MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #19

Riemann Integration:

Terminology:

1. Partition of $[a, b]$ (denoted by P)

   A finite set of numbers $x_0, x_1, x_2, x_3, .., x_n$ such that $a = x_0 < x_1 < \cdots < x_n = b$

   (i.e. We cut the interval into **finitely many parts**)

2. For each partition P, we define

   $M_i = \sup\{f(x): x_{i-1} \le x \le x_i\}$

   $m_i = \inf\{f(x): x_{i-1} \le x \le x_i\}$

   $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$ (where $\Delta x_i = x_i - x_{i-1}$)

   $L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$

3. Upper Intergal and Lower Integral of $f(x)$

   $(U)\int_a^b f(x)dx = \inf\{U(P, f): \text{P is any partition of } [a, b]\}$

   $(L)\int_a^b f(x)dx = \sup\{L(P, f): \text{P is any partition of } [a, b]\}$

4. Refinement of P

   The partition P' is said to be refinement of P if $P \subseteq P'$

   Properties: $\mathbf{L(f, P) \le L(f, P')}$ **and** $\mathbf{U(f, P) \ge U(f, P')}$

Basic Definition:

Given a bounded function $f: [a, b] \to R$, we say $f(x)$ is **Riemann-Integrable** if and only if $(U)\int_a^b f(x)dx = (L)\int_a^b f(x)dx$

We denote this common value by $\int_a^b f(x)dx$

Theorem: (Integral Criterion)

$f(x)$ is **Riemann-Integrable** if and only if for any $\varepsilon > 0$, there exists a partition P on $[a, b]$ such that $|U(P, f) - L(P, f)| < \varepsilon$

(Remark: The theorem says that if an integral of a function exists if and only if we can find a partition such that the difference between the upper integral and lower integral is very small.)

Example 1

Show that the function $f(x) = e^x$ is Riemann-Integrable on $[0,1]$

Solution:

Consider a partition of $[0,1]$ $(0 < \frac{1}{n} < \frac{2}{n} < \frac{3}{n} < \cdots < \frac{n-1}{n} < 1)$

For each interval $\left[\frac{k}{n},\frac{k+1}{n}\right]$ (for $k = 0,1,2,..,n-1$)

We get $\sup\{f(x)\} = e^{\frac{k+1}{n}}$ and $\inf\{f(x)\} = e^{\frac{k}{n}}$

$$U(P,f) = \sum_{k=0}^{n-1} e^{\frac{k+1}{n}}\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{k=0}^{n-1} e^{\frac{k+1}{n}}$$

$$L(P,f) = \sum_{k=0}^{n-1} e^{\frac{k}{n}}\left(\frac{1}{n}\right) = \frac{1}{n}\sum_{k=0}^{n-1} e^{\frac{k}{n}}$$

$$|U(P,f) - L(P,f)| = \left|\frac{1}{n}\sum_{k=0}^{n-1} e^{\frac{k+1}{n}} - \frac{1}{n}\sum_{k=0}^{n-1} e^{\frac{k}{n}}\right| = \frac{1}{n}\left(e^{\frac{n}{n}} - e^{\frac{0}{n}}\right) = \frac{1}{n}(e-1) < \varepsilon$$

Hence we can pick large n, $n > \frac{e-1}{\varepsilon}$ (By Archimedean Property). With this partition, we get $|U(P,f) - L(P,f)| < \varepsilon$

Hence $f(x) = e^x$ is Riemann Integrable.

(Remark: In normal situation, for a continuous function in bounded interval, we can consider the partition with "uniform cutting"

Example 2 Show that the function defined on $[a,b]$ by $$f(x) = \begin{cases} 2 & \text{if } x = x_1, x_2, .., x_n \\ 1 & \text{otherwise} \end{cases}$$ is Riemann integrable. (Note: $x_1, x_2, \ldots x_n$ are points inside the (a,b))

Solution:

Here there are discontinuities at $x_1, x_2, \ldots \ldots x_n$ (Assume $x_1 < x_2 < \cdots < x_n$). To deal with this situation, for $\varepsilon > 0$, since the function is bounded, we consider this partition

$$a \le x_1 - \delta \le x_1 + \delta \le x_2 - \delta \le x_2 + \delta \le \cdots \le x_n - \delta \le x_n + \delta \le b$$

(That is, for each discontinuous point, we give an small interval)

Then the corresponding upper bound and lower bound are given by

$$U(P,f) = (x_1 - \delta - a)(1) + \sum_{k=2}^{n}(1)(x_k - \delta - x_{k-1} - \delta) + n(2\delta)(2) + (b - x_n - \delta)(1)$$

$$L(P,f) = (x_1 - \delta - a)(1) + \sum_{k=2}^{n}(1)(x_k - \delta - x_{k-1} - \delta) + n(2\delta)(1) + (b - x_n - \delta)(1)$$

$$|U(P,f) - L(P,f)| = |2n\delta|$$

For any $\varepsilon > 0$, by picking $\delta < \frac{\varepsilon}{2n}$. we get $|U(P,f) - L(P,f)| = |2n\delta| < \varepsilon$

Hence $f(x)$ is Riemann Integrable.

Example 3

Show that the function defined on $[0,1]$

$$f(x) = \begin{cases} 1 & \text{if x is rational} \\ 0 & \text{if x is irrational} \end{cases}$$

is not **Riemann-Integrable**

Solution:

For any partition P (say $0 = x_0 < x_1 < \cdots < x_n = 1$), by density of rational number, we can find a rational number q_i such that $x_{i-1} < q_i < x_i$ (for $i = 1,2,\ldots,n$). Hence the upper bound and lower bound are given by

$$U(P,f) = \sum_{i=1}^{n} (1)(x_i - x_{i-1}) = x_n - x_0 = 1 - 0 = 1$$

$$L(P,f) = \sum_{i=1}^{n} (0)(x_i - x_{i-1}) = 0$$

Hence $|U(P,f) - L(P,f)| = 1$, so the difference between the upper integral and lower integral cannot be arbitrarily small. (Say $\varepsilon < 1$). Hence $f(x)$ is not Riemann Integrable.

Example 4

Show that the function defined on $[0,1]$ by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is rational} \\ 0 & \text{if x is irrational} \end{cases}$$

is **Riemann-Integrable**

Solution:

To show this, pick integer N such that $N > \frac{2}{\varepsilon}$ (i.e. $\frac{1}{N} < \frac{\varepsilon}{2}$)

consider a set $Q_N = \{\text{All rational } \frac{p}{q} \in [0,1]: q \le N\}$. We first see that Q_N is finite (say there are n elements $x_1, x_2, x_3, \ldots x_n$ and assume $x_1 < x_2 < \cdots < x_n$) and let $x_i = \frac{p_i}{q_i}$). Consider the partition similar to Example 2.

$$0 \le x_1 - \delta \le x_1 + \delta \le x_2 - \delta \le x_2 + \delta \le \cdots \le x_n - \delta \le x_n + \delta \le 1$$

Then the upper bound is given by

$$U \le (x_1 - \delta - 0)\left(\frac{1}{N}\right) + \sum_{k=2}^{n}\left(\frac{1}{N}\right)(x_k - \delta - x_{k-1} - \delta) + \sum_{k=1}^{n}\left(\frac{1}{q_k}\right)(2\delta) + (1 - x_n - \delta)(\frac{1}{N})$$

The lower bound is given by

$$L = (x_1 - \delta - 0)(0) + \sum_{k=2}^{n}(0)(x_k - \delta - x_{k-1} - \delta) + \sum_{k=1}^{n}(0)(2\delta) + (1 - x_n - \delta)(0) = 0$$

So $|U(f, P) - L(f, P)|$

$$\leq \left|(x_1 - \delta - 0)\left(\frac{1}{N}\right) + \sum_{k=2}^{n}\left(\frac{1}{N}\right)(x_k - \delta - x_{k-1} - \delta) + \sum_{k=1}^{n}\left(\frac{1}{q_k}\right)(2\delta) + (1 - x_n - \delta)\left(\frac{1}{N}\right)\right|$$

$$= \left|(x_1 - \delta - 0)\left(\frac{1}{N}\right) + \sum_{k=2}^{n}\left(\frac{1}{N}\right)(x_k - \delta - x_{k-1} - \delta) + \sum_{k=1}^{n}(1)(2\delta) + (1 - x_n - \delta)\left(\frac{1}{N}\right)\right|$$

$$= \left|\frac{1}{N}(1 - 2n\delta) + 2n\delta\right|$$

$$= \left|\frac{1}{N} + 2n\delta\left(1 - \frac{1}{N}\right)\right| \leq \left|\frac{1}{N}\right| + |2n\delta|\left|\left(1 - \frac{1}{N}\right)\right|$$

Pick $\delta < \frac{\varepsilon}{4n}$, then

$$< \left|\frac{1}{N}\right| + \frac{\varepsilon}{2}\left|1 - \frac{1}{N}\right| < \frac{\varepsilon}{2} + \frac{\epsilon}{2} = \varepsilon.$$

So $f(x)$ is Riemann Integrable.

Example 5

Let f, h are bounded function and Riemann Integrable on $[a, b]$ and let $g: \{a, b\} \to \mathbf{R}$ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in [a, b]$. Suppose $\int_a^b f(x)dx = \int_a^b h(x)dx = A$. Show that $g(x)$ is also Riemann Integrable on $[a, b]$

Solution:

For any $\varepsilon > 0$

Since $f(x)$ is Riemann Integrable, then there exists partition P_1 such that

$$A - \frac{\varepsilon}{2} < L(P_1, f) < A < U(P_1, f) < A + \frac{\varepsilon}{2}$$

Similarly $h(x)$ is Riemann Integrable, then there exists partition P_2 such that

$$A - \frac{\varepsilon}{2} < L(P_2, h) < A < U(P_2, h) < A + \frac{\varepsilon}{2}$$

Consider the partition P which is **the refinement of P, $\mathbf{P_1} \cup \mathbf{P_2}$**. Then for this partition P

$$U(P, g) \leq U(P, h) \leq U(P_2, h) < A + \frac{\varepsilon}{2}$$

$$L(P, g) \geq L(P, f) \geq L(P_1, f) > A - \frac{\varepsilon}{2}$$

Then $A - \frac{\varepsilon}{2} < L(P, g) < U(P, g) < A + \frac{\varepsilon}{2}$

So

$$|U(P, g) - L(P, g)| = U(P, g) - L(P, g)$$

$$< A + \frac{\varepsilon}{2} - \left(A - \frac{\varepsilon}{2}\right)$$

$$= \varepsilon$$

So $g(x)$ is Riemann Integrable on $[a, b]$.

Try to work on the following exercises to understand the material, you are welcome to give your solution to me for comments.

☺Exercise 1

Show that the following functions are Riemann integrable on specific intervals

a) $f(x) = 5x^4$ on $[0,1]$

b) $g(x) = \sin x$ on $\left[0, \frac{\pi}{2}\right]$

(Hint: The following inequality may be useful, for $a > b$, by mean value theorem $\mathbf{\sin a - \sin b = \cos c(a - b)} \le (\mathbf{a - b})$ (where $c \in (a, b)$)

c) $h(x) = |x - 1|$ on $\left[\frac{1}{2}, \frac{3}{2}\right]$

d) $u(x) = \begin{cases} a + 1 & \text{if } x = \frac{1}{2} \\ a & \text{otherwise} \end{cases}$ on $[0,1]$ (Hint: The partition is similar to Example 2)

☺Exercise 2 (Important Eercise)

Show that the function defined on $[0,1]$ by

$$f(x) = \begin{cases} (-1)^{n-1} & \text{if } \dfrac{1}{n+1} < x \le \dfrac{1}{n} \quad \text{for } n = 1,2,3,4 \dots. \\ 0 & \text{if } x = 0 \end{cases}$$

is Riemann Integrable.

(Hint: For any $\varepsilon > 0$, there exists K such that $\frac{1}{K} < \frac{\varepsilon}{2}$, use the partition

$\{0, \frac{1}{K}, \frac{1}{K-1}, \frac{1}{K-2}, \dots., \frac{1}{3}, \frac{1}{2}, 1\}$)

(Remark: You may ask why we cannot use the partition $\{0, \dots, \frac{1}{n}, \frac{1}{n-1}, \dots, \frac{1}{3}, \frac{1}{2}, 1\}$, it is because, we need to find a partition with finitely many points but not infinitely many)

☺Exercise 3 (*More Difficult)

Show that the function $\frac{1}{x}$ defined on $[a, b]$ (where $b > a > 0$) is Riemann Integrable by considering the partition $\{a, ar, ar^2, ar^3, \dots., ar^n = b\}$

(Hint: Note that $r > 1$ and $ar^n = b \to n = \frac{\ln\left(\frac{b}{a}\right)}{\ln r}$)

(Remark: In fact, we may use partition with uniform cutting since $1/x$ is continuous, but the computation is more tenious.)

☺Exercise 4

Show that the function defined on $[0,1]$ by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbf{Q} \\ -x & \text{if } x \in \mathbf{R}\backslash\mathbf{Q} \end{cases}$$

is NOT Riemann Integrable.

(Hint: The proof is same as Example 3, in any sub-interval $[a,b]$, the supremum and infinmum of $f(x)$ are given by $\sup f(x) = \frac{1}{b}$ and $\inf f(x) = -\frac{1}{b}$ (Why? Try to explain it). In the calculation, you may use the fact that $2a(a-b) = (a-b)^2 + a^2 - b^2$).

☺Exercise 5

Let $f: [a,b] \to \mathbf{R}$ be bounded and Riemann integrable function on $[a,b]$ and let $g: [a,b] \to \mathbf{R}$ be a function which is obtained by alternating the values of $f(x)$ at a finite number of points. (i.e. It means that $g(x) \neq f(x)$ for finitely many points $(x_1, x_2, \dots x_k)$ and for other points $f(x) = g(x)$. Show that $g(x)$ is also Riemann Integrable on $[a,b]$ and $\int_a^b f(x)dx = \int_a^b g(x)dx$.

☺Exercise 6

Show if $f(x)$ is Riemann Integrable on $[a,b]$ and let $[c,d] \subseteq [a,b]$. Show that $f(x)$ is Riemann Integrable on $[c,d]$

(Hint: Try to draw a simple graph to help you)

(Note: It means that if $f(x)$ is integrable on bigger interval, then it is also integrable on smaller interval)

☺Exercise 7 (*More Difficult)

Let f be strictly increasing on $[a,b]$. Let $Q = \{f(a) = y_0 < y_1 < \cdots < y_n = f(b)\}$ be a partition of $[f(a), f(b)]$ and $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ be the corresponding partition of $[a,b]$. (i.e. $f(x_j) = y_j \;\; j = 0,1,2,..,n$). (Try to draw the picture first)

a) Use the fact that any monotonic function is integrable to argue $f(x)$ and its inverse $f^{-1}(x)$ are integrable.

b) Show that $U(f,P) + L(f^{-1},Q) = bf(b) - af(a)$

c) Deduce that $\int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a) - \int_a^b f(x)dx$