MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #24

More Examples in Integration

Example 1 (Limit Comparison Test)

Discuss the convergence of the following improper integral

$$\int_0^1 \frac{\mathrm{dx}}{\sqrt{1-\mathrm{x}^3}}$$

Solution:

The function $\mathrm{f(x)} = \frac{1}{\sqrt{1-\mathrm{x}^3}}$ is undefined at $\mathrm{x} = 1$ and continuous on $[0,1)$,

Note that $\frac{1}{\sqrt{1-\mathrm{x}^3}} = \frac{1}{\sqrt{(1-x)(1+x+x^2)}}$, and $1+\mathrm{x}+\mathrm{x}^2 \to 3$ as $\mathrm{x} \to 1^-$.

Hence we apply limit comparison test with $\mathrm{g(x)} = \frac{1}{\sqrt{1-\mathrm{x}}}$

Note $\lim_{\mathrm{x}\to 1^-} \frac{g(x)}{f(x)} = \lim_{x\to 1^-} \sqrt{1+x+x^2} = \sqrt{3}$

$$\int_0^1 \frac{1}{\sqrt{1-\mathrm{x}}} \mathrm{dx} = -2(1-\mathrm{x})^{\frac{1}{2}}|_0^1 = 2 < \infty$$

So $\int_0^1 \frac{1}{\sqrt{1-\mathrm{x}}} \mathrm{dx}$ converges,

Using limit comparison test, we conclude $\int_0^1 \frac{\mathrm{dx}}{\sqrt{1-\mathrm{x}^3}}$ also converges

Example 2 (Change of Variable Technique)

Discuss the convergence of

$$\int_0^1 \sin\left(\frac{1}{x}\right) dx$$

Solution:

Once again, $\sin\left(\frac{1}{x}\right)$ is undefined at 0, (since when $\mathrm{x} \to 0^+$, $\frac{1}{\mathrm{x}} \to \infty$).

Here we apply a trick, we make a substitution $\mathrm{u} = \frac{1}{\mathrm{x}}$,

then $\mathrm{du} = -\frac{1}{\mathrm{x}^2}\mathrm{dx} \to \mathrm{du} = -\mathrm{u}^2 \mathrm{dx} \to \mathrm{dx} = -\frac{1}{\mathrm{u}^2}\mathrm{du}$

$$\int_0^1 \sin\left(\frac{1}{x}\right) dx = \lim_{c\to 0^+} \int_c^1 \sin\left(\frac{1}{x}\right) dx = \lim_{c\to 0^+} \int_{\frac{1}{c}}^1 -\frac{sinu}{u^2} du = \lim_{c\to 0^+} \int_1^{\frac{1}{c}} \frac{sinu}{u^2} du$$

$$= \int_1^\infty \frac{\mathrm{sinu}}{\mathrm{u}^2} du \ldots\ldots (*)$$

We are going to check the convergence of $\int_1^\infty \frac{\sin u}{u^2}\,du$

Note that $\sin u$ can be negative, so we first consider

$$\int_1^\infty \left|\frac{\sin u}{u^2}\right| du \le \int_1^\infty \frac{1}{u^2}\,du = -\frac{1}{u}\Big|_1^\infty = 1 < \infty$$

By comparison test, $\int_1^\infty \left|\frac{\sin u}{u^2}\right| du$ converges

By absolute convergence test $\int_1^\infty \frac{\sin u}{u^2}\,du$ converges.

Hence from $(*)$, $\int_0^1 \sin\left(\frac{1}{x}\right) dx$ also converges.

Example 3

Let $f: [0,\infty) \to \mathbf{R}$, be a twice differentiable function with $|f''(x)| \le 1/(x+1)^4$ for all $x \in [0,\infty)$, $\lim_{x\to\infty} xf(x) = 0$ *and* $\lim_{x\to\infty} x^2 f'(x) = 0$. Show that the integral

$$\int_0^\infty f(x)dx$$

Converges

Solution:

Applying integration by part two times, we get

$$\int_0^\infty f(x)dx = \lim_{c\to\infty}\int_0^c f(x)dx = \lim_{c\to\infty}\left(xf(x)|_0^c - \int_0^c xf'(x)dx\right) = \lim_{c\to\infty} -\int_0^c xf'(x)dx$$

$$= \lim_{c\to\infty} -\frac{1}{2}\int_0^c f'(x)d(x^2) = \lim_{c\to\infty}\left(-\frac{1}{2}x^2 f'(x)|_0^c + \frac{1}{2}\int_0^c x^2 f''(x)dx\right)$$

$$= \frac{1}{2}\int_0^\infty x^2 f''(x)dx$$

Consider

$$\int_0^\infty |x^2 f''(x)|dx \le \int_0^\infty \frac{x^2}{(x+1)^4}dx$$

and $\int_0^\infty \frac{x^2}{(x+1)^4}dx = \lim_{c\to\infty}\int_0^c \frac{x^2}{(x+1)^4}dx = \lim_{c\to\infty}\int_0^c \frac{(x+1-1)^2}{(x+1)^4}dx$

$$= \lim_{c\to\infty}\int_0^c \frac{(x+1)^2 - 2(x+1) + 1}{(x+1)^4}dx = \lim_{c\to\infty}\int_0^c \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3} + \frac{1}{(x+1)^4}dx$$

$$= \lim_{c\to\infty} -\frac{1}{x+1} + \frac{1}{(x+1)^2} - \frac{1}{3(x+1)^3}\Big|_0^c = \frac{1}{3} < \infty$$

Hence $\int_0^\infty |x^2 f''(x)|dx$ converges $\to \int_0^\infty x^2 f''(x)dx$ converges $\to \int_0^\infty f(x)dx$ converges

Example 4a (Integration by Part)

Prove that the following integral converges

$$\int_1^{\infty} \frac{\sin x}{x} dx$$

☹Important Note:

Some students may use the following method:

Consider (since $\sin x$ may be negative for some $x \in [1, \infty)$)

$$\int_1^{\infty} \left|\frac{\sin x}{x}\right| dx \le \int_1^{\infty} \frac{1}{x} dx = \lim_{c\to\infty} \int_1^{c} \frac{1}{x} dx = \lim_{c\to\infty} \ln c = \infty$$

We see R.H.S. diverges, **so no conclusion can be drawn from comparison test**.

In this case, when the simple method is not applicable, we need to change the integral a bit. One possible way is to use integration by parts.

$$\int_1^{\infty} \frac{\sin x}{x} dx = -\int_1^{\infty} \frac{1}{x} d(\cos x) = -\frac{\cos x}{x}\Big|_1^{\infty} - \int_1^{\infty} \frac{\cos x}{x^2} dx = \cos 1 - \int_1^{\infty} \frac{\cos x}{x^2} dx$$

Now consider the integral

$$\int_1^{\infty} \left|\frac{\cos x}{x^2}\right| dx \le \int_1^{\infty} \frac{1}{x^2} dx = -\frac{1}{x}\Big|_1^{\infty} = 1 < \infty$$

So $\int_1^{\infty} \left|\frac{\cos x}{x^2}\right| dx$ converges→ $\int_1^{\infty} \frac{\cos x}{x^2} dx$ converges → $\int_1^{\infty} \frac{\sin x}{x} dx$ converges

Example 4b (Important!!)

Show that the integral

$$\int_1^{\infty} \left|\frac{\sin x}{x}\right| dx$$

diverges

The technique we will show here is to compare the integral with some diverging series (It is quite useful skill when proving some integral diverges)

To do so, we apply the following trick (Note that $|\sin x|$ has period π)

$$\int_1^{\infty} \left|\frac{\sin x}{x}\right| dx \ge \int_{\pi}^{\infty} \left|\frac{\sin x}{x}\right| dx = \sum_{k=1}^{\infty} \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin x}{x}\right| dx \ge \sum_{k=1}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{1}{(k+1)\pi} |\sin x| dx$$

$$= \sum_{k=1}^{\infty} \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x| dx = \sum_{k=1}^{\infty} \frac{2}{(k+1)\pi} = \frac{2}{\pi} \sum_{k=2}^{\infty} \frac{1}{k}$$

(Note that $\int_{k\pi}^{(k+1)\pi} |\sin x| dx = \int_0^{\pi} |\sin x| dx = \int_0^{\pi} \sin x dx = -\cos x|_0^{\pi} = 2$)

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by p-test, therefore $\int_1^{\infty} \left|\frac{\sin x}{x}\right| dx$ also diverges by comparison test.

Example 5 (Estimate the bound of series)

For any positive integer n, prove that

$$\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n-1} \geq \ln 3$$

Note: Of course, you may use induction to verify the inequality. But here, we will make use of integration trick instead. (The proof is much easier)

$$\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n-1}$$

$$=\sum_{k=n}^{3n-1}\frac{1}{k}=\sum_{k=n}^{3n-1}\frac{1}{k}\int_{k}^{k+1}dx=\sum_{k=n}^{3n-1}\int_{k}^{k+1}\frac{1}{k}dx \geq \sum_{k=n}^{3n-1}\int_{k}^{k+1}\frac{1}{x}dx$$

$$=\int_{n}^{3n}\frac{1}{x}dx=\ln x|_{n}^{3n}=\ln 3$$

Try to work on the following exercises, you may submit your solutions to me for comments.

☺Exercise 1

Discuss the convergence of following integrals:

i) $\int_0^{\infty}\frac{x}{\sqrt{x^6+1}}dx$

ii) $\int_0^{\infty}\frac{2x+1}{3x^2+4\sqrt{x}+7}dx$

iii) $\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\sin x}}$

iv) $\int_{-1}^{1}\frac{1}{1-x^4}dx$

(Note: The function is undefined at $x=1$ and $x=-1$,pick a point between -1 and 1, and split the integral)

☺Exercise 2

a) Show that the integral

$$\int_1^{\infty}\frac{\sin x}{x^p}dx$$

Converges for $p>0$

b) Show that the integral

$$\int_1^{\infty}\left|\frac{\sin x}{x^p}\right|dx$$

Converges for $p>1$ and diverges for $0<p\leq 1$

(Hint: For divergent part, apply similar method as in Example 4b)

☺Exercise 3

Let $f: [0, \infty) \to \mathbf{R}$ be continuous and periodic with period a (i.e. $f(x + a) = f(x)$),

Suppose $\int_0^a |f(x)| dx \neq 0$. Show that the integral

$$\int_0^\infty \left|\frac{f(x)}{x^p}\right| dx$$

Diverges for $0 < p \leq 1$.

(Hint: The method is similar to Example 4b)

☺Exercise 4

Show that the integral $\int_0^\infty \sin(x^2)\, dx$ and $\int_0^\infty \cos(x^2)\, dx$ are both convergent and these 2 integrals are called Fresnel Integrals.

(Hint: For $\int_0^\infty \sin(x^2)\, dx$, split the integral $\int_0^\infty \sin(x^2)\, dx = \int_0^1 \sin(x^2)\, dx + \int_1^\infty \sin(x^2)\, dx$. The first is clearly converges since it is a proper integral, for second one apply similar method as in Example 2 and use Exercise 2 to help you.)

(Remark: The reason why we need to split the integral in 2 parts is to avoid the impropriety at $x = 0$, you will know why when you do the problem)

☺Exercise 5

Let $h(x)$ be differentiable on $[0,1]$ with $h(0) = 0$ and for all $x \in (0,1)$, $0 < h'(x) < 1$. Prove that

$$\left(\int_0^1 h(x)dx\right)^2 > \int_0^1 h(x)^3 dx$$

By using Generalized Mean Value Theorem

(Hint: To apply generalized mean value theorem, you need to construct suitable $F(x)$ and $G(x)$, rearrange the term first and note that $\int_0^0 h(x)dx = 0$. You may need to apply the generalized mean value theorem twice)

☺Exercise 6

Show that for any positive integer n

$$\ln\left(\frac{(100n+1)^{100(n+1)}}{(n+1)^{n+1}}\right) - 99n \geq \ln(n+1) + \ln(n+2) + \cdots + \ln(100n) \geq \ln\left(\frac{(100n)^{100n}}{n^n}\right) - 99n$$

(Hint: Prove the inequality side by side)