Supremum and Infimum

Supremum Limit Theorem (p.53)

Let S be a nonempty subset of $\mathbb{R}$ and let c be an upper bound of S. Then

$$c = \sup S \iff \exists\, s_n \in S \text{ such that } \lim_{n\to\infty} s_n = c$$

Infimum Limit Theorem (p.53)

Let S be a nonempty subset of $\mathbb{R}$ and let d be a lower bound of S. Then

$$d = \inf S \iff \exists\, t_n \in S \text{ such that } \lim_{n\to\infty} t_n = d$$

x irrational or rational

$$x = d_1 \cdots d_k . a_1 a_2 a_3 \cdots$$

$$10^n x = d_1 \cdots d_k a_1 \cdots a_n . a_{n+1} a_{n+2} \cdots$$

$$[10^n x] = d_1 \cdots d_k a_1 \cdots a_n$$

$$\frac{[10^n x]}{10^n} = d_1 \cdots d_k . a_1 \cdots a_n$$

$\in \mathbb{Q}$

Let $n \to \infty$, $\lim_{n\to\infty} \frac{[10^n x]}{10^n} = x$.

$$0 < \frac{1}{\sqrt[n+1]{2}} < 1$$

$\in \mathbb{R} - \mathbb{Q}$

$\parallel$

$$2^{-\frac{1}{n+1}} \to 2^{-\frac{1}{\infty}} = 2^0 = 1$$

2007 Fall Midterm

Let c be a positive rational number. Determine (with proof) the supremum and infimum of

$$B = \{x+y \,:\, x \in [0, c\sqrt{2}]\cap\mathbb{Q},\ y \in [0,c]\setminus\mathbb{Q}\}$$

Solution For $x \in [0,c\sqrt{2}]\cap\mathbb{Q}$, $y \in [0,c]\setminus\mathbb{Q}$, we have $0 \le x+y \le c\sqrt{2}+c$.

Let $x_n = 0$, $y_n = \frac{1}{n\sqrt{2}}c$. Then $x_n + y_n \in B$ and $\lim\limits_{n\to\infty}(x_n+y_n) = 0$. By infimum limit theorem, we get $\inf B = 0$.

Let $x_n' = \frac{[10^n c\sqrt{2}]}{10^n}$, $y_n' = \frac{1}{\sqrt[n+1]{2}}c = 2^{-1/(n+1)}c$.
Then $x_n' + y_n' \in B$ and $\lim\limits_{n\to\infty}(x_n'+y_n') = c\sqrt{2}+c$.
By supremum limit theorem, we get $\sup B = c\sqrt{2}+c$.

2009 Fall Midterm

Let D be a nonempty bounded subset of $\mathbb{R}$ such that $\inf D = 3$ and $\sup D = 5$. Let

$$A = \{xy + xy^3 \,:\, x \in (2,\pi]\cap\mathbb{Q},\ y \in D\}.$$

Show that A is bounded. Determine (with proof) the infimum and supremum of A.

Solution $\inf D = 3$ and $\sup D = 5 \Rightarrow D \subseteq [3,5]$.

$$\begin{array}{l} 2 < x \le \pi \\ 3 \le y \le 5 \end{array} \Rightarrow 60 = 2(3+3^3) < xy+xy^3 = x(y+y^3) \le \pi(5+5^3) = 130\pi.$$

So A is bounded below by 60 and above by 130π.

Let $x_n = 2+\frac{1}{n} \in (2,\pi]\cap\mathbb{Q}$. By infimum limit theorem, $\inf D = 3 \Rightarrow \exists\, y_n \in D$ such that $\lim\limits_{n\to\infty} y_n = 3$.
Then $x_n y_n + x_n y_n^3 \in A$ and $\lim\limits_{n\to\infty} x_n y_n + x_n y_n^3 = 2\times 3 + 2\times 3^3 = 60$.
By infimum limit theorem, $\inf A = 60$.

Let $x_n' = \frac{[10^n\pi]}{10^n} \in (2,\pi]\cap\mathbb{Q}$. By supremum limit theorem, $\sup D = 5 \Rightarrow \exists\, y_n' \in D$ such that $\lim\limits_{n\to\infty} y_n' = 5$.
Then $x_n' y_n' + x_n' {y_n'}^3 \in A$ and $\lim x_n' y_n' + x_n' {y_n'}^3 = \pi\times 5 + \pi\times 5^3 = 130\pi$.
By supremum limit theorem, $\sup A = 130\pi$.

2011 Midterm Problem 2

A and B are nonempty bounded subsets of $\mathbb{R}$ such that $\inf A = 1$, $\sup A = 5$, $\inf B = 0$, $\sup B = 1$.

Let $C = \left\{ \frac{y}{3-x} - \frac{1}{y} : x \in B, y \in A \right\}$.

Prove C is bounded. Determine $\inf C$ and $\sup C$.

Solution $\inf A = 1, \sup A = 5 \Rightarrow \forall y \in A,\ 1 \le y \le 5$

$\Rightarrow -1 \le -\frac{1}{y} \le -\frac{1}{5}$.

$\inf B = 0, \sup B = 1 \Rightarrow \forall x \in B,\ 0 \le x \le 1$

$\Rightarrow 2 \le 3 - x \le 3$

$\Rightarrow \frac{1}{3} \le \frac{1}{3-x} \le \frac{1}{2}$

$$-\frac{2}{3} = 1\left(\tfrac{1}{3}\right) - 1 \le \frac{y}{3-x} - \frac{1}{y} \le 5\left(\tfrac{1}{2}\right) - \frac{1}{5} = \frac{23}{10}$$ $\therefore C$ is bounded

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By infimum limit theorem and supremum limit theorem,

$\inf A = 1 \Rightarrow \exists y_n \in A$ with $\lim_{n\to\infty} y_n = 1$

$\sup A = 5 \Rightarrow \exists y_n' \in A$ with $\lim_{n\to\infty} y_n' = 5$.

$\inf B = 0 \Rightarrow \exists x_n \in B$ with $\lim_{n\to\infty} x_n = 0$

$\sup B = 1 \Rightarrow \exists x_n' \in B$ with $\lim_{n\to\infty} x_n' = 1$

Then $\frac{y_n}{3-x_n} - \frac{1}{y_n} \in C$, $\lim_{n\to\infty} \frac{y_n}{3-x_n} - \frac{1}{y_n} = \frac{1}{3-0} - \frac{1}{1} = -\frac{2}{3}$

$\frac{y_n'}{3-x_n'} - \frac{1}{y_n'} \in C$, $\lim_{n\to\infty} \frac{y_n'}{3-x_n'} - \frac{1}{y_n'} = \frac{5}{3-1} - \frac{1}{5} = \frac{23}{10}$.

$\therefore \inf C = -2/3$ and $\sup C = 23/10$.

2008 Fall Final

Let $A_1, A_2, A_3, \ldots$ be subsets of $[0,1]$ such that $\bigcap_{n=1}^{\infty} A_n$ is nonempty. If

$$\sup\{\inf A_n : n=1,2,3,\ldots\} = \inf\{\sup A_n : n=1,2,3,\ldots\},$$

then prove that $\bigcap_{n=1}^{\infty} A_n$ has exactly one element.

Solution 1

Let $\sup\{\inf A_n : n=1,2,3,\cdots\} = \inf\{\sup A_n : n=1,2,3,\cdots\} = a$.

By supremum limit theorem, $\exists \inf A_{n_k} \to a$ as $k\to\infty$.

By infimum limit theorem, $\exists \sup A_{m_j} \to a$ as $j \to \infty$.

$x \in \bigcap_{n=1}^{\infty} A_n \Rightarrow \forall k,\ x \in A_{n_k} \Rightarrow x \ge \inf A_{n_k} \Rightarrow x \ge a$ (let $k\to\infty$)

$\Rightarrow \forall j,\ x \in A_{m_j} \Rightarrow x \le \sup A_{m_j} \Rightarrow x \le a$ (let $j \to \infty$)

$\therefore x = a$. $\therefore \bigcap_{n=1}^{\infty} A_n = \{a\}$.

Solution 2

Let $\sup\{\inf A_n : n=1,2,3,\ldots\} = \inf\{\sup A_n : n=1,2,3,\ldots\} = a$.

$x \in \bigcap_{n=1}^{\infty} A_n \Rightarrow \forall n=1,2,3,\cdots,\ x\in A_n \Rightarrow \forall n=1,2,3,\cdots,\ \inf A_n \le x \le \sup A_n$

$\Rightarrow x$ is an upper bound of $\{\inf A_n : n=1,2,3,\ldots\}$

x is a lower bound of $\{\sup A_n : n=1,2,3,\ldots\}$

$\Rightarrow x \ge \sup\{\inf A_n : n=1,2,3,\ldots\} = a$

$x \le \inf\{\sup A_n : n=1,2,3,\ldots\} = a \Rightarrow x = a$. $\therefore \bigcap_{n=1}^{\infty} A_n = \{a\}$.

Fall 2006 Problem 2

Let $(0,\frac{1}{2})\cap\mathbb{Q}\subseteq A_1\subseteq[0,1)$. For $n=1,2,3,\cdots$,

let $A_{n+1}=\{\sqrt{x}: x\in A_n\}$.

Determine the supremum and infimum of $\bigcup_{k=1}^{\infty}A_k$.

Solution $A_1\subseteq[0,1)$. If $A_n\subseteq[0,1)$, then

$A_{n+1}=\{\sqrt{x}: x\in A_n\}\subseteq[\sqrt{0},\sqrt{1})=[0,1)$.

$\therefore A_k\subseteq[0,1)$ for $k=1,2,3,\cdots$. $\therefore \bigcup_{k=1}^{\infty}A_k\subseteq[0,1)$.

So 0 is a lower bound of $\bigcup_{k=1}^{\infty}A_k$ and 1 is an upper bound of $\bigcup_{k=1}^{\infty}A_k$.

Since $(0,\frac{1}{2})\cap\mathbb{Q}\subseteq A_1\subseteq\bigcup_{k=1}^{\infty}A_k$, we have $\frac{1}{3},\frac{1}{4},\frac{1}{5},\cdots\in\bigcup_{k=1}^{\infty}A_k$ and $\frac{1}{3},\frac{1}{4},\frac{1}{5},\cdots\to 0$.

Next $\frac{1}{3}\in A_1$, $\frac{1}{\sqrt{3}}\in A_2$, $\frac{1}{\sqrt[4]{3}}\in A_3,\cdots$

So $\frac{1}{3},\frac{1}{\sqrt{3}},\frac{1}{\sqrt[4]{3}},\cdots\in\bigcup_{k=1}^{\infty}A_k$

and $\lim_{n\to\infty}\frac{1}{\sqrt[2^n]{3}}=\lim_{n\to\infty}3^{-1/2^n}=3^0=1$

$\therefore \inf\bigcup_{k=1}^{\infty}A_k=0$, $\sup\bigcup_{k=1}^{\infty}A_k=1$ by infimum limit theorem and supremum limit theorem respectively

Relevant Theorems for Sequences defined by Recurrence Relations

Monotone Sequence theorem (p.54)

If $\{x_n\}$ is increasing and bounded above, then $\lim_{n\to\infty}x_n=\sup\{x_1,x_2,x_3,\ldots\}$.

If $\{x_n\}$ is decreasing and bounded below, then $\lim_{n\to\infty}x_n=\inf\{x_1,x_2,x_3,\cdots\}$

Subsequence theorem (p.54)

If $\lim_{n\to\infty}x_n=x$, then $\forall$ subsequence $\{x_{n_j}\}$ ($n_1<n_2<n_3<\cdots$), $\lim_{j\to\infty}x_{n_j}=x$.

Intertwining Sequence theorem (p.55)

If $x_1,x_3,x_5,x_7,\cdots\to x$ and $x_2,x_4,x_6,x_8,\cdots\to x$, then $\lim_{n\to\infty}x_n=x$.

Nested Interval Theorem (p.55)

If $I_n=[a_n,b_n]$ and $I_n\supseteq I_{n+1}$ for $n=1,2,3,\cdots$ then $\bigcap_{n=1}^{\infty}I_n=[a,b]$, where $a=\lim_{n\to\infty}a_n$ and $b=\lim_{n\to\infty}b_n$.

If $\lim_{n\to\infty}(b_n-a_n)=0$, then $\bigcap_{n=1}^{\infty}I_n=\{x\}$

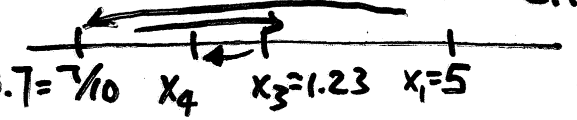
2010 Fall Midterm

Prove the sequence $\{x_n\}$ converges, where

$$x_1=5 \quad\text{and}\quad x_{n+1}=\frac{7}{x_n+5},$$

and find its limit. Show work!

Solution. (Scratch Work: $x_1=5$, $x_2=\frac{7}{10}$, $x_3=\frac{7}{5.7}\approx 1.23\ldots$, $x_4=\frac{7}{6.23\ldots}\approx 1.12$



)

Define $I_n=[x_{2n},x_{2n-1}]$. Claim: $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$

For this, we will prove $0.7\le x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}\le 5$ for all $n=1,2,3,\ldots$

Case $n=1$ is done above. Assume $x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}$.

Then $x_{2n}+5\le x_{2n+2}+5\le x_{2n+1}+5\le x_{2n-1}+5$

$$\Rightarrow x_{2n+1}=\frac{7}{x_{2n}+5}\ge x_{2n+3}=\frac{7}{x_{2n+2}+5}\ge x_{2n+2}=\frac{7}{x_{2n+1}+5}\ge x_{2n}=\frac{7}{x_{2n-1}+5}$$

$$\Rightarrow x_{2n+1}+5\ge x_{2n+3}+5\ge x_{2n+2}+5\ge x_{2n}+5$$

$$\Rightarrow 0.7\le x_{2n}\le x_{2n+2}=\frac{7}{x_{2n+1}+5}\le x_{2n+4}=\frac{7}{x_{2n+3}+5}\le x_{2n+3}=\frac{7}{x_{2n+2}+5}\le x_{2n+1}=\frac{7}{x_{2n}+5}\le x_{2n-1}\le 5$$

By M.I., we proved the claim.

By the nested interval theorem, $\lim\limits_{n\to\infty}x_{2n}=a$ and $\lim\limits_{n\to\infty}x_{2n-1}=b$.

We have $a=\lim\limits_{n\to\infty}x_{2n}=\lim\limits_{n\to\infty}\frac{7}{x_{2n-1}+5}=\frac{7}{b+5}\Rightarrow ab+5a=7$

and $b=\lim\limits_{n\to\infty}x_{2n+1}=\lim\limits_{n\to\infty}\frac{7}{x_{2n}+5}=\frac{7}{a+5}\Rightarrow ab+5b=7$

$\therefore a=b$. By intertwining sequence theorem, $\lim\limits_{n\to\infty}x_n=a$

Then $a=\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}\frac{7}{x_n+5}=\frac{7}{a+5}\Rightarrow a^2+5a-7=0$

$\Rightarrow a=\frac{-5\pm\sqrt{53}}{2}$. Since $a\in I_1=[0.7,5]$, $a=\frac{-5+\sqrt{53}}{2}$.

2011 Midterm Problem 1

Prove $\{x_n\}$ converges, where

$$x_1=27 \quad\text{and}\quad x_{n+1}=8-\sqrt{28-x_n}, \quad n=1,2,3,\ldots$$

Find its limit.

Solution (Scratch Work: Note $x_1=27>x_2=7>x_3=8-\sqrt{21}\approx\underbrace{8-4.5\ldots}_{=3.5\ldots}$

$x=8-\sqrt{28-x}\Rightarrow(x-8)^2=28-x\Rightarrow x^2-15x+36=0$, $(x-12)(x-3)$, $x=3$)

Claim: $27=x_1\ge x_n>x_{n+1}>3$.

For $n=1$, $27=x_1>x_2=7>3$. Suppose $27\ge x_n>x_{n+1}>3$.

Then $1=28-27\le 28-x_n<28-x_{n+1}<28-3=25$.

So $1\le\sqrt{28-x_n}<\sqrt{28-x_{n+1}}<\sqrt{25}=5$.

$$\therefore 27\ge 7=8-1\ge \underbrace{8-\sqrt{28-x_n}}_{=x_{n+1}}>\underbrace{8-\sqrt{28-x_{n+1}}}_{=x_{n+2}}>\underbrace{8-5}_{=3}$$

By M.I., we are done.

By monotone sequence theorem, $\{x_n\}$ converges, say to x. Then by subsequence theorem, $x_{n+1}\to x$. $\therefore$

$$x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}(8-\sqrt{28-x_n})=8-\sqrt{28-x}$$

$$(x-8)^2=28-x\iff x^2-15x+36=0=(x-12)(x-3)$$

Since $12>7=x_2>x_3>\cdots$, $x\ne 12$. $\therefore x=3$.

2011 Fall Final

Let $x_1=0$, $x_2=3$ and $x_{n+2}=\sqrt{\frac{4}{9}x_{n+1}^2+\frac{5}{9}x_n^2}$ for $n=1,2,3,\dots$. Prove $x_1,x_2,x_3,\dots$ converges and find its limit.

Solution (Scratch work: $x_3=2<x_4=\sqrt{6\frac{7}{9}}<3=x_2$
$x_1<x_3<x_4<x_2$)

Claim: $0\le x_{2n-1}<x_{2n+1}<x_{2n+2}<x_{2n}$ for $n=1,2,3,\dots$.

For $n=1$, $x_1=0<x_3=2<x_4=\sqrt{6\frac{7}{9}}<x_2=3$. Suppose we have $0\le x_{2n-1}<x_{2n+1}<x_{2n+2}<x_{2n}$. Then

$$x_{2n+1}=\sqrt{\frac{4}{9}x_{2n+1}^2+\frac{5}{9}x_{2n+1}^2}<\sqrt{\frac{4}{9}x_{2n+2}^2+\frac{5}{9}x_{2n+1}^2}=x_{2n+3}$$

$0\le x_{2n-1}\le$ used $x_{2n+1}<x_{2n+2}$ $\rightarrow$ $<\sqrt{\frac{4}{9}x_{2n+2}^2+\frac{5}{9}x_{2n+2}^2}=x_{2n+2}$

$$0\le x_{2n+3}=\sqrt{\frac{4}{9}x_{2n+3}^2+\frac{5}{9}x_{2n+3}^2}<\sqrt{\frac{4}{9}x_{2n+3}^2+\frac{5}{9}x_{2n+2}^2}=x_{2n+4}$$

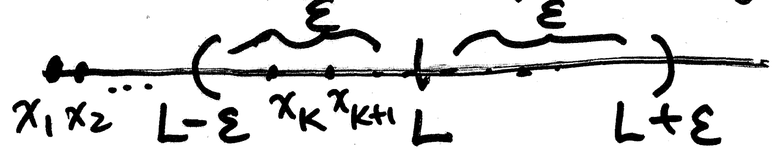
used $x_{2n+3}<x_{2n+2}$ $\rightarrow$ $<\sqrt{\frac{4}{9}x_{2n+2}^2+\frac{5}{9}x_{2n+2}^2}=x_{2n+2}$

So $x_{2n+1}<x_{2n+3}<x_{2n+4}<x_{2n+2}$. By M.I., the claim is proved.

By nested interval theorem, $\lim\limits_{n\to\infty}x_{2n-1}=a\ge0$, $\lim\limits_{n\to\infty}x_{2n}=b\ge0$.
Then $a=\lim\limits_{n\to\infty}x_{2n+1}=\lim\limits_{n\to\infty}\sqrt{\frac{4}{9}x_{2n}^2+\frac{5}{9}x_{2n-1}^2}=\sqrt{\frac{4}{9}b^2+\frac{5}{9}a^2}$.
So $a^2=\frac{4}{9}b^2+\frac{5}{9}a^2$. Hence $a^2=b^2$. $\therefore a=b$.
By intertwining sequence theorem, $x_1,x_2,x_3,\dots$ converges.

Limit of Sequences

(P.48) **Definition** of $x_1,x_2,x_3,\dots$ **converges** to L



$\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that

$$x_K,x_{K+1},x_{K+2},\dots\in(L-\varepsilon,L+\varepsilon)$$

$$n\ge K\Rightarrow \underbrace{|x_n-L|}_{\text{distance between }x_n\text{ and }L}<\varepsilon$$

For different ε, K will change!

Find limit

$$x_3^2=\frac{4}{9}x_2^2+\frac{5}{9}x_1^2$$
$$x_4^2=\frac{4}{9}x_3^2+\frac{5}{9}x_2^2$$
$$x_5^2=\frac{4}{9}x_4^2+\frac{5}{9}x_3^2$$
$$\vdots$$
$$x_{n-1}^2=\frac{4}{9}x_{n-2}^2+\frac{5}{9}x_{n-3}^2$$
$$+\quad x_n^2=\frac{4}{9}x_{n-1}^2+\frac{5}{9}x_{n-2}^2$$

$$x_3^2+x_4^2+\dots+x_{n-1}^2+x_n^2=\frac{5}{9}x_1^2+x_2^2+x_3^2+\dots+x_{n-2}^2+\frac{4}{9}x_{n-1}^2$$

Adding and cancelling common terms, we get

$$\frac{5}{9}x_{n-1}^2+x_n^2=x_2^2+\frac{5}{9}x_1^2=9$$

As $n\to\infty$, we get $\frac{14}{9}a^2=9$.
$\therefore a=9/\sqrt{14}$.

② 2007 Fall Final

$$y_n = \frac{4n^2-\sqrt{n}}{2n^2+n} + \frac{n-1}{n}$$

Prove $\lim_{n\to\infty} y_n = 3$ by checking definition

Scratch

$\frac{4n^2-\sqrt{n}}{2n^2+n} \to 2 \quad \frac{n-1}{n} \to 1$

$\left|\frac{4n^2-\sqrt{n}}{2n^2+n} - 2\right| = \frac{2n+\sqrt{n}}{2n^2+n} \overset{\sqrt{n}\le n}{\le} \frac{3n}{2n^2} = \frac{3}{2}\frac{1}{n}$

$\left|\frac{n-1}{n} - 1\right| = \frac{1}{n} \qquad \frac{5}{2}\frac{1}{n} = \frac{3}{2}\frac{1}{n} + \frac{1}{n} < \varepsilon$

$\Rightarrow n > \frac{5}{2\varepsilon}$

Solution

$\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{5}{2}\frac{1}{\varepsilon}$.

Then $n \ge K \Rightarrow \left|\frac{4n^2-\sqrt{n}}{2n^2+n} + \frac{n-1}{n} - 3\right|$ $\quad 3 = 2+1$

$= \left|\left(\frac{4n^2-\sqrt{n}}{2n^2+n} - 2\right) + \left(\frac{n-1}{n} - 1\right)\right|$

$\le \left|\frac{4n^2-\sqrt{n}}{2n^2+n} - 2\right| + \left|\frac{n-1}{n} - 1\right|$

$= \frac{2n+\sqrt{n}}{2n^2+n} + \frac{1}{n} \le \frac{3n}{2n^2} + \frac{1}{n} = \frac{5}{2}\frac{1}{n}$

$\le \frac{5}{2}\frac{1}{K} < \varepsilon$.

2010 Final Problem 2

Let $a_1, a_2, a_3, \dots$ be real numbers that converge to 1. Prove that $\lim_{n\to\infty}\left(\frac{3+a_n^2}{a_n+1} + \frac{2n}{4+n}\right) = 4$ by checking the definition of limit of sequence.

Scratch work

$\frac{3+a_n^2}{a_n+1} \to \frac{3+1^2}{1+1} = 2, \quad \frac{2n}{4+n} \to 2$

$\left|\frac{3+a_n^2}{a_n+1} - 2\right| = \left|\frac{a_n^2 - 2a_n + 1}{a_n+1}\right| = \frac{|a_n-1|^2}{|a_n+1|} < \frac{|a_n-1|^2}{1} < \varepsilon/2$

when $|a_n - 1| < 1 \Rightarrow a_n \in (0,2)$
$\Rightarrow a_n + 1 \in (1,3)$

$\left|\frac{2n}{4+n} - 2\right| = \left|\frac{-8}{4+n}\right| = \frac{8}{4+n} < \frac{8}{n} < \frac{\varepsilon}{2} \Leftrightarrow n > 16/\varepsilon$.

Solution Since $\lim_{n\to\infty} a_n = 1$, for $1 > 0$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |a_n - 1| < 1 \Leftrightarrow a_n \in (0,2)$

$\forall \varepsilon > 0$, $\exists K_2 \in \mathbb{N}$ such that $n \ge K_2 \Rightarrow |a_n - 1| < \sqrt{\frac{\varepsilon}{2}}$.

Let $K > \max\{K_1, K_2, \frac{16}{\varepsilon}\}$. Then

$n \ge K \Rightarrow n \ge K_1$ and $n \ge K_2$ and $n > \frac{16}{\varepsilon}$

$\Rightarrow \left|\left(\frac{3+a_n^2}{a_n+1} + \frac{2n}{4+n}\right) - 4\right| = \left|\left(\frac{3+a_n^2}{a_n+1} - 2\right) + \left(\frac{2n}{4+n} - 2\right)\right|$

$\le \left|\frac{3+a_n^2}{a_n+1} - 2\right| + \left|\frac{2n}{4+n} - 2\right| = \frac{|a_n-1|^2}{|a_n+1|} + \frac{8}{4+n}$

$< \frac{|a_n-1|^2}{1} + \frac{8}{n} < \left(\sqrt{\frac{\varepsilon}{2}}\right)^2 + \frac{\varepsilon}{2} = \varepsilon.$

2011 Fall Final

Let $a_1, a_2, a_3, \ldots$ be a sequences of real numbers that converges to 3. Prove that

$$\lim_{n\to\infty}\left(\frac{a_n}{a_n^2+3}+\frac{3n^2}{1+4n^2}+\frac{a_n}{n}\right)=1$$

by checking the definition of limit of sequence.

Solution

Scratch work: $\frac{a_n}{a_n^2+3}\to\frac{1}{4}$, $\frac{3n^2}{1+4n^2}\to\frac{3}{4}$, $\frac{a_n}{n}\to 0$.

① $\left|\frac{a_n}{a_n^2+3}-\frac{1}{4}\right| = \frac{|a_n^2-4a_n+3|}{4a_n^2+12} \le \frac{|(a_n-1)(a_n-3)|}{12} \le \frac{3|a_n-3|}{12} < \frac{\varepsilon}{3}$

when $|a_n-3|<1 \Leftrightarrow a_n\in(2,4) \Leftrightarrow a_n-1\in(1,3)$

when $|a_n-3|<4\varepsilon/3$

② $\left|\frac{3n^2}{1+4n^2}-\frac{3}{4}\right| = \frac{3}{4+16n^2} < \frac{3}{16n^2} < \frac{\varepsilon}{3}$

when $n>\frac{3}{4\sqrt{\varepsilon}}$.

③ $\left|\frac{a_n}{n}-0\right| < \frac{4}{n} < \frac{\varepsilon}{3}$ when $n>12/\varepsilon$.

when $|a_n-3|<1 \Leftrightarrow a_n\in(2,4)$

$\forall\varepsilon>0$, since $\lim_{n\to\infty}a_n=3$,

for $1>0$, $\exists K_1\in\mathbb{N}$ such that $n\ge K_1 \Rightarrow |a_n-3|<1$

for $4\varepsilon/3>0$, $\exists K_2\in\mathbb{N}$ such that $n\ge K_2\Rightarrow |a_n-3|<\frac{4\varepsilon}{3}$.

By Archimedean principle, $\exists K\in\mathbb{N}$ such that

$$K>\max\left\{K_1, K_2, \frac{3}{4\sqrt{\varepsilon}}, \frac{12}{\varepsilon}\right\}.$$

$n\ge K \Rightarrow n>K_1,\ n>K_2,\ n>\frac{3}{4\sqrt{\varepsilon}},\ n>\frac{12}{\varepsilon}$

$\Rightarrow \left|\frac{a_n}{a_n^2+3}+\frac{3n^2}{1+4n^2}+\frac{a_n}{n}-1\right| = \left|\left(\frac{a_n}{a_n^2+3}-\frac{1}{4}\right)+\left(\frac{3n^2}{1+4n^2}-\frac{3}{4}\right)+\left(\frac{a_n}{n}-0\right)\right|$

$\le \left|\frac{a_n}{a_n^2+3}-\frac{1}{4}\right|+\left|\frac{3n^2}{1+4n^2}-\frac{3}{4}\right|+\left|\frac{a_n}{n}-0\right| \underset{\text{by ①, ②, ③ above}}{<} \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$