MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #26

Pointwise Convergence and Uniform Convergence

Part I: Pointwise Convergence

Definition:

1) Pointwise Convergence of a function

Given a sequence of function $f_n: E \to \mathbf{R}$ (where E is a set), we say f_n converges pointwise to f iff for **each $x \in E$**, $\lim_{n\to\infty} f_n(x) = f(x)$

2) Pointwise Convergence of a series of function

We say a series of function $\sum_{k=1}^{\infty} f_k(x)$ converges pointwise if and only if **For each x**, the partial sum $\sum_{k=1}^{n} f_k(x)$ converges pointwise to $\sum_{k=1}^{\infty} f_k(x)$

Example 1

Discuss the pointwise convergence of

$$f_n(x) = (\sin x)^n \text{ for } x \in [0, \pi]$$

Solution:

We can see $\sin x < 1$ for $x \in \left[0, \frac{\pi}{2}\right)$ and $(\frac{\pi}{2}, \pi]$ and $\sin x = 1$ for $x = \frac{\pi}{2}$. Hence

$$\lim_{n\to\infty} f_n(x) = \begin{cases} \lim_{n\to\infty} (\sin x)^n = 0 & for\ x \in \left[0, \frac{\pi}{2}\right) \cup (\frac{\pi}{2}, \pi] \\ \lim_{n\to\infty} 1^n = 1 & for\ x = \frac{\pi}{2} \end{cases}$$

Example 2

Discuss the pointwise convergence of

$$\sum_{n=1}^{\infty} \frac{x^n e^x}{n} \text{ for } x \in [0, \infty)$$

Solution:

We apply root test

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{x e^{\frac{x}{n}}}{\sqrt[n]{n}} = x \qquad (\text{Note: } \lim_{n\to\infty} \sqrt[n]{n} = 1\ and\ \lim_{n\to\infty} e^{\frac{x}{n}} = e^0 = 1)$$

From the root test, the series converges when $0 \le x < 1$ and diverges when $x > 1$

For $x = 1$ (which the root has no conclusion)

$$\sum_{n=1}^{\infty} \frac{x^n e^x}{n} = \sum_{n=1}^{\infty} \frac{1^n e^1}{n} = e \sum_{n=1}^{\infty} \frac{1}{n} \quad \text{which diverges by p} - \text{test } (p = 1)$$

Part 2: Uniform Convergence

Definition: (Uniform Convergence of Function)

Given a sequence of function $f_n : E \to \mathbf{R}$, we say f_n **converges uniformly** to f iff

$$\lim_{n\to\infty}\left(\sup_{x\in E}|f_n(x) - f(x)|\right) = 0$$

In other word, $\sup_{x\in E}|f_n(x) - f(x)| \to 0$ as $n \to \infty$

(Note: $\sup_{x\in E}|g(x)|$ is called **sup-norm** of $g(x)$ on E)

Definition: (Uniform Convergence of Series of Function)

Let $g_n : E \to \mathbf{R}$ be a sequence of functions, we say the series $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly to function $S(x)$ on E iff the partial sum $S_n(x) = \sum_{k=1}^{n} g_k(x)$ converges uniformly to $S(x)$ on E

Example 3

Show that the following functions

$$f_n(x) = \frac{\sin nx}{1 + nx}$$

Converges uniformly on $[c, \infty)$ where c is a positive number.

(Step 1: Find the limit first)

For any $x \in [c, \infty)$, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin(nx)}{1 + nx} = 0 \quad (\text{Since } 1 + nx \to \infty)$$

(Step 2: Compute the sup-norm)

$$0 \le \sup_{x\in[c,\infty)} |f_n(x) - f(x)| = \sup_{x\in[c,\infty)} \frac{|\sin nx|}{|1 + nx|} \le \sup_{x\in[c,\infty)} \frac{1}{|1 + nx|} = \frac{1}{1 + nc}$$

Taking limit on both side $(n \to \infty)$ and note that $\lim_{n\to\infty} \frac{1}{1+nc} = 0$, we have

$$\lim_{n\to\infty}\left(\sup_{x\in[c,\infty)} |f_n(x) - f(x)|\right) = 0$$

Hence $f_n(x)$ converges uniformly on $[c, \infty)$.

Example 4

Show that the sequence of functions

$$f_n(x) = x^n$$

Converges uniformly on $[0, b]$ where $b < 1$, does not converge uniformly on $[0,1]$ where $n \in \mathbf{N}$.

For $[\mathbf{0}, \boldsymbol{b}]$***, (where*** $\mathbf{b} < 1$***)***

(Step 1: Find the Limit First)

$$\mathrm{f(x)} = \lim_{n\to\infty} \mathrm{f_n(x)} = \lim_{n\to\infty} \mathrm{x^n} = 0 \quad \text{for all } \mathrm{x} \in [0,\mathrm{b}]$$

(Step 2: Compute the sup-norm)

$$\sup_{x\in[0,b]} |f_n(x) - f(x)| = \sup_{x\in[0,b]} |x^n| = \mathrm{b^n}$$

$$\lim_{n\to\infty}\left(\sup_{x\in[0,b]} |f_n(x) - f(x)|\right) = \lim_{n\to\infty} b^n = 0$$

Hence $\mathrm{f_n(x)}$ converges uniformly on $[0,\mathrm{b}]$

For $[\mathbf{0}, \mathbf{1}]$

(Step 1: Find the limit first)

$$\mathrm{f(x)} = \lim_{n\to\infty} \mathrm{f_n(x)} = \lim_{n\to\infty} \mathrm{x^n} = \begin{cases} \lim_{n\to\infty} \mathrm{x^n} = 0 & \text{for } \mathrm{x} \in [0,1) \\ \lim_{n\to\infty} 1^n = 1 & \text{for } \mathrm{x} = 1 \end{cases}$$

(Step 2: Compute the sup-norm)

Note that

$$|\mathrm{f_n(x)} - \mathrm{f(x)}| = \begin{cases} \mathrm{x^n} - 0 = \mathrm{x^n} & \text{for } \mathrm{x} \in [0,1) \\ 1^n - 1 = 0 & \text{for } \mathrm{x} = 1 \end{cases}$$

So $\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1^n = 1$, therefore

$$\lim_{n\to\infty}\left(\sup_{x\in[0,1]} |f_n(x) - f(x)|\right) = \lim_{n\to\infty} 1 = 1 \neq 0$$

Hence $\mathrm{f_n(x)}$ is not uniformly convergent on $[0,1]$.

Remark: From the above example, we see the uniformly convergence of a function also depends on the interval.

Example 5
Let $\mathrm{f}: \mathbf{R} \to \mathbf{R}$ be uniformly continuous on $\mathbf{R}$ and let $\mathrm{f_n(x)} = \mathrm{f}\left(\mathrm{x} + \frac{1}{n}\right)$ for $\mathrm{x} \in \mathrm{R}$.
Show that $\mathrm{f_n(x)}$ converges uniformly on $\mathbf{R}$ to $\mathrm{f(x)}$

(Step 1: Find the limit first)

Since $\mathrm{f(x)}$ is uniformly continuous on $\mathbf{R}$, then $\mathrm{f(x)}$ is continuous on $\mathbf{R}$

$$\lim_{n\to\infty} \mathrm{f_n(x)} = \lim_{n\to\infty} \mathrm{f}\left(\mathrm{x} + \frac{1}{n}\right) = \mathrm{f}\left(\lim_{n\to\infty}\left(\mathrm{x} + \frac{1}{n}\right)\right) = \mathrm{f(x)}$$

(Step 2: Compute the sup-norm)

$$\sup_{x\in\mathbf{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbf{R}} \left|f\left(x + \frac{1}{n}\right) - f(x)\right|$$

Since $\mathrm{f(x)}$ is uniformly continuous on R,

then when $n \to \infty$, $\left|\mathrm{x} + \frac{1}{n} - \mathrm{x}\right| = \left|\frac{1}{n}\right| \to 0$, $\mathrm{f}\left(\mathrm{x} + \frac{1}{n}\right) - \mathrm{f(x)} \to 0$. Hence we have

$$\lim_{n\to\infty}\sup_{x\in\mathbf{R}}|f_n(x)-f(x)| = \lim_{n\to\infty}\sup_{x\in\mathbf{R}}\left|f\left(x+\frac{1}{n}\right)-f(x)\right| = 0$$

Hence $f_n(x)$ converges uniformly to $f(x)$

Example 6

Show that if f_n, g_n are bounded and converges uniformly to f, g on E respectively, show that $f_n g_n$ converges uniformly to fg on E

Solution:

Clearly, if f_n and g_n are bounded, then $f(x)$ and $g(x)$ are also bounded

Note that

$$|f_n(x)g_n(x) - f(x)g(x)| \le |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)|$$
$$\le |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)|$$
$$\le M|g_n(x) - g(x)| + N|f_n(x) - f(x)|$$

$$\sup_{x\in E}|f_n(x)g_n(x) - f(x)g(x)| \le M\sup_{x\in E}|g_n(x) - g(x)| + N\sup_{x\in E}|f_n(x) - f(x)|$$

$$\lim_{n\to\infty}\left(\sup_{x\in E}|f_n(x)g_n(x) - f(x)g(x)|\right) \le M(0) + N(0) = 0$$

So $\lim_{n\to\infty}(\sup_{x\in E}|f_n(x)g_n(x) - f(x)g(x)|) = 0$

Therefore $f_n g_n$ converges uniformly to fg on E.

Part 3: Power Series and radius of convergence

Definition:

A power series is a function of the form

$$a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x-c)^n$$

Where $c, a_1, a_2, ..$ are numbers and c is called **center** of the series

One interesting thing is about the convergence of power series

Given a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, the domain of convergence of the series is an non-empty interval (E) which $E \subseteq [c-R+c+R]$ where $R = \frac{1}{\limsup_{n\to\infty}\sqrt[n]{a_n}}$ is so called **radius of convergence** of the series

Example 7

Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{n^3}{3^n}(x-1)^n$$

Solution:

We can apply root test

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{\sqrt[n]{n^3}}{3}|x-1|=\lim_{n\to\infty}\frac{\left(\sqrt[n]{n}\right)^3}{3}|x-1|=\frac{|x-1|}{3}$$

The series converges when $\frac{|x-1|}{3}<1\rightarrow|x-1|<3\rightarrow-2<x<4$

The series diverges when $\frac{|x-1|}{3}>1\rightarrow|x-1|>3\rightarrow x<-2\ and\ x>4$

Hence the domain of convergence $E\subseteq[-2,4]$ (we do not know the convergence at $x=2,4$). So $R=3$.

Example 8 Find the radius of convergence of the following power series $$\sum_{n=1}^{\infty}\frac{2^n}{n!}x^n$$

Solution:

Since the terms involves factorial, instead of using root test, it may better for us to use ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}x^{n+1}}{(n+1)!}}{\frac{2^nx^n}{n!}}\right|=\lim_{n\to\infty}\frac{2x}{n+1}=0<1$$

So the series converges for all $x\in\mathbf{R}$, hence $R=\infty$

Try to work on the following exercises. You may submit your work to me for comments.

☺Exercise 1

Discuss the pointwise convergence of following series of functions

a) $\sum_{n=1}^{\infty}\frac{x^n}{n!}$

b) $$\sum_{n=1}^{\infty}a^{n^2}x^n\quad where\ a<1$$

c) $\sum_{n=1}^{\infty}n(\tan x)^n$

d) $\sum_{n=1}^{\infty}\frac{x^{2^{n-1}}}{1-x^{2^n}}$

(Hint: $\frac{x^{2^{n-1}}}{1-x^{2^n}}=\frac{1}{1-x^{2^{n-1}}}-\frac{1}{1-x^{2^n}}$)

☺Exercise 2

Define $f_n(x) = \frac{x}{x+n}$ for $x \geq 0$

Show that if $a > 0$, $f_n(x)$ converges uniformly on $[0, a]$ but does not conveges uniformly on $[0, \infty)$

☺Exercise 3

Define $f_n(x) = \frac{x^n}{1+x^n}$

Show that if $0 < b < 1$, $f_n(x)$ converges uniformly on $[0, b]$ but does not converges uniformly on $[0,1]$

☺Exercise 4

For $x \geq 0$, define $f_n(x) = e^{nx}$ and $g_n(x) = xe^{nx}$

Show that $f_n(x)$ **does not converges uniformly** on $[0, \infty)$ but $g_n(x)$ **converges uniformly** on $[0, \infty)$

☺Exercise 5

Show that if f_n and g_n converges uniformly to f, g respectively on a set A, then $f_n + g_n$ converges uniformly on A.

☺Exercise 6

Find the radius of convergence of the following power series

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+3)^n}{(n+2)2^n}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n+1}}{4^n}$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2(2n+2)!}{(2n)!\,[(n+1)!]^2} x^n$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2 4^n}$

e) $\displaystyle\sum_{n=1}^{\infty} a_n x^n$ where $a_n = \begin{cases} \dfrac{1}{2^n} & \text{if n is odd} \\ 1 & \text{if n is even} \end{cases}$