Math 2033 Problems appeared in Past paper by Topics

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Countability

1. (2002 L3 Midterm) Let S be the set of all ordered pairs (p, C), where $p = (x, y) \in \mathbb{Q} \times \mathbb{Q}$ and C is the circle with center p and radius $|xy| + 1$. Determine if S is a countable set or not.

2. (2003 Final) Let P be a countable set of points in $\mathbb{R}^2$. Prove that there exists a circle C with the origin as center and positive radius such that every point of the circle C is not in P. (Note points inside the circle do not belong to the circle.)

3. (2004 L2 Midterm) Let S be the set of all intersection points $(x, y) \in \mathbb{R}^2$ of the graphs of the equations $x^2 + my^2 = 1$ and $mx^2 + y^2 = 1$, where $m \in \mathbb{Z}\backslash\{-1, 1\}$. Determine if S is countable or uncountable. Provide a proof of your answer.

4. (2007 Fall Midterm) Let S be a nonempty countable subset of the interval $(0, +\infty)$. Prove that there exists a positive real number which is not the area of any triangle whose three sides have lengths in S.

5. (2008 Fall Midterm) Prove that there exists a positive real number c which does <u>not</u> equal to any number of the form $2^{a+b\sqrt{2}}$, where $a, b \in \mathbb{Q}$.

6. (2009 Fall Midterm) Let S be the set of all points (x, y) in the coordinate plane that satisfy the equations

$$x^2 + y^2 = a^2 \text{ and } y = x^2 - x^3 + b$$

for some $a, b \in \mathbb{Q}$ with $a \neq b$. Determine (with proof) if S is countable or not.

7. (2012 Fall Midterm) Let S be the set of all points $(x, y) \in \mathbb{R}^2$ that satisfy the system of equations

$$x + y = mx^2 - x^3 \text{ and } mx + y^4 = x^6 - 7mx^3 + 2$$

for some $m \in \mathbb{Q}$. Determine (with proof) if S is countable or not.

8. (2012 Fall Midterm) Let S be a nonempty countable subset of $\mathbb{R}$. Prove that there exists a positive real number r such that the equation

$$5^x + 7^y = \sqrt{r}$$

does not have any solution with $x, y \in S$.

9. (2014 Spring Midterm) Prove that there exist infinitely many real numbers r such that the equation $10^{xy} + r - y^3 = xy$ does not have any solution with $x, y \in \mathbb{Q}$.

Supremum/Infimum

1. (2006 Fall Exam) Let $\left(0, \frac{1}{2}\right) \cap \mathbb{Q} \subseteq A_1 \subseteq [0, 1)$. For $n = 1, 2, 3, \cdots$, let

$$A_{n+1} = \{\sqrt{x} : x \in A_n\}.$$

Determine the supremum and infimum of $\bigcup_{k=1}^{\infty} A_k$ with proof.

2. (2007 Fall Midterm) Let c be a positive rational number. Determine (with proof) the supremum and infimum of

$$B = \{x + y : x \in [0, c\sqrt{2}] \cap \mathbb{Q},\ y \in [0, c]\backslash\mathbb{Q}\}$$

(with proof) the infimum and supremum of C.

3. (2008 Fall Final) Determine (with proof) the infimum of the set

$$S = \{x : x \in \mathbb{R} \text{ and there exist } b, c \in [-1, 1) \text{ such that } x^2 + bx + c = 0\}.$$

4. (2008 Fall Final) Let $A_1, A_2, A_3, \cdots$ be subsets of $[0,1]$ such that $\bigcap_{n=1}^{\infty} A_n$ is nonempty. If

$$\sup\{\inf A_n : n = 1, 2, 3, \cdots\} = \inf\{\sup A_n : n = 1, 2, 3, \cdots\},$$

then prove that $\bigcap_{n=1}^{\infty} A_n$ has exactly one element.

5. (2009 Fall Midterm) Let D be a nonempty bounded subset of R such that $\inf D = 3$ and $\sup D = 5$. Let

$$A = \{xy + xy^3 : x \in (2, \pi] \cap \mathbb{Q},\ y \in D\}.$$

Show that A is bounded. Determine (with proof) the infimum and supremum of A.

6. (2011 Fall Midterm) Suppose A and B are two nonempty bounded subsets of $\mathbb{R}$ such that $\inf A = 1, \sup A = 5, \inf B = 0$ and $\sup B = 1$. Let

$$C = \left\{\frac{y}{3-x} - \frac{1}{y} : x \in B,\ y \in A\right\}$$

Prove that C is bounded. Determine (with proof) the infimum and supremum of C.

7. (2014 Spring Midterm) Let A be a nonempty bounded subset of R such that $\inf A = 0$ and $\sup A = 3$.
Let

$$B = \{x + 2^{xy} + y : x \in [1, 2]\backslash\mathbb{Q},\ y \in A\}$$

Prove that B is bounded. Determine (with proof) the infimum and supremum of B.

8. (2015 Spring Midterm) Let A be a nonempty bounded subset of R such that $\inf A = 1$ and $\sup A = 2$. Let

$$B = \left\{\sqrt{y}\cos x : x \in \left(0, \frac{\pi}{3}\right] \cap \mathbb{Q},\ y \in A\right\}$$

Prove that B is bounded. Determine (with proof) the infimum and supremum of B.

Sequences defined by recurrence relations

1. (2007 Fall Final) Let $x_1, x_2, x_3, \cdots$ be a sequence of real numbers such that $x_{n+1} = \dfrac{x_1 - 2}{10 + x_n}$ for $n = 1, 2, 3, \cdots$.
   (a) If $x_1 = -7$, then prove that $x_1, x_2, x_3, \cdots$ converges and find its limit.
   (b) If $x_1 = 26$, then prove that $x_1, x_2, x_3, \cdots$ converges and find its limit.

2. (2010 Fall Midterm) Prove the sequence $\{x_n\}$ converges, where

$$x_1 = 5 \text{ and } x_{n+1} = \frac{7}{x_n + 5} \text{ for } n = 1, 2, 3, \cdots$$

and find its limit. Show work!

3. (2011 Fall Midterm) Prove the sequence $\{x_n\}$ converges, where

$$x_1 = 27 \text{ and } x_{n+1} = 8 - \sqrt{28 - x_n} \text{ for } n = 1, 2, 3, \cdots$$

and find its limit. Show work!

4. (2011 Fall Final) Let $x_1 = 0$, $x_2 = 3$ and for $n = 1, 2, 3\cdots$, let $x_{n+2} = \sqrt{\dfrac{4}{9}x_{n+1}^2 + \dfrac{5}{9}x_n^2}$.
   Prove that the sequence $x_1, x_2, x_3, \cdots$ converges and find its limit.

5. (2014 Spring Midterm) Prove that the sequence $\{x_n\}$ converges, where

$$x_1 = 11 \text{ and for } n = 1, 2, 3, \cdots,\ x_{n+1} = \frac{18}{x_n + 7}$$

and find its limit. Show all details.

6. (2015 Spring Midterm) (a) Prove that the sequence $\{w_n\}$ converges, where

$$w_1 = 6 \text{ and for } n = 1, 2, 3, \cdots, \; w_{n+1} = 6 - \frac{9}{w_n}$$

and find its limit. Show all details.

(b) Prove that the sequence $\{x_n\}$ converges, where

$$x_1 = 60 \text{ and for } n = 1, 2, 3, \cdots, \; x_{n+1} = 8 + \frac{120}{x_n}$$

and find its limit. Show all details.

Limit of Sequences

1. (2006 Fall Exam) Let $x_1, x_2, x_3, \cdots$ and $y_1, y_2, y_3, \cdots$ be sequences of positive numbers such that

$$\lim_{n\to\infty} x_n = 1 = \lim_{n\to\infty} y_n.$$

Prove that

$$\lim_{n\to\infty} \left(4x_n + \frac{1}{y_n}\right) = 5$$

by checking the definition of limit. (Do not use the computation formulas for limits, sandwich theorem or l'Hopital's rule, otherwise you will get 0 mark for this problem!)

2. (2007 Fall Final) For $n = 1, 2, 3, \cdots$, let

$$y_n = \frac{4n^2 - \sqrt{n}}{2n^2 + n} + \frac{n-1}{n}.$$

Prove that $\lim_{n\to\infty} y_n = 3$ by checking the definition of limit of a sequence only. (Do not use computation formulas, sandwich theorem or l'Hopital's rule! Otherwise, you will get zero mark for this problem.)

3. (2010 Fall Final) Let $a_1, a_2, a_3, \cdots$ be a sequence of real numbers that converges to 1. Prove that

$$\lim_{n\to\infty} \left(\frac{3 + a_n^2}{a_n + 1} + \frac{2n}{4 + n}\right) = 4$$

by checking the definition of limit of a sequence. (Do not use computation formulas, sandwich theorem or L'Hopital's rule, otherwise you will get zero mark on this problem!)

4. (2011 Fall Final) Let $a_1, a_2, a_3, \cdots$ be a sequence of real numbers that converges to 3. Prove that

$$\lim_{n\to\infty} \left(\frac{a_n}{a_n^2 + 3} + \frac{3n^2}{1 + 4n^2} + \frac{a_n}{n}\right) = 1$$

by checking the definition of limit of a sequence. (Do not use computation formulas, sandwich theorem or L'Hopital's rule, otherwise you will get zero mark on this problem!)

5. (2014 Spring Midterm) Prove that

$$\lim_{n\to\infty} \left(\frac{6n^2 + n - 3}{1 + 2n^2} + \frac{n + 5\sqrt{n} + \sqrt[3]{n}}{6 + n}\right) = 4$$

by checking the definition of limit of a sequence only.

(Do not use computation formulas, sandwich theorem or l'Hopital's rule! Otherwise, you will get zero mark for this problem.)

6. (2015 Spring Midterm) Let $y_1, y_2, y_3, \cdots$ and $z_1, z_2, z_3, \cdots$ be sequences of real numbers such that both converge to 4. Prove that

$$\lim_{n\to\infty} \left(\frac{9}{z_n^2 + 2} + \frac{5}{y_n - 2}\right) = 3$$

by checking the definition of limit of a sequence only.

(Do not use computation formulas, sandwich theorem or l'Hopital's rule! Otherwise, you will get zero mark for this problem.)