MATH2033 HW3 Name: Leung Ko Tsun SID:20516287

1. Given $\lim_{n\to\infty} x_n = 0$,

$\Rightarrow \forall \varepsilon > 0, \exists K \in \mathbb{N}$, s.t. $n \geq K \Rightarrow |x_n| < \varepsilon$

For $\varepsilon = \frac{1}{2}$, $\exists K_{1/2}$ s.t. $n \geq K_{1/2} \Rightarrow |x_n| < 1/2$

$x_n \in (-\frac{1}{2}, \frac{1}{2})$

$1 + x_n \in (1/2, 3/2)$

$\frac{1}{1+x_n} \in (2/3, 2)$

For $\varepsilon > 0$, $\exists K_{\varepsilon/2}$ s.t. $n \geq K_{\varepsilon/2} \Rightarrow |x_n| < \varepsilon/2$

For $\varepsilon > 0$, $\exists K = \max\{K_{1/2}, K_{\varepsilon/2}\}$ s.t.

$n \geq K \Rightarrow \left|\frac{x_n}{1+x_n}\right| \leq \frac{\varepsilon}{2} \cdot 2 = \varepsilon$

$\therefore \lim_{n\to\infty} \frac{x_n}{1+x_n} = 0$

2. Given $a_1 = 9$, $a_{n+1} = \frac{\sqrt{a_n} + 2a_n}{3}$

Now we will prove that $1 < a_{n+1} < a_n$. $\forall n \in N$

Base Case: when $n=1$,

$$a_2 = \frac{\sqrt{9} + 2\times 9}{3} = \frac{3+18}{3} = 7$$

$$a_1 = 9$$

$\therefore \quad 1 < a_2 < a_1$.

Assume $n=k$ is true $\Rightarrow$ $1 < a_{k+1} < a_k$

$$a_{k+2} = \frac{\sqrt{a_{k+1}} + 2a_{k+1}}{3}$$

$$a_{k+2} < \frac{a_{k+1} + 2a_{k+1}}{3} \quad \left(\because \sqrt{a_{k+1}} < a_{k+1} \quad \forall a_{k+1} > 0\right)$$

$\Rightarrow \quad a_{k+2} < a_{k+1}$

and $a_{k+2} = \frac{\sqrt{a_{k+1}} + 2a_{k+1}}{3}$

$$a_{k+2} > \frac{\sqrt{1} + 2\times 1}{3} \quad (\because a_{k+1} > 1)$$

$\Rightarrow \quad a_{k+2} > a_1$

$\Rightarrow 1 < a_{k+2} < a_{k+1}$

Hence $n=k+1$ is also true.

By the principle of mathematical induction,

$$1 < a_{n+1} < a_n \quad \forall n \in \mathbb{N}.$$

Hence, By the monotone sequence theorem,

$\lim_{n\to\infty} a_n = L$ exists.

$$\Rightarrow \lim_{n\to\infty} a_{n+1} = L = \lim_{n\to\infty} \frac{\sqrt{a_n} + 2a_n}{3} = \frac{\sqrt{L}+2L}{3}$$

$$\Rightarrow L = \sqrt{L}$$

$$\Rightarrow \sqrt{L}(\sqrt{L}-1) = 0$$

$$\Rightarrow \sqrt{L} = 0 \text{ or } \sqrt{L} = 1$$

$$\Rightarrow L = 0 \text{ or } L = 1$$

Since we showed that $a_n > 1 \quad \forall n \in \mathbb{N}$,

Hence $L = \lim_{n\to\infty} a_n = 1$.

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3. For $m, n \in \mathbb{N}$, where $m \geq n$.

$$|w_m - w_n| = |w_m - w_{m-1} + w_{m-1} - w_{m-2}$$
$$+ w_{m-2} - \cdots - w_{n+1} + w_{n+1} - w_n|$$

$$|w_m - w_n| \leq |w_m - w_{m-1}| + |w_{m-1} - w_{m-2}| + \cdots$$
$$+ |w_{n+1} - w_n| \quad \text{(Triangle ineq.)}$$

$$< \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^n}$$

$$= \frac{1}{2^n}\left[\frac{1}{2^{m-n-1}} + \frac{1}{2^{m-n-2}} + \cdots + \frac{1}{2} + 1\right]$$

$$\leq \frac{1}{2^n}\left[\sum_{i=0}^{\infty} \frac{1}{2^i}\right]$$

$$= \frac{1}{2^n}(2) = \frac{1}{2^{n-1}}$$

$\forall \varepsilon > 0, \exists K \in \mathbb{N}$ s.t. $K > 1 + \log_2\left(\frac{1}{\varepsilon}\right)$, $m \geq n > K$

s.t. $|w_m - w_n| < \frac{1}{2^{n-1}} < \varepsilon$

$\Rightarrow w_n$ is a Cauchy sequence.

4. $\forall t \in \mathbb{R}$, we can construct a strictly increasing sequence which converge to the given t by:

1. Choose $t_1 \in (\mathbb{R} \setminus \mathbb{Q}) \cap (t-1, t)$

$\forall n \in \mathbb{N}$, we choose $t_{n+1} \in (\mathbb{R} \setminus \mathbb{Q}) \cap (t_n, t) \cap (t - \frac{1}{n}, t)$

($\because$ irrational number is dense in $\mathbb{R}$)

So, this sequence can be built recursively.

$\forall n \in \mathbb{N}$, we have $t_n < t_{n+1} < t$, —①

and $t - t_n < \frac{1}{n}$ —②

From ①, we can see that it is a strictly increasing sequence.

From ②, $\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$,

s.t. $K > \frac{1}{\varepsilon}$, then $n > K \Rightarrow |t - t_n| = \frac{1}{n} < \varepsilon$

Hence t_n converges to t. $|t - t_n| = \frac{1}{n} < \varepsilon$.

$\Rightarrow$ there is a strictly increasing sequence of irrational numbers $t_1, t_2, t_3, \ldots$ converging to t.