Homework 2

Sunday, 3 October 2021 4:07 AM

Math2033Hw2

Math 2033 (Homework 2) L1 Fall 2021

Problems (Due Oct. 11 at 11:59 pm)

① Prove that there is a bijection from $[0,1]$ to $(0,$

② Determine if the set A of all intersection points in $\mathbb{R}$
of the family of lines $\{y=mx : m\in\mathbb{Z}\}$ with the
family of circles $\{x^2+y^2=r^2 : r\in\mathbb{Q}\}$ is countable or
uncountable. Here A is the set of all points in $\mathbb{R}^2$ tha
are on at least one of the lines $y=mx$ $(m\in\mathbb{Z})$ an
at least one of the circles $x^2+y^2=r^2$ $(r\in\mathbb{Q})$.

③ Prove that there exist infinitely many positive real numb
r such that the equation $2^x+3^y+5^z=r$ has no
Solution $(x,y,z)\in\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$.

④ Let T be a nonempty subset of the interval $(0,1)$. If
every finite subset $\{x_1,x_2,\dots,x_n\}$ of T (with no tw
of $x_1, x_2, \dots, x_n$ equal) has the property that

1].
2
t
d
ers
)
0

... of $x_1, x_2, \cdots, x_n$ equal) has the property that $x_1^2 + x_2^2 + \cdots + x_n^2 < 1$, then prove that T is a countab[le]
set.

Name: Leung Ko Tsun

SID: 20516287

1. Bijection from $[0,1]$ to $(0,1]$

We can define a function $f: [0,1] \to (0,1]$ as

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n},\ n \in \mathbb{N} \\ x & \text{if } x \in (0,1) \text{ and } x \neq \frac{1}{n},\ n \in \mathbb{N} \end{cases}$$

It is obvious that $f(x)$ is one-to-one.

Onto: $(0,1]$ contains all elements that $[0,1]$ has excep[t]

if we define f like this, 1 in $[0,1]$ is mapped to $\frac{1}{2}$

2 in $[0,1]$ is mapped to $\frac{1}{3}$ in $(0,1]$, all other values not

the form of $\frac{1}{n}$ will be mapped to itself, and finally

is mapped to 1 in $(0,1]$. Therefore no element of B ...

le

s

pt 0.

in [0, 1],

f in

0 in [0,1]

will be

omitted as a value.

$\Rightarrow$ It is a one-to-one correspondence

$\Rightarrow$ There exists a bijection from $[0,1]$ to $(0,1]$.

2.

② Determine if the set A of all intersection points in $\mathbb{R}^2$ of the family of lines $\{y=mx : m\in\mathbb{Z}\}$ with the family of circles $\{x^2+y^2=r^2 : r\in\mathbb{Q}\}$ is countable or uncountable. Here A is the set of all points in $\mathbb{R}^2$ that are on at least one of the lines $y=mx$ $(m\in\mathbb{Z})$ and at least one of the circles $x^2+y^2=r^2$ $(r\in\mathbb{Q})$.

Let L_m be the set of lines with equation $y=mx$, m

C_r be the set of circles with equation $x^2+y^2=r^2$,

The union of L_m, $\bigcup_{m\in\mathbb{Z}} L_m$ is countable by countable u

The union of C_r. $\bigcup_{r\in\mathbb{Q}} C_r$ is countable by the countable union

intersection points.

$$\begin{cases} L_m: y=mx \\ C_r: x^2+y^2=r^2 \end{cases} \Rightarrow x^2+(mx)^2=r^2 \Rightarrow (m^2+1)x^2=r^2 \Rightarrow x=\pm$$

$\in \mathbb{Z}$,

, $r \in \mathbb{Q}$.

nion
theorem.

theorem.

$\frac{r}{\sqrt{m^2+1}}$

So there are at most 2 intersection points between the
and the circle C_r.

Given $A = \{(x,y) \in \mathbb{R}^2, (x,y)$ is the intersection point b
L_m and C_r, $m \in \mathbb{Z}$, $r \in \mathbb{Q}\}$.

Define $f: \mathbb{Z} \times \mathbb{Q} \to A$, $f(m,r)$ maps (m,r) to th
(at most 2) intersection points between L_m and C_r.

f is injective, as if $(m_1, r_1) \neq (m_2, r_2)$, they will h
different intersection points, $f(m_1,r_1) \neq f(m_2,r_2)$.

it is obvious that f is surjective as every ele
A is covered by f.

$\Rightarrow$ f is bijective.

$\mathbb{Z} \times \mathbb{Q}$ is countable by product theorem since $\mathbb{Z}, \mathbb{Q}$

$\Rightarrow$ By bijection theorem, A is countable.

3.

③ Prove that there exist infinitely many positive real numbers r such that the equation $2^x + 3^y + 5^z = r$ has no solution $(x,y,z) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$.

line L_m

between

e

.

ave

ment in

comtelle.
are

Let $S = \{2^x + 3^y + 5^z : (x, y, z) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}\}$.

Since $\mathbb{Q}$ is countable, $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ is countable by pro[...]

theorem.

$$S = \bigcup_{(x,y,z) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}} \{2^x + 3^y + 5^z\}$$

is countable by countable union theorem.

Let $g : \mathbb{R} \to \mathbb{R}^+$, $g(x) = e^x$.

$g^{-1} : \mathbb{R}^+ \to \mathbb{R}$, $g^{-1}(x) = \ln(x)$.

g is a bijection from $\mathbb{R}$ to $\mathbb{R}^+$.

Since $\mathbb{R}$ is uncountable, $\mathbb{R}^+$ is uncountable by bij[...]

theorem.

Since $\mathbb{R}^+$ is uncountable, $\mathbb{R}^+ \setminus S$ is also uncoun[...]

$\Rightarrow \mathbb{R}^+ \setminus S$ is an infinite set.

$\Rightarrow$ There exist infinitely many positive real numbers r ject

jection

ntable.

such

$\Rightarrow$ There exist infinitely many positive ...

that $2^x+3^y+5^z=r$ has no solution with $(x,y,z) \in$

4.

④ Let T be a nonempty subset of the interval $(0,1)$. If every finite subset $\{x_1, x_2, \cdots, x_n\}$ of T (with no two of $x_1, x_2, \cdots, x_n$ equal) has the property that $x_1^2+x_2^2+\cdots+x_n^2<1$, then prove that T is a countable set.

Prove that T is a countable set.

First, showing that the number of elements is

than $(k+1)^2$ in every finite subset of T in $[\frac{1}{k+1}, \frac{1}{k}$

Assume the above statement is false.

Let $S_k = \{x_1, x_2, \ldots, x_{(k+1)^2}\} \subseteq T \quad \forall x_i \in [\frac{1}{k+1}, \frac{1}{k}$

$\inf S_k = \frac{1}{k+1}$.

$\forall i$ in $\{1, 2, \cdots, (k+1)^2\}$, $\quad x_i \geqslant \frac{1}{k+1}$

$\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$.

les

$), k \in \mathbb{N}$.

$), k \in \mathbb{N}$

$$\Rightarrow \quad x_i^2 \geq \frac{1}{(k+1)^2}$$

$$\sum_{i=1}^{(k+1)^2} x_i^2 = x_1^2 + x_2^2 + \dots + x_{(k+1)^2}^2 \geq \underbrace{\frac{1}{(k+1)^2} + \dots + \frac{1}{(k}}_{(k+1)^2 \text{ ter}}$$

$$= \frac{(k+1)^2}{(k+1)^2}$$

$$= 1$$

This contradicts with the given condition of

$$x_1^2 + x_2^2 + \dots + x_n^2 < 1$$

$\Rightarrow$ S_k has less than $(k+1)^2$ elements in $[\frac{1}{k+1}, \frac{1}{k})$.

$\Rightarrow$ S_k has finite elements in $[\frac{1}{k+1}, \frac{1}{k})$, $k \in \mathbb{N}$.

$\Rightarrow$ S_k is countable on $[\frac{1}{k+1}, \frac{1}{k})$

Now, rewrite $(0,1)$ as:

$$(0,1) = [\tfrac{1}{2}, 1) \cup [\tfrac{1}{3}, \tfrac{1}{2}) \cup [\tfrac{1}{4}, \tfrac{1}{3}) \dots$$

$$= \bigcup^{\infty} [\tfrac{1}{k+1}, \tfrac{1}{k})$$

$+1)^2$

ns

, KEIN

$_{k=1} \; [\frac{1}{k+1}, \frac{1}{k})$

$$T = \bigcup_{k=1}^{\infty} \left(S_k \cap [\tfrac{1}{k+1}, \tfrac{1}{k}) \right), \; k \in \mathbb{N}$$

Since S_k is countable on $[\frac{1}{k+1}, \frac{1}{k})$, by countable un

theorem, T is countable.

nion