Lecture 5

21-02-2019

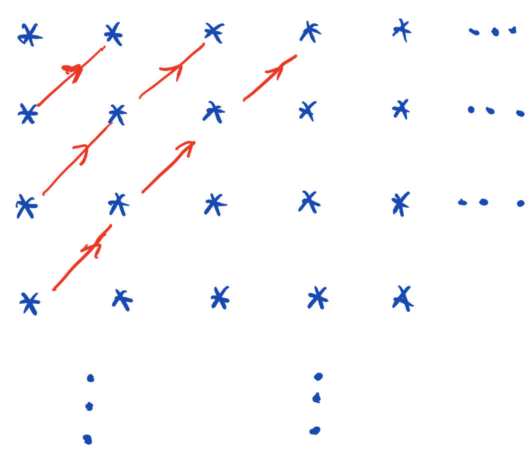
Review :

① Def: A set S is countable iff either S is finite or $\exists$ a bijection $f: \mathbb{N} \to S$

② Bijection theorem : If $S \sim T$. Then

(S is countable $\Leftrightarrow$ T is countable)

③ Important counting method : Diagonal counting scheme



Remark : the case when some rows or columns are finite is also applicable , by letting the

Corresponding "*" to be blank.

Consequences: ① Countable union theorem

② product theorem: $A_1 \times A_2$ is countable if both A_1 and A_2 is. eg. $\mathbb{N} \times \mathbb{N}$.

④ Useful method to show countability

THM 1. Countable subset theorem: $A \subseteq B$, then B countable $\Rightarrow$ A countable

THM 2. Injection theorem: $f: A \to B$ injective, then B countable $\Rightarrow$ A countable

Proof: $A \sim f(A) \subseteq B$. then use countable subset theorem.

THM 3. Surjective theorem: $f: A \to B$ surjective, then A countable $\Rightarrow$ B countable

Proof: $B = f(A) = \bigcup_{x \in A} \{f(x)\}$, then use countable union theorem

THM 4. Countable union theorem: $\bigcup_{i \in \Lambda} A_i$ is countable if Λ and each A_i is countable.

THM 5. Product theorem: $A_1 \times A_2$ is countable if A_1, A_2 are

⑤ Uncountable sets.

Example A : the interval $(0,1)$

Standard proof : By contradiction, assume countable,

$\Rightarrow$ a listing of all elements in $(0,1)$. Using the base-10 numeral representation of real numbers to construct a real number in $(0,1)$ which is not in the list.

Example B : $\bigtimes_{i\in\mathbb{N}} A_i$, where $A_i = \{0,1\}$

Proof : Construct a surjective $f: \bigtimes_{i\in A_i} A_i \to (0,1)$

using the binary representation of real numbers

Example C : $P(\mathbb{N}) \simeq 2^{\mathbb{N}}$.

Proof : Construct a bijection $f: 2^{\mathbb{N}} \to \bigtimes_{i\in\mathbb{N}} A_i$

where $A_i = \{0,1\}$.

Example 11. Is every real number a root of some polynomial with integer coefficients?

Solution: Let S be the set of all polynomials with integer coefficients. [See example 6]. Then S is countable. For each $f \in S$, denote R_f the set of roots of f. Then R_f is finite.

Therefore $T = \bigcup_{f \in S} R_f$ is countable by countable union theorem. Hence $\mathbb{R} \setminus T$ is uncountable.

Note that T is the set of all roots of polynomials with integer coefficients. We see that there are uncountable many real numbers which are not root of polynomials of integer coefficients. Such numbers are call transcendental, such as π, e.

Other examples.

12. Every interval (a, b) with $a<b$ is uncountable.

13 $A = \{ r\sqrt{m} : m \in \mathbb{N}, r \in (0,1) \}$ is uncountable.

14. $B = \{ r\sqrt{m} : m \in \mathbb{N}, r \in (0,1) \cap \mathbb{Q} \}$ is countable

15. If A_1 is uncountable, then $A_1 \times A_2$ is uncountable if $A_2 \neq \phi$.

Continuum Hypothesis. [1-st in Hilbert's 23 problems]

Recall that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \overset{?}{\subseteq} \mathbb{R} \subseteq \mathbb{C}$

$\underbrace{\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}}_{\text{Countable}}$; $\mathbb{R}$ uncountable

$$\mathrm{Card}\,\mathbb{Q} = \aleph_0 \overset{?}{<} \mathrm{Card}\,\mathbb{R} = \mathfrak{c}.$$

Continuum Hypothesis: For every uncountable set S, $\exists$ injective

$$f : \mathbb{R} \to S$$

[or $\mathrm{Card}\, S \geq \mathrm{Card}\,\mathbb{R} = \mathfrak{c}$, or $\mathrm{Card}\,\mathbb{R}$ is the smallest among all cardinal numbers of uncountable set]

Question: Is the C-H true or not.

1940: Kurt Gödel proved that $\sim$ C-H would not lead to any contradiction [in ZFC set theory : Zermelo - Fraenkel set theory with the axiom of choice included]

1964 : Paul Cohen proved that C-H would not lead to any contradiction [in ZFC set theory]

(He won the Fields' medal for this work)

Gödel and Cohen's work show that C-H cannot be proved nor disproved from ZFC. Hence C-H is independent of ZFC.

Chapter 4 Real number

Def : The set of all real numbers (denoted by $\mathbb{R}$) is a set satisfying the following axioms:

① Field Axiom ② Order Axiom

③ Well-ordering Axiom ④ Competeness Axiom

Remark : ① an axiom is a self-evident statement that is assumed to be foundamental in order to obtain consequences (such as theorem) by deduction.

② An Axiomatic system consists of a set of axioms. Theorems are deduced from axioms using rules of inference. All the derived theorems and the axioms form a theory, such as "theory of real numbers", "theory of Eclidean geometry".

Field Axiom

Field Axiom : $\mathbb{R}$ has two operations $+$ and $\cdot$ such that

$\forall\ a, b, c \in \mathbb{R}$

(i) $a+b$, $a \cdot b \in \mathbb{R}$

(ii) $a+b = b+a$, $a \cdot b = b \cdot a$

(iii) $(a+b)+c = a+(b+c)$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(iv) $\exists$ unique elements $0, 1 \in \mathbb{R}$ with $1 \neq 0$

such that $a+0 = a$, $a \cdot 1 = a \quad \forall\ a \in \mathbb{R}$.

(v) $\exists\ -a \in \mathbb{R}$ such that $a+(-a) = 0$;

if $a \neq 0$, then $\exists\ a^{-1} \in \mathbb{R}$, such that $a \cdot a^{-1} = 1$

[One can use (ii), (iii) to show that $-a$, a^{-1} is unique]

(vi). $a \cdot (b+c) = a \cdot b + a \cdot c$.

Order Axiom

Order Axiom : $\mathbb{R}$ has an ordering relation " $<$ " such that $\forall a, b, c \in \mathbb{R}$,

(i) exactly one of the following is true: $a < b$, $a = b$, $b < a$.

(ii) if $a < b$, $b < c$, then $a < c$.

(iii) if $a < b$, then $a + c < b + c$.

(iv) if $a < b$, and $0 < c$, then $ac < bc$.

Remark : ① Order Axiom are not satisfied for $\mathbb{C}$.

② We say that a is strictly smaller than b if $a < b$.

We also write $a > b$ iff $b < a$, and say that a is strictly greater than b.

We define $a \leq b$ iff $a < b$ or $a = b$

$a \geq b$ iff $a > b$ or $a = b$

③ Let $\{a_1, a_2, \cdots a_n\} \subset \mathbb{R}$. We denote

by $\text{Max}(a_1, \cdots a_n)$ to be the greatest element in

the set, and $\text{Min}(a_1, \cdots a_n)$ the smallest element.

Exercise : show that $\text{Max}(a_1, \cdots a_n)$ exists.

④ We define $|x| = \max(-x, x)$

$$|x| \leq a \iff x \leq a \text{ and } -x \leq a$$

$$\iff -a \leq x \leq a$$

We have the triangle inequality :

$$\forall\, x, y \in \mathbb{R}, \quad |x+y| \leq |x| + |y|$$

$$\left.\begin{array}{l} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{array}\right\} \quad \begin{array}{l} -|x|-|y| \leq x+y \leq |x|+|y| \\ = -(|x|+|y|) \end{array}$$

$$\Rightarrow \quad |x+y| \leq |x| + |y|$$

Thm:

⑤ We can show that $0 < 1$.

Proof: By contradiction, assume $\sim(0<1)$, then

$0=1$ or $0>1$. But $0 \neq 1$ So $\underline{0>1}$. or $1<0$.

Note $0 = 1 + (-1) < 0 + (-1) = -1$

So $0 < -1$.

Using $0<-1$ again, $\Rightarrow \underbrace{0\cdot(-1)}_{0} < -1\cdot(-1) = 1$

$\Rightarrow 0 < 1$. Contradiction. $\Rightarrow 0<1$.

Exercise: show that $(-1)\cdot(-1) = 1$

As a consequence, we can further show that

$0 < 1 < 2 < 3 < \cdots$

Well-ordering Axiom

W-O-A : $\mathbb{N} = \{1, 2, 3, \cdots\}$ is well-ordered in the sense that " $\forall$ non-empty set $S \subseteq \mathbb{N}$, $\boxed{\exists}$ $m \in S$ such that $m \le x$ for all $x \in S$. " This m is the least element (or the minimum) of S.

Remark : S may be infinite.

Example : ① S = the set of all prime numbers , $m = 2$.

② S = the set of all odd numbers , $m = 1$

③ $S = (\pi, \sqrt{99}) \cap \mathbb{N}$, $m = 4$.

④ $S = \{ q : q$ is a prime number,

$q > 2019^{2019} \}$

[W-O-A is non-trivial in some sense]

Remark: ① W-O-A holds for $\mathbb{N}$, BUT DOES NOT holds for $\mathbb{Q}$.

Counterexample : $S = \{ q : q$ is a rational number

$q^2 > 2 \}$

② W-O-A is the foundation of "proof by induction."