Appendix 1: Proof of Lebesgue's Theorem

Lemma Let $f:[a,b]\to\mathbb{R}$ be continuous on $K=[a,b]\setminus\bigcup_{j=1}^{\infty}(a_j,b_j)$

Then $\forall \varepsilon>0\ \exists\delta>0$ such that $\forall \begin{matrix} x\in K \\ t\in[a,b]\end{matrix}$ $|x-t|<\delta \Rightarrow |f(x)-f(t)|<\varepsilon$.

Proof. Assume the lemma is false. Then

$\exists\varepsilon>0\ \forall\delta>0\ \exists x\in K, t\in[a,b]$, $|x-t|<\delta$ and $|f(x)-f(t)|\ge\varepsilon$.

For $\delta=\frac{1}{n}$, $\exists x_n\in K, t_n\in[a,b]$, $|x_n-t_n|<\frac{1}{n}$ and $|f(x_n)-f(t_n)|\ge\varepsilon$

By Bolzano-Weierstrass theorem, $\exists x_{n_j}\to w\in[a,b]$. Then

$|t_{n_j}-w|\le|t_{n_j}-x_{n_j}|+|x_{n_j}-w|\le\frac{1}{n_j}+|x_{n_j}-w|\to 0$ as $j\to\infty$.

So $t_{n_j}\to w$.

Claim: $w\in K$ Assume $w\notin K$. Then $w\in(\alpha_i,\beta_i)$ for some i. Since $x_{n_j}\to w$, so $\exists x_{n_p}\in(\alpha_i,\beta_i)$. However, $x_{n_p}\in K=[a,b]\setminus\bigcup_{j=1}^{\infty}(\alpha_j,\beta_j)$, a contradiction.

Since f is continuous on K, hence at w, we have

$$0=|f(w)-f(w)|=\lim_{j\to\infty}|f(x_{n_j})-f(t_{n_j})|\ge\varepsilon>0,$$

a contradiction. $\therefore$ lemma is true.

Recall

Lebesgue's Theorem Let $f:[a,b]\to\mathbb{R}$ be bounded. Then f is Riemann integrable on $[a,b]$

$\Leftrightarrow$ $S_f=\{x\in[a,b]: f$ is discontinuous at $x\}$ is of measure 0.

Proof of Lebesgue's Theorem ($\Leftarrow$) Direction

($\Leftarrow$) Suppose S_f is of measure 0. Since f bounded on $[a,b]$, let $|f(x)|\le N$ on $[a,b]$.

For every $\varepsilon>0$, let $\varepsilon_0=\frac{\varepsilon}{3(N+b-a)}$. Since S_f is of measure 0, $\exists$ intervals (α_i,β_i) such that

$$S_f\subseteq\bigcup_{i=1}^{\infty}(\alpha_i,\beta_i) \text{ and } \sum_{i=1}^{\infty}|\alpha_i-\beta_i|<\varepsilon_0.$$

Then f is continuous on $K=[a,b]\setminus\bigcup_{i=1}^{\infty}(\alpha_i,\beta_i)$.

By lemma, for ε_0, $\exists\delta>0$ such that

$\forall x\in K,\ t\in[a,b]\quad |x-t|<\delta\Rightarrow|f(x)-f(t)|<\varepsilon_0$. (*)

Take a partition $P=\{x_0,x_1,\dots,x_n\}$ with $|x_{j-1}-x_j|<\delta$ for $j=1,2,\dots,n$.

Two Cases

① If $K\cap[x_{j-1},x_j]=\phi$, then $[x_{j-1},x_j]\subseteq\bigcup_{i=1}^{\infty}(\alpha_i,\beta_i)$.

② If $x\in K\cap[x_{j-1},x_j]$, then

$$M_j-m_j=\sup\{|f(t_0)-f(t_1)|: t_0,t_1\in[x_{j-1},x_j]\}$$
$$\le\sup\{|f(t_0)-f(x)|+|f(x)-f(t_1)|: t_0,t_1\in[x_{j-1},x_j]\}$$
by (*) $\to$ $\le\varepsilon_0+\varepsilon_0=2\varepsilon_0$.

So

$$U(f,P)-L(f,P)=\sum_{\substack{K\cap[x_{j-1},x_j]=\phi \\ \text{case ①}}}(M_j-m_j)\Delta x_j+\sum_{\substack{K\cap[x_{j-1},x_j]\ne\phi \\ \text{case ②}}}(M_j-m_j)\Delta x_j$$
$$\le 2N\sum_{K\cap[x_{j-1},x_j]=\phi}\Delta x_j+2\varepsilon_0\sum_{K\cap[x_{j-1},x_j]\ne\phi}\Delta x_j$$
$$\le 2N\sum_{i=1}^{\infty}|\alpha_i-\beta_i|+2\varepsilon_0(b-a)$$
$$\le 2N\varepsilon_0+2\varepsilon_0(b-a)=2\varepsilon/3<\varepsilon.$$

Integral criterion $\Rightarrow f$ integrable on $[a,b]$.

Proof of Lebesgue's Theorem ($\Rightarrow$) Direction

($\Rightarrow$) Let f be integrable on $[a,b]$. Need to show

$S_f = \{x \in [a,b] : f \text{ is discontinuous at } x\}$ is of measure 0.

For $k = 1, 2, 3, \cdots$, define

$$D_k = \{x \in [a,b] : \forall \text{ open interval } I \text{ containing } x,\ \exists z \in I \cap [a,b] \text{ such that } |f(x) - f(z)| > \tfrac{1}{k}\}.$$

Claims: ① $S_f = \bigcup_{k=1}^{\infty} D_k$ ② Each D_k is of measure 0

(① and ② $\Rightarrow$ S_f is of measure 0.)

For ①, $x \in D_k \Rightarrow \exists z_n \in I = (x - \frac{1}{n}, x + \frac{1}{n})$ with $|f(x) - f(z_n)| > \frac{1}{k}$.

$\Rightarrow z_n \to x$, but $\lim_{n\to\infty} |f(z_n) - f(x)| \neq 0$

$\Rightarrow x \in S_f$ $\therefore \boxed{\bigcup_{k=1}^{\infty} D_k \subseteq S_f.}$

Conversely, $x \in S_f \Rightarrow \exists \varepsilon > 0\ \forall \delta > 0\ \exists z \in (x-\delta, x+\delta) \cap [a,b]$ such that $|f(x) - f(z)| \geq \varepsilon$.

By Archimedean principle, $\exists k \in \mathbb{N}$ such that $\varepsilon > \frac{1}{k}$.

$\forall$ open interval I containing x, we have $x \in (x-\delta, x+\delta) \subseteq I$ for some $\delta > 0$.

So $\exists z \in I \cap [a,b]$ such that $|f(x) - f(z)| \geq \varepsilon > \frac{1}{k}$.

then $x \in D_k$. $\therefore x \in S_f \Rightarrow x \in \bigcup_{k=1}^{\infty} D_k$.

$\therefore \boxed{S_f \subseteq \bigcup_{k=1}^{\infty} D_k}$. Therefore, ① is true.

For ②, to show D_k is of measure 0, let $\varepsilon > 0$.

By integral criterion, $\exists P = \{x_0, x_1, \cdots, x_n\}$ partition of $[a,b]$ such that $U(f,P) - L(f,P) < \frac{\varepsilon}{2k}$.

If $x \in D_k \cap (x_{j-1}, x_j)$, then $\exists z \in (x_{j-1}, x_j)$ such that

$$(*) \quad M_j - m_j \geq |f(x) - f(z)| > \frac{1}{k}.$$

Let $J = \{j : D_k \cap (x_{j-1}, x_j) \neq \emptyset\}$. Then J is a finite set.

$$D_k \setminus \{x_0, x_1, \cdots, x_n\} = D_k \cap ([a,b] \setminus \{x_0, x_1, \cdots, x_n\}) = D_k \cap \bigcup_{j=1}^{n} (x_{j-1}, x_j) \subseteq \bigcup_{j \in J} (x_{j-1}, x_j).$$

Now $$\sum_{j\in J} \underbrace{|x_{j-1} - x_j|}_{=\Delta x_j} \leq \sum_{j \in J} \underbrace{k(M_j - m_j)}_{>1 \text{ by } (*)} \Delta x_j \leq k(U(f,P) - L(f,P)) < \varepsilon/2.$$

Next $D_k \cap \{x_0, x_1, \cdots, x_n\}$ is a finite set. So around each x_j, we can take open interval $I_j = (x_j - \frac{\varepsilon}{4(n+1)}, x_j + \frac{\varepsilon}{4(n+1)})$.

Then $D_k \cap \{x_0, x_1, \cdots, x_n\} \subseteq \bigcup_{j=0}^{n} I_j$ and sum of length I_j is less than $\varepsilon/2$.

Therefore, $D_k \subseteq \left(\bigcup_{j\in J} (x_{j-1}, x_j)\right) \cup \left(\bigcup_{j=0}^{n} I_j\right)$ and the sum of lengths of all these open interval is less than ε.

$\therefore D_k$ is of measure 0. $\therefore$ ② is true.

Appendix 2: Riemann's Definition of the Integral

Our definition: following Darboux
$$\int_a^b f(x)\,dx = \sup L(f,P) = \inf U(f,P) \quad \leftarrow P \text{ partition of } [a,b]$$

Riemann's definition: $\displaystyle\lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j$

What is the meaning of this?

Recall for $g: S \to \mathbb{R}$, $\displaystyle\lim_{x\to 0} g(x) = L$ means
$\forall \varepsilon>0\ \exists\delta>0$ such that $\forall x\in S$, $\underset{|x|}{|x-0|}<\delta \Rightarrow |g(x)-L|<\varepsilon$.

Definition. Let $f(x)$ be bounded on $[a,b]$. Write
$\displaystyle\lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = I$ iff $\forall\varepsilon>0\ \exists\delta>0$ such that
$\forall$ partition $P=\{\underset{a}{x_0}, x_1, \cdots, \underset{b}{x_n}\}$ of $[a,b]$, $t_j\in[x_{j-1},x_j]$, $j=1,2,\cdots,n$,
$$\|P\|<\delta \Rightarrow \left|\sum_{j=1}^{n} f(t_j)\Delta x_j - I\right|<\varepsilon.$$

Stieltjes's definition: $\displaystyle\lim_{Q\to\infty}\sum_{j=1}^{n} f(t_j)\Delta x_j$

What is the meaning of this?

Recall $\displaystyle\lim_{n\to\infty} a_n = L$ means $\forall\varepsilon>0\ \exists K$ such that $n\ge K \Rightarrow |a_n-L|<\varepsilon$.

Definition $\displaystyle\lim_{Q\to\infty}\sum_{j=1}^{n} f(t_j)\Delta x_j = I$ iff $\forall\varepsilon>0\ \exists P$ partition of $[a,b]$ such that $Q\supseteq P$, $Q =$ partition $\{x_0,x_1,\cdots,x_n\}$ of $[a,b]$, $t_j\in[x_{j-1},x_j]$ $j=1,2,\cdots,n$
$$\Rightarrow \left|\sum_{j=1}^{n} f(t_j)\Delta x_j - I\right|<\varepsilon.$$
(Note: $Q\supseteq P \Leftrightarrow Q$ is a refinement of P.)

Theorem Let $f(x)$ be bounded on $[a,b]$.
The following are equivalent (TFAE)

(a) $\displaystyle\int_a^b f(x)\,dx = I$

(b) $\displaystyle\lim_{\|P\|\to 0}\sum_{j=1}^{n} f(t_j)\Delta x_j = I$

(c) $\displaystyle\lim_{Q\to\infty}\sum_{j=1}^{n} f(t_j)\Delta x_j = I$.

$$\begin{array}{ccc} & a & \\ \Downarrow & & \Uparrow \\ b & \Rightarrow & c \end{array}$$

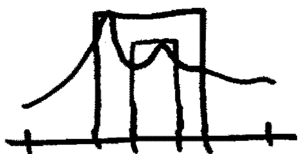
Proof. We will show (a)⇒(b), (b)⇒(c), (c)⇒(a).

(a)⇒(b) Let $\int_a^b f(x)\,dx = I$. $\forall\varepsilon>0$,

$$I-\varepsilon/2 \quad L(f,P_\varepsilon) \quad I \quad U(f,P_\varepsilon) \quad I+\varepsilon/2$$
$I =$ sup of lowersums $=$ inf of uppersums

$\exists$ partition $P_\varepsilon=\{w_0,w_1,\cdots,w_q\}$ of $[a,b]$ such that
$I-\varepsilon/2 < L(f,P_\varepsilon)\le U(f,P_\varepsilon) < I+\varepsilon/2$. Let $\delta = \dfrac{\varepsilon}{4qK}$,
where $K=\sup\{|f(x)| : x\in[a,b]\}$.

$\forall$ partition $P=\{x_0,x_1,\cdots,x_n\}$ of $[a,b]$ with $\|P\|<\delta$,
$$U(f,P) = \sum_{i=1}^{n} M_i\Delta x_i = \underbrace{\sum_{[x_{i-1},x_i]\cap P_\varepsilon=\phi} M_i\Delta x_i}_{S_1} + \underbrace{\sum_{[x_{i-1},x_i]\cap P_\varepsilon\ne\phi} M_i\Delta x_i}_{S_2}$$



$\underset{x_0}{w_0=a}$ w_{j-1} x_{i-1} x_i w_j $\underset{x_n}{b=w_n}$

$S_1 < U(f,P_\varepsilon) < I+\varepsilon/2$

S_2 has at most $2q$ terms $\le 2qK\|P\| < 2qK\delta = \varepsilon/2$

So $U(f,P)<I+\varepsilon$. Similarly, $I-\varepsilon<L(f,P)$.

Then $I-\varepsilon < L(f,P) \le \sum_{i=1}^{n} f(t_j)\Delta x_j \le U(f,P) < I+\varepsilon$

$\Rightarrow -\varepsilon < \sum_{i=1}^{n} f(t_j)\Delta x_j - I < \varepsilon \Rightarrow \left|\sum_{i=1}^{n} f(t_j)\Delta x_j - I\right| < \varepsilon.$

(b) $\Rightarrow$ (c) If (b) is true, then $\forall \varepsilon>0$, $\exists \delta>0$ such that $\forall$ partition P of $[a,b]$ with $\|P\|<\delta$, we have $\left|\sum_{j=1}^{n} f(t_{j,P})\Delta x_{j,P} - I\right| < \varepsilon$. If $Q \supseteq P$, then $\|Q\| \le \|P\| < \delta$ and so $\left|\sum_{j=1}^{n} f(t_{j,Q})\Delta x_{j,Q} - I\right| < \varepsilon$.

(c) $\Rightarrow$ (a) If (c) is true, then $\forall \varepsilon>0$, $\exists$ partition P of $[a,b]$ such that $\underset{t_j\in[x_{j-1},x_j]}{Q \supseteq P} \Rightarrow \left|\sum_{j=1}^{n} f(t_j)\Delta x_j - I\right| < \varepsilon/3$.

By supremum limit theorem, $\exists$ sequence $t_{j,k}$ such that $\lim_{k\to\infty} f(t_{j,k}) = M_j = \sup\{f(x) : x\in[x_{j-1},x_j]\}$. Then

$$|U(f,P)-I| = \left|\sum_{j=1}^{n} M_j\Delta x_j - I\right| = \lim_{k\to\infty}\left|\sum_{j=1}^{n} f(t_{j,k})\Delta x_j - I\right| \le \frac{\varepsilon}{3}$$

Similarly, $|L(f,P)-I| \le \varepsilon/3$. Then

$$U(f,P)-L(f,P) \le \big(U(f,P)-I\big)+\big(I-L(f,P)\big) \le \frac{2\varepsilon}{3} < \varepsilon.$$

By integral criterion, $\int_a^b f(x)\,dx$ exists, say it is I'.

Using (a) $\Rightarrow$ (b), we get $\lim_{\|P\|\to 0} \sum f(t_j)\Delta x_j = I'$

Using (b) $\Rightarrow$ (c), we get $\lim_{Q\to\infty} \sum f(t_j)\Delta x_j = I'$. Uniqueness of limit

Since we are given (c) $\lim_{Q\to\infty}\sum_{j=1}^{n} f(t_j)\Delta x_j = I$, $\therefore I' = I$.

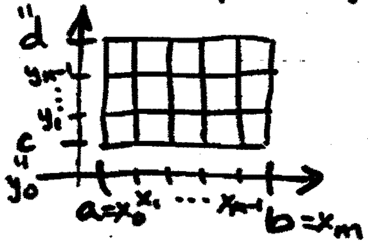
Multiple Integration Theory

We will outline the theory on $\mathbb{R}^2$ and leave the theory on $\mathbb{R}^n$ to the interested students.

Let P_1 be a partition of $[a,b]$ and P_2 be a partition of $[c,d]$. Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be a bounded function. Let $P_1=\{\underset{a''}{x_0},x_1,\dots,\underset{b''}{x_m}\}$, $P_2=\{\underset{c''}{y_0},y_1,\dots,\underset{=d}{y_n}\}$.

For $i=1,2,\dots,m$; $j=1,2,\dots,n$, define

$$m_{ij}=\inf\{f(x,y): x\in[x_{i-1},x_i],\ y\in[y_{j-1},y_j]\}$$
$$M_{ij}=\sup\{f(x,y): x\in[x_{i-1},x_i],\ y\in[y_{j-1},y_j]\}$$



Let $P=P_1\times P_2$, $\|P\|=\max(\|P_1\|,\|P_2\|)$,

$$L(f,P)=\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\,\Delta x_i\,\Delta y_j$$
$$U(f,P)=\sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\,\Delta x_i\,\Delta y_j$$

$$(L)\int_{[a,b]\times[c,d]} f(x,y)\,dx\,dy=\sup\left\{L(f,P): P=P_1\times P_2,\ \begin{matrix}P_1 \text{ partition of } [a,b]\\ P_2 \text{ partition of } [c,d]\end{matrix}\right\}$$

$$(U)\int_{[a,b]\times[c,d]} f(x,y)\,dx\,dy=\inf\{U(f,P): \quad ..\quad ..\quad ..\quad ..\}$$

<u>Definition</u> f is <u>Riemann integrable</u> iff $(L)\int_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = (U)\int_{[a,b]\times[c,d]} f(x,y)\,dx\,dy$. In that case, $\int_{[a,b]\times[c,d]} f(x,y)\,dx\,dy$ is the common value.

<u>Remarks</u> Let $P=P_1\times P_2$ and $Q=Q_1\times Q_2$.

Q is a refinement of $P \iff Q\geq P \iff \begin{matrix}Q_1\geq P_1 \text{ and}\\ Q_2\geq P_2.\end{matrix}$

<u>Theorem</u> Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be bounded. T.F.A.E. (the following are equivalent):

① $\displaystyle\int_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = I$

② $\displaystyle\lim_{\|P\|\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n} f(s_i,t_j)\,\Delta x_i\,\Delta y_j = I$

③ $\displaystyle\lim_{Q\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(s_i,t_j)\,\Delta x_i\,\Delta y_j = I$.

<u>Theorem</u> Continuous functions are Riemann integrable.

<u>Fubini's Theorem</u> If $f:[a,b]\times[c,d]\to\mathbb{R}$ is integrable, then

$$\int_{[a,b]\times[c,d]} f(x,y)\,dx\,dy=\int_a^b\left(\begin{smallmatrix}(L)\\ \text{or}\\ (U)\end{smallmatrix}\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\begin{smallmatrix}(L)\\ \text{or}\\ (U)\end{smallmatrix}\int_a^b f(x,y)\,dx\right)dy.$$

In particular, if f is continuous, then

$$\int_{[a,b]\times[c,d]} f(x,y)\,dx\,dy=\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy.$$

For proof, see Apostol's book, Theorem 14.6.

Definition A set S in $\mathbb{R}^2$ is of measure 0 iff $\forall \varepsilon > 0$ $\exists\ (a_1,b_1)\times(c_1,d_1),\ (a_2,b_2)\times(c_2,d_2)\times\cdots$ such that $S \subseteq \bigcup_{i=1}^{\infty}(a_i,b_i)\times(c_i,d_i)$ and $\sum_{i=1}^{\infty}|a_i-b_i||c_i-d_i| < \varepsilon$.

Lebesgue's Theorem For bounded $f:[a,b]\times[c,d]\to\mathbb{R}$, f is integrable $\Longleftrightarrow S_f=\{(x,y): f \text{ is discontinuous at } (x,y)\}$ is of measure 0.

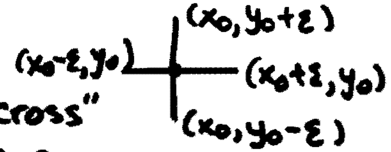
Tonelli-Hobson's Theorem For bounded $f:[a,b]\times[c,d]\to\mathbb{R}$, if $\int_a^b\left(\int_c^d |f(x,y)|\,dy\right)dx$ or $\int_c^d\left(\int_a^b |f(x,y)|\,dx\right)dy$ exists, then f is integrable and

$$\int_{[a,b]\times[c,d]} f(x,y)\,dx\,dy = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy.$$

In particular, if $f(x,y)\ge 0$ $\forall (x,y)\in[a,b]\times[c,d]$ and either $\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx$ or $\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$ exists, then the same conclusion as above is true.

For proof, see Apostol's book, Theorem 15.8.

Partial Differentiation



Let $S\subseteq\mathbb{R}^2$ and S contain a "cross". Define partial derivatives of $f:S\to\mathbb{R}$ at (x_0,y_0) to be

$$\frac{\partial f}{\partial x}(x_0,y_0)=\left(\frac{d}{dx}f(x,y_0)\right)_{\text{at } x=x_0}$$

and

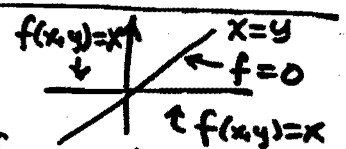
$$\frac{\partial f}{\partial y}(x_0,y_0)=\left(\frac{d}{dy}f(x_0,y)\right)_{\text{at } y=y_0}$$

Examples ① Let $f(x,y)=x\cos(xy)e^y$ on $\mathbb{R}^2$. For $(x_0,y_0)\in\mathbb{R}^2$,

$$\frac{\partial f}{\partial x}(x_0,y_0)=\left(\frac{d}{dx}x\cos(xy_0)e^{y_0}\right)_{\text{at } x=x_0} = e^{y_0}\left(\cos(x_0y_0)-x_0\sin(x_0y_0)\,y_0\right)$$

$$\frac{\partial f}{\partial y}(x_0,y_0)=\left(\frac{d}{dy}x_0\cos(x_0y)e^y\right)_{\text{at } y=y_0} = x_0\left(-\sin(x_0y_0)x_0e^{y_0}+\cos(x_0y_0)e^{y_0}\right)$$

② Let $f(x,y)=\begin{cases} x & \text{if } x\ne y\\ 0 & \text{if } x=y\end{cases}$



① For $(x_0,y_0)\in\mathbb{R}^2$, if $x_0\ne y_0$, then $\frac{\partial f}{\partial x}(x_0,y_0)=\left(\frac{d}{dx}x\right)_{x=x_0}=1$, $\frac{\partial f}{\partial y}(x_0,y_0)=0$

② if $x_0=y_0\ne 0$, then $f(x,y_0)$ and $f(x_0,y)$ are discontinuous at $x=x_0$ and at $y=y_0$ respectively, so $\frac{\partial f}{\partial x}(x_0,y_0)$ and $\frac{\partial f}{\partial y}(x_0,y_0)$ do not exist.

③ if $x_0=y_0=0$, then $\frac{\partial f}{\partial x}(0,0)=\left(\frac{d}{dx}x\right)_{x=x_0}=1$, $\frac{\partial f}{\partial y}(0,0)=\left(\frac{d}{dy}0\right)_{y=0}=0$.

Theorem (Differentiation Under the Integral)
Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be continuous. $\forall y\in[c,d]$, let $g(y)=\int_a^b f(x,y)\,dx$. If function $\frac{\partial f}{\partial y}$ is continuous on $[a,b]\times[c,d]$, then $\forall y\in[c,d]$,

$$\frac{d}{dy}\left(\int_a^b f(x,y)\,dx\right)=\frac{dg}{dy}=\int_a^b \frac{\partial f}{\partial y}(x,y)\,dx .$$

Proof. By the fundamental theorem of calculus,

$$\int_c^y \frac{\partial f}{\partial y}(x,y)\,dy=\underline{f(x,y)}-f(x,c).$$

So $g(y)=\int_a^b \underline{f(x,y)}\,dx=\int_a^b\left(\int_c^y \frac{\partial f}{\partial y}(x,y)\,dy\right)dx+\int_a^b f(x,c)\,dx$

By Fubini's theorem,

$$g(y)=\int_c^y\left(\int_a^b \frac{\partial f}{\partial y}(x,y)\,dx\right)dy+\int_a^b f(x,c)\,dx.$$

By the fundamental theorem of calculus,

$$g'(y)=\int_a^b \frac{\partial f}{\partial y}(x,y)\,dx.$$

Question If $h(y)=\int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx$, then what is $h'(y)$?

Define $g(y,v,u)=\int_u^v f(x,y)\,dx$. If $u=\alpha(y)$, $v=\beta(y)$, then by the chain rule in multivariable calculus,

$$h'(y)=\frac{d}{dy}g(y,v,u)=\frac{\partial g}{\partial y}\frac{dy}{dy}+\frac{\partial g}{\partial v}\frac{dv}{dy}+\frac{\partial g}{\partial u}\frac{du}{dy}$$

$$=\int_{u=\alpha(y)}^{v=\beta(y)}\frac{\partial f}{\partial y}(x,y)\,dx+f(\underset{\beta(y)}{\underset{\shortparallel}{v}},y)\beta'(y)-f(\underset{\alpha(y)}{\underset{\shortparallel}{u}},y)\alpha'(y).$$

Examples ① Find $\int_0^1 \frac{x^\pi-x^e}{\ln x}\,dx$.

Solution Recall $\frac{d}{dt}(a^t)=a^t\ln a$. So $\int a^t\,dt=\frac{a^t}{\ln a}+C$

$\int_e^\pi x^y\,dy=\frac{x^y}{\ln x}\Big|_e^\pi=\frac{x^\pi-x^e}{\ln x}$. Now $f(x,y)=x^y$ is continuous on $[0,1]\times[e,\pi]$. By Fubini's theorem,

$$\int_0^1\frac{x^\pi-x^e}{\ln x}\,dx=\int_0^1\left(\int_e^\pi x^y\,dy\right)dx=\int_e^\pi\left(\int_0^1 x^y\,dx\right)dy$$

$$=\int_e^\pi\left(\frac{x^{y+1}}{y+1}\Big|_0^1\right)dy=\int_e^\pi\frac{1}{y+1}\,dy=\ln(y+1)\Big|_e^\pi=\ln\frac{\pi+1}{e+1}.$$

Remarks $\lim_{x\to0^+}\frac{x^\pi-x^e}{\ln x}=0$ and $\lim_{x\to1^-}\frac{x^\pi-x^e}{\ln x}=\lim_{x\to1^-}\frac{\pi x^{\pi-1}-ex^{e-1}}{1/x}=\pi-e$

So $\int_0^1\frac{x^\pi-x^e}{\ln x}\,dx$ is a proper integral.

__Alternatively__ we can also do it the following way. Let $g(y)=\int_0^1\frac{x^y-x^e}{\ln x}\,dx$, then differentiating under the integral,

$$g'(y)=\int_0^1\frac{\partial}{\partial y}\left(\frac{x^y-x^e}{\ln x}\right)dx=\int_0^1 x^y\,dx=\frac{x^{y+1}}{y+1}\Big|_0^1=\frac{1}{y+1}$$

$\Rightarrow g(y)=\ln(y+1)+C.$

Since $g(e)=0$, so $C=-\ln(e+1)$. Therefore,

$$\int_0^1\frac{x^\pi-x^e}{\ln x}\,dx=g(\pi)=\ln(\pi+1)-\ln(e+1)=\ln\frac{\pi+1}{e+1}.$$

Remarks With additional conditions, both differentiation under the integral and the Tonelli-Hobson theorem can be extended to improper integral settings.

Examples ② Compute $I=\int_0^{+\infty} e^{-x^2}dx$ formally.

Solution. (Note $0 \le e^{-x^2} \le e^{-x}$ on $[1,+\infty)$ and $\int_1^{+\infty} e^{-x}dx = \lim_{c\to+\infty}(-e^{-x})\Big|_0^c = \lim_{c\to+\infty}(-e^{-c}+1)=1$. By the Comparison test, $\int_1^{+\infty} e^{-x^2}dx<\infty$. $\therefore \int_0^{+\infty} e^{-x^2}dx<\infty$, Since $\int_0^1 e^{-x^2}dx$ is a proper Riemann integral.)

To find $I=\int_0^{+\infty} e^{-x^2}dx$ formally, we first substitute $x=ut$ for constant $u>0$ to get $dx=u\,dt$

$$I=\int_0^{+\infty} e^{-x^2}dx = \int_0^{+\infty} u e^{-u^2t^2}dt.$$

Now

$$I^2=\int_0^{+\infty} I e^{-u^2}du = \int_0^{+\infty}\left(\int_0^{+\infty} u e^{-u^2t^2}dt\right)e^{-u^2}du$$

$$=\int_0^{+\infty}\int_0^{+\infty} \underbrace{u e^{-(1+t^2)u^2}}_{\ge 0}dt\,du$$

Tonelli-Hobson

$$=\int_0^{+\infty}\int_0^{+\infty} u e^{-(1+t^2)u^2}du\,dt$$

$$=\int_0^{+\infty}\left(\frac{e^{-(1+t^2)u^2}}{-2(1+t^2)}\Big|_0^{\infty}\right)dt$$

$$=\int_0^{+\infty}\frac{1}{2}\frac{1}{1+t^2}dt = \frac{1}{2}\operatorname{Arctan} t\Big|_0^{+\infty} = \frac{1}{2}\frac{\pi}{2}.$$

$$\therefore I=\frac{\sqrt{\pi}}{2}.$$

③ Compute $\int_0^{+\infty}\frac{\sin x}{x}dx$ formally.

Solution. For $t\ge 0$, define $I(t)=\int_0^{+\infty} e^{-tx}\frac{\sin x}{x}dx$.

Differentiating under the integral, we get

$$I'(t)=\int_0^{+\infty}\frac{\partial}{\partial t}\left(e^{-tx}\frac{\sin x}{x}\right)dx$$

$$=\int_0^{+\infty} -e^{-tx}\sin x\,dx \quad \text{integration by parts twice}$$

$$=-\frac{1}{1+t^2}$$

Then $I(t)=-\operatorname{Arctan} t + C$. Now

$$|I(t)|\le\int_0^{+\infty}\left|e^{-tx}\frac{\sin x}{x}\right|dx \le \int_0^{+\infty} e^{-tx}dx = \frac{1}{t}.$$

As $t\to\infty$, $I(t)\to -\frac{\pi}{2}+C$ and $I(t)\to 0$. $\therefore C=\frac{\pi}{2}$.

Then $\int_0^{+\infty}\frac{\sin x}{x}dx = I(0) = C = \frac{\pi}{2}$.