MATH2033 Mathematical Analysis
Suggestion Solution of Problem Set 1

Problem 1

Write down the negation of the following statements:

(a) x is divisible by 3 or 4.

(b) If x and y are positive, then $x + y > 0$.

(c) There exists a differentiable function $f(x)$ such that $\frac{df}{dx} + 2x = 0$ for all $x \in \mathbb{R}$.

(d) For any $\varepsilon > 0$, there exists a positive integer K such that $|x_n - L| < \varepsilon$ for all $n \geq K$

(*In this problem, $\{x_1, x_2, x_3, \ldots\}$ denotes a sequence of real number)

☺Solution

(a) The negation is "x is not divisible by 3 and x is not divisible by 4" ($\sim(A\ or\ B) = (\sim A)\ and\ (\sim B)$).

(b) The negation is "x and y are positive and $x + y \leq 0$" (since $\sim(p \Rightarrow q) = (p\ \ and\ \sim q)$).

(c) The negation is "For any differentiable function $f(x)$, $\frac{df}{dx} + 2x \neq 0$ for some $x \in \mathbb{R}$." (since $\sim\left(\exists x, P(x)\right) = \left(\forall x, \sim P(x)\right)$ and $\sim\left(\forall x, Q(x)\right) = \left(\exists x, \sim P(x)\right)$)

(d) The negation is "There exists $\varepsilon > 0$ such that for any positive integer K, there exists $n \geq K$ such that $|x_n - L| \geq \varepsilon$."

Problem 2

(a) We let $\{x_1, x_2, x_3, \ldots\}$ be a sequence of real numbers defined by $x_1 = 2$ and $x_{n+1} = 2x_n + 1$. Is it true that x_n is prime number for all positive integers n. Explain your answer. (☺Hint: Calculate x_2, x_3, x_4, x_5, x_6)

(b) We let n be a positive integer.

(i) If n^2 is multiple of 4, is it true that n is multiple of 4? Explain your answer.

(ii) If n^2 is multiple of 3, is it true that n is multiple of 3? Explain your answer.

(c) We let $f(x)$ be a function. Prove or disprove the following statement

"If $f(0) = 0$, then $f'(0) = 0$."

☺Solution

(a) By taking $n = 1,2,3,4,5$ in the recursive formula, we get

$$x_2 = 2(2) + 1 = 5, \qquad x_3 = 2(5) + 1 = 11, \qquad x_4 = 2(11) + 1 = 23,$$
$$x_5 = 2(23) + 1 = 47, \qquad x_6 = 2(47) + 1 = 95.$$

Since $x_6 = 95 = 19 \times 5$ is not prime number, so the statement is false (since its negation "x_n is not prime for some positive integer n" is true).

(b) **(i)** The statement is false. To see this, we take $n = 6$. We observe that $n^2 = 36$ is multiple of 4 but $n = 6$ is *not* multiple of 4 (i.e. the negation p and $\sim q$ is true.)

(ii) The statement is true. One can prove this using proof by contradiction.

Suppose that n is not multiple of 3, then either $n = 3p + 1$ or $n = 3q + 2$ for some integers p and q.

- If $n = 3p + 1$, then $n^2 = (3p + 1)^2 = 9p^2 + 6p + 1$ which is not multiple of 3.
- If $n = 3q + 2$, then $n^2 = (3q + 2)^2 = 9q^2 + 12q + 2$ which is not multiple of 3.

Both cases imply that n^2 is not multiple of 3 and there is contradiction. So the negation cannot be true and the statement holds.

(c) The statement is false. To see this, we take $f(x) = x$. One can show that

$$f(0) = 0 \quad but \quad f'(0) = 1.$$

So the negation (p and $\sim q$) is true.

Problem 3

We let $f(x)$ be a function.

Determine if each of the following statements is correct or not.

(a) Suppose that $f(x) > 0$ for all $x \in (1,4)$ (i.e. $1 < x < 4$), then $f(2)f(3) > 0$.

(b) Suppose that $f(x) > 0$ for some $x \in (1,4)$, then $f(2)f(3) > 0$.

😊Solution

(a) Since $f(x) > 0$ for <u>all</u> $x \in (1,4)$, we have $f(2) > 0$ and $f(3) > 0$, thus $f(2)f(3) > 0$.

(b) The statement may not be true. To see this, we take $f(x) = x^2 - 4$. We observe

- $f(3) = 5 > 0$, so $f(x) > 0$ for some $x \in (1,4)$
- But $f(2)f(3) = 0$.

Remark: Note that $f(x) > 0$ for some $x \in (1,4)$ does not guarantee the sign of $f(2)$ and $f(3)$, thus $f(2)f(3) > 0$ may not hold.

Problem 4

Prove that $\sqrt[3]{3}$ is an irrational number.

😊Solution

We will prove the claim using proof by contradiction.

Suppose that $\sqrt[3]{3}$ is an rational number. We write $\sqrt[3]{3} = \frac{m}{n}$ where m, n are some integers.

Here, we assume that $\frac{m}{n}$ is already in simplest form in the sense that m, n are relatively prime. Next, we note that

$$\sqrt[3]{3} = \frac{m}{n} \Rightarrow 3 = \frac{m^3}{n^3} \Rightarrow m^3 = 3n^3.$$

This implies that m^3 is multiple of 3.

Next, we argue that m is also multiple of 3. If m is not multiple of 3, then either $m = 3k + 1$ or $m = 3k + 2$ for some integer k.

- If $m = 3k + 1$, we have $m^3 = (3k + 1)^3 = 27k^3 + 27k^2 + 9k + 1$ which is not multiple of 3

- If $m = 3k + 2$, we have $m^3 = (3k + 2)^3 = 27k^3 + 54k^2 + 36k + 8$ which is not multiple of 3.

So m^3 is not multiple of 3 in both caes and there is contradiction. So m must be multiple of 3. We write $m = 3p$ for some integer p. Then we have

$$(3p)^3 = 3n^3 \Rightarrow n^3 = 9p^3.$$

Then n^3 is multiple of 3 and hence n is also multiple of 3.
Then m, n will have a common factor 3 and are no longer relatively prime. So there is a contradiction and the result follows.

Problem 5

Prove that there does *not* exist integers a and b such that $21a + 30b = 1$.

😊Solution
Suppose that there exists integers a, b such that $21a + 30b = 1$. Since L.H.S. is a multiple of 3 and R.H.S. is not a multiple of 3, so the equality does not hold and there is a contradiction. The negation cannot be true and the result follows.

Problem 6

We let a and b be two real numbers. Prove that if $a, b > 0$, then $\frac{2}{a} + \frac{2}{b} \neq \frac{4}{a+b}$.

😊Solution
We shall prove this statement using proof by contradiction.
Suppose that $\frac{2}{a} + \frac{2}{b} = \frac{4}{a+b}$ and $a, b > 0$, one can deduce that

$$\frac{2}{a} + \frac{2}{b} = \frac{4}{a+b} \Rightarrow \frac{2(a+b)}{ab} = \frac{4}{a+b} \Rightarrow 2(a+b)^2 = 4ab \Rightarrow \underbrace{2a^2 + 2b^2}_{>0 \text{ as } a>0, b>0} = 0.$$

The equality does not hold and there is contradiction. So the negation cannot be true and the result follows.

Problem 7

We let x be a non-zero rational number and y be an irrational number, show that $x + y$ and xy are both irrational.

😊Solution
We first argue that $x + y$ is irrational using proof by contradiction.
Suppose that $x + y$ is rational, then $x + y = \frac{m}{n}$ for some integer m and positive integer n.
Since x is also rational, we have $x = \frac{p}{q}$ for some integer p and positive integer q.
Then we have

$$y = (x + y) - x = \frac{m}{n} - \frac{p}{q} = \frac{mq - np}{nq},$$

which is also rational. It leads to contradiction. So $x + y$ must be irrational.

Next, we argue that xy is also irrational.

Suppose that xy is rational, we write $xy = \frac{m'}{n'}$ for some integer m' and positive integer n'.
Since $x \neq 0$, we have

$$y = \frac{xy}{x} = \frac{m'q}{n'p}$$

which is a rational number. It leads to contradiction. So xy must be irrational.

Remark: The condition $x \neq 0$ is required in order that xy is irrational since $xy = 0$ will be rational number if $x = 0$.

Problem 8 (Harder)

We let x, y, z be three positive integers satisfying $x^2 + y^2 = z^2$. Show that if x and y are relatively prime (i.e. H.C.F. of x and y is 1), then one of them is odd and another one is even.

☺Solution

We shall prove this by proof by contradiction. Suppose that x, y are either both odd or both even. Since x, y are relatively prime, it must be that both x, y are odd (otherwise the H.C.F. of x and y is at least 2).

We write $x = 2p + 1$ and $y = 2q + 1$, where p, q are some integers. Then one can deduce from the given equation that

$$(2p+1)^2 + (2q+1)^2 = z^2 \Rightarrow \underbrace{4p^2 + 4p + 4q^2 + 4q + 2}_{even} = z^2.$$

So z^2 is even number and it implies that z is even number.

By writing $z = 2k$ for some integer k, we deduce that

$$4p^2 + 4p + 4q^2 + 4q + 2 = (2k)^2 = \underbrace{2p^2 + 2p + 2q^2 + 2q + 1}_{odd} = \underbrace{2k^2}_{even}$$

So the equality does not hold and there is a contradiction.

Thus the negation cannot be true and the result follows.

Problem 9

We let $f(x)$ be a function satisfying $f(ax + by) = af(x) + bf(y)$ for all real numbers a, b, x, y. Show that $f(z_1) = 0$ and $f(z_2) = 0$ if and only if $f(z_1 + z_2) = 0$ and $f(z_1 - z_2) = 0$.

☺Solution

($\Rightarrow$ part)

If $f(z_1) = f(z_2) = 0$, then it follows from the given property that

$$f(z_1 + z_2) = \underbrace{f(z_1)}_{=0} + \underbrace{f(z_2)}_{=0} = 0,$$

$$f(z_1 - z_2) = f(z_1 + (-1)z_2) = \underbrace{f(z_1)}_{=0} - \underbrace{f(z_2)}_{=0} = 0.$$

($\Leftarrow$ part)

Since $f(z_1 + z_2) = 0$ and $f(z_1 - z_2) = 0$, we have

$$\underbrace{f(z_1) + f(z_2)}_{f(z_1+z_2)} = 0 \quad and \quad \underbrace{f(z_1) - f(z_2)}_{f(z_1-z_2)} = 0$$

By solving these two equations, we get

$$2f(z_1) = 0 \Rightarrow f(z_1) = 0 \quad and \quad f(z_2) = 0.$$

Problem 10

Prove that a positive integer n is divisible by 9 if and only if the sum of digits of n is divisible by 9.

(😊Hint: We write $n = d_r d_{r-1} \dots d_1 d_0$ in decimal representation, where each d_i represents a digit of n. Then n can be expressed as

$$n = d_r \times 10^r + d_{r-1} \times 10^{r-1} + \cdots + d_1 \times 10 + d_0.)$$

😊Solution

According to the hint, one can express the positive integer n as

$$n = d_r \times 10^r + d_{r-1} \times 10^{r-1} + \cdots + d_1 \times 10 + d_0.$$

For any positive integer m, one can see that $10^m - 1 = \underbrace{99 \cdots 9}_{m \text{ "9"s}}$ is divisible by 9. Thus, one can express n as

$$n = \underbrace{d_r \times (10^r - 1) + \cdots + d_1 \times (10 - 1)}_{multiple\ of\ 9} + d_r + d_{r-1} + \cdots + d_2 + d_1 + d_0$$

$$= 9k + d_r + d_{r-1} + \cdots + d_1 + d_0 \dots (*)$$

(⇒ part)

If n is divisible by 9, then

$$n - 9k = d_r + d_{r-1} + \cdots + d_1 + d_0$$

is also divisible by 9.

(⇐ part)

If the sum of digit $d_r + d_{r-1} + \cdots + d_1 + d_0$ is divisible by 9, then it follows from the equation $(*)$ that

$$n = 9k + \underbrace{d_r + d_{r-1} + \cdots + d_1 + d_0}_{=9m\ \ for\ some\ integer\ m}$$

is also divisible by 9.