Solutions of Exercises on Proof by Contradiction Notes

① Assume x is rational. So there exist $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $x = \frac{m}{n}$ and after cancelling common factors, <u>m, n will have no common factor greater than 1</u>.

We have $\left(\frac{m}{n}\right)^3 + 4\left(\frac{m}{n}\right) - 4 = 0$. Multiplying by n^3 on both sides, we get $m^3 + 4mn^2 - 4n^3 = 0$. Then $m^3 = -4mn^2 + 4n^3$ is even, hence m is even. Then there exists $k \in \mathbb{Z}$ such that $m = 2k$. So $(2k)^3 + 4(2k)n^2 - 4n^3 = 0$. Then $4n^3 = -(8k^3 + 8kn^2)$. Cancelling 4 on both sides, we get $n^3 = -2k^3 - 2kn^2$ is even. Hence, n is even. As m, n are both even, they have a common factor 2, contradiction (to the underlined statement). Therefore, x is irrational.

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② Assume there are only finitely many prime numbers, say in increasing order, they are $p_1 = 2, p_2 = 3, p_3 = 5, \cdots, p_n$. Let $M = p_1 p_2 \cdots p_n + 1$. Since $M > p_n$, M is not a prime number. Then checking $2, 3, 4, \ldots, M$, we see there is a smallest integer p (between 2 and M) that is a divisor of M. Being smallest, there cannot be any integer q (with $1 < q < p$) that is a divisor of p (which would also be a divisor of M). Hence p is a prime number. Then $p = p_i$ for some $i = 1, 2, \cdots, n$. Hence, $M - 1 = p_1 p_2 \cdots p_n$ and M are divisible by p_i. Then $M - (M-1) = 1$ will be divisible by $p_i > 1$, which contradicts the only positive divisor of 1 is 1. Therefore, there are infinitely many prime numbers.

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③ Assume it is possible. Since $i \neq 0$, by (a) either $i > 0$ or $0 > i$.

<u>Case 1</u> $(i > 0)$. Then by (c), $i > 0$ and $i > 0$ implies $i^2 > 0$. So $-1 > 0$. By (b), we add 1 to both sides to get $0 > 1$. Also by (c), $-1 > 0$ and $-1 > 0$ implies $(-1)^2 > 0$. So $1 > 0$. Now $0 > 1$ and $1 > 0$ contradicts (a).

<u>Case 2</u> $(0 > i)$. Then by (b), we add $-i$ to both sides to get $-i > 0$. By (c), $-i > 0$ and $-i > 0$ implies $(-i)^2 > 0$. So $-1 > 0$. By (b), we add 1 to both sides to get $0 > 1$. Also by (c), $-1 > 0$ and $-1 > 0$ implies $(-1)^2 > 0$. So $1 > 0$. Now $0 > 1$ and $1 > 0$ contradicts (a). Both cases led to contradiction. We are done.