MATH 2031 Introduction to Real Analysis

September 19, 2012

Tutorial Note 2

Logic

(I) Negation $\sim$ (or $\neg$),

(II) De Morgan's laws $\begin{array}{ccc} \sim (p \text{ and } q) & \Leftrightarrow & (\sim p) \text{ or } (\sim q) \\ \sim (p \text{ or } q) & \Leftrightarrow & (\sim p) \text{ and } (\sim q) \end{array}$,

(III) Quantifiers for all $\forall$, there exist $\exists$

(IV) Conditional statements $p \Rightarrow q$ and $\sim (p \Rightarrow q) = p$ and $(\sim q)$

(V) Contrapositive $(\sim q) \Rightarrow (\sim p) = p \Rightarrow q$

Set

(a) Common sets in Maths $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$

(b) Basic Set Operation
Let $S_1, S_2, \cdots$ be subsets of A,

(1) Union

$$S_1 \cup S_2 \cup \cdots = \bigcup_{n=1}^{\infty} S_n = \{x \in A : x \in S_i \text{ for some } i \in \mathbb{N}\}$$

(2) Intersection

$$S_1 \cap S_2 \cap \cdots = \bigcap_{n=1}^{\infty} S_n = \{x \in A : x \in S_i \text{ for all } i \in \mathbb{N}\}$$

(3) Cartesian Product

$$S_1 \times S_2 \times \cdots = \{(x_1, x_2, x_3, \cdots) : \text{ each } x_i \in S_i \text{ for all } i \in \mathbb{N}\}$$

(4) Complement

$$S_1 \setminus S_2 = \{x \in A : x \in S_1 \text{ and } x \notin S_2\}$$

Sometimes we denote $A \setminus S_i$ as $(S_i)^c$,i.e.$(S_i)^c = A \setminus S_i$

(c) $S_1, S_2, \cdots$ are disjoint iff $\bigcap_{n=1}^{\infty} S_n = \emptyset$. $S_1, S_2, \cdots$ are mutually disjoint iff $S_i \cap S_j = \emptyset$ for all distinct $i, j \in \mathbb{N}$.

(d) Subset $A \subseteq B$, proper subset $A \subset B$

(e) Equality of sets $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

Problem 1 Negate each of the following statements

(i) $\forall x \exists y \exists z$, $x^3 + y^7 = z$ and $x + y < z$.

(ii) $w < 0$ implies w has no real square root.

Solution:

(i) The statement says that "For all x, there is a y and there is a z such that the properties $x^3 + y^7 = z$ and $x + y < z$ are satisfied." The negation is that "There is a x such that for any y and any z, the properties $x^3 + y^7 = z$ and $x + y < z$ are not satisfied".

$$\begin{aligned} &\exists x \forall y \forall z, \quad \sim (x^3 + y^7 = z \text{ and } x + y < z) \\ i.e \quad &\exists x \forall y \forall z, \quad x^3 + y^7 \neq z \text{ or } x + y \geq z \end{aligned}$$

(ii) The statement in symbols is that $w < 0 \Rightarrow \sqrt{w} \notin \mathbb{R}$. Negating the statement, we get

$$\sim (w < 0 \Rightarrow \sqrt{w} \notin \mathbb{R}) = [(w < 0) \text{ and } \sim (\sqrt{w} \notin \mathbb{R})] = [(w < 0) \text{ and } \sqrt{w} \in \mathbb{R}].$$

Problem 2 Give the contrapositive of the following statements.

(a) If $AB = AC$ in $\triangle ABC$, then $\angle B = \angle C$ in $\triangle ABC$.

(b) If a function is differentiable, then it is continuous.

(c) If $\lim\limits_{x\to\infty} f(x) = a$ and $\lim\limits_{x\to\infty} g(x) = b$, then $\lim\limits_{x\to\infty} (f(x) + g(x)) = (a + b)$.

(d) If $ax^2 + bx + c = 0$, then $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Solution:

(a) $\sim (\angle B = \angle C$ in $\triangle ABC) \Rightarrow \sim (AB = AC$ in $\triangle ABC)$, which is equivalent to $\angle B \neq \angle C$ in $\triangle ABC \Rightarrow AB \neq AC$ in $\triangle ABC$.

(b) If a function is not continuous, then it is not differentiable.

(c) $\lim\limits_{x\to\infty} (f(x) + g(x)) \neq (a + b) \Rightarrow (\lim\limits_{x\to\infty} f(x) \neq a$ or $\lim\limits_{x\to\infty} g(x) \neq b)$.

(d) $(x \neq \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x \neq \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}) \Rightarrow ax^2 + bx + c \neq 0$.

Remark:
Since the contrapositives are equivalent to the original statements, they say the same thing. This is useful because sometimes its contrapositive is more helpful.
For example, if we want to show a function is not differentiable, we could first check whether it's continuous, which is usually easier to check. In the contrapositive of part (b), if a function is not continuous, then it is not differentiable.

Problem 3 Compute the following sets.

(a) $(\{x, y, z\} \cup \{w, z\}) \setminus \{u, v, w\}$.(Here$u, v, w, x, y, z$ are distinct objects)

(b) $\{1, 2\} \times \{3, 4\} \times \{5\}$

(c) $\mathbb{Z} \cap [-2, 10] \cap \{n^2 + 1 : n \in \mathbb{N}\}$

(d) $\{n \in \mathbb{N} : 5 < n < 9\} \setminus \{2m : m \in \mathbb{N}\}$

(e) $([0, 2] \setminus [1, 3]) \cup ([1, 3] \setminus [0, 2])$

Solution:

(a) $(\{x, y, z\} \cup \{w, z\}) \setminus \{u, v, w\} = \{w, x, y, z\} \setminus \{u, v, w\} = \{x, y, z\}$

(b) $\{1, 2\} \times \{3, 4\} \times \{5\} = \{(x, y, z) : x \in \{1, 2\}$ and $y \in \{3, 4\}$ and $z \in \{5\}\} = \{(1, 3, 5), (1, 4, 5), (2, 3, 5), (2, 4, 5)\}$

(c) $\mathbb{Z} \cap [-2, 10] \cap \{n^2 + 1 : n \in \mathbb{N}\} = \{-1, -2, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \cap \{2, 5, 10, 17, \cdots\} = \{2, 5, 10\}$

(d) $\{n \in \mathbb{N} : 5 < n \leq 9\} \setminus \{2m : m \in \mathbb{N}\} = \{6, 7, 8, 9\} \cap \{2, 4, 6, 8, \cdots\} = \{6, 8\}$

(e) $([0, 2] \setminus [1, 3]) \cup ([1, 3] \setminus [0, 2]) = [0, 1) \cup (2, 3]$

Problem 4 Let A, B and C be subsets of X. Proof the following properties

(a) $A \setminus B = A \cap B^c$

(b) $A \cup A^c = X$

(c) $A \cap A^c = \emptyset$

(d) $(A^c)^c = A$

(e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(f) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(g) $(A \cup B)^c = A^c \cap B^c$

(h) $(A \cap B)^c = A^c \cup B^c$

Solution:

(a) $x \in A \setminus B \Leftrightarrow (x \in A \text{ and } x \notin B) \Leftrightarrow (x \in A \text{ and } (x \in X \text{ and } x \notin B)) \Leftrightarrow (x \in A \text{ and } x \in B^c) \Leftrightarrow x \in A \cap B^c$

(b) $x \in X \Leftrightarrow (x \in A \text{ or } (x \in X \text{ and } x \notin A)) \Leftrightarrow (x \in A \text{ or } x \in A^c) \Leftrightarrow x \in A \cup A^c$

(c) Clearly, $A \cap A^c \supseteq \emptyset$. Conversely, we prove the reverse direction by contradiction.
Suppose $\exists x \in A \cap A^c$, then we have $x \in A$ and $x \notin A$ contradiction. Hence $A \cap A^c \subseteq \emptyset$.

(d) $x \in (A^c)^c \Leftrightarrow x \in X$ and $x \notin A^c$. By definition, $A^c = \{x \in X : x \notin A\}$. So $x \notin A^c$ means that $x \in A$. Conversely, $x \in A \Rightarrow x \notin A^c$ and thus, $x \in (A^c)^c$.

(e) $x \in A \cup (B \cap C) \Leftrightarrow (x \in A \text{ or } x \in (B \cap C)) \Leftrightarrow (x \in A \text{ or } (x \in B \text{ and } x \in C)) \Leftrightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \Leftrightarrow x \in (A \cup B) \text{ and } (x \in A \cup C) \Leftrightarrow (x \in A \cup B) \cap (A \cup C)$.

(f) It's similar to part (e) and is left as an exercise.

(g) $x \in (A \cup B)^c \Leftrightarrow (x \in X \text{ and } x \notin (A \cup B)) \Leftrightarrow (x \in X \text{ and } ((x \notin A) \text{and } x \notin B)) \Leftrightarrow (x \in X \text{ and } (x \notin A)) \text{ and } ((x \in X \text{ and } x \notin B)) \Leftrightarrow x \in A^c \cap B^c$.

(h) It's similar to part (g) and is left as an exercise.

Remark:
Parts (e) and (f) are the distributive rules for sets and parts (g) and (h) are De Morgan's law for sets.

Problem 5 Let A and B be subsets of X. Define the symmetric difference $A \triangle B = (A \cup B) \setminus (A \cap B)$.
Prove

$$A \triangle B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$$

(So we can see that in **Problem 3**(e) $([0,2] \setminus [1,3]) \cup ([1,3] \setminus [0,2]) = [0,2] \triangle [1,3]$.)
Solution:

$$\begin{aligned}
A \triangle B = (A \cup B) \setminus (A \cap B) &= (A \cup B) \cap (A \cap B)^c && \text{(by 4(a))} \\
&= (A \cap (A \cap B)^c) \cup (B \cap (A \cap B)^c) && \text{(by 4(f))} \\
&= (A \cap (A^c \cup B^c)) \cup (B \cap (A^c \cup B^c)) && \text{(by 4(h))} \\
&= ((A \cap A^c) \cup (A \cap B^c)) \cup ((B \cap A^c) \cup (B \cap B^c)) && \text{(by 4(f))} \\
&= (\emptyset \cup (A \setminus B)) \cup ((B \setminus a) \cup \emptyset) && \text{(by 4(a) and 4(c))} \\
&= (A \setminus B) \cup (B \setminus A)
\end{aligned}$$

Problem 6 For any subset $Y$of $\mathbb{R}$, define $\mathcal{I}(Y) = \{f : f \text{ is a continuous real function such that } f(p) = 0\ \forall p \in Y\}$.
Prove

(a) If $A \subseteq B$ are subsets of $\mathbb{R}$, then $\mathcal{I}(A) \supseteq \mathcal{I}(B)$.

(b) If C and D are subsets of $\mathbb{R}$, then $\mathcal{I}(C \cup D) = \mathcal{I}(C) \cap \mathcal{I}(D)$.

Solution:

(a) From the definition of $I(Y)$, $f \in \mathcal{I}(B) \Leftrightarrow f$ is continuous real function and $\forall p \in B, f(p) = 0$.
As $A \subseteq B$, $q \in A \Rightarrow q \in B$, thus $\forall q \in A, f(q) = 0$ and f is continuous real function, so $f \in \mathcal{I}(A)$.

(b) By part (a), since C and D are subsets of $C \cup D$, $\mathcal{I}(C) \supseteq \mathcal{I}(C \cup D)$ and $\mathcal{I}(D) \supseteq \mathcal{I}(C \cup D)$. Thus $\mathcal{I}(C) \cap \mathcal{I}(D) \supseteq \mathcal{I}(C \cup D)$.
Conversely, $f \in \mathcal{I}(C) \cap \mathcal{I}(D) \Rightarrow (f$ is continuous real function and $\forall p \in C$, $f(p) = 0$ and $\forall q \in D$, $f(q) = 0) \Rightarrow (f$ is continuous real function and $\forall p \in C \cup D$, $f(p) = 0 \Rightarrow f \in \mathcal{I}(C \cup D)$.