6 4. (14 points) .

Let $f \in C^1(\mathbb{R})$ and $M > 0$.

4.1 Assume that $|f'(x)| < M$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous.

8' 4.2 Assume that $\lim_{x\to\infty} f'(x) = 0$. Define

$$\phi(x) = \frac{\int_0^x f(t)dt}{x^2} \quad \text{for} \quad x \geq 1.$$

Does the limit $\lim_{x\to\infty} \phi(x)$ exists? if so, find the limit and justify your answer.

Solution:

4.1 $\forall \epsilon > 0$, let $\delta = \epsilon/M$, then $\forall |x - y| < \delta$, we have 6'

$$|f(x) - f(y)| = |f'(\xi)||x - y| < M\delta = \epsilon.$$

3'

By definition, f is uniformly continuous.

4.2 By L'Hospital's rule,

3'

$$\lim_{x\to\infty} \frac{\int_0^x f(t)dt}{x^2} = \lim_{x\to\infty} \frac{f(x)}{2x} = \lim_{x\to\infty} \frac{f'(x)}{2} = 0.$$ 8'