MATH 2031 Introduction to Real Analysis

October 21, 2012

Tutorial Note 6

Real Numbers

(I) **Definition:**
A supremum (least upper bound) of a non-empty bounded above set S, denoted by $\sup S$, is an upper bound $\widetilde{M}$ of S such that $\tilde{M} \leq M$ for any upper bound of S
A infimum (greatest lower bound) of a non-empty bounded below set S, denoted by $\inf S$, is an lower bound $\widetilde{m}$ of S such that $\tilde{m} \geq m$ for any lower bound of S

(II) Infinitesimal Principle:
Let $x, y \in \mathbb{R}$, $x < y + \varepsilon$ for all $\varepsilon > 0 \iff x \leq y$.

(III) Supremum Property:
If a set S has a supremum in $\mathbb{R}$ and $\varepsilon > 0$, then $\exists x \in S$ such that

$$\sup S - \varepsilon < x \leq \sup S$$

Infimum Property:
If a set S has a infimum in $\mathbb{R}$ and $\varepsilon > 0$, then $\exists x \in S$ such that

$$\inf S \leq x < \inf S + \varepsilon$$

(IV) Archimedean Principle:
$\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > x$

(V) Density of $\mathbb{Q}$:
If $x < y$, then $\exists \frac{m}{n} \in \mathbb{Q}$ such that $x < \frac{m}{n} < y$.

Density of $\mathbb{R} \setminus \mathbb{Q}$:
If $x < y$, then $\exists w \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < w < y$.

(VI) **Definition:**
A sequence $x_1, x_2, \cdots$ converges to a number x, written as $\lim_{n\to\infty} x_n = x$, iff $\forall \varepsilon > 0, \exists K \in \mathbb{N}$(depends on ε) such that $n \geq K \Rightarrow |x_n - x| < \varepsilon$.

(VII) Supremum Limit Theorem:
Let c be an upper bound of S, then

$$\left(\exists w_n \in S \text{ such that } \lim_{n\to\infty} w_n = c\right) \iff c = \sup S.$$

Infimum Limit Theorem:
Let c be an lower bound of S, then

$$\left(\exists w_n \in S \text{ such that } \lim_{n\to\infty} w_n = c\right) \iff c = \inf S.$$

Problem 1 Find the sup and inf of each of the following sets of real numbers.

(i) $S = \{x + y | x, y \in \left[\frac{1}{2}, 1\right)\} \setminus \{2 - \frac{1}{n} | n \in \mathbb{N}\}$

(ii) $S = \{\frac{k}{n!} | k, n \in \mathbb{N}, \frac{k}{n!} < \sqrt{2}\}$

Solution:

(i) First we find the upper bound and lower bound of S, $1 = \frac{1}{2} + \frac{1}{2} \leq x + y < 1 + 1 = 2$, so $S \subseteq [1, 2)$.
Then we try to find a sequence in S converges to the upper bound 2 and a sequence in S converges to the lower bound 1.
Here we should beware that $\{2 - \frac{1}{n} | n \in \mathbb{N}\}$ is not in S, so we could not simply take $x_n = y_n = 1 - \frac{1}{2n}$, which gives $x_n + y_n = 2 - \frac{1}{n}$. Instead, we could take $x_n = y_n = 1 - \frac{1}{2n\pi}$, since π is irrational, $w_n = x_n + y_n = 2 - \frac{1}{n\pi}$ which is not in $\{2 - \frac{1}{n} | n \in \mathbb{N}\}$, so $w_n \in S$. And $\lim_{n\to\infty} w_n = \lim_{n\to\infty} \left(2 - \frac{1}{n\pi}\right) = 2$.

Also by similar argument, we could take $x_n = y_n = \frac{1}{2} + \frac{1}{2n\pi}$, so $w_n = x_n + y_n = 1 + \frac{1}{n\pi} \in S$ and $\lim_{n\to\infty} w_n = \lim_{n\to\infty} \left(1 + \frac{1}{n\pi}\right) = 1$.
Therefore the $\sup S = 2$ and $\inf S = 1$.

(ii) Since $k, n \in \mathbb{N}$, both k, n are positive and $\frac{k}{n!} < \sqrt{2}$, we get that the lower bound of S is 0 and upper bound is $\sqrt{2}$.
For infimum, we could take $w_n = \frac{1}{n!}$, since $n! \geq 1!$ for all $n \in \mathbb{N}$, we get $w_n = \frac{1}{n!} \leq \frac{1}{1!} = 1 < \sqrt{2}$, so $w_n \in S$ and $\lim_{n\to\infty} w_n = \lim_{n\to\infty} \frac{1}{n!} = 0$.
Thus $\inf S = 0$.
For supremum, as the upper bound is irrational while elements in S are rational, it would be hard to take a sequence in S converges to $\sqrt{2}$. So we would claim $\sup S = \sqrt{2}$ and prove it by contradiction.
Suppose $\sup S \neq \sqrt{2}$, then by definition of supremum $\sup S < \sqrt{2}$, by the density of $\mathbb{Q}$, there exist $\frac{m}{n} \in \mathbb{Q}$ such that $\sup S < \frac{m}{n} < \sqrt{2}$.
However, $\frac{m}{n} = \frac{m(n-1)!}{n!} \in S$ contradict the fact that $\sup S$ is the least upper bound of S.
Thus we get $\sup S = \sqrt{2}$.

Problem 2 Let $A, B \subset \mathbb{R}$ be bounded sets and $\inf A = -2,\ \sup A = 4,\ \inf B = 0,\ \sup B = 1$.
Then find the sup and inf of the following set.

$$S = \{x + e^y | x \in A, y \in B\}$$

Solution:
Since $\inf A = -2,\ \sup A = 4,\ \inf B = -1,\ \sup B = 1$, we get that $x \in A$, $-2 \leq x \leq 4$ and $y \in B$, $0 \leq y \leq 1$.
Also e^y is an increasing function, we get $1 = e^0 \leq e^y \leq e^1 = e$. Then the upper bound of $S = -2 + 1 = -1$ and lower bound of $S = 4 + e$
Here we should beware that -2 and 4 may not in A and 0 and 1 may not in B.

Since $\inf A = -2$ and $\inf B = 0$, we get that there is a sequence $x_n \in A$ converges to -2 and a sequence $y_n \in B$ converges to 0.
Since e^y is continuous, then e^{y_n} is sequence converge to $e^0 = 1$.
Then we get $w_n = x_n + e^{y_n}$ which is in S and $\lim_{n\to\infty} w_n = \lim_{n\to\infty} (x_n + e^{y_n}) = -2 + 1 = -1$.
So the $\inf S = -1$.
By similar argument,$\sup A = 4$ and $\sup B = 1$ give us the sequences $x'_n \in A$ converges to 4 and $y'_n \in B$ converges to 1, and take $w'_n = x'_n + e^{y'_n}$ which is in S and $\lim_{n\to\infty} w'_n = \lim_{n\to\infty} (x'_n + e^{y'_n}) = 4 + e$.
So the $\sup S = 4 + e$.

Problem 3 Let $A, B \subset \mathbb{R}$ be bounded sets such that $\inf B < \inf A < \sup A < \sup B$, then

$$\inf(B \setminus A) = \inf B \text{ and } \sup(B \setminus A) = \sup B$$

Solution:
Here I would prove $\inf(B \setminus A) = \inf B$, the other part is essentially the same.

Since we have $\inf B < \inf A$, then let $k = \inf A - \inf B > 0$, then by infimum property, we have for each $n \in \mathbb{N}$, there exists $x_n \in B$ such that

$$\inf B \leq x_n < \inf B + \frac{k}{n} \leq \inf B + k < \inf B + \inf A - \inf B = \inf A$$

By definition, $\inf A$ is the greatest lower bound, so $x_n \in B \setminus A$ for each $n \in \mathbb{N}$.

Also,by sandwich theorem, $\inf B \leq x_n < \inf B + \frac{k}{n}$ implies that $\lim\limits_{n\to\infty} \inf B \leq \lim\limits_{n\to\infty} x_n \leq \lim\limits_{n\to\infty} \left(\inf B + \dfrac{k}{n}\right)$.
As $\lim\limits_{n\to\infty} \inf B = \inf B$ and $\lim\limits_{n\to\infty} \left(\inf B + \dfrac{k}{n}\right) = \inf B$, we get $\lim\limits_{n\to\infty} x_n = \inf B$.