MATH2033 Mathematical Analysis (2021 Spring)
Assignment 2

Submission deadline of Assignment 2: 11:59p.m. of 26th Mar, 2020 (Fri)

Instruction: Please complete all required problems. Full details (including description of methods used and explanation, key formula and theorem used and final answer) must be shown **clearly** to receive full credits. Marks can be deducted for incomplete solution or unclear solution.

Please submit your completed work via the submission system in canvas before the deadline. Late assignment will not be accepted.

Your submission must (1) be hand-written (typed assignment will not be accepted), (2) in a single pdf. file (other file formats will not be accepted) and (3) contain your full name and student ID on the first page of the assignment.

Problem 1

(a) Find the supremum and infimum of the following set:

$$S = \left\{ e^{\sqrt{x}} | x \in \mathbb{Q} \cap (0,1) \right\}.$$

(b) We consider a set defined by

$$T = \left\{ n \cos \frac{n\pi}{2} | n \in \mathbb{N} \right\}.$$

Show that the infimum of T does not exist in $\mathbb{R}$.

Problem 2

(a) We let $A \subseteq \mathbb{R}$ be a bounded non-empty subset of real numbers and let $S \subseteq A$ be non-empty subset of real numbers. Prove that

$$\inf A \leq \inf S \leq \sup S \leq \sup A.$$

(b) We let A, B be two bounded subsets of *positive real numbers*. We define

$$C = \{ab | a \in A, b \in B\}.$$

(i) Show that $\sup C = \sup A \sup B$.

(ii) Is the result (i) valid if either A or B contain negative number? Explain your answer.
(*Note: If your answer is yes, give a mathematical proof. If your answer is no, you need to give a counter-example.)

Problem 3

We let $a \in \mathbb{R}$ be a real number. Show that there exists a sequence of rational number $\{q_n\}$ (where $q_n \in \mathbb{Q}$) such that $\{q_n\}$ converges to a (i.e. $\lim_{n\to\infty} q_n = a$).

Problem 4

Prove the following fact using the definition of limits

(a) $\lim_{n\to\infty} \cos\left(a+\frac{b}{n}\right) = \cos a$, where a, b are positive number.

(b) $\lim_{n\to\infty} \sqrt{b_n} = \sqrt{b}$, where $\{b_n\}$ is a convergent sequence with $\lim_{n\to\infty} b_n = b > 0$.

Problem 5

We let $\{x_n\}$ be a sequence defined by

$$x_1 = 0.4, \qquad x_{n+1} = \frac{x_n^3+2}{3} \ for\ n \in \mathbb{N}.$$

Show that $\{x_n\}$ converges and find the limits.

Problem 6 (Harder)

We let $\{x_n\}$ be a sequence of positive real numbers.

(a) Suppose that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L < 1$, show that $\{x_n\}$ converges and $\lim_{n\to\infty} x_n = 0$.
(☺Hint: We let $L < r < 1$ be a number. One can apply the definition of limits to $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$ with $\varepsilon < r - L$ and argue that $\frac{x_{n+1}}{x_n} < r$ when n is greater than some positive integer $K \in \mathbb{N}$.)

(b) Suppose that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L > 1$, show that $\{x_n\}$ does not converge.

(c) Suppose that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L = 1$,

(i) Find an example of $\{x_n\}$ which $\{x_n\}$ converges

(ii) Find another example of $\{x_n\}$ which $\{x_n\}$ does not converges.

****End of Assignment 2****