Extra – Corrected proof of (3) of Property 7

We let P_1, P_2 be any partitions of $[a,b]$ and consider the partition $P = P_1 \cup P_2$ (which is refinement of both P_1 and P_2).

Recall the following facts

$$U(f+g,P) \le U(f,P) + U(g,P) \quad \text{and} \quad L(f+g,P) \ge L(f,P) + L(g,P),$$

and

$$L(f,P_i) \le L(f,P) \le U(f,P) \le U(f,P_i) \quad \text{for } i=1,2.$$

Then

$$\int_a^b [f(x)+g(x)]\,dx \le U(f+g,P) \le U(f,P) + U(g,P) \le U(f,P_1) + U(g,P_2) \quad (*)$$

and

$$\int_a^b [f(x)+g(x)]\,dx \ge L(f+g,P) \ge L(f,P) + L(g,P) \ge L(f,P_1) + L(g,P_2) \quad (**)$$

By taking infimum on $(*)$ over the partitions P_1, P_2 respectively and noting the fact that f, g are integrable, we get

$$\underline{\int_a^b [f(x)+g(x)]\,dx} \le \inf_{P_1} U(f,P_1) + \inf_{P_2} U(g,P_2) = \overline{\int_a^b} f(x)\,dx + \overline{\int_a^b} g(x)\,dx$$
$$= \underline{\int_a^b f(x)\,dx + \int_a^b g(x)\,dx}.$$

By taking supermum on $(**)$ over the partitions P_1, P_2 respectively, we get

$$\underline{\int_a^b [f(x)+g(x)]\,dx} \ge \sup_{P_1} L(f,P_1) + \sup_{P_2} L(f,P_2) = \underline{\int_a^b} f(x)\,dx + \underline{\int_a^b} g(x)\,dx$$
$$= \underline{\int_a^b f(x)\,dx + \int_a^b g(x)\,dx}$$

It follows from sandwich theorem that

$$\boxed{\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx}$$