(705) Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and decreasing. Prove that there exists a unique element $(a,b,c)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ such that $a=f(b)$, $b=f(c)$ and $c=f(a)$.

Solution We need to prove existence and uniqueness.

(Existence) If $\exists x_0\in\mathbb{R}$ such that $f(x_0)=x_0$, then we can set $a=b=c=x_0$, then $f(a)=f(b)=f(c)=f(x_0)=x_0$

$\therefore a=f(b)$, $b=f(c)$ and $c=f(a)$.

To show such x_0 exists, consider $h(x)=\underset{\text{continuous}}{f(x)-x}$.

$\boxed{\text{Assume}}$ $h(x)\neq 0$ for all $x\in\mathbb{R}$. Then ($h(x)>0$ for all $x\in\mathbb{R}$) or ($h(x)<0$ for all $x\in\mathbb{R}$).

Case 1 ($h(x)>0$ for all $x\in\mathbb{R}$). Then $f(x)>x$ for all $x\in\mathbb{R}$. In particular, $f(0)>0$. Since f is decreasing $f(f(0))\le f(0)$. Let $x=f(0)$, then $f(x)\le x$, contradict.

Case 2 ($h(x)<0$ for all $x\in\mathbb{R}$). Then $f(x)<x$ for all $x\in\mathbb{R}$. In particular, $f(0)<0$. Since f is decreasing $f(f(0))\ge f(0)$. Let $x=f(0)$, then $f(x)\ge x$, contradiction.

$\therefore$ the assumption is false. $\therefore \exists x_0\in\mathbb{R}$, $h(x_0)=0$, $f(x_0)=x_0$.

(Uniqueness) Suppose $a=f(b)$, $b=f(c)$, $c=f(a)$ and $a'=f(b')$, $b'=f(c')$, $c'=f(a')$. Then a,b,c,a',b',c' satisfy $f(f(f(x)))=x$. Let $r=\max\{a,b,c,a',b',c'\}$ and $s=\min\{a,b,c,a',b',c'\}$. We have $r\ge s$. Since f is decreasing, $f(r)\le f(s)$, $f(f(r))\ge f(f(s))$ and $r=f(f(f(r)))\le f(f(f(s)))=s$. $\therefore r=s$. $\therefore a=b=c=a'=b'=c'$.

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(807) Let $f:\mathbb{R}\to\mathbb{R}$ be twice differentiable. If $f'(0)=2=f'(1)$ and for all $x\in[0,1]$, $|f''(x)|\le 4$, then prove that $|f(1)-f(0)|\le 3$.

Thoughts From $f'(0)=2=f'(1)$, this suggests the center may be 0 or 1.

Solution By Taylor's theorem, $\forall x\in[0,1]$,

$$f(x)=f(1)+f'(1)(x-1)+\frac{f''(\theta_1)}{2}(x-1)^2 \text{ and}$$

$$f(x)=f(0)+f'(0)(x-0)+\frac{f''(\theta_0)}{2}(x-0)^2$$

for some θ_1 between x and 1, θ_0 between x and 0.

Subtracting these and solving for $f(1)-f(0)$,

$$0=f(1)-f(0)+2(-1)+\frac{f''(\theta_1)}{2}(x-1)^2-\frac{f''(\theta_0)}{2}x^2$$

$$f(1)-f(0)=2-\frac{f''(\theta_1)}{2}(x-1)^2+\frac{f''(\theta_0)}{2}x^2$$

$$|f(1)-f(0)|\le 2+2(x-1)^2+2x^2$$ use $|f''(x)|\le 4$

Need $2(x-1)^2+2x^2=1$. By quadratic formula,

$\Leftrightarrow 4x^2-4x+1=0$ $\quad x=\frac{1}{2}\in[0,1]$.

Let $x=\frac{1}{2}$, then $|f(1)-f(0)|\le 3$.

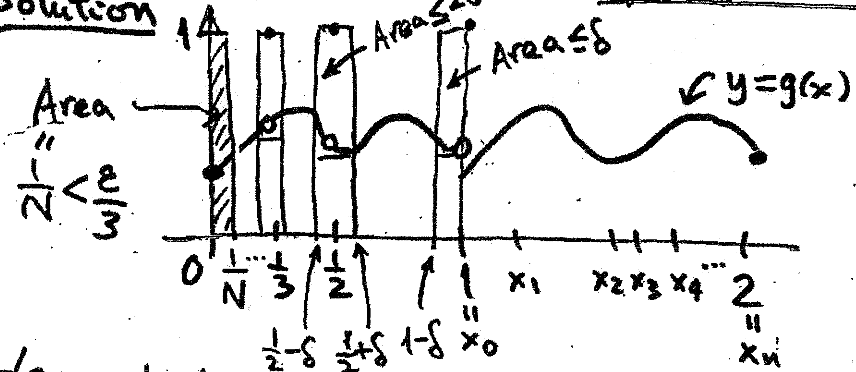
Remark Minimum of $2(x-1)^2+2x^2=4(x-\frac{1}{2})^2+1$ is 1 at $x=\frac{1}{2}$.

304 Let $g:[1,2]\to[0,1]$ be Riemann integrable. Prove that $G:[0,1]\to[0,1]$ defined by

$$G(x)=\begin{cases} g(x+1) & \text{if } x\in[0,1]\setminus\{1,\frac12,\frac13,\dots\} \\ 1 & \text{if } x\in\{1,\frac12,\frac13,\dots\}\end{cases}$$

is Riemann integrable on $[0,1]$ by integral criterion.

Solution



$\forall\varepsilon>0$, by Archimedian principle, $\exists N\in\mathbb{N}$ such that $N>\frac{3}{\varepsilon}$.

Partition $[\frac1N,1]$ by $P_1=\{\frac1N<\frac1N+\delta<\frac{1}{N-1}-\delta<\frac{1}{N-1}+\delta<\dots<\frac12-\delta<\frac12+\delta<1-\delta<1\}$ with $0<\delta<\min\{\frac12(\frac{1}{N-1}-\frac1N),\frac{\varepsilon}{6(N-1)}\}$

Since g is integrable on $[1,2]$, $\exists\ P_2=\{1=x_0<x_1<\dots<x_n=2\}$ such that $U(g,P_2)-L(g,P_2)<\frac{\varepsilon}{3}$. on $[1,2]$

Let $P_3=\{0<x_1-1<\dots<x_n-1=1\}=\{x_i-1:x_i\in P_2\}$.

Then $\underbrace{U(g(x+1),P_3)-L(g(x+1),P_3)}_{\text{on }[0,1]}=\underbrace{U(g(x),P_2)-L(g(x),P_2)}_{\text{on }[1,2]}<\frac{\varepsilon}{3}$

Let $P_4=P_1\cup P_3$. Then P_4 is a refinement of P_3. So

$$\underbrace{U(G,P_4)-L(G,P_4)}_{\text{on }[0,1]}\le\frac{\varepsilon}{3}+\left(U(g(x+1),P_3)-L(g(x+1),P_3)\right)+2\delta(N-1)<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

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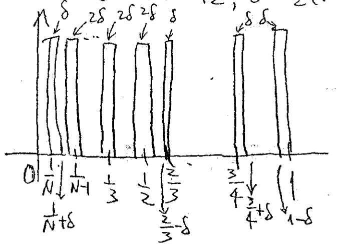
(a) h is given to be bounded. $\forall x\in[0,\frac23)\cup(\frac34,1]$, if $x\notin S_f\cup\{1,\frac12,\frac13,\dots\}\cup\{0\}$, then $h(x)=f(x)$ is continuous at x. $\forall x\in[\frac23,\frac34]$, if $x\notin S_g\cup\{\frac23,\frac34\}$, then $h(x)=g(x)$ is continuous at x. So $S_h\subseteq\underbrace{S_f}_{\text{measure }0}\cup S_g\cup\underbrace{\{\frac23,\frac34\}\cup\{1,\frac12,\frac13,\dots\}\cup\{0\}}_{\text{countable}\Rightarrow\text{measure }0}$.
This implies S_h is of measure 0.
Therefore h is Riemann integrable on $[0,1]$ by Lebesgue's theorem.

(b) Note $0\le h(x)\le1$ for all $x\in[0,1]$. $\forall\varepsilon>0$, choose $N\in\mathbb{N}$ such that $\frac1N<\frac{\varepsilon}{4}$ ($\Leftrightarrow N>\frac4\varepsilon$). Next choose $\delta>0$ such that $2N\delta<\frac{\varepsilon}{4}$ and $\delta<\frac12(\frac{1}{N-1}-\frac1N)$, $\delta<\frac12(\frac23-\frac12)=\frac1{12}$, $\delta<\frac12(1-\frac34)=\frac18$ ($\Leftrightarrow\delta<\min\{\frac12(\frac1{N-1}-\frac1N),\frac{\varepsilon}{8N},\frac1{12}\}$).



Since f is Riemann integrable, $\exists$ partition P_1 of $[0,1]$ such that $U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{4}$

Next partition $[\frac23,\frac34]$ into k subintervals of lengths $\frac1k(\frac34-\frac23)=\frac1{12k}$ with $P_2=\{\frac23+\frac{j}{12k}:j=0,1,\dots,k\}$

Then $U(g,P_2)-L(g,P_2)=\sum_{j=0}^{k-1}\left(g(\frac23+\frac{j+1}{12k})-g(\frac23+\frac{j}{12k})\right)\frac1{12k}$
$<(1-0)\frac1{12k}=\frac1{12k}<\frac{\varepsilon}{4}$ for $k>\frac1{3\varepsilon}$

Let $P_3=\{0,\frac1N,\frac1N+\delta,\frac1{N-1}-\delta,\frac1{N-1}+\delta,\dots,\frac12+\delta,\frac23-\delta,\frac23,\frac34,\frac34+\delta,1-\delta,1\}$ and $P=P_1\cup P_2\cup P_3$. We have

$$U(h,P)-L(h,P)<(1-0)\frac1N+2N\delta+\left(U(f,P_1)-L(f,P_1)\right)+\left(U(g,P_2)-L(g,P_2)\right)$$
$$<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon.$$

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Since f is continuous on $[0,1]\setminus\mathbb{Q}$, $S_f=\{x\in[0,1]:f\text{ is discontinuous at }x\}\subseteq\underbrace{[0,1]\cap\mathbb{Q}}_{\text{Countable}\Rightarrow\text{measure }0}$.
S_f is of measure 0. Therefore, f is Riemann integrable on $[0,1]$.
For $h(x)=f(\frac{x}{\sqrt2})$, $S_h\subseteq\{x\in[0,1]:w=\frac{x}{\sqrt2}\text{ for some }w\in S_f\}=[0,1]\cap\left(\bigcup_{w\in S_f}\{\sqrt2 w\}\right)$ is countable since S_f is countable. Countable by union theorem
Then $h(x)=f(x/\sqrt2)$ is Riemann integrable on $[0,1]$. So $g(x)=f(x)f(x/\sqrt2)$ is Riemann integrable on $[0,1]$.

<u>Alternatively</u> we can also point out $S_g\subseteq S_f\cup S_h$ (Since $x\notin S_f\cup S_h\Rightarrow x\notin S_f$ and $x\notin S_h\Rightarrow f$ and h are continuous at $x\Rightarrow g$ is continuous at $x\Rightarrow x\notin S_g$).