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① Prove that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{2+3x}{x^2+4}$ is continuous at $x=2$ by checking the ε-δ definition of a function continuous at a point.

First, evaluate $f(2)$.

$$f(2) = \frac{2+3(2)}{(2)^2+4} = \frac{8}{8} = 1$$

Scratch work:

$$|f(x) - 1| = \left|\frac{2+3x}{x^2+4} - 1\right| = \left|\frac{2+3x-x^2-4}{x^2+4}\right| = \left|\frac{-x^2+3x-2}{x^2+4}\right|$$

$$= \frac{|x^2-3x+2|}{x^2+4}$$

$$= \frac{|x-1|\,|x-2|}{x^2+4}$$

$$\leq \frac{2\,|x-2|}{4}$$

$$\begin{pmatrix} \text{Set } |x-2| \leq 1 \\ 1 \leq x \leq 3 \\ 0 \leq x-1 \leq 2 \end{pmatrix} \quad \leq \frac{1}{2}\delta \quad (0 < |x-2| < \delta)$$

So $\delta < 2\varepsilon$ is enough.

Then, $\forall \varepsilon > 0$, let $\delta = \min(1, 2\varepsilon)$, $\forall x \in \mathbb{R}$,

$$0 < |x-2| < \min\{1, 2\varepsilon\} \Rightarrow \left|\frac{2+3x}{x^2+4} - 1\right| = \left|\frac{-x^2+3x-2}{x^2+4}\right|$$

$$= \frac{|x-1|\,|x-2|}{x^2+4} < \frac{2\delta}{4} < \varepsilon$$

$\lim_{x\to 2} \frac{2+3x}{x^2+4} = 1 = \frac{2+3(2)}{2^2+4}$. Therefore, f is continuous at x.

② Prove that there does not exist any continuous function $f: \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) + x = 0$ for every $x \in \mathbb{R}$.

$f(f(x)) = -x$

f is a bijection, since otherwise we would have for $x \neq y$, $f(x) = f(y)$

but $-x = f(f(x)) = f(f(y)) = -y$, leads to a contradiction.

By the continuous injection theorem,

if f is continuous and injective on $[a,b]$, then f is strictly monotone on $[a,b]$

If f is strictly increasing, let $x<y$, then $f(x) < f(y)$,

$f(f(x)) < f(f(y))$ thus $-x<-y$ and $x>y$, leads to a contradiction.

If f is strictly decreasing, then $x>y$, $f(x) > f(y)$,

then $f(f(x)) < f(f(y))$, thus $-x<-y$ and $x>y$, leads to a contradiction.

$\therefore$ There does not exist any continuous function $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(f(x)) + x = 0$ for every $x \in \mathbb{R}$.

③ Let $f: \mathbb{R} \to \mathbb{R}$ be continuous such that $|f(x) - f(y)| \leq \frac{1}{2}|x-y|$ for every $x, y \in \mathbb{R}$.

(a) Let $w \in \mathbb{R}$. Define $x_1 = w$ and $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$. Show that $x_1, x_2, x_3, \dots$ is a Cauchy sequence.

Let $n > m \geq N$, $x_n = f(x_{n-1})$

$$x_m = f(x_{m-1}) \quad \text{for } m \in \mathbb{N}$$

$$|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \dots + x_{m+1} - x_m|$$

$$|x_n - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \dots + |x_{m+1} - x_m| \quad -(*)$$

Then, we prove that $|x_{n+1} - x_n| \leq \frac{1}{2^{n-1}}|x_2 - x_1|$:

$$|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}|$$

$$|x_{n+1} - x_n| \leq \frac{1}{2^2}|x_{n-1} - x_{n-2}|$$

$$\vdots$$

$$|x_{n+1} - x_n| \leq \frac{1}{2^{n-1}}|x_2 - x_1|$$

From $(*)$,

$$|x_n - x_m| \leq \frac{1}{2^{n-2}}|x_2 - x_1| + \frac{1}{2^{n-3}}|x_2 - x_1| + \dots + \frac{1}{2^{m-1}}|x_2 - x_1|$$

$$= |x_2 - x_1|\left(\frac{1}{2^m} - \frac{1}{2^{n-2}}\right)$$

$$\leq |x_2 - x_1|\left(\frac{1}{2^m}\right)$$

$$\leq |x_2 - x_1|\left(\frac{1}{2^N}\right) \quad (\because m \geq N)$$

We want $\frac{|x_2 - x_1|}{2^N} < \varepsilon$, $2^N > \frac{|x_2 - x_1|}{\varepsilon}$, $N > \log_2\left(\frac{|x_2 - x_1|}{\varepsilon}\right)$

So, $\forall \varepsilon > 0$, $N \geq \lceil \log_2 \left(\frac{|x_2 - x_1|}{\varepsilon} \right) \rceil$, $n \geq m \geq N \Rightarrow$

$$|x_n - x_m| < \varepsilon$$

$\Rightarrow$ $x_1, x_2, x_3, \dots$ is Cauchy sequence.