182. (2006 Spring Exam) Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be continuous. Also, let $f(x)$ be differentiable for all $x \in (a, b)$. Prove that if the graph of f is not a line segment, then there exist numbers x_1 and x_2 in the open interval (a, b) such that

$$f'(x_1) < \frac{f(b) - f(a)}{b - a} < f'(x_2).$$

183. (2006 Spring Exam) Let $f, g : [0, 1] \to \mathbb{R}$ be continuous. If there exists a sequence of numbers $x_1, x_2, x_3, \ldots \in [0, 1]$ such that $g(x_n) = f(x_{n+1})$ for $n = 1, 2, 3, \ldots$, then prove that there exists $w \in [0, 1]$ such that $g(w) = f(w)$.

<u>*Caution*</u> Be careful, x_{n_i} converges does not imply x_{n_i+1} converges !!!

184. (2006 Fall Exam) (a) Determine the set of <u>*all*</u> the positive numbers b such that $\sum_{k=1}^{\infty} \frac{k}{(k+b)^2}$ converges. Be sure to prove you have gotten all such b.

(b) Determine the set of <u>*all*</u> the positive numbers c such that $\sum_{k=1}^{\infty} \frac{(-1)^k k}{(k+c)^2}$ converges. Be sure to prove you have gotten all such c.

185. (2006 Fall Exam) Let $\left(0, \frac{1}{2}\right) \cap \mathbb{Q} \subseteq A_1 \subseteq [0, 1)$. For $n = 1, 2, 3, \ldots$, let

$$A_{n+1} = \{\sqrt{x} : \ x \in A_n\}.$$

Determine the supremum and infimum of $\bigcup_{k=1}^{\infty} A_k$ with proof.

186. (2006 Fall Exam) (a) State the definition of a sequence $a_1, a_2, a_3, \ldots$ of real numbers <u>*converges*</u> to a number L.

(b) Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ be sequences of positive numbers such that $\lim_{n\to\infty} x_n = 1 = \lim_{n\to\infty} y_n$. Prove that

$$\lim_{n\to\infty} \left(4x_n + \frac{1}{y_n}\right) = 5$$

by checking the definition of limit. Do not use the computation formulas for limits, sandwich theorem or l'Hopital's rule, otherwise you will get 0 mark for this problem!

187. (2006 Fall Exam) Let

$$x_1 = 2, \quad x_2 = 4 \quad \text{and} \quad x_{n+2} = \sqrt{10x_n - 9} \quad \text{for } n = 1, 2, 3, \ldots.$$

Determine if the sequence $x_1, x_2, x_3, \ldots$ converges or not with proof. In case of convergence, also find the limit.

188. (2007 Spring Exam) (a) Let $f : S \to \mathbb{R}$ be a function and x_0 be an accumulation point of S. State the definition of $\lim_{x\to x_0} f(x) = L$ (or "$f(x)$ converges to L as x tends to x_0").

(b) Let $f : (0, +\infty) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{\sqrt{x} + 1}$. Prove that $\lim_{x\to 1} f(x) = \frac{1}{2}$ by checking the definition. (Zero mark will be given to those who used computation formulas, sandwich theorem or l'Hopital's rule!)

189. (2007 Spring Exam (a) State the definition of $x_1, x_2, x_3, \ldots$ is a Cauchy sequence.

(b) Define $a_1 = 1$ and $a_{n+1} = 2a_n + \sin a_n$ for $n = 1, 2, 3, \ldots$. Prove that $\frac{a_1}{2}, \frac{a_2}{4}, \ldots, \frac{a_n}{2^n}, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence. (Zero mark will be given to those who used the fact that convergent sequences are Cauchy sequences!)

190. (2007 Spring Exam) Let $f : [0,1] \to [0,1]$ be continuous such that $f(0) = 0,\ f(1) = 1$ and $f(f(x)) = x$ for all $x \in [0,1]$. Prove that $f(x) = x$ for all $x \in [0,1]$.

191. (2007 Spring Exam) Let $f : [0,1] \to \mathbb{R}$ be continuous and let it be differentiable on $(0,1)$. Also, $f(0) = 0$ and $f(1) = 1$. Let a and b be positive real numbers.

(a) Prove that there exist $x_0 \in (0,1)$ such that $f(x_0) = \dfrac{a}{a+b}$.

(b) Prove that there exist distinct $x_1, x_2 \in (0,1)$ such that

$$\frac{a}{f'(x_1)} + \frac{b}{f'(x_2)} = a + b.$$

(c) Prove that if $c_1, c_2, \ldots, c_n > 0$ and $c_1 + c_2 + \cdots + c_n = 1$, then there exist distinct $t_1, t_2, \ldots, t_n \in (0,1)$ such that

$$\frac{c_1}{f'(t_1)} + \frac{c_2}{f'(t_2)} + \cdots + \frac{c_n}{f'(t_n)} = 1.$$

192. (2007 Spring Exam) Determine the domain (of convergence) of $\sum\limits_{k=1}^{\infty} \dfrac{1}{2^k k}(3x-1)^k$.

193. (2007 Spring Exam) Determine whether the improper integral $\displaystyle\int_{-1}^{1} \frac{dx}{x \cos x}$ converges or not. Also, determine whether the principal value integral $P.V. \displaystyle\int_{-1}^{1} \frac{dx}{x \cos x}$ converges or not.

194. (2007 Spring Exam) Prove that the series of functions $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^2(e^{kx} + e^{-kx})}$ converges uniformly on $\mathbb{R}$.

195. (2007 Spring Exam) For $n = 1, 2, 3, \ldots$, let $x_n, y_n \in (0, +\infty)$ and let $\{x_n\}, \{y_n\}$ be Cauchy sequences. Prove that $\left\{\dfrac{y_n}{x_n + 1}\right\}$ is also a Cauchy sequence by checking the definition of Cauchy sequence.

196. (2007 Spring Exam) State Lebesgue's theorem.

(b) Let $f, g : [0,1] \to \mathbb{R}$ be monotone functions. Prove that $h : [0,1] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} f(x) - g(x) & \text{if } x \in [0, 1/2) \\ f(x) + g(x) & \text{if } x \in [1/2, 1] \end{cases}$$

is bounded and Riemann integrable on $[0,1]$.

197. (2007 Spring Exam) Let $a_1 > 0$ and $a_{n+1} = a_n + \dfrac{1}{a_n}$ for $n = 1, 2, 3, \ldots$. Show that $\lim\limits_{n \to +\infty} \dfrac{a_n^2}{n} = 2$.

198. (2007 Spring Exam) Prove that the equation $1 - x + \dfrac{x^2}{2} - \dfrac{x^3}{3} + \cdots + \dfrac{x^{2006}}{2006} - \dfrac{x^{2007}}{2007} = 0$ has a positive solution.

199. (2007 Spring Exam) State Taylor's theorem with Lagrange remainder.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a three-times differentiable function. If $f(x)$ and $f'''(x)$ are bounded functions on $\mathbb{R}$, then prove that $f'(x)$ and $f''(x)$ are also bounded functions on $\mathbb{R}$.

200. (2007 Fall Exam) (a) Determine (with proof) if $\sum_{k=1}^{\infty} \frac{2^k k^2}{(2k)!}$ converges.

(b) Determine (with proof) if $\sum_{k=3}^{\infty} \frac{\cos k}{k(\ln k)^2}$ converges.

201. (2007 Fall Exam) (a) Let D be a nonempty subset of $\mathbb{R}$ such that $\inf D = 2$ and $\sup D = 5$. Determine (with proof) the supremum and infimum of the set

$$A = \Big\{\frac{x}{y} : x, y \in D\Big\}.$$

(b) (6 marks) Let c be a positive rational number. Determine (with proof) the supremum and infimum of

$$B = \big\{x + y : x \in [0, c\sqrt{2}] \cap \mathbb{Q},\ y \in [0, c] \setminus \mathbb{Q}\big\}.$$

202. (2007 Fall Exam) Let S be a nonempty countable subset of the interval $(0, +\infty)$. Prove that there exists a positive real number which is not the area of any triangle whose three sides have lengths in S.

203. (2007 Fall Exam) Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers such that

$$x_{n+1} = \frac{x_1 - 2}{10 + x_n} \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

(a) If $x_1 = -7$, then prove that $x_1, x_2, x_3, \ldots$ converges and find its limit.

(b) If $x_1 = 26$, then prove that $x_1, x_2, x_3, \ldots$ converges and find its limit.

204. (2007 Fall Exam) For $n = 1, 2, 3, \ldots$, let

$$y_n = \frac{4n^2 - \sqrt{n}}{2n^2 + n} + \frac{n-1}{n}\ .$$

Prove that $\lim_{n\to\infty} y_n = 3$ by checking the definition of limit of a sequence <u>only</u>.

205. (2007 Fall Exam) Let A and B be nonempty subsets of $\mathbb{R}$. Both A and B are bounded above. Let

$$C = (A \setminus B) \cup (B \setminus A).$$

(a) Give an example of such sets A and B so that C is nonempty and $\sup C \neq \max\{\sup A, \sup B\}$.

(b) If C is nonempty and $\sup C \neq \max\{\sup A, \sup B\}$, then prove that

$$\sup(A \cap B) = \max\{\sup A, \sup B\}.$$

(c) If C is nonempty and $\sup A \neq \sup B$, then prove that

$$\sup C = \max\{\sup A, \sup B\}.$$

206. (2007 Fall Exam) (a) State the definition of a sequence $x_1, x_2, x_3, \ldots$ of real numbers converging to a real number L.

(b) (15 marks) Let $a_1, a_2, a_3, \ldots$ be positive numbers such that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}+a_{n+2}} = 0$. Prove that $a_1, a_2, a_3, \ldots$ cannot be bounded above.

207. (2008 Spring Exam) Prove that $\lim_{x\to 1} \frac{3x}{x^2+2} = 1$ by checking the ε-δ definition of limit of function.

(Do not use any computation formula, sandwich theorem or l'Hopital's rule, otherwise, you will get zero mark.)

208. (2008 Spring Exam) Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of real numbers. Let $b_n \in \mathbb{R}$ satisfy

$$a_n \leq b_n \leq a_n + \frac{1}{n} \quad \text{for} \quad n = 1, 2, 3, \ldots.$$

Prove that $b_1, b_2, b_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

(Do not use Cauchy theorem that said a sequence converges if and only if it is a Cauchy sequence, otherwise you will get zero mark.)

209. (2008 Spring Exam) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$. Prove that there exists at least one $x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$.

210. (2008 Spring Exam) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. There are $a, b \geq 0$ such that for all $x \in [0, 1]$, we have $|f(x)| \leq a$ and $|f''(x)| \leq b$. Prove that for every $c \in (0, 1)$ we have

$$|f'(c)| \leq 2a + \frac{1}{2}b.$$

211. (2008 Spring Exam) Determine the domain (of convergence) of $f(x) = \sum_{k=1}^{\infty} \frac{k^2}{3^k}(\pi - 2x)^k$.

212. (2008 Spring Exam) Determine whether the improper integral $\int_{-1}^{1} \frac{x\,dx}{\sin^2 x}$ converges or not. Also, determine whether the principal value integral $P.V. \int_{-1}^{1} \frac{x\,dx}{\sin^2 x}$ converges or not.

213. (2008 Spring Exam) Prove the series of functions $\sum_{k=1}^{\infty} \left(\frac{kx}{1+k^2x^2}\right)^k$ converges uniformly on $\mathbb{R}$.

214. (2008 Spring Exam) Let $\{x_n\}, \{y_n\}$ be two Cauchy sequences of real numbers. Prove that $\sqrt{x_n^2 + y_n^2}$ is also a Cauchy sequence by checking the definition of Cauchy sequence.

215. (2008 Spring Exam) (a) State Lebesgue's theorem.

(b) For $n = 1, 2, 3, \ldots$, let $f_n : [0, 1] \to [0, 1]$ be Riemann integrable functions. Prove that $g : [0, 1] \to \mathbb{R}$ defined by $g(0) = 0$ and

$$g(x) = f_n(x) \quad \text{for } n = 1, 2, 3, \ldots \quad \text{and} \quad x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$$

is Riemann integrable on $[0, 1]$.

216. (2008 Spring Exam) Let $a_1, a_2, a_3, \ldots \in \mathbb{R}$ and s_n be the n-th partial sum of the convergent series $\sum_{k=1}^{\infty} a_k$. Prove that $\lim_{n\to\infty} \frac{a_1 + 2a_2 + 3a_3 + \cdots + na_n}{n} = 0$.

217. (2008 Spring Exam) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f''(x)$ is continuous and $|f''(x)| \le 1$ for all $x \in [0,1]$. If $f\Big(\dfrac{1}{2}\Big) = 0$, then prove that $\left|\displaystyle\int_0^1 f(x)\,dx\right| \le \dfrac{1}{24}$.

218. (2008 Fall Exam) (a) Determine if the series $\sum\limits_{k=1}^{\infty} (\cos k)\sin\Big(\dfrac{1}{k^2+\sqrt{2}}\Big)$ converges.

219. (2008 Fall Exam) (b) Prove the sequence $\{x_n\}$ converges, where

$$x_1 = 1 \qquad \text{and} \qquad x_{n+1} = \frac{4\sqrt{x_n} + x_n}{3}$$

and find its limit.

220. (2008 Fall Exam) (a) Determine (with proof) the supremum and infimum of

$$B = \big\{\cos x + \sin y : \ x, y \in (0, \pi/2] \cap \mathbb{Q}\big\}.$$

(b) Let D and E be nonempty bounded subsets of $\mathbb{R}$ such that

$$\inf D = 3, \quad \sup D = 5, \quad \inf E = 7 \quad \text{and} \quad \sup E = 9.$$

Determine (with proof) the supremum and infimum of the set

$$A = \Big\{x + \frac{1}{y} : \ x \in D, \ y \in E\Big\}.$$

221. (2008 Fall Exam) Prove that there exists a positive real number c which does *not* equal to any number of the form $2^{a+b\sqrt{2}}$, where $a, b \in \mathbb{Q}$.

222. (2008 Fall Exam) (a) Prove that $\lim\limits_{x\to 1} \dfrac{x+8}{x^2+3} = \dfrac{9}{4}$ by checking the definition of limit of a function or the limit of a sequence via the sequential limit theorem.

(b) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that both $\{a_n\}$ and $\{b_n\}$ converge to 1. Prove that $\lim\limits_{n\to\infty} \Big(2b_n^3 + \dfrac{a_n}{2n}\Big) = 2$ by checking the definition of limit of a sequence.

223. (2008 Fall Exam) (a) Determine (with proof) the infimum of the set

$$S = \{x : x \in \mathbb{R} \text{ and there exist } b, c \in [-1, 1) \text{ such that } x^2 + bx + c = 0\}.$$

(b) (12 marks) Let $A_1, A_2, A_3, \ldots$ be subsets of $[0,1]$ such that $\bigcap\limits_{n=1}^{\infty} A_n$ is nonempty. If

$$\sup\{\inf A_n : n = 1, 2, 3, \ldots\} = \inf\{\sup A_n : n = 1, 2, 3, \ldots\},$$

then prove that $\bigcap\limits_{n=1}^{\infty} A_n$ has exactly one element.

224. (2008 Fall Exam) For all $k \in \mathbb{N}$, let $a_k > 0$ and $\sum\limits_{k=1}^{\infty} a_k = 1$. For all $n \in \mathbb{N}$, let $s_n = \sum\limits_{k=1}^{n} a_k$ and $t_n = \sum\limits_{k=n}^{\infty} a_k$.

(a) (10 marks) Prove that $\sum\limits_{n=1}^{\infty} (-1)^n t_n$ and $\sum\limits_{n=1}^{\infty} \dfrac{a_n}{\sqrt{s_n}}$ both converge.

(b) Prove that $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{t_n}}$ converges.

225. (2009 Spring Exam) Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of positive real numbers. For $n = 1, 2, 3, \ldots$, let

$$b_n = \sin(a_n^2) + \sqrt[3]{7a_n}.$$

Prove that $b_1, b_2, b_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

(Do not use Cauchy theorem that said a sequence converges if and only if it is a Cauchy sequence, otherwise you will get zero mark. However, you may use the fact Cauchy sequences are bounded.)

226. (2009 Spring Exam) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable and $f''(x)$ be continuous. If

$$f(-1) = 0, \quad f(0) = 2, \quad f(1) = 5 \quad \text{and} \quad f'(0) = 0,$$

then prove that there exists $c \in \mathbb{R}$ such that $f''(c) = \sqrt{2}$.

227. (2009 Spring Exam) Prove that there exists a unique continuous function $f : [0,1] \to [0,1]$ such that $f\Big(f\big(f(x)\big)\Big) + f(x) = 2x$ for all $x \in [0,1]$.

228. (2009 Spring Exam) Determine the domain (of convergence) of $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^4}(3x-2)^k$.

229. (2009 Spring Exam) Determine whether the improper integral $\int_{-1}^{1} \frac{\sin x}{x^2 \cos^2 x}\, dx$ converges or not. Determine whether the principal value integral $P.V. \int_{-1}^{1} \frac{\sin x}{x^2 \cos^2 x}\, dx$ converges or not.

230. (2009 Spring Exam) Prove that $\sum_{k=1}^{\infty} k\Big(\frac{x^2}{1+x^3}\Big)^k$ converges uniformly on $[0, +\infty)$.

231. (2009 Spring Exam) Determine $\lim_{n\to\infty} \frac{1^{1/n} + 2^{1/n} + \cdots + n^{1/n}}{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}$ and $\lim_{x\to 0+} \frac{\sin(x^2) - x^2\cos(\sqrt{x})}{e^{x^3} - 1}$.

232. (2009 Spring Exam) Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence in $\mathbb{R}$ and let

$$y_n = x_{n+1} + x_n^2 + \cos(x_n) \quad \text{for } n = 1, 2, 3, \ldots.$$

Prove that $y_1, y_2, y_3, \ldots$ is also a Cauchy sequence by checking the definition of Cauchy sequence.

233. (2009 Spring Exam) (a) State Lebesgue's theorem.

(b) Let S be a set of measure 0. Prove that $T = \{2x : x \in S\}$ is also a set of measure 0. Let $f : [0,1] \to [0,1]$ be a Riemann integrable function. Prove that $g : [0,2] \to [0,1]$ defined by $g(x) = f(x/2)$ is Riemann integrable on $[0,1]$.

234. (2009 Spring Exam) Let $f : \mathbb{R} \to \mathbb{R}$ be a 7-times differentiable function such that for all $x \in \mathbb{R}$, $f^{(7)}(x) + f(x) = 0$ and $f(0) = f'(0) = \cdots = f^{(7)}(0) = 0$. Prove that f is n-times differentiable for every integer $n > 7$. Prove that $f(t) = 0$ for all $t \in \mathbb{R}$.

235. (2009 Fall Exam) (a) Determine (with proof) <u>all</u> nonnegative real number b such that the series $\sum_{k=1}^{\infty} \frac{2^{k+3}}{k^2(b+1)^k}$ converges. (This means for the <u>remaining</u> nonnegative real number b, you also have to explain why the series diverges.) Show details!

(b) Let $a_1, a_2, a_3, \dots$ be real numbers in the open interval $(0,1)$ such that $\sum\limits_{k=1}^{\infty} a_k$ converges. Determine (with proof) whether $\sum\limits_{k=1}^{\infty} \dfrac{\sin a_k}{1-a_k}$ converges or not.

236. (2009 Fall Exam) Let D be a nonempty bounded subset of $\mathbb{R}$ such that $\inf D = 3$ and $\sup D = 5$. Let

$$A = \{xy + xy^3 : \ x \in (2,\pi] \cap \mathbb{Q}, \quad y \in D\}.$$

Show that A is bounded. Determine (with proof) the infimum and supremum of A.

237. (2009 Fall Exam) Let S be the set of <u>all</u> points (x, y) in the coordinate plane that satisfy the equations

$$x^2 + y^2 = a^2 \qquad \text{and} \qquad y = x^2 - x^3 + b$$

for some $a, b \in \mathbb{Q}$ with $a \neq b$. Determine (with proof) if S is countable or not.

238. (2009 Fall Exam) Let $x_1 = 1$ and for $n = 1, 2, 3, \dots$, let $x_{n+1} = \dfrac{x_n^3 + x_n}{5}$.

(a) Prove that the sequence $x_1, x_2, x_3, \dots$ converges and find its limit.

(b) Prove that the series $\sum\limits_{n=1}^{\infty} x_n$ converges.

239. (2009 Fall Exam) Let $x_1 = 2$ and for $n = 1, 2, 3, \dots$, let $x_{n+1} = \dfrac{22}{3} + \dfrac{16}{3x_n}$.

(a) Prove that the sequence $x_1, x_2, x_3, \dots$ converges and find its limit.

(b) Prove that the series $\sum\limits_{n=1}^{\infty} (x_n - x_{n+1})$ converges and determine its sum.

240. (2009 Fall Exam) Let $a_1, a_2, a_3, \dots$ and $b_1, b_2, b_3, \dots$ be two sequences of real numbers such that both of them converge to 1. Prove that $\lim\limits_{n\to\infty} \left(\dfrac{3a_n^2 + 1}{a_n^2 + 1} + \dfrac{nb_n}{n+2} \right) = 3$ by checking the definition of limit of a sequence.
Do not use computation formulas, sandwich theorem or L'Hopital's rule, otherwise you will get zero mark on this problem!

241. (2009 Fall Exam) Let $a_1, a_2, a_3, \dots$ be a sequence of positive real numbers. For $n = 1, 2, 3, \dots$, let

$$P_n(x) = (x+1)(x+2)\cdots(x+n) \qquad \text{and} \qquad Q_n(x) = (x+a_1)(x+a_2)\cdots(x+a_n).$$

(a) For every $x \in \mathbb{R}$, determine whether $\sum\limits_{n=1}^{\infty} \dfrac{P_n(x)}{n!} x^n$ converges or not.

(b) Prove that $\lim\limits_{n\to\infty} \dfrac{a_n}{Q_n(1)} = 0$.

242. (2010 Spring Exam) Let $f : [1,3] \to \mathbb{R}$ be defined by $f(x) = \dfrac{1}{\sqrt[4]{x^2 + 6x}}$. Prove that $\lim\limits_{x\to 2} f(x) = \dfrac{1}{2}$ by checking the ε-δ definition of limit of function.

243. (2010 Spring Exam) Let $a_1, a_2, a_3, \dots$ be a Cauchy sequence of real numbers. For $n = 1, 2, 3, \dots$, let

$$b_n = \sin^2(a_n + a_{2n}).$$

Prove that $b_1, b_2, b_3, \dots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

244. (2010 Spring Exam) Prove that there does not exist any contnuous function $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)$ is rational if and only if $f(x+1)$ is irrational.

245. (2010 Spring Exam) Let $f:\mathbb{R}\to\mathbb{R}$ be twice differentiable and for all $x\in[0,1]$, $|f''(x)|\le 2010$. If there exists $c\in(0,1)$ such that $f(c)>f(0)$ and $f(c)>f(1)$, then prove that

$$|f'(0)|+|f'(1)|\le 2010.$$

246. (2010 Spring Exam) Determine the domain (of convergence) of $f(x)=\sum\limits_{k=1}^{\infty}\frac{1}{k\sqrt{k}}(2x-1)^k$.

247. (2010 Spring Exam) Determine whether the improper integral $\displaystyle\int_{-1}^{1}\frac{\cos x}{x(x^2+1)}\,dx$ converges or not. Determine whether the principal value integral $P.V.\displaystyle\int_{-1}^{1}\frac{\cos x}{x(x^2+1)}\,dx$ converges or not.

248. (2010 Spring Exam) Let $a_1,a_2,a_3,\ldots$ and $b_1,b_2,b_3,\ldots$ be Cauchy sequences in $[0,+\infty)$ and let

$$c_n=a_n^2+\sqrt{b_n}+\sin(a_n+b_n)\quad\text{for } n=1,2,3,\ldots.$$

Prove that $c_1,c_2,c_3,\ldots$ is also a Cauchy sequence by checking the definition of Cauchy sequence.

249. (2010 Spring Exam) (a) State Lebesgue's theorem.

(b) Let $f:[0,1]\to[0,1]$ be a Riemann integrable function. Let $r_1,r_2,r_3,\ldots$ be a strictly increasing sequence in $(0,1]$. Prove that $g:[0,1]\to[0,1]$ defined by

$$g(x)=\begin{cases}1-f(x) & \text{if } x\notin\{r_1,r_2,r_3,\ldots\}\\ \cos x & \text{if } x\in\{r_1,r_2,r_3,\ldots\}\end{cases}$$

is Riemann integrable on $[0,1]$.

250. (2010 Spring Exam) Let $f:(-1,1)\to\mathbb{R}$ be four times differentiable such that for all $c\in(-1,1)$, $|f^{(4)}(c)|\le 1$. Prove that for all $x\in(0,1)$, we have

$$\left|f''(0)-\frac{f(x)-2f(0)+f(-x)}{x^2}\right|\le\frac{x^2}{12}.$$

251. (2010 Spring Exam) For $n=1,2,3,\ldots$, let $x_n=\sum\limits_{k=n+1}^{\infty}\frac{1}{k^2}$. Prove that $\lim\limits_{n\to\infty}nx_n=1$.

252. (2010 Fall Exam) Let A be a nonempty bounded subset of $\mathbb{R}$ such that $\inf A=1$ and $\sup A=3$. Let

$$B=\{\sqrt{2x(15+xy)}:\ x\in(2,4)\cap\mathbb{Q},\quad y\in A\}.$$

Prove that B is bounded. Determine (with proof) the infimum and supremum of B.

253. (2010 Fall Exam) Prove the sequence $\{x_n\}$ converges, where

$$x_1=5\qquad\text{and}\qquad x_{n+1}=\frac{7}{x_n+5},$$

and find its limit.

254. (2010 Fall Exam) Do either (a) or (b) below:

(a) Determine (with proof) all positive irrational numbers b such that

$$\sum_{k=1}^{\infty} \frac{\cos(k-3b)}{(2k-b)\big((\ln k)^2+1\big)}$$

converges.

(b) Determine (with proof) whether the set

$$S = \Big\{ b: \ b \in (0,+\infty) \setminus \mathbb{Q} \text{ and } \sum_{k=1}^{\infty} \frac{\cos(k-3b)}{(2k-b)\big((\ln k)^2+1\big)} \text{ converges} \Big\}$$

is countable or not.

255. (2010 Fall Exam) Let $x_1 = \dfrac{1}{4}$ and for $n = 1, 2, 3, \ldots$, let $x_{n+1} = \dfrac{\sqrt{x_n} + 3x_n}{4}$.

(a) Prove that the sequence $x_1, x_2, x_3, \ldots$ converges and find its limit.

(b) Determine (with proof) the supremum of $A = \Big\{ \sqrt{x_n - \dfrac{1}{4n}} : n = 1, 2, 3, \ldots \Big\}$.

256. (2010 Fall Exam) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers that converges to 1. Prove that $\lim\limits_{n\to\infty} \Big(\dfrac{3 + a_n^2}{a_n + 1} + \dfrac{2n}{4+n} \Big) = 4$ by checking the definition of limit of a sequence.
Do not use computation formulas, sandwich theorem or L'Hopital's rule, otherwise you will get zero mark on this problem!

257. (2010 Fall Exam) (a) Let S be the set of all intersection points (x, y) that lie on the graphs of at least one pair of equations $y = x^3 + mx + n$ and $mx^2 - ny^2 = 1$, where $m, n \in \mathbb{Q}$. Determine (with proof) whether S is a countable set or not.

(b) Prove that there exist infinitely many positive real numbers that are not equal to any number of the form $a + b(2^c \pi^d)$, where $a, b \in \mathbb{Q} \cap (0, +\infty)$ and $c, d \in \mathbb{Q} \cap [0, +\infty)$.

258. (2011 Spring Exam) Let $f : [0, +\infty) \to \mathbb{R}$ be defined by $f(x) = \sin^2 \Big(\dfrac{1}{1 + \sqrt[4]{x}} \Big)$. Prove that $\lim\limits_{x\to 1} f(x) = \sin^2 \dfrac{1}{2}$ by checking the ε-δ definition of limit of function.

259. (2011 Spring Exam) Let $A_1, A_2, A_3, \ldots$ be a Cauchy sequence of decreasing positive real numbers. For $n = 1, 2, 3, \ldots$, let B_n be a real number such that

$$\sqrt{A_{n+2011}} \le B_n \le \sqrt{A_n}.$$

Prove that $B_1, B_2, B_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

260. (2011 Spring Exam) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathbb{R}$. If $f(0) = f(1) = 0$ and $\max\{f(x) : x \in [0,1]\} = 2$, then prove that there exists $\theta \in (0,1)$ such that $f''(\theta) \le -16$.

261. (2011 Spring Exam) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and decreasing. Prove that there exists a unique element $(a, b, c) \in \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ such that

$$a = f(b), \quad b = f(c) \quad \text{and} \quad c = f(a).$$

262. (2011 Spring Exam) Determine the domain (of convergence) of $f(x) = \sum\limits_{k=1}^{\infty} \dfrac{(-1)^k}{k} (x-3)^{k+2}$.

263. (2011 Spring Exam) Determine whether the improper integral $\int_{-1}^{1} \frac{\tan x}{x^2}\,dx$ converges or not. Determine whether the principal value integral $P.V. \int_{-1}^{1} \frac{|\tan x|}{x^2}\,dx$ converges or not.

264. (2011 Spring Exam) Determine (with proof) if $\sum_{k=1}^{\infty} \sqrt{k}\left(\frac{x}{e^{kx}}\right)$ converges uniformly on $[2, +\infty)$.

265. (2011 Spring Exam) Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be Cauchy sequences in $[0, +\infty)$. Let

$$c_n = \sqrt{a_n + b_n} + \frac{a_n^2}{n} \quad \text{for } n = 1, 2, 3, \ldots.$$

Prove that $c_1, c_2, c_3, \ldots$ is also a Cauchy sequence by checking the definition of Cauchy sequence.

266. (2011 Spring Exam) Let $f : [0, 1] \to [0, 1]$ be a Riemann integrable function. Prove that $F : [0, 2] \to [0, 1]$ defined by

$$F(x) = \begin{cases} |f(x) - 1| & \text{if } x \in [0, 1) \\ f(x-1) & \text{if } x \in [1, 2] \end{cases}$$

is Riemann integrable on $[0, 2]$.

267. (2011 Spring Exam) Let $g : [1, 2] \to [0, 1]$ be a Riemann integrable function. Prove that $G : [0, 1] \to [0, 1]$ defined by

$$G(x) = \begin{cases} g(x+1) & \text{if } x \in [0, 1] \setminus \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \\ \\ 1 & \text{if } x \in \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \end{cases}$$

is Riemann integrable on $[0, 1]$ by checking the integral criterion.

268. (2011 Fall Exam) Prove the sequence $\{x_n\}$ converges, where

$$x_1 = 27 \qquad \text{and} \qquad x_{n+1} = 8 - \sqrt{28 - x_n} \quad \text{for } n = 1, 2, 3, \ldots$$

and find its limit.

269. (2011 Fall Exam) Suppose A and B are two nonempty bounded subsets of $\mathbb{R}$ such that $\inf A = 1,\ \sup A = 5,\ \inf B = 0$ and $\sup B = 1$. Let

$$C = \left\{ \frac{y}{3-x} - \frac{1}{y} :\ x \in B,\quad y \in A \right\}.$$

Prove that C is bounded. Determine (with proof) the infimum and supremum of C.

270. (2011 Fall Exam) (a) Give an example of <u>real numbers</u> $c_1, c_2, c_3, \ldots$ such that

$$\sum_{k=1}^{\infty} c_k \text{ converges, but } \sum_{k=1}^{\infty} (-1)^k c_k \text{ diverges.}$$

Be sure to explain the convergence and divergence of these series.

(b) Let $0 < a_k < 1$ for $k = 1, 2, 3, \ldots$. Suppose $\sum_{k=1}^{\infty} a_k$ converges. Determine (with proof) at least one real number b such that

$$\sum_{k=1}^{\infty} \frac{b - \cos a_k}{\sin a_k} \text{ converges.}$$

Determine (with proof) all such real number b.

271. (2011 Fall Exam) Let $x_1 = 0$, $x_2 = 3$ and for $n = 1, 2, 3, \ldots$, let $x_{n+2} = \sqrt{\frac{4}{9}x_{n+1}^2 + \frac{5}{9}x_n^2}$.

   (a) Prove that the sequence $x_1, x_2, x_3, \ldots$ converges.

   (b) Determine (with proof) the limit of the sequence $x_1, x_2, x_3, \ldots$.

272. (2011 Fall Exam) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers that converges to 3. Prove that

$$\lim_{n\to\infty}\left(\frac{a_n}{a_n^2+3} + \frac{3n^2}{1+4n^2} + \frac{a_n}{n}\right) = 1$$

   by checking the definition of limit of a sequence.

273. (2011 Fall Exam) Prove that there exist infinitely many positive irrational numbers that are <u>not</u> equal to any number of the form $\dfrac{a\sqrt{2}+b}{c+d\pi}$, where $a, b, c, d \in \mathbb{Q} \cap (0, +\infty)$.

   (b) Let $f : \mathbb{R} \to \mathbb{Q}$ be a function. Prove that there exists an uncountable subset S of $\mathbb{R}$ such that for all $x, y \in S$, we have $f(x) = f(y)$.

274. (2012 Spring Exam) Let $f : [0, +\infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{\dfrac{1}{2+\sqrt{x}}}$. Prove that $\lim\limits_{x\to 4} f(x) = \dfrac{1}{2}$ by checking the ε-δ definition of limit of function.

275. (2012 Spring Exam) Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of real numbers. For $n = 1, 2, 3, \ldots$, let $b_n = a_n \sin a_n$. Prove that $b_1, b_2, b_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

276. (2012 Spring Exam) Let $f : [0, 2] \to \mathbb{R}$ be continuous and $f(2) = 0$. If $\lim\limits_{x\to 1} \dfrac{f(x)-2}{\sqrt{x}-1} = 1$, then prove that there exists $x \in [0, 2]$ such that $f(x) = x^2$.

277. (2012 Spring Exam) Let $f : \mathbb{R} \to \mathbb{R}$ be three-times differentiable on $\mathbb{R}$. If $\dfrac{f(0)+f(2)}{2} = f(1)$, then prove that there exist $a, b, c \in \mathbb{R}$ such that

$$f'''(a) - f'''(b) = 6f''(c).$$

278. (2012 Spring Exam) Determine whether the improper integral $\displaystyle\int_{-1}^{1} \frac{\sin x}{\sin(x^2)}\,dx$ converges or not. Determine whether the principal value integral $P.V. \displaystyle\int_{-1}^{1} \frac{\sin x}{\sin(x^2)}\,dx$ converges or not.

279. (2012 Spring Exam) Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence in $[0, +\infty)$ and let

$$c_n = \sin(a_n^2 + \sqrt{a_n}) \quad \text{for } n = 1, 2, 3, \ldots.$$

   Prove that $c_1, c_2, c_3, \ldots$ is also a Cauchy sequence by checking the definition of Cauchy sequence.

280. (2012 Spring Exam) Let $f : \mathbb{R} \to \mathbb{R}$ be three-time differentiable. If

$$f(0) = 5, \quad f(2) = 7 \quad \text{and} \quad \text{for all } x \in [0, 2], \quad |f'''(x)| \le 6,$$

then prove that $|f'(1)| \leq 2$.

281. (2012 Spring Exam) Let $f : [0,1] \to [0,1]$ be a Riemann integrable function. Let $g : [0,1] \to [0,1]$ be an increasing function. Define $h : [0,1] \to [0,1]$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \notin \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} \cup \left[\frac{2}{3}, \frac{3}{4}\right], \\ g(x) & \text{if } x \in \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} \cup \left[\frac{2}{3}, \frac{3}{4}\right] \end{cases}.$$

(a) Use Lebesgue's theorem to prove $h(x)$ is Riemann integrable on $[0,1]$.

(b) Use the integral criterion to prove $h(x)$ is Riemann integrable on $[0,1]$.

282. (2012 Fall Exam) Let S be the set of all points $(x,y) \in \mathbb{R}^2$ that satisfy the system of equations

$$x + y = mx^2 - x^3 \qquad \text{and} \qquad mx + y^4 = x^6 - 7mx^3 + 2$$

for some $m \in \mathbb{Q}$. Determine (with proof) if S is countable or not.

283. (2012 Fall Exam) Determine (with proof) <u>all</u> positive real number b such that the series $\sum\limits_{k=1}^{\infty} \dfrac{2^{k+3}}{\sqrt{k}(\sqrt{b}+1)^k}$ converges. Be sure to prove you have gotten all such b.

284. (2012 Fall Exam) Let S be a nonempty countable subset of $\mathbb{R}$. Prove that there exists a positive real number r such that the equation $5^x + 7^y = \sqrt{r}$ does not have any solution with $x, y \in S$.

285. (2012 Fall Exam) If $x_1 = -2$ and $x_{n+1} = \sqrt{6 + x_n}$ for $n = 1, 2, 3, \ldots$, then prove that $x_1, x_2, x_3, \ldots$ converges and find its limit.

(b) (14 marks) If $y_1 = 0$ and $y_{n+1} = \dfrac{2}{2 + y_n}$ for $n = 1, 2, 3, \ldots$, then prove that $y_1, y_2, y_3, \ldots$ converges and find its limit.

286. (2012 Fall Exam) (a) Let D be a nonempty subset of $\mathbb{R}$ with $\inf D = 1$ and $\sup D = 5$. Determine (with proof) the supremum of the set

$$E = \left\{ x(y + \sqrt{2}) - \frac{1}{x} \ : \ x \in D, \quad y \in [0, \sqrt{2}) \cap \mathbb{Q} \right\}.$$

(b) Let A, B, C be nonempty subsets of $\mathbb{R}$ such that $A \subseteq B \subseteq C$. Suppose C is bounded above in $\mathbb{R}$. If $\sup A = w = \sup C$, then prove that $\sup B = w$.

287. (2012 Fall Exam) Let $b_1, b_2, b_3, \ldots$ be a sequence of positive real numbers with $\lim\limits_{n\to\infty} b_n = 2$. Prove that

$$\lim_{n\to\infty} \left(\frac{4n-1}{n+3} - \frac{2}{b_n} + \frac{b_n}{n} \right) = 3$$

by checking the definition of limit of a sequence <u>only</u>.

288. (2013 Spring Exam) Let $a_1, a_2, a_3, \ldots$ be a Cauchy sequence of <u>positive</u> real numbers. For $n = 1, 2, 3, \ldots$, let $b_n = \sqrt{\dfrac{a_n}{a_n + 3} + 5}$. Prove that $b_1, b_2, b_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

289. (2013 Spring Exam) Let $f:[0,+\infty)\to\mathbb{R}$ be defined by $f(x)=\dfrac{x}{1+2x}+\dfrac{2}{2+\sqrt{x}}$. Prove that $\lim\limits_{x\to 1} f(x)=1$ by checking the ε-δ definition of limit of function.

290. (2013 Spring Exam) Let $f:\mathbb{R}\to\mathbb{R}$ be twice differentiable. If $f'(0)=2=f'(1)$ and for all $x\in[0,1]$, $|f''(x)|\le 4$, then prove that $|f(1)-f(0)|\le 3$.

291. (2013 Spring Exam) Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable such that for all $x\in\mathbb{R}$,

$$f\big(1-f(x)\big)=1-x^9.$$

If $f(1)=0$ and $f'(1)<0$, then prove that there exists $r\in\mathbb{R}$ such that $f(r)=r^{2013}$.

292. (2013 Spring Exam) Determine the domain (of convergence) of $f(x)=\sum\limits_{k=3}^{\infty}\dfrac{\ln k}{\ln(k+1)}(2x-1)^k$.

293. (2013 Spring Exam) Determine whether the principal value integral $P.V.\displaystyle\int_{-1}^{1}\frac{\cos x}{(x\sin x)^{1/4}}\,dx$ converges or not.

294. (2013 Spring Exam) Let $x_1,x_2,x_3,\ldots$ be a Cauchy sequence in $(0,+\infty)$ and let

$$y_n=e^{-x_n}+\Big(\frac{x_n}{n}\Big)^2 \quad \text{for } n=1,2,3,\ldots.$$

Prove that $y_1,y_2,y_3,\ldots$ is also a Cauchy sequence by checking the definition of Cauchy sequence. (Do not use the theorem that asserts a sequence is a Cauchy sequence if and only if it converges. Otherwise you will get 0 mark for this problem!)

295. (2013 Spring Exam) Let $f:[0,1]\to[0,1]$ be a function that is continuous at all $x\in[0,1]\setminus\mathbb{Q}$. Let $g:[0,1]\to[0,1]$ be defined by $g(x)=f(x)f\Big(\dfrac{x}{\sqrt{2}}\Big)$ for all $x\in[0,1]$. Prove that $g(x)$ is Riemann integrable on $[0,1]$ by Lebesgue's theorem.

296. (2013 Spring Exam) Let the sequence $a_1,a_2,a_3,\ldots$ converge to $\sqrt{3}$, where all $a_n\in\mathbb{R}$. Define

$$b_n=\frac{1}{n^2}\sum_{k=1}^{n}(n+1-k)a_k \quad \text{for } n=1,2,3,\ldots.$$

Prove that the sequence $b_1,b_2,b_3,\ldots$ converges and find its limit.

297. (2013 Spring Exam) Let $h:[0,2]\to[0,1]$ be Riemann integrable. Define $p:[0,2]\to[0,2]$ by

$$p(x)=\begin{cases} 1 & \text{if } x\in\big\{\frac{n}{n+1}:n=1,2,3,\ldots\big\} \\[2ex] h(x)+1 & \text{if } x\in[0,2]\setminus\big\{\frac{n}{n+1}:n=1,2,3,\ldots\big\}\end{cases}$$

Prove that $p(x)$ is Riemann integrable on $[0,2]$ using the integral criterion.

298. (2014 Exam) Prove that there exist infinitely many real numbers r such that the equation $10^{xy}+r-y^3=xy$ does not have any solution with $x,y\in\mathbb{Q}$.

299. (2014 Exam) Let A be a nonempty bounded subset of $\mathbb{R}$ such that $\inf A=0$ and $\sup A=3$. Let

$$B=\{x+2^{xy}+y : x\in[1,2]\setminus\mathbb{Q},\ y\in A\}.$$

Prove that B is bounded. Determine (with proof) the infimum and supremum of B.

300. (2014 Exam) Prove that the sequence $\{x_n\}$ converges, where

$$x_1 = 11 \quad \text{and} \quad \text{for } n = 1, 2, 3, \ldots,\ x_{n+1} = \frac{18}{x_n + 7}$$

and find its limit. Show all details.

301. (2014 Exam) Prove that

$$\lim_{n\to\infty}\Big(\frac{6n^2+n-3}{1+2n^2} + \frac{n+5\sqrt{n}+\sqrt[3]{n}}{6+n}\Big) = 4$$

by checking the definition of limit of a sequence <u>only</u>.

302. (2014 Exam) Let $J : \mathbb{R} \to \mathbb{R}$ be defined by $J(x) = \sin\Big(\dfrac{x-1}{|x|+2}\Big) + \dfrac{3-x}{x^2+3}$. Prove that $\lim\limits_{x\to 1} J(x) = \dfrac{1}{2}$ by checking the ε-δ definition of limit of function.

303. (2014 Exam) (a) Give the name of a theorem that was taught in class that you would use to solve part (c) of this problem.

(b) Describe the theorem you named in part (a) and state the reason(s) you want to use it for solving part (c).

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable. For every $x \in [1,3]$, $|f''(x)| > 1$. If $f(1) = 0 = f(3)$, then prove that there exists at least one $w \in [1,3]$ such that $|f(w)| \geq \frac{1}{2}$.

304. (2014 Exam) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers such that $a_1, a_3, a_5, \ldots$ is a Cauchy sequence and for $j = 1, 2, 3, \ldots$, $a_{2j} = a_{2j-1} + \dfrac{1}{j}$. Prove that $a_1, a_2, a_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

305. (2014 Exam) Let $g : \mathbb{R} \to \mathbb{R}$ be three times differentiable such that

$$g(0) > 0, \quad g'(0) < 0 \quad \text{and} \quad g''(0) = 0.$$

If for all $x > 0$, $g'''(x) < 0$, then prove that there exists some $r \in (0, +\infty)$ such that $g(r) = 0$.

306. (2014 Exam) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}$, $3F(F(x)) = F(x) + x$. Prove that $F(0) = 0$.

307. (2014 Exam) Let $p : [1/2, 1] \to [0,1]$ be Riemann integrable such that $p(1/2) = p(1) = 0$. Define $h : [0,1] \to [-1,1]$ as follows: $h(0) = h(1) = 0$ and

$$\text{for all } n = 0, 1, 2, \ldots \quad \text{and} \quad x \in [1/2^{n+1}, 1/2^n), \quad h(x) = (-1)^n p(2^n x).$$

Prove that h is Riemann integrable on $[0,1]$ by Lebesgue's theorem.

308. (2015 Exam) Let A be a nonempty bounded subset of $\mathbb{R}$ such that $\inf A = 1$ and $\sup A = 2$. Let

$$B = \Big\{\sqrt{y}\cos x : x \in \Big(0, \frac{\pi}{3}\Big] \cap \mathbb{Q},\ y \in A\Big\}.$$

Prove that B is bounded. Determine (with proof) the infimum and supremum of B.

309. (2015 Exam) (a) Prove that the sequence $\{w_n\}$ converges, where

$$w_1 = 6 \quad \text{and} \quad \text{for } n = 1, 2, 3, \ldots,\ w_{n+1} = 6 - \frac{9}{w_n}$$

and find its limit.

(b) Prove that the sequence $\{x_n\}$ converges, where

$$x_1 = 60 \quad \text{and} \quad \text{for } n = 1, 2, 3, \ldots,\ x_{n+1} = 8 + \frac{120}{x_n}$$

and find its limit.

310. (2015 Exam) Let $y_1, y_2, y_3, \ldots$ and $z_1, z_2, z_3, \ldots$ be sequences of real numbers such that both converge to 4. Prove that

$$\lim_{n\to\infty} \left(\frac{9}{z_n^2 + 2} + \frac{5}{y_n - 2} \right) = 3$$

by checking the definition of limit of a sequence <u>only</u>.

311. (2015 Exam) Prove that there exist infinitely many positive real numbers r such that the equation $2^x r^3 = \pi^y$ does not have any solution with $x, y \in \mathbb{Q}$.

312. (2015 Exam) Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence of real numbers in $[1, +\infty)$. For every positive integer n, let $y_n = x_{2n} - \dfrac{x_n}{x_n + 1}$. Prove that $y_1, y_2, y_3, \ldots$ is a Cauchy sequence by checking the definition of Cauchy sequence.

313. (2015 Exam) Let $f : \mathbb{R} \to (0, +\infty)$ be a function such that $\lim_{x\to 1} f(x) = 1$. Prove that

$$\lim_{x\to 1} \sin\left(\frac{5\pi}{4} \sqrt[3]{2f(x) + 6} \right) = 1$$

by checking the ε-δ definition of limit of function.

314. (2015 Exam) Let $f : [0, 1] \to [0, 1]$ be continuous and injective with $f(0) < f(1)$. Determine how many solution(s) the equation $\dfrac{1 - f(x)}{1 + f(x)} = \dfrac{x^2}{2 - x^2}$ has and prove your answer is correct.

315. (2015 Exam) Let $f : \mathbb{R} \to \mathbb{R}$ be n-time differentiable for $n = 1, 2, 3$. If $f(2) = 4$, $f(1) = 2$ and $f''(0) = 1$, then prove that there exist $a, b, c \in [0, 2]$ such that

$$8f'''(a) - 3f''(b) + 6f'(c) = 0.$$

316. (2015 Exam) Let $f : [0, 1] \to [0, 1]$ be an increasing function. Define $g : [0, 1] \to [0, 1]$ by

$$g(x) = \begin{cases} f(2x) & \text{if } x \in [0, 1/2) \\ 1 - f(2x - 1) & \text{if } x \in [1/2, 1] \end{cases}.$$

Prove that g is Riemann integrable on $[0, 1]$ by checking the integral criterion.