Tutorial lecture 2 (Corresponding to lec3 + lec4):

Outline:

I. Set Properties. ← see Jiayu's note.

II. Equivalent relationship.

III. Cardinality. ← this note.

II. Equivalence relation.

Def (Equivalence relation): An equivalence relation over S, denoted often by $\sim$, is a subset of $S \times S$. Satisfying

① $(x,x) \in \sim$ $(x \sim x)$. $\forall x \in S$.

② $(x,y) \in \sim \Leftrightarrow (y,x) \in \sim$ $(x \sim y \Leftrightarrow y \sim x)$. $\forall x,y \in S$.

③ $(x,y), (y,z) \in \sim \Rightarrow (x,z) \in \sim$ $(x \sim y, y \sim z \Rightarrow x \sim z)$. $\forall x,y,z \in S$.

After defining a equivalence relation $\sim$, we sometimes concern the equivalence class induced by $\sim$:

Def (Equivalence class): Given $x \in S$, we define

a set $\leftarrow$ $[x] := \{y : x \sim y\}$, and say that $[x]$ is a equivalent class with representive element x. If $z \sim x$, then we have $[x] = [z]$.

of course $[x] = [y] \Leftrightarrow x \sim y$.

Notation: we denote the collection of all equivalent classes by $S/\sim$, i.e.

$$S/\sim = \{ [x] : x \in S \}.$$

Important properties: ① $[x]=[y]$ iff $x \sim y$.

② $[x] \neq [y] \iff [x] \cap [y] = \phi$.

Example 1: Check that $\sim$ defined by

$x \sim y$ iff $x-y \in \mathbb{Z} = \{\text{integers}\}$.

is a equivalent relation over $\mathbb{R} := \{\text{real numbers}\}$.

In general

$A = \{1,2\}$

$B = \{2,3\}$.

$A \neq B$ but $A \cap B = \{2\}$.

Proof: We need only to check three conditions:

① $x \sim x$, $\forall x$: $\iff$ show that $x - x \in \mathbb{Z}$.

Proof: It is obivous, because $x-x=0$.

② $x \sim y \iff y \sim x$: $\iff$ show that $(x-y \in \mathbb{Z} \iff y-x \in \mathbb{Z})$.

Proof: By the fact that (z is an integer iff $-z$ is an integer).

③ $x \sim y$, $y \sim z \Rightarrow x \sim z$: $\iff$ show that ($x-y \in \mathbb{Z}$ and $y-z \in \mathbb{Z}$ $\Rightarrow x-z \in \mathbb{Z}$).

Proof: By the fact if $z_1, z_2 \in \mathbb{Z}$, then z_1+z_2 is an integer.

$x-z = (x-y)+(y-z)$

Finding $\mathbb{R}/\sim := \{[x] : x \in \mathbb{R}\}$:

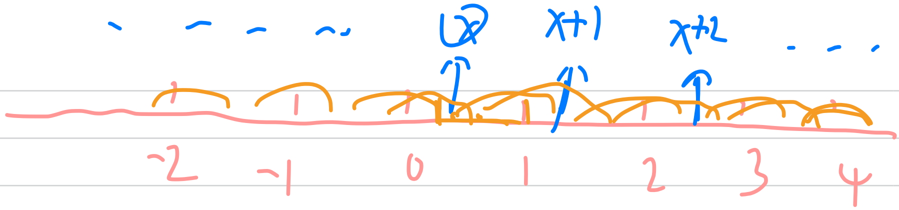
$\mathbb{R}/\mathbb{Z}$.

denoted by

for every $x \in \mathbb{R}$, we have

$[x] = \{x+m : m \in \mathbb{Z}\}$, so $[x] \neq [y]$ iff $x-y \notin \mathbb{Z}$. so we can represent $\mathbb{R}/\sim$ as

$$\mathbb{R}/\sim = \{[x] : x \in [0,1)\}.$$


(x)
x+1
x+2
-2
-1
0
1
2
3
4

III. Cardinality.

The cardinality of a set measures how "large" a set is.

We can consider a special equivalence relation over the set of all sets:

For sets S_1, S_2, we define $\sim$ via:

$S_1 \sim S_2$ iff exists a bijective function f from S_1 to S_2.

Recall (See also Jiayu's note for a detailed proof):

① $f: S_1 \to S_2$ is bijective iff $\underbrace{\exists g: S_2 \to S_1}_{}$ s.t. (g is bijective)

$f \circ g(x) = x \quad \forall x \in S_1, \quad g \circ f(x) = x \quad \forall x \in S_2.$

② If $f: S_1 \to S_2$, $g: S_2 \to S_3$ are bijections, then

$f \circ g: S_1 \to S_3$ is also a bijection.

Check that is a equivalence relation using ①, ②:

1. $S \sim S$, $\forall S$: Just consider function $f: S \to S$, $f(x) = x$, $\forall x \in S$.
2. $S_1 \sim S_2 \Rightarrow S_2 \sim S_1$. $S_1 \sim S_2 \Rightarrow \exists f: S_1 \to S_2$ is bijective $\overset{①}{\Rightarrow}$ exists $g: S_2 \to S_1$ s.t. g is bijective
3. $\underbrace{S_1 \sim S_2}_{\exists f_1: S_1 \to S_2 \text{ bijective}}$, $\underbrace{S_2 \sim S_3}_{\exists f_2: S_2 \to S_3 \text{ bijective.}} \Rightarrow \underline{S_1 \sim S_3}$. Consider $\underbrace{f_1 \circ f_2}_{\text{bijective.}}$,

Example 2: Show that $\mathbb{R}/\mathbb{Z}$ (see example 1) is equivalent to $[0,1)$ under $\sim$ defined above.

Hint: Using the fact that $\mathbb{R}/\mathbb{Z} = \{[x] : x \in [0,1)\}$.

Proof: Consider $f : [0,1) \mapsto \mathbb{R}/\mathbb{Z}$, $f(x) = [x]$.

We will show that f is bijective: $[x_1] \neq [x_2]$

① Injective: If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ ✓.

② Surjective: $\forall [x] \in \mathbb{R}/\mathbb{Z}$, $\exists x_0$ s.t. $f(x_0) = [x]$. #.

Consider $x_0 = x$.

Example 3: denote $m\mathbb{Z} := \{mx : x \in \mathbb{Z}\}$. then

$m_1\mathbb{Z} \sim m_2\mathbb{Z}$ $\forall m_1, m_2 \neq 0$, $m_1, m_2 \in \mathbb{R}$.

Hint: Construct a bijection from $\mathbb{Z}$ to $m\mathbb{Z}$ and using the fact that $\mathbb{Z} \sim m_1\mathbb{Z}$, $\mathbb{Z} \sim m_2\mathbb{Z} \Rightarrow m_1\mathbb{Z} \sim m_2\mathbb{Z}$.

Proof: We need only show that $\mathbb{Z} \sim m\mathbb{Z}$, $\forall m \neq 0$:

We can consider the map $f : \mathbb{Z} \to m\mathbb{Z}$,

$f(x) = mx$. #.

Remark: If $m = 0$, then $\mathbb{Z}$ is an infinite set

$m\mathbb{Z} = \{0\}$.

III.1 Countability

Def (Countability): We say a set S is countable if $S \sim \mathbb{N}$ or S is finite.

all natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$.

Countably infinite.

Remark: By $\sim$ is an equivalence relation. we have if $S_1 \sim S_2$ and either S_1 or S_2 is countable, then both of them are countable.

Proof: If $S_1 \sim S_2$. W.L.O.G. assume that S_1 is countable, then if S_1 is finite. S_2 must be finite, too. If $S_1 \sim \mathbb{N}$, then $S_2 \sim \mathbb{N}$. #

If S is not countable, then we say it is uncountable.

(Equivalent def of countability):

We say S is countable iff S can be listed as a (infinite or finite) sequence

$S = \{x_1, x_2, \cdots, x_n, \cdots\}$.

Example 4: $m\mathbb{Z}$ is countable for all $m \in \mathbb{R}$.

Proof: Case 1: $m = 0$: finite

Case 2: $m \neq 0$: see example 3. $m\mathbb{Z} \sim \mathbb{Z}$. $\mathbb{Z} \sim \mathbb{N}$. #

Example 5: Check that the product of two countable sets is still countable.

$X \times Y$. X, Y.

Proof: Case 1: Both is finite. ✓. $\Rightarrow$ $X \times Y$ is also finite.

$X = (x_1, \cdots, x_m)$ $\quad$ $Y = (y_1, \cdots, y_n, \cdots)$.

Case 2: One finite + one infinite.

$$\begin{array}{ccccc}
(x_1,y_1) & \to & (x_2,y_1) & \to \cdots \to & (x_m,y_1) \\
(x_1,y_2) & \to & (x_2,y_2) & \to \cdots \to & (x_m,y_2) \\
\vdots & & \vdots & & \vdots \\
(x_1,y_n) & & (x_2,y_n) & \cdots & (x_m,y_n) \\
\vdots & & \vdots & & \vdots
\end{array}$$

Case 3: $X = (x_1, \cdots, x_n, \cdots)$

$Y = (y_1, \cdots, y_n, \cdots)$.

diagonal scheme:

$$\begin{array}{ccccccc}
(x_1,y_1) & & (x_2,y_1) & \to & (x_3,y_1) & \cdots \\
\downarrow & & & & & \\
(x_1,y_2) & & (x_2,y_2) & & (x_3,y_2) & \cdots \\
(x_1,y_3) & & (x_2,y_3) & & (x_3,y_3) & \cdots \\
\vdots & & \vdots & & \vdots &
\end{array}$$

#.

V. Countable subset theorem

$\boxed{\text{Theorem 4}}$: Let $A \subseteq B$. if B is countable. then A is countable. (its contrapositive statement is also important)

Corollary: $A \subseteq B$, if A is uncountable, then B is also uncountable.

VI. Countable union theorem

$\boxed{\text{Theorem 5}}$: $\bigcup_{n \in \mathbb{N}} A_n$ is countable if A_n is countable for $n \geq 1$

nothing but the diagonal scheme!

VII. Product theorem

$\boxed{\text{Theorem 6}}$: Any finite product of countable sets is still countable.

also use the diagonal scheme.

Exercise 4: What happened to the infinite case?

Can you give an uncountable example?

See example 6. In fact, the countable product of finite sets can be uncountable.

VIII. Injection theorem

$\boxed{\text{Theorem 7}}$ Let $f: A \to B$ be injective. if B is countable.

Example6: $[0,1)$ is uncountable.

Proof: By theorem 4, it suffice to show that a subset of $[0,1)$ is uncountable:

$[0,1)$: digital form

$0.a_1 a_2 \cdots a_n \cdots$

$a_i \in \{0,1,\cdots,9\}$.

Consider the following subset:

$$S := \{ 0.a_1 a_2 \cdots a_m \cdots : a_i \in \{1,\cdots,9\} \}.$$

then $\forall s_1 = 0.a_{11} a_{21} \cdots a_{m1} \cdots$ $\in S$,
$\forall s_2 = 0.a_{12} a_{22} \cdots a_{m2} \cdots$

we have $s_1 = s_2$ iff $a_{i1} = a_{i2}$ $\forall i \geq 1$. ← didn't hold for $a_i \in \{0,\cdots,9\}$: $0.09999\cdots = 0.1$.

then if S is countable, we have

$S = \{ s_1, \cdots, s_n, \cdots \}$. and if we denote

$s_j = 0.a_{1j} \cdots a_{mj} \cdots$. and construct

$\tilde{s} = 0.b_1 \cdots b_m \cdots$ s.t. $b_i \neq a_{ii}$, we have then
$b_j \neq s_{jj}$, $b_i \in \{1,\cdots,9\}$.

① $\tilde{s} \in S$.

② $\tilde{s} \neq s_n$ for any n.

① + ② ⇒ $\tilde{s}$ is not listed, a contradiction. #.

⇒ S is uncountable. $S \subset [0,1) \subseteq \mathbb{R}$.
is uncountable.

Here $\underset{\sim}{S} \sim \underline{A \times A \times \cdots A \times \cdots}$

$A = \{1, \cdots, 9\}$.

$f : S \to A \times A \times \cdots \times A \times \cdots$

$f(0.a_1 \cdots a_2 \cdots) = (a_1, a_2, \cdots, a_n, \cdots)$

then A is countable.

Consequence: $(0,1)$ uncountable $\Rightarrow \mathbb{R}$ uncountable.

Rmk: if $f: A \to B$ is injective. we can say A is embeded in B by f.

2X. Surjection theorem.

$\boxed{\text{Theorem 8}}$: Let $g: A \to B$ be surjective. If A is countable. then B is countable.

Just write $B = \bigcup_{x \in A} \{f(x)\}$.

Example: $\mathbb{Q}$ is countable.

2 ways to show that.

1) $\mathbb{Q}$ can be seen as a subset of $\mathbb{Z} \times \mathbb{Z}$

$f: q = \frac{m}{n} \to (m, n)$ injection

2) $\mathbb{Q} = \bigcup_{n \in \mathbb{Z}} S_n = \bigcup_{n \in \mathbb{Z}} \{\frac{n}{m} : m \in \mathbb{Z}, m \neq 0\}$

Obviously S_n is countable.

Exercise 5: Show that (the set of all finite subsets of $\mathbb{N}$) countable.

Hint: ① Show that the collection of all subsets with m elements is countable. and denote it by A_m.

② By the collection of all finite subsets can be represented as $\bigcup_{m \geq 0} A_m$, we have the claim holds. #.

Example: $D = \{x \in \mathbb{R} : x^7 + x^6 + x^5 + \cdots + x + 1 \in \mathbb{Q}\}$
countable or not?

Seems tricky. but just let:

$$D_r \triangleq \{x \in \mathbb{R} : x^7 + x^6 + \cdots + x + 1 = r\} \quad (r \in \mathbb{Q})$$

Then D_r contains at most 7 elements.

$\Rightarrow D = \bigcup_{r \in \mathbb{Q}} D_r$ is countale (a countable union of countable sets is still countable).

Thm

Exercise 6. Show that.

$$F \triangleq \{a \in \mathbb{R} : x^5 + ax^3 + 1 = 0 \text{ has a rational root}\}.$$

is countable or not.

Hint: Trying to show that for every $r \in \mathbb{Q}$,

$r^5 + ar^3 + 1 = 0$ holds only for finitely many a. #.