Chapter 6 Limits ε = epsilon δ = delta

We say a sequence $x_1, x_2, x_3, \cdots$ is in S iff every term $x_1, x_2, x_3, \cdots$ is an element of the set S.

A sequence $x_1, x_2, x_3, \cdots$ in $\mathbb{R}$ is bounded above iff the set $\{x_1, x_2, x_3, \cdots\}$ is bounded above in $\mathbb{R}$. Similarly, one can define sequence bounded below or bounded in $\mathbb{R}$.

Notations $\forall x, y \in \mathbb{R}$, let $d(x,y) = |x-y|$. This is the distance between x and y.

$\forall \varepsilon > 0, c \in \mathbb{R}$, $|x-c| < \varepsilon \iff -\varepsilon < x-c < \varepsilon$
$\iff c-\varepsilon < x < c+\varepsilon \iff x \in (c-\varepsilon, c+\varepsilon)$

ε-neighborhood of c.

Intuitive meaning of limit of sequences

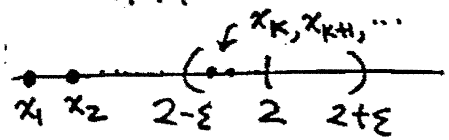
A sequence $x_1, x_2, x_3, \cdots$ in $\mathbb{R}$ has $c \in \mathbb{R}$ as limit means "x_n may be as close to c as desired when n is sufficiently large" or more loosely "as n tends to ∞, $d(x_n, c) = |x_n - c|$ goes to 0."

Warning The words "close", "large", "tends", "goes to" are not precise as they involve personal judgements.

Example Let $x_n = \dfrac{2n^2-1}{n^2+1}$. We may think its limit is 2.

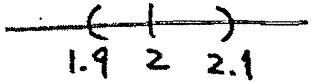
For every $\varepsilon > 0$, Consider the open interval $(2-\varepsilon, 2+\varepsilon)$. If the limit is 2, then we should be able to see $x_n, x_{n+1}, x_{n+2}, \cdots$ in $(2-\varepsilon, 2+\varepsilon)$ eventually!

$x_n = \frac{2n^2-1}{n^2+1}$ has 2 as limit should mean that for every interval $(2-\varepsilon, 2+\varepsilon)$, $\varepsilon>0$,



the sequence $x_1, x_2, x_3, \dots$ will <u>get into</u> the interval and <u>stay</u> in the interval when <u>n is sufficiently large</u>.

<u>Checking</u> For $\varepsilon = 0.1$, how large should n be so x_n will be in $(2-\varepsilon, 2+\varepsilon) = (1.9, 2.1)$?



Note $x_n \in (2-\varepsilon, 2+\varepsilon)$

$$\iff 2-\varepsilon < x_n < 2+\varepsilon$$
$$\iff -\varepsilon < x_n - 2 < \varepsilon$$
$$\iff |x_n - 2| < \varepsilon.$$

$$|x_n - 2| = \left|\frac{2n^2-1}{n^2+1} - 2\right| = \left|\frac{2n^2-1-2(n^2+1)}{n^2+1}\right| = \frac{3}{n^2+1} < \varepsilon$$

$$\iff \frac{3}{\varepsilon} < n^2+1 \iff \frac{3}{\varepsilon} - 1 < n^2 \iff n > \sqrt{\frac{3}{\varepsilon}-1}.$$

For $\varepsilon = 0.1$, $n > \sqrt{\frac{3}{0.1}-1} = \sqrt{29} \approx 5.\cdots$, so $n \ge 6 = K$ is enough.

For $\varepsilon = 0.01$; $n > \sqrt{\frac{3}{0.01}-1} = \sqrt{299} \approx 17.\cdots$, so $n \ge 18 = K$ is enough.

For $\varepsilon = 4$, $n^2 > \frac{3}{4} - 1 = -\frac{1}{4}$, so $n \ge 1 = K$ is enough.

So $\forall \varepsilon > 0$, let $K = \left\lceil \sqrt{\max\left(\frac{3}{\varepsilon}-1, 1\right)} \right\rceil$, then

$n \ge K \Rightarrow x_n \in (2-\varepsilon, 2+\varepsilon)$

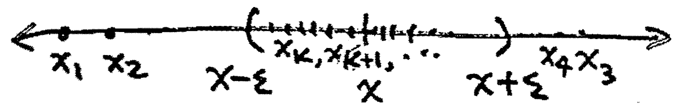
$\therefore x_K, x_{K+1}, x_{K+2}, \dots \in (2-\varepsilon, 2+\varepsilon)$.

Note for different ε, the value of K will be different. We say <u>K depends on ε</u> in such situation.

<u>**Definition**</u> A sequence $x_1, x_2, x_3, \dots$ <u>converges</u> to a number x (or has <u>limit</u> x) iff

$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ (depends on ε) such that

$$x_K, x_{K+1}, x_{K+2}, \dots \in (x-\varepsilon, x+\varepsilon)$$



equivalently,

$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ such that $n \ge K \Rightarrow |x_n - x| < \varepsilon$

This version is easier to do computations.

<u>Remarks</u> ① For simple sequences, given ε, it may be easy to compute K exactly. However, for complicated sequences, all we need to do is to show such K <u>exists</u>.

② If we are <u>given</u> $x_1, x_2, \dots$ has limit x, then we may set any positive ε and there is a K for us to use.

If we are asked to <u>prove</u> $x_1, x_2, \dots$ has limit x, then for every positive ε, we have to find a K or show such a K exists as in the definition.

Examples ① $v_n = c$. Prove $\{v_n\}$ converges to c.
↑ sequence $v_1, v_2, v_3, \cdots$

Solution. $\forall \varepsilon > 0$, $\quad c-\varepsilon \quad c \quad c+\varepsilon \quad$ let $K = 1$, then
$n \ge K \Rightarrow |v_n - c| = |c - c| = 0 < \varepsilon$.

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② $w_n = c - \frac{1}{n}$. Prove $\{w_n\}$ converges to c.

Solution $\forall \varepsilon > 0$, $\quad w_1 = c-1 \cdots c-\varepsilon \quad c \quad c+\varepsilon \qquad$ why is such n?

(**Scratch works**: $|w_n - c| = \frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$.)
By Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\varepsilon}$. Then $n \ge K \Rightarrow |w_n - c| = \frac{1}{n} \le \frac{1}{K} < \varepsilon$.

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③ $x_n = \frac{n}{(\cos n) - n}$. Prove $\{x_n\}$ converges to -1.

Solution $\forall \varepsilon > 0$, $\quad x_1 \cdots c-\varepsilon \quad c \quad c+\varepsilon$

(**Scratch works**: $|x_n - (-1)| = \left|\frac{\cos n}{(\cos n) - n}\right|$ $< \varepsilon$ ← difficult to solve for n
$\le \frac{1}{n-1} < \varepsilon \iff n > 1 + \frac{1}{\varepsilon}$.)

By Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > 1 + \frac{1}{\varepsilon}$. Then $n \ge K \Rightarrow |x_n - (-1)| = \left|\frac{\cos n}{(\cos n) - n}\right| \le \frac{1}{n-1}$
$\le \frac{1}{K-1} < \varepsilon$.

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④ $y_n = (-1)^n$. Prove $\{y_n\}$ does not converge.

-1: $y_1, y_3, \cdots \qquad 1$: $y_2, y_4, \cdots$

Solution Assume $\{y_n\}$ converges to y. For $\varepsilon = 0.1$, $\exists K$ such that $n \ge K \Rightarrow |(-1)^n - y| < \varepsilon$
n odd $\Rightarrow y \in (-1-\varepsilon, -1+\varepsilon) = (-1.1, -0.9)$
n even $\Rightarrow y \in (1-\varepsilon, 1+\varepsilon) = (0.9, 1.1)$. No y satisfies both.

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⑤ $z_n = n^{1/n}$. Prove $\{z_n\}$ converges to 1.

Scratch. (Hard to solve $|z_n - 1| = n^{1/n} - 1 < \varepsilon$.)
(Let $u_n = |z_n - 1| = n^{1/n} - 1 \ge 0$. Then $n^{1/n} = 1 + u_n$,
$n = (1+u_n)^n = 1 + n u_n + \frac{n(n-1)}{2} u_n^2 + \cdots \ge \frac{n(n-1)}{2} u_n^2$.
Solving for u_n, we get $u_n \le \sqrt{\frac{2}{n-1}} < \varepsilon \iff n > 1 + \frac{2}{\varepsilon^2}$.)

Solution $\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > 1 + \frac{2}{\varepsilon^2}$. Then
$n \ge K \Rightarrow |z_n - 1| \le \sqrt{\frac{2}{n-1}} \le \sqrt{\frac{2}{K-1}} < \varepsilon$.

Uniqueness of Limit If $\{x_n\}$ converges to x and y, then $x = y$. (So we may introduce the notation $\lim_{n\to\infty} x_n = x$.)

Given: ① $\{x_n\}$ converges to x ($\forall \varepsilon_1 > 0\ \exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |x_n - x| < \varepsilon_1$)
② $\{x_n\}$ converges to y ($\forall \varepsilon_2 > 0\ \exists K_2 \in \mathbb{N}$ such that $n \ge K_2 \Rightarrow |x_n - y| < \varepsilon_2$)

To Prove: $x = y$ ($\iff \forall \varepsilon > 0$, $|x - y| < \varepsilon$ Infinitesimal Principle)

Proof. $\forall \varepsilon > 0$, let $\varepsilon_1 = \frac{\varepsilon}{2} > 0$ and $\varepsilon_2 = \frac{\varepsilon}{2} > 0$. Then
$\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |x_n - x| < \varepsilon_1 = \frac{\varepsilon}{2}$
$\exists K_2 \in \mathbb{N}$ such that $n \ge K_2 \Rightarrow |x_n - y| < \varepsilon_2 = \frac{\varepsilon}{2}$.
Let $n = \max(K_1, K_2)$. Then $n \ge K_1$ and $n \ge K_2$.
So $|x - y| = |x - x_n + x_n - y| \le |x - x_n| + |x_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
$\therefore$ by infinitesimal principle, $x = y$. (by triangle inequality)

Example

⑥ Prove $\left\{\frac{2n}{n+5}+\frac{n^8}{n^8+n^5+1}\right\}$ converges to 3 by checking the definition of limit.

Scratch ① When n is large, $\frac{2n}{n+5}\approx 2$, $\frac{n^8}{n^8+n^5+1}\approx 1$.

② $\forall \varepsilon>0$,

$\left|\frac{2n}{n+5}+\frac{n^8}{n^8+n^5+1}-3\right|<\varepsilon$ is hard to solve for n

$=\left|\frac{2n}{n+5}-2+\frac{n^8}{n^8+n^5+1}-1\right|$

$\leq\left|\frac{2n}{n+5}-2\right|+\left|\frac{n^8}{n^8+n^5+1}-1\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$ is easier to solve

③ $\left|\frac{2n}{n+5}-2\right|=\left|\frac{-10}{n+5}\right|=\frac{10}{n+5}<\frac{\varepsilon}{2}$ if $n>\frac{20}{\varepsilon}-5$

$\left|\frac{n^8}{n^8+n^5+1}-1\right|=\left|\frac{-n^5-1}{n^8+n^5+1}\right|=\frac{n^5+1}{n^8+n^5+1}<\frac{2n^5}{n^8}=\frac{2}{n^3}<\frac{\varepsilon}{2}$ if $n>\sqrt[3]{4/\varepsilon}$.

Solution $\forall\varepsilon>0$, by Archimedean Principle, $\exists K_1\in\mathbb{N}$ such that $K_1>\frac{20}{\varepsilon}-5$ and $\exists K_2\in\mathbb{N}$ such that $K_2>\sqrt[3]{4/\varepsilon}$.

Let $K=\max(K_1,K_2)$. Then

$n\geq K \Rightarrow n\geq K_1>\frac{20}{\varepsilon}-5$ and $n\geq K_2>\sqrt[3]{4/\varepsilon}$

$\Rightarrow \left|\frac{2n}{n+5}+\frac{n^8}{n^8+n^5+1}-3\right|\leq\left|\frac{2n}{n+5}-2\right|+\left|\frac{n^8}{n^8+n^5+1}-1\right|$

$=\frac{10}{n+5}+\frac{n^5+1}{n^8+n^5+1}<\frac{10}{n+5}+\frac{2}{n^3}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.

Boundedness Theorem If $\{x_n\}$ converges, then the set $\{x_1,x_2,x_3,\dots\}$ is bounded (above and below).

Given: $\{x_n\}$ converges to some $x\in\mathbb{R}$ ($\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $n\geq K\Rightarrow |x_n-x|<\varepsilon$)

To Prove: $\{x_1,x_2,x_3,\dots\}$ is bounded ($\Leftrightarrow \exists M\in\mathbb{R}$ $\forall x_n$, $|x_n|\leq M$)

Proof. Let $x=\lim_{n\to\infty}x_n$. For $\varepsilon=1$, $\exists K\in\mathbb{N}$ such that

$n\geq K\Rightarrow |x_n-x|<1 \Rightarrow |x_n|=|x_n-x+x|$
$\leq |x_n-x|+|x|$
$<1+|x|$.

Let $M=\max(|x_1|,|x_2|,\dots,|x_{K-1}|,1+|x|)$. Then

$\forall n\in\mathbb{N}$, $n\geq K\Rightarrow |x_n|<1+|x|\leq M$
$n<K\Rightarrow x_n=x_1$ or x_2 or $\dots=x_{K-1}\Rightarrow |x_n|\leq M$.

Remarks The converse of the boundedness theorem is false. $x_n=(-1)^n$ $\{x_1,x_2,x_3,\dots\}=\{-1,1\}$ is bounded but $\{x_n\}$ does not converge by example ④.

Remarks The following are equivalent:

① $\{x_n\}$ converges to x ($\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $n\geq K\Rightarrow |x_n-x|<\varepsilon$)

② $\{x_n-x\}$ converges to 0 ($\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $n\geq K\Rightarrow |(x_n-x)-0|<\varepsilon$)

③ $\{|x_n-x|\}$ converges to 0 ($\forall\varepsilon>0\ \exists K\in\mathbb{N}$ such that $n\geq K\Rightarrow \big||x_n-x|-0\big|<\varepsilon$.)

Computation Formulas for Limits

Given: ① $\lim_{n\to\infty} x_n = x$ $(\forall \varepsilon_1 > 0, \exists K_1^{\in N}, n \ge K_1 \Rightarrow |x_n - x| < \varepsilon_1)$

② $\lim_{n\to\infty} y_n = y$ $(\forall \varepsilon_2 > 0, \exists K_2^{\in N}, n \ge K_2 \Rightarrow |y_n - y| < \varepsilon_2)$

To prove: $\lim_{n\to\infty} (x_n + y_n) = x + y$

$(\forall \varepsilon > 0, \exists K^{\in N}, n \ge K \Rightarrow |(x_n+y_n) - (x+y)| < \varepsilon)$

Idea: $|(x_n+y_n) - (x+y)| = |(x_n - x) + (y_n - y)|$

$\le \underbrace{|x_n - x|}_{<\varepsilon/2} + \underbrace{|y_n - y|}_{<\varepsilon/2}$

Proof. $\forall \varepsilon > 0$, let $\varepsilon_1 = \varepsilon/2 > 0$ and $\varepsilon_2 = \varepsilon/2 > 0$.

By ①, $\exists K_1$, $n \ge K_1 \Rightarrow |x_n - x| < \varepsilon_1 = \varepsilon/2$.

By ②, $\exists K_2$, $n \ge K_2 \Rightarrow |y_n - y| < \varepsilon_2 = \varepsilon/2$.

Max trick Let $K = \max(K_1, K_2) \in \mathbb{N}$.

$$n \ge K \Rightarrow \begin{cases} n \ge K_1 \\ \text{and} \\ n \ge K_2 \end{cases} \Rightarrow \begin{cases} |x_n - x| < \varepsilon/2 \\ \text{and} \\ |y_n - y| < \varepsilon/2 \end{cases}$$

$$\Rightarrow |(x_n + y_n) - (x+y)| = |(x_n - x) + (y_n - y)| \qquad (*)$$

$$\le |x_n - x| + |y_n - y| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Given: ① and ② above

To Prove: $\lim_{n\to\infty} (x_n - y_n) = x - y$

$(\forall \varepsilon > 0, \exists K^{\in N}, n \ge K \Rightarrow |(x_n - y_n) - (x-y)| < \varepsilon)$

Proof. Just change the 3 + signs in $(*)$ to − signs above.

Lemma

If (a) $\{a_n\}$ is bounded $(\exists M > 0$ such that $\forall n, |a_n| \le M)$

and (b) $\lim_{n\to\infty} b_n = 0$ $(\forall \varepsilon_1 > 0, \exists K_1^{\in \mathbb{N}}, n \ge K_1 \Rightarrow |b_n - 0| < \varepsilon_1)$

then $\lim_{n\to\infty} a_n b_n = 0$ $(\forall \varepsilon > 0, \exists K^{\in \mathbb{N}}, n \ge K \Rightarrow |a_n b_n - 0| < \varepsilon)$

Idea $|a_n b_n - 0| = |a_n b_n| \le M |b_n| = M |b_n - 0| < M\varepsilon_1 = \varepsilon$

should choose $\varepsilon_1 = \varepsilon/M$.

Proof. $\forall \varepsilon > 0$, let $\varepsilon_1 = \varepsilon/M$, where M is as in (a).

By (b), $\exists K = K_1 \in \mathbb{N}$, $n \ge K \Rightarrow |b_n - 0| < \varepsilon_1 = \varepsilon/M$

$\Rightarrow |a_n b_n - 0| = |a_n b_n| \le M |b_n| = M |b_n - 0| < M\varepsilon_1 = \varepsilon$.

Given: ① $\lim_{n\to\infty} x_n = x$ $(\Leftrightarrow \lim_{n\to\infty} (x_n - x) = 0)$, ② $\lim_{n\to\infty} y_n = y$ $(\Leftrightarrow \lim_{n\to\infty} (y_n - y) = 0)$

To Prove: $\lim_{n\to\infty} x_n y_n = xy$ $(\Leftrightarrow \lim_{n\to\infty} (x_n y_n - xy) = 0$ by earlier remark$)$

Proof.
$$\left.\begin{aligned} x_n y_n - xy &= x_n y_n - x_n y + x_n y - xy \\ &= x_n (y_n - y) + y (x_n - x) \end{aligned}\right\} (\Delta)$$

Since $\{x_n\}$ converges, $\{x_n\}$ is bounded by boundedness theorem.

Constant sequence $\{y\}$ is bounded

$\therefore \lim_{n\to\infty} (x_n y_n - xy) = \lim_{n\to\infty} (x_n(y_n - y) + y(x_n - x))$ by (Δ)

$= \lim_{n\to\infty} x_n(y_n - y) + \lim_{n\to\infty} y(x_n - x)$ by $\lim(a_n + b_n) = \lim a_n + \lim b_n$

$= 0 + 0$ by lemma

$= 0$

Given: ① $\lim_{n\to\infty} x_n = x$, ② $\forall n\in\mathbb{N}$, $y_n \neq 0$
and ③ $\lim_{n\to\infty} y_n = y \neq 0$ $(\forall \varepsilon_1>0, \exists K_1^{\in\mathbb{N}}, n\ge K_1 \Rightarrow |y_n - y| < \varepsilon_1)$

To Prove: $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{x}{y}$.

Proof (Step 1) We will show $\lim_{n\to\infty}\frac{1}{y_n} = \frac{1}{y}$ first.

$$(\forall \varepsilon>0, \exists K^{\in\mathbb{N}}, n\ge K \Rightarrow |\tfrac{1}{y_n} - \tfrac{1}{y}| < \varepsilon)$$

Since $\frac12|y|>0$, by ③, $\exists K_0^{\in\mathbb{N}}$, $n\ge K_0 \Rightarrow |y_n - y| < \frac12|y|$

$\Rightarrow |y| = |y_n - (y_n - y)| \le |y_n| + |y_n - y| < |y_n| + \frac12|y|$

$\Rightarrow \frac12|y| < |y_n|$

$\Rightarrow \frac{1}{|y_n|} < \frac{1}{\frac12|y|}$.

$\forall \varepsilon>0$, let $\varepsilon_1 = \frac12|y|^2\varepsilon > 0$. By ③, $\exists K_1^{\in\mathbb{N}}$ such that $n\ge K_1 \Rightarrow |y_n - y| < \varepsilon_1 = \frac12|y|^2\varepsilon$.

Max trick Let $K = \max(K_0, K_1)$. Then

$$n\ge K \Rightarrow \begin{cases} n\ge K_0 \\ \text{and} \\ n \ge K_1 \end{cases} \Rightarrow \begin{cases} \frac{1}{|y_n|} \le \frac{1}{\frac12|y|} \\ |y_n - y| < \frac12|y|^2\varepsilon \end{cases}$$

$$\Rightarrow \left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{y-y_n}{y_n y}\right| = \frac{|y_n - y|}{|y_n||y|} < \frac{\frac12|y|^2\varepsilon}{\frac12|y||y|} = \varepsilon.$$

(Step 2)

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty} x_n \frac{1}{y_n} = x\frac{1}{y} = \frac{x}{y}.$$

by $\lim(a_nb_n) = (\lim a_n)(\lim b_n)$ and step 1.

Recall $|a-b|<r \Leftrightarrow a\in(b-r, b+r)$.

Sandwich Theorem (or **Squeeze Limit Theorem**)

If ① $\forall n\in\mathbb{N}$, $x_n \le w_n \le y_n$
and ② $\lim_{n\to\infty} x_n = z = \lim_{n\to\infty} y_n$

$(\forall\varepsilon>0\ \exists K_1^{\in\mathbb{N}}, n\ge K_1 \Rightarrow |x_n - z| < \varepsilon$
$\forall \varepsilon>0\ \exists K_2^{\in\mathbb{N}}, n\ge K_2 \Rightarrow |y_n - z| < \varepsilon)$

then $\lim_{n\to\infty} w_n = z$ $(\forall\varepsilon>0\ \exists K^{\in\mathbb{N}}, n\ge K \Rightarrow |w_n - z| < \varepsilon)$

Proof. $\forall \varepsilon>0$, let $K = \max(K_1, K_2)$, where K_1, K_2 are as in ②. Then

$$n\ge K \Rightarrow \begin{cases} n\ge K_1 \\ \text{and} \\ n\ge K_2\end{cases} \Rightarrow \begin{cases} |x_n - z| < \varepsilon \\ \text{and} \\ |y_n - z| < \varepsilon \end{cases} \Leftrightarrow \begin{cases} x_n \in (z-\varepsilon, z+\varepsilon) \\ y_n\in(z-\varepsilon, z+\varepsilon)\end{cases}$$

$$\xRightarrow{\text{by ①}} w_n \in (z-\varepsilon, z+\varepsilon) \Leftrightarrow |w_n - z| < \varepsilon.$$

Example Let $w_n = \frac{[10^n\sqrt{2}]}{10^n} \in \mathbb{Q}$ for all $n\in\mathbb{N}$.

(Note $w_1 = 1.4$, $w_2 = 1.41$, $w_3 = 1.414$, $w_4 = 1.4142, \ldots$)

Then $10^n\sqrt{2} - 1 < [10^n\sqrt{2}] \le 10^n\sqrt{2}$ and so

$$\frac{10^n\sqrt2 - 1}{10^n} < \frac{[10^n\sqrt2]}{10^n} = w_n \le \sqrt2.$$

Since $\lim_{n\to\infty}\frac{10^n\sqrt2 - 1}{10^n} = \sqrt2$, by sandwich theorem, $\lim_{n\to\infty} w_n = \sqrt2$.

Remark We may replace $\sqrt2$ by any real number. Every real number is the limit of a sequence in $\mathbb{Q}$.

Limit Inequality

If ① $\forall n \in \mathbb{N}$, $a_n \geq 0$

and ② $\lim_{n\to\infty} a_n = a$ ($\forall \varepsilon > 0\ \exists K^{\in \mathbb{N}}, n \geq K \Rightarrow |a_n - a| < \varepsilon$)

then $a \geq 0$.

Proof. Assume $a < 0$. Then let $\varepsilon = |a| = -a > 0$.

By ②, $\exists K^{\in \mathbb{N}}$, $n \geq K \Rightarrow |a_n - a| < \varepsilon = -a$

$\Rightarrow a_n - a < -a$

$\Rightarrow a_n < 0$, contradiction to ①.

Remarks ① If $\forall n \in \mathbb{N}$, $x_n \leq y_n$ and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then ($a_n = y_n - x_n \geq 0$, $\lim_{n\to\infty} a_n = y - x \geq 0$) $x \leq y$.

② If $\forall n \in \mathbb{N}$, $a \leq x_n \leq b$ and $\lim_{n\to\infty} x_n = x$, then ($a = \lim_{n\to\infty} a \leq \lim_{n\to\infty} x_n = x$, $x = \lim_{n\to\infty} x_n \leq \lim_{n\to\infty} b = b$) $a \leq x \leq b$.

Equivalently, if $\forall n \in \mathbb{N}$, $x_n \in [a,b]$ and $\lim_{n\to\infty} x_n = x$, then $x \in [a,b]$. This is not true for open intervals !!!

$\frac{1}{n} > 0$, $\lim_{n\to\infty} \frac{1}{n} = 0$. $\frac{1}{n} \in (0, +\infty)$, $\lim_{n\to\infty} \frac{1}{n} = 0 \notin (0, +\infty)$

Supremum Limit Theorem

Let c be an upper bound of a nonempty set S. Then ($\exists w_n \in S$ such that $\lim_{n\to\infty} w_n = c$) $\Leftrightarrow$ $c = \sup S$.

Proof. ($\Rightarrow$) $\exists w_n \in S$, $\lim_{n\to\infty} w_n = c$. Since $w_n \in S$, $w_n \leq \sup S \leq c$ ← c is an upper bound

Taking limit, $c \leq \sup S \leq c$. $\therefore c = \sup S$.

($\Leftarrow$) $c = \sup S$. By supremum property, $\forall n \in \mathbb{N}$, $\exists w_n \in S$ such that $\underbrace{c - \frac{1}{n} = \sup S - \frac{1}{n}}_{\to c} < w_n \leq \underbrace{\sup S = c}_{\to c}$. Sandwich $\Rightarrow \lim_{n\to\infty} w_n = c$.

Infimum Limit Theorem

Let c be a lower bound of a nonempty set S. Then ($\exists w_n \in S$ such that $\lim_{n\to\infty} w_n = c$) $\Leftrightarrow$ $c = \inf S$.

Proof is similar to the proof of supremum limit theorem.

Examples ① Let $S = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$.

$$\left.\begin{array}{l} 0 \leq \frac{1}{n}\ \forall n \in \mathbb{N} \Rightarrow 0 \text{ is a lower bound of } S \\ w_n = \frac{1}{n} \in S,\ \lim_{n\to\infty} w_n = 0 \end{array}\right\} \Rightarrow \inf S = 0.$$

---

② Let $S = \{x\pi + \frac{1}{y} : x \in \mathbb{Q} \cap (0,1], y \in [1,2]\}$.

$\forall x \in \mathbb{Q} \cap (0,1], y \in [1,2]$, $x\pi + \frac{1}{y} > 0\pi + \frac{1}{2} = \frac{1}{2}$

$$\left.\begin{array}{l} \Rightarrow \frac{1}{2} \text{ is a lower bound of } S \\ w_n = \frac{1}{n}\pi + \frac{1}{2} \in S,\ \lim_{n\to\infty} w_n = \frac{1}{2} \end{array}\right\} \Rightarrow \inf S = \frac{1}{2}.$$

---

③ Let A and B be bounded sets in $\mathbb{R}$.

Let $A - 2B = \{a - 2b : a \in A, b \in B\}$.

Prove $\sup(A - 2B) = \sup A - 2\inf B$.

Solution. Since A bounded, $\sup A$ exists in $\mathbb{R}$. Since B bounded, $\inf B$ exists in $\mathbb{R}$. $\forall a \in A, b \in B$, we have $a \leq \sup A$, $\inf B \leq b$ $\Rightarrow a - 2b \leq \sup A - 2\inf B$.

$\therefore$ $c = \sup A - 2\inf B$ is an upper bound of $A - 2B$.

By supremum limit theorem, $\exists a_n \in A$, $\lim_{n\to\infty} a_n = \sup A$.

By infimum limit theorem, $\exists b_n \in B$, $\lim_{n\to\infty} b_n = \inf B$.

Then $a_n - 2b_n \in A - 2B$ and $\lim_{n\to\infty}(a_n - 2b_n) = \sup A - 2\inf B$.

$\therefore$ by supremum limit theorem, $\sup(A - 2B) = \sup A - 2\inf B$.

Question: How can we show a sequence has a limit if it is given by a recurrent relation? For example, $x_1 = 2$ and $x_{n+1} = \sqrt{3x_n - 2}$ for $n = 1, 2, 3, \cdots$

Definition Let $\{x_n\}$ be a sequence of numbers. $x_{n_1}, x_{n_2}, x_{n_3}, \cdots$ is a <u>subsequence</u> of $\{x_n\}$ iff $n_1 < n_2 < n_3 < \cdots$ and $n_j \in \mathbb{N}$ $\forall j = 1, 2, 3, \cdots$

Examples For sequence $x_1, x_2, x_3, \cdots$, if we set $n_j = j^2$, then we get $x_1, x_4, x_9, x_{16}, \cdots$, which is a subsequence because $1 < 4 < 9 < 16 < \cdots$.
If we set $n_j = 2j+1$, then we get $x_3, x_5, x_7, x_9, \cdots$ which is a subsequence because $3 < 5 < 7 < 9 < \cdots$.

Remarks $n_1 < n_2 < n_3 < \cdots$ and $n_j \in \mathbb{N}$ $\forall j = 1, 2, 3, \cdots$
$\Rightarrow n_j \geq j$ $\forall j = 1, 2, 3, \cdots$
We can prove this by mathematical induction. For $j = 1$, $n_1 \in \mathbb{N} \Rightarrow n_1 \geq 1$. If $n_j \geq j$, then $n_{j+1} > n_j \geq j$ and $n_{j+1} \in \mathbb{N} \Rightarrow n_{j+1} \geq j+1$.

Subsequence Theorem If $\lim_{n\to\infty} x_n = x$, then for every subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \cdots$, we have $\lim_{j\to\infty} x_{n_j} = x$.

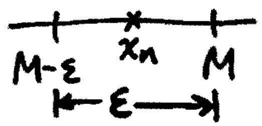
Proof. $\forall \varepsilon > 0$, since $\lim_{n\to\infty} x_n = x$, $\exists K \in \mathbb{N}$ such that $n \geq K \Rightarrow |x_n - x| < \varepsilon$. Then
$j \geq K \Rightarrow n_j \geq j \geq K \Rightarrow |x_{n_j} - x| < \varepsilon$.

Terminologies Let $\{x_n\}$ be a sequence of real numbers.
$\{x_n\}$ is <u>increasing</u> iff $x_1 \leq x_2 \leq x_3 \leq \cdots$.
$\{x_n\}$ is <u>decreasing</u> iff $x_1 \geq x_2 \geq x_3 \geq \cdots$.
$\{x_n\}$ is <u>strictly increasing</u> iff $x_1 < x_2 < x_3 < \cdots$.
$\{x_n\}$ is <u>strictly decreasing</u> iff $x_1 > x_2 > x_3 > \cdots$.
$\{x_n\}$ is <u>monotone</u> iff $\{x_n\}$ is increasing or decreasing.
$\{x_n\}$ is <u>strictly montone</u> iff $\{x_n\}$ is strictly increasing or strictly decreasing.

Monotone Sequence Theorem If $\{x_n\}$ is increasing and bounded above, then $\lim_{n\to\infty} x_n = \sup\{x_1, x_2, x_3, \cdots\}$. (Similarly, if $\{x_n\}$ is decreasing and bounded below, then $\lim_{n\to\infty} x_n = \inf\{x_1, x_2, x_3, \cdots\}$.)

Proof. Since $\{x_n\}$ is bounded above, $M = \sup\{x_1, x_2, x_3, \cdots\}$ exists. $\forall \varepsilon > 0$, by the supremum property, $\exists x_K$ such that $M - \varepsilon < x_K \leq M$. Then $x_K \in (M-\varepsilon, M]$. So

$$n \geq K \Rightarrow x_K \leq x_n \leq \sup\{x_1, x_2, x_3, \cdots\} = M$$
$$\Rightarrow x_n \in (M-\varepsilon, M]$$
$$\Rightarrow |x_n - M| < \varepsilon$$



The decreasing case is similar.

Examples ① Let $0<c<1$ and $x_n = c^{1/n}$ for $n=1,2,3,\cdots$. Then $x_n<1$ $\forall n$. Also,

$$c^{n+1}<c^n \Rightarrow x_n = c^{1/n} = (c^{n+1})^{\frac{1}{n(n+1)}} < (c^n)^{\frac{1}{n(n+1)}} = c^{\frac{1}{n+1}} = x_{n+1}$$

By the monotone sequence theorem, $\{x_n\}$ has a limit x. Now $x_{2n}^2 = (c^{\frac{1}{2n}})^2 = c^{\frac{1}{n}} = x_n$. Taking limits on both sides, by subsequence theorem, $x^2=x$. So $x=0$ or 1. Since $0<c=x_1\le x=\sup\{x_1,x_2,x_3,\cdots\}$, we get $x=1$. (Similarly, if $c\ge 1$, then $x_n=c^{1/n}$ will decrease to the limit 1.)

② Does $\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}$ represent a real number? Here, the question is if $x_1=\sqrt{2}$ and $x_{n+1}=\sqrt{2+x_n}$ for $n=1,2,3,\cdots$ converges to a real numbers.

Scratch Work $x_1=\sqrt{2} < x_2=\sqrt{2+\sqrt{2}} < x_3=\sqrt{2+\sqrt{2+\sqrt{2}}}$

We suspect $\{x_n\}$ is strictly increasing.

Assume $\lim_{n\to\infty} x_n = x$. Then $x^2 = \lim_{n\to\infty} x_{n+1}^2 = \lim_{n\to\infty}(2+x_n) = 2+x$

$\Rightarrow x^2-x-2 = (x-2)(x+1)=0 \Rightarrow x=2$ or ~~-1~~ reject.

Solution. We will show $x_n<x_{n+1}<2$ $\forall n\in\mathbb{N}$ by math induction. For $n=1$, $x_1=\sqrt{2}<x_2=\sqrt{2+\sqrt{2}}<2$.

If $x_n<x_{n+1}<2$, then $x_n+2<x_{n+1}+2<2+2=4$

$\Rightarrow x_{n+1}=\sqrt{x_n+2}<\sqrt{x_{n+1}+2}=x_{n+2}<\sqrt{2+2}=\sqrt{4}=2$.

So monotone sequence theorem $\Rightarrow \lim_{n\to\infty} x_n = x$ exists.

As in scratch work, since $\sqrt{2}=x_1\le x$, we get $x=2$.

Note If $\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} b_n = x$, then we expect $a_1,b_1,a_2,b_2,a_3,b_3,\cdots$ converges to x.

Intertwining Sequence Theorem

If ① $\lim_{m\to\infty} x_{2m-1} = x$ $(\forall\varepsilon>0\ \exists K_1^{\in\mathbb{N}}, m\ge K_1 \Rightarrow |x_{2m-1}-x|<\varepsilon)$

and ② $\lim_{m\to\infty} x_{2m} = x$ $(\forall\varepsilon>0\ \exists K_2^{\in\mathbb{N}}, m\ge K_2 \Rightarrow |x_{2m}-x|<\varepsilon)$

then $\lim_{n\to\infty} x_n = x$ $(\forall\varepsilon>0\ \exists K^{\in\mathbb{N}}, n\ge K \Rightarrow |x_n-x|<\varepsilon)$

Proof. $\forall\varepsilon>0$, let K_1, K_2 be as in conditions ①, ②. Let $K=\max(2K_1-1, 2K_2)$. Then

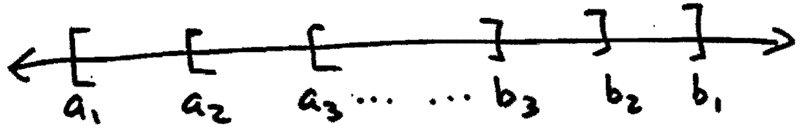
$$n\ge K \Rightarrow \begin{cases} n\ge 2K_1-1 \\ \text{and} \\ n\ge 2K_2 \end{cases} \Rightarrow \begin{cases} n \text{ odd} \Rightarrow n=2m-1 \text{ with } m\ge K_1 \\ n \text{ even} \Rightarrow n=2m \text{ with } m\ge K_2 \end{cases}$$

$$\Rightarrow \begin{cases} n \text{ odd} \Rightarrow |x_n-x| = |x_{2m-1}-x|<\varepsilon \\ n \text{ even} \Rightarrow |x_n-x| = |x_{2m}-x|<\varepsilon \end{cases}.$$

Nested Interval Theorem $a_n, b_n\in\mathbb{R}$

If $\forall n\in\mathbb{N}$, $I_n=[a_n,b_n]$ and $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$, then $\bigcap_{n=1}^{\infty} I_n = [a,b]$, where $a=\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n = b$.

If $\lim_{n\to\infty}(b_n-a_n)=0$, then $\bigcap_{n=1}^{\infty} I_n = \{x\}$ for some $x\in\mathbb{R}$.



Proof. $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ implies $\{a_n\}$ is increasing and bounded above by b_1 and $\{b_n\}$ is decreasing and bounded below by a_1. By monotone sequence theorem, $\lim_{n\to\infty} a_n = \sup\{a_1, a_2, a_3, \dots\} = a$, $\lim_{n\to\infty} b_n = \inf\{b_1, b_2, b_3, \dots\} = b$. Since $a_n \le b_n\ \forall n$, taking limit on both sides, get $a \le b$. Hence $x \in \bigcap_{n=1}^{\infty} [a_n, b_n] \iff \forall n \in \mathbb{N},\ a_n \le x \le b_n$

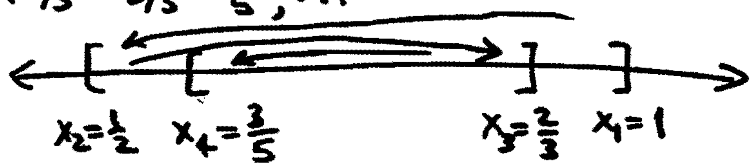
$\iff a \le x \le b \iff x \in [a,b]$.

If $0 = \lim_{n\to\infty}(b_n - a_n) = b - a$, then $a = b$, $\bigcap_{n=1}^{\infty} I_n = \{a\}$.

Example. Does $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$ represent a real number?

Here the question is if $x_1 = 1$ and $x_{n+1} = \frac{1}{1+x_n}$ for $n = 1, 2, 3, \cdots$ converges to a real number.

Scratch Work $x_1 = 1$, $x_2 = \frac{1}{1+1} = \frac{1}{2}$, $x_3 = \frac{1}{1+\frac{1}{2}} = \frac{1}{3/2} = \frac{2}{3}$, $x_4 = \frac{1}{1+2/3} = \frac{1}{5/3} = \frac{3}{5}, \dots$



Solution (Step 1: Form I_n and show $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$)

Define $I_n = [x_{2n}, x_{2n-1}]$ for $n = 1, 2, 3, \dots$.

We will show $I_n \supseteq I_{n+1}$ by math induction for $n \in \mathbb{N}$.

$I_n \supseteq I_{n+1} \iff x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}\ \forall n \in \mathbb{N}$.

For $n = 1$, $x_2 = \frac{1}{2} \le x_4 = \frac{3}{5} \le x_3 = \frac{2}{3} \le x_1 = 1$.

If $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$, then case n is true

$1 + x_{2n} \le 1 + x_{2n+2} \le 1 + x_{2n+1} \le 1 + x_{2n-1}$

$\Rightarrow \frac{1}{1+x_{2n}} = x_{2n+1} \ge \frac{1}{1+x_{2n+2}} = x_{2n+3} \ge \frac{1}{1+x_{2n+1}} = x_{2n+2} \ge \frac{1}{1+x_{2n-1}} = x_{2n}$

$\Rightarrow 1 + x_{2n+1} \ge 1 + x_{2n+3} \ge 1 + x_{2n+2} \ge 1 + x_{2n}$

$\Rightarrow \frac{1}{1+x_{2n+1}} = x_{2n+2} \le \frac{1}{1+x_{2n+3}} = x_{2n+4} \le \frac{1}{1+x_{2n+2}} = x_{2n+3} \le \frac{1}{1+x_{2n}} = x_{2n+1}$

case $n+1$ is true. So $\bigcap_{n=1}^{\infty} I_n = [a,b]$, where $\lim_{n\to\infty} x_{2n} = a$ and $\lim_{n\to\infty} x_{2n-1} = b$.

(Step 2: Show $\lim_{n\to\infty} |x_{2n} - x_{2n-1}| = 0$ and compute limit.)

$$|x_{m+1} - x_m| = \left|\frac{1}{1+x_m} - \frac{1}{1+x_{m-1}}\right| = \frac{|x_m - x_{m-1}|}{(1+x_m)(1+x_{m-1})}$$

$$\le \frac{|x_m - x_{m-1}|}{(1+\frac{1}{2})(1+\frac{1}{2})} = \frac{4}{9}|x_m - x_{m-1}|$$

So $|x_{2n} - x_{2n-1}| \le (\frac{4}{9})^1 |x_{2n-1} - x_{2n-2}| \le (\frac{4}{9})^2 |x_{2n-2} - x_{2n-3}|$

$\le \cdots \le (\frac{4}{9})^{2n-2} |x_2 - x_1| = (\frac{4}{9})^{2n-2} \frac{1}{2} \to 0$ as $n \to \infty$.

By Sandwich theorem, $\lim_{n\to\infty} |x_{2n} - x_{2n-1}| = 0$. By the nested interval theorem, $\lim_{n\to\infty} x_{2n} = x = \lim_{n\to\infty} x_{2n-1}$. By the intertwining sequence theorem, $\lim_{n\to\infty} x_n = x$. Taking limit of $x_{n+1} = \frac{1}{1+x_n}$, we get $x = \frac{1}{1+x} \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$. Since $\frac{-1-\sqrt{5}}{2} \notin I_1$, so $x = \frac{-1+\sqrt{5}}{2}$.

Alternative way to do step 2

From end of step 1, we have $\lim_{n\to\infty} x_{2n}=a$, $\lim_{n\to\infty} x_{2n-1}=b$.

Now $x_{2n}=\dfrac{1}{1+x_{2n-1}} \Rightarrow a=\dfrac{1}{1+b}$ by taking limit.

Also $x_{2n+1}=\dfrac{1}{1+x_{2n}} \Rightarrow b=\dfrac{1}{1+a}$ by taking limit.

$$\left.\begin{aligned} a&=\frac{1}{1+b} \\ b&=\frac{1}{1+a}\end{aligned}\right\} \Rightarrow a(1+b)=1=b(1+a) \Rightarrow a+ab=b+ab$$
$$\Rightarrow a=b$$

Then $\left.\begin{aligned} &\lim_{n\to\infty} x_n=a \text{ and } a=\frac{1}{1+a} \\ &x_n\in I_1 \Rightarrow a\in I_1\end{aligned}\right\} \Rightarrow a=\dfrac{-1+\sqrt{5}}{2}$

as $\dfrac{-1-\sqrt{5}}{2}\notin I_1$.

Question How can we prove a sequence converges without identifying the limit?

In the 19th century, Cauchy introduced the following

Definition $\{x_n\}$ is a Cauchy sequence iff $\forall \varepsilon > 0$ $\exists K \in \mathbb{N}$ such that $n, m \geq K \Rightarrow |x_n - x_m| < \varepsilon$.

Remarks This means the terms are as close as desired when the indices are sufficiently large.

Example Let $x_n = \frac{1}{n^2}$. Show $\{x_n\}$ is Cauchy.

Scratch Work Say $m \geq n$, $|x_n - x_m| = \frac{1}{n^2} - \frac{1}{m^2} < \frac{1}{n^2} < \varepsilon$

$n > \frac{1}{\sqrt{\varepsilon}}$ is enough.

Solution. $\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\sqrt{\varepsilon}}$. Then

$n, m \geq K \Rightarrow |x_n - x_m| = \left|\frac{1}{n^2} - \frac{1}{m^2}\right| < \frac{1}{K^2} < \varepsilon$.

Cauchy's Theorem $\{x_n\}$ converges $\Leftrightarrow$ $\{x_n\}$ is Cauchy.

Proof ($\Rightarrow$) Given: $\forall \varepsilon_0 > 0$ $\exists K_0 \in \mathbb{N}$, $n \geq K_0 \Rightarrow |x_n - x| < \varepsilon_0$

To prove: $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$, $m, n \geq K \Rightarrow |x_m - x_n| < \varepsilon$.

$$\boxed{\text{Idea: } |x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x - x_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon}$$

$\forall \varepsilon > 0$, let $\varepsilon_0 = \varepsilon/2$. We are given that $\exists K_0 \in \mathbb{N}$, $n \geq K_0 \Rightarrow |x_n - x| < \varepsilon/2$. Set $K = K_0$. Then

$m, n \geq K \Rightarrow |x_m - x_n| \leq |x_m - x| + |x - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

($\Leftarrow$) We are given $\{x_n\}$ is Cauchy. We are to prove $\{x_n\}$ converges to some x. We will do this in 3 steps.

Step 1 $\{x_n\}$ is Cauchy $\Rightarrow$ $\{x_1, x_2, x_3, \cdots\}$ is bounded

Step 2 (Bolzano-Weierstrass Theorem)

$\{x_1, x_2, x_3, \cdots\}$ is bounded $\Rightarrow$ $\exists$ subsequence $\{x_{n_k}\}$ which converges.

Step 3 $\left.\begin{array}{l}\{x_n\} \text{ is Cauchy} \\ \text{and a subsequence } \{x_{n_k}\} \\ \text{converges to } x\end{array}\right\} \Rightarrow \{x_n\}$ converges to x.

For step 1, we modify the proof of the boundedness theorem.

For $\varepsilon = 1$, $\exists K \in \mathbb{N}$, $n, m \geq K \Rightarrow |x_n - x_m| < \varepsilon = 1$.

So the case $m = K$ means $n \geq K \Rightarrow |x_n - x_K| < 1$

$\Rightarrow |x_n| = |x_n - x_K + x_K| \leq |x_n - x_K| + |x_K| < 1 + |x_K|$.

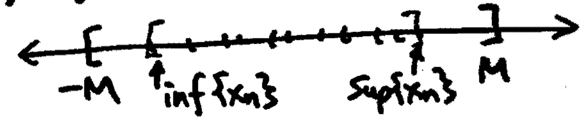
Let $M = \max\{|x_1|, |x_2|, \cdots, |x_{K-1}|, 1 + |x_K|\}$.

Then $\forall n \in \mathbb{N}$, $n = 1, 2, \cdots, K-1 \Rightarrow |x_n| \leq M$

$n \geq K \Rightarrow |x_n| < 1 + |x_K| \leq M$.

$\therefore \{x_1, x_2, x_3, \ldots\}$ is bounded.

For step 2



Let $a_1 = \inf\{x_n\}$, $b_1 = \sup\{x_n\}$ and $I_1 = [a_1, b_1]$.

Let m_1 be the midpoint of I_1.

If $[a_1, m_1]$ contains infinitely many terms of $\{x_n\}$, then let $a_2 = a_1$, $b_2 = m_1$ and $I_2 = [a_2, b_2]$.
Otherwise, $[m_1, b_1]$ contains infinitely many terms of $\{x_n\}$, then let $a_2 = m_1$, $b_2 = b_1$ and $I_2 = [a_2, b_2]$.
Let m_2 be the midpoint of I_2. Keep repeating, we get $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ and since I_{n+1} is either the left or the right half of I_n, we have $\lim_{n\to\infty} \underbrace{(b_n - a_n)}_{\text{length of } I_n} = \lim_{n\to\infty} \frac{b_1 - a_1}{2^{n-1}} = 0$. $\therefore \bigcap_{n=1}^{\infty} I_n = \{x\}$.

Take $n_1 = 1$, then $x_{n_1} = x_1 \in I_1$. Since I_2 has infinitely many terms, $\exists\, x_{n_2} \in I_2$ with $n_2 > n_1$.
Keep repeating, we get $x_{n_k} \in I_k$ and $n_1 < n_2 < n_3 < \cdots$
So $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$.
Now $x_{n_k}, x \in I_k \Rightarrow |x_{n_k} - x| \le \underbrace{b_k - a_k}_{\text{length of } I_k} \to 0$ as $k \to \infty$
$\therefore \{x_{n_k}\}$ converges to x.

For step 3 $\forall \varepsilon > 0$, $\{x_n\}$ Cauchy $\Rightarrow \exists K_1 \in \mathbb{N}$
$m, n \ge K_1 \Rightarrow |x_n - x_m| < \varepsilon/2$

$\{x_{n_j}\}$ converges to $x \Rightarrow \exists K_2 \in \mathbb{N}$, $j \ge K_2 \Rightarrow |x_{n_j} - x| < \varepsilon/2$.
Let $K = \max(K_1, K_2)$. Then

$$n \ge K \Rightarrow \begin{cases} n_K \ge K \ge K_1 \Rightarrow |x_n - x_{n_K}| < \varepsilon/2 \\ K \ge K_2 \quad\ \Rightarrow |x_{n_K} - x| < \varepsilon/2 \end{cases}$$
$$\Rightarrow |x_n - x| = |x_n - x_{n_K} + x_{n_K} - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Example Let $x_1 = \sin 1$ and $x_k = x_{k-1} + \frac{\sin k}{k^2}$.
Prove $\{x_n\}$ converges.

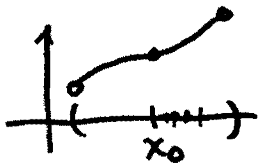
Solution (Scratch Work) Check Cauchy Condition

$$\begin{aligned} m > n \Rightarrow |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} \cdots - x_n| \\ &\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &= \left|\frac{\sin m}{m^2}\right| + \left|\frac{\sin(m-1)}{(m-1)^2}\right| + \cdots + \left|\frac{\sin(n+1)}{(n+1)^2}\right| \\ &\le \frac{1}{m^2} + \frac{1}{(m-1)^2} + \cdots + \frac{1}{(n+1)^2} \\ &< \frac{1}{m(m-1)} + \frac{1}{(m-1)(m-2)} + \cdots + \frac{1}{(n+1)n} \\ &= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots + \left(\frac{1}{m-2} - \frac{1}{m-1}\right) + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ &= \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \varepsilon \leftarrow n > \frac{1}{\varepsilon} \text{ is enough} \end{aligned}$$

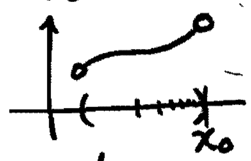
$\forall \varepsilon > 0$, by Archimedean principle, $\exists K \in \mathbb{N}$ such that $K > \frac{1}{\varepsilon}$. Then $n, m \ge K \Rightarrow |x_m - x_n| < \left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{K} < \varepsilon$.
$\therefore \{x_n\}$ is a Cauchy sequence. $\therefore \{x_n\}$ converges.

Limit of Functions

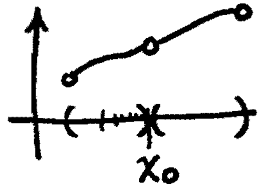
Question Let S be an interval (more generally a set). Let $f: S \to \mathbb{R}$ be a function. At which number x_0 can we consider $\lim_{x \to x_0} f(x)$?



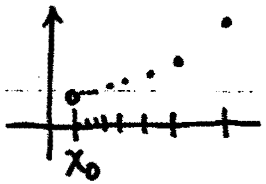
$x_0 \in S$



$x_0 \notin S$



$x_0 \notin S$



S is a sequence
x_0 is a limit

What do these cases have in common about x_0 and S ?

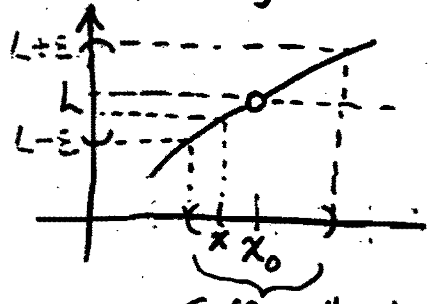
Definition x_0 is an accumulation point (or limit point or cluster point) of S iff $\exists\, w_n \in S$, $w_n \neq x_0$ such that $\lim_{n\to\infty} w_n = x_0$.

Remarks Accumulation points may or may not be in S.

Notation We write $w_n \to x_0$ in $S \setminus \{x_0\}$ to mean $w_n \in S$, $w_n \neq x_0$ and $\lim_{n\to\infty} w_n = x_0$.

Convention When discussing $\lim_{x\to x_0} f(x)$, we will assume x_0 is an accumulation point of the domain of f.

Let $f: S \to \mathbb{R}$, $\lim_{x\to x_0} f(x) = L$ roughly means for any desired distance $\varepsilon > 0$, when $x \in S, x \neq x_0$ is sufficiently close to x_0, we can obtain



$$\underbrace{d(f(x), L)}_{\text{distance between } f(x) \text{ and } L} < \varepsilon .$$

Precise Definition $\boxed{\lim_{x\to x_0} f(x) = L}$ iff $\forall \varepsilon > 0\ \exists \delta > 0$

$$\forall x \in S, \quad \left.\begin{array}{l} x \in (x_0 - \delta, x_0 + \delta) \\ x \neq x_0 \end{array}\right\} \Rightarrow |f(x) - L| < \varepsilon .$$

Equivalently, $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in S$,

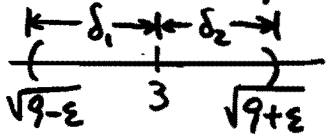
$$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon .$$

This is easier to do computations.

Examples ① Let $f(x) = \dfrac{x^3 - 3x^2}{x-3} = x^2 \dfrac{(x-3)}{x-3}$. Check $\lim_{x\to 3} f(x) = 9$

Scratch Work $x \neq 3 \Rightarrow f(x) = x^2$ $\quad |f(x) - 9| = |x^2 - 9| < \varepsilon$

$\Leftrightarrow x^2 \in (9-\varepsilon, 9+\varepsilon) \Leftrightarrow x \in (\sqrt{9-\varepsilon}, \sqrt{9+\varepsilon})$ for $\varepsilon < 9$



Let $\delta = \min(\delta_1, \delta_2) = \min(3 - \sqrt{9-\varepsilon}, \sqrt{9+\varepsilon} - 3)$

Then $0 < |x - 3| < \delta \Rightarrow x \in (\sqrt{9-\varepsilon}, \sqrt{9+\varepsilon})$

$\Rightarrow |f(x) - 9| = |x^2 - 9| < \varepsilon$.

② Let $g:[0,\infty)\to\mathbb{R}$ be defined by $g(x)=\sqrt{x}$.

Check: $\lim_{x\to 0} g(x)=0$ and $\lim_{x\to 4} g(x)=2$.

Solution. (Scratch Work) $|g(x)-0|=\sqrt{x}<\varepsilon$ $\underset{|x-0|}{\overset{\shortparallel}{x}}<\varepsilon^2$ is enough

$\forall \varepsilon>0$, let $\delta=\varepsilon^2$, then $\forall x\in[0,\infty)$, $0<|x-0|=x<\delta=\varepsilon^2$

$\Rightarrow |g(x)-0|=\sqrt{x}<\varepsilon$.

(Scratch Work) $|g(x)-2|=|\sqrt{x}-2|=\dfrac{|x-4|}{\sqrt{x}+2}\leq\dfrac{|x-4|}{2}<\varepsilon$

$|x-4|<2\varepsilon$ is enough.

$\forall\varepsilon>0$, let $\delta=2\varepsilon$, then $\forall x\in[0,\infty)$,

$0<|x-4|<\delta=2\varepsilon \Rightarrow |g(x)-2|\leq\dfrac{|x-4|}{2}<\varepsilon$.

③ Let $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be defined by $f(x)=\dfrac{1}{5x}$

Check: $\lim_{x\to 2} f(x)=\dfrac{1}{10}$.

Solution (Scratch Work) x is close to 2

$\left|f(x)-\frac{1}{10}\right|=\left|\frac{1}{5x}-\frac{1}{10}\right|=\dfrac{|x-2|}{10x}\leq\dfrac{|x-2|}{10}<\varepsilon$

min trick

$\left.\begin{array}{l}|x-2|<10\varepsilon\\ \text{and } x\geq 1\end{array}\right\}$ are enough. if $x\geq 1$ $\leftarrow \delta<1$

1 2

$\forall\varepsilon>0$, let $\delta=\min(1,10\varepsilon)$, then $\forall x\in\mathbb{R}\setminus\{0\}$,

$0<|x-2|<\delta \Rightarrow \left\{\begin{array}{l}|x-2|<1\\ \text{and}\\ |x-2|<10\varepsilon\end{array}\right. \Rightarrow \left\{\begin{array}{l}x\in(1,3) \leftarrow x\geq 1\\ \text{and}\\ |x-2|<10\varepsilon\end{array}\right.$

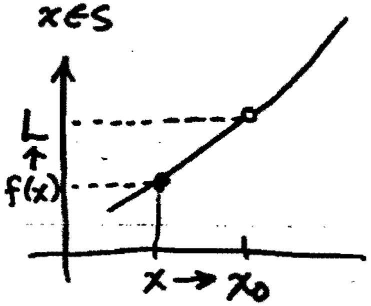
$\Rightarrow \left|f(x)-\frac{1}{10}\right|=\dfrac{|x-2|}{10x}\leq\dfrac{|x-2|}{10}<\varepsilon$.

Recall "$x_n \to x_0$ in $S\setminus\{x_0\}$" means $\forall n \in \mathbb{N}$, $x_n \in S$, $x_n \neq x_0$, $\lim_{n\to\infty} x_n = x_0$

Sequential Limit Theorem (S.L.T.)

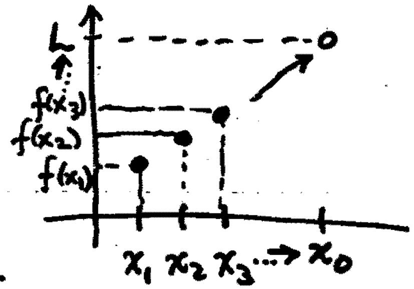
Let $f: S \to \mathbb{R}$ be a function and x_0 be an accumulation point of S. Then

$$\lim_{\substack{x\to x_0\\ x\in S}} f(x) = L \iff \forall x_n \to x_0 \text{ in } S\setminus\{x_0\},\ \lim_{n\to\infty} f(x_n) = L.$$



$\Rightarrow$

$\Leftarrow$



$\forall x_n \to x_0$ in $S\setminus\{x_0\}$

Proof ($\Rightarrow$)

Given: $\lim_{\substack{x\to x_0\\ x\in S}} f(x) = L$ $\left(\forall \varepsilon>0\ \exists \delta>0, \forall x\in S\ \ 0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon\right)$

$x_n \to x_0$ in $S\setminus\{x_0\}$ $\left(\forall \delta>0\ \exists K\in\mathbb{N}\ \ n\geq K \Rightarrow |x_n-x_0|<\delta\right)$

So $\forall \varepsilon>0\ \exists K\in\mathbb{N}$ $\ n\geq K \Rightarrow \underbrace{0<}_{\text{as } x_n\neq x_0}|x_n-x_0|<\delta \Rightarrow |f(x_n)-L|<\varepsilon$

$\therefore \lim_{n\to\infty} f(x_n) = L$

($\Leftarrow$) Assume $\lim_{x\to x_0} f(x) \neq L$.

$\sim(\forall\varepsilon>0\ \exists\delta>0, \forall x\in S, 0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon)$

$= \exists \varepsilon>0\ \forall\delta>0\ \exists x\in S\ \ 0<|x-x_0|<\delta$ and $|f(x)-L|\geq\varepsilon$.

for $\delta=1$ $\exists x_1\in S$ $0<|x_1-x_0|<1$ and $|f(x_1)-L|\geq\varepsilon$

for $\delta=\frac12$ $\exists x_2\in S$ $0<|x_2-x_0|<\frac12$ and $|f(x_2)-L|\geq\varepsilon$

$\vdots$

for $\delta=\frac1n$ $\exists x_n\in S$ $0<|x_n-x_0|<\frac1n$ and $|f(x_n)-L|\geq\varepsilon$

$\vdots$

$\therefore x_n\in S$, $0<|x_n-x_0|<\frac1n \Rightarrow x_n\neq x_0$ and $\lim_{n\to\infty} x_n = x_0$

$\therefore x_n \to x_0$ in $S\setminus\{x_0\}$. Then $\lim_{n\to\infty} f(x_n) = L$.

$|f(x_n)-L|\geq\varepsilon \Rightarrow 0=|L-L|=\lim_{n\to\infty}|f(x_n)-L|\geq\varepsilon$

Contradicting $\varepsilon>0$.

Application ① $\lim_{x\to 0}(1+x)^{\frac1x}=e \underset{x_n=\frac1n\to 0}{\overset{SLT}{\Longrightarrow}} \lim_{n\to\infty}\left(1+\frac1n\right)^n = e$

$\Leftarrow$ need not just $x_n=\frac1n$, need all $x_n\to 0$, $x_n\neq 0$ to have $\lim_{n\to\infty}(1+x_n)^{1/x_n}=e$

② If $\underbrace{\lim_{\substack{x\to x_0\\ x\in S}} f(x)=L_1}_{(*)}$, $\underbrace{\lim_{\substack{x\to x_0\\ x\in S}} g(x)=L_2}_{(**)}$, then prove $\lim_{\substack{x\to x_0\\ x\in S}}(f(x)+g(x))=L_1+L_2$

Solution 1 $\forall x_n\to x_0$ in $S\setminus\{x_0\}$,

by S.L.T., $(*) \Rightarrow \lim_{n\to\infty} f(x_n)=L_1$

$(**) \Rightarrow \lim_{n\to\infty} g(x_n)=L_2$.

By computation formula, $\lim_{n\to\infty}(f(x_n)+g(x_n))=L_1+L_2$

By S.L.T., $\lim_{\substack{x\to x_0\\ x\in S}}(f(x)+g(x))=L_1+L_2$.

Solution 2

$\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x)=L_1$ $\left(\forall \varepsilon_1>0\ \exists \delta_1>0\ \forall x\in S\ \ 0<|x-x_0|<\delta_1 \Rightarrow |f(x)-L_1|<\varepsilon_1\right)$

$\lim\limits_{\substack{x\to x_0\\ x\in S}} g(x)=L_2$ $\left(\forall \varepsilon_2>0\ \exists \delta_2>0\ \forall x\in S\ \ 0<|x-x_0|<\delta_2 \Rightarrow |g(x)-L_2|<\varepsilon_2\right)$

$\forall \varepsilon>0$, let $\varepsilon_1=\frac{\varepsilon}{2}$ and $\varepsilon_2=\frac{\varepsilon}{2}$. From above, get $\delta_1,\delta_2>0$

Set $\delta=\min(\delta_1,\delta_2)$. Then $\forall x\in S$,

$0<|x-x_0|<\delta \Rightarrow$ $0<|x-x_0|<\delta_1 \Rightarrow |f(x)-L_1|<\frac{\varepsilon}{2}$
$\qquad\qquad\qquad\quad$ $0<|x-x_0|<\delta_2 \Rightarrow |g(x)-L_2|<\frac{\varepsilon}{2}$

$$\Rightarrow |(f(x)+g(x))-(L_1+L_2)| = |f(x)-L_1+g(x)-L_2|$$
$$\le |f(x)-L_1|+|g(x)-L_2| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

$\therefore \lim\limits_{\substack{x\to x_0\\ x\in S}} (f(x)+g(x)) = L_1+L_2$

Similarly

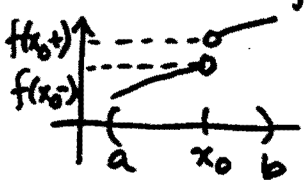
If $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x)=L_1$, $\lim\limits_{\substack{x\to x_0\\ x\in S}} g(x)=L_2$, then
$\lim (f(x)-g(x))=L_1-L_2$
$\lim f(x)g(x)=L_1L_2$
$\lim f(x)/g(x)=L_1/L_2$ (provided $g(x)\ne 0$, $L_2\ne 0$)

If $f(x)\le g(x)\le h(x)$ for all $x\in S$, $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x)=L=\lim\limits_{\substack{x\to x_0\\ x\in S}} h(x)$, then $\lim\limits_{\substack{x\to x_0\\ x\in S}} g(x)=L$.

If $f(x)\ge 0$ for all $x\in S$ and $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x)=L$, then $L\ge 0$.

One-sided Limits

Definitions For $f:(a,b)\to\mathbb{R}$ and $x_0\in(a,b)$,



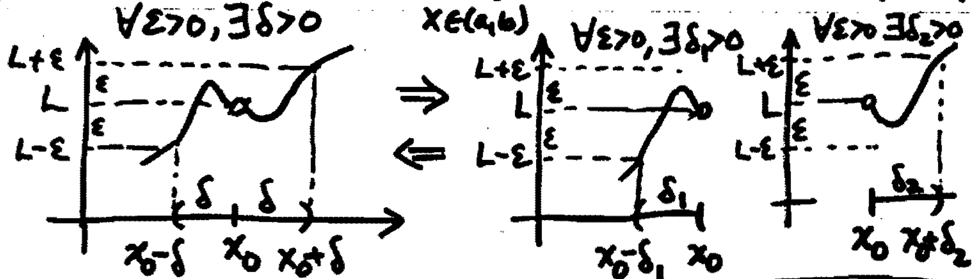
left hand limit of f at x_0

$$f(x_0-)=\lim_{x\to x_0^-} f(x)=\lim_{\substack{x\to x_0\\ x\in(a,x_0)}} f(x)$$

right hand limit of f at x_0

$$f(x_0+)=\lim_{x\to x_0^+} f(x)=\lim_{\substack{x\to x_0\\ x\in(x_0,b)}} f(x).$$

Theorem For $x_0\in(a,b)$, $\lim\limits_{\substack{x\to x_0\\ x\in(a,b)}} f(x)=L \Leftrightarrow f(x_0-)=L=f(x_0+)$



Proof

$\boxed{\lim\limits_{\substack{x\to x_0\\ x\in(a,b)}} f(x)=L \text{ "}\forall\varepsilon>0\ \exists\delta>0\ \forall x\in(a,b),\ 0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon\text{"}}$

$\Longrightarrow$ let $\delta_1=\delta$, $\delta_2=\delta$

$\boxed{f(x_0-)=L = \lim\limits_{\substack{x\to x_0\\ x\in(a,x_0)}} f(x) \text{ "}\forall\varepsilon>0\ \exists\delta_1>0\ \forall x\in(a,x_0)\ 0<|x-x_0|<\delta_1 \Rightarrow |f(x)-L|<\varepsilon\text{"}}$

$\boxed{f(x_0+)=L = \lim\limits_{\substack{x\to x_0\\ x\in(x_0,b)}} f(x) \text{ "}\forall\varepsilon>0\ \exists\delta_2>0\ \forall x\in(x_0,b)\ 0<|x-x_0|<\delta_2 \Rightarrow |f(x)-L|<\varepsilon}$

$\Longleftarrow$ let $\delta=\min(\delta_1,\delta_2)$

Definitions Let $f: S \to \mathbb{R}$ be a function.

① f is <u>increasing</u> on S iff $\forall x, y \in S$, $x<y \Rightarrow f(x) \le f(y)$

② f is <u>decreasing</u> on S iff $\forall x, y \in S$, $x<y \Rightarrow f(x) \ge f(y)$

③ f is <u>strictly increasing</u> on S iff $\forall x, y \in S$, $x<y \Rightarrow f(x) < f(y)$

④ f is <u>strictly decreasing</u> on S iff $\forall x, y \in S$, $x<y \Rightarrow f(x) > f(y)$.

⑤ f is <u>monotone</u> on S iff f is increasing or decreasing on S.

⑥ f is <u>strictly monotone</u> on S iff f is strictly increasing or strictly decreasing on S.

⑦ f is <u>bounded above</u> on S iff $\{f(x): x \in S\}$ is bounded above.

⑧ f is <u>bounded below</u> on S iff $\{f(x): x \in S\}$ is bounded below.

⑨ f is <u>bounded</u> on S iff f is bounded above and below.

Monotone Function Theorem

Ⅰ If f is increasing on (a,b), then $\forall x_0 \in (a,b)$,

$$\left.\begin{array}{l} f(x_0-) = \sup\{f(x): a<x<x_0\} \\ \text{and } f(x_0+) = \inf\{f(x): x_0<x<b\} \end{array}\right\} \Rightarrow f(x_0-) \le f(x_0) \le f(x_0+)$$

If f is bounded below, then $f(a+) = \inf\{f(x): a<x<b\}$.

If f is bounded above, then $f(b-) = \sup\{f(x): a<x<b\}$.

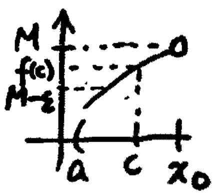
Ⅱ f has countably many discontinuous points on (a,b)

$J = \{x_0 : x_0 \in (a,b), f(x_0-) \ne f(x_0+)\}$ is countable.

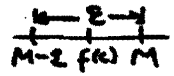
Remarks Similarly, the theorem is true for decreasing functions and all other kinds of intervals.

Proof. Ⅰ If $a<x<x_0<b$, then $f(x) \le f(x_0)$ as f is increasing. So $\{f(x): a<x<x_0\}$ is bounded above by $f(x_0)$. Hence, $M = \sup\{f(x): a<x<x_0\}$ exists by completeness axiom.

To show $f(x_0-) = M$, $\forall \varepsilon > 0$, by the supremum property, $\exists c \in (a, x_0)$ such that $M - \varepsilon < f(c) \le M$. Let $\delta = x_0 - c$.



Then $\forall x \in (a, x_0)$,



$$0<|x-x_0|<\delta \Rightarrow x \in (x_0-\delta, x_0) = (c, x_0)$$
$$\Rightarrow c < x < x_0 \Rightarrow f(c) \le f(x) \le M$$
$$\Rightarrow |f(x) - M| = M - f(x) \le M - f(c) < \varepsilon$$

$$\therefore f(x_0-) = \lim_{\substack{x \to x_0 \\ x \in (a, x_0)}} f(x) = M = \sup\{f(x): a<x<x_0\} \le f(x_0).$$

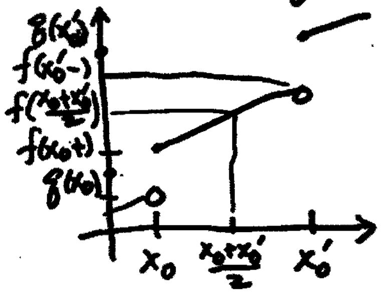
The other parts of Ⅰ are similarly proved.

Ⅱ f is discontinuous at $x_0 \in (a,b) \Leftrightarrow f(x_0-) < f(x_0+)$.

$\Rightarrow \exists g(x_0) \in \mathbb{Q}$ such that $f(x_0-) < g(x_0) < f(x_0+)$ by density of $\mathbb{Q}$.

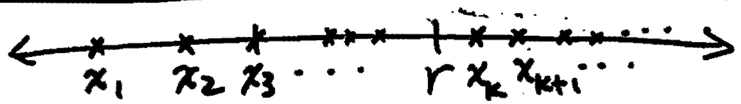
The function $g: J = \{x_0 : f(x_0-) < f(x_0+)\} \to \mathbb{Q}$ is injective because $\forall x_0, x_0' \in J$,

$$x_0 < x_0' \Rightarrow g(x_0) < f(x_0+) \le f\left(\frac{x_0+x_0'}{2}\right) \le f(x_0'-) < g(x_0')$$

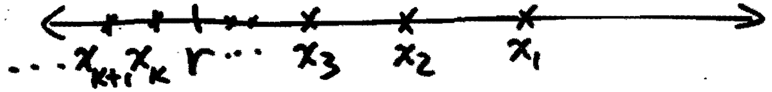


By injection theorem, J is countable.

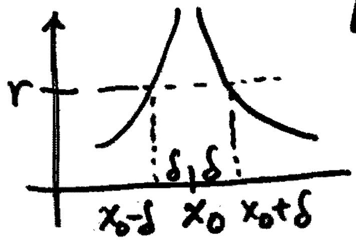
Infinite Limits



Definitions $\{x_n\}$ diverges to $+\infty$ (or $\lim\limits_{n\to\infty} x_n = +\infty$) iff $\forall r\in\mathbb{R}, \exists K\in\mathbb{N}$ such that $n\geq K \Rightarrow x_n > r$.



$\{x_n\}$ diverges to $-\infty$ (or $\lim\limits_{n\to\infty} x_n = -\infty$) iff $\lim\limits_{n\to\infty} -x_n = +\infty$, $\forall r\in\mathbb{R}, \exists K\in\mathbb{N}$ such that $n\geq K \Rightarrow x_n < r$.



Let $f: S\to\mathbb{R}$ and x_0 be an accumulation point of S.

f diverges to $+\infty$ as x tends to x_0 (or $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x) = +\infty$) iff

$\forall r\in\mathbb{R}\ \exists \delta>0$ such that $\forall x\in S$

$$\underbrace{x\neq x_0 \text{ and } x\in(x_0-\delta, x_0+\delta)}_{0<|x-x_0|<\delta} \Rightarrow f(x) > r.$$

f diverges to $-\infty$ as x tends to x_0 (or $\lim\limits_{\substack{x\to x_0\\ x\in S}} f(x) = -\infty$) iff $\lim\limits_{\substack{x\to x_0\\ x\in S}} -f(x) = +\infty$,

$\forall r\in\mathbb{R}\ \exists\delta>0$ such that $\forall x\in S$

$0<|x-x_0|<\delta \Rightarrow f(x)<r.$

Limit at Infinity

Recall $\lim\limits_{n\to\infty} a_n = L$ iff $\forall \varepsilon>0\ \exists K\in\mathbb{N}\ n\geq K \Rightarrow |x_n - L|<\varepsilon$.

Let $f: S\to\mathbb{R}$ be a function and $+\infty, -\infty$ are accumulation points of S (that is, $\exists$ sequences in S with $+\infty, -\infty$ as limits). $L\in\mathbb{R}$.

Definitions $\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(x) = L$ iff $\forall\varepsilon>0\ \exists K\in\mathbb{R}$ $x\geq K \Rightarrow |f(x)-L|<\varepsilon$.

$\lim\limits_{\substack{x\to-\infty\\ x\in S}} f(x) = L$ iff $\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(-x) = L$ iff $\forall\varepsilon>0\ \exists K\in\mathbb{R}$ $x\leq K \Rightarrow |f(x)-L|<\varepsilon$

Recall $\lim\limits_{n\to\infty} a_n = +\infty$ iff $\forall r\in\mathbb{R}, \exists K\in\mathbb{N}$ $n\geq K \Rightarrow a_n > r$.

Definitions $\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(x) = +\infty$ iff $\forall r\in\mathbb{R}, \exists K\in\mathbb{R}$ $x\geq K\Rightarrow f(x)>r$.

$\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(x) = -\infty$ iff $\lim\limits_{\substack{x\to+\infty\\ x\in S}} -f(x) = +\infty$ iff $\forall r\in\mathbb{R}, \exists K\in\mathbb{R}$ $x\geq K\Rightarrow f(x)<r$.

$\lim\limits_{\substack{x\to-\infty\\ x\in S}} f(x) = +\infty$ iff $\lim\limits_{\substack{x\to+\infty\\ x\in S}} f(-x) = +\infty$ iff $\forall r\in\mathbb{R}, \exists K\in\mathbb{R}$ $x\leq K\Rightarrow f(x)>r$.

$\lim\limits_{\substack{x\to-\infty\\ x\in S}} f(x) = -\infty$ iff $\lim\limits_{\substack{x\to+\infty\\ x\in S}} -f(-x) = +\infty$ iff $\forall r\in\mathbb{R}, \exists K\in\mathbb{R}$ $x\leq K\Rightarrow f(x)<r$.

Chapter 7 Continuity

Definition A function $f: S \to \mathbb{R}$ is continuous at $x_0 \in S$ iff $\lim\limits_{\substack{x \to x_0 \\ x \in S}} f(x) = f(x_0)$ (more precisely, $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall x \in S$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$)

For $E \subseteq S$, we say f is continuous on E iff f is continuous at every element of E. Also, we say f is continuous iff f is continuous on the domain S.

Sequential Continuity Theorem (S.C.T.)

$f: S \to \mathbb{R}$ is continuous at $x_0 \in S$ $\iff$ $\forall x_n \to x_0$ in S, $\lim\limits_{n\to\infty} f(x_n) = f(x_0) = f(\lim\limits_{n\to\infty} x_n)$

Drop $x_n \neq x_0$ requirement.

Proof. Just modify the proof of the sequential limit theorem by replacing ① L by $f(x_0)$

② $0 < |x - x_0| < \delta$ by $|x - x_0| < \delta$

③ $x_n \to x_0$ in $S \setminus \{x_0\}$ by $x_n \to x_0$ in S (delete $x_n \neq x_0$ requirement)

Examples ① Since $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$, so $f: \mathbb{R} \to \mathbb{R}$ defined by $f(\theta) = \begin{cases} \frac{\sin\theta}{\theta} & \text{if } \theta \neq 0 \\ 1 & \text{if } \theta = 0 \end{cases}$ is continuous at $x_0 = 0$.

Also $\lim\limits_{n\to\infty} n \sin\frac{1}{n} = \lim\limits_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim\limits_{n\to\infty} f(\frac{1}{n}) \overset{\text{by S.C.T.}}{=} f(0) = 1$

② $\exists f: \mathbb{R} \to \mathbb{R}$ discontinuous (not continuous) at every $x \in \mathbb{R}$. Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$

$\forall x_0 \in \mathbb{R}$, $n \in \mathbb{N}$, by density of $\mathbb{Q}$ and density of $\mathbb{R} \setminus \mathbb{Q}$, $\exists r_n \in \mathbb{Q}$, $s_n \in \mathbb{R} \setminus \mathbb{Q}$, both $r_n, s_n \in (x_0, x_0 + \frac{1}{n})$

So $\lim\limits_{n\to\infty} r_n = x_0 = \lim\limits_{n\to\infty} s_n$ by Sandwich theorem.

Then $\lim\limits_{n\to\infty} f(r_n) = \lim\limits_{n\to\infty} 1 = 1$, $\lim\limits_{n\to\infty} f(s_n) = \lim\limits_{n\to\infty} 0 = 0$, so $\lim\limits_{x \to x_0} f(x)$ cannot exist by S.C.T. $\therefore f$ is discontinuous at x_0.

Theorem If $f, g: S \to \mathbb{R}$ are continuous at $x_0 \in S$, then $f \pm g$, fg, f/g (provided $g(x_0) \neq 0$) are continuous at x_0.

Proof f, g continuous at $x_0 \in S$ $\iff$ $\lim\limits_{x\to x_0} f(x) = f(x_0)$, $\lim\limits_{x \to x_0} g(x) = g(x_0)$

by definition of continuity at x_0

$\Longrightarrow \lim\limits_{x\to x_0} (f \pm g)(x) = (f \pm g)(x_0)$

see applications of S.L.T.

$\lim\limits_{x \to x_0} (fg)(x) = (fg)(x_0)$

$\lim\limits_{x\to x_0} (\frac{f}{g})(x) = (\frac{f}{g})(x_0)$

$\iff f \pm g$, fg, $\frac{f}{g}$ is continuous at x_0

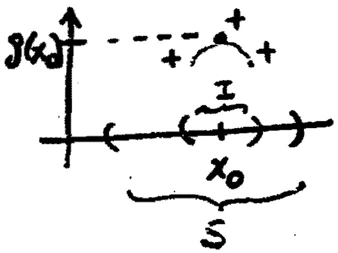
<u>Theorem</u> If $f: S \to \mathbb{R}$ is continuous at x_0, $f(S) \subseteq S'$ and $g: S' \to \mathbb{R}$ is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Proof. By S.C.T., we need to show $\forall x_n \to x_0$ in S, $\lim_{n\to\infty} (g\circ f)(x_n) = (g \circ f)(x_0)$. By S.C.T., since f is continuous at x_0, $\lim_{n\to\infty} f(x_n) = f(x_0)$. So $f(x_n) \to f(x_0)$ in S'. Since g is continuous at $f(x_0)$, by S.C.T.,

$$\lim_{n\to\infty}(g\circ f)(x_n) = \lim_{n\to\infty} g(f(x_n)) = g(f(x_0)) = (g\circ f)(x_0).$$

Below, S will denote an interval of positive length.

<u>Sign Preserving Property</u>



If $g: S \to \mathbb{R}$ is continuous and $g(x_0) > 0$, then $\exists$ an interval $I = (x_0 - \delta, x_0 + \delta)$ with $\delta > 0$ such that

$g(x) > 0$ for all $x \in S \cap I$.

(Similarly for the case $g(x_0) < 0$.)

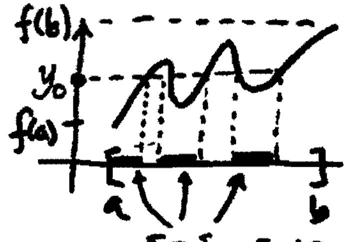
<u>Proof</u> Let $\varepsilon = g(x_0) > 0$. Note $(g(x_0)-\varepsilon, g(x_0)+\varepsilon) = (0, 2g(x_0))$. Since g is continuous at x_0, $\exists \delta > 0$ such that

$$\forall x \in S,\ \underbrace{|x - x_0| < \delta}_{x \in S \cap \underbrace{(x_0-\delta, x_0+\delta)}_{=I}} \Rightarrow \underbrace{|g(x) - g(x_0)| < \varepsilon}_{g(x) \in (\underbrace{g(x_0)-\varepsilon}_{=0},\, g(x_0)+\varepsilon)}$$

$$\Rightarrow g(x) > 0$$

<u>Intermediate Value Theorem</u>

If $f: [a,b] \to \mathbb{R}$ is continuous and y_0 is between $f(a)$ and $f(b)$, then $\exists x_0 \in [a,b]$ such that $f(x_0) = y_0$.



$S = \{x \in [a,b] : f(x) \le y_0\}$

<u>Proof</u> <u>Case 1</u>: $y_0 = f(a)$. Take $x_0 = a$.

<u>Case 2</u>: $y_0 = f(b)$. Take $x_0 = b$.

<u>Case 3</u>: $f(a) < y_0 < f(b)$.

(Case $f(a) > y_0 > f(b)$ is similar.) Let $S = \{x \in [a,b] : f(x) \le y_0\}$. $S \ne \emptyset$ as $a \in S$. S is bounded above by b. By completeness axiom, $x_0 = \sup S$ exists.

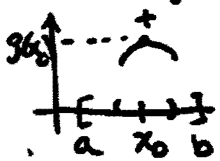
By supremum limit theorem, $\exists x_n \in S$ such that $\lim_{n\to\infty} x_n = x_0$.

$x_n \in [a,b] \Rightarrow a \le x_n \le b \Rightarrow a \le x_0 \le b \Rightarrow x_0 \in [a,b]$.

By S.C.T., $f(x_0) = \lim_{n\to\infty} f(x_n) \le y_0$.

Assume $f(x_0) < y_0$. Then $x_0 \ne b$ since $y_0 < f(b)$.

Define $g(x) = y_0 - f(x)$ on $[a,b]$. Then $g(x_0) = y_0 - f(x_0) > 0$.



By sign preserving property, there is interval $I = (x_0 - \delta, x_0 + \delta)$ such that $\forall x \in [a,b] \cap I$, $g(x) > 0$.

Now $x_0 < b$ and $x_0 < x_0 + \delta$ $\Rightarrow \exists x_1 \in (x_0, b] \cap (x_0, x_0+\delta) \subseteq [a,b] \cap I$

$\Rightarrow g(x_1) = y_0 - f(x_1) > 0$

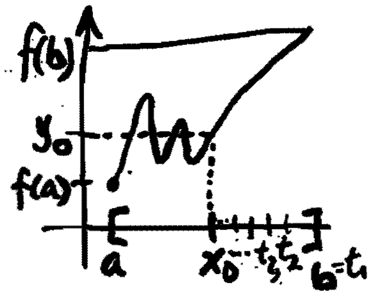
$\Rightarrow f(x_1) < y_0$, but $x_1 > x_0$, $x_1 \in [a,b]$

contradict $x_0 = \sup S$.

$\therefore f(x_0) = y_0$.

Alternative Ending (Avoiding sign preserving property)

As in the previous proof, we get $f(x_0) \le y_0$.



Since $y_0 < f(b)$, we get $x_0 \ne b$.

Let $t_n = x_0 + \frac{1}{n}(b - x_0) \in (x_0, b]$.

$\lim_{n\to\infty} t_n = x_0$ and $t_n > x_0 = \sup S$

So $t_n \notin S \Rightarrow f(t_n) > y_0$.

By S.C.T., $f(x_0) = \lim_{n\to\infty} f(t_n) \ge y_0$. $\therefore f(x_0) = y_0$.

Exercise

Let $f: [0,1] \to [0,1]$ be an increasing function (perhaps discontinuous). Suppose $0 < f(0)$ and $f(1) < 1$. Show that f has at least one fixed point.

(A fixed point of f is an element r in the domain of f such that $f(r) = r$.)

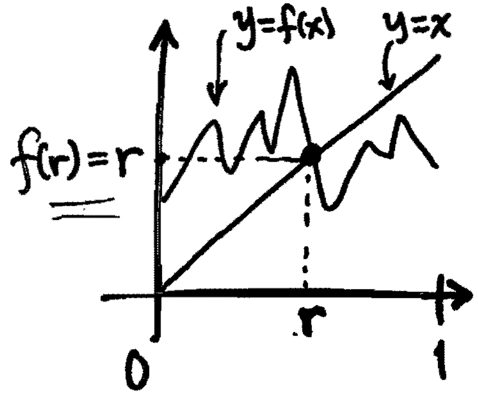
Hint: Sketch the graph of f and consider

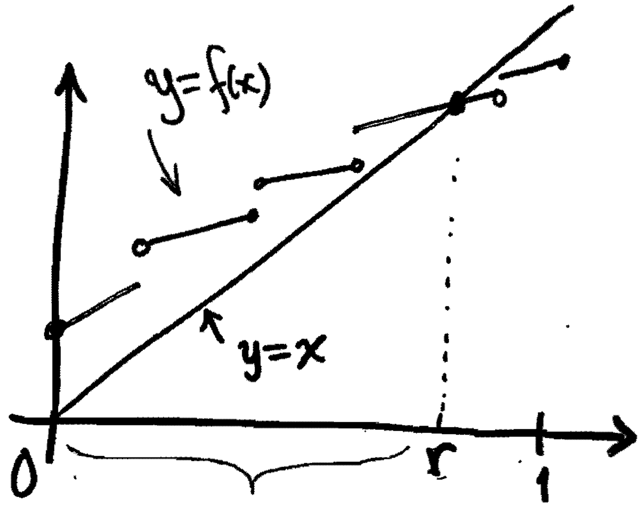
$$S = \{t \in [0,1] : t \le f(t)\}.$$

Does it have a supremum?

Use monotone function theorem and S.L.T.

Fixed Point





$S = \{t \in [0,1] : t \le f(t)\}$

Examples ① The equation $x^5+3x+\sin x=\cos x+10$ has a solution.

Let $f(x)=x^5+3x+\sin x-\cos x-10$. Then f is continuous

$f(0)=-11$ and $26=2^5+3\cdot 2-1-1-10\le f(2)$

So 0 is between $f(0)$ and $f(2)$. By intermediate value theorem, $\exists x_0\in[0,2]$ such that $f(x_0)=0$. Then

$x_0^5+3x_0+\sin x_0=\cos x_0+10$. $\leftarrow$ x_0 is a solution of equation.

② Every odd degree polynomial with real coefficients has at least one real root.

Let $P(x)=x^n+a_1x^{n-1}+\cdots+a_n$ with n odd.

Let $x_0=1+|a_1|+\cdots+|a_n|\ge 1$. Then

$P(x_0)=x_0^n+a_1x_0^{n-1}+\cdots+a_n \Rightarrow x_0^n-P(x_0)=-a_1x_0^{n-1}-\cdots-a_n$

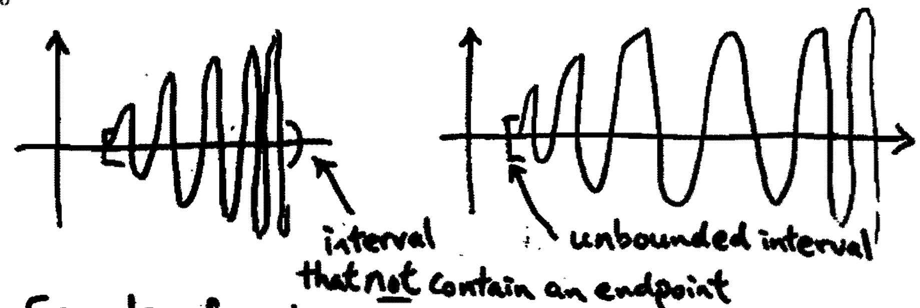
$P(-x_0)=-x_0^n+a_1x_0^{n-1}-\cdots+a_n \Rightarrow x_0^n+P(-x_0)=a_1x_0^{n-1}-\cdots+a_n$

$$\Rightarrow \left.\begin{matrix} x_0^n-P(x_0) \\ x_0^n+P(-x_0)\end{matrix}\right\} \le |a_1x_0|^{n-1}+\cdots+|a_n|$$
$$\le |a_1|x_0^{n-1}+\cdots+|a_n|x_0^{n-1}$$
$$=\underbrace{(|a_1|+\cdots+|a_n|)}_{=x_0-1}x_0^{n-1}$$
$$<x_0^n$$

$\Rightarrow$ $P(x_0)>0$ and $P(-x_0)<0$

$\Rightarrow$ 0 is between $P(-x_0)$ and $P(x_0)$

$\Rightarrow$ P has a real root between $-x_0$ and x_0.



Examples of continuous function with no maximum nor minimum values.

Extreme Value Theorem Let $a,b\in\mathbb{R}$ with $a\le b$.

If $f:[a,b]\to\mathbb{R}$ is continuous, then $\exists x_0,w_0\in[a,b]$ such that $f(w_0)\le f(x)\le f(x_0)$ $\forall x\in[a,b]$.

So range of $f=\{f(x):x\in[a,b]\}=f([a,b])$ is the interval $[f(w_0),f(x_0)]$. In particular, f is bounded on $[a,b]$. $f(x_0)=\sup\{f(x):x\in[a,b]\}=\max\limits_{x\in[a,b]}f(x)$ and $f(w_0)=\inf\{f(x):x\in[a,b]\}=\min\limits_{x\in[a,b]}f(x)$.

Proof. Assume $f([a,b])$ is not bounded above. Then every $n\in\mathbb{N}$ is not an upper bound. So $\exists z_n\in[a,b]$ with $f(z_n)>n$. By Bolzano-Weierstrass theorem, $\{z_n\}$ has a subsequence $\{z_{n_j}\}$ converging to some $z_0\in[a,b]$. Since f is continuous at z_0, $\lim\limits_{n\to\infty}f(z_{n_j})=f(z_0)$ by SCT. By boundedness theorem, $\{f(z_{n_j})\}$ is bounded. However, $f(z_{n_j})>n_j\ge j \Rightarrow \{f(z_{n_j})\}$ is unbounded, a contradiction.

$\therefore f([a,b])$ is bounded above and $M=\sup f([a,b])$ exists.

By supremum limit theorem, $\exists x_n \in [a,b]$ such that $M = \lim_{n\to\infty} f(x_n)$. By Bolzano-Weierstrass theorem, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ converging to some $x_0 \in [a,b]$. By S.C.T., $f(x_0) = f(\lim_{i\to\infty} x_{n_i}) = \lim_{i\to\infty} f(x_{n_i}) = M$.

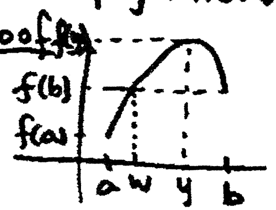
Similarly, $\exists w_0 \in [a,b]$ with $f(w_0) = \inf f([a,b])$.

Application. Let $f:[a,b]$ be continuous. Then $-\infty < \int_a^b f(x)\,dx < +\infty$ because $\exists x_0, w_0 \in [a,b]$ such that $f(w_0) \le f(x) \le f(x_0) \Rightarrow \int_a^b f(w_0)\,dx \le \int_a^b f(x)\,dx \le \int_a^b f(x_0)\,dx \Rightarrow -\infty < f(w_0)(b-a) \le \int_a^b f(x)\,dx < f(x_0)(b-a) < +\infty$.

Continuous Injection Theorem

If f is continuous and injective on $[a,b]$, then f is strictly monotone on $[a,b]$ and $f([a,b]) = [f(a), f(b)]$ or $[f(b), f(a)]$. (The theorem is true for any other nonempty interval.)

Proof.



Since f is injective, either $f(a) < f(b)$ or $f(a) > f(b)$. Suppose $f(a) < f(b)$. $\forall y \in (a,b)$, $f(y) > f(b)$ is false for otherwise, by intermediate value theorem, $\exists w \in (a,y)$ with $f(w) = f(b)$, contradicting injectivity of f. Similarly, $f(y) < f(a)$ is false. So $a < y < b \Rightarrow f(a) < f(y) < f(b)$. Then similarly, $a \le x < y \le b \Rightarrow f(a) \le f(x) < f(y) \le f(b)$.

$\therefore f$ is strictly increasing on $[a,b]$ and $f([a,b]) = [f(a), f(b)]$.

The case $f(a) > f(b)$ is similar.

Application The continuous injection theorem will be used to prove the following theorem, which will be used to prove the $dx/dy = 1\big/\frac{dy}{dx}$ rule for differentiation.

Continuous Inverse Theorem

If f is continuous and injective on $[a,b]$, then

$$f^{-1}: f([a,b]) \to [a,b] \text{ is continuous and surjective.}$$

(The theorem is true for any other nonempty interval.)

Proof. f^{-1} is surjective because $c \in [a,b] \Rightarrow f(c) \in f([a,b])$ and $f^{-1}(f(c)) = c$.

By continuous injection theorem, f is strictly monotone, say strictly increasing. Then f^{-1} is also strictly increasing.

Assume f^{-1} is discontinuous at some $y_0 = f(x_0) \in f([a,b])$.

Then either $a \le f^{-1}(y_0-) < f^{-1}(y_0) = x_0 \le b$

or $a \le x_0 = f^{-1}(y_0) < f^{-1}(y_0+) \le b$.

by Monotone function theorem.

This implies either the interval $(f^{-1}(y_0-), f^{-1}(y_0))$ or the interval $(f^{-1}(y_0), f^{-1}(y_0+))$ is not in the range of f^{-1}. This contradicts f^{-1} is surjective.

$\therefore f^{-1}$ is continuous.