DEPARTMENT OF MATHEMATICS

Math2033 TA note 6

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1 LIMIT

Example 1. *For each of the following sequence* $\{x_n\}$*, show it converges and find its limit.*

$$x_1 = 1, x_2 = 2 \text{ and } x_{n+1} = \sqrt{x_n} + \sqrt{x_{n-1}} \text{ for } n = 2, 3, \cdots$$

Solution: We have

$$x_{n+1} - x_n = \sqrt{x_n} + \sqrt{x_{n-1}} - (\sqrt{x_{n-1}} + \sqrt{x_{n-2}}) = \sqrt{x_n} - \sqrt{x_{n-2}}. \tag{1.1}$$

Then we state

$$x_n > x_{n-1}, x_n > x_{n-2}.$$

Firstly, $x_3 = \sqrt{2} + 1 > x_2, x_3 > x_1$. Then if for $n = k$, the statement is true, i.e.,

$$x_k > x_{k-1}, x_k > x_{k-2}.$$

by equation (1.1), we have

$$x_{k+1} - x_k = \sqrt{x_k} - \sqrt{x_{k-2}} > 0 \implies x_{k+1} > x_k$$

and by $x_k > x_{k-1}$ and $x_{k+1} > x_k$, we have

$$x_{k+1} > x_{k-1}.$$

So the statement is true when $n = k+1$. By induction principle, we prove the statement is true for all $n = 3, 4, \cdots$. Then we prove the sequence is monotone increasing. If $x_n < 4, x_{n-1} < 4$ then $x_{n+1} < 4$. So the sequence is bounded. Then by monotone convergence theorem, there is a limit denoted as a. By

$$a = \sqrt{a} + \sqrt{a},$$

we have $a = 4$. This is the limit.

Example 2. *For each of the following sequence $\{x_n\}$, show it converges and find its limit.*

$$x_1 = 1 \textit{ and } x_{n+1} = \tfrac{2-x_n}{3+x_n} \textit{ for } n = 1,2,3,\cdots$$

Solution: Firstly calculate that

$$x_{n+1} = -1 + \frac{5}{3+x_n}.$$

When $0 < x_n < 2$, by $x_{n+1} = -1 + \frac{5}{3+x_n}$, we can show that $0 < x_{n+1} < 2$. By induction principle, $0 < x_n < 2, \forall n = 1,2,3,\cdots$.

Secondly, we consider the monotone property of the sequence.

$$x_{n+1} - x_n = \frac{5}{3+x_n} - \frac{5}{3+x_{n-1}}$$

indicates that if $x_n > x_{n-1}$ for some fixed n, then $x_{n+1} < x_n$ as a consequence, which doesn't have the monotone property. But we can consider the odd sequence and even sequence, they may have monotone property. That is

$$\begin{aligned}
x_{n+2} - x_n &= \frac{5}{3+x_{n+1}} - \frac{5}{3+x_{n-1}} \\
&= \frac{5(x_{n-1} - x_{n+1})}{(3+x_{n+1})(3+x_{n-1})} \\
&= \frac{5}{(3+x_{n+1})(3+x_{n-1})}\left(\frac{5}{3+x_{n-2}} - \frac{5}{3+x_n}\right) \\
&= \frac{5}{(3+x_{n+1})(3+x_{n-1})}\frac{5}{(3+x_n)(3+x_{n-2})}(x_n - x_{n-2})
\end{aligned} \tag{1.2}$$

By $0 < x_n < 2$, we have $\frac{5}{(3+x_{n+1})(3+x_{n-1})}\frac{5}{(3+x_n)(3+x_{n-2})} > 0$. Then by equation (1.2),

$$x_{n+2} > x_n \quad \text{if} \quad x_n > x_{n-2}$$

or

$$x_{n+2} < x_n \quad \text{if} \quad x_n < x_{n-2}.$$

Then by $x_1 = 1, x_2 = \frac{1}{4}, x_3 = \frac{7}{13}, x_4 = \frac{19}{46}$, for sequence

$$x_1, x_3, x_5, x_7, \cdots,$$

because of $x_1 > x_3$, according to our analysis we have $x_5 < x_3, x_7 < x_5, \cdots$ and it's a monotone decreasing sequence. For the sequence

$$x_2, x_4, x_6, x_8, \cdots,$$

in the same fashion, it's a monotone increasing sequence. Because $0 < x_n < 2$ and the monotone property of odd sequence and even sequence, there is a limit for each sequence based on monotone convergence theorem. Suppose a is the limit of odd sequence and b is the limit of even sequence, then by

$$a = -1 + \frac{5}{3+b} \quad \text{and} \quad b = -1 + \frac{5}{3+a},$$

we can solve that $a = b = \sqrt{6}-2$ as $0 < x_n < 2$. By intertwining theorem, the sequence converges and the limit is $\sqrt{6}-2$.

Example 3. *Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$, $M_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}$ and $m_n = \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}$ for $n \in \mathbb{N}$.*

(a) Prove that both sequence $\{M_n\}$ and $\{m_n\}$ converge. (The limit of M_n is called the limit superior of x_n and is denoted by $\limsup_{n\to\infty} x_n$, while the limit of m_n is called the limit inferior of x_n and is denoted by $\liminf_{n\to\infty} x_n$.)

(b) Prove that $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} M_n = x = \lim_{n\to\infty} m_n$. (i.e. $\limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$.)

Solution: (a) Since $\{x_n\}$ is bounded, $\{M_n\}$ and $\{m_n\}$ are bounded. M_n is an upper bound of $\{x_{n+1}, x_{n+2}, \dots\}$ and m_n is a lower bound of $\{x_{n+1}, x_{n+2}, \dots\}$ imply $M_{n+1} \le M_n$ and $m_n \le m_{n+1}$. So $\{M_n\}$ is decreasing, $\{m_n\}$ is increasing. By the monotone limit theorem, both sequence $\{M_n\}$ and $\{m_n\}$ converge.

(b) Since $m_n \le x_n \le M_n$, $\lim_{n\to\infty} M_n = x = \lim_{n\to\infty} m_n \Rightarrow \lim_{n\to\infty} x_n = x$ by sandwich theorem.

Conversely, if $\lim_{n\to\infty} x_n = x$, then $\forall\epsilon > 0$, $\exists K$ such that $x - \epsilon/2 < x_n < x + \epsilon/2$ holds for all $n \ge K$. Thus, for all $n \ge K$, we have $M_n, m_n \in [x - \epsilon/2, x + \epsilon/2] \subseteq (x - \epsilon, x + \epsilon)$. Therefore, $\lim_{n\to\infty} M_n = x = \lim_{n\to\infty} m_n$.

Remark 4. *We can use this example to prove the Bolzano-Weierstrass theorem: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof. Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$, then by the example, the sequence M_n converges to some real number M. So, $\forall\epsilon > 0$, there exists $K(\epsilon)$ such that $n \ge K(\epsilon) \Rightarrow |M_n - M| < \epsilon$. We choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ as follow: let $x_{n_1} = x_1$; suppose $x_{n_1}, \dots, x_{n_i}$ have been selected, we choose $j \ge max\{n_i + 1, K(\frac{1}{2i})\}$, then $|M_j - M| < \frac{1}{2i}$. Since M_j is the supreme of $\{x_j, x_{j+1}, \dots\}$, there exists $m \ge j$ such that $|x_m - M_j| < \frac{1}{2i}$. We let $x_{n_{i+1}} = x_m$, then $|x_{n_{i+1}} - M| \le |x_{n_{i+1}} - M_j| + |M_j - M| < 1/i$ for all $i \ge 2$. Therefore, $\{x_{n_i}\}$ converges to M. □

Example 5. *Prove that if $\{a_n\}$ converges to A, then $\{\alpha_n\}$ converges to A, where $\alpha_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$. Show that the converse false.*

Solution: Let $b_n = a_n - A$ and $\beta_n = \frac{b_1 + b_2 + \cdots + b_n}{n}$, then $\lim\limits_{n\to\infty} \alpha_n = A \Leftrightarrow \lim\limits_{n\to\infty} (\alpha_n - A) \Leftrightarrow \lim\limits_{n\to\infty} \beta_n = 0$, which is to be shown.

Since $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} (a_n - A) = 0$, by boundedness theorem, $\{b_n\}$ is bounded, say $|b_n| \le M, M > 0$ for all $n \in \mathbb{N}$. Moreover, for any $\epsilon > 0$, there is $K_1 \in \mathbb{N}$ such that $n \ge K_1$ implies

$$|b_n - 0| < \frac{\epsilon}{2}$$

Let $K = [max(K_1, \frac{2(K_1 - 1)M}{\epsilon})]$, then for $n \ge K$, we have

$$|\beta_n - 0| = |\frac{b_1 + b_2 + \cdots + b_{K_1 - 1}}{n} + \frac{b_{K_1} + \cdots + b_n}{n}| < \frac{(K_1 - 1)M}{n} + \frac{(n - K_1 + 1)\frac{\epsilon}{2}}{n} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, $\lim\limits_{n\to\infty} \beta_n = 0$.

To see the converse is false, take $a_n = (-1)^n$. Then

$$\alpha_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$$

So $\lim_{n\to\infty} \alpha_n = 0$ but $\{a_n\}$ doesn't converge.