MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #7

Real Number

Summary of terminology and theorems:

Definition: (Supremum & infimum)

A supremum (or least upper bound) of a non-empty subset S of **R** is the upper bound (M) such that $M \leq L$ for all upper bound L of S

A infimum (or greatest lower bound) of a non-empty subset S of **R** is the lower bound (P) such that $Q \leq P$ for all lower bound Q of S

Theorem 1: (Supremum Property)

If a set S has a supermum in $\mathbf{R}$, then for any $\varepsilon > 0$, there is a $x \in S$, such that

$$\sup S - \varepsilon < x < \sup S$$

Theorem 2: (Infimum Property)

If a set S has a n infinmum in $\mathbf{R}$, then for any $\varepsilon > 0$, there is a $x \in S$, such that

$$\inf S + \varepsilon > x > \inf S$$

Theorem 3: (Archimedean Property)

For any $x \in R$, then exist $n \in \mathbf{N}$, such that $n > x$

"Another Version"

For any $x, y \in R$ and $x > 0$, there exist $n \in \mathbf{N}$, such that $nx > y$

Theorem 4: (Density of Rational Number)

If $x < y$, there exist $\frac{m}{n} \in \mathbf{Q}$, such that $x < \frac{m}{n} < y$

Theorem 5: (Density of Irrational Number)

If $x < y$, there exist irrational $w \in \mathbf{R} \backslash \mathbf{Q}$, such that $x < w < y$

Theorem 6: (Supremum Limit Theorem)

$$\begin{matrix} \text{c is upper bound of S} \\ \text{There exists } \{w_n\} \in S, \text{such that } \lim_{n\to\infty} w_n = c \end{matrix} \leftrightarrow c = \sup S$$

Theorem 7: (Infimum Limit Theorem)

$$\begin{matrix} \text{c is lower bound of S} \\ \text{There exists } \{w_n\} \in S, \text{such that } \lim_{n\to\infty} w_n = c \end{matrix} \leftrightarrow c = \inf S$$

Given a set, to compute the supremum and infimum of the set,

(Step 1) Find out the upper bound and lower bound (if any) of the set

(Step 2) Show they are the desired supremum and infimum by applying the theorems

Example 1
Find the infimum and supremum of the set $$S = \{3x - y^2 : x \in \mathbf{Q} \cap (0,1), y \in [2,8)\}$$

Solution:

(Step 1: Finding upper bound and lower bound)

Note that $0 < x < 1$ and $2 \le y < 8$, so

Since $3x - y^2 > 3(0) - 8^2 = -64$, then the lower bound is -64

Since $3x - y^2 < 3(1) - 2^2 = -1$, then the upper bound is -1

(Step 2)

We would like to show -64 and -1 are infimum and supremum of set S respectively.

<u>Use Limit Theorem</u>

We first show the infimum is -64

Pick $\mathbf{x_n = \frac{1}{n}}$ and $\mathbf{y_n = 8 - \frac{1}{n}}$ for $\mathbf{n = 1, 2, 3, \ldots}$

Note $\frac{1}{n} \in \mathbf{Q}$, so $x_n \in \mathbf{Q} \cap (0,1)$ and $y_n \in [2,8)$

Thus $3x_n - y_n^2 \in S$

Now $\lim_{n\to\infty} 3x_n - y_n^2 = \lim_{n\to\infty} 3\left(\frac{1}{n}\right) - \left(8 - \frac{1}{n}\right)^2 = 0 - 8^2 = -64$

Hence by infimum limit theorem, we conclude $\inf S = -64$

Next we show the supremum is -1

Pick $\mathbf{x_n = 1 - \frac{1}{n}}$ and $\mathbf{y_n = 2}$ for $\mathbf{n = 1, 2, 3, \ldots}$

(Since y can be 2, so we just simply pick all y_n to be 2)

Note $1 - \frac{1}{n} \in Q$, then $x_n \in Q \cap (0,1)$

Now $\lim_{n\to\infty} 3x_n - y_n^2 = \lim_{n\to\infty} 3\left(1 - \frac{1}{n}\right) - (2)^2 = 3 - 4 = -1$

Hence by supremum limit theorem, we conclude $\sup S = -1$

(In the examples, the x_n and y_n are arbitrarily chosen, you may use different choices of x_n and y_n, but you need to make sure that the elements formed is still in the set S.

Example 2

Find the supremum and infimum of the set

$$S = \left\{x - \frac{1}{n} : x \in [0,1) \cap \mathbf{Q}, n \in \mathbf{N}\right\} \backslash [-1, \frac{1}{2})$$

(Step 1: Finding upper bound and lower bound)

Note that $0 \leq x < 1$ and $n \geq 1 \rightarrow 1 \geq \frac{1}{n} > 0$

Since $x - \frac{1}{n} < 1 - 0 = 1$, then 1 is the upper bound of S

Therefore $x - \frac{1}{n} > 0 - 1 = -1$, however the set does not include anything in

$[-1, \frac{1}{2})$, then one possible lower bound will be $\frac{1}{2}$

(Step 2)

We first show $\sup S = 1$

Pick $x_k = 1 - \frac{1}{4k}$ and $n = 4k$

Then our $w_k = \left(1 - \frac{1}{4k}\right) - \frac{1}{4k} = 1 - \frac{1}{2k}$,

since $w_k = 1 - \frac{1}{2k} \geq \frac{1}{2}$ for $k = 1,2,3, \ldots$, therefore $w_k \in S$

By simple computation, we get $\lim_{k \to \infty} w_k = 1$

Therefore by supermum limit theorem, we get $\sup S = 1$

We next show $\inf S = \frac{1}{2}$

Pick $x_k = 1$ and $n = 2$, then $w_k = \frac{1}{2}$ and clearly $w_k \in S$

and $\lim_{k \to \infty} w_k = 1/2$

Therefore by infimum limit theorem, $\inf S = \frac{1}{2}$

(Note: If the sup/inf can be achieved by some elements in S, then we can set w_k to be that number. If not, we need to construct a sequence of w_k so that w_k tends to our sup/inf.)

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☺Exercise 1

Compute the infimum and supremum of the set

$$S = \left\{x + y : x, y \in [\frac{1}{2}, 1)\right\} \backslash \{2 - \frac{1}{n} : n \in \mathbf{N}\}$$

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Difficult Situation:

In some cases, when you apply the limit theorem, it may be difficult for you to construct the sequence $\{w_k\}$. Then in this case, we need to use other theorems.

Example 3 (Practice Exercise 90m))   Determine the supremum and infimum of the set $$S = \left\{\frac{k}{n!} : k, n \in \mathbf{N}, \frac{k}{n!} < \sqrt{2}\right\}$$

(Step 1: Find the upper bound and lower bound)

Note that k,n are positive, therefore one possible lower bound is 0

Note since $\frac{k}{n!} < \sqrt{2}$ (by definition of set S), so a possible upper bound is $\sqrt{2}$.

(Step 2)

We first claim $\inf S = 0$

Fix $k = 1$, pick $w_n = \frac{1}{n!}$

Since $\frac{1}{n!} \leq \frac{1}{1!} = 1 < \sqrt{2}$, so $w_n \in S$

Because $\lim_{n\to\infty} \frac{1}{n!} = 0$, by infimum limit theorem, $\inf S = 0$

Next we claim $\sup S = \sqrt{2}$ ***(☹The difficult part)***

(Note: Since all elements in S are rational but $\sqrt{2}$ is irrational, so in this stage, we are unable to construct a sequence in S converges to $\sqrt{2}$.)

Here we will proof by contradiction,

Suppose $\sup S \neq 2$, then there exists $M < 2$, such that $\sup S = M$

By density of rational number,

There exists a rational $\frac{m}{n}$ such that $M < \frac{m}{n} < \sqrt{2}$

Note that $\frac{m}{n} = \frac{m(n-1)!}{n(n-1)!} = \frac{m(n-1)!}{n!}$, so $\frac{m}{n} \in S$, then M is not upper bound

Which leads to contradiction, so $\sup S = \sqrt{2}$

Example 4 (Unknown set)

Let B be the bounded subset of $\mathbf{R}$ which $\sup B = 3$ and $\inf B = -3$, Find the supremum and infimum of the set

$$S = \{x + \sin y : x \in B, y \in (\frac{\pi}{2}, \pi)$$

(Step 1: Find upper bound and lower bound)

Note that $x \in B \to -3 \leq x \leq 3$, since $y \in (\frac{\pi}{2}, \pi)$, therefore $\sin y$ is positive

<u>(Caution: $-3, 3$ MAY NOT be in B)</u>

Then $x + \sin y > -3 + 0 = -3$, so -3 is lower bound.

Next $x + \sin y < 3 + \sin\left(\frac{\pi}{2}\right) = 3 + 1 = 4$, therefore the upper bound is 4.

(Step 2)

We first claim $\inf S = -3$

First since $\inf B = -3$, therefore by infimum limit theorem(another direction), there exists $\{x_n\} \in B$ such that $\lim_{n\to\infty} x_n = -3$

Pick $y_n = \pi - \frac{1}{n}$, then $w_n = x_n + \sin y_n = x_n + \sin\left(\pi - \frac{1}{n}\right) \in S$

Now $\lim_{n\to\infty} w_n = \lim_{n\to\infty}(x_n + \sin\left(\pi - \frac{1}{n}\right)) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} \sin\left(\pi - \frac{1}{n}\right) = -3$

By infimum limit theorem, we conclude $\inf S = -3$

We then claim $\sup S = 4$.

Since $\sup B = 3$, therefore by supreme limit theorem(another direction), there exists $\{x_n\} \in B$ such that $\lim_{n\to\infty} x_n = 3$

Pick $y_n = \frac{\pi}{2} + \frac{1}{n}$ for $n = 1,2,3,\ldots$, $y_n \leq \frac{\pi}{2} + 1 < 2.57\ldots < \pi$

Then $w_n = x_n + \sin y_n = x_n + \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) \in S$

Now $\lim_{n\to\infty} w_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) = 3 + 1 = 4$

By supremum limit theorem, we get $\sup S = 4$

Example 5

Let A be a non-empty subset of **R** such that $\inf B = -1$ and $\sup B = 1$, find the supremum and infimum (if any) of the set

$$S = \{b^4 - 5b + 20 : b \in B\}$$

Solution:

(Step 1: Find the upper bound and lower bound)

Let $f(b) = b^4 - 5b + 20$

Then $f'(b) = 4b^3 - 5 < 0$ for $-1 \le b \le 1$

So f is decreasing, therefore $f(1) \le f(b) \le f(-1) \to 16 \le f(b) \le 26$

Therefore the upper bound and lower bound of S are 26 and 16 respectively.

(Step 2)

We first claim $\sup S = 24$

First since $\inf B = -1$, therefore by infimum limit theorem(another direction), there exists $\{b_n\} \in B$ such that $\lim_{n\to\infty} b_n = -1$

Pick $w_n = b_n^4 - 5b_n + 20 \in S$

Then $\lim_{n\to\infty} w_n = (-1)^4 - 5(-1) + 20 = 26$

Hence by supreme limit theorem, $\sup S = 26$

Since $\sup B = 1$, therefore by supreme limit theorem(another direction), there exists $\{b_n\} \in B$ such that $\lim_{n\to\infty} b_n = 1$

Pick $w_n = b_n^4 - 5b_n + 20 \in S$

Then $\lim_{n\to\infty} w_n = (1)^4 - 5(1) + 20 = 16$

Hence by infimum limit theorem, $\inf S = 16$

I will not post the solutions of the following exercises, try to do it by your own. You may submit your solution to me and I can give some comments to you.

☺Exercise 2

Find the supreme and infimum of the following sets

$$S = \left\{\frac{x-\pi}{x+\pi} : x \in Q \cap [0,\infty)\right\} \text{ and } T = \left\{\frac{\sqrt{2}}{m+n} + \frac{1}{k\sqrt{2}} : m, n, k \in N\right\}$$

☺Exercise 3

Find the supreme and infimum of the following set

$$S = \left\{x + y : x, y \in \left[\frac{1}{2}, 1\right)\right\} \backslash \{2 - \frac{1}{n} : n \in \mathbf{N}\}$$

☺Exercise 4 (2003 Final)

Let A be a nonempty subset of R such that $\inf A = 0$ and $\sup A = 1$. Determine the infimum and supremum of $S = \{a^3 - 4a + 1 : a \in A\}$

☺Exercise 5 (2005 Fall Exam)

Let A be a non-empty bounded subset of R such that $\sup A = 1$ and $\inf A = 0$. Find the supremum and infimum (if any) of the set $S = \{x - y\sqrt{2} : x \in A, y \in [-2,2)\}$

☺Exercise 6 (2005 Spring Final)

Determine the supremum of

$$S = \bigcup_{n=1}^{\infty} \{\frac{1}{x} + \frac{1}{n\sqrt{2}} : x \in (2,3] \backslash \mathbf{Q}\}$$