MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)
Tutorial Note #13

Introduction of Cauchy Sequence
(The material below will NOT be covered in final)

Definition (Cauchy Sequence):
A sequence of real number $\{x_n\}$ is Cauchy if and only if for any $\varepsilon > 0$, there is positive integer K such that $m, n > K \rightarrow |x_m - x_n| < \varepsilon$

Roughly speaking, if a sequence is Cauchy, then when K is sufficient large, all x_n $(n > K)$ should be very close to each other. (i.e. the distance of two points x_m, x_n is very small). One interesting thing about the definition of Cauchy is, we need not know what the limit of sequence is since sometimes it may be difficult (or impossible) for us to compute the limit.

In Real Number Space, if a sequence is Cauchy, then the sequence will converges to some number L in $\mathbf{R}$. This is so called Cauchy Theorem.

Theorem: (Cauchy Theorem)
The sequence is Cauchy if and only if $\{x_n\}$ converges to some real number L.

So it provides an alternative way to show the sequence converges, all we need to do is to check the sequence is Cauchy.

Let us show you some examples about the use of Cauchy Sequence

Example 1 (Tutorial Note #12 Example 1)
Show that the sequence $\{y_n\}$ defined by

$$y_1 = \frac{1}{1+2}, y_{n+1} = y_n + \frac{1}{2^{n+1}+1} \text{ for } n \geq 1$$

converges

In previous tutorial, we show y_n using monotonic sequence theorem. Here we solve it by proving y_n is Cauchy

IDEA: We would like to show for any $\varepsilon > 0$, there exists K such that for $m, n > K$, $|y_m - y_n| < \varepsilon$. Assume $m > n$

$$|y_m - y_n| = |(y_m - y_{m-1}) + (y_{m-1} - y_{m-2}) + \cdots \ldots + (y_{n+2} - y_{n+1}) + (y_{n+1} - y_n)$$

$$\leq |y_m - y_{m-1}| + |y_{m-1} - y_{m-2}| + \cdots .. + |y_{n+2} - y_{n+1}| + |y_{n+1} - y_n|$$

$$= \frac{1}{2^m+1} + \frac{1}{2^{m-1}+1} + \cdots \ldots + \frac{1}{2^{n+2}+1} + \frac{1}{2^{n+1}+1}$$

$$\leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \cdots \ldots . + \frac{1}{2^m} \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \cdots \ldots \ldots \ldots$$

$$= \frac{\frac{1}{2^{n+1}}}{1-\frac{1}{2}} = \frac{1}{2^n} < \varepsilon \to n > \frac{\ln\left(\frac{1}{\varepsilon}\right)}{ln2}$$

Solution:

By Archimedean property, there exists K such that $n > K,\ n > \frac{\ln\left(\frac{1}{\varepsilon}\right)}{ln\,2}$.

Then for $m, n > K$, from the previous calculation, we have

$|y_m - y_n| < \varepsilon$

Therefore y_n is Cauchy and therefore converges by Cauchy Theorem.(i.e. $\lim_{n\to\infty} y_n$ exists)

Why we need Cauchy so much? Sometimes given a sequence, we may not know what exactly the sequence is. So it may be difficult for us to show the sequence converges by using theorems we learned before. Here Cauchy Sequence sometimes can help us to deal with such difficult situation.

Example 2

Suppose a sequence of distinct positive numbers $\{x_n\}$ satisfy

$$|x_{n+1} - x_n| < \frac{1}{3}|x_n - x_{n-1}|$$

So $\{x_n\}$ converges

IDEA: We will show $\{x_n\}$ is indeed Cauchy. i.e. for any $\varepsilon > 0$, there exists K such that for any $m, n > K$, $|x_m - x_n| < \varepsilon$

Note that

$$|x_{n+1} - x_n| < \frac{1}{3}|x_n - x_{n-1}| < \frac{1}{3^2}|x_{n-1} - x_{n-2}| < \frac{1}{3^3}|x_{n-2} - x_{n-3}| \ldots \ldots \ldots ..$$

$$\to |x_{n+1} - x_n| < \frac{1}{3^{n-1}}|x_2 - x_1|$$

For $n = 1,2,3,4, \ldots, \ldots.$

So assume $m > n$

$$|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots .. + |x_{n+2} - x_{n+1}| + |x_{n+1} - x_n|$$

$$= \left(\frac{1}{3^{m-2}} + \frac{1}{3^{m-3}} + \cdots \ldots . + \frac{1}{3^n} + \frac{1}{3^{n-1}}\right)|x_2 - x_1|$$

$$\leq \left(\frac{1}{3^{n-1}} + \frac{1}{3^n} + \frac{1}{3^{n+1}} + \cdots \ldots \ldots\right)|x_2 - x_1| = \frac{1}{2(3)^{n-2}}|x_2 - x_1| < \varepsilon$$

$$\to n > \frac{\ln\frac{|x_2 - x_1|}{2\varepsilon}}{\ln 3} + 2 \text{ (what a terrible term =.= ")}$$

Solution:

First, we have

$$|x_{n+1} - x_n| < \frac{1}{3^{n-1}}|x_2 - x_1|$$

For any $\varepsilon > 0$

By Archimedean property, there exists K such that $K > \frac{\ln\frac{|x_2 - x_1|}{2\varepsilon}}{\ln 3} + 2$, then for any

$m > n > K$, $n > \frac{\ln\frac{|x_2 - x_1|}{2\varepsilon}}{\ln 3} + 2$

From the previous calculation, we have

$|x_m - x_n| < \varepsilon$

Therefore x_n is Cauchy and therefore by Cauchy Theorem, x_n converges.

So from the examples above, we see Cauchy Sequence is useful in proving the sequence converges.

Further Comment about Cauchy Sequence:

In real number space, if a sequence is Cauchy, then the sequence converges. However in some other space H, it may happen that a Cauchy sequence does not converges to any point in that space H.

For consider space $\mathbf{Q}$ which is the set of rational number, let us consider our "old friends" $x_n = \frac{[10^n\sqrt{2}]}{10^n}$, one can show that x_n is a Cauchy but it turns out the limit $\sqrt{2}$ is not in $\mathbf{Q}$. (In that case, we say $\mathbf{Q}$ is not complete).

So the concept of Cauchy sequence and convergence of sequence is quite different even though in real number, it looks the same. (It is just because $\mathbf{R}$ is a complete space.)

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