Tutorial 11:

1. (a) if $f:(a,b)\mapsto \mathbb{R}$ is differentiable and $\sup\limits_{x\in(a,b)} |f'(x)| \leq M$ for some $M>0$, show that f is uniformly continuous

(b) $\forall\, a<b \in \mathbb{R}$, show that if $f:(a,b)\mapsto\mathbb{R}$ is uniformly continuous, then $\sup\limits_{x\in(a,b)} |f(x)| < +\infty$.

Proof of 1 (a): By definiation, we need to show that

$$\boxed{\begin{array}{l}\forall \varepsilon>0,\ \exists\, \delta>0 \ \text{s.t.} \\ \forall\, x,y \in (a,b) \ \text{s.t.}\ |x-y|<\delta, \ \text{we have}\ |f(x)-f(y)| \leq \varepsilon.\end{array}}$$

In fact, $\forall \varepsilon>0$, set $\delta := \frac{\varepsilon}{M}$. we have then $\forall\, x,y \in (a,b)$ s.t. $|x-y|\leq\delta$,

$$|f(x)-f(y)| \overset{\text{mean-value thm}}{=} |x-y|\cdot|f'(\xi)| \overset{|f'(\xi)|\leq M}{\leq} |x-y|\cdot M \overset{\delta=\frac{\varepsilon}{M}.}{=} M\cdot\frac{\varepsilon}{M} = \varepsilon.$$

Thus the claim holds. #

Proof of (1) b:

By f is continuous. select $\varepsilon_0 = 1$, there exists some δ_0 s.t.

$\forall x, y \in (a,b)$, $|x-y| \leq \delta_0$ we have $|f(x)-f(y)| \leq 1$.

Now selecting N large enough so that $N\delta_0 > \frac{b-a}{2}$,

and set $x_0 := \frac{a+b}{2}$, then for any $x \in (a,b)$, if

① $x > x_0$, we can construct a sequence $x_i := x_0 + i \cdot \frac{\delta}{2}$, $i \geq 1$.

we will have there exists some i_0 s.t. $x_{i_0} \leq x < x_{i_0+1}$, $i_0 \leq N$.

thus

$$|f(x) - f(x_0)| \leq |f(x) - f(x_{i_0})| + \sum_{i=1}^{i_0} |f(x_i) - f(x_{i+1})|$$

$$\leq 1 + \sum_{i=1}^{i_0} 1 = i_0 + 1 \leq N+1.$$

② $x < x_0$. the similar argument with the constructed sequence

$$x_i' := x_0 - i \cdot \frac{\delta}{2}, \quad i \geq 1$$

implies that $|f(x)| \leq |f(x_0)| + N+1$.

thus $\sup_{x \in (a,b)} |f(x)| \leq |f(x_0)| + N+1$. #.

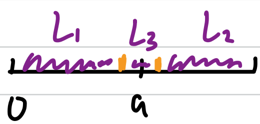
2. Show that if $f:[0,1]\to\mathbb{R}$ is bounded by $\frac{M}{2}$ and continuous in $[0,a)$ and $(a,1]$ for some $a\in(0,1)$, then f is integrable.

Here we would check it by the following integrable criterion

(See lecture 22)

$$\boxed{f \text{ is integrable } \Leftrightarrow \forall \varepsilon>0,\ \exists P \text{ s.t. } U(f,P)-L(f,P)<\varepsilon.}$$

Proof. Now $\forall \varepsilon>0$, we need only find some P of $[0,1]$ s.t. $U(f,P)-L(f,P)<\varepsilon$.



Firstly W.L.O.G. we can assume that ε is small enough so that $0<a-\frac{\varepsilon}{3M}<a+\frac{\varepsilon}{3M}<1$.

then set
$$L_1 := [0,\ a-\tfrac{\varepsilon}{3M}].$$
$$L_2 := [a-\tfrac{\varepsilon}{3M},\ a+\tfrac{\varepsilon}{3M}]$$
$$L_3 := [a+\tfrac{\varepsilon}{3M},\ 1].$$

we have f is continuous in L_1, L_3, thus is integrable, so we can find partitions
$$\begin{cases} P_1: 0=x_0<x_1<\cdots<x_n=a-\frac{\varepsilon}{3M} & \text{of } L_1 \\ P_2: a+\frac{\varepsilon}{3M}=y_0<y_1<\cdots<y_m=1 & \text{of } L_3 \end{cases}$$

such that $\begin{cases} U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{3}. \\ U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{3} \end{cases}$

Now in L_2, we consider the trivial partition P_3:

$a - \frac{\varepsilon}{3M} = z_0 < z_1 = a + \frac{\varepsilon}{3M}$. we will have

$$U(f,P_3) - L(f,P_3) = \left(\sup_{x\in L_2} f(x) - \inf_{x\in L_2} f(x)\right)\frac{\varepsilon}{3M}$$

$$\leq M \cdot \frac{\varepsilon}{3M} = \frac{\varepsilon}{3}.$$

thus if we consider the partition P given by

$$\overbrace{x_0 < x_1 < \cdots < x_n}^{P_1} \overbrace{< y_1}^{P_3} \overbrace{< y_2 < \cdots < y_m}^{P_2}$$

$x_n = z_0$, $y_1 = z_1$

we will have

$$U(f,P) - L(f,P)$$

$$= U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2)$$

$$+ U(f,P_3) - L(f,P_3)$$

$$< \frac{\varepsilon}{3} \cdot 3 = \varepsilon.$$

Thus P is the desired partition. #.

Remark: The arguement can be extended to bounded f with finitely many dis-continuous points. see the next problem.

3. Let $f:[0,1] \longmapsto \mathbb{R}$ be bounded by $\frac{M}{2}$ and

$\{x \in [0,1] : f \text{ is dis-continuous at } x\} = \{x_n : n \in \mathbb{N}\}$

with $(x_n : n \in \mathbb{N})$ a sequence in $[0,1]$ such that $\lim_{n \to \infty} x_n = 0$.

Show that $f(x)$ is integrable in $[0,1]$.

Proof: As in Ex.2, we want to show that $\forall \varepsilon > 0$, exists

a partition P s.t. $U(f,P) - L(f,P) < \varepsilon$.

Firstly by $\lim_{n \to \infty} x_n = 0$, for $\varepsilon_0 < \frac{\varepsilon}{2M}$, exists some N_0

s.t. $\forall n \geq N_0$ $x_n < \varepsilon_0$. So if we denote

$$L_1 := [0, \varepsilon_0], \quad L_2 := [\varepsilon_0, 1].$$

we then have f has only finitely many dis-continuous points

in L_2. Denote these discontinuous points in L_2 by

$$\frac{\varepsilon}{2M} < x_1 < x_2 < \cdots < x_N \leq 1$$

and select $\delta < \frac{\varepsilon}{4NM}$ s.t. $[x_i - \delta, x_i + \delta]$ are disjoint for $1 \leq i \leq N$.

then the interval L_2 is divided into following intervals:

$$L_2 = \left(\bigcup_{i=1}^{N} I_i\right) \cup \left(\bigcup_{i=1}^{N} J_i\right), \text{ with } I_i := L_2 \cap [x_i - \delta, x_i + \delta], \ 1 \leq i \leq N,$$

$$J_i := \begin{cases} [\frac{\varepsilon}{2M}, x_1 - \delta] & i = 1 \\ [x_i + \delta, x_{i+1} - \delta] & 1 < i < N \\ [x_N + \delta, 1] & i = N. \end{cases}$$

then ① by f is integrable in every non-empty J_i, we have exists partitions P_i of J_i s.t. $U(f, P_i) - L(f, P_i) < \frac{\varepsilon}{4N}$.

② Consider trivial partition Q_i of I_i:

$\max\{\varepsilon_0, x_i - \delta\} = z_{0,i} < z_{1,i} = \min\{x_i + \delta, 1\}$.

then we have $U(f, Q_i) - L(f, Q_i) < \frac{M \cdot \varepsilon}{4N/M} = \frac{\varepsilon}{4N}$.

thus if we construct $\hat{P}$ of $[\varepsilon_0, 1]$ by letting

$$\hat{P} = \bigcup_{i=1}^{N} P_i \cup \bigcup_{i=1}^{N} Q_i, \text{ then}$$

$$U(f, \hat{P}) - L(f, \hat{P}) = \sum_i [U(f, P_i) - L(f, P_i) + U(f, Q_i) - L(f, Q_i)] < \frac{\varepsilon}{2}.$$

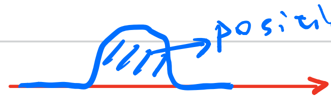
Now by $\varepsilon_0 < \frac{\varepsilon}{2M}$, we have for the partition $\hat{Q}$ of $[0, \varepsilon_0]$ given by $0 = y_0 < y_1 = \varepsilon_0$, we have

$$U(f, \hat{Q}) - L(f, \hat{Q}) \leq \varepsilon_0 \cdot M = \frac{\varepsilon}{2}.$$

thus if we set P as the union of $\hat{P}, \hat{Q}$, we will have then

$U(f, P) - L(f, P) < \varepsilon$, thus P is the desired partition of $[0, 1]$. #.

4. If f is continuous in $[a,b]$, $f(x)\geq 0$ for all $x\in[a,b]$, and $\int_a^b f(x)=0$. Show that $f(x)=0$ for all $x\in[a,b]$.

idea: 

Proof:

Suppose $\exists x_0\in[a,b]$ s.t. $f(x_0)>0$. From the symbol-preserving property of continuous function, we know:

$$\exists B_r(x_0):=(x_0-r,\ x_0+r)\subset[a,b]\ \text{s.t. } f(x)>0 \text{ for } x\in B_r(x_0)$$

(here we assume $x_0\in(a,b)$; the case for $x_0=a$ or b is similar). If we further narrow the neighborhood, we can have:

$$f(x)>0 \text{ for } x\in[x_0-r,\ x_0+r]$$

→ closed.

Moreover, since f continuous, f can attain its minimum on $\overline{B_r(x_0)}$, say: $f(x)\geq m:=\min\limits_{x\in\overline{B_r(x_0)}} f(x)>0$.

Hence, we observe that.

$$\int_a^b f(x)\,dx = \int_a^{x_0-r} \underset{\geq 0}{f(x)}\,dx + \int_{x_0-r}^{x_0+r} f(x)\,dx + \int_{x_0+r}^{b} \underset{\geq 0}{f(x)}\,dx$$

$$\geq \int_{x_0-r}^{x_0+r} f(x)\,dx$$

$\geq m$ (constant function: integrable)

$$\geq \int_{x_0-r}^{x_0+r} m\,dx = m\cdot 2r>0.$$

a contradiction. $\square$