Mathematical Induction

Mathematical Induction (M. I.) For every positive integer n, let $P(n)$ be a statement which is either true or false.

(a) We check $P(1)$ is true.

(b) For every $n = 1, 2, 3, \ldots$, if $P(n)$ is true, then $P(n+1)$ is true.

Then $P(1), P(2), P(3), \ldots$ are all true.

(*Reason* By (a), $P(1)$ is true. By (b), if $P(1)$ is true, then $P(2)$ is true. So $P(2)$ is true. By (b), if $P(2)$ is true, then $P(3)$ is true. So $P(3)$ is true. Keep on repeating this. We get $P(1), P(2), P(3), \ldots$ are all true.)

Examples of Mathematical Induction

(1) Prove that for every $n = 1, 2, 3, \ldots$, we have $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Solution. Let $P(n)$ be the statement $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

(a) $P(1)$ is $1^2 = \dfrac{1(1+1)(2 \cdot 1+1)}{6}$, which is $1 = 1$, hence true.

(b) If $P(n)$ is true, then $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$. Adding $(n+1)^2$ to both sides, we have

$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = (n+1)\Big[\frac{n(2n+1)}{6} + n + 1\Big] \\ &= (n+1)\Big[\frac{2n^2 + n + 6n + 6}{6}\Big] = \frac{(n+1)(n+2)(2n+3)}{6}. \end{aligned}$$

Then $P(n+1)$ is true. By M.I., we are done.

For the following examples, we will use some facts about inequalities of real numbers. Let us list these facts:

(i) If $a \in \mathbb{R}$ and $c < d$, then we can add or subtract both sides by a to get $c + a < d + a$ and $c - a < d - a$.

(ii) If $p \leq q$ and $r < s$, then $p + r < q + s$. However, we may not subtract both sides to get $p - r < q - s$ (for example, $0 < 1$ and $0 < 2$, but $0 - 0 < 1 - 2$ is wrong!)

(iii) If $a > 0$ and $c < d$, then $ac < ad$. If $b < 0$, and $c < d$, then $bc > bd$.

(iv) If $x > y > 0$, then $\dfrac{1}{x} < \dfrac{1}{y}$. However, $a > b > 0$ and $c > d > 0$ do not imply $\dfrac{a}{c} > \dfrac{b}{d}$ (for example, $10 > 1$ and $100 > 1$, but $\dfrac{10}{100} > \dfrac{1}{1}$ is wrong!)

(v) If $x > y > 0$, then $\sqrt{x} > \sqrt{y} > 0$.

With these facts about ineqiality, we will do two more examples of mathematical induction.

(2) Let $x_1 = 1$ and $x_{n+1} = \dfrac{x_n}{2} + \sqrt{x_n}$ for $n = 1, 2, 3, \ldots$. Prove that $0 < x_n < x_{n+1}$ for $n = 1, 2, 3, \ldots$.

Solution. Let $P(n)$ be the statement $0 < x_n < x_{n+1}$.

(a) We check $P(1)$, namely $0 < x_1 < x_2$, which is $0 < x_1 = 1 < x_2 = \dfrac{x_1}{2} + \sqrt{x_1} = \dfrac{1}{2} + 1 = \dfrac{3}{2}$, hence true.

(b) If $0 < x_n < x_{n+1}$, then $0 < x_{n+1}$. Also, $\dfrac{x_n}{2} < \dfrac{x_{n+1}}{2}$ and $\sqrt{x_n} < \sqrt{x_{n+1}}$. Adding both sides, we have $\dfrac{x_n}{2} + \sqrt{x_n} < \dfrac{x_{n+1}}{2} + \sqrt{x_{n+1}}$, which is $x_{n+1} < x_{n+2}$. Then $0 < x_{n+1} < x_{n+2}$. By M.I., we are done.

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(3) Let $x_1 = 1$ and $x_{n+1} = \dfrac{2 - x_n}{3 + x_n}\Big(= \dfrac{5}{3 + x_n} - 1\Big)$. Prove that for all $k = 1, 2, 3, \ldots$, we have $0 < x_{2k} < x_{2k+2} < x_{2k+1} < x_{2k-1}$.

__Solution.__ (a) We check the case $k = 1$. Now $x_1 = 1,\ x_2 = \dfrac{2-1}{3+1} = \dfrac{1}{4},\ x_3 = \dfrac{2 - \frac{1}{4}}{3 + \frac{1}{4}} = \dfrac{7}{13},\ x_4 = \dfrac{2 - \frac{7}{13}}{3 + \frac{7}{13}} = \dfrac{19}{46}$. Then $0 < \dfrac{1}{4} < \dfrac{19}{46} < \dfrac{7}{13} < 1$. So $0 < x_2 < x_4 < x_3 < x_1$.

(b) If $0 < x_{2k} < x_{2k+2} < x_{2k+1} < x_{2k-1}$, then we need to show $0 < x_{2(k+1)} < x_{2(k+1)+2} < x_{2(k+1)+1} < x_{2(k+1)-1}$, i.e. $0 < x_{2k+2} < x_{2k+4} < x_{2k+3} < x_{2k+1}$.

From $0 < x_{2k} < x_{2k+2} < x_{2k+1} < x_{2k-1}$, we have $0 < 3 + x_{2k} < 3 + x_{2k+2} < 3 + x_{2k+1} < 3 + x_{2k-1}$. Taking reciprocal of the positive parts, we have $\dfrac{1}{3 + x_{2k}} > \dfrac{1}{3 + x_{2k+2}} > \dfrac{1}{3 + x_{2k+1}} > \dfrac{1}{3 + x_{2k-1}}$. Multiplying by 5 on all parts, we get $\dfrac{5}{3 + x_{2k}} > \dfrac{5}{3 + x_{2k+2}} > \dfrac{5}{3 + x_{2k+1}} > \dfrac{5}{3 + x_{2k-1}}$. Subtracting 1 in all parts, we get $\dfrac{5}{3 + x_{2k}} - 1 > \dfrac{5}{3 + x_{2k+2}} - 1 > \dfrac{5}{3 + x_{2k+1}} - 1 > \dfrac{5}{3 + x_{2k-1}} - 1$.

Using the definition of x_{n+1} and recalling $x_{2k} > 0$, we get $x_{2k+1} > x_{2k+3} > x_{2k+2} > x_{2k} > 0$. Repeating all the steps in the last paragraph once more, we get $0 < x_{2k+2} < x_{2k+4} < x_{2k+3} < x_{2k+1}$. By M.I., we are done.

__Exercises__

(1) Prove that for every positive integer n, we have $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$.

(2) Let $x_1 = 1$ and $x_{n+1} = 1 - \dfrac{1}{4x_n}$ for all $n = 1, 2, 3, \ldots$. Prove that for all $n = 1, 2, 3, \ldots$, we have $x_n > x_{n+1} > \dfrac{1}{2}$.

(3) Let $x_1 = 5$ and $x_{n+1} = 3 + \dfrac{4}{x_n}$ for all $n = 1, 2, 3, \ldots$. Prove that $x_{2k} < x_{2k+2} < x_{2k+1} < x_{2k-1}$ for all $k = 1, 2, 3, \ldots$.