MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)
Tutorial Note #6

Series (Part 2)

In this note, we will go over some tests which are useful for arbitrary series

(Alternating Series Test) If $c_1 \geq c_2 \geq c_3 \geq \cdots ....$ and $\lim_{k\to\infty} c_k = 0$, then $\sum_{k=1}^{\infty} c_k$ converges
(Absolute Convergence Test) If $\sum_{k=1}^{\infty} \lvert a_k \rvert$ converges, then $\sum_{k=1}^{\infty} a_k$ converges

In many case, the series may not be non-negative series (i.e. some terms are positive and some terms are negative) which the integral test, comparison test, limit comparison test cannot be applied. The use of this test is to help you to convert this series into non-negative series, which we can use all the previous tests.
(Note: This only works when the series is suspected to be convergent)

Example 1 Check whether the series $$\sum_{k=1}^{\infty} (\sin^3 k) \sin^3\left(\frac{1}{k}\right)$$ Converges or not

Solution:
Since $\sin^3 k$ can be negative, let first consider

$$\sum_{k=1}^{\infty} \left|(\sin^3 k) \sin^3\left(\frac{1}{k}\right)\right| = \sum_{k=1}^{\infty} |\sin^3 k| \sin^3\left(\frac{1}{k}\right) \leq \sum_{k=1}^{\infty} \sin^3\left(\frac{1}{k}\right) \qquad (\sin k \leq 1)$$

We can apply previous tests now

Note $\sin\left(\frac{1}{k}\right) \approx \frac{1}{k}$ when k is large, then $\sin^3\left(\frac{1}{k}\right) \approx \frac{1}{k^3}$ when k is large

Note that $\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges by p-test

Then by limit comparison test, $\sum_{k=1}^{\infty} \sin^3\left(\frac{1}{k}\right)$ converges

Next by comparison test, $\sum_{k=1}^{\infty} \left|(\sin^3 k) \sin^3\left(\frac{1}{k}\right)\right|$ converges

Finally by absolute convergence test, $\sum_{k=1}^{\infty} (\sin^3 k) \sin^3\left(\frac{1}{k}\right)$ converges.

☺Exercise 1

Check whether the series $\sum_{k=1}^{\infty} \sin^k k\, e^{1-k}$ converges

Lastly, we will provide two useful tests in checking convergence of series

(Root Test)

If $\lim_{k\to\infty} \sqrt[k]{|a_k|}$ exists, then

(1) $\lim_{k\to\infty} \sqrt[k]{|a_k|} < 1 \rightarrow \sum_{k=1}^{\infty} a_k$ converges absolutely

(2) $\lim_{k\to\infty} \sqrt[k]{|a_k|} > 1 \rightarrow \sum_{k=1}^{\infty} a_k$ diverges

(3) $\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1$, the test does not tell whether the series converges or not

(Ratio Test)

If $a_k \neq 0$ for every k and $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$ exists, then

(1) $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| < 1 \rightarrow \sum_{k=1}^{\infty} a_k$ converges absolutely

(2) $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1 \rightarrow \sum_{k=1}^{\infty} a_k$ diverges

(3) $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = 1$, the test does not tell whether the series converges or not

Example 2

Check whether the series

$$\sum_{k=1}^{\infty} \frac{k \sin^k k}{e^{k^2}}$$

Converges or not

Solution:

$$\lim_{k\to\infty} \left|\sqrt[k]{a_k}\right| = \lim_{k\to\infty} \left|\sqrt[k]{\frac{k \sin^k k}{e^{k^2}}}\right| = \lim_{k\to\infty} \left|\frac{\sqrt[k]{k} \sin k}{e^k}\right| = 0 < 1$$

Therefore by root test, the series converges absolutely and therefore converges.

Example 3

Check whether the series

$$\sum_{k=1}^{\infty} \frac{(2k)!}{e^{2k}}$$

Converges or not

Solution:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left(\frac{\left|\frac{(2(k+1))!}{e^{2(k+1)}}\right|}{\left|\frac{(2k)!}{e^{2k}}\right|}\right) = \lim_{k\to\infty}\frac{(2k+2)(2k+1)}{e^2} = \infty > 1$$

Therefore by ratio test, the series diverges

☺Exercise 2

Check whether the series converges or not

a) $\sum_{k=1}^{\infty} \cos^k\left(1+\frac{1}{k}\right)$

b) $\sum_{k=1}^{\infty} C_{k+1}^{2k}$

Difficult Situation

Example 4

Check whether the series

$$\sum_{k=1}^{\infty} \frac{k^2 \sin\left(\frac{1}{k}\right)}{(2k+1)!}$$

Converges or not

Solution:

We first simplify the series first, by limit comparison test, (note that all the terms are non-negative)

(Step 1: Use limit comparison)

Note that $\sin\left(\frac{1}{k}\right) \approx \frac{1}{k}$ when k is large, then $\frac{k^2 \sin\left(\frac{1}{k}\right)}{(2k+1)!} \approx \frac{k}{(2k+1)!}$

$$\lim_{k\to\infty}\frac{\frac{k^2\sin\left(\frac{1}{k}\right)}{(2k+1)!}}{\frac{k}{(2k+1)!}}=1$$

(Step 2: Use Ratio test)

Since the series $\sum_{k=1}^{\infty}\frac{k}{(2k+1)!}$ has factorial, we can apply ratio test here

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|=\lim_{k\to\infty}\frac{\left(\frac{k+1}{(2k+3)!}\right)}{\left(\frac{k}{(2k+1)!}\right)}=\lim_{k\to\infty}\frac{k+1}{k(2k+2)(2k+3)}=0<1$$

So the series $\sum_{k=1}^{\infty}\frac{k}{(2k+1)!}$ converges by ratio test.

Therefore by limit comparison test, $\sum_{k=1}^{\infty}\frac{k^2\sin\left(\frac{1}{k}\right)}{(2k+1)!}$ converges

Example 5 Find all possible real number p such that the series $$\sum_{k=1}^{\infty}k^n\tan^{n^2+p}\left(\frac{1}{k}\right)$$ Converges (where n is some real number)

Solution:

(Step 1)

Note that $\tan\left(\frac{1}{k}\right)\approx\frac{1}{k}$ for k is large, so $k^n\tan^{n^2+p}\left(\frac{1}{k}\right)\approx k^n\left(\frac{1}{k}\right)^{n^2+p}=k^{n-n^2-p}$

Note $\lim_{k\to\infty}\frac{k^n\tan^{n^2+p}\left(\frac{1}{k}\right)}{k^{n-n^2-p}}=1$

By limit comparison test

$\sum_{k=1}^{\infty}k^n\tan^{n^2+p}\left(\frac{1}{k}\right)$ converges if and only if $\sum_{k=1}^{\infty}k^{n-n^2-p}$ converges

(Step 2)

Now by p-test, $\sum_{k=1}^{\infty}k^{n-n^2-p}$ converges if $n-n^2-p<-1\to p>n-n^2+1$

So when $p>n-n^2+1$, the series $\sum_{k=1}^{\infty}k^n\tan^{n^2+p}\left(\frac{1}{k}\right)$ converges

(Note: For $p\le n-n^2+1$, the series will diverges)

Remarks: Here all we need to do is to apply the test directly and find out our answer.

Example 6 (Rudin P.78 #7)

If $x_n \geq 0$ and $\sum_{k=1}^{\infty} x_k$ converges, show that $\sum_{k=1}^{\infty} \frac{\sqrt{x_k}}{k}$ converges

Solution:

Recall the Cauchy-Schwarz Inequality

$$\left(\sum_{k=1}^{\infty} |a_k b_k|\right)^2 \leq \left(\sum_{k=1}^{\infty} |a_k|^2\right)\left(\sum_{k=1}^{\infty} |b_k|^2\right)$$

By putting $a_k = \sqrt{x_k}$, $b_k = \frac{1}{k}$, we have

$$\left(\sum_{k=1}^{\infty} \left|\frac{\sqrt{x_k}}{k}\right|\right) \leq \left(\sum_{k=1}^{\infty} \left|\sqrt{x_k}\right|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \left|\frac{1}{k}\right|^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{\infty} x_k\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{\frac{1}{2}}$$

Since both $(\sum_{k=1}^{\infty} x_k)$ and $\left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)$ both converges, so R.H.S converges,

By comparison test, $\left(\sum_{k=1}^{\infty} \left|\frac{\sqrt{x_k}}{k}\right|\right)$ converges

Finally, by absolute convergence test, $\sum_{k=1}^{\infty} \frac{\sqrt{x_k}}{k}$ converges.

Example 7 (Ratio Comparison Test)

If $a_n > 0$, $b_n > 0$ for all n, if $\sum a_n$ converges and if $\frac{b_{n+1}}{b_n} \leq \frac{a_{n+1}}{a_n}$ for all n

then $\sum b_n$ converges

Solution:

Consider $\left(\frac{b_2}{b_1}\right)\left(\frac{b_3}{b_2}\right)\left(\frac{b_4}{b_3}\right) \ldots\ldots \left(\frac{b_n}{b_{n-1}}\right) = \left(\frac{b_n}{b_1}\right)$

Since $\left(\frac{b_n}{b_1}\right) = \left(\frac{b_2}{b_1}\right)\left(\frac{b_3}{b_2}\right)\left(\frac{b_4}{b_3}\right) \ldots\ldots \left(\frac{b_n}{b_{n-1}}\right) \leq \left(\frac{a_2}{a_1}\right)\left(\frac{a_3}{a_2}\right)\left(\frac{a_4}{a_3}\right) \ldots\ldots \left(\frac{a_n}{a_{n-1}}\right) = \left(\frac{a_n}{a_1}\right)$

So $b_n \leq b_1 \left(\frac{a_n}{a_1}\right) = \left(\frac{b_1}{a_1}\right) a_n$

Therefore $\sum b_n \leq \sum \left(\frac{b_1}{a_1}\right) a_n = \left(\frac{b_1}{a_1}\right) \sum a_n$

Since $\sum a_n$ converges, by comparison test, $\sum b_n$ converges

No solution will be provided for the following exercises, you are welcome to submit your solution to me and I will give comments to your work.

☺Exercise 3 (Practice Exercise #139)

Check whether the series

$$\sum_{k=1}^{\infty}\frac{2^k}{k^3(3k)!} \text{ and } \sum_{k=1}^{\infty}\left(\frac{1}{e}+\frac{1}{k}\right)^k \sin k$$

Converges or not.

(Hint: For 2nd series, the term can be negative due to the effect of $\sin x$

☺Exercise 4 (Practice Exercise #90h)

Check whether the series

$$\sum_{k=1}^{\infty}\frac{k^{\pi}+\cos k\pi}{3+k^4} \text{ and } \sum_{k=1}^{\infty}\frac{k^{\pi}\cos k\pi}{3k^4}$$

Converges or not. (Hint: Choose the appropriate tests for each series)

☺Exercise 5 (Practice Exercise #151)

Let $a_k > 0$ for $k = 1,2,3,\ldots$ and $\sum_{k=1}^{\infty} a_k$ converges. Determine <u>all positive real number b</u> such that the series $\sum_{k=1}^{\infty}\frac{(b+a_k)^k}{k}$ converges.

(Hint: Use root test)

☺Exercise 6

Determine all possible real number b such that the series

$$\sum_{k=1}^{\infty}\sin^b\left(\frac{1}{k}\right)(k^7-k^3+2)$$

i) converges ii) diverges

☺Exercise 7

If $\sum|a_n|$ converges and $\{b_n\}$ is a bounded sequence, show that the series $\sum a_n b_n$ converges.

(Note: Here $\{b_n\}$ is bounded $\leftrightarrow$ there exist $M \in R$ such that $|b_n| < M$ for all n)

(Hint: Consider $\sum|a_n b_n|$)