Lecture 17 04-04-2019

Review :

1. $f: S \to \mathbb{R}$ is differentiable at $x_0 \in S$ iff

$\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists. This limit is denoted by $f'(x_0)$

and is call the derivative of f at x_0

2. f is differentiable at x_0 $\iff$ $f(x_0+h) = f(x_0) + f'(x_0)h + R(x_0,h)$

with $\lim\limits_{h \to 0} \dfrac{R(x_0,h)}{h} = 0$

3. f is differentiable at x_0 $\Rightarrow$ $\begin{cases} ① \ f \text{ is continuous at } x_0 \\ ② \ g(x) = \begin{cases} \dfrac{f(x)-f(x_0)}{x-x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0 \end{cases} \text{ is continuous at } x_0 \end{cases}$

average rate of change of f from x_0 to x.

4. f, g are differentiable $\Rightarrow$ $f \pm g$, $f \cdot g$, f/g are differentiable

5. $y = f(x)$ is differentiable in x, $x = g(t)$ is differentiable in t

then $y = f(g(t))$ is differentiable in t, and $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$

Remark: the derivative f' of a differentiable function may not be continuous.

Exercise: ① let $g(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0 & x = 0 \end{cases}$

Show that g is differentible in $\mathbb{R}$, so $g'(x)$ is well-defined for all $x \in \mathbb{R}$. As a function of x, $g'(x)$ is continuous at any $x \neq 0$. But $g'(x)$ is not continuous at $x = 0$.

② let $h(x) = \begin{cases} 0 & , x \neq 0 \\ x & x = 0. \end{cases}$

Find $h'(x) \equiv 0$

Notation

let S be an interval of positive length.

$C^0(S) = C(S)$: the set of all continuous functions on S

$C^n(S)$: the set of functions $f: S \to \mathbb{R}$ s.t. $f^{(n)}$ is continuous.

$C^\infty(S) = \bigcap_{n \in \mathbb{N}} C^{(n)}(S)$: the set of functions having n-th derivatives for all $n \in \mathbb{N}$, or the set of infinitely smooth functions

Especially, functions in $C^1(S)$ are said to be continuously differentiable on S

Inverse Function Theorem

THM : If f is continuous and injective on (a,b) and $f'(x_0) \neq 0$ for some $x_0 \in (a,b)$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

[Notation : $y = f(x) \Leftrightarrow x = f^{-1}(y)$, $\left.\frac{dx}{dy}\right|_{y=y_0} \cdot \left.\frac{dy}{dx}\right|_{x=x_0} = 1$]

Proof: Step 1. f is continuous and injective on (a,b), by the continuous inverse thm, f^{-1} is well-defined and is continuous.

Step 2. Consider

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{f(f^{-1}(y)) - f(f^{-1}(y_0))}$$

Define

$$g(x) = \begin{cases} \dfrac{x - x_0}{f(x) - f(x_0)}, & x \neq x_0 \\ \dfrac{1}{f'(x_0)}, & x = x_0 \end{cases}$$

then $\lim\limits_{x\to x_0} g(x) = \lim\limits_{x\to x_0} \dfrac{x-x_0}{f(x)-f(x_0)} = \lim\limits_{x\to x_0} \dfrac{1}{\frac{f(x)-f(x_0)}{x-x_0}} = \dfrac{1}{f'(x_0)}$

So $g(x)$ is continuous at $x_0 = f^{-1}(y_0)$

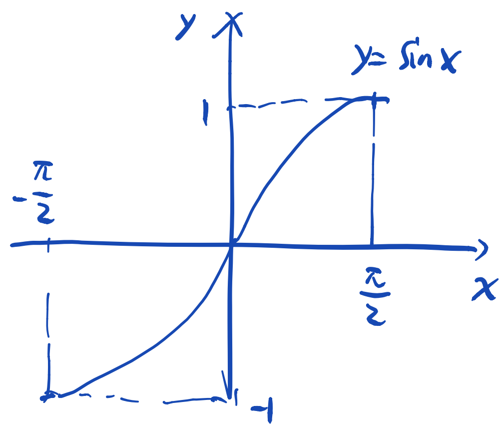
then $\lim\limits_{y\to y_0} \dfrac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} = \lim\limits_{y\to y_0} \dfrac{f^{-1}(y)-f^{-1}(y_0)}{f(f^{-1}(y))-f(f^{-1}(y_0))} = \lim\limits_{y\to y_0} g(f^{-1}(y))$

$= g(f^{-1}(y_0)) = g(x_0) = \dfrac{1}{f'(x_0)}$

$g\circ f^{-1}$ is continuous at y_0 by the composite rule for continuous function

$\Rightarrow$ f^{-1} is differentiable at y_0 and the derivative is $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$

Example: $y = f(x) = \sin x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. then f is differentiable and injective on $(-\frac{\pi}{2}, \frac{\pi}{2})$.



$$x = f^{-1}(y) = \sin^{-1} y = \arcsin y$$

$$\frac{dx}{dy} = \frac{d\, f^{-1}}{dy} = \frac{d \sin^{-1} y}{dy} = \frac{d \arcsin y}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cos x}$$

Since $y = \sin x$, $\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x + y^2 = 1$

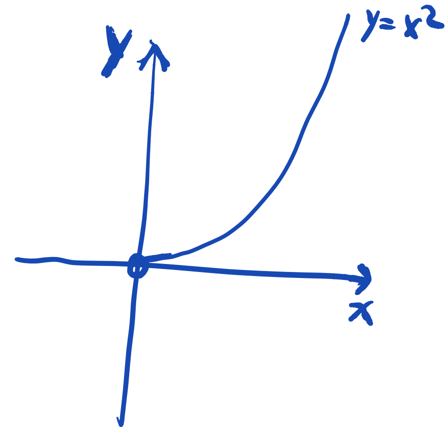
$\Rightarrow \cos^2 x = 1 - y^2$. since $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\cos x \geq 0$

$\Rightarrow \cos x = \sqrt{1-y^2}$

$\Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{1-y^2}}$ or $\frac{d \sin^{-1} y}{dy} = \frac{1}{\sqrt{1-y^2}}$

Example: $y = x^2$. $x > 0$.

$x = \sqrt{y}$ $y > 0$.



Method ①: Apply inverse Function theorem

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}.$$

Method ②: $\frac{dx}{dy} = \frac{d\sqrt{y}}{dy} = \frac{1}{2}y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}.$

Note that $\frac{dy}{dx}\Big|_{x=0} = 0$

Which $\frac{dx}{dy}$ is not differentiable

at $y=0$

Local Extremum Theorem

THM: (Femat's principal). let $f: (a, b) \to \mathbb{R}$ be differentiable,

If $f(x_0) = \min\limits_{x \in (a,b)} f(x)$ or $f(x_0) = \max\limits_{x \in (a,b)} f(x)$,

then $f'(x_0) = 0$.

Proof: If $f(x_0) = \min\limits_{x \in (a,b)} f(x)$. then $\dfrac{f(x) - f(x_0)}{x - x_0} \geq 0$ for $x > x_0$,

By limiting inequality $\lim\limits_{x \to x_0^+} \dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \geq 0$.

Similarly, $\dfrac{f(x) - f(x_0)}{x - x_0} \leq 0$ for $x < x_0$,

$\Rightarrow \lim\limits_{x \to x_0^-} \dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \leq 0 \Rightarrow f'(x_0) = 0$

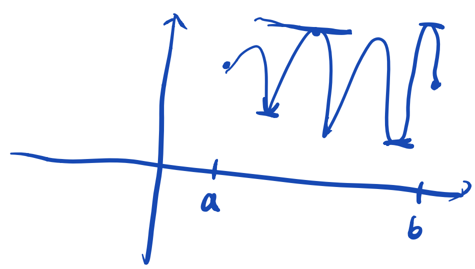
The case for $f(x_0) = \max\limits_{x \in (a,b)} f(x)$ is similar.

Remark: the above thm is false when (a, b) is replaced by $[a, b]$

Example: $f(x) = x$, $0 \leq x \leq 1$.

Rolle's theorem

THM: Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$, then $\exists\ z_0 \in (a,b)$ s.t $f'(z_0)=0$.



Pf: If f is a constant function, then $f'(x)=0$. $\forall x \in (a,b)$.

Otherwise, by the extrema value theorem for continuous functions, $\exists\ x_0, w_0$ s.t $f(x_0)=\max\limits_{x\in[a,b]} f(x)$, $f(w_0)=\min\limits_{x\in[a,b]} f(x)$

then we have either $f(x_0)\neq f(a)$, or $f(w_0)\neq f(a)$

Since $f(a)=f(b)$, So either $x_0\neq a,b$, or $w_0\neq a,b$,

i.e either $x_0\in(a,b)$ or $w_0\in(a,b)$. By the local extrema theorem, either $f'(x_0)=0$ or $f'(w_0)=0$.