MATH2033 Mathematical Analysis (2021 Spring)
Problem Set 3

Problem 1

Prove that the following sets are countably infinite using the definition.

(a) $A = \{n \in \mathbb{N} |\ n \text{ is not multiple of } 5\}$.

(b) $B = \{n \in \mathbb{Z} |\ n \text{ is odd number}\}$.

Problem 2

We let $A_1, A_2, A_3, \ldots$ be subsets of $\mathbb{R}$. Suppose that the set

$$S = A_1 \times A_2 \times A_3 \times \ldots$$

is countable. Prove that there are only finitely many sets that have more than one elements.

Problem 3

We let $f_1, f_2, f_3, \ldots : \mathbb{R} \to [0, \infty)$ be a collection of functions from $\mathbb{R}$ to $[0, \infty)$. For any positive integer n, we define

$$A_n = \{x \in \mathbb{R}: f_n(x) = 0\}.$$

We consider a set defined by

$$S = \left\{ x \in \mathbb{R} \middle| \sum_{n=1}^{\infty} f_n(x) = 0 \right\}$$

(a) Show that if A_n is countable for some $n \in \mathbb{N}$, then S is countable.

(b) Suppose A_n is uncountable for all $n \in \mathbb{N}$,

(i) Is it true that S is always uncountable?

(😊Note: If your answer is yes, give a proof. If your answer is no, give a counter-example.)

(ii) Is it true that S is always countable?

(😊Note: If your answer is yes, give a proof. If your answer is no, give a counter-example.)

Problem 4

Determine if the set C defined by

$$C = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x^2 + y^2 = r^2\}$$

is countable.

(😊Hint: Recall that $x^2 + y^2 = r^2$ represents the equation of circle with radius r in $\mathbb{R}^2$-plane)

Problem 5

(a) Determine if the set D defined by

$$D = \{x \in \mathbb{R}: \tan^{10} x - 3\tan^3 x + 1 = 0\}$$

is countable.

(b) Determine if the set E defined by

$$E = \{x \in \mathbb{R} | a\cos 2x + b\cos x + c = 0 \quad for\ some\ a, b, c, \in \mathbb{Q}\backslash\{0\}\}$$

is countable.

Problem 6

We let $f: A \to B$ be a function, where A, B are non-empty set.

(a) If A is countable, determine if $f(A)$ is countable.

(b) We consider the case when A is uncountable

(i) If f is injective, show that $f(A)$ is also uncountable by mimicking the proof of the injection theorem.

(ii) Is $f(A)$ always uncountable if the function f is not injective? Explain your answer.

Problem 7

We let $A, B \subseteq \mathbb{R}$ be two uncountable sets.

(a) Is it always true that $A\backslash B$ is uncountable?

(b) Is it always true that $A\backslash B$ is countable?

(😊Note: If your answer is yes, give a proof. If your answer is no, give a counter-example.)

Problem 8 (Harder)

We let A be set of all functions from the set $\{0,1\}$ to the set of positive integers $\mathbb{N}$. That is,

$$A = \{f | f: \{0,1\} \to \mathbb{N}\}.$$

Show that A is countable.

Problem 9

Show that for any open interval (a, b), there are infinitely many irrational numbers that lie in this interval.