MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #11

Limit (Part I)

WARNING: Starting from this Chapter, many problems will require a lot of thinking. So you need to learn how to think by doing problems.

Part I) Limit of sequence

Definition: Given a sequence of real numbers $\{x_n\}$, we say $\lim_{n\to\infty} x_n = x$ if and only if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n > N$, we have $|x_n - x| < \varepsilon$.

When using definition to show the limit of sequence, for any $\varepsilon > 0$, we need to find the corresponding N (which should depend on ε), such that for $n > N$, we have $|x_n - x| < \varepsilon$.

One commonly used theorem is so called Archimedean Property.
"For any $x \in R$, then there exists $n \in \mathbf{N}$, such that $n > x$"

Example 1 Show that $\lim_{n\to\infty} \frac{3n-4}{n-1} = 3$ by checking the definition of limit

(Preliminary: Find our N) (No need to write this in your solution)

Now we need for any $\varepsilon > 0$, we have $\left|\frac{3n-4}{n-1} - 3\right| < \varepsilon$

Note $\left|\frac{3n-4}{n-1} - 3\right| = \left|-\frac{1}{n-1}\right| = \frac{1}{n-1} < \varepsilon \leftrightarrow n > \frac{1}{\varepsilon} + 1$

Solution: **<u>(Write the solution officially)</u>**

For any $\varepsilon > 0$,

By Archimedean Property, there exists $N \in \mathbf{N}$, such that for $n > N$, $n > \frac{1}{\varepsilon} + 1$

We have $\left|\frac{3n-4}{n-1} - 3\right| = \left|-\frac{1}{n-1}\right| = \frac{1}{n-1} < \varepsilon$

Hence $\lim_{n\to\infty} \frac{3n-4}{n-1} = 3$

Example 2

Show $\lim_{n\to\infty} \frac{[10^n\sqrt{2}]}{10^n} = \sqrt{2}$ by checking definition of limit (officially)

Where $[x]$ means the greatest integer less than or equal to x.

(Preliminary: Find our N)

We hope for any $\varepsilon > 0$, we have $\left|\frac{[10^n\sqrt{2}]}{10^n} - \sqrt{2}\right| < \varepsilon$

(Now we need to use some inequality so that we can find N easier)

Note that $10^n\sqrt{2} - 1 < [10^n\sqrt{2}] < 10^n\sqrt{2}$ (by definition of greatest integer)

$$\left|\frac{[10^n\sqrt{2}]}{10^n} - \sqrt{2}\right| = \sqrt{2} - \frac{[10^n\sqrt{2}]}{10^n} < \sqrt{2} - \frac{10^n\sqrt{2} - 1}{10^n} = \frac{1}{10^n} < \varepsilon \to n > \log\left(\frac{1}{\varepsilon}\right)$$

Solution: <u>(Write the solution officially)</u>

For any $\varepsilon > 0$,

By Archimedean Property, there exists $N \in \mathbb{N}$, such that for $n > N$, $n > \log\left(\frac{1}{\varepsilon}\right)$

Then $\left|\frac{[10^n\sqrt{2}]}{10^n} - \sqrt{2}\right| = \sqrt{2} - \frac{[10^n\sqrt{2}]}{10^n} < \sqrt{2} - \frac{10^n\sqrt{2}-1}{10^n} = \frac{1}{10^n} < \varepsilon$.

Therefore $\lim_{n\to\infty} \frac{[10^n\sqrt{2}]}{10^n} = \sqrt{2}$.

☺Exercise 1

Show $\lim_{n\to\infty} \frac{4n^3+5}{n^3} = 0$ by definition of limit

☺Exercise 2

Show officially $\lim_{n\to\infty} \frac{[10^n\pi]+1}{10^n} = \pi$ by checking the definition of limit

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Example 3

Show that $\lim_{n\to\infty} \frac{4n+5}{n^3} = 0$ by checking the definition

(Preliminary: Find N)

Note $\left|\frac{4n+5}{n^3} - 0\right| = \left|\frac{4}{n^2} + \frac{5}{n^3}\right| \leq \frac{4}{n^2} + \frac{5}{n^3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

We need $\frac{4}{n^2} < \frac{\varepsilon}{2} \to n > \sqrt{\frac{8}{\varepsilon}}$ and $\frac{5}{n^3} < \frac{\varepsilon}{2} \to n > \sqrt[3]{\frac{10}{\varepsilon}}$

Solution:

By Archimedean property, where exist K_1 and K_2 such that $n > \sqrt{\frac{8}{\varepsilon}} \to \frac{4}{n^2} < \frac{\varepsilon}{2}$

for $n > K_1$ and $n > \sqrt[3]{\frac{10}{\varepsilon}} \to \frac{5}{n^3} < \frac{\varepsilon}{2}$ for $n > K_2$.

To satisfy both inequality, pick $K = \max\{K_1, K_2\}$, then for $n > K$

$$\left|\frac{\mathbf{4n+5}}{\mathbf{n^3}}-\mathbf{0}\right|=\left|\frac{\mathbf{4}}{\boldsymbol{n^2}}+\frac{\mathbf{5}}{\boldsymbol{n^3}}\right|\le\frac{\mathbf{4}}{\boldsymbol{n^2}}+\frac{\mathbf{5}}{\boldsymbol{n^3}}<\frac{\boldsymbol{\varepsilon}}{\mathbf{2}}+\frac{\boldsymbol{\varepsilon}}{\mathbf{2}}=\boldsymbol{\varepsilon}$$

Hence $\mathbf{lim}_{\mathbf{n}\to\infty}\frac{\boldsymbol{4n+5}}{\boldsymbol{n^3}}=\mathbf{0}$

Example 4

If a_n converges to A, then show that $\left\{\frac{(a_n+a_{n+1}\}}{2}\right\}$ converges to A by checking the definition

(Preliminary: Find out N)

$$\left|\frac{(a_n+a_{n+1}\}}{2}-A\right|=\left|\frac{(a_n-A)+(a_{n+1}-A)}{2}\right|\le\left|\frac{a_n-A}{2}\right|+\left|\frac{a_{n+1}-A}{2}\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

We need N such that for $n>N$ $|a_n-A|<\varepsilon$ (Then $|a_{n+1}-A|<\varepsilon$)

Solution:

For any $\boldsymbol{\varepsilon}>0$

Since $\mathbf{lim}_{\mathbf{n}\to\infty}\boldsymbol{a_n}=\boldsymbol{A}$**, for this** $\boldsymbol{\varepsilon}$**, there exists positive integer K such that for** $\mathbf{n}>K$**,** $|\mathbf{a_n}-\mathbf{A}|<\varepsilon$ **and we have**

$$\left|\frac{(\mathbf{a_n}+\mathbf{a_{n+1}}\}}{\mathbf{2}}-\mathbf{A}\right|=\left|\frac{(\mathbf{a_n}-\mathbf{A})+(\mathbf{a_{n+1}}-\mathbf{A})}{\mathbf{2}}\right|\le\left|\frac{\mathbf{a_n}-\mathbf{A}}{\mathbf{2}}\right|+\left|\frac{\mathbf{a_{n+1}}-\mathbf{A}}{\mathbf{2}}\right|<\frac{\boldsymbol{\varepsilon}}{\mathbf{2}}+\frac{\boldsymbol{\varepsilon}}{\mathbf{2}}=\boldsymbol{\varepsilon}$$

Hence $\left\{\frac{(\mathbf{a_n}+\mathbf{a_{n+1}}\}}{\mathbf{2}}\right\}$ **converges to A**

Example 5

If $\lim_{n\to\infty}a_n=16$, check $\lim_{n\to\infty}\sqrt[4]{a_n}=2$ by checking the definition of limit

(Preliminary: Find N)

$$|\sqrt[4]{a_n}-2|=\frac{|a_n-2^4|}{\left|(a_n)^{\frac{3}{4}}+2(a_n)^{\frac{2}{4}}+2^2(a_n)^{\frac{1}{4}}+2^3\right|}<\frac{|a_n-2^4|}{2^3}<\frac{\varepsilon}{8}<\varepsilon$$

We need N such that for $n>N$, $a_n>0\leftarrow|a_n-16|<16$ and $|a_n-2^4|<\varepsilon$

Solution:

For any $\boldsymbol{\varepsilon}>0$**,**

For this $\boldsymbol{\varepsilon}$**, there exists** $\mathbf{N_1}$ **such that, for** $\mathbf{n}>\mathbf{N_1}$**,** $\left|\mathbf{a_n}-\mathbf{2^4}\right|<\boldsymbol{\varepsilon}$**.......(*)**

There exists $\mathbf{N_2}$ **such that** $\mathbf{n}>\mathbf{N_2}$**,** $|\mathbf{a_n}-\mathbf{16}|<\mathbf{16}$**, therefore** $\mathbf{a_n}>\mathbf{0}$**.....(**)**

Pick N= $\mathbf{max}\{\mathbf{N_1},\mathbf{N_2}\}$**, then for** $\mathbf{n}>N$

$$|\sqrt[4]{\mathbf{a_n}}-\mathbf{2}|=\frac{|\mathbf{a_n}-\mathbf{2^4}|}{\left|(\mathbf{a_n})^{\frac{3}{4}}+\mathbf{2}(\mathbf{a_n})^{\frac{2}{4}}+\mathbf{2^2}(\mathbf{a_n})^{\frac{1}{4}}+\mathbf{2^3}\right|}<\frac{|\mathbf{a_n}-\mathbf{2^4}|}{\mathbf{2^3}}<\frac{\boldsymbol{\varepsilon}}{\mathbf{8}}<\varepsilon$$

Therefore $\mathbf{lim}_{\mathbf{n}\to\infty}\sqrt[4]{\boldsymbol{a_n}}=\mathbf{2}$**.**

Remark: In the problem above, we have make use of the following identity

$$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots\ldots+b^{n-1})$$

☺Exercise 3 (Practice Exercise #102,103)

Show the following by checking definition of limit

a) $\lim_{n\to\infty}\left(\frac{1}{n^2}-\frac{\sqrt{2}}{n^3}\right)=0$ b) $\lim_{n\to\infty}\left(\frac{2}{n+1}-\frac{1}{n^2}\right)=0$

☺Exercise 4 (Practice Exercise #106)

Given $\lim_{n\to\infty} x_n = 8$. Prove $\lim_{n\to\infty} \sqrt[3]{x_n} = 2$ by checking the definition.

Part 2: Recurrence Relation

In many situation, the sequence will appear in the following mode,

$$x_1 = 2, x_2 = 3, x_{n+2} = 2x_{n+1} - x_n \text{ for } n = 1,2,3,\ldots$$

The third equation, give us the way to generate $x_3, x_4, \ldots.$ and is called recurrence relation. To prove the convergence of such sequence, checking the definition may not be so efficient at all. Hence, we need some theorems to help us

Case i) Monotone Sequence

Theorem 1 (Monotone Sequence Theorem)
If $\{x_n\}$ is increasing and bounded from above, then $\{x_n\}$ converges and $\lim_{n\to\infty} x_n = \sup\{x_n\}$. Similarly, if $\{x_n\}$ is decreasing and bounded from below, then $\{x_n\}$ converges and $\lim_{n\to\infty} x_n = \inf\{x_n\}$.

Example 6
Let $\{s_n\}$ be the sequence defined by $s_1 = 1$ and $s_{n+1} = \sqrt{1+s_n}$ for $n \geq 1$. Show s_n converges and find the limit.

By computing $s_2, s_3, s_4, \ldots\ldots$, we get $s_2 = \sqrt{2} = 1.414$, $s_3 = \sqrt{1+\sqrt{2}} = 1.554$ and $s_4 = 1.598\ldots$ Now, we see s_n is increasing and it should be bounded from above by 2.

Solution:

(Step 1: Show s_n is bounded by 2)

For $n = 1$, $s_1 = 1 < 2$, assume $s_k < 2$, then $s_{k+1} = \sqrt{1+s_k} < \sqrt{1+2} = \sqrt{3} < 2$

Hence by induction, $s_n < 2$ for $n = 1,2,3,4\ldots\ldots$

(Step 2: Show s_n is increasing, namely $s_{n+1} > s_n$ for $n = 1,2,3,\ldots\ldots$)

For $n = 1$, note $s_2 = \sqrt{2} > 1 = s_1$, suppose $s_{k+1} > s_k$, now consider $s_{k+2} = \sqrt{1+s_{k+1}} > \sqrt{1+s_k} = s_{k+1}$. Hence by induction, we completes the proof.

(Step 3)

By monotone sequence theorem, s_n converges. Let $\lim_{n\to\infty} s_n = s$. From the relation $s_{n+1} = \sqrt{1+s_n}$. Taking limit to both sides, we have $s = \sqrt{1+s} \to s^2 - s - 1 = 0 \to s = (1 \pm \sqrt{5})/2$, since $s > 1$, we have $s = (1+\sqrt{5})/2$

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☺Exercise 5

Consider a sequence $\{y_n\}$ defined by $y_1 = 1$ and $y_{n+1} = \frac{y_n + 1}{4}$. Show that y_n converges and find the limit

☺Exercise 6

Show the following sequence converges and find the limit

$$x_1 = 1 \text{ and } x_{n+1} = \frac{x_n}{2} + \sqrt{x_n}$$

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<u>Case ii) Back and Forth Sequence</u>

Theorem 2: (Nested Interval Theorem)
If $I_n = [a_n, b_n]$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$., then $\bigcap_{n=1}^{\infty} I_n = [a, b]$ where $a = \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n = b$,
Furthermore, if $\lim_{n\to\infty}(b_n - a_n) = 0$ (i.e. $a = b$), then $\bigcap_{n=1}^{\infty} I_n$ contains exactly one number

Theorem 3: (Intertwining Sequence Theorem)
If $\{x_{2m}\}$ and $\{x_{2m-1}\}$ converge to x, then $\{x_n\}$ converges to x.

Example 7 (Practice Exercise #96)
Show that the sequence $\{x_n\}$ given by

$$x_1 = 1, x_2 = 2 \text{ and } x_{n+1} = \frac{1}{3}x_n + \frac{2}{3}x_{n-1}$$

Converges and find the limit
(Note: $a \le ta + (1-t)b \le b$ for $a \le b$ and $0 \le t \le 1$….(*))

By computing $x_2, x_3, x_4, \ldots$., we get $x_3 = 1.333$, $x_4 = 1.778$, $x_5 = 1.481$, $x_6 = 1.679$. $x_7 = 1.547$. By plotting all this points on the real number line, we have

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---------------x--------------------------x------x-----x-----------------x----x----x-----------------
              x1                         x3     x5    x7   ……….       x6   x4  x2
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Solution:

(Step 1: Now we need to show both $\lim_{n\to\infty} x_{2n}$ and $\lim_{n\to\infty} x_{2n+1}$ exists, by making use of Nested Interval Theorem)

From the graph, we define $\mathbf{I_1 = [x_1, x_2]}$, $\mathbf{I_2 = [x_3, x_4]}$, $\mathbf{I_3 = [x_5, x_6]}$…..$\mathbf{I_k = [x_{2k-1}, x_{2k}]}$.

(We need to check $I_k \supseteq I_{k+1}$, it is same as checking $x_{2k-1} \le x_{2k+1} \le x_{2k+2} \le x_{2k}$)

First, by induction, we have $x_{2k-1} \le x_{2k}$ (Left as exercise)

By making use of (*), we have

$$x_{2k-1} \le \frac{2}{3}x_{2k-1} + \frac{1}{3}x_{2k} \le x_{2k} \to x_{2k-1} \le x_{2k+1} \le x_{2k}$$

Apply same trick on $x_{2k+1} \le x_{2k}$, we have

$$x_{2k+1} \le \frac{1}{3}x_{2k+1} + \frac{2}{3}x_{2k} \le x_{2k} \to x_{2k+1} \le x_{2k+2} \le x_{2k}$$

Therefore we have $x_{2k-1} \le x_{2k+1} \le x_{2k+2} \le x_{2k}$, so $I_k \supseteq I_{k+1}$.

Hence by Nested Interval Theorem, $\lim_{k\to\infty} x_{2k+1} = a$ and $\lim_{k\to\infty} x_{2k} = b$.

From the relation $x_{2k+1} = \frac{1}{3}x_{2k} + \frac{2}{3}x_{2k-1}$ (Put $n = 2k$), by taking limit, we have

$a = \frac{1}{3}b + \frac{2}{3}a \to a = b$. So by Intertwining Sequence Theorem, x_n converges.

To find the limit, we let $\lim_{n\to\infty} x_n = x$ and consider

$$x_3 = \frac{1}{3}x_2 + \frac{2}{3}x_1$$

$$x_4 = \frac{1}{3}x_3 + \frac{2}{3}x_2$$

$$x_5 = \frac{1}{3}x_4 + \frac{2}{3}x_3$$

$$x_6 = \frac{1}{3}x_5 + \frac{2}{3}x_6$$

$$x_7 = \frac{1}{3}x_6 + \frac{2}{3}x_5$$

.....

$$x_k = \frac{1}{3}x_{k-1} + \frac{2}{3}x_{k-2}$$

By summing and some cancellation, we get $x_k + x_{k-1} = x_2 + \frac{2}{3}x_1 + \frac{1}{3}x_{k-1}$, taking limit, we get $x + x = 2 + \frac{2}{3}(1) + \frac{1}{3}x \to x = \frac{8}{5} = 1.6$

☺Exercise 7 (Practice Exercise #45)

Let $x_1 = 1$, $x_2 = 2$ and $x_{n+1} = \frac{x_n + x_{n-1}}{2}$ for $n = 2,3,4,\ldots$. Prove the sequence $\{x_n\}$ converges and find the limit. (Hint: Plot a few point on the real line first and for the limit, use the similar method as in Example 7)

☺Exercise 8 (Practice Exercise #96)

Show that the sequence $\{x_n\}$ given by

$$x_0 = 0, x_1 = 1 \text{ and } x_{n+1} = \sqrt{\frac{1}{4}x_n^2 + \frac{3}{4}x_{n-1}^2}$$

Converges and find the limit.