ such that $f(x_0)=0$. Then $x_0^5+3x_0+\sin x_0=\cos x_0+10$. ← x_0 is a solution of equation.

② Every odd degree polynomial with real coefficients has at least one real root.

Let $P(x)=x^n+a_1x^{n-1}+\cdots+a_n$ with n odd.

Let $x_0=1+|a_1|+\cdots+|a_n|\ge1$. Then

$P(x_0)=x_0^n+a_1x_0^{n-1}+\cdots+a_n\Rightarrow x_0^n-p(x_0)=-a_1x_0^{n-1}-\cdots-a_n$

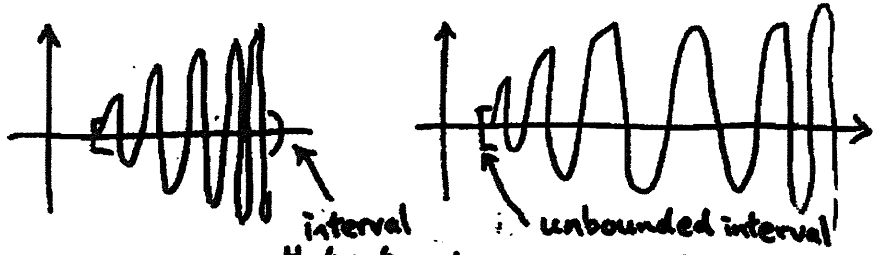
$P(-x_0)=-x_0^n+a_1x_0^{n-1}-\cdots+a_n\Rightarrow x_0^n+p(-x_0)=a_1x_0^{n-1}-\cdots+a_n$

$$\Rightarrow \left.\begin{matrix}x_0^n-p(x_0)\\ x_0^n+p(-x_0)\end{matrix}\right\}\le|a_1x_0|^{n-1}+\cdots+|a_n| \le|a_1|x_0^{n-1}+\cdots+|a_n|x_0^{n-1} =(\underbrace{|a_1|+\cdots+|a_n|}_{=x_0-1})x_0^{n-1} < x_0^n$$

$\Rightarrow p(x_0)>0$ and $p(-x_0)<0$

$\Rightarrow 0$ is between $p(-x_0)$ and $p(x_0)$

$\Rightarrow P$ has a real root between $-x_0$ and x_0.



Examples of continuous function with no maximum nor minimum values.

Extreme Value Theorem Let $a,b\in\mathbb{R}$ with $a\le b$. If $f:[a,b]\to\mathbb{R}$ is continuous, then $\exists x_0,w_0\in[a,b]$ such that $f(w_0)\le f(x)\le f(x_0)$ $\forall x\in[a,b]$.

So range of $f=\{f(x):x\in[a,b]\}=f([a,b])$ is the interval $[f(w_0),f(x_0)]$. In particular, f is bounded on $[a,b]$. $f(x_0)=\sup\{f(x):x\in[a,b]\}=\max_{x\in[a,b]}f(x)$ and $f(w_0)=\inf\{f(x):x\in[a,b]\}=\min_{x\in[a,b]}f(x)$.

Proof. Assume $f([a,b])$ is not bounded above. Then every $n\in\mathbb{N}$ is not an upper bound. So $\exists z_n\in[a,b]$ with $f(z_n)>n$. By Bolzano-Weierstrass theorem, $\{z_n\}$ has a subsequence $\{z_{n_j}\}$ converging to some $z_0\in[a,b]$. Since f is continuous at z_0, $\lim_{n\to\infty}f(z_{n_j})=f(z_0)$ by SCT. By boundedness theorem, $\{f(z_{n_j})\}$ is bounded. However, $f(z_{n_j})>n_j\ge j\Rightarrow\{f(z_{n_j})\}$ is unbounded, a contradiction.

$\therefore f([a,b])$ is bounded above and $M=\sup f([a,b])$ exists.

By supremum limit theorem, $\exists\, x_n \in [a,b]$ such that $M = \lim_{n\to\infty} f(x_n)$. By Bolzano-Weierstrass theorem, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ converging to some $x_0 \in [a,b]$. By S.C.T., $f(x_0) = f(\lim_{i\to\infty} x_{n_i}) = \lim_{i\to\infty} f(x_{n_i}) = M$.

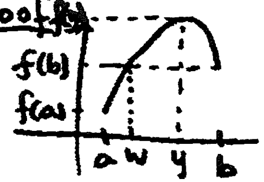
Similarly, $\exists\, w_0 \in [a,b]$ with $f(w_0) = \inf f([a,b])$.

Application. Let $f:[a,b]$ be continuous. Then $-\infty < \int_a^b f(x)\,dx < +\infty$ because $\exists\, x_0, w_0 \in [a,b]$ such that $f(w_0) \le f(x) \le f(x_0) \Rightarrow \int_a^b f(w_0)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(x_0)\,dx \Rightarrow -\infty < f(w_0)(b-a) \le \int_a^b f(x)\,dx < f(x_0)(b-a) < +\infty$.

Continuous Injection Theorem

If f is continuous and injective on $[a,b]$, then f is strictly monotone on $[a,b]$ and $f([a,b]) = [f(a), f(b)]$ or $[f(b), f(a)]$. (the theorem is true for any other nonempty interval.)

Proof.



Since f is injective, either $f(a) < f(b)$ or $f(a) > f(b)$. Suppose $f(a) < f(b)$. $\forall y \in (a,b)$, $f(y) > f(b)$ is false for otherwise, by intermediate value theorem, $\exists\, w \in (a,y)$ with $f(w) = f(b)$, contradicting injectivity of f. Similarly, $f(y) < f(a)$ is false. So $a<y<b \Rightarrow f(a) < f(y) < f(b)$. Then similarly, $a \le x < y \le b \Rightarrow f(a) \le f(x) < f(y) \le f(b)$. $\therefore$ f is strictly increasing on $[a,b]$ and $f([a,b]) = [f(a), f(b)]$. The case $f(a) > f(b)$ is similar.

Application The continuous injection theorem will be used to prove the following theorem, which will be used to prove the $\frac{dx}{dy} = 1\Big/\frac{dy}{dx}$ rule for differentiation.

Continuous Inverse Theorem

If f is continuous and injective on $[a,b]$, then

$$f^{-1}: f([a,b]) \to [a,b] \text{ is continuous and surjective.}$$

(The theorem is true for any other nonempty interval.)

Proof. f^{-1} is surjective because $c \in [a,b] \Rightarrow f(c) \in f([a,b])$ and $f^{-1}(f(c)) = c$.

By continuous injection theorem, f is strictly monotone, say strictly increasing. Then f^{-1} is also strictly increasing.

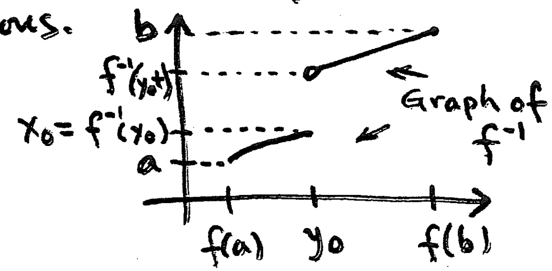
Assume f^{-1} is discontinuous at some $y_0 = f(x_0) \in f([a,b])$.

Then either $a \le f^{-1}(y_0-) < f^{-1}(y_0) = x_0 \le b$

or $a \le x_0 = f^{-1}(y_0) < f^{-1}(y_0+) \le b$ by Monotone function theorem.

This implies either the interval $(f^{-1}(y_0-), f^{-1}(y_0))$ or the interval $(f^{-1}(y_0), f^{-1}(y_0+))$ is not in the range of f^{-1}. This contradicts f^{-1} is surjective.

$\therefore$ f^{-1} is continuous.



Brief Descriptions of Facts

Completeness Axiom In $\mathbb{R}$,
every set that is bounded above has a supremum;
every set that is bounded below has an infimum.

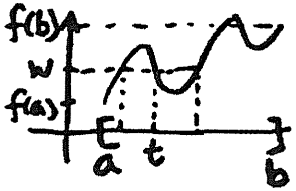
Supremum Property Let S be a set that is bounded above. Then $\forall \varepsilon > 0$, $\exists x \in S$ such that

$$\sup S - \varepsilon < x \leq \sup S.$$

Supremum Limit Theorem Let S be bounded above and c is an upper bound of S. Then

$$c = \sup S \iff \exists x_n \in S \text{ with } \lim_{n\to\infty} x_n = c.$$

Intermediate Value Theorem Let f be continuous on $[a,b]$ and w is between $f(a)$ and $f(b)$. Then



$\exists t \in [a,b]$ such that

$$f(t) = w.$$

Monotone Function Theorem Let f be monotone on (a,b). Then ① $\forall x_0 \in (a,b)$, $f(x_0-) = \lim_{x\to x_0^-} f(x)$, $f(x_0+) = \lim_{x\to x_0^+} f(x)$ both exist

② f has countably many discontinuities on (a,b).

Continuous Injection Theorem, Continuous Inverse Theorem

f continuous and injective on $[a,b]$ $\Rightarrow$ f is strictly monotone
$\Rightarrow$ f^{-1} is continuous on $f([a,b])$

Monotone Sequence Theorem

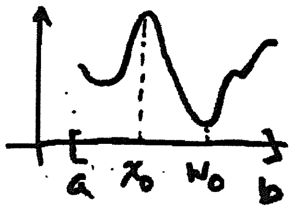
$$x_1 \leq x_2 \leq x_3 \leq \cdots \leq M \Rightarrow \lim_{n\to\infty} x_n = \sup\{x_1, x_2, x_3, \ldots\}$$

$$x_1 \geq x_2 \geq x_3 \geq \cdots \geq m \Rightarrow \lim_{n\to\infty} x_n = \inf\{x_1, x_2, x_3, \ldots\}$$

Bolzano-Weierstrass Theorem

If $x_1, x_2, x_3, \ldots \in [a,b]$, then $\exists$ subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ having a limit in $[a,b]$.
(indices $n_1 < n_2 < n_3 < \cdots$

Extreme Value Theorem Let f be continuous on $[a,b]$. Then $\exists$ $x_0, w_0 \in [a,b]$ such that



$f(x_0) = \sup\{f(x) : x \in [a,b]\}$
$= $ maximum of $f(x)$ on $[a,b]$
$f(w_0) = \inf\{f(x) : x \in [a,b]\}$
$=$ minimum of $f(x)$ on $[a,b]$.

Question How can we prove a sequence converges without identifying the limit?

In the 19th century, Cauchy introduced the following

Definition $\{x_n\}$ is a *Cauchy sequence* iff $\forall \varepsilon > 0$ $\exists K \in \mathbb{N}$ such that $n, m \geq K \Rightarrow |x_n - x_m| < \varepsilon$.

Remarks This means the terms are as close as desired when the indices are sufficiently large.

Example Let $x_n = \frac{1}{n^2}$. Show $\{x_n\}$ is Cauchy.

Scratch Work Say $m \geq n$, $|x_n - x_m| = \frac{1}{n^2} - \frac{1}{m^2} < \frac{1}{n^2} < \varepsilon$

$n > \frac{1}{\sqrt{\varepsilon}}$ is enough.

Solution. $\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\sqrt{\varepsilon}}$. Then

$n, m \geq K \Rightarrow |x_n - x_m| = \left|\frac{1}{n^2} - \frac{1}{m^2}\right| < \frac{1}{K^2} < \varepsilon$.

Topics to be Covered

(0) Differentiation

(1) Big-Oh and Small-Oh Notations
Stolz' Theorem (L'Hopital's Rule for sequences)

(2) Riemann Integration and Improper Integrals

(3) Preview of
Sequence and Series of Functions
- Limit Superior and Limit Inferior
- Pointwise and Uniform Convergence

(4) Introduction to Metric Space Theory
- Open, Closed, Compact, Connected Sets

OR

(4') Introduction to Fourier Series

Chapter 8 Differentiation

Definitions Let S be an interval of positive length. A function $f: S \to \mathbb{R}$ is differentiable at $x_0 \in S$ iff $f'(x_0) = \lim_{\substack{x \to x_0 \\ x \in S}} \frac{f(x)-f(x_0)}{x-x_0}$ exists in $\mathbb{R}$. Also, f is differentiable iff f is differentiable at every element of S.

Theorem If $f: S \to \mathbb{R}$ is differentiable at $x_0 \in S$, then it is continuous at x_0.

Proof. Since $f(x) = \frac{f(x)-f(x_0)}{x-x_0}(x-x_0) + f(x_0)$, so $\lim_{x \to x_0} f(x) = f'(x_0) \cdot 0 + f(x_0) = f(x_0)$.

Theorem (Differentiation Formulas)
If $f, g: S \to \mathbb{R}$ is differentiable at x_0, then $f+g$, $f-g$, fg, f/g (when $g(x_0) \neq 0$) are differentiable at x_0.
In fact, $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$$

Proof. $\frac{(f \pm g)(x) - (f \pm g)(x_0)}{x-x_0} = \frac{f(x)-f(x_0)}{x-x_0} \pm \frac{g(x)-g(x_0)}{x-x_0}$.

Take limit as $x \to x_0$ on both sides, $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$.

$$\frac{(fg)(x)-(fg)(x_0)}{x-x_0} = \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x-x_0}$$

$$= \frac{f(x)-f(x_0)}{x-x_0} g(x) + f(x_0) \frac{g(x)-g(x_0)}{x-x_0}.$$

Take limit as $x \to x_0$, $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

$$\frac{(f/g)(x) - (f/g)(x_0)}{x-x_0} = \frac{1}{x-x_0}\left[\frac{f(x)g(x_0) - f(x_0)g(x)}{g(x)g(x_0)}\right]$$

$$= \frac{1}{g(x)g(x_0)}\left[\frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{x-x_0}\right]$$

$$= \frac{1}{g(x)g(x_0)}\left[\frac{f(x)-f(x_0)}{x-x_0} g(x_0) - f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\right].$$

Take limit as $x \to x_0$, $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$.

Theorem (Chain Rule)
If $f: S \to \mathbb{R}$ is differentiable at x_0, $f(S) \subseteq S'$ and $g: S' \to \mathbb{R}$ is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$.

Proof. Define $h: S' \to \mathbb{R}$ by $h(y) = \begin{cases} \dfrac{g(y)-g(f(x_0))}{y-f(x_0)} & \text{if } y \neq f(x_0) \\ g'(f(x_0)) & \text{if } y = f(x_0) \end{cases}$

Then h is continuous at $f(x_0)$ because

$$\lim_{y \to f(x_0)} h(y) = \lim_{y \to f(x_0)} \frac{g(y)-g(f(x_0))}{y-f(x_0)} = g'(f(x_0)) = h(f(x_0)).$$

Now $g(y) - g(f(x_0)) \overset{(*)}{=} h(y)(y-f(x_0))$ if $y \neq f(x_0)$ and also if $y = f(x_0)$. So it is true for all $y \in S'$.

$$\underbrace{\lim_{x \to x_0} \frac{(g\circ f)(x) - (g \circ f)(x_0)}{x - x_0}}_{(g\circ f)'(x_0)} = \lim_{x \to x_0} \frac{h(f(x))(f(x)-f(x_0))}{x - x_0} \quad \text{by } (*)$$
$$= h(f(x_0)) f'(x_0) = g'(f(x_0)) f'(x_0).$$

Remarks f differentiable at x_0 does not imply f' is continuous at x_0. Here is an example.

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

As $x \to 0$, $|f(x)| \leq |x^2 \sin\frac{1}{x}| \leq x^2 \to 0 \Rightarrow \lim_{x\to 0} f(x) = 0$ by sandwich theorem. So f is continuous.

For $x \neq 0$, $f'(x) = (x^2 \sin\frac{1}{x})' = 2x \sin\frac{1}{x} + x^2 \cos(\frac{1}{x})(-\frac{1}{x^2})$
$$= 2x\sin\tfrac{1}{x} - \cos(\tfrac{1}{x}).$$

For $x = 0$, $f'(0) = \lim_{x \to 0} \dfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x \sin\frac{1}{x} = 0.$

So f is differentiable. as $x \to 0$, $|x \sin\frac{1}{x}| \leq |x| \to 0$

Finally, $\lim_{x \to 0} f'(x) = \lim_{x\to 0}(2x\sin\frac{1}{x} - \cos\frac{1}{x})$ doesn't exist ($\neq f'(0)$).

$\therefore f'$ is not continuous at 0 and hence f'' doesn't exist at 0.

Exercise $g(x) = \begin{cases} x^2 \sin\frac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is differentiable, but $g'(x)$ is not continuous at 0 and $g'(x)$ is unbounded on every open interval containing 0.

Example If $h(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ x & \text{if } x = 0, \end{cases}$ is $h'(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$?

The answer is no! $h(x) = 0$ for all x. So $h'(x) = 0$ for all x. In particular, $h'(0) = 0 \neq 1$.

Notations Let S be an interval of positive length.

$C^0(S) = C(S)$ is the set of all continuous functions on S.

$\forall n \in \mathbb{N}$, $C^n(S)$ is the set of all functions $f: S \to \mathbb{R}$ such that the n-th derivative $f^{(n)}$ is continuous on S.

$C^\infty(S)$ is the set of all functions having n^{th} derivatives for all $n \in \mathbb{N}$. Functions in $C^1(S)$ are said to be continuously differentiable on S.

Inverse Function Theorem If f is continuous and injective on (a,b) and $f'(x_0) \neq 0$ for some $x_0 \in (a,b)$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = 1/f'(x_0).$$

If $y = f(x)$, then $x = f^{-1}(y)$ and so $\left.\dfrac{dx}{dy}\right|_{\text{at } y_0} = 1 \Big/ \left.\dfrac{dy}{dx}\right|_{\text{at } x_0}$.

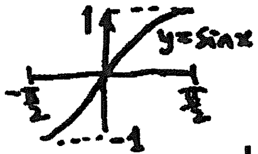
Proof. Define $g(x)=\begin{cases}\dfrac{x-x_0}{f(x)-f(x_0)} & \text{if } x\neq x_0\\ 1/f'(x_0) & \text{if } x=x_0\end{cases}$. Then g is continuous at x_0 because $\lim\limits_{x\to x_0} g(x)=\lim\limits_{x\to x_0}\dfrac{x-x_0}{f(x)-f(x_0)}=\dfrac{1}{f'(x_0)}=g(x_0)$.

Since f is continuous and injective on (a,b), by the continuous inverse theorem, f^{-1} is continuous.

So $\lim\limits_{y\to y_0} f^{-1}(y)=f^{-1}(y_0)=x_0$. For $y\neq y_0$, $g(f^{-1}(y))=\dfrac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}$.

$$\therefore\ (f^{-1})'(y_0)=\lim_{y\to y_0}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}=\lim_{y\to y_0} g(f^{-1}(y))=g(f^{-1}(y_0))$$
$$=g(x_0)=1/f'(x_0).$$

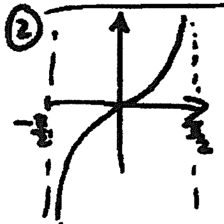
Examples ① If $y=f(x)=\sin x$ on $(-\frac{\pi}{2},\frac{\pi}{2})$, then f is differentiable and injective on $(-\frac{\pi}{2},\frac{\pi}{2})$.



We have $x=f^{-1}(y)=\sin^{-1}y=\text{Arcsin}\,y$ for $y\in(-1,1)$ and

$$\frac{d}{dy}(\text{Arcsin}\,y)=\frac{d}{dy}(\sin^{-1}y)=\frac{dx}{dy}=1\Big/\frac{dy}{dx}=1/\cos x=1/\sqrt{1-y^2}$$

②



$y=f(x)=\tan x$ on $(-\frac{\pi}{2},\frac{\pi}{2})$ is differentiable and injective on $(-\frac{\pi}{2},\frac{\pi}{2})$.

$x=f^{-1}(y)=\tan^{-1}y=\text{Arctan}\,y$ for $y\in(-\infty,\infty)$ and

$$\frac{d}{dy}(\text{Arctan}\,y)=\frac{d}{dy}(\tan^{-1}y)=\frac{dx}{dy}=1\Big/\frac{dy}{dx}=1/\sec^2x=1/(1+\tan^2x)=1/(1+y^2).$$

Local Extremum Theorem

Let $f:(a,b)\to\mathbb{R}$ be differentiable. If $f(x_0)=\min_{x\in(a,b)} f(x)$ or $f(x_0)=\max_{x\in(a,b)} f(x)$, then $f'(x_0)=0$.

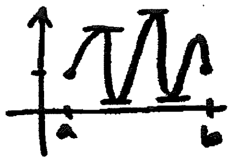
Proof. If $f(x_0)=\min_{x\in(a,b)} f(x)$, then

$$0\le \lim_{x\to x_0^+} \frac{\overbrace{f(x)-f(x_0)}^{\ge 0}}{\underbrace{x-x_0}_{>0}} = f'(x_0)=\lim_{x\to x_0^-}\frac{\overbrace{f(x)-f(x_0)}^{\ge 0}}{\underbrace{x-x_0}_{<0}}\le 0.$$

$\therefore f'(x_0)=0$. The case $f(x_0)=\max_{x\in(a,b)} f(x)$ is similar.

Remark The theorem is false in general for closed interval, for example, $f(x)=x$ on $[-1,1]$. $f(1)=\max_{x\in[-1,1]} f(x)$, but $f'(1)=1\neq 0$.

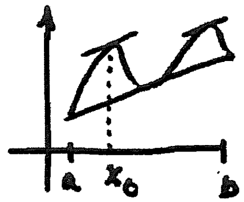
Rolle's Theorem Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$, then there is (at least one) $z_0\in(a,b)$ such that $f'(z_0)=0$.



Proof. If f is a constant function, then $f'(x)=0$ for any $x\in(a,b)$. Otherwise, by the extreme value theorem, $\exists\, x_0, w_0\in[a,b]$ such that

$$f(x_0)=\max_{x\in[a,b]} f(x) > \min_{x\in[a,b]} f(x)=f(w_0).$$

Then either $f(x_0)\neq f(a)$ or $f(w_0)\neq f(a)$.
Then x_0 or $w_0\in(a,b)$. By last theorem, $f'(x_0)=0$ or $f'(w_0)=0$.



Mean-Value Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b), then $\exists x_0\in(a,b)$ such that

$$f(b)-f(a)=f'(x_0)(b-a).$$

Proof. Define $F(x)=f(x)-\Big(\underbrace{\frac{f(b)-f(a)}{b-a}(x-a)+f(a)}_{\text{linear function through } (a,f(a)),\,(b,f(b))}\Big)$.

Then $F(a)=0=F(b)$.

By Rolle's theorem, $\exists x_0\in(a,b)$ such that $F'(x_0)=0$.

Since $F'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$, we get $f'(x_0)=\frac{f(b)-f(a)}{b-a}$.

Examples. ① $\forall a,b\in\mathbb{R}$, prove $|\sin b-\sin a|\le|b-a|$.

Solution. The case $a=b$ is clear. If $a<b$, then by mean-value theorem, for $f(x)=\sin x$, $\exists x_0\in(a,b)$ such that $|\sin b-\sin a|=|\cos(x_0)(b-a)|\le|b-a|$. The case $b<a$ is similar.

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② Prove $(1+x)^\alpha\ge 1+\alpha x$ for $x>-1$ and $\alpha\ge 1$.

Solution. Let $f(x)=(1+x)^\alpha-1-\alpha x$. Then $f(0)=0$.

Case 1: $x>0$
$\exists x_0\in(0,x)$ such that

$$(1+x)^\alpha-1-\alpha x=f(x)-f(0)=f'(x_0)(x-0)=\alpha\big(\underbrace{(1+x_0)^{\alpha-1}-1}_{\ge 0}\big)\underbrace{x}_{>0}\ge 0$$

Case 2: $-1<x<0$
$\exists x_0\in(x,0)$ such that

$$(1+x)^\alpha-1-\alpha x=f(x)-f(0)=f'(x_0)(x-0)=\alpha\big(\underbrace{(1+x_0)^{\alpha-1}-1}_{\le 0}\big)\underbrace{x}_{<0}\ge 0$$

③ Prove that $\ln x \le x-1$ for $x>0$.

Solution. Let $f(x)=\ln x - x + 1$, then $f(1)=0$.

If $x>1$, then $\exists\, x_0 \in (1,x)$ such that

$$\ln x - x + 1 = f(x) = f(x) - f(1) = f'(x_0)(x-1)$$
$$= \underbrace{\left(\tfrac{1}{x_0}-1\right)}_{<0}\underbrace{(x-1)}_{>0} < 0.$$

The case $0<x<1$ is similar.

④ Approximate $\sqrt{16.1}$.

Let $f(x)=\sqrt{x}$. Then $f(16.1)-f(16)=f'(c)(16.1-16)$ for some $c\in(16,16.1)$. Now $c \approx 16$. So

$$f(16.1)-f(16) \approx f'(16)(16.1-16) = \frac{1}{2\sqrt{16}}(0.1) = 0.0125$$

$\therefore\ \sqrt{16.1}-4 \approx 0.0125$, $\sqrt{16.1} \approx 4.0125$.

Theorem (for Curve Tracing)

$$\text{If } \left\{\begin{array}{l} f'\ge 0 \\ f'>0 \\ f'\le 0 \\ f'<0 \\ f'\ne 0 \\ f'\equiv 0 \end{array}\right\} \text{ everywhere on } (a,b), \text{ then } f \text{ is } \left\{\begin{array}{l} \text{increasing} \\ \text{strictly increasing} \\ \text{decreasing} \\ \text{strictly decreasing} \\ \text{injective} \\ \text{constant} \end{array}\right\} \text{ respectively}$$

Proof. If $x,y\in(a,b)$, $x<y$, then by mean value theorem, $\exists\, x_0 \in (x,y)$ such that

$$f(y)-f(x) = f'(x_0)\underbrace{(y-x)}_{>0} \left\{\begin{array}{l} \ge 0 \\ >0 \\ \le 0 \\ <0 \\ \ne 0 \\ =0 \end{array}\right\}. \quad \therefore \left\{\begin{array}{l} f(x)\le f(y) \\ f(x)<f(y) \\ f(x)\ge f(y) \\ f(x)>f(y) \\ f(x)\ne f(y) \\ f(x)=f(y) \end{array}\right\}.$$

Remarks For differentiable function f,

$$\text{if } f \text{ is } \left\{\begin{array}{l} \text{strictly increasing} \\ \text{strictly decreasing} \\ \text{injective} \end{array}\right\}, \text{ then } \left\{\begin{array}{l} f'>0 \\ f'<0 \\ f'\ne 0 \end{array}\right\} \text{ everywhere}$$

may be false!

Examples ① $f(x)=x^3$ is strictly increasing and injective, but $f'(0)=0$. ② $f(x)=-x^3$ is strictly decreasing, but $f'(0)=0$.

For differentiable function $f:(a,b)\to\mathbb{R}$,

$$\text{if } f \text{ is } \left\{\begin{array}{l} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{array}\right\}, \text{ then } \left\{\begin{array}{l} f'\ge 0 \\ f'\le 0 \\ f'\equiv 0 \end{array}\right\} \text{ everywhere on } (a,b)$$

is true.

Proof. For $x, x_0 \in (a,b)$,

$$f \text{ is } \left\{\begin{array}{l} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{array}\right\} \Longrightarrow \frac{f(x)-f(x_0)}{x-x_0}\left\{\begin{array}{l} \ge 0 \\ \le 0 \\ =0 \end{array}\right\}$$

$$\Longrightarrow f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\left\{\begin{array}{l} \ge 0 \\ \le 0 \\ =0 \end{array}\right\}.$$

Local Tracing Theorem

If $f:[a,b]\to\mathbb{R}$ is continuous and $f'(c)>0$ for some $c\in[a,b]$, then $\exists\, c_0, c_1\in\mathbb{R}$ such that

$c_0<c<c_1$ and $f(x)<f(c)<f(y)$ $\forall\, x,y\in[a,b]$ and $c_0<x<c$, $c<y<c_1$.

c_0 $\quad x \quad$ c $\quad y \quad$ c_1

A similar result for the case $f'(c)<0$ is true and the inequality becomes $f(x)>f(c)>f(y)$.

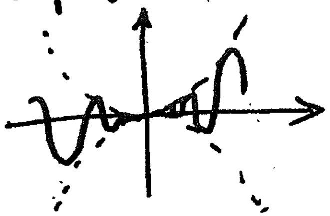
Proof. Let $f'(c)>0$. Assume there is no such c_0. Then $\forall\, n=1,2,3,\dots$, $\exists\, x_n\in[a,b]$ and $c-\frac{1}{n}<x_n<c$ satisfying $f(x_n)\ge f(c)$. This will lead to

$$f'(c)=\lim_{n\to\infty}\frac{\overbrace{f(x_n)-f(c)}^{\ge 0}}{\underbrace{x_n-c}_{<0}}\le 0,\ \text{contradiction.}$$

The other parts are similar.

Remarks. If we only know $f'(c)\ge 0$, we do not have a similar result. For example, let

$$f(x)=\begin{cases} x^2\sin\frac{1}{x} & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$$



We have $f'(0)=0$, but on every open interval $(c_0,0)$ or $(0,c_1)$, $f(x)$ takes both positive and negative values.

Generalized Mean-Value Theorem

If f,g are continuous on $[a,b]$ and are differentiable on (a,b), then $\exists\, x_0\in(a,b)$ such that

$$g'(x_0)\big(f(b)-f(a)\big)=f'(x_0)\big(g(b)-g(a)\big).\quad (*)$$

Proof. Define $F(x)=g(x)\big(f(b)-f(a)\big)-f(x)\big(g(b)-g(a)\big)$. Then $F(a)=g(a)f(b)-f(a)g(b)=F(b)$. By Rolle's theorem, $\exists\, x_0\in(a,b)$ such that $F'(x_0)=0$. So we get $(*)$.

Remark. If $g(b)\ne g(a)$, then $(*)$ can be put as

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(x_0)}{g'(x_0)}.$$

($\frac{0}{0}$ form of L'Hôpital's Rule)

① Let f,g be differentiable on (a,b)

② $g(x), g'(x)\ne 0$ $\forall\, x\in(a,b)$

③ $\lim_{x\to a^+}f(x)=0=\lim_{x\to a^+}g(x)$

④ $\lim_{x\to a^+}\frac{f'(x)}{g'(x)}=L$, where $L\in\mathbb{R}$ or $L=-\infty$ or $L=+\infty$.

Then $\lim_{x\to a^+}\frac{f(x)}{g(x)}=\lim_{x\to a^+}\frac{f'(x)}{g'(x)}$.

$\boxed{\text{The case } x\to b^- \text{ is similar.}}$

Proof. Define $f(a)=0$ and $g(a)=0$. $\forall\, x\in(a,b)$, f,g are continuous on $[a,x]$ and differentiable on (a,x). By generalized mean value theorem, $\exists\, x_0\in(a,x)$ such that

$\frac{f(x)}{g(x)}=\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(x_0)}{g'(x_0)}$. As $x\to a^+$, $x_0\to a^+$. $\therefore \frac{f(x)}{g(x)}=\frac{f'(x_0)}{g'(x_0)}\to L$.

($\frac{*}{\infty}$ form of L'Hôpital's Rule)

① Let f, g be differentiable on (a,b)

② $g(x), g'(x) \neq 0 \quad \forall x \in (a,b)$

③ $\lim_{x\to a^+} g(x) = \infty$ ← No need $\lim_{x\to a^+} f(x)$ exists!

④ $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$, where $L \in \mathbb{R}$ or $L = -\infty$ or $L = +\infty$

Then $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$. The case $x \to b^-$ is similar.

Proof. We do the case $L \in \mathbb{R}$ first. By ④, $\exists$ interval $I = (a, a+\delta_0)$ such that $t \in I \Rightarrow \left|\frac{f'(t)}{g'(t)} - L\right| < \varepsilon/2$.
Let $y \in I$. $\forall x \in I$, by generalized mean-value theorem, $\exists t \in I$ such that $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)}$.

Multiply by $\frac{g(x)-g(y)}{g(x)}$, add $\frac{f(y)}{g(x)}$, then subtract $\frac{f'(t)}{g'(t)}$ on both sides. Get $\frac{f(x)}{g(x)} - \frac{f'(t)}{g'(t)} = -\frac{g(y)}{g(x)}\frac{f'(t)}{g'(t)} + \frac{f(y)}{g(x)}$.

So $\left|\frac{f(x)}{g(x)} - \frac{f'(t)}{g'(t)}\right| \le \left|\frac{g(y)}{g(x)}\right|\left(|L| + \frac{\varepsilon}{2}\right) + \left|\frac{f(y)}{g(x)}\right|$. ← consider $x \to a^+$

By ③, the right side has limit 0. So $\exists$ interval $J = (a, a+\delta_1)$ so that $\forall x \in J$, the right side is at most $\varepsilon/2$.
Then $\forall x \in I \cap J = (a, a + \min\{\delta_0, \delta_1\})$

$$\left|\frac{f(x)}{g(x)} - L\right| \le \left|\frac{f(x)}{g(x)} - \frac{f'(t)}{g'(t)}\right| + \left|\frac{f'(t)}{g'(t)} - L\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The cases $L = \pm\infty$ follow by making simple modifications.

Examples ① Let $f(x) = x^2 \sin\frac{1}{x}$ and $g(x) = \sin x$ on $(0, \frac{\pi}{2})$.
Since $\underset{\to 0}{-x^2} \le x^2 \sin\frac{1}{x} \le \underset{\to 0}{x^2}$, so $\lim_{x\to 0^+} f(x) = 0$. $\lim_{x\to 0^+} g(x) = 0$.

$\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{2x\sin\frac{1}{x} - \cos\frac{1}{x}}{\underset{\to 1}{\cos x}}$ doesn't exist as $\lim_{x\to 0^+} \cos\frac{1}{x}$ does not exist.

$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{x}{\sin x}\left(x \sin\frac{1}{x}\right) = 1 \cdot 0 = 0 \neq \lim_{x\to 0^+} \frac{f'(x)}{g'(x)}$.

② $\forall r \in \mathbb{R}$, $\lim_{x\to+\infty} \frac{x^r}{e^x} = 0$. (To see this, choose $n > |r|$. Then $x^r \le x^n$ on $[1,\infty)$. So $0 \le \frac{x^r}{e^x} \le \frac{x^n}{e^x}$ on $[1,\infty)$. Since $\frac{d^n}{dx^n} x^n = n!$ and $\lim_{x\to+\infty} \frac{n!}{e^x} = 0$, applying L'Hopital's rule n-times, we see $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$. $\therefore \lim_{x\to+\infty} \frac{x^r}{e^x} = 0$.)

③ Let $f: (a, +\infty) \to \mathbb{R}$ be differentiable. Then

$$\lim_{x\to+\infty} (f'(x) + f(x)) = 0 \Rightarrow \lim_{x\to+\infty} f(x) = 0 = \lim_{x\to+\infty} f'(x).$$

(To see this, we apply ($\frac{*}{\infty}$)-form of L'Hôpital's rule as follow:

$$\lim_{x\to\infty} f(x) = \lim_{x\to+\infty} \frac{f(x)e^x}{\underset{\to\infty}{e^x}} = \lim_{x\to\infty} \frac{f'(x)e^x + f(x)e^x}{e^x} = \lim_{x\to+\infty} f'(x) + f(x) = 0$$

and $\lim_{x\to+\infty} f'(x) = \lim_{x\to+\infty} \big((f'(x) + f(x)) - f(x)\big) = 0 - 0 = 0$.)

Remarks In O.D.E., if $\lim_{x\to+\infty} g(x) = 0$, then every solution $y = f(x)$ of $\frac{dy}{dx} + y = g(x)$ satisfies $\lim_{x\to\infty} f(x) = 0$ by the reason above.

④ Let $f(x)=2x+\sin x$ and $g(x)=2x-\sin x$ on $(-\infty,+\infty)$.
As $x\to+\infty$, $f(x), g(x)\to+\infty$.

$\lim\limits_{x\to+\infty}\frac{f'(x)}{g'(x)}=\lim\limits_{x\to+\infty}\frac{2+\cos x}{2-\cos x}$ doesn't exist

$\hookrightarrow\left(\begin{array}{ll} x=2n\pi & \text{limit}=3 \\ x=(2n+1)\pi & \text{limit}=1/3 \end{array}\right)$

but $\lim\limits_{x\to+\infty}\frac{f(x)}{g(x)}=\lim\limits_{x\to+\infty}\frac{2x+\sin x}{2x-\sin x}=\lim\limits_{x\to+\infty}\frac{2+\frac{\sin x}{x}}{2-\frac{\sin x}{x}}=\frac{2}{2}=1$

Taylor's Theorem Let $f:(a,b)\to\mathbb{R}$ be n-times differentiable. $\forall x,c\in(a,b)$, $\exists x_0$ between x and c such that

$$f(x)=\underbrace{f(c)+\frac{f'(c)}{1!}(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots+\frac{f^{(n-1)}(c)}{(n-1)!}(x-c)^{n-1}}_{(n^{th}\text{ Taylor expansion of } f \text{ about } c)}+\underbrace{\frac{f^{(n)}(x_0)}{n!}(x-c)^n}_{R_n(x)\text{ Lagrange form of the remainder.}}$$

Proof. Let I be the closed interval with x and c as endpoints. For $t\in I$, define $g(t)=(n-1)!\sum\limits_{k=0}^{n-1}\frac{f^{(k)}(t)}{k!}(x-t)^k$, where $f^{(0)}=f$ and define $p(t)=-\frac{(x-t)^n}{n}$. We have

$g'(t)=f^{(n)}(t)(x-t)^{n-1}$ and $p'(t)=(x-t)^{n-1}$.

By generalized mean value theorem, $\exists x_0$ between x and c such that

$$\underbrace{g'(x_0)}_{f^{(n)}(x_0)(x-x_0)^{n-1}}\Big(\underbrace{p(x)-p(c)}_{(x-c)^n/n}\Big)=\underbrace{p'(x_0)}_{(x-x_0)^{n-1}}\Big(\underbrace{g(x)-g(c)}_{(n-1)!f(x)}\Big)$$

$$\Rightarrow f(x)=\frac{g(c)}{(n-1)!}+\frac{f^{(n)}(x_0)}{n!}(x-c)^n=\sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k+\frac{f^{(n)}(x_0)}{n!}(x-c)^n.$$

Taylor Expansions of Common Functions at $c=0$

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+R_{n+1}(x)=\sum_{k=0}^{n}\frac{x^k}{k!}+R_{n+1}(x)$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+R_{2n+2}(x)$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+R_{2n+3}(x)$$

$$(1+x)^a=1+\sum_{k=1}^{n}\underbrace{\frac{a(a-1)\cdots(a-k+1)}{k!}}_{=\binom{a}{k}=C_a^k}x^k+R_{n+1}(x)$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+\frac{(-1)^n x^n}{n}+R_{n+1}(x)$$

$$\operatorname{Arctan}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots+\frac{(-1)^n x^{2n+1}}{2n+1}+R_{2n+3}(x)$$

$$\operatorname{Arcsin}x=x+\sum_{k=1}^{n}\underbrace{\frac{1\cdot3\cdot5\cdots(2k-1)}{2\cdot4\cdot6\cdots(2k)}}_{=\frac{(2k-1)!!}{(2k)!!}}\frac{x^{2k+1}}{2k+1}+R_{2n+3}(x)$$

Notation:

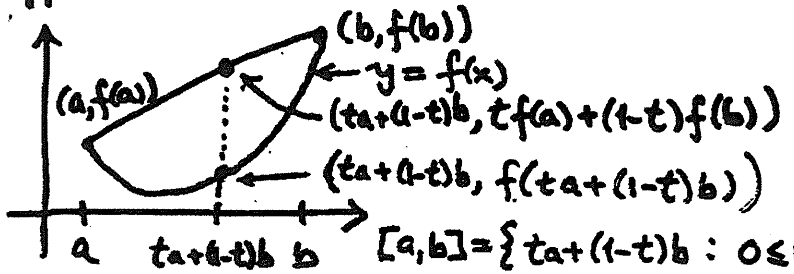
$$m!!=\begin{cases}1\cdot3\cdot5\cdots m & \text{if } m \text{ is odd}\\ 2\cdot4\cdot6\cdots m & \text{if } m \text{ is even}\end{cases}$$

n^{th} Taylor expansion is also called n^{th} Taylor Polynomial.

If we let $n\to\infty$, the n^{th} Taylor expansion goes to $\sum\limits_{k=0}^{\infty}\frac{f^{(k)}(c)}{k!}(x-c)^k$. This is called the Taylor series of $f(x)$ about c.

$f^{(0)}(c)=f(c)$ $\quad (x-c)^0=1$ even if $x=c$.

Appendix 1: Convex and Concave Functions



$[a,b] = \{ta+(1-t)b : 0 \le t \le 1\}$

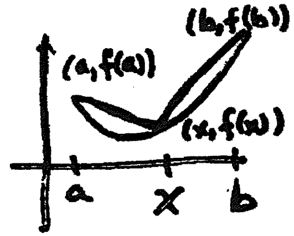
Definitions ① Let I be an interval and $f: I \to \mathbb{R}$. We say f is a convex function on I iff
$\forall a,b \in I$, $0 \le t \le 1$, $f(ta+(1-t)b) \le tf(a)+(1-t)f(b)$.

② f is a concave function on I iff
$\forall a,b \in I$, $0 \le t \le 1$, $f(ta+(1-t)b) \ge tf(a)+(1-t)f(b)$

Remarks ① A function is convex on I iff every chord joining $(a,f(a))$ and $(b,f(b))$ with $a,b \in I$ is always above or on the curve $y=f(x)$. A function is concave on I iff every chord is below or on the curve.

② f is strictly convex iff $f(ta+(1-t)b) < tf(a)+(1-t)f(b)$ for $0<t<1$
Similarly for strictly concave.
Strictly convex functions are the ones whose chords are above the curve (except the endpoints are on the curve, of course). Similarly for strictly concave functions.

③ f is convex $\Leftrightarrow$ $-f$ is concave.
f is strictly convex $\Leftrightarrow$ $-f$ is strictly concave.



Theorem f is convex on I iff the slope of the chords always increase in the sense that
$\forall a,x,b \in I$,
$$a<x<b \Rightarrow \frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(x)}{b-x}.$$

Proof. Note $x = ta+(1-t)b$ for some $t \in [0,1]$ $\Leftrightarrow$ $0 \le t = \frac{b-x}{b-a} \le 1$.
$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(x)}{b-x} \Leftrightarrow f(x) \le \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)$$
$$\Leftrightarrow f(ta+(1-t)b) \le tf(a)+(1-t)f(b).$$

Theorem For f differentiable on I, f is convex on I $\Leftrightarrow$ f' is increasing on I ($\Leftrightarrow f'' \ge 0$ on I for f twice differentiable on I). ← from curve tracing theorem.

Proof ($\Rightarrow$) $\forall a,b \in I$ with $a<b$, by last theorem,
$$f'(a) = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \le \lim_{x\to a^+} \frac{f(b)-f(x)}{b-x} = \frac{f(b)-f(a)}{b-a}$$
$$= \lim_{x\to b^-} \frac{f(x)-f(a)}{x-a} \le \lim_{x \to b^-} \frac{f(b)-f(x)}{b-x} = f'(b).$$

($\Leftarrow$) $\forall a,x,b \in I$ with $a<x<b$, by the mean-value theorem, $\exists r,s$ such that $a<r<x<s<b$ and
$$\frac{f(x)-f(a)}{x-a} = f'(r) \le f'(s) = \frac{f(b)-f(x)}{b-x}.$$
By last theorem, f is convex on I.

Theorem If f is convex on (a,b), then f is continuous on (a,b).

Proof. $\forall x_0 \in (a,b)$, consider $u, v, w \in (a,b)$ such that $u < x_0 < v < w$. Then

$$\frac{f(x_0)-f(u)}{x_0-u} \le \frac{f(v)-f(x_0)}{v-x_0} \le \frac{f(w)-f(v)}{w-v}.$$

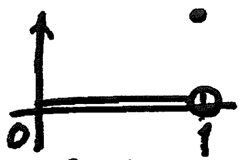
Solving for $f(v)$, we get

$$\frac{f(x_0)-f(u)}{x_0-u}(v-x_0)+f(x_0) \le f(v) \le \frac{f(w)-f(v)}{w-v}(v-x_0)+f(x_0).$$

Take limit as $v \to x_0^+$, we get $f(x_0) \le f(x_0+) \le f(x_0)$. So $f(x_0+) = f(x_0)$. Similarly, $f(x_0-) = f(x_0)$ by taking $u < v < x_0 < w$. Therefore, f is continuous on (a,b).

Remark and Example The above theorem may not be true for $[a,b]$. For example,

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$



is convex on $[0,1]$ by checking the definition or checking the slope of chords is increasing. However, f is not continuous at 1.

Example Prove that if $a, b \ge 0$ and $0 < r < 1$, then $|a^r - b^r| \le |a-b|^r$.

In particular, $|\sqrt[n]{a} - \sqrt[n]{b}| \le \sqrt[n]{|a-b|}$ $(*)$

for $n = 2, 3, 4, \ldots$.

Solution. We may assume $a \ge b$, otherwise interchange them.

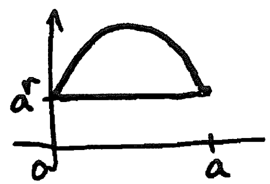
Define $f : [0,a] \to \mathbb{R}$ by $f(x) = x^r + (a-x)^r$.

Since $r - 1 < 0$, so

$$f''(x) = r(r-1)(x^{r-2} + (a-x)^{r-2}) \le 0.$$

So f is concave on $[0,a]$.

Since $f(0) = a^r = f(a)$, we get



$f(x) = x^r + (a-x)^r \ge a^r \quad \forall x \in [0,a]$.

If $b \in [0,a]$ (i.e. $0 \le b \le a$), then

$$f(b) = b^r + (a-b)^r \ge a^r \Rightarrow |a^r - b^r| = a^r - b^r \le (a-b)^r = |a-b|^r.$$

Remark $(*)$ is the case $r = \frac{1}{n}$ for $n = 2, 3, 4, \ldots$

$(*)$ is useful in some exercises.

Extra Examples for the Chapter on Differentiation

① If $\{x_n\}$ is Cauchy, and $x_n \geq 0$ for all n, then prove $\{\sqrt[5]{x_n}\}$ is Cauchy.

Solution. Since $\{x_n\}$ is Cauchy, for every $\varepsilon > 0$, $\exists K \in \mathbb{N}$ such that $m, n \geq K \Rightarrow |x_m - x_n| < \varepsilon^5$.

Then $|\sqrt[5]{x_m} - \sqrt[5]{x_n}| \leq \sqrt[5]{|x_m - x_n|} < \sqrt[5]{\varepsilon^5} = \varepsilon$.

$\therefore \{\sqrt[5]{x_n}\}$ is Cauchy.

---

② $f: \mathbb{R} \to \mathbb{R}$ is three-times differentiable.

If $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$, then prove that $\sup\{f^{(3)}(x) : -1 < x < 1\} \geq 3$.

Thoughts $f^{(3)}, f', f$ suggest Taylor's theorem

c should be $-1, 1$ or 0, more likely $c = 0$.

Solution By Taylor's theorem, using $c = 0$,

$$f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(\theta_x)}{6}(x-0)^3$$
$$= f(0) + \frac{f''(0)}{2}x^2 + \frac{f'''(\theta_x)}{6}x^3$$

for some θ_x between x and 0. Setting $x = 1, -1$,

$$1 = f(1) = f(0) + \frac{f''(0)}{2} + \frac{f'''(\theta_1)}{6} \text{ for some } \theta_1 \in (0,1)$$
$$0 = f(-1) = f(0) + \frac{f''(0)}{2} - \frac{f'''(\theta_{-1})}{6} \text{ for some } \theta_{-1} \in (-1,0).$$

Subtracting yields $f'''(\theta_1) + f'''(\theta_{-1}) = 6$.

$\sup\{f^{(3)}(x) : -1 < x < 1\} \geq \max\{f'''(\theta_1), f'''(\theta_{-1})\} \geq 3$.

③ Let f be twice differentiable on $[0,2]$.

$\forall x \in [0,2]$, $|f(x)| \leq 1$, $|f''(x)| \leq 1$.

Prove that $\forall x \in [0,2]$, $|f'(x)| \leq 2$.

Solution By Taylor's theorem, let $x \in [0,2]$, $a \in [0,2]$

$$f(a) = f(x) + f'(x)(a-x) + \frac{f''(\theta_a)}{2}(a-x)^2$$

for some θ_a between a and x. Setting $a = 0, 2$,

$$f(0) = f(x) - f'(x)x + \frac{f''(\theta_0)}{2}x^2 \text{ for some } \theta_0 \in (0,x).$$
$$f(2) = f(x) + f'(x)(2-x) + \frac{f''(\theta_2)}{2}(2-x)^2$$

for some $\theta_2 \in (x,2)$.

Subtracting these, we get

$$f(2) - f(0) = 2f'(x) + \frac{f''(\theta_2)}{2}(2-x)^2 - \frac{f''(\theta_0)}{2}x^2.$$

Solving for $f'(x)$, we see

$$|f'(x)| = \frac{1}{2}\left|f(2) - f(0) + \frac{f''(\theta_0)}{2}x^2 - \frac{f''(\theta_2)}{2}(2-x)^2\right|$$
$$\leq \frac{1}{2}\left(1 + 1 + \frac{1}{2}x^2 + \frac{1}{2}(2-x)^2\right)$$
$$= \frac{1}{2}(x^2 - 2x + 4)$$
$$\leq \frac{1}{2}((x-1)^2 + 3) \qquad \begin{cases} x \in [0,2] \\ |x-1| \leq 1 \end{cases}$$
$$\leq \frac{1}{2}(1+3) = 2$$

④ Let $f:[1,2]$ be continuous and f be differentiable on $(1,2)$. Prove there exists $\theta\in(1,2)$ such that
$$f(2)-f(1)=\tfrac{1}{2}\theta^2 f'(\theta).$$
(Note mean value theorem only gives $f(2)-f(1) = f'(\theta_0)(2-1) = f'(\theta_0)$ for some $\theta_0\in(1,2)$.)

Solution Write
$$\theta^2 f'(\theta) = \frac{f'(\theta)}{1/\theta^2} = \frac{f'(\theta)}{g'(\theta)}, \text{ where } g(\theta) = -\frac{1}{\theta}.$$
By generalized mean value theorem,
$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(\theta)}{g'(\theta)} \text{ for some } \theta\in(1,2).$$
This is just
$$\frac{f(2)-f(1)}{-\frac{1}{2}-(-1)} = \frac{f'(\theta)}{1/\theta^2} \iff f(2)-f(1) = \tfrac{1}{2}\theta^2 f'(\theta).$$

⑤ Let $0<a<b$ and let $f:[a,b]\to\mathbb{R}$ be continuous with f differentiable on (a,b). Prove $\exists\,\theta\in(a,b)$ such that
$$\frac{1}{b-a}\begin{vmatrix} f(a) & f(b) \\ a & b\end{vmatrix} = f(\theta)-\theta f'(\theta).$$

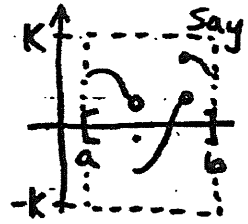
Solution. Expanding the left side, we get
$$\frac{1}{b-a}\begin{vmatrix} f(a) & f(b) \\ a & b\end{vmatrix} = \underbrace{\frac{bf(a)-af(b)}{b-a}}_{\text{dividing by } ab \text{ in numerator + denominator.}} = \frac{f(a)/a - f(b)/b}{1/a - 1/b}$$
This suggest we consider $F(x) = f(x)/x$, $G(x) = 1/x$.
By the generalized mean value theorem,
$$\frac{f(a)/a - f(b)/b}{1/a-1/b} = \frac{F(a)-F(b)}{G(a)-G(b)} = \frac{F'(\theta)}{G'(\theta)} \text{ for some } \theta\in(a,b)$$
$$= \frac{\theta f'(\theta)-f(\theta)}{\theta^2}\Big/\left(-\frac{1}{\theta^2}\right)$$
$$= f(\theta)-\theta f'(\theta).$$
The result follows.

PAGES 84 to 90
Used to contain extra materials
when we had a longer semester.
As it is, we do not have time to cover
these materials nowadays.

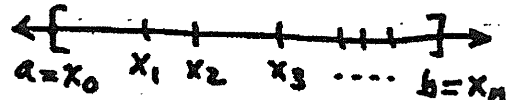
Chapter 9 Riemann Integral

Proper Integral

Setting: Let $f(x)$ (may be discontinuous) be a bounded function on a closed and bounded interval $[a,b]$,

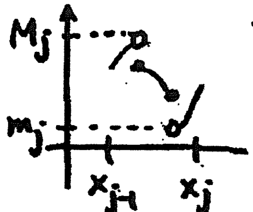


Say $\exists K>0, \forall x\in[a,b], |f(x)|\le K$.



$P=\{x_0,x_1,\cdots,x_n\}$ is a partition of $[a,b]$ iff $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$.

Let $\Delta x_j = x_j - x_{j-1}$. $\|P\| = \max\{\Delta x_1,\cdots,\Delta x_n\}$ is called the mesh of P.

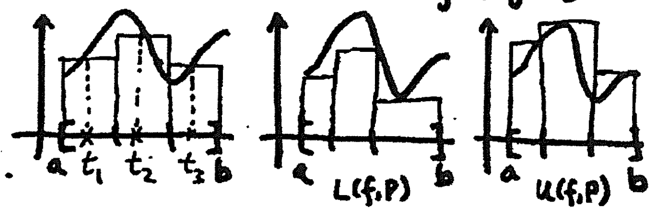


Since $f(x)$ may be discontinuous on $[a,b]$, we consider

$$m_j = \inf\{f(x) : x\in[x_{j-1},x_j]\}$$
$$M_j = \sup\{f(x) : x\in[x_{j-1},x_j]\}.$$

Definitions

Let $t_j\in[x_{j-1},x_j]$ for $j=1,2,\cdots,n$.



$S=\sum_{j=1}^{n} f(t_j)\Delta x_j$ is a Riemann sum with respect to P.

$L(f,P)=\sum_{j=1}^{n} m_j\Delta x_j$ is the lower Riemann sum w.r.t. P.

$U(f,P)=\sum_{j=1}^{n} M_j\Delta x_j$ is the upper Riemann sum w.r.t. P.

Riemann considered $\lim\limits_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j$ as the integral of $f(x)$ on $[a,b]$.

However, this limit is <u>not</u> a limit of a sequence nor a limit of a function. Since the t_j's may be arbitrary, it is not clear how the Riemann sums are approximating the integral.

Observe that $\forall j=1,2,\dots,n,\ -K \le m_j \le f(t_j) \le M_j \le K$.

So $$\underbrace{\sum_{j=1}^{n} -K\Delta x_j} \le \underbrace{\sum_{j=1}^{n} m_j\Delta x_j} \le \underbrace{\sum_{j=1}^{n} f(t_j)\Delta x_j} \le \underbrace{\sum_{j=1}^{n} M_j\Delta x_j} \le \underbrace{\sum_{j=1}^{n} K\Delta x_j}$$

i.e. $$-K(b-a) \le L(f,P) \le S \le U(f,P) \le K(b-a)$$

In particular, $U(f,P)$ and $L(f,P)$ are bounded above and below. Also, $\forall$ partition P,

$$L(f,P) \le U(f,P).$$

How about if we have 2 partitions P_1 and P_2 ?

First, note $P_1 \cup P_2$ is also a partition.

<u>Definitions</u>

① For partitions P, P', we say P' is a <u>refinement</u> of P (or P' is <u>finer</u> than P) iff $P \subseteq P'$.

② For partitions P_1, P_2, we say $P_1 \cup P_2$ is the <u>common refinement</u> of P_1 and P_2.

<u>Refinement Theorem</u> If $P \subseteq P'$, then

$$L(f,P) \le L(f,P') \le U(f,P') \le U(f,P).$$

<u>Proof.</u> It suffices to consider the case $P' = P \cup \{w\}$, say $P = \{\underset{a}{\overset{\shortparallel}{x_0}}, x_1, \dots, x_{j-1}, x_j, \dots, \underset{b}{\overset{\shortparallel}{x_n}}\}$ and $x_{j-1} < w < x_j$.

Since $[x_{j-1}, w], [w, x_j] \subseteq [x_{j-1}, x_j]$, so

$m' = \inf\{f(x) : x \in [x_{j-1}, w]\} \ge m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\}$

and $m'' = \inf\{f(x) : x \in [w, x_j]\} \ge m_j$. Then

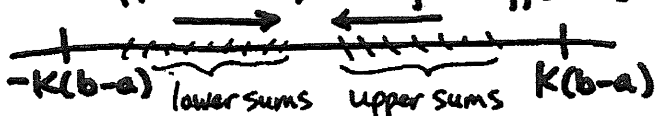
$$L(f,P) = \cdots + m_j \underbrace{\Delta x_j}_{=(w-x_{j-1})+(x_j-w)} + \cdots \le \underbrace{\cdots + m'(w-x_{j-1}) + m''(x_j-w) + \cdots}_{=L(f,P')}$$

Similarly, $U(f,P') \le U(f,P)$.

For partitions P_1, P_2, since $P_1, P_2 \subseteq P_1 \cup P_2$,

$$L(f,P_1) \le L(f, P_1 \cup P_2) \le U(f, P_1 \cup P_2) \le U(f, P_2).$$

So lower sums $\le$ upper sums, even for different partitions!



$(L)\displaystyle\int_a^b f(x)\,dx = \sup\{L(f,P) : P \text{ partition of } [a,b]\} = \underline{\int_a^b} f(x)\,dx$

is the <u>lower integral</u> of $f(x)$ on $[a,b]$.

$(U)\displaystyle\int_a^b f(x)\,dx = \inf\{U(f,P) : P \text{ partition of } [a,b]\} = \overline{\int_a^b} f(x)\,dx$

is the <u>upper integral</u> of $f(x)$ on $[a,b]$.

Definitions

$f(x)$ is (Riemann) integrable on $[a,b]$ iff

$$(L)\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx.$$

In that case, we write $\int_a^b f(x)\,dx$ for this number.

If $b \le a$, define $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

In particular, $\int_a^a f(x)\,dx = 0$.

Questions Are there integrable functions? Are there non-integrable functions?

Examples ① $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ is nonintegrable on $[a,b]$ where $a<b$.

On any $[x_{j-1}, x_j]$, $m_j = 0$ by density of irrationals and $M_j = 1$ by density of rationals. So for all partition P of $[a,b]$,

$$L(f,P) = \sum_{j=1}^n m_j \Delta x_j = 0, \quad U(f,P) = \sum_{j=1}^n M_j \Delta x_j = \sum_{j=1}^n \Delta x_j = b-a$$

$$(L)\int_a^b f(x)\,dx = \sup_P L(f,P) = 0, \quad (U)\int_a^b f(x)\,dx = \inf_P U(f,P) = b-a$$

different

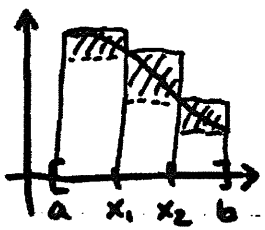
② $f(x) = c \ (\forall x \in [a,b])$ is integrable on $[a,b]$.

On any $[x_{j-1}, x_j]$, $m_j = c = M_j$. So $\forall$ partition P,

$$L(f,P) = \sum m_j \Delta x_j = \underbrace{\sum c\Delta x_j}_{=c(b-a)} = \sum M_j \Delta x_j = U(f,P).$$

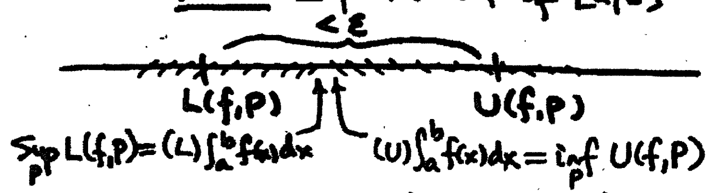
$$\therefore (L)\int_a^b f(x)\,dx = c(b-a) = (U)\int_a^b f(x)\,dx.$$

Continuous functions on $[a,b]$ are integrable. For that we need



Theorem (Integral Criterion)
Let $f(x)$ be bounded on $[a,b]$.
$f(x)$ is Riemann integrable on $[a,b]$
$\iff \forall \varepsilon > 0 \ \exists$ partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

Proof ($\Leftarrow$) $\forall \varepsilon > 0$ $\exists$ partition P of $[a,b]$

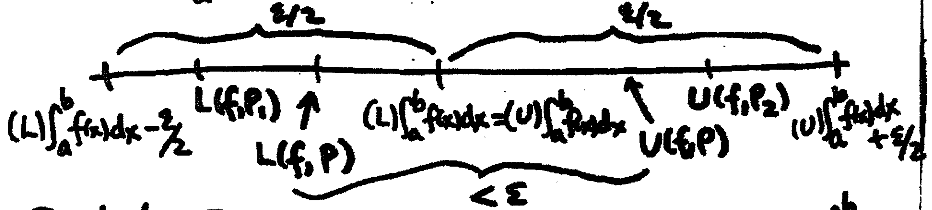


$$\varepsilon > U(f,P) - L(f,P) \ge (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx \ge 0.$$

$\therefore (U)\int_a^b f(x)\,dx - (L)\int_a^b f(x)\,dx = 0$ by infinitesimal principle.

($\Rightarrow$) $\forall \varepsilon > 0$, by supremum property, $\exists P_1$ such that

$$(L)\int_a^b f(x)\,dx - \frac{\varepsilon}{2} < L(f,P_1) \le (L)\int_a^b f(x)\,dx$$



Similarly, $\exists P_2$ such that $(U)\int_a^b f(x)\,dx \le U(f,P_2) \le (U)\int_a^b f(x)\,dx + \varepsilon/2$.

Let $P = P_1 \cup P_2$, then

$$U(f,P) - L(f,P) < \left((U)\int_a^b f(x)\,dx + \frac{\varepsilon}{2}\right) - \left((L)\int_a^b f(x)\,dx - \frac{\varepsilon}{2}\right) = \varepsilon.$$

<u>Recall</u> $f: S \to \mathbb{R}$ is continuous at $t \in S$ means $\forall \varepsilon > 0$, $\exists \delta > 0$ (δ depends on ε and t) such that $\forall x \in S$ $|x-t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon$.

<u>Definition</u> $f: S \to \mathbb{R}$ is <u>uniformly continuous</u> iff $\forall \varepsilon > 0$, $\exists \delta > 0$ (δ depends only on ε) such that $\forall x, t \in S$, $|x-t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon$.

<u>Remark</u> For any set S, $f: S \to \mathbb{R}$ uniformly continuous $\Rightarrow$ f is continuous (at every $t \in S$) because the δ in the definition can be used for every $t \in S$.

For closed and bounded intervals, the <u>converse</u> is true.

<u>Uniform Continuity theorem</u>

If $f: [a,b] \to \mathbb{R}$ is continuous, then it is uniformly continuous.

Proof. Assume f is not uniformly continuous. Then

$\exists \varepsilon > 0$ $\forall \delta > 0$ $\exists x, t \in [a,b]$, $|x-t| < \delta$ and $|f(x) - f(t)| \geq \varepsilon$

$\delta = 1$ $\quad \exists x_1, t_1 \in [a,b]$ $\quad |x_1 - t_1| < \delta = 1$ and $|f(x_1) - f(t_1)| \geq \varepsilon$

$\delta = \frac{1}{2}$ $\quad x_2, t_2$ $\quad |x_2 - t_2| < \delta = \frac{1}{2}$ $\quad |f(x_2) - f(t_2)| \geq \varepsilon$

$\vdots$

$\delta = \frac{1}{n}$ $\quad x_n, t_n$ $\quad |x_n - t_n| < \delta = \frac{1}{n}$ $\quad |f(x_n) - f(t_n)| \geq \varepsilon$.

By Bolzano-Weierstrass theorem, $\exists x_{n_j} \to w \in [a,b]$. Then

$|t_{n_j} - w| \leq |t_{n_j} - x_{n_j}| + |x_{n_j} - w| < \frac{1}{n_j} + |x_{n_j} - w| \to 0$ as $j \to \infty$.

So $t_{n_j} \to w$. Then f continuous implies

$$0 = |f(w) - f(w)| = \lim_{j \to \infty} |f(x_{n_j}) - f(t_{n_j})| \geq \varepsilon$$ contradicts $\varepsilon > 0$.

<u>Theorem</u> If $f: [a,b] \to \mathbb{R}$ is continuous, then it is integrable.

<u>Proof</u>. We check the integral criterion. $\forall \varepsilon > 0$, since f is uniformly continuous, $\exists \delta > 0$ $\forall x, t \in [a,b]$,

$$|x - t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon/(b-a).$$

Let P be a partition of $[a,b]$ such that $\max_{1 \leq j \leq n} |x_j - x_{j-1}| < \delta$

$a \underbrace{\text{|||||||||||}}_{\text{all } < \delta} b$ On $[x_{j-1}, x_j]$, <u>by extreme value theorem</u>, $\exists w_j, u_j \in [x_{j-1}, x_j]$ such that
use f continuous here also

$f(w_j) = \sup\{f(x) : x \in [x_{j-1}, x_j]\} = M_j$ and $f(u_j) = m_j$.

Then

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} (M_j - m_j) \Delta x_j = \sum_{j=1}^{n} (f(w_j) - f(u_j)) \Delta x_j$$

$$< \sum_{j=1}^{n} \frac{\varepsilon}{b-a} \Delta x_j = \frac{\varepsilon}{b-a} \sum_{j=1}^{n} \Delta x_j = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

$\therefore$ by integral criterion, f is integrable on $[a,b]$.

<u>Remarks</u> (Exercise) If $f: [a,b] \to \mathbb{R}$ is continuous except at finitely many $c_1, c_2, \ldots, c_n \in [a,b]$, then f is integrable on $[a,b], [a,c_1], \ldots, [c_i, c_{i+1}], \ldots, [c_n, b]$ and

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots + \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^{b} f(x)\,dx.$$

Questions How bad can an integrable function be discontinuous? Which functions are integrable?

Answer f integrable on $[a,b] \Leftrightarrow S_f = \{x \in [a,b] : f \text{ discontinuous at } x\}$ is a zero-length set

Questions What is a zero-length set? Which sets are zero-length?

Definitions ① A set $S \subseteq \mathbb{R}$ is of measure 0 (or has zero-length or is a null set) iff $\forall \varepsilon > 0$,
$\exists$ intervals $(a_1,b_1), (a_2,b_2), (a_3,b_3), \cdots$ such that
$S \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$ and $\sum_{n=1}^{\infty} |a_n - b_n| < \varepsilon$.

a.e.

② A property is said to hold almost everywhere (or almost surely) iff the property holds except on a set of measure 0.

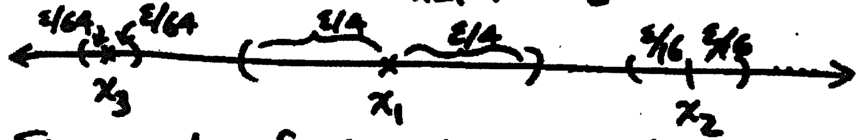
a.s.

Lebesgue's Theorem (1902)
Let $f : [a,b] \to \mathbb{R}$ be a bounded function.
f is integrable on $[a,b] \Leftrightarrow f$ is continuous a.e. on $[a,b]$
(that means f is continuous on $[a,b]$ except on a set of measure 0).

Remarks So all we need to check is that
$S_f = \{x \in [a,b] : f \text{ is discontinuous at } x\}$
is of measure 0.

Examples ① Empty set $\emptyset$ is of measure 0 because $\emptyset \subseteq \bigcup_{n=1}^{\infty} (0,0)$ and $\sum_{n=1}^{\infty} |0-0| = 0 < \varepsilon$. So Lebesgue's theorem implies every continuous function on $[a,b]$ is integrable.

② A countable set $\{x_1, x_2, \dots\}$ is of measure 0 because $\{x_1, x_2, \dots\} \subseteq \bigcup_{n=1}^{\infty} \left(x_n - \frac{\varepsilon}{4^n}, x_n + \frac{\varepsilon}{4^n}\right)$ and
$\sum_{n=1}^{\infty} \left|\left(x_n - \frac{\varepsilon}{4^n}\right) - \left(x_n + \frac{\varepsilon}{4^n}\right)\right| = \sum_{n=1}^{\infty} \frac{2\varepsilon}{4^n} = \frac{2\varepsilon}{3} < \varepsilon$.



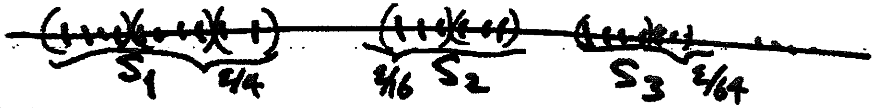
Since montone functions have countably many jumps by the monotone function theorem, so Lebesgue's theorem implies monotone functions are integrable on $[a,b]$.

③ Uncountable sets may or may not be of measure 0. The Cantor set is uncountable (by exercise 29) and is of measure 0. At stage n, there are 2^n subintervals of $[0,1]$, each of length $\frac{1}{3^n}$. So $\lim_{n\to\infty} 2^n\left(\frac{1}{3^n}\right) = 0$.

For $a<b$, $[a,b]$ is uncountable, but its length $b-a>0$.
Since $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ is discontinuous everywhere on $[a,b]$, $S_f = [a,b]$, which is not of measure 0, so Lebesgue's theorem implies f is not integrable on $[a,b]$.

④ A countable union of sets of measure 0 is of measure 0.

To see this, let $S_1, S_2, S_3, \dots$ be sets of measure 0 and $S = \bigcup_{n=1}^{\infty} S_n$. Use the idea in example 2 !



$\forall \varepsilon > 0$, since S_n is of measure 0 and $\frac{\varepsilon}{4^n} > 0$, by the definition of measure 0, $\exists$ open intervals $(a_{n,1}, b_{n,1})$, $(a_{n,2}, b_{n,2}), (a_{n,3}, b_{n,3}), \dots$ such that

$$S_n \subseteq \bigcup_{i=1}^{\infty} (a_{n,i}, b_{n,i}) \text{ and } \sum_{i=1}^{\infty} |a_{n,i} - b_{n,i}| \le \frac{\varepsilon}{4^n}.$$

Then $S \subseteq \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} (a_{n,i}, b_{n,i})$ and $\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} |a_{n,i} - b_{n,i}| \le \sum_{n=1}^{\infty} \frac{\varepsilon}{4^n} = \varepsilon/3 < \varepsilon$

$\therefore$ S is of measure 0.

⑤ If S is of measure 0 and $S' \subseteq S$, then S' is of measure 0.

To see this, $\forall \varepsilon > 0$, since S is of measure 0, $\exists$ open interval $(a_1, b_1), (a_2, b_2), (a_3, b_3), \dots$ such that

$$S \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n) \text{ and } \sum_{n=1}^{\infty} |a_n - b_n| < \varepsilon$$

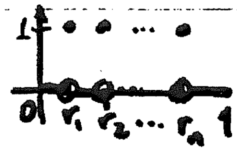
Since $S' \subseteq S$, $S' \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$ and $\sum_{n=1}^{\infty} |a_n - b_n| < \varepsilon$.

$\therefore$ S' is of measure 0.

⑥ The limit of a sequence of Riemann integrable functions on $[a,b]$ may not be Riemann integrable on $[a,b]$.

To see this, note $\mathbb{Q} \cap [0,1]$ is countable. So we can arrange its elements as $r_1, r_2, r_3, \dots$ without repetition and without omission. Define

$$f_n(x) = \begin{cases} 1 & \text{if } x = r_1, r_2, \dots \text{ or } r_n \\ 0 & \text{otherwise} \end{cases}$$



$f_n(x)$ is discontinuous only at $r_1, r_2, \dots, r_n$. $S_{f_n} = \{r_1, r_2, \dots, r_n\}$ is countable, hence S_{f_n} is of measure 0. By Lebesgue's theorem, f_n is Riemann integrable on $[0,1]$. Next,

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = r_1, r_2, r_3, \dots \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{if } x \notin \mathbb{Q} \cap [0,1], \end{cases}$$

which is not Riemann integrable on $[0,1]$.

We will postpone the proof of Lebesgue's Theorem. Here we will use it to prove some basic facts.

Theorem For $c \in [a,b]$, f is integrable on $[a,b]$ $\Longleftrightarrow$ f is integrable on $[a,c]$ and on $[c,b]$.

Proof. Note f bounded on $[a,b]$ $\Leftrightarrow$ f bounded on $[a,c], [c,b]$.

Let S, S_1, S_2 be the sets of discontinuous points of f on $[a,b]$, $[a,c]$, $[c,b]$, respectively. Note $S_1, S_2 \subseteq S$.

$(\Rightarrow)$ f integrable on $[a,b]$ $\underset{\text{by Lebesgue}}{\Longleftrightarrow}$ S is of measure 0 $\underset{\text{by example 5}}{\Longrightarrow}$ S_1, S_2 are of measure 0 $\underset{\text{by Lebesgue}}{\Longleftrightarrow}$ f is integrable on $[a,c], [c,b]$

$(\Leftarrow)$ Note $S \subseteq S_1 \cup S_2$. Since S_1 and S_2 are of measure 0, by examples 4 and 5, S is of measure 0. $\therefore$ f is integrable on $[a,b]$.

Theorem If $f, g : [a,b] \to \mathbb{R}$ are integrable on $[a,b]$, then $f+g$, $f-g$, fg are also integrable on $[a,b]$.

Proof. f,g integrable on $[a,b]$ $\Rightarrow$ f,g bounded on $[a,b]$ $\Rightarrow$ $f+g, f-g, fg$ are bounded on $[a,b]$.

Next, note that if f,g are continuous at x, then $f+g$ is also continuous at x. Taking contrapositive, if $f+g$ is discontinuous at x, then f or g is discontinuous at x.

So $x \in S_{f+g} \Rightarrow x \in S_f \cup S_g$. $\therefore S_{f+g} \subseteq S_f \cup S_g$

f,g integrable on $[a,b]$ $\Leftrightarrow$ S_f, S_g are of measure 0 $\Rightarrow$ S_{f+g} is of measure 0 $\Leftrightarrow$ $f+g$ is integrable on $[a,b]$.

Similarly, $f-g$, fg are integrable on $[a,b]$.

Theorem If $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ and g is bounded and continuous on $f([a,b])$, then $g \circ f$ is integrable on $[a,b]$. (In particular, taking $g(x) = |x|, x^2, e^x, \cos x, \ldots$ respectively, we see f integrable on $[a,b]$ $\Rightarrow$ $|f|, f^2, e^f, \cos f$ integrable on $[a,b]$.)

Proof. Note g bounded on $f([a,b])$ implies $g \circ f$ is bounded on $[a,b]$.

Note g is continuous on $f([a,b])$. So if f is continuous at $x \in [a,b]$, then $g \circ f$ is continuous at $x \in [a,b]$.

Taking contrapositive, we see that $S_{g \circ f} \subseteq S_f$.

f integrable on $[a,b]$ $\Leftrightarrow$ S_f is of measure 0 $\Rightarrow$ $S_{g\circ f}$ is of measure 0 $\Leftrightarrow$ $g \circ f$ is integrable on $[a,b]$.

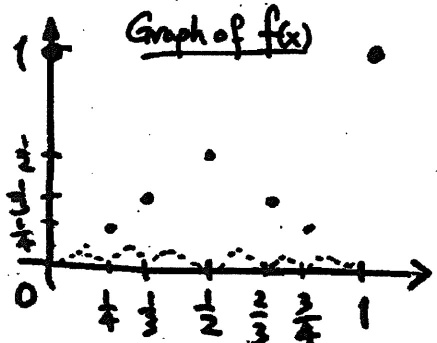
Remarks Even if $f : [a,b] \to [c,d]$ is integrable on $[a,b]$ and $g : [c,d] \to \mathbb{R}$ is integrable on $[c,d]$, $g \circ f$ may <u>not</u> be integrable on $[a,b]$. Here is an example.

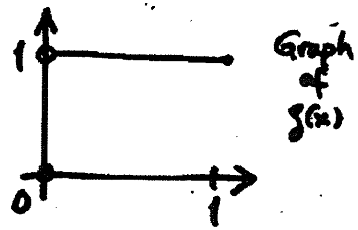
Define $f : [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{m}{n} \in (0,1] \\ & m,n \text{ positive integers with no common prime factor} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \\ 1 & \text{if } x = 0 \end{cases}$$

and define $g : [0,1] \to [0,1]$ by

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in (0,1] \end{cases}.$$





<u>Exercise</u>: $S_f = [0,1] \cap \mathbb{Q}$ $\quad$ $S_g = \{0\}$

Countable $\Rightarrow$ measure 0

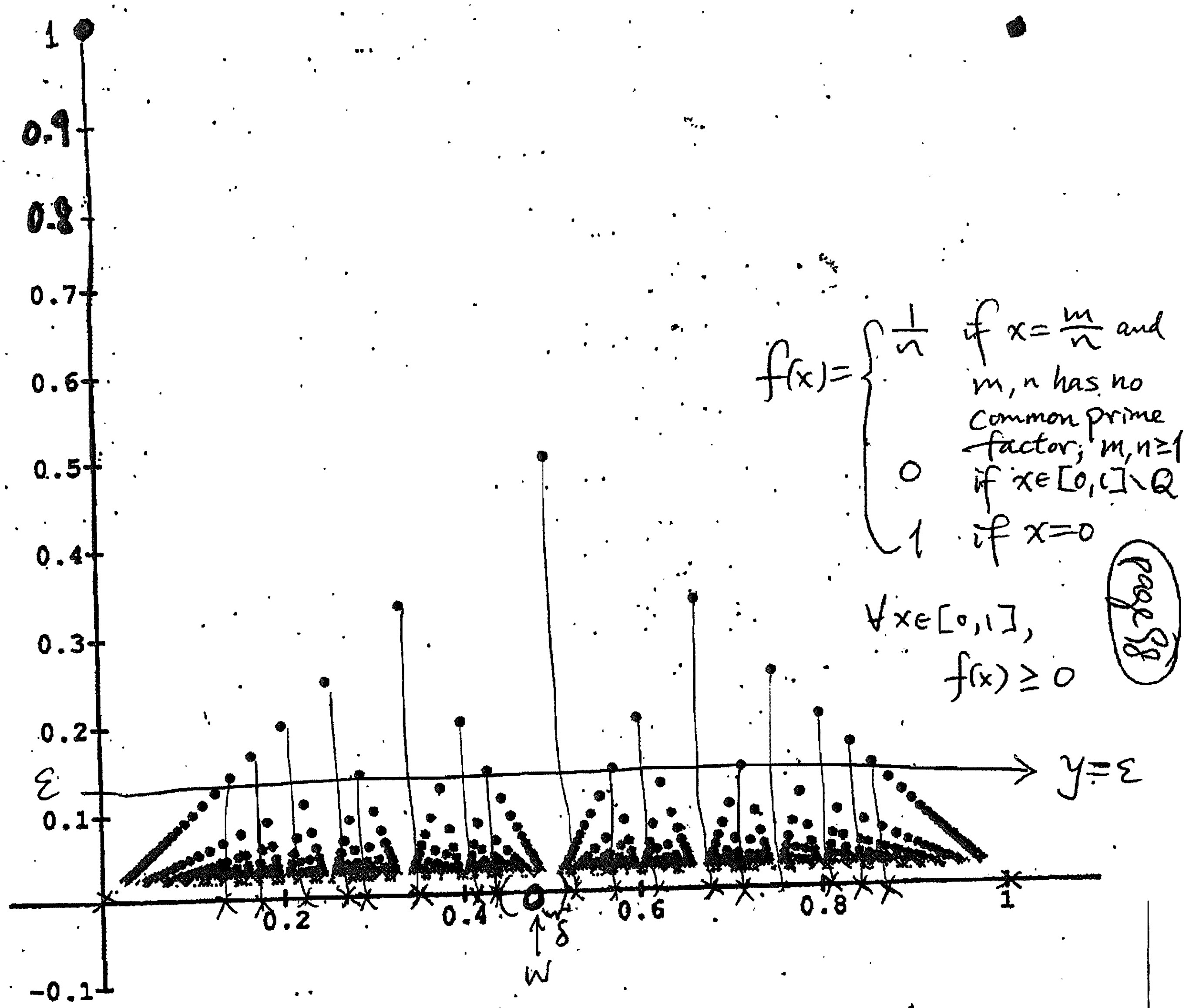
By Lebesgue's theorem, f,g are integrable on $[0,1]$.

However,

$$(g \circ f)(x) = \begin{cases} 1 & \text{if } x = \frac{m}{n} \text{ or } 0 \in [0,1] \cap \mathbb{Q} \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

is <u>not</u> integrable on $[0,1]$.

Graph of f(x)



$$f(x)=\begin{cases}\frac{1}{n} & \text{if } x=\frac{m}{n} \text{ and } m,n \text{ has no common prime factor; } m,n\geq 1\\ 0 & \text{if } x\in[0,1]\setminus\mathbb{Q}\\ 1 & \text{if } x=0\end{cases}$$

$\forall x\in[0,1]$, $f(x)\geq 0$

Exercise: $S_f=[0,1]\cap\mathbb{Q}$ (i.e. f discontinuous at only rational numbers on $[0,1]$)

Solution $\boxed{\text{We will show for every } w\in[0,1],\ \lim_{x\to w} f(x)=0.}$
By definition, we have to show $\forall\varepsilon>0,\ \exists\delta>0$ such that $0<|x-w|<\delta$ and $x\in[0,1]$ imply $\underbrace{|f(x)-0|}_{=f(x)}<\varepsilon$.

For every $\varepsilon>0$, we have $\frac{1}{N+1}<\varepsilon\leq\frac{1}{N}$, where $N=\left[\frac{1}{\varepsilon}\right]$ ← greatest integer $\leq 1/\varepsilon$.
Let $S=\{0,1,\frac{1}{2},\frac{1}{3},\frac{2}{3},\ldots,\frac{1}{N},\frac{2}{N},\ldots,\frac{N-1}{N}\}$. Let $\delta=\min\{|x-w|: x\in S, x\neq w\}$
Then S finite $\Rightarrow\delta>0$. Now $0<|x-w|<\delta\Rightarrow x\notin S\Rightarrow f(x)\neq 1,\frac{1}{2},\ldots,\frac{1}{N}$
$\Rightarrow |f(x)-0|=f(x)\leq\frac{1}{N+1}<\varepsilon$. $\therefore \underbrace{\lim_{x\to w}f(x)=f(w)}_{f \text{ continuous at } w}\Leftrightarrow f(w)=0\Leftrightarrow w\in[0,1]\setminus\mathbb{Q}$.

Simple Properties of Riemann Integrals

Let f and g be integrable on $[a,b]$.

① $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

$\forall c \in \mathbb{R}$, $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

Proof. Recall f integrable means

$$\int_a^b f(x)dx = \sup\{L(f,P) : P \text{ partition of } [a,b]\}$$
$$= \inf\{U(f,P) : P \text{ partition of } [a,b]\}.$$

So by supremum property and infimum property, $\forall \varepsilon > 0$, $\exists P_1, P_2, P_3, P_4$ such that

$$\int_a^b f(x)dx - \varepsilon/2 < L(f,P_1) \le \int_a^b f(x)dx$$
$$\int_a^b g(x)dx - \varepsilon/2 < L(g,P_2) \le \int_a^b g(x)dx$$
$$\int_a^b f(x)dx \le U(f,P_3) < \int_a^b f(x)dx + \frac{\varepsilon}{2}$$
$$\int_a^b g(x)dx \le U(g,P_4) < \int_a^b g(x)dx + \frac{\varepsilon}{2}.$$

Then for the common refinement $P = P_1 \cup P_2 \cup P_3 \cup P_4$.

$$\int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon < L(f,P) + L(g,P)$$
$$\le L(f+g,P) \le \int_a^b (f(x)+g(x))dx \le U(f+g,P)$$
$$< U(f,P) + U(g,P) < \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

$U(f+g,P) < U(f,P) + U(g,P)$ is similar to $L(f,P) + L(g,P) \le L(f+g,P)$

Next $\int_a^b -f(x)dx = \inf\{U(-f,P) : P \text{ partition of } [a,b]\}$

$-\sup S = \inf(-S)$ $\quad \sup S \quad 0 \quad -S \quad S$

$$= \inf\{-L(f,P) : P \text{ partition of } [a,b]\}$$
$$= -\sup\{L(f,P) : P \text{ partition of } [a,b]\}$$
$$= -\int_a^b f(x)dx.$$

So $\int_a^b (f(x) - g(x))dx = \int_a^b (f(x) + (-g(x)))dx$

$$= \int_a^b f(x)dx + \int_a^b -g(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

For $\int_a^b cf(x)dx = c\int_a^b f(x)dx$,

Case $c=0$: $\int_a^b 0 f(x)dx = \int_a^b 0\, dx = 0 = 0 \cdot \int_a^b f(x)dx$.

Case $c>0$: $\int_a^b cf(x)dx = \sup\{L(cf,P) : P \text{ partition of } [a,b]\}$

$$= \sup\{c\,L(f,P) : \text{" " " "}\}$$
$$= c\sup\{L(f,P) : \text{" " " "}\}$$
$$= c\int_a^b f(x)dx.$$

Case $c<0$: $\int_a^b cf(x)dx = \int_a^b -(-cf(x))dx$

$$= -\int_a^b -cf(x)dx$$

$-c>0 \rightarrow$ $= -(-c)\int_a^b f(x)dx$

$$= c\int_a^b f(x)dx.$$

$L(f,P) + L(g,P) \le L(f+g,P)$

Proof. $L(f,P) = \sum_{i=1}^n m_i \Delta x_i$, $L(g,P) = \sum_{i=1}^n n_i \Delta x_i$, $L(f+g,P) = \sum_{i=1}^n k_i \Delta x_i$

where $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$, $k_i = \inf\underbrace{\{f(x)+g(x) : x \in [x_{i-1}, x_i]\}}_{\text{Call this set } T}$

$n_i = \inf\{g(x) : x \in [x_{i-1}, x_i]\}$

$m_i + n_i \le f(x) + g(x)$ for all $x \in [x_{i-1}, x_i]$

$\Rightarrow m_i + n_i$ is a lower bound of $T \Rightarrow m_i + n_i \le \underbrace{\inf}_{\text{greatest lower bound}} T = k_i$

$\Rightarrow \sum m_i \Delta x_i + \sum n_i \Delta x_i \le \sum k_i \Delta x_i$.

② If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$.
Also, $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$.

Proof. We have $g-f \ge 0$ on $[a,b]$, which implies $L(g-f,P) \ge 0$ $\forall$ partition P of $[a,b]$. So

$$\int_a^b (g(x)-f(x))dx = \sup\{L(g-f,P) : P \text{ partition of } [a,b]\} \ge 0$$

$\therefore$ $\int_a^b g(x)dx - \int_a^b f(x) \ge 0$, i.e. $\int_a^b f(x)dx \le \int_a^b g(x)dx$.

Next $-|f| \le f \le |f|$ on $[a,b]$. So
$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$, which is the same as $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$.

---

③ $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for $c \in [a,b]$.

Proof. $\forall$ partition P of $[a,b]$, let $P' = P \cup \{c\}$, $P_1 = P' \cap [a,c]$ and $P_2 = P' \cap [c,b]$. Then $P \subseteq P'$, P_1 is a partition of $[a,c]$ and P_2 is a partition of $[c,b]$.

Let $A = \{L(f,P) : P \text{ partition of } [a,b]\}$
and $B = \{L(f,P') : P \text{ partition of } [a,b] \text{ and } P' = P \cup \{c\}\}$

P' is also partition of $[a,b] \Rightarrow B \subseteq A \Rightarrow \sup B \le \sup A$

$P \subseteq P' \Rightarrow L(f,P) \le L(f,P') \Rightarrow \sup A \le \sup B$
(refinement theorem) $\therefore \sup A = \sup B$.

$$\begin{aligned}
\int_a^b f(x)\,dx &= \sup A = \sup B \\
&= \sup\left\{ L(f,P_1) + L(f,P_2) : \begin{array}{l} P_1 \text{ partition of } [a,c] \\ P_2 \text{ partition of } [c,b] \end{array} \right\} \\
&= \sup\{L(f,P_1) : P_1 \text{ partition of } [a,c]\} \\
&\quad + \sup\{L(f,P_2) : P_2 \text{ partition of } [c,b]\} \\
&= \int_a^c f(x)dx + \int_c^b f(x)dx.
\end{aligned}$$

Definition For an integrable function $f(x)$ on $[a,b]$ and $c \in [a,b]$, the function $F(x) = \int_c^x f(t)dt$ is called an antiderivative (or a primitive function) of f.

Example For $x \in [-1,1]$, define $f(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0. \end{cases}$

f is discontinuous at 0.
$S_f = \{0\} \Rightarrow f$ is integrable on $[-1,1]$.

$$F(x) = \int_0^x f(t)dt = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases} = |x|$$

is continuous on $[-1,1]$, but not differentiable at 0.
(hence uniformly continuous on $[-1,1]$ by the uniform continuity theorem)

Theorem If f is integrable on $[a,b]$ and $c \in [a,b]$, then $F(x) = \int_c^x f(t)dt$ is uniformly continuous on $[a,b]$.

Proof. f integrable $\Rightarrow f$ bounded $\Rightarrow \exists K > 0$ $|f(x)| \le K$ on $[a,b]$.
$\forall \varepsilon > 0$, let $\delta = \varepsilon/K$, then
$|x-w| < \delta \Rightarrow |F(x)-F(w)| = \left|\int_w^x f(t)dt\right| \le K|x-w| < K\delta = \varepsilon$.

Fundamental Theorem of Calculus

Let $c, x_0 \in [a,b]$.

(1) If f is integrable on $[a,b]$, continuous at x_0 and $F(x) = \int_c^x f(t)\,dt$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$. $\quad \left(\frac{d}{dx}\int_c^x f(t)\,dt\right)(x_0) = f(x_0)$

Proof. f cont. at $x_0 \Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x \in [a,b],\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

then
$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| = \left|\frac{\int_{x_0}^x f(t)\,dt}{x - x_0} - \frac{\int_{x_0}^x f(x_0)\,dt}{x - x_0}\right|$$
$$\leq \frac{1}{|x - x_0|}\left|\int_{x_0}^x (f(t) - f(x_0))\,dt\right| < \frac{1}{|x - x_0|}\varepsilon |x - x_0| = \varepsilon.$$

By definition of limit, $\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$.

$\therefore F'(x_0) = f(x_0)$.

(2) If G is differentiable on $[a,b]$ and G' integrable on $[a,b]$, then $\int_a^b G'(x)\,dx = G(b) - G(a)$.

(Note G' may not be continuous.) $\int_a^b \frac{d}{dx}G(x)\,dx = G\Big|_a^b$

Proof. $\forall \varepsilon > 0$, by integral criterion, $\exists$ partition P of $[a,b]$ such that $U(G', P) - L(G', P) < \varepsilon$.

$a = x_0 \quad x_1 \quad x_2 \quad \cdots \quad b = x_n$

By mean value theorem, $\exists\, t_j \in [x_{j-1}, x_j]$ such that $G(x_j) - G(x_{j-1}) = G'(t_j)(x_j - x_{j-1})$. Then
$$L(G', P) \leq \sum_{j=1}^n G'(t_j)(x_j - x_{j-1}) = G(b) - G(a) \leq U(G', P).$$

$L(G',P) \quad \int_a^b G'(x)\,dx \quad G(b) - G(a) \quad U(G',P)$

$\therefore \left|\int_a^b G'(x)\,dx - (G(b) - G(a))\right| < \varepsilon$. Let $\varepsilon \to 0$.

Integration by Parts

If f, g are differentiable on $[a,b]$ and f', g' are integrable on $[a,b]$, then $\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx$.

Proof. $\int_a^b (fg)'(x)\,dx = f(b)g(b) - f(a)g(a)$
$$\int_a^b (f(x)g'(x) + f'(x)g(x))\,dx$$

Subtracting $\int_a^b f'(x)g(x)\,dx$ from both sides, we get formula.

Change of Variable Formula

If $\phi : [a,b] \to \mathbb{R}$ is differentiable, ϕ' integrable on $[a,b]$ and f continuous on $\phi([a,b])$, then
$$\int_{\phi(a)}^{\phi(b)} f(t)\,dt = \int_a^b f(\phi(x))\,\phi'(x)\,dx.$$

Proof. Let $g(x) = \int_{\phi(a)}^{u = \phi(x)} f(t)\,dt$. By part (1) of the Fundamental Theorem of Calculus and Chain Rule, $g'(x) = \frac{dg}{du}\frac{du}{dx} = f(\phi(x))\,\phi'(x)$, which is integrable on $[a,b]$.

So
$$\int_a^b f(\phi(x))\,\phi'(x)\,dx = \int_a^b g'(x)\,dx = g(b) - g(a) = \int_{\phi(a)}^{\phi(b)} f(t)\,dt.$$

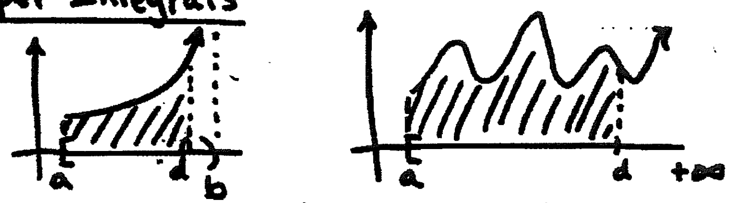
Improper Setting: f is an unbounded function or f is defined on an interval that is not closed or not bounded

Definition Let I be an interval. A function $f: I \to \mathbb{R}$ is locally integrable iff f is integrable on every closed and bounded subintervals of I. We denote this by $f \in L_{loc}(I)$.

Example. If f is continuous on an interval I, then f is locally integrable because f is continuous on every closed and bounded subinterval of I, hence integrable there.

Improper Integrals

Case 1



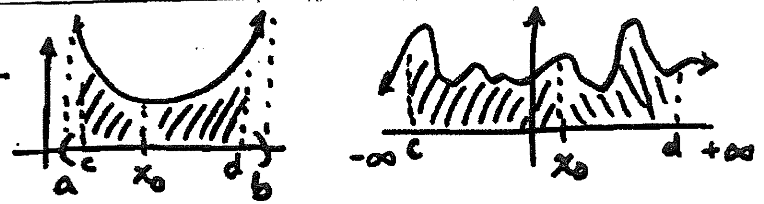
Let $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$, $I = [a,b)$, $f \in L_{loc}(I)$. The improper integral of f on $[a,b)$ is

$$\int_a^b f(x)\,dx = \lim_{d \to b^-} \int_a^d f(x)\,dx$$ provided the limit exists in $\mathbb{R}$.

In this case, we say f is improper integrable on $[a,b)$.

The case $I = (a,b]$ with $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R}$ is similarly defined: $\int_a^b f(x)\,dx = \lim_{d \to a^+} \int_d^b f(x)\,dx$.

Case 2



Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$, $I = (a,b)$, $f \in L_{loc}(I)$. The improper integral of f on (a,b) is $x_0 \in I$

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^{x_0} f(x)\,dx + \lim_{d \to b^-} \int_{x_0}^d f(x)\,dx$$

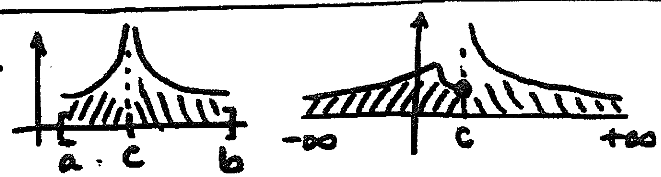
provided the limits exist in $\mathbb{R}$. In this case, we say f is improper integrable on (a,b).

Remark The answer does not depend on x_0. For another x_0', the first term

$$\lim_{c \to a^+} \int_c^{x_0'} f(x)\,dx = \lim_{c \to a^+} \int_c^{x_0} f(x)\,dx + \underbrace{\int_{x_0}^{x_0'} f(x)\,dx}_{\text{number because } f \text{ integrable on } [x_0, x_0']}$$

So left side is a number iff the right side is a number. The second term is similar.

Case 3

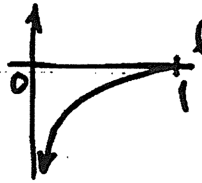


Let $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$, I be an interval with endpoints a, b, $I_0 = I \cap (-\infty, c)$, $I_1 = I \cap (c, +\infty)$ for $c \in (a,b)$, $f \in L_{loc}(I_0)$, $f \in L_{loc}(I_1)$. The improper integral of f on I is

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$ provided both integrals are numbers. In that case, f is improper integrable on I.

In each case, if the improper integral is a number, then we say the improper integral <u>converges</u>, otherwise we say it <u>diverges</u>.

<u>Examples</u> ① Consider $\int_0^1 \ln x \, dx$.



$\ln x$ continuous on $(0,1] \Rightarrow \ln x \in L_{loc}((0,1])$

$$\int_0^1 \ln x\,dx = \lim_{c \to 0^+} \int_c^1 \ln x\,dx$$
$$= \lim_{c\to 0^+} (x\ln x - x)\Big|_c^1 \quad \Big\}\text{ integration by parts}$$
$$= \lim_{c\to 0^+} (-1 - c\ln c + c) = -1$$

$(\ln c)/(\frac{1}{c}) \to 0$ by l'Hopital's rule

$\ln x$ improper integrable on $(0,1]$.

$\therefore \int_0^1 \ln x\,dx$ converges to -1.

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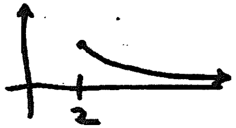
② Consider $\int_2^{+\infty} \frac{1}{x^2}dx$.



$\frac{1}{x^2}$ continuous on $[2,+\infty) \Rightarrow \frac{1}{x^2} \in L_{loc}([2,+\infty))$

$$\int_2^{+\infty} \frac{1}{x^2}dx = \lim_{d\to+\infty} \int_2^d \frac{1}{x^2}dx$$
$$= \lim_{d\to+\infty} -\frac{1}{x}\Big|_2^d = \lim_{d\to+\infty}\left(-\frac{1}{d}+\frac{1}{2}\right) = \frac{1}{2}$$

$\therefore \int_2^{+\infty} \frac{1}{x^2}dx$ converges to $\frac{1}{2}$.

---

③ Consider $\int_{-\infty}^{+\infty} e^x dx$. $\quad e^x \in L_{loc}((-\infty,+\infty))$.

Take $x_0 = 0$. $\int_0^{+\infty} e^x dx = \lim_{d\to+\infty} \int_0^d e^x dx = \lim_{d\to+\infty} e^x\Big|_0^d$

$= \lim_{d\to+\infty} (e^d - 1) = +\infty$, not a number.

$\therefore e^x$ is not improper integrable on $(-\infty,+\infty)$.

$\int_{-\infty}^{\infty} e^x dx$ diverges.

<u>Question</u> What if the improper integral cannot be computed?

<u>P-test</u> For $0<a<\infty$, $\int_a^{+\infty} \frac{1}{x^p}dx < \infty \Leftrightarrow p>1$.

Also, $\int_0^a \frac{1}{x^p}dx < \infty \Leftrightarrow p<1$.

<u>Comparison Test</u> Suppose $0 \le f(x) \le g(x)$ on interval I and $f, g \in L_{loc}(I)$. If g is improper integrable on I, then f is improper integrable on I. (Taking contrapositive, if f is not improper integrable on I, then g is not improper integrable on I.)

<u>Limit Comparison Test</u> Suppose $f(x), g(x) > 0$ on $(a,b]$ and $f, g \in L_{loc}((a,b])$.

If $\lim_{x\to a^+} \frac{g(x)}{f(x)}$ is a positive number L, then either (both $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converges) or (both diverges).

If $\lim_{x\to a^+}\frac{g(x)}{f(x)} = 0$, then $\left(\int_a^b f(x)dx \text{ converges} \Rightarrow \int_a^b g(x)dx \text{ converges}\right)$.

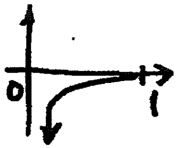
If $\lim_{x\to a^+}\frac{g(x)}{f(x)} = +\infty$, then $\left(\int_a^b f(x)dx \text{ diverges} \Rightarrow \int_a^b g(x)dx \text{ diverges}\right)$.

In the case $[a,b)$, we take $\lim_{x\to b^-}\frac{g(x)}{f(x)}$. Results are similar.

<u>Absolute Convergence Test</u> Let $f \in L_{loc}(I)$.

If $|f|$ is improper integrable on I, then f is improper integrable on I.

Examples ④ Consider $\int_0^1 \frac{\ln x}{1+x^2}dx$. $\frac{\ln x}{1+x^2}\in L_{loc}((0,1])$.



On $(0,1]$, $\left|\frac{\ln x}{1+x^2}\right| = \frac{|\ln x|}{1+x^2} \le |\ln x| = -\ln x$

By example ①, $\int_0^1 |\ln x|\,dx = -\int_0^1 \ln x\,dx = 1$.

By comparison test, $\int_0^1 \left|\frac{\ln x}{1+x^2}\right| dx$ converges.

By absolute convergence test, $\int_0^1 \frac{\ln x}{1+x^2}dx$ converges.

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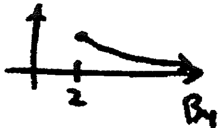
⑤ Consider $\int_2^{+\infty}\frac{dx}{\sqrt{x^2-1}}dx$.



On $[2,+\infty)$, $0<\frac{1}{x}<\frac{1}{\sqrt{x^2-1}}$. $\int_2^{+\infty}\frac{1}{x}dx=\infty$ by p-test

By comparison test, $\int_2^{+\infty}\frac{dx}{\sqrt{x^2-1}}$ diverges.

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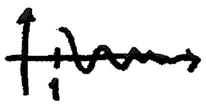
⑥ Consider $\int_1^{+\infty}\frac{\sin x}{x}dx$.



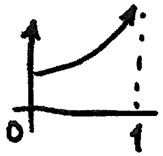
$$\int_1^c \frac{\sin x}{x}dx = \frac{-\cos x}{x}\Big|_1^c - \int_1^c\frac{\cos x}{x^2}dx = \frac{-\cos c}{c}+\cos 1 - \int_1^c \frac{\cos x}{x^2}dx$$

Since $|\cos c|\le 1$, $\lim_{c\to+\infty}\frac{-\cos c}{c}=0$.

On $[1,+\infty)$, $\left|\frac{\cos x}{x^2}\right|\le\frac{1}{x^2}$ and $\int_1^{+\infty}\frac{1}{x^2}dx<\infty$ by p-test.

By comparison test, $\int_1^{+\infty}\left|\frac{\cos x}{x^2}\right|dx<\infty$. By absolute convergence test, $\int_1^{+\infty}\frac{\cos x}{x^2}dx$ converges. $\therefore \int_1^{+\infty}\frac{\sin x}{x}dx$ converges.

⑦ Consider $\int_0^1\frac{dx}{1-x^3}$.



$\frac{1}{1-x^3}=\frac{1}{1-x}\left(\frac{1}{1+x+x^2}\right)$, $\lim_{x\to1}\frac{1}{1+x+x^2}=\frac{1}{3}$

So $\lim_{x\to1^-}\frac{\frac{1}{1-x^3}}{\frac{1}{3(1-x)}}=1$.

$$\int_0^1\frac{1}{3(1-x)}dx=\lim_{d\to1^-}\int_0^d\frac{1}{3(1-x)}dx=\lim_{d\to1^-}-\frac{1}{3}\ln(1-d)=+\infty.$$

By limit comparison test, $\int_0^1\frac{1}{1-x^3}dx$ diverges.

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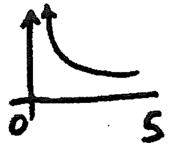
⑧ Consider $\int_0^5\frac{dx}{\sqrt[3]{7x+2x^4}}$.



$\frac{1}{\sqrt[3]{7x+2x^4}}=\frac{1}{\sqrt[3]{x}}\left(\frac{1}{\sqrt[3]{7+2x^3}}\right)$, $\lim_{x\to0^+}\frac{1}{\sqrt[3]{7+2x^3}}=\frac{1}{\sqrt[3]{7}}$

So $\lim_{x\to0^+}\left(\frac{1}{\sqrt[3]{7x+2x^4}}\right)/\frac{1}{\sqrt[3]{7x}}=1$.

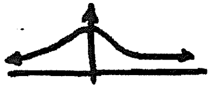
$\int_0^5\frac{1}{\sqrt[3]{7x}}dx=\frac{1}{\sqrt[3]{7}}\int_0^5\frac{1}{x^{1/3}}dx<\infty$ by p-test.

By limit comparison test, $\int_0^5\frac{dx}{\sqrt[3]{7x+2x^4}}$ converges.

Cauchy Principal Value of Integrals

Definition. Let $f \in L_{loc}(\mathbb{R})$. The principal value of $\int_{-\infty}^{\infty} f(x)dx$ is $P.V.\int_{-\infty}^{\infty} f(x)dx = \lim_{c\to+\infty} \int_{-c}^{c} f(x)dx$.

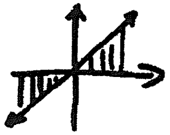
Examples ① Consider $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$ and $P.V.\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$.



$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \int_{-\infty}^{0}\frac{1}{1+x^2}dx + \int_{0}^{+\infty}\frac{1}{1+x^2}dx$$
$$= \lim_{c\to-\infty}\int_{c}^{0}\frac{1}{1+x^2}dx + \lim_{d\to+\infty}\int_{0}^{d}\frac{1}{1+x^2}dx = \lim_{c\to-\infty}(-\arctan c) + \lim_{d\to+\infty}(\arctan d)$$
$$= \frac{\pi}{2}+\frac{\pi}{2} = \pi.$$

$$P.V.\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \lim_{c\to+\infty}\int_{-c}^{c}\frac{1}{1+x^2}dx = \lim_{c\to+\infty}(2\arctan c) = 2\cdot\frac{\pi}{2} = \pi.$$

② Consider $\int_{-\infty}^{\infty} x\,dx$ and $P.V.\int_{-\infty}^{\infty} x\,dx$.



$$\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{+\infty} x\,dx = \lim_{c\to-\infty}\left(-\frac{c^2}{2}\right) + \lim_{d\to+\infty}\left(\frac{d^2}{2}\right)$$
$$= -\infty + \infty, \text{ not exist.}$$

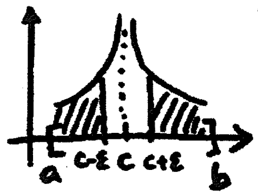
$$P.V.\int_{-\infty}^{\infty} x\,dx = \lim_{c\to+\infty}\int_{-c}^{c} x\,dx = \lim_{c\to+\infty}\frac{x^2}{2}\Big|_{-c}^{c} = \lim_{c\to+\infty} 0 = 0.$$

So $\int_{-\infty}^{\infty} f(x)dx$ and $P.V.\int_{-\infty}^{\infty} f(x)dx$ may be different.

Theorem If the improper integral $\int_{-\infty}^{\infty} f(x)dx$ exists in $\mathbb{R}$, then $P.V.\int_{-\infty}^{\infty} f(x)dx$ exists and equals the improper integral $\int_{-\infty}^{\infty} f(x)dx$. The converse is false by example ②.

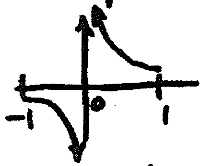
Proof. If $\int_{-\infty}^{\infty} f(x)dx$ exists, then $\lim_{d\to-\infty}\int_{d}^{0} f(x)dx$ and $\lim_{c\to+\infty}\int_{0}^{c} f(x)dx$ both exist as numbers. So

$$P.V.\int_{-\infty}^{\infty} f(x)dx = \lim_{c\to+\infty}\int_{-c}^{c} f(x)dx = \lim_{c\to+\infty}\left(\int_{-c}^{0} f(x)dx + \int_{0}^{c} f(x)dx\right)$$
$$= \lim_{d\to-\infty}\int_{d}^{0} f(x)dx + \lim_{c\to+\infty}\int_{0}^{c} f(x)dx = \int_{-\infty}^{\infty} f(x)dx.$$



Definition Let I be an interval with endpoints a and b, let $c\in(a,b)$, $I_0 = I\cap(-\infty,c)$ and $I_1 = I\cap(c,+\infty)$. Let $f\in L_{loc}(I_0)$ and $f\in L_{loc}(I_1)$. Define the principal value of $\int_a^b f(x)dx$ as $P.V.\int_a^b f(x)dx = \lim_{\varepsilon\to0^+}\left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^{b} f(x)dx\right)$.

Example Consider $\int_{-1}^{1}\frac{1}{x}dx$ and $P.V.\int_{-1}^{1}\frac{1}{x}dx$.



$$\int_0^1\frac{1}{x}dx = \lim_{c\to0^+}\int_c^1\frac{1}{x}dx = \lim_{c\to0^+}(-\ln c) = +\infty \text{ not a number.}$$

So $\int_{-1}^{1}\frac{1}{x}dx$ diverges.

$$P.V.\int_{-1}^{1}\frac{1}{x}dx = \lim_{\varepsilon\to0^+}\left(\int_{-1}^{-\varepsilon}\frac{1}{x}dx + \int_{\varepsilon}^{1}\frac{1}{x}dx\right) = \lim_{\varepsilon\to0^+}(\ln|-\varepsilon| - \ln|\varepsilon|) = 0.$$

Remarks ① $\int_{-1}^{1}\frac{1}{x}dx = \ln|x|\Big|_{-1}^{1} = \ln 1 - \ln 1 = 0$ is incorrect as the fundamental theorem of calculus requires $f(x)=\ln|x|$ differentiable on the whole interval $[-1,1]$ and $f'(x)=\frac{1}{x}$ bounded integrable!

② There is a theorem "$\int_a^b f(x)dx$ converges $\Rightarrow P.V.\int_a^b f(x)dx = \int_a^b f(x)dx$." The proof is similar to the one above.

Proof of p-test Since $\int_a^1 \frac{1}{x^p}dx<\infty$, so

integral test

$$\int_a^\infty \frac{1}{x^p}dx<\infty \iff \int_1^\infty \frac{1}{x^p}dx<\infty \iff \sum_{n=1}^\infty \frac{1}{n^p}<\infty \iff p>1.$$

$$\int_0^a \frac{1}{x^p}dx<\infty \iff \int_0^1 \frac{1}{x^p}dx<\infty$$

$$\int_c^1 \frac{1}{x^p}dx = \int_1^{1/c}\frac{1}{y^{2-p}}dy \qquad y=1/x,\quad dy=-1/x^2\,dx$$

$$\iff \int_1^\infty \frac{1}{y^{2-p}}dy<\infty$$

$$\iff 2-p>1 \iff p<1.$$

Proof of Comparison Test For the case $I=[a,b)$, if

$$\left.\begin{array}{l} 0\le f\le g \text{ on } [a,b) \\ g \text{ improper integrable on } [a,b) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \int_a^d f(x)dx \text{ is increasing when } d\nearrow b \\ \int_a^d f(x)dx \le \int_a^b g(x)dx<\infty \end{array}\right.$$

$$\xrightarrow{\text{Monotone function theorem}} \int_a^b f(x)dx = \lim_{d\to b^-}\int_a^d f(x)dx<\infty.$$

$\therefore f$ is improper integrable on $[a,b)$.

The cases $(a,b]$ and (a,b) are similar.

Proof of Limit Comparison Test On $(a,b]$, $f(x),g(x)>0$.

Case $\lim_{x\to a^+}\frac{g(x)}{f(x)}=L$ positive number. For $\varepsilon=\frac{L}{2}>0$, $\exists\delta>0$ such that

$$\forall x\in(a,a+\delta) \Rightarrow \frac{L}{2}=L-\varepsilon<\frac{g(x)}{f(x)}<L+\varepsilon=\frac{3L}{2}. \text{ Then}$$

$$\frac{L}{2}\int_a^{a+\delta}f(x)dx \le \int_a^{a+\delta}g(x)dx \le \frac{3L}{2}\int_a^{a+\delta}f(x)dx. \leftarrow \text{by comparison test.}$$

So $\int_a^{a+\delta}f(x)dx<\infty \iff \int_a^{a+\delta}g(x)dx<\infty$.

Since $f,g\in L_{loc}((a,b])$ and $[a+\delta,b]\subseteq(a,b]$, so

$\int_{a+\delta}^b f(x)dx<\infty$, $\int_{a+\delta}^b g(x)<\infty$. Therefore,

$$\int_a^b f(x)dx<\infty \iff \int_a^b g(x)dx<\infty.$$

Case $\lim_{x\to a^+}\frac{g(x)}{f(x)}=0$ For $\varepsilon=1>0$, $\exists\delta'>0$ such that

$$\forall x\in(a,a+\delta') \Rightarrow 0<\frac{g(x)}{f(x)}<1 \Rightarrow 0<g(x)<f(x). \text{ Then}$$

$$0\le\int_a^{a+\delta'}g(x)dx\le\int_a^{a+\delta'}f(x)dx. \leftarrow \text{by comparison test}$$

So $\int_a^b f(x)dx<\infty \Rightarrow \int_a^{a+\delta'}f(x)dx<\infty \Rightarrow \int_a^{a+\delta'}g(x)dx<\infty$

$\Rightarrow \int_a^b g(x)dx<\infty$.

Case $\lim_{x\to a^+}\frac{g(x)}{f(x)}=+\infty$ For $r=1$, $\exists\delta''>0$ such that

$$\forall x\in(a,a+\delta'') \Rightarrow \frac{g(x)}{f(x)}>1 \Rightarrow g(x)>f(x). \text{ Then}$$

$$\int_a^{a+\delta''}g(x)dx \ge \int_a^{a+\delta''}f(x)dx. \leftarrow \text{by comparison test}$$

So $\int_a^b f(x)dx=+\infty \Rightarrow \int_a^b g(x)dx=+\infty$.

Proof of Absolute Convergence Test

$-|f|\le f\le|f|$ on I $\Rightarrow$ $0\le f+|f|\le 2|f|$ on I

$|f|$ improper integrable on I $\Rightarrow$ $f+|f|$ improper integrable on I by comparison test

$\Rightarrow f=(f+|f|)-|f|$ improper integrable on I.