Lecture 16   02-04-2019

Review on continuity

1. Definition : f is continuous at x_0 iff $\lim_{x\to x_0} f(x) = f(x_0)$

2. Sequential limit theorem :

$$\lim_{x\to x_0} f(x) = f(x_0) \iff \forall\, x_n \to x_0,\ \lim_{n\to\infty} f(x_n) = f(x_0)$$

3. Rules of generating continuous function : f, g continuous $\Rightarrow$

   $f \pm g$, $f \cdot g$, f/g, $f \circ g$ are continuous

4. Sign preserving property : f continuous at x_0, $f(x_0) \neq 0$

   $\Rightarrow$ $f(x) \cdot f(x_0) > 0$ for x in a sufficiently small neighborhood of x_0.

5. Intermediate value thm : $f : [a,b] \to \mathbb{R}$ continuous.

$$\Rightarrow \quad [f(a), f(b)] \subseteq f([a,b]) \text{ or } [f(b), f(a)] \subseteq f([a,b]).$$

The proof is based on a search algorithm and nested interval thm.

Many other proofs.

6. Extreme value thm: $f: [a,b] \to \mathbb{R}$ continuous

$\Rightarrow$ f achieves its Maximum and Minimum at points in $[a,b]$ repectively.

Proof is based on applying Bolzano-Weierstrass thm to the minimizing / maximizing sequence.

7. Continuous injection and inverse thm:

Let $f: S \to \mathbb{R}$ be continuous, where S is an interval, then

f is injective $\Leftrightarrow$ f is strictly monotone $\Leftrightarrow$

$f^{-1}: f(S) \to S$ exists and is continuous

Chapter 7 Differentiation

Def: let S be an interval ($S=(a,b)$ or $[a,b]$, or $(a,b]$, or $[a,b)$ with $a<b$).

A function $f: S \to \mathbb{R}$ is differentiable at $x_0 \in S$ [x_0 may be the two end points] iff

$$\lim_{\substack{x \to x_0 \\ x \in S}} \frac{f(x)-f(x_0)}{x-x_0} \text{ exists.}$$

This limit is denoted by $f'(x_0)$.

f is called differentiable iff f is differentiable at every element in S.

Example: $f(x)=x^2$ is differentiable

$f(x)=\sqrt{x^3} : [0,\infty) \to \mathbb{R}$ is differentiable.

especially differentiable at $x=0$. $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x} = 0$

Equivalent definition of Differentiability

f is differentiable at x_0 iff $\lim\limits_{x \to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}$ $\exists$,

iff $\lim\limits_{h \to 0} \dfrac{f(x_0+h)-f(x_0)}{h}$ $\exists$, iff

$\exists$ a real number, denoted by $f'(x_0)$, s.t

$f(x_0+h) = f(x_0) + f'(x_0)h + R(x_0,h)$ with

$$\lim_{h \to 0} \frac{R(x_0,h)}{h} = 0$$

One can show that $f'(x_0)$ is unique if exists

$f'(x_0)$ is called the derivative at f at x_0.

Proof: Exercise

Differentiability $\Rightarrow$ Continuity

THM : If $f : S \to \mathbb{R}$ is differentiable at $x_0 \in S$,

then it is continuous at x_0

Proof : $f(x) = f(x) - f(x_0) + f(x_0)$

$$= \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) + f(x_0)$$

letting $x \to x_0$

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) + f(x_0)$$

$$= f'(x_0) \cdot 0 + f(x_0) = f(x_0)$$

$\Rightarrow$ f is continuous at x_0

Differentiability and continuity

Lemma: Let $f : S \to \mathbb{R}$ be continuous.

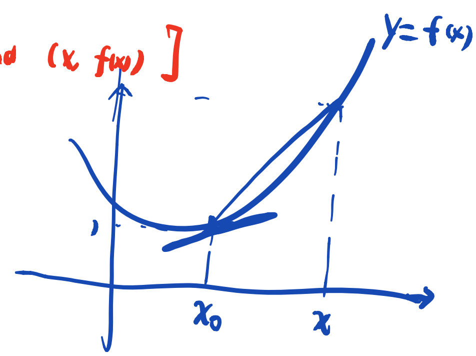
Assume that f is differentiable at $x_0 \in S$.

Then the function

$$g(x) = \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0 \end{cases}$$

is continuous at x_0

[$g(x)$ is called the average speed or average rate of change of f between x_0 and x, or the slope of the line connecting $(x_0, f(x_0))$ and $(x, f(x))$]



proof: Exercise

Application: $f(x) = f(x_0) + g(x)(x - x_0)$.

Differentiation Formula

THM : If $f, g : S \to \mathbb{R}$ is differentiable at x_0,

then $f+g$, cf, $f \cdot g$, $\frac{f}{g}$ (when $g(x_0) \neq 0$)

are differentiable at x_0. Moreover,

$$(f+g)'(x_0) = f'(x_0) + g'(x_0) \quad \cdots \quad (1)$$

$$(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) \quad \cdots \quad (2)$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)} \quad \cdots \quad (3)$$

Pf : ①. $$\frac{(f+g)(x)-(f+g)(x_0)}{x-x_0} = \underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\downarrow x \to x_0 \; f'(x_0)} + \underbrace{\frac{g(x)-g(x_0)}{x-x_0}}_{\downarrow x \to x_0 \; g'(x_0)}$$

So $$\lim_{x \to x_0} \frac{(f+g)(x)-(f+g)(x_0)}{x-x_0} = f'(x_0) + g'(x_0)$$

$\Rightarrow$ $f+g$ is differentiable at x_0 and $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

② $$\frac{f\cdot g(x) - f\cdot g(x_0)}{x-x_0} = \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x-x_0}$$

$$= \frac{g(x)[f(x)-f(x_0)]}{x-x_0} + \frac{f(x_0)[g(x)-g(x_0)]}{x-x_0}$$

$$\downarrow x\to x_0 \qquad \downarrow x\to x_0$$

$$g(x_0)\cdot f'(x_0) + f(x_0)g'(x_0)$$

$\Rightarrow$ $f\cdot g$ is differentiable at $x=x_0$ and $(f\cdot g)'(x_0) = g(x_0)f'(x_0) + f(x_0)g'(x_0)$

③ $$\frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x-x_0} = \frac{g(x_0)-g(x)}{g(x)g(x_0)(x-x_0)} = \frac{-1}{g(x_0)g(x)}\frac{[g(x)-g(x_0)]}{x-x_0}$$

$$\downarrow x\to x_0$$

$$\frac{-1}{g(x_0)^2}\cdot g'(x_0)$$

$\Rightarrow$ $\frac{1}{g}$ is differentiable at $x=x_0$ and $(\frac{1}{g})'(x_0) = \frac{-g'(x_0)}{g(x_0)^2}$

Using ②, we get

$\frac{f}{g}$ is differentiable at $x=x_0$ and $(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$

Chain Rule

THM: If $f: S \to R$ is differentiable at x_0,

$f(S) \subset S'$ and $g: S' \to R$ is differentiable at

$f(x_0)$. Then $g \circ f$ is differentiable at x_0, and

$$(g \circ f)'(x_0) = g'(f(x_0))\, f'(x_0)$$

Pf: Let $y_0 = f(x_0)$, define $h: S' \to R$ by

$$h(y) = \begin{cases} \dfrac{g(y) - g(y_0)}{y - y_0}, & y \neq y_0 \\ g'(y_0), & y = y_0. \end{cases}$$

Then h is continuous at y_0 and

$$g(y) - g(y_0) = h(y) \cdot (y - y_0)$$

$$\Rightarrow \frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{h(f(x))\,(f(x) - f(x_0))}{x - x_0}$$

letting $x \to x_0$, $h(f(x)) \to h(f(x_0))$ since $h \circ f$ is

continuous at y_0 by the composite rule for continuous function,

$$\frac{f(x)-f(x_0)}{x-x_0} \to f'(x_0) \quad \text{since } f \text{ is differentiable at } x=x_0.$$

$$\Rightarrow \lim_{x\to x_0} \frac{g(f(x))-g(f(x_0))}{x-x_0} = h(f(x_0))\cdot f'(x_0) = h(y_0)\cdot f'(x_0)$$

$$= g'(y_0)\cdot f'(x_0) = g'(f(x_0))\cdot f'(x_0)$$

$\Rightarrow$ $g \circ f$ is differentiable at $x=x_0$ and

$$(g\circ f)'(x_0) = g'(f(x_0))\cdot f'(x_0).$$

Differentiability and Continuity

Remark : f is differentiable $\Rightarrow$ f' exists

However, f' may be discontinuous.

Example : $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Since $\left| x^2 \sin \frac{1}{x} \right| \leq x^2$. $\Rightarrow \lim_{x \to 0} f(x) = 0 = f(0)$

by sandwich theorem. So f is continuous at 0

and hence everywhere in $\mathbb{R}$

For $x \neq 0$, $f'(x) = \left(x^2 \sin \frac{1}{x} \right)' = 2x \sin \frac{1}{x} + x^2 \left(\sin \frac{1}{x} \right)'$

$$= 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

For $x=0$, $f'(0) = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} x \sin \frac{1}{x} = 0$

So f is differentiable in $\mathbb{R}$.

However, $\lim_{x \to 0} f'(x) = \lim_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x}\right)$

$= \lim_{x \to 0} \cos \frac{1}{x}$ Does not exist.

therefore f' is not continuous at 0.