MATH2033 Mathematical Analysis (2021 Spring)
Assignment 4

Submission deadline of Assignment 4: <u>11:59p.m. of 5th May, 2020 (Wed)</u> Instruction: Please complete all required problems. Full details (including description of methods used and explanation, key formula and theorem used and final answer) must be shown **<u>clearly</u>** to receive full credits. Marks can be deducted for incomplete solution or unclear solution. <u>Please submit your completed work via the submission system in canvas</u> before the deadline. Late assignment will not be accepted. Your submission must (1) be hand-written (typed assignment will not be accepted), (2) in a single pdf. file (other file formats will not be accepted) and (3) contain your full name and student ID on the first page of the assignment.

Problem 1

We consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^n \sin \frac{1}{x} & if \;\; x \neq 0 \\ 0 & if \; x = 0 \end{cases},$$

where $n \in \mathbb{N}$.

(a) Find the values of n which $f(x)$ is differentiable at $x = 0$.

(b) Find the values of n which $f(x)$ is continuous differentiable at $x = 0$.

Problem 2

We let $f: (a, b) \to \mathbb{R}$ be a function and let $x_0 \in (a, b)$.

(a) If f is differentiable at $x = x_0$, show that

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} = f'(x_0) \dots \dots (*)$$

(b) If $\lim\limits_{h \to 0} \frac{f(x_0+h)-f(x_0-h)}{2h}$ exists, is it necessary that $f(x)$ is differentiable at $x = x_0$? Explain your answer.

(☺Hint: If your answer is yes, give a mathematical proof. If your answer is no, give a counter example).

Problem 3

We let $f: (0,1] \to \mathbb{R}$ be a differentiable function on $(0,1]$ such that $|f'(x)| < M$ for all $x \in (0,1]$, where $M > 0$ is a positive number. For any $n \in \mathbb{N}$, we define

$$a_n = f\left(\frac{1}{n}\right).$$

Show that the sequence $\{a_n\}$ converges.

(☺Hint: Be careful that $f(0)$ is not defined since the domain of f is $(0,1]$. On the other hand, you can prove the convergence without finding the limits.)

Problem 4

We let $f: [a, b] \to \mathbb{R}$ be n-times differentiable function which $f(x) = 0$ has $n + 1$ distinct roots over $[a, b]$. Show that there exists $c \in (a, b)$ such that $f^{(n)}(c) = 0$.

Problem 5

Show that for any $x > 0$,

$$1 - x + \frac{x^2}{2} > e^{-x} > 1 - x.$$

Problem 6 (Harder)

We let $f: [0,1] \to \mathbb{R}$ be a twice differentiable function on $[0,1]$ and $f''(x)$ is continuous on $[0,1]$. Suppose that

- $f(0) = f(1) = 0$ and
- $|f''(x)| \leq A$ for all $x \in [0,1]$, where $A > 0$ is a constant.

Show that $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{A}{4}$.

(☺Hint: Apply Taylor theorem with suitable choice of a.)