MATH2033 Mathematical Analysis (2021 Spring)
Suggested Solution of Assignment 3

Problem 1

Prove the following statement using the definition of limits:

(a) $\lim_{x\to c}\frac{1}{x}=\frac{1}{c}\quad for\;\; c\neq 0.$

(b) We let $f:(c,\infty)\to\mathbb{R}$ be a function with $f(x)>0$ for all $x\in(c,\infty)$. Show that $\lim_{x\to c}f(x)=\infty$ if and only if $\lim_{x\to c}\frac{1}{f(x)}=0$.

(c) We let $f:(0,\infty)\to\mathbb{R}$ be a function. Show that $\lim_{x\to\infty}f(x)=L$ if and only if $\lim_{x\to 0^+}f\left(\frac{1}{x}\right)=L$.

😊Solution

(a) For any $\varepsilon>0$, we take $\delta=\min\left(\frac{|c|}{2},\frac{|c|^2}{2}\varepsilon\right)$. Then for any x satisfying $0<|x-c|<\delta$, we have

$$\left|\frac{1}{x}-\frac{1}{c}\right|=\frac{|x-c|}{|c||x|}<\frac{|x-c|}{|c|\left(\frac{|c|}{2}\right)}<\left(\frac{1}{\frac{|c|^2}{2}}\right)\frac{|c|^2}{2}\varepsilon=\varepsilon.$$

So $\lim_{x\to c}\frac{1}{x}=\frac{1}{c}$ by definition of limits.

(b) "⇒" part

For any $\varepsilon>0$, since $\lim_{x\to c}f(x)=\infty$, then there exists $\delta>0$ such that for any x satisfying $0<|x-c|<\delta$,

$$f(x)>M>\frac{1}{\varepsilon}.$$

This implies that for $|x-c|<\delta$

$$\left|\frac{1}{f(x)}-0\right|\overset{f(x)>0}{\cong}\frac{1}{f(x)}<\varepsilon.$$

So we have $\lim_{x\to c}\frac{1}{f(x)}=0$.

"⇐" part

For any $M>0$, we note that $\lim_{x\to c}\frac{1}{f(x)}=0$. By picking $\varepsilon=\frac{1}{M}$, there exists $\delta>0$ such that when $0<|x-c|<\delta$, we have

$$\left|\frac{1}{f(x)}-0\right|<\varepsilon=\frac{1}{M}\Rightarrow|f(x)|>M.$$

This implies that for $|x-c|<\delta$,

$$f(x)=|f(x)|>M.$$

So we have $\lim_{x\to c}f(x)=\infty$

(c) "⇒" part

For any $\varepsilon > 0$, since $\lim_{x\to\infty} f(x) = L$, then there exists M> 0 such that for any x satisfying $x > M$,

$$|f(x) - L| < \varepsilon.$$

By taking $\delta = \frac{1}{M}$, then for any $0 < x < \delta = \frac{1}{M}$, we have $\frac{1}{x} > M$ and

$$\left|f\left(\frac{1}{x}\right) - L\right| < \varepsilon.$$

So we have $\lim_{x\to 0^+} f\left(\frac{1}{x}\right) = L$.

"$\Leftarrow$" part

For any $\varepsilon > 0$, since $\lim_{x\to 0^+} f\left(\frac{1}{x}\right) = L$, there exists $\delta > 0$ such that when $0 < x < \delta$ (or $\frac{1}{x} > \frac{1}{\delta}$), we have

$$\left|f\left(\frac{1}{x}\right) - L\right| < \varepsilon.$$

By taking $M = \frac{1}{\delta}$, then for any $x > M = \frac{1}{\delta}$, we have

$$|f(x) - L| < \varepsilon.$$

So we conclude that $\lim_{x\to\infty} f(x) = L$.

Problem 2

We consider a function $f: \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x & if\ \ x \in \mathbb{Q} \\ 0 & if\ \ x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}.$$

(a) Show that $\lim_{x\to 0} f(x) = 0$. Is $f(x)$ continuous at $x = 0$?

(b) For any $c \neq 0$, show that $\lim_{x\to c} f(x)$ does not exist.

Solution

(a) For any $\varepsilon > 0$, we take $\delta = \varepsilon$. Then for any $0 < |x - 0| < \varepsilon$,

$$|f(x) - 0| = |f(x)| \overset{f(x)=x\ \ or\ 0}{\leqq} |x| < \varepsilon.$$

Thus $\lim_{x\to 0} f(x) = 0$ by definition. Since $\lim_{x\to 0} f(x) = f(0) = 0$, so $f(x)$ is continuous at $x = 0$.

(b) For any $n \in \mathbb{N}$, we can deduce from density of rational number and density of irrational number that there exists $r_n \in \mathbb{Q}$ and $q_n \in \mathbb{R}\backslash\mathbb{Q}$ such that

$$c - \frac{1}{n} < r_n < c \quad and \quad c - \frac{1}{n} < q_n < c$$

Since $\lim_{n\to\infty}\left(c - \frac{1}{n}\right) = c$, then we get $\lim_{n\to\infty} r_n = c$ and $\lim_{n\to\infty} q_n = c$ by sandwich theorem.

Note that

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} r_n = c \quad and \quad \lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} 0 = 0 \neq c$$

Since $\lim_{n\to\infty} f(r_n) \neq \lim_{n\to\infty} f(q_n)$, so it follows from sequential limit theorem that $\lim_{x\to c} f(x)$ does not exist.

Problem 3

(a) We let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function and define a set

$$S = \{x \in \mathbb{R} | f(x^2) \geq f(x)\}.$$

Suppose that there exists a sequence $\{y_n\}$ which $y_n \in S$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = y$, show that $y \in S$.

(b) We let $f, g: \mathbb{R} \to \mathbb{R}$ be two continuous functions on $\mathbb{R}$ such that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Is it true that $f(x) = g(x)$ for all $x \in \mathbb{R}$?

😊Solution

(a) Since x^2 is continuous n $\mathbb{R}$, it follows from $f(x^2)$ is also continuous. Then $g(x) = f(x^2) - f(x)$ is also continuous on $\mathbb{R}$.
On the other hand, since $y_n \in S$, we have

$$f(y_n^2) \geq f(y_n) \Leftrightarrow g(y_n) = f(y_n^2) - f(y_n) \geq 0.$$

Since $\lim_{n\to\infty} y_n = y$ and $g(x)$ is continuous at $x = y$, it follows from sequential limit theorem and limit inequality that

$$g(y) = \lim_{n\to\infty} g(y_n) \overbrace{\geq}^{g(y_n)\geq 0} 0.$$

This implies $f(y^2) \geq f(y)$ and $y \in S$.

(b) For any $x \in \mathbb{R}$, it follows from the result in Problem 2 that there exists a sequence of rational number $\{r_n\}$ such that $\lim_{n\to\infty} r_n = x$.
Since $f(x), g(x)$ are continuous at any $x \in \mathbb{R}$, it follows from sequential limit theorem that

$$\lim_{n\to\infty} f(r_n) = f(x) \quad and \quad \lim_{n\to\infty} g(r_n) = g(x)$$

Since $f(r_n) = g(r_n)$, it follows that

$$f(x) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} g(r_n) = g(x).$$

Problem 4

We let f be a continuous function over $[0,2]$ (i.e. $f \in C([0,2])$) with $f(0) = f(2) = 0$. Show that there exists $c \in [0,1]$ such that $f(c) = f(c+1)$.
(☺Hint: Do the analysis by considering the *sign* of $f(1)$).

😊Solution

We let a function $g: [0,1] \to \mathbb{R}$ be

$$g(x) = f(x) - f(x+1).$$

Since $f(x)$ is continuous on $[0,2]$, then $g(x)$ is also continuous on $[0,1]$.

If $f(1) = 0$, then one can see that $f(0) = f(0+1)$ and we are done. When $f(1) \neq 0$, we consider the following two cases:

- <u>Case 1: $f(1) > 0$</u>
  Since $g(0) = \underbrace{f(0)}_{=0} - f(1) < 0$ and $g(1) = f(1) - \underbrace{f(2)}_{=0} > 0$, then it follows from intermediate value theorem that there exists $c \in (0,1)$ such that

$$g(c) = 0 \Rightarrow f(c) = f(c+1).$$

- Case 2: $f(1) < 0$
  Since $g(0) = \underbrace{f(0)}_{=0} - f(1) > 0$ and $g(1) = f(1) - \underbrace{f(2)}_{=0} < 0$, then it follows from intermediate value theorem that there exists $c \in (0,1)$ such that
  $$g(c) = 0 \Rightarrow f(c) = f(c+1).$$

Problem 5

We let $f: [a,b] \to \mathbb{R}$ be a continuous function such that for any $x \in [a,b]$, there exists $y \in [a,b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Show that there exists $c \in [a,b]$ such that $f(c) = 0$.

😊Solution

Since $f(x)$ and $|x|$ are continuous on $[a,b]$, so $|f(x)|$ is also continuous on $[a,b]$. By extreme value theorem, there exists $x_L \in [a,b]$ such that

$$|f(x_L)| = \inf\{|f(x)|: x \in [a,b]\}.$$

We shall argue that $|f(x_L)| = 0$. Suppose that $|f(x_L)| > 0$, then there exists $y \in [a,b]$ such that

$$|f(y)| \leq \frac{1}{2}|f(x_L)| < |f(x_L)| = \inf\{|f(x)|: x \in [a,b]\}.$$

This leads to contradiction. Thus we must have $|f(x_L)| = 0$ and the proof is completed.

Problem 6 (Harder)

We let $f: [a,b] \to \mathbb{R}$ be a continuous function over $[a,b]$. We define two functions $M(x)$ and $m(x)$ as

$$M(x) = \sup\{f(t) | t \in [a,x]\}$$
$$m(x) = \inf\{f(t) | t \in [a,x]\}$$

Show that both $M(x)$ and $m(x)$ are continuous at any $x_0 \in [a,b]$.

(☺Hint: Note that both functions $M(x)$ and $m(x)$ are monotone functions.)

😊Solution

Since f is continuous on $[a,x]$ for any $x \in [a,b]$, it follows from extreme value theorem that $f(x)$ is bounded over $[a,x]$, so $M(x)$ and $m(x)$ exists.

For any $x, y \in [a,b]$ with $x < y$, we have $[a,x] \subseteq [a,y]$ so that

$$M(x) = \sup\{f(t) | t \in [a,x]\} \leq \sup\{f(t) | t \in [a,y]\} = M(y)$$
$$m(x) = \inf\{f(t) | t \in [a,x]\} \geq \inf\{f(t) | t \in [a,y]\} = m(y)$$

So $M(x)$ is increasing and $m(x)$ is decreasing.

Next, we shall prove that $M(x)$ is continuous at $x_0 \in (a,b)$.

- Since $M(x)$ is monotone, it follows from monotone function theorem that the one-sided limit exists and
  $$\lim_{x \to x_0^-} M(x) \leq M(x_0) \leq \lim_{x \to x_0^+} M(x)$$

- Next, we argue that $\lim_{x \to x_0^-} M(x) = M(x_0)$.

  By applying extreme value theorem on $[a, x_0]$, there exists $x_U \in [a, x_0]$ such that $f(x_U) = \sup\{f(t) | t \in [a, x_0]\} = M(x_0)$.

  - ✓ If $x_U < x_0$, then it follows that $M(x) = f(x_U) = M(x_0)$ for all $x \geq x_U$. So we have $\lim_{x \to x_0^-} M(x) = \lim_{x \to x_0^-} M(x_0) = M(x_0)$.
  - ✓ If $x_U = x_0$, since $f(x)$ is continuous at $x = x_0$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $0 < |x - x_0| < \delta$, we have
    $$|f(x) - f(x_0)| < \varepsilon \Rightarrow f(x) > f(x_0) - \varepsilon.$$
    Then for this δ, we deduce that for $0 < |x - x_0| < \delta$ and $x < x_0$,
    $$|M(x) - M(x_0)| = \underbrace{M(x_0)}_{=f(x_0)} - M(x) < f(x_0) - f(x) < \varepsilon.$$
    So we also have $\lim_{x \to x_0^-} M(x) = M(x_0)$ by definition of limits.

- Finally, we argue that $\lim_{x \to x_0^+} M(x) = M(x_0)$.

  Since $|f(x) - f(x_0)| < \varepsilon \Rightarrow f(x) < f(x_0) + \varepsilon$ for $0 < |x - x_0| < \delta$. It follows that for $0 < |x - x_0| < \delta$ and $x > x_0$,
  $$|M(x) - M(x_0)| = M(x) - M(x_0)$$
  $$< \begin{cases} f(x_0) + \varepsilon - f(x_0) & if \ \ f(x_0) + \varepsilon \geq M(x_0) \\ M(x_0) - M(x_0) & if \ \ f(x_0) + \varepsilon < M(x_0) \end{cases} < \varepsilon.$$
  So $\lim_{x \to x_0^+} M(x) = M(x_0)$ by definition of limits.

Therefore, we conclude that $\lim_{x \to x_0} M(x) = M(x_0)$. Similarly, one can show that $\lim_{x \to x_0} m(x) = m(x_0)$ and we omit the details here.