MATH 2031 Introduction to Real Analysis

February 19, 2013

Tutorial Note 11

Cauchy sequence

(I) **Definition:**
$\{x_n\}$ is a Cauchy sequence iff $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ such that $n, m \geq K \Rightarrow |x_n - x_m| < \varepsilon$.

(II) **Cauchy Theorem**
$\{x_n\}$ converges iff it is a Cauchy sequence.
The proof consists of 4 steps

Step.1 If $\{x_n\}$ converges, then it is a Cauchy sequence.

Step.2 If $\{x_n\}$ is a Cauchy sequence, then it is bounded.

Step.3 (Bolzano-Weierstrass Theorem) Every bounded sequence has a convergent subsequence.

Step.4 If $\{x_n\}$ is a Cauchy sequence and some subsequence $\{x_{n_k}\}$ converges to x, then $\{x_n\}$ converges to x.

Limit of Functions

(F.I) **Definition:**
Let $f : S \to \mathbb{R}$ be a function.

$$\lim_{x \to x_0} f(x) = L \iff \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in S,\ |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

(F.II) **Sequential Limit Theorem (S.L.T.)**
Let $f : S \to \mathbb{R}$ be a function and x_0 be an accumulation point of S. Then

$$\lim_{\substack{x \to x_0 \\ x \in S}} f(x) = L \iff \text{for every sequence } \{x_n\} \subset S \setminus \{x_0\} \text{ that converges to } x_0,\ \lim_{n \to \infty} f(x_n) = L$$

(F.III) **Monotone Function Theorem**
If f is increasing on (a, b), then

(I) $\forall x_0 \in (a, b)$,

$$\left.\begin{array}{r} f(x_0-) = sup\{f(x) | a < x < x_0\} \\ \text{and } f(x_0+) = inf\{f(x) | x_0 < x < b\} \end{array}\right\} \Rightarrow f(x_0-) \leq f(x_0) \leq f(x_0+)$$

If f is bounded below, then $f(a+) = inf\{f(x) | a < x < b\}$.

If f is bounded above, then $f(b-) = sup\{f(x) | a < x < b\}$.

(II) f has countably many discontinuous point on (a, b). i.e.

$$J = \{x_0 | x_0 \in (a, b), f(x_0-) \neq f(x_0+)\} \text{ is countable.}$$

(F.IV) **One-sided Limits:**

(i) **Definition:** For $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$,
left hand limit of f at x_0: $f(x_0-) = \lim_{x \to x_0^-} f(x) = \lim_{\substack{x \to x_0 \\ x \in (a, x_0)}} f(x)$;
right hand limit of f at x_0: $f(x_0+) = \lim_{x \to x_0^+} f(x) = \lim_{\substack{x \to x_0 \\ x \in (x_0, b)}} f(x)$.

(ii) **Theorem:**
For $x_0 \in (a,b)$, $\lim\limits_{\substack{x\to x_0 \\ x\in(a,b)}} f(x) = L \iff f(x_0-) = L = f(x_0+)$

Problem 1 Let $\{x_n\}$ be a sequence with the following property

$$|x_{n+2} - x_{n+1}| < r|x_{n+1} - x_n|$$

for all $n \in \mathbb{N}$ and for some $r < 1$.
Show that $\{x_n\}$ converges.

Scratch:
Notice that

$$|x_n - x_{n-1}| < r|x_{n-1} - x_{n-2}| < r^2|x_{n-2} - x_{n-3}| < \cdots < r^{n-2}|x_2 - x_1|.$$

Assume that $m > n$, then

$$\begin{aligned}
|x_m - x_n| &\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
&< (r^{m-2} + r^{m-3} + \cdots + r^{n-1})|x_2 - x_1| \\
&\le (r^{n-1} + r^n + \cdots)|x_2 - x_1| \\
&= |x_2 - x_1|(r^{n-1}) \left(\sum_{k=0}^{\infty} r^k\right) \\
&= \frac{r^{n-1}|x_2 - x_1|}{1-r}.
\end{aligned}$$

Requiring $n > \dfrac{\ln\left(\frac{(1-r)\varepsilon}{|x_2-x_1|}\right)}{\ln r} + 1$, then we have $\dfrac{r^{n-1}|x_2 - x_1|}{1-r} < \varepsilon$

Solution:

$\forall\ \varepsilon > 0$, by Archimedean principle, there exists $K \in \mathbb{N}$ such that $K > \dfrac{\ln\left(\frac{(1-r)\varepsilon}{|x_2-x_1|}\right)}{\ln r} + 1$. Then for any $m, n > K$, we have from the above and the symmetry of m, n,

$$|x_m - x_n| < \frac{r^{n-1}|x_2 - x_1|}{1-r} \qquad \text{or} \qquad |x_m - x_n| < \frac{r^{m-1}|x_2 - x_1|}{1-r}.$$

Since $m, n > K$, both $\dfrac{r^{n-1}|x_2 - x_1|}{1-r}$ and $\dfrac{r^{m-1}|x_2 - x_1|}{1-r}$ are strictly less than ε. Thus $|x_m - x_n| < \varepsilon$.
Therefore $\{x_n\}$ is Cauchy, and by Cauchy theorem, $\{x_n\}$ converges.

Problem 2 Let $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ be defined by $f(x) = \dfrac{x^2}{x+1} + \dfrac{\sqrt{x}}{2}$.
Prove that $\lim\limits_{x\to 1} f(x) = 1$ by checking definition.
Scratch:
"Want: To find a positive δ such that $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in S,\ |x-1| < \delta\ \Rightarrow \left|\dfrac{x^2}{x+1} + \dfrac{\sqrt{x}}{2} - 1\right| < \varepsilon$"

$$\begin{aligned}
\left|\frac{x^2}{x+1} + \frac{\sqrt{x}}{2} - 1\right| &= \left|\frac{x^2}{x+1} - \frac{1}{2} + \frac{\sqrt{x}}{2} - \frac{1}{2}\right| \\
&\le \left|\frac{x^2}{x+1} - \frac{1}{2}\right| + \left|\frac{\sqrt{x}}{2} - \frac{1}{2}\right| \\
&= \left|\frac{x^2 - x - 1}{2(x+1)}\right| + \frac{|\sqrt{x} - \sqrt{1}|}{2} \\
&\le \underbrace{\frac{|2x+1||x-1|}{2|x+1|}}_{\text{(i)}} + \underbrace{\frac{\sqrt{|x-1|}}{2}}_{\text{(ii)}}
\end{aligned}$$

If each of the (i) and (ii) strictly less than $\frac{\varepsilon}{2}$, we could get the inequality we want.

For (i)
By requiring $|x-1|<1$, we have $0<x<2$, then $|2x+1|<5$ and $1<|x+1|$. Then

$$\frac{|2x+1||x-1|}{2|x+1|}<\frac{5|x-1|}{2}$$

and

$$\frac{5|x-1|}{2}<\frac{\varepsilon}{2}\iff 5|x-1|<\varepsilon\iff |x-1|<\frac{\varepsilon}{5}.$$

For (ii)

$$\frac{\sqrt{|x-1|}}{2}<\frac{\varepsilon}{2}\iff \sqrt{|x-1|}<\varepsilon\iff |x-1|<\varepsilon^2$$

Thus, requiring $\delta<\min\left\{1,\frac{\varepsilon}{5},\varepsilon^2\right\}$ then $|x-1|<\delta$ implies each of the (i) and (ii) strictly less than $\frac{\varepsilon}{2}$ and the desired inequality holds.

Solution:
For any $\varepsilon>0$, take $\delta>0$ and $\delta<\min\left\{1,\frac{\varepsilon}{5},\varepsilon^2\right\}$. Then for all $x\in\mathbb{R}\setminus\{-1\}$ and $|x-1|<\delta$,

$$\begin{aligned}\left|\frac{x^2}{x+1}+\frac{\sqrt{x}}{2}-1\right| &= \left|\frac{x^2}{x+1}-\frac{1}{2}+\frac{\sqrt{x}}{2}-\frac{1}{2}\right|\\ &\le \left|\frac{x^2}{x+1}-\frac{1}{2}\right|+\left|\frac{\sqrt{x}}{2}-\frac{1}{2}\right|\\ &= \left|\frac{x^2-x-1}{2(x+1)}\right|+\frac{|\sqrt{x}-\sqrt{1}|}{2}\\ &\le \frac{|2x+1||x-1|}{2|x+1|}+\frac{\sqrt{|x-1|}}{2}\\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

Thus by definition, $\lim_{x\to 1} f(x)=1$.

Problem 3 Define $g:(-1,5)\to\mathbb{R}$ by

$$g(x)=\begin{cases} x^3 & \text{for } x\in(-1,5)\cap\mathbb{Q}\\ 4x^2-x-6 & \text{for } x\in(-1,5)\setminus\mathbb{Q}\end{cases}$$

For which x_0 does $\lim_{x\to x_0} g(x)$ exist?

Solution:
Suppose $\lim_{x\to x_0} g(x)$ exists.
By density of rational numbers, there exists $r_n\in\mathbb{Q}$ such that $x_0-\frac{1}{n}<r_n<x_0$.
Similarly, by density of irrational numbers, there exists $w_n\in\mathbb{Q}$ such that $x_0-\frac{1}{n}<w_n<x_0$.
These imply that $\lim_{n\to\infty} r_n=x_0=\lim_{n\to\infty} w_n$.
By sequential limit theorem,

$$\lim_{n\to\infty} g(r_n)=\lim_{x\to x_0} g(x)=\lim_{n\to\infty} g(w_n)$$

By uniqueness of limit,

$$x_0^3=4x_0^2-x_0-6$$

Then, by long division,

$$0=x_0^3-4x_0^2+x_0+6=(x_0-2)(x_0^2-2x_0-3)=(x_0-2)(x_0-3)(x_0+1)$$

So, if $\lim_{x \to x_0} g(x)$ exists, x_0 can only be $2, 3$ and -1. Even -1 is not in the domain of g, there are still sequence converges to -1, so x_0 could also be -1.

Beware that what we have done is just finding some candidates x_0 such that $\lim_{x \to x_0} g(x)$ may exist, but we haven't really checked the existence of limit at these points.
Since $g(x) = \begin{cases} x^3 & \text{for } x \in (-1,5) \cap \mathbb{Q} \\ 4x^2 - x - 6 & \text{for } x \in (-1,5) \setminus \mathbb{Q} \end{cases}$

- Check that $\lim_{x \to 2} g(x)$ exists (the limit should be $2^3 = 8 = 4(2^2) - 2 - 6$):

  Consider
  $$0 \leq |g(x) - 8| \leq |x^3 - 8| + |(4x^2 - x - 6) - 8|$$
  Taking limit on both sides,
  $$0 = \lim_{x \to 2} 0 \leq \lim_{x \to 2} |g(x) - 8| \leq \lim_{x \to 2} \big(|x^3 - 8| + |(4x^2 - x - 6) - 8|\big) = 0$$
  (From the definition of g, $g(x)$ is either x^3 or $4x^2 - x - 6$ and adding something positive is greater or equal to the original one)
  Then by sandwich theorem, we get $\lim_{x \to 2} |g(x) - 8| = 0$. i.e. $\lim_{x \to 2} g(x) = 8$.

- Similarly, check that $\lim_{x \to 3} g(x)$ exists (the limit should be $3^3 = 27 = 4(3^2) - 3 - 6$):

  Consider
  $$0 \leq |g(x) - 27| \leq |x^3 - 27| + |(4x^2 - x - 6) - 27|$$
  Taking limit on both sides,
  $$0 = \lim_{x \to 3} 0 \leq \lim_{x \to 3} |g(x) - 27| \leq \lim_{x \to 3} \big(|x^3 - 27| + |(4x^2 - x - 6) - 27|\big) = 0$$
  Then by sandwich theorem, we get $\lim_{x \to 2} |g(x) - 27| = 0$. i.e. $\lim_{x \to 3} g(x) = 27$.

- Check that $\lim_{x \to -1} g(x)$ exists (the limit should be $(-1)^3 = -1 = 4((-1)^2) - (-1) - 6$):

  Consider
  $$0 \leq |g(x) - 8| \leq |x^3 + 1| + |(4x^2 - x - 6) + 1|$$
  Taking limit on both sides,
  $$0 = \lim_{x \to -1} 0 \leq \lim_{x \to -1} |g(x) + 1| \leq \lim_{x \to -1} \big(|x^3 + 1| + |(4x^2 - x - 6) + 1|\big) = 0$$
  Then by sandwich theorem, we get $\lim_{x \to -1} |g(x) + 1| = 0$. i.e. $\lim_{x \to 2} g(x) = 8$.