Lecture 2.

12-02-2019

1. Review on Basics of logic

Notation p, q : mathematical statement

quantifiers $\begin{cases} \forall : \text{for all, for any, for every} \\ \exists : \text{there exists, there is (at least one)} \end{cases}$

$\sim p$: the opposite of statement p.

Rules of negation :

① $\sim(\sim p) = p$ ② $\sim(p \text{ and } q) = \sim p \text{ or } \sim q$

③ $\sim(p \text{ or } q) = \sim p \text{ and } \sim q$.

④ $\sim(\forall x, p) = \exists x, \sim p$

⑤ $\sim(\exists y, p) = \forall y, \sim p$

⑥ $\sim(\forall x, \exists y, p) = \exists x, \forall y, \sim p$

(derived from ④, ⑤)

Conditional statement (If - then statement)

Notation $p \Rightarrow q$

- If p, then q
- p imlies q
- p is sufficient for q
- q is nessary for p

rule : $p \Rightarrow q = \sim p$ or q (★)

$\sim(p \Rightarrow q) = p$ and $\sim q$

Converse / Contrapositive statement

For the statement " If p, then q " or ($p \Rightarrow q$)

Its converse is " If q, then p " or ($q \Rightarrow p$)

Its contrapositive is " If $\sim q$, then $\sim p$ " or ($\sim q \Rightarrow \sim p$)

Example : (1) Statement : If $x = -3$, then $x^2 = 9$. **T**

Converse : if $x^2 = 9$, then $x = -3$ **F**

Contrapositive : If $x^2 \neq 9$, then $x \neq -3$. **T**

(2) Statement : $x = -3 \Rightarrow 2x = -6$. **T**

Converse : $2x = -6 \Rightarrow x = -3$ **T**

Contrapositive : $2x \neq -6 \Rightarrow x \neq -3$. **T**

Remark : ① Contrapositive = statement [Foundation of proof by contradiction]

Pf : Contrapositive : if $\sim q$, then $\sim p$ $= \sim(\sim q)$ or $\sim p$ (by rule ✱)

$= q$ or $\sim p$ $= \sim p$ or q

$= p \Rightarrow q$

② If " $p \Rightarrow q$ " and " $q \Rightarrow p$ " both are True, we will write $p \Leftrightarrow q$, and say that

p if and only if q or use

Abbreviation : p iff q.

③ $\forall \alpha, \forall \beta = \forall \beta, \forall \alpha$

$\exists \alpha, \exists \beta = \exists \beta, \exists \alpha$

$\forall \alpha, \exists \beta \neq \exists \beta, \forall \alpha$

Example : $(\forall \alpha, \quad \exists \beta > 0, \quad \alpha + \beta > 0)$ True

$(\exists \beta > 0, \forall \alpha, \quad \alpha + \beta > 0)$ False

Exercise : Negate each of the following

① If ΔABC is a right triangle, then $a^2+b^2=c^2$.

② $\forall \varepsilon>0$, $\exists \delta>0$ such that

$$0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon$$

Solution : ① $\sim(p \Rightarrow q) \;=\; p$ and $\sim q$

Opposite : ΔABC is a right triangle, and $a^2+b^2\neq c^2$

② Recall : $\sim(\forall \varepsilon, \exists \delta, p \Rightarrow q) = \exists \varepsilon, \forall \delta, \sim(p \Rightarrow q)$.

$\exists \varepsilon>0$, $\forall \delta>0$, we have

$0<|x-x_0|<\delta$ and $|f(x)-L|\geq \varepsilon$.

Chapter 2 Basics of set theory [Language to communicate math efficiently and precisely]

modern theory founded by G. Cantor in 1870's to resolve paradoxes in naive set theory.

Def : a set is a collection of math "objects" (usually numbers, functions, ordered pair of numbers, ...)

short for definition

the objects in the set are the elements of the set.

Notation : We write $x \in S$ to say that x is an element of set S.

We write $x \bar{\in} S$ to say that x is NOT an element of set S.

Example : let $\mathbb{Z}$ be the set of all integers, then $-1 \in \mathbb{Z}$, $-2^{100} \in \mathbb{Z}$, but $\sqrt{2} \bar{\in} \mathbb{Z}$.

Def : a set is finite if it has finitely many elements

a set is infinite if it has infinitely many elements.

The empty set is the set having no element and is denoted by ϕ.

Example: Common sets in math

$\mathbb{N}$: the set of all positive integers

$\mathbb{Z}$: the set of all integers

$\mathbb{Q}$: the set of all rational numbers

$\mathbb{R}$: the set of all real numbers

$\mathbb{C}$: the set of all complex numbers

Set descriptions

① List all elements

$S = \{1, 2, 3\}$, $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

$\phi = \{\ \}$, $\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, \dots\}$

② Write the form of the elements

$\mathbb{Q} = \{ \frac{m}{n} : m \in \mathbb{Z},\ n \in \mathbb{N} \}$

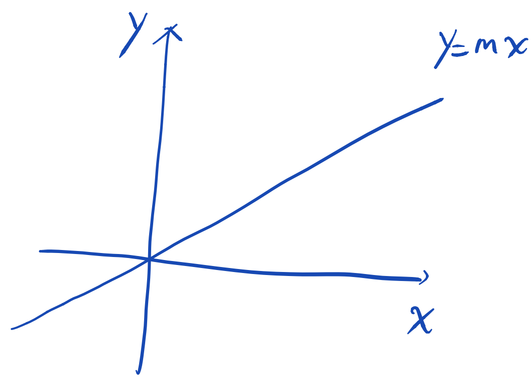
$\mathbb{R} = \{ x : x \text{ is a real number} \}$

$\mathbb{C} = \{ x + iy : x \in \mathbb{R},\ y \in \mathbb{R},\ i = \sqrt{-1} \}$

$[a, b] = \{ x : x \in \mathbb{R} \text{ and } a \leq x \leq b \}$

$(a, b) = \{ x : x \in \mathbb{R} \text{ and } a < x < b \}$

Let ℓ_m be straight line with equation $y = mx$ in the xy-plane [or the set of order pairs (x, y) satisfying the equation $y = mx$].



We can describe ℓ_m by

$$\ell_m = \{ (x, y) : x, y \in \mathbb{R} \text{ and } y = mx \}$$

or $\ell_m = \{ (x, mx) : x \in \mathbb{R} \}$

Set relations

Let A, B be two sets.

Def: We say that A is a subset of B if every element of A is also an element of B. In this case, we write

$$A \subseteq B.$$

Using math notations: $A \subseteq B \equiv (\forall x \in A,\ x \in B.)$

Def: We say that $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Using math notations: $A = B \equiv (A \subseteq B \text{ and } B \subseteq A)$

$\equiv (x \in A \Leftrightarrow x \in B)$

Def: A is a proper subset of B if $A \subseteq B$ and $A \neq B$

We write $A \subset B$.

Remark: $\phi \subseteq S$ for any set S.

Example. Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$

$$C = \{1, 1, 2, 3\}.$$

then $A \subset B = C$

Remark: repeated elements counts only once in the set.

for example $\{1, 1, 1\} = \{1\}$.

If $X \subseteq Y$, then the number of elements of X

$\leq$ the number of elements of Y

Power Set

Let S be a set, the power set of S is the set of all subsets of S. It is denoted by $P(S)$ or 2^S.

Example:

$S = \phi$, then ϕ is the only subset of S. So

$$P(\phi) = 2^{\phi} = \{\phi\} \neq \phi$$

Note that ϕ has no elements, but 2^{ϕ} has one element.

$S = \{x\}$, then $\phi, \{x\} \subseteq S$. $P(S) = 2^S = \{\phi, \{x\}\}$.

$S = \{x, y\}$, then $\phi, \{x\}, \{y\}, \{x, y\} \subseteq S$.

$$P(S) = 2^S = \{\phi, \{x\}, \{y\}, \{x, y\}\}.$$

Remark: If S has n elements, then $P(S)$ has 2^n elements.

Set operators

1. Set union. Let $A, B, C, D, \dots$ be sets,

their union is

$A \cup B \cup C \cup \cdots = \{ x : x$ is an element in at least one of the sets $A, B, C, \cdots \}$

$= \{ x : x \in A, \text{ or } x \in B, \text{ or } x \in C, \text{ or} \cdots \}$

Example: $\{p, q\} \cup \{r\} = \{p, q, r\}$

$\{x, y, z\} \cup \{v, w, x, y\} = \{x, y, z, v, w\}$

$\mathbb{R} \cup \mathbb{Q} = \mathbb{R}$, $\mathbb{N} \cup \mathbb{Z} \cup \mathbb{Q} = \mathbb{Q}$.

$S \cup \emptyset = S$

The intersection of $A, B, C, \dots$ is

$$A \cap B \cap C \cap \cdots = \{ x : x \text{ is an element in every one of the sets } A, B, C, \dots \}$$

$$= \{ x : x \in A, \text{ and } x \in B, \text{ and } x \in C, \text{ and } \dots \}$$

Example: $\{p, q\} \cap \{r\} = \emptyset$

$$\{x, y, z\} \cap \{v, w, x, y, z\} = \{x, y, z\}$$

$$\{x, y, z\} \cap \{v, w, x, y, z\} \cap \{u, v, w, x\} = \{x\}$$

$$\mathbb{R} \cap \mathbb{Q} \cap [0, 1] = \{ x : x \in \mathbb{Q}, \ 0 \le x \le 1 \}$$

$S \cap \emptyset = \emptyset$ for every set S.

Def: The Cartesian product of $A, B, C, \dots$ is

$$A \times B \times C \times \dots = \{ (a, b, c, \dots) : a \in A,\ b \in B,\ c \in C,\ \dots \}$$

Example: $\mathbb{R} \times \mathbb{R} = \{ (x, y) : x, y \in \mathbb{R} \} = \mathbb{R}^2$

$$\mathbb{N} \times \mathbb{Z} \times \{0, 1\} = \{ (x, y, z) : x \in \mathbb{N},\ y \in \mathbb{Z},\ z = 0, \text{or } 1 \}$$

Remark: $S \times \phi = \{ (x, y) : x \in S,\ y \in \phi \} = \phi = \phi \times S$

for all sets S.

If $A \neq B$, then $A \times B \neq B \times A$

Def: the complement of B in A is

$$A \setminus B = \{ x : x \in A,\ x \notin B \}$$

Example: $\mathbb{R} \setminus \mathbb{Q} = \{ x : x \text{ is an irrational number} \}$

$\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q}) = \{ (u, v) : u \in \mathbb{Q},\ v \text{ is irrational number} \}$

$S \setminus \phi = S$, $\phi \setminus S = \phi$ for every set S.

Def: We say that sets $A, B, C, \ldots$ are disjoints if

$$A \cap B \cap C \cap \cdots = \phi.$$

We say that sets $A, B, C, \ldots$ are mutually disjoint if the intersection of every two of the sets is empty.

Example: $A = \{ x, y \}$, $B = \{ y, z \}$, $C = \{ z, x \}$

then $A \cap B \cap C = \phi$, A, B, C are disjoint

But NOT mutually disjoint since $A \cap B \neq \phi$.

Some notations : n is a positive number

$$S_1 \cup S_2 \cup \cdots \cup S_n = \bigcup_{k=1}^{n} S_k$$

$$S_1 \cap S_2 \cap \cdots \cap S_n = \bigcap_{k=1}^{n} S_k$$

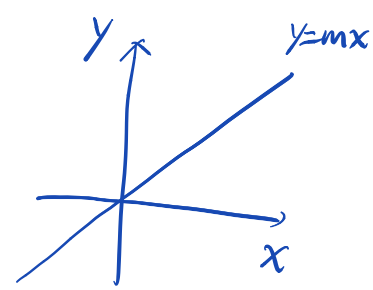
$$S_1 \times S_2 \times \cdots \times S_n = \bigtimes_{k=1}^{n} S_k$$

$$S_1 \cup S_2 \cup S_3 \cup \cdots = \bigcup_{k=1}^{\infty} S_k \quad \left(\text{or } \bigcup_{k\in\mathbb{N}} S_k\right)$$

Example : If $m \in \mathbb{R}$, let ℓ_m be the line $y = mx$ in the x-y plane.

then $$\bigcup_{m\in\mathbb{R}} \ell_m = \mathbb{R}^2 \setminus \{(0,y) : y \neq 0\}$$

$$\bigcap_{m\in\mathbb{R}} \ell_m = \{(0,0)\}.$$



Set properties

1. If $A \subseteq B$ and $C \subseteq D$, prove that

$$A \cap C \subseteq B \cap D$$

Idea: According to the definition of "$\subseteq$", we need to check that $\forall\, x \in A \cap C$, $x \in B \cap D$ also.

proof: $\forall\, x \in A \cap C$, $x \in A$ and $x \in C$

Since $A \subseteq B$, $C \subseteq D$,

We have $x \in B$, $x \in D$ (according to the definition of $\subseteq$)

Hence $x \in B \cap D$ (according to the definition of "$\cap$")

Example Two

Prove that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$

Strategy : to get "$=$", check "$\subseteq$" and "$\supseteq$".

Proof : Step 1, we show that $(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$.

$\forall\, x \in (A \cup B) \setminus C$, we have $x \in A \cup B$ and $x \notin C$.

[by the definition of "$\setminus$"]

Since $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

We have $(x \in A$ and $x \notin C)$ or $(x \in B$ and $x \notin C)$.

It follows that $x \in A \setminus C$ or $x \in B \setminus C$

Therefore $x \in (A \setminus C) \cup (B \setminus C)$

[Using math symbols : $\forall\, x \in (A \cup B) \setminus C \Rightarrow (x \in A \cup B)$ and $(x \notin C)$

$\Rightarrow (x \in A$ or $x \in B)$ and $x \notin C \Rightarrow (x \in A$ and $x \notin C)$

or $(x \in B$ and $x \notin C) \Rightarrow x \in A \setminus C$ or $x \in B \setminus C$

$\Rightarrow x \in (A \setminus C) \cup (B \setminus C)$]

Step 2. we show that $(A \cup B) \setminus C \supseteq (A \setminus C) \cup (B \setminus C)$

$\forall x \in (A \setminus C) \cup (B \setminus C)$, we have $x \in A \setminus C$ or $x \in B \setminus C$.

or equivalently $x \in A$ and $x \notin C$, or $x \in B$ and $x \notin C$.

therefore $x \in A \cup B$ and $x \notin C$. Hence $x \in (A \cup B) \setminus C$.

Remark: actually, one can prove "=" in one step

Proof: $x \in (A \cup B) \setminus C \Leftrightarrow (x \in A \cup B)$ and $(x \notin C)$

$\Leftrightarrow (x \in A$ or $x \in B)$ and $x \notin C \Leftrightarrow (x \in A$ and $x \notin C)$

or $(x \in B$ and $x \notin C) \Leftrightarrow x \in A \setminus C$ or $x \in B \setminus C$

$\Leftrightarrow x \in (A \setminus C) \cup (B \setminus C)$

Functions

Def : a function (or map or mapping) f from a set A to a set B (denoted by $f: A \to B$) is a rule of assigning to every $a \in A$ exactly one $b \in B$. This b (denoted by $f(a)$), is called the value of f at a. A is called the domain of f, and B is called the codomain of f.

$f(A) = \{ y : y = f(x) \text{ for some } x \in A \}$ is called the range (or image) of f. We may say that f is a function on A or f is a B-valued function. The set

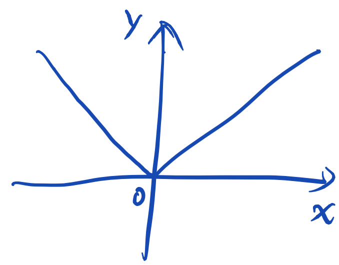
$\{ (x, f(x)) : x \in A \}$ is called the graph of f.

Two functions are equal iff their graphs are the same.

Examples

① $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

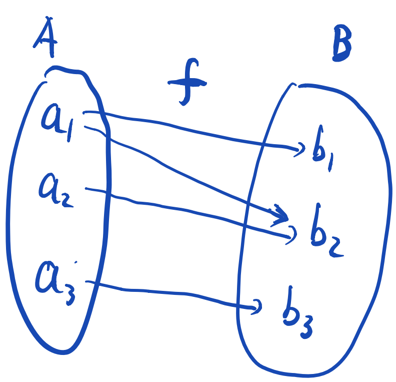


[or $f(x) = |x|$].

The domain of f is $\mathbb{R}$. the range of f is $[0, \infty)$

the graph of f is $\{ (x, |x|) : x \in \mathbb{R} \}$.

②



f is not a map.

Types of functions

Def : ① The identity function on a set S, denoted by $I_S : S \to S$, is given by $I(x) = x$ for any $x \in S$.

② Let $f : A \to B$, $g : B \to C$ be functions. The composition of g by f is denoted by $g \circ f : A \to C$ and is given by

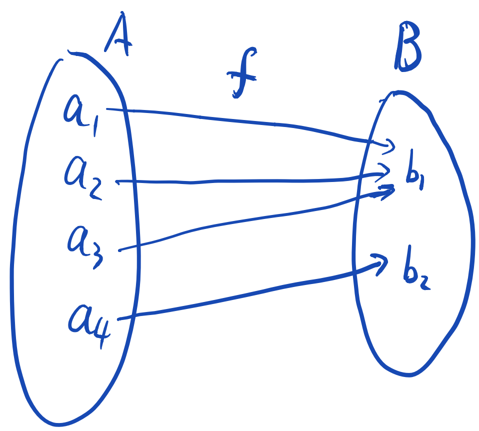
$$g \circ f\,(x) = g(f(x)) \text{ for any } x \in A.$$

③ Let $f : A \to B$ be a function and $C \subseteq A$. The restriction of f to C is denoted by $f|_C : C \to B$ and is given by

$$f|_C(x) = f(x), \ \forall x \in C.$$

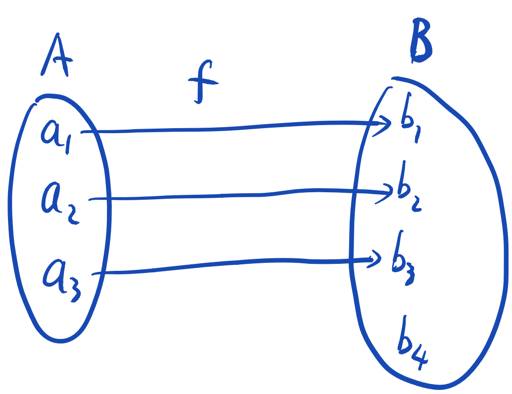
④ $f: A \to B$ is surjective (or onto) iff

$f(A) = B$.



⑤ $f: A \to B$ is injective (or one-to-one) iff

$f(x) = f(y)$ implies that $x = y$.

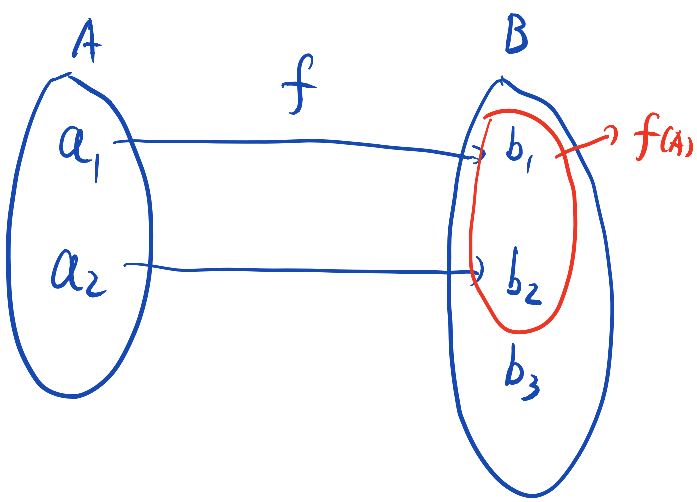


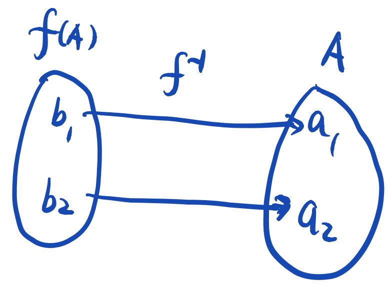
⑥ For an injective function $f : A \to B$

the inverse function of f is denoted by $f^{-1} : f(A) \to A$

and is given by $f^{-1}(y) = x$ where x is

such that $x = f(y)$.





⑦ $f : A \to B$ is bijective (or a one-to-one correspondence)

iff f is injective and surjective

Example:

Show that $f: [0,1] \to [3,4]$ defined by $f(x) = x^3 + 3$ is a bijective.

Proof: Step 1, we show that f is injective.

If $f(x) = f(y)$ for $x, y \in [0,1]$,

then $x^3 + 3 = y^3 + 3$, $x^3 = y^3$.

Since $x, y \geq 0 \Rightarrow x = y$. This shows that f is injective.

Step 2. we show that f is surjective.

$\forall\ y \in [3,4]$, Let $f(x) = x^3 + 3 = y$. then

$x^3 = y - 3$, $x = \sqrt[3]{y-3} \in [0,1]$. This show that f is surjective