Lecture 10

12-03-2019

Review: The basic facts about convergent sequence

$\lim_{n\to\infty} x_n = x$. (several equivalent definitions)

① Given $\{x_n\}$ and x, check that $\lim x_n = x$

② boundedness theorem ($\{x_n\}$ converges $\Rightarrow$ $\{x_n : n \geq 1\}$ is bounded)

③ Computational rule for limit ($\lim(x_n + y_n) = \lim x_n + \lim y_n$)

④ limit inequality ($x_n \leq y_n \Rightarrow \lim x_n \leq \lim y_n$)

⑤ Sandwich theorem $\left[(a_n \leq b_n \leq c_n) + (\lim a_n = \lim c_n) \Rightarrow \lim b_n = \lim a_n = \lim c_n \right]$

⑥ Supremum (Infimum) limit theorem

$\sup S = \lim s_n \quad s_n \in S.$

In all above, we are given both the sequence and the limit.

Question: Given a sequence $\{x_n\}$, how can we know that it is convergent (or has a limit)? For example, given $\{x_n\}$ defined by $x_1 = 2$, $x_{n+1} = \sqrt{3x_n - 2}$ for $n = 1, 2, \cdots$

Definition: Let $\{x_n\}$ be a sequence. $x_{n_1}, x_{n_2}, \cdots$ is a subsequence of $\{x_n\}$ iff $n_1 < n_2 < n_3 < \cdots$ and $n_j \in \mathbb{N}$, $\forall j \in \mathbb{N}$.

$$x_1, x_2, \cdots \boxed{x_{n_1}}, x_{n_1+1}, \cdots \boxed{x_{n_2}}, x_{n_2+1}, \cdots \boxed{x_{n_3}}, x_{n_3+1} \cdots,$$

Example: For a sequence $x_1, x_2, x_3, \cdots$, if we set $\boxed{n_j} = j^2$, then we get $x_1, x_4, x_9, x_{16}, \cdots$ which is a subsequence.

If we set $n_j = 2j+1$, then we get

$x_3, x_5, x_7, x_9, \cdots$

which is another subsequence.

Remark: If $n_1 < n_2 < n_3 < \cdots$ and $n_j \in \mathbb{N}$ $\forall j \in \mathbb{N}$.

Then $n_j \geq j$. $\forall j \in \mathbb{N}$.

We can prove by mathematical induction.

For $j=1$, $n_1 \in \mathbb{N}$, so $n_1 \geq 1$.

Now, assume $n_j \geq j$, then $n_{j+1} > n_j \geq j$.

Since $n_{j+1} \in \mathbb{N}$, so $n_{j+1} \geq j+1$.

By mathematical induction, $n_j \geq j$ for all $j \in \mathbb{N}$.

Subsequence Theorem

THM: If $\lim_{n\to\infty} x_n = x$, then for every subsequence $\{x_{n_j}\}$

we have $\lim_{j\to\infty} x_{n_j} = x$.

proof: $\forall \varepsilon > 0$, Since $\lim x_n = x$

$\exists\ K$ s.t $|x_n - x| < \varepsilon$ for all $n \geq K$.

Now, since $n_j \geq j$, ..

for $j \geq K$, we have $n_j \geq j \geq K$

So $|x_{n_j} - x| < \varepsilon$.

Therefore $\lim x_{n_j} = x$

Def : $\{x_n\}$ is increasing iff $x_n \leq x_m \quad \forall\, n < m$.

($\text{or } \ x_n \leq x_{n+1} \quad \forall\, n \in \mathbb{N}$)

$\{x_n\}$ is decreasing iff $x_n \leq x_m \quad \forall\, n > m$.

($\text{or } \ x_{n+1} \leq x_n \quad \forall\, n \in \mathbb{N}$)

$\{x_n\}$ is strictly increasing iff $x_n < x_m, \ \forall\, n < m$.

($\text{or } \ x_n < x_{n+1}, \ \forall\, n \in \mathbb{N}$)

$\{x_n\}$ is strictly decreasing iff $x_n < x_m, \ \forall\, n > m$

($\text{or } \ x_{n+1} < x_n, \ \forall\, n \in \mathbb{N}$)

$\{x_n\}$ is monotone if $\{x_n\}$ is increasing or decreasing.

$\{x_n\}$ is strictly monotone if $\{x_n\}$ is strictly increasing or strictly decreasing.

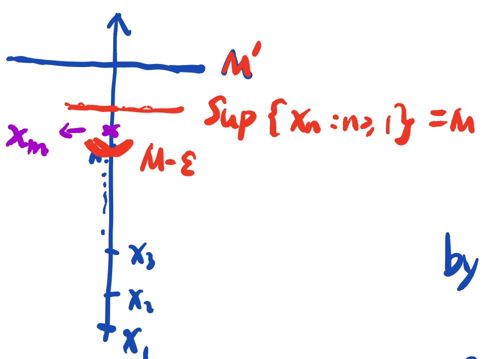
Monotone Sequence Theorem

THM : ① If $\{x_n\}$ is increasing and bounded above,

then $\lim_{n\to\infty} x_n$ exists and $\lim_{n\to\infty} x_n = \sup\{x_n : n\geq 1\}$

② If $\{x_n\}$ is decreasing and bounded below,

then $\lim_{n\to\infty} x_n$ exists and $\lim_{n\to\infty} x_n = \inf\{x_n : n\geq 1\}$



Proof : ① Since $\{x_n\}$ is bound above,

$M = \sup\{x_n : n\geq 1\}$ exists. $\forall \varepsilon > 0$,

by the Supremum property, $\exists\ \boxed{x_m} \in \{x_n : n\geq 1\}$

s.t $M-\varepsilon < x_m \leq M$

For $\boxed{n\geq m}$, we have $M \geq x_n \geq x_m$

So $M-\varepsilon < x_m \leq x_n \leq M$, or $|x_n - M| < \varepsilon$

Therefore $\lim x_n = M$

Proof of ② Similar.

Example ①. Let $0 < c < 1$, $x_n = c^{\frac{1}{n}}$, $n = 1, 2, \cdots$

Show that $\lim x_n = 1$.

Proof: Step 1. $\{x_n\}$ is bounded.

Since $0 < x_n < 1^{\frac{1}{n}} = 1$ for all n.

Step 2. We show that $\{x_n\}$ is increasing.

Indeed $x_{n+1} = c^{\frac{1}{n+1}} = \left(c^{\frac{1}{n}}\right)^{\frac{n}{n+1}}$

$= x_n^{\frac{n}{n+1}} = x_n^{1-\frac{1}{n+1}} = x_n \cdot x_n^{-\frac{1}{n+1}}$

$\geq x_n$ for all n.

Step 3. By the monotone sequence thm, $\{x_n\}$ converges.

Let $x = \lim\limits_{n\to\infty} x_n$. We need to find x

Observe that $x_{2n} = c^{\frac{1}{2n}} = \left(c^{\frac{1}{n}}\right)^{\frac{1}{2}} = x_n^{\frac{1}{2}}$, or $x_{2n}^2 = x_n$

let $n \to \infty$, using the fact that $\lim x_{2n} = \lim x_n = x$, we have

$$x^2 = x$$

$\Rightarrow$ $x = 0$ or $x = 1$.

On the other hand, $x = \sup\{x_n : n \geq 1\} \geq x_1 > 0$

So $x = 1$.

② What is $\sqrt{2+\sqrt{2+\sqrt{\cdots}}}$? More precisely, let $x_1=\sqrt{2}$, and $x_{n+1}=\sqrt{2+x_n}$ for $n\geq 1$. Does $\{x_n\}$ converge to some number ?

Solution : Step 1. we show that $\{x_n\}$ is increasing and bounded above.

Actually, we have

$$x_n < x_{n+1} < 2 \quad \text{for all} \quad n\geq 1$$

We show by mathematical induction (applies to sequence defined by iteration)

For $n=1$, we have $x_1=\sqrt{2}<2$.

$$x_2=\sqrt{2+\sqrt{2}} > \sqrt{2} = x_1 \quad \text{and} \quad x_2<2.$$

Now, assume $x_n < x_{n+1} < 2$,

then $x_{n+2}=\sqrt{2+x_{n+1}} > \sqrt{2+x_n} = x_{n+1}$

$$x_{n+2}=\sqrt{2+x_{n+1}} < \sqrt{2+2} = 2 \quad , \text{ i.e}$$

$$x_{n+1} < x_{n+2} < 2$$

So $\quad x_n < x_{n+1} < 2$ holds for all $n \in \mathbb{N}$.

Step 2. By monotone sequence thm, $\lim x_n = x$ exists

Observe that

$$x_{n+1} = \sqrt{x_n + 2}$$

or $\quad x_{n+1}^2 = x_n + 2$.

let $n \to \infty$, $\quad x^2 = x+2$

$$\Rightarrow \quad x = 2 \quad \text{or} \quad x = -1$$

On the other hand $\quad x = \sup\{x_n : n \geq 1\} \geq x_1 > 0$

So $\quad x = 2$

Intertwining Sequence Theorem

THM: If ① $\lim_{m\to\infty} x_{2m-1} = x$ ② $\lim_{m\to\infty} x_{2m} = x$

then $\lim_{n\to\infty} x_n = x$

[Or if $\lim_{n\to\infty} a_n = x$, $\lim_{n\to\infty} b_n = x$, then $a_1, b_1, a_2, b_2, \cdots$ converges to x]

Proof: $\forall\ \varepsilon > 0$, need to find K s.t

$|x_n - x| < \varepsilon$ for all $n \geq K$. $\cdots (\star)$

By ①, $\exists\ k_1$ s.t $|x_{2m-1} - x| < \varepsilon$ for all $m \geq k_1$

By ②, $\exists\ k_2$, s.t $|x_{2m} - x| < \varepsilon$ for all $m \geq k_2$.

let $K = \max(2k_1 - 1,\ 2k_2)$, then $(\star)$ is satisfied.

Example: $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ converges to 0.

$-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \cdots$ converges to 0

Then $1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \cdots$ converges to 0.

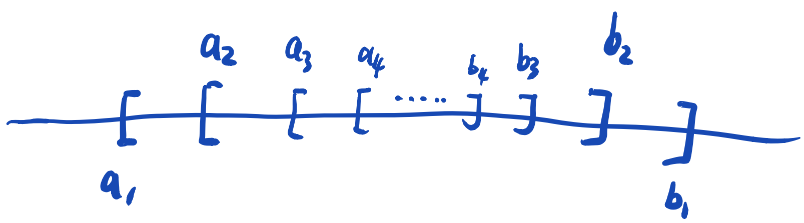
Nested Interval Theorem

THM: If $\forall n \in \mathbb{N}$, $I_n = [a_n, b_n]$ where $a_n \leq b_n$ s.t

$I_1 \supseteq I_2 \supseteq I_3 \cdots$ Then $\bigcap_{n=1}^{\infty} I_n = [a, b]$, where

$a = \lim_{n\to\infty} a_n$, $b = \lim_{n\to\infty} b_n$.

If $\lim_{n\to\infty} b_n - a_n = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{a\}$.



Proof: $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ implies that

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq b_1$$

$$b_1 \geq b_2 \geq b_3 \geq \cdots \geq a_1$$

By the monotone theorem, $\lim a_n = \sup\{a_n : n \geq 1\} = a$

$\lim b_n = \inf\{b_n : n \geq 1\} = b$

Since $a_n \le b_n \quad \forall n,$

$a \le b$ by the limit inequality.

We now show that $\bigcap_{n \in \mathbb{N}} I_n = [a, b]$

Note that $a_n \le a$, $b \le b_n$, $\Rightarrow$ $[a, b] \subseteq [a_n, b_n] = I_n$, $\forall n \in \mathbb{N}$.

$$\Rightarrow [a, b] \subseteq \bigcap_{n\in\mathbb{N}} I_n.$$

On the other hand, $\forall x \in \bigcap_{n\in\mathbb{N}} I_n$, $a_n \le x \le b_n$

Taking limit $n\to\infty$, we get $a \le x \le b$, i.e $\bigcap_{n\in\mathbb{N}} I_n \subseteq [a, b]$.

So we have $[a, b] = \bigcap_{n\in\mathbb{N}} I_n$.

Finally, if $0 = \lim_{n\to\infty}(b_n - a_n)$, then $0 = b - a$ $\Rightarrow$ $a = b$

$$\Rightarrow [a, b] = \{a\} = \bigcap_{n\in\mathbb{N}} I_n.$$

R.k: Nested interval theorem $\Leftrightarrow$ Monotone sequence theorem