Tutorial class 1 Logic & Sets.

(Corresponds to lec 1 + lec 2).

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I. Logic

Notation : We say x_0 satisfies $q(x)$, if $q(x_0)$ is true.

I.1 Statements : denoted by p usually.

E.g. ① 1 is a real number. True

② 1 is not a real number. False

quantifier notation : $\forall$, $\exists$.

forall exist.

③ $\exists x$ such that $x^2 = 1$. True

④ $\forall x$, $\exists y$ such that $x = 2y$. True.

⑤ $\exists y$, $\forall x$ such that $x = 2y$. False.

Statement with variable x: $q(x)$.

e.g. $q(x) := x = 0$.

p: for all $x > 1$, $\underbrace{q(x)}_{x=0}$.

I.2 Basic Operation:

For given (mathematical) statements p, q, we have following operations :

① negation : $\sim p$.

② and : $p \wedge q$.

③ or : $p \vee q$.

Check that ⑤ is false:

For every fixed real number y_0, we have

$x_0 = 2y_0$ is the unique real number such that $x_0 = 2 \cdot y_0$.

but the statement is true only when for every x we have

$x = 2y_0$.

Example1: $p: x \geq 0$ $\quad q: x<1.$

$\sim p: x<0$ $\quad \sim q: x \geq 1.$

$p \wedge q$: $x \geq 0$ and $x<1.$ $\quad p \vee q$: $x \geq 0$ or $x<1.$ $\quad$ x is a real number.

Rule: $\sim$($\exists x$ such that x satisfies $\overset{p}{q}$).

$=$ $\forall x$, we have x satisfies $\sim q$

$\sim$($\forall x$, we have x satisfies q)

$=$ $\exists x$ such that x satisfies $\sim q$

Example2: p: $\exists x>0$, such that $x^2=1.$

q: $\forall x \leq 0$, we have $x^2 \neq 1.$

Finding $\sim p$: define a new statement w: $x^2=1.$

then $p = \exists x>0$ such that x satisfies w.

thus $\sim p$: $\forall x>0$ we have x satisfies $\sim w$. $\quad x^2 \neq 1.$

$\sim p = \forall x>0$ we have $x^2 \neq 1.$

(1 min) Question: What is $\sim q$? $\quad \sim w: x^2=1.$

Answer: define $w: x^2 \neq 1$, $\quad q: \forall x \leq 0$. x satisfies w.

$\overset{\text{rule}}{\Longrightarrow}$ $\exists x \leq 0$, such that x satisfies $\sim w$ $\quad x^2=1.$

$(\sim p) \wedge q$: $\forall x>0$. we have $x^2 \neq 1$ and $\forall x \leq 0$, we have $x^2 \neq 1.$

$\Vert$: $\forall x$, we have $x^2 \neq 1.$

I.3 Rules of operations

① $\sim(\sim p) = p$

② $\sim(p \wedge q) = (\sim p) \vee (\sim q)$

③ $\sim(p \vee q) = (\sim p) \wedge (\sim q)$.

Example 3: $p: \exists x > 0,\ x^2 = 1.$

$q: \exists x \leq 0,\ x^2 = 1.$

$p \vee q = \exists x > 0, x^2 = 1$ or $\exists x \leq 0, x^2 = 1 = \exists x,\ x^2 = 1.$

$\sim(p \vee q)$

approach1: $= \sim(\exists x > 0,\ x^2 = 1)$ and $\sim(\exists x \leq 0,\ x^2 = 1)$

$= \forall x > 0,\ x^2 \neq 1$ and $\forall x \leq 0,\ x^2 \neq 1$

approach2: $= \sim(\exists x, x^2 = 1) = \forall x,\ x^2 \neq 1.$

$\forall x > 0$ or $\forall x \leq 0 \overset{?}{\Leftrightarrow} \forall x.$

Question: $p: \forall x > 0, x^2 = 1.$ $q: \forall x \leq 0,\ x^2 = 1.$

$\sim(p \vee q) = ?$ Answer. $(p \vee q) \neq \forall x,\ x^2 = 1.$

$\sim(p \vee q) = \sim p \wedge \sim q = \exists x > 0, x^2 \neq 1$ and $\exists x \leq 0, x^2 \neq 1.$

I.4 Conditional Structure of Statements.

For statements p, q. we can consider a new statement w, written as

w: If p, then q (p implies q). Notation: $w: p \Rightarrow q$.

① Conditional structure can be represented using 'negation' and 'or':

$$p \Rightarrow q = (\sim p) \vee q.$$

② As a corollary:

$$\sim(p \Rightarrow q) = \sim((\sim p) \vee q)$$
$$= p \wedge (\sim q).$$

defined similarly, by "if q, then p".

Converse Statement of $p \Rightarrow q$ is defined as $q \Rightarrow p$. and by ① we have

$$q \Rightarrow p = (\sim q) \vee p \neq p \wedge (\sim q) = \sim(p \Rightarrow q) \;!!$$

Converse is different with negate!

and we can discuss the converse statement only when the statement has conditional structure!

③ (Contrapositive Statement): Contrapositive of $p \Rightarrow q$ is defined as $(\sim q) \Rightarrow (\sim p)$

$$p \Rightarrow q = (\sim q) \Rightarrow (\sim p).$$

Proof: By def $(\sim q) \Rightarrow (\sim p) = \sim(\sim q) \vee (\sim p) = q \vee (\sim p)$
$= p \Rightarrow q$

Example 4.

$p: x=1.$ $\quad q: x^2=1.$

$\forall x, p \Rightarrow q$: $\forall x$, if $x=1$, then $x^2=1$. ← true

Or equivalently: $= (\sim p) \vee q = (x \neq 1$ or $x^2=1)$

converse

$\forall x, q \Rightarrow p$: $\forall x$, if $x^2=1$, then $x=1$. ← false

$= (\sim q) \vee p = (x^2 \neq 1$ or $x=1)$

negation

$\sim(\sim p \vee q) = p \wedge \sim q$

$\sim(\forall x, p \Rightarrow q)$: $\exists x$, such that $\sim(p \Rightarrow q)$ false

$= \exists x$ such that $x=1$ and $x^2 \neq 1$.

Question: Write down $\forall x, \sim q \Rightarrow \sim p$. Whether it is true or false?

Answer: $\forall x, x^2 \neq 1 \Rightarrow x \neq 1$. it is true.

II. Set theory

II.1 Basic def of a set.

Def: a set is a collection of objects, and we say the objects in the set are the elements of the set.

$x \in A$

(Classical paradox: $A = \{x : x \text{ not in set } A\}$), $x \notin A$

To avoid such problem, we require that any set A should satisfy:

$\forall x$, the statement $x \in A$ should be either true or false.

Size of a set $\begin{cases} \text{empty set} & \phi. \\ \text{finite set} & \\ \text{infinite set} & \end{cases}$

How to describe a set? $\begin{cases} A = \{x : x \text{ satisfies } p\}. \\ A = \{x_1, \cdots, x_n\}. \\ A = \{1, 2, 3, 4, 5, \cdots\} \end{cases}$

describe the property of elements.

← list all elements

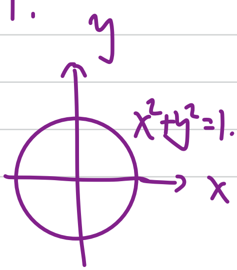
Example 0:

$\mathbb{R} := \{x : x \text{ is a real number}\}$

$\mathbb{Z} := \{x : x \text{ is a integer}\}$ or $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$

$[a,b] := \{x : \underbrace{x \in \mathbb{R}}_{p} \ \underbrace{\text{and}}_{\wedge} \ \underbrace{a \le x \le b}_{q}\}$

Example 1:



Graph

$S := \{(x,y) : x^2 + y^2 = 1,\ x, y \in \mathbb{R}\}.$

II.2 Relation of sets.

For two sets A, B. we say

① A is a subset of B ($A \subseteq B$), if

$\forall x \in A$, we have $x \in B$ holds.

② $A = B$ if $A \subseteq B$ and $B \subseteq A$.

③ A is a proper subset of B ($A \subset B$), if $A \subseteq B$ and $A \neq B$.

① In fact, $A \subset B \Leftrightarrow \exists x \in B$ such that $x \notin A$.

Example 2: $A = \mathbb{Z}$, $B = \mathbb{R}$, then $A \subset B$.

$-1 \in A$, $-1 \in B$. $3/2 \in B$ not in A.

$A = \mathbb{Z}$, $B = \mathbb{R}_+ := \{x : x \in \mathbb{R}, x \geq 0\}$, then

we have neither $A \subseteq B$ nor $B \subseteq A$.

II.3 Power set

a collection of sets

Def of power set: Let S be a set, then power set of S, denoted by 2^S or $P(S)$, is given by

$$2^S := \{A : A \text{ is a subset of } S\}$$

[305] Question: $S = \{1\}$. $2^S = \{\phi, 1\}$ or $\{\phi, \{1\}\}$?

$\{1\}$. ✗ ✓.

$2^S = \{\phi, \{1\}\}$.

II.4 Operations of sets

Consider sets $A_1, A_2, \cdots$

① Union: $\bigcup_{n=1}^{+\infty} A_n := \{x : x \text{ is an element of } A_i \text{ for some } i\}$

② Intersection: $\bigcap_{n=1}^{+\infty} A_n := \{x : x \text{ is an element of } A_i \text{ for all } i\}$

③ Cartesian product:

$$\prod_{n=1}^{+\infty} A_n = A_1 \times \cdots \times A_n \times \cdots = \{(x_1, \cdots, x_n, \cdots) : x_i \in A_i \text{ for all } i\}$$

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

④ Complement of A_2 in A_1:

$$A_1 \setminus A_2 := \{x : x \in A_1, x \notin A_2\}$$

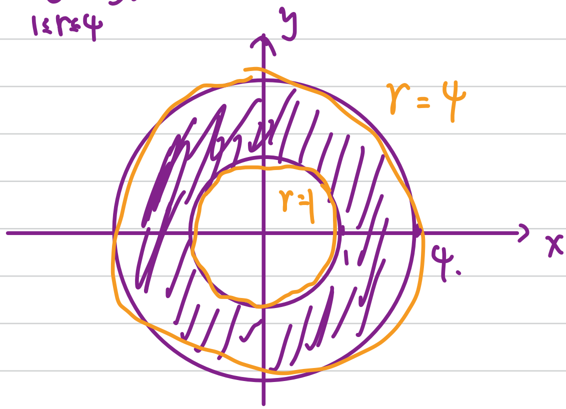
Example 3:

Denote $A := \{n \in \mathbb{Z} : \underbrace{1 < n < 5}_{\{2,3,4\}}\}$ $B := \{2m : m \in \mathbb{Z}, m \geq 0\}$.
then compute $A \cap B$, $A \cup B$, $A \setminus B$, $B \setminus A$:

$A \cap B = \{2, 4\}$, $A \cup B = \{2, 3, 4, 6, 8, 10, 12, \cdots\}$

$A \setminus B = \{3\}$ $B \setminus A = \{0, 6, 8, 10, \cdots\}$

Example 4: denote $S_r := \{(x,y) : x^2+y^2=r^2\}$, then plot the graph of $\bigcup_{1\le r\le 4} S_r$:



Example 5: Show that for any sets A, B, C, we have

$$(A\setminus B)\setminus C = (A\setminus C)\setminus B.$$

we have

$x \in (A\setminus B)\setminus C$ if and only if

$x \in A\setminus B$ and $x \notin C$ if and only if

$x \in A$ and $x \notin B, x \notin C$ if and only if

$x \in A\setminus C$ and $x \notin B$ if and only if

$x \in (A\setminus C)\setminus B$.

II.5 functions

Def: Given two sets A, B, we say f is a function from A to B, if f is a rule that assign every $a \in A$ exactly to a value $b \in B$, such b is called the value of f at a denoted by $f(a)$.

domain: A

$b = f(a)$.

codomain: B

range: $= \{ f(x) : x \in A \}$

graph: $= \{ (x, f(x)) , x \in A \}$

Properties:

We say f is

① injective : if $\forall a, a' \in A, a \neq a'$ we have $f(a) \neq f(a')$

② surjective : if $\forall b \in B$, we can find $a \in A$ such that $f(a) = b$.

range $(f) = B$.

③ bijective : if f is both injective and surjective

Example 6:

Consider $A = \mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$

$B = \mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$.

$f(x) = x^2$, then

① whether f is surjective?

② whether f is injective?

③ what about ①, ② when $A = \mathbb{R}$?

Answer: ① Yes, because for all $b \in B$, $\sqrt{b}$ is in A, and

$f(\sqrt{b}) = b$.

② Yes, for every $b_1, b_2 \geq 0$, $b_1^2 = b_2^2 \Leftrightarrow b_1 = b_2$, since if $b_1 \neq b_2$, then $f(b_1) \neq f(b_2)$. thus f is an injection.

③ : ① still is Yes, because $\forall b \in B$, $\sqrt{b}$ is still in A.

② is No, because for $a \in A$ such that $a \neq 0$, we have $-a \neq a$ but $f(a) = f(-a)$

Example 7:

Suppose A, B are subsets of $\mathbb{R}$ and $f: A \to B$ is a function.

If for every $b \in B$, the horizontal line given by

$\{(x,y): y=b\}$ satisfies it intersect with graph(f) at least once, show that f is surjective:

By the claim, we have $\forall b \in B$, there exists a point

$(x_0, y_0) \in \{(x,y): y=b\} \cap \underbrace{\{(x, f(x)): x \in A\}}_{\text{graph}(f)}$, by

$(x_0, y_0) \in$ graph(f), we have

$(x_0, y_0) = (x_0, f(x_0))$, thus $y_0 = f(x_0)$. $(*1)$

On the other hand.

$(x_0, y_0) \in \{(x,y): y=b\}$ implies that $y_0 = b$ $(*2)$.

Combining $(*1)$, $(*2)$, we get $f(x_0) = b$.

By b is aribrtary, we get f is surjective.

Quesition: If "at least once" is replaced by "at most once", show that f is injective.

Answer:

Consider $a_1, a_2 \in A$ s.t. $a_1 \neq a_2$, we have

$(a_1, f(a_1)), (a_2, f(a_2)) \in graph(f)$. and denote $b_2 = f(a_2) \in B$,

$graph(f)$ intersects with the line $\{(x,y) : y = b_2\}$ at most once, we have

$(a_2, f(a_2))$ is the unique point in $\{(x,y) : y = b_2\} \cap graph(f)$,

that implies $f(a_1) \neq b_2$: otherwise, we must have

$(a_1, f(a_1)) \in graph(f) \cap \{(x,y) : y = b_2\}$ as well, but by

$(a_1, f(a_1))$ and $(a_2, f(a_2))$ are different points, we get

a contradiction (with the fact "$(a_2, f(a_2))$ is the unique point in $\{(x,y) : y = b_2\} \cap graph(f)$".

by a_1, a_2 is selected arbirtary in A, we have

$a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ for all $a_1, a_2 \in A$.

That implies f is injective.