Solutions

(201) $\forall x \in S$, $\exists\, b, c \in [-1,1)$ such that $x^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2-4c}}{2}$

$\Rightarrow x \geq \frac{-1-\sqrt{1^2-4(-1)}}{2} = \frac{-1-\sqrt{5}}{2}$. So $\frac{-1-\sqrt{5}}{2}$ is a lower bound of S.

Let $b_n = 1 - \frac{1}{n}$ and $c_n = -1$ for every $n \in \mathbb{N}$. Then $b_n, c_n \in [-1,1)$

$x_n = \frac{-b_n - \sqrt{b_n^2 - 4c_n}}{2} \in S$ and $\lim_{n\to\infty} x_n = \frac{-1-\sqrt{5}}{2}$.

By infimum limit theorem, $\inf S = \frac{-1-\sqrt{5}}{2}$.

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(202) (a) $\forall x, y \in (0, \frac{\pi}{2}] \cap \mathbb{Q}$, $0 = 0 + 0 \leq \cos x + \sin y \leq 1 + 1 = 2$. So B is bounded below by 0 and bounded above by 2.

Let $a_n = \frac{1}{n}$ and $b_n = \frac{[n\pi/2]}{n}$, then $a_n, b_n \in (0, \frac{\pi}{2}] \cap \mathbb{Q}$ for $n = 1, 2, 3, \ldots$

So $\cos a_n + \sin b_n \in B$ and $\lim_{n\to\infty} \cos a_n + \sin b_n = \cos 0 + \sin\frac{\pi}{2} = 2$

and $\cos b_n + \sin a_n \in B$ and $\lim_{n\to\infty} \cos b_n + \sin a_n = \cos\frac{\pi}{2} + \sin 0 = 0$.

By supremum limit theorem, $\sup B = 2$. By infimum limit theorem, $\inf B = 0$.

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(202) (b) $\left.\begin{array}{l} \inf D = 3,\ \sup D = 5 \Rightarrow D \subseteq [3,5] \\ \inf E = 7,\ \sup E = 9 \Rightarrow E \subseteq [7,9] \end{array}\right\} \Rightarrow \begin{array}{l} \forall x \in D,\ y \in E, \\ 3 + \frac{1}{9} \leq x + \frac{1}{y} \leq 5 + \frac{1}{7} \end{array}$

So A is bounded above by $5\frac{1}{7}$ and bounded below by $3\frac{1}{9}$.

By infimum limit theorem, $\exists\, a_n \in D$ such that $\lim_{n\to\infty} a_n = 3$ and $\exists\, b_n \in E$ such that $\lim_{n\to\infty} b_n = 7$.

By supremum limit theorem, $\exists\, c_n \in D$ such that $\lim_{n\to\infty} c_n = 5$ and $\exists\, d_n \in E$ such that $\lim_{n\to\infty} d_n = 9$.

Then $a_n + \frac{1}{d_n} \in A$ and $\lim_{n\to\infty} a_n + \frac{1}{d_n} = 3\frac{1}{9}$. By infimum limit theorem, $\inf A = 3\frac{1}{9}$.

Also $c_n + \frac{1}{b_n} \in A$ and $\lim_{n\to\infty} c_n + \frac{1}{b_n} = 5\frac{1}{7}$. By supremum limit theorem, $\sup A = 5\frac{1}{7}$.

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(301) Sketch $2b_n^3 \to 2$, $\frac{a_n}{2n} \to 0 \Rightarrow \left|(2b_n^3 + \frac{a_n}{2n}) - 2\right| = \left|(2b_n^3 - 2) + (\frac{a_n}{2n} - 0)\right|$

$\leq 2\underbrace{|b_n^3 - 1|}_{(b_n-1)(b_n^2+b_n+1)} + \frac{|a_n|}{2n}$

$< 14|b_n - 1| + \frac{2}{2n}$

$\exists K_1\ n \geq K_1 \Rightarrow |b_n - 1| < 1 \Rightarrow b_n \in (0,2) \Rightarrow b_n^2 + b_n + 1 \in (1,7)$

$\exists K_2\ n \geq K_2 \Rightarrow |a_n - 1| < 1 \Rightarrow a_n \in (0,2)$

$\exists K_3\ n \geq K_3 \Rightarrow |b_n - 1| < \frac{\varepsilon}{28}$ $\quad \boxed{\frac{2}{2n} = \frac{1}{n} < \frac{\varepsilon}{2} \Leftrightarrow \frac{2}{\varepsilon} < n.}$

Solution! $\forall \varepsilon > 0$, since $b_n \to 1$, $a_n \to 1$,

$\exists K_1$ such that $n \geq K_1 \Rightarrow |b_n - 1| < 1 \Rightarrow b_n \in (0,2) \Rightarrow b_n^2 + b_n + 1 \in (1,7)$

$\exists K_2$ such that $n \geq K_2 \Rightarrow |a_n - 1| < 1 \Rightarrow a_n \in (0,2)$

$\exists K_3$ such that $n \geq K_3 \Rightarrow |b_n - 1| < \frac{\varepsilon}{28}$

By Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \max\{K_1, K_2, K_3, \frac{2}{\varepsilon}\}$.

Then $n \geq K \Rightarrow n \geq K_1,\ n \geq K_2,\ n \geq K_3,\ n > \frac{2}{\varepsilon}$

$\Rightarrow \left|(2b_n^3 + \frac{a_n}{2n}) - 2\right| = \left|(2b_n^3 - 2) + (\frac{a_n}{2n} - 0)\right| \leq 2|b_n^3 - 1| + \frac{|a_n|}{2n}$

$\leq 2|b_n^2 + b_n + 1||b_n - 1| + \frac{1}{n} < 14|b_n - 1| + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Solution 2 Since $\{a_n\}$ and $\{b_n\}$ converge, by the boundedness theorem, $\exists L, M > 0$ such that $\forall n \in \mathbb{N}$, $|a_n| \le L$ and $|b_n| \le M$.

Since $\lim\limits_{n\to\infty} b_n = 1$ and $\frac{\varepsilon}{4(M^2+M+1)} > 0$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |b_n - 1| < \frac{\varepsilon}{4(M^2+M+1)}$

Let $K_2 = \lceil \frac{L}{\varepsilon} \rceil$, then $n \ge K_2 \Rightarrow n \ge \frac{L}{\varepsilon} \Leftrightarrow \frac{L}{2n} < \frac{\varepsilon}{2}$.

Let $K = \max(K_1, K_2)$, then $n \ge K \Rightarrow n \ge K_1$ and $n \ge K_2$

$\Rightarrow |2b_n^3 + \frac{a_n}{2n} - 2| \le |2b_n^3 - 2| + |\frac{a_n}{2n}| = 2|b_n - 1||b_n^2 + b_n + 1| + \frac{|a_n|}{2n} < 2|b_n - 1|(|b_n|^2 + |b_n| + 1) + \frac{L}{2n}$

$\le 2|b_n - 1|(M^2+M+1) + \frac{\varepsilon}{2} < 2\frac{\varepsilon}{4(M^2+M+1)}(M^2+M+1) + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

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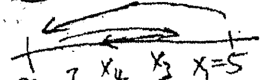
(403) (Scratch: $x_1 = 5$; $x_2 = \frac{7}{10}$, $x_3 = \frac{7}{5.7} \approx 1.23$, $x_4 = \frac{7}{6.23} \approx 1.12$



$x = \frac{7}{x+5} \Leftrightarrow x^2 + 5x - 7 = 0 \Leftrightarrow x = \frac{-5 \pm \sqrt{53}}{2}$)

Define $I_n = [x_{2n}, x_{2n-1}]$ for $n = 1, 2, 3, \dots$ We claim $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$.

For this, we will prove $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$ for $n = 1, 2, 3, \dots$.

For $n = 1$, we have $x_2 = \frac{7}{10} = 0.7 < x_4 \approx 1.12 < x_3 \approx 1.23 < x_1 = 5$.

Assume case n. Then $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$

$\Rightarrow x_{2n} + 5 \le x_{2n+2} + 5 \le x_{2n+1} + 5 \le x_{2n-1} + 5 \Rightarrow \underset{x_{2n+1}}{\underset{\parallel}{\frac{7}{x_{2n}+5}}} \ge \underset{x_{2n+3}}{\underset{\parallel}{\frac{7}{x_{2n+2}+5}}} \ge \underset{x_{2n+2}}{\underset{\parallel}{\frac{7}{x_{2n+1}+5}}} \ge \underset{x_{2n}}{\underset{\parallel}{\frac{7}{x_{2n-1}+5}}}$

$\Rightarrow x_{2n+1} + 5 \ge x_{2n+3} + 5 \ge x_{2n+2} + 5 \ge x_{2n} + 5$

$\Rightarrow \frac{7}{x_{2n+1}+5} \le \frac{7}{x_{2n+3}+5} \le \frac{7}{x_{2n+2}+5} \le \frac{7}{x_{2n}+5} \Leftrightarrow x_{2n+2} \le x_{2n+4} \le x_{2n+3} \le x_{2n+1}$,

which completed the case $n+1$. $\therefore$ the claim is true.

By the nested interval theorem, $\lim\limits_{n\to\infty} x_{2n} = a$ and $\lim\limits_{n\to\infty} x_{2n-1} = b$ exist.

Then $a = \lim\limits_{n\to\infty} x_{2n} = \lim\limits_{n\to\infty} \frac{7}{x_{2n-1}+5} = \frac{7}{b+5}$ and $b = \lim\limits_{n\to\infty} x_{2n+1} = \lim\limits_{n\to\infty} \frac{7}{x_{2n}+5} = \frac{7}{a+5}$.

So $a(b+5) = 7 = b(a+5) \Rightarrow ab + 5a = ab + 5b \Rightarrow 5a = 5b \Rightarrow a = b$.

So $\lim\limits_{n\to\infty} x_n = x$ exists by intertwining theorem. Then $x = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \frac{7}{x_n+5} = \frac{7}{x+5}$

Then $x^2 + 5x - 7 = 0 \Rightarrow x = \frac{-5 \pm \sqrt{53}}{2}$. Since $x \in I_1$, $x = \frac{-5 + \sqrt{53}}{2}$.

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(404) $x_1 = \frac{1}{4} < x_2 = \frac{\frac{1}{2} + \frac{3}{4}}{4} = \frac{5}{16}$. $\boxed{\text{Scratch Work } x = \frac{\sqrt{x}+3x}{4} \Leftrightarrow 4x = \sqrt{x} + 3x \Leftrightarrow x = \sqrt{x} \Leftrightarrow x = 0 \text{ or } 1}$

We claim $x_n < x_{n+1} < 1$. The case $n=1$ follows from $x_1 = \frac{1}{4} < x_2 = \frac{5}{16} < 1$.

Suppose case n holds. So $x_n < x_{n+1} < 1$. Then $\sqrt{x_n} < \sqrt{x_{n+1}} < \sqrt{1} = 1$ and so

$x_{n+1} = \frac{\sqrt{x_n} + 3x_n}{4} < \frac{\sqrt{x_{n+1}} + 3x_{n+1}}{4} = x_{n+2} < \frac{\sqrt{1} + 3 \times 1}{4} = 1$, which completes induction.

Then $\lim\limits_{n\to\infty} x_n = x$ exists and $x = \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \frac{\sqrt{x_n} + 3x_n}{4} = \frac{\sqrt{x} + 3x}{4}$.

So $4x = \sqrt{x} + 3x \Rightarrow x = \sqrt{x} \Rightarrow 0 = x - \sqrt{x} = \sqrt{x}(\sqrt{x} - 1) \Rightarrow x = 0$ or 1.

Since $\frac{1}{4} \le x_n$, $\lim\limits_{n\to\infty} x_n = 1$.