MATH 2033 HW 5

1) We want to prove $\overline{\int_0^2} f(x)dx = \underline{\int_0^2} f(x)dx$.

a)

let partition $P = \{x_0, x_1, \cdots, x_{2n}\}$, where $x_j = \frac{r}{n}$, $j = \{1, \cdots, 2n\}$

$f(x) = |x-1|$, $U(f,P) = \sum_j M_j \Delta x_j = \sum_{j=1}^{2n} \sup(|x-1|) \cdot (x_j - x_{j-1})$, ...

$x \in [x_{r-1}, x_r]$

$$= \sum_{j=1}^{2n} |x_r - 1| \cdot (x_j - x_{r-1})$$

$$U(f,P) = |x_1 - 1| \cdot \left(\frac{1}{n} - 0\right) + |x_2 - 1| \cdot \left(\frac{2}{n} - \frac{1}{n}\right) + \cdots + |x_{2n} - 1| \cdot \left(\frac{2n}{n} - \frac{2n-1}{n}\right)$$

$$= \frac{1}{n}\left[|x_1 - 1| + |x_2 - 1| \cdot (2-1) + \cdots + |x_{2n} - 1| \cdot (2n - 2n + 1)\right]$$

$$= \frac{1}{n}\left(\left|\frac{1}{n} - 1\right| + \left|\frac{2}{n} - 1\right| + \cdots + \left|\frac{2n}{n} - 1\right|\right)$$

$$= \frac{1}{n}\left(\left(1 - \frac{1}{n}\right) + \left(1 - \frac{2}{n}\right) + \cdots + \left(1 - \frac{n-1}{n}\right) + 0 + \left(\frac{n+1}{n} - 1\right) + \cdots + \left(\frac{2n}{n} - 1\right)\right)$$

$$= \frac{1}{n}\left(\frac{n+1}{n} - \frac{1}{n} + \frac{n+2}{n} - \frac{2}{n} + \cdots + \frac{2n-1}{n} - \frac{n-1}{n} + 1\right)$$

$$= \frac{n}{n} = 1 \qquad \text{Inf}\{U(f,P)\} = 1 = \overline{\int_0^2} f(x)dx$$

$$L(f,P) = \sum_{j=1}^{2n} m_j \Delta x_j = \sum_{j=1}^{2n} \inf(|x-1|) \cdot (x_j - x_{j-1}),\ x \in [x_{j-1}, x_j]$$

$$= \sum_{j=1}^{2n} |x_{j-1} - 1| \cdot (x_j - x_{j-1}) = |0-1| \cdot \left(\frac{1}{n} - 0\right) + \left|\frac{1}{n} - 1\right| \cdot \left(\frac{2}{n} - \frac{1}{n}\right) + \cdots + \left|\frac{2n-1}{n} - 1\right| \cdot \left(\frac{2n}{n} - \frac{2n-1}{n}\right)$$

$$= \frac{1}{n}\left[|0-1| + \left|\frac{1}{n} - 1\right| + \left|\frac{2}{n} - 1\right| + \cdots + \left|\frac{2n-1}{n} - 1\right|\right]$$

$$= \frac{1}{n}\left(1 + \left(1 - \frac{1}{n}\right) + \cdots + \left(1 - \frac{n-1}{n}\right) + 0 + \left(\frac{n+1}{n} - 1\right) + \cdots + \left(\frac{2n-1}{n} - 1\right)\right)$$

$$= \frac{1}{n}\left(1 + \left(\frac{n+1}{n} - \frac{1}{n}\right) + \left(\frac{n+2}{n} - \frac{2}{n}\right) + \cdots + \left(\frac{2n-1}{n} - \frac{n-1}{n}\right)\right) = \frac{n}{n} = 1$$

$$\underline{\int_0^2} f(x)dx = \sup\{L(f,P)\} = 1 = \overline{\int_0^2} f(x)dx$$

$\therefore f(x) = |x-1|$ is integrable on $[0,2]$.

1b). let partition $P = \{0, \frac{1}{n}, \cdots, \frac{n}{n}\}$, $x_j = \frac{j}{n}$, $j = \{0, 1, \cdots, n\}$

$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ $M_j = x$ by density of ration numbers

$m_j = -x$ by density of irrational numbers.

$$U(f,P) = \sum_{j=1}^{n} M_j \Delta x_j = \sum_{j=1}^{n} x_j(x_j - x_{j-1}) = \frac{1}{n}\left(\frac{1}{n} - 0\right) + \frac{2}{n}\left(\frac{2}{n} - \frac{1}{n}\right) + \cdots + \frac{n}{n}\left(\frac{n}{n} - \frac{n-1}{n}\right)$$

$$= \frac{1}{n^2}(1 + 2 + \cdots + n) = \frac{(1+n)(n)}{2n^2} = \frac{1}{2n} + \frac{1}{2}$$

$$L(f,P) = \sum_{j=1}^{n} m_j \Delta x_j = \sum_{j=1}^{n} -x_j(x_j - x_{j-1}) = \frac{-1}{n}\left(\frac{1}{n} - 0\right) + \frac{-2}{n}\left(\frac{2}{n} - \frac{1}{n}\right) + \cdots + \frac{-n}{n}\left(\frac{n}{n} - \frac{n-1}{n}\right)$$

$$= \frac{-1}{n^2}(1 + \cdots + n) = \frac{-1}{2} + \frac{-1}{2n}$$

$\overline{\int_0^1} f(x)\,dx = \inf\{U(f,P)\} = \inf\{\frac{1}{2n} + \frac{1}{2}\} = \frac{1}{2}$, $\underline{\int_0^1} f(x)\,dx = \sup\{L(f,P)\} = \sup\{-\frac{1}{2} - \frac{1}{2n}\} = -\frac{1}{2}$

Hence $f: [0,1] \to \mathbb{R}$ is not integrable on $[0,1]$

2a) $f(x)$ and $h(x)$ are Integrable and with Partion P_1 and P_2 on $[a,b]$

Since $f(x) \leq g(x) \leq h(x)$, and take $P = P_1 \cup P_2$

$$M_{f_j} \leq M_{g_j} \leq M_{h_j} \text{ and } m_{f_j} \leq m_{g_j} \leq m_{h_j}$$

$$\Rightarrow M_{f_j} - m_{h_j} \leq M_{g_j} - m_{g_j} \leq M_{h_j} - m_{f_j}$$

So, $U(f, P_1) - L(h, P_2) \leq U(g, P) - L(g, P) \leq U(h, P_2) - L(f, P_1) < \varepsilon$

b) Since $f(x) \leq g(x) \leq h(x)$

By Infimum property, By Supremum property,

$\overline{\int} g(x)dx \leq U(g, P) \leq U(h, P) < \overline{\int} h(x)dx + \varepsilon$ $\quad \int g(x)dx \geq L(g, P) \geq L(f, P) > \underline{\int} f(x)dx - \varepsilon$

$\Rightarrow \underline{\int} f(x)dx - \varepsilon < \underline{\int} g(x)dx = \overline{\int} g(x)dx < \overline{\int} h(x)dx + \varepsilon \Rightarrow \left| \int g(x)dx - \int h(x)dx \right| < \varepsilon$

Hence, $\int g(x)dx = \int h(x)dx = \int f(x)dx$

3a) Using the identity of min, $\min\{f,g\} = \frac{1}{2}(f(x)+g(x) - |f(x)-g(x)|)$

By thm of computational rule of integral, $f(x)+g(x)$ and $f(x)-g(x)$ are integrable

By thm of integrability of $|f|$, $|f(x)-g(x)|$ is integrable

Hence, $\frac{1}{2}(f(x)+g(x) - |f(x)-g(x)|)$ is integrable.

b) i) No, it is false, let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$ such that $f(x)$ is bounded.

By density of rational number and density of irrational number

$U(f,P) = 1$, $L(f,P) = -1 \Rightarrow \overline{\int_a^b} f(x)\,dx \neq \underline{\int_a^b} f(x)\,dx$

$f(x)$ is not riemann integral, but $f^2(x) = 1$ $\forall x \in \mathbb{R}$

$f^2(x)$ is riemann integral.

ii) Using thm of integrability of composite function,

let $g(x) = x^{\frac{1}{3}}$, $g: \mathbb{R} \to \mathbb{R}$, s.t. $|g(x)-g(y)| \leq M|x-y|$ for some $M>0$ and $\forall x,y \in \mathbb{R}$

$g \circ f^3 = (f^3(x))^{\frac{1}{3}} = f(x)$, $f(x)$ is integrable on $[a,b]$