Math 2033 Solutions of Final Exam (Version 2) Spring 2018

① **Sketch** $|y_n - y_m| \le |x_n - x_m| + \left|\frac{\sqrt{2}}{x_n} - \frac{\sqrt{2}}{x_m}\right| = |x_n - x_m| + \sqrt{2}\left|\frac{x_m - x_n}{x_m x_n}\right|$
$\le |x_n - x_m| + \sqrt{2}\,\frac{|x_m - x_n|}{\sqrt{2}\sqrt{2}} = \left(1 + \frac{1}{\sqrt{2}}\right)|x_m - x_n| < \varepsilon$

Solution For every $\varepsilon > 0$, since $\{x_n\}$ is Cauchy, there exist $K \in \mathbb{N}$ such that $m, n \ge K \Rightarrow |x_m - x_n| < \varepsilon / (1 + \frac{1}{\sqrt{2}})$. Then $|y_n - y_m| < \varepsilon$ as showed above.

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② **Sketch** $|\cos(\sin \pi\sqrt{x}) + \sqrt{25 - x^2} - 4| \le |\cos(\sin\pi\sqrt{x}) - \underset{\cos(\sin 2\pi)}{1}| + |\sqrt{25-x^2} - \underset{\sqrt{9}}{3}|$
$\le |\sin\pi\sqrt{x} - \sin 2\pi| + \sqrt{|25 - x^2 - 9|} \le |\pi\sqrt{x} - 2\pi| + \sqrt{|16 - x^2|} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
due to $|\pi\sqrt{x} - 2\pi| = \pi|\sqrt{x} - \sqrt{4}| \le \pi\sqrt{|x-4|}$ $\le \varepsilon/2$ if $|x-4| < \left(\frac{\varepsilon}{2\pi}\right)^2$ and $\sqrt{|16-x^2|} = \sqrt{|4+x|}\sqrt{|x-4|}$ $\le \sqrt{5}\sqrt{|x-4|} < \frac{\varepsilon}{2}$ for $x \in (3,5)$, $|x-4| < \frac{\varepsilon^2}{20}$

Solution For every $\varepsilon > 0$, let $\delta = \min\left\{\left(\frac{\varepsilon}{2\pi}\right)^2, 1, \frac{\varepsilon^2}{20}\right\}$, then $|x - 4| < \delta \Rightarrow |x-4| < \left(\frac{\varepsilon}{2\pi}\right)^2$, $\underbrace{|x-4| < 1}_{\Leftrightarrow x \in (3,5)}$, $|x-4| < \frac{\varepsilon^2}{20}$
$\Rightarrow |\cos(\sin\pi\sqrt{x}) + \sqrt{25 - x^2} - 4| < \varepsilon$ as showed above.

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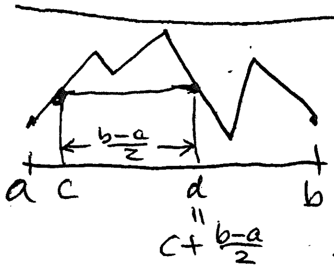
③ 

Define $H: \mathbb{R} \to \mathbb{R}$ by $H(x) = h(x + \frac{b-a}{2}) - h(x)$. We need $c \in [a,b]$ satisfies $H(c) = 0$ ($\Leftrightarrow h(c + \frac{b-a}{2}) = h(c)$). We try $x = a$ and get $H(a) = h(a + \frac{b-a}{2}) - h(a) = h(\frac{a+b}{2}) - h(a)$. Then we try $x = \frac{a+b}{2}$ and get $H(\frac{a+b}{2}) = \underset{= h(a)}{h(b)} - h(\frac{a+b}{2}) = -H(a)$

So $H(a)$ and $-H(a)$ are in the range of H. By the intermediate value theorem, there exists $c \in [a, \frac{a+b}{2}]$ such that $H(c) = 0$ ($\Leftrightarrow h(c) = h(c + \frac{b-a}{2}) = h(d)$).

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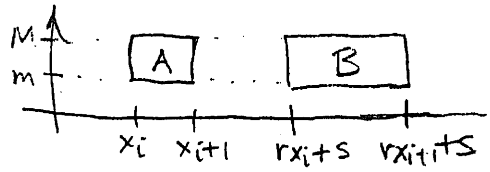
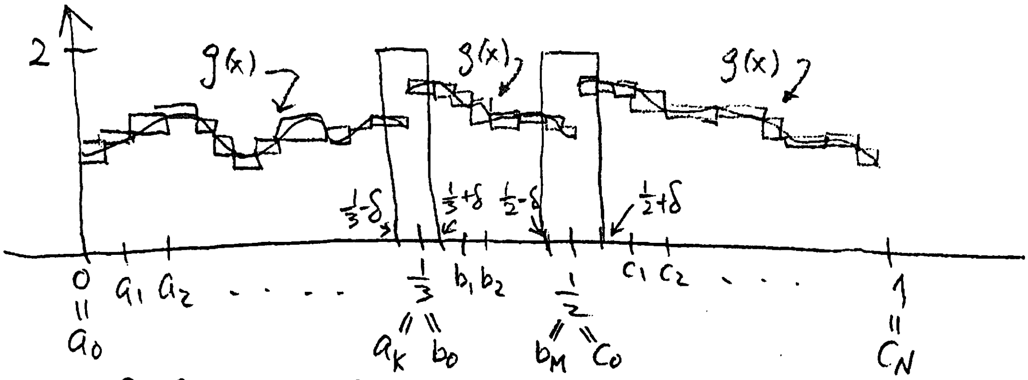
④ Observe that $7F'(r)/F(r) - 4F'(1-r)/F(1-r)$ can be integrated to $7\ln F(r) + 4\ln F(1-r) = \ln F(r)^7 F(1-r)^4$. We can consider this function, Better yet, we can consider $G(x) = F(x)^7 F(1-x)^4$. Then $G(0) = F(0)^7 F(1-0)^4 = 0$ and $G(1) = F(1)^7 F(1-1)^4 = 0$. By mean value theorem, there exist $\theta \in (0,1)$ such that $0 = G(1) - G(0) = G'(\theta) = 7F(\theta)^6 F'(\theta) F(1-\theta)^4 - 4F(\theta) F(1-\theta)^3 F'(1-\theta)$. Since $F(\theta)$ and $F(1-\theta) > 0$, Cancelling $F(\theta)^7 F(1-\theta)^4$, we get $0 = 7\frac{F'(\theta)}{F(\theta)} - 4\frac{F'(1-\theta)}{F(1-\theta)}$. We can let $r = \theta$ and $s = 1 - r = 1 - \theta$. Then $7F'(r)/F(r) = 4F'(s)/F(s)$.

⑤ Observe that for $r>0$



$$\underbrace{\text{Area } A = \frac{1}{r}\text{Area } B}_{(*)}$$



$Q_1=\{a_0,a_1,\cdots,a_K\}$ $Q_2=\{b_0,b_1,\cdots,b_M\}$ $Q_3=\{c_0,c_1,\cdots,c_N\}$

Let P_1 be a partition of $[0,\frac{2}{3}]$, where $U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{5}$.
Then $Q_1=\{\frac{1}{2}x : x\in P_1\}$ is a partition of $[0,\frac{1}{3}]$ and $U(f_1,Q_1)-L(f_1,Q_1)$ $\overset{\text{by }(*)}{=}\frac{1}{2}(U(f,P_1)-L(f,P_1))<\frac{\varepsilon}{10}$, where $f_1(x)=f(2x)$ for $x\in[0,\frac{1}{3}]$.

Let P_2 be a partition of $[\frac{1}{3},\frac{1}{2}]$, where $U(f,P_2)-L(f,P_2)<\frac{\varepsilon}{5}$.
Then $Q_2=\{\frac{y+1}{4} : y\in P_2\}$ is a partition of $[\frac{1}{3},\frac{1}{2}]$ and we have $U(f_2,Q_2)-L(f_2,Q_2)\overset{\text{by }(*)}{=}\frac{1}{4}(U(f,P_2)-L(f,P_2))<\frac{\varepsilon}{20}$, where $f_2(x)=f(4x-1)$ for $x\in[\frac{1}{3},\frac{1}{2}]$.

Let P_3 be a partition of $[\frac{1}{2},1]$, where $U(f,P_3)-L(f,P_3)<\frac{\varepsilon}{5}$.
Then $Q_3=\{z : z\in P_3\}$ is a partition of $[\frac{1}{2},1]$ and we have $U(f_3,Q_3)-L(f_3,Q_3)\overset{\text{by }(*)}{=}U(f,P_3)-L(f,P_3)<\frac{\varepsilon}{5}$, where $f_3(x)=f(x-0.25)$ for $x\in[\frac{1}{2},1]$.

(In Q, $\frac{1}{3}-\delta,\frac{1}{3}+\delta$ are closest to $\frac{1}{3}$; $\frac{1}{2}-\delta,\frac{1}{2}+\delta$ are closest to $\frac{1}{2}$.)

Let $Q=Q_1\cup Q_2\cup Q_3\cup\{\frac{1}{3}-\delta,\frac{1}{3}+\delta,\frac{1}{2}-\delta,\frac{1}{2}+\delta\}$, where $\delta<\frac{13}{160}\varepsilon$.
Then $U(g,Q)-L(g,Q)<\sum_{i=1}^{3}(U(f_i,Q_i)-L(f_i,Q_i))+2\delta\times2+2\delta\times2$

$<\frac{\varepsilon}{10}+\frac{\varepsilon}{20}+\frac{\varepsilon}{5}+8\delta<\varepsilon.$