Definitions

A set S is countably infinite $\Leftrightarrow \exists$ bijection $f: \mathbb{N} \to S$.

A set S is countable $\Leftrightarrow$ S is finite or countably infinite.

A set S is uncountable $\Leftrightarrow$ S is not countable.

Basic Examples of Countable Sets

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{N}\times\mathbb{N}, \mathbb{Q}\times\mathbb{Q}, \dots$

Basic Examples of Uncountable Sets

$\mathbb{R}$, intervals with more than 1 elements, $\mathbb{R}\setminus\mathbb{Q}$

$\mathbb{C}$, $P(\mathbb{N})$, $\{0,1\}\times\{0,1\}\times\{0,1\}\times\cdots$ $\quad \underset{\text{uncountable}}{A} \setminus \underset{\text{countable}}{B}$

Countable Union Theorem

① $A_1, A_2, A_3, \dots$ are countable $\Rightarrow \bigcup_{n=1}^{\infty} A_n$ is countable.

② S countable and $\forall s \in S$, A_s is countable

$\Rightarrow \bigcup_{\underset{\text{countable}}{s\in S}} \underbrace{A_s}_{\text{countable}}$ is countable.

Countable Subset Theorem

Let $A \subseteq B$. If B is countable, then A is countable.

If A is uncountable, then B is uncountable.

Product Theorem

If $A_1, A_2, \dots, A_n$ are countable, $n \in \mathbb{N}$, then $A_1\times A_2\times\cdots\times A_n$ is countable.

Bijection Theorem

If $\exists$ bijection $f: A \to B$, then either both A and B are countable or both A and B are uncountable.

2012 Fall Midterm

① Let S be the set of all points $(x,y)\in\mathbb{R}^2$ that satisfy the system of equations

$$x+y = mx^2 - x^3 \quad \text{and} \quad mx + y^4 = x^6 - 7mx^3 + 2$$

for some $m \in \mathbb{Q}$. Determine (with proof) if S is countable or not.

Solution If $m \in \mathbb{Q}$, then

$$\left.\begin{aligned} x+y &= mx^2 - x^3 \\ mx+y^4 &= x^6 - 7mx^3 + 2 \end{aligned}\right\} \Leftrightarrow \begin{aligned} y &= mx^2 - x^3 - x \\ mx + \underbrace{(mx^2 - x^3 - x)^4}_{\text{degree } 12} &= x^6 - 7mx^3 + 2 \end{aligned}$$

So there are at most 12 x's and 1 y for each x.

$$\therefore S = \left\{(x,y) : (x,y)\in\mathbb{R}^2,\ m\in\mathbb{Q},\ \begin{aligned} x+y &= mx^2 - x^3 \\ mx+y^4 &= x^6 - 7mx^3 + 2 \end{aligned}\right\}$$

$$= \bigcup_{\underset{\text{Countable}}{m\in\mathbb{Q}}} \underbrace{\left\{(x,y) : (x,y)\in\mathbb{R}^2,\ \begin{aligned} x+y &= mx^2 - x^3 \\ mx+y^4 &= x^6 - 7mx^3 + 2 \end{aligned}\right\}}_{\text{at most } 12\ (x,y)\text{'s} \Rightarrow \text{finite} \Rightarrow \text{countable}}$$

is countable by the countable union theorem.

2009 Fall Midterm

Let S be the set of all points (x,y) in the coordinate plane that satisfy the equations

$$x^2+y^2=a^2 \text{ and } y=x^2-x^3+b$$

for some $a,b \in \mathbb{Q}$ with $a \neq b$. Determine (with proof) if S is countable or not.

Solution ① Let $T=\{(a,b): a,b\in\mathbb{Q}, a\neq b\}$.
Then $T \subseteq \mathbb{Q}\times\mathbb{Q}$. Since $\mathbb{Q}\times\mathbb{Q}$ is countable by product theorem, so T is countable by countable subset theorem.

② For $a,b\in\mathbb{Q}$ with $a\neq b$ ($\Leftrightarrow (a,b)\in T$),

$$\left.\begin{aligned} x^2+y^2&=a^2 \\ y&=x^2-x^3+b \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x^2+(x^2-x^3+b)^2&=a^2 \\ y&=x^2-x^3+b \end{aligned}\right\} \Rightarrow \begin{array}{l} \text{at most } \cancel{4}\ 6\ x\text{'s} \\ \text{One } y \text{ for each } x. \end{array}$$

Let $S_{(a,b)}=\{(x,y): x^2+y^2=a^2 \text{ and } y=x^2-x^3+b\}$.
Then $S_{(a,b)}$ has at most $\cancel{4}$ 6 elements, hence $S_{(a,b)}$ is countable.

$$S=\{(x,y): x^2+y^2=a^2 \text{ and } y=x^2-x^3+b \text{ for some } (a,b)\in T\}$$

$$=\bigcup_{\underbrace{(a,b)\in T}_{\text{countable by ①}}} \underbrace{S_{(a,b)}}_{\text{countable by ②}} \text{ is countable by countable union theorem.}$$

2008 Midterm

Prove that there exists a positive real number c which does not equal to any number of the form $2^{a+b\sqrt{2}}$, where $a,b\in\mathbb{Q}$.

Solution. $S=\{2^{a+b\sqrt{2}}: a,b\in\mathbb{Q}\}$

$$=\bigcup_{\underbrace{(a,b)\in\mathbb{Q}\times\mathbb{Q}}_{\text{Countable}}} \underbrace{\{2^{a+b\sqrt{2}}\}}_{\substack{1 \text{ number} \\ \Rightarrow \text{Countable}}}$$

By countable union theorem, S is countable.
Also, $\mathbb{R}^+=(0,+\infty)$ is uncountable since $\mathbb{R}^+ \supseteq \underbrace{(0,1)}_{\text{uncountable}}$

$\therefore$ $\mathbb{R}^+\setminus S$ is uncountable

$\therefore$ $\mathbb{R}^+\setminus S$ is nonempty

$\therefore$ $\exists c\in\mathbb{R}^+$ and $c\notin S$,

that is c is not of the form $2^{a+b\sqrt{2}}$, where $a,b\in\mathbb{Q}$.

③ Let S be a nonempty countable subset of $\mathbb{R}$. Prove that there exists a positive real number r such that the equation $5^x + 7^y = \sqrt{r}$ does not have any solution with $x, y \in S$. (2012 Fall Midterm)

<u>Solution</u> $5^x + 7^y = \sqrt{r} \Leftrightarrow (5^x + 7^y)^2 = r$.

Let $T = \{(5^x + 7^y)^2 : x, y \in S\}$. Then

$$T = \bigcup_{(x,y) \in \underbrace{S \times S}_{\text{countable}}} \underbrace{\{(5^x+7^y)^2\}}_{1 \text{ element} \Rightarrow \text{finite} \Rightarrow \text{countable}} \text{ is countable.}$$

$\therefore (0, \infty) \setminus T$ is uncountable

$\therefore \exists\, r \in (0, \infty)$ and $r \notin T$

$\therefore$ there exists $r > 0$ and $r \neq (5^x + 7^y)^2$ with $x, y \in S$.

$\Leftrightarrow 5^x + 7^y \neq \sqrt{r}$

2011 Fall Final

Let $f : \mathbb{R} \to \mathbb{Q}$ be a function. Prove that there exists an uncountable subset S of $\mathbb{R}$ such that for all $x, y \in S$, we have $f(x) = f(y)$.

<u>Solution</u> $\forall c \in \mathbb{Q}$, let $S_c = \{x : x \in \mathbb{R}, f(x) = c\}$.

Then $\mathbb{R} = \bigcup_{c \in \mathbb{Q}} S_c$ (because $r \in \mathbb{R} \Rightarrow f(r) = a \in \mathbb{Q}$ $\Rightarrow r \in S_a$).

Assume all S_c are countable.

Then $\underbrace{\bigcup_{c \in \mathbb{Q}}}_{\text{countable}} S_c$ is countable by countable union theorem

$\mathbb{R} = \bigcup_{c \in \mathbb{Q}} S_c$ Contradicting $\mathbb{R}$ is uncountable.

$\therefore \exists\, S_c$ uncountable. Then $\forall x, y \in S_c$, $f(x) = c = f(y)$.

$\therefore S_c$ is such a set.

Supremum and Infimum

Supremum Limit Theorem (p.53)

Let S be a nonempty subset of $\mathbb{R}$ and let c be an upper bound of S. Then

$$c = \sup S \iff \exists\, s_n \in S \text{ such that } \lim_{n\to\infty} s_n = c$$

Infimum Limit Theorem (p.53)

Let S be a nonempty subset of $\mathbb{R}$ and let d be a lower bound of S. Then

$$d = \inf S \iff \exists\, t_n \in S \text{ such that } \lim_{n\to\infty} t_n = d$$

Convergent Sequences to Endpoints

Examples

For interval $(2,7)$,

$$a_n = 2 + \frac{1}{n} \in (2,7)\cap\mathbb{Q} \text{ and } \lim_{n\to\infty} a_n = 2$$

$$b_n = 2 + \frac{1}{n\sqrt{2}} \in (2,7)\setminus\mathbb{Q} \text{ and } \lim_{n\to\infty} b_n = 2$$

$$c_n = 7 - \frac{1}{n} \in (2,7)\cap\mathbb{Q} \text{ and } \lim_{n\to\infty} c_n = 7$$

$$d_n = 7 - \frac{1}{n\sqrt{2}} \in (2,7)\setminus\mathbb{Q} \text{ and } \lim_{n\to\infty} d_n = 7$$

For interval $(\pi, \sqrt{11})$,

$$e_n = \frac{-[-10^n\pi]}{10^n} \in (\pi,\sqrt{11})\cap\mathbb{Q}, \ \lim_{n\to\infty} e_n = \pi$$

$$f_n = \pi + \frac{1}{n} \in (\pi,\sqrt{11})\setminus\mathbb{Q}, \ \lim_{n\to\infty} f_n = \pi$$

$$g_n = \frac{[10^n\sqrt{11}]}{10^n} \in (\pi,\sqrt{11})\cap\mathbb{Q}, \ \lim_{n\to\infty} g_n = \sqrt{11}$$

$$h_n = \sqrt{11} - \frac{1}{n} \in (\pi,\sqrt{11})\setminus\mathbb{Q}, \ \lim_{n\to\infty} h_n = \sqrt{11}.$$

Recall $x - 1 < [x] \le x$

$$10^n x - 1 < [10^n x] \le 10^n x \qquad -10^n x - 1 < [-10^n x] \le -10^n x$$

$$x - \frac{1}{10^n} \le \frac{[10^n x]}{10^n} \le x \qquad 10^n x + 1 > -[-10^n x] \ge 10^n x$$

$$\rightarrow x \leftarrow \qquad x + \frac{1}{10^n} > \frac{-[-10^n x]}{10^n} \ge x$$

$$\rightarrow x \leftarrow$$

2008 Fall Final Problem 2

(a) Determine the infimum of the set

$$S = \{ x : x \in \mathbb{R} \text{ and } \exists\, b, c \in [-1,1) \text{ such that } x^2 + bx + c = 0 \}.$$

Solutions

(a) $\forall x \in S$, $\exists\, b, c \in [-1,1)$ such that $x^2 + bx + c = 0$

$\Leftrightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4c}}{2}$. Now $b < 1$ and $b^2 - 4c < 1^2 - 4(-1) = 5$

Then $x \geq \dfrac{-1 - \sqrt{1^2 - 4(-1)}}{2} = \dfrac{-1-\sqrt{5}}{2}$. a lower bound of S.

Let $b_n = 1 - \frac{1}{n} \in [-1,1)$ and $c_n = -1 \in [-1,1)$.

Then $x_n = \dfrac{-b_n - \sqrt{b_n^2 - 4c_n}}{2} \in S$ and $\lim\limits_{n\to\infty} x_n = \dfrac{-1-\sqrt{5}}{2}$

By infimum limit theorem, $\inf S = \dfrac{-1-\sqrt{5}}{2}$.

2008 Fall Midterm

Find (with proof) the supremum and infimum of

$$B = \{ \cos x + \sin y : x, y \in (0, \tfrac{\pi}{2}] \cap \mathbb{Q} \}$$

Solution. $x, y \in (0, \frac{\pi}{2}] \Rightarrow \begin{matrix} 0 \leq \cos x < 1 \\ 0 < \sin y \leq 1 \end{matrix}$

$\therefore 0 < \cos x + \sin y < 2$

lower bound of B upper bound of B.

Let $x_n = \frac{1}{n} \in (0, \frac{\pi}{2}] \cap \mathbb{Q}$, $\lim\limits_{n\to\infty} x_n = 0$

$y_n = \dfrac{[10^n \frac{\pi}{2}]}{10^n} \in (0, \frac{\pi}{2}] \cap \mathbb{Q}$, $\lim\limits_{n\to\infty} y_n = \frac{\pi}{2}$

Then $\cos x_n + \sin y_n \in B$, $\lim\limits_{n\to\infty} (\cos x_n + \sin y_n) = \cos 0 + \sin \frac{\pi}{2} = 2$

By supremum limit theorem, $\sup B = 2$.

Also $\cos y_n + \sin x_n \in B$, $\lim\limits_{n\to\infty} (\cos y_n + \sin x_n) = \cos \frac{\pi}{2} + \sin 0 = 0$

By infimum limit theorem, $\inf B = 0$.

2009 Fall Midterm

Let D be a nonempty bounded subset of $\mathbb{R}$ such that $\inf D = 3$ and $\sup D = 5$. Let

$$A = \{xy + xy^3 : x \in (2,\pi] \cap \mathbb{Q},\ y \in D\}.$$

Show that A is bounded. Determine (with proof) the infimum and supremum of A.

Solution $\inf D = 3$ and $\sup D = 5 \Rightarrow D \subseteq [3,5]$.

$$\begin{matrix} 2 < x \le \pi \\ 3 \le y \le 5 \end{matrix} \Rightarrow \quad 60 = 2(3+3^3) < xy + xy^3 = x(y+y^3) \le \pi(5+5^3) = 130\pi.$$

So A is bounded below by 60 and above by 130π.

Let $x_n = 2 + \frac{1}{n} \in (2,\pi] \cap \mathbb{Q}$. By infimum limit theorem, $\inf D = 3 \Rightarrow \exists\, y_n \in D$ such that $\lim_{n\to\infty} y_n = 3$.

Then $x_n y_n + x_n y_n^3 \in A$ and $\lim_{n\to\infty} x_n y_n + x_n y_n^3 = 2\times 3 + 2\times 3^3 = 60$.

By infimum limit theorem, $\inf A = 60$.

Let $x_n' = \frac{[10^n \pi]}{10^n} \in (2,\pi] \cap \mathbb{Q}$. By supremum limit theorem, $\sup D = 5 \Rightarrow \exists\, y_n' \in D$ such that $\lim_{n\to\infty} y_n' = 5$.

Then $x_n' y_n' + x_n' y_n'^3 \in A$ and $\lim x_n' y_n' + x_n' y_n'^3 = \pi\times 5 + \pi\times 5^3 = 130\pi$.

By supremum limit theorem, $\sup A = 130\pi$.

2011 Midterm Problem 2

A and B are nonempty bounded subsets of $\mathbb{R}$ such that $\inf A = 1$, $\sup A = 5$, $\inf B = 0$, $\sup B = 1$.

Let $C = \left\{ \frac{y}{3-x} - \frac{1}{y} : x \in B,\ y \in A \right\}$.

Prove C is bounded. Determine $\inf C$ and $\sup C$.

Solution $\inf A = 1$, $\sup A = 5 \Rightarrow \forall y \in A,\ 1 \le y \le 5$

$\Rightarrow -1 \le -\frac{1}{y} \le -\frac{1}{5}$.

$\inf B = 0$, $\sup B = 1 \Rightarrow \forall x \in B,\ 0 \le x \le 1$

$\Rightarrow 2 \le 3 - x \le 3$

$\Rightarrow \frac{1}{3} \le \frac{1}{3-x} \le \frac{1}{2}$

$$-\frac{2}{3} = 1\left(\frac{1}{3}\right) - 1 \le \frac{y}{3-x} - \frac{1}{y} \le 5\left(\frac{1}{2}\right) - \frac{1}{5} = \frac{23}{10} \quad \therefore C \text{ is bounded}$$

By infimum limit theorem and supremum limit theorem,

$\inf A = 1 \Rightarrow \exists\, y_n \in A$ with $\lim_{n\to\infty} y_n = 1$

$\sup A = 5 \Rightarrow \exists\, y_n' \in A$ with $\lim_{n\to\infty} y_n' = 5$.

$\inf B = 0 \Rightarrow \exists\, x_n \in B$ with $\lim_{n\to\infty} x_n = 0$

$\sup B = 1 \Rightarrow \exists\, x_n' \in B$ with $\lim_{n\to\infty} x_n' = 1$

Then $\frac{y_n}{3-x_n} - \frac{1}{y_n} \in C$, $\lim_{n\to\infty} \frac{y_n}{3-x_n} - \frac{1}{y_n} = \frac{1}{3-0} - \frac{1}{1} = -\frac{2}{3}$

$\frac{y_n'}{3-x_n'} - \frac{1}{y_n'} \in C$, $\lim_{n\to\infty} \frac{y_n'}{3-x_n'} - \frac{1}{y_n'} = \frac{5}{3-1} - \frac{1}{5} = \frac{23}{10}$.

$\therefore \inf C = -2/3$ and $\sup C = 23/10$.

Problem (2012 Fall Final)

Let $A \subseteq B \subseteq C \subseteq \mathbb{R}$ with $A \neq \emptyset$ and C bounded above.
If $\sup A = \sup C = L$, then $\sup B = L$.

Solution. Since $B \supseteq A \neq \emptyset$, so $B \neq \emptyset$.

$\forall x \in B$, $B \subseteq C \Rightarrow x \in C \Rightarrow x \leq \sup C = L$.

So L is an upper bound of B.

Since $\sup A = L$, by supremum limit theorem,
$\exists\, a_n \in A$ such that $\lim_{n\to\infty} a_n = L$.

$A \subseteq B \Rightarrow a_n \in B$. By supremum limit theorem,
$\sup B = L$.

Relevant Theorems for Sequences defined by Recurrence Relations

Monotone Sequence theorem (p.54)

If $\{x_n\}$ is increasing and bounded above, then
$\lim_{n\to\infty} x_n = \sup\{x_1, x_2, x_3, \dots\}$.

If $\{x_n\}$ is decreasing and bounded below, then
$\lim_{n\to\infty} x_n = \inf\{x_1, x_2, x_3, \dots\}$

Subsequence theorem (p.54)

If $\lim_{n\to\infty} x_n = x$, then $\forall$ subsequence $\{x_{n_j}\}$ ($n_1 < n_2 < n_3 < \dots$),
$\lim_{j\to\infty} x_{n_j} = x$.

Intertwining Sequence theorem (p.55)

If $x_1, x_3, x_5, x_7, \dots \to x$
and $x_2, x_4, x_6, x_8, \dots \to x$, then $\lim_{n\to\infty} x_n = x$.

Nested Interval theorem (p.55)

If $I_n = [a_n, b_n]$ and $I_n \supseteq I_{n+1}$ for $n = 1, 2, 3, \dots$
then $\bigcap_{n=1}^{\infty} I_n = [a, b]$, where $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$.

If $\lim_{n\to\infty}(b_n - a_n) = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{x\}$

2011 Midterm Problem 1

Prove $\{x_n\}$ converges, where

$x_1 = 27$ and $x_{n+1} = 8 - \sqrt{28 - x_n}$, $n = 1, 2, 3, \dots$

Find its limit.

Solution Note $x_1 = 27 > x_2 = 7 > x_3 = 8 - \sqrt{21} \approx 8 - 4.5\ldots = 3.5\ldots$

$x = 8 - \sqrt{28 - x} \Rightarrow (x-8)^2 = 28 - x \Rightarrow x^2 - 15x + 36 = 0$, $(x-12)(x-3)$, $x = 3$

Claim: $27 = x_1 \geq x_n > x_{n+1} > 3$.

For $n = 1$, $27 = x_1 > x_2 = 7 > 3$. Suppose $27 \geq x_n > x_{n+1} > 3$.

Then $1 = 28 - 27 \leq 28 - x_n < 28 - x_{n+1} < 28 - 3 = 25$.

So $1 \leq \sqrt{28 - x_n} < \sqrt{28 - x_{n+1}} < \sqrt{25} = 5$.

$$\therefore\ 27 \geq 7 = 8 - 1 \geq \underbrace{8 - \sqrt{28 - x_n}}_{= x_{n+1}} > \underbrace{8 - \sqrt{28 - x_{n+1}}}_{= x_{n+2}} > \underbrace{8 - 5}_{= 3}$$

By M.I., we are done.

By monotone sequence theorem, $\{x_n\}$ converges, say to x. Then by subsequence theorem, $x_{n+1} \to x$. $\therefore$

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(8 - \sqrt{28 - x_n}\right) = 8 - \sqrt{28 - x}$$

$$(x-8)^2 = 28 - x \iff x^2 - 15x + 36 = 0 = (x-12)(x-3)$$

Since $12 > 7 = x_2 > x_3 > \cdots$, $x \neq 12$. $\therefore x = 3$.

Fall 2007 Final, Problem 1(a)

If $x_1 = -7$ and $x_{n+1} = \dfrac{x_1 - 2}{10 + x_n}$ for $n = 1, 2, 3, \dots$, then prove $x_1, x_2, x_3, \dots$ converges and find its limit.

Scratch $x_{n+1} = \dfrac{-9}{10 + x_n}$ $\quad x_1 = -7,\ x_2 = -3,\ x_3 = -\dfrac{9}{7}$

$x = -\dfrac{9}{10 + x} \Rightarrow x^2 + 10x + 9 = 0$, $(x+1)(x+9) = 0$, $x = -1$ $\quad -7 \to -3 \to -9/7$

Solution **Claim:** $x_1 \leq x_n \leq x_{n+1} \leq -1$.

For $n = 1$, $x_1 = -7 \leq x_2 = -3 \leq -1$. Suppose case n is true.

Then $3 = 10 - 7 \leq 10 + x_n \leq 10 + x_{n+1} \leq 9 \Rightarrow$

$$x_1 \leq \underbrace{\frac{-9}{10 + x_n}}_{= x_{n+1}} \leq \underbrace{\frac{-9}{10 + x_{n+1}}}_{= x_{n+2}} \leq \underbrace{\frac{-9}{9}}_{= -1}$$

$\therefore$ Case $n+1$ is true.

By the monotone sequence theorem, $\lim_{n\to\infty} x_n = x$ exists.

Then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} -\dfrac{9}{10 + x_n} = -\dfrac{9}{10 + x}$.

So $x^2 + 10x + 9 = 0 \Rightarrow x = -1$ or -9
$(x+1)(x+9)$

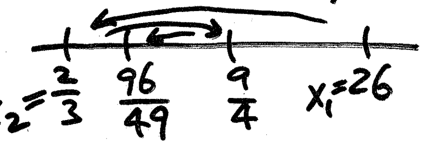
Since $x_1 = -7$ and x_n increasing, so $x \neq -9$. $\therefore x = -1$.

Fall 2007 Final, Problem 1 (b)

If $x_1=26$ and $x_{n+1}=\frac{x_1-2}{10+x_n}$ for $n=1,2,3,\dots$, then prove $x_1,x_2,x_3,\dots$ converge and find its limit.

Scratch $x_{n+1}=\frac{24}{10+x_n}$, $x_1=26$, $x_2=\frac{2}{3}$, $x_3=\frac{9}{4}$

$x_4=\frac{96}{49}$



$x=\frac{24}{10+x}$

$\Leftrightarrow x^2+10x-24=0$

$(x+12)(x-2)=0$

$x=-12$ or 2

rej.

Solution Let $I_n=[x_{2n},x_{2n-1}]$.

Claim: $I_n\supseteq I_{n+1}$ ($x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}$)

For $n=1$, $x_2=\frac{2}{3}\le x_4=\frac{96}{49}\le x_3=\frac{9}{4}\le x_1=26$.

Suppose case n is true. Then $x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}$

So $10+x_{2n}\le 10+x_{2n+2}\le 10+x_{2n+1}\le 10+x_{2n-1}$

$\Rightarrow x_{2n+1}=\frac{24}{10+x_{2n}}\ge x_{2n+3}=\frac{24}{10+x_{2n+2}}\ge x_{2n+2}\ge x_{2n+1}$

$\Rightarrow 10+x_{2n+1}\ge 10+x_{2n+3}\ge 10+x_{2n+2}\ge 10+x_{2n+1}$

$\Rightarrow x_{2n+2}\le x_{2n+4}\le x_{2n+3}\le x_{2n+2}$

$\therefore$ Case $n+1$ is true. By MI, claim is true.

By nested interval theorem, $\lim_{n\to\infty}x_{2n}=a$, $\lim_{n\to\infty}x_{2n+1}=b$

$\Rightarrow a=\lim_{n\to\infty}x_{2n}=\lim_{n\to\infty}\frac{24}{10+x_{2n-1}}=\frac{24}{10+b}$, $b=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}\frac{24}{10+x_{2n}}=\frac{24}{10+a}$

$\Rightarrow a(10+b)=24=b(10+a)\Rightarrow 10a+ab=10b+ab\Rightarrow a=b$

$\therefore \lim_{n\to\infty}x_n=a=b$. Then $a=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\frac{24}{10+x_n}=\frac{24}{10+a}$.

$\therefore a^2+10a-24=0\Rightarrow a=-12$ or 2. $a\in I_1\Rightarrow a=2$.

2010 Fall Midterm

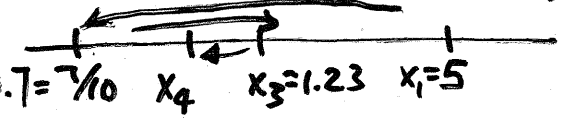
Prove the sequence $\{x_n\}$ converges, where

$$x_1=5 \quad\text{and}\quad x_{n+1}=\frac{7}{x_n+5},$$

and find its limit. Show work!

Solution. (Scratch work: $x_1=5$, $x_2=\frac{7}{10}$, $x_3=\frac{7}{5.7}\approx1.23$

$x_4=\frac{7}{6.23\ldots}\approx1.12$



)

Define $I_n=[x_{2n},x_{2n-1}]$. Claim: $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$

For this, we will prove $0.7\le x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}\le 5$ for all $n=1,2,3,\dots$

Case $n=1$ is done above. Assume $0.7\le x_{2n}\le x_{2n+2}\le x_{2n+1}\le x_{2n-1}\le 5$.

Then $0<x_{2n}+5\le x_{2n+2}+5\le x_{2n+1}+5\le x_{2n-1}+5$

$\Rightarrow x_{2n+1}=\frac{7}{x_{2n}+5}\ge x_{2n+3}=\frac{7}{x_{2n+2}+5}\ge x_{2n+2}=\frac{7}{x_{2n+1}+5}\ge x_{2n}=\frac{7}{x_{2n-1}+5}$

$\Rightarrow x_{2n+1}+5\ge x_{2n+3}+5\ge x_{2n+2}+5\ge x_{2n}+5$

$\Rightarrow 0.7\le x_{2n}\le x_{2n+2}=\frac{7}{x_{2n+1}+5}\le x_{2n+4}=\frac{7}{x_{2n+3}+5}\le x_{2n+3}=\frac{7}{x_{2n+2}+5}\le x_{2n+1}=\frac{7}{x_{2n}+5}\le x_{2n-1}\le 5$

By M.I., we proved the claim.

By the nested interval theorem, $\lim_{n\to\infty}x_{2n}=a$ and $\lim_{n\to\infty}x_{2n-1}=b$.

We have $a=\lim_{n\to\infty}x_{2n}=\lim_{n\to\infty}\frac{7}{x_{2n-1}+5}=\frac{7}{b+5}\Rightarrow ab+5a=7$

and $b=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}\frac{7}{x_{2n}+5}=\frac{7}{a+5}\Rightarrow ab+5b=7$

$\therefore a=b$. By intertwining sequence theorem, $\lim_{n\to\infty}x_n=a$

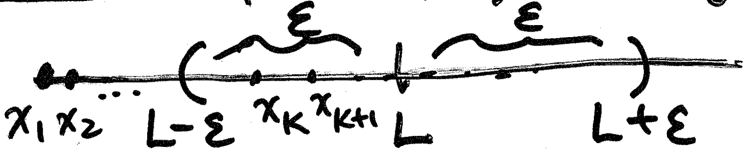
Then $a=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\frac{7}{x_n+5}=\frac{7}{a+5}\Rightarrow a^2+5a-7=0$

$\Rightarrow a=\frac{-5\pm\sqrt{53}}{2}$ Since $a\in I_1$, $\lim_{n\to\infty}x_n=a=\frac{-5+\sqrt{53}}{2}$

Limit of Sequences

(P.48)

Definition of $x_1, x_2, x_3, \dots$ converges to L



$\forall \varepsilon > 0 \quad \exists K \in \mathbb{N}$ such that

$$x_K, x_{K+1}, x_{K+2}, \dots \in (L-\varepsilon, L+\varepsilon)$$

$$n \geq K \Rightarrow \underbrace{|x_n - L|}_{\text{distance between } x_n \text{ and } L} < \varepsilon$$

For different ε, K will change !

---

② 2007 Fall Final

$$y_n = \frac{4n^2 - \sqrt{n}}{2n^2+n} + \frac{n-1}{n}$$

Prove $\lim\limits_{n\to\infty} y_n = 3$ by checking definition

Scratch

$$\frac{4n^2-\sqrt{n}}{2n^2+n} \to 2 \qquad \frac{n-1}{n} \to 1$$

$$\left|\frac{4n^2-\sqrt{n}}{2n^2+n} - 2\right| = \frac{2n+\sqrt{n}}{2n^2+n} \overset{\sqrt{n}\leq n}{\leq} \frac{3n}{2n^2} = \frac{3}{2}\frac{1}{n}$$

$$\left|\frac{n-1}{n} - 1\right| = \frac{1}{n} \qquad \frac{5}{2}\frac{1}{n} = \frac{3}{2}\frac{1}{n} + \frac{1}{n} < \varepsilon$$

$$\Rightarrow n > \frac{5}{2\varepsilon}$$

Solution

$\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{5}{2}\frac{1}{\varepsilon}$ (or let $K = \lceil \frac{5}{2}\frac{1}{\varepsilon} \rceil$).

Then $n \geq K \Rightarrow \left| \frac{4n^2-\sqrt{n}}{2n^2+n} + \frac{n-1}{n} - 3 \right|$ $\quad 3 = 2+1$

$$= \left| \left(\frac{4n^2-\sqrt{n}}{2n^2+n} - 2\right) + \left(\frac{n-1}{n} - 1\right) \right|$$

$$\leq \left| \frac{4n^2-\sqrt{n}}{2n^2+n} - 2 \right| + \left| \frac{n-1}{n} - 1 \right|$$

$$= \frac{2n+\sqrt{n}}{2n^2+n} + \frac{1}{n} \overset{\sqrt{n}\leq n}{\leq} \frac{3n}{2n^2} + \frac{1}{n} = \frac{5}{2}\frac{1}{n}$$

$$\leq \frac{5}{2}\frac{1}{K}$$

$$< \varepsilon.$$

2010 Final Problem 2

Let $a_1, a_2, a_3, \dots$ be real numbers that converge to 1. Prove that $\lim\limits_{n\to\infty}\left(\frac{3+a_n^2}{a_n+1}+\frac{2n}{4+n}\right)=4$ by checking the definition of limit of sequence.

Scratch work $\frac{3+a_n^2}{a_n+1} \to \frac{3+1^2}{1+1}=2$, $\frac{2n}{4+n}\to 2$

$$\left|\frac{3+a_n^2}{a_n+1}-2\right| = \left|\frac{a_n^2-2a_n+1}{a_n+1}\right| = \frac{|a_n-1|^2}{|a_n+1|} < \frac{|a_n-1|^2}{1} < \varepsilon/2$$

when $|a_n-1|<1 \Rightarrow a_n\in(0,2)$ $\Rightarrow a_n+1\in(1,3)$

$$\left|\frac{2n}{4+n}-2\right| = \left|\frac{-8}{4+n}\right| = \frac{8}{4+n} < \frac{8}{n} < \frac{\varepsilon}{2} \Leftrightarrow n > 16/\varepsilon.$$

Solution Since $\lim\limits_{n\to\infty} a_n = 1$, for $1>0$, $\exists K_1\in\mathbb{N}$ such that $n\ge K_1 \Rightarrow |a_n-1|<1 \Leftrightarrow a_n\in(0,2)$

$\forall \varepsilon>0$, $\exists K_2\in\mathbb{N}$ such that $n\ge K_2\Rightarrow |a_n-1|<\sqrt{\frac{\varepsilon}{2}}$.

Let $K>\max\{K_1,K_2,\frac{16}{\varepsilon}\}$. Then

$n\ge K \Rightarrow n\ge K_1$ and $n\ge K_2$ and $n>\frac{16}{\varepsilon}$

$$\Rightarrow \left|\left(\frac{3+a_n^2}{a_n+1}+\frac{2n}{4+n}\right)-4\right| = \left|\left(\frac{3+a_n^2}{a_n+1}-2\right)+\left(\frac{2n}{4+n}-2\right)\right|$$

$$\le \left|\frac{3+a_n^2}{a_n+1}-2\right|+\left|\frac{2n}{4+n}-2\right| = \frac{|a_n-1|^2}{|a_n+1|}+\frac{8}{4+n}$$

$$< \frac{|a_n-1|^2}{1}+\frac{8}{n} < \left(\sqrt{\frac{\varepsilon}{2}}\right)^2+\frac{\varepsilon}{2}=\varepsilon.$$

2011 Fall Final

Let $a_1,a_2,a_3,\dots$ be a sequences of real numbers that converges to 3. Prove that

$$\lim_{n\to\infty}\left(\frac{a_n}{a_n^2+3}+\frac{3n^2}{1+4n^2}+\frac{a_n}{n}\right)=1$$

by checking the definition of limit of sequence.

Solution (Scratch work: $\frac{a_n}{a_n^2+3}\to\frac{1}{4}$, $\frac{3n^2}{1+4n^2}\to\frac{3}{4}$, $\frac{a_n}{n}\to 0$

① $\left|\frac{a_n}{a_n^2+3}-\frac{1}{4}\right| = \frac{|a_n^2-4a_n+3|}{4a_n^2+12} \le \frac{|(a_n-1)(a_n-3)|}{12} \le \frac{3|a_n-3|}{12} < \frac{\varepsilon}{3}$

when $|a_n-3|<4\varepsilon/3$

when $|a_n-3|<1 \Leftrightarrow a_n\in(2,4)$ $\Leftrightarrow a_n-1\in(1,3)$

② $\left|\frac{3n^2}{1+4n^2}-\frac{3}{4}\right| = \frac{3}{4+16n^2} < \frac{3}{16n^2} < \frac{\varepsilon}{3}$

when $n>\frac{3}{4\sqrt{\varepsilon}}$.

③ $\left|\frac{a_n}{n}-0\right| < \frac{4}{n} < \frac{\varepsilon}{3}$ when $n>12/\varepsilon$.

when $|a_n-3|<1 \Leftrightarrow a_n\in(2,4)$

$\forall\varepsilon>0$, since $\lim\limits_{n\to\infty} a_n=3$,

for $1>0$, $\exists K_1\in\mathbb{N}$ such that $n\ge K_1 \Rightarrow |a_n-3|<1$

for $4\varepsilon/3>0$, $\exists K_2\in\mathbb{N}$ such that $n\ge K_2\Rightarrow |a_n-3|<\frac{4\varepsilon}{3}$.

By Archimedean principle, $\exists K\in\mathbb{N}$ such that

$K>\max\{K_1,K_2,\frac{3}{4\sqrt{\varepsilon}},\frac{12}{\varepsilon}\}$.

$n\ge K \Rightarrow n>K_1, n>K_2, n>\frac{3}{4\sqrt{\varepsilon}}, n>\frac{12}{\varepsilon}$

$$\Rightarrow \left|\frac{a_n}{a_n^2+3}+\frac{3n^2}{1+4n^2}+\frac{a_n}{n}-1\right| = \left|\left(\frac{a_n}{a_n^2+3}-\frac{1}{4}\right)+\left(\frac{3n^2}{1+4n^2}-\frac{3}{4}\right)+\left(\frac{a_n}{n}-\right.\right.$$

$$\le \left|\frac{a_n}{a_n^2+3}-\frac{1}{4}\right|+\left|\frac{3n^2}{1+4n^2}-\frac{3}{4}\right|+\left|\frac{a_n}{n}-0\right| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

by ①, ②, ③ above.