Math 2033 (Math Analysis)

Calculus Review Exercises

Notations: In these exercises, x takes on real values and n takes on positive integer values.

1. Draw the graph of the function $f(x) = \dfrac{x}{x}$.

2. Find $\displaystyle\lim_{x\to+\infty} \frac{x+2\cos x}{3+4x}$.

3. Let $f(x) = \begin{cases} x^2 & \text{if } x \neq 3 \\ 3x & \text{if } x = 3 \end{cases}$. Is it true that $f'(x) = \begin{cases} 2x & \text{if } x \neq 3 \\ 3 & \text{if } x = 3 \end{cases}$?

4. Must $1^\infty = 1$? More precisely, let $a_1, a_2, a_3, \ldots$ be positive real numbers. Must it be true that if $\displaystyle\lim_{n\to+\infty} a_n = 1$, then $\displaystyle\lim_{n\to+\infty} a_n^n = 1$?

5. We know that $\displaystyle\lim_{x\to+\infty} \sin x$ doesn't exist. If $a_1, a_2, a_3, \ldots$ are positive real numbers with $\displaystyle\lim_{n\to\infty} a_n = +\infty$, then must it be true that $\displaystyle\lim_{n\to+\infty} \sin a_n$ doesn't exist?

More difficult ↘

6. Show $\displaystyle\lim_{n\to+\infty} \sin n \neq 0$. (By exercise 5, this is <u>not</u> because $\displaystyle\lim_{x\to\infty} \sin x$ fails to exist!)

7. Let
$$g(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number in } [0,1] \\ 0 & \text{if } x \text{ is an irrational number in } [0,1] \end{cases}.$$
For every positive integer n, divide $[0,1]$ into intervals $\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right]$. On the j-th interval $\left[\frac{j-1}{n}, \frac{j}{n}\right]$, let x_j be its midpoint. Since x_j is rational, we have $g(x_j) = 1$. Now, adding areas of rectangles over the intervals and taking limit, we set
$$\lim_{n\to+\infty} \underbrace{g(x_1)}_{=1}\left(\frac{1}{n} - 0\right) + \underbrace{g(x_2)}_{=1}\left(\frac{2}{n} - \frac{1}{n}\right) + \cdots + \underbrace{g(x_n)}_{=1}\left(1 - \frac{n-1}{n}\right) = 1.$$
So $\displaystyle\int_0^1 g(x)\,dx = 1$. Is this correct?

8. Let $h(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Graph $h(x)$. Find $h'(x)$. What is $h''(0)$?