Exercise 1.

Let $\{f_i\}_{i=1}^n [0,2] \to [0,1]$ be integrable. Prove that the function

$$h(x) := \begin{cases} \max_i \{f_i(x)\}, & x \in [0,1) \\ \min_i \{f_i(x)\}, & x \in [1,2]. \end{cases}$$ is integrable.

Proof:

We already show that the glue of two integrable functions is still integrable. So we only have to show $\max_{1\le i\le n}\{f_i\}$ is integrable. We do this by induction.

1) When $n=2$, we have the formula

$$\max\{f_1, f_2\} = \frac{f_1+f_2+|f_1-f_2|}{2}$$

since $|f|$, $f+g$, $f/2$ are integrable when f, g are, we're done.

2) When $n=k+1$, we prove that

$$\max_{1\le i\le k+1}\{f_i\} = \max\{\max_{1\le i\le k}\{f_i\}, f_{k+1}\}.$$

In fact, if $f_p = \max_{1\le i\le k+1}\{f_i\}$, we have

$$\begin{cases} f_p \ge f_i \ (1\le i\le k) \Rightarrow f_p \ge \max_{1\le i\le k}\{f_i\} \\ f_p \ge f_{k+1} \end{cases}$$

$\Rightarrow f_p \ge \max\{\max_{1\le i\le k}\{f_i\}, f_{k+1}\} \Rightarrow f_p = \max\{\max_{1\le i\le k}\{f_i\}, f_{k+1}\}.$

By induction, we know $\max_{1\le i\le n}\{f_i\}$ is integrable. Similar for min.

Exercise 2.

(2004 Final) Let $f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \setminus \mathbb{Q} \\ 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \text{ for } m, n \in \mathbb{N} \text{ with no common prime factor.} \end{cases}$

Prove that there exists a Riemann integrable function $g : [0,1] \to [0,1]$ such that the composition function $g \circ f : [0,1] \to [0,1]$ is not Riemann integrable on $[0,1]$.

Proof:

idea: observe the image of f is $\{1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots, 0\}$

Let $g(x) := \begin{cases} 1, & x = \frac{1}{n} \ (n \in \mathbb{N}^+) \\ 0, & \text{else} \end{cases}$. then we find

$(g \circ f)(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] \setminus \mathbb{Q} \end{cases}$. which is no integrable

(since for any partition P, $U(f, P) = \sum_i \overset{1}{M_i} |\Delta x_i| = 1$, $L(f, P) = \sum_i \underset{0}{m_i} |\Delta x_i| = 0$, $U - L = 1 > \varepsilon$). But at the same time, g is integrable (see the proof given in Tutorial 11).

$\square$

Exercise 3.

(A complementary proof for Lecture 22.)

For the following condition:

(1): f is Darboux integrable and $\int_a^b f(x)\,dx = I$

(2) f is Riemann integrable and $\lim\limits_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j = I$

(3) $\forall \varepsilon>0$, $\exists$ partition $P=\{x_0=a, x_1, \cdots x_n=b\}$ of $[a,b]$ s.t. $\left|\sum_{j=1}^{n} f(t_j)\Delta x_j - I\right| < \varepsilon$ for all $t_j \in [x_{j-1}, x_j]$

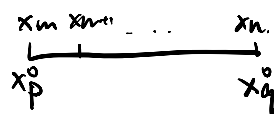
(4). $\forall \varepsilon>0$, $\exists$ partition P s.t. $|U(f,P) - L(f,P)| < \varepsilon$.

(f bounded by M on $[a,b]$).

a) Show that (3) $\Rightarrow$ (2) b) (3) $\Leftrightarrow$ (4)

Proof:

a) Denote P_0 as the partition given in (3).
For each P. if $P_0 \subseteq P$ (recall the definition). Then we consider $x_p^0 = x_m < \cdots < x_n = x_{p+1}^0$ $\Rightarrow$



where $\{x_p^0, x_{p+1}^0\} \subset P_0$ and $\{x_m \cdots x_n\} \subset P$. we know:

$$\sum_{j=m+1}^{n} f(t_j)\Delta x_j \le \max_{x\in[x_m, x_{m+1}]} f(x) \cdot \sum_j \Delta x_j = f(\bar{t}_m)\Delta x_m^0$$

$$\sum_j \Delta x_j = |x_{m+1} - x_m|$$

$$\sum_{j=m+1}^{n} f(t_j)\Delta x_j \ge \min_{x\in[x_m, x_{m+1}]} f(x) \cdot \sum_j \Delta x_j = f(\underline{t_m})\Delta x_m^0$$

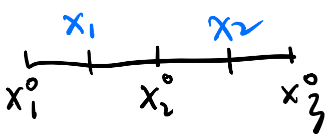
where $\bar{t}_m$, $\underline{t_m}$ are the extreme points on $[x_m, x_{m+1}]$.
Then we know

$$-\varepsilon < \sum_i f(\underline{t_i})\Delta x_i^0 - I < \sum_j f(t_j)\Delta x_j - I \le \sum_i f(\bar{t}_i)\Delta x_i^0 - I < \varepsilon$$

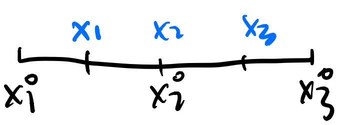
(pay attention to the subscript). So we know

$$\left|\sum_{j=1} f(t_j)\Delta x_j - I\right| < \varepsilon.$$

Let $\|P\| < \|P_0\|$. then we know every subinterval in P can intersect 2 subintervals of P_0 at most. When we have this shape:



we refine the partition and re-rank it.
$\hat{P}$ to $\bar{P}$



Then we come back the the $P_0 \subset \bar{P}$ case. Now we measure the difference between P and $\bar{P}$. Let $\|P\| < \frac{\varepsilon}{4M(N+1)}$ where N is the number of mesh in P_0.

$$|f(t_1)\Delta x_1 + f(t_2)\Delta x_2 - f(t_1)(\Delta x_1 + \Delta x_2)|$$
$$= |f(t_2) - f(t_1)|\,\Delta x_2 \le \|P\|\cdot 2M < \frac{\varepsilon}{2(N+1)}$$

the difference comes at most N times we we take the total sum. so.

$$\left|\sum_{x_i \in P} f(t_i)\Delta x_i - \sum_{x_i \in P} f(t_i)\Delta x_i\right| < \frac{\varepsilon}{2(N+1)}\cdot N < \frac{\varepsilon}{2}.$$

So if $\left|\sum_{x_i \in P_0} f(t_i)\Delta x_i - 2\right| < \varepsilon/2$ and we select $\delta = \min\{\|P_0\|, \frac{\varepsilon}{4M(N+1)}\}$. we know

$\left|\sum_{x_i \in P} f(t_i)\Delta x_i - 2\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

so we're done

$\square$

Exercise 4

Give another example different from the one in the lecture satisfying that: f is integrable but not continuous, but $F(x)=\int_c^x f(t)dt$. namely the anti-derivative is differentiable.

(In lecture, we give $f(x)=\begin{cases}0, & -1\le x<0\\ 1, & 0\le x\le 1\end{cases}$)

Proof:

(Since continuity is a sufficient condition to garantee the differentiability, we show that it's not necessary.)

Let $f(x):=\begin{cases}1, & x\in\{1,\frac{1}{2},\cdots,\frac{1}{n},\cdots\}\\ 0, & \text{else}\end{cases}$ on $[0,1]$, we've

already shown f is integrable and obviously not continuous.

But what is its anti-derivative?

In fact, $F(x)=\int_0^x f(t)\,dt=0$ for $\forall x\in[0,1]$. WLOG, we only need to show $F(1)=0$ ($0\le F(x)\le F(1)=0 \Rightarrow F(x)\equiv 0$). Since Riemann integrability meets Darboux integrability, we know:

$$F(1)=\int_0^1 f(t)\,dt=\max_P L(f,P)=0$$

$\Rightarrow$ we can use it since we already know f is integrable

since every subinterval $I_i\subset[0,1]$ s.t. $I_i\setminus\{\frac{1}{n}:n\in\mathbb{N}\}\neq\phi$. So we are done.

$\square$