Review for Spring Final Exam

Definition A set S is of measure 0 iff $\forall \varepsilon > 0$, $\exists$ open intervals $(a_1,b_1), (a_2,b_2), (a_3,b_3), \cdots$ such that $S \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)$ and $\sum_{n=1}^{\infty} |a_n - b_n| < \varepsilon$.

Known Examples

1. Every countable set is of measure 0. There also exist uncountable sets of measure 0.
2. If $S_1, S_2, S_3, \cdots$ are sets of measure 0, then $\bigcup_{n=1}^{\infty} S_n$ is also of measure 0.
3. If $A \subseteq B$ and B is of measure 0, then A is of measure 0.

Lebesgue's Theorem

A bounded function $f: [a,b] \to \mathbb{R}$ is Riemann integrable iff the set

$$S_f = \{x \in [a,b] : f \text{ is not continuous at } x\}$$

is a set of measure 0.

Known Facts

1. Continuous functions on $[a,b]$ are integrable. Monotone functions on $[a,b]$ are integrable.
2. If f_1, f_2 are integrable on $[a,b]$, then f_1+f_2, f_1-f_2, $f_1 f_2$ are integrable on $[a,b]$.
3. If f is integrable on $[a,b]$ and $[c,d] \subseteq [a,b]$, then f is integrable on $[c,d]$.
4. If f is integrable on $[a,b]$ and g is continuous on $f([a,b])$, then $g \circ f$ is integrable on $[a,b]$.

2007 Final Exam

#5 Let $f, g : [0,1] \to \mathbb{R}$ be monotone. Prove that $h : [0,1] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} f(x) - g(x) & \text{if } x \in [0, 1/2) \\ f(x) + g(x) & \text{if } x \in [1/2, 1] \end{cases}$$

is bounded and Riemann integrable on $[0,1]$.

Solution.

Show h bounded

f, g montone $\Rightarrow$ f bounded between $f(0)$ and $f(1)$, g bounded between $g(0)$ and $g(1)$.

$\Rightarrow \exists M_1, M_2 > 0$ such that $\forall x \in [0,1]$, $|f(x)| \le M_1$ and $|g(x)| \le M_2$

$\Rightarrow \exists M_1, M_2 > 0$ such that $\forall x \in [0,1]$, $|f(x) \pm g(x)| \le M_1 + M_2$

$\Rightarrow h$ is bounded on $[0,1]$.

Show S_h is of measure 0

f, g monotone $\Rightarrow f, g$ Riemann integrable on $[0,1]$

$\Rightarrow p = f - g$, $q = f + g$ Riemann integrable on $[0,1]$

$$S_h \subseteq \underbrace{(S_p \cap [0, 1/2))}_{\text{measure } 0} \cup \underbrace{(S_q \cap [1/2, 1])}_{\text{measure } 0} \cup \underbrace{\{1/2\}}_{\text{measure } 0}$$

$\Rightarrow S_h$ is of measure 0

$\Rightarrow h$ is Riemann integrable on $[0,1]$.

⑤ (a) State the integral criterion 2006

(b) State Lebesgue's theorem

(c) Let $f:[0,1]\to[0,1]$ be Riemann integrable. g is given to be bounded!

Prove that $g:[1,2]\to[0,1]$ defined by

$$g(x)=f(2-x) \text{ is Riemann integrable}$$

by integral criterion.

Prove that $h:[0,2]\to[0,1]$ defined by

$$h(x)=\begin{cases} f(x) & \text{if } x\in[0,1) \\ g(x) & \text{if } x\in[1,2]\end{cases}$$

is Riemann integrable by Lebesgue's theorem.

<u>Solution</u>

(a) Let $f:[a,b]\to\mathbb{R}$ be a bounded function.

f is Riemann integrable iff

$\forall \varepsilon>0$, $\exists$ partition P of $[a,b]$

Such that $U(f,P)-L(f,P)<\varepsilon$.

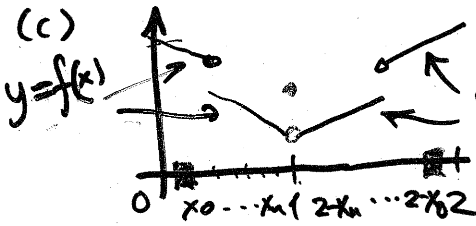
(b) For a bounded function $f:[a,b]\to\mathbb{R}$,

f is Riemann integrable iff

$S_f=\{x\in[a,b] : f \text{ is discontinuous at } x\}$

is of measure 0 (i.e. f is continuous almost everywhere).

(c)



$g(1)=f(1)$ $g(2)=f(0)$

$g(1\tfrac{1}{2})=f(\tfrac{1}{2})$

Since f is Riemann integrable on $[0,1]$, by integral criterion, $\exists$ partition $P=\{0=x_0<x_1<\cdots<x_n=1\}$ Such that $U(f,P)-L(f,P)<\varepsilon$.

Let $x_i'=2-x_{n-i}$, then $P'=\{1=x_0'<x_1'<\cdots<x_n'=2\}$ is a partition of $[1,2]$. Since $g(x)=f(2-x)$ and $x_i'=2-x_{n-i}$, we have

$$\sup\{g(x): x\in[x_i',x_{i+1}']\}$$
$$=\sup\{f(t): t\in[x_{n-i-1},x_{n-i}]\}$$

Similarly,

$$\inf\{g(x): x\in[x_i',x_{i+1}']\}$$
$$=\inf\{f(t): t\in[x_{n-i-1},x_{n-i}]\}$$

Hence $U(g,P')-L(g,P')$

$$=U(f,P)-L(f,P)$$
$$<\varepsilon.$$

By integral criterion, g is Riemann integrable on $[1,2]$.

Next, on $[0,1)$, since $h(x) = f(x)$, so

h is discontinuous at x

$\iff f$ is discontinuous at x.

On $[1,2]$, since $h(x) = g(x)$, so

h is discontinuous at x

$\iff g$ is discontinuous at x.

So $$S_h \subseteq (S_f \cap [0,1]) \cup (S_g \cap (1,2]) \cup \{1\}$$
$$\subseteq S_f \cup S_g \cup \{1\}.$$

Since f is integrable on $[0,1]$ and

g is integrable on $[1,2]$,

S_f, S_g are measure 0 sets.

$\therefore$ S_h (being a subset of $S_f \cup S_g \cup \{1\}$)

is of measure 0.

$\therefore$ h is Riemann integrable on $[0,2]$

by Lebesgue's theorem.

2008 Final

5 (a) State Lebesgue's Theorem.

(b) For $n = 1, 2, 3, \cdots$, let $f_n : [0,1] \to [0,1]$ be integrable.

Prove that $g : [0,1] \to \mathbb{R}$ defined by $g(0) = 0$ and

$$g(x) = f_n(x) \text{ for } n = 1,2,3,\cdots \text{ and } x \in \left(\tfrac{1}{n+1}, \tfrac{1}{n}\right]$$

is Riemann integrable on $[0,1]$.

Solution.

(a) A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable

iff $S_f = \{x \in [a,b] : f \text{ is discontinuous at } x\}$

is of measure 0

(i.e. f is continuous almost everywhere).

(b) Since f_n is Riemann integrable on $[0,1]$, S_{f_n} is of measure 0. Then $S_{f_n} \cap \left(\tfrac{1}{n+1}, \tfrac{1}{n}\right]$ is of measure 0.

Now
$$\underbrace{S_g \subseteq \underbrace{\left\{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots\right\}}_{\substack{\text{countable} \\ \Rightarrow \text{ measure } 0}} \cup \underbrace{\bigcup_{n=1}^{\infty} \underbrace{\left(S_{f_n} \cap \left(\tfrac{1}{n+1}, \tfrac{1}{n}\right]\right)}_{\text{measure } 0}}_{\text{measure } 0}}_{\text{measure } 0}$$

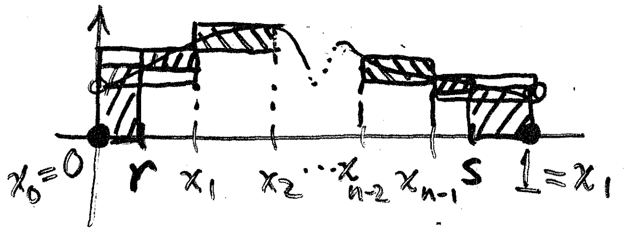
$\therefore$ S_g is of measure 0.

$\therefore$ g is Riemann integrable on $[0,1]$.

<u>Remark</u> Since we are given $g(x) = f_n(x) \in [0,1]$, we see g is bounded.

(2003 Final) Let $f : [0,1] \to [-1,1]$ be Riemann integrable. Using the integral criterion, prove that

$$g(x) = \begin{cases} f(x) & \text{if } 0<x<1 \\ 0 & \text{if } x=0 \text{ or } 1 \end{cases}$$ is also Riemann integrable on $[0,1]$.



<u>Solution</u> Since f is Riemann integrable on $[0,1]$, $\forall \varepsilon>0, \exists$ partition $P_1 = \{0=x_0<x_1<\cdots<x_n=1\}$ such that $\boxed{U(f,P_1)-L(f,P_1)<\varepsilon/3}$ by the integral criterion.

Choose $r \in (0,x_1)$ and $r<\varepsilon/6$. Also choose $s\in(x_{n-1},1)$ and $1-s<\varepsilon/6$. Let $P_2 = P_1 \cup \{r,s\}$. By refinement theorem, $L(f,P_1)\le L(f,P_2)\le U(f,P_2) \le U(f,P_1)$

So $U(f,P_2)-L(f,P_2)\le U(f,P_1)-L(f,P_1)<\varepsilon/3$.

Since $g(x)\in[-1,1]$,

$$\begin{aligned} U(g,P_2)-L(g,P_2) &\le r\left(\sup\{g(x):x\in[0,r]\}-\inf\{g(x):x\in[0,r]\}\right) \\ &\quad + \left(U(f,P_2)-L(f,P_2)\right)+(1-s)\left(\sup\{g(x):x\in[s,1]\}-\inf\{g(x):x\in[s,1]\}\right) \\ &\le \frac{\varepsilon}{6}(1-(-1))+\frac{\varepsilon}{3}+\frac{\varepsilon}{6}(1-(-1)) = \varepsilon. \end{aligned}$$

By integral criterion, g is Riemann integrable on $[0,1]$.