Problem 1.

(a) I won't (watch a movie and have a dinner outside) no matter tomorrow is rainy or not.

$\Downarrow$ since 'rainy' includes 'sunny'

(b). $\exists\, \varepsilon>0$ such that for $\forall\, \delta>0$. $\exists\, x, y$ satisfying $|x-y|<\delta$ but $|f(x)-f(y)|\geq \varepsilon$.

(c) $\exists\, \varepsilon>0$ such that. for $\forall\, \delta>0$ $\exists\, x$ satisfying $|x-y|<\delta$ but $|f(x)-f(y)|\geq\varepsilon$

(d) $\exists\, \varepsilon>0$ such that for $\forall N>0$. here $\exists\, n, m\geq N$ and $|f_n(x)-f_m(x)|\geq\varepsilon$ $(x\in\mathbb{R})$.

(e) $\exists\, \varepsilon>0$. such that for $\forall N>0$. $|f_n(x)-f_m(x)|\geq\varepsilon$ for some $x\in\mathbb{R}$ and $\forall\, n, m\geq N$. $\square$

Problem 2.

(a) For $\forall\, x\in U$. $f(x)\in f(u) \Rightarrow x\in f^{-1}(f(u)) \Rightarrow U\subseteq f^{-1}(f(u))$

eg. of "$U\subset f^{-1}(f(u))$": $U=[0,+\infty)$. $f(x): \mathbb{R}\to\mathbb{R}$ by $x\mapsto x^2$.

then $f^{-1}(f(u)) = f^{-1}([0,+\infty)) = \mathbb{R} \supset U$.

(b) For $\forall\, y\in f(f^{-1}(v))$. $\exists\, x\in f^{-1}(v)$ s.t. $f(x)=y$. ; but $x\in f^{-1}(v)\Rightarrow f(x)\in V$.

thus. $y=f(x)\in V$. $\Rightarrow f(f^{-1}(v))\subseteq V$

eg. of "$f(f^{-1}(v))\subset V$": again. $f(x)=x^2$. take $V=\mathbb{R}$. then $f(f^{-1}(v)) = f(\mathbb{R}) = [0,+\infty) \subset V$

$f(\cup X_\alpha) = \cup f(X_\alpha)$

(c) "$\subseteq$": Take any $y\in f(\bigcup_{\alpha\in I} X_\alpha)$. then $y=f(x)$ for some $x\in\bigcup_\alpha X_\alpha \Rightarrow x\in X_\alpha$ for some α. Collapse: $y\in f(X_\alpha)$ for some α $\Rightarrow y\in\bigcup_{\alpha\in I} f(X_\alpha)$.

"$\supseteq$": Take any $y\in\bigcup_{\alpha\in I} f(X_\alpha)$. $\Rightarrow \exists\,\alpha$ s.t. $y\in f(X_\alpha)$

$\Rightarrow y=f(x)$ for some $x\in X_\alpha$ $\to$ $x\in\bigcup_\alpha X_\alpha$

$\Rightarrow y=f(x)$ for some $x\in\bigcup_\alpha X_\alpha \Rightarrow y\in f(\bigcup_{\alpha\in I} X_\alpha)$

Thus. $f(\bigcup_\alpha X_\alpha) = \bigcup_\alpha f(X_\alpha)$.

$f^{-1}(\bigcup_{\alpha\in L} Y_\alpha) = \bigcup_{\alpha\in L} f^{-1}(Y_\alpha)$

"$\subseteq$": For $\forall\, x \in f^{-1}(\bigcup_{\alpha\in L} Y_\alpha)$. $\exists\, y \in \bigcup_\alpha Y_\alpha$ s.t. $f(x) = \alpha$

$\Rightarrow \exists$ some α such that $f(x) = y \in Y_\alpha$

$\Rightarrow x \in f^{-1}(Y_\alpha)$ for some α $\Rightarrow x \in \bigcup_\alpha f^{-1}(Y_\alpha)$

"$\supseteq$": For $\forall\, x \in \bigcup_{\alpha\in L} f^{-1}(Y_\alpha)$. $\exists$ some α s.t. $x \in f^{-1}(Y_\alpha)$

$\Rightarrow \exists$ some α s.t. $f(x) \in Y_\alpha$

$\Rightarrow f(x) \in \bigcup_{\alpha\in L} Y_\alpha \Rightarrow x \in f^{-1}(\bigcup_\alpha Y_\alpha)$.

Hence. "=" is proved.

(d) $f(\bigcap_{\alpha\in L} X_\alpha) \subseteq \bigcap_{\alpha\in L} f(X_\alpha)$

Take any $y \in f(\bigcap_\alpha X_\alpha)$. then $\exists\, x \in \bigcap_\alpha X_\alpha$ s.t. $f(x) = y$

$\Rightarrow$ for each α. we have $y = f(x) \in f(X_\alpha)$ (since $x \in X_\alpha$, $\forall \alpha$)

$\Rightarrow y \in \bigcap_\alpha f(X_\alpha)$

$f^{-1}(\bigcap_{\alpha\in L} Y_\alpha) = \bigcap_{\alpha\in L} f^{-1}(Y_\alpha)$

"$\subseteq$": Take any $x \in f^{-1}(\bigcap_\alpha Y_\alpha)$. then $f(x) \in \bigcap_\alpha Y_\alpha \Rightarrow f(x) \in Y_\alpha$ for all α

$\Rightarrow x \in f^{-1}(Y_\alpha)$ for all α $\Rightarrow x \in \bigcap_{\alpha\in L} f^{-1}(Y_\alpha)$

"$\supseteq$": For $\forall\, x \in \bigcap_{\alpha\in L} f^{-1}(Y_\alpha)$. we have $x \in f^{-1}(Y_\alpha)$ for all α

$\Rightarrow f(x) \in Y_\alpha$ for all α $\Rightarrow f(x) \in \bigcap_\alpha Y_\alpha \Rightarrow x \in f^{-1}(\bigcap_\alpha Y_\alpha)$

Hence. "=" is proved. $\square$

Problem 3.

Proof:

"$\Rightarrow$": Proved in 2(d): $f(A\cap B)\subseteq f(A)\cap f(B)$. Can "$\supseteq$" be true?

suppose $y\in f(A)\cap f(B)$ but $y\notin f(A\cap B)$. From $y\in f(A)\cap f(B)$ we know

$\exists\, a\in A$ and $b\in B$ s.t. $f(a)=y=f(b)$. Then the injectivity implies $a=b$.

But then we have $a\in A\cap B \Rightarrow y=f(a)\in f(A\cap B)$. which is a contradiction

"$\Leftarrow$": If the converse is true. say $f(A\cap B)=f(A)\cap f(B)$ (all $A,B\subseteq X$) but f is not injective. here $\exists\ x_1,x_2\in X$ s.t. $x_1\neq x_2$ and $f(x_1)=f(x_2)=y$

Let $A=\{x_1\}$. $B=\{x_2\}$. then

$$\{y\}=f(A)\cap f(B)=f(A\cap B)=f(\phi)=\phi.$$

which is a contradiction. Thus. the sufficiency holds.

Hence. the "iff" relation is proved. $\square$

Problem 4.

Proof:

(a) We claim that $S_n=\bigcup_{k=1}^{n}A_k$. Then the conclusion comes easily from the countable union theorem.

Let's we prove the claim as follows:

"$\supseteq$": For $\forall\, x\in\bigcup_{k=1}^{n}A_k$. here $\exists$ some k_0 s.t. $x\in A_{k_0}\Rightarrow f_{k_0}(x)=0$

$$\Rightarrow \prod_{k=1}^{n}f_k(x)=f_1(x)\cdots \underset{0}{\underset{\parallel}{f_{k_0}(x)}}\cdots f_n(x)=0.\quad\Rightarrow x\in S_n\Rightarrow S_n\supseteq\bigcup_{k=1}^{n}A_k$$

"$\subseteq$": For $\forall\, x\in S_n$. we have $f_1(x)\cdots f_n(x)$. then there must be some k_0 s.t. $f_{k_0}(x)=0$ (which doesn't hold when $n=\infty$). $\Rightarrow x\in A_{k_0}\Rightarrow x\in\bigcup_{k=1}^{n}A_k\Rightarrow S_n\subseteq\bigcup_{k=1}^{n}A_k$.

(b). May not be countable.

eg: $f_k(x) := x$. then $A_k = \{0\}$. and $S = (-1, 1)$ ($\prod_{k=1}^{\infty} f_k(x) = \lim_{n\to\infty} \prod_{k=1}^{n} f_k(x) = \lim_{n\to\infty} x^n = 0$ when $|x|<1$.) $\square$

Problem 5.

Proof: Make the following conventions: we write all numbers in $[0,1]$ in binary system. where $0.\underbrace{00\cdots}_{\text{all zeroes}}$ for $0.\underbrace{111\cdots}_{\text{all 1's}}$ for 1. Then we can define a function

$$\varphi : P(\mathbb{N}) \to [0, 1]$$

in the following rules: (For $\forall S \in P(\mathbb{N})$. the i-th digit of $\varphi(S)$ after the decimal dot is 1 if $i \in S$. otherwise zero) e.g. $\varphi(\{1,3\}) = 0.101\underbrace{00\cdots0\cdots}_{\text{all zeroes}}$

In particular. $\varphi(\phi) = 0$ and $\varphi(\mathbb{N}) = 1$.

It's easy to show φ is well-defined. Now show it's bijective. For Injectivity. if $\varphi(S_1) = \varphi(S_2)$, we have $S_1 = S_2$; since otherwise. we can suppose $\exists s \in \mathbb{N}$ s.t. $s \in S_1$ but $s \notin S_2$ (or converse. WLOG). then the s-th digit of $\varphi(S_1) = 1$ but 0 for $\varphi(S_2)$. a contradiction to $\varphi(S_1) = \varphi(S_2)$.

For surjectivity, we just can recover a subset of $\mathbb{N}$ from $\forall$ number in $[0,1]$, according to the mapping rule.

Hence. $P(\mathbb{N}) \cong [0,1]$ by φ. which implies that $P(\mathbb{N})$ is uncountable. $\square$