Solution to Presentation Exercises

(91)(m) Since $0<\frac{k}{n!}<\sqrt{2}$ for all $\frac{k}{n!}\in S$, so S is bounded below by 0 and bounded above by $\sqrt{2}$. For every $n\in\mathbb{N}$, let $k=(n-1)!\in\mathbb{N}$, then $\frac{1}{n}=\frac{k}{n!}\in S$ and $\lim_{n\to\infty}\frac{1}{n}=0$. So $\inf S=0$ by infimum limit theorem.

Next for every $n\in\mathbb{N}$, let $k=(n-1)!\,[n\sqrt{2}]\in\mathbb{N}$, then $\frac{[n\sqrt{2}]}{n}=\frac{k}{n!}\in S$.
Now $\sqrt{2}-\frac{1}{n}=\frac{n\sqrt{2}-1}{n}<\frac{[n\sqrt{2}]}{n}<\frac{n\sqrt{2}}{n}=\sqrt{2}$. Since $\lim_{n\to\infty}\sqrt{2}-\frac{1}{n}=\sqrt{2}$, by sandwich theorem, $\lim_{n\to\infty}\frac{[n\sqrt{2}]}{n}=\sqrt{2}$. So $\sup S=\sqrt{2}$ by supremum limit theorem.

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(91)(n) Note $S=\bigcup_{n=1}^{10}\left([\frac{1}{n\sqrt{2}},2-\frac{1}{n}]\setminus\mathbb{Q}\right)=[\frac{1}{10\sqrt{2}},1.9]\setminus\mathbb{Q}$. So S is bounded below by $\frac{1}{10\sqrt{2}}$ and above by 1.9. We will show $\inf S=\frac{1}{10\sqrt{2}}$ and $\sup S=1.9$.
Since $\frac{1}{10\sqrt{2}}\in S$, every lower bound $m\le\frac{1}{10\sqrt{2}}$, so $\inf S=\frac{1}{10\sqrt{2}}$.
Next, let $w_n=1.9-\frac{1}{n\sqrt{2}}$, then $\frac{1}{10\sqrt{2}}<1<1.9-\frac{1}{\sqrt{2}}\le w_n<1.9$. So $w_n\in S$. Since $\lim_{n\to\infty}w_n=1.9$, by the supremum limit theorem, $\sup S=1.9$.

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(92)(a) (Note $x_1=1<x_2=\frac{1}{2}+\sqrt{1}=\frac{3}{2}<x_3=\frac{3}{4}+\sqrt{\frac{3}{2}}=\frac{3+2\sqrt{6}}{4}$. Also $x=\frac{x}{2}+\sqrt{x}\Rightarrow x=0$ or 4.)
We will show $x_n\le x_{n+1}\le 4$ by induction. For $n=1$, $1\le\frac{3}{2}\le 4$. Next suppose $x_n\le x_{n+1}\le 4$. Then $\frac{x_n}{2}\le\frac{x_{n+1}}{2}\le 2$ and $\sqrt{x_n}\le\sqrt{x_{n+1}}\le\sqrt{4}\Rightarrow x_{n+1}=\frac{x_n}{2}+\sqrt{x_n}\le x_{n+2}=\frac{x_{n+1}}{2}+\sqrt{x_{n+1}}\le 2+\sqrt{4}=4$. Therefore, $\{x_n\}$ is increasing and bounded above. By the monotone sequence theorem, $\lim_{n\to\infty}x_n=x$ exists. Then $x=\frac{x}{2}+\sqrt{x}\Rightarrow x=0$ or 4. Since $x_1>1$, $\lim_{n\to\infty}x_n=x=4$.

(b) (Note $x_1=1<x_2=2<x_3=\sqrt{2}+\sqrt{1}=\sqrt{2}+1$, so we suspect $\{x_n\}$ is increasing.)
We will show $x_n\le x_{n+1}$ for all $n\in\mathbb{N}$ by induction. The cases $n=1,2$ are true as shown above. Assume the cases $n<k$ are true. For the case $n=k$, we have $x_k\le x_{k+1}\Leftrightarrow\sqrt{x_{k-1}}+\sqrt{x_{k-2}}\le\sqrt{x_k}+\sqrt{x_{k-1}}\Leftrightarrow x_{k-2}\le x_k$, which is true by cases $n=k-2$ ($x_{k-2}\le x_{k-1}$) and $n=k-1$ ($x_{k-1}\le x_k$). So $\{x_n\}$ is increasing.

Next we will show $x_n\le 4$ for all $n\in\mathbb{N}$. For $n=1,2$, this is clear. Assume the cases $n<k$ are true, then $x_k=\sqrt{x_{k-1}}+\sqrt{x_{k-2}}\le\sqrt{4}+\sqrt{4}=4$. So by induction, $x_n\le 4$ for all $n\in\mathbb{N}$. By the monotone sequence theorem, $\{x_n\}$ converges. Let $x=\lim_{n\to\infty}x_n$, then $x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}(\sqrt{x_n}+\sqrt{x_{n-1}})=2\sqrt{x}\Rightarrow x=0$ or 4. Since $1=x_1\le x$, $x=4$.

(92) (i) $x_1=2$, $x_2=\frac{3}{2}=1.5$, $x_3=\frac{4}{3}=1.33\cdots$. We suspect $\{x_n\}$ is decreasing.

Thoughts: If $\{x_n\}$ converges to x, then $x=\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}(2-\frac{1}{x_n})=2-\frac{1}{x}$, which leads to $x=2-\frac{1}{x}$, hence $x=1$.

Solution: We claim $1\le x_{n+1}\le x_n$ for $n=1,2,3,\cdots$. For $n=1$, $1\le x_2=1.5\le x_1=2$. Suppose the n-th case is true (that is $1\le x_{n+1}\le x_n$). Then $\frac{1}{1}\ge\frac{1}{x_{n+1}}\ge\frac{1}{x_n}$ and so $2-\frac{1}{1}\le 2-\frac{1}{x_{n+1}}\le 2-\frac{1}{x_n}$ yielding $1\le x_{n+2}\le x_{n+1}$. By M.I., the claim is true. Now the claim implies $\{x_n\}$ is decreasing and is bounded below by 1, So $\lim\limits_{n\to\infty}x_n=x$ exists by monotone Sequence theorem. Then $x=\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}(2-\frac{1}{x_n})=2-\frac{1}{x}\Rightarrow x=1$.

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(101) Let $x\in\mathbb{R}$. For every positive integer n, since $x-\frac{1}{n}<x-\frac{1}{n+1}$, by the density of irrational numbers, there exists $x_n\in\mathbb{R}\setminus\mathbb{Q}$ such that $x-\frac{1}{n}<x_n<x-\frac{1}{n+1}$. Since $\lim\limits_{n\to\infty}(x-\frac{1}{n})=x=\lim\limits_{n\to\infty}(x-\frac{1}{n+1})$, by sandwich theorem, $\lim\limits_{n\to\infty}x_n=x$. Finally, x_n is strictly increasing because $x_n<x-\frac{1}{n+1}<x_{n+1}<x-\frac{1}{n+2}$.

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(102) Note $\frac{1}{n^2}<\frac{\varepsilon}{2}\Leftrightarrow\sqrt{\frac{2}{\varepsilon}}<n$ and $\frac{\sqrt{2}}{n^3}<\frac{\varepsilon}{2}\Leftrightarrow\sqrt[3]{\frac{2\sqrt{2}}{\varepsilon}}<n$. For every $\varepsilon>0$, by the Archimedean principle, there exists $K\in\mathbb{N}$ such that $K>\max(\sqrt{\frac{2}{\varepsilon}},\sqrt[3]{\frac{2\sqrt{2}}{\varepsilon}})$. Then $n\ge K\Rightarrow\left|(\frac{1}{n^2}-\frac{\sqrt{2}}{n^3})-0\right|\le\frac{1}{n^2}+\frac{\sqrt{2}}{n^3}\underset{\text{from Note above}}{<}\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. So $\lim\limits_{n\to\infty}(\frac{1}{n^2}-\frac{\sqrt{2}}{n^3})=0$ by definition.

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(104) For every $\varepsilon>0$, since $\lim\limits_{n\to\infty}x_n=0$, there is $K_1\in\mathbb{N}$ such that $n\ge K_1\Rightarrow|x_n-0|<\frac{\varepsilon}{2}$. By the Archimedean principle, there is $K_2\in\mathbb{N}$ such that $K_2>\frac{2}{\varepsilon}$. Let $K=\max(K_1,K_2)$. Then $n\ge K\Rightarrow|(x_n+\frac{1}{n})-0|\le|x_n-0|+\frac{1}{n}<\frac{\varepsilon}{2}+\frac{1}{K_2}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. Therefore, $\lim\limits_{n\to\infty}(x_n+\frac{1}{n})=0$ by definition.

$n\ge K_1$ $n\ge K_2\Rightarrow\frac{1}{n}\le\frac{1}{K_2}<\frac{\varepsilon}{2}$

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(105) Since $\lim\limits_{n\to\infty}x_n=\frac{1}{2}$, so for $\varepsilon_0=\frac{1}{3}$, there is $K_1\in\mathbb{N}$ such that $n\ge K_1\Rightarrow|x_n-\frac{1}{2}|<\varepsilon_0=\frac{1}{3}$ $\Rightarrow-\frac{1}{3}<x_n-\frac{1}{2}<\frac{1}{3}\Rightarrow\frac{1}{6}<x_n<\frac{5}{6}\Rightarrow|x_n^n-0|<(\frac{5}{6})^n$. So for every $\varepsilon>0$, let $K=\max(K_1,\lceil\frac{\ln 1/\varepsilon}{\ln 6/5}\rceil)$, then $n\ge K\Rightarrow|x_n^n-0|<(\frac{5}{6})^n\le\varepsilon$.