Math2033 TA note 12

Yang Yunfei, Chen Yipei, Liu Ping

May 5, 2019

Example 1. *Let* $f:(0,+\infty)\to\mathbb{R}$ *be twice differentiable,* $M_0=\sup\{|f(x)|:x>0\}<\infty, M_1=\sup\{|f'(x)|:x>0\}<\infty$ *and* $M_2=\sup\{|f''(x)|:x>0\}<\infty$. *Show that* $M_1^2\le 4M_0M_2$. *(Hint: Let* $h>0$. *Apply Taylor's theorem to* $f(x)$ *with* $c+h$*, then solve for* $f'(c)$*.)*

Solution: Let $h>0$. By Taylor's theorem, there is a $\xi\in(c,c+h)$ such that

$$f(c+h)=f(c)+f'(c)h+\frac{f''(\xi)}{2}h^2 \implies f'(c)=\frac{f(c+h)-f(c)}{h}-f''(\xi)\frac{h}{2}$$
$$\implies |f'(c)|\le\frac{|f(c+h)|+|f(c)|}{h}+\frac{|f''(\xi)|h}{2}$$
$$\implies |f'(c)|\le\frac{2M_0}{h}+\frac{M_2h}{2}\qquad\forall h>0.$$

Since $\frac{2M_0}{h}+\frac{M_2h}{2}\ge 2\sqrt{\frac{2M_0}{h}\frac{M_2h}{2}}=2\sqrt{M_0M_2}$ and the equality hold when $h=2\sqrt{\frac{M_0}{M_2}}$, we have $|f'(c)|\le 2\sqrt{M_0M_2}$ for every $c\in\mathbb{R}$. Therefore, $M_1\le 2\sqrt{M_0M_2}$, i.e., $M_0^2\le 4M_0M_2$.

Example 2. *Show that* $f:(0,+\infty)$ *defined by* $f(x)=\sin\frac{1}{x}$ *is not uniformly continuous.*

Solution: Let $x_n=\frac{1}{n\pi}, y_n=\frac{1}{(n+\frac{1}{2})\pi}$. Take $\epsilon=1$, for every $\delta>0$, by archimedian principle, $\exists N=[\sqrt{\frac{1}{\pi\delta}}]+1$, such that $|x_n-y_n|=|\frac{1}{n\pi}-\frac{1}{(n+\frac{1}{2})\pi}|=\frac{1}{2n(n+\frac{1}{2})\pi}<\frac{1}{n^2\pi}<\delta$,but

$$|f(x_n)-f(y_n)|=|\sin n\pi-\sin(n+\frac{1}{2})\pi|=1$$

Example 3. $f:[a,b]\to\mathbb{R}$ *is bounded.* f *is continuous except points* $\{x_1,\cdots,x_n\}$ *where* $x_1<x_2<\cdots<x_n$. *Show that* f *is integrable on* $[a,b]$.

Solution: Because f is bounded, $\forall x, |f(x)| < M$ for some M. For $\epsilon > 0$, We find interval $[a_j, b_j], j = 1, \cdots, n$ such that $x_j \in (a_j, b_j), j = 1, \cdots, n$ and $|b_j - a_j| < \frac{\epsilon}{2nM}$. Then the interval $[a, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_n, b]$ is closed and f in continuous in this closed interval. So f is integrable in the region. Let P be a partition of $[a, a_1] \cup [b_1, a_2] \cup \cdots \cup [b_n, b]$ s.t

$$U(f, P) - L(f, P) \leq \frac{\epsilon}{2}.$$

Then the partition P is also a partition of $[a, b]$ s.t

$$U(f, P) - L(f, P) \leq M \sum_{j=1}^{n} (b_j - a_j) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, f is integrable on $[a, b]$.

Example 4. *Suppose $a < c < d < b$ and f is integrable on $[a, b]$, then f is integrable on $[c, d]$.*

Solution: Since f is integrable on $[a, b]$, for any $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. We can assume that $c, d \in P$, otherwise we can consider $P \cup \{c, d\}$. Let $P_1 = P \cap [c, d]$, then P_1 is a partition of $[c, d]$. And we have

$$U(f, P_1) - L(f, P_1) \leq U(f, P) - L(f, P) < \epsilon.$$

Therefore, f is integrable on $[c, d]$.