Integral Criterion for 215 (b)

May 6, 2016

215(b). (2008 Spring Exam) (b) For $n = 1, 2, 3, \cdots$, let $f_n : [0, 1] \to [0, 1]$ be Riemann integrable functions.
Prove that $g : [0, 1] \to \mathbb{R}$ defined by

$$g(0) = 0 \text{ and } g(x) = f_n(x) \text{ for } n = 1, 2, 3, \cdots \text{ and } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$$

is Riemann integrable on $[0, 1]$.

Proof. For any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $k + 1 > \dfrac{2}{\epsilon}$.

Since for each $n \in \mathbb{N}$, f_n is Riemann Integrable on $[0, 1]$, $f_n|_{[\frac{1}{n+1}, \frac{1}{n}]}$ is also Riemann Integrable on $\left[\dfrac{1}{n+1}, \dfrac{1}{n}\right]$. There exists a partition P_n of $\left[\dfrac{1}{n+1}, \dfrac{1}{n}\right]$ such that $U(f_n, P_n) - L(f_n, P_n) \leq \dfrac{\epsilon}{2^{n+2}}$

Let $\delta = \min\left\{\dfrac{\epsilon}{4k}, \dfrac{1}{2}\left(\dfrac{1}{k} - \dfrac{1}{k+1}\right)\right\} > 0$,

Consider a partition, $P = \left\{0, \dfrac{1}{k+1}\right\} \cup P'$, where $P' = P_k \cup \cdots \cup P_2 \cup P_1 \cup \left\{\dfrac{1}{i+1} + \delta : i = 2, \cdots, k\right\}$.

Note that $g = f_j$ on $\left(\dfrac{1}{j+1}, \dfrac{1}{j}\right]$,

$$U(g, P_j \cup \left\{\frac{1}{j+1} + \delta\right\} \setminus \left\{\frac{1}{j+1}\right\}) - L(g, P_j \cup \left\{\frac{1}{j+1} + \delta\right\} \setminus \left\{\frac{1}{j+1}\right\}) \leq U(f_j, P_j) - L(f_j, P_j)$$

Note that $0 \leq g \leq 1$ since $0 \leq f_n \leq 1$ for all $n \in \mathbb{N}$ and $g(0) = 0$, we get

$$U(g, \left\{\frac{1}{j+1}, \frac{1}{j+1} + \delta\right\}) - L(g, \left\{\frac{1}{j+1}, \frac{1}{j+1} + \delta\right\} \leq (1 - 0) \times \left(\frac{1}{j+1} + \delta - \frac{1}{j+1}\right) = \delta$$

Also,

$$\begin{aligned}
&U(g, P') - L(g, P') \\
&= \sum_{j=1}^{k} \left(U(g, P_j \cup \left\{\frac{1}{j+1} + \delta\right\} \setminus \left\{\frac{1}{j+1}\right\}) - L(g, P_j \cup \left\{\frac{1}{j+1} + \delta\right\} \setminus \left\{\frac{1}{j+1}\right\})\right) \\
&\quad + \sum_{j=1}^{k} \left(U(g, \left\{\frac{1}{j+1}, \frac{1}{j+1} + \delta\right\}) - L(g, \left\{\frac{1}{j+1}, \frac{1}{j+1} + \delta\right\})\right)
\end{aligned}$$

Then

$$
\begin{aligned}
U(g,P)-L(g,P) \leq& 1 \times \left(\frac{1}{k+1} - 0\right) + U(g,P') - L(g,P') \\
\leq& \frac{1}{k+1} + U(g,P') - L(g,P') \\
\leq& \frac{1}{k+1} + \sum_{j=1}^{k} \left(U(f_j,P_j) - L(f_j,P_j)\right) + \sum_{j=1}^{k} \delta \\
\leq& \frac{\epsilon}{2} + \sum_{j=1}^{k} \frac{\epsilon}{2^{j+2}} + \sum_{j=1}^{k} \frac{\epsilon}{4k} \\
\leq& \frac{\epsilon}{2} + \sum_{j=1}^{\infty} \frac{\epsilon}{2^{j+2}} + \sum_{j=1}^{k} \frac{\epsilon}{4k} \\
=& \frac{\epsilon}{2} + \frac{\epsilon}{2^3} \frac{1}{1-\frac{1}{2}} + \sum_{j=1}^{k} \frac{\epsilon}{4k} \\
=& \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\
=& \epsilon
\end{aligned}
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