MATH2033 Mathematical Analysis
Suggested Solution of Problem Set 5

Problem 1

Prove the following limits using the definition of limits

(a) $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$

(b) $\lim_{n\to\infty}\sqrt{x_n+y_n}=2$, where $\{x_n\}$ and $\{y_n\}$ are two sequences of positive real number with $\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=2$.

☺Solution

(a) For any $\varepsilon>0$, we pick $K=\left\lfloor\frac{1}{4\varepsilon^2}\right\rfloor+1$. then for any $n\geq K>\frac{1}{4\varepsilon^2}$, we have

$$\left|\sqrt{n+1}-\sqrt{n}-0\right|=\left|\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}\right|=\frac{1}{\sqrt{n+1}+\sqrt{n}}\leq\frac{1}{2\sqrt{n}}<\varepsilon$$

So it follows from the definition of limits that $\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$.

(b) For any $\varepsilon>0$, there exists $K_1\in\mathbb{N}$ and $K_2\in\mathbb{N}$ such that

- For $n\geq K_1$, $|x_n-2|<\varepsilon$
- For $n\geq K_2$, $|y_n-2|<\varepsilon$

We take $K=\max(K_1,K_2)$. Then for $n\geq K$, we have

$$\left|\sqrt{x_n+y_n}-2\right|=\left|\frac{x_n+y_n-4}{\sqrt{x_n+y_n}+2}\right|=\frac{|x_n+y_n-4|}{\sqrt{x_n+y_n}+2}\overset{\substack{x_n>0\\y_n>0}}{<}\frac{|(x_n-2)+(y_n-2)|}{2}$$
$$\leq\frac{1}{2}\underbrace{|x_n-2|}_{<\varepsilon}+\frac{1}{2}\underbrace{|y_n-2|}_{<\varepsilon}<\varepsilon.$$

So it follows from the definition of limits that $\lim_{n\to\infty}\sqrt{x_n+y_n}=2$.

Problem 2

We let $\{x_n\}$ and $\{y_n\}$ be two sequence of real number with $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}y_n=y$. Suppose that $xy>0$, show that there exists $K\in\mathbb{N}$ such that x_n and y_n have the same sign (either both positive or both negative) when $n\geq K$.

☺Solution

Since $xy>0$, this implies that either both $x,y>0$ or both $x,y<0$.

We first consider the case when $x>0$ and $y>0$.

- Note that $\lim_{n\to\infty}x_n=x>0$. We pick $\varepsilon_1=\frac{|x|}{2}=\frac{x}{2}$, then there exists $K_1\in\mathbb{N}$ such that $|x_n-x|<\varepsilon_1\Rightarrow x_n>x-\varepsilon_1=\frac{x}{2}>0$ for $n\geq K_1$
- Note that $\lim_{n\to\infty}y_n=y>0$. We pick $\varepsilon_2=\frac{|y|}{2}=\frac{y}{2}$, then there exists $K_2\in\mathbb{N}$ such that $|y_n-y|<\varepsilon_2\Rightarrow y_n>y-\varepsilon_2=\frac{y}{2}>0$ for $n\geq K_2$

By taking $K = \max(K_1, K_2)$, we deduce that $x_n > 0$ and $y_n > 0$ for all $n \geq K$.
Next, we consider the case when $x < 0$ and $y < 0$.

- Note that $\lim_{n\to\infty} x_n = x < 0$. We pick $\varepsilon_3 = \frac{|x|}{2} = -\frac{x}{2}$, then there exists $K_3 \in \mathbb{N}$ such that $|x_n - x| < \varepsilon_1 \Rightarrow x_n < x + \varepsilon_3 = \frac{x}{2} < 0$ for $n \geq K_1$
- Note that $\lim_{n\to\infty} y_n = y > 0$. We pick $\varepsilon_4 = \frac{|y|}{2} = -\frac{y}{2}$, then there exists $K_4 \in \mathbb{N}$ such that $|y_n - y| < \varepsilon_4 \Rightarrow y_n < y + \varepsilon_4 = \frac{y}{2} < 0$ for $n \geq K_2$

By taking $K^* = \max(K_3, K_4)$, we deduce that $x_n < 0$ and $y_n < 0$ for all $n \geq K^*$.
Combining the two cases, we conclude that x_n, y_n will have the same sign when n is sufficiently large.

Problem 3

(a) Give an example of two divergent sequences $\{x_n\}, \{y_n\}$ such that the sequence $\{x_n + y_n\}$ converges.

(b) Give an example of two divergent sequences $\{x_n\}, \{y_n\}$ such that the sequence $\{x_n y_n\}$ converges.

☺Solution

(a) We take $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. We observe that

- Both x_n and y_n diverges (as shown in Example 4)
- $x_n + y_n = (-1)^n(1-1) = 0$ for all $n \in \mathbb{N}$. So $\{x_n + y_n\}$ converges to 0.

(b) We take $x_n = (-1)^n$ and $y_n = (-1)^n$. We observe that

- Both x_n and y_n diverges and
- $x_n y_n = (-1)^n(-1)^n = 1$ for all $n \in \mathbb{N}$. So $\{x_n + y_n\}$ converges to 1.

Problem 4

Show that the sequence $\{x_n\}$ defined by $x_n = n^2 - n$ diverges to $+\infty$ using the definition.

☺Solution

For any $M > 0$, we note that

$n^2 - n > M \Leftrightarrow n^2 - n - M > 0$

$$\Leftrightarrow n < \frac{1 - \sqrt{(-1)^2 - 4(1)(-M)}}{2(1)} \quad or \quad n > \frac{1 + \sqrt{(-1)^2 - 4(1)(-M)}}{2(1)}$$

$$\Leftrightarrow n < \frac{1 - \sqrt{1+4M}}{2} \quad or \quad n > \frac{1 + \sqrt{1+4M}}{2}$$

So by picking $K = \left\lfloor \frac{1+\sqrt{1+4M}}{2} \right\rfloor + 1$, then for any $n \geq K > \frac{1+\sqrt{1+4M}}{2}$, we have

$$x_n = n^2 - n > M.$$

So we conclude that $\lim_{n\to\infty} x_n = +\infty$.

Problem 5

We let $\{x_n\}$ be a sequence of positive real number which $\lim\limits_{n\to\infty} x_n = +\infty$. Show that $\lim\limits_{n\to\infty} \frac{1}{x_n} = 0$.

😊Solution

For any $\varepsilon > 0$, we take $M = \frac{1}{\varepsilon} > 0$.

Since $\lim\limits_{n\to\infty} x_n = +\infty$, then there exists $K \in \mathbb{N}$ such that $x_n > M$ for $n \geq K$.

This implies that for $n \geq K$

$$0 < \frac{1}{x_n} < \frac{1}{M} = \varepsilon \Rightarrow \left|\frac{1}{x_n}\right| = \left|\frac{1}{x_n} - 0\right| < \varepsilon.$$

So we conclude that $\lim\limits_{n\to\infty} \frac{1}{x_n} = 0$.

Problem 6

Show that the sequence $\{x_n\}$ defined by $x_n = (-1)^n \left(2 + \frac{1}{n}\right)$ does not converge.

😊Solution

We consider the subsequence $\{x_{2n}\}$ and $\{x_{2n-1}\}$, note that

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} \left(2 + \frac{1}{2n}\right) = 2 \quad and \quad \lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} -\left(2 + \frac{1}{2n-1}\right) = -2.$$

Since two subsequence have different limits, so we conclude that $\{x_n\}$ does not converge.

Problem 7 (Amended)

We let $x_1 > \sqrt{a}$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for $n \in \mathbb{N}$, where $a > 0$. Show that the sequence $\{x_n\}$ converges.

(😊Hint: Show that $\{x_n\}$ is decreasing by considering $x_{n+1} - x_n$.)

😊Solution

To prove the convergence, we shall argue that

(1) $x_n > \sqrt{a}$ for all $n \in \mathbb{N}$ and

(2) $\{x_n\}$ is decreasing, i.e. $x_{n+1} \leq x_n$.

To prove (1), we note that $x_1 > 0$. Assuming that $x_k > \sqrt{a} > 0$, then we deduce that

$$x_{k+1} - \sqrt{a} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right) - \sqrt{a} = \frac{x_k^2 - 2x_k\sqrt{a} + a}{2x_k} = \frac{\left(x_k - \sqrt{a}\right)^2}{2x_k} > 0.$$

So we have $x_{k+1} > \sqrt{a}$. It follows from mathematical induction that $x_n > \sqrt{a}$ for all $n \geq N$.

To prove (2), we note that for any $n \in \mathbb{N}$

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - x_n = \frac{a - x_n^2}{2x_n} \overset{x_n > \sqrt{a}}{\lesssim} 0.$$

So $\{x_n\}$ is decreasing.

Since the sequence is decreasing and bounded from below, it follows from monotone sequence theorem that $\{x_n\}$ converges.

Problem 8

We let $\{x_n\}$ be a bounded sequence of real numbers. For any $n \in \mathbb{N}$, we define

$$y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}.$$

Show that $\{y_n\}$ converges.

☺Solution

Since $\{x_n\}$ is bounded, so that $|x_n| \le M$ for some positive constant M. Then

$$y_n = \sup\{x_n. x_{n+1}, x_{n+2}, \dots\} \ge \sup\{-M, -M, -M, \dots\} = -M.$$

So $\underline{y_n \text{ is bounded below by } -M}$.

Also for any $n \in \mathbb{N}$, we have

$$y_{n+1} = \sup\{x_{n+1}, x_{n+2}, \dots\} \le \sup\{x_n, \sup\{x_{n+1}, x_{n+2}, \dots\}\} = \sup\{x_n. x_{n+1}, x_{n+2}, \dots\} = y_n.$$

So $\underline{\{y_n\} \text{ is decreasing}}$.

It follows from monotone sequence theorem that $\{y_n\}$ converges.

Problem 9

We let $\{x_n\}$ is a sequence of positive real numbers. For any $n \in \mathbb{N}$, we define

$$y_n = \max\{x_1, x_2, \dots, x_n\}.$$

(a) If $\{x_n\}$ is bounded, show that $\{y_n\}$ converges.

(b) If $\{x_n\}$ is unbounded, show that $\{y_n\}$ diverges to $+\infty$.

(a) Note that for any $n \in \mathbb{N}$, we have

$$y_n = \max\{x_1, x_2, \dots, x_n\} \le \max\{x_{n+1}, \max\{x_1, x_2, \dots, x_n\}\}$$
$$= \max\{x_1, x_2, \dots, x_n, x_{n+1}\} = y_{n+1}.$$

So $\{y_n\}$ is increasing.

Since $\{x_n\}$ is bounded, we have $|x_n| \le M$ or $x_n \le M$ for some positive number M. Thus, we have $y_n = \max\{x_1, x_2, \dots, x_n\} \le \max(M, M, \dots, M) = M$. So $\{y_n\}$ is bounded from above by M.

It follows from monotone sequence theorem that $\{y_n\}$ converges.

(b) If $\{x_n\}$ is unbounded, then for any $M > 0$, there exists x_K such that $x_K > M$. This implies that

$$y_K = \max\{x_1, x_2, \dots, x_K\} \ge x_K > M.$$

Since $\{y_n\}$ is increasing, it follows that for any $n \ge K$

$$y_n \ge y_K > M.$$

So $\lim_{n\to\infty} y_n = +\infty$ using the definition of limits (to infinity)

Problem 10

Show that a sequence $\{x_n\}$ defined by $x_n = (-1)^n$ is not Cauchy sequence.

☺Solution

Suppose that $\{x_n\}$ is Cauchy, we take $\varepsilon = 1$, then there exists $K \in \mathbb{N}$ such that for any $m, n \ge K$, $|x_m - x_n| < \varepsilon = 1$.

However, if we take $m = n + 1$ and $n \ge K$, it follows that

Problem 11

Show that if $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequence, then $\{x_n + y_n\}$ and $\{x_n y_n\}$ are both Cauchy sequence using the definition of Cauchy sequence.

☺Solution

We first prove that $\{x_n + y_n\}$ is Cauchy sequence.

- For any $\varepsilon > 0$, note that both $\{x_n\}$ and $\{y_n\}$ are Cauchy, then
  - ✓ There exists $K_1 \in \mathbb{N}$ such that $|x_n - x_m| < \frac{\varepsilon}{2}$ for all $m, n \geq K_1$ and
  - ✓ There exists $K_2 \in \mathbb{N}$ such that $|y_n - y_m| < \frac{\varepsilon}{2}$ for all $m, n \geq K_2$.
- We pick $K = \max(K_1, K_2)$, then for any $m, n \geq K$, we have

$$|(x_n + y_n) - (x_m + y_m)| = |(x_n - x_m) + (y_n - y_m)| \leq |x_n - x_m| + |y_n - y_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So we conclude that $\{x_n + y_n\}$ is Cauchy sequence by definition.

Then we proceed to prove that $\{x_n y_n\}$ is Cauchy sequence.

- For any $\varepsilon > 0$, note that both $\{x_n\}$ and $\{y_n\}$ are Cauchy, then
  - ✓ Both $\{x_n\}$ and $\{y_n\}$ are bounded, we write $|x_n| \leq M_x$ and $|y_n| \leq M_y$ for all $n \in \mathbb{N}$, where M_x, M_y are some positive constants.
  - ✓ There exists $K_3 \in \mathbb{N}$ such that $|x_n - x_m| < \frac{\varepsilon}{2M_y}$ for all $m, n \geq K_3$ and
  - ✓ There exists $K_4 \in \mathbb{N}$ such that $|y_n - y_m| < \frac{\varepsilon}{2M_x}$ for all $m, n \geq K_4$.
- We pick $K = \max(K_1, K_2)$, then for any $m, n \geq K$, we have

$$\begin{aligned} |x_n y_n - x_m y_m| &= |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\ &\leq |x_n||y_n - y_m| + |y_m||x_n - x_m| < M_x\left(\frac{\varepsilon}{2M_x}\right) + M_y\left(\frac{\varepsilon}{2M_y}\right) \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So we conclude that $\{x_n y_n\}$ is Cauchy sequence by definition.

Problem 12 (Harder)

We let $\{x_n\}$ be a sequence of real number with $\lim_{n\to\infty} x_n = x$. Show that

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = x.$$

(☺Hint: Note that $\lim_{n\to\infty} x_n = x$. Then for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for $n \geq K$.)

☺Solution

- Note that $\lim_{n\to\infty} x_n = x$. For any $\varepsilon > 0$, there exists $K_1 \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$ for $n \geq K_1$.
- By Archimedean property, there exists $K_2 \in \mathbb{N}$ such that

$$K_2 > \frac{2\sum_{m=1}^{K}|x_m - x|}{\varepsilon} \Leftrightarrow \frac{\sum_{m=1}^{K}|x_m - x|}{K_2} < \frac{\varepsilon}{2}.$$

By choosing $K = \max(K_1, K_2)$, then we have for any $n \geq K$,

$$\left|\frac{x_1+x_2+\cdots+x_n}{n}-x\right| = \left|\sum_{m=1}^{n}\frac{x_m-x}{n}\right| \le \frac{1}{n}\sum_{m=1}^{n}|x_m-x|$$

$$= \frac{1}{n}\left(\sum_{m=1}^{K}|x_m-x| + \sum_{m=K+1}^{n}\underbrace{|x_m-x|}_{\frac{\varepsilon}{2}}\right) \le \underbrace{\frac{1}{n}\sum_{m=1}^{K}|x_m-x|}_{<\frac{\varepsilon}{2}\ as\ n\ge K_2} + \underbrace{\frac{n-K}{n}}_{<1}\left(\frac{\varepsilon}{2}\right)$$

$$\le \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

So $\lim_{n\to\infty}\frac{x_1+x_2+\cdots+x_n}{n} = x$ by definition.

Problem 13 (Harder)

We let $\{x_n\}$ be a bounded sequence and let $s = \sup\{x_n | x \in \mathbb{N}\}$. Show that if $s \notin \{x_n | n \in \mathbb{N}\}$, then there exists a subsequence of $\{x_n\}$ which converges to s.

(😊Hint: You need to construct such subsequence. Using the property of supremum and the fact that $s \notin \{x_n | n \in \mathbb{N}\}$, argue that for any $\varepsilon > 0$, there exists infinitely many $x_n s$ such that $s > x_n > s - \varepsilon$. Construct the subsequence by taking $\varepsilon = \frac{1}{k}$ for $k \in \mathbb{N}$.)

☺Solution

We first argue that for any $\varepsilon > 0$, there are infinitely many $x_n s$ such that $s > x_n > s - \varepsilon$.

- Suppose that for some $\varepsilon_0 > 0$, there are finitely many such $x_n s$ (we write them as $x_{K_1}, x_{K_2}, \dots, x_{K_m}$).
- We take $M = \max(s-\varepsilon_0, x_{K_1}, x_{K_2}, \dots, x_{K_m}) < s$ (as $x_n \ne s$), then we deduce that $x_n \le M < s$ for all $n \in \mathbb{N}$ and M is also the upper bound which contradicts to the fact that s is the least upper bound.

Next, we construct a subsequence $\{x_{n_k}\}$ as follows:

- We take $\varepsilon = 1$, there is x_{n_1} such that $s > x_{n_1} > s - 1$.
- We take $\varepsilon = \frac{1}{2}$, there are infinitely many $x_n s$ such that $s > x_n > s - \frac{1}{2}$. Among those $x_n s$, we choose x_{n_2} which $n_2 > n_1$ (it is feasible since there are infinitely many x_ns.)
- We take $\varepsilon = \frac{1}{3}$, there are infinitely many $x_n s$ such that $s > x_n > s - \frac{1}{3}$. Among those $x_n s$, we choose x_{n_3} which $n_3 > n_2$
- By repeating this process for all $\varepsilon = \frac{1}{k}$ where $k \in \mathbb{N}$, we obtain a subsequence $\{x_{n_k}\}$ which
  - ✓ $s > x_{n_k} > s - \frac{1}{k}$ for all $k \in \mathbb{N}$ and
  - ✓ $n_{k+1} > n_k$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} n_k = +\infty$.

Since $\lim_{k\to\infty} s = \lim_{k\to\infty}\left(s - \frac{1}{k}\right) = s$, it follows from sandwich theorem that

$$\lim_{k\to\infty} x_{n_k} = s.$$