MATH 2031 Introduction to Real Analysis

May 9, 2013

Tutorial Note 21
Review on Riemann Integral

Proper integral
Let $f(x)$ be a function which is bounded on a closed and bounded interval $[a,b]$.

(I) **Definition (partition):**

(i) A partition P of $[a,b]$ is a finite set $\{x_0, x_1, \cdots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_n = b$.

(ii) Denote $m_j = \inf\{f(x) | x \in [x_{j-1}, x_j]\}$ and $M_j = \sup\{f(x) | x \in [x_{j-1}, x_j]\}$.

(II) **Definition (Riemann sums):**
Given a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ and a function f bounded on $[a,b]$,

(i) A Riemann sum of f is $S = \sum_{j=1}^{n} f(t_j)\Delta x_j$, where every $t_j \in [x_{j-1}, x_j]$.

(ii) A lower Riemann sum of f is $L(f,P) = \sum_{j=1}^{n} m_j \Delta x_j$, where every $t_j \in [x_{j-1}, x_j]$.

(iii) A upper Riemann sum of f is $U(f,P) = \sum_{j=1}^{n} M_j \Delta x_j$, where every $t_j \in [x_{j-1}, x_j]$.

Remark:
Since f is bounded, $|f(x)| < K$ on $[a,b]$. So we have

$$-K \le m_j \le f(t_j) \le M_j \le K \quad \Rightarrow \quad -K(b-a) \le L(f,P) \le S \le U(f,P) \le K(b-a).$$

(III) **Definition (refinement):**

(i) Given partitions P_1, P_2 of the same interval $[a,b]$, we say that P_2 is a refinement of P_1 iff $P_1 \subseteq P_2$.

(ii) Given partitions P_1, P_2 of the same interval $[a,b]$, we say that $P_1 \cup P_2$ is the common refinement of P_1 and P_2.

(IV) **Refinement theorem:**
If $P \subseteq \widetilde{P}$, then

$$\underbrace{L(f,P) \le L(f,\widetilde{P})}_{\text{Lower sum increasing}} \quad \le \quad \underbrace{U(f,\widetilde{P}) \le U(f,P)}_{\text{Upper sum decreasing}}$$

Remark:
The above inequality gives

$$U(f,\widetilde{P}) - L(f,\widetilde{P}) \le U(f,P) - L(f,P).$$

(V) **Integral criterion:**
Let $f(x)$ be bounded on $[a, b]$. Then

$$f(x) \text{ is Riemann integrable on } [a,b] \iff \begin{pmatrix} \forall \varepsilon > 0 \ \exists \text{ partition } P \text{ of } [a,b] \text{ such that} \\ U(f,P) - L(f,P) < \varepsilon \end{pmatrix}$$

Remark:
We may rewrite $U(f, P) - L(f, P)$ as follows:

$$U(f,P) - L(f,P) = \sum_{j=1}^{n} M_j \Delta x_j - \sum_{j=1}^{n} m_j \Delta x_j = \sum_{j=1}^{n} (M_j - m_j) \Delta x_j$$

(VI) **Definition:**

(i) A set $S \subseteq \mathbb{R}$ is of measure zero iff $\begin{pmatrix} \forall \varepsilon > 0, \ \exists \text{ intervals } (a_1, b_1), (a_2, b_2) \cdots \text{ such that} \\ S \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \text{ and } \sum_{k=1}^{\infty} (b_k - a_k) < \varepsilon \end{pmatrix}$.

(ii) A property is said to hold almost everywhere (a.e.) iff it holds except on a set of measure zero.

Remarks:

(i) Countable sets are of measure zero, but uncountable sets may or may not be of measure zero. (The set of irrational numbers in $[a, b]$ for $a < b$ has measure $b - a$, while the Cantor set is uncountable but of measure zero)

(ii) A countable union of measure zero sets is also of measure zero.

(iii) Subsets of a measure zero set are again of measure zero.

(iv) The limit of a sequence of Riemann integrable functions on $[a, b]$ may not be a Riemann integrable function.

(VII) **Lebesgue's Theorem**
Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then

$$f \text{ is integrable on } [a,b] \iff f \text{ is continuous a.e. on } [a,b]$$

(VIII) **Monotone Function Theorem**
If f is increasing on (a, b), then f has countably many points of discontinuity on (a, b). Hence we have

$$S_f = \{x_0 \in [a,b] | \ f \text{ is discontinuous at } x_0\} \text{ is countable.}$$

(IX) **Fundamental theorem of Calculus**

(i) If f is integrable on $[a, b]$, continuous at $x_0 \in [a, b]$ and $F(x) = \displaystyle\int_c^x f(t)dt$, where $c \in [a, b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

(ii) If G is differentiable on $[a, b]$ and G' is integrable on $[a, b]$, then $\displaystyle\int_a^b G'(x)dx = G(x)\Big|_a^b = G(b) - G(a)$. ($G'$ may not be continuous.)

Problem 1 (2008 Q5) (c.f. 2011 Q6)

For $n = 0, 1, 2, \ldots$, let $f_n : [0,1] \to [0,1]$ be Riemann integrable functions. Prove that $g : [0,1] \to \mathbb{R}$ defined by $g(0) = 0$ and

$$g(x) = f_n(x) \quad \text{for } n \in \mathbb{N} \text{ and } \quad x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$$

is Riemann integrable on $[0,1]$ by using integrable criterion.

Solution:

For every $\varepsilon > 0$, by Archimedean principle, there exists $N \in \mathbb{N}$ such that $n > \frac{3}{\varepsilon}$.

Choose a number $0 < \delta < \min\left\{\frac{1}{2}\left(\frac{1}{N-1} - \frac{1}{N}, \frac{\varepsilon}{6(N-1)}\right)\right\}$ and take the partition

$$P_0 = \left\{\frac{1}{N} < \frac{1}{N} + \delta < \frac{1}{N-1} - \delta < \cdots < \frac{1}{2} + \delta < 1 - \delta < 1\right\}.$$

Since for all $n \in \{1, 2, \cdots, N-1\}$, f_n is integrable on $\left(\frac{1}{n+1}, \frac{1}{n}\right]$, there exist partitions P_n on $\left(\frac{1}{n+1}, \frac{1}{n}\right]$ such that

$$U(f_n, P_n) - L(f_n, P_n) < \frac{\varepsilon}{3(N-1)}.$$

Now consider the partition $P = \bigcup_{n=0}^{N-1} P_n$. Then

$$\begin{aligned}
U(g,P) - L(g,P) &\le \frac{1}{N} + 2\delta(N-1)(1-0) + \sum_{n=1}^{N-1} \left(U(f_n, P_n) - L(f_n, P_n)\right) \\
&< \frac{\varepsilon}{3} + 2(N-1)\frac{\varepsilon}{6(N-1)} + (N-1)\frac{\varepsilon}{3(N-1)} \\
&= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Thus by integrable criterion, we get g is integrable.

Problem 2 (adopted form 2010 Q4)

Let $f : [0,1] \to [0,1]$ be a Riemann integrable function. Let $\{r_n | n \in \mathbb{N}\}$ be a strictly increasing sequence on $(0,1]$. Prove that $g : [0,1] \to [0,1]$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in [0,1] \setminus \{r_n | n \in \mathbb{N}\} \\ 1 & \text{if } x \in \{r_n | n \in \mathbb{N}\} \end{cases}$$

is Riemann integrable on $[0,1]$.

Solution:

Since the sequence $\{r_n | n \in \mathbb{N}\}$ is increasing and bounded, by monotonic sequence theorem, $\lim_{n\to\infty} r_n$ exists, say $\lim_{n\to\infty} r_n = r$. i.e.

$$\text{For all } \varepsilon_0 > 0, \exists N_0 \in \mathbb{N} \text{ such that for every } n > n_0, |r - r_n| < \varepsilon_0.$$

$$(\text{If } r = 1, \text{ then replace } |r - r_n| < \varepsilon_0 \text{ by } 1 - r_n < \varepsilon_0).$$

For every $\varepsilon > 0$, by definition of limit, there exists $N \in \mathbb{N}$ such that $|r - r_n| < \frac{\varepsilon}{3}$.

Choose a number $0 < \delta < \min\left\{\frac{1}{2}(r_{N-1} - r_N), \frac{\varepsilon}{6(N-1)}\right\}$ and take a partition

$$P_0 = \{r_1 - \delta < r_1 + \delta < r_2 - \delta < \cdots < r_N - \delta < r_N < r < 1\}.$$

$$(\text{Or } P_0 = \{r_1 - \delta < r_1 + \delta < r_2 - \delta < \cdots < r_N - \delta < r_N < r = 1\} \text{ if } r = 1.)$$

Since $f(x)$ is integrable on $[0,1]$, there exists a partition P_1 on $[0,1]$ such that

$$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{3}.$$

Now consider the partition $P = P_0 \cup P_1$. Then

$$\begin{aligned} U(g,P) - L(g,P) &\leq r - r_N + 2\delta(N-1)(1-0) + (U(f,P_1) - L(f,P_1)) \\ &< \frac{\varepsilon}{3} + 2(N-1)\frac{\varepsilon}{6(N-1)} + \frac{\varepsilon}{3} \\ &= \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Then by integrable criterion, we get g is integrable.

Improper integral

Now we focus on functions $f(x)$ which are either unbounded or defined on an interval which is not closed or not bounded.

(I) **Definition (Local Integrability):**
Let I be an interval. A function $f : I \to \mathbb{R}$ is locally integrable if f is integrable on every closed and bounded subinterval of I. We denote this by $f \in L_{loc}(I)$.

(II) **Definition (Improper Integrals):**

Case 1: Let $a \in \mathbb{R},\ b \in \mathbb{R} \cup \{+\infty\},\ I = [a,b),\ f \in L_{loc}(I)$.
The improper integral of f on $[a,b)$ is

$$\int_a^b f(x)dx = \lim_{d \to b^-} \int_a^d f(x)dx$$

provided that the limit exists in $\mathbb{R}$.
In this case, we say that f is improper integrable on $[a,b)$.

Case 2: Let $a \in \mathbb{R} \cup \{-\infty\},\ b \in \mathbb{R} \cup \{+\infty\},\ I = (a,b),\ x_0 \in I,\ f \in L_{loc}(I)$.
The improper integral of f on (a,b) is

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^{x_0} f(x)dx + \lim_{d \to b^-} \int_{x_0}^d f(x)dx$$

provided that the limit exists in $\mathbb{R}$.
In this case, we say that f is improper integrable on (a,b).

Remark:
This definition is independent of the choice of x_0.

Case 3: Let $a \in \mathbb{R} \cup \{-\infty\},\ b \in \mathbb{R} \cup \{+\infty\},\ I$ be an interval with endpoints $a, b,\ I_0 = I \cap (-\infty, c)$, $I_1 = I \cap (c, +\infty)$ for $c \in (a,b)$. $f \in L_{loc}(I_0)$, $f \in L_{loc}(I_1)$
The improper integral of f on I is

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

provided that both integrals on the RHS exist in $\mathbb{R}$.
In this case, we say that f is improper integrable on I.

In each case, if the improper integral is a number, we say that the improper integral converges, otherwise it diverges.

(III) **Tests for Improper integral:**

***p*-test:**

For $0 < a < \infty$, $\displaystyle\int_a^{+\infty} \frac{1}{x^p} dx < +\infty \iff p > 1$;

Also $\displaystyle\int_0^a \frac{1}{x^p} dx < +\infty \iff p < 1$.

Comparison test:

Suppose $0 \leq f(x) \leq g(x)$ on interval I and $f, g \in L_{loc}(I)$.
If g is improper integrable on I, then f is also improper integrable on I.

Limit Comparison test:

Suppose $f(x), g(x) > 0$ on $(a, b]$ and $f, g \in L_{loc}((a, b])$.

If $\displaystyle\lim_{x \to a^+} \frac{g(x)}{f(x)} = \left\{ \begin{array}{l} L > 0 \\ 0 \\ +\infty \end{array} \right\}$, then $\left\{ \begin{array}{l} \text{either both } \int_a^b f(x)dx, \int_a^b g(x)dx \text{ converge or both diverge} \\ \int_a^b f(x)dx \text{ converges} \Rightarrow \int_a^b g(x)dx \text{ converges} \\ \int_a^b f(x)dx \text{ diverges} \Rightarrow \int_a^b g(x)dx \text{ diverges} \end{array} \right\}$

For an interval $[a, b)$, we take $\displaystyle\lim_{x \to b^-} \frac{g(x)}{f(x)}$ and the results are similar.

Absolute Convergence test:

Let $f \in L_{loc}(I)$. If $|f|$ is improper integrable on I, then f is improper integrable on I.

<u>Cauchy Principal Value of Integrals</u>

P.V. (I) **Definition:**

Let $f \in L_{loc}(\mathbb{R})$. The principal value of $\displaystyle\int_{-\infty}^{\infty} f(x)dx$ is

$$P.V. \int_{-\infty}^{\infty} f(x)dx = \lim_{c \to \infty} \int_{-c}^{c} f(x)dx$$

P.V. (II) **Theorem:**

If the improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)dx$ exists in $\mathbb{R}$,

then $P.V. \displaystyle\int_{-\infty}^{\infty} f(x)dx$ exists and equals the improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)dx$.

P.V. (III) **Definition:**

Let I be an integral with endpoints a, b, let $c \in (a, b)$, $I_0 = I \cap (-\infty, c)$, $I_1 = I \cap (c, +\infty)$. Let $f \in L_{loc}(I_0)$, $f \in L_{loc}(I_1)$.

Define the principal value of the improper integral $\displaystyle\int_a^b f(x)dx$ as

$$P.V. \int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^{b} f(x)dx \right)$$

Remark:

You should check carefully before applying the Fundamental theorem, which require that the primitive function of $f(x)$ is differentiable. It may happen that your integral is improper.

Problem 3 Determine whether the following improper integral converges or not, then determine whether their principal value converges or not.

(i) $\displaystyle\int_{-1}^{1} \frac{\sin(\sin(x))}{x}\, dx$

(ii) $\displaystyle\int_{-1}^{1} \frac{\sin(\sin(x))}{x^2}\, dx$

Solution:

(i) Since $\dfrac{\sin(\sin(x))}{x}$ is continuous on $[-1,1]\setminus\{0\}$,

$$\int_{-1}^{1} \frac{\sin(\sin(x))}{x}\, dx = \int_{-1}^{0} \frac{\sin(\sin(x))}{x}\, dx + \int_{0}^{1} \frac{\sin(\sin(x))}{x}\, dx.$$

Consider $\displaystyle\int_{0}^{1} \frac{\sin(\sin(x))}{x}\, dx$. As $x \to 0$, $\dfrac{\sin(\sin(x))}{x} \sim \dfrac{\sin(x)}{x}$.

$$\lim_{x\to 0} \frac{\left(\frac{\sin(\sin(x))}{x}\right)}{\left(\frac{\sin(x)}{x}\right)} = \lim_{x\to 0} \frac{\sin(\sin(x))}{\sin(x)} = \lim_{y\to 0} \frac{\sin(y)}{y} = 1.$$

As

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1,$$

then consider

$$\int_{0}^{1} dx = x\Big|_0^1 = 1.$$

Thus by limit comparison test, $\displaystyle\int_{0}^{1} dx$ converges implies $\displaystyle\int_{0}^{1} \frac{\sin(x)}{x}\, dx$ converges and this then implies $\displaystyle\int_{0}^{1} \frac{\sin(\sin(x))}{x}\, dx$ converges.

Since

$$\frac{\sin(\sin(-x))}{-x} = 1 \cdot \frac{\sin(-\sin(x))}{x} = \frac{\sin(\sin(x))}{x},$$

the integrand is even. Then

$$\int_{-1}^{0} \frac{\sin(\sin(x))}{x}\, dx = \int_{0}^{1} \frac{\sin(\sin(x))}{x}\, dx$$

also converges.
So $\displaystyle\int_{-1}^{1} \frac{\sin(\sin(x))}{x}\, dx$ converges and hence the principle value converges $\left(\text{to } 2\displaystyle\int_{0}^{1} \frac{\sin(\sin(x))}{x}\, dx\right)$.

(ii) Similar to the above, we get

$$\int_{-1}^{1} \frac{\sin(\sin(x))}{x^2}\, dx = \int_{-1}^{0} \frac{\sin(\sin(x))}{x^2}\, dx + \int_{0}^{1} \frac{\sin(\sin(x))}{x^2}\, dx.$$

Consider $\displaystyle\int_{0}^{1} \frac{\sin(\sin(x))}{x^2}\, dx$. As $x \to 0$, $\dfrac{\sin(\sin(x))}{x^2} \sim \dfrac{\sin(x)}{x^2}$.

$$\lim_{x\to 0} \frac{\left(\frac{\sin(\sin(x))}{x^2}\right)}{\left(\frac{\sin(x)}{x^2}\right)} = \lim_{x\to 0} \frac{\sin(\sin(x))}{\sin(x)} = \lim_{y\to 0} \frac{\sin(y)}{y} = 1.$$

And consider the limit

$$\lim_{x\to 0}\frac{\left(\frac{\sin(x)}{x^2}\right)}{\left(\frac{1}{x}\right)}=\lim_{x\to 0}\frac{\sin(x)}{x}=1.$$

Since

$$\int_0^1 \frac{1}{x}\,dx=\ln(x)\Big|_0^1=+\infty,$$

by limit comparison test, $\displaystyle\int_0^1 \frac{\sin(x)}{x^2}\,dx$ diverges and so $\displaystyle\int_0^1 \frac{\sin(\sin(x))}{x^2}\,dx$ diverges. Hence $\displaystyle\int_{-1}^1 \frac{\sin(\sin(x))}{x^2}\,dx$ diverges.

Since

$$\frac{\sin(\sin(-x))}{(-x)^2}=\frac{\sin(-\sin(x))}{x^2}=-\frac{\sin(\sin(x))}{x^2},$$

the integrand is odd.
So we have

$$\begin{aligned}
P.V.\int_{-1}^1 \frac{\sin(\sin(x))}{x^2}\,dx &= \lim_{c\to 0}\left(\int_c^1 \frac{\sin(\sin(x))}{x^2}\,dx+\int_{-1}^{c}\frac{\sin(\sin(x))}{x^2}\,dx\right)\\
&= \lim_{c\to 0}\left(\int_c^1 \frac{\sin(\sin(x))}{x^2}\,dx+\int_{-1}^{-c}\frac{\sin(\sin(-x))}{(-x)^2}\,d(-x)\right)\\
&= \lim_{c\to 0}\left(\int_c^1 \frac{\sin(\sin(x))}{x^2}\,dx-\int_{c}^{1}\frac{\sin(\sin(x))}{x^2}\,dx\right)\\
&= 0.
\end{aligned}$$

Problem 4 Determine the value of the improper integral $\displaystyle\int_0^\infty \frac{dt}{1-t^4}$.

Solution:
Consider $\displaystyle\int_1^\infty \frac{dt}{1-t^4}$. Using the substitution $t=\frac{1}{x}$,

$$\int_1^\infty \frac{dt}{1-t^4}=\int_1^0 \frac{-\frac{1}{x^2}}{1-\left(\frac{1}{x^4}\right)}\,dx=\int_0^1 \frac{x^2}{x^4-1}\,dx=\int_0^1 \frac{-t^2}{1-t^4}\,dt.$$

Therefore we have

$$\begin{aligned}
\int_0^\infty \frac{dt}{1-t^4} &= \int_0^1 \frac{dt}{1-t^4}+\int_1^\infty \frac{dt}{1-t^4}\\
&= \int_0^1 \frac{dt}{1-t^4}+\int_0^1 \frac{-t^2}{1-t^4}\,dt\\
&= \int_0^1 \frac{1-t^2}{1-t^4}\,dt\\
&= \int_0^1 \frac{1}{1+t^2}\,dt\\
&= \arctan x\Big|_0^1\\
&= \arctan 1-\arctan 0\\
&= \frac{\pi}{4}.
\end{aligned}$$

Thus, the improper integral $\displaystyle\int_0^\infty \frac{dt}{1-t^4}$ converges to $\dfrac{\pi}{4}$.