# Solution of Math 2033 Final Exam (Spring 2015)

① $2^x r^3 = \pi^y \Leftrightarrow r = \sqrt[3]{\frac{\pi^y}{2^x}}$. The set $S = \left\{\sqrt[3]{\frac{\pi^y}{2^x}} : x, y \in \mathbb{Q}\right\} = \underbrace{\bigcup_{(x,y)\in\mathbb{Q}\times\mathbb{Q}}}_{\text{countable}} \underbrace{\left\{\sqrt[3]{\frac{\pi^y}{2^x}}\right\}}_{1\text{ element} \Rightarrow \text{countable}}$ is countable. Then $(0,+\infty)\setminus S$ is uncountable. So there are infinitely many such $r \in (0,+\infty)\setminus S$.

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② Sketch $|y_n - y_m| = \left|(x_{2n}-x_{2m}) - \left(\frac{x_n}{x_n+1} - \frac{x_m}{x_m+1}\right)\right| \le |x_{2n}-x_{2m}| + \left|\frac{x_n}{x_n+1} - \frac{x_m}{x_m+1}\right|$

$\le |x_{2n}-x_{2m}| + \frac{|x_n-x_m|}{(x_n+1)(x_m+1)} \le |x_{2n}-x_{2m}| + \frac{|x_n-x_m|}{(1+1)(1+1)}$

For every $\varepsilon > 0$, since $x_1, x_2, x_3, \dots$ is Cauchy in $[1,+\infty)$, there exists $K \in \mathbb{N}$ such that $n, m \ge K \Rightarrow |x_n - x_m| < \frac{4\varepsilon}{5}$. Then $n, m, 2n, 2m \ge K \Rightarrow |y_n - y_m| \le |x_{2n}-x_{2m}| + \frac{1}{4}|x_n - x_m| < \frac{4\varepsilon}{5} + \frac{1}{4}\frac{4\varepsilon}{5} = \varepsilon$. $\therefore \{y_n\}$ is Cauchy.

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③ Sketch As $x \to 1$, $f(x) \to 1$, $\frac{5\pi}{4}\sqrt[3]{2f(x)+6} \to \frac{5\pi}{4}\sqrt[3]{2+6} = \frac{5\pi}{2}$, $\sin\left(\frac{5\pi}{4}\sqrt[3]{2f(x)+6}\right) \to \sin\frac{5\pi}{2} = 1$.

$\left|\sin\left(\frac{5\pi}{4}\sqrt[3]{2f(x)+6}\right) - 1\right| = \left|\sin\left(\frac{5\pi}{4}\sqrt[3]{2f(x)+6}\right) - \sin\frac{5\pi}{2}\right| \le \left|\frac{5\pi}{4}\sqrt[3]{2f(x)+6} - \frac{5\pi}{2}\right| = \frac{5\pi}{4}\left|\sqrt[3]{2f(x)+6} - \underbrace{2}_{\sqrt[3]{8}}\right|$

$\le \frac{5\pi}{4}\sqrt[3]{|2f(x)+6-8|} = \frac{5\pi}{4}\sqrt[3]{2}\sqrt[3]{|f(x)-1|} < \varepsilon \Leftrightarrow |f(x)-1| < \left(\frac{4\varepsilon}{5\pi\sqrt[3]{2}}\right)^3 = \frac{32\varepsilon^3}{125\pi^3}$

For every $\varepsilon > 0$, since $\lim_{x\to 1} f(x) = 1$, there exists $\delta > 0$ such that $0 < |x-1| < \delta \Rightarrow |f(x)-1| < \frac{32\varepsilon^3}{125\pi^3} \Rightarrow \left|\sin\left(\frac{5\pi}{4}\sqrt[3]{2f(x)+6}\right) - 1\right| < \varepsilon$. $\therefore \lim_{x\to 1}\sin\left(\frac{5\pi}{4}\sqrt[3]{2f(x)+6}\right) = 1$.

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④ Since $f: [0,1] \to [0,1]$ is continuous injective, $f$ is strictly monotone. Since $f(0) < f(1)$, $f$ is strictly increasing. Cross-multiplying $\frac{1-f(x)}{1+f(x)} = \frac{x^2}{2-x^2}$ and simplifying, we get $f(x) = 1-x^2$. Now $g(x) = 1-x^2$ is strictly decreasing and continuous on $[0,1]$. So $h(x) = f(x) - (1-x^2)$ is strictly increasing and continuous. Using $0 \le f(0) < f(1) \le 1$, we have $h(0) = f(0) - 1 < 0$ and $h(1) = f(1) > 0$. By the intermediate value theorem, $h(x) = 0$ for some $x \in [0,1]$. Since $h$ is strictly increasing, there is exactly 1 solution.

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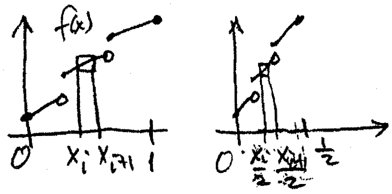
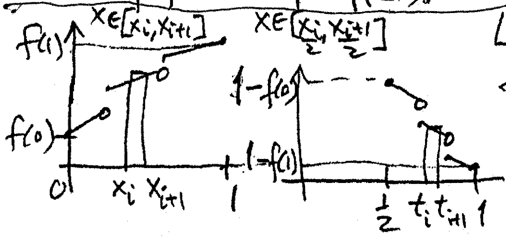
⑤ By Taylor's theorem, $\exists\, a \in (0,2)$ such that $f(2) = f(0) + f'(0)(2-0) + \frac{f''(0)}{2}(2-0)^2 + \frac{f'''(a)}{6}(2-0)^3$ and $\exists\, b \in (0,1)$ such that $f(1) = f(0) + f'(0)(1-0) + \frac{f''(b)}{2}(1-0)^2$. Subtracting these, we get $f(2) - f(1) = f'(0) + 2f''(0) - \frac{f''(b)}{2} + \frac{4}{3}f'''(a)$. Since $f(2) - f(1) = 2 = 2f''(0)$, we have $\frac{4}{3}f'''(a) - \frac{1}{2}f''(b) + f'(0) = 0$. Let $c = 0$, then $8f'''(a) - 3f''(b) + 6f'(c) = 0$.

⑥ 

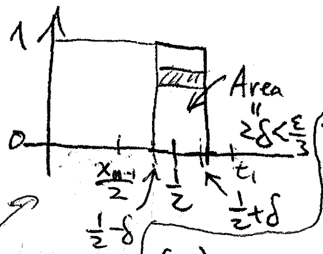



Since $f:[0,1]\to[0,1]$ is increasing, $f$ is Riemann integrable on $[0,1]$. So for $\varepsilon>0$, by the integral criterion, there exists a partition $P=\{0=x_0<x_1<\cdots<x_n=1\}$ such that on $[0,1]$, $U(f(x),P)-L(f(x),P)<\frac{2\varepsilon}{3}$. Let $P_1=\{0=\frac{x_0}{2}<\frac{x_1}{2}<\cdots<\frac{x_n}{2}=\frac{1}{2}\}$. Then on $[0,\frac{1}{2}]$ $U(f(2x),P_1)-L(f(2x),P_1)<\frac{\varepsilon}{3}$ since $\left(\frac{x_{i+1}}{2}-\frac{x_i}{2}\right)=\frac{1}{2}(x_{i+1}-x_i)$, $\sup\limits_{x\in[x_i,x_{i+1}]} f(x)=\sup\limits_{x\in[\frac{x_i}{2},\frac{x_{i+1}}{2}]} f(2x)$ and $\inf\limits_{x\in[x_i,x_{i+1}]} f(x)=\inf\limits_{x\in[\frac{x_i}{2},\frac{x_{i+1}}{2}]} f(2x)$.




Let $t_i=\frac{1}{2}+\frac{x_i}{2}$ for $i=0,1,\cdots,n$ and $P_2=\{\frac{1}{2}=t_0<t_1<\cdots<t_n=1\}$. Then on $[\frac{1}{2},1]$, $U(1-f(2x-1),P_2)-L(1-f(2x-1),P_2)<\frac{\varepsilon}{3}$ Since $(t_{i+1}-t_i)=\left(\frac{x_{i+1}}{2}-\frac{x_i}{2}\right)$, $\sup\limits_{x\in[t_i,t_{i+1}]}1-f(2x-1)=1-\inf\limits_{x\in[t_i,t_{i+1}]}f(2x-1)$ $=1-\inf\limits_{x\in[x_i,x_{i+1}]}f(x)$ and $\inf\limits_{x\in[t_i,t_{i+1}]}1-f(2x-1)=1-\sup\limits_{x\in[x_i,x_{i+1}]}f(x)$.

Let $Q=P_1\cup\{\frac{1}{2}-\delta\}\cup P_2\cup\{\frac{1}{2}+\delta\}$, where we want $\frac{1}{2}<\frac{1}{2}+\delta<t_1$ and $2\delta<\frac{\varepsilon}{3}$ So $0<\delta<\min\left(\frac{\varepsilon}{6},t_1-\frac{1}{2},\frac{1-x_{n-1}}{2}\right)$. $\frac{x_{n-1}}{2}<\frac{1}{2}-\delta<\frac{1}{2}$




Then $U(f(2x),Q\cap[0,\frac{1}{2}-\delta])-L(f(2x),Q\cap[0,\frac{1}{2}-\delta])=(\mathrm{I})$
$\le U(f(2x),Q\cap[0,\frac{1}{2}])-L(f(2x),Q\cap[0,\frac{1}{2}])$
$\le U(f(2x),P_1)-L(f(2x),P_1)<\frac{\varepsilon}{3}$.

$(\mathrm{II})=$

Similarly, $U(1-f(2x-1),Q\cap[\frac{1}{2}+\delta,1])-L(1-f(2x-1),Q\cap[\frac{1}{2}+\delta,1])\le U(1-f(2x-1),P_2)-L(1-f(2x-1),P_2)$ $<\varepsilon/3$. Then $U(g(x),Q)-L(g(x),Q)\le(\mathrm{I})+2\delta\Big(\underbrace{\sup\limits_{x\in[\frac{1}{2}-\delta,\frac{1}{2}+\delta]}g(x)-\inf\limits_{x\in[\frac{1}{2}-\delta,\frac{1}{2}+\delta]}g(x)}_{\le 1-0}\Big)+(\mathrm{II})<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}$ $=\varepsilon$.