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1/a/ I will not watch a movie or have a dinner outside if tomorrow is sunny or not rainy.

b/ $\exists \varepsilon > 0, \forall \delta > 0$ such that $|x-y|<\delta$, and $|f(x)-f(y)| \geq \varepsilon$.

c/ $\exists x \in S, \exists \varepsilon > 0, \forall \delta > 0$ such that $|x-y|<\delta$, and $|f(x)-f(y)| \geq \varepsilon$.

d/ $\exists \varepsilon > 0, \forall N > 0, \exists m,n \geq N$, $x \in R$ such that $|f_n(x) - f_m(x)| \geq \varepsilon$.

e/ $\exists x \in R, \exists \varepsilon > 0, \forall N > 0, \exists m,n \geq N$ such that $|f_n(x) - f_m(x)| \geq \varepsilon$.

2/a/ Given that $U \subseteq A$, $\forall x,y \in U$ s.t $x,y \in A$. Under $f: A \to B$, $f(x), f(y) \in f(U)$, Also, $f(U) \subseteq B$.

Since $\forall x,y \in U$, $f(x), f(y) \in B$. $f^{-1}(f(x)) = \{x \in A \mid f(x) \in Y\}$. $\forall x \in U$, $x \in f^{-1}(f(U))$.

$\therefore U \subseteq f^{-1}(f(U))$.

Example: Let $U = \{2\}$, $f(x) = x^2$, $f^{-1}(x) = \sqrt{x}$, then $f(U) = \{4\}$, $f^{-1}(f(U)) = \{-2, 2\}$. $U \subset f^{-1}(f(U))$.

b/ Given that $V \subseteq B$, $\forall x \in V$ s.t $x \in B$. $f^{-1}(x) \in A$, Since $\forall x \in V$, $f^{-1}(x) \in f^{-1}(V)$.

$f(f^{-1}(x)) \in B$, $f(f^{-1}(x)) = x$ which is in V. Since $\forall x \in V$, $x \in f(f^{-1}(V))$, $f(f^{-1}(V)) \subseteq V$.

Example: Let $V = \{-2, 2\}$, $f^{-1} = x^2$, $f = |\sqrt{x}|$, then $f^{-1}(V) = \{4\}$, $f(f^{-1}(V)) = \{2\}$. $f(f^{-1}(V)) \subset V$.

c/ $\forall x_i \in X_i$ where $i \in I$, $x_i \in \bigcup_{a \in I} X_a$. $f(x_i) \in f(\bigcup_{a \in I} X_a)$. Since $\forall f(x_i) \in f(\bigcup_{a \in I} X_a)$,

$\bigcup_{a \in I} f(X_a) \subseteq f(\bigcup_{a \in I} X_a)$. Given that $X_a \subseteq A$.

$\forall x_i \in X_i$ where $i \in I$, $f(x_i) \in \bigcup_{a \in I} f(X_a)$, also in $f(\bigcup_{a \in I} X_a)$. $f(\bigcup_{a \in I} X_a) \subseteq \bigcup_{a \in I} f(X_a)$.

$f(\bigcup_{a \in I} X_a) = \bigcup_{a \in I} f(X_a)$.

Given that $Y_a \subseteq B, a \in I$, $\forall y_i \in Y_i$ where $i \in I$. Since $y_i \in \bigcup_{a \in I} Y_a$, $f^{-1}(y_i) \in \bigcup_{a \in I} f^{-1}(Y_a)$.

$f^{-1}(y_i) \in f^{-1}(\bigcup_{a \in I} Y_a) \subseteq \bigcup_{a \in I} f^{-1}(Y_a)$, $f^{-1}(y_i) \in \bigcup_{a \in I} f^{-1}(Y_a) \subseteq f^{-1}(\bigcup_{a \in I} Y_a)$

$\therefore f^{-1}(\bigcup_{a \in I} Y_a) = \bigcup_{a \in I} f^{-1}(Y_a)$.

d/ $\forall y \in f(\bigcap_{a \in I} X_a)$, $\exists x \in \bigcap_{a \in I} X_a$ s.t $f(x) = y$. ~~Since $x \in X_1 \cap X_2 \cap \dots \cap X$~~

Since $x \in \bigcap_{a \in I} X_a$, $f(x) \in f(X_a)$, $a \in I$. Also, $f(x) \in \bigcap_{a \in I} f(X_a)$. Since for $\forall y \in f(\bigcap_{a \in I} X_a)$, $y \in \bigcap_{a \in I} f(X_a)$.

$\therefore f(\bigcap_{a \in I} X_a) \subseteq \bigcap_{a \in I} f(X_a)$.

To prove $f^{-1}(\bigcap_{a \in I} Y_a) = \bigcap_{a \in I} f^{-1}(Y_a)$.

$\forall x \in f^{-1}(\bigcap_{a \in I} Y_a)$, $\exists y \in \bigcap_{a \in I} Y_a$ such that $x = f^{-1}(y)$. Since $y \in \bigcap_{a \in I} Y_a$, $f^{-1}(y) \in f^{-1}(Y_a), a \in I$. It impiles that

$f^{-1}(y) \in \bigcap_{a \in I} f^{-1}(Y_a)$. Therefore, for $\forall x \in f^{-1}(\bigcap_{a \in I} Y_a)$, $x \in \bigcap_{a \in I} f^{-1}(Y_a)$. $f^{-1}(\bigcap_{a \in I} Y_a) \subseteq \bigcap_{a \in I} f^{-1}(Y_a)$.

$\forall x \in \bigcap_{a \in I} f^{-1}(Y_a)$, $x \in f^{-1}(Y_a), a \in I$. By definition of f function, there exist a $y \in Y_a$ such that $x = f^{-1}(y)$.

Since $x \in f^{-1}(Y_a), a \in I$, $y \in Y_a, a \in I \Rightarrow y \in \bigcap_{a \in I} Y_a$. Therefore, $\forall x \in \bigcap_{a \in I} f^{-1}(Y_a)$, $x \in f^{-1}(\bigcap_{a \in I} Y_a)$.

$\therefore f^{-1}(\bigcap_{a \in I} Y_a) \subseteq \bigcap_{a \in I} f^{-1}(Y_a)$ and $\bigcap_{a \in I} f^{-1}(Y_a) \subseteq f^{-1}(\bigcap_{a \in I} Y_a) \Rightarrow f^{-1}(\bigcap_{a \in I} Y_a) = \bigcap_{a \in I} f^{-1}(Y_a)$

3/ $\forall x \in A\cap B$, $\exists y \in f(A\cap B)$ s.t $y=f(x)$. Since $x\in A$ and $x \in B$, $y=f(x)\in f(A)$ and $y\in f(B)$, then $y \in f(A)\cap f(B)$. So $f(A\cap B)\subseteq f(A)\cap f(B)$.

$\forall y \in f(A)\cap f(B)$, $\exists x\in A$ and $x\in B$ s.t $y=f(x)$. Since $x\in A\cap B$, $y\in f(A\cap B)$. So $f(A)\cap f(B)\subseteq f(A\cap B)$

$\therefore f(A\cap B)=f(A)\cap f(B)$. We assume that $f(A\cap B)=f(A)\cap f(B)$ and f is not injective.

Since f is not injective, $\exists x, y \in X$ s.t $x\neq y$ but $f(x)=f(y)$. ~~Assume Let $x,y \in A$ and also $\in B$~~

~~Let $x,y \in A\cap B$ such that $x\neq y$, $\exists f(x), f(y) \in f(A\cap B)$~~.

Let $x\in A$, $y\in B$, $\exists f(x), f(y) \in Y$, s.t $x\neq y$. Assume $A\cap B=\{\emptyset\}$, $f(A\cap B)=\{\emptyset\}$.

Since $f(x)=f(y)$ for some $x,y\in X$ s.t $x\neq y$, $f(A)\cap f(B)=\{f(x)\}$. $f(A\cap B)\neq f(A)\cap f(B)$ which is a contradiction. Since the statement $f(A\cap B)=f(A)\cap f(B)$ and f is not injective is false.

We can prove that $f(A\cap B)=f(A)\cap f(B) \Rightarrow f$ is injective is true.

$\therefore f$ is injective if and only if $f(A\cap B)=f(A)\cap f(B)$.

4/a/ By MI, for $n=1$, $S_n=\{x\in R \mid f_1(x)=0\}=A_1$ it is countable. Assume $n=k, k\in N$. $S_n=\{x\in R \mid \prod_{k=1}^{n} f_k(x)=0\}$.

Since $\prod_{k=1}^{n} f_k(x)=0 \Rightarrow f_1(x)f_2(x)\dots f_n(x)=0$. There exists a $f_i(x)=f_1(x)f_2(x)\dots f_n(x)=0$, where $i\in N$,

By definition of A_k, there exists a set $A_i=\{x\in R \mid f_i(x)=0\}=S_n$ which is countable.

b/ ~~$\prod_{k=1}^{\infty} f_k(x) = f_1(x)f_2(x)\dots = 0$ implies that $\prod_{k=1}^{\infty} f_k(x)=0$~~ implies that $f_k(x)=0$ for some $k\in N$. $A_k=\{x\in R \mid f_k(x)=0\}$ also implies

$S_\infty=\{x\in R \mid \prod_{k=1}^{\infty} f_k(x)=0\}$ implies that $f_k(x)=0$ for some $k\in N$. $A_k=\{x\in R\mid f_k(x)=0\}$ also implies $\forall x\in S_\infty \Rightarrow x\in A_k$, Also $\forall x\in A_k \Rightarrow x\in S_\infty$. for $k\in N$

that $f_k(x)=0$ for $k\in N$. Therefore $\forall x \in S_\infty \Rightarrow x\in A_k$, Also $\forall x\in A_k\Rightarrow x\in S_\infty$. for $k\in N$

$S_\infty=\bigcup_{k=1}^{\infty} A_k$

By Countable union theorem, $\bigcup_{k=1}^{\infty} A_k$ is countable since A_k is countable for $k\in N$.

5/ To prove $P(N)$ is uncountable, $\exists$ ~~injection $f: P(N)\to A_1\times A_2\times\dots$~~ ~~(all $A_i=\{0,1\}$)~~

Let $f(P(N))=(a_1,a_2,\dots)$ where $a_m \begin{cases} 1 & \text{if } m\in S \\ 0 & \text{if } m\notin S \end{cases}$, $m\in N$.

For example: $f(\{1,4,5\})=(1,0,0,1,1)$. , $f^{-1}((a_1,a_2,\dots))=\{m : a_m=1\}$.

Since $\forall x\in P(N)$, $\exists f(x)\in A_1\times A_2\times\dots$, it is surjective.

By Surjection theorem, $P(N)$ is uncountable since $A_1\times A_2\times\dots$ is uncountable.