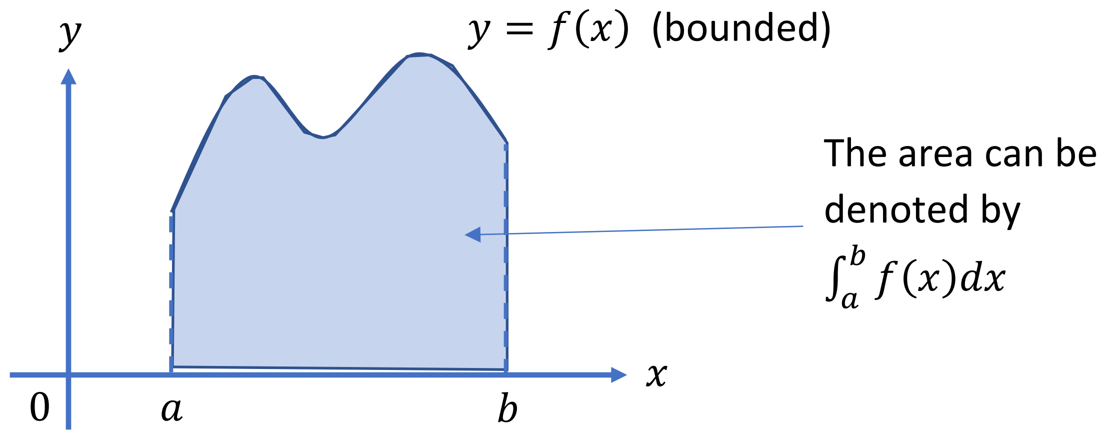
MATH2033 Mathematical Analysis

Lecture Note 8

Integration and Integrability

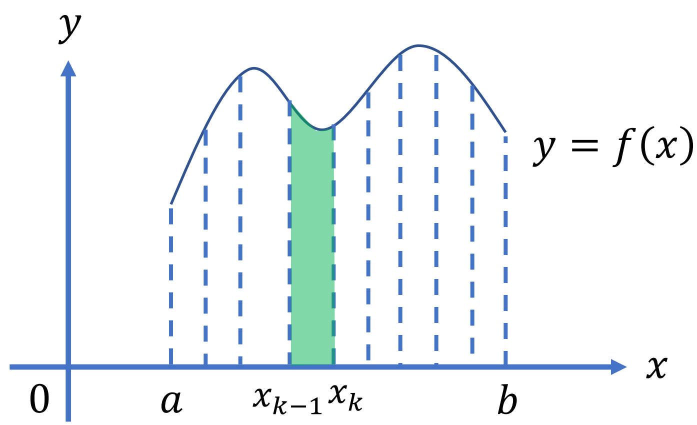
Integrability of bounded function

Suppose that we would like to find the area of the region (i.e. shaded region below) bounded by a bounded function $y = f(x)$ and x-axis over the interval $[a, b]$.



Since the region has irregular shape, one can only estimate the area of the region through *estimation*.

- To do so, we first divide the region into a number of small parts by partitioning the interval $[a, b]$ into n subintervals (denoted by $[a, x_1]$, $[x_1, x_2], . .,$ $[x_{n-1}, b]$). Here, the set of nodes $\{a, x_1, x_2, \dots, x_{n-1}, b\}$ is called *partition* of $[a, b]$ and is denoted by $\mathcal{P}$.
  (*Note: In general, a partition of $[a, b]$ is defined as a <u>finite</u> set of points $\{x_0, x_1, \dots, x_{n-1}, x_n\}$ which $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.



- Given the partition $\mathcal{P}$, we estimate the area of the region by finding its *upper bound* and *lower bound*.
  - ✓ For every subinterval $[x_{k-1}, x_k]$ (where $k = 1,2,\dots,n$), we let $M_k = \sup\{f(x) | x \in [x_{k-1}, x_k]\}$ be the "maximum value" of the function over the subinterval, then the upper bound of the area of the region is given by

$$U(\mathcal{P}, f) = \sum_{k=1}^{n} M_k (x_k - x_{k-1}).$$

  - ✓ Similarly, we let $m_k = \inf\{f(x) | x \in [x_{k-1}, x_k]\}$ be the "minimum value" of the function over the subinterval $[x_{k-1}, x_k]$, then the lower bound of the area of the region is given by

$$L(\mathcal{P}, f) = \sum_{k=1}^{n} m_k (x_k - x_{k-1}).$$

Example 1

We let $f(x) = x^2$ over the interval $[0,1]$ and consider the partition $\mathcal{P} = \left[0\,, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right]$. Compute $U(\mathcal{P}, f)$ and $L(\mathcal{P}, f)$.

😋Solution

Since $f(x) = x^2$ is increasing over $[0,1]$, we have

$$M_k = \sup\left\{x^2 | x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]\right\} = \left(\frac{k}{n}\right)^2 \quad and \quad m_k = \inf\left\{x^2 | x \in \left[\frac{k-1}{n}, \frac{k}{n}\right]\right\} = \left(\frac{k-1}{n}\right)^2.$$

Thus the upper sum and lower sum are given by

$$U(\mathcal{P}, f) = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{(n+1)(2n+1)}{6n^2}.$$

$$L(\mathcal{P}, f) = \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 = \frac{1}{n^3}\left(\frac{(n-1)(n)(2n-1)}{6}\right)$$
$$= \frac{(n-1)(2n-1)}{6n^2}.$$

As an example, we take $n = 100$. Then the area of the region A can be estimated as

$$0.32835 \overset{n=100}{\stackrel{\frown}{=}} L(\mathcal{P}, f) \le A \le U(\mathcal{P}, f) \overset{n=100}{\stackrel{\frown}{=}} 0.33835.$$

Obtaining a better estimation – Refinement of partition

From Example 1 (with $n = 100$), one can improve the estimation by further dividing each subinterval into 100 equal parts so that new partition will be $P^* =$ $\left\{0, \frac{1}{10000}, \frac{2}{1000}, \frac{3}{10000}, \dots, \frac{9999}{10000}, 1\right\} \supseteq \left\{0, \frac{1}{100}, \frac{2}{100}, \dots, \frac{9999}{10000}, 1\right\}$. Then the corresponding upper sum and lower sum are computed as

$$U(\mathcal{P}^*, f) = \sum_{k=1}^{10000} \left(\frac{k}{10000}\right)^2 \left(\frac{k}{10000} - \frac{k-1}{10000}\right) = \cdots = 0.333383 \ \ and$$

$$L(\mathcal{P}^*, f) = \sum_{k=1}^{10000} \left(\frac{k-1}{10000}\right)^2 \left(\frac{k}{10000} - \frac{k-1}{10000}\right) = 0.333283.$$

Then the new prediction will be

$$(0.32835 <) \ \ 0.333283 = L(\mathcal{P}, f) \leq A \leq U(\mathcal{P}^*, f) = 0.333383 \ \ (< 0.33835)$$

which is better than the previous estimation.

So the new partition $\mathcal{P}^*$ is called *refinement* of partition P.

Definition (Refinement)

Given a partition $\mathcal{P}$ of $[a, b]$, a refinement $\mathcal{P}^*$ is a partitional which $\mathcal{P}^* \supseteq P$

As inspired from the example, we expect the refinement should yield a better estimation on the area of the region.

Theorem 1

We let $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. Then for any refinement $\mathcal{P}^* \supseteq \mathcal{P}$, we have $L(\mathcal{P}, f) \leq L(\mathcal{P}^*, f)$ and $U(P^*, f) \leq U(\mathcal{P}, f)$

😊Solution

We first prove the case when $\mathcal{P}^* = \mathcal{P} \cup \{c\}$ (i.e. we place one more "cutting point" in the partition), where $c \neq x_i$ for any $i = 0,1, \dots, n$. We let $x_{k-1} < c < x_k$ for some $k = 1,2, \dots, n$.

Note that $w_1 = \inf\{f(x) | x \in [x_{k-1}, c]\} \geq m_k = \inf\{f(x) | x \in [x_{k-1}, x_k]\}$ and $w_2 = \inf\{f(x) | x \in [c, x_k]\} \geq m_k = \inf\{f(x) | x \in [x_{k-1}, x_k]\}$. Then we deduce that

$$L(\mathcal{P}, f) - L(\mathcal{P}^*, f)$$

$$= m_k(x_k - x_{k-1}) - w_1(c - x_{k-1}) - w_2(x_k - c)$$

$$\leq m_k(x_k - x_{k-1}) - m_k(c - x_{k-1}) - m_k(x_k - c) = 0$$

So we deduce that $L(\mathcal{P}, f) \leq L(\mathcal{P}^*, f)$. $U(P^*, f) \leq U(\mathcal{P}, f)$ can be proved similarly.

For the general case when $\mathcal{P}$ has n more cutting point, one can deduce the theorem by using the above result n times.

Obtaining the best estimate – Upper integral and lower integral

By considering various possible partition $\mathcal{P}$, we obtain a set of lower sum $L(\mathcal{P}, f)$ and upper sum $U(\mathcal{P}, f)$. Using these estimates, one can define

$$\overline{\int_a^b} f(x)dx = \inf_{\mathcal{P}}\{U(\mathcal{P}, f)\} \quad and \quad \underline{\int_a^b} f(x)dx = \sup_{\mathcal{P}}\{L(\mathcal{P}, f)\}$$

be the *upper integral* and *lower integral* of $f(x)$ which represents the *best estimate* on the upper bound and lower bound of the are of the region (or the integral $\int_a^b f(x)dx$).

If $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx = L$, one can deduce that the area of the region is L so that $\int_a^b f(x)dx = L$. Then we say $f(x)$ is *Riemann integrable* over $[a, b]$.

Definition

We let $f: [a, b] \to \mathbb{R}$ be a *bounded* function. We say f is Riemann integrable if and only if $\overline{\int_a^b} f(x)dx = \int_a^b f(x)dx = L$. We denote the common value L by $L = \int_a^b f(x)dx$.

Remark: If $f(x)$ is bounded (i.e. $m \leq f(x) \leq M$), this implies that all of $L(\mathcal{P}, f)$, $U(\mathcal{P}, f)$, $\overline{\int_a^b} f(x)dx$ and $\underline{\int_a^b} f(x)dx$ exist.

Example 2

Show that $f(x) = x^2$ in Example 1 is integrable over $[0,1]$.

😊Solution

We consider the partition $\mathcal{P}_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\right\}$ for $n \in \mathbb{N}$.

Using the result in Example 1, the upper sum and lower sum are given by

$$L(\mathcal{P}_n, f) = \frac{(n-1)(2n-1)}{6n^2}, \qquad U(\mathcal{P}_n, f) = \frac{(n+1)(2n+1)}{6n^2}.$$

This implies that

$$\frac{(n-1)(2n-1)}{6n^2} = L(\mathcal{P}_n, f) \le \underline{\int_0^1} f(x)dx \le \overline{\int_0^1} f(x)dx \le U(\mathcal{P}_n, f) = \frac{(n+1)(2n+1)}{6n^2}$$

By taking $n \to \infty$, we have $\lim_{n\to\infty} \frac{(n-1)(2n-1)}{6n^2} = \lim_{n\to\infty} \frac{2n^2-3n+1}{6n^2} = \lim_{n\to\infty} \frac{2-\frac{3}{n}+\frac{1}{n^2}}{6} = \frac{1}{3}$ and $\lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \cdots = \frac{1}{3}$.

So it follows from sandwich theorem that $\underline{\int_0^1} f(x)dx \le \overline{\int_0^1} f(x)dx = \frac{1}{3}$. Hence, $f(x)$ is integrable on $[0,1]$.

Example 3

We consider a function $f : [a, b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & if \;\; x \in [a, b] \cap \mathbb{Q} \\ 0 & if \;\; x \in [a, b] \backslash \mathbb{Q} \end{cases}.$$

Show that $f(x)$ is not integrable on $[a, b]$.

😊Solution

We consider a partition $\mathcal{P} = \{x_0, x_1, x_2, \dots, x_n\}$, where $x_0 = a$ and $x_n = b$.

For each subinterval $[x_{k-1}, x_k]$, we note the followings:

- By the density of rational number, there exists $q \in \mathbb{Q}$ such that $x_{k-1} < q < x_k$ and $f(q) = 1$.
- By the density of irrational number, there exists $r \in \mathbb{R} \backslash \mathbb{Q}$ such that $x_{k-1} < r < x_k$ and $f(r) = 0$.

So we have $M_k = \sup\{f(x) | x \in [x_{k-1}, x_k]\} = 1$ and $m_k = \inf\{f(x) | x \in [x_{k-1}, x_k]\} = 0$. This implies that

$$U(\mathcal{P}, f) = \sum_{k=1}^{n} M_k (x_k - x_{k-1}) = 1 \sum_{k=1}^{n} (x_k - x_{k-1}) = b - a, \;\; L(\mathcal{P}, f) = \sum_{k=1}^{n} m_k (x_k - x_{k-1}) = 0$$

This implies that $\overline{\int_0^1} f(x)dx = 1$ and $\underline{\int_0^1} f(x)dx = 0$.

As $\overline{\int_0^1} f(x)dx \neq \underline{\int_0^1} f(x)dx$, it follows that $f(x)$ is not integrable on $[a, b]$.

A shortcut of verifying integrability – Integral criterion

Practically, it is not quite efficient to verify the integrability of a function by computing the upper integral $\overline{\int_a^b} f(x)dx$ and lower integral $\underline{\int_a^b} f(x)dx$ because one needs to consider all possible partitions over $[a, b]$.

As inspired from the Example 2, it appears that it is sufficient to <u>find a partition $\mathcal{P}$ which the upper sum $U(\mathcal{P}, f)$ and lower sum $L(\mathcal{P}, f)$ are sufficiently close in the sense that $U(\mathcal{P}, f) - L(\mathcal{P}, f)$ is very small</u>. If such partition exists, this would imply that the upper integral and lower integral are also very close and eventually "equal".

The following theorem confirms our conjecture:

Theorem 2 (Integral criterion)

A bounded function $f: [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for any $\varepsilon > 0$, there exists a partition $\mathcal{P}$ of $[a, b]$ such that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon.$$

Proof of Theorem 2

"⇒" part

Since the function is integrable, we have $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx = K$.

- Note that $\overline{\int_a^b} f(x)dx = \inf_{\mathcal{P}}\{U(\mathcal{P}, f)\} = K$. Then for any $\varepsilon > 0$, there exists a partition $\mathcal{P}_1$ such that
$$U(\mathcal{P}_1, f) < K + \frac{\varepsilon}{2}.$$
- Since $\underline{\int_a^b} f(x)dx = \sup_{\mathcal{P}}\{L(\mathcal{P}, f)\} = K$. Then there exists a partition $\mathcal{P}_2$ such that
$$L(\mathcal{P}_2, f) > K - \frac{\varepsilon}{2}.$$

By taking $\mathcal{P}^* = \mathcal{P}_1 \cup \mathcal{P}_2$ (which can be seen as a refinement to both $\mathcal{P}_1$ and $\mathcal{P}_2$), it follows that $U(\mathcal{P}_1, f) \geq U(\mathcal{P}^*, f)$ and $L(\mathcal{P}_2, f) \leq L(\mathcal{P}^*, f)$. Hence, we have
$$U(\mathcal{P}^*, f) - L(\mathcal{P}^*, f) \leq U(\mathcal{P}_1, f) - L(\mathcal{P}_2, f) < K + \frac{\varepsilon}{2} - \left(K - \frac{\varepsilon}{2}\right) = \varepsilon.$$

"$\Leftarrow$" part

For any $\varepsilon > 0$, there exists $\mathcal{P}$ such that $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon$.

Since $\overline{\int_a^b} f(x)dx = \inf_{\mathcal{P}}\{U(\mathcal{P}, f)\} \leq U(\mathcal{P}, f)$ and $\underline{\int_a^b} f(x)dx = \sup_{\mathcal{P}}\{L(\mathcal{P}, f)\} \geq L(\mathcal{P}, f)$, then
$$0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon.$$

It follows from infinitesimal property that $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx = 0$. SO $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$ and $f(x)$ is Riemann integrable.

Example 4

Show that $f(x) = \sin x$ is integrable over $\left[0, \frac{\pi}{2}\right]$.

Solution

We consider the partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\} = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots, \frac{n\pi}{2n}\right\}$, where $x_k = \frac{k\pi}{2n}$.

Since $\sin x$ is increasing over $\left[0, \frac{\pi}{2}\right]$, so that $M_k = \sup\{f(x) | x \in [x_{k-1}, x_k]\} = \sin \frac{k\pi}{2n}$ and $m_k = \sup\{f(x) | x \in [x_{k-1}, x_k]\} = \sin \frac{(k-1)\pi}{2n}$. Then the upper sum and lower sum are given by

$$U(\mathcal{P}, f) = \sum_{k=1}^{n} \underbrace{\sin \frac{k\pi}{2n}}_{M_k} \times \underbrace{\left(\frac{k\pi}{2n} - \frac{(k-1)\pi}{2n}\right)}_{x_k - x_{k-1}} = \frac{\pi}{2n} \sum_{k=1}^{n} \sin \frac{k\pi}{2n} \quad and \quad L(\mathcal{P}, f) = \frac{\pi}{2n} \sum_{k=1}^{n} \underbrace{\sin \frac{(k-1)\pi}{2n}}_{m_k}$$

Using the mean value theorem, we deduce that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) = \frac{\pi}{2n} \sum_{k=1}^{n} \left(\sin \frac{k\pi}{2n} - \sin \frac{(k-1)\pi}{2n}\right) = \frac{\pi}{2n} \sum_{k=1}^{n} (\cos c_k) \left(\frac{k\pi}{2n} - \frac{(k-1)\pi}{2n}\right)$$
$$< \frac{\pi}{2n} \sum_{k=1}^{n} \frac{\pi}{2n} = \left(\frac{\pi}{2n}\right)^2 (n) = \frac{\pi^2}{4n}.$$

By Archimedean property, there exists $K \in \mathbb{N}$ such that $K > \frac{\pi^2}{4\varepsilon} \Leftrightarrow \frac{\pi^2}{4K} < \varepsilon$. By taking $n = K$, we have $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \frac{\pi^2}{4K} < \varepsilon$. So $f(x)$ is integrable.

Example 5 (Integrability of discontinuous functions)

Show that the function defined on $[a, b]$ by

$$f(x) = \begin{cases} 2 & if\ x = x_1, x_2, .., x_n \\ 1 & otherwise \end{cases}$$

is Riemann integrable. Here, $a < x_1 < x_2 < \cdots < x_n < b$.

😊Solution

Since the function has discontinuities (jumps) at points $x_1, x_2, .., x_n$, we consider this partition

$$\mathcal{P} = \{a, x_1 - \delta, x_1 + \delta, x_2 - \delta, x_2 + \delta, \dots, x_n - \delta, x_n + \delta, b\}$$

Then the corresponding upper bound and lower bound are given by

$$U(\mathcal{P}, f) = (x_1 - \delta - a)(1) + \sum_{k=2}^{n} (1)(x_k - \delta - x_{k-1} - \delta) + n(2\delta)(2) + (b - x_n - \delta)(1)$$

$$L(\mathcal{P}, f) = (x_1 - \delta - a)(1) + \sum_{k=2}^{n} (1)(x_k - \delta - x_{k-1} - \delta) + n(2\delta)(1) + (b - x_n - \delta)(1)$$

This implies that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) = 2n\delta$$

For any $\varepsilon > 0$, by picking $\delta < \frac{\varepsilon}{2n}$. we get $|U(P, f) - L(P, f)| = |2n\delta| < \varepsilon$

Hence $f(x)$ is Riemann Integrable.

Some properties of integrability

In this section, we shall present some basic properties related to integrability. To facilitate the presentation, we let

$$\sup_{x\in[a,b]} f(x) = \sup\{f(x)|x \in [a,b]\}, \quad \inf_{x\in[a,b]} f(x) = \inf\{f(x)|x \in [a,b]\}$$

Property 1 (Continuous function is integrable)

We let $f: [a,b] \to \mathbb{R}$ be a continuous function, then $f(x)$ is Riemann integrable over $[a,b]$.

In order to prove the property, one needs the concept of uniform continuity:

- We say $f(x)$ is uniform continuous on $[a,b]$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for any x, y satisfying $|x - y| < \delta$

One can prove that any continuous function f over $[a,b]$ is also uniform continuous on $[a,b]$.

- To see this, suppose that f is *not* uniform continuous, it follows that there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$, there exists x, y such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_0$.
- We pick $\delta = \frac{1}{n}$ for any $n \in \mathbb{N}$, there exists a pair x_n, y_n satisfying

$$|x_n - y_n| < \frac{1}{n} \text{ and } |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

- Since $x_n, y_n \in [a,b]$ and $\{x_n\}$, $\{y_n\}$ are bounded, it follows from Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = w \in [a,b]$.

- Note that $\left|x_{n_k} - y_{n_k}\right| < \frac{1}{n_k} \Rightarrow \underbrace{x_{n_k} - \frac{1}{n_k}}_{\to w\ as\ k\to\infty} < y_{n_k} < \underbrace{x_{n_k} + \frac{1}{n_k}}_{\to w\ as\ k\to\infty}$, it follows from sandwich theorem that $\lim_{k\to\infty} y_{n_k} = w$.
- Since $f(x)$ is continuous at $x = w \in [a, b]$, it follows that

$$\lim_{k\to\infty} |f(x_n) - f(y_n)| \geq \varepsilon_0 \Rightarrow \underbrace{|f(w) - f(w)|}_{=0} \geq \varepsilon_0.$$

  This leads to the contradiction.

Hence, we conclude that $f(x)$ is also uniformly continuous on $[a, b]$.

Proof of property 1

Since $f(x)$ is uniformly continuous on $[a, b]$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a} \quad for\ \ |x - y| < \delta.$$

We choose a positive integer N such that $N > \frac{b-a}{\delta} \Rightarrow \frac{b-a}{N} < \delta$ and consider the partition

$$\mathcal{P} = \{a, x_1, x_2, \dots, x_{N-1}, b\},\ \ where\ \ x_k = a + \frac{k(b-a)}{N}.$$

We consider the interval $[x_{k-1}, x_k]$. By extreme value theorem, there are $x_L, x_U \in [x_{k-1}, x_k]$ such that $\sup_{x\in[x_{k-1},x_k]} f(x) = f(x_U)\ \ and\ \inf_{x\in[x_{k-1},x_k]} f(x) = f(x_L)$.

Since $|x_U - x_L| \le |x_k - x_{k-1}| = \frac{b-a}{N}$ and $f(x)$ is uniformly continuous, it follows that

$$\sup_{x\in[x_{k-1},x_k]} f(x) - \inf_{x\in[x_{k-1},x_k]} f(x) = f(x_U) - f(x_L) < \frac{\varepsilon}{b-a}.$$

Then it follows that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) < \frac{\varepsilon}{b-a}\sum_{k=1}^{n}(x_k - x_{k-1}) = \frac{\varepsilon}{b-a}(b-a)$$
$$= \varepsilon.$$

So we conclude that f is integrable.

Property 2 (Integrability of monotone function)

We let f be a bounded function on $[a,b]$. Suppose that $f(x)$ is monotone, then $f(x)$ is integrable over $[a,b]$.

Proof of theorem 2

To facilitate the analysis, we shall consider the case when $f(x)$ is monotonic increasing.

We consider the partition

$$\mathcal{P} = \{x_0, x_1, \dots, x_n\}, \qquad where\ \ x_k = a + \frac{b-a}{n}.$$

(*We will determine the value of n later.)

Over the interval $[x_{k-1}, x_k]$, we note that

$$\sup_{x\in[x_{k-1},x_k]} f(x) = f(x_k) \quad and \quad \inf_{x\in[x_{k-1},x_k]} f(x) = f(x_{k-1}).$$

It follows that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) = \sum_{k=1}^{n} \underbrace{[f(x_k) - f(x_{k-1})]}_{\sup_{x\in[x_{k-1},x_k]} f(x) - \inf_{x\in[x_{k-1},x_k]} f(x)} (x_k - x_{k-1})$$

$$= \frac{b-a}{n} \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] = \frac{b-a}{n} [f(x_n) - f(x_1)]$$

$$= \frac{b-a}{n} \big(f(b) - f(a)\big).$$

By Archimedean property, we choose a positive integer n such that

$$n > \frac{b-a}{\varepsilon\big(f(b) - f(a)\big)} \Leftrightarrow \frac{b-a}{n} \big(f(b) - f(a)\big) < \varepsilon.$$

It follows that $U(\mathcal{P}, f) - L(\mathcal{P}, f) = \frac{b-a}{n}\big(f(b) - f(a)\big) < \varepsilon$. This proves that $f(x)$ is integrable over $[a, b]$.

Property 3 (Integrability of composite function)

We let f be a bounded function on $[a, b]$ and is integrable. We let g be a continuous function on $[c, d]$ (where $[c, d] \supseteq f([a, b])$). Then $g\big(f(x)\big)$ is also integrable on $[a, b]$

Remark of Example 3

- If we take $g(x) = |x|$ (which is continuous over $\mathbb{R}$), then it follows that $|f|$ is integrable.
- If we take $g(x) = x^n$ (where $n \in \mathbb{N}$), then we have f^n is also integrable.

Proof of property 3 (Quite technical)

For any $\varepsilon > 0$,

<u>Step 1: Some preparations</u>

- Since $g(x)$ is continuous over $[c, d]$, it follows from extreme value theorem that $g(x)$ is bounded and we write $-K \leq g(x) \leq K$ for some positive number K.
- As $g(x)$ is also uniform continuous over $[c, d]$, then there exists $\delta > 0$ such that
$$|g(x) - g(y)| < \varepsilon' = \frac{\varepsilon}{b - a + 2K} \quad for \ \ |x - y| < \delta.$$
For technical purpose, we choose δ such that $\delta < \frac{\varepsilon}{b-a+2K}$.
- Since f is integrable on $[a, b]$, then there exists a partition $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ such that
$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon'' = \delta^2.$$

Step 2: Argue that $U\big(g\big(f(x)\big),\mathcal{P}\big) - L(g(f(x),\mathcal{P}) < \varepsilon$ (so that $g\big(f(x)\big)$ will be integrable by integral criterion)

We let $\widetilde{M}_k = \sup\{g\big(f(x)\big): x \in [x_{k-1}, x_k]\}$ and $\widetilde{m}_k = \inf\{g\big(f(x)\big): x \in [x_{k-1}, x_k]\}$, Then

$$U\big(g\big(f(x)\big),\mathcal{P}\big) - L(g(f(x),\mathcal{P}) = \sum_{k=1}^{n} \big(\widetilde{M}_k - \widetilde{m}_k\big)(x_k - x_{k-1}) \dots\dots (*)$$

We let $M_k = \sup\{f(x): x \in [x_{k-1}, x_k]\}$ and $m_k = \inf\{f(x): x \in [x_{k-1}, x_k]\}$

Note that the magnitude of $\widetilde{M}_k - \widetilde{m}_k$ depends on the difference $M_k - m_k$. So we let

$$A = \{k|\ M_k - m_k < \delta\} \quad and \quad B = \{k|M_k - m_k \geq \delta\}.$$

Then

$$U\big(g\big(f(x)\big),\mathcal{P}\big) - L(g(f(x),\mathcal{P}) = \sum_{k=1}^{n} \big(\widetilde{M}_k - \widetilde{m}_k\big)(x_k - x_{k-1})$$

$$= \sum_{k\in A} \underbrace{\big(\widetilde{M}_k - \widetilde{m}_k\big)}_{\substack{<\frac{\varepsilon}{b-a+2K} \\ (as\ |f(x)-f(y)|<\delta)}} (x_k - x_{k-1}) + \sum_{k\in B} \underbrace{\big(\widetilde{M}_k - \widetilde{m}_k\big)}_{<2K} (x_k - x_{k-1})$$

$$< \frac{\varepsilon}{b-a+2K} \sum_{k\in A} (x_k - x_{k-1}) + 2K \sum_{k\in B} (x_k - x_{k-1})$$

$$< \frac{\varepsilon}{b-a+2K}(b-a) + \frac{2K}{\delta}\sum_{k\in B}\underbrace{(M_k - m_k)}_{as\ M_k - m_k \geq \delta}(x_k - x_{k-1})$$

$$= \frac{\varepsilon}{b-a+2K}(b-a) + \frac{2K}{\delta}\big(U(f,\mathcal{P}) - L(f,\mathcal{P})\big)$$

$$< \frac{\varepsilon}{b-a+2K}(b-a) + \frac{2K}{\delta}\delta^2 \overset{\delta<\varepsilon'}{\lesssim} \frac{\varepsilon}{b-a+2K}(b-a) + \frac{2K}{\delta}\left(\frac{\varepsilon}{b-a+2K}\right) = \varepsilon.$$

So $g\big(f(x)\big)$ is integrable by integral criterion.

Property 4

We let $f, g: [a,b] \to \mathbb{R}$ be two bounded function. Suppose that both functions are integrable, then cf, $f+g$, $f-g$ and fg are integrable, where c is some constant.

Proof of property 4

We first prove the integrability of cf.

- When $c = 0$, we have $cf = 0$ which is clearly integrable.
- When $c > 0$
  - Since f is integrable, then for any $\varepsilon > 0$, there exists a partition $\mathcal{P}$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \frac{\varepsilon}{|c|}$$

- Since $\sup_{x\in[x_{k-1},x_k]} cf(x) = c \sup_{x\in[x_{k-1},x_k]} f(x)$ and $\inf_{x\in[x_{k-1},x_k]} cf(x) = c \inf_{x\in[x_{k-1},x_k]} f(x)$ for $c > 0$, it follows that

$$U(cf,\mathcal{P}) = cU(f,\mathcal{P}) \quad and \quad L(cf,\mathcal{P}) = cL(f,\mathcal{P})$$

- So we deduce that

$$U(cf,\mathcal{P}) - L(cf,\mathcal{P}) = c\big(U(f,\mathcal{P}) - L(f,\mathcal{P})\big) < c\frac{\varepsilon}{|c|} \overset{c>0}{\triangleq} \varepsilon$$

- When $c < 0$
  - Since f is integrable, then for any $\varepsilon > 0$, there exists a partition $\mathcal{P}$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \frac{\varepsilon}{|c|}$$

  - Since $\sup_{x\in[x_{k-1},x_k]} cf(x) = c \inf_{x\in[x_{k-1},x_k]} f(x)$ and $\inf_{x\in[x_{k-1},x_k]} cf(x) = c \sup_{x\in[x_{k-1},x_k]} f(x)$ for $c < 0$, it follows that

$$U(cf,\mathcal{P}) = cL(f,\mathcal{P}) \quad and \quad L(cf,\mathcal{P}) = cU(f,\mathcal{P})$$

  - So we deduce that

$$U(cf,\mathcal{P}) - L(cf,\mathcal{P}) = c\big(L(f,\mathcal{P}) - U(f,\mathcal{P})\big) < (-c)\frac{\varepsilon}{|c|} \overset{c>0}{\triangleq} \varepsilon$$

So combining all cases, we deduce that cf is integrable.

Next, we first prove the integrability of $f+g$.

- For any partition $\mathcal{P}=\{x_0,x_1,\dots,x_n\}$, we note that
$$\sup_{x\in[x_{k-1},x_k]}[f(x)+g(x)]\le\sup_{x\in[x_{k-1},x_k]}f(x)+\sup_{x\in[x_{k-1},x_k]}g(x)\quad and$$
$$\inf_{x\in[x_{k-1},x_k]}[f(x)+g(x)]\ge\inf_{x\in[x_{k-1},x_k]}f(x)+\inf_{x\in[x_{k-1},x_k]}g(x)$$
It follows that (why?)
$$U(f+g,\mathcal{P})\le U(f,\mathcal{P})+U(g,\mathcal{P})\quad and\quad L(f+g,\mathcal{P})\ge L(f,\mathcal{P})+L(g,\mathcal{P})$$
- Note that f,g are integrable on $[a,b]$. Then for any $\varepsilon>0$, there exists partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ such that
$$U(f,\mathcal{P}_1)-L(f,\mathcal{P}_1)<\frac{\varepsilon}{2}\quad and\quad U(g,\mathcal{P}_2)-L(g,\mathcal{P}_2)<\frac{\varepsilon}{2}.$$
- We consider the partition $\mathcal{P}^*=\mathcal{P}_1\cup\mathcal{P}_2$ (which is the refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$, it follows that
$$U(f+g,\mathcal{P}^*)-L(f+g,\mathcal{P}^*)\le\big(U(f,\mathcal{P}^*)+U(g,\mathcal{P}^*)\big)-\big(L(f,\mathcal{P}^*)+L(g,\mathcal{P}^*)\big)$$
$$=\big(U(f,\mathcal{P}^*)-L(f,\mathcal{P}^*)\big)+\big(U(g,\mathcal{P}^*)-L(g,\mathcal{P}^*)\big)$$
$$\overset{\mathcal{P}^*\ is\ refinement}{\lessapprox}\quad\big(U(f,\mathcal{P}_1)-L(f,\mathcal{P}_1)\big)+\big(U(g,\mathcal{P}_2)-L(g,\mathcal{P}_2)\big)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

So we conclude that $f+g$ is integrable.

Since $f-g=f+(-1)g$, it follows from the above result that $f-g$ is integrable.

To prove the product fg is integrable, we note that

- f^2 is integrable for any integrable function f (by remark of property 3)
- The product fg can be written as
$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2].$$
- Since f, g are integrable, it follows that $f+g$ and $f-g$ are integrable. Then it implies that both $(f+g)^2$, $(f-g)^2$ are integrable. Therefore, it follows that fg is also integrable.

Property 5

We let $f: [a,b] \to \mathbb{R}$ be a function and let $c \in (a,b)$. Then f is Riemann integrable on $[a,b]$ if and only if f is Riemann integrable on both $[a,c]$ and $[c,b]$

Proof of property 5

("⇒" part) If f is Riemann integrable, then for any $\varepsilon > 0$, there exists a partition $\mathcal{P}$ on $[a,b]$such that
$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon.$$
To prove the integrability on $[a,c]$, we take $\mathcal{P}_1 = [a,c] \cap (\mathcal{P} \cup \{c\})$, then it follows that
$$U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1) \leq U(f,\mathcal{P} \cup \{c\}) - L(f,\mathcal{P} \cup \{c\}) \leq U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon.$$
So f is integrable on $[a,c]$. Similarly, one can show that f is integrable on $[c,b]$

(" $\Leftarrow$ " part).

Given that f is Riemann integrable on $[a, c]$ and $[c, b]$, for any $\varepsilon > 0$, there exists a partition $\mathcal{P}_1$ on $[a, c]$ and a partition $\mathcal{P}_2$ on $[c, b]$ such that

$$\underbrace{U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1)}_{on\ [a,c]} < \frac{\varepsilon}{2}, \qquad \underbrace{U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2)}_{on\ [c,b]} < \frac{\varepsilon}{2}$$

We pick $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, then we have

$$\underbrace{U(f, \mathcal{P}) - L(f, \mathcal{P})}_{on\ [a,b]} = \underbrace{\sum_{[x_{k-1}, x_k] \subseteq \mathcal{P}_1} (M_k - m_k)(x_k - x_{k-1})}_{= U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1)} + \underbrace{\sum_{[x_{k-1}, x_k] \subseteq \mathcal{P}_2} (M_k - m_k)(x_k - x_{k-1})}_{= U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So f is also integrable on $[a, b]$.

Property 6 (Integrability of discontinuous function)

We let $f: [a, b] \to \mathbb{R}$ be a bounded function. Suppose that f is discontinuous at finitely many points $x_1, x_2, \ldots, x_n \in [a, b]$, then f is Riemann integrable on (a, b).

Proof of property 6

We shall consider the case when f has one discontinuity at $x = c$.

Since f has discontinuity at $x = c$, we shall construct the Partition by first considering the region near $x = c$.

- Since f is bounded, we have $|f(x)| \leq K$ for all $x \in [a, b]$.
- For any $\varepsilon > 0$, we pick an interval $\left[c - \frac{\varepsilon}{6K}, c + \frac{\varepsilon}{6K}\right]$ so that the difference of upper sum and lower sum over this interval is at most $(K - (-K))\left(c + \frac{\varepsilon}{6K} - \left(c - \frac{\varepsilon}{6K}\right)\right) = \frac{\varepsilon}{3}$.

Since f is integrable over $[a, b]$ and hence integrable on both $\left[a, c - \frac{\varepsilon}{6K}\right]$ and $\left[c + \frac{\varepsilon}{6K}, b\right]$, there exists partition $\mathcal{P}_1$ on $\left[a, c - \frac{\varepsilon}{6K}\right]$ and partition $\mathcal{P}_2$ on $\left[c + \frac{\varepsilon}{6K}, b\right]$ such that

$$\underbrace{U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1)}_{on\ \left[a, c - \frac{\varepsilon}{6K}\right]} < \frac{\varepsilon}{3}, \qquad \underbrace{U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2)}_{on\ \left[c + \frac{\varepsilon}{6K}, b\right]} < \frac{\varepsilon}{3}.$$

So we consider the partition $\mathcal{P} = \mathcal{P}_1 \cup \underbrace{\left\{c - \frac{\varepsilon}{6K}, c + \frac{\varepsilon}{6K}\right\}}_{\mathcal{P}_3} \cup \mathcal{P}_2$, then we deduce that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \underbrace{[U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1)]}_{\mathcal{P}_1} + \underbrace{\frac{\varepsilon}{3}}_{\mathcal{P}_3} + \underbrace{[U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2)]}_{\mathcal{P}_2} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So the proof is complete. The general case when f has n discontinuities can be proved in a similar fashion.

Properties of integral

In this section, we shall present the derivation of some computational formula of integral which we have used a lot in Calculus course.

Property 7 (Simple properties of Riemann integral)

We let $f, g: [a, b] \to \mathbb{R}$ be two Riemann integrable function over $[a, b]$. Then

$$(1)\quad \int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$(2)\quad \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,\quad for\ \ c \in (a, b)$$

$$(3)\quad \int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

$$(4)\quad If\ \ f(x) \le g(x),\ \ then \int_a^b f(x)dx \le \int_a^b g(x)dx,$$

$$(5)\quad \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx.$$

Remark:

To establish these properties, one can use the integral criterion and infinitesimal property.

Proof of property 7

To prove (1), we consider the case when $k > 0$. The case for $k < 0$ can be proved similarly.

Note that $kf(x)$ is integrable by property 4.

For any $\varepsilon > 0$, there exists partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ on $[a, b]$ such that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \frac{\varepsilon}{k}.$$

Since $L(\mathcal{P}, f) \leq \int_a^b f(x)dx \leq U(\mathcal{P}, f)$, $U(\mathcal{P}, kf) = kU(\mathcal{P}, f)$ and $L(\mathcal{P}, kf) = kL(\mathcal{P}, f)$ for any partition $\mathcal{P}$, it follows that

$$kL(\mathcal{P}, f) \leq L(\mathcal{P}, kf) \leq \int_a^b kf(x)dx \leq U(\mathcal{P}, kf) \leq kU(\mathcal{P}, f)$$

By taking supremum on the first inequality and taking infimum on the second inequality over all partition $\mathcal{P}$, it follows that

$$k\underline{\int_a^b} f(x)dx = k\sup\{L(\mathcal{P}, f)|\mathcal{P}\} \leq \int_a^b kf(x)dx \leq k\inf\{U(\mathcal{P}, f)|\mathcal{P}\} = k\overline{\int_a^b} f(x)dx$$

As f is integrable with $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx = \int_a^b f(x)dx$, it follows from sandwich theorem that $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

To prove (2), we let $\mathcal{P}$ be a partition of $[a,b]$ and define $\mathcal{P}' = \mathcal{P} \cup \{c\}$ (which is refinement of $\mathcal{P}$). Then it follows that

$$\int_a^c f(x)dx + \int_c^b f(x)dx \le U(\mathcal{P}' \cap [a,c], f) + U(\mathcal{P}' \cap [c,b], f) = U(\mathcal{P}', f) \le U(\mathcal{P}, f) \ldots (*)$$

Similarly, one can show that

$$\int_a^c f(x)dx + \int_c^b f(x)dx \ge L(\mathcal{P}' \cap [a,c], f) + L(\mathcal{P}' \cap [c,b], f) = L(\mathcal{P}', f) \ge L(\mathcal{P}, f) \ldots (**)$$

By taking infimum on $(*)$ and taking supremum on $(**)$, we deduce that

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx \le \int_a^c f(x)dx + \int_c^b f(x)dx \le \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx$$

Thus it follows from sandwich theorem that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

To prove (3), we recall the following facts:

$$U(f+g,\mathcal{P}) \le U(f,\mathcal{P}) + U(g,\mathcal{P}) \quad and \quad L(f+g,\mathcal{P}) \ge L(f,\mathcal{P}) + L(g,\mathcal{P}).$$

Then for any partition $\mathcal{P}$, we have

$$\int_a^b [f(x)+g(x)]dx \le U(f+g,\mathcal{P}) \le U(f,\mathcal{P}) + U(g,\mathcal{P}) \ldots (*) \quad and$$

$$\int_a^b [f(x)+g(x)]dx \geq L(f+g,\mathcal{P}) \geq L(f,\mathcal{P})+L(g,\mathcal{P}) \ldots (**).$$

By taking infimum on $(*)$ and taking supremum on $(**)$ and using the fact that both f, g are integrable., we deduce that

$$\underbrace{\underline{\int_a^b} f(x)dx + \underline{\int_a^b} g(x)dx}_{=\int_a^b f(x)dx+\int_a^b g(x)dx} \leq \int_a^b [f(x)+g(x)]dx \leq \underbrace{\overline{\int_a^b} f(x)dx + \overline{\int_a^b} g(x)dx}_{=\int_a^b f(x)dx+\int_a^b g(x)dx}$$

So we deduce from sandwich theorem that $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

To prove (4), for any partition $\mathcal{P}$, we have

$$\int_a^b f(x)dx \leq U(\mathcal{P},f) \leq U(\mathcal{P},g).$$

By taking infimum and noting that $g(x)$ is integrable, we get $\int_a^b f(x)dx \leq \overline{\int_a^b} g(x)dx = \int_a^b g(x)dx$.

Finally, we note that $-|f(x)| \leq f(x) \leq |f(x)|$. Then (5) can be proved using the result of (4).

Property 8 (Continuity of Riemann integral)

We let $f: [a, b] \to \mathbb{R}$ a bounded Riemann integrable function over $[a, b]$. Then for any $c \in [a, b]$, $F(x) = \int_c^x f(y)dy$ is continuous (and hence uniformly continuous) over $[a, b]$

Proof of property 8

For any $\varepsilon > 0, x_0 \in [a, b]$, we note that $|f(x)| \leq K$ since $f(x)$ is bounded. We take $\delta = \frac{\varepsilon}{K}$, then for any $0 < |x - x_0| < \delta$, we have

$$|F(x) - F(x_0)| = \left|\int_c^x f(x)dx - \int_c^{x_0} f(x)dx\right| = \left|\int_{x_0}^x f(x)dx\right| \leq K \underbrace{|x - x_0|}_{<\delta = \frac{\varepsilon}{K}} < \varepsilon.$$

So $F(x)$ is continuous at any $x = x_0 \in [a, b]$. It follows that $F(x)$ is uniformly continuous on $[a, b]$.

Property 9 (Fundamental theorem of Calculus)

We let $f: [a, b] \to \mathbb{R}$ a bounded Riemann integrable function over $[a, b]$.

(a) If $f(x)$ is continuous at $x = x_0$ and $F(x) = \int_c^x f(y)dy$, then $F'(x) = f(x_0)$.

(b) If $F(x)$ is differentiable on $[a, b]$, $F'(x) = f(x)$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof property 9

To prove (a), we shall argue that $F'(x_0) = f(x_0)$ using the definition of derivative and limits.

- Since $f(x)$ is continuous at $x = x_0$, then for any $\varepsilon > 0$, there exists δ such that for $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$.
- On the other hand, we note that

$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{\int_c^x f(y)dy - \int_c^{x_0} f(y)dy - f(x_0)(x - x_0)}{x - x_0}$$

$$= \frac{\int_{x_0}^x f(y)dy - f(x_0)(x - x_0)}{x - x_0} \dots (*)$$

- Since for any $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| < \varepsilon \Leftrightarrow f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon.$$

  Then, we get $(f(x_0) - \varepsilon)(x - x_0) < \int_{x_0}^x f(y)dy < (f(x_0) + \varepsilon)(x - x_0)$.

- It follows from equation $(*)$ that

$$-\varepsilon < -\frac{\varepsilon(x - x_0)}{x - x_0} < \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) < \frac{\varepsilon(x - x_0)}{x - x_0} = \varepsilon$$

$$\Rightarrow \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon.$$

  So $F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0)$ by the definition of limits.

To prove **(b)**,

We consider a partition $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$

For any subinterval $I_k = [x_{k-1}, x_k]$, one can deduce from mean value theorem that there exists $t_k \in (x_{k-1}, x_k)$ such that

$$\frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}} = F'(t_k) \Rightarrow F(x_k) - F(x_{k-1}) = \underbrace{F'(t_k)}_{=f(t_k)}(x_k - x_{k-1}).$$

It follows that

$$U(f, \mathcal{P}) \overset{\sup_{x\in I_k} f(x) \geq f(t_k)}{\geq} \underbrace{\sum_{k=1}^{n} [F(x_k) - F(x_{k-1})]}_{=F(b)-F(a)} = \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \overset{f(t_k) \geq \inf_{x\in I_k} f(x)}{\geq} L(f, \mathcal{P}).$$

By taking infimum and supremum on each side of the inequality, we get

$$\int_a^b f(x)dx \overset{f \text{ is integrable}}{=} \underbrace{\overline{\int_a^b} f(x)dx}_{\inf\{U(f,\mathcal{P})|\mathcal{P}\}} \geq F(b) - F(a) \geq \underbrace{\underline{\int_a^b} f(x)dx}_{\sup\{L(f,\mathcal{P})|\mathcal{P}\}} = \int_a^b f(x)dx.$$

So we deduce that $\int_a^b f(x)dx = F(b) - F(a)$ by sandwich theorem.

In fact, fundamental theorem of calculus is one of the important theorem in integration.

- It allows us to compute the integral $\int_a^b f(x)dx$ by finding a function $F(x)$ (also called *anti-derivative*) such that $F'(x) = f(x)$. For example, since $\frac{d}{dx}\sin x = \cos x$, it follows that $\int_a^b \cos x \, dx = \sin b - \sin a$.
- On the other hand, one can use this theorem to establish other useful theorems such as "integration by parts" and "method of substitution".

Property 10 (Integration by parts)

If $f(x)$, $g(x)$ are integrable on $[a, b]$ and their derivatives $f'(x), g'(x)$ are also integrable on $[a, b]$, then

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

Property 11 (Method of substitution)

We let $\phi: [a, b] \to \mathbb{R}$ be differentiable and ϕ' is integrable on $[a, b]$. If f is continuous on $\phi([a, b])$, then

$$\int_a^b f(\phi(x))\phi'(x)dx = \int_{\phi(a)}^{\phi(b)} f(y)dy.$$

Proof of property 10

We let $F(x) = f(x)g(x)$. One can see that $F'(x) = \underbrace{f'(x)g(x)}_{integrable} + \underbrace{f(x)g'(x)}_{integrable}$ is integrable. It follows from fundamental theorem of calculus that

$$\int_a^b [f'(x)g(x) + f(x)g'(x)]dx = \int_a^b F'(x)dx = F(b) - F(a) = f(b)g(b) - f(a)g(a)$$

$$\Rightarrow \int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx.$$

Proof of property 11

We let $g(x) = \int_{\phi(a)}^{\phi(x)} f(y)dy$ for $x \in [a, b]$.

Since $f(x)$ is continuous over $\phi(x) \in \phi([a, b])$, it follows from fundamental theorem of calculus (1st statement) that

$$g'(x) = \frac{d}{dx} f\big(\phi(x)\big) = f\big(\phi(x)\big)\phi'(x) \quad for \ \ any \ x \in [a, b].$$

Then it follows that

$$\int_a^b f\big(\phi(x)\big)\phi'(x)dx = \int_a^b g'(x)dx = g(b) - g(a) \overset{g(a)=0}{\triangleq} g(b) = \int_{\phi(a)}^{\phi(b)} f(y)dy$$