MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)
Tutorial Note #20

More Examples on Continuity
(It is the continuation of Extra Tutorial Note "Midterm Review")

Question about Continuous Injective Theorem
What condition(s) will the function become injective?

Of course, there are many possible cases, to find out the answer, one may try to look back the practice exercises. There are several possible conditions which can imply the function is injective.

$$|f(a) - f(b)| \geq |a - b|$$

(Proof: Suppose $f(x_1) = f(x_2) \to f(x_1) - f(x_2) = 0 \to 0 \geq |x_1 - x_2| \to x_1 = x_2$

$f\big(f(x)\big) = g(x)$ where $g(x)$ is any injective function
(Example: $f\big(f(x)\big) = x$, $f\big(f(x)\big) = -x^9$ or $f\big(f(x)\big) = e^x$)
(Proof: Suppose $f(x_1) = f(x_2) \to f\big(f(x_1)\big) = f(f(x_2) \to g(x_1) = g(x_2) \to x_1 = x_2$ since $g(x)$ is injective)

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| Example 1 |
| Suppose $f: [0,1] \to \mathbf{R}$ is a continuous function such that $f\Big(f\big(f(x)\big)\Big) = e^x$ and $f(0) < 0$ and $f(1) > 0$, show that there exist an unique solution for $f(x) = 0$ in $[0,1]$. |

Solution:
(Existence of solution)
Since $f(0) < 0$ and $f(1) > 0$, then by intermediate value function, there is $x \in (0,1)$ such that $f(x) = 0$

(Uniqueness of solution)
We prove by contradiction, suppose there is $x_1 \neq x_2$ (say $x_1 < x_2$) such that $f(x_1) = f(x_2) = 0$.

Given a condition $f\Big(f\big(f(x)\big)\Big) = e^x$, we will claim $f(x)$ is injective,

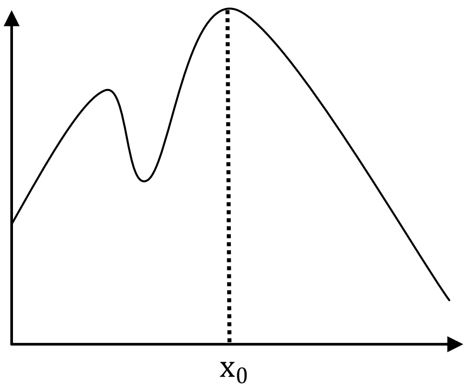
$$f(a) = f(b) \to f\big(f(a)\big) = f\big(f(b)\big) \to f\Big(f\big(f(a)\big)\Big) = f\Big(f\big(f(b)\big)\Big) \to e^a = e^b \to a = b$$

Hence $f(x)$ is injective and hence by continuous injection theorem, $f(x)$ is increasing or decreasing. Since $f(1) > 0 > f(0)$. So $f(x)$ is increasing, then $x_2 > x_1 \to f(x_2) > f(x_1) = 0$. Contradict to $f(x_2) = 0$.

One more example in continuity

Example 2 (Practice Exercise #61)
For a function $f: R \to R$, we say f has a local (or relative) maximum at $x_0$ if there exists an open interval $(a, b)$ containing $x_0$ such that $f(x) \le f(x_0)$ and we say f has a local (or relative) minimum at $x_0$ if there exists an open interval $(c, d)$ containing $x_0$ such that $f(x) \ge f(x_0)$.
If $f: \mathbf{R} \to \mathbf{R}$ is continuous and has a local maximum or local minimum at every real number. Show that $f(x)$ must be a constant function.

Solution:
Note that for every $x_0 \in \mathbf{R}$, since it is either local maximum or minimum. Let say it is local maximum, then there exists an interval $(a, b)$ containing $x_0$ such that $f(x) \le f(x_0)$. Then at each interval $(a, x_0)$ and $(x_0, b)$, we pick a rational number $r_1$ and $r_2$ such that $a < r_1 < x_0$ and $x_0 < r_2 < b$, then $f(x_0)$ is still a local maximum in $(r_1, r_2)$




Then each $x_0$ (local maximum) can be represented by an open interval with rational endpoint $(r_1, r_2)$. Then the image $f(\mathbf{R})$

$$f(\mathbf{R}) = \{f(x): x \in \mathbf{R}\} = \{f(x): \text{x is local maximum or local minimum}\}$$

$$\subseteq \left\{\max_{r_1<x<r_2} f(x) : r_1, r_2 \in \mathbf{Q}\right\} \cup \left\{\min_{r_3<x<r_4} f(x) : r_3, r_4 \in \mathbf{Q}\right\}$$

$$= \left[\cup_{r_1 \in \mathbf{Q}} \cup_{r_2 \in \mathbf{Q}} \left\{\max_{r_1<x<r_2} f(x)\right\}\right] \cup \left[\cup_{r_3 \in \mathbf{Q}} \cup_{r_4 \in \mathbf{Q}} \left\{\min_{r_3<x<r_4} f(x)\right\}\right]$$

R. H. S. is countable by countable union theorem$\to$ $f(\mathbf{R})$ is countable.
By Exercise 6 of Tutorial Note #15. $f(x)$ is constant.

Appendix: Proof of Theorem

Prove that a monotone function only have countably many discontinuous points

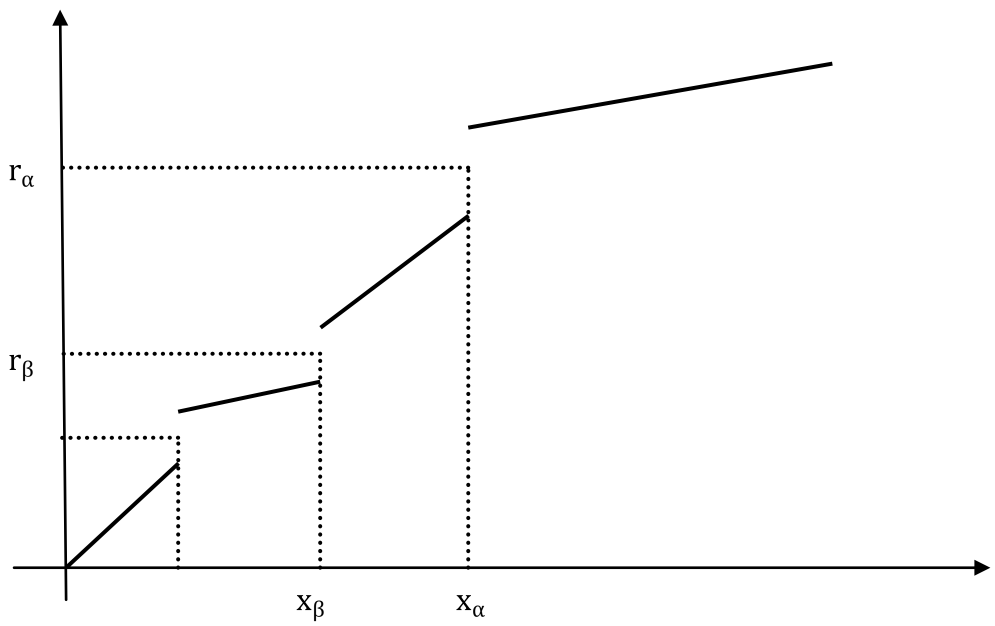
Solution:

We consider the case when the function is increasing. Let $x_\alpha$ be a discontinuous points, then since the function is increasing, hence we have

$$\lim_{x \to x_\alpha^-} f(x) < \lim_{x \to x_\alpha^+} f(x)$$

By density of rational number, we can pick a rational number $r_\alpha$ such that

$$\lim_{x \to x_\alpha^-} f(x) < r < \lim_{x \to x_\alpha^+} f(x)$$




Continuing this process, we obtain a set of rational number $\{r_\alpha, r_\beta, \dots.\}$, each $r_\alpha$ corresponding to a discontinuous points $x_\alpha$. Hence $\{x_\alpha, x_\beta, \dots\}$ and $\{r_\alpha, r_\beta, \dots\}$ should have "same number of elements" (one to one correspondent). Since $\{r_\alpha, r_\beta, \dots\} \subseteq \mathbf{Q}$ and **Q** is countable, then $\{r_\alpha, r_\beta, \dots\}$ is countable and hence $\{x_\alpha, x_\beta, \dots\}$ is also countable.