Math 202 (Introduction to Real Analysis) Spring 2008

Final Examination – (Duration: 120 minutes)

Directions: This is a closed book exam. Works (including scratch works) must be shown legibly to receive credits. Answers alone are worth very little. Calculators are allowed.

Notations: $\mathbb{R}$ denotes the set of all real numbers.

Problems

1. (5 marks) Determine the domain (of convergence) of $f(x) = \sum_{k=1}^{\infty} \frac{k^2}{3^k}(\pi - 2x)^k$. Be sure to show work.

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2. (10 marks) Determine whether the improper integral $\int_{-1}^{1} \frac{x\,dx}{\sin^2 x}$ converges or not. Also, determine whether the principal value integral $P.V. \int_{-1}^{1} \frac{x\,dx}{\sin^2 x}$ converges or not. (Make sure works for each step are shown clearly!)

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3. (10 marks) Prove the series of functions $\sum_{k=1}^{\infty} \left(\frac{kx}{1+k^2x^2}\right)^k$ converges uniformly on $\mathbb{R}$.

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4. (10 marks) Let $\{x_n\}, \{y_n\}$ be two Cauchy sequences of real numbers. Prove that $\sqrt{x_n^2 + y_n^2}$ is also a Cauchy sequence by checking the definition of Cauchy sequence. (Do not use the theorem that asserts a sequence is a Cauchy sequence if and only if it converges. Otherwise 0 mark will be given for this problem!)

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5. (a) (5 marks) State Lebesgue's theorem.

   (b) (10 marks) For $n = 1, 2, 3, \ldots$, let $f_n : [0,1] \to [0,1]$ be Riemann integrable functions. Prove that $g : [0,1] \to \mathbb{R}$ defined by $g(0) = 0$ and

$$g(x) = f_n(x) \quad \text{for } n = 1, 2, 3, \ldots \quad \text{and} \quad x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$$

   is Riemann integrable on $[0,1]$.

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6. (8 marks) Let $a_1, a_2, a_3, \ldots \in \mathbb{R}$ and s_n be the n-th partial sum of the convergent series $\sum_{k=1}^{\infty} a_k$. Prove that $\lim_{n\to\infty} \frac{a_1 + 2a_2 + 3a_3 + \cdots + na_n}{n} = 0$.

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7. (8 marks) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f''(x)$ is continuous and $|f''(x)| \le 1$ for all $x \in [0,1]$. If $f\left(\frac{1}{2}\right) = 0$, then prove that $\left|\int_0^1 f(x)\,dx\right| \le \frac{1}{24}$.

–End of Paper–