Solutions of Math 2033 Past Exam Problems

Countability

(101) Let $W=\{(a,b): a\in\mathbb{Q}, b\in\mathbb{Q}, a\neq b\}$. Then $W\subseteq \underbrace{\mathbb{Q}\times\mathbb{Q}}_{\text{Countable by product theorem}}$. So W is countable by countable subset theorem.

Let $S_{(a,b)}=\{(x,y): x^2+y^2=a^2,\ y=x^2-x^3+b\}$.

Now $\left.\begin{array}{l} x^2+y^2=a^2 \\ y=x^2-x^3+b \end{array}\right\} \Rightarrow \left.\begin{array}{l} \underbrace{x^2+(x^2-x^3+b)^2=a^2}_{\text{degree 6}} \\ y=x^2-x^3+b \end{array}\right\} \Rightarrow$ $\exists$ at most 6 possible x's. Each such x gives one y.

Hence, $S_{(a,b)}$ has at most 6 elements $\Rightarrow S_{(a,b)}$ is countable.

Finally, $S=\bigcup_{(a,b)\in W} S_{(a,b)}$ is countable by the countable union theorem. (W countable)

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(102) For $m\in\mathbb{Q}$, let $S_m=\{(x,y): x+y=mx^2-x^3,\ mx+y^4=x^6-7mx^3+2\}$.

Now $\left.\begin{array}{l} x+y=mx^2-x^3 \\ mx+y^4=x^6-7mx^3+2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y=mx^2-x^3-x \\ \underbrace{mx+(mx^2-x^3-x)^4-x^6+7mx^3-2=0}_{\text{degree 12}} \end{array}\right\} \Rightarrow$ $\exists$ at most 12 possible x's. Each such x gives one y.

Hence S_m has at most 12 elements $\Rightarrow S_m$ is countable.

Finally, $S=\bigcup_{m\in\mathbb{Q}} S_m$ is countable by the countable union theorem. ($\mathbb{Q}$ countable)

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(103)(a) For $(m,n)\in\mathbb{Q}\times\mathbb{Q}$, let $S_{(m,n)}=\{(x,y): y=x^3+mx+n,\ mx^2-ny^2=1\}$.

Now $\left.\begin{array}{l} y=x^3+mx+n \\ mx^2-ny^2=1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} y=x^3+mx+n \\ \underbrace{mx^2-n(x^3+mx+n)^2-1=0}_{\text{degree 6}} \end{array}\right\} \Rightarrow$ $\exists$ at most 6 possible x's. Each such x gives on y.

Hence $S_{(m,n)}$ has at most 6 elements $\Rightarrow S_{(m,n)}$ is countable.

Finally, $S=\bigcup_{(m,n)\in\mathbb{Q}\times\mathbb{Q}} S_{(m,n)}$ is countable by the countable union theorem. ($\mathbb{Q}\times\mathbb{Q}$ countable)

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(103)(b) Let $S=\mathbb{Q}\cap(0,+\infty)$ and $T=\mathbb{Q}\cap[0,+\infty)$. Since $S,T\subseteq\mathbb{Q}$ and $\mathbb{Q}$ is countable, so S,T are countable by the countable subset theorem.

Let $W=\{a+b(2^c\pi^d): a,b\in S,\ c,d\in T\}$. For $(a,b,c,d)\in S\times S\times T\times T$, let $W_{(a,b,c,d)}=\{a+b(2^c\pi^d)\}$, then $W_{(a,b,c,d)}$ has 1 element $\Rightarrow W_{(a,b,c,d)}$ is countable.

Then $W=\bigcup_{(a,b,c,d)\in \underbrace{S\times S\times T\times T}_{\text{Countable by product theorem}}} W_{(a,b,c,d)}$ ($\leftarrow$ countable) is countable by the countable union theorem.

Finally, $\underbrace{(0,+\infty)}_{\text{uncountable}}\setminus \underbrace{W}_{\text{Countable}}$ is uncountable $\Rightarrow$ infinite. So there exist infinitely many positive real numbers that are not equal to any number of the form $a+b(2^c\pi^d)$, where $a,b\in\mathbb{Q}\cap(0,+\infty)$ and $c,d\in\mathbb{Q}\cap[0,+\infty)$.

(104) Let $W=\{2^{a+b\sqrt{2}} : a,b\in\mathbb{Q}\}$. For $(a,b)\in\mathbb{Q}\times\mathbb{Q}$, let $W_{(a,b)}=\{2^{a+b\sqrt{2}}\}$, then $W_{(a,b)}$ has 1 element $\Rightarrow W_{(a,b)}$ is countable.

Then $W=\bigcup_{(a,b)\in\mathbb{Q}\times\mathbb{Q}} \underbrace{W_{(a,b)}}_{\text{countable}}$ is countable by the countable union theorem.

Finally, $\underbrace{(0,\infty)}_{\text{uncountable}}\setminus \underbrace{W}_{\text{countable}}$ is uncountable $\Rightarrow$ nonempty. So there exists a positive real number, which doesn't equal to any number of the form $2^{a+b\sqrt{2}}$, where $a,b\in\mathbb{Q}$.

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(105) Note $5^x+7^y=\sqrt{r} \Leftrightarrow (5^x+7^y)^2=r$. Since S is countable, $S\times S$ is countable by product theorem.

Let $W=\{(5^x+7^y)^2 : x,y\in S\}$. For $(x,y)\in S\times S$, let $W_{(x,y)}=\{(5^x+7^y)^2\}$, then $W_{(x,y)}$ has 1 element $\Rightarrow W_{(x,y)}$ is countable.

Then $W=\bigcup_{(x,y)\in S\times S} \underbrace{W_{(x,y)}}_{\text{countable}}$ is countable by the countable union theorem.

Finally, $\underbrace{(0,\infty)}_{\text{uncountable}}\setminus \underbrace{W}_{\text{countable}}$ is uncountable $\Rightarrow$ nonempty. So there exists a positive real number r such that $5^x+7^y=\sqrt{r}$ has no solution with $x,y\in S$.

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(106) (a) Let $S=\mathbb{Q}\cap(0,+\infty)$, then $S\subseteq\mathbb{Q}$ and $\mathbb{Q}$ countable $\Rightarrow S$ is countable by the countable subset theorem.

Let $W=\left\{\frac{a\sqrt{2}+b}{c+d\pi} : a,b,c,d\in S\right\}$. For $(a,b,c,d)\in S\times S\times S\times S$, let $W_{(a,b,c,d)}=\left\{\frac{a\sqrt{2}+b}{c+d\pi}\right\}$, then $W_{(a,b,c,d)}$ has 1 element $\Rightarrow W_{(a,b,c,d)}$ is countable.

Then $W=\bigcup_{(a,b,c,d)\in S\times S\times S\times S} \underbrace{W_{(a,b,c,d)}}_{\text{countable}}$ is countable by the countable union theorem.

Finally, $\underbrace{(\underbrace{(0,+\infty)}_{\text{uncountable}}-\underbrace{\mathbb{Q}}_{\text{countable}})}_{\text{uncountable}}\setminus \underbrace{W}_{\text{countable}}$ is uncountable $\Rightarrow$ infinite.

So there exist infinitely many positive irrational numbers that are not equal to any number of the form $\frac{a\sqrt{2}+b}{c+d\pi}$, where $a,b,c,d\in S$.

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(106) (b) For every $c\in\mathbb{Q}$, let $W_c=\{t : t\in\mathbb{R} \text{ and } f(t)=c\}$. For every real number r, $f(r)\in\mathbb{Q}$ and $r\in W_{f(r)}$. So $\mathbb{R}=\bigcup_{c\in\mathbb{Q}} W_c$.

Assume every W_c is countable, then $\bigcup_{c\in\mathbb{Q}} \underbrace{W_c}_{\text{countable}}$ is countable by the countable union theorem, contradicting $\mathbb{R}$ is uncountable.

Therefore, there exists an uncountable $S=W_c$ and $\forall x,y\in S=W_c$, we have $f(x)=c=f(y)$.

Infimum – Supremum

(203) $\inf D = 3$ and $\sup D = 5 \Rightarrow D \subseteq [3,5]$

$$\begin{matrix} 2 < x < \pi \\ 3 \leq y \leq 5 \end{matrix} \Rightarrow 60 = 2(3+3^3) < xy + xy^3 = x(y+y^3) < \pi(5+5^3) = 130\pi$$

So D is bounded below by 60 and bounded above by 130π.

Let $x_n = 2 + \frac{1}{n} \in (2,\pi] \cap \mathbb{Q}$. By the infimum limit theorem, since $\inf D = 3$, there are $y_n \in D$ with $\lim_{n\to\infty} y_n = 3$. Then $x_n y_n + x_n y_n^3 \in A$ and $\lim_{n\to\infty} x_n y_n + x_n y_n^3 = 2\cdot 3 + 2\cdot 3^3 = 60$. $\therefore \inf A = 60$.

Let $x_n' = \frac{[10^n \pi]}{10^n} \in (2,\pi] \cap \mathbb{Q}$. By the supremum limit theorem, since $\sup D = 5$, there are $y_n' \in D$ with $\lim_{n\to\infty} y_n' = 5$. Then $x_n' y_n' + x_n' y_n'^3 \in A$ and $\lim_{n\to\infty} x_n' y_n' + x_n' y_n'^3 = \pi \cdot 5 + \pi \cdot 5^3 = 130\pi$. $\therefore \sup A = 130\pi$.

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(204) $\inf A = 1$ and $\sup A = 3 \Rightarrow A \subseteq [1,3]$

$$\begin{matrix} x \in (2,4) \cap \mathbb{Q} \\ y \in A \end{matrix} \Rightarrow \begin{matrix} 2 < x < 4 \\ 1 \leq y \leq 3 \end{matrix} \Rightarrow \underbrace{\sqrt{2\times 2(15+2\cdot 1)}}_{=2\sqrt{17}} < \sqrt{2x(15+xy)} < \underbrace{\sqrt{2\times 4(15+4\times 3)}}_{=6\sqrt{6}}$$

$\Rightarrow B$ is bounded.

It remains to find sequences in B with limits $2\sqrt{17}$ and $6\sqrt{6}$ respectively.

Let $x_n = 2 + \frac{1}{n}$ and $x_n' = 4 - \frac{1}{n}$. By infimum limit theorem, since $\inf A = 1$, $\exists\, y_n \in A$ such that $\lim_{n\to\infty} y_n = 1$. By supremum limit theorem, since $\sup A = 3$, $\exists\, y_n' \in A$ such that $\lim_{n\to\infty} y_n' = 3$.

Then $\sqrt{2x_n(15 + x_n y_n)} \in B$ and $\lim_{n\to\infty} \sqrt{2x_n(15+x_n y_n)} = \sqrt{2\times 2(15+2\times 1)} = 2\sqrt{17}$

and $\sqrt{2x_n'(15 + x_n' y_n')} \in B$ and $\lim_{n\to\infty} \sqrt{2x_n'(15+x_n' y_n')} = \sqrt{2\times 4(15+4\times 3)} = 6\sqrt{6}$

$\therefore \inf B = 2\sqrt{17}$ and $\sup B = 6\sqrt{6}$.

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(205) Since $\inf A = 1$ and $\sup A = 5$, so $y \in A \Rightarrow 1 \leq y \leq 5 \Rightarrow -1 \leq -\frac{1}{y} \leq -\frac{1}{5}$

Since $\inf B = 0$ and $\sup B = 1$, so $x \in B \Rightarrow 0 \leq x \leq 1 \Rightarrow 2 \leq 3 - x \leq 3$.

$\Rightarrow \frac{1}{3} \leq \frac{1}{3-x} \leq \frac{1}{2}$

$$-\frac{2}{3} = 1\left(\frac{1}{3}\right) - 1 \leq \frac{y}{3-x} - \frac{1}{y} \leq 5\left(\frac{1}{2}\right) - \frac{1}{5} = \frac{23}{10}.$$

So C is bounded above by $\frac{23}{10}$ and below by $-\frac{2}{3}$.

By infimum limit theorem and supremum limit theorem,

$\inf A = 1 \Rightarrow \exists\, y_n \in A$ with $\lim_{n\to\infty} y_n = 1$

$\sup A = 5 \Rightarrow \exists\, y_n' \in A$ with $\lim_{n\to\infty} y_n' = 5$

$\inf B = 0 \Rightarrow \exists\, x_n \in B$ with $\lim_{n\to\infty} x_n = 0$

$\sup B = 1 \Rightarrow \exists\, x_n' \in B$ with $\lim_{n\to\infty} x_n' = 1$

(205 cont.) Then $\frac{y_n}{3-x_n}-\frac{1}{y_n}\in C$ and has limit $\frac{1}{3-0}-\frac{1}{1}=-\frac{2}{3}$

and $\frac{y_n'}{3-x_n'}-\frac{1}{y_n'}\in C$ and has limit $\frac{5}{3-1}-\frac{1}{5}=\frac{23}{10}$.

$\therefore$ $\inf C=-2/3$ and $\sup C=\frac{23}{10}$.

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(206) (a) $\inf D=1$ and $\sup D=5 \Rightarrow D\subseteq[1,5] \Rightarrow \forall x\in D,\ 1\le x\le 5,\ \frac{1}{5}\le\frac{1}{x}\le 1,\ -1\le -\frac{1}{x}\le -\frac{1}{5}$

$y\in[0,\sqrt{2})\cap\mathbb{Q} \Rightarrow 0\le y<\sqrt{2},\ \sqrt{2}\le y+\sqrt{2}<2\sqrt{2},\ x(y+\sqrt{2})-\frac{1}{x}\le 5(2\sqrt{2})-\frac{1}{5}=10\sqrt{2}-\frac{1}{5}$

Hence $10\sqrt{2}-\frac{1}{5}$ is an upper bound of E.

Since $\sup D=5$, by supremum limit theorem, $\exists x_n\in D$ such that $\lim_{n\to\infty}x_n=5$.

Let $y_n=\frac{[10^n\sqrt{2}]}{10^n}$, then $y_n\in[0,\sqrt{2})\cap\mathbb{Q}$. Hence $x_n(y_n+\sqrt{2})-\frac{1}{x_n}\to 5(\sqrt{2}+\sqrt{2})-\frac{1}{5}$
$=10\sqrt{2}-\frac{1}{5}$.

By supremum limit theorem, we get $\sup E=10\sqrt{2}-\frac{1}{5}$.

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(206) (b) To show $\sup B=w$, by supremum limit theorem, it is enough to show ① w is an upper bound of B and ② $\exists w_n\in B$ such that $\lim_{n\to\infty}w_n=w$.

For ①, we have $\forall b\in B$, since $B\subseteq C$, so $b\in C$, then $b\le\sup C=w$.
So w is an upper bound of B.

For ②, since $\sup A=w$, by supremum limit theorem, $\exists w_n\in A$ such that $\lim_{n\to\infty}w_n=w$. Since $A\subseteq B$, $w_n\in B$ and $\lim_{n\to\infty}w_n=w$. We are done.

Limit of Sequences

(302) $\frac{3a_n^2+1}{a_n^2+1} \to \frac{4}{2}=2$, $\frac{nb_n}{n+2} \to 1\cdot 1=1$ $\left|\frac{3a_n^2+1}{a_n^2+1}-2\right| = \frac{|a_n^2-1|}{a_n^2+1} = \frac{|a_n+1|}{a_n^2+1}|a_n-1| < \frac{3}{1}|a_n-1|$

$|a_n-1|<1 \Rightarrow a_n \in (0,2)$
$\Rightarrow a_n+1 \in (1,3)$

$\left|\frac{nb_n}{n+2}-1\right| = \left|\frac{nb_n-n}{n+2}-\frac{2}{n+2}\right| \le \frac{n}{n+2}|b_n-1| + \frac{2}{n+2} \le |b_n-1| + \frac{2}{n}$

$\frac{1}{n+2}<\frac{1}{n}$, $\frac{n}{n+2}<1$

For every $\varepsilon>0$, since $\lim_{n\to\infty} a_n = 1$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |a_n-1|<1$ $\Rightarrow a_n \in (0,2) \Rightarrow a_n+1 \in (1,3)$. $\exists K_2 \in \mathbb{N}$ such that $n \ge K_2 \Rightarrow |a_n-1| < \frac{\varepsilon}{9}$

Since $\lim_{n\to\infty} b_n = 1$, $\exists K_3 \in \mathbb{N}$ such that $n \ge K_3 \Rightarrow |b_n-1| < \frac{\varepsilon}{3}$.

By Archimedean Principle, $\exists K_4 \in \mathbb{N}$ such that $K_4 > \frac{6}{\varepsilon}$.

Let $K = \max\{K_1, K_2, K_3, K_4\}$. Then $n \ge K \Rightarrow n \ge K_1, K_2, K_3, K_4$

$\Rightarrow a_n+1 \in (1,3)$, $|a_n-1|<\frac{\varepsilon}{9}$, $|b_n-1|<\varepsilon/3$, $n > \frac{6}{\varepsilon} \Leftrightarrow \frac{2}{n} < \frac{\varepsilon}{3}$

$$\Rightarrow \left|\left(\frac{3a_n^2+1}{a_n^2+1}+\frac{nb_n}{n+2}\right)-3\right| = \left|\left(\frac{3a_n^2+1}{a_n^2+1}-2\right)+\left(\frac{nb_n}{n+2}-1\right)\right|$$

$$\le \left|\frac{3a_n^2+1}{a_n^2+1}-2\right| + \left|\frac{nb_n}{n+2}-1\right| \le 3|a_n-1| + |b_n-1| + \frac{2}{n} < 3\left(\frac{\varepsilon}{9}\right)+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$$

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(303) Scratch Work $\frac{3+a_n^2}{a_n+1} \to \frac{3+1}{1+1}=2$, $\frac{2n}{4+n} \to 2$ $\frac{3+a_n^2}{a_n+1}-2 = \frac{a_n^2-2a_n+1}{a_n+1} = \frac{(a_n-1)^2}{a_n+1}$

$\varepsilon=1$ $\exists K_1 \in \mathbb{N}$ such that $|a_n-1|<1 \Leftrightarrow a_n \in (0,2) \Rightarrow a_n+1 \in (1,3) \Rightarrow \frac{1}{a_n+1} \in (\frac{1}{3},1)$

$\left|\frac{(a_n-1)^2}{a_n+1}\right| < 1\,|a_n-1|^2 < \frac{\varepsilon}{2}$ $\Leftrightarrow |a_n-1| < \sqrt{\frac{\varepsilon}{2}}$.

$\left|\frac{2n}{4+n}-2\right| = \left|\frac{-8}{4+n}\right| = \frac{8}{4+n} < \frac{8}{n} < \frac{\varepsilon}{2} \Leftrightarrow n > \frac{16}{\varepsilon}$.

Since $\lim_{n\to\infty} a_n = 1$, for $1>0$, $\exists K_1 \in \mathbb{N}$ such that $|a_n-1|<1 \Rightarrow a_n \in (0,2)$ $\Rightarrow \frac{1}{a_n+1} \in (\frac{1}{3},1)$. $\forall \varepsilon>0$, $\exists K_2 \in \mathbb{N}$ such that $n \ge K_2 \Rightarrow |a_n-1| < \sqrt{\frac{\varepsilon}{2}}$.

Let $K > \max\{K_1, K_2, \frac{16}{\varepsilon}\}$. Then $n \ge K \Rightarrow n \ge K_1, n \ge K_2, n > \frac{16}{\varepsilon}$

$$\Rightarrow \left|\left(\frac{3+a_n^2}{a_n+1}+\frac{2n}{4+n}\right)-4\right| = \left|\left(\frac{3+a_n^2}{a_n+1}-2\right)+\left(\frac{2n}{4+n}-2\right)\right| \le \left|\frac{3+a_n^2}{a_n+1}-2\right| + \left|\frac{2n}{4+n}-2\right|$$

$$= \frac{|a_n-1|^2}{a_n+1} + \frac{8}{4+n} < 1\times|a_n-1|^2 + \frac{8}{n} < \left(\sqrt{\frac{\varepsilon}{2}}\right)^2 + \frac{\varepsilon}{2} = \varepsilon.$$

(304) **Sketch** $\frac{a_n}{a_n^2+3} \to \frac{1}{4}$, $\frac{3n^2}{1+4n^2} \to \frac{3}{4}$, $\frac{a_n}{n} \to 0$

① $\left|\frac{a_n}{a_n^2+3} - \frac{1}{4}\right| = \frac{|a_n^2-4a_n+3|}{4(a_n^2+3)} = \frac{|(a_n-1)(a_n-3)|}{4a_n^2+12} < \frac{3|a_n-3|}{12} = \frac{|a_n-3|}{4} < \frac{\varepsilon}{3}$

$|a_n-3|<1 \Rightarrow a_n \in (2,4)$
$\Rightarrow a_n - 1 \in (1,3)$

$|a_n-3| < \frac{4\varepsilon}{3}$

② $\left|\frac{3n^2}{1+4n^2} - \frac{3}{4}\right| = \frac{3}{4(1+4n^2)} < \frac{3}{16n^2} < \frac{\varepsilon}{3} \Leftrightarrow \frac{9}{16\varepsilon} < n^2 \Leftrightarrow n > \frac{3}{4\sqrt{\varepsilon}}$.

③ $\left|\frac{a_n}{n}\right| < \frac{4}{n} < \varepsilon/3 \Leftrightarrow n > \frac{12}{\varepsilon}$.

$|a_n-3|<1 \Rightarrow a_n \in (2,4)$

Solution $\forall \varepsilon > 0$, since $\lim_{n\to\infty} a_n = 3$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |a_n - 3| < 1$ and $\exists K_2 \in \mathbb{N}$ such that $n \ge K_2 \Rightarrow |a_n - 3| < \frac{4\varepsilon}{3}$. By Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \max\{K_1, K_2, \frac{3}{4\sqrt{\varepsilon}}, \frac{12}{\varepsilon}\}$. Then

$n \ge K \Rightarrow$ $n \ge K_1$, $n \ge K_2$, $n > \frac{3}{4\sqrt{3}}$, $n > \frac{12}{\varepsilon}$ $\Rightarrow \left|\frac{a_n}{a_n^2+3} + \frac{3n^2}{1+4n^2} + \frac{a_n}{n} - 1\right| = \left|\left(\frac{a_n}{a_n^2+3} - \frac{1}{4}\right) + \left(\frac{3n^2}{1+4n^2} - \frac{3}{4}\right) + \left(\frac{a_n}{n} - 0\right)\right|$

$\le \left|\frac{a_n}{a_n^2+3} - \frac{1}{4}\right| + \left|\frac{3n^2}{1+4n^2} - \frac{3}{4}\right| + \left|\frac{a_n}{n}\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. see sketch above.

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(305) **Sketch Work** As $n \to \infty$, $\frac{4n-1}{n+3} \to 4$, $b_n \to 2 \Rightarrow \frac{-2}{b_n} \to -1$, $\frac{b_n}{n} \approx \frac{2}{n} \to 0$.

$\left|\frac{4n-1}{n+3} - 4\right| = \frac{13}{n+3} < \frac{13}{n} < \frac{\varepsilon}{3}$ for $n > \frac{39}{\varepsilon}$

$\left|-\frac{2}{b_n} - (-1)\right| = \frac{|b_n - 2|}{b_n > 0} < \frac{|b_n-2|}{1} < \varepsilon/3$

when $|b_n - 2| < 1$; $b_n \in (1,3)$

$\left|\frac{b_n}{n} - 0\right| = \frac{|b_n|}{n} < \frac{3}{n} < \frac{\varepsilon}{3}$ for $n > \frac{9}{\varepsilon}$.

$\forall \varepsilon > 0$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |b_n - 2| < \frac{\varepsilon}{3}$.
$1 > 0 \Rightarrow \exists K_2 \in \mathbb{N}$ such that $n \ge K_2 \Rightarrow |b_n - 1| < 1 \Rightarrow b_n \in (1,3)$

By Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \max\{K_1, K_2, \frac{39}{\varepsilon}, \frac{9}{\varepsilon}\}$.

Then $n \ge K \Rightarrow n \ge K_1$, $n \ge K_2$, $n > \frac{39}{\varepsilon}$ and $n > \frac{9}{\varepsilon}$

$$\Rightarrow \left|\left(\frac{4n-1}{n+3} - \frac{2}{b_n} + \frac{b_n}{n}\right) - 3\right| = \left|\left(\frac{4n-1}{n+3} - 4\right) + \left(-\frac{2}{b_n} - (-1)\right) + \left(\frac{b_n}{n} - 0\right)\right|$$

$$\le \left|\frac{4n-1}{n+3} - 4\right| + \left|-\frac{2}{b_n} - (-1)\right| + \left|\frac{b_n}{n} - 0\right| < \frac{13}{n+3} + \frac{|b_n-2|}{1} + \frac{3}{n}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Limit of Recurrence Relations

(401) $x_1=1 > x_2=\frac{1+1}{5}=0.4 > x_3=\frac{0.4^3+0.4}{5}=\frac{0.464}{5}$, $x=\frac{x^3+x}{5} \Rightarrow x^3-4x=x(x^2-4)=0$
$\Rightarrow x=0,\pm 2$

(a) We claim $0<x_{n+1}\le x_n$ for $n=1,2,3,\ldots$.

Case $n=1$: $0<x_2=0.4<x_1=1$.

Suppose case n is true, i.e. $0<x_{n+1}\le x_n$. Then $0<x_{n+1}^3\le x_n^3$ and so $0<x_{n+2}=\frac{x_{n+1}^3+x_{n+1}}{5}\le x_{n+1}=\frac{x_n^3+x_n}{5}$. By M.I., the claim is true.

By monotone sequence theorem, $\lim_{n\to\infty} x_n=x$ exists. Then $x=\frac{x^3+x}{5}$.

So $x=0$ or 2 or -2. Since $0<x_n\le x_1=1$, the claim implies $x=0$.

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(402) $x_1=2$, $x_2=\frac{22}{3}+\frac{8}{3}=10$, $x_3=\frac{22}{3}+\frac{16}{30}=\frac{236}{30}=7.866\ldots$ $\quad 2 \quad 7\frac{26}{30} \quad 10$

$x=\frac{22}{3}+\frac{16}{3x}$
$\Rightarrow 3x^2-22x-16=0$
$(3x+2)(x-8)=0$
$\Rightarrow x=8$ or $-2/3$

We claim $0<x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$ for $n=1,2,3,\ldots$

Case $n=1$: $x_1=2\le x_3=7.866\ldots\le x_4=8.047\ldots\le x_2=10$.

Suppose case n is true, then $\frac{16}{3x_{2n-1}}\ge\frac{16}{3x_{2n+1}}\ge\frac{16}{3x_{2n+2}}\ge\frac{16}{3x_{2n}}>0$.

Adding $\frac{22}{3}$ to all parts, we get $x_{2n}\ge x_{2n+2}\ge x_{2n+3}\ge x_{2n+1}$.

Then $\frac{16}{3x_{2n}}\le\frac{16}{3x_{2n+2}}\le\frac{16}{3x_{2n+3}}\le\frac{16}{3x_{2n+1}}$. Adding $\frac{22}{3}$ to all parts, we get $x_{2n+1}\le x_{2n+3}\le x_{2n+4}\le x_{2n+2}$. By MI, the claim is true.

By nested interval theorem, $\lim_{n\to\infty} x_{2n+1}=a$ and $\lim_{n\to\infty} x_{2n}=b$ exist.

$a=\lim_{n\to\infty} x_{2n+1}=\lim_{n\to\infty}\left(\frac{22}{3}+\frac{16}{3x_{2n}}\right)=\frac{22}{3}+\frac{16}{3b}$, $b=\lim_{n\to\infty} x_{2n}=\lim_{n\to\infty}\left(\frac{22}{3}+\frac{16}{3x_{2n-1}}\right)=\frac{22}{3}+\frac{16}{3a}$

$\Rightarrow 3ab=22b+16$ and $3ab=22a+16 \Rightarrow a=b$. So $\lim_{n\to\infty} x_n=a$ by intertwining sequence theorem. Then $3a^2-22a-16=0\Rightarrow a=8$ or $-\frac{2}{3}$.

Since $2\le x_n\le 10$, so $\lim_{n\to\infty} x_n=8$.

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(405) Note $x_1=27>x_2=7>x_3=8-\sqrt{21}=3.4$ Suspect decreasing $\quad x=3$ or 12
$x=8-\sqrt{28-x}\Rightarrow (x-8)^2=28-x\Rightarrow x^2-15x+36=(x-12)(x-3)=0$

Claim: $27=x_1\ge x_n>x_{n+1}>3$

For $n=1$, $27=x_1=x_1>x_2=7>3$. Suppose $27\ge x_n>x_{n+1}>3$.

Then $1\le 28-x_n<28-x_{n+1}<25\Rightarrow 1\le\sqrt{28-x_n}<\sqrt{28-x_{n+1}}<5$

$27\ge 7=8-1\ge 8-\sqrt{28-x_n}=x_{n+1}>8-\sqrt{28-x_{n+1}}=x_{n+2}>8-5=3$.

By M.I., the claim holds.

By the monotone sequence theorem, $\lim_{n\to\infty} x_n=x$ exists. Then $x=\lim_{n\to\infty} x_{n+1}=\lim_{n\to\infty} 8-\sqrt{28-x_n}=\lim_{n\to\infty} 8-\sqrt{28-x}$. As above, $x=3$ or 12.

Since $x_2=7>x_3>\cdots>x$, $x=3$.

(406)(a) Sketch $x_1=0, x_2=3, x_3=2, x_4=\sqrt{6\frac{7}{9}}$ $x_1<x_3<x_4<x_2$

Claim For $n=1,2,3,\cdots$, $x_{2n-1}<x_{2n+1}<x_{2n+2}<x_{2n}$.

Proof For $n=1$, $x_1=0<x_3=2<x_4=\sqrt{6\frac{7}{9}}<x_2=3$. Suppose $x_{2n-1}<x_{2n+1}<x_{2n+2}<x_{2n}$.

We have $x_{2n+1}=\sqrt{\frac{4}{9}x_{2n+1}^2+\frac{5}{9}x_{2n+1}^2}<\sqrt{\frac{4}{9}x_{2n+2}^2+\frac{5}{9}x_{2n+1}^2}=x_{2n+3}<\sqrt{\frac{4}{9}x_{2n+2}^2+\frac{5}{9}x_{2n+2}^2}=x_{2n+2}$

$x_{2n+1}<x_{2n+2}$ $\qquad$ $x_{2n+1}<x_{2n+2}$

and $x_{2n+3}=\sqrt{\frac{4}{9}x_{2n+3}^2+\frac{5}{9}x_{2n+3}^2}<\sqrt{\frac{4}{9}x_{2n+3}^2+\frac{5}{9}x_{2n+2}^2}=x_{2n+4}<\sqrt{\frac{4}{9}x_{2n+2}^2+\frac{5}{9}x_{2n+2}^2}=x_{2n+2}$

Combining, we have $x_{2n+1}<x_{2n+3}<x_{2n+4}<x_{2n+2}$. This proved the claim.

By nested interval theorem, let $\lim_{n\to\infty}x_{2n-1}=a$ and $\lim_{n\to\infty}x_{2n}=b$. Then

$a=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}\sqrt{\frac{4}{9}x_{2n}^2+\frac{5}{9}x_{2n-1}^2}=\sqrt{\frac{4}{9}b^2+\frac{5}{9}a^2}\Rightarrow a^2=\frac{4}{9}b^2+\frac{5}{9}a^2\Rightarrow a^2=b^2$.

Since $0=x_1<x_n$ for $n>1$, so $a,b\ge 0\Rightarrow a=b$. So $\{x_n\}$ converges by intertwining sequence theorem.

(406)(b)

$x_3^2=\frac{4}{9}x_2^2+\frac{5}{9}x_1^2$

$x_4^2=\frac{4}{9}x_3^2+\frac{5}{9}x_2^2$

$x_5^2=\frac{4}{9}x_4^2+\frac{5}{9}x_3^2$

$\vdots$

$x_{n-1}^2=\frac{4}{9}x_{n-2}^2+\frac{5}{9}x_{n-3}^2$

$x_n^2=\frac{4}{9}x_{n-1}^2+\frac{5}{9}x_{n-2}^2$

Adding the equations and cancelling the common terms, we get $\frac{5}{9}x_{n-1}^2+x_n^2=x_2^2+\frac{5}{9}x_1^2=9$

Taking limit as $n\to\infty$, $\frac{5}{9}a^2+a^2=9\Rightarrow a=\frac{9}{\sqrt{14}}$

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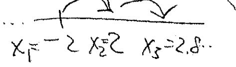
(407) (a) Scratch Work $x_1=-2$, $x_2=2$, $x_3=\sqrt{8}=2\sqrt{2}\approx 2.82\ldots$



Suspect increasing. $x=\sqrt{6+x}\Rightarrow x^2=6+x\Rightarrow x^2-x-6=(x-3)(x+2)\Rightarrow x=3$

Claim: $x_n<x_{n+1}<3$

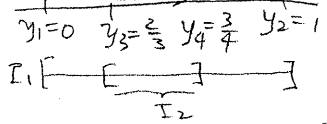
Proof Case $n=1$ is $x_1=-2<x_2=2<3$. Suppose $x_n<x_{n+1}<3$. Then $6+x_n<6+x_{n+1}<9$. $0<6+x_1$

So $x_{n+1}=\sqrt{6+x_n}<x_{n+2}=\sqrt{6+x_{n+1}}<3$, completing induction.

By monotone sequence theorem, $\lim_{n\to\infty}x_n=x$ exists. Then $x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\sqrt{6+x_n}$ $=\sqrt{6+x}$. So $x^2=6+x\Rightarrow x^2-x-6=(x-3)(x+2)=0$. Since $x>x_2=2$, $x=3$.

(407)(b) Scratch Work $y_1=0$, $y_2=1$, $y_3=\frac{2}{3}$, $y_4=\frac{3}{4}$

$I_1=[y_1,y_2]\supseteq I_2=[y_3,y_4]\supseteq\cdots\supseteq I_n=[y_{2n-1},y_{2n}]$



Claim: Let $I_n=[y_{2n-1},y_{2n}]$, then $I_n\supseteq I_{n+1}$, i.e. $y_{2n-1}\le y_{2n+1}\le y_{2n+2}\le y_{2n}$.

Proof. Case $n=1$ is $y_1=0<y_3=\frac{2}{3}<y_4=\frac{3}{4}<y_2=1$. Suppose $0\le y_{2n-1}\le y_{2n+1}\le y_{2n+2}\le y_{2n}$.

So $2+y_{2n-1}\le 2+y_{2n+1}\le 2+y_{2n+2}\le 2+y_{2n}$, $\frac{2}{2+y_{2n-1}}\ge\frac{2}{2+y_{2n+1}}\ge\frac{2}{2+y_{2n+2}}\ge\frac{2}{2+y_{2n}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ \ \|\ y_{2n}\qquad \|\ y_{2n+2}\qquad \|\ y_{2n+3}\qquad \|\ y_{2n+1}$

$2+y_{2n}\ge 2+y_{2n+2}\ge 2+y_{2n+3}\ge 2+y_{2n+1}$, $\frac{2}{2+y_{2n}}\le\frac{2}{2+y_{2n+2}}\le\frac{2}{2+y_{2n+3}}\le\frac{2}{2+y_{2n+1}}$

This completes the induction. $\qquad$ $\|\ y_{2n+1}\qquad \|\ y_{2n+3}\qquad \|\ y_{2n+4}\qquad \|\ y_{2n+2}$

By nested interval theorem, $\lim_{n\to\infty}y_{2n-1}=a$ and $\lim_{n\to\infty}y_{2n}=b$ exist. Then

$a=\lim_{n\to\infty}y_{2n+1}=\lim_{n\to\infty}\frac{2}{2+y_n}=\frac{2}{2+b}$ and $b=\lim_{n\to\infty}y_{2n}=\lim_{n\to\infty}\frac{2}{2+y_{2n-1}}=\frac{2}{2+a}$.

Combining, we have $a(2+b)=2=b(2+a)$. So $2a+ab=2b+ab\Rightarrow 2a=2b\Rightarrow a=b$.

By intertwining sequence theorem, $\lim_{n\to\infty}y_n=y$ exists, then

$y=\lim_{n\to\infty}y_{n+1}=\lim_{n\to\infty}\frac{2}{2+y_n}=\frac{2}{2+y}$. So $y(2+y)=2$, $y^2+2y-2=0$, $y=-1\pm\sqrt{3}$

$y_n\in I_1=[0,1]\Rightarrow y\in[0,1]$. $\therefore$ $y=-1+\sqrt{3}$.