MATH 2031 Introduction to Real Analysis

February 26, 2013

Tutorial Note 12

Continuity

(C.I) **Definition:**
A function $f : S \to \mathbb{R}$ is continuous at $x_0 \in S$ iff $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x) = f(x_0)$.

$$\lim_{\substack{x\to x_0\\ x\in S}} f(x) = f(x_0) \iff \forall\varepsilon > 0\ \exists\delta > 0\ \forall x \in S,\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

(C.II) **Sequential Continuity Theorem (S.C.T)**

$$\begin{aligned} & f : S \to \mathbb{R} \text{ is continuous at } x_0 \in S \\ \iff & \text{ for every sequence } \{x_n\} \subset S \setminus \{x_0\} \text{ that converges to } x_0,\ \lim_{n\to\infty} f(x_n) = f(x_0) = f\left(\lim_{n\to\infty} x_n\right) \end{aligned}$$

(C.III) **Theorem** (to construct continuous functions from given continuous functions)

- If $f, g : S \to \mathbb{R}$ are continuous at $x_0 \in S$, then $f \pm g$, fg, f/g (provided that $g(x_0) \neq 0$) are continuous.
- If $f : S \to \mathbb{R}$ is continuous at x_0, $f(S) \subseteq T$ and $g : T \to \mathbb{R}$ is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

(C.IV) **Sign Preserving Property**
If $g : S \to \mathbb{R}$ is continuous and $g(x_0) > 0$, then $\exists$ an interval $I = (x_0 - \delta, x_0 + \delta)$ with $\delta > 0$ such that $g(x) > 0$ for all $x \in x \cap I$.

(C.V) **Intermediate Value Theorem (I.V.T.)**
Let $a, b \in \mathbb{R}$ with $a \leq b$.
If $f : [a, b] \to \mathbb{R}$ is continuous and y_0 is between $f(a)$ and $f(b)$, then $\exists\ x_0 \in [a, b]$ such that $f(x_0) = y_0$.

(C.VI) **Extreme Value Theorem (E.V.T.)**
Let $a, b \in \mathbb{R}$ with $a \leq b$.
If $f : [a, b] \to \mathbb{R}$ is continuous, then $\exists\ x_0, w_0 \in [a, b]$ such that $f(w_0) \leq f(x) \leq f(x_0)\ \forall\ x \in [a, b]$.
So the range of $f = \{f(x)|\ x \in [a, b]\} = f([a, b])$ is the interval $[f(w_0), f(x_0)]$.

(C.VII) **Continuous Injection Theorem**
If f is continuous and injective on $[a, b]$, then f is strictly monotone on $[a, b]$ and $f([a, b]) = [f(a), f(b)]$ or $[f(b), f(a)]$.

(C.VIII) **Continuous Inverse Theorem**
If f is continuous and injective on $[a, b]$, then $f^{-1} : f([a, b]) \to [a, b]$ is continuous and surjective.

Problem 1 (Adapted from Rudin) Let f be a real valued function on $\mathbb{R}$. Prove that f is continuous on $\mathbb{R}$ implies for every $x \in \mathbb{R}$, $\lim_{h \to 0} f(x+h) - f(x-h) = 0$.

How about the converse?

Solution:

The trick here is to take a particular value of x to interpret the 2 statements.

"$\Rightarrow$"

$$f \text{ is continuous on } \mathbb{R} \iff \begin{pmatrix} \forall\, y \in \mathbb{R}, \forall\, \varepsilon > 0, \exists\, \delta_1 > 0 \text{ such that} \\ \forall h \in \mathbb{R}, |h| = |h-0| < \delta_1 \Rightarrow |f(y+h) - f(y)| < \varepsilon \end{pmatrix}.$$

Now $\forall\, x \in \mathbb{R}, \forall\, \varepsilon > 0$, we take $\delta = \frac{\delta_1}{2}$. Then $\forall h \in \mathbb{R}, |h| = |h-0| < \delta$ implies $|2h| < \delta_1$. Since y was arbitrary in the above statement, we take $y = x - h$ to get

$$|f(x+h) - f(x-h)| = |f(y+2h) - f(y)| < \varepsilon.$$

By definition of limit, $\lim_{h \to 0} f(x+h) - f(x-h) = 0$ for any $x \in \mathbb{R}$.

The converse is not true, we may take the following as an counterexample.

$$f(x) = \begin{cases} x & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Problem 2 Let $f, g : [0,1] \to \mathbb{R}$ be continuous. If there exists a sequence of numbers $\{x_n\} \subseteq [0,1]$ such that $g(x_n) = f(x_{n+1})$ for all $n \in \mathbb{N}$, prove that there exists $w \in [0,1]$ such that $g(w) = f(w)$.

Solution:

In the problem we can see that f and g are continuous and we are looking for certain w such that $g(w) = f(w)$ (or $g(w) - f(w) = 0$). We should immediately think of the Intermediate value theorem (I.V.T.) or the Extreme value theorem (E.V.T.).

Define $h(x) = g(x) - f(x)$. Since both f and g are continuous, h is also continuous. Then by the Extreme value theorem, $\exists a, b \in [0,1]$ such that for all $x \in [0,1]$, $h(a) \le h(x) \le h(b)$. In particular, for each $n \in \mathbb{N}$, we have

$$h(a) \le h(x_n) = g(x_n) - f(x_n) = f(x_{n+1}) - f(x_n) \le h(b).$$

From the middle equality about $h(x_n)$, we have $h(x_1) + \cdots + h(x_n) = f(x_{n+1}) - f(x_1)$, thus we get

$$h(a) \le \frac{h(x_1) + \cdots + h(x_n)}{n} = \frac{f(x_{n+1}) - f(x_1)}{n} \le h(b).$$

Now write $y_n = \dfrac{h(x_1) + \cdots + h(x_n)}{n} = \dfrac{f(x_{n+1}) - f(x_1)}{n}$. Applying the Intermediate value theorem to the above, we see that $\exists w_n \in [0,1]$ such that $h(w_n) = y_n$.

Note that $0 \le |y_n| \le \dfrac{1}{n}\left[2 \max_{x \in [0,1]} f(x)\right]$ for each $n \in \mathbb{N}$. Since $\lim_{n \to \infty} \dfrac{1}{n}\left[2 \max_{x \in [0,1]} f(x)\right] = 0$, so by Sandwich theorem, $\lim_{n \to \infty} |y_n| = 0$, and so $\lim_{n \to \infty} y_n = 0$.

Also, $\{w_n\} \subseteq [0,1]$ is a bounded sequence. So by Bolzano-Weierstrass theorem, there is a subsequence $\{w_{n_k}\}$ that converges to some $w \in [0,1]$.

Finally we have

$$h(w) = h\left(\lim_{k \to \infty} w_{n_k}\right) = \lim_{k \to \infty} h(w_{n_k}) = \lim_{k \to \infty} y_{n_k} = 0,$$

thus $0 = h(w) = g(w) - f(w)$, i.e $g(w) = f(w)$.

Problem 3 If $f(x) = x^3$, then $f(f(x)) = x^9$. Is there any continuous function $g : [-1, 1] \to [-1, 1]$ such that $g(g(x)) = -x^9$?

Solution:

Since we don't have much information about g, we may assume that such g exists and see what properties it must have.

Is it injective?
For $x, y \in [-1, 1]$ and $g(x) = g(y)$, then

$$-x^9 = g(g(x)) = g(g(y)) = -y^9 \qquad \Rightarrow \qquad x = y$$

Thus, g is injective.

Then Continuous Injection Theorem asserts that g is strictly monotone. i.e. g is strictly increasing or strictly decreasing.

g is strictly increasing :
For $x < y$, we have $g(x) < g(y)$ and also $g(g(x)) < g(g(y))$.
So $g(g(x))$ is also strictly increasing.

g is strictly decreasing :
For $x < y$, we have $g(x) > g(y)$ and then $g(g(x)) < g(g(y))$.
So $g(g(x))$ is again strictly increasing.

In both cases, $g(g(x))$ must be a strictly increasing function.
However, $g(g(x)) = -x^9$ is a decreasing function, contradiction.

Therefore, such g does not exist.