MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #15

Continuity and its properties

Recall

f(x) is continuous at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$

(Definition of Continuous Function)

$\lim_{x \to x_0} f(x) = L$ iff For any $\varepsilon > 0$, there exists $\delta > 0$, such that $|x - x_0| < \delta \rightarrow |f(x) - L| < \varepsilon$

(Definition of limit of function)

f(x) is continuous at x_0 iff for any $\varepsilon > 0$, there exists $\delta > 0$, such that $|x - x_0| < \delta \rightarrow |f(x) - f(x_0)| < \varepsilon$

($\varepsilon - \delta$ definition of continuous Function)

Example 1

Use $\varepsilon - \delta$ definition to show $f(x) = \sqrt{x^2 + 3}$ is continuous at $x = 1$

IDEA: It is easy to see $f(1) = \sqrt{1 + 3} = 2$ and we just need to check, by definition that $\lim_{x \to 1} f(x) = 2$

Note that using $\left|\sqrt{x} - \sqrt{y}\right| < \left|\sqrt{|x - y|}\right|$

We get

$$\left|\sqrt{x^2 + 3} - 2\right| = \left|\sqrt{x^2 + 3} - \sqrt{4}\right| < \left|\sqrt{|x^2 + 3 - 4|}\right| = \left|\sqrt{|x^2 - 1|}\right|$$

$$= \left|\sqrt{|(x - 1)(x + 1)|}\right| \leq^{(*)} \left|\sqrt{|x - 1|(2 + 1)}\right| = \left|\sqrt{3|x - 1|}\right| <^{(**)} \varepsilon$$

So we require $(*) |x - 1| < 1 \rightarrow 0 < x < 2$ and $(**) |x - 1| < \varepsilon^2/3 \rightarrow \sqrt{|x - 1|} < \frac{\varepsilon}{\sqrt{3}}$

Solution:

First, $f(1) = 2$

For any $\varepsilon > 0$, pick $\delta = \min\{1, \varepsilon^2\}$, then for $|x - 1| < \delta$. From the previous steps, we get $\left|\sqrt{x^2 + 3} - 2\right| < \varepsilon$. So f is continuous at $x = 1$.

Example 2 (Application of $\varepsilon - \delta$ definition)

Let $f(x)$ and $g(x)$ be continuous functions, show that $h(x) = g(f(x))$ is continuous.

IDEA: To show h is continuous, we need to show $h(x)$ is continuous at every x_0.

Note that

$|h(x) - h(x_0)| = |g(f(x)) - g(f(x_0))| <^{(*)} \varepsilon$

(*)We first require $|g(f(x)) - g(f(x_0))| < \varepsilon \rightarrow |f(x) - f(x_0)| < \delta'$ (for some $\delta' > 0$)

Next to make $|f(x) - f(x_0)| < \delta'$,
We require $|x - x_0| < \delta"$ for some $\delta" > 0$

Solution:
For any $\varepsilon > 0$, since $g(y)$ is continuous at y_0, there exists $\delta' > 0$ such that for $|y - y_0| < \delta' \rightarrow |g(y) - g(y_0)| < \varepsilon$
Next since $f(x)$ is continuous at x_0, for this δ', there exists $\delta" > 0$ such that $|x - x_0| < \delta" \rightarrow |f(x) - f(x_0)| < \delta'$.

Pick $\boldsymbol{\delta} = \boldsymbol{\delta}" > 0$, then for $|x - x_0| < \delta" \rightarrow |f(x) - f(x_0)| < \delta'$. From the steps above, we get
$|h(x) - h(x_0)| = |g\big(f(x)\big) - g(f(x_0)| < \varepsilon$
Hence $h = g(f(x))$ is continuous.
(Note: Another Proof of Example 2 (using sequential continuity theorem) can be found in Kin Li's Lecture Note P.36)

Theorem: (Sequential Continuity Theorem)
$f(x)$ is continuous at x_0 if and only if for every sequence $x_n \rightarrow x_0$, we have $\lim_{n\rightarrow\infty} f(x_n) = f(x_0)$

Example 3
Define $f(x) = \begin{cases} 1 \text{ if x is rational} \\ 0 \text{ if x is irrational} \end{cases}$, show that $f(x)$ is not continuous at $x_0 \in \mathbf{R}$

Solution:
First, using the density of rational number and irrational number, one can take a sequence of rational $\{r_n\}$ and irrational $\{q_n\}$ such that $\lim_{n\rightarrow\infty} r_n = x_0$ and $\lim_{n\rightarrow\infty} q_n = x_0$.

Note that $\lim_{n\rightarrow\infty} f(r_n) = \lim_{n\rightarrow\infty} 1 = 1$ and $\lim_{n\rightarrow\infty} f(q_n) = \lim_{n\rightarrow\infty} 0 = 0$.
We see $\lim_{n\rightarrow\infty} f(r_n) \neq \lim_{n\rightarrow\infty} f(q_n)$. By the sequential continuity theorem, $f(x)$ is not continuous at any $x_0 \in \mathbf{R}$.

Example 4
Define a function $f(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ 1 - 2x & \text{if } x \in (\mathbf{R}\backslash\mathbf{Q}) \end{cases}$. Find all $x_0 \in \mathbf{R}$ such that $f(x)$ is continuous at x_0

Solution:
(Step 1: We first find out all such x_0)
If $f(x)$ is continuous at x_0, then $\lim_{x\rightarrow x_0} f(x)$ exists (say $\lim_{x\rightarrow x_0} f(x) = L$). By Sequential Limit Theorem, for every sequence $x_n \rightarrow x_0$, we have $\lim_{n\rightarrow\infty} f(x_n) = L$.

Consider a rational $\{r_n\}$ and irrational $\{q_n\}$ sequences which both limits are x_0, by the theorem, we get $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} f(q_n)$

$$\rightarrow \lim_{n\to\infty} r_n = \lim_{n\to\infty} 1 - 2q_n \rightarrow x_0 = 1 - 2x_0 \rightarrow x_0 = \frac{1}{3}$$

☹Caution: Many students may end the solution here(Step 1) and it is not correct.
It is because we have only shown if $f(x)$ is continuous at x_0, then $x_0 = \frac{1}{3}$. But
it cannot imply $f(x)$ is continuous at this point.

(Step 2: Next we need to show $f(x)$ is continuous at $\frac{1}{3}$)

IDEA: First $f\left(\frac{1}{3}\right) = \frac{1}{3}$, since x can be rational or irrational,

For the case x is rational,

$\left|f(x) - \frac{1}{3}\right| = \left|x - \frac{1}{3}\right| < \varepsilon$. (We require $\left|x - \frac{1}{3}\right| < \varepsilon$)

For the case x is irrational,

$\left|f(x) - \frac{1}{3}\right| = \left|1 - 2x - \frac{1}{3}\right| = \left|\frac{2}{3} - 2x\right| = 2\left|\frac{1}{3} - x\right| < \varepsilon$ (We require $\left|x - \frac{1}{3}\right| < \frac{\varepsilon}{2}$)

To make $\left|f(x) - \frac{1}{3}\right| < \varepsilon$ whenever x is rational or not,

pick $\delta = \min\left\{\varepsilon, \frac{\varepsilon}{2}\right\} = \frac{\varepsilon}{2}$, then for $|x - 0| < \delta$, from the steps above, we get

$\left|f(x) - \frac{1}{3}\right| < \varepsilon$.

Hence $f(x)$ is continuous at $x_0 = \frac{1}{3}$.

Some properties of Continuous Function

Theorem: (Intermediate Value Theorem) Let $f(x)$ be a continuous real function on the interval $[a, b]$, and suppose $f(a) < f(b)$ (or $f(a) > f(b)$). Then for every y_0 which $f(a) < y_0 < f(b)$, there exists $x_0 \in (a, b)$ such that $f(x_0) = y_0$

In fact, Intermediate Value Theorem is one of useful theorems, next we will show you some applications of this theorem.

Example 5 Let $f: \mathbf{R} \rightarrow \mathbf{Z}$ be a continuous function, show that $f(x)$ is a constant function.

Solution:

We prove by contradiction, suppose f is non-constant,

There exists two points a,b $(a \neq b)$ such that $f(a) \neq f(b)$,

assume $f(a) < f(b)$, since $f(a)$ and $f(b)$ are integers, we can pick a non-integer c such that $f(a) < c < f(b)$.

Because f is continuous, by intermediate value theorem, there exists $x \in \mathbf{R}$ such that $f(x) = c$. It leads to contradiction

(Note: The statement is still true if $\mathbf{Z}$ is replaced by $\mathbf{Q}$ or any countable set A, see the exercise for detail, the argument is slightly different)

Example 6 (Fixed Point Theorem) Let $f(x)$ be a continuous function from $[0,1]$ to $[0,1]$, show that $f(x) = x$ for at least one x in [0,1] (This is called fixed point)

IDEA: We can try to sketch the graph $y = f(x)$ and $y = x$.

Solution:

Consider function $g(x) = f(x) - x$ which is also continuous,

If $f(0) = 0$ or $f(1) = 1$, then we are done.

If $f(0) \neq 0$ and $f(1) \neq 1$, then $f(0) > 0$ and $f(1) < 1$ (since $f(x)$ lie between 0 and 1). We get $g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$.

Therefore by intermediate value theorem, there exists a point $x \in [0,1]$ such that $g(x) = 0 \rightarrow f(x) = x$

Example 7 (Practice Exercise #112c) Show that there is a non-zero continuous function $g: \mathbf{R} \rightarrow \mathbf{R}$ such that $g(t) + g(2t) + g(3t) = g(4t) + g(5t)$ (Hint: Try $g(t) = \lvert t\rvert^r$)

Solution:

According to the hint, try $g(t) = |t|^r$, substitute into the equation, we get

$g(t) + g(2t) + g(3t) = g(4t) + g(5t)$

$\rightarrow |t|^r + 2^r|t|^r + 3^r|t|^r = 4^r|t|^r + 5^r|t|^r$

$\rightarrow 1 + 2^r + 3^r = 4^r + 5^r$ (for $|t|^r \neq 0$)

Consider $f(r) = 1 + 2^r + 3^r - 4^r - 5^r$

Pick $r = 0$, $f(0) = 1 > 0$

Pick $r = 1$, $f(1) = 1 + 2 + 3 - 4 - 5 = -3 < 0$

Then by intermediate value theorem, there exists $c \in (0,1)$ such that $1 + 2^c + 3^c = 4^c + 5^c$. Then the function $g(t) = |t|^c$ is our desired function.

Given a continuous function $f(x)$, suppose $f(x)$ is also injective, then there are 2 properties about such function

Theorem: (Continuous Injection Theorem) Suppose $f(x)$ defined on $[a,b]$ is continuous and injective, then f is either monotonic increasing or monotonic decreasing on $[a,b]$
Theorem (Continuous Inverse Theorem) Suppose $f(x)$ defined on $[a,b]$ is continuous and injective, then f^{-1} is also continuous on [f(a),f(b)]

Example 8
Is there any continuous function $f(x)$ such that $f\left(f(x)\right) = -x^9$?

Solution:
Suppose there is such function $\mathrm{f(x)}$ satisfying $\mathrm{f}\left(\mathrm{f(x)}\right) = -\mathrm{x}^9$
We will first show f is injective
Proof: If $\mathrm{f(a)} = \mathrm{f(b)} \to \mathrm{f}\left(\mathrm{f(a)}\right) = \mathrm{f}\left(\mathrm{f(b)}\right) \to -\mathrm{a}^9 = -\mathrm{b}^9 \to \mathrm{a} = \mathrm{b}$
Next, since f is continuous and injective, by Continuous Injection Theorem, we know f is either monotonic increasing or decreasing

<u>Case i) If $f(x)$ is monotonic increasing</u>
Then $a > b \to f(a) > f(b) \to f\left(f(a)\right) > f(f(b))$, which imply $f(f(x))$ is increasing. But $-x^9$ is decreasing function, so it leads to contradiction
<u>Case ii) If $f(x)$ is monotonic decreasing</u>
Then $a > b \to f(a) < f(b) \to f\left(f(a)\right) > f(f(b))$, which also imply $f(f(x))$ is increasing. Same contradiction occurs.

Finally, there is a theorem about continuous function

Theorem: (Extreme Value Theorem)
If $f(x)$ is continuous on $[a, b]$, then $f(x)$ is bounded. Furthermore there exists w_0 and y_0 in [a,b] such that $f(w_0) \leq f(x) \leq f(y_0)$.

In the following, there are some suggested exercises, you should try to do them in order to understand the material. If you have any questions about them, you are welcome to find me during office hours. You are also welcome to submit your work to me and I can give some comments to your work.

☺Exercise 1

a) Use $\varepsilon - \delta$ definition, show that x^5 is continuous at $x = 1$.
b) Use $\varepsilon - \delta$ definition, show that $f(x) = \frac{2+3x}{x^2+4}$ is continuous at $x = 2$ (Practice Exercise #169)
c) Use $\varepsilon - \delta$ definition, show that if $\mathrm{f(x)} \geq 0$ is continuous, then $\sqrt{\mathrm{f(x)}}$ is also continuous.

☺Exercise 2
Find a function $f(x)$ such that f is NOT continuous at any $x_0 \in R$, but $|f|$ is continuous at EVERY $x_0 \in R$ (Hint: You may use the function in Example 3, but you need to change some number to satisfy the condition)

☺Exercise 3 (Modified from Practice Exercise #58)

Define $f: \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} 8x & \text{if x is rational} \\ 2x^2 + 8 & \text{if x is irrational} \end{cases}$$

Find all $x_0 \in \mathbf{R}$ such that $f(x)$ is continuous at $x = x_0$.

☺Exercise 4 (Practice Exercise #59)

If $f: R \to R$ is continuous and $f(x) = 0$ for every rational number, show that $f(x) = 0$ for every $x \in \mathbf{R}$.

(Hint: Use sequential continuity theorem, use the fact that for each irrational x, there exists a sequence of rational number $\{q_n\}$ such that $q_n \to x$.)

☺Exercise 5 (Practice Exercise #112b)

Let $f: [0,2] \to R$ be continuous and $f(0) = f(2)$. Show that there exists $c \in [0,1]$ such that $f(c) = f(c + 1)$.

(Hint: Think about $f(1)$, consider the following 3 cases $f(0) = f(1), f(0) < f(1), f(0) > f(1)$, and consider $g(x) = f(x) - f(x + 1)$, plot the graph to help you)

☺Exercise 6

In Example 5, we have mentioned that the statement is still true if $\mathbf{Z}$ can be replaced by $\mathbf{Q}$ and any non-empty countable set A. Try to prove them.

(Hint: For the case of $\mathbf{Q}$, use density of rational number, for the case of A, think about what will $f(x)$ contain if f is non-constant.

☺Exercise 7 (Practice Exercise #155)

Prove that there does not exist any continuous function $f: \mathbf{R} \to \mathbf{R}$ such that $f(f(x)) + x = 0$ (See Example 8)

☺Exercise 8

Show that there exists a non-zero function $f: \mathbf{R} \to \mathbf{R}$ such that

$$f(t) + \frac{df(t)}{dt} + \frac{d^2 f(t)}{dt^2} + \frac{d^3 f(t)}{dt^3} + \cdots \ldots + \frac{d^{2007} f(t)}{dt^{2007}} = 2008 \frac{d^{2008} f(t)}{dt^{2008}}$$

(Hint: Try $f(t) = e^{rt}$)