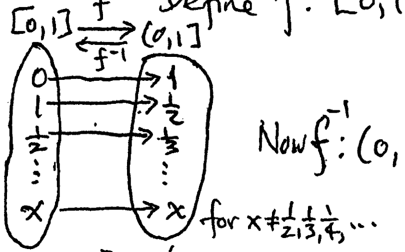
Solution of Math 2033 Homework #1

① Let $X=\{0,1,\frac{1}{2},\frac{1}{3},\cdots\}$. Then $[0,1]\setminus X=(0,1)\setminus\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots\}=(0,1]\setminus X$.

Define $f:[0,1]\to(0,1]$ by
$$f(x)=\begin{cases} x & \text{if } x\in(0,1)\setminus\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots\}\\ 1 & \text{if } x=0\\ \frac{1}{n+1} & \text{if } x=\frac{1}{n},\ n=1,2,3,\cdots\end{cases}$$



Now $f^{-1}:(0,1]\to[0,1]$ given by
$$f^{-1}(x)=\begin{cases} x & \text{if } x\in(0,1)\setminus\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots\}\\ 0 & \text{if } x=1\\ \frac{1}{n-1} & \text{if } x=\frac{1}{n},\ n=2,3,4,\cdots\end{cases}$$
is the inverse of f. $\therefore$ f is a bijection.

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② For every $(m,r)\in\mathbb{Z}\times\mathbb{Q}$, $\begin{cases}y=mx\\ x^2+y^2=r^2\end{cases} \Rightarrow \begin{cases}x^2+(mx)^2=r^2\\ y=mx\end{cases} \Rightarrow \begin{cases}x^2=\frac{r^2}{1+m^2}\\ y=mx\end{cases}$

has at most 2 solutions since $x^2=r^2/(1+m^2)$ has at most 2 solutions and at most 1 y for each solution of x.

Let $S_{(m,r)}$ be the solutions of the system $\begin{cases}y=mx\\ x^2+y^2=r^2\end{cases}$ for $(m,r)\in\mathbb{Z}\times\mathbb{Q}$.

Then $S=\bigcup\limits_{\underbrace{(m,r)\in\mathbb{Z}\times\mathbb{Q}}_{\text{Countable}}}\underbrace{S_{(m,r)}}_{\text{finite, hence countable}}$ is countable by countable union theorem.

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③ For every $(x,y,z)\in\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$, let $S_{(x,y,z)}=\{2^x+3^y+5^z\}$ ← 1 element set.

then $S=\{2^x+3^y+5^z:(x,y,z)\in\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}\}=\bigcup\limits_{(x,y,z)\in\underbrace{\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}}_{\text{Countable}}}\underbrace{S_{(x,y,z)}}_{\text{finite, hence countable}}$

is countable by countable union theorem.

By example 6 (on transparencies page 21) $(0,+\infty)\setminus S$ is uncountable.

Hence $(0,+\infty)\setminus S$ is not a finite set (finite sets are countable).

So $(0,+\infty)\setminus S$ is infinite. $\therefore$ there are infinitely many $r\in(0,\infty)$ and r is not in S.

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④ Note $(0,1)=\bigcup\limits_{k=1}^{\infty}[\frac{1}{k+1},\frac{1}{k})$

So every element of T $\subseteq(0,1)$ is in at least one of the interval $[\frac{1}{k+1},\frac{1}{k})$.

If $x_1,x_2,\cdots,x_n$ are in T and $[\frac{1}{k+1},\frac{1}{k})$, then $1>x_1^2+\cdots+x_n^2\geq\underbrace{\frac{1}{(k+1)^2}+\cdots+\frac{1}{(k+1)^2}}_{n\text{ times}}$.

From this we get $1>\frac{n}{(k+1)^2}$. So $n<(k+1)^2$.

Let $T_k=T\cap[\frac{1}{k+1},\frac{1}{k})$, then T_k has less than $(k+1)^2$ elements.

So T_k is a finite set. $\therefore$ $T=\bigcup\limits_{k=1}^{\infty}\underbrace{T_k}_{\text{finite, hence countable}}$ is countable by the countable union theorem.

⑤. $\inf D = 1$ and $\sup D = 5 \Rightarrow D \subseteq [1,5] \Rightarrow \forall x \in D,\ 1 \le x \le 5,\ \frac{1}{5} \le \frac{1}{x} \le 1,\ -1 \le -\frac{1}{x} \le -\frac{1}{5}$

$y \in [0, \sqrt{2}) \cap \mathbb{Q} \Rightarrow 0 \le y < \sqrt{2},\ \sqrt{2} \le y + \sqrt{2} < 2\sqrt{2},\ x(y+\sqrt{2}) - \frac{1}{x} \le 5(2\sqrt{2}) - \frac{1}{5} = 10\sqrt{2} - \frac{1}{5}$

Hence $10\sqrt{2} - \frac{1}{5}$ is an upper bound of E.

Since $\sup D = 5$, by supremum limit theorem, $\exists x_n \in D$ such that $\lim_{n\to\infty} x_n = 5$.

Let $y_n = \frac{[10^n \sqrt{2}]}{10^n}$, then $y_n \in [0, \sqrt{2}) \cap \mathbb{Q}$, Hence $x_n(y_n + \sqrt{2}) - \frac{1}{x_n} \to 5(\sqrt{2}+\sqrt{2}) - \frac{1}{5} = 10\sqrt{2} - \frac{1}{5}$.

By supremum limit theorem, we get $\sup E = 10\sqrt{2} - \frac{1}{5}$.

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⑥ To show $\sup B = w$, by supremum limit theorem, it is enough to show ① w is an upper bound of B and ② $\exists w_n \in B$ such that $\lim_{n\to\infty} w_n = w$.

For ①, we have $\forall b \in B$, since $B \subseteq C$, so $b \in C$, then $b \le \sup C = w$.

So w is an upper bound of B.

For ②, since $\sup A = w$, by supremum limit theorem, $\exists w_n \in A$ such that $\lim_{n\to\infty} w_n = w$. Since $A \subseteq B$, $w_n \in B$ and $\lim_{n\to\infty} w_n = w$. We are done.