MATH2033 Mathematical Analysis (2021 Spring)
Suggested Solution of Assignment 2

Problem 1

(a) Find the supremum and infimum of the following set:

$$S = \{e^{\sqrt{x}} | x \in \mathbb{Q} \cap (0,1)\}.$$

(b) We consider a set defined by

$$T = \left\{n \cos\frac{n\pi}{2} | n \in \mathbb{N}\right\}.$$

Show that the infimum of T does not exist in $\mathbb{R}$.

😊Solution

(a) For any $x \in \mathbb{Q} \cap (0,1)$ which $0 < x < 1$, we have

$$1 = e^{\sqrt{0}} < e^{\sqrt{x}} < e^{\sqrt{1}} = e.$$

So 1 and e are lower bound and upper bound of S respectively. By completeness axiom, the supremum and infimum exists.

- We first argue that $\sup S = e$. For any $\varepsilon > 0$, it follows from density of rational number that there exists $x \in \mathbb{Q}$ such that

  $$0 < (\ln(e - \varepsilon))^2 < x < 1 \Leftrightarrow e^{\sqrt{x}} > e - \varepsilon.$$

  for any $e > \varepsilon > 0$.
  It also implies that $e^{\sqrt{x}} > e - \varepsilon \geq e - \varepsilon^*$ for any $\varepsilon^* > e > \varepsilon > 0$
  So $e - \varepsilon$ cannot be upper bound for any $\varepsilon > 0$. Thus $\sup S = e$.

- We then argue that $\inf S = 1$. For any $\varepsilon > 0$, it follows from density of rational number that there exists $x \in \mathbb{Q}$ such that

  $$0 < x < (\ln(1 + \varepsilon))^2 < 1 \Leftrightarrow e^{\sqrt{x}} < 1 + \varepsilon.$$

  for any $e - 1 > \varepsilon > 0$.
  It also implies that $e^{\sqrt{x}} < 1 + \varepsilon \leq 1 + \varepsilon^*$ for any $\varepsilon^* > e - 1$
  So $1 + \varepsilon$ cannot be lower bound for any $\varepsilon > 0$. Thus $\inf S = 1$.

(b) Since for $n = 2$, we have $2\cos\frac{2\pi}{2} = -2$. So $M \geq 0$ cannot be lower bound.
For any $M < 0$, it follows Archimedean principle that there exists $K \in \mathbb{N}$ such that $4K + 2 > -M$.
This implies that when $n = 4K + 2$

$$(4K + 2)\cos\frac{(4K + 2)\pi}{2} = -(4K + 2) < M.$$

So $M < 0$ is not lower bound as well.
Hence, T has no lower bound and thus $\inf T$ does not exist as real number.

Problem 2

(a) We let $A \subseteq \mathbb{R}$ be a bounded non-empty subset of real numbers and let $S \subseteq A$ be non-empty subset of real numbers. Prove that

$$\inf A \leq \inf S \leq \sup S \leq \sup A.$$

(b) We let A, B be two bounded subsets of *positive real numbers*. We define

$$C = \{ab | a \in A, b \in B\}.$$

(i) Show that $\sup C = \sup A \sup B$.

(ii) Is the result (i) valid if either A or B contain negative number? Explain your answer.
(*Note: If your answer is yes, give a mathematical proof. If your answer is no, you need to give a counter-example.)

😊Solution

(a) We first argue that $\inf A \leq \inf S$.
Suppose that $\inf A > \inf S$, note that $\inf A$ is not lower bound of S, there exists $x \in S$ such that
$$\inf A > x \geq \inf S$$
As $S \subseteq A$, it follows that $x \in S \subseteq A$. So $\inf A$ is not lower bound of A and there is contradiction.
Next, we argue that $\inf A \leq \inf S$.
Suppose that $\inf A > \inf S$, note that $\inf A$ is not lower bound of S, there exists $x \in S$ such that
$$\inf A > x \geq \inf S$$
As $S \subseteq A$, it follows that $x \in S \subseteq A$. So $\inf A$ is not lower bound of A and there is contradiction.

(b) **(i)** Since both A, B are bounded so that $\sup A$ and $\sup B$ both exists. For any $ab \in C$, we have (as $a, b > 0$)
$$ab \leq a \sup B \leq \sup A\,(\sup B)$$
So $\sup A\,(\sup B)$ is the upper bound of C.

Next, we argue that $\sup C = \sup A\,(\sup B)$.
For any $\varepsilon > 0$

- We pick $\varepsilon_1 = \min\left(\sup A, \frac{\varepsilon}{2 \sup B}\right)$, there exists $a \in A$ such that $a > \sup A - \varepsilon_1 > 0$
- We pick $\varepsilon_2 = \min\left(\sup B, \frac{\varepsilon}{2 \sup A}\right)$, there exists $b \in B$ such that $B > \sup B - \varepsilon_2 > 0$

It follows that
$$ab > (\sup A - \varepsilon_1)(\sup B - \varepsilon_2)$$
$$> \sup A \sup B - \varepsilon_1 \sup B - \varepsilon_2 \sup A + \varepsilon_1 \varepsilon_2$$
$$\overset{\varepsilon_1, \varepsilon_2 > 0}{\gtrsim} \sup A \sup B - \varepsilon_1 \sup B - \varepsilon_2 \sup A$$
$$> \sup A \sup B - \left(\frac{\varepsilon}{2 \sup B}\right) \sup B - \left(\frac{\varepsilon}{2 \sup A}\right) \sup A$$
$$> \sup A \sup B - \varepsilon.$$
So $\sup A \sup B - \varepsilon$ is not upper bound of C. So we conclude that $\sup C = \sup A\,(\sup B)$.

(ii) We take $A = [-1,0]$ and $B = [-1,0]$, we have $\sup A = \sup B = 0$ But $\sup C \neq \sup A \sup B = 0$ since $\underbrace{(-1)(-1)}_{\in C} = 1 > 0$ so that 0 is not upper bound of C. (*In fact, one can verify that $\sup C = 1$)

Problem 3

We let $a \in \mathbb{R}$ be a real number. Show that there exists a sequence of rational number $\{q_n\}$ (where $q_n \in \mathbb{Q}$) such that $\{q_n\}$ converges to a (i.e. $\lim_{n\to\infty} q_n = a$).

Solution

For any $\varepsilon = \frac{1}{n}$ with $n \in \mathbb{N}$, one can deduce from density of rational number that there exists $q_n \in \mathbb{Q}$ such that

$$a - \frac{1}{n} < q_n < a.$$

By repeating this process for any positive integer n, we obtain a sequence of rational number $\{q_n\}$. Since $\lim_{n\to\infty}\left(a - \frac{1}{n}\right) = \lim_{n\to\infty} a = a$, it follows from sandwich theorem that $\{q_n\}$ converges and $\lim_{n\to\infty} q_n = a$.

Problem 4

Prove the following fact using the definition of limits

(a) $\lim_{n\to\infty} \cos\left(a + \frac{b}{n}\right) = \cos a$, where a, b are positive number.

(b) $\lim_{n\to\infty} \sqrt{b_n} = \sqrt{b}$, where $\{b_n\}$ is a convergent sequence with $\lim_{n\to\infty} b_n = b > 0$.

Solution

(a) For any $\varepsilon > 0$, we deduce from Archimedean property that there exists $K \in \mathbb{N}$ such that

$$K > \max\left(\frac{\pi}{2b}, \frac{\pi}{b \sin^{-1}\frac{\varepsilon}{2}}\right) \Leftrightarrow \frac{b}{n} < \frac{\pi}{2} \;\; and \;\; \frac{b}{n} < \sin^{-1}\frac{\varepsilon}{2} \Rightarrow \sin\frac{b}{n} < \frac{\varepsilon}{2}.$$

It follows that for $n \geq K$

$$\left|\cos\left(a + \frac{b}{n}\right) - \cos a\right| =^{(*)} \left|-2\sin\left(2a + \frac{b}{n}\right)\sin\frac{b}{n}\right| \leq 2\left|\sin\frac{b}{n}\right| \overset{\frac{b}{n}<\frac{\pi}{2}}{\cong} 2\sin\frac{b}{n}$$
$$< 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$

So $\lim_{n\to\infty} \cos\left(a + \frac{b}{n}\right) = \cos a$ by definition.

(*Note: The equality follows from sum-to-product formula which states that

$$\cos A - \cos B = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2}.)$$

(b) (We take $b_n \geq 0$ in order that $\sqrt{b_n}$ is well-defined as real number)

For any $\varepsilon > 0$, as $\lim_{n\to\infty} b_n = b$, there exists $K \in \mathbb{N}$ such that

$$|b_n - b| < \varepsilon\sqrt{b} \;\; for \;\; n \geq K$$

Then for $n \geq K$, we have

$$\left|\sqrt{b_n} - \sqrt{b}\right| = \left|\frac{b_n - b}{\sqrt{b_n} + \sqrt{b}}\right| \leq \frac{|b_n - b|}{\sqrt{b}} < \frac{\varepsilon\sqrt{b}}{\sqrt{b}} = \varepsilon.$$

So $\lim_{n\to\infty} \sqrt{b_n} = \sqrt{b}$ by definition.

Problem 5

We let $\{x_n\}$ be a sequence defined by

$$x_1 = 0.4, \qquad x_{n+1} = \frac{x_n^3 + 2}{3} \ \ for \ \ n \in \mathbb{N}.$$

Show that $\{x_n\}$ converges and find the limits.

😊Solution

To prove the convergence, we shall argue that

- $\{x_n\}$ is increasing (i.e. $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$) and
- $0.4 \leq x_n \leq 1$ for all $n \in \mathbb{N}$.

To prove the second statement, we note that $0.4 \leq x_1 = 0.4 \leq 1$. Assuming that $0.4 \leq x_k \leq 1$ for some $k \in \mathbb{N}$, then for $n = k + 1$, we have

$$0.4 < 0.688 = \frac{0.4^3 + 2}{3} \leq x_{k+1} = \frac{x_k^3 + 2}{3} \leq \frac{1^3 + 2}{3} = 1.$$

So the case for $n = k + 1$ holds. It follows from mathematical induction that $0.4 \leq x_n \leq 1$ for all $n \in \mathbb{N}$.

To prove the first statement, we note that

$$x_2 = \frac{x_1^3 + 2}{3} = \frac{0.4^3 + 2}{3} = 0.688 \geq 0.4 = x_1$$

Assuming that $x_{k+1} \geq x_k$ for some $k \in \mathbb{N}$, then for $n = k + 1$, we consider

$$x_{k+2} - x_{k+1} = \frac{x_{k+1}^3 + 2}{3} - \frac{x_k^3 + 2}{3} = \frac{x_{k+1}^3 - x_k^3}{3}$$

$$= \frac{\overbrace{(x_{k+1} - x_k)}^{\geq 0}\overbrace{(x_{k+1}^2 + x_k x_{k+1} + x_k^2)}^{\geq 0 \ \ as \ \ x_k \geq 0.4}}{3} \geq 0$$

So we have $x_{k+2} \geq x_{k+1}$ and the statement is valid for $n = k + 1$. . It follows from mathematical induction that $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.

Since the sequence is increasing and bounded from above, it follows from montone sequence theorem that the sequence $\{x_n\}$ converges.

To get the limits, we let $x = \lim_{n\to\infty} x_n$. From the recurrence relation (take $n \to \infty$), we get

$$x = \frac{x^3 + 2}{3} \Rightarrow x^3 - 3x + 2 = 0 \Rightarrow (x - 1)(x^2 + x - 2) = 0$$

$$\Rightarrow (x - 1)(x + 2) = 0$$

$$\Rightarrow x = 1 \ \ or \ \ x = -2$$

$$\Rightarrow x = 1$$

(*Note: The last case is rejected since $0.4 \leq x_n \leq 1$ for all $n \in \mathbb{N}$ so that $0.4 \leq x = \lim_{n\to\infty} x_n \leq 1$ by limit inequality.)

Problem 6 (Harder)

We let $\{x_n\}$ be a sequence of positive real numbers.

(a) Suppose that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L < 1$, show that $\{x_n\}$ converges and $\lim_{n\to\infty} x_n = 0$.

(☺Hint: We let $L < r < 1$ be a number. One can apply the definition of limits to $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$ with $\varepsilon < r - L$ and argue that $\frac{x_{n+1}}{x_n} < r$ when n is greater than some positive integer $K \in \mathbb{N}$.)

(b) Suppose that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L > 1$, show that $\{x_n\}$ does not converge.

(c) Suppose that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L = 1$,

(i) Find an example of $\{x_n\}$ which $\{x_n\}$ converges

(ii) Find another example of $\{x_n\}$ which $\{x_n\}$ does not converges.

☺Solution

(a) We let $L < r < 1$ be a number. Since $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$, we take $\varepsilon < r - L$ and There exists $K \in \mathbb{N}$ such that for $n \geq K$

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon = r - L \Leftrightarrow -(r - L) < \frac{x_{n+1}}{x_n} - L < r - L \Rightarrow \frac{x_{n+1}}{x_n} < r$$

$$\Rightarrow x_{n+1} < rx_n.$$

Using this inequality, one can deduce that for any $n > K$

$$x_n < rx_{n-1} < r^2 x_{n-2} < \cdots < r^{n-K} x_K$$

By Archimedean principle, there exists $K_1 \in \mathbb{N}$ such that

$$K_1 > K - \frac{\ln \frac{\varepsilon}{x_K}}{\ln r} \Leftrightarrow r^{K_1 - K} x_K < \varepsilon$$

By taking $K^* = \max(K, K_1)$, then we have for $n \geq K^*$,

$$|x_n - 0| = x_n < r^{n-K} x_K < r^{K_1 - K} x_K < \varepsilon.$$

So $\lim_{n\to\infty} x_n = 0$ by definition.

(b) It suffices to prove that the sequence is not bounded from above (since any convergent sequence must be bounded).

We let $r \in (1, L)$ be a number. Since $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$, we take $\varepsilon < L - r$ and There exists $K \in \mathbb{N}$ such that for $n \geq K_3$

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon = L - r \Leftrightarrow -(L - r) < \frac{x_{n+1}}{x_n} - L < L - r \Rightarrow \frac{x_{n+1}}{x_n} > r$$

$$\Rightarrow x_{n+1} > rx_n.$$

Using this inequality, one can deduce that for any $n > K_3$

$$x_n > rx_{n-1} > r^2 x_{n-2} > \cdots > r^{n-K_3} x_{K_3}$$

For any $M > 0$, one can deduce from Archimedean principle that there exists $K_4 \in \mathbb{N}$ such that

$$K_4 > K_3 + \frac{\ln \frac{M}{x_{K_3}}}{\ln r} \Leftrightarrow r^{K_4 - K_3} x_{K_3} > M.$$

This implies that $x_{K_4} r^{K_4 - K_3} x_{K_3} > M$.

It shows that any $M > 0$ is not upper bound for $\{x_n\}$. As $x_n > 0$, $M \leq 0$ cannot be upper bound also. So $\{x_n\}$ is not bounded and hence does not converge.

(c) (i) We take $x_n = 1$ for all $n \in \mathbb{N}$. One can verify that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{1}{1} = 1$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} 1 = 1$.

(ii) We take $x_n = n$ for all $n \in \mathbb{N}$. One can verify that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$ but $\lim_{n\to\infty} x_n = \lim_{n\to\infty} n = \infty$. So $\{x_n\}$ diverges to ∞.

****End of Assignment 2****