Lecture 7 28-02-2019

Review:

① Three Axioms hold for the rational numbers:

I: Field Axiom; II: Order Axiom; III: Well-ordering Axiom;

The fundamental difference between $\mathbb{Q}$ and $\mathbb{R}$ lies in IV: Completeness Axiom.

② Sup S : (1) upper bound of S; ($x \leq \sup S$, $\forall x \in S$)

(2) the least upper bound; ($\sup S \leq M$, $\forall$ upper bound M)

Inf S : (1) lower bound of S; ($x \geq \inf S$, $\forall x \in S$)

(2) the greatest lower bound; ($\inf S \geq m$, $\forall$ lower bound m)

③ IV: Completeness axiom (for supremum): Sup S exists in $\mathbb{R}$

if $\emptyset \neq S \subset \mathbb{R}$ is bounded above.

Foundation to take limit and do analysis with $\mathbb{R}$.

④ $\inf B = -\sup(-B)$ if one side exists. (Dual between Sup and Inf)

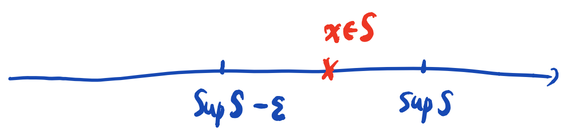
⑤ Infinetesimal principal : $x \leq y \Leftrightarrow x < y + \varepsilon$, $\forall \varepsilon > 0$

Supremum Property

THM : If S has a supremum in $\mathbb{R}$, then $\forall\, \varepsilon>0$,

$\exists\, x \in S$ such that

$$\sup S - \varepsilon < x \leq \sup S.$$



Proof : ① By definition of Sup, $x \leq \sup S$, $\forall\, x \in S$.

② Since $\sup S - \varepsilon < \sup S$, by the definition of Supremum, $\sup S - \varepsilon$ is not an upper bound of S.

[$\sup S - \varepsilon$ is an upper bound $\Leftrightarrow$ $\forall\, x \in S$, $x \leq \sup S - \varepsilon$]

Then $\exists\, x \in S$, such that $x > \sup S - \varepsilon$.

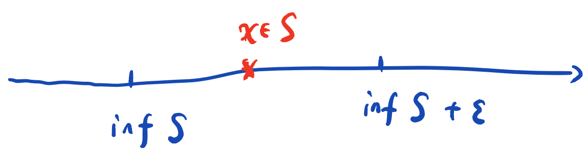
For this x, we have $\sup S - \varepsilon < x \leq \sup S$

Infimum Property

THM : If a set S has an infimum in $\mathbb{R}$, then $\forall\ \varepsilon > 0$,

$\exists\ x \in S$ such that

$$\inf S \leq x < \inf S + \varepsilon$$



Proof is similar to supremum property (Exercise)

Archimedean Principal

THM: $\forall x \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$ such that $n > x$.

Proof: By contradiction, assume $\sim(\forall x \in \mathbb{R}, \exists\, n \in \mathbb{N}, \text{ s.t } n > x)$.

$= (\exists\, x \in \mathbb{R}, \forall n \in \mathbb{N}, n \leq x)$. $\Rightarrow$ x is an upper bound for $\mathbb{N} \subset \mathbb{R}$. By C.A, $\sup \mathbb{N}$ exists.

By the property of Sup, $\exists\, m \in \mathbb{N}$, s.t $\sup \mathbb{N} - 1 < m$

then $\sup \mathbb{N} < m+1$, Contradicting to the def of $\sup \mathbb{N}$.

This contradiction proves our theorem.

Remark: The above thm $\Leftrightarrow$ $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-\infty, n)$

A consequence of Archimedean Principal

Lemma: $\forall\, x \in \mathbb{R}$, $\exists$ a greatest integer $\leq x$.

Moreover, for this integer, denoted by $[x]$, we have

$$[x] \leq x < [x]+1.$$

$\big[$ Example: $[1.1] = 1$, $[-1.1] = -2$. $\big]$

Proof I: By Archimedean Principal, $\exists\, n \in \mathbb{N}$ s.t $n > |x|$.

Then $-n \leq x \leq n$. $\Rightarrow$ $0 < x+n < 2n$

Define $S = \{ k : k \in \mathbb{N},\ k > x+n \}$. Then

$S \neq \emptyset$ (Since $2n \in S$). By Well-ordering axiom, $\operatorname{Min} S$

exists and $\operatorname{Min} S \in S \subset \mathbb{N}$. We have $\operatorname{Min} S > x+n$

and $\operatorname{Min} S - 1 \leq x+n$. So $\operatorname{Min} S - 1$ is the

greatest element $\leq x+n$ $\Rightarrow$ $\operatorname{Min} S - 1 - n$ is the

greatest element $\leq x$. We may define $[x] = \operatorname{Min} S - 1 - n$.

R.K: The Lemma above $\Leftrightarrow$ $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n+1)$

Proof II : Define $S = \{ n : n \in \mathbb{Z} , n > x \}$

By Archimedean Principal, $S \neq \phi$. Since $x < n, \forall n \in S$

S is bounded below. By the completeness axiom, $\text{Inf } S$ exists.

by the Inf property,

$\exists\ n_0 \in S$, s.t $n_0 - 1 < \text{Inf } S \leq n_0$.

$\Rightarrow n_0 - 1 \notin S$, but $n_0 \in S$. So the n_0 must be

the minimum element of S, or equivalently $n_0 = \text{Inf } S$.

As a result, $x < \text{Inf } S$. but $\underline{x \geq \text{Inf } S - 1}$

i.e, $\text{Inf } S - 1 \leq x < \text{Inf } S$

Therefore $\text{Inf } S - 1$ is the largest integer $\leq x$

We define $[x] = \text{Inf } S - 1$. then $[x] \leq x < [x] + 1$

R.K : The Lemma above $\Leftrightarrow \mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n+1)$

Proof III : Case 1: $x=0$, then $[x]=0$ is the greatest integer $\leq x$.

Case 2: $x>0$. Define $S=\{n : n\in\mathbb{N},\ n>x\}$

Then $S\neq\emptyset$ by Archimedean Principle. By the

Well-ordering axiom, $\operatorname{Min} S$ exists and $\operatorname{Min} S\in S\subset\mathbb{N}$.

We have $x<\operatorname{Min} S$. (Since $\operatorname{Min} S\in S$). On the other hand,

Since $\operatorname{Min} S-1$ is an integer not in S, $x\geq\operatorname{Min} S-1$.

Therefore $\operatorname{Min} S-1$ is the greatest integer $\leq x$. We may define $[x]=\operatorname{Min} S-1$

Case 3: $x<0$. Since $-x>0$, by preceeding argument, $[-x]$ is

well-defined and $[-x]\leq -x<[-x]+1$. So

$-[-x]-1<x\leq -[-x]$. If $x=-[-x]$, then $-[-x]$ is

the desired integer. If $x<-[-x]$, then $-[-x]-1$ is

the desired integer.

Proof II : Define $S = \{ n : n \in \mathbb{Z} , n > x \}$

By Archimedean Principal, $S \neq \phi$. Since $x < n, \forall n \in S$

S is bounded below. By the completeness axiom, $\operatorname{Inf} S$ exists.

by the Inf property,

$\exists\, n_0 \in S$, s.t $n_0 - 1 < \operatorname{Inf} S \leq n_0$.

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We define $[x] = \operatorname{Inf} S - 1$. then $[x] \leq x < [x] + 1$

R.K : The Lemma above $\Leftrightarrow \mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n+1)$

The Density of $\mathbb{Q}$ in $\mathbb{R}$

THM: If $x < y$, then $\exists\ \frac{m}{n} \in \mathbb{Q}$, s.t $x < \frac{m}{n} < y$.

Proof: Consider first the case when $y - x > 1$.

then $x < [x] + 1 \leq x + 1 < y$

For the general case, by Archimedean Principal, $\exists\ n \in \mathbb{N}$, s.t

$\frac{1}{y-x} < n \quad \Rightarrow \quad 1 < ny - nx$.

$\Updownarrow$

$\frac{1}{n} < y - x \qquad$ then $nx < [nx] + 1 \leq nx + 1 < ny$

i.e $\quad nx < [nx] + 1 < ny$

$$\Rightarrow \quad x < \frac{\overset{m}{[nx]+1}}{n} < y$$

Density of $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$

THM: If $x<y$, then $\exists\ w \in \mathbb{R}\setminus\mathbb{Q}$ s.t $x < w < y$.

Proof: Since $x<y$, $\sqrt{2}x < \sqrt{2}y$.

By the preview theorem (density of $\mathbb{Q}$), $\exists\ \frac{m}{n} \in \mathbb{Q}$ s.t

$$\sqrt{2}x < \frac{m}{n} < \sqrt{2}y \quad \Rightarrow \quad x < \frac{m}{n\sqrt{2}} < y$$

If $m \neq 0$, then $\frac{m}{n\sqrt{2}} \in \mathbb{R}\setminus\mathbb{Q}$, we can set $w = \frac{m}{n\sqrt{2}}$.

If $m=0$, then $\sqrt{2}x < 0 < \sqrt{2}y$. By the density of $\mathbb{Q}$ again,

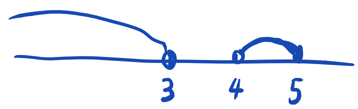
$\exists\ \frac{\tilde{m}}{\tilde{n}} \in \mathbb{Q}$ s.t $0 < \frac{\tilde{m}}{\tilde{n}} < \sqrt{2}y$. Then

$0 < \frac{\tilde{m}}{\tilde{n}\sqrt{2}} < y$. We may take $w = \frac{\tilde{m}}{\tilde{n}\sqrt{2}}$.

Remark: The above two theorems show that both $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in $\mathbb{R}$. They also show that there are gaps in rational numbers and irrational numbers. Especially, to fill gaps in $\mathbb{Q}$, we need to add uncountable numbers in order to form a continuum.

Examples of Sup and Inf

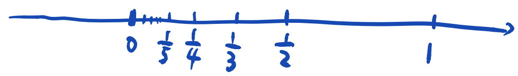
Example ①, consider $S = (-\infty, 3) \cup (4, 5]$



S is not bounded below. So S has no infimum.

On the other hand, S is bounded above. It's obvious that 5 is an upper bound of S. Since $5 \in S$, 5 is also the least upper bound. Hence $\sup S = 5$

Example ②. $S = \{ \frac{1}{n} : n \in \mathbb{N} \} = \{ 1, \frac{1}{2}, \frac{1}{3}, \cdots \}$.



$\forall\ n \geq 1,\quad \frac{1}{n} \leq 1 \quad \Rightarrow \quad 1$ is an upper bound of S.

$1 \in S \quad \Rightarrow \quad \forall$ upper bound M of S, $M \geq 1$.

$\Rightarrow$

1 is the least upper bound $\quad \Rightarrow \quad \text{Sup } S = 1$.

Next, we claim that $\text{Inf } S = 0$.

Proof of the claim : First, $\forall\ \frac{1}{n} \in S,\quad 0 < \frac{1}{n}$,

So 0 is a lower bound of S

By C-A, $\text{Inf } S$ exists and $0 \leq \inf S$.

We show $\text{Inf } S = 0$ by Contradiction.

If $\text{Inf } S \neq 0$, then $\text{Inf } S > 0$.. By A-P. $\exists\ n \in \mathbb{N}$

S.t $\quad n > \frac{1}{\text{Inf } S}$. Then $\text{Inf } S > \frac{1}{n}$.. Contradicts

to the definition of $\text{Inf } S$. So $\text{Inf } S = 0$

③ $S = [2, 6) \cap \mathbb{Q}$.

Step 1. $\forall x \in S$, $2 \le x \Rightarrow 2$ is a lower bound
$2 \in S \Rightarrow$ every lower bound ≤ 2
$\Big\} \Rightarrow 2$ is the greatest lower bound $\Rightarrow \text{Inf } S = 2$.

Step 2. We show that $\text{Sup } S = 6$.

First since $x \le 6$ $\forall x \in S$. So 6 is an upper bound.

So $\text{Sup } S$ exists and $\text{Sup } S \le 6$.

By contradiction, if $\text{Sup } S < 6$. By density of $\mathbb{Q}$, $\exists$ $r \in \mathbb{Q}$ s.t $\text{Sup } S < r < 6$. It's clear that $r > 2$

So $r \in S$. This contradicts to the inequality $\text{Sup } S < r$.

So $\text{Sup } S = 6$

Exercise (For your interest)

Def: $\tilde{\mathbb{R}} = \{\tilde{0}, \tilde{1}\}$.

We define " + " and " . " in the following way

$$\tilde{0} + \tilde{0} = \tilde{0}, \quad \tilde{0} + \tilde{1} = \tilde{1} + \tilde{0} = \tilde{1}, \quad \tilde{1} + \tilde{1} = \tilde{0}.$$

$$\tilde{0} \cdot \tilde{0} = \tilde{0}, \quad \tilde{0} \cdot \tilde{1} = \tilde{1} \cdot \tilde{0} = 0, \quad \tilde{1} \cdot \tilde{1} = \tilde{1}.$$

Define the set of positive integers $\tilde{\mathbb{N}} = \{1\}$

We define the order in $\tilde{\mathbb{R}}$ by

$$\tilde{0} < \tilde{1}.$$

Question: Can $\tilde{\mathbb{R}}$ be viewed as a real number system, i.e, do the Four Axioms hold for $\tilde{\mathbb{R}}$?