Complex-valued Functions and Euler's Formula

For every $z=x+iy\in\mathbb{C}$, $|z|=\sqrt{x^2+y^2}\in\mathbb{R}$. Since

$$\sum_{n=0}^{\infty}\frac{|z|^n}{n!}=e^{|z|},\quad \sum_{k=0}^{\infty}\frac{|z|^{2k}}{(2k)!}<e^{|z|},\quad \sum_{k=0}^{\infty}\frac{|z|^{2k+1}}{(2k+1)!}<e^{|z|},$$

by the absolute convergence test, we may define

$$\boxed{\forall z\in\mathbb{C},\ e^{z}=\sum_{n=0}^{\infty}\frac{z^n}{n!},\ \cos z=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k)!},\ \sin z=\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)!}}$$

Facts ① $\forall z\in\mathbb{C}$, since $i, i^2=-1, i^3=-i, i^4=1, \cdots$ are periodic,

$$e^{iz}=\sum_{n=0}^{\infty}\frac{i^n z^n}{n!}=\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)+i\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)$$
$$=\cos z+i\sin z.$$

② $e^{-iz}=\cos(-z)+i\sin(-z)=\cos z-i\sin z$. Plug $-z$ into cos and sin series.

$$\cos z=\frac{e^{iz}+e^{-iz}}{2},\quad \sin z=\frac{e^{iz}-e^{-iz}}{2i}$$

$$\cos^2 z+\sin^2 z=\frac{(e^{iz}+e^{-iz})^2-(e^{iz}-e^{-iz})^2}{4}=1 \text{ after expansion}$$

However, $\cos(iy)=\dfrac{e^{-y}+e^{y}}{2}\to\infty$ as $y\to\infty$

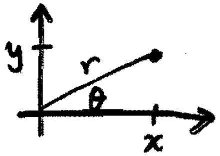
$\sin(iy)=\dfrac{e^{-y}-e^{y}}{2i}\to\infty$ as $y\to\infty$

So $\cos z$ and $\sin z$ are not bounded!!!

Euler's Formula

$$e^{i\pi}=\cos\pi+i\sin\pi=-1\Rightarrow\boxed{e^{i\pi}+1=0}$$

In my opinion, this is the most beautiful formula in mathematics. It connects 5 of the most important constants $1, 0, \pi, i, e$ in mathematics.



For every $z\in\mathbb{C}\setminus\{0\}$, $r=|z|$ is called the norm or modulus of z. If $z=x+iy$, then $\exists\,\theta\in\mathbb{R}$ such that $x=r\cos\theta$, $y=r\sin\theta$ so that $z=re^{i\theta}$. The particular $\theta\in(-\pi,\pi]$ is called the principal argument of z and is denoted by $\operatorname{Arg} z$.

The numbers $\theta=\operatorname{Arg} z+2n\pi$, $n\in\mathbb{Z}$, are called argument of z and we write $\arg z$ for them.

③ $e^{w+z}=e^{w}e^{z}$ for all $w,z\in\mathbb{C}$. (de Moivre's formula)

($w=u+iv$, $z=x+iy\Rightarrow e^{w+z}=e^{(u+x)+i(v+y)}$

$e^{i(v+y)}=\cos(v+y)+i\sin(v+y)=(\cos v\cos y-\sin v\sin y)+i(\sin v\cos y+\cos v\sin y)$
$=(\cos v+i\sin v)(\cos y+i\sin y)=e^{iv}e^{iy}$

$\therefore e^{w+z}=e^{u}e^{x}e^{iv}e^{iy}=e^{u+iv}e^{x+iy}=e^{w}e^{z}$.)

Logarithm

For a fixed $z\in\mathbb{C}\setminus\{0\}$, consider the equation $z=e^{w}$. Suppose $w=x+iy$ is a solution. Then $|z|=|e^{w}|=|e^{x}e^{iy}|=e^{x}$

$\Rightarrow x=\ln|z|$ and $z=e^{w}=e^{x}e^{iy}=|z|e^{iy}\Rightarrow y=\arg z$.

$\therefore$ all solutions are $w=\ln|z|+i\arg z$.

Definitions ① For $z\in\mathbb{C}\setminus\{0\}$, we define $\log z=\ln|z|+i\arg z$. Since $\arg z=\operatorname{Arg} z+2n\pi$, there are infinitely many choices of $\arg z$. For the choice $n=0$, we call $\operatorname{Log} z=\ln|z|+i\operatorname{Arg} z$ the principal logarithm of z.

② For $z\in\mathbb{C}\setminus\{0\}$ and $w\in\mathbb{C}$, define $z^{w}=e^{w\operatorname{Log} z}$.

$$z^{w_1+w_2}=e^{(w_1+w_2)\operatorname{Log} z}=e^{w_1\operatorname{Log} z}e^{w_2\operatorname{Log} z}=z^{w_1}z^{w_2}.$$

Recall for $|x|<1$, we have

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

For the case $x=-1$, the left side is $-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots$ which converges by the alternating series test. In that case, the answer is $-\ln 2$. So for some case when $|x|=1$, the above equation is also true.

> For complex z with $|z|<1$, we also have
> $$-\operatorname{Log}(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

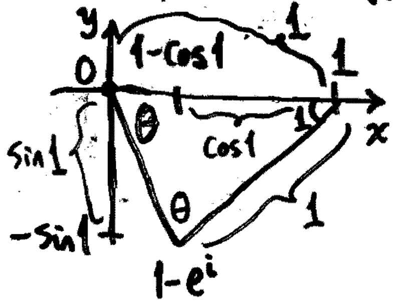
For the case $z=e^i$, $z^k=(e^i)^k=e^{ik}=\cos k + i\sin k$, the left side is $\displaystyle\sum_{k=1}^{\infty}\frac{\cos k + i\sin k}{k} = \sum_{k=1}^{\infty}\frac{\cos k}{k} + i\sum_{k=1}^{\infty}\frac{\sin k}{k}$.

We checked $\displaystyle\sum_{k=1}^{\infty}\frac{\sin k}{k}$ converges by summation by part. Similarly, $\displaystyle\sum_{k=1}^{\infty}\frac{\cos k}{k}$ converges. To find these sums, it turns out the equation is also true for $z=e^i$. So we have to work out

$$-\operatorname{Log}(1-e^i) = -\ln|1-e^i| - i\operatorname{Arg}(1-e^i)$$

Now $1-e^i = (1-\cos 1) - i\sin 1$. So $\quad \frac{1-\cos 1}{2} = \sin^2\frac{1}{2}$

$$-\ln|1-e^i| = -\ln\sqrt{(1-\cos 1)^2+\sin^2 1} = -\ln\sqrt{2-2\cos 1} = -\ln\left(2\sin\tfrac{1}{2}\right)$$



$$\theta = \frac{\pi-1}{2} > 0 \qquad \operatorname{Arg}(1-e^i) = -\left(\frac{\pi-1}{2}\right)$$

$$\therefore \sum_{k=1}^{\infty}\frac{\cos k}{k} = -\ln|1-e^i| = -\ln\left(2\sin\tfrac{1}{2}\right)$$

and $$\sum_{k=1}^{\infty}\frac{\sin k}{k} = -\operatorname{Arg}(1-e^i) = \frac{\pi-1}{2}.$$

Fourier Series

Definition Any function $P(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)$, where $a_n, b_n\in\mathbb{R}$, is called a trigonometric series.

Let $f:\mathbb{R}\to\mathbb{R}$ be a 2π-periodic function such that f is Riemann integrable on $[-\pi,\pi]$. Define

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx,\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx,\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

for $n=1,2,3,\ldots$. The trigonometric series resulted by using these numbers as coefficients is the Fourier series of f.

> **Theorem** If f is 2π-periodic on $\mathbb{R}$, Riemann integrable on $[-\pi,\pi]$ and $f_+'(\theta) = \lim_{h\to0^+}\frac{f(\theta+h)-f(\theta)}{h}$, $f_-'(\theta) = \lim_{h\to0^-}\frac{f(\theta+h)-f(\theta)}{h}$ both exist, then the Fourier series of f at $x=\theta$ equals $\frac{f(\theta+)+f(\theta-)}{2}$. (Recall $f(\theta+) = \lim_{x\to\theta+}f(x)$, $f(\theta-) = \lim_{x\to\theta-}f(x)$.)

Example Let $f(x)=x^2$ on $[-\pi,\pi]$ and extend it to $\mathbb{R}$ by making it 2π-periodic (i.e. $f(x+2\pi)=f(x)\ \forall x\in\mathbb{R}$). Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\,dx = \frac{2}{3}\pi^2,\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,dx = \begin{cases}\frac{4}{n^2} & \text{if } n=2k\\ -\frac{4}{n^2} & \text{if } n=2k-1\end{cases}\quad k=1,2,3,\ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{x^2\sin nx}_{\text{odd}}\,dx = 0$$

Since $f_+'(x)$ and $f_-'(x)$ exist for all $x\in[-\pi,\pi]$, by the theorem, $\forall x\in[-\pi,\pi]$, we have

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}.$$

Taking $x=\pi$, we get

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{4}\left(\pi^2-\frac{\pi^2}{3}\right)=\frac{\pi^2}{6}.$$

Also, on $[-\pi,\pi]$, $\left|\frac{(-1)^n\cos nx}{n^2}\right|\le\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty$.

By Weierstrass M-test, $\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}$ converges uniformly on $[-\pi,\pi]$. By the integration theorem, $\forall x\in[-\pi,\pi]$,

$$\frac{x^3}{3}=\int_0^x t^2\,dt=\int_0^x\left(\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nt}{n^2}\right)dt$$

$$=\frac{\pi^2}{3}x+4\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n^3}.$$

Taking $x=\frac{\pi}{2}$, we get

$$\frac{\pi^3}{24}=\frac{\pi^3}{6}+4\left(-1+\frac{1}{3^3}-\frac{1}{5^3}+\frac{1}{7^3}-\cdots\right)$$

$$\Rightarrow 1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{1}{4}\left(\frac{\pi^3}{6}-\frac{\pi^3}{24}\right)=\frac{\pi^3}{32}.$$

Open Problem

What is $\zeta(3)=1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots$?