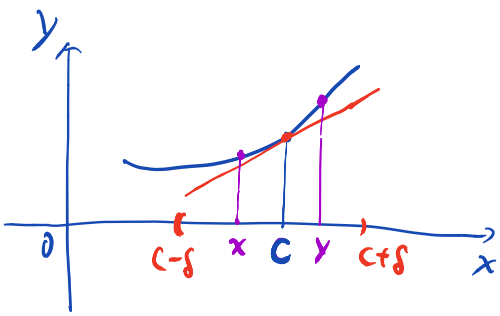
Lecture 20

16-04-2019

Review

1. local tracing thm: $f'(c) > 0 \Rightarrow \exists$ δ-neighborhood of c

   s.t $f(x) < f(c) < f(y)$ $\forall$ $c-\delta < x < c < y < c+\delta$



$f'(c) \neq 0 \Rightarrow f(c)$ is NOT a local extremum

The contrapositive is:

$f(c)$ is a local extremum $\Rightarrow f'(c) = 0$

2. $\frac{0}{0}$ form of L'Hôpital's rule

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0, \quad \lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

Proof: Generalized Mean-Value thm.

3. $\frac{*}{\infty}$ form of LH's rule: $\lim_{x\to a^+} g(x) = \infty$, $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x\to a^+} \frac{f(x)}{g(x)} = L$

Proof: Generalized mean-value thm + techniques.

Example 1. $\lim_{x\to 0^+} x \ln x$

Solution: $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{(\ln x)'}{(\frac{1}{x})'}$

$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$

$\Rightarrow \lim_{x\to 0^+} x \ln x = 0$

Remark: $\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{x}{\frac{1}{\ln x}}$. However,

the $\frac{0}{0}$ L-H rule does not work.

Example 2. $\lim_{x\to\infty} \frac{x^n}{e^x}$, $n \in \mathbb{N}$ (Tutorial)

Proof : By induction. For $n=1$, $\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$

Assume that $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for $n \in \mathbb{N}$.

then $\lim_{x\to\infty} \frac{x^{n+1}}{e^x} = \lim_{x\to\infty} \frac{(x^{n+1})'}{e^x} = \lim_{x\to\infty} \frac{(n+1)\, x^n}{e^x} = 0$

So $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for all $n \in \mathbb{N}$

As a result, $\lim_{x\to\infty} \frac{x^r}{e^x} = 0$ for $r \in \mathbb{R}$.

Since $\exists$ n s.t $r \leq n$

$\Rightarrow$ $x^r < x^n$ for $x \geq 1$

$\Rightarrow$ $0 < \frac{x^r}{e^x} < \frac{x^n}{e^x}$. letting $x \to \infty$, by Sandwitch thm

$\lim_{x\to\infty} \frac{x^r}{e^x} = 0$

Limitation of L-H's rule

Example: Consider $\lim\limits_{x\to+\infty} \dfrac{2x+\sin x}{2x-\sin x}$

Let $f(x) = 2x + \sin x$

$g(x) = 2x - \sin x$

As $x \to +\infty$, $f(x), g(x) \to +\infty$

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{2x+\sin x}{2x-\sin x} = \lim_{x\to+\infty} \frac{2+\frac{\sin x}{x}}{2-\frac{\sin x}{x}} = 1$$

But $\lim\limits_{x\to+\infty} \dfrac{f'(x)}{g'(x)} = \lim\limits_{x\to+\infty} \dfrac{2+\cos x}{2-\cos x}$ Does not $\exists$.

Taylor's Theorem

THM: let $f:(a,b)\to\mathbb{R}$ be n-times differentiable.

then $\forall\ x, x_0\in(a,b)$, $\exists$ c between x_0 and x s.t

$$f(x)=\underbrace{f(x_0)+\frac{f'(x_0)}{1}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots\frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}}_{\text{(n-1)-th Taylor expansion of } f \text{ at } x_0}$$

$$+\underbrace{\frac{f^{(n)}(c)}{n!}(x-x_0)^n}_{\text{Lagrange form of the reminder}}$$

proof: Step 1 Define $P_{n-1}(x)=f(x_0)+\frac{f'(x_0)}{1}(x-x_0)+\cdots\frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}$

then the error in approximating $f(x)$ by $P_{n-1}(x)$ is

$$E(x)=f(x)-P_{n-1}(x).$$

We aim to estimate $E(x)$

Observe that $P_{n-1}(x_0)=f(x_0)$. $P_{n-1}'(x_0)=f'(x_0)$

$$P_{n-1}^{(2)}(x_0) = f''(x_0), \quad \cdots \quad P_{n-1}^{(n-1)}(x_0) = f^{(n-1)}(x_0), \quad P_{n-1}^{(n)}(x) \equiv 0.$$

$$\Rightarrow \quad E(x_0) = 0, \quad E'(x_0) = 0, \quad \cdots \quad E^{(n-1)}(x_0) = 0, \quad E^{(n)}(x) = f^{(n)}(x)$$

Step 2. Define $Q(x) = (x - x_0)^n$, $Q^{(n)}(x) = n!$

Consider $\dfrac{E(x)}{Q(x)} = \dfrac{E(x) - E(x_0)}{Q(x) - Q(x_0)} = \dfrac{E'(x_1)}{Q'(x_1)}$ for some x_1 inbetween x_0 and x by the generalized mean-value thm.

but $\dfrac{E'(x_1)}{Q'(x_1)} = \dfrac{E'(x_1) - E'(x_0)}{Q'(x_1) - Q'(x_0)} = \dfrac{E''(x_2)}{Q''(x_2)}$ for some x_2 inbetween x_0 and x_1

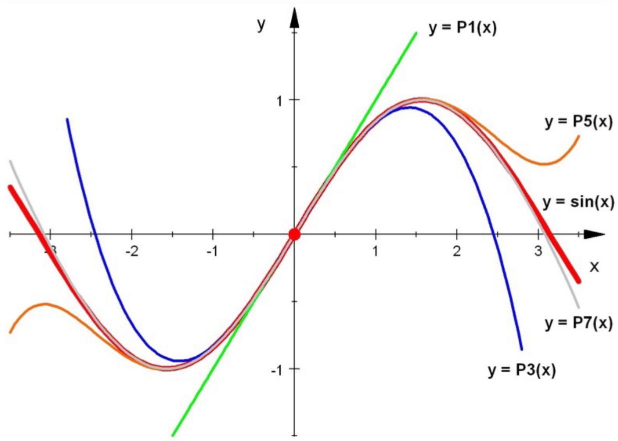
Continuing the above argument n times:

$$\frac{E(x)}{Q(x)} = \frac{E'(x_1)}{Q'(x_1)} = \frac{E''(x_2)}{Q''(x_2)} = \frac{E'''(x_3)}{Q'''(x_3)} = \cdots \frac{E^{(n)}(x_n)}{Q^{(n)}(x_n)} = \frac{f^{(n)}(x_n)}{n!}.$$

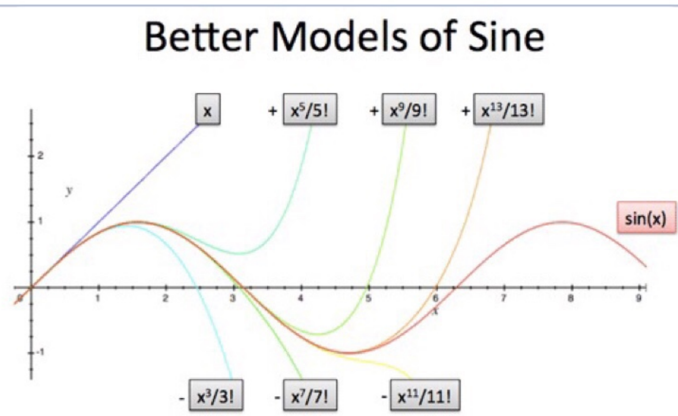
Take $c = x_n$, then c is inbetween x_0 and x

and $\dfrac{E(x)}{Q(x)} = \dfrac{f(x) - P_{n-1}(x)}{(x - x_0)^n} = \dfrac{f^{(n)}(c)}{n!} \Rightarrow f(x) = P_{n-1}(x) + \dfrac{f^{(n)}(c)}{n!}(x - x_0)^n$

Example : Taylor expansion of $\sin x$



near $x=0$



① An equivalent form of Taylor's expansion

$$f(x+h) = \underbrace{f(x) + f'(x)h + \frac{f^{(2)}(x)}{2!}h^2 + \cdots \frac{1}{n!}f^{(n)}(x)\,h^n}_{\text{n-th Taylor expansion at } x}$$

$$+ \underbrace{\frac{1}{(n+1)!} f^{(n+1)}(\xi)\, h^{n+1}}_{\text{Reminder term (or error term)}}$$

② If f is C^∞, then we can formally define a series

$$f(x_0) + f'(x_0)h + \cdots \frac{f^{(n)}(x_0)}{n!}h^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} h^n$$

and call it the Taylor series of f at x_0

Assume that $\exists\, M>0, R>0$, $\sup\{ |f^{(n)}(x)| : |x-x_0| < R \} \le M^n$ for all $n \in \mathbb{N}$. Then

$$f(x_0+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} h^n \qquad \text{for all } |h| < R.$$

Taylor expansion of Common functions at $x_0=0$

① $e^x = 1+x+\frac{1}{2!}x^2+\frac{x^3}{3!}+\cdots\frac{x^n}{n!}+\cdots$

② $\cos x = 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\cdots+\frac{(-1)^n x^{2n}}{(2n)!}x^{2n}+\cdots$

$x_0=0$, R arbitrary

③ $\sin x = x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}x^{2n+1}+\cdots$

$x_0=0$, R arbitrary

④ $\ln(1+x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3+\cdots+\frac{(-1)^n x^n}{n}+\cdots$

$x_0=0$, $R=1$

⑤ $(1+x)^\alpha = 1+\sum_{k=1}^{\infty}\frac{\alpha\cdot(\alpha-1)\cdots(\alpha-k+1)}{k!}x^k+\cdots$

$\forall\ \alpha\in\mathbb{R}$

$x_0=0$, $R=1$

⑥ $\arctan x = x-\frac{x^3}{3}+\frac{x^5}{5}+\cdots\frac{(-1)^n}{(2n+1)}x^{2n+1}+\cdots$ $x_0=R$. R arbitrary

Example: use 2nd order Taylor expansion to

approximate $\sqrt{16.1}$ $= (4.0124805295 5\ldots)$

and estimate the approximation error.

Solution: $f(x) = \sqrt{x}$, $\quad f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $\quad f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$

$$f(16.1) = f(16) + f'(16)(16.1-16)$$

$$+ \frac{f''(16)}{2!}(16.1-16)^2 + \frac{f'''(\xi)}{3!}(16.1-16)^3$$

The 2nd order approximation is $f(16) + f'(16)\cdot 0.1 + \frac{f''(16)}{2}\cdot 0.1^2$

$$= \sqrt{16} + \frac{1}{2\sqrt{16}}\cdot 0.1 + \frac{1}{2}\times\left(-\frac{1}{4}\right)\cdot 16^{-\frac{3}{2}}\cdot 0.1^2$$

$$= 4 + \frac{0.1}{8} - \frac{1}{8}\times\frac{1}{4^3}\cdot 0.1^2 = 4.01248046875$$

$\xi \in (16, 16.1)$

The error is $\frac{f'''(\xi)}{3!}0.1^3 = \frac{1}{6}\times\frac{1}{4}\times\frac{3}{2}\times\xi^{-\frac{5}{2}}\cdot 0.1^3 = \frac{10^{-3}}{16}\times\left(\frac{1}{\sqrt{\xi}}\right)$

$$< \frac{10^{-3}}{16}\times\left(\frac{1}{\sqrt{16}}\right)^5 = 6.104\times 10^{-8}$$

Example 2. Estimate the maximum error made in approximating $f(x) = e^x$ by its 2nd order Taylor expansion $P_2(x) = 1 + x + \frac{1}{2}x^2$ over the interval $[-1, 1]$

Solution : $f(x) = e^x$, $f^{(n)}(x) = e^x$. The 2nd expansion at $x = 0$ is

$$f(x) = P_2(x) + R_2(x) \quad \text{where}$$

$$R_2(x) = \frac{f^{(3)}(\xi)}{3!} x^3 = \frac{e^{\xi}}{3!} x^3$$

for some ξ in between 0 and x.

$$\Rightarrow \quad |R_2(x)| \leq \frac{e^{\xi}}{6} \cdot |x|^3 \leq \frac{e}{6}$$

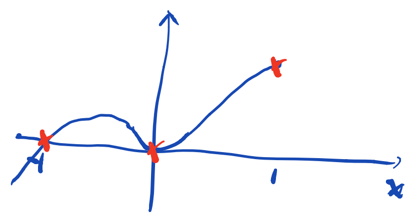
$\Rightarrow$ The maximum error $\leq \frac{e}{6}$

Example 3. Let $f \in C^3(\mathbb{R})$.

If $f'(0)=0$, $f(1)=1$, $f(-1)=0$, prove that

$$\sup\{ f^{(3)}(x) : -1<x<1 \} \geq 3.$$

Solution:



Taylor expand f at $x=0$

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2!} f''(0)\,x^2 + \frac{1}{3!} f^{(3)}(\xi)\,x^3$$

for some ξ in between 0 and x.

For $x=1$, $1 = f(1) = f(0) + f'(0)\cdot 1 + \frac{1}{2!} f''(0)\,1^2 + \frac{1}{3!} f^{(3)}(\xi_1)\cdot 1^3$

$$= f(0) + \frac{1}{2} f''(0) + \frac{1}{6} f^{(3)}(\xi_1) \quad \text{for some } \xi_1 \in (0,1)$$

For $x=-1$, $0 = f(-1) = f(0) + f'(0)(-1) + \frac{1}{2!} f''(0)(-1)^2 + \frac{1}{3!} f^{(3)}(\xi_2)(-1)^3$

$$= f(0) + \frac{1}{2} f''(0) - \frac{1}{6} f^{(3)}(\xi_2) \quad \text{for some } \xi_2 \in (-1,0)$$

$$\Rightarrow \begin{cases} f(0) + \frac{1}{2} f''(0) + \frac{1}{6} f'''(\xi_1) = 1 & \cdots ① \\ f(0) + \frac{1}{2} f''(0) - \frac{1}{6} f'''(\xi_2) = 0 & - ② \end{cases}$$

$$① - ② \Rightarrow \frac{1}{6}\left(f'''(\xi_1) + f'''(\xi_2) \right) = 1$$

$$\Rightarrow \operatorname{Max}\{ f'''(\xi_1), f'''(\xi_2) \} \geq \frac{f'''(\xi_1) + f'''(\xi_2)}{2} = 3 \Rightarrow \cdots$$

Review on differentiation

Key thms and concept :

Definition of differentiable , Differentiation formula,

Chain rule , Inverse Function thm ,

Local extremum thm , Rolle's thm ,

Mean-value thm , L-Hôpital's rule

Taylor's thm