Practice Exercises

The starred problems are difficult and involve more work or deeper thinking.

For exercises 1 to 7, negate each of the following expressions or statements.

1. ($x > 0$ and $x < 1$) or $x = -1$

2. $x > 0$ and ($x < 1$ or $x = -1$)

3. For every triangle ABC, $\angle A + \angle B + \angle C = 180°$.

4. There exists a man who does not have any wife.

5. For each x, there is a y such that $x + y = 0$.

6. $\exists \alpha \;\; \forall \beta \;\; \exists \gamma$ such that $|\alpha - \beta| < \gamma$.

7. If x and y are positive, then $x + y > 0$.

---

8. Give the contrapositive of the following statements. (Since the contrapositives are equivalent to the statements, they say the *same* thing.)

   (a) If $AB = AC$ in $\triangle ABC$, then $\angle B = \angle C$ in $\triangle ABC$.

   (b) If a function is differentiable, then it is continuous.

   (c) If $\lim\limits_{x\to 0} f(x) = a$ and $\lim\limits_{x\to 0} g(x) = b$, then $\lim\limits_{x\to 0}(f(x) + g(x)) = a + b$.

   (d) If $x^2 + bx + c = 0$, then $x = \dfrac{-b + \sqrt{b^2 - 4c}}{2}$ or $x = \dfrac{-b - \sqrt{b^2 - 4c}}{2}$.

9. Compute the following sets.

   (a) $\big(\{x, y, z\} \cup \{w, z\}\big) \setminus \{u, v, w\}$. (Here u, v, w, x, y, z are distinct objects.)

   (b) $\{1, 2\} \times \{3, 4\} \times \{5\}$.

   (c) $\mathbb{Z} \cap [0, 10] \cap \{n^2 + 1 : n \in \mathbb{N}\}$.

   (d) $\{n \in \mathbb{N} : 5 < n < 9\} \setminus \{2m : m \in \mathbb{N}\}$.

   (e) $([0, 2] \setminus [1, 3]) \cup ([1, 3] \setminus [0, 2])$.

10. (i) Let $A = [0, 1]$ and $B = [0, 1] \cup [2, 3]$. Plot the graphs of $A \times A$ and $B \times B$ on the plane.

    (ii) If A, B are sets that are not the empty set and $A \times B = B \times A$, what can be said about A and B?

11. (a) If $B \subseteq C$, then prove that $A \cup B \subseteq A \cup C$.

    (b) For sets X, Y, Z, prove that $(X \setminus Y) \setminus Z = (X \setminus Z) \setminus Y$.

12. (i) For all sets A, B, C, is it always true that $(A \cup B) \cap C = A \cup (B \cap C)$?

    (ii) For all sets A, B, C, is it always true that if $A \cup B = A \cup C$, then $B = C$?

    (iii) For all sets A, B, C, is it always true that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$?

13. (i) Show that if $A \subseteq B$ and $C \subseteq D$, then $A \cup C \subseteq B \cup D$.

(ii) Is it always true that if $A \subset B$ and $C \subset D$, then $A \cup C \subset B \cup D$?

(iii) For $a < b$, let

$$(a, b)_Q = \{x : x \in \mathbb{Q} \text{ and } a < x < b\} \quad \text{and} \quad [a, b)_Q = \{x : x \in \mathbb{Q} \text{ and } a \leq x < b\}.$$

Does $\bigcup_{n=1}^{\infty}[\frac{1}{n}, 2)_Q = \bigcup_{n=1}^{\infty}(\frac{1}{n}, 2)_Q$?

14. Define functions $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x \leq 0 \end{cases}$ and $g(x) = 1 - 2x$. For f and g, determine if each is injective or surjective. Compute $f \circ g$ and $g \circ f$.

15. (i) Let $f : A \to B$ be a function. Show that if there is a function $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$, then f is a bijection. (*Comment*: Such a g is f^{-1}.)

    (ii) Show that if $f : A \to B$ and $h : B \to C$ are bijections, then $h \circ f : A \to C$ is a bijection.

16. Let A, B be subsets of $\mathbb{R}$ and $f : A \to B$ be a function. If for every $b \in B$, the horizontal line $y = b$ intersects the graph of f at most once, then show that f is injective. If 'at most once' is replaced by 'at least once', what can be said about f?

For each of the sets in exercises 17 to 26, determine if it is countable or uncountable:

17. intervals (a, b) and $[a, b]$, where we assume $a < b$;

18. $\mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$;

19. $A = \{\frac{1}{2^n} + \frac{1}{3^m} : n, m \in \mathbb{Z}\}$;

20. $B = \{x + \sqrt{2}y : x, y \in \mathbb{N}\}$;

21. the set C of all lines in $\mathbb{R}^2$ passing through the origin;

22. $D = \{x \in \mathbb{R} : x^5 + x + 2 \in \mathbb{Q}\}$;

23. the set E of all circles in $\mathbb{R}^2$ with centers at rational coordinate points and positive rational radius.

24. the set $F = \{a : x^4 + ax - 5 = 0 \text{ has a rational root}\}$;

25. the set $G = \{a^3 + b^3 : a \in X, b \in Y\}$, where X is a nonempty countable subet of $\mathbb{R}$ and Y is an uncountable subet of $\mathbb{R}$;

26. the set $H = (X \setminus Y) \cup (Y \setminus X)$, where X is a countable set and Y is an uncountable set. (*Remark:* The set $(X \setminus Y) \cup (Y \setminus X)$ is called the *symmetric difference* of X and Y and is usually denoted by $X \triangle Y$. This concept will appear in other algebra and analysis courses later.)

27. Show that the set F of all finite subsets of $\mathbb{N}$ is countable.

28. If S is a countable subset of $\mathbb{R}^2$, show that for any two points $x, y \in \mathbb{R}^2 \setminus S$, there is a parallelogram in $\mathbb{R}^2 \setminus S$ having x, y as opposite vertices. Here parallelogram means only the 4 edges (including the 4 vertices, but not including any interior point).

*29. From $K_0 = [0,1]$, remove the middle thirds to get $K_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Then remove the middle thirds of the 2 subintervals of K_1 to get $K_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Inductively, remove the middle thirds of the 2^n subintervals of K_n to get K_{n+1}. The set $K = K_0 \cap K_1 \cap K_2 \cap K_3 \cap \cdots$ is called the *Cantor set*. Prove that K is uncountable.
(*Hint:* Consider base 3 representations, i.e. representations of the form

$$(.a_1a_2a_3\ldots)_3 = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \cdots,$$

where each $a_i = 0, 1$ or 2. What do the base 3 representations of numbers in K_n have in common? Note some numbers have 2 representations, *e.g.* $\frac{1}{3} = (.1000\ldots)_3$ and $(.0222\ldots)_3$.)

---

30. For each of the following series, determine if it converges or diverges.

(a) $\sum\limits_{k=1}^{\infty} \cos\left(\sin\dfrac{1}{k}\right)$ (term test)

(b) $\sum\limits_{k=1}^{\infty} \dfrac{1}{\sqrt{k(k+1)(k+2)}}$

(c) $\sum\limits_{k=1}^{\infty} \ln\left(1+\dfrac{1}{k}\right)$ (find sum)

(d) $\sum\limits_{k=1}^{\infty} \left(\dfrac{1}{2}+\dfrac{1}{k}\right)^k$

(e) $\sum\limits_{k=2}^{\infty} \dfrac{\ln k}{k}$

(f) $\sum\limits_{k=1}^{\infty} \dfrac{\cos 2^k}{k^2}$

(g) $\sum\limits_{k=1}^{\infty} \dfrac{k+2}{k+1}\left(\dfrac{2}{3}\right)^k$

(h) $\sum\limits_{k=1}^{\infty} \dfrac{\cos k\pi}{\sqrt{k}}$

(i) $\sum\limits_{k=1}^{\infty} ke^{-k^2}$

(j) $\sum\limits_{k=1}^{\infty} \dfrac{k}{(k+1)!}$

(k) $\sum\limits_{k=1}^{\infty} \dfrac{\operatorname{Arctan} k}{k^2+1}$

(l) $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^{1+\frac{1}{k}}}$ (compare with a p-series)

(m) $\sum\limits_{k=1}^{\infty} \tan^k\left(\dfrac{k+1}{k}\right)$

(n) $\sum\limits_{k=1}^{\infty} \left(1-\cos\dfrac{1}{k}\right)$ (compare with a p-series)

(o) $\sum\limits_{k=1}^{\infty} k^2\sin^p\left(\dfrac{1}{k}\right)$ (depends on p)

(p) $\sum\limits_{k=1}^{\infty} (-1)^{k+1}(\sqrt{k+1}-\sqrt{k})$

31. Let $a_1 \geq a_2 \geq a_3 \geq \ldots \geq 0$. Prove that $\sum\limits_{k=1}^{\infty} a_k$ converges if and only if $\sum\limits_{k=1}^{\infty} 2^k a_{2^k}$ converges. (This is called *Cauchy's condensation test.*) Use this test to determine if $\sum\limits_{k=3}^{\infty} \dfrac{1}{k\ln k\ln(\ln k)}$ converges.

32. Show that $\sum\limits_{k=2}^{\infty} \dfrac{k}{2^{k-1}} = \dfrac{2}{2} + \dfrac{3}{2^2} + \dfrac{4}{2^3} + \cdots$ converges and find the sum. (*Hint*: Compare the same series with index from 1 to ∞.)

(Probabilistic interpretation: the sum is the expected number of births to get babies of both sex. The probability of the k-th birth finally resulted in babies of both sex is $\dfrac{2}{2^k} = \dfrac{1}{2^{k-1}}$.)

*33. Let p_n be the n-th prime number, i.e. $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \cdots$. Show that $\sum\limits_{k=1}^{\infty} \dfrac{1}{p_k}$ diverges. (*Hint:* Suppose it converges to s. Then the partial sum s_n has limit s, as $n \to \infty$. So for some n, $s - s_n = \sum\limits_{k=n+1}^{\infty} \dfrac{1}{p_k} < \dfrac{1}{2}$. Let $Q = p_1p_2\cdots p_n$. Show, by considering the prime factorization of $1+mQ$, that $\sum\limits_{m=1}^{N} \dfrac{1}{1+mQ} \leq \sum\limits_{j=1}^{\infty}\left(\sum\limits_{k=n+1}^{\infty} \dfrac{1}{p_k}\right)^j$ for every positive integer N, which will lead to a contradiction.)

For exercises 34 to 36, use definitions, Archimedean principle, density of rationals, density of irrationals, etc. to support your reasonings.

34. For each of the following sets, if it is bounded above, give an upper bound and find its supremum with proof. If it is bounded below, give a lower bound and find its infimum with proof.

(a) $A = \{\sqrt{m} + \sqrt{n} : m, n \in \mathbb{N}\}$

(b) $B = (-\infty, \pi] \cup \left\{4 - \dfrac{1}{n} : n \in \mathbb{N}\right\}$

(c) $C = \left\{\dfrac{1}{n} + \dfrac{1}{2^m} : m, n \in \mathbb{N}\right\}$

(d) $D = \mathbb{Q} \cap (0, \sqrt{2}]$

35. Let A and B be nonempty subsets of $\mathbb{R}$, which are bounded above. Let

$$S = \{xy : x \in A, y \in B\} \quad \text{and} \quad T = \{x - y : x \in A, y \in B\}.$$

Must S or T be bounded above? Give a proof if you think the answer is 'yes' or give a counterexample if you think the answer is 'no'.

36. Let A and B be nonempty subsets of $\mathbb{R}$, which are bounded above. Let

$$C = \{x + y : x \in A, y \in B\}.$$

Show that C is bounded above and $\sup C = \sup A + \sup B$. (*Hint*: If $\sup C < \sup A + \sup B$, then consider $\varepsilon = (\sup A + \sup B - \sup C)/2$. Apply supremum property to get a contradiction.)

---

37. Let $w_n = \dfrac{4n+5}{n^3}$, then $\{w_n\}$ should converge to 0. For a given $\varepsilon > 0$, show there is a positive integer K such that if $n \geq K$, then $|w_n - 0| < \varepsilon$. If $\varepsilon = 0.1$, give one such positive integer K.

38. Let x be positive and $a_n = \dfrac{[x] + [2x] + \cdots + [nx]}{n^2}$. Show that $\{a_n\}$ converges to $\frac{x}{2}$ by the squeeze limit theorem. (Here $[y]$ is the greatest integer less than or equal to y.)

39. Show that if x is a real number, then there is a sequence of rational numbers converging to x.

40. If $\lim\limits_{n\to\infty} a_n = A$, then show that $\lim\limits_{n\to\infty} |a_n| = |A|$. (*Hint*: Show $\big||x| - |y|\big| \leq |x - y|$ for $x, y \in \mathbb{R}$ first.) Is the converse true?

41. If $\{a_n\}$ converges to A, then $\left\{\dfrac{a_n + a_{n+1}}{2}\right\}$ should converge to $\dfrac{A + A}{2} = A$. Prove this by checking the definition.

42. Let $x_1 = 4$ and $x_{n+1} = \dfrac{4(1 + x_n)}{4 + x_n}$ for $n = 1, 2, 3, \ldots$. Plot the first 3 terms on the real line. Then prove the sequence $\{x_n\}$ converges.

43. Show that the sequence $\left\{\left(1 + \dfrac{1}{n}\right)^n\right\}$ is increasing and bounded above.

44. Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$, $M_n = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}$ and $m_n = \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}$ for $n \in \mathbb{N}$.

(a) Prove that both sequence $\{M_n\}$ and $\{m_n\}$ converge. (The limit of M_n is called the *limit superior* of x_n and is denoted by $\limsup\limits_{n\to\infty} x_n$, while the limit of m_n is called the *limit inferior* of x_n and is denoted by $\liminf\limits_{n\to\infty} x_n$.)

(b) Prove that $\lim\limits_{n\to\infty} x_n = x$ if and only if $\lim\limits_{n\to\infty} M_n = x = \lim\limits_{n\to\infty} m_n$ (i.e. $\limsup\limits_{n\to\infty} x_n = x = \liminf\limits_{n\to\infty} x_n$).

45. Let $x_1 = 1, x_2 = 2$ and $x_{n+1} = \dfrac{x_n + x_{n-1}}{2}$ for $n = 2, 3, 4, \ldots$. Plot the first 4 terms on the real line. Then prove the sequence $\{x_n\}$ converges.

46. If $\sum\limits_{k=2}^{\infty} |x_k - x_{k-1}|$ converges, then show the sequence $\{x_n\}$ is a Cauchy sequence by checking the definition of Cauchy sequence.

47. If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\{a_n\}$ is a Cauchy sequence, then show that $\{\sqrt{a_n}\}$ is also a Cauchy sequence by checking the definition of Cauchy sequence.

48. Let $0 < k < 1$. If $|x_{n+1} - x_n| \leq k|x_n - x_{n-1}|$ for $n = 2, 3, 4, \ldots$, then prove that $\{x_n\}$ is a Cauchy sequence.

49. Prove that if $\{a_n\}$ converges to A, then $\{\alpha_n\}$ converges to A, where $\alpha_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$. Show that the converse is false.

    (*Hint*: Let $b_n = a_n - A$ and $\beta_n = \dfrac{b_1 + b_2 + \cdots + b_n}{n}$, then the first part of the problem becomes showing $\{\beta_n\}$ converges to 0. Here use $\{b_n\}$ converges to 0 and so $|b_{K_0}|, |b_{K_0+1}|, \ldots$ is small when K_0 is large. For $n \geq K_0$, write
    $$\beta_n = \frac{b_1 + b_2 + \cdots + b_{K_0-1}}{n} + \frac{b_{K_0} + \cdots + b_n}{n}.)$$

50. If $\{x_n\}$ is bounded and all its convergent subsequences have the same limit x, then prove that $\lim\limits_{n\to\infty} x_n = x$.

51. Let $S = \{2^{-n} : n \in \mathbb{N}\}$ and $f : \mathbb{N} \to S$ is injective. Show that $\lim\limits_{n\to\infty} f(n) = 0$.

*52. Let $\{a_n\}$ be a sequence satisfying $\lim\limits_{n\to\infty} (a_{n+1} - \dfrac{a_n}{2}) = 0$. Prove that $\lim\limits_{n\to\infty} a_n = 0$.

*53. Let $\{x_n\}$ be a sequence satisfying $\lim\limits_{n\to\infty} (x_n - x_{n-2}) = 0$. Prove that $\lim\limits_{n\to\infty} \dfrac{x_n - x_{n-1}}{n} = 0$.

*54. Let $\{x_n\}$ be a sequence and let $y_1 = 0$ and $y_n = x_{n-1} + 2x_n$ for $n = 2, 3, 4, \ldots$. If $\{y_n\}$ converges, prove that $\{x_n\}$ also converges.

55. For a sequence $\{a_n\}$ of nonzero numbers, we say the *infinite product* $\prod\limits_{n=1}^{\infty} a_n$ *converges* to a nonzero number L (or has *value* L) if $\lim\limits_{k\to\infty} a_1 a_2 \cdots a_k = L$. We say it *diverges* if the limit is 0 or does not exist. Determine if each of the following infinite products converges or diverges. Find the values of the infinite products that converge.

    (a) $\displaystyle\prod_{n=2}^{\infty}\left(1 - \frac{2}{n(n+1)}\right)$ (b) $\displaystyle\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right)$

    (c) $\displaystyle\prod_{n=2}^{\infty}\frac{n^3 - 1}{n^3 + 1}$ (d) $\displaystyle\prod_{n=0}^{\infty}(1 + z^{2^n})$ for $|z| < 1$.

    Remarks: In Apostol's book, the following theorem is proved: In the case every $a_n \geq 0$, we have $\prod\limits_{n=1}^{\infty}(1 + a_n)$, $\prod\limits_{n=1}^{\infty}(1 - a_n)$, $\sum\limits_{n=1}^{\infty} a_n$ all converge or all diverge. Try to do the above exercises without using this theorem.

---

56. Prove that every bounded infinite subset of $\mathbb{R}$ has an accumulation point. (This is also often called the *Bolzano-Weierstrass* theorem.)

57. Let $f : (0, +\infty) \to \mathbb{R}$ be defined by $f(x) = \dfrac{x}{x+1}$. Show that $\lim\limits_{x \to 1} f(x) = \dfrac{1}{2}$ by checking the definition.

58. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 8x & \text{if } x \text{ is rational,} \\ 2x^2 + 8 & \text{if } x \text{ is irrational.} \end{cases}$$

For which x_0, does $\lim\limits_{x \to x_0} f(x)$ exist? (*Hint*: Sequential limit theorem.)

59. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) = 0$ for every rational number x, show that $f(x) = 0$ for all x.

60. (a) Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{Q}$.
(b) Find all strictly increasing functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

61. For a function $f : \mathbb{R} \to \mathbb{R}$, we say f has a *local (or relative) maximum* at x_0 if there exists an open interval (a, b) containing x_0 such that $f(x) \leq f(x_0)$ for every $x \in (a, b)$. Similarly, we say f has a *local (or relative) minimum* at x_1 if there exists an open interval (c, d) containing x_1 such that $f(x) \geq f(x_1)$ for every $x \in (c, d)$. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and has a local maximum or a local minimum at every real number, show that f is a constant function.

62. If $f(x) = x^3$, then $f(f(x)) = x^9$. Is there a continuous function $g : [-1, 1] \to [-1, 1]$ such that $g(g(x)) = -x^9$ for all $x \in [-1, 1]$? (*Hint*: If such a function g exists, then it is injective.)

63. A *fixed point* of a function f is a number w such that $f(w) = w$. Show that if $f : [0, 1] \to [0, 1]$ is continuous, then f has at least one fixed point. (*Hint*: Consider $g(x) = f(x) - x$.)

64. Let $f : [0, 1] \to [0, 1]$ be an increasing function (perhaps discontinuous). Suppose $0 < f(0)$ and $f(1) < 1$, show that f has at least one fixed point. (*Hint*: Sketch the graph of f and consider the set $\{t \in [0, 1] : t < f(t)\}$. Does it have a supremum?)

65. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Show that f is bijective. (*Hint*: Easy to show f is injective. To show f is surjective, let $w \in \mathbb{R}$ and $M = |w - f(0)|$, show that w and $f(0)$ are between $f(-M)$ and $f(M)$.)

*66. Let $f : [0, 1] \to (0, +\infty)$ be continuous and $M = \sup\{f(x) : x \in [0, 1]\}$. Show that

$$\lim_{n \to \infty} \left(\int_0^1 f(x)^n dx \right)^{\frac{1}{n}} = M$$

if the limit exists. (*Hint*: $M = f(x_0)$. For every $k \in \mathbb{N}$, use the sign preserving property to show that $f(x_0) - \frac{1}{k} \leq f(x) \leq M$ on an interval containing x_0. Squeeze the integral.)

---

67. Find the derivatives of the functions $f(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ x & \text{if } x = 0 \end{cases}$ and $g(x) = |\cos x|$.

68. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at c and $I_n = [a_n, b_n]$ be such that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ and $\bigcap\limits_{n=1}^{\infty} [a_n, b_n] = \{c\}$. Prove that if $a_n < b_n$ for all $n \in \mathbb{N}$, then $f'(c) = \lim\limits_{n \to \infty} \dfrac{f(b_n) - f(a_n)}{b_n - a_n}$.

69. Let $f(x) = |x|^3$ for every $x \in \mathbb{R}$. Show that $f \in C^2(\mathbb{R})$. However, $f'''(0)$ does not exist.

70. Let $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(a) - f(b)| \leq |a - b|^2$ for every $a, b \in \mathbb{R}$. Show that f is a constant function. If the exponent 2 in the inequality is replaced by a number greater than 1, must f be a constant function? If 2 is replaced by 1, must f be a constant function?

71. Let n be a positive integer and $f(x) = (x^2 - 1)^n$. Show that the n-th derivative of f has n distinct roots.

72. Let $f : [0, \infty) \to \mathbb{R}$ be continuous and $f(0) = 0$. If $f'(x) \leq f(x)$ for every $x > 0$, then show that $f(x) \leq 0$ for every $x \in [0, \infty)$. (*Hint*: What if $f'(x) = f(x)$?)

73. If $f : (0, +\infty) \to \mathbb{R}$ is differentiable and $|f'(x)| \leq 2$ for all $x > 0$, then show that the sequence $x_n = f\left(\frac{1}{n}\right)$ converges. Also, show $\lim\limits_{x\to 0+} f(x)$ exists. (*Hint*: Check $\{x_n\}$ is a Cauchy sequence. For the second part, consider the sequential limit theorem and the remarks following it.)

74. For $0 < x < \frac{\pi}{2}$, prove that $|\ln(\cos x)| < x \tan x$.

75. Let $f : [0, \infty) \to \mathbb{R}$ be continuous and $f(0) = 0$. If $|f'(x)| \leq |f(x)|$ for every $x > 0$, show that $f(x) = 0$ for every $x \in [0, \infty)$. (*Hint*: Let $|f|$ has maximum value M on $[0, \frac{1}{2}]$. Apply the mean value theorem to f on $[0, \frac{1}{2}]$.)

76. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. If f' is differentiable at x_0, show that

$$\lim_{h\to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = f''(x_0).$$

77. Prove that if $0 \leq \theta \leq \frac{\pi}{2}$, then

$$1 - \frac{\theta^2}{2} \leq \cos\theta \leq 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

(*Hint*: Apply Taylor's theorem to the four times differentiable function $\cos\theta$.)

78. Let $f : (0, +\infty) \to \mathbb{R}$ be twice differentiable, $M_0 = \sup\{|f(x)| : x > 0\} < \infty$, $M_1 = \sup\{|f'(x)| : x > 0\} < \infty$ and $M_2 = \sup\{|f''(x)| : x > 0\} < \infty$. Show that $M_1^2 \leq 4M_0M_2$. (*Hint*: Let $h > 0$. Apply Taylor's theorem to $f(x)$ with $x = c + 2h$, then solve for $f'(c)$.)

---

79. (a) If $f : (a, b) \to \mathbb{R}$ is differentiable and $|f'(x)| \leq 2$ for all $x \in (a, b)$, then show that f is uniformly continuous.

    (b) Show that $f : (0, +\infty) \to \mathbb{R}$ defined by $f(x) = \sin\frac{1}{x}$ is not uniformly continuous.

80. (a) Prove that if the union of a collection of open intervals contains $[a, b]$, then there are finitely many of these intervals, whose union also contains $[a, b]$. (*Hint*: Suppose this is false. Let $m_1 = (a + b)/2$. Then one of $[a, m_1]$ or $[m_1, b]$ is not contained in the union of finitely many of these open intervals. Proceed as in the proof of the Bolzano Weierstrass theorem.)

    (b) Give a proof of the uniform continuity theorem using part (a).

81. If f is continuous on $[a, b]$, $f(x) \geq 0$ for all $x \in [a, b]$ and $\displaystyle\int_a^b f(x)\,dx = 0$, then prove that $f(x) = 0$ for all $x \in [a, b]$.

82. Let $f : [a, d]$ be a bounded function and $a \leq b \leq c \leq d$.

    (i) Use the integral criterion to show that if f is integrable on $[a, b]$ and $[b, c]$, then f is integrable on $[a, c]$.

    (ii) Use the integral criterion to show that if f is integrable on $[a, d]$, then f is integrable on $[b, c]$.

83. Let $f : [a, b] \to \mathbb{R}$ be bounded and $\{x \in [a, b] : f \text{ is discontinuous at } x\} = \{x_1, x_2, \ldots, x_n\}$, where $x_1 < x_2 < \cdots < x_n$. Use the integral criterion to show that f is integrable on $[a, b]$. (*Hint*: Divide $[a, b]$ into subintervals where f is continuous except at one endpoint and note part (i) of problem above.)

84. (i) Let $f, g : [a, b] \to \mathbb{R}$ be bounded and P is a partition of $[a, b]$. Show that

$$L(f, P) + L(g, P) \le L(f + g, P) \le U(f + g, P) \le U(f, P) + U(g, P).$$

(ii) If f and g are integrable on $[a, b]$, show that

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(*Hint*: First show RHS $- \varepsilon < L(f, P) + L(g, P) \le L(f + g, P)$ for some P using supremum property.)

85. Determine if each of the following improper integrals exists or not.

(a) $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{e^x}}$ (b) $\displaystyle\int_0^{\infty} \sin x\,dx$

(c) $\displaystyle\int_0^1 \frac{dx}{x^2 + 5x}$ (d) $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt[3]{x}}$

(e) $\displaystyle\int_0^1 \frac{dx}{x(x-1)}$ (f) $\displaystyle\int_0^{\infty} \frac{\cos x}{1 + x^2}dx$

86. Find the principal value of each of the following integrals if it exists.

(a) P.V. $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{e^{x^2}}dx$ (b) P.V. $\displaystyle\int_0^2 \frac{dx}{x^2 - 1}$

87. Prove that for $0 < x < \infty$, the improper integral $\Gamma(x) = \displaystyle\int_0^{\infty} t^{x-1}e^{-t}\,dt$ exists. This is called the *gamma function*. (*Hint*: Consider the cases $x \in (0, 1)$ and $x \in [1, \infty)$ separately.)

Past Exam Problems

88. Define the following terms:
   (a) S is a *countably infinite* set
   (b) S is a *countable* set
   (c) a series $\sum_{k=1}^{\infty} a_k$ *converges* to a number S
   (d) a nonempty subset S of $\mathbb{R}$ is *bounded above*
   (e) the *supremum* of a subset S of $\mathbb{R}$ that is bounded above
   (f) a sequence $\{x_n\}$ *converges* to a number x
   (g) a sequence $\{x_n\}$ is a *Cauchy* sequence
   (h) x is an *accumulation point* of a set S
   (i) $f : S \to \mathbb{R}$ has a *limit* L at x_0
   (j) $f : S \to \mathbb{R}$ is *continuous* at $x_0 \in S$ (the ε-δ definition)

89. For each of the following sets S, determine if it is countable or uncountable. Be sure to give reasons to support your answer.
   (a) S is the set of all intersection points (x, y) of the line $y = \pi x$ with the graphs of all equations $y = x^3 + x + m$, where $m \in \mathbb{Z}$.
   (b) S is the set of all intersection points (x, y) of the graph of $y = x^3 + x + 1$ with all lines $y = mx$, where $m \in \mathbb{Z}$.
   (c) S is the set of all intersection points (x, y) of the circle $x^2 + y^2 = 1$ with all hyperbolas $xy = \dfrac{1}{m}$, where $m \in \mathbb{N}$.

(d) $S=\{a+b\in\mathbb{R}: \quad |a|\in M, b\in\mathbb{Q}\}$, where M is a *uncountable* subset of the positive real numbers.
(e) $S=\{a+b\in\mathbb{R}: \quad |a|\in M, b\in\mathbb{Q}\}$, where M is a *countable* subset of the positive real numbers.
(f) $S=\mathbb{Q}(\sqrt{2})=\left\{\dfrac{a+b\sqrt{2}}{c+d\sqrt{2}}: a,b,c,d\in\mathbb{Q}, \quad c+d\sqrt{2}\neq 0\right\}$.
(g) $S=\{x^2+y^2+z^2: \quad x\in A\cap B, \quad y\in\mathbb{Q}\cap A, \quad z\in B\cap\mathbb{Q}\}$, where A is a nonempty countable subset of $\mathbb{R}$ and B is an uncountable subset of $\mathbb{R}$.
(h) $S=\{x-y: x,y\in A\}$, where A is an *uncountable* subset of $\mathbb{R}$.
(i) $S=\{x^2+y^2: x,y\in A\}$, where A is a nonempty countable subset of $\mathbb{R}$.
(j) $S=\{x+\sin y: x\in\mathbb{R}\setminus A, y\in\mathbb{Z}\}$, where A is a nonempty countable subset of $\mathbb{R}$.
(k) $S=\{(x,y)\in\mathbb{R}^2: x\in A, y\in\mathbb{R}\setminus A\}$, where A is a nonempty countable subset of $\mathbb{R}$.
(l) $S=\{x+y\sqrt{2}: x\in\mathbb{Z}, y\in A\}$, where A is a nonempty countable subset of $\mathbb{R}$.
(m) $S=\mathbb{R}\setminus\{a+b\sqrt{2}-c\sqrt{3}: a,b,c\in T\}$, where $T=\{r\pi: r\in\mathbb{Q}\}$.
(n) $S=T\cap U$, where $T=\mathbb{R}\setminus\mathbb{Q}$ and $U=\mathbb{R}\setminus\{\sqrt{m}+\sqrt{n}: m,n\in N\}$. (*Hint*: Consider $\mathbb{R}\setminus(T\cap U)$.)
(o) S is the set of all squares on the plane that can be circumscribed by circles with rational radii and centers with rational coordinates.
(p) S is the set of all nonconstant polynomials with coefficients in G, where $i=\sqrt{-1}$ and $G=\{a+bi: \ a,b\in\mathbb{Z}\}$.

90. Determine if each of the following series converges or diverges. Be sure to give reasons to support your answer.
(a) $\sum\limits_{k=1}^{\infty}\dfrac{\cos k\pi}{k^2+2^k}$ and $\sum\limits_{k=1}^{\infty}\dfrac{e^{\sqrt{k}}}{\sqrt{k}}$
(b) $\sum\limits_{k=1}^{\infty}\dfrac{(2k)!}{3^k k^4}$ and $\sum\limits_{k=1}^{\infty}\dfrac{(\cos k)(\sin 2k)}{2^k}$
(c) $\sum\limits_{k=1}^{\infty}\dfrac{1}{2}\left(\cos\dfrac{1}{k}+\sin\dfrac{1}{k}\right)$ and $\sum\limits_{k=1}^{\infty}\sin\left(\dfrac{1}{k}-\dfrac{1}{k+1}\right)$
(d) $\sum\limits_{k=1}^{\infty}\dfrac{2^k+3^k}{1^k+4^k}$ and $\sum\limits_{k=1}^{\infty}\cos\left(\sin\dfrac{1}{k}\right)$
(e) $\sum\limits_{k=1}^{\infty}\dfrac{2^{1/k}+3^{1/k}}{1^{1/k}+4^{1/k}}$ and $\sum\limits_{k=1}^{\infty}(\cos k\pi)\left(\sin\dfrac{1}{k\pi}\right)$
(f) $\sum\limits_{k=1}^{\infty}\dfrac{(k!)^2}{(k^2)!}$ and $\sum\limits_{k=1}^{\infty}\left(\cos\dfrac{1}{k}\right)\left(\sin\dfrac{1}{k}\right)\left(\tan\dfrac{1}{k}\right)$
(g) $\sum\limits_{k=2}^{\infty}\dfrac{2^k\cos k}{(k-1)!}$ and $\sum\limits_{k=2}^{\infty}\dfrac{\sin(\frac{1}{k})}{\ln k}$
(h) $\sum\limits_{k=1}^{\infty}\dfrac{k^{\pi}+\cos k\pi}{3+k^4}$ and $\sum\limits_{k=1}^{\infty}\dfrac{k^{\pi}\cos k\pi}{3k^4}$
(i) $\sum\limits_{k=2}^{\infty}\dfrac{(2k)!}{(k+1)!(k-1)!}$ and $\sum\limits_{k=1}^{\infty}k\cos\left(\dfrac{1}{k^2}\right)$
(j) $\sum\limits_{k=1}^{\infty}\dfrac{(3k)!}{k!(2k)!}$ and $\sum\limits_{k=2}^{\infty}\dfrac{\cos(1/k)}{k^2-1}$
(k) $\sum\limits_{k=1}^{\infty}\dfrac{k!}{(2k-1)!}$ and $\sum\limits_{k=1}^{\infty}\dfrac{\cos k\pi}{\sqrt{k}+1}$
(l) $\sum\limits_{k=1}^{\infty}\dfrac{2^k k^2}{k!}$ and $\sum\limits_{k=1}^{\infty}\dfrac{1}{\sqrt{k}}\sin\left(\dfrac{1}{\sqrt{k}}\right)$
(m) $\sum\limits_{k=1}^{\infty}\dfrac{1}{k\cos k\pi}$ and $\sum\limits_{k=1}^{\infty}\dfrac{k^2\sin(1/k)}{(2k+1)!}$
(n) $\sum\limits_{k=1}^{\infty}\cos^k(1+\dfrac{1}{k})$ and $\sum\limits_{k=1}^{\infty}\dfrac{\cos(\sin(1/k))}{\sin(\cos(1/k))}$

91. Determine if each of the following set has an infimum and a supremum. If they exists, find them and give reasons to support your answers.
(a) $S=\left\{\dfrac{1}{m}+\dfrac{1}{n}: m,n\in\mathbb{N}\right\}\setminus\left\{\dfrac{2}{k}: k\in\mathbb{N}\right\}$

(b) $S = \left\{x + y : x, y \in \left[\frac{1}{2}, 1\right)\right\} \setminus \left\{2 - \frac{1}{n} : n \in \mathbb{N}\right\}$

(c) $S = \left\{x - \frac{1}{n} : x \in [0, 1] \cap \mathbb{Q}, \quad n \in \mathbb{N}\right\} \setminus \left[-1, \frac{1}{2}\right)$

(d) $S = \left\{\frac{x - \pi}{x + \pi} : x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [\pi, \infty)\right\}$

(e) $S = \left\{\frac{x - \pi}{x + \pi} : x \in \mathbb{Q} \cap [0, \infty)\right\}$

(f) $S = \{x^3 + y^3 : x \in \mathbb{Q} \cap [0, 1], \quad y \in (-1, 0]\}$

(g) $S = \left\{\frac{\sqrt{2}}{m + n} + \frac{1}{k\sqrt{2}} : m, n, k \in \mathbb{N}\right\}$

(h) $S = \bigcup_{k=1}^{\infty} [1 - \frac{1}{2k - 1}, 1 - \frac{1}{2k})$

(i) $S = \left\{\sqrt{x} + y^2 : x, y \in (0, 1] \cap \mathbb{Q}\right\}$

(j) $S = \left\{\frac{1}{n} + x : x \in [0, 1] \cap \mathbb{Q}, n = 1, 2, 3, \ldots\right\}$

(k) $S = \{x + y : x \in [0, 1] \cap \mathbb{Q}, y \in [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})\}$

(l) $S = \{x \in \mathbb{R} : x(x + 1) \leq 0 \text{ and } x \in \mathbb{R} \setminus \mathbb{Q}\}$

(m) $S = \{\frac{k}{n!} : k, n \in \mathbb{N}, \ \frac{k}{n!} < \sqrt{2}\}$

(n) $S = \bigcup_{n=1}^{10} \left([\frac{1}{n\sqrt{2}}, 2 - \frac{1}{n}] \setminus \mathbb{Q} \right)$

(o) $S = \{x^2 + y^3 + z^4 : \ x \in (-1, 0) \setminus \mathbb{Q}, \ y \in (0, 1) \cap \mathbb{Q}, \ z \in (-1, 1)\}$

92. For each of the following sequences $\{x_n\}$, show it converges and find its limit.

(a) $x_1 = 1$ and $x_{n+1} = \frac{x_n}{2} + \sqrt{x_n}$ for $n = 1, 2, 3, \ldots$.

(b) $x_1 = 1, \quad x_2 = 2 \quad$ and $\quad x_{n+1} = \sqrt{x_n} + \sqrt{x_{n-1}} \quad$ for $\quad n = 2, 3, 4, \cdots$.

(c) $x_1 = 1$ and $x_{n+1} = \frac{2 - x_n}{3 + x_n}$ for $n = 1, 2, 3, \ldots$.

(d) $x_1 = \frac{15}{16}$ and $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n = 1, 2, 3, \cdots$. Also, do the sequence $\left\{\frac{x_{n+1}}{x_n}\right\}$.

(e) $x_n = \frac{a_{n+1}}{a_n}$, for $n = 1, 2, 3, \ldots$, where $a_1 = 1, a_2 = 2$ and $a_{n+1} = a_n + a_{n-1}$ for $n = 2, 3, 4, \cdots$.

(f) $x_1 = 1$ and $x_{n+1} = 1 - \frac{1}{4x_n}$ for $n = 1, 2, 3, \ldots$.

(g) $x_1 = 4$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{4}{x_n}\right)$ for $n = 1, 2, 3, \ldots$. (*Hint*: Sketch the graph of $f(x) = x + \frac{4}{x}$ for $x \geq 2$.)

(h) $x_1 = 5$ and $x_{n+1} = 3 + \frac{4}{x_n}$ for $n = 1, 2, 3, \cdots$.

(i) $x_1 = 2$ and $x_{n+1} = 2 - \frac{1}{x_n}$ for $n = 1, 2, 3, \cdots$.

(j) $x_1 = 0$ and $x_{n+1} = \frac{x_n^2 + 4}{5}$ for $n = 1, 2, 3, \ldots$.

(k) $x_1 = 1$ and $x_{n+1} = \sqrt{x_n} - \frac{1}{4}$ for $n = 1, 2, 3, \ldots$.

(l) $x_1 = 3$ and $x_{n+1} = \sqrt{1 - \frac{1}{x_n + 1}}$ for $n = 1, 2, 3, \ldots$.

(m) $0 < x_1 < 1$ and $7x_{n+1} = x_n^3 + 6$ for $n = 1, 2, 3, \ldots$. (*Hint*: $x^3 - 7x + 6 = (x - 1)(x - 2)(x + 3)$.)

(n) $x_1 \in [1, \infty)$ and $x_{n+1} = \sqrt{3x_n - 2}$ for $n = 1, 2, 3, \ldots$.

(o) $x_1 = 0, x_2 \in [0, \frac{1}{2}]$ and $x_{n+1} = \frac{1}{3}(1 + x_n + x_{n-1}^3)$ for $n = 2, 3, 4, \ldots$.

93. If $\{a_n\}$ is a decreasing sequence (of nonnegative real numbers) converging to 0, show that the sequence $\{x_n\}$

converges, where

$$x_n = \sum_{k=1}^{n} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - \cdots + (-1)^{n+1} a_n.$$

94. Let $0 < a < b$ and $a_1 = a,\ \ b_1 = b,\ \ a_{n+1} = \dfrac{a_n + b_n}{2},\ \ b_{n+1} = \sqrt{\dfrac{a_n^2 + b_n^2}{2}}$ for $n = 1, 2, 3 \ldots$. Show that $\{a_n\}$ and $\{b_n\}$ both converge and their limits are the same.

95. (i) If $a \le b$ and $0 < t < 1$, then show that $a \le ta + (1-t)b \le b$.

(ii) Let

$$x_1 = 1, \quad x_2 = 2 \quad \text{and} \quad x_{n+1} = \frac{1}{3}x_n + \frac{2}{3}x_{n-1} \quad \text{for} \quad n = 2, 3, 4, \cdots.$$

Show that the sequence $\{x_n\}$ converges.

96. Show that the sequence $\{x_n\}$ given by

$$x_0 = 0, \quad x_1 = 1 \quad \text{and} \quad x_{n+1} = \sqrt{\frac{1}{4}x_n^2 + \frac{3}{4}x_{n-1}^2} \quad \text{for } n \in \mathbb{N}$$

converges and find its limit.

97. Let S be a set of real numbers such that every sequence in S has a convergent subsequence, show that S is bounded.

98. Let A be a nonempty subset of the *nonnegative* real numbers. If A is bounded above and $B = \{x^2 + y^2 : x, y \in A\}$, show B is bounded above and $\sup B = 2(\sup A)^2$.

99. Suppose $A_n \subseteq (-\infty, 2)$ and $x_n = \sup A_n$ for $n = 1, 2, 3, \ldots, 10$. Prove that

$$\sup\Big(\bigcup_{k=1}^{10} A_n\Big) = \max(x_1, x_2, \ldots, x_{10}).$$

100. Use the definitions of infimum and supremum to explain the statement: Let $f : \mathbb{R} \times \mathbb{R} \to [0, 1]$ be a function, $g(x) = \sup\{f(x, y) : y \in \mathbb{R}\}$ and $h(y) = \inf\{f(x, y) : x \in \mathbb{R}\}$. Show that $\sup\{h(y) : y \in \mathbb{R}\} \le \inf\{g(x) : x \in \mathbb{R}\}$.

101. Show that for every $x \in \mathbb{R}$, there is a strictly increasing sequence of irrational numbers $\{x_n\}$ converging to x.

102. Prove that $\lim\limits_{n\to\infty} \big(\frac{1}{n^2} - \frac{\sqrt{2}}{n^3}\big) = 0$ by checking the definition of limit.

103. Prove that $\lim\limits_{n\to\infty} \big(\frac{2}{n+1} - \frac{1}{n^2}\big) = 0$ by checking the definition of limit.

104. Given $\lim\limits_{n\to\infty} x_n = 0$. Prove that $\lim\limits_{n\to\infty} \big(x_n + \frac{1}{n}\big) = 0$ by checking the definition of limit.

105. Given $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$. Prove that $\lim\limits_{n\to\infty} x_n^n = 0$ by checking the definition of limit.

106. Given $\lim\limits_{n\to\infty} x_n = 8$. Prove that $\lim\limits_{n\to\infty} \sqrt[3]{x_n} = 2$ by checking the definition of limit.

107. Given sequences $\{x_n\}$ and $\{y_n\}$ both converge to A. For $n \in \mathbb{N}$, let $z_n = \max(x_n, y_n)$. Show that $\{z_n\}$ converges to A by checking the definition.

108. If $\{x_n\}$ is a sequence such that $|x_{k+1} - x_k| < \dfrac{1}{2^k}$ for $k = 1, 2, 3, \ldots$, then show that $\{x_n\}$ is a Cauchy sequence.

109. (a) Let S be an open interval and $x_0 \in S$. For a function $f : S \to \mathbb{R}$, state the definition of $f(x)$ *converges to* L (or has *limit* L) as x tends to x_0 in S.
(b) Let $f : (1, 3) \to \mathbb{R}$ be defined by $f(x) = x^2 + \dfrac{1}{x}$. Prove that $\lim\limits_{x\to 2} f(x) = \dfrac{9}{2}$ by checking the definition.
(c) Let $f : (1, 4) \to \mathbb{R}$ be defined by $f(x) = |x^2 - 9|$. Prove that $\lim\limits_{x\to 2} f(x) = 5$ by checking the definition.

110. If $f, g : (0, 1) \to \mathbb{R}$ are increasing functions, show that $h(x) = \max(f(x), g(x))$ has countably many jumps on the interval $(0, 1)$.

111. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at $x \in \mathbb{Z}$, but discontinuous at every $x \notin \mathbb{Z}$. Be sure to give reasons to support your example.

112. (a) State the intermediate value theorem.
(b) Let $f : [0, 2] \to \mathbb{R}$ be continuous and $f(0) = f(2)$. Show that there exists $c \in [0, 1]$ such that $f(c) = f(c+1)$.
(c) Show that there is a nonzero continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(t) + g(2t) + g(3t) = g(4t) + g(5t)$ for every $t \in \mathbb{R}$. (*Hint*: Try $g(t) = |t|^r$ for some constant r.)

113. (a) Show that the set $T = \{x : \sin x \in \mathbb{Q}\}$ is countable. (*Hint:* For a fixed rational r, how many solutions of $\sin x = r$ are there in the interval $[k\pi, k\pi + 2\pi)$?)
(b) If $f : [0, 1] \to \mathbb{R}$ is continuous and $\sin f(x) \in \mathbb{Q}$ for every $x \in [0, 1]$, then show that f is a constant function.

114. Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that for every $c \in \mathbb{R}$, $f(x) = c$ has exactly 2 solutions.

115. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f(x + y) = f(x) + f(y)$ for every $x, y \in \mathbb{R}$ and $|f(x)| \le x^4/|x|$ for every $x \neq 0$.
(a) Show that f is continuous at some $x \in \mathbb{R}$.
(b) Show that f is continuous at every $x \in \mathbb{R}$.
(c) Give an example of such a function.

116. Suppose $f, g : [1, 2] \to [3, 4]$ are continuous functions and also $\{g(x) : x \in [1, 2]\} = [3, 4]$. Show that there is $c \in [1, 2]$ such that $f(c) = g(c)$.

117. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $|f(x) - f(y)| \le \frac{1}{2}|x - y|$ for every $x, y \in \mathbb{R}$.
(a) Let $w \in \mathbb{R}$. Define $x_1 = w$ and $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$. Show that $\{x_n\}$ is a Cauchy sequence. (*Hint*: How is $|x_{k+1} - x_k|$ compared to $|x_2 - x_1|$?)
(b) Show that there is $x \in \mathbb{R}$ such that $f(x) = x$.

118. Give an example of a function $f : (0, 2) \to \mathbb{R}$ that is differentiable for every $x \in (0, 2)$, but $f'(x)$ is not continuous at $x = 1$. Be sure to give reasons to support your example.

119. Let $f : (0, \pi) \to \mathbb{R}$ satisfies $\sqrt{|f(a) - f(b)|} \le \sin|a - b|$ for every $a, b \in (0, \pi)$. Show that f is a constant function.

120. (a) State the mean value theorem.
(b) For the function $f(x) = \sin x$, find the smallest constant K such that $|f(b) - f(a)| \le K|b - a|$ holds for every $a, b \in \mathbb{R}$.

121. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $\lim\limits_{x\to 0} f'(x)$ exists, then show that f' is continuous at 0.

122. Use the mean value theorem to prove that $f(x) = \sin 5x$ is uniformly continuous.

123. Prove that if $f : \mathbb{R} \to (0, +\infty)$ is uniformly continuous, then the function $g(x) = \sqrt{f(x)}$ is also uniformly continuous.

124. (a) State Lebesgue's theorem.
(b) Let $f, g : [0, 2] \to [0, 1]$ be Riemann integrable. Prove that the function $h : [0, 2] \to [0, 1]$ defined by $h(x) = \begin{cases} f(x) & \text{if } x \in [0, 1) \\ g(x) & \text{if } x \in [1, 2] \end{cases}$ is Riemann integrable.

125. If $f, g : [0, 1] \to \mathbb{R}$ are Riemann integrable, show that the function $h(x) = \min(f(x), g(x))$ is Riemann integrable on $[0, 1]$.

126. For every positive integer n, give an example of a Riemann integrable function $f_n : [0, 1] \to [0, 1]$ such that $\lim_{n\to+\infty} f_n(x)$ exists for every $x \in [0, 1]$, but the function $f(x) = \lim_{n\to+\infty} f_n(x)$ is not Riemann integrable on $[0, 1]$. Be sure to give reasons to support your example.

127. (a) Determine if the improper integral $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 3x}{1+x^2}\,dx$ exists or not.
(b) Determine if the principal value P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 3x}{1+x^2}\,dx$ exists or not.
(c) Determine if the improper integral $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\,dx$ exists or not.
(d) Determine if the principal value P.V. $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt[3]{x}}\,dx$ exists or not.
(e) Determine if the improper integral $\displaystyle\int_{-\infty}^{\infty} \sin x\,dx$ exists or not.
(f) Determine if the principal value P.V. $\displaystyle\int_{-\infty}^{\infty} \sin x\,dx$ exists or not.

128. (2002 L1 Midterm) Let $f(x) = x^2 + 3$. Determine if the set $S = \{f(w+z) : w \in [0, 1] \cap \mathbb{Q}, z \in [2, 3] \setminus \mathbb{Q}\}$ has an infimum and a supremum. If they exist, find them and give reasons to support your answers.

129. (2002 L1 Midterm) Determine if each of the series $\sum\limits_{k=1}^{\infty} \dfrac{2^k\sqrt{k}}{(2k)!}$ and $\sum\limits_{k=1}^{\infty} (\cos k)\big(\sin \dfrac{1}{k^2}\big)$ converges or diverges.

130. (2002 L1 Midterm) Let S be the set of all lines L on the coordinate plane such that L passes through two distinct points in $\mathbb{Q} \times \mathbb{Q}$ and T be the set of all points, each of which is the intersection of a pair of distinct lines in S. Determine if T is a countable set or not.

131. (2002 L1 Midterm) Given $x_n \neq -1$ for all $n \in \mathbb{N}$. If $\lim\limits_{n\to\infty} x_n = 0$, then show that $\lim\limits_{n\to\infty} \dfrac{x_n}{1+x_n} = 0$ by checking the definition of limit.

132. (2002 L1 Midterm) Given $\{x_n\}$ converges to $w \in \mathbb{R}$ and $x_n < w$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, let

$$y_n = \sup\Big\{x_{2k} : k \in \mathbb{N}, k \le \frac{n+1}{2}\Big\}.$$

Show that $\{y_n\}$ converges to w.

133. (2002 L2 Midterm) Let $f(x) = \sin x$. Determine if the set $S = \Big\{f(w) - \dfrac{1}{n} : w \in \big(\dfrac{\pi}{4}, \dfrac{\pi}{3}\big) \setminus \mathbb{Q}, n \in \mathbb{N}\Big\}$ has an infimum and a supremum. If they exist, find them and give reasons to support your answers.

134. (2002 L2 Midterm) Determine if each of the series $\sum\limits_{k=1}^{\infty} \dfrac{(2k+1)^5}{k!}$ and $\sum\limits_{k=1}^{\infty} \dfrac{\cos k}{k^4+k+1}$ converges or diverges.

135. (2002 L2 Midterm) Let S be the set of all circles on the coordinate plane that pass through $(1, 1)$ and another point $(x\sqrt{2}, x\sqrt{2})$ for some $x \in \mathbb{Q}$. Determine if S is a countable set or not.

136. (2002 L2 Midterm) Given $x_n \neq 1$ for all $n \in \mathbb{N}$. If $\lim\limits_{n\to\infty} x_n = 0$, then show that $\lim\limits_{n\to\infty} \dfrac{x_n}{x_n - 1} = 0$ by checking the definition of limit.

137. (2002 L2 Midterm) Given $\{x_n\}$ converges to $w \in \mathbb{R}$ and $x_n > w$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, let

$$y_n = \inf\left\{x_{2k} : k \in \mathbb{N}, k \le \frac{n+1}{2}\right\}.$$

Show that $\{y_n\}$ converges to w.

138. (2002 L3 Midterm) Let $f(x, y) = x^2 + y^2$. Determine if $S = \{f(1 + (-1)^n, w) : n \in \mathbb{N}, w \in [1, \sqrt{2}) \setminus \mathbb{Q}\}$ has an infimum and a supremum. If they exist, find them and give reasons to support your answers.

139. (2002 L3 Midterm) Determine if each of the series $\sum\limits_{k=1}^{\infty} \dfrac{2^k}{k^3(3k)!}$ and $\sum\limits_{k=1}^{\infty} \left(\dfrac{1}{e} + \dfrac{1}{k}\right)^k \sin k$ converges or diverges.

140. (2002 L3 Midterm) Let $x_1 = 1$ and $x_{n+1} = \dfrac{1}{2}\sqrt{x_n^2 + 4x_n}$ for $n = 1, 2, 3, \ldots$. Show that $\{x_n\}$ converges. Find the limit of $\{x_n\}$.

141. (2002 L3 Midterm) Let S be the set of all ordered pairs (p, C), where $p = (x, y) \in \mathbb{Q} \times \mathbb{Q}$ and C is the circle with center p and radius $|xy| + 1$. Determine if S is a countable set or not.

142. (2002 L3 Midterm) If $\lim\limits_{n\to\infty} x_n = 1$, then show that $\lim\limits_{n\to\infty} (x_n^2 - 1) = 0$ by checking the definition of limit.

143. (2002 L3 Midterm) Let $\{x_n\}$ be a bounded sequence of real numbers. Let

$$S = \{w : \text{there exist infinitely many } n \in \mathbb{N} \text{ such that } w < x_n\}.$$

(a) Show that S is nonempty and bounded above.

(b) Show that $\{x_n\}$ has at least one subsequence $\{x_{n_k}\}$ converging to $s = \sup S$.

144. (2004 L1 Midterm) Find (with proof) all positive numbers b such that the series $\sum\limits_{k=1}^{\infty} \left(b + \dfrac{1}{k}\right)^k$ converges.

145. (2004 L1 Midterm) Let A be a nonempty countable subset of $\mathbb{R}$. Let

$$S = \{\theta : \theta \in \mathbb{R},\ \sin\theta \in A\} \quad \text{and} \quad T = \{\theta : \theta \in \mathbb{R},\ \sin\theta \notin A\}.$$

Determine (with proof) if each of the sets S and T is countable or uncountable.

146. (2004 L1 Midterm) Show that the sequence $\{x_n\}$ given by

$$x_1 = 6 \quad \text{and} \quad x_{n+1} = \frac{x_n^2 + 4}{2x_n + 3} \quad \text{for } n = 1, 2, 3, \ldots$$

converges and find its limit.

147. (2004 L1 Midterm) If $x_n \neq -1$ for all n and $\lim_{n\to\infty} \frac{x_n}{x_n+1} = \frac{1}{2}$, then prove that $\lim_{n\to\infty} x_n = 1$ by checking the definition of limit.

148. (2004 L1 Midterm) Let $S \subseteq [0, \frac{\pi}{2}]$, $T = \{\cos^2 a + \cos^2 b : a, b \in S\}$ and $U = \{\sin c : c \in S\}$. If $\sup T = \frac{1}{2}$, then find the infimum of U with proof.

149. (2004 L2 Midterm) Let S be the set of all intersection points $(x, y) \in \mathbb{R}^2$ of the graphs of the equations $x^2 + my^2 = 1$ and $mx^2 + y^2 = 1$, where $m \in \mathbb{Z} \setminus \{-1, 1\}$. Determine if S is countable or uncountable. Provide a proof of your answer.

150. (2004 L2 Midterm) Let $x_1 = 9$ and $x_{n+1} = \frac{\sqrt{x_n} + 2x_n}{3}$ for $n = 1, 2, 3, \ldots$. Prove that $\{x_n\}$ converges. Find the limit of $\{x_n\}$.

151. (2004 L2 Midterm) Let $a_k > 0$ for $k = 1, 2, 3, \ldots$ and $\sum_{k=1}^{\infty} a_k$ converges. Determine all positive real number b such that the series $\sum_{k=1}^{\infty} \frac{(b + a_k)^k}{k}$ converges. Be sure to give a proof of your answers.

152. (2004 L2 Midterm) Prove that if $\lim_{k\to\infty} x_{2k} = 0.5$ and $\lim_{k\to\infty} x_{2k+1} = 0.6$, then the sequence $\lim_{n\to\infty} x_n^n = 0$ by checking the definition of limit.

153. (2004 L2 Midterm) Let I be a nonempty set. For every $t \in I$, let A_t be a nonempty subset of $[0, 1]$. Let $x_t = \sup A_t$. Prove that if $A = \bigcup_{t\in I} A_t$, then $\sup A = \sup\{x_t : t \in I\}$.

154. (2002 Final) Determine if the improper integral $\int_0^1 \frac{\cos 3x}{\sqrt{x}}\, dx$ exists or not.

155. (2002 Final) Prove that there does not exist any continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f\big(f(x)\big) + x = 0$ for every $x \in \mathbb{R}$.

156. (2002 Final) Let $\{x_n\}$ be a Cauchy sequence of real numbers. Prove that $\{\sin 5x_n\}$ is also a Cauchy sequence by checking the definition of a Cauchy sequence.

157. (2002 Final) Let $f, g : [0, 2] \to \mathbb{R}$ be Riemann integrable. Prove that $h : [0, 2] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} \max\big(f(x), g(x)\big) & \text{if } x \in [0, 1] \\ \min\big(f(x), g(x)\big) & \text{if } x \in (1, 2] \end{cases}$$

is also Riemann integrable on $[0, 2]$.

158. (2003 Final) Determine (with proof) if each of $\sum_{k=1}^{\infty} \sin^k\big(1 + \frac{1}{k}\big)$ and $\sum_{k=1}^{\infty} \frac{1 - \cos(1/k)}{1/k^2}$ converges or not.

159. (2003 Final) Let A be a nonempty subset of $\mathbb{R}$ such that $\inf A = 0$ and $\sup A = 1$. Determine the infimum and supremum of $S = \{a^3 - 4a + 1 : a \in A\}$.

160. (2003 Final) Let P be a countable set of points in $\mathbb{R}^2$. Prove that there exists a circle C with the origin as center and positive radius such that every point of the circle C is not in P. (Note points inside the circle do not belong to the circle.)

161. (2003 Final) Let $f : \mathbb{R} \to [u, v]$ be a function and $w \in \mathbb{R}$. For every $r > 0$, let

$$M(r) = \sup\{f(t) : 0 < |t - w| < r\} \quad \text{and} \quad m(r) = \inf\{f(t) : 0 < |t - w| < r\}.$$

(a) Prove that $\lim_{r\to 0+} m(r)$ and $\lim_{r\to 0+} M(r)$ exist.

(b) Prove that $\lim_{x\to w} f(x) = L$ if and only if $\lim_{r\to 0+} m(r) = L = \lim_{r\to 0+} M(r)$.

162. (2003 Final) Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Prove that $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \dfrac{1}{1+f(x)^2}$ is also uniformly continuous.

163. (2003 Final) Let $f : [0,1] \to [-1,1]$ be Riemann integrable. Using the integral criterion, prove that $g(x) = \begin{cases} f(x) & \text{if } 0 < x < 1 \\ 0 & \text{if } x = 0 \text{ or } 1 \end{cases}$ is also Riemann integrable on $[0,1]$.

164. (2004 Final) Determine whether the improper integral $\int_0^\infty \frac{\sin x}{x^{3/2}}\,dx$ converge or not.

165. (2004 Final) Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function and $g : \mathbb{R} \to \mathbb{R}$ be a decreasing function. Prove that $f(x)g(x)$ is discontinuous for only countably many $x \in \mathbb{R}$.

166. (2004 Final) Let $f(x) = \begin{cases} 0 & \text{if } x \in [0,1]\setminus\mathbb{Q} \\ 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \text{ for } m, n \in \mathbb{N} \text{ with no common prime factor.} \end{cases}$

Prove that there exists a Riemann integrable function $g : [0,1] \to [0,1]$ such that the composition function $g \circ f : [0,1] \to [0,1]$ is not Riemann integrable on $[0,1]$.

167. (2004 Final) Let $f : (0,+\infty) \to \mathbb{R}$ satisfy $|f(x) - f(y)| \le |\sin(x^2) - \sin(y^2)|$ for all $x, y > 0$. Prove that the sequence $x_1, x_2, x_3, \ldots$ given by $x_n = f(1/n)$ is a Cauchy sequence.

168. (2005 Spring Final) Determine the supremum of $S = \bigcup_{n=1}^{\infty} \left\{ \frac{1}{x} + \frac{1}{n\sqrt{2}} : x \in (2,3]\setminus\mathbb{Q} \right\}$.

169. (2005 Spring Final) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{2+3x}{x^2+4}$ is continuous at 2 by checking the ε-δ definition of a function continuous at a point.

170. (2005 Spring Final) Let $x_n > 0$ for $n = 1, 2, 3, \ldots$. If $\{x_n\}$ is a Cauchy sequence, then prove that $\{e^{-x_n}\}$ is a Cauchy sequence by checking the definition of a Cauchy sequence.

171. (2005 Spring Final) (a) Let $S \subseteq [0,1]$. If S is a set of measure 0, then prove that $T = \{x^2 : x \in S\}$ is a set of measure 0.

(b) Let $f : [0,1] \to [0,1]$ be integrable. Prove that $h : [0,1] \to [0,1]$ defined by $h(x) = f(\sqrt{x})$ is integrable. (*Caution*: In general it is false that f integrable and g continuous on $[a,b]$ imply $h = f \circ g$ integrable on $[a,b]$.)

172. (2005 Fall Exam, Version 1) (a) Determine with proof if $\sum_{k=1}^{\infty} \frac{3^k}{(2k)!k!}$ converges or not.

(b) Let $a_k \ne -1$ for $k = 1, 2, 3, \ldots$ and $\sum_{k=1}^{\infty} |a_k|$ converges. Determine with proof if $\sum_{k=1}^{\infty} \frac{a_k}{1+a_k}$ converges or not.

173. (2005 Fall Exam, Version 1) Define $x_1 = 4$ and $x_{n+1} = 4 - \dfrac{4}{x_n}$ for $n = 1, 2, 3, \ldots$. Prove that $x_1, x_2, x_3, \ldots$ converges and find its limit.

174. (2005 Fall Exam, Version 1) Let A be a nonempty bounded subset of $\mathbb{R}$ such that $\sup A = 1$ and $\inf A = 0$. Let

$$S = \{x - y\sqrt{2} : x \in A,\ y \in [-2, 2)\}.$$

Determine the supremum and infimum of S with proof.

175. (2005 Fall Exam, Version 1) If $\lim_{n\to\infty} a_n = 2$, $\lim_{n\to\infty} b_n = 3$ and all $b_n \neq -2$, then prove that $\lim_{n\to\infty} \frac{a_n + 3}{b_n + 2} = 1$ by checking the definition of limit.

176. (2005 Fall Exam, Version 2) (a) Determine with proof if $\sum_{k=1}^{\infty} \frac{(3k)!}{k!2^k}$ converges or not.

(b) Let $\cos a_k \neq 0$ for $k = 1, 2, 3, \ldots$ and $\sum_{k=1}^{\infty} |a_k|$ converges. Determine with proof if $\sum_{k=1}^{\infty} \frac{a_k}{\cos a_k}$ converges or not.

177. (2005 Fall Exam, Version 2) Define $x_1 = 1$ and $x_{n+1} = \frac{x_n^2 + 15}{8}$ for $n = 1, 2, 3, \ldots$. Prove that $x_1, x_2, x_3, \ldots$ converges and find its limit.

178. (2005 Fall Exam, Version 2) Let A be a nonempty bounded subset of $\mathbb{R}$ such that $\sup A = 6$ and $\inf A = 2$. Let

$$S = \{\frac{x}{y} - xy : x \in A,\ y \in \big[\frac{1}{2}, 1\big)\}.$$

Determine the supremum and infimum of S with proof.

179. (2005 Fall Exam, Version 2) If $\lim_{n\to\infty} a_n = 1$ and all $a_n \neq n$, then prove that $\lim_{n\to\infty} \frac{a_n^2 + n}{n - a_n} = 1$ by checking the definition of limit.

180. (2006 Spring Exam)(a) Let $f : S \to \mathbb{R}$ be a function and x_0 be an accumulation point of S. State the definition of $f(x)$ <u>*converges*</u> to a real number L as x tends to x_0.

(b) Let $f : (0.5, +\infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x + \frac{1}{x}}$. Prove that $\lim_{x\to 1} f(x) = \sqrt{2}$ by checking the definition.

181. (2006 Spring Exam) Define $a_1 = 1$ and $a_{n+1} = \frac{n}{n+1} a_n + \frac{\cos n}{(n+1)^3}$ for $n = 1, 2, 3, \ldots$. Prove that $\lim_{n\to\infty} n a_n$ exists in $\mathbb{R}$.

182. (2006 Spring Exam) Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be continuous. Also, let $f(x)$ be differentiable for all $x \in (a, b)$. Prove that if the graph of f is not a line segment, then there exist numbers x_1 and x_2 in the open interval (a, b) such that

$$f'(x_1) < \frac{f(b) - f(a)}{b - a} < f'(x_2).$$

183. (2006 Spring Exam) Let $f, g : [0, 1] \to \mathbb{R}$ be continuous. If there exists a sequence of numbers $x_1, x_2, x_3, \ldots \in [0, 1]$ such that $g(x_n) = f(x_{n+1})$ for $n = 1, 2, 3, \ldots$, then prove that there exists $w \in [0, 1]$ such that $g(w) = f(w)$.

<u>*Caution*</u> Be careful, x_{n_i} converges does not imply x_{n_i+1} converges !!!