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Q1) a) Tomorrow is sunny or not rainy, and I will not watch a movie or not have a dinner outside

b) $\exists \varepsilon > 0, \forall \delta > 0$ such that $|x-y| < \delta$, and $|f(x) - f(y)| \geq \varepsilon$

c) $\exists x \in S, \exists \varepsilon > 0, \forall \delta > 0$ such that $|x-y| < \delta$, and $|f(x) - f(y)| \geq \varepsilon$

d) $\exists \varepsilon > 0, \forall N > 0$ such that $|f_n(x) - f_m(x)| \geq \varepsilon$ exist $m, n \geq N$ or $x \notin \mathbb{R}$.

e) $\exists x \in \mathbb{R}, \exists \varepsilon > 0, \forall N > 0$ such that $|f_n(x) - f_m(x)| \geq \varepsilon$ exist $m, n \geq N$.

2a) Let $x \in U$,

then $f(x) \in f(U)$

therefore $x \in f^{-1}(f(U))$

Hence $U \subseteq f^{-1}(f(U))$

Let $A = \{1, 2\}$, $B = \{b\}$

and $f: A \to B$, $f(1) = f(2) = b$

Let $U = \{1\}$,

then $f(U) = B$

therefore $f^{-1}(f(U)) = f^{-1}(B) = A$

Since, $A \nsubseteq U$, so that $f^{-1}(f(U)) \nsubseteq U$

Hence, $U \subset f^{-1}(f(U))$

2b) Let $y \in f(f^{-1}(V)) \Rightarrow \exists x \in f^{-1}(V)$ s.t. $f(x) = y$

$\Rightarrow f(x) \in V$ s.t. $f(x) = y$

$\Rightarrow y \in V$

Hence $f(f^{-1}(V)) \subseteq V$

Let $A = \{1, -1\}$, $V = \{1, -1\}$, $f(x) = |x| = y$

$f(1) = 1$, $f(-1) = 1$

$f^{-1}(V) = \{x \in A, f(x) \in V\} = \{1, -1\}$

$f(f^{-1}(V)) = \{1\}$

~~and $f(A) = f(f^{-1}(V)) =$~~

Therefore $V \nsubseteq f(f^{-1}(V))$

Hence, $f(f^{-1}(V)) \subset V$.

c) Let $Y \in f(\bigcup_{\alpha\in I} X_\alpha) \Leftrightarrow f^{-1}(Y) \in \bigcup_{\alpha\in I} X_\alpha$

$\Leftrightarrow f^{-1}(Y) \in X_\alpha$ for $\alpha\in I$

$\Leftrightarrow Y \in f(X_\alpha)$ for $\alpha\in I$

$\Leftrightarrow Y \in \bigcup_{\alpha\in I} f(X_\alpha)$

$\therefore f(\bigcup_{\alpha\in I} X_\alpha) \subseteq \bigcup_{\alpha\in I} f(X_\alpha)$

Let $Y \in \bigcup_{\alpha\in I} f(X_\alpha) \Leftrightarrow x \in f(X_\alpha)$ for $\alpha\in I$

$\Leftrightarrow f^{-1}(Y) \in X_\alpha$ for $\alpha\in I \Leftrightarrow f^{-1}(Y) \in \bigcup_{\alpha\in I} X_\alpha$

$\Leftrightarrow Y \in f(\bigcup_{\alpha\in I} X_\alpha)$

$\therefore \cancel{f} \bigcup_{\alpha\in I} f(X_\alpha) \subseteq f(\bigcup_{\alpha\in I} X_\alpha)$

$\therefore \bigcup_{\alpha\in I} f(X_\alpha) = f(\bigcup_{\alpha\in I} X_\alpha)$

Let $X \in f^{-1}(\bigcup_{\alpha\in I} Y_\alpha) \Leftrightarrow f(X) \in \bigcup_{\alpha\in I} Y_\alpha$

$\Leftrightarrow f(x) \in Y_\alpha$ for $\alpha\in I \Leftrightarrow X \in f^{-1}(Y_\alpha)$ for $\alpha\in I$

$\Leftrightarrow x \in \bigcup_{\alpha\in I} f^{-1}(Y_\alpha)$

$\therefore f^{-1}(\bigcup_{\alpha\in I} Y_\alpha) \subseteq \bigcup_{\alpha\in I} f^{-1}(Y_\alpha)$

Let $X \in \bigcup_{\alpha\in I} f(Y_\alpha) \Leftrightarrow X \in f(Y_\alpha)$ for $\alpha\in I$

$\Leftrightarrow f^{-1}(X) \in Y_\alpha$ for $\alpha\in I \Leftrightarrow f^{-1}(X) \in \bigcup_{\alpha\in I} Y_\alpha$

$\Leftrightarrow X \in f(\bigcup_{\alpha\in I} Y_\alpha)$

$\therefore \bigcup_{\alpha\in I} f(Y_\alpha) \subseteq f(\bigcup_{\alpha\in I} Y_\alpha)$

$\therefore \bigcup_{\alpha\in I} f(Y_\alpha) = f(\bigcup_{\alpha\in I} Y_\alpha)$

d) $\cap_{\alpha \in I} X_\alpha = \{x \in X_\alpha \mid x$ is common element for all set $X_\alpha\}$

$\bigcap_{\alpha \in I} f(X_\alpha) = \{f(x) \in Y_\alpha \mid f(x)$ is common element for all set $f(X_\alpha)\}$

Let $X_1 = \{a, b\}$, $X_2 = \{a\}$, $f(a) = 1$, $f(b) = 2$

$f(X_1) = \{1, 2\}$, $f(X_2) = \{1\}$, $X_1 \cap X_2 = \{a\}$

$f(X_1 \cap X_2) = 1$ $f(X_1) \cap f(X_2) = 1$

$\therefore f(\bigcap_{\alpha \in I} X_\alpha) \subseteq \bigcap_{\alpha \in I} f(X_\alpha)$

d cont) Let $X \in f^{-1}(\bigcap_{\alpha \in I} Y_\alpha)$

$\Leftrightarrow f(X) \in \bigcap_{\alpha \in I} Y_\alpha$

$\Leftrightarrow f(X) \in Y_\alpha$ for $\alpha \in I$

$\Leftrightarrow X \in f^{-1}(Y_\alpha)$ for $\alpha \in I$

$\Leftrightarrow X \in \bigcap_{\alpha \in I} f^{-1}(Y_\alpha)$

$f^{-1}(\bigcap_{\alpha \in I} Y_\alpha) \subseteq \bigcap_{\alpha \in I} f^{-1}(Y_\alpha)$

Let $X \in \bigcap_{\alpha \in I} f^{-1}(Y_\alpha)$

$\Leftrightarrow X \in f^{-1}(Y_\alpha)$ for $\alpha \in I$

$\Leftrightarrow f(X) \in Y_\alpha$ for $\alpha \in I$

$\Leftrightarrow f(X) \in \bigcap_{\alpha \in I} Y_\alpha$

$\Leftrightarrow X \in f^{-1}(\bigcap_{\alpha \in I} Y_\alpha)$

$\bigcap_{\alpha \in I} f^{-1}(Y_\alpha) \subseteq f^{-1}(\bigcap_{\alpha \in I} Y_\alpha)$

$\Rightarrow f^{-1}(\bigcap_{\alpha \in I} Y_\alpha) = \bigcap_{\alpha \in I} f^{-1}(Y_\alpha)$

Q3) first we assume f is injective.

Let $y \in f(A) \cap f(B)$,

then $y \in f(A)$, $y \in f(B)$

$y = f(a)$, where $a \in A$

$y = f(b)$, where $b \in B$

As $f(a) = f(b)$ and f is injective,

$a = b$.

Therefore, $y = f(a) = f(b) \in A \cap B$

$f(y) \in f(A \cap B)$

Thus $f(A) \cap f(B) \subseteq f(A \cap B)$ — ①

Let $y \in f(A \cap B)$, $y = f(x)$, where $x \in A \cap B$

Then $x \in A$ and $x \in B$

So $y = f(x) \in f(A)$, $y = f(x) \in f(B)$

Hence, $y = f(x) \in f(A) \cap f(B)$

$f(A \cap B) \subseteq f(A) \cap f(B)$ — ②

By ① and ②, $f(A \cap B) = f(A) \cap f(B)$ if f is injective.

Q3 cont) Second we assume $f(A \cap B) = f(A) \cap f(B)$.

Let $x_1, x_2 \subseteq X$, $A = \{x_1\}$, $B = \{x_2\}$, where $x_1 \neq x_2$

Then, $A \cap B = \phi$ because $x_1 \neq x_2$

So, $f(A) \cap f(B) \subseteq f(A \cap B) = f(\phi) = \phi$

Thus, $f(A) \cap f(B) = \phi$

Hence, $f(x_1) \neq f(x_2)$

By contrapositive, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, we conclude if $f(A \cap B) = f(A) \cap f(B)$, then f is injective

By two parts prove above, we conclude f is injective iff $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq X$

4 a) $A_k = \{x \in \mathbb{R} \mid f_k(x) = 0\}$

$$S_n = \{x \in \mathbb{R} \mid \prod_{k=1}^{n} f_k(x) = 0\}$$

$$= \{x \in \mathbb{R} \mid f_1(x) f_2(x) \cdots f_n(x) = 0\}$$

Then, $x \in S_n$ iff $f_1(x) f_2(x) \cdots f_n(x) = 0$

$\Leftrightarrow f_k(x) = 0$ for $k = (1, 2, \cdots, n)$

Therefore,

$S_n = \bigcup_{k=1}^{n} A_k$ is finite set because

A_k is countable, so the union of

A_k, $k = (1, 2, \cdots, n)$ is also countable.

b) $S = \{x \in \mathbb{R} \mid \prod_{k=1}^{\infty} f_k(x) = 0\}$

$$= \{x \in \mathbb{R} \mid f_1(x) f_2(x) \cdots = 0\}$$

Same situation In a),

$x \in S$ iff $f_1(x) f_2(x) \cdots = 0$

$\Leftrightarrow f_k(x) = 0$ for $k = (1, 2, \cdots)$

If some of A_k is empty, we could leave them out.

Then then remaining countable set, let

$A_1 = \{a_{10}, a_{11}, a_{12}, \ldots\}$

$A_2 = \{a_{20}, a_{21}, a_{22}, \cdots\}$

$\vdots$

Following the diagonals, $\bigcup_{k=1}^{\infty} A_k = \{a_{10}, a_{11}, a_{20}, a_{12}, a_{21}, a_{30}, \cdots\}$

Therefore, $\bigcup_{k=1}^{\infty} A_k$ is a countable set.

5) We need to prove $P(N)$ is not finite or countably infinite set

First, N is not a finite set.

Thus, $P(N)$ is not a finite set

Second, $card(P(N)) = 2^{card(N)}$

$$= 2^{\aleph_0}$$

By Cantor's theorem,

$$c = 2^{\aleph_0} > \aleph_0$$

Therefore,

$P(N)$ is not Countably infinite set.

Hence,

$P(N)$ is a uncountable set.