MATH2033 Mathematical Analysis

Lecture Note 6

Limits of functions and Continuity

Limits of functions

Practically, limits of function allows us to investigate the behavior of a function $f(x)$ near a point where $f(x)$ is not well-defined.

- $f(x) = \frac{\sin x}{x}$ is not defined at $x = 0$. However, one can study the value of $f(x)$ when $x \to 0$. In fact, we have $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

On the other hand, it is also important for studying the continuity of a function.

Mathematical definition of limits of functions

Similar to the limits of sequence, we say $\lim_{x\to x_0} f(x) = L$ if and only if $f(x)$ is getting close to L when x is sufficiently close to x_0.

Definition (Limits of function)

We let $f: S \to \mathbb{R}$ be a function, where S is an interval (more generally, a set). We say $\lim_{x\to x_0} f(x) = L$ (or $f(x)$ converges to L) if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for any $x \in S$ satisfying $0 < |x - x_0| < \delta$

Remark about the definition

- According to the definition of limits, it examines the behavior of $f(x)$ at the point x near x_0 since we only consider the point which $|x - x_0| > 0$. So it is *not* necessary that $f(x)$ is defined at $x = x_0$.
- Similar to case for limits of sequence, δ depends on the choice of ε.

Example 1

Show that $\lim\limits_{x\to 1} \frac{x^2}{x+1} = \frac{1}{2}$ using the definition of limits.

😊Solution

For any $\varepsilon > 0$, we take $\delta = \min\left(1, \frac{6}{5}\varepsilon\right)$. Then for x which $|x - 1| < \delta$, we have

$$\left|\frac{x^2}{x+1} - \frac{1}{2}\right| = \left|\frac{2x^2 - x - 1}{2(x+1)}\right| = \left|\frac{(2x+1)(x-1)}{2x+2}\right| = \left|\frac{2x+1}{2x+2}\right| |x-1|$$

$$= \left|1 - \frac{1}{2x+2}\right| |x-1| \overset{|x-1|<1}{<} \frac{5}{6}|x-1| \overset{|x-1|<\frac{6}{5}\varepsilon}{<} \varepsilon.$$

So we conclude that $\lim\limits_{x\to 1} \frac{x^2}{x+1} = \frac{1}{2}$.

Example 2

We let $f: \mathbb{R} \to \mathbb{R}$ be a function which $\lim_{x\to 1} f(x) = L > 0$. Show that $\lim_{x\to 1} \sqrt[3]{f(x)} = \sqrt[3]{L}$ by checking the definition of limits.

Solution

For any $\varepsilon > 0$, note that $\lim_{x\to 1} f(x) = L$, then

- There exists $\delta_1 > 0$ such that $|f(x) - L| < L \Rightarrow f(x) > 0$ if $|x-1| < \delta_1$.
- There exists $\delta_2 > 0$ such that $|f(x) - L| < \varepsilon L^{\frac{2}{3}}$ if $|x-1| < \delta_2$.

We take $\delta = \min(\delta_1, \delta_2)$. Then for any $|x-1| < \delta$, we have

$$\left|\sqrt[3]{f(x)} - \sqrt[3]{L}\right| = \left|\frac{f(x)-L}{\left(\sqrt[3]{f(x)}\right)^2 + \left(\sqrt[3]{f(x)}\right)\left(\sqrt[3]{L}\right) + \left(\left(\sqrt[3]{L}\right)\right)^2}\right| \overset{\substack{|x-1|<\delta_1 \\ \Rightarrow f(x)>0}}{<} \frac{1}{L^{\frac{2}{3}}}|f(x)-L|$$

$$\overset{|x-1|<\delta_2}{<} \varepsilon.$$

So we deduce that $\lim_{x\to 1} \sqrt[3]{f(x)} = \sqrt[3]{L}$.

Example 3

We let $f: S \to \mathbb{R}$, $g: S \to \mathbb{R}$ be two functions which $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$. Show that $\lim_{x \to x_0} \big(f(x) + g(x)\big) = L + M$.

😊Solution

For any $\varepsilon > 0$, note that $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$, then

- ✓There exists $\delta_1 > 0$ such that $|f(x) - L| < \frac{\varepsilon}{2}$ when $|x - x_0| < \delta_1$
- ✓There exists $\delta_2 > 0$ such that $|g(x) - M| < \frac{\varepsilon}{2}$ when $|x - x_0| < \delta_2$

We pick $\delta = \min(\delta_1, \delta_2)$. Then for any $x \in S$ which $0 < |x - x_0| < \delta$, we have

$$\left|\big(f(x) + g(x)\big) - (L + M)\right| = |(f(x) - L) + (g(x) - M)|$$
$$\leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, we conclude that $\lim_{x \to x_0} \big(f(x) + g(x)\big) = L + M$.

Infinite limit

In some cases, it is possible that the function $f(x)$ may approach to infinity (either $+\infty$ or $-\infty$) when $x \to x_0$. In this case, we say $f(x)$ diverges to infinity when $x \to x_0$ and is denoted by $\lim_{x \to x_0} f(x) = +\infty$ or $-\infty$.

Definition (Infinite limit)

We let $f: S \to \mathbb{R}$ be a function. We say $\lim_{x \to x_0} f(x) = \infty$ (resp. $\lim_{x \to x_0} f(x) = -\infty$) if and only if for any $M > 0$, there exists $\delta > 0$ such that $f(x) > M$ (resp. $f(x) < -M$) for any $x \in S$ satisfying $0 < |x - x_0| < \delta$.

Example 4

Show that $\lim_{x \to 0} \frac{1}{x^2} = +\infty$ using the definition.

😊Solution

For any $M > 0$, we take $\delta = \frac{1}{\sqrt{M}}$. Then for any x which $0 < |x - 0| < \delta$, we have $\frac{1}{x^2} = \frac{1}{|x|^2} > M$. So we conclude that $\lim_{x \to 0} \frac{1}{x^2} = +\infty$.

Limit to infinity

In some cases, we are interested in examining the behavior of $f(x)$ when x tends to either $+\infty$ or $-\infty$. This can be done by examining the limits $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$.

Definition (Limits to positive infinity)

We let $f: \mathbb{R} \to \mathbb{R}$ be a function. We say $\lim_{x\to\infty} f(x) = L$ (or $f(x)$ converges to L when $x \to \infty$) if and only if for any $\varepsilon > 0$, there exists $K > 0$ such that $|f(x) - L| < \varepsilon$ for $x > K$.

Example 5

Show that $\lim_{x\to\infty} \frac{x+1}{3x} = \frac{1}{3}$ using the definition.

😊Solution

For any $\varepsilon > 0$, we take $K = \frac{1}{3\varepsilon}$. Then for any $x > K = \frac{1}{3\varepsilon} \Leftrightarrow \frac{1}{3x} < \varepsilon$, we have

$$\left|\frac{x+1}{3x} - \frac{1}{3}\right| = \left|\frac{1}{3x}\right| = \frac{1}{3x} < \varepsilon$$

Some properties of limits

The following theorem summarizes some basic properties of limits of functions. These properties are similar to those of limits of sequences.

Theorem 1 (Properties of Limits)

1. **(Computational formula)** We let $f: S \to \mathbb{R}$ and $g: S \to \mathbb{R}$ be two functions which $\lim_{x \to x_0} f(x) = L \in \mathbb{R}$ and $\lim_{x \to x_0} g(x) = M \in \mathbb{R}$. Then we have
   (a) $\lim_{x \to x_0} [f(x) \pm g(x)] = L \pm M$.
   (b) $\lim_{x \to x_0} f(x)g(x) = LM$
   (c) $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$, provided that $g(x) \neq 0$ for all $x \in S$ and $M \neq 0$.
2. **(Sandwich theorem)** We let f, g, h be 3 functions from S to $\mathbb{R}$ which $f(x) \leq g(x) \leq h(x)$ for all $x \in S$. Suppose that $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L$, then $\lim_{x \to x_0} g(x) = L$.
3. **(Limit inequality)** If $f(x) \leq g(x)$ for all $x \in S$, then $\lim_{x \to x_0} f(x) \leq \lim_{x \to x_0} g(x)$ provided that both limits exists.

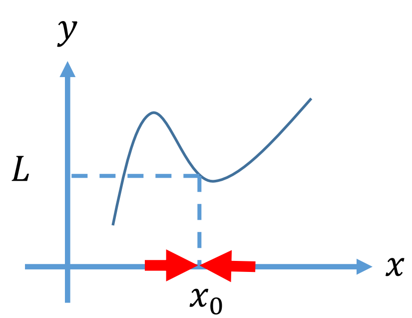
One useful tools for studying limits of function – Sequential Limit Theorem

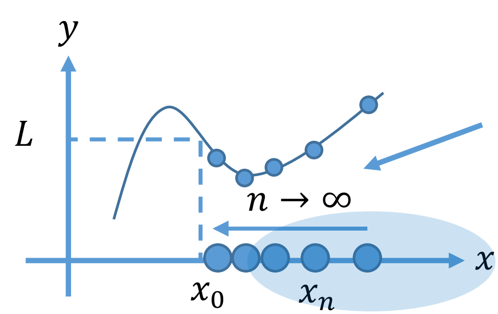
Although one can prove the above properties using the $\varepsilon - \delta$ definition, the derivation may be a bit tedious.

The following theorem, known as *sequential limit theorem*, allows us to study the limit of functions by studying limits of corresponding sequences. One advantage of this approach is that some theorems such as Bolzano-Weierstrass theorem can be applied.

Theorem 2 (Sequential limits theorem)

We let $f: S \to \mathbb{R}$ be a function. Then $\lim_{x \to x_0} f(x) = L$ if and only if $\lim_{n \to \infty} f(x_n) = L$ for *every convergent sequence* $\{x_n\}$ (with $x_n \in S \backslash \{x_0\}$) with $\lim_{n \to \infty} x_n = x_0$.





The sequence $\{x_n\}$ can be viewed as one of the paths that approaches to x_0. We expect that $f(x_n)$ should also approaches to L when $n \to \infty$.

Proof of sequential limits theorem

("$\Rightarrow$" part)

Suppose that $\lim_{x \to x_0} f(x) = L$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for any $|x - x_0| < \delta$.

Since $\lim_{n \to \infty} x_n = x_0$, then there exists $K \in \mathbb{N}$ such that $|x_n - x_0| < \delta$ for $n \geq K$.

This implies that $|f(x_n) - L| < \varepsilon$ for $n \geq K$.

So we conclude that $\lim_{n \to \infty} f(x_n) = L$.

("$\Leftarrow$" part)

We shall prove this by contradiction. Assume that $\lim_{x \to x_0} f(x) \neq L$, it follows that there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$, there exists $x \neq x_0$ such that $|x - x_0| < \delta$ and $|f(x) - L| > \varepsilon_0$ (i.e. negation of the definition).

😊Key step: In order to arrive a contradiction (contradict to given condition), one needs to construct a suitable convergent sequence $\{x_n\}$.

- For any $n \in \mathbb{N}$, we pick $\delta = \frac{1}{n}$, then there exists $x_n \neq x_0$ such that $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - L| > \varepsilon_0$.
- By repeating this for all $n \in \mathbb{N}$, we obtain a sequence $\{x_n\}$ which
  - ✓ $\lim_{n\to\infty} x_n = x_0$; (Since $x_0 - \frac{1}{n} < x_n < x_0 + \frac{1}{n}$ and $\lim_{n\to\infty}\left(x_0 - \frac{1}{n}\right) = \lim_{n\to\infty}\left(x_0 + \frac{1}{n}\right) = L$, it follows from sandwich theorem that $\lim_{n\to\infty} x_n = x_0$)
  - ✓ $|f(x_n) - L| > \varepsilon_0$ for all $n \in \mathbb{N}$.

Note that $\lim_{n\to\infty} f(x_n) = L$ (given condition). We pick $\varepsilon = \varepsilon_0$, there exists $K \in \mathbb{N}$ such that for $n \geq K$

$$|f(x_n) - L| < \varepsilon = \varepsilon_0.$$

This contradicts to the fact that $|f(x_n) - L| > \varepsilon_0$ for all $n \in \mathbb{N}$.

So we conclude that $\lim_{x\to x_0} f(x) = L$.

Application of sequential limit theorem 1 – Proving theorem 1

Practically, sequential limit theorem provides an alternative proof to the theorem 1 without using ε-δ definition.

Proof of (1)

We first prove $\lim_{x\to x_0}[f(x) \pm g(x)] = L + M$.

- We consider a sequence $\{x_n\}$ that converges to x_0. As $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, it follows from sequential limit theorem that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$.
- It follows from computational formula that $\lim_{n\to\infty}[f(x_n) + g(x_n)] = L + M$.

Since the argument is true for all $\{x_n\}$ converges to x_0, so $\lim_{x\to x_0}[f(x) \pm g(x)] = L + M$. The proof of $\lim_{x\to x_0}[f(x)g(x)] = LM$ and $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{L}{M}$ can be done similarly.

Proof of (2)

We consider a sequence $\{x_n\}$ that converges to x_0, we have

$$f(x_n) \leq g(x_n) \leq h(x_n)$$

- As $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} h(x_n) = L$ by sequence limit theorem (since $\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} h(x) = L$), it follows from sandwich theorem (sequence version) that $\lim\limits_{n\to\infty} g(x_n) = L$.
- Since the argument is true for all $\{x_n\}$ converges to x_0, so it follows from sequential limit theorem that $\lim\limits_{x\to x_0} g(x) = L$.

Proof of (3)

We consider a sequence $\{x_n\}$ that converges to x_0, we have

$$f(x_n) \leq g(x_n).$$

- Since $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{x\to x_0} f(x) = L$ and $\lim\limits_{n\to\infty} g(x_n) = \lim\limits_{x\to x_0} g(x) = M$ exists by sequential limits theorem, it follows from limits inequality that $\lim\limits_{x\to x_0} f(x) = \lim\limits_{n\to\infty} f(x_n) \leq \lim\limits_{n\to\infty} g(x_n) = \lim\limits_{x\to x_0} g(x)$.

<u>Application 2 – Proving some limits does not exist</u>

Example 6

Determine if the limits $\lim_{x\to 0} \sin\frac{1}{x}$ exists.

😊Solution

We consider two sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_n = \frac{1}{n\pi}$ for $n \in \mathbb{N}$ and $y_n = \frac{2}{(4n+1)\pi}$ for $n \in \mathbb{N}$.

- One can see that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n\pi} = 0$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{2}{(4n+1)\pi} = 0$.
- On the other hand, one can verify that

$$\lim_{n\to\infty} \sin\frac{1}{x_n} = \lim_{n\to\infty} \underbrace{\sin(n\pi)}_{=0} = 0 \quad \text{and} \quad \lim_{n\to\infty} \sin\frac{1}{y_n} = \lim_{n\to\infty} \underbrace{\sin\frac{(4n+1)\pi}{2}}_{=1} = 1.$$

Since $\lim_{n\to\infty} \sin\frac{1}{x_n} \neq \lim_{n\to\infty} \sin\frac{1}{y_n}$, it follows from the contrapositive of sequential limits theorem that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

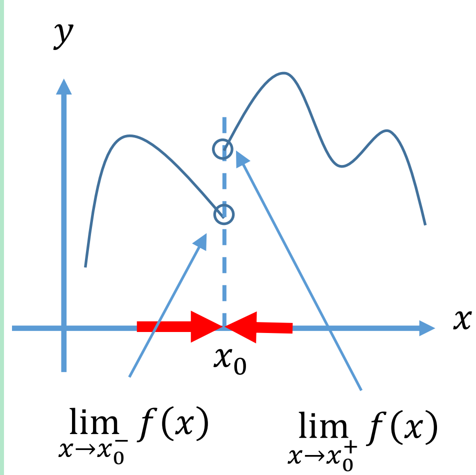
One sided limits -- Left-hand limits and Right-hand limits

Roughly speaking, these limits study the behavior of $f(x)$ when x approaches to x_0 in a particular direction (i.e. either from the left ($x \to x_0^-$) or from the right ($x \to x_0^+$)). Practically, one will use it to examine the existence of limits $\lim_{x \to x_0} f(x)$ or examine the limits of $f(x)$ when $f(x)$ is not defined on one side.

Definition (Left-hand limits and right-hand limits)

We let $f: (a, b) \to \mathbb{R}$ be a function.

- We say $\lim_{x \to x_0^-} f(x) = L \in \mathbb{R}$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for any $x \in (-\infty, x_0) \cap S$ satisfying $|x - x_0| < \delta$.
- We say $\lim_{x \to x_0^+} f(x) = L \in \mathbb{R}$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for any $x \in (x_0, \infty) \cap S$ satisfying $|x - x_0| < \delta$.



Theorem 3

We let $f: (a, b) \to \mathbb{R}$ be a function. Then

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L.$$

Proof of Theorem 3

("⇒" part) Given that $\lim_{x \to x_0} f(x) = L$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ if $0 < |x - x_0| < \delta$.

This implies that if $x < x_0$ and $|x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$. So $\lim_{x \to x_0^-} f(x) = L$

This implies that if $x > x_0$ and $|x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$. So $\lim_{x \to x_0^+} f(x) = L$

("⇐" part) Note that $\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = L$. For any $\varepsilon > 0$, there exists $\delta_1, \delta_2 > 0$ such that

- If $x < x_0$ and $|x - x_0| < \delta_1$, then $|f(x) - L| < \varepsilon$ (from $\lim_{x \to x_0^-} f(x) = L$).
- If $x > x_0$ and $|x - x_0| < \delta_2$, then $|f(x) - L| < \varepsilon$ (from $\lim_{x \to x_0^+} f(x) = L$).

Take $\delta = \min(\delta_1, \delta_2)$, then we have $|f(x) - L| < \varepsilon$ for any x satisfying $|x - x_0| < \delta$. So we deduce that $\lim_{x \to x_0} f(x) = L$.

Example 7 (Application of one-sided limits)

We let $f(x) = \frac{|x|}{x}$. Does the limits $\lim_{x \to 0} f(x)$ exist? How about $\lim_{x \to 2} f(x)$?

☺Solution:

Recall that $|x| = \begin{cases} x & if \ \ x \geq 0 \\ -x & if \ \ x < 0 \end{cases}$. We consider the left hand limit and right hand limit:

$$\lim_{x \to 0^+} \frac{|x|}{x} = \underbrace{\lim_{x \to 0^+} \frac{x}{x}}_{\substack{x>0 \\ \Rightarrow |x| = x}} \underbrace{=}_{x \neq 0} \lim_{x \to 0^+} 1 = 1, \qquad \lim_{x \to 0^-} \frac{|x|}{x} = \underbrace{\lim_{x \to 0^-} \frac{-x}{x}}_{\substack{x>0 \\ \Rightarrow |x| = -x}} = \lim_{x \to 0^-} (-1) = -1.$$

Since $\lim_{x \to 0^+} \frac{|x|}{x} \neq \lim_{x \to 0^-} \frac{|x|}{x}$, so $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

On the other hand, note that $|x| = x$ for x is near 2, thus

$$\lim_{x \to 2} \frac{|x|}{x} = \lim_{x \to 2} \frac{x}{x} = \lim_{x \to 2} 1 = 1.$$

Remark: Since the limits consider the behavior of $f(x)$ when x is close to x_0, thus we only consider the value of $f(x)$ when x is near 2 (i.e. $x > 0$) when computing the second limits.

Example 8

Let $[x]$ be the greatest integer less or equal to x (e.g. $[7.2] = 7, [7.9] = 7, [7] = 7$). Determine whether the limits $\lim_{x\to 3}[x]$ exist.

☺Solution:

We consider the left-hand limits and the right-hand limits again.

When x approaches to 3 from the left, $x \to$ 2.xx... so that $[x] = 2$. Then

$$\lim_{x\to 3^-}[x] = \lim_{x\to 3^-} 2 = 2.$$

When x approaches to 3 from the right, $x \to$ 3.xx... so that $[x] = 3$. Then

$$\lim_{x\to 3^+}[x] = \lim_{x\to 3^+} 3 = 3.$$

Since $\lim_{x\to 3^-}[x] \neq \lim_{x\to 3^+}[x]$, we conclude that $\lim_{x\to 3}[x]$ does not exist.

*Note: Similar to earlier example, we just need to consider the value of $[x]$ when x is close to 3 when computing the limits.

Monotone function

Roughly speaking, monotonic function is a function $f: S \to \mathbb{R}$ which is either increasing or decreasing over the entire domain S. More precisely, we say

- $f(x)$ is increasing on S if and only if $f(x) \le f(y)$ for any $x, y \in S$ with $x < y$ (We say $f(x)$ is strictly increasing if the strict inequality holds)
- $f(x)$ is decreasing on S if and only if $f(x) \ge f(y)$ for any $x, y \in S$ with $x < y$ (We say $f(x)$ is strictly decreasing if the strict inequality holds)
- $f(x)$ is monotone (resp. strictly monotone) if and only if $f(x)$ is either increasing (resp. strictly increasing) or decreasing (resp. strictly decreasing).

Theorem 4 (Monotone function theorem)

Suppose that a function $f: (a, b) \to \mathbb{R}$ is monotonic, then for any $x_0 \in (a, b)$, the one-sided limits $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ both exists. Furthermore, we have either $\lim_{x \to x_0^-} f(x) \le f(x_0) \le \lim_{x \to x_0^+} f(x)$ or $\lim_{x \to x_0^-} f(x) \ge f(x_0) \ge \lim_{x \to x_0^+} f(x)$.

Proof of Theorem 4

We consider the case when $f(x)$ is increasing (The case when $f(x)$ is decreasing can be proved in a similar manner).

- We consider the set $A = \{f(x) | x < x_0\}$.
  - ✓ Since $f(x) \leq f(x_0)$ for any $x < x_0$, so $f(x_0)$ is the upper bound of the set A and $\sup A$ exists.

  Next, we argue that $\lim_{x \to x_0^-} f(x) = \sup A$.
  - ✓ For any $\varepsilon > 0$, $\sup A - \varepsilon$ is not the upper bound of A and there is $x^* < x_0$ such that $\sup A - \varepsilon < f(x^*) \leq \sup A$.
  - ✓ We take $\delta = x_0 - x^*$. Then for any $x < x_0$ satisfying $|x - x_0| < \delta$ (or $x > x_0 - \delta = x^*$), we get

$$|f(x) - \sup A| = \sup A - f(x) < \sup A - f(x^*)$$
$$< \sup A - (\sup A - \varepsilon) = \varepsilon.$$

  Therefore, we conclude that $\lim_{x \to x_0^-} f(x) = \sup A$.

- We consider the set $B = \{f(x) | x > x_0\}$.
  - ✓ Since $f(x) \geq f(x_0)$ for any $x > x_0$, so $f(x_0)$ is the lower bound of the set B and $\inf B$ exists.

  Next, we argue that $\lim_{x \to x_0^+} f(x) = \inf B$.
  - ✓ For any $\varepsilon > 0$, $\inf B + \varepsilon$ is not the lower bound of B and there is $x^{**} > x_0$ such that $\inf B + \varepsilon > f(x^{**}) \geq \inf B$.
  - ✓ We take $\delta = x^{**} - x_0 > 0$.

    Then for any $x > x_0$ satisfying $|x - x_0| < \delta$ (or $x < x_0 + \delta = x^{**}$), we get

    $$|f(x) - \inf B| = f(x) - \inf B < f(x^{**}) - \inf B < (\inf B + \varepsilon) - \inf B = \varepsilon.$$

  Therefore, we conclude that $\lim_{x \to x_0^+} f(x) = \inf B$.

Finally, $f(x_0)$ is upper bound of A and $f(x_0)$ us also the lower bound of B, it follows that $\lim_{x \to x_0^-} f(x) = \sup A \leq f(x_0) \leq \inf B = \lim_{x \to x_0^+} f(x)$.

Continuity of function

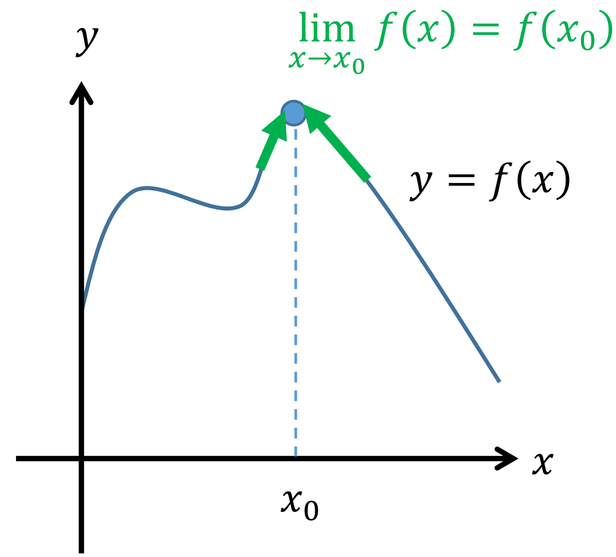
Roughly speaking, a continuous function $f(x)$ is a function which the graph $y = f(x)$ is continuous (with no jumps, no breaks).

More precisely, we say a function $f: S \to \mathbb{R}$ is continuous at $x = x_0 \in S$ if the value of $f(x)$ approaches to $f(x_0)$ when x approaches to x_0.

Definition (Continuity of function $\boldsymbol{f(x)}$)
We say a function $f: S \to \mathbb{R}$ is continuous at $x = x_0 \in S$ if and only if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Furthermore, we say a function is <u>continuous</u> on S if it is continuous at every point of its domain S.

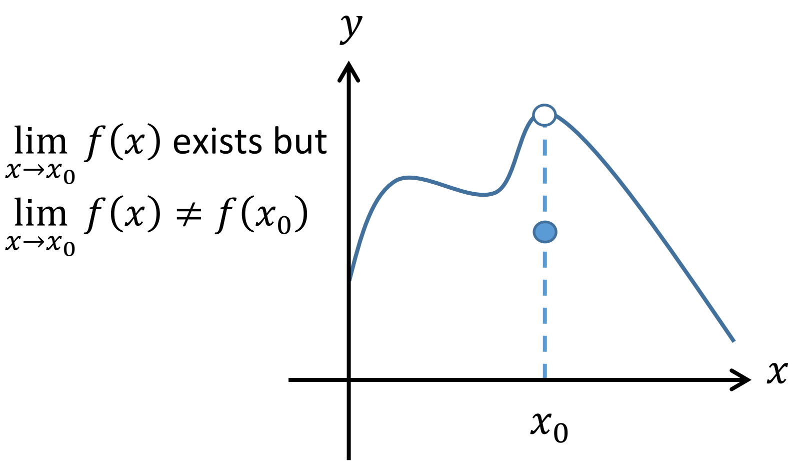


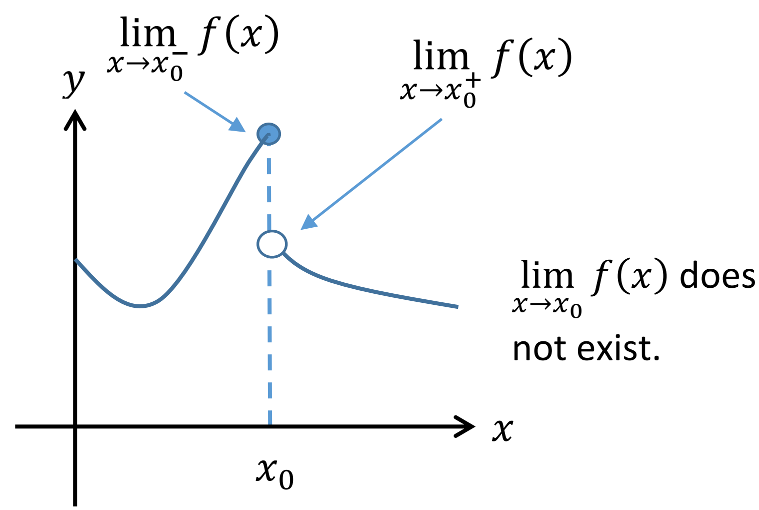
Remark:

- (Another definition of continuity) Using the definition of limits, one can rephrase the definition of continuity as follows:

> We say a function $f: S \to \mathbb{R}$ is continuous at $x = x_0 \in S$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for any $x \neq x_0$ with $0 < |x - x_0| < \delta$.

- If $\lim\limits_{x \to x_0} f(x) = f(x_0)$ does not hold, we say $f(x)$ is *discontinuous* at $x = x_0$. Here are some examples:





Example 9

Consider the function

$$f(x) = \begin{cases} \dfrac{\sin 3x}{x} & if \;\; x \neq 0 \\ 2 & if \;\; x = 0 \end{cases}.$$

Determint if $f(x)$ continuous at $x = 0$?

☺Solution:

Step 1: First, note that $f(0) = 2$ by definition.

Step 2: $\lim\limits_{x\to 0} f(x) \overset{x\neq 0}{\triangleq} \lim\limits_{x\to 0} \dfrac{\sin 3x}{x} = \lim\limits_{x\to 0} 3\left(\dfrac{\sin 3x}{3x}\right) \overset{\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta}=1}{\triangleq} 3 \times 1 = 3.$

Step 3: $\lim\limits_{x\to 0} f(x) = 3 \neq 2 = f(0)$.

Therefore, we conclude that $f(x)$ is not continuous at $x = 0$.

Example 10

We consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & if \;\; x \in \mathbb{Q} \\ 0 & if \;\; x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}.$$

Show that $f(x)$ is discontinuous at any point x_0 on $\mathbb{R}$.

😊Solution

For any $\varepsilon = \frac{1}{n} > 0$ (where $n \in \mathbb{N}$), one can deduce that there exists a rational number $q_n \in \mathbb{Q}$ and an irrational number $r_n \in \mathbb{R}\backslash\mathbb{Q}$ such that

$$x_0 - \frac{1}{n} < q_n < x_0 \quad and \quad x_0 - \frac{1}{n} < r_n < x_0.$$

By repeating this for all $n \in \mathbb{N}$, we obtain two sequences $\{q_n\}$ and $\{r_n\}$. Since $\lim_{n\to\infty}\left(x_0 - \frac{1}{n}\right) = \lim_{n\to\infty} x_0 = x_0$, it follows from sandwich theorem that $\lim_{n\to\infty} q_n = \lim_{n\to\infty} r_n = x_0$.

- For the sequence $\{q_n\}$ and $q_n \in \mathbb{Q}$, we have $\lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} 1 = 1$;
- For the sequence $\{r_n\}$ and $r_n \in \mathbb{R}\backslash\mathbb{Q}$, we have $\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 0 = 0$;

Since $\lim_{n\to\infty} f(q_n) \neq \lim_{n\to\infty} f(r_n)$, it follows from sequential limit theorem that $\lim_{x\to x_0} f(x)$ does not exist. Hence, the function is not continuous at any $x_0 \in \mathbb{R}$.

Example 11 (Discontinuity of monotone function)

We let $f: \mathbb{R} \to \mathbb{R}$ be a monotone function. Show that $f(x)$ has countably many discontinuous points.

😊Solution

We consider the case when $f(x)$ is increasing (the case when $f(x)$ is decreasing is similar).We let $x_\alpha \in \mathbb{R}$ be discontinuous point of $f(x)$. It follows from monotone function theorem that

$$\lim_{x \to x_\alpha^-} f(x) \le f(x_\alpha) \le \lim_{x \to x_\alpha^+} f(x).$$

Since $f(x)$ is not continuous at $x = x_\alpha$, it must be that $\lim_{x \to x_\alpha^-} f(x) < \lim_{x \to x_\alpha^+} f(x)$ (Otherwise, we get $\lim_{x \to x_\alpha^-} f(x) = f(x_0) = \lim_{x \to x_\alpha^+} f(x)$ and $f(x)$ is continuous at $x = x_\alpha$)

By density of rational number, there is $q_\alpha \in \mathbb{Q}$ such that $\lim_{x \to x_\alpha^-} f(x) < q_\alpha < \lim_{x \to x_\alpha^+} f(x)$.

So, we "*label*" the discontinuity point as x_{q_α}. Then the set D can be expressed as

$$D = \{x_{q_\alpha} | \alpha \in I\}, \ \ where \ \ I \ is \ some \ index \ set.$$

- Since $A = \{q_\alpha | \alpha \in I\} \subseteq \mathbb{Q}$ and $\mathbb{Q}$ is countable, it follows from countable subset theorem that A is also countable.
- By countable union theorem, the set $D = \{x_{q_\alpha} | \alpha \in I\} = \bigcup_{q_\alpha \in I} \underbrace{\{x_{q_\alpha}\}}_{\substack{1 \ element \\ (countable)}}$ is also countable.

Basic properties of continuous function

In this section, we shall explore various property of continuous functions.

Property 1

We let $f: S \to \mathbb{R}$ and $g: S \to \mathbb{R}$ be two functions which are continuous at $x = x_0$. Then the function (i) $f(x) \pm g(x)$, (ii) $f(x)g(x)$ and (iii) $\frac{f(x)}{g(x)}$ (provided that $g(x_0) \neq 0$) are all continuous at $x = x_0$.

Proof of property 1

Since both $f(x)$ and $g(x)$ are continuous at $x = x_0$, we have $\lim\limits_{x \to x_0} f(x) = f(x_0)$ and $\lim\limits_{x \to x_0} g(x) = g(x_0)$. It follows from properties of limits that

$$\lim_{x \to x_0} \big(f(x) \pm g(x)\big) = f(x_0) \pm g(x_0), \quad \lim_{x \to x_0} \big(f(x)g(x)\big) = f(x_0)g(x_0)$$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f(x_0)}{g(x_0)}.$$

This prove that those three functions are continuous at $x = x_0$.

Property 2 (Composite function)

We let $f: S \to \mathbb{R}$ be a function which is continuous at x_0 and $g: S' \to \mathbb{R}$ (with $f(S) \subseteq S'$) be another function which is continuous at $f(x_0)$. Then the function $(g \circ f)(x) = g\big(f(x)\big)$ is also continuous at $x = x_0$.

Proof of property 2:

We shall verify $\lim\limits_{x \to x_0} g\big(f(x)\big) = g\big(f(x_0)\big)$ using sequential limit theorem.

For any sequence $\{x_n\}$ with $\lim\limits_{n \to \infty} x_n = x_0$,

- Note that $f(x)$ is continuous at $x = x_0$, we have $\lim\limits_{n \to \infty} f(x_n) = \lim\limits_{x \to x_0} f(x) = f(x_0)$.
- On the other hand, $g(y)$ is continuous at $y = f(x_0)$, then for any sequence $\{y_n\}$ with $\lim\limits_{n \to \infty} y_n = f(x_0)$, we have $\lim\limits_{n \to \infty} g(y_n) = g\big(f(x_0)\big)$
- Since the sequence $\{f(x_n)\}$ converges to $f(x_0)$, one can take $y_n = f(x_n)$ and get
$$\lim_{n \to \infty} g\underbrace{\big(f(x_n)\big)}_{y_n} = g\big(f(x_0)\big).$$
So it follows sequential limit theorem that $\lim\limits_{x \to x_0} g\big(f(x)\big) = g\big(f(x_0)\big)$ and $g\big(f(x)\big)$ is continuous at $x = x_0$.

Example 12 (Application of propery 2)

We let $f, g: S \to \mathbb{R}$ be two continuous functions on S.

(a) Show that $|f(x)|$ is continuous on S.

(b) Show that the function $h(x) = \max\big(f(x), g(x)\big)$ is also continuous on S.

😊Solution

(a) We let $p: \mathbb{R} \to \mathbb{R}$ be $p(x) = |x|$. Since $p(x)$ is continuous on $\mathbb{R}$ and $f(S) \subseteq \mathbb{R}$, it follows from property 2 that $|f(x)| = p\big(f(x)\big)$ is also continuous on S.

(b) One can verify that

$$\max\big(f(x), g(x)\big) = \frac{f(x) + g(x)}{2} + \frac{1}{2}|f(x) - g(x)|.$$

✓ Since $f(x)$, $g(x)$ are continuous, then both $f(x) + g(x)$ and $f(x) - g(x)$ are continuous from property 1.

✓ According the result of (a), it follows that $|f(x) - g(x)|$ is also continuous on S.

✓ It follows from property 1 that $\max\big(f(x), g(x)\big) = \frac{f(x)+g(x)}{2} + \frac{1}{2}|f(x) - g(x)|$ is also continuous on S.

Property 3 (Sign preserving property)

We let $f: S \to \mathbb{R}$ be a continuous function on S with $f(x_0) > 0$ for some $x_0 \in S$. Then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in S \cap (x_0 - \delta, x_0 + \delta)$.

Proof of property 3

Since $f(x)$ is continuous at $x = x_0$, we have $\lim_{x \to x_0} f(x) = f(x_0) > 0$.

By taking $\varepsilon = \frac{f(x_0)}{2} > 0$, then it follows from definition of limits that there exists $\delta > 0$ such that for any x satisfying $|x - x_0| < \delta$,

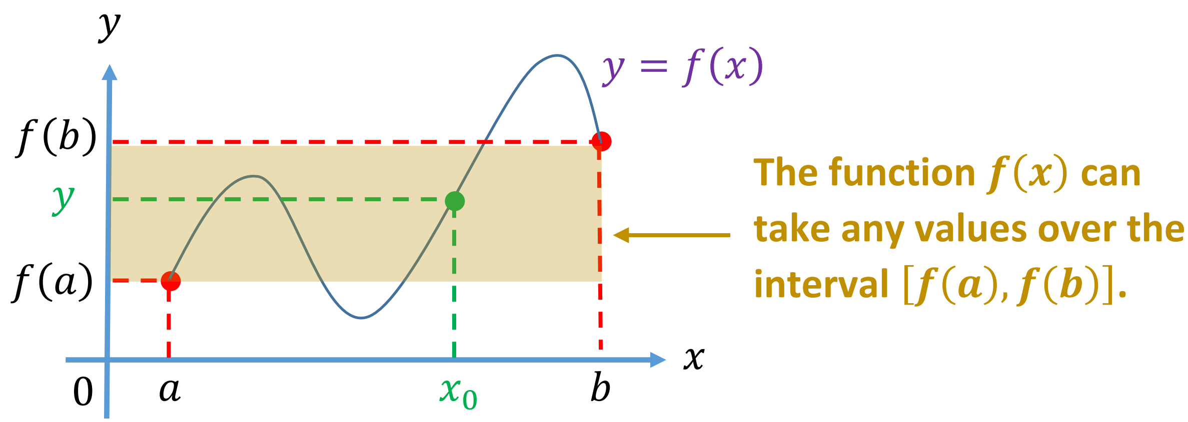
$$|f(x) - f(x_0)| < \varepsilon = \frac{f(x_0)}{2} \Rightarrow f(x) > f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2} > 0.$$

Remark of property 3

- If $f(x_0) < 0$, one can show that there exists $\delta > 0$ such that $f(x) < 0$ for all $x \in S \cap (x_0 - \delta, x_0 + \delta)$
- One can extend the property to a more general case: If $f(x_0) > c$ (where c is some real number), then there exists $\delta > 0$ such that $f(x) > c$ for $x \in S \cap (x_0 - \delta, x_0 + \delta)$. (To prove this, we take $g(x) = f(x) - c$ and use property 2)

Property 4 (Intermediate value theorem)

We let $f: [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ with $f(a) \neq f(b)$. We let y be any point between $f(a)$ and $f(b)$, where $y \neq f(a), f(b)$. Then there exists $x_0 \in (a, b)$ such that $f(x_0) = y$.



Proof of property 4

We shall prove the theorem by <u>constructing</u> x_0 which $f(x_0) = y$. To facilitate the analysis, we assume that $f(a) < f(b)$ (the case for $f(a) > f(b)$ can be proved in a similar way). To do so, we consider the set

$$S = \{x \in [a, b] | f(x) > y\}.$$

Since $y < f(b)$, then S is non-empty. As $x \geq a$ for all $x \in S$, so S is bounded below by a and the infimum of S exists.

We take $x_0 = \inf S$. It remains to argue that $f(x_0) = y$.

Suppose that $f(x_0) \neq y$, we consider the following two cases:

- If $f(x_0) < y$, then it follows from sign-preserving property (as $f(x)$ is continuous) that there exists $\delta > 0$ such that $f(x) < y$ for *all* $x \in (x_0 - \delta, x_0 + \delta)$. On the other hand, $x_0 + \delta$ is not lower bound of S (as $x_0 = \inf S$) and there exists $x_1 \in S$ such that $x_0 \leq x_1 < x_0 + \delta$ and $f(x_1) > y$, this leads to contradiction.
- If $f(x_0) > y$, then it follows from sign-preserving property (as $f(x)$ is continuous) that there exists $\delta > 0$ such that $f(x) > y$ for *all* $x \in (x_0 - \delta, x_0 + \delta)$. However, it implies that $(x_0 - \delta, x_0) \subseteq S$ and $x_0 = \inf S$ is not the lower bound. This leads to contradiction.

So it follows that $f(x_0) = y$. The proof is completed.

Remark (Application of intermediate value theorem)

Technically, the intermediate value theorem is often used to prove the existence of solution of an equation. On the other hand, it is also used in developing some numerical methods for solving an equation numerically (e.g. bisection method).

Example 13

We consider the equation $3e^{x^2} - x^3 - 10 = 0$.

(a) Show that the equation has a solution in $[0, 2]$.

(b) Show that the equation has at least two real solutions.

😊Solution

(a) We let $f(x) = 3e^{x^2} - x^3 - 10$. Our goal is to locate the point x such that $f(x) = 0$. Note that

- $f(0) = -10 < 0$ and $f(2) = 145.79 > 0$.

As $0 \in (-10, 145.79)$, it follows from intermediate value theorem that there is $x_0 \in (0, 2)$ such that $f(x_0) = 0$ (which is the solution of the equation).

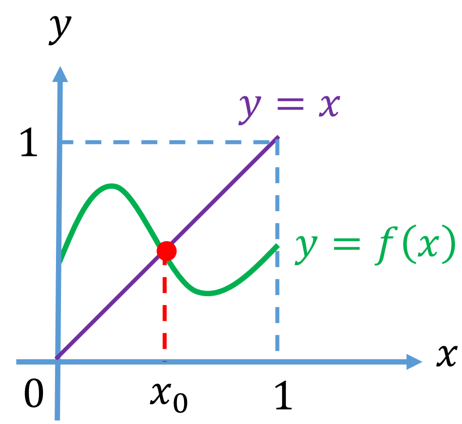
(b) It suffices to find another solution $x_1 \neq x_0$ which $f(x_1) = 0$. We consider the interval $(-2,0)$. Note that

- $f(0) = -10 < 0$ and $f(-2) = 161.79 > 0$.

As $0 \in (-10, 161.79)$, it follows that there is $x_1 \in (-2, 0)$ and $x_1 \neq x_0$ such that $f(x_1) = 0$. Combining with the result in (a), we conclude that the equation has at least two solutions (namely, x_0 and x_1).

Example 14

We let $f: [0,1] \to \mathbb{R}$ be a continuous function which $f(0) > 0$ and $f(1) < 1$. Prove that there exists $x_0 \in (0,1)$ such that $f(x_0) = x_0$. (*Here, x_0 is called *fixed point of the function* f)



😊Solution

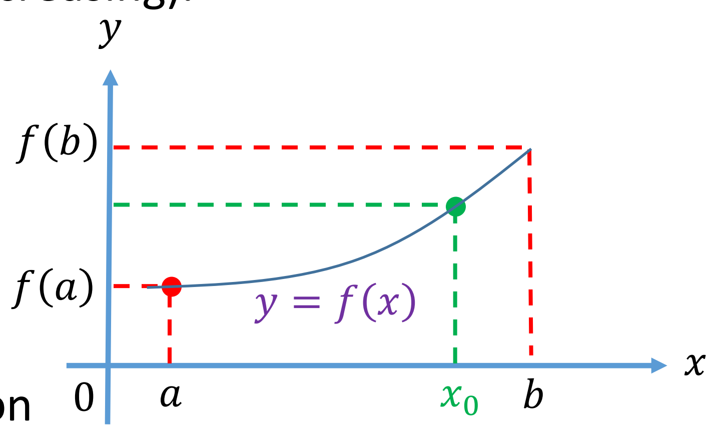
We let $g(x) = f(x) - x$ be a function. Since both $f(x)$ and x are continuous over $[0,1]$, then $g(x)$ is also continuous over $[0,1]$.

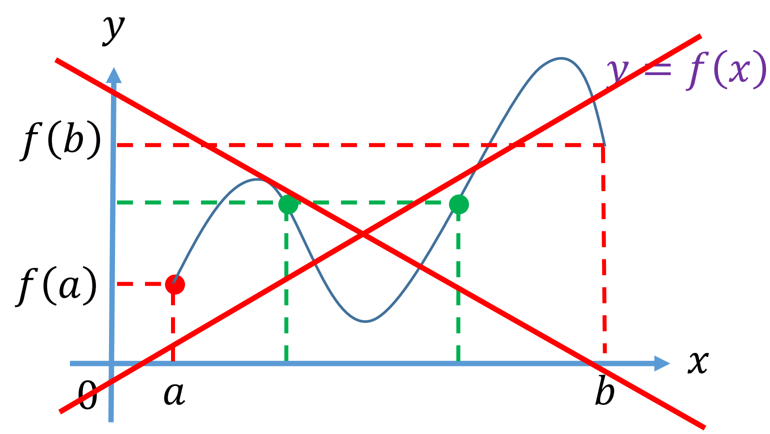
Since $g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$, it follows from intermediate value theorem that there exists $x_0 \in (0,1)$ such that

$$g(x_0) = 0 \Rightarrow f(x_0) = x_0.$$

Example 15 (Harder)

We let $f: [a, b] \to \mathbb{R}$ be a continuous function. Suppose that f is also injective, prove that f is strictly monotone (i.e. the function is either strictly increasing or strictly decreasing).





😊Solution

Since $f(x)$ injective, it must be that either $f(a) < f(b)$ or $f(a) > f(b)$.

We first consider the case when $f(a) < f(b)$ and we shall argue that $f(x)$ is strictly increasing over $[a, b]$. It consists of two steps:

- For any $x \in (a, b)$, we argue that $f(a) < f(x) < f(b)$.
  - Suppose that $f(x) \geq f(b) > f(a)$, it must be that $f(x) > f(b)$ (since $x \neq b$ and f is injective).

- We take $y_0 \in \big(f(b), f(x)\big)$, then it follows from intermediate value theorem that there exists $x_1 \in (a, x)$ and $x_2 \in (x, b)$ such that $f(x_1) = y_0$ and $f(x_2) = y_0$. As $x_1 \neq x_2$, so $f(x)$ is not injective which leads to contradiction.

➢ Next, we argue that $f(x) < f(y)$ for any $a < x < y < b$.

- Suppose that $f(x) \geq f(y)$ for some $x < y$ ($x, y \in (a, b)$), then it must be that $f(x) > f(y)$ (or f is not injective).
- We take $z_0 \in \big(f(y), f(x)\big)$ and since $z_0 > f(y) > f(a)$ and $z_0 < f(x) < f(b)$ (as shown in the first part) , then it follows from intermediate value theorem that there exists $x_3 \in (a, x)$ and $x_4 \in (x, y)$ such that $f(x_3) = z_0$ and $f(x_4) = z_0$. As $x_3 \neq x_4$, so $f(x)$ is not injective which leads to contradiction.

➢ So it follows that $f(x)$ is strictly increasing if $f(a) < f(b)$.

- Using similar method, one can deduce that $f(x)$ is strictly decreasing if $f(a) > f(b)$ (left as exercise).

Property 5 (Extreme value theorem)

We let $f: [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ (where $a < b$). Then $f(x)$ is bounded and there exists $x_L, x_U \in [a, b]$ such that

$$f(x_L) = \inf\{f(x)|x \in [a, b]\}, \qquad f(x_U) = \sup\{f(x)|x \in [a, b]\}$$

Remark of property 5

- Note that the theorem may not hold if the interval $[a, b]$ is replaced by other intervals such as (a, b) (open interval), (a, ∞) (bounded interval). For example,
  - ✓ We let $f: (0,1] \to \mathbb{R}$ to be $f(x) = \frac{1}{x^2}$. Although $f(x) = \frac{1}{x^2}$ is continuous over $(0,1]$, $f(x)$ is not bounded from above as $\lim_{x \to 0^+} f(x) = \infty$.
  - ✓ We let $g: [0, \infty) \to \mathbb{R}$ to be $g(x) = e^{-x}$. We observe that $g(x)$ is decreasing and $\lim_{x \to \infty} e^{-x} = 0$. One cannot find $x_L \in \mathbb{R}$ which $e^{-x_L} = 0$. It is because the interval $[0, \infty)$ is *unbounded*.
- Using the intermediate value theorem, one can show that the range of f is $f([a, b]) = \{f(x)|x \in [a, b]\} = [f(x_L), f(x_U)]$ (why?).

Proof of property 5

We first argue that $f(x)$ is bounded over $[a, b]$ so that both $\inf\{f(x)|x \in [a, b]\}$ and $\sup\{f(x)|x \in [a, b]\}$ exist.

- ✓ Suppose that $f(x)$ is not bounded from above (i.e. no upper bound), then for *every* $M = n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $f(x_n) \geq n$.
- ✓ Since the sequence $\{x_n\}$ is bounded (as $x_n \in [a, b]$), it follows from Bolzano-Weierstress theorem that there exists a convergent subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = x^* \in [a, b]$ and $f(x_{n_k}) > n_k$.
- ✓ As $f(x)$ is continuous at $x = x^*$, it follows that $f(x^*) = \lim_{k\to\infty} f(x_{n_k})$. However since $f(x_{n_k}) > n_k$ for all $k \in \mathbb{N}$, so $f(x^*) = \lim_{k\to\infty} f(x_{n_k}) = +\infty$ and this leads to contradiction.
- ✓ So $\{f(x)|x \in [a, b]\}$ must be bounded from above. Using similar method, one can show that $\{f(x)|x \in [a, b]\}$ must be bounded from below. So $f(x)$ is bounded.

Next, we argue that there exists $x_L, x_U \in [a,b]$ such that

$$f(x_L) = \underbrace{\inf\{f(x) | x \in [a,b]\}}_{m}, \qquad f(x_U) = \underbrace{\sup\{f(x) | x \in [a,b]\}}_{M}.$$

- ✓ Using infimum property (with $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$), there exists $x'_n \in [a,b]$ such that $m \le f(x'_n) < m + \varepsilon = m + \frac{1}{n}$.
- ✓ Since the sequence $\{x'_n\}$ is bounded (as $x'_n \in [a,b]$), it follows from Bolzano-Weierstress theorem that there exists a convergent subsequence $\{x'_{n_k}\}$ such that $\lim_{k\to\infty} x'_{n_k} = x_L \in [a,b]$ and $m \le f\left(x'_{n_k}\right) < m + \frac{1}{n_k}$.
- ✓ Since $\lim_{k\to\infty} m + \frac{1}{n_k} = m$ and $f(x)$ is continuous at $x = x_L$, it follows from sandwich theorem that

$$f(x_L) = \lim_{k\to\infty} f\left(x'_{n_k}\right) = m = \inf\{f(x) | x \in [a,b]\}.$$

- ✓ Using similar method, one can deduce that there exists $x_U \in [a,b]$ such that

$$f(x_U) = \sup\{f(x) | x \in [a,b]\}.$$

Example 16

We let $f: \mathbb{R} \to [0, \infty)$ be a continuous function which $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$. Show that $f(x)$ must be bounded.

😊Solution

It is clear that $f(x)$ is bounded from below by 0. It remains to show $f(x)$ is bounded from above.

Note that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$.

We pick $\varepsilon = 1 > 0$, there exists $K_1 > 0$ and $K_2 > 0$ such that

$$|f(x) - 0| < \varepsilon = 1 \Leftrightarrow 0 \leq f(x) < 1 \quad for \;\; x < -K_2 \quad and$$

$$|f(x) - 0| < \varepsilon = 1 \Leftrightarrow 0 \leq f(x) < 1 \quad for \;\; x > K_1.$$

On the other hand, $f(x)$ is continuous over the interval $[-K_2, K_1]$. It follows from extreme value theorem that there exists $x^* \in [-K_2, K_1]$ such that

$$f(x^*) = \sup\{f(x)|x \in [-K_2, K_1]\}.$$

By taking $M = \max(f(x^*), 1)$, we can deduce that $f(x) \leq M$ for all $x \in \mathbb{R}$ so that $f(x)$ is also bounded from above.

Continuous function and inverse function

We let $f:[a,b]\to\mathbb{R}$ be a function. Recall that if f is injective, then there exists an inverse function $f^{-1}: f([a,b])\to[a,b]$.

Suppose that f is continuous on $[a,b]$, the following theorem confirms that the corresponding inverse is also continuous.

Property 6 (Continuous inverse theorem)

Suppose that $f:[a,b]\to\mathbb{R}$ is continuous and injective on $[a,b]$ (where $a<b$), then the inverse function f^{-1} exists and is continuous over $f([a,b])$. Here, $f([a,b])=[f(a),f(b)]$ or $[f(b),f(a)]$

Remark:

For example, we take $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$ to be $f(x)=\sin x$. Note that $\sin x$ is continuous and injective over $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so the inverse $\sin^{-1}x$ is also continuous on $f\left(\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right)=[-1,1]$.

Proof of property 6

According to the result in Example 15, we note that $f(x)$ is strictly monotone. On the other hand, since $f(x)$ is injective over $[a,b]$, then there exists an inverse function $f^{-1}: f([a,b])\to[a,b]$.

To facilitate the analysis, we consider the case when $f(x)$ is strictly increasing (the case when $f(x)$ is strictly decreasing can be deduced in a similar manner). In this case, we have $f([a,b]) = [f(a), f(b)]$.

😊IDEA: To show f^{-1} is continuous at any $y_0 \in \left(f(a), f(b)\right)$, one needs to verify that $\lim_{y\to y_0} f^{-1}(y) = f^{-1}(y_0)$. We first prove the existence of the limits $\lim_{y\to y_0} f^{-1}(y)$.

We first argue that f^{-1} is also strictly increasing over $y \in [f(a), f(b)]$.

- For any $f(a) \leq y_1 < y_2 \leq f(b)$, there exists $x_1, x_2 \in [a,b]$ such that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$.
- If $x_1 \geq x_2$, it follows that $f(x_1) \geq f(x_2) \Rightarrow y_1 \geq y_2$ which leads to contradiction. So we have $f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2)$ and f^{-1} is strictly increasing.

Since f^{-1} is strictly increasing, then the one-sided limits $\lim_{y\to y_0^-} f^{-1}(y)$ and $\lim_{y\to y_0^+} f^{-1}(y)$ both exists by monotone function theorem.

- As $\lim_{y\to y_0^-} f^{-1}(y) = L$. For any sequence $\{y_n\}$ (with $y_n < y_0$) converges to y_0, we have $\lim_{n\to\infty} f^{-1}(y_n) = L$. Since $x_n = f^{-1}(y_n) \in [a,b]$, $\{x_n\}$ converges to L and f is continuous, it follows from the definition that

$$\lim_{n\to\infty} f(x_n) = f(L) \overset{\lim_{n\to\infty} f(x_n)=\lim_{n\to\infty} f\left(f^{-1}(y_n)\right)=\lim_{n\to\infty} y_n=y_0}{\Rightarrow} y_0 = f(L).$$

$$\Rightarrow \lim_{y\to y_0^-} f^{-1}(y) = L = f^{-1}(y_0).$$

- As $\lim_{y\to y_0^+} f^{-1}(y) = M$. For any sequence $\{y_n'\}$ (with $y_n' > y_0$) converges to y_0, we have $\lim_{n\to\infty} f^{-1}(y_n') = M$. Since $x_n' = f^{-1}(y_n') \in [a, b]$, $\{x_n'\}$ converges to M and f is continuous, it follows from the definition that

$$\lim_{n\to\infty} f(x_n') = f(M) \overset{\lim_{n\to\infty} f(x_n')=\lim_{n\to\infty} f\left(f^{-1}(y_n')\right)=\lim_{n\to\infty} y_n'=y_0}{\Rightarrow} y_0 = f(M).$$

$$\Rightarrow \lim_{y\to y_0^+} f^{-1}(y) = M = f^{-1}(y_0).$$

- Since $\lim_{y\to y_0^+} f^{-1}(y) = \lim_{y\to y_0^-} f^{-1}(y) = f^{-1}(y_0)$, so it follows that the limits $\lim_{y\to y_0} f^{-1}(y)$ exists and

$$\lim_{y\to y_0} f^{-1}(y) = f^{-1}(y_0).$$

So we conclude that f^{-1} is continuous at any $y_0 \in [f(a), f(b)]$.