MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #23

Riemann Integration (Part 3): Improper Integral

For improper integral, we deal with some integrals which (1) $f(x)$ is unbounded (somewhere within the interval), (2) integrate on unbounded interval (e.g. $[a,\infty)$].
In this case, we treat such integral as a limit of proper integration (provided that **$f(x)$ is locally integrable on that interval**) and see whether the limit exists or not.

If the limit is a number, we say the integral converges and $f(x)$ is improper integrable and if the limit does not exists or $=\infty$, we say the integral diverges and the function is not improper integrable.

Example 1 (Unbounded interval)
Discuss the convergence of $\int_0^\infty e^{-ax}\,dx$ (where $a>0$)

Solution:
First, e^{-ax} is continuous on $\mathbf{R}\rightarrow$ integrable on $[0,c]$ for any $c\in\mathbf{R}$. So

$$\int_0^\infty e^{-ax}\,dx=\lim_{c\to\infty}\int_0^c e^{-ax}\,dx=\lim_{c\to\infty}-\frac{e^{-ax}}{a}\Big|_0^c=\lim_{c\to\infty}\left(\frac{1}{a}-\frac{e^{-ac}}{a}\right)=\frac{1}{a}$$

Hence the integral converges

☺Exercise 0
In example 1, how about the case when $a\le 0$?

Example 2 (Unbounded interval)
Discuss the convergence of $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Solution:
We first split the integral into 2 parts

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx=\int_0^\infty \frac{1}{1+x^2}\,dx+\int_{-\infty}^0 \frac{1}{1+x^2}\,dx=\lim_{c\to\infty}\int_0^c \frac{1}{1+x^2}\,dx+\lim_{d\to-\infty}\int_d^0 \frac{1}{1+x^2}\,dx$$

$$=\lim_{c\to\infty}\tan^{-1}c-\lim_{d\to-\infty}\tan^{-1}d=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi$$

Hence the integral converges

(Caution: It is wrong for students to use

$$\int_0^\infty \frac{1}{1+x^2}\,dx+\int_{-\infty}^0 \frac{1}{1+x^2}\,dx=\lim_{c\to\infty}\left(\int_0^c \frac{1}{1+x^2}\,dx+\int_{-c}^0 \frac{1}{1+x^2}\,dx\right)$$

Example 3

Discuss the convergence of $\int_{-\infty}^{\infty} \cos x dx$

$$\int_{-\infty}^{\infty} \sin x dx = \int_{0}^{\infty} \sin x dx + \int_{-\infty}^{0} \sin x dx = \lim_{c\to\infty} \int_{0}^{c} \sin x dx + \lim_{d\to-\infty} \int_{d}^{0} \sin x dx$$

$$= \lim_{c\to\infty} (1 - \cos c) - \lim_{d\to-\infty} (1 - \cos d)$$

Which both limits does not exists, hence the integral diverges

Which the limit does not exists, hence the integral diverges

(☹**Caution:**

It is wrong for students to consider

$$\int_{-\infty}^{\infty} \sin x dx = \int_{0}^{\infty} \sin x dx + \int_{-\infty}^{0} \sin x dx = \lim_{c\to\infty} \left(\int_{0}^{c} \sin x dx + \int_{-c}^{0} \sin x dx \right)$$

$$= \lim_{c\to\infty} (1 - \cos c - 1 + \cos(-c)) = \lim_{c\to\infty} 0 = 0)$$

Example 4

Discuss the convergence of $\int_{-1}^{1} \frac{e^{\frac{1}{x}}}{x^2} dx$

Solution:

Since the function is undefined at $x = 0$, hence we split the integral in the following ways:

$$\int_{-1}^{1} \frac{e^{\frac{1}{x}}}{x^2} dx = \int_{0}^{1} \frac{e^{\frac{1}{x}}}{x^2} dx + \int_{-1}^{0} \frac{e^{\frac{1}{x}}}{x^2} dx = \lim_{c\to 0^+} \int_{c}^{1} \frac{e^{\frac{1}{x}}}{x^2} dx + \lim_{d\to 0^-} \int_{-1}^{d} \frac{e^{\frac{1}{x}}}{x^2} dx$$

Note

$$\lim_{c\to 0^+} \int_{c}^{1} \frac{e^{\frac{1}{x}}}{x^2} dx = \lim_{c\to 0^+} \left(-e + e^{\frac{1}{c}} \right) = +\infty$$

$$\lim_{d\to 0^-} \int_{-1}^{d} \frac{e^{\frac{1}{x}}}{x^2} dx = \lim_{d\to 0^-} \left(-e^{\frac{1}{d}} + e^{-1} \right) = e^{-1}$$

Since one of the integral diverges, hence the original diverges also.

<u>Theorem 1: Comparison Test</u>

Let $\boldsymbol{f(x), g(x) \geq 0}$ and $f(x) \leq g(x)$ on I, suppose $f(x), g(x)$ are locally integrable on I, then

a) If $g(x)$ is improper integrable, then $f(x)$ is improper integrable

b) If $f(x)$ is not improper integrable, then $g(x)$ is notimproper integrable

Theorem 2: Limit Comparison Test

Suppose $f(x), g(x) > 0$ on $(a, b]$ and are locally integrable on $(a, b]$,

Case i) If $\lim_{x\to a^+} \frac{g(x)}{f(x)} = L > 0$,

then either both $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converges or both diverges.

Case ii) If $\lim_{x\to a^+} \frac{g(x)}{f(x)} = 0$,

Then $\int_a^b f(x)dx$ converges $\rightarrow$ $\int_a^b g(x)dx$ converges

and $\int_a^b g(x)dx$ diverges $\rightarrow$ $\int_a^b f(x)dx$ diverges

Case iii) If $\lim_{x\to a^+} \frac{g(x)}{f(x)} = \infty$,

Then $\int_a^b g(x)dx$ converges $\rightarrow$ $\int_a^b f(x)dx$ converges

and $\int_a^b f(x)dx$ diverges $\rightarrow$ $\int_a^b g(x)dx$ diverges

(Note: Similar result can be obtained for $[a, b)$ by taking $\lim_{x\to b^-} \frac{g(x)}{f(x)}$

In case the function is not non-negative, taking absolute value can help us to convert the function into non-negative one.

Theorem 3 (Absolute Convergence Test)

If $f(x)$ is locally integrable on I and $|f(x)|$ is improper integrable on I, then $f(x)$ is also is improper integrable on I.

Example 5

Discuss the convergence of $\int_1^\infty \frac{1}{\sqrt{x^3+e^{-2x}+\ln x+1}} dx$

Solution:

Note that in $[1, \infty)$, $e^{-2x} \geq 0$, $\ln x \geq \ln 1 = 0$, then

$$\frac{1}{\sqrt{x^3 + e^{-2x} + \ln x + 1}} \leq \frac{1}{\sqrt{x^3 + 0 + 0 + 1}} < \frac{1}{\sqrt{x^3}} = \frac{1}{x^{\frac{3}{2}}}$$

Note that

$$\int_1^\infty \frac{1}{x^{\frac{3}{2}}} dx = \lim_{c\to\infty} \int_1^c \frac{1}{x^{\frac{3}{2}}} dx = \lim_{c\to\infty} -2x^{-\frac{1}{2}}|_1^c = \lim_{c\to\infty} -2c^{-\frac{1}{2}} + 2 = 2 < \infty$$

We see this integral converges, so by comparison test, $\int_1^\infty \frac{1}{\sqrt{x^3+e^{-2x}+\ln x+1}} dx$ converges.

☹**<u>Important Note</u>**

Some students may consider (for $x^3 \geq 1 > 0$)

$$\frac{1}{\sqrt{x^3 + e^{-2x} + \ln x + 1}} \le \frac{1}{\sqrt{0 + e^{-2x} + 0 + 1}} < \frac{1}{\sqrt{e^{-2x}}} = e^x$$

The inequality is OK, but one can check that $\int_1^\infty e^x dx = \infty$, then we can not draw any conclusion from comparison test. So this approach is NOT useful.

Example 6
Discuss the convergence of $\int_0^1 \frac{1}{(1+x^3)\ln(1+x)} dx$

Solution:

Since $f(x) = \frac{1}{(1+x^3)\ln(1+x)}$ is undefined at $x = 0$, and $f(x)$ is continuous on $(0,1]$ and therefore locally continuous on $(0,1]$. Pick $g(x) = \frac{1}{x}$, then

$$\lim_{x\to 0^+} \frac{g(x)}{f(x)} = \lim_{x\to 0^+} \frac{(1+x^3)\ln(1+x)}{x} = \cdots = 1$$

Now $\int_0^1 \frac{1}{x} dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x} dx = \lim_{c\to 0^+} -\ln c = +\infty$

Hence the integral diverges, so by limit comparison test, $\int_0^1 \frac{1}{(1+x^3)\ln(1+x)} dx$ diverges.

Example 7
Discuss the convergence of $\int_0^1 \frac{1}{\sqrt{x}(1-x)^{\frac{1}{3}}} dx$

Note that the function is undefined on $x = 0,1$, then the function is locally integrable on (0,1). If we wish to apply limit comparison test, we need a bit adjustment because the limit comparison test can only apply to $(a,b]$ or $[a,b)$ but not (a,b). So we need to split the integral into two parts.

$$\int_0^1 \frac{1}{\sqrt{x}(1-x)^{\frac{1}{3}}} dx = \underbrace{\int_{\frac{1}{2}}^1 \frac{1}{\sqrt{x}(1-x)^{\frac{1}{3}}} dx}_{(*)} + \underbrace{\int_0^{\frac{1}{2}} \frac{1}{\sqrt{x}(1-x)^{\frac{1}{3}}} dx}_{(**)}$$

For (*), since the function is locally integrable on $[\frac{1}{2}, 1)$We apply limit comparison test and use $g(x) = \frac{1}{(1-x)^{\frac{1}{3}}}$ and $\lim_{x\to 1^-} \frac{g(x)}{f(x)} = \lim_{x\to 1^-} \sqrt{x} = 1$.

$$\int_{\frac{1}{2}}^1 \frac{1}{(1-x)^{\frac{1}{3}}} dx = \lim_{c\to 1^-} \int_{\frac{1}{2}}^c \frac{1}{(1-x)^{\frac{1}{3}}} dx = \lim_{c\to 1^-} \frac{3(1-c)^{\frac{2}{3}}}{2} - \frac{3}{2}\left(\frac{1}{2}\right)^{\frac{2}{3}} = -\frac{3}{2}\left(\frac{1}{2}\right)^{\frac{2}{3}} < \infty$$

So by limit comparison test, integral (*) converges

For (**), since the function is locally integrable on $(0,\frac{1}{2}]$. We apply limit comparison test and use $g(x)=\frac{1}{\sqrt{x}}$ and $\lim_{x\to 0^+}\frac{g(x)}{f(x)}=\lim_{x\to 0^+}(1-x)^{\frac{1}{3}}=1$.

$$\int_0^{\frac{1}{2}}\frac{1}{\sqrt{x}}dx=\lim_{c\to 0^+}\int_c^{\frac{1}{2}}\frac{1}{\sqrt{x}}dx=\lim_{c\to 0^+}\left(2\sqrt{\frac{1}{2}}-2\sqrt{c}\right)=\sqrt{2}<\infty$$

Hence by limit comparison test, the integral (**) converges.

So $\int_0^1\frac{1}{\sqrt{x}(1-x)^{\frac{1}{3}}}dx$ converges.

Example 8 Discuss the convergence of $\int_1^\infty \frac{\sin x^p}{1+x^p}dx$ (where $p>1$)

Since $\sin x$ can be negative from $[1,\infty)$, in order to apply the comparison test (or limit comparison test), we need to first take the absolute value first.

Consider the function $\left|\frac{\sin x^p}{1+x^p}\right|$ which is clearly continuous on $[1,\infty)$ and therefore locally integrable. Now $\left|\frac{\sin x^p}{1+x^p}\right|\le\left|\frac{1}{1+x^p}\right|<\frac{1}{1+x^p}<\frac{1}{x^p}$

Note that $\int_1^\infty\frac{1}{x^p}dx=\lim_{c\to\infty}\int_1^c\frac{1}{x^p}dx=\lim_{c\to\infty}(1-p)(c^{1-p}-1)=p-1$

Hence $\int_1^\infty\frac{1}{x^p}dx$ converges and by comparison test, $\int_1^\infty\left|\frac{\sin x^p}{1+x^p}\right|dx$ converges. Finally by absolute convergence test, $\int_1^\infty\frac{\sin x^p}{1+x^p}dx$ converges.

Try to work on the following exercises. You are welcome to submit your work to me so that I can give you some comments.

☺Exercise 1

Discuss the convergence of the following integrals: (By checking the definition directly)

a) $\int_2^\infty\frac{1}{x(\ln^p x)}dx \quad (p\in\mathbf{R})$

b) $\int_{-\infty}^\infty\frac{x}{\sqrt{2x^2+5}}dx$

c) $\int_0^2\frac{1}{x(x-1)(x-2)}dx$ (Hint: Since the function is undefined on 0,1,2, split the integral into 2 parts: $\int_0^2\frac{1}{x(x-1)(x-2)}dx=\int_0^1\frac{1}{x(x-1)(x-2)}dx+\int_1^2\frac{1}{x(x-1)(x-2)}dx$)

d) $\int_0^\infty e^{\alpha x}\sin\beta x\,dx$ (for $\alpha,\beta\in\mathbf{R}$) (Hint: Integration by parts MAY be useful)

e) $\int_0^2 \frac{e^{\frac{x}{x-1}}}{(x-1)^2} dx$ (Hint: $e^{\frac{x}{x-1}} = e^{1+\frac{1}{x-1}} = e \times e^{\frac{1}{x-1}}$)

☺Exercise 2

Discuss the convergence of the following integrals

a) $\int_{-\infty}^{\infty} \frac{\cos 3x}{1+x^2} dx$ (Practice Exercise #127a) (Hint: $\cos 3x$ can be negative!)

b) $\int_0^1 \frac{\cos 3x}{\sqrt{x}} dx$ (Practice Exercise #154)

c) $\int_0^{\infty} \frac{\sin x}{x^{\frac{3}{2}}} dx$ (Practice Exercise #164) (Hint: $\lim_{x\to 0} \frac{\sin x}{x} = 1$ and split the integral into 2 parts $\int_0^{\infty} \frac{\sin x}{x^{\frac{3}{2}}} dx = \int_0^1 \frac{\sin x}{x^{\frac{3}{2}}} dx + \int_1^{\infty} \frac{\sin x}{x^{\frac{3}{2}}} dx$, for 1st interval, use limit comparison and use comparison test for 2nd integral)

d) $\int_{-1}^{1} \frac{1}{x\cos x} dx$ (2007 Spring Final)

☺Exercise 3

Show that $\int_1^{\infty} \frac{\sin x}{x^p + \sin x} dx$ i) diverges when $p \le \frac{1}{2}$, ii) converges $p > \frac{1}{2}$

(Note: For ii), the absolute convergence **FAILS** for the case $\frac{1}{2} < p \le 1$)

☺Exercise 4

Show that the integral

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} dx$$

converges if and only if $0 < \alpha < 1$.

☺Exercise 5

Determine all possible p, q such that

$$\int_0^{\infty} \frac{1}{x^p + x^q} dx$$

a) Converges b) diverges (Hint: Divide it into two cases: $p = q$ and $p \neq q$. For $p \neq q$, assume $p > q$ and $x^p + x^q = x^q(x^{p-q} + 1)$)

☺Exercise 6

Suppose $f(x) \ge 0$

a) If $[a, b]$ is a bounded interval and $\int_a^b f(x)^2 dx$ converges, then $\int_a^b f(x) dx$ converges

(Hint: Verify the inequality $f(x) \le \frac{1+f(x)^2}{2}$)

b) If $f(x)$ is locally integrable on $[a, \infty)$, $\lim_{x\to\infty} f(x) = 0$ and $\int_a^{\infty} f(x)\, dx$ converges, then $\int_a^{\infty} f(x)^2 dx$ converges also (Hint: Limit Comparison Test with $g(x) = f(x)^2$)