MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #21

Midterm Review

Example 1 (Definition of Limit of Function)

Show by the definition of limit of function,

$$\lim_{x\to 3}\sqrt[5]{x^3+5}=2$$

For any $\varepsilon>0$, we need to find $\delta>0$ such that $|x-3|<\delta \to \left|\sqrt[5]{x^3+5}-2\right|<\varepsilon$

$$\left|\sqrt[5]{x^3+5}-2\right|=\left|\sqrt[5]{x^3+5}-\sqrt[5]{32}\right|$$

$$\le\left|\sqrt[5]{x^3+5-32}\right| \qquad (\text{Note } \left|\sqrt[5]{a}-\sqrt[5]{b}\right|\le\left|\sqrt[5]{|a-b|}\right|)$$

$$=\left|\sqrt[5]{|x^3-27|}\right|=\left|\sqrt[5]{|(x-3)(x^2+3x+9)|}\right|$$

$$<\left|\sqrt[5]{|x-3|(4^2+3(4)+9)}\right|=\left|\sqrt[5]{37|x-3|}\right| \qquad (\text{Require } |x-3|<1)$$

$$<\varepsilon \qquad (Require\ \sqrt[5]{37|x-3|}<\varepsilon\to|x-3|<\frac{\varepsilon^5}{37})$$

Solution:

For any $\varepsilon>0$, pick $\delta=\min\left\{1,\frac{\varepsilon^5}{37}\right\}$, then for $|x-3|<\delta$, from the previous argument, $\left|\sqrt[5]{x^3+5}-2\right|<\varepsilon$

Example 2

For $a>0$, show by definition of limit of function that

$$\lim_{x\to 0}\frac{a(x+a)}{x-a}=-a$$

IDEA: For any $\varepsilon>0$

$$\left|\frac{a(x+a)}{x-a}-(-a)\right|=\left|\frac{a(x+a)+a(x-a)}{x-a}\right|=\left|\frac{2ax}{x-a}\right|$$

$$<\frac{|2ax|}{\frac{a}{2}} \qquad (\text{Require } |x-0|<\frac{a}{2})$$

$$=4|x|<\varepsilon \qquad (Require\ |x-0|<\frac{\varepsilon}{4})$$

Solution:

For any $\varepsilon>0$, pick $\delta=\left\{\frac{a}{2},\frac{\varepsilon}{4}\right\}$, then for $|x-0|<\delta$, we have $\left|\frac{a(x+a)}{x-a}-(-a)\right|<\varepsilon$

Example 3 (2007 Final)

Given a Cauchy sequence $\{x_n\}$, $\{y_n\} \in (0, \infty)$, show that $\left\{\frac{x_n}{1+y_n}\right\}$ is Cauchy using the definition of Cauchy (The use of Cauchy Theorem is not allowed)

IDEA:

$$\left|\frac{x_n}{1+y_n} - \frac{x_m}{1+y_m}\right| = \left|\frac{x_n(1+y_m) - x_m(1+y_n)}{(1+y_n)(1+y_m)}\right|$$

$< |x_n(1+y_m) - x_m(1+y_n)| \qquad$ since $y_n > 0$

$= |x_n - x_m + x_n y_m - x_m y_n|$

$\leq |x_n - x_m| + |x_n y_m - x_m y_n|$

$\leq |x_n - x_m| + |x_n y_m - x_m y_m + x_m y_m - x_m y_n|$

$\leq |x_n - x_m| + |y_m||x_n - x_m| + |x_m||y_m - y_n|$

$< |x_n - x_m| + N|x_n - x_m| + M|y_m - y_n|$

$= (1+N)|x_n - x_m| + M|y_m - y_n| \quad (|x_n| < M \ and \ |y_n| < N)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad$ (Require $|x_n - x_m| < \frac{\varepsilon}{2(1+N)}, |y_m - y_n| < \frac{\varepsilon}{2M}$)

Solution:

For any $\varepsilon > 0$, since x_n, y_n are Cauchy and therefore bounded, so $|x_n| < M \ and \ |y_n| < N$. There exists K_1 and K_2 such that

For $m, n > K_1$, $|x_n - x_m| < \frac{\varepsilon}{2(1+N)}$

For $m, n > K_2$, $|y_m - y_n| < \frac{\varepsilon}{2M}$

Pick $K = \max\{K_1, K_2\}$, then for $m, n > K$, we get

$$\left|\frac{x_n}{1+y_n} - \frac{x_m}{1+y_m}\right| < \varepsilon$$

Example 4

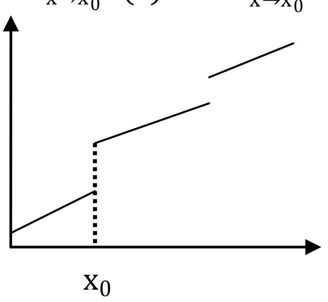
Let $f(x)$ be a montone function on $[a,b]$ and suppose $f[a,b]$ is an interval. Show that $f(x)$ is continuous on $[a,b]$

Solution:

We prove by contradiction.

We first assume $f(x)$ is increasing one (The decreasing case is similar)

Suppose there is x_0 such that $f(x)$ is not continuous at $x = x_0$, there is a “jump” at $x = x_0$ which $\lim_{x \to x_0^-} f(x) < \lim_{x \to x_0^+} f(x)$. (As Shown)



Then the resulting image is $\left[f(a), \lim_{x\to x_0^-} f(x)\right) \cup \left(\lim_{x\to x_0^+} f(x), ?\right) \cup \ldots$ and is no longer to be interval, contradicted to the image is interval.

Example 5
Let $f, g: [0,1] \to [0,1]$ be continuous and $f(g(x)) = g(f(x))$ for $x \in [0,1]$. Prove that there exists $x_0 \in [0,1]$ such that $f(x_0) = g(x_0)$.

Solution:
Suppose $f(x) \neq g(x) \to f(x) - g(x) \neq 0$, since $f - g$ is continuous, then either $f(x) - g(x) > 0$ or $f(x) - g(x) < 0$

Consider the case $f(x) - g(x) > 0$, since $f(x) - g(x)$ is continuous on $[0,1]$, by extreme value theorem, there is a minimum x_0 such that

$$f(x) - g(x) \geq f(x_0) - g(x_0) = d > 0$$

Then consider another given condition

$$f(g(x)) = g(f(x))$$

Using $f(x) - g(x) \geq d$, we have

$$g(g(x)) + d < f(g(x)) = g(f(x)) < f(f(x)) - d$$
$$\to f(f(x)) > g(g(x)) + 2d$$

Next, $f\left(f(f(x))\right) > g\left(g(f(x))\right) + 2d = f\left(g(g(x)) + 2d > g\left(g(g(x))\right)\right) + 3d.$

Then by induction, we obtain $f^{(n)}(x) > g^{(n)}(x) + nd \to f^{(n)}(x) > nd$

Since n can be arbitrarily chosen, pick $n > \frac{1}{d}$, we get $f^{(n)}(x) > 1$, contradicted to $f: [0,1] \to [0,1]$.

Example 6
Let $f: [0,1] \to R$ be continuous. If $f(x)$ is differentiable on $(0,1)$ and $f(0) = 0$, but $f(x) \neq 0$ for all x, then prove that there exists $x_0 \in (0,1)$ such that

$$\frac{f'(1-x_0)}{f(1-x_0)} = \frac{2f'(x_0)}{f(x_0)}$$

IDEA: We first rearrange the term

$$\frac{f'(1-x_0)}{f(1-x_0)} = \frac{2f'(x_0)}{f(x_0)} \to f(x_0)f'(1-x_0) = 2f'(x_0)f(1-x_0)$$
$$\to 2f'(x_0)f(1-x_0) - f(x_0)f'(1-x_0) = 0$$
$$\to 2f(x_0)f'(x_0)f(1-x_0) - f(x_0)^2 f'(1-x_0) = 0$$
$$\to \frac{d}{dx} f(x)^2 f(1-x)\Big|_{x=x_0} = 0$$

Solution:
Consider $g(x) = f(x)^2 f(1-x)$, apply mean value theorem on $[0,1]$, we have

$$\frac{g(1)-g(0)}{1-0} = g'(x_0) \qquad \text{for some } x_0 \in (0,1)$$

$\rightarrow g'(x_0) = 0$

Expand $g'(x_0)$ and rearrange the terms, we get

$$\rightarrow \frac{f'(1-x_0)}{f(1-x_0)} = \frac{2f'(x_0)}{f(x_0)}$$

Example 7 (Spring 2007 Exam 2)

Let $f: [0,1] \rightarrow R$ be continuous and let it differentiable on $(0,1)$. Also $f(0) = 0$ and $f(1) = 1$. Let a, b are positive real numbers.

a) Prove that there is x_0 such that $f(x_0) = \frac{a}{a+b}$

b) Prove that there exist distinct $x_1, x_2 \in (0,1)$ such that

$$\frac{a}{f'(x_1)} + \frac{b}{f'(x_2)} = a + b$$

c) Prove that if $c_1, c_2, \dots c_n > 0$ and $c_1 + c_2 + \cdots + c_n = 1$. Then there exists distinct $t_1, t_2, \dots, t_n \in (0,1)$ such that

$$\frac{c_1}{f'(t_1)} + \frac{c_2}{f'(t_2)} + \cdots + \frac{c_n}{f'(t_n)} = 1$$

Solution:

a) Note that $0 < \frac{a}{a+b} < 1$, $f(0) = 0$ and $f(1) = 1$, then by intermediate value theorem, there is x_0 such that $f(x_0) = \frac{a}{a+b}$

b) (IDEA: Even though, the conclusion suggests us to use mean value theorem, but since the x_1, x_2 are different, hence we need to apply mean value theorem for several times.

Rearrange the conclusion first,

$$\frac{a}{f'(x_1)} + \frac{b}{f'(x_2)} = a + b \rightarrow \frac{\frac{a}{a+b}}{f'(x_1)} + \frac{\frac{b}{a+b}}{f'(x_2)} = 1$$

$$\rightarrow \frac{\frac{a}{a+b}}{f'(x_1)} + \frac{1 - \frac{a}{a+b}}{f'(x_2)} = 1 \rightarrow \frac{f(x_0) - 0}{f'(x_1)} + \frac{f(1) - f(x_0)}{f'(x_2)} = 1 \text{)}$$

<u>Solution of b)</u>

Apply Mean Value Theorem on $f(x)$ on interval $[0, x_0]$ and $[x_0, 1]$ respectively

$$\frac{f(x_0) - f(0)}{x_0 - 0} = f'(x_1) \rightarrow \frac{\frac{a}{a+b}}{f'(x_1)} = x_0$$

$$\frac{f(1) - f(x_0)}{1 - x_0} = f'(x_2) \rightarrow \frac{1 - \frac{a}{a+b}}{f'(x_2)} = 1 - x_0 \rightarrow \frac{\frac{b}{a+b}}{f'(x_2)} = 1 - x_0$$

Add these 2 equations up and rearrange the terms, we get

$$\frac{\frac{a}{a+b}}{f'(x_1)}+\frac{\frac{b}{a+b}}{f'(x_2)}=1 \rightarrow \frac{\mathrm{a}}{\mathrm{f}'(\mathrm{x}_1)}+\frac{b}{f'(x_2)}=a+b$$

c) Note that $\frac{\mathrm{c}_1}{\mathrm{f}'(\mathrm{t}_1)}+\frac{\mathrm{c}_2}{\mathrm{f}'(\mathrm{t}_2)}+\cdots+\frac{\mathrm{c}_\mathrm{n}}{\mathrm{f}'(\mathrm{t}_\mathrm{n})}=1$

$$\rightarrow \frac{\mathrm{c}_1}{\mathrm{f}'(\mathrm{t}_1)}+\frac{\mathrm{c}_2}{\mathrm{f}'(\mathrm{t}_2)}+\cdots+\frac{\mathrm{c}_\mathrm{n}}{\mathrm{f}'(\mathrm{t}_\mathrm{n})}=c_1+c_2+\cdots+c_n$$

The method is same as b) and note that
$\mathrm{c}_\mathrm{k}=(\mathrm{c}_1+\mathrm{c}_2+\cdots+\mathrm{c}_\mathrm{k})-(\mathrm{c}_1+\mathrm{c}_2+\cdots+\mathrm{c}_{\mathrm{k}-1})$

Solution of c)
First, by intermediate value theorem, there is $\mathrm{x}_1,\mathrm{x}_2,\ldots\mathrm{x}_{\mathrm{n}-1}$ such that

$$\mathrm{f}(\mathrm{x}_\mathrm{k})=\frac{\mathrm{c}_1+\mathrm{c}_2+\cdots+\mathrm{c}_\mathrm{k}}{\mathrm{c}_1+\mathrm{c}_2+\cdots+\mathrm{c}_\mathrm{n}} \qquad \text{for } \mathrm{k}=1,2,3,\ldots,\mathrm{n}-1$$

Then apply mean value theorem, on $[0,\mathrm{x}_1],[\mathrm{x}_1,\mathrm{x}_2],\ldots.,[\mathrm{x}_{\mathrm{n}-1},1]$,we get

$$\frac{\mathrm{f}(\mathrm{x}_\mathrm{k})-\mathrm{f}(\mathrm{x}_{\mathrm{k}-1})}{\mathrm{x}_\mathrm{k}-\mathrm{x}_{\mathrm{k}-1}}=f'(t_k)\rightarrow \frac{\left(\frac{\mathrm{c}_\mathrm{k}}{\mathrm{c}_1+\mathrm{c}_2+\cdots+\mathrm{c}_\mathrm{n}}\right)}{f'(t_k)}=x_k-x_{k-1}\rightarrow \frac{c_k}{f'(t_k)}=x_k-x_{k-1}$$

For $\mathrm{k}=1,2,3,\ldots,\mathrm{n}$ and $\mathrm{x}_0=0,\mathrm{x}_\mathrm{n}=1$

Add these n equations, we get

$$\sum_{k=1}^{n}\frac{c_k}{f'(t_k)}=\sum_{k=1}^{n}x_k-x_{k-1}=x_n-x_0=1$$

$$\rightarrow \frac{\mathrm{c}_1}{\mathrm{f}'(\mathrm{t}_1)}+\frac{\mathrm{c}_2}{\mathrm{f}'(\mathrm{t}_2)}+\cdots+\frac{\mathrm{c}_\mathrm{n}}{\mathrm{f}'(\mathrm{t}_\mathrm{n})}=1$$

Try to do the following exercise if you have time

☺Exercise 1
Prove by definition of limit that

a) $\lim_{\mathrm{x}\to 2}|3x^2-2|=10$
b) $\lim_{\mathrm{x}\to 3}\sqrt[4]{7+x^2}=2$
c) $\lim_{\mathrm{x}\to \mathrm{c}}\frac{c^2}{x^2-c}=\frac{c}{c-1}$ (where c is constant and $\mathrm{c}>1$)

☺Exercise 2

Define $a_1 = 1$ and $a_{n+1} = 2a_n + \cos a_n$,

Prove that the sequence $\frac{a_1}{2}, \frac{a_2}{2^2} \ldots \ldots \frac{a_n}{2^n} \ldots$ is Cauchy Sequence .

(Hint: Let $b_n = \frac{a_n}{2^n}$ and rewrite the equation above)

☺Exercise 3

Suppose $\{x_n\}$ and $\{y_n\}$ is Cauchy and $\lim_{n\to\infty}(x_n - y_n) = 0$. Show that

$z_n = \min\{x_n, y_n\}$ is also Cauchy. (Hint: $\min\{a,b\} = \frac{1}{2}(a+b) - |a-b|$)

☺Exercise 4

Show that if $\{x_n\}$ is Cauchy, then $\{\tan^{-1} x_n\}$ is also Cauchy by checking the definition of Cauchy (The use of Cauchy Theorem is not allowed)

☺Exercise 5

Let $f: [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and three-times differentiable on (a,b), suppose $f(a) = f(b) = f'(a) = f'(b) = 0$, prove that there exists $c \in (a,b)$ such that $f'''(c) = 0$.

(Hint: There is enough information for you to apply mean value theorem to get the conclusion. Of course, if you wish, you may also use Taylor Theorem.)

☺Exercise 6

Let $f: [a,b] \to \mathbf{R}$ be n-times differentable with $\left|f^{(k)}(c)\right| \le M$ for some $c \in (a,b)$, for $k = 1,2,3,\ldots,n-1$ and $\left|f^{(n)}(x)\right| \le M$ for all x. Show that $|f(x)| \le Me^{b-a}$

(Hint: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$)

☺Exercise 7

Let $f: [0,b] \to \mathbf{R}$ be continuous and let it be differentiable on (a,b), suppose $f(x) \neq 0$ and $f(0) = f(b)$. Show that there exists $x_0 \in (a,b)$ such that

$$f'(x_0)f(b-x_0) + 2f'(x_0)f'(b-x_0) = 0$$

(Hint: Divide both side by $\left(f(b-x_0)\right)^3$)