Math2033 TA note 3

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1 COUNTABILITY

Theorem 1 (Bijection theorem)**.** *Let $g : S \to T$ be a bijection. S is countable if and only if T is countable.*

Example 2. *Every interval (a, b) with $a < b$ is uncountable.*

Solution: Let $f : (0,1) \to (a, b)$ be given by $f(x) = a + (b - a)x$, $\forall x \in (0,1)$.
Clearly, f is both injective and surjective, i.e. f is a bijection and $f^{-1}(y) = \frac{y-a}{b-a}$.
Taking the contrapositive of Theorem 1, (a, b) is uncountable iff $(0,1)$ is uncountable.
Since the open interval $(0,1)$ is uncountable (Proof by base-10 numeral representation of real number and Cantor's diagonal argument in lecture note), we show that every interval (a, b) with $a < b$ is uncountable.
Notation: The cardinality of a set A is usually denoted $|A|$. Alternatively, the cardinality of a set A may be denoted by $card(A)$ or $\#A$. In general, if there exists an injection $f : A \to B$, then we denote $card(A) \le card(B)$.

Example 3. *$\mathbb{R}$ and $\mathbb{C}$ has the same cardinality.*

To prove this result, we need to use the Cantor-Bernstein theorem:

Theorem 4 (Cantor-Bernstein)**.** *If $card(A) \le card(B)$ and $card(B) \le card(A)$ then $card(A) = card(B)$.*

More explicitly, this theorem states that if there exist injections $f : A \to B$ and $g : B \to A$ then there exists bijection $h : A \to B$. The proof of this theorem is technical. For those who are interested, many references could be found on the Internet, for example,
http://www.cs.cornell.edu/courses/cs2800/2017fa/lectures/lec14-cantor.html

Solution: To show $card(\mathbb{R}) = card(\mathbb{C})$, we need to show $card(\mathbb{R}) \le card(\mathbb{C})$ and $card(\mathbb{C}) \le card(\mathbb{R})$.

Part I: Since $\mathbb{R} \subset \mathbb{C}$, we have $card(\mathbb{R}) \le card(\mathbb{C})$.

Part II: Since there exists a bijection $f : (0,1) \to \mathbb{R}$ given by $f(x) = tan(x - \frac{1}{2})\pi$, open interval $(0,1)$ has the same cardinality as $\mathbb{R}$. On the other hand, $\forall a + bi \in \mathbb{C}$, there exits a bijection $g : \mathbb{C} \to \mathbb{R} \times \mathbb{R}$ given by $g(a + bi) = (a, b)$, showing that $\mathbb{C}$ has the same cardinality as $\mathbb{R} \times \mathbb{R}$.

In order to show $card(\mathbb{C}) \le card(\mathbb{R})$, i.e. there exists an injection $h : \mathbb{C} \to \mathbb{R}$, we only need to show there is an injection $h : (0,1) \to (0,1) \times (0,1)$.

For all $(a, b) \in (0,1) \times (0,1)$, a, b have binary representation $a = 0.a_1a_2a_3\cdots, b = 0.b_1b_2b_3\cdots$, where a_n and b_n is either 0 or 1. Also, it is forbidden that $1 = a_k = a_{k+1} = a_{k+2} = \cdots$ and $1 = b_k = b_{k+1} = b_{k+2} = \cdots$ for some $k \in \mathbb{N}$.

We can thereby construct an injection $h((a, b)) = 0.a_1b_1a_2b_2a_3b_3\cdots \ \forall (a, b) \in (0,1) \times (0,1)$. For $(a, b) \ne (\tilde{a}, \tilde{b})$, $h((a, b))$ differs from $h((\tilde{a}, \tilde{b}))$ for at least one numeral by the uniqueness of binary representation.

Combined part I and part II, we show that $\mathbb{R}$ and $\mathbb{C}$ has the same cardinality.

Theorem 5 (Countable union theorem). *$\bigcup_{i \in \Lambda} A_i$ is countable if Λ and each A_i is countable.*

Example 6. *$A = \{r\sqrt{m} : m \in \mathbb{N}, r \in \mathbb{Q} \cap (0,1)\}$ is countable.*

Solution: We observe that

$$A = \bigcup_{m \in \mathbb{N}} A_m, \quad \text{where } A_m = \{r\sqrt{m} : r \in \mathbb{Q} \cap (0,1)\} = \bigcup_{r \in \mathbb{Q} \cap (0,1)} \{r\sqrt{m}\}.$$

Since $\mathbb{Q} \cap (0,1)$ is countable and $\{r\sqrt{m}\}$ has 1 element for each $r \in \mathbb{Q} \cap (0,1)$, A_m is countable by the countable union theorem. Finally, since $\mathbb{N}$ is countable and A_m is countable for every $m \in \mathbb{N}$, A is countable by the countable union theorem.

Example 7. *Let $T = \mathbb{R} \setminus \mathbb{Q}$ and $U = \mathbb{R} \setminus \{\sqrt{m} + \sqrt{n} : m, n \in \mathbb{N}\}$, then $T \cap U$ is uncountable.*

Solution: Denote $V = \{\sqrt{m} + \sqrt{n} : m, n \in \mathbb{N}\}$, then $U = \mathbb{R} \setminus V$ and

$$T \cap U = (\mathbb{R} \setminus \mathbb{Q}) \cap (\mathbb{R} \setminus V) = \mathbb{R} \setminus (\mathbb{Q} \cup V).$$

We observe that

$$V = \bigcup_{m \in \mathbb{N}} V_m \quad \text{where } V_m = \{\sqrt{m} + \sqrt{n} : n \in \mathbb{N}\}.$$

Since V_m is countable for every $m \in \mathbb{N}$, V is countable by the countable union theorem. Since $\mathbb{Q}$ and V are countable, $\mathbb{Q} \cup V$ is countable by the countable union theorem. Since $\mathbb{R}$ is uncountable, $T \cap U = \mathbb{R} \setminus (\mathbb{Q} \cup V)$ is uncountable.

Theorem 8 (Countable subset theorem). *$A \subset B$, then B countable $\implies$ A is countable.*

Example 9. *If A_1 is uncountable then $A_1 \times A_2$ is uncountable if $A_2 \ne \phi$.*

Solution: We assume A_1 is uncountable and $A_1 \times A_2$ is countable. Because $A_2 \neq \phi$, we choose one element α in A_2. Then easy to see $A_1 \times \{\alpha\} \subset A_1 \times A_2$. By countable subset theorem and $A_1 \times A_2$ countable, we have $A_1 \times \{\alpha\}$ is countable. We now construct the map $f : A_1 \times \{\alpha\} \to A_1$ as $f((b,\alpha)) = b$. It's easy to see f is bijective and by bijection theorem A_1 is countable which contradicts the assumption A_1 is uncountable. So the assumption is false, we prove if A_1 is uncountable then $A_1 \times A_2$ is uncountable if $A_2 \neq \phi$.

Example 10. *S is the set of all circles on the plane with rational radii and centers with rational coordinates. Then it is countable. But all squares on the plane with rational side length and centers with rational coordinates is uncountable.*

Solution: Let $f : S \to \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ defined as f map the circle with radius r and center with coordinates (a,b) into (r,a,b). It's easy to see f is surjective because every element $(r,a,b) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ can be mapped by circle with radius r and center with coordinates (a,b). For any different circle $T_1, T_2 \in S$. $f(T_1) \neq f(T_2)$. Then f is injective. So f is bijective. By bijection theorem, S and $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ have the same cardinality and hence S is countable.
For the square case, because the square is different after rotate some angle. We thus can easily show it is uncountable by constructing some map.