**Math 2033 (Mathematical Analysis)** Spring 2018

**Final Examination (Version Y) – (Duration: 150 minutes)**

**Directions**: **Work must be shown in full details legibly to receive credits. Answers alone are worth very little. Calculators are not allowed.**

**Notations**: $\mathbb{R}$ is the set of all real numbers.

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## Problems

1. (15 marks) Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence of real numbers in $[\sqrt{7}, +\infty)$. For every positive integer $n$, let $y_n = x_n + \sqrt{7}/x_n$. Prove that $y_1, y_2, y_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

   (Do not use Cauchy's theorem that said a sequence converges if and only if it is a Cauchy sequence, otherwise you will get zero mark.)

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2. (20 marks) Let Prove that $\lim_{x\to 4}(\cos(\sin \pi\sqrt{x}) + \sqrt{25 - x^2}) = 4$ by checking the $\varepsilon$-$\delta$ definition of limit of function.

   (Do not use any computation formula for limits, sandwich theorem or l'Hopital's rule. Otherwise, you will get zero mark.)

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3. (20 marks) Let $a, b \in \mathbb{R}$ with $a < b$ and $h : \mathbb{R} \to \mathbb{R}$ be continuous with $h(a) = h(b)$. Prove that there exist $c, d \in [a, b]$ with $d - c = (b - a)/2$ and $h(c) = h(d)$.

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4. (20 marks) Let $F : [0, 1] \to \mathbb{R}$ be continuous. Suppose $F$ is differentaiable on $(0, 1)$, $F(0) = 0$ and $F(x) > 0$ for all $x \in (0, 1)$. Prove that there exist $r, s \in (0, 1)$ such that $r + s = 1$ and $8F'(r)/F(r) = 5F'(s)/F(s)$.

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5. (25 marks) Let $f : [0, 1] \to [0, 2]$ be an Riemann integrable function. Define $g : [0, 1] \to [0, 2]$ by

   $$g(x) = \begin{cases} f(2x + 0.1) & \text{if } x \in [0, 1/3) \\ f(4x - 1) & \text{if } x \in [1/3, 1/2] \\ f(x) & \text{if } x \in (1/2, 1]\ . \end{cases}$$

   Prove that $g$ is Riemann integrable on $[0, 1]$ by checking the integral criterion.

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–End of Paper–