Recall that

Supremum Limit Theorem Let c be an upper bound of a nonempty set $S \subseteq \mathbb{R}$.
Then ($\exists\, w_n \in S$ with $\lim_{n\to\infty} w_n = c$) $\Leftrightarrow$ $c = \sup S$.

Infimum Limit Theorem Let b be a lower bound of a nonempty set $S \subseteq \mathbb{R}$.
Then ($\exists\, x_n \in S$ with $\lim_{n\to\infty} x_n = b$) $\Leftrightarrow$ $b = \inf S$.

Extra Examples on Supremum/Infimum of Sets

Exercise 91 (a) Determine the supremum and infimum of

$$S = \left\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\right\} \setminus \left\{\frac{2}{k} : k \in \mathbb{N}\right\}$$

Step 1 Try to find the largest element of S or the "right" endpoint of S
$\frac{1}{m} + \frac{1}{n}$ is largest when $m=1, n=1$, then $\frac{1}{m}+\frac{1}{n} = \frac{1}{1}+\frac{1}{1} = 2$, but $2 = \frac{2}{1} \notin S$
The next element we try is $\frac{1}{m}+\frac{1}{n} = \frac{1}{2}+\frac{1}{1} = \frac{3}{2} \neq \frac{2}{k}$ for $k \in \mathbb{N}$.

We have $\frac{1}{m}+\frac{1}{n} \le \frac{3}{2}$ for all $m, n \in \mathbb{N}$ and $\frac{3}{2} \neq \frac{2}{k}$ for $k \in \mathbb{N}$
So $\frac{3}{2}$ is an upper bound of S.

Step 2 Find a sequence in S with limit equal to the upper bound.

Let $w_n = \frac{3}{2} = \frac{1}{2} + \frac{1}{1} \in S$ (as $\frac{3}{2} \neq \frac{2}{k}$). Now $\lim_{n\to\infty} w_n = \frac{3}{2}$.
By supremum limit theorem, $\sup S = \frac{3}{2}$.

For infimum of S, we have $0 < \frac{1}{m}+\frac{1}{n}$ for all $m, n \in \mathbb{N}$. So 0 is a lower bound of S. Let $x_n = \frac{1}{n} + \frac{1}{n+1}$. Since $\frac{2}{n+1} < \frac{1}{n} + \frac{1}{n+1} < \frac{2}{n}$, we see $\frac{1}{n}+\frac{1}{n+1} \neq \frac{2}{k}$ for $k \in \mathbb{N}$.
So $x_n \in S$. Now $\lim_{n\to\infty} x_n = 0$. By infimum limit theorem, $\inf S = 0$.

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91 (i) Determine the supremum/Infimum of $S = \{\sqrt{x} + y^2 : x, y \in (0,1] \cap \mathbb{Q}\}$.

We have $\sqrt{x} + y^2 \le \sqrt{1} + 1^2 = 2$ for all $x, y \in (0,1] \cap \mathbb{Q}$.
So 2 is an upper bound of S.
Let $w_n = \sqrt{1} + (1)^2 \in S$ since $1 \in (0,1] \cap \mathbb{Q}$. Now $\lim_{n\to\infty} w_n = \sqrt{1} + 1^2 = 2$.
By supremum limit theorem, $\sup S = 2$.

For infimum of S, we have $0 < \sqrt{x} + y^2$ for all $x, y \in (0,1] \cap \mathbb{Q}$.
So 0 is a lower bound of S. Let $x_n = \sqrt{\frac{1}{n}} + \left(\frac{1}{n}\right)^2 \in S$ since $\frac{1}{n} \in (0,1] \cap \mathbb{Q}$.
Now $\lim_{n\to\infty} x_n = 0$. By infimum limit theorem, $\inf S = 0$.