MATH2033 Mathematical Analysis (2021 Spring)
Midterm Examination

Time allowed: 90 minutes (7:30p.m.-9:00p.m.)

Instructions: Answer ALL problems. Full details must be clearly shown to receive full credits. Please submit your work via the submission system in canvas before 9:15p.m.. Late submission will not be accepted. Your submission must be

- **100% handwritten (typed solution will not be accepted)**
- **In a single pdf. files (other file format will not be accepted)**
- **With your full name (as shown in your student ID card) and student ID number on the first page of your submission.**

Problem 1 (30 marks)

10' **(a)** **(i)** State the definition of limits of sequence (i.e. $\lim_{n\to\infty} x_n = L \in \mathbb{R}$)

(ii) State the definition of Cauchy sequence.

10' **(b)** Using the definition of limits, prove that $\lim_{n\to\infty} \frac{1}{n^4-4n+10} = 0$.

10' **(c)** We let $\{x_n\}$ be a sequence of real number such that $\lim_{n\to\infty} x_n = a$, where a is a positive real number. Using the definition of limits, show that $\lim_{n\to\infty} \sqrt[3]{x_n + a} = \sqrt[3]{2a}$.

Problem 2 (20 marks)

We consider a sequence of real number $\{x_n\}$ defined by

$$x_1 = 2 \;\; and \;\; x_{n+1} = 1 + \frac{x_n^2}{1 + x_n^2} \;\; for\, n \in \mathbb{N}.$$

12' **(a)** Show that the sequence $\{x_n\}$ is monotone.

8' **(b)** Hence, show that the sequence $\{x_n\}$ converges and find its limits.

Problem 3 (22 marks)

Recall that the cubic root of 2 (denoted by $\sqrt[3]{2}$) is defined as a real number x satisfying

$$x^3 = 2.$$

In this problem, you are asked to prove the existence of the cubic root $\sqrt[3]{2}$. To do so, we consider the set $S = \{r \in \mathbb{Q} | \, r > 0 \;\; and \;\; r^3 < 2\}$.

6' **(a)** Prove that $x = \sup S$ exists.

16' **(b)** Show that the supremum x satisfy $x^3 = 2$.

Problem 4 (28 marks)

(a) **(6 marks)** We let $[x]$ denotes the greatest integer less than or equal to x. For example, $[7.2] = 7$, $[7.9] = 7$, $[7] = 7$. We consider the set

$$T = \left\{ \frac{[x]^2}{y} \,|\, x \in \mathbb{R}\backslash\mathbb{Q} \;\; and \;\; y \in \mathbb{Z}\backslash\{0\} \right\}.$$

Determine if the set T is countable.

(b) We let m be a real number. We consider a set S which is the collection of all sequences of integers $\{x_n\}$ that converges to m. That is,

$$S = \left\{ \{x_n\} | x_n \in \mathbb{Z} \;\; for\, all\, n \in \mathbb{N} \;\; and \;\; \lim_{n\to\infty} x_n = m \right\}.$$

(i) **(10 marks)** If m is *not* integer, show that S must be an empty set.

(ii) **(12 marks)** If m is an integer, show that S is countable.

(😊Hint: If the sequence $\{x_n\}$ converges, what will happen to x_n when n is large?)

End of paper

Problem 1 (30 marks)

(a) **(i)** State the definition of limits of sequence (i.e. $\lim_{n\to\infty} x_n = L \in \mathbb{R}$)

(ii) State the definition of Cauchy sequence.

(b) Using the definition of limits, prove that $\lim_{n\to\infty} \frac{1}{n^4-4n+10} = 0$.

(c) We let $\{x_n\}$ be a sequence of real number such that $\lim_{n\to\infty} x_n = a$, where a is a positive real number. Using the definition of limits, show that $\lim_{n\to\infty} \sqrt[3]{x_n + a} = \sqrt[3]{2a}$.

Proof:

(a) (i) Say $\lim_{n\to\infty} x_n = L$ iff

5'

"For any given $\varepsilon>0$. $\exists N>0$ s.t. when $n>N$. $|x_n - L| < \varepsilon$".

(ii) Say $\{x_n\} \subset \mathbb{R}$ is Cauchy iff

5'

"For any given $\varepsilon>0$. $\exists N>0$. s.t. when $n, m > N$. $|x_n - x_m| < \varepsilon$".

(b) First. $\exists N_1 > 0$ s.t. $n > N_1$, we have $n^4 - 4n + 10 > \frac{1}{2} n^4 > 0$ (e.g. $N_1 = 10$). Moreover. for $\varepsilon > 0$. let $N_2 = [(\frac{2}{\varepsilon})^{\frac{1}{4}}] + 1$. 5'

and $N = \max\{N_1, N_2\}$ ~~then.~~ when $n > N$. we have

5'

$|n^4 - 4n + 10| > \frac{1}{2} n^4 > \frac{1}{2} N_2^4 > \frac{1}{2} \cdot \frac{2}{\varepsilon} = \frac{1}{\varepsilon}$

$\Rightarrow \left| \frac{1}{n^4 - 4n + 10} - 0 \right| < \varepsilon$

So we draw the conclusion.

(c) $\left| \sqrt[2]{x_n + a} - \sqrt[3]{2a} \right| = \left| \frac{(x_n + a) - 2a}{(x_n+a)^{3/2} + \sqrt[3]{2a(x_n+a)} + (2a)^{3/2}} \right|$ 2'

Since $x_n \to a > 0$. we know $\begin{cases} \text{for any } \varepsilon > 0. \exists N_1 \text{ s.t. } |x_n - a| < (2a)^{\frac{3}{2}} \cdot \varepsilon \text{ when } n > N_1 \\ \exists N_2 \text{ s.t. } x_n > 0 \text{ when } n > N_2 \end{cases}$

5'

let $N = \max\{N_1, N_2\}$. then if $n > N$. we have

$\left| \sqrt[3]{x_n + a} - \sqrt[3]{2a} \right| \le \frac{1}{(2a)^{3/2}} |x_n - a| < \varepsilon$ 3'

So we draw the conclusion. $\square$

Problem 2 (20 marks)

We consider a sequence of real number $\{x_n\}$ defined by

$$x_1 = 2 \quad and \quad x_{n+1} = 1 + \frac{x_n^2}{1+x_n^2} \quad for\, n \in \mathbb{N}.$$

(a) Show that the sequence $\{x_n\}$ is monotone.

(b) Hence, show that the sequence $\{x_n\}$ converges and find its limits.

Proof:

3'

By a simple induction. we know $1 < x_n \le 2$ for all n. And again. by induction:

① $X_2 = 1 + \frac{4}{5} < 2 = X_1$ 2'

② Assume $X_k < X_{k-1}$. then

$$X_{k+1} - X_k = \frac{x_k^2}{1+x_k^2} - \frac{x_{k-1}^2}{1+x_{k-1}^2}$$

7'

since $f(x) = \frac{x^2}{1+x^2}$ is increasing. so $f(x_k) < f(x_{k-1})$ due to $x_k < x_{k-1}$. hence $x_{k+1} < x_k$.

Thus. $\{x_n\}$ is decreasing and bounded below. which implies $\{x_n\}$ converges. 2'

Let $x_0 = \lim x_n$. then

$$x_0 = \lim x_{n+1} = \lim \frac{x_n^2}{1+x_n^2} + 1 = \frac{x_0^2}{x_0^2+1} + 1$$

$\Rightarrow x_0$ is the root of $x^3 - 2x^2 + x - 1 = 0$. (which greater than 1) 6'

□

Problem 3 (22 marks)

Recall that the cubic root of 2 (denoted by $\sqrt[3]{2}$) is defined as a real number x satisfying

$$x^3 = 2.$$

In this problem, you are asked to prove the existence of the cubic root $\sqrt[3]{2}$. To do so, we consider the set $S = \{r \in \mathbb{Q} | r > 0 \text{ and } r^3 < 2\}$.

(a) Prove that $x = \sup S$ exists.

(b) Show that the supremum x satisfy $x^3 = 2$.

Proof:

(a) It's obvious since S is bounded above by 2 (if $r<1$ obvious; if $r \geq 1$, $r \leq r^3 < 2$) and nonempty ($1 \in S$). Hence $x = \sup S$ exist.

6'

(b) ① $x^3 \leq 2$.

Otherwise $x^3 > 2$. we claim that if positive ε small enough, we have $(x-\varepsilon)^3 > 2$. ($(x-\varepsilon)^3 = x^3 - 3x^2\varepsilon + 3x\varepsilon^2 - \varepsilon^3 \geq x^3 - (3x^2+\varepsilon^2)\varepsilon > x^3 - 4x^2\varepsilon > 2$.

8'

where we set $\varepsilon = \min\{\frac{1}{2}x, \frac{x^3-2}{8x^2}\}$). Hence, from $x = \sup S$, we know $\exists y \in S$ s.t. $y > x - \varepsilon \Rightarrow y^3 > (x-\varepsilon)^3 > 2$. a contradiction.

② $x^3 \geq 2$.

Otherwise, $x^3 < 2$. Let's try constructing some $r \in \mathbb{Q}$ s.t. $r > x$ but $r^3 < 2$. Similar to ①, we claim that if $\varepsilon > 0$ small enough,

8'

we have $(x+\varepsilon)^3 < 2$. ($(x+\varepsilon)^3 = x^3 + 3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3 \leq x^3 + 5x^2\varepsilon < 2$.

where we set $\varepsilon = \min\{\frac{x}{6}, \frac{2-x^3}{10x^2}\}$). Hence, from $\mathbb{Q} \overset{dense}{\hookrightarrow} \mathbb{R}$,

we know $\exists r \in \mathbb{Q}$ s.t. $x < r < x+\varepsilon$, then

$r^3 < (x+\varepsilon)^3 < 2$ but $r > x$.

which contradicts to $x = \sup S$.

Thus, we conclude that $x^3 = 2$. $\square$

Problem 4 (28 pts)

(a) We let $[x]$ denotes the greatest integer less than or equal to x. For example, $[7.2] = 7$, $[7.9] = 7$, $[7] = 7$. We consider the set

$$T = \left\{ \frac{[x]^2}{y} \mid x \in \mathbb{R} \backslash \mathbb{Q} \;\; and \;\; y \in \mathbb{Z} \backslash \{0\} \right\}.$$

Determine if the set T is countable.

(b) We let m be a real number. We consider a set S which is the collection of all sequences of integers $\{x_n\}$ that converges to m. That is,

$$S = \left\{ \{x_n\} | x_n \in \mathbb{Z} \;\; for \; all \; n \in \mathbb{N} \;\; and \;\; \lim_{n\to\infty} x_n = m \right\}.$$

(i) If m is *not* integer, show that S must be an empty set.

(ii) If m is an integer, show that S is countable.

End of paper

Proof:

(a) Since $[x]^2 \in \mathbb{Z}$ and $y \in \mathbb{Z}$, $\frac{[x]^2}{y} \in \mathbb{Q}$. So $T \subseteq \mathbb{Q}$. thus countable.

6'

(b)

(i) Since m is not integer, we can select $[m] < m < [m]+1$. define $\varepsilon_0 = \min\{m-[m], [m]+1-m\}$. then for every integer sequence $\{x_n\}$, we have

$|x_n - m| \geq \alpha$

So $x_n \to m$ is not hold $\Rightarrow S = \phi$.

10'

Otherwise, $|x_n - m| \geq 1 > \varepsilon$, for any $\varepsilon < 1$

(ii) Since $x_n \to m$, here $\exists$ a N s.t. $x_n = m$ when $n \geq N$ (prove by contradiction). So each $\{x_n\} \in S$ has the form $\{x_1, x_2, \cdots, x_{N-1}, m, m, \cdots\}$.

6'

which means we can identify $\{x_n\}$ with a set consisting of finitely many integers. Since the class of such sets can be

6'

identified with the countable union of countable sets (in fact, $\bigcup_{n=1}^{\infty} S_n$, where $S_n = \mathbb{Z}^n$. countable). So S is countable.

$\square$