MATH2033 Mathematical Analysis

Lecture Note 1

Logic

In order to do proof in analysis, it is important to know some logic. This is essential for making a convincing argument and understanding the amount of message can be drawn from a statement/formula/theorem etc.

Quantifier

Most of statements in the analysis will specify the scenarios which a given property is valid. For example,

$$\text{“}\mathit{For\ every}\ x > 0,\ |x| = x\text{”},\ \ \text{“}x_{n+1} = 3x_n - 2\ \ \mathit{for\ all}\ n = 1,2,3,\ldots\text{”}.$$

Roughly speaking, quantifier specifies how many elements/objects that has certain property. In analysis, there are two types of quantifiers:

- ✓ Universal Quantifier (denoted by “∀”) – It represents *“for all”*, *“for every”*
  - ✓ For example, the statement “For every $x > 0$, $|x| = x$” can be expressed as “$\forall x > 0, |x| = x$”.
- ✓ Existential Quantifier (denoted by “∃”) – It represents *“for some”*, *“there exists”*, *“for at least one”*
  - ✓ For example, the statement “$f(x) > 0$ for some real number x” can be expressed as “$\exists x, f(x) > 0$”.

Negation

When we judge whether a given statement P (for example: $f(x) \geq 0$ for all $x > 0$) is true, we actually determine which of the following two statements hold:

1. $f(x) \geq 0$ for all $x > 0$ (statement P)
2. $f(x) < 0$ for some $x > 0$ (opposite of statement P)

Given a statement P, negation refer to *opposite of* the statement P and is noted by "$\sim P$". In logic, only <u>exactly one</u> of "P" and "$\sim P$" is true.

Example 1 (Examples of Negation)

1. We consider the statement P as "$f(x) > 0, \exists x \in \mathbb{R}$", then the negation of P will be "$f(x) \leq 0, \forall x \in \mathbb{R}$".

2. We consider the statement Q as "For every $x \geq 0$, there is $y \geq 0$ such that $y^2 = x$." Then the negation of Q will be
"There is $x \geq 0$ such that for any $y \geq 0$, $y^2 \neq x$."

Remarks about negation

- We let A, B, $C(x)$ be there statements. The following table summarizes the negations of some type of statements:

Statement P	Negation $(\sim P)$
$\forall \boldsymbol{x},\ \ \boldsymbol{C}(\boldsymbol{x})$	$\exists x,\ \ \sim C(x)$
$\exists \boldsymbol{x},\ \ \boldsymbol{C}(\boldsymbol{x})$	$\forall x,\ \ \sim C(x)$
$\boldsymbol{A}$ **or** $\boldsymbol{B}$	$(\sim A)\ \ and\ \ (\sim B)$
$\boldsymbol{A}$ **and** $\boldsymbol{B}$	$(\sim A)\ \ or\ \ (\sim B)$

- (Application of negation) In practice, negation is an important technique in proving/disproving a statement.
  - If we want to disprove certain statement P (i.e. statement P is false), then we do this by arguing that the statement $(\sim P)$ is true.
  - Suppose that we want to prove a statement P is true. Besides proving it directly, one can do this by considering its negation $(\sim P)$ and argue that it is *false* (this technique is known as “prove by contradiction”)

Example 2

We let $f(x) = \ln x$ for all $x > 0$. Determine if the statement "$\forall x > 0, f(x) > 0$" is correct?

Solution

By sketching the graph of $\ln x$, one would expect that the statement is not true. To argue this formally, we shall prove that its negation "$\exists x > 0, f(x) \leq 0$" is true. It suffices to find a $x > 0$ such that $f(x) = \ln x \leq 0$.

- We take $x = 0.5 > 0$, then $f(0.5) = \ln 0.5 = -0.69315 < 0$.

So the negation is correct and the statement is incorrect.

Remark of Example 2

The technique demonstrated in the example is a common technique to disprove the statement of the form $P =$"$\forall x, Q(x)$". The principal is to provide a specific example of x (known as *counter*-example) which $Q(x)$ is false.

Example 3 (Prove by contradiction)

(a) We let m be an integer such that m^2 is an even number. Show that m is an even number.

(b) Show that $\sqrt{2}$ is an irrational number.

😊Solution

(a) Suppose that m is odd number (negation is true), we have $m = 2p + 1$ for some integer p. Then

$$m^2 = (2p + 1)^2 = 4p^2 + 4p + 1,$$

which is an odd number. This contradicts to the fact that m^2 is even number. Thus, the negation is false and m is even number.

(b) Suppose that $\sqrt{2}$ is rational number (negation if true), then $\sqrt{2} = \frac{m}{n}$, where m is integer and n is positive integer. Here, we assume that $\frac{m}{n}$ is in simplified form in the sense that the H.C.F of m and n is 1.

- Since $\sqrt{2} = \frac{m}{n}$, we have $m^2 = \underbrace{2n^2}_{even}$.

  It follows from the result of **(a)** that m is even so that $m = 2q$ where q is an integer.

- Substitute $m = 2q$ in the equation, we have

$$(2q)^2 = 2n^2 \Rightarrow n^2 = \underbrace{4q^2}_{even}.$$

  So n is also an even number from the result of (a).

Because m and n are both even number, it follows that the H.C.F. of m and n is at least 2 and it contradicts to the fact that H.C.F. of m and n is 1. Therefore the negation is false and $\sqrt{2}$ is an irrational number.

If-then statement

Roughly speaking, *if-then statement* is a statement of the form (If p, then q) which says that the statement q is true if the statement p is true. Some examples of *if-then statement* are given below:

1. If $\underbrace{x \geq 3}_{p}$, then $\underbrace{x^2 \geq 9}_{q}$
2. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a}\big(f(x) + g(x)\big) = L + M$.

Some terminologies

- A *if-then statement* (If p, then q) can also be expressed as "p implies q" or "$p \Rightarrow q$".
- The statement p is called *sufficient condition* and the statement q is called *necessary condition*. Given these terminologies, the *if-then* statement can also be described as "p is sufficient for q" or "q is necessary for p".

Some important remarks regarding if-then statement

Since if-then statement appears in many aspects of mathematics, it is essential to understand the messages/conclusion can be drawn from the statement.

- One has to be careful that the if-then statement <u>*does not say anything* on the validity of the statement q if the statement p is *false*</u>.

  To see this, we consider a *if-then statement*

  "If $x > 3$, then $x^2 > 9$"

  Suppose that $x \leq 3$ (p is false),

  - If we take $x = 2$, then $x^2 = 4 \leq 9$ (q is false)
  - If we take $x = -5$, then $x^2 = 25 > 9$ (q is true).
- (The negation of *if-then* statement) The negation of a if-then statement ($p \Rightarrow q$) is given by $\sim(p \Rightarrow q) = (p \;\; and \;\; \sim q)$.

  One can justify it mathematically as follows:

  - Note that a if-then statement can be expressed as

    $$(p \Rightarrow q) = (q \quad or \;\; \sim p)$$
  - Using the property of negation, we get

    $$\sim(p \Rightarrow q) = \sim(q \quad or \;\; \sim p) = \big(\sim q \;\; and \;\; \sim(\sim p)\big) = (p \;\; and \sim q).$$

As an example, we consider the statement "If $x > 1$, then $\ln x > 0$" (true statement). Then the negation of this statement is

$$(x > 1 \quad and \quad \ln x \leq 0) \quad (false\ statement)$$

- (Contrapositive of if-then statement) Given a if-then statement ($p \Rightarrow q$), one can deduce that <u>p is false if the q is false</u>, then we can express the if-then statement as

  $$\text{"}(\sim q) \Rightarrow (\sim p)\text{"}$$

  This equivalent statement is known as *contrapositive* of the if-then statement.

  Mathematically, one can deduce that

  $$\big((\sim q) \Rightarrow (\sim p)\big) = \big(\sim p \quad or \quad \sim(\sim q)\big) = (\sim p \quad or \quad q) = (p \Rightarrow q)$$

  Therefore, these two statements (original statement and its contradictive) are proven to be equivalent.

- (Converse of if-then statement) Given a if-then statement ($p \Rightarrow q$), the *converse* of the statement is defined as ($q \Rightarrow p$).
  - Suppose that the statement "$p \Rightarrow q$" is true, it is not necessary that the converse "$q \Rightarrow p$" is also true. As an example, we consider the statement "If $x \geq 3$, then $x^2 \geq 9$". Note that $x^2 \geq 9$ does *not* always imply $x \geq 3$ (e.g. $x = -4$)

- (if and only if statement) We let p and q be two statements. If both "$p \Rightarrow q$" and "$\underbrace{q \Rightarrow p}_{\substack{converse\ of \\ p \Rightarrow q}}$" are true, then we say "p if and only if q" or "$p \Leftrightarrow q$".
  - Here, we say p (resp. q) is *necessary and sufficient condition* for q (resp. p)
  - If $p \Leftrightarrow q$, we say two statements p and q are *equivalent* in the sense that two statements are either *both correct* or *both incorrect*.

Example 4

We let a, b be two real number. Using proof by contrapositive, show that if $a \neq 0$ and $b \neq 0$, then $\sqrt{a^2+b^2} \neq a+b$.

🙂Solution

One can prove this statement by proving the corresponding contrapositive, i.e.

$$\underbrace{\sqrt{a^2+b^2}=a+b}_{\sim q} \Rightarrow \underbrace{a=0 \ \ or \ \ b=0}_{\sim p}.$$

Note that

$$\sqrt{a^2+b^2}=a+b \Rightarrow a^2+b^2=(a+b)^2 \Rightarrow 2ab=0.$$

This implies that $a=0$ or $b=0$. The result follows.

Remark of Example 4

By taking contrapositive, one can convert the statement into another equivalent form which can be proved easily.

Example 5

We let x, y be two integers. Prove that if x and y are odd integers, then there *does not* exist an integer z such that $x^2 + y^2 = z^2$.

😊Solution

We shall prove this statement using "prove by contradiction".

Suppose that there is an integer z such that $x^2 + y^2 = z^2$.

Since both x and y are odd number, so $x = 2p + 1$ and $y = 2q + 1$ for some integers p, q. Then we have

$$\underbrace{(2p+1)^2 + (2q+1)^2}_{x^2+y^2} = z^2 \Rightarrow \underbrace{4p^2 + 4p + 4q^2 + 4q + 2}_{even} = z^2.$$

Since z^2 is even, so z is also even by Example 3(a).

Write $z = 2k$ where k is integer, we get

$$4p^2 + 4p + 4q^2 + 4q + 2 = (2k)^2 \Rightarrow \underbrace{2p^2 + 2p + 2q^2 + 2q + 1}_{odd} = \underbrace{2k^2}_{even}$$

It leads to contradiction. So the original statement is valid.

(Question: Can we prove this by considering contrapositive?)

Example 6 (Proving a "if and only if" statement)

We let m, n be two integers. Show that $m - n$ is even if and only if $m^3 - n^3$ is even.

😊Solution

To prove the statement "$p \Leftrightarrow q$", we need to show "$p \Rightarrow q$" *and* "$q \Rightarrow p$"

"$\Rightarrow$" part

Recall the identity $m^3 - n^3 = (m-n)(m^2 + mn + n^2)$.

Since $m - n$ is even (so $m - n = 2k$) and $m^2 + mn + n^2$ is integer, so $m^3 - n^3$ is also even according to the above identity.

"$\Leftarrow$" part

We prove this by "prove by contradiction". Suppose that $m - n$ is not even, then it must be that one of m and n is even and one of m, n is odd.

- If m is even and n is odd, we write $m = 2p$ and $n = 2q + 1$ (where p, q are integers). Then $m^3 = (2p)^3 = 8p^3$ is even and $n^3 = (2q+1)^3 = 8q^3 + 12q^2 + 6q + 1$ is odd, so $\underline{m^3 - n^3 \text{ is odd}}$. This leads to contradiction.
- If m is odd and n is even, one can use the similar method and deduce the contradiction.

Therefore, $m - n$ must be even.