Review

- Sequential limit Thm:

$$\lim_{x\to x_0} f(x) = L \iff \lim_{n\to\infty} f(x_n) = L \text{ for any } \{x_n\} \text{ with } x_n \to x_0 \text{ in } S\setminus\{x_0\}$$

- Thm: Let $f, g : S \to \mathbb{R}$. Then

$$\lim_{x\to x_0} f(x) = L_1, \quad \lim_{x\to x_0} g(x) = L_2$$

$$\Rightarrow \lim_{x\to x_0} f(x) + g(x) = L_1 + L_2$$

- Thm: Let $f: S \to \mathbb{R}$

$$\lim_{x\to x_0} f(x) = L \iff f(x_0^+) = f(x_0^-) = L$$

- Thm: Let $f : (a,b) \to \mathbb{R}$ be increasing

Then $\forall x_0 \in (a,b)$

$$f(x_0^-) \le f(x_0) \le f(x_0^+)$$

# Chapter 6

Def: Let $S$ be an interval of $\mathbb{R}$ (or more generally a subset of $\mathbb{R}$), A function $f : S \to \mathbb{R}$ is continuous at $x_0 \in S$ iff

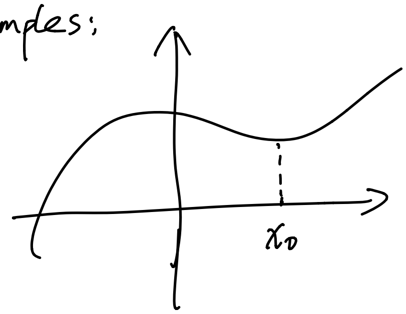
$$\lim_{x\to x_0} f(x) = f(x_0),$$

$$\Updownarrow$$

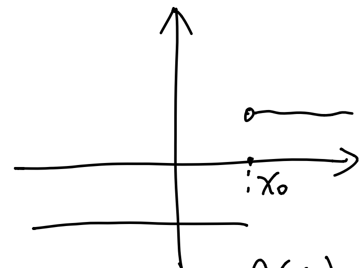
Or, more precisely, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in S$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

We say that $f$ is continous iff $f$ is continous at every points in $S$.

Examples:




Continuous




$f(x_0^-) \neq f(x_0^+)$

Not continuous

$f(x) = \sum_{n=0}^{N} a_n x^n \qquad f(x) = e^x \qquad f(x) = \sin x$

are continuous

Sequential Continuity Thm:

Thm: $f: S \to \mathbb{R}$ is continuous at $x_0 \in S$ (i.e., $\lim_{x \to x_0} f(x) = f(x_0)$)

$\iff$ $\forall\, x_n \to x_0$ in $S$ (i.e., $x_n \in S$, $\lim_{n\to\infty} x_n = x_0$)

$$\lim_{n\to\infty} f(x_n) = f(x_0) = f\left(\lim_{n\to\infty} x_n\right)$$

Proof. Use the sequential limit thm, which states that

$\lim_{x \to x_0} f(x) = L \iff \lim_{n\to\infty} f(x_n) = L$ for any $x_n \to x_0$ in $S \setminus \{x_0\}$

We need only replace $L$ by $f(x_0)$, and notice that

$\forall\, x_n \to x_0$ in $S \setminus \{x_0\}$, $\lim_{n\to\infty} f(x_n) = f(x_0)$

$\Updownarrow$

$\forall\, x_n \to x_0$ in $S$, $\lim_{n\to\infty} f(x_n) = f(x_0)$. $\boxtimes$

Example 1: Consider $f(x) = \dfrac{\sin x}{x}$.

$f$ is well-defined on $\mathbb{R} \setminus \{0\}$

and

$f$ is continuous on $\mathbb{R}\setminus\{0\}$.

Since $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, we define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Obviously, $\tilde{f}: \mathbb{R} \to \mathbb{R}$, and

$$\lim_{x\to 0} \tilde{f}(x) = \tilde{f}(0) = 1$$

$$\left.\begin{array}{l} \text{So, } \tilde{f} \text{ is continous at } 0, \\ \tilde{f} \text{ is continuous on } \mathbb{R}\setminus\{0\} \\ \quad \text{because } \tilde{f} = f \text{ on } \mathbb{R}\setminus\{0\} \end{array}\right\} \Longrightarrow \tilde{f} \text{ is continuous}$$

$\tilde{f}$ is called the continuous extension of $f$.

Example 2: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$ ($\mathbb{Q}$ means the set of all rational numbers)

The $f$ is discontinous at every $x \in \mathbb{R}$.

Indeed, $\forall\, x_0 \in \mathbb{R}$, for each $n \in \mathbb{N}$, by the density of $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$,

$\exists\; q_n, r_n \in (x_0, x_0 + \frac{1}{n})$ s.t. $q_n \in \mathbb{Q}$, $r_n \in \mathbb{R}\setminus\mathbb{Q}$

Obviously, $q_n \to x_0$, $r_n \to x_0$

By the sequential continuity thm,

$\lim\limits_{n\to\infty} f(q_n) = \lim\limits_{n\to\infty} 1 = 1$ Not the same

$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 0 = 0$$

$\Longrightarrow$ $f$ is discontinuous at $x_0$.

Thm: If $f, g : S \to \mathbb{R}$ are continuous at $x_0 \in S$,

then $f \pm g$, $f \cdot g$, $f/g$ (provided $g(x_0) \neq 0$)

are all continuous at $x_0$.

Proof. This is an immediate consequence of the sequential continuity thm. $\boxtimes$

Composite rule.

Thm: If $f : S \to \mathbb{R}$ is continuous at $x_0$.

$f(S) \subset S'$ and $g : S' \to \mathbb{R}$ is continuous at $f(x_0) \in S'$,

then $g \circ f$ is continuous at $x_0$

proof. $\forall \varepsilon > 0$, since $g$ is cont. at $f(x_0)$,

$\exists\, \delta_1 > 0$ s.t. $\forall y \in S'$ $|y - f(x_0)| < \delta_1 \Rightarrow |g(y) - g(f(x_0))| < \varepsilon$.

Since $f$ is cont. at $x_0$, for any $\varepsilon' > 0$,

$\exists\, \delta_2 > 0$, s.t. $\forall x \in S$ $|x - x_0| < \delta_2 \Rightarrow |f(x) - f(x_0)| < \varepsilon'$.

Choose $\varepsilon' = \delta_1$ above, then we can find a $\delta_2 > 0$

s.t. $\forall x \in S$, $|x - x_0| < \delta_2 \Rightarrow |\underbrace{f(x)}_{y} - f(x_0)| < \delta_1$

$$\Longrightarrow |g(f(x)) - g(f(x_0))| < \varepsilon$$

In summary,

$\exists\, \varepsilon > 0$, $\exists\, \delta_2 > 0$ s.t. $\forall x \in S$ $|x - x_0| < \delta_2$

$$\Longrightarrow |g(f(x)) - g(f(x_0))| < \varepsilon.$$

$$\Updownarrow$$

$g \circ f$ is continuous at $x_0$. $\boxtimes$

Sign preserving property

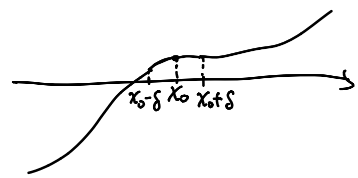
Thm: Let $f: S \to \mathbb{R}$ be continuous

If $f(x_0) > 0$, then $\exists$ an interval

$I = [x_0 - \delta, x_0 + \delta]$ with $\delta > 0$

s.t.

$f(x) > 0$ for all $x \in S \cap I$.




proof. Let $\varepsilon = \frac{f(x_0)}{2} > 0$,

Since $f$ is cont. at $x_0$,

$\exists\, \delta > 0$ s.t. $|x - x_0| < \delta$, $x \in S$,

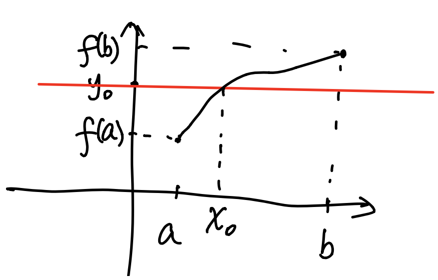
$$\Rightarrow \quad |f(x) - f(x_0)| < \varepsilon$$

$$\Rightarrow \quad f(x) > f(x_0) - \varepsilon = f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2} > 0$$

for all $x: \underbrace{|x - x_0| < \delta \text{ and } x \in S}_{x \in S \cap I}$. $\boxtimes$

Intermediate Value Theorem (IVT)

Thm: If $f: [a,b] \to \mathbb{R}$ is continuous, and $y_0$ is between $f(a)$ and $f(b)$, then $\exists\ x_0 \in [a,b]$ s.t. $f(x_0) = y_0$




Remark: $x_0$ is NOT necessarily unique.

Proof. Case I: $f(a) = f(b)$.

Then $y_0 = f(a)$ and we take $x_0 = a$.

Case II: $f(a) < f(b)$

If $y_0 = f(a)$ or $f(b)$, the we take $x_0 = a$ or $x_0 = b$.

So we only need to consider

$$f(a) < y_0 < f(b)$$

We will use the nested interval thm.

Let $a_1 = a$, $b_1 = b$ and $I_1 = [a_1, b_1]$

Let $m_1 = \frac{a_1 + b_1}{2}$.

If $f(m_1) = y_0$, then we are done. by taking $x_0 = m_1$

If $f(m_1) > y_0$, then we construct:

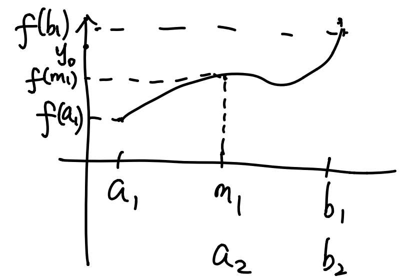
$a_2 = a_1$, $b_2 = m_1$

$I_2 = [a_2, b_2]$

If $f(m_1) < y_0$, the

$a_2 = m_1$, $b_2 = b_1$

$I_2 = [a_2, b_2]$




After n steps, we have

either we stop by $x_0 = a_n$ or $x_0 = b_n$

or $f(a_n) < y_0 < f(b_n)$

and $b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1})$

$= \cdots = \frac{1}{2^{n-1}}(b_1 - a_1) \to 0$

as $n \to \infty$

By nested interval theorem,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x_0$$

Also

$$f\left(\lim_{n\to\infty} a_n\right) = \lim_{n\to\infty} f(a_n) \le y_0 \le \lim_{n\to\infty} f(b_n) \overset{\text{f is continuous}}{=} f\left(\lim_{n\to\infty} b_n\right) = f(x_0)$$

$$f\left(\lim_{n\to\infty} a_n\right) = f(x_0)$$

$$\Longrightarrow \quad f(x_0) = y_0 . \qquad \boxtimes$$

Example 1: Let $f : [0,1] \to [0,1]$ ~~an~~ ~~an~~ a continuous function. Suppose $f(0) > 0$ and $f(1) < 1$.

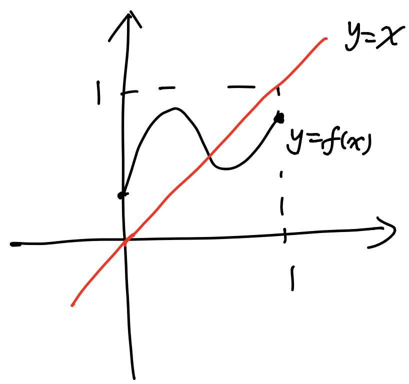
Then, there exists a fixed point of $f$,

i.e., $\exists\, r$ s.t. $f(r) = r$

Solution:

Let $g(x) = f(x) - x$

$g$ is continuous.




Also $g(0) = f(0) - 0 > 0$

$g(1) = f(1) - 1 < 0$

Therefore $0$ is in between of $g(0)$ and $g(1)$

By IVT, $\exists\, x_0$ s.t. $g(x_0) = 0$,

i.e., $f(x_0) - x_0 = 0$

$f(x_0) = x_0$.

Example 2: Show that the equation

$$x^5 + 3x + \sin x = \cos x + 10$$

has a solution

Proof.

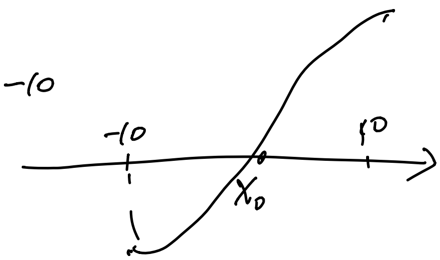
Let $f(x) = x^5 + 3x \overset{+\sin x}{\cancel{+3x}} - \cos x - 10$

$f$ is obviously continuous,

The equation has a solution $\Longleftrightarrow$ $\exists\, x_0$ s.t $f(x_0)=0$

~~Let~~

$f(10) = 10^5 + 3\cdot 10 + \sin 10 - \cos 10 - 10$
$> 0$




$f(-10) = -10^5 - 3\cdot 10 + \sin(-10) - \cos(-10) - 10$
$< 0$

0 is between $f(10)$ and $f(-10)$

$\exists\, x_0 \in [-10, 10]$ s.t $f(x_0) = 0$ $\boxtimes$

Example 3:
Every odd degree polynomial with real coefficients has at least one real root.

proof. Let $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ with $n$ odd

$a_0 \neq 0$

Then $p(x) = 0 \Longleftrightarrow \underbrace{x^n + \frac{a_1}{a_0}x^{n-1} + \cdots + \frac{a_n}{a_1}}_{q(x)} = 0$

when $x \to +\infty$, $q(x) \to +\infty$

$x \to -\infty$, $q(x) \to -\infty$

For sufficiently large $x_1$, $q(x_1) > 0$

... small $x_2$, $q(x_2) < 0$

$0$ is between $g(x_1)$, $g(x_2)$.

$\exists\, x_0 \in [x_2, x_1]$ s.t. $g(x_0) = 0$. by IVT.

Exetrem Value Theorem:

Theorem: If $f: [a,b] \to \mathbb{R}$ is continuous,

then $\exists\ x_0, w_0 \in [a,b]$ such that

$$f(x_0) = \inf\{f(x) : a \le x \le b\} = \min_{a \le x \le b} f(x)$$

$$f(w_0) = \sup\{f(x) : a \le x \le b\} = \max_{a \le x \le b} f(x).$$

Proof.

Step 1: We show that $f([a,b])$ is bounded above.

By contradiction.

Suppose $f([a,b])$ is not bounded above.

Then $\exists \{x_n\} \subset [a,b]$ s.t. $f(x_n) > n$.

By Bolzano-Weierstrauss Theorem,

$\{x_n\}$ has a subsequence $\{x_{n_j}\}_j$ s.t

$$\lim_{j\to\infty} x_{n_j} = c \in [a,b].$$

Since $f$ is continuous,

$$\lim_{j\to\infty} f(x_{n_j}) \underset{\uparrow}{=} f\left(\lim_{j\to\infty} x_{n_j}\right) = f(c)$$

Sequential Continuity Thm

Therefore $\{f(x_{n_j})\}_j$ is bounded.

This contradicts with

$$f(x_{n_j}) > n_j > j$$

Step 2: By Supereme limit theorem

$\exists \ \{z_n\} \subset [a,b]$ s.t.

$$\lim_{n\to\infty} f(z_n) = \sup\{f(x) : a \le x \le b\}$$

Again $\{z_n\}$ is bounded,

$\exists \ \{z_{n_j}\} \subset \{z_n\}$ s.t. $\lim_{j\to\infty} z_{n_j} = w_0 \in [a,b]$

$$f(w_0) = f\left(\lim_{j\to\infty} z_{n_j}\right) = \lim_{j\to\infty} f(z_{n_j}) = \sup\{f(x) : a \le x \le b\}$$

i.e., $\exists \ w_0 \in [a,b]$ s.t.

$$f(w_0) = \sup\{f(x) : a \le x \le b\}$$

Step 3: The inf part is done similarly. $\boxtimes$

Continuous Injection Theorem.

Theorem: If $f$ is continuous and injective on $[a,b]$, then $f$ is strictly monotone and

$$f([a,b]) = [f(a), f(b)] \quad \text{if } f \nearrow$$

$$f([a,b]) = [f(b), f(a)] \quad \text{if } f \searrow$$

proof. We consider only the case when $f(a) < f(b)$.
The other case $f(a) > f(b)$ can be done similarly.

Step 1: We show $f$ is strictly $\nearrow$.

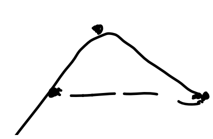
$\forall \ a < y < b$, we show that $f(y) < f(b)$.

By contradiction.

Suppose $f(y) \ge f(b)$

Since $f$ is injective, and $y \ne b$ $\Big\} \Rightarrow f(y) > f(b)$

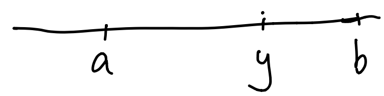
By IVT (Intermediate Value Theorem)



$\exists\ c \in [a, y]$ s.t. $f(c) = f(b)$

(because $f(b)$ is in between $f(a)$ and $f(y)$)

Contradict with the injectivity.




Similarly, $f(y) > f(a)$

Altogether, $a < y < b \implies f(a) < f(y) < f(b)$.

Now $\forall\ a < x < y \le b$,

the same argument implies $f(x) < f(y)$.

Therefore, $f$ is strictly $\nearrow$.

Step 2: $f(a) \le f(x) \le f(b) \quad \forall\ x \in [a, b]$

$\implies f([a,b]) \subset [f(a), f(b)]$

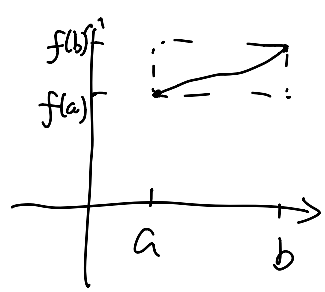
To show the reverse,

$\forall\ y_0 \in (f(a), f(b))$, IVT implies $\exists\ x_0 \in [a, b]$ s.t. $f(x_0) = y_0$

If $y_0 = f(a)$ or $f(b)$, then $\exists\ x_0 \in [a,b]$ s.t. $f(x_0) = y_0$

$\implies [f(a), f(b)] \subset f([a,b])$

$\implies f([a,b]) = [f(a), f(b)]$. $\boxtimes$




Continuous Inverse Theorem.

Thm: If $f$ is continuous and injective on $[a, b]$,

then $f^{-1} : f([a,b]) \to [a, b]$ is continuous and surjective.

Proof. Step 1:

$\forall\ c \in [a,b]$,

$$f(c) \in f([a,b]) \ . \quad c = f^{-1}(f(c))$$

So, $f^{-1}$ is surjective.

Step 2: By Continuous injection theorem,

$f$ is strictly monotone.

WLOG, assume $f$ is $\nearrow$

Then $f^{-1}$ is also strictly $\nearrow$

becaus $f(x) < f(y) \quad \forall\ a \le x < y \le b$

$$\Updownarrow$$

$$x < y$$

Step 3: We show $f^{-1}$ is continuous.

By contradiction.

Assume $f^{-1}$ is discontinuous at $y_0 = f(x_0) \in [f(a), f(b)]$

WLOG, Assume $f(a) < y_0 < f(b)$

Then

$$f^{-1}(y_0^-) = \lim_{\substack{y \to y_0 \\ y < y_0}} f^{-1}(y)$$ exists because monotone function theorem.

$$f^{-1}(y_0^+) = \lim_{\substack{y \to y_0 \\ y > y_0}} f^{-1}(y)$$ exists

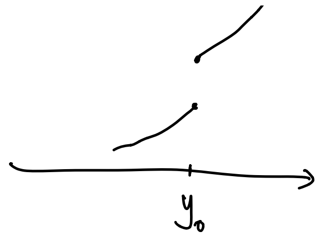
$f^{-1}$ is discontinuous at $y_0$

$$\Rightarrow \ f^{-1}(y_0^-) \neq f^{-1}(y_0) \ \text{ or } \ f^{-1}(y_0^+) \neq f^{-1}(y_0)$$

If $f^{-1}(y_0^-) \neq f^{-1}(y_0)$, then $f^{-1}(y_0^-) < f^{-1}(y_0)$

For $y \geq y_0$, $\quad f^{-1}(y) \geq f^{-1}(y_0)$ because $f^{-1} \nearrow$.

For $y < y_0$, $\quad f^{-1}(y) \leq \sup\{f^{-1}(y) : y < y_0\} = f^{-1}(y_0^-)$




This contracti with the fact that $f^{-1}$ is surjective. $\boxtimes$