MATH 2031 Introduction to Real Analysis

December 7, 2012

Tutorial Note 10

Supremum and Infimum

(S.I) **Definition:**
The supremum (least upper bound) of a non-empty set S bounded above, denoted by $\sup S$, is the upper bound $\widetilde{M}$ of S such that $\widetilde{M} \leq M$ for any upper bound of S

The infimum (greatest lower bound) of a non-empty set S bounded below, denoted by $\inf S$, is the lower bound $\widetilde{m}$ of S such that $\widetilde{m} \geq m$ for any lower bound of S

(S.II) Supremum Property:
If a set S has supremum in $\mathbb{R}$ and $\varepsilon > 0$, then $\exists x \in S$ such that

$$\sup S - \varepsilon < x \leq \sup S$$

Infimum Property:
If a set S has infimum in $\mathbb{R}$ and $\varepsilon > 0$, then $\exists x \in S$ such that

$$\inf S \leq x < \inf S + \varepsilon$$

(S.III) Supremum Limit Theorem:
If c is an upper bound of S, then

$$\left(\exists w_n \in S \text{ such that } \lim_{n\to\infty} w_n = c\right) \iff c = \sup S.$$

Infimum Limit Theorem:
If c is an lower bound of S, then

$$\left(\exists w_n \in S \text{ such that } \lim_{n\to\infty} w_n = c\right) \iff c = \inf S.$$

Remark:

(1) There are a few steps in using the above to find the Supremum/Infimum of S;

Step 1 : Find the bound of S. (Upper bound for supremum, lower bound for infimum);

Step 2 : Construct a sequence in S that converges to the bound. (Make sure you have checked that the sequence is in S. If S is given explicitly, then write the direct expression of the sequence; if S is given abstractly (only the supremum and/or infimum is given), then you should apply the above theorem to obtain a sequence and construct the required sequence from it)

Step 3 : Compute the limit of the sequence you constructed above. (It should converge to the bound.)

(2) When S is given abstractly, please beware that the given supremum and infimum may not be in S.

(S.IV) Infinitesimal Principle:
Let $x, y \in \mathbb{R}$. Then ($x < y + \varepsilon$ for all $\varepsilon > 0$) $\iff$ $x \leq y$.

(S.V) Archimedean Principle:
$\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ such that $n > x$

(S.VI) Density of $\mathbb{Q}$:
If $x < y$, then $\exists \frac{m}{n} \in \mathbb{Q}$ such that $x < \frac{m}{n} < y$.

Density of $\mathbb{R} \setminus \mathbb{Q}$:
If $x < y$, then $\exists w \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < w < y$.

Examples

Problem 1 Let $S = \{\frac{a}{x} + \sqrt{a} | \; a \in A \text{ and } x \in (1, 5]\}$, with $\sup A = 3$ and $\inf A = 2$. Determine $\sup S$ and $\inf S$ with proof.

Solution:

(Finding the bound of S)
Since $\sup A = 2$, $\inf A = 3$ and $x \in (1, 5]$, we get $S \subseteq [\frac{2}{5} + \sqrt{2}, 3 + \sqrt{3})$.

(Construct the required sequences)
By Supremum Limit Theorem, there exists a sequence $\{a_n\}$ in A such that $\lim\limits_{n\to\infty} a_n = 3$;
By Infimum Limit Theorem, there exists a sequence $\{\widetilde{a_n}\}$ in A such that $\lim\limits_{n\to\infty} \widetilde{a_n} = 2$;

Define $\{w_n\}$ by $w_n = \dfrac{a_n}{(1 + \frac{1}{n})} + \sqrt{a_n}$. Since for any $n \in \mathbb{N}$, $1 + \frac{1}{n} \in [0, 5)$ and $a_n \in A$, $\{w_n\}$ is a sequence in S.
Define $\{\widetilde{w_n}\}$ by $w_n = \dfrac{\widetilde{a_n}}{5} + \sqrt{\widetilde{a_n}}$. Since $5 \in [0, 5)$ and $\widetilde{a_n} \in A$, $\{\widetilde{w_n}\}$ is a sequence in S.

(Compute the limit of the constructed sequences)
$\lim\limits_{n\to\infty} w_n = \lim\limits_{n\to\infty} \left(\dfrac{a_n}{(1 + \frac{1}{n})} + \sqrt{a_n} \right) = 3 + \sqrt{3}$. So by Supremum Limit Theorem, $\sup S = 3 + \sqrt{3}$.
$\lim\limits_{n\to\infty} \widetilde{w_n} = \lim\limits_{n\to\infty} \left(\dfrac{\widetilde{a_n}}{5} + \sqrt{\widetilde{a_n}} \right) = \dfrac{2}{5} + \sqrt{2}$. So by Infimum Limit Theorem, $\inf S = \dfrac{2}{5} + \sqrt{2}$.

Problem 2 Let $U = \{xy^2 | \; x \in (0, 2) \text{ and } y \in [-3, 2)\} \setminus (\{\frac{1}{k} | \; k \in \mathbb{N}\} \cup \{\frac{18n-1}{n} | \; n \in \mathbb{N}\})$, determine $\sup U$ and $\inf U$ with proof.

Solution:
Since $-3 \leq y < 2$, we get $0 \leq y^2 \leq 9$ and as $0 < x < 2$, so $0 < xy^2 < 18$.

Since y^2 attains both 0 and 9, we can focus on how to pick up a sequence for the bound of x.
If we directly pick $a_n = \dfrac{1}{n}$ and $b_n = 2 - \dfrac{1}{n}$, then we may get the candidate sequences $c_n = \dfrac{1}{n}$ and $d_n = \dfrac{18n - 1}{n}$.
However, they are both not in U, so we need modification.

Here we may consider the following:
Define $w_n = \dfrac{\pi}{4n}$. Then the terms in $\{w_n\}$ are irrational, so $\{w_n\}$ is in U;
Define $\widetilde{w_n} = 9\left(2 - \dfrac{\pi}{4n}\right)$. Then the terms in $\widetilde{w_n}$ are also irrational, so $\widetilde{w_n}$ is in U.

$\lim\limits_{n\to\infty} w_n = \lim\limits_{n\to\infty} \dfrac{\pi}{4n} = 0$. So by Infimum Limit Theorem, $\inf U = 0$.

$\lim\limits_{n\to\infty} \widetilde{w_n} = \lim\limits_{n\to\infty} 9\left(2 - \dfrac{\pi}{4n}\right) = 18$. So by Supremum Limit Theorem, $\sup U = 18$.

Check Limit by definition

(L.I) **Definition:**

A sequence $\{x_1, x_2, \cdots\}$ (or written as $\{x_n\}_{n\in\mathbb{N}}$) converges to a number x (or has limit x) iff

$$\underbrace{\forall\varepsilon > 0}_{1}, \underbrace{\exists K \in \mathbb{N}}_{2} \underbrace{\text{such that } \forall n \geq K}_{3} \Rightarrow \underbrace{|x_n - x| < \varepsilon}_{4}$$

Remark:

When you check the limit by definition, please make sure you have written the above 4 items in your solution.
In item 2, you should determine (obtain from stratch) and state what K should be.
In item 4, you should check whether the inequality holds.

Examples

Problem 3 For every $n \in \mathbb{N}$ and for a fixed positive k, let

$$x_n = \frac{8n^3 - \sqrt[3]{n}}{4n^3 + n} + \frac{1 - kn}{1 + kn},$$

Prove that $\lim_{n\to\infty} x_n = 1$ by checking the definition of limit of a sequence **only**.

Scratch:

We want

$$\text{“}\left| \underbrace{\frac{8n^3 - \sqrt[3]{n}}{4n^3 + n}}_{\text{“tends to” } 2} + \underbrace{\frac{1 - kn}{1 + kn}}_{\text{“tends to” } -1} - 1 \right| < \varepsilon\text{”},$$

so we may “split” the term in the absolute sign as follows:

$$\left| \frac{8n^3 - \sqrt[3]{n}}{4n^3 + n} - 2 + \frac{1 - kn}{1 + kn} + 1 \right| \leq \left| \frac{8n^3 - \sqrt[3]{n}}{4n^3 + n} - 2 \right| + \left| \frac{1 - kn}{1 + kn} + 1 \right|.$$

The desired inequality will hold if we have each terms on the right to be strictly less than $\frac{\varepsilon}{2}$.
Since

$$\begin{aligned}
\left| \frac{8n^3 - \sqrt[3]{n}}{4n^3 + n} - 2 \right| &= \left| \frac{8n^3 - \sqrt[3]{n} - 8n^3 - 2n}{4n^3 + n} \right| \\
&= \left| \frac{-\sqrt[3]{n} - 2n}{4n^3 + n} \right| \\
&= \frac{\sqrt[3]{n} + 2n}{4n^3 + n} \\
&\leq \frac{n + 2n}{3n^3} \qquad \text{because} n \in \mathbb{N} \\
&= \frac{1}{n^2},
\end{aligned}$$

if we require that $\dfrac{1}{n^2} < \dfrac{\varepsilon}{2}$, or equivalently $\sqrt{\dfrac{\varepsilon}{2}} < n$, then $\left| \dfrac{8n^3 - \sqrt[3]{n}}{4n^3 + n} - 2 \right| \leq \dfrac{1}{n^2} < \dfrac{\varepsilon}{2}$.
On the other hand, since

$$\left| \frac{1 - kn}{1 + kn} + 1 \right| = \left| \frac{1 - kn + 1 + kn}{1 - kn} \right| = \frac{2}{1 + kn},$$

if we require that $\dfrac{2}{1 + kn} < \dfrac{\varepsilon}{2}$, or equivalently $\dfrac{\frac{4}{\varepsilon} - 1}{k} < n$, then $\left| \dfrac{1 - kn}{1 + kn} + 1 \right| = \dfrac{2}{1 + kn} < \dfrac{\varepsilon}{2}$.

Solution:
$\forall\, \varepsilon > 0$, by Archimedean Principle, $\exists\, K \in \mathbb{N}$ such that $K > max\left\{\sqrt{\frac{\varepsilon}{2}}, \frac{\frac{4}{\varepsilon}-1}{k}\right\}$. Then for any $n \geq K$, we have

$$\sqrt{\frac{\varepsilon}{2}} < n \text{ and } \frac{\frac{4}{\varepsilon}-1}{k} < n, \qquad \text{i.e} \left|\frac{8n^3 - \sqrt[3]{n}}{4n^3+n} - 2\right| < \frac{\varepsilon}{2} \text{ and } \left|\frac{1-kn}{1+kn}+1\right| < \frac{\varepsilon}{2}$$

Then
$$\left|\frac{8n^3 - \sqrt[3]{n}}{4n^3+n} + \frac{1-kn}{1+kn} - 1\right| = \left|\frac{8n^3 - \sqrt[3]{n}}{4n^3+n} - 2 + \frac{1-kn}{1+kn} + 1\right| \leq \left|\frac{8n^3 - \sqrt[3]{n}}{4n^3+n} - 2\right| + \left|\frac{1-kn}{1+kn}+1\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Then by definition of limit, we have $\lim\limits_{n\to\infty} x_n = 1$.

Problem 4 Given that $\lim\limits_{n\to\infty} y_n = 3$, check by definition that $\lim\limits_{n\to\infty} \sqrt[5]{y_n^3+5} = 2$.
Scratch:
We want
$$\text{“}\left|\sqrt[5]{y_n^3+5} - 2\right| < \varepsilon\text{”}.$$
Since
$$\begin{aligned}
|\sqrt[5]{y_n^3+5} - 2| &= \left|\sqrt[5]{y_n^3+5} - \sqrt[5]{32}\right| \\
&\leq \left|\sqrt[5]{y_n^3+5-32}\right| && \text{since } \left|\sqrt[p]{a} - \sqrt[p]{b}\right| \leq \sqrt[p]{|a-b|} \text{ for } p \in \mathbb{N} \\
&= \sqrt[5]{|y_n^3 - 27|} \\
&= \sqrt[5]{|y_n^3 - 3^3|} \\
&= (\sqrt[5]{|y_n - 3||(y_n^2 + 3y_n + 9)|}) \\
&\leq (\sqrt[5]{|y_n - 3||((6)^2 + 3(6) + 9)|}) && \text{if } |y_n - 3| < 3 \\
&= \sqrt[5]{63|y_n - 3|},
\end{aligned}$$
if we require that $\sqrt[5]{63|y_n-3|} < \varepsilon$, or equivalently $|y_n - 3| < \dfrac{\varepsilon^5}{63}$, then $|\sqrt[5]{y_n^3+5} - 2| \leq |\sqrt[5]{63|y_n-3|}| < \varepsilon$.
Solution:
Since $\lim\limits_{n\to\infty} y_n = 3$, by definition, for $3 > 0$, there exists $K_0 \in \mathbb{N}$ such that for any $n \geq K_0$, we have $|y_n - 3| < 3 \iff 0 < y_n < 6$.
And $\forall\, \varepsilon > 0$, since $\dfrac{\varepsilon^5}{63} > 0$, there exists $K_1 \in \mathbb{N}$ such that for any $n \geq K_1$, we have $|y_n - 3| < \dfrac{\varepsilon^5}{63}$.

By Archimedean Principle, $\exists\, K \in \mathbb{N}$ such that $K > max\{K_0, K_1\}$. Then for any $n \geq K$, we have $0 < y_n < 6$ and $|y_n - 3| < \dfrac{\varepsilon^5}{63}$. Then,
$$\begin{aligned}
\left|\sqrt[5]{y_n^3+5} - 2\right| &= \left|\sqrt[5]{y_n^3+5} - \sqrt[5]{32}\right| \\
&\leq \left|\sqrt[5]{y_n^3+5-32}\right| && \text{since } \left|\sqrt[p]{a} - \sqrt[p]{b}\right| \leq \sqrt[p]{|a-b|} \text{ for } p \in \mathbb{N} \\
&= \sqrt[5]{|y_n^3 - 27|} \\
&= \sqrt[5]{|y_n^3 - 3^3|} \\
&= \sqrt[5]{|y_n - 3||(y_n^2 + 3y_n + 9)|} \\
&\leq \sqrt[5]{|y_n - 3||((6)^2 + 3(6) + 9)|} \\
&= \sqrt[5]{63|y_n - 3|} \\
&< \sqrt[5]{63(\frac{\varepsilon^5}{63})} = \varepsilon
\end{aligned}$$
Thus by definition of limit, $\lim\limits_{n\to\infty} \sqrt[5]{y_n^3+5} = 2$.

Sequences Defined by Recurrence Relations

(R.I) **Monotone Sequence Theorem**
If $\{x_n\}$ is increasing and bounded above, then $\lim_{n\to\infty} x_n = \sup\{x_1, x_2, x_3 \cdots\}$.
Similarly, if $\{x_n\}$ is decreasing and bounded below, then $\lim_{n\to\infty} x_n = \inf\{x_1, x_2, x_3 \cdots\}$.

(R.II) **Intertwining Sequence Theorem**
If $\lim_{m\to\infty} x_{2m-1} = x$ and $\lim_{m\to\infty} x_{2m} = x$, then $\lim_{n\to\infty} x_n = x$.

(R.III) **Nested Interval Theorem**
If $\forall n \in \mathbb{N}$, $I_n = [a_n, b_n]$ and $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, then $\bigcap_{n=1}^{\infty} I_n = [a, b]$, where $a = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n = b$.

If $\lim_{n\to\infty} (b_n - a_n) = 0$, then $\bigcap_{n=1}^{\infty} I_n = \{x\}$ for some $x \in \mathbb{R}$.

Remark:
There are usually two types of sequences, monotone sequences and intertwining sequences. You should try the first few terms (usually 4 terms are enough), to guess which type of the sequence defined by the recurrence relation is of.

Examples

Problem 5 Prove that the sequence $\{x_n\}$ converges, where

$$x_1 = 3, \qquad x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

for n$\in \mathbb{N}$

and find its limit.

Scratch:
(Compute the first few terms)
$x_1 = 3,\ x_2 = \frac{1}{2}\left(3 + \frac{2}{3}\right) = \frac{11}{6} \approx 1.83,\ x_3 = \frac{1}{2}\left(\frac{11}{6} + \frac{2}{\frac{11}{6}}\right) \approx 1.46$, so it seems to be a monotonic decreasing sequence.

(Try to figure out the lower bound for $\{x_n\}$)
Assume that the limit exists, say $\lim_{n\to\infty} x_n = x$, then $x = \frac{1}{2}\left(x + \frac{2}{x}\right) \iff (x^2 - 2) = 0 \iff x = \sqrt{2}$ or $x = -\sqrt{2}$.
We may reject $x = -\sqrt{2}$, because $\{x_n\}$ is a positive sequence.
Solution:
By induction we can show that $x_n > 0$ for all $n \in \mathbb{N}$. Then by the AM-GM inequality,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq \sqrt{x_n \frac{2}{x_n}} = \sqrt{2}.$$

Consider

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) - x_n = \frac{1}{x_n} - \frac{1}{2}x_n < \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{2} = 0,$$

so the sequence $\{x_n\}$ is decreasing and bounded below by $\sqrt{2}$.
By the monotone sequence theorem, $\{x_n\}$ converges. So we let $\lim_{n\to\infty} x_n = x$. then

$$x = \frac{1}{2}\left(x + \frac{2}{x}\right) \iff (x^2 - 2) = 0 \iff x = \sqrt{2} \text{ or } x = -\sqrt{2}.$$

Since the sequence is positive, we get $\lim_{n\to\infty} x_n = \sqrt{2}$.

Problem 6 Let $x_1 = 2$ and define $x_{n+1} = \frac{1}{4}\left(3 + \frac{1}{x_n}\right)$. Prove the sequence $\{x_n\}$ converges and find its limit.

Scratch:

$x_1 = 2, x_2 = \frac{1}{4}\left(3+\frac{1}{2}\right) = \frac{7}{8} = 0.875, x_3 = \frac{1}{4}\left(3 + \frac{1}{(\frac{7}{8})}\right) \approx 1.036, x_4 \approx 0.99.$

Since $x_2 \le x_4 \le x_3 \le x_1$, it seems to be an intertwining sequence.

Solution:

It is clear from the definition that $x_n > 0$ for all $n \in \mathbb{N}$.

Claim: For any $n \in \mathbb{N}$, $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$.

From the scratch above, we see that the inequalities hold for $n = 1$, i.e. $x_2 \le x_4 \le x_3 \le x_1$.

Now assume that the inequalities hold for $n = k$, i.e $x_{2k} \le x_{2k+2} \le x_{2k+1} \le x_{2k-1}$.

Since x_n is positive for all $n \in \mathbb{N}$, we get

$$
\begin{aligned}
& x_{2k} \le x_{2k+2} \le x_{2k+1} \le x_{2k-1} \\
& \Rightarrow \frac{1}{x_{2k}} \ge \frac{1}{x_{2k+2}} \ge \frac{1}{x_{2k+1}} \ge \frac{1}{x_{2k-1}} \\
& \Rightarrow 3 + \frac{1}{x_{2k}} \ge 3 + \frac{1}{x_{2k+2}} \ge 3 + \frac{1}{x_{2k+1}} \ge 3 + \frac{1}{x_{2k-1}} \\
& \Rightarrow x_{2k+1} \ge x_{2k+3} \ge x_{2k+2} \ge x_{2k} \\
& \Rightarrow x_{2k+1} \ge x_{2k+3} \ge x_{2k+2} \ge x_{2k} \\
& \Rightarrow \frac{1}{x_{2k+1}} \le \frac{1}{x_{2k+3}} \le \frac{1}{x_{2k+2}} \le \frac{1}{x_{2k}} \\
& \Rightarrow 3 + \frac{1}{x_{2k+1}} \le 3 + \frac{1}{x_{2k+3}} \le 3 + \frac{1}{x_{2k+2}} \le 3 + \frac{1}{x_{2k}} \\
& \Rightarrow x_{2k+2} \le x_{2k+4} \le x_{2k+3} \le x_{2k+1}.
\end{aligned}
$$

So, the inequalities also hold for $n = k + 1$.

Thus by mathematical induction, the inequalities $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$ hold for all $n \in \mathbb{N}$.

Define $I_n = [x_{2n}, x_{2n-1}]$ for any $n \in \mathbb{N}$. From the above induction, we have $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Then by nested interval theorem, the limit $\lim_{n\to\infty} x_{2n}$ and $\lim_{n\to\infty} x_{2n+1}$ exist, say $\lim_{n\to\infty} x_{2n} = a$ and $\lim_{n\to\infty} x_{2n+1} = b$, and $a \le b$.

Since $\begin{cases} x_{2n+1} = \frac{1}{4}\left(3 + \frac{1}{x_{2n}}\right) \\ x_{2n} = \frac{1}{4}\left(3 + \frac{1}{x_{2n-1}}\right) \end{cases}$, we get $\begin{cases} b = \frac{1}{4}\left(3 + \frac{1}{a}\right) \\ a = \frac{1}{4}\left(3 + \frac{1}{b}\right) \end{cases}$.

Consider

$$b - a = \frac{1}{4}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{4}\left(\frac{b-a}{ab}\right),$$

from which we get $(b-a)\left(1 - \frac{1}{4ab}\right) = 0$. So $b = a$ or $\left(1 - \frac{1}{4ab}\right) = 0$.

Since we have $\bigcap_{n\in\mathbb{N}} I_n = [a, b]$, i.e $a \in I_n$ and $b \in I_n$ for all $n \in \mathbb{N}$, in particular, $a, b \in I_1 = [\frac{7}{8}, 2]$. Thus

$$\frac{49}{16} \le 4ab \le 16 \Rightarrow 0 < 1 - \frac{1}{16} \le 1 - \frac{1}{4ab} \le 1 - \frac{16}{49}.$$

So, $\left(1 - \frac{1}{4ab}\right) \ne 0$.

Now $\lim_{n\to\infty} x_{2n} = a = b = \lim_{n\to\infty} x_{2n+1}$. By intertwining sequence theorem, $\lim_{m\to\infty} x_m$ exist, say $\lim_{m\to\infty} x_m = x$.

Then taking limit on both sides of the recurrence relation, we get $x = \frac{1}{4}\left(3 + \frac{1}{x}\right)$,

then we have $(4x+1)(x-1) = 0$, so $x = 1$ or $x = -\frac{1}{4}$. Since $x \in [\frac{7}{8}, 2]$, we get $\lim_{m\to\infty} x_m = x = 1$.