MATH 2031 Introduction to Real Analysis

September 28, 2012

Tutorial Note 3

Functions

(I) **Definition:**
A function f from a set A to a set B(denoted by $f : A \to B$) is an assignment of every $a \in A$ to **exactly ONE** $b \in B$ which we denote by $f(a)$, the value of f at a. A function f is well-defined if $a = a'$, then $f(a) = f(a')$.
The range or image of f is $f(A) = \{f(x) : x \in A\}$

(II) Identity function
The identity function on a set S is $I_S : S \to S$ given by $I(x) = x$ for all $x \in S$.

(III) Composition
Let $f : A \to B$, $g : B' \to C$ be functions and $f(A) \subseteq B'$.
The composition $g \circ f : A \to C$ given by $g \circ f(x) = g(f(x))$.

(IV) Restriction
Let $f : A \to B$ be a function and $C \subseteq A$.
The restriction of f to C is $f|_C : C \to B$ given by $f|_C(x) = f(x)$ for all $x \in C$

(V) Surjective (onto) function
A function $f : A \to B$ is surjective iff $f(A) = B$.

(VI) Injective (one-to-one or 1-1) function
A function $f : A \to B$ is injective iff $f(x) = f(y) \Rightarrow x = y$.
We may also check its contrapositive $x \neq y \Rightarrow f(x) \neq f(y)$.

(VII) Inverse function
For an injective function $f : A \to B$,
the inverse function of f is $f^{-1} : f(A) \to A$ given by $f^{-1}(y) = x \Leftrightarrow f(x) = y$.

(VIII) Bijective function
A function f is bijective iff f is injective and surjective.

Problem 1 Determine whether the following functions are injective, surjective or bijective.

(i) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$

(ii) $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x$

(iii) $\chi_{[0,1]} : \mathbb{R} \to \mathbb{R}$ given by $\chi_{[0,1]} = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{if } x \notin [0,1] \end{cases}$

Solution:

(i) Check injective:
From calculus, we know that $f(x) = e^x$ is strictly increasing, i.e. $x < y \Rightarrow f(x) < f(y)$,thus injective.
Check Surjective:
Since $f(x) = e^x$ is always positive, i.e $f(\mathbb{R}) = (0, +\infty) \subset \mathbb{R}$, it's not surjective.
Check Bijective :
Since f is only injective but not surjective, it's not a bijection.

(ii) Check injective:
By definition, $g(x) = g(y) \Rightarrow x = y$, so injective.
Check Surjective:
For every y in the codomain $\mathbb{R}$, we see $y \in \mathbb{R}$ in the domain and $g(y) = y$, so g is surjective.
Check Bijective :
Since g is both injective and surjective, it's a bijection.

(iii) Check injective:
Since there exist 2 distinct elements, namely 0.3 and 0.4, with the same function value, $\chi_{[0,1]}(0.3) = 1 = \chi_{[0,1]}(0.4)$ as both belongs to $[0, 1]$, it's not injective. Check Surjective:
Since the range of $\chi_{[0,1]}$ has only 2 elements $\{0, 1\} \subset \mathbb{R}$, it's not surjective.
Check Bijective :
Since $\chi_{[0,1]}$ is neither injective nor sujective, it could not be a bijection.

Problem 2 Let $f(x) = \tan(x)$

(i) If f defined above is a function $f : \mathbb{R} \setminus \{\frac{(2n+1)\pi}{2} : n \in \mathbb{Z}\} \to \mathbb{R}$, then is f injective, surjective or bijective?

(ii) What if we changed the domain of f to $(-\frac{\pi}{2}, \frac{\pi}{2})$ is f injective, surjective or bijective?

Solution:

(i) As we know that $\tan(x)$ is a periodic function with period π, i.e $\tan(x) = \tan(x+\pi)$, $f(x) = \tan(x)$ is not injective.
For every $y \in \mathbb{R}$, there exist $x = \arctan(y)$ and $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $f(x) = \tan(x) = y$, so it's surjective.
Since f is only surjective but not injective, it's not a bijection.

(ii) From calculus, $f'(x) = \dfrac{d}{dx}\tan(x) = \sec^2(x) > 0$, so $f(x) = \tan(x)$ is strictly increasing,thus injective.
By the same argument in part (i),it's surjective.
Now, $f(x) = \tan(x) = y$ is both injective and surjective, it's a bijection.

Remark:
From the above, we could see that the domain, also the codmain, could affect how the function behaves, like injectivity, surjectivity, bijectivity, positivity, etc.

Problem 3

(i) Let $f : A \to B$ be a function. Show that there is a function $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$ iff f is bijective.

(ii) Show that if $f : A \to B$ and $h : B \to C$ are bijections, then $h \circ f : A \to C$ is a bijection.

Solution:

(i) "$\Leftarrow$" f is surjective implies $f(A) = B$, i.e. for each $b \in B$ there exist an $a \in A$ such that $f(a) = b$. Since f is injective, such a is unique. then we define $g : B \to A$ by $g(b) = a$ for which $f(a) = b$.
From the definition of g, we check to get $g \circ f = I_A$ and $f \circ g = I_B$.
"$\Rightarrow$" direction, we will prove it by contradiction.
Assume f is not injective, i.e there exist 2 distinct elements in A, say $x, y \in A$ such that $x \neq y$, and $f(x) = f(y)$.
Then,$x = I_A(x) = g \circ f(x) = g(f(x)) = g(f(y)) = g \circ f(y) = I_A(y) = y$, but $x \neq y$, contradiction.
Hence f is injective.
Assume f is not surjective, i.e there exist an element $z \in B \setminus f(A)$.
However, $z = I_B(z) = f(g(z))$, which means that $z \in f(A)$ contradiction.
Hence f is surjective.

Combining the above we get that f is a bijection.

(ii) Surjective: For all $z \in C$, there is a $y \in B$ such that $h(y) = z$ as h is surjective. For such y, there is a $x \in A$ such that $f(x) = y$as f is surjective. Then For each $z \in C$, we have a $x \in A$, such that $h \circ f(x) = h(f(x)) = h(y) = z$.
Thus $h \circ f$ is surjective.

Injective: For any $x, y \in A$ and $x \neq y$.
Assume $h \circ f(x) = h \circ f(y)$, then $f(x) = f(y)$ since h is injective. Also by the injectivity of f, $f(x) = f(y) \Rightarrow x = y$, contradiction.
Hence $h \circ f$ is injective.

Problem 4 Constrict a bijection between $\mathbb{N}$ and positive even number denoted as $A = \{a : a \text{ is a positive even number}\}$.

Solution:
Let $f : \mathbb{N} \to A$ be a function given by $f(n) = 2n$ for all $n \in \mathbb{N}$.Then from problem 3(i), if we can find the inverse function of f, then f is bijective. Clearly then $f^{-1} : A \to \mathbb{N}$ given by $f^{-1}(a) = \dfrac{a}{c}$ is the required inverse function. Thus, f is bijective.

Remark:
From this problem, we can see that even the one set "looks" much smalled then another set, there may exist a bijective between them, which telling us that they have the same cardinality.

Problem 5 Construct a bijection between the following pair of set

(i) $(0, 1)$ and $\mathbb{R}$

(ii) $[0, 1)$ and $(0, 1]$

(iii) $(0, 1)$ and $(0, 1]$

(iv) $[0, 1]$ and $[0, 1)$

Solution:

(i) According to Problem 3(ii), we can see the composition of bijections is again a bijection, so sometimes we could construct a bijection between 2 sets step by step if it's hard to work it out directly.
For this part, I would construct it from the following steps
$(0,1) \xrightarrow{\quad f \quad} (0,\pi) \xrightarrow{\quad g \quad} (-\frac{\pi}{2}, \frac{\pi}{2}) \xrightarrow{\quad h \quad} \mathbb{R}$
Define $f : (0, 1) \to (0, \pi)$ is a function given by $f(x) = \pi x$.
We can see that f has an inverse $f^{-1} : (0, \pi) \to (0, 1)$ given by $f^{-1}(a) = \dfrac{a}{\pi}$. So f is a bijection.

Define $g : (0, \pi) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ is a function given by $g(y) = y - \frac{\pi}{2}$.
Clearly that $g^{-1} : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (0, \pi)$ given by $g^{-1}(b) = b + \frac{\pi}{2}$ is the inverse of g, thus g is a bijection.

Then from problem 2(ii),define $h : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ to be the function given by $h(z) = \tan(z)$ is a bijection.

Thus, define $f_1 : (0, 1) \to \mathbb{R}$ given by $f_1(x) = h \circ g \circ f(x)$ is a bijection.

(ii) Define $f_2 : [0, 1) \to (0, 1]$ is a function given by $f_2(x) = 1-x$ then by direct checking we can see that $f_2^{-1} = 1-x$, so f_2 is bijection.

(iii) Here we will construct a bijection, by splitting the domain and codomain into 2 parts and map them separately.
First, split the domain $(0, 1)$ into $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} = \{\frac{1}{n+1} : n \in \mathbb{N}\}$ and $(0, 1) \setminus \{\frac{1}{n+1} : n \in \mathbb{N}\}$.
For the codomain $(0, 1]$ we split it into $\{1, \frac{1}{2}, \frac{1}{3}, , \cdots\} = \{\frac{1}{n+1} : n \in \mathbb{N}\}$ and $(0, 1] \setminus \{\frac{1}{n} : n \in \mathbb{N}\}$.
Then we come up with the function $f_3 : (0, 1) \to (0, 1]$ given as follow,

$$f_3(x) = \begin{cases} \frac{x}{1-x} & \text{if } x \in \{\frac{1}{n+1} : n \in \mathbb{N}\}; \\ x & \text{if } x \notin \{\frac{1}{n+1} : n \in \mathbb{N}\} \end{cases}$$

So $f_3\left(\dfrac{1}{n+1}\right) = \dfrac{\left(\frac{1}{n+1}\right)}{1-\left(\frac{1}{n+1}\right)} = \left(\dfrac{1}{n+1}\right)\left(\dfrac{n+1}{n}\right) = \dfrac{1}{n}$.
One can directly check it's a bijection.

(iv) Here we use the same trick as above then $[0, 1] = \{0\} \cup (0, 1]$ and $[0, 1) = \{0\} \cup (0, 1)$.
Then we define $f_4 : [0, 1] \to [0, 1)$ given by $f_4(x) = \begin{cases} 0 & \text{if } x = 0; \\ f_3^{-1}(x) & \text{if } x \in (0, 1] \end{cases}$
Then as f_3 is bijective then so is f_3^{-1}, therefore f_4 is a bijection.

Remark:
As mention above, composition of bijections is also a bijection, then by composing the suitable bijections, we can see the all 5 sets $(0, 1), (0, 1], [0, 1), [0, 1]$ and $\mathbb{R}$ are all of the same cardinality.

Problem 6 Let $A = \{a_i : i \in \mathbb{N}\}$,$B = \{a_i : j \in \mathbb{N}\}$ and $C = \{c_k : k \in \mathbb{N}\}$ with all a_i, b_j and c_k are distinct. Constract a bijection between A and $B \cup C$.

Solution:
We can define $f : A \to B \cup C$ as follow:

$$f(a_i) = \begin{cases} b_{\frac{i+1}{2}} & \text{if } i \text{ is odd number} \\ c_{\frac{i}{2}} & \text{if } i \text{ is even number} \end{cases}.$$

We can check $g : B \cup C \to A$ defined by

$$g(x) = \begin{cases} a_{2n-1} & \text{if } x = b_n \in B \\ a_{2n} & \text{if } x = c_n \in C \end{cases}$$

is f^{-1} by showing $g \circ f = I_A$ and $f \circ g = I_{B \cup C}$.

Remark:
With problem 6, we can see that even the set of all odd numbers and the set of all even numbers are proper subsets of $\mathbb{Z}$, but there exist a bijection between the set of all odd numbers and Z and a bijection between the set of all even numbers and Z.
Thus, they all are of the same cardinality.