MATH2033 Mathematical Analysis
Suggested Solution of Problem Set 2

Problem 0

Do problem #9 and problem #10 in practice exercise.

😊Answer

Problem #9

(a) $(\{x, y, z\} \cup \{w, z\}) \backslash \{u, v, w\} = \{w, x, y, z\} \backslash \{u, v, w\} = \{x, y, z\}$.

(b) $\{1,2\} \times \{3,4\} \times \{5\} = \{(1,3,5), (1,4,5), (2,3,5), (2,4,5)\}$.

(c) $\mathbb{Z} \cap [0,10] \cap \{n^2 + 1 | n \in \mathbb{N}\} = \mathbb{Z} \cap [0,10] \cap \{2,5,10,17, \ldots\} = \mathbb{Z} \cap \{2,5,10\} = \{2,5,10\}$

(d) $\{n \in \mathbb{N} | 5 < n < 9\} \backslash \{2m | m \in \mathbb{N}\} = \{6,7,8\} \backslash \{2,4,6,8,10, \ldots\} = \{7\}$.

(e) $([0,2] \backslash [1,3]) \cup ([1,3] \backslash [0,2]) = [0,1) \cup (2,3]$.

Problem #10

(i) (Omitted)

(ii) We claim $A = B$. Suppose that $A \neq B$, then either (1) there exists $x \in A$ which $x \notin B$ or (2) these exists $x \in B$ which $x \notin A$.
Suppose that the case (1) happens, we consider the element $(x, b) \in A \times B$. As $x \notin B$, then $(x, b) \notin B \times A$. This implies that $A \times B \neq B \times A$ and it leads to contradiction. One can also show that a similar contradiction is induced if the case (2) happens. Thus, we have $A = B$.

Problem 1

We let A, B be any two sets. The symmetric difference of A and B (denoted by $A \Delta B$) is defined by

$$A \Delta B = (A \backslash B) \cup (B \backslash A).$$

(a) Show that $A \Delta B = (A \cup B) \backslash (A \cap B)$

(b) Show that $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$ for any sets A, B, C.

☺Solution

(a) For any $x \in A \Delta B$, we have

$x \in A \Delta B = (A \backslash B) \cup (B \backslash A)$
$\Rightarrow x \in A \backslash B \;\; or \;\; x \in B \backslash A$
$\Rightarrow (x \in A \;\; and \;\; x \notin B) \;\; or \;\; (x \in B \;\; and \;\; x \notin A)$
$\Rightarrow x \in A \cup B \;\; and \;\; x \notin A \cap B$
$\Rightarrow x \in (A \cup B) \backslash (A \cap B)$.
So $A \Delta B \subseteq (A \cup B) \backslash (A \cap B)$.

For any $x \in (A \cup B) \backslash (A \cap B)$, we have
$x \in (A \cup B) \backslash (A \cap B) \Rightarrow x \in A \cup B \;\; and \;\; x \notin A \cap B$

$\overset{\substack{x\in A\cup B \\ \Leftrightarrow x\in A \ or \ x\in B}}{\Rightarrow}$ $(x \in A \ \ and \ x \notin A \cap B) \ \ or \ \ (x \in B \ \ and \ x \notin A \cap B)$

$\overset{\substack{x\notin A\cap B \\ \Leftrightarrow x\notin A \ or \ x\notin B}}{\Rightarrow}$ $(x \in A \ \ and \ \ x \notin B)$ or $(x \in B \ \ and \ \ x \notin A)$

$\Rightarrow x \in A\backslash B \ \ or \ \ x \in B\backslash A \Rightarrow x \in A\Delta B = (A\backslash B) \cup (B\backslash A)$

So we have $(A \cup B)\backslash(A \cap B) \subseteq \ A\Delta B$.

Hence, we conclude that $A \, \Delta \, B = (A \cup B)\backslash \, (A \cap B)$.

(b) For any $x \in A \cap (B \, \Delta \, \mathrm{C})$, we have

$x \in A \cap (B \, \Delta \, \mathrm{C})$

$\Rightarrow x \in A \ \ and \ \ x \in B \, \Delta \, \mathrm{C}$

$\Rightarrow x \in A \ \ and \ \ (x \in B\backslash C \ \ or \ \ x \in C\backslash B)$

$\Rightarrow (x \in A \ \ and \ \ x \in B\backslash C) \ \ \ or \ \ (x \in A \ \ and \ \ x \in C\backslash B)$

$\Rightarrow (x \in A \ and \ x \in B \ and \ x \notin C)$ or $(x \in A \ and \ x \in C \ and \ x \notin B)$

$\overset{x\notin C \Rightarrow x\notin A\cap C}{\Rightarrow}$ $(x \in A \cap B \ \ and \ \ x \notin A \cap C)$ or $(x \in A \cap C \ \ and \ \ x \notin A \cap B)$

$\Rightarrow x \in (A \cap B)\backslash(A \cap C) \ \ \ or \ \ x \in (A \cap C)\backslash(A \cap B)$

$\Rightarrow x \in (A \cap B) \, \Delta \, (A \cap C)$.

So $A \cap (B \, \Delta \, \mathrm{C}) \subseteq (A \cap B) \, \Delta \, (A \cap C)$.

For any $x \in (A \cap B) \, \Delta \, (A \cap C)$, we have

$x \in (A \cap B) \, \Delta \, (A \cap C)$

$\Rightarrow x \in (A \cap B)\backslash(A \cap C) \ \ \ or \ \ x \in (A \cap C)\backslash(A \cap B)$

$\Rightarrow (x \in A \cap B \ \ and \ \ x \notin A \cap C)$ or $(x \in A \cap C \ \ and \ \ x \notin A \cap B)$

$\Rightarrow (x \in A \ and \ x \in B \ and \ x \notin C)$ or $(x \in A \ and \ x \in C \ and \ x \notin B)$

$\Rightarrow (x \in A \ \ and \ \ x \in B\backslash C) \ \ \ or \ \ (x \in A \ \ and \ \ x \in C\backslash B)$

$\Rightarrow x \in A \ \ and \ \ (x \in B\backslash C \ \ or \ \ x \in C\backslash B)$

$\Rightarrow x \in A \ \ and \ \ x \in B \, \Delta \, \mathrm{C}$

$\Rightarrow x \in A \cap (B \, \Delta \, \mathrm{C})$

So we have $(A \cap B) \, \Delta \, (A \cap C) \subseteq \ A \cap (B \, \Delta \, \mathrm{C})$.

Hence, we deduce that $A \cap (B \, \Delta \, \mathrm{C}) = (A \cap B) \, \Delta \, (A \cap C)$.

Problem 2

We let A, B, C be 3 sets, prove that

(a) $A\backslash(B\backslash A) = A$

(b) $(A \cap B)\backslash C = (A\backslash C) \cap (B\backslash C)$.

☺Solution

(a) For any $x \in A\backslash(B\backslash A)$, we have

$x \in A\backslash(B\backslash A) \Rightarrow x \in A \ \ and \ \ x \notin (B\backslash A)$

$\Rightarrow x \in A \ \ and \ \ (x \notin B \ \ or \ \ x \in A)$
$\Rightarrow x \in A$
So $A\backslash(B\backslash A) \subseteq A$.

For any $x \in A$, we must have $x \notin B\backslash A$ (since if $x \in B\backslash A$, the $x \notin A$ by the definition of $B\backslash A$). So we have
$x \in A \Rightarrow x \in A \ \ and \ \ x \notin (B\backslash A) \Rightarrow x \in A\backslash(B\backslash A)$
So $A \subseteq A\backslash(B\backslash A)$.

Therefore $A\backslash(B\backslash A) = A$.

(b) For any $x \in (A \cap B)\backslash C$, we have
$x \in (A \cap B)\backslash C \Rightarrow x \in A \cap B \ \ and \ \ x \notin C$
$\Rightarrow x \in A \ \ and \ \ x \in B \ \ and \ \ x \notin C$
$\Rightarrow (x \in A \ \ and \ \ x \notin C) \ \ and \ \ (x \in B \ \ and \ \ x \notin C)$
$\Rightarrow x \in A\backslash C$ and $x \in B\backslash C$
$\Rightarrow x \in (A\backslash C) \cap (B\backslash C)$
So $(A \cap B)\backslash C \subseteq (A\backslash C) \cap (B\backslash C)$.

For any $x \in (A\backslash C) \cap (B\backslash C)$, we have
$x \in (A\backslash C) \cap (B\backslash C)$
$\Rightarrow \ x \in A\backslash C$ and $x \in B\backslash C$
$\Rightarrow (x \in A \ \ and \ \ x \notin C) \ \ and \ \ (x \in B \ \ and \ \ x \notin C)$
$\Rightarrow x \in A \ \ and \ \ x \in B \ \ and \ \ x \notin C \Rightarrow x \in A \cap B \ \ and \ \ x \notin C$
$\Rightarrow \ x \in (A \cap B)\backslash C$.
So $(A\backslash C) \cap (B\backslash C) \subseteq (A \cap B)\backslash C$.

Thus, $(A \cap B)\backslash C = (A\backslash C) \cap (B\backslash C)$.

Problem 3

Prove that the following statements are incorrect.

(a) $A\backslash B = B\backslash A$ for all non-empty sets A and B
(b) $(A\backslash B)\backslash C = A\backslash(B\backslash C)$ for any non-empty sets A, B and C
(c) $(A\backslash B) \cup C = (A \cup C)\backslash(B \cup C)$ for any non-empty sets A, B and C.
(d) If $A \cap B = A \cap C$, then $B = C$. Here, A, B, C are non-empty sets.

☺Solution

(a) We take $A = \{1,2,3\}$ and $B = \{2,3,4\}$. Then
$$A\backslash B = \{1\} \quad and \quad B\backslash A = \{4\}.$$
So $A\backslash B \neq B\backslash A$ in this case.

(b) We let $A = \{1,2,3\}$, $B = \{2,3,4\}$ and $C = \{2\}$. Then
$$(A\backslash B)\backslash C = \{1\}\backslash\{2\} = \{1\} \quad and \quad A\backslash(B\backslash C) = \{1,2,3\}\backslash\{3,4\} = \{1,2\}.$$

So $(A\backslash B)\backslash C \neq A\backslash(B\backslash C)$ in this case.

(c) We take $A = \{1\}$, $B = \{1\}$ and $C = \{1,2\}$.

Then $\underbrace{(A\backslash B)}_{=\phi} \cup C = \{1,2\}$ and $(A \cup C)\backslash(B \cup C) = \{1,2\}\backslash\{1,2\} = \phi$.

So $(A\backslash B) \cup C \notin (A \cup C)\backslash(B \cup C)$.

(d) We take $A = \{1,2,3\}$, $B = \{4\}$ and $C = \{5\}$, we have

$$A \cap B = A \cap C = \phi$$

But $B = \{4\} \neq \{5\} = C$.

Problem 4 (Harder)

We let A, B be two subsets of a set Ω. If there exists a subset $C \subseteq \Omega$ such that

$$B \cap C = A \cap C \quad and \quad B \cup C = A \cup C.$$

Show that $A = B$.

☺Solution

We shall prove it by contradiction. Assume that $A \neq B$, then either

1. There exists $a \in A$ which $a \notin B$ (i.e. A is not subset of B) or
2. There exists $b \in B$ which $b \notin A$ (i.e. B is not subset of A)

Suppose that the first case happens,

- Since $B \cup C = A \cup C$ and $a \in A \cup C$, it follows that $a \in B \cup C$. As $a \notin B$, then it must be that $a \in C$
- Since $a \in A$ and $a \in C$, then $a \in A \cap C$. It follows from the fact $A \cap C = B \cap C$ that $a \in B \cap C$. This implies that $a \in B$ and it leads to the contradiction.

Using similar argument, one can argue that similar contradiction occurs when the second case happens. Therefore, the negation is never true and we can conclude that $A = B$.

Problem 5

We consider a function $f: \mathbb{R} \to \mathbb{R}$. Suppose that the function is strictly increasing (i.e. $f(x_1) < f(x_2)$ for any x_1, x_2 which $x_1 < x_2$),

(a) Show that $f(x)$ is injective.

(b) Determine if $f(x)$ is always surjective. (😊Hint: The answer is false)

☺Solution

(a) We consider $f(x_1) = f(x_2)$.

Suppose that $x_1 \neq x_2$, we assume $x_1 < x_2$ (the case for $x_1 > x_2$ will be similar)

Since f is strictly increasing, it follows that $f(x_1) < f(x_2)$ which contradicts to $f(x_1) = f(x_2)$. So it must be that $x_1 = x_2$ and the function is injective.

(b) We consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$. We observe that $f(x)$ is strictly increasing since $f'(x) = e^x > 0$ for all $x \in \mathbb{R}$. However, the function is not surjective since if we pick $y = -1 \in \mathbb{R}$, there is no x such that $f(x) = e^x = -1$ as $e^x \geq 0$ for all $x \in \mathbb{R}$.

Problem 6

We consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$.

(a) Write down the range of $f(x)$.

(b) Determine if the function is injective. Also, determine if the function is surjective

(c) Suppose that the domain of f is changed to $\left[0, \frac{\pi}{2}\right]$, determine if the function is injective.

☺Solution

(a) Range of $f(x)$ is $[-1,1]$ (as $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$.)

(b) We pick $x_1 = 0$ and $x_2 = \pi$ (which $x_1 \neq x_2$), we have $f(x_1) = 0 = f(x_2)$. So the function is not injective

The function is not surjective since if we take $y = 2$, there is no $x \in \mathbb{R}$ such that $f(x) = \sin x = 2$ (as $\sin x \leq 1$ for all $x \in \mathbb{R}$).

(c) Suppose that the domain is changed to $\left[0, \frac{\pi}{2}\right]$, we see that the function $f(x) = \sin x$ is strictly increasing since $f(x)$ is continuous and $f'(x) = \cos x > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$. So it follows from the result of Problem 5 that f is injective.

Problem 7

We let $f: \mathbb{R} \to \mathbb{R}$ be a function satisfying $|f(x_1) - f(x_2)| = |x_1 - x_2|$ for any $x_1, x_2 \in \mathbb{R}$. Show that f is injective.

☺Solution

For any $x_1, x_2 \in \mathbb{R}$ which $f(x_1) = f(x_2)$, we have

$$|x_1 - x_2| = |f(x_1) - f(x_2)| = 0 \Rightarrow x_1 = x_2.$$

So f is injective.

Problem 8

We let $f: \mathbb{R}\backslash\{-1,1\} \to \mathbb{R}$ be a function defined by $f(x) = \frac{x}{1-x^2}$.

(a) Determine if the function is injective.

(b) Determine if the function is surjective.

☺Solution

(a) By taking $x_1 = \frac{-1+\sqrt{5}}{2}$ and $x_2 = \frac{-1-\sqrt{5}}{2}$, we get

$$f(x_1) = 1 = f(x_2).$$

So f is not injective.

(b) For any $y \in \mathbb{R}\backslash\{0\}$, we consider

$$y = f(x) = \frac{x}{1-x^2} \Rightarrow yx^2 + x - y = 0.$$

Using quadratic formula, we deduce that

$$x = \frac{-1 \pm \sqrt{1^2 + 4y^2}}{2y} \in \mathbb{R}\backslash\{1, -1\}.$$

For $y = 0$, we take $x = 0$ and get $f(x) = f(0) = 0 = y$.

So f is surjective.

Problem 9 (Harder)

We let $f_1: A \to B$, $f_2: A \to B$, $g: B \to C$, $h_1: C \to D$ and $h_2: C \to D$ be five functions (Here, A, B, C and D ae subsets of $\mathbb{R}$).

(a) Suppose that g is injective and $g\big(f_1(x)\big) = g\big(f_2(x)\big)$ for all $x \in A$, show that $f_1(x) = f_2(x)$ for all $x \in A$.

(b) Suppose that g is surjective and $h_1\big(g(x)\big) = h_2\big(g(x)\big)$ for all $x \in B$, show that $h_1(x) = h_2(x)$ for all $x \in C$.

☺Solution

(a) Since $f_1(x), f_2(x) \in B$ for any $x \in A$ and g is injective, so

$$g\big(f_1(x)\big) = g\big(f_2(x)\big) \Rightarrow f_1(x) = f_2(x).$$

(b) For any $x \in C$, since g is surjective, there exists $z \in B$ such that $g(z) = x$.
So we have

$$h_1(x) = h_1\big(g(z)\big) = h_2\big(g(z)\big) = h_2(x).$$

Problem 10

We consider a relation $\sim$ on $\mathbb{Z}$ which

$$m \sim n \ \Leftrightarrow m + n \ \textit{is even}$$

(a) Prove that $\sim$ is an equivalent relation.

(b) Determine all equivalent classes.

☺Solution

(a) (Reflexive) Since $x + x = 2x$ is even, so $x \sim x$.
(Symmetric) If $x \sim y$, we have $x + y = y + x$ is even. Then it follows that $y \sim x$.
(Transitive) If $x \sim y$ and $y \sim z$, we get $x + y$ and $y + z$ are even.
Suppose that $x + z$ is odd, then one of x and z (say x) is even and another one (say z) is odd.

- Since $x + y$ is even and x is even, then $y = (x + y) - x$ is also even.
- Since $y + z$ is even and z is odd, then $y = (y + z) - z$ is odd. This contradicts to the first statement.

Thus $x + z$ must also be even and $x \sim z$.
Therefore $\sim$ is an equivalent relation by definition.

(b) If x is odd number, then $x + y$ is even ($x \sim y$) if and only if y is also odd number (as $y = (x + y) - x$). Thus, the equivalent class $[x]$ will contain all odd integers.
If x is even number, then $x + y$ is even ($x \sim y$) if and only if y is even number (as $y = (x + y) - x$). Thus, the equivalent class $[x]$ will contain all even integers.
Summing up, there are two equivalent classes. That is, $[0] = \{\dots, -4, -2, 0, 2, 4, 6, \dots\}$ and $[1] = \{\dots, -3, -1, 1, 3, 5, \dots\}$.

Problem 11

We consider a relation $\sim$ on $\mathbb{Z}$

$$m \sim n \Leftrightarrow |m-5| = |n-5|$$

(a) Prove that $\sim$ is an equivalent relation.

(b) Determine all equivalent classes.

(a) (Reflexive) Since $|x-5| = |x-5|$, so $x \sim x$.
(Symmetric) If $x \sim y$, we have $|x-5| = |y-5|$. Then it follows that $y \sim x$.
(Transitive) If $x \sim y$ and $y \sim z$, we get

$$|x-5| = |y-5| \;\; and \;\; |y-5| = |z-5| \Rightarrow |x-5| = |z-5|.$$

Thus $x \sim z$.
Therefore $\sim$ is an equivalent relation by definition.

(b) Based on the equivalent relation, all elements in a equivalent class have same distance from 5. Thus, the equivalent classes are

$$[a] = \begin{cases} \{5\} & if \;\; a = 5 \\ \{a, 10-a\} & if \;\; a > 5 \\ \{a, 10-a\} & if \;\; a < 5 \end{cases}.$$

Problem 12 (Harder)

We consider an non-empty set A and let $\sim_1$ and $\sim_2$ be two equivalent relations on A.

(a) We let $\sim$ be another relation on A which

$$x \sim y \Leftrightarrow x \sim_1 y \;\; and \;\; x \sim_2 y.$$

(i) Prove that $\sim$ is an equivalent relation.

(ii) We consider an element $a \in A$ and let $[a]$, $[a]_1$ and $[a]_2$ be the equivalence class of a with respect to the equivalent relations $\sim$, $\sim_1$ and $\sim_2$ respectively. Show that $[a] = [a]_1 \cap [a]_2$.

(b) We let $\sim^*$ be another relation on A which

$$x \sim^* y \Leftrightarrow x \sim_1 y \;\; or \;\; x \sim_2 y.$$

Determine if $\sim^*$ is an equivalence relation for any $A, \sim_1, \sim_2$.

(😊Hint: If your answer is yes, provide a mathematical proof. If your answer is no, you need to provide a counter-example and you need to specify the set A and the equivalent relations $\sim_1$ and $\sim_2$.)

😊Solution

(c) (i) (Reflexive) Since $\sim_1$ and $\sim_2$ are equivalent relation, we have $x \sim^1 x$ and $x \sim^2 x$. So $x \sim x$.
(Symmetric) If $x \sim y$, we have $x \sim^1 y \Rightarrow y \sim^1 x$ and $x \sim^2 y \Rightarrow y \sim^2 x$. Then it follows that $y \sim x$.
(Transitive) If $x \sim y$ and $y \sim z$, we get

$$x \sim^1 y, \;\; y \sim^1 z \Rightarrow x \sim^1 z \;\; and$$
$$x \sim^2 y, \;\; y \sim^2 z \Rightarrow x \sim^2 z$$

Thus $x \sim z$.

Therefore $\sim$ is an equivalent relation by definition.

(ii) For any $x \in [a]$, we have $a \sim x$. It follows that $a \sim^1 x$ and $a \sim^2 x$. So $x \in [a]_1$ and $x \in [a]_2$ and thus $x \in [a]_1 \cap [a]_2$. So we get $[a] \subseteq [a]_1 \cap [a]_2$.
For any $x \in [a]_1 \cap [a]_2$, we get $x \in [a]_1$ and $x \in [a]_2$ so that $a \sim^1 x$ and $a \sim^2 x$.
Therefore, $a \sim x$ and $x \in [a]$. So we get $[a]_1 \cap [a]_2 \subseteq [a]$.
Hence, we conclude that $[a] = [a]_1 \cap [a]_2$.

(d) The answer is no. We consider $A = \mathbb{N}$ and define the relations $\sim^1$ and $\sim^2$ as

$$x \sim^1 y \Rightarrow x - y \ \ is\ \ a\, multiple\, of\ 3$$
$$x \sim^2 y \Rightarrow x - y \ \ is\ \ a\, multiple\, of\ 5$$

One can show that both $\sim^1$ and $\sim^2$ are equivalent relations (see Example 13 of Lecture Note 2). On the other hand, we take $x = 3$, $y = 6$ and $z = 1$. We see that

- $x \sim y$ since $x - y = -3 \Rightarrow x \sim^1 y$
- $y \sim z$ since $y - z = 5 \Rightarrow x \sim^2 y$.

However, $x - z = 2$ which is not multiple of 3 or 5. So $x \nsim^1 z$ and $x \nsim^2 z$.
Therefore, $x \nsim z$ and the transitivity property is not satisfied.