**Problem 7**

$\lim_{n\to+\infty}$ $\lim_{n\to+\infty}$ $\Rightarrow$ $x = \frac{1}{2}(x + \frac{a}{x})$.

We let $x_1 > 0$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for $n \in \mathbb{N}$, where $a > 0$. Show that the sequence $\{x_n\}$ converges.

(🙂Hint: Show that $\{x_n\}$ is decreasing by considering $x_{n+1} - x_n$.)

Proof: $x_1 > 0 \Rightarrow x_n > 0$, for $n \in \mathbb{N}$ $\quad a+b \geq 2\sqrt{ab}$, $a, b \geq 0$.

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \geq \frac{1}{2} \cdot 2 \cdot \sqrt{x_n \cdot \frac{a}{x_n}} = \sqrt{a}$. $\Rightarrow x_n \geq \sqrt{a}$, $n \geq 1$.

$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - x_n$

$= \frac{1}{2}\left(\frac{a}{x_n} - x_n\right)$

$= \frac{1}{2x_n}\left(a - x_n^2\right) \leq 0$.

$\Rightarrow \{x_n\}$ is decreasing. $\{x_n\}$ is bounded.

$\Rightarrow x_n$ converges. to $x$.

$\Rightarrow \lim_{n\to+\infty} x_n = \frac{1}{2} \lim_{n\to+\infty} \left(x_n + \frac{a}{x_n}\right) \Rightarrow x = \frac{1}{2}\left(x + \frac{a}{x}\right) \Rightarrow x = \sqrt{a}$.

$\Rightarrow \lim_{n\to+\infty} x_n = \sqrt{a}$.

**Problem 8**

We let $\{x_n\}$ be a bounded sequence of real numbers. For any $n \in \mathbb{N}$, we define $\quad \lim_{n\to\infty} y_n = \lim_{n} \sup_{m\geq n} x_m$

$$y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}.$$

Show that $\{y_n\}$ converges. $\quad z_n = \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}$. $\{z_n\}$.

Proof: $\{x_n\}$ is bounded $\Rightarrow \exists M > 0$. s.t. $-M \leq x_n \leq M$. for $n \in \mathbb{N}$.

$\Rightarrow -M \leq y_n \leq M$. $\Rightarrow \{y_n\}$ is bounded.

Relationship of $y_n$ and $y_{n+1}$?

$\{x_{n+1}, x_{n+2}, \dots\} \subseteq \{x_n, x_{n+1}, x_{n+2}, \dots\}$

$\Rightarrow \sup\{x_{n+1}, x_{n+2}, \dots\} \leq \sup\{x_n, x_{n+1}, x_{n+2}\}$.

$\Rightarrow y_{n+1} \leq y_n$. $\Rightarrow \{y_n\}$ is decreasing.

$\Rightarrow \{y_n\}$ converges.

**Problem 9**

We let $\{x_n\}$ is a sequence of positive real numbers. For any $n \in \mathbb{N}$, we define

$$y_n = \max\{x_1, x_2, \dots, x_n\}.$$

**(a)** If $\{x_n\}$ is bounded, show that $\{y_n\}$ converges.

**(b)** If $\{x_n\}$ is unbounded, show that $\{y_n\}$ diverges to $+\infty$.

(a). $\{x_n\}$ is bounded. $\Rightarrow$ $\exists M>0$ s.t. $0 \le x_n \le M$. for any $n \in \mathbb{N}$.

$\Rightarrow$ $0 \le y_n \le M$ $\Rightarrow$ $\{y_n\}$ is bounded.

Note that

$\{x_1, x_2, \dots, x_n\} \subseteq \{x_1, x_2, \dots, x_n, x_{n+1}\}$

Relationship of $y_n$ and $y_{n+1}$?

$\Rightarrow$ $\max\{x_1, \dots, x_n\} \le \max\{x_1, \dots, x_{n+1}\}$

$\Rightarrow$ $y_n \le y_{n+1}$

$\Rightarrow$ $\{y_n\}$ is increasing,

$\Rightarrow$ $\{y_n\}$ converges.

Definition, / Negation of bounded,

(b), If $\{x_n\}$ is unbounded, $\Rightarrow$ For any $K>0$, For any $N \in \mathbb{N}$.

$\exists n_K > N$ s.t. $x_{n_K} > K$.

Choose $K=1$, $\exists n_1$ s.t. $x_{n_1} > 1 \to y_{n_1} > 1$. $y_{n_1} = \max\{x_1, x_2, \dots, x_{n_1}\}$.

Choose $K=2$ $\exists n_2 > n_1$ s.t. $x_{n_2} > 2$ $\to y_{n_2} > 2$. $\boxed{\{y_n\}\text{. is increasing.}}$

Choose $K=3$. $\exists n_3 > n_2$ s.t. $x_{n_3} > 3 \to y_{n_3} > 3$

... ...

$\Rightarrow$ $y_{n_1} > 1$, $y_{n_2} > 2$ .. $y_{n_3} > 3$ ..... $y_{n_K} > K$. $\Rightarrow$ $\boxed{\lim_{K \to +\infty} y_{n_K} = +\infty}$

$\{y_n\}$ is increasing. $\Rightarrow$ $\lim_{n \to +\infty} y_n = +\infty$.

**Problem 10**

Show that a sequence $\{x_n\}$ defined by $x_n = (-1)^n$ is not Cauchy sequence.

Definition,

Proof: Take $\varepsilon = 1$.

$x_n$ : $n = 1, 2, 3, 4, \dots$

$x_n = -1, 1, -1, 1,$

$\forall N \in \mathbb{N}$. Choose $n = N+1 > N$.

$|x_{n+1} - x_n| = 2 > \varepsilon = 1$.

$\Rightarrow$ $\{x_n\}$ is not Cauchy sequence

Definition of Cauchy Sequence. Bounded.

$\{x_n\}$ is Cauchy Sequence. $\Leftrightarrow$ For any $\varepsilon > 0$ $\exists k \in \mathbb{N}$

such that $|x_m - x_n| < \varepsilon$ for any $m, n \geq k$.

Take $\varepsilon = 1$. $\exists k \in \mathbb{N}$, $\forall m \geq k$, we have $|x_m - x_{k+1}| < \varepsilon$.

$\Rightarrow$ $|x_m| \leq \boxed{\varepsilon + |x_{k+1}|}$ $\Rightarrow$ $\max_n \{|x_n|\} = \max\{|x_1|, \ldots, |x_k|, |x_{k+1}| + \varepsilon\}$

Negation:

$\{x_n\}$ is not a Cauchy Sequence $\Leftrightarrow$ $\exists \varepsilon > 0$. $\forall k \in \mathbb{N}$.

$\exists$ $m, n \geq k$. such that $|x_m - x_n| \geq \varepsilon$.

**Problem 11**

Show that if $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequence, then $\{x_n + y_n\}$ and $\{x_n y_n\}$ are both Cauchy sequence using the definition of Cauchy sequence.

What we have:

$\{x_n\}$, $\{y_n\}$

$x_m - x_n$, $x_m - y_n$,

Write new, or unknown things into the form of $x_m - x_n$ $y_m - y_n$

Proof: (1) $\forall \varepsilon > 0$. $\exists N_1$. s.t. $\forall m, n > N_1$ $|x_n - x_m| < \frac{\varepsilon}{2}$

$\exists N_2$ s.t. $\forall m, n > N_2$ $|y_n - y_m| < \frac{\varepsilon}{2}$

$\Rightarrow$ Take $N = \max\{N_1, N_2\}$

$\forall$ $m, n > N$. $|x_n - x_m| < \frac{\varepsilon}{2}$, $|y_n - y_m| < \frac{\varepsilon}{2}$

$\Rightarrow$ $|(x_n + y_n) - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m|$
$< \varepsilon$.

$\Rightarrow$ $\{x_n + y_n\}$ is Cauchy Sequence.

(2) Note that. $\{x_n\}$, $\{y_n\}$ are bounded.

(because they are Cauchy sequences)

$\Rightarrow$ $\exists M > 0$. s.t. $|x_n| \leq M$. $|y_n| \leq M$.

? C

$\forall \varepsilon > 0$. $\exists N_1$ s.t. $\forall m, n > N_1$. $|x_n - x_m| < \frac{\varepsilon}{2M}$

$\exists N_2$ s.t. $\forall m,n > N_2$, $|y_n - y_m| < \frac{\varepsilon}{2M}$. c,

$x_n y_n - x_m y_m$, ?

$= x_n y_n - x_n y_m + x_n y_m - x_m y_m$

$\{x_n y_n\}$,

$= \underbrace{x_n}_{\le M}\underbrace{(y_n - y_m)}_{0} + \underbrace{(x_n - x_m)}_{0}\underbrace{y_m}_{\le M}$.

Take $N = \max\{N_1, N_2\}$.

$\Rightarrow \forall m,n > N$. $|x_n - x_m| < \frac{\varepsilon}{2M}$. $|y_n - y_m| < \frac{\varepsilon}{2M}$

$\Rightarrow |x_n y_n - x_m y_m| \le |x_n y_n - x_m y_n| + |x_m y_n - x_m y_m|$

$\le c \cdot M + c \cdot M$.

$\le \varepsilon$

$\Rightarrow c < \frac{\varepsilon}{2M}$.

$\le |x_n - x_m|\,|y_n| + |x_m|\,|y_n - y_m|$

$\le M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon$.

$\Rightarrow \{x_n y_n\}$ is Cauchy sequence

**Problem 12 (Harder)**

We let $\{x_n\}$ be a sequence of real number with $\lim_{n\to\infty} x_n = x$. Show that

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = x.$$

(😊Hint: Note that $\lim_{n\to\infty} x_n = x$. Then for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for $n \ge K$.)

Proof: $\boxed{\forall \varepsilon > 0,\ \exists K \in \mathbb{N},\ \text{s.t. } |x_n - x| < \frac{\varepsilon}{2}.\ n \ge K.}$

Consider $x_1 + x_2 + \dots + x_{k-1} - (k-1)x$ is a constant

$\exists \tilde{k} > K$, $\tilde{k} \in \mathbb{N}$. s.t. $\left| \frac{\sum_{i=1}^{k-1} x_i - (k-1)x}{\tilde{k}} \right| < \frac{\varepsilon}{2}$.

$x_1 + x_2 + \dots + x_n$.

$x_k$. $k$ large, $x_k$. close to $x$

$\Rightarrow \forall n > \tilde{k}$.

$\left| \frac{x_1 + x_2 + \dots + x_n}{n} - x \right|$

$\le \left| \frac{x_1 + x_2 + \dots + x_{k-1} - (k-1)x}{n} \right| + \left| \frac{\sum_{i=k}^{n} (x_i - x)}{n} \right|$.

$\le \left| \frac{x_1 + x_2 + \dots + x_{k-1} - (k-1)x}{\tilde{k}} \right| + \frac{1}{n} \sum_{i=k}^{n} |x_i - x|$

?

$\le \frac{\varepsilon}{2} + \frac{1}{n} \cdot (n-k+1) \cdot \frac{\varepsilon}{2}$

$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$\Rightarrow \lim_{n\to+\infty} \frac{x_1 + x_2 + \dots + x_n}{n} = x$.

**Problem 13 (Harder)**

We let $\{x_n\}$ be a bounded sequence and let $s = \sup\{x_n | x \in \mathbb{N}\}$. Show that if $s \notin \{x_n | n \in \mathbb{N}\}$, then there exists a subsequence of $\{x_n\}$ which converges to $s$.

(😊Hint: You need to construct such subsequence. Using the property of supremum and the fact that $s \notin \{x_n | n \in \mathbb{N}\}$, argue that for any $\varepsilon > 0$, there exists infinitely many $x_n s$ such that $s > x_n > s - \varepsilon$. Construct the subsequence by taking $\varepsilon = \frac{1}{k}$ for $k \in \mathbb{N}$.)

Proof:

Take $\varepsilon_1 = 1$. $\exists\, n_1 \in \mathbb{N}$. s.t. $s - \varepsilon_1 \le x_{n_1} \le s$.

Take $\varepsilon_2 = \frac{1}{2}$ $\exists\, n_2 > n_1$ s.t. $s - \varepsilon_2 \le x_{n_2} \le s$

why? Not hard.

Take $\varepsilon_3 = \frac{1}{3}$ $\exists\, n_3 > n_2$. s.t. $s - \varepsilon_3 \le x_{n_3} \le s$

$\vdots$

$\varepsilon_k = \frac{1}{k}$. $\exists\, n_k > n_{k-1}$ s.t. $s - \varepsilon_k \le x_{n_k} \le s$.

$\Rightarrow$. $\lim_{k \to \infty} x_{n_k} = s$.

why? ( For any $\varepsilon > 0$, $\exists\, K$. s.t. $\varepsilon_K < \varepsilon$.

Then, $\forall m > K$.

$s - \varepsilon \le x_{n_m} \le s$

$\Rightarrow |x_{n_m} - s| \le \varepsilon$. )