Chapter 4 Infinite Series

An infinite series is of the form

$$a_1 + a_2 + a_3 + \cdots \quad \text{or} \quad \sum_{k=1}^{\infty} a_k$$

where $a_1, a_2, a_3, \ldots$ are numbers.

For $n \in \mathbb{N}$, $S_n = \sum_{k=1}^{n} a_k$ is the n^{th} partial sum of the series.

Examples ① $\sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$,

$$S_n = \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 2 - \frac{1}{2^n} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = \lim_{n\to\infty} S_n = 2$$

We say the series converges to 2 in this case.

② $\sum_{k=1}^{\infty} 1 = 1+1+1+1+\cdots$, $\quad S_n = \sum_{k=1}^{n} 1 = n$

We have $\lim_{n\to\infty} S_n = \infty$. We say the series diverges to ∞

③ $\sum_{k=1}^{\infty} (-1)^{k-1} = 1 + (-1) + 1 + (-1) + \cdots$; $S_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$

$\lim_{n\to\infty} S_n$ doesn't exist. We say the series diverges.

Definitions For an infinite series $\sum_{k=1}^{\infty} a_k$,

① it converges to a number S iff $\lim_{n\to\infty} S_n = S$.
(S is the sum of the series.)

② it diverges to ∞ iff $\lim_{n\to\infty} S_n = \infty$.

③ it diverges iff $\lim_{n\to\infty} S_n$ doesn't exist.

Facts

① For $\sum_{k=1}^{\infty} a_k$ with partial sums S_n, we have

$\boxed{a_1 = S_1}$, $a_2 = (a_1+a_2) - a_1 = S_2 - S_1, \dots$

$\underline{k>1} \Rightarrow a_k = (a_1 + \dots + a_k) - (a_1 + \dots + a_{k-1})$
$\Rightarrow \boxed{a_k = S_k - S_{k-1}}$

② For $m \in \mathbb{N}$, $\sum_{k=1}^{\infty} a_k$ converges to A iff $\sum_{k=m}^{\infty} a_k$ converges to

$$B = \lim_{n\to\infty}(a_m + \dots + a_n) = \lim_{n\to\infty}(S_n - (a_1 + \dots + a_{m-1}))$$
$$= \lim_{n\to\infty} S_n - (a_1 + \dots + a_{m-1}) = A - (a_1 + \dots + a_{m-1}).$$

To check $\sum_{k=1}^{\infty} a_k$ converge, it is enough to check $\sum_{k=m}^{\infty} a_k$ converge for some $m \in \mathbb{N}$.

③ If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, where A, B numbers

then $$\sum_{k=1}^{\infty}(a_k + b_k) = A + B = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$
$$\sum_{k=1}^{\infty}(a_k - b_k) = A - B = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$$

$\forall c \in \mathbb{R}$, $\sum_{k=1}^{\infty}(c\, a_k) = c \sum_{k=1}^{\infty} a_k$.

Geometric Series Test

$$\sum_{k=0}^{\infty} r^k = \lim_{n\to\infty}(1 + r + r^2 + \dots + r^n) = \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r}$$
$$= \begin{cases} \frac{1}{1-r} & \text{if } |r|<1 \\ \text{doesn't exist} & \text{if } |r| \geq 1. \end{cases}$$

Example $$0.999\dots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots$$
$$= \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \dots\right)$$
$$= \frac{9}{10} \cdot \frac{1}{1-\frac{1}{10}} = 1 = 1.000\dots$$

Telescoping Series Test

$(b_1-b_2)+(b_2-b_3)+(b_3-b_4)+\dots$

$$= \sum_{k=1}^{\infty}(b_k - b_{k+1}) = \lim_{n\to\infty}((b_1-b_2)+(b_2-b_3)+\dots+(b_n-b_{n+1}))$$
$$= \lim_{n\to\infty}(b_1 - b_{n+1}) = b_1 - \lim_{n\to\infty} b_{n+1}$$

Examples ① $$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) \qquad b_k = \frac{1}{k}$$
$$= \left(1-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}-\tfrac{1}{3}\right) + \left(\tfrac{1}{3}-\tfrac{1}{4}\right) + \dots = 1 - \lim_{n\to\infty}\frac{1}{n+1} = 1$$

② $$\sum_{k=1}^{\infty}\left(5^{1/k} - 5^{1/(k+1)}\right) = (5-\sqrt{5}) + (\sqrt{5}-\sqrt[3]{5}) + \dots$$
$$= 5 - \lim_{n\to\infty} 5^{1/(n+1)} = 5 - 5^0 = 5 - 1 = 4.$$

Term Test If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} a_n = 0$.

(**Contrapositive**: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.)

Reason $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = S \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty}(S_n - S_{n-1})$

$$= S - S = 0.$$

Examples ① $1+1+1+\cdots = \sum_{k=1}^{\infty} 1 \quad a_n = 1, \ \lim_{n\to\infty} a_n = 1 \neq 0$

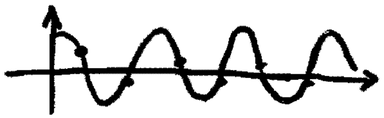
↖ series diverges.

② $\sum_{k=1}^{\infty} \cos\left(\frac{1}{k}\right) \quad a_n = \cos\frac{1}{n}, \ \lim_{n\to\infty} a_n = \cos 0 = 1 \neq 0$

↖ Series diverges

③ $\sum_{k=1}^{\infty} \cos k \quad a_n = \cos n, \ \lim_{n\to\infty} a_n \neq 0$

↖ series diverges ↖ why?



Assume $\lim_{n\to\infty} \cos n = 0$.

Then $\cos 1, \cos 2, \cos 3, \dots \to 0$

So $\cos 2, \cos 3, \cos 4, \dots \to 0 \Leftrightarrow \lim_{n\to\infty} \cos(n+1) = 0$

$\lim_{n\to\infty} |\sin n| = \lim_{n\to\infty} \sqrt{1-\cos^2 n} = \sqrt{1-0^2} = 1$

$0 = \lim_{n\to\infty} |\cos(n+1)| = \lim_{n\to\infty} |\underbrace{\cos n}_{0} \cos 1 - \sin n \sin 1| = |\sin 1| \neq 0$

Contradiction.

Question What if $\lim_{n\to\infty} a_n = 0$?

Answer $\sum_{k=1}^{\infty} a_k$ may or may not converge.

Examples ④ $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k$

$a_n = \left(-\frac{1}{2}\right)^n, \ \lim_{n\to\infty} a_n = 0,$ ↖ series converges by geometric series test

$r = -\frac{1}{2}, \ |r| = \frac{1}{2} < 1.$

---

⑤ $1 + \underbrace{\frac{1}{2} + \frac{1}{2}}_{2 \text{ times}} + \underbrace{\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}}_{4 \text{ times}} + \underbrace{\frac{1}{8} + \cdots + \frac{1}{8}}_{8 \text{ times}} + \cdots$

$a_1 \geq a_2 \geq a_3 \geq \cdots \quad a_n$ is "decreasing to 0"

$\lim_{n\to\infty} a_n = 0$

$S_1 \leq S_2 \leq S_3 \leq \cdots \quad S_{2^n - 1} = n \quad \lim_{n\to\infty} S_n = \infty$

Series diverges to ∞. S_n is "increasing" to ∞.

Nonnegative Series $\sum_{k=1}^{\infty} a_k$ with $a_k \geq 0 \ \forall k$

$\Rightarrow \forall n,\ S_{n+1} = S_n + a_{n+1} \geq S_n$

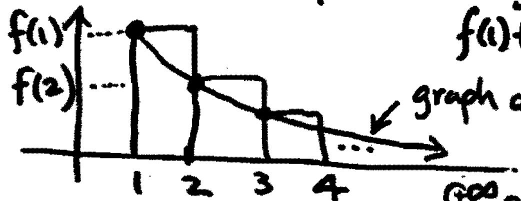
$\Rightarrow S_1 \leq S_2 \leq S_3 \leq \cdots \Rightarrow \lim_{n\to\infty} S_n =$ number or $+\infty$

$\Rightarrow \sum_{k=1}^{\infty} a_k$ converges or $\sum_{k=1}^{\infty} a_k$ diverges to $+\infty$.

Integral Test Let $f: [1,\infty) \to \mathbb{R}$ decrease to 0 as $x \to \infty$. Then

$$\sum_{k=1}^{\infty} f(k) \text{ converges} \iff \underbrace{\int_1^{+\infty} f(x)\,dx}_{=\lim_{t\to\infty}\int_1^t f(x)dx} < \infty$$

Reason ($\Rightarrow$) "$\sum_{k=1}^{\infty} f(k)$ converges" means



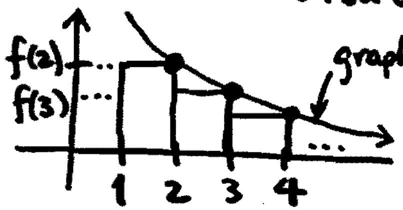
$f(1)+f(2)+f(3)+f(4)+\cdots < \infty$

$$\int_1^{+\infty} f(x)\,dx \leq f(1)+f(2)+\cdots < \infty$$

area under graph of f on $[1,\infty)$ area of rectangles on $[1,\infty)$

($\Leftarrow$) "$\int_1^{\infty} f(x)\,dx < \infty$" means

area under graph of f on $[1,\infty) < \infty$



$$f(2)+f(3)+\cdots \leq \int_1^{+\infty} f(x)\,dx < \infty$$

$$\Rightarrow f(1) + \underbrace{(f(2)+f(3)+\cdots) < \infty}_{=\sum_{k=1}^{\infty} f(k) \text{ converges.}}$$

$f(1) \in \mathbb{R}$

Examples (1) Consider $\sum_{k=1}^{\infty} \frac{1}{1+k^2}$. $f(x) = \frac{1}{1+x^2}$

As $x \nearrow \infty$, $1+x^2 \nearrow \infty$, so $\frac{1}{1+x^2} \searrow 0$.

$$\int_1^{\infty} \frac{1}{1+x^2}\,dx = \operatorname{Arctan} x \Big|_1^{\infty} = \operatorname{Arctan}\infty - \operatorname{Arctan} 1 = \frac{\pi}{2} - \frac{\pi}{4} < \infty.$$

$\therefore \sum_{k=1}^{\infty} \frac{1}{1+k^2}$ converges by integral test.

---

(2) Consider $\sum_{k=2}^{\infty} \frac{1}{k\ln k}$ and $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$.

As $x \nearrow \infty$, $\ln x \nearrow \infty$, $x\ln x \nearrow \infty$, $x(\ln x)^2 \nearrow \infty$

so $\frac{1}{x\ln x} \searrow 0$, $\frac{1}{x(\ln x)^2} \searrow 0$.

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx \underset{u=\ln x}{=} \int_{\ln 2}^{+\infty} \frac{1}{u}\,du = \ln u \Big|_{\ln 2}^{+\infty} = \infty - \ln(\ln 2) = \infty$$

$\therefore \sum_{k=2}^{\infty} \frac{1}{k\ln k}$ diverges to ∞

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx \underset{u=\ln x}{=} \int_{\ln 2}^{+\infty} \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_{\ln 2}^{+\infty} = 0 - \left(-\frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$$

$\therefore \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ converges.

<u>p-test</u> For $p \in \mathbb{R}$, p constant

$$\zeta(p) = \sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots \text{ converges}$$

"p-series" $\iff p > 1$.

<u>Reason</u> For $p \le 0$, $\frac{1}{k^p} = k^{-p} = k^{|p|} \ge k^0 = 1$

$\Rightarrow \lim_{k\to\infty} \frac{1}{k^p} \ne 0 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges by term test.

For $p > 0$, as $x \nearrow \infty$, $x^p \nearrow \infty$, so $\frac{1}{x^p} \searrow 0$.

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \int_1^{+\infty} x^{-p}\,dx = \begin{cases} \ln x \Big|_1^{\infty} & p = 1 \\ \frac{x^{-p+1}}{-p+1}\Big|_1^{\infty} & 0 < p < 1 \text{ or } p > 1 \end{cases}$$

$$= \begin{cases} +\infty & p = 1 \\ +\infty & 0 < p < 1 \\ \frac{1}{p-1} & p > 1 \end{cases} \qquad \therefore \sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges } \iff p > 1.$$

<u>Known Cases</u> In 1736, Euler showed

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

$$\zeta(4) = 1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \cdots = \frac{\pi^4}{90}$$

$\vdots$

$$\zeta(2n) = r_n \pi^{2n}, \quad r_n \in \mathbb{Q}$$

In 1980, Apery showed

$\zeta(3) = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots$ is <u>irrational</u>.

<u>Comparison Test</u> Given $v_k \ge u_k \ge 0$ $\forall k \in \mathbb{N}$.

$\sum_{k=1}^{\infty} v_k$ converges $\Rightarrow \sum_{k=1}^{\infty} u_k$ converges.

(<u>Contrapositive</u>: $\sum_{k=1}^{\infty} u_k$ diverges $\Rightarrow \sum_{k=1}^{\infty} v_k$ ~~converges~~ diverges)

<u>Reason</u> $v_k \ge u_k \ge 0$ $\forall k \Rightarrow \sum_{k=1}^{\infty} v_k \ge \sum_{k=1}^{\infty} u_k \ge 0$

If $\sum_{k=1}^{\infty} v_k$ is a number, then $\sum_{k=1}^{\infty} u_k$ is a number.

If $\sum_{k=1}^{\infty} u_k = +\infty$, then $\sum_{k=1}^{\infty} v_k = +\infty$.

<u>Limit Comparison Test</u> Given $u_k, v_k \ge 0$ $\forall k \in \mathbb{N}$.

$\lim_{k\to\infty} \frac{v_k}{u_k} =$ positive number $\Rightarrow$ Both Σu_k, Σv_k converges or both Σu_k, Σv_k diverges

$\forall$ large k, $v_k \approx c\, u_k$, $c > 0$

$$\lim_{k\to\infty} \frac{v_k}{u_k} = 0 \Rightarrow \begin{cases} \Sigma u_k \text{ converges} \Rightarrow \Sigma v_k \text{ converges} \\ \Sigma v_k \text{ diverges} \Rightarrow \Sigma u_k \text{ diverges} \end{cases}$$

$\forall$ large k, $\frac{v_k}{u_k} < 1 \Rightarrow v_k < u_k$

$$\lim_{k\to\infty} \frac{v_k}{u_k} = +\infty \Rightarrow \begin{cases} \Sigma u_k \text{ diverges} \Rightarrow \Sigma v_k \text{ diverges} \\ \Sigma v_k \text{ converges} \Rightarrow \Sigma u_k \text{ converges.} \end{cases}$$

$\forall$ large k, $\frac{v_k}{u_k} > 1 \Rightarrow v_k > u_k$

Examples (1) Consider $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}\underbrace{\cos\left(\frac{1}{k}\right)}_{>0 \text{ as } \cos 1,\, \cos\frac{1}{2},\, \dots > 0}$

$$\left.\begin{array}{l} 0 \le \dfrac{1}{k^2}\cos\left(\dfrac{1}{k}\right) \le \dfrac{1}{k^2} \\ \displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2} \text{ converges} \\ \text{p series, } p=2>1 \end{array}\right\} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2}\cos\left(\frac{1}{k}\right) \text{ converges by comparison test}$$

(2) Consider $\displaystyle\sum_{k=2}^{\infty} \underbrace{\frac{3^k}{k^2-1}}_{>0}$ When k large, $\dfrac{3^k}{k^2-1}$ is dominated by 3^k

$0 \le \left(\frac{3}{2}\right)^k < \dfrac{3^k}{k^2-1}$ because $k^2-1 < 2^k$ for $k \ge 2$.

$\displaystyle\sum_{k=1}^{\infty}\left(\frac{3}{2}\right)^k$ diverges, geometric series $r = \frac{3}{2} > 1$ $\Rightarrow \displaystyle\sum_{k=1}^{\infty}\frac{3^k}{k^2-1}$ diverges by comparison test

(3) Consider $\displaystyle\sum_{k=1}^{\infty} \underbrace{\frac{\sqrt{k+1}}{k^2+5k}}_{>0}$ When k large, $\dfrac{\sqrt{k+1}}{k^2+5k} \approx \dfrac{\sqrt{k}}{k^2} = \dfrac{1}{k^{3/2}}$

Set $u_k = \dfrac{\sqrt{k}}{k^2} = \dfrac{1}{k^{3/2}}$ and $v_k = \dfrac{\sqrt{k+1}}{k^2+5k}$. $u_k, v_k > 0$.

$$\lim_{k\to\infty}\frac{v_k}{u_k} = \lim_{k\to\infty}\frac{\sqrt{k+1}}{k^2+5k}\cdot\frac{k^2}{\sqrt{k}} = \lim_{k\to\infty}\sqrt{\frac{k+1}{k}}\,\frac{k^2}{k^2+5k} = 1\cdot 1 = 1$$

$\sum u_k = \sum \dfrac{1}{k^{3/2}}$ converges, p-series $p = 3/2 > 1$ $\Rightarrow \displaystyle\sum v_k = \sum_{k=1}^{\infty}\frac{\sqrt{k+1}}{k^2+5k}$ Converges by limit comp. test.

(4) Consider $\displaystyle\sum_{k=1}^{\infty} \underbrace{\sin\left(\frac{1}{k}\right)}_{>0}$ When k large $\sin\left(\frac{1}{k}\right) \approx \frac{1}{k}$ as $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

$\theta = \frac{1}{k} \to 0$, $\sin\theta \approx \theta$

Set $u_k = \dfrac{1}{k}$, $v_k = \sin\dfrac{1}{k}$, $u_k, v_k > 0$

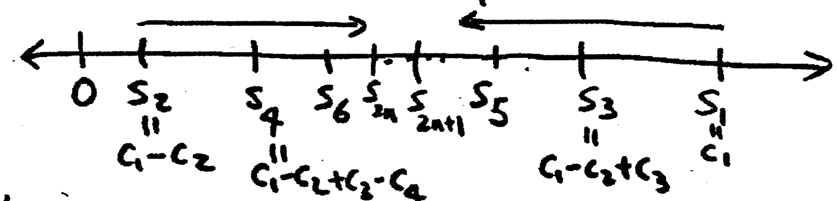
$$\lim_{k\to\infty}\frac{v_k}{u_k} = \lim_{k\to\infty}\frac{\sin(1/k)}{1/k} \underset{\theta = \frac{1}{k}\to 0}{=} \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$\sum u_k = \sum \frac{1}{k}$ diverges, p-series $p=1$ $\Rightarrow \displaystyle\sum v_k = \sum_{k=1}^{\infty}\sin\left(\frac{1}{k}\right)$ diverges by limit comp. test

Alternating Series Test If c_k decreases to 0 as $k\to\infty$ (i.e. $c_1 \ge c_2 \ge c_3 \ge \dots$ and $\displaystyle\lim_{k\to\infty} c_k = 0$), then

$$\sum_{k=1}^{\infty}(-1)^{k+1}c_k = \underbrace{c_1 - c_2 + c_3 - c_4 + c_5 - c_6 + \cdots}_{\text{"alternating series"}} \text{ converges.}$$

Reason For these series, partial sums are as follow



$$\lim_{n\to\infty}|S_{2n} - S_{2n+1}| = \lim_{n\to\infty}(S_{2n+1} - S_{2n}) = \lim_{n\to\infty} c_{2n+1} = 0$$

$\Rightarrow \displaystyle\lim_{n\to\infty} S_n$ is a number $\Rightarrow \displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}c_k$ Converges

Examples Consider $\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k}$ and $\sum_{k=1}^{\infty} e^{-k}\underbrace{\cos(k\pi)}_{=(-1)^k}$

For $c_k = \frac{1}{k\ln k}$, as $k\nearrow\infty$, $\ln k\nearrow\infty$, $k\ln k\nearrow\infty$

so $\frac{1}{k\ln k}\searrow 0$. $\therefore \sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k}$ converges by alt. series test.

For $c_k = e^{-k}$, as $k\nearrow\infty$, $-k\searrow-\infty$, $e^{-k}\searrow 0$

$\therefore \sum_{k=1}^{\infty} e^{-k}\cos(k\pi) = \sum_{k=1}^{\infty} (-1)^k e^{-k}$ converges by alt. series test.

Tests for general series $a_k\in\mathbb{R}\ \forall k\in\mathbb{N}$

Absolute Convergence Test $\underbrace{\sum_{k=1}^{\infty} |a_k|}_{\text{converges}} \Rightarrow \underbrace{\sum_{k=1}^{\infty} a_k}_{\text{converges}}$

(**Converse is false**: $\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k}$ converges from above example

$\sum_{k=2}^{\infty} \left|\frac{(-1)^k}{k\ln k}\right| = \sum_{k=2}^{\infty} \frac{1}{k\ln k}$ diverges from integral test)

page 31, example (2)

Reason for Absolute Convergence Test

$\forall k\in\mathbb{N}$, $-|a_k|\le a_k\le |a_k| \Rightarrow 0\le |a_k|+a_k\le 2|a_k|$

Add $|a_k|$ to all parts

$\underbrace{\sum|a_k| \text{ converges}}_{\text{Given}} \Rightarrow \sum 2|a_k|$ converges

$\Rightarrow \sum(|a_k|+a_k)$ converges by comparison test

$\Rightarrow \underbrace{\sum a_k}_{\text{Converges}} = \sum((|a_k|+a_k)-|a_k|) = \underbrace{\sum(|a_k|+a_k)}_{\text{Converges}} - \underbrace{\sum|a_k|}_{\text{Converges}}$

Definitions $\sum_{k=1}^{\infty} a_k$ converges absolutely iff $\sum_{k=1}^{\infty} |a_k|$ converges

$\sum_{k=1}^{\infty} a_k$ converges conditionally iff $\sum_{k=1}^{\infty} |a_k|$ diverges and $\sum_{k=1}^{\infty} a_k$ converges

Facts to be presented later

Dirichlet proved that for absolute convergent $\sum_{k=1}^{\infty} a_k$,

$\forall$ bijection $f:\mathbb{N}\to\mathbb{N}$, $\underbrace{\sum_{k=1}^{\infty} a_{f(k)}}_{\text{permutation of terms}} = \sum_{k=1}^{\infty} a_k$. same sum

Riemann proved that for condition convergent $\sum_{k=1}^{\infty} a_k$,

$\forall -\infty\le c\le\infty$, $\exists$ bijection $f:\mathbb{N}\to\mathbb{N}$,

$\underbrace{\sum_{k=1}^{\infty} a_{f(k)}}_{\text{permutation of terms}} = c$ ← sum may be arbitrary

Examples Consider $\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$ and $\sum_{k=1}^{\infty} \underbrace{\frac{\cos k\pi}{1+k}}_{=\frac{(-1)^k}{1+k}}$

$$\left.\begin{array}{l} 0\le\left|\frac{\cos k}{k^3}\right|\le\frac{1}{k^3} \\ \sum\frac{1}{k^3} \text{ converges} \\ \text{p-series, } p=3>1 \end{array}\right\} \Rightarrow \sum_{k=1}^{\infty}\left|\frac{\cos k}{k^3}\right| \text{ converges}$$

$\therefore \sum_{k=1}^{\infty}\frac{\cos k}{k^3}$ converges absolutely.

$\sum\left|\frac{\cos k\pi}{1+k}\right| = \sum\frac{1}{1+k}$ As $x\nearrow\infty$, $1+x\nearrow\infty$, so $\frac{1}{1+x}\searrow 0$

$\int_1^\infty \frac{1}{1+x}dx = \ln(1+x)\Big|_1^\infty = \infty \Rightarrow \sum\frac{1}{1+k}$ diverges

Alt. series test

$\frac{1}{1+k}\searrow 0 \Rightarrow \sum(-1)^k\frac{1}{1+k} = \sum_{k=1}^{\infty}\frac{\cos k\pi}{1+k}$ converges (hence conditionally)

Ratio Test If $\forall k$, $a_k \neq 0$ and $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ exists, then

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| \begin{cases} <1 \Rightarrow \sum a_k \text{ converges absolutely} \\ =1 \Rightarrow \sum a_k \text{ may or may not converge} \\ >1 \Rightarrow \sum a_k \text{ diverges} \end{cases}$$

Reason Let $r = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$. Then $\forall k$ large,

$$\left|\frac{a_{k+1}}{a_k}\right|, \left|\frac{a_{k+2}}{a_{k+1}}\right|, \ldots, \left|\frac{a_{k+n}}{a_{k+n-1}}\right| \approx r \Rightarrow \left|\frac{a_{k+n}}{a_k}\right| = r^n$$

$$\Rightarrow |a_{k+n}| \approx |a_k| r^n$$

$$\Rightarrow |a_k| + |a_{k+1}| + |a_{k+2}| + \cdots \approx |a_k|(1 + r + r^2 + \cdots)$$

So for $r<1$, $|a_k| + |a_{k+1}| + |a_{k+2}| + \cdots \approx \frac{|a_k|}{1-r}$

"hence" $\sum |a_k|$ converges

For $r>1$, $1+r+r^2+\cdots$ diverges, "so" $\lim_{k\to\infty} a_k \neq 0$

"hence" $\sum a_k$ diverges.

Root Test If $\lim_{k\to\infty}\sqrt[k]{|a_k|}$ exists, then

$$\lim_{k\to\infty}\sqrt[k]{|a_k|} \begin{cases} <1 \Rightarrow \sum a_k \text{ converges absolutely} \\ =1 \Rightarrow \sum a_k \text{ may or may not converge} \\ >1 \Rightarrow \sum a_k \text{ diverges.} \end{cases}$$

Reason Let $r = \lim_{k\to\infty}\sqrt[k]{|a_k|}$. Then $\forall k$ large,

$$\sqrt[k]{|a_k|} \approx r \Rightarrow |a_k| \approx r^k \Rightarrow \sum |a_k| \approx \sum r^k.$$

Examples Consider (1) $\sum_{k=1}^{\infty} \frac{1}{3^k - 2^k}$ (2) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$

(1) **Ratio Test**

$$\lim_{k\to\infty} \frac{\frac{1}{3^{k+1}-2^{k+1}}}{\frac{1}{3^k-2^k}} = \lim_{k\to\infty} \frac{3^k-2^k}{3^{k+1}-2^{k+1}} \times \frac{\frac{1}{3^{k+1}}}{\frac{1}{3^{k+1}}} = \lim_{k\to\infty} \frac{\frac{1}{3} - \left(\frac{2}{3}\right)^k \frac{1}{3}}{1 - \left(\frac{2}{3}\right)^{k+1}}$$

$$= 1/3 < 1$$

$\therefore$ series converges.

Root Test

$$\lim_{k\to\infty} \sqrt[k]{\frac{1}{3^k-2^k}} = \lim_{k\to\infty} \frac{1}{\sqrt[k]{3^k-2^k}} = \lim_{k\to\infty} \frac{1}{3\sqrt[k]{1-\left(\frac{2}{3}\right)^k}} = \frac{1}{3}$$

$\therefore$ series converges.

(2) **Ratio Test**

$$\lim_{k\to\infty} \frac{(k+1)!}{(k+1)^{k+1}} \frac{k^k}{k!} = \lim_{k\to\infty} \overbrace{\frac{k^k}{(k+1)^k}}^{=\frac{1}{\frac{(k+1)^k}{k^k}}} = \lim_{k\to\infty} \frac{1}{\underbrace{\left(1+\frac{1}{k}\right)^k}_{1^\infty \neq 1}} = \frac{1}{e} < 1.$$

$\therefore$ Series $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ converges.

Theorem Let $a_k > 0$. If $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r \in \mathbb{R}$, then $\lim_{k\to\infty} \sqrt[k]{a_k} = r$. Converse is false.

Examples (1) $a_k = k \Rightarrow \lim_{k\to\infty} \frac{k+1}{k} = 1 \Rightarrow \lim_{k\to\infty} \sqrt[k]{k} = 1$

(2) $a_k = \frac{k!}{k^k}$, $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \frac{1}{e} \Rightarrow \lim_{k\to\infty} \sqrt[k]{\frac{k!}{k^k}} = \frac{1}{e}$

Stirling's Formula

$\forall k$ large, $\sqrt[k]{\frac{k!}{k^k}} \approx \frac{1}{e} \Rightarrow \frac{k!}{k^k} \approx \left(\frac{1}{e}\right)^k \Rightarrow k! \approx \left(\frac{k}{e}\right)^k$.

Application Find the number of digits of 100! approximately.

$100! \approx \left(\frac{100}{e}\right)^{100}$ $\quad \log_{10} \frac{100}{e} \approx 1.566 \Rightarrow \frac{100}{e} \approx 10^{1.566}$

$\Rightarrow 100! \approx \left(\frac{100}{e}\right)^{100} \approx 10^{156.6}$

100! has about 157 digits.

Summation by Parts Let $S_j = a_1 + a_2 + \cdots + a_j$ and $\Delta b_k = b_{k+1} - b_k$. Then

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \\ &= S_1 b_1 + (S_2 - S_1) b_2 + \cdots + (S_n - S_{n-1}) b_n \\ &= S_n b_n - S_1(b_2 - b_1) - \cdots - S_{n-1}(b_n - b_{n-1}) \\ &= S_n b_n - \sum_{k=1}^{n-1} S_k \Delta b_k .\end{aligned}$$

Example Consider $\sum_{k=1}^{\infty} \frac{\sin k}{k}$. $\quad \frac{\sin k}{k} = \underbrace{(\sin k)}_{a_k} \underbrace{\frac{1}{k}}_{b_k}$

$$\sin m \sin\tfrac{1}{2} = \tfrac{1}{2}\left(\cos(m - \tfrac{1}{2}) - \cos(m + \tfrac{1}{2})\right)$$

$$\begin{aligned} S_k = \sum_{m=1}^{k} \sin m &= \sum_{m=1}^{k} \frac{\cos(m-\frac{1}{2}) - \cos(m+\frac{1}{2})}{2\sin\frac{1}{2}} \\ &= \frac{\cos\frac{1}{2} - \cos(k+\frac{1}{2})}{2\sin\frac{1}{2}}\end{aligned}$$

$$|S_k| \le \frac{1+1}{2\sin\frac{1}{2}} = \frac{1}{\sin\frac{1}{2}} \Rightarrow \lim_{n\to\infty} S_n b_n = \lim_{n\to\infty} \frac{S_n}{n} = 0$$

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\sin k}{k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin k}{k} &= \lim_{n\to\infty} \left(\frac{S_n}{n} - \sum_{k=1}^{n-1} S_k\left(\frac{1}{k+1} - \frac{1}{k}\right)\right) \\ &= \sum_{k=1}^{\infty} S_k\left(\frac{1}{k} - \frac{1}{k+1}\right)\end{aligned}$$

$$\sum_{k=1}^{\infty} \left|S_k\left(\frac{1}{k} - \frac{1}{k+1}\right)\right| \le \frac{1}{\sin\frac{1}{2}} \underbrace{\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)}_{\text{example of telescoping series}} = \frac{1}{\sin\frac{1}{2}}$$

$\therefore \sum_{k=1}^{\infty} \frac{\sin k}{k} = \sum_{k=1}^{\infty} S_k\left(\frac{1}{k} - \frac{1}{k+1}\right)$ converges.

Summary

Geometric Series Test for geometric series only

$1+a+a^2+\cdots=\sum_{k=1}^{\infty} a^k$

Telescoping Series Test for telescoping series only

$b_1-\lim_{n\to\infty} b_{n+1}=\sum_{k=1}^{\infty}(b_k-b_{k+1})$

Term Test
1. Use to show series diverges only
2. May use in the beginning to scan for divergent series

Integral Test for $a_k=f(k)$, $f(x)$ integrable and decreases to 0

p-test
1. Use this for p-series only
2. Use to do Comparison with other series

Comparison Test

Use when you can do inequalities to compare a_k with known examples.

Limit Comparison Test

Use when there are dominated terms in a_k (when k is large) that can be singled out for comparison

Alternating Series Test

for alternating series only with $|a_k|\searrow 0$.

Absolute Convergence Test

for series with positive and negative terms.

Ratio Test for a_k involving $k!$, polynomials in k, k-th power expressions $a_k=(\cdots)^k$

Root Test for k-th power expressions $a_k=(\cdots)^k$

Summation by Parts

for series of the form $\sum a_k b_k$ with $S_n b_n=(a_1+\cdots+a_n)b_n$ having a limit.

$a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9+a_{10}+\cdots$

$$\underbrace{(a_1+a_2)}_{=b_1}+\underbrace{(a_3+a_4+a_5+a_6)}_{=b_2}+\underbrace{(a_7)}_{=b_3}+\underbrace{(a_8+a_9+a_{10})}_{=b_4}+\cdots$$

k_1 terms k_2 terms k_3 terms k_4 terms

$\sum_{k=1}^{\infty} b_k$ is obtained from $\sum_{k=1}^{\infty} a_k$ by inserting parentheses.

Grouping Theorem Let $\sum_{k=1}^{\infty} b_k$ be obtained from $\sum_{k=1}^{\infty} a_k$ by inserting parentheses.

- If $\sum_{k=1}^{\infty} a_k$ converges to S, then $\sum_{k=1}^{\infty} b_k$ converges to S.

The converse is false.

Examples ① $\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 1$

$\Rightarrow \frac{1}{2}+\left(\frac{1}{4}+\frac{1}{8}\right)+\left(\frac{1}{16}+\frac{1}{32}+\frac{1}{64}\right)+\cdots = 1$

② $(1-1)+(1-1)+(1-1)+\cdots = 0+0+0+\cdots = 0$,

but $1-1+1-1+1-1+\cdots$ diverges by term test.

$(1-1)+\left(\frac{1}{2}+\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}\right)+\cdots = 0$,

but $1-1+\frac{1}{2}+\frac{1}{2}-\frac{1}{2}-\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}+\cdots$

diverges since $S_{n^2}=1$ and $S_{n^2+n}=0$ so that $\lim_{n\to\infty} S_n$ doesn't exist.

- If $\lim_{n\to\infty} a_n = 0$, $\underbrace{k_n \text{ is bounded}}$, $\sum_{k=1}^{\infty} b_k$ converges to S, then $\sum_{k=1}^{\infty} a_k$ converges to S. $\forall n,\ k_n \le \text{Constant}$

Example ③ $\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots = \sum_{j=1}^{\infty}\left(\frac{1}{2j-1}-\frac{1}{2j}\right)$

$= \sum_{j=1}^{\infty} \frac{1}{2j(2j-1)}$ converges by limit comparison test with $\sum_{j=1}^{\infty}\frac{1}{j^2}$.

Since $\frac{1}{2j-1}, \frac{1}{2j} \to 0$, $\underbrace{\left(\frac{1}{2j-1}-\frac{1}{2j}\right)}_{2 \text{ terms}}$ $\forall n,\ k_n = 2$

we get $\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

Note $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges by alternating series test.

It converges conditionally because $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots = \sum_{k=1}^{\infty}\frac{1}{k}$ diverges by p-test.

To find the sum of $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$,

define $f(x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots$ converge for $x\in[0,1]$ by ratio test

Then $f'(x) = 1-x+x^2-x^3+\cdots = \frac{1}{1+x}$ for $x\in[0,1)$

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = f(1)$

$= f(1)-f(0) = \int_0^1 f'(t)\,dt = \int_0^1 \frac{1}{1+t}\,dt$

$= \ln(1+t)\Big|_0^1 = \ln 2 - \ln 1 = \underline{\underline{\ln 2}}$

Definition Let $f: \mathbb{N} \to \mathbb{N}$ be a bijection.
$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_{f(k)}$ is a rearrangement of $\sum_{k=1}^{\infty} a_k$.

Example $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

Terms are $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \cdots$

Rearrange terms to $1, \frac{1}{3}, -\frac{1}{2}, \frac{1}{5}, \frac{1}{7}, -\frac{1}{4}, \cdots$

every term appears exactly once.

By Grouping Theorem

$$
\begin{array}{rcccccccl}
 & (1-\frac{1}{2}) & + & (\frac{1}{3}-\frac{1}{4}) & + & (\frac{1}{5}-\frac{1}{6}) & + & (\frac{1}{7}-\frac{1}{8}) & +\cdots = \ln 2 \\
+ & \frac{1}{2} & & -\frac{1}{4} & & +\frac{1}{6} & & -\frac{1}{8} & +\cdots = \frac{1}{2}\ln 2 \\
\hline
 & 1 & + & (\frac{1}{3}-\frac{1}{2}) & + & \frac{1}{5} & + & (\frac{1}{7}-\frac{1}{4}) & +\cdots = \frac{3}{2}\ln 2
\end{array}
$$

$$= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$$

By Grouping theorem (terms $\to 0$, $k_n \le 2$)

Riemann's Rearrangement Theorem

Let $a_k \in \mathbb{R}$ $\forall k$ and $\sum_{k=1}^{\infty} a_k$ converges conditionally.
$\forall x \in \mathbb{R} \cup \{+\infty, -\infty\}$, $\exists$ a rearrangement $\sum_{k=1}^{\infty} b_k$ of $\sum_{k=1}^{\infty} a_k$ such that $\sum_{k=1}^{\infty} b_k = x$.

Dirichlet's Rearrangement Theorem

Let $a_k \in \mathbb{R}$ $\forall k$ and $\sum_{k=1}^{\infty} a_k$ converges absolutely.
$\forall$ rearrangement $\sum_{k=1}^{\infty} b_k$ of $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_k$.

Example $\sum_{k=1}^{\infty} (-\frac{1}{2})^k = -\frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \cdots = \frac{-\frac{1}{2}}{1-(-\frac{1}{2})} = -\frac{1}{3}$

So by Dirichlet's rearrangement theorem,

$$-\frac{1}{2} + \frac{1}{2^2} + \underbrace{\frac{1}{2^4} - \frac{1}{2^3}}_{\text{switched 2 terms}} + \underbrace{\frac{1}{2^8} - \frac{1}{2^7} + \frac{1}{2^6} - \frac{1}{2^5}}_{\text{switched 4 terms}} + \underbrace{\cdots}_{\text{switched } 2^n \text{ terms}} = -\frac{1}{3}$$

Complex Series $z_1+z_2+z_3+\cdots = \sum_{k=1}^{\infty} z_k$, $z_k \in \mathbb{C}$

- $z = a+ib \Rightarrow |z| = \sqrt{a^2+b^2}$
- $S_n = u_n + i v_n$

  Definition of Limit

  $\lim_{n\to\infty} S_n = u + iv \Leftrightarrow \lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$
- $z_k = x_k + i y_k$ $\quad S_n = z_1 + z_2 + \cdots + z_n$

  $\sum_{k=1}^{\infty} z_k = \lim_{n\to\infty} S_n = x+iy \Leftrightarrow \sum_{k=1}^{\infty} x_k = x$ and $\sum_{k=1}^{\infty} y_k = y$
- Definitions of absolute convergence and conditional convergence for series are the same.
- Geometric series test, telescoping series test, term-test, absolute convergence test, ratio test and root test are true for complex series for the same reasons.

Examples (1) Since $|i| = 1$, $\lim_{n\to\infty} |i^n| = \lim_{n\to\infty} 1 = 1 \neq 0$
so $\sum_{k=1}^{\infty} i^k$ diverges by term test.

(2) If $|z| \le 1$, then $\left|\frac{z^k}{k^2}\right| \le \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p-test
so $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ converges absolutely.

If $|z| > 1$, then $\lim_{k\to\infty} \left| \frac{z^{k+1}}{(k+1)^2} \frac{k^2}{z^k} \right| = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} |z| = \underbrace{|z|}_{>1}$.

By ratio test, $\sum_{k=1}^{\infty} \frac{z^k}{k^2}$ diverges.

Chapter 5 Real Numbers

The set of all real numbers (denoted by $\mathbb{R}$) satisfies the following axioms:

1. Field Axiom
2. Order Axiom
3. Well-ordering Axiom
4. Completeness Axiom

An axiom is a self-evident statement that is assumed to be foundational in order to obtain more important consequences by deduction.

Field Axiom $\mathbb{R}$ has 2 operations $+$ and $\cdot$ such that $\forall a,b,c \in \mathbb{R}$,

(i) $a+b,\ a\cdot b \in \mathbb{R}$

(ii) $a+b=b+a$, $a\cdot b = b\cdot a$

(iii) $(a+b)+c = a+(b+c)$, $(a\cdot b)\cdot c = a\cdot(b\cdot c)$

(iv) $\exists$ unique elements $0, 1 \in \mathbb{R}$ with $1 \neq 0$ such that $a+0=a$, $a\cdot 1 = a$

(v) $\exists\ -a \in \mathbb{R}$ such that $a+(-a)=0$; if $a \neq 0$, then $\exists\ a^{-1} \in \mathbb{R}$ such that $a\cdot(a^{-1}) = 1$

(vi) $a\cdot(b+c) = a\cdot b + a\cdot c$.

$\boxed{-a \text{ and } a^{-1} \text{ are unique}}$

Remarks ① From this axiom, we can define

$a-b = a+(-b)$ ← definition of subtraction

$ab = a\cdot b$ ← shorthand notation of multiplication

$\frac{a}{b} = a\cdot(b^{-1})$ for $b \neq 0$ ← definition of division

$2 = 1+1$, $3 = 2+1$, ...

② $x\cdot 0 \overset{\text{by (v) and def of subtraction}}{=} x(1-1) \overset{\text{by (vi)}}{=} x\cdot 1 - x\cdot 1 \overset{\text{by (iv)}}{=} x - x \overset{\text{by (v)}}{=} 0$. $\forall x \in \mathbb{R}$.

Order Axiom $\mathbb{R}$ has an (ordering) relation $<$ such that $\forall a,b,c \in \mathbb{R}$,

(i) exactly one of the following $a<b$, $a=b$, $b<a$ is true

(ii) if $a<b$ and $b<c$, then $a<c$

(iii) if $a<b$, then $a+c < b+c$

(iv) if $a<b$ and $0<c$, then $ac<bc$.

Remarks We also write $a>b \Leftrightarrow b<a$,

$a \leq b \Leftrightarrow a<b$ or $a=b$, $a \geq b \Leftrightarrow b \leq a$.

$[a,b] = \{x : x \in \mathbb{R} \text{ and } a \leq x \leq b\}$

$(a,b) = \{x : x \in \mathbb{R} \text{ and } a < x < b\}$

$\max(a_1,\cdots,a_n)$ or $\max\{a_1,\cdots,a_n\}$ denote the maximum of $a_1,\cdots,a_n$ (similarly for $\min(a_1,\cdots,a_n)$)

$|x| = \max(x,-x)$ (then $\underbrace{x \leq |x| \text{ and } -x \leq |x|}_{\Leftrightarrow\ -|x| \leq x \leq |x|}$)

$|x| \leq a \Leftrightarrow x \leq a$ and $-x \leq a \Leftrightarrow -a \leq x \leq a$.

Triangle Inequality $\forall x,y \in \mathbb{R}$, $|x+y| \leq |x|+|y|$

(Adding $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$, we get $-|x|-|y| \leq x+y \leq |x|+|y|$. So $|x+y| \leq |x|+|y|$.)

$0<1$ Since $1 \neq 0$, by (i), $0<1$ or $1<0$.
Assume $1<0$. Then $0 = 1+(-1) < 0+(-1) = -1$.
By (iv), $0 = 0\cdot(-1) < (-1)\cdot(-1) = 1$, contradiction to (i).

CAUTIONS ① $a<b$ and $c<d$ does not imply $a-c < b-d$

② $a<b$ does not imply $|a|<|b|$.

Well-ordering Axiom $\mathbb{N}=\{1,2,3,\cdots\}$ is well-ordered which means "$\forall$ nonempty $S\subseteq\mathbb{N}$, $\exists\, m\in S$ such that $m\le x$ for all $x\in S$." This m is the least element (or the minimum) of S.

Examples ① S = set of all prime numbers, $m=2$

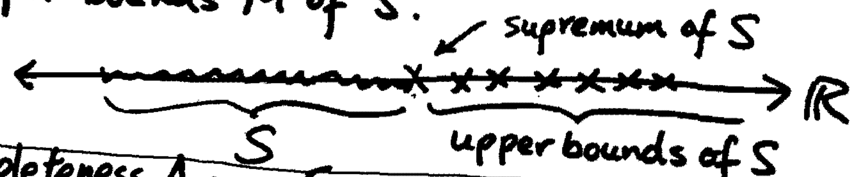
② S = set of all 4-digit positive integers, $m=1000$

③ $S=(\pi,\sqrt{99})\cap\mathbb{N}$, $m=4$

Definitions For a nonempty subset S of $\mathbb{R}$, we say S is bounded above iff $\exists\, M\in\mathbb{R}$ such that $M\ge x$ for all $x\in S$. ↖ M may not be in S

Such an M is called an upper bound of S.

A supremum or least upper bound of S (denoted by $\sup S$ or $\operatorname{lub} S$) is ① an upper bound $\tilde{M}$ of S such that ② $\tilde{M}\le M$ for all upper bounds M of S.



$$\boxed{\text{Completeness Axiom: Every nonempty subset of } \mathbb{R} \text{ which is bounded above has a supremum in } \mathbb{R}.}$$

↑ The supremum may or may not be in the set !!!

Examples ① $S=\{\frac{1}{n}: n\in\mathbb{N}\}=\{1,\frac{1}{2},\frac{1}{3},\cdots\}$

Upper bounds of S: every real number $M\ge 1$

Supremum of S is 1. ← the least number among upper bounds of S. In this case $\sup S=1\in S$.

② $S=\{x: x\in\mathbb{R} \text{ and } x<0\}=(-\infty,0)$

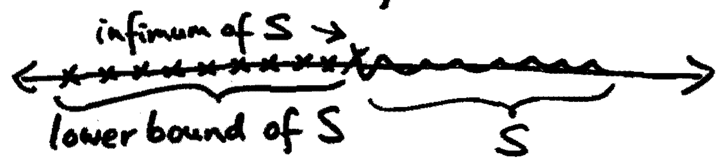
Upper bounds of S: every real number $M\ge 0$

Supremum of S is 0. However, $\sup S=0\notin S$.

Definitions For a nonempty subset S of $\mathbb{R}$, we say S is bounded below iff $\exists\, m\in\mathbb{R}$ such that $m\le x$ for all $x\in S$.

Such an m is called a lower bound of S.

An infimum or greatest lower bound of S (denoted by $\inf S$ or $\operatorname{glb} S$) is ① a lower bound $\tilde{m}$ of S such that ② $m\le\tilde{m}$ for all lower bounds m of S.



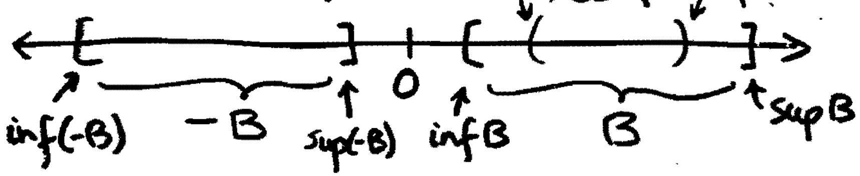
Exercises Let $c\in\mathbb{R}$. Let A,B be nonempty subsets of $\mathbb{R}$. Define

$-B=\{-x: x\in B\}$, $c+B=\{c+x: x\in B\}$,

$cB=\{cx: x\in B\}$,

$A+B=\{x+y: x\in A \text{ and } y\in B\}$.

① B is bounded above $\Leftrightarrow$ $-B$ is bounded below
$\inf(-B) = -\sup B$.
B is bounded below $\Leftrightarrow$ $-B$ is bounded above
$\sup(-B) = -\inf B$.



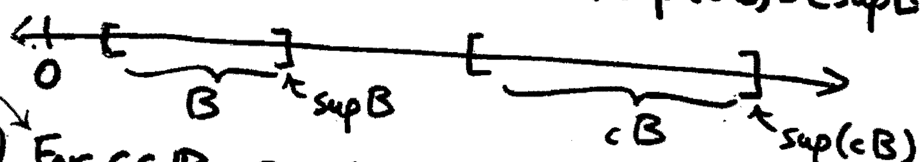
If $\emptyset \neq A \subseteq B$, then $\inf B \leq \inf A$ (when B is bounded below) and $\sup A \leq \sup B$ (when B is bounded above).

Remarks From ① and completeness axiom, we get

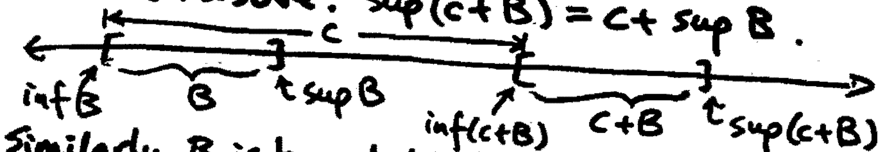
Completeness Axiom for Infimum Every nonempty subset of $\mathbb{R}$ which is bounded below has an infimum in $\mathbb{R}$.

Exercises

② If B is bounded above and $c \geq 0$, then cB is bounded above and $\sup(cB) = c\sup B$.



③ For $c \in \mathbb{R}$, B is bounded above $\Leftrightarrow$ $c+B$ is bounded above. $\sup(c+B) = c + \sup B$.



Similarly, B is bounded below $\Leftrightarrow$ $c+B$ is bounded below. $\inf(c+B) = c + \inf B$.

More generally, if A and B are bounded above and below, then $A+B = \{x+y : x \in A, y \in B\}$ is bounded above and below, $\sup(A+B) = \sup A + \sup B$ and $\inf(A+B) = \inf A + \inf B$. ← Exercises

Definition Let S be a nonempty subset of $\mathbb{R}$. S is <u>bounded</u> iff S is bounded above and below.

Remarks

① S is bounded $\overset{②}{\Rightarrow}$ $\forall x \in S$, $x \leq \sup S$, $\inf S \leq x$ ($\Leftarrow$ $-x \leq -\inf S$)

$\overset{③}{\Rightarrow}$ $\forall x \in S$, $|x| \leq \max(\sup S, -\inf S)$

$\overset{④}{\Rightarrow}$ $\exists c \in \mathbb{R}, \forall x \in S$, $|x| \leq c$. ($\Leftarrow$ $-c \leq x \leq c$)

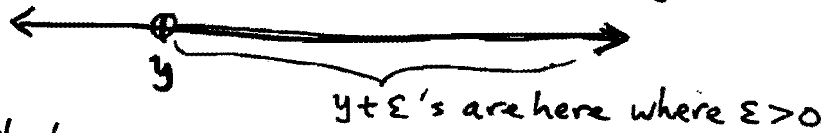
$\therefore$ all 4 statements are equivalent.

Consequences of Axioms

$\alpha, \beta, \gamma, \delta, \varepsilon$ ← epsilon, ↑ delta

theorem (Infinitesimal Principle) Let $x, y \in \mathbb{R}$.

(*) $x < y + \varepsilon$ for all $\varepsilon > 0 \iff x \le y$



Similarly $y - \varepsilon < x$ for all $\varepsilon > 0 \iff y \le x$.)

Proof. ($\Leftarrow$) If $x \le y$, then $\forall \varepsilon > 0$, $x \le y \underset{\text{Field Axiom}}{=} y + 0 \overset{\text{Order Axiom}}{<} y + \varepsilon$.

($\Rightarrow$) If $\forall \varepsilon > 0$, $x < y + \varepsilon$, then assume $x > y$. By order axiom, $\varepsilon_0 = x - y > y - y = 0$. Then $x < y + \varepsilon_0$ ← by (*)
But also $x = y + \varepsilon_0$, contradicting (i) of order axiom
$\therefore x \le y$.

Remarks Letting $x = |a-b|$ and $y = 0$, we have

$|a-b| < \varepsilon$ for all $\varepsilon > 0 \iff |a-b| \le 0 \iff a = b$.

The principle is often used this way to show expressions are equal.

theorem (Mathematical Induction Principle)

(1) $\forall n \in \mathbb{N}$, $A(n)$ is a statement that is either true or false
(2) $A(1)$ is true
(3) $\forall k \in \mathbb{N}$ $A(k)$ true $\Rightarrow A(k+1)$ true

Then $\forall n \in \mathbb{N}$, $A(n)$ is true.

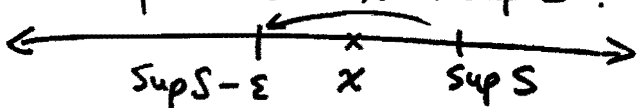
Proof. Assume $\sim(\forall n \in \mathbb{N}, A(n)$ is true$) = \exists n \in \mathbb{N}$ such that $A(n)$ is false. Then $S = \{n : A(n)$ is false$\}$ is a nonempty subset of $\mathbb{N}$.

By the well-ordering axiom, S has a least element m in S. So $A(m)$ is false and if $A(n)$ is false, then $m \le n$. Taking contrapositive, <u>if $n < m$, then $A(n)$ is true.</u>

Since $A(1)$ is true, $m \ne 1$. Now $m \in \mathbb{N}$ and $m \ne 1$
$\Rightarrow m \ge 2 \Rightarrow m - 1 \ge 1 \Rightarrow m - 1 \in \mathbb{N}$.
Now $m - 1 < m$. So $A(m-1)$ is true. By (3), we get $A(m)$ is true, contradiction

theorem (Supremum Property) If a set S has a supremum in $\mathbb{R}$ and $\varepsilon > 0$, then $\exists x \in S$ such that

$$\sup S - \varepsilon < x \le \sup S.$$



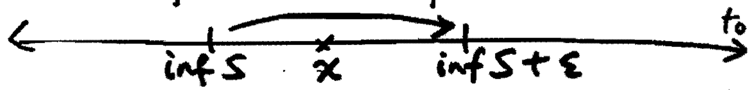
Recall M is an upper bound of S
$\iff \forall x \in S$, $x \le M$.

Proof of Supremum Property Since $\sup S - \varepsilon < \sup S$, $\sup S - \varepsilon$ is <u>not</u> an upper bound of S. So $\exists x \in S$ such that $\sup S - \varepsilon < x$. Since $x \in S$, $x \le \sup S$.
$\therefore \sup S - \varepsilon < x \le \sup S$.

Theorem (Infimum Property) If a set S has an infimum in $\mathbb{R}$ and $\varepsilon > 0$, then $\exists x \in S$ such that

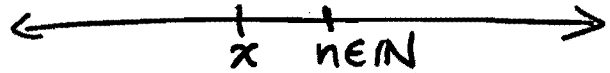
$$\inf S \le x \le \inf S + \varepsilon.$$

(Proof is similar to supremum property.)



Theorem

(Archimedean Principle) $\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > x$.



Proof. Assume $\sim(\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > x)$ $= \exists x \in \mathbb{R}, \forall n \in \mathbb{N}, n \le x$. Then $\mathbb{N}$ is bounded above by x. By the completeness axiom, $\mathbb{N}$ has a supremum in $\mathbb{R}$. By supremum property, $\exists n \in \mathbb{N}$ such that $\sup \mathbb{N} - 1 < n \le \sup \mathbb{N}$. Then $\sup \mathbb{N} < n+1 \in \mathbb{N}$, a contradiction to $\sup \mathbb{N}$ is an upper bound of $\mathbb{N}$.

Questions How is $\mathbb{Q}$ contained in $\mathbb{R}$? How is $\mathbb{R} \setminus \mathbb{Q}$ contained in $\mathbb{R}$?

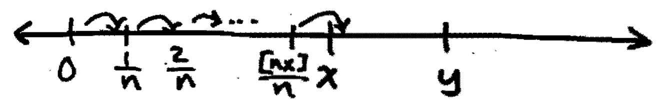
Lemma $\forall x \in \mathbb{R}$, $\exists$ a least integer (denoted by $\lceil x \rceil$ ← ceiling of x) greater than or equal to x. Similarly, $\exists$ a greatest integer (denoted by $\lfloor x \rfloor$ or $[x]$ ← floor of x) less than or equal to x.

Proof. By Archimedean principle, $\exists n \in \mathbb{N}$ such that $n > |x|$. Then $-n < x < n$. By order axiom, $0 < x+n < 2n$. So $S = \{k : k \in \mathbb{N}, k \ge x+n\}$ is a nonempty subset of $\mathbb{N}$ because $2n \in S$. By the well-ordering axiom, $\exists$ a least positive integer $m \ge x+n$. Then $m-n$ is the least ~~positive~~ integer $\ge x$. So $\lceil x \rceil$ exists. Next, let k be the least ~~positive~~ integer $\ge -x$. Then $-k$ is the greatest ~~positive~~ integer $\le x$. So $\lfloor x \rfloor$ exists.

Theorem

(Density of $\mathbb{Q}$) If $x<y$, then $\exists \frac{m}{n} \in \mathbb{Q}$ such that $x < \frac{m}{n} < y$.

Proof. By Archimedean principle, $\exists n \in \mathbb{N}$ such that $n > \frac{1}{y-x}$. So $ny - nx > 1$. Hence $nx + 1 < ny$. Let $m = [nx]+1$, then $m-1 = [nx] \le nx < [nx]+1 = m$. So $nx < m \le nx+1 < ny$. $\therefore x < \frac{m}{n} < y$.

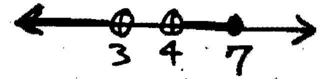


Theorem (Density of $\mathbb{R} \setminus \mathbb{Q}$) If $x<y$, then $\exists w \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < w < y$.

Proof. Let $w_0 \in \mathbb{R} \setminus \mathbb{Q}$ (e.g. $w_0 = \sqrt{2}$). By density of $\mathbb{Q}$, $\exists \frac{m}{n} \in \mathbb{Q}$ such that $\frac{x}{|w_0|} < \frac{m}{n} < \frac{y}{|w_0|}$. (If $\frac{m}{n} = 0$, then pick another rational number between 0 and $\frac{y}{|w_0|}$. So we may take $\frac{m}{n} \ne 0$.) Let $w = \frac{m}{n}|w_0|$, then $w \in \mathbb{R} \setminus \mathbb{Q}$ and $x < w < y$.

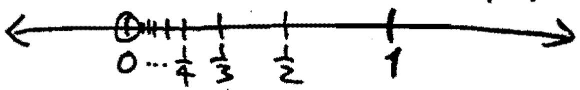
Examples of Supremum and Infimum

① Consider $S = (-\infty, 3) \cup (4, 7]$



S is not bounded below. So S has no infimum. S is bounded above by 7 and every upper bound of S is greater than or equal to 7 because $7 \in S$. So 7 is an upper bound and is the least among upper bounds. $\therefore \sup S = 7$.

② Consider $S=\{\frac{1}{n}: n\in\mathbb{N}\}=\{1,\frac{1}{2},\frac{1}{3},\cdots\}$



$\forall n\in\mathbb{N}$,

$\left.\begin{array}{l}\frac{1}{n}\le 1 \Rightarrow 1 \text{ is an upper bound} \\ 1\in S \Rightarrow \text{every upper bound} \ge 1\end{array}\right\} \Rightarrow \sup S = 1.$

Next we claim $\inf S=0$.

$\forall n\in\mathbb{N}$, $0<\frac{1}{n} \Rightarrow 0$ is a lower bound of S.

(However, $0\notin S$, so we cannot say "every lower bound ≤ 0.") Assume S has a lower bound $t>0$. (To get a contradiction, we will try to get a $\frac{1}{n}\in S$ such that $\frac{1}{n}<t$.) By the Archimedean principle, $\exists n\in\mathbb{N}$ such that $n>\frac{1}{t}$. Then $\frac{1}{n}\in S$ and $\frac{1}{n}<t$, contradicting t is a lower bound of S. So every lower bound $t\le 0$.

$\therefore \inf S=0$.

---

③ Consider $S=[2,6)\cap\mathbb{Q}$

$\left.\begin{array}{l}\forall x\in S,\ 2\le x \Rightarrow 2 \text{ is a lower bound} \\ 2\in S \Rightarrow \text{every lower bound} \le 2\end{array}\right\} \Rightarrow \inf S = 2.$

Next we claim $\sup S=6$.

$\forall x\in S$, $x<6 \Rightarrow 6$ is an upper bound of S. Note $6\notin S$.

Assume S has an upper bound $u<6$. Since $2\in S$, $2\le u$. By the density of $\mathbb{Q}$, $\exists r\in\mathbb{Q}$ such that $u<r<6$. Then $r\in[2,6)\cap\mathbb{Q}=S$.

Now $u<r$ contradicts u is an upper bound of S. So every upper bound $u\ge 6$. $\therefore \sup S=6$.

Supremum Limit Theorem

Let c be an upper bound of a nonempty set S. Then $(\exists\, w_n\in S$ such that $\lim_{n\to\infty} w_n = c) \Leftrightarrow c=\sup S$.

Infimum Limit Theorem

Let c be a lower bound of a nonempty set S. Then $(\exists\, w_n\in S$ such that $\lim_{n\to\infty} w_n = c) \Leftrightarrow c=\inf S$.

Proofs will be given in the next chapter.

Examples ① Let $S=\{\frac{1}{n}: n\in\mathbb{N}\}=\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots\}$.

$\left.\begin{array}{l}0\le\frac{1}{n}\ \forall n\in\mathbb{N} \Rightarrow 0 \text{ is a lower bound of } S \\ w_n=\frac{1}{n}\in S,\ \lim_{n\to\infty} w_n = 0\end{array}\right\} \Rightarrow \inf S \overset{\shortparallel}{} 0.$

---

② Let $S=\{x\pi+\frac{1}{y}: x\in\mathbb{Q}\cap(0,1],\ y\in[1,2]\}$.

$\left.\begin{array}{l}\forall x\in\mathbb{Q}\cap(0,1],\ y\in[1,2],\ x\pi+\frac{1}{y}>0\pi+\frac{1}{2}=\frac{1}{2} \\ \quad\Rightarrow \frac{1}{2} \text{ is a lower bound of } S \\ w_n=\frac{1}{n}\pi+\frac{1}{2}\in S,\ \lim_{n\to\infty} w_n=\frac{1}{2}\end{array}\right\} \Rightarrow \inf S=\frac{1}{2}.$

---

③ Let A and B be bounded sets in $\mathbb{R}$.
Let $A-2B=\{a-2b: a\in A, b\in B\}$.
Prove $\sup(A-2B)=\sup A-2\inf B$.

Solution. Since A bounded, $\sup A$ exists in $\mathbb{R}$. Since B bounded, $\inf B$ exists in $\mathbb{R}$. $\forall a\in A, b\in B$, we have $a\le\sup A$, $\inf B\le b \Rightarrow a-2b\le\sup A-2\inf B$.

$\therefore$ $c=\sup A-2\inf B$ is an upper bound of $A-2B$.

By supremum limit theorem, $\exists\, a_n\in A$, $\lim_{n\to\infty} a_n=\sup A$.

By infimum limit theorem, $\exists\, b_n\in B$, $\lim_{n\to\infty} b_n=\inf B$.

Then $a_n-2b_n\in A-2B$ and $\lim_{n\to\infty}(a_n-2b_n)=\sup A-2\inf B$.

$\therefore$ by supremum limit theorem, $\sup(A-2B)=\sup A-2\inf B$.