Solution to Math 2033 Final Exam (Spring 2014)

$|x-1|<1 \Rightarrow x\in(0,2) \Rightarrow x+3\in(3,5)$

① Sketch $x\to 1 \Rightarrow \sin\left(\frac{x-1}{|x|+2}\right)\to 0$, $\frac{3-x}{x^2+3}\to\frac{1}{2}$ // $\frac{6-2x-x^2-3}{2(x^2+3)}=\frac{-(x-1)(x+3)}{2(x^2+3)}$

$$\left|\sin\left(\frac{x-1}{|x|+2}\right)+\frac{3-x}{x^2+3}-\frac{1}{2}\right|\le\left|\sin\left(\frac{x-1}{|x|+2}\right)\right|+\left|\frac{3-x}{x^2+3}-\frac{1}{2}\right|\le\left|\frac{x-1}{|x|+2}\right|+\frac{|x-1||x+3|}{2(x^2+3)}<\frac{|x-1|}{2}+\frac{5}{6}|x-1|=\frac{4}{3}|x-1|<\varepsilon$$

Solution $\forall\varepsilon>0$, let $\delta=\min\{1,\frac{3}{4}\varepsilon\}$, then

$0<|x-1|<\delta \Rightarrow \begin{matrix}|x-1|<1\\ |x-1|<\frac{3}{4}\varepsilon\end{matrix} \Rightarrow \left|\sin\left(\frac{x-1}{|x|+2}\right)+\frac{3-x}{x^2+3}-\frac{1}{2}\right|<\varepsilon.$ See sketch work above

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② (a) Taylor's theorem

(b) It is about n-times differentiable function f that you can expand as $f(x)=f(c)+f'(c)(x-c)+\cdots+\frac{f^{(n)}(\theta_x)}{n!}(x-c)^n$. Part (c) is about twice differentiable functions.

(c) f has maximum or minimum value at some $w\in[1,3]$, where $f'(w)=0$.
By Taylor's theorem, $0=f(1)=f(w)+\underbrace{f'(w)(1-w)}_{=0}+\frac{f''(\theta_1)}{2}(1-w)^2$ and
$0=f(3)=f(w)+\underbrace{f'(w)}_{=0}(3-w)+\frac{f''(\theta_3)}{2}(3-w)^2$ for some $\theta_1\in[1,w]$, $\theta_3\in[w,3]$.
Since $w\in[1,3]$, $m=\max\{3-w, w-1\}\ge 1$. Solving for $|f(w)|$, we get
$|f(w)|\ge\frac{1}{2}m^2\ge\frac{1}{2}$.

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③ Sketch Consider $|a_m-a_n|$. There are 3 cases: ① m,n both even, ② m,n one odd one even, ③ m,n both odd

① $|a_m-a_n|=|a_{2i}-a_{2j}|=|a_{2i-1}+\frac{1}{i}-a_{2j-1}-\frac{1}{j}|\le|a_{2i-1}-a_{2j-1}|+\frac{1}{i}+\frac{1}{j}$

② $|a_m-a_n|=|a_{2i-1}-a_{2j}|=|a_{2i-1}-a_{2j-1}-\frac{1}{j}|\le|a_{2i-1}-a_{2j-1}|+\frac{1}{j}$

③ $|a_m-a_n|=|a_{2i-1}-a_{2j-1}|$

$m=2i$ or $2i-1$, $n=2j$ or $2j-1$

Solution $\forall\varepsilon>0$, by Archimedian axiom, $\exists K_1\in\mathbb{N}$ such that $K_1>\frac{3}{\varepsilon}$ $(\Leftrightarrow\frac{1}{K_1}<\frac{\varepsilon}{3})$.
Since $a_1,a_3,a_5,\ldots$ is Cauchy, $\exists K_2\in\mathbb{N}$ such that $i,j\ge K_2\Rightarrow|a_{2i-1}-a_{2j-1}|<\varepsilon/3$.
Let $K=\max\{2K_1,2K_2\}$, then $m,n\ge K\Rightarrow\begin{matrix}m,n\ge 2K_1>2K_1-1\\ m,n\ge 2K_2>2K_2-1\end{matrix}\Rightarrow\begin{matrix}i,j\ge K_1\\ i,j\ge K_2\end{matrix}$

$\Rightarrow|a_m-a_n|\le|a_{2i-1}-a_{2j-1}|+\frac{1}{i}+\frac{1}{j}<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$
from sketch above

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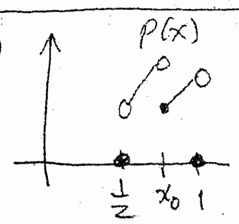
④ For $x>0$, by Taylor's theorem, $g(x)=g(0)+g'(0)x+\frac{g'''(\theta_x)}{6}x^3$ for some $\theta_x\in(0,x)$. Let $x=-g(0)/g'(0)>0$, then $g(x)=\frac{g'''(\theta_x)}{6}x^3<0$. Since $g(0)>0$, by intermediate value theorem, $\exists r\in(0,x)\subseteq(0,+\infty)$ Such that $g(r)=0$. (Alternatively, $g'''(\theta_x)<0\Rightarrow g(x)<g(0)+g'(0)x$
$\Rightarrow\lim_{x\to+\infty}g(x)\le\lim_{x\to+\infty}g(0)+\underbrace{g'(0)}_{<0}x=-\infty\Rightarrow g(x)<0$ for large x.)

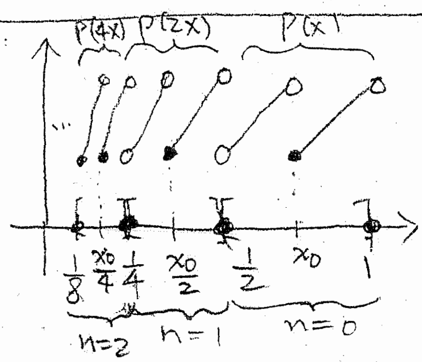
⑤ We check F is injective. If $F(a)=F(b)$, then $3F(F(a))=3F(F(b))$ and so $F(a)+a=F(b)+b$. Subtracting $F(a)=F(b)$, we get $a=b$. By the Continuous Injection theorem, F is strictly increasing or strictly decreasing on $\mathbb{R}$.

Case 1 (F is strictly increasing). If $F(0)>0$, then $F(F(0))>F(0) \Rightarrow F(0)+0=3F(F(0))>3F(0) \Rightarrow 0>F(0)$, contradiction. If $F(0)<0$, then $F(F(0))<F(0) \Rightarrow F(0)+0=3F(F(0))<3F(0) \Rightarrow 0<F(0)$, contradiction, $\therefore F(0)=0$.

Case 2 (F is strictly decreasing). Assume $F(0)\neq 0$. Now $F(x)-x$ is continuous on $\mathbb{R}$. By intermediate value theorem, either (*) $\forall x\in\mathbb{R}$, $F(x)-x>0$ or (**) $\forall x\in\mathbb{R}$, $F(x)-x<0$. In the former case, $\forall x\in\mathbb{R}$, $F(x)>x$. So $F(0)>0$. Take some $x_0>F(0)>0$. F strictly decreasing $\Rightarrow F(x_0)<F(0)<x_0$ $\Rightarrow F(x_0)-x_0<0$ Contradicting (*). In the latter case, $\forall x\in\mathbb{R}$, $F(x)<x$. So $F(0)<0$. Take some $x_1<F(0)<0$. F strictly decreasing $\Rightarrow$ $F(x_1)>F(0)>x_1 \Rightarrow F(x_1)-x_1>0$ Contradicting (**). $\therefore F(0)=0$.

⑥





$P(x)$ is Riemann integrable on $[\frac{1}{2},1]$ $\Leftrightarrow$ S_p is of measure 0 by Lebesgue's theorem.

Note $S_p\subseteq[\frac{1}{2},1]$.

For every $n\in\{0,1,2,3,\dots\}$ and $W\in(\frac{1}{2^{n+1}},\frac{1}{2^n})$, we have $2^nW\in(\frac{1}{2},1)$. If $p(x)$ is continuous at $2^nW\in(\frac{1}{2},1)$, then $h(x)=p(2^nx)$ is continuous at $W\in(\frac{1}{2^{n+1}},\frac{1}{2^n})$. Taking Contrapositive, if h is discontinuous at W, then p is discontinuous at 2^nW. This means $W\in S_h\cap(\frac{1}{2^{n+1}},\frac{1}{2^n})\Rightarrow 2^nW\in S_p$. Letting $W=\frac{x_0}{2^n}$, $S_h\cap(\frac{1}{2^{n+1}},\frac{1}{2^n})\subseteq\{W: W\in(\frac{1}{2^{n+1}},\frac{1}{2^n}) \text{ and } 2^nW\in S_p\}\subseteq\{\frac{x_0}{2^n}: x_0\in S_p\}$.

Since S_p is of measure 0, $\forall\varepsilon>0$ $\exists (a_1,b_1),(a_2,b_2),\dots$ such that $S_p\subseteq\bigcup_{i=1}^{\infty}(a_i,b_i)$ and $\sum_{i=1}^{\infty}|a_i-b_i|<\varepsilon$. Then $S_h\cap(\frac{1}{2^{n+1}},\frac{1}{2^n})\subseteq\bigcup_{i=1}^{\infty}(\frac{1}{2^n}a_i,\frac{1}{2^n}b_i)$ and $\sum_{i=1}^{\infty}|\frac{1}{2^n}a_i-\frac{1}{2^n}b_i|=\frac{1}{2^n}\sum_{i=1}^{\infty}|a_i-b_i|<\frac{1}{2^n}\varepsilon\leq\varepsilon$. Hence, $S_h\cap(\frac{1}{2^{n+1}},\frac{1}{2^n})$ is of measure 0.

Finally, $S_h\subseteq\underbrace{\bigcup_{n=0}^{\infty}}_{\text{Countable}}\underbrace{\left(S_h\cap(\frac{1}{2^{n+1}},\frac{1}{2^n})\right)}_{\text{measure } 0}\cup\underbrace{\{\frac{1}{2},\frac{1}{4},\frac{1}{8},\dots\}\cup\{0,1\}}_{\text{Countable}\Rightarrow\text{measure } 0}$

$\Rightarrow S_h$ is of measure 0. $\therefore h$ is Riemann integrable on $[0,1]$.