Math2033 TA note 8

Yang Yunfei, Chen Yipei, Liu Ping

March 31, 2019

1 CONTINUITY

Example 1. *Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ that is continuous at $x\in\mathbb{Z}$, but discontinuous at every $x\notin\mathbb{Z}$.*

Solution: Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous function such that $g(x)=0$ for every $x\in\mathbb{Z}$ and $g(x)\neq 0$ for every $x\notin\mathbb{Z}$. For example, take $g(x)=\sin\pi x$. Define $f(x)=g(x)$ if $x\in\mathbb{Q}$ and $f(x)=0$ if $x\notin\mathbb{Q}$. Then for every $m\in\mathbb{Z}$, $|f(x)|\le|g(x)|\to 0$ as $x\to m$. Therefore, f is continuous at every $m\in\mathbb{Z}$. For every $x_0\notin\mathbb{Z}$, let $a_n\in\mathbb{Q}$ and $b_n\notin\mathbb{Q}$ be sequences such that $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=x_0$. Then $\lim\limits_{n\to\infty}f(a_n)=g(x_0)\neq 0=\lim\limits_{n\to\infty}f(b_n)$. So f is not continuous at x_0 by the sequential continuity theorem.

Example 2. *Let $f:\mathbb{R}\to\mathbb{R}$ be continuous such that $|f(x)-f(y)|\le\frac{1}{2}|x-y|$ for every $x,y\in\mathbb{R}$.*
(a) Let $x_1\in\mathbb{R}$. Define $x_{n+1}=f(x_n)$ for $n\in\mathbb{N}$. Show that $\{x_n\}$ is a Cauchy sequence.
(b) Show that there is $x\in\mathbb{R}$ such that $f(x)=x$.

Solution: (a) Observe that $|x_{k+1}-x_k|=|f(x_k)-f(x_{k-1})|\le\frac{1}{2}|x_k-x_{k-1}|$. Repeating this, we have

$$|x_{k+1}-x_k|\le\frac{1}{2}|x_k-x_{k-1}|\le\frac{1}{2^2}|x_{k-1}-x_{k-2}|\le\cdots\le\frac{1}{2^{k-1}}|x_2-x_1|.$$

Thus, for $m>n$, we have

$$\begin{aligned}|x_m-x_n| &\le |(x_m-x_{m-1})+(x_{m-1}-x_{m-2})+\cdots+(x_{n+1}-x_n)|\\ &\le |x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n|\\ &\le\left(\frac{1}{2^{m-2}}+\frac{1}{2^{m-3}}+\cdots+\frac{1}{2^{n-1}}\right)|x_2-x_1|\\ &\le\sum_{j=n-1}^{\infty}\frac{1}{2^j}|x_2-x_1|=\frac{1}{2^{n-2}}|x_2-x_1|.\end{aligned}$$

If $x_1 = x_2$ then $x_m = x_n$ for all m, n. So $\{x_n\}$ is a Cauchy sequence. If $x_1 \neq x_2$, then for every $\epsilon > 0$, let $K \in \mathbb{N}$ such that $K > 2 - \log_2 \frac{\epsilon}{|x_2 - x_1|}$, then for all $m, n > K$, we have $|x_m - x_n| \leq \frac{1}{2^{K-2}} |x_2 - x_1| < \epsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence.

(b) Define $\{x_n\}$ as in (a), then $\{x_n\}$ is a Cauchy sequence. So $\{x_n\}$ converges to some $x \in \mathbb{R}$. We have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x).$$

Example 3. *Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for every $c \in \mathbb{R}$, $f(x) = c$ has exactly 2 solutions.*

Solution: We proof by assuming the opposite. Suppose such a function exists. Let a, b be the solutions of $f(x) = 0$ with $a < b$.

Case I: $\max\limits_{x\in[a,b]} f(x) = f(x_0) \neq 0$. Let y_0 be another solution of $f(y_0) = f(x_0)$.
If $y_0 \notin [a, b]$, then by the intermediate value theorem, there will be 3 solutions of $f(x) = \frac{1}{2} f(x_0)$, one in (a, x_0), one in (x_0, b) and one between the closer endpoint of $[a, b]$ to y_0.
If $y_0 \in [a, b]$, without loss of generality, let $x_0 < y_0$ and $f(z_0) = \min\limits_{z\in[x_0,y_0]} f(x)$. Let $w = \max\{f(z_0), 0\}$, then by the intermediate value theorem, there are at least 3 solutions of $f(z) = w$, one in (a, x_0), one in (x_0, y_0), one in (y_0, b) .
Thus, whether $y_0 \notin [a, b]$ or $y_0 \in [a, b]$ will lead to a contradiction.

Case II: $\min\limits_{x\in[a,b]} f(x) \neq 0$. Let $g(x) = -f(x)$, from case I we know this will lead to a contradiction as well.

Case III: $\max\limits_{x\in[a,b]} f(x) = 0 = \min\limits_{x\in[a,b]} f(x)$. Then $f(x) \equiv 0$ on $[a, b]$,a contradiction.

Example 4. .
(a) State the intermediate value theorem.
(b) Let $f : [0, 2] \to \mathbb{R}$ be continuous and $f(0) = f(2)$. Show that there exists $c \in [0, 1]$ such that $f(c) = f(c + 1)$.
(c) Show that there is a nonzero continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(t) + g(2t) + g(3t) = g(4t) + g(5t)$ for every $t \in \mathbb{R}$.

Solution:.
(a) If a continuous function $f : [a, b] \to \mathbb{R}$ takes values $f(a)$ and $f(b)$ at each end of the interval, then it also takes any value between $f(a)$ and $f(b)$ at some point within the interval.
(b) Define $g(x) = f(x) - f(x + 1)$. Then $g(0) + g(1) = f(0) - f(2) = 0$. If $g(0) = 0$ or $g(1) = 0$, we find the c such that

$$g(c) = 0 \implies f(c) = f(c + 1).$$

Otherwise, $g(0), g(1)$ has different signs which means $\exists c \in [0, 1]$ s.t $g(c) = 0$ by intermediate theorem. So

$$\exists c \in [0, 1], \quad f(c) = f(c + 1).$$

(c) suppose $g = t^r$ for some r, by the constraint we have

$$t^r + 2^r t^r + 3^r t^r = 4^r t^r + 5^r t^r.$$

That is

$$1+2^r+3^r=4^r+5^r.$$

Define $f(r)=1+2^r+3^r-(4^r+5^r)$, and we have

$$f(0)=1 \quad f(1)=-3.$$

By intermediate value theorem, $\exists r_0 \in [0,1], f(r_0)=0$. So

$$g(x)=t^{r_0}$$

which satisfies the constraint.