Additional Practice Exercises

A1. Find the sum of $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(k-1)k(k+1)}$ by taking the limit of the partial sums.

A2. Let $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. If $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, then prove that $\displaystyle\lim_{n\to\infty} na_n = 0$. (*Hint*: Show ka_{2k} and ka_{2k+1} have limit 0.)

A3. If $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges, then prove that $\displaystyle\sum_{k=1}^{\infty} ka_k$ diverges. Note each a_k may be positive or negative so that comparison tests cannot be used! Letting $c_k = ka_k$, the contrapositive statement asserts that if $\displaystyle\sum_{k=1}^{\infty} c_k$ converges, then $\displaystyle\sum_{k=1}^{\infty} \frac{c_k}{k}$ converges. (*Hint*: Prove the contrapositive using summation by part.)

A4. Prove that $\displaystyle\sum_{k=1}^{\infty} \left|\frac{\sin k}{k}\right|$ diverges. (*Hint*: Show that there is a constant $c > 0$ such that for every real number x, $|\sin x| + |\sin(x+1)| \geq c$. Group the series two terms at a time.)

A5. (a) Let $-B = \{-x : x \in B\}$. If B is nonempty and bounded below, then prove that $-B$ is bounded above and $\sup(-B) = -\inf B$.

(b) For a nonempty set B that is bounded above and $c \geq 0$, let $cB = \{cx : x \in B\}$. Prove that $\sup cB = c \sup B$.

(c) If $\emptyset \neq A \subseteq B$ and B is bounded below, then prove that A is bounded below and $\inf B \leq \inf A$.

A6. Since none of the axioms of $\mathbb{R}$ asserts the existence for square roots, in this exercise we will introduce $\sqrt{2}$ by using supremum concept. We begin with the fact that for $a, b > 0$, if $a \geq b$, then $a^2 \geq b^2$ (by the order axiom). Taking contrapositive, if $a^2 < b^2$, then $a < b$. Next, let

$$S = \{x : x \in \mathbb{R},\ x > 0,\ x^2 < 2\}.$$

Note $1 \in S$ so that $S \neq \emptyset$. For all $x \in S$, $x^2 < 2 < 2^2$ implies $x < 2$. So S is bounded above by 2. By the completeness axiom, $j = \sup S$ exists.

(a) Prove that $j^2 = 2$ by showing $j^2 < 2$ and $j^2 > 2$ are false. (Hint: Consider $j' = (2j+2)/(j+2)$.) (Remark: In particular, $j^2 = 2$ implies $j \notin S$. Also, $1^2 < j^2 = 2 < 2^2$ implies $1 < j < 2$.) So we may denote $\sqrt{2}$ for j.

(b) Prove that $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ by assuming $\sqrt{2} = m/n$ with $m \in \mathbb{Z}, n \in \mathbb{N}$ and considering

$$T = \{k : k \in \mathbb{N},\ k\sqrt{2} \in \mathbb{Z}\}$$

to get a contradiction from the well-ordering axiom. (Remarks: Since we have not proved any fact about factorization of integers, do not use any such fact like $m^2 = 2n^2$ implies m is even. We are avoiding the usual proof that $j = \sqrt{2}$ is not rational!)

(c) Let $S' = \{x : x \in \mathbb{Q},\ x > 0,\ x^2 < 2\}$. Note S' is nonempty ($1 \in S'$) and bounded above (by 2). Prove that if $x \in S'$, then $x' = (2x+2)/(x+2) \in S'$ and $x < x'$. Prove that if $M \in \mathbb{Q}$ is an upper bound for S', then $M' = (2M+2)/(M+2) \in \mathbb{Q}$ is also upper bound for S' with $M' < M$. (Remark: So S' has no least upper bound in $\mathbb{Q}$. Hence, there is no the completeness axiom for $\mathbb{Q}$.)

A7. Use the summation by parts formula to prove the so-called *Dirichlet's Test* : if there exists a number $M > 0$ such that for every $n \in \mathbb{N}$, $\left|\sum_{k=1}^{n} a_k\right| \le M$ and $b_k \searrow 0$ as $k \to \infty$, then $\sum_{k=1}^{\infty} a_k b_k$ converges.

Remarks. (1) The case $a_k = (-1)^{k+1}$ and $M = 1$ is the alternating series test.

(2) Similarly, there is *Abel's Test* : If $\sum_{k=1}^{\infty} a_k$ converges and $d_k \searrow d$, then $\sum_{k=1}^{\infty} a_k d_k$ converges. (Since $\sum_{k=1}^{\infty} a_k$ converges implies its partial sum sequence $S_n = \sum_{k=1}^{n} a_k$ has a limit and there exists a number $M > 0$ such that for every $n \in \mathbb{N}$, $|S_n| = \left|\sum_{k=1}^{n} a_k\right| \le M$, so by taking $b_k = d_k - d$, Dirichlet's test implies Abel's test. The case $d_k = 1/k$ is exercise A3 above.)

A8. Show $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} a_n$ diverges, where $a_n = \begin{cases} 1/n & \text{if } n \text{ is not a multiple of } 3 \\ -1/n & \text{if } n \text{ is a multiple of } 3 \end{cases}$.

A9. (a) Let S be the set of all numbers in $[0, 1]$ having decimal representations of the form $0.a_1a_2a_3\ldots$, where $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$. Show that S is countable.

(b) Let T be the set of all numbers in $[0, 1]$ having decimal representations of the form $0.a_1a_2a_3\ldots$, where $|a_n - a_{n+1}| \le 1$ for all $n \in \mathbb{N}$. Determine (with proof) if T is countable or not.

(A1) $\frac{1}{(n-1)n(n+1)} = \frac{1}{2n}\left(\frac{1}{n-1}-\frac{1}{n+1}\right) = \frac12\left(\frac{1}{(n-1)n}-\frac{1}{n(n+1)}\right) = \frac12\left(\left(\frac{1}{n-1}-\frac1n\right)-\left(\frac1n-\frac{1}{n+1}\right)\right)$

$= \frac12\left(\frac{1}{n-1}-\frac2n+\frac{1}{n+1}\right)$.

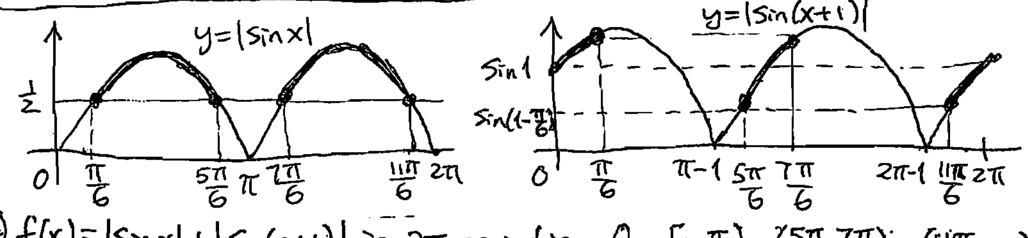
$$S_n = \begin{cases} \frac12\left(\frac11-\frac22+\cancel{\frac13}\right) \\ +\frac12\left(\frac12-\cancel{\frac23}+\cancel{\frac14}\right) \\ +\frac12\left(\cancel{\frac13}-\cancel{\frac24}+\cancel{\frac15}\right) \\ \vdots \\ +\frac12\left(\cancel{\frac{1}{n-2}}-\cancel{\frac{2}{n-1}}+\frac1n\right) \\ +\frac12\left(\cancel{\frac{1}{n-1}}-\frac2n+\frac{1}{n+1}\right) \end{cases} = \frac12\left(\frac11-\frac22+\frac12+\frac1n-\frac2n+\frac{1}{n+1}\right) = \frac12\left(\frac12-\frac1n+\frac{1}{n+1}\right)$$

$$\sum_{n=2}^{\infty}\frac{1}{(n-1)n(n+1)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac12\left(\frac12-\frac1n+\frac{1}{n+1}\right) = \frac14.$$

(A2) Let $S_n = a_1+\dots+a_n$. Since $\sum_{k=1}^{\infty}a_k$ converges to some number a, $\lim_{n\to\infty}S_n = a$.
If n is even, then $n=2k$ and $0 \le na_n = 2ka_{2k} \le 2(a_{k+1}+\dots+a_{2k}) = 2(S_{2k}-S_k)$
$= 2(S_n - S_{[n/2]})$. If n is odd, then $n=2k+1$ and
$0 \le na_n \le (2k+2)a_{2k+1} = 2(k+1)a_{2k+1} \le 2(a_{k+1}+\dots+a_{2k+1}) = 2(S_{2k+1}-S_k) = 2(S_n - S_{[\frac n2]})$.
So $0 \le na_n \le 2(S_n - S_{[n/2]}) \to 2(a-a) = 0$, By Sandwich theorem, $\lim_{n\to\infty} na_n = 0$.

(A3) Let $c_n = na_n$. Assume $\sum_{n=1}^{\infty}c_n = \sum_{n=1}^{\infty}na_n$ converges to some number c.
Then $S_n = c_1+\dots+c_n \to c$. Applying Summation by Parts, we have
$$\sum_{n=1}^{\infty}a_n = \lim_{N\to\infty}\sum_{n=1}^{N}c_n\frac1n = \lim_{N\to\infty}\left(S_N\frac1N - \sum_{n=1}^{N-1}S_n\left(\frac{1}{n+1}-\frac1n\right)\right) = \sum_{n=1}^{\infty}S_n\left(\frac1n-\frac{1}{n+1}\right).$$
Now $\left|S_n\left(\frac1n-\frac{1}{n+1}\right)\right| \le (|S_n|+1)\left(\frac1n-\frac{1}{n+1}\right)$ and $\lim_{n\to\infty}\frac{(|c|+1)\left(\frac1n-\frac{1}{n+1}\right)}{(|S_n|+1)\left(\frac1n-\frac{1}{n+1}\right)} = 1$. So
$$\sum_{n=1}^{\infty}(|c|+1)\left(\frac1n-\frac{1}{n+1}\right) = (|c|+1)\underbrace{\sum_{n=1}^{\infty}\left(\frac1n-\frac{1}{n+1}\right)}_{=1 \text{ by telescoping series test}} = |c|+1 < \infty \Rightarrow \sum_{n=1}^{\infty}(|S_n|+1)\left(\frac1n-\frac{1}{n+1}\right) \text{ converges by limit comparison test}$$
$\underset{\text{by Comparison test}}{\Rightarrow} \sum_{n=1}^{\infty}\left|S_n\left(\frac1n-\frac{1}{n+1}\right)\right|$ converges $\underset{\text{by abs. conv. test}}{\Rightarrow} \sum_{n=1}^{\infty}a_n = \sum_{n=1}^{\infty}S_n\left(\frac1n-\frac{1}{n+1}\right)$ converges, a contradiction.
Therefore, $\sum_{n=1}^{\infty}na_n$ diverges.

<u>Remarks</u> Using the boundedness theorem, we can say since S_n has limit, $\{S_1, S_2, S_3, \dots\}$ is bounded. So $\exists M$ such that $\forall n$, $|S_n| \le M$. Then
$\left|S_n\left(\frac1n-\frac{1}{n+1}\right)\right| \le M\left(\frac1n-\frac{1}{n+1}\right)$ and $\sum_{n=1}^{\infty}M\left(\frac1n-\frac{1}{n+1}\right) = M\sum_{n=1}^{\infty}\left(\frac1n-\frac{1}{n+1}\right) = M$.
So $\sum_{n=1}^{\infty}S_n\left(\frac1n-\frac{1}{n+1}\right)$ converges by comparison test and absolute convergence test.



(A4) $f(x) = |\sin x| + |\sin(x+1)|$ is 2π-periodic. On $[0,\frac{\pi}{6}) \cup (\frac{5\pi}{6}, \frac{7\pi}{6}) \cup (\frac{11\pi}{6}, 2\pi)$,
$|\sin(x+1)| \ge \sin(1-\frac{\pi}{6}) > 0$ and on $[\frac{\pi}{6}, \frac{5\pi}{6}] \cup [\frac{7\pi}{6}, \frac{11\pi}{6}]$, $|\sin x| \ge \sin\frac{\pi}{6} > \sin(1-\frac{\pi}{6})$.
So $\forall x\in\mathbb{R}$, $f(x) \ge c = \sin(1-\frac{\pi}{6}) > 0$. Assume $\sum_{n=1}^{\infty}\left|\frac{\sin n}{n}\right|$ converges.
then $$\sum_{n=1}^{\infty}\left|\frac{\sin n}{n}\right| = \sum_{k=1}^{\infty}\left(\frac{|\sin(2k-1)|}{2k-1}+\frac{|\sin(2k)|}{2k}\right) \ge \sum_{k=1}^{\infty}\frac{|\sin(2k-1)|+|\sin(2k)|}{2k} \ge \frac{c}{2}\sum_{k=1}^{\infty}\frac1k.$$
Since $\sum_{k=1}^{\infty}\frac1k$ diverges by p-test, $\sum_{n=1}^{\infty}\left|\frac{\sin n}{n}\right|$ diverges, a contradiction.

(A5) (a) Since B is nonempty and bounded below, by the completeness axiom for infimum $\inf B$ exists. Then $\forall x \in B$, $\inf B \le x$, so $-x \le -\inf B$. Then $-\inf B$ is an upper bound of $-B$.

Assume $\exists$ an upper bound t of $-B$ with $t < -\inf B$. Then $-t > \inf B$ and $\forall -x \in B$, $\overset{\in -B}{-x} \le t$, so $-t \le x$. Then $-t$ is a lower bound of B and is greater than $\inf B$ = greatest lower bound of B, contradiction.

$\therefore -\inf B$ = least upper bound of $-B = \sup(-B)$.

(b) Since B is nonempty and bounded above, by the completeness axiom $\sup B$ exists. Then $\forall x \in B$, $x \le \sup B$, so $cx \le c\sup B$ because $c \ge 0$. Then $c\sup B$ is an upper bound of cB.

If $c = 0$, then $cB = \{0\}$ and so $\sup cB = 0 = c\sup B$.

If $c > 0$, then assume $\exists$ an upper bound t of cB with $t < c\sup B$. Then $\frac{t}{c} < \sup B$ and $\forall x \in B$, $\overset{\in cB}{cx} \le t$, so $x \le \frac{t}{c}$. Then $\frac{t}{c}$ is an upper bound of B and $\frac{t}{c} < \sup B$ = least upper bound of B, contradiction. $\therefore c\sup B$ = least upper bound of $cB = \sup(cB)$.

(c) Since $B \ne \emptyset$ and B is bounded below, $\inf B$ exists. Then $\forall x \in A$, $x \in B$ and so $\inf B \le x$. So $\inf B$ is a lower bound for A. Then A is bounded below and $\inf A$ exists. Also,

$\inf B \le$ greatest lower bound of $A = \inf A$.

(A8) Show $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ diverges. Here the series always has two positive terms followed by one negative term.

Solution Note $S_3 = 1 + \left(\frac{1}{2} - \frac{1}{3}\right) > 1$, $S_6 = 1 + \left(\frac{1}{2} - \frac{1}{3}\right) + \frac{1}{4} + \left(\frac{1}{5} - \frac{1}{6}\right) > 1 + \frac{1}{4}$, $\ldots$, $S_{3n} = 1 + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \frac{1}{3n-2} + \left(\frac{1}{3n-1} - \frac{1}{3n}\right) > 1 + \frac{1}{4} + \cdots + \frac{1}{3n-2}$.

Now $\lim_{n\to\infty}\left(1 + \frac{1}{4} + \cdots + \frac{1}{3n-2}\right) = \sum_{k=1}^{\infty} \frac{1}{3k-2} = \infty$ (because $\lim_{k\to\infty} \frac{\frac{1}{3k-2}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{3k-2} = \frac{1}{3}$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to ∞ by p-test). Therefore, $\lim_{n\to\infty} S_{3n} = +\infty$. By definition of convergence and divergence, $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ diverges.

(A6) Let $x>0$ and $y=\frac{2x+2}{x+2}$. Then $x-y=\frac{x^2-2}{x+2}$ and $y^2-2=\frac{2(x^2-2)}{(x+2)^2}$.

① If $x^2<2$, then $x<y$ and $y^2<2$.

② If $x^2>2$, then $x>y$ and $y^2>2$.

$j'^2<2$

(a) We have $j>1>0$. Assume $j^2<2$. By ①, $j<j'=\frac{2j+2}{j+2}\in S$, contradict j is an upper bound of S

Assme $j^2>2$. By ②, $j>j'$ and $j'^2>2>x^2$ for all $x\in S$
$\Rightarrow \forall x\in S$, $j'>x \Rightarrow j'$ is an upper bound of S, contradicting $j=\sup S$.

See Comment on next page

(b) Assume $j=\sqrt{2}=\frac{m}{n}$ for some $m\in\mathbb{Z}$, $n\in\mathbb{N}$. Then $n\sqrt{2}=m\in\mathbb{Z}$.
So $n\in T=\{k: k\in\mathbb{N},\ k\sqrt{2}\in\mathbb{Z}\}$. By the well-ordering axiom, T has a least element K. Then $K\sqrt{2}\in\mathbb{Z}$. Let $K_0=\underbrace{K\sqrt{2}}_{\in\mathbb{Z}}-K$.
Then $K_0\in\mathbb{Z}$ and $1<j=\sqrt{2}<2 \Rightarrow 0<\sqrt{2}-1<1 \Rightarrow 0<K_0<K$
$\Rightarrow K_0\in\mathbb{N}$. Also $K_0\sqrt{2}=2K-K\sqrt{2}\in\mathbb{Z}$. So $K_0\in T$.
Since $K_0<K$, this contradicts K is least in T.

(c) $x\in S' \overset{\text{by ①}}{\Longrightarrow} x<x'=\frac{2x+2}{x+2}\in\mathbb{Q}$ and $x'\in S'$. Now $1\in S'$.
Let $M\in\mathbb{Q}$ be an upper bound of S'. Then $M>1>0$. If $M^2<2$, then $M\in S'$
$\Rightarrow M<M'\in S'$, contradicting M is an upper bound of S'.
By (b), $M\in\mathbb{Q}$ and $M^2=2$ is not possible.
So $M^2>2$. By ②, $M>M'=\frac{2M+2}{M+2}\in\mathbb{Q}$ and $M'^2>2>x^2\ \forall x\in S'$
$\Rightarrow \forall x\in S'$, $M'>x \Rightarrow M'$ is an upper bound of S'.

(A7) $\sum_{k=1}^{\infty}a_kb_k=\lim_{n\to\infty}\sum_{k=1}^{n}a_kb_k=\lim_{n\to\infty}\left(S_nb_n-\sum_{k=1}^{n-1}S_k\Delta b_k\right)$. Since $|S_nb_n|=\left|\sum_{k=1}^{n}a_kb_n\right|\le M|b_n|$ and $\lim_{k\to\infty}b_k=0$, by Sandwich theorem, $\lim_{n\to\infty}S_nb_n=0$. Next $|S_k\Delta b_k|\le M(b_k-b_{k+1})$.
By the telescoping test, $\sum_{k=1}^{\infty}M(b_k-b_{k+1})=M(b_1-\lim_{n\to\infty}b_{n+1})=Mb_1$. By comparison test, $\sum_{k=1}^{\infty}|S_k\Delta b_k|$ converges. By the absolute convergence test, $\sum_{k=1}^{\infty}S_k\Delta b_k=\lim_{n\to\infty}\sum_{k=1}^{n-1}S_k\Delta b_k$ converges. Therefore, $\sum_{k=1}^{\infty}a_kb_k=\lim_{n\to\infty}S_nb_n-\lim_{n\to\infty}\sum_{k=1}^{n-1}S_k\Delta b_k=-\sum_{k=1}^{\infty}S_k\Delta b_k$ converges.

(A9) (a) Solution 1 Since $0 \le a_i \le 9$ and a_i's are increasing, so there is a_N such that $a_N = a_{N+1} = a_{N+2} = \cdots$
Then $0.a_1a_2a_3\cdots \in S \Rightarrow 0.a_1a_2a_3\cdots = \frac{a_1\cdots a_{N-1}}{10^{N-1}} + \frac{a_N}{9\times 10^{N-1}} \in \mathbb{Q}$. So $S \subseteq \mathbb{Q}$.
Since $\mathbb{Q}$ is countable, by the countable subset theorem, S is countable.

Solution 2 Let $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Define $f: S \to \underbrace{\mathbb{N}_0 \times \cdots \times \mathbb{N}_0}_{10 \text{ factors}}$ by $f(0.a_1a_2a_3\cdots) = (n_0, n_1, \ldots, n_9)$ where for $j = 0, 1, \ldots, 9$, we assign $n_j = 0$ if all $a_i \ne j$, otherwise $n_j = i$, where i is the least index such that $a_i = j$. (For example, $f(0.1115599\cdots) = (0,1,0,0,0,4,0,0,0,7)$ with $\overset{\shortparallel}{a_1}$, $\overset{\shortparallel}{a_4}$, $\overset{\shortparallel}{a_7}$. Given $(0,1,0,0,0,4,0,0,0,7)$, we can get back $0.1115599\ldots$)
Then f is injective. Since $\mathbb{N}_0 \times \cdots \times \mathbb{N}_0$ is countable, S is countable.

(b) Consider the subset $T_0 = \{0.a_1a_2a_3\cdots \mid \text{every } a_i = 0 \text{ or } 1\}$ of T. The function $g: \{0,1\} \times \{0,1\} \times \cdots \to T_0$ defined by $g(a_1, a_2, \cdots) = 0.a_1a_2a_3\cdots$ is a bijection. Since $\{0,1\} \times \{0,1\} \times \cdots$ is uncountable, so T_0 is uncountable. Then T is uncountable by the countable subset theorem.

Comment on A6 part (b): In place of $\sqrt{2}$, you can modify the proof of A6 part (b) to show "$\sqrt{m}$ is irrational" for every positive integer m that is not the square of any integer. For example, "$\sqrt{61}$ is irrational" can be shown by considering

$$T = \{k : k \in \mathbb{N} \text{ and } k\sqrt{61} \in \mathbb{Z}\}$$

and since $7 < \sqrt{61} < 8$, we have $0 < \sqrt{61} - 7 < 1$ so for the "least k" in T, we can use $k_0 = k\sqrt{61} - 7k < k$ and $k_0\sqrt{61} = 61k - 7(k\sqrt{61}) \in \mathbb{Z}$ to see $k_0 \in T$, contradicting k least in T.

Extra Exercises on Cauchy Sequences and Limit of Functions

Notation: In the following exercises, a sequence $x_1, x_2, x_3, \dots$ may also be briefly denoted by $\{x_n\}$. This is a common notation in many math books or courses (including Math 202). It should not be confused with a set of one element, since we rarely discuss any set with one element (specifically of the form x_n).

1. Let $0 < a_n < 1$ for all $n = 1, 2, 3, \dots$. If sequence $b_n = \dfrac{1}{a_n}$ is a Cauchy sequence, then prove that sequence $c_n = b_n + \dfrac{1}{b_n}$ is also a Cauchy sequence by checking the definition of Cauchy sequence. (You will get 0 mark if you use Cauchy's theorem!)

2. If sequence $\{a_n\}$ is a Cauchy sequence, then prove that sequence $\{\sin(a_n^3)\}$ is also a Cauchy sequence by checking the definition of Cauchy sequence. (You will get 0 mark if you use Cauchy's theorem!) *Hint*: Recall Cauchy sequences are bounded!

3. If sequence $\{a_n\}$ is consisted of distinct real numbers (i.e. $i \neq j$ implies $a_i \neq a_j$) and $\left|\dfrac{a_{n+1} - a_n}{a_{n+2} - a_{n+1}}\right| > 2$ for all $n = 1, 2, 3, \dots$, then prove that sequence $\{a_n\}$ is a Cauchy sequence by checking the definition of Cauchy sequence.

4. If sequence $\{a_n\}$ is a Cauchy sequence, then prove that sequence $\left\{a_n^2 + \dfrac{1}{n^2}\right\}$ is also a Cauchy sequence by checking the definition of Cauchy sequence. (You will get 0 mark if you use Cauchy's theorem!)

5. Prove that $\displaystyle\lim_{x \to 1}\left(x + \frac{1}{x^3 + 1}\right) = 1 + \frac{1}{2}$ by checking the definition of limit. (You will get 0 mark if you use computation formulas, sandwich theorem or l'Hopital's rule!)

6. Prove that $\displaystyle\lim_{x \to 1} \sin\left(\frac{x}{x^2 + 1}\right) = \sin\frac{1}{2}$ by checking the definition of limit. (You will get 0 mark if you use computation formulas, sandwich theorem or l'Hopital's rule!)

7. For every nonzero $a \in \mathbb{R}$, prove that $\displaystyle\lim_{x \to a} \arctan\left(\frac{1}{x}\right) = \arctan\left(\frac{1}{a}\right)$ by checking the definition of limit. (You will get 0 mark if you use computation formulas, sandwich theorem or l'Hopital's rule!)

8. (a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x) \neq 1$ for all $x \in \mathbb{R}$ and $\displaystyle\lim_{x \to 1} f(x) = 1$, then $\displaystyle\lim_{x \to 1} f\big(f(x)\big) = 1$ by checking the definition of limit.

   (b) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $\displaystyle\lim_{x \to 1} f(x) = 1$ and $\displaystyle\lim_{x \to 1} f\big(f(x)\big) \neq 1$.

9. Prove that $\lim_{x \to 2} \sqrt[3]{x} = \sqrt[3]{2}$ by checking the definition of limit.

10. Prove that $\lim_{x \to 1} \dfrac{x^3+1}{x+2} = \dfrac{2}{3}$ by checking the definition of limit.

11. If $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = 0$, then prove that $f(x)^3$ and $\cos f(x)$ are continuous at $x = 0$ by checking the definition of continuity at a point.

12. Let $a_1 = 1$ and $a_n = a_{n-1} + \dfrac{\cos n}{(1+a_n^2)n^{50}}$ for $n = 2, 3, 4, \ldots$. Prove that the sequence $\{a_n\}$ converges by checking the definition that it is a Cauchy sequence .

13. Let $\{a_n\}$ be a sequence such that $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are Cauchy sequences. If also $\lim_{n \to \infty} (a_{2n+1} - a_{2n}) = 0$, then prove that $\{a_n\}$ is a Cauchy sequence by checking the definition of Cauchy sequence.

Solutions to Extra Exercises on Cauchy Sequences and Limit of Functions

① Note $0<a_n<1 \Rightarrow b_n=\frac{1}{a_n}>1$. For every $\varepsilon>0$, since b_n is Cauchy, $\exists K\in\mathbb{N}$ such that $m,n\geq K \Rightarrow |b_m-b_n|<\frac{\varepsilon}{2}$.

Then $m,n\geq K$ implies

$$|c_m-c_n|=\left|b_m+\frac{1}{b_m}-b_n-\frac{1}{b_n}\right|\leq |b_m-b_n|+\left|\frac{1}{b_m}-\frac{1}{b_n}\right|\leq |b_m-b_n|+\frac{|b_m-b_n|}{|b_mb_n|}$$

$$\underset{|b_m|,|b_n|>1}{\leq} |b_m-b_n|+|b_m-b_n|=2|b_m-b_n|<\varepsilon.$$

Therefore, c_n is a Cauchy sequence.

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② Since $\{a_n\}$ Cauchy $\Rightarrow$ $\{a_n\}$ bounded, so there is a constant C such that $|a_n|<C$ for all n.

$|\sin a-\sin b|\leq|a-b|$

Note
$$|\sin(a_m^3)-\sin(a_n^3)|\leq|a_m^3-a_n^3|=|a_m-a_n|\,|a_m^2+a_ma_n+a_n^2|$$
$$\leq|a_m-a_n|(|a_m|^2+|a_m||a_n|+|a_n|^2)$$
$$\leq 3C^2|a_m-a_n|.$$

Since $\{a_n\}$ is Cauchy, for every $\varepsilon>0$, there is $K\in\mathbb{N}$ such that $m,n\geq K\Rightarrow|a_m-a_n|<\frac{\varepsilon}{3C^2}$. Then $m,n\geq K\Rightarrow|\sin(a_m^3)-\sin(a_n^3)|<\varepsilon$.

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③ Note $\left|\frac{a_{n+1}-a_n}{a_{n+2}-a_{n+1}}\right|>2 \Leftrightarrow |a_{n+2}-a_{n+1}|<\frac{|a_{n+1}-a_n|}{2}$ for $n=1,2,3,\ldots$

$$\Rightarrow |a_n-a_{n-1}|<\frac{|a_{n-1}-a_{n-2}|}{2^1}<\frac{|a_{n-2}-a_{n-3}|}{2^2}<\cdots<\frac{|a_2-a_1|}{2^{n-2}}.$$

Now for every $\varepsilon>0$, let $\boxed{K>\log_2\frac{4|a_2-a_1|}{\varepsilon}}$

Then $m,n\geq K$ (say $m>n$) implies

$$|a_m-a_n|\leq|a_m-a_{m-1}|+|a_{m-1}-a_{m-2}|+\cdots+|a_{n+1}-a_n|$$
$$<\frac{|a_2-a_1|}{2^{m-2}}+\frac{|a_2-a_1|}{2^{m-3}}+\cdots+\frac{|a_2-a_1|}{2^{n-1}}$$
$$\leq|a_2-a_1|\left(\frac{1}{2^{n-1}}+\frac{1}{2^n}+\frac{1}{2^{n+1}}+\cdots\right)$$
$$=\frac{|a_2-a_1|}{2^{n-2}}\leq\frac{|a_2-a_1|}{2^{K-2}}<\varepsilon$$

solve for K.

Therefore, $\{a_n\}$ is a Cauchy sequence.

④ Since $\{a_n\}$ Cauchy $\Rightarrow$ $\{a_n\}$ bounded, so there is a constant C such that $|a_n| < C$ for all n.

Note
$$\begin{aligned}
\left|\left(a_m^2+\tfrac{1}{m^2}\right)-\left(a_n^2+\tfrac{1}{n^2}\right)\right| &= \left|\left(a_m^2-a_n^2\right)+\left(\tfrac{1}{m^2}-\tfrac{1}{n^2}\right)\right| \\
&\le |a_m^2-a_n^2| + \left|\tfrac{1}{m^2}-\tfrac{1}{n^2}\right| \\
&\le |a_m+a_n|\,|a_m-a_n| + \tfrac{1}{K^2} \\
&\le 2C\,|a_m-a_n| + \tfrac{1}{K^2}.
\end{aligned}$$

Since $\{a_n\}$ is Cauchy, for every $\varepsilon>0$, there is $K_1\in\mathbb{N}$ such that $m,n\ge K_1 \Rightarrow |a_m-a_n| < \frac{\varepsilon}{4C}$. Let $K > \max\left(K_1, \sqrt{\frac{2}{\varepsilon}}\right)$,

then
$$\begin{aligned}
m,n\ge K \Rightarrow \left|\left(a_m^2+\tfrac{1}{m^2}\right)-\left(a_n^2+\tfrac{1}{n^2}\right)\right| &\le 2C\,|a_m-a_n| + \tfrac{1}{K^2} \\
&\underset{K>K_1 \text{ and } K>\sqrt{\frac{2}{\varepsilon}}}{<} 2C\,\tfrac{\varepsilon}{4C} + \left(\sqrt{\tfrac{\varepsilon}{2}}\right)^2 = \varepsilon
\end{aligned}$$

---

⑤ $\forall \varepsilon>0$, set $\delta = \min\left(1, \frac{2}{9}\varepsilon\right)$. Then
$$\begin{aligned}
0<|x-1|<\delta \Rightarrow \left|\left(x+\tfrac{1}{x^3+1}\right)-\left(1+\tfrac{1}{2}\right)\right| &= \left|x-1+\left(\tfrac{1}{x^3+1}-\tfrac{1}{2}\right)\right| \\
&\le |x-1| + \left|\tfrac{1}{x^3+1}-\tfrac{1}{2}\right| \\
&= |x-1| + \left|\tfrac{1-x^3}{2(x^3+1)}\right| \\
&= |x-1|\left(1+\tfrac{|1+x+x^2|}{2\,|x^3+1|}\right) \\
&\le |x-1|\left(1+\tfrac{7}{2}\right) \\
&< \varepsilon
\end{aligned}$$

$|x-1|<1 \Rightarrow x\in(0,2) \rightarrow$

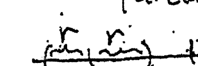
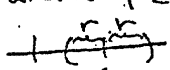
$|x-1| < \frac{2}{9}\varepsilon \rightarrow$

---

⑥ $\forall\varepsilon>0$, set $\delta=\sqrt{2\varepsilon}$. Then
$$\begin{aligned}
0<|x-1|<\delta \Rightarrow \left|\sin\left(\tfrac{x}{x^2+1}\right)-\sin\tfrac{1}{2}\right| &\le \left|\tfrac{x}{x^2+1}-\tfrac{1}{2}\right| = \left|\tfrac{-x^2+2x-1}{2(x^2+1)}\right| \\
&= \tfrac{|x-1|^2}{2(x^2+1)} \le \tfrac{|x-1|^2}{2} < \varepsilon
\end{aligned}$$

$|x-1|<\sqrt{2\varepsilon}$

⑦ Solution 1. Note $\left|\frac{d}{dx}\arctan x\right| = \left|\frac{1}{1+x^2}\right| \le 1 \Rightarrow |\arctan v - \arctan w| \overset{\text{by mean-value theorem}}{=} \frac{1}{1+c^2}|v-w| \le |v-w|$ for some c between v and w.

For every nonzero $a \in \mathbb{R}$,  or  let $0 < r < |a|$.

For every $\varepsilon > 0$, let $\delta = \min(r, |a|(|a|-r)\varepsilon)$.

Then $0 < |x-a| < \delta \Rightarrow \begin{cases} |x-a| < r \Rightarrow |a| - |x| < r \Rightarrow |a| - r < |x| \\ \text{and } \left|\arctan\left(\frac{1}{x}\right) - \arctan\left(\frac{1}{a}\right)\right| \\ \quad \le \left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|x-a|}{|a||x|} < \frac{|x-a|}{|a|(|a|-r)} < \varepsilon \end{cases}$

Solution 2. Note $\left|\frac{d}{dx}\arctan\left(\frac{1}{x}\right)\right| = \left|\frac{1}{1+\left(\frac{1}{x}\right)^2}\left(-\frac{1}{x^2}\right)\right| = \frac{1}{x^2+1} < 1$. If $0 \notin [v, w]$,

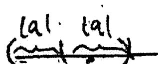
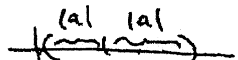
then $\arctan\frac{1}{x}$ is differentiable on $[v, w]$ and $\left|\arctan\frac{1}{v} - \arctan\frac{1}{w}\right| \le \frac{1}{c^2+1}|v-w| \le |v-w|$

for some $c \in (v, w)$

For every nonzero $a \in \mathbb{R}$, let $0 < r < |a|$. For every $\varepsilon > 0$, let $\delta = \min(r, \varepsilon)$

Then $0 < |x-a| < \delta \Rightarrow \begin{cases} |x-a| < r \Rightarrow 0 \notin [a,x], [x,a] \text{ (as in solution 1).} \\ \left|\arctan\left(\frac{1}{x}\right) - \arctan\left(\frac{1}{a}\right)\right| \le |x-a| < \varepsilon. \end{cases}$

Solution 3

For every $\varepsilon > 0$, let $\delta = \min(|a|, \tan\varepsilon)$.  or 

then $0 < |x-a| < \delta \Rightarrow \begin{cases} 0 < |x-a| < |a| \Rightarrow a, x \text{ both negative or both positive} \\ \left|\arctan\left(\frac{1}{x}\right) - \arctan\left(\frac{1}{a}\right)\right| = \left|\arctan\left(\frac{\frac{1}{x} - \frac{1}{a}}{1 + \frac{1}{x}\frac{1}{a}}\right)\right| \\ \quad = \left|\arctan\left(\frac{a-x}{ax+1}\right)\right| = \arctan\left|\frac{a-x}{ax+1}\right| \\ \quad < \arctan\left|\frac{\delta}{1}\right| = \arctan\delta \le \varepsilon \end{cases}$

arctan is strictly increasing

$|a-x| < \delta$

$\underset{>0}{ax+1} > 1$

$\left|\frac{a-x}{ax+1}\right| < |a-x| < \delta$

$\delta \le \tan\varepsilon$ and $\arctan(\tan\varepsilon) \le \varepsilon$

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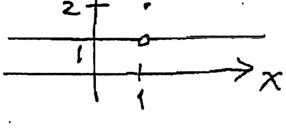
⑧

(a) For every $\varepsilon > 0$, since $\lim_{t \to 1} f(t) = 1$, there exists $\delta_0 > 0$ such that $0 < |t-1| < \delta_0$ implies $|f(t) - 1| < \varepsilon$. Since $\delta_0 > 0$ and $\lim_{x\to 1} f(x) = 1$, there exists $\delta > 0$ such that $0 < |x-1| < \delta$ implies $|f(x) - 1| < \delta_0$.

Now $\delta > 0$ satisfies $0 < |x-1| < \delta \Rightarrow \underset{0 <}{} |f(x) - 1| < \delta_0 \underset{t = f(x)}{\Longrightarrow} |f(f(x)) - 1| < \varepsilon$.

$f(x) \ne 1$ is given

Therefore, $\lim_{x \to 1} f(f(x)) = 1$.

(b) 

Define $f(x) = \begin{cases} 1 & \text{if } x \ne 1 \\ 2 & \text{if } x = 1 \end{cases}$. Then $\lim_{x\to 1} f(x) = 1$

and $\lim_{x \to 1} f(f(x)) = \lim_{x\to 1} f(1) = \lim_{x \to 1} 2 = 2 \ne 1$.

(9) $\forall \varepsilon > 0$, set $\delta = \min(1, 2^{2/3}\varepsilon)$. Then $|x-2|<1 \Rightarrow x \in (1,3)$

$0 < |x-2| < \delta \Rightarrow |\sqrt[3]{x} - \sqrt[3]{2}| = \dfrac{|x-2|}{|x^{2/3} + 2^{1/3}x^{1/3} + 2^{2/3}|} < \dfrac{|x-2|}{2^{2/3}} < \varepsilon$ $|x-1| < 2^{2/3}\varepsilon$

(10) $\forall \varepsilon > 0$, set $\delta = \min(1, \frac{6}{19}\varepsilon)$. Then $|x-1|<1 \Rightarrow x \in (0,2)$ $|x-1| < \frac{6}{19}\varepsilon$

$0 < |x-1| < \delta \Rightarrow \left|\dfrac{x^3+1}{x+2} - \dfrac{2}{3}\right| = \dfrac{|3x^3 - 2x - 1|}{3|x+2|} = \dfrac{|x-1||3x^2+3x+1|}{3|x+2|} \leq \dfrac{19}{6}|x-1| < \varepsilon$

(11) Since $f(x)$ is continuous at 0, $\exists \delta_1$ such that $0 < |x-0| < \delta_1 \Rightarrow |f(x) - f(0)| < 1$

Then $|f(x)| = |f(x) - f(0) + f(0)| \leq |f(x) - f(0)| + |f(0)| \leq 1 + |f(0)|$. Now

$|f(x)^3 - f(0)^3| = |f(x) - f(0)||f(x)^2 + f(x)f(0) + f(0)^2|$ and

$|f(x)^2 + f(x)f(0) + f(0)^2| \leq |f(x)|^2 + |f(x)||f(0)| + |f(0)|^2 \leq \overbrace{(1+|f(0)|)^2 + (1+|f(0)|)|f(0)| + |f(0)|^2}^{\text{Call this } L}$.

Again since f is continuous at 0, $\exists \delta_2$ such that $0 < |x-0| < \delta_2 \Rightarrow |f(x) - f(0)| < \frac{\varepsilon}{L}$.

Let $\delta = \min(\delta_1, \delta_2)$. Then $0 < |x-0| < \delta \Rightarrow |f(x)^3 - f(0)^3| \leq L \cdot |f(x) - f(0)| < \varepsilon$.

Next, since f is continuous at 0, $\forall \varepsilon > 0$, $\exists \delta_0 > 0$ such that $0 < |x-0| < \delta_0$

$\Rightarrow |f(x) - f(0)| < \varepsilon$. Then $|\cos f(x) - \cos f(0)| \leq |f(x) - f(0)| < \varepsilon$.

$|\cos a - \cos b| \leq |a-b|$.

(12) $\forall \varepsilon > 0$, choose $K > \frac{1}{\varepsilon}$. Then $m > n \geq K$ implies

$|a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| = \left|\dfrac{\cos m}{(1+a_m^2)m^{50}}\right| + \cdots + \left|\dfrac{\cos(n+1)}{(1+a_{n+1}^2)(n+1)^{50}}\right|$

$\leq \dfrac{1}{m^{50}} + \cdots + \dfrac{1}{(n+1)^{50}} \leq \dfrac{1}{m^2} + \cdots + \dfrac{1}{(n+1)^2} < \dfrac{1}{m(m-1)} + \cdots + \dfrac{1}{(n+1)n} = \left(\dfrac{1}{m-1} - \dfrac{1}{m}\right) + \cdots + \left(\dfrac{1}{n} - \dfrac{1}{n+1}\right)$

$= \dfrac{1}{n} - \dfrac{1}{m} < \dfrac{1}{n} \leq \dfrac{1}{K} < \varepsilon$.

(13) $\{a_{2n}\}$ Cauchy $\Rightarrow \forall \varepsilon > 0\ \exists K_1 \in \mathbb{N}$ such that $p, q \geq K_1 \Rightarrow |a_{2p} - a_{2q}| < \varepsilon/2$.

$\{a_{2n+1}\}$ Cauchy $\Rightarrow \forall \varepsilon > 0\ \exists K_2 \in \mathbb{N}$ such that $p, q \geq K_2 \Rightarrow |a_{2p+1} - a_{2q+1}| < \varepsilon/2$.

$\lim_{n\to\infty}(a_{2n+1} - a_{2n}) = 0 \Rightarrow \forall \varepsilon > 0\ \exists K_3 \in \mathbb{N}$ such that $r \geq K_3 \Rightarrow |a_{2r+1} - a_{2r}| < \varepsilon/2$

To show $\{a_n\}$ is Cauchy, $\forall \varepsilon > 0$, let $K = \max(2K_1, 2K_2+1, 2K_3)$. Then

for $m, n \geq K$, there are 4 cases:

(1) m even, n even (2) m even, n odd (3) m odd, n even (4) m odd, n odd.

For case (1), say $m = 2p$, $n = 2q$. Then $m, n \geq K \Rightarrow p, q \geq K_1 \Rightarrow |a_m - a_n| = |a_{2p} - a_{2q}| < \varepsilon/2 < \varepsilon$.

For case (2), say $m = 2p+1$, $n = 2q$. Then $m, n \geq K \Rightarrow p, q \geq K_1, K_3 \Rightarrow |a_m - a_n| = |a_{2p+1} - a_{2q}|$

$= |a_{2p+1} - a_{2p} + a_{2p} - a_{2q}| \leq |a_{2p+1} - a_{2p}| + |a_{2p} - a_{2q}| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Case (3) follows from case (2) as $|a_m - a_n| = |a_n - a_m|$.

For case (4), say $m = 2p+1$, $n = 2q+1$. Then $m, n \geq K \Rightarrow p, q \geq K_2 \Rightarrow |a_m - a_n| = |a_{2p+1} - a_{2q+1}| < \frac{\varepsilon}{2} < \varepsilon$.

Continuity and Differentiation Problems

Notation. In the following problems, $f^{(n)}$ will denote the composition $\underbrace{f \circ f \circ \cdots \circ f}_{n\ f's}$.

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous and $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$. Prove that if the equation $f^{(2)}(x) = g^{(2)}(x)$ has a solution, then $f(x) = g(x)$ also has a solution.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If there exist $a \in \mathbb{R}$ and $c > 0$ such that $|f^{(n)}(a)| < c$ for $n = 1, 2, 3, \ldots,$ then prove that f has a fixed point (i.e. there exists $w \in \mathbb{R}$ such that $f(w) = w$.)

3. A function $f : [0, 1] \to \mathbb{R}$ satisfies $f(0) < 0$ and $f(1) > 0$, and there exists a function g continuous on $[0, 1]$ and such that $f + g$ is decreasing. Prove that the equation $f(x) = 0$ has a solution in the open interval $(0, 1)$. (*Hint*: Consider $A = \{x : f(x) \geq 0\}$.)

4. Show that every bijection $f : \mathbb{R} \to [0, +\infty)$ has infinitely many points of discontinuity.

5. Let n be a positive integer and $a_0, a_1, \ldots, a_n \in \mathbb{R}$ be such that $a_0 + \dfrac{a_1}{2} + \dfrac{a_2}{3} + \cdots + \dfrac{a_n}{n+1} = 0$. Prove that the polynomial $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ has a root in the open interval $(0, 1)$.

6. Let $f : [1, +\infty)$ be continuous. Also, f is differentiable on $(1, +\infty)$. If $e^{-x} f'(x)$ is bounded on $(1, +\infty)$, then prove that $e^{-x} f(x)$ is also bounded on $(1, +\infty)$. (*Hint*: $(f(x) - f(1))/(e^x - e)$ is bounded on $(1, +\infty)$.)

7. Let $f : [a, b] \to \mathbb{R}$ be continuous such that both $f'(x)$ and $f''(x)$ are continuous on (a, b). If $f(a) = f(b) = 0$ and there exists a function $g : (a, b) \to \mathbb{R}$ such that $f''(x) + f'(x)g(x) - f(x) = 0$ for all $x \in (a, b)$, then prove that $f(x) = 0$ for all $x \in (a, b)$. (*Hint*: Prove that if the maximum of f is greater than 0, then it is a minimum to get a contradiction.)

8. Let $f : [a, b] \to \mathbb{R}$ be continuous. Also, f is differentiable on (a, b). If $\theta \in (a, b)$ and $f'(\theta)$ is not the supremum nor the infimum of $\{f'(x) : x \in (a, b)\}$, then prove that there exists distinct $c, d \in (a, b)$ such that $f'(\theta) = \dfrac{f(c) - f(d)}{c - d}$. Give an example to show that if $f'(\theta)$ is the supremum or infimum of $\{f'(x) : x \in (a, b)\}$, then there may not be any such $c, d \in (a, b)$. (*Hint*: Consider $g(x) = f(x) - f'(\theta)x$. First show g cannot be injective.)

9. Let $f(x)$ be a polynomial on $\mathbb{R}$ with real coefficients. If for every polynomial $g(x)$ on $\mathbb{R}$ with real coefficients, we have $f(g(x)) = g(f(x))$ for all $x \in \mathbb{R}$, then prove that $f(x) = x$ for all $x \in \mathbb{R}$.

10. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Prove that there exists $c \in [0, 1]$ such that $f(c) = \sqrt[3]{\displaystyle\int_0^1 f^3(t)\, dt}$.

11. Let $f : [0, 1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. For every positive integer n, prove that there exists $t \in \left[0, 1 - \dfrac{1}{n}\right]$ such that $f(t + \dfrac{1}{n}) = f(t)$.

12. For $f \in C^2(\mathbb{R})$ (i.e. f' and f'' exist and are continuous on $\mathbb{R}$), if f is bounded, then prove that there exists x_0 such that $f''(x_0) = 0$. (*Hint*: Assume $f''(x) > 0$ for all x, then do Taylor expansion of f at a center c such that $f'(c) \neq 0$ to get a contradiction.)

13. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. If $\displaystyle\lim_{x \to +\infty} f(x)$ and $\displaystyle\lim_{x \to +\infty} f'(x)$ exist, then prove that $\displaystyle\lim_{x \to +\infty} f'(x) = 0$.

14. Let $f : (a, b) \to [0, +\infty)$ be twice differentiable and $f''(x) \geq 0$ for all $x \in (a, b)$. For every nonempty open subinterval, $f(x)$ is not the zero function. Prove that $f(x)$ has at most one root on (a, b). (*Hint*: If f has 2 roots x_1 and x_2, then explain there is a critical point on (x_1, x_2) and use it as center for a Taylor expansion of f.)

15. Let $f, g : (a, b) \to \mathbb{R}$ be differentiable such that $f(x)g'(x) - f'(x)g(x) \neq 0$ for all $x \in (a, b)$. If there exist x_0, x_1 such that $a < x_0 < x_1 < b$ and $f(x_0) = f(x_1) = 0$, then prove that there exists $c \in (x_0, x_1)$ such that $g(c) = 0$. (*Hint*: Assume opposite of conclusion and study $h(x) = f(x)/g(x)$.)

16. If $|f''(x)| \leq |f'(x)| + |f(x)|$ for all $x \in (a, b)$ and there exists $c \in (a, b)$ such that $f(c) = f'(c) = 0$, then $f(x) = 0$ for all $x \in (a, b)$.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and bounded. If $|f(x) + f'(x)| \leq 1$ for all $x \in \mathbb{R}$, then prove that $|f(x)| \leq 1$ for all $x \in \mathbb{R}$. (*Hint*: Show $e^x(f(x) \pm 1)$ are monotone on $\mathbb{R}$ and consider $x \to -\infty$.)

18. Let $f : [0, 1] \to \mathbb{R}$ be continuous. If $f(0) = 0$ and f is differentiable on $(0, 1)$ with f' decreasing on $(0, 1)$, then prove that for $0 \leq a \leq b \leq a + b \leq 1$, we have $f(a + b) \leq f(a) + f(b)$. (*Hint*: Draw chords to graphs of $f(x)$ over $[0, a]$ and $[b, a + b]$.)

19. Let $f : [a, b] \to [a, b]$ be such that $|f(x) - f(y)| < |x - y|$ for all $a \leq x < y \leq b$, then there exists $c \in [a, b]$ such that $f(c) = c$. (*Hint*: Minimize $|f(x) - x|$.)

20. Let f be bounded on $[a, b]$, g be differentiable, $g(a) = 0$ and c is a nonzero constant such that $|g(x)f(x) + cg'(x)| \leq |g(x)|$ for all $x \in [a, b]$, then prove that $g(x) = 0$ for all $x \in [a, b]$.

21. Let $f : \mathbb{R} \to \mathbb{R}$ be three times differentiable. If f and f''' are bounded on $\mathbb{R}$, then prove that f' and f'' are also bounded on $\mathbb{R}$. (*Hint*: Use x as center for a Taylor expansion of f. Evaluate f at $x + 1$ on the left side.)

22. Let $f : [0, 1] \to \mathbb{R}$ be continuous and $f(0) = f(1)$. If f is twice differentiable on $(0, 1)$ and there is $M > 0$ such that $|f''(x)| \leq M$ for all $x \in (0, 1)$, then prove that $|f'(x)| \leq \frac{1}{2}M$ for all $x \in (0, 1)$.

23. Let $f : \mathbb{R} \to \mathbb{R}$ be three times differentiable and satisfy $\lim\limits_{x \to +\infty} f(x) = c \in \mathbb{R}$, $\lim\limits_{x \to +\infty} f'''(x) = 0$. Prove that $\lim\limits_{x \to +\infty} f'(x) = 0$ and $\lim\limits_{x \to +\infty} f''(x) = 0$. (*Hint*: Use x as center for a Taylor expansion of f. Evaluate f at $x + 1$ on the left side.)

24. Let $f : [-2, 2] \to \mathbb{R}$ be continuous. Also, let f be twice differentiable on $(-2, 2)$, $|f(x)| \leq 1$ for all $x \in [-2, 2]$ and $f^2(0) + f'^2(0) = 4$. Prove that there exists $\theta \in (-2, 2)$ such that $f(\theta) + f''(\theta) = 0$.

Solutions to Sample Problems for Extra Credit Open Book Quizzes

① Assume $f(x)=g(x)$ has no solution. Then $h(x)=f(x)-g(x)\neq 0$. Since $h(x)$ is continuous, either $\forall x$, $h(x)>0$ or $\forall x$, $h(x)<0$. Then $\forall x$,
$$0\neq h(f(x))+h(g(x))=f^{(2)}(x)-g(f(x))+f(g(x))-g^{(2)}(x)=f^{(2)}(x)-g^{(2)}(x),$$ contradiction.

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② Assume $\forall w\in\mathbb{R}$, $f(w)\neq w$. Since $f(w)-w$ is continuous and never 0, we may assume $f(w)-w>0$. Then $a<f(a)<f^{2}(a)<\cdots$ and $f^{n}(a)<c$.
So $\lim_{n\to\infty} f^{(n)}(a)=x$. By continuity of f, $f(x)=f(\lim_{n\to\infty} f^{(n)}(a))=\lim_{n\to\infty} f^{(n+1)}(a)=x$, a contradiction.

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③ Let $A=\{x\in[0,1]: f(x)\geq 0\}$, then $1\in A$ and A is bounded below by 0. So $s=\inf A$ exists. Now $h=f+g$ decreasing $\Rightarrow h(s)\geq h(x)\geq g(x)$ for all $x\in A$.
Since g is continuous and $s=\lim_{n\to\infty} x_n$ for some $x_n\in A$, so $h(s)\geq \lim_{n\to\infty} g(x_n)=g(s)$.
Hence $f(s)=h(s)-g(s)\geq 0$. Now $g(0)=h(0)-f(0)>h(0)\geq h(s)\geq g(s)$.
Since g is continuous, by the intermediate value theorem, $\exists t\in[0,s]$ such that $g(t)=h(s)$. Then $h(t)\geq h(s)=g(t)$, which implies $f(t)=h(t)-g(t)\geq 0$.
By definition of s, $t=s$. So $g(s)=h(s)$, i.e. $f(s)=0$.

---

④ Step 1 (f is not continuous). Assume f is continuous. As a bijection, f is injective. So f is strictly monotone, say strictly increasing. Now $f(x_0)=0$ for some x_0, but then $f(x_0-1)<0$, contradiction. (For strictly decreasing case, $f(x_0+1)<0$.)
Step 2 (f has infinitely many discontinuities). Assume f is discontinuous at $x_1<x_2<\cdots<x_n$ only. Then f is continuous and injective, hence strictly monotone, on each of the intervals $(-\infty,x_1),(x_1,x_2),\ldots,(x_n,\infty)$. By injectivity of f and the intermediate value theorem, their ranges $f(-\infty,x_1)$, $f(x_1,x_2),\ldots,f(x_n,\infty)$ are pairwise disjoint open intervals. Now
$\mathbb{R}\setminus((-\infty,x_1)\cup(x_1,x_2)\cup\cdots\cup(x_n,\infty))=\{x_1,x_2,\ldots,x_n\}$, but
$[0,\infty)\setminus(\underbrace{f(-\infty,x_1)\cup f(x_1,x_2)\cup\cdots\cup f(x_n,\infty)}_{n+1\text{ open interval}})$ contains 0 and n other numbers.
Then f cannot be bijective, contradiction.

---

⑤ Let $Q(x)=a_0x+\frac{a_1x^2}{2}+\frac{a_2x^3}{3}+\cdots+\frac{a_nx^{n+1}}{n+1}$. Since $Q(0)=0$ and $Q(1)=a_0+\frac{a_1}{2}+\cdots+\frac{a_n}{n+1}=0$.
By Rolle's theorem, $Q'(x)=P(x)=0$ for some $x\in(0,1)$.

(6) Suppose $|e^{-x}f'(x)| \le M$ for all $x \in (1,+\infty)$. Then for $x>1$,

$$|e^{-x}f(x)| = \frac{|f(x)-f(1)+f(1)|}{e^x} \le \frac{|f(x)-f(1)|}{e^x} + \frac{|f(1)|}{e^x} \le \frac{|f(x)-f(1)|}{e^x-e^1} + \frac{|f(1)|}{e}.$$

By mean-value theorem, $\left|\frac{f(x)-f(1)}{e^x-e^1}\right| = \left|\frac{f'(\theta)}{e^\theta}\right| = |e^{-\theta}f'(\theta)| \le M$ for some θ between x and 1. Therefore, $|e^{-x}f(x)| \le M + \frac{|f(1)|}{e}$.

---

(7) We have $m = \min_{x\in[a,b]} f(x) \le f(a) = 0 = f(b) \le \max_{x\in[a,b]} f(x) = M$. Assume $f(x) \not\equiv 0$, then $M>0$ or $m<0$. If $M>0$, then $\exists\, c \in (a,b)$ such that $f(c) = M > 0$ so that $f'(c)=0$. Then $f''(c) - 0\,g(c) - f(c) = 0 \Rightarrow f''(c) = f(c) = M > 0$. This implies M is $\min_{x\in[a,b]} f(x) = m$, contradiction. Similarly, $m<0$ leads to m is $\max_{x\in[a,b]} f(x) = M$.

---

(8) The function $g(x) = f(x) - f'(0)x$ is continuous. Assume $g(x)$ is injective. Then $g(x)$ is strictly increasing or strictly decreasing. In the former case, since $f'(0) \ne \inf\{f'(x) : x\in(a,b)\}$, $\exists\, t \in (a,b)$ such that $f'(t) < f'(0)$. For $x > t$, $g(x) > g(t) \Rightarrow f(x) - f'(0)x > f(t) - f'(0)t \Rightarrow \frac{f(x)-f(t)}{x-t} > f'(0)$. As $x \to t^+$, we get $f'(t) \ge f'(0)$, a contradiction. Similarly, the case $g(x)$ is strictly decreasing will lead to a contradiction.

Therefore, $g(x)$ is not injective. Then $\exists$ distinct $c,d \in (a,b)$ such that $g(c) = g(d)$ $\Leftrightarrow f(c) - f'(0)c = f(d) - f'(0)d \Leftrightarrow f'(0) = \frac{f(c)-f(d)}{c-d}$.

For the example, let $f(x) = x^3$ for $x \in (-1,1)$, then $\inf_{x\in(-1,1)} f'(x) = 0 = f'(0)$, but $\frac{f(c)-f(d)}{c-d} \ne 0$ when $c \ne d$ because f is injective. For supremum, use $f(x) = -x^3$ on $(-1,1)$.

---

(9) Let $g(x) = x+h$. Then $f(g(x)) = g(f(x))$ implies $f(x+h) = f(x)+h$. So $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{h}{h} = 1 \Rightarrow f(x) = x + c$. Let $g(x)=0$, then $f(0)=0$. So $c=0$. Therefore, $f(x)=x$.

---

(10) Since f is continuous on $[0,1]$, $\exists\, a, b \in [0,1]$ such that $\min_{x\in[0,1]} f(x) = f(a)$ and $\max_{x\in[0,1]} f(x) = f(b)$. Now $f(a) = \sqrt[3]{\int_0^1 f^3(a)\,dt} \le \sqrt[3]{\int_0^1 f^3(t)\,dt} \le \sqrt[3]{\int_0^1 f^3(b)\,dx} = f(b)$. By the intermediate value theorem, $\exists\, c \in [0,1]$ such that $f(c) = \sqrt[3]{\int_0^1 f^3(t)\,dt}$.

(11) Solution 1

Define $F: [0, 1-\frac{1}{n}] \to \mathbb{R}$ by $F(x) = f(x+\frac{1}{n}) - f(x)$. Note F is continuous and $F(0) + F(\frac{1}{n}) + F(\frac{2}{n}) + \cdots + F(\frac{n-1}{n}) = f(1) - f(0) = 0$. If $F(\frac{i}{n}) = 0$ for some $i \in \{0, 1, \cdots, n-1\}$, then $f(t+\frac{1}{n}) = f(t)$ for $t = \frac{i}{n}$. Otherwise, at least one $F(\frac{i}{n}) > 0$ and at least one $F(\frac{j}{n}) < 0$. By the intermediate value theorem, $\exists\, t \in [0, 1-\frac{1}{n}]$ such that $F(t) = 0 \Rightarrow f(t+\frac{1}{n}) = f(t)$.

Solution 2 Define $F: [0, 1-\frac{1}{n}] \to \mathbb{R}$ by $F(x) = f(x+\frac{1}{n}) - f(x)$. Note F is continuous. Assume $F(x) \neq 0$, then $F(x) > 0$ on $[0, 1-\frac{1}{n}]$ or $F(x) < 0$ on $[0, 1-\frac{1}{n}]$. In the former case, $f(x+\frac{1}{n}) > f(x)$ on $[0, 1-\frac{1}{n}] \Rightarrow f(0) < f(\frac{1}{n}) < f(\frac{2}{n}) < \cdots < f(1)$ Contradicting $f(0) = f(1)$.
The latter case is similar. So $F(t) = 0$ for some $t \in [0, 1-\frac{1}{n}]$, i.e. $f(t+\frac{1}{n}) = f(t)$.

(12) Assume $f''(x) > 0$ for all $x \in \mathbb{R}$. Then $f'(x)$ is strictly increasing. So $\exists\, c$ such that $f'(c) \neq 0$. By Taylor's theorem, $f(x) = f(c) + f'(c)(x-c) + \frac{f''(\theta)}{2}(x-c)^2$.
If $f'(c) > 0$, then $\lim_{x \to +\infty} f(x) = +\infty$. If $f'(c) < 0$, then $\lim_{x \to -\infty} f(x) = +\infty$. These contradict f is bounded. The case $f''(x) < 0$ for all $x \in \mathbb{R}$ similarly leads to contradiction. So $f''(x)$ takes on positive and negative values. Therefore, $\exists\, x_0$ such that $f''(x_0) = 0$.

(13) Let $\lim_{x \to +\infty} f'(x) = L$. Then $0 = \lim_{x \to +\infty} f(x+1) - \lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} f(x+1) - f(x) \overset{\text{mean value theorem}}{=} \lim_{x \to +\infty} f'(\theta_x) = L$

$x \to \infty \Rightarrow \theta_x \to \infty$.

(14) Assume $\exists\, x_1, x_2$ such that $a < x_1 < x_2 < b$ and $f(x_1) = f(x_2) = 0$. Since $f(x) \not\equiv 0$ on (x_1, x_2), so $\exists\, x' \in (x_1, x_2)$ such that $f(x') = \max_{x \in (x_1, x_2)} f(x) > 0$.
Then $f'(x') = 0$. By Taylor's theorem,

$$f(x_2) = f(x') + f'(x')(x_2 - x') + \frac{f''(\theta)}{2}(x_2 - x')^2 = f(x') + \frac{f''(\theta)}{2}(x_2 - x')^2 > 0,$$

a contradiction. So f has at most one root on (a, b).

(15) Assume $g(x) \neq 0$ for all $x \in (x_0, x_1)$. Note $\underbrace{f(x_0)}_{=0} g'(x_0) - f'(x_0) g(x_0) \neq 0 \Rightarrow g(x_0) \neq 0$.
Similarly, $g(x_1) \neq 0$. Let $h(x) = \frac{f(x)}{g(x)}$ on $[x_0, x_1]$. Then $h(x)$ is continuous on $[x_0, x_1]$, differentiable on (x_0, x_1). Since $h(x_0) = h(x_1) = 0$, by Rolle's theorem, $\exists\, \theta \in (x_0, x_1)$ such that $0 = h'(\theta) = \frac{f'(\theta) g(\theta) - f(\theta) g'(\theta)}{g(\theta)^2}$, contradiction.

(16) By Taylor's theorem, $f(x)=f(c)+f'(c)(x-c)+\frac{f''(\theta)}{2}(x-c)^2=\frac{f''(\theta)}{2}(x-c)^2$ for some θ between x and c. Also, $f'(x)=f'(c)+f''(\sigma)(x-c)=f''(\sigma)(x-c)$ for some σ between x and c. Let $M=\max\{|f(t)|+|f'(t)| : t\in[c-\delta,c+\delta]\}$, where $0<\delta<\frac{1}{2}$ and $[c-\delta,c+\delta]\subseteq(a,b)$. Then $M=|f(x_0)|+|f'(x_0)|$ for some $x_0\in[c-\delta,c+\delta]$. Now

$$M=|f(x_0)|+|f'(x_0)|=\frac{1}{2}|f''(\theta_0)||x_0-c|^2+|f''(\sigma_0)||x_0-c|\le\delta\left(|f''(\theta_0)|+|f''(\sigma_0)|\right)$$
$$\le\delta\left(|f(\theta_0)|+|f'(\theta_0)|+|f(\sigma_0)|+|f'(\sigma_0)|\right)\le 2\delta M \Rightarrow M=0 \text{ since } 2\delta<1$$

So $f(x)=0$ $\forall x\in[c-\delta,c+\delta]$. We can repeat this argument to cover the whole interval (a,b) since $0<\delta<\frac{1}{2}$.

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(17) Let $g(x)=f(x)-1$, then $g(x)+g'(x)=f(x)-1+f'(x)\le 0$.
So $\frac{d}{dx}(e^x g(x))=e^x g(x)+e^x g'(x)\le 0 \Rightarrow e^x g(x)$ is decreasing on $\mathbb{R}$.
Then $e^x g(x)\le \lim_{x\to-\infty} e^x g(x)=0$ (since $|g(x)|\le|f(x)|+1\Rightarrow g(x)$ is bounded).
Hence $g(x)\le 0 \Leftrightarrow f(x)\le 1$. Similarly, considering $h(x)=f(x)+1$,
$h(x)+h'(x)=f(x)+1+f'(x)\ge 0 \Rightarrow h(x)\ge 0\Leftrightarrow f(x)\ge -1$.
Combining, we get $|f(x)|\le 1$.

---

(18) The case $a=0$ is clear. For $a>0$, $\frac{f(a)}{a}=\frac{f(a)-f(0)}{a-0}=f'(\theta_0)$ for some $\theta_0\in(0,a)$ and $\frac{f(a+b)-f(b)}{a}=f'(\theta_1)$ for some $\theta_1\in(b,a+b)$. Since $\theta_0<a\le b<\theta_1$, $f'(\theta_0)\ge f'(\theta_1)$ and so $\frac{f(a)}{a}\ge\frac{f(a+b)-f(b)}{a}\Rightarrow f(a+b)\le f(a)+f(b)$.

---

(19) Assume $f(x)\ne x$ for all $x\in[a,b]$. Then $|f(x)-x|$ is continuous. By the extreme value theorem, $\min_{x\in[a,b]}|f(x)-x|=|f(x_0)-x_0|\ne 0$ for some $x_0\in[a,b]$. However the inequality implies $|f(f(x_0))-f(x_0)|<|f(x_0)-x_0|=\min_{x\in[a,b]}|f(x)-x|$, which is a contradiction. Therefore, $f(c)=c$ for some $c\in[a,b]$.

(20) Since f is bounded on $[a,b]$, $\exists K>0$ such that $|f(x)|\le K$ $\forall x\in[a,b]$. Then $|g'(x)|\le\frac{1}{|c|}|g(x)f(x)+g'(x)|+\frac{1}{|c|}|g(x)f(x)|\le\frac{|g(x)|}{|c|}(1+|f(x)|)=\frac{1+K}{|c|}|g(x)|$. Call $L=\frac{1+K}{|c|}$.
So $|g'(x)|\le L|g(x)|$ $\forall x\in[a,b]$. For $x\in[a,a+\frac{1}{2L}]$, we have $|g(x)|\le\max\limits_{t\in[a,a+\frac{1}{2L}]}|g(t)|=M$ $=|g(t_0)|$ for some $t_0\in[a,a+\frac{1}{2L}]$. By the mean-value theorem,

$$M=|g(t_0)|=|g(t_0)-g(a)|=|g'(\theta)||t_0-a|\le L|g(\theta)|\frac{1}{2L}\le\frac{M}{2}\Rightarrow M=0.$$

So $g(x)=0$ for $x\in[a,a+\frac{1}{2L}]$. Since $\frac{1}{2L}>0$, we can repeat this $\left\lceil\frac{b-a}{1/2L}\right\rceil$ times to get $g(x)=0$ for all $x\in[a,b]$.

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(21) Assume f' is not bounded above. Then $\exists x_n\in\mathbb{R}$ such that $f'(x_n)\to+\infty$.
By Taylor's theorem, $f(x_n+1)=f(x_n)+f'(x_n)+\frac{1}{2}f''(x_n)+\frac{1}{6}f'''(\theta_n)$ (1)
and $f(x_n-1)=f(x_n)-f'(x_n)+\frac{1}{2}f''(x_n)-\frac{1}{6}f'''(\sigma_n)$, (2)
where $x_n-1<\sigma_n<x_n<\theta_n<x_n+1$. Since f and f''' are bounded, equation (1) implies $\underset{\substack{\parallel \\ 2(f(x_{n+1})-f(x_n)-f'(x_n)-\frac{1}{6}f'''(\theta_n))}}{f''(x_n)}\to-\infty$, but equation (2) implies $\underset{\substack{\parallel \\ 2(f(x_n-1)-f(x_n)+f'(x_n)+\frac{1}{6}f'''(\sigma_n))}}{f''(x_n)}\to+\infty$, contradiction.
Similarly, f' is bounded below. So f' is bounded. Similarly, f'' is bounded.

---

(22) By Taylor's theorem, $f(1)=f(x)+f'(x)(1-x)+\frac{f''(\theta)}{2}(1-x)^2$ for some $\theta\in(x,1)$ and $f(0)=f(x)-f'(x)x+\frac{f''(\sigma)}{2}x^2$ for some $\sigma\in(0,x)$. Since $f(1)=f(0)$, subtracting these equations, we get $f'(x)=\frac{f''(\sigma)}{2}x^2-\frac{f''(\theta)}{2}(1-x)^2$. Then $|f'(x)|\le\frac{M}{2}(x^2+(1-x)^2)\le\frac{M}{2}$.

(23) By Taylor's theorem, $f(x+1)=f(x)+f'(x)+\frac{1}{2}f''(x)+\frac{1}{6}f'''(\theta_x)$ for some $\theta_x\in(x,x+1)$ and $f(x-1)=f(x)-f'(x)+\frac{1}{2}f''(x)-\frac{1}{6}f'''(\sigma_x)$ for some $\sigma_x\in(x-1,x)$. Adding and subtracting these, we get $f''(x)=f(x+1)-2f(x)+f(x-1)-\frac{1}{6}f'''(\theta_x)+\frac{1}{6}f'''(\sigma_x)\to0$ as $x\to+\infty$ and
$2f'(x)=f(x+1)-f(x-1)-\frac{1}{6}f'''(\theta_x)-\frac{1}{6}f'''(\sigma_x)\to0$ as $x\to+\infty$
Since $x\to+\infty\Rightarrow\theta_x,\sigma_x\to+\infty$.

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(24) By the mean-value theorem, $\frac{f(0)-f(-2)}{2}=f'(a)$ and $\frac{f(2)-f(0)}{2}=f'(b)$ for some $a\in(-2,0)$ and $b\in(0,2)$. Since $|f(x)|\le1$, we get $|f'(a)|\le1$ and $|f'(b)|\le1$.
Define $g(x)=f^2(x)+f'^2(x)$. Then $\underbrace{g(a)\le2 \text{ and } g(b)\le2}$, Since $g(0)=4$, we get $g(\theta)=\max\{g(x):x\in[a,b]\}\ge4$ with $\theta\in(a,b)$, $g'(\theta)=0$. If $f'(\theta)=0$, then $f^2(\theta)=g(\theta)\ge4\Rightarrow f(\theta)\ge2$, contradiction. So $f'(\theta)\ne0$. Then $0=g'(\theta)=2f'(\theta)(f(\theta)+f''(\theta))\Rightarrow f(\theta)+f''(\theta)=0$.

Additional Integration Exercises

351. (a) Let $f(x)$ have a continuous derivative on $[0,1]$. If $f(0)=f(1)=0$, then prove by using $dx = d(x-\frac{1}{2})$ that $\left|\int_0^1 f(x)\,dx\right| \le \frac{1}{4}\max\{|f'(x)| : x\in[0,1]\}$.

(b) Let $g(x)$ have a continuous second derivative on $[0,2]$. If $g(1)=0$, then prove by using Taylor's theorem that $\left|\int_0^2 g(x)\,dx\right| \le \frac{1}{3}\max\{|g''(x)| : x\in[0,2]\}$.

352. Give an example of a bounded continuous function $f:[0,+\infty)\to[0,+\infty)$ such that $\int_0^{+\infty} f(x)\,dx<\infty$, but $\lim_{x\to+\infty} f(x)\ne 0$. So there is no term test for improper integral.

353. (a) Prove that $\int_0^{\pi/2} \ln(\sin x)\,dx$ and $\int_0^{\pi/2} \ln(\tan x)\,dx$ converge.

(b) Find the value of $\int_0^{\pi/2} \ln(\sin x)\,dx$ by substituting $x=2t$ and using $\sin 2t = 2\sin t\cos t$. Find the value of $\int_0^{\pi/2} \ln(\tan x)\,dx$ by substituting $t=\tan x$, writing $\int_0^{+\infty} = \int_0^1 + \int_1^{+\infty}$ and using $\ln\frac{1}{t} = -\ln t$.

(c) Find the value of $\int_0^1 \frac{\arcsin x}{x}\,dx$ by substituting $x=\sin t$ and integrating by parts.

354. (a) Let $f(x)$ have a continuous derivative on $[0,1]$. Prove that

$$\int_0^1 |f(x)|\,dx \le \max\left\{\int_0^1 |f'(x)|\,dx, \left|\int_0^1 f(x)\,dx\right|\right\}.$$

(*Hint*: Does f have a root?)

(b) Let $g(x)$ be decreasing on $[0,1]$. Prove that for every $c\in(0,1)$, $\int_0^c g(x)\,dx \ge c\int_0^1 g(x)\,dx$.

(c) Let $h(x)$ be differentiable on $[0,1]$, $h(0)=0$ and for all $x\in(0,1)$, $0<h'(x)<1$. Prove that $\left(\int_0^1 h(x)\,dx\right)^2 > \int_0^1 h(x)^3\,dx$ by using the generalized mean value theorem.

355. (a) Prove that $\int_0^{+\infty} e^{-x}\ln x\,dx$ converges. Determine if $\int_1^{+\infty} \frac{\ln x}{x+e^{-x}}\,dx$ converges.

(b) Determine if $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$ conveges. Determine if P.V. $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$ conveges.

356. (a) Prove the *Dirichlet test*: if $f(x)$ is continuous on $[a,+\infty)$, $F(t)=\int_a^t f(x)\,dx$ is bounded on $[a,+\infty)$ and $g(t)$ is monotone and continuously differentiable on $[a,+\infty)$ with $\lim_{t\to+\infty} g(t)=0$, then $\int_a^{+\infty} f(x)g(x)\,dx$ converges.

(b) Prove that $\int_{100}^{+\infty} \frac{\sqrt{x}\cos x}{100+x}\,dx$ converges, but $\int_{100}^{+\infty} \left|\frac{\sqrt{x}\cos x}{100+x}\right|\,dx$ diverges by using $|\cos x|\ge\cos^2 x = (1+\cos 2x)/2$.

(c) Prove the *Abel test*: if $f(x)$ is continuous on $[a, +\infty)$, $\int_a^{+\infty} f(x)dx$ converges and $g(t)$ is monotone, continuously differentiable and bounded on $[a, +\infty)$, then $\int_a^{+\infty} f(x)g(x)dx$ converges. Prove that $\int_0^{+\infty} \frac{\arctan x}{x} \sin x \, dx$ converges.

357. (a) Determine if each of the following improper integral converges :

$$\int_1^5 \frac{dx}{\sqrt{(5-x)(x-1)}}, \quad \int_0^1 \frac{1}{x} \sin \frac{1}{x} \, dx, \quad \int_0^{\pi/2} \sqrt{\sin x \tan x} \, dx.$$

(b) The *Beta function* is defined as $B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} \, dx$. Determine for which real numbers p and q the integeral converges.

Solutions to Additional Integration Exercises

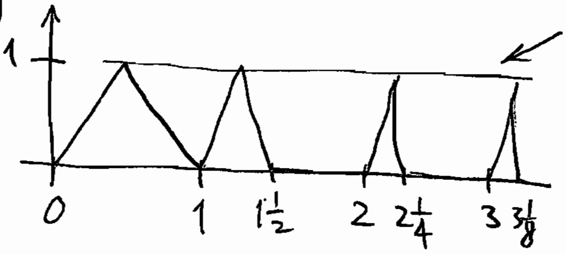
(351) (a) Let $M = \max\{|f'(x)| : x \in [0,1]\}$. Integrating by parts using $dx = d(x-\frac{1}{2})$, we get

$$\left|\int_0^1 f(x)\,dx\right| = \left|\underbrace{(x-\tfrac{1}{2})f(x)\Big|_0^1}_{=0 \text{ at } x=0,1} - \int_0^1 (x-\tfrac{1}{2})f'(x)\,dx\right| \le \int_0^1 |x-\tfrac{1}{2}|\,M\,dx = \frac{1}{4}M.$$

(b) By Taylor's theorem, $g(x) = \underbrace{g(1)}_{=0} + g'(1)(x-1) + \frac{g''(\theta)}{2}(x-1)^2$. Let $N = \max_{x\in[0,2]} |g''(x)|$

$$\left|\int_0^2 g(x)\,dx\right| = \left|g'(1)\underbrace{\int_0^2 (x-1)\,dx}_{=0} + \int_0^2 \frac{g''(\theta)}{2}(x-1)^2\,dx\right| \le \frac{N}{2}\int_0^2 (x-1)^2\,dx = \frac{N}{3}.$$

(352)



Define f so the graph of f is as on the left.

$$\int_0^\infty f(x)\,dx = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

$\lim\limits_{x\to+\infty} f(x)$ does not exist.

(353) (a) ($\ln\sin x$ is unbounded near 0) Since $\sin x \sim x$ near 0, we compare with $\ln x$.

Since $\lim\limits_{x\to0^+}\frac{\ln\sin x}{\ln x} = \lim\limits_{x\to0^+}\frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} = \lim\limits_{x\to0^+}\frac{x}{\sin x}\cos x = 1$ and $\int_0^{\pi/2}|\ln x|\,dx < \infty$ (see class examples),

so $\int_0^{\pi/2}\ln\sin x\,dx$ converges by the limit comparison test

($\ln\tan x$ is unbounded near 0 and near $\frac{\pi}{2}$) Since

$$\lim_{x\to0^+}\left|\frac{\ln\tan x}{\ln x}\right| = \left|\lim_{x\to0^+}\left(\frac{\ln\sin x}{\ln x} - \frac{\ln\cos x}{\ln x}\right)\right| = 1 - 0 = 1 \Rightarrow \int_0^1 \ln\tan x\,dx < \infty \text{ and}$$

$$\lim_{x\to\frac{\pi}{2}}\left|\frac{\ln\tan x}{\ln(\frac{\pi}{2}-x)}\right| = \left|\lim_{x\to\frac{\pi}{2}^-}\frac{\frac{1}{\tan x}\sec^2 x}{\frac{1}{\pi/2-x}(-1)}\right| = \left|\lim_{x\to\frac{\pi}{2}^-}\frac{x-\frac{\pi}{2}}{\cos x}\frac{1}{\sin x}\right| = \left|\lim_{x\to\frac{\pi}{2}^-}\frac{1}{(-\sin x)}\right| = 1,\quad \int_1^{\pi/2}\ln(\tfrac{\pi}{2}-x)\,dx \overset{t=\frac{\pi}{2}-x}{=} \int_0^{\pi/2-1}\ln t\,dt < \infty$$

$\Rightarrow \int_1^{\pi/2}\ln\tan x\,dx < \infty$, so $\int_0^{\pi/2}\ln\tan x\,dx$ converges.

(b) $$I = \int_0^{\pi/2}\ln\sin x\,dx \overset{x=2t}{=} 2\int_0^{\pi/4}\ln\sin 2t\,dt = 2\left(\int_0^{\pi/4}\ln 2\,dt + \int_0^{\pi/4}\ln\sin t\,dt + \int_0^{\pi/4}\ln\cos t\,dt\right)$$

$$\int_0^{\pi/4}\ln\cos t\,dt \underset{t=\frac{\pi}{2}-y}{=} \int_{\pi/4}^{\pi/2}\ln\sin y\,dy \Rightarrow I = \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/2}\ln\sin x\,dx = \frac{\pi}{2}\ln 2 + 2I$$

$$\Rightarrow I = -\frac{\pi}{2}\ln 2.$$

$$J = \int_0^{\pi/2}\ln\tan x\,dx \overset{t=\tan x}{=} \int_0^{+\infty}\frac{\ln t}{1+t^2}\,dt = \int_0^1\frac{\ln t}{1+t^2}\,dt + \int_1^{+\infty}\frac{\ln t}{1+t^2}\,dt \overset{t=1/u}{=} \int_0^1\frac{\ln t}{1+t^2}\,dt - \int_0^1\frac{\ln u}{u^2+1}\,du = 0.$$

(c) $$\int_0^1\frac{\arcsin x}{x}\,dx \underset{x=\sin t}{=} \int_0^{\pi/2}\frac{t}{\sin t}\cos t\,dt = \underbrace{t\ln\sin t\Big|_0^{\pi/2}}_{=0-0=0} - \int_0^{\pi/2}\ln\sin t\,dt = \frac{\pi}{2}\ln 2.$$

Proper integral since $\lim\limits_{x\to0^+}\frac{\arcsin x}{x} = \lim\limits_{t\to0^+}\frac{t}{\sin t} = 1$

(354) (a) If f doesn't have a root on $[0,1]$, then $f \geq 0$ or $f \leq 0$ everywhere on $[0,1]$

$$\Rightarrow \int_0^1 |f(x)|dx = \left|\int_0^1 f(x)dx\right| \leq \max\left\{\int_0^1 |f'(x)|dx, \left|\int_0^1 f(x)dx\right|\right\}.$$

If f has a root at $x_0 \in [0,1]$, then

$$|f(x)| = |f(x) - f(x_0)| = \left|\int_x^{x_0} f'(t)dt\right| \leq \int_0^1 |f'(t)|dt$$

$$\Rightarrow \int_0^1 |f(x)|dx \leq \int_0^1 \left(\int_0^1 |f'(t)|dt\right) dx = \int_0^1 |f'(t)|dt \leq \max\left\{\int_0^1 |f'(x)|dx, \left|\int_0^1 f(x)dx\right|\right\}.$$

(b) Note $c\int_0^1 g(x)dx = c\int_0^c g(x)dx + c\int_c^1 g(x)dx$. So $\int_0^c g(x)dx \geq c\int_0^1 g(x)dx \Leftrightarrow$

$(1-c)\int_0^c g(x)dx \geq c\int_c^1 g(x)dx \Leftrightarrow \frac{1}{c}\int_0^c g(x)dx \geq \frac{1}{1-c}\int_c^1 g(x)dx$.

The last inequality is true because g is decreasing on $[0,1]$ so that

$$\frac{1}{c}\int_0^c g(x)dx \geq \frac{1}{c}\int_0^c g(c)dx = g(c) = \frac{1}{1-c}\int_c^1 g(c)dx \geq \frac{1}{1-c}\int_c^1 g(x)dx,$$

(c) Let $F(t) = \left(\int_0^t h(x)dx\right)^2$ and $G(t) = \int_0^t h(x)^3dx$, then $F(0)=0$ and $G(0)=0$.

By generalized mean value theorem,

$$\frac{F(1)}{G(1)} = \frac{F(1)-F(0)}{G(1)-G(0)} \underset{\text{for some }\theta\text{ between }0\text{ and }1}{=} \frac{F'(\theta)}{G'(\theta)} = \frac{2\left(\int_0^\theta h(x)dx\right)h(\theta)}{h(\theta)^3} = \frac{2\int_0^\theta h(x)dx - 2\int_0^0 h(x)dx}{h(\theta)^2 - h(0)^2} \underset{\text{for some }\sigma\text{ between }0\text{ and }\theta}{=} \frac{2h(\sigma)}{2h(\sigma)h'(\sigma)} > 1$$

So $F(1) > G(1)$

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(355) (a) ($e^{-x}\ln x$ is unbounded near 0 and $+\infty$) Since

$\lim_{x\to 0^+} \frac{e^{-x}\ln x}{\ln x} = e^{-0} = 1$ and $\int_0^1 \ln x\,dx$ Converges $\Rightarrow \int_0^1 e^{-x}\ln x\,dx$ converges and

$\lim_{x\to\infty} \frac{e^{-x}\ln x}{1/x^2} = \lim_{x\to+\infty} \frac{x^2\ln x}{e^x} = \lim \frac{2x\ln x + x}{e^x} = \lim_{x\to+\infty} \frac{2\ln x + 3}{e^x} = 0$ and $\int_1^{+\infty} 1/x^2 dx < \infty$ by p-test

$\Rightarrow \int_1^{+\infty} e^{-x}\ln x\,dx$ converges, so $\int_0^{+\infty} e^{-x}\ln x\,dx$ Converges.

($\frac{\ln x}{x+e^{-x}}$ is unbounded near $+\infty$) Since

$\lim_{x\to+\infty} \frac{\frac{\ln x}{x+e^{-x}}}{\frac{\ln x}{x}} = \lim_{x\to+\infty} \frac{x}{x+e^{-x}} = \lim_{x\to+\infty} \frac{1}{1-e^{-x}} = 1$ and $\int_1^{+\infty} \frac{\ln x}{x}dx = \frac{1}{2}(\ln x)^2\Big|_1^{+\infty} = +\infty$,

so $\int_1^{+\infty} \frac{\ln x}{x+e^{-x}}dx$ diverges.

355 (b) ($\frac{1}{\sqrt{|x-1|}}$ is unbounded near 1).

$$\int_0^2 \frac{dx}{\sqrt{|x-1|}} = \lim_{c\to 1^-}\int_0^c \frac{dx}{\sqrt{1-x}} + \lim_{b\to 1^+}\int_b^2 \frac{dx}{\sqrt{x-1}} = \underbrace{\lim_{c\to 1^-}\left(-2(1-x)^{\frac{1}{2}}\Big|_0^c\right)}_{=0+2} + \underbrace{\lim_{b\to 1^+} 2(x-1)^{\frac{1}{2}}\Big|_b^2}_{=2+0} = 4$$

$\therefore$ $\int_0^2 \frac{dx}{\sqrt{|x-1|}}$ converges and hence also $PV\int_0^2 \frac{dx}{\sqrt{|x-1|}}$ converges.

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356 (a) $$\int_a^{+\infty} f(x)g(x)\,dx = \int_a^{+\infty} F'(x)g(x)\,dx = \lim_{t\to+\infty}\int_a^t F'(x)g(x)\,dx = \lim_{t\to+\infty}\left(F(x)g(x)\Big|_a^t - \int_a^t F(x)g'(x)\,dx\right)$$

$\lim_{t\to+\infty} \underbrace{F(t)}_{\text{bounded}}g(t) = 0$ and $\int_a^{+\infty}|F(x)g'(x)|\,dx \le K\left|\int_a^{+\infty} g'(x)\,dx\right| = K|0-g(a)| < \infty$.

where $|F(x)| \le K$ on $[a,+\infty)$ $\quad g'\ge 0$ or $g'\le 0$ everywhere on $[a,+\infty)$

So $\lim_{t\to+\infty}\left(F(x)g(x)\Big|_a^t\right)$ and $\int_a^{+\infty} F(x)g'(x)\,dx$ converges. $\therefore$ $\int_a^{+\infty} f(x)g(x)\,dx$ converges.

(b) Let $f(x)=\cos x$ and $g(x)=\frac{\sqrt{x}}{100+x}$. Then $F(t)=\int_{100}^t \cos x\,dx = \sin t - \sin 100$ is bounded and $g'(x) = \frac{100-x}{2\sqrt{x}(100+x)^2} < 0$ on $(100,+\infty)$, $0 < g(x) < \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \to 0$ as $x\to\infty$

$\underset{\text{Dirichlet test}}{\Longrightarrow}$ $\int_{100}^{+\infty}\frac{\sqrt{x}\cos x}{100+x}\,dx$ converges.

Using $|\cos x| \ge \cos^2 x = (1+\cos 2x)/2$, we have

$$\int_{100}^{+\infty}\left|\frac{\sqrt{x}\cos x}{100+x}\right|dx \ge \underbrace{\frac{1}{2}\int_{100}^{+\infty}\frac{\sqrt{x}}{100+x}\,dx}_{\text{diverges}} + \underbrace{\frac{1}{2}\int_{100}^{+\infty}\frac{\sqrt{x}\cos 2x}{100+x}\,dx}_{\text{converges by Dirichlet test as } \int_{100}^t \cos 2x\,dx = \frac{1}{2}\sin 2x\big|_{100}^t \text{ is bounded}} \text{ diverges.}$$

since $\frac{\sqrt{x}}{100+x} \sim \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$, $\int_{100}^{+\infty}\frac{1}{\sqrt{x}}\,dx$ diverges by p-test $p=\frac{1}{2}\le 1$.

$\therefore$ $\int_{100}^{+\infty}\left|\frac{\sqrt{x}\cos x}{100+x}\right|dx$ diverges.

(c) $\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx = \lim_{t\to+\infty} F(t)$ exists $\Rightarrow F(t)=\int_a^t f(x)\,dx$ bounded on $[a,+\infty)$.

$g(t)$ monotone and bounded $\Rightarrow \lim_{t\to a} g(t) = c$ exists $\Rightarrow g(t)-c$ monotone and has limit 0 as $t\to+\infty$.

By Dirichlet test, $\int_a^{+\infty} f(x)(g(x)-c)\,dx$ converges. Then

$\int_a^{+\infty} f(x)g(x)\,dx = \int_a^{+\infty} f(x)(g(x)-c)\,dx + c\int_a^{+\infty} f(x)\,dx$ converges.

Let $f(x)=\frac{\sin x}{x}$ and $g(x)=\arctan x$, then $\int_0^{+\infty}\frac{\sin x}{x}\,dx = \underbrace{\int_0^1\frac{\sin x}{x}\,dx}_{\text{proper integral}} + \underbrace{\int_1^{+\infty}\frac{\sin x}{x}\,dx}_{\text{converges (class example)}} < \infty$

and $g(x) \nearrow \frac{\pi}{2}$ as $x\to+\infty$.

$\therefore$ by Abel test, $\int_0^{+\infty}\frac{\arctan x}{x}\sin x\,dx$ converges.

357 (a) ($\frac{1}{\sqrt{(5-x)(x-1)}}$ is unbounded near 1 and 5) Applying the limit comparison test,

$$\lim_{x\to 1^+}\frac{\frac{1}{\sqrt{(5-x)(x-1)}}}{\frac{1}{\sqrt{x-1}}}=\lim_{x\to 1^+}\frac{1}{\sqrt{5-x}}=\frac{1}{2} \text{ and } \int_1^3\frac{dx}{\sqrt{x-1}}=2\sqrt{x-1}\Big|_1^3=2\sqrt{2} \Rightarrow \int_1^3\frac{dx}{\sqrt{(5-x)(x-1)}} \text{ converges}$$

and $$\lim_{x\to 5^-}\frac{\frac{1}{\sqrt{(5-x)(x-1)}}}{\frac{1}{\sqrt{5-x}}}=\lim_{x\to 5^-}\frac{1}{\sqrt{x-1}}=\frac{1}{2} \text{ and } \int_3^5\frac{dx}{\sqrt{5-x}}=-2\sqrt{5-x}\Big|_3^5=2\sqrt{2}\Rightarrow\int_3^5\frac{dx}{\sqrt{(5-x)(x-1)}} \text{ converges,}$$

so $\int_1^5\frac{dx}{\sqrt{(5-x)(x-1)}}$ converges.

(b) ($\frac{1}{x}\sin\frac{1}{x}$ is unbounded near 0)

$$\int_0^1\frac{1}{x}\sin\frac{1}{x}dx=\lim_{b\to 0^+}\int_b^1\frac{1}{x}\sin\frac{1}{x}dx=\lim_{b\to 0^+}\int_{1/b}^1 t\sin t\left(-\frac{1}{t^2}dt\right)=\int_1^{+\infty}\frac{\sin t}{t}dt \text{ converges}$$

by class example

(c) ($\sqrt{\sin x\tan x}$ is unbounded near $\frac{\pi}{2}$) On $[0,\frac{\pi}{2})$, $\sqrt{\sin x\tan x}=\frac{\sin x}{\sqrt{\cos x}}$. Since

$$\lim_{x\to\frac{\pi}{2}^-}\frac{\frac{\sin x}{\sqrt{\cos x}}}{\frac{1}{\sqrt{\frac{\pi}{2}-x}}}=\lim_{x\to\frac{\pi}{2}^-}\sin x\sqrt{\frac{\frac{\pi}{2}-x}{\cos x}}=\lim_{x\to\frac{\pi}{2}}\sqrt{\frac{-1}{-\sin x}}=1 \text{ and } \int_0^{\pi/2}\frac{dx}{\sqrt{\frac{\pi}{2}-x}}=-2\sqrt{\frac{\pi}{2}-x}\Big|_0^{\frac{\pi}{2}}=2\sqrt{\frac{\pi}{2}},$$

so $\int_0^{\pi/2}\sqrt{\sin x\tan x}\,dx$ converges.

(b) ($x^{p-1}(1-x)^{q-1}$ may be unbounded near 0 and 1)

$$\lim_{x\to 0^+}\frac{x^{p-1}(1-x)^{q-1}}{x^{p-1}}=\lim_{x\to 0^+}(1-x)^{q-1}=1 \text{ and } \int_0^1 x^{p-1}dx \text{ converges} \overset{\text{by p-test}}{\Leftrightarrow} 1-p<1\Leftrightarrow p>0$$

and $$\lim_{x\to 1^-}\frac{x^{p-1}(1-x)^{q-1}}{(1-x)^{q-1}}=\lim_{x\to 1^-}x^{p-1}=1 \text{ and } \int_0^1(1-x)^{q-1}dx=\int_0^1 t^{q-1}dt \overset{\text{by p-test}}{\Leftrightarrow} 1-q<1\Leftrightarrow q>0,$$

so $B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx$ converges $\Leftrightarrow p>0$ and $q>0$.