Lecture 23 02-05-2019

Review :

1. Integrable criterion : f is integrable $\Leftrightarrow \forall \varepsilon > 0, \exists P$ s.t $U(f, P) - L(f, P) < \varepsilon$

2. Uniform continuity : $\forall \varepsilon > 0, \exists \delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

3. Uniformly continuity theorem : $f \in C[a, b] \Rightarrow f$ is uniformly continuous.

4. Theorem : $f \in C[a, b] \Rightarrow f$ is integrable

5. Riemann's definition of integral : $\int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j) \Delta x_j$.

6. Riemann integral = Darboux integral

7. Theorem : f integrable on $[a, b]$, $a < c < b$

$$\Rightarrow \begin{cases} ① \ f \text{ integrable on } [a, c]. \text{ and } [c, b] \\ ② \ \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \end{cases}$$

THM: Let $c \in (a,b)$, f is integrable on $[a,c]$, $[c,b]$.

then f is integrable on $[a,b]$. [Self - study]

Pf. f is integrable on $[a,c]$ and $[c,b]$

$$\Rightarrow \exists \; P_1 = \{x_0 = a, \cdots x_k = c\}$$

$$P_2 = \{x_k = c, \cdots x_n = b\} \quad \text{s.t}$$

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}$$

$$U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2}$$

Let $P = P_1 \cup P_2$, then

$$U(f,P) - L(f,P) = U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2)$$

$$< \varepsilon$$

$\Rightarrow$ f is integrable on $[a,b]$

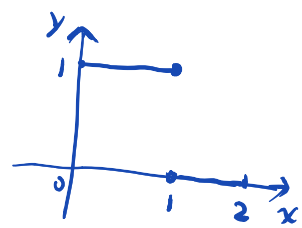
A consequence.

THM: If $f: [a,b] \to \mathbb{R}$ is bounded and is continuous except at finitely many points, say $a < c_1 < c_2 < \cdots < c_n < b$, then f is integrable on $[a,b]$, and

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots \int_{c_n}^{b} f(x)dx$$

Example: the function $f(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 0, & 1 < x < 2 \end{cases}$

is integrable on $[0, 2]$.



Computation rule of integral

THM: Let $f, g : [a, b] \to \mathbb{R}$ be integrable

① $f+g$ is integrable and

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)dx + \int_a^b f(x)dx$$

② cf is integrable and $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

Proof: ① Step 1, we show that $f+g$ is integrable.

$\forall \varepsilon > 0$, since f, g are integrable, $\exists\ P_1, P_2$ s.t

$$\begin{cases} U(f, P_1) - L(f, P_1) < \varepsilon/2 \\ U(g, P_2) - L(g, P_2) < \varepsilon/2 \end{cases} \xRightarrow{\text{refinement theorem}} \begin{cases} U(f, P) - L(f, P) < \varepsilon/2 \\ U(g, P) - L(g, P) < \varepsilon/2 \end{cases}$$

where $P = P_1 \cup P_2 = \{x_0 = a, x_1, \cdots x_n = b\}$

Step 2. We show that $U(f+g, P) - L(f+g, P) < \varepsilon$

So that $f+g$ is integrable.

Indeed, $U(f+g,P) = \sum_{j=1}^{n} \sup\{f(x)+g(x) : x \in [x_{j-1}, x_j]\} \Delta x_j$

$$\le \sum_{j=1}^{n} \left[\sup\{f(x) : x \in [x_{j-1}, x_j]\} + \sup\{g(x) : x \in [x_{j-1}, x_j]\}\right] \Delta x_j$$

$$= U(f,P) + U(g,P)$$

Similarly, $L(f+g, P) = \sum_{j=1}^{n} \inf\{f(x)+g(x) : x \in [x_{j-1}, x_j]\} \Delta x_j$

$$\ge \sum_{j=1}^{n} \left[\inf\{f(x) : x \in [x_{j-1}, x_j]\} + \inf\{g(x) : x \in [x_{j-1}, x_j]\}\right] \Delta x_j$$

$$= L(f,P) + L(g,P)$$

$$\Rightarrow U(f+g,P) - L(f+g,P) < U(f,P) + U(g,P) - L(f,P) - L(g,P) < \varepsilon$$

Step 3. We show that $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

$$\int_a^b (f+g)(x)\,dx \le U(f+g,P) \le U(f,P) + U(g,P) \le L(f,P) + \frac{\varepsilon}{2} + L(g,P) + \frac{\varepsilon}{2}$$
$$\le \int_a^b f(x)\,dx + \frac{\varepsilon}{2} + \int_a^b g(x)\,dx + \frac{\varepsilon}{2} = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \varepsilon$$

By infinetesimal principal, $\int_a^b (f+g)(x)\,dx \le \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Similarly, we can show that $\int_a^b (f+g)(x)dx \geq \int_a^b f(x)dx + \int_a^b g(x)dx$

[Exercise]

$$\Rightarrow \quad \int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

② Exercise.

Another proof of ① : [Self-study]

$$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx = \sup \{ L(f, P) : P \text{ partition of } [a,b] \}$$

$$= \overline{\int_a^b} f(x)dx = \inf \{ U(f, P) : P \cdots \quad \cdots \quad \cdots \}$$

By the infimum property, $\forall \varepsilon > 0$, $\exists$ P_1, P_2, s.t

$$\int_a^b f(x)dx - \frac{\varepsilon}{2} < L(f, P_1) < \int_a^b f(x)dx$$

$$\int_a^b g(x)dx - \frac{\varepsilon}{2} < L(g, P_2) < \int_a^b g(x)dx$$

Then for the refinement $P = P_1 \cup P_2 = \{x_0 = a, \cdots x_n = b\}$,

$$L(f+g, P) = \sum_{j=1}^{n} \inf\{f(x)+g(x) : x \in [x_{j-1}, x_j]\} \Delta x_j$$

$$\geq \sum_{j=1}^{n} \inf\{f(x) : x \in [x_{j-1}, x_j] \Delta x_j$$

$$+ \sum_{j=1}^{n} \inf\{g(x) : x \in [x_{j-1}, x_j] \Delta x_j$$

$$= L(f, P) + L(g, P) \geq L(f, P_1) + L(g, P_2)$$

$$\geq \int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon$$

$$\Rightarrow \underline{\int_a}^b (f+g)(x)dx = \sup\{L(f+g, P) : P\}$$

$$\geq \int_a^b f(x)dx + \int_a^b g(x)dx - \varepsilon$$

By the infinetesimal principal, we have

$$\underline{\int_a}^b (f+g)(x)dx \geq \int_a^b f(x)dx + \int_a^b g(x)dx$$

Similarly, we can show that

$$\overline{\int_a^b} (f+g)(x)\,dx \leq \int_a^b f(x)dx + \int_a^b g(x)dx$$

Since $\quad \underline{\int_a^b} (f+g)\,dx \leq \overline{\int_a^b} (f+g)\,dx$

$$\Rightarrow \quad \underline{\int_a^b} (f+g)\,dx = \overline{\int_a^b} (f+g)\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$\Rightarrow \quad f+g$ is integrable and

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Integral inequality I

THM: Let $f, g : [a, b] \to \mathbb{R}$ be integrable

If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

Proof: Let $h(x) = g(x) - f(x) \ge 0$. By the computation rule,

h is integrable. Since $h(x) \ge 0$,

$L(h, P) \ge 0 \quad \forall$ partition P on $[a,b]$

$$\Rightarrow \int_a^b h(x)dx = \sup\{L(h,P) : P\} \ge 0$$

$$\Rightarrow \int_a^b g(x)\,dx - \int_a^b f(x)dx = \int_a^b h(x)dx \ge 0$$

Integrability of composite function

THM: Let $f:[a,b]\to[c,d]$ be integrable. let $g:[c,d]\to\mathbb{R}$ be s.t $|g(x)-g(y)| \leq M|x-y|$ for some $M>0$ and all $x,y\in[c,d]$, then $g\circ f$ is integrable on $[a,b]$.

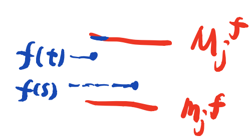
Proof:

Step 1. $\forall\, \varepsilon>0$, since f is integrable, $\exists$ Partion $P=\{t_0,\cdots t_n\}$ of $[a,b]$ s.t

$$U(f,P)-L(f,P)=\sum_{j=1}^{n}(M_j^f-m_j^f)\Delta t_j < \frac{1}{2M}\varepsilon.$$

where $M_j^f = \sup\{f(t): t\in[t_{j-1},t_j]\}$

$m_j^f = \inf\{f(t): t\in[t_{j-1},t_j]\}$



Step 2. We show that $|f(t)-f(s)| \leq M_j^f - m_j^f$ $(\star)$ for all $t,s\in[t_{j-1},t_j]$.

Indeed, we have $f(t) \le M_j^f$, $f(s) \ge m_j^f$ $\left(-f(s) \le -m_j^f \right)$

$$\Rightarrow f(t) - f(s) \le M_j^f - m_j^f$$

Similarly $f(s) - f(t) \le M_j^f - m_j^f$, or $-\left(f(t) - f(s)\right) \le M_j^f - m_j^f$

Therefore ($\star$) holds.

Recall :

$|a| < b \Leftrightarrow a < b, \ -a < b$

Step 3. Let $M_j = \sup \{ g\circ f(t) : t \in [t_{j-1}, t_j] \}$

$$m_j = \inf \{ g\circ f(t) : t \in [t_{j-1}, t_j] \}$$

We claim that $M_j - m_j \le M(M_j^f - m_j^f)$ ($\star\star$)

Indeed, $\forall$ t, $s \in [t_{j-1}, t_j]$

$$|g\circ f(t) - g\circ f(s)| \le M |f(t) - f(s)| \overset{\text{by}(\star)}{\le} M (M_j^f - m_j^f)$$

Step 4. By ($\star\star$), $U(g\circ f, P) - L(g\circ f, P)$

$$= \sum_{j=1}^{n} (M_j - m_j) \Delta t_j \le \sum_{j=1}^{n} M (M_j^f - m_j^f) \Delta t_j = M (U(f,P) - L(f,P))$$

$$< M \cdot \frac{\varepsilon}{M} = \varepsilon$$

$\Rightarrow$ $g\circ f$ is integrable

Integrability of product of functions

THM: Let f, g be integrable on $[a, b]$, then $f \cdot g$ is integrable on $[a, b]$.

pf: Observe that $f \cdot g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$

We need only show that $(f \pm g)^2$ are integrable.

Since f, g are integrable, f, g are bounded,

i.e $\exists\ M > 0$, s.t $|f(x)| \le M$, $|g(x)| \le M$ for all $x \in [a, b]$.

$\Rightarrow \quad -2M \le f(x) \pm g(x) \le 2M \quad$ for all $x \in [a, b]$

Let $\varphi(y) = y^2$, $\forall\ y_1, y_2 \in [-2M, 2M]$

$$|\varphi(y_1) - \varphi(y_2)| = |\varphi'(\xi)|\,|y_1 - y_2| = |2\xi| \cdot |y_1 - y_2| \le 4M\,|y_1 - y_2|$$

$\Rightarrow \quad (f \pm g)^2 = \varphi \circ (f \pm g)$ are integrable on $[a, b]$ by the composite rule.

Integrability of $|f|$.

THM: If f is integrable on $[a,b]$, then

① $|f|$ is integrable on $[a,b]$.

② $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

proof ①: Let $\varphi(x) = |x|$, then we can show that

$$|\varphi(x) - \varphi(y)| \leq |x-y|, \ \forall\, x, y \in \mathbb{R}. \quad [\text{Exercise}]$$

$$\big(\,\big||x|-|y|\big| \leq |x-y|\,\big)$$

By the composite rule, $|f| = \varphi \circ f$ is integrable.

② $-|f(x)| \leq f(x) \leq |f(x)|$, by the integral inequality,

$$-\int_a^b |f(x)|\,dx = \int_a^b -|f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

$$\Rightarrow \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

THM: If f, g are integrable on $[a,b]$,

then:

$$\left.\begin{aligned} h_1(x) &= \max\{f(x), g(x)\} \\ h_2(x) &= \min\{f(x), g(x)\} \end{aligned}\right\} \text{ are also integrable.}$$

proof: use the identity

$$\max\{a, b\} = \frac{a+b+|a-b|}{2}$$

$$\min\{a, b\} = \frac{a+b-|a-b|}{2}.$$

Mean-value theorem for integrals

THM: Assume further that $g(x) \geq 0$ for all $x \in [a, b]$.

Let $m = \inf_{x \in [a,b]} f(x)$, $M = \sup_{x \in [a,b]} f(x)$.

then $\exists$ $c \in [m, M]$ s.t $\int_a^b f(x) g(x)\, dx = c \int_a^b g(x)\, dx$

Proof: Since $g(x) \geq 0$, $m g(x) \leq f(x) g(x) \leq M g(x)$, by integral inequality,

$$m \int_a^b g(x)\, dx \leq \int_a^b f(x) g(x)\, dx \leq M \int_a^b g(x)\, dx$$

If $\int_a^b g(x)\, dx = 0 \Rightarrow \int_a^b f(x) g(x) dx = 0 = \int_a^b g(x) dx$.

If $\int_a^b g(x)\, dx \neq 0$, we take $c = \dfrac{\int_a^b f(x) g(x) dx}{\int_a^b g(x) dx}$

then $m \leq c \leq M$.

Remark: If one assumes that f is continuous, then one can replace c by $f(\xi)$ for some $\xi \in [a, b]$ in the above theorem

Uniform Continuity of anti-derivative

THM: If f is integrable on $[a, b]$ and $c \in [a, b]$. then

$$F(x) = \int_c^x f(t)\,dt \quad \text{is uniformly continuous on } [a, b].$$

Pf: Let M be the bound of f, then

$$|F(x) - F(y)| = \left| \int_c^x f(t)\,dt - \int_c^y f(t)\,dt \right|$$

$\forall\, \varepsilon > 0$, let $\delta = \frac{\varepsilon}{M}$, then $|x-y| < \delta \Rightarrow$

$|F(x) - F(y)| < \varepsilon \Rightarrow F$ is uniformly continuous.