

Extra Examples for the Chapter on Differentiation

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① If $\{x_n\}$ is Cauchy, then prove $\{\sqrt[5]{x_n}\}$ is Cauchy.

Solution. Since $\{x_n\}$ is Cauchy, ^{and $x_n \geq 0$ for all n} for every $\varepsilon > 0$, $\exists K \in \mathbb{N}$ such that $m, n \geq K \Rightarrow |x_m - x_n| < \varepsilon^5$.

Then $|\sqrt[5]{x_m} - \sqrt[5]{x_n}| \leq \sqrt[5]{|x_m - x_n|} < \sqrt[5]{\varepsilon^5} = \varepsilon$.
 $\therefore \{\sqrt[5]{x_n}\}$ is Cauchy.

② $f: \mathbb{R} \rightarrow \mathbb{R}$ is three-times differentiable.

If $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$, then prove that $\sup \{f^{(3)}(x) : -1 < x < 1\} \geq 3$.

Thoughts $f^{(3)}$, f' , f suggest Taylor's theorem. c should be -1 , 1 or 0 , more likely $c = 0$.

Solution By Taylor's theorem, using $c = 0$,

$$\begin{aligned} f(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(\theta_x)}{6}(x-0)^3 \\ &= f(0) + \frac{f''(0)}{2}x^2 + \frac{f'''(\theta_x)}{6}x^3 \end{aligned}$$

for some θ_x between x and 0 . Setting $x = 1, -1$,

$$1 = f(1) = f(0) + \frac{f''(0)}{2} + \frac{f'''(\theta_1)}{6} \text{ for some } \theta_1 \in (0, 1)$$

$$0 = f(-1) = f(0) + \frac{f''(0)}{2} - \frac{f'''(\theta_{-1})}{6} \text{ for some } \theta_{-1} \in (-1, 0).$$

Subtracting yields $f'''(\theta_1) + f'''(\theta_{-1}) = 6$.

$$\sup \{f^{(3)}(x) : -1 < x < 1\} \geq \max \{f'''(\theta_1), f'''(\theta_{-1})\} \geq 3.$$

③ Let f be twice differentiable on $[0, 2]$.

$$\forall x \in [0, 2], |f(x)| \leq 1, |f''(x)| \leq 1.$$

Prove that $\forall x \in [0, 2], |f'(x)| \leq 2$.

Solution By Taylor's theorem, let $x \in [0, 2], a \in [0, 2]$

$$f(a) = f(x) + f'(x)(a-x) + \frac{f''(\theta_a)}{2}(a-x)^2$$

for some θ_a between a and x . Setting $a = 0, 2$,

$$f(0) = f(x) - f'(x)x + \frac{f''(\theta_0)}{2}x^2 \text{ for some } \theta_0 \in (0, x).$$

$$f(2) = f(x) + f'(x)(2-x) + \frac{f''(\theta_2)}{2}(2-x)^2 \text{ for some } \theta_2 \in (x, 2).$$

Subtracting these, we get

$$f(2) - f(0) = 2f'(x) + \frac{f''(\theta_2)}{2}(2-x)^2 - \frac{f''(\theta_0)}{2}x^2.$$

Solving for $f'(x)$, we see

$$|f'(x)| = \frac{1}{2} |f(2) - f(0) + \frac{f''(\theta_0)}{2}x^2 - \frac{f''(\theta_2)}{2}(2-x)^2|$$

$$\leq \frac{1}{2} (1 + 1 + \frac{1}{2}x^2 + \frac{1}{2}(2-x)^2)$$

$$= \frac{1}{2} (x^2 - 2x + 4) \quad \left\{ \begin{array}{l} x \in [0, 2] \\ |x-1| \leq 1 \end{array} \right.$$

$$\leq \frac{1}{2} ((x-1)^2 + 3)$$

$$\leq \frac{1}{2} (1 + 3) = 2$$

- ④ Let $f: [1, 2]$ be continuous and f be differentiable on $(1, 2)$. Prove there exists $\theta \in (1, 2)$ such that $f(2) - f(1) = \frac{1}{2} \theta^2 f'(\theta)$.

(Note mean value theorem only gives $f(2) - f(1) = f'(\theta_0)(2-1) = f'(\theta_0)$ for some $\theta_0 \in (1, 2)$.)

Solution Write

$$\theta^2 f'(\theta) = \frac{f'(\theta)}{1/\theta^2} = \frac{f'(\theta)}{g'(\theta)}, \text{ where } g(\theta) = -\frac{1}{\theta}.$$

By generalized mean value theorem,

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(\theta)}{g'(\theta)} \text{ for some } \theta \in (1, 2).$$

This is just

$$\frac{f(2) - f(1)}{-\frac{1}{2} - (-1)} = \frac{f'(\theta)}{1/\theta^2} \Leftrightarrow f(2) - f(1) = \frac{1}{2} \theta^2 f'(\theta).$$



- ⑤ Let $0 < a < b$ and let $f: [a, b] \rightarrow \mathbb{R}$ be continuous with f differentiable on (a, b) . Prove $\exists \theta \in (a, b)$ such that $\frac{1}{b-a} \left| \frac{f(a)}{a} - \frac{f(b)}{b} \right| = f'(\theta)$.

Solution. Expanding the left side, we get $\frac{1}{b-a} \left| \frac{f(a)}{a} - \frac{f(b)}{b} \right| = \frac{bf(a) - af(b)}{b-a} = \frac{f(a)/a - f(b)/b}{1/a - 1/b}$ dividing by ab in numerator & denominator.

This suggests we consider $F(x) = f(x)/x$, $G(x) = 1/x$. By the generalized mean value theorem,

$$\begin{aligned} \frac{f(a)/a - f(b)/b}{1/a - 1/b} &= \frac{F(a) - F(b)}{G(a) - G(b)} = \frac{F'(\theta)}{G'(\theta)} \text{ for some } \theta \in (a, b) \\ &= \frac{\theta f'(\theta) - f(\theta)}{\theta^2} / \left(-\frac{1}{\theta^2}\right) \\ &= f'(\theta). \end{aligned}$$

The result follows.

Landau's Big-Oh and Little-Oh Notations

Definitions Let $c \in \mathbb{R}$ or $c = +\infty$ or $c = -\infty$.

Let I be an interval containing c or with c as an endpoint.

Let $f(x)$ and $g(x)$ be functions on I , $g(x) \neq 0$ on $I \setminus \{c\}$.

① We write $f(x) = O(g(x))$ iff $\exists A \in \mathbb{R}$ such that
 $\forall x \in I, |f(x)| \leq A |g(x)|$. \checkmark $b(x) = f(x)/g(x)$

So $f(x) = b(x)g(x)$ for some $b(x)$ bounded on I .

Example, $(\sin x) \ln x = O(\ln x)$ on $I = (0, +\infty)$.

② We write $f(x) = o(g(x))$ as $x \rightarrow c$ iff $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = 0$.

So $f(x) = s(x)g(x)$ for some $s(x)$ with $\lim_{x \rightarrow c} s(x) = 0$.

Example, $(\sin x)e^x = o(e^x)$ as $x \rightarrow 0$.

③ We write $f(x) = O^*(g(x))$ as $x \rightarrow c$ iff $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = k$
 is a nonzero number.

So $f(x) = kg(x) + s(x)g(x)$ for some $s(x)$ with $\lim_{x \rightarrow c} s(x) = 0$.

Example, $\sin 2x = O^*(x)$ as $x \rightarrow 0$.

④ We write $f(x) \sim g(x)$ as $x \rightarrow c$ iff $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = 1$.

Example, $\sin x \sim x$ as $x \rightarrow 0$.

Read " \sim " as "asymptotically equal to"
 or "asymptotically equivalent to"

CAUTION In equations involving big-oh and little-oh notations, $A = B$ means "A is of type B", which does not imply "B is of type A".

Example For every $g(x)$, $g(x) = O^*(g(x))$ because $\lim_{x \rightarrow c} \frac{g(x)}{g(x)} = 1 \neq 0$, but $O^*(g(x)) \neq g(x)$ since the $O^*(g(x))$ function on the left may be $2g(x)$ and $2g(x) \neq g(x)$ in general.

Remarks ① The phrase "as $x \rightarrow c$ " may be omitted when the context is clear.

② For sequences a_n, b_n as $n \rightarrow \infty$, there are similar concepts and notations.

③ $f(x) = O(1)$ means $f(x)$ is bounded on I
 $f(x) = o(1)$ means $\lim_{x \rightarrow c} f(x) = 0$.

④ Extending notations, we write
 $f(x) = h(x) + O(g(x))$ to mean $f(x) - h(x) = O(g(x))$
 $f(x) = h(x) + o(g(x))$ to mean $f(x) - h(x) = o(g(x))$.

Basic Facts As $x \rightarrow c$,

① $O(g(x)) \pm O(g(x)) = O(g(x))$ (meaning if $f_1(x) = O(g(x))$ and $f_2(x) = O(g(x))$, then $f_1(x) \pm f_2(x) = O(g(x))$ since

$$\left. \begin{array}{l} |f_1(x)| \leq A_1 |g(x)| \\ |f_2(x)| \leq A_2 |g(x)| \end{array} \right\} \Rightarrow |f_1(x) \pm f_2(x)| \leq (A_1 + A_2) |g(x)|$$

② $o(g(x)) \pm o(g(x)) = o(g(x))$; $o(g(x)) + O^*(g(x)) = O^*(g(x))$

$$\left. \begin{array}{l} O(g_1(x))O(g_2(x)) = O(g_1(x)g_2(x)) \\ O^*(g_1(x))O^*(g_2(x)) = O^*(g_1(x)g_2(x)) \\ o(g_1(x))o(g_2(x)) = o(g_1(x)g_2(x)) \\ o(g_1(x))O^*(g_2(x)) = o(g_1(x)g_2(x)) \\ o(g_1(x))O(g_2(x)) = o(g_1(x)g_2(x)) \\ o(g_1(x))g_2(x) = o(g_1(x)g_2(x)) \end{array} \right\} \begin{array}{l} \forall p > 0, \\ O(g(x))^p = O(g(x)^p) \\ \forall r \in \mathbb{R} \\ O^*(g(x))^r = O^*(g(x)^r) \end{array}$$

④ $o(o(g(x))) = o(g(x))$; $O^*(o(g(x))) = o(g(x))$
 $o(O^*(g(x))) = o(g(x))$

⑤ $a < b \Rightarrow O^*(x^a) \pm O^*(x^b) = O^*(x^a)$ as $x \rightarrow 0$
 $o(x^n) \pm o(x^m) = o(x^{\min(n,m)})$ as $x \rightarrow 0$

$$\lim_{x \rightarrow 0} \frac{1 - e^x}{-x} = \lim_{x \rightarrow 0} \frac{-e^x}{-1} = 1$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$$

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \frac{1/(1+x)}{1} = 1$$

Remainder of Taylor Expansion

$$R_n(x) = \frac{f^{(n)}(\xi)}{n!} (x-c)^n = \begin{cases} O((x-c)^n) & \text{if } f^{(n)}(x) \text{ is bounded on } I \\ o((x-c)^{n-1}) & \text{as } x \rightarrow c \\ O^*((x-c)^n) & \text{as } x \rightarrow c \end{cases} \left. \begin{array}{l} f^{(n)}(x) \\ \text{contin} \\ \text{at } x=c \end{array} \right\}$$

Taylor Expansions at $c=0$ As $x \rightarrow 0$, if $f^{(n)}(c) \neq 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + R_{n+1}(x) = \sum_{k=0}^n \frac{x^k}{k!} + R_{n+1}(x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + R_{2n+2}(x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + R_{2n+3}(x)$$

$$(1+x)^a = 1 + \sum_{k=1}^n \frac{a(a-1)\dots(a-k+1)}{k!} x^k + R_{n+1}(x)$$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \dots + \frac{(-1)^{n-1} x^n}{n!} + R_{n+1}(x)$$

$$\text{Arctan } x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + R_{2n+3}(x)$$

$$\text{Arcsin } x = x + \sum_{k=1}^n \frac{1 \cdot 3 \cdot 5 \dots (2k-1)}{2 \cdot 4 \cdot 6 \dots (2k)} \frac{x^{2k+1}}{2k+1} + R_{2n+3}(x)$$

Notation:

$$m!! = \begin{cases} 1 \cdot 3 \dots m & \text{if } m \text{ is odd} \\ 2 \cdot 4 \dots m & \text{if } m \text{ is even} \end{cases} \rightarrow \frac{(2k-1)!!}{(2k)!!}$$

Remarks We have $1 - e^x \sim -x$ as $x \rightarrow 0$, also note $\sin x \sim x$, $\ln(1+x) \sim x$, $\tan x \sim x$, $\text{Arctan } x \sim x$, $\text{Arcsin } x \sim x$ as $x \rightarrow 0$. These are often useful.

Examples

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① Consider the convergence of $\sum_{n=1}^{\infty} (\sin \frac{1}{n} - \text{Arctan}(\frac{1}{n}))$

Solution Let $x = \frac{1}{n} \in [0, 1] = I$. We have

$$\sin x = x - \frac{1}{6}x^3 + O(x^5)$$

$$\text{Arctan } x = x - \frac{1}{3}x^3 + O(x^5)$$

$$\sin x - \text{Arctan } x = \frac{1}{6}x^3 + O(x^5) - O(x^5)$$

$$\begin{aligned} \text{On } I = (0, 1] \\ O(x^5) \leq C|x^5| \leq C|x^3| &= O(x^3) \quad \text{by fact ①} \\ O(x^5) &= O(x^3) \quad \text{by box on left} \end{aligned}$$

$$\sum_{n=1}^{\infty} \left| \sin\left(\frac{1}{n}\right) - \text{Arctan}\left(\frac{1}{n}\right) \right| = \sum_{n=1}^{\infty} O\left(\frac{1}{n^3}\right) \leq \sum_{n=1}^{\infty} A \frac{1}{n^3} < \infty$$

$$\therefore \sum_{n=1}^{\infty} (\sin(\frac{1}{n}) - \text{Arctan}(\frac{1}{n})) \text{ converges absolutely.}$$

For later solution, we record the fact in the box above

Fact 6 On $I = [-1, 1]$, if $0 \leq a < b$, then
 $O(x^a) + O(x^b) = O(x^a)$.

Proof $f(x) = O(x^b) \Leftrightarrow |f(x)| \leq C|x^b|$

$$\text{On } I = [0, 1] \Rightarrow |f(x)| \leq C|x^a|$$

$$|x| \leq 1 \Rightarrow |x^b| \leq |x^a| \Leftrightarrow f(x) = O(x^a) \quad \text{by fact 1.}$$

$$\text{So } O(x^a) + O(x^b) = O(x^a) + O(x^a) = O(x^a)$$

② Let $p \in \mathbb{R}$ and $a_n = (e - (1 + \frac{1}{n})^n)^p$. For which p , will $\sum_{n=1}^{\infty} a_n$ converges?

Solution As $n \rightarrow \infty$, $x = \frac{1}{n} \rightarrow 0$, $O^*(\frac{1}{n}) = \frac{1}{n} \frac{O^*(\frac{1}{n})}{\frac{1}{n}} \rightarrow 0 \times \frac{1}{n} = 0$

① $\ln(1+x) = x + O^*(x^2)$

② $1 - e^x \sim -x$ since $\lim_{x \rightarrow 0} \frac{1 - e^x}{-x} = 1$.

$$(1 + \frac{1}{n})^n = e^{n \ln(1 + \frac{1}{n})} = e^{n(\frac{1}{n} + O^*(\frac{1}{n^2}))} \text{ by ①}$$

$$e - (1 + \frac{1}{n})^n = e - e^{1 + O^*(\frac{1}{n})} = e(1 - e^{O^*(\frac{1}{n})})$$

$$\text{by ② } e(-O^*(\frac{1}{n})) = O^*(\frac{1}{n})$$

$$a_n = (e - (1 + \frac{1}{n})^n)^p \sim O^*(\frac{1}{n})^p = O^*(\frac{1}{n^p}) = b_n \quad \text{Fact ③}$$

$$a_n \sim b_n \Leftrightarrow \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1 \Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{1/n^p} = K' \neq 0$$

by $\Rightarrow \sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ both converges or both diverges
 limit comparison test

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges iff } p > 1$$

by p-test

$$\begin{aligned}
 \textcircled{3} \quad & \lim_{x \rightarrow \infty} (\sqrt[6]{x^6 + x^5} - \sqrt[6]{x^6 - x^5}) \\
 &= \lim_{x \rightarrow \infty} x \left(\sqrt[6]{1 + \frac{1}{x}} - \sqrt[6]{1 - \frac{1}{x}} \right) \\
 &= \lim_{x \rightarrow \infty} x \left(\left(1 + \frac{1}{x}\right)^{1/6} - \left(1 - \frac{1}{x}\right)^{1/6} \right) \\
 &= \lim_{x \rightarrow \infty} x \left(1 + \frac{1}{6x} + o\left(\frac{1}{x}\right) - \left(1 - \frac{1}{6x} + o\left(\frac{1}{x}\right)\right) \right) \quad \text{by fact } \textcircled{3} \\
 &= \lim_{x \rightarrow \infty} \left(\frac{1}{3} + o(1) \right) = \frac{1}{3}
 \end{aligned}$$

As $x \rightarrow \infty$, $w = \frac{1}{x} \rightarrow 0$,
 $(1+w)^a = 1 + aw + o(w)$
 $x \cdot o\left(\frac{1}{x}\right) = o(1)$

$$\textcircled{4} \quad \text{Find } \lim_{x \rightarrow 0} \frac{\cos(\sin x) - \cos x}{\sin^4 x}$$

$$\begin{aligned}
 \text{As } x \rightarrow 0, \quad & \cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4) \\
 \sin x &= x - \frac{1}{6}x^3 + o(x^3)
 \end{aligned}$$

$$\begin{aligned}
 \cos(\sin x) &= 1 - \frac{1}{2} \sin^2 x + \frac{1}{24} \sin^4 x + o(\sin^4 x) \\
 &= 1 - \frac{1}{2} \left(x - \frac{1}{6}x^3 + o(x^3) \right)^2 + \frac{1}{24} \left(x - \frac{1}{6}x^3 + o(x^3) \right)^4 + o(x^4) \\
 &\quad \begin{matrix} = o(x^4) \\ = o(x^4) \end{matrix} \quad \text{by fact } \textcircled{4} \\
 &= 1 - \frac{1}{2} \left(x^2 - \frac{1}{3}x^4 + o(x^4) \right) + \frac{1}{24} (x^4 + o(x^4)) + o(x^4) \\
 &= 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + o(x^4)
 \end{aligned}$$

$$\text{So } \cos(\sin x) - \cos x = \frac{1}{6}x^4 + o(x^4)$$

$$\frac{\cos(\sin x) - \cos x}{\sin^4 x} = \frac{\frac{1}{6}x^4 + o(x^4)}{x^4} \cdot \frac{x^4}{\sin^4 x} \rightarrow \left(\frac{1}{6} + 0\right) \cdot 1 = \frac{1}{6}$$

Proof of facts $\textcircled{1}, \textcircled{2}, \textcircled{3}$ follow easily from the definitions of big-oh, little-oh notations. We will leave them as exercises later.

Proofs of Facts $\textcircled{4}$ and $\textcircled{5}$

$$\begin{aligned}
 & o(o(g(x))) = o(g(x)) \quad \text{Let } f(x) = o(o(g(x))), \text{ then} \\
 & f(x) = o(h(x)), \text{ where } h(x) = o(g(x)). \text{ So we have} \\
 & \lim_{x \rightarrow c} \frac{f(x)}{h(x)} = 0, \lim_{x \rightarrow c} \frac{h(x)}{g(x)} = 0 \Rightarrow \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = 0 \Leftrightarrow f(x) = o(g(x))
 \end{aligned}$$

$$\begin{aligned}
 & O^*(o(g(x))) = o(g(x)) \quad \text{Let } f(x) = O^*(o(g(x))), \text{ then} \\
 & f(x) = O^*(h(x)), \text{ where } h(x) = o(g(x)). \text{ So we have} \\
 & \lim_{x \rightarrow c} \frac{f(x)}{h(x)} = k \neq 0, \lim_{x \rightarrow c} \frac{h(x)}{g(x)} = 0 \Rightarrow \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = 0 \Leftrightarrow f(x) = o(g(x))
 \end{aligned}$$

$$\begin{aligned}
 & o(O^*(g(x))) = o(g(x)) \quad \text{Let } f(x) = o(O^*(g(x))), \text{ then} \\
 & f(x) = o(h(x)), \text{ where } h(x) = O^*(g(x)). \text{ So we have} \\
 & \lim_{x \rightarrow c} \frac{f(x)}{h(x)} = 0, \lim_{x \rightarrow c} \frac{h(x)}{g(x)} = k \neq 0 \Rightarrow \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = 0 \Leftrightarrow f(x) = o(g(x))
 \end{aligned}$$

$$a < b \Rightarrow O^*(x^a) \pm O^*(x^b) = O^*(x^a) \text{ as } x \rightarrow 0$$

Let $f_1(x) = O^*(x^a)$, $f_2(x) = O^*(x^b)$, then we have

$$\lim_{x \rightarrow 0} \frac{f_1(x)}{x^a} = k_1 \neq 0, \lim_{x \rightarrow 0} \frac{f_2(x)}{x^b} = k_2 \neq 0$$

\swarrow $b-a > 0$ since $a < b$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{f_1(x) \pm f_2(x)}{x^a} = \lim_{x \rightarrow 0} \left(\frac{f_1(x)}{x^a} \pm x^{b-a} \frac{f_2(x)}{x^b} \right) = k_1 \pm 0 \cdot k_2 = k_1 \neq 0$$

$$\Leftrightarrow f_1(x) \pm f_2(x) = O^*(x^a)$$

$$O(x^n) \pm O(x^m) = O(x^{\min(n,m)}) \text{ as } x \rightarrow 0$$

Let $f_1(x) = O(x^n)$, $f_2(x) = O(x^m)$, then we have

$$\lim_{x \rightarrow 0} \frac{f_1(x)}{x^n} = 0, \lim_{x \rightarrow 0} \frac{f_2(x)}{x^m} = 0. \text{ Suppose } \min(n,m) = n.$$

\swarrow $m-n \geq 0$

$$\text{Then } \lim_{x \rightarrow 0} \frac{f_1(x) \pm f_2(x)}{x^n} = \lim_{x \rightarrow 0} \left(\frac{f_1(x)}{x^n} \pm x^{m-n} \frac{f_2(x)}{x^m} \right) = 0 \pm 0 = 0$$

$$\Leftrightarrow f_1(x) \pm f_2(x) = O(x^n).$$

Related to fact ④, we also have

$$O(O(g(x))) = O(g(x)) \text{ Let } f(x) = O(O(g(x))), \text{ then } f(x) = O(h(x)),$$

where $h(x) = O(g(x))$. So we have

$$|f(x)| \leq A|h(x)|, \lim_{x \rightarrow c} \frac{h(x)}{g(x)} = 0 \Rightarrow \left| \frac{f(x)}{g(x)} \right| \leq A \left| \frac{h(x)}{g(x)} \right| \rightarrow 0 \text{ as } x \rightarrow c$$

$$\Rightarrow \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = 0 \Leftrightarrow f(x) = o(g(x)).$$

$$o(O(g(x))) = o(g(x)) \text{ Let } f(x) = o(O(g(x))), \text{ then } f(x) = o(h(x)),$$

where $h(x) = O(g(x))$. So we have

$$\lim_{x \rightarrow c} \frac{f(x)}{h(x)} = 0, |h(x)| \leq A|g(x)| \Rightarrow \left| \frac{f(x)}{g(x)} \right| \leq A \left| \frac{f(x)}{h(x)} \right| \rightarrow 0 \text{ as } x \rightarrow c.$$

$$\Rightarrow \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = 0 \Leftrightarrow f(x) = o(g(x)).$$

Stolz' Theorem

Let b_1, b_2, b_3, \dots be a strictly monotone sequence.

If either $\lim_{n \rightarrow \infty} a_n = 0 = \lim_{n \rightarrow \infty} b_n$

or $\lim_{n \rightarrow \infty} b_n = \pm \infty$,

then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$

provided the right side exists as a number or $\pm \infty$.

Examples ① Find $\lim_{n \rightarrow \infty} \frac{1^p + 2^p + \dots + n^p}{n^{p+1}}$, where p is a positive integer.

$$a_n = 1^p + 2^p + \dots + n^p, \quad b_n = n^{p+1} \nearrow +\infty$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)^p}{(n+1)^{p+1} - n^{p+1}} = \lim_{n \rightarrow \infty} \frac{(n+1)^p}{(p+1)n^p + o(n^p)} \\ &= \lim_{n \rightarrow \infty} \frac{(1 + \frac{1}{n})^p}{p+1 + o(1)} = \frac{1}{p+1}. \quad \therefore \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{1}{p+1}. \end{aligned}$$

② If $\lim_{n \rightarrow \infty} x_n = x$, then prove that $\lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = x$.
(Here x is a number or $\pm \infty$).

$$a_n = x_1 + x_2 + \dots + x_n, \quad b_n = n \nearrow +\infty$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n \rightarrow \infty} \frac{x_{n+1}}{1} = x. \quad \therefore \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = x.$$

③ Let $a_n = 1 - 2 + 3 - 4 + \dots + (-1)^{n-1}n$ and $b_n = n^2 \nearrow +\infty$.
then $\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n \rightarrow \infty} \frac{(-1)^n(n+1)}{2n+1}$ doesn't exist, but $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$
since $|a_n| \leq \frac{n+1}{2}$.

④ Let $0 < x_1 < 1$ and $x_{n+1} = x_n(1 - x_n)$ for $n=1, 2, 3, \dots$.
Prove that x_1, x_2, x_3, \dots decreases to 0 and $\lim_{n \rightarrow \infty} nx_n = 1$.

We will show $0 < x_n < 1$ and $x_{n+1} < x_n$ by math induction. We are given $0 < x_1 < 1$. Now

$0 < x_n < 1 \Rightarrow 0 < 1 - x_n < 1 \Rightarrow 0 < x_{n+1} = x_n(1 - x_n) < 1$,
completing induction step for $0 < x_n < 1$.

Next $x_{n+1} = x_n - x_n^2 < x_n$.

By monotone sequence theorem, $\lim_{n \rightarrow \infty} x_n = x$ exists.

$$x_{n+1} = x_n(1 - x_n) \Rightarrow x = x(1 - x) = x - x^2 \Rightarrow x = 0.$$

By Stolz' theorem, since $1/x_n \nearrow +\infty$,

$$\begin{aligned} \lim_{n \rightarrow \infty} nx_n &= \lim_{n \rightarrow \infty} \frac{n}{1/x_n} = \lim_{n \rightarrow \infty} \frac{1}{1/x_{n+1} - 1/x_n} \\ &= \lim_{n \rightarrow \infty} \frac{x_{n+1}x_n}{x_n - x_{n+1}} = \lim_{n \rightarrow \infty} \frac{x_n(1 - x_n)x_n}{x_n^2} \\ &= \lim_{n \rightarrow \infty} (1 - x_n) = 1. \end{aligned}$$

Remarks For example ④, alternatively we can show $\lim_{n \rightarrow \infty} nx_n = 1$ by the $\frac{0}{0}$ version of Stolz' Theorem as follow. As before, we have $x_n \neq 0$.

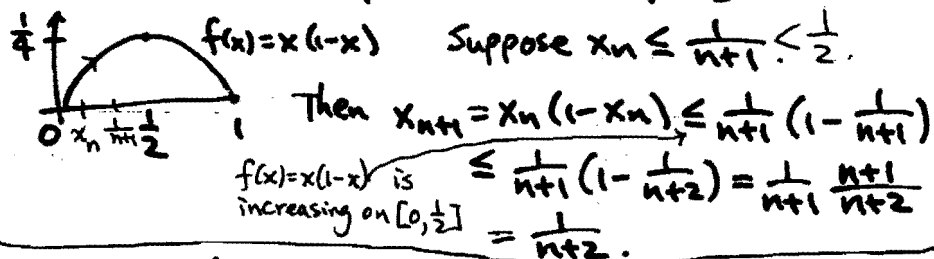
First we show $n \times n$ is increasing and bounded above.

Note $(n+1)x_{n+1} - nx_n = (n+1)x_n(1-x_n) - nx_n$
 $= x_n((n+1)(1-x_n) - n)$
 $= x_n(1 - (n+1)x_n) \geq 0$

$$\Leftrightarrow 1 - (n+1)x_n \geq 0 \Leftrightarrow x_n \leq \frac{1}{n+1}$$

Next, we show $x_n \leq \frac{1}{n+1}$ for $n \geq 1$ by math induction.

$$x_2 = x_1(1-x_1) = \frac{1}{4} - (x_1 - \frac{1}{2})^2 \leq \frac{1}{4} < \frac{1}{3}$$



Now $x_n \leq \frac{1}{n+1} \Rightarrow nx_n \leq (n+1)x_n < 1$. So $\lim_{n \rightarrow \infty} nx_n$ exists.

Let $L = \lim_{n \rightarrow \infty} nx_n$. Then recall $x_n \geq 0$. So

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} n x_n = \lim_{n \rightarrow \infty} \frac{x_n}{\frac{1}{n}} \stackrel{\frac{0}{0}}{=} \lim_{n \rightarrow \infty} \frac{x_{n+1} - x_n}{\frac{1}{n+1} - \frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} -n(n+1)(x_{n+1} - x_n) = \lim_{n \rightarrow \infty} n(n+1)x_n^2 \\ &= \lim_{n \rightarrow \infty} (nx_n)^2 \frac{n+1}{n} = L^2 \Rightarrow L=1. \end{aligned}$$

Proof of Stolz' Theorem Suffices to consider $\underbrace{b_n < b_{n+1}}_{(*)}$
for $n=1, 2, 3, \dots$. Let $\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L$.

First, suppose $\lim_{n \rightarrow \infty} a_n = 0 = \lim_{n \rightarrow \infty} b_n$. Then $b_n \nearrow 0$.

Case LER

$$\textcircled{1} \forall \varepsilon > 0 \exists K \in \mathbb{N} \quad n \geq K \Rightarrow L - \varepsilon < \frac{a_{n+1} - a_n}{b_{n+1} - b_n} < L + \varepsilon.$$

② by (*), $b_{n+1} - b_n > 0$
 $\Rightarrow (L - \varepsilon)(b_{n+1} - b_n) < a_{n+1} - a_n < (L + \varepsilon)(b_{n+1} - b_n)$

$$\textcircled{3} \Rightarrow (L-\varepsilon) \underbrace{\sum_{j=n}^{\infty} (b_{j+1} - b_j)}_{= 0 - b_n} < \underbrace{\sum_{j=n}^{\infty} (a_{j+1} - a_j)}_{= 0 - a_n} < (L+\varepsilon) \underbrace{\sum_{j=n}^{\infty} (b_{j+1} - b_j)}_{= 0 - b_n}$$

$$\textcircled{4} \quad -b_n > 0 \implies L - \varepsilon < \frac{a_n}{b_n} < L + \varepsilon \quad \therefore \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L.$$

Case $L = +\infty$ $\forall r \in \mathbb{R} \exists K, n \geq K \Rightarrow r < \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$

Follow steps (2) to (4) above to get $r < \frac{a_n}{b_n}$.

$$\therefore \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = +\infty = L.$$

Case $L = -\infty$ $\forall r \in \mathbb{R} \exists K, n \geq K \Rightarrow \frac{a_{n+1} - a_n}{b_{n+1} - b_n} < r$.

Again follow steps (2) to (4) above to get $\frac{a_n}{b_n} < r$.

$$\therefore \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = -\infty = L.$$

Next, suppose $\lim_{n \rightarrow \infty} b_n = +\infty$ (Since $b_n \uparrow$, $-\infty$ is not possible)

Case $L \in \mathbb{R}$

Modify step ① with $\frac{\varepsilon}{2}$ replacing ε .

Modify step ③ with $\{m \text{ replacing } \infty\}$ to get $\{K \text{ replacing } n\}$

$$(L - \frac{\varepsilon}{2})(b_m - b_K) < a_m - a_K < (L + \frac{\varepsilon}{2})(b_m - b_K)$$

$$\Rightarrow \left| \frac{a_m - a_K}{b_m - b_K} - L \right| < \frac{\varepsilon}{2}$$

By expansion, we can check that ^{less than $\varepsilon/2$}

$$\frac{a_m}{b_m} - L = \underbrace{\frac{a_K - L b_K}{b_m}}_{\text{doesn't depend on } m} + \underbrace{\left(1 - \frac{b_K}{b_m}\right)}_{< 1 \text{ as } b_n \uparrow +\infty} \left(\frac{a_m - a_K}{b_m - b_K} - L \right)$$

As $b_m \uparrow +\infty$, $\exists M$, $m \geq M \Rightarrow \frac{2}{\varepsilon} |a_K - L b_K| < b_m$

Then $\left| \frac{a_m}{b_m} - L \right| < \frac{\varepsilon}{2} + 1 \cdot \frac{\varepsilon}{2} = \varepsilon \therefore \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$

Case $L = +\infty$

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} \rightarrow +\infty \Rightarrow \frac{a_{n+1} - a_n}{b_{n+1} - b_n} > 1$$

$$\Rightarrow a_{n+1} - a_n > b_{n+1} - b_n > 0 \quad \text{for } n \text{ large}$$

$$\Rightarrow a_n \uparrow +\infty \text{ since } \lim_{n \rightarrow \infty} (b_{n+1} - b_n) = \sum_{n=1}^{\infty} (b_{n+1} - b_n) \text{ is } +\infty$$

$$\text{Now } \lim_{n \rightarrow \infty} \frac{b_{n+1} - b_n}{a_{n+1} - a_n} = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = 0 \text{ by above...}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = +\infty$$

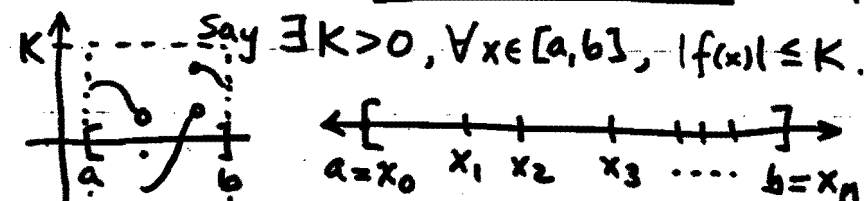
Case $L = -\infty$ is similar to Case $L = +\infty$.

Chapter 9 Riemann Integral

Proper Integral

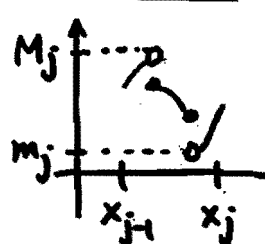
may be discontinuous

Setting: Let $f(x)$ be a bounded function on a closed and bounded interval $[a, b]$,



$P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$ iff $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$.

Let $\Delta x_j = x_j - x_{j-1}$. $\|P\| = \max \{\Delta x_1, \dots, \Delta x_n\}$ is called the mesh of P .



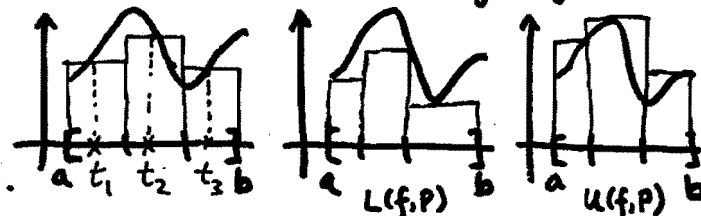
Since $f(x)$ may be discontinuous on $[a, b]$, we consider

$$m_j = \inf \{f(x) : x \in [x_{j-1}, x_j]\}$$

$$M_j = \sup \{f(x) : x \in [x_{j-1}, x_j]\}$$

Definitions

Let $t_j \in [x_{j-1}, x_j]$ for $j = 1, 2, \dots, n$.



$S = \sum_{j=1}^n f(t_j) \Delta x_j$ is a Riemann sum with respect to P .

$L(f, P) = \sum_{j=1}^n m_j \Delta x_j$ is the lower Riemann sum w.r.t. P .

$U(f, P) = \sum_{j=1}^n M_j \Delta x_j$ is the upper Riemann sum w.r.t. P .