Math 2033 Past Exam Problems

Cauchy Sequences

Useful Facts: Definition of a Cauchy Sequence is $x_1, x_2, x_3, \ldots$ (or $\{x_n\}$) is a Cauchy sequence if and only if <u>for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $m, n > K$ implies $|x_m - x_n| < \varepsilon$.</u> The $\varepsilon/2+\varepsilon/2 = \varepsilon$ and $\max\{K_1, K_2\}$ trick are useful when we have two sequences. Also, the fact <u>*Cauchy sequences are bounded*</u> is used sometimes. Inequalities like $|\sin a - \sin b| \le |a - b|$ and $|\sqrt[n]{a} - \sqrt[n]{b}| \le \sqrt[n]{|a-b|}$ are useful.

501. (2009SM) Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of positive real numbers. For $n = 1, 2, 3, \ldots$, let

$$b_n = \sin(a_n^2) + \sqrt[3]{7a_n}.$$

Prove that $b_1, b_2, b_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

You may use the fact Cauchy sequences are bounded. However, do not use Cauchy theorem that said a sequence converges if and only if it is a Cauchy sequence, otherwise you will get 0 mark on this problem!

502. (2009SF) Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence in $\mathbb{R}$ and let

$$y_n = x_{n+1} + x_n^2 + \cos(x_n) \quad \text{for } n = 1, 2, 3, \ldots.$$

Prove that $y_1, y_2, y_3, \ldots$ is also a Cauchy sequence by checking the definition of Cauchy sequence.

Do not use the theorem that asserts a sequence is a Cauchy sequence if and only if it converges. Otherwise you will get 0 mark for this problem!

503. (2010SF) Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be Cauchy sequences in $[0, +\infty)$ and let

$$c_n = a_n^2 + \sqrt{b_n} + \sin(a_n + b_n) \quad \text{for } n = 1, 2, 3, \ldots.$$

Prove that $c_1, c_2, c_3, \ldots$ is also a Cauchy sequence by checking the definition of Cauchy sequence.

Do not use the theorem that asserts a sequence is a Cauchy sequence if and only if it converges. Otherwise you will get 0 mark for this problem!

504. (2011SM) Let $A_1, A_2, A_3, \ldots$ be a Cauchy sequence of decreasing <u>positive</u> real numbers. For $n = 1, 2, 3, \ldots$, let B_n be a real number such that

$$\sqrt{A_{n+2011}} \le B_n \le \sqrt{A_n}.$$

Prove that $B_1, B_2, B_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

Do not use Cauchy theorem that said a sequence converges if and only if it is a Cauchy sequence, otherwise you will get 0 mark for this problem!

505. (2012SM) Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of real numbers. For $n = 1, 2, 3, \ldots$, let $b_n = a_n \sin a_n$. Prove that $b_1, b_2, b_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

Do not use Cauchy theorem that said a sequence converges if and only if it is a Cauchy sequence, otherwise you will get 0 mark on this problem!

506. (2013SM) Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of <u>positive</u> real numbers. For $n = 1, 2, 3, \ldots$, let $b_n = \sqrt{\dfrac{a_n}{a_n + 3}} + 5$. Prove that $b_1, b_2, b_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

Do not use Cauchy's theorem that said a sequence converges if and only if it is a Cauchy sequence, otherwise you will get 0 mark on this problem!

Limit of Functions

601. (2007SM) Let $f:(0,+\infty)\to\mathbb{R}$ be defined by $f(x)=\dfrac{1}{\sqrt{x}+1}$. Prove that $\lim_{x\to 1} f(x)=\dfrac{1}{2}$ by checking the definition.

     Do not use computation formulas, sandwich theorem or L'Hopital's rule, otherwise you will get zero mark on this problem!

602. (2008FF) Prove that $\lim_{x\to 1}\dfrac{x+8}{x^2+3}=\dfrac{9}{4}$ by checking the definition of limit of a function or the limit of a sequence via the sequential limit theorem.

     Do not use computation formulas, sandwich theorem or L'Hopital's rule, otherwise you will get zero mark on this problem!

603. (2010SM) Let $f:[1,3]\to\mathbb{R}$ be defined by $f(x)=\dfrac{1}{\sqrt[4]{x^2+6x}}$. Prove that $\lim_{x\to 2} f(x)=\dfrac{1}{2}$ by checking the ε-δ definition of limit of function.

     Do not use any computation formula, sandwich theorem or l'Hopital's rule, otherwise, you will get 0 mark on this problem.

604. (2011SM) Let $f:[0,+\infty)\to\mathbb{R}$ be defined by $f(x)=\sin^2\left(\dfrac{1}{1+\sqrt[4]{x}}\right)$. Prove that $\lim_{x\to 1} f(x)=\sin^2\dfrac{1}{2}$ by checking the ε-δ definition of limit of function.

     <u>Do not use any computation formula, sandwich theorem or l'Hopital's rule,</u> otherwise, you will get zero mark.

605. (2012SM) Let $f:[0,+\infty)\to\mathbb{R}$ be defined by $f(x)=\sqrt{\dfrac{1}{2+\sqrt{x}}}$. Prove that $\lim_{x\to 4} f(x)=\dfrac{1}{2}$ by checking the ε-δ definition of limit of function.

     <u>Do not use any computation formula, sandwich theorem or l'Hopital's rule,</u> otherwise, you will get 0 mark.

606. (2013SM) Let $f:[0,+\infty)\to\mathbb{R}$ be defined by $f(x)=\dfrac{x}{1+2x}+\dfrac{2}{2+\sqrt{x}}$. Prove that $\lim_{x\to 1} f(x)=1$ by checking the ε-δ definition of limit of function.

     <u>Do not use any computation formula, sandwich theorem or l'Hopital's rule,</u> otherwise, you will get 0 mark.

Continuity

Useful Facts: Intermediate Value Theorem for showing solution of equations exist, Extreme Value Theorem, Continuous Injection Theorem for problem involving composition of functions.

701. (2007SM) Let $f : [0,1] \to [0,1]$ be continuous such that $f(0) = 0,\ f(1) = 1$ and $f(f(x)) = x$ for all $x \in [0,1]$. Prove that $f(x) = x$ for all $x \in [0,1]$.

702. (2008SM) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$. Prove that there exists at least one $x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$.

703. (2009SM) Prove that there exists a unique continuous function $f : [0,1] \to [0,1]$ such that $f\Big(f\big(f(x)\big)\Big) + f(x) = 2x$ for all $x \in [0,1]$.

704. (2010SM) Prove that there does not exist any contnuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ is rational if and only if $f(x+1)$ is irrational.

705. (2010SM) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and decreasing. Prove that there exists a unique element $(a, b, c) \in \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ such that $a = f(b),\ b = f(c)$ and $c = f(a)$.

706. (2012SM) Let $f : [0,2] \to \mathbb{R}$ be continuous and $f(2) = 0$. If $\lim\limits_{x \to 1} \dfrac{f(x) - 2}{\sqrt{x} - 1} = 1$, then prove that there exists $x \in [0,2]$ such that $f(x) = x^2$.

707. (2013SM) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that for all $x \in \mathbb{R}$,
$$f\big(1 - f(x)\big) = 1 - x^9.$$
If $f(1) = 0$ and $f'(1) < 0$, then prove that there exists $r \in \mathbb{R}$ such that $f(r) = r^{2013}$.

Differentiability

<u>**Useful Facts:**</u> Mean Value Theorem, Taylor's Theorem.

801. (2009SM) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and $f''(x)$ be continuous. If

$$f(-1) = 0, \quad f(0) = 2, \quad f(1) = 5 \quad \text{and} \quad f'(0) = 0,$$

then prove that there exists $c \in \mathbb{R}$ such that $f''(c) = \sqrt{2}$.

802. (2010SM) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and for all $x \in [0, 1]$, $|f''(x)| \leq 2010$. If there exists $c \in (0, 1)$ such that $f(c) > f(0)$ and $f(c) > f(1)$, then prove that

$$|f'(0)| + |f'(1)| \leq 2010.$$

803. (2010SF) Let $f : (-1, 1) \to \mathbb{R}$ be four times differentiable such that for all $c \in (-1, 1)$, $|f^{(4)}(c)| \leq 1$. Prove that for all $x \in (0, 1)$, we have

$$\left| f''(0) - \frac{f(x) - 2f(0) + f(-x)}{x^2} \right| \leq \frac{x^2}{12}.$$

804. (2011SM) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathbb{R}$. If $f(0) = f(1) = 0$ and $\max\{f(x) : x \in [0, 1]\} = 2$, then prove that there exists $\theta \in (0, 1)$ such that $f''(\theta) \leq -16$.

805. (2012SM) Let $f : \mathbb{R} \to \mathbb{R}$ be three-times differentiable on $\mathbb{R}$. If $\dfrac{f(0) + f(2)}{2} = f(1)$, then prove that there exist $a, b, c \in \mathbb{R}$ such that

$$f'''(a) - f'''(b) = 6f''(c).$$

806. (2012SF) Let $f : \mathbb{R} \to \mathbb{R}$ be three-time differentiable. If

$$f(0) = 5, \quad f(2) = 7 \quad \text{and} \quad \text{for all } x \in [0, 2], \quad |f'''(x)| \leq 6,$$

then prove that $|f'(1)| \leq 2$.

807. (2013SM) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. If $f'(0) = 2 = f'(1)$ and for all $x \in [0, 1]$, $|f''(x)| \leq 4$, then prove that $|f(1) - f(0)| \leq 3$.

Riemann Integrability

<u>**Useful Facts:**</u> Monotone Function Theorem, Integral Criterion, Lebesgue's Theorem. Definition and Examples of Sets of Measure 0.

901. (2009SF) (a) State Lebesgue's theorem.

(b) Let S be a set of measure 0. Prove that $T = \{2x : x \in S\}$ is also a set of measure 0. Let $f : [0,1] \to [0,1]$ be a Riemann integrable function. Prove that $g : [0,2] \to [0,1]$ defined by $g(x) = f(x/2)$ is Riemann integrable on $[0,1]$.

902. (2010SF) (a) State Lebesgue's theorem.

(b) Let $f : [0,1] \to [0,1]$ be a Riemann integrable function. Let $r_1, r_2, r_3, \ldots$ be a strictly increasing sequence in $(0,1]$. Prove that $g : [0,1] \to [0,1]$ defined by

$$g(x) = \begin{cases} 1 - f(x) & \text{if } x \notin \{r_1, r_2, r_3, \ldots\} \\ \cos x & \text{if } x \in \{r_1, r_2, r_3, \ldots\} \end{cases}$$

is Riemann integrable on $[0,1]$.

903. (2011SF) Let $f : [0,1] \to [0,1]$ be a Riemann integrable function. Prove that $F : [0,2] \to [0,1]$ defined by

$$F(x) = \begin{cases} |f(x) - 1| & \text{if } x \in [0,1) \\ f(x-1) & \text{if } x \in [1,2] \end{cases}$$

is Riemann integrable on $[0,2]$.

904. (2011SF) Let $g : [1,2] \to [0,1]$ be a Riemann integrable function. Prove that $G : [0,1] \to [0,1]$ defined by

$$G(x) = \begin{cases} g(x+1) & \text{if } x \in [0,1] \setminus \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \\ \\ 1 & \text{if } x \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \end{cases}$$

is Riemann integrable on $[0,1]$ by checking the integral criterion. Do not use Lebesgue's theorem in any part of your solution. Otherwise you will get 0 mark for this problem!

905. (2012SF) Let $f : [0,1] \to [0,1]$ be a Riemann integrable function. Let $g : [0,1] \to [0,1]$ be an increasing function. Define $h : [0,1] \to [0,1]$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \notin \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} \cup \left[\frac{2}{3}, \frac{3}{4}\right], \\ \\ g(x) & \text{if } x \in \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} \cup \left[\frac{2}{3}, \frac{3}{4}\right] \end{cases}.$$

(a) Use Lebesgue's theorem to prove $h(x)$ is Riemann integrable on $[0,1]$.

(b) Use the integral criterion to prove $h(x)$ is Riemann integrable on $[0,1]$.

906. (2013SF) Let $f : [0,1] \to [0,1]$ be a function that is continuous at all $x \in [0,1] \setminus \mathbb{Q}$. Let $g : [0,1] \to [0,1]$ be defined by $g(x) = f(x) f\Big(\dfrac{x}{\sqrt{2}}\Big)$ for all $x \in [0,1]$. Prove that $g(x)$ is Riemann integrable on $[0,1]$ by Lebesgue's theorem.