

## Real Number (Part 2)

Note: For the theorems covered in Chapter 5, please refer to P.1 of tutorial note #7

### Part 1: Finding Supremum and infimum of a given set (Review)

#### Example 1

Find the infimum and supremum of the set

$$S = \left\{ \frac{1}{p} + \frac{2}{q} + \frac{3}{r} : p, q, r \in \mathbf{N} \right\}$$

(Step 1: Find the upper bound and lower bound)

Note that  $\frac{1}{p} + \frac{2}{q} + \frac{3}{r} \leq \frac{1}{1} + \frac{2}{1} + \frac{3}{1} = 6$  (Since  $p, q, r \geq 1$ )

Which gives the upper bound is 6.

Since  $\frac{1}{p} + \frac{2}{q} + \frac{3}{r} > 0$ , so one possible lower bound is 0

(Step 2)

Since the upper bound 6 can be achieved by an elements in  $S$  (Put  $p = q = r = 1$ , we have  $\frac{1}{p} + \frac{2}{q} + \frac{3}{r} = 6$ ). Hence the supreme is 6.

Next, note that 0 is lower bound of  $S$ ,

**(to show  $\inf S = 0$ , here we use infimum limit theorem)**

Pick  $w_k = \frac{1}{k} + \frac{2}{k} + \frac{3}{k} \in S$  (Just simply set  $p = q = r = k$ )

Then by simple computation gives  $\lim_{k \rightarrow \infty} w_k = 0$

Therefore by infimum limit theorem, we conclude  $\inf S = 0$

### Difficult Situations

The set involves set operations such as intersection, union or complement.

#### Example 2

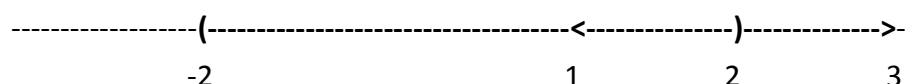
Find the infimum and supremum of the set

$$S = \left\{ 1 + \frac{2}{n} : n \in \mathbf{N} \right\} \cup \left\{ \frac{4 - 2q}{q} : q \in \mathbf{Q} \cap (1, \infty) \right\}$$

(Step 1) IDEA: We first find out the upper bound and lower bound of each set and combine them

	Upper Bound	Lower Bound
$A = \left\{1 + \frac{2}{n} : n \in \mathbf{N}\right\}$	3 (when $n = 1$ )	1 (when $n \rightarrow \infty$ )
$B = \left\{\frac{4-2q}{q} : q \in \mathbf{Q} \cap (1, \infty)\right\}$	2 (when $q = 1$ )	-2 (when $q \rightarrow \infty$ )

For  $S = A \cup B$ ,



We can see all elements in  $A \cup B$  lies between -2 and 3. Therefore the lower bound and upper bound is -2 and 3 respectively.

(Step 2)

To show  $\inf S = -2$

From the graph, we see this lower bound is achieved by elements in B. Since every elements in B must be in  $A \cup B = S$ , it is suffice to find  $\{w_n\} \in B$  (and it should be in S) such that  $\lim_{n \rightarrow \infty} w_n = -2$

Pick  $q = 1 + n$  and set  $w_n = \frac{4-2(1+n)}{1+n} = \frac{2-2n}{1+n} \in B$  (and  $w_n \in A \cup B$ )

And  $\lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} \frac{2-2n}{1+n} = \lim_{n \rightarrow \infty} \frac{\frac{2}{n}-2}{\frac{1}{n}+1} = -2$

Therefore by infimum limit theorem, we conclude  $\inf S = -2$

Similarly to show  $\sup S = 3$ , we see that the upper bound is achieved by elements in A. Therefore we just need to concentrate on A

Since  $3 = 1 + \frac{2}{1} \in A$ , the upper bound is in A and therefore also in  $A \cup B$ , hence

$\sup S = 3$

Example 3

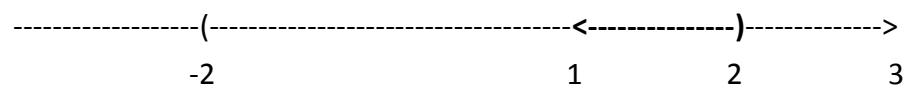
Find the supremum and infimum of the set

$$S = \left\{1 + \frac{2}{n} : n \in \mathbf{N}\right\} \cap \left\{\frac{4-2q}{q} : q \in \mathbf{Q} \cap (1, \infty)\right\}$$

(Step 1) From Example 2, we have

	Upper Bound	Lower Bound
$A = \left\{1 + \frac{2}{n} : n \in \mathbf{N}\right\}$	3 (when $n = 1$ )	1 (when $n \rightarrow \infty$ )
$B = \left\{\frac{4-2q}{q} : q \in \mathbf{Q} \cap (1, \infty)\right\}$	2 (when $q = 1$ )	-2 (when $q \rightarrow \infty$ )

For  $S = A \cap B$ ,



We can see all elements in  $A \cap B$  lies between 1 and 2. Therefore the lower bound and upper bound is 1 and 2 respectively.

(Step 2)

If we use infimum limit theorem and supremum limit theorem, it will be difficult for us to find  $w_n$  since we need to ensure ALL  $w_n$ 's are in both sets. Of course, you can try to do it. But here I will use proof by contradiction.

(Show  $\inf S = 1$ , note this lower bound is achieved by A)

Suppose  $\inf S > 1$ ,

Since  $\inf A = 1$  (**☺Left as exercise for you, you need to prove it in your midterm**)

Then there exists an element  $x \in A$ , such that  $\inf S > x > 1$ .

(Let  $x = 1 + \frac{2}{n}$ , since 2 is upper bound, therefore  $1 + \frac{2}{n} \leq 2 \rightarrow n \geq 2$ )

Next, we will show  $x \in B$  (Then it will imply  $x \in A \cap B$ , which will give the contradiction that  $\inf S$  is being lower bound)

**IDEA: We need to find  $q$  such that  $x = 1 + \frac{2}{n} = \frac{4-2q}{q}$**

Set  $1 + \frac{2}{n} = \frac{4-2q}{q} \rightarrow q = \frac{4n-2}{3n} > 1$  for  $n \geq 2$  (so  $q \in \mathbf{Q} \cap (1, \infty)$ )

Hence  $x = 1 + \frac{2}{n} = \frac{4-2q}{q} \in B$ , so  $x \in A \cap B$ , it leads to contradiction.

Therefore  $\inf S = 1$

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☺Exercise 1

a) In Example 3, show that  $\inf A = 1$  (where  $A = \left\{1 + \frac{2}{n} : n \in \mathbf{N}\right\}$ )

b) In Example 3, show that  $\sup S = \frac{5}{3}$  (which completes the solution)

(Comment: It is quite difficult for you)

Example 4 (Example 2 of Tutorial Note #7)

Find the infimum and supremum of the set

$$S = \left\{ x - \frac{1}{n} : x \in [0, 1] \cap \mathbf{Q}, n \in \mathbf{N} \right\} \setminus \left[ -1, \frac{1}{2} \right)$$

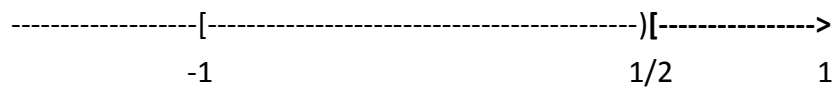
(Step 1: Finding upper bound and lower bound)

For the set  $S = \left\{ x - \frac{1}{n} : x \in [0, 1] \cap \mathbf{Q}, n \in \mathbf{N} \right\} \setminus \left[ -1, \frac{1}{2} \right)$

Since  $x - \frac{1}{n} < 1 - 0 = 1$ , then 1 is the upper bound of S

Therefore  $x - \frac{1}{n} > 0 - 1 = -1$ ,

For S



From the graph, we see the upper bound and lower bound are 1 and 1/2 respectively.

(Step 2)

We may use limit theorem to show  $\sup S = 1$  and  $\inf S = 1/2$ , however it is quite difficult for some students to construct the sequence. (Please refer to Example 2 of Tutorial Note 7 for detail) Here I present another alternative.

(Show  $\inf S = 1/2$ )

Since  $1/2 \in S$ , (we can pick  $x = \frac{3}{4}$  and  $n = 4$ , then  $x - \frac{1}{n} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$ )

Therefore,  $\inf S = 1/2$

(Show  $\sup S = 1$ )

Suppose  $\sup S < 1$ , by density of rational number, there exists a rational number  $p/q$  such that  $\sup S < \frac{p}{q} < 1$ , since  $\frac{p}{q}$  is positive, then  $p$  and  $q$  are both positive integers\*.

(Show  $\frac{p}{q} \in S$ ). Note that  $\frac{p}{q} = \frac{p+1}{q} - \frac{1}{q}$ , by letting  $x = \frac{p+1}{q} \geq 0$  and  $x \leq 1$  (why?)

and  $n = q$ , then  $\frac{p}{q} = x - \frac{1}{n} \in S$ . Contradict to the fact that  $\sup S$  is the upper bound.

Therefore  $\sup S = 1$

(\*Remark: since the lower bound of S is  $1/2$ , therefore it is OK for us to assume  $p/q$  is positive.)

### Unknown Set

#### Example 5

Let A and B be subsets of non-negative real number with  $\sup A = 7$ ,  $\sup B = 2$ ,

- a) Find the supreme of  $S = \{x^2 + 3y^3 : x \in A, y \in B\}$
- b) Will the supreme in a) remain the same if the word “non-negative” is omitted?

a) Note that  $0 \leq x \leq 7$  and  $0 \leq y \leq 2$

Then  $x^2 + 3y^3 \leq 7^2 + 3(2)^3 = 73$ , the upper bound is 73

To show  $\sup S = 73$ ,

Since we do not what exactly A and B are, so we can't construct our  $w_n$  directly. Instead, we can use the converse of supermum limit theorem to obtain such sequence.

Note that  $\sup A = 7$

There exist  $\{a_1, a_2, \dots, a_n, \dots\} \in S$  such that  $\lim_{n \rightarrow \infty} a_n = 7$

Similarly for  $\sup B = 2$

There exist  $\{b_1, b_2, \dots, b_n, \dots\} \in S$  such that  $\lim_{n \rightarrow \infty} b_n = 2$

Now take our  $w_n = a_n^2 + 3b_n^3 \in S$  and  $\lim_{n \rightarrow \infty} w_n = 7^2 + 3(2^3) = 73$

Therefore by supermum limit theorem, we get  $\sup S = 73$

b) If the word “non-negative” is omitted,  $\sup S$  may not be 73.

Example: Pick  $A = (-10, 7)$  and  $B = (0, 2)$ , it is clear that  $\sup A = 7, \sup B = 2$

But  $x^2 + 3y^3 \leq (-10)^2 + 3(2)^3 = 124$  and  $\sup S = 124 \neq 73$

(Note: That's why the word non-negative is important in this problem)

#### Example 6

Let A be a non-empty subset of  $\mathbf{R}$  such that  $\inf B = -1$  and  $\sup B = 1$ , find the supremum and infimum (if any) of the set

$$S = \{b^4 - 5b + 20 : b \in B\}$$

(Step 1: Find the upper bound and lower bound)

Let  $f(b) = b^4 - 5b + 20$

Then  $f'(b) = 4b^3 - 5 < 0$  for  $-1 \leq b \leq 1$

So f is decreasing, therefore  $f(1) \leq f(b) \leq f(-1) \rightarrow 16 \leq f(b) \leq 26$

Therefore the upper bound and lower bound of S are 26 and 16 respectively.

(Step 2)

We first claim  $\sup S = 24$

First since  $\inf B = -1$ , therefore by infimum limit theorem (another direction), there exists  $\{b_n\} \in B$  such that  $\lim_{n \rightarrow \infty} b_n = -1$

Pick  $w_n = b_n^4 - 5b_n + 20 \in S$

Then  $\lim_{n \rightarrow \infty} w_n = (-1)^4 - 5(-1) + 20 = 26$

Hence by supreme limit theorem,  $\sup S = 26$

Since  $\sup B = 1$ , therefore by supreme limit theorem (another direction), there exists  $\{b_n\} \in B$  such that  $\lim_{n \rightarrow \infty} b_n = 1$

Pick  $w_n = b_n^4 - 5b_n + 20 \in S$

Then  $\lim_{n \rightarrow \infty} w_n = (1)^4 - 5(1) + 20 = 16$

Hence by infimum limit theorem,  $\inf S = 16$

(You can also refer to Tutorial Note #7 for more exercises)

### ☺ Exercise 2

Find the supremum and infimum of the sets

$$S = \left\{ \frac{2}{n} - x^2 : n \in \mathbf{N}, x \in \mathbf{Q} \right\}, T = \left\{ 3x^2 - \frac{1}{y} : x \in (0,2) \cap \mathbf{Q}, y \in (1,5] \right\}$$

$$U = \{ \sqrt{2}x + \sqrt{3}y : x, y \in [1,4] \setminus \mathbf{Q} \}$$

(Hint: in set U,  $[1,4] \setminus \mathbf{Q}$  means the set of all irrational (non-rational) number between 1 and 4)

### ☺ Exercise 3

Find the supremum and infimum of the sets

$$S = \bigcup_{n=1}^5 \left[ \frac{1}{n}, 2 - \frac{1}{n} \right], T = \{ x^2 + y^2 : x, y \in (-2,1) \cap \mathbf{Q} \} \cap (2,7]$$

### ☺ Exercise 4

Suppose  $A_n \subseteq (-\infty, 2)$  and  $\sup A_n = x_n$  for  $n = 1, 2, 3, \dots, 10$ . Prove that

$$\sup \left( \bigcup_{k=1}^{10} A_k \right) = \max \{ x_1, x_2, \dots, x_{10} \}$$

(Hint: With loss of generality, you may assume  $x_1 \leq x_2 \leq \dots \leq x_{10}$ )

### ☺ Exercise 5

Let  $S = \{ \pi - x^2 : x \in (-2,2) \cap \mathbf{Q} \}$  and  $T = \left\{ \frac{x-y+z}{2x+y+z} : x, y, z \in (2,9) \cap \mathbf{Q} \right\}$

Find the supreme and infimum (if any) of  $S \cap T$

(Hint: Show that all elements in S are irrational and show all elements in T are rational)