Solution of Midterm

① Note $10^{xy}+r-y^3=xy \Leftrightarrow r=xy-10^{xy}+y^3$

Let $W=\{xy-10^{xy}+y^3 : x,y\in\mathbb{Q}\}$. For $(x,y)\in\mathbb{Q}\times\mathbb{Q}$, let $W_{(x,y)}=\{xy-10^{xy}+y^3\}$, then $W_{(x,y)}$ has 1 element $\Rightarrow W_{(x,y)}$ is countable.

Then $W=\bigcup_{\underbrace{(x,y)\in\mathbb{Q}\times\mathbb{Q}}_{\text{countable}}} W_{(x,y)}$ is countable by countable union theorem.

Finally, $\underbrace{\mathbb{R}}_{\text{uncountable}}\setminus\underbrace{W}_{\text{countable}}$ is uncountable $\Rightarrow$ infinite. So there exist infinitely many real numbers r such that the equation $10^{xy}+r-y^3=xy$ does not have any solution with $x,y\in\mathbb{Q}$.

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② $\inf A=0$ and $\sup A=3 \Rightarrow A\subseteq[0,3]$.

$$\left.\begin{array}{l} x\in[1,2]\setminus\mathbb{Q} \\ y\in A\end{array}\right\} \Rightarrow \begin{array}{l} 1\le x\le 2 \\ 0\le y\le 3\end{array} \Rightarrow \underbrace{1+2^0+0}_{=2}\le x+2^{xy}+y\le \underbrace{2+2^6+3}_{=69}$$

$\Rightarrow B$ is bounded.

Let $x_n=1+\frac{1}{n\sqrt2}$. Since $\inf A=0$, $\exists\, y_n\in A$ such that $\lim_{n\to\infty} y_n=0$ by infimum limit theorem. Then $x_n+2^{x_ny_n}+y_n\in B$ and $\lim_{n\to\infty} x_n+2^{x_ny_n}+y_n=2$.

By infimum limit theorem, $\inf B=2$.

Let $x_n'=2-\frac{1}{n\sqrt2}$. Since $\sup A=3$, $\exists\, y_n'\in A$ such that $\lim_{n\to\infty} y_n'=3$ by supremum limit theorem. Then $x_n'+2^{x_n'y_n'}+y_n'\in B$ and $\lim_{n\to\infty} x_n'+2^{x_n'y_n'}+y_n'=69$.

By supremum limit theorem, $\sup B=69$.

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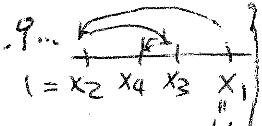
③ **Sketch** $x_1=11$, $x_2=\frac{18}{11+7}=1$, $x_3=\frac{18}{1+7}=\frac{9}{4}=2.25$, $x_4=\frac{18}{\frac94+7}=\frac{72}{37}=1.9\ldots$



We claim $0<x_{2n}<x_{2n+2}<x_{2n+1}<x_{2n-1}$ for $n=1,2,3,\ldots$

Case $n=1$: $0<x_2=1<x_4=\frac{72}{37}<x_3=\frac94<x_1=11$.

Suppose case n is true. Then $0<x_{2n}<x_{2n+2}<x_{2n+1}<x_{2n-1}$. Adding 7 to all parts, we get $7+x_{2n}<7+x_{2n+2}<7+x_{2n+1}<7+x_{2n-1}$. Taking reciprocal and multiplying by 18, we get

$$\underbrace{\frac{18}{7+x_{2n}}}_{=x_{2n+1}}>\underbrace{\frac{18}{7+x_{2n+2}}}_{=x_{2n+3}}>\underbrace{\frac{18}{7+x_{2n+1}}}_{=x_{2n+2}}>\underbrace{\frac{18}{7+x_{2n-1}}}_{=x_{2n}}$$

Adding 7 to all parts, we get $7+x_{2n+1}>7+x_{2n+3}>7+x_{2n+2}>7+x_{2n}$.

Taking reciprocal and multiply by 18, we get $18/(7+x_{2n+1})<18/(7+x_{2n+3})<18/(7+x_{2n+2})<18/(7+x_{2n})$

So $x_{2n+2} < x_{2n+4} < x_{2n+3} < x_{2n+1}$. By MI, the claim is true.

By the nested interval theorem, $\lim_{n\to\infty} x_{2n} = a$ and $\lim_{n\to\infty} x_{2n+1} = b$ exist.

$b = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} \frac{18}{7+x_{2n}} = \frac{18}{7+a}$ and $a = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} \frac{18}{7+x_{2n+1}} = \frac{18}{7+b}$.

$\Rightarrow b(7+a) = 18 = a(7+b) \Rightarrow 7b + ab = 7a + ab \Rightarrow a = b$.

So $\lim_{n\to\infty} x_n = a$ by the intertwining sequence theorem. Then $a = \frac{18}{7+a}$

$\Rightarrow a^2 + 7a - 18 = 0 \Rightarrow (a+9)(a-2) = 0 \Rightarrow a = -9$ or 2. $\therefore \lim_{n\to\infty} x_n = 2$.
since $x_n > 0$

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④ Sketch $\frac{6n^2+n-3}{1+2n^2} \approx \frac{6n^2}{2n^2} = 3$, $\frac{n+5\sqrt{n}+\sqrt[3]{n}}{6+n} \approx \frac{n}{n} = 1$ $\quad 1 \le \sqrt[3]{n} \le \sqrt{n}$

$\left|\frac{6n^2+n-3}{1+2n^2} - 3\right| = \frac{|n-6|}{1+2n^2} \le \frac{n+6n}{2n^2} = \frac{7}{2n} < \frac{\varepsilon}{2}$ if $n > \frac{7}{\varepsilon}$ $\quad$ if $n > \left(\frac{24}{\varepsilon}\right)^2$

$\left|\frac{n+5\sqrt{n}+\sqrt[3]{n}}{6+n} - 1\right| = \frac{|5\sqrt{n}+\sqrt[3]{n}-6|}{6+n} \le \frac{5\sqrt{n}+\sqrt[3]{n}+6}{n} \le \frac{5\sqrt{n}+\sqrt{n}+6\sqrt{n}}{n} = \frac{12}{\sqrt{n}} < \frac{\varepsilon}{2}$

$\forall \varepsilon > 0$, by Archimedian principle, $\exists K \in \mathbb{N}$ such that $K > \max\left(\frac{7}{\varepsilon}, \left(\frac{24}{\varepsilon}\right)^2\right)$.

Then $n \ge K \Rightarrow n > \frac{7}{\varepsilon}$ and $n > \left(\frac{24}{\varepsilon}\right)^2$

$$\Rightarrow \left|\left(\frac{6n^2+n-3}{1+2n^2} + \frac{n+5\sqrt{n}+\sqrt[3]{n}}{6+n}\right) - 4\right| = \left|\left(\frac{6n^2+n-3}{1+2n^2} - 3\right) + \left(\frac{n+5\sqrt{n}+\sqrt[3]{n}}{6+n} - 1\right)\right|$$

$$\le \left|\frac{6n^2+n-3}{1+2n^2} - 3\right| + \left|\frac{n+5\sqrt{n}+\sqrt[3]{n}}{6+n} - 1\right| = \frac{|n-6|}{1+2n^2} + \frac{|5\sqrt{n}+\sqrt[3]{n}-6|}{6+n}$$

$$\le \frac{n+6n}{2n^2} + \frac{5\sqrt{n}+\sqrt[3]{n}+6}{n} \le \frac{7}{2n} + \frac{5\sqrt{n}+\sqrt{n}+6\sqrt{n}}{n} = \frac{7}{2n} + \frac{12}{\sqrt{n}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
