Review for Math 2033 Final Exam

Definition — Remember this!

$a_1, a_2, a_3, \ldots$ is a Cauchy sequence iff
$\forall \varepsilon > 0 \ \exists K \in \mathbb{N}$ such that
$m, n \geq K \Rightarrow |a_m - a_n| < \varepsilon$.

Review on Cauchy Sequences

To prove a sequence $\{x_n\}$ is Cauchy

- telescoping expansion (for recurrent sequences)

$$|x_m - x_n| = |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)|$$
$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$

#46, 48, 108, 117

- use mean value theorem (for $x_n = f(t_n)$, $|f'(t)| \leq M$)

$$|x_m - x_n| = |f(t_m) - f(t_n)| = |f'(\theta)(t_m - t_n)|$$
$$\leq M|t_m - t_n|$$

#73, 156, 167, 170

- special inequalities such as $|\sqrt[n]{a} - \sqrt[n]{b}| \leq \sqrt[n]{|a-b|}$ (for $a, b \geq 0$)

$|\sin a - \sin b| \leq |a - b|$

#47

- use boundedness of given Cauchy sequences
(see step 1 on right half of p.58 of transparencies)

Practice Exercise 108

If $\{x_n\}$ is a sequence such that $|x_{k+1} - x_k| < \frac{1}{2^k}$ for $k = 1, 2, 3, \ldots$, then prove $\{x_n\}$ is a Cauchy sequence.

Sketch Say $m > n \geq K$

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n|$$
$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$
$$< \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^n}$$
$$< \frac{1}{2^K} + \frac{1}{2^{K+1}} + \frac{1}{2^{K+2}} + \cdots = \frac{1}{2^K} \Big/ \left(1 - \frac{1}{2}\right)$$
$$= \frac{1}{2^{K-1}} < \varepsilon \iff 2^{K-1} > \frac{1}{\varepsilon}$$
$$\iff (K-1)\ln 2 > -\ln \varepsilon$$
$$K > 1 - (\ln \varepsilon)/(\ln 2)$$

Solution $\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > 1 - (\ln\varepsilon)/(\ln 2)$. By sketch above,

$m, n \geq K \Rightarrow |x_m - x_n| < \varepsilon$.

($m = n \Rightarrow |x_m - x_n| = 0 < \varepsilon$)
($m > n$, $n > m$ are similar cases.)

2010 Midterm

Problem 2 Let $a_1, a_2, a_3, \cdots$ be a Cauchy sequence of real numbers. Let $b_n = \sin^2(a_n + a_{2n})$. Prove that $b_1, b_2, b_3, \cdots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

Scratch Work

$$|b_n - b_m| = |\sin^2(a_n + a_{2n}) - \sin^2(a_m + a_{2m})|$$
$$= |\sin(a_n + a_{2n}) + \sin(a_m + a_{2m})||\sin(a_n + a_{2n}) - \sin(a_m + a_{2m})|$$
$$\le 2|(a_n + a_{2n}) - (a_m + a_{2m})|$$
$$\le 2(|a_n - a_m| + |a_{2n} - a_{2m}|)$$

Solution $\forall \varepsilon > 0$, since $\{a_n\}$ is Cauchy, $\exists K \in \mathbb{N}$ such that $n, m \ge K \Rightarrow |a_n - a_m| < \varepsilon/4$.

Then $n, m \ge K \Rightarrow n, m, 2n, 2m \ge K$

$$\Rightarrow |b_n - b_m| \le 2(|a_n - a_m| + |a_{2n} - a_{2m}|)$$
$$< 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \varepsilon.$$

Variation Let $f(x) = \sin^2 x$, then $f'(x) = 2\sin x \cos x$

By mean-value theorem, $|f(c) - f(d)| = |f'(\theta)(c-d)|$
$$\le 2|c-d|$$

$$|b_n - b_m| = |f(a_n + a_{2n}) - f(a_m + a_{2m})|$$
$$\le 2|(a_n + a_{2n}) - (a_m + a_{2m})| \le 2\left(\begin{matrix}|a_n - a_m| \\ + |a_{2n} - a_{2m}|\end{matrix}\right)$$

Example If $\{x_n\}$ is Cauchy, then prove $\{x_n^2\}$ is Cauchy.

Solution $\{x_n\}$ Cauchy $\Rightarrow$ $\{x_n\}$ bounded, say $|x_n| \le C$.

Since $\{x_n\}$ is Cauchy,

$\forall \varepsilon > 0\ \exists K \in \mathbb{N}$ such that $m, n \ge K \Rightarrow |x_m - x_n| < \frac{\varepsilon}{2C}$.

Then $m, n \ge K \Rightarrow |x_m^2 - x_n^2| = |x_m + x_n||x_m - x_n|$
$$\le (|x_m| + |x_n|)|x_m - x_n|$$
$$\le 2C|x_m - x_n|$$
$$< 2C\frac{\varepsilon}{2C} = \varepsilon.$$

2009 Midterm

① Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of positive real numbers. For $n = 1, 2, 3, \cdots$, let
$$b_n = \sin(a_n^2) + \sqrt[3]{7a_n}.$$
Prove that $b_1, b_2, b_3, \cdots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

Solution. Observe that

$$|b_n - b_m| = |\sin(a_n^2) - \sin(a_m^2) + \sqrt[3]{7a_n} - \sqrt[3]{7a_m}|$$
$$\le |\sin(a_n^2) - \sin(a_m^2)| + |\sqrt[3]{7a_n} - \sqrt[3]{7a_m}|$$
$$\le |a_n^2 - a_m^2| + \sqrt[3]{|7a_n - 7a_m|}$$
$$\le |a_n + a_m||a_n - a_m| + \sqrt[3]{7|a_n - a_m|}.$$

For $|a_n + a_m|$, we need to use $\{a_n\}$ Cauchy $\Rightarrow$ $\{a_n\}$ bounded

(Continued on next page)

Since $\{a_n\}$ is Cauchy, $\exists M>0$ such that $\forall n \in \mathbb{N}$, $|a_n| \le M$.

Also, $\forall \varepsilon>0$, $\exists K_1 \in \mathbb{N}$ such that $n,m \ge K_1$ implies
$$|a_n - a_m| < \frac{\varepsilon}{4M}$$
and $\exists K_2 \in \mathbb{N}$ such that $n,m \ge K_2$ implies $|a_n - a_m| < \frac{\varepsilon^3}{56}$

Let $K = \max\{K_1, K_2\}$. Then

$n,m \ge K \Rightarrow n,m \ge K_1$ and K_2
$$\Rightarrow |b_n - b_m| \le 2M\left(\frac{\varepsilon}{4M}\right) + \sqrt[3]{7\left(\frac{\varepsilon^3}{56}\right)} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Review on Limit of Functions

Solutions to Math 202 Exam 2 (Spring 2006)

① (a) $f(x)$ converges to L as x tends to x_0 ($\lim_{x\to x_0} f(x) = L$)

iff $\forall \varepsilon>0$ $\exists \delta>0$ such that for every $x \in S$,
$$0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon.$$

① (b) $f:(0.5,+\infty)\to\mathbb{R}$, $f(x)=\sqrt{x+\frac{1}{x}}$.

Prove $\lim_{x\to 1} f(x) = \sqrt{2}$ by checking definition.

$\forall \varepsilon>0$, let $\delta = \varepsilon/\sqrt{2} > 0$

$\forall x \in (0.5,+\infty)$, Use $|\sqrt{a}-\sqrt{b}| \le \sqrt{|a-b|}$
$$0<|x-1|<\delta \Rightarrow \left|\sqrt{x+\tfrac{1}{x}} - \sqrt{2}\right| \le \sqrt{\left|x+\tfrac{1}{x}-2\right|}$$
$$= \sqrt{\left|\frac{x^2-2x+1}{x}\right|} = \frac{\sqrt{|x-1|^2}}{\sqrt{x}}$$
$$= \frac{|x-1|}{\sqrt{x}} < \sqrt{2}\,|x-1| < \varepsilon$$

$x > 0.5 = \frac{1}{2}$
$\Rightarrow \frac{1}{\sqrt{x}} < \sqrt{2}$

Practice Exercise 169 (2005 Spring Final)

Prove $\lim_{x\to 2} \frac{2+3x}{x^2+4} = 1$ by checking the ε-δ definition.

Solution Observe that if $|x-2|<1$, then $x\in(1,3)$, $x-1\in(0,2)$

$$\left|\frac{2+3x}{x^2+4}-1\right| = \frac{|x^2-3x+2|}{x^2+4} \le \frac{|x-2|\,|x-1|}{4} < \frac{2|x-2|}{4} < |x-2|$$

$\forall \varepsilon>0$, let $\delta=\min\{1,\varepsilon\}$. Then

$$0<|x-2|<\delta \Rightarrow \begin{matrix}|x-2|<1\\ \text{and}\\ |x-2|<\varepsilon\end{matrix} \Rightarrow \begin{matrix}x\in(1,3)\\ \text{and}\\ |x-2|<\varepsilon\end{matrix} \Rightarrow \begin{matrix}x-1\in(0,2)\\ \text{and}\\ |x-2|<\varepsilon\end{matrix}$$

$$\Rightarrow \left|\frac{2+3x}{x^2+4}-1\right| < |x-2| < \varepsilon.$$

↑ by observation above.

Example

Prove $\lim_{x\to 2}\left(\frac{2}{x^2}+\frac{3x}{4}\right)=2$ by checking definition.

Sketch $x\to 2 \Rightarrow \frac{2}{x^2}\to\frac{1}{2}$, $\frac{3x}{4}\to\frac{3}{2}$

$$\left|\left(\frac{2}{x^2}+\frac{3x}{4}\right)-2\right| = \left|\left(\frac{2}{x^2}-\frac{1}{2}\right)+\left(\frac{3x}{4}-\frac{3}{2}\right)\right| \le \left|\frac{2}{x^2}-\frac{1}{2}\right| + \left|\frac{3x}{4}-\frac{3}{2}\right|$$

$$= \left|\frac{x^2-4}{2x^2}\right| + \frac{3|x-2|}{4} = \frac{|x+2|\,|x-2|}{2x^2} + \frac{3}{4}|x-2| < \frac{5}{2}|x-2| + \frac{3}{4}|x-2|$$

$$= \frac{13}{4}|x-2|$$

if $|x-2|<1$, $x\in(1,3)$, $x+2\in(3,5)$

Solution $\forall\varepsilon>0$, let $\delta=\min\{1,\frac{4}{13}\varepsilon\}$. Then $0<|x-2|<\delta$ (<1, $<\frac{4}{13}\varepsilon$)

$$\Rightarrow \left|\left(\frac{2}{x^2}+\frac{3x}{4}\right)-2\right| < \frac{13}{4}|x-2| < \varepsilon.$$

↑ by sketch above

2011 Math 202 Spring Midterm

Problem 1 Let $f:[0,+\infty)\to\mathbb{R}$ be defined by $f(x)=\sin^2\left(\frac{1}{1+\sqrt[4]{x}}\right)$. Prove that $\lim_{x\to 1} f(x) = \sin^2\frac{1}{2}$ by checking ε-δ definition.

Solution $\forall\varepsilon>0$, let $\delta=\varepsilon^4>0$. Then $\forall x\in[0,+\infty)$, $0<|x-1|<\delta$ implies

$$\left|\sin^2\left(\frac{1}{1+\sqrt[4]{x}}\right)-\sin^2\frac{1}{2}\right| = \left|\sin\left(\frac{1}{1+\sqrt[4]{x}}\right)+\sin\frac{1}{2}\right|\left|\sin\left(\frac{1}{1+\sqrt[4]{x}}\right)-\sin\frac{1}{2}\right|$$

$$\le 2\left|\frac{1}{1+\sqrt[4]{x}}-\frac{1}{2}\right| = 2\left|\frac{1-\sqrt[4]{x}}{2(1+\sqrt[4]{x})}\right|$$

$$= 2\,\frac{|1-\sqrt[4]{x}|}{2(1+\sqrt[4]{x})} \boxed{\le \frac{|1-\sqrt[4]{x}|}{1+0} \le \sqrt[4]{|1-x|} < \sqrt[4]{\delta} = \varepsilon}$$

switch

Variation

$$\frac{|1-\sqrt[4]{x}|}{1+\sqrt[4]{x}} = \frac{|1-\sqrt[4]{x}|(1+\sqrt[4]{x})}{(1+\sqrt[4]{x})(1+\sqrt[4]{x})} = \frac{|1-\sqrt{x}|}{(1+\sqrt[4]{x})^2}\,\frac{(1+\sqrt{x})}{(1+\sqrt{x})}$$

$$= \frac{|1-x|}{(1+\sqrt[4]{x})^2(1+\sqrt{x})} \le \frac{|1-x|}{(1+0)^2(1+0)} = |1-x| < \delta$$

$\forall\varepsilon>0$, let $\delta=\varepsilon$ in this case.

Review on Continuous Functions

Intermediate Value theorem Let $f:[a,b]\to\mathbb{R}$ be continuous.

c between $f(a), f(b) \Rightarrow \exists x_0 \in [a,b]$ such that $f(x_0)=c$.

- Use to show equation has solution:

#63 $f(x)=x$ fixed point $\iff g(x)=f(x)-x=0$

#112 $f(x)=f(x+1) \iff g(x)=f(x)-f(x+1)=0$

- Use to show f is constant function by showing the range is countable. This is because if f is not constant, then range of f contains interval between $f(a), f(b)$ (hence range is uncountable)

#113(b), 61

- use to show f surjective in special situation.

#65

Extreme Value Theorem Let $f:[a,b]\to\mathbb{R}$ be continuous.

Then $\exists\, x_0, x_1 \in [a,b]$ such that

$$f(x_0)=\min\{f(x): x\in[a,b]\}$$
$$f(x_1)=\max\{f(x): x\in[a,b]\}$$

and the range of f is $f([a,b])=[f(x_0), f(x_1)]$.

2006 Final #2

Let $f:[0,1]\to\mathbb{R}$ be continuous. Prove that $\exists\, c\in[0,1]$ such that $f(c)=\sqrt[3]{\int_0^1 f^3(t)\,dt}$.

Solution. Since f is continuous, by the extreme value theorem, $\exists\, a\in[0,1]$ and $b\in[0,1]$ such that

$$f(a)=\max\{f(x): x\in[0,1]\}$$
and
$$f(b)=\min\{f(x): x\in[0,1]\}.$$

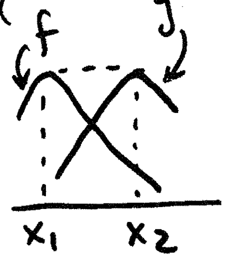
Now $f(b)\le f(t)\le f(a)$ for all $t\in[0,1]$. Then

$$f(b)=\sqrt[3]{\int_0^1 f^3(b)\,dt} \le \underbrace{\sqrt[3]{\int_0^1 f^3(t)\,dt}}_{\text{Constant}} \le \sqrt[3]{\int_0^1 f^3(a)\,dt}=f(a)$$

By the intermediate value theorem, $\exists\, c\in[0,1]$ such that

$$f(c)=\sqrt[3]{\int_0^1 f^3(t)\,dt}.$$

Let $f, g : [a,b] \to \mathbb{R}$ be continuous and $\sup\{f(x) : x \in [a,b]\} = \sup\{g(x) : x \in [a,b]\}$. Prove $\exists x_0 \in [a,b]$ such that $f(x_0) = g(x_0)$.



Solution Since f, g are continuous, by the extreme value theorem, $\exists x_1, x_2 \in [a,b]$ such that

$$f(x_1) = \sup\{f(x) : x \in [a,b]\} = \sup\{g(x) : x \in [a,b]\} = g(x_2).$$

Let $h(x) = f(x) - g(x)$. Then h is continuous on $[a,b]$.

$$h(x_1) = f(x_1) - g(x_1) \geq f(x_1) - g(x_2) = 0$$

$$h(x_2) = f(x_2) - g(x_2) \leq f(x_1) - g(x_2) = 0.$$

By the intermediate value theorem, $\exists x_0$ between x_1 and x_2 such that $h(x_0) = 0$. So $f(x_0) = g(x_0)$.

Let $f : [0,1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. $\forall n \in \mathbb{N}$, prove that $\exists t \in [0, 1-\frac{1}{n}]$ such that $f(t + \frac{1}{n}) = f(t)$.

Solution Define $F : [0, 1-\frac{1}{n}] \to \mathbb{R}$ by

$$F(x) = f(x + \tfrac{1}{n}) - f(x).$$

Then F is continuous. We will show $\exists t \in [0, 1-\frac{1}{n}]$ such that $F(t) = 0$.

Assume $F(t) \neq 0$. By the contrapositive of the Intermediate value theorem,

either $F(x) > 0$ for all $x \in [0, 1-\frac{1}{n}]$

or $F(x) < 0$ for all $x \in [0, 1-\frac{1}{n}]$.

In the former case, $f(x + \frac{1}{n}) > f(x)$ holds for all $x \in [0, 1-\frac{1}{n}]$. Then

$$f(0) < f(\tfrac{1}{n}) < f(\tfrac{2}{n}) < \cdots < f(1).$$

take $x = 0$ take $x = \frac{1}{n}$ ...

this contradicts $f(0) = f(1)$.

The latter case is similar (by reversing inequality signs).

③ Let $f:[0,2]\to\mathbb{R}$ be continuous and $f(2)=0$. If $\lim_{x\to 1}\frac{f(x)-2}{\sqrt{x}-1}=1$, then prove that $\exists\, x\in[0,2]$ such that $f(x)=x^2$. (2012 Spring Midterm)

Solution

Let $g(x)=f(x)-x^2$. Then g is continuous on $[0,2]$ since f is continuous on $[0,2]$.

$g(2)=f(2)-2^2=0-4<0$.

Next, $f(x)-2=\frac{f(x)-2}{\sqrt{x}-1}(\sqrt{x}-1)$ for $x\neq 1$.

Then $f(1)-2=\lim_{x\to 1}(f(x)-2)=\lim_{x\to 1}\frac{f(x)-2}{\sqrt{x}-1}\cdot\lim_{x\to 1}(\sqrt{x}-1)$
$=1\cdot 0=0$

$\therefore f(1)=2$

Then $g(1)=f(1)-1^2=2-1>0$.

By intermediate value theorem, $\exists\, x\in[1,2]$ such that $g(x)=0$, so $f(x)=x^2$.

Let $f:[0,1]\to[0,+\infty)$ be continuous. If for every $x\in[0,1]$, $e^{-\sqrt{f(x)}}\in\mathbb{Q}$, then prove that f is a constant function.

Solution Assume f is not a constant function. Then $\exists\, 0\le a<b\le 1$ such that $f(a)\neq f(b)$.

Since $\sqrt{x}$ is strictly increasing, $\sqrt{f(a)}\neq\sqrt{f(b)}$

Since e^{-x} is strictly decreasing, $e^{-\sqrt{f(a)}}\neq e^{-\sqrt{f(b)}}$

Since f, $\sqrt{x}$, e^{-x} are continuous, so composing them $g(x)=e^{-\sqrt{f(x)}}$ is continuous. Now $g(a)\neq g(b)$.

So the range of g contains the interval I with $g(a)$ and $g(b)$ as endpoints. By density of irrational, $\exists$ irrational w in I. By the intermediate value theorem, w is between $g(a)$ and $g(b)$ $\Rightarrow w=g(x)$ for some $x\in[a,b]$.

Then $e^{-\sqrt{f(x)}}=g(x)=w\notin\mathbb{Q}$, contradiction.

$\therefore f$ is a constant function.

Continuous Injection Theorem On any nonempty interval,

Continuity + Injectivity $\Rightarrow$ Strictly monotoneity.

Use when a continuous function or inequality

#62, 65, 155 → satisfies an equation and equation $\Rightarrow$ injective

2006 HW3 prob 3 → derivative is never zero

Exercise 62

Is there a continuous function $g : [-1,1] \to [-1,1]$ such that $g(g(x)) = -x^9$ for all $x \in [-1,1]$?

Solution Assume $\exists$ continuous $g : [-1,1] \to [-1,1]$ such that $g(g(x)) = -x^9 \ \forall x \in [-1,1]$.

Then g is injective because

$$g(a) = g(b) \Rightarrow g(g(a)) = g(g(b)) \Rightarrow -a^9 = -b^9 \Rightarrow a = b.$$

Now g is continuous and injective.

By continuous injection theorem, ① g is strictly increasing or ② g is strictly decreasing.

① g strictly increasing

$$\Rightarrow a<b \Rightarrow g(a) < g(b) \Rightarrow g(g(a)) < g(g(b)) \Rightarrow -a^9 < -b^9 \Rightarrow a > b, \text{ contradiction.}$$

② g strictly decreasing

$$\Rightarrow a<b \Rightarrow g(a) > g(b) \Rightarrow g(g(a)) < g(g(b)) \Rightarrow -a^9 < -b^9 \Rightarrow a > b, \text{ contradiction}$$

$\therefore$ no such g exists.

2007 Midterm

③ (This is similar to $g(g(x)) = -x^9$ problem #62)

$f : [0,1] \to [0,1]$ continuous

$f(0) = 0,\ f(1) = 1$

$f(f(x)) = x \quad \forall x \in [0,1]$

Prove $f(x) = x \quad \forall x \in [0,1]$

f is injective since $f(a) = f(b) \Rightarrow \underset{a}{\underset{\shortparallel}{f(f(a))}} = \underset{b}{\underset{\shortparallel}{f(f(b))}}$

By continuous injection theorem, f is strictly monotone.

Since $f(0) = 0, f(1) = 1$, f is strictly increasing.

$$x \le f(x) \Rightarrow f(x) \le f(f(x)) = x \Rightarrow x = f(x)$$

$$f(x) \le x \Rightarrow x = f(f(x)) \le f(x) \Rightarrow f(x) = x.$$

$\therefore \forall x \in [0,1],\ f(x) = x.$

2012-2013 Spring Homework 3 #5

Let $a<b$ and $f:[a,b]\to\mathbb{R}$ be differentiable.
If $f'(a)<w<f'(b)$, then prove that $\exists\, c\in(a,b)$ such that $f'(c)=w$. (Hint: Consider $w=0$ first.)

Solution

Case 1 ($w=0$) Assume no $c\in(a,b)$ satisfy $f'(c)=0$.
Then $f'(x)\neq 0$ for all $x\in[a,b]$. This imply
f is continuous and injective

$$\boxed{\begin{matrix}x_0\neq x_1\\ \text{in } [a,b]\end{matrix} \Rightarrow f(x_0)-f(x_1)=\underbrace{f'(\theta)}_{\neq 0}\underbrace{(x_0-x_1)}_{\neq 0}\neq 0 \Rightarrow f(x_0)\neq f(x_1)}$$

By continuous injection theorem, f is strictly increasing or f is strictly decreasing.
Then $f'(x)\geq 0$ for all $x\in[a,b]$
or $f'(x)\leq 0$ for all $x\in[a,b]$.
This contradicts $f'(a)<w=0<f'(b)$.
$\therefore\ \exists\, c\in(a,b)$ such that $f'(c)=0$.

Case 2 ($w\neq 0$). Let $g(x)=f(x)-wx$. Then
$g'(x)=f'(x)-w$. $f'(a)<w<f'(b) \Leftrightarrow f'(a)-w<0<f'(b)-w$
$\Leftrightarrow g'(a)<0<g'(b)$.
By case 1, $\exists\, c\in(a,b)$ such that $g'(c)=0$.
Finally, $g'(c)=0 \Leftrightarrow f'(c)=w$.

Mean Value Theorem Let f be continuous on $[a,b]$ and differentiable on (a,b). Then $\exists\, x_0\in(a,b)$ such that

$$f(a)-f(b)=f'(x_0)(a-b).$$

Use when you have $f(\ )-f(\ )$ expression and f is differentiable. ← #74, 75, 120
Use to prove inequalities ← examples on p. 38

~~Generalized Mean Value Theorem Let $f, g:[a,b]\to\mathbb{R}$ be continuous. Let f, g be differentiable on (a,b). Then $\exists\,\theta\in(a,b)$ such that~~

$$\cancel{(f(b)-f(a))g'(\theta)=(g(b)-g(a))f'(\theta).}$$

Taylor's Theorem For n-times diff $f:(a,b)\to\mathbb{R}$, $x,c\in(a,b)$,
$$f(x)=f(c)+\frac{f'(c)}{1!}(x-c)+\cdots+\frac{f^{(n-1)}(c)}{(n-1)!}(x-c)^{n-1}+R_n(x)$$
where $R_n(x)=\frac{f^{(n)}(x_0)}{n!}(x-c)^n$ for some x_0 between x and c.

- Use when functions are n-times differentiable, $n>1$
- center c should be
  1. something we know about $f'(c)$

  or 2. among $f(a), f(b), \ldots$ given, $f(c)$ is slightly special

  otherwise 3. try a variable for c.

  Or 4. local max or min, then $f'(c)=0$.

2009 Midterm

② Let $f:\mathbb{R}\to\mathbb{R}$ be twice differentiable and $f''(x)$ be continuous. If

$$f(-1)=0,\ f(0)=2,\ f(1)=5 \text{ and } f'(0)=0,$$

then prove that there exists $c\in\mathbb{R}$ such that $f''(c)=\sqrt{2}$.

Extra Credit Try to do this without using $f''(x)$ is continuous.

Solution By Taylor's theorem,

$$f(x)=f(0)+f'(0)x+\frac{f''(\theta_x)}{2}x^2 \text{ for some } \theta_x \text{ between } 0 \text{ and } x.$$

Letting $x=-1$ and 1, we get

$$0=f(-1)=2+\frac{f''(\theta_{-1})}{2} \text{ and } 5=f(1)=2+\frac{f''(\theta_1)}{2}$$

This implies $f''(\theta_{-1})=-4$ and $f''(\theta_1)=6$.

Since $f''(x)$ is continuous, by the intermediate value theorem, $\exists\, c\in\mathbb{R}$ such that $f''(c)=\sqrt{2}$.

For extra credit, use Homework 3, Exercise #5

Remark 0 is special, we know $f'(0)$.

③ Let f be twice differentiable on $[0,2]$. $\forall x\in[0,2]$, $|f(x)|\le 1$, $|f''(x)|\le 1$.

Prove that $\forall x\in[0,2]$, $|f'(x)|\le 2$.

Solution By Taylor's theorem, let $x\in[0,2]$, $a\in[0,2]$

$$f(a)=f(x)+f'(x)(a-x)+\frac{f''(\theta_a)}{2}(a-x)^2$$

for some θ_a between a and x. Setting $a=0,2$,

$$f(0)=f(x)-f'(x)x+\frac{f''(\theta_0)}{2}x^2 \text{ for some } \theta_0\in(0,x).$$

$$f(2)=f(x)+f'(x)(2-x)+\frac{f''(\theta_2)}{2}(2-x)^2 \text{ for some } \theta_2\in(x,2).$$

Subtracting these, we get

$$f(2)-f(0)=2f'(x)+\frac{f''(\theta_2)}{2}(2-x)^2-\frac{f''(\theta_0)}{2}x^2.$$

Solving for $f'(x)$, we see

$$|f'(x)|=\frac{1}{2}\left|f(2)-f(0)+\frac{f''(\theta_0)}{2}x^2-\frac{f''(\theta_2)}{2}(2-x)^2\right|$$

$$\le\frac{1}{2}\left(1+1+\frac{1}{2}x^2+\frac{1}{2}(2-x)^2\right)$$

$$=\frac{1}{2}(x^2-2x+4)$$

$$\le\frac{1}{2}((x-1)^2+3) \qquad \left\{\begin{array}{l} x\in[0,2] \\ |x-1|\le 1\end{array}\right.$$

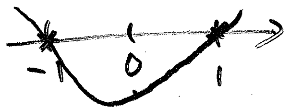
$$\le\frac{1}{2}(1+3)=2$$

Remark We use x as center because we have information about $f(x)$, $f''(x)$ for general x, but no specific x.

Let $f:\mathbb{R}\to\mathbb{R}$ be twice differentiable such that $f(-1)=0=f(1)$, $f(0)<0$ and $\forall x\in[-1,1]$, $f''(x)\geq 2$. Prove that $\exists\, b\in[-1,1]$ satisfying $f(b)\leq -(1+b^2)$.

Note we don't know $f(0)$

f has a minimum in $(-1,1)$.



Solution Let $c\in[-1,1]$ such that $f(c)=\min\{f(x): x\in[-1,1]\}$ by extreme value theorem. By Taylor's theorem,

$$0=f(-1)=f(c)+\underbrace{f'(c)}_{=0}(-1-c)+\frac{f''(\theta_{-1})}{2}(-1-c)^2$$

$$0=f(1)=f(c)+\underbrace{f'(c)}_{=0}(1-c)+\frac{f''(\theta_1)}{2}(1-c)^2$$

Moving $f(c)$ to the left side and adding equations, we get

$$-2f(c)=\frac{f''(\theta_{-1})}{2}(1+c)^2+\frac{f''(\theta_1)}{2}(1-c)^2$$
$$\geq (1+c)^2+(1-c)^2=2+2c^2$$

$\Rightarrow f(c)\leq -(1+c^2)$. So c is such b.

Let $f:\mathbb{R}\to\mathbb{R}$ be three times differentiable. If $f(x)$ and $f'''(x)$ are bounded functions, then prove that $f'(x)$ and $f''(x)$ are bounded functions.

Solution By Taylor's theorem,

① $$f(x+1)=f(x)+f'(x)((x+1)-x)+\frac{f''(x)}{2}((x+1)-x)^2+\frac{f'''(\theta)}{6}((x+1)-x)^3$$
$$=f(x)+f'(x)+\frac{f''(x)}{2}+\frac{f'''(\theta)}{6} \quad \text{for some } \theta\in(x,x+1)$$

② $$f(x-1)=f(x)-f'(x)+\frac{f''(x)}{2}-\frac{f'''(\sigma)}{6} \quad \text{for some } \sigma(x-1,x).$$

Since $f(x)$, $f'''(x)$ are bounded, $\exists\, M_0, M_3>0$ such that $\forall x\in\mathbb{R}$, $|f(x)|\leq M_0$ and $|f'''(x)|\leq M_3$.

Adding equations ① and ②, we get

$$f(x+1)+f(x-1)=2f(x)+f''(x)+\frac{f'''(\theta)}{6}-\frac{f'''(\sigma)}{6}$$

$$\Rightarrow f''(x)=f(x+1)+f(x-1)-2f(x)-\frac{f'''(\theta)}{6}+\frac{f'''(\sigma)}{6}$$

$$\Rightarrow |f''(x)|\leq M_0+M_0+2M_0+\tfrac{1}{6}M_3+\tfrac{1}{6}M_3$$
$$=4M_0+\tfrac{1}{3}M_3=M_2$$

Solving for $f'(x)$ in equation 1, we get

$$f'(x)=f(x+1)-f(x)-\frac{f''(x)}{2}-\frac{f'''(\theta)}{6}$$

$$\Rightarrow |f'(x)|\leq M_0+M_0+\tfrac{1}{2}M_2+\tfrac{1}{6}M_3$$

$\therefore$ $f'(x)$ and $f''(x)$ are bounded functions.

Let $f:[0,1]\to\mathbb{R}$ be continuous and $f(0)=f(1)$.
If f is twice differentiable on $(0,1)$ and there is $M>0$ such that $|f''(x)|\le M$ for all $x\in(0,1)$, then prove that $|f'(x)|\le \frac{1}{2}M$ for all $x\in(0,1)$.

<u>Thoughts</u>: Taylor theorem problem because higher derivatives are involved. Although $f(0)=f(1)$ is given, we have <u>no information on $f'(0)$ and $f'(1)$</u>.

<u>Solution</u> By Taylor's theorem,

$$f(1)=f(x)+f'(x)(1-x)+\frac{f''(\theta)}{2}(1-x)^2 \quad \text{for some } \theta\in(x,1)$$

$$f(0)=f(x)+f'(x)(0-x)+\frac{f''(\sigma)}{2}(0-x)^2 \quad \text{for some } \sigma\in(0,x).$$

Since $f(1)=f(0)$, we subtract the equations above to get

$$0=f'(x)+\frac{f''(\theta)}{2}(1-x)^2-\frac{f''(\sigma)}{2}x^2.$$

So $f'(x)=\frac{f''(\sigma)}{2}x^2-\frac{f''(\theta)}{2}(1-x)^2$.

Then
$$\begin{aligned}|f'(x)| &\le \frac{|f''(\sigma)|}{2}x^2+\frac{|f''(\theta)|}{2}(1-x)^2\\ &\le \frac{M}{2}x^2+\frac{M}{2}(1-x)^2\\ &= \frac{M}{2}(x^2+(1-x)^2)\le\frac{M}{2}\end{aligned}$$

On $[0,1]$, $x^2+(1-x)^2=2x^2-2x+1$ has <u>maximum value</u> 1 by calculus

or $2(x-\frac{1}{2})^2+\frac{1}{2}\le 2(\frac{1}{2})^2+\frac{1}{2}=1$

Let $f(x)$ have 2^{nd} derivative at every $x\in[a,b]$.
If $f'(a)=f'(b)=0$, then prove that $\exists\, c\in(a,b)$ such that $|f''(c)|\ge\frac{4}{(b-a)^2}|f(b)-f(a)|$.

<u>Solution</u> By Taylor's theorem,

$$f(x)=f(b)+f'(b)(x-b)+\frac{f''(\theta_1)}{2}(x-b)^2$$

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(\theta_2)}{2}(x-a)^2.$$

To get $f(b)-f(a)$, we subtract these equations

$$0=f(b)-f(a)+\frac{1}{2}\left(f''(\theta_1)(x-b)^2-f''(\theta_2)(x-a)^2\right).$$

Setting $x=\frac{a+b}{2}$, then $(x-b)^2=(x-a)^2=\left(\frac{b-a}{2}\right)^2$

so $0=f(b)-f(a)+\frac{(b-a)^2}{8}(f''(\theta_1)-f''(\theta_2))$

$$\Rightarrow |f(b)-f(a)|\frac{4}{(b-a)^2}=\frac{1}{2}|f''(\theta_1)-f''(\theta_2)|$$

$$\le \underbrace{\frac{1}{2}\left(|f''(\theta_1)|+|f''(\theta_2)|\right)}_{\le |f''(c)|}$$

If $|f''(\theta_1)|\le|f''(\theta_2)|$, then take $c=\theta_2$

If $|f''(\theta_2)|\le|f''(\theta_1)|$, then take $c=\theta_1$