DEPARTMENT OF MATHEMATICS

Math2033 TA note 4

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1 COUNTABILITY

Theorem 1 (Bijection theorem)**.** *Let $g: S \to T$ be a bijection. S is countable if and only if T is countable.*

Theorem 2 (Countable union theorem)**.** *$\bigcup_{i\in\Lambda} A_i$ is countable if Λ and each A_i is countable.*

Theorem 3 (Countable subset theorem)**.** *$A \subset B$, then B countable $\implies$ A is countable.*

Example 4. *For $a < b$, let*

$$(a,b)_{\mathbb{Q}} = \{x : x \in \mathbb{Q} \text{ and } a < x < b\} \quad \text{and} \quad [a,b)_{\mathbb{Q}} = \{x : x \in \mathbb{Q} \text{ and } a \le x < b\}$$

Does $\bigcup_{n=1}^{\infty} [\frac{1}{n}, 2)_{\mathbb{Q}} = \bigcup_{n=1}^{\infty} (\frac{1}{n}, 2)_{\mathbb{Q}}$?

Solution: Yes!

Since $(\frac{1}{n}, 2)_{\mathbb{Q}} \subset [\frac{1}{n}, 2)_{\mathbb{Q}}$ for each n, $\bigcup_{n=1}^{\infty} (\frac{1}{n}, 2)_{\mathbb{Q}} \subset \bigcup_{n=1}^{\infty} [\frac{1}{n}, 2)_{\mathbb{Q}}$.

On the other hand, $[\frac{1}{n}, 2)_{\mathbb{Q}} \subset (\frac{1}{n+1}, 2)_{\mathbb{Q}}$ for each n, we have $\bigcup_{n=1}^{\infty} [\frac{1}{n}, 2)_{\mathbb{Q}} \subset \bigcup_{n=2}^{\infty} (\frac{1}{n}, 2)_{\mathbb{Q}} = \bigcup_{n=1}^{\infty} (\frac{1}{n}, 2)_{\mathbb{Q}}$.

Therefore, $\bigcup_{n=1}^{\infty} [\frac{1}{n}, 2)_{\mathbb{Q}} = \bigcup_{n=1}^{\infty} (\frac{1}{n}, 2)_{\mathbb{Q}}$.

Example 5. *Let A, B be subsets of $\mathbb{R}$ and $f: A \to B$ be a function. If for every $b \in B$, the horizontal line $y = b$ intersects the graph of f at most once, show that f is injective. If 'at most once' is replaced by 'at least once', what can be said about f?*

Solution: To show f is injective for first part, we assume $f(a_1) = f(a_2) = b \in B$ for some $a_1, a_2 \in A$.

Since for every $b \in B$, the horizontal line $y = b$ intersects the graph of f at most once, we have $(a_1, f(a_1)) = (a_2, f(a_2))$, which implies $a_1 = a_2$ and hence f is injective.

As for second part, f is surjective but no necessary injective.

Since for $\forall b \in B$, there exists at least one $a \in A$ such that $f(a) = b$, we have $B \subset f(A)$. Combined $f(A) \subset B$, we have $f(A) = B$ and thus f is surjective.

Example 6. *Show that the set $F = \{a : x^4 + ax - 5 = 0$ has a rational root$\}$ is countable.*

Solution: Since the equation $x^4 + ax - 5 = 0$ has a root $x \in \mathbb{Q}$ and $x \neq 0$, we have $a = \frac{5-x^4}{x} \in \mathbb{Q}$. By countable subset theorem, $F \subset \mathbb{Q}$ and $\mathbb{Q}$ countable implies F countable.

Example 7. *Show that the set F of all finite subsets of $\mathbb{N}$ is countable.*

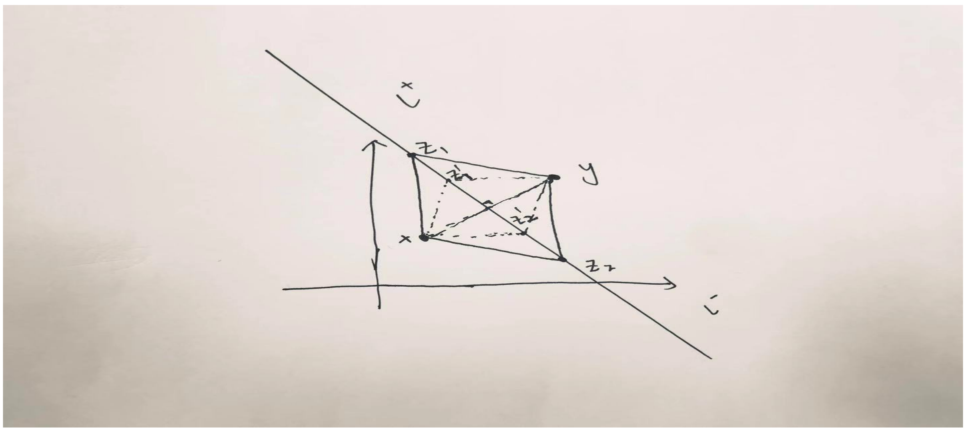
Solution: For $k = 0, 1, 2, \ldots$, let S_k be the set of all subsets of $\mathbb{N}$ having exactly k elements. Thus,

$$S_k = \{S \subseteq \mathbb{N} : card(S) = k\}, \qquad F = S_0 \bigcup \left(\bigcup_{k \in \mathbb{N}} S_k \right).$$

Since $S_0 = \{\emptyset\}$, S_0 is countable. For $k \in \mathbb{N}$, the function $f_k : S_k \to \mathbb{N}^k$ defined by $f(\{n_1, n_2, \ldots, n_k\}) = (n_1, n_2, \ldots, n_k)$ is injective, where we require that $n_1 < n_2 < \cdots < n_k$ so that f_k is well-defined. Since $\mathbb{N}^k$ is countable by the product theorem, we conclude that S_k is countable by the injection theorem. Then $F = S_0 \bigcup (\bigcup_{k \in \mathbb{N}} S_k)$ is countable by the countable union theorem.

Example 8. *If S is a countable subset of $\mathbb{R}^2$, show that for any two points $x, y \in \mathbb{R}^2 \setminus S$, there is a parallelogram in $\mathbb{R}^2 \setminus S$ having x, y as opposite vertices. Here parallelogram means only the 4 edges.*

Solution: As the figure shown, for fixed $x, y \in \mathbb{R}^2 \setminus S$, we construct a line l passing through



the middle point of line segment xy. And we denote the segment of l above xy as l^+ and the segment below xy as l^{-1}. Then for point $z_1 \in l^+$, we can find a point $z_2 \in l^{-1}$ to have a parallelogram with vertex x, z_1, y, z_2. We denote the parallelogram with vertex x, z_1, y, z_2 (only consider 4 edges) as $P(xz_1yz_2)$. And we first see that

$$P(xz_1yz_2) \cap P(xz_1'yz_2') = \{x, y\}, \text{for } z_1 \neq z_1', \ z_1, z_1' \in l^+ \tag{1.1}$$

Then we assume that there is no parallelogram in $\mathbb{R}^2 \setminus S$ having x, y as opposite vertices. Then we have

$$S \cap P(xz_1yz_2) \neq \phi, \forall z_1 \in l^+$$

For $z_1 \in l^+$, we can choose one element in $S \cap P(xz_1yz_2)$ denoted as $s(z_1)$. And by equation (1.1), we have

$$s(z_1) \neq s(z_1'), \text{for } z_1 \neq z_1', \ z_1, z_1' \in l^+$$

Then

$$card(\{s(z_1), z_1 \in l^+\}) = card(l^+)$$

Because l^+ is uncountable, then $\{s(z_1), z_1 \in l^+\}$ is uncountable. By $\{s(z_1), z_1 \in l^+\} \subset S$, we show that S is uncountable which contradicts the problem condition. So the assumption is not right and we must have some parallelogram in $\mathbb{R}^2 \setminus S$ having x, y as opposite vertices.