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Problem 1

a/ For $\forall y \in S$, $y \leq e^{\sqrt{1}}$, e is an upper bound of S. $\sup S \leq e$. ~~Prove~~ Prove by contradiction, if $\sup S < e$, By density of Q, $\exists z \in Q$ s.t ~~sup S < z < e~~ $x < z < 1$, where $\sup S = e^{\sqrt{x}}$. So $\sup S = e^{\sqrt{x}} < e^{\sqrt{z}} < e^1$. Since $e^{\sqrt{z}} \in S$ and $e^{\sqrt{z}} > \sup S$ which is a contradiction. So $\sup S = e$.

For $\forall y \in S$, $e^0 = 1 < y$. 1 is an lower bound of S. $1 \leq \inf S$. Prove by contradiction, if $1 < \inf S$, By density of Q, $\exists z \in Q$ s.t $0 < z < x$, where $\inf S = e^{\sqrt{x}}$. So $1 < e^{\sqrt{z}} < e^{\sqrt{x}} = \inf S$, Since $e^{\sqrt{z}} \in S$ and $e^{\sqrt{z}} < \inf S$, which is a contradiction. So $\inf = 1$.

$\therefore$ The supremum is e, The infimum is 1.

b/ $T = \{n \cos\frac{n\pi}{2} \mid n \in N\}$. Since $\cos\frac{n\pi}{2} \in [-1, 1]$, $n \in N$, it is obvious that the set $T = \{n\cos\frac{n\pi}{2} \mid n \in N\}$ is unbounded. To prove it is unbounded below if it is bounded below, it converge to a point. Since $n \in N$, $\lim_{n \to -\infty} n \cos\frac{n\pi}{2}$ is still $-\infty$, it means T is unbounded below, hence its infimum doesn't exist.

Problem 2

a/ Since A is bounded, $S \subseteq A$ implie that S is also bounded. Since $S \subseteq A$, $\forall s \in S$, $s \geq \inf A$, also $s \leq \sup A$. by Theorem 2 of the dual between Sup and Inf. By the ~~property~~ definition of Infimum and Supremum, $\inf S \leq \sup S$. Therefore, $\inf A \leq \inf S \leq \sup S \leq \sup A$.

b/i/ Since A and B are both subsets of positive number. C is also a subsets of positive number. ~~Let~~ Let $a \in A, b \in B, c \in C$. $\inf A \leq a \leq \sup A$, $\inf B \leq b \leq \sup B$, By the definition of Infimum and Supremum. $(\inf A)(\inf B) \leq ab \leq (\sup A)(\sup B)$. Since $(\sup A)(\sup B)$ is an upper bound of C and $\sup A \sup B \in C$. By definition of supremum. $\sup C = \sup A \sup B$.

ii/ No, if $A = \{-3, -2, -1\}$, $B = \{1, 2, 3\}$, $C = \{-9, -6, -4, -3, -2, -1\}$

In this example, $\sup A = -1$, $\sup B = 3$, result (i) will show that $\sup C = (-1)(3) = -3$.

However, $\sup C = -1 \neq -3$, which shows that $\sup C = \sup A \sup B$ is not valid if either A or B contain negative number.

3/

4/ a/ $\lim_{n\to\infty} \frac{b}{n}$. take $K_1 = [\frac{b}{\varepsilon}]+1$, then $n \geq K_1 \Rightarrow |\frac{b}{n} - 0| < \varepsilon$, So $\lim_{n\to\infty} \frac{b}{n} = 0$.

To prove $\lim_{n\to\infty} \cos(a+\frac{b}{n}) = \cos a$. $\forall \varepsilon > 0$, we need to find $K_2 \in \mathbb{N}$ s.t $|\cos(a+\frac{b}{n}) - \cos a| < \varepsilon$ for all $n \geq K_2$.

$|\cos(a+\frac{b}{n}) - \cos a| = |\cos(a)\cos(\frac{b}{n}) - \sin(a)\sin(\frac{b}{n}) - \cos a| \leq |\cos(a) - \cos(a)| = 0 < \varepsilon$ for $n \geq K$,

Therefore, $\lim_{n\to\infty} \cos(a+\frac{b}{n}) = \cos a$.

b/ Since $\{b_n\}$ is a convergent sequence with $\lim_{n\to\infty} b_n = b > 0$. $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t $n \geq K$, $|b_n - b| < \varepsilon$.

To prove $\lim_{n\to\infty} \sqrt{b_n} = \sqrt{b}$, $|\sqrt{b_n} - \sqrt{b}| = |\sqrt{b} - \sqrt{b}| = 0 < \varepsilon$ for $n \geq K$. Therefore, $\lim_{n\to\infty} \sqrt{b_n} = \sqrt{b}$

5/ By MI, For $n=1$, $X_1 = 0.4 < 1$, $X_2 = 0.688$. $X_1 < X_2 < 1$. Assume $X_n < X_{n+1} < 1$,

then $X_n^3 < X_{n+1}^3 < 1^3 \Rightarrow X_n^3 + 2 < X_{n+1}^3 + 2 < 3 \Rightarrow \frac{X_n^3+2}{3} < \frac{X_{n+1}^3+2}{3} < 1$

$\Rightarrow X_{n+1} < X_{n+2} < 1$. By MI, $X_n < X_{n+1} < 1$. which show that $\{X_n\}$ is strictly increasing and also bounded above.

By monotone sequence theorem, $\lim_{n\to\infty} X_n$ exists. $\frac{X_n^3+2}{3} = X_{n+1}$, Let $n \to \infty$, $\frac{X^3+2}{3} = X \Rightarrow X^3 - 3X + 2 = 0$

$\Rightarrow (X-1)(X^2-2) = 0 \Rightarrow X = \sqrt{2}$ or $-\sqrt{2}$ or 1. By monotone sequence theorem, $\lim_{n\to\infty} X_n = \sup\{X_n : n \geq 1\}$.

Therefore, $X = 1$, $\therefore \lim_{n\to\infty} X_n = 1$, $\{X_n\}$ converges to 1.

6/ a/ Since $\lim_{n\to\infty} \frac{X_{n+1}}{X_n} = L < 1$, $\forall \varepsilon$, $\exists k \in N$ s.t $\left|\frac{X_{n+1}}{X_n} - L\right| < \varepsilon \Rightarrow \left|\frac{X_{n+1}}{X_n}\right| - |L| < \left|\frac{X_{n+1}}{X_n} - L\right| < \varepsilon$

$\cancel{\Rightarrow \left|\frac{X_{n+1}}{X_n}\right| < \varepsilon + |L|}$. We let $L < r < 1$, with $\varepsilon < r - L$. $\left|\frac{X_{n+1}}{X_n}\right| - |L| < \varepsilon \Rightarrow \left|\frac{X_{n+1}}{X_n}\right| - |L| < \varepsilon < r - L$

$\Rightarrow \left|\frac{X_{n+1}}{X_n}\right| < r$.

$\left|\frac{X_2}{X_1}\right| \left|\frac{X_3}{X_2}\right| \cdots \left|\frac{X_n}{X_{n-1}}\right| < r^{n-1} \Rightarrow \left|\frac{X_n}{X_1}\right| < r^{n-1} \Rightarrow |X_n| < \frac{|X_1|}{r} r^n$. Since $0 < r < 1$, $\lim_{n\to\infty} r^n = 0$

Therefore, $\lim_{n\to\infty} |X_n| = 0$. $\Rightarrow \{X_n\}$ is convergent.

b/ $\lim_{n\to\infty} \frac{X_{n+1}}{X_n} = L > 1$. We let $L - \varepsilon > r > 1$. Since $\lim_{n\to\infty} \frac{X_{n+1}}{X_n} = L > 1 \Rightarrow \left|\frac{X_{n+1}}{X_n} - L\right| < \varepsilon$, $\forall n > k \in N$

$\Rightarrow L - \varepsilon < \frac{X_{n+1}}{X_n} < L + \varepsilon \Rightarrow r < \frac{X_{n+1}}{X_n} < L + \varepsilon$. $r < \frac{X_{n+1}}{X_n}$. $\forall n > k \in N$

$r^{n-1} < \left(\frac{X_n}{X_{n-1}}\right)\left(\frac{X_{n-1}}{X_{n-2}}\right) \cdots \left(\frac{X_2}{X_1}\right) \Rightarrow r^{n-1} < \frac{X_n}{X_1} \Rightarrow r^n \frac{X'}{r} < X_n$. Since $r > 1$, $\lim_{n\to\infty} r^n = \infty$.

$\lim_{n\to\infty} X_n = \infty$. Hence $\{X_n\}$ does not converge.

c/ i/ $\{X_n\} = \{1, 1, 1, \ldots, 1\}$ then $\{X_n\}$ converges to 1

ii/ $\{X_n\} = n$, where $n \in N$. $\lim_{n\to\infty} \frac{n+1}{n} \Rightarrow \frac{1 + \frac{1}{\infty}}{1} \Rightarrow 1$. $\{X_n\}$ does not converges.