MATH2033 Mathematical Analysis

Lecture Note 7

Differentiability

Differentiation and derivatives

Roughly speaking, differentiation aims to study the *trend* of a function (e.g. monotonicity, curvature etc.) and has a wide approximation in mathematics such as optimalization, numerical analysis etc.

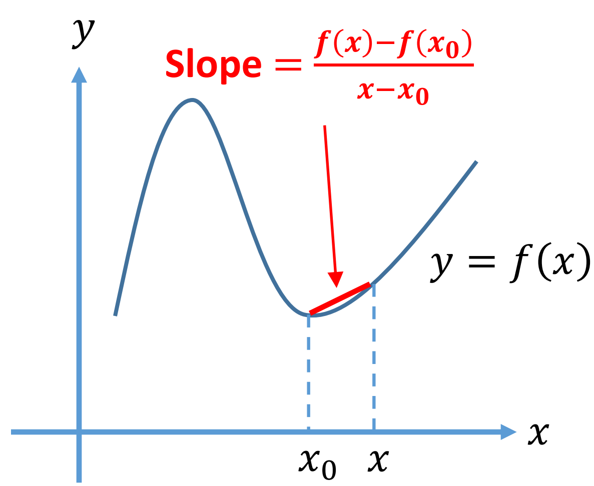
Mathematical definition of derivatives

We let $f: S \to \mathbb{R}$ be a function and let $x_0 \in S$. We would like to study the *rate of change* of the function at $x = x_0$. This can be done by considering the slope $\frac{f(x)-f(x_0)}{x-x_0}$ which reflects the change of $f(x)$ when we move from x_0 to x.

By taking x to be close to x_0 (i.e. $x \to x_0$), then the resulting limits

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is called *derivatives* of a function $f(x)$ at $x = x_0$ which measures the rate of change of f at $x = x_0$.



Definition (Differentiability and derivative)

We let $f: S \to \mathbb{R}$ be a function. We say f is differentiable at $x = x_0 \in S$ if and only if the limits $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ exists as <u>a real number</u>. (Here, $f'(x_0)$ is called *derivative* of $f(x)$ at $x = x_0$), We say f is differentiable on S if and only if it is differentiable at *any* $x \in S$.

Remark about the definition

- One has to be careful that there is no guarantee that $f'(x_0)$ exists for any function f. For example, we consider $f(x) = |x|$ and let $x_0 = 0$. Then one can show that the limits $\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{|x|}{x}$ does not exists since two one-sided limits are unequal, i.e.
$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1, \qquad \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = -1.$$
In this case, we say $f(x)$ is not differentiable at $x = 0$.
- In general, we say $f(x)$ is not differentiable at $x = x_0$ if either $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ does not exist or $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \pm\infty$ (see Example 2).

Differentiability and continuity

- If a function $f(x)$ is differentiable at $x = x_0$, then $f(x)$ is continuous at $x = x_0$. To see this, we consider

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left[f(x_0) + \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\to f'(x_0)} \underbrace{(x - x_0)}_{\to 0} \right] = f(x_0).$$

  So $f(x)$ is continuous at $x = x_0$ by definition.
- However, the converse of the statement is not true in general. For example, we take $f(x) = |x|$ and take $x_0 = 0$.
  - One can show that $|x|$ is continuous at $x = 0$ since $\lim_{x \to 0} |x| = 0$.
  - However, $|x|$ is not differentiable at $x = 0$ as shown in p.3.
- Therefore, differentiability is a "stronger" condition than continuity.
- By taking the contrapositive of the statement, one can deduce that

> "If $f(x)$ is not continuous at $x = x_0$, then $f(x)$ is not differentiable at $x = x_0$"

  This fact provides an alternative to verify a function is not differentiable.

Example 1 (Quick example of checking differentiability)

(a) Show that $f(x) = x^3$ is differentiable at any $x = x_0 \in \mathbb{R}$.

(b) Show that $g(x) = \sin x$ is differentiable at any $x = x_0 \in \mathbb{R}$.

😊Solution

> **Recall that**
> $\boldsymbol{a^3 - b^3 = (a-b)(a^2 + ab + b^2)}$

(a) Note that for any $x_0 \in \mathbb{R}$, we have

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{x^3 - x_0^3}{x-x_0} = \lim_{x\to x_0}\frac{(x-x_0)(x^2+xx_0+x_0^2)}{x-x_0}$$

$$\overbrace{=}^{x\neq x_0} \lim_{x\to x_0}(x^2+xx_0+x_0^2) = 3x_0^2.$$

So $f(x)$ is differentiable at any $x = x_0$ with $f'(x_0) = 3x_0^2$.

(b) Note that for any $x_0 \in \mathbb{R}$, we have

$$\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{\sin x - \sin x_0}{x-x_0} = \lim_{x\to x_0}\frac{2\cos\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)}{x-x_0}$$

$$\overbrace{=}^{x\neq x_0} \lim_{x\to x_0}\underbrace{\cos\left(\frac{x+x_0}{2}\right)}_{\cos x_0}\underbrace{\frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}}}_{\to 1} = \cos x_0.$$

So $g(x)$ is differentiable at any $x = x_0$ with $g'(x_0) = \cos x_0$.

Example 2

We consider the function $f(x) = \sqrt[3]{x}$, determine whether the function is differentiable at $x = c$ $(c \neq 0)$ and $x = 0$ respectively.

☺Solution:

To check the differentiability of $f(x)$ when $c \neq 0$, we consider

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{\sqrt[3]{x} - \sqrt[3]{c}}{x - c} = \lim_{x \to c} \frac{\left(\sqrt[3]{x} - \sqrt[3]{c}\right)\left[\left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{x}\right)\left(\sqrt[3]{c}\right) + \left(\sqrt[3]{c}\right)^2\right]}{(x - c)\left[\left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{x}\right)\left(\sqrt[3]{c}\right) + \left(\sqrt[3]{c}\right)^2\right]}$$

$$= \lim_{x \to c} \frac{\left(\sqrt[3]{x}\right)^3 - \left(\sqrt[3]{c}\right)^3}{(x - c)\left[\left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{x}\right)\left(\sqrt[3]{c}\right) + \left(\sqrt[3]{c}\right)^2\right]} = \lim_{x \to c} \frac{1}{\left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{x}\right)\left(\sqrt[3]{c}\right) + \left(\sqrt[3]{c}\right)^2}$$

$$= \frac{1}{3c^{2/3}} \in \mathbb{R}.$$

The function is differentiable at $x = c \neq 0$.

When $c = 0$, we have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\sqrt[3]{x}}{x} = \lim_{x \to 0} \frac{1}{\left(\sqrt[3]{x}\right)^2} = +\infty.$$

Hence, the function is not differentiable at $c = 0$.

Computation formula for derivatives

The following properties summarize some useful facts for computing the derivatives/ checking differentiability of some complicated functions.

Property 1

We let $f, g: S \to \mathbb{R}$ be two functions. Suppose that both f and g are differentiable at $x = x_0$, then the functions (1) $(f \pm g)(x) = f(x) \pm g(x)$, (2) $(fg)(x) = f(x)g(x)$ and (3) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (provided that $g(x) \neq 0$) are all differentiable at $x = x_0$ and

(1) $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$

(2) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0) \quad (\textit{product rule})$

(3) $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2} \quad (\textit{quotient rule})$

Property 2 (Chain rule)

If $f: S \to \mathbb{R}$ is differentiable at $x = x_0$ and $g: S' \to \mathbb{R}$ is differentiable at $y = f(x_0)$ (where $f(S) \subseteq S'$), then the function $(g \circ f)(x) = g\big(f(x)\big)$ is differentiable at $x = x_0$ and $(g \circ f)'(x_0) = g'\big(f(x_0)\big)f(x_0)$.

Proof of property 1

Since $f(x)$ and $g(x)$ are differentiable at $x = x_0$, we have

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \quad and \quad \lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} = g'(x_0).$$

It follows from property of limits that

$$\lim_{x\to x_0}\frac{[f(x)\pm g(x)]-[f(x_0)\pm g(x_0)]}{x-x_0} = \lim_{x\to x_0}\left[\left(\frac{f(x)-f(x_0)}{x-x_0}\right)\pm\left(\frac{g(x)-g(x_0)}{x-x_0}\right)\right]$$
$$= f'(x_0)+g'(x_0);$$

$$\lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}$$
$$= \lim_{x\to x_0}\frac{f(x)g(x)+f(x_0)g(x)-f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0}$$
$$= \lim_{x\to x_0}\left[\underbrace{g(x)}_{\substack{\to g(x_0)\\(why?)}}\underbrace{\left(\frac{f(x)-f(x_0)}{x-x_0}\right)}_{\to f'(x_0)}+f(x_0)\underbrace{\left(\frac{g(x)-g(x_0)}{x-x_0}\right)}_{\to g'(x_0)}\right]$$
$$= f'(x_0)g(x_0)+f(x_0)g'(x_0) \quad and$$

$$
\begin{aligned}
\lim_{x\to x_0} \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(x_0)}{g(x_0)}}{x - x_0} &= \lim_{x\to x_0} \frac{f(x)g(x_0) - f(x_0)g(x)}{(x - x_0)g(x)g(x_0)} \\
&= \lim_{x\to x_0} \frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{(x - x_0)g(x)g(x_0)} \\
&= \lim_{x\to x_0} \frac{g(x_0)\left(\dfrac{f(x) - f(x_0)}{x - x_0}\right) + f(x_0)\left(\dfrac{g(x) - g(x_0)}{x - x_0}\right)}{g(x)g(x_0)} \\
&= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{[g(x_0)]^2}
\end{aligned}
$$

This shows the functions $(f \pm g)$, fg and $\frac{f}{g}$ are all differentiable at $x = x_0$.

Proof of property 2

Intuitively, one may prove the rule by expressing the limits as

$$
\lim_{x\to x_0} \frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{x - x_0} = \lim_{x\to x_0} \underbrace{\left[\frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{f(x) - f(x_0)}\right]}_{\to g'(f(x_0)) \text{ as } f(x)\to f(x_0)} \underbrace{\left(\frac{f(x) - f(x_0)}{x - x_0}\right)}_{\to f'(x_0)}
$$

However, this approach may not work well since $f(x)$ can equal to $f(x_0)$ even $x \neq x_0$.

To resolve this problem, we need to modify the proof as follows:

- Define a function $G: S' \to \mathbb{R}$ as
$$G(y) = \begin{cases} \dfrac{g(y) - g\big(f(x_0)\big)}{y - f(x_0)} & if \;\; y \neq f(x_0) \\ g'\big(f(x_0)\big) & if \;\; y = f(x_0) \end{cases}$$
Since g is differentiable at $y = f(x_0)$ so that $g'\big(f(x_0)\big) = \lim_{y \to f(x_0)} \frac{g(y) - g(f(x_0))}{y - f(x_0)}$. Thus G is continuous at $y = f(x_0)$.
- So one can deduce from the definition of $G(y)$ that
$$g(y) - g\big(f(x_0)\big) = G(y)\big(y - f(x_0)\big)$$
for all $y \in S'$. (*Note: When $y = f(x_0)$, both sides equal 0.)
- Hence, one can deduce that
$$\lim_{x \to x_0} \frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{x - x_0} \overset{y = f(x)}{\cong} \lim_{x \to x_0} \frac{G\big(f(x)\big)\big(f(x) - f(x_0)\big)}{x - x_0}$$
$$= \lim_{x \to x_0} \underbrace{G\big(f(x)\big)}_{\to G(f(x_0)) = g'(f(x_0))} \underbrace{\left(\frac{f(x) - f(x_0)}{x - x_0}\right)}_{\to f'(x_0)} = g'\big(f(x_0)\big)f'(x_0).$$

Therefore, we conclude that $g\big(f(x)\big)$ is differentiable at $x = x_0$.

Example 2

We consider $f: \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & if \;\; x \neq 0 \\ 0 & if \;\; x = 0 \end{cases}.$$

(a) Show that $f(x)$ is differentiable at $x = x_0 \neq 0$ and find $f'(x_0)$ for $x_0 \neq 0$.

(b) Determine if $f(x)$ is differentiable at $x = 0$.

😊Solution

(a) Since $\frac{1}{x}$ is differentiable for all $x \neq 0$ and $\sin x$ is differentiable for all $x \in \mathbb{R}$, so $\sin\frac{1}{x}$ is also differentiable at all $x \neq 0$.

On the other hand, x^2 is differentiable at all $x \in \mathbb{R}$, so $x^2 \sin\frac{1}{x}$ (and hence) is also differentiable at $x = x_0 \neq 0$.

Using product rule and chain rule, we deduce that

$$f'(x_0) = 2x_0 \sin\frac{1}{x_0} + x_0^2\left(-\frac{1}{x_0^2}\cos\frac{1}{x_0}\right) = 2x_0 \sin\frac{1}{x_0} - \cos\frac{1}{x_0}.$$

(b) At $x_0 = 0$, one need to compute $f'(0)$ using the definition. That is,

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} \overset{x\neq 0}{\triangleq} \lim_{x\to 0}\frac{x^2\sin\frac{1}{x}-0}{x-0} \overset{x\neq 0}{\triangleq} \lim_{x\to 0} x\sin\frac{1}{x} =^{(*)} 0$$

(*Note: The last equality follows from the fact that $|\sin y| \leq 1$ so that

$$\left|x\sin\frac{1}{x}\right| \leq |x| \Rightarrow -|x| \leq x\sin\frac{1}{x} \leq |x|.$$

As $\lim\limits_{x\to 0}|x| = 0$, so the result follows by sandwich theorem.)

So we conclude that $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

Remark of Example 2

- Note that we cannot compute $f'(0)$ by considering $\frac{d}{dx}x^2\sin\frac{1}{x}$ because $f(0)$ does not take the form of $x^2\sin\frac{1}{x}$.
- Although the function $f(x)$ is differentiable at all $x \in \mathbb{R}$, we observe that $f'(x)$ is not necessarily to be a continuous function. In particular, $f(x)$ is not continuous at $x = 0$ since the limits $\lim\limits_{x\to 0} f'(x) = \lim\limits_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$ does not exist.
- If $f'(x)$ is also continuous, we say $f(x)$ is *continuously differentiable*.

Differentiation of inverse function – Inverse function theorem

We let $f: [a, b] \to \mathbb{R}$ be a function. Suppose that the function is continuous and injective, it follows that there exists an inverse function $f^{-1}: f([a, b]) \to (a, b)$.

In previous chapter, we have shown that f^{-1} is continuous. One would like to ask whether f^{-1} is differentiable if f is differentiable.

Theorem 1 (Inverse function theorem)

If $f(x)$ is continuous and injective on (a, b) and f is differentiable at $x_0 \in (a, b)$ with $f'(x_0) \neq 0$. Then the inverse function f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} \quad \left(i.e. \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \right).$$

Remark of theorem 1

- The theorem may not hold if $f'(x_0) = 0$. To see this, we consider $f(x) = x^3$.
  - Note that x^3 is continuous and strictly increasing over $\mathbb{R}$ so that $f(x)$ is continuous and injective. The inverse function of f is $f^{-1}(x) = \sqrt[3]{x}$.
  - Take $x_0 = 0$. We observe that $f'(0) = [3x^2]_{x=0} = 0$.

- On the other hand, f^{-1} is not differentiable at $x_0 = 0$ since
$$\lim_{x\to 0} \frac{f^{-1}(x) - f^{-1}(0)}{x - 0} = \lim_{x\to 0} \frac{\sqrt[3]{x}}{x} = \lim_{x\to 0} \frac{1}{x^{\frac{2}{3}}} = +\infty.$$

- As an example, we take $f(x) = e^x$ for $x \in \mathbb{R}$.
  - Note that e^x is continuous and injective (since e^x is strictly increasing). So the inverse function $f^{-1}(y) = \ln y$ exists and is continuous over $(0, \infty)$ (i.e. range of e^x).
  - Since e^x is differentiable at any $x \in \mathbb{R}$ and $\frac{d}{dx} e^x = e^x \neq 0$, it follows from inverse function theorem that $\ln y$ is also differentiable at any $y \in (0, \infty)$ and
$$\frac{d}{dy} \ln y = \frac{1}{e^x} = \frac{1}{e^{\ln y}} = \frac{1}{y}.$$
- We consider another example and take $f(x) = \cos x$ for $x \in (0, \pi)$
  - Note that $\cos x$ is continuous and strictly decreasing (hence injective) over $x \in (0, \pi)$. So the inverse function $f^{-1}(y) = \cos^{-1} y$ exists and is continuous over $(-1,1)$.
  - As $\cos x$ is differentiable on $(0, \pi)$ and $\frac{d}{dx} \cos x = -\sin x \neq 0$ for $x \in (0, \pi)$, it follows that $f^{-1}(y) = \cos^{-1} y$ is also differentiable over $(-1,1)$ and
$$\frac{d}{dy} \cos^{-1} y = \frac{1}{-\sin x} = \frac{1}{-\sqrt{1 - \cos^2 x}} = \frac{1}{-\sqrt{1 - y^2}}$$

Proof of inverse function theorem

Since $f(x)$ is continuous and injective over (a,b), then the inverse function f^{-1} exists and continuous over $\big(f(a),f(b)\big)$ (assume that $f(x)$ is strictly increasing).

We let $y_0 = f(x_0) \in \big(f(a),f(b)\big)$ and consider

$$\lim_{y\to y_0}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} = \lim_{y\to y_0}\frac{f^{-1}(y)-f^{-1}(y_0)}{f\big(f^{-1}(y)\big)-f\big(f^{-1}(y_0)\big)} \ldots (*)$$

To compute the limits, we define the function

$$g(x) = \begin{cases} \dfrac{x-x_0}{f(x)-f(x_0)} & if\ \ x \neq x_0 \\ \dfrac{1}{f'(x_0)} & if\ \ x = x_0 \end{cases}.$$

Given that $f'(x_0) \neq 0$, one can show that $\lim\limits_{x\to x_0} g(x) = \lim\limits_{x\to x_0}\frac{x-x_0}{f(x)-f(x_0)} = \frac{1}{f'(x_0)}$ and g is continuous at $x = x_0$.

By taking $x = f^{-1}(y)$ and $x_0 = f^{-1}(y_0)$, the limits $(*)$ can be computed as

$$\lim_{y\to y_0}\frac{f^{-1}(y)-f^{-1}(y_0)}{f\big(f^{-1}(y)\big)-f\big(f^{-1}(y_0)\big)} = \lim_{y\to y_0} g\big(f^{-1}(y)\big) =^{(*)} g\big(f^{-1}(y_0)\big) = \frac{1}{f'(x_0)}.$$

(*Note: The equality follows by the fact that $g\big(f^{-1}(y)\big)$ is continuous at $y = f(x_0)$.)

Mean-value theorem and its application

Recall that the derivative of a function describes the “trend” of the function which can reflect the behavior of a function.

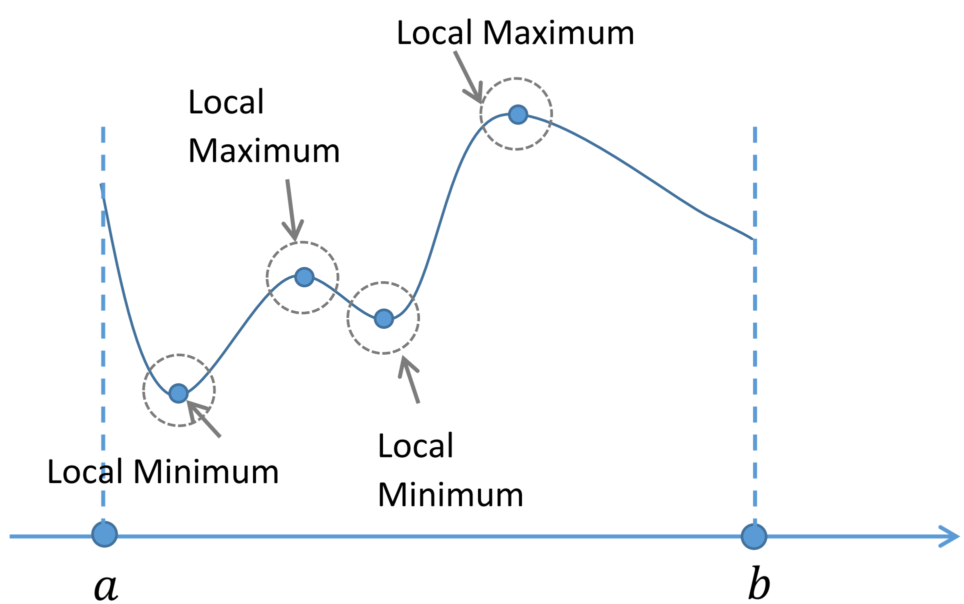
In this section, we shall introduce an important theorem, known as *mean value theorem*, that describes the relation between the function and its derivative.

In order to develop this theorem, we introduce the concept of *local extrema*. Recall that local extrema is often used to find the maximum/ minimum of a function over a certain range.

<u>How to find max/min of $f(x)$ over $[a, b]$?</u>

Step 1: Find all local minimum and local maximum of $f(x)$.

Step 2: Compare the function values of all local minimum, local maximum and boundary points to find the maximum and minimum of $f(x)$.



Definition (Local maximum and local minimum)

We let $f: S \to \mathbb{R}$ be a function. We say $f(x)$ has local maximum (resp. local minimum) at $x = x_0$ if and only if there exists $\delta > 0$ such that $f(x_0) \geq f(x)$ (resp. $f(x_0) \leq f(x)$) for all $x \in S \cap (x_0 - \delta, x_0 + \delta)$.

The following theorem suggests that it is possible to identify the local maximum and local minimum by considering the derivative of $f(x)$.

Theorem 2 (Properties of local maxima and local minima)

We let $f: [a, b] \to \mathbb{R}$ be a function. Suppose that $f(x)$ has a local maximum (or local minimum) at $x = x_0 \in (a, b)$ (i.e. $x_0 \neq a, b$) and $f'(x_0)$ exists, then $f'(x_0) = 0$.

Remark of theorem 2

- The theorem may not hold if $x_0 = a$ or $x_0 = b$. To see this, we consider $f: [0,1] \to \mathbb{R}$ as $f(x) = (x + 1)^2$.
  - Since the function $(x + 1)^2$ is increasing over $[0,1]$, then $f(x)$ has maximum at $x = 1$ and has minimum at $x = 0$.
  - However, we observe that $f'(1) = [2(x + 1)]_{x=1} = 4 \neq 0$ and $f'(0) = 2 \neq 0$.

Proof of theorem 2

We shall prove the case for local maximum and the case for local minimum can be proved in a similar manner.

Since $f(x)$ has local maximum at $x = x_0 \in (a, b)$, then there exists $\delta > 0$ such that

$$f(x_0) > f(x) \quad for\ \ all\ x \in (x_0 - \delta, x_0 + \delta).$$

Since $f'(x_0)$ exists so that $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$, then we deduce that

$$f'(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \overset{x_0 - \delta < x < x_0}{\widetilde{\geq}} \lim_{x \to x_0^-} \frac{f(x_0) - f(x_0)}{x - x_0} = 0.$$

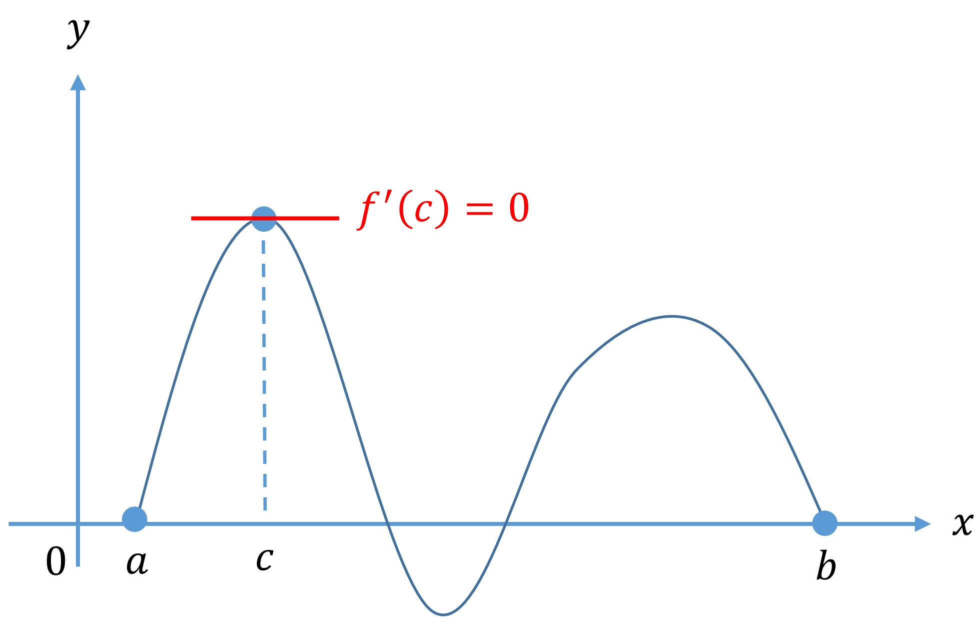
$$f'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \overset{x_0 < x < x_0 + \delta}{\widetilde{\leq}} \lim_{x \to x_0^+} \frac{f(x_0) - f(x_0)}{x - x_0} = 0.$$

It follows that $f'(x_0) = 0$ and the proof is completed.

With this concept, we can establish the following theorem which is essential for deriving the mean value theorem.

Theorem 3 (Rolle's theorem)

Suppose that a function $f(x)$ is continuous on $[a, b]$, is differentiable over (a, b) and $f(a) = f(b) = 0$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.



Proof of theorem 3

We first consider the special case when $f(x) = 0$ for all $x \in [a, b]$. Then for any $c \in (a, b)$, we have $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{0 - 0}{x - c} = 0$.

Next, we consider the case when $f(x) \neq 0$ for some $x \in (a, b)$.

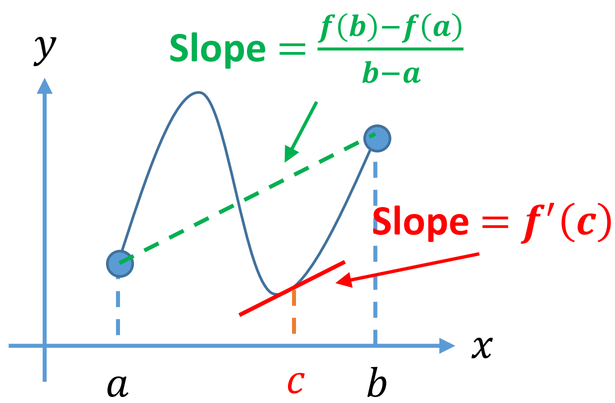
- <u>Case 1: Suppose that $f(x) > 0$ for some $x \in (a, b)$</u>
  By extreme value theorem, there exists $c \in (a, b)$ such that $f(x)$ attains maximum at $x = c$ and
  $$f(c) = \sup\{f(x) | x \in [a, b]\} > 0.$$
  Since $f(c) \geq f(x)$ for all $x \in [a, b]$ and c is local maxima, it follows from theorem 2 that $f'(c) = 0$.
- <u>Case 2: Suppose that $f(x) \leq 0$ for all $x \in (a, b)$</u>
  As $f(x) \neq 0$ for some $x \in (a, b)$, it must be that $f(x) < 0$ for some $x \in (a, b)$. By extreme value theorem, there exists $d \in (a, b)$ such that $f(x)$ attains minimum at $x = d$ and
  $$f(d) = \inf\{f(x) | x \in [a, b]\} < 0.$$
  Since $f(d) \leq f(x)$ for all $x \in [a, b]$ and d is local minima, it follows from theorem 2 that $f'(d) = 0$.

Using Rolle's theorem, one can establish the mean value theorem.

Theorem 4 (Mean value theorem)

Suppose that a function $f(x)$ is continuous on $[a, b]$, is differentiable over (a, b), then there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$



Proof of theorem 4

We consider a function $F: [a, b] \to \mathbb{R}$ defined by

$$F(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right].$$

One can see that $F(x)$ is continuous on $[a, b]$ and is differentiable on (a, b). On the other hand, one can verify that $F(a) = F(b) = 0$. Thus, it follows from Rolle's theorem that there exists $c \in (a, b)$ such that

$$F'(c) = 0 \Leftrightarrow f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \Leftrightarrow \frac{f(b) - f(a)}{b - a} = f'(c).$$

Some applications of mean value theorem

Application 1: Study the monotonicity of a function

Theoretically, mean value theorem allows us to draw conclusions about the behavior of the function from its derivative.

Lemma 1:

A function $f(x)$ is continuous on $[a, b]$ and is differentiable on (a, b). Suppose that $f'(x) = 0$ for all $x \in (a, b)$, then $f(x)$ is a constant function over $[a, b]$.

Proof:

We shall argue that $f(x) = f(a)$ for all $x \in (a, b]$. Note that $[a, x] \subseteq [a, b]$, it follows from mean value theorem that there exists $c \in (a, x)$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(c) = 0 \Rightarrow f(x) = f(a).$$

So $f(x)$ is a constant function over $[a, b]$ with $f(x) = f(a)$.

Monotonicity and derivative

Recall that a function is monotonic increasing (resp. monotonic decreasing) over an interval I if and only if $f(x_1) \leq f(x_2)$ (resp. $f(x_1) \geq f(x_2)$) for any $x_1, x_2 \in I$ satisfying $x_1 < x_2$. In fact, one can study the monotonicity using the derivative.

Lemma 2

Suppose that a function $f(x)$ is differentiable over (a, b), then

1. $f(x)$ is monotonic increasing over (a, b) if and only if $f'(x) \geq 0$ for any $x \in (a, b)$.
2. $f(x)$ is monotonic decreasing over (a, b) if and only if $f'(x) \leq 0$ for any $x \in (a, b)$.

Proof of Lemma 2

We shall prove the case when $f(x)$ is monotonic increasing.

- ("⇒" part) Note that for any $x_0 \in (a, b)$, we have

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \overset{\substack{x > x_0 \\ \Rightarrow f(x) \geq f(x_0)}}{\overbrace{\geq}} \lim_{x \to x_0^+} \frac{f(x_0) - f(x_0)}{x - x_0} = 0$$

- ("⇐" part)
  For any $x_1, x_2 \in I$ with $x_1 < x_2$, one can apply mean-value theorem over $[x_1, x_2]$ and deduce that there is $c \in (x_1, x_2)$ such that
  $$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \geq 0 \Rightarrow f(x_2) - f(x_1) \geq 0 \Rightarrow f(x_2) \geq f(x_1).$$
  So we conclude that $f(x)$ is monotonic increasing.

The case when $f(x)$ is monotonic decreasing can be proved in a similar way.

Remark of Lemma 2 -- How about the case for strictly monotone function?

- Using similar method, one can show that
  - ✓ If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is strictly increasing over (a, b)
  - ✓ If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is strictly decreasing over (a, b)
- However, the converse of the statement may not be true in general. That is, <u>a function is strictly increasing may *not* imply $f'(x) > 0$</u>.
- To see this, we consider $f(x) = x^3$.
  - ✓ It is clear that x^3 is strictly increasing since $x < y$ implies $x^3 < y^3$.
  - ✓ However, one can see that $f'(x) = 3x^2$ and $f'(0) = 0$.

Local derivative

Suppose that f is differentiable at some $x = c \in \mathbb{R}$, one can study the behavior of $f(x)$ near $x = c$.

Lemma 3 ("Local tracing theorem")

A function $f(x)$ is differentiable at $x = c$.

(a) If $f'(c) > 0$, then there is $\delta > 0$ such that

$$f(x) < f(c) < f(y) \text{ for any } c - \delta < x < c < y < c + \delta.$$

(b) If $f'(c) < 0$, then there is $\delta > 0$ such that

$$f(x) > f(c) > f(y) \text{ for any } c - \delta < x < c < y < c + \delta.$$

Proof of Lemma 3

We shall proof the statement (a) (the statement (b) can be proved in a similar manner)

Note $\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = f'(c) > 0$. By picking $\varepsilon = \frac{f'(c)}{2} > 0$, there exists $\delta > 0$ such that for $|x - c| < \delta$ (or $c - \delta < x < c + \delta$),

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \varepsilon = \frac{f'(c)}{2} \Rightarrow \frac{f(x) - f(c)}{x - c} > f'(c) - \frac{f'(c)}{2} = \frac{f'(c)}{2} > 0.$$

This implies that

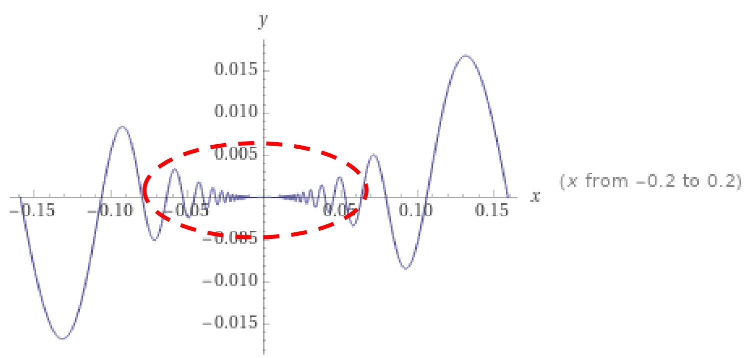
- When $c - \delta < x < c$, we have $x - c < 0$ and
$$\frac{f(x) - f(c)}{x - c} > 0 \Rightarrow f(x) - f(c) < 0 \Rightarrow f(x) < f(c).$$
- When $c < y < c + \delta$, we have $y - c > 0$ and
$$\frac{f(y) - f(c)}{y - c} > 0 \Rightarrow f(y) - f(c) > 0 \Rightarrow f(y) > f(c).$$

Remark of Lemma 3

When $f'(c) = 0$, one cannot draw any conclusion on the monotonicity of $f(x)$ near c.

As an example, we consider $f(x) = x^2 \sin\frac{1}{x}$ and take $c = 0$.

- Note that $f(0) = 0$. On the other hand, $f'(0) = 0$ as shown in Example 2.
- However, $f(x)$ can take positive or negative value near $x = 0$



Application 2 – Deriving some inequalities

Example 3

(a) For any $a, b \in \mathbb{R}$, show that

$$|\sin b - \sin a| \leq |b - a|.$$

Hence, prove that the sequence $\{y_n\}$ defined by $y_n = \sin x_n$ is Cauchy sequence where $\{x_n\}$ is a Cauchy sequence.

(b) Show that $(1 + x)^{\alpha} \geq 1 + \alpha x$ for any $x \geq -1$ and $\alpha \geq 1$.

☺Solution

(a) The inequality holds trivially when $a = b$ (since L.H.S. = R.H.S = 0). We consider the case when $a \neq b$. We assume that $a < b$.

Note that $\sin x$ is continuous and differentiable on $\mathbb{R}$. By applying mean-value theorem with $f(x) = \sin x$ over the interval $[a, b]$, then there exists $c \in (a, b)$ such that

$$|\sin b - \sin a| = \left|\frac{\sin a - \sin b}{b - a}\right| |b - a| = |\underbrace{\cos c}_{f'(c)}| \, |b - a| \overset{|\cos x| \leq 1}{\lesssim} |b - a|.$$

Next, we use this result and argue that $\{y_n\}$ is Cauchy sequence.

- Since $\{x_n\}$ is Cauchy sequence, then for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for any $m, n \geq K$.

- Using the above inequality, we deduce that for any $m, n \geq K$
$$|y_n - y_m| = |\sin x_n - \sin x_m| \leq |x_n - x_m| < \varepsilon.$$

Thus, we deduce that $\{y_n\}$ is also Cauchy sequence.

(b) We let $g(x) = (1+x)^\alpha - 1 - \alpha x$. We have $g(0) = 1^\alpha - 1 = 0$. Next, we consider the following two cases:

- If $x > 0$, it follows from mean-value theorem that there exists $c \in (0, x)$ such that
$$\frac{g(x) - g(0)}{x - 0} = g'(c) \Leftrightarrow (1+x)^\alpha - 1 - \alpha x = \underbrace{[\alpha(1+c)^{\alpha-1} - \alpha]}_{g'(c)} x$$
$$= \alpha \underbrace{[(1+c)^{\alpha-1} - 1]}_{>0 \text{ as } \alpha \geq 1 \text{ and } c>0} \underbrace{x}_{>0} > 0.$$
- If $x < 0$, it follows from mean-value theorem that there exists $c \in (x, 0)$ such that
$$\frac{g(x) - g(0)}{x - 0} = g'(c) \Leftrightarrow \alpha \underbrace{[(1+c)^{\alpha-1} - 1]}_{<0 \text{ as } \alpha \geq 1 \text{ and } c<0} \underbrace{x}_{<0} > 0.$$
- On the other hand, the inequality holds trivially for $x = 0$ since $L.H.S. = R.H.S. = 0$.

Example 4

(a) Show that

$$\frac{\tan x}{x} > 1 \quad for\ \ all\ \ 0 < x < \frac{\pi}{2}$$

(b) Hence, show that

$$\frac{2}{\pi} \le \frac{\sin x}{x} < 1 \quad for\ all\ \ 0 < x \le \frac{\pi}{2}$$

☺Solution

(a) For any $0 < x < \frac{\pi}{2}$, we can apply mean value theorem on $f(x) = \tan x$ over $[0, x]$ and deduce that there exists $c \in (0, x)$ such that

$$\frac{\tan x - \tan 0}{x - 0} = \underbrace{\sec^2 c}_{f'(c)} = 1 + \underbrace{\tan^2 c}_{>0\ \ for\ c \in (0,x)} > 1.$$

(b) We consider a function $f: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & if\ \ 0 < x \le \dfrac{\pi}{2} \\ \lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1 & if\ \ x = 0 \end{cases}$$

- ✓ We observe that $f(x)$ is continuous on $\left[0, \frac{\pi}{2}\right]$ and is differentiable on $\left(0, \frac{\pi}{2}\right)$ (because $f(x) = \frac{\sin x}{x}$ over $\left(0, \frac{\pi}{2}\right)$).
- ✓ On the other hand, one can use the result of (a) and deduce that

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \underbrace{\frac{\cos x}{x}}_{>0} \underbrace{\left(1 - \frac{\tan x}{x}\right)}_{<0} < 0$$

  for any $0 < x < \frac{\pi}{2}$.
- ✓ It follows that $f(x)$ is strictly decreasing over $0 < x < \frac{\pi}{2}$. Then for any $0 < a < x < b < \frac{\pi}{2}$

$$f(b) < f(x) = \frac{\sin x}{x} < f(a).$$

  By taking the limits $b \to \frac{\pi}{2}^-$ (left-hand limits) and $a \to 0^+$, we have

$$\underbrace{\frac{2}{\pi} = f\left(\frac{\pi}{2}\right) = \lim_{b \to \frac{\pi}{2}^-} f(b)}_{f(x)\ is\ continuous\ at\ x=\frac{\pi}{2}} < f(x) = \frac{\sin x}{x} < \underbrace{\lim_{a \to 0^+} f(a) = f(0) = 1}_{\substack{f(x)\ is\ continuous\ at\ x=0 \\ (by\ construction)}}.$$

Generalized mean value theorem (or *Cauchy mean value theorem)*

Technically, one can extend the mean value theorem into the following form which is useful in deriving L'Hopital rule.

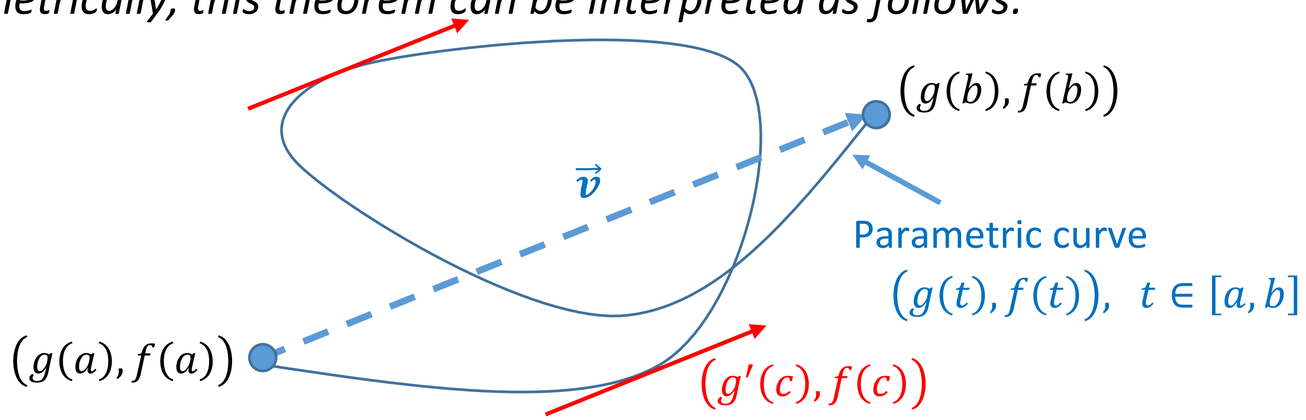
Theorem 4 (Generalized mean value theorem)

Suppose that a function $f(x), g(x)$ is continuous on $[a, b]$, is differentiable over (a, b), then there exists $c \in (a, b)$ such that

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$$

$$or\ \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}\ \ \big(provided\ that\ g(a) \neq g(b)\big).$$

Remark: Geometrically, this theorem can be interpreted as follows:



We consider a parametric curve defined by $\big(g(t), f(t)\big)$, where $t \in [a, b]$. Here, one can treat the parameter t as time. Then $\big(g(t), f(t)\big)$ represents the position at time t.

- ✓ Starting from $\big(g(a), f(a)\big)$, the point moves to $\big(g(b), f(b)\big)$ as t moves from a to b. It traces out a curve in 2-D plane.
- ✓ One can observe from the above figure that there exists a time c such that the *tangent vector* $(g'(c), f'(c))$ at that point is *parallel* to the vector $\vec{v}$, which is a vector joining $\big(g(a), f(a)\big)$ and $\big(g(b), f(b)\big)$.
- ✓ Note that $\vec{v} = \big(g(b) - g(a), f(b) - f(a)\big)$, then we have

$$(g'(c), f'(c)) = \lambda\vec{v} \Leftrightarrow (g'(c), f'(c)) = \Big(\lambda\big(g(b) - g(a)\big), \lambda\big(f(b) - f(a)\big)\Big).$$

This implies that

$$g'(c) = \lambda\big(g(b) - g(a)\big) \ \ and \ \ f'(c) = \lambda\big(f(b) - f(a)\big)$$

By eliminating λ, we deduce that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)},$$

which is the generalized mean value theorem.

Proof of generalized mean value theorem

When $g(x) = x$, the theorem is reduced to standard mean value theorem. So we shall prove this theorem by mimicking the proof of mean value theorem.

We consider the function

$$F(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{g(b) - g(a)} \big(g(x) - g(a) \big) \right]$$

or more generally (to avoid the possibility of $g(b) - g(a) = 0$),

$$G(x) = \big(f(x) - f(a) \big)\big(g(b) - g(a) \big) - \big(g(x) - g(a) \big)\big(f(b) - f(a) \big).$$

Note that

- ✓ Since f, g are continuous on $[a, b]$ and differentiable on (a, b), so does $G(x)$.
- ✓ One can verify that $G(a) = G(b) = 0$.
- ✓ $G'(x) = f'(x)\big(g(b) - g(a) \big) - g'(x)\big(f(b) - f(a) \big)$

It follows from Rolle's theorem that there exists $c \in (a, b)$ such that

$$G'(c) = 0 \Leftrightarrow f'(c)\big(g(b) - g(a) \big) - g'(c)\big(f(b) - f(a) \big) \Leftrightarrow \underbrace{\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}}_{if\ g(b) - g(a) \neq 0}.$$

L'Hopital's Rule

It is one of the powerful tools for computing the limits of the forms of $\lim_{x\to a}\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are some functions.

Theorem 5 (L'Hoptial's Rule, $\frac{0}{0}$ version)

We let f, g be two differentiable functions on (a, b) and $g(x), g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq \infty$ (i.e. a can be $-\infty$ and b can be $+\infty$). If

(1) $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$

(2) $\lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L, \quad where\ L \in \mathbb{R}\ \ or\ \ L = \pm\infty.$

$$Then \quad \lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L$$

Remark of Theorem 5

- The statement holds if all "$x \to a^+$" are replaced by "$x \to b^-$"
- The statement alslo holds if all "$x \to a^+$" are replaced by "$x \to c$" ($c \in (a, b)$) and the interval (a, b) in the theorem is replaced by $(a, b)\backslash\{c\}$.

Similarly, one can also deduce that a similar rule if the limit $\lim_{x\to a}\frac{f(x)}{g(x)}$ is of the form of $\frac{\infty}{\infty}$.

Theorem 6 (L'Hoptial's Rule, $\frac{*}{\infty}$ version)

We let f, g be two differentiable functions on (a, b) and $g(x), g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq \infty$ (i.e. a can be $-\infty$ and b can be $+\infty$). If

(1) $\lim_{x\to a^+} g(x) = +\infty$ (or $-\infty$)

(2) $\lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L, \quad where\ L \in \mathbb{R}\ \ or\ \ L = \pm\infty.$

$$Then \quad \lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)} = L$$

Remark of theorem 6

- Similar to the earlier version, the statement holds if all "$x \to a^+$" is replaced by "$x \to b^-$".
- On the other hand, the statement holds if all "$x \to a^+$" are replaced by "$x \to c$" (where $c \in (a, b)$) and the interval (a, b) is replaced by $(a, b)\backslash\{c\}$).

Example 5

Compute the limits

$(a)\quad \lim_{x\to 0}\frac{\sin 3x}{\sin x},\qquad (b)\quad \lim_{x\to\infty}\frac{e^x}{x^2+x-1},\qquad (c)\quad \lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

😊Solution

(a) We consider the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\backslash\{0\}$ (so that and $\sin x\neq 0$).

- $\lim_{x\to 0}\sin 3x=\lim_{x\to 0}\sin x=0$ and
- $\lim_{x\to 0}\frac{\frac{d}{dx}\sin 3x}{\frac{d}{dx}\sin x}=\lim_{x\to 0}\frac{3\cos 3x}{\cos x}=3\in\mathbb{R}.$

So it follows from L' Hopital rule that

$$\lim_{x\to 0}\frac{\sin 3x}{\sin x}=\lim_{x\to 0}\frac{3\cos 3x}{\cos x}=3.$$

One can show that the result obtained is consistent to that obtained without using L'Hopital rule. That is,

$$\lim_{x\to 0}\frac{\sin 3x}{\sin x}=\lim_{x\to 0}3\frac{\left(\frac{\sin 3x}{3x}\right)}{\left(\frac{\sin x}{x}\right)}=3.$$

(b) We consider the interval $(-\infty, \infty)$. Note that $e^x \to \infty$ and $x^2 + x - 1 \to \infty$ as $x \to \infty$. It follows from L'Hopital's rule that

$$\lim_{x\to\infty} \frac{e^x}{x^2 + x - 1} = \lim_{x\to\infty} \frac{e^x}{2x + 1},$$

provided that <u>the limits on R.H.S exists</u>. To compute the limits $\lim_{x\to\infty} \frac{e^x}{2x+1}$, we note that $e^x \to \infty$ and $2x + 1 \to \infty$, it follows from L'Hopital's rule that

$$\lim_{x\to\infty} \frac{e^x}{2x + 1} = \underbrace{\lim_{x\to\infty} \frac{e^x}{2} = +\infty.}_{The\ limits\ \lim\limits_{x\to\infty}\frac{f'(x)}{g'(x)}\ exists}$$

Then it follows that $\lim_{x\to\infty} \frac{e^x}{x^2+x-1} = \lim_{x\to\infty} \frac{e^x}{2x+1} = +\infty$.

(c) Note that the limits can be written as

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0^+} \frac{\sin x - x}{x \sin x}$$

Since both $\sin x - x \to 0$ and $x \sin x \to 0$ as $x \to 0^+$, it follows that

$$\lim_{x\to 0^+} \frac{\sin x - x}{x \sin x} = \lim_{x\to 0^+} \frac{\overbrace{\cos x - 1}^{\to 0}}{\underbrace{\sin x + x \cos x}_{\to 0}} = \underbrace{\lim_{x\to 0^+} \frac{-\sin x}{2\cos x - x \sin x} = \frac{0}{2}}_{the\ limits\ exists} = 0.$$

Important remark: About the use of L'Hopital Rule

One has to be careful that the validity of L'Hopital rule relies on the existence of the limits $\lim_{x\to a^+}\frac{f'(x)}{g'(x)}$. The theorem *does not* reveal any information if $\lim_{x\to a^+}\frac{f'(x)}{g'(x)}$ does not exist.

- As an example, we consider $\lim_{x\to\infty}\frac{x-\sin x}{x+\cos x}$. One can see that
  - Both $x-\sin x\to\infty$ and $x+\cos x\to\infty$ as $x\to\infty$.
  - $\lim_{x\to\infty}\frac{\frac{d}{dx}(x-\sin x)}{\frac{d}{dx}(x+\cos x)}=\lim_{x\to\infty}\frac{1-\cos x}{1-\sin x}$ does not exist

    (This can be verified mathematically sequential limits theorem with $\{x_n\}=\left\{2n\pi+\frac{3\pi}{2}\right\}$ and $\{y_n\}=\{2n\pi\}$).
  - But $\lim_{x\to\infty}\frac{x-\sin x}{x+\cos x}=\lim_{x\to\infty}\frac{1-\frac{\sin x}{x}}{1+\frac{\cos x}{x}}=1$.

Therefore, one needs to confirm that the limits $\lim_{x\to a^+}\frac{f'(x)}{g'(x)}$ exits before using L'Hopial rule.

Proof of L'Hopital rule (Proof of Theorem 5 and Theorem 6)

We shall prove the theorem using generalized mean value theorem since it gives a connection between $\frac{f(x)}{g(x)}$ and $\frac{f'(x)}{g'(x)}$.

<u>Proof of Theorem 5 (A rough proof)</u>

To demonstrate the idea of the proof, we shall consider the case when $a, b \in \mathbb{R}$.

We define $f(a) = \lim_{x\to a^+} f(x) = 0$ and $g(a) = \lim_{x\to a^+} g(x) = 0$ (so that both functions are also well-defined at $x = a$). Then f, g are continuous at $x = 0$.

For any $x \in (a, b)$, we note that both f, g are continuous over $[a, x] \subseteq [a, b)$ and is differentiable over $(a, x) \subseteq (a, b)$, it follows from generalized mean value theorem that there exists $c \in (a, x)$ such that (note that $g(x) \neq 0 = g(a)$)

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)} \Leftrightarrow \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \dots \dots (*)$$

Next, we show $\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$ using the definition of limits. Since L can be either finite number or infinity, we shall consider the following two cases:

- <u>Case 1: If $L \in \mathbb{R}$,</u>

  Since $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L$, then for any $\varepsilon > 0$, there exists $\delta^* > 0$ such that
  $$\left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon \quad when \quad 0 < |x-a| < \delta^* \quad and \quad x > a.$$
  We pick $\delta = \delta^*$, we deduce that for any $0 < |x-a| < \delta$ and $x > a$,
  $$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| \overbrace{<}^{\substack{c \in (a,x) \\ |c-a|<|x-a|<\delta^*}} \varepsilon.$$
  So $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$ by definition.
- <u>Case 2: If $L = +\infty$ (*the case for $L = -\infty$ would be similar)</u>

  Since $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = +\infty$, then for any $M > 0$, there exists $\delta^* > 0$ such that
  $$\frac{f'(x)}{g'(x)} > M \quad when \quad 0 < |x-a| < \delta^* \quad and \quad x > a.$$
  We pick $\delta = \delta^*$, we deduce that for any $0 < |x-a| < \delta$ and $x > a$,
  $$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} > M \quad as \quad c \in (a,x)$$
  So $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$ by definition.

Addition remark about the proof of theorem 5

For the case when $a = -\infty$, the argument above cannot used directly since $f(a) = f(-\infty)$ is not defined.

To resolve this problem, we note that $\lim_{x\to-\infty} f(x) = L \Leftrightarrow \lim_{x\to0^+} f\left(-\frac{1}{x}\right) = L$. Given this equivalence, one can prove the statement by proving $\lim_{x\to0^+} \frac{f\left(-\frac{1}{x}\right)}{g\left(-\frac{1}{x}\right)} = L$.

Since $\lim_{x\to0^+} f\left(-\frac{1}{x}\right) = 0$ and $\lim_{x\to0^+} g\left(-\frac{1}{x}\right) = 0$ (as given), it follows from L'Hopital's rule and chain rule that

$$\lim_{x\to0^+} \frac{f\left(-\frac{1}{x}\right)}{g\left(-\frac{1}{x}\right)} = \lim_{x\to0^+} \frac{\frac{1}{x^2} f'\left(-\frac{1}{x}\right)}{\frac{1}{x^2} g'\left(-\frac{1}{x}\right)} = \lim_{x\to0^+} \frac{f'\left(-\frac{1}{x}\right)}{g'\left(-\frac{1}{x}\right)} = \lim_{x\to-\infty} \frac{f'(x)}{g'(x)} = L.$$

So we can conclude that $\lim_{x\to-\infty} \frac{f(x)}{g(x)} = L$.

The case for $b = +\infty$ can be proved in a similar fashion and is left as exercise.

Proof of theorem 6

We shall focus on the case when $a, b \in \mathbb{R}$ and $L \in \mathbb{R}$.

Similar to theorem 5, we shall prove $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$ using definition of limits.

For any $\varepsilon > 0$,

- Since $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, there exists $\delta_1 > 0$ such that
$$\left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\varepsilon}{2} \quad for \;\; 0 < |x - x_0| < \delta_1 \;\; and \;\; x > a \dots (*)$$
- We take $x_0 = a + \frac{\delta_1}{2} < a + \delta_1$. From generalized mean value theorem, we deduce that for any $x \in (a, x_0)$, there exists $c_x \in (x, x_0)$ such that
$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c_x)}{g'(c_x)} \dots \dots (**)$$
(*Note that $x_0, x, c_x \in (a, a + \delta_1)$)

 Our goal is to estimate the difference $\left|\frac{f(x)}{g(x)} - \frac{f'(c_x)}{g'(c_x)}\right|$. By rearranging the equation $(**)$, we deduce that

$$\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\frac{f'(c_x)}{g'(c_x)} \Rightarrow \frac{f(x)-f(x_0)}{g(x)}=\frac{f'(c_x)}{g'(c_x)}\left(\frac{g(x)-g(x_0)}{g(x)}\right)$$

$$\Rightarrow \frac{f(x)}{g(x)}-\frac{f'(c_x)}{g'(c_x)}=\frac{f(x_0)}{g(x)}-\frac{f'(c_x)}{g'(c_x)}\left(\frac{g(x_0)}{g(x)}\right).$$

- Note that $\lim_{x\to a^+} g(x)=+\infty$ and $L-\frac{\varepsilon}{2}<\frac{f'(c)}{g'(c)}<L+\frac{\varepsilon}{2}$ (since $c\in(a,a+\delta_1)$. So we can deduce that $\lim_{x\to a^+}\left(\frac{f(x)}{g(x)}-\frac{f'(c_x)}{g'(c_x)}\right)=\lim_{x\to a^+}\frac{f(x_0)}{g(x)}-\frac{f'(c_x)}{g'(c_x)}\left(\frac{g(x_0)}{g(x)}\right)=0.$
- It follows that there exists $\delta_2>0$ such that

$$\left|\frac{f(x)}{g(x)}-\frac{f'(c_x)}{g'(c_x)}\right|<\frac{\varepsilon}{2} \quad for \;\; 0<|x-x_0|<\delta_2 \;\; and \;\; x>a \ldots (***)$$

➢ By choosing $\delta=\min\left(\frac{\delta_1}{2},\delta_2\right)$, we deduce that for any $0<|x-a|<\delta$ and $x>a$,

$$\left|\frac{f(x)}{g(x)}-L\right|\le\left|\frac{f(x)}{g(x)}-\frac{f'(c_x)}{g'(c_x)}\right|+\underbrace{\left|\frac{f'(c_x)}{g'(c_x)}-L\right|}_{c_x\in(a,x_0)\subseteq(a,a+\delta_1)}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

So $\lim_{x\to a^+}\frac{f(x)}{g(x)}=L$ by definition.

Higher-order derivatives and Taylor's theorem

Suppose that f is differentiable over an open interval (a, b), then the derivative $f'(x)$ over this open interval (a, b).

- If the function $f'(x)$ is differentiable at $x_0 \in (a, b)$, one can denote the derivative of $f'(x)$ by $f''(x_0) = (f')'(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0}$. Here, $f''(x_0)$ is called *second derivative* of $f(x)$ at $x = x_0$.
- If $f''(x)$ is differentiable at $x_0 \in (a, b)$, we can define the third derivative of $f(x)$ by $f'''(x_0) = (f'')'(x_0)$.
- By repeating this process, one can obtain a series of derivatives (i.e. $f'(x_0), f''(x_0), \ldots, f^{(n)}(x_0)$), where each of the derivative is the derivative of the prec. Here, $f^{(n)}(x)$ is called n-th derivative of $f(x)$.

In order that the n-th derivative $f^{(n)}(x_0)$ exists at $x = x_0$, it must be that $f^{(n-1)}(x)$, $f^{(n-2)}(x_0)$,…, $f'(x)$ exists *near* $x = x_0$. So we say f is n-times differentiable at $x = x_0$.

✓ Recall that differentiability implies continuity, so if f is n-times differentiable at $x = x_0$, it follows that the derivatives $f', f'', \ldots, f^{(n-1)}$ are continuous at $x = x_0$.

Example 6

We consider a function $f(x) = |x|^3$. Show that $|x^3|$ is twice differentiable at $x = 0$ but is not three-times differentiable at $x = 0$.

😊Solution

Note that $|x|^3 = \begin{cases} x^3 & if \;\; x \geq 0 \\ (-x)^3 = -x^3 & if \;\; x < 0 \end{cases}$.

We first compute the first derivative:

- For $x > 0$, we have $f(x) = x^3$ near x. So $f'(x) = \frac{d}{dx}x^3 = 3x^2$
- For $x < 0$, we have $f(x) = -x^3$ near x. So $f'(x) = \frac{d}{dx}(-x^3) = -3x^2$.
- For $x = 0$, we have

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{x^3}{x} = \lim_{x\to 0^+} x^2 = 0,$$
$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{-x^3}{x} = \lim_{x\to 0^-} -x^2 = 0$$

So $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$. This implies $f'(0) = 0$.

Next, we compute the second derivative

- For $x > 0$, we have $f'(x) = 3x^2$ near x. So $f''(x) = \frac{d}{dx}(3x^2) = 6x$
- For $x < 0$, we have $f'(x) = -3x^2$ near x. So $f''(x) = \frac{d}{dx}(-3x^2) = -6x$.
- For $x = 0$, we have

$$\lim_{x\to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0^+} \frac{3x^2}{x} = 3 \lim_{x\to 0^+} x = 0,$$
$$\lim_{x\to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0^-} \frac{-3x^2}{x} = -3 \lim_{x\to 0^-} x = 0$$

  So $\lim_{x\to 0} \frac{f'(x)-f'(0)}{x-0} = 0$. This implies $f''(0) = 0$.

Finally, we note that

$$\lim_{x\to 0^+} \frac{f''(x) - f''(0)}{x - 0} = \lim_{x\to 0^+} \frac{6x}{x} = 6 \;\; and \;\; \lim_{x\to 0^-} \frac{f''(x) - f''(0)}{x - 0} = \lim_{x\to 0^+} \frac{-6x}{x} = -6.$$

Since $\lim_{x\to 0^+} \frac{f''(x)-f''(0)}{x-0} \neq \lim_{x\to 0^-} \frac{f''(x)-f''(0)}{x-0}$, so the limits $f'''(0) = \lim_{x\to 0} \frac{f''(x)-f''(0)}{x-0}$ does not exists. Thus we conclude that $f(x)$ is 2-times differentiable at $x = 0$ but is not 3-times differentiable at $x = 0$.

Taylor theorem

Similar to mean value theorem, Taylor theorem gives a connection between a function $f(x)$ and its derivatives ($f'(x)$, $f''(x)$, ... etc.). On the other hand, it allows us to approximate the function by a polynomial.

Motivation of the theorem – first order approximation

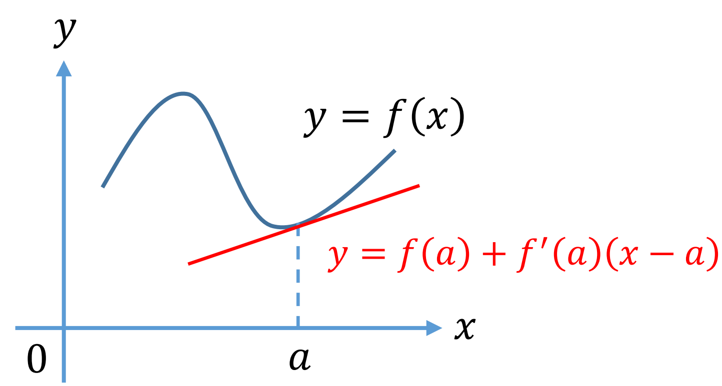
If a function f is differentiable at $x = a$, one can draw a tangent line which touches $y = f(x)$ at $x = a$. The equation of the tangent line is known to be

$$y = f(a) + f'(a)(x - a).$$

As we observe from the figure that the tangent line is very close to $y = f(x)$ when x is closed to a so that one can approximate $f(x)$ by

$$f(x) \approx f(a) + f'(a)(x - a)$$

for x is near a. This is known as *first-order approximation*.



However, this approximation works well only when $x = a$ and works poorly if x is far away from a.

To improve the accuracy of the approximation, one can approximate the function by a polynomial with degree n. That is,

$$f(x) \approx a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n,$$

where n is a positive integer.

It remains to determine the coefficients a_k:

- We take $x = a$, we get $f(a) = a_0$.
- Next, we differentiate $f(x)$ with respect to x and get
$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1}.$$
By taking $x = a$, we get $f'(a) = a_1$.
- To find the coefficient a_k, we differentiate $f(x)$ for k times and get
$$f^{(k)}(x) = k!\,a_k + \frac{(k+1)!}{1!}a_{k+1}(x-a) + \cdots + \frac{n!}{(n-k)!}a_n(x-a)^{n-k}.$$
We take $x = a$, we get
$$f^{(k)}(a) = k!\,a_k \Rightarrow a_k = \frac{f^{(k)}(a)}{k!}, \qquad k = 1,2,\ldots,n.$$

Hence, the function $f(x)$ can be approximated by

$$f(x) \approx f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The polynomial on the right is sometimes called Taylor's polynomial.

It remains to estimate the error of the approximation $E(x)$ which is defined as

$$E(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k,$$

We let $E(x) = M(x-a)^{n+1}$, where M is a constant depending on x. Using Rolle's theorem, one can deduce the following theorem:

Theorem 7 (Taylor theorem)

We let $f\colon [\alpha, \beta] \to \mathbb{R}$ be a function which $f^{(n)}(x)$ is continuous over $[\alpha, \beta]$ and $f^{(n+1)}(x)$ exists for all $x \in (\alpha, \beta)$. For any $a \in [\alpha, \beta]$ and $x \in [\alpha, \beta]$ (with $a \neq x$), there exists $c \in (a, x)$ (or $c \in (x, a)$) such that

$$E(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad or \quad f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Proof of Theorem 7

The goal of the proof is to find the formula for the "constant" M.

To do so, we define a function $G: [a, x] \to \mathbb{R}$ as

$$G(y) = f(y) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(y-a)^k - M(y-a)^{n+1}.$$

Since f is $(n+1)$-times differentiable on $[a, x]$, it follows that $G(y)$ is also $(n+1)$-times differentiable on $[a, x]$.

One can show that

- $G(a) = f(a) - f(a) = 0,$
- $G(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k - M(x-a)^{n+1} = 0$
- For any $m = 1,2, \dots n,$

  $$G^{(m)}(a) = \left[f^{(m)}(a) - \sum_{k=m}^{n} \frac{f^{(k)}(a)}{(k-m)!}(y-a)^{k-m} - \frac{(n+1)!\,M}{(n-m+1)!}(y-a)^{n-m+1} \right]_{y=a}$$

  $$= f^{(m)}(a) - f^{(m)}(a) = 0.$$

Given the properties of $G(y)$, we determine the value of M as follows:

- As $G(a) = G(x) = 0$, it follows from Rolle's theorem that there exists $c_1 \in (a, x)$ such that $G'(c_1) = 0$.
- Since $G'(a) = G'(c_1) = 0$, we deduce that there exists $c_2 \in (a, c_1)$ such that $G'(c_2) = 0$.
- Since $G''(a) = \cdots = G^{(n)}(a) = 0$, one can repeat the above argument and deduce that there exists $c_2 \in (a, c_1), c_3 \in (a, c_2), \dots, c_n \in (a, c_{n-1})$ and $c^* \in (a, c_n)$ such that
$$G^{(2)}(c_2) = 0, \;\; G^{(3)}(c_3) = 0, \dots, \;\; G^{(n)}(c_n) = 0 \;\; and$$
$$G^{(n+1)}(c^*) = 0$$
- From $G^{(n+1)}(c_{n+1}) = 0$, we deduce that
$$G^{(n+1)}(c^*) = 0 \Rightarrow f^{(n+1)}(c^*) - (n+1)!\, M = 0 \Rightarrow M = \frac{f^{(n+1)}(c^*)}{(n+1)!}.$$

So we conclude that $E(x) = \frac{f^{(n+1)}(c^*)}{(n+1)!}(x-a)^{n+1}$ and the proof is completed.

Example 7 (Some Taylor expansion of some common functions)

(a) We let $f(x) = e^x$. Since $f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = e^0 = 1$ for all $n \in \mathbb{N}$. It follows from Taylor theorem that

$$e^x = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \sum_{k=0}^{n} \frac{x^k}{k!} + \underbrace{\frac{e^c}{(n+1)!} x^{n+1}}_{error\ term}$$

$$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \underbrace{\frac{e^c}{(n+1)!} x^{n+1}}, \quad c \in (0, x)$$

(b) We let $f(x) = \sin x$. Since $f^{(2n-1)}(x) = (-1)^{n-1} \cos x$ and $f^{(2n)}(x) = (-1)^n \sin x$, we have $f^{(2n-1)}(0) = (-1)^{n-1}$ and $f^{(2n)}(0) = 0$. It follows from Taylor theorem that

$$\sin x = \sum_{k=0}^{2n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(2n+1)}(c)}{(2n+1)!} x^{2n+1} = \sum_{k=1}^{n} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!} + \frac{(-1)^n \cos c}{(2n+1)!} x^{2n+1}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} + \frac{(-1)^n \cos c}{(2n+1)!} x^{2n+1}$$

Example 8 (Optimization, Second derivative test)

We let $f(x)$ be a twice differentiable function on $\mathbb{R}$ which $f''(x)$ is continuous on $\mathbb{R}$.

(a) Suppose that $f'(x_0) = 0$ and $f''(x_0) > 0$ for some $x_0 \in \mathbb{R}$, show that $f(x)$ has local minimum at $x = x_0$.

(b) Suppose that $f'(x_0) = 0$ and $f''(x_0) < 0$, show that $f(x)$ has local minimum at $x = x_0$.

Solution

(a) Note that $f''(x_0) > 0$ and $f''(x)$ is continuous. We take $\varepsilon = \frac{f''(x_0)}{2} > 0$, then there exists $\delta > 0$ such that

$$|f''(x) - f''(x_0)| < \varepsilon = \frac{f''(x_0)}{2} \Rightarrow f''(x) > \frac{f''(x_0)}{2} > 0$$

for $0 < |x - x_0| < \delta$

For any $x \in (x_0 - \delta, x_0 + \delta)$, one can use Taylor theorem (with $a = x_0$ and $n = 1$) and deduce that there exists $c \in (x_0, x)$ such that

$$f(x) = f(x_0) + \underbrace{f'(x_0)}_{=0}(x - x_0) + \frac{f''(c)}{2!}(x - x_0)^2 \overset{f''(c)>0}{\geq} f(x_0).$$

So it reveals that $f(x)$ achieves local minimum at $x = x_0$.

(b) Note that $f''(x_0) < 0$ and $f''(x)$ is continuous. We take $\varepsilon = \frac{|f''(x_0)|}{2} = -\frac{f''(x_0)}{2} > 0$, then there exists $\delta > 0$ such that

$$|f''(x) - f''(x_0)| < \varepsilon = -\frac{f''(x_0)}{2} \Rightarrow f''(x) < \frac{f''(x_0)}{2} < 0$$

for $0 < |x - x_0| < \delta$

For any $x \in (x_0 - \delta, x_0 + \delta)$, one can use Taylor theorem (with $a = x_0$ and $n = 1$) and deduce that there exists $c \in (x_0, x)$ such that

$$f(x) = f(x_0) + \underbrace{f'(x_0)}_{=0}(x - x_0) + \frac{f''(c)}{2!}(x - x_0)^2 \overset{f''(c)<0}{\widehat{<}} f(x_0).$$

So it reveals that $f(x)$ achieves local maximum at $x = x_0$.

Remark of Example 8

- In general, the second derivative test *does not* require the continuity of second derivative. To prove the general case, one needs to use the local tracing theorem and first derivative test.

Example 9 (Jensen inequality)

We let $f(x)$ be twice differentiable function. Suppose that $f''(x) \geq 0$ for all $x \in \mathbb{R}$ (or $f(x)$ is convex), show that

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n).$$

for any $x_1, x_2, \ldots, x_n \in \mathbb{R}$, $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$.

☺Solution

We let $a = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$ and take $x = x_i$, it follows from Taylor theorem that there exists $c_i \in (0, x_i)$ such that

$$f(x_i) = f(a) + f'(a)(x_i - a) + \frac{f''(c_i)}{2!}(x_i - a)^2 \overbrace{\geq}^{f''(x) \geq 0} f(a) + f'(a)(x_i - a).$$

Then it follows that

$$\lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n) \geq \sum_{i=1}^{n} \lambda_i [f(a) + f'(a)(x_i - a)]$$

$$= f(a) \underbrace{\sum_{i=1}^{n} \lambda_i}_{=1} + f'(a) \left(\underbrace{\sum_{i=1}^{n} \lambda_i x_i}_{=a} - a \underbrace{\sum_{i=1}^{n} \lambda_i}_{=1} \right) = f(a) = f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n).$$

Example 10

We let $f(x)$ be a 3-times differentiable function over $[-1,1]$ such that

$$f(-1)=0,\ \ f(0)=0,\ \ f(1)=1,\ \ f'(0)=0$$

Prove that there exists $c^* \in (-1,1)$ such that $f^{(3)}(c^*) \geq 3$.

😊Solution

By applying Taylor theorem on $f(x)$ (with $a=0$), we get

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(c)}{3!}x^3 \quad for\ \ some\ \ c \in (0,x).$$

We take $x=1$ and $x=-1$ respectively, we get

$$f(1)=f(0)+f'(0)+\frac{f''(0)}{2!}+\frac{f^{(3)}(c_1)}{3!} \quad for\ \ some\ c_1 \in (0,1) \ldots (1)$$

$$f(-1)=f(0)-f'(0)+\frac{f''(0)}{2!}-\frac{f^{(3)}(c_2)}{3!} \quad for\ \ some\ c_2 \in (-1,0) \ldots (2)$$

By $(2)-(1)$, we get

$$\underbrace{f(1)}_{=1}-\underbrace{f(-1)}_{=-1}=2\underbrace{f'(0)}_{=0}+\frac{f^{(3)}(c_1)}{3!}+\frac{f^{(3)}(c_2)}{3!} \Rightarrow f^{(3)}(c_1)+f^{(3)}(c_2) \geq 6.$$

So we have $f^{(3)}(c_1) \geq 3$ and $f^{(3)}(c_2) \geq 3$ (otherwise, we will have $f^{(3)}(c_1)+f^{(3)}(c_2) < 6$)