MATH 2033 Assignment 2. Ho Chuen Ho 20601111.

1(a) $S = \{ e^{\sqrt{x}} \mid x \in \mathbb{Q} \cap (0,1) \}$

$e^{\sqrt{x}}$ ↑, increasing function

$\Rightarrow e^{\sqrt{x}}$ also increasing when $x \in \mathbb{Q} \cap (0,1)$

The supremum is

$$e^{\sqrt{1}} = e$$

The infimum is

$$e^{\sqrt{0}} = 1$$

b) $T = \{ n \cos\frac{n\pi}{2} \mid n \in \mathbb{N} \}$

Start with $n=1$

$T_1 = \cos\frac{\pi}{2} = 0$

$n=2$, $T_2 = 2\cos\pi = -2$

$n=3$, $T_3 = 3\cos\frac{3\pi}{2} = 0$

$n=4$, $T_4 = 4\cos\frac{4\pi}{2} = 4$

$n=5$, $T_5 = 5\cos\frac{5\pi}{2} = 0$

We list out the series, we can conclude that T does not have infimum.

Because cosine function is oscillating function, the series would not converge as n is set of natural number which is countably infinite.

2 a) We know $\inf S \leq \sup S$ and $\inf A \leq \sup A$

Let $x = \sup A$, Then $x \geq a$, $\forall a \in A$

As $S \subseteq A$, $x \geq s$, $\forall s \in S$

x is upper bound of $S \Rightarrow S$ has supremum.

Therefore $\sup A \geq \sup S$

Let $y = \inf A$. Then $y \leq a$, $\forall a \in A$

As $S \subseteq A$, $y \leq s$, $\forall s \in S$

y is lower bound of $S \Rightarrow S$ is lower bounded

Therefore $\inf A \leq \inf S$

$\Rightarrow \inf A \leq \inf S \leq \sup S \leq \sup A$

b i) $a \leq \sup A$, $\forall a \in A$, $b \leq \sup B$, $\forall b \in B$

$\Rightarrow ab \leq \sup A \sup B$, $\forall a \in A$ $\forall b \in B$

Let $\varepsilon > 0$, such that ~~$(\sup A - \varepsilon)(\sup B - \varepsilon)$~~

$\sup A - \varepsilon < a$, $\sup B - \varepsilon < b$

Then, $(\sup A - \varepsilon)(\sup B - \varepsilon) < ab$

$\Rightarrow \sup A \sup B - \varepsilon \sup A - \varepsilon \sup B + \varepsilon^2 < ab$

Let $\varepsilon' = \varepsilon \sup A + \varepsilon \sup B - \varepsilon^2$

$\Rightarrow \sup A \sup B - \varepsilon' < ab \leq \sup A \sup B$

$\sup A \sup B$ is least upper bound of ab

Therefore $\sup C = \sup A \sup B$

2b iii) Let $A = \{-a \mid a \in \mathbb{N}\}$, $B = \{b \mid b \in \mathbb{N} \cap [1, 3]$

Then $C = \{ab \mid a \in A, b \in B\}$

$\sup A = -1$, $\sup B = 3$

However, $\sup C = -1 \neq \sup A \sup B$

3. Define a sequence q. $q_n = \sum_{i=0}^{n} \left(\frac{1}{i!}\right)$

$q_1 = \frac{1}{0!} = 1 \in \mathbb{Q}$, $q_2 = 1 + \frac{1}{1!}$, $q_3 = 1 + \frac{1}{1!} + \frac{1}{2!}$

$q_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \in \mathbb{Q}$

$\lim_{n\to\infty} q_n = \lim_{n\to\infty}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) = e.$

4a) $\lim_{n\to\infty} \cos\left(a+\frac{b}{n}\right) = \lim_{n\to\infty} \cos a \cos\frac{b}{n} - \sin a \sin\frac{b}{n}$

$= \cos a \cos\frac{b}{\infty} - \sin a \sin\frac{b}{\infty}$

$= \cos a - 0 = \cos a$

b) $\lim_{n\to\infty} \sqrt{b_n} = \lim_{n\to\infty} (b_n)^{\frac{1}{2}} = \left(\lim_{n\to\infty} b_n\right)^{\frac{1}{2}}$

$= (b)^{\frac{1}{2}} = \sqrt{b}$.

5 $x_1 = 0.4$, $x_{n+1} = \dfrac{x_n^3 + 2}{3}$

$x_1 = 0.4 < 1$, $x_2 = \dfrac{0.4^3 + 2}{3} < \dfrac{1+2}{3} = 1$

$x_3 = \dfrac{x_2^3 + 2}{3} < \dfrac{1+2}{3} = 1$

Then similiarly, $x_n < 1$, $\forall n \in \mathbb{N}$

$\Rightarrow S_n$ is upper bounded

$x_{n+1} - x_n = \dfrac{x_n^3 + 2}{3} - x_n = \dfrac{x_n^3 - 3x_n + 2}{3}$

$= \dfrac{(x_n - 1)^2 (x_n + 2)}{3} > 0$

Therefore the sequence is straightly increasing function

Let limit be L,

$\lim_{n\to\infty} x_{n+1} = L$, $\lim_{n\to\infty} x_n = L$

$\lim_{n\to\infty} \dfrac{x_n^3 + 2}{3} = L$

$L^3 - 3L + 2 = 0$

$(L-1)^2 (L+2) = 0$

$L = 1$ or -2

As the sequence always longer than 0.

the limit of sequence is 1.

6 a) $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L < 1$, Let $L < r < 1$ and $r = L + \varepsilon$

where $\varepsilon > 0$

Then $$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon$$

$$\left|\frac{x_{n+1}}{x_n}\right| - |L| \leq \left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon$$

$$\left|\frac{x_{n+1}}{x_n}\right| < \varepsilon + |L| = r$$

Consider 1 to $n-1$

$$\left|\frac{x_2}{x_1}\right| \cdot \left|\frac{x_3}{x_2}\right| \cdot \ldots \cdot \left|\frac{x_{n-1}}{x_{n-2}}\right| \cdot \left|\frac{x_n}{x_{n-1}}\right| < r^n$$

$$\Rightarrow \frac{|x_n|}{|x_1|} < r^n \Rightarrow |x_n| < r^n \cdot |x_1|$$

Take $\lim_{n\to\infty}$, $\lim_{n\to\infty} r^n = 0$

$\therefore \lim_{n\to\infty} |x_n| = 0 \Rightarrow \lim_{n\to\infty} x_n = 0$.

Therefore, $\{x_n\}$ is convergence series.

b) $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L > 1$, Let $r = L - \varepsilon > 1$ where $\varepsilon > 0$.

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon$$

$$\Rightarrow L - \varepsilon < \frac{x_{n+1}}{x_n} \text{ and } \frac{x_{n+1}}{x_n} < \varepsilon + L$$

$$\Rightarrow r < \frac{x_{n+1}}{x_n} < \varepsilon + L$$

Condside 1 to $n-1$ for $\{x_n\}$,

$$\frac{x_2}{x_1} \cdot \frac{x_3}{x_2} \cdot \ldots \cdot \frac{x_n}{x_{n-1}} > r^n$$

$$x_n > x_1 \cdot r^n$$

Since $r > 1$, $\lim_{n\to\infty} r^n = \infty \Rightarrow \lim_{n\to\infty} x_n = \infty$, $\{x_n\}$ does not converge.

6(i) Let a series $\{x_n\}$ be, $\{1, 1, \cdots\}$ all one.

Then clearly, $x_{n+1} = 1$, $x_n = 1$, $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$

Also, $\lim_{n\to\infty} x_n = 1$, which $\{x_n\}$ converges.

ii) Let a series of $\{x_n\}$ be $\{1, 2, \cdots, n\}$

$x_n = n$, $x_{n+1} = n+1$

Then $\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$

but $\lim_{n\to\infty} x_n = \lim_{n\to\infty} n = \infty$

which shows $\{x_n\}$ does not converges.