Lecture One 31-01-2019

Welcome to Math 2033 (Mathematical analysis)

Main items in the Syllabus.

Instructor : Prof. Hai Zhang

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Office hour : Tue / Thu : 10 am - 11 am

or by appointment

prerequisite : Math 1014 / 1018 / 1020 / 1024

Lecture Notes : Prof. Kin Y. Li 's lecture notes

available on canvas

All the course related materials will be posted on canvas.

Grading Scheme :

Homework Type I : 10 % (proofs of theorems)

Submit to TA before mid-term / Final exam.

Homework Type II : 10% (regular problems, available on canvas)

Submit to TA on specified dates.

Midterm exam : 30% (2 hours. will be arranged in week 6-8)

Final Exam : 50% (2 hours. will be arranged by ARRO)

Have questions : contact instructor or TAs

Course Overview

Algebra : Matrix , quadratic equations

Geometry : planes , curves , surfaces

Number Theory : integers, prime numbers

Calculus : Compute derivatives and integrals , and applications

Analysis : A rigorous foundation of playing with infinity : rigorous definition of limit , continuity , differentiation integration .

Example 1 : what is $\sqrt{2}$.

Solution to $x^2 = 2$

$\sqrt{2} \approx 1.41421356237\ldots$

Approximate value of $\sqrt{2}$, not the real $\sqrt{2}$.

History : In 1800-1600 BC, Babylonian provided an approximation

$\sqrt{2} \approx 1.41421\overline{296}$

Story : In 5 BC, Hippasus discovered that $\sqrt{2}$ is an irrational number. But Pythagoras, believed in the absoluteness of numbers, did not accept the fact and sentenced Hippasus to death.

We need the concept of limit (or infinity) to understand the precise meaning of $\sqrt{2}$; For rational number, the concept of limit (infinity) is NOT needed.

$$\sqrt{2} = \lim_{n \to \infty} a_n$$

where $a_1 = 1$, $a_2 = \frac{a_1}{2} + \frac{1}{a_1}$, $a_3 = \frac{a_2}{2} + \frac{1}{a_2}$

$\ldots$, $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$

Questions: what are real numbers?

Example 2: What is the meaning of

$$\lim_{n\to\infty} a_n = A$$

Intuitively: a_n tends to A as n tends to ∞.

or a_n is closed to A as desired when n is sufficiently large

Question: how large is sufficiently large?

$n > 10000$? $n > 10^{10}$? $n > 100^{100}$?

how close is close?

$|a_n - A| < 10^{-3}$? $|a_n - A| < 10^{-10}$?

One need precise definition to aviod these ambiguity.

We will introduce $\varepsilon - N$ language to make things rigorous !!.

Example 3 : what is continuity.

Hueristically : a function $f(x)$ is continuous at x_0

if $f(x)$ may be close to $f(x_0)$ as desired

as x is sufficiently close to x_0.

Rigorous definition using $\varepsilon - \delta$ language

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Example 4. What is derivative

or "speed" or "tangent line"?

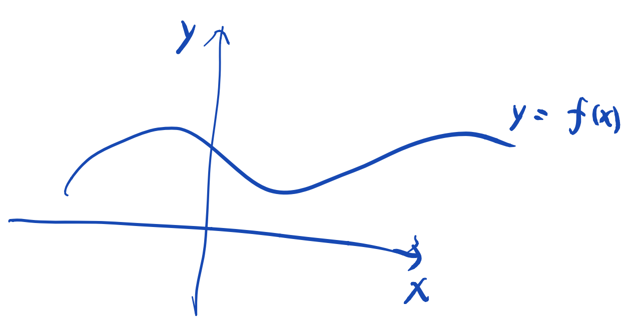
Remark: differentiability $\neq$ continuity

(visually looks the same)

There exists a continuous function

$f: R \to R$, which is not

differentiable everywhere



Chapter I : logic

Rules for reasoning

1. Mathematical statement (expression , phrase...)

Example : 2 is / is not an even number

R.K. A statement may be false.

Notation : use p or q to denote a statement

Example : p : $x \geq 0$.

2. quantifiers : ① for all , for any, for every

Notation : $\forall$

$\forall x$: for all x

②. There exists, there is (at least one)

Notation : $\exists$

Example : $\exists\ x > 0$: there exists $x > 0$

3. quantified statement : statement involving quantifier

Example : ① $\forall\ x > 0$, x has a square root.

② $\exists\ x > 0$, x does not have a square root.

③ $\forall\ x > 0$, $\exists\ y > 0$, (such that) $x = y^2$.

4. Negating a statement : taking the opposite of a statement.

Let P be a statement, then negation of P is denoted by $\sim P$.

Example : P : $x \geq 0$; $\sim P$: $x < 0$

5. rules of negation :

Rule ①. $\sim(\sim p) = p$

Example : $p: x>0$, $\sim p: x\leq 0$, $\sim(\sim p): x>0$

Rule ② $\sim(p \text{ and } q) = \sim p \text{ or } \sim q$

Example : $p: x>0$, $q: x<1$, $p \text{ and } q: 0<x<1$

$$\sim(p \text{ and } p) = \sim(0<x<1) = \underbrace{x\leq 0}_{\sim p} \text{ or } \underbrace{x\geq 1}_{\sim q}$$

Rule ③ $\sim(p \text{ or } q) = (\sim p) \text{ and } (\sim q)$

Example : $p: x>0$, $q: x<1$

$p \text{ or } q: x>0 \text{ or } x<1$ = x can be any number

$$\sim(p \text{ or } q) = \sim(x>0 \text{ or } x<1) = \underbrace{x\leq 0 \text{ and } x\geq 1}_{\text{x cannot be a number}}$$

Excercise : $p: x<0$, $q: x>1$.

$$\sim(p \text{ or } q) \stackrel{?}{=} \sim p \text{ and } \sim q$$

Rule ④. $\sim(\forall x, p) = \exists x, \sim p$

Example : $\forall x>0, \overbrace{2x>0}^{p}$. : True

the opposite is : $\exists x>0, 2x\leq 0$ False

Rule ⑤ $\sim(\exists x, p) = \forall x, \sim p$

Example : $\exists x>0$, x has a square root, True

the opposite is : $\forall x>0$, x has no square root : False

Rule ⑥ $\sim(\forall x, \underbrace{\exists y ; p}_{q}) = \exists x, \forall y, \sim p$

proof : $\sim(\forall x, q) = \exists x, \sim q$ (rule ④)

$= \exists x, \sim(\exists y, p)$

$= \exists x, \forall y, \sim p$ rule ⑤

Example : $\forall x>0, \exists y>0$, (such that) $x = y^2$. [true]

opposite : $\exists x>0, \forall y>0, x \neq y^2$ False

Remark : If statement p is true, then

$\sim p$ is False ;

6. Conditional statements (If - then statements)

If p, then q

p implies q

p is sufficient for q

q is necessary for p

Notation.

$\boxed{p \Rightarrow q}$

Example: If $\underbrace{x>0}_{p}$, then $\underbrace{x = |x|}_{q}$. True

Rule: $p \Rightarrow q \equiv \sim p$ or q. (*)

Example: If $x>0$, then $|x| = x$.

$(\overbrace{x>0}^{p}) \Rightarrow (\overbrace{|x| = x}^{q})$ True

$\underbrace{x \leq 0}_{\sim p}$ or $\underbrace{|x| = x}_{q}$ True

Rule ⑦ : $\sim(p \Rightarrow q) = p$ and $(\sim q)$

Pf : $\sim(p \Rightarrow q) = \sim(\sim p$ or $q)$

$= \sim(\sim p)$ and $\sim q$

$= p$ and $\sim q$

Example : $x > 0 \Rightarrow x = |x|$ True

Opposite :

Converse / contrapositive statement

For the statement " If p, then q " or ($p \Rightarrow q$)

Its converse is " If q, then p " or ($q \Rightarrow p$)

Its contrapositive is " If $\sim q$, then $\sim p$ " or ($\sim q \Rightarrow \sim p$)

Example : (1) Statement : If $x = -3$, then $x^2 = 9$.

Converse : If $x^2 = 9$, then $x = -3$

Contrapositive : If $x^2 \neq 9$, then $x \neq -3$.

(2) Statement : $x = -3 \Rightarrow 2x = -6$.

Converse : $2x = -6 \Rightarrow x = -3$

Contrapositive : $2x \neq -6 \Rightarrow x \neq -3$.

Remark : ① Contrapoistive = statement

Pf :

② If "$p \Rightarrow q$" and "$q \Rightarrow p$" both are True, we will write $p \Leftrightarrow q$, and say that

p if and only if q or use

Abbreviation : p iff q.

③ $\forall \alpha, \forall \beta = \forall \beta, \forall \alpha$

$\exists \alpha, \exists \beta = \exists \beta, \exists \alpha$

$\forall \alpha, \exists \beta \neq \exists \beta, \forall \alpha$

Example : $(\forall \alpha > 0, \exists \beta, \alpha\beta > 0)$ True

$\neq (\exists \beta, \forall \alpha, \alpha\beta > 0)$ False

Exercise : Negate each of the following

① If ΔABC is a right triangle, then $a^2+b^2=c^2$.

② $\forall \varepsilon>0$, $\exists \delta>0$ such that

$$0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon$$

Solution : ①