MATH2033 Mathematical Analysis
Problem Set 6

Limits of function

Problem 1

Prove the following limits using the definition of limits (ε-δ definition)

(a) $\lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}$.

(b) $\lim_{x\to c} x^3 = c^3$, where $c \in \mathbb{R}$.

(c) $\lim_{x\to 0} x\sin\frac{1}{x} = 0$ and $\lim_{x\to \frac{\pi}{2}} x\cos x = 0$.

Problem 2

Prove the following limits using the definition of limits

(a) $\lim_{x\to \infty} \cos\frac{1}{x} = 1$ (😊Hint: Recall that $\frac{1}{x} \to 0$ when $x \to \infty$, so $\frac{1}{x} < \frac{\pi}{2}$ when x is large).

(b) $\lim_{x\to -\infty} e^x = 0$

(c) $\lim_{x\to \infty} e^x = \infty$

Problem 3

We let $[x]$ denotes the greatest integer less than or equal to x.

(a) We let c be an integer. Determine if the limits $\lim_{x\to c}[x]$ exists.

(😊Hint: Try an example when $c = 3$)

(b) We let d be a non-integer. Determine if the limits $\lim_{x\to d}[x]$ exists

Problem 4

We let $f\colon \mathbb{R} \to \mathbb{R}$ be a function which $\lim_{x\to 0} f(x) = L \in \mathbb{R}$. Let $a > 0$ be a positive number and define $g\colon \mathbb{R} \to \mathbb{R}$ as $g(x) = f(ax)$.

(a) Show that $\lim_{x\to 0} g(x) = L$ using the definition of limits.

(b) Redo **(a)** using the sequential limits theorem.

Problem 5

(a) We let $f\colon \mathbb{R} \to \mathbb{R}$ be a function which $\lim_{x\to x_0} f(x) = L$. Show that there exists $\delta > 0$ and $M > 0$ such that $|f(x)| < M$ for all $|x - x_0| < \delta$.

(😊Hint: You can consider the definition of limits)

(b) We let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be two functions which $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Using the definition of limits (ε-δ definition), prove that $\lim_{x\to a} f(x)g(x) = LM$.

(😊Hint: Write $f(x)g(x) - LM = f(x)g(x) - f(x)M + f(x)M - LM$. Also, the result in (a) is also useful.)

Problem 6

We let $f: \mathbb{R} \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} x & if \;\; x \in \mathbb{Q} \\ 0 & if \; otherwise \end{cases}.$$

(a) Show that $\lim_{x \to 0} f(x)$ exists.

(b) Show that $\lim_{x \to c} f(x)$ does not exist for any $c \neq 0$.

Continuity

Problem 7

We consider a function $f: (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{[x]}{x}$, where $[x]$ denotes the greatest integer less than or equal to x.

(a) Determine if $f(x)$ is continuous at $x = 1$.

(b) Determine if $f(x)$ is continuous at $x = 2.5$.

Problem 8

(a) We let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be two continuous functions on $\mathbb{R}$, show that the function $h(x) = \min\big(f(x), g(x)\big)$ is continuous on $\mathbb{R}$.

(b) We let $f_1, f_2, \dots, f_n: \mathbb{R} \to \mathbb{R}$ be n continuous functions on $\mathbb{R}$. Using the result of (a), show that $p(x) = \min\big(f_1(x), f_2(x), \dots, f_n(x)\big)$ is continuous on $\mathbb{R}$.

(😊Hint: You can try mathematical induction)

Problem 8

We consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x - x^3 & if \;\; x \in \mathbb{Q} \\ 0 & if \; otherwise \end{cases}.$$

(a) Show that the function is continuous at $x = 0$ and $x = \pm 1$.

(b) Show that the function is not continuous at point $x = x_0$ where $x_0 \neq 0, -1, 1$.

Problem 9

Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ which f is discontinuous at any point on $\mathbb{R}$ but $|f|$ is continuous on $\mathbb{R}$.

Problem 10

We let $f, g: \mathbb{R} \to \mathbb{R}$ be two continuous functions on $\mathbb{R}$ such that $f(r) = g(r)$ for all $r \in \mathbb{Q}$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Problem 11

(a) Show that the equation $x = \cos x$ has a solution in the interval $\left[0, \frac{\pi}{2}\right]$.

(b) Show that the equation $x^4 + 7x^3 - 9 = 0$ has at least two real solutions.

Problem 12

We let $L > 0$ be a positive number and let $f: [a, b] \to \mathbb{R}$ be a continuous function. Suppose that for any $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that

$$|f(x_n) - L| < \frac{1}{2^n}.$$

Show that there exists $x^* \in [a, b]$ such that $f(x^*) = L$.

Problem 13

We let $f: [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Suppose that there exists $c \in (a, b)$ such that $f(c) > f(x)$ for all $x \in [a, b]$, show that $f(x)$ is not injective.

(☺Hint: Draw a figure and get some idea)