Math 202 (Introduction to Real Analysis) Fall 2010

Midterm

Directions: This is a closed book exam. Every student must show work in every problem with *full* details legibly to receive marks. *Answers alone are worth very little!!!*

Notations: $\mathbb{R}$ denotes the set of all real numbers. $\mathbb{Q}$ denotes the set of all rational numbers. The variable n in the problems below takes on positive integer values $1, 2, 3, \ldots$.

1. (11 Marks) Let A be a nonempty bounded subset of $\mathbb{R}$ such that $\inf A = 1$ and $\sup A = 3$. Let
$$B = \{\sqrt{2x(15 + xy)} : \ x \in (2, 4) \cap \mathbb{Q}, \quad y \in A\}.$$
Prove that B is bounded. Determine (with proof) the infimum and supremum of B.

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2. (11 Marks) Prove the sequence $\{x_n\}$ converges, where
$$x_1 = 5 \qquad \text{and} \qquad x_{n+1} = \frac{7}{x_n + 5},$$
and find its limit. Show work!

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3. (11 marks) Do either (a) or (b) below:

(a) Determine (with proof) all positive irrational numbers b such that
$$\sum_{k=1}^{\infty} \frac{\cos(k - 3b)}{(2k - b)\big((\ln k)^2 + 1\big)}$$
converges.

(b) Determine (with proof) whether the set
$$S = \Big\{ b : \ b \in (0, +\infty) \setminus \mathbb{Q} \text{ and } \sum_{k=1}^{\infty} \frac{\cos(k - 3b)}{(2k - b)\big((\ln k)^2 + 1\big)} \text{ converges} \Big\}$$
is countable or not.

–End of Paper–

Math 202 (Introduction to Real Analysis) Fall 2009

Midterm

Directions: This is a closed book exam. Every student must show work in every problem with correct details legibly to receive marks. *Answers alone are worth very little!!!*

Notations: $\mathbb{R}$ denotes the set of all real numbers. $\mathbb{Q}$ denotes the set of all rational numbers.

1. (a) (5 marks) Determine (with proof) all nonnegative real number b such that the series $\displaystyle\sum_{k=1}^{\infty} \frac{2^{k+3}}{k^2(b+1)^k}$ converges. (This means for the remaining nonnegative real number b, you also have to explain why the series diverges.) Show details!

   (b) (5 marks) Let $a_1, a_2, a_3, \ldots$ be real numbers in the open interval $(0,1)$ such that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges. Determine (with proof) whether $\displaystyle\sum_{k=1}^{\infty} \frac{\sin a_k}{1-a_k}$ converges or not.

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2. (12 Marks) Let D be a nonempty bounded subset of $\mathbb{R}$ such that $\inf D = 3$ and $\sup D = 5$. Let
$$A = \{xy + xy^3 : \ x \in (2,\pi] \cap \mathbb{Q}, \quad y \in D\}.$$
   Show that A is bounded. Determine (with proof) the infimum and supremum of A.

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3. (11 marks) Let S be the set of all points (x,y) in the coordinate plane that satisfy the equations
$$x^2 + y^2 = a^2 \qquad \text{and} \qquad y = x^2 - x^3 + b$$
   for some $a, b \in \mathbb{Q}$ with $a \neq b$. Determine (with proof) if S is countable or not.

–End of Paper–

Math 202 (Introduction to Real Analysis) Fall 2008

Midterm

Directions: This is a closed book exam. Works (including scratch works) must be shown legibly to receive credits. *Answers alone are worth very little!!!*

Notations: $\mathbb{R}$ denotes the set of all real numbers. $\mathbb{Q}$ denotes the set of all rational numbers.

1. (a) (6 marks) Determine if the series $\displaystyle\sum_{k=1}^{\infty}(\cos k)\sin\Big(\frac{1}{k^2+\sqrt{2}}\Big)$ converges. Show work!

   (b) (8 marks) Prove the sequence $\{x_n\}$ converges, where

   $$x_1 = 1 \qquad \text{and} \qquad x_{n+1} = \frac{4\sqrt{x_n}+x_n}{3}$$

   and find its limit. Show work!

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2. (a) (6 marks) Determine (with proof) the supremum and infimum of

   $$B = \big\{\cos x + \sin y : \ x, y \in (0, \pi/2] \cap \mathbb{Q}\big\}.$$

   (b) (8 marks) Let D and E be nonempty bounded subsets of $\mathbb{R}$ such that

   $$\inf D = 3, \quad \sup D = 5, \quad \inf E = 7 \quad \text{and} \quad \sup E = 9.$$

   Determine (with proof) the supremum and infimum of the set

   $$A = \Big\{x + \frac{1}{y} : \ x \in D, \ y \in E\Big\}.$$

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3. (5 marks) Prove that there exists a positive real number c which does *not* equal to any number of the form $2^{a+b\sqrt{2}}$, where $a, b \in \mathbb{Q}$.

–End of Paper–

Math 202 (Introduction to Real Analysis) Fall 2007

Midterm

Directions: This is a closed book exam. Works (including scratch works) must be shown legibly to receive credits. *Answers alone are worth very little!!!*

Notations: $\mathbb{R}$ denotes the set of all real numbers. $\mathbb{Q}$ denotes the set of all rational numbers.

1. (a) (6 marks) Determine (with proof) if $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k^2}{(2k)!}$ converges.

   (b) (6 marks) Determine (with proof) if $\displaystyle\sum_{k=3}^{\infty} \frac{\cos k}{k(\ln k)^2}$ converges.

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2. (a) (7 marks) Let D be a nonempty subset of $\mathbb{R}$ such that $\inf D = 2$ and $\sup D = 5$. Determine (with proof) the supremum and infimum of the set

$$A = \Big\{\frac{x}{y} : x, y \in D\Big\}.$$

   (b) (6 marks) Let c be a positive rational number. Determine (with proof) the supremum and infimum of

$$B = \big\{x + y : x \in [0, c\sqrt{2}] \cap \mathbb{Q},\ y \in [0, c] \setminus \mathbb{Q}\big\}.$$

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3. (8 marks) Let S be a nonempty countable subset of the interval $(0, +\infty)$. Prove that there exists a positive real number which is not the area of any triangle whose three sides have lengths in S.

–End of Paper–

Math 202 (Introduction to Real Analysis) Fall 2006

Examination

Directions: This is a closed book exam. Works (including scratch works) must be shown legibly to receive credits. *Answers alone are worth very little!!!*

Notations: $\mathbb{R}$ denotes the set of all real numbers. $\mathbb{Q}$ denotes the set of all rational numbers.

1. (a) (10 marks) Determine the set of *all* the positive numbers b such that $\sum_{k=1}^{\infty} \frac{k}{(k+b)^2}$ converges. Be sure to prove you have gotten all such b.

   (b) (15 marks) Determine the set of *all* the positive numbers c such that $\sum_{k=1}^{\infty} \frac{(-1)^k k}{(k+c)^2}$ converges. Be sure to prove you have gotten all such c.

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2. (25 marks) Let $\left(0, \frac{1}{2}\right) \cap \mathbb{Q} \subseteq A_1 \subseteq [0,1)$. For $n = 1, 2, 3, \ldots$, let

$$A_{n+1} = \{\sqrt{x} : \ x \in A_n\}.$$

   Determine the supremum and infimum of $\bigcup_{k=1}^{\infty} A_k$ with proof.

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3. (a) (5 marks) State the definition of a sequence $a_1, a_2, a_3, \ldots$ of real numbers *converges* to a number L.

   (b) (25 marks) Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ be sequences of positive numbers such that $\lim_{n\to\infty} x_n = 1 = \lim_{n\to\infty} y_n$. Prove that

$$\lim_{n\to\infty} \left(4x_n + \frac{1}{y_n}\right) = 5$$

   by checking the definition of limit. Do not use the computation formulas for limits, sandwich theorem or l'Hopital's rule, otherwise you will get 0 mark for this problem!

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4. (20 marks) Let

$$x_1 = 2, \quad x_2 = 4 \quad \text{and} \quad x_{n+2} = \sqrt{10x_n - 9} \quad \text{for } n = 1, 2, 3, \ldots.$$

   Determine if the sequence $x_1, x_2, x_3, \ldots$ converges or not with proof. In case of convergence, also find the limit.

–End of Paper–