MATH202 Introduction to Analysis (2007 Fall & 2008 Spring)

Tutorial Note #2

Review on sets

Set is a fundamental part in many branches of Mathematics. In this note, we will review some properties and operations of set. Also we will show how to write proof about set problems.

Operations of Set

Example 1

Compute $\mathbf{Z} \cap [0,10] \cap \{\mathrm{n}^2 + 1 : \mathrm{n} \in \mathbf{N}\}$

Solution:

$\boxed{\boldsymbol{Note\ \ that\ x \in\ A \cap B\ \ means\ \ x \in A\ \ and\ \ x \in B}}$

First $\mathrm{Z} \cap [0,10] = \{\text{integers from 0 to 10}\} = \{0,1,2,3,4,5,6, \dots ,10\}$

Next $\{\mathrm{n}^2 + 1 : \mathrm{n} \in \mathrm{N}\} = \{1^2 + 1, 2^2 + 1, 3^2 + 1, 4^2 + 1, \dots\} = \{2,5,10,17, \dots\}$

Therefore

$\mathrm{Z} \cap [0,10] \cap \{\mathrm{n}^2 + 1 : \mathrm{n} \in \mathrm{N}\} = \{0,1,2,3, \dots ,10\} \cap \{2,5,10,17 \dots\} = \{2,5,10\}$

Example 2

Compute $\{\mathrm{n} \in \mathbf{N} : 5 < n < 9\} \backslash \{2\mathrm{m} : \mathrm{m} \in \mathbf{N}\}$

Solution:

First $\{\mathbf{n} \in \mathbf{N} : \mathbf{5} < n < 9\} = \{\mathbf{6}, \mathbf{7}, \mathbf{8}\}$

$\{\mathbf{2m} : \mathbf{m} \in \mathbf{N}\} = \{\mathbf{2}, \mathbf{4}, \mathbf{6}, \mathbf{8}, \mathbf{10}, \dots\}$

$\boxed{\textbf{\textit{Note:}}\ \ \boldsymbol{x \in A \backslash B}\ \ \textbf{\textit{means}}\ \ \boldsymbol{x \in A}\ \ \textbf{\textit{and}}\ \ \boldsymbol{x \notin B}}$

Therefore

$\{\mathrm{n} \in \mathbf{N} : 5 < n < 9\} \backslash \{2\mathrm{m} : \mathrm{m} \in \mathbf{N}\} = \{6,7,8\} \backslash \{2,4,6,8,10, \dots\} = \{7\}$

Example 3

Find $([0,5] \cup [2,7]) \times ([-2,3] \cap [2,4])$

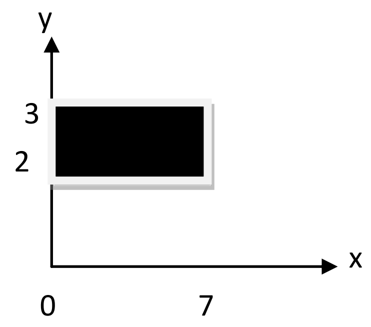
Solution:

First we deal with the set in the brackets first,

$[\mathbf{0}, \mathbf{5}] \cup [\mathbf{2}, \mathbf{7}] = [\mathbf{0}, \mathbf{7}]$ ***and*** $[-\mathbf{2}, \mathbf{3}] \cap [\mathbf{2}, \mathbf{4}] = [\mathbf{2}, \mathbf{3}]$

Then the resultant set will be $[\mathbf{0}, \mathbf{7}] \times [\mathbf{2}, \mathbf{3}]$ ***which is a rectangle in x-y plane***

$\boxed{\textbf{\textit{Note:}}\ \ \mathbf{A \times B = \{(a, b) : a \in A\ and\ b \in B\}}}$



☺Exercise

Compute the following sets

a) $(\{x, y, z\} \cup \{w, z\}) \backslash \{u, v, w, x\}$

b) $([0,2] \cup [1,3]) \backslash ([1,3] \cap [0,2])$

How to prove things?

Example 4

Let $D_r = \{(x, y) \in R^2 | x^2 + y^2 \leq r^2\}$ and $S_r = \{(x, y) \in R^2 : |x| \leq r \text{ and } |y| \leq r\}$

Show that $D_r \subset S_r$

By drawing pictures, it is easy to see the fact. But we need to prove it precisely,

Proof:

For any $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{D_r}$***. we consider***

$$|\boldsymbol{x}| = \sqrt{\boldsymbol{x}^2} \leq \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2} = \sqrt{\boldsymbol{r}^2} = \boldsymbol{r}$$

Similar argument shows $|\boldsymbol{y}| \leq \boldsymbol{r}$***, then*** $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{S_r}$

So $\boldsymbol{D_r} \subseteq \boldsymbol{S_r}$

(However we need to show $\boldsymbol{D_r} \subset \boldsymbol{S_r}$***, therefore after this, we need to show***

$D_r \neq S_r$)

Pick $(r,r) \in S_r$, then $x^2+y^2=r^2+r^2=2r^2>r^2$, hence $(r,r) \notin D_r$

It shows $D_r \neq S_r$, therefore $D_r \subset S_r$.

☺Exercise

Let $A=\{(x,y)\in R^2: x^4-81y^4=0\}$ and $B=\{(x,y)\in R^2: \frac{x}{y}=3 \text{ or } -3\}$

Show that $B \subseteq A$. Is it true that $A=B$?

In some situations, we need to prove some set “formula” like

$X\backslash(Y\cap Z)=(X\backslash Y)\cup(X\backslash Z)$

We can easily verified it by drawing figure, however, in MATH202, we need to show it precisely. In fact, the proof actually bases on the definitions.

Example 5

If $B\subseteq C$, prove that $A\cup B\subseteq A\cup C$

Proof:

Note $A\subseteq B \leftrightarrow$ for any $a\in A$, we have $a \in B$

(*Note: $B\subseteq C$ means $x\in B \to x\in C$)

For any $x\in A\cup B$

$\to x\in A$ ***or*** $x\in B$

$\to x\in A$ ***or*** $x\in C$ ***(Since $B\subseteq C$)***

$\to x\in A\cup C$

So $A\cup B\subseteq A\cup C$.

For more complicated formula, we can show it step by step. No tricks are needed.

Example 6

Show $X\backslash(Y\cap Z)=(X\backslash Y)\cup(X\backslash Z)$

Proof:

To show $A=B$, we need to show $A\subseteq B$ and $B\subseteq A$

(Step 1: $\boldsymbol{A \subseteq B}$***)***

For any $\boldsymbol{x \in X\backslash(Y \cap Z)}$

$\rightarrow \boldsymbol{x \in X}$ ***and*** $\boldsymbol{x \notin Y \cap Z}$

$\rightarrow \boldsymbol{x \in X\ and\ (x \notin Y\ or\ x \notin Z)}$ ***(Recall in Logic: ~(x and y) = ~x or ~y)***

$\rightarrow \boldsymbol{(x \in X\ or\ x \notin Y) and\ (x \in X\ or\ x \notin Z)}$

$\rightarrow \boldsymbol{x \in X\backslash Y}$ ***or*** $\boldsymbol{x \in X\backslash Z}$

$\rightarrow \boldsymbol{x \in (X\backslash Y) \cup (X\backslash Z)}$

$\rightarrow \boldsymbol{X\backslash(Y \cap Z) \subseteq (X\backslash Y) \cup (X\backslash Z)}$

(Step 2: $\boldsymbol{A \supseteq B}$***)***

By reversing the step in Step 1, we can show $\boldsymbol{X\backslash(Y \cap Z) \supseteq (X\backslash Y) \cup (X\backslash Z)}$

So $\boldsymbol{X\backslash(Y \cap Z) = (X\backslash Y) \cup (X\backslash Z)}$***.***

Example 8

Show that $(X\backslash Y) \cup (Y\backslash X) = (X \cup Y)\backslash(X \cap Y)$

Proof:

(Step 1: $\boldsymbol{A \subseteq B}$***)***

$\boldsymbol{x \in (X\backslash Y) \cup (Y\backslash X)}$

$\rightarrow \boldsymbol{x \in X\backslash Y\ \ or\ \ x \in Y\backslash X}$

$\rightarrow \boldsymbol{(x \in X\ and\ x \notin Y)\ or\ (x \in Y\ and\ x \notin X)}$

$\rightarrow \boldsymbol{\big(x \in X\ and\ (x \notin X\ or\ x \notin Y)\big)\ or\ (x \in Y\ and\ (x \notin X\ or\ x \notin Y))}$

$\rightarrow \boldsymbol{(x \in X\ or\ x \in Y)\ and\ (x \notin X\ or\ x \notin Y)}$

$\rightarrow \boldsymbol{x \in X \cup Y\ \ and\ \ x \notin X \cap Y}$

$\rightarrow \boldsymbol{x \in (X \cup Y)\backslash(X \cap Y)}$

$\rightarrow \boldsymbol{(X\backslash Y) \cup (Y\backslash X) \subseteq (X \cup Y)\backslash(X \cap Y)}$

(Step 2: $\boldsymbol{A \supseteq B}$***)***

By reversing the steps, we can show $\boldsymbol{(X\backslash Y) \cup (Y\backslash X) \supseteq (X \cup Y)\backslash(X \cap Y)}$

Therefore $\boldsymbol{(X\backslash Y) \cup (Y\backslash X) = (X \cup Y)\backslash(X \cap Y)}$

☺Exercise:

Show $X\backslash(Y \cup Z) = (X\backslash Y) \cap (X\backslash Z)$

Given a set formula, how can we say whether it is true or not? If the formula is simple, we can first check by drawing diagrams.

Case i) If your answer is TRUE, then prove it using technique shown above Case ii) If your answer is FALSE, then provide a counter-example

Example 9

a) Is it always true that $(\mathrm{A} \cup \mathrm{B}) \cap \mathrm{C} = \mathrm{A} \cup (\mathrm{B} \cap \mathrm{C})$

b) Is it always true that if $\mathrm{A} \cup \mathrm{B} = \mathrm{A} \cup \mathrm{C}$, then $\mathrm{B} = \mathrm{C}$

c) Is it always true that $\mathrm{A} \backslash (\mathrm{B} \cap \mathrm{C}) = (\mathrm{A} \backslash \mathrm{B}) \cup (\mathrm{A} \backslash \mathrm{C})$

Solution:

a) FALSE

Let $\boldsymbol{A} = \{\mathbf{1}, \mathbf{2}\}$***,*** $\boldsymbol{B} = \{\mathbf{2}, \mathbf{3}\}$ ***and*** $\boldsymbol{C} = \{\mathbf{1}\}$***, then***

$(\boldsymbol{A} \cup \boldsymbol{B}) \cap \boldsymbol{C} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\} \cap \{\mathbf{1}\} = \{\mathbf{1}\}$

$\boldsymbol{but}\ \boldsymbol{A} \cup (\boldsymbol{B} \cap \boldsymbol{C}) = \{\mathbf{1}, \mathbf{2}\} \cup \boldsymbol{\phi} = \{\mathbf{1}, \mathbf{2}\}$

b) FALSE

Let $\boldsymbol{A} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ $\quad \boldsymbol{B} = \{\mathbf{1}, \mathbf{2}\}$ ***and*** $\boldsymbol{C} = \{\mathbf{2}, \mathbf{3}\}$

It is easy to see $\boldsymbol{A} \cup \boldsymbol{B} = \boldsymbol{A} \cup \boldsymbol{C} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$

But $\boldsymbol{B} \neq \boldsymbol{C}$

c) TRUE

The proof is in Example 6.

Remark: When constructing counter-example, it is an good idea to use some simple sets which are easy to handle.

☺Exercise

Decide whether the formulas below are true or not. Provide Reason

a) $(\mathrm{X} \cap \mathrm{Y}) \cup \mathrm{Z} = (\mathrm{X} \cup \mathrm{Z}) \cap (\mathrm{X} \cup \mathrm{Y})$

b) $\mathrm{X} \cap (\mathrm{Y} \backslash \mathrm{Z}) = (\mathrm{X} \cap \mathrm{Y}) \backslash (\mathrm{X} \cap \mathrm{Z})$

More Exercises about Set

(Try to write a short solution for each problem below. You may submit your solutions to me and I will give some comments to it. It is NOT HOMEWORK, JUST FOR PRACTICE ONLY.)

Operation of Set

1. Compute the following set
   a) $\{1,2\} \times \{3,4\} \times \{5\}$
   b) $\big(\{3n: n \in \mathbf{N}\} \cap [2,28]\big) \backslash \{5n: n \in \mathbf{N}\}$
   c) $[(0,3) \cup (2,5)] \setminus [(0,3) \cap (2,5)]$
      (Note: The resultant set is so called ***symmetric difference*** of two sets)

How to prove things?

2. Let $A = \{3x + 7y: \text{x and y are integers}\}$ and
   $B = \{2x - 3y: \text{x and y are integers}\}$

   Prove that $A = B$.
   (Hint: To show $A = B$, we need to show $A \subseteq B$ and $B \subseteq A$)

3. Show that if $A \subseteq B$ and $C \subseteq D$, then $A \cup C \subseteq B \cup D$

4. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

5. Show $A \subset B$ if and only if $A \cup B = B$ (It needs a little bit more work)

6. Decide which of the following are true or not
   a) $(A \cap B)\backslash C = (A\backslash C) \cap (B\backslash C)$
   b) $A\backslash(A\backslash B) = B\backslash(B\backslash A)$
   c) $A \cup (B\backslash A) = B$
   d) If $A \subseteq B$, then $A \cup (B\backslash A) = B$