MATH2033 Mathematical Analysis (2021 Spring)
Assignment 2

Submission deadline of Assignment 1: **<u>11:59p.m. of 25th Mar, 2020 (Thurs)</u>**

Instruction: Please complete all required problems. Full details (including description of methods used and explanation, key formula and theorem used and final answer) must be shown **clearly** to receive full credits. Marks can be deducted for incomplete solution or unclear solution.

Please submit your completed work via the submission system in canvas before the deadline. Late assignment will not be accepted.

Your submission must (1) be hand-written (typed assignment will not be accepted), (2) in a single pdf. file (other file formats will not be accepted) and (3) contain yo
full name and student ID on the first page of the assignment.

Problem 1

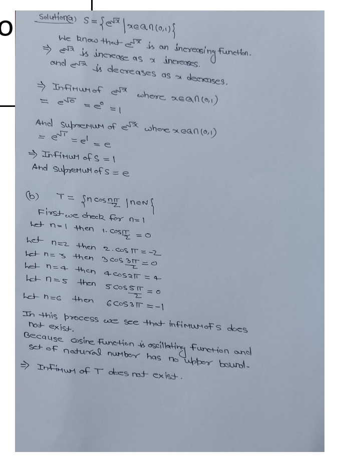
(a) Find the supremum and infimum of the following set:

$$S = \{e^{\sqrt{x}} | x \in \mathbb{Q} \cap (0,1)\}.$$

(b) We consider a set defined by

$$T = \left\{n \cos\frac{n\pi}{2} | n \in \mathbb{N}\right\}.$$

Show that the infimum of T does not exist in $\mathbb{R}$.


Solution(a) $S = \{e^{\sqrt{x}} | x \in \mathbb{Q} \cap (0,1)\}$
We know that $e^{\sqrt{x}}$ is an increasing function.
$\Rightarrow$ $e^{\sqrt{x}}$ is increase as x increases.
and $e^{\sqrt{x}}$ is decreases as x decreases.
$\Rightarrow$ Infimum of $e^{\sqrt{x}}$ where $x \in \mathbb{Q} \cap (0,1)$
$= e^{\sqrt{0}} = e^0 = 1$
And supremum of $e^{\sqrt{x}}$ where $x \in \mathbb{Q} \cap (0,1)$
$= e^{\sqrt{1}} = e^1 = e$
$\Rightarrow$ Infimum of $S = 1$
And supremum of $S = e$

(b) $T = \{n \cos\frac{n\pi}{2} | n \in \mathbb{N}\}$
First we check for $n=1$
Let $n=1$ then $1 \cdot \cos\frac{\pi}{2} = 0$
Let $n=2$ then $2 \cdot \cos\pi = -2$
Let $n=3$ then $3\cos\frac{3\pi}{2} = 0$
Let $n=4$ then $4\cos 2\pi = 4$
Let $n=5$ then $5\cos\frac{5\pi}{2} = 0$
Let $n=6$ then $6\cos 3\pi = -1$
In this process we see that infimum of S does not exist.
Because cosine function is oscillating function and set of natural number has no upper bound.
$\Rightarrow$ Infimum of T does not exist.

Problem 2

(a) We let $A \subseteq \mathbb{R}$ be a bounded non-empty subset of real numbers and let
non-empty subset of real numbers. Prove that

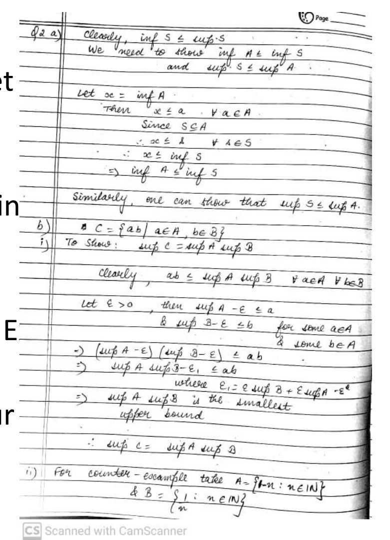
$$\inf A \leq \inf S \leq \sup S \leq \sup A.$$

(b) We let A, B be two bounded subsets of *positive real numbers*. We defin

$$C = \{ab | a \in A, b \in B\}.$$

(i) Show that $\sup C = \sup A \sup B$.

(ii) Is the result (i) valid if either A or B contain negative number? E
your answer.
(*Note: If your answer is yes, give a mathematical proof. If your
no, you need to give a counter-example.)


Q2 a) Clearly, $\inf S \leq \sup S$
We need to show $\inf A \leq \inf S$
and $\sup S \leq \sup A$.
Let $x = \inf A$
Then $x \leq a \quad \forall a \in A$
Since $S \subseteq A$
$\therefore x \leq s \quad \forall s \in S$
$\therefore x \leq \inf S$
$\Rightarrow \inf A \leq \inf S$
Similarly, one can show that $\sup S \leq \sup A$.
b) $C = \{ab | a \in A, b \in B\}$
i) To show: $\sup C = \sup A \sup B$
Clearly, $ab \leq \sup A \sup B \quad \forall a \in A \; \forall b \in B$
Let $\varepsilon > 0$, then $\sup A - \varepsilon \leq a$
& $\sup B - \varepsilon \leq b$ for some $a \in A$ & some $b \in A$
$\Rightarrow (\sup A - \varepsilon)(\sup B - \varepsilon) \leq ab$
$\Rightarrow \sup A \sup B - \varepsilon_1 \leq ab$
where $\varepsilon_1 = \varepsilon \sup B + \varepsilon \sup A - \varepsilon^2$
$\Rightarrow \sup A \sup B$ is the smallest upper bound
$\therefore \sup C = \sup A \sup B$
ii) For counter-example take $A = \{1-n : n \in \mathbb{N}\}$
& $B = \{\frac{1}{n} : n \in \mathbb{N}\}$
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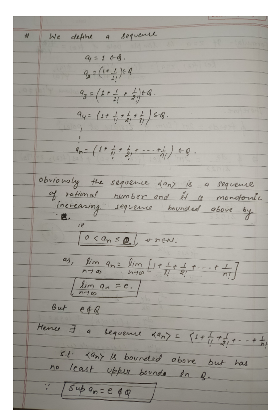
Problem 3

We let $a \in \mathbb{R}$ be a real number. Construct a sequence of rational number $\{q_n\}$ (where $q_n \in \mathbb{Q}$) such that $\{q_n\}$ converges to a (i.e. $\lim_{n\to\infty} q_n = a$).

e or pi


We define a sequence
$a_1 = 1 \in \mathbb{Q}$.
$a_2 = (1 + \frac{1}{1!}) \in \mathbb{Q}$
$a_3 = (1 + \frac{1}{1!} + \frac{1}{2!}) \in \mathbb{Q}$.
$a_4 = (1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}) \in \mathbb{Q}$.
$\vdots$
$a_n = (1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}) \in \mathbb{Q}$.
Obviously the sequence $\langle a_n \rangle$ is a sequence of rational number and it is monotonic increasing sequence bounded above by e.
ie $\boxed{0 < a_n \leq e}, \forall n \in \mathbb{N}$.
as, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right]$
$\boxed{\lim_{n\to\infty} a_n = e.}$
But $e \notin \mathbb{Q}$
Hence $\exists$ a sequence $\langle a_n \rangle = \{1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$
s.t. $\langle a_n \rangle$ is bounded above but has no least upper bound in $\mathbb{Q}$.
$\therefore \boxed{\sup a_n = e \notin \mathbb{Q}}$

(☺Hint: First consider the case when 0<a<1. Construct the sequen
representation of a using the operation [x] which takes the greates
smaller than or equal to x).

Problem 4

Prove the following fact using the definition of limits

(a) $\lim_{n\to\infty} \cos\left(a + \frac{b}{n}\right) = \cos a$, where a, b are positive number.

(b) $\lim_{n\to\infty} \sqrt{b_n} = \sqrt{b}$, where $\{b_n\}$ is a convergent sequence with

Problem 5

We let $\{x_n\}$ be a sequence defined by

$$x_1 = 0.4, \qquad x_{n+1} = \frac{x_n^3 + 2}{3} \quad for\ n \in \mathbb{N}.$$

Show that $\{x_n\}$ converges and find the limits.

Solution: (Problem 5)

The sequence is given by $x_1 = 0.4$, $x_{n+1} = \frac{x_n^3+2}{3} \to (1)$

Since $x_1 < 1$, therefore $x_1^3 < 1$

Therefore from (1) we get $x_2 = \frac{x_1^3+2}{3} < \frac{1+2}{3} = 1$

i.e. $x_2 < 1$.

Similarly $x_3 = \frac{x_2^3+1}{3} < \frac{1+1}{3} = 1$

i.e. $x_3 < 1$

By similar argument we can say that $x_n < 1 \ \forall n \in \mathbb{N}$.

So $\{x_n\}$ is a bounded sequence.

Now $x_{n+1} - x_n = \frac{x_n^3+2}{3} - x_n = \frac{x_n^3 - 3x_n + 2}{x_n}$

$\Rightarrow (x_{n+1} - x_n) = \frac{(x_n+2)(x_n-1)^2}{x_n} > 0$

since $x_n + 2 > 0 \ \forall n \in \mathbb{N}$

and $(x_n - 1)^2 > 0 \ \forall n \in \mathbb{N}$.

So the sequence $\{x_n\}$ is monotone increasing.

As we have the sequence is monotone and bdd, so it must converges. Let l be the limit of the sequence.

Then $l = \frac{l^3+2}{3} \Rightarrow l^3 + 2 - 3l = 0$

$\Rightarrow (l+2)(l-1)^2 = 0$

So either $l = -2$ or $l = 1$.

As $\{x_n\}$ is a sequence of positive terms, the limit can't be negative. So limit of the sequence must be equal to 1.

i.e. $\lim_{n\to\infty} x_n = 1$.

Problem 6 (Harder)

We let $\{x_n\}$ be a sequence of real numbers.

(a) Suppose that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L < 1$, show that $\{x_n\}$ converges and $\lim_{n\to\infty} x_n = 0$.

(☺Hint: We let $L < r < 1$ be a number. One can apply the definition of limits to $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$ with $\varepsilon < r - L$ and argue that $\frac{x_{n+1}}{x_n} < r$ when n is greater than some positive integer $K \in \mathbb{N}$.)

(b) Suppose that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L > 1$, show that $\{x_n\}$ does not converge.

(c) Suppose that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L = 1$,

(i) Find an example of $\{x_n\}$ which $\{x_n\}$ converges

(ii) Find another example of $\{x_n\}$ which $\{x_n\}$ does not converges.

End of Assignment 2

Problem-6

(a) Let $\langle x_n \rangle$ be a sequence of real number s.t $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$, To prove if $|L| < 1$ then $\langle x_n \rangle$ convergent and converges to '0'. i.e. $\lim_{n\to\infty} x_n = 0$.

Since $|L| < 1$, we can choose a positive number ϵ, so small that $|L| + \epsilon < 1$

take $|L| + \epsilon = K$

$\Rightarrow K < 1 \longrightarrow (1)$

Now $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$

$\Rightarrow \exists$ a positive integer m such that $\forall n \geq m$

$\left| \frac{x_{n+1}}{x_n} - L \right| < \epsilon$

$\Rightarrow \left| \frac{x_{n+1}}{x_n} \right| - |L| < \left| \frac{x_{n+1}}{x_n} - L \right| < \epsilon$ (as $|a| - |b| \leq |a-b|$ if $a, b \in \mathbb{R}$)

$\Rightarrow \left| \frac{x_{n+1}}{x_n} \right| < |L| + \epsilon = K$

$\Rightarrow \left| \frac{x_{n+1}}{x_n} \right| < K \longrightarrow (2)$.

changing n to $m, m+1, \ldots, (n-1)$ in eq (2) and multiply, we get

$\left| \frac{x_{m+1}}{x_m} \right| \times \left| \frac{x_{m+2}}{x_{m+1}} \right| \times \cdots \times \left| \frac{x_n}{x_{n-1}} \right| < K \cdots K$ (n-m) factor.

$\Rightarrow \left| \frac{x_n}{x_m} \right| < K^{n-m} \Rightarrow |x_n| < \frac{|x_m|}{K^m} \cdot K^n \longrightarrow (3)$

Since $0 < K < 1$, so $\lim_{n\to\infty} K^n = 0$.

$\therefore$ by (3), $\lim_{n\to\infty} |x_n| = 0$

$\Rightarrow \lim_{n\to\infty} x_n = 0$

$\Rightarrow \langle x_n \rangle$ is convergent.

(b) Suppose $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L > 1$.

Since, $L > 1$, we can choose a (+ve) number ϵ so small that $L - \epsilon > 1$

$\Rightarrow K > 1$, where $K = L - \epsilon$.

Now $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$

$\Rightarrow \exists$ a (+ve) integer m such that

$\left| \frac{x_{n+1}}{x_n} - L \right| < \epsilon \ \forall n \geq m$.

$\Rightarrow L - \epsilon < \frac{x_{n+1}}{x_n} < L + \epsilon \ \forall n \geq m$.

$\Rightarrow K < \frac{x_{n+1}}{x_n} < L + \epsilon \ \forall n \geq m$.

Thus $\frac{x_{n+1}}{x_n} > K$, $\forall n \geq m$.

Changing n to $m, m+1, \ldots (n-1)$ and multiply, we get

$\frac{x_{m+1}}{x_m} \times \frac{x_{m+2}}{x_{m+1}} \times \cdots \times \frac{x_n}{x_{n-1}} > K \cdot K \cdots K$ (n-m) factor.

$\Rightarrow \frac{x^n}{x^m} > K^{n-m}$

$\Rightarrow x^n > \frac{x^m}{K^m} \cdot K^n$

since $K > 1$, $\lim_{n\to\infty} K^n = \infty$

$\Rightarrow \lim_{n\to\infty} x_n = \infty$.

$\Rightarrow \langle x_n \rangle$ does not convergence

(c) $\lim \frac{x_{n+1}}{x_n} = L = 1$

Take $x_n = 1$

i.e. $x_n = \langle 1, 1, 1, \ldots \rangle$ i.e. a constant sequence.

Now $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$.

And clearly $\lim_{n\to\infty} x_n = 1$ i.e. convergent

Take $x_n = n$.

$\Rightarrow \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+1}{n} = 1$

but $\lim_{n\to\infty} x_n = \lim_{n\to\infty} n = \infty$

i.e. $\langle x_n \rangle$ is not convergent.