MATH 2031 Introduction to Real Analysis

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Tutorial Note 18

Riemann Integral Con't

Improper Integral

In this part, we focus on functions $f(x)$ which are either unbounded or defined on an interval which is not closed or not bounded.

(I) **Definition:**
Let I be an interval. A function $f : I \to \mathbb{R}$ is locally integrable if f is integrable on every closed and bounded subinterval of I. We denote this by $f \in L_{loc}(I)$.

(II) **Improper Integrals:**

Case 1: Let $a \in \mathbb{R},\ b \in \mathbb{R} \cup \{+\infty\},\ I = [a, b),\ f \in L_{loc}(I)$.
The improper integral of f on $[a, b)$ is

$$\int_a^b f(x)dx = \lim_{d \to b^-} \int_a^d f(x)dx$$

provided that the limit exists in $\mathbb{R}$.
In this case, we say that f is improper integrable on $[a, b)$.

Case 2: Let $a \in \mathbb{R} \cup \{-\infty\},\ b \in \mathbb{R} \cup \{+\infty\},\ I = (a, b),\ x_0 \in I,\ f \in L_{loc}(I)$.
The improper integral of f on (a, b) is

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^{x_0} f(x)dx + \lim_{d \to b^-} \int_{x_0}^d f(x)dx$$

provided that the limit exists in $\mathbb{R}$.
In this case, we say that f is improper integrable on (a, b).

Remark:
This definition is independent of the choice of x_0.

Case 3: Let $a \in \mathbb{R} \cup \{-\infty\},\ b \in \mathbb{R} \cup \{+\infty\},\ I$ be an interval with endpoints $a, b,\ I_0 = I \cap (-\infty, c)$, $I_1 = I \cap (c, +\infty)$ for $c \in (a, b)$. $f \in L_{loc}(I_0)$, $f \in L_{loc}(I_1)$
The improper integral of f on I is

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

provided that both integrals on the RHS exist in $\mathbb{R}$.
In this case, we say that f is improper integrable on I.

In each case, if the improper integral is a number, we say that the improper integral converges, otherwise it diverges.

(III) **Tests for Improper integral:**

p-test:

For $0 < a < \infty$, $\displaystyle\int_a^{+\infty} \frac{1}{x^p} dx < +\infty \iff p > 1$;

Also $\displaystyle\int_0^a \frac{1}{x^p} dx < +\infty \iff p < 1$.

Comparison test:

Suppose $0 \leq f(x) \leq g(x)$ on interval I and $f, g \in L_{loc}(I)$.
If g is improper integrable on I, then f is also improper integrable on I.

Limit Comparison test:

Suppose $f(x), g(x) > 0$ on $(a, b]$ and $f, g \in L_{loc}((a, b])$.

If $\displaystyle\lim_{x\to a^+} \frac{g(x)}{f(x)} = \left\{ \begin{array}{l} L > 0 \\ 0 \\ +\infty \end{array} \right\}$, then $\left\{ \begin{array}{l} \text{either both } \int_a^b f(x)dx, \int_a^b g(x)dx \text{ converge or both diverge} \\ \int_a^b f(x)dx \text{ converges} \Rightarrow \int_a^b g(x)dx \text{ converges} \\ \int_a^b f(x)dx \text{ diverges} \Rightarrow \int_a^b g(x)dx \text{ diverges} \end{array} \right\}$

For an interval $[a, b)$, we take $\displaystyle\lim_{x\to b^-} \frac{g(x)}{f(x)}$ and the results are similar.

Absolute Convergence test:

Let $f \in L_{loc}(I)$. If $|f|$ is improper integrable on I, then f is improper integrable on I.

Cauchy Principal Value of Integrals

P.V. (I) **Definition:**

Let $f \in L_{loc}(\mathbb{R})$. The principal value of $\displaystyle\int_{-\infty}^{\infty} f(x)dx$ is

$$P.V. \int_{-\infty}^{\infty} f(x)dx = \lim_{c\to\infty} \int_{-c}^{c} f(x)dx$$

P.V. (II) **Theorem:**

If the improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)dx$ exists in $\mathbb{R}$,

then $P.V. \displaystyle\int_{-\infty}^{\infty} f(x)dx$ exists and equals the improper integral $\displaystyle\int_{-\infty}^{\infty} f(x)dx$.

P.V. (III) **Definition:**

Let I be an integral with endpoints a, b, let $c \in (a, b)$, $I_0 = I \cap (-\infty, c)$, $I_1 = I \cap (c, +\infty)$. Let $f \in L_{loc}(I_0)$, $f \in L_{loc}(I_1)$.

Define the principal value of the improper integral $\displaystyle\int_a^b f(x)dx$ as

$$P.V. \int_a^b f(x)dx = \lim_{\varepsilon\to 0^+} \left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^{b} f(x)dx \right)$$

Remark:

You should check carefully before applying the Fundamental theorem, which require that the primitive function of $f(x)$ is differentiable. It may happen that your integral is improper.

Problem 1 Determine the convergence of the following improper integrals:

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{e^{x^2}} \sin \frac{1}{e^{x^2}} dx$

(ii) $\displaystyle\int_{0}^{\infty} \frac{dx}{x^{(1+\frac{1}{x})}}$

Solution:

(i) Since

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x}{e^{x^2}} \sin \frac{1}{e^{x^2}} dx &= \frac{1}{2}\left[\lim_{c\to\infty}\int_0^c \frac{1}{e^{x^2}} \sin \frac{1}{e^{x^2}} dx^2 + \lim_{d\to-\infty}\int_d^0 \frac{1}{e^{x^2}} \sin \frac{1}{e^{x^2}} dx^2\right] \\
&= \frac{1}{2}\left[-\lim_{c\to\infty}\int_0^c \sin \frac{1}{e^{x^2}} de^{-x^2} - \lim_{d\to-\infty}\int_d^0 \sin \frac{1}{e^{x^2}} de^{-x^2}\right] \\
&= \frac{1}{2}\left[-\lim_{c\to\infty} \cos e^{-x^2}\Big|_0^c - \lim_{d\to-\infty} \cos e^{-x^2}\Big|_d^0\right] \\
&= \frac{1}{2}\left[-\lim_{c\to\infty} (\cos e^{-c^2} - \cos 1) - \lim_{d\to-\infty} (\cos 1 - \cos e^{-d^2})\right] \\
&= \frac{1}{2}[-\cos 0 + \cos 1 - \cos 1 + \cos 0] \\
&= 0,
\end{aligned}$$

the improper integral converges (and equals 0).

(ii) Notice that

$$\lim_{x\to\infty} \frac{\left(\frac{1}{x^{(1+\frac{1}{x})}}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{1}{x^{\frac{1}{x}}} = 1 > 0$$

Also, by p-test, we know that $\displaystyle\int_0^{\infty} \frac{dx}{x}$ diverges.

Then by limit comparison test, the improper integral diverges.

Problem 2 Determine if the improper integral $\displaystyle\int_{-1}^{1} \frac{\ln|x|}{x^4+2x^2+1} dx$ converges. Also determine whether the principle value $P.V. \displaystyle\int_{-1}^{1} \frac{\ln|x|}{x^4+2x^2+1} dx$ converges or not.

Solution:

Notice that the denominator x^4+2x^2+1 is continuous and positive on the interval $[-1, 1]$, so the "problematic point" of the integrand is only 0.

Then we have

$$\int_{-1}^{1} \frac{\ln|x|}{x^4+2x^2+1} dx = \lim_{c\to 0^+}\int_c^1 \frac{\ln|x|}{x^4+2x^2+1} dx + \lim_{d\to 0^-}\int_{-1}^d \frac{\ln|x|}{x^4+2x^2+1} dx$$

Since $\displaystyle\frac{\ln|-x|}{(-x)^4+2(-x)^2+1} = \frac{\ln|x|}{x^4+2x^2+1}$, it suffices to consider $\displaystyle\lim_{c\to 0^+}\int_c^1 \frac{\ln|x|}{x^4+2x^2+1} dx$.

Now since

$$\lim_{x\to 0^+} \frac{\frac{\ln|x|}{x^4+2x^2+1}}{\ln|x|} = \lim_{x\to 0^+} \frac{1}{x^4+2x^2+1} = 1 > 0,$$

and $\ln|x|$ is integrable on $[0, 1]$ ($\displaystyle\int_0^1 \ln|x| dx = \int_{-\infty}^0 e^x dx = 1$), by limit comparison test, $\displaystyle\int_0^1 \frac{\ln|x|}{x^4+2x^2+1} dx$ is improper integrable.

Then by symmetry $\lim_{d\to 0^-}\int_{-1}^{d}\frac{\ln|x|}{x^4+2x^2+1}dx$ also converges.

Thus the improper integral $\int_{-1}^{1}\frac{\ln|x|}{x^4+2x^2+1}dx$ converges.

For the principal value, we can consider the following:

$$P.V.\int_{-1}^{1}\frac{\ln|x|}{x^4+2x^2+1}dx=\lim_{\varepsilon\to 0^+}\left(\int_{\varepsilon}^{1}\frac{\ln|x|}{x^4+2x^2+1}dx+\int_{-1}^{\varepsilon}\frac{\ln|x|}{x^4+2x^2+1}dx\right)=\lim_{\varepsilon\to 0^+}2\int_{\varepsilon}^{1}\frac{\ln|x|}{x^4+2x^2+1}dx$$

Then by similar argument as before, it also converges.

Problem 3 Determine if the improper integral $\int_{-\infty}^{\infty}\frac{e^x-e^{-x}}{2}dx$ converges. Also determine whether the principle value $P.V.\int_{-\infty}^{\infty}\frac{e^x-e^{-x}}{2}dx$ converges or not.

Solution:
Since $\frac{e^x-e^{-x}}{2}$ is continuous on $\mathbb{R}$, we have

$$\int_{-\infty}^{\infty}\frac{e^x-e^{-x}}{2}dx=\lim_{c\to\infty}\int_{0}^{c}\frac{e^x-e^{-x}}{2}dx+\lim_{d\to-\infty}\int_{d}^{0}\frac{e^x-e^{-x}}{2}dx.$$

But then

$$\begin{aligned}\lim_{c\to\infty}\int_{0}^{c}\frac{e^x-e^{-x}}{2}dx&=\lim_{c\to\infty}\left.\frac{e^x+e^{-x}}{2}\right|_{0}^{c}\\&=\lim_{c\to\infty}\left[\frac{e^c+e^{-c}}{2}-1\right]\\&=\infty\end{aligned}$$

and

$$\begin{aligned}\lim_{d\to-\infty}\int_{d}^{0}\frac{e^x-e^{-x}}{2}dx&=\lim_{d\to-\infty}\left.\frac{e^x+e^{-x}}{2}\right|_{d}^{0}\\&=\lim_{d\to-\infty}\left[1-\frac{e^d+e^{-d}}{2}\right]\\&=-\infty\end{aligned}$$

Thus the improper integral does not exist.

As for the principal value,

$$\begin{aligned}P.V.\int_{-\infty}^{\infty}\frac{e^x-e^{-x}}{2}dx&=\lim_{c\to\infty}\int_{-c}^{c}\frac{e^x-e^{-x}}{2}dx\\&=\lim_{c\to\infty}\left[\int_{0}^{c}\frac{e^x-e^{-x}}{2}dx+\int_{-c}^{0}\frac{e^x-e^{-x}}{2}dx\right]\\&=\lim_{c\to\infty}\left[\frac{e^c+e^{-c}}{2}-1+1-\frac{e^c+e^{-c}}{2}\right]\\&=\lim_{c\to\infty}0\\&=0,\end{aligned}$$

so the principal value exists and equals 0.