DEPARTMENT OF MATHEMATICS

Math2033 TA note 10

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1 Mean value theorem

Example 1. *Calculate* $\lim_{x\to\infty}\frac{x^n}{e^x}$*, where* $n\in\mathbb{N}$.

Solution: We show that $\lim_{x\to\infty}\frac{x^n}{e^x}=0$ by induction.

For $n=1$, $\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{1}{e^x}=0$.

Suppose $\lim_{x\to\infty}\frac{x^k}{e^x}=0$, then

$$\lim_{x\to\infty}\frac{x^{k+1}}{e^x}=\lim_{x\to\infty}\frac{(x^{k+1})'}{e^x}=\lim_{x\to\infty}\frac{(k+1)x^k}{e^x}=0.$$

Example 2. *Let* $f:[1,2]\to\mathbb{R}$ *be continuous. If* f *is differentiable on* $(1,2)$*, prove that there exists* $\theta\in(1,2)$ *such that* $f(2)-f(1)=\frac{1}{2}\theta^2 f'(\theta)$.

Remark 3. *Mean value theorem only gives* $f(2)-f(1)=f'(\theta_0)(2-1)=f'(\theta_0)$ *for some* $\theta_0\in(1,2)$.

Solution: Let $g(\theta)=-\frac{1}{\theta}$, then for all $\theta\in(1,2)$,

$$\theta^2 f'(\theta)=\frac{f'(\theta)}{1/\theta^2}=\frac{f'(\theta)}{g'(\theta)}.$$

By generalized mean value theorem, there exists $\theta\in(1,2)$ such that

$$\frac{f(2)-f(1)}{g(2)-g(1)}=\frac{f'(\theta)}{g'(\theta)}=\theta^2 f'(\theta).$$

Since $f(2)-f(1)=\frac{1}{2}$, we have $f(2)-f(1)=\frac{1}{2}\theta^2 f'(\theta)$.

Example 4. *Let f be twice differentiable on* $[0,2]$*. For any* $x \in [0,2], |f(x)| \le 1, |f''(x)| \le 1$*, prove that* $\forall x \in [0,2], |f'(x)| \le 2$.

Solution: By Taylor's theorem, let $x \in [0,2], a \in [0,2]$

$$f(a) = f(x) + f'(x)(a-x) + \frac{f''(\theta_a)}{2}(a-x)^2$$

for some θ_a between a and x. Setting $a = 0, 2$.

$$f(0) = f(x) - f'(x)x + \frac{f''(\theta_0)}{2}x^2$$

$$f(2) = f(x) + f'(x)(2-x) + \frac{f''(\theta_2)}{2}(2-x)^2$$

Subtracting these, we get

$$f(2) - f(0) = 2f'(x) + \frac{f''(\theta_2)}{2}(2-x)^2 - \frac{f''(\theta_0)}{2}x^2$$

Solving for $f'(x)$, we see

$$\begin{aligned}
|f'(x)| &= \frac{1}{2}|f(2) - f(0) + \frac{f''(\theta_0)}{2}x^2 - \frac{f''(\theta_2)}{2}(2-x)^2| \\
&\le \frac{1}{2}(1 + 1 + \frac{1}{2}x^2 + \frac{1}{2}(2-x)^2) \\
&= \frac{1}{2}(x^2 - 2x + 4) \qquad (1.1) \\
&\le \frac{1}{2}((x-1)^2 + 3) \\
&\le \frac{1}{2}(1+3) = 2
\end{aligned}$$

Example 5. *For* $a > b > 0$*, prove* $\frac{a-b}{a} < \ln\frac{a}{b} < \frac{a-b}{b}$.

Solution: $f(x) = \ln x, f'(x) = \frac{1}{x}$. Using mean value theorem in $[b,a]$, there exists $\xi \in (b,a)$ such that

$$\ln a - \ln b = \frac{1}{\xi}(a-b).$$

That is

$$\ln\frac{a}{b} = \frac{1}{\xi}(a-b).$$

Because $b < \xi < a$,

$$\frac{a-b}{a} < \ln\frac{a}{b} < \frac{a-b}{b}$$

Interesting exercises:
1:for $x > 0$, prove $\frac{x}{1+x} < \ln(1+x) < x$
2:for $a > b > 0, n > 1$, prove $nb^{n-1}(a-b) < a^n - b^n < na^{n-1}(a-b)$