HO Chuen H. 20601111 MATH 2033 EXAM.

(a) we need to show $\lim_{x\to0} f(x) = 0$,

for any $\varepsilon > 0$, take $\varepsilon = \delta^m$, $0 < |x-0| < \delta$,

$|f(x) - 0| = |f(x)|$, Case 1: $f(x) = 0$, then $|f(x)| = 0 < \varepsilon$

Case 2: $f(x) = x^m$, then $|f(x)| = |x^m| = \overbrace{|x|\cdot|x|\cdots|x|}^{m} = |x|^m < (\sqrt[m]{\varepsilon})^m$

$= \varepsilon$

Thus, $\lim_{x\to0} f(x) = 0$ by definition, since $\lim_{x\to0} f(x) = f(0) = 0$,

all $m > 0$, $f(x)$ is continuous at $x=0$.

(b) Check differentiability, $\lim_{x\to0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0} \frac{x^m}{x}$

consider $x \to 0^+$, $\lim_{x\to0^+} \frac{x^m}{x} = \lim_{x\to0^+} x^{m-1} = 0^+$

consider $x \to 0^+$, $\lim_{x\to0^-} \frac{x^m}{x}$ Case 1: m is odd.

let $m = 2n+1$, $\lim_{x\to0^-} \frac{x^{2n}\cdot x}{x} = \lim_{x\to0^+} x^{2n} = 0^+$, m is odd is differentiable

Case 2: m is even, $\lim_{x\to0^-} \frac{x^{2n-1}\cdot x}{x} = \lim_{x\to0^-} x^{2n-1} = 0^-$

let $m = 2n$

m is even is differentiable

so $m > 0$ $f(x)$ is differentiable.

2) let $H(x) = f(x+1) - f(x)$, $H(x)$ continuous $\in [0,1]$

$H(0) = f(1) - f(0)$, $H(1) = f(2) - f(1)$

$\frac{1}{2}(H(0) + H(1)) = (f(2) - f(0)) \cdot \frac{1}{2}$

As a mid-point either $H(0) \leq \frac{1}{2}(H(0) + H(1)) \leq H(1)$

or $H(1) \leq \frac{1}{2}(H(0) + H(1)) \leq H(0)$

By intermediate value thm, $\exists\, c \in [0,1]$ such that

$H(c) = \frac{1}{2}(H(0) + H(1)) \Rightarrow f(c+1) - f(c) = \frac{1}{2}(f(2) - f(0))$ $\square$

3a) As x_n is cauchy's sequence, $|x_n - x_m| < \varepsilon \quad \forall \varepsilon > 0$

$\lim_{n\to\infty} x_n = x_0$, we need to show $|y_n - y_m| < \varepsilon$

$|y_n - y_m| = |f(x_n) - f(x_m)| \Rightarrow \lim_{n,m\to\infty} |f(x_n) - f(x_m)|$

$\Rightarrow \left|\lim_{n\to\infty} f(x_n) - \lim_{m\to\infty} f(x_m)\right|$ By sequential limit thm,

$\Rightarrow |f(x_0) - f(x_0)| = 0 < \varepsilon$. Hence, $\{y_n\}$ is also a cauchy's sequence.

3b)

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \frac{1}{6}f'''(x_0)(x-x_0)^3 + \frac{1}{24}f^{(4)}(x_0)(x-x_0)^4$$

$f^{(4)}(x_0) \leq M$, $x_0 \in \mathbb{R}$

let $x = x_0 + 2h$,

$$f(x) - f(x_0) - f'(x_0)(2h) - \frac{1}{2}f''(x_0)(4h^2) - \frac{1}{6}f'''(x_0)(8h^3) = \frac{1}{24}f^{(4)}(x_0)(16h^4)$$

$$|f^{(4)}(x_0)| \leq \frac{24}{16h^4}(|f(x)-f(x_0)|) - \frac{24(2h)}{16h^4}|f'(x_0)| - \frac{24h^2}{16h^4} \times (|f''(x_0)|) - \frac{24(8h^3)}{6(16h^4)}|f'''(x_0)|$$

$$\lim_{h\to 0} f(x_0+h) + f(x_0-h) - 2f(x_0) = 0, \quad \lim_{h\to 0} h^2 = 0$$

By l Hospital rule, $\lim_{h\to 0} \frac{f'(x_0+h) - f'(x_0-h)}{2h}$

$$= \frac{1}{2}\lim_{h\to 0}\left(\frac{f'(x_0+h) - f(x_0)}{h} + \frac{f'(x_0-h) - f(x_0)}{h}\right) = \frac{1}{2}(f''(x_0) + f''(x_0)) = f''(x_0)$$

so $\left|\frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2} - f''(x_0)\right| < \varepsilon$.

4a) As $f(x)$ is n-times differentiable, it has ~~least~~ at least m roots $\in (a,b)$ there exist $c_1, c_2, \cdots, c_m$ be roots of $f^{(k)}(x)=0$ where $0 \le k \le n-1$. By Rolle's thm, in $f^{(k)}(x)$ each of the interval (c_1, c_2), (c_2, c_3), $\cdots$, (c_{m-1}, c_m) such that

$$f^{(k)'}(d_j) = f^{(k+1)}(d_j) = 0 \quad , \quad d_j \in (c_j, c_{j-1})$$

Then $f(x)$ has at most $n+1$ roots,

And $f^{(n)}(x) > 0$, by the polynomial rules n-degree should have at most n roots.

4b i) ~~let~~ Let $f(x) = 4x^2 - 8x + 5 - 2^x$

$f(0) = 5 > 0$, $f(1) = -1 < 0$

By intermediate value thm, $\exists c \in (0,1)$ such that

$f(c) = 0$, so at least one solution for $f(x) = 0$ $x \in (0,1)$

b ii) $f(2) = 1 > 0$

From a) $\exists c \in (0,1)$ $f(c) = 0$, by intermediate value thm again

$\exists d \in (1,2)$ such that $f(d) = 0$

so at least two solution over $(0,2)$.

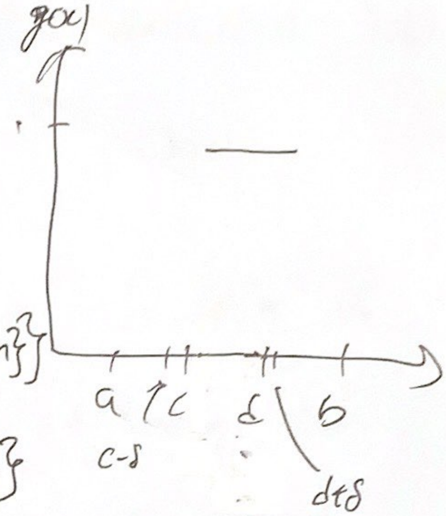
And, 2^x is strictly ~~increasing~~ increasing function $\in (0,2)$

the degree of polynomial is 2.

By polynomial definition, we could have at most 2 solution

Then, we have exact two ~~sol~~ solution for $f(x) = 0$.

5 a) we need to show



$U(g,P) - L(g,P) < \varepsilon$

let P_1 be ~~the~~ partition $\{c = x_0 < x_1 = c + \frac{d-c}{n} < \cdots < x_n = d\}$

~~$U(g,P_1)=$~~ since $\sup\{g(x) : x \in [x_{i-1}, x_i], i = \{1, \dots, n\}\}$

is equal to $\inf\{g(x) : x \in [x_{i-1}, x_i], i = \{1, \dots, n\}\}$

Then $U(g,P_1) - L(g,P_1) < \frac{\varepsilon}{3}$

let $P = \{c-\delta\} \cup P \cup \{d+\delta\}$, where $c-\delta < c < x_1$ and $x_{n-1} < d < d+\delta$

and $\delta < \frac{\varepsilon}{6} \Rightarrow 0 < \delta = \min\{\frac{\varepsilon}{6}, x_1 - c, d - x_{n-1}\}$

Then $U(g,P) - L(g,P) \le U(g, P \cap [c-\delta, d-\delta]) - L(g, P \cap [c-\delta, d-\delta])$

$$+ 2\delta\left(\sup_{x\in[c-\delta,c+\delta]} g(x) - \inf_{x\in[c-\delta,c+\delta]} g(x)\right) + 2\delta\left(\sup_{x\in[d-\delta,d+\delta]} (g(x)) - \inf_{x\in[d-\delta,d+\delta]} (g(x))\right)$$

$\le U(g,P_1) - L(g,P_1) + 4\delta \le \frac{\varepsilon}{3} + 4\frac{\varepsilon}{6} = \varepsilon$

5b i) Let $h(x) = g(x) - f(x)$,

$$\cancel{h(x) =} \quad h(x) = \begin{cases} g(x_i) - f(x_i) & i = \{1, \cdots, n\}, x_i \in [a,b) \\ 0 & \text{otherwise} \end{cases}$$

As $g(x)$, $f(x)$ bounded $g(x_i) - f(x_i)$ bounded.

Similar in part a) $\delta < \frac{\varepsilon}{2(2n+1)}$

$$U(h(x), P) - L(h(x), P) \leq \sum_{i=1}^{n} [U(h, P_i) - L(h, P_i)] + (n+1)(2\delta)$$

$$< n \cdot \frac{\varepsilon}{2n+1} + (n+1)(2)\frac{\varepsilon}{2(2n+1)} = \varepsilon.$$

$h(x)$ is riemann integrable.

Since $h(x)$ and $f(x)$ is riemann integrable,
$g(x)$ is also riemann integrable.

5b ii) $$\int_a^b h(x) = \begin{cases} \int_a^b [g(x_i) - f(x_i)] & i = \{1, \cdots, n\}, x_i \in (a,b) \\ 0 & \text{otherwise.} \end{cases}$$

For $\int_a^b h(x) = 0$, then $f(x) = g(x) \Rightarrow \int_a^b f(x) = \int_a^b g(x)$

For $\int_a^b h(x) \neq 0$.

6(a) If $\lim_{x\to+\infty} f(x)=L$, $\forall \varepsilon>0$, $\delta>0$ such that for all x

$0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon$

If $\lim_{n\to\infty} x_n=+\infty$ and $x_n\neq x_0$, $\exists k\in\mathbb{N}$ $n\geq k \Rightarrow 0<|x_n-x_0|<\delta$

such that $|f(x_n)-L|<\varepsilon$, so $\lim_{n\to\infty} f(x_n)=L$

If $\lim_{x\to x_0} f(x)\neq+\infty$ there exist $\varepsilon>0$, $\forall\delta>0$

$0<|x-x_0|<\delta$, $|f(x)-L|\geq\varepsilon$ If $\delta=\frac{1}{n}$

$0<|x_n-x_0|<\delta \Rightarrow |f(x_n)-L|\geq\varepsilon$ By sandwich thm,

$\lim_{n\to\infty} f(x_n)=+\infty$, then $0=\lim_{n\to\infty}|f(x_n)-L|\geq\varepsilon$

Prove by contradiction.

b) $\lim_{x\to\infty} \frac{\sin x}{x}\cdot\frac{x}{2+\cos x} = 1\cdot\lim_{x\to\infty}\frac{x}{2+\cos x}$

By L Hospital rule,

$\lim_{x\to\infty} \frac{1}{-\sin x} = \lim_{x\to\infty}\frac{x}{\sin x}\cdot\frac{-1}{x}$

$=0$. So the limit converge to real number.