MATH 2031 Introduction to Real Analysis

November 26, 2012

Tutorial Note 9

Limits of Functions

(I) **Definition:**
Let $f : S \to \mathbb{R}$ be a function.

$$\lim_{x \to x_0} f(x) = L \iff \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in S, \; |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

(II) **Sequential Limit Theorem (S.L.T.)**
Let $f : S \to \mathbb{R}$ be a function and x_0 be an accumulation point of S. Then

$$\lim_{\substack{x \to x_0 \\ x \in S}} f(x) = L \iff \text{for every sequence } \{x_n\} \subset S \setminus \{x_0\} \text{ that converges to } x_0, \; \lim_{n \to \infty} f(x_n) = L$$

(III) **Monotone Function Theorem**
If f is increasing on (a, b), then

(I) $\forall x_0 \in (a, b)$,

$$\left.\begin{array}{r} f(x_0-) = sup\{f(x)|a < x < x_0\} \\ \text{and } f(x_0+) = inf\{f(x)|x_0 < x < b\} \end{array}\right. \Rightarrow f(x_0-) \le f(x_0) \le f(x_0+)$$

If f is bounded below, then $f(a+) = inf\{f(x)|a < x < b\}$.

If f is bounded above, then $f(b-) = sup\{f(x)|a < x < b\}$.

(II) f has countably many discontinuous point on (a, b). i.e.

$$J = \{x_0 | x_0 \in (a, b), f(x_0-) \neq f(x_0+)\} \text{ is countable.}$$

(IV) **One-sided Limits:**

(i) **Definition:** For $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$,
left hand limit of f at x_0: $f(x_0-) = \lim_{x \to x_0^-} f(x) = \lim_{\substack{x \to x_0 \\ x \in (a, x_0)}} f(x)$;
right hand limit of f at x_0: $f(x_0+) = \lim_{x \to x_0^+} f(x) = \lim_{\substack{x \to x_0 \\ x \in (x_0, b)}} f(x)$.

(ii) **Theorem:**
For $x_0 \in (a, b)$, $\lim_{\substack{x \to x_0 \\ x \in (a, b)}} f(x) = L \iff f(x_0-) = L = f(x_0+)$

(V) **Infinite Limits:**
Definition for sequence:

(i) $\{x_n\}$ diverges to $+\infty \iff \forall r \in \mathbb{R}, \; \exists K \in \mathbb{N}$ such that $n \ge K \Rightarrow x_n > r$
(ii) $\{x_n\}$ diverges to $-\infty \iff \forall r \in \mathbb{R}, \; \exists K \in \mathbb{N}$ such that $n \ge K \Rightarrow x_n < r$

Definition for function: Let $f : S \to \mathbb{R}$ and x_0 be an accumulation point of S

(i) $\forall r \in \mathbb{R}, \exists \delta > 0$ such that $\forall x \in S \; 0 < |x - x_0| < \delta \Rightarrow f(x) > r$
(ii) $\forall r \in \mathbb{R}, \exists \delta > 0$ such that $\forall x \in S \; 0 < |x - x_0| < \delta \Rightarrow f(x) < r$

(VI) **Limit at Infinity:**

(i) (a) $\lim_{x\to+\infty} f(x) = L \iff \forall\varepsilon > 0, \exists K \in \mathbb{R}$ such that $x \geq K \Rightarrow |f(x) - L| < \varepsilon$,

(b) $\lim_{x\to-\infty} f(x) = L \iff \lim_{x\to+\infty} f(-x) = L \iff \forall\varepsilon > 0, \exists K \in \mathbb{R}$ such that $x \leq K \Rightarrow |f(x) - L| < \varepsilon$.

(ii) (a) $\lim_{\substack{x\to+\infty\\x\in S}} f(x) = +\infty \iff \forall r \in \mathbb{R}, \exists K \in \mathbb{R}$ such that $x \geq K \Rightarrow f(x) > r$;

(b) $\lim_{\substack{x\to+\infty\\x\in S}} f(x) = -\infty \iff \lim_{\substack{x\to+\infty\\x\in S}} -f(x) = +\infty \iff \forall r \in \mathbb{R}, \exists K \in \mathbb{R}$ such that $x \geq K \Rightarrow f(x) < r$;

(c) $\lim_{\substack{x\to-\infty\\x\in S}} f(x) = +\infty \iff \lim_{\substack{x\to+\infty\\x\in S}} f(-x) = +\infty \iff \forall r \in \mathbb{R}, \exists K \in \mathbb{R}$ such that $x \leq K \Rightarrow f(x) > r$;

(d) $\lim_{\substack{x\to-\infty\\x\in S}} f(x) = -\infty \iff \lim_{\substack{x\to+\infty\\x\in S}} -f(-x) = +\infty \iff \forall r \in \mathbb{R}, \exists K \in \mathbb{R}$ such that $x \leq K \Rightarrow f(x) < r$.

Problem 1 Let $f : (0, +\infty) \to \mathbb{R}$ be defined as $f(x) = \dfrac{2x^2+1}{x^2+x+1}$. Show that $\lim_{x\to1} f(x) = 1$ by definition.

Scratch:

"Want: To find a positive δ such that $\forall\varepsilon > 0\ \exists\delta > 0\ \forall x \in S,\ |x - 1| < \delta\ \Rightarrow \left|\dfrac{2x^2+1}{x^2+x+1} - 1\right| < \varepsilon$"

$$\begin{aligned}
\left|\frac{2x^2+1}{x^2+x+1} - 1\right| &= \left|\frac{2x^2+1-x^2-x-1}{x^2+x+1}\right| \\
&= \left|\frac{x^2-x}{x^2+x+1}\right| \\
&= \left|\frac{x(x-1)}{x^2+x+1}\right| \\
&\overset{(?)}{<} \left|\frac{x(x-1)}{x^2+x}\right| \\
&= \frac{|x-1|}{|x+1|}
\end{aligned}$$

Want to hold :

1. the inequality (?)
2. $|x + 1|$ is bound below by some constant, say c. This implies the following :

$$\frac{|x-1|}{|x+1|} \leq \frac{|x-1|}{c}$$

Then take $\delta = c\varepsilon$ and get

$$\frac{|x-1|}{|x+1|} \leq \frac{|x-1|}{c} < \frac{\delta}{c} = \varepsilon$$

To make these two hold, what we need is $|x - 1| < 1$, i.e. $-1 < x - 1 < 1$, or $0 < x < 2$. Then x is positive and thus $x^2 + x + 1 > 0$, so (?) holds.

With same condition, we get $1 < x + 1 < 3$ then $|x + 1|$ is bound below by 3.

Therefore, we need to take $\delta < min\{1, \varepsilon\}$.

Solution:
For any $\varepsilon > 0$, take $\delta > 0$ and $\delta < min\{1, \varepsilon\}$. Then for all $x \in (0, +\infty)$ and $|x - 1| < \delta$,

$$\begin{aligned}\left|\frac{2x^2+1}{x^2+x+1} - 1\right| &= \left|\frac{2x^2+1-x^2-x-1}{x^2+x+1}\right| \\ &= \left|\frac{x^2-x}{x^2+x+1}\right| \\ &= \frac{|x(x-1)|}{x^2+x+1} \\ &< \frac{x|x-1|}{x^2+x} \\ &= \frac{|x-1|}{x+1} \\ &< |x_1| \\ &< \delta \\ &\leq \varepsilon.\end{aligned}$$

Thus by definition, $\lim_{x \to 1} f(x) = 1$.

Problem 2 Let $g : \mathbb{R} \to \mathbb{R}$ be defined as $g(x) = \dfrac{1}{x}$.

(a) Show that $\lim_{x \to +\infty} g(x) = 0$ by definition;

(b) Show that $\lim_{x \to 0^+} g(x) = +\infty$ and $\lim_{x \to 0^-} g(x) = -\infty$ by definition.

Solution:

(a) **Scratch:**
"Want: To find K such that $\forall \varepsilon > 0, \exists K \in \mathbb{R}$ such that $x \geq K \Rightarrow \left|\dfrac{1}{x} - 0\right| < \varepsilon$"
Here it is clear that we should take $K > \dfrac{1}{\varepsilon}$.

Solution:
For all $\varepsilon > 0$, take $K > \dfrac{1}{\varepsilon}$. Then for $x \geq K$,

$$\left|\frac{1}{x}\right| < \frac{1}{(\frac{1}{\varepsilon})} = \varepsilon.$$

Thus by definition, $\lim_{x \to +\infty} g(x) = 0$.

(b) I will only work out $\lim_{x \to 0^+} g(x) = +\infty$, the other one is similar and is left as an exercise.
Scratch:
"Want: To find $\delta > 0$ such that $\forall r \in \mathbb{R}, \exists \delta > 0$ such that $\forall x \in \mathbb{R}$, $\underbrace{0 < x - 0 < \delta}_{\text{one-sided limit}} \Rightarrow \dfrac{1}{x} > r$"
Here we can see that $0 < \delta < \frac{1}{r}$.
Solution:
$\forall r \in \mathbb{R}$, take $0 < \delta$ such that $\delta < \frac{1}{r}$. Then $\forall x \in \mathbb{R}$ $0 < x - 0 < \delta \Rightarrow \dfrac{1}{x} > \dfrac{1}{\delta} > \dfrac{1}{\frac{1}{r}} = r$.
Thus by definition, $\lim_{x \to 0^+} g(x) = +\infty$.

Problem 3 Let

$$h(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Does $\lim_{x \to \pi} h(x)$ exist?

Solution:
Here we can apply the S.L.T. (more precisely it should be the converse of S.L.T.), because we can easily find a rational sequence and an irrational sequence both converge to π but the function values at the rational ones converge to 1 and the other converge to 0.

Consider $x_n = \pi - \dfrac{1}{n}$ and $y_n = \dfrac{[10^n \pi]}{10^n}$. Then clearly we have

$$\lim_{n \to \infty} x_n = \pi = \lim_{n \to \infty} y_n.$$

However, $\lim_{n \to \infty} h(x_n) = \lim_{n \to \infty} 0 = 0$ and $\lim_{n \to \infty} h(y_n) = \lim_{n \to \infty} 1 = 1$ which is not equal.
Thus by S.L.T., $\lim_{x \to \pi} h(x)$ does not exist.