MATH2033 Mathematical Analysis

Lecture Note 3

Countability

Basic definition of countability – An intuitive approach

In order to develop a formal definition of countability, we first review how to count the elements in a set S.

We let $S = \{$A, B, C, D, #, \$, ☺, ☹$\}$

Intuitive approach

The easiest way to count the elements in the set is to assign positive integers (from 1 to n) to each of the elements. For example,

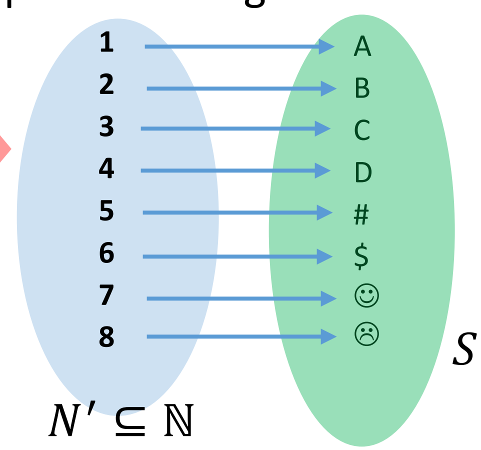
1 2 3 4 5 6 7 8

$S = \{$A, B, C, D, #, \$, ☺, ☹$\}$

So we can conclude that there are 8 elements in the set S.

Mathematical formulation

The counting scheme can be described by a mapping f from the set S and the subset of positive integer $\mathbb{N}$:



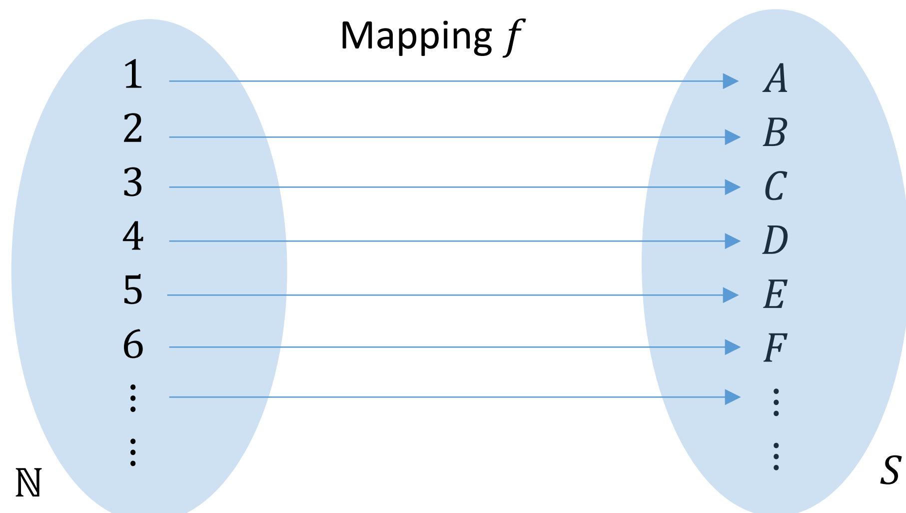
$f(1) = A$
$f(2) = B$
$f(3) = C$
$f(4) = D$
$f(5) = \#$
$f(6) = \$$
$f(7) =$ ☺
$f(8) =$ ☹

We say the set S is *countable* if there is such a counting scheme.

If the set S is finite in the sense that it contains *finitely many elements*, then one can construct the counting scheme (or mapping) easily so that <u>every finite set is always countable</u>.

If the set S contains infinitely many elements, then the set is countable (or more formally *countably* infinite) if we are able to construct a counting scheme such that one can count *all* elements in the set.

The mapping f is seen to be *bijective*.



Definition (Countability)

An *infinite set* S is said to be *countably infinite* if and only if there exists a bijection map f from $\mathbb{N}$ to S, where $\mathbb{N}$ is the set of positive integers.

A set S is said to be countable if and only if it is either finite or countably infinite.

Remarks

- By considering the opposite of the statement, then we say a set is *uncountable* if such bijection map does not exist.
- If an infinite set S is countably infinite, then there is *one-to-one correspondence* between the elements in S and the set of positive integers $\mathbb{N}$. In this case, one can express the set S as

$$S = \{a_1, a_2, a_3, \dots \dots\}.$$

Example 1

Prove that the set of integers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is countable.

☺Solution

The key step is to construct a bijection between $\mathbb{Z}$ and $\mathbb{N}$. One can consider the following mapping $f: \mathbb{N} \to \mathbb{Z}$ defined by

$$f(1) = 0, \qquad f(2n) = n, \qquad f(2n+1) = -n.$$

where $n = 1, 2, 3, \dots$.

$\mathbb{Z}$	$\cdots$	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	$\cdots$
$\mathbb{N}$	$\cdots$	9	7	5	3	1	2	4	6	8	$\cdots$

One can show that the mapping f is bijective. Hence, the set $\mathbb{Z}$ is countable.

Example 2

Prove that the set defined by $\mathbb{N}\times\mathbb{N}=\{(a,b)|a\in\mathbb{N},b\in\mathbb{N}\}$ is countable.

☺Solution

To construct the counting scheme, we list the elements in $\mathbb{N}\times\mathbb{N}$ as an *array* and consider the following strategy:

Step 1: Count the elements in the first diagonal ($m+n=2$)

Step 2: Count the elements in the second diagonal ($m+n=3$)

Step 3: Count the elements in the third diagonal ($m+n=4$)

$$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Then the corresponding mapping $f:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ can be described as

$$f\left(\sum_{k=2}^{m+n-1}(k-1)+n\right)=(m,n).$$

One can check that the mapping is bijective (left as exercise). So the set $\mathbb{N}\times\mathbb{N}$ is countable.

Example 3 (Uncountable set)

Show that the interval $(0,1)$ is *uncountable*.

☺Solution

We shall prove this using "*proof by contradiction*". Suppose that the set is countable and there is a bijection $f: \mathbb{N} \to (0,1)$, then we can list the elements in the following way:

$$
\begin{aligned}
f(1) &= 0.a_{11}a_{12}a_{13}a_{14}a_{15}a_{16} \dots \dots \\
f(2) &= 0.a_{21}a_{22}a_{23}a_{24}a_{25}a_{26} \dots \dots \\
f(3) &= 0.a_{31}a_{32}a_{33}a_{34}a_{35}a_{36} \dots \dots \\
f(4) &= 0.a_{41}a_{42}a_{43}a_{44}a_{45}a_{46} \dots \dots \\
f(5) &= 0.a_{51}a_{52}a_{53}a_{54}a_{55}a_{56} \dots \dots \\
f(6) &= 0.a_{61}a_{62}a_{63}a_{64}a_{65}a_{66} \dots \dots \\
&\vdots
\end{aligned}
$$

Here, a_{ij} denotes digit (0 ~9) of i^{th} element which not all of them are 0.

Next, we argue that this counting scheme *cannot* cover all elements in the set.

We consider an element $x \in [0,1)$

$$x = 0.b_1 b_2 b_3 \dots,$$

where the digits $b_1, b_2, \dots$ are chosen such that $b_k \neq a_{kk}$ (i.e. the k^{th} decimal place of x must be different from that of $f(k)$) for all $k = 1,2, \dots$.

More precisely, we set

$$b_k = \begin{cases} 1 & if \;\; a_{kk} \neq 1 \\ 2 & if \;\; a_{kk} = 1 \end{cases}.$$

Since $b_k \neq a_{kk}$ for $k \in \mathbb{N}$, so $x \neq f(k)$ for all $k = 1,2, \dots$. Hence, the mapping f is *not* surjective. It leads to contradiction. Thus, $[0,1)$ is uncountable.

Remark of Example

In this example, we can see the difference between countably infinite set and uncountable set.

- For countably infinite set, we are still able to find a way to count all elements (although we cannot finish within finite time)
- For uncountable set, there are too many elements which we cannot find a counting method that can count *every element* in the set.

Example 4

Show that the set $A = \{0,1\} \times \{0,1\} \times \{0,1\} \times \cdots$ is uncountable.

😊Solution

Suppose that A is countable and there is a bijection $f: \mathbb{N} \to A$ such that

$$f(1) = (a_{11}, a_{12}, a_{13}, a_{14}, \cdots\cdots)$$
$$f(2) = (a_{21}, a_{22}, a_{23}, a_{24}, \cdots\cdots)$$
$$f(3) = (a_{31}, a_{32}, a_{33}, a_{34}, \cdots\cdots)$$
$$f(4) = (a_{41}, a_{42}, a_{43}, a_{44}, \cdots\cdots)$$
$$\vdots$$

Next, we consider an element $x = (b_1, b_2, b_3, \cdots\cdots)$ which $b_k = \begin{cases} 1 & if \quad a_{kk} = 0 \\ 0 & if \quad a_{kk} = 1 \end{cases}$ for $k = 1,2,3, \ldots.$

Since $b_k \neq a_{kk}$ for all $k = 1,2, \ldots$, it implies that $x \neq f(k)$ for all $k \in \mathbb{N}$ and the function f is not surjective (and hence not bijective). So it leads to contradiction and the set A should be uncountable.

Some properties of countability

In this section, we shall explore some properties of countability which are useful in analyzing the countability of some complicated sets.

Theorem 1

We let A, B be two sets and let $f: A \to B$ be a function.

- **(Injection theorem)** Suppose that f is injective and B is countable, then A is countable.
- **(Surjection theorem)** Suppose that f is surjective and A is countable, then B is countable
- **(Bijection theorem)** Suppose that f is bijective, then A is countable if and only if B is countable.

This theorem suggests that one can examine the countability of a set A by considering another set B which is known to be countable or uncountable and constructing a mapping f (injective/surjective/bijective) between these two sets.

Example 5

Prove that the set of rational numbers $\mathbb{Q} = \left\{\frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\right\}$ is countable.

😊Solution

We consider a mapping $f: \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ as

$$f(m,n) = \frac{m}{n}.$$

One can show that f is surjective. Since for any $r \in \mathbb{Q}$, r can be written as $\frac{m}{n}$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. By taking $(m,n) \in \mathbb{Z} \times \mathbb{N}$, we get $f(m,n) = \frac{m}{n} = r$.

Since $\mathbb{Z}$ is countable, one can write the elements of $\mathbb{Z}$ as $\mathbb{Z} = \{a_1, a_2, a_3, \ldots\}$. Then one can use the similar method in Example 2 and show that $\mathbb{Z} \times \mathbb{N}$ is countable.

Therefore, we deduce from surjection theorem that $\mathbb{Q}$ is countable.

Remark of Example 5

Note that the mapping f is not injective. For example, $f(3,6) = f(2,4) = \frac{1}{2}$.

Example 6

We learnt from Example 3 that the open interval $(0,1)$ is uncountable. Using this fact and show that any open interval (a, b) with $a < b$ is also uncountable.

☺Solution

To see this, we consider a mapping $f\colon \underbrace{(0,1)}_{uncountable} \rightarrow \underbrace{(a,b)}_{"target"}$ defined by

$$f(x) = a + (b-a)x.$$

One can show that f is bijective:

- **(Injective)** For any $x_1, x_2 \in (0,1)$ such that $f(x_1) = f(x_2)$, we have
$$f(x_1) = f(x_2) \Rightarrow a + (b-a)x_1 = a + (b-a)x_2 \Rightarrow x_1 = x_2.$$
- **(Surjective)** For any $y \in (a,b)$, we take $x = \frac{y-a}{b-a} \in (0,1)$. One can see that
$$f(x) = a + (b-a)\left(\frac{y-a}{b-a}\right) = y.$$

  So f is injective and surjective and hence bijective.

Since $(0,1)$ is uncountable, it follows from bijection theorem that (a, b) is also uncountable.

☺Exercise: Show that $[a,b)$, $(a,b]$ and $[a,b]$ are uncountable.

Example 7

We let A_1 and A_2 be two non-empty sets which A_1 is uncountable, show that

$$A_1 \times A_2 = \{(a_1, a_2) | a_1 \in A_1 \ and \ a_2 \in A_2\}.$$

is also uncountable.

☺Solution

We let c be an element in A_2 and consider a mapping $f: A_1 \to A_1 \times A_2$ defined by

$$f(a_1) = (a_1, c).$$

One can see that f is injective since

$$f(a_1) = f(a_2) \Rightarrow (a_1, c) = (a_2, c) \Rightarrow a_1 = a_2.$$

(*On the other hand, f may not be surjective)

Since the domain A_1 is uncountable, it follows from the *contrapositive* of injection theorem that $A_1 \times A_2$ is also uncountable.

Proof of Theorem 1

Part 1: Proof of injection theorem

It is sufficient to consider the case when A is an infinite set since A must be countable if A is finite. Note that

- If A is an infinite set, then B must also be infinite set (why?).
- Since B is countable, then there is a bijective function $g: \mathbb{N} \to B$ and the set B can be expressed as $B = \{b_1, b_2, b_3, \dots\}$.
- Since $f: A \to B$ is injective, then there exists an inverse function $f^{-1}: f(A) \to A$. Since $f(A) \subseteq B$, we can write $f(A) = \{b_{n_1}, b_{n_2}, b_{n_3}, \dots\}$. Thus, $A = \{a_{n_1}, a_{n_2}, a_{n_3}, \dots.\}$, where $a_{n_k} = f^{-1}(b_{n_k})$.

Then, one can construct a bijective function $h: \mathbb{N} \to A$ as $h(k) = a_{n_k} = f^{-1}(b_{n_k})$.

Therefore, the set A is countable.

Part 2: Proof of surjection theorem

Similar to part 1, we just need to consider the case when B is an infinite set.

Since f is surjective, then for any $y \in B$, there exists $x \in A$ such that $f(x) = y$. So we get $B = f(A)$.

As A is countable, so A can be expressed as $A = \{a_1, a_2, a_3, \dots\}$ and

$$B = f(A) = \{f(a_1), f(a_2), f(a_3), \dots\}.$$

(*Minor technical step) By removing repeated elements, we have

$$B = \{f(a_{n_1}), f(a_{n_2}), f(a_{n_3}), \dots \dots\}.$$

Hence, we can construct a bijective mapping $h: \mathbb{N} \to B$ as $h(k) = f(a_{n_k})$.

So we can conclude that B is countable.

<u>Part 3: Proof of bijection theorem</u>

We divide the proof into two parts:

"$\Rightarrow$" part:

If A is countable, since f is surjective (as it is bijective), it follows from surjection theorem that B is countable.

"$\Leftarrow$" part:

If B is countable, since f is injective (as it is bijective), it follows from injection theorem that A is countable.

Theorem 2

1. **(Countable subset theorem)** We let A, B be two sets which $A \subseteq B$. If B is countable, then A is also countable.
2. **(Countable union theorem)** We let $A_1, A_2, A_3, \ldots$ be countable collection of sets which are countable, then $\bigcup_{n=1}^{\infty} A_n$ is countable.
3. We let $A_1, A_2, \ldots, A_n$ be n countable sets, then $A_1 \times A_2 \times \ldots \times A_n$ is also countable.
4. If A is uncountable and B is countable, then $A \backslash B$ is uncountable.

Remark of Theorem 2

- By taking the contrapositive of countable subset theorem, we have A is uncountable implies B is uncountable.
- The countable union theorem can be rephrased as follows:
  "We let S be a countable and suppose the set A_s is countable for *all* $s \in S$, then $\bigcup_{s\in S} A_s$ is countable."
- The 3rd statement is valid for *finite* product and it may *not* hold for infinite product (see Example 4 for a counter example)

Proof of Theorem 2

Proof of (1): It suffices to consider the case when A is an infinite set (since A is automatically countable if it is finite). Then B must also be infinite set as $B \supseteq A$. Since B is countably infinite, we write $B = \{b_1, b_2, b_3, \dots\}$. As $A \subseteq B$, then $A = \{b_{n_1}, b_{n_2}, b_{n_3}, \dots\}$. Then one can construct a bijection function $f: \mathbb{N} \to A$ as $f(k) = b_{n_k}$. So A is countable.

Proof of (2)

We will consider the case when all $A_k s$ are countably infinite, i.e. $A_k = \{a_{k1}, a_{k2}, a_{k3}, \dots\}$. We express the elements of $S = A_1 \cup A_2 \cup A_3 \cup \dots$ as an array, i.e.

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

k^{th} row contains the elements of A_k

Using similar method as in Example 2, we construct the mapping $f: \mathbb{N} \to S$ as

$$f\left(\sum_{k=1}^{m+n-1} (k-1) + n\right) = a_{mn}, \qquad m, n \in \mathbb{N}.$$

Provided that all $a_{ij}s$ are distinct (*see remark below), the mapping f will be bijective. Thus, S is countable.

*Remark: If there is common element, then this element will be counted more than once and the mapping f will only be surjective. But since $\mathbb{N}$ is countable, it follows from surjection theorem that S is countable.

__Proof of (3)__

We first argue that if A and B are countable, then $A \times B$ is also countable.

Assume that both A and B are infinite sets, then the two sets A and B can be expressed as $A = \{a_1, a_2, a_3, \dots \dots\}$ and $B = \{b_1, b_2, b_3, \dots \dots\}$.

To construct the bijective function from $\mathbb{N}$ to $A \times B$, we first express the elements in $A \times B$ as an array:

$$\begin{array}{ccccc}
(a_1, b_1) & (a_1, b_2) & (a_1, b_3) & (a_1, b_4) & \cdots \\
(a_2, b_1) & (a_2, b_2) & (a_2, b_3) & (a_2, b_4) & \cdots \\
(a_3, b_1) & (a_3, b_2) & (a_3, b_3) & (a_3, b_4) & \cdots \\
(a_4, b_1) & (a_4, b_2) & (a_4, b_3) & (a_4, b_4) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Using similar method as in Example 2, we construct the mapping $f: \mathbb{N} \to A \times B$ as

$$f\left(\sum_{k=1}^{m+n-1}(k-1)+n\right)=(a_m,b_n), \qquad m,n\in\mathbb{N}.$$

Since f is seen to be bijective (by construction), so $A \times B$ is countable.

The case for n countable sets $A_1, A_2, \dots, A_n$ can be proved using *induction* as follows:

- When $n = 1$, we have $A_1 \times \dots \times A_n = A_1$ which is clearly countable.
- Suppose that the statement is true for $n = k$ (i.e. $A_1 \times A_2 \times \dots \times A_k$ is countable). Since A_{k+1} is countable, it follows from the above result that the set

$$A_1 \times A_2 \times \dots \times A_{k+1} = \underbrace{(A_1 \times A_2 \times \cdots \times A_k)}_{countable} \times \underbrace{A_{k+1}}_{countable}$$

  is countable.

By induction, we conclude that $A_1 \times A_2 \times \dots \times A_n$ is countable for all $n \in \mathbb{N}$.

__Proof of (4)__

Suppose that $A \backslash B$ is countable. Note that

$$A = (A\backslash B) \cup (A \cap B).$$

Since $A \cap B \subseteq B$ is countable by countable subset theorem, it follows from countable union theorem that A is also countable and there is a contradiction.

In the context of examining countability, the theorem 2 suggests that one can check the countability of a set A by expressing the set as a combination (union, intersection, complement etc.) of some simple sets which their countabilities are known.

Example 8

Determine if a set defined by

$$A = \{x \in \mathbb{R} | x^6 + 6x - 3 \in \mathbb{Q}\}$$

is countable.

😊Solution

One can express the set A into

$$A = \{x \in \mathbb{R} | x^6 + 6x - 3 = y,\;\; y \in \mathbb{Q}\} = \bigcup_{y \in \mathbb{Q}} \underbrace{\{x \in \mathbb{R} | x^6 + 6x - 3 = y\}}_{B_y}.$$

For a fixed value of y, the equation $x^6 + 6x - 3 - y = 0$ has at most 6 real roots. So the set B_y has at most 6 elements and must be countable.

Since $\mathbb{Q}$ is also countable, it follows from countable union theorem that the set $A = \bigcup_{y \in \mathbb{Q}} B_y$.

Example 9

(a) Show that the set of *irrational number* is uncountable.

(b) We let $A_1, A_2, A_3, \dots$ be a countable collection of non-empty sets which A_1 is countable and A_2 is uncountable. Show that

(i) $\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \dots$ is countable

(ii) $\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \dots$ is uncountable

😊Solution

(a) Note that the set of irrational number can be expressed as $\mathbb{R}\backslash\mathbb{Q}$. Since $\mathbb{Q}$ is countable and $\mathbb{R} \supseteq (0,1)$ is uncountable (since $(0,1)$ is uncountable), it follows from the 4th statement that $\mathbb{R}\backslash\mathbb{Q}$ is uncountable.

(b) **(i)** Note that $\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap A_3 \cap \dots \subseteq A_1$ and A_1 is countable, it follows from countable subset theorem that $\bigcap_{n=1}^{\infty} A_n$ is countable.

(ii) Since $\bigcup_{n=1}^{\infty} A_n \supseteq A_2$ and A_2 is uncountable, it follows from the countable subset theorem (see remark) that $\bigcup_{n=1}^{\infty} A_n$ is uncountable.

Example 10

We say a real number x is an algebraic number if and only if it is the solution of some polynomial with integer coefficients. That is, x satisfies

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0,$$

where $a_n \in \mathbb{Z}\backslash\{0\}$, $a_0, a_1, \dots, a_{n-1} \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Otherwise, we say x is transcendental number if it is not algebraic number.

Show that there exists a transcendental number.

☺Solution

One can prove the existence by studying the set of transcendental number. We let A be the set of algebraic number and it can be expressed as

$$A = \left\{ x \in \mathbb{R} \middle| \begin{array}{c} a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0 \;\; for\ some\ n \\ for\ some\ n \in \mathbb{N}, (a_n, a_{n-1}, \dots, a_1, a_0) \in (\mathbb{Z}\backslash\{0\}) \times \mathbb{Z} \times \dots \mathbb{Z} \times \mathbb{Z} \end{array} \right\}$$

Then the set of transcendental number is seen to be $T = \mathbb{R}\backslash A$.

We first examine the set A. Note that the set A can be expressed as

$$A = \bigcup_{n\in\mathbb{N}} \underbrace{\left\{x \in \mathbb{R} \middle| \begin{array}{c} a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0 \ \ for\ some\ n \\ for\ some\ (a_n, a_{n-1}, \dots, a_1, a_0) \in (\mathbb{Z}\backslash\{0\}) \times \mathbb{Z} \times \dots \mathbb{Z} \times \mathbb{Z} \end{array}\right\}}_{A_n}$$

$$= \bigcup_{n\in\mathbb{N}} \underbrace{\left(\bigcup_{\substack{(a_n, a_{n-1} \dots, a_0) \\ \in (\mathbb{Z}\backslash\{0\}) \times \dots \mathbb{Z} \times \mathbb{Z}}} \{x \in \mathbb{R} | a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0\} \right)}_{A_n}$$

Note that $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0$ has at most n roots, then the set $\{x \in \mathbb{R} | a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0\}$ is finite (has at most n elements) and therefore countable.

Since both $\mathbb{Z}$ and $\mathbb{Z}\backslash\{0\} \subseteq \mathbb{Z}$ are countable, then $(\mathbb{Z}\backslash\{0\}) \times \dots \mathbb{Z} \times \mathbb{Z}$ is countable. It follows that countable union theorem that A_n is countable.

As $\mathbb{N}$ is countable, it implies that $A = \cup_{n\in\mathbb{N}} A_n$ is also countable.

Since $\mathbb{R}$ is uncountable, then it follows that $T = \mathbb{R}\backslash A$ is uncountable and is non-empty. So there exists transcendental number x.

Example 11

Determine if the set defined by

$$S = \{3x^3 + 2y + 2 |\, x \in \mathbb{R},\ \ y \in \mathbb{Q}\}$$

is countable.

😊Solution

One can observe that the set is likely to be uncountable since the number of choices of x is uncountable. To prove this claim, we prick an element $y_0 \in \mathbb{Q}$ and consider a subset

$$S^* = \{3x^3 + 2y_0 + 2 |\, x \in \mathbb{R}\} \subseteq S.$$

Next, we consider a mapping $f: \mathbb{R} \to S^*$ by

$$f(x) = 3x^3 + 2y_0 + 2.$$

Since $f(x)$ is increasing function, we can argue that f is injective. On the other hand, for any $z \in S^*$ and $z = 3x_0^3 + 2y_0 + 2$, we can take $x = x_0$ such that $f(x) = z$. So f is also surjective. Therefore, $f(x)$ is bijection. Hence, we deduce that

$$\mathbb{R} \text{ is uncountable} \quad \overset{\substack{bijection\\ theorem}}{\Longrightarrow} \quad S^* \text{ is uncountable} \quad \overset{\substack{countable\\ subset\ theorem}}{\Longrightarrow} \quad S \supseteq S^* \text{ is uncountable.}$$