MATH 2031 Introduction to Real Analysis

April 16, 2013

Tutorial Note 19

<u>**Sequences and Series of Functions**</u>

(I) **Extended real number system**

(i) **Definition:**
$[-\infty,+\infty]=\mathbb{R}\cup\{-\infty,+\infty\}$ is called the extended real number system.

(ii) **Ordering in** $[-\infty,+\infty]$

- Usual ordering in $\mathbb{R}$
- $\forall x\in\mathbb{R}$, $-\infty<x<+\infty$

(iii) **Supremum and Infimum in** $[-\infty,+\infty]$
For any non empty set $S\subseteq[-\infty,+\infty]$, $+\infty$ is an upper bound of S and $-\infty$ is a lower bound of S.
$\sup S=$ least upper bound of S in $[-\infty,+\infty]$ and $\inf S=$ greatest lower bound of S in $[-\infty,+\infty]$

(iv) **Arithmetics in** $[-\infty,+\infty]$

- Usual arithmetics in $\mathbb{R}$
- For all $x\in\mathbb{R}\cup\{+\infty\}$ and $c>0$,

$$\begin{aligned}x+(+\infty)&=+\infty=(+\infty)+x\\ c(+\infty)&=+\infty=(+\infty)c\\ c(-\infty)&=-\infty=(-\infty)c\end{aligned}$$

- For all $x\in\mathbb{R}\cup\{-\infty\}$ and $c<0$,

$$\begin{aligned}x+(-\infty)&=-\infty=(+\infty)-x\\ c(-\infty)&=+\infty=(-\infty)c\\ c(+\infty)&=-\infty=(+\infty)c\end{aligned}$$

- $|+\infty|=+\infty=|-\infty|$

(II) **Infinite Limit**

(i) Let $x_1,x_2\cdots$ be sequence in $[-\infty,+\infty]$.
Define

$$\lim_{n\to\infty}x_n=+\infty\iff\forall\text{ real }r>0,\exists K\in\mathbb{N}\text{ such that }n\geq K\ \Rightarrow\ x_n>r.$$
$$\lim_{n\to\infty}x_n=-\infty\iff\forall\text{ real }r>0,\exists K\in\mathbb{N}\text{ such that }n\geq K\ \Rightarrow\ x_n<-r.$$

(ii) **Subsequence:**
Let $x_1,x_2,x_3,\cdots$ be a sequence, $x_{n_1},x_{n_2},x_{n_3},\cdots$ is a subsequence of $x_1,x_2,x_3,\cdots$ if $n_1<n_2<\cdots$.

$$\mathcal{L}=\left\{z\in[-\infty,+\infty]\,\middle|\,\exists\text{ subsequence }x_{n_1},x_{n_2},x_{n_3},\cdots\text{ such that }\lim_{k\to\infty}x_{n_k}=z\right\}$$

is called the set of subsequential limits of $x_1,x_2,x_3,\cdots$.

(iii) **Definition (limsup and liminf):**

$$\limsup_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n = \sup \mathcal{L} \text{ in } [-\infty, +\infty]$$
$$\liminf_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n = \inf \mathcal{L} \text{ in } [-\infty, +\infty]$$

(iv) **Properties of limsup and liminf:**

(i) For every sequence $x_1, x_2, x_3, \cdots$, $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$ always exist since $\mathcal{L} \neq \varnothing$. Also

$$\liminf_{n\to\infty} x_n = \inf \mathcal{L} \leq \sup \mathcal{L} = \limsup_{n\to\infty} x_n$$

(ii) If $\{x_n\}$ is not bounded above in $\mathbb{R}$, then $+\infty \in \mathcal{L}$, so $\limsup\limits_{n\to\infty} x_n = \sup \mathcal{L} = +\infty$.
If $\{x_n\}$ is not bounded below in $\mathbb{R}$, then $-\infty \in \mathcal{L}$, so $\liminf\limits_{n\to\infty} x_n = \inf \mathcal{L} = +\infty$.

(iii) $\lim\limits_{n\to\infty} x_n = z \in [-\infty, +\infty] \iff \mathcal{L} = \{z\} \iff \liminf\limits_{n\to\infty} x_n = z = \limsup\limits_{n\to\infty} x_n$

(iv)

$$\begin{aligned}
&\limsup_{n\to\infty}(-x_n) = \sup(-\mathcal{L}) = -\inf \mathcal{L} = -\liminf_{n\to\infty} x_n \\
&\forall c > 0, \\
&\limsup_{n\to\infty}(cx_n) = \sup(c\mathcal{L}) = c \inf \mathcal{L} = c \liminf_{n\to\infty} x_n \\
&\liminf_{n\to\infty}(cx_n) = \inf(c\mathcal{L}) = c \sup \mathcal{L} = c \limsup_{n\to\infty} x_n \\
&\forall c \in \mathbb{R}, \\
&\limsup_{n\to\infty}(c + x_n) = \sup(c + \mathcal{L}) = c + \inf \mathcal{L} = c + \liminf_{n\to\infty} x_n \\
&\liminf_{n\to\infty}(c + x_n) = \inf(c + \mathcal{L}) = c + \sup \mathcal{L} = c + \limsup_{n\to\infty} x_n
\end{aligned}$$

(v) M_k**-Theorem**
For every sequence $\{x_n\}$, define $M_k = \sup\{x_k, x_{k+1}, x_{k+2}, \cdots\}$. Then

(i) $M_1 \geq M_2 \geq M_3 \geq \cdots$ and
(ii) $\limsup\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} M_k \in \mathcal{L}$.

(vi) m_k**-Theorem**
For every sequence $\{x_n\}$, define $m_k = \inf\{x_k, x_{k+1}, x_{k+2}, \cdots\}$. Then

(i) $m_1 \leq m_2 \leq m_3 \leq \cdots$ and
(ii) $\liminf\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} m_k \in \mathcal{L}$.

(vii) **Strong Form of Root test**
For a sequence $\{a_n\}$ of real numbers,

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \begin{cases} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges absolutely;} \\ > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges} \end{cases}$$

(viii) **Strong Form of Ratio test**
For a sequence $\{a_n\}$ of non-zero real numbers,

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges absolutely;}$$
$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Problem 1 Find the limit superior and limit inferior of each of the following sequence:

(i) $x_n = n\sin(e^{-n})$ for $n \in \mathbb{N}$

(ii) $y_n = 2^{n\cos(\frac{n\pi}{7})}$ for $n \in \mathbb{N}$

Solution:

(i) Notice that

$$\lim_{n\to\infty} n\sin(e^{-n}) = \lim_{n\to\infty} \frac{n}{e^n}\frac{\sin(e^{-n})}{e^{-n}} = \lim_{n\to\infty} \frac{n}{e^n} \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = \lim_{n\to\infty} \frac{1}{e^n} = 0,$$

in which we have let $\theta = e^{-}n$. Since the limit exists, we get

$$\limsup_{n\to\infty} x_n = 0 = \liminf_{n\to\infty} x_n.$$

(ii) Since $0 < x_n < +\infty$, $\mathcal{L} \subseteq [0, +\infty]$.
As 2^x is increasing and $|\cos x| \leq 1$, we see that

If $n = 7(2k+1)$, then $x_n = 2^{-7(2k+1)}$ and this subsequence $\{x_n\}$ converges to zero.
Thus $\liminf\limits_{n\to\infty} x_n = 0$.

If $n = 14k$, then $x_n = 2^{14k}$ and this subsequence $\{x_n\}$ converges to $+\infty$.
Thus $\limsup\limits_{n\to\infty} x_n = +\infty$.

Problem 2 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

Solution:
Let $M_k = \sup\{a_k, a_{k+1}, \cdots\}$ and $N_k = \sup\{b_k, b_{k+1}, \cdots\}$, then we get

$$\sup\{a_k + b_k, a_{k+1} + b_{k+1}\} \leq M_k + N_k.$$

Taking limit as $k \to \infty$ on both sides, we have

$$\lim_{k\to\infty} \sup\{a_k + b_k, a_{k+1} + b_{k+1}\} \leq \lim_{k\to\infty}(M_k + N_k) = \lim_{k\to\infty} M_k + \lim_{k\to\infty} N_k$$

Then by M_k-theorem, we have

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Problem 3 Find $\limsup\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, $\liminf\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$, $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}$ and $\liminf\limits_{n\to\infty} \sqrt[n]{|a_n|}$, where

$$a_n = \begin{cases} \frac{n}{2^n} & \text{if } n \text{ is odd} \\ \frac{n^2}{2^n} & \text{if } n \text{ is even} \end{cases}$$

Solution:
Note that

$$0 < \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \dfrac{(n+1)^2}{2n} & \text{if } n \text{ is odd} \\[2ex] \dfrac{n+1}{2n^2} & \text{if } n \text{ is even} \end{cases} < +\infty,$$

and that for odd n, $\left|\frac{a_{n+1}}{a_n}\right| \to +\infty$ as $n \to \infty$; while for even n, $\left|\frac{a_{n+1}}{a_n}\right| \to 0$ as $n \to \infty$.
So $\limsup\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = +\infty$ and $\liminf\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$.

Now consider

$$0 < \sqrt[n]{|a_n|} = \begin{cases} \dfrac{\sqrt[n]{n}}{2} & \text{if } n \text{ is odd} \\ \\ \dfrac{(\sqrt[n]{n})^2}{2} & \text{if } n \text{ is even} \end{cases} < +\infty$$

and note that $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. Then since

$$\lim_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{2},$$

By intertwining sequence theorem, $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \dfrac{1}{2}$.

Thus, $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} = \dfrac{1}{2} = \liminf\limits_{n\to\infty} \sqrt[n]{|a_n|}$.

Remark:

If we consider a series $\sum\limits_{n=1}^{\infty} a_n$ where a_n is as in this problem, then we cannot get any conclusion about the convergence of $\sum\limits_{n=1}^{\infty} a_n$ from the Strong form of ratio test.

However, from the Strong form of root test, we can see that $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely.