Tutorial 3

all positive integers

1. Let A be the collection of functions from the set $\{0,1\}$ to $\mathbb{N}$.

That is,

$$A = \{f \mid f: \{0,1\} \to \mathbb{N}\}.$$

Show that A is countable.

Proof:

Firstly, we note that every function $f: \{0,1\} \to \mathbb{N}$ is determined by only two values $f(0)$ and $f(1)$.

And two functions f, g in A equals to each other if and only if

$$(f(0), f(1)) = (g(0), g(1)).$$

That is, $f = g \iff (f(0), f(1)) = (g(0), g(1)).$ (1.1).

Thus we can construct a map φ from A to $\mathbb{N} \times \mathbb{N}$. written as

$$\varphi(f) = (f(0), f(1)).$$

by (1.1) we know that φ is an injection.

Then by $\mathbb{N}\times\mathbb{N}$ is countable and the injection theorem, we know that A must be a countable set. #.

2. Find the sup and inf of the set

$$S := \{\tfrac{1}{n} : n \in \mathbb{N}\}.$$

Solution:

① Supremum:

Observing that $\frac{1}{n}$ is decreasing as n increases:

$$1 > \tfrac{1}{2} > \tfrac{1}{3} > \cdots > \tfrac{1}{n} > \cdots.$$

Thus we have 1 is an upper bound of S.

On the other hand, by $1 \in S$, for any upper M of S, we have $M \geq 1$, that leads to 1 is the smallest upper bound of S, thus is the supremum of S.

② Infimum:

Observing that for every $n \in \mathbb{N}$, we have $\frac{1}{n} > 0$, thus 0 is a lower bound of S.

So by definition, we have $\inf S \geq 0$. Now we show that

$\inf S$ cannot larger than 0:

Otherwise suppose in contradiction that $\inf S > 0$.

Then $\frac{1}{\inf S} \in \mathbb{R}$ and $\frac{1}{\inf S} > 0$. By Archimedean's principle we have then $\exists n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\inf S}$, thus

$$\inf S > \frac{1}{n_0}. \quad (2.1)$$

Noticing that $\frac{1}{n_0} \in S$, so (2.1) contradicts to $\inf S$ is a lower bound of S.

Thus $\inf S$ must equal to 0. #.

3. Let S be a bounded set and $S_0 \subseteq S$ be a subset of S, check the following statements:

① both $\sup S_0$ and $\inf S_0$ exists and $\inf S_0 \geq \inf S$. $\sup S_0 \leq \sup S$.

② Suppose S_0 is a proper subset of S (i.e. $S_0 \neq S$, $S_0 \neq \phi$), is it always true that $\inf S_0 > \inf S$ and $\sup S_0 < \sup S$?

Proof:

① To show $\sup S_0$ and $\inf S_0$ exists, we need only show that S_0 is upper bounded and lower bounded. In fact, by S is bounded, $\sup S$ and $\inf S$ exists, and

$$\forall x \in S, \quad x \leq \sup S. \quad (3.1)$$
$$\forall x \in S. \quad x \geq \inf S. \quad (3.2).$$

By (3.1), (3.2), we directly have

$$\forall x \in S_0, \quad x \leq \sup S \qquad (3.1')$$
$$\forall x \in S_0, \quad x \geq \inf S \qquad (3.2')$$

(3.1') and (3.2') implies that $\begin{cases} \sup S \text{ is a upper bound of } S_0 \\ \inf S \text{ is a lower bound of } S_0. \end{cases}$

thus $\sup S_0$, $\inf S_0$ exists and

$$\begin{cases} \sup S_0 \leq \sup S \\ \inf S_0 \geq \inf S. \end{cases}$$

#.

(2.1) Exercise: Firstly consider the case that S is a finite set, in which case we have $\sup S_0 = \sup S$ and $\inf S_0 = \inf S$? (1 min).

(2.2) For general S, we also don't have $\sup S > \sup S_0$ in general.

Counter example:

$S = \{\frac{1}{n} : n \in \mathbb{N}\}$. $S_0 = \{\frac{1}{2n} : n \in \mathbb{N}\} \cup \{1\}$

$S_1 = \{\frac{1}{n} : n \geq 5, n \in \mathbb{N}\} \cup \{1\}$.

Both S_0, S_1 are proper subsets of S, and we have

$\inf S_0 = \inf S_1 = \inf S$.

$\sup S_0 = \sup S_1 = \sup S$. #.

Remark (A condition on $\sup S_0 < \sup S$):

Check that if $\sup S \notin S$ (or $\inf S \notin S$), and $S_0 \subset S$ is a finite subset of S, then we must have $\sup S_0 < \sup S$ (or $\inf S_0 > \inf S$).

Proof: For finite S_0, we have $\sup S_0 = \max\{s : s \in S_0\}$, so $\sup S_0 \in S_0 \subseteq S$. On the other hand, $\sup S \notin S$, thus $\sup S_0 \neq \sup S$.

By $\sup S_0 \leq \sup S$, we know it must be $\sup S_0 < \sup S$. #.

4. Find the sup and inf of following sets,

(a) $D = \{ \frac{1}{n} - \frac{1}{m} : m, n \in \mathbb{N} \}$.

(b) $E = \{ a+b : a \in (0,1) \cap \mathbb{Q},\ b \in (1,2) \cap \mathbb{Q} \}$.

We will use this result (see the end of lec6 note):

if A and B are bounded, then $A+B$ defined by

$$A+B = \{ a+b : a \in A,\ b \in B \}$$ is also bounded.

Moreover, we have

$$\inf(A+B) = \inf A + \inf B \qquad (4.1)$$

$$\sup(A+B) = \sup A + \sup B. \qquad (4.2)$$

Solution of (a):

By prob2, we know $S = \{ \frac{1}{n} : n \in \mathbb{N} \}$ is bounded and $\inf S = 0$, $\sup S = 1$.

So if we set $A = S$, $B = -S$ in the above result, then $D = A+B$.

And by the dual property we have

$$\sup B = \sup(-S) = -\inf S = 0.$$

$$\inf B = \inf(-S) = -\sup S = -1.$$

thus $\sup D = \sup(A+B) = \sup A + \sup B = 1.$

$\inf D = \inf(A+B) = \inf A + \inf B = -1.$ #.

1b):

To use the above theorem, we set

$A = [0,1) \cap \mathbb{Q}$,

$B = (1,2) \setminus \mathbb{Q}$

then we would compute $\sup A$. $\sup B$ and $\inf A$. $\inf B$

using the following lemma:

Lemma: If $S \subseteq \mathbb{R}$ is a densed set, i.e. $\forall a, b \in \mathbb{R}$, with $a<b$, $\exists s \in S$ such that $a<s<b$, then for any interval $E=(u,v) \subseteq \mathbb{R}$, we have

$$\sup(S \cap E) = \sup E. \qquad (4.3)$$
$$\inf(S \cap E) = \inf E. \qquad (4.4)$$

If the lemma holds, we have by

① $\mathbb{Q}$ is dense in $\mathbb{R}$. (Lec7 note p10)

② $\mathbb{R}\setminus\mathbb{Q}$ is dense in $\mathbb{R}$. (Lec7 note p11)

then $\sup A = \sup [0,1) = 1.$ $\inf A = \inf [0,1) = 0.$

$\sup B = \sup (1,2) = 2,$ $\inf B = \inf (1,2) = 1.$

thus $\sup(A+B) = 1+2 = 3.$

$\inf(A+B) = 0+1 = 1.$ #.

So we need only prove the above lemma:

Proof of (4.3):

For $E = (u, v)$, we know that $\sup E = v$,
and $\sup(E \cap S) \leq v$.

suppose in contradiction that $\sup(E \cap S) < v$.

then we set $w := \frac{1}{2}(\sup(E \cap S) + v)$, we have

$\sup(E \cap S) < w < v$. Thus w is an upper bound of $E \cap S$.

On the other hand by S is dense. exists some $z \in S$ such that

$w < z < v$.

Thus ① $z > \sup(E \cap S)$.
② $z \in S$
③ $z \in E$

that leads to a contradiction. so we must have

$\sup(E \cap S) = \sup E$.

The proof of inf case is similar. #.

Remark on the lemma:

The lemma doesn't hold for general subset $E \subseteq \mathbb{R}$. consider the case that

$$E = \mathbb{R} \setminus \mathbb{Q} \cap [0,1], \ S = \mathbb{Q} \text{ then}$$

$$\sup E = 1, \ \inf E = 0, \ \text{but } E \cap S = \phi.$$

5. Using the mathematical induction to show that the claim $P(n)$:

$$\forall x \geq -1, \ (1+x)^n \geq 1+nx.$$

holds for all $n \in \mathbb{N}$.

Proof: To prove $P(n)$ by induction, we need following steps:

Step 1: Prove that $P(1)$ is true.

When $n=1$, we have $P(1)$ turns to "$\forall x \geq -1, \ 1+x \geq 1+x$." that is obviously true.

Step 2: Prove that if $P(n_0)$ is true for some $n_0 \in \mathbb{N}$, then $P(n_0+1)$ is also true.

if $P(n_0)$ is true, we have

$$\forall x \geq -1,\ (1+x)^{n_0} \geq 1+n_0 x$$

thus $\forall x \geq -1$,

$$\begin{aligned}(1+x)^{n_0+1} &= (1+x)(1+x)^{n_0} \\ &\geq (1+x)(1+n_0 x) \quad (\text{by } P(n_0) \text{ is true}) \\ &= 1+n_0 x+x+\underbrace{n_0 x^2}_{\geq 0} \\ &\geq 1+(n_0+1)x.\end{aligned}$$

that is, $P(n_0+1)$ is true. #.