Tutorial 5.

Prove by defination for (1) (2).

defination of $\lim X_n = a$

$\forall \varepsilon>0$. $\exists K \in \mathbb{N}$ when $n>K$

$|X_n - a| < \varepsilon$.

(1). $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

Let $\sqrt[n]{n} - 1 = X_n$. $\sqrt[n]{n} = X_n + 1$

so $n = (X_n+1)^n = 1 + nX_n + \frac{n(n-1)}{2} X_n^2 + \dots$

$X_n > 0$.

so $n = (X_n+1)^n > 1 + \frac{n(n-1)}{2} X_n^2$

so $\frac{n(n-1)}{2} X_n^2 < n-1$

$X_n^2 < \frac{2}{n}$. $|X_n| < \sqrt{\frac{2}{n}}$

This step just tell you how to use defination. ~~not ness~~

for $\forall \varepsilon>0$ when $n > \frac{2}{\varepsilon^2}$. we have $|X_n| < \varepsilon$

so $\lim\limits_{n\to\infty} X_n = 0$ $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

you can directly prove from $|X_n| < \sqrt{\frac{2}{n}}$ and let $n\to\infty$

(2). $\lim\limits_{n\to\infty} \frac{n}{a^n} = 0$ $(a>1)$

Let $a = 1+h$. $(h>0)$

$\frac{n}{(1+h)^n} < \frac{n}{\frac{n(n-1)}{2} h^2} = \frac{2}{(n-1)h^2}$

just tell you how to use defination.

you can directly prove from. $\frac{n}{a^n} < \frac{2}{(n-1)h^2}$ and let $n\to\infty$

for $\forall \varepsilon>0$. when. $n > \frac{2}{\varepsilon h^2} + 1$

we have. $\frac{2}{(n-1)h^2} < \varepsilon$.

so $\frac{n}{(1+h)^n} < \varepsilon$.

so $\lim\limits_{n\to\infty} \frac{n}{a^n} = 0$

you can prove. $\lim\limits_{n\to\infty} \frac{n^k}{a^n} = 0$ $k \geq 1$

(3). If $\lim_{n\to\infty} x_n = a$. $a \in \mathbb{R}$. Then $\{x_n\}$ is bdd $\exists M$, $|x_n| \le M$ for all n.

for $\forall \varepsilon > 0$. $\exists k_1$ when $n \geq k_1$ $|x_n - a| < \varepsilon$.

$$\text{so } \left|\frac{x_1 + \cdots + x_n}{n} - a\right| = \left|\frac{(x_1-a) + (x_2-a) + \cdots + (x_n-a)}{n}\right|$$

$$= \left|\frac{(x_1-a) + \cdots + (x_{k_1-1}-a) + (x_{k_1}-a) + \cdots + (x_n-a)}{n}\right|$$

$$\leq \frac{|x_1-a| + \cdots + |x_{k_1-1}-a| + |x_{k_1}-a| + \cdots + |x_n-a|}{n}$$

$$\leq \frac{M\cdot(k_1-1) + \varepsilon\cdot(n-k_1)}{n}$$

Let $K = \max\{k_1, \frac{(k_1-1)M}{\varepsilon}\}$.

when $n > K$.

$n > k_1$, $n > \frac{(k_1-1)M}{\varepsilon}$

$\frac{M(k_1-1)}{n} < \varepsilon$. $\frac{\varepsilon(n-k_1)}{n} < \varepsilon$.

so $\left|\frac{x_1+\cdots+x_n}{n} - a\right| < 2\varepsilon$. $\lim \frac{x_1+\cdots+x_n}{n} = a$.

If $\lim_{n\to\infty} x_n = +\infty$ Then ~~$\exists M$. k_1 when $n \geq k_1$ for $\forall M > 0$.~~

for $\forall N > 0$, $\exists k_1 > 0$, when $n \geq k_1$ ~~when $|x_n| \geq M$ $x_n \geq M$~~

$x_n > N$

when $n < k_1$ $|x_n|$ is bdd by $\max\{|x_1|, |x_2|, \ldots |x_{k_1-1}|\} = M$

$$\frac{x_1+\cdots+x_n}{n} = \frac{x_1+\cdots+x_{k_1-1}+x_{k_1}+\cdots+x_n}{n} > \frac{-|x_1|-|x_2|-\cdots-|x_{k_1-1}|}{n} + \frac{(n-k_1)N}{n}$$

$$\geq -\frac{M}{n} + \frac{(n-K_1)}{n} N$$

~~for $\forall N>0$~~ when $n \geq 2K_1$ $\quad \frac{n-K_1}{n} = 1 - \frac{K_1}{n} \geq \frac{1}{2}$

$\forall \varepsilon > 0$ $\exists K_2$, when $n > K_2$. $\left| -\frac{M}{n} \right| < \varepsilon$.

Let $K = \max\{2K_1, K_2\}$

when $n > K$. $\frac{x_1 + \cdots + x_n}{n} > -\varepsilon + \frac{N}{2}$

so $\lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} = +\infty$

If $\frac{x_1 + \cdots + x_n}{n} \to a$

we can't get $x_n \to a$.

$x_n = (-1)^n$

If $\lim_{n\to\infty} x_n = -\infty$ similarly.

(4). If $x_n > 0$ $\lim_{n\to\infty} x_n = a$ $\quad (a > 0$ or $a = +\infty)$

so. $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{a}$ $\quad \left(\frac{1}{a} > 0 \text{ or } \frac{1}{x_n} \to 0\right)$

$$\lim_{n\to\infty} \frac{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}{n} = \frac{1}{a}$$

$$\lim_{n\to\infty} \left(\frac{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}{n}\right)^{-1} = a.$$

(5). $x_n > 0$ $\lim_{n\to\infty} x_n = a$ $\quad a \geq 0$ or $a = +\infty$

If $a = 0$. $\quad 0 < \sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}$

If $a > 0$ or $a = +\infty$ $\quad \left(\frac{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}{n}\right)^{-1} \leq \sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}$

(6). If $\lim\limits_{n\to\infty} \frac{X_{n+1}}{X_n} = a$ $X_n > 0$ $a \geqslant 0$ or $a = +\infty$

$$\lim_{n\to\infty} \sqrt[n]{\frac{X_2}{X_1} \cdot \frac{X_3}{X_2} \cdots \frac{X_{n+1}}{X_n}} = a.$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{X_{n+1}}{X_1}} = a. \qquad \lim_{n\to\infty} \sqrt[n]{\frac{1}{X_1}} = 1$$

$$\lim_{n\to\infty} \sqrt[n]{X_{n+1}} = \lim_{n\to\infty} \left(X_{n+1}^{\frac{1}{n+1}}\right)^{\frac{n+1}{n}} = a$$

$$\lim_{n\to\infty} X_{n+1}^{\frac{1}{n+1}} = a.$$

because If $\lim X_n = a$. $a \geqslant 0$ ~~or $a = +\infty$~~

Then $\lim X_n^{\frac{n}{n+1}} = \lim\limits_{n\to\infty} X_n \cdot X_n^{-\frac{1}{n+1}} = a \cdot 1 = a$

If $\lim X_n = +\infty$ Then $\lim X_n^{\frac{n}{n+1}} = +\infty$

If $\lim \sqrt[n]{X_n} = a$ we can't get $\lim \frac{X_{n+1}}{X_n} = a$.

~~(7). In series. root test can be applied more than ratio test~~

(7). Let $X_n = \cancel{\frac{n}{\sqrt[n]{n!}}} \frac{n^n}{n!}$ $\frac{X_{n+1}}{X_n} = \left(\frac{n+1}{n}\right)^n$

$$\lim_{n\to\infty} \frac{X_{n+1}}{X_n} = e \qquad \lim_{n\to\infty} \sqrt[n]{X_n} = \frac{n}{\sqrt[n]{n!}} = e.$$

(8). $\sqrt[n]{n} \to 1$

$$\sqrt[n]{a^{\log_a n}} \to 1. \qquad a^{\frac{\log_a n}{n}} \to 1 \qquad a > 1$$

$\frac{\log_a n}{n} \to 0 = \log_a 1$ because. $\log_a X$ is a countious function

(9) Let $C_n = \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{n} - \ln n$ $\lim\limits_{n\to\infty} C_n = \gamma$

$$C_{2n} - C_n = \frac{1}{n+1} + \dots + \frac{1}{2n} - \ln 2$$

$$\lim_{n\to\infty} \frac{1}{n+1} + \dots + \frac{1}{2n} = \ln 2.$$