MATH2033 Mathematical Analysis (2021 Spring)
Suggested Solution of Assignment 4

Problem 1

We consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^n \sin\frac{1}{x} & if \ \ x \neq 0 \\ 0 & if \ x = 0 \end{cases},$$

where $n \in \mathbb{N}$.

(a) Find the values of n which $f(x)$ is differentiable at $x = 0$.

(b) Find the values of n which $f(x)$ is continuous differentiable at $x = 0$.

☺Solution

(a) To check the differentiability of $f(x)$ at $x = 0$, we consider

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{x^n \sin\frac{1}{x}}{x} = \lim_{x\to 0} x^{n-1}\sin\frac{1}{x} = \begin{cases} does\ not\ exist & if\ n = 1 \\ 0 & if\ \ n \geq 2 \end{cases}.$$

So we conclude that $f(x)$ is differentiable at $x = 0$ when $n \geq 2$.

(b) Since the function is not differentiable at $x = 0$ when $n = 1$, so $f(x)$ is not continuously differentiable at $x = 0$ as well. So we just need to consider the case when $n \geq 2$.

For $x \neq 0$, one can deduce that

$$f'(x) = \frac{d}{dx}x^n \sin\frac{1}{x} = nx^{n-1}\sin\frac{1}{x} - x^{n-2}\cos\frac{1}{x}.$$

- When $n = 2$, we have $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$.

  One can show that $\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$ does not exists.

- When $n \geq 3$, we have

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(n\underbrace{x^{n-1}}_{\to 0}\underbrace{\sin\frac{1}{x}}_{bounded} - \underbrace{x^{n-2}}_{\to 0}\underbrace{\cos\frac{1}{x}}_{bounded}\right) = 0 = f'(0).$$

So we deduce that $f(x)$ is continuously differentiable at $x = 0$ only when $n \geq 3$.

Problem 2

We let $f: (a, b) \to \mathbb{R}$ be a function and let $x_0 \in (a, b)$.

(a) If f is differentiable at $x = x_0$, show that

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h} = f'(x_0) \ldots\ldots (*)$$

(b) If $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}$ exists, is it necessary that $f(x)$ is differentiable at $x = x_0$? Explain your answer.

(☺Hint: If your answer is yes, give a mathematical proof. If your answer is no, give a counter example).

☺Solution

(a) We define a function $g(x)$ by

$$g(x)=\begin{cases}\dfrac{f(x)-f(x_0)}{x-x_0} & for\ \ x\neq x_0\\ f'(x_0) & for\ \ x=x_0\end{cases}.$$

Since $f(x)$ differentiable at $x=x_0$ so that $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=f'(x_0)\Rightarrow$ $\lim_{x\to x_0}g(x)=g(x_0)$, then g is continuous at $x=x_0$.

Then it follows that

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}=\lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)}{2h}-\frac{f(x_0-h)-f(x_0)}{2h}\right]$$
$$=\lim_{h\to 0}\left[\frac{1}{2}\frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0}+\frac{1}{2}\frac{f(x_0-h)-f(x_0)}{(x_0-h)-x_0}\right]$$
$$=\frac{1}{2}\lim_{h\to 0}[g(x_0+h)+g(x_0-h)]=\frac{1}{2}\big(g(x_0)+g(x_0)\big)=f'(x_0).$$

(b) The answer is no. To see this, we consider $f(x)=|x|$.

- One can show that $f(x)=|x|$ is not differentiable at $x_0=0$ since
$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x}{x}=\lim_{x\to 0^+}1=1\quad and$$
$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}(-1)=-1,$$
So that the limits $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exist.
- On the other hand, we can deduce that
$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0-h)}{2h}=\lim_{h\to 0}\frac{f(h)-f(-h)}{2h}\overset{|-h|=|h|}{=}\lim_{h\to 0}\frac{|h|-|h|}{2h}$$
$$=0.$$

Problem 3

We let $f:(0,1]\to\mathbb{R}$ be a differentiable function on $(0,1]$ such that $|f'(x)|<M$ for all $x\in(0,1]$, where $M>0$ is a positive number. For any $n\in\mathbb{N}$, we define

$$a_n=f\left(\frac{1}{n}\right).$$

Show that the sequence $\{a_n\}$ converges.

(☺Hint: Be careful that $f(0)$ is not defined since the domain of f is $(0,1]$. On the other hand, you can prove the convergence without finding the limits.)

☺Solution

For any $\varepsilon>0$, we can deduce from Archimedean property that there exists $K\in\mathbb{N}$ such that $K>\frac{M}{\varepsilon}\Leftrightarrow\frac{1}{K}<\frac{\varepsilon}{M}$.

For any $m>n\geq K$, one can apply mean value theorem on $f(x)$ over the interval $\left[\frac{1}{m},\frac{1}{n}\right]$ and deduce that

$$|a_m - a_n| = \left|f\left(\frac{1}{m}\right) - f\left(\frac{1}{n}\right)\right| = \left|\frac{f\left(\frac{1}{m}\right) - f\left(\frac{1}{n}\right)}{\frac{1}{m} - \frac{1}{n}}\right|\left|\frac{1}{m} - \frac{1}{n}\right| = \underbrace{|f'(c)|}_{\substack{where \\ c\in\left(\frac{1}{m},\frac{1}{n}\right)}}\left|\frac{1}{m} - \frac{1}{n}\right|$$

$$< M\left|\frac{1}{m} - \frac{1}{n}\right| < M\left(\frac{1}{n} - \frac{1}{m}\right) < \frac{M}{n} \leq \frac{M}{K} < M\left(\frac{\varepsilon}{M}\right) = \varepsilon.$$

So we deduce that $\{a_n\}$ is Cauchy sequence and hence converges.

Problem 4

We let $f: [a, b] \to \mathbb{R}$ be n-times differentiable function which $f(x) = 0$ has $n + 1$ distinct roots over $[a, b]$. Show that there exists $c \in (a, b)$ such that $f^{(n)}(c) = 0$.

☺Solution

We shall prove the following lemma.

Lemma:

Suppose that $f^{(k)}(x) = 0$ has at least m distinct roots over $[a, b]$ (where $m \geq 2$ and $0 \leq k \leq n - 1$), then there exists $c_1, c_2, \dots, c_{m-1} \in (a, b)$ such that $f^{(k+1)}(c_i) = 0$ for $i = 1,2, \dots, m - 1$.

Proof

We let $a_1, a_2, \dots, a_m$ (with $a_1 < a_2 < \cdots < a_m$) be m roots of $f^{(k)}(x) = 0$.

For any $j = 1,2, \dots, m - 1$, we apply the Rolle's theorem on the function $f^{(k)}(x)$ over the interval $[a_j, a_{j+1}]$ and deduce that there exists $c_j \in (a_j, a_{j-1})$ such that

$$\left(f^{(k)}\right)'(c_j) = f^{(k+1)}(c_j) = 0.$$

Since $f(x)$ has $(n + 1)$ distinct roots, then it follows that $f'(x)$ has at least n roots by the above lemma. Then this implies that $f''(x)$ has at least $n - 1$ roots. By repeating this argument, one can deduce that $f^{(n)}(x)$ has at least one root and there exists $c \in (a, b)$ such that $f^{(n)}(c) = 0$.

Problem 5

Show that for any $x > 0$,

$$1 - x + \frac{x^2}{2} > e^{-x} > 1 - x.$$

☺Solution

We take $f(x) = e^{-x}$, we have $f'(x) = -e^{-x}$, $f''(x) = e^{-x}$ and $f'''(x) = -e^{-x}$.

- By applying Taylor theorem with $n = 1$, we have

  $$e^{-x} = f(x) = f(0) + f'(0)x + \frac{f''(c_1)}{2!}x^2 = 1 - x + \frac{e^{-c_1}}{2}x^2, \quad c_1 \in (0, x)$$

  As $x > 0$ and $e^{-c_1} > 0$, we have

  $$e^{-x} > 1 - x.$$

- By applying Taylor theorem with $n = 2$. We have

  $$e^{-x} = f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(c_2)}{3!}x^3$$

  $$= 1 - x + \frac{x^2}{2} - \frac{e^{-c_2}}{3!}x^3, \qquad c_2 \in (0, x).$$

As $x > 0$ and $e^{-c_1} > 0$, we have

$$e^{-x} < 1 - x + \frac{x^2}{2}.$$

Problem 6 (Harder)

We let $f: [0,1] \to \mathbb{R}$ be a twice differentiable function on $[0,1]$ and $f''(x)$ is continuous on $[0,1]$. Suppose that

- $f(0) = f(1) = 0$ and
- $|f''(x)| \leq A$ for all $x \in [0,1]$, where $A > 0$ is a constant.

Show that $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{A}{4}$.

(☺Hint: Apply Taylor theorem with suitable choice of a.)

☺Solution

By applying Taylor expansion with $a = \frac{1}{2}$, we deduce that there exists $c_x \in (a, x)$ such that

$$f(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{f''(c_x)}{2!}\left(x - \frac{1}{2}\right)^2.$$

- By taking $x = 0$, we get

$$\underbrace{f(0)}_{=0} = f\left(\frac{1}{2}\right) - \frac{1}{2}f'\left(\frac{1}{2}\right) + \frac{f''(c_0)}{8} \ldots\ldots (1)$$

- By taking $x = 1$, we get

$$\underbrace{f(1)}_{=0} = f\left(\frac{1}{2}\right) + \frac{1}{2}f'\left(\frac{1}{2}\right) + \frac{f''(c_1)}{8} \ldots\ldots (2)$$

By $(2) - (1)$, we have

$$0 = f'\left(\frac{1}{2}\right) + \frac{f''(c_1)}{8} - \frac{f''(c_0)}{8} \Rightarrow f'\left(\frac{1}{2}\right) = \frac{f''(c_0)}{8} - \frac{f''(c_1)}{8}.$$

Since $|f''(x)| \leq A \Rightarrow -A \leq f''(x) \leq A$, we get

$$-\frac{A}{4} = \frac{-A}{8} - \frac{A}{8} \leq f'\left(\frac{1}{2}\right) \leq \frac{A}{8} - \frac{-A}{8} = \frac{A}{4}.$$

So we have $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{A}{4}$.