Exercises.

Calculate the following limits.

1) $\lim\limits_{x\to 0} \frac{1}{x^2}(\sqrt{1+x}+\sqrt{1-x}-2)$

2) $\lim\limits_{x\to 0^+} (\cot x)^{\frac{1}{\ln x}}$

$\cot x = \frac{\cos x}{\sin x}$

Solution:

1) Notice $\begin{cases} \lim\limits_{x\to 0} \sqrt{1+x}+\sqrt{1-x}-2 = 0. \\ \lim\limits_{x\to 0} x^2 = 0. \end{cases}$

we have, by

$$\lim_{x\to 0} \frac{1}{(x^2)'}(\sqrt{1+x}+\sqrt{1-x}-2)' = \lim_{x\to 0} \frac{1}{2x}\left(\frac{\frac{1}{2}}{\sqrt{1+x}} - \frac{\frac{1}{2}}{\sqrt{1-x}}\right)$$

$$= \lim_{x\to 0} \frac{1}{2}\left(\frac{\frac{1}{\sqrt{1+x}}-1}{x} + \frac{\frac{1}{\sqrt{1-x}}-1}{-x}\right)$$

$$= -\frac{1}{2}.$$

exists.

derivative of $f(x) = \frac{1}{\sqrt{1+x}}$ at $x=0$.

notice that

$f'(x) = -\frac{1}{2}(1+x)^{-\frac{3}{2}}$

$$\lim_{x\to 0} \frac{1}{x^2}(\sqrt{1+x}+\sqrt{1-x}-2) \overset{\text{(L'Hospital)}}{=} -\frac{1}{2} \quad \#$$

2) $$\lim_{x\to 0^+} (\cot x)^{\frac{1}{\ln x}} = \lim_{x\to 0^+} \exp\left(\frac{\ln \cot x}{\ln x}\right).$$

$$= \exp\left(\lim_{x\to 0^+} \frac{\ln \cot x}{\ln x}\right).$$

Notice that $\lim\limits_{x\to 0^+} \ln x = \infty$, so by $\lim_{x\to 0^+} \frac{(\ln \cot x)'}{(\ln x)'} = \lim_{x\to 0^+} \frac{\frac{1}{\cot x}\cdot\left(-\frac{1}{(\sin^2 x)}\right)}{\frac{1}{x}}$

Check that:

$(\cot x)' = -\frac{1}{\sin^2 x}.$

$$= \lim_{x\to 0^+} -x \cdot \frac{\sin x}{\cos x\,(\sin^2 x)}$$

$$= -\lim_{x\to 0^+} \frac{x}{\cos x \cdot \sin x}$$

$$= -\lim_{x\to 0^+} \frac{1}{\cos x} \cdot \lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)^{-1}$$

$$= -1$$

exists, we have by L'Hospital's rule.

$$\lim_{x\to 0^+} (\cot x)^{\frac{1}{\ln x}} = \exp\left(\lim_{x\to 0^+} \frac{\ln \cot x}{\ln x}\right).$$

$$= \exp(-1). \quad \#.$$

Exercise 2.

(Complementary proof of L'hospital rule in the lecture note).

Let f, g differentiable on $\mathbb{R}$. s.t.

① $g, g' \neq 0$ on $\mathbb{R}$. ② $\lim_{x\to+\infty} f(x) = +\infty = \lim_{x\to+\infty} g(x)$

③ $\lim_{x\to+\infty} \frac{f'}{g'} = L = +\infty$

then $\lim_{x\to+\infty} \frac{f}{g} = \lim_{x\to+\infty} \frac{f'}{g'} = L = +\infty$

Proof:

Idea: weaken the assumptions one by one

First we show that the assumption "$x\to\infty$" makes no difference (compared with the lecture note)

$\Rightarrow$ Suppose $\lim_{x\to\infty} \frac{f'}{g'} = L \in \mathbb{R}$, $\lim_{x\to\infty} f = \lim_{x\to\infty} g = 0$. Define $\hat{f}(x) = f(\frac{1}{x})$ and $\hat{g}(x) = g(\frac{1}{x})$. Hence ① $\hat{f}, \hat{g}$ is differentiable on $(0, +\infty)$ ② $\lim_{x\to 0} \tilde{f} = \lim_{x\to 0} \tilde{g} = 0$ ③ $\tilde{g}, \tilde{g}' \neq 0$ on $(0, a)$ $(a>0)$. So

we can apply the L'hospital rule:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\hat{f}(\frac{1}{x})}{\hat{g}(\frac{1}{x})} = \lim_{x\to 0} \frac{-\frac{1}{x^2}\hat{f}'(\frac{1}{x})}{-\frac{1}{x^2}\hat{g}'(\frac{1}{x})} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$$

Second, we show that the assumption "$L \in \mathbb{R}$" makes no difference

Assume all assumptions in the lecture not except $L \in \mathbb{R}$.

Let $L = +\infty$. Then.

$$\left|\frac{f(x)}{g(x)}\right| = \left|\frac{f(x)-0}{g(x)-0} - \frac{f'(x)}{g'(x)} + \frac{f'(x)}{g'(x)}\right|$$

$$\geq \left|\frac{f'(x)}{g'(x)}\right| - \left|\frac{f'(\xi)}{g'(\eta)} - \frac{f'(x)}{g'(x)}\right|$$

$$\geq N - \varepsilon$$

when x closed enough to 0. So $\frac{f(x)}{g(x)} \to L = +\infty$ as $x \to 0$.

Finally, show that the assumption "$\frac{0}{0}$-form" can be replaced by "$\frac{\infty}{\infty}$-form".

Since $\lim\limits_{x\to 0} \frac{f'(x)}{g'(x)} = L$. $\forall \varepsilon > 0$. $\exists (-\delta, \delta)$ s.t. $L - \varepsilon < \frac{f'(x)}{g'(x)} < L + \varepsilon$. $(x \in (-\delta, \delta))$

then for $\forall (x_1, x_2) \subset (0, \delta)$ we know, from MVT.

$$L - \varepsilon < \frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} = \frac{f'(\xi)}{g'(\eta)} < L + \varepsilon$$

different from ξ, but can be controlled,

On the other hand,

$$\frac{f(x_1) - f(x_2)}{g(x_1) - g(x_2)} = \left(\frac{f(x_1)}{g(x_1)} - \frac{f(x_2)}{f(x_1)}\right) \cdot \frac{g(x_1)}{g(x_1) - g(x_2)} < L + \varepsilon$$

Fix x_2 and let $x_1 \to 0$, we have.

$$\limsup_{x\to 0} \frac{f(x)}{g(x)} \leq L + \varepsilon \xRightarrow{\varepsilon \text{ arbitrary}} \limsup_{x\to 0} \frac{f(x)}{g(x)} \leq L \qquad (1)$$

Similary, we know

$$\liminf_{x\to 0} \frac{f(x)}{g(x)} \geq L \qquad (2)$$

From (1), (2), we draw the conclusion.

$\square$

Exererse3.

Let $f : [0, \infty) \to \mathbb{R}$ be continuous and $f(0) = 0$. If $|f'(x)| \le |f(x)|$ for every $x > 0$, show that $f(x) = 0$ for every $x \in [0, \infty)$. (*Hint*: Let $|f|$ has maximum value M on $[0, \frac{1}{2}]$. Apply the mean value theorem to f on $[0, \frac{1}{2}]$.)

Proof:

First. we claim that $f(x) \equiv 0$ on $[0, \frac{1}{2}]$.

$\Rightarrow$ Otherwise. $M = \max_{x \in [0,\frac{1}{2}]} |f(x)| > 0$. From the assumption we have.

$$-|f(x)| \le f'(x) \le |f(x)| \qquad -(1)$$

and for $\forall x \in [0, \frac{1}{2}]$. $\exists \xi \in (0, x)$ s.T. $\Rightarrow$ MVT.

$$\underbrace{f(x) - f(0)}_{f(0)=0} = f'(\xi)\underbrace{(x-0)}_{=x} \Rightarrow f(x) = f'(\xi)x \qquad -(2)$$

Collapse (1)(2). we have

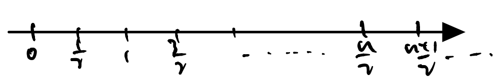
$$|f(x)| = |f'(\xi)| \cdot x \le x|f(x)| \le \frac{1}{2} \cdot M \quad (\forall x \in [0, \tfrac{1}{2}])$$

$\Rightarrow M = \max_{x \in [0,\frac{1}{2}]} |f(x)| \le \frac{1}{2} M$. a contradiction.

Next. we can extend this argument to $\mathbb{R}$.

$\Rightarrow$ By induction. we know $f|_{[\frac{n}{2}, \frac{n+1}{2}]} \equiv 0$ $(n \in \mathbb{Z})$ (eg. $f|_{[0,\frac{1}{2}]} \equiv 0$. so $f(\frac{1}{2}) = 0$. since $|f'(x)| \le |f(x)|$ on $[\frac{1}{2}, 1]$. we draw the same conclusion on $[\frac{1}{2}, 1]$). So. if here $\exists x_0 \in \mathbb{R}$ s.t. $f(x_0) \neq 0$. x_0 must lie in some $[\frac{n}{2}, \frac{n+1}{2}]$. Hence $\max_{x \in [\frac{n}{2}, \frac{n+1}{2}]} |f(x)| > 0$. a contradiction.

$\square$



Exercise 4.

Prove that if $0 \le \theta \le \frac{\pi}{2}$, then

$$1 - \frac{\theta^2}{2} \le \cos\theta \le 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

(*Hint*: Apply Taylor's theorem to the four times differentiable function $\cos\theta$.)

Proof:

From the Taylor expansion. first expand $\cos\theta$ at $\theta = 0$ to 2-order:

$$\cos\theta = \cos 0 + (\cos\theta)'\big|_0 \cdot \theta + \frac{1}{2!}(\cos\theta)''\big|_0 \theta^2 + \frac{1}{3!}(\cos\theta)'''\big|_\xi \theta^3$$

non-negative since $0 \le \xi \le \frac{\pi}{2}$ $\quad (0 \le \theta \le \frac{\pi}{2}.\ \xi \in (0, \theta))$

$$\Rightarrow \cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{6}\sin\xi \cdot \theta^3 \ge 1 - \frac{1}{2}\theta^2 \quad — (1)$$

Then expand $\cos\theta$ at $\theta = 0$ to 4 order.

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 - \frac{1}{120}\sin\xi\, \theta^5 \quad (0 \le \theta \le \frac{\pi}{2}.\ \xi \in (0, \theta))$$

$$\le 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 \quad — (2)$$

From (1). (2). we can draw the conclusion.

$\square$

Exercise 5.

Let $f, g : [a, b] \to \mathbb{R}$ be bounded and P is a partition of $[a, b]$. Show that

$$L(f, P) + L(g, P) \le L(f + g, P) \le U(f + g, P) \le U(f, P) + U(g, P).$$

Proof:

Fix the partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Then we show the inequalities one by one.

1) $L(f+g, P) \le U(f+g, P)$

On every subinterval $I_i = [x_i, x_{i+1})$. $m_i^{f+g} = \min_{x \in I_i} \{f(x) + g(x)\}$

$\le \max_{x \in I_i} \{f(x) + g(x)\} = M_i^{f+g}$. So.

$$L(f+g, P) = \sum_i m_i^{f+g} \overset{x_{i+1}-x_i}{\overset{\parallel}{\Delta x_i}} \le \sum_i M_i^{f+g} \Delta x_i = U(f+g, P)$$

2) $L(f, P) + L(g, P) \le L(f+g, P)$

As in 1), we define the terms m_i^f, m_i^g, m_i^{f+g} $(0 \le i \le n-1)$

Since $\begin{cases} f(x) \ge m_i^f \\ g(x) \ge m_i^g \end{cases}$ when $x \in I_i$. So we have.

$m_i^f + m_i^g \le f(x) + g(x)$ $(x \in I_i)$

$\xrightarrow[\text{to both sides}]{\text{take inf}}$ $m_i^f + m_i^g \le m_i^{f+g} := \sup_{x \in I_i} (f+g)$

$$\Rightarrow L(f, P) + L(g, P) = \sum_i m_i^f \Delta x_i + \sum_i m_i^g \Delta x_i$$
$$\le \sum_i m_i^{f+g} \Delta x_i = L(f+g, P)$$

3) $U(f+g, P) \leq U(f, P) + U(g, P)$

Similar to 2).

Collapse 1) ~ 3). we draw the conclusion.

$\square$