MATH 2033 L1 Mathematical Analysis
Midterm Exam
Spring 2019

29 March 2019

Student Name: ____________________

Student Id: ____________________

Tutorial Section: ____________________

Instructions

Please read the following and sign in the blank provided below and sign in the blank below.

1. You **MUST** seat according to the **Seating Plan**.
2. **DO NOT OPEN** the exam until you are told to.
3. This is a **CLOSED BOOK** exam.
4. All mobile phones and communication devices should be switched **OFF**.
5. Only calculators approved by **HKEAA** allowed.
6. Answer **ALL** 17 questions.
7. You must **SHOW YOUR WORK** to receive credits in all questions except Multiple Questions. Answers alone (whether correct or not) will not receive any credit.
8. Some questions are structured into several parts. The results stated in the preceding parts can be quoted for doing the next parts, regardless of your answers in the preceding parts. However, different parts in the same question may not be co-related.

Score Summary (Examiner Only)

Question No	Points	Scores
1	15	
2	25	
3	25	
4	35	
Total	100	

Integrity Statement

I have neither given nor received any unauthorised aid during this exam. The answers submitted are my own work. I understand that sanctions will be imposed if I am found to have violated the University's regulations governing academic integrity.

Signature: ____________________

1. (15 points) .

1.1 Negating the following statements.

$$\forall \epsilon > 0, \exists \delta > 0, 0 < |x - x_0| < \delta \implies |\frac{f(x) - f(x_0)}{x - x_0} - L| < \epsilon$$

1.2 Find $\cap_{n \in \mathbb{N}} (0, \frac{1}{n})$ and justify your answer.

Solution:

1.1

$$\exists \epsilon > 0, \forall \delta > 0, \exists x, 0 < |x - x_0| < \delta, |\frac{f(x) - f(x_0)}{x - x_0} - L| \geq \epsilon$$

1.2

$$\cap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset.$$

For $x \leq 0, x \geq 1, x \notin (0, \frac{1}{n}), \forall n \in \mathbb{N}$, thus $x \notin \cap_{n \in \mathbb{N}} (0, \frac{1}{n})$ for $x \leq 0$ and $x \geq 1$. For $0 < x < 1$, if $n > [\frac{1}{x}] + 1$, then $x \notin (0, \frac{1}{n})$. This implies $x \notin \cap_{n \in \mathbb{N}} (0, \frac{1}{n})$. Therefore,

$$\cap_{n \in \mathbb{N}} (0, \frac{1}{n}) = \emptyset.$$

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2. (25 points) .
2.1 Write down the definition of infimum. State and prove the infimum property.
2.2 Determine if the following set A has an infimum. If it exists, find it and justify your answer.

$$A = \{x + y^2 : x \in [0, 1] \cap \mathbb{Q}, y \in [0, 1] \setminus \mathbb{Q}\}$$

Solution:
2.1 Definition: An infimum of S, denoted by $InfS$ is a lower bound such that $K \leq InfS$ for all bounds K of S.
Theorem:(Infimum property)
$InfS$ is an infimum of S in $\mathbb{R}$ if and only if $InfS$ is a lower bound of S and

$$\forall \epsilon > 0, \exists x \in S, \text{ s.t } InfS \leq x \leq InfS + \epsilon.$$

Proof " $\Longrightarrow$ " If $InfS$ is an infimum of S in $\mathbb{R}$, by definition of infimum, we have $InfS$ is a lower bound of S. Since $InfS$ is the largest lower bound of S, for all $\epsilon > 0$, $InfS + \epsilon$ cannot be a lower bound of S. Hence, for all $\epsilon > 0$, we can find $x \in S$ such that

$$InfS \leq x \leq InfS + \epsilon.$$

" $\Longleftarrow$ " We only need to show $InfS$ is the largest bound. Otherwise, there is another lower bound m such that $m > InfS$. Then $\epsilon = m - InfS > 0$, for this ϵ, we can find $x \in S$ such that

$$x < InfS + \epsilon = m.$$

This contradicts m is a lower bound of S. So $InfS$ is the largest lower bound and the infimum.

2.2 Since $x \in [0, 1] \cap \mathbb{Q}, y \in [0, 1] \setminus \mathbb{Q}$, we can see $0 \leq x + y$ and thus 0 is a lower bound of A. To show 0 is an infimum of A, we note that $x_n = \frac{1}{n} \in [0, 1] \cap \mathbb{Q}, y_n = \frac{1}{\sqrt{2}n} \in [0, 1] \setminus \mathbb{Q}$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0.$$

Hence, we find a sequence $\{x_n + y_n^2\} \subset A$ such that

$$\lim_{n\to\infty} x_n + y_n^2 = 0.$$

By Infimum-limit theorem, we conclude that $InfA = 0$.

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3. (25 points) .
3.1 Let $C \geq 1$, prove that
$$\lim_{n\to+\infty} C^{\frac{1}{n}} = 1.$$
3.2 Determine whether the sequence x_n defined by
$$x_1 = 3, x_{n+1} = 3 + \frac{4}{x_n} \quad \text{for } n \geq 1,$$
converges or not. If it converges, prove its convergence and find the limit.

Solution:
3.1 $\forall \epsilon > 0$, take $K = [\frac{C-1}{\epsilon}] + 1 \in \mathbb{N}$. We can see, $\forall n > K, n \in \mathbb{N}, C < 1 + n\epsilon < (1+\epsilon)^n$. Therefore, for any $\epsilon > 0, n > K, n \in \mathbb{N}$, we have
$$C^{\frac{1}{n}} < 1 + \epsilon.$$
On the other hand, $C^{\frac{1}{n}} \geq 1^{\frac{1}{n}} = 1$. Then we have
$$|C^{\frac{1}{n}} - 1| < \epsilon, \forall n > K.$$
That is $\lim_{n\to\infty} C^{\frac{1}{n}} = 1$.

3.2 The sequence converges. If $3 \leq x_n \leq 5$, then by $x_{n+1} = 3 + \frac{4}{x_n}$, we have
$$3 \leq 3 + \frac{4}{5} \leq x_{n+1} \leq 3 + \frac{4}{3} \leq 5.$$
By mathematical induction principle, we have
$$3 \leq x_n \leq 5, \forall n \in \mathbb{N}.$$
Then by
$$\begin{aligned} x_{n+2} - x_n &= \frac{4}{x_{n+1}} - \frac{4}{x_{n-1}} \\ &= \frac{4}{x_{n+1}x_{n-1}}(x_{n-1} - x_{n+1}) \\ &= \frac{4}{x_{n+1}x_{n-1}}(\frac{4}{x_{n-2}} - \frac{4}{x_n}) \\ &= \frac{4}{x_{n+1}x_{n-1}}\frac{4}{x_n x_{n-2}}(x_n - x_{n-2}), \end{aligned}$$
because $\frac{4}{x_{n+1}x_{n-1}}\frac{4}{x_n x_{n-2}} > 0$, then
$$x_{n+2} > x_n \text{ if } x_n > x_{n-2}$$
and
$$x_{n+2} < x_n \text{ if } x_n < x_{n-2}.$$
Because $x_1 = 3, x_2 = \frac{13}{3}, x_3 = \frac{51}{13}, x_4 = \frac{205}{51}$, then
$$\{x_{2n+1}, \in\in \mathbb{N}\} \text{ increasing}$$
and
$$\{x_{2n}, n \in \mathbb{N}\} \text{ dereasing.}$$

Suppose $x_{2n+1} \to a$ and $x_{2n} \to b$ as $n \to \infty$. Then

$$a = 3 + \frac{4}{b} \quad b = 3 + \frac{4}{a}.$$

The solution and the constraint $3 \leq a, b \leq 5$ gives

$$a = b = 4.$$

By interwining sequence theorem, the sequence $\{x_n\}$ converges and

$$\lim_{n \to +\infty} x_n = 4.$$

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4. (35 points) .

4.1 Write down the definition of Cauchy sequence.
4.2 Let $\{x_n\}$, $\{y_n\}$ be Cauchy sequence, show that $\{x_n - y_n\}$ is a Cauchy sequence.
4.3 Let S be the set of all Cauchy sequences in $\mathbb{Q}$. More precisely,

$$S = \{\{x_n\} : \{x_n\} \text{ is a Cauchy sequence s.t } x_n \in \mathbb{Q} \text{ for all } n \in \mathbb{N}\}.$$

Determine if S is countable and justify your answer.
4.4 Let S be defined above, let $\{x_n\} \in S$, we say that $\{x_n\}$ is positive iff there exists $\delta > 0, \delta \in \mathbb{Q}$ and $k \in \mathbb{N}$ s.t $x_n > \delta$ for all $n \geq k$. We say that $\{x_n\} < \{y_n\}$ iff $\{y_n - x_n\}$ is positive. Show that

$$\forall \{x_n\}, \{y_n\}, \{z_n\} \in S,$$

if $\{x_n\} < \{y_n\}$, and $\{z_n\}$ is positive, then

$$\{x_n z_n\} < \{y_n z_n\}.$$

Solution:
4.1 Definition: A sequence $\{x_n\}$ is a Cauchy sequence iff $\forall \epsilon > 0, \exists K \in \mathbb{N}$ such that $m, n > K$,

$$|x_m - x_n| < \epsilon.$$

4.2 By definition, $\forall \epsilon > 0, \exists K_1, K_2 \in \mathbb{N}$ such that $m, n > K_1, m', n' > K_2$

$$|x_m - x_n| < \frac{\epsilon}{2}$$
$$|y_{m'} - y_{n'}| < \frac{\epsilon}{2}$$

Therefore, taking $K = \max(K_1, K_2)$, we have $p, q > K$ implies

$$|(x_p - y_p) - (x_q - y_q)| \leq |x_p - x_q| + |y_p - y_q| < \epsilon$$

which means $\{x_n - y_n\}$ is a Cauchy sequence.
4.3 We can construct a surjective mapping $f : S \to \mathbb{R}$ given by

$$f(\{x_n\}) = \lim_{n\to\infty} x_n.$$

This mapping is well defined because $\{x_n\} \subset \mathbb{Q} \subset \mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$ and by Cauchy theorem, the $\lim_{n\to\infty} x_n$ exists in $\mathbb{R}$. Next, we show this map is surjective. For $\forall x \in \mathbb{R}$, we can construct a sequence $\{x_n\}$ as follow:

$$x_n \in (x - \frac{1}{n}, x + \frac{1}{n})$$

where $x_n \in \mathbb{Q}$. Clearly, it is a Cauchy sequence since $\forall \epsilon > 0, \exists K = [\frac{2}{\epsilon}] + 1$, for $m, n > K$,

$$|x_m - x_n| \leq |x_m - x| + |x_n - x| \leq \frac{1}{n} + \frac{1}{m} < \epsilon.$$

Finally, by surjective theorem, S is uncountable since $\mathbb{R}$ is uncountable.
4.4 Since $\{x_n\} < \{y_n\}$, there exist $\delta_1 > 0, \delta_1 \in \mathbb{Q}$ and $K_1 \in \mathbb{N}$ such that

$$y_n - x_n > \delta_1, \quad \forall n \geq K_1.$$

Similarly, since $\{z_n\}$ positive, there exists $\delta_2 > 0, \delta_2 \in \mathbb{Q}$ and $K_2 \in \mathbb{N}$ such that

$$z_n > \delta_2, \quad \forall n \geq K_2.$$

Take $K = \max\{K_1, K_2\}, \delta = \delta_1\delta_2 > 0, \delta \in \mathbb{Q}$ and for all $n > K$, we have

$$z_n y_n - z_n x_n = z_n(y_n - x_n) > \delta$$

which means

$$\{x_n z_n\} < \{y_n z_n\}.$$

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