Exercise 1. Determine the following series converge or not.

(1) $\sum_{k=1}^{\infty} \ln(1+\frac{1}{k})$

(2) $\sum_{k=1}^{\infty} \frac{1}{k^{1+1/k}}$

Proof:

(1) Directly check the sum:

$$S_n = \sum_{k=1}^{n} \ln(1+\frac{1}{k}) = \ln \prod_{k=1}^{n}(1+\frac{1}{k}) = \ln(\frac{2}{1}\cdot\frac{3}{2}\cdot \cdots \cdot \frac{n+1}{n}) = \ln(n+1)$$

Since $\ln(n+1) \to +\infty$ when $n \to +\infty$. S_n diverges.

(A rigorous proof: For each $M>0$. let $N = [e^M]$. then for $\forall n>N$. we have $S_n = \ln(n+1) > \ln(N+1) > \ln(e^M) = M$. So by definition. we have $S_n \to \infty$)

$[e^M]$

(how to find that 'N' corresponding to the given 'M'? Try to do it on yourself)

(2) Recall the comparison principle.

First intuition: compare with $\sum \frac{1}{n}$. say. the harmonic series. which is a classical divergent series. but $\frac{1}{k^{1+\frac{1}{k}}} / \frac{1}{k} = \frac{1}{k^{\frac{1}{k}}} < 1$. we cannot use it to draw the conclusion. But $\lim \frac{1}{k^{\frac{1}{k}}} = 1$. So we can do something with the sign-preserving property of limit.

Rigorous proof:

Let $a_k = 1/k^{1+\frac{1}{k}}$. $b_k = 1/k$. then we know

$$\lim_{k\to\infty}\frac{a_k}{b_k}=\lim_{k\to\infty}\frac{1}{\sqrt[k]{k}}=1$$

(How to do it? Let $\varepsilon=\frac{1}{2}$, $\Rightarrow \exists N$ s.t. $\left|\frac{a_n}{b_n}-1\right|<\varepsilon$ ---)

the symbol-preserving property

Hence here $\exists N_1>0$ s.t. when $n>N_1$, $\frac{1}{2}<\frac{a_n}{b_n}<\frac{3}{2}$. On the other hand. Since $\sum b_n$ diverges, here $\exists\, \varepsilon_0>0$ such that for $\forall N_2>0$, here $\exists$ $n, m>N_2$ such that.

$$|P_m-P_n|\geq 2\varepsilon_0$$

where $P_n:=\sum_{k=1}^{n}\frac{1}{k}$. Then we let $N=\max\{N_1, N_2\}$. We have the following statement:

i.e. find $n, m>N$

"Here $\exists\,\varepsilon_0>0$, s.t. for all $N_2>0$, if we let $n, m>\max\{N, N_2\}\geq N$,

we have
$$|S_m-S_n|=\left|\underbrace{\frac{1}{(n+1)^{1+\frac{1}{n+1}}}}_{\to a_{n+1}}+\cdots+\frac{1}{m^{1+\frac{1}{m}}}\right|$$
$$\geq\frac{1}{2}\left|\underbrace{\frac{1}{n+1}}_{\to b_{n+1}}+\cdots+\frac{1}{m}\right|$$
$$\geq\frac{1}{2}\cdot 2\varepsilon_0=\varepsilon_0."$$

($a_n>\frac{1}{2}b_n$ when $n>N_1$)

So we prove that $\sum\frac{1}{k^{1+1/k}}$ diverges.

Remark: A important moral: (only the 'remainder' of a series determines the convergence). regardless of any finite terms.

$\square$

Exercise 2.

If $\sum_{k=2}^{\infty} |x_k - x_{k-1}|$ converges, then show the sequence $\{x_n\}$ is a Cauchy sequence by checking the definition of Cauchy sequence.

Proof:

According to the convergence of $\Sigma |x_k - x_{k-1}|$

For each $\varepsilon > 0$. $\exists N > 0$ s.t. when $n, m > N$. we have

$$|S_m - S_n| = \big|\, |x_{n+1} - x_n| + |x_{n+2} - x_{n+1}| + \cdots + |x_m - x_{m-1}| \,\big| < \varepsilon$$

$$\Rightarrow |x_{n+1} - x_n| + \cdots + |x_m - x_{m-1}| < \varepsilon$$

On the other hand, using the trigonometric inequality.

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} \cdots + x_{n+1} - x_n|$$
$$\leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n|$$
$$< \varepsilon.$$

So we prove that 'For each $\varepsilon > 0$. $\exists N > 0$ s.t. when $n, m > N$. we have $|x_n - x_m| < \varepsilon$'. which ensures that $\{x_n\}$ is Cauchy. $\square$

Exercise 3.

(1)

If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\{a_n\}$ is a Cauchy sequence, then show that $\{\sqrt{a_n}\}$ is also a Cauchy sequence by checking the definition of Cauchy sequence.

(2)

Let $0 < k < 1$. If $|x_{n+1} - x_n| \leq k|x_n - x_{n-1}|$ for $n = 2, 3, 4, \ldots$, then prove that $\{x_n\}$ is a Cauchy sequence.

Proof:

(1) **Intuition**: $|\sqrt{a_m} - \sqrt{a_n}| = \left|\frac{(\sqrt{a_n} - \sqrt{a_m})(\sqrt{a_n} + \sqrt{a_m})}{\sqrt{a_n} + \sqrt{a_m}}\right| = \left|\frac{a_n - a_m}{\sqrt{a_n} + \sqrt{a_m}}\right| < C \cdot \varepsilon$

but we should consider the case $\sqrt{a_n} = 0$. How to discuss?

Rigorous proof:

Case 1: $\lim a_n = 0$.

Since $\lim a_n = 0$. for $\forall \varepsilon > 0$. $\exists N > 0$ s.t. when $n > N$. we have $|a_n| = |a_n - 0| < \frac{1}{4}\varepsilon^2$. Then we conclude that. "for $\forall \varepsilon > 0$. $\exists N > 0$. s.t. when $n, m > N$. we have

$$|\sqrt{a_m} - \sqrt{a_n}| \leq \sqrt{a_m} + \sqrt{a_n} < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Case 2: $\lim a_n > 0$.

From the symbol-preserving property. here $\exists$ a constant $c > 0$ and $N_1 > 0$ such that when $n > N_1$. we have

$$a_n = |a_n| \geq c \Rightarrow$$

purpose: make the denominator in $\frac{a_n - a_m}{\sqrt{a_n} + \sqrt{a_m}}$ away from 0.

On the other hand. since $\{a_n\}$ is Cauchy. for any $\varepsilon > 0$. here $\exists$ $N_2 > 0$ such that when $n > N_2$. we have

$$|a_m - a_n| < 2\sqrt{c} \cdot \varepsilon.$$

So collapse all discussions. we conclude that " for any $\varepsilon>0$. here $\exists\ N=\max\{N_1, N_2\}$ such that when $n, m>N$. we have

$$|\sqrt{a_m}-\sqrt{a_n}| = \left|\frac{a_m-a_n}{\sqrt{a_m}+\sqrt{a_n}}\right| \le \frac{|a_m-a_n|}{2\sqrt{c}} \le \frac{2\sqrt{c}\cdot\varepsilon}{2\sqrt{c}} = \varepsilon \text{ "}$$

So $\{\sqrt{a_n}\}$ is also Cauchy.

□

(2) Iterate the contraction. we have.

$$|x_{n+1}-x_n| \le k|x_n-x_{n-1}| \le k^2|x_{n-1}-x_{n-2}| \le \cdots \le k^{n-1}|x_2-x_1|$$

So for $n, n+p$ ($p>0$).

$$\begin{aligned}|x_{n+p}-x_n| &= |x_{n+p}-x_{n+p-1}+x_{n+p-1}\cdots+x_{n+1}-x_n| \\ &\le |x_{n+p}-x_{n+p-1}|+\cdots+|x_{n+1}-x_n| \\ &\le (k^{n+p-2}+\cdots+k^{n-1})|x_2-x_1| \\ &= k^{n-1}\frac{1-\underbrace{k^p}_{<1}}{1-k}|x_2-x_1| \\ &\le \frac{k^{n-1}}{1-k}|x_2-x_1|\end{aligned}$$

Hence. for each $\varepsilon>0$. set $N=\left[\frac{\ln\varepsilon'}{\ln k}\right]+1$, where $\varepsilon'=\frac{1-k}{|x_2-x_1|}\varepsilon$ (we assume $x_2\neq x_1$. otherwise. $x_1=x_2=\cdots=x_n=\cdots$, trivial). then if $n>N$. we have

$$|x_{n+p}-x_n| \le \frac{k^{n-1}}{1-k}|x_2-x_1| \le \frac{\varepsilon'}{1-k}|x_2-x_1| = \varepsilon.$$

□

Exercise 4. Let $f:(0,+\infty)\to\mathbb{R}$ be $f(x)=\frac{x}{x+1}$. Check that $\lim\limits_{x\to 1} f(x)=\frac{1}{2}$ by definition.

$$\left|\frac{x}{x+1}-\frac{1}{2}\right| = \left|\frac{x-1}{2x+2}\right|$$

$$< \frac{1}{3}|x-1| < \varepsilon$$

Proof: We need to check that $\forall \varepsilon>0$, $\exists\delta>0$ s.t.

$|f(x)-\frac{1}{2}|<\varepsilon$ holds for all $x\neq 1$ s.t. $|x-1|<\delta$.

Now notice that

$$\left|\frac{x}{x+1}-\frac{1}{2}\right| = \left|\frac{x-1}{2x+2}\right|.$$

For every $0<c<1$, $|x-1|<c$, $x\neq 1$, we have

$1-c<x<1$ or $1<x<1+c$, thus

$$\left|\frac{1}{2x+2}\right| < \frac{1}{2(1+c)} \text{ and}$$

$$\left|\frac{x-1}{2x+2}\right| < \frac{1}{2(1+c)}|x-1|.$$

Now selecting $\delta=\min\{\frac{1}{2},3\varepsilon\}$.

we have, $\forall x$ s.t. $|x-1|<\delta$, $x\neq 1$.

$$\left|\frac{x}{x+1}-\frac{1}{2}\right| = \left|\frac{x-1}{2x+2}\right| \underset{\delta\le\frac{1}{2}}{<} \frac{1}{3}|x-1| \underset{\delta\le 3\varepsilon}{<} \varepsilon.$$

#.

Exercise 5.

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 8x & \text{if } x \text{ is rational,} \\ 2x^2 + 8 & \text{if } x \text{ is irrational.} \end{cases}$$

For which x_0, does $\lim_{x \to x_0} f(x)$ exist? (*Hint*: Sequential limit theorem.)

Suppose $\lim_{x \to x_0} f(x)$ exists at x_0. we have. $x_n \to x_0$ (rational), $y_n \to x_0$ (irrational).

sequential limit / sequential limit.

$\therefore$ $f(x_n) \to f(x_0) \leftarrow f(y_n)$

$f(x_n) = 8x_n \to 8x_0 = 2x_0^2 + 8 \leftarrow 2y_n^2 + 8$

$\Downarrow$

$x_0 = 2$

$\Downarrow$

a necessary condition

prove $\lim_{x \to x_0 = 2} f(x) = f(x_0)$

(Since we can't count every sequence $\{x_n\} \to x_0$. so we directly use the definition)

$|f(x) - f(x_0)| \leq |8x - 16| + |2x^2 - 8|$

(x rational or irrational)

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Proof: By sequential limit theorem, a necessary condition for $\lim_{x \to x_0} f(x)$ exists is that for any sequence $(x_n)_{n=1}^{+\infty}$ s.t. $\lim_n x_n = x_0$, $x_n \neq x_0$, $x_n \in \mathbb{Q}$ and any sequence $(y_n)_{n=1}^{+\infty}$ s.t. $y_n \neq x_0$, $\lim_n y_n = x_0$, $y_n \neq y_0$, $y_n \notin \mathbb{Q}$, we have

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(y_n)$$

Only a necessary condition.

Now by for any such (x_n) and (y_n), we have

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} 8x_n = 8x_0,$$

$$\lim_{n \to \infty} f(y_n) = \lim_{n \to \infty} 2y_n^2 + 8 = 2x_0^2 + 8.$$

we know that $\lim_{x \to x_0} f(x)$ exists only possibly for x_0 s.t. $8x_0 = 2x_0^2 + 8$

$\Updownarrow$

$x_0 = 2$

$x_0^2 - 4x_0 + 4$

$x_0 = 2$

Now, we need to use ε-δ language to check whether $\lim\limits_{x\to 2} f(x)$ exists:

Firstly, we have

$$|f(x)-f(2)| \le \max\{|8x-16|,\ |2x^2+8-16|\}$$

$$= \max\{8|x-2|,\ 2|x+2|\cdot|x-2|\}.$$

So $\forall \varepsilon>0$, if we pick $\delta = \min\{\frac{\varepsilon}{8}, 1\}$, then

$\forall x$ s.t. $|x-2|<\delta$, $x\neq 2$, we have $|x+2| \le 3$ by $\delta \le 1$

$$|f(x)-f(2)| < \max\{8\cdot\underbrace{\delta}_{\delta\le\frac{\varepsilon}{8}},\ 2\underbrace{|x+2|}_{\le 3}\,\underbrace{\delta}_{\le\frac{\varepsilon}{8}}\}.$$

$$\le \max\{\varepsilon,\ \frac{6}{8}\varepsilon\}$$

$$= \varepsilon.$$

That leads to $\lim\limits_{x\to 2} f(x)$ exists.

So $\lim\limits_{x\to x_0} f(x)$ exists, and $\lim\limits_{x\to x_0} f(x) = f(2) = 16$. #.