Mean Value Theorem Let f be continuous on $[a,b]$ and differentiable on (a,b). Then $\exists\, x_0 \in (a,b)$ such that

$$f(a)-f(b)=f'(x_0)(a-b).$$

Use when you have $f(\ )-f(\ )$ expression and f is differentiable. ↖ #74, 75, 120

Use to prove inequalities ↖ examples on p. 38

Generalized Mean Value Theorem Let $f,g:[a,b]\to\mathbb{R}$ be continuous. Let f,g be differentiable on (a,b). Then $\exists\, \theta\in(a,b)$ such that

$$(f(b)-f(a))\,g'(\theta)=(g(b)-g(a))\,f'(\theta).$$

Taylor's Theorem For $\underline{\text{n-times diff}}$ $f:(a,b)\to\mathbb{R}$, $x,c\in(a,b)$, $f(x)=f(c)+\frac{f'(c)}{1!}(x-c)+\cdots+\frac{f^{(n-1)}(c)}{(n-1)!}(x-c)^{n-1}+R_n(x)$

where $R_n(x)=\frac{f^{(n)}(x_0)}{n!}(x-c)^n$ for some x_0 between x and c.

- Use when functions are n-times differentiable, $n>1$
- center c should be

① something we know about $f'(c)$

or ② among $f(a), f(b),\dots$ given, $f(c)$ is slightly special

otherwise ③ try a variable for c.

Or ④ local max or min, then $f'(c)=0$.

Examples of Mean Value Theorem or Generalized Version

① Suppose $a_0,a_1,\dots,a_n\in\mathbb{R}$ such that $a_0+\frac{a_1}{2}+\dots+\frac{a_n}{n+1}=0$. Prove that $P(x)=a_0+a_1x+\dots+a_nx^n$ has a root in $[0,1]$.

Solution Define $Q(x)=a_0x+\frac{a_1}{2}x^2+\dots+\frac{a_n}{n+1}x^{n+1}$. Then $Q(0)=0=Q(1)$. So $0=Q(1)-Q(0)=Q'(\theta)(1-0)=P(\theta)$ for some $\theta\in(0,1)$.

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② Let $a,b,c\in\mathbb{R}$. Prove that the equation $e^x=ax^2+bx+c$ has at most 3 solutions in $\mathbb{R}$.

Solution Assume the equation has $\underbrace{x_1,x_2,x_3,x_4}_{\text{with } x_1<x_2<x_3<x_4}\in\mathbb{R}$ as solutions. Then let $f(x)=e^x-ax^2-bx-c$.

For $i=1,2,3$, $0=\underbrace{f(x_i)}_{=0}-\underbrace{f(x_{i+1})}_{=0}=f'(y_i)\underbrace{(x_i-x_{i+1})}_{<0}\Rightarrow f'(y_i)=0$. $x_i<y_i<x_{i+1}$

For $j=1,2$, $0=f'(y_j)-f'(y_{j+1})=f''(z_j)(y_j-y_{j+1})$

$\Rightarrow f''(z_j)=0$. Then $0=f''(z_1)-f''(z_2)=f'''(w)\underbrace{(z_1-z_2)}_{<0}$ where $y_j<z_j<y_{j+1}$

So $0=f'''(w)=e^w\neq 0$, Contradiction.

---

③ Let $0\le a<b$. If $f:[a,b]\to\mathbb{R}$ is continuous, $f(a)\neq f(b)$ and f differentiable on (a,b), then prove $\exists\, r,s\in(a,b)$ such that $f'(r)=\frac{b+a}{2s}f'(s)$.

Solution The equation is the same as $\boxed{f'(r)(b-a)=\frac{f'(s)(b^2-a^2)}{2s}}$.

By mean value theorem and generalized version, $\exists\, r,s\in(a,b)$ such that $\frac{f(b)-f(a)}{b-a}=f'(r)$ and $\frac{f(b)-f(a)}{b^2-a^2}=\frac{f'(s)}{2s}$. Then

$f'(r)(b-a)=f(b)-f(a)=\frac{f'(s)}{2s}(b^2-a^2)$.

Examples of Taylor's Theorem

Let $f(x)$ have 2^{nd} derivative at every $x \in [a,b]$.
If $f'(a) = f'(b) = 0$, then prove that $\exists\, c \in (a,b)$
such that $|f''(c)| \geq \frac{4}{(b-a)^2}|f(b)-f(a)|$.

<u>Solution</u> By Taylor's theorem,

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(\theta_1)}{2}(x-b)^2$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(\theta_2)}{2}(x-a)^2.$$

To get $f(b)-f(a)$, we subtract these equations

$$0 = f(b) - f(a) + \tfrac{1}{2}\left(f''(\theta_1)(x-b)^2 - f''(\theta_2)(x-a)^2\right).$$

Setting $x = \frac{a+b}{2}$, then $(x-b)^2 = (x-a)^2 = \left(\frac{b-a}{2}\right)^2$

so $0 = f(b) - f(a) + \frac{(b-a)^2}{8}\left(f''(\theta_1) - f''(\theta_2)\right)$

$$\Rightarrow |f(b)-f(a)|\frac{4}{(b-a)^2} = \tfrac{1}{2}|f''(\theta_1) - f''(\theta_2)|$$

$$\leq \underbrace{\tfrac{1}{2}\left(|f''(\theta_1)| + |f''(\theta_2)|\right)}_{\leq |f''(c)|}$$

If $|f''(\theta_1)| \leq |f''(\theta_2)|$, then take $c = \theta_2$
If $|f''(\theta_2)| \leq |f''(\theta_1)|$, then take $c = \theta_1$

Let $f: [0,1] \to \mathbb{R}$ be continuous and $f(0) = f(1)$.
If f is twice differentiable on $(0,1)$ and there is $M > 0$
such that $|f''(x)| \leq M$ for all $x \in (0,1)$,
then prove that $|f'(x)| \leq \frac{1}{2}M$ for all $x \in (0,1)$.

<u>Thoughts</u>: Taylor theorem problem because higher derivatives are involved. Although $f(0) = f(1)$ is given, we have <u>no information on $f'(0)$ and $f'(1)$</u>.

<u>Solution</u> By Taylor's theorem,

$$f(1) = f(x) + f'(x)(1-x) + \frac{f''(\theta)}{2}(1-x)^2 \quad \text{for some } \theta \in (x,1)$$

$$f(0) = f(x) + f'(x)(0-x) + \frac{f''(\sigma)}{2}(0-x)^2 \quad \text{for some } \sigma \in (0,x).$$

Since $f(1) = f(0)$, we subtract the equations above to get

$$0 = f'(x) + \frac{f''(\theta)}{2}(1-x)^2 - \frac{f''(\sigma)}{2}x^2.$$

So $f'(x) = \frac{f''(\sigma)}{2}x^2 - \frac{f''(\theta)}{2}(1-x)^2$.

Then
$$|f'(x)| \leq \frac{|f''(\sigma)|}{2}x^2 + \frac{|f''(\theta)|}{2}(1-x)^2$$
$$\leq \frac{M}{2}x^2 + \frac{M}{2}(1-x)^2$$
$$= \frac{M}{2}\left(x^2 + (1-x)^2\right) \leq \frac{M}{2}$$

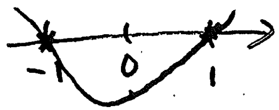
On $[0,1]$,
$x^2 + (1-x)^2 = 2x^2 - 2x + 1$
has <u>maximum value</u> 1 by calculus
or $2(x-\frac{1}{2})^2 + \frac{1}{2} \leq 2(\frac{1}{2})^2 + \frac{1}{2} = 1$.

0 $\frac{1}{2}$ 1

Let $f:\mathbb{R}\to\mathbb{R}$ be twice differentiable such that $f(-1)=0=f(1)$, $f(0)<0$ and $\forall x\in[-1,1]$, $f''(x)\geq 2$. Prove that $\exists\, b\in[-1,1]$ satisfying $f(b)\leq -(1+b^2)$.

Note we don't know $f(0)$

f has a minimum in $(-1,1)$.



Solution Let $c\in(-1,1)$ such that $f(c)=\min\{f(x): x\in[-1,1]\}$ by extreme value theorem. By Taylor's theorem,

$$0=f(-1)=f(c)+\underbrace{f'(c)}_{=0}(-1-c)+\frac{f''(\theta_{-1})}{2}(-1-c)^2$$

$$0=f(1)=f(c)+\underbrace{f'(c)}_{=0}(1-c)+\frac{f''(\theta_{1})}{2}(1-c)^2$$

Moving $f(c)$ to the left side and adding equations, we get

$$-2f(c)=\frac{f''(\theta_{-1})}{2}(1+c)^2+\frac{f''(\theta_{1})}{2}(1-c)^2$$
$$\geq (1+c)^2+(1-c)^2=2+2c^2$$

$\Rightarrow f(c)\leq -(1+c^2)$. So c is such b.

Main Facts on Riemann Integration

Definition A set S is of measure 0 iff $\forall \varepsilon>0$, $\exists$ open intervals $(a_1,b_1),(a_2,b_2),(a_3,b_3),\cdots$ such that $S\subseteq\bigcup_{n=1}^{\infty}(a_n,b_n)$ and $\sum_{n=1}^{\infty}|a_n-b_n|<\varepsilon$.

Known Examples

1. Every countable set is of measure 0. There also exist uncountable sets of measure 0.
2. If $S_1,S_2,S_3,\dots$ are sets of measure 0, then $\bigcup_{n=1}^{\infty}S_n$ is also of measure 0.
3. If $A\subseteq B$ and B is of measure 0, then A is of measure 0.

Lebesgue's Theorem

A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable iff the set

$$S_f=\{x\in[a,b]: f \text{ is not continuous at } x\}$$

is a set of measure 0.

Known Facts

1. Continuous functions on $[a,b]$ are integrable. Monotone functions on $[a,b]$ are integrable.
2. If f_1,f_2 are integrable on $[a,b]$, then f_1+f_2, f_1-f_2, f_1f_2 are integrable on $[a,b]$.
3. If f is integrable on $[a,b]$ and $[c,d]\subseteq[a,b]$, then f is integrable on $[c,d]$.
4. If f is integrable on $[a,b]$ and g is continuous on $f([a,b])$, then $g\circ f$ is integrable on $[a,b]$.

Examples of Lebesgue's Theorem

① Let $f:[0,1]\to[0,1]$ be Riemann integrable. For all $x\in[0,1]$, let $g(x)=f(\sqrt[3]{x})$. Prove that g is Riemann integrable on $[0,1]$.

Solution For all $x\in[0,1]$, $g(x)=f(\sqrt[3]{x})\in[0,1]$, which implies g is bounded.

Since $f(x)$ is Riemann integrable on $[0,1]$, the set $S_f=\{x\in[0,1]: f \text{ is discontinuous at } x\}$ is a set of measure 0. Now f is discontinuous at $w \Leftrightarrow g$ is discontinuous at w^3. So we need to show $S_g=\{w^3: w\in S_f\}$ is a set of measure 0.

For every $\varepsilon>0$, $\exists (a_i,b_i)$ such that $S_f\subseteq \bigcup_{i=1}^{\infty}(a_i,b_i)$ and $\sum_{i=1}^{\infty}|a_i-b_i|<\frac{\varepsilon}{12}$. We may replace each (a_i,b_i) by $(a_i,b_i)\cap(-1,2)$ to ensure $-1\le a_i\le b_i\le 2$. In that case, $|a_i^3-b_i^3|=|a_i-b_i|\underbrace{|a_i^2+a_ib_i+b_i^2|}_{\le 2^2+2^2+2^2=12}\le 12|a_i-b_i|$

Then $S_g\subseteq\bigcup_{i=1}^{\infty}(a_i^3,b_i^3)$ and $\sum_{i=1}^{\infty}|a_i^3-b_i^3|\le 12\sum_{i=1}^{\infty}|a_i-b_i|$ $<12\left(\frac{\varepsilon}{12}\right)=\varepsilon$. $\therefore$ S_g is of measure 0 and g is Riemann integrable on $[0,1]$.

② Let $f,g:[0,1]\to\mathbb{R}$ be Riemann integrable. Define $h:[0,1]\to\mathbb{R}$ by

$$h(x)=\begin{cases} f(x) & \text{if } x\in\bigcup_{n=1}^{\infty}\left[\frac{1}{2n},\frac{1}{2n-1}\right] \\ g(x) & \text{if } x\in\{0\}\cup\bigcup_{n=1}^{\infty}\left(\frac{1}{2n+1},\frac{1}{2n}\right).\end{cases}$$

Prove that $h(x)$ is Riemann integrable on $[0,1]$.

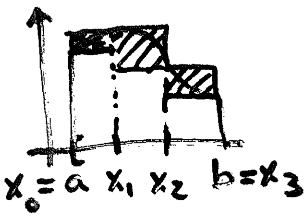
Solution f,g Riemann integrable on $[0,1]$ $\Rightarrow$ f,g bounded on $[0,1]$
$\Rightarrow \exists K_1,K_2>0$ such that $\forall x\in[0,1]$, $|f(x)|\le K_1$, $|g(x)|\le K_2$
$\Rightarrow \forall x\in[0,1]$, $|h(x)|\le\max\{|f(x)|,|g(x)|\}\le\max\{K_1,K_2\}$
$\Rightarrow h$ is bounded on $[0,1]$.

If $x\in\left(\frac{1}{2n},\frac{1}{2n-1}\right)$, then f is continuous at $x \Leftrightarrow$ h is continuous at x because $f(x)=h(x)$ on $\left(\frac{1}{2n},\frac{1}{2n-1}\right)$.
If $x\in\left(\frac{1}{2n+1},\frac{1}{2n}\right)$, then g is continuous at $x \Leftrightarrow$ h is continuous at x because $g(x)=h(x)$ on $\left(\frac{1}{2n+1},\frac{1}{2n}\right)$.
Also, h may be discontinuous at $\{1,\frac{1}{2},\frac{1}{3},\cdots\}\cup\{0\}$.

So $S_h\subseteq S_f\cup S_g\cup\{1,\frac{1}{2},\frac{1}{3},\cdots\}\cup\{0\}$.

measure 0 sets

$\therefore$ S_h is a set of measure 0.
$\therefore$ h is Riemann integrable on $[0,1]$.



Theorem (Integral Criterion)

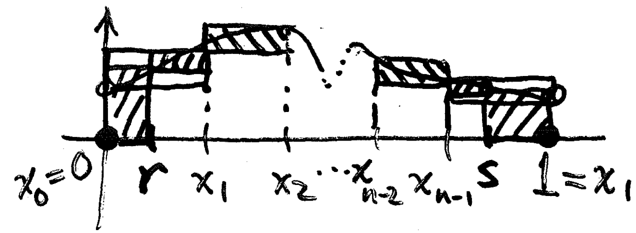
Let $f(x)$ be bounded on $[a,b]$. $f(x)$ is Riemann integrable on $[a,b]$ $\Longleftrightarrow$ $\forall \varepsilon>0$, $\exists$ partition P of $[a,b]$ such that $U(f,P)-L(f,P)<\varepsilon$.

Examples of Integral Criterion

(2003 Final) Let $f:[0,1]\to[-1,1]$ be Riemann integrable. Using the integral criterion, prove that

$$g(x)=\begin{cases} f(x) & \text{if } 0<x<1 \\ 0 & \text{if } x=0 \text{ or } 1 \end{cases}$$

is also Riemann integrable on $[0,1]$.



__Solution__ Since f is Riemann integrable on $[0,1]$, $\forall \varepsilon>0$, $\exists$ partition $P_1=\{0=x_0<x_1<\cdots<x_n=1\}$ such that $\boxed{U(f,P_1)-L(f,P_1)<\varepsilon/3}$ by the integral criterion.

Choose $r\in(0,x_1)$ and $r<\varepsilon/6$. Also choose $s\in(x_{n-1},1)$ and $1-s<\varepsilon/6$. Let $P_2=P_1\cup\{r,s\}$.

By refinement theorem, $L(f,P_1)\le L(f,P_2)\le U(f,P_2)\le U(f,P_1)$

So $U(f,P_2)-L(f,P_2)\le U(f,P_1)-L(f,P_1)<\varepsilon/3$.

Since $g(x)\in[-1,1]$,

$$U(g,P_2)-L(g,P_2)\le r\left(\sup\{g(x):x\in[0,r]\}-\inf\{g(x):x\in[0,r]\}\right)+\left(U(f,P_1)-L(f,P_1)\right)+(1-s)\left(\sup\{g(x):x\in[s,1]\}-\inf\{g(x):x\in[s,1]\}\right)$$

$$\le \frac{\varepsilon}{6}(1-(-1))+\frac{\varepsilon}{3}+\frac{\varepsilon}{6}(1-(-1))=\varepsilon.$$

By integral criterion, g is Riemann integrable on $[0,1]$.

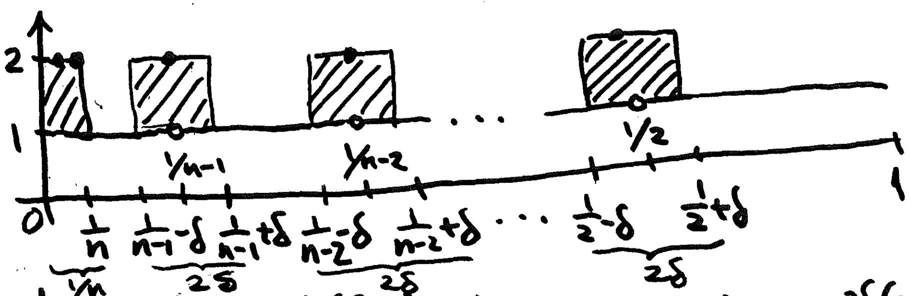
Prove that $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases} 2 & \text{if } x\in\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots\} \\ 1 & \text{if } x\in[0,1]\setminus\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots\} \end{cases}$$

is Riemann integrable by integral criterion.

__Solution__ $\forall\varepsilon>0$, by Archimedean principle, $\exists n\in\mathbb{N}$ such that $n>2/\varepsilon$. So $\frac{1}{n}<\varepsilon/2$. Let $\delta\le\min\{\frac{1}{2}(\frac{1}{n-1}-\frac{1}{n}),\frac{\varepsilon}{4(n-2)}\}$. So $2\delta\le\frac{1}{n-1}-\frac{1}{n},\cdots,\frac{1}{2}-\frac{1}{3},1-\frac{1}{2}$ and $2\delta(n-2)<\frac{\varepsilon}{2}$.

Let $P=\{0,\frac{1}{n},\frac{1}{n-1}-\delta,\frac{1}{n-1}+\delta,\cdots,\frac{1}{2}-\delta,\frac{1}{2}+\delta,1\}$.



Then $U(f,P)-L(f,P)=\frac{1}{n}(2-1)+\underbrace{2\delta(2-1)+\cdots+2\delta(2-1)}_{n-2\text{ terms}}$

$$=\frac{1}{n}+2\delta(n-2)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

$\therefore$ f is Riemann integrable by integral criterion.