Math 2033 (Mathematical Analysis) **Fall Spring 2014**

Homework # 1 – Due Tuesday, March 11, 2014 at 3:00pm

Be sure to write your name (as shown on your student ID card) and your tutorial session number on the homework! Show work. Answers are worth very little. **Make a copy of your homework and submit the original.**

1. (a) Prove that if A, B, C are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   (b) Prove that if X, Y, Z are sets, then $X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z)$.

   Remarks. The equation in (a) is called the *distributive law of sets* and the equation in (b) is called *De Morgan's law of sets*. There are two other similar laws $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $X \setminus (Y \cap Z) = (X \setminus Y) \cup (X \setminus Z)$, which you don't have to prove.

2. (a) If S is an infinite set, prove by mathematical induction that S contains a countably infinite subset.

   (b) Show there is a bijection from $[0, 1]$ to $(0, 1]$. (*Hint*: Consider omiting $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots$ from the domain and codomain first.)

   (c) Prove that every infinite set S contains a proper subset S' such that there is a bijection $g : S \to S'$.

3. Determine if the set A of all intersection points in $\mathbb{R}^2$ of the family of lines $\{y = mx : m \in \mathbb{Z}\}$ with the family of circles $\{x^2 + y^2 = r^2 : r \in \mathbb{Q}\}$ is countable or uncountable. Here A is the set of all points in $\mathbb{R}^2$ that are in at least one of the lines $y = mx$ ($m \in \mathbb{Z}$) and at least one of the circles $x^2 + y^2 = r^2$ ($r \in \mathbb{Q}$).

4. Prove that there exist infinitely many positive real numbers r such that the equation $2^x + 3^y + 5^z = r$ has no solution $(x, y, z) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$.
   (*Hint*: Is the set $S = \{2^x + 3^y + 5^z : (x, y, z) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}\}$ countable?)

5. Let T be a nonempty subset of the interval $(0, 1)$. If every finite subset $\{x_1, x_2, \ldots, x_n\}$ of T (with no two of $x_1, x_2, \ldots, x_n$ equal) has the property that $x_1^2 + x_2^2 + \cdots + x_n^2 < 1$, then prove that T is a countable set.

   (*Hint:* For every $k \in \mathbb{N}$, how many elements of T can be in $\left[\frac{1}{k+1}, \frac{1}{k}\right)$? Do you use the Archimedean principle anywhere?)