Problem 1

Write down the opposite statement (negation) for each of the following statements

(a) I will watch a movie and have a dinner outside if tomorrow is sunny or not rainy.

(b) $\forall \varepsilon > 0, \exists \delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. $\rightarrow$ $|f(x)-f(y)| \geq \varepsilon$.

(c) $\forall x \in S, \forall \varepsilon > 0, \exists \delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

(d) $\forall \varepsilon > 0, \exists N > 0$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $m, n \geq N$ and $x \in \mathbb{R}$.

(e) $\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists N > 0$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $m, n \geq N$.

$<$ $>$ $\geq$

(a). I won't watch a movie or have a dinner outside if tomorrow is sunny or not rainy

(b) $\exists \varepsilon > 0$, such that $\forall \delta > 0$ there exist x, y satisfying $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

(c). $\exists x \in S$. $\exists \varepsilon > 0$ such that $\forall \delta > 0$ there exist x, y satisfying $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

(d) $\exists \varepsilon > 0$ $\forall N > 0$ there exist $m, n \geq N$ $x \in \mathbb{R}$ such that $|f_n(x) - f_m(x)| \geq \varepsilon$.

(e). $\exists x \in \mathbb{R}$. $\exists \varepsilon > 0$ $\forall N > 0$ there exist $m, n \geq N$ such that $|f_n(x) - f_m(x)| \geq \varepsilon$.

Negation of "If-then". statement.

$(P \Rightarrow Q.)$

$\sim(P \Rightarrow Q) = (P \text{ and } \sim Q)$

Statement P. $\forall x, C(x)$.

Negation $\sim P$. $\exists x, \sim C(x)$

$\exists x, C(x).$ $\forall x, \sim C(x).$

Problem 2

We let $f: A \to B$ be a function. For any subset $Y \subseteq B$, we define the *inverse image* of Y under f (denoted by $f^{-1}(Y)$) as the collection of elements in the domain A that maps to elements in $f(Y)$. That is, $f: A \to B.$

$$f^{-1}(Y) = \{x \in A | f(x) \in Y\}.$$

Prove the following statements

(a) $U \subseteq f^{-1}(f(U))$ for any subset $U \subseteq A$. Give an example which $U \subset f^{-1}(f(U))$

(b) $f(f^{-1}(V)) \subseteq V$ for any subset $V \subseteq B$. Give an example which $f(f^{-1}(V)) \subset V$

(c) $f(\bigcup_{\alpha\in I} X_\alpha) = \bigcup_{\alpha\in I} f(X_\alpha)$ and $f^{-1}(\bigcup_{\alpha\in I} Y_\alpha) = \bigcup_{\alpha\in I} f^{-1}(Y_\alpha)$.

Here, X_α is subset of A and Y_α is subset of B for all $\alpha \in I$

(d) $f(\bigcap_{\alpha\in I} X_\alpha) \subseteq \bigcap_{\alpha\in I} f(X_\alpha)$ and $f^{-1}(\bigcap_{\alpha\in I} Y_\alpha) = \bigcap_{\alpha\in I} f^{-1}(Y_\alpha)$.

(*Note: In (c) and (d), I is called index set and

$$\bigcup_{\alpha\in I} X_\alpha = \{x | x \in X_\alpha \ \textit{for some } \alpha \in I\} \quad \textit{and} \quad \bigcap_{\alpha\in I} X_\alpha = \{x | x \in X_\alpha \ \textit{for all } \alpha \in I\}$$

(*Note 2: Here, $A \subset B$ means that A is proper subset of B in the sense that $A \subseteq B$ but $A \neq B$)

$A \subseteq B$

$\forall x \in A \Rightarrow x \in B.$

$\Rightarrow A \subseteq B$

$A = B$

① $A \subseteq B$

② $B \subseteq A.$

(a). $\forall t \in U.$ Definition.

$f^{-1}(f(t)) = \{x \in A \mid f(x) = f(t)\}$

$t \in f^{-1}(f(t)) \subseteq f^{-1}(f(U))$

$\Rightarrow U \subseteq f^{-1}(f(U))$

Example of $U \subset f^{-1}(f(U))$

Define. $f: \mathbb{R} \to \mathbb{R}.$

$f(x) = 0, \quad \forall x \in \mathbb{R}.$

Constant function.

$U = \{1\}$ $\quad f(U) = \{0\},$

$f^{-1}(\{0\}) = \mathbb{R},$

$U \subsetneq f^{-1}(f(U)) = \mathbb{R}.$

(b). $\forall y \in f(f^{-1}(V))$

$\exists x \in f^{-1}(V)$ such that $y = f(x)$.

Note that $x \in f^{-1}(V) \Rightarrow f(x) \in V$

$\Rightarrow y \in V$

$\Rightarrow f(f^{-1}(V)) \subseteq V$.

Example of $f(f^{-1}(V)) \subsetneq V$.

Define $f: \mathbb{R} \longrightarrow \mathbb{R}$.

$f(x) = x^2$. $x \in \mathbb{R}$.

$f(x) = x^2$.

$V = (-4, 4)$

$f^{-1}(V) = (-2, 2)$

$f(f^{-1}(V)) = [0, 4) \subsetneq V$.

$f: (-2, 2) \longmapsto [0, 4)$

$\neq V = (-4, 4)$.

(c). Proof of $f(\bigcup_{\alpha \in I} X_\alpha) = \bigcup_{\alpha \in I} f(X_\alpha)$.

First Prove $f(\bigcup_{\alpha \in I} X_\alpha) \subseteq \bigcup_{\alpha \in I} f(X_\alpha)$.

$\forall y \in f(\bigcup_{\alpha \in I} X_\alpha)$.

$\Rightarrow \exists x \in \bigcup_{\alpha \in I} X_\alpha$, $f(x) = y$.

$\Rightarrow \exists \alpha_0 \in I$ s.t. $x \in X_{\alpha_0}$.

$\Rightarrow y = f(x) \in f(X_{\alpha_0}) \subseteq \bigcup_{\alpha \in I} f(X_\alpha)$

$\Rightarrow f(\bigcup_{\alpha \in I} X_\alpha) \subseteq \bigcup_{\alpha \in I} f(X_\alpha)$.

Then prove $\bigcup_{\alpha \in I} f(X_\alpha) \subseteq f(\bigcup_{\alpha \in I} X_\alpha)$.

$\forall y \in \bigcup_{\alpha \in I} f(X_\alpha)$.

$\Rightarrow \exists \alpha_0 \in I$ s.t. $y \in f(X_{\alpha_0})$. $X_{\alpha_0} \subseteq \bigcup_{\alpha \in I} X_\alpha$

$\Rightarrow$ $f(X_{\alpha_0}) \subseteq$ $f(\bigcup_{\alpha \in I} X_\alpha)$

$\Rightarrow y \in f(X_{\alpha_0}) \subseteq f(\bigcup_{\alpha \in I} X_\alpha)$.

$\Rightarrow \bigcup_{\alpha \in I} f(X_\alpha) \subseteq f(\bigcup_{\alpha \in I} X_\alpha)$.

$\Rightarrow \bigcup_{\alpha \in I} f(X_\alpha) = f(\bigcup_{\alpha \in I} X_\alpha)$.

Similar proof for the remaining part.

$f: \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2.$

Problem 3

We let $f: X \to Y$ be a function, prove that f is injective if and only if $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq X$.

(☺Hint: To prove " $\Leftarrow$ " (i.e. $f(A \cap B) = f(A) \cap f(B)$ implies f is injective) part, you can consider "proof by contradiction" and derive a contradiction by considering suitable choices of A and B).

Proof: "$\Rightarrow$" If f is injective

$A \cap B \subseteq A, \quad A \cap B \subseteq B$

"$\subseteq$". $\Rightarrow f(A \cap B) \subseteq f(A), \quad f(A \cap B) \subseteq f(B)$

$\Rightarrow f(A \cap B) \subseteq f(A) \cap f(B).$

"$\supseteq$" $\forall y \in f(A) \cap f(B).$

$\exists!\, x \in X$ s.t. $f(x) = y$

$\Rightarrow x \in A, \ x \in B \Rightarrow x \in A \cap B$

$\Rightarrow f(x) = y \in f(A \cap B)$

$\Rightarrow f(A) \cap f(B) \subseteq f(A \cap B)$

$\Rightarrow f(A) \cap f(B) = f(A \cap B).$

"$\Leftarrow$". Prove by contradiction.

What will happen!

Assume f is not injective.

$\exists x_1, x_2 \in X$ s.t. $f(x_1) = f(x_2) = y$

Construct Contraction.

$x_1 \neq x_2$.

$x_1 \neq x_2$

$\boxed{f(A) \cap f(B) = f(A \cap B) \text{ for all } A, B \subset X.}$ ✓

$A = \{x_1\}$ $B = \{x_2\}$

$A \cap B = \phi$.

$f(A) = f(B) = \{y\}$

$f(A \cap B) = \phi$.

$f(A) \cap f(B) = \{y\}$

$\Rightarrow f(A \cap B) \neq f(A) \cap f(B)$

It contradicts the condition that any A, B. $f(A \cap B) = f(A) \cap f(B)$.

Problem 4

We let $f_1(x), f_2(x), f_3(x), \ldots$ be functions (where $f_k: \mathbb{R} \to \mathbb{R}$ for all $k \in \mathbb{N}$). It is given that

$$A_k = \{x \in \mathbb{R} | f_k(x) = 0\}$$

is countable for any $k \in \mathbb{N}$.

(a) Show that for any $n \in \mathbb{N}$, the set

$$S_n = \left\{x \in \mathbb{R} \middle| \prod_{k=1}^{n} f_k(x) = 0\right\}$$

$f_1(x) = 0$ or $f_2(x) = 0 \ldots$ or $f_n(x) = 0$

$\exists k = 1, \ldots, n$, $f_k(x) = 0$

is countable.

(b) Determine if the set

$$S = \left\{x \in \mathbb{R} \middle| \prod_{k=1}^{\infty} f_k(x) = 0\right\}$$

? if $\prod_{k=1}^{\infty} f_k(x) = 0$, then $f_k(x) \neq 0$ for all k.

$\neq \bigcup_{k=1}^{+\infty} A_k$

is countable.

(☺Hint: If your answer is yes, please give a mathematical proof. If your answer is no, please give a counter-example (you need to specify the functions $f_k(x)$ in your answer).

? possible.

(*Note: Here,

$$\prod_{k=1}^{n} f_k(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot \ldots \cdot f_n(x)$$

$$\prod_{k=1}^{\infty} f_k(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot \ldots$$

(a). $S_n = \{x \in \mathbb{R} \mid \prod_{k=1}^{n} f_k(x) = 0\}$ Definition.

① $x \in S_n$, $\prod_{k=1}^{n} f_k(x) = 0$. $\exists m = 1, \dots, n$ s.t. $f_m(x) = 0$. $x \in \bigcup_{k=1}^{n} \{x \in \mathbb{R} \mid f_k(x) = 0\}$

② $x \in \bigcup_{k=1}^{n} \{x \in \mathbb{R} \mid f_k(x) = 0\}$. $\prod_{k=1}^{n} f_k(x) = 0$.

$$= \bigcup_{k=1}^{n} \{x \in \mathbb{R} \mid f_k(x) = 0\}.$$

$$= \bigcup_{k=1}^{n} A_k.$$

Because A_k is countable,

$S_n = \bigcup_{k=1}^{n} A_k$ is countable.

(b). Define $f_k : \mathbb{R} \to \mathbb{R}$,

$f_k(x) = \frac{1}{k}$. a constant function.

$S \neq \bigcup_{k=1}^{+\infty} A_k$

If $f_k(x) \neq 0$, $\forall k$. $\prod_{k=1}^{+\infty} f_k(x) \neq 0$?

$f_k(x) = \frac{1}{2^k}$

$\prod_{k=1}^{n} f_k(x) = \frac{1}{2^{1+2+\dots+n}} < \frac{1}{2^n}$

$n \to +\infty$,

$0 \leq \lim_{n\to+\infty} \prod_{k=1}^{n} f_k(x) \leq \lim_{n\to+\infty} \frac{1}{2^n} = 0$

$\Rightarrow \prod_{k=1}^{+\infty} f_k(x) = 0$.

$1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} \cdots$

$a_n = \prod_{k=1}^{n} \frac{1}{k}$. ↓

$0 < a_{n+1} < a_n$.

$\lim_{n\to\infty} a_n = 0$.

$\exp(\ln a_n) = -\sum_{k=1}^{n} \ln k$.

$\exp(\to -\infty.)$

$\Rightarrow a_n \to 0$.

$\Rightarrow S = \mathbb{R}$.

$S_k = \phi$. $k = 1, 2, \dots$

$\Rightarrow S$ is uncountable.

S_k is countable for all k.

Problem 5

Prove that the power set $\mathcal{P}(\mathbb{N})$, which is a collection of all subsets (including empty set) of $\mathbb{N}$, is uncountable. Here, $\mathbb{N}$ is the set of positive integers (natural numbers).

(*Note: Mathematically, we can express the power set $\mathcal{P}(\mathbb{N})$ as

$$\mathcal{P}(\mathbb{N}) = \{A |\ A \subseteq \mathbb{N}\}.)$$

Proof: $\mathbb{N}$: 1, 2, 3, 4, ...

Denote: $A = \{0, 1\}$

Consider: $T = A \times A \times A \times \dots$

$T = \{ (x_1, x_2, \dots, x_n, \dots) \mid x_n = 0 \text{ or } 1 \}$.

Define $f : P(\mathbb{N}) \mapsto T$

S can be $\{1, 3, 5\}$, $\{2, 8\}$.

$S \in P(\mathbb{N})$. $f(S) = (x_1, x_2, \dots, x_n, \dots)$,

where $x_n = \begin{cases} 0 & \text{if } n \notin S \\ 1 & \text{if } n \in S \end{cases}$

f is bijective.

PPT of Proof → Problem Set. general. → $T = \{0,1\} \times \{0,1\} \times \{0,1\} \times \dots$ is uncountable

$\Rightarrow P(\mathbb{N})$ is uncountable.

$\{ n\cos\frac{n\pi}{2} \mid n \in \mathbb{N} \}$.

By contradiction.

If the infimum exists, $a \in \mathbb{R}$.

$n\cos\frac{n\pi}{2} \geq a$. $\quad n = 1, 2, \ldots$

$N_0 = [|a|] + 1 \geq |a|$.

integer.

when $\frac{n\pi}{2} = 2k\pi + \pi$. $\quad k \in \mathbb{Z}$.

$\cos\frac{n\pi}{2} = -1$.

$n = \cancel{2}4k + 2$.

$\cos\frac{n\pi}{2} = -1$. $\quad n = 4k+2$. $\quad 2, 6, 10$.

$N = 4N_0 + 2$.

$\cos\frac{N\pi}{2} = -1$.

$N > |a|$

$N\cos\frac{N\pi}{2} < -|a| \leq a$.