Lecture 24

07-05-2019

Review :

① $f \in C[a,b] \Rightarrow f$ is integrable

② f is integrable on $[a,b]$, $a<c<b$.

$\Leftrightarrow f$ is integrable on $[a,c]$ and $[c,b]$.

③ f, g are integrable on $[a,b] \Rightarrow$

$f+g$, $f\cdot g$, cf are integrable on $[a,b]$

and $\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx$, $\int_a^b cf\,dx = c\int_a^b f\,dx$

④ $\begin{cases} f \text{ integrable on } [a,b], \\ |g(x)-g(y)| < M|x-y|, \ \forall\, x,y \in f[a,b], \end{cases} \Rightarrow g\circ f$ is integrable on $[a,b]$.

④ $g(x) \geq 0 \Rightarrow \int_a^b f(x)g(x)\,dx = c\int_a^b g(x)\,dx$

where $\inf\{f(x) : a \leq x \leq b\} \leq c \leq \sup\{f(x) : a \leq x \leq b\}$

Question: So far, we have discussed integrability, but how to compute the integrals ?

By definition ?

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \sup \{ L(f, P) : P \text{ partition of } [a,b] \}$$

$$= \overline{\int_a^b} f(x)dx = \inf \{ U(f, P) : P \cdots \quad \cdots \quad \cdots \}$$

or $$\int_a^b f(x)\, dx = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j) \Delta x_j .$$

It works for special functions such as $f(x) = 1, x, x^2, \ldots$

To compute $\int_a^b f(x)dx$ for general functions, we have to resort to "The fundamental theorem of Calculas".

Anti-derivative of f : $F(x) = \int_a^x f(t)\, dt$

Uniform Continuity of anti-derivative

THM: If f is integrable on $[a, b]$ and $c \in [a, b]$, then

$F(x) = \int_c^x f(t)\,dt$ is uniformly continuous on $[a, b]$.

Pf: Let M be a bound of f, then

$$|F(x) - F(y)| = \left| \int_c^x f(t)\,dt - \int_c^y f(t)\,dt \right|$$

$$= \left| \int_y^x f(t)\,dt \right| \le \left| \int_y^x |f(t)|\,dt \right|$$

$$\le \left| \int_y^x M\,dt \right| = M\,|x-y|.$$

$\forall\, \varepsilon > 0$, let $\delta = \frac{\varepsilon}{M}$, then $|x-y| < \delta \Rightarrow$

$|F(x) - F(y)| < \varepsilon \Rightarrow F$ is uniformly continuous.

Fundamental theorem of Calculas

THM 1. Let f be integrable on $[a,b]$ and continuous at $x_0 \in [a,b]$

then $F(x) = \int_a^x f(t)\,dt$ is differentiable at x_0.

Moreover, $F'(x_0) = f(x_0)$

$f(x_0) + f(t) - f(x_0)$

Pf :
$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt}{x - x_0} = \frac{\int_{x_0}^x f(t)\,dt}{x - x_0}$$

$$= \frac{\int_{x_0}^x f(x_0)\,dt + \int_{x_0}^x [f(t) - f(x_0)]\,dt}{x - x_0} = f(x_0) + \frac{\int_{x_0}^x [f(t) - f(x_0)]\,dt}{x - x_0}$$

$\forall\, \varepsilon > 0$, f is continuous at $x_0 \Rightarrow \exists\, \delta > 0$ s.t

$\forall\, x \in [a,b]$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$$\Rightarrow \left| \frac{\int_{x_0}^x [f(t) - f(x_0)]\,dt}{x - x_0} \right| < \frac{\left|\int_{x_0}^x \varepsilon\,dt\right|}{|x - x_0|} = \frac{\varepsilon\,|x - x_0|}{|x_0 - x|} = \varepsilon$$

$$\Rightarrow \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon \Rightarrow F'(x_0) = f(x_0)$$

Fundamental theorem of Calculus

Thm 2. If F is differentiable on $[a,b]$ and F' is integrable on $[a,b]$, then $\int_a^b F'(x)dx = F(b) - F(a)$.

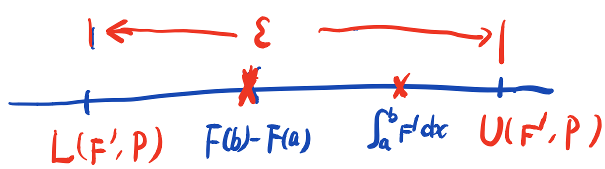
pf: $\forall \varepsilon > 0$, by integral criterion, $\exists$ partion P of $[a,b]$

s.t $U(F', P) - L(F', P) < \varepsilon$

By mean-value theorem $F(x_j) - F(x_{j-1}) = F'(t_j)(x_j - x_{j-1})$

for some $t_j \in [x_{j-1}, x_j]$

$$\Rightarrow L(F', P) \leq \sum_{j=1}^{n} F'(t_j)\Delta x_j = \sum_{j=1}^{n}\left[F(x_j) - F(x_{j-1})\right] = F(b) - F(a) \leq U(F', P)$$



$$\Rightarrow \left| F(b) - F(a) - \int_a^b F'(x)dx \right| \leq U(F', P) - L(F', P) < \varepsilon$$

By infinetesimal principal, $F(b) - F(a) = \int_a^b F'(x)dx$

Integration by parts

THM : Let f, g be differentiable on $[a, b]$ and f', g' be integrable on $[a, b]$. then $\int_a^b f(x) g'(x) dx = f(b) g(b) - f(a) g(a) - \int_a^b f'(x) g(x) dx$

Pf : $\int_a^b (fg)'(x) dx = f(b) g(b) - f(a) g(a)$

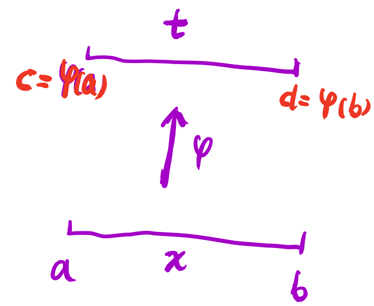
But $(f \cdot g)'(x) = f'(x) g(x) + f(x) g'(x)$

$\Rightarrow \int_a^b f'(x) g(x) dx + \int_a^b f(x) g'(x) dx = f(b) g(b) - f(a) g(a)$

Change of Variable Formula

THM: Let $\phi : [a, b] \to \mathbb{R}$ be differentiable, ϕ' be integrable on $[a, b]$ and f continuous on $\phi([a, b])$, then

$$\int_c^d f(t)\,dt \overset{t=\varphi(x)}{=\!=} \int_a^b f(\phi(x))\,\phi'(x)\,dx .$$



Pf: Define $F(t) = \int_{\phi(a)}^{t} f(s)\,ds$

and $g(x) = \int_{\phi(a)}^{\phi(x)} f(s)\,ds = F \circ \phi(x)$

then $F'(t) = f(t)$, $g(a) = 0$ and

$$g'(x) = F'(\phi(x))\,\phi'(x) = f(\phi(x))\,\phi'(x).$$

$$\Rightarrow \int_{\phi(a)}^{\phi(b)} f(t)\,dt = g(b) = g(b) - g(a) = \int_a^b g'(x)\,dx$$

$$= \int_a^b f(\phi(x))\,\phi'(x)\,dx$$

Lebesgue's thm (Additional topics)

Question : What functions are integrable?

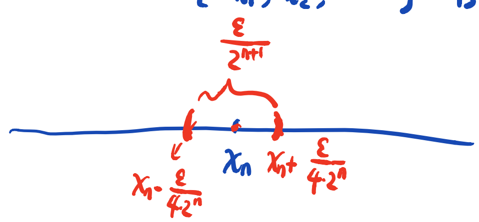
Def : A set $S \subseteq \mathbb{R}$ is of measure 0 (or has zero-length)

iff $\forall \varepsilon > 0$, $\exists$ intervals (a_1, b_1), (a_2, b_2), ...

s.t $S \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$ and

$\sum_{n=1}^{\infty} |a_n - b_n| < \varepsilon$

Example : ① A countable set $\{x_1, x_2, \cdots\}$ is of measure 0.



Pf : $\forall \varepsilon > 0$, $x_n \in \left(x_n - \frac{\varepsilon}{4 \cdot 2^n} , x_n + \frac{\varepsilon}{4 \cdot 2^n} \right)$

$\Rightarrow \{x_1, x_2, \cdots\} \subset \bigcup_{n=1}^{\infty} \left(x_n - \frac{\varepsilon}{4 \cdot 2^n} , x_n + \frac{\varepsilon}{4 \cdot 2^n} \right)$

$\sum_{n=1}^{\infty} \left| x_n + \frac{\varepsilon}{4 \cdot 2^n} - \left(x_n - \frac{\varepsilon}{4 \cdot 2^n} \right) \right| = \sum_{n=1}^{\infty} \frac{\varepsilon}{2 \cdot 2^n} = \frac{1}{2} \varepsilon < \varepsilon.$

Example ②. Let (a, b), $a<b$ be an interval.

then (a, b) is not a 0-measure set.

Remark: ① the above measure 0 set is actually called

Lebesgue-measure 0 set. Lebesgue-measure is one type of measure

which is an extension of the notion of size (length, area, volume)

to more complicated shapes than, for example, rectangles, triangles.

Lebesgue (1875-1941) is a French mathematician, he developed his theory of

integral in his thesis in 1902.

② The area we defined by using Riemann integral is also one type of

measure, called Jordan measure.

Def: A property is said to hold almost everywhere (a.e)

or almost surely, iff the property holds except

on a set of Lebesgue measure 0.

Lebesgue's thm

Lebesgue's thm (1902) : Let $f : [a, b] \to \mathbb{R}$ be a bounded function

then f is integrable $\Leftrightarrow$ f is continuous a.e

(or f is continuous on $[a, b]$ except a set of measure 0).

Remark : Lebesgue's thm gives a complete charaterization for

Riemanian integrability. The disadvantage of this type of integrability

is that it is not "complete" in the following sense :

the limit of R-integrable functions

on $[a, b]$ may not be R-integrable on $[a, b]$.

Counterexample: Let $\{r_n : n=1,2,\cdots\}$ be a listing of $\mathbb{Q}\cap[0,1]$ without repetition and omission. Define $f_n(x)=\begin{cases} 1, & x\in\{r_1,\cdots r_n\} \\ 0, & \text{otherwise} \end{cases}$.

then $\lim\limits_{n\to\infty} f_n(x) = \begin{cases} 1, & x\in \mathbb{Q}\cap[0,1] \\ 0, & \text{otherwise} \end{cases}$.

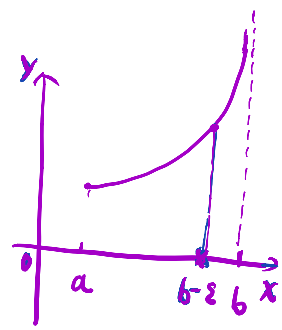
Using integral criterion, we can show that f_n is R-integrable, but $\lim\limits_{n\to\infty} f_n$ is not.

Remark: To make integrability "complete" under limit, Lebesgue introduced another type of integral which is call "Lebesgue integral". This is the main topic in real analysis.

Improper integrable

Question: how to define integrals for unbounded functions and functions defined in an unbounded interval, say for example: $\int_0^1 \ln x \, dx$, $\int_1^\infty \frac{1}{x^2} dx$.

Def: Let $f: [a, b) \to \mathbb{R}$ be unbounded such that f is integrable on $[a, b-\varepsilon]$ for all $0 < \varepsilon < b-a$, we define the improper integral of f on $[a, b)$, denoted by $\int_a^b f(x)\,dx$, to be the following limit

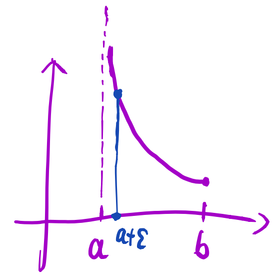


$\lim\limits_{c \to b^-} \int_a^c f(x)\,dx$ (or $\lim\limits_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)\,dx$), if the limit exists.

In that case, we say that f is improper integrable on $[a, b)$

otherwise, we say f is not improper integrable.

Similarly, we may define improper integrals for $f: [a, b] \to \mathbb{R}$ which is unbounded and integrable on $[a+\varepsilon, b]$ for all $0 < \varepsilon < b-a$.



Example ① $\int_0^1 \ln x \, dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \ln x \, dx$

$$= \lim_{\varepsilon \to 0^+} (x \ln x - x) \Big|_\varepsilon^1$$

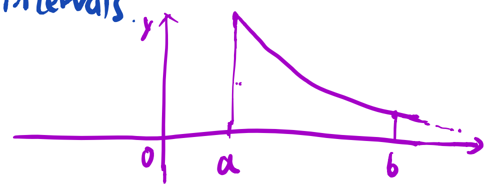
$$= \lim_{\varepsilon \to 0^+} -1 - \varepsilon \ln \varepsilon + \varepsilon = -1$$

Therefore, $\ln x$ is improper integrable on $(0, 1]$.

Example ② $\int_0^1 \frac{1}{x} \, dx$. (Exercise).

$\frac{1}{x}$ is not improper integrable on $(0, 1]$.

Improper integrals on unbounded intervals.



Def: Let $f : [a, \infty)$ be such that

f is integrable on $[a, b]$ for all $b > a$. We define

the improper integrable of f on $[a, \infty)$, denoted by

$\int_a^\infty f(x)\,dx$, to be the limit $\lim_{b \to \infty} \int_a^b f(x)\,dx$,

if the limit exists. In that case, We call

that the improper integral converges. Otherwise we

say that it diverges.

Similarly, we can define improper integrals for

$f : (-\infty, a]$ by

$$\int_{-\infty}^a f(x)\,dx = \lim_{c \to -\infty} \int_c^a f(x)\,dx.$$

Example : ① $\int_2^\infty \frac{1}{x^2}\,dx = \lim_{b\to+\infty} \int_2^b \frac{1}{x^2}\,dx$

$$= \lim_{b\to+\infty} -\frac{1}{x}\Big|_2^b$$

$$= \lim_{b\to+\infty} \left(-\frac{1}{b}+\frac{1}{2}\right)$$

$$= \frac{1}{2}$$

$\Rightarrow$ the improper integral of $\frac{1}{x^2}$ converges on $[2, \infty)$.

② $\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x}\,dx$

$$= \lim_{b\to+\infty} \ln x \Big|_1^b = \lim_{b\to\infty} (\ln b - 1)$$

$\Rightarrow$ the improper integral $\int_1^\infty \frac{1}{x}\,dx$ diverges.