⑧ (i) Solution 1 Since $\frac{d}{dx}(xe^{-x^2}) = e^{-x^2}(1-2x^2) < 0$ for $x \geq 1$ and $\lim_{x\to\infty} \frac{x}{e^{x^2}} = \lim_{x\to\infty}\frac{1}{2xe^{x^2}} = 0$, xe^{-x^2} decreases to 0 as $x\to\infty$. Now $\int_1^\infty xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}\Big|_0^\infty = 0-(-\frac{1}{2}e^{-1}) < \infty$. By the integral test, $\sum_{k=1}^{\infty} ke^{-k^2}$ converges.

Solution 2 (Ratio test) $\lim_{k\to\infty} \frac{(k+1)e^{-(k+1)^2}}{k e^{-k^2}} = \lim_{k\to\infty}\frac{k+1}{k}e^{-2k-1} = 0 < 1 \Rightarrow \sum_{k=1}^{\infty} ke^{-k^2}$ converges.

(j) Solution 1 (Ratio test) $\lim_{k\to\infty} \frac{\frac{k+1}{(k+2)!}}{\frac{k}{(k+1)!}} = \lim_{k\to\infty}\frac{k+1}{k}\frac{(k+1)!}{(k+2)!} = \lim_{k\to\infty}\frac{k+1}{k}\lim_{k\to\infty}\frac{1}{k+2} = 1\cdot 0 = 0 < 1$, Series converges.

Solution 2 $\sum_{k=1}^{\infty}\frac{k}{(k+1)!} = \sum_{k=1}^{\infty}\frac{(k+1)-1}{(k+1)!} = \underbrace{\sum_{k=1}^{\infty}\left(\frac{1}{k!}-\frac{1}{(k+1)!}\right)}_{\text{telescoping series}} = \lim_{n\to\infty}\left(\frac{1}{1!}-\frac{1}{(n+1)!}\right) = 1.$

(k) Solution 1 Since $0 \leq \frac{\operatorname{Arctan} k}{k^2+1} \leq \frac{\frac{\pi}{2}}{k^2}$ and $\sum_{k=1}^{\infty}\frac{\frac{\pi}{2}}{k^2} = \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by p-test, so $\sum_{k=1}^{\infty}\frac{\operatorname{Arctan} k}{k^2+1}$ converges by the comparison test.

Solution 2 $0 \leq \frac{\operatorname{Arctan} k}{k^2+1} \leq \frac{\frac{\pi}{2}}{k^2+1} \xRightarrow{\text{sandwich thm}} \lim_{k\to\infty}\frac{\operatorname{Arctan} k}{k^2+1} = 0$. Now $\frac{d}{dx}\left(\frac{\operatorname{Arctan} x}{x^2+1}\right) = \frac{1-2x\operatorname{Arctan} x}{(x^2+1)^2}$

Since x, $\operatorname{Arctan} x$ are increasing, $1-2x\operatorname{Arctan} x \leq 1-2\operatorname{Arctan} 1 < 0$ for $x \geq 1$.
So $\frac{\operatorname{Arctan} x}{x^2+1}$ decreases to 0 as $x\to\infty$. Now $\int_1^\infty \frac{\operatorname{Arctan} x}{x^2+1}dx = \frac{1}{2}(\operatorname{Arctan} x)^2\Big|_1^\infty = \frac{1}{2}(\frac{\pi}{2})^2 - \frac{1}{2}(\frac{\pi}{4})^2 < \infty$. By the integral test, $\sum_{k=1}^{\infty}\frac{\operatorname{Arctan} k}{k^2+1}$ converges.

(l) Since $\lim_{k\to\infty}\frac{\frac{1}{k^{1+1/k}}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{1}{k^{1/k}} = 1$ and $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges by p-test, so $\sum_{k=1}^{\infty}\frac{1}{k^{1+1/k}}$ diverges by the limit comparison test.

(m) By the root test, $\lim_{k\to\infty}\sqrt[k]{\left|\tan^k\left(\frac{k+1}{k}\right)\right|} = \lim_{k\to\infty}\left|\tan\left(\frac{k+1}{k}\right)\right| = \tan 1 > \tan\frac{\pi}{4} = 1$
$\Rightarrow$ Series diverges

(n) $\lim_{k\to\infty}\frac{1-\cos\frac{1}{k}}{\frac{1}{k^p}} \underset{\theta=\frac{1}{k}}{=} \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^p} = \lim_{\theta\to 0}\frac{\sin\theta}{p\theta^{p-1}} = \frac{1}{2}$ (if we set $p=2$). Since $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by p-test, so $\sum_{k=1}^{\infty}(1-\cos\frac{1}{k})$ converges by the limit comparison test.

(o) $\lim_{k\to\infty}\frac{k^2\sin^p(\frac{1}{k})}{k^2(\frac{1}{k})^p} = \lim_{k\to\infty}\left(\frac{\sin(\frac{1}{k})}{\frac{1}{k}}\right)^p \underset{\theta=\frac{1}{k}}{=} \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^p = 1$. Since $\sum_{k=1}^{\infty} k^2(\frac{1}{k})^p = \sum_{k=1}^{\infty}\frac{1}{k^{p-2}}$ converges iff $p-2>1$ by the p-test, so $\sum_{k=1}^{\infty}k^2\sin^p(\frac{1}{k})$ converges iff $p>3$.

(p) $\sqrt{k+1}-\sqrt{k} = (\sqrt{k+1}-\sqrt{k})\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}} = \frac{1}{\sqrt{k+1}+\sqrt{k}}$. As $k\nearrow\infty$, $\sqrt{k+1}+\sqrt{k}\nearrow\infty$, $\sqrt{k+1}-\sqrt{k} = \frac{1}{\sqrt{k+1}+\sqrt{k}}\searrow 0$.
So $\sum_{k=1}^{\infty}(-1)^{k+1}(\sqrt{k+1}-\sqrt{k})$ converges by the alternating series test.

(31) Let $s_n = a_1 + a_2 + \cdots + a_n$ and $t_k = 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$. By definition, $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$ and $\sum_{k=1}^{\infty} 2^k a_{2^k} = \lim_{k\to\infty} t_k$. Since $a_1 \ge a_2 \ge a_3 \ge \cdots \ge 0$, s_n and t_k's are increasing. Their limits are either numbers or $+\infty$. Now
$$S_{2^k-1} = a_1 + (a_2+a_3) + (a_4 + \cdots + a_7) + \cdots + (a_{2^{k-1}} + \cdots + a_{2^k-1})$$
$$\le a_1 + 2a_2 + 4a_4 + \cdots + 2^{k-1} a_{2^{k-1}} = a_1 + t_{k-1}$$

So if $\lim_{k\to\infty} t_k < \infty$, then $\lim_{n\to\infty} S_n \underset{\uparrow \text{ } S_n \text{ increasing}}{=} \lim_{k\to\infty} S_{2^k-1} \le a_1 + \lim_{k\to\infty} t_k < \infty$.

Conversely, $t_k = 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = 2(a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k}) \le 2(a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}))$
$$= 2(S_{2^k} - a_1).$$
So if $\lim_{k\to\infty} S_n = \lim_{k\to\infty} S_{2^k} < \infty$, then $\lim_{k\to\infty} t_k \le 2(\lim_{k\to\infty} S_{2^k} - a_1) < \infty$.

For the second part, $\sum_{k=3}^{\infty} 2^k \frac{1}{2^k \ln 2^k \ln(\ln 2^k)} = \sum_{k=3}^{\infty} \frac{1}{k \ln 2 (\ln k + \ln \ln 2)}$. We compare this with $\sum_{k=3}^{\infty} \frac{1}{k \ln k}$. Since $\lim_{k\to\infty} \dfrac{\frac{1}{k\ln 2(\ln k + \ln\ln 2)}}{\frac{1}{k\ln k}} = \lim_{k\to\infty} \dfrac{1}{\ln 2 + \frac{\ln \ln 2}{\ln k}} = \dfrac{1}{\ln 2}$ and $\sum_{k=3}^{\infty} \frac{1}{k \ln k}$ diverges by example of integral test, so $\sum_{k=3}^{\infty} 2^k \frac{1}{2^k \ln 2^k \ln(\ln 2^k)}$ diverges by the limit comparison test. By first part, $\sum_{k=3}^{\infty} \frac{1}{k \ln k \ln(\ln k)}$ diverges.

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(32) Since $\lim_{k\to\infty} \frac{k+1}{2^k} \Big/ \frac{k}{2^{k-1}} = \lim_{k\to\infty} \frac{k+1}{k} \frac{1}{2} = \frac{1}{2} < 1$, the series converges by the ratio test.

Now
$$S = \sum_{k=2}^{\infty} \frac{k}{2^{k-1}} = \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \cdots$$
$$\frac{1}{2} S = \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots$$
$$\frac{1}{2} S = S - \frac{1}{2} S = \frac{2}{2} + \left(\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\right) = \frac{2}{2} + \frac{\frac{1}{2^2}}{1 - \frac{1}{2}} = \frac{3}{2}$$

Therefore, $S = 3$.

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(33). Suppose $\sum_{k=1}^{\infty} \frac{1}{p_k} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ converges to s. Then $s_n = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p_n}$ has limit s as $n \to \infty$, i.e. $\lim_{n\to\infty} (s - s_n) = 0$. So for some n, $s - s_n = \frac{1}{p_{n+1}} + \frac{1}{p_{n+2}} + \cdots = \sum_{k=n+1}^{\infty} \frac{1}{p_k} < \frac{1}{2}$.

Let $Q = p_1 p_2 \cdots p_n$, then the numbers $1 + mQ$ cannot be divisible by $p_1, p_2, \ldots, p_n$.

So $1 + mQ = p_{n+1}^{e_{n+1}} p_{n+2}^{e_{n+2}} \cdots$, where the exponents e_k are nonnegative integers.

Let $j = e_{n+1} + e_{n+2} + \cdots$ (only finitely many $e_k \ne 0$), then
$$\frac{1}{1+mQ} = \frac{1}{p_{n+1}^{e_{n+1}}} \cdot \frac{1}{p_{n+2}^{e_{n+2}}} \cdots \text{ is a term in } \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^{e_{n+1}} \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^{e_{n+2}} \cdots = \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^j$$
So the numbers $\frac{1}{1+Q}, \frac{1}{1+2Q}, \ldots, \frac{1}{1+NQ}$ will correspond to N terms of $\sum_{j=1}^{\infty} \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^j$, which is less than $\sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = 1$. Then
$$\sum_{m=1}^{N} \frac{1}{1+mQ} \le \sum_{j=1}^{\infty} \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^j < \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = 1 \quad \text{for every positive integer } N.$$

Since $\frac{1}{2mQ} \le \frac{1}{1+mQ}$ and $\sum_{m=1}^{\infty} \frac{1}{2mQ} = \frac{1}{2Q} \sum_{m=1}^{\infty} \frac{1}{m}$ diverges to $+\infty$ by p-test, so $\sum_{m=1}^{\infty} \frac{1}{1+mQ}$ diverges to $+\infty$ by the comparison test. Therefore $\sum_{m=1}^{N} \frac{1}{1+mQ} < 1$ cannot hold for all positive integer N, a contradiction.

(34) (a) $A=\{\sqrt{1}+\sqrt{1}, \sqrt{2}+\sqrt{1}, \sqrt{1}+\sqrt{2}, \dots\}$ is not bounded above. However, A has 2 as a lower bound because $\sqrt{m}+\sqrt{n} \ge \sqrt{1}+\sqrt{1}=2$ for every $m,n\in\mathbb{N}$. In fact, $\inf A=2$ because 2 is a lower bound and every lower bound $b \le \sqrt{1}+\sqrt{1} \in A$.

(b) $B=(-\infty,\pi]\cup\{3, 3\frac{1}{2}, 3\frac{2}{3}, \dots\}$ is not bounded below. However, B has 4 as an upper bound because $\pi \le 4$ and $4-\frac{1}{n}\le 4$ for all $n\in\mathbb{N}$. (Note $4\notin B$) We will show $\sup B=4$. Assume there is an upper bound $t<4$. By the Archimedian principle, there is $n\in\mathbb{N}$ such that $n>\frac{1}{4-t}$. Then $4-\frac{1}{n}>t$ and $4-\frac{1}{n}\in B$, which contradicts t being an upper bound.

(c) For $n,m\in\mathbb{N}$, $0<\frac{1}{n}+\frac{1}{2^m}\le\frac{1}{1}+\frac{1}{2^1}=\frac{3}{2}$. So C has 0 as a lower bound and $\frac{3}{2}$ as an upper bound. In fact, $\sup C=\frac{3}{2}$ because $\frac{1}{1}+\frac{1}{2^1}=\frac{3}{2}\in C$ and every upper bound $M\ge\frac{1}{1}+\frac{1}{2^1}$. Also, we can show $\inf C=0$ as follow. Assume there is a lower bound $t>0$. By Archimedean principle, there is $k\in\mathbb{N}$ such that $k>\frac{1}{t}$. Then taking $m=n=2k$, we have $t>\frac{1}{k}=\underset{n}{\frac{1}{2k}}+\underset{m}{\frac{1}{2k}}\ge\frac{1}{n}+\frac{1}{2^m}\in C$, contradicting t being a lower bound.

(d) For $x\in D$, $0<x<\sqrt{2}$. So D has 0 as a lower bound and $\sqrt{2}$ as an upper bound. In fact, $\sup D=\sqrt{2}$ because if there is an upper bound $t<\sqrt{2}$, then by density of rationals, there will be $\frac{m}{n}\in\mathbb{Q}$ such that $\max(t,0)<\frac{m}{n}<\sqrt{2}$, which means $t<\frac{m}{n}\in D$, contradicting t being an upper bound.

Next, $\inf D=0$ because if there is a lower bound $s>0$, then by the density of rationals, there will be $\frac{p}{q}\in\mathbb{Q}$ such that $0<\frac{p}{q}<\min(s,\sqrt{2})$, which means $\frac{p}{q}\in D$ and $\frac{p}{q}<s$, contradicting s being a lower bound.

Remarks If supremum limit theorem and infimum limit theorem are allowed, then the proofs by contradiction above can be avoided.

For (b), taking $w_n=4-\frac{1}{n}\in B$, we have $\lim_{n\to\infty}w_n=4$. Since 4 is an upper bound, $\sup B=4$ by the supremum limit theorem.

For (c), taking $w_n=\frac{1}{n}+\frac{1}{2^n}\in C$, we have $\lim_{n\to\infty}w_n=0$. Since 0 is a lower bound, $\inf C=0$ by the infimum limit theorem.

For (d), taking $w_n=\frac{1}{n}\in D$ and $z_n=\frac{[10^n\sqrt{2}]}{10^n}\in D$, we have $\lim_{n\to\infty}w_n=0$ and $\lim z_n=\sqrt{2}$. Since 0 is a lower bound and $\sqrt{2}$ is an upper bound, so $\inf D=0$ and $\sup D=\sqrt{2}$ by the infimum limit theorem and the supremum limit theorem.

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(35) Let $A=(-\infty,0)=B$, then both A and B are bounded above by 0, but $S=(0,+\infty)$ is not bounded above, $T=(-\infty,\infty)$ is not bounded above.

(36) For every $x \in A$, $y \in B$, we have $x \le \sup A$ and $y \le \sup B$. So $x+y \le \sup A + \sup B$.
$\therefore$ C is bounded above by $\sup A + \sup B$. As $\sup A+\sup B$ is an upper bound of C, we have $\sup C \le \sup A + \sup B$. Assume $\sup C < \sup A + \sup B$. Let $\varepsilon = \frac{\sup A + \sup B - \sup C}{2} > 0$.
By the supremum property, $\exists\, x \in A$ such that $\sup A - \varepsilon < x \le \sup A$ and $\exists\, y \in B$ such that $\sup B - \varepsilon < y \le \sup B$. Adding these, we get $\sup C = \sup A + \sup B - 2\varepsilon < x+y \in C$, a contradiction. Therefore, $\sup C = \sup A + \sup B$.

Another Solution As in the first solution, $\sup C \le \sup A + \sup B$.
Conversely, for every $x \in A, y \in B$, $x+y \le \sup C$, so $x \le \sup C - y$.
Then $\sup C - y$ is an upper bound of A. So $\sup A \le \sup C - y$. Then $y \le \sup C - \sup A$. This implies $\sup C - \sup A$ is an upper bound of B.
So $\sup B \le \sup C - \sup A$. Then $\sup A + \sup B \le \sup C$. $\therefore \sup C = \sup A + \sup B$.

Alternate Solution (using Supremum Limit theorem) As above, we have $\sup C \le \sup A + \sup B$.
By Supremum limit theorem, $\exists\, a_n \in A$ with $\lim_{n\to\infty} a_n = \sup A$ and $\exists\, b_n \in B$ with $\lim_{n\to\infty} b_n = \sup B$.
Then $a_n + b_n \in C$ and $\lim_{n\to\infty}(a_n+b_n) = C$. By the supremum limit theorem, the upper bound $\sup A + \sup B$ of set C is the supremum of C.

(37) Given $\varepsilon > 0$. (Consider the inequalities $\frac{4}{n^2} < \frac{\varepsilon}{2}$ and $\frac{5}{n^3} < \frac{\varepsilon}{2}$. If n satisfies these, then $\frac{4n+5}{n^3} = \frac{4}{n^2} + \frac{5}{n^3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.) So let $K = \left[\max\left(\sqrt{\frac{8}{\varepsilon}}, \sqrt[3]{\frac{10}{\varepsilon}}\right)\right] + 1$, then
$n \ge K \Rightarrow n > \sqrt{\frac{8}{\varepsilon}}$ and $n > \sqrt[3]{\frac{10}{\varepsilon}} \Rightarrow \left|\frac{4n+5}{n^3} - 0\right| = \frac{4}{n^2} + \frac{5}{n^3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
For $\varepsilon = 0.1$, we can choose $K = \left[\max\left(\sqrt{\frac{8}{0.1}}, \sqrt[3]{\frac{10}{0.1}}\right)\right]$, e.g. $K=9$ will do.

(38) We have $y - 1 < [y] \le y$. So $\frac{(x-1)+(2x-1)+\cdots+(nx-1)}{n^2} < a_n \le \frac{x+2x+\cdots+nx}{n^2}$,
i.e. $\frac{\frac{n(n+1)}{2}x - n}{n^2} = \frac{(n+1)x}{2n} - \frac{1}{n} < a_n \le \frac{\frac{n(n+1)}{2}x}{n^2} = \frac{(n+1)x}{2n}$.
Since $\lim_{n\to\infty}\left(\frac{(n+1)x}{2n} - \frac{1}{n}\right) = \frac{x}{2} = \lim_{n\to\infty}\frac{(n+1)x}{2n}$, by squeeze limit theorem, $\lim_{n\to\infty} a_n = \frac{x}{2}$.

(39) Let $x \in \mathbb{R}$. For every $n \in \mathbb{N}$, by the density of rational numbers, there is $r_n \in \mathbb{Q}$ such that $x - \frac{1}{n} < r_n < x$. Since $\lim_{n\to\infty}(x - \frac{1}{n}) = x = \lim_{n\to\infty} x$, by the squeeze limit theorem or Sandwich theorem, $\lim_{n\to\infty} r_n = x$.

(40) Let $r = |x-y|$. By triangle inequality, $|x| = |(x-y)+y| \le |x-y| + |y| = r + |y|$ and so $|x| - |y| \le r$. Also $|y| = |(y-x)+x| \le |y-x| + |x| = r + |x|$ and so $-r \le |x| - |y|$. Then $-r \le |x| - |y| \le r$. Therefore, $\big||x| - |y|\big| \le r = |x-y|$.
Next we will show if $\lim_{n\to\infty} a_n = A$, then $\lim_{n\to\infty}|a_n| = |A|$. For $\varepsilon > 0$, since $\lim_{n\to\infty} a_n = A$, by definition of convergence, there is $K \in \mathbb{N}$ such that $n \ge K \Rightarrow |a_n - A| < \varepsilon$. Then $n \ge K \Rightarrow \big||a_n| - |A|\big| \le |a_n - A| < \varepsilon$.
(Alternatively, $\lim_{n\to\infty} a_n = A \Leftrightarrow \lim_{n\to\infty}|a_n - A| = 0$. Since $0 \le \big||a_n| - |A|\big| \le |a_n - A|$ and $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty}|a_n - A|$, by squeeze limit theorem, $\lim_{n\to\infty}\big||a_n| - |A|\big| = 0 \Leftrightarrow \lim_{n\to\infty}|a_n| = |A|$.) The converse is false. Take $a_n = (-1)^n$. Then $\lim_{n\to\infty}|a_n| = 1$, but $\lim_{n\to\infty} a_n$ doesn't exist.