MATH202 Introduction to Analysis (2007 Fall-2008 Spring)

Tutorial Note #1

Calculus Review

Problem 1

Draw the graph of the function

$$f(x) = \frac{x}{x}$$

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☹ ***Wrong Solution***

$$\boldsymbol{f(x) = \frac{x}{x} = 1}$$

then the graph should be a continuous line

The problem here is $\frac{x}{x} = 1$ is only true when $x \neq 0$, for x = 0, $\frac{0}{0}$ is undefined and could not be 1. Therefore,

☺***Right Solution***

$$\boldsymbol{for\ x \neq 0, f(x) = \frac{x}{x} = 1}$$

$$\boldsymbol{for\ x = 0, f(0) = \frac{0}{0}\ is\ undefined}$$

There the graph is a continuous line but with a cut at $\mathbf{x = 0}$

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Problem 2

Find $\lim_{x\to\infty} \frac{x + 2\cos x}{3 + 4x}$

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☹***Wrong Solution"***

Since both denominator and numerator tends to infinity, therefore applying L'Hospital Rule, we have

$$\boldsymbol{\lim_{x\to\infty} \frac{x + 2\cos x}{3 + 4x} = \lim_{x\to\infty} \frac{1 - 2\sin x}{4}}$$

Which the limit does not exist as sinx oscillates.

The trouble here is to apply the L'Hospital Rule

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

We need to make sure the limit at R.H.S. exists. But in this case, we see the limit does not exist. So it does not represent the limit of original expression.

☺Right Solution

$$\lim_{x\to\infty}\frac{x+2cosx}{3+4x}=\lim_{x\to\infty}\frac{1+\frac{2cosx}{x}}{\frac{3}{x}+4}=\frac{1}{4}\quad since\ \ cosx\ is\ bounded\ by\ 1$$

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Problem 3

Let $f(x)=\begin{cases}x^2 \text{ if } x\neq 3\\ 3x \text{ if } x=3\end{cases}$. Is it true that $f'(x)=\begin{cases}2x \text{ if } x\neq 3\\ 3 \text{ if } x=3\end{cases}$?

The answer is wrong here, for $x\neq 3$, of course we can apply direct differentiation to get the derivative as shown. But there is trouble at $x=3$, it is because according to the definition of derivative (i.e. first principle), we have

$$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$

Since $h\neq 0$, so $3+h\neq 3$, then $f(3+h)=(3+h)^2$, but if we take the direct differentiation, then we have assumed $f(3+h)=3(3+h)$ which is not true.

☺Right Solution

$$\boldsymbol{For\ x\neq 3, f'(x)=\frac{d}{dx}x^2=2x}$$

$$\boldsymbol{For\ x=3,}$$

$$\boldsymbol{f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}=\lim_{h\to 0}\frac{(3+h)^2-3(3)}{h}=\lim_{h\to 0}\frac{6h+h^2}{h}=6}$$

(Remarks: Many students have made this mistake in the past MATH202)

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Problem 4

Let $a_1, a_2, a_3, \ldots\ldots$ be positive real numbers. Must it be true that if $\lim_{n\to\infty} a_n=1$, then $\lim_{n\to\infty} a_n^n=1$?

Indeed, it is not true in general, to disprove the wrong statement, all we need to do is to provide the counter-example.

☺Right Solution:

$$\boldsymbol{Let\ a_n=1+\frac{1}{n}}$$

Then $\lim_{n\to\infty} a_n^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 1$

Problem 5

We know that $\lim_{x\to\infty} \sin x$ doesn't exist. If $a_1, a_2, a_{3,\ldots}$ are positive real numbers with $\lim_{n\to\infty} a_n = \infty$, then must it be true that $\lim_{n\to\infty} \sin a_n$ doesn't exist?

Once again, it is not true, so all we need to do is to find a sequence which makes the above limit exists.

☺Right Solution:

Pick $a_n = n\pi$***, then*** $\sin a_n = \sin n\pi = 0$ ***for*** $n = 1,2,3,\ldots.$

Therefore $\lim_{n\to\infty} \sin a_n = \lim_{n\to\infty} 0 = 0$

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Problem 6

Show $\lim_{n\to\infty} \sin n \neq 0$

By exercise 5, we see that it is not because the $\lim_{x\to\infty} \sin x$ does not exist, since we can still find a sequence which the limit exists. The actual reason is the following:

☺Right Solution:

We proof by contradiction, suppose $\lim_{n\to\infty} \sin n = 0$***, then*** $\lim_{n\to\infty} \sin(n+1) = 0$***, by compound angle formula*** $\sin(A+B) = \sin A\cos B + \cos A\sin B$***, we have***

$$\lim_{n\to\infty} |\sin(n+1)| = \lim_{n\to\infty} |\sin n\cos 1 + \sin 1\cos n| = |\sin 1| \neq 0$$

(Note $\lim_{n\to\infty} |\cos n| = \lim_{n\to\infty} |\sqrt{1-\sin^2 n}| = 1$ ***and*** $\lim_{n\to\infty} \sin n = 0$***)***

Which leads to contradiction, hence $\lim_{n\to\infty} \sin n \neq 0$

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Problem 7

$$\text{Let } g(x) = \begin{cases} 1 \text{ if } x \text{ is rational number in } [0,1] \\ 0 \text{ if } x \text{ is irrational number in } [0,1] \end{cases}$$

For every positive integer n, divide [0,1] into interval

$$\left[\frac{0}{n},\frac{1}{n}\right],\left[\frac{1}{n},\frac{2}{n}\right],\left[\frac{2}{n},\frac{3}{n}\right],\ldots\left[\frac{n-1}{n},1\right]$$

On the j-th interval $[\frac{j-1}{n},\frac{j}{n}]$, pick the mid-point and let it to be x_j. Since x_j is rational,we have $g(x_j) = 1$. Now

$$\lim_{n\to\infty} g(x_1)\left(\frac{1}{n}-0\right) + g(x_2)\left(\frac{2}{n}-\frac{1}{n}\right) + \cdots\ldots + g(x_n)\left(1-\frac{n-1}{n}\right) = 1$$

Therefore $\int_0^1 g(x)\,dx = 1$, is it correct?

Of course, it is not true, if we just pick another point $x_j = \frac{j-1}{n} + \frac{1}{\sqrt{2}}\left(\frac{1}{n}\right)$ which is irrational, then $g(x_j) = 0$, then the

$$\lim_{n\to\infty} g(x_1)\left(\frac{1}{n} - 0\right) + g(x_2)\left(\frac{2}{n} - \frac{1}{n}\right) + \cdots \ldots + g(x_n)\left(1 - \frac{n-1}{n}\right) = 0$$

Which means $\int_0^1 g(x)\,dx = 0$,

In fact, to conclude the integral equal to 1, we need to make sure that whenever which point we take for the function value. The limit above should be the same.

Actually, to find out the actual value of the integral, one can use Lebesgue Integration, the answer turns out to be 0. This subject will be discussed in MATH301.

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Problem 8

Let $h(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & if\ x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, Find $h'(x)$. What is $h''(0)$?

To compute the derivative for case-by-case function, for the "safer" part, we can apply direct differentiation to get the derivative, for the bottleneck part, we need to use the definition of derivative (or first principle)

☺Right Solution:

$$For\ x \neq 0, h'(x) = \frac{d}{dx} x^2 \sin\left(\frac{1}{x}\right) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

$$For\ x = 0, h'(0) = \lim_{c\to 0} \frac{h(c) - h(0)}{c} = \lim_{c\to 0}\left(\frac{c^2 \sin\left(\frac{1}{c}\right)}{c}\right) = \lim_{c\to 0} c\sin\left(\frac{1}{c}\right) = 0$$

$$h''(0) = \lim_{c\to 0} \frac{h'(c) - h'(0)}{c} = \lim_{c\to 0} \frac{2c\sin\left(\frac{1}{c}\right) - \cos\left(\frac{1}{c}\right)}{c}$$

$= \lim_{c\to 0} 2\sin\left(\frac{1}{c}\right) - \frac{\cos\left(\frac{1}{c}\right)}{c}$ ***which means the limit does not exist***

Remark for Q8: (Comment from Prof. Kin Li)

It is ok to do differentiation of formula on open intervals (say $x>0$ and $x<0$), but at endpoint 0, we have to see the left and right sides of the derivative at the point agree, that is why the point at 0 is more special. This is also the case of the point $x=3$ in problem 3.