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Problem (a) we want to show $|\frac{1}{x}-\frac{1}{c}|<\varepsilon$, for all $\varepsilon>0$.

$$\left|\frac{1}{x}-\frac{1}{c}\right| = \left|\frac{c-x}{cx}\right| = \left|\frac{1}{c}\cdot\left(\frac{c}{x}-1\right)\right|$$

let $k=\left[\frac{c}{c\varepsilon+1}\right]+1$ for all $x\geq k$,

$$\left|\frac{1}{x}-\frac{1}{c}\right| = \left|\frac{1}{c}\cdot\left(\frac{c}{x}-1\right)\right| < \left|\frac{1}{c}\left(\frac{c}{k}-1\right)\right| < \varepsilon$$

By definition of limit, $\lim_{x\to c}\frac{1}{x}=\frac{1}{c}$

(b) When $\lim_{x\to c} f(x)=\infty$, $0<|x-c|<\delta \Rightarrow |f(x)-\infty|<\varepsilon$

We need to show $0<|x-c|<\delta \Rightarrow \left|\frac{1}{f(x)}-0\right|<\varepsilon$

$$\left|\frac{1}{f(x)}-0\right| = \frac{|\infty-f(x)|}{\infty\cdot f(x)} < \frac{\varepsilon}{\infty\, f(x)} < \varepsilon$$

Therefore, If $\lim_{x\to c} f(x)=\infty$, then $\lim_{x\to c}\frac{1}{f(x)}=0$.

When $\lim_{x\to c}\frac{1}{f(x)}=0$, $0<|x-c|<\delta \Rightarrow \left|\frac{1}{f(x)}-0\right|<\varepsilon$

we need to show $0<|x-c|<\delta \Rightarrow |f(x)-\infty|<\varepsilon$

$$|f(x)-\infty| = \left|\frac{1}{\frac{1}{f(x)}}-\frac{1}{\varepsilon}\right| < \varepsilon$$

There, $\lim_{x\to c} f(x)=\infty$ iff $\lim_{x\to c}\frac{1}{f(x)}=0$.

1c) when $\lim_{x \to \infty} f(x) = L$, $0 < |x - \infty| < \delta \Rightarrow |f(x) - L| < \varepsilon$

let $\exists K > 0$, s.t. $x > K$, ie $\frac{1}{x} < \frac{1}{K}$

then $|f(\frac{1}{x}) - L| < \varepsilon$, whenever $\frac{1}{x} < \frac{1}{K}$

So $\lim_{x \to 0^+} f(\frac{1}{x}) = L$.

2 a) let $a \in \mathbb{R}$ and a_n be the irrational number in the neighberhood of a and b_n be the rational number in the neighberhood of a. Such that.

$$|a_n - a| < \varepsilon$$

and $|b_n - a| < \varepsilon$, for all $\varepsilon > 0$.

$\Rightarrow \lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = a$

ie $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a$

By Sequential limit thm,

$\lim_{n\to\infty} f(a_n) = f(a) = 0$ since a_n is irrational number

$\lim_{n\to\infty} f(b_n) = f(a) = a$ since b_n is rational number.

So, when $a=0$,

In a_n, $\lim_{n\to\infty} f(a_n) = f(0) = 0$

and In b_n, $\lim_{n\to\infty} f(b_n) = f(0) = 0$

Hence, $\lim_{x\to 0} f(x) = 0$ and $f(x)$ continuous at $x=0$

b) When $x = c \neq 0$, $\lim_{n\to\infty} f(a_n) = f(c) = 0$ and $\lim_{n\to\infty} f(b_n) = c = f(c)$

Therefore $\lim_{x\to c} f(x)$ does not exist

3a) By Sequential limit thm,

$$\lim_{n\to\infty} f(y_n) = f(y)$$

Then, $f(y_n^2) \geq f(y_n)$

$$\Rightarrow \lim_{n\to\infty} f(y_n^2) \geq \lim_{n\to\infty} f(y_n) \Rightarrow \lim_{n\to\infty} f(y_n^2) \geq f(y)$$

Since, $\lim_{n\to\infty} y_n = y \Rightarrow \lim_{n\to\infty} y_n^2 = y^2$

$$\Rightarrow \lim_{n\to\infty} f(y_n^2) = f(y^2) \geq f(y)$$

Thus, $y \in S$.

b) Prove by contradiction, Assume there exist $c \in \mathbb{R} \setminus \mathbb{Q}$ such that $f(c) \neq g(c)$, f and g are continuous at c.

Let a rational sequence $\{r_n\}$, where $\lim_{n\to\infty} r_n = c$

Then $f(r_n) = g(r_n) \Rightarrow \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n)$

As $\lim_{n\to\infty} f(r_n) = f(c)$ and $\lim_{n\to\infty} g(r_n) = g(c)$

we have $f(c) = g(c)$ which contradict to assumption.

therefore, It is true for $f(x) = g(x)$ for all $x \in \mathbb{R}$

4) Define $g(c) = f(c+1) - f(c)$,

$g(0) = f(1) - f(0) = f(1)$

$g(1) = f(2) - f(1) = -f(1)$

Assume $f(1) > 0$, then $g(0) > 0$, $g(1) < 0$

By intermediate interval thm,

$\exists c \in [0,1]$ s.t $g(c) = 0$

Therefore $f(c+1) = f(c)$, $c \in [0,1]$

5.) Let $|f(x)| = l > 0$, $\exists y_1 \in [a,b]$ Such that $|f(y_1)| \leq \frac{l}{2}$

If $f(y_1) = 0$, then $c = y_1$.

otherwise $f(y_1) \neq 0 \Rightarrow |f(y_1)| > 0$, $\exists y_2 \in [a,b]$

$|f(y_2)| \leq \frac{1}{2}|f(y_1)| \leq \frac{l}{2^2}$

And so we construct a sequence $\{y_n\}$, $|f(y_n)| > 0$

$|f(y_n)| \leq \frac{l}{2^n}$, $|f(y_n)| \leq \frac{1}{2}|f(y_{n-1})| < |f(y_{n-1})|$

$\Rightarrow \{|f(y_n)|\}$ is a decreasing sequence and bounded

By Bolzano-Wei thm,

Sub-sequence of $\{y_n\}$, called $\{y_{n_k}\}$ converge to $y \in [a,b]$

as $n \to \infty$. then we have $f(y_{n_k}) \to f(y)$ as $n \to \infty$

So, $0 < |f(y_{n_k})| \leq \frac{l}{2^{n_k}}$, $\frac{l}{2^{n_k}} \to 0$ as $n \to \infty$

Hence, $|f(y_{n_k})| \to 0$, as $n \to \infty$

$\Rightarrow f(y) = 0$, $y \in [a,b]$

b) $\text{Sup}(f(t)) = f(x^-)$, where $a<t<x$

$f(x^-) \le f(x) \le f(x^+)$

$f(x^+) = \text{Inf}(f(t))$, where $x<t<b$

Let $A = \{f(t) \mid a<t<x\}$, $B = \{f(t) \mid x<t<b\}$

$f(x) \ge f(t) \in A$, $f(x) \le f(t) \in B$

(Since the function $f(\cdot)$ is monotone)

So, set A is bounded above and set B is bounded below.

We can say $\text{Sup}\, A$ and $\text{Inf}\, B$ exist.

Let $\text{Sup}\, A = u$, $\forall \varepsilon > 0$, there exist $x_1 \in (a, x)$

s.t. $u - \varepsilon < f(x_1) \le u \Rightarrow u - \varepsilon < f(t) \le u$, $t \in (x_1, x)$

$u - \varepsilon < f(t) < u + \varepsilon \Rightarrow -\varepsilon < f(t) - u < \varepsilon$

So, $|f(t) - u| < \varepsilon \Rightarrow f(x^-) = u$

Let $\text{Inf}\, B = v$, $\forall \varepsilon > 0$ exist $x_2 \in (x, b)$

s.t $v \le f(x_2) < v + \varepsilon \Rightarrow v \le f(t) < v + \varepsilon$, $t \in (x, x_2)$

$v - \varepsilon < f(t) < v + \varepsilon \Rightarrow -\varepsilon < f(t) - v < \varepsilon$

So, $|f(t) - v| < \varepsilon \Rightarrow f(x^+) = v$

Hence, $\text{Sup}\, A \le f(x) \le \text{Inf}\, B$, $M(x)$ and $m(x)$ both continuous at any $x_0 \in [a, b]$