Ex.1 : If $f:(0,+\infty)\to\mathbb{R}$ is differentiable and $|f'(x)|\le 2$, $\forall x>0$.

① Show that the limit of sequence $x_n := f(\frac{1}{n})$ exists,

② Show that $\lim_{x\to 0+} f(x)$ exists.

Proof :

① Set $z_n := \frac{1}{n}$, by $\lim_{n\to+\infty} z_n = 0$ we know it is a Cauchy sequence, now we would show that $x_n = f(z_n)$ is also a Cauchy sequence:

$\forall \varepsilon>0$, if we choose N s.t. $|z_n - z_m| \le \frac{\varepsilon}{2}$, $\forall n,m \ge N$. then we have

$$|x_n - x_m| = |f(z_n) - f(z_m)| \le |f'(\underline{\xi_n})|\ |z_n - z_m| \le \frac{\varepsilon}{2}\cdot 2 = \varepsilon, \quad \forall n,m \ge N.$$

mean-value theorem $\qquad$ $|f'(\xi_n)| \le 2.$

Thus the claim holds. #

② Notice that for any sequence (y_n) with $\begin{cases} \lim_{n\to+\infty} y_n = 0, \\ y_n > 0, \ \forall n. \end{cases}$

we have $f(y_n)$ is a Cauchy sequence by the same argument in ①, thus $\lim_{n\to+\infty} f(y_n)$ exists.

Now by the sequenital limit theorem, we need only show that $\lim_n f(y_n)$ converges to the same limit for any (y_n) s.t.

$$\begin{cases} \lim_{n\to\infty} y_n = 0, & (1.1) \\ y_n > 0,\ \forall n. & (1.2) \end{cases}$$

Proof by contradiction: Suppose exists two sequence (y_n), $(\tilde{y}_n)$ satisfying (1.1) and (1.2), but

$\lim_n f(y_n) \neq \lim_n f(y_n')$.

Then we define a new sequence w_n with

$$w_n = \begin{cases} y_m & n = 2m-1 \\ \tilde{y}_m & n = 2m \end{cases}$$

and obviously w_n satisfies (1.1) and (1.2).

Thus $\lim_n f(w_n)$ exists, which contradicts to the fact that

$$\lim_{n\to\infty} f(w_{2n}) = \lim_{n\to\infty} f(\tilde{y}_n) \neq \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(w_{2n-1}).$$

#.

Ex.2 : Let $f: \mathbb{R} \mapsto \mathbb{R}$ be differentiable, if f' is differentiable at x_0, show that

$$\lim_{h \to 0} \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2} = f''(x_0).$$

Proof: Notice that by f is continuous,

$$\lim_{h \to 0} f(x_0+h) + f(x_0-h) - 2f(x_0) = 0$$

thus we can use L'Hospital's rule:

$$\lim_{h \to 0} \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2}$$

$$= \lim_{h \to 0} \frac{f'(x_0+h) - f'(x_0-h)}{2h}$$

$$= \frac{1}{2} \lim_{h \to 0} \frac{f'(x_0+h) - f'(x_0) + f'(x_0) - f'(x_0-h)}{h}$$

$$= \frac{1}{2} \cdot 2 f'(x_0) = f'(x_0) \quad \#.$$

Ex.3: For $a, b, c > 0$, show that the equation

$$e^x = ax^2 + bx + c$$

has at most 3 distinct roots.

The equation has at most 3 distinct roots $\Leftrightarrow f(x)$ has at most 3 distinct zero-points.

Proof: Set $f(x) := e^x - ax^2 - bx - c$, then

$$f'(x) = e^x - 2ax - b.$$

Notice that $f''(x) = e^x - 2a$ is $\begin{cases} > 0 & x > \log 2a \\ \leq 0 & \text{otherwise} \end{cases}$

thus $f'(x)$ is $\begin{cases} \uparrow & x > \log 2a \\ \downarrow & \text{otherwise} \end{cases}$

as a result $f'(x)$ has at most 2 zero-points.

Now suppose in contradiction that f has more than 3 distinct zero-points, then we can find

$$x_1 < x_2 < x_3 < x_4 \quad \text{s.t.}$$

$$f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0.$$

Then by Rolle's theorem, there exists

$$z_1 \in (x_1, x_2),\ z_2 \in (x_2, x_3),\ z_3 \in (x_3, x_4) \quad \text{s.t.}$$

$f'(z_1) = f'(z_2) = f'(z_3) = 0$, which leads to a contradiction. #.

Ex 4 For the function $\delta(x) := e - (1+x)^{\frac{1}{x}}$, $x > 0$,

show that $\lim_{x \to 0_+} \frac{\delta(x)}{x} := \frac{e}{2}$ using L'Hospital's rule.

Remark: We know that $\lim_{n \to \infty} (1+\frac{1}{n})^n = e$, this exercise gives (setting $x_n := \frac{1}{n}$) the rate of convergence in this limit: $|(1+\frac{1}{n})^n - e| \leq C \cdot \frac{e}{2n}$ for n large enough.

Proof: $\lim_{x \to 0_+} \frac{\delta(x)}{x} = \lim_{x \to 0_+} \frac{e - (1+x)^{\frac{1}{x}}}{x}$

$\overset{\text{L'Hospital}}{=} \lim_{x \to 0_+} (e - (1+x)^{\frac{1}{x}})'$

compute derivative:

$((1+x)^{\frac{1}{x}})' = (\exp \frac{1}{x} \log(1+x))'$

$= (\frac{\log(1+x)}{x})' \cdot \exp \frac{1}{x} \log(1+x)$

$= [-\frac{1}{x^2} \log(1+x) + \frac{1}{x(1+x)}](1+x)^{\frac{1}{x}}$

$= \underbrace{\lim_{x \to 0^+} (1+x)^{\frac{1}{x}}}_{e} \cdot \lim_{x \to 0^+} \frac{1}{x^2}[\log(1+x) - \frac{x}{1+x}]$

$\overset{\text{L'Hospital}}{=} e \cdot \lim_{x \to 0^+} \frac{\frac{1}{1+x} - \frac{1}{1+x} + \frac{x}{(1+x)^2}}{2x}$

$= e \cdot \lim_{x \to 0^+} \frac{1}{2(1+x)^2} = \frac{e}{2}$. #.

Ex.5 : Let $f: \mathbb{R} \to \mathbb{R}$ satisfies $|f(a)-f(b)| \le |a-b|^2$ for every $a, b \in \mathbb{R}$,

① Show that f is a constant function.

② If 2 is replaced by $\alpha > 1$, must f be a constant? Yes.

③ if 2 is replaced by $\beta \le 1$, must f be a constant? No.

Proof:

① and ②: Claim 1: f is differentiable in $\mathbb{R}$ and $f'(x)=0$, $\forall x \in \mathbb{R}$.

Proof of the claim: $\forall x_0, h \in \mathbb{R}$, we have

$$\left| \frac{f(x_0+h)-f(x_0)}{h} \right| \le |h|^{\alpha-1}.$$

Since RHS turns to 0 as $h \to 0$, we have

$$\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} = 0.$$

thus $f'(x_0)=0$. #.

Claim 2: If $f'(x)=0$, $\forall x \in \mathbb{R}$, then $f(x)=f(0)$, $\forall x \in \mathbb{R}$.

Proof: By $|f(x)-f(0)| \underset{\text{mean-value theorem.}}{=} |f'(\xi_x)| \, |x| = 0$. We have $f(x)=0$. #.

③ In that case. f need not be a constant function.

for example $f(x)=x$ is not a constant function while satisfying the condition in ③. #