Math 202 (Introduction to Real Analysis) Fall 2007

Final Examination

Directions: This is a closed book exam. Works (including scratch works) must be shown legibly to receive credits. *Answers alone are worth very little!!!*

Notations: $\mathbb{R}$ denotes the set of all real numbers.

Part I (Concrete Problems)

1. Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers such that
$$x_{n+1} = \frac{x_1 - 2}{10 + x_n} \quad \text{for} \quad n = 1, 2, 3, \ldots.$$
(a) (11 marks) If $x_1 = -7$, then prove that $x_1, x_2, x_3, \ldots$ converges and find its limit.
(b) (11 marks) If $x_1 = 26$, then prove that $x_1, x_2, x_3, \ldots$ converges and find its limit.

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2. (11 marks) For $n = 1, 2, 3, \ldots$, let
$$y_n = \frac{4n^2 - \sqrt{n}}{2n^2 + n} + \frac{n-1}{n}.$$
Prove that $\lim_{n\to\infty} y_n = 3$ by checking the definition of limit of a sequence only.
(Do not use computation formulas, sandwich theorem or l'Hopital's rule! Otherwise, you will get zero mark for this problem.)

Part II (Abstract Problems)

3. Let A and B be nonempty subsets of $\mathbb{R}$. Both A and B are bounded above. Let
$$C = (A \setminus B) \cup (B \setminus A).$$
(a) (3 marks) Give an example of such sets A and B so that C is nonempty and $\sup C \neq \max\{\sup A, \sup B\}$.
(b) (6 marks) If C is nonempty and $\sup C \neq \max\{\sup A, \sup B\}$, then prove that
$$\sup(A \cap B) = \max\{\sup A, \sup B\}.$$
(c) (6 marks) If C is nonempty and $\sup A \neq \sup B$, then prove that
$$\sup C = \max\{\sup A, \sup B\}.$$

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4. (a) (3 marks) State the definition of a sequence $x_1, x_2, x_3, \ldots$ of real numbers converging to a real number L.
(b) (15 marks) Let $a_1, a_2, a_3, \ldots$ be positive numbers such that $\lim_{n\to\infty} \frac{a_n}{a_{n+1} + a_{n+2}} = 0$. Prove that $a_1, a_2, a_3, \ldots$ cannot be bounded above.

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