DEPARTMENT OF MATHEMATICS

Math2033 TA note 11

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1 MEAN VALUE THEOREM

Example 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Show that f is bijective.*

Solution: Firstly, if $f(x) = f(y)$ and $x \neq y$, then $0 = |f(x) - f(y)| < |x - y|$ which contradicts to the condition. Hence, f is injective. So $f(x)$ is monotone. For any $w \in \mathbb{R}$, we define $M = |w - f(0)|$. Then by $|f(M) - f(0)| \geq |M| = |w - f(0)|$ and $|f(-M) - f(0)| \geq |M| = |w - f(0)|$. Together with f is monotone, we have w is contained in $[f(-M), f(M)]$. So f is surjective. Therefore, f is bijective.

Example 2. *Find the derivatives of the function* $f(x) = \begin{cases} x^2 & if x \neq 0 \\ x & if x = 0 \end{cases}$, $g(x) = |\cos x|$ *and* $h(x) = \begin{cases} \frac{\sin x}{x} & if x \neq 0 \\ 1 & if x = 0 \end{cases}$.

Solution: For $x \neq 0$, using derivative formula. For $x = 0$, using the definition of derivative.

Example 3. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. If f' is differentiable at x_0, show that*

$$\lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = f''(x_0)$$

Solution: Using L'Hopital's rule

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}\\
=&\lim_{h\to 0}\frac{f'(x_0+h)-f'(x_0-h)}{2h}\\
=&\lim_{h\to 0}\frac{f'(x_0+h)-f'(x_0)+f'(x_0)-f'(x_0-h)}{2h}\\
=&f''(x_0) \qquad \text{(by defintion of derivative)}
\end{aligned}$$

Example 4. *Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable at c and $I_n=[a_n,b_n]$ be such that $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$ and $\cap_{n=1}^{\infty}[a_n,b_n]=\{c\}$. Prove that if $a_n<b_n$ for all $n\in\mathbb{N}$, then $f'(c)=\lim_{n\to\infty}\frac{f(b_n)-f(a_n)}{b_n-a_n}$.*

Solution: It is easy to see that $\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=c$. By sequential limit theorem,

$$f'(c)=\lim_{n\to\infty}\frac{f(b_n)-f(c)}{b_n-c}=\lim_{n\to\infty}\frac{f(c)-f(a_n)}{c-a_n}.$$

We calculate

$$\begin{aligned}
\frac{f(b_n)-f(a_n)}{b_n-a_n}-f'(c)&=\left(\frac{f(b_n)-f(c)}{b_n-a_n}+\frac{f(c)-f(a_n)}{b_n-a_n}\right)-f'(c)\\
&=\frac{f(b_n)-f(c)}{b_n-c}\frac{b_n-c}{b_n-a_n}+\frac{f(c)-f(a_n)}{c-a_n}\frac{c-a_n}{b_n-a_n}-f'(c)\left(\frac{b_n-c}{b_n-a_n}+\frac{c-a_n}{b_n-a_n}\right)\\
&=\left(\frac{f(b_n)-f(c)}{b_n-c}-f'(c)\right)\frac{b_n-c}{b_n-a_n}+\left(\frac{f(c)-f(a_n)}{c-a_n}-f'(c)\right)\frac{c-a_n}{b_n-a_n}
\end{aligned}$$

Therefore,

$$\left|\frac{f(b_n)-f(a_n)}{b_n-a_n}-f'(c)\right|\le\left|\frac{f(b_n)-f(c)}{b_n-c}-f'(c)\right|+\left|\frac{f(c)-f(a_n)}{c-a_n}-f'(c)\right|\to 0.$$

Example 5. *Let n be a positive integer and $f(x)=(x^2-1)^n$. Show that $f^{(n)}$ has n distinct roots.*

Solution: By induction, we can prove that $f^{(j)}(x)=(x^2-1)^{n-j}P_j(x)$, where $1\le j\le n$ and $P_j(x)$ is a polynomial. So $f(\pm1)=f'(\pm1)=\cdots=f^{(n-1)}(\pm1)=0$. Since $f(1)=f(-1)=0$, by Roller's theorem, there is $x_0\in(-1,1)$ such that $f'(x_0)=0$. Thus, f' has at least 3 distinct roots. By Roller's theorem, f'' has at least 4 distinct roots. By induction, we can show that $f^{(n-1)}$ has at least $n+1$ distinct roots. So by Roller's theorem, $f^{(n)}$ has at least n distinct roots. Since $f^{(n)}$ is a polynomial of degree n, it has exactly n distinct roots.

Example 6. *If $f:(0,+\infty)\to\mathbb{R}$ is differentiable and $|f'(x)|\le 2$ for all $x>0$, then show that the sequence $x_n=f(\frac{1}{n})$ converges. Also, show $\lim_{x\to 0_+}f(x)$ exits.*

Solution: We first show $x_n=f(\frac{1}{n})$ is a Cauchy sequence. For every $\epsilon>0$, let $K\in\mathbb{N}$ such that $K>\frac{2}{\epsilon}$ (by Archimedian principle). By mean-value theorem, $m,n>K$ implies

$$|x_m-x_n|=\left|f(\frac{1}{m})-f(\frac{1}{n})\right|=|f'(x_0)|\left|\frac{1}{m}-\frac{1}{n}\right|\le 2\left|\frac{1}{m}-\frac{1}{n}\right|\le 2\left(\frac{1}{K}-0\right)=\frac{2}{K}<\epsilon.$$

Next, to show $\lim_{x\to 0_+} f(x)$ exits, it is enough to show $\lim_{n\to\infty} f(t_n)$ exits for every $t_n \to 0$ in $(0,+\infty)$ by the sequential limit theorem.
Otherwise, if there are two sequences in $(0,+\infty)$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0$, but $\lim_{n\to\infty} f(y_n) \neq \lim_{n\to\infty} f(z_n)$. We can construct a sequence such that $t_n = y_n$, if n is odd, $t_n = z_n$, if n is even and it converged to 0 but $f(t_n)$ is a divergent sequence.
For every $t_n \to 0$ in $(0,+\infty)$, $\{t_n\}$ is a Cauchy sequence by Cauchy's theorem. We will show $\lim_{n\to\infty} f(t_n)$ exits by showing $\{f(t_n)\}$ is a Cauchy sequence. For every $\epsilon > 0$, since $\{t_n\}$ is a Cauchy sequence, $\exists K_1 \in \mathbb{N}$ such that $m, n > K_1$

$$|t_m - t_n| < \frac{\epsilon}{2} \implies |f(t_m) - f(t_n)| = |f'(t_c)(t_m - t_n)| \leq 2|t_m - t_n| < 2\frac{\epsilon}{2} = \epsilon.$$

Example 7. *Prove that if* $0 \leq \theta \leq \frac{\pi}{2}$, *then*

$$1 - \frac{\theta^2}{2} \leq \cos\theta \leq 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

Solution: Since $\frac{\mathrm{d}^4 \cos\theta}{\mathrm{d}\theta^4} = \cos\theta$, by Taylor's theorem, there is $\theta_0 \in (0,\theta)$ such that

$$\cos\theta = 1 + 0(\theta - 0) - \frac{1}{2!}(\theta-0)^2 + \frac{0}{3!}(\theta-0)^3 + \frac{\cos\theta_0}{4!}(\theta-0)^4.$$

Since $0 \leq \theta_0 \leq \theta \leq \frac{\pi}{2}$, so $0 \leq \cos\theta_0 \leq 1$. Therefore,

$$1 - \frac{\theta^2}{2} \leq \cos\theta \leq 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$