MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #25

Limit Superior and Limit Interior

Definition: (Limit superior and Limit inferior)

Given a sequence $\{a_1, a_2, a_3, \ldots.\}$, we define the limit superior and limit inferior denoted by $\mathrm{limsup}_{n\to\infty} a_n$ and $\mathrm{liminf}_{n\to\infty} a_n$ by

$$\mathrm{limsup}_{n\to\infty} a_n = \sup\{z: z \text{ is limit of some subsequences } a_{n_i}\}$$

$$\mathrm{liminf}_{n\to\infty} a_n = \inf\{z: z \text{ is limit of some subsequences } a_{n_i}\}$$

Using $M_k$ and $m_k$ theorem, one can rewrite the limsup and liminf in another way.

Theorem: ($M_k, m_k$ theorem)

$$\mathrm{limsup}_{n\to\infty} a_n = \lim_{k\to\infty} \sup\{x_k, x_{k+1}, \ldots.\}$$

$$\mathrm{liminf}_{n\to\infty} a_n = \lim_{k\to\infty} \inf\{x_k, x_{k+1}, \ldots.\}$$

(Remark: In many textbook use these as the definition of limsup and liminf).

Furthermore, one can show the following

$$\lim_{n\to\infty} a_n \ \textit{exists} \ \leftrightarrow \mathrm{limsup}_{n\to\infty} a_n = \mathrm{liminf}_{n\to\infty} a_n$$

Example 1

Find the limit superior and limit inferior of the following sequences

a) $x_n = (-1)^n$

b) $x_n = \sin\left(\frac{n\pi}{3}\right)$

c) $x_n = 2^{(-1)^n}\left(1 + \frac{1}{n^2}\right) + 3^{(-1)^{n+1}}$

d) $x_n = 2^{n\cos\left(\frac{2n\pi}{3}\right)}$

Solution:

a) Note that $\sup\{x_k, x_{k+1}, x_{k+2}, \ldots\} = \sup\{(-1)^k, (-1)^{k+1}, \ldots\} = \sup\{-1,1\} = 1$

Thus $\mathrm{limsup}_{n\to\infty} x_n = \lim_{k\to\infty} \sup\{x_k, x_{k+1}, \ldots\} = \lim_{k\to\infty} 1 = 1$

Similarly $\inf\{x_k, x_{k+1}, x_{k+2}, \ldots\} = \inf\{(-1)^k, (-1)^{k+1}, \ldots\} = \inf\{-1,1\} = -1$

Thus $\mathrm{liminf}_{n\to\infty} x_n = \lim_{k\to\infty} \inf\{x_k, x_{k+1}, \ldots\} = \lim_{k\to\infty} -1 = -1$

b) We first write out some terms,

$$\left\{\sin\left(\frac{\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right), \sin\left(\frac{3\pi}{3}\right), \sin\left(\frac{4\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right), \sin\left(\frac{6\pi}{3}\right), \sin\left(\frac{7\pi}{3}\right), \sin\left(\frac{8\pi}{3}\right), \ldots\right\}$$

$$= \left\{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, \ldots.\right\}$$

We see the sequence is repeated every 6 terms.

Hence $\sup\{x_k, x_{k+1}, x_{k+2}, \ldots\} = \sup\left\{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\right\} = \frac{\sqrt{3}}{2}$

Thus $\limsup_{n\to\infty} x_n = \lim_{k\to\infty} \sup\{x_k, x_{k+1}, \ldots\} = \lim_{k\to\infty} \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

Hence $\inf\{x_k, x_{k+1}, x_{k+2}, \ldots\} = \inf\left\{\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0\right\} = -\frac{\sqrt{3}}{2}$

Thus $\liminf_{n\to\infty} x_n = \lim_{k\to\infty} \inf\{x_k, x_{k+1}, \ldots\} = \lim_{k\to\infty} -\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$

c) Once we try to write out a few terms

$$\left\{2^{(-1)^1}\left(1+\frac{1}{1^2}\right) + 3^{(-1)^2}, 2^{(-1)^2}\left(1+\frac{1}{2^2}\right) + 3^{(-1)^3}, 2^{(-1)^3}\left(1+\frac{1}{3^2}\right) + 3^{(-1)^4}, \ldots\right\}$$

$$= \left\{\frac{1}{2}\left(1+\frac{1}{1^2}\right) + 3, 2\left(1+\frac{1}{2^2}\right) + \frac{1}{3}, \frac{1}{2}\left(1+\frac{1}{3^2}\right) + 3, 2\left(1+\frac{1}{4^2}\right) + \frac{1}{3}, \ldots..\right\}$$

$\sup\{x_k, x_{k+1}, x_{k+2}, \ldots\}$

$$= \begin{cases} \sup\left\{\frac{1}{2}\left(1+\frac{1}{k^2}\right) + 3, 2\left(1+\frac{1}{(k+1)^2}\right) + \frac{1}{3}\right\} & \textit{if } k \textit{ is odd} \\ \sup\left\{\frac{1}{2}\left(1+\frac{1}{(k+1)^2}\right) + 3, 2\left(1+\frac{1}{k^2}\right) + \frac{1}{3}\right\} & \textit{if } k \textit{ is even} \end{cases}$$

$$\limsup_{n\to\infty} x_n = \lim_{k\to\infty} \sup\{x_k, x_{k+1}, \ldots\} = \sup\left\{\frac{1}{2} + 3, 2 + \frac{1}{3}\right\} = \frac{7}{2}$$

One can show the $\liminf_{n\to\infty} x_n = \frac{7}{3}$. (We omitted the proof here)

d) Note that $\cos\left(\frac{2\pi n}{3}\right)$ is repeated for every 3 terms, hence it suggests we can divide the sequence into 3 parts

$$\{x_1, x_4, x_7, x_{10}, \ldots\} = \left\{2^{1\cos\left(\frac{2\pi}{3}\right)}, 2^{4\cos\left(\frac{2\pi}{3}\right)}, 2^{7\cos\left(\frac{2\pi}{3}\right)}, \ldots\right\} = \left\{2^{-\frac{1}{2}}, 2^{-\left(\frac{4}{2}\right)}, 2^{-\left(\frac{7}{2}\right)}, \ldots\right\}$$

$$\{x_2, x_5, x_8, x_{11} \ldots\} = \left\{2^{2\cos\left(\frac{4\pi}{3}\right)}, 2^{5\cos\left(\frac{4\pi}{3}\right)}, 2^{8\cos\left(\frac{4\pi}{3}\right)}, \ldots\right\} = \left\{2^{-\frac{2}{2}}, 2^{-\left(\frac{5}{2}\right)}, 2^{-\left(\frac{8}{2}\right)}, \ldots\right\}$$

$$\{x_3, x_6, x_9, x_{12} \ldots\} = \left\{2^{3\cos\left(\frac{6\pi}{3}\right)}, 2^{6\cos\left(\frac{6\pi}{3}\right)}, 2^{9\cos\left(\frac{6\pi}{3}\right)}, \ldots\right\} = \{2^3, 2^6, 2^9, \ldots\}$$

Then $\sup\{x_k, x_{k+1}, \dots\} = \begin{cases} \sup\{x_{3k}, x_{3k+1}, \dots\} \\ \sup\{x_{3k+1}, x_{3k+2}, \dots\} \\ \sup\{x_{3k+2}, x_{3k+3}, \dots\} \end{cases} = +\infty$

And $\inf\{x_k, x_{k+1}, \dots\} = \begin{cases} \inf\{x_{3k}, x_{3k+1}, \dots\} \\ \inf\{x_{3k+1}, x_{3k+2}, \dots\} \\ \inf\{x_{3k+2}, x_{3k+3}, \dots\} \end{cases} = 0$

So $\limsup_{n\to\infty} x_n = \infty$ and $\liminf_{n\to\infty} x_n = 0$

Example 2

Find the limit superior and limit inferior of

$$a_n = n\sin(e^{-n})$$

Solution:

Note that $e^{-n} < e^{-1} < 1$, so $\sin e^{-n} > 0$

$$0 \le \lim_{n\to\infty} n\sin(e^{-n}) \le \lim_{n\to\infty} ne^{-n} = \lim_{n\to\infty} \frac{n}{e^n} = \lim_{n\to\infty} \frac{1}{e^n} = 0$$

Hence $\lim_{n\to\infty} n\sin(e^{-n}) = 0$

So $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = 0$

(Remark: When the limit of the sequence exists, then the limsup and liminf will equal to the limit of the sequence.)

Theorem: (Strong Root Test)

Given $\sum_{n=1}^{\infty} a_n$, if

a) $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely

b) $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

c) $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then NO information is given

Theorem: (Strong Ratio Test)

Given $\sum_{n=1}^{\infty} a_n$, if

a) $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, $\sum_{n=1}^{\infty} a_n$ converges absolutely

b) $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, $\sum_{n=1}^{\infty} a_n$ diverges

These two roots are stronger than the one we learnt in Chapter 4 (Series). It can deal with some series which $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ may not exists.

To see this, let us look at the following examples

| Example 3 |
|---|
| Discuss the convergence of the series |
| $$\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots$$ |

| **Before** | **After** |
|---|---|
| (Root test) | (Strong Root test) |
| Since consider two subsequences | Note that |
| $\{a_1, \sqrt[3]{a_3}, \sqrt[5]{a_5}, \dots.\}$ and $\{\sqrt{a_2}, \sqrt[4]{a_4}, \sqrt[6]{a_6}, \dots\}$ | $\{a_1, \sqrt[2]{a_2}, \sqrt[3]{a_3}, \sqrt[4]{a_4}, \sqrt[5]{a_5}, \sqrt[6]{a_6} \dots\}$ |
| $\lim_{k\to\infty} \sqrt[2k-1]{a_{2k-1}} = \lim_{k\to\infty} \sqrt[2k-1]{\frac{1}{2^k}}$ | $= \{\frac{1}{2}, \sqrt{\frac{1}{3}}, \frac{1}{2^{\frac{2}{3}}}, \sqrt{\frac{1}{3}}, \frac{1}{2^{\frac{3}{5}}}, \sqrt{\frac{1}{3}} \dots.\}$ |
| $= \lim_{k\to\infty} 2^{-\frac{k}{2k-1}} = 1/\sqrt{2}$ | We see for large k |
| $\lim_{k\to\infty} \sqrt[2k]{a_{2k}} = \cdots = 1/\sqrt{3}$ | $\sup\{\sqrt[k]{a_k}, \sqrt[k+1]{a_{k+1}}, \dots.\} = \frac{1}{\sqrt{2}}$ |
| $\therefore \lim_{n\to\infty} \sqrt[n]{a_n}$ does not exist | $\limsup_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\sqrt{2}} < 1$ |
| (No conclusion) | (Hence the series converges) |
| (Ratio Test) | (Strong Ratio Test) |
| Consider two subsequences | Note that |
| $\left\{\frac{a_2}{a_1}, \frac{a_4}{a_3}, \frac{a_6}{a_5}, \dots.\right\}$ and $\left\{\frac{a_3}{a_2}, \frac{a_4}{a_3}, \frac{a_6}{a_5}, \dots.\right\}$ | $\left\{\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \frac{a_5}{a_4} \dots.\right\}$ |
| $\lim_{k\to\infty} \frac{a_{2k}}{a_{2k-1}} = \lim_{k\to\infty} \frac{\frac{1}{3^k}}{\frac{1}{2^k}} = \lim_{k\to\infty} \left(\frac{2}{3}\right)^k = 0$ | $= \left\{\frac{2}{3}, \frac{3}{2^2}, \frac{2^2}{3^2}, \frac{3^2}{2^3}, \frac{2^3}{3^3}, \dots.\right\}$ |
| $\lim_{k\to\infty} \frac{a_{2k+1}}{a_{2k}} = \lim_{k\to\infty} \frac{\frac{1}{2^{k+1}}}{\frac{1}{3^k}} = \lim_{k\to\infty} \frac{1}{2}\left(\frac{3}{2}\right)^k = \infty$ | We see |
| So $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exists | $\sup\left\{\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \frac{a_5}{a_4} \dots.\right\} = \infty$ |
| (No conclusion) | $\limsup_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = +\infty > 1$ |
| | (No conclusion about convergence) |
| | And $\liminf_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = 0 < 1$ |
| | (No conclusion also) |

In the above result, we see strong root test gives a more powerful result. Another interesting discovery is between strong root test and strong ratio test, only strong root test can give a result while another test does not give any conclusion.

See one more example for this issue.

Example 4

Discuss the convergence of the series

$$\frac{1}{2}+1+\frac{1}{8}+\frac{1}{4}+\frac{1}{32}+\frac{1}{16}+\frac{1}{128}+\frac{1}{64}+\cdots.$$

Solution:

(If we use strong root test)

$$\{a_1, \sqrt{a_2}, \sqrt[3]{a_3}, \sqrt[4]{a_4}, \ldots\ldots\} = \{\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2^{\frac{1}{2}}}, \frac{1}{2}, \frac{1}{2^{\frac{2}{3}}}, \ldots\ldots\}$$

We see $\sup\{\sqrt[k]{a_k}, \sqrt[k+1]{a_{k+1}}, \ldots.\} = \begin{cases} \frac{1}{2^{\frac{k}{k+1}}} & \text{if } n \text{ is even} \\ \frac{1}{2^{\frac{k+1}{k+2}}} & \text{if } n \text{ is odd} \end{cases}$

And $\limsup_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sup\{\sqrt[k]{a_k}, \sqrt[k+1]{a_{k+1}}, \ldots.\} = \frac{1}{2} < 1$

So the series converges.

(If we use strong ratio test)

$$\left\{\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \frac{a_5}{a_4}, \ldots\right\} = \{2, \frac{1}{8}, 2, \frac{1}{8}, 2, \frac{1}{8}, \ldots.\}$$

One can easily show

$$\sup\left\{\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \frac{a_5}{a_4} \ldots.\right\} = 2 \ \ and \ \ \inf\left\{\frac{a_2}{a_1}, \frac{a_3}{a_2}, \frac{a_4}{a_3}, \frac{a_5}{a_4} \ldots.\right\} = \frac{1}{8}$$

Hence $\limsup_{k\to\infty} \left(\frac{a_{k+1}}{a_k}\right) = 2 > 1$ *and* $\liminf_{k\to\infty} \left(\frac{a_{k+1}}{a_k}\right) = \frac{1}{8} < 1$

So no conclusion can be drawn.

From the above two examples, we see root test can be applied to more series than ratio test although ratio test does not require too much computation.

Try to work on the following exercises. You may submit your work to me for comments.

☺Exercise 1

Find the limit superior and limit inferior of the following sequences

a) $a_n = \tan(\frac{2n\pi}{5})$

b) $a_n = n^{\sin\left(\frac{n\pi}{2}\right)}$

c) $a_n = 3\sin\left(\frac{n\pi}{2}\right) + (-1)^n\left(2 + \frac{1}{n}\right)$

d) $a_n = -\frac{n}{4} + \left[\frac{n}{4}\right] + (-1)^n$ where $[n]$ is the greatest integer less than or equal to n.

e) $\{a_n\} = \left\{\frac{2}{3}, \frac{1}{3}, \frac{3}{4}, \frac{1}{4}, \frac{4}{5}, \frac{1}{5}, \frac{5}{6}, \dots\dots\right\}$

f) $a_n = \left(1 + \frac{(-1)^n}{n}\right)^n$

☺Exercise 2

Check whether the following series converges or not

(Using both strong root test and strong ratio test)

a) $a + 1 + a^3 + a^2 + a^5 + a^4 + a^7 + \cdots \dots$

b) $1 - \frac{1}{3\times 2^2} + \frac{1}{5\times 3^2} - \frac{1}{7\times 4^2} + \cdots$

c) $0 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} + \frac{2}{3^2} - \frac{1}{4} + \frac{3}{4^2} - \frac{1}{5} + \frac{4}{5^2} - \frac{1}{6} + \frac{5}{6^2} - \cdots$

☺Exercise 3

Find all possible $r \in \mathbf{R}$ such that

$$1 + 2r + r^2 + 2r^3 + r^4 + 2r^5 + r^6 + 2r^7 + \cdots$$

(Hint: use strong root test or ratio test first, then for the remaining r which root test does not give any conclusion (i.e. $\limsup_{n\to\infty} a_n = 1$). Check the convergence using other test.

☺Exercise 4

Find all possible $a, b \in \mathbf{R}$ such that the series

$$1 + a + ab + a^2b + a^2b^2 + a^3b^2 + a^3b^3 + a^4b^3 + \cdots$$

Converges?

(Hint: The idea is same as Exercise 3)