Lecture 19

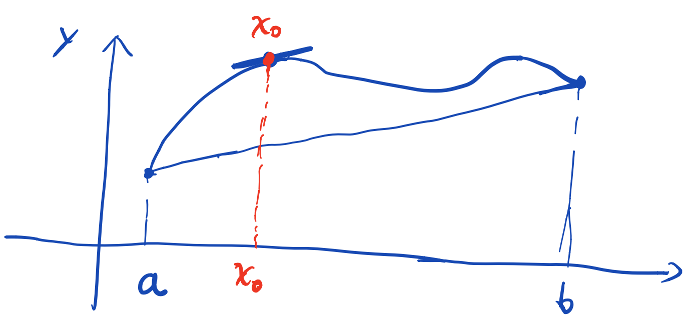
11-04-2019

Review :

1. Mean value thm : $f \in C[a,b]$ and differentiable on (a,b)

$\Rightarrow$ $f(b) - f(a) = f'(x_0)(b-a)$ for some $x_0 \in (a,b)$

or $\dfrac{f(b)-f(a)}{b-a} = f'(x_0)$



Proof : Rolle's theorem.

2. Generalized mean-value thm : $f, g \in C[a,b]$ and differentiable on (a, b) $\Rightarrow$

$$g'(x_0)\big(f(b)-f(a)\big) = f'(x_0)\big(g(b)-g(a)\big)$$

③ Local extremum thm $\Rightarrow$ Rolle's thm

$\Leftrightarrow$ $\boxed{\text{mean value thm}}$ $\Leftrightarrow$ generalize mean value thm

④ Curve Tracing thm :

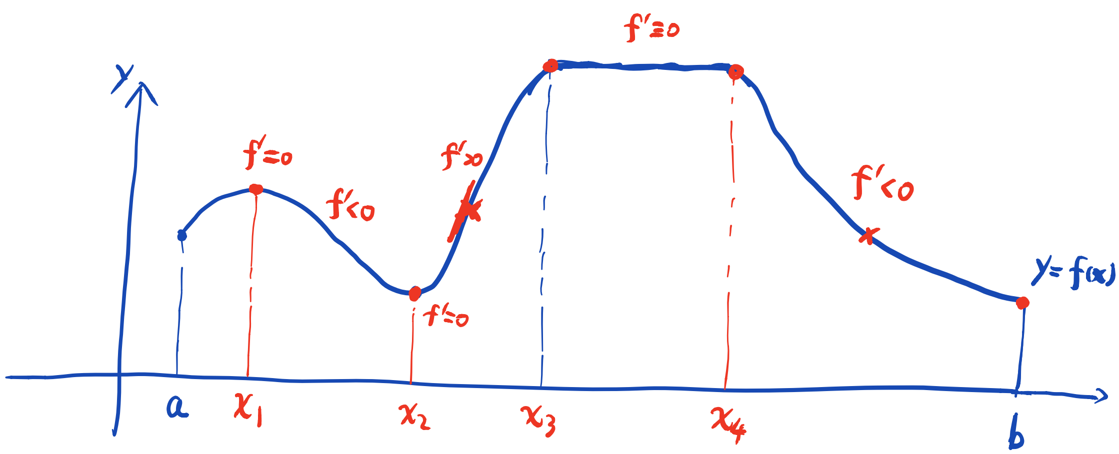
$f' > 0 \Rightarrow f$ strictly $\nearrow$ [proof : mean-value thm]

$f' \geq 0 \Rightarrow f \nearrow$

$f' \equiv 0 \Rightarrow f =$ constant

Converse, f strictly $\nearrow \Rightarrow f' \geq 0$. [proof : definition of $f'(x)$]

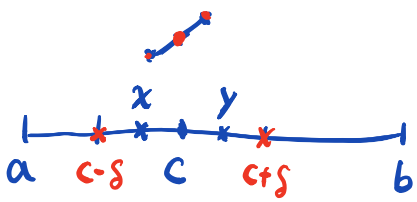
$f \nearrow \Rightarrow f' \geq 0$



Counter-example : $f(x) = x^3$. f strictly $\nearrow$, but $f'(0) = 0$.

Local tracing theorem

THM: Let $f \in C[a,b]$, assume that $f'(c) > 0$ for some $c \in (a,b)$, then $\exists$ δ-neighborhood of c $(c-\delta, c+\delta)$ s.t $f(x) < f(c) < f(y)$ for all $x, y \in [a,b]$ with $c-\delta < x < c < y < c+\delta$



A similar result holds for $f'(c) < 0$ and the inequality becomes

$$f(x) > f(c) > f(y).$$

proof: $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0$

$\Rightarrow$ for $\varepsilon = \frac{f'(c)}{2} > 0$, $\exists\ \delta > 0$, s.t $0 < |x-c| < \delta$, $x \in [a,b]$

$\Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon = \frac{f'(c)}{2}$.

$\Rightarrow \frac{f(x) - f(c)}{x - c} - f'(c) > -\varepsilon = -\frac{f'(c)}{2}$

$|A-B| < \varepsilon$

$-\varepsilon < A-B < \varepsilon$

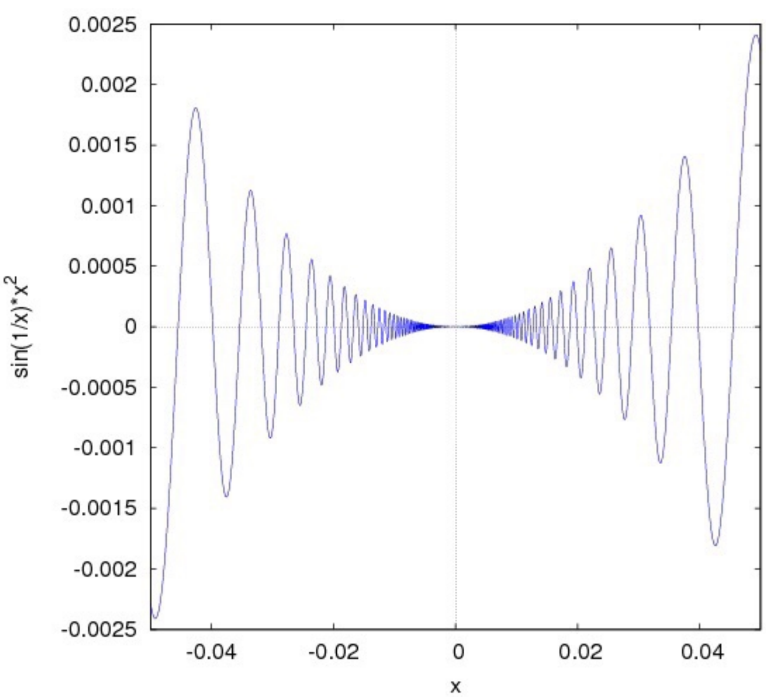
$\Rightarrow A-B > -\varepsilon$

$$\Rightarrow \frac{f(x)-f(c)}{x-c} > \frac{f'(c)}{2} > 0 \quad \Rightarrow \begin{cases} f(x) > f(c) & \text{for} \quad c < x < c+\delta \\ f(x) < f(c) & \text{for} \quad c-\delta < x < c \end{cases}$$

Remark: The conclusion is not true for $f'(c)=0$.

Counterexample: $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x=0 \end{cases}$. $f'(0)=0$.

$f'(0)=0$ but 0 is not an local extreme point of f.



$\frac{0}{0}$ form of L'Hôpital's Rule

Thm: Let f, g be differentiable on an interval (a, b)

① $g(t), g'(t) \neq 0$, $\forall\, t \in (a,b)$ ② $\lim_{t \to a^+} f(t) = 0 = \lim_{t \to a^+} g(t)$

③ $\lim_{t \to a^+} \frac{f'(t)}{g'(t)} = L$. where $L \in \mathbb{R}$, or $L = +\infty$ or $L = -\infty$

Then $\lim_{t \to a^+} \frac{f(t)}{g(t)} = \lim_{t \to a^+} \frac{f'(t)}{g'(t)} = L$.

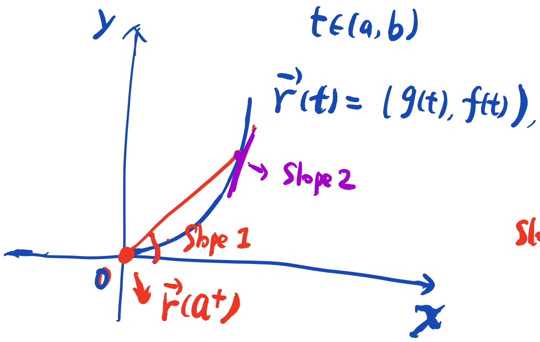
The above theorem also holds if we replace

$t \to a^+$ by $t \to b^-$

or $t \to c$ for some $a < c < b$

or $t \to +\infty$, or $t \to -\infty$

Geometric interpretation :



Velocity vector at $\vec{r}(t)$: $\vec{r}'(t) = (g'(t), f'(t))$

$$\lim_{t\to a^+} \vec{r}(t) = \lim_{t\to a^+} (g(t), f(t)) = 0$$

Slope 1 of secant line connecting $(0,0)$ and $\vec{r}(t)$

is : $\dfrac{f(t)-0}{g(t)-0} = \dfrac{f(t)}{g(t)}$

Slope 2 of the tangent line at $\vec{r}(t)$ is : $\dfrac{f'(t)}{g'(t)}$.

As $t \to a^+$, Slope 1 $\to$ Slope 2

Proof: We only consider the case $L \in \mathbb{R}$, and $t \to a^+$.

Define $f(a) = 0$, $g(a)$. $\forall\, t \in (a,b)$, the function f, g are continuous on $[a, t]$ and are differentiable on (a, t). By the generalized mean-value,

$$\frac{f(t)}{g(t)} = \frac{f(t)-f(a)}{g(t)-g(a)} = \frac{f'(t_0)}{g'(t_0)} \quad \text{for some } t_0 \in (a, t)$$

let $t \to a^+$, since $a < t_0 < t$, $t_0 \to a^+$

$$\Rightarrow \lim_{t\to a^+} \frac{f(t)}{g(t)} = \lim_{t\to a^+} \frac{f'(t)}{g'(t)} = L$$

Example 1. $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution: $f(x) = \sin x$, $g(x) = x$

$\lim_{x\to 0} f(x) = f(0) = 0$, $\lim_{x\to 0} g(x) = g(0) = 0$

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{\cos x}{1} = \lim_{x\to 0} \cos x = \cos 0 = 1$$

By L-H's rule, $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 1$

Example 2. $\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}}$

$$\lim_{x\to 0^+} \frac{(\sin x)'}{(\sqrt{x})'} = \lim_{x\to 0^+} \frac{\cos x}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} 2\sqrt{x}\cos x = 0$$

By L-H's rule $\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+} \frac{(\sin x)'}{(\sqrt{x})'} = 0$

Example 3: $\lim_{x \to 0} \frac{\sin x - x}{x^2}$ (Exercise)

Let $f(x) = \sin x - x$, $g(x) = x^2$

then $\lim_{x \to 0} f(x) = f(0) = 0$, $\lim_{x \to 0} g(x) = g(0) = 0$

$$\Rightarrow \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\cos x - 1}{2x}.$$

Since $\lim_{x \to 0} (\cos x - 1) = \cos 0 - 1 = 0$, $\lim_{x \to 0} 2x = 0$

Apply L-H rule again, we have

$$\lim_{x \to 0} \frac{\cos x - 1}{2x} = \lim_{x \to 0} \frac{(\cos x - 1)'}{(2x)'} = \lim_{x \to 0} \frac{-\sin x}{2}$$

$$= 0$$

Example 4 $\quad \lim_{x\to 0} \dfrac{x^2 \sin\frac{1}{x}}{\sin x}$

Solution: $\lim_{x\to 0} \dfrac{x^2 \sin\frac{1}{x}}{\sin x} = \lim_{x\to 0} x \sin\frac{1}{x} \cdot \dfrac{1}{\frac{\sin x}{x}}$

$\left[\lim_{x\to 0} x \sin\frac{1}{x} = 0 \qquad \lim_{x\to 0} \dfrac{1}{\frac{\sin x}{x}} = \dfrac{1}{\lim_{x\to 0} \frac{\sin x}{x}} = \dfrac{1}{1} = 1 \right]$

$= 0 \cdot 1 = 0$

But $\lim_{x\to 0} \dfrac{(x^2 \sin\frac{1}{x})'}{(\sin x)'} = \lim_{x\to 0} \dfrac{2x \sin\frac{1}{x} - \cos\frac{1}{x}}{\cos x} = \lim_{x\to 0} (\underline{2x \sin\frac{1}{x}} - \cos\frac{1}{x})$

$= 0 - \lim_{x\to 0} \cos\frac{1}{x}$ Limit of $\cos\frac{1}{x}$ Does not exists

$\Rightarrow \lim_{x\to 0} \dfrac{(x^2 \sin\frac{1}{x})'}{(\sin x)'}$ Does not exists

RK. Example 4 Shows the limitation of L-H rule.

$\frac{*}{\infty}$ form of L'Hôpital's Rule

THM: let f, g be differentiable on (a,b). If ① $g(t), g'(t) \neq 0 \ \forall t \in (a,b)$

② $\lim_{t \to a^+} g(t) = \infty$, ③ $\lim_{t \to a^+} \frac{f'(t)}{g'(t)} = L$, where $L \in \mathbb{R}$ or $L = +\infty$, or $L = -\infty$

[No condition on f]

Then $\lim_{t \to a^+} \frac{f(t)}{g(t)} = \lim_{t \to a^+} \frac{f'(t)}{g'(t)} = L$

The theorem also holds if we replace $t \to a^+$ by $t \to b^-$

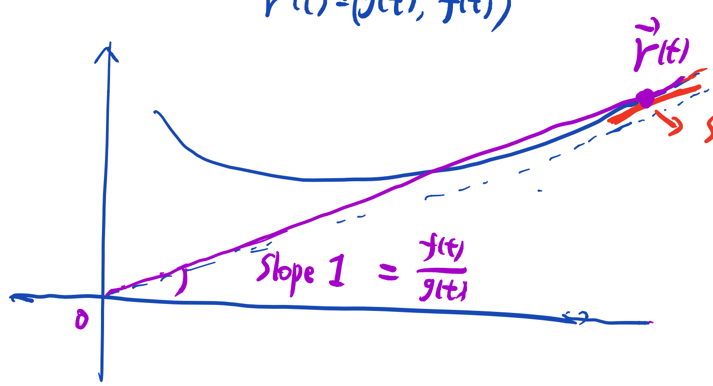
or $t \to c$ for $a < c < b$, and (a,b) being replaced

by $(a,b) \setminus \{c\}$.

or $t \to +\infty$, or $t \to -\infty$.

$\boxed{y = x + \frac{1}{x}}$

$\vec{r}(t) = (g(t), f(t))$



as $t \to a^+$, $\vec{r}(t) \to \infty$

Slope 2 $\to L$

$\Rightarrow$ Slope 1 $\to L$

Proof: We consider the case $L \in \mathbb{R}$ and $x \to a^+$.

Step 1. $\forall \varepsilon > 0$, since $\lim_{t \to a^+} \frac{f'(t)}{g'(t)} = L$, $\exists\ \delta_0 > 0$, s.t

$$\left| \frac{f'(t)}{g'(t)} - L \right| < \frac{\varepsilon}{2} \quad \text{for all} \quad a < t < a + \delta_0$$

Step 2. Let $t_0 = a + \frac{\delta_0}{2}$, $\forall\ t \in (a, t_0) = (a, a + \frac{\delta_0}{2})$, by the G-M-V-T

$\exists\ \xi \in (a, t_0)$ s.t

$$\frac{f(t) - f(t_0)}{g(t) - g(t_0)} = \frac{f'(\xi)}{g'(\xi)} \Rightarrow \frac{f(t) - f(t_0)}{g(t)} = \frac{g(t) - g(t_0)}{g(t)} \cdot \frac{f'(\xi)}{g'(\xi)}$$

$$\Rightarrow \frac{f(t)}{g(t)} - \frac{f(t_0)}{g(t)} = \frac{f'(\xi)}{g'(\xi)} - \frac{g(t_0)}{g(t)} \cdot \frac{f'(\xi)}{g'(\xi)}$$

$$\Rightarrow \frac{f(t)}{g(t)} = \frac{f'(\xi)}{g'(\xi)} + \frac{f(t_0)}{g(t)} - \frac{g(t_0)}{g(t)} \frac{f'(\xi)}{g'(\xi)} \quad \cdots (\bigstar)$$

Step 3. Since $\lim_{t \to a^+} g(t) = \infty$, $\Rightarrow \lim_{t \to a^+} \frac{f(t_0)}{g(t)} = 0$, $\lim_{t \to a^+} \frac{g(t_0)}{g(t)} = 0$

Since $\left| \frac{f'(\xi)}{g'(\xi)} - L \right| < \frac{\varepsilon_0}{2}$, $\frac{f'(\xi)}{g'(\xi)}$ is bounded for $\xi \in (a, a + \frac{\delta_0}{2})$

$$\Rightarrow \lim_{x\to a^+} \frac{f(t_0)}{g(t)} - \frac{g(t_0)}{g(t)} \cdot \frac{f'(\xi)}{g'(\xi)} = 0$$

$\Rightarrow \exists\ \delta_1 > 0$, s.t $\left| \frac{f(t_0)}{g(t)} - \frac{g(t_0)}{g(t)} \frac{f'(\xi)}{g'(\xi)} \right| < \frac{\varepsilon}{2}$ for

all $a < t < a + \delta_1$

Step 4. Take $\delta = \min\{\delta_1, \frac{\delta_0}{2}\}$.

for $a < t < \delta \Rightarrow a < t < x_0 = a + \frac{\delta_0}{2}$ and $a < t < a + \delta_1$

$$\Rightarrow \left| \frac{f'(\xi)}{g'(\xi)} - L \right| < \frac{\varepsilon}{2}, \quad \left| \frac{f(t_0)}{g(t)} - \frac{g(t_0)}{g(t)} \frac{f'(\xi)}{g'(\xi)} \right| < \frac{\varepsilon}{2}$$

$$(\star) \Rightarrow \left| \frac{f(t)}{g(t)} - L \right| \leq \left| \frac{f'(\xi)}{g'(\xi)} - L \right| + \left| \frac{f(t_0)}{g(t)} - \frac{g(t_0)}{g(t)} \frac{f'(\xi)}{g'(\xi)} \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \lim_{t\to a^+} \frac{f(t)}{g(t)} = L$$