Math 2033 (Mathematical Analysis) Spring 2014

<u>Final Examination – (Duration: 180 minutes)</u>

Directions: This is a closed book exam. **For every problem of this exam, detailed written works supported by correct reasons must be shown legibly to receive credits**. Answers alone are worth very little. Calculators are not allowed.

Notations: $\mathbb{R}$ is the set of all real numbers.

Problems

1. (10 marks) Let $J : \mathbb{R} \to \mathbb{R}$ be defined by $J(x) = \sin\left(\dfrac{x-1}{|x|+2}\right) + \dfrac{3-x}{x^2+3}$. Prove that $\lim\limits_{x\to 1} J(x) = \dfrac{1}{2}$ by checking the ε-δ definition of limit of function.

   <u>(Do not use any computation formula for limits, sandwich theorem or l'Hopital's rule,</u> otherwise, you will get zero mark.)

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2. (a) (2 marks) Give the name of a theorem that was taught in class that you would use to solve part (c) of this problem.

   (b) (6 marks) Describe the theorem you named in part (a) and state the reason(s) you want to use it for solving part (c).

   (c) (12 marks) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. For every $x \in [1,3]$, $|f''(x)| > 1$. If $f(1) = 0 = f(3)$, then prove that there exists at least one $w \in [1,3]$ such that $|f(w)| \geq \frac{1}{2}$.

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3. (15 marks) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers such that $a_1, a_3, a_5, \ldots$ is a Cauchy sequence and for $j = 1, 2, 3, \ldots$, $a_{2j} = a_{2j-1} + \dfrac{1}{j}$. Prove that $a_1, a_2, a_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

   (<u>Do not use Cauchy's theorem</u> that said a sequence converges if and only if it is a Cauchy sequence, otherwise you will get zero mark.)

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4. (15 marks) Let $g : \mathbb{R} \to \mathbb{R}$ be three times differentiable such that

   $$g(0) > 0, \quad g'(0) < 0 \quad \text{and} \quad g''(0) = 0.$$

   If for all $x > 0$, $g'''(x) < 0$, then prove that there exists some $r \in (0, +\infty)$ such that $g(r) = 0$.

–Please turn to reverse side for more problems–

5. (20 marks) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}$, $3F(F(x)) = F(x) + x$. Prove that $F(0) = 0$.

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6. (20 marks) Let $p : [1/2, 1] \to [0, 1]$ be Riemann integrable such that $p(1/2) = p(1) = 0$. Define $h : [0, 1] \to [-1, 1]$ as follows: $h(0) = h(1) = 0$ and

$$\text{for all} \quad n = 0, 1, 2, \ldots \quad \text{and} \quad x \in [1/2^{n+1}, 1/2^n), \quad h(x) = (-1)^n p(2^n x).$$

Prove that h is Riemann integrable on $[0, 1]$ by Lebesgue's theorem.

–End of Paper–