Math 2033 (Homework 4) Solutions

① Prove that the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{2+3x}{x^2+4}$ is continuous at $x=2$ by checking the ε–δ definition of a function continuous at a point.

Solution We have $f(2) = \frac{2+6}{4+4} = 1$. Observe that

10 $|f(x) - 1| = \left|\frac{2+3x}{x^2+4} - 1\right| = \frac{|x^2-3x+2|}{x^2+4} = \frac{|x-2||x-1|}{x^2+4}$.

5 If $x \in (1,3)$, then $x-1 \in (0,2)$. For every $\varepsilon > 0$, let $\delta = \min(1, 2\varepsilon) > 0$, then

$$|x-2| < \delta \Rightarrow |x-2| < 1 \text{ and } |x-2| < 2\varepsilon$$

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$$\Rightarrow x \in (1,3),\ |x-2| < 2\varepsilon$$

$$\Rightarrow |f(x) - 1| = \frac{|x-2||x-1|}{x^2+4} \leq \frac{|x-2|2}{4} < \varepsilon.$$

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② Prove that there does not exist any continuous function $f: \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) + x = 0$ for every $x \in \mathbb{R}$.

Solution Assume such continuous function f exists. If $f(a) = f(b)$, then $-a = f(f(a)) = f(f(b)) = -b$. So $a = b$. Then f is

10 injective. By the continuous injection function, f is strictly increasing, then $x < y \Rightarrow f(x) < f(y) \Rightarrow f(f(x)) < f(f(y))$.

10 If f is strictly decreasing, then $x < y \Rightarrow f(x) > f(y) \Rightarrow f(f(x)) < f(f(y))$. In both cases, we have $f(f(x))$ is strictly increasing. However, $f(f(x)) = -x$ is <u>not</u>

10 strictly increasing, a contraction. Therefore, no such continuous function f exists.

③ Let $f: \mathbb{R} \to \mathbb{R}$ be continuous such that $|f(x)-f(y)| \le \frac{1}{2}|x-y|$ for every $x, y \in \mathbb{R}$.

(a) Let $w \in \mathbb{R}$. Define $x_1 = w$ and $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$. Show that $x_1, x_2, x_3, \ldots$ is a Cauchy sequence.

~~(b) Show that there is $x \in \mathbb{R}$ such that $f(x) = x$.~~
~~No need to give a solution!~~

Solution

(a) Observe that $|x_{k+1} - x_k| = |f(x_k) - f(x_{k-1})| \le \frac{1}{2}|x_k - x_{k-1}|$.
Repeating this, we get $|x_{k+1} - x_k| \le \frac{1}{2}|x_k - x_{k-1}| \le \left(\frac{1}{2}\right)^2 |x_{k-1} - x_{k-2}|$
10 $\le \cdots \le \left(\frac{1}{2}\right)^{k-1} |x_2 - x_1|$. So for $m > n$, we have

$$|x_m - x_n| = |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)|$$
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$$\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$
$$\le \left(\left(\tfrac{1}{2}\right)^{m-2} + \left(\tfrac{1}{2}\right)^{m-3} + \cdots + \left(\tfrac{1}{2}\right)^{n-1}\right)|x_2 - x_1| \le \left(\tfrac{1}{2}\right)^{n-2} |x_2 - x_1|.$$

If $x_1 = x_2$, then $x_m = x_n$ for all m, n and $x_1, x_2, x_3, \ldots$ is a
10 constant sequence. Hence, $x_1, x_2, x_3, \ldots$ converges and is a Cauchy sequence. If $x_1 \ne x_2$, then for every $\varepsilon > 0$, by the Archimedean principle, there is $K \in \mathbb{N}$ such that
15 $K > 2 - \log_2 \frac{\varepsilon}{|x_2 - x_1|}$, which implies $\left(\frac{1}{2}\right)^{K-2} |x_2 - x_1| < \varepsilon$.
So $m, n \ge K \Rightarrow |x_m - x_n| \le \left(\frac{1}{2}\right)^{K-2} |x_2 - x_1| < \varepsilon$.
Therefore, $x_1, x_2, x_3, \ldots$ is a Cauchy sequence.

This is true.

~~(b) Let $w \in \mathbb{R}$. Define $x_1, x_2, x_3, \ldots$ as in part (a). Then $x_1, x_2, x_3, \ldots$ is a Cauchy sequence by (a). By Cauchy's theorem, $x_1, x_2, x_3, \ldots$ converges to some $x \in \mathbb{R}$. We have~~

$$\cancel{x = \underbrace{\lim_{n\to\infty} x_{n+1}}_{\text{by subsequence theorem p. 54}} = \lim_{n\to\infty} f(x_n) \underbrace{=}_{\text{sequential continuity theorem p. 66}} f\left(\lim_{n\to\infty} x_n\right) = f(x).}$$