Math 2033 Midterm Review

March 23, 2016

Countability

For the problems about Countability, the problems appears in the midterm may test whether you manage to use those theorem:

(i) Injection/Surjection/Bijection Theorem

(ii) Countable subset Theorem

(iii) Countable Union Theorem, etc.

Example 1:

Determine whether the following are countable or not

$$M = \{y^2 - 7x : y \in A \text{ and } x \in \mathbb{R}\backslash A\},$$

where A is countable.

$$N = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 3 \text{ and } dx^3 + y = 4 \text{ for some integer } d\}$$

Proof. For set B, there are two cases:

Case (i): $A = \emptyset$, then we cannot assign any value to y. Thus M is an empty set and hence M is countable.

Remark: Student always forget the case (i).

Case (ii): $A \neq \emptyset$, then $\mathbb{R}$ is uncountable and A is countable, implies $\mathbb{R}\backslash A$ is uncountable.
(Otherwise, if $\mathbb{R}\backslash A$ is countable, then $\mathbb{R} = A \cup (\mathbb{R}\backslash A)$ is also countable by Countable Union Theorem, which is a contradiction.)
Since variable x lies in $\mathbb{R}\backslash A$ which is uncountable, we suspect M is uncountable.

To show this, we fix y and let $M_y = \{y^2 - 7x : x \in \mathbb{R}\backslash A\}$.
Define $f : M_y \to \mathbb{R}\backslash A$ by $f(y^2 - 7x) = x$, which is a bijection since $g(x) = y^2 - 7x$ is the inverse map of f.
By Bijection Theorem, M_y is uncountable.
Therefore, $M_y \subseteq M$, by Countable Subset Theorem, we get M is uncountable.

For set N, note that

$$\begin{aligned} N =& \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 3 \text{ and } dx^2 + y = 4 \text{ for some integer } d\} \\ =& \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 3 \text{ and } dx^2 + y = 4, \exists\, d \in \mathbb{Z}\} \\ =& \bigcup_{d\in\mathbb{Z}} \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 3 \text{ and } dx^2 + y = 4\} \end{aligned}$$

For the set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 3 \text{ and } dx^2 + y = 4\}$, substitute $y = 4 - dx^2$ into $x^2 + y^2 = 3$, we get $d^2x^4 + (1 - 8d)x^2 + 13 = 0$, which has at most 4 real roots for x, hence the set $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 3 \text{ and } dx^2 + y = 4\}$ has at most 4 elements and is countable.

Since $\mathbb{Z}$ is countable, by Countable Union Theorem, N is also countable. □

Example 2: (2004 Midterm)

Let A be a non-empty countable subset of $\mathbb{R}$. Let $S = \{\theta \in \mathbb{R} : \sin\theta \in A\}$ and $T = \{\theta \in \mathbb{R} : \sin\theta \notin A\}$.
Determine (with proof) if each of the sets S and T is countable or uncountable.

Proof. For S, we have

$$S = \{\theta \in \mathbb{R} : \sin\theta \in A\} \overset{(*)}{=} \{\theta \in \mathbb{R} : \sin\theta = a, \exists a \in A\} = \bigcup_{a\in A}\{\theta \in \mathbb{R} : \sin\theta = a\}$$

Note that $\sin\theta = a$ iff $\theta = n\pi + (-1)^n \sin^{-1} a$ for some $n \in \mathbb{Z}$.
So,

$$\{\theta \in \mathbb{R} : \sin\theta = a\} = \{\theta = n\pi + (-1)^n \sin^{-1} a : n \in \mathbb{Z}\} = \bigcup_{n\in\mathbb{Z}}\{\theta = n\pi + (-1)^n \sin^{-1} a\}$$

which is countable by **Countable Union Theorem**.
By **Countable Union Theorem** again, we get S is countable.

For T, we have $T = \mathbb{R}\backslash S$ as the description of T is simply the opposite of that of S.
Since $\mathbb{R}$ is uncountable, S is countable, we must have $T = \mathbb{R}\backslash S$ is uncountable.
(Otherwise, if T is countable, then $\mathbb{R} = S \cup T$ is also countable by Countable Union Theorem, which is a contradiction.) □

Remark 1: Sometimes, if the description of a set is similar to that of set S above, namely, $\{x \in C : f(x) \in D\}$ for some sets C, D and function f, it may better to rewrite it into $\{x \in C : f(x) \in D\} = \{x \in C : f(x) = d, \exists d \in D\}$, just like $(*)$ above the second equality. So that we may able to use Countable Union Theorem, just like what we did to the set S as above.

Remark 2: Note that

$$S = \bigcup_{a\in A}\bigcup_{n\in\mathbb{Z}}\{\theta = n\pi + (-1)^n \sin^{-1} a\} = \bigcup_{(a,n)\in A\times\mathbb{Z}}\{\theta = n\pi + (-1)^n \sin^{-1} a\}$$

Instead of using Countable Union Theorem Twice, one may use the fact that $A\times\mathbb{Z}$ is countable and Countable Union Theorem once.

Example 3: (2003 Spring).

Let P be a countable set of points in $\mathbb{R}^2$.
Prove that there exists a circle C with the origin as center and positive radius such that every point of the circle C is not in P. (Note points inside the circle do not belong to the circle)

Proof. Note that the problem is not trivial since countable set in $\mathbb{R}^2$, as in that in $\mathbb{R}$, can be very "dense"
(Imagine how $\mathbb{Q}\times\mathbb{Q}$ is distributed in $\mathbb{R}^2$).
Let's denote for $r \geq 0$, $C_r = \{(x,y) : \sqrt{x^2+y^2} = r\}$.

In other words, C_r is a circle of radius r centered at $(0,0)$ and $C_0 = \{0\}$.

<u>Method 1</u>

Note that the set of circles with nonempty intersection with P must be countable. Indeed, let $P = \{\vec{p_1}, \vec{p_2}, \cdots\}$ and let

$$I = \{r : C_r \cap P \neq \emptyset\}$$

Suppose $x \in I$, then $C_x \cap P \neq \emptyset$, so there is an i such that $\vec{p_i} \in C_x$, thus $x = \|\vec{p_i}\|$ for some i,
therefore the implication means the following set inclusion

$$I \subseteq \{\|\vec{p_i}\| : i = 1, 2, 3, \cdots\}$$

As $\{\|\vec{p_i}\| : i = 1, 2, 3, \cdots\}$ is countable, so is I, by **Countable Subset Theorem**.

As $(0,\infty)\backslash I$ is uncountable, hence nonempty, thus there is an $r > 0$, $r \notin I$,
i.e., $C_r \cap P = \emptyset$.

<u>Method 2</u>

We have $\mathbb{R}^2 = \bigcup_{r\geq 0} C_r$, since every point $(x,y) \in \mathbb{R}^2$ must lie in some C_r, namely, $C_{\sqrt{x^2+y^2}}$.
Thus C_r 's give a decomposition of $\mathbb{R}^2$, and therefore give a decomposition of P because

$$P = P \cap \mathbb{R}^2 = P \cap \Big(\bigcup_{r\geq 0} C_r\Big) = \bigcup_{r\geq 0}(P \cap C_r)$$

WANT TO SHOW: $P \cap C_r = \emptyset$; for some $r > 0$.

Suppose not, then $P \cap C_r \neq \emptyset$, for every $r > 0$.
This means we can pick an element $a_r \in P \cap C_r$ for each $r > 0$.
Note that $\{a_r : r > 0\}$ is uncountable since these are extracted from each of uncountably many circles.

In other words, the function $f : (0, \infty) \to \{a_r : r > 0\}$ given by $f(r) = a_r$ is bijective.
The fact that f is surjective follows from the way we parametrize the set $\{a_r : r > 0\}$.
While f is injective because
$x \neq y \implies a_x \neq a_y$ in view of radius.

Now $P = \bigcup_{r\geq 0}(P \cap C_r) \supseteq \bigcup_{r>0}(P \cap C_r) \supseteq \bigcup_{r>0}\{a_r\} = \{a_r : r > 0\}$, contradicts to the fact that P is countable.
Therefore $P \cap C_r = \emptyset$, for some $r > 0$. □

Example 4:

> Given a function $f : \mathbb{R}^3 \to (0, +\infty)$.
> Prove that there exist infinitely many positive real numbers r such that
> the equation $f(x, y, z) = r$ has no solution $(x, y, z) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$.

For problem like example 4, we usually show $(0, +\infty)\backslash S$ or $\mathbb{R}\backslash S$ is uncountable (which implies infinite), by showing S is countable.

Proof. For every $(x, y, z) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$, let $S_{(x,y,z)} = \{f(x, y, z)\}$, which is a one element set, by the definition of function.

Then $S = \{f(x, y, z) : (x, y, z) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}\} = \bigcup_{(x,y,z)\in\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}} S_{(x,y,z)} \ \left(= \bigcup_{x\in\mathbb{Q}} \bigcup_{y\in\mathbb{Q}} \bigcup_{z\in\mathbb{Q}} S_{(x,y,z)}\right)$.

Since $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ is countable and $S_{(x,y,z)}$ is finite (implies $S_{(x,y,z)}$ is countable), by Countable Union Theorem, we get S is countable.

Then $(0, +\infty)\backslash S$ is uncountable.
(Otherwise, if $(0, +\infty)\backslash S$ is countable, we get $(0, +\infty) = S \cup \big((0, +\infty)\backslash S\big)$ is countable by Countable Union Theorem,
which is a contradiction.)

Note that $S = \{r \in (0, +\infty) : f(x, y, z) = r$ have a solution $(x, y, z) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}\}$.
This implies $\{r \in (0, +\infty) : f(x, y, z) = r$ have no solution $(x, y, z) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}\} = (0, +\infty)/S$, is uncountable, and thus infinite.

Hence, there exist infinitely many positive real numbers r such that the equation $f(x, y, z) = r$ has no solution $(x, y, z) \in \mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$. □

Supremum and Infimum

For the problems asking you to find the Supremum and Infimum of a certain set S, there are two common ways to do it.

(i) Using limit supremum/infimum Theorem by Constructing a suitable sequence

Example 1: (Practice Exercise 91(m))

> Determine the supremum and infimum of the set $S = \left\{\dfrac{k}{n!} : k, n \in \mathbb{N}, \dfrac{k}{n!} < \sqrt{2}\right\}$.

Proof. (Step 1) Find an upper bound and a lower bound of S.
Note that $k, n > 0$, thus one possible lower bound of S is 0.
Note that since $\dfrac{k}{n!} < \sqrt{2}$ by the definition of S, so a possible upper bound is $\sqrt{2}$.

(Step 2) Show $\inf S = 0$ and $\sup S = \sqrt{2}$ by constructing suitable sequences.
For infimum, fix $k = 1$, we get $w_n = \dfrac{1}{n!} \in S$ since $w_n = \dfrac{1}{n!} \leq \dfrac{1}{1!} = 1 < \sqrt{2}$.
Since $\lim_{n\to\infty} w_n = 0$, by Infimum Limit Theorem, $\inf S = 0$.

For supremum, let $w_n = \dfrac{[\sqrt{2}n!]}{n!}$, where $[x]$ is the greatest integer less than or equal to x.
Since for any $x \in \mathbb{R}$, $x - 1 < [x] \leq x$, with equality holds iff $x \in \mathbb{Z}$, put $x = \sqrt{2}n!$, which is clearly irrational, we get

$$\sqrt{2} - \frac{1}{n!} = \frac{\sqrt{2}n! - 1}{n!} < \frac{[\sqrt{2}n!]}{n!} < \frac{\sqrt{2}n!}{n!} = \sqrt{2}$$

In particular, $w_n < \sqrt{2}$ and in the form of $\dfrac{k}{n!}$, for some $k, n \in \mathbb{N}$. Thus $w_n \in S$.
By Sandwich Theorem, we get $\lim_{n\to\infty} w_n = \sqrt{2}$.
By Supremum Limit Theorem, we get $\sup S = \sqrt{2}$.

Remark: May see the Example 1 on p.10 under the section about limit of a sequence by checking definition of limit. □

Example 2: (Practice Exercises 99 and)

> (a) Suppose $A_n \subseteq (-\infty, 2)$ and $x_n = \sup A_n$ for $n = 1, 2, 3, \cdots, 10$, show that $\sup\left(\bigcup_{k=1}^{10} A_k\right) = \max\{x_1, x_2, x_3 \cdots, x_{10}\}$.
>
> (b) Let I be a non-empty set. For every $t \in I$, let A_t be a nonempty subset of $[0, 1]$. Let $x_t = \sup A_t$.
> Prove that if $A = \bigcup_{t\in I} A_t$, then $\sup A = \sup\{x_t : t \in I\}$.

Proof. (a) (Step 1) Find an upper bound of S.

Let $x_i = \max\{x_1, x_2, \cdots, x_{10}\}$.
For any $x \in \bigcup_{k=1}^{10} A_k$ implies $x \in A_k$ for some $k = 1, 2, \cdots, 10$, then $x \le \sup A_k = x_k \le x_i$.
So, x_i is an upper bound of $\bigcup_{k=1}^{10} A_k$.

(Step 2) Show $\sup\left(\bigcup_{k=1}^{10} A_k\right) = x_i$ by constructing a suitable sequence.

By Supremum Limit Theorem, there exists $\{a_n\}$ in A_i such that $\lim_{n\to\infty} a_n = x_i$. Pick $w_n = a_n \in A_i \subseteq \bigcup_{k=1}^{10} A_k$, and $\lim_{n\to\infty} w_n = x_i$. Hence by Supremum Limit Theorem again, we get $\sup\left(\bigcup_{k=1}^{10} A_k\right) = x_i = \max\{x_1, x_2, \cdots, x_{10}\}$.

Remark 1: The number 10 is nothing special, one may replace every 10 by m instead.
Remark 2: There is another way to prove Step 2, see week 6 solution.

(b) (Step 1) Show $\sup\{x_t : t \in I\} \le \sup A$.
This is because $A_t \subseteq A \implies x_t = \sup A_t \le \sup A \implies \sup\{x_t : t \in I\} \le \sup A$
(The last implication is due to the fact that $\sup\{x_t : t \in I\}$ is the least upper bound of $\{x_t : t \in I\}$.)

(Step 2) Show $\sup\{x_t : t \in I\} \ge \sup A$.
Consider the set A, by Supremum Limit Theorem, there exists a sequence $\{a_n\}$ in A, such that $\lim_{n\to\infty} a_n = \sup A$.

Since $a_n \in A = \bigcup_{t\in I} A_t$, this implies $a_n \in A_t$, for some $t \in I$. This implies $a_n \le \sup A_t = x_t \le \sup\{x_t : t \in I\}$.

By taking limit on both sides, we get

$$\sup A = \lim_{n\to\infty} a_n \le \sup\{x_t : t \in I\}$$

Hence we get $\sup A = \sup\{x_t : t \in I\}$. □

Example 3:

> For $n = 1, 2, 3, \cdots$, let $x_n, y_n, z_n \in \mathbb{R}$ such that $x_n \le y_n \le z_n$. Suppose $\sup\{x_1, x_2, x_3, \cdots\} = \sup\{z_1, z_2, z_3, \cdots\} = a \in \mathbb{R}$.
> Show that $\sup\{y_1, y_2, y_3, \cdots\} = a$.

Proof. (Step 1: We first show a is an upper bound of $\{y_1, y_2, \cdots\}$.)
Since $y_k \le z_k \le \sup\{z_1, z_2, z_3, \cdots\} = a$.
Hence a is an upper bound of $\{y_1, y_2, \cdots\}$.

(Step 2: Construct a suitable sequence)
$\sup\{x_1, x_2, x_3, \cdots\} = a$. By Supremum Limit Theorem, there exists $x_{n_1}, x_{n_2}, x_{n_3} \cdots$ such that $\lim_{k\to\infty} x_{n_k} = a$.
(Note that $\{x_{n_k}\}$ is a sequence in $\{x_1, x_2, x_3, \cdots\}$.)

Consider $w_k = y_{n_k}$, i.e., $\{w_k\}$ is a sequence in $\{y_1, y_2, y_3, \cdots\}$. We also have $x_{n_k} \le y_{n_k} \le z_{n_k} \le \sup\{z_1, z_2, z_3, \cdots\} = a$.
By taking limit $k \to \infty$, we get $a = \lim_{k\to\infty} x_{n_k} \le \lim_{k\to\infty} w_k \le a$ implies $\lim_{k\to\infty} w_k = a$.
Hence by Supremum Limit Theorem, $\sup\{y_1, y_2, y_3, \cdots\} = a$. □

Example 4:

> Let $A \subseteq \mathbb{R}$ which $\sup A = \sqrt{3}$, find the supremum of the set $B = \{x^5 + 6y : x, y \in A\}$

Proof. (Step 1) Find out an upper bound of B.

Note that $x, y \in A$ and $\sup A = \sqrt{3}$, we get $x, y \le \sqrt{3}$.
Thus, $x^5 + 6y \le (\sqrt{3})^5 + 6(\sqrt{3}) = 15\sqrt{3}$. So S is bounded above by $15\sqrt{3}$.

(Step 2) To show $\sup B = 15\sqrt{3}$.

Since $\sup A = \sqrt{3}$, by Supremum Limit Theorem, there exists a sequence $\{a_n\}$ in A such that $\lim_{n\to\infty} a_n = \sqrt{3}$.
Pick $x_n = y_n = a_n$, (so that $x_n, y_n \in A$)

Then $w_n = x_n^5 + 6y_n = a_n^5 + 6a_n \in B$ and $\lim_{n\to\infty} w_n = (\sqrt{3})^5 + 6(\sqrt{3}) = 15\sqrt{3}$.
By the Supremum Limit Theorem again, we conclude that $\sup B = 15\sqrt{3}$.

> Remark: Some students may set $x_n = y_n = \sqrt{3} - \dfrac{1}{n}$ or $x_n = y_n = \sqrt{3}$, which is NOT correct as B is a unknown set and we do not know What B Exactly Contain.

□

(ii) Proof by contradiction

When we try to use Proof by contradiction to show $\sup S$ is some certain number, we may make use of

(I) The density of $\mathbb{R}\backslash\mathbb{Q}$ or $\mathbb{Q}$

(II) The Property of Supremum/ Infimum

to yield some contradiction.

For example, if one want to show that $\sup S = a$ for some $a \in \mathbb{R}$, then one first show a is an upper bound of S and this implies $\sup S \le a$, then suppose $\sup S \ne a$ will give us $\sup S < a$. At this stage, we may make use of (I) or (II) to guarantee there is some number $b \in \mathbb{R}$ such that $\sup S < b < a$ or $\sup S < b \le a$, which may then lead to a contradiction.

Example 1: Practice Exercise 91(m)

> Determine the supremum and infimum of the set $S = \left\{\dfrac{k}{n!} : k, n \in \mathbb{N}, \dfrac{k}{n!} < \sqrt{2}\right\}$.

Proof. (Step 1) Find an upper bound and a lower bound of S.

Note that $k, n > 0$, thus one possible lower bound of S is 0.
Note that since $\dfrac{k}{n!} < \sqrt{2}$ by the definition of S, so a possible upper bound is $\sqrt{2}$.

(Step 2) Show $\inf S = 0$ and $\sup S = \sqrt{2}$ by using proof by contradiction.

If $\inf S > 0$, by the density of rational, there is $\dfrac{m}{n} \in \mathbb{Q}$ such that $0 < \dfrac{m}{n} < \inf S$. (Note that $\dfrac{m}{n} > 0$, we may assume $m, n \in \mathbb{N}$.)
Since $\dfrac{m}{n} = \dfrac{m(n-1)!}{n!} \in S$, which contradicts to the fact that $\inf S$ is a lower bound of S. Thus $\inf S = 0$.

If $\sup S < \sqrt{2}$, by the density of rational, there is $\dfrac{p}{q} \in \mathbb{Q}$ such that $\sup S < \dfrac{p}{q} < \sqrt{2}$.
(Note that $\dfrac{p}{q} > \sup S \ge \inf S > 0$, we may assume $p, q \in \mathbb{N}$.)
Since $\dfrac{p}{q} = \dfrac{p(q-1)!}{q!} \in S$, which contradicts to the fact that $\sup S$ is a upper bound of S. Thus $\sup S = \sqrt{2}$.

Remark: Note that $\inf S \ge 0$ as 0 is a lower bound of S. In this situation, $\inf S \ne 0$ is equivalent to $\inf S > 0$.
Similarly, $\sup S \le \sqrt{2}$ as $\sqrt{2}$ is a upper bound of S. In this case, $\sup S \ne \sqrt{2}$ is equivalent to $\sup S < \sqrt{2}$. □

Example 2: Practice Exercises 99 and

(a) Suppose $A_n \subseteq (-\infty, 2)$ and $x_n = \sup A_n$ for $n = 1, 2, 3, \cdots, 10$, show that $\sup\left(\bigcup_{k=1}^{10} A_k\right) = \max\{x_1, x_2, x_3 \cdots, x_{10}\}$.

(b) Let I be a non-empty set. For every $t \in I$, let A_t be a nonempty subset of $[0, 1]$. Let $x_t = \sup A_t$.
Prove that if $A = \bigcup_{t \in I} A_t$, then $\sup A = \sup\{x_t : t \in I\}$.

Proof. (a) (Step 1) Find an upper bound of S.

Let $x_i = \max\{x_1, x_2, \cdots, x_{10}\}$.
For any $x \in \bigcup_{k=1}^{10} A_k$ implies $x \in A_k$ for some $k = 1, 2, \cdots, 10$, then $x \leq \sup A_k = x_k \leq x_i$.
So, x_i is an upper bound of $\bigcup_{k=1}^{10} A_k$.

(Step 2) Show $\sup\left(\bigcup_{k=1}^{10} A_k\right) = x_i$ by making use of proof by contradiction.

Suppose $\sup\left(\bigcup_{k=1}^{10} A_k\right) \neq x_i$, that means $\sup\left(\bigcup_{k=1}^{10} A_k\right) < x_i$ by Step 1.

Since $x_i = \sup A_i$, then by supremum propetry, there exists $a \in A_i$ such that $\sup\left(\bigcup_{k=1}^{10} A_k\right) < a \leq x_i = \sup A_i$.

(Take $\epsilon = x_i - \sup\left(\bigcup_{k=1}^{10} A_k\right) > 0$.)

Since $a \in A_i$ implies $a \in \bigcup_{k=1}^{10} A_k$, this implies $a \leq \sup\left(\bigcup_{k=1}^{10} A_k\right) < a$, which is a contradiction.

Thus, $\sup\left(\bigcup_{k=1}^{10} A_k\right) = x_i = \max\{x_1, x_2, \cdots, x_{10}\}$.

(b) (Step 1: Show $\sup\{x_t : t \in I\}$ is an upper bound of A.)

For any $a \in A$, $a \in A_t$ for some $t \in I$. This implies $a \leq x_t \leq \sup\{x_t : t \in I\}$. Thus $\sup\{x_t : t \in I\}$ is a upper bound of A.

(Step 2: Prove by Contradiction to show $\sup\{x_t : t \in I\}$ is the supremum of A.)

Suppose $\sup A < \sup\{x_t : t \in I\}$, then there exists x_t such that $\sup A < x_t \leq \sup\{x_t : t \in I\}$, by supremum property.
(Take $\epsilon = \sup\{x_t : t \in I\} - \sup A > 0$.)

Note that $x_t = \sup A_t$.

By supremum property, there exists $a \in A_t$ such that $\sup A < a \leq \sup A_t = x_t$. (Take $\epsilon = \sup A_t - \sup A > 0$.)
Since $a \in A_t$ implies $a \in A$, which contradicts to the fact that $\sup A$ is a upper bound of A.

Therefore, $\sup A = \sup\{x_t \in I\}$. □

Example 3:

For $n = 1, 2, 3, \cdots$, let $x_n, y_n, z_n \in \mathbb{R}$ such that $x_n \leq y_n \leq z_n$. Suppose $\sup\{x_1, x_2, x_3, \cdots\} = \sup\{z_1, z_2, z_3, \cdots\} = a \in \mathbb{R}$.
Show that $\sup\{y_1, y_2, y_3, \cdots\} = a$.

Proof. (Step 1: We first show a is an upper bound of $\{y_1, y_2, y_3, \cdots\}$.)
Since $y_k \leq z_k \leq \sup\{z_1, z_2, z_3, \cdots\} = a$.
Hence a is an upper bound of $\{y_1, y_2, y_3, \cdots\}$.

(Step 2: Prove by Contradiction)
Since a is a upper bound of $\{y_1, y_2, y_3, \cdots\}$, we get $\sup\{y_1, y_2, y_3, \cdots\} \leq a$.
Suppose $\sup\{y_1, y_2, y_3, \cdots\} \neq a$, that means $\sup\{y_1, y_2, y_3, \cdots\} < a$.
Since $\sup\{x_1, x_2, \cdots\} = a$, by supremum property, there exists x_k such that $\sup\{y_1, y_2, \cdots\} < x_k \leq a$. (Take $\epsilon = a - \sup\{y_1, y_2, \cdots\} > 0$.)
This implies $y_k \leq \sup\{y_1, y_2, \cdots\} < x_k$, contradicts to $x_k \leq y_k$. □

Sequences

There are two types of problem about sequences usually appears in the past Midterm.

(i) Find limit of a sequence involving Recurrence Relation

Monotonic Sequence

Example 1: (Adapted from Problem in Mathematical Analysis I, 2.1.9)

Let $x_1 = 0$, $x_{n+1} = \sqrt{6 + x_n}$ for $n \geq 1$. Prove that $\{x_n\}$ converges and find its limit.

Proof. (Step 1) Show $\{x_n\}$ is bounded (from above).
Claim: $0 \leq x_n \leq 3$ for $n \geq 1$.
Let $P(n)$ be the proposition: $0 \leq x_n \leq 3$.
$P(1)$ is true since $0 \leq x_1 = 0 \leq 3$.

Assume $P(k)$ is true, $k \geq 1$, i.e., $0 \leq x_k \leq 3$.
$x_{k+1} = \sqrt{6 + x_k} \geq 0$ and $x_{k+1} \leq \sqrt{6 + 3} = 3$.
Thus $P(k + 1)$ is true.
By MI, $P(n)$ is true for all $n \in \mathbb{N}$.

(Step 2) Show $\{x_n\}$ is monotonic (increasing).
Claim: $x_{n+1} \geq x_n$ for $n \geq 1$.
$x_{n+1}^2 - x_n^2 = (6 + x_n) - x_n^2 = (2 + x_n)(3 - x_n) \geq 0$ by Step 1.
The claim follows.

(Step 3) Find the limit by algebraic operation.
By Monotonic sequence Theorem, $\{x_n\}$ converges.
Let $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{6 + x_n}$ implies $x = \sqrt{6 + x}$.
This implies $x^2 - x - 6 = 0$ iff $(x - 3)(x + 2) = 0$ iff $x = -2$ or 3.
Since $x_n \geq 0$ for $n \geq 1$, implies $x \geq 0$ by taking limit, we get $\lim_{n\to\infty} x_n = 3$. □

Example 2: (Adapted from Problem in Mathematical Analysis I, 2.1.10)

Let $x_1 = 0$, $x_2 = \frac{1}{2}$, $x_{n+1} = \frac{1}{3}(1 + x_n + x_{n-1}^3)$ for $n \geq 2$. Prove that $\{x_n\}$ converges and find its limit.

Proof. (Step 1) Show $\{x_n\}$ is bounded (from above).
Claim: $0 \leq x_n < \frac{2}{3}$ for $n \geq 1$.
Let $S(n)$ be the proposition $0 \leq x_n < \frac{2}{3}$.
Note that $0 \leq x_1 = 0 < \frac{2}{3}$ and $0 \leq x_2 = \frac{1}{2} < \frac{2}{3}$.
So, $S(1)$ and $S(2)$ are true.

Assume $S(k)$, $S(k + 1)$ are true for $k \geq 1$, i.e., $0 \leq x_k < \frac{2}{3}$ and $0 \leq x_{k+1} < \frac{2}{3}$.
Then $0 \leq x_{k+2} = \frac{1}{3}\left(1 + x_{k+1} + x_k^3\right) < \frac{1}{3}\left(1 + \frac{2}{3} + (\frac{2}{3})^3\right) = \frac{53}{81} < \frac{2}{3}$.
So, $S(k + 2)$ is true. By MI, $S(n)$ is true for all $n \in \mathbb{N}$.

(Step 2) Show $\{x_n\}$ is monotonic (increasing).
Claim: $x_{n+1} \geq x_n$ for $n \geq 1$.

Let $P(n)$ be the proposition $x_{n+1} \geq x_n$.

$x_2 = \frac{1}{2} > 0 = x_1$ and $x_3 = \frac{1}{3}\left(1 + \frac{1}{2} + 0^3\right) = \frac{1}{2} = x_2$.
So, $P(1)$, $P(2)$ are true.

Assume $P(k)$, $P(k + 1)$ are true for $k \geq 1$, i.e., $x_{k+1} \geq x_k$ and $x_{k+2} \geq x_{k+1}$.
Thus $x_{k+3} - x_{k+2} = \frac{1}{3}\left(x_{k+2} - x_{k+1} + x_{k+1}^3 - x_k^3\right) \geq 0$ as $f(x) = x^3$ is a increasing function.
Thus $P(k + 2)$ is true. By MI, P(n) is true for all $n \in \mathbb{N}$.

(Step 3) Find the limit by algebraic operation.
By Monotonic sequence Theorem, $\{x_n\}$ converges.

Let $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(\frac{1}{3}(1 + x_n + x_{n-1}^3)\right)$. Then $x = \frac{1}{3}(1 + x + x^3)$.

Thus $3x = 1+x+x^3$ implies $x^3-2x+1 = 0$, note that $x = 1$ is a root of $x^3-2x+1 = 0$, then $x^3-2x+1 = 0$ iff $(x-1)(x^2+x-1) = 0$. Thus $x = 1$ or $x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$.

Note that $0 \le x_n \le \frac{2}{3} < 1$, which implies $0 \le x \le \frac{2}{3}$ by taking limit, hence we get $\lim_{n\to\infty} x_n = \frac{-1+\sqrt{5}}{2}$.

Remark 1: $\frac{2}{3}$ is not that special, as long as you find a number r such that $0 \le x_n \le r < 1$,

we can reject the possibility of $x = 1$ and $x = \frac{-1-\sqrt{5}}{2}$.

Remark 2: Also see Practice exercise 92 (o). □

Interwining Sequence

Example 1: (Adapted from Rudin)

Fix $\gamma > 1$. Take $x_1 > \sqrt{\gamma}$, and define

$$x_{n+1} = \frac{\gamma + x_n}{1 + x_n} = x_n + \frac{\gamma - x_n^2}{1 + x_n}.$$

Show that $\{x_n\}$ converges and find its limit.

Idea and Observation:

$\gamma > 1$, $x_1 > \sqrt{\gamma} > 1$ and $x_{n+1} = \frac{\gamma + x_n}{1 + x_n} > 1$.

$$\begin{aligned}
x_2 - \sqrt{\gamma} &= \frac{\gamma + x_1}{1 + x_1} - \sqrt{\gamma} \\
&= \frac{\gamma + x_1 - \sqrt{\gamma}(1 + x_1)}{1 + x_1} \\
&= \frac{\gamma + x_1 - \sqrt{\gamma} - \sqrt{\gamma}x_1}{1 + x_1} \\
&= \frac{-\sqrt{\gamma}(1-\sqrt{\gamma}) + x_1(1-\sqrt{\gamma})}{1 + x_1} \\
&= \frac{(x_1 - \sqrt{\gamma})(1-\sqrt{\gamma})}{1 + x_1} \\
&< 0 \qquad\qquad \text{as } 1 + x_1 > 0,\ x_1 > \sqrt{\gamma} \text{ and } 1 < \sqrt{\gamma}
\end{aligned}$$

i.e $1 < x_2 < \sqrt{\gamma} < x_1$.

Also, $x_3 = x_2 + \frac{\gamma - x_2^2}{1 + x_2}$ and $x_2 < \sqrt{\gamma}$, $x_3 > x_2$ and

$$\begin{aligned}
x_1 - x_3 &= x_1 - \frac{\gamma + x_2}{1 + x_2} \\
&= x_1 - \frac{\gamma + \left(\frac{\gamma + x_1}{1 + x_1}\right)}{1 + \left(\frac{\gamma + x_1}{1 + x_1}\right)} \\
&= \frac{2(x_1^2 - \gamma)}{(1 + \gamma) + 2x_1} \\
&> 0
\end{aligned}$$

By similar argument as above we can get $x_2 < x_4 < \sqrt{\gamma} < x_3 < x_1$.
We suspect that $\{x_n\}$ is intertwining.

So we are going to show that $\forall n \in \mathbb{N}$,

$$x_{2n} < x_{2n+2} < \sqrt{\gamma} < x_{2n+3} < x_{2n+1}.$$

Proof. The above statement can be proved by mathematical induction on $n \in \mathbb{N}$ and by the following equality:

$$x_{k+2} - x_k = \frac{2(\gamma - x_k^2)}{(1+\gamma) + 2x_k}$$

We then get a collection of nested intervals $\{I_n = [x_{2n}, x_{2n+1}] \mid n \in \mathbb{N}\}$ and that $\{x_{2n}\}$ is increasing and bounded above by $\sqrt{\gamma}$; $\{x_{2n+1}\}$ is decreasing and bounded below by $\sqrt{\gamma}$.

Thus $\lim\limits_{n\to\infty} x_{2n}$ and $\lim\limits_{n\to\infty} x_{2n+1}$ exist, say $\lim\limits_{n\to\infty} x_{2n} = a$ and $\lim\limits_{n\to\infty} x_{2n+1} = b$.

Follow from the above identity with $k = 2n$, $x_{2n+2} - x_{2n} = \dfrac{2(\gamma - x_{2n}^2)}{(1+\gamma) + 2x_{2n}}$, we get $0 = a - a = \dfrac{2(\gamma - a^2)}{(1+\gamma) + 2a}$.
i.e., $a^2 = \gamma$ as for all $n \in \mathbb{N}$, $x_n > 1$, thus $a = \sqrt{\gamma}$.

Consider

$$\begin{aligned} b - a &= \lim_{n\to\infty}(x_{2n+1} - x_{2n}) \\ &= \lim_{n\to\infty}\left(\frac{\gamma - x_{2n}^2}{1 + x_{2n}}\right) \\ &= \frac{\gamma - a^2}{1+a} \\ &= 0 \end{aligned}$$

By Nested Interval Theorem, $\lim\limits_{n\to\infty} x_{2n} = a = \sqrt{\gamma} = b = \lim\limits_{n\to\infty} x_{2n+1}$.
Then by Intertwining Theorem, $\lim\limits_{m\to\infty} x_m = \sqrt{\gamma}$. □

Example 2: (Practice Exercise 95)

> Show that the sequence $\{x_n\}$ given by $x_1 = 1$, $x_2 = 2$ and $x_{n+1} = \dfrac{1}{3}x_n + \dfrac{2}{3}x_{n-1}$ for $n \geq 2$, converges and find the limit.
> (Hint: $a \leq ta + (1-t)b \leq b$ for $a \leq b$ and $0 \leq t \leq 1$.............(*))

Idea: Compute first few terms and observe whether the sequence is intertwining or simply monotonic.

Proof. (Step 1: Now we need to show $\lim_{n\to\infty} x_{2n}$ and $\lim_{n\to\infty} x_{2n+1}$ exists, by using Nested Interval Theorem)

Let $I_1 = [x_1, x_2]$, $I_2 = [x_3, x_4]$, $\cdots$, $I_n = [x_{2n-1}, x_{2n}]$.

In order to make use of Nested Interval Theorem, we need to check $I_{n+1} \subseteq I_n$,
which is same as $x_{2n-1} \leq x_{2n+1} \leq x_{2n+2} \leq x_{2n}$ for all $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, note that by the hint, we get

$$x_{2n-1} \leq x_{2n+1} = \frac{1}{3}x_{2n} + \frac{2}{3}x_{2n-1} = \frac{2}{3}x_{2n-1} + \frac{1}{3}x_{2n} \leq x_{2n}$$

Also, by the hint again

$$x_{2n+1} \leq x_{2n+2} = \frac{1}{3}x_{2n+1} + \frac{2}{3}x_{2n} \leq x_{2n}$$

For any $n \in \mathbb{N}$, we get

$$x_{2n-1} \leq x_{2n+1} \leq x_{2n+2} \leq x_{2n}$$

Since $x_1 = 1$ and $x_2 = 2$, we get $1 \leq x_{2n-1} \leq x_{2n} \leq 2$ for any $n \in \mathbb{N}$.

By Nested Interval Theorem(or by Monotonic sequence Theorem, if you like),
we get $\{x_{2n-1}\}$ converges to a, and $\{x_{2n}\}$ converges to b for some $a, b \in \mathbb{R}$.

Since $x_{2n+1} = \dfrac{1}{3}x_{2n} + \dfrac{2}{3}x_{2n-1}$, taking limit $n \to \infty$, we get $a = \dfrac{1}{3}b + \dfrac{2}{3}a$, which implies $a = b$.

By the Intertwining Sequence Theorem, $\{x_n\}$ converges.

(Step 2:) Find out the limit of $\{x_n\}$.

Let $\lim_{n\to\infty} x_n = x$, consider

$$\begin{aligned}
x_3 &= \frac{1}{3}x_2 + \frac{2}{3}x_1 \\
x_4 &= \frac{1}{3}x_3 + \frac{2}{3}x_2 \\
x_5 &= \frac{1}{3}x_4 + \frac{2}{3}x_3 \\
x_6 &= \frac{1}{3}x_5 + \frac{2}{3}x_4 \\
x_7 &= \frac{1}{3}x_6 + \frac{2}{3}x_5 \\
&\vdots \\
x_n &= \frac{1}{3}x_{n-1} + \frac{2}{3}x_{n-2}
\end{aligned}$$

By adding and subtracting, we get $x_n + x_{n-1} = x_2 + \frac{2}{3}x_1 + \frac{1}{3}x_{n-1}$.

Taking limit $n \to \infty$, we get $x + x = 2 + \frac{2}{3}(1) + \frac{1}{3}x$, which implies $x = \frac{8}{5}$.

Thus $\lim_{n\to\infty} x_n = \frac{8}{5}$. □

(ii) Show the limit of a sequence is equal to a certain number by checking definition

Example 1:

By checking definition of limit, show part (a) and part (b). And then show part (c) by using part (b):

(a) $\lim\limits_{n\to\infty} \frac{[10^n \sqrt{2}]}{10^n} = \sqrt{2}$, where $[x]$ means the greatest integer less than or equal to x.

(b) If $\{x_n\}$ be a sequence such that $\lim\limits_{n\to\infty} \frac{1}{x_n} = 0$ and $x_n > 0$ for all $n \in \mathbb{N}$, show that for any $y \in \mathbb{R}$, $\lim_{n\to\infty} \frac{[x_n y]}{x_n} = y$.

(c) For any $y \in \mathbb{R}$, construct a sequence of rational number and a sequence of irrational number which both converge to y.

Remark: Note that the definition of $[x]$ means that

$$x - 1 < [x] \leq x \text{ with equality holds iff } x \in \mathbb{Z}$$

Since $[x]$ being the greatest integer less than or equal to x, it is clear to see $[x] \leq x$. Suppose we get $x - 1 \geq [x]$ which is same as $[x] + 1 \leq x$, which violates $[x]$ is the greatest integer less than or equal to x, a contradiction. Thus $x - 1 < [x]$.

Conversely, if $r \in \mathbb{Z}$ satisfies $x - 1 < r \leq x$ with equality holds iff $x \in \mathbb{Z}$, if $r + 1 \leq x$, we get $x - 1 \geq r$, a contradiction, that r is the greatest integer less than or equal to x.

Idea for part (a): Given any $\epsilon > 0$, Want to find out a suitable $N \in \mathbb{N}$. (Note that N depends on ϵ)

We hope for $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $n \geq N \implies \left|\frac{10^n \sqrt{2}}{10^n} - \sqrt{2}\right| < \epsilon$.

Now we make use of some inequality so that we can find N easier.

Note that $10^n \sqrt{2} - 1 < [10^n \sqrt{2}] < 10^n \sqrt{2}$ (by definition of greatest integer.)

$$\left|\frac{[10^n \sqrt{2}]}{10^n} - \sqrt{2}\right| = \sqrt{2} - \frac{[10^n \sqrt{2}]}{10^n} < \sqrt{2} - \frac{10^n \sqrt{2} - 1}{10^n} = \frac{1}{10^n} < \epsilon$$

The last inequality can be achieved if $n > \log\left(\frac{1}{\epsilon}\right)$.

Proof. (a) For any $\epsilon > 0$, by Archimedean's Property, there exists $N \in \mathbb{N}$ such that $N > \log\left(\frac{1}{\epsilon}\right)$.

Then

$$n \geq N \implies n > \log\left(\frac{1}{\epsilon}\right) \implies \left|\frac{[10^n\sqrt{2}]}{10^n} - \sqrt{2}\right| = \sqrt{2} - \frac{[10^n\sqrt{2}]}{10^n} < \sqrt{2} - \frac{10^n\sqrt{2} - 1}{10^n} = \frac{1}{10^n} < \epsilon$$

Therefore, $\lim_{n\to\infty} \dfrac{[10^n\sqrt{2}]}{10^n} = \sqrt{2}$.

Remark: Part (a) is just the special case of part (b), and also one can use Sandwich Theorem to prove part (a) or (b), if the problem DO NOT require you to show it by checking the definition of limit.

(b) For any $\epsilon > 0$, by definition of $\lim_{n\to\infty} \dfrac{1}{x_n} = 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\left|\dfrac{1}{x_n} - 0\right| < \epsilon$.

Note that $x_n y - 1 < [x_n y] \leq x_n y$. Then $\left|\dfrac{[x_n y]}{x_n} - y\right| = y - \dfrac{[x_n y]}{x_n} < y - \dfrac{x_n y - 1}{x_n} = \dfrac{1}{x_n} = \left|\dfrac{1}{x_n} - 0\right| < \epsilon$

Therefore, $\lim_{n\to\infty} \dfrac{[x_n y]}{x_n} = y$.

(c) For a sequence of rational number which converges to y.

Take $x_n = 10^n$, we get $w_n = \dfrac{[10^n y]}{10^n} \in \mathbb{Q}$ which converges to y, by part (b).

For a sequence of irrational number which converges to y.

Take $x_n = 10^n\sqrt{2}$, we get $z_n = \dfrac{[10^n\sqrt{2}y]}{10^n\sqrt{2}} \in \mathbb{R}\backslash\mathbb{Q}$, which converges to y, part (b).

Remark: It is not necessary to take $x_n = 10^n$, as long as your sequence $\{x_n\}$ satisfying $\lim\limits_{n\to\infty} \dfrac{1}{x_n} = 0$ and $x_n > 0$ for all $n \in \mathbb{N}$, it works. Let's say taking $x_n = n!$ or $x_n = \sqrt{2}n!$, we still get a sequence $\left\{\dfrac{[yn!]}{n!}\right\}$ of rational number converging to y, and a sequence $\left\{\dfrac{[y\sqrt{2}n!]}{\sqrt{2}n!}\right\}$ of irrational number converging to y.

$\square$

Example 2:

Suppose $\lim_{n\to\infty} x_n = 0.6$, show that $\lim_{n\to\infty} x_n^n = 0$

IDEA: Since x_n and n vary at the same time, we need to simplify the problem by "restricting" the range of x_n.
Since $\lim_{n\to\infty} x_n = 0.6$, for large n, all x_n should lie in the interval $[0.5, 0.7]$.
Then $|x_n^n - 0| < |0.7^n| < \epsilon$ iff $n > \dfrac{\ln\epsilon}{\ln(0.7)}$.

Proof. For any $\epsilon > 0$, by Archimedean's Principle, there exists a $K_1 \in \mathbb{N}$ such that $K_1 > \dfrac{\ln\epsilon}{\ln(0.7)}$.

By the definition of $\lim_{n\to\infty} x_n = 0.6$, there exists an $K_2 \in \mathbb{N}$ such that $n \geq K_2$, $|x_n - 0.6| < 0.1$, which implies $0.5 < x_n < 0.7$.

Pick $K = \max\{K_1, K_2\}$, then

$$n \geq K \implies n > \frac{\ln\epsilon}{\ln(0.7)} \text{ and } |x_n| < 0.7 \implies |x_n^n - 0| < |0.7^n| < \epsilon$$

Thus $\lim_{n\to\infty} x_n^n = 0$.

Remark: In the tutorial on Tuesday, I forgot to mention, there exists an $K_2 \in \mathbb{N}$ such that $n \geq K_2$, implies $0.5 < x_n < 0.7$

$\square$

Example 3:

Let $a_1, a_2, \cdots \in \mathbb{R}$ be such that $\lim_{n\to\infty} a_n = 1$, prove that

$$\lim_{n\to\infty}\Big(\frac{a_n^{\frac{2}{3}}-1}{a_n-\frac{2}{3}}+\frac{n^2}{n^2+2016}\Big)=1$$

by checking the definition of limit of a sequence. **Do not** use computation formulas, Sandwich theorem or L'Hopital's rule.

Proof. By observation, we get $\dfrac{a_n^{\frac{2}{3}}-1}{a_n-\frac{2}{3}} \to 0$ and $\dfrac{n^2}{n^2+2016} \to 1$, thus in order to prove the convergence by checking definition, we split the terms in the following manner:

$$L_n := \Big|\Big(\frac{a_n^{\frac{2}{3}}-1}{a_n-\frac{2}{3}}+\frac{n^2}{n^2+2016}\Big)-1\Big| \le \Big|\frac{a_n^{\frac{2}{3}}-1}{a_n-\frac{2}{3}}-0\Big|+\Big|\frac{n^2}{n^2+2016}-1\Big| \overset{(*)}{\le} \frac{|a_n-1|^{\frac{2}{3}}}{|a_n-\frac{2}{3}|}+\frac{2016}{n}$$

Then for any $\epsilon > 0$, by the definition of $\lim_{n\to\infty} a_n = 1$,
there is an $N_1 \in \mathbb{N}$ such that

$$n \ge N_1 \implies |a_n - 1| < \frac{1}{6}$$

It follows from the triangular inequality $|x-y| \ge \big||x|-|y|\big|$ and $|x+y|-|x| \le |y|$, we get

$$n \ge N_1 \implies |a_n-\frac{2}{3}| = |a_n-1+\frac{1}{3}| \ge \Big||a_n-1|-|\frac{1}{3}|\Big| \ge \frac{1}{3}-|a_n-1| > \frac{1}{3}-\frac{1}{6}=\frac{1}{6}$$

and there is an $N_2 \in \mathbb{N}$ such that $n \ge N_2 \implies |a_n-1| < \big(\frac{\epsilon}{12}\big)^{\frac{3}{2}}$

and by Archimedean principle, there is an $N_3 \in \mathbb{N}$ such that $N_3 > \dfrac{4032}{\epsilon}$.

Therefore,

$$n \ge \max\{N_1,N_2,N_3\} \implies L_n \le \frac{|a_n-1|^{\frac{2}{3}}}{|a_n-\frac{2}{3}|}+\frac{2016}{n} < \frac{\Big(\big(\frac{\epsilon}{12}\big)^{\frac{3}{2}}\Big)^{\frac{2}{3}}}{\frac{1}{6}}+\frac{2016}{\frac{4032}{\epsilon}}=\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

Remark:

For inequality labeled with $(*)$, we make use of the following fact:
Let $p \in (0, 1]$, for every $x, y \ge 0$, $|x^p - y^p| \le |x - y|^p$.
For the proof, see example 1 in the last section. $\square$

Other problems

Example 1:

Let $p \in (0, 1]$, show that for every $x, y \ge 0$, $|x^p - y^p| \le |x - y|^p$.

Proof. WLOG, we may assume $y \le x$. Then we get

$$x^p - y^p \le (x-y)^p \iff 1-\Big(\frac{y}{x}\Big)^p \le \Big(1-\frac{y}{x}\Big)^p$$

It suffices to show

$$1-u^p \le (1-u)^p, \forall u \in [0,1]$$

and then take $u = \dfrac{y}{x} \in [0, 1]$.

For any $z \in [0, 1]$, we get $z \le z^p$.

This is because the following: $z - z^p = z^p(z^{1-p} - 1) \le 0$ as $z^p \ge 0$ and $z^{1-p} \le 1$ (note that $1 - p \ge 0$)

Thus for any $u \in [0, 1]$, we get $u \le u^p$ and since $1 - u \in [0, 1]$, we get $(1 - u) \le (1 - u)^p$.

By adding up two inequalities, we get

$$1 = u + (1-u) \le u^p + (1-u)^p$$

which is equivalent to $1 - u^p \le (1 - u)^p$. $\square$

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The examples in this Midterm Review are mostly borrowed directly or adapted with modification from the Tutorial Notes written by L.C.M, C.C.Lee, T.F.Hung, and also from the practice Exercise and Past midterms, and books: Rudin, Problem in Mathematical Analysis.

Rudin: https://notendur.hi.is/vae11/%C3%9Eekking/principles_of_mathematical_analysis_walter_rudin.pdf

Problem in Mathematical Analysis:
https://lethuc92.files.wordpress.com/2014/08/problems-in-mathematical-analysis1.pdf