MATH202 Introduction to Analysis (2007 Fall-2008 Spring)
Extra Practice Problems for Final

You may try to work on the following exercises to get familiar with the materials.

Integral Criterion and Lebesgue Theorem

☺Exercise 1

Show that the function

$$f(x) = |\cos x|$$

is Riemann Integrable on $[0,2\pi]$ using integral criterion. (Hint: Draw graph first)

☺Exercise 2

Determine whether the function

$$f(x) = \begin{cases} 2x & \text{if } x \in \left[0, \frac{1}{2}\right) \\ x - 2 & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

is Riemann Integrable or not on $[0,1]$ by **using a) Lebesgue Theorem, b) Integral Criterion**

(Hint: You may draw the figure to help you and here note that $f(x)$ is not continuous at $x = \frac{1}{2}$. Use Lebesgue Theorem first to see whether the function is Riemann Integrable or not (It is an important technique!!). Next, when using integral criterion, when you draw your partition, note that the function is discontinuous at $x = \frac{1}{2}$, so you need to first draw a $\delta - \text{interval}$ around $x = \frac{1}{2}$ and for the remaining part, apply uniform cutting (since the function is continuous)

☺Exercise 3

Determine whether the function defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \in \mathbf{R} \backslash \mathbf{Q} \end{cases}$$

is Riemann Integrable or not on $[0,1]$ by **using a) Lebesgue Theorem, b) Integral Criterion .**

☺Exercise 4

Let $f_1(x), f_2(x), f_3(x)$ be bounded Riemann Integrable function on $[a,b]$ and define

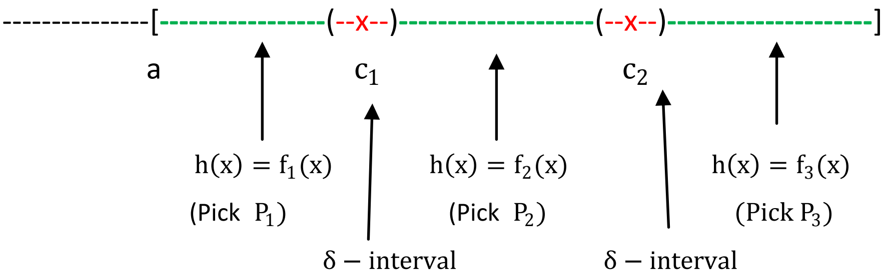
$$h(x) = \begin{cases} f_1(x) \text{ if } x \in [a, c_1] \\ f_2(x) \text{ if } x \in (c_1, c_2] \\ f_3(x) \text{ if } x \in (c_2, b] \end{cases}$$

where $a < c_1 < c_2 < b$. (Draw the simple graph).

Determine (using BOTH **a) Lebesgue Theorem, b) Integral Criterion)** whether $h(x)$ is Riemann Integrable on not

Hint: In part b), when you draw the partition, since $h(x)$ may be discontinuous at $x = c_1$ or $x = c_2$. So you first draw $\delta -$ intervals around $x = c_1$ and $x = c_2$ respectively. Then for other parts, since f_1, f_2 and f_3 are Riemann Integrable, there exists P_1, P_2, P_3 on $[0,1]$ such that $U(P_i, f_i) - L(P_i, f_i) < \frac{\varepsilon}{5}$ for $i = 1,2,3$

Draw you partition by fill in the blank below



☺Exercise 5

Let $f(x)$ be Riemann Integrable on $[0,2]$ and $g(x)$ is monotone function on $[0,2]$, define

$$h(x) = \begin{cases} f(x) + g(x) & \text{if } x \in [0,,1) \\ |f(x)| & \text{if } x \in [1,2] \end{cases}$$

Determine with proof whether the $f(x)$ is Riemann Integrable on $[0,2]$

Improper Integral and Cauchy principal Value

☺Exercise 6

Determine the convergence of the following integrals

a) $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$

b) $\int_{-2}^2 \frac{dx}{\sqrt{4-x^2}}$

c) $\int_0^\infty \frac{\sqrt{x}+1}{\sqrt[3]{(x^{12}-3\sqrt{x}+2}} dx$

d) $\int_0^{2\pi} \frac{1}{\sin x} dx$ (Hint: $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$)

e) $\int_0^{\infty} x\sin(e^x) dx$ (Hint: Let $u = e^x$)

☺Exercise 7

Determine all possible α such that the improper integral

$$\int_0^{\pi} \frac{1}{x(\sin x)^\alpha} dx$$

Converges.

(Hint: Since there are two trouble points (at $x = 0, x = \pi$), so we split the integral into 2 parts:

$$\int_0^{\pi} \frac{1}{x(\sin x)^\alpha} dx = \int_0^{\frac{\pi}{2}} \frac{1}{x(\sin x)^\alpha} dx + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{x(\sin x)^\alpha} dx$$

Note that $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$. For second integral, use the substitution $u = \pi - x$ and note that $\sin(\pi - x) = \sin x$. (Try to understand why we need to make this substitution)

☺Exercise 8

Determine all possible $\alpha, \beta \in \mathbf{R}$ such that the integral

$$\int_0^{\frac{\pi}{2}} \frac{x^\beta}{(\cos x)^\alpha} dx$$

Converges.

(Hint: Once again, find all trouble points and split the integral if necessary, note that $\cos\left(\frac{\pi}{2} - x\right) = \sin x$, the techniques used in Exercise 8 may be useful)

☺Exercise 9

Determine all possible α such that

$$\int_0^{2} \frac{|1 - x^\alpha|}{x^\alpha(2 - x)} dx$$

Converges

☺Exercise 10

Compute the Cauchy Principal Value of the following integrals

a) $\int_{-\infty}^{\infty} \frac{1+x}{1+x^2} dx$ b) $\int_{\frac{1}{2}}^{2} \frac{1}{x\ln x} dx$ c) $\int_{-\infty}^{+\infty} \frac{x}{(\cos x)(1+x^2)} dx$ (Note that it is odd function)

Domain of Convergence and M-test

☺Exercise 11 (Tutorial Note #26 Exercise 1)

Find the domain of convergence and radius of convergence (if the series is power series) of the following power series

a) $\sum_{n=1}^{\infty} \frac{(x+3)^n}{(n+2)2^n}$

b) $\sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n}$

c) $\sum_{n=1}^{\infty} \frac{(n!)^2(2n+2)!}{(2n)!\,[(n+1)!]^2} x^n$

d) $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2 4^n}$

$*$ e) $\sum_{n=1}^{\infty} a_n x^n$ where $a_n = \begin{cases} \frac{1}{2^n} & \text{if n is even} \\ \frac{1}{3} & \text{if n is odd} \end{cases}$

f) $\sum_{k=1}^{\infty} \frac{e^{kx}}{k+1}$

☺Execise 12 (Tutorial Note #27 Exercise 2)

Show that the following series of functions converges uniformly on indicated intervals

a) $\sum_{n=1}^{\infty} n^2 x^n$ on $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

b) $\sum_{n=1}^{\infty} \frac{\cos nx}{5^n}$ on $x \in R$

c) $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ on $x \in [0,1]$

(Use Calculus to help you)

d) $\sum_{k=1}^{\infty} \frac{kx}{e^{kx}}$ on $[r, \infty)$, where $r > 0$

(Use Calculus to help you)

e) $\sum_{k=1}^{\infty} k^2 x^k$ on $[-r, r]$ where $0 < r < 1$

Big-O, Small-O notation and stole's theorem

☺Exercise 13

Work on some of Practice Exercise #302~#310

☺Exercise 14

Work on Practice Exercise #311-#317 (If you have time, work on #318, #319)

Cauchy Sequence

☺Exercise 15

Let $\{x_n\}$ and $\{y_n\}$ are two Cauchy Sequence

Show, by definition of Cauchy Sequence that the sequence $\left\{x_n + \frac{y_n}{3}\right\}$ and $\left\{\frac{x_n}{1+y_n}\right\}$ are also Cauchy

☺Exercise 16

Suppose $\{a_n\}$ be Cauchy, and let $\{b_n\}$ be a sequence satisfying

$$a_n - \frac{1}{2^n} < b_n < a_n + \frac{1}{2^n}$$

Show that $\{b_n\}$ is also Cauchy by checking the definition

(Hint: Rearrange the inequality, to get $? < b_n - a_n < ?$

☺Exercise 17

Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $\{a_n + b_n\}$ and $\{a_n - b_n\}$ are both Cauchy, show by definition by Cauchy that $\{a_n\}$ is Cauchy.

(Hint: $a_n = \frac{1}{2}(a_n + b_n) + \frac{1}{2}(a_n - b_n)$

Taylor Theorem

Redo the problem in Extra Credit Quiz, Midterm and some presentation problems.