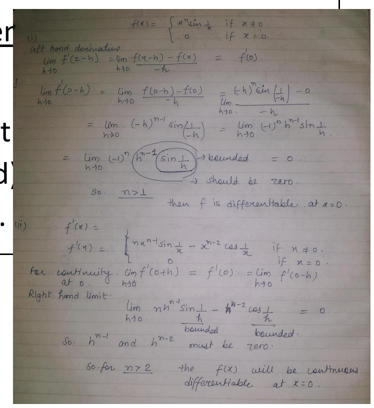
MATH2033 Mathematical Analysis (2021 Spring)
Assignment 4

Submission deadline of Assignment 4: <u>11:59p.m. of 5th May, 2020 (Wed)</u>

Instruction: Please complete all required problems. Full details (including description of methods used and explanation, key formula and theorem used and final answer) must be shown **<u>clearly</u>** to receive full credits. Marks can be deducted for incomplete solution or unclear solution.

<u>Please submit your completed work via the submission syster</u>
deadline. Late assignment will not be accepted.

Your submission must (1) be hand-written (typed assignment
(2) in a single pdf. file (other file formats will not be accepted)
full name and student ID on the first page of the assignment.



Problem 1

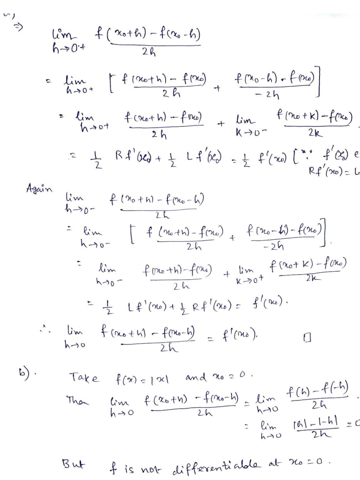
We consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^n \sin\frac{1}{x} & if \ \ x \neq 0 \\ 0 & if \ x = 0 \end{cases},$$

where $n \in \mathbb{N}$.

(a) Find the values of n which $f(x)$ is differentiable at $x = 0$.

(b) Find the values of n which $f(x)$ is continuous differentiable at $x = 0$



Problem 2

We let $f: (a, b) \to \mathbb{R}$ be a function and let $x_0 \in (a, b)$.

(a) If f is differentiable at $x = x_0$, show that

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h)}{2h} = f'(x_0) \ldots \ldots (*)$$

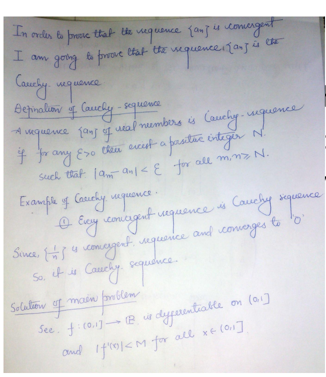
(b) If $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{2h}$ exists, is it necessary that $f(x)$ is differentiable
Explain your answer.
(☺Hint: If your answer is yes, give a mathematical proof. If your ansv
give a counter example).

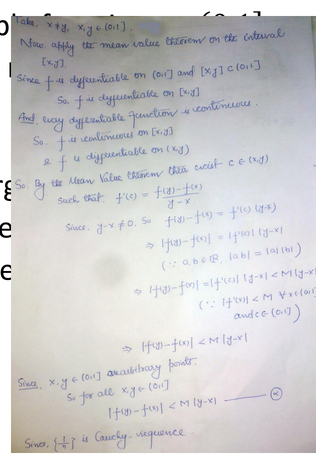
Problem 3

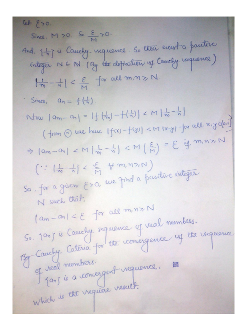
We let $f: (0,1] \to \mathbb{R}$ be a differentiab … h that $|f'(x)| < M$ for all $x \in (0,1]$, where $M > 0$ is a positive … we define

… hat the sequence $\{a_n\}$ converg …

: Be careful that $f(0)$ is not de … of f is $(0,1]$. On the other
ou can prove the convergence … its.)

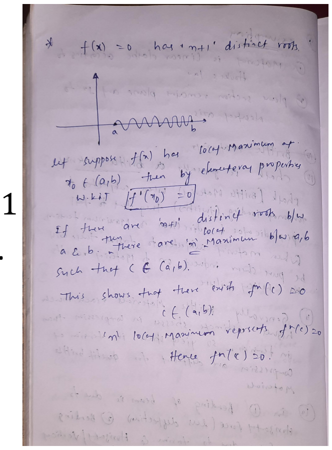






Problem 4

We let $f: [a, b] \to \mathbb{R}$ be n-times differentiable function which $f(x) = 0$ has $n + 1$ distinct roots over $[a, b]$. Show that there exists $c \in (a, b)$ such that $f^{(n)}(c) = 0$.



* $f(x) = 0$ has '$n+1$' distinct roots.

Let suppose $f(x)$ has local Maximum at $x_0 \in (a,b)$ then by elementary properties w.k.T $\boxed{f'(x_0) = 0}$

If there are '$n+1$' distinct roots b/w a & b then there are 'n' local Maximum b/w a, b such that $c \in (a,b)$.

This shows that there exists $f^n(c) = 0$ $c \in (a,b)$.

'n' local Maximum represents $f^n(c) = 0$

Hence $f^n(c) = 0$.

Problem 5

Show that for any $x > 0$,

$$1 - x + \frac{x^2}{2} > e^{-x} > 1 - x.$$

Problem 6 (Harder)

We let $f: [0,1] \to \mathbb{R}$ be a twice differentiable function on $[0,1]$ and $f''(x)$ is continuous on $[0,1]$. Suppose that

- $f(0) = f(1) = 0$ and
- $|f''(x)| \leq A$ for all $x \in [0,1]$, where $A > 0$ is a constant.

Show that $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{A}{4}$.

(☺Hint: Apply Taylor theorem with suitable choice of a.)

To prove for any $x > 0$, $1 - x + \frac{x^2}{2} > e^{-x} > 1 - x$.

Let $f(x) = 1 - x + \frac{x^2}{2} - e^{-x}$, $x \geq 0$.

f is continuous on $[0, \infty)$. $f'(x) = -1 + x + e^{-x} > 0$ for all $x > 0$

Therefore f is strictly increasing function on $[0, \infty)$.

So, $f(x) > f(0)$ for all $x > 0$

$\Rightarrow 1 - x + \frac{x^2}{2} - e^{-x} > 0$

$\Rightarrow 1 - x + \frac{x^2}{2} > e^{-x}$ for all $x > 0$ ....(i)

Let $g(x) = e^{-x} \ldots$

g is continuous on $[0, \infty)$. $g'(x) = -e^{-x} + 1 > 0$ for all $x > 0$

Therefore g is strictly increasing function on $[0, \infty)$.

So, $g(x) > g(0)$ for all $x > 0$.

Consequently $e^{-x} - 1 + x > 0$

$\Rightarrow e^{-x} > 1 - x$ ...... (ii)

From (i) & (ii) it follows that $1 - x + \frac{x^2}{2} > e^{-x} > 1 - x$ (Proved).

Problem 6: Let $f: [0,1] \to \mathbb{R}$ be a twice differentiable function on $[0,1]$ and $f''(x)$ is continuous on $[0,1]$.

Suppose that $f(0) = f(1) = 0$ and $|f''(x)| \leq A$ $\forall x \in [0,1]$, where A is positive constant.

Now using Taylor's th. about $x = 1/2$.

$$f(x) = f(1/2) + (x - 1/2) f'(1/2) + \frac{(x - 1/2)^2}{2!} f''(1/2 + \theta_1 (x - 1/2))$$

for $0 < \theta_1 < 1$