Lecture 13 21-03-2019

Review:

1. Cauchy's theorem: $\{x_n\}$ converges $\Leftrightarrow$ $\{x_n\}$ is Cauchy

   A metric space is call Cauchy complete if every Cauchy sequence converges. For example: $\mathbb{R}$ is Cauchy complete.

   The space of Cauchy sequence in $\mathbb{Q}$ is Cauchy complete.

2. Limit of functions.

   $f: S \to \mathbb{R}$, x_0 is a cluster point of S.

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \left(\forall \varepsilon > 0, \exists \delta > 0, 0 < |x - x_0| < \delta, x \in S \Rightarrow |f(x) - L| < \varepsilon\right)$$

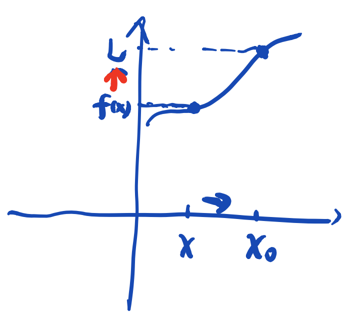
3. Sequential limit theorem: $\lim_{x \to x_0} f(x) = L \Leftrightarrow$

   $\lim_{n \to \infty} f(x_n) = L$ for any $\{x_n\}$ with $x_n \to x_0$ in $S \setminus \{x_0\}$

Sequential limit theorem (S.L.T)

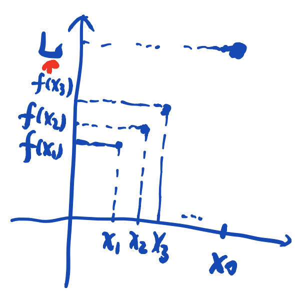
THM: Let $f: S \to \mathbb{R}$ be a function and x_0 be an accumulation point of S. Then

$$\lim_{x \to x_0} f(x) = L \iff \forall x_n \to x_0 \text{ in } S \setminus \{x_0\}, \quad \lim_{n \to \infty} f(x_n) = L.$$



$\Rightarrow$

$\Leftarrow$



Proof: $\Leftarrow$. Given that $\forall x_n \to x_0$ in $S \setminus \{x_0\}$, $\lim_{n \to \infty} f(x_n) = L$, we show that $\lim_{x \to x_0} f(x) = L$. We prove by contradiction. Assume that $\sim\left(\lim_{x \to x_0} f(x) = L\right)$

$$= \sim\left(\forall \varepsilon > 0, \exists \delta > 0, \forall x \in S, 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon\right)$$

$$= \exists \varepsilon_0 > 0, \forall \delta > 0, \sim\left(\forall x \in S, 0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon\right)$$

$= \exists\, \varepsilon_0 > 0,\ \forall\, \delta > 0,\ \exists\, x \in S,\ 0 < |x - x_0| < \delta$ and $|f(x) - L| \geq \varepsilon_0$

For $\delta = 1$, $\exists\, x_1 \in S$, $0 < |x_1 - x_0| < 1$ and $|f(x_1) - L| \geq \varepsilon_0$

$\delta = \frac{1}{2}$, $\exists\, x_2 \in S$, $0 < |x_2 - x_0| < \frac{1}{2}$ and $|f(x_2) - L| \geq \varepsilon_0$

$\vdots$

$\delta = \frac{1}{n}$, $\exists\, x_n \in S$, $0 < |x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon_0$

Consider the sequence $\{x_n\}$. we have $\lim\limits_{n \to \infty} x_n = x_0$

Since $0 < |x_n - x_0| < \frac{1}{n}$. Therefore, $\lim\limits_{n \to \infty} f(x_n) = L$.

On the other hand, we have $|f(x_n) - L| \geq \varepsilon_0$.

letting $n \to \infty$, by using the limit inequality, we get

$\lim\limits_{n \to \infty} |f(x_n) - L| = |L - L| = 0 \geq \varepsilon_0$, which

Contradicts to the fact that $\varepsilon_0 > 0$.

This contradiction shows that $\lim\limits_{x \to x_0} f(x) = L$

Example : ① $f(x) = \sqrt{x}$. $\forall\, x_0 \geq 0$, $\lim_{x \to x_0} \sqrt{x} = \sqrt{x_0}$

$$\overset{S.L.T}{\Longrightarrow} \quad \lim_{n \to \infty} \sqrt{x_n} = \sqrt{x_0} \quad \forall\; x_n \to x_0$$

② $\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e$

$$\overset{SLT}{\Longrightarrow} \quad \lim (1+\tfrac{1}{n})^n = e$$

by taking $x_n = \frac{1}{n}$.

Actually for all $x_n \to 0$, we have

$$\lim (1+x_n)^{\frac{1}{x_n}} = e.$$

For instance, $x_n = \frac{1}{n^3}$

$$\lim_{n \to \infty} (1+\tfrac{1}{n^3})^{n^3} = e$$

THM: Let $f, g : S \to \mathbb{R}$. Then

$$\lim_{x \to x_0} f(x) = L_1 \qquad \lim_{x \to x_0} g(x) = L_2$$

$$\Rightarrow \quad \lim_{x \to x_0} f(x) + g(x) = L_1 + L_2$$

proof (1). $\forall\ x_n \to x_0$ in $S \setminus \{x_0\}$.

By S.L.T, $\lim_{n \to \infty} f(x_n) = L_1$, $\lim_{n \to \infty} g(x_n) = L_2$

By the sum of limit of sequence, we get

$$\lim_{n \to \infty} f(x_n) + g(x_n) = L_1 + L_2$$

By S.L.T again, $\lim_{x \to x_0} f(x) + g(x) = L_1 + L_2$

proof (2).: direct proof using definition (similar to the sum rule for sequences.) Exercise.

THM: ① $\lim_{x\to x_0} f(x) = L_1$, $\lim_{x\to x_0} g(x) = L_2$, then

$$\lim_{x\to x_0} f(x)\cdot g(x) = L_1\cdot L_2 \qquad \lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$$ (provided $g(x)\neq 0$ and $L_2 \neq 0$)

② If $f(x) \le g(x) \le h(x)$ for all $x\in S$,

and $\lim_{x\to x_0} f(x) = L = \lim_{x\to x_0} h(x)$, then $\lim_{x\to x_0} g(x) = L$.

③ If $f(x) \geq 0$ for all $x \in S$, and $\lim_{x\to x_0} f(x) = L$, then $L \geq 0$

④ If $f(x) \le g(x) \le h(x)$. for all $x\in S$, and

$\lim_{x\to x_0} f(x) = L_1$, $\lim_{x\to x_0} g(x) = L_2$, $\lim_{x\to x_0} h(x) = L_3$

then $L_1 \le L_2 \le L_3$

Proof : Similar to the previous one . [Exercise]

One sided limit

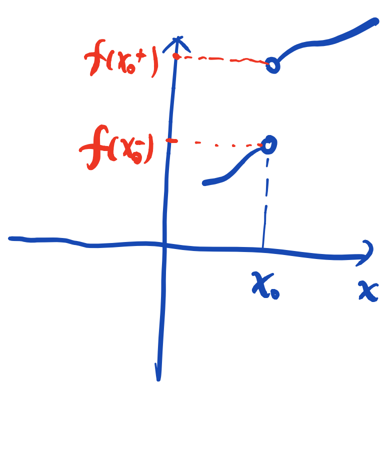
Def : For $f: S \longrightarrow \mathbb{R}$ and $x_0 \in S$, the

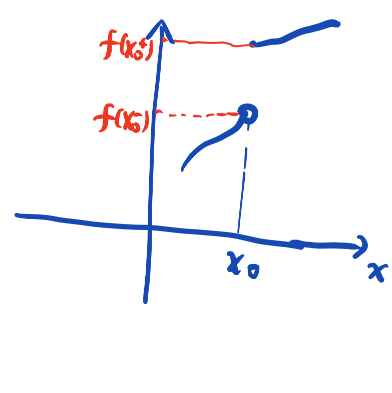
left hand limit of f at x_0 is

$$\boxed{f(x_0^-)} = \lim_{x \to x_0^-} f(x) = \lim_{\substack{x \to x_0 \\ x < x_0}} f(x)$$

right hand limit of f at x_0 is

$$\boxed{f(x_0^+)} = \lim_{x \to x_0^+} f(x) = \lim_{\substack{x \to x_0 \\ x > x_0}} f(x)$$





Limit and Sided limit

Theorem : let $f : S \to \mathbb{R}$. Then

$$\lim_{x \to x_0} f(x) = L \iff f(x_0^+) = f(x_0^-) = L$$

proof:

$\lim_{x \to x_0} f(x) = L$

$\Updownarrow$

$\forall \varepsilon > 0, \ \exists \delta > 0,$

$\forall \ x \in S,$

$0 < |x - x_0| < \delta$

$\Rightarrow |f(x) - L| < \varepsilon$

$f(x_0^-) = L$

$\Updownarrow$

$\forall \varepsilon > 0, \ \exists \delta_1 > 0$

$\forall \ x \in S,$

$x_0 - \delta_1 < x < x_0$

$\Rightarrow |f(x) - L| < \varepsilon$

$f(x_0^+) = L$

$\Updownarrow$

$\forall \varepsilon > 0, \ \exists \delta_2 > 0$

$\forall \ x \in S,$

$x_0 < x < x_0 + \delta_2$

$\Rightarrow |f(x) - L| < \varepsilon$

$\Rightarrow$ let $\delta_1 = \delta$, $\delta_2 = \delta$

$\Leftarrow$ let $\delta = \min\{\delta_1, \delta_2\}$

Definition: let $f: S \to \mathbb{R}$, then

f is

1. increasing
2. decreasing
3. strictly increasing
4. strictly decreasing

iff $\forall x, y \in S$, $x < y \Rightarrow$

1. $f(x) \leq f(y)$
2. $f(x) \geq f(y)$
3. $f(x) < f(y)$
4. $f(x) > f(y)$

f is

5. monotone iff f is increasing or decreasing
6. strictly monotone iff f is strictly increasing or strictly decreasing
7. bounded above iff $\{f(x) : x \in S\}$ is bounded above
8. bounded below iff $\{f(x) : x \in S\}$ is bounded below
9. bounded iff $\{f(x) : x \in S\}$ is bounded

Monotone function theorem

THM: ① Let $f:(a,b)\to\mathbb{R}$ be increasing. Then $\forall\, x_0\in(a,b)$

$$f(x_0^-) = \sup\{f(x) : a<x<x_0\}$$

$$f(x_0^+) = \inf\{f(x) : x_0<x<b\}$$

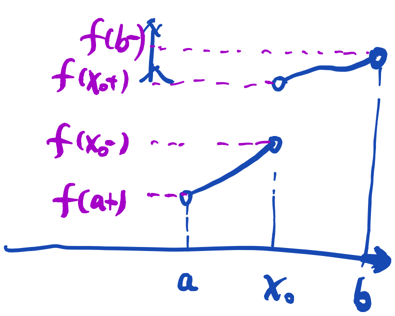
and $f(x_0-) \le f(x_0) \le f(x_0+)$

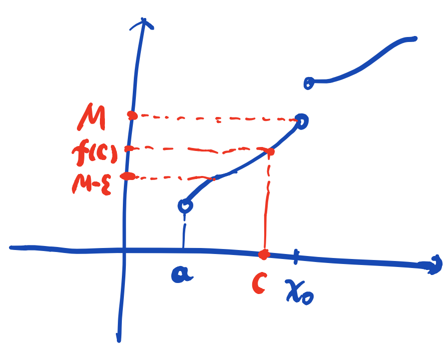
Moreover, if f is bounded below, then

$$f(a^+) = \inf\{f(x) : a<x<b\}$$

if f is bounded above, then

$$f(b^-) = \sup\{f(x) : a<x<b\}$$





proof: ① Step 1. Since $f(x) \le f(x_0)$ $\forall$ $a < x < x_0$

$f(x_0)$ is an upper bound of $\{f(x) : a < x < x_0\}$.

By the completeness axiom, $\sup\{f(x) : a < x < x_0\}$ exists.

Step 2. Let $M = \sup\{f(x) : a < x < x_0\}$. We show that

$f(x_0^-) = M$. $\forall \varepsilon > 0$, by the supremum property,

$\exists$ $c \in (a, x_0)$ s.t $M - \varepsilon < f(c) \le M$. f is $\nearrow$

$\Rightarrow$ $\forall$ $x \in (c, x_0)$, $M - \varepsilon < f(c) \le f(x) \le M$.

let $\delta = x_0 - c$, then $\forall x \in (x_0 - \delta, x_0) \Rightarrow$

$|f(x) - M| < \varepsilon$. Therefore $\lim\limits_{x \to x_0^-} f(x_0) = M$.

the other parts: Exercise.

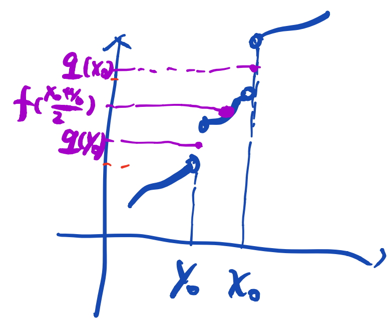
Continuity of monotone functions

THM : Let $f : (a,b) \to \mathbb{R}$ be increasing. Then

f has countable many discontinous points on (a,b)

that is $J = \{ x_0 : x_0 \in (a,b),\ f(x_0^-) \neq f(x_0^+) \}$

is countable.



proof : $\forall\ x_0 \in J$,

$f(x_0^-) < f(x_0^+)$

$\Rightarrow \exists\ q(x_0) \in \mathbb{Q}$ s.t $f(x_0^-) < q(x_0) < f(x_0^+)$

by the density of rational numbers.

In this way, we obtain a map $q : J \to \mathbb{Q}$

We show that q is injective. $\forall\ y_0 \in J$, $x_0 \in J$, $y_0 < x_0$

$$q(y_0) < f(y_0^+) \le f\left(\frac{y_0+x_0}{2}\right) \le f(x_0^-) < q(x_0)$$

So q is injective. By injection theorem, J is countable

Infinite limits

Def : $\{x_n\}$ diverges to $+\infty$ or

$\lim x_n = +\infty$, iff $\forall M \in \mathbb{R}$, $\exists K \in \mathbb{N}$

s.t $n \geq K \Rightarrow x_n > M$.

$\{x_n\}$ diverges to $-\infty$ or

$\lim x_n = -\infty$, iff $\forall m \in \mathbb{R}$, $\exists K \in \mathbb{N}$

s.t $n \geq K \Rightarrow x_n < m$

Example : $x_n = n$. $\lim\limits_{n\to\infty} x_n = \infty$

$x_n = -n^2$, $\lim\limits_{n\to\infty} x_n = -\infty$

Def: Let $f: S \to \mathbb{R}$.

f diverges to $+\infty$ (or $-\infty$) as $x \to x_0$,

denoted by $\lim_{x \to x_0} f(x) = +\infty$ (or $\lim_{x \to x_0} f(x) = -\infty$)

iff $\forall\ C \in \mathbb{R}$, $\exists\ \delta > 0$ such that

$\forall\ x \in S$, $0 < |x - x_0| < \delta \Rightarrow$

$f(x) > C$ (or $f(x) < C$).

Example: $f: (0, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$.

then $\lim_{x \to 0} f(x) = +\infty$

$g: (0, \infty) \to \mathbb{R}$, $g(x) = -\frac{1}{x^2}$

$\lim_{x \to 0} g(x) = -\infty$

Limit at Infinity

Let $f: S \to \mathbb{R}$ be a function.

Let $+\infty$ (or $-\infty$) be an accumulation point of S. i.e $\exists$ sequence $\{x_n\} \subset S$ s.t

$\lim_{n\to\infty} x_n = +\infty$ (or $\lim x_n = -\infty$).

We call $L \in \mathbb{R}$ the limit of f at $+\infty$ (or $-\infty$)

denoted by $\lim_{x\to+\infty} f(x) = L$ (or $\lim_{x\to-\infty} f(x) = L$)

iff $\forall \varepsilon > 0$, $\exists$ $K \in \mathbb{R}$ s.t $x \geq K \Rightarrow |f(x) - L| < \varepsilon$

(or $x \leq K \Rightarrow |f(x) - L| < \varepsilon$)

Example: $f(x) = \frac{1}{x}$, $\lim_{x\to+\infty} f(x) = 0$.

Infinity limit at infinity

Let $f: S \to R$.

$\lim_{x\to\infty} f(x) = +\infty$ iff $\forall M \in \mathbb{R}, \exists r \in \mathbb{R}$

s.t $\forall x \in S, x > r \Rightarrow f(x) > M$.

$\lim_{x\to\infty} f(x) = -\infty$ iff $\forall M \in \mathbb{R}, \exists r \in \mathbb{R}$

s.t $\forall x \in S, x > r \Rightarrow f(x) < M$

$\lim_{x\to-\infty} f(x) = +\infty$ iff $\forall M \in \mathbb{R}, \exists r \in \mathbb{R}$,

s.t $\forall x \in S, x < r \Rightarrow f(x) > M$

$\lim_{x\to-\infty} f(x) = -\infty$ iff $\forall M \in \mathbb{R}, \exists r \in \mathbb{R}$

s.t $\forall x \in S. x < r \Rightarrow f(x) < M$.

Example: $f(x) = x$, $\lim_{x\to+\infty} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = -\infty$