Homework 1

Monday, 20 September 2021 5:04 AM

Math203...

Math 2033 (Homework 1) L1 Fall 2021

Problems (Due September 27 at 11:59 pm)

1. Prove that the set $W = \{x \in \mathbb{R} : x^3 - 2x + 5 \in \mathbb{Q}\}$ is countable. (Here $\mathbb{R}$ is the set of all real numbers and $\mathbb{Q}$ is the set of all rational numbers.)

~~2~~ Prove that the set $S = \{b : x^4 + bx - 5 = 0$ has a rational root$\}$ is countable.

3. Determine if the set $B = \{x + \sqrt{2}y : x, y \in \mathbb{N}\}$ is countable or not. (Here $\mathbb{N} = \{1, 2, 3, \cdots\}$).

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① Prove that the set $W = \{x \in \mathbb{R} : x^3 - 2x + 5 \in \mathbb{Q}\}$ is countable.

$$W = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{Q}, y = x^3 - 2x + 5\}$$

$y = x^3 - 2x + 5$ has at most 3 solutions $\{x_1, x_2, x_3\}$ for a given y.

Since $y \in \mathbb{Q}$ is countable, we can list it as $q_1, q_2, q_3 \dots$

For a given q_i, it corresponds to $\{x_1, x_2, x_3\}$.

$\forall q_i \in \mathbb{Q}$, We can write it as: $(x_1, q_1), (x_2, q_1), (x_3, q_1),$
$(x_1, q_2), (x_2, q_2), (x_3, q_2),$
$(x_1, q_3), (x_2, q_3), (x_3, q_3) \dots$

which is still countable.

Then, we can remove any duplicates and empty solutions and non-real values for x_1, x_2, x_3.

After that we get W which is countable.

③ $B = \{x + \sqrt{2}y : x, y \in \mathbb{N}\}$.

For every $(x,y) \in \mathbb{N} \times \mathbb{N}$, Define $f : \mathbb{N} \times \mathbb{N} \to B$ by letting $f(x,y) = x + \sqrt{2}y$, (with f^{-1} sending $x + \sqrt{2}y$ back to (x,y)) is a bijection. Since $\mathbb{N} \times \mathbb{N}$ is countable by the product theorem, so B is countable by the bijection theorem.