MATH 2031 Introduction to Real Analysis

March 25, 2013

Tutorial Note 16

Riemann Integral

Proper Integral
In this section, we focus on functions $f(x)$ which are bounded on a closed and bounded interval $[a, b]$. (f is bounded on $[a, b]$ if there exists $K \in \mathbb{R},\ K \geq 0$ such that $|f(x)| \leq K$ for all $x \in [a, b]$.)

(I) **Definition (partition):**

(i) A partition P of $[a, b]$ is a set $\{x_0, x_1, \cdots, x_n\}$, for some $n \in \mathbb{N}$, such that $a = x_0 < x_1 < \cdots < x_n = b$.

(ii) The length of $[x_{j-1}, x_j]$ is $\Delta x_j = x_j - x_{j-1}$.

(iii) The mesh of P is $||P|| = \max\{\Delta x_1, \Delta x_2, \cdots, \Delta x_n\}$

(iv) Denote $m_j = \inf\{f(x)|x \in [x_{j-1}, x_j]\}$ and $M_j = \sup\{f(x)|x \in [x_{j-1}, x_j]\}$

(II) **Definition:**
Given a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ and a function f bounded on $[a, b]$

(i) A Riemann sum of f is $S = \displaystyle\sum_{j=1}^{n} f(t_j)\Delta x_j$, where every $t_j \in [x_{j-1}, x_j]$.

(ii) A lower Riemann sum of f is $L(f, P) = \displaystyle\sum_{j=1}^{n} m_j \Delta x_j$, where every $t_j \in [x_{j-1}, x_j]$.

(iii) A upper Riemann sum of f is $U(f, P) = \displaystyle\sum_{j=1}^{n} M_j \Delta x_j$, where every $t_j \in [x_{j-1}, x_j]$.

Remark:
Since f is bounded, we get

$$-K \leq m_j \leq f(t_j) \leq M_j \leq K \quad \Rightarrow \quad K(b-a) \leq L(f, P) \leq S \leq U(f, P) \leq K(b-a)$$

(III) **Definition (refinement):**

(i) For partition P_1, P_2, we say that P_2 is a refinement of P_1 iff $P_1 \subseteq P_2$.

(ii) For partition P_1, P_2, we say that $P_1 \cup P_2$ is the common refinement of P_1 and P_2.

(IV) **Refinement theorem:**
If $P \subseteq \widetilde{P}$, then

$$\underbrace{L(f, P) \leq L(f, \widetilde{P})}_{\text{Lower sum increasing}} \quad \leq \quad \underbrace{U(f, \widetilde{P}) \leq U(f, P)}_{\text{Upper sum decreasing}}$$

Remark:
Follows from Refinement theorem (and common refinement), even with different partitions, lower sum $\leq$ upper sum.

(V) **Definition (Riemann integrable):**

(i) • The lower integral of $f(x)$ on $[a,b]$ is

$$(L)\int_a^b f(x)dx = \sup\{L(f,P) | P \text{ partition of } [a,b]\} = \underline{\int_a^b} f(x)dx$$

• The upper integral of $f(x)$ on $[a,b]$ is

$$(U)\int_a^b f(x)dx = \inf\{U(f,P) | P \text{ partition of } [a,b]\} = \overline{\int_a^b} f(x)dx$$

Remark:
With the remark above, we get that $(L)\int_a^b f(x)dx \leq (U)\int_a^b f(x)dx$

(ii) $f(x)$ is Riemann integrable on $[a,b]$ iff

$$(L)\int_a^b f(x)dx = (U)\int_a^b f(x)dx$$

In that case we write $\displaystyle\int_a^b f(x)dx$ for that value.

(VI) **Integral criterion:**
Let $f(x)$ be bounded on $[a,b]$.

$$f(x) \text{ is Riemann integrable on } [a,b] \iff \begin{pmatrix} \forall\varepsilon > 0\ \exists \text{ partition } P \text{ of } [a,b] \text{ such that} \\ U(f,P) - L(f,P) < \varepsilon \end{pmatrix}$$

(VII) **Definition (uniform continuity):**
$f : S \to \mathbb{R}$ is uniform continuous iff $\forall\varepsilon > 0,\ \exists\delta > 0$ (δ depends only on ε) such that

$$\forall x, t \in S, \qquad |x - t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon$$

(VIII) **Uniform Continuity Theorem:**
If f is continuous in a closed and bounded intervals, then it's uniformly continuous.

(IX) **Theorem:**
If $f : [a,b] \to \mathbb{R}$ is continuous, then it is integrable.

Problem 1 Let $f : [0,1] \to \mathbb{R}$ be given by $f(x) = x^3$, prove that f is Riemann integrable.

Scratch:
Consider the partition $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\right\}$.
On each $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ for $k \in \{1, 2, \cdots, n\}$, we have
$m_j = \inf\{f(x)\} = \left(\frac{k-1}{n}\right)^3$ and $M_j = \sup\{f(x)\} = \left(\frac{k}{n}\right)^3$.
We get

$$L(f,P) = \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k-1}{n}\right)^3 \qquad \text{and} \qquad U(f,P) = \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k}{n}\right)^3$$

then

$$U(f,P) - L(f,P) = \frac{1}{n}\left[\sum_{k=1}^{n}\left(\frac{k}{n}\right)^3 - \sum_{k=1}^{n}\left(\frac{k-1}{n}\right)^3\right] = \frac{1}{n}\left(\frac{n}{n}\right)^3 = \frac{1}{n}.$$

So if $n > \dfrac{1}{\varepsilon}$, then $U(f,P) - L(f,P) < \varepsilon$.

Solution:
Consider a partition $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\right\}$ with $n > \dfrac{1}{\varepsilon}$.

On each $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ for $k \in \{1, 2, \cdots, n\}$,
$m_j = \inf\{f(x)\} = \left(\frac{k-1}{n}\right)^3$ and $M_j = \sup\{f(x)\} = \left(\frac{k}{n}\right)^3$.
We get

$$L(f,P) = \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k-1}{n}\right)^3 \qquad \text{and} \qquad U(f,P) = \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k}{n}\right)^3,$$

thus

$$U(f,P) - L(f,P) = \frac{1}{n}\left(\sum_{k=1}^{n}\left(\frac{k}{n}\right)^3 - \sum_{k=1}^{n}\left(\frac{k-1}{n}\right)^3\right) = \frac{1}{n}\left(\frac{n}{n}\right)^3 = \frac{1}{n} < \varepsilon.$$

By Integral criterion, $f(x) = x^3$ is Riemann integrable.
(Of course we can directly state that $f(x) = x^3$ is Riemann integrable since f is continuous. The above is just an example on how to apply the Integral criterion.)

Problem 2 Let $x_1, x_2, \cdots, x_n$ be distinct points in $[0,1]$ such that $x_1 < x_2 < \cdots < x_n$.
Define $g : [0,1] \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 1 & \text{for } x = x_1, x_2, \cdots, x_n \\ 0 & \text{otherwise} \end{cases}$$

Prove that g is Riemann integrable.

Scratch:
Consider a partition $P = \{0, x_1 - \delta, x_1 + \delta, x_2 - \delta, x_2 + \delta \cdots, x_n - \delta, x_n + \delta, 1\}$.
It is clear that $\inf\{f(x)|x \in [0,1]\} = 0$ for any x, and $\sup\{f(x)|x \in [0,1]\} = 1$ on those intervals containing one of the $x_1, x_2, \cdots, x_n$.
Then $m_j = 0$ and $M_j = \begin{cases} 1 & \text{if the } j^{\text{th}} \text{ interval is of the form } (x_k - \delta, x_k + \delta) \text{ for some } k \in \{1, 2, \cdots, n\} \\ 0 & \text{otherwise} \end{cases}$

We get

$$L(f,P) = 0 \qquad \text{and} \qquad U(f,P) = \sum_{k=1}^{n}(x_k + \delta - (x_k - \delta)) = 2\delta\sum_{k=1}^{n} 1 = 2\delta\left(\frac{n(n+1)}{2}\right) = n(n+1)\delta$$

Then

$$U(f,P) - L(f,P) = n(n+1)\delta$$

If $\delta < \dfrac{\varepsilon}{n(n+1)}$, then $U(f,P) - L(f,P) < \varepsilon$.

Solution:
Consider a partition $P = \{0, x_1 - \delta, x_1 + \delta, x_2 - \delta, x_2 + \delta \cdots, x_n - \delta, x_n + \delta, 1\}$ with $\delta < \dfrac{\varepsilon}{n(n+1)}$.
Clearly that for any x, $\inf\{f(x)|x \in [0,1]\} = 0$, and on those intervals containing one of the $x_1, x_2, \cdots, x_n$, $\sup\{f(x)|x \in [0,1]\} = 1$.
Then $m_j = 0$ and $M_j = \begin{cases} 1 & \text{if the } j^{th} \text{ intervals is of the form } (x_k - \delta, x_k + \delta) \text{ for some } k \in \{1, 2, \cdots, n\} \\ 0 & \text{otherwise} \end{cases}$

We get

$$L(f,P) = 0 \qquad \text{and} \qquad U(f,P) = \sum_{k=1}^{n}(x_k + \delta - (x_k - \delta)) = 2\delta\sum_{k=1}^{n} 1 = 2\delta\left(\frac{n(n+1)}{2}\right) = n(n+1)\delta$$

Then $U(f,P) - L(f,P) = n(n+1)\delta < \varepsilon$.
By Integral criterion, $g(x)$ is Riemann integrable.

Problem 3 Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = ax + b$, where $a, b \in \mathbb{R}$ and $a \neq 0$, prove that f is uniformly continuous.

Solution:
$\forall \varepsilon > 0$, take $\delta = \frac{\varepsilon}{|a|} > 0$, then $\forall x, t \in \mathbb{R}$ $|x - t| < \delta$,

$$|f(x) - f(t)| = |ax + b - (at + b)| = |a||x - t| < |a|\delta < \varepsilon$$

By definition, f is uniformly continuous.

Problem 4 Let $g : (0, \infty) \to \mathbb{R}$ be given by $g(x) = x^2$, prove that g is continuous but not uniformly continuous.

Solution:

Continuous

For any $t \in \mathbb{R}$, $\forall \varepsilon > 0$, take $\delta = \min\left\{1, \dfrac{\varepsilon}{2(t+1)}\right\} > 0$.
Then $\forall x \in (0, \infty)$ with $|x - t| < \delta$, notice that $|x - t| < 1 \Rightarrow\ x < t + 1$,

$$|g(x) - g(t)| = |x^2 - t^2| = |x + t||x - t| < 2(t + 1)\delta < \varepsilon$$

Then by definition, g is continuous at every $t \in (0, \infty)$.

Not uniformly continuous

What we need to prove is that

$$\exists \varepsilon > 0,\ \forall \delta > 0,\ \exists x, t \in (0, \infty) \text{ such that} \qquad |x - t| < \delta \text{ and } |x^2 - t^2| \geq \varepsilon$$

Let $\varepsilon = 1$, $\forall \delta > 0$, consider $t = \dfrac{1}{\delta}$ and $x = t + \dfrac{\delta}{2}$, so $|x - t| = \dfrac{\delta}{2} < \delta$.
However,

$$|g(x) - g(t)| = |x^2 - t^2| = \left|\left(\frac{1}{\delta} + \frac{\delta}{2}\right)^2 - \left(\frac{1}{\delta}\right)^2\right| = 1 + \frac{\delta^2}{4} \geq 1 = \varepsilon$$

Therefore by definition, g is not uniformly continuous.