Solutions to Presentation Exercises

(186) (a) Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Then f is Riemann integrable $\Leftrightarrow$ f is continuous on $[a,b]$ except on a set of measure 0.

(b) f,g monotone $\Rightarrow$ f bounded between $f(0)$ and $f(1)$
g bounded between $g(0)$ and $g(1)$

$\Rightarrow \exists\, M_1, M_2>0$ such that $\forall x\in[0,1]$, $|f(x)|\le M_1$, $|g(x)|\le M_2$

$\Rightarrow \exists\, M_1, M_2>0$ such that $|f(x)\pm g(x)|\le |f(x)|+|g(x)|\le M_1+M_2$

$\Rightarrow h$ is bounded.

by an example of measure 0 set.

f,g monotone $\Rightarrow f,g$ Riemann integrable $\Rightarrow p=f-g,\ q=f+g$ Riemann integrable

Now $S_h\subseteq \big(S_p\cap[0,1/2)\big)\cup\big(S_q\cap[1/2,1]\big)\cup\{\tfrac12\}$ $\Rightarrow S_h$ is of measure 0

measure 0 $\therefore h$ is Riemann integrable.

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(215) (b) Since f_n is Riemann integrable on $[0,1]$, S_{f_n} is of measure 0. Then $S_{f_n}\cap(\frac{1}{n+1},\frac1n]$ is also of measure 0. Now $S_g\subseteq\{0,\frac12,\frac13,\dots\}\cup\bigcup_{n=1}^{\infty}\big(S_{f_n}\cap(\frac{1}{n+1},\frac1n]\big)$. Since $\underbrace{\{0,\frac12,\frac13,\dots\}}_{\text{countable}}$, $S_{f_n}\cap(\frac{1}{n+1},\frac1n]$ are of measure 0, so $\{0,\frac12,\frac13,\dots\}\cup\bigcup_{n=1}^{\infty}\big(S_{f_n}\cap(\frac{1}{n+1},\frac1n]\big)$ is of measure 0. Then S_g is of measure 0, $\therefore g$ is Riemann integrable by Lebesgue's theorem

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(217) By Taylor's theorem, $f(x)=\underbrace{f(\tfrac12)}_{=0}+f'(\tfrac12)(x-\tfrac12)+\frac{f''(\theta_x)}{2}(x-\tfrac12)^2$ for some θ_x between x and $\frac12$. Then

$$\int_0^1 f(x)\,dx=f'(\tfrac12)\underbrace{\int_0^1(x-\tfrac12)\,dx}_{=\frac{x^2}{2}-\frac12 x\big|_0^1=\frac12-\frac12=0}+\int_0^1\frac{f''(\theta_x)}{2}(x-\tfrac12)^2\,dx$$

$$\therefore\ \left|\int_0^1 f(x)\,dx\right|=\left|\int_0^1\frac{f''(\theta_x)}{2}(x-\tfrac12)^2\,dx\right|\le\int_0^1\left|\frac{f''(\theta_x)}{2}(x-\tfrac12)^2\right|dx\le\frac12\int_0^1(x-\tfrac12)^2\,dx$$

$$=\frac12\,\frac{(x-\frac12)^3}{3}\Big|_0^1=\frac1{24}.$$

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(226) By Taylor's Theorem, $f(x)=\underbrace{f(0)}_{=2}+\underbrace{f'(0)}_{0}x+\frac{f''(\theta)}{2}x^2$ for some θ_x between x and 0. Letting $x=-1$ and 1, we get

$$0=f(-1)=2+\frac{f''(\theta_{-1})}{2}\quad\text{and}\quad 5=f(1)=2+\frac{f''(\theta_1)}{2}.$$

This implies $f''(\theta_{-1})=-4$ and $f''(\theta_1)=6$. Since $f''(x)$ is continuous and $-4<\sqrt2<6$, by the intermediate value theorem, there exists $c\in\mathbb{R}$ such that $f''(c)=\sqrt2$.

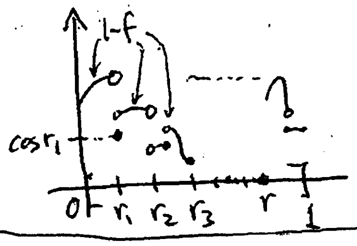
(249) (a) A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable if and only if the set $S_f=\{x\in[a,b]: f \text{ is discontinuous at } x\}$ is of measure 0.

(b) Since $\{r_n\}$ is strictly increasing and bounded above by 1, $\lim_{n\to\infty} r_n = r$ exists.

Next f is continuous at $x \Leftrightarrow 1-f$ is continuous at x. Hence $S_f = S_{1-f}$.

For g, we have $S_g \subseteq \underset{\underset{S_f \text{ measure } 0}{\parallel}}{S_{1-f}} \cup \underbrace{\{r_1, r_2, r_3, \dots\}}_{\text{Countable hence measure } 0} \cup \{r\}$.

$\therefore S_g$ is of measure 0 and g is Riemann integrable.



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(266) If $f(x)$ is continuous at $x=x_0\in[0,1)$, then $|f(x)-1|$ is continuous at $x=x_0$. Taking contrapositive, we get $S_{|f-1|}\subseteq S_f$. Since f is integrable on $[0,1]$, so S_f and $S_{|f-1|}$ are of measure 0.

If $f(x)$ is continuous at $x=x_1\in[0,1]$, then $h(x)=f(x-1)$ is continuous at $x=x_1+1\in[1,2]$. Taking contrapositive, we get $x_1+1\in[1,2]\cap S_h$ implies $x_1\in[0,1]\cap S_f\subseteq S_f$. So $[1,2]\cap S_h\subseteq\{1+x: x\in S_f\}$.

We will show $T=\{1+x: x\in S_f\}$ is of measure 0. Since S_f is of measure 0, $\forall \varepsilon>0, \exists (a_n,b_n)$ such that $S_f\subseteq\bigcup_{n=1}^{\infty}(a_n,b_n)$ and $\sum_{n=1}^{\infty}|a_n-b_n|<\varepsilon$. Then $T\subseteq\bigcup_{n=1}^{\infty}(1+a_n,1+b_n)$ and $\sum_{n=1}^{\infty}|(1+a_n)-(1+b_n)|=\sum_{n=1}^{\infty}|a_n-b_n|<\varepsilon$. $\therefore T$ is of measure 0.

Finally, $S_F\subseteq S_{|f-1|}\cup([1,2]\cap S_h)\cup\{1\}\subseteq S_{|f-1|}\cup T\cup\{1\}$
measure 0

$\therefore S_F$ is of measure 0 and F is Riemann integrable on $[0,2]$

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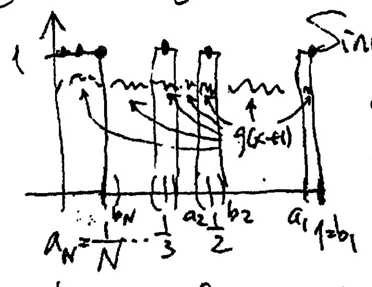
(267) For every $\varepsilon>0$, by Archimedian Principle, $\exists N\in\mathbb{N}$ such that $N>\frac{3}{\varepsilon}$.



Since g is Riemann integrable on $[1,2]$, $\exists$ partition $P_1=\{\overset{1}{\overset{\parallel}{x_0}}<x_1<\dots<\overset{2}{\overset{\parallel}{x_n}}\}$ of $[1,2]$ such that $U(g,P_1)-L(g,P_1)<\frac{\varepsilon}{3}$. Let $P_0=\{x-1: x\in P_1\}$

For $k=1,2,\dots,N$, choose $a_k<\frac{1}{k}\le b_k$ so $b_k-a_k<\frac{\varepsilon}{3N}$ and $0<a_N=\frac{1}{N}<b_N<a_{N-1}<\frac{1}{N-1}<b_{N-1}<\dots<a_2<\frac{1}{2}<b_2<a_1<1=b_1$

Let $P_2=\{0<a_N<b_N<a_{N-1}<b_{N-1}<\dots<a_1<1=b_1\}$. Let $P=P_0\cup P_2$. Let $h(x)=g(x+1)$.

Then $L(g,P_1)=L(h,P_0)\le L(h,P)\le U(h,P)\le U(h,P_0)=U(g,P_1)$. So $U(h,P)-L(h,P)<\frac{\varepsilon}{3}$

and $U(G,P)-L(G,P)\le U(h,P)-L(h,P)+\sum_{k=1}^{N}1(b_k-a_k)+\underbrace{1(a_N-0)}_{=\frac{1}{N}}$

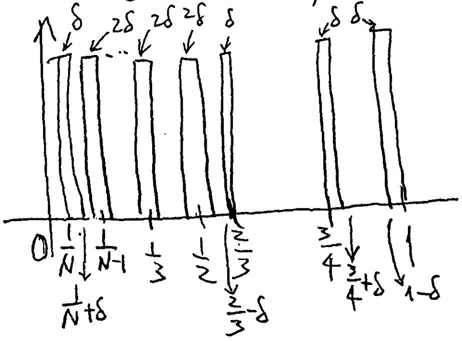
$$<\frac{\varepsilon}{3}+N\left(\frac{\varepsilon}{3N}\right)+\frac{\varepsilon}{3}=\varepsilon.$$

(281) (a) h is given to be bounded. $\forall x\in[0,\frac{2}{3})\cup(\frac{3}{4},1]$, if $x\notin S_f\cup\{1,\frac{1}{2},\frac{1}{3},\cdots\}\cup\{0\}$, then $h(x)=f(x)$ is continuous at x. $\forall x\in[\frac{2}{3},\frac{3}{4}]$, if $x\notin S_g\cup\{\frac{2}{3},\frac{3}{4}\}$, then $h(x)=g(x)$ is continuous at x. So $S_h\subseteq \underbrace{S_f}_{\text{measure }0}\cup \underbrace{S_g\cup\{\frac{2}{3},\frac{3}{4}\}\cup\{1,\frac{1}{2},\frac{1}{3},\cdots\}\cup\{0\}}_{\text{countable}\Rightarrow\text{measure }0}$.

This implies S_h is of measure 0.

Therefore h is Riemann integrable on $[0,1]$ by Lebesgue's theorem.

(b) Note $0\le h(x)\le 1$ for all $x\in[0,1]$. $\forall\varepsilon>0$, choose $N\in\mathbb{N}$ such that $\frac{1}{N}<\frac{\varepsilon}{4}$ ($\Leftrightarrow N>\frac{4}{\varepsilon}$). Next choose $\delta>0$ such that $2N\delta<\frac{\varepsilon}{4}$ and $\delta<\frac{1}{2}(\frac{1}{N-1}-\frac{1}{N})$, $\delta<\frac{1}{2}(\frac{2}{3}-\frac{1}{2})=\frac{1}{12}$, $\delta<\frac{1}{2}(1-\frac{3}{4})=\frac{1}{8}$ ($\Leftrightarrow \delta<\min\{\frac{1}{2}(\frac{1}{N-1}-\frac{1}{N}),\frac{\varepsilon}{8N},\frac{1}{12}\}$).



Since f is Riemann integrable, $\exists$ partition P_1 of $[0,1]$ such that $U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{4}$.

Next partition $[\frac{2}{3},\frac{3}{4}]$ into k subintervals of lengths $\frac{1}{k}(\frac{3}{4}-\frac{2}{3})=\frac{1}{12k}$ with $P_2=\{\frac{2}{3}+\frac{j}{12k}: j=0,1,\cdots,k\}$

Then $U(g,P_2)-L(g,P_2)=\sum_{j=0}^{k-1}\left(g(\frac{2}{3}+\frac{j+1}{12k})-g(\frac{2}{3}+\frac{j}{12k})\right)\frac{1}{12k}$

$<(1-0)\frac{1}{12k}=\frac{1}{12k}<\frac{\varepsilon}{4}$ for $k>\frac{1}{3\varepsilon}$

Let $P_3=\{0,\frac{1}{N},\frac{1}{N}+\delta,\frac{1}{N-1}-\delta,\frac{1}{N-1}+\delta,\cdots,\frac{1}{2}+\delta,\frac{2}{3}-\delta,\frac{2}{3},\frac{3}{4},\frac{3}{4}+\delta,1-\delta,1\}$ and $P=P_1\cup P_2\cup P_3$. We have

$$U(h,P)-L(h,P)<(1-0)\frac{1}{N}+2N\delta+\left(U(f,P_1)-L(f,P_1)\right)+\left(U(g,P_2)-L(g,P_2)\right)$$
$$<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon.$$

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(295) Since f is continuous on $[0,1]\setminus\mathbb{Q}$, $S_f=\{x\in[0,1]: f \text{ is discontinuous at } x\}\subseteq \underbrace{[0,1]\cap\mathbb{Q}}_{\text{countable}\Rightarrow\text{measure }0}$.

S_f is of measure 0. Therefore, f is Riemann integrable on $[0,1]$.

For $h(x)=f(\frac{x}{\sqrt{2}})$, $S_h\subseteq\{x\in[0,1]: w=\frac{x}{\sqrt{2}} \text{ for some } w\in S_f\}=[0,1]\cap\left(\bigcup_{w\in S_f}\{\sqrt{2}w\}\right)$ is countable since S_f is countable. Then $h(x)=f(x/\sqrt{2})$ is Riemann integrable on $[0,1]$. So $g(x)=f(x)f(x/\sqrt{2})$ is Riemann integrable on $[0,1]$.

Countable by union theorem

Alternatively we can also point out $S_g\subseteq S_f\cup S_h$. (Since $x\notin S_f\cup S_h \Rightarrow x\notin S_f$ and $x\notin S_h\Rightarrow f$ and h are continuous at $x\Rightarrow g$ is continuous at $x\Rightarrow x\notin S_g$).