1. (15 points). Recall that on $\mathbb{R}^2$, the metrics d_p and d_∞ are defined as follows: for all $x = (x_1, x_2) \in \mathbb{R}^2, y = (y_1, y_2) \in \mathbb{R}^2$,

$$d_p(x, y) = \Big(|x_1 - y_1|^p + |x_2 - y_2|^p\Big)^{\frac{1}{p}} \quad \forall\, p \in [1, \infty),$$
$$d_\infty(x, y) = \max\Big(|x_1 - y_1|, |x_2 - y_2|\Big).$$

Draw the regions of the unit ball $B_1(0)$ in $\mathbb{R}^2$ under the metric d_1, d_2 and d_∞, respectively. NO justification is needed.

2. (10 points). Let (X, d) be a metric space.

(i) Write down the **negation** of the definition that (X, d) is complete.

(ii) Provide an example of a metric space that is NOT complete. NO justification is needed.

3. (10 points). Let $(X, d), (X', d')$ be two nonempty metric spaces, and $f : X \to X'$ be a continuous function. Let A be a subset of X.

(i) Write down the definition of A being compact.

(ii) Suppose A is compact. Prove that $f(A)$ is a compact set in X'.

4. (15 points). Let

$$f_n(x) = \sum_{k=1}^{n} \frac{\cos(kx)}{k^2}.$$

(i): Prove that for each $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x)$ exists. (*Hint: use the fact that* $|\cos y| \leq 1$ *for all* $y \in \mathbb{R}$).

(ii): We define f as the pointwise limit of f_n, i.e., $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in \mathbb{R}$. Prove that f_n converges uniformly to f on $\mathbb{R}$.

(iii): Is f a continuous function on $\mathbb{R}$? Explain your answer.

Note that every metric in this problem is the usual one defined by the absolute value.