MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #10

Midterm Review (Real Number)

Recall the two main theorems

Theorem 1: (Supremum Limit Theorem)

$$\left.\begin{array}{c}\text{c is upper bound of S}\\ \text{There exists } \{w_n\} \in S, \text{such that } \lim\limits_{n\to\infty} w_n = c\end{array}\right. \leftrightarrow \quad c = \sup S$$

Theorem 2: (Infimum Limit Theorem)

$$\left.\begin{array}{c}\text{c is lower bound of S}\\ \text{There exists } \{w_n\} \in S, \text{such that } \lim\limits_{n\to\infty} w_n = c\end{array}\right. \leftrightarrow \quad c = \inf S$$

Example 1

Compute the supremum and infimum of the set

$$S = \{x + 2y: x \in (-1,1), y \in (-2,5)$$

Solution:

(Step 1) First, since $-1 < x < 1$ and $-2 < y < 5$, thus we have

$$-5 = -1 + 2(-2) < x + 2y < 1 + 2(5) = 11$$

So the upper bound and lower bound are 11 and -5 respectively.

(Step 2)

To show $\sup S = 11$ (This maximum is obtained when $x \to 1$ and $y \to 5$)

We construct our $w_n = x_n + 2y_n$

Pick $x_n = 1 - \frac{1}{n}$ and $y_n = 5 - \frac{1}{n}$, (note $x_n \in (0,1)$ and $y_n \in (-2,5)$)

Hence $w_n = 1 - \frac{1}{n} + 2\left(5 - \frac{1}{n}\right) = 11 - \frac{3}{n} \in S$ and $\lim_{n\to\infty} w_n = 11$

Therefore by supremum limit theorem, we get $\sup S = 11$

To show $\inf S = -5$ (This minimum is obtained when $x \to -1$ and $y \to -2$)

Pick $x_n = -1 + \frac{1}{n}$ and $y_n = -2 + \frac{1}{n}$, (note $x_n \in (0,1)$ and $y_n \in (-2,5)$)

So $w_n = -1 + \frac{1}{n} + 2\left(-2 + \frac{1}{n}\right) = -5 + \frac{3}{n} \in S$ and $\lim_{n\to\infty} w_n = -5$

By infimum limit theorem, we get $\inf S = -5$

Example 2

Find the supreme and infimum of the set

$$S = \{3x^2 - y^3 : x \in [-2,3], y \in (0,3)\}$$

(Step 1) Note that $0 \le x^2 \le 9$ and $0 < y^3 < 3^3 = 27$

Hence

$$-27 = 3(0) - 27 < 3x^2 - y^3 < 3(9) - (0)^3 = 27$$

So the upper bound and lower bound are 27 and -27 respectively

(Step 2) To show $\sup S = 27$ (It is obtained when $x^2 = 9 \to x = 3$ and $y \to 0$)

Pick $x_n = 3$ and $y_n = \frac{1}{n}$ (then $x_n \in [-2,3]$ and $y_n \in (0,3)$)

Then $w_n = 3x_n^2 - y_n^3 = 3(3)^2 - \left(\frac{1}{n}\right)^3 = 27 - \frac{1}{n^3} \in S$ and $\lim_{n\to\infty} w_n = 27$

Therefore by supreme limit theorem, $\sup S = 27$

To show $\inf S = -27$ (obtained when $x^2 = 0 \to x = 0$ and $y \to 3$

Pick $x_n = 0$ and $y_n = 3 - \frac{1}{n}$ (then $x_n \in [-2,3]$ and $y_n \in (0,3)$)

So $w_n = 3x_n^2 - y_n^3 = 3(0)^2 - \left(3 - \frac{1}{n}\right)^3 \in S$ and $\lim_{n\to\infty} w_n = (-3)^3 = -27$

Therefore by infimum limit theorem, $\inf S = -27$

Example 3

Find the supreme and infimum of the set

$$A = \{x^6 + \frac{1}{y^3} : x \in (1,2)\backslash\mathbf{Q}, y \in (2,4) \cap \mathbf{Q}\}$$

(Note: Hence x need to be irrational and y need to be rational)

(Step 1) Note that $1 < x < 2$ and $2 < y < 4$, so

$$\frac{65}{64} = 1^6 + \frac{1}{4^3} < x^6 + \frac{1}{y^3} < 2^6 + \frac{1}{2^3} = \frac{513}{8}$$

So the upper bound and lower bound are $\frac{513}{8}$ and $\frac{65}{64}$ respectively

(Step 2) To show $\sup S = \frac{513}{8}$ (when $x \to 2$ and $y \to 2$)

Pick $\mathbf{x_n} = \mathbf{2} - \frac{\mathbf{1}}{\sqrt{\mathbf{2}}\mathbf{n}}$ (But not $x_n = 2 - \frac{1}{n+1}$) and $y_n = 2 + \frac{1}{n}$

So $w_n = \left(2 - \frac{1}{\sqrt{2}n}\right)^6 + \frac{1}{\left(2+\frac{1}{n}\right)^3} \in S$ and $\lim_{n\to\infty} w_n = 2^6 + \frac{1}{2^3} = \frac{513}{8}$

By supreme limit theorem, $\sup A = \frac{513}{8}$.

To show $\inf S = \frac{65}{64}$ (when $x \to 1$ and $y \to 4$)

Pick $x_n = 1 + \frac{1}{\sqrt{2}n}$ (once again not $x_n = 1 + \frac{1}{2n}$) and $y_n = 4 - \frac{1}{n}$

Then $w_n = \left(1 + \frac{1}{\sqrt{2}n}\right)^6 + \frac{1}{\left(4 - \frac{1}{n}\right)^3} \in S$ and $\lim_{n\to\infty} w_n = 1^6 + \frac{1}{4^3} = \frac{65}{64}$

So by infimum limit theorem, $\inf S = \frac{65}{64}$

Example 4 Find the supreme and infimum of the set $S = \{x - \sqrt{y} : x \in \mathbf{Q} \cap (0, \sqrt{3}), y \in \mathbf{Q} \cap (2, \pi)$ Hence we require x and y are both rational numbers.

Solution:

(Step 1) Note that $0 < x < \sqrt{3}$ and $-2 < y < \pi$

Then $-\sqrt{\pi} = 0 - \sqrt{\pi} < x - \sqrt{y} < \sqrt{3} - \sqrt{2}$

Hence the upper bound and lower bound are $\sqrt{3} - \sqrt{2}$ and $-\sqrt{\pi}$ respectively.

(Step 2) To show $\sup S = \sqrt{3} - \sqrt{2}$ (when $x \to \sqrt{3}$ and $y \to 2$)

Pick $\mathbf{x_n} = \frac{[\mathbf{10^n}\sqrt{\mathbf{3}}]}{\mathbf{10^n}} \in \mathbf{Q}$ (not $x_n = \sqrt{3} - \frac{1}{n}$!!!!) and $\mathbf{y_n} = \mathbf{2} + \frac{\mathbf{1}}{\mathbf{n}}$

Then $w_n = x_n - \sqrt{y_n} = \frac{[10^n\sqrt{3}]}{10^n} - \sqrt{2 + \frac{1}{n}} \in S$ and $\lim_{n\to\infty} w_n = \sqrt{3} - \sqrt{2}$

By supreme limit theorem, $\sup S = \sqrt{3} - \sqrt{2}$

To show $\inf S = -\sqrt{\pi}$ (when $x \to 0$ and $y \to \pi$)

Pick $\mathbf{x_n} = \frac{\mathbf{1}}{\mathbf{n}}$ and $\mathbf{y_n} = \frac{[\mathbf{10^n}\boldsymbol{\pi}]}{\mathbf{10^n}} \in \mathbf{Q}$

Then $w_n = x_n - \sqrt{y_n} = \frac{1}{n} - \sqrt{\frac{[10^n\pi]}{10^n}} \in S$ and $\lim_{n\to\infty} w_n = -\sqrt{\pi}$

By infimum limit theorem, $\inf S = -\sqrt{\pi}$

Difficult situation:

A) Unknown Set

Example 5

Let A be the subset of real numbers which $\sup A = \sqrt{7}$, find the supreme of the set

$$B = \{x^3 + 7y : x, y \in A\}$$

Solution:

(Step 1) Note $x, y \in A$ and $\sup A = \sqrt{7}$, so $x \leq \sqrt{7}$ and $y \leq \sqrt{7}$

Therefore $x^3 + 7y \leq \left(\sqrt{7}\right)^3 + 7\left(\sqrt{7}\right) = 14\sqrt{7}$. So the upper bound is $14\sqrt{7}$

(Step 2) To show $\sup B = 14\sqrt{7}$

Since $\sup A = \sqrt{7}$. By supreme limit theorem, there exist a sequence $\{a_n\} \in A$ such that $\lim_{n\to\infty} a_n = \sqrt{7}$

Pick $x_n = y_n = a_n$ (so $x_n, y_n \in A$)

Then $w_n = x_n^3 + 7y_n = a_n^3 + 7a_n \in B$ and $\lim_{n\to\infty} w_n = 14\sqrt{7}$

By supremum limit theorem, we conclude $\sup B = 14\sqrt{7}$.

(Remark: Some students may set $x_n = y_n = \sqrt{7} - \frac{1}{n}$ or $x_n = y_n = \sqrt{7}$,which is not right since B is unknown set and we do not know what B exactly contains!)

Example 6

Let C be the subset of rational number which $\inf C = \frac{1}{2}$, find the infimum of the set

$$D = \{p^3 - q : p \in C, q \in [0,1]\backslash\mathbf{Q}\}$$

(Here q is an irrational number)

Solution:

(Step 1) Note that $p \in \mathbf{Q} \to p \geq \frac{1}{2}$ and $0 \leq q \leq 1$

Hence $p^3 - q \geq \left(\frac{1}{2}\right)^3 - 1 = -\frac{7}{8}$, so the lower bound is $-\frac{7}{8}$

(Step 2) To show $\inf D = -\frac{7}{8}$ (when $p \to \frac{1}{2}$ and $q \to 1$)

Since $\inf C = \frac{1}{2}$, by infimum limit theorem, there exist $\{c_n\} \in C$, such that

$\lim_{n\to\infty} c_n = \frac{1}{2}$

Pick $p_n = c_n$ and $q_n = 1 - \frac{1}{\sqrt{2}n}$ (so $p_n \in C$ and $q_n \in [0,1]\backslash\mathbf{Q}$)

Then $w_n = p_n^3 - q_n = c_n^3 - (1 - \frac{1}{\sqrt{2}n}) \in D$ and $\lim_{n\to\infty} w_n = -\frac{7}{8}$.

By infimum limit theorem, $\inf D = -\frac{7}{8}$.

Example 7 Let A_1, A_2 and A_3 be the subsets of real numbers such that $\inf A_1 = 2$, $\inf A_2 = 6$ and $\inf A_3 = 3$. Find the infimum of the set $S = A_1 \cup A_2 \cup A_3$

(Step 1)

For any $x \in S$,then $x \in A_1$ or $x \in A_2$ or $x \in A_3$,

but we must have $x \geq 2$, so the lower bound of S is 2.

(Step 2)

To show $\inf S = 2$ (happen when $x \to 2$ by elements in A_1)

By infimum limit theorem, there exists $a_n \in A_1$ such that $\lim_{n\to\infty} a_n = 2$.

Pick $w_n = a_n$, since $A_1 \subseteq A_1 \cup A_2 \cup A_3$, then $w_n \in A_1 \cup A_2 \cup A_3)$

and $\lim_{n\to\infty} w_n = 2$.

By infimum limit theorem, we conclude $\inf S = 2$

Example 8 Let D be a subset of real number such that $\sup E = 1$, find the supremum of the set $$S = \{x^3 - 3y + 2z^5 : x \in \mathbf{Q} \cap (-1, e), y \in (1,3) \backslash \mathbf{Q}, z \in D)\}$$ (Note that x is rational and y is irrational)

(Step 1)

Since $-1 < x < e$, $1 < y < 3$ and $z \leq 1$

So $x^3 - 3y + 2z^5 < e^3 - 3(1) + 2(1)^5 = e^3 - 1$

The upper bound is $e^3 - 1$

(Step 2)

To show $\sup S = e^3 - 2$ (when $x \to e$, $y \to 1$ and $z \to 1$)

We construct $w_n = x_n^3 - 3y_n + 2z_n^5$ as follows:

For x_n, we pick $x_n = \frac{[10^n e]}{10^n} \in \mathbf{Q} \cap (-1, e)$

For y_n,we pick $y_n = 1 + \frac{1}{\sqrt{2}n} \in (1,3) \backslash \mathbf{Q}$

For z_n, by supreme limit theorem, there exist $\{e_n\} \in E$ which $\lim_{n\to\infty} e_n = 1$, pick $z_n = e_n$

So $w_n = x_n^3 - 3y_n + 2z_n^5 = \left(\frac{[10^n e]}{10^n}\right)^3 - 3\left(1 + \frac{1}{\sqrt{2}n}\right) + 2e_n^5 \in S$, and $\lim_{n\to\infty} w_n =$

$e^3 - 1$. By supreme limit theorem, $\sup S = e^3 - 1$

Try to finish the following exercises, if you have any questions about the exercises, please feel free to find me.

☺Exercise 1

Find the supreme and infimum of the sets by using limit theorem.

a) $\{x - 2y : x \in (2,3) \text{and } y \in [2,5)\}$

b) $\left\{\frac{3}{n} - \frac{4}{m} : n \in \mathbf{N}, m \in \mathbf{Q} \cap (1,2)\right\}$

c) $\{2 - x^3 + y - z : x \in (2,3)\backslash\mathbf{Q}, , y, z \in (1,2) \cap \mathbf{Q}\}$

d) $\{|x - y| : x, y \in (0, \sqrt{3}) \cap \mathbf{Q}\}$

e) $\{e^x - y : x \in (-2, \ln 3) \cap \mathbf{Q}, y \in [2, \infty)\}$

f) $\{4x^5 + 2y^2 : x \in (2,5)\backslash\mathbf{Q} \text{ and } y \in (-3,2)\}$

☺Exercise 2

Let A be the non-empty subset of the real number, which $\sup A = 3$ and $\inf A = -2$. Find the supreme of the following sets

a) $\left\{\frac{1}{x+5} - y^3, x, y \in A\right\}$

b) $\{3x - 2y : x \in A, y \in \mathbf{Q} \cap (-2,2)\}$

c) $\{6x^3 - 2y + z : x, y \in A \text{ and } z \in (0,2)\backslash\mathbf{Q}\}$

d) $\left\{\sqrt{5 + x} + y : x \in A, y \in Q \cap (1, \pi]\right\}$

☺Exercise 3

a) Find the supreme and infimum of the following sets

$A = \{1 + \frac{2}{n} : n \in \mathbf{N}\}$, $B = \{3 - p : p \in \mathbf{Q} \cap (2,3)\}$, $C = \{x + \sqrt{2}y : x, y \in [1,2]\backslash\mathbf{Q}\}$

b) Find the supreme and infimum of the set

$$S = A \cup B \cup C$$

☺Exercise 4

Find the infimum of the set

$$S = \{x \in \mathbf{Q} : \sum_{k=1}^{\infty} \sin^{\frac{x}{e}}\left(\frac{1}{k}\right) \text{ converges}\}$$

$$T = \{b \in \mathbf{R}\backslash\mathbf{Q} : \sum_{k=1}^{\infty} \ln\left(1 + \frac{b}{k}\right) \text{ diverges}, b \geq 0\}$$