MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Some Proof about function:

The examples are the exercise in your lecture note (Chapter 2)

Example 1

Let $f: A \to B$ be a function. We have f is bijection ***if and only if*** there exists a function $g: B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$ (where I_A and I_B are identity function)

Proof:

<u>(Step 1:$\to$ part)</u>

(*To proof the existence, the common way is to construct a function g)

Since f is bijective and therefore surjective, for any $y \in B$, there exists $x \in A$, such that $f(x) = y$.

Construct a map $g: B \to A$, which $g(y) = x$, we need to show g is a well-defined function** (i.e. $y \in B$ only map to only one $x \in A$).

(**A function is said to be well-defined if $x_1 = x_2 \to f(x_1) = f(x_2)$)

Since $y_1 = y_2$

$\to f(x_1) = f(x_2)$ for some x_1, x_2 (since f is surjective)

$\to x_1 = x_2$ (since f is injective)

$\to g(y_1) = g(y_2)$ (by definition of our g)

Then $(g \circ f)(x) = g(f(x)) = g(y) = x \to g \circ f = I_A$,

Similarly $f \circ g = I_B$

<u>(Step 2: $\leftarrow$ part)</u>

(*We just need to check f is injective and surjective, then it will be bijective)

(Injective?)

Since there exist $g: B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$

For $f(x_1) = f(x_2)$

$\to g(f(x_1)) = g(f(x_2))$

$\to (g \circ f)(x_1) = (g \circ f)(x_2)$

$\to x_1 = x_2$ (It is injective)

(Surjective?)

For any $y \in B$, pick $x = g(y) \in A$,

Then $f(x) = f(g(y)) = (f \circ g)(y) = y$ (It is surjective)

Example 2

If $f: A \to B$ and $h: B \to C$ are bijection, then $h \circ f: A \to C$ is also bijection

Proof: (There are 2 ways to prove it)

(Method 1: We show $h \circ f$ is injective and surjective)

(Injective?)

If $\boldsymbol{(h \circ f)(x_1) = (h \circ f)(x_2)}$

$\rightarrow \boldsymbol{h(f(x_1)) = h(f(x_2))}$

$\rightarrow \boldsymbol{f(x_1) = f(x_2)}$ ***(since h is injective)***

$\rightarrow \boldsymbol{x_1 = x_2}$ ***(since f is injective)***

Therefore $\boldsymbol{h \circ f}$ ***is injective***

(Surjective?)

For any element $\boldsymbol{c \in C}$

Since h is surjective, there exists $\boldsymbol{b \in B}$***, such that*** $\boldsymbol{h(b) = c}$

For this b, since f is also surjective, there exists $\boldsymbol{a \in A}$ ***such that*** $\boldsymbol{f(a) = b}$

Choose this a, we have

$\boldsymbol{(h \circ f)(a) = h(f(a)) = h(b) = c}$

So $\boldsymbol{h \circ f}$ ***is surjective***

(Method 2: We can use the result in Example 1, then all we need to do is to find a function g)

Since f and h are bijection, there exist function $\mathbf{g_f: B \to A}$ **and** $\mathbf{g_h: C \to B}$**, such that**

$\mathbf{f \circ g_f = I_B, g_f \circ f = I_A}$ **and** $\mathbf{h \circ g_h = I_C, g_h \circ h = I_B}$

pick $\mathbf{g = g_f \circ g_h}$

then we have

$\mathbf{(h \circ f) \circ g = (h \circ f) \circ (g_f \circ g_h) = h \circ f \circ g_f \circ g_h}$

$\mathbf{= h \circ (f \circ g_f) \circ g_h = h \circ I_B \circ g_h}$

$\mathbf{= h \circ g_h = I_C}$ **(since** $\mathbf{I_B}$ **is identity element)**

Similarly

$\mathbf{g \circ (h \circ f) = I_A}$ **(Try to work out the detail by yourself!!)**

Therefore by the result of Example 1, $h \circ f$ is bijection

Example 3

Let A and B be subsets of R and $\mathrm{f}: \mathrm{A} \rightarrow \mathrm{B}$ be a function. If for every $\mathrm{b} \in \mathrm{B}$, the horizontal line $\mathrm{y}=\mathrm{b}$ intersects the graph of f $\{(\mathrm{x}, \mathrm{f}(\mathrm{x})): \mathrm{x} \in \mathrm{A}\}$ exactly once, then f is bijection

The question looks complicated, but all we need to show are f is injective and surjective.

Proof:

(Injective?)

If $\boldsymbol{f(x_1)=f(x_2)=b}$

Then consider these two points $\boldsymbol{(x_1, f(x_1))=(x_1, b)}$ ***and*** $\boldsymbol{(x_2, f(x_2)=(x_2, b)}$ ***on the graph of f.***

Since these 2 points lies on $\boldsymbol{y=b}$***,***

And by the condition, $\boldsymbol{y=b}$ ***intersects the graph of f exactly once.***

So $\boldsymbol{(x_1, b)=(x_2, b)}$

$\rightarrow \boldsymbol{x_1=x_2}$

So f is injective

(Surjective?)

For any element $\boldsymbol{b \in B}$***,***

For this b, by condition, we know $\boldsymbol{y=b}$ ***intersects the graph of f.***

i.e. $\boldsymbol{(x, b)=(x, f(x))}$ ***for some*** $\boldsymbol{x \in A}$

Therefore $\boldsymbol{f(x)=b}$

So f is surjective

(Comment: Even though the condition given is just strange, but you needn't to handle it first, all you need to do is to focus on WHAT YOU WANT TO SHOW and make use of condition to arrive your desired result)