MATH2033 Mathematical Analysis (2021 Spring)
Suggested Solution of Assignment 5

Problem 1

(a) Using the definition of integrability or integral criterion, prove that $f(x) = |x-1|$ is integrable on $[0,2]$.

(b) Using the definition of integrability or integral criterion, prove that the function $f: [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & if \;\; x \in \mathbb{Q} \\ -x & if \;\; x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

is not integrable on $[0,1]$.

☺Solution of (a)

For any $\varepsilon > 0$, we consider the partition

$$\mathcal{P} = \{x_0, x_1, x_2, \ldots, x_{2n}\} = \left\{0, \frac{2}{2n}, \frac{4}{2n}, \frac{6}{2n}, \ldots, 2\right\}, \qquad where \; x_k = \frac{2k}{2n} = \frac{k}{n}.$$

Next, we compute the upper sum and lower sum of this function:

- Note that $|x-1| = \begin{cases} x-1 & if \;\; x \geq 1 \\ 1-x & if \;\; x < 1 \end{cases}$ so that $|x-1|$ is decreasing over $[0,1]$ and is increasing over $[1,2]$.
- It follows that for any subinterval $[x_{k-1}, x_k]$, we have

$$M_k = \sup_{x \in [x_{k-1}, x_k]} f(x) = \begin{cases} f(x_{k-1}) = 1 - x_{k-1} = \dfrac{n-k+1}{n} & if \;\; k \leq n \\ f(x_k) = x_k - 1 = \dfrac{k-n}{n} & if \;\; k \geq n+1 \end{cases}$$

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x)$$
$$= \begin{cases} f(x_k) = 1 - x_k = \dfrac{n-k}{n} & if \;\; k \leq n \\ f(x_{k-1}) = x_{k-1} - 1 = \dfrac{k-1-n}{n} & if \;\; k \geq n+1 \end{cases}$$

Then we deduce that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) = \sum_{k=1}^{2n} (M_k - m_k)(x_k - x_{k-1})$$
$$= \sum_{k=1}^{n} \underbrace{\left(\frac{n-k+1}{n} - \frac{n-k}{n}\right)}_{=\frac{1}{n}} \left(\frac{1}{n}\right) + \sum_{k=n+1}^{2n} \underbrace{\left(\frac{k-n}{n} - \frac{k-1-n}{n}\right)}_{=\frac{1}{n}} \left(\frac{1}{n}\right)$$
$$= \frac{1}{n^2}(n) + \frac{1}{n^2}(n) = \frac{2}{n}.$$

By choosing n such that $n > \frac{2}{\varepsilon} \Leftrightarrow \frac{2}{n} < \varepsilon$ (guaranteed by Archimedean property), we deduce that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon.$$

So f is integrable by integral criterion.

☺Solution of (b)

We let $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ (where $x_0 = 0$ and $x_n = 1$) be a partition of $[0,1]$.

For every subinterval $I_k = [x_{k-1}, x_k] \subseteq [0,1]$, we note that

- $f(x) \le x \le x_k$ for all $x \in [x_{k-1}, x_k]$ and x_k is upper bound of $\{f(x) | x \in I_k\}$. For any $\varepsilon > 0$, we deduce from density of rational number that there exists a rational number q such that $x_k - \varepsilon < q < x_k$ which $f(q) = q > x_k - \varepsilon$. Then it follows from supremum property that $M_k = \sup\{f(x) | x \in I_k\} = x_k$
- $f(x) \ge -x \ge -x_k$ for all $x \in [x_{k-1}, x_k]$ and $-x_k$ is lower bound of $\{f(x) | x \in I_k\}$. For any $\varepsilon > 0$, we deduce from density of irrational number that there exists a irrational number q such that $x_k - \varepsilon < r < x_k$ which $f(r) = -r < -x_k + \varepsilon$. Then it follows from infimum property that $m_k = \inf\{f(x) | x \in I_k\} = -x_k$

Thus, it follows from the upper sum and lower sum are given by

$$U(\mathcal{P}, f) = \sum_{k=1}^{n} M_k (x_k - x_{k-1}) = \sum_{k=1}^{n} x_k (x_k - x_{k-1}).$$

$$L(\mathcal{P}, f) = \sum_{k=1}^{n} M_k (x_k - x_{k-1}) = \sum_{k=1}^{n} (-x_k)(x_k - x_{k-1}).$$

Using the fact that $2b(b-a) = b^2 - a^2 + (b-a)^2$, we deduce that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) = \sum_{k=1}^{n} 2x_k (x_k - x_{k-1}) = \underbrace{\sum_{k=1}^{n} (x_k^2 - x_{k-1}^2)}_{= x_n^2 - x_0^2 = b^2 - a^2} + \sum_{k=1}^{n} \underbrace{(x_k - x_{k-1})^2}_{\ge 0}$$

$$\ge b^2 - a^2 > 0.$$

Then it follows from the *negation* of integral criterion (with $\varepsilon = b^2 - a^2$) that the function is not integrable.

(*Note: The negation of integral criterion states that f is not integrable on $[a, b]$ if there exists $\varepsilon > 0$ such that for any partition $\mathcal{P}$ on $[a, b]$, $U(\mathcal{P}, f) - L(\mathcal{P}, f) \ge \varepsilon$.)

Problem 2

We let f, g, h be three bounded functions on $[a, b]$ such that $f(x) \le g(x) \le h(x)$ for all $x \in [a, b]$. Suppose that f, h are integrable on $[a, b]$ and $\int_a^b f(x)dx = \int_a^b h(x)dx$.

(a) Show that g is integrable on $[a, b]$.

(b) Show that $\int_a^b g(x)dx = \int_a^b f(x)dx$.

☺Solution

(a) We note the following facts:

- For any functions f_1, f_2 which $f_1(x) \le f_2(x)$, we have
$$U(\mathcal{P}, f_1) \le U(\mathcal{P}, f_2) \quad and \quad L(\mathcal{P}, f_1) \le L(\mathcal{P}, f_2).$$
- Since f, h are integrable over $[a, b]$, it follows that for any partition $\mathcal{P}$,
$$L(P, f) \le \int_a^b f(x)dx \le U(P, f) \quad and$$

$$L(P,h) \leq \int_a^b h(x)dx \leq U(P,h)$$

- For any $\varepsilon > 0$, there exists partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that
$$U(\mathcal{P}_1,f) - L(\mathcal{P}_1,f) < \frac{\varepsilon}{2} \Rightarrow \int_a^b f(x)dx - L(\mathcal{P}_1,f) < \frac{\varepsilon}{2}$$
and
$$U(\mathcal{P}_2,h) - L(\mathcal{P}_2,h) < \frac{\varepsilon}{2} \Rightarrow U(\mathcal{P}_2,h) - \int_a^b h(x)dx < \frac{\varepsilon}{2}.$$

By taking the partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ (which is the refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$, we deduce that

$$U(\mathcal{P},g) - L(\mathcal{P},g) \leq U(\mathcal{P}_2,g) - L(\mathcal{P}_1,g) \leq U(\mathcal{P}_2,h) - L(\mathcal{P}_1,f)$$
$$< \left(\int_a^b h(x)dx + \frac{\varepsilon}{2}\right) - \left(\int_a^b f(x)dx - \frac{\varepsilon}{2}\right) \overset{\int_a^b f(x)dx=\int_a^b h(x)dx}{\cong} \varepsilon.$$

So $g(x)$ is integrable on $[a,b]$ by integral criterion.

(b) Recall that for any partition $\mathcal{P}$
$$\int_a^b g(x)dx \leq U(\mathcal{P},g) \leq U(\mathcal{P},h) \ldots\ldots (*)$$
$$\int_a^b g(x)dx \geq L(\mathcal{P},g) \geq L(\mathcal{P},f) \ldots\ldots (**)$$

By taking infimum on the inequality $(*)$,, we have
$$\int_a^b g(x)dx \leq \inf_{\mathcal{P}} U(\mathcal{P},h) = \overline{\int_a^b} h(x)dx = \int_a^b h(x)dx = \int_a^b f(x)dx$$

By taking supremum on the inequality $(**)$, we have
$$\int_a^b g(x)dx \geq \sup_{\mathcal{P}} L(\mathcal{P},f) = \underline{\int_a^b} f(x)dx = \int_a^b f(x)dx.$$

Then it follows from sandwich theorem that $\int_a^b g(x)dx = \int_a^b f(x)dx$.

Problem 3

(a) We let $f,g: [a,b] \to \mathbb{R}$ be two bounded Riemann integrable function on $[a,b]$, show that the function $h(x) = \min(f(x), g(x))$ is also Riemann integrable on $[a,b]$.

(b) We let $f: [a,b] \to \mathbb{R}$ be a bounded function on $[a,b]$.

(i) Suppose that f^2 is Riemann integrable, is it true that f is Riemann integrable? Explain your answer.

(ii) Suppose that f^3 is Riemann integrable, is it true that f is Riemann integrable? Explain your answer.

(*Note: If your answer is yes, please give a proof. If your answer is no, please provide a counter-example.)

☺Solution

(a) Note that

$$\min\big(f(x), g(x)\big) = \frac{f(x)+g(x)}{2} - \frac{|f(x)-g(x)|}{2}.$$

Since f, g are integrable, it follows that

- $\frac{f+g}{2}$ and $f-g$ are integrable.
- Since $|x|$ is continuous over $\mathbb{R}$, so $|f(x)-g(x)|$ is also integrable.

Hence, we conclude that $\min\big(f(x), g(x)\big) = \frac{f(x)+g(x)}{2} - \frac{|f(x)-g(x)|}{2}$ is also integrable.

(b) **(i)** No. We consider a function $f: [a,b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & if \;\; x \in \mathbb{Q} \\ -1 & if \;\; x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}.$$

We let $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a,b]$. For any subinterval $I_k = [x_{k-1}, x_k]$, we deduce from density of rational number and density of irrational number that there exists $q \in \mathbb{Q}$ and $r \in \mathbb{R}\backslash\mathbb{Q}$ such that

$$x_{k-1} < q < x_k \quad and \quad x_{k-1} < r < x_k.$$

Since $f(q) = 1$ and $f(r) = -1$, we have

$$M_k = \sup_{x \in I_k} f(x) = 1 \quad and \quad m_k = \inf_{x \in I_k} f(x) = -1.$$

Then the upper sum and lower sum are given by

$$U(\mathcal{P}, f) = \sum_{k=1}^{n} \underbrace{M_k}_{=1} (x_k - x_{k-1}) = x_n - x_0 = b - a$$

$$L(\mathcal{P}, f) = \sum_{k=1}^{n} \underbrace{m_k}_{=-1} (x_k - x_{k-1}) = -(x_n - x_0) = -(b-a).$$

So we deduce that

$$\overline{\int_a^b} f(x)dx = \inf_{\mathcal{P}} U(\mathcal{P}, f) = b - a \quad and$$

$$\underline{\int_a^b} f(x)dx = \sup_{\mathcal{P}} L(\mathcal{P}, f) = -(b-a)$$

As $\overline{\int_a^b} f(x)dx \neq \underline{\int_a^b} f(x)dx$, so $f(x)$ is not integrable.

However, we see that

$$f^2 = \begin{cases} 1 & if \;\; x \in \mathbb{Q} \\ (-1)^2 = 1 & if \;\; x \in \mathbb{R}\backslash\mathbb{Q} \end{cases} = 1.$$

So f^2 is constant function and therefore integrable.

(ii) Since f is bounded, we have $m \leq f(x) \leq M$ for all $x \in [a,b]$. Note that the function $g(x) = \sqrt[3]{x}$ is continuous over $[m, M]$ and f^3 is integrable, it follows that

$$f(x) = \sqrt[3]{f^3(x)} = g\big(f^3(x)\big)$$

is integrable.