MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #12

Limit (Part 2)

Recurrence Relation:

Type 1: Monotone Sequence (Increasing/ Decreasing sequence)

Theorem 1: Monotone Sequence Theorem

f $\{x_n\}$ is increasing and bounded from above, then $\{x_n\}$ converges. Similarly, if $\{x_n\}$ is decreasing and bounded from below, then $\{x_n\}$ converges.

Example 1

Given a sequence y_n such that

$$y_1 = \frac{1}{2+1} \text{ and } y_{n+1} = y_n + \frac{1}{2^{n+1}+1}$$

Show $\{y_n\}$ is convergent.

By computing first 3-4 terms, we see that

$y_2 = \frac{1}{3} + \frac{1}{5}, y_3 = \frac{1}{3} + \frac{1}{5} + \frac{1}{9}, y_4 = \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \frac{1}{17}$, we suspect y_n is increasing. And we also see y_n is bounded from by 1 (or any number greater than 1)

(Step 1: Show y_n is increasing, i.e. $y_{n+1} \geq y_n$)

By induction, for $n = 1$, $y_2 = \frac{1}{3} + \frac{1}{5} > \frac{1}{3} = y_1$. Assume $y_{k+1} > y_k$, for $n = k+1$,

$y_{k+2} = y_{k+1} + \frac{1}{2^{n+1}+1} > y_{k+1}$ which completes the proof.

(Step 2: Show y_n is bounded from above by 1)

Note that $y_n = \frac{1}{2^n+1} + y_{n-1} = \frac{1}{2^n+1} + \frac{1}{2^{n-1}+1} + y_{n-2} = \frac{1}{2^n+1} + \frac{1}{2^{n-1}+1} + \frac{1}{2^{n-2}+1} + y_{n-3}$

$= \ldots\ldots = \frac{1}{2^n+1} + \frac{1}{2^{n-1}+1} + \frac{1}{2^{n-2}+1} + \cdots.. + \frac{1}{2+1}$

Note that $y_n = \frac{1}{2+1} + \frac{1}{2^2+1} \ldots.. + \frac{1}{2^{n-1}+1} + \frac{1}{2^n+1} < \frac{1}{2} + \frac{1}{2^2} + \cdots. + \frac{1}{2^n} < \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \ldots.$

$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$. So y_n is bounded from above by 1.

(Step 3)

Applying Monotone Sequence Theorem, we see y_n converges.

☺Exercise 1:

Let a sequence $\{x_n\}$ which $x_1 = 0$ and $x_{n+1} = \sin\left(\frac{2+x_n}{2}\right)$ for $n = 1,2,3, \ldots.$

Show $\{x_n\}$ converges. (One can use Newton's Method to approximate the limit)

Type 2: Intertwining Sequence

Theorem 2: (Nested Interval Theorem)
If $I_n = [a_n, b_n]$ such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, then $\bigcap_{n=1}^{\infty} I_n = [a, b]$ where $a = \lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n = b$,
Furthermore, if $\lim_{n\to\infty}(b_n - a_n) = 0$ (i.e. $a = b$), then $\bigcap_{n=1}^{\infty} I_n$ contains exactly one number

Theorem 3: (Intertwining Sequence Theorem)
If $\{x_{2m}\}$ and $\{x_{2m-1}\}$ converge to x, then $\{x_n\}$ converges to x.

Remark: In normal situation, we use first theorem to show $\lim_{n\to\infty} x_{2n-1}$ and $\lim_{n\to\infty} x_{2n}$ exists. Then we show $\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} x_{2n}$. Finally, use the second theorem to conclude the sequence converges.

Example 2
A sequence $\{x_n\}$ is defined as follows:

$$x_1 = 1, x_{n+1} = \frac{2}{x_n} + 1$$

Show x_n converges and compute the limit.

Solution:

By computing 7 more terms, we get $x_2 = 3, x_3 = \frac{5}{3}, x_4 = 2.2,\ x_5 = 1.909,\ x_6 = 2.048,$ $x_7 = 1.977,\ x_8 = 2.012$, we see the sequence is intertwining type. By plotting the points on the real line,

------------X------------------------------X-------X-------X------X--------X------X-----------------X--------

x_1 x_3 x_5 x_7 x_8 x_6 x_4 x_2

Let $I_1 = [x_1, x_2], I_2 = [x_3, x_4], I_3 = [x_5, x_6\}, \ldots\ldots, I_k = [x_{2k-1}, x_{2k}]$

We need to show $I_k \supseteq I_{k+1}$, i.e. $x_{2k-1} \le x_{2k+1} \le x_{2k+2} \le x_{2k}$
(We can try to prove by induction)

For $n = 1$, it is true that $x_1 \le x_3 \le x_4 \le x_2$
Assume $x_{2k-1} \le x_{2k+1} \le x_{2k+2} \le x_{2k}$

$\mathrm{x}_{2k-1} \leq \mathrm{x}_{2k+1} \leq \mathrm{x}_{2k+2} \leq \mathrm{x}_{2k} \rightarrow \frac{1}{\mathrm{x}_{2k-1}} \geq \frac{1}{x_{2k+1}} \geq \frac{1}{x_{2k+2}} \geq \frac{1}{x_{2k}}$

$\rightarrow 1+\frac{2}{\mathrm{x}_{2k-1}} \geq 1+\frac{2}{x_{2k+1}} \geq 1+\frac{2}{x_{2k+2}} \geq 1+\frac{2}{x_{2k}}$ (Multiply both side by 2 and then add 1)

$\rightarrow \mathrm{x}_{2k} \geq \mathrm{x}_{2k+2} \geq \mathrm{x}_{2k+3} \geq \mathrm{x}_{2k+1}$
$\rightarrow \frac{1}{\mathrm{x}_{2k+1}} \geq \frac{1}{x_{2k+3}} \geq \frac{1}{x_{2k+2}} \geq \frac{1}{x_{2k}} \rightarrow 1+\frac{2}{\mathrm{x}_{2k+1}} \geq 1+\frac{2}{x_{2k+3}} \geq 1+\frac{2}{x_{2k+2}} \geq 1+\frac{2}{x_{2k}}$
$\rightarrow \mathrm{x}_{2k+2} \geq \mathrm{x}_{2k+4} \geq \mathrm{x}_{2k+3} \geq \mathrm{x}_{2k+1}$
which completes our induction.

Therefore by nested interval theorem, we see $\lim_{k\to\infty} \mathrm{x}_{2k}$ and $\lim_{k\to\infty} \mathrm{x}_{2k-1}$ exists, let $\lim_{k\to\infty} \mathrm{x}_{2k} = \mathrm{a}$ and $\lim_{k\to\infty} \mathrm{x}_{2k-1} = \mathrm{b}$, from the recurrence relation $\mathrm{x}_{2k+1} = \frac{2}{\mathrm{x}_{2k}} + 1$ (By putting $\mathrm{n} = 2\mathrm{k}$), take $\mathrm{k} \to \infty$, we get $\mathrm{a} = \frac{2}{\mathrm{b}} + 1$, also $\mathrm{x}_{2k+2} = \frac{2}{\mathrm{x}_{2k+1}} + 1$ (By putting $\mathrm{n} = 2\mathrm{k} + 1$). Take $\mathrm{k} \to \infty$, we get $\mathrm{b} = \frac{2}{\mathrm{a}} + 1$. Solving 2 equations, we get $\mathrm{a} = \mathrm{b} = 2$

Hence by intertwining Sequence Theorem, $\{\mathrm{x}_n\}$ converges and $\lim_{n\to\infty} \mathrm{x}_n = 2$.

(One more example is given in Example 7 of Tutorial Note #11)

☺Exercise 2

In Example 2, solve $\mathrm{a} = \frac{2}{\mathrm{b}} + 1$ and $\mathrm{b} = \frac{2}{\mathrm{a}} + 1$ to get $\mathrm{a} = \mathrm{b} = 2$

☺Exercise 3

Given a sequence which defined as

$$\mathrm{x}_1 = 2 \text{ and } \mathrm{x}_{n+1} = \frac{1}{4}\left(3 + \frac{1}{\mathrm{x}_n^2}\right)$$

Show $\{\mathrm{x}_n\}$ converges and find $\lim_{n\to\infty} x_n$

More difficult sequence

Example 3

Given a sequence defined by

$$\mathrm{x}_1 = 1.5, \mathrm{x}_2 = 2, \mathrm{x}_{n+2} = \sqrt[3]{4\mathrm{x}_n - 3}$$

Determine whether $\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \ldots$ converges or not. In case the sequence converges, find the limit also.

By testing a few terms, we see $\mathrm{x}_3 = 1.442$, $\mathrm{x}_4 = 1.710$, $\mathrm{x}_5 = 1.404$, $\mathrm{x}_6 = 1.566$, $\mathrm{x}_7 = 1.378$, $\mathrm{x}_8 = 1.483$, $\mathrm{x}_9 = 1.359$….. We see the sequence does not belong one of the above types.

However when we look at the sequence "separately", we look at the sequence $\{x_1, x_3, x_5, \ldots\}$ and $\{x_2, x_4, x_6, \ldots\}$. We see each of them is decreasing sequence (monotone sequence).

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--------------------X-------X--------X-------X----X----X------X-----------X---------------------X-------
                    x9      x7       x5      x3   x8   x1     x6          x4                    x2
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So our proof is as follows:

Step 1: Show $\{x_1, x_3, x_5, \ldots\}$ converges and find the limit

--Show the sequence is decreasing and bounded from below by ??

Step 2: Show $\{x_2, x_4, x_6, \ldots.\}$ converges and find the limit

--Show the sequence is decreasing and bounded from below by ??

Step 3: If both limit (in step 1 and step 2) are the same, we can conclude the sequence converges by intertwining sequence theorem.

Solution:

(Step 1)

First, we show x_{2n-1} is bounded from below by 1

For $n = 1,\ x_1 = 1.5 > 1$

Assume $x_{2k-1} > 1$, then $x_{2k+1} = \sqrt[3]{4x_{2k-1} + 3} > \sqrt[3]{4(1) + 3} = \sqrt[3]{7} > 1$

By induction, $x_{2n-1} > 3$ (bounded from below)

Second, we show x_{2n-1} is decreasing (i.e. $x_{2n+1} < x_{2n-1}$). We use induction to show this.

For $n = 1,\ x_3 = 1.442 < 1.5 = x_1$, assume $x_{2k+1} < x_{2k-1}$,

Consider $x_{2k+3} - x_{2k+1} = \sqrt[3]{4x_{2k+1} - 3} - \sqrt[3]{4x_{2k-1} - 3}$

$$= \frac{\left(\left(\sqrt[3]{4x_{2k+1} - 3}\right)^3 - \left(\sqrt[3]{4x_{2k-1} - 3}\right)^3\right)}{\left(\sqrt[3]{4x_{2k+1} - 3}\right)^2 + \left(\sqrt[3]{4x_{2k+1} - 3}\right)\left(\sqrt[3]{4x_{2k-1} - 3}\right) + \left(\sqrt[3]{4x_{2k-1} - 3}\right)^2} \ldots\ldots\ldots(*)$$

$$= \frac{(4x_{2k+1} - 3) - (4x_{2k-1} - 3)}{\Delta} = \frac{4(x_{2k+1} - x_{2k-1})}{\Delta} < 0 \quad (\text{Note } \Delta > 0)$$

(*Note: We have used the formula: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$)

Therefore, x_{2n-1} is decreasing sequence.

So by monotone theorem, $\{x_{2n-1}\}$ converges, similar argument also shows $\{x_{2n}\}$ Converges (left as exercise).

Now we write $\lim_{n\to\infty} x_{2n-1} = a$ and $\lim_{n\to\infty} x_{2n} = b$.

From $x_{2k+1} = \sqrt[3]{4x_{2k-1} - 3}$, let $k \to \infty$, we get $a^3 - 4a + 3 = 0 \to (a - 1)(a^2 + a -$

$3) = 0 \rightarrow a = 1$ and $a^2 - a - 3 = 0 \rightarrow a = \frac{-1\pm\sqrt{13}}{2}$. So we get $a = \frac{-1+\sqrt{13}}{2}$, similarly

We get $b = \frac{-1+\sqrt{13}}{2}$.

Therefore, since $a = b$, so by intertwining sequence theorem, the sequence $\{x_n\}$ converges and $\lim_{n\to\infty} x_n = \frac{1+\sqrt{13}}{2}$.

☺Exercise 4

In Example 3, show the sequence $\{x_{2n}\}$ converges.

☺Exercise 5 (2006 Fall Final)

Given a sequence $\{x_n\}$ defined as

$$x_1 = 2, x_2 = 4, x_{n+2} = \sqrt{10x_n - 9}$$

Show $\{x_n\}$ converges and compute the limit

Appendix:

In fact, the intertwining sequence theorem can be extended to three or more subsequences.

Theorem 4: (Extension of Intertwining sequence theorem, 3 subsequences) Suppose $\{x_{3n}\}$, $\{x_{3n+1}\}$, $\{x_{3n+2}\}$ converges to x, then $\{x_n\}$ converges to x

Proof:

for any $\varepsilon > 0$,

We would like to show there exists N such that $n > N$, $\lvert x_n - x\rvert < \varepsilon$

Since $\lim_{n\to\infty} x_{3n} = x$, there exist M_1, such that $n > M_1$, $\lvert x_{3n} - x\rvert < \varepsilon$

Similarly there exists M_2, such that $n > M_2$, $\lvert x_{3n+1} - x\rvert < \varepsilon$

There exists M_3, such that $n > M_3$, $\lvert x_{3n+2} - x\rvert < \varepsilon$

Pick $N = \max\{M_1, M_2, M_3\}$, then for any $n > N$,

$\lvert x_n - x\rvert = \lvert x_{3n} - x\rvert$ or $\lvert x_{3n+1} - x\rvert$ or $\lvert x_{3n+2} - x\rvert$, in any case, we must have $\lvert x_n - x\rvert < \varepsilon$.

Therefore by definition of limit, $\{x_n\}$ converges to x

So one can make use the theorem and use the similar method as in Example 3 to do the following:

<u>Try to do it if you have time</u>

☺Exercise 6a

Given a sequence, defined as

$$x_1 = 1, x_2 = 2, x_3 = 3 \text{ and } x_{n+3}^2 = 7x_n - 3$$

Show x_n converges and compute the limit.

Of course, you may try to extend the theorem further to k subsequences. $k = 4,5,6, \ldots$

☺Exercise 6b (Try to write the proof by yourselves)

Suppose $\{x_{kn}\}$, $\{x_{kn+1}\}$,……$\{x_{kn+(k-1)}\}$ converges to x, then $\{x_n\}$ converges to x.