Math 2033 Tutorial Exercises (on Supremum / Infimum)

For each of the following sets, if it is bounded above, give an upper bound and find its supremum with proof. If it is bounded below, give a lower bound and find its infimum with proof. ($\mathbb{N}$ denotes positive integers, $\mathbb{Q}$ denotes rational numbers)

(a) $A = \{\sqrt{m} + \sqrt{n} : m, n \in \mathbb{N}\}$

(b) $B = (-\infty, \pi] \cup \{4 - \frac{1}{n} : n \in \mathbb{N}\}$

(c) $C = \{\frac{1}{n} + \frac{1}{2^m} : m, n \in \mathbb{N}\}$

(d) $D = \mathbb{Q} \cap (0, \sqrt{2}]$

See page 46

You may use supremum limit theorem, infimum limit theorem, then the proofs by contradiction can be avoided.

Supremum Limit Theorem

Let c be an upper bound of a nonempty set S. Then (there exists $w_n \in S$ such that $\lim_{n\to\infty} w_n = c$) if and only if $c = \sup S$.

Infimum Limit Theorem

Let c be a lower bound of a nonempty set S. Then (there exists $w_n \in S$ such that $\lim_{n\to\infty} w_n = c$) if and only if $c = \inf S$.

Math 2033 Solutions of Tutorial Exercises (on Supremum/Infimum)

(a) $A = \{\sqrt{m} + \sqrt{n} : m, n \in \mathbb{N}\}$

Solution $A = \{\sqrt{1}+\sqrt{1}, \sqrt{2}+\sqrt{1}, \sqrt{1}+\sqrt{2}, \ldots\}$ is not bounded above. However, A has 2 as a lower bound because $\sqrt{m}+\sqrt{n} \geq \sqrt{1}+\sqrt{1} = 2$ for every $m, n \in \mathbb{N}$. In fact $\inf A = 2$ (because 2 is a lower bound and every lower bound $b \leq \sqrt{1}+\sqrt{1} = 2 \in A$).

(b) $B = (-\infty, \pi] \cup \{4 - \frac{1}{n} : n \in \mathbb{N}\}$

Solution 1 $B = (-\infty, \pi] \cup \{3, 3\frac{1}{2}, 3\frac{2}{3}, \dots\}$ is not bounded below. However, B has 4 as an upper bound because $\pi \leq 4$ and $4 - \frac{1}{n} \leq 4$ for all $n \in \mathbb{N}$. (Note $4 \notin B$) We will show $\sup B = 4$.

Assume there is an upper bound $t < 4$. By the Archimedian principle, there is $n \in \mathbb{N}$ such that $n > \frac{1}{4-t}$. Then $4 - \frac{1}{n} > t$ and $4 - \frac{1}{n} \in B$, which contradicts t being an upper bound.

Solution 2 Taking $w_n = 4 - \frac{1}{n} \in B$, we have $\lim_{n \to \infty} w_n = 4$. Since 4 is an upper bound of B, $\sup B = 4$ by the supremum limit theorem (see p.46, top right)

(c) $C=\left\{\frac{1}{n}+\frac{1}{2^m} : m,n\in\mathbb{N}\right\}$

<u>Solution 1</u> For $n,m\in\mathbb{N}$, $0<\frac{1}{n}+\frac{1}{2^m}\le\frac{1}{1}+\frac{1}{2^1}=\frac{3}{2}$.
So C has 0 as a lower bound and $\frac{3}{2}$ as an upper bound. In fact, $\sup C=\frac{3}{2}$ because $\frac{1}{1}+\frac{1}{2^1}=\frac{3}{2}\in C$ and every upper bound $M\ge\frac{1}{1}+\frac{1}{2^1}$. Also, we can show $\inf C=0$ as follows.

Assume there is a lower bound $t>0$. By the Archimedian principle, there is $k\in\mathbb{N}$ such that $k>\frac{1}{t}$. Taking $m=n=2k$, we have

$$t>\frac{1}{k}=\underbrace{\frac{1}{2k}}_{1/n}+\underbrace{\frac{1}{2k}}_{1/m}\ge\frac{1}{n}+\frac{1}{2^m}\in C,$$

Contradicting t being lower bound.

<u>Solution 2</u> Taking $w_n=\frac{1}{n}+\frac{1}{2^n}\in C$, we have $\lim\limits_{n\to\infty}w_n=0$.
Since 0 is a lower bound, $\inf C=0$ by the infimum limit theorem.

(d) $D = \mathbb{Q} \cap (0, \sqrt{2}]$

Solution 1 For $x \in D$, $0 < x < \sqrt{2}$. So D has 0 as a lower bound and $\sqrt{2}$ as an upper bound.

In fact, $\sup D = \sqrt{2}$ because if there is an upper bound $t < \sqrt{2}$, then by density of rational numbers, there will be $\frac{m}{n} \in \mathbb{Q}$ such that $\max\{t, 0\} < \frac{m}{n} < \sqrt{2}$, which means $t < \frac{m}{n} \in D$, Contradicting t being an upper bound.

Next, $\inf D = 0$ because if there is a lower bound $s > 0$, then by the density of rational numbers, there will be $\frac{p}{q} \in \mathbb{Q}$ such that $0 < \frac{p}{q} < \min\{s, \sqrt{2}\}$, which means $\frac{p}{q} \in D$ and $\frac{p}{q} < s$, Contradicting s being a lower bound.

Solution 2 Taking $w_n = \frac{1}{n} \in D$ and $z_n = \frac{[10^n \sqrt{2}]}{10^n} \in D$, we have $\lim_{n \to \infty} w_n = 0$ and $\lim_{n \to \infty} z_n = \sqrt{2}$. Since 0 is a lower bound and $\sqrt{2}$ is an upper bound, so $\inf D = 0$ and $\sup D = \sqrt{2}$ by the infimum limit theorem and Supremum limit theorem.