MATH2033 Mathematical Analysis (2021 Spring)
Suggested Solution of Midterm Examination

Problem 1 (30 marks)

(a) **(i)** State the definition of limits of sequence (i.e. $\lim_{n\to\infty} x_n = L \in \mathbb{R}$)

(ii) State the definition of Cauchy sequence.

(b) Using the definition of limits, prove that $\lim_{n\to\infty} \frac{1}{n^4-4n+10} = 0$.

(c) We let $\{x_n\}$ be a sequence of real number such that $\lim_{n\to\infty} x_n = a$, where a is a positive real number. Using the definition of limits, show that $\lim_{n\to\infty} \sqrt[3]{x_n + a} = \sqrt[3]{2a}$.

😊Solution

(a) (i) We say $\lim_{n\to\infty} x_n = L$ if and only if for any $\varepsilon > 0$, there exists a positive integer $K \in \mathbb{N}$ such that $|x_n - L| < \varepsilon$ for any $n \geq K$.

(ii) We say $\{x_n\}$ is Cauchy sequence if and only if for any $\varepsilon > 0$, there exists a positive integer $K \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for any $m, n \geq K$.

(b) We first note that for any $n \geq 2$

$$n^4 - 4n + 10 > n^4 - 4n = n\underbrace{(n^3 - 4)}_{>1 \ for\ n\geq 2} > n.$$

For any $\varepsilon > 0$, we take $K = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1 > \frac{1}{\varepsilon}$. Then for any $n \geq K$, we have

$$\left|\frac{1}{n^4 - 4n + 10} - 0\right| \overset{see\ remark}{\widehat{=}} \frac{1}{n^4 - 4n + 10} < \frac{1}{n} \leq \frac{1}{K} < \varepsilon.$$

So it follows from definition of limits (see (a)(i)) that $\lim_{n\to\infty} \frac{1}{n^4-4n+10} = 0$.

(*Remark: Note that for $n = 1$, $n^4 - 4n + 10 = 7 > 0$ and for $n \geq 2$, $n^4 - 4n + 10 > n > 0$. So $n^4 - 4n + 10$ is always positive.)

(c) We note that $\lim_{n\to\infty} x_n = a$.

- By picking $\varepsilon_0 = a$, there exists $K_1 \in \mathbb{N}$ such that for any $n \geq K_1$,
$$|x_n - a| < \varepsilon_0 = a \Leftrightarrow -a < x_n - a < a \Leftrightarrow 0 < x_n < 2a.$$
- On the other hand, for any $\varepsilon > 0$, there exists $K_2 \in \mathbb{N}$ such that for any $n \geq K_2$,
$$|x_n - a| < a^{\frac{2}{3}}\left(1 + \sqrt[3]{2} + \left(\sqrt[3]{2a}\right)^2\right)\varepsilon \ldots (*).$$

By choose $K = \max(K_1, K_2)$, then for any $n \geq K$, we have

(*Note that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$)

$$\left|\sqrt[3]{x_n + a} - \sqrt[3]{2a}\right| = \frac{\left|\left(\sqrt[3]{x_n + a}\right)^3 - \left(\sqrt[3]{2a}\right)^3\right|}{\left|\left(\sqrt[3]{x_n + a}\right)^2 + \left(\sqrt[3]{x_n + a}\right)\left(\sqrt[3]{2a}\right) + \left(\sqrt[3]{2a}\right)^2\right|}$$

$$\overset{\substack{x_n>0 \\ a>0}}{\widehat{<}} \frac{|x_n - a|}{\left|\left(\sqrt[3]{0 + a}\right)^2 + \left(\sqrt[3]{0 + a}\right)\left(\sqrt[3]{2a}\right) + \left(\sqrt[3]{2a}\right)^2\right|} = \frac{|x_n - a|}{a^{\frac{2}{3}}\left(1 + \sqrt[3]{2} + \left(\sqrt[3]{2a}\right)^2\right)} \overset{from\ (*)}{\widehat{<}} \varepsilon$$

So it follows from definition of limits (see (a)(i)) that $\lim_{n\to\infty} \sqrt[3]{x_n + a} = \sqrt[3]{2a}$.

Problem 2 (20 marks)

We consider a sequence of real number $\{x_n\}$ defined by

$$x_1 = 2 \quad and \quad x_{n+1} = 1 + \frac{x_n^2}{1 + x_n^2} \quad for\, n \in \mathbb{N}.$$

(a) Show that the sequence $\{x_n\}$ is monotone.

(b) Hence, show that the sequence $\{x_n\}$ converges and find its limits.

😊Solution

(a) We shall prove that

(1) $1 \leq x_n \leq 2$ for any $n \in \mathbb{N}$

(2) $\{x_n\}$ is decreasing. That is, $x_{n+1} \leq x_n$ for any $n \in \mathbb{N}$

This can be done using mathematical induction.

Proof of (1)

When $n = 1$, it is clear that $1 < x_1 = 2 \leq 2$.

Assuming that $1 \leq x_k \leq 2$ for some $k \in \mathbb{N}$, then for $n = k + 1$,

$$1 < \frac{3}{2} = 2 - \frac{1}{1 + 1^2} < x_{k+1} = 1 + \frac{x_k^2}{1 + x_k^2} = 2 - \frac{1}{1 + x_k^2} < 2 - \frac{1}{1 + 2^2} = \frac{9}{5} < 2$$

(3) So $1 \leq x_{k+1} \leq 2$. It follows from mathematical induction that $1 \leq x_n \leq 2$ for any $n \in \mathbb{N}$

Proof of (2)

When $n = 1$, we have $x_2 = 1 + \frac{x_1^2}{1+x_1^2} = \frac{9}{5} < 2 = x_1$.

Assuming $x_{k+1} \leq x_k$ for some $k \in \mathbb{N}$, then for $n = k + 1$, we consider

$$x_{k+2} - x_{k+1} = \left(1 + \frac{x_{k+1}^2}{1 + x_{k+1}^2}\right) - \left(1 + \frac{x_k^2}{1 + x_k^2}\right) = \frac{x_{k+1}^2(1 + x_k^2) - x_k^2(1 + x_{k+1}^2)}{(1 + x_{k+1}^2)(1 + x_k^2)}$$

$$= \frac{x_{k+1}^2 - x_k^2}{(1 + x_{k+1}^2)(1 + x_k^2)} = \frac{\overbrace{(x_{k+1} - x_k)}^{\leq 0\ (given(}\overbrace{(x_{k+1} + x_k)}^{\geq 1+1=2>0}}{\underbrace{(1 + x_{k+1}^2)}_{\geq 1>0}\underbrace{(1 + x_k^2)}_{\geq 1>0}} \leq 0$$

So we have $x_{k+2} \leq x_{k+1}$. It follows from mathematical induction that $x_{n+1} \leq x_n$ for any $n \in \mathbb{N}$ and $\{x_n\}$ is decreasing (and thus monotone).

(b) Since the sequence $\{x_n\}$ is decreasing and bounded from below by 1 (i.e. $x_n \geq 1$), it follows from monotone sequence theorem that $\{x_n\}$ converges.

We let $\lim\limits_{n\to\infty} x_n = x$. By taking limits on the recursive formula $x_{n+1} = 1 + \frac{x_n^2}{1+x_n^2}$, the limit x satisfies

$$x = 1 + \frac{x^2}{1 + x^2} = x^3 - 2x^2 + x - 1 = 0.$$

Problem 3 (22 marks)

Recall that the cubic root of 2 (denoted by $\sqrt[3]{2}$) is defined as a real number x satisfying

$$x^3 = 2.$$

In this problem, you are asked to prove the existence of the cubic root $\sqrt[3]{2}$. To do so, we consider the set $S = \{r \in \mathbb{Q} | \, r > 0 \ \ and \ \ r^3 < 2\}$.

(a) Prove that $x = \sup S$ exists.

(b) Show that the supremum x satisfy $x^3 = 2$.

😊Solution

(a) Since $2^3 > 2 > r^3$, we argue that S is bounded from above by 2.

Suppose that 2 is not upper bound and there is $r \in S$ such that $r > 2$, it follows that (order axiom) $r^3 > 2^3 = 8$ and $r \notin S$. This leads to contradiction.

So 2 is upper bound of S and $\sup S$ exists by completeness axiom.

(b) We shall prove it by contradiction. Suppose that $x^3 = (\sup S)^3 \neq 2$, we consider two cases:

- <u>Case 1: $x^3 < 2$</u>

  We first argue that there exists a small $0 < \varepsilon < 1$ such that $(x+\varepsilon)^3 < 2$.

  - Since $x = \sup S \leq 2$ (recall that $\sup S$ is least upper bound and 2 is upper bound), we take $\varepsilon \in (0,1)$ and get

    $$(x+\varepsilon)^3 = x^3 + 3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3 \overset{x\leq 2}{\leq} x^3 + 12\varepsilon + 6\varepsilon^2 + \varepsilon^3$$

    $$\overset{\substack{\varepsilon<1 \\ \varepsilon^k \leq \varepsilon \ for\ k\in\mathbb{N}}}{<} x^3 + 12\varepsilon + 6\varepsilon + \varepsilon = x^3 + 19\varepsilon.$$

  - By taking $0 < \varepsilon < \frac{2-x^3}{19}$ (the existence is guaranteed by density of rational number), then we have $(x+\varepsilon)^3 < x^3 + 19\varepsilon < 2$.

  By choosing ε such that $(x+\varepsilon)^3 < 2$, it follows from density of rational number that there exists $r \in \mathbb{Q}$ such that

  $$x < r < x + \varepsilon \Leftrightarrow x^3 < r^3 < (x+\varepsilon)^3 < 2.$$

  So $r \in S$ and $x = \sup S$ is not upper bound. It leads to contradiction.

- <u>Case 2: $x^3 > 2$</u>

  We first argue that there exists a small $0 < \varepsilon < 1$ such that $(x-\varepsilon)^3 > 2$.

  - Since $1^3 = 1 < 2$, so $1 \in S$, then $x = \sup S \geq 1$. We take $1 > \varepsilon > 0$ and get

    $$(x-\varepsilon)^3 = x^3 - 3x^2\varepsilon + \underbrace{3x\varepsilon^2}_{>0} - \varepsilon^3 > x^3 - 12\varepsilon - \varepsilon = x^3 - 13\varepsilon.$$

  - By taking $0 < \varepsilon < \frac{x^3-2}{13}$ (the existence is guaranteed by density of rational number), then we have $(x-\varepsilon)^3 > x^3 - 13\varepsilon > 2$.

  By choosing ε such that $(x-\varepsilon)^3 > 2$, it follows from supremum property that there exists $r \in S$ such that

  $$r > x - \varepsilon \Leftrightarrow r^3 > (x-\varepsilon)^3 > 2.$$

  So $r \notin S$. It leads to contradiction.

Therefore, we conclude that $x^3 = 2$.

*Special Note: Since the problem aims to prove the existence of $\sqrt[3]{2}$, so one cannot use $\sqrt[3]{2}$ in the argument (unless you have proved the existence successfully). For example

$$r^3 < 2 \Leftrightarrow r < \sqrt[3]{2}.$$

Problem 4 (28 marks)

(a) **(6 marks)** We let $[x]$ denotes the greatest integer less than or equal to x. For example, $[7.2] = 7$, $[7.9] = 7$, $[7] = 7$. We consider the set

$$T = \left\{\frac{[x]^2}{y} \mid x \in \mathbb{R}\backslash\mathbb{Q} \ and \ y \in \mathbb{Z}\backslash\{0\}\right\}.$$

Determine if the set T is countable.

(b) We let m be a real number. We consider a set S which is the collection of all sequences of integers $\{x_n\}$ that converges to m. That is,

$$S = \left\{\{x_n\} \middle| x_n \in \mathbb{Z} \ for\ all\ n \in \mathbb{N} \ and \ \lim_{n\to\infty} x_n = m\right\}.$$

(i) **(10 marks)** If m is *not* integer, show that S must be an empty set.

(ii) **(12 marks)** If m is an integer, show that S is countable.

(😊Hint: If the sequence $\{x_n\}$ converges, what will happen to x_n when n is large?)

😊Solution

(a) For any $s = \frac{[x]^2}{y} \in T$, since $[x]$ is an integer and y is non-zero integer, it follows $\frac{[x]^2}{y}$ is rational number so that $s \in \mathbb{Q}$. Thus, we have

$$T \subseteq \mathbb{Q}.$$

Since $\mathbb{Q}$ is countable, it follows from countable subset theorem that T is countable.

(b) **(i)** We can prove this by contradiction. Suppose that there exists a sequence of integers $\{x_n\}$ such that $\lim_{n\to\infty} x_n = m \notin \mathbb{Z}$, we note that

- Since m is not integer, then we have $[m] < m < [m] + 1$. (For example, we take $m = 3.4$. Then we have $[m] = 3 < m = 3.4 < [m] + 1 = 4$.
- By picking $\varepsilon = \min\left\{\underbrace{m - [m]}_{\substack{distance \\ between \\ m \ and \ [m]}}, \underbrace{([m] + 1) - m}_{\substack{distance \\ between \\ m \ and \ [m]+1}}\right\} > 0$, then there exists $K \in \mathbb{N}$ such that for any $n \geq K$, we have
  $|x_n - m| < \varepsilon$
  $\Leftrightarrow [m] = m - (m - [m]) = m - \varepsilon < x_n < m + \varepsilon < m + ([m] + 1) - m = [m] + 1$
  $\Leftrightarrow [m] < x_n < [m] + 1.$

Since there is no integer between $[m]$ and $[m] + 1$, it follows that x_n is not integer for $n \geq K$. It leads to contradiction. Thus, there is no such convergent sequence and the set S is empty.

(ii) Suppose that $\lim_{n\to\infty} x_n = m \in \mathbb{Z}$. We take $\varepsilon = 0.5$, then there exists $K \in \mathbb{N}$ such that

$$|x_n - m| < 0.5 \Leftrightarrow m - 0.5 < x_n < m + 0.5 \quad for \ \ n \geq K.$$

Since m is the only integer in the interval $(m - 0.5, m + 0.5)$ and $x_n \in \mathbb{Z}$, it follows that

$$x_n = m \ \ for \ n \geq K$$

Therefore, the set S can be expressed as

$$S = \left\{\{x_n\} \middle| \ x_1, \dots, x_{K-1} \in \mathbb{Z} \ \ and \ \ x_K = x_{K+1} = x_{K+2} \dots = m \quad for \ \ some \ K \in \mathbb{N}\right\}.$$

For any $K \in \mathbb{N}$, we define

$$\begin{aligned} S_K &= \left\{\{x_n\} \middle| \ x_1, \dots, x_{K-1} \in \mathbb{Z} \ \ and \ \ x_K = x_{K+1} = x_{K+2} \dots = m\right\} \\ &= \bigcup_{(x_1, x_2, \dots, x_{K-1}) \in \mathbb{Z}\times\mathbb{Z}\times\dots\times\mathbb{Z}} \{x_1, x_2, \dots, x_{K-1}, m, m, m, \dots\}. \end{aligned}$$

Then $S = \bigcup_{K\in\mathbb{N}} S_K$.

- Since $\{x_1, x_2, \dots, x_{K-1}, m, m, m, \dots\}$ is a single element, so it is finite and countable.
- Since K is finite and $\mathbb{Z}$ is countable, then the product $\mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z}$ is countable. It follows from countable union theorem that

  $$S_K = \bigcup_{(x_1, x_2, \dots, x_{K-1}) \in \mathbb{Z}\times\mathbb{Z}\times\dots\times\mathbb{Z}} \{x_1, x_2, \dots, x_{K-1}, m, m, m, \dots\}$$

  is countable.
- As $\mathbb{N}$ is countable, it follows from countable union theorem that $S = \bigcup_{K\in\mathbb{N}} S_K$ is also countable.