# Math 2033 Final Review

May 11, 2016

In general, you may be asked to state the definitions of various concepts.

The Final Exam Coverage should be as follow:

1. Cauchy sequence,
2. limit of functions,
3. continuity theorems (intermediate value theorem, extreme value theorem, continuous injection theorem),
4. differentiation theorems(mean-value theorem,Taylor's theorem),
5. integration theorems (integral criterion, Lebesgue's theorem).

## 1. Cauchy sequence

**Cauchy sequence by checking the definition of Cauchy sequence**

1. Suppose $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, show that the sequence $\{x_n(1+y_n)\}$ is also Cauchy by checking the definition only.

*Proof.* Since $\{x_n\}$ and $\{y_n\}$ is Cauchy, then they must be bounded. i.e., $\exists M, N \in \mathbb{N}$, $|x_n| < M$ , $|y_n| < N$, for all $n \in \mathbb{N}$.

For any $\epsilon > 0$, since $\{x_n\}$ and $\{y_n\}$ are Cauchy, then there exists $K_1, K_2 \in \mathbb{N}$ such that

$$m, n > K_1 \implies |x_m - x_n| < \frac{\epsilon}{2(1+N)} \text{ and } m, n > K_2 \implies |y_m - y_n| < \frac{\epsilon}{2M}.$$

Pick $K = \max\{K_1, K_2\}$, then for $n > K$, we get

$$\begin{aligned}
|x_m(1+y_m) - x_n(1+y_n)| &= |x_m + x_m y_m - x_n - x_n y_n| \\
&= |(x_m - x_n) + (x_m y_m - x_n y_n)| \\
&\leq |x_m - x_n| + |x_m y_m - x_n y_n| \\
&\leq |x_m - x_n| + |x_m y_m - x_m y_n + x_m y_n - x_n y_n| \\
&\leq |x_m - x_n| + |x_m y_m - x_m y_n| + |x_m y_n - x_n y_n| \\
&\leq |x_m - x_n| + |x_m||y_m - y_n| + |y_n||x_m - x_n| \\
&\leq |x_m - x_n| + M|y_m - y_n| + N|x_m - x_n| \\
&\leq (1+N)|x_m - x_n| + M|y_m - y_n| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Thus the sequence is Cauchy. □

2. Let $\{x_n\}$, $\{y_n\}$ be Cauchy, define a sequence $\{z_n\}$ to be

$$z_n = \begin{cases} x_n & \text{when } n \text{ is odd} \\ y_n & \text{when } n \text{ is even} \end{cases}$$

Suppose $\lim_{n\to\infty} |x_n - y_n| = 0$. Show that $\{z_n\}$ is Cauchy.

*Proof.* For any $\epsilon > 0$, since $\{x_n\}$ and $\{y_n\}$ are Cauchy, then there exists $K_1, K_2 \in \mathbb{N}$ such that

$$n > K_1 \implies |x_m - x_n| < \frac{\epsilon}{2} \text{ and } n > K_2 \implies |y_m - y_n| < \frac{\epsilon}{2}.$$

Since $\lim\limits_{n\to\infty} |x_n - y_n| = 0$, then there exists $K_3 \in \mathbb{N}$ such that $n > K_3 \implies |x_n - y_n| < \dfrac{\epsilon}{2}$.

Since values of $z_n$ depends on whether $n$ is even or odd, we will split into the four cases, for W.L.O.G., $m > n > K = \max\{K_1, K_2, K_3\}$,

(i) Both $m, n$ are odd:

$$|z_m - z_n| = |x_m - x_n| < \epsilon$$

(ii) Both $m, n$ are even:

$$|z_m - z_n| = |y_m - y_n| < \epsilon$$

(iii) $m$ is odd and $n$ is even:

$$|z_m - z_n| = |x_m - y_n| = |x_m - y_m| + |y_m - y_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(iv) $m$ is even and $n$ is odd:

$$|z_m - z_n| = |y_m - x_n| = |y_m - x_m| + |x_m - x_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence $\{z_n\}$ is Cauchy. □

3. Let $\{a_n\}$ and $\{b_n\}$ are Cauchy Sequence and $\lim_{n\to\infty}(a_n - b_n) = 0$. Show that $c_n = \max\{a_n, b_n\}$ is also Cauchy.

*Proof.* One can verify the following identity

$$\max\{a, b\} = \frac{a + b + |a - b|}{2}$$

For any $\epsilon > 0$, since $\{a_n\}$, $\{b_n\}$ are Cauchy, there exists $K_1, K_2 \in \mathbb{N}$ such that

$$m, n > K_1 \implies |a_m - a_n| < \frac{\epsilon}{2} \text{ and } m, n > K_2 \implies |b_m - b_n| < \frac{\epsilon}{2}$$

Since $\lim\limits_{n\to\infty} |a_n - b_n| = 0$, then there exists $K_3 \in \mathbb{N}$ such that $n > K_3 \implies |a_n - b_n| < \dfrac{\epsilon}{2}$.

Pick $K = \max\{K_1, K_2, K_3\}$, then for $m, n > K$,

$$\begin{aligned}
\implies |c_m - c_n| = &\left| \frac{a_m + b_m + |a_m - b_m|}{2} - \frac{a_n + b_n + |a_n - b_n|}{2} \right| \\
=&\frac{1}{2} \left| (a_m - a_n) + (b_m - b_n) + |a_m - b_m| - |a_n - b_n| \right| \\
\leq&\frac{1}{2}|a_m - a_n| + \frac{1}{2}|b_m - b_n| + \frac{1}{2}|a_m - b_m| + \frac{1}{2}|a_n - b_n| \\
\leq&\frac{1}{2}\left(\frac{\epsilon}{2}\right) + \frac{1}{2}\left(\frac{\epsilon}{2}\right) + \frac{1}{2}\left(\frac{\epsilon}{2}\right) + \frac{1}{2}\left(\frac{\epsilon}{2}\right) = \epsilon
\end{aligned}$$

Hence $\{c_n\}$ is Cauchy. □

To prove a sequence is Cauchy, Mean Value Theorem is a useful tool.

4. If $\{x_n\}$ is Cauchy and $x_n > 0$, show that $y_n = x_n^2 \sin\left(\frac{1}{x_n}\right)$ is Cauchy (by the definition).

*Proof.* For any $\epsilon > 0$, since $\{x_n\}$ is Cauchy, there exists $K \in \mathbb{N}$ such that

$$m, n > K \implies |x_m - x_n| < \frac{\epsilon}{2M+1}$$

Then

$$\begin{aligned}
|y_m - y_n| &= \left| x_m^2 \sin\left(\frac{1}{x_m}\right) - x_n^2 \sin\left(\frac{1}{x_n}\right) \right| \\
&= \left| 2c \sin\left(\frac{1}{c}\right) - \cos\left(\frac{1}{c}\right) \right| |x_m - x_n| \quad \text{for some } c \text{ between } x_m, x_n \\
&\le \left( \left| 2c \sin\left(\frac{1}{c}\right) \right| + \left| \cos\left(\frac{1}{c}\right) \right| \right) |x_m - x_n| \\
&\le (2|c| + 1)|x_m - x_n| \\
&< (2M+1)|x_m - x_n| \quad \text{since } \{x_n\} \text{ is bounded by } M \text{ and } c \text{ between } x_m, x_n \\
&< \epsilon
\end{aligned}$$

Hence $\{y_n\}$ is Cauchy. $\square$

5. If $a_1 = 1$, $a_{n+1} = \frac{n}{n+1} a_n + \frac{\cos n}{(1+n)^3}$. Prove that the sequence $\{n a_n\}$ is Cauchy.

*Proof.* Let $b_n = n a_n$,

$$a_{n+1} = \frac{n}{n+1} a_n + \frac{\cos n}{(1+n)^3}$$

iff

$$(n+1) a_{n+1} = n a_n + \frac{\cos n}{(1+n)^2}$$

iff

$$b_{n+1} = b_n + \frac{\cos n}{(1+n)^2}$$

iff

$$b_{n+1} - b_n = \frac{\cos n}{(1+n)^2}$$

This implies $|b_{n+1} - b_n| \le \frac{1}{(1+n)^2}$.

For any $\epsilon > 0$, by Archimedean Principle, there exists $K \in \mathbb{N}$
such that $K > \frac{2}{\epsilon}$, then $m, n > K \implies m > \frac{2}{\epsilon}$ and $n > \frac{2}{\epsilon}$.
W.L.O.G., assume $m > n$,

Then

$$\begin{aligned}
|b_m - b_n| &= |b_n - b_m| \\
&= |(b_n - b_{n+1}) + \cdots + (b_{m-2} - b_{m-1}) + (b_{m-1} - b_m)| \\
&\le |b_n - b_{n+1}| + \cdots + |b_{m-2} - b_{m-1}| + |b_{m-1} - b_m| \\
&\le \frac{1}{(n+1)^2} + \cdots + \frac{1}{(m-1)^2} + \frac{1}{m^2} \\
&\le \frac{1}{(n+1)n} + \cdots + \frac{1}{(m-1)(m-2)} + \frac{1}{m(m-1)} \\
&\le \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots + \left(\frac{1}{m-2} - \frac{1}{m-1}\right) + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\
&\le \frac{1}{n} - \frac{1}{m} \\
&\le \frac{1}{m} + \frac{1}{n} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Thus $b_n = na_n$ is Cauchy. $\square$

## 2. Limit of Functions

**Checking the $\epsilon$-$\delta$ definition of limit of function**

1. Show by the definition of limit of function, $\lim\limits_{x\to 3} \sqrt[5]{x^3+2} = 2$.

*Proof.* For any $\epsilon > 0$, pick $\delta = \min\left\{1, \frac{\epsilon^5}{37}\right\}$, then $|x-3| < \delta \implies \begin{cases} |x-3| < 1 \\ |x-3| < \frac{\epsilon^5}{37} \end{cases} \implies \begin{cases} 2 < x < 4 \\ |x-3| < \frac{\epsilon^5}{37} \end{cases}$

Then

$$\begin{aligned}
|\sqrt[5]{x^3+5} - 2| =& |\sqrt[5]{x^3+5} - \sqrt[5]{32}| \\
\le& \left|\sqrt[5]{x^3+5-32}\right| \quad (\text{since } |\sqrt[n]{a} - \sqrt[n]{b}| \le \sqrt[n]{|a-b|}.) \\
=& \left|\sqrt[5]{|x^3-27|}\right| \\
=& \left|\sqrt[5]{|(x-3)(x^2+3x+9)|}\right| \\
=& \left|\sqrt[5]{|x-3||x^2+3x+9|}\right| \\
\le& \left|\sqrt[5]{|x-3||4^2+3(4)+9|}\right| \\
=& \left|\sqrt[5]{37|x-3|}\right| \\
<& \epsilon
\end{aligned}$$

$\square$

2. For $a > 0$, show by definition of limit of function that $\lim\limits_{x\to 0} \frac{a(x+a)}{x-a} = -a$.

*Proof.* For any $\epsilon > 0$, pick $\delta = \min\left\{\frac{a}{2}, \frac{\epsilon}{4}\right\}$, then $|x - 0| < \delta \implies \begin{cases} |x| < \frac{a}{2} \\ |x| < \frac{\epsilon}{4} \end{cases} \implies \begin{cases} \frac{a}{2} < |x-a| < \frac{3a}{2} \\ |x| < \frac{\epsilon}{4} \end{cases}$

Then

$$\begin{aligned}
\left|\frac{a(x+a)}{x-a} - (-a)\right| &= \left|\frac{a(x+a)+a(x-a)}{x-a}\right| \\
&= \left|\frac{2ax}{x-a}\right| \\
&\leq \left|\frac{2ax}{\frac{a}{2}}\right| \\
&= 4|x| \\
&< 4\delta \\
&< \epsilon
\end{aligned}$$

□

3. Show that $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^3$ is continuous on $\mathbb{R}$, by checking the $\epsilon$-$\delta$ definition of continuity.

Let $x_0 \in \mathbb{R}$, we analyse $|g(x) - g(x_0)| = |x^3 - x_0^3| = |x - x_0||x^2 + x_0x + x_0^2|$.
We expect when $x$ is close to $x_0$, $|x - x_0|$ is small, but $|x^2 + x_0x + x_0^2|$ can be large, and small$\times$ big can be big!
We need to be careful in using the $\epsilon$-$\delta$ definition.
Indeed the term $|x^2 + x_0x + x_0^2|$ causes no trouble even it is large!
The choice of $x_0$ is fixed, it cannot be arbitrarily large.
As $x$ is supposed to be close to $x_0$, let's say when $|x - x_0| < 1$, we have $|x| < 1 + |x_0|$,
$|x|$ cannot be arbitrarily large when $|x - x_0| < 1$

Therefore, the whole term

$$|x^2 + x_0x + x_0^2|$$

cannot be arbitrarily large as by triangle inequality
It has a bound

$$(1 + |x_0|)^2 + (1 + |x_0|)|x_0| + |x_0|^2 =: C(x_0)$$

when $|x - x_0| < 1$, where $C(x_0)$ is a constant depending on $x_0$, which is really a constant since $x_0$ is fixed.

*Proof.* For any $\epsilon > 0$, choose $\delta = \min\left\{1, \frac{\epsilon}{C(x_0)}\right\} > 0$, where $C(x_0) = (1+|x_0|)^2 + (1+|x_0|)|x_0| + |x_0|^2 > 0$, for $|x - x_0| < \delta \implies \begin{cases} |x - x_0| < 1 \\ |x - x_0| < \frac{\epsilon}{C(x_0)} \end{cases} \implies \begin{cases} |x| < 1 + |x_0| \\ |x - x_0| < \frac{\epsilon}{C(x_0)} \end{cases}$

Then

$$|g(x) - g(x_0)| = |x^3 - x_0^3| = |x - x_0||x^2 + x_0x + x_0^2| < \delta\left(|x|^2 + |x_0||x| + |x_0|^2\right) < \delta C(x_0) \leq \epsilon$$

□

## 3. Continuity Theorems

### Intermediate Value Theorem

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period $T > 0$. Prove that there is $x_0$ such that

$$f\left(x_0 + \frac{T}{2}\right) = f(x_0)$$

*Proof.* Let $g(x) = f\left(x + \frac{T}{2}\right) - f(x)$, which is continuous, $g(0) = f\left(\frac{T}{2}\right) - f(0)$ and $g\left(\frac{T}{2}\right) = f(T) - f\left(\frac{T}{2}\right) = f(0) - f\left(\frac{T}{2}\right)$.

By Intermediate Value Theorem, there exists $x_0 \in \left[0, \frac{T}{2}\right]$ such that $g(x_0) = 0$,

hence $f\left(x_0 + \frac{T}{2}\right) = f(x_0)$. □

2. A function $f : (a, b) \to \mathbb{R}$ is continuous. Prove that, given $x_1, x_2, \cdots, x_n$ in $(a, b)$, there exists $x_0 \in (a, b)$ such that

$$f(x_0) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

*Proof.* Let $M = \max\{f(x_1), f(x_2), \cdots, f(x_n)\}$ and $m = \min\{f(x_1), f(x_2), \cdots, f(x_n)\}$.

Then $m \leq \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \leq M$.
Consequently, there is $x_0 \in (a, b)$ such that

$$f(x_0) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

□

3. Show that the function

$$f(x) = (x - a)^2(x - b) + x$$

has a value $f(c) = \frac{a + b}{2}$ for a number $c$.

*Proof.* Since $f$ is a polynomial, $f$ is continuous.

Note that $f(a) = a$ and $f(b) = b$,

For $a = b$, we get $f(a) = a = \frac{a + b}{2}$. Simply take $c = a$.

For $a \neq b$, since $\frac{a + b}{2}$ lies between $a$ and $b$, by Intermediate Value Theorem, there is a $c$ lying between $a$ and $b$ such that $f(c) = \frac{a + b}{2}$. □

4. Let $f, g : [a, b] \to [0, +\infty)$ be continuous functions satisfying $\sup_{x\in[a,b]} f(x) = \sup_{x\in[a,b]} g(x)$, prove that there is a $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

*Proof.* Method 1: Suppose $f(x) \neq g(x)$ for all $x \in [a, b]$, we get two cases:

(i) $f(x) - g(x) > 0$ for all $x \in [a, b]$.

(ii) $f(x) - g(x) < 0$ for all $x \in [a, b]$.

(Otherwise, there are $x_1, x_2 \in [a, b]$ such that $f(x_1) - g(x_1) > 0$ and $f(x_2) - g(x_2) < 0$, by Intermediate Value Theorem, there is $x_0$ between $x_1, x_2$ such that $f(x_0) - g(x_0) = 0$.)

For case (i):
Note that $\sup\{g(x) : x \in [a, b]\} = M$, by extreme value theorem, there exists $x_0 \in [a, b]$ such that $g(x) \leq g(x_0) = M$.
Note that $f(x) > g(x)$ for all $x \in [a, b]$, then $f(x_0) > g(x_0) = M$ which imply $\sup\{f(x) : x \in [a, b]\} > M$, contradicts to the fact that $\sup\{f(x) : x \in [a, b]\} = M$.

For case (ii):
Note that $\sup\{f(x) : x \in [a, b]\} = M$, by extreme value theorem, there exists $x_0 \in [a, b]$ such that $f(x) \leq f(x_0) = M$.
Note that $f(x) < g(x)$ for all $x \in [a, b]$, then $M = f(x_0) < g(x_0)$ which imply $\sup\{g(x) : x \in [a, b]\} > M$, contradicts to the fact that $\sup\{g(x) : x \in [a, b]\} = M$.

Method 2: Since $[a, b]$ is closed and bounded interval and $f(x)$, $g(x)$ are continuous,
then by extreme value theorem, there exists $x_1, x_2 \in [a, b]$ such that $f(x_1) = M = g(x_2)$.
By the property of supremum, $f(x) \leq M$ and $g(x) \leq M$ for all $x \in [a, b]$.
Let $h(x) = f(x) - g(x)$.
Then $h(x_1) = f(x_1) - g(x_1) = M - g(x_1) \geq 0$.
And $h(x_2) = f(x_2) - g(x_2) = f(x_2) - M \leq 0$.
By Intermediate Value Theorem, there exists $x_0$ between $x_1, x_2$, hence $x_0 \in [a, b]$
such that $h(x_0) = 0$, thus $f(x_0) = g(x_0)$. □

5. Let $f(x)$ and $g(x)$ be continuous function and $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$, prove that if the equation $f(f(x)) = g(g(x))$ has a solution, then $f(x) = g(x)$ must have a solution.

*Proof.* Suppose $f(x) \neq g(x)$ for all $x \in \mathbb{R}$, we get two cases:

(i) $f(x) - g(x) > 0$ for all $x \in [a, b]$.

(ii) $f(x) - g(x) < 0$ for all $x \in \mathbb{R}$.

(Otherwise, there are $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) - g(x_1) > 0$ and $f(x_2) - g(x_2) < 0$, by Intermediate Value Theorem, there is $x_0$ between $x_1, x_2$ such that $f(x_0) - g(x_0) = 0$.)

Case (i):
Since we know $f(f(x)) = g(g(x))$ has solution, let say $x = c$, is one of the solutions.
Thus $f(f(c)) - g(g(c)) = 0$.

$$\implies f(f(c)) - g(f(c)) + g(f(c)) - g(g(c)) = 0$$

$$\implies f(f(c)) - g(f(c)) + f(g(c)) - g(g(c)) = 0$$

Let $p = f(c)$, $q = g(c)$,

$$\implies 0 = 0 + 0 < (f(p) - g(p)) + (f(q) - g(q)) = 0$$

which is a contradiction.

Case (ii) is similar and yield similar contradiction.

Hence $f(x) = g(x)$ should have a solution. □

6. Let $f(x)$ be a continuous function on $[a, b]$ and $f'(a)$ exists. Let $\xi$ be a number such that

$$f'(a) > \xi > \frac{f(b) - f(a)}{b - a}$$

Prove that there is a $c \in (a, b)$ such that $\dfrac{f(c) - f(a)}{c - a} = \xi$.

*Proof.* Consider $h(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} & \text{if } x \neq a \\ f'(a) & \text{if } x = a \end{cases}$ on $[a, b]$, since $f$ is continuous on $[a, b]$,

then $\dfrac{f(x) - f(a)}{x - a}$ is continuous on $(a, b]$. As a result, in order to show $h$ is continuous on $[a, b]$, it suffices to show $\lim\limits_{x \to a^+} h(x) = h(a)$ which is true due to the following

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = f'_+(a) = f'(a)$$

Apply Intermediate Value Theorem to $h$ on $[a, b]$, note that

$$f'(a) > \xi > \frac{f(b) - f(a)}{b - a} \text{ means } h(a) > \xi > h(b)$$

We get there is a $c \in (a, b)$ such that $h(c) = \xi$, i.e., $\dfrac{f(c) - f(a)}{c - a} = \xi$. □

7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and suppose that there is some real number $a$ such that $f(f(f(a))) = a$. Show that there is some real number $b$ such that $f(b) = b$.

*Proof.* Consider $g(x) = f(x) - x$. If $g(x)$ has a root $b$, then $f(b) = b$ and we are done.

Suppose for contradiction that $g(x)$ does not have a root,
we get two cases: $g(x)$ is everywhere positive or $g(x)$ is everywhere negative.
(If not, then $g(x)$ takes both positive and negative values, then by the intermediate value theorem, it has a root.)

In the former case, this gives $f(x) > x$ everywhere, so $f(f(f(a))) > f(f(a)) > f(a) > a$, contradicting the fact that $f(f(f(a))) = a$.

In the latter case, we similarly have $f(x) < x$ and thus $f(f(f(a))) < a$, again a contradiction.

In every case we have a contradiction, so in fact $g(x)$ has a root and thus there exists $b \in \mathbb{R}$ with $f(b) = b$. □

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous decreasing function.
Prove that there exists unique $(x, y, z) \in \mathbb{R}^3$ such that $x = f(y)$, $y = f(z)$, $z = f(x)$.

*Proof.* The fact that $f$ is decreasing implies immediately that

$$\lim_{x \to -\infty} (f(x) - x) = +\infty \quad \text{and} \quad \lim_{x \to +\infty} (f(x) - x) = -\infty$$

By the Intermediate Value Theorem, there is $x_0$ such that $f(x_0) - x_0 = 0$, i.e., $f(x_0) = x_0$. The decreasing function $f$ has unique fixed point as if both $x$ and $y$ are fixed point, with $x < y$, then $x = f(x) \geq f(y) = y$, a contradiction.

The triple $(x, y, z) = (x_0, x_0, x_0)$ is a solution to the system $\begin{cases} x = f(y) \\ y = f(z) \\ z = f(x) \end{cases}$.

If $(x, y, z)$ is a solution of the system, then $f(f(f(x))) = x$ and $f(f(f(y))) = y$ and $f(f(f(z))) = z$, i.e., $x, y, z$ are fixed point of $f \circ f \circ f$. In particular, we get $f(f(f(x_0))) = x_0$, i.e., $x_0$ is a fixed point of $f \circ f \circ f$.

Since $f$ is continuous and decreasing, we get $f \circ f \circ f$ is also continuous and decreasing, so $f \circ f \circ f$ has a unique fixed point, which can only be $x_0$. Thus $x = y = z = x_0$. This proves that the solution to the system is unique. $\square$

**Extreme Value Theorem**

1. Let $f : [a, b] \to \mathbb{R}$ be a function, continuous on $[a, b]$ and differentiable on $(a, b)$.

   Prove that if there exists $c \in (a, b)$ such that $\dfrac{f(b) - f(c)}{f(c) - f(a)} < 0$, then there exists $\xi \in (a, b)$ such that $f'(\xi) = 0$.

*Proof.* If $f(b) > f(c)$, hence $f(a) > f(c)$. Let $\xi$ be an absolute minimum of $f$ on $[a, b]$, which exists since $f$ is continuous, by extreme value theorem.

Since $f(b) > f(c)$ and $f(a) > f(c)$, the absolute minimum of $f$ does not occur at end point.

Then $\xi \in (a, b)$ and then by local extremum theorem, therefore $f'(\xi) = 0$.

If $f(b) < f(c)$, replace $f$ by $-f$. $\square$

2. Can there be a continuous function $f(x)$ from $[0, 1]$ onto $(0, 1)$?

*Proof.* If $f$ is continuous and onto, then $f([0, 1]) = (0, 1)$ has a maximum and minimum by Extreme Value Theorem, this is impossible. $\square$

**Remark:**

By homework 1, there is a bijection from $[0, 1]$ to $(0, 1]$, but there cannot be <u>continuous</u> bijection map from $[0, 1]$ to $(0, 1]$ since $(0, 1]$ has no minimum.

3. (Mean-Value Theorem for Integrals). Let $f(x)$ be continuous on $[a, b]$ and $g(x) \geq 0$ be integrable on $[a, b]$. Show that there is $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

*Proof.* Let $m = \min\limits_{x\in[a,b]} f(x)$ and $M = \max\limits_{x\in[a,b]} f(x)$, such min and max exist by Extreme Value Theorem.

Since $g(x) \geq 0$, direct computation gives

$$m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx. \qquad (*)$$

Case 1: Suppose that $\int_a^b g(x)\,dx > 0$, then

$$m \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leq M,$$

and therefore by Intermediate Value Theorem, there is $c \in [a, b]$ such that

$$\frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} = f(c).$$

Case 2: Suppose that $\int_a^b g(x)\,dx = 0$, then the method in case 1 fails since we cannot divide a number by zero, but by $(*)$, we have

$$\int_a^b f(x)g(x)\,dx = 0,$$

therefore there is $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = 0 = f(c)\int_a^b g(x)\,dx.$$

Note that Any choice $c \in [a,b]$ will do. $\square$

4. (Darboux's theorem) Prove that if $f$ is differentiable on an interval $I$, then $f'$ enjoys the intermediate value property.

*Proof.* W.L.O.G., for $[a,b] \subseteq I$, with $f'(a) < f'(b)$.
Let $f'(a) < \lambda < f'(b)$. Consider $g(x) = f(x) - \lambda x$, then $g'(a) < 0$ and $g'(b) > 0$.
Consequently, $g$ attains its minimum on $[a,b]$ at $x_0 \in (a,b)$.
By local extremum theorem, $g'(x_0) = 0$, hence $f'(x_0) = \lambda$. $\square$

**Remark:** Continuity implies Intermediate Value Property, however, the above exercise suggests that the converse is Not True.

More Precisely, consider $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. One can show this $f$ is differentiable everywhere but $f'$ is not continuous at 0, then by the above exercise, we get a non-continuous function $f'$ with Intermediate Value Property.

**Continuous Injection Theorem**

1. Is there any continuous function $f(x)$ such that $f(f(x)) = -x^9$?

*Proof.* Suppose there is such $f$, satisfying $f(f(x)) = -x^9$.

If $f(a) = f(b) \implies f(f(a)) = f(f(b)) \implies -a^9 = -b^9 \implies a = b$. Therefore, $f$ is injective.
Since $f$ is also continuous, by Continuous Injection Theorem, we know that $f$ is either strictly increasing or strictly decreasing.

(i) If $f$ is strictly increasing:
If $a > b \implies f(a) > f(b) \implies f(f(a)) > f(f(b))$, this implies $f(f(x))$ is strictly increasing, while $-x^9$ is a decreasing function, so it yields a contradiction.

(ii) If $f$ is strictly decreasing:
If $a > b \implies f(a) < f(b) \implies f(f(a)) > f(f(b))$, this also implies $f(f(x))$ is strictly increasing, so it yields the same contradiction as above. $\square$

**Remark:**

In general, suppose $f \circ g = h$.

If $h$ is injective, we get $g$ is injective. (If in case, $h$ is injective and g is continuous, this may be a hint for you to use Continuous Injection Theorem. See example 2 below.)
If $h$ is surjective, we get $f$ is surjective.

2. (2007 Spring Midterm) Let $f : [0,1] \to [0,1]$ be continuous with $f(0) = 0$ and $f(1) = 1$ and $f(f(x)) = x$ for all $x \in [0,1]$. Show that $f(x) = x$ for all $x \in [0,1]$.

*Proof.* Since $f(f(x)) = x$, we get

$$f(a) = f(b) \implies f(f(a)) = f(f(b)) \implies a = b$$

This implies $f$ is injective.

Since $f$ is continuous, by Continuous Injection Theorem, we get $f$ is either strictly increasing or strictly decreasing.

Since $f(0) = 0$ and $f(1) = 1$ implies $f(1) > f(0)$ and thus $f$ is strictly increasing.

Suppose $f(x_0) \neq x_0$ for some $x_0 \in [0, 1]$,

if $f(x_0) > x_0$, since $f$ is strictly increasing, we get $x_0 = f(f(x_0)) > f(x_0) > x_0$, a contradiction.

if $f(x_0) < x_0$, since $f$ is strictly increasing, we get $x_0 = f(f(x_0)) < f(x_0) < x_0$, a contradiction.

Hence $f(x) = x$ for all $x \in [0, 1]$. $\square$

3. Let $f : [a, b] \to \mathbb{R}$ be continuous function and $f(x)$ is differentiable on $(a, b)$.
   Let $\theta \in (a, b)$ and $f'(\theta)$ is not the supremum and infinmum of $f'(x)$ among $(a, b)$.
   Then prove that there exists distinct $c, d \in (a, b)$ such that $f'(\theta) = \dfrac{f(c) - f(d)}{c - d}$.
   (Hint: Consider $g(x) = f(x) - f'(\theta)x$)

*Proof.* Since $f'(\theta)$ is not the supremum and infinmum of $\{f'(x) : x \in (a, b)\}$, there exists $t_1, t_2 \in (a, b)$ such that

$$\sup\{f'(x)\} > f'(t_1) > f'(\theta) > f'(t_2) > \inf\{f'(x)\}$$

Suppose $g(x)$ is injective, on contrary, since $f(x)$ is continuous, $g(x)$ is also continuous.
By Continuous Injection Theorem, we get either $g(x)$ is strictly increasing or $g(x)$ is strictly decreasing.

(i) For $g(x)$ is strictly increasing,

$$\begin{aligned} & \text{For any } x > t_2, \\ \implies & g(x) > g(t_2) \\ \implies & f(x) - f'(\theta)x > f(t_2) - f'(\theta)t_2 \\ \implies & \frac{f(x) - f(t_2)}{x - t_2} > f'(\theta) \end{aligned}$$

Taking $x \to t_2^+$, we get $f'(t_2) = f'_+(t_2) \geq f'(\theta)$, which contradicts to $f'(\theta) > f'(t_2)$.

(ii) For $g(x)$ is strictly decreasing,

$$\begin{aligned} & \text{For any } x < t_1, \\ \implies & g(x) > g(t_1) \\ \implies & f(x) - f'(\theta)x > f(t_1) - f'(\theta)t_1 \\ \implies & \frac{f(x) - f(t_1)}{x - t_1} < f'(\theta) \end{aligned}$$

Taking $x \to t_1^-$, we get $f'(t_1) = f'_-(t_1) \leq f'(\theta)$, which contradicts to $f'(\theta) < f'(t_1)$.

Thus $g(x)$ is not injective, there exists $c \neq d$ such that $g(c) = g(d)$, this implies $f'(\theta) = \dfrac{f(c) - f(d)}{c - d}$. $\square$

## 4. Differentiation Theorems

### Mean Value Theorem

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function, with positive second derivative.
Prove that $f(x + f'(x)) \geq f(x)$, for any real number $x$

*Proof.* If $x$ is such that $f'(x) = 0$, then the relation holds with equality.

If for certain $x$, $f'(x) < 0$, then the Mean Value Theorem applied on the interval $[x + f'(x), x]$ yields

$$f(x) - f(x + f'(x)) = f'(c)\,(x - (x + f'(x))) = f'(c)(-f'(x))$$

for some $c$ with $x + f'(x) < c < x$. Since the second derivative is positive, $f'$ is increasing; hence $f'(c) < f'(x) < 0$. Thus $f(x) - f(x + f'(x)) < 0$, i.e., $f(x) < f(x + f'(x))$.

If for certain $x$, $f'(x) > 0$, then the Mean Value Theorem applied on the interval $[x, x + f'(x)]$ yields

$$f(x + f'(x)) - f(x) = f'(c)\,((x + f'(x)) - x) = f'(c)f'(x)$$

for some $c$ with $x < c < x + f'(x)$. Since the second derivative is positive, $f'$ is increasing; hence $0 < f'(x) < f'(c)$. Thus $f(x + f'(x)) - f(x) > 0$, i.e., $f(x) < f(x + f'(x))$. □

2. Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions and $f'(x)g(x) - f(x)g'(x) \neq 0$ for all $x \in (a, b)$. If there exists $a < x_0 < x_1 < b$ such that $f(x_0) = f(x_1) = 0$. Show that there exists $c \in (a, b)$ such that $g(c) = 0$.

*Proof.* Suppose $g(x) \neq 0$ for all $x \in (a, b)$, then consider $h(x) = \dfrac{f(x)}{g(x)}$, which is well-defined as $g(x) \neq 0$ for all $x \in (a, b)$.

Apply Mean Value Theorem on $h(x)$ over $[x_0, x_1]$, we get

$$\begin{aligned}
\frac{h(x_1) - h(x_0)}{x_1 - x_0} &= h'(c) \text{ for some } c \in (x_0, x_1) \\
0 &= h'(c) \text{ for some } c \in (x_0, x_1) \\
0 &= \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2} \\
f'(c)g(c) - f(c)g'(c) &= 0
\end{aligned}$$

which contradicts to the fact that $f'(x)g(x) - f(x)g'(x) \neq 0$ for all $x \in (a, b)$.

Thus we get there exists $c \in (a, b)$ such that $g(c) = 0$. □

3. Let $n > 1$ be an integer, and let $f : [a, b] \to \mathbb{R}$ be a continuous function, $n$-times differentiable on $(a, b)$, with the property that the graph of f has $n + 1$ collinear points. Prove that there exists a point $c \in (a, b)$ with the property that $f^{(n)}(c) = 0$.

*Proof.* Let $\alpha$ be the slope of the line passing through the collinear points $(a_i, f(a_i))$, $i = 0, 1, \cdots, n$, on the graph of $f$. Then

$$\frac{f(a_i) - f(a_{i-1})}{a_i - a_{i-1}} = \alpha, \quad i = 1, 2, \cdots, n.$$

From the mean value theorem it follows that there exists points $c_i \in (a_{i-1}, a_i)$ such that $f'(c_i) = \alpha$, $i = 1, 2, \cdots, n$. Consider the function $F : [a_0, a_n] \to \mathbb{R}$ defined by $F(x) = f'(x) - \alpha$.

$F$ is continuous, $(n - 1)$-times differentiable, and has $n$ zeros in $[a_0, a_n]$. Applying successively Rolle's Theorem, we can see that $F'$ has $n - 1$ zeros in $[a_0, a_n]$, $F''$ has $n - 2$ zeros in $[a_0, a_n]$, and eventually $F^{(n-1)}$ has 1 zeros in $[a_0, a_n]$. We conclude that $F^{(n-1)} = f^{(n)}$ has a zero in $[a, b]$. □

**Talyor's Theorem**

1. Let $f(x)$ be three times differentiable and satisfy $\lim_{x\to\infty} f(x) = c \in \mathbb{R}$ and $\lim_{x\to\infty} f^{(3)}(x) = 0$, show that $\lim_{x\to\infty} f''(x) = 0$.

*Proof.* For any $x_0 \in \mathbb{R}$, using the Taylor Theorem (up to $x^3$ term), we get

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(c_x)}{3!}(x - x_0)^3, \quad \text{for some } c_x \text{ between } x \text{ and } x_0.$$

Put $x = x_0 + 1$, we get

$$f(x_0 + 1) = f(x_0) + f'(x_0) + \frac{f''(x_0)}{2!} + \frac{f^{(3)}(c_{x_0+1})}{3!}, \quad \text{for some } c_{x_0+1} \in (x_0, x_0 + 1).......(*)$$

Put $x = x_0 - 1$, we get

$$f(x_0 - 1) = f(x_0) - f'(x_0) + \frac{f''(x_0)}{2!} - \frac{f^{(3)}(c_{x_0-1})}{3!}, \quad \text{for some } c_{x_0-1} \in (x_0 - 1, x_0).......(**)$$

Adding these equations, we get

$$f(x_0 + 1) + f(x_0 - 1) = 2f(x_0) + f''(x_0) + \frac{f^{(3)}(c_{x_0+1})}{3!} - \frac{f^{(3)}(c_{x_0-1})}{3!}$$

$$\implies f''(x_0) = f(x_0 + 1) + f(x_0 - 1) - 2f(x_0) - \frac{f^{(3)}(c_{x_0+1})}{3!} + \frac{f^{(3)}(c_{x_0-1})}{3!}$$

By taking $x_0 \to \infty$, (then $x_0 + 1 \to \infty$, $x_0 - 1 \to \infty$, $c_{x_0+1} \to \infty$, $c_{x_0-1} \to \infty$), we get

$$\lim_{x_0\to\infty} f''(x_0) = c + c - 2c - 0 + 0 = 0$$

$(*) - (**)$, we get

$$f(x_0 + 1) - f(x_0 - 1) = 2f'(x_0) + \frac{f^{(3)}(c_{x_0+1})}{3!} + \frac{f^{(3)}(c_{x_0-1})}{3!}$$

$$\implies 2f'(x_0) = f(x_0 + 1) - f(x_0 - 1) - \frac{f^{(3)}(c_{x_0+1})}{3!} - \frac{f^{(3)}(c_{x_0-1})}{3!}$$

By taking $x_0 \to \infty$, (then $x_0 + 1 \to \infty$, $x_0 - 1 \to \infty$, $c_{x_0+1} \to \infty$, $c_{x_0-1} \to \infty$), we get

$$2\lim_{x_0\to\infty} f'(x_0) = c - c - 0 - 0 = 0 \implies \lim_{x_0\to\infty} f'(x_0) = 0$$

□

2. Let $f(x)$ be a function defined on an open interval containing $[a, b]$ and $f(x)$ have second derivative at all $x \in [a, b]$. If $f'(a) = f'(b) = 0$, then prove that there exists $c \in (a, b)$ such that

$$|f''(c)| \geq \frac{4}{(b - a)^2}|f(b) - f(a)|$$

*Proof.* Applying Taylor Theorem at $c = a$ and $c = b$ respectively,

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(x_0)}{2!}(x - a)^2 = f(a) + \frac{f''(x_0)}{2!}(x - a)^2 \quad \text{for some } x_0 \text{ between } x \text{ and } a.......(*)$$

$$f(x) = f(b) + f'(b)(x - b) + \frac{f''(x_1)}{2!}(x - b)^2 = f(b) + \frac{f''(x_1)}{2!}(x - b)^2 \quad \text{for some } x_1 \text{ between } x \text{ and } b.......(**)$$

Put $x = \dfrac{a+b}{2}$, we get

$$f\left(\frac{a+b}{2}\right) = f(a) + \frac{f''(x_0)}{2!}\left(\frac{a+b}{2} - a\right)^2 = f(a) + \frac{f''(x_0)}{2!}\left(\frac{b-a}{2}\right)^2 \text{ for some } x_0 \in \left[a, \frac{a+b}{2}\right] ......(***)$$

$$f\left(\frac{a+b}{2}\right) = f(b) + \frac{f''(x_1)}{2!}\left(\frac{a+b}{2} - b\right)^2 = f(b) + \frac{f''(x_1)}{2!}\left(\frac{a-b}{2}\right)^2 \text{ for some } x_1 \in \left[\frac{a+b}{2}, b\right] ......(****)$$

Consider $(***) - (****)$, we get

$$f(a) - f(b) + \frac{f''(x_0)}{2!}\left(\frac{b-a}{2}\right)^2 - \frac{f''(x_1)}{2!}\left(\frac{a-b}{2}\right)^2 = 0$$

Then

$$\begin{aligned}
|f(a) - f(b)| &= \left|\left(\frac{f''(x_1)}{2!} - \frac{f''(x_0)}{2!}\right)\left(\frac{b-a}{2}\right)^2\right| \\
&\le \left(\frac{b-a}{2}\right)^2 \left|\frac{f''(x_1) - f''(x_0)}{2}\right| \\
&\le \left(\frac{b-a}{2}\right)^2 \frac{|f''(x_1)| + |f''(x_0)|}{2} \\
&\le \left(\frac{b-a}{2}\right)^2 \frac{2\max\{|f''(x_1)|, |f''(x_0)|\}}{2} \\
&\le \left(\frac{b-a}{2}\right)^2 |f''(c)| \quad \text{where } |f''(c)| = \max\{|f''(x_0)|, |f''(x_1)|\}
\end{aligned}$$

Then

$$|f''(c)| \ge \frac{4}{(b-a)^2}|f(b) - f(a)|$$

$\square$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable such that $M_k := \sup_{x\in\mathbb{R}} |f^{(k)}(x)| < \infty$ for $k = 0, 1, 2$.

   Show that
$$M_1 \le \sqrt{2M_0M_2}.$$

*Proof.* For every $x \in \mathbb{R}$ and $h \in \mathbb{R}$, we still use the following form of Taylor series:

$$f(x+h) = f(x) + f'(x)h + f''(x+\theta h)\frac{h^2}{2}, \text{ for some } \theta \in (0,1)$$

and

$$f(x-h) = f(x) - f'(x)h + f''(x+\theta' h)\frac{h^2}{2}, \text{ for some } \theta' \in (-1,0).$$

By subtracting them, we have

$$f(x+h) - f(x-h) = 2f'(x)h + (f''(x+\theta h) - f''(x+\theta' h))\frac{h^2}{2}.$$

Using the definition of $M_0, M_1$ and $M_2$, together with Triangular inequality, we have

$$0 = \left|f(x-h) - f(x+h) + 2f'(x)h + (f''(x+\theta h) - f''(x+\theta' h))\frac{h^2}{2}\right| \le 2M_0 + 2M_1h + M_2h^2$$

Since this holds for every $h \in \mathbb{R}$ (as the domain of $f$ is $\mathbb{R}$), clearly, $M_k \geq 0$ for all $k = 0, 1, 2, \cdots$, we get the following situations.

If $M_2 > 0$, it follows that the quadratic equation in $h$

$$M_2 h^2 + 2M_1 h + 2M_0 = 0$$

either has only one solution or has no solution, i.e.,

$$\Delta = (2M_1)^2 - 4(M_2)(2M_0) \leq 0$$

iff

$$M_1 \leq \sqrt{2M_0 M_2}.$$

For $M_2 = 0$ this implies

$$0 \leq 2M_0 + 2M_1 h$$

If $M_1 \neq 0$, then $M_1 > 0$, taking limit $h \to -\infty$, we get a contradiction. Thus

$$M_1 = 0 \leq \sqrt{2M_0(0)} = \sqrt{2M_0 M_2}.$$

□

## 5. Integration Theorems

**Integral Criterion**

1. Let $f : [0, 1] \to \mathbb{R}$ be given by $f(x) = x^3$, prove that $f$ is Riemann integrable.

*Proof.* Consider a partition $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\right\}$ with $n > \frac{1}{\epsilon}$.

On each $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, for $k \in \{1, 2, \cdots, n\}$, $m_j = \inf\limits_{x \in [x_{j-1}, x_j]} f(x)$ and $M_j = \sup\limits_{x \in [x_{j-1}, x_j]} f(x)$.

We get

$$L(f, P) = \sum_{k=1}^{n} \frac{1}{n} \left(\frac{k-1}{n}\right)^3 \quad \text{and} \quad U(f, P) = \sum_{k=1}^{n} \frac{1}{n} \left(\frac{k}{n}\right)^3,$$

Thus

$$U(f, P) - L(f, P) = \frac{1}{n} \left( \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^3 - \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^3 \right) = \frac{1}{n} \left(\frac{n}{n}\right)^3 = \frac{1}{n} < \epsilon$$

By Integral criterion, $f(x) = x^3$ is Riemann integrable.

(Of course we can directly state that $f(x) = x^3$ is Riemann integrable since $f$ is continuous. The above is just an example on how to apply the Integral criterion.) □

2. (Thomae's function) Show that the function defined on $[0, 1]$ by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is rational in lowest terms with } q > 0 \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is Riemann-Integrable.

*Proof.* The irrational numbers are dense.
Thus for any partition $P = \{x_0, \cdots, x_n\}$, there is always an irrational in every interval $[x_{i-1}, x_i]$.
Thus $L(f, P) = 0$.

To prove that $f$ is integrable,
it is enough to show that for every $\epsilon > 0$, there is a partition $P$ with $U(f, P) < \epsilon$.

For any $\epsilon > 0$, by Archimedean Principle, there is $N \in \mathbb{N}$ such that $N > \dfrac{2}{\epsilon}$.
Consider the set $\mathbb{Q}_N = \left\{\dfrac{p}{q} \in [0,1] \cap \mathbb{Q} : q \leq N\right\}$.
One can see that $\mathbb{Q}_N$ is finite. Say there are $n$ elements $x_i = \dfrac{p_i}{q_i}$ for $i = 1, 2, \cdots, n$.
W.L.O.G., assume $x_1 < x_2 < \cdots < x_n$.
Consider the partition with $\delta = \min\left\{x_1, \dfrac{x_{i+1} - x_i}{2}, 1 - x_n, \dfrac{\epsilon}{8n} \;\middle|\; i = 1, 2, \cdots, n\right\} > 0$

$$0 \leq x_1 - \delta \leq x_1 + \delta \leq x_2 - \delta \leq x_2 + \delta \leq \cdots \leq x_n - \delta \leq x_n + \delta \leq 1$$

$$U(P, f) \leq (x_1 - \delta - 0)\left(\frac{1}{N}\right) + \sum_{k=2}^{n}\left(\frac{1}{N}\right)((x_k - \delta) - (x_{k-1} + \delta)) + \sum_{k=1}^{n}\left(\frac{1}{q_k}\right)(2\delta) + (1 - (x_k + \delta))\left(\frac{1}{N}\right)$$

$$\begin{aligned}
|U(f,P) - L(f,p)| =& |U(f,P)| \\
\leq& (x_1 - \delta - 0)\left(\frac{1}{N}\right) + \sum_{k=2}^{n}\left(\frac{1}{N}\right)((x_k - \delta) - (x_{k-1} + \delta)) + \sum_{k=1}^{n}\left(\frac{1}{q_k}\right)(2\delta) + (1 - (x_k + \delta))\left(\frac{1}{N}\right) \\
\leq& (x_1 - \delta - 0)\left(\frac{1}{N}\right) + \sum_{k=2}^{n}\left(\frac{1}{N}\right)((x_k - \delta) - (x_{k-1} + \delta)) + \sum_{k=1}^{n}(2\delta) + (1 - (x_k + \delta))\left(\frac{1}{N}\right) \\
=& \frac{1}{N}(1 - 2n\delta) + 2n\delta \\
=& \frac{1}{N} + 2n\delta\left(1 - \frac{1}{N}\right) \\
<& \frac{1}{N} + 2n\delta \\
<& \frac{\epsilon}{2} + 2n\frac{\epsilon}{4n} \quad (\text{Note that } \delta < \frac{\epsilon}{4n}) \\
=& \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
=& \epsilon
\end{aligned}$$

So, $f(x)$ is Riemann Integrable. □

3. Let $f, h$ are bounded function and Riemann Integrable on $[a, b]$
and let $g : \{a, b\} \to \mathbb{R}$ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in [a, b]$.
Suppose
$$\int_a^b f(x)\,dx = \int_a^b h(x)\,dx = A.$$
Show that $g(x)$ is also Riemann Integrable on $[a, b]$.

*Proof.* For any $\epsilon > 0$, since $f(x)$ is Riemann Integrable, then there exists partition $P_1$ such that

$$A - \frac{\epsilon}{2} < L(P_1, f) \leq A \leq U(P_1, f) < A + \frac{\epsilon}{2}$$

Similarly, $h(x)$ is Riemann Integrable, then there exists partition $P_2$ such that

$$A - \frac{\epsilon}{2} < L(P_2, h) \leq A \leq U(P_2, h) < A + \frac{\epsilon}{2}$$

Consider the partition $P$ which is the refinement of $P$, $P = P_1 \cup P_2$.

Then for this partition $P$,

$$U(P, g) \leq U(P, h) \leq U(P_2, h) < A + \frac{\epsilon}{2}$$

and

$$L(P, g) \geq L(P, f) \geq L(P_1, f) > A - \frac{\epsilon}{2}$$

Then

$$A - \frac{\epsilon}{2} < L(P, g) \leq U(P, g) < A + \frac{\epsilon}{2}$$

This implies

$$|U(P, g) - L(P, g)| = U(P, g) - L(P, g) < \left(A + \frac{\epsilon}{2}\right) - \left(A - \frac{\epsilon}{2}\right) = \epsilon$$

Hence $g(x)$ is Riemann Integrable on $[a, b]$. □

**Lebesgue's theorem**

1. (2002 Final). Let $f, g : [0, 2] \to \mathbb{R}$ be Riemann integrable.
   Prove that $h : [0, 2] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if } x \in [0, 1] \\ \min\{f(x), g(x)\} & \text{if } x \in (1, 2] \end{cases}$$

is also Riemann integrable on $[0, 2]$.

*Proof.* We try to show $S_h = \{x \in [0, 2] : h \text{ is not continuous at } x\}$ has measure zero.

Due to the way we define $h$, let's decompose $S_h$ into $S_h = (S_h \cap [0, 1)) \cup (S_h \cap (1, 2]) \cup (S_h \cap \{1\})$
Since $h = \max\{f, g\}$ on $[0, 1)$, we have $S_h \cap [0, 1) = S_{\max\{f,g\}} \cap [0, 1)$.
Similarly, $S_h \cap (1, 2] = S_{\min\{f,g\}} \cap (1, 2]$.

Therefore we conclude $S_h \subseteq S_{\max\{f,g\}} \cup S_{\min\{f,g\}} \cup \{1\}$.

Now it is enough to show $S_{\max\{f,g\}}$ and $S_{\min\{f,g\}}$ have measure zero.

This can be done in two ways:

Method 1:

By the formula

$$\max\{x, y\} = \frac{x + y + |x - y|}{2} \quad \text{and} \quad \min\{x, y\} = \frac{x + y - |x - y|}{2}$$

we see that both $\max\{f, g\}$ and $\min\{f, g\}$ are Riemann integrable, hence both $S_{\max\{f,g\}}$, $S_{\min\{f,g\}}$ have measure zero by Lebesgue Theorem.

Method 2:

Since $f$ is continuous at $x$ and $g$ is continuous at $x$
this implies $\max\{f, g\}$ is continuous at $x$,
By taking contrapositive, we have $S_{\max\{f,g\}} \subseteq S_f \cup S_g$

Since a union of two measure zero sets are of measure zero, $S_f \cup S_g$ has measure zero.
$S_{\max\{f,g\}}$ being a subset of measure zero set is also of measure zero.

Similarly, since $S_{\min\{f,g\}} \subseteq S_f \cup S_g$, so $S_{\min\{f,g\}}$ has measure zero.

Hence $h$ is Riemann Integrable on $[0, 2]$. □

2. Let $g$ be an integrable function on $[a, b]$ for $a, b \in \mathbb{R}$, define $f : [a, b] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is a prime number} \\ g(x) & \text{otherwise} \end{cases}$$

Prove that $f$ is integrable on $[a, b]$.

*Proof.* Since $g$ is integrable on $[a, b]$, the set $S_g$ of discontinuous points of $g$ on $[a, b]$ is of measure zero.
Also, the set of all prime numbers on $[a, b]$, denoted as $P_{[a,b]}$, is finite and thus again of measure zero.
Then we get $S_f \subseteq S_g \cup P_{[a,b]}$.
Since $S_f$ is a subset of a union of two measure zero sets, $S_f$ is also of measure zero.
Hence $f$ is integrable on $[a, b]$ by Lebesgue Theorem. □

3. (Thomae's function) Show that the function defined on $[0, 1]$ by

$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ is rational in lowest terms with } q > 0 \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is Riemann-Integrable.

*Proof.* For $x_0 \in \mathbb{R}\backslash\mathbb{Q}$, $f(x_0) = 0$.

For any $\epsilon > 0$, by Archimedean Principle, there is $K \in \mathbb{N}$ such that $K > \dfrac{1}{\epsilon}$.

Then $\dfrac{1}{q} < \epsilon$ for all $q \geq K$.

The only $x$ with $|f(x)| = |f(x) - f(x_0)| \geq \epsilon$ is $\dfrac{p}{q}$ for $q < K$.

There are only finitely many $x = \dfrac{p}{q}$ with $q < K$ as $0 \leq p < q < K$.

Let's say $\dfrac{p_1}{q_1}, \cdots, \dfrac{p_n}{q_n}$. Take $\delta = \min\left\{ \left| \dfrac{p_i}{q_i} - x_0 \right| \mid i = 1, 2, \cdots, n \right\} > 0$, then $|x - x_0| < \delta$ implies $x \neq \dfrac{p_i}{q_i}$ for $i = 1, 2, \cdots, n$, then either $x$ is irrational or $x = \dfrac{p}{q}$ with $\dfrac{1}{q} \leq \dfrac{1}{K} < \epsilon$.

In either case, we get

$$|f(x) - f(x_0)| = |f(x)| < \epsilon$$

This shows $f$ is continuous at $x \in \mathbb{R}\backslash\mathbb{Q}$, that means if $x \in \mathbb{R}\backslash\mathbb{Q}$, then $f$ is continuous at $x$.

In other words, $\mathbb{R}\backslash\mathbb{Q} \subseteq \mathbb{R}\backslash S_f$.

Taking contrapositive, this shows $S_f \subseteq \mathbb{Q}$, which is countable and hence of measure zero.
By Lebesgue's Theorem, $f$ is Riemann Integrable. □


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Rudin: `https://notendur.hi.is/vae11/%C3%9Eekking/principles_of_mathematical_analysis_walter_rudin.pdf`

Problem in Mathematical Analysis:
`https://lethuc92.files.wordpress.com/2014/08/problems-in-mathematical-analysis1.pdf`

Putnam and Beyond:
`http://www-bcf.usc.edu/~lototsky/PiMuEp/PutnamAndBeyond-Andreescu.pdf`