DEPARTMENT OF MATHEMATICS

# Math2033 TA note 5

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## 1 LIMIT

**Theorem 1** (Supremum property)**.** *If a set $S$ has a supremum in $\mathbb{R}$ and $\epsilon > 0$, then there is $x \in S$ such that* $\sup S - \epsilon < x \leq \sup S$.

**Definition 2.** *A sequence $\{x_n\}$ converges to a number $x$ (or has limit $x$) iff for every $\epsilon > 0$, there is $K \in \mathbb{N}$ such that for every $n > K$, it implies $d(x_n, x) = |x_n - x| < \epsilon$ (which means $x_K, x_{K+1}, x_{K+2}, \ldots \in (x - \epsilon, x + \epsilon)$).*

**Theorem 3** (Supremum Limit Theorem)**.** *Let $S$ be a nonempty set with upper bound $c$. There is a sequence $\{\omega_n\}$ in $S$ converging to $c$ if and only if $c = \sup S$.*

**Example 4.** *Determine if each of the following set has an infimum and a supremum. If they exists, find them and give reasons to support your answers.*

$$S = \left\{ x + y : x, y \in [\frac{1}{2}, 1) \right\} \setminus \left\{ 2 - \frac{1}{n} : n \in \mathbb{N} \right\}$$

*Solution:* We first observe that 1 and 2 is the lower bound and upper bound of $S$ respectively, because $\forall x, y \in [\frac{1}{2}, 1), 1 \leq x + y < 2$.

Next, we show that 1 is the infimum of $S$ using Theorem 1. For any $\epsilon > 0$, we can find a $n > 4$ and $n > \frac{1}{[\frac{\epsilon}{2}]} + 1$. Hence, if we take $x = y = \frac{1}{2} + \frac{1}{n} \in [\frac{1}{2}, 1)$, we have $x + y \in S$ and $1 < x + y < 1 + \frac{2}{n} < 1 + \frac{2}{\frac{1}{[\frac{\epsilon}{2}]} + 1} < 1 + \epsilon$, which shows $\inf S = 1$.

To show $\sup S = 2$, we can construct a sequence $x_n = 2 - \frac{1}{\sqrt{2}n}$. Since $x_n \in S$ and $\lim\limits_{n \to \infty} 2 - \frac{1}{\sqrt{2}n} = 2$, by Theorem 3, we prove 2 is the supremum of $S$.

**Example 5.** *Determine if each of the following set has an infimum and a supremum. If they exists, find them and give reasons to support your answers.*

$$S = \{\frac{k}{n!} : k, n \in \mathbb{N}, \frac{k}{n!} < \sqrt{2}\}$$

*Solution:*(1)
The infimum is 0 and supremum $\sqrt{2}$. We prove it by definition. Firstly $\frac{k}{n!} > 0$ for $k, n \in \mathbb{N}$. Then if $\inf S = a > 0$, fix $k = 1$, we can choose $n$ large enough to make $\frac{1}{n!} < a$ which contradicts $a$ is the infimum of $S$. So $\inf S = 0$. For the supremum, we have $\sup S \le \sqrt{2}$ because $\frac{k}{n!} < \sqrt{2}, k, n \in \mathbb{N}$. If $\sqrt{2}$ is not the supremum and $\sup S = a < \sqrt{2}$. Then we can find a rational number $a < \frac{p}{q} < \sqrt{2}$. Then choose $n = q, k = (q-1)!p$, we have $\frac{k}{n!} = \frac{p}{q}$ which contradicts $a$ is the supremum of $S$. So $\sup S = \sqrt{2}$.
*Solution:*(2)
The infimum is 0 and supremum $\sqrt{2}$. Firstly $\frac{k}{n!} > 0$ for $k, n \in \mathbb{N}$ and the sequence $\frac{1}{n!} \to 0$ as $n \to +\infty$. By infimum limit theorem we have $\inf S = 0$. In the same fashion, we first have $\sup S \le \sqrt{2}$ because $\frac{k}{n!} < \sqrt{2}, k, n \in \mathbb{N}$. We can have a sequence of rational number $\frac{p_i}{q_i} \to \sqrt{2}$ as $i \to +\infty$. Then choose $n_i = q_i, k_i = p_i(q_i - 1)!$, we have $\frac{k_i}{n_i!} \to \sqrt{2}$ as $i \to +\infty$. Then by supremum limit theorem we have $\sup S = \sqrt{2}$.

**Example 6.** *Let $z_n = n^{1/n}$. Show that $\{z_n\}$ converges to* 1 *by checking the definition.*

*Solution:* For every $\epsilon > 0$ and $n \ge 2$, by the binomial theorem,

$$(1+\epsilon)^n = 1 + n\epsilon + \frac{n(n-1)}{2}\epsilon^2 + \cdots + \epsilon^n \ge \frac{n(n-1)}{2}\epsilon^2.$$

By the Archimedean principle, there exists integer $K(\epsilon) \ge \max(2, 1+\frac{2}{\epsilon^2})$. Thus, for each $n > K(\epsilon)$, we have $\epsilon^2 > \frac{2}{n-1}$ and

$$(1+\epsilon)^n \ge \frac{n(n-1)}{2}\epsilon^2 > n.$$

Therefore, $n > K(\epsilon)$ implies $1 \le n^{1/n} < 1+\epsilon$. So $\{z_n\}$ converges to 1.