Math 2033 (Mathematical Analysis) **Spring 2016**

Optional Homework # 3 – Due Thursday, May 5, 2016 at noon

Please submit your homework to my office in Room 3471 (by slipping it under the door if I am not in the office). *Be sure to write your name (as shown on your student ID card) and your tutorial section number on the homework! Give full details of your solutions.* **Take pictures of your homework and submit the original.**

1. (2013 Spring Exam) Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of <u>positive</u> real numbers. For $n = 1, 2, 3, \ldots$, let $b_n = \sqrt{\dfrac{a_n}{a_n + 3} + 5}$. Prove that $b_1, b_2, b_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

2. (2013 Spring Exam) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that for all $x \in \mathbb{R}$,
$$f\big(1 - f(x)\big) = 1 - x^9.$$
If $f(1) = 0$ and $f'(1) < 0$, then prove that there exists $r \in \mathbb{R}$ such that $f(r) = r^{2013}$.

3. (2010 Spring Exam) Let $f : (-1, 1) \to \mathbb{R}$ be four times differentiable such that for all $c \in (-1, 1)$, $|f^{(4)}(c)| \le 1$. Prove that for all $x \in (0, 1)$, we have
$$\Big| f''(0) - \frac{f(x) - 2f(0) + f(-x)}{x^2} \Big| \le \frac{x^2}{12}.$$

4. (2009 Spring Exam) Let S be a set of measure 0. Prove that $T = \{2x : x \in S\}$ is also a set of measure 0. Let $f : [0, 1] \to [0, 1]$ be a Riemann integrable function. Prove that $g : [0, 2] \to [0, 1]$ defined by $g(x) = f(x/2)$ is Riemann integrable on $[0, 1]$.