Math 2033 (Homework 1) Solutions Fall 2021

① For $r \in \mathbb{Q}$, $W_r = \{x \in \mathbb{R} : x^3 - 2x + 5 = r\}$ has at most

10 marks

3 elements. So W_r is countable. Now $W = \bigcup_{r \in \mathbb{Q}} W_r$,

5 marks

$\mathbb{Q}$ is countable and each W_r is countable for $r \in \mathbb{Q}$.

15 marks

By the countable union theorem, W is countable.

10 marks

~~② We are given $x^4 + bx - 5 = 0$ has a rational root r. (If $r = 0$, then $r^4 + br - 5 \neq 0$.) We get $r \neq 0$ and $r^4 + br - 5 = 0$. Then $b = \frac{5 - r^4}{r} \in \mathbb{Q}$. Also, $b \in S$. Hence $b \in \mathbb{Q} \cap S$. Then $\mathbb{Q} \cap S$ is countable.~~

③ For $x \in \mathbb{N}$, let $B_x = \{x + \sqrt{2}y : y \in \mathbb{N}\}$. The

10 marks

function $f : \mathbb{N} \to B_x$ is defined by $f(y) = x + \sqrt{2}y$

15 marks

f is a bijection. So B_x is countable. Now

5 marks

$B = \bigcup_{x \in \mathbb{N}} B_x$, $\mathbb{N}$ is countable, each B_x is countable

5 marks

for $x \in \mathbb{N}$. Due to the countable union theorem,

so B is countable.

15 marks

10 marks