MATH2033 Mathematical Analysis (2021 Spring)
Assignment 3

Submission deadline of Assignment 3: **11:59p.m. of 26th Mar, 2020 (Fri)**

Instruction: Please complete all required problems. Full details (including description of methods used and explanation, key formula and theorem used and final answer) must be shown **clearly** to receive full credits. Marks can be deducted for incomplete solution or unclear solution.

Please submit your completed work via the submission system in canvas before the deadline. Late assignment will not be accepted.

Your submission must (1) be hand-written (typed assignment will not be accepted), (2) in a single pdf. file (other file formats will not be accepted) and (3) contain your full name and student ID on the first page of the assignment.

Problem 1

Prove the following statement using the definition of limits:

(a) $\lim_{x\to c}\frac{1}{x}=\frac{1}{c} \quad for\ \ c\neq 0.$

(b) We let $f:(c,\infty)\to\mathbb{R}$ be a function with $f(x)>0$ for all $x\in(c,\infty)$. Show that $\lim_{x\to c} f(x)=\infty$ if and only if $\lim_{x\to c}\frac{1}{f(x)}=0$.

(c) We let $f:(0,\infty)\to\mathbb{R}$ be a function. Show that $\lim_{x\to\infty} f(x)=L$ if and only if $\lim_{x\to 0^+} f\left(\frac{1}{x}\right)=L$.

Problem 2

We consider a function $f:\mathbb{R}\to\mathbb{R}$ defined as

$$f(x)=\begin{cases} x & if\ \ x\in\mathbb{Q} \\ 0 & if\ \ x\in\mathbb{R}\backslash\mathbb{Q} \end{cases}.$$

(a) Show that $\lim_{x\to 0} f(x)=0$. Is $f(x)$ continuous at $x=0$?

(b) For any $c\neq 0$, show that $\lim_{x\to c} f(x)$ does not exist.

Problem 3

(a) We let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function and define a set

$$S=\{x\in\mathbb{R}|f(x^2)\geq f(x)\}.$$

Suppose that there exists a sequence $\{y_n\}$ which $y_n\in S$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty} y_n=y$, show that $y\in S$.

(b) We let $f,g:\mathbb{R}\to\mathbb{R}$ be two continuous functions on $\mathbb{R}$ such that $f(r)=g(r)$ for all $r\in\mathbb{Q}$. Is it true that $f(x)=g(x)$ for all $x\in\mathbb{R}$?

Problem 4
We let f be a continuous function over $[0,2]$ (i.e. $f \in C([0,2])$) with $f(0) = f(2) = 0$.
Show that there exists $c \in [0,1]$ such that $f(c) = f(c+1)$.
(☺Hint: Do the analysis by considering the *sign* of $f(1)$).

Problem 5
We let $f: [a, b] \to \mathbb{R}$ be a continuous function such that for any $x \in [a, b]$, there exists $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Show that there exists $c \in [a, b]$ such that $f(c) = 0$.

Problem 6 (Harder)
We let $f: [a, b] \to \mathbb{R}$ be a continuous function over $[a, b]$. We define two functions $M(x)$ and $m(x)$ as

$$M(x) = \sup\{f(t) | t \in [a, x]\}$$
$$m(x) = \inf\{f(t) | t \in [a, x]\}$$

Show that both $M(x)$ and $m(x)$ are continuous at any $x_0 \in [a, b]$.
(☺Hint: Note that both functions $M(x)$ and $m(x)$ are monotone functions.)