Problem 1

Prove the following limits using the definition of limits (ε-δ definition)

(a) $\lim_{x\to1}\frac{x}{x+1}=\frac{1}{2}$.

(b) $\lim_{x\to c}x^3=c^3$, where $c\in\mathbb{R}$.

(c) $\lim_{x\to0}x\sin\frac{1}{x}=0$ and $\lim_{x\to\frac{\pi}{2}}x\cos x=0$.

Problem 2

Prove the following limits using the definition of limits

(a) $\lim_{x\to\infty}\cos\frac{1}{x}=1$ (😊Hint: Recall that $\frac{1}{x}\to0$ when $x\to\infty$, so $\frac{1}{x}<\frac{\pi}{2}$ when x is large).

(b) $\lim_{x\to-\infty}e^x=0$

(c) $\lim_{x\to\infty}e^x=\infty$

Problem 1: (a). $\forall\varepsilon>0$.

$$\frac{x}{x+1}-\frac{1}{2}$$

$$=\frac{x-1}{2(x+1)}$$

$$\left|\frac{x}{x+1}-\frac{1}{2}\right|=\frac{|x-1|}{2|x+1|}$$

Choose $\delta=\min\{\frac{1}{2},3\varepsilon\}$. Then. $\forall x\in O(1,\delta)$. $|x+1|>\frac{3}{2}$

$$\left|\frac{x}{x+1}-\frac{1}{2}\right|=\frac{|x-1|}{\underbrace{2|x+1|}_{>\frac{3}{2}.}}<\frac{3\varepsilon}{2\cdot\frac{3}{2}}=\varepsilon\quad\Rightarrow\lim_{x\to1}\frac{x}{x+1}=\frac{1}{2}$$

$=\varepsilon$

(b). $x^3-c^3=(x-c)(x^2+xc+c^2)$.

$\forall\varepsilon>0$.

Choose $\delta=\min\{\boxed{\frac{|c|}{2}},\frac{1}{10c^2}\varepsilon\}$. Then $\forall x\in O(c,\delta)$.

$x^2+xc+c^2<10c^2$. $\quad \boxed{\frac{1}{2}|c|<|x|<\frac{3}{2}|c|}$

$$|x^3-c^3|\le\boxed{|x-c|}\,|x^2+xc+c^2|$$

$$<\frac{1}{10c^2}\varepsilon\cdot10c^2=\varepsilon\qquad\Rightarrow\lim_{x\to c}x^3=c^3.$$

$\forall\varepsilon>0$.

(c). $|x\sin\frac{1}{x}|\le|x|$.

Choose $\delta=\varepsilon$. $\Rightarrow$. $|x\sin\frac{1}{x}|\le|x|<\varepsilon$

$\forall x\in O(0,\delta)$

$$\Rightarrow\lim_{x\to0}x\sin\frac{1}{x}=0.$$

$\cos x$ is continuous.

$\exists \delta > 0.\quad \forall x \in O(\frac{\pi}{2}, \delta),\quad |\cos x| < \varepsilon \cdot \frac{1}{\pi}$

Choose $\hat{\delta} = \min\{\delta, \frac{\pi}{2}\}$.

$\Rightarrow \forall x \in O(\frac{\pi}{2}, \hat{\delta})\quad |x \cos x| = |x|\,|\cos x| < \pi \cdot \varepsilon \cdot \frac{1}{\pi} = \varepsilon$

$\Downarrow$

$|x| < \pi$

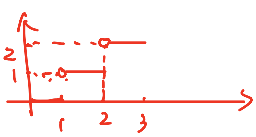
Problem 3

We let $[x]$ denotes the greatest integer less than or equal to x.

(a) We let c be an integer. Determine if the limits $\lim_{x\to c}[x]$ exists.

(😊Hint: Try an example when $c = 3$)

(b) We let d be a non-integer. Determine if the limits $\lim_{x\to d}[x]$ exists



(a). c is an integer. $[c] = c$.

$\{c - \frac{1}{n}\}_{n=1}^{+\infty}$ $\quad [c - \frac{1}{n}] = c-1,\quad \lim_{n\to\infty} [c - \frac{1}{n}] = c-1,$

$\lim_{n\to\infty} c - \frac{1}{n} = c$

$\{c + \frac{1}{n}\}_{n=2}^{+\infty}$. $\quad [c + \frac{1}{n}] = c,\quad \lim_{n\to\infty} [c + \frac{1}{n}] = c,$

$\lim_{n\to\infty} c + \frac{1}{n} = c.$

(b). d: not integer.

Choose $\delta = \min\{d - [d],\ [d]+1-d\}$.

Then $\forall x \in O(d, \delta)$. $\quad [x] = [d]$.

Problem 4

We let $f: \mathbb{R} \to \mathbb{R}$ be a function which $\lim_{x\to 0} f(x) = L \in \mathbb{R}$. Let $a > 0$ be a positive number and define $g: \mathbb{R} \to \mathbb{R}$ as $g(x) = f(ax)$.

(a) Show that $\lim_{x\to 0} g(x) = 0$ using the definition of limits.

(b) Redo **(a)** using the sequential limits theorem.

(a). Because $\lim_{x\to 0} f(x) = L$.

$\Rightarrow \forall \varepsilon > 0.\ \exists \delta > 0.\quad \forall x \in O(0, \delta)\quad |f(x) - L| < \varepsilon.$

$$\Rightarrow \forall x \in O(0, \tfrac{\delta}{a}),\ |g(x) - L| = |f(ax) - L| = |f(y) - L| < \varepsilon$$

$$y = ax.\ y \in O(0, \delta)$$

(b). $\lim_{x \to 0} g(x) = L.$

$\Leftrightarrow$. For any $\{a_n\}$. $\lim_{n \to \infty} a_n = 0$ $\lim_{n \to \infty} g(a_n) = \lim_{n \to \infty} f(a a_n) = L$

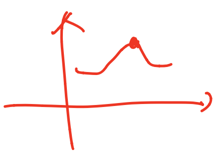
$\Leftrightarrow$ For any $\{a_n\}$ $\lim_{n \to \infty} a_n = 0 \Rightarrow \left(\lim_{n \to \infty} a a_n = 0\right).$

$\lim_{n \to \infty} f(a a_n) = L.$

$\Leftrightarrow$ For any $\{b_n\}$ $\lim_{n \to \infty} b_n = 0,$ $\lim_{n \to \infty} f(b_n) = L.$

Problem 13

We let $f: [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Suppose that there exists $c \in (a, b)$ such that $f(c) > f(x)$ for all $x \in [a, b]$, show that $f(x)$ is not injective.
(☺Hint: Draw a figure and get some idea)



Should be : for all $x \in [a,b] \setminus \{c\}$.

$f(c) > f(a),$

$f(c) > f(b).$

Choose y. such that. $f(c) > y > \max\{f(a), f(b)\}$.

By Intermediate Value Theorem:

$\exists\, x_1 \in (c, b)$ s.t. $f(x_1) = y.$

$\exists\, x_2 \in (a, c)$ s.t. $f(x_2) = y.$

$\Rightarrow f(x_1) = f(x_2) = y$

$\Rightarrow f$ is not injective.