Solution of Homework 2

① (Sketch: $x_1=1,\quad x_2=\frac13\sqrt{5+11}=\frac43,\quad x_3=\frac13\sqrt{5(\frac43)^2+11(\frac43)}=\frac13\sqrt{\frac{212}{9}}=\frac{14.\cdots}{9}$

$x_1=1 \quad x_2=4/3 \quad x_3=\frac{14\ldots}{9}$ $\quad x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\frac13\sqrt{5x_n^2+11x_n}=\frac13\sqrt{5x^2+11x}\Rightarrow 9x^2=5x^2+11x\Rightarrow x=\overset{\text{reject}}{0}$ or $11/4$)

Solution We will prove $0<x_n\le x_{n+1}\le\frac{11}{4}$. For $n=1$, $0<x_1=1<x_2=\frac43<\frac{11}{4}$.

Suppose $0<x_n\le x_{n+1}\le\frac{11}{4}$. Then $0<x_n^2\le x_{n+1}^2\le(\frac{11}{4})^2$. Hence

$$0\le x_{n+1}=\frac13\sqrt{5x_n^2+11x_n}<\frac13\sqrt{5x_{n+1}^2+11x_{n+1}}\le\frac13\sqrt{5(\tfrac{11}{4})^2+11(\tfrac{11}{4})}=\frac13\sqrt{\frac{9\times11^2}{4^2}}=\frac{11}{4}.$$

By math induction, we proved $0<x_n\le x_{n+1}\le\frac{11}{4}$.

By the monotone sequence theorem, $x=\lim_{n\to\infty}x_n$ exists. By subsequence theorem,

$$x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\frac13\sqrt{5x_n^2+11x_n}=\frac13\sqrt{5x^2+11x}\Rightarrow(3x)^2=5x^2+11x$$

$\Rightarrow x=0$ or $11/4$. Since $x_n\ge x_1=1>0$, so $x=11/4$.

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② (Sketch: $x_1=1,\quad x_2=\frac74+\frac12=\frac94=2\frac14,\quad x_3=\frac74+\frac29=\frac{71}{36}=1\frac{35}{36},\quad x_4=\frac74+\frac{18}{71}=\frac{569}{284}=2\frac{1}{284}$

$x_1=1\quad x_3\quad x_4\quad x_2=2\frac14$ $\quad x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}(\frac74+\frac1{2x_n})=\frac74+\frac1{2x}\Rightarrow \underbrace{4x^2-7x-2}_{(4x+1)(x-2)}=0\Rightarrow \underline{x=2}$ or $-\frac14$ reject ($\notin[1,2\frac14]$))

Solution Let $I_n=[x_{2n-1},x_{2n}]$. We claim $I_n\supseteq I_{n+1}$ ($\Leftrightarrow x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$) for $n\in\mathbb{N}$.

For $n=1$, $x_1=1\le x_3=1\frac{35}{36}\le x_4=2\frac{1}{284}\le x_2=2\frac14$. Next suppose $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$.

Then $\frac{1}{2x_{2n-1}}\ge\frac{1}{2x_{2n+1}}\ge\frac{1}{2x_{2n+2}}\ge\frac{1}{2x_{2n}}$ and $\underbrace{\frac74+\frac{1}{2x_{2n-1}}}_{=x_{2n}}\ge\underbrace{\frac74+\frac{1}{2x_{2n+1}}}_{=x_{2n+2}}\ge\underbrace{\frac74+\frac{1}{2x_{2n+2}}}_{=x_{2n+3}}\ge\underbrace{\frac74+\frac{1}{2x_{2n}}}_{=x_{2n+1}}$

Also $\frac{1}{2x_{2n}}\le\frac{1}{2x_{2n+2}}\le\frac{1}{2x_{2n+3}}\le\frac{1}{2x_{2n+1}}$ and $\underbrace{\frac74+\frac{1}{2x_{2n}}}_{=x_{2n+1}}\le\underbrace{\frac74+\frac{1}{2x_{2n+2}}}_{=x_{2n+3}}<\underbrace{\frac74+\frac{1}{2x_{2n+3}}}_{=x_{2n+4}}\le\underbrace{\frac74+\frac{1}{2x_{2n+1}}}_{=x_{2n+2}}$

This completes the induction for the claim.

By the nested interval theorem, $\lim_{n\to\infty}x_{2n-1}=a$ and $\lim_{n\to\infty}x_{2n}=b$ exist. By the subsequence theorem, $a=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}(\frac74+\frac{1}{2x_{2n}})=\frac74+\frac1{2b}$ and $b=\lim_{n\to\infty}x_{2n}=\lim_{n\to\infty}(\frac74+\frac{1}{2x_{2n-1}})=\frac74+\frac1{2a}$.

Then $7b+2=4ab=7a+2\Rightarrow 7b=7a\Rightarrow a=b$. By intertwining sequence theorem, $\lim_{n\to\infty}x_n=a$. Then $a=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}(\frac74+\frac1{2x_n})=\frac74+\frac1{2a}\Rightarrow \underbrace{4a^2-7a-2}_{(4a+1)(a-2)}=0\Rightarrow a=2$ as $-\frac14\notin I_1$.

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③ (Sketch: $\frac{n+\sqrt n}{4n-3}\to\frac14,\quad\frac{7n^2}{4n^2+1}\to\frac74$

$$\left|\frac{n+\sqrt n}{4n-3}+\frac{7n^2}{4n^2+1}-2\right|=\left|\left(\frac{n+\sqrt n}{4n-3}-\frac14\right)+\left(\frac{7n^2}{4n^2+1}-\frac74\right)\right|\le\left|\frac{n+\sqrt n}{4n-3}-\frac14\right|+\left|\frac{7n^2}{4n^2+1}-\frac74\right|$$

$$=\frac{4\sqrt n+3}{4(4n-3)}+\frac{7}{4(4n^2+1)}\le\frac{7\sqrt n}{4n}+\frac{7}{16n^2}<\frac\varepsilon2+\frac\varepsilon2=\varepsilon$$

$3\le3\sqrt n$, $\frac{1}{4n-3}\le\frac{1}{4n-3n}$ $\qquad \frac{1}{4n^2+1}\le\frac{1}{4n^2}$

if $\frac{7}{4\sqrt n}<\frac\varepsilon2$ ($\Leftrightarrow n>(\frac{7}{2\varepsilon})^2$) and $\frac{7}{16n^2}<\frac\varepsilon2$ ($\Leftrightarrow n>\sqrt{\frac{7}{8\varepsilon}}$))

Solution For every $\varepsilon>0$, by the Archimedean principle, $\exists K\in\mathbb{N}$ such that $K>\max\{(\frac{7}{2\varepsilon})^2,\sqrt{\frac{7}{8\varepsilon}}\}$. Then $n\ge K\Rightarrow n>(\frac{7}{2\varepsilon})^2$ and $n>\sqrt{\frac{7}{8\varepsilon}}$

$\Rightarrow\left|\frac{n+\sqrt n}{4n-3}+\frac{7n^2}{4n^2+1}-2\right|<\varepsilon$ as shown in the box above.

④ (Sketch: $w_n \to 3 \Rightarrow \sqrt{\frac{w_n}{2w_n+6}} \to \sqrt{\frac{3}{12}} = \frac{1}{2}$, $\frac{2w_n+n}{2n+w_n} \approx \frac{6+n}{2n+3} \to \frac{1}{2}$ $= \sqrt{\frac{1}{4}}$

$$\left|\sqrt{\frac{w_n}{2w_n+6}} + \frac{2w_n+n}{2n+w_n} - 1\right| = \left|\left(\sqrt{\frac{w_n}{2w_n+6}} - \frac{1}{2}\right) + \left(\frac{2w_n+n}{2n+w_n} - \frac{1}{2}\right)\right| \le \left|\sqrt{\frac{w_n}{2w_n+6}} - \frac{1}{2}\right| + \left|\frac{2w_n+n}{2n+w_n} - \frac{1}{2}\right|$$

$$\underset{|\sqrt{a}-\sqrt{b}| \le \sqrt{|a-b|}}{\le} \sqrt{\left|\frac{w_n}{2w_n+6} - \frac{1}{4}\right|} + \frac{3w_n}{2(2n+w_n)} \underset{2 \le w_n \le 4,\ \frac{1}{2n+w_n} \le \frac{1}{2n}}{\le} \sqrt{\frac{2|w_n-3|}{4(2w_n+6)}} + \frac{12}{4n} \underset{\frac{1}{2w_n+6} \le \frac{1}{6}}{\le} \sqrt{\frac{|w_n-3|}{12}} + \frac{3}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

if $|w_n - 3| < 3\varepsilon^2$ and $n > 3/\varepsilon$.

Solution For every $\varepsilon > 0$, Since $w_n \to 3$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |w_n - 3| < 3\varepsilon^2$. By Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \max\{K_1, 3\varepsilon^2\}$. Then $n \ge K \Rightarrow n \ge K_1$ and $n > 3\varepsilon^2 \Rightarrow \left|\sqrt{\frac{w_n}{2w_n+6}} + \frac{2w_n+n}{2n+w_n} - 1\right| < \varepsilon$ as shown in the box above.

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⑤ Let $a_n = x_{2n-1}$ and $b_n = x_{2n}$. Then $a_1 = 2$ and $a_{n+1} = \sqrt{10a_n - 9}$. Also, $b_1 = 4$ and $b_{n+1} = \sqrt{10a_n - 9}$.

Claim: $1 < a_n < a_{n+1} < 9$ and $1 < b_n < b_{n+1} < 9$. Check by math induction.

Proof $1 < a_1 = 2 < a_2 = \sqrt{11} < 9$. If $1 < a_n < a_{n+1} < 9$, then $1 < 10a_n - 9 < 10a_{n+1} - 9 < 81$, so $1 < a_{n+1} = \sqrt{10a_n - 9} < a_{n+2} = \sqrt{10a_{n+1} - 9} < 9$.

Similarly, $1 < b_1 = 4 < b_2 = \sqrt{31} < 9$. If $1 < b_n < b_{n+1} < 9$, then $1 < 10b_n - 9 < 10b_{n+1} - 9 < 81$, so $1 < b_{n+1} < b_{n+2} < 9$.

By the monotone sequence theorem, $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ exist. Then $a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{10a_n - 9} = \sqrt{10a - 9} \Rightarrow a^2 = 10a - 9 \Rightarrow a = 1$ or 9 (reject 1 since a_n increases).

So $a = 9$. Similarly, $b = 9$. By the intertwining sequence theorem, x_n converges to 9.