Review: Countability, countably infinite.

$\{0,1\}\times\{0,1\}\times\ldots\times\{0,1\}\times\ldots$ irrational

Uncountability. (how to prove $(0,1)$ is uncountable).

uncountable: $(0,1)$, $\underline{\mathbb{R}}$, $\mathbb{R}\setminus\mathbb{Q}$,

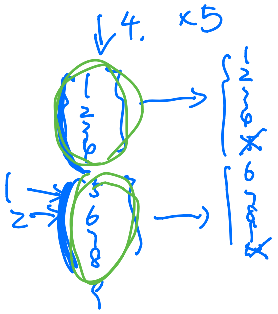
countable: $\mathbb{N}$, $\mathbb{Z}$, $2\mathbb{Z}$, ...

Problem1: Prove that the following sets are <u>countably infinite</u> using the definition.

$f: \mathbb{N} \mapsto \underline{A} / B$ bijective.

(a) $A = \{n \in \mathbb{N} \mid n \text{ is not multiple of } 5\}$

(b) $B = \{n \in \mathbb{Z} \mid n \text{ is odd number}\}$



Proof: (a). Construct a bijective function from $\mathbb{N} \mapsto A$.

$A = \{\underline{1, 2, 3, 4}, \underline{6, 7, 8, 9}, 11, \ldots\}$ $12, 13, 14, \cancel{15}, 16$

$\mathbb{N} = \{\underline{1, 2, 3, 4}, \underline{5, 6, 7, 8}, 9, \ldots\}$ $10, 11, 12, 13$

$n \div 4 = [\frac{n}{4}] \ldots a$

$a = 0, 1, 2, 3.$

$[\frac{n}{4}]$: the largest integer $< \frac{n}{4}$.

$f: \mathbb{N} \mapsto A$.

$n = 13$, $[\frac{n}{4}] = 3 \times 5$ $\quad 13 - 3\cdot 4$ $\quad n - [\frac{n}{4}]\cdot 4 = 1.$ $\quad 16.$

$f(n) = [\frac{n}{4}]\cdot 5 + (n - [\frac{n}{4}]\cdot 4) = n + [\frac{n}{4}]$.

$n = 2$, $[\frac{2}{4}] = 0$, $2 - [\frac{2}{4}]\cdot 4 = 2$, $= 2$

(b). Construct a bijective function from $\mathbb{N} \to B$.

$B = \{1, -1, 3, -3, 5, -5, \ldots\}$

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$.

$g: \mathbb{N} \mapsto B$

$$g(n) = \begin{cases} n & n \text{ is odd.} \\ -n+1 & n \text{ is even.} \end{cases}$$

Problem2: We let $A_1, A_2, A_3, \ldots$ be subsets of $\mathbb{R}$. Suppose that the set

$$S = A_1 \times A_2 \times A_3 \times \ldots$$

is countable. Prove that there are only finitely many sets that have more than one elements.

$\{0,1\}\times\{0,1\}\times\ldots\times\{0,1\}\times\ldots$ uncountable.

similar.

Proof: Prove by contradiction:

① $(0,1)$

$a_1 = 0.a_{11}a_{12}\cdots a_{1n}\cdots$

$a_2 = 0.a_{21}a_{22}\cdots a_{2n}\cdots$

$\vdots$

$a_k = 0.a_{k1}a_{k2}\cdots a_{kn}\cdots$

$\vdots$

② p/q, $p\in\mathbb{Z}$, $q\in\mathbb{Z}$, $q\neq 0$.

$1/1 \quad 1/2 \quad 1/3$

$2/1 \quad 2/2 \quad 2/3$

$\vdots$

③. construct a map to a known set.

$S \leftrightarrow T$.

Suppose there are infinite many sets that have more than one elements: $A_{k_1}, A_{k_2}, \dots, A_{k_n}, \dots$

$A_{k_1} = \{a_{k_1,1}, a_{k_1,2}, \dots\}$ $\qquad a_{k_i,1} \neq a_{k_i,2}$.

$A_{k_2} = \{a_{k_2,1}, a_{k_2,2}, \dots\}$.

$\vdots$

$A_{k_n} = \{a_{k_n,1}, a_{k_n,2}, \dots\}$

S is countable:

$S = \{s_1, s_2, \dots, s_n, \dots\}$,

$s_1 = (\dots, s_{k_1}^{(1)}, \dots, s_{k_2}^{(1)}, \dots, \dots, s_{k_n}^{(1)}, \dots)$

$\vdots$

$s_n = (\dots, s_{k_1}^{(n)}, \dots, s_{k_2}^{(n)}, \dots, s_{k_n}^{(n)}, \dots)$.

$\vdots$

Construct:

$t \in S$,

$$t_{k_i} = \begin{cases} a_{k_i,1} & \text{if } s_{k_i}^{(i)} \neq a_{k_i,1} \\ a_{k_i,2} & \text{if } s_{k_i}^{(i)} = a_{k_i,1} \end{cases}$$

Then $t \neq s_k$ for $k = 1, 2, \dots, n, \dots$

$\Rightarrow$ $t \notin S = \{s_1, s_2, \dots, s_k, \dots\}$.

$\Rightarrow$ Our assumption is not right.

$\Rightarrow$ There are only finitely many sets that have more than one elements.

$\{0,1\} \times \{0,1\} \times \dots \times \{0,1\} \times \dots$ is uncountable.

Problem 3: $f_1, f_2, f_3, \dots : \mathbb{R} \mapsto [0, +\infty)$ are a collection of functions from $\mathbb{R}$ to $[0,+\infty)$. For any positive integer n, we define

$$A_n = \{x \in \mathbb{R} : f_n(x) = 0\}.$$

We consider a set defined by

$$S = \{x \in \mathbb{R} \mid \sum_{n=1}^{\infty} f_n(x) = 0\}.$$

(a) Show that if A_n is countable for some $n \in \mathbb{N}$, then S is countable.

(b) Suppose A_n is uncountable for all $n \in \mathbb{N}$. Is S always countable or uncountable?

(a). $f_k : \mathbb{R} \mapsto [0, +\infty)$. $k = 1, 2, \ldots$

If $\sum_{n=1}^{+\infty} f_n(x) = 0$, then $f_n(x) = 0$, $n = 1, 2, \ldots$

$$\Rightarrow S = \{x \in \mathbb{R} \mid \sum_{n=1}^{\infty} f_n(x) = 0\} = \{x \in \mathbb{R} \mid f_n(x) = 0, \forall n = 1, 2, \ldots\}$$

$$= \{x \in \mathbb{R} \mid f_1(x) = 0, f_2(x) = 0, \ldots, f_n(x) = 0, \ldots\}$$

$$= A_1 \cap A_2 \cap \ldots \cap A_n \cap \ldots$$

$$= \bigcap_{n=1}^{+\infty} A_n.$$

If A_n is countable for some n, then S is countable.

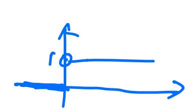
(b).

- Construct $f_n = 0$, $n = 1, 2, \ldots$

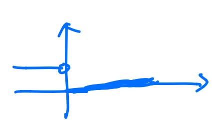
  $A_n = \mathbb{R}$. $S = \mathbb{R}$. S is uncountable.

  $\forall x \in \mathbb{R}$, $f_n(x) \geq 0$.

- $f_1(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$



$f_k(x) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0. \end{cases}$



$k = 2, 3, \ldots$

A_n is uncountable.

$S = \bigcap_{n=1}^{+\infty} A_n$ is countable

$= \{0\}$.

Problem 4: Determine if the set C defined by

$$C = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid x^2+y^2=r^2\}$$

is countable.

$f: [0, 2\pi) \longmapsto C$

$\theta \longmapsto (x,y)$

$f(\theta) = (r\cos\theta, r\sin\theta)$

f is a bijective function.

For $[0, 2\pi)$ is uncountable

$\Rightarrow$ C is uncountable.

Problem 5: (a) Determine if the set D defined by

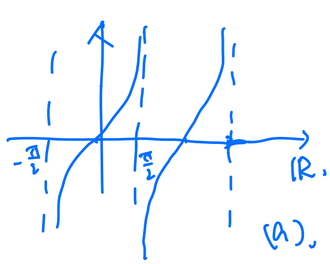
$$D = \{x \in \mathbb{R};\ \tan^{10}x - 3\tan^3 x + 1 = 0\}$$

is countable.

(b) Determine if the set E defined by

$$E = \{x \in \mathbb{R} \mid a\cos 2x + b\cos x + c = 0 \text{ for some } a, b, c \in \mathbb{Q}\setminus\{0\}\}$$

is countable.



(a).

$F = \{t \in \mathbb{R},\ t^{10} - 3t + 1 = 0\} \longrightarrow$ Finite set.

$D = \bigcup_{n=0}^{+\infty} \left(D_n^{(1)} \cup D_n^{(2)}\right)$ $\quad D_n^{(1)} = \{x \mid x \in (n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}),\ \tan x \in F\}$

$D_n^{(3)} = \{x \mid x \in (-n\pi - \frac{\pi}{2}, -n\pi + \frac{\pi}{2}),\ \tan x \in F\}.$

$D_n^{(1)} \cup D_n^{(3)}$: Finite set.

$\Rightarrow$ D : countable set.

(b). $E = \{x \in \mathbb{R} \mid a\cos 2x + b\cos x + c = 0$, for some $a, b, c \in \mathbb{Q} \setminus \{0\}\}$

$$\begin{aligned} & a\cos 2x + b\cos x + c \\ &= a(2\cos^2 x - 1) + b\cos x + c \\ &= 2a\cos^2 x + b\cos x + c - 2a. \end{aligned}$$

$\cos x = t.$

$G_{a,b,c} = \{t \in \mathbb{R} \mid 2at^2 + bt + c - 2a = 0\}$. $\longrightarrow$ finite.

$$E = \bigcup_{a \in \mathbb{Q}} \bigcup_{b \in \mathbb{Q}} \bigcup_{c \in \mathbb{Q}} \bigcup_{n=-\infty}^{+\infty} E_{n,a,b,c}.$$

$E_{n,a,b,c} = \{x \mid x \in [2n\pi, 2n\pi + 2\pi),\ \cos x \in G_{a,b,c}\}.$

$E_{n,a,b,c}$ is finite set.

$\Rightarrow$ E is countable.

Problem 8: We let A be collection of functions from the set of positive integers $\mathbb{N}$. That is

$$A = \{f \mid f : \{0,1\} \mapsto \mathbb{N}\}.$$

Show that A is countable.

Proof: $F : A \mapsto \mathbb{N} \times \mathbb{N}.$

$f \in A$

$F(f) = (f(0), f(1)).$

F is a bijective function.

$\mathbb{N} \times \mathbb{N}$: countable

$\Rightarrow$ A : countable

Open problem:

A: set

$\mathcal{P}(A)$: power set of A.

Does there exist a bijective function:

$$f : A \mapsto \mathcal{P}(A). ?$$