MATH 2031 Introduction to Real Analysis

May 9, 2013

Tutorial Note 20

Sequences and Series of Functions (Con't)

(I) **Definition (Pointwise Convergence):**
Let E be a set. Then a sequence of functions $S_n : E \to \mathbb{R}$ is said to converge pointwise on E to a function $S : E \to \mathbb{R}$
if $\forall x \in E$, $\lim\limits_{n\to\infty} S_n(x) = S(x)$.

In this case, we say that $S(x)$ is the pointwise limit of the sequence $S_n(x)$.

(II) **Definition (Pointwise Convergence of series of functions):**

Given functions $f_k : E \to \mathbb{R}$, the series $\sum\limits_{k=1}^{\infty} f_k$ is said to converge pointwise on E to a function $S : E \to \mathbb{R}$

if $\forall x \in E$, $\sum\limits_{k=1}^{\infty} f_k = \lim\limits_{n\to\infty}(f_1(x) + f_2(x) + \cdots + f_n(x)) = S(x)$, i.e. $\underbrace{S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)}_{\text{sequence of partial sums}}$ converges pointwise on E to $S(x)$.

(III) **Power series:**

(i) **Definition:** A power series is a function of the form $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$,
where $c, a_0, a_1, \cdots$ are numbers and c is called the center of the power series.
$E = \left\{ x \in \mathbb{R} \middle| \sum\limits_{k=0}^{\infty} a_k(x-c)^k \text{ converges} \right\}$ is the domain of convergence of the power series.

(ii) **Domain theorem for Power series**

The domain of a power series $f(x) = \sum\limits_{k=0}^{\infty} a_k(x-c)^k$ is a non-empty interval with midpoint c.

The half-length of the interval is the radius of convergence $R = \dfrac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}$.

Remark:
Both of the endpoints may or may not be in the domain.

(iii) **Definition (Taylor series of functions):**
If a function $f(x)$ is infinitely differentiable at c, then the Taylor series of f about c is the series

$$\sum_{k=0}^{\infty} a_k(x-c)^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

(iv) **Taylor series theorem**
If $f : (a,b) \to \mathbb{R}$ is infinitely differentiable, $c \in (a,b)$ and $\exists$ constants $M, \alpha > 0$ such that $|f^{(n)}(x)| \le \alpha M^n$ for every $x \in (a,b)$ and $n \in \mathbb{N}$,
then $\sum\limits_{k=0}^{\infty} \dfrac{f^{(k)}(c)}{k!}(x-c)^k$ converges pointwise on (a,b) to $f(x)$.

(v) **Taylor Formula with Integral Remainder**
Let f be n-times differentiable on (a,b). Then for every $x, c \in (a,b)$, if $f^{(n)}$ is integrable on the closed interval with endpoints x and c, then

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x),$$

where $R_n(x) = \dfrac{1}{(n-1)!}\displaystyle\int_c^x (x-c)^{n-1} f^{(n)}(t)dt$.

(vi) **Mean Value Theorem for Integral**
Let f be continuous on $[a,b]$ and $g \geq 0$ be integrable on $[a,b]$. Then $\exists x_0 \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(x_0)\int_a^b g(x)dx.$$

(vii) **Taylor Formula with Cauchy Form Remainder**
Let f be n-times differentiable on (a,b). For every $x, c \in (a,b)$, if $f^{(n)}$ is continuous (hence, integrable) on the closed interval with x, c as endpoints, then there exists x_n between x and c such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x)$$

where $R_n(x) = \dfrac{1}{(n-1)!}\displaystyle\int_c^x (x-c)^{n-1} f^{(n)}(t)dt = \underbrace{\dfrac{(x-c)(x-x_n)^{n-1}f^{(n)}(x_n)}{(n-1)!}}_{\text{Cauchy form remainder}}$

Problem 1 Define $S_n : [0,1] \to \mathbb{R}$ by $S_n(x) = x^n$, find the pointwise limit of S_n.
Solution:
For $0 \leq x < 1$, $\lim_{n\to\infty} x^n = 0$ and for $x = 1$, $\lim_{n\to\infty} x^n = 1$, so the pointwise limit of S_n is

$$S(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

Note that even though every S_n is continuous, the pointwise limit S may not necessarily be continuous.
When you learn the concept of uniform continuity later, this will serve as an example of a sequence of functions which converges pointwise but not uniformly.

Problem 2 Find the domain of convergence of the series of functions $\displaystyle\sum_{k=2}^{\infty} \frac{x^k}{\ln k}$.

Solution:
$\displaystyle\sum_{k=2}^{\infty} \frac{x^k}{\ln k}$ converges only if $\displaystyle\lim_{k\to\infty}\left|\frac{x^{k+1}}{\ln(k+1)}\frac{\ln k}{x^k}\right| < 1$. Then for $|x| < 1$,

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{x^{k+1}}{\ln(k+1)}\frac{\ln k}{x^k}\right| &= \lim_{k\to\infty}\frac{|x|\ln k}{\ln(k+1)} \\ &= \lim_{k\to\infty}\frac{|x|(k+1)}{k} \\ &= |x| < 1\end{aligned}$$

We also need to check the boundary points $x = \pm 1$.

For $x = 1$ $\displaystyle\sum_{k=2}^{\infty}\frac{1}{\ln k} \geq \sum_{k=2}^{\infty}\frac{1}{k}$ which diverges by p-test and comparison test.

For $x = -1$ Since $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{\ln k}$ is an alternating series and $\dfrac{1}{\ln k}$ decreases to zero as $k \to \infty$, by alternating series test it converges.

Thus, the domain of convergence of $\displaystyle\sum_{k=2}^{\infty}\frac{x^k}{\ln k}$ is $[-1, 1)$.