Math 2033 (Mathematical Analysis) **Spring 2020**

Final Examination (Part B) – (Duration: 40 minutes)

Directions: **For every problem of this exam, detailed written works supported by correct reasons must be shown legibly to receive credits**.

Notations: $\mathbb{R}$ is the set of all real numbers.

Problems

4. (15 marks) Let $f, g : [0,1] \to \mathbb{R}$ be continuous and let $h(x) = f(x)e^{g(x)}$ for $x \in [0,1]$. Let f and g be differentiable on the interval $(0,1)$. If $f(0) = f(1) = 0$, then prove that there exists $t \in (0,1)$ such that

$$f'(t) + f(t)g'(t) = 0.$$

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5. (15 marks) Let $F : [a,b] \to \mathbb{R}$ be an increasing function. Prove that F is integrable on $[a,b]$ by checking the integral criterion.

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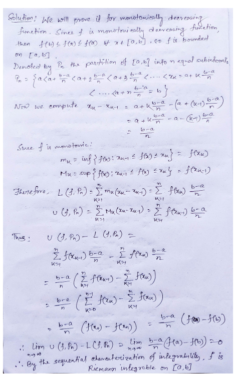
6. (20 marks) Let $f : (a,b) \to \mathbb{R}$ be twice differentiable such that for all $x \in (a,b)$,

$$|f''(x)| \leq |f(x)| + |f'(x)|$$

and there exists $c \in (a,b)$ satisfying $f(c) = f'(c) = 0$. Using Taylor's theorem, prove that $f(x) = 0$ for all $x \in (a,b)$.

– End of Part B –



Solution: We will prove it for monotonically decreasing function. Since f is monotonically decreasing function, then $f(b) \le f(x) \le f(a)$ $\forall x \in [a,b]$, so f is bounded on $[a,b]$.

Denoted by P_n the partition of $[a,b]$ into n equal subintervals

$$P_n = \left\{a < a + \frac{b-a}{n} < a + 2\frac{b-a}{n} < a + 3\frac{b-a}{n} < \cdots < x_k = a + k\frac{b-a}{n} < \cdots < a + n\frac{b-a}{n} = b\right\}$$

Now we compute $x_k - x_{k-1} = a + k\frac{b-a}{n} - \left(a + (k-1)\frac{b-a}{n}\right)$

$$= a + k\frac{b-a}{n} - a - (k-1)\frac{b-a}{n}$$

$$= \frac{b-a}{n}$$

Since f is monotonic:

$$m_k = \inf\{f(x) : x_{k-1} \le f(x) \le x_k\} = f(x_k)$$

$$M_k = \sup\{f(x) : x_{k-1} \le f(x) \le x_k\} = f(x_{k-1})$$

Therefore, $L(f, P_n) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_k)\frac{b-a}{n}$

$$U(f, P_n) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_{k-1})\frac{b-a}{n}$$

Thus: $U(f, P_n) - L(f, P_n) =$

$$\sum_{k=1}^{n} f(x_{k-1})\frac{b-a}{n} - \sum_{k=1}^{n} f(x_k)\frac{b-a}{n}$$

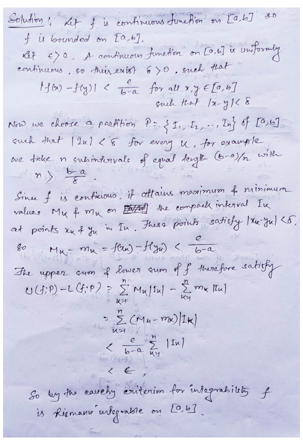
$$= \frac{b-a}{n}\left(\sum_{k=1}^{n} f(x_{k-1}) - \sum_{k=1}^{n} f(x_k)\right)$$

$$= \frac{b-a}{n}\left(\sum_{k=0}^{n-1} f(x_k) - \sum_{k=1}^{n} f(x_k)\right)$$

$$= \frac{b-a}{n}\left(f(x_0) - f(x_n)\right) = \frac{b-a}{n}\left(f(a) - f(b)\right)$$

$\therefore \lim_{n\to\infty} U(f, P_n) - L(f, P_n) = \lim_{n\to\infty} \frac{b-a}{n}\left(f(a) - f(b)\right) = 0$

$\therefore$ By the sequential characterization of integrability, f is Riemann integrable on $[a,b]$



Solution: Let f is continuous function on $[a,b]$ so f is bounded on $[a,b]$.

Let $\epsilon > 0$. A continuous function on $[a,b]$ is uniformly continuous, so their exist $\delta > 0$, such that

$$|f(x) - f(y)| < \frac{\epsilon}{b-a} \text{ for all } x, y \in [a,b] \text{ such that } |x-y| < \delta$$

Now we choose a partition $P = \{I_1, I_2, \ldots, I_n\}$ of $[a,b]$ such that $|I_k| < \delta$ for every k, for example we take n subintervals of equal length $(b-a)/n$ with $n > \frac{b-a}{\delta}$.

Since f is continuous, it attains maximum & minimum values M_k & m_k on the compact interval I_k at points x_k & y_k in I_k. These points satisfy $|x_k - y_k| < \delta$, so $M_k - m_k = f(x_k) - f(y_k) < \frac{\epsilon}{b-a}$

The upper sum & lower sum of f therefore satisfy

$$U(f;P) - L(f;P) = \sum_{k=1}^{n} M_k|I_k| - \sum_{k=1}^{n} m_k|I_k|$$

$$= \sum_{k=1}^{n} (M_k - m_k)|I_k|$$

$$< \frac{\epsilon}{b-a}\sum_{k=1}^{n} |I_k|$$

$$< \epsilon,$$

So by the cauchy criterion for integrability f is Riemann integrable on $[a,b]$.