MATH2033 Mathematical Analysis
Problem Set 5

Problem 0

Try the practice exercises #37-#48, #101-#106, #131, #136, #142, #143, #146, #150, #152.
(*Note: The solution is available in canvas.)

Problem 1

Prove the following limits using the definition of limits

(a) $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$

(b) $\lim_{n\to\infty}\sqrt{x_n+y_n}=2$, where $\{x_n\}$ and $\{y_n\}$ are two sequences of positive real number with $\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=2$.

Problem 2

We let $\{x_n\}$ and $\{y_n\}$ be two sequence of real number with $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}y_n=y$. Suppose that $xy>0$, show that there exists $K\in\mathbb{N}$ such that x_n and y_n have the same sign (either both positive or both negative) when $n\geq K$.

Problem 3

(a) Give an example of two divergent sequences $\{x_n\}$, $\{y_n\}$ such that the sequence $\{x_n+y_n\}$ converges.

(b) Give an example of two divergent sequences $\{x_n\}$, $\{y_n\}$ such that the sequence $\{x_ny_n\}$ converges.

Problem 4

Show that the sequence $\{x_n\}$ defined by $x_n=n^2-n$ diverges to $+\infty$ using the definition.

Problem 5

We let $\{x_n\}$ be a sequence of positive real number which $\lim_{n\to\infty}x_n=+\infty$. Show that $\lim_{n\to\infty}\frac{1}{x_n}=0$.

Problem 6

Show that the sequence $\{x_n\}$ defined by $x_n=(-1)^n\left(2+\frac{1}{n}\right)$ does not converge.

Problem 7

We let $x_1>\sqrt{a}$ and $x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)$ for $n\in\mathbb{N}$, where $a>0$. Show that the sequence $\{x_n\}$ converges.

(😊Hint: Show that $\{x_n\}$ is decreasing by considering $x_{n+1}-x_n$.)

Problem 8
We let $\{x_n\}$ be a bounded sequence of real numbers. For any $n \in \mathbb{N}$, we define
$$y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}.$$
Show that $\{y_n\}$ converges.

Problem 9
We let $\{x_n\}$ is a sequence of positive real numbers. For any $n \in \mathbb{N}$, we define
$$y_n = \max\{x_1, x_2, \dots, x_n\}.$$
- **(a)** If $\{x_n\}$ is bounded, show that $\{y_n\}$ converges.
- **(b)** If $\{x_n\}$ is unbounded, show that $\{y_n\}$ diverges to $+\infty$.

Problem 10
Show that a sequence $\{x_n\}$ defined by $x_n = (-1)^n$ is not Cauchy sequence.

Problem 11
Show that if $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequence, then $\{x_n + y_n\}$ and $\{x_n y_n\}$ are both Cauchy sequence using the definition of Cauchy sequence.

Problem 12 (Harder)
We let $\{x_n\}$ be a sequence of real number with $\lim\limits_{n\to\infty} x_n = x$. Show that
$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = x.$$
(😊Hint: Note that $\lim\limits_{n\to\infty} x_n = x$. Then for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for $n \geq K$.)

Problem 13 (Harder)
We let $\{x_n\}$ be a bounded sequence and let $s = \sup\{x_n | x \in \mathbb{N}\}$. Show that if $s \notin \{x_n | n \in \mathbb{N}\}$, then there exists a subsequence of $\{x_n\}$ which converges to s.
(😊Hint: You need to construct such subsequence. Using the property of supremum and the fact that $s \notin \{x_n | n \in \mathbb{N}\}$, argue that for any $\varepsilon > 0$, there exists infinitely many $x_n s$ such that $s > x_n > s - \varepsilon$. Construct the subsequence by taking $\varepsilon = \frac{1}{k}$ for $k \in \mathbb{N}$.)