MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #18

Differentiability (Part 3)

Theorem: (Taylor Theorem)

Let $f:(a,b)\to R$ be n-time differentiable on (a,b), then for any $x, c \in (a,b)$, there exists $x_0 \in (a,b)$ such that

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots..+\frac{f^{(n-1)}(c)}{(n-1)!}(x-c)^{n-1}+\frac{f^{(n)}(x_0)}{n!}(x-c)^n$$

Using Taylor Theorem, one can express some differentiable functions into series. Some Examples are shown (The derivation is left as exercise)

Example 1

(1) $\frac{1}{1+x}=1-x+x^2-x^3+\cdots\ldots..=\sum_{k=0}^{\infty}(-1)^k x^k$ for $|x|<1$

(2) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots..=\sum_{k=0}^{\infty}\frac{x^k}{k!}$

(3) $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots..=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$

(4) $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots..=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$

(5) $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots..=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{k}$ for $-1<x\le 1$

(6) $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots..=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1}$ for $-1<x\le 1$

Solution:

(1) Let $f(x)=\frac{1}{1+x}$, by computing a first few derivatives, we get

$$f'(x)=-\frac{1}{(1+x)^2},\ f''(x)=\frac{2}{(1+x)^3},\ f^{(3)}(x)=\frac{3!}{(1+x)^4}\ \ldots\ f^{(n)}(x)=\frac{(-1)^n n!}{(1+x)^{n+1}}$$

Therefore by Taylor Theorem (at $c=0$), we have

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots..+\frac{f^{(n)}(0)}{(n)!}x^n+\cdots..$$

$$\to \frac{1}{1+x}=1-x+x^2-x^3+\cdots\ldots..=\sum_{k=0}^{\infty}(-1)^k x^k$$

(2)-(6) are left as exercises

Example 2

Let $f: I\to \mathbf{R}$ be $(n+1)$-times differentiable on I (where I is any interval). If $f^{(n+1)}(x)=0$ for all $x\in I$, then on the interval I, f is a polynomial with degree at most n

Solution:

Applying Taylor Theorem up to x^{n+1} terms (around any $c \in I$, we get

$$f(x) = f(c) + f'(c)(x-c) + \cdots. + \frac{f^{(n)}(c)(x-c)^n}{n!} + \frac{f^{(n+1)}(x_0)(x-c)^{n+1}}{(n+1)!}$$

Since $f^{(n+1)}(x_0) = 0$, then

$$f(x) = f(c) + f'(c)(x-c) + \cdots. + \frac{f^{(n)}(c)(x-c)^n}{n!}$$

which is a polynomial with degree at most n.

Example 3 (Modified From Rudin P.116 #17)

Suppose $f(x)$ is real, three times differentiable function on $[-1,1]$, such that

$$f(-1) = 0, \qquad f(1) = 1, \quad f'(0) = 0$$

Prove that $f^{(3)}(c) \geq 3$ for some $x \in (-1,1)$

(Hint: Apply Taylor Theorem about $x = 0$)

Solution:

Applying Taylor Theorem on $f(x)$ around $x = 0$ (up to x^3 terms)

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(s)}{3!}x^3 \text{ for some } s \in (-1,1)(\text{depend on } x)$$

Putting $x = -1$ and $x = 1$, we get

$$f(1) = f(0) + f'(0)(1) + \frac{f''(0)}{2!}1^2 + \frac{f^{(3)}(t_1)}{3!}1^3$$

$$\rightarrow 1 = f(0) + \frac{f''(0)}{2!}1^2 + \frac{f^{(3)}(t_1)}{3!}1^3 \ldots\ldots\ldots\ldots\ldots (*)$$

$$f(-1) = f(0) - f'(0)(1) + \frac{f''(0)}{2!}1^2 - \frac{f^{(3)}(t_2)}{3!}1^3$$

$$\rightarrow 0 = f(0) + \frac{f''(0)}{2!}1^2 - \frac{f^{(3)}(t_2)}{3!}1^3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (**)$$

Subtract (*) from (**), we get

$$1 = \frac{f^{(3)}(t_1)}{3!} + \frac{f^{(3)}(t_2)}{3!} \rightarrow f^{(3)}(t_1) + f^{(3)}(t_2) = 6$$

Then one of $f^{(3)}(t_1)$ and $f^{(3)}(t_2)$ must be ≥ 3

Hence $f^{(3)}(c) \geq 3$ (where $c = t_1 \text{ or } t_2$ and $c \in (-1,1)$)

(Remark: There is a general form of this statement, see Exercise for detail)

In differentiation, when we differentiate an expression which is a product of functions. (Namely: $\frac{d}{dx}f(x)g(x)$), we apply product rule and get $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$. Next, when we compute $\frac{d^2}{dx^2}f(x)g(x)$, we apply product rule again to get $\frac{d^2}{dx^2}f(x)g(x) = f(x)g(x)+2f'(x)g'(x)+g''(x)$. However, when we compute higher derivative, then the computation can be tedious. In fact, there is a general formula of $\frac{d^n}{dx^n}f(x)g(x)$ which is so called Leibniz Rule. Next Example, we will show you one derviation of this formula

Example 4 (Leibniz Rule)
Let f and g be infinitely differentiable on (a,b). Then the product of fg is also differentiable on (a,b) and

$$(fg)^{(n)}(x_0) = \sum_{k=0}^{n} C_k^n f^{(k)}(x_0)g^{(n-k)}(x_0)$$

(Remark: The formula looks like a binomial expression.

Solution:

There are two ways to prove this theorem, one is by induction (in exercise), another one is by Taylor Theorem

First, apply Taylor Theorem on $f(x)g(x)$ at point $x = x_0$, we get

$$f(x)g(x) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(x_0)}{n!}(x-x_0)^n \dots\dots(*)$$

next, we obtain another expression for $f(x)g(x)$, applying taylor theorem, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \text{ and } g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(x_0)}{n!}(x-x_0)^n$$

Then, multiply them, we get

$$f(x)g(x) = \sum_{n=0}^{\infty}\left[\left(\frac{f^{(n)}(x_0)}{n!}\right)\left(\frac{g(x_0)}{0!}\right) + \left(\frac{f^{(n-1)}(x_0)}{(n-1)!}\right)\left(\frac{g'(x_0)}{1!}\right) + \cdots\left(\frac{f(x_0)}{0!}\right)\left(\frac{g^{(n)}(x_0)}{n!}\right)\right](x-x_0)^n$$

$$\to f(x)g(x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left[\left(\frac{f^{(n-k)}(x_0)}{(n-k)!}\right)\left(\frac{g^{(k)}(x_0)}{k!}\right)\right](x-x_0)^n \dots\dots(**)$$

Compute the coefficient of $(x-x_0)^n$

$$\frac{(fg)^{(n)}(x_0)}{n!} = \sum_{k=0}^{n}\left[\left(\frac{f^{(n-k)}(x_0)}{(n-k)!}\right)\left(\frac{g^{(k)}(x_0)}{k!}\right)\right]$$

$$(fg)^{(n)}(x_0) = \sum_{k=0}^{n}\frac{n!}{(n-k)!\,k!}f^{(n-k)}(x_0)g^{(k)}(x_0) = \sum_{k=0}^{n} C_k^n\, f^{(n-k)}(x_0)g^{(k)}(x_0)$$

Try to work on the following exercises to understand the material, you are welcome to give your solution to me for comments.

(Exercise 1 and 2 have appeared in Tutorial Note #17)

☺Exercise 1

Derive the formula $(2)-(6)$ in Example 4

(Hint: For (5), note that $\frac{d}{dx}\ln(1+x)=\frac{1}{1+x}$, expand the R.H.S.

For (6), the method is similar to (5))

☺Exercise 2

Let $f: I \to \mathbf{R}$ and assume that $f^{(n)}(x)=0$ for all $x \in I$ and $f^{(k)}(x_0)=0$ for $0 \le k \le n-1$ and some $x_0 \in I$. Show that f is a constant function.

(Hint: Apply Taylor Theorem around $c=???$)

☺Exercise 3

Suppose $f(x)$ is real, $(2n+1)$ times differentiable function on $[-1,1]$, such that $f(-1)=0,\ f(1)=1,\quad f^{(k)}(0)=0$ for $k=1,3,5,\dots,(2n-1)$

Prove that $f^{(2n+1)}(c) \ge \frac{(2n+1)!}{2}$ for some $x \in (-1,1)$

☺Exercise 4

Let f be an odd function, that is $f(x)=f(-x)$ for all x, and suppose f can be expanded in an infinite taylor series at $c=0$. Show that the terms of this series are all have odd degree (i.e. $f(x)=a_1x+a_3x^3+\cdots$). State and prove a similar result for even function.

(It gives a intuitive reasons why they are called odd (or even) function)

(*Note: Exercise 5 and 6 are more difficult problems)

☺Exercise 5 (2007 Spring Final)

Let $f: \mathbf{R} \to \mathbf{R}$ be a three-times differentiable function. If $f(x)$ and $f'''(x)$ are bounded functions on $\mathbf{R}$, show that $f'(x)$ and $f''(x)$ are also bounded functions on R.

☺Exercise 6 (Cauchy's Generalized Mean Value Theorem)

If $f(x)$ and $g(x)$ has n times differentiable and $f^{(n-1)}(x), g^{(n-1)}(x)$ are both continuous in $[a,b]$. Then there exists a number $c \in (a,b)$ such that

$$\frac{f(b)-f(a)-\frac{b-a}{1!}f'(a)-\cdots.-\frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)}{g(b)-g(a)-\frac{b-a}{1!}g'(a)-\cdots.-\frac{(b-a)^{m-1}}{(n-1)!}g^{(m-1)}(a)}=\frac{(m-1)!}{(n-1)!}(b-c)^{n-m}\left(\frac{f^{(n)}(c)}{g^{(m)}(c)}\right)$$