Lecture 12 19-03-2019

Review:

Question: What sequences are convergent?

1. Monotone sequence thm.

$x_n \uparrow$, $\{x_n\}$ bounded above $\Rightarrow$ $\{x_n\}$ converges

$x_n \downarrow$, $\{x_n\}$ bounded below $\Rightarrow$ $\{x_n\}$ converges

2. Nested interval thm:

$$\left.\begin{array}{l} I_n = [a_n, b_n], \quad I_1 \supseteq I_2 \supseteq \cdots \\ \text{or } a_n \le a_{n+1} \le b_{n+1} \le b_n \quad \forall n \in \mathbb{N} \end{array}\right\} \Rightarrow \bigcap_{n \in \mathbb{N}} I_n \neq \phi.$$

Actually $\bigcap_{n \in \mathbb{N}} I_n = [a, b]$

with $a = \sup\{a_n\}$, $b = \sup\{b_n\}$

3. Bolzano-Weierstrass thm:

$\{x_n\}$ is bounded $\Rightarrow$ $\exists$ subsequence $\{x_{n_k}\}$ converges.

Proof is based on a searching algorithm for the limiting point

4. Cauchy sequence : $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t

$|x_n - x_m| < \varepsilon$ for all $m, n \geq K$.

5. Boundedness theorem of Cauchy sequence :

$\{x_n\}$ is Cauchy $\Rightarrow$ $\{x_n\}$ is bounded

6. Cauchy's theorem

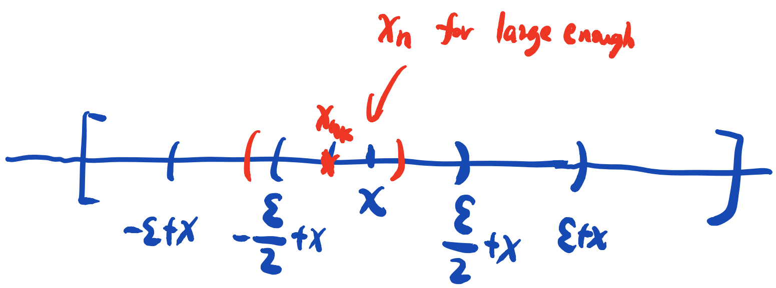
$\{x_n\}$ converges $\Leftrightarrow$ $\{x_n\}$ is Cauchy

Last time, we showed $(\Rightarrow)$.

Cauchy's Theorem

THM: $\{x_n\}$ converges $\Leftrightarrow$ $\{x_n\}$ is Cauchy.

Pf: $(\Leftarrow)$. Given $\{x_n\}$ is Cauchy, we need to show that $\{x_n\}$ Converges.



Since $\{x_n\}$ is Cauchy, $\{x_n\}$ is bounded by the boundedness thm.

By Bolzano-Weistrass thm, $\exists$ subsequence $\{x_{n_k}\}$ which converges to x. We show that $\{x_n\}$ converges to x.

$\forall\ \varepsilon>0$, $\{x_n\}$ is Cauchy $\Rightarrow \exists\ K_1 \in \mathbb{N}$ s.t. $|x_n - x_m| < \frac{\varepsilon}{2}$ $\forall\ m,n \geq K_1$

$\{x_{n_k}\}$ converges to $x \Rightarrow \exists\ K_2 \in \mathbb{N}$ s.t. $|x_{n_j} - x| < \frac{\varepsilon}{2}$ $\forall\ j \geq K_2$

let $k = \max(K_1, K_2)$. Then $\forall\ n \geq k$.

fixed

$n_k \geq k \geq K_1 \Rightarrow |x_n - x_{n_k}| < \frac{\varepsilon}{2}$; $k \geq K_2 \Rightarrow |x_{n_k} - x| < \frac{\varepsilon}{2}$.

$\Rightarrow |x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \varepsilon \Rightarrow \lim x_n = x$

Remark 1: One can use Cauchy sequence to construct $\mathbb{R}$ from $\mathbb{Q}$

Remark 2. The following versions of completeness are "equivalent".

(1) Completeness Axiom

(2) Every Cauchy sequence in $\mathbb{R}$ has a limit in $\mathbb{R}$

(3): Every increasing/decreasing sequence in $\mathbb{R}$ which is bounded above/below has a limit in $\mathbb{R}$

(4) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence with a limit in $\mathbb{R}$.

(5) Nested interval thm: $I_n = [a_n, b_n]$, $I_1 \supseteq I_2 \supseteq \cdots \Rightarrow \bigcap_{n \in \mathbb{N}} I_n \neq \phi$

Example : let $x_1 = \sin 1$. $x_n = x_{n-1} + \frac{\sin n}{n^2}$ $n \geq 1$.

Prove that $\{x_n\}$ converges $\sum_{n\geq 1} \frac{\sin n}{n^2}$

Solution : we show that $\{x_n\}$ is Cauchy.

let $n < m$,

$$|x_m - x_n| = |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + (x_{m-2} - x_{m-3}) + \cdots (x_{n+1} - x_n)|$$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$

$$= \left|\frac{\sin m}{m^2}\right| + \left|\frac{\sin(m-1)}{(m-1)^2}\right| + \cdots \left|\frac{\sin(n+1)}{(n+1)^2}\right|$$

$$< \frac{1}{m^2} + \frac{1}{(m-1)^2} + \cdots \frac{1}{(n+1)^2}$$

$$< \frac{1}{m(m-1)} + \frac{1}{(m-1)(m-2)} + \cdots \frac{1}{(n+1)n}$$

$$= \left(\cancel{\frac{1}{m-1}} - \frac{1}{m}\right) + \left(\frac{1}{m-2} - \cancel{\frac{1}{m-1}}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{n} - \frac{1}{m} < \frac{1}{n}$$

$\forall \varepsilon > 0$, let $K = [\frac{1}{\varepsilon}] + 1$,

$\Rightarrow$ $|x_m - x_n| < |\frac{1}{m} - \frac{1}{n}| < \frac{1}{K} < \varepsilon$, for all $m, n \geq K$.

$\Rightarrow$ $\{x_n\}$ is Cauchy $\Rightarrow$ $\{x_n\}$ converges.

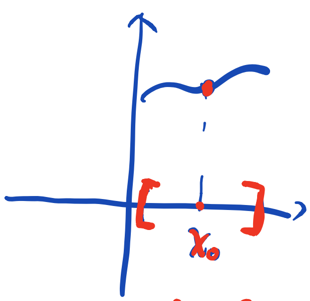
Limit of functions

We aim to give a rigorous meaning to $\lim_{x \to x_0} f(x) = L$

Question: Let $f: S \to \mathbb{R}$ be a function, where S is an interval such as $[a,b]$, (a,b), $[a,b)$, $(a,b]$ with $a<b$, or union of several intervals,

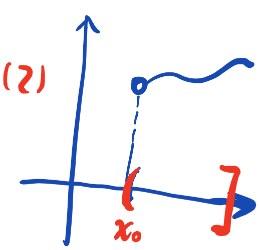
At what point can we talk about $x \to x_0$?

(1)



$x_0 \in S$

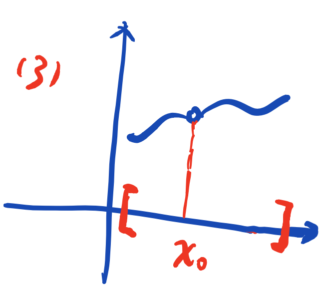
(2)



$x_0 \notin S$

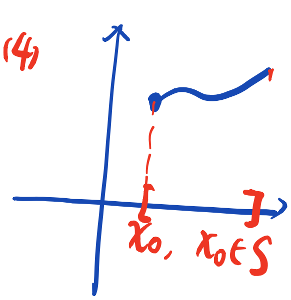
x_0 is the limit of a sequence of points in S

(3)

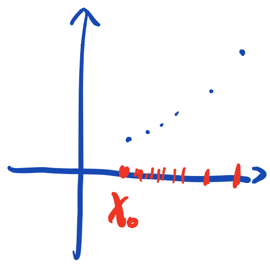


$x_0 \in S$

(4)



S can also be a very general subset of $\mathbb{R}$.



$S = \{x_n : n \geq 1\}$, $\lim x_n = x_0$

Def: x_0 is an accumulation point (or limit point, or cluster point) of S iff $\exists$ $x_n \in S \setminus \{x_0\}$ $n \geq 1$, ($x_n \in S$, $x_n \neq x_0$) s.t. $\lim x_n = x_0$

Remark: Accumulation points may or may not be in S.

Notation : We write $x_n \to x_0$ in $S \setminus \{x_0\}$ to mean that $x_n \in S \setminus \{x_0\}$ (or $x_n \in S$, $x_n \neq x_0$) and $\lim x_n = x_0$

Convention : let $f: S \to \mathbb{R}$ (domain of f), when discussing $\lim_{x \to x_0} f(x)$, we will assume that x_0 is an accumulation point of S.

Remark : a more complete notation is $\lim_{\substack{x \to x_0 \\ x \in S}} f(x)$. For simplicity, we write $\lim_{x \to x_0} f(x)$ with in mind that x should be in S, the domain of f.

Remark : In this course, we focus on the case when S is an interval, or union of intervals. But most of results apply to general S.

Definition of $\lim_{x \to x_0} f(x) = L$.

roughly: $f(x)$ is as close to L as desired

when x is sufficient close to x_0

$\Downarrow$

$\forall \varepsilon > 0$, $|f(x) - L| < \varepsilon$ if $|x - x_0|$ is

small enough.

$\Downarrow$

$\forall \varepsilon > 0$, $|f(x) - L| < \varepsilon$ if $|x - x_0| < \delta$

for some $\delta > 0$.

$\Downarrow$

The ε-δ language

Def: $\lim\limits_{x \to x_0} f(x) = L$ iff $\forall \varepsilon > 0$, $\exists \delta > 0$ (depending on ε)

s.t $|f(x) - L| < \varepsilon$ for all $x \in S \setminus \{x_0\}$ with $|x - x_0| < \delta$.

Or equivalently

$$\boxed{\begin{array}{l} \forall \varepsilon > 0,\ \exists \delta > 0 \text{ s.t } \forall x \in S \\ 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon. \end{array}}$$

Remark: Compare with " $\lim x_n = x$. " iff

$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$, s.t

$|x_n - x| < \varepsilon$ for all $n \geq K$.

Remark: In the definition above, the value of $f(x_0)$ plays no role.

Actually, it is ok that f is even not defined at x_0.

Example ① $f(x) = \dfrac{x^3 - 3x^2}{x-3} = x^2 \cdot \dfrac{x-3}{x-3}$.

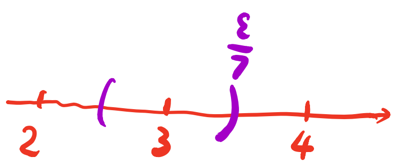
Check that $\lim\limits_{x \to 3} f(x) = 9$.

(Remark : f is not defined at $x=3$.) $S = \mathbb{R} \setminus \{3\}$

Solution: $\boxed{\forall\, \varepsilon > 0,}$ need to find $\delta > 0$ s.t $\forall\, x \neq 3$,

$0 < |x-3| < \delta \Rightarrow |f(x) - 9| < \varepsilon$.

For $x \neq 3$, $f(x) = x^2$. $|f(x) - 9| = |x^2 - 9| = \underbrace{|x-3|}_{< \frac{\varepsilon}{7}} \cdot \underbrace{|x+3|}_{< 7} < \varepsilon$



take $\boxed{\delta = \min\left\{ \frac{\varepsilon}{7}, 1 \right\}}$

then $\boxed{0 < |x-3| < \delta} \Rightarrow \begin{cases} |x-3| < \frac{\varepsilon}{7} \\ |x+3| < 7 \end{cases}$

$\Rightarrow \boxed{|f(x) - 9| < \varepsilon} \Rightarrow \lim\limits_{x \to 3} f(x) = 9$

Example 2. Let $g: [0, \infty) \to \mathbb{R}$ be defined by

$$g(x) = \sqrt{x}.$$

Check that $\lim_{x \to 0} g(x) = 0$.

Solution: $\forall \varepsilon > 0$, need to find $\delta > 0$ s.t

$$\forall x \in \underset{\overset{\downarrow}{S}}{[0, \infty)}, \quad 0 < |x - 0| < \delta \Rightarrow |g(x) - 0| < \varepsilon$$

$$|g(x) - 0| = \sqrt{x} < \varepsilon \quad \Rightarrow \quad x < \varepsilon^2$$

Take $\delta = \varepsilon^2$. then $\forall x \in [0, \infty)$, $0 < |x - 0| < \underset{= \varepsilon^2}{\delta}$

$$\Rightarrow \quad |g(x) - 0| = \sqrt{x} < \sqrt{\varepsilon^2} = \varepsilon$$

$$\Rightarrow \quad \lim_{x \to 0} g(x) = 0$$

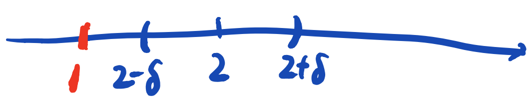
Example 3. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by

$f(x) = \frac{1}{5x}$. Check $\lim_{x \to 2} f(x) = \frac{1}{10}$

Solution: $\forall\, \varepsilon > 0$, need to find $\delta > 0$ s.t.

$\forall\, x \neq 0$, $0 < |x-2| < \delta \Rightarrow \left| f(x) - \frac{1}{10} \right| < \varepsilon$.

$$\left| f(x) - \frac{1}{10} \right| = \left| \frac{1}{5x} - \frac{1}{10} \right| = \frac{|x-2|}{10x} \underset{\text{if } x>1}{<} \frac{|x-2|}{10} \underset{\text{if } |x-2|<\varepsilon}{<} \frac{\varepsilon}{10} < \varepsilon$$



Take $\delta = \min\{1, \varepsilon\}$, then $\forall\, x \in \mathbb{R} \setminus \{0\}$

$0 < |x-2| < \delta \Rightarrow x > 1$ and $|x-2| < \delta$, so

$$\left| f(x) - \frac{1}{10} \right| = \left| \frac{1}{5x} - \frac{1}{10} \right| = \frac{|x-2|}{10x} < \frac{|x-2|}{10} < \frac{\varepsilon}{10} < \varepsilon.$$

$\Rightarrow \lim_{x \to 2} f(x) = \frac{1}{10}$

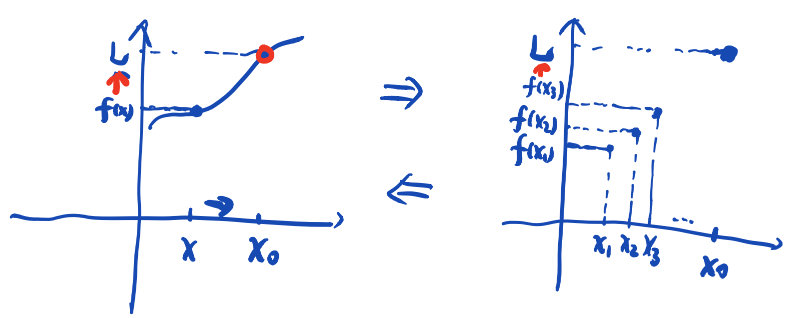
Sequential limit theorem (S.L.T)

Recall: " $x_n \to x_0$ in $S \setminus \{x_0\}$ " $\Leftrightarrow$ $x_n \in S \setminus \{x_0\}$

$\lim x_n = x_0$

THM: Let $f: S \to \mathbb{R}$ be a function and x_0 be an accumulation point of S. Then

$$\lim_{x \to x_0} f(x) = L \iff \forall\, x_n \to x_0 \text{ in } S\setminus\{x_0\},\quad \lim_{n\to\infty} f(x_n) = L.$$



Pf: $\Rightarrow$ $\forall\, \varepsilon > 0$, $\lim\limits_{x \to x_0} f(x) = L \Rightarrow \Big($ $\exists\, \delta > 0$, s.t

$\forall\, x \in S$, $0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

Since $\lim_{n\to\infty} x_n = x_0$, for $\delta > 0$, $\exists\, K \in \mathbb{N}$ s.t

$$(n \geq K \Rightarrow |x_n - x_0| < \delta)$$

Then, for $n \geq K$, we have $0 < |x_n - x_0| < \delta$

$$\Rightarrow |f(x_n) - L| < \varepsilon \quad \Rightarrow \lim_{n\to\infty} f(x_n) = L$$