# Tutorial class 2. Set theory & Countability.

(corresponding to Lec 3+4)

## 1. Set property

During lectures. we've proved the following propositions.

$\boxed{\text{Prop 1}}$: If $A \subseteq B$ and $C \subseteq D$. then $A \cap C \subseteq B \cap D$

$\boxed{\text{Prop 2}}$: $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.

Now we give the proof of some others.

$\boxed{\text{Prop 3}}$: If $S_\alpha \subseteq T_\alpha$ for every $\alpha$. then

$$\bigcup_\alpha S_\alpha \subseteq \bigcup_\alpha T_\alpha \quad \text{and} \quad \bigcap_\alpha S_\alpha \subseteq \bigcap_\alpha T_\alpha$$

Proof:

For the 1st part. consider $\forall x \in \bigcup_\alpha S_\alpha$, then here $\exists$ some $\alpha$ such that $x \in S_\alpha$. Since we have $S_\alpha \subseteq T_\alpha$ for all indice. we know $x \in S_\alpha \subseteq T_\alpha \Rightarrow x \in T_\alpha$. Thus according to the definition. $x \in \bigcup_\alpha T_\alpha$. which implies $\bigcup_\alpha S_\alpha \subseteq \bigcup_\alpha T_\alpha$.

Similarly. let $x \in \bigcap_\alpha S_\alpha$ (suppose $\bigcap S_\alpha$ is nonempty : otherwise the result comes from the trivial fact that $\phi \subseteq \bigcap_\alpha T_\alpha$). Due to the definition. we know $x \in S_\alpha$ for all $\alpha$; thus $x \in T_\alpha$ for all $\alpha$ (since $S_\alpha \subseteq T_\alpha$), which implies that $x \in \bigcap_\alpha T_\alpha$ and moreover.
$\bigcap_\alpha S_\alpha \subseteq \bigcap_\alpha T_\alpha$. $\square$

$\boxed{\text{Prop 4.}}$ For $S \subseteq \mathbb{R}$. $\mathbb{R} \setminus (\mathbb{R} \setminus S) = S$. If $A \subseteq B$. then for every set $C$. $C \setminus B \subseteq C \setminus A$.

Proof:

Consider the 1st part. We first prove the "$\subseteq$" relation. For $\forall s \in \mathbb{R}\setminus(\mathbb{R}\setminus S)$, we know $s \in \mathbb{R}$ and $s \notin \mathbb{R}\setminus S$, which further implies $s \in S$. Thus we've reached that $\mathbb{R}\setminus(\mathbb{R}\setminus S) \subseteq S$. For the "$\supseteq$" relation, suppose $s \in S$, then we must have $s \notin \mathbb{R}\setminus S$; on the other hand, since $s \in S \subseteq \mathbb{R}$, according to the definition, we have $s \in \mathbb{R}\setminus(\mathbb{R}\setminus S)$, which ensures $\mathbb{R}\setminus(\mathbb{R}\setminus S) \supseteq S$. Collapse all results above, we attain $\mathbb{R}\setminus(\mathbb{R}\setminus S) = S$.

For the second part, for any $x \in C\setminus B$, we have $x \in C$ and $x \notin B$, then actually, $x \in C$ and $x \notin A$ (otherwise, $x \in A \subset B$ $\Rightarrow x \in B$, a contradiction), which implies $x \in C\setminus A$ and $C\setminus B \subseteq C\setminus A$, as promised. $\square$

$\boxed{\text{Prop 5}}$: (Distributive law)

$$A \cap \left(\bigcup_\alpha S_\alpha\right) = \bigcup_\alpha (A \cap S_\alpha), \quad A \cup \left(\bigcap_\alpha S_\alpha\right) = \bigcap_\alpha (A \cup S_\alpha)$$

Proof: (For your convenience, I write blue for 1st part and red for 2nd part).

Let me show $\overset{V}{\overbrace{A \cap \left(\bigcup_\alpha S_\alpha\right)}} \subseteq \overset{W}{\overbrace{\bigcup (A \cap S_\alpha)}}$ first. Suppose $x \in V$, then

$$\begin{aligned} x \in A \text{ and } x \in \bigcup_\alpha S_\alpha &\Rightarrow x \in A \text{ and } x \in S_\alpha \text{ for some } \alpha \\ &\Rightarrow x \in (A \cap S_\alpha) \text{ for some } \alpha \\ &\Rightarrow x \in \bigcup_\alpha (A \cap S_\alpha) \\ &\Rightarrow V \subseteq W. \end{aligned}$$

For the converse direction, suppose $x \in \bigcup (A \cap S_\alpha)$, then we have $x \in A \cap S_\alpha$ for some $\alpha$

$\Rightarrow x \in A$ and $x \in S_\alpha$ for some $\alpha$

$\Rightarrow x \in A$ and $x \in \bigcup_\alpha S_\alpha$

$\Rightarrow x \in A \cap (\bigcup_\alpha S_\alpha)$

$\Rightarrow V \supseteq W$

In conclusion. we have $V = W \Leftrightarrow A \cap (\bigcup_\alpha S_\alpha) = \bigcup_\alpha (A \cap S_\alpha)$.

Now we comes to the 2nd Law. As usuall. Let me show $\underbrace{A \cup (\bigcap_\alpha S_\alpha)}_{= V} \subseteq \underbrace{\bigcap_\alpha (A \cup S_\alpha)}_{= W}$. Suppose $x \in V$. then we have $x \in A$ or $x \in \bigcap_\alpha S_\alpha$. If $x \in A$. then $x \in A \cup S_\alpha$ for all $\alpha$. which promises $x \in \bigcap_\alpha (A \cup S_\alpha) \Rightarrow V \subseteq W$; if $x \in \bigcap_\alpha S_\alpha$. then $x \in S_\alpha$ for all $\alpha$ $\Rightarrow x \in A \cup S_\alpha$ for all $\alpha$

$\Rightarrow x \in \bigcap_\alpha (A \cup S_\alpha) \Rightarrow V \subseteq W$.

Consider the converse direction. Suppose $x \in W$. then $x \in A \cup S_\alpha$ for all $\alpha$. If $x \in A$, $x \in A \cup (\bigcap_\alpha S_\alpha) \Rightarrow V \supseteq W$; If $x \notin A$, we must have $x \in S_\alpha$ for all $\alpha$. $\Rightarrow x \in \bigcap_\alpha S_\alpha$ $\Rightarrow x \in A \cup (\bigcap_\alpha S_\alpha) \Rightarrow V \supseteq W$.

In a word. $V = W \Leftrightarrow A \cup (\bigcap_\alpha S_\alpha) = \bigcap_\alpha (A \cup S_\alpha)$. $\square$

$\boxed{\text{Prop 6}}$: (de Morgan's Law)

$$A \setminus (\bigcup_\alpha S_\alpha) = \bigcap_\alpha (A \setminus S_\alpha). \quad A \setminus (\bigcap_\alpha S_\alpha) = \bigcup_\alpha (A \setminus S_\alpha)$$

Proof:

Suppose $x \in A \setminus (\bigcup_\alpha S_\alpha)$. then we have $x \in A$ and $x \notin \bigcup_\alpha S_\alpha$

$\Rightarrow x \in A$ and $x \notin S_\alpha$ for all $\alpha$

$\Rightarrow x \in (A \setminus S_\alpha)$ for all $\alpha$

$\Rightarrow x \in \bigcap_\alpha (A \setminus S_\alpha) \Rightarrow A \setminus (\bigcup_\alpha S_\alpha) \subseteq \bigcap_\alpha (A \setminus S_\alpha)$

On the other hand. suppose $x \in \bigcap_\alpha (A \setminus S_\alpha)$. then

$x \in A \setminus S_\alpha$ for all $\alpha$ $\Rightarrow x \in A$ and $x \notin S_\alpha$ for all $\alpha$

$\Rightarrow x \in A$ and $x \notin \bigcup_\alpha S_\alpha$

$\Rightarrow x \in A \setminus (\bigcup_\alpha S_\alpha) \Rightarrow A \setminus (\bigcup_\alpha S_\alpha) \supseteq \bigcap_\alpha (A \setminus S_\alpha)$

In conclusion. we've reached that $A \setminus (\bigcup_\alpha S_\alpha) = \bigcap_\alpha (A \setminus S_\alpha)$.

Exercise 1: Try to prove the other law. i.e.

$$A \setminus (\bigcap_\alpha S_\alpha) = \bigcup_\alpha (A \setminus S_\alpha).$$

## II. Functions.

we've already introduced the definition of a function. Now list some fondamental types of functions.

1. Identity function.
2. Composite function.
3. Restriction function.
4. surjective function.
5. Injective function
6. Inverse function.
7. Bijective function.

Now we prove a theorem left in our lecture.

$\boxed{\text{Thm 1}}$ ① $f: A \to B$ is bijective iff. $\exists g: B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$.

② $f: A \to B$ and $g: B \to C$ are two bijections. then $g \circ f: A \to C$ is a bijection.

③ Let $A, B \subseteq \mathbb{R}$ and $f: A \to B$ be a function. If $\forall$ $b \in B$, the horizontal line $y = b$ intersects the graph of $f$ exactly once. then $f$ is a bijection.

**Proof:**

① "$\Rightarrow$": Since $f$ is bijective. for each $b \in B$. here

$\exists!\ a \in A$ such that $f(a) = b$. Hence, we can define a function $f^{-1}: B \to A$ by

$$f^{-1}: b \to a \quad ( f(a) = b )$$

which is well-defined. Now we check that.

1) for $\forall\ a \in A$. $f^{-1}(f(a)) = f^{-1}(b) = a$

$$\Rightarrow f^{-1} \circ f = id_A$$

2) for $\forall\ b \in B$. $f(f^{-1}(b)) = f(a) = b$.

$$\Rightarrow f \circ f^{-1} = id_B$$

So we've already constructed the $g = f^{-1}$ which we need.

"$\Leftarrow$": we check that:

1) $f$ is injective.

For $a_1, a_2 \in A$. if $f(a_1) = f(a_2)$, the apply $g$ at BHS of this Eq. then:

$$a_1 = g(f(a_1)) = g(f(a_2)) = a_2$$

which confirms that $f$ is injective.

2) $f$ is surjective.

For each $b \in B$. let $a = g(b) \in A$. Then we have

$$f(a) = f(g(b)) = b$$

which implies that $f$ is surjective.

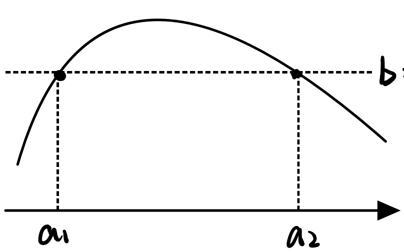
Hence. $f$ is bijective.

② Exercise 2. Try to check it!

③ As usual, we check that:

1) $f$ is injective:

Suppose that $f(a_1) = f(a_2)$ for $a_1, a_2 \in A$. Can $a_1 \neq a_2$ happen? If so, the piece of graph will look like:




Intuition: intersect 2 times!

Formally speaking:

since $a_1 \neq a_2$ and $f(a_1) = f(a_2) \triangleq b$,

$$\begin{cases} (a_1, b), (a_2, b) \in G(f) \\ (a_1, b), (a_2, b) \in Y_b \triangleq \{(x,y) : y = b\} \end{cases}$$

(Recall $G(f) := \{(x,y) : y = f(x)\}$)

↳ horizontal line with height b

$\Rightarrow (a_1, b), (a_2, b) \in Y_b \cap G(f)$

$\xRightarrow[\text{are different!}]{\text{but they}}$ a contradiction.

2) $f$ is surjective.

It's also obvious if you imagine the graph. Formally speaking, for $\forall b \in B$. $Y_b \cap G(f) \neq \phi$. Select a point $(x, b) \in Y_b \cap G(f)$. then we have $f(x)=b$; in other words. $\exists x \in A$ s.t. $f(x)=b$, which confirms that $f$ is surjective.

Collapse 1). 2). we reach the proof. $\square$

## III. Equivalent relation.

Def: A relation on a set $S$ is a subset of $S \times S$.

A relation $R$ on a set $S$ is an equivalent relation iff

① $\forall x \in S$, $(x, x) \in R$ (reflective)

② $(x, y) \in R \Rightarrow (y, x) \in R$ (symmetry)

③ $(x, y) \in R$. $(y, z) \in R \Rightarrow (x, z) \in R$ (transitive)

notation: write $x \sim y$ when $(x, y) \in R$.

$$[x] \triangleq \{y \in S : x \sim y\}$$

equivalence class containing (represented by) $x$.

Some propositions:

① $S$ can be decomposed with respect to equivalence, say

$$S = \bigcup_{x \in S} [x]$$

② The representative element is unambiguous. say

$x \sim y \Rightarrow [x] = \{z : z \sim x\} = \{z : z \sim y\} = [y]$

③ The . say

$x \nsim y \Rightarrow [x] \cap [y] = \phi$

(otherwise, $\exists z \in [x] \cap [y] \Rightarrow z \sim x, z \sim y \Rightarrow x \sim y$).

## IV. Cardinality.

Based on previous discussions. we know the bijective function can produce an equivalent relation. say:

$$\text{sets } S_1 \sim S_2 \quad \text{iff} \quad \exists \text{ a bijection } f : S_1 \to S_2.$$

Then. we say $S_1$ and $S_2$ have the same cardinality if ($S_1 \sim S_2$). Call the number of elements in $S$ the (cardinal number) of $S$. denoted by Card $S$ or $|S|$.

Rmk: $|S|$ may be infinite. but still can be compared!

Exercise 3: Does $\mathbb{Z} \sim \text{Even}_{\mathbb{Z}}$ ?

all even integers.

Example: We can define a equivalence relation on $\mathbb{R}$ by setting.

$$x \sim y \quad \text{iff} \quad x - y \in \mathbb{Z}$$

Check that

① Reflectivity

For $\forall x \in \mathbb{R}$. $x - x = 0 \in \mathbb{Z} \Leftrightarrow x \sim x$

② Symmetry

If $x \sim y$. we have $x - y = z \in \mathbb{Z}$.

$\Rightarrow y - x = -z \in \mathbb{Z} \Leftrightarrow y \sim x$

③ Transitivity.

Suppose $x \sim y$. $y \sim z$. then $\begin{cases} x - y = a \in \mathbb{Z} \\ y - z = b \in \mathbb{Z} \end{cases}$

$\Rightarrow x - z = (x - y) + (y - z) = a + b \in \mathbb{Z} \Leftrightarrow x \sim z$

Rmk: If we let $R/\mathbb{Z} \triangleq \{[x] : x \in \mathbb{R}\}$ (the collection of all equivalence classes). then $R/\mathbb{Z} \sim [0, 1)$ !

## Ch3. Countability

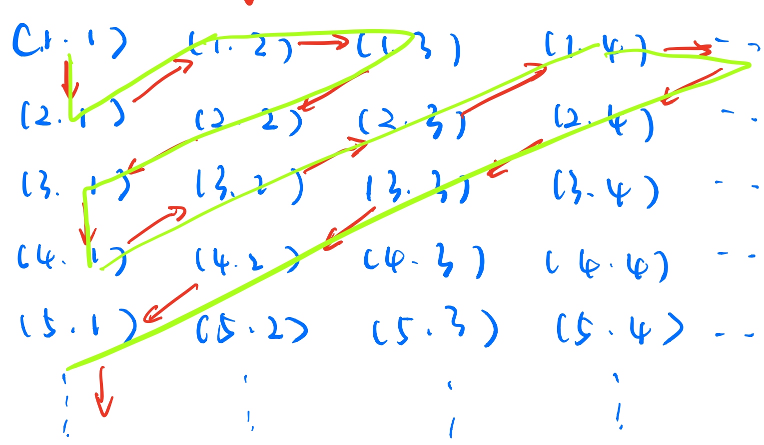
Def: ① A set $S$ is called countably infinite iff $S \sim \mathbb{N}$. (i.e. $\text{Card } S = \text{Card } \mathbb{N} = \aleph_0$)

② A set $S$ is countable iff $S$ is finite or countably infinite.

$\boxed{\text{Theorem 2}}$ $S$ is countable $\Leftrightarrow$ $S$ can be listed as $S = \{s_1, s_2, \dots\}$ ($s_i \neq s_j$ if $i \neq j$)

$\boxed{\text{Theorem 3}}$ (Bijection theorem) Let $g: S \to T$ be a bijection. $S$ is countable $\Leftrightarrow$ $T$ is countable

Example 1: $\mathbb{N} \times \mathbb{N} = \{(m, n) : m, n \in \mathbb{N}\}$ is countably infinite: diagonal scheme!




Example 2: open interval (e.g. $(0, 1)$) is uncountable.

(idea: construct an element which does not belong to $(0, 1)$)

Exercise 3: Try to construct a bijection between $(0, 1)$ and $[0, 1)$

## V. Countable subset theorem

$\boxed{\text{Theorem 4}}$: Let $A \subseteq B$. if $B$ is countable. then $A$ is countable. (its contrapositive statement is also important)

## VI. Countable union theorem.

$\boxed{\text{Theorem 5}}$: $\bigcup_{n\in\mathbb{N}} A_n$ is countable if $A_n$ is countable for $n \geq 1$

nothing but the diagonal scheme!

## VII. Product theorem.

$\boxed{\text{Theorem 6}}$: Any finite product of countable sets is still countable.

also use the diagonal scheme.

Exercise 4: What happened to the infinite case?

Can you give an uncountable example?

## VIII. Injection theorem.

$\boxed{\text{Theorem 7}}$ Let $f: A \to B$ be injective. if $B$ is countable.

then $A$ is countable.

Consequence: $(0,1)$ uncountable $\Rightarrow \mathbb{R}$ uncountable.

Rmk: if $f: A \to B$ is injective, we can say $A$ is embeded in $B$ by $f$.

## IX. Surjection theorem.

$\boxed{\text{Theorem 8}}$: Let $g: A \to B$ be surjective. If $A$ is countable, then $B$ is countable.

Just write $B = \bigcup_{x \in A} \{f(x)\}$.

Example: $\mathbb{Q}$ is countable.

2 ways to show that.

1) $\mathbb{Q}$ can be seen as a subset of $\mathbb{Z} \times \mathbb{Z}$

$f: q = \frac{m}{n} \to (m, n)$ <u>injection</u>

2) $\mathbb{Q} = \bigcup_{n \in \mathbb{Z}} S_n = \bigcup_{n \in \mathbb{Z}} \{\frac{n}{m} : m \in \mathbb{Z}, m \neq 0\}$

Obviously $S_n$ is countable.

<u>Exercise 5</u>: Show that (the set of all finite subsets of $\mathbb{N}$) countable.

Example: $D = \{x \in \mathbb{R} : x^7 + x^6 + x^5 + \cdots + x + 1 \in \mathbb{Q}\}$ countable or not?

Seems tricky. but just let:

$$D_r \triangleq \{x \in \mathbb{R} : x^7 + x^6 + \cdots + x + 1 = r\} \quad (r \in \mathbb{Q})$$

Then $D_r$ contains at most 7 elements.

$\Rightarrow D = \bigcup_{r \in \mathbb{Q}} D_r$ is countale (a countable union of countable

Thm

sets is still countable).

Exercise 6. Show that

$$F \triangleq \{a \in \mathbb{R} : x^5 + ax^3 + 1 = 0 \text{ has a rational root}\}.$$

is countable or not.