6' 3. (22 points) .

3.1 State the Taylor's theorem for a n-times differentiable function f.

8' 3.2 Approximate $\cos 0.1$ by using ~~the second~~ order Taylor expansion and estimate the error in the approximation.

8' 3.3 Let $f \in C^2(\mathbb{R})$, prove that f achieves a local maximum at $x = a$ if

$$f'(a) = 0 \quad \text{and} \quad f''(a) < 0.$$

Solution:

3.1 Let $f : (a, b) \to \mathbb{R}$ be n-times differentiable, then $\forall x_0, x \in (a, b)$, $\exists \xi$ between x_0 and x such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + \frac{f^{(n)}(\xi)}{n!}(x - x_0)^n.$$

3.2 For $x > 0$, expand $\cos x$ at point $x = 0$, we have

-3 5'

$$\boxed{\cos 0.1 = 1 - \frac{1}{2}0.1^2 + \frac{\cos\xi}{4!}0.1^4} \quad \xi \in (0, 0.1).$$

$1 - \frac{1}{2} \times 0.1^2$

Hence the second order approximation of $\cos 0.1$ is 0.995. The error can be estimate as

$$\left|\frac{\cos\xi}{4!}0.1^4\right| \le \frac{1}{4!}0.1^4 \approx 4.167 \times 10^{-6}.$$

3'

8' 3.3 We have Taylor expansion at $x = a$

a -4

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(\xi)}{2}(x - a)^2 = f(a) + \frac{f''(\xi)}{2}(x - a)^2,$$

for some ξ between x and a.

Since $f \in C^2(\mathbb{R})$, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall |x - a| < \delta$, $|f''(x) - f''(a)| < \epsilon$. Choose $\epsilon = \frac{|f''(a)|}{2}$, then when $|x - a| < \delta$, we have $|f''(x) - f''(a)| < \frac{|f''(a)|}{2}$, which implies $f''(x) < \frac{f''(a)}{2} < 0$.

So $\forall |x - a| < \delta$, $\exists \xi$ between x and a such that

$$f(x) - f(a) = \frac{f''(\xi)}{2}(x - a)^2 < 0.$$

Therefore, f achieves a local maximum at $x = a$.

$f'(a) < 0$