1. Suppose $f \in C[a,b]$ and $f([a,b]) \subset [a,b]$. Prove: $\exists \xi \in [a,b]$ such that $f(\xi)=\xi$. (Fixed point)

$f(x)=x$, $x$, fixed point, domain of $f$: $[a,b]$, range of $f \subseteq [a,b]$.

Proof:

Consider $F(x) := f(x) - x$.

From $f([a,b]) \subset [a,b]$, we know

$$F(a) = f(a) - a \geq 0. \qquad F(b) = f(b) - b \leq 0.$$

Thus $F(a)F(b) \leq 0$. ① If $F(a) = 0$ OR $F(b)=0$, then $f(a)=a$ OR $f(b)=b$.

Intermediate value theorem. ② $F(a)F(b) < 0$, $\Rightarrow \exists \xi \in (a,b)$ $F(\xi)=0$. $\Rightarrow f(\xi)=\xi$.

2. Suppose $f \in C[a,b]$. $a \leq x_1 < x_2 < \cdots < x_n \leq b$. Prove: $\exists \xi \in [x_1, x_n]$ such that

$$f(\xi) = \frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}.$$

$f(x_i) <$ $<f(x_j)$

Proof: If $f(x_1)=f(x_2)=\cdots=f(x_n)$,

then $\xi = x_1$. $f(\xi) = \frac{1}{n}(f(x_1)+\cdots+f(x_n)) = f(x_1)$.

If $f(x_1)=f(x_2)=\cdots=f(x_n)$ does not hold.

Suppose $f(x_i) \geq f(x_k)$, $k=1,2,\ldots,n$.

($f(x_i)$ is the largest one)

$f(x_j) \leq f(x_k)$ $k=1,2,\ldots,n$.

($f(x_j)$ is the smallest one).

$f(x_j) < f(x_i)$ ( because ✓ )

Then $f(x_j) < \frac{1}{n}(f(x_1)+\dots+f(x_n)) < f(x_i)$.

By intermediate value theorem,

$\subseteq [x_1, x_n]$.

$\exists\, \xi$ in $[x_i, x_j]$ or $[x_j, x_i]$ such that

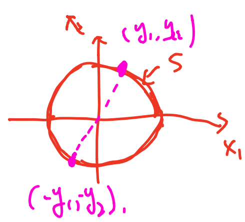
$f(\xi) = \boxed{\frac{1}{n}(f(x_1)+\dots+f(x_n))}$

Notice $\xi \in [x_1, x_n]$.

3. Suppose: $S = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1\}$. $f$ is a function defined on $S$:

$f: S \mapsto \mathbb{R}$. Prove: $\exists\, (y_1, y_2) \in S$ such that

$f(y_1, y_2) = f(-y_1, -y_2)$




$f: S \mapsto \mathbb{R}$. $(x_1, x_2) \in S$

$f(x_1, x_2) \to \mathbb{R}$,

$F(x_1, x_2) = f(x_1, x_2) - f(-x_1, -x_2)$.

$(x_1, x_2) = (1, 0)$, $F(1, 0) = f(1, 0) - f(-1, 0)$.

if $f(1,0) = f(-1,0) \Rightarrow$, $y_1 = 1$, $y_2 = 0$

if $f(1,0) \neq f(-1,0)$. $F(1,0) > 0$ or $F(1,0) < 0$,

$(x_1, x_2) = (-1, 0)$, $F(-1,0) = f(-1,0) - f(1,0)$

$= -F(1,0)$. $F(-1,0) < 0$. $F(-1,0) > 0$

so set, $F(1,0) \cdot F(-1,0) < 0$

Consider. $\{(x_1, x_2) \mid x_1 \in (-1, 1),\ x_2 = \sqrt{1 - x_1^2}\}$

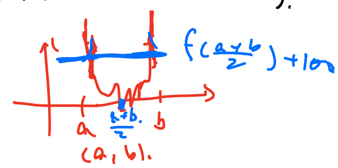
$\exists\, (y_1, y_2) \in \{(x_1, x_2) \mid x_1 \in (-1, 1),\ x_2 = \sqrt{1 - x_1^2}\}$

s.t. $F(y_1, y_2) = 0 \Rightarrow f(y_1, y_2) = f(-y_1, -y_2)$.

4. Suppose $f \in C(a,b)$. ~~[a,b]~~ and $f(a^+) = f(b^-) = +\infty$.

$\lim_{x \to a^+} f(x) = \lim_{x \to b^-} f(x) = +\infty$,

Prove : minimum value of $f$ in $(a,b)$ exists (not $-\infty$).




Proof:

Because $f(a^+) = f(b^-) = +\infty$.

$\exists\ 0 < \delta < \frac{b-a}{2}$ such that $\forall\ x \in O(a,\delta) \cap (a,b)$

$f(x) > f(\frac{a+b}{2}) + 100$

$\forall x \in O(b,\delta) \cap (a,b)$

$f(x) > f(\frac{a+b}{2}) + 100$.

Then Consider $[a+\delta,\ b-\delta]$. ( $\frac{a+b}{2} \in [a+\delta, b-\delta]$ ).

$f$ is continuous in $[a+\delta,\ b-\delta]$.

$\Rightarrow f$ has a minimum value. ( Extreme Value Theorem).

$\Rightarrow \exists\ x_0 \in [a+\delta, b-\delta]$, $f(x_0) = \inf\{f(x) \mid x \in [a+\delta, b-\delta]\}$

$f(x_0) \le f(\frac{a+b}{2}) < f(\frac{a+b}{2}) + 100$.

$\Rightarrow$ For any $x \in (O(a,\delta) \cap (a,b)) \cup (O(b,\delta) \cap (a,b))$

$f(x) > f(\frac{a+b}{2}) + 100 > f(x_0)$.

$\Rightarrow$ For any $x \in (a,b)$

$f(x) \geq f(x_0)$.

5. Suppose $f_n(x) = x^n + x$ . $n \in \mathbb{N}_+$. Prove:

① For any $n > 1$. $f_n(x) = 1$ has only one root in $(\frac{1}{2}, 1)$.

② If $C_n \in (\frac{1}{2}, 1)$ is root of $f_n(x) = 1$. then $\lim_{n\to\infty} C_n$ exists.

What is $\lim_{n\to\infty} C_n$.

①. $f_n(x)$ is an increasing function in $(\frac{1}{2}, 1)$.

$f_n(\frac{1}{2}) < 1$. $f_n(1) > 1$. $\Rightarrow \exists!\ c_n \in (\frac{1}{2}, 1)$ such that $f_n(c_n) = 1$.

$\overset{\shortparallel}{(\frac{1}{2})^n + \frac{1}{2} < 1}$, $\overset{\shortparallel}{2}$.

②. $c_n^n + c_n = 1$

$c_{n+1}^{n+1} + c_{n+1} = 1$.

$\underbrace{f_{n+1}(c_n)}_{<1} = c_n^{n+1} + c_n < c_n^n + c_n = 1$, $c_n < 1$.

$f_{n+1}(c_{n+1}) = 1$. increasing function? $\Rightarrow c_{n+1} > c_n$.

$f_{n+1}(c_n) = c_n^{n+1} + c_n < c_n^n + c_n = 1$. Because $f_{n+1}(x)$ is increasing of $x$.

$\Rightarrow c_{n+1} > c_n$.

$\Rightarrow \{c_n\}$ is monotonically increasing and also bounded.

$\Rightarrow \lim_{n\to+\infty} c_n$ exists.

$c_{n+1} > c_n$. $c_{n+1}^{n+1} + c_{n+1} = c_n^n + c_n \overset{=1}{}$ $\Rightarrow c_{n+1}^{n+1} < c_n^n$.

Note that $c_{n+1}^{n+1} < c_n^n$. $\{c_n^n\}$ is decreasing $c_n^n \in (0, \frac{1}{2})$.

$\{c_n^n\}_{n=1}^{+\infty}$.

$\lim_{n\to+\infty} c_n^n$ exists.

$\lim_{n\to+\infty} \overset{a}{c_n} + \lim_{n\to+\infty} \overset{1-a}{c_n^n} = 1$. $\leftarrow$ $c_n + c_n^n = 1$, take limit!

if $\lim_{n\to\infty} c_n = a < 1$. then $c_n \leq a < 1$ $\lim_{n\to+\infty} c_n^n = 1 - a > 0$

however $\lim_{n\to\infty} c_n^n \leq \lim_{n\to\infty} a^n = 0$.

Contradiction!

Then $\lim_{n\to+\infty} c_n = 1$.