Math 202 (Introduction to Real Analysis) Spring 2008

Spring Midterm

Directions: This is a closed book exam. Detailed works must be shown legibly to receive credits. Answers alone are worth very little. Calculators are allowed.

Notations: $\mathbb{R}$ denotes the set of all real numbers.

1. (8 marks) Prove that $\lim_{x\to 1} \frac{3x}{x^2+2} = 1$ by checking the ε-δ definition of limit of function.

   (Do not use any computation formula, sandwich theorem or l'Hopital's rule, otherwise, you will get zero mark.)

---

2. (8 marks) Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of real numbers. Let $b_n \in \mathbb{R}$ satisfy

$$a_n \leq b_n \leq a_n + \frac{1}{n} \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

   Prove that $b_1, b_2, b_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

   (Do not use Cauchy theorem that said a sequence converges if and only if it is a Cauchy sequence, otherwise you will get zero mark.)

---

3. (8 marks) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$. Prove that there exists at least one $x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$.

---

4. (9 marks) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. There are $a, b \geq 0$ such that for all $x \in [0, 1]$, we have $|f(x)| \leq a$ and $|f''(x)| \leq b$. Prove that for every $c \in (0, 1)$ such that

$$|f'(c)| \leq 2a + \frac{1}{2}b.$$

–End of Paper–

Solutions to Math 202 Spring Midterm (2008)

① (Scratch $\left|\frac{3x}{x^2+2}-1\right| = \frac{|x^2-3x+2|}{x^2+2} \le \frac{|x-2||x-1|}{2} < \frac{2|x-1|}{2} = |x-1| < \varepsilon$

$|x-1|<1 \Rightarrow x\in(0,2) \Rightarrow x-2\in(-2,0)$.)

$\forall \varepsilon>0$, let $\delta=\min(1,\varepsilon)$. Then

$$0<|x-1|<\delta \Rightarrow \begin{matrix}|x-1|<1\\ \text{and}\\ |x-1|<\varepsilon\end{matrix} \Rightarrow \begin{matrix}x\in(0,2)\\ \text{and}\\ |x-1|<\varepsilon\end{matrix} \Rightarrow \left|\frac{3x}{x^2+2}-1\right| = \frac{|x^2-3x+2|}{x^2+2} \le \frac{|x-2||x-1|}{2} < \frac{2}{2}|x-1| = |x-1| < \varepsilon.$$

($\Rightarrow x-2\in(-2,0)$)

② (Scratch $|b_m-b_n| \le |b_n-a_n|+|a_n-a_m|+|a_m-b_m| \le \frac{1}{n}+|a_n-a_m|+\frac{1}{m}$.)

$\forall\varepsilon>0$, since $\{a_n\}$ is Cauchy, $\exists K_1\in\mathbb{N}$ such that $m,n\ge K_1 \Rightarrow |a_n-a_m|<\frac{\varepsilon}{3}$.

Next let $K_2>\frac{3}{\varepsilon}$, then $m,n\ge K_2 \Rightarrow \frac{1}{m},\frac{1}{n}\le\frac{1}{K_2}<\frac{\varepsilon}{3}$.

Let $K=\max(K_1,K_2)$. Then $m,n\ge K \Rightarrow m,n\ge K_1$ and $m,n\ge K_2$

$$\Rightarrow |b_m-b_n| \le |b_n-a_n|+|a_n-a_m|+|a_m-b_m| \le \frac{1}{n}+|a_n-a_m|+\frac{1}{m} < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

③ By the extreme value theorem, $\sup\{f(x): x\in[0,2\pi]\}=M$, $\inf\{f(x):x\in[0,2\pi]\}=m$ exist in $\mathbb{R}$. Since $f(x+2\pi)=f(x)$, so $\forall x\in\mathbb{R}$, $m\le f(x)\le M$.

Now $g(t)=f(t)-t$ is continuous on $\mathbb{R}$. We have

$g(M)=f(M)-M\le 0$ and $g(m)=f(m)-m\ge 0$.

By the intermediate value theorem, $\exists x_0\in[m,M]$ such that $g(x_0)=0$.

Then $f(x_0)=x_0$.

④ $\forall c\in(0,1)$, by Taylor's theorem, there are $\theta_0\in(0,c)$ and $\theta_1\in(c,1)$ such that

$$f(1)=f(c)+f'(c)(1-c)+\frac{f''(\theta_1)}{2}(1-c)^2$$

and

$$f(0)=f(c)+f'(c)(0-c)+\frac{f''(\theta_0)}{2}(0-c)^2.$$

Subtracting these, we get

$$f(1)-f(0)=f'(c)+\frac{1}{2}\left(f''(\theta_1)(1-c)^2-f''(\theta_0)c^2\right).$$

Solving for $f'(c)$, we get

$$f'(c)=f(1)-f(0)+\frac{1}{2}\left(f''(\theta_0)c^2-f''(\theta_1)(1-c)^2\right).$$

Then

$$|f'(c)| \le |f(1)|+|f(0)|+\frac{1}{2}\left(|f''(\theta_0)|c^2+|f''(\theta_1)|(1-c)^2\right) \le 2a+\frac{1}{2}b\left(c^2+(1-c)^2\right).$$

Now $0<c<1 \Rightarrow c^2<c$ and $(1-c)^2<(1-c) \Rightarrow c^2+(1-c)^2<c+(1-c)=1$

$\Rightarrow |f'(c)| \le 2a+\frac{1}{2}b$