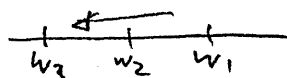


Solution of Math 2033 (Spring 2015) Midterm

- ① $\inf A = 1$ and $\sup A = 2 \Rightarrow A \subseteq [1, 2]$. For $x \in (0, \frac{\pi}{3}] \cap \mathbb{Q}$, $y \in A$, we have $\sqrt{y} \in [1, \sqrt{2}]$, $\cos x \in [\frac{1}{2}, 1)$. So $\frac{1}{2} \leq \sqrt{y} \cos x \leq \sqrt{2}$, B is bounded.
- $\inf A = 1 \Rightarrow \exists y_n \in A$ such that $\lim_{n \rightarrow \infty} y_n = 1$ by infimum limit theorem.
- $\sup A = 2 \Rightarrow \exists y'_n \in A$ such that $\lim_{n \rightarrow \infty} y'_n = 2$ by supremum limit theorem.
- Let $x_n = \frac{[10^n \frac{\pi}{3}]}{10^n} \in (0, \frac{\pi}{3}] \cap \mathbb{Q}$, then $\lim_{n \rightarrow \infty} x_n = \frac{\pi}{3}$.
- Let $x'_n = \frac{1}{n} \in (0, \frac{\pi}{3}] \cap \mathbb{Q}$, then $\lim_{n \rightarrow \infty} x'_n = 0$.
- Then $\sqrt{y_n} \cos x_n \in B$ and $\lim_{n \rightarrow \infty} \sqrt{y_n} \cos x_n = \frac{1}{2}$. Also, $\sqrt{y'_n} \cos x'_n \in B$, $\lim_{n \rightarrow \infty} \sqrt{y'_n} \cos x'_n = \sqrt{2}$.
- Therefore, $\inf B = \frac{1}{2}$ and $\sup B = \sqrt{2}$.

- ② (a) Sketch $w_1 = 6$, $w_2 = 6 - \frac{9}{6} = 4.5$, $w_3 = 6 - \frac{9}{4.5} = 4$ 
- $w = 6 - \frac{9}{w} \Rightarrow w^2 - 6w + 9 = 0 \Rightarrow (w-3)^2 = 0 \Rightarrow w = 3$

Claim: $\forall n=1, 2, 3, \dots, w_n \geq w_{n+1} \geq 3$.

For $n=1$, $w_1 = 6 \geq w_2 = 4.5 \geq 3$. Suppose $w_n \geq w_{n+1} \geq 3$. Then $\frac{9}{w_n} \leq \frac{9}{w_{n+1}} \leq \frac{9}{3}$.

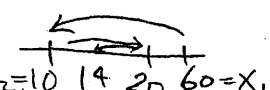
So $w_{n+1} = 6 - \frac{9}{w_n} \geq w_{n+2} = 6 - \frac{9}{w_{n+1}} \geq 6 - \frac{9}{3} = 3$. By M.I., the claim is true.

Since $\{w_n\}$ is decreasing and bounded below, $\lim_{n \rightarrow \infty} w_n = w$ exists.

Then $w = \lim_{n \rightarrow \infty} w_{n+1} = \lim_{n \rightarrow \infty} (6 - \frac{9}{w_n}) = 6 - \frac{9}{w}$, which implies $w^2 - 6w + 9 = 0$.

So $w = 3$.

- (b) Sketch $x_1 = 60$, $x_2 = 8 + \frac{120}{60} = 10$, $x_3 = 8 + \frac{120}{10} = 20$, $x_4 = 8 + \frac{120}{20} = 14$

 $x = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} (8 + \frac{120}{x_n}) = 8 + \frac{120}{x} \Rightarrow x^2 - 8x - 120 = 0$

$\Rightarrow x = \frac{8 \pm \sqrt{64 + 480}}{2} \Rightarrow x = \frac{8 + \sqrt{544}}{2}$

$x \in [10, 60]$ $x > 0$

Claim: For $I_n = [x_{2n}, x_{2n+1}]$, we have $x_{2n} \leq x_{2n+2} \leq x_{2n+1} \leq x_{2n-1}$ for $n=1, 2, 3, \dots$.

For $n=1$, $x_2 = 10 \leq x_4 = 14 \leq x_3 = 20 \leq x_1 = 60$. Suppose $x_{2n} \leq x_{2n+2} \leq x_{2n+1} \leq x_{2n-1}$.

Then $\frac{120}{x_{2n}} \geq \frac{120}{x_{2n+2}} \geq \frac{120}{x_{2n+1}} \geq \frac{120}{x_{2n-1}}$. Also $x_{2n+1} = 8 + \frac{120}{x_{2n}} \geq x_{2n+3} = 8 + \frac{120}{x_{2n+2}} \geq x_{2n+2} = 8 + \frac{120}{x_{2n+1}} \geq x_{2n+1}$.

Then $\frac{120}{x_{2n+1}} \leq \frac{120}{x_{2n+3}} \leq \frac{120}{x_{2n+2}} \leq \frac{120}{x_{2n}}$. So $x_{2n+2} = 8 + \frac{120}{x_{2n+1}} \leq x_{2n+4} = 8 + \frac{120}{x_{2n+3}} \leq x_{2n+3} = 8 + \frac{120}{x_{2n+2}} \leq x_{2n+2} = 8 + \frac{120}{x_{2n+1}} \leq x_{2n+1}$.

By M.I., the claim is true.

By nested interval theorem, $\lim_{n \rightarrow \infty} x_{2n} = a$ and $\lim_{n \rightarrow \infty} x_{2n+1} = b$ exist. Then we have

$a = \lim_{n \rightarrow \infty} x_{2n} = \lim_{n \rightarrow \infty} 8 + \frac{120}{x_{2n-1}} = 8 + \frac{120}{b}$ and $b = \lim_{n \rightarrow \infty} x_{2n+1} = \lim_{n \rightarrow \infty} (8 + \frac{120}{x_{2n}}) = 8 + \frac{120}{a}$.

So $8b + 120 = ab = 8a + 120 \Rightarrow a = b$. Then $\lim_{n \rightarrow \infty} x_n = x$ exists and $x \in [10, 60]$.

$x > 0$, $x = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} (8 + \frac{120}{x_n}) = 8 + \frac{120}{x} \Rightarrow x^2 - 8x - 120 = 0 \Rightarrow x = \frac{8 + \sqrt{544}}{2}$ (as $\frac{8 - \sqrt{544}}{2} < 0$).

③ Sketch $\frac{9}{z_n^2+2} \rightarrow \frac{9}{18} = \frac{1}{2}$, $\frac{5}{y_n-2} \rightarrow \frac{5}{2}$. $\left| \frac{9}{z_n^2+2} - \frac{1}{2} \right| = \frac{|z_n^2-16|}{2(z_n^2+2)} \leq \frac{|z_n+4||z_n-4|}{4} \leq \frac{9}{4}|z_n-4|$
 $\left| \frac{5}{y_n-2} - \frac{5}{2} \right| = \left| \frac{20-5y_n}{2(y_n-2)} \right| = \frac{5|y_n-4|}{2|y_n-2|} \leq \frac{5}{2}|y_n-4| < \frac{\varepsilon}{2} \Leftrightarrow |y_n-4| < \frac{\varepsilon}{5}$
 $(z_n-4) < 1 \Rightarrow z_n \in (3,5) \Rightarrow \frac{9}{4}|z_n-4| < \frac{9}{4} \cdot \frac{\varepsilon}{5} = \frac{9\varepsilon}{20}$
 $|y_n-4| < 1 \Rightarrow y_n \in (3,5) \Rightarrow y_n-2 \in (1,3) \Rightarrow \frac{5}{2}|y_n-4| < \frac{5}{2} \cdot \frac{\varepsilon}{5} = \frac{\varepsilon}{2}$
 $\Rightarrow z_n+4 \in (7,9)$

Since $\lim_{n \rightarrow \infty} z_n = 4$, $\forall \varepsilon > 0 \Rightarrow \exists K_1 \in \mathbb{N}$ such that $n \geq K_1 \Rightarrow |z_n-4| < 1 \Rightarrow z_n \in (3,5)$

Since $\lim_{n \rightarrow \infty} y_n = 4$, $\forall \varepsilon > 0 \Rightarrow \exists K_2 \in \mathbb{N}$ such that $n \geq K_2 \Rightarrow |y_n-4| < 1 \Rightarrow y_n \in (3,5)$

$\forall \varepsilon > 0$, $\frac{2}{9}\varepsilon > 0 \Rightarrow \exists K_3 \in \mathbb{N}$ such that $n \geq K_3 \Rightarrow |z_n-4| < \frac{2}{9}\varepsilon \Rightarrow y_n-2 \in (1,3)$

$\frac{\varepsilon}{5} > 0 \Rightarrow \exists K_4 \in \mathbb{N}$ such that $n \geq K_4 \Rightarrow |y_n-4| < \frac{\varepsilon}{5}$.

Let $K = \max\{K_1, K_2, K_3, K_4\}$. Then $n \geq K$ implies $n \geq K_1, n \geq K_2, n \geq K_3, n \geq K_4$

$$\Rightarrow \left| \frac{9}{z_n^2+2} + \frac{5}{y_n-2} - 3 \right| = \left| \left(\frac{9}{z_n^2+2} - \frac{1}{2} \right) + \left(\frac{5}{y_n-2} - \frac{5}{2} \right) \right| \leq \left| \frac{9}{z_n^2+2} - \frac{1}{2} \right| + \left| \frac{5}{y_n-2} - \frac{5}{2} \right|$$

$$= \frac{|z_n^2-16|}{2(z_n^2+2)} + \frac{5|y_n-4|}{2|y_n-2|} \leq \frac{|z_n+4||z_n-4|}{4} + \frac{5}{2}|y_n-4| < \frac{9}{4}|z_n-4| + \frac{5}{2}|y_n-4| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

see sketch