MATH2033 Mathematical Analysis (2021 Spring)
Suggested Solution of Problem Set 4

Problem 1

Find the supremum and infimum, if exists as a number, for the following sets

(a) $A = \{e^{-x} | x \in (0,1) \cap \mathbb{Q}\}$.

(b) $B = \left\{\cos\frac{1}{n} | n \in \mathbb{N}\right\}$ (😊Hint: The function $\cos x$ is decreasing over $\left[0, \frac{\pi}{2}\right]$)

(c) $C = \left\{1 - \frac{(-1)^n}{n} | n \in \mathbb{N}\right\}$

😊Solution

(a) Note that for any $x \in (0,1)$, we have

$$e^{-1} \leq e^{-x} \leq e^{-0} = 1$$

So 1 and e^{-1} are upper bound and lower bound of the set $A = \{e^{-x} | x \in (0,1) \cap \mathbb{Q}\}$ respectively.

- We argue that $\sup A = 1$. For any $\varepsilon > 0$, by density of rational number, there exists $x \in \mathbb{Q}$ such that
$$0 < x < \underbrace{\min(1, -\ln(1-\varepsilon))}_{\in(0,1)} < -\ln(1-\varepsilon).$$
So $x \in (0,1) \cap \mathbb{Q}$. It follows that
$$e^{-x} > e^{-(-\ln(1-\varepsilon))} = 1 - \varepsilon.$$
Hence, $1 - \varepsilon$ is not the upper bound of A so that $\sup A = 1$.
- Next, we argue that $\inf A = e^{-1}$. For any $\varepsilon > 0$, by density of rational number, there exists $x \in \mathbb{Q}$ such that
$$x > \underbrace{\max(0, -\ln(e^{-1}+\varepsilon))}_{\in(0,1)} > -\ln(e^{-1}+\varepsilon).$$
So $x \in (0,1) \cap \mathbb{Q}$. It follows that
$$e^{-x} > e^{-(-\ln(e^{-1}+\varepsilon))} = e^{-1} + \varepsilon.$$
Hence, $e^{-1} + \varepsilon$ is not the lower bound of A so that $\inf A = e^{-1}$.

(b) Since $0 < \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$ and $\cos x$ is strictly decreasing over $\left[0, \frac{\pi}{2}\right]$, it follows that

$$\cos 1 \leq \cos\frac{1}{n} < \cos 0 = 1$$

So 1 and $\cos 1$ are upper bound and lower bound of the set $B = \left\{\cos\frac{1}{n} | n \in \mathbb{N}\right\}$ respectively.

- We argue that $\sup B = 1$. For any $\varepsilon > 0$, we deduce from Archimedean property that there exists $K \in \mathbb{N}$ such that
$$K > \frac{1}{\cos^{-1}(1-\varepsilon)} \Rightarrow \cos^{-1}(1-\varepsilon) > \frac{1}{K} \Rightarrow \cos\frac{1}{K} > 1 - \varepsilon.$$
So $1 - \varepsilon$ is not the upper bound of the set B and $\sup B = 1$.

- We argue that $\inf B = \cos 1$. Since $\cos 1 \in B$, so we can deduce that for any $\varepsilon > 0$, $\cos 1 < \cos 1 + \varepsilon$. It reveals that $\cos 1 + \varepsilon$ is not the lower bound of B and hence $\inf B = \cos 1$.

(c) For any $n \in \mathbb{N}$, we note that

$$1 - \frac{(-1)^n}{n} = \begin{cases} 1 + \dfrac{1}{n} & if \;\; n \, is \, odd \\ 1 - \dfrac{1}{n} & if \;\; n \, is \, even \end{cases}$$

So we deduce that

$$\frac{1}{2} = \underbrace{1 - \frac{1}{2}}_{when\, n=2} \leq 1 - \frac{(-1)^n}{n} \leq \underbrace{1+1}_{when\, n=1} = 2.$$

So 2 and $\frac{1}{2}$ are upper bound and lower bound of the set C.

- Since $2 = 1 + \frac{1}{1} \in C$, it follows that $2 - \varepsilon < 2$ and $2 - \varepsilon$ is not upper bound of the set C. So we conclude that $\sup C = 2$.
- Since $\frac{1}{2} = 1 - \frac{1}{2} \in C$, it follows that $\frac{1}{2} < \frac{1}{2} + \varepsilon$ and $\frac{1}{2} + \varepsilon$ is not lower bound of the set C. So we conclude that $\inf C = \frac{1}{2}$.

Problem 2

Find the supremum and infimum, if exists as a number, for the following sets

(a) (A bit harder) $D = \left\{\frac{1}{n} - \frac{1}{m} | m \in \mathbb{N}, n \in \mathbb{N}\right\}$

(b) (A bit harder) $E = \{a + b | a \in (0,1) \cap \mathbb{Q},\; b \in (1,2) \backslash \mathbb{Q}\}$.

😊Solution

(a) Recall that $0 < \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$. It follows that

$$-1 = 0 - 1 < \frac{1}{n} - \frac{1}{m} < 1 - 0 = 1.$$

So 1 and -1 are upper bound and lower bound of the set D respectively.

- We argue that $\sup D = 1$. For any $\varepsilon > 0$, we deduce from Archimedean property that there exists $K \in \mathbb{N}$ such that

$$K > \frac{1}{\varepsilon} \Rightarrow \frac{1}{K} < \varepsilon \Rightarrow \underbrace{\frac{1}{1} - \frac{1}{K}}_{\in D} > 1 - \varepsilon.$$

So $1 - \varepsilon$ is not the upper bound of the set D and $\sup D = 1$.

- We argue that $\inf D = -1$. For any $\varepsilon > 0$, we deduce from Archimedean property that there exists $K \in \mathbb{N}$ such that

$$K > \frac{1}{\varepsilon} \Rightarrow \frac{1}{K} < \varepsilon \Rightarrow \underbrace{\frac{1}{K} - \frac{1}{1}}_{\in D} < -1 + \varepsilon.$$

So $-1 + \varepsilon$ is not the lower bound of the set D and $\inf D = -1$.

(b) Since $0 < a < 1$ and $1 < b < 2$, it follows that

$$1 = 0 + 1 < a + b < 1 + 2 = 3.$$

So 3 and 1 are upper bound and lower bound of the set E respectively.

- We argue that $\sup E = 3$. For any $\varepsilon > 0$, we deduce from density of rational number and density of irrational number that there exists $p \in \mathbb{Q}$ and $q \in \mathbb{R}\backslash\mathbb{Q}$ such that
$$1 - \frac{\varepsilon}{2} < \max\left(0, 1 - \frac{\varepsilon}{2}\right) < p \leq 1$$
$$2 - \frac{\varepsilon}{2} < \max\left(1, 2 - \frac{\varepsilon}{2}\right) < q \leq 2$$
It follows that
$$\underbrace{p + q}_{E} > \left(1 - \frac{\varepsilon}{2}\right) + \left(2 - \frac{\varepsilon}{2}\right) = 3 - \varepsilon.$$
Hence, $3 - \varepsilon$ is not upper bound of E. So $\sup E = 3$.
- We argue that $\inf E = 1$. For any $\varepsilon > 0$, we deduce from density of rational number and density of irrational number that there exists $r \in \mathbb{Q}$ and $s \in \mathbb{R}\backslash\mathbb{Q}$ such that
$$0 < r < \min\left(1, \frac{\varepsilon}{2}\right) < \frac{\varepsilon}{2}$$
$$1 < s < \min\left(2, 1 + \frac{\varepsilon}{2}\right) < 1 + \frac{\varepsilon}{2}$$
It follows that
$$\underbrace{r + s}_{E} > \left(\frac{\varepsilon}{2}\right) + \left(1 + \frac{\varepsilon}{2}\right) = 1 + \varepsilon.$$
So $1 + \varepsilon$ is not the lower bound of the set E and $\inf E = 1$.

Problem 3

We let S be a bounded subset in $\mathbb{R}$ and let $S_0 \subseteq S$ be a subset of S_0

(a) Show that the supremum and infimum of S_0 exist and satisfy $\inf S_0 \geq \inf S$ and $\sup S_0 \leq \sup S$.

(b) Suppose that $S_0 \subset S$ (i.e. S_0 is proper subset of S), is it always true that $\inf S_0 > \inf S$ and $\sup S_0 < \sup S$? Explain your answer.

😊Solution
(This is one of the problems of Assignment 2. The solution will be posted after the due date of Assignment 2)

Problem 4

Prove the following statements using Archimedean property.

(a) We let $I_n = \left[0, \frac{1}{n}\right]$ for every $n \in \mathbb{N}$. If $x > 0$, prove that $x \notin \bigcap_{n=1}^{\infty} I_n$.

(b) We let $J_n = \left(0, \frac{1}{n}\right)$ for every $n \in \mathbb{N}$, prove that $\bigcap_{n=1}^{\infty} J_n = \phi$.

(c) We let $K_n = [n, \infty)$ for every $n \in \mathbb{N}$, prove that $\bigcap_{n=1}^{\infty} K_n = \phi$

😊Solution

(a) By Archimedean property, there exists $K \in \mathbb{N}$ such that
$$K > \frac{1}{x} \Rightarrow \frac{1}{K} < x.$$

It follows that $x \notin \left[0, \frac{1}{K}\right] = I_K$. So $x \notin \bigcap_{n=1}^{\infty} I_n$.

(b) For any $x \leq 0$, we have $x \notin J_1 = (0,1)$ and hence $x \notin \bigcap_{n=1}^{\infty} J_n$.
For any $x > 0$, it follows from Archimedean property that there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x \Rightarrow x \notin J_n \Rightarrow x \notin \bigcap_{n=1}^{\infty} J_n$.

(c) For any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ (by Archimedean property) such that
$$n > x \Rightarrow x \notin [n, \infty) = K_n \Rightarrow x \notin \bigcap_{n=1}^{\infty} K_n.$$
So we conclude that $\bigcap_{n=1}^{\infty} K_n = \phi$.

Problem 5

We let X be a non-empty set. We let $f, g: X \to \mathbb{R}$ be two functions which the ranges $f(X)$ and $g(X)$ are both bounded.

(a) Show that $\sup\{f(x) + g(x) | x \in X\} \leq \sup\{f(x) | x \in X\} + \sup\{g(x) | x \in X\}$. Provide an example which the strict inequality holds.

(b) Show that $\inf\{f(x) + g(x) | x \in X\} \geq \inf\{f(x) | x \in X\} + \inf\{g(x) | x \in X\}$. Provide an example which the strict inequality holds.

😊Solution

Since $f(X)$ and $g(X)$ are bounded, so the supremum and infimum of $f(X)$ and $g(X)$ exist due to the completeness axiom.
We let $M_f = \sup\{f(x) | x \in X\}$, $M_g = \sup\{g(x) | x \in X\}$, $m_f = \inf\{f(x) | x \in X\}$ and $m_g = \inf\{g(x) | x \in X\}$.

(a) For any $x \in X$, we have
$$f(x) + g(x) \leq \sup\{f(x) | x \in X\} + \sup\{g(x) | x \in X\} = M_f + M_g.$$
So $M_f + M_g$ is the upper bound of the set $\{f(x) + g(x) | x \in X\}$.
Since the supremum represents the least upper bound of the set, thus
$$\sup\{f(x) + g(x) | x \in X\} \leq M_f + M_g.$$

To see whether the strictly inequality can hold, we take $X = \mathbb{R}$ and define $f(x) = \begin{cases} 1 & if \ \ x \geq 0 \\ 0 & if \ \ x < 0 \end{cases}$ and $g(x) = \begin{cases} 0 & if \ \ x \geq 0 \\ 1 & if \ \ x < 0 \end{cases}$. One can show that

- $f(x) + g(x) = \begin{cases} 1 + 0 = 1 & if \ \ x \geq 0 \\ 0 + 1 = 1 & if \ \ x < 0 \end{cases} = 1 \ \ for \ \ all \ x \in \mathbb{R}$ so that $\sup\{f(x) + g(x) | x \in X\} = 1$.
- $\sup\{f(x) | x \in X\} = \sup\{g(x) | x \in X\} = 1$

This implies that
$$\underbrace{\sup\{f(x) + g(x) | x \in X\}}_{=1} < \underbrace{\sup\{f(x) | x \in X\} + \sup\{g(x) | x \in X\}}_{=1+1=2}.$$

(b) For any $x \in X$, we have
$$f(x) + g(x) \geq \inf\{f(x) | x \in X\} + \inf\{g(x) | x \in X\} = m_f + m_g.$$
So $m_f + m_g$ is the lower bound of the set $\{f(x) + g(x) | x \in X\}$.
Since the infimum represents the greatest lower bound of the set, thus
$$\inf\{f(x) + g(x) | x \in X\} \geq m_f + m_g.$$

To see whether the strictly inequality can hold, we consider the same example adopted in (a). One can show that

- $\inf\{f(x) + g(x)|x \in X\} = 1.$
- $\inf\{f(x)|x \in X\} = \inf\{g(x)|x \in X\} = 0$

This implies that

$$\underbrace{\inf\{f(x) + g(x)|x \in X\}}_{=1} > \underbrace{\inf\{f(x)|x \in X\} + \inf\{g(x)|x \in X\}}_{=0+0=0.}$$

Problem 6

We let X and Y be two non-empty sets. We let $h: X \times Y \to \mathbb{R}$ be a function which $h(X \times Y)$ is bounded (*Note: Here, $h = h(x, y)$ is a function of two variables where $x \in X$ and $y \in Y$.)

We define two functions $f: X \to \mathbb{R}$ and $g: Y \to \mathbb{R}$ to be

$$f(x) = \sup\{h(x, y)|y \in Y\} \quad and \quad g(y) = \sup\{h(x, y)|x \in X\}.$$

(a) Suppose that $h(x, y) = 2x + y$ and $X = Y = [0,1]$, compute $f(x)$ and $g(y)$.

(b) (Independent of (a)) Prove that

$$\sup\{h(x, y)|x \in X, y \in Y\} = \sup\{f(x)|x \in X\} = \sup\{g(y)|y \in Y\}.$$

(*Note: The above equation is known as *principle of the iterated supremum*. The principle suggests that the supremum of a function $h(x, y)$ can be found through the following two steps procedure:

- For each $x \in X$, we first find the supremum ("maximum") of $h(x, y)$ over all possible of y and call this maximum be $f(x)$.
- Given $f(x)$ obtained, we find the final supremum by finding the supremum of $f(x)$ over all possible values of X.

☺Solution

(a) $f(x) = \sup\{h(x, y)|y \in Y\} = \sup\{2x + y|y \in [0,1]\} = 2x + 1.$

$g(x) = \sup\{h(x, y)|x \in X\} = \sup\{2x + y|x \in [0,1]\} = y + 2.$

(b) We first argue that $\sup\{f(x)|x \in X\} = \sup\{h(x, y)|x \in X, y \in Y\}$.

- For any $x \in X$, we have

$$f(x) = \sup\{h(x, y)|y \in Y\} \leq \underbrace{\sup\{h(x, y)|x \in X, y \in Y\}}_{denoted\ by\ M}$$

So $\sup\{h(x, y)|x \in X, y \in Y\}$ is upper bound of $\{f(x)|x \in X\}$.

- Then for any $\varepsilon > 0$, $M - \varepsilon$ is not upper bound of $\{h(x, y)|x \in X, y \in Y\}$, there exists $(x, y) \in X \times Y$ such that

$$h(x, y) > M - \varepsilon.$$

For this x, we deduce that

$$f(x) = \sup\{h(x, y)|y \in Y\} \geq h(x, y) > M - \varepsilon.$$

It shows that $M - \varepsilon$ is also not upper bound for $\{f(x)|x \in X\}$.

Therefore, we conclude that $\sup\{f(x)|x \in X\} = \sup\{h(x, y)|x \in X, y \in Y\}$.

Using similar method, we deduce that

$$\sup\{g(y)|y \in Y\} = \sup\{h(x, y)|x \in X, y \in Y\}.$$

Problem 7 (A bit harder)

We consider the nested interval theorem (see Theorem 6 of Lecture Note 4) as follows:

> **Nested Interval Theorem**
> We let $\{I_n = [a_n, b_n] | n \in \mathbb{N}\}$ be a set of closed intervals such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Then $\bigcap_{n=1}^{\infty} I_n = [a, b]$, where $a = \sup\{a_n | n \in \mathbb{N}\}$ and $b = \inf\{b_n | n \in \mathbb{N}\}$.

Suppose that $\inf\{b_n - a_n | n \in \mathbb{N}\} = 0$, prove that $\bigcap_{n=1}^{\infty} I_n$ contains a single element.

(😊Hint: It suffices to argue that $a = b$. This can be done by first showing $0 \le b - a < \varepsilon$ for any $\varepsilon > 0$ and conclude that $a = b$ using infinitesimal property.)

😊Solution

For any $\varepsilon > 0$,

Note that $\inf\{b_n - a_n | n \in \mathbb{N}\} = 0$, there exists $K \in \mathbb{N}$ such that

$$b_K - a_K < \varepsilon.$$

As $[a, b] = \bigcap_{n=1}^{\infty} I_n \subseteq I_K$, it follows that

$$0 \le b - a = (b_K - a_K) < \varepsilon.$$

Since the above inequality is true for all $\varepsilon > 0$, it follows from infinitesimal property that $0 \le b - a \le 0 \Rightarrow b - a = 0 \Rightarrow a = b$.

So $[a, b] = \{a\}$ becomes a single element set.

Problem 8

(a) Using mathematical induction, prove that

$$\cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\cos\left(\frac{n+1}{2}\theta\right)\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

for all positive integer n. Here, $\theta \neq k\pi$ for any $k \in \mathbb{Z}$.

(b) We let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers defined by

$$a_0 = \sqrt{2}, \qquad a_n = \sqrt{2 + a_{n-1}} \;\; for \;\; n = 1,2, \ldots$$

Using mathematical induction, prove that

$$a_n = 2\cos\frac{\pi}{2^{n+2}}$$

for all $n = 0,1,2, \ldots$.

(c) We let $A = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$. Using mathematical induction, prove that

$$A^n = \begin{pmatrix} 2^n & 3(2^n - 1) \\ 0 & 1 \end{pmatrix}$$

for any positive integer n.

😊Solution

(a) For $n = 1$, we have $L.H.S. = \cos\theta$ and

$$R.H.S. = \frac{\cos\theta \sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cos\theta = L.H.S.$$

The statement is true for $n = 1$.

Assume that the statement is true for $n = k$, then for $n = k + 1$,

$\cos\theta + \cos 2\theta + \cdots + \cos k\theta + \cos(k+1)\theta$

$$= \frac{\cos\left(\frac{k+1}{2}\theta\right)\sin\frac{k\theta}{2}}{\sin\frac{\theta}{2}} + \cos(k+1)\theta = \frac{\cos\left(\frac{k+1}{2}\theta\right)\sin\frac{k\theta}{2} + \cos(k+1)\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= \frac{\frac{1}{2}\left(\sin\frac{2k+1}{2}\theta - \sin\frac{\theta}{2}\right) + \frac{1}{2}\left(\sin\frac{2k+3}{2}\theta - \sin\frac{2k+1}{2}\theta\right)}{\sin\frac{\theta}{2}}$$

$$= \frac{\sin\frac{2k+3}{2}\theta - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \frac{2\cos\frac{2k+4}{2}\theta\sin\frac{2k+2}{2}\theta}{2\sin\frac{\theta}{2}} = \frac{\cos(k+2)\theta\sin(k+1)\theta}{\sin\frac{\theta}{2}}$$

So the statement is true for $n = k+1$. It follows from mathematical induction that the statement is true for all $n \in \mathbb{N}$.

(b) For $n = 0$, a direct calculation shows that

$$a_0 = 2\cos\frac{\pi}{2^2} = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}.$$

The statement is true for $n = 0$.

Suppose that the statement is also true for $n = k$ (where $k \geq 0$), then for $n = k+1$,

$$a_{k+1} = \sqrt{2 + a_k} = \sqrt{2 + 2\cos\frac{\pi}{2^{k+2}}} =^{(*)} \sqrt{2 + 2\left(2\cos^2\frac{\pi}{2^{k+3}} - 1\right)} = 2\cos\frac{\pi}{2^{k+3}}.$$

(*Note: The equality follows from the compound angle formula. That is,

$$\cos(A+B) = \cos A\cos B - \sin A\sin B)$$

So the statement is true for $n = k+1$.

It follows from mathematical induction that the statement is true for non-negative integer n.

(c) For $n = 1$, a direct calculation shows that

$$A^1 = \begin{pmatrix} 2^1 & 3(2^1 - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}.$$

The statement is true for $n = 1$.

Suppose that the statement is also true for $n = k$ (where $k \in \mathbb{N}$), then for $n = k+1$,

$$A^{k+1} = A^k(A) = \begin{pmatrix} 2^k & 3(2^k - 1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 3(2^k) + 3(2^k - 1) \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2^{k+1} & 3(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}.$$

So the statement is true for $n = k+1$.

It follows from mathematical induction that the statement is true for non-negative integer n.

Problem 9

Using mathematical induction, prove that

(a) $(1+x)^n \geq 1+nx$ for any positive integer n, where $x \geq -1$ is real number.

(b) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1}-1\right)$ for all positive integer n.

Solution

(a) For $n=1$, we can verify that

$$L.H.S. = (1+x)^1 = 1+(1)x = R.H.S.$$

So the inequality is valid for $n=1$.

Assume that $(1+x)^k \geq 1+kx$ for some $k \in \mathbb{N}$, then for $n=k+1$,

$$\begin{aligned}(1+x)^{k+1} = (1+x)^k(1+x) &\overset{\text{As } x\geq -1}{\geq} (1+kx)(1+x) = 1+(k+1)x+kx^2 \\ &\geq 1+(k+1)x.\end{aligned}$$

So the inequality is also valid for $n=k+1$.

By mathematical induction, we conclude that $(1+x)^n \geq 1+nx$ for all $n \in \mathbb{N}$

(b) For $n=1$, we can verify that

$$L.H.S. = 1 > 0.828 \approx 2\left(\sqrt{2}-1\right) = R.H.S.$$

So the inequality is valid for $n=1$.

Assume that $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}} > 2\left(\sqrt{k+1}-1\right)$ for some $k \in \mathbb{N}$, then for $n = k+1$,

$$\begin{aligned}1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}} &> 2\left(\sqrt{k+1}-1\right)+\frac{1}{\sqrt{k+1}} = \frac{2k+3}{\sqrt{k+1}}-2 \\ &= \frac{2\sqrt{\left(k+\frac{3}{2}\right)^2}}{\sqrt{k+1}}-2 >^{(*)} 2\frac{\sqrt{(k+2)(k+1)}}{\sqrt{k+1}}-2 = 2\sqrt{k+2}-2\end{aligned}$$

(*Note: One can show that

$$\left(k+\frac{3}{2}\right)^2-(k+2)(k+1) = \left(k^2+3k+\frac{9}{4}\right)-(k^2+3k+2) = \frac{1}{4} > 0.$$

Hence, $\left(k+\frac{3}{2}\right)^2 > (k+2)(k+1)$.)

So the inequality is also valid for $n=k+1$.

By mathematical induction, we conclude that $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1}-1\right)$ for all $n \in \mathbb{N}$.

Problem 10

We let $P(n)$ be a statement which depends on the positive integer n. The second principle of mathematical induction states that $P(n)$ is true for all positive integer n if all of the following conditions hold:

- $P(1)$ and $P(2)$ are true
- If $P(k)$ and $P(k+1)$ are true for some integer k, then $P(k+2)$ is also true.

(a) Prove the principle using well-ordering principle.

(b) Using the second principle of mathematical induction, prove the following statement:

We let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers defined by

$$a_1 = 1, \qquad a_2 = 7, \qquad a_{n+2} - 4a_{n+1} + 3a_n = 0 \quad for \ \ n = 1,2,3, \ldots$$

Then $a_n = 3^n - 2$ for all $n \in \mathbb{N}$.

Solution

(a) Suppose that $P(n)$ is false for some $n \in \mathbb{N}$. Since $P(1)$ and $P(2)$ are known to true, so $n \geq 3$.
Next, we consider the set $A = \{n \in \mathbb{N} | P(n) \ is \ false\} \subseteq \mathbb{N}$. By well-ordering property, there exists a minimal element $n_0 \in A$ in A, where $n_0 \geq 3$.
By the definition of the set A and n_0, we have $P(n_0 - 1)$ and $P(n_0 - 2)$ are true.
This implies from second condition (with $k = n_0 - 2$) that $P(n_0)$ is true and this leads to contradiction.
So the proof is completed.

(b) By direct calculation, we get

$$a_1 = 3^1 - 2 = 1 \ \ and \ \ a_2 = 3^2 - 2 = 7.$$

The statement is true for both $n = 1$ and $n = 2$.
Assume that the statement is true for $n = k$ and $n = k + 1$, then for $n = k + 2$,

$$a_{k+2} = 4a_{k+1} - 3a_k = 4(3^{k+1} - 2) - 3(3^k - 2\ ) = 12(3^k) - 8 - 3(3^k) + 6$$
$$= 9(3^k) - 2 = 3^{k+2} - 2.$$

So the statement is also true for $n = k + 2$.
It follows from mathematical induction that the statement is true for all positive integer n.

Remark of Problem 10

The classical mathematical induction (first principle) does not work in this problem. It is because the second statement only assumes that the statement is true for $n = k$ which is not sufficient in this problem (because the argument requires the assumption that the statements $P(k)$ and $P(k + 1)$ are both true).