MATH 2031 Introduction to Real Analysis

March 25, 2013

Tutorial Note 1

Calculus Review Exercises

Problem 1. Draw the graph of the function $f(x) = \frac{x}{x}$.
Solution:

$$f(x) = \frac{x}{x} = \begin{cases} 1 & : \text{if } x \neq 0 \\ \text{Undefined} & : \text{if } x = 0 \end{cases}$$

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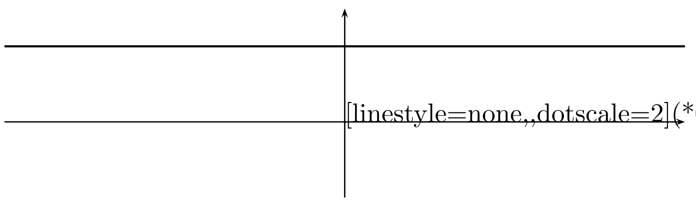

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Remark:
We should beware of how and where the function $f(x)$ is defined. A priori, $f(x) = \frac{x}{x}$ "seems to be the same" as the constant function $f(x) = 1$. However, it's **NOT** true, since $f(x)$ is not defined at $x = 0$(everything divided by 0 is not well-defined).

Problem 2. Find $\lim\limits_{x \to +\infty} \dfrac{x + 2\cos x}{3 + 4x}$.
Solution:

$$\lim_{x \to +\infty} \frac{x + 2\cos x}{3 + 4x} = \lim_{x \to +\infty} \frac{1 + 2(\frac{\cos x}{x})}{(\frac{3}{x}) + 4} = \frac{1 + 0}{0 + 4} = \frac{1}{4}$$

Here $\cos x$ is a bounded function ($-1 \leq \cos x \leq 1$).

Remark:
The reason why we do not apply L'Hospital rule is that not all the conditions are satisfied.
The conditions for L'Hospital rule are

1. $\left(\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0\right)$ or $\left(\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = \infty\right)$

2. $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists and $g'(x) \neq 0$ for x near a

then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$

In this problem condition 2 falls as $\lim\limits_{x \to +\infty} \dfrac{1 - 2\sin x}{4}$ does not exist due to $\sin x$ oscillating.

Problem 3. Let $f(x) = \begin{cases} x^2 & : \text{if } x \neq 3 \\ 3x & : \text{if } x = 3 \end{cases}$. Is it true that $f'(x) = \begin{cases} 2x & : \text{if } x \neq 3 \\ 3 & : \text{if } x = 3 \end{cases}$?
Solution:

For $x \neq 3$, $f(x) = x^2$, so $f'(x) = 2x$,
For $x = 3$,

$$f'(x) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2 - 3(3)}{h} = \lim_{h\to 0} \frac{6h + h^2}{h} = 6$$

i.e. $f'(x) = \begin{cases} 2x & : \text{if } x \neq 3 \\ 6 & : \text{if } x = 3 \end{cases} = 2x \neq \begin{cases} 2x & : \text{if } x \neq 3 \\ 3 & : \text{if } x = 3 \end{cases}$.

Remark:
In general, $f(x) = \begin{cases} g(x) & : \text{if } x \notin S \\ h(x) & : \text{if } x \in S \end{cases}$ **DOES NOT** imply $f'(x) = \begin{cases} g'(x) & : \text{if } x \notin S \\ h'(x) & : \text{if } x \in S \end{cases}$

Problem 4. Must $1^\infty = 1$ More precisely, let $a_1, a_2, a_3, \cdots$ be positive real numbers. Must it be true that if $\lim_{n\to\infty} a_n = 1$, then $\lim_{n\to\infty} a_n^n = 1$.
Solution:
It's not true in general, the following are counter-examples.

Take $a_n = \sqrt[n]{n}$, then $\ln(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} \ln(n^{\frac{1}{n}}) = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$. Hence $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{n} = 1$.

However, $\lim_{n\to\infty} a_n^n = \lim_{n\to\infty} (\sqrt[n]{n})^n = \lim_{n\to\infty} n = \infty \neq 1$.

Also, Take $a_n = 1 + \frac{1}{n}$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1 + \frac{1}{n}) = 1$, but $\lim_{n\to\infty} a_n^n = \lim_{n\to\infty} (1 + \frac{1}{n})^n = e \neq 1$.

Problem 5. We know that $\lim_{x\to+\infty} \sin x$ doesn't exist. If $a_1, a_2, a_3, \cdots$ are positive real numbers with $\lim_{n\to+\infty} a_n = +\infty$, then must it be true that $\lim_{n\to+\infty} \sin a_n$ doesn't exist?
Solution:
It's not always true, so again we will give a counter-example to disprove it.
Take $a_n = n\pi$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\pi = \infty$, but $\sin a_n = \sin(n\pi) = 0$ for all $n = 1, 2, 3, \cdots$, i.e. $\lim_{n\to+\infty} \sin a_n = \lim_{n\to+\infty} 0 = 0$ which exists.

Problem 6. Show $\lim_{n\to\infty} \sin n \neq 0$
Solution:
As shown above that we may not simply conclude that $\lim_{x\to+\infty} \sin x$ doesn't exist so $\lim_{n\to+\infty} \sin n$ also doesn't exist. We will prove the statement by contradiction.

Assume $\lim_{n\to\infty} \sin n = 0$, then $\lim_{n\to\infty} \sin(n+1) = 0$ and $\lim_{n\to\infty} |\cos n| = \lim_{n\to\infty} \sqrt{1 - \sin^2(n)} = \sqrt{1 - 0^2} = 1$.
However,

$$\lim_{n\to\infty} |sin(n+1)| = \lim_{n\to\infty} |\underbrace{\sin n}_{\to 0} \cos 1 + \sin 1 \underbrace{\cos n}_{|\cos n| \to 1}| = \sin 1 \neq 0$$

contradiction.

Hence, $\lim_{n\to\infty} \sin n \neq 0$

Problem 7.
Let

$$g(x) = \begin{cases} 1 & : \text{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & : \text{if } x \notin [0,1] \cap \mathbb{Q} (i.e. [0,1] \setminus \mathbb{Q}) \end{cases}$$

For every positive integer n, divide $[0,1]$ into intervals $\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \cdots, \left[\frac{n-1}{n}, 1\right]$.
On the j-th interval $\left[\frac{j-1}{n}, \frac{j}{n}\right]$, let x_j be its midpoint. Since x_j is rational, we have $g(x_j) = 1$. Now

$$\lim_{n\to+\infty} \left(g(x_1) \left(\frac{1}{n} - 0 \right) + g(x_2) \left(\frac{2}{n} - \frac{1}{n} \right) + \cdots + g(x_n) \left(1 - \frac{n-1}{n} \right) \right) = 1$$

So $\int_0^1 g(x)dx = 1$. Is this correct?

Solution:

It's not true. As we can see if we pick those points in another way we may get other value for the above limit. Say, we take $x_j = \frac{j-1}{n} + \frac{1}{n\pi}$, then x_j is an irrational number, then we have $g(x_j) = 0$, thus the value for the above limit is 0.

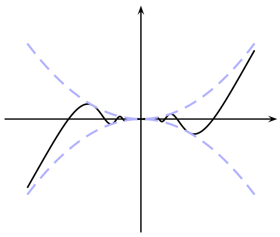
In fact, the problem here is that the argument presented in the problem is not sufficient to conclude that $\int_0^1 g(x)dx = 1$.

Recall, the definition of integral required that we have to check the above expression for **all possible points chosen from each intervals**, instead of only checking on the midpoint. Also the choice of intervals may vary as long as the maximum length among them tends to zero.

Problem 8. Let $h(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & : \text{if } x \neq 0 \\ 0 & : \text{if } x = 0 \end{cases}$. Graph $h(x)$. Find $h'(x)$. What is $h''(0)$

Solution:

Since $-1 \leq \sin\frac{1}{x} \leq 1$, we get $-x^2 \leq h(x) = x^2 \sin\frac{1}{x} \leq x^2$. Thus we get the graph below.



Since $h(x)$ is a piecewise defined function, similar to Problem 3, we will do it case-by-case.

For $x \neq 0$, $h'(x) = \frac{d}{dx}(x^2 \sin(\frac{1}{x})) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$,

For $x = 0$, $h'(0) = \lim_{x\to 0} \frac{h(x) - h(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x}) - 0}{x} = \lim_{x\to 0} x \underbrace{\sin\left(\frac{1}{x}\right)}_{\text{bounded}} = 0.$

i.e.

$$h'(x) = \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos(\frac{1}{x}) & : \text{if } x \neq 0 \\ 0 & : \text{if } x = 0 \end{cases}$$

$$\begin{aligned} h''(0) &= \lim_{x\to 0} \frac{h'(x) - h'(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) - 0}{x} \\ &= \lim_{x\to 0} \left(2\sin\left(\frac{1}{x}\right) - \frac{\cos\left(\frac{1}{x}\right)}{x}\right) \end{aligned}$$

which does not exist.

From the problem, we can see that even if the first derivative of a function exist, it DOES NOT imply that the second derivative of the function exist.