MATH202 Introduction to Analysis

Note on Countability

Part I: How to Count Thing?

Before we discuss about countability, we need to first understand what is counting?

Given a set A={a,f,y,b,c,w,e,%,&}

How many elements are there? A fast counting tells us there are 9 elements in this set. But how do we get this? Many people will answer that they count from left to right: 1,2,3,4,5,6,7,8,9. So we get 9.

From this simple example, we see that when we count thing, we have assign a number (1,2,3,4,…. i.e. positive integer) to each elements. That's is

$$
\begin{array}{cccccccccc}
A: & a & f & y & b & c & w & e & \% & \& \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
\mathbf{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}
$$

Mathematically, we say there is a map (bijective) which maps A to **N**

Why we need countability concept?

Given a set with finitely many elements (for example: students in MATH202?), we can always count it by some methods. For an infinite set, of course we can not tell people how many elements are there, but if we can find out the method to count it (i.e. we can assign each element to a positive integr), then we say it is **countable (or countable infinite)**.

Definition: (Countability) Given a set S, we say S is countable if and only if there exists a bijection map $f: S \to \mathbf{N}$ (or $f: \mathbf{N} \to S$**)** between S and **N** (positive integer set)

(*Note: The bijection map can be **from S to N** or **from N to S**. Both direction are OK because when we have the bijection of one direction (e.g. $N \to S$). Then there exists an inverse function (i.e. $N \to S$), this inverse, of course is bijective)

To show a set is countable, all we need to do is to find a bijective map from S to N (or from N to S)

Let look at some simple examples:

Example 1

Define a set

$$S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \dots\}$$

Show that it is countable

Solution:

Define a function $S \to \mathbf{N}$**,** such that

$$f\left(\frac{1}{n}\right) = n$$

which is a bijection (left to readers as exercise)

Therefore S is countable.

Example 2

Show that $N \times N$ is countable

Solution:

We first list out the elements

(1,1)	(1,2)	(1,3)	(1,4)	....
(2,1)	(2,2)	(2,3)	(2,4)	....
(3,1)	(3,2)	(3,3)	(3,4)	....
(4,1)	(4,2)	(4,3)	(4,4)	....

We can’t count in neither horizontal nor vertical direction, it is because suppose we count it horizontally, (1,1,),(1,2),(1,3),…. and since it is endless, then we are unable to count (2,1),(2,2)…etc….

So to count it, we count diagonally, i.e.

$x + y = 2$	(1,1)	(1,2)	(1,3)	(1,4)	....
$x + y = 3$	(2,1)	(2,2)	(2,3)	(2,4)	....
$x + y = 4$	(3,1)	(3,2)	(3,3)	(3,4)	....
	(4,1)	(4,2)	(4,3)	(4,4)	....

Starting from (1,1) and following by (2,1),(1,2) and so on……

Now we know how to count it. We still need to find a bijective map.

Consider an element $(m,n) \in \mathbf{N} \times \mathbf{N}$

(m,n) ← nth elements in this "row"

....

(m,1)

$x + y = m + n$

$f(m,n) = (\text{Total Number of elements in } x + y = 2, x + y = 3, \ldots, x + y = m + n - 1) + n$

(Since there are 1 elements in $x + y = 2$, 2 in $x + y = 3$ and 3 in $x + y = 4$....).

So we have

$$f(m,n) = 1 + 2 + 3 + \cdots .. + (m + n - 2) + n = \frac{(m + n - 1)(m + n - 2)}{n} + n$$

which is bijective.

(In Kin Li's Note: The map is $(m,n) = f(\frac{(m+n-1)(m+n-2)}{n} + n)$ which is same as this one.)

Therefore $\mathbf{N} \times \mathbf{N}$ is countable

☺Exercise 1

Show that $A = \{\text{the set of square number}\} = \{1,4,9,16,25,36, \ldots .\}$ is countable.

☺Exercise 2 (More Difficult)

Show that the set $B = \bigcup_{n=1}^{\infty}\{x \in R : x^2 - n = 0\}$ is countable by constructing the bijective function $f: B \to N$ (or $f: N \to B$)

How to prove a set is uncountable?

If a set S is uncountable, then by definition, there is NO bijective map from S to N.

How can we show there is NO bijective map? We can use proof by contradiction method. We first assume the set S is countable, then there will be a bijective map $\mathrm{f}: \mathrm{S} \rightarrow \mathrm{N}$. In another word, we can arrange the elements in set S into

$$\mathrm{S} = \{s_1, s_2, s_3, s_4, \dots \dots .\}$$

Then try to arrive contradiction by considering some elements.

Example 3 Show that an open interval (0,1) is uncountable

Proof:

Suppose (0,1) is countable, then there exists a bijective map $\mathrm{f}: \mathrm{N} \rightarrow (0,1)$, then we can rearrange the elements into

$$\begin{gathered} f(1) = 0.\mathbf{a_{11}}a_{12}a_{13}a_{14} \dots \dots .. \\ f(2) = 0.a_{21}\mathbf{a_{22}}a_{23}a_{24} \dots \dots .. \\ f(3) = 0.a_{31}a_{32}\mathbf{a_{33}}a_{34} \dots \dots .. \\ \dots\dots \\ f(k) = 0.a_{k1}a_{k2}a_{k3}a_{k4} \dots \dots \dots \\ \dots.. \end{gathered}$$

Now

We construct an element $b = 0.b_1b_2b_3b_4 \dots \dots$ and $b_k \neq 0$ or 9* which

$b_1 \neq a_{11}, b_2 \neq a_{22}, b_3 \neq a_{33}, \dots .. b_k \neq a_{kk} \dots\dots$

Now we see this $b \neq f(1), b \neq f(2), b \neq f(3), \dots.$ because it must differ by at least one digit.

Therefore, there is no n which $f(n) = b$, where say f is not surjective, then it contradicts to f is bijective.

So (0,1) is uncountable.

(*Note: we need to require $b_k \neq 0$ or 9, the purpose is to prevent the cases likes $0.23140000 \dots .. = 0.2314$ or $0.32889999999 \dots \dots = 0.3289$. Then the decimal representation of each number is UNIQUE.

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☺Exercise

Prove that the set $S = \{0.a_1a_2a_3 \dots \dots : a_k = 1 \text{ or } 2\}$ is uncountable.

Part 2: How to deal with difficult sets?

Now we have some basic ideas to prove some sets are countable or uncountable using definition. But in many cases, the set we investigate are complicated. So in order to check the set is countable or not, we need some special methods. In practice, we have two main methods.

Method I: By set theory

We first introduce one of the two useful and basic theorems.

Theorem 1A Let A, B be two sets and $A \subseteq B$, if B is countable, then A is also countable

By taking contrapositive ($p \to q = \sim q \to \sim p$), we have another theorem

Theorem 1B Suppose $A \subseteq B$, if A is uncountable, then B is also uncountable

These two theorems said (1) a set is subset of countable set is countable and (2) a set contains an countable subset is uncountable)

Let us look at the examples below, to see how to apply the theorems

Example 4 Show that $A = \{a \in \mathbf{R}: x^2 + 3ax + a = 0$ has integer solution$\}$ is countable.

IDEA: Here we will show A is subset of some countable set.

Solution:

For any $a \in A$,

Then $x^2 + 3ax + a = 0$ has integer solution (Let this solution be N),

$\to N^2 + 3aN + a = 0$ (We substitute $x = N$)

$\rightarrow \mathrm{a} = -\dfrac{\mathrm{N}^2}{3\mathrm{N}+1}$

Since N is integer, therefore a is rational number. <u>So $a \in Q$</u>

Therefore: <u>$A \subseteq Q$</u> (Recall: $A \subseteq B \leftrightarrow$ for any $a \in A$, we have $a \in B$

Since Q is countable, therefore A is countable**.**

Example 5 Show that $\mathbf{R}$ is uncountable

Solution:

Since $\mathbf{R} \supseteq (0,1)$, from Example 3, we know (0,1) is uncountable, therefore $\mathbf{R}$ is uncountable.

☺Exercise 3 (Practice Exercise #24)

Prove that the set $F = \{a: x^4 + ax - 5 = 0 \text{ has a rational root}\}$ is countable

☺Exercise 4 (Practice Exercise #25)

Prove that the set $G = \{a + b{:}a \in \mathbf{Q}, b \in \mathbf{R}\}$ is uncountable.

(Hint: Set $a = 0$ and see what happens)

Another method is using Set Operation Method.

Theorem 2A If $A_1, A_2, A_3, A_4, \ldots..$ are countable sets, then the following sets are countable (1) $A_1 \cup A_2 \cup A_3 \cup \ldots . \cup A_k$ and $\bigcup_{n=1}^{\infty} A_n$ (2) $A_1 \cap A_2 \cap A_3 \cap \ldots . \cap A_k$ and $\bigcap_{n=1}^{\infty} A_n$ (3) $A_1 \times A_2 \times A_3 \times \ldots \ldots \times A_k$ (But it is not true for infinite many A_k)

A more general result will be helpful

Theorem 2B: If S is countable set and A_s is countable for every $s \in S$. Then $\bigcup_{s \in S} A_s$ is countable.

What is the use of these theorem? Sometimes if the set is too complicated, by using some set operations, we can express the set into the union/intersection/product of some simple sets which are easy to determine whether it is countable or not.

Example 6 Show that the set $A = \{am + bn : m \in \mathbf{N}, n \in \mathbf{N}\}$ is countable, where a and b are real numbers

Solution:

Note (Put $m = 1,2,3,4, \dots \dots$)

$A = \{\mathbf{a(1) + bn: n \in N}\} \cup \{\mathbf{a(2) + bn: n \in N}\} \cup \{\mathbf{a(3) + bn: n \in N}\} \cup \dots \dots ..$

$\rightarrow A = \bigcup_{m=1}^{\infty}\{am + bn : n \in \mathbf{N}\}$

Let $P_m = \{am + bn : n \in N\}$

Define a map $f: P_m \rightarrow \mathbf{N}$, which $f(am + bn) = n$,

This is a bijective function, so P_m is countable

So $A = \bigcup_{m=1}^{\infty}(am + bn : n \in \mathbf{N})$ is also countable by Theorem 2A (1)

Example 7 Show that the set L of all lines with equation $y = mx + b$, where $m, b \in \mathbf{Q}$ is countable. (Note: It gives another solution other than the one in Lecture Note)

Solution:

Using similar argument as Example 6, we have

$L = \{y = mx + b : m, b \in Q\}$

$$L = \bigcup_{m \in Q}\{y = mx + b : b \in \mathbf{Q}\}$$

$$L = \bigcup_{m \in Q}\bigcup_{b \in Q}\{y = mx + b\}$$

Now $\{y = mx + b\}$ has only 1 element for fix m and b, therefore it is countable.

Then $\bigcup_{b \in Q}\{y = mx + b\}$ is countable (By theorem 2B, since $\mathbf{Q}$ is countable)

Finally $L = \bigcup_{m \in Q}\bigcup_{b \in Q}\{y = mx + b\}$ is also countable. (By theorem 2B)

Example 8

Let A and B are countable subsets in $\mathbf{R}$, show that the set

$S = \{a + b\sqrt{7} : a \in A \text{ and } b \in B\}$ is countable

Solution:

Define a function $f: A \times B \to S$ which $f(a, b) = a + b\sqrt{7}$,

We can show f is bijective (left as exercise for you).

Since A and B are countable, then $A \times B$ is countable,

As f is bijective, so S is also countable.

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☺Exercise 5 (Practice Exercise #19)

Prove that $A = \left\{\frac{1}{2^n} + \frac{1}{3^m} : n, m \in N\right\}$ is countable

☺Exercise 6 (Practice Exercise #23)

Prove that the set E of all circles in $\mathbf{R}^2$ with centers at rational coordinate points and positive rational radius is countable.

☺Exercise 7

Prove that the set $\{x \in R: x^9 + x^2 + 2 \in \mathbf{Z}\}$