MATH 2031 Introduction to Real Analysis

January 31, 2013

Tutorial Note 14

Differentiation (con't)

(I) **Generalized Mean Value Theorem**
If f, g are continuous on $[a,b]$ and differentiable on (a,b), then $\exists x_0 \in (a,b)$ such that

$$g'(x_0)(f(b)-f(a)) = f'(x_0)(g(b)-g(a))$$

Remark:
If $g(b) \neq g(a)$, then the above expression can be written as

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

(II) **Taylor's Theorem**
Let $f : (a,b) \to \mathbb{R}$ be n-times differentiable. Then $\forall x, c \in (a,b)$, $\exists x_0$ between x and c such that

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n-1)}(c)}{(n-1)!}(x-c)^{n-1} + \underbrace{\frac{f^n(x_0)}{n!}(x-c)^n}_{\text{the remainder in Lagrange form}}$$

We call it the n^{th} Taylor expansion of f about c

The following are some expansions at $c = 0$ that you should remember:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_{n+1}(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + R_{n+1}(x) \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{2n!} + R_{2n+2}(x) \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + R_{2n+3}(x) \\
(1+x)^a &= 1 + \sum_{k=1}^{n} \binom{a}{k} x^k + R_{n+1}(x) \qquad\qquad \text{where } a \in \mathbb{N} \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^n x^n}{n} + R_{n+1}(x) \\
\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + R_{2n+3}(x) \\
\arcsin x &= x + \sum_{k=1}^{n} \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 5 \cdots (2k)} \frac{x^{2k+1}}{2k+1} + R_{2n+3}(x)
\end{aligned}$$

Problem 1 Show that for $x \geq 0$, $\tan^{-1} x \leq e^x - 1$.

Solution:

We can do it by 2 approaches.

Solution 1: Note that $e^x - 1 - \tan^{-1} x = (e^x - \tan^{-1} x) - (e^0 - \tan^{-1} 0)$. Clearly it's differentiable for $x > 0$. Applying mean value theorem, there exists $c \in (0, x)$ such that

$$e^x - 1 - \tan^{-1} x = (e^x - \tan^{-1} x) - (e^0 - \tan^{-1} 0) = \left(e^c - \frac{1}{1+c^2}\right)(x - 0).$$

Then we consider $h : [0, \infty) \to \mathbb{R}$ defined by $h(y) = e^y - \dfrac{1}{1+y^2}$.

Since $h'(y) = e^y + \dfrac{2y}{(1+y^2)^2} > 0$ on $[0, \infty)$, $h(y)$ is increasing. Thus, $h(y) \geq h(0)$ for $y > 0$.

Since $c > 0$, we have

$$e^c - \frac{1}{1+c^2} = h(c) \geq h(0) = e^0 - \frac{1}{1+0^2} = 0.$$

Thus, for $x \geq 0$, $\tan^{-1} x \leq e^x - 1$.

Solution 2: It's clear that $\tan^{-1} x = e^x - 1$ for $x = 0$. For $x > 0$, we may consider the generalized mean value theorem on $(0, x)$.

For $x > 0$, take $f : [0, x] \to \mathbb{R}$ given by $f(y) = \tan^{-1} y$ and $g : [0, x] \to \mathbb{R}$ given by $g(y) = e^y$. Then f, g are continuous on $[0, x]$ and differentiable on $(0, x)$. By generalized mean value theorem, there exist $c \in (0, x)$ such that,

$$\frac{\tan^{-1} x}{e^x - 1} = \frac{\tan^{-1} x - \tan^{-1}(0)}{e^x - 1} = \frac{\left(\frac{1}{1+c^2}\right)}{e^c} = \frac{1}{e^c(1+c^2)}$$

Since e^x is an increasing function and $c > 0$, we see that $e^c \geq e^0 = 1$, so

$$\frac{\tan^{-1} x}{e^x - 1} = \frac{1}{e^c(1+c^2)} \leq 1.$$

Therefore for every $x \geq 0$, $\tan^{-1} x \leq e^x - 1$.

Problem 2

(i) Suppose $0 < a < b$, let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b). Prove that $\exists c \in (a, b)$ such that

$$f(b) - f(a) = cf'(c) \ln\left(\frac{b}{a}\right).$$

(ii) For $b > 1$, define $x_n = n(b^{\frac{1}{n}} - 1)$. Prove that x_n converges to $\ln b$.

Solution:

(i) Note that

$$f(b) - f(a) = cf'(c) \ln\left(\frac{b}{a}\right) \iff f(b) - f(a) = cf'(c)(\ln b - \ln a)$$
$$\iff \frac{f(b) - f(a)}{\ln b - \ln a} = \frac{f'(c)}{\frac{1}{c}}$$

We simply apply the generalized mean value theorem to $f(x)$ and $g(x) = \ln x$. Then there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{\ln b - \ln a} = \frac{f'(c)}{(\ln x)'|_{x=c}} = \frac{f'(c)}{\frac{1}{c}} \Rightarrow f(b) - f(a) = cf'(c) \ln\left(\frac{b}{a}\right)$$

(ii) **Scratch:**
We may check it directly by definition.

$$\begin{aligned}|x_n - \ln b| &= |n(b^{\frac{1}{n}} - 1) - \ln b| \\ &= |n(b^{\frac{1}{n}} - 1) - n(1^{\frac{1}{n}} - 1) - \ln b| \\ &= |f(b) - f(1) - \ln b| \qquad\qquad \text{take } f(x) = n(x^{\frac{1}{n}} - 1) \text{ and apply (i)} \\ &= \left| c\left[n\left(\frac{1}{n}\right)c^{\frac{1}{n}-1}\right]\ln b - \ln b\right| \\ &= (c^{\frac{1}{n}} - 1)\ln b \\ &\le (b^{\frac{1}{n}} - 1)\ln b\end{aligned}$$

Requiring $n > \dfrac{\ln b}{\frac{\varepsilon}{\ln b} + 1}$, we get $(b^{\frac{1}{n}} - 1)\ln b < \varepsilon$.
Solution:
For each fixed n, let $f : [1, b] \to \mathbb{R}$ given by $f(x) = n(x^{\frac{1}{n}} - 1)$. Clearly f is continuous on $[1, b]$ and differentiable on $(1, b)$.
Then by part (i), there exists $c \in (1, b)$ such that

$$f(b) - f(1) = c\left(n\left(\frac{1}{n}\right)c^{\frac{1}{n}-1}\right)\ln b = c(c^{\frac{1}{n}-1})\ln b = c^{\frac{1}{n}}\ln b.$$

$\forall \varepsilon > 0$, by Archimedean principle, there exist $K \in \mathbb{N}$, such that $K > \dfrac{\ln b}{\frac{\varepsilon}{\ln b} + 1}$. Then for any $n \ge K$,

$$\begin{aligned}|x_n - \ln b| &= |n(b^{\frac{1}{n}} - 1) - \ln b| \\ &= |n(b^{\frac{1}{n}} - 1) - n(1^{\frac{1}{n}} - 1) - \ln b| \\ &= |f(b) - f(1) - \ln b| \\ &= \left|c^{\frac{1}{n}}\ln b - \ln b\right| \\ &= (c^{\frac{1}{n}} - 1)\ln b \\ &\le (b^{\frac{1}{n}} - 1)\ln b \\ &< \varepsilon\end{aligned}$$

By definition of limit, x_n converges to $\ln b$.

Problem 3 (Adapted from Rudin) Supposed f is real, three-time differentiable function on $[0, 1]$, such that

$$f(-1) = 0, \qquad f(0) = 0, \qquad f(1) = 1, \qquad f'(0) = 0.$$

Prove that $f'''(x) \ge 3$ for some $x \in (-1, 1)$.
Solution:
Since we don't have enough information about all higher derivatives as in the previous problem, we should try Taylor's theorem.

We have the value of f' at 0, so we expand f about 0 and evaluate it at ± 1.
By Taylor's theorem, there exist $r \in (-1, 0)$ and $s \in (0, 1)$ such that

$$\begin{aligned}1 = f(1) &= f(0) + \frac{f'(0)}{1!}(1-0) + \frac{f''(0)}{2!}(1-0)^2 + \frac{f'''(r)}{3!}(1-0)^3 \\ 0 = f(-1) &= f(0) + \frac{f'(0)}{1!}(-1-0) + \frac{f''(0)}{2!}(-1-0)^2 + \frac{f'''(s)}{3!}(-1-0)^3\end{aligned}$$

In other words,

$$1 = \frac{f''(0)}{2} + \frac{f'''(r)}{6} \tag{1}$$

$$0 = \frac{f''(0)}{2} - \frac{f'''(s)}{6} \tag{2}$$

(1) − (2) gives

$$f'''(r) + f'''(s) = 6$$

This implies at least one of the $f'''(r), f'''(s)$ is greater than or equal to 3. (Otherwise $f'''(r) + f'''(s) < 6$.) Therefore, $f'''(x) \geq 3$ for some $x \in (-1, 1)$.

Problem 4 (Adapted from Rudin) Suppose $a \in \mathbb{R}$, f is a twice-differentiable real valued function on (a, ∞) with $M_0, M_1, M_2 < \infty$ are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$ respectively on (a, ∞).
Prove that $M_1^2 \leq M_0 M_2$.
Solution:
For any $h > 0$, consider the Taylor expansion at $x + 2h$ about x. Then there exist $\xi \in (x, x + 2h)$ such that

$$\begin{aligned}
& f(x+2h) = f(x) + \frac{f'(x)}{1!}(2h) + \frac{f''(\xi)}{2!}(2h)^2 \\
\Rightarrow \qquad & f'(x) = \frac{f(x+2h) - f(x)}{2h} - f''(\xi)2h \\
\Rightarrow \qquad & |f'(x)| \leq \left| \frac{f(x+2h) - f(x)}{2h} \right| + |f''(\xi)|2h \\
& \qquad \leq \frac{M_0}{h} + M_2 h
\end{aligned}$$

Then consider $g : (0, \infty) \to \mathbb{R}$ given by $g(h) = \dfrac{M_0}{h} + M_2 h$. Since $g'(h) = -\dfrac{M_0}{h^2} + M_2$ and $g''(h) = \dfrac{M_0}{h^3} > 0$, $g(h) = \dfrac{M_0}{h} + M_2 h$ attains its minimum $2\sqrt{M_0 M_2}$ at $h = \sqrt{\dfrac{M_0}{M_2}}$ $\left(\text{as } 0 = g'(h) = -\dfrac{M_0}{h^2} + M_2 \Rightarrow \; h = \sqrt{\dfrac{M_0}{M_2}}\right)$.
So $|f'(x)| \leq 2\sqrt{M_0 M_2}$ for all $x \in \mathbb{R}$.
Thus, $M_1 \leq 2\sqrt{M_0 M_2}$, i.e. $M_1^2 \leq 4 M_0 M_2$.

To see the equality can actually hold, we consider the following example.

Take $a = -1$, and define

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0) \\ \dfrac{x^2 - 1}{x^2 + 1} & (0 \leq x < \infty) \end{cases}$$

We could check that $M_0 = 1$, $M_1 = 4$ and $M_2 = 4$. (This verification is left as an exercise.)