DEPARTMENT OF MATHEMATICS

Math2033 TA note 9

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April 7, 2019

1 CONTINUITY

Example 1. *(a) Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$.*

(b) Find all strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Solution: (a) The function $f(x) = cx$ with $c \in \mathbb{R}$ satisfies the requirement. We show that all functions we want to find have this form.

Observe that $f(0) = f(0) + f(0)$ and $f(x) + f(-x) = f(0)$. So $f(0) = 0$ and $f(-x) = f(x)$. By mathematical induction, we can show that $f(nx) = nf(x)$ for any $n \in \mathbb{Z}$. If we take $x = \frac{1}{n}$, then $f(1) = nf(\frac{1}{n})$ i.e. $f(\frac{1}{n}) = \frac{1}{n}f(1)$. Therefore, for any $x \in \mathbb{Q}$, $x = p/q$ for some $p, q \in \mathbb{Z}$ and we have

$$f(x) = f(\frac{p}{q}) = pf(\frac{1}{q}) = \frac{p}{q}f(1) = f(1)x.$$

So f have the form $f(x) = cx$ with $c = f(1) \in \mathbb{R}$.

(b) Since f satisfies the condition in (a), $f(x) = f(1)x$ for $x \in \mathbb{Q}$. For any $x \in \mathbb{R}$, there exist $a_n, b_n \in \mathbb{Q}$ such that $a_n \le x \le b_n$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = x$. Since f is strictly increasing, $f(1) > 0$, and

$$a_n f(1) = f(a_n) \le f(x) \le f(b_n) = b_n f(1).$$

Taking limit, we have $f(x) = f(1)x$. So the function f must have the form $f(x) = cx$ with $c = f(1) > 0$.

Example 2. *f is differentiable and $\lim_{x\to 0} f'(x)$ exists, prove that f' is continuous at 0.*

Solution: Denote $\lim_{x\to 0} f'(x) = L$. We suppose $f'(0) = a < L$, and for suitable $\epsilon > 0$ such that

$$a < L - \epsilon < L,$$

by $\lim_{x\to 0} f'(x) = L$, there $\exists\delta > 0, \forall 0 < |x| \le \delta, |f'(x) - L| < \epsilon$. That is in the region $0 < |x| \le \delta$,

$$|f'(x)| > L - \epsilon.$$

For b such that $a < b < L - \epsilon$, we define

$$g(x) = f(x) - bx.$$

g is continuous and differentiable, g can attain its minimum value in $[0,\delta]$. By $g'(0) = a - b < 0, g'(\delta) > L - \epsilon - b > 0$, we see g cannot attain the minimum value at boundary. It attains minimum value in $(0,\delta)$. Then $g'(\xi) = 0$ for some $\xi \in (0,\delta)$. Because $0 < |x| \le \delta, |f'(x)| > L - \epsilon$, this contradicts to $f'(\xi) = b < L - \epsilon$. So the assumption is false. In the same fashion, we can prove $f'(0)$ can not large than L. So $f'(0) = L$ and f' is continuous at 0.

Example 3. *Suppose $f, g : [1,2] \to [3,4]$ are continuous functions and also $\{g(x) : x \in [1,2]\} = [3,4]$. Show that there is $c \in [1,2]$ such that $f(c) = g(c)$.*

Solution: Since $\{g(x) : x \in [1,2]\} = [3,4]$, there are $x_0, x_1 \in [1,2]$ such that $g(x_0) = 3$ and $g(x_1) = 4$. Further, since $f : [1,2] \to [3,4]$, we have $(f - g)(x_0) = f(x_0) - 3 \ge 0$ and $(f - g)(x_1) = f(x_1) - 4 \le 0$. Since $f - g$ is continuous on $[1,2]$, by intermediate value theorem, there is c between x_0, x_1 and hence $c \in [1,2]$ such that $(f - g)(c) = 0$, i.e., $f(c) = g(c)$.

Example 4 (**Equivalent definition of Differentibility**). *f is differentiable at x_0 $\iff$ $f(x_0 + h) = f(x_0) + f'(x_0)h + R(x_0, h)$ with $\lim\limits_{h\to 0} \frac{R(x_0,h)}{h} = 0$.*

Solution: By definition, f is differentiable at x_0
$\iff$ $\forall\epsilon > o, \exists\delta > 0$, such that $0 < |h| < \delta$,

$$\left|\frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0)\right| < \epsilon$$

$\iff$ $\forall\epsilon > o, \exists\delta > 0$, such that $0 < |h| < \delta$,

$$\left|\frac{R(x_0, h)}{h}\right| = \left|\frac{f(x_0 + h) - f(x_0) - f'(x_0)h}{h}\right| < \epsilon$$

$\iff$ $f(x_0 + h) = f(x_0) + f'(x_0)h + R(x_0, h)$ with $\lim\limits_{h\to 0} \frac{R(x_0,h)}{h} = 0$.