2008 Spring Final

① Determine the domain of convergence of

$$f(x)=\sum_{k=1}^{\infty}\frac{k^2}{3^k}(\pi-2x)^k.$$

Solution.

$$\lim_{k\to\infty}\left|\frac{(k+1)^2}{3^{k+1}}\frac{(\pi-2x)^{k+1}3^k}{k^2(\pi-2x)^k}\right|=\lim_{k\to\infty}\left(\frac{k+1}{k}\right)^2\left|\frac{\pi-2x}{3}\right|=\left|\frac{\pi-2x}{3}\right|<1$$

$$\Leftrightarrow |x-\tfrac{\pi}{2}|<\tfrac{3}{2} \Leftrightarrow x\in\left(\tfrac{\pi}{2}-\tfrac{3}{2},\ \tfrac{\pi}{2}+\tfrac{3}{2}\right).$$

When $\frac{\pi-2x}{3}=\pm1$, $f(x)=\sum_{k=1}^{\infty}\underbrace{k^2(\pm1)^k}_{\text{limit}\neq 0}$ diverges by term test.

So domain $=\left(\frac{\pi}{2}-\frac{3}{2},\ \frac{\pi}{2}+\frac{3}{2}\right)$.

② Determine if improper integral $\int_{-1}^{1}\frac{x\,dx}{\sin^2 x}$ converges.

Determine if P.V. $\int_{-1}^{1}\frac{x\,dx}{\sin^2 x}$ converges.

Solution. (Near 0, $\sin x\sim x$, $\frac{x}{\sin^2 x}\sim\frac{1}{x}$.)

$$\lim_{x\to0}\frac{x/\sin^2x}{1/x}=\lim\left(\frac{x}{\sin x}\right)^2=1,\quad \int_0^1\frac{1}{x}dx=\ln x\Big|_0^1=\infty$$

$$\underset{\text{limit comparison test}}{\overset{\uparrow}{\Rightarrow}} \int_0^1\frac{x}{\sin^2x}dx=\infty \Rightarrow \int_{-1}^{1}\frac{x\,dx}{\sin^2x}=\underbrace{\int_{-1}^{0}\frac{x\,dx}{\sin^2x}}_{\text{diverges}}+\underbrace{\int_0^1\frac{x\,dx}{\sin^2x}}_{\text{diverges}}$$

$$\text{P.V.}\int_{-1}^{1}\frac{x}{\sin^2x}dx=\lim_{\varepsilon\to0^+}\Bigg(\underbrace{\int_{-1}^{-\varepsilon}\frac{x}{\sin^2x}dx+\int_{\varepsilon}^{1}\frac{x}{\sin^2x}dx}_{x/\sin^2x\text{ is an odd function.}}\Bigg)=0$$

③ Prove $\sum_{k=1}^{\infty} \left(\frac{kx}{1+k^2x^2}\right)^k$ converges uniformly on $\mathbb{R}$.

Solution. $\forall x \in \mathbb{R}, x \neq 0$, $\left|\frac{kx}{1+k^2x^2}\right| \underset{\text{AM-GM inequality}}{\leq} \frac{k|x|}{2k|x|} = \frac{1}{2}$. Also $\left|\frac{k0}{1+k^2 0^2}\right| \leq \frac{1}{2}$.

(**Alternatively**, $\frac{d}{dx}\left(\frac{kx}{1+k^2x^2}\right) = \frac{k(1-k^2x^2)}{(1+k^2x^2)^2} = 0 \Leftrightarrow x = \pm\frac{1}{k}$

$\lim_{x\to\pm\infty} \frac{kx}{1+k^2x^2} = 0$, $\frac{k(\pm 1/k)}{1+k^2(1/k)^2} = \pm\frac{1}{2}$. $\therefore \left|\frac{kx}{1+k^2x^2}\right| \leq \frac{1}{2}$)

So $\left|\left(\frac{kx}{1+k^2x^2}\right)^k\right| \leq \left(\frac{1}{2}\right)^k = M_k$

$\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ converges by geometric series test.

$\therefore \sum_{k=1}^{\infty}\left(\frac{kx}{1+k^2x^2}\right)^k$ converges uniformly on $\mathbb{R}$ by the Weierstrass M-test.

④ Let $\{x_n\}$, $\{y_n\}$ be two Cauchy sequences of real numbers. Prove that $\{\sqrt{x_n^2+y_n^2}\}$ is a Cauchy sequence.

Solution. Note $|\sqrt{a}-\sqrt{b}| \leq \sqrt{|a-b|}$ $\quad a \geq 0, b \geq 0$

$$\left|\sqrt{x_n^2+y_n^2} - \sqrt{x_m^2+y_m^2}\right| \leq \sqrt{|(x_n^2+y_n^2)-(x_m^2+y_m^2)|} \qquad (*)$$

$$\leq \sqrt{\underbrace{|x_n^2-x_m^2|}_{=|x_n+x_m||x_n-x_m|} + \underbrace{|y_n^2-y_m^2|}_{|y_n+y_m||y_n-y_m|}}$$

Since Cauchy sequences are bounded, $\exists M_1, M_2 > 0$ such that $|x_n| \leq M_1$ and $|y_n| \leq M_2$ for all n.

For every $\varepsilon > 0$, $\exists K_1, K_2 \in \mathbb{N}$ such that

$m,n \geq K_1 \Rightarrow |x_n - x_m| \leq \varepsilon^2/4M_1$

$m,n \geq K_2 \Rightarrow |y_n - y_m| \leq \varepsilon^2/4M_2$

Let $K = \max\{K_1, K_2\}$. Then

$m,n \geq K \Rightarrow m,n \geq K_1$ and $m,n \geq K_2$

$$\Rightarrow \left|\sqrt{x_n^2+y_n^2} - \sqrt{x_m^2+y_m^2}\right|$$

$$\underset{\text{by }(*)\text{ above}}{\leq} \sqrt{|x_n^2-x_m^2| + |y_n^2-y_m^2|}$$

$$< \sqrt{2M_1\frac{\varepsilon^2}{4M_1} + 2M_2\frac{\varepsilon^2}{4M_2}} = \varepsilon.$$

(5) (a) State Lebesgue's Theorem.

(b) For $n=1,2,3,\cdots$, let $f_n:[0,1]\to[0,1]$ be integrable. Prove that $g:[0,1]\to\mathbb{R}$ defined by $g(0)=0$ and $g(x)=f_n(x)$ for $n=1,2,3,\cdots$ and $x\in(\frac{1}{n+1},\frac{1}{n}]$ is Riemann integrable on $[0,1]$.

Solution.

(a) A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable iff $S_f=\{x\in[a,b]: f \text{ is discontinuous at } x\}$ is of measure 0 (i.e. f is continuous almost everywhere).

(b) Since f_n is Riemann integrable on $[0,1]$, S_{f_n} is of measure 0. Then $S_{f_n}\cap(\frac{1}{n+1},\frac{1}{n}]$ is of measure 0. Now

$$\underbrace{S_g \subseteq \underbrace{\{0,1,\tfrac{1}{2},\tfrac{1}{3},\cdots\}}_{\substack{\text{countable}\\ \Rightarrow \text{measure } 0}} \cup \underbrace{\bigcup_{n=1}^{\infty}\underbrace{\left(S_{f_n}\cap\left(\tfrac{1}{n+1},\tfrac{1}{n}\right]\right)}_{\text{measure } 0}}_{\text{measure } 0}}_{\text{measure } 0}$$

$\therefore$ S_g is of measure 0.

$\therefore$ g is Riemann integrable on $[0,1]$.

(6) Let $a_1,a_2,a_3,\cdots\in\mathbb{R}$ and s_n be the n^{th} partial sum of the convergent series $\sum_{k=1}^{\infty}a_k = a\in\mathbb{R}$. Prove that $\lim_{n\to\infty}\frac{a_1+2a_2+3a_3+\cdots na_n}{n}=0$.

Solution. Note $\sum_{k=1}^{\infty}a_k=\lim_{n\to\infty}s_n\in\mathbb{R}$ since $\sum_{k=1}^{\infty}a_k$ converges.

$$\lim_{n\to\infty}\frac{a_1+2a_2+3a_3+\cdots+na_n}{n}$$

$$=\lim_{n\to\infty}\frac{s_1+2(s_2-s_1)+3(s_3-s_2)+\cdots+n(s_n-s_{n-1})}{n}$$

$$=\lim_{n\to\infty}\frac{ns_n-s_1-s_2-\cdots-s_{n-1}}{n}$$

$$=\lim_{n\to\infty}\left(s_n-\frac{s_1+s_2+\cdots+s_{n-1}}{n}\right)$$

$$=\lim_{n\to\infty}s_n-\lim_{n\to\infty}\frac{s_1+s_2+\cdots+s_{n-1}}{n}$$

by Stolz' theorem, $n\nearrow\infty$

$$=\lim_{n\to\infty}s_n-\lim_{n\to\infty}\frac{s_n}{1}$$

$$=0. \quad =a-a =0.$$

⑦ Let $f:\mathbb{R}\to\mathbb{R}$ be a twice differentiable function such that $f''(x)$ is continuous and $|f''(x)|\le 1$ for all $x\in[0,1]$. If $f(\frac{1}{2})=0$, then prove that $\left|\int_0^1 f(x)\,dx\right|\le\frac{1}{24}$.

Solution. By Taylor's theorem, $\exists\,\theta_x$ between x and $\frac{1}{2}$ such that

$$f(x)=\underbrace{f(\tfrac{1}{2})}_{=0}+f'(\tfrac{1}{2})(x-\tfrac{1}{2})+\frac{f''(\theta_x)}{2}(x-\tfrac{1}{2})^2.$$

then

Not constant

$$\int_0^1 f(x)\,dx=\underbrace{\int_0^1 f'(\tfrac{1}{2})(x-\tfrac{1}{2})\,dx}_{=f'(\frac{1}{2})(\frac{x^2}{2}-\frac{1}{2}x)\big|_0^1=0}+\int_0^1\frac{f''(\theta_x)}{2}(x-\tfrac{1}{2})^2\,dx$$

$$\therefore\left|\int_0^1 f(x)\,dx\right|=\left|\int_0^1\frac{f''(\theta_x)}{2}(x-\tfrac{1}{2})^2\,dx\right|$$

$$\le\int_0^1\left|\frac{f''(\theta_x)}{2}(x-\tfrac{1}{2})^2\right|dx$$

$$\le\frac{1}{2}\int_0^1(x-\tfrac{1}{2})^2\,dx$$

$$=\frac{1}{2}\,\frac{(x-\frac{1}{2})^3}{3}\Big|_0^1$$

$$=\frac{1}{24}.$$