

# MATH2033 Mathematical Analysis (2021 Spring)

## Assignment 4

**Submission deadline of Assignment 4: 11:59p.m. of 5<sup>th</sup> May, 2020 (Wed)**

**Instruction:** Please complete all required problems. Full details (including description of methods used and explanation, key formula and theorem used and final answer) must be shown **clearly** to receive full credits. Marks can be deducted for incomplete solution or unclear solution.

**Please submit your completed work via the submission system by the deadline.** Late assignment will not be accepted.

Your submission must (1) be hand-written (typed assignment) (2) in a single pdf. file (other file formats will not be accepted) full name and student ID on the first page of the assignment.

$$\begin{aligned}
 f(x) &= \begin{cases} x^n \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \\
 \text{Left-hand derivative: } &\lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{-h} = f'(x) \\
 &= \lim_{h \rightarrow 0} \frac{x^{-n} \sin \frac{1}{x-n} - x^n \sin \frac{1}{x}}{-h} \\
 &= \lim_{h \rightarrow 0} (-h)^{n-1} \sin \frac{1}{\frac{1}{x-n}} = \lim_{h \rightarrow 0} (-h)^{n-1} \sin \frac{1}{\frac{1}{h}} \\
 &\approx \lim_{h \rightarrow 0} (-h)^{n-1} \left( \sin \frac{1}{h} \right) \xrightarrow{\text{bounded}} 0 \\
 &\text{So, } n > 1 \text{ then } f \text{ is differentiable at } x=0. \\
 f'(x) &= \\
 f'(x) &= \begin{cases} nx^{n-1} \sin \frac{1}{x} - x^{n-2} \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \\
 \text{For continuity at } 0: &\lim_{h \rightarrow 0} f'(0+h) = f'(0) = \lim_{h \rightarrow 0} f'(0-h) \\
 \text{Right-hand limit: } &\lim_{h \rightarrow 0^+} nh^{n-1} \sin \frac{1}{h} = h^{n-2} \cos \frac{1}{h} = 0 \\
 \text{So, } h^{n-1} \text{ and } h^{n-2} \text{ must be zero.} \\
 \text{So for } n \geq 2 \text{ the } f(x) \text{ will be continuous differentiable at } x=0.
 \end{aligned}$$

### Problem 1

We consider a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} x^n \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

where  $n \in \mathbb{N}$ .

(a) Find the values of  $n$  which  $f(x)$  is differentiable at  $x = 0$ .

(b) Find the values of  $n$  which  $f(x)$  is continuously differentiable at  $x = 0$

### Problem 2

We let  $f: (a, b) \rightarrow \mathbb{R}$  be a function and let  $x_0 \in (a, b)$ .

(a) If  $f$  is differentiable at  $x = x_0$ , show that

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} = f'(x_0) \dots \dots (*)$$

(b) If  $\lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}$  exists, is it necessary that  $f(x)$  is differentiable

Explain your answer.

( $\ominus$  Hint: If your answer is yes, give a mathematical proof. If your answer is no, give a counter example).

$$\begin{aligned}
 &\stackrel{h \rightarrow 0^+}{\lim} \frac{f(x_0 + h) - f(x_0 - h)}{2h} \\
 &= \lim_{h \rightarrow 0^+} \left[ \frac{f(x_0 + h) - f(x_0)}{2h} + \frac{f(x_0 - h) - f(x_0)}{-2h} \right] \\
 &= \lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{2h} + \lim_{h \rightarrow 0^+} \frac{f(x_0 - h) - f(x_0)}{2h} \\
 &= \frac{1}{2} Rf'(x_0) + \frac{1}{2} Lf'(x_0) \stackrel{Rf'(x_0) = Lf'(x_0)}{=} f'(x_0) \\
 \text{Again } &\stackrel{h \rightarrow 0^-}{\lim} \frac{f(x_0 + h) - f(x_0 - h)}{2h} \\
 &= \lim_{h \rightarrow 0^-} \left[ \frac{f(x_0 + h) - f(x_0)}{2h} + \frac{f(x_0 - h) - f(x_0)}{-2h} \right] \\
 &= \lim_{h \rightarrow 0^-} \frac{f(x_0 + h) - f(x_0)}{2h} + \lim_{h \rightarrow 0^+} \frac{f(x_0 - h) - f(x_0)}{2h} \\
 &= \frac{1}{2} Lf'(x_0) + \frac{1}{2} Rf'(x_0) = f'(x_0). \\
 \therefore \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} &= f'(x_0). \quad \square
 \end{aligned}$$

$$\begin{aligned}
 &\text{Take } f(x) = |x| \text{ and } x_0 = 0. \\
 &\text{Then } \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} = \lim_{h \rightarrow 0} \frac{|x_0 + h| - |x_0 - h|}{2h} \\
 &= \lim_{h \rightarrow 0} \frac{|h| - |-h|}{2h} = 0
 \end{aligned}$$

But  $f$  is not differentiable at  $x_0 = 0$ .

### Problem 3

We let  $f: (0, 1] \rightarrow \mathbb{R}$  be a differentiable function such that  $|f'(x)| < M$  for all  $x \in (0, 1]$ , where  $M > 0$  is a positive constant.

In order to prove that the sequence  $\{a_n\}$  is convergent I am going to prove that the sequence  $\{a_n\}$  is Cauchy sequence. Definition of Cauchy sequence: A sequence  $\{a_n\}$  of real numbers is Cauchy sequence if for any  $\epsilon > 0$  there exist a positive integer  $N$  such that  $|a_m - a_n| < \epsilon$  for all  $m, n \in \mathbb{N}$ . Example of Cauchy sequence: Every convergent sequence is Cauchy sequence. Since  $\{\frac{1}{n}\}$  is convergent sequence and converges to 0, so it is Cauchy sequence. Solution of main problem: See,  $f: (0, 1] \rightarrow \mathbb{R}$  is differentiable on  $(0, 1]$  and  $|f'(x)| < M$  for all  $x \in (0, 1]$ .

that the sequence  $\{a_n\}$  converges. Be careful that  $f(0)$  is not defined. You can prove the convergence

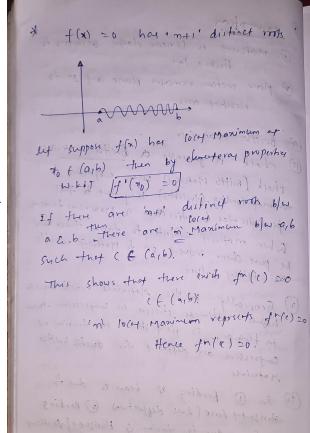
Now applying the mean value theorem on the interval  $[x_0, x_1]$ . Since  $f$  is differentiable on  $(x_0, x_1)$  So,  $f$  is differentiable on  $[x_0, x_1]$ . And every differentiable function is continuous. So,  $f$  is continuous on  $[x_0, x_1]$ . So,  $f$  is differentiable on  $(x_0, x_1)$  by the Mean Value Theorem there exist  $c \in (x_0, x_1)$  such that  $f(x_1) - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ . Since,  $x_1 \neq 0$ , So,  $f(x_1) - f(x_0) = f'(c)(x_1 - x_0)$ .  $\Rightarrow |f(x_1) - f(x_0)| = |f'(c)| |x_1 - x_0|$ .  $(\because c \in (x_0, x_1))$   $\Rightarrow |f(x_1) - f(x_0)| = |f'(c)| |x_1 - x_0| < M|x_1 - x_0|$ .  $(\because |f'(c)| < M \forall x \in (0, 1])$  and  $c \in (x_0, x_1)$ .  $\Rightarrow |f(x_1) - f(x_0)| < M|x_1 - x_0|$ . Since,  $x, y \in (0, 1)$  arbitrary points. So, for all  $x, y \in (0, 1)$ .  $|f(x) - f(y)| < M|x - y|$ .  $\therefore$   $\{a_n\}$  is Cauchy sequence.

Now that  $|f'(x)| < M$  for all  $x \in (0, 1]$  we define

Let  $\epsilon > 0$ . Since  $M > 0$   $\Rightarrow \frac{1}{M} > 0$ . And if  $\{a_n\}$  is Cauchy sequence, we need to find  $N \in \mathbb{N}$  (by the definition of Cauchy sequence)  $\frac{1}{M} < \frac{1}{N} \Rightarrow \frac{1}{N} < \frac{1}{M} \Rightarrow N > M$  for all  $m, n > N$ .  $(\because \frac{1}{m-1} < \frac{1}{N} \forall m > N)$ . Since,  $a_n = f(n)$ . Now  $|a_m - a_n| = |f(m) - f(n)| < M|\frac{1}{m-1}|$  for all  $m, n > N$ .  $\Rightarrow |a_m - a_n| < M|\frac{1}{m-1}| < M|\frac{1}{N}| < \epsilon$ .  $\therefore$   $|a_m - a_n| < \epsilon$  for all  $m, n > N$ . So,  $\{a_n\}$  is Cauchy sequence of real numbers. By Cauchy criterion for the convergence of the sequence of real numbers if  $\{a_n\}$  is a convergent sequence, then it is the required result.

### Problem 4

We let  $f: [a, b] \rightarrow \mathbb{R}$  be  $n$ -times differentiable function which  $f(x) = 0$  has  $n + 1$  distinct roots over  $[a, b]$ . Show that there exists  $c \in (a, b)$  such that  $f^{(n)}(c) = 0$ .



### Problem 5

Show that for any  $x > 0$ ,

$$1 - x + \frac{x^2}{2} > e^{-x} > 1 - x.$$

### Problem 6 (Harder)

We let  $f: [0, 1] \rightarrow \mathbb{R}$  be a twice differentiable function on  $[0, 1]$  and  $f''(x)$  is continuous on  $[0, 1]$ . Suppose that

- $f(0) = f(1) = 0$  and
- $|f''(x)| \leq A$  for all  $x \in [0, 1]$ , where  $A > 0$  is a constant.

Show that  $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{A}{4}$ .

( $\textcircled{S}$ Hint: Apply Taylor theorem with suitable choice of  $a$ .)

To prove for any  $x > 0$ ,  $1-x + \frac{x^2}{2} > e^{-x} > 1-x$ .

$$\text{let } f(x) = 1-x + \frac{x^2}{2} - e^{-x}, \quad x > 0.$$

$f$  is continuous on  $[0, \infty)$ .  $f'(x) = -1+x+e^{-x} > 0$  for all  $x > 0$ .

Therefore  $f$  is strictly increasing function on  $[0, \infty)$ . So,  $f(x) > f(0)$  for all  $x > 0$ .

$$\Rightarrow 1-x + \frac{x^2}{2} - e^{-x} > 0 \\ \Rightarrow 1-x + \frac{x^2}{2} > e^{-x} \text{ for all } x > 0 \dots \text{(i)}$$

Let  $g(x) = e^x$ .  $g$  is continuous on  $[0, \infty)$ .  $g'(x) = e^{-x} + 1 > 0$  for all  $x > 0$ .

Therefore  $g$  is strictly increasing function on  $[0, \infty)$ . So,  $g(x) > g(0)$  for all  $x > 0$ .

$$\text{consequently } e^{-x} - 1 + x > 0 \\ \Rightarrow e^{-x} > 1 - x \dots \text{(ii)}$$

From (i) & (ii) it follows that

$$1-x + \frac{x^2}{2} > e^{-x} > 1-x$$

(Proved).

Problem 6: Let  $f: [0, 1] \rightarrow \mathbb{R}$  be a twice differentiable function on  $[0, 1]$  and  $f''(x)$  is continuous on  $[0, 1]$ .

Suppose that  $f(0) = f(1) = 0$  and  $|f''(x)| \leq A \quad \forall x \in [0, 1]$ ,

where  $A$  is positive constant.

Now using Taylor's th. about  $x = y_2$ ,

$$f(x) = f(y_2) + (x-y_2)f'(y_2) + \frac{(x-y_2)^2}{2!}f''(y_2 + \theta(x-y_2))$$

for  $0 < \theta < 1$