Math 2033 Past Exam Problems

Countability

Useful Facts: Bijection Theorem, *Countable Union Theorem*, Countable Subset Theorem, Product Theorem, Injection Theorem, Surjection Theorem. Examples of Countable Sets. Examples of Uncountable Sets. For existence problems, use the fact that *if A is uncountable and B is countable, then $A \setminus B$ is uncountable.*

101. (2009FM) Let S be the set of all points (x, y) in the coordinate plane that satisfy the equations

$$x^2 + y^2 = a^2 \qquad \text{and} \qquad y = x^2 - x^3 + b$$

for some $a, b \in \mathbb{Q}$ with $a \neq b$. Determine (with proof) if S is countable or not.

102. (2012FM) Let S be the set of all points $(x, y) \in \mathbb{R}^2$ that satisfy the system of equations

$$x + y = mx^2 - x^3 \qquad \text{and} \qquad mx + y^4 = x^6 - 7mx^3 + 2$$

for some $m \in \mathbb{Q}$. Determine (with proof) if S is countable or not.

103. (2010FF) (a) Let S be the set of all intersection points (x, y) that lie on the graphs of at least one pair of equations $y = x^3 + mx + n$ and $mx^2 - ny^2 = 1$, where $m, n \in \mathbb{Q}$. Determine (with proof) whether S is a countable set or not.

(b) Prove that there exist infinitely many positive real numbers that are not equal to any number of the form $a + b(2^c\pi^d)$, where $a, b \in \mathbb{Q} \cap (0, +\infty)$ and $c, d \in \mathbb{Q} \cap [0, +\infty)$.

104. (2008FM) Prove that there exists a positive real number c which does *not* equal to any number of the form $2^{a+b\sqrt{2}}$, where $a, b \in \mathbb{Q}$.

105. (2012FM) Let S be a nonempty countable subset of $\mathbb{R}$. Prove that there exists a positive real number r such that the equation $5^x + 7^y = \sqrt{r}$ does not have any solution with $x, y \in S$.

106. (2011FF) (a) Prove that there exist infinitely many positive irrational numbers that are not equal to any number of the form $\dfrac{a\sqrt{2} + b}{c + d\pi}$, where $a, b, c, d \in \mathbb{Q} \cap (0, +\infty)$.

(b) Let $f : \mathbb{R} \to \mathbb{Q}$ be a function. Prove that there exists an uncountable subset S of $\mathbb{R}$ such that for all $x, y \in S$, we have $f(x) = f(y)$.

Infimum-Supremum

<u>Useful Facts:</u> Supremum Limit Theorem, Infimum Limit Theorem. The fact that for $x \in \mathbb{R}$, $w_n = \dfrac{[nx]}{n} \in \mathbb{Q}$ converges to x. Density of rational numbers and density of irrational numbers are also useful sometimes.

201. (2008FF) Determine (with proof) the infimum of the set

$$S = \{x : x \in \mathbb{R} \text{ and there exist } b, c \in [-1, 1) \text{ such that } x^2 + bx + c = 0\}.$$

202. (2008FM) (a) Find (with proof) the supremum and infimum of $B = \big\{\cos x + \sin y : \ x, y \in (0, \pi/2] \cap \mathbb{Q}\big\}$.

(b) Let D and E be nonempty bounded subsets of $\mathbb{R}$ such that

$$\inf D = 3, \quad \sup D = 5, \quad \inf E = 7 \quad \text{and} \quad \sup E = 9.$$

Determine (with proof) the supremum and infimum of the set $A = \{x + \frac{1}{y} : \ x \in D, \ y \in E\}$.

203. (2009FM) Let D be a nonempty bounded subset of $\mathbb{R}$ such that $\inf D = 3$ and $\sup D = 5$. Let

$$A = \{xy + xy^3 : \ x \in (2, \pi] \cap \mathbb{Q}, \quad y \in D\}.$$

Show that A is bounded. Determine (with proof) the infimum and supremum of A.

204. (2010FM) Let A be a nonempty bounded subset of $\mathbb{R}$ such that $\inf A = 1$ and $\sup A = 3$. Let

$$B = \{\sqrt{2x(15 + xy)} : \ x \in (2, 4) \cap \mathbb{Q}, \quad y \in A\}.$$

Prove that B is bounded. Determine (with proof) the infimum and supremum of B.

205. (2011FM) Suppose A and B are two nonempty bounded subsets of $\mathbb{R}$ such that $\inf A = 1, \ \sup A = 5, \ \inf B = 0$ and $\sup B = 1$. Let

$$C = \Big\{\frac{y}{3 - x} - \frac{1}{y} : \ x \in B, \quad y \in A\Big\}.$$

Prove that C is bounded. Determine (with proof) the infimum and supremum of C.

206. (2012FF) (a) Let D be a nonempty subset of $\mathbb{R}$ with $\inf D = 1$ and $\sup D = 5$. Determine (with proof) the supremum of the set

$$E = \Big\{x(y + \sqrt{2}) - \frac{1}{x} \ : \ x \in D, \quad y \in [0, \sqrt{2}) \cap \mathbb{Q}\Big\}.$$

(b) Let A, B, C be nonempty subsets of $\mathbb{R}$ such that $A \subseteq B \subseteq C$. Suppose C is bounded above in $\mathbb{R}$. If $\sup A = w = \sup C$, then prove that $\sup B = w$.

Limit of Sequences

Useful Facts: Definition of Limit of a Sequence is $x_1, x_2, x_3, \ldots$ (or $\{x_n\}$) converges to (or has limit) x if and only if for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $n \geq K$ implies $|x_n - x| < \varepsilon$. The $\varepsilon/2 + \varepsilon/2 = \varepsilon$ and $\max\{K_1, K_2\}$ trick are useful when we have two sequences. Also, the fact *convergent sequences are bounded* is used sometimes.

Inequalities like $|\sin a - \sin b| \leq |a - b|$ and $|\sqrt[n]{a} - \sqrt[n]{b}| \leq \sqrt[n]{|a-b|}$ are useful. Identity $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$ is sometimes useful.

301. (2008FF) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that both $\{a_n\}$ and $\{b_n\}$ converge to 1. Prove that $\lim_{n\to\infty}\left(2b_n^3 + \frac{a_n}{2n}\right) = 2$ by checking the definition of limit of a sequence.

    Do not use computation formulas, sandwich theorem or L'Hopital's rule, otherwise you will get zero mark on this problem!

302. (2009FF) Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be two sequences of real numbers such that both of them converge to 1. Prove that $\lim_{n\to\infty}\left(\frac{3a_n^2+1}{a_n^2+1} + \frac{nb_n}{n+2}\right) = 3$ by checking the definition of limit of a sequence.

    Do not use computation formulas, sandwich theorem or L'Hopital's rule, otherwise you will get zero mark on this problem!

303. (2010FF) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers that converges to 1. Prove that

$$\lim_{n\to\infty}\left(\frac{3+a_n^2}{a_n+1} + \frac{2n}{4+n}\right) = 4$$

    by checking the definition of limit of a sequence.

    Do not use computation formulas, sandwich theorem or L'Hopital's rule, otherwise you will get zero mark on this problem!

304. (2011FF) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers that converges to 3. Prove that

$$\lim_{n\to\infty}\left(\frac{a_n}{a_n^2+3} + \frac{3n^2}{1+4n^2} + \frac{a_n}{n}\right) = 1$$

    by checking the definition of limit of a sequence.

    Do not use computation formulas, sandwich theorem or L'Hopital's rule, otherwise you will get zero mark on this problem!

305. (2012FF) Let $b_1, b_2, b_3, \ldots$ be a sequence of positive real numbers with $\lim_{n\to\infty} b_n = 2$. Prove that

$$\lim_{n\to\infty}\left(\frac{4n-1}{n+3} - \frac{2}{b_n} + \frac{b_n}{n}\right) = 3$$

    by checking the definition of limit of a sequence only.

    Do not use computation formulas, sandwich theorem or l'Hopital's rule! Otherwise, you will get zero mark for this problem.

Limit of Recurrence Relations

Useful Facts: Compute the first three to four terms to see if sequence is monotone or back-and-forth type.

For monotone sequences, find limit by taking limit of the recurrence relation. Then use limit as a bound. Do mathematical induction. Apply monotone sequence theorem.

For back-and-forth type, do mathematical induction to confirm intervals are nested. Let a be the limit of $x_1, x_3, x_5, \ldots$ and b be the limit of $x_2, x_4, x_6, \ldots$. Take two limits of the recurrence relation for case n odd and case n even. This should give two equations on a and b. Show $a = b$. By intertwing sequence theorem, a is the limit of the whole sequence. Then take limit of the recurrence relation to get an equation on a. Solve for a.

401. (2009FF) Let $x_1 = 1$ and for $n = 1, 2, 3, \ldots$, let $x_{n+1} = \dfrac{x_n^3 + x_n}{5}$. Prove that the sequence $x_1, x_2, x_3, \ldots$ converges and find its limit.

402. (2009FF) Let $x_1 = 2$ and for $n = 1, 2, 3, \ldots$, let $x_{n+1} = \dfrac{22}{3} + \dfrac{16}{3x_n}$. Prove that the sequence $x_1, x_2, x_3, \ldots$ converges and find its limit.

403. (2010FM) Prove the sequence $\{x_n\}$ converges, where $x_1 = 5$ and $x_{n+1} = \dfrac{7}{x_n + 5}$, and find its limit.

404. (2010FF) Let $x_1 = \dfrac{1}{4}$. For $n = 1, 2, 3, \ldots$, let $x_{n+1} = \dfrac{\sqrt{x_n} + 3x_n}{4}$. Prove that the sequence $x_1, x_2, x_3, \ldots$ converges and find its limit.

405. (2011FM) Prove the sequence $\{x_n\}$ converges, where

$$x_1 = 27 \qquad \text{and} \qquad x_{n+1} = 8 - \sqrt{28 - x_n} \quad \text{for } n = 1, 2, 3, \ldots$$

and find its limit. Show work!

406. (2011FF) Let $x_1 = 0$, $x_2 = 3$ and for $n = 1, 2, 3, \ldots$, let $x_{n+2} = \sqrt{\dfrac{4}{9}x_{n+1}^2 + \dfrac{5}{9}x_n^2}$.

(a) Prove that the sequence $x_1, x_2, x_3, \ldots$ converges.

(b) Determine (with proof) the limit of the sequence $x_1, x_2, x_3, \ldots$.

407. (2012FF) (a) If $x_1 = -2$ and $x_{n+1} = \sqrt{6 + x_n}$ for $n = 1, 2, 3, \ldots$, then prove that $x_1, x_2, x_3, \ldots$ converges and find its limit.

(b) If $y_1 = 0$ and $y_{n+1} = \dfrac{2}{2 + y_n}$ for $n = 1, 2, 3, \ldots$, then prove that $y_1, y_2, y_3, \ldots$ converges and find its limit.