(78) Let $h>0$ and $x=c+2h$. By Taylor's theorem, there is $x_0\in(c,x)$ such that

$$f(x)=f(c)+f'(c)\underbrace{(x-c)}_{2h}+\frac{f''(x_0)}{2}\underbrace{(x-c)^2}_{4h^2} \Rightarrow f'(c)=\frac{f(x)-f(c)}{2h}-f''(x_0)h$$

$$\Rightarrow |f'(c)|\le \frac{1}{2h}(|f(x)|+|f(c)|)+|f''(x_0)|h$$

$$\le \frac{M_0}{h}+M_2h \text{ for every } h>0.$$

By calculus, $\frac{M_0}{h}+M_2h$ has minimum value $2\sqrt{M_0M_2}$ when $h=\sqrt{\frac{M_0}{M_2}}$, so $|f'(c)|\le 2\sqrt{M_0M_2}$ for every $c\in\mathbb{R}$. Then $M_1\le 2\sqrt{M_0M_2}$, i.e. $M_1^2\le 4M_0M_2$.

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(79) (a) For every $\varepsilon>0$, let $\delta=\frac{\varepsilon}{2}$, then $|x-t|<\delta \Rightarrow |f(x)-f(t)| \underset{\text{mean-value theorem}}{=} |f'(c_0)(x-t)|\le 2|x-t| < 2\delta=\varepsilon$. $\therefore$ f is uniformly continuous.

(b) Suppose $f(x)=\sin\frac{1}{x}$ is uniformly continuous on $(0,\infty)$. Then for every $\varepsilon>0$ (in particular $\varepsilon=1$), there is $\delta>0$ such that $\forall x,t\in(0,\infty)$, $|x-t|<\delta$ $\Rightarrow |f(x)-f(t)|<\varepsilon=1$. By Archimedian principle, $\exists\, n\in\mathbb{N}$ such that $n>\sqrt{\frac{1}{\pi\delta}}$. Now let $x=\frac{1}{n\pi}$ and $t=\frac{1}{(n+\frac12)\pi}$, then $|x-t|=\left|\frac{1}{n\pi}-\frac{1}{(n+\frac12)\pi}\right|=\frac{1}{2n(n+\frac12)\pi}<\frac{1}{n^2\pi}<\delta$, but $|f(x)-f(t)|=|\sin n\pi-\sin(n+\frac12)\pi|=1$, a contradiction.

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(80) (a) Suppose the statement is false. Let $m_1=(a+b)/2$. Then one of $[a,m_1]$ or $[m_1,b]$ is not contained in the union of finitely many of these open intervals, call that interval I_1. Again, we divide I_1 into two using its midpoint. Then one of these two, call it I_2, is not contained in the union of finitely many of these open intervals. Continuing this process, we get closed intervals $[a,b]\supseteq I_1\supseteq I_2\supseteq\cdots$ and length of I_n goes to 0. So by nested interval theorem, $\bigcap_{n=1}^{\infty} I_n=\{x\}$. Since $x\in[a,b]$, one of the open intervals will contain x. Since length of I_n goes to 0, this open interval containing x will contain some I_n, contradicting the definition of I_n. Therefore, the statement must be true.

(b). If $f:[a,b]\to\mathbb{R}$ is continuous, then $\forall\varepsilon>0$, $\forall t\in[a,b]$, $\underset{f \text{ continuous at } t \text{ implies}}{\exists}$ $\delta_t>0$ such that $x\in(t-\delta_t,t+\delta_t)\Rightarrow |f(x)-f(t)|<\frac{\varepsilon}{2}$. Since $[a,b]\subseteq\bigcup_{t\in[a,b]}(t-\frac{\delta_t}{2},t+\frac{\delta_t}{2})$, by part (a) $\exists\, t_1,\dots,t_n\in[a,b]$ such that $[a,b]\subseteq(t_1-\frac{\delta_{t_1}}{2},t_1+\frac{\delta_{t_1}}{2})\cup\dots\cup(t_n-\frac{\delta_{t_n}}{2},t_n+\frac{\delta_{t_n}}{2})$. Let $\delta=\frac12\min\{\delta_{t_1},\dots,\delta_{t_n}\}>0$. Now for every $x,y\in[a,b]$ with $|x-y|<\delta$, we have $x\in(t_i-\frac{\delta_{t_i}}{2},t_i+\frac{\delta_{t_i}}{2})$ for some i. So $|x-t_i|<\frac{\delta_{t_i}}{2}<\delta_{t_i}$ and $|y-t_i|\le|y-x|+|x-t_i|<\delta+\frac{\delta_{t_i}}{2}\le\frac{\delta_{t_i}}{2}+\frac{\delta_{t_i}}{2}=\delta_{t_i}$. Then $|f(x)-f(y)|\le|f(x)-f(t_i)|+|f(t_i)-f(y)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.

Therefore, f is uniformly continuous on $[a,b]$.

(81) Solution 1 Assume $f(x_0)>0$ for some $x_0\in[a,b]$. Since f is continuous at x_0, for $\varepsilon=\frac{f(x_0)}{2}$, there is a $\delta>0$ such that $x\in[a,b]\cap(x_0-\delta,x_0+\delta)$ implies $|f(x)-f(x_0)|<\varepsilon=\frac{f(x_0)}{2}$. Then $-\frac{f(x_0)}{2}<f(x)-f(x_0)$ so that $f(x)>\frac{f(x_0)}{2}>0$.
Now $[a,b]\cap(x_0-\delta,x_0+\delta)$ contains a closed interval $[c,d]$ of positive length. Then $0<\int_c^d \frac{f(x_0)}{2}\,dx<\int_c^d f(x)\,dx\le\int_a^b f(x)\,dx=0$, contradiction. So $f(x)\equiv0$ $\forall x\in[a,b]$.

Solution 2 Define $g(t)=\int_a^t f(x)\,dx$. Since f is continuous on $[a,b]$, by the fundamental theorem of calculus, $g'(x)=f(x)\ge0$ for all $x\in[a,b]$. So g is increasing on $[a,b]$. Since $0=g(a)\le g(t)\le g(b)=\int_a^b f(x)\,dx=0$, we must have $g(x)=0$ for all $x\in[a,b]$. Then $f(x)=g'(x)=0$ for all $x\in[a,b]$.

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(82) (i) For $\varepsilon>0$, since f is integrable on $[a,b]$ and $[b,c]$, by the integral criterion, there are partition P_1 of $[a,b]$ such that $U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{2}$ and partition P_2 of $[b,c]$ such that $U(f,P_2)-L(f,P_2)<\frac{\varepsilon}{2}$. Then $P=P_1\cup P_2$ is a partition of $[a,c]$ and

$$\begin{aligned}U(f,P)-L(f,P)&=(U(f,P_1)+U(f,P_2))-(L(f,P_1)+L(f,P_2))\\&=(U(f,P_1)-L(f,P_1))+(U(f,P_2)-L(f,P_2))\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

So by the integral criterion, f is integrable on $[a,c]$.

(ii) For $\varepsilon>0$, since f is integrable on $[a,d]$, by the integral criterion, there is a partition P_1 of $[a,d]$ such that $U(f,P_1)-L(f,P_1)<\varepsilon$. Then $P_2=P_1\cup\{b,c\}$ is finer partition of P_1 so that $L(f,P_1)\le L(f,P_2)\le U(f,P_2)\le U(f,P_1)$. Then $U(f,P_2)-L(f,P_2)\le U(f,P_1)-L(f,P_1)<\varepsilon$. Now $P=P_2\cap[b,c]$ is a partition of $[b,c]$ and $\underbrace{U(f,P)-L(f,P)}\le U(f,P_2)-L(f,P_2)<\varepsilon$.

Only the terms of $U(f,P_2)-L(f,P_2)=\sum_{i=1}^{n}\underbrace{(M_i-m_i)}_{\ge0}\Delta x_i$ in $[b,c]$ are used to compute $U(f,P)-L(f,P)$

So by the integral criterion, f is integrable on $[b,c]$.

(83) Consider the subintervals $[a, x_1]$, $[x_1, \frac{x_1+x_2}{2}]$, $[\frac{x_1+x_2}{2}, x_2]$, $\cdots$, $[\frac{x_{n-1}+x_n}{2}, x_n]$, $[x_n, b]$. By exercise 82(i), it is enough to show f is integrable on each of these intervals. (If $a=x_1$, then ignore $[a,x_1]$. If $x_n=b$, then ignore $[x_n,b]$.) In each of the subinterval $[u,v]$ above, either f is discontinuous only at u or f is discontinuous only at v. In the former case, since f is bounded on $[a,b]$, there is $K>0$ such that $|f(x)|\le K$ for every $x\in[a,b]$. For $\varepsilon>0$, choose $w\in(u,v)$ such that $2K(w-u)<\frac{\varepsilon}{2}$ ($\Leftrightarrow w<u+\frac{\varepsilon}{4K}$). Since f is continuous on $[w,v]$, f is integrable on $[w,v]$. By the integral criterion, there is a partition P_1 of $[w,v]$ such that $U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{2}$. Let $P=\{u\}\cup P_1$, then P is a partition of $[u,v]$ and $U(f,P)-L(f,P)=(M_1-m_1)(w-u)+U(f,P_1)-L(f,P_1)\le 2K(w-u)+\frac{\varepsilon}{2}<\varepsilon$. So by the integral criterion, f is integrable on $[u,v]$. The latter case when f is discontinuous only at v is similar.

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(84) (i) Since $\inf\limits_{x\in[x_{i-1},x_i]} f(x)+\inf\limits_{x\in[x_{i-1},x_i]} g(x)$ is a lower bound of $\{f(x)+g(x): x\in[x_{i-1},x_i]\}$, we get

$$\inf_{x\in[x_{i-1},x_i]} f(x)+\inf_{x\in[x_{i-1},x_i]} g(x)\le \inf_{x\in[x_{i-1},x_i]}(f(x)+g(x)). \text{ So } L(f,P)+L(g,P)\le L(f+g,P).$$

Similarly, $U(f+g,P)\le U(f,P)+U(g,P)$ since $\sup\limits_{x\in[x_{i-1},x_i]}(f(x)+g(x))\le \sup\limits_{x\in[x_{i-1},x_i]} f(x)+\sup\limits_{x\in[x_{i-1},x_i]} g(x)$.

(ii) For $\varepsilon>0$, since $\int_a^b f(x)\,dx=\sup\{L(f,P): P \text{ partition of } [a,b]\}$, by the supremum property, there is a partition P_1 such that $\int_a^b f(x)\,dx-\frac{\varepsilon}{2}<L(f,P_1)\le\int_a^b f(x)\,dx$. Similarly, there is a partition P_2 such that $\int_a^b g(x)\,dx-\frac{\varepsilon}{2}<L(g,P_2)\le\int_a^b g(x)\,dx$. Letting $P=P_1\cup P_2$, then $P_1,P_2\subseteq P$. So

$$\int_a^b f(x)\,dx+\int_a^b g(x)\,dx-\varepsilon<L(f,P_1)+L(g,P_2)\le L(f,P)+L(g,P)\overset{\text{by part (i)}}{\le} L(f+g,P)\le (L)\int_a^b (f(x)+g(x))\,dx.$$

By the infinitesimal principle, $\int_a^b f(x)\,dx+\int_a^b g(x)\,dx\le (L)\int_a^b (f(x)+g(x))\,dx$. Similarly, the inequality $\int_a^b f(x)\,dx+\int_a^b g(x)\,dx\ge (U)\int_a^b (f(x)+g(x))\,dx$ can be obtained by using the infimum property. Combining, we get

$$\int_a^b f(x)\,dx+\int_a^b g(x)\,dx\le (L)\int_a^b (f(x)+g(x))\,dx\le (U)\int_a^b (f(x)+g(x))\,dx\le\int_a^b f(x)\,dx+\int_a^b g(x)\,dx.$$

Therefore, equality must hold throughout, i.e. $f+g$ is integrable and $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx+\int_a^b g(x)\,dx$.

(85) (a) $\int_0^\infty \frac{dx}{\sqrt{e^x}} = \int_0^\infty e^{-\frac{1}{2}x}dx = \lim_{d\to+\infty} \int_0^d e^{-\frac{1}{2}x}dx = \lim_{d\to+\infty}\left(-\frac{1}{2}e^{-\frac{1}{2}x}\Big|_0^d\right) = \lim_{d\to+\infty}\left(-2e^{-\frac{1}{2}d}+2\right) = 2$

Integral exists

(b) $\int_0^\infty \sin x\,dx = \lim_{d\to+\infty}\int_0^d \sin x\,dx = \lim_{d\to+\infty}\left(-\cos x\Big|_0^d\right) = \lim_{d\to+\infty}(-\cos d+1)$ does not exist.

(c) Note $0 \le \frac{1}{6x} \le \frac{1}{x^2+5x}$ for $x\in(0,1]$. $\int_0^1 \frac{1}{6x}dx = \lim_{c\to0^+}\int_c^1 \frac{1}{6x}dx = \lim_{c\to0^+}\left(\frac{1}{6}\ln x\Big|_c^1\right)$
$= \lim_{c\to0^+}\left(-\frac{1}{6}\ln c\right)$ does not exists. By comparison test, $\int_0^1 \frac{dx}{x^2+5x}$ does not exist.

(d) $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = \int_{-1}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{d\to0^-}\int_{-1}^d \frac{dx}{\sqrt[3]{x}} + \lim_{c\to0^+}\int_c^1 \frac{dx}{\sqrt[3]{x}} = \lim_{d\to0^-}\left(\frac{3}{2}x^{\frac{2}{3}}\Big|_{-1}^d\right) + \lim_{c\to0^+}\left(\frac{3}{2}x^{\frac{2}{3}}\Big|_c^1\right)$
$= \lim_{d\to0^-}\left(\frac{3}{2}d^{\frac{2}{3}}-\frac{3}{2}\right) + \lim_{c\to0^+}\left(\frac{3}{2}-\frac{3}{2}c^{\frac{2}{3}}\right) = -\frac{3}{2}+\frac{3}{2} = 0$. Integral exists.

(e) $\int_0^1 \frac{dx}{x(x-1)} = \int_0^{\frac{1}{2}} \frac{dx}{x(x-1)} + \int_{\frac{1}{2}}^1 \frac{dx}{x(x-1)}$. $\int_0^{\frac{1}{2}} \frac{dx}{x(x-1)} = \lim_{c\to0^+}\int_c^{\frac{1}{2}} \frac{dx}{x(x-1)} = \lim_{c\to0^+}\int_c^{\frac{1}{2}}\left(\frac{1}{x-1}-\frac{1}{x}\right)dx$
$= \lim_{c\to0^+}\left(\ln|x-1| - \ln|x|\right)_c^{\frac{1}{2}} = \lim_{c\to0^+}\left(-\ln|c-1| + \ln|c|\right) = 0-\infty$ does n't exist (as a number). So $\int_0^1 \frac{dx}{x(x-1)}$ does not exist.

(f) For $x\in(0,+\infty)$, $\left|\frac{\cos x}{1+x^2}\right| \le \frac{1}{1+x^2}$. Since $\int_0^{+\infty}\frac{1}{1+x^2}dx = \lim_{b\to+\infty}\int_0^b \frac{1}{1+x^2}dx = \lim_{b\to+\infty}\tan^{-1}b = \frac{\pi}{2}$,
$\int_0^{+\infty}\left|\frac{\cos x}{1+x^2}\right|dx$ exists by the comparison test. Then $\int_0^{+\infty}\frac{\cos x}{1+x^2}dx$ exists by the absolute convergence test.

(86) (a) $\text{P.V.}\int_{-\infty}^\infty \frac{x}{e^{x^2}}dx = \lim_{b\to+\infty}\int_{-b}^b xe^{-x^2}dx = \lim_{b\to+\infty}\left(-\frac{1}{2}e^{-x^2}\Big|_{-b}^b\right) = \lim_{b\to+\infty}\left(\underbrace{-\frac{1}{2}e^{-b^2}+\frac{1}{2}e^{-b^2}}_{=0}\right) = 0$

(b) $\text{P.V.}\int_0^2 \frac{dx}{x^2-1} = \lim_{\varepsilon\to0^+}\left(\int_0^{1-\varepsilon}\frac{dx}{x^2-1} + \int_{1+\varepsilon}^2 \frac{dx}{x^2-1}\right) = \lim_{\varepsilon\to0^+}\left(\int_0^{1-\varepsilon}\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx + \int_{1+\varepsilon}^2 \frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx\right)$
$= \lim_{\varepsilon\to0^+}\left(\cancel{\frac{1}{2}\ln\varepsilon} - \frac{1}{2}\ln(2-\varepsilon) - \cancel{\frac{1}{2}\ln\varepsilon} - \frac{1}{2}\ln3 + \frac{1}{2}\ln(2+\varepsilon)\right) = -\frac{1}{2}\ln3$.

(87) We have $\int_0^\infty t^{x-1}e^{-t}dt = \int_0^1 t^{x-1}e^{-t}dt + \int_1^\infty t^{x-1}e^{-t}dt$.

For $\int_0^1 t^{x-1}e^{-t}dt$, since $\lim_{t\to0^+}\frac{t^{x-1}e^{-t}}{t^{x-1}} = \lim_{t\to0^+}e^{-t} = 1$, by the limit comparison tes
$\int_0^1 t^{x-1}e^{-t}dt$ converges $\Leftrightarrow \int_0^1 t^{x-1}dt = \int_0^1 \frac{1}{t^{1-x}}dt$ converges $\overset{\text{p-test}}{\Longleftrightarrow} 1-x<1 \Leftrightarrow x>$

For $\int_1^\infty t^{x-1}e^{-t}dt$, note that $\lim_{t\to+\infty}\frac{t^{x-1}e^{-t}}{\frac{1}{t^2}} = \lim_{t\to+\infty}\frac{t^{x+1}}{e^t} = 0$ by example 1 on p. 39.
Since $\int_1^\infty \frac{1}{t^2}dt$ converges by p-test, so by the limit comparison test, $\int_1^\infty t^{x-1}e^{-t}dt$ converges. Therefore, $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ converges for $x>0$.