Tutorial 2.

(1) $Q \quad Q^2 \quad Q^3$ countable.

$A = \{$all circles in R^2 with rational centre and radius$\}$

~~proof~~ prove A is countable.

$X \in A$. X is a circle centre (a, b) radius r.

$a, b, r \in Q$.

$f: A \to Q^3$ f is injective. $f: X \longrightarrow (a, b, r) \in Q^3$.

so A is countable

(2) prove $\exists$ a circle without rational point on it

by contradiction assume $\forall$ circle has a rational point on it

$S_r = \{x \in R^2 : |x| = r\}$ $(r > 0)$

S_r is a circle with centre $(0,0)$ radius r.

$\exists\, X_r \in Q^2$. X_r is a rational point.

X_r is on S_r. $X_r \in S_r$.

If $0 < r < s < \infty$ we can prove $X_r \neq X_s$.

so we have a mapping $f: \quad A = \{X_r, r > 0\} \to (0, +\infty)$

$f: X_r \to r$.

f is bijective.

$(0, \infty)$ uncountable so A uncountable.

$A \subseteq Q^2$. contradiction.

(3). $A \subseteq R^n$ A is countable.

prove $\exists x \in R^n$. $A \cap (A+x) = \phi$

$A+x = \{a+x : a \in A\}$

prove by contradiction.

assume $\forall x \in R^n$. $A \cap (A+x) \neq \phi$.

assume $a \in A \cap (A+x)$ then $a \in A$ $a \in A+x$

$\exists b \in A$ $a = b + x$. $x = a - b$

define $A - A = \{a-b : a, b \in A\}$

$x \in A-A$ for $\forall x \in R^n$ $x \in A-A$

so $R^n \subseteq A-A$

R^n is uncountable so $A-A$ is ~~unco~~ uncountable

But from A is countable we can prove $A-A$ is countable.

so contradiction.

(4). for $a_i > 0$. $i = 1, 2, \dots n$. $\frac{a_1 + a_2 + \dots + a_n}{n} \geq (a_1 \cdots a_n)^{\frac{1}{n}}$

when $a_1 = a_2 = \dots = a_n$ "=" holds

prove by mathematical induction. a little difference.

first we prove it is true for all $n = 2^k$ $k = 1, 2, \dots$

check $k=1$, $n=2$, $\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$

suppose. $n = 2^k$ is true.

when $n = 2^{k+1}$

$$\frac{a_1 + \dots + a_{2^k} + a_{2^k+1} + \dots + a_{2^{k+1}}}{2^{k+1}} = \frac{\frac{a_1 + \dots + a_{2^k}}{2^k} + \frac{a_{2^k+1} + \dots + a_{2^{k+1}}}{2^k}}{2}$$

$$\geq \frac{(a_1 \cdots a_{2^k})^{\frac{1}{2^k}} + (a_{2^k+1} \cdots a_{2^{k+1}})^{\frac{1}{2^k}}}{2}.$$

(use the assumption)

$$\geq (a_1 \cdots a_{2^{k+1}})^{\frac{1}{2^{k+1}}} \quad \text{(use when } n=2\text{)}.$$

so. it is true for all $n = 2^k$ $k = 1, 2, \cdots$

$\forall m \in \mathbb{N}^+$ find $2^k > m$. let $2^k - m = l$ so $2^k = m + l$

for $m+l$ we have $\dfrac{b_1 + \cdots + b_m + b_{m+1} + \cdots + b_{m+l}}{m+l} \geq (b_1 \cdots b_{m+l})^{\frac{1}{m+l}}$

choose $b_1 = a_1 \cdots b_m = a_m$ $b_{m+1} = \cdots = b_{m+l} = \dfrac{a_1 + \cdots + a_m}{m}$

we can get $\dfrac{a_1 + a_2 + \cdots + a_m + \frac{l(a_1 + \cdots + a_m)}{m}}{m+l} \geq \left(a_1 \cdots a_m \left(\frac{a_1 + \cdots + a_m}{m}\right)^l\right)^{\frac{1}{m+l}}$

left side $= \dfrac{\frac{m(a_1 + \cdots + a_m)}{m} + \frac{l(a_1 + \cdots + a_m)}{m}}{m+l} = \dfrac{a_1 + \cdots + a_m}{m}$

so. $\dfrac{a_1 + \cdots + a_m}{m} \geq \left[a_1 \cdots a_m \left(\frac{a_1 + \cdots + a_m}{m}\right)^l\right]^{\frac{1}{m+l}}$

$\Leftrightarrow \left(\dfrac{a_1 + \cdots + a_m}{m}\right)^{m+l} \geq a_1 \cdots a_m \left(\dfrac{a_1 + \cdots + a_m}{m}\right)^l$

$\Leftrightarrow \left(\dfrac{a_1 + \cdots + a_m}{m}\right)^m \geq a_1 \cdots a_m.$

$\Leftrightarrow \dfrac{a_1 + \cdots + a_m}{m} \geq (a_1 \cdots a_m)^{\frac{1}{m}}$

prove is done !

(5). let $a_1 = a_2 = \cdots = a_n = (1+\frac{1}{n})$ $a_{n+1} = 1$

we have $\frac{n(1+\frac{1}{n})+1}{n+1} \geqslant \left[(1+\frac{1}{n})^n\right]^{\frac{1}{n+1}}$

left $= \frac{n+2}{n+1} = 1+\frac{1}{n+1}$

right $= (1+\frac{1}{n})^{\frac{n}{n+1}}$

so. $1+\frac{1}{n+1} \geqslant (1+\frac{1}{n})^{\frac{n}{n+1}}$

$\Leftrightarrow (1+\frac{1}{n+1})^{n+1} \geqslant (1+\frac{1}{n})^{n}$. "=" cannot hold.

so. $(1+\frac{1}{n+1})^{n+1} > (1+\frac{1}{n})^{n}$

Let $a_1 = a_2 = \cdots = a_{n+1} = \frac{n}{n+1}$ $a_{n+2} = 1$

$\frac{\frac{n}{n+1}\cdot(n+1)+1}{n+2} \geqslant \left[\left(\frac{n}{n+1}\right)^{n+1}\right]^{\frac{1}{n+2}}$

left $= \frac{n+1}{n+2}$ right $= \left(\frac{n}{n+1}\right)^{\frac{n+1}{n+2}}$

so $\frac{n+1}{n+2} \geqslant \left(\frac{n}{n+1}\right)^{\frac{n+1}{n+2}}$

$\Leftrightarrow \frac{n+2}{n+1} \leq \left(\frac{n+1}{n}\right)^{\frac{n+1}{n+2}}$

$\Leftrightarrow (1+\frac{1}{n+1}) \leq (1+\frac{1}{n})^{\frac{n+1}{n+2}}$

$\Leftrightarrow (1+\frac{1}{n+1})^{n+2} \leq (1+\frac{1}{n})^{n+1}$ "=" can not hold

so. $(1+\frac{1}{n})^{n+1} > (1+\frac{1}{n+1})^{n+2}$

so. $(1+\frac{1}{n})^{n} \uparrow$ $(1+\frac{1}{n})^{n+1} \downarrow$.

$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$ $\lim_{n\to\infty}(1+\frac{1}{n})^{n+1} = e$.