MATH2033 Mathematical Analysis

Lecture Note 4

Real number and its properties

Introduction -- A quick review on number system and the limitation of rational number

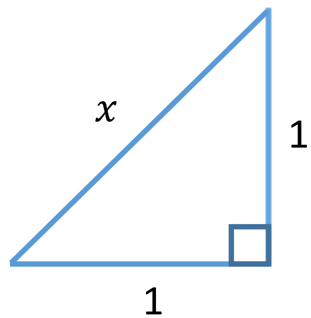
In reality, we used numbers to describe quantities.

- Positive integers $\mathbb{N} = \{1, 2, 3, \dots\}$ -- It is often used for counting and ordering.
- Integers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ – It is an extension from positive integer which includes "zero" and "negative numbers".
- Rational numbers $\mathbb{Q} = \left\{\frac{m}{n} \mid m \in \mathbb{Z} \ \ and \ \ n \in \mathbb{N}\right\}$ -- It is an extension from integer which also includes "fraction" (e.g. $\frac{1}{2}, \frac{2}{3}$ etc.)

Although the rational numbers $\mathbb{Q}$ can describe most of the quantities that we have seen in the real world, it is not a good number system for studying various concepts in real analysis which involves limits (e.g. convergence, continuity, differentiability etc.).

Scenario 1: Geometry

Suppose that we would like to find the length of hypotenuse of a right triangle as follows:



Using Pythagorean theorem, we get the length x satisfies the equation

$$x^2 = 1^2 + 1^2 = 2.$$

However, one can show that there is <u>no rational number</u> x that satisfies $x^2 = 2$.

- To see this, suppose that there is such a $x = \frac{p}{q}$ (where $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and they are relatively prime) that satisfies $x^2 = 2$.
- So we get $p^2 = 2q^2$. Since p^2 is even, p is also even. By writing $p = 2k$, we get $(2k)^2 = 2q^2 \Rightarrow q^2 = 2k^2$. So q is also even as well (as q^2 is even). This implies that p, q have common factor 2 and it leads to contradiction.

Scenario 2: Limit of a sequence of rational number

We consider a sequence of rational number $\{x_1, x_2, x_3, \dots\}$ defined by

$$x_n = \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n}, \qquad for\ n = 1,2,3, \dots.$$

(*Note 1: $\lfloor x \rfloor$ denotes greatest integer less than or equal to x.)

(*Note 2: One can observe that $x_1 = 1.4$, $x_2 = 1.41$, $x_3 = 1.414$, $x_4 = 1.4142$. So x_n denotes the approximation of $\sqrt{2}$ by keeping the first n decimal places of $\sqrt{2} = 1.414213562\dots$.)

Since $x - 1 < \lfloor x \rfloor \le x$, one can show that

$$\sqrt{2} - \frac{1}{10^n} = \frac{10^n\sqrt{2} - 1}{10^n} < x_n \le \frac{10^n\sqrt{2}}{10^n} = \sqrt{2}$$

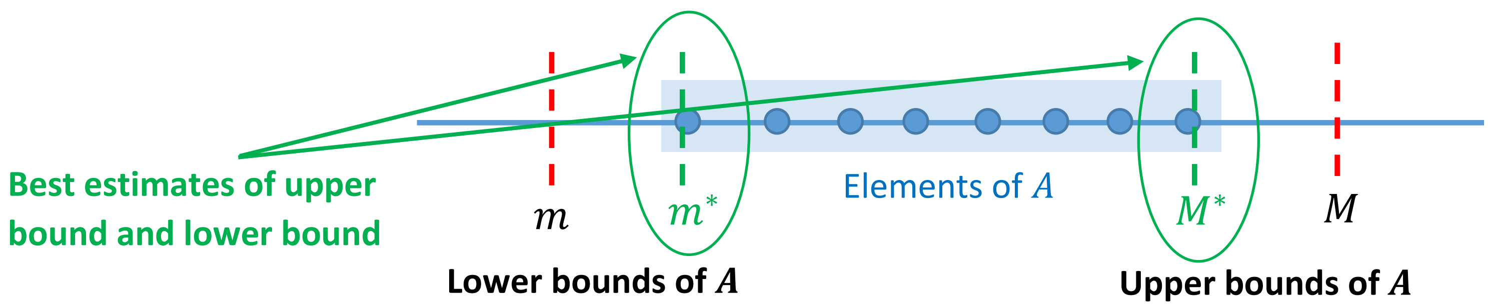
$$\Rightarrow \sqrt{2} = \lim_{n\to\infty}\left(\sqrt{2} - \frac{1}{10^n}\right) \le \lim_{n\to\infty} x_n \le \lim_{n\to\infty} \sqrt{2} = \sqrt{2}.$$

So $\lim_{n\to\infty} x_n = \sqrt{2}$ (by sandwich theorem, will be discussed in Topic 5). But it is known that $\sqrt{2}$ is not rational number (see Lecture Note 1).

Scenario 3: Estimating the range of a set – Supremum and Infimum

Given a set of numbers A, one would like to study the range of set A by studying the upper bound and lower bound of the set.

- ✓ We say M is *upper bound* of the set A if $x \le M$ for any $x \in A$
- ✓ We say m is lower bound of the set A if $x \ge m$ for any $x \in A$



- ✓ In order to estimate the range of set A accurately, we should choose the upper bound M such that it is *least upper bound* M^* among all possible upper bounds of A. So M^* is called supremum of A and is denoted by $M^* = \sup A$.
- ✓ On the other hand, we should choose the lower bound M such that it is *greatest lower bound* m^* among all possible lower bounds of A. So m^* is called infimum of A and is denoted by $m^* = \inf A$.

Definition (Supremum and Infimum)

We let A be a set which is bounded from above (i.e. upper bound exists), the supremum of A, denoted by $\sup A$, is defined as the least upper bound of A. That is,

- $\sup A$ is upper bound of A and
- For any upper bound M of A, we have $M \geq \sup A$

Suppose that A is bounded from below (i.e. lower bound exists), the infimum of A, denoted by $\inf A$, is defined as the greatest lower bound of A. That is,

- $\inf A$ is lower bound of A and
- For any lower bound m of A, we have $\inf A \geq m$.

Remark about supremum and infimum

- One can show that $M^* = \sup A$ (resp. $m^* = \inf A$) if and only if (i) M^* is upper bound (resp. lower bound) and (ii) for any $\varepsilon > 0$, there exists $x \in A$ such that $x > \sup A - \varepsilon$ (resp. $x < \inf A + \varepsilon$).
- In certain sense, supremum and infimum can represent "maximum" and "minimum" of a set. However, it is not necessary to have an element x in the set which the value equals to supremum or infimum.

Question: Can we always represent $\sup A$ *and* $\inf A$ *using rational number?*

We consider a set defined by

$$A = \{x \in \mathbb{Q} |\, x^2 < 2\}.$$

One can show that the set is bounded above. We would like to know whether the supremum $\sup A$ can be represents by a rational number. Suppose that $\sup A = p \in \mathbb{Q}$.

- If $p = \sup A \in A$ (i.e. $p^2 < 2$), we consider a rational number q defined by
$$q = p + \frac{2 - p^2}{p + 2} = \frac{2p + 2}{p + 2} > p.$$
One can show that $q^2 - 2 = \frac{(2p+2)^2}{(p+2)^2} - 2 = \frac{2(p^2-2)}{(p+2)^2} < 0$ so that $q^2 < 2$ and $q \in A$.
So p is not upper bound and there is a contradiction.
- If $p \notin A$ (i.e. $p^2 \geq 2$), then $p^2 > 2$ (since $p^2 = 2$ has no rational number solution).
One can see that $q = p + \frac{2-p^2}{p+2} < p$ and $q^2 > 2 > x^2 \Rightarrow q > x$ for all $x \in A$ so that q is an upper bound of A smaller than p, this contradicts to the fact that p is the least upper bound of the set A.

Hence, we conclude that $\sup A$ cannot always be represented by a rational number $\mathbb{Q}$.

Real number system

Roughly speaking, set of real numbers (denoted by $\mathbb{R}$) is a number system extended from rational number $\mathbb{Q}$ which satisfies the following axioms:

1. *Field Axiom* – There are two operations $+$ and $\cdot$ such that for any $a, b, c \in \mathbb{R}$

   (a) $a + b \in \mathbb{R}\ \ and\ \ a \cdot b \in \mathbb{R}$

   (b) $a + b = b + a\ \ \ and\ \ a \cdot b = b \cdot a$

   (c) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

   (d) There are two elements $0 \in \mathbb{R}$ and $1 \in \mathbb{R}$ such that $a + 0 = a$ and $a \cdot 1 = a$.

   (e) For any $a \in \mathbb{R}$, there exists $(-a) \in \mathbb{R}$ such that $a + (-a) = 0$.

   For any $a \in \mathbb{R}\backslash\{0\}$, there exists $a^{-1} = \frac{1}{a}$ such that $a \cdot (a^{-1}) = 1$.

   (f) $a \cdot (b + c) = a \cdot b + a \cdot c$

   (*Remark: These axioms are mainly for algebraic operations)

2. *Order Axiom* – There is an order relation (denoted by $<$) such that for any $a, b \in \mathbb{R}$
   (a) exactly one of the following $a < b$, $a = b$, $b < a$ are true.
   (b) If $a < b$ and $b < c$, then $a < c$
   (c) If $a < b$, then $a + c < b + c$
   (d) If $a < b$ and $c > 0$, then $ac < bc$.

   (*Remark: These axioms are mainly used for comparing numbers or inequalities)

3. *Completeness axiom* -- Every non-empty subset $A \subseteq \mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$.
   (*Remark: This axiom also implies that every non-empty subset $B \subseteq \mathbb{R}$ which is bounded below has an infimum in $\mathbb{R}$. To see this, we consider the set $-B = \{-x |\ x \in B\}$. One can show that the set $-B$ is bounded above and has a supremum (say β) in $\mathbb{R}$, then $-\beta \in \mathbb{R}$ can be shown to be infimum of B)

Example 1

(a) Find the infimum and supremum of the set

$$S = \left\{1 + \frac{1}{n^2} | n \in \mathbb{N}\right\}.$$

(b) We let T be a set of real numbers which $\sup T = 3$ and $\inf T = 1$. Fin the supremum of the set $W = \{3 - x^3 | x \in T\}$.

😊Solution

(a) Note that for any $n \in \mathbb{N}$ (and $1 \leq n < \infty$), we have $1 < 1 + \frac{1}{n^2} \leq 2$. So S is bounded above by 2 and bounded below by 1. By completeness axiom, $\sup S$ and $\inf S$ exist in $\mathbb{R}$.

✓ (Claim $\inf S = 1$) For any $\varepsilon > 0$, we choose a positive integer n which $n > \sqrt{\frac{1}{\varepsilon}}$ (or $\frac{1}{n^2} < \varepsilon$). We get $\underbrace{1 + \frac{1}{n^2}}_{\in S} < 1 + \varepsilon$. So $1 + \varepsilon$ is not lower bound for any $\varepsilon > 0$ and $\inf S = 1$.

✓ (Claim $\sup S = 2$). Since when $n = 1$, we have $1 + \frac{1}{n^2} = 2 \in S$. Then $2 - \varepsilon$ cannot be upper bound for any $\varepsilon > 0$. So we conclude that $\sup S = 2$.

(b) For any $x \in T$, we have

$$1 = \inf T \leq x \leq \sup T = 3.$$
$$\Rightarrow -24 = 3 - 3^3 \leq \underbrace{3 - x^3}_{\in W} \leq 3 - 1^3 = 2.$$

So W is bounded above by 2.

Next, we claim $\sup W = 2$.

For any $\varepsilon > 0$, since $\inf T = 1$, there exists $x \in T$ such that $x < \sqrt[3]{1 + \varepsilon}$ (as $\sqrt[3]{1 + \varepsilon} > 1$ as not lower bound of T).

Then it implies that

$$\underbrace{3 - x^3}_{\in T} > 3 - \left(\sqrt[3]{1 + \varepsilon}\right)^3 = 2 - \varepsilon.$$

So $2 - \varepsilon$ is not upper bound, thus $\sup W = 2$.

Example 2 (Some properties of supremum and infimum)

We let $A \subseteq \mathbb{R}, B \subseteq R$ be two non-empty subsets of real numbers which are bounded above.

(a) We define $A + B = \{a + b |\ a \in A\ and\ \ b \in B\}$.
Prove that $\sup(A + B) = \sup A + \sup B$.

(b) We define $-A = \{-a |\ a \in A\}$. Prove that $\inf(-A) = -\sup A$.

(c) We define $cA = \{ca | a \in A\}$. If $c > 0$, show that $\sup(cA) = c \sup A$.

😊Solution

(a) Note that $a \leq \sup A$ and $b \leq \sup B$ for any $a \in A$ and $b \in B$. Then for any $a + b \in A + B$, we have

$$a + b \leq \sup A + \sup B.$$

So $\sup A + \sup B$ is the upper bound of $A + B$. By completeness axiom, $\sup(A + B)$ exists.

For any $\varepsilon > 0$, we can deduce from supremum property that there exists $a_0 \in A$ and $b_0 \in B$ such that

$$a_0 > \sup A - \frac{\varepsilon}{2}, \qquad b_0 > \sup B - \frac{\varepsilon}{2}$$

This implies $\underbrace{a_0 + b_0}_{\in A+B} > \sup A + \sup B - \varepsilon$. Then it follows from supremum property that $\sup(A + B) = \sup A + \sup B$.

(b) For any $-a \in -A$, we have

$$-a \geq -\sup A.$$

So $-A$ is bounded below by $-\sup A$ and the infimum $\inf(-A)$ exists.

It remains to argue $\inf(-A) = -\sup A$.

For any $\varepsilon > 0$, there exists $a \in A$ such that

$$a > \sup A - \varepsilon.$$

This implies $-a < -(\sup A - \varepsilon) < -\sup A + \varepsilon$.

By infimum property, we conclude that $\inf(-A) = -\sup A$.

(c) It is left as exercise.

Example 3

We let $A_1, A_2, A_3, \ldots$ be subsets of real numbers which $A_n \subseteq (a, b)$ for all $n \in \mathbb{N}$ (Here, $a < b$). For any $n \in \mathbb{N}$, we let $M_n = \sup A_n$ and $m_n = \inf A_n$.

We let $S = \bigcup_{n=1}^{\infty} A_n$. Show that $\sup S = \sup\{M_n | n \in \mathbb{N}\}$ and $\inf S = \inf\{m_n | n \in \mathbb{N}\}$.

😊Solution

$$\boxed{\text{Prove } \sup S = \underbrace{\sup\{M_n | n \in \mathbb{N}\}}_{\textit{denoted by } M}}$$

Since $A_n \subseteq (a, b)$, so A_n is bounded above by b and $M_n \leq b$ for all $n \in \mathbb{N}$. So the supremum $M = \sup\{M_n | n \in \mathbb{N}\}$ exists.

For any $x \in S$, we have $x \in A_k$ for some $k \in \mathbb{N}$. Then $x \leq M_k \leq M$. So S is bounded above by M, so $\sup S$ exists.

It remains to argue that $\sup S = M$. For any $\varepsilon > 0$,

- since $M - \varepsilon$ is *not* upper bound of the set $\{M_n | n \in \mathbb{N}\}$, there exists M_k such that $M_k > M - \varepsilon$.
- Since $M_k = \sup A_k$ and $M - \varepsilon$ is not upper bound of A_k, there is $x_k \in A_k$ such that $x_k > M - \varepsilon$.

- As $x_k \in A_k \subseteq \bigcup_{n=1}^{\infty} A_n = S$ and $x_k > M - \varepsilon$, then $M - \varepsilon$ is not upper bound of S.

Hence, we conclude that $\sup S = M$.

Prove $\inf S = \underbrace{\inf\{m_n | n \in \mathbb{N}\}}_{denoted\ by\ m}$

Since $A_n \subseteq (a, b)$, so A_n is bounded below by a and $m_n \geq a$ for all $n \in \mathbb{N}$. So the infimum $m = \inf\{m_n | n \in \mathbb{N}\}$ exists.

For any $x \in S$, we have $x \in A_k$ for some $k \in \mathbb{N}$. Then $x \geq m_k \geq m$. So S is bounded below by m, so $\inf S$ exists.

It remains to argue that $\inf S = m$. For any $\varepsilon > 0$,

- since $m + \varepsilon$ is *not* lower bound of the set $\{m_n | n \in \mathbb{N}\}$, there exists m_k such that $m + \varepsilon > m_k$.
- We choose a real number which $m_k < m + \varepsilon$. Since $m_k = \inf A_k$ and $m + \varepsilon$ is not lower bound of A_k, there is $x_k \in A_k$ such that $x_k < m + \varepsilon$.
- As $x_k \in A_k \subseteq S$ and $x_k < m + \varepsilon$, then $m + \varepsilon$ is not lower bound of S.

Hence, we conclude that $\inf S = m$.

Some properties about real numbers

In this section, we present some properties of real numbers which are useful in deriving some important theorems in analysis. These properties can be derived using the axioms of real numbers described above.

Property 1: Infinitesimal Principle

For any $x \in \mathbb{R}$ and $y \in \mathbb{R}$, we have

$$x < y + \varepsilon \text{ for any } \varepsilon > 0 \text{ if and only if } x \leq y$$

Proof:

"$\Rightarrow$ part": We shall prove by contradiction.

Suppose that $x < y + \varepsilon$ for any $\varepsilon > 0$ and $x > y$ for some $x \in \mathbb{R}$ and $y \in \mathbb{R}$ (i.e. negation is true).

We let $\varepsilon_0 = x - y > 0$, then we have $x = y + \varepsilon_0 \geq y + \varepsilon_0$. It contradicts to the fact that $x < y + \varepsilon_0$ (based on the initial assumption by taking $\varepsilon = \varepsilon_0$). So it follows that $x \leq y$.

"$\Leftarrow$ part"

Since $x \leq y$ and $\varepsilon > 0$, it follows from order axiom that

$$x \leq y \Rightarrow x = x + 0 < y + \varepsilon \quad for\ any\ \varepsilon > 0.$$

The proof is completed.

Remark about infinitesimal property

- By subtracting the inequality both sides by ε, the statement can be rephrased as

$$x - \varepsilon < y \text{ for any } \varepsilon > 0 \text{ if and only if } x \leq y$$

- One can use the property and establish that for any $x \in \mathbb{R}$,

$$\underbrace{|x| < \varepsilon}_{-\varepsilon < x < \varepsilon} \ for\ any\ \varepsilon > 0 \Leftrightarrow x = 0.$$

  When proving two numbers are the same (i.e. $a = b$), the above fact suggests that we can establish this by proving $|a - b| < \varepsilon$ for all $\varepsilon > 0$.

Property 2: Well ordering property of positive integers

Every non-empty subset $S \subseteq \mathbb{N}$ of positive integers must have a least element.

Proof:

Since $n \geq 1$ for every positive integer n, so S must be bounded below by 1. As $S \subseteq \mathbb{N} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, $\inf S$ must exist in $\mathbb{R}$ by the completeness axiom.

By the infimum property (with $\varepsilon = 0.5$), there exists an unique $n \in S$ such that

$$\inf S \leq n < \inf S + 0.5.$$

It remains to argue that $\inf S = n$ (so that $\inf S \in S$). This can be proved by contradiction.

- Suppose that $\inf S \neq n$, then $n = \inf S + \varepsilon_0$ for some $\varepsilon_0 < 0.5$.
- Using the infimum property again (with $\varepsilon = \varepsilon_0$), there exists another element $m \in S$ (and $m \neq n$) such that
$$\inf S < m < \inf S + \varepsilon_0 = n.$$
- Since $m \neq n$ and both m, n are positive integer, so $n \geq m + 1 > \inf S + 1$, which contradicts to the fact that $n < \inf S + 0.5$.

So $\inf S = n \in S$. $n \in S$ is the minimum of the set S.

Application of well-ordering property – Mathematical induction

Suppose that we would like to prove the following statements which depend on a positive integer n.

Statement 1:

$$1^3 + 2^3 + 3^3 + \cdots + (2n)^3 = n^2(2n+1)^2, \qquad for\ any\ n \in \mathbb{N}.$$

Statement 2:

Suppose that $a_1 = 6$ and $a_{n+1} = 3(n+1)(n+2) + a_n$ for any $n \in \mathbb{N}$. Then

$$a_n < (n+1)^3 \text{ for any } n \in \mathbb{N}$$

Clearly, it is not practical to verify the statement by testing the validity of the statement for each positive integer n. Mathematical induction provides an efficient method to prove these statements:

Mathematical induction

For any $n \in \mathbb{N}$, we let $A(n)$ be a statement. Suppose that

- $A(1)$ is true
- $A(k)$ is true implies $A(k+1)$ is true for all $k \in \mathbb{N}$

Then $A(n)$ is true for all $n \in \mathbb{N}$.

Proof

We shall prove this by contradiction. Suppose that $A(n)$ is false for *some* $n \in \mathbb{N}$.

We consider a set $S = \{n \in \mathbb{N}: A(n)\ is\ false\}$.

- Since $S \neq \phi$ and $S \subseteq N$, it follows from well-ordering property that S has the least element and we denote this least element by $\inf S = m$.
- Since $A(1)$ is true, thus $m \geq 2$.

Note that $\underbrace{m-1}_{\in \mathbb{N}} \notin S$ by the definition of $\inf S$, then $A(m-1)$ is true. It follows from the second condition that $A(m)$ is true and it leads to contradiction.

(*Comment: It is essential to show $m \geq 2$ to ensure that $m-1$ is positive integer)

Example 4

Using mathematical induction, prove that for any positive integer $n \in \mathbb{N}$

$$\frac{d^n}{dx^n}\left(\frac{1}{1-x^2}\right) = \frac{n!}{2}\left[\frac{1}{(1-x)^{n+1}} + \frac{(-1)^n}{(1+x)^{n+1}}\right].$$

😋Solution

When $n = 1$, we have

$$L.H.S. = \frac{d}{dx}\left(\frac{1}{1-x^2}\right) = \frac{2x}{(1-x^2)^2}$$

$$R.H.S. = \frac{1}{2}\left[\frac{1}{(1-x)^2} + \frac{-1}{(1+x)^2}\right] = \frac{(1+x)^2 - (1-x)^2}{2(1-x)^2(1+x)^2} = \frac{2x}{(1-x^2)} = L.H.S.$$

The statement is true for $n = 1$.

Assume that the statement is true for $n = k$, then for $n = k+1$,

$$\frac{d^{k+1}}{dx^{k+1}}\left(\frac{1}{1-x^2}\right) = \frac{d}{dx}\left(\frac{d^k}{dx^k}\left(\frac{1}{1-x^2}\right)\right) = \frac{d}{dx}\left(\frac{k!}{2}\left[\frac{1}{(1-x)^{k+1}} + \frac{(-1)^k}{(1+x)^{k+1}}\right]\right)$$

$$= \frac{k!}{2}\left[\frac{k+1}{(1-x)^{k+2}} + \frac{(-1)^{k+1}(k+1)}{(1+x)^{k+2}}\right] = \frac{(k+1)!}{2}\left[\frac{1}{(1-x)^{k+2}} + \frac{(-1)^{k+1}}{(1+x)^{k+2}}\right].$$

It implies that the statement is also true for $n = k+1$.

By mathematical induction, we conclude that the statement is true for all $n \in \mathbb{N}$.

Example 5 (Triangle inequality)

(a) For any $x, y \in \mathbb{R}$, show that $|x + y| \leq |x| + |y|$.

(b) Prove that $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$ for any positive integer n and any $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

😊Solution

(a) We note that $|x| = \begin{cases} x & if \;\; x \geq 0 \\ -x & if \;\; x < 0 \end{cases}$, then we get $|x|^2 = x^2$. Then

$$(|x + y|)^2 = (x + y)^2 = x^2 + 2xy + y^2$$

$$\overbrace{\leq}^{xy \leq |xy|} |x|^2 + 2|xy| + |y|^2$$

$$\overbrace{=}^{|xy| = |x||y|} |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2$$

Note that $|x| + |y| \geq 0$, $|x + y| \geq 0$, we have $|x + y| \leq |x| + |y|$.

Remark:
In the last inequality, we have used the fact that for any $a, b \geq 0$,

$$\text{“}a \leq b \Leftrightarrow a^2 \leq b^2\text{”}$$

(b) One can prove this using mathematical induction as follows:

- For $n = 1$, it is clear that $|x_1| \leq |x_1|$ and the inequality holds trivially.
- Assume that the inequality holds for $n = k$, i.e.
$$|x_1 + x_2 + \cdots + x_k| \leq |x_1| + |x_2| + \cdots + |x_k|.$$
For $n = k + 1$, we apply the inequality in (a) with $x = x_1 + x_2 + \cdots + x_k$ and $y = x_{k+1}$ and obtain
$$|x_1 + x_2 + \cdots + x_{k+1}| = \left|\underbrace{(x_1 + x_2 + \cdots + x_k)}_{x} + \underbrace{x_{k+1}}_{y}\right|$$
$$\leq |x_1 + x_2 + \cdots + x_k| + |x_{k+1}|$$
$$\leq |x_1| + |x_2| + \cdots + |x_k| + |x_{k+1}|$$
This implies that the statement remains valid for $n = k + 1$.
- By mathematical induction, we conclude that
$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$
For all $n \in \mathbb{N}$

Archimedean Principle

Property 3: Archimedean Principle

For any $x \in \mathbb{R}$, there exists a positive integer $n \in \mathbb{N}$ such that $n > x$.

(*Note: This property implies that $\mathbb{N}$ is unbounded in $\mathbb{R}$)

Proof of property 3

Suppose that there exists $x_0 \in \mathbb{R}$ such that $n \leq x_0$ for all $n \in \mathbb{N}$. This implies that x_0 is an upper bound of the set $\mathbb{N}$.

By the completeness axiom of real number, the *supremum* of $\mathbb{N}$ (i.e. $\sup \mathbb{N}$) exists and it satisfies $n \leq \sup \mathbb{N}$ for all $n \in \mathbb{N}$.

To get the contradiction, we consider a number $m = \sup \mathbb{N} - 1 < \sup \mathbb{N}$. Since m is *not* the upper bound of $\mathbb{N}$, then there exists $n_0 \in \mathbb{N}$ such that

$$n_0 > \sup \mathbb{N} - 1 \Rightarrow n_0 + 1 > \sup \mathbb{N}.$$

However, $n_0 + 1 \in \mathbb{N}$ as $n_0 \in \mathbb{N}$. So this contradicts to the fact that $\sup \mathbb{N}$ is an upper bound of $\mathbb{N}$.

Remark of Archimedean property

- A more general version of Archimedean property is as follows:
  "For any $x, y \in \mathbb{R}$ with $y > 0$, there exists a positive integer $n \in \mathbb{N}$ such that $ny > x$."
  - This general version can be derived by replacing x by $\frac{x}{y}$ in the property 2.
- By replacing x by $\frac{1}{x}$ (where $x > 0$) in the property, one can deduce that there exists $n \in \mathbb{N}$ such that
$$n > \frac{1}{x} \Leftrightarrow \frac{1}{n} < x.$$
- Technically, Archimedean property is an useful tool in proving various mathematical properties. Some applications are
  - Finding supremum and infimum of some sets
  - Density of rational number (which describes the distribution of rational number over the set of real numbers).

Example 6

We let $A = \left\{ e^{\frac{1}{n}} \middle| \, n \in \mathbb{N} \right\}$. Prove that $\inf A = 1$.

😊Solution

<u>Step 1: Verify that 1 is the lower bound of A</u>

Since $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$ and e^x is increasing with respect to x, it follows that

$$e^{\frac{1}{n}} > e^0 = 1 \quad for\ all\ n \in \mathbb{N}.$$

So 1 is the lower bound of A.

<u>Step 2: Verify that $\inf A = 1$ using infimum property.</u>

For any $\varepsilon > 0$, it follows from Archimedean property that there is $n \in \mathbb{N}$ which

$$n > \frac{1}{\ln(1+\varepsilon)} \Rightarrow \ln(1+\varepsilon) > \frac{1}{n} \Rightarrow 1+\varepsilon > \underbrace{e^{\frac{1}{n}}}_{A}.$$

So $\inf A = 1$ by infimum property.

Example 7

Argue that the supremum of a set defined by $S = \left\{\sqrt{n}e^{2-\frac{1}{n}} | n \in \mathbb{N}\right\}$ does not exist in $\mathbb{R}$.

😊Solution

Our goal is to argue that S is *not* bounded from above.

Suppose that S is bounded from above, then $\sup S$ exists and $\sup S = M \in \mathbb{R}$.

So that $\sqrt{n}e^{2-\frac{1}{n}} \leq M$ for all $n \in \mathbb{N}$.

By Archimedean property, there exists a positive integer k such that $k > \left(\frac{M}{e}\right)^2$.

Then it follows that

$$\underbrace{\sqrt{k}e^{2-\frac{1}{k}}}_{\in S} \geq \sqrt{k}e^{2-1} > \left(\frac{M}{e}\right)e = M,$$

which contradicts to the fact that S is bounded above by M.

Density of rational numbers and irrational numbers

Each real number is either rational (if it can be written as a form of $\frac{m}{n}$, where m is integer and n is positive integer) or irrational. One would like to know the distribution of rational number and irrational number in real number line.

The following theorem confirms that rational number (resp. irrational) exists everywhere in the real number line in the sense that there exists at least one rational number (resp. irrational number) in any interval in $\mathbb{R}$.

Property 4: Density of rational number

For any $x \in \mathbb{R}$ and $y \in \mathbb{R}$ which $x < y$, there exists a rational number $q \in \mathbb{Q}$ such that

$$x < q < y.$$

Property 5: Density of irrational number

For any $x \in \mathbb{R}$ and $y \in \mathbb{R}$ which $x < y$, there exists an irrational number $p \in \mathbb{R}\backslash\mathbb{Q}$ such that

$$x < p < y.$$

Proof of property 4

The key step in proving property 4 is to construct a suitable rational number q. This can be done by the following steps:

- ✓ <u>Step 1:</u> Choose a positive integer $m \in \mathbb{N}$ such that $m > y$. The feasibility can be confirmed by Archimedean property.
- ✓ <u>Step 2:</u> Using Archimedean property again, one can pick a positive integer $n \in \mathbb{N}$ such that $\frac{1}{n} < y - x$.
- ✓ <u>Step 3:</u> By Archimedean property, there exists a positive integer $k \in \mathbb{N}$ such that
$$k > n(m - y) \Rightarrow m - \frac{k}{n} < y.$$
We take the smallest positive integer k_0 such that $k_0 > n(m - y) \Leftrightarrow m - \frac{k_0}{n} < y$ (guaranteed by well-ordering axiom).
- ✓ <u>Step 4:</u> We take let $q = m - \frac{k_0}{n} = \frac{mn - k_0}{n} \in \mathbb{Q}$. It remains to argue that $q > x$. Suppose that $q \leq x$. Note that $k_0 - 1 \leq n(m - y) \Leftrightarrow m - \frac{k_0 - 1}{n} \geq y$. This implies
$$\frac{1}{n} = \left(m - \frac{k_0 - 1}{n}\right) - q \geq y - x.$$
This contradicts to the fact that $\frac{1}{n} < y - x$. So $x < q < y$ and the result follows.

Proof of property 5

Note that $(x, y)\backslash\mathbb{Q}$ denotes the set of irrational number over the interval (x, y). Since (x, y) is uncountable and $\mathbb{Q}$ is countable, so $(x, y)\backslash\mathbb{Q}$ must be uncountable and therefore non-empty. So there is an irrational number $p \in (x, y)\backslash\mathbb{Q}$ which $x < p < y$.

Example 8 (Application of density of rational number)

We let (a, b) be an open interval (where $a < b$). Prove that there are *infinitely many* rational numbers which lies within the interval (a, b).

😊Solution

One can prove this by contradiction. Suppose that there are *finitely many* rational numbers lie within (a, b), we denote those numbers as $q_1, q_2, \dots, q_K$ where $a < q_1 < q_2 < \cdots < q_K < b$.

We consider an open interval (a, q_1). Since $q_1, q_2, \dots, q_K$ are the only rational number in (a, b), then there is *no* rational number that lies within (a, q_1).

However, one can deduce from the density of rational number that there exists a rational number $q^* \in \mathbb{Q}$ such that $a < q^* < q_1$ which contradicts to our earlier conjecture. Thus the negation is never true and the result follows.

Example 9

We let $S = (a, b) \cap \mathbb{Q}$, where $a < b$.

(a) Show that S is non-empty.

(b) Show that $\inf S = a$ and $\sup S = b$.

😊Solution

(a) Using the density of rational number, there exists $x \in \mathbb{Q}$ such that $a < x < b$ (or $x \in (a, b)$). So $x \in S$ and S is non-empty.

(b) For any $x \in S$, we have $x \in (a, b)$ so that

$$a < x < b.$$

So the set S is bounded above by b and bounded below by a. Thus $\inf S$ and $\sup S$ exists by completeness axiom.

We first argue $\inf S = a$. Suppose that $\inf S \neq a$ and there is a lower bound L such that $\inf S < L$. Since b is upper bound of S, so $L \leq b$. Then by density of rational number, there exists $r \in \mathbb{Q}$ such that

$$a < r < L$$

Since $L \leq b$, so $r \in (a, b)$ and hence $r \in S$. This implies that L is not lower bound and there is contradiction. The proof of $\sup S = b$ is similar.

Property 6: Nested Interval theorem

We let $\{I_n = [a_n, b_n] | n \in \mathbb{N}\}$ be a set of closed intervals such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Then $\bigcap_{n=1}^{\infty} I_n = [a, b]$, where $a = \sup\{a_n | n \in \mathbb{N}\}$ and $b = \inf\{b_n | n \in \mathbb{N}\}$.

Proof of property 6

We first prove the existence of a and b. For any $n \in \mathbb{N}$, since $I_n \subseteq I_1$, then we have $a_1 \leq a_n \leq b_n \leq b_1$. So a_n is bounded above by b_1 and b_n is bounded below by a_1. Therefore, the supremum a and the infimum b exists.

Next, we argue that $\bigcap_{n=1}^{\infty} I_n = [a, b]$.

- For any $x \in \bigcap_{n=1}^{\infty} I_n$, we have $x \in I_n$ for all n so that $a_n \leq x \leq b_n$.
  - Suppose that $x < a$, one can deduce from supremum property that there is $k \in \mathbb{N}$ such that $x < a_k \leq a$, then it follows that $x \notin I_k$. It contradicts to $x \in \bigcap_{n=1}^{\infty} I_n$. So it must be that $x \geq a$.
  - Suppose that $x > b$, one can deduce from infimum property that there is $m \in \mathbb{N}$ such that $b \leq b_m < x$, then it follows that $x \notin I_m$. It contradicts to $x \in \bigcap_{n=1}^{\infty} I_n$. So it must be that $x \leq b$.
  - Combining the result, we have $x \in [a, b]$ so that $\bigcap_{n=1}^{\infty} I_n \subseteq [a, b]$

- For any $x \in [a, b]$, we have
$$a_n \le \sup\{a_n | n \in \mathbb{N}\} = a \le x \le b = \inf\{b_n | n \in \mathbb{N}\} \le b_n$$
for any positive integer n.
Thus, $x \in I_n$ for all n and hence $x \in \bigcap_{n=1}^{\infty} I_n$ and $[a, b] \subseteq \bigcap_{n=1}^{\infty} I_n$.

Therefore, we conclude that $\bigcap_{n=1}^{\infty} I_n = [a, b]$.

Remark of the proof

One can prove that $a \le b$ by contradiction as follows.

Suppose that $a > b$, one can deduce from supremum property that there exists a_n such that $a \ge a_n > b$. By infimum property, there exists b_m such that $a_n > b_m \ge b$.

It implies that $b_n \ge a_n > b_m \ge a_m$ and $I_n \cap I_m = \phi$. This contradict to the assumption that either $I_n \subseteq I_m$ or $I_m \subseteq I_n$.