MATH 2033 Mathematical Analysis
Midterm Exam
Spring 2019

29 March 2019

Student Name: ______________________________

Student Id: ______________________________

Tutorial Section: ______________________________

Instructions

Please read the following and sign in the blank provided below and sign in the blank below.

1. You **MUST** seat according to the **Seating Plan**.
2. **DO NOT OPEN** the exam until you are told to.
3. This is a **CLOSED BOOK** exam.
4. All mobile phones and communication devices should be switched **OFF**.
5. Only calculators approved by **HKEAA** allowed.
6. Answer **ALL** the questions.
7. You must **SHOW YOUR WORK** to receive credits in all questions. Answers alone (whether correct or not) will not receive any credit.

Score Summary (Examiner Only)

Question No	Points	Scores
1	15	
2	25	
3	25	
4	35	
Total	100	

Integrity Statement

I have neither given nor received any unauthorised aid during this exam. The answers submitted are my own work. I understand that sanctions will be imposed if I am found to have violated the University's regulations governing academic integrity.

Signature: ______________________________

1. (15 points) .

1.1 Negating the following statements.

$$\forall \epsilon > 0, \exists \delta > 0, \text{such that } \forall\, x, 0 < |x - x_0| < \delta \implies |\frac{f(x) - f(x_0)}{x - x_0} - L| < \epsilon.$$

1.2 Find $\cap_{n\in\mathbb{N}}(0, \frac{1}{n})$ and justify your answer.

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2. (25 points) .
   2.1 Write down the definition of infimum. State and prove the infimum property.
   2.2 Determine if the following set A has an infimum. If it exists, find it and justify your answer.

$$A = \{x + y^2 : x \in [0, 1] \cap \mathbb{Q}, y \in [0, 1] \setminus \mathbb{Q}\}$$

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3. (25 points) .

3.1 Let $C \geq 1$, prove that

$$\lim_{n \to +\infty} C^{\frac{1}{n}} = 1.$$

3.2 Determine whether the sequence x_n defined by

$$x_1 = 3, x_{n+1} = 3 + \frac{4}{x_n} \quad \text{for } n \geq 1,$$

converges or not. If it converges, justify the convergence and find the limit.

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4. (35 points) .

4.1 Write down the definition of Cauchy sequence.

4.2 Let $\{x_n\}$, $\{y_n\}$ be two Cauchy sequences, show that both $\{x_n - y_n\}$ and $\{x_n y_n\}$ are Cauchy sequences by checking the definition of Cauchy sequence.

4.3 Let S be the set of all Cauchy sequences in $\mathbb{Q}$. More precisely,

$$S = \{\{x_n\} : \{x_n\} \text{ is a Cauchy sequence s.t } x_n \in \mathbb{Q} \text{ for all } n \in \mathbb{N}\}.$$

Determine if S is countable and justify your answer.

4.4 Let S be defined above, let $\{x_n\} \in S$, we say that $\{x_n\}$ is positive iff there exists $\delta > 0, \delta \in \mathbb{Q}$ and $k \in \mathbb{N}$ s.t $x_n > \delta$ for all $n \geq k$. We say that $\{x_n\} < \{y_n\}$ iff $\{y_n - x_n\}$ is positive. Show that

$$\forall \{x_n\}, \{y_n\}, \{z_n\} \in S,$$

if $\{x_n\} < \{y_n\}$, and $\{z_n\}$ is positive, then

$$\{x_n z_n\} < \{y_n z_n\}.$$

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