Chapter 10 Sequences and Series of Functions

$[-\infty,+\infty]=\mathbb{R}\cup\{-\infty,+\infty\}$ is the extended real number system.

Ordering in $[-\infty,+\infty]$

① usual ordering among real numbers

② $\forall x\in\mathbb{R}$, $-\infty<x<+\infty$

Supremum and infimum in $[-\infty,+\infty]$

① $\forall$ nonempty $S\subseteq[-\infty,\infty]$,

$+\infty$ is an upper bound of S

$-\infty$ is a lower bound of S

② $\sup S$ = least upper bound of S in $[-\infty,+\infty]$

$\inf S$ = greatest lower bound of S in $[-\infty,+\infty]$

Remark For nonempty subset S of $[-\infty,+\infty]$, $\sup S$ and $\inf S$ always exist in $[-\infty,+\infty]$.

Arithmetics in $[-\infty,+\infty]$ usual arithmetic on $\mathbb{R}$

plus ① $\forall x\in\mathbb{R}\cup\{+\infty\}$ and $c>0$,

$$x+(+\infty)=+\infty=(+\infty)+x$$
$$c(+\infty)=+\infty=(+\infty)c$$
$$c(-\infty)=-\infty=(-\infty)c$$

② $\forall x\in\mathbb{R}\cup\{-\infty\}$ and $c<0$,

$$x+(-\infty)=-\infty=(-\infty)+x$$
$$c(-\infty)=+\infty=(-\infty)c$$
$$c(+\infty)=-\infty=(+\infty)c$$

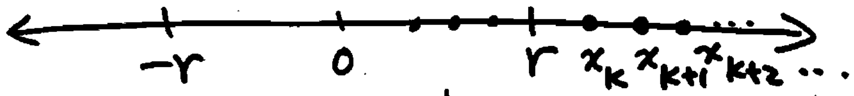
③ $|+\infty|=+\infty=|-\infty|$

Infinite Limits

Let $x_1,x_2,x_3,\cdots$ be a sequence in $[-\infty,+\infty]$. Define

$\lim_{n\to\infty}x_n=+\infty$ iff $\forall$ real $r>0$, $\exists K\in\mathbb{N}$ such that $n\geq K\Rightarrow x_n>r$.

$\lim_{n\to\infty}x_n=-\infty$ iff $\forall$ real $r>0$, $\exists K\in\mathbb{N}$ such that $n\geq K\Rightarrow x_n<-r$.



$\lim_{n\to\infty}x_n=+\infty$

Subsequences

Let $x_1,x_2,x_3,\ldots$ be a sequence. $x_{n_1},x_{n_2},x_{n_3},\cdots$ is a subsequence of $x_1,x_2,\cdots$ iff $n_1<n_2<n_3<\cdots$

Examples
$x_3,x_6,x_9,x_{12},\cdots$ $(n_i=3i)$
$x_1,x_4,x_9,x_{16},\cdots$ $(n_i=i^2)$
$x_2,x_3,x_5,x_7,\cdots$ $(n_i=i^{th}$ prime$)$

are subsequences of $x_1,x_2,x_3,x_4,\ldots$

Set of Subsequential Limits of $x_1,x_2,x_3,\cdots$

$$\mathcal{L}=\{z\in[-\infty,+\infty]:\exists\text{ subsequence }x_{n_1},x_{n_2},\cdots\text{ such that }\lim_{i\to\infty}x_{n_i}=z\}$$

Such z's are called subsequential limits of $x_1,x_2,x_3,\cdots$

If $\{x_1, x_2, x_3, \cdots\}$ is bounded in $\mathbb{R}$, then it has a convergent subsequence by the Bolzano-Weierstrass theorem.

If $\{x_1, x_2, x_3, \cdots\}$ is not bounded above in $\mathbb{R}$, then some subsequence will have limit $+\infty$.

If $\{x_1, x_2, x_3, \cdots\}$ is not bounded below in $\mathbb{R}$, then some subsequence will have limit $-\infty$.

Therefore, $\mathcal{L} \neq \phi$ and so $\sup \mathcal{L}$, $\inf \mathcal{L}$ exist in $[-\infty, +\infty]$.

<u>Definitions</u>

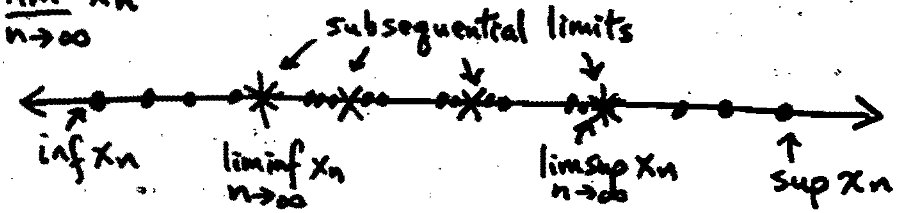
$$\overline{\lim_{n\to\infty}}\, x_n = \limsup_{n\to\infty} x_n = \sup \mathcal{L} \quad \text{in } [-\infty, +\infty]$$

← limit superior or upper limit of $x_1, x_2, x_3, \cdots$

$$\liminf_{n\to\infty} x_n = \inf \mathcal{L} \quad \text{in } [-\infty, +\infty]$$

$$\| \qquad \underline{\lim}_{n\to\infty}\, x_n$$

← limit inferior or lower limit of $x_1, x_2, x_3, \cdots$



<u>Remarks</u>

① $\forall$ sequence $x_1, x_2, x_3, \cdots$, $\limsup_{n\to\infty} x_n$, $\liminf_{n\to\infty} x_n$ always exist in $[-\infty, +\infty]$ since $\mathcal{L} \neq \phi$.

$$\liminf_{n\to\infty} x_n = \inf \mathcal{L} \leq \sup \mathcal{L} = \limsup_{n\to\infty} x_n$$

② If $\{x_1, x_2, x_3, \cdots\}$ is not bounded above in $\mathbb{R}$, then $+\infty \in \mathcal{L}$. So $\limsup_{n\to\infty} x_n = \sup \mathcal{L} = +\infty$.

If $\{x_1, x_2, x_3, \cdots\}$ is not bounded below in $\mathbb{R}$, then $-\infty \in \mathcal{L}$. So $\liminf_{n\to\infty} x_n = \inf \mathcal{L} = -\infty$.

③ $\lim_{n\to\infty} x_n = z \in [-\infty, +\infty] \iff \mathcal{L} = \{z\}$

$$\iff \limsup_{n\to\infty} x_n = \sup \mathcal{L} = z = \inf \mathcal{L} = \liminf_{n\to\infty} x_n.$$

④ $\limsup_{n\to\infty} (-x_n) = \sup(-\mathcal{L}) = -\inf \mathcal{L} = -\liminf_{n\to\infty} x_n$.

$\forall c > 0$,

$$\limsup_{n\to\infty} (c x_n) = \sup(c\mathcal{L}) = c \sup \mathcal{L} = c \limsup_{n\to\infty} x_n$$

$$\liminf_{n\to\infty} (c x_n) = \inf(c\mathcal{L}) = c \inf \mathcal{L} = c \liminf_{n\to\infty} x_n$$

$\forall c \in \mathbb{R}$

$$\limsup_{n\to\infty} (c + x_n) = \sup(c + \mathcal{L}) = c + \sup \mathcal{L} = c + \limsup_{n\to\infty} x_n$$

$$\liminf_{n\to\infty} (c + x_n) = \inf(c + \mathcal{L}) = c + \inf \mathcal{L} = c + \liminf_{n\to\infty} x_n$$

<u>M_k-Theorem</u> For sequence $x_1, x_2, x_3, \cdots$, define

$$M_k = \sup\{x_k, x_{k+1}, x_{k+2}, \cdots\}.$$

Then ① $M_1 \ge M_2 \ge M_3 \ge \cdots$

and ② $\limsup_{n\to\infty} x_n = \lim_{k\to\infty} M_k \in \mathcal{L}$.

<u>Proof</u> ① $M_k \ge x_{k+1}, x_{k+2}, x_{k+3}, \cdots$

$\forall k = 1,2,3,\cdots \quad \Rightarrow M_k \ge \sup\{x_{k+1}, x_{k+2}, x_{k+3}, \cdots\} = M_{k+1}$

② M_k monotone $\Rightarrow \lim_{k\to\infty} M_k = M$ exists in $[-\infty, +\infty]$.

$\forall z \in \mathcal{L}$, $z = \lim_{j\to\infty} x_{n_j} \le \lim_{j\to\infty} M_{n_j} = M \Rightarrow \underline{\sup \mathcal{L} \le M}.$

Conversely, if $M_1 = \sup\{x_1, x_2, x_3, \cdots\} = +\infty$, then $\{x_1, x_2, x_3, \cdots\}$ is not bounded above, $\sup \mathcal{L} = +\infty = M$.
Otherwise, $M_1 < +\infty$. By supremum property, $\exists\, n_1 \ge 1$ such that $M_1 - 1 < x_{n_1} \le M_1$. Consider

$$M_{1+n_1} = \sup\{x_{1+n_1}, x_{2+n_1}, x_{3+n_1}, \cdots\}$$
$$= \sup\{x_j : j > n_1\}.$$

By supremum property, $\exists\, n_2 > n_1$ with $M_{1+n_1} - \frac{1}{2} < x_{n_2} \le M_{1+n_1}$
Keep repeating this. $\exists\, n_{k+1} > n_k$ such that

$$M_{1+n_k} - \frac{1}{k+1} < x_{n_{k+1}} \le M_{1+n_k}.$$

Sandwich theorem $\Rightarrow M = \lim_{k\to\infty} x_{n_k} \in \mathcal{L} \Rightarrow \underline{M \le \sup \mathcal{L}}$. $\therefore M = \sup \mathcal{L}$

Similarly, there is

<u>m_k-Theorem</u> For sequence $x_1, x_2, x_3, \cdots$, define

$$m_k = \inf\{x_k, x_{k+1}, x_{k+2}, \cdots\}.$$

Then ① $m_1 \le m_2 \le m_3 \le \cdots$

and ② $\liminf_{n\to\infty} x_n = \lim_{k\to\infty} m_k \in \mathcal{L}$.

<u>Corollary</u> If $x_n \le y_n \ \forall n \in \mathbb{N}$, then

(a) $\limsup_{n\to\infty} x_n \le \limsup_{n\to\infty} y_n$ and (b) $\liminf_{n\to\infty} x_n \le \liminf_{n\to\infty} y_n$.

<u>Proof</u>. M_k for $x_n \le M_k$ for $y_n \Rightarrow$ (a) by M_k-thm

m_k for $x_n \le m_k$ for $y_n \Rightarrow$ (b) by m_k-thm.

<u>Examples</u> ① $\lim_{n\to\infty} \frac{1+n}{3+4n} = \frac{1}{4} \Rightarrow \limsup_{n\to\infty} \frac{1+n}{3+4n} = \frac{1}{4}$

by remark ③ $\quad \liminf_{n\to\infty} \frac{1+n}{3+4n} = \frac{1}{4}$

② Let $x_n = e^{(-1)^n n} = \begin{cases} e^{-n} & \text{if } n \text{ is odd} \\ e^{n} & \text{if } n \text{ is even} \end{cases}$

We have $0 < x_n < +\infty \Rightarrow \mathcal{L} \subseteq [0, +\infty]$

$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} e^{-(2n+1)} = 0 \in \mathcal{L} \Rightarrow \liminf_{n\to\infty} x_n = 0$

$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} e^{2n} = +\infty \in \mathcal{L} \Rightarrow \limsup_{n\to\infty} x_n = +\infty$.

<u>Alternative Solution</u>

$$e^{-1}, e^{2}, e^{-3}, e^{4}, e^{-5}, e^{6}, \cdots$$
$$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6$$

k odd $\Rightarrow x_k = e^{-k} \Rightarrow M_k = \sup\{e^{-k}, e^{k+1}, e^{-(k+2)}, \cdots\} = +\infty$

k even $\Rightarrow x_k = e^{k} \Rightarrow M_k = \sup\{e^{k}, e^{-(k+1)}, e^{k+2}, \cdots\} = +\infty$

$\therefore \ \forall k,\ M_k = +\infty \Rightarrow \limsup\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} M_k = +\infty$

Similarly, k odd $\Rightarrow m_k = 0$, k even $\Rightarrow m_k = 0 \Rightarrow \liminf\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} m_k = 0$.

③ Let $x_n = \begin{cases} -(3+\frac{1}{j+1}) & \text{if } n = 3j+1 \\ 0 & \text{if } n = 3j+2 \\ \frac{1}{j+1} & \text{if } n = 3j+3 \end{cases}$.

$$-4, 0, 1, -3\tfrac{1}{2}, 0, \tfrac{1}{2}, -3\tfrac{1}{3}, 0, \tfrac{1}{3}, -3\tfrac{1}{4}, \cdots$$
$$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9 \quad x_{10}$$

k	1	2	3	4	5	6	7	$\cdots$
M_k	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\cdots \frac{1}{j+1}$
m_k	-4	$-3\frac{1}{2}$	$-3\frac{1}{2}$	$-3\frac{1}{2}$	$-3\frac{1}{3}$	$-3\frac{1}{3}$	$-3\frac{1}{3}$	$\cdots -(3+\frac{1}{j+1})$

$\limsup\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} M_k = 0$, $\liminf\limits_{n\to\infty} x_n = \lim\limits_{k\to\infty} m_k = -3$

Many theorems require limit conditions. If limits do not exist, the theorems cannot be applied. However, some theorems have strong forms, where limits are replaced by limsups or liminfs.

<u>Strong Form of Root Test</u> For a real sequence $a_1, a_2, \cdots$,

① $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} < 1 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ converges absolutely;

② $\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|} > 1 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ diverges.

<u>Strong Form of Ratio Test</u> For a real sequence $a_1, a_2, \cdots$ and $a_n \neq 0\ \forall n$,

① $\limsup\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ converges absolutely;

② $\liminf\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow \sum\limits_{n=1}^{\infty} a_n$ diverges.

Their proofs require the following.

<u>Theorem</u> For a real sequence $a_1, a_2, a_3, \cdots$ and $a_n \neq 0\ \forall n$,

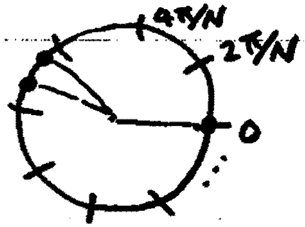
$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \underset{①}{\le} \liminf_{n\to\infty} \sqrt[n]{|a_n|} \underset{②}{\le} \limsup_{n\to\infty} \sqrt[n]{|a_n|} \underset{③}{\le} \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

(If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L$.
<u>Converse is false</u>, see practice exercise $\boxed{32}$.)

Examples (4) Find the set $\mathcal{L}$ of all subsequential limits of $x_n = \sin n$. Find $\liminf\limits_{n\to\infty} \sin n$ and $\limsup\limits_{n\to\infty} \sin n$.

Solution. We need an observation first.

Observation $\forall \varepsilon > 0$, take integer $N > \frac{2\pi}{\varepsilon}$. Divide the unit circle into N disjoint arcs of equal lengths.



For every positive integer m, Consider the $N+1$ marks on the unit circle corresponding to the angles $0, m, 2m, \cdots, Nm$ (in radians)

In one of the arcs, there will be at least 2 marks.
So there are $i<j$ among $0, 1, 2, \cdots, N$ such that

$$n = jm - im = 2\pi k + c^{\neq 0}, \text{ where } k\in\mathbb{Z} \text{ and } |c| < \frac{2\pi}{N} < \varepsilon.$$

Then $\exists\, n = (j-i)m \geq 1m = m$ and $|\sin n| = |\sin c| \leq |c| < \varepsilon$.

Next we will show $\mathcal{L} = [-1, 1]$ so that $\liminf\limits_{n\to\infty} \sin n = -1$ and $\limsup\limits_{n\to\infty} x_n = 1$.

Clearly, $\mathcal{L} \subseteq [-1,1]$ since $-1 \leq \sin x \leq 1$.

Conversely, for every $z = \sin\theta \in [-1,1]$, we will Construct strictly increasing integers $n_1, n_2, n_3, \cdots$ with $|\sin n_k - z| \leq \frac{2}{k}$ so that $\sin n_k \to z$.

Let $n_1 = 1$, then $|\sin 1 - z| \leq 2$. Assuming we have n_k satisfying $|\sin n_k - z| \leq \frac{2}{k}$, we will construct n_{k+1} with required properties.

Apply the observation with $\varepsilon = \frac{2}{k+1}$, $m = n_k + 1$. We get $n \geq m$ such that $|\sin n| < \varepsilon$ and n is of the form $2\pi k + c$ with $|c| < \varepsilon$.

Since interval $(\theta - \varepsilon, \theta + \varepsilon)$ has length 2ε, there is a positive integer t such that $tc \in (\theta-\varepsilon, \theta+\varepsilon)$

Let $n_{k+1} = tn$. Then $n_{k+1} = tn \geq n \geq m = n_k + 1 > n_k$ and $|tc - \theta| < \varepsilon$ implies

$$|\sin n_{k+1} - z| = |\sin tn - \sin\theta| \overset{tn = 2\pi k t + tc}{=} |\sin tc - \sin\theta|$$
$$\leq |tc - \theta| < \varepsilon = \frac{2}{k+1}.$$

$\therefore \sin n_k \to z$ and $\mathcal{L} = [-1, 1]$.

Remarks If $c > 0$, then θ should be taken to be positive. If $c < 0$, then θ should be taken to be negative. This is for ensuring $\exists\, tc \in (\theta - \varepsilon, \theta+\varepsilon)$.

(5) Does $\sum\limits_{n=1}^{\infty} \tan^n(\sin n)$ converge?

Solution. $\limsup\limits_{n\to\infty} \sqrt[n]{|\tan^n(\sin n)|} = \limsup\limits_{n\to\infty} |\tan(\sin n)|$ $= \tan 1 > \tan\frac{\pi}{4} = 1$. Series diverges by strong form of root test.

⑥ Does $1+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2^2 3}+\frac{1}{2^2 3^2}+\cdots$ Converge?

$\times\frac{1}{2}\quad \times\frac{1}{3}\quad \times\frac{1}{2}\quad \times\frac{1}{3}\quad \cdots$

Ratio test doesn't apply.

Solution. $\frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd} \\ \frac{1}{3} & \text{if } n \text{ is even} \end{cases}$ $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ doesn't exist.

The $\mathcal{L}$ set for $\frac{a_{n+1}}{a_n}$ is $\{\frac{1}{2}, \frac{1}{3}\}$. So $\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2} < 1$.
By strong form of ratio test, the series converges.

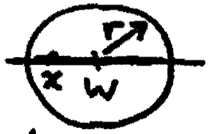
Limit Superior and Limit Inferior for Functions

Let $f: S \to \mathbb{R}$ be a function and w be an accumulation point of S. Define the sequential limit set

$$\mathcal{L} = \{ z : \exists\, x_n \in S,\ x_n \to w \text{ and } \lim_{n\to\infty} f(x_n) = z \}.$$

Either $+\infty \in \mathcal{L}$ or $-\infty \in \mathcal{L}$ or $\exists\, z \in \mathbb{R} \cap \mathcal{L}$ by Bolzano-Weierstrass theorem. $\therefore\ \mathcal{L} \neq \emptyset$ in $[-\infty, +\infty]$.

Definitions $\limsup\limits_{x\to w} f(x) = \sup \mathcal{L}$, $\liminf\limits_{x\to w} f(x) = \inf \mathcal{L}$.



M(r) Theorem For every $r > 0$, define

$$M(r) = \sup\{ f(x) : x \in S,\ 0 < |x-w| < r \}$$

Then ① $M(r)$ is an increasing function of r,
and ② $\limsup\limits_{x\to w} f(x) = \lim\limits_{r\to 0^+} M(r) \in \mathcal{L}$.

Remarks (a) Similarly, there is a $m(r)$ theorem for $\liminf\limits_{x\to w} f(x)$.
The proofs are similar to the M_K-theorem and m_K-theorem.
(b) There are similar remarks and corollary for the limit superior and inferior of functions as those for sequences.

Pointwise Convergence

Let E be a set. $\forall n \in \mathbb{N}$, let $S_n: E \to \mathbb{R}$ be a function.

Definitions Sequence $S_n: E \to \mathbb{R}$ converges pointwise on E to a function $S: E \to \mathbb{R}$ iff $\forall x \in E$, $\lim\limits_{n\to\infty} S_n(x) = S(x)$
(i.e. $\forall x \in E$, $\forall \varepsilon > 0$ $\exists K \in \mathbb{N}$ (K depends on x and ε) such that $n \geq K \Rightarrow |S_n(x) - S(x)| < \varepsilon$.)
In this case, $S(x)$ is the pointwise limit of $S_n(x)$.

Example Consider $S_n(x) = (1-x^2)^n$ on $E = [0,1]$.

$$\forall x \in [0,1],\ \lim_{n\to\infty} S_n(x) = \begin{cases} 0 & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x = 0 \end{cases}$$

$\therefore$ $S_n(x)$ converges pointwise on E to $S(x) = \begin{cases} 0 & 0 < x \leq 1 \\ 1 & x = 0. \end{cases}$

Definition For functions $f_k: E \to \mathbb{R}$, series $\sum\limits_{k=1}^{\infty} f_k(x)$ converges pointwise on E to a function $S: E \to \mathbb{R}$ iff $\forall x \in E$, $\sum\limits_{k=1}^{\infty} f_k(x) = \lim\limits_{n\to\infty} (f_1(x) + \cdots + f_n(x)) = S(x)$,
i.e. $\underbrace{S_n(x) = f_1(x) + \cdots + f_n(x)}_{\text{partial sum sequence}}$ converges pointwise on E to $S(x)$.

Example Consider $\sum\limits_{k=1}^{\infty} e^{kx} \cos x$ on $E_1 = (-\infty, 0)$ and on $E_2 = (-1, 1]$.

$$\sum_{k=1}^{\infty} e^{kx}\cos x = \cos x \underbrace{\sum_{k=1}^{\infty}(e^x)^k}_{\substack{\text{geometric series}\\ r=e^x}} = \begin{cases} \cos x \dfrac{e^x}{1-e^x} & \text{if } |e^x|<1 \\ \text{diverges} & \text{if } |e^x|\geq 1 \end{cases}$$

$|e^x| = e^x < 1 \iff x<0$

$|e^x| = e^x \geq 1 \iff x \geq 0$

$\therefore \sum_{k=1}^{\infty} e^{kx}\cos x$ converges pointwise on $E_1 = (-\infty, 0)$ to $\cos x \dfrac{e^x}{1-e^x}$, but it doesn't converge pointwise on $E_2 = (-1, 1]$.

Definitions A <u>power series</u> is a function of the form

$$a_0 + a_1(x-c) + a_2(x-c)^2 + \dots = \sum_{k=0}^{\infty} a_k (x-c)^k$$

where $c, a_0, a_1, a_2, \dots$ are numbers.

c is called the <u>center</u> of the power series.

$E = \{x : \sum_{k=0}^{\infty} a_k(x-c)^k \text{ converges}\}$ is the <u>domain</u> (of convergence) of the power series.

$\therefore$ power series converges pointwise on its domain.

Examples (see Taylor series theorem)

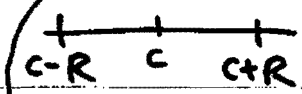
$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$ domain $= (-\infty, +\infty)$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ domain $= (-\infty, +\infty)$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{k=0}^{\infty} x^k$ domain $= (-1, 1)$

Domain Theorem for Power Series

The domain of a power series $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$ is a <u>nonempty</u> interval with c as midpoint.



The half-length of the interval is the <u>radius of convergence</u>

$f(c) = a_0 \Rightarrow c$ in domain

$$R = 1 \Big/ \limsup_{k\to\infty} \sqrt[k]{|a_k|}$$

Each endpoint of the interval may or may not be in the domain.

The power series converges absolutely on $(c-R, c+R)$.

<u>Proof</u>. By the strong form of root test,

$$\limsup_{k\to\infty} \sqrt[k]{|a_k(x-c)^k|} = |x-c| \limsup_{k\to\infty}\sqrt[k]{|a_k|} < 1$$

$$\iff |x-c| < 1 \Big/ \limsup_{k\to\infty}\sqrt[k]{|a_k|} = R$$

$$\iff x \in (c-R, c+R)$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k(x-c)^k \text{ converges absolutely on } (c-R, c+R).$$

$$\limsup_{k\to\infty} \sqrt[k]{|a_k(x-c)^k|} > 1 \iff |x-c| > R$$

$$\iff x \in \underbrace{(-\infty, c-R) \cup (c+R, +\infty)}_{= \mathbb{R} \setminus [c-R, c+R]}$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k(x-c)^k \text{ diverges on } \mathbb{R} \setminus [c-R, c+R].$$

It is easier to find domain by the ratio test.

Examples (1) Consider $f(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}$. $c=0$

$\lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)!}\frac{k!}{x^k}\right|=\lim_{k\to\infty}\frac{|x|}{k+1}=0<1 \Rightarrow \sum_{k=0}^{\infty}\frac{x^k}{k!}$ converge for all x.

Domain of $f(x)$ is $\mathbb{R}=(-\infty,+\infty)$, $R=+\infty$.

(2) Consider $f(x)=\sum_{k=0}^{\infty}k!(x-\pi)^k$. $c=\pi$

$\lim_{k\to\infty}\left|\frac{(k+1)!(x-\pi)^{k+1}}{k!(x-\pi)^k}\right|=\lim_{k\to\infty}(k+1)|x-\pi|=\begin{cases}0 & \text{if } x=\pi\\ \infty & \text{if } x\neq\pi\end{cases}$

$\Rightarrow$ Domain of $f(x)$ is $\{\pi\}=[\pi,\pi]$, $R=0$.

(3) Consider $f(x)=\sum_{k=1}^{\infty}\frac{(-1)^k(x-25)^k}{k}$. $c=25$

$\lim_{k\to\infty}\left|\frac{(-1)^{k+1}(x-25)^{k+1}}{k+1}\frac{k}{(-1)^k(x-25)^k}\right|=\lim_{k\to\infty}|x-25|\frac{k}{k+1}$

$=|x-25|<1 \Rightarrow 24<x<26$ series converges.

$>1 \Rightarrow x<24$ or $x>26$ series diverges.

At $x=24$, $\sum\frac{(-1)^k(24-25)^k}{k}=\sum\frac{1}{k}$ diverges by p-test

At $x=26$, $\sum\frac{(-1)^k(26-25)^k}{k}=\sum\frac{(-1)^k}{k}$ converges by alt. series test

$\therefore$ Domain of $f(x)$ is $(24,26]$, $R=1$.

Question How do we expand functions into series?

IF $f(x)=a_0+a_1(x-c)+a_2(x-c)^2+a_3(x-c)^3+\cdots$,

then formally

$f'(x)=a_1+2a_2(x-c)+3a_3(x-c)^2+4a_4(x-c)^3+\cdots$

$f''(x)=2a_2+6a_3(x-c)+12a_4(x-c)^2+\cdots$

$f'''(x)=6a_3+24a_4(x-c)+\cdots$,

so $f(c)=a_0$, $f'(c)=a_1$, $f''(c)=2a_2$, $f'''(c)=6a_3$

$a_0=f(c)$, $a_1=f'(c)$, $a_2=\frac{f''(c)}{2}$, $a_3=\frac{f'''(c)}{6}$,

$\ldots$, $a_n=\frac{f^{(n)}(c)}{n!}$, $\ldots$

Definition For a function $f(x)$ that is infinitely differentiable at c,

$a_0+a_1(x-c)+a_2(x-c)^2+\cdots$, where $a_n=\frac{f^{(n)}(c)}{n!}$

is the Taylor series of f about c.

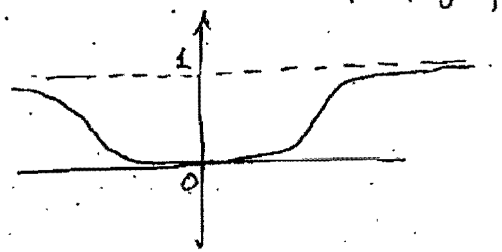
Note $f(x)$ equals its Taylor series when $x=c$ as $f(c)=a_0$

Example Let $f(x)=\begin{cases}e^{-1/x^2} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$. Then $f(0)=0$

$f^{(n)}(0)=0\ \forall n\in\mathbb{N} \Rightarrow$ Taylor series of f about 0 is the zero series $0+0x+0x^2+\cdots$
(Exercise)

$\Rightarrow f(x)=$ its Taylor series about 0 only when $x=0$.

Graph of $y = f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$



Question When does Taylor series converge pointwise on an interval to the function? $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ is the Taylor series of $f(x)$

Taylor Series Theorem

If $f:(a,b) \to \mathbb{R}$ is infinitely differentiable, $c \in (a,b)$ and $\exists$ constants $M, \alpha > 0$ such that $|f^{(n)}(x)| \leq \alpha M^n$ for every $x \in (a,b)$, $n \in \mathbb{N}$,

then $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ converges pointwise on (a,b) to $f(x)$.

Proof By Taylor's Theorem, $\forall x \in (a,b)$, $\exists \theta \in (a,b)$

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x), \text{ where}$$

$$R_n(x) = \frac{f^{(n)}(\theta)}{n!}(x-c)^n. \quad \text{Then } |R_n(x)| \leq \frac{\alpha M^n}{n!}|x-c|^n$$

$$\underset{y = M|x-c|}{=} \alpha \frac{y^n}{n!}$$

By example (1) on domain of power series, $\sum_{n=0}^{\infty} \frac{y^n}{n!}$ Converges. By term test, $\lim_{n\to\infty} \frac{y^n}{n!} = 0$.

$\therefore \lim_{n\to\infty} R_n(x) = 0$ by Sandwich theorem.

$$\therefore f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty}\left(\sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x)\right)$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \quad \forall x \in (a,b).$$

Examples (1) For $f(x)=\sin x$ on $\mathbb{R}=(-\infty,+\infty)$,

$$f^{(n)}(x)=\begin{cases}(-1)^k\cos x & \text{if } n=2k+1\\ (-1)^k\sin x & \text{if } n=2k\end{cases} \Rightarrow |f^{(n)}(x)|\le 1=1\cdot 1^n$$

$\alpha=1,\ M=1$

Taking $c=0$, by Taylor series theorem

$$\sin x=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}\quad \forall x\in\mathbb{R}.$$

Remarks For $x\in[0,\frac{\pi}{2}]$, $|R_{18}(x)|\le\frac{|x|^{18}}{18!}\le\frac{(\pi/2)^{18}}{18!}$ which is less than 6×10^{-13}.

So $\sin x$ is approximated by $\underbrace{x-\frac{x^3}{3!}+\cdots+\frac{x^{17}}{17!}}_{9\text{ terms}}$ to 10 decimal places on $[0,\frac{\pi}{2}]$.

---

(2) For $f(x)=e^x$ on $(-w,w)$, w is a positive number

$$f^{(n)}(x)=e^x\Rightarrow|f^{(n)}(x)|\le e^w=e^w\cdot 1^n$$

$\alpha=e^w$, $M=1$

Taking $c=0$, by Taylor series theorem

$$e^x=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{1}{n!}x^n\quad\forall x\in(-w,w)$$

Since w can be any positive number,

$$e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n\quad\forall x\in(-\infty,\infty).$$

---

(3) For $f(x)=\cos x$ on $\mathbb{R}=(-\infty,+\infty)$, we can imitate example (1) to get $\cos x=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}\ \forall x\in\mathbb{R}$.
The same remarks apply.

For positive integer a,

$$(1+x)^a=1+\binom{a}{1}x+\binom{a}{2}x^2+\cdots+\binom{a}{a-1}x^{a-1}+x^a$$

where $\binom{a}{k}=\frac{a!}{k!(a-k)!}=\frac{a(a-1)\cdots(a-k+1)}{k!}$

Binomial Theorem $\forall a\in\mathbb{R}$, $x\in\underbrace{(-1,1)}_{\text{open interval}}$

$$(1+x)^a=1+ax+\frac{a(a-1)}{2!}x^2+\cdots=1+\sum_{k=1}^{\infty}\frac{a(a-1)\cdots(a-k+1)}{k!}x^k$$

Examples (1) $\binom{-\frac12}{k}=\frac{(-\frac12)(-\frac32)\cdots(-\frac{2k-1}{2})}{k!}=\frac{(-1)^k 1\cdot3\cdots(2k-1)}{2\cdot4\cdot6\cdots(2k)}$

For $-1<x<1$,

$$\frac{1}{\sqrt{1-x^2}}=(1+(-x^2))^{-\frac12}=1+\sum_{k=1}^{\infty}\frac{(-1)^k 1\cdot3\cdots(2k-1)}{2\cdot4\cdot6\cdots(2k)}\underbrace{(-x^2)^k}_{x^{2k}}$$

$$=1+\frac12x^2+\frac{1\cdot3}{2\cdot4}x^4+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6+\cdots$$

$$\operatorname{Arcsin}x=\int_0^x\frac{1}{\sqrt{1-t^2}}dt\underset{\text{to be justified later}}{=}\text{“}x+\frac12\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\frac{x^5}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{x^7}{7}+\cdots\text{”}$$

Similarly,

$$(1+x)^{-1}=\frac{1}{1+x}=1-x+x^2-x^3+x^4-\cdots=\sum_{k=0}^{\infty}(-1)^kx^k$$

$$\ln(1+x)=\int_0^x\frac{1}{1+t}dt=\text{“}x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots\text{”}=\sum_{k=0}^{\infty}\frac{(-1)^kx^{k+1}}{k+1}$$

$$(1+x^2)^{-1}=\frac{1}{1+x^2}=1-x^2+x^4-x^6+\cdots=\sum_{k=0}^{\infty}(-1)^kx^{2k}$$

$$\operatorname{Arctan}x=\int_0^x\frac{1}{1+t^2}dt=\text{“}x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots\text{”}=\sum_{k=0}^{\infty}\frac{(-1)^kx^{2k+1}}{2k+1}$$

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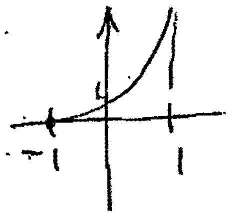
From S. Barnard and J. M. Child's Higher Algebra, 1936



← $y=\frac{1+x}{1-x}$ on $(-1,1)$

$\forall y \in (0,+\infty)\ \exists$ unique $x \in (-1,1)$

Such that $y=\frac{1+x}{1-x} \Leftrightarrow x=\frac{y-1}{y+1}$.

pp. 316–317.

John Napier invented $\ln x$

Henry Briggs invented $\log_{10} x$.

17. Logarithmic Series. If $-1<x<1$, we have

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots \quad \text{.........(A)}$$

Changing the sign of x,

$$\log(1-x)=-\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}+\ldots\right), \quad \text{..............(B)}$$

and since $\log\frac{1+x}{1-x}=\log(1+x)-\log(1-x)$, it follows that

$$\boxed{\log\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right). \quad \text{........................(C)}}$$

Writing $\frac{1+x}{1-x}=\frac{n+1}{n}$, so that $x=\frac{1}{2n+1}$, we have

$$\log(n+1)-\log n=2\left(\frac{1}{2n+1}+\frac{1}{3}\cdot\frac{1}{(2n+1)^3}+\frac{1}{5}\cdot\frac{1}{(2n+1)^5}+\ldots\right). \quad \text{...(D)}$$

Ex. 1. If $0<x<1$ *and* $s_n=x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots$ *to* n *terms, show that the error in taking* $2s_n$ *as the value of* $\log\frac{1+x}{1-x}$ *is less than* $\frac{2x^{2n+1}}{2n+1}\cdot\frac{1}{1-x^2}$.

If R_n is the remainder after n terms of the series,

$$R_n=\frac{x^{2n+1}}{2n+1}+\frac{x^{2n+3}}{2n+3}+\frac{x^{2n+5}}{2n+5}+\ldots<\frac{x^{2n+1}}{2n+1}(1+x^2+x^4+\ldots);$$

$$\therefore \text{ the error}=2R_n<\frac{2x^{2n+1}}{2n+1}\cdot\frac{1}{1-x^2}.$$

19. Calculation of Napierian Logarithms. The number e is chosen as the base of the system of logarithms used in theoretical work. Such logarithms are called *Napierian*, after Napier, the inventor of logarithms. In theoretical work, $\log N$ means $\log_e N$; just as, in practical reckoning, $\log N$ means $\log_{10} N$.

The method of applying the equations of Art. 17 to the calculation of Napierian logarithms is exhibited in the following example.

Ex. 1. Calculate $\log_e 2$ *to seven places of decimals.*

Let $\frac{1+x}{1-x}=2$; $\therefore x=\frac{1}{3}$; $\therefore \log_e 2=2\left\{\frac{1}{3}+\frac{1}{3}\cdot\frac{1}{3^3}+\frac{1}{5}\cdot\frac{1}{3^5}+\ldots\right\}$.

Carrying the reckoning to nine places, we have

$1/3$	$=0{\cdot}333$	333	333	$1/3$	$=0{\cdot}333$	333	333
$1/3^3$	$=0{\cdot}037$	037	037	$1/(3\,.\,3^3)$	$=0{\cdot}012$	345	679
$1/3^5$	$=0{\cdot}004$	115	226	$1/(5\,.\,3^5)$	$=0{\cdot}000$	823	045
$1/3^7$	$=0{\cdot}000$	457	247	$1/(7\,.\,3^7)$	$=0{\cdot}000$	065	321
$1/3^9$	$=0{\cdot}000$	050	805	$1/(9\,.\,3^9)$	$=0{\cdot}000$	005	645
$1/3^{11}$	$=0{\cdot}000$	005	645	$1/(11\,.\,3^{11})$	$=0{\cdot}000$	000	513
$1/3^{13}$	$=0{\cdot}000$	000	627	$1/(13\,.\,3^{13})$	$=0{\cdot}000$	000	048
$1/3^{15}$	$=0{\cdot}000$	000	070	$1/(15\,.\,3^{15})$	$=0{\cdot}000$	000	005
					$=0{\cdot}346$	573	589
							2
					$0{\cdot}693$	147	178

$\therefore \log_e 2=0{\cdot}693147178$ *nearly.*

The proof of the binomial theorem need the following

Theorem (Taylor's Formula with Integral Remainder)
Let f be n times differentiable on (a,b).
For every $x, c \in (a,b)$, if $f^{(n)}$ is integrable on the closed interval with endpoints x and c, then

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x), \text{ where}$$

$$R_n(x) = \frac{1}{(n-1)!}\int_c^x (x-t)^{n-1} f^{(n)}(t)\,dt.$$

Proof. Note $\frac{d}{dt}(-(x-t)) = \frac{d}{dt}(-x+t) = 1$.
Integration by parts $n-1$ times to get

$$f(x) - f(c) = \int_c^x f'(t)\cdot 1\,dt = \int_c^x f'(t)(-(x-t))'\,dt$$

$$= -f'(t)(x-t)\Big|_c^x + \int_c^x (x-t) f''(t)\,dt$$

$$= f'(c)(x-c) + \left(-f''(t)\frac{(x-t)^2}{2}\Big|_c^x + \frac{1}{2!}\int_c^x (x-t)^2 f'''(t)\,dt\right)$$

$$= \cdots$$

$$= f'(c)(x-c) + \cdots + \frac{f^{(n-1)}(c)}{(n-1)!}(x-c)^{n-1}$$

$$+ \frac{1}{(n-1)!}\int_c^x (x-t)^{n-1} f^{(n)}(t)\,dt.$$

There is another common form of the remainder. For that, we need the following fact.

Mean-Value Theorem for Integral Let f be continuous on $[a,b]$ and $g \ge 0$ be integrable on $[a,b]$. Then $\exists\, x_0 \in [a,b]$ such that $\int_a^b f(x)g(x)\,dx = f(x_0)\int_a^b g(x)\,dx$.

For the case $g(x) \equiv 1$, we get $\int_a^b f(x)\,dx = f(x_0)(b-a)$.

Proof. Since f is continuous on $[a,b]$, by the extreme value theorem, $M = \max_{x\in[a,b]} f(x) = f(u)$ and $m = \min_{x\in[a,b]} f(x) = f(v)$. $\forall x \in [a,b]$, $f(v) \le f(x) \le f(u)$.
Since $g \ge 0$, so $f(v)g(x) \le f(x)g(x) \le f(u)g(x)$ and

$$f(v)\int_a^b g(x)\,dx = \int_a^b f(v)g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le \int_a^b f(u)g(x)\,dx = f(u)\int_a^b g(x)\,dx.$$

Then $f(v) \le \int_a^b f(x)g(x)\,dx \Big/ \int_a^b g(x)\,dx \le f(u)$.
The intermediate value theorem implies

$$\int_a^b f(x)g(x)\,dx \Big/ \int_a^b g(x)\,dx = f(x_0) \text{ for some } x_0 \in [a,b].$$

Taylor's Formula with Cauchy Form Remainder.
In Taylor's theorem, if $f^{(n)}$ is continuous on the closed interval with x, c as endpoints, then $\exists\, x_n$ between x and c such that

$$R_n(x) = \frac{1}{(n-1)!}\int_c^x \underbrace{(x-t)^{n-1} f^{(n)}(t)}_{\text{continuous}}\,dt = \underbrace{\frac{(x-c)(x-x_n)^{n-1} f^{(n)}(x_n)}{(n-1)!}}_{\text{Cauchy form remainder}}.$$

<u>Examples</u> ① Show that if $-1<x<1$, then
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

<u>Solution 1</u> Let $f(x)=\ln(1+x)$, then $f'(x)=\frac{1}{1+x}$,
$f''(x) = -\frac{1}{(1+x)^2}$, $f'''(x)=\frac{2}{(1+x)^3}$, $\cdots$, $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$.

Consider the Cauchy form remainder $R_n(x)$ with $c=0$.
$$|R_n(x)| = \left|\frac{x(x-x_n)^{n-1}}{(1+x_n)^n}\right| = \frac{|x|}{1+x_n}\left|\frac{x-x_n}{1+x_n}\right|^{n-1} \underset{(*)}{\le} \frac{|x|}{1-|x|}|x|^{n-1} \xrightarrow[\text{as } n\to\infty]{|x|<1} 0$$
x_n between 0 and x $\Rightarrow x_n \ge -|x|$

where $(*)$ is because $g(t)=\frac{x-t}{1+t}$ for t between 0 and x
$\Rightarrow g'(t) = \frac{-1-x}{(1+t)^2} < 0 \Rightarrow g(t)$ is decreasing
$\Rightarrow g(t)$ is between $g(0)=x$ and $g(x)=0$.

<u>Solution 2</u> $$\frac{1}{1+t} = 1 - t + t^2 - \cdots + (-1)^n t^{n-2} + (-1)^{n+1}\frac{t^{n-1}}{1+t}$$
Integration from 0 to x on both sides, we get
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^n\frac{x^{n-1}}{n-1} + (-1)^{n+1}\int_0^x \frac{t^{n-1}}{1+t}\,dt.$$
By mean-value theorem for integrals, $\exists x_n$ between 0 and x such that
$$\left|\int_0^x \underbrace{\frac{t^{n-1}}{1+t}}_{\text{continuous}}dt\right| = \left|\frac{x_n^{n-1}}{1+x_n}(x-0)\right| \le \frac{|x|^{n-1}}{1-|x|}|x| \xrightarrow[\text{as } n\to\infty]{|x|<1} 0$$

② From $\frac{1}{1+t^2} = 1 - t^2 + t^4 - \cdots + (-1)^n t^{2n-4} + (-1)^{n+1}\frac{t^{2n-2}}{1+t^2}$,
we get $\operatorname{Arctan} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n\frac{x^{2n-3}}{2n-3} + (-1)^{n+1}\int_0^x \frac{t^{n-2}}{1+t^2}dt$.
For $|x|<1$, $\left|\int_0^x \frac{t^{n-2}}{1+t^2}dt\right| \le \frac{|x|^{n-2}}{1+0}|x| \to 0$ as $n\to\infty$.
$\therefore \operatorname{Arctan} x = x - x^3/3 + x^5/5 - \cdots$ for $-1<x<1$.

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THE BINOMIAL EXPANSION

Let m be an *arbitrary* real number. For $x > 0$ we have

$$\frac{d^n}{dx}(x^m) = m(m-1)\dots(m-n+1)x^{m-n};$$

the Taylor formula of order n about the point $x = 0$ for the function $(1+x)^m$ shows that for every $x > -1$

$$(1+x)^m = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \dots + \binom{m}{n}x^n + r_n(x) \tag{19}$$

with

$$r_n(x) = \frac{m(m-1)\dots(m-n)}{n!}\int_0^x \left(\frac{x-t}{1+t}\right)^n (1+t)^{m-1}\,dt$$

where we put $\binom{m}{n} = \frac{m(m-1)\dots(m-n+1)}{n!}$. The formula (19) reduces to the binomial formula when m is an integer > 0 and $n \geqslant m$; by extension, we again call it the *binomial formula*, and the coefficients $\binom{m}{n}$ are called the *binomial coefficients*, when m is an *arbitrary* real number and n is an arbitrary integer > 0.

The remainder in (19) has the same sign as $\binom{m}{n+1}$ if $x > 0$, and the sign of $(-1)^{n+1}\binom{m}{n+1}$ if $-1 < x < 0$. Since $\left|\frac{x-t}{1+t}\right| \leqslant |x|$ for $t > -1$ in the interval with endpoints 0 and x, we have the following bound for the remainder, for m and n arbitrary and $x > -1$:

$$\begin{aligned}\left|\frac{m(m-1)\dots(m-n)}{n!}\int_0^x \left(\frac{x-t}{1+t}\right)^n (1+t)^{m-1}\,dt\right| \\ \leqslant \left|\binom{m-1}{n}x^n\left((1+x)^m - 1\right)\right|.\end{aligned} \tag{20}$$

If we suppose $x \geqslant 0$, and $n \geqslant m-1$, then $(1+t)^{n-m+1} \geqslant 1$ on the interval of integration, so

$$0 \leqslant \int_0^x \frac{(x-t)^n}{(1+t)^{n-m+1}}\,dt \leqslant \int_0^x (x-t)^n\,dt = \frac{x^{n+1}}{n+1}$$

which gives the estimate

$$|r_n(x)| \leqslant \left|\binom{m}{n+1}\right| x^{n+1} \qquad (x \geqslant 0,\ n \geqslant m-1) \tag{21}$$

for the remainder. On the other hand, suppose that $-1 \leqslant m < 0$; if one makes the change of variable $u = \frac{x-t}{x(1+t)}$ in the integral (19) one obtains

$$r_n(x) = \frac{m(m-1)\dots(m-n)}{n!}(1+x)^m x^{n+1}\int_0^1 \frac{u^n\,du}{(1+ux)^{m+1}}. \tag{22}$$

To estimate the integral for $x > -1$ we remark that, since $m + 1 < 1$, the integral $\int_0^1 \frac{u^n\,du}{(1-u)^{m+1}}$ converges and bounds the right-hand side of (22) since $1 + ux > 1 - u$. Now, for $-1 < x < 0$ the hypothesis on m implies that all the terms $\binom{m}{1}x, \binom{m}{2}x^2, \dots, \binom{m}{n}x^n$ which appear in the right-hand side of (19) are $\geqslant 0$, and hence $r_n(x) \leqslant (1+x)^m$, from which, on dividing by $(1+x)^m$,

$$\frac{m(m-1)\dots(m-n)}{n!}x^{n+1}\int_0^1 \frac{u^n\,du}{(1+ux)^{m+1}} \leqslant 1.$$

Moreover, for $-1 < x < 0$ the factor in front of the integral is $\geqslant 0$, so, letting x approach -1,

$$\left|\frac{m(m-1)\dots(m-n)}{n!}\int_0^1 \frac{u^n\,du}{(1-u)^{m+1}}\right| \leqslant 1$$

and consequently for $-1 \leqslant m < 0$ and $x > -1$ we have

$$|r_n(x)| \leqslant (1+x)^m\,|x|^{n+1}. \tag{23}$$

From these inequalities we can, for a start, deduce that for $|x| < 1$ we have

$$(1+x)^m = \sum_{n=0}^{\infty}\binom{m}{n}x^n \tag{24}$$

the right-hand side (called the *binomial series*) being *absolutely and uniformly convergent* on every *compact* subset of $]-1, +1[$. Indeed one can write

$$\binom{m}{n} = (-1)^n\left(1 - \frac{m+1}{1}\right)\left(1 - \frac{m+1}{2}\right)\dots\left(1 - \frac{m+1}{n}\right) \tag{25}$$

whence

$$\left|\binom{m}{n}\right| \leqslant \left(1 + \frac{|m+1|}{1}\right)\left(1 + \frac{|m+1|}{2}\right)\dots\left(1 + \frac{|m+1|}{n}\right).$$

If $|x| \leqslant r < 1$ there is an n_0 such that $1 + \frac{|m|}{n_0} < \frac{1}{r'}$, where $r < r' < 1$; whence, putting

$$k = \left(1 + \frac{|m|}{1}\right)\left(1 + \frac{|m|}{2}\right)\dots\left(1 + \frac{|m|}{n_0}\right)$$

we have

$$\left|\binom{m-1}{n}x^n\right| \leqslant k\,|x|^{n_0}\left(\frac{r}{r'}\right)^{n-n_0},$$

which proves the proposition. On the other hand, for $x > 1$, the absolute value of the general term of the series (24) increases indefinitely with n if m is not an integer $\geqslant 0$; indeed, from (25), we have for $n > n_1 \geqslant |m+1|$

$$\left|\binom{m}{n}\right| \geqslant \left|\left(1-\frac{m+1}{1}\right)\left(1-\frac{m+1}{2}\right)\cdots\left(1-\frac{m+1}{n_1}\right)\right| \left(1-\frac{|m+1|}{n_1+1}\right)\cdots\left(1-\frac{|m+1|}{n}\right).$$

Let $n_0 \geqslant n_1$ be such that for $n \geqslant n_0$ we have $1-\dfrac{|m+1|}{n} > \dfrac{1}{x'}$, where $1 < x' < x$. If we put

$$k' = \left|\left(1-\frac{m+1}{1}\right)\cdots\left(1-\frac{m+1}{n_1}\right)\right|\left(1-\frac{|m+1|}{n_1+1}\right)\cdots\left(1-\frac{|m+1|}{n_0}\right)$$

then, for $n > n_0$,

$$\left|\binom{m}{n}x^n\right| \geqslant k'\,|x|^{n_0}\left(\frac{x}{x'}\right)^{n-n_0}$$

from which the proposition follows.

We remark that for $m = -1$ the algebraic identity

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^{n-1}x^{n-1} + (-1)^n\frac{x^n}{1+x} \qquad (26)$$

gives the expression for the remainder in the general formula (19) without having to integrate; the formula (23) reduces in this case to the expression for the sum of the *geometric series* (or *progression*).

In the second place let us study the convergence of the binomial series for $x = 1$ or $x = -1$ (excluding the trivial case $m = 0$):

a) $m \leqslant -1$. The product with general term $1 - \dfrac{m+1}{n}$ converges to $+\infty$ if $m < -1$, to 1 if $m = -1$, so it follows from (25) that for $x = \pm 1$ the general term of the binomial series does not tend to 0. The binomial series diverges at $x=\pm 1$.

b) $-1 < m < 0$. This time the product with general term $1 - \dfrac{m+1}{n}$ converges to 0, so the inequality (21) shows that $r_n(1)$ tends to 0. Thus the binomial series converges for $x = 1$ and has sum 2^m; moreover, the binomial series is uniformly convergent on every interval $]x_0, 1]$ with $-1 < x_0 \leqslant 1$, by virtue of what we saw above and of (21). On the other hand, for $x = -1$ all the terms on the right-hand side of (24) are $\geqslant 0$; if this series were convergent one could deduce that the binomial series would be normally convergent on $[-1, 1]$ and so would have for its sum a continuous function on this interval, which is absurd because $(1+x)^m$ is not bounded on $]-1, 1]$ for $m < 0$. We conclude that also for $x = 1$ the binomial series is not absolutely convergent. The binomial series converges conditionally at $x=1$ and diverges at $x=-1$.

c) $m > 0$. The definition of $r_n(x)$ shows that $r_n(x)$ tends to the limit $r_n(-1)$ when x tends to -1; on passing to the limit in (20) one concludes that $|r_n(-1)| \leqslant \left|\dbinom{m-1}{n}\right|$, and since $m - 1 > -1$ we see that for $x = -1$ the binomial series is convergent. Furthermore, for $n > m + 1$ all the terms of this series have the same sign; thus the binomial series is *normally convergent* on the interval $[-1, 1]$ and has sum $(1+x)^m$ on this interval. the binomial series converges absolutely at $x=\pm 1$.

See definition of normally convergent in the last page.

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EXPANSIONS OF $\log(1+x)$, OF $\operatorname{Arc\,tan} x$ AND OF $\operatorname{Arc\,sin} x$

Let us integrate the two sides of (26) between 0 and x; we obtain the Taylor expansion of order n of $\log(1+x)$, valid for $x > -1$

$$\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + (-1)^n\int_0^x \frac{t^n\,dt}{1+t}. \qquad (27)$$

The remainder has the same sign as $(-1)^n$ if $x > 0$, and is < 0 if $-1 < x < 0$; further, when $x > 0$, we have $1 + t \geqslant 1$ for $0 \leqslant t \leqslant x$, and, when $-1 < x < 0$, we have $1 + t \geqslant 1 - |x|$ for $x \leqslant 0$; whence the estimates for the remainder

$$\left|\int_0^x \frac{t^n\,dt}{1+t}\right| \leqslant \frac{|x|^{n+1}}{n+1} \qquad \text{for } x \geqslant 0 \qquad (28)$$

$$\left|\int_0^x \frac{t^n\,dt}{1+t}\right| \leqslant \frac{|x|^{n+1}}{(n+1)(1-|x|)} \qquad \text{for } -1 < x \leqslant 0. \qquad (29)$$

From these last two formulae one deduces immediately that for $-1 < x \leqslant 1$ one has

$$\log(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} \qquad (30)$$

the series being *uniformly convergent* on every compact interval contained in $]-1, +1]$, and *absolutely convergent* for $|x| < 1$.

Similarly, let us replace x by x^2 in (26) and integrate both sides between 0 and x; we obtain the Taylor expansion of order $2n - 1$ for $\operatorname{Arc\,tan} x$, valid for all real x

$$\operatorname{Arc\,tan} x = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + (-1)^n\int_0^x \frac{t^{2n}\,dt}{1+t^2}. \qquad (31)$$

The remainder has the sign of $(-1)^n x$, and since $1 + t^2 \geqslant 1$ for all t we have the estimate

$$\left|\int_0^x \frac{t^{2n}\,dt}{1+t^2}\right| \leqslant \frac{|x|^{2n+1}}{2n+1} \qquad (32)$$

from which one deduces that, for $|x| \leqslant 1$,

$$\operatorname{Arc\,tan} x = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} \qquad (33)$$

the series being *uniformly convergent* on $[-1, +1]$, and *absolutely convergent* for $|x| < 1$.

In particular, for $x = 1$ one obtains the formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots + (-1)^n\frac{1}{2n+1} + \cdots. \qquad (34)$$

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Finally, for the Taylor expansion of Arc sin x we start from the expansion of its derivative $(1-x^2)^{-1/2}$; this last expansion is obtained by replacing x by $-x^2$ in the expansion of $(1+x)^{-1/2}$ as a binomial series; for $|x|<1$ this gives

$$(1-x^2)^{-1/2} = 1+\frac{1}{2}x^2+\frac{1.3}{2.4}x^4+\cdots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}x^{2n}+r_n(x)$$

with, by (23), the bound

$$0 \leqslant r_n(x) \leqslant \frac{x^{2n+2}}{\sqrt{1-x^2}}$$

for the remainder.

On taking the primitive of the preceding expansion we obtain

$$\operatorname{Arc}\sin x = x+\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\cdots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\frac{x^{2n+1}}{2n+1}+R_n(x) \quad (35)$$

where $R_n(x)$ has the sign of x and satisfies the inequality

$$|R_n(x)| \leqslant \int_0^x \frac{t^{2n+2}\,dt}{\sqrt{1-t^2}}. \quad (36)$$

Further, the relation (35) shows that $R_n(x)$ tends to a limit when x approaches 1 or -1, so one has

$$|R_n(1)| \leqslant \int_0^1 \frac{t^{2n+2}\,dt}{\sqrt{1-t^2}}. \quad (37)$$

But the right-hand side of (37) tends to 0 when n tends to $+\infty$: for, since the integral $\int_0^1 dt/\sqrt{1-t^2}$ is convergent, for every $\varepsilon>0$ there is an a such that $0<a<1$ and $\int_a^1 dt/\sqrt{1-t^2}\leqslant\varepsilon$; on the other hand we have

$$\int_0^a \frac{t^{2n+2}\,dt}{\sqrt{1-t^2}} \leqslant \frac{1}{\sqrt{1-a^2}}\int_0^a t^{2n+2}\,dt = \frac{a^{2n+3}}{(2n+3)\sqrt{1-a^2}}$$

and so there is an n_0 such that for $n\geqslant n_0$ one has $\dfrac{a^{2n+3}}{(2n+3)\sqrt{1-a^2}}\leqslant\varepsilon$, whence, finally, $|R_n(x)|\leqslant 2\varepsilon$ for $|x|\leqslant 1$ and $n\geqslant n_0$. Thus one has

$$\operatorname{Arc}\sin x = \sum_{n=0}^{\infty}\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\frac{x^{2n+1}}{2n+1} \quad (38)$$

the right-hand side being *normally convergent* on the compact interval $[-1, 1]$.

On putting $x=\frac{1}{2}$, for example, in (38) we obtain a new expression for the number π;

$$\frac{\pi}{6} = \sum_{n=0}^{\infty}\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\frac{1}{(2n+1)2^{2n+1}}$$

which is much better suited than formula (34) to calculating approximations to π one thus obtains

$\pi = 3.141\,592\,653\ldots$ accurate to within $1/10^9$.

Let I be a nonempty interval.

Definition A power series $\sum_{n=0}^{\infty} a_n x^n$ converges normally on I iff there are nonnegative constants $b_1, b_2, b_3, \cdots$ such that

① for every $n=1,2,3,\cdots$ and every $x\in I$, we have $|a_n x^n|\leq b_n$

and ② $\sum_{n=0}^{\infty} b_n$ converges.

So a power series converges normally on I implies it converges absolutely for every $x\in I$ and it converges uniformly on I.

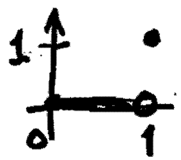
Uniform Convergence

$$\sum_{k=1}^{\infty} g_k(x) = \lim_{n\to\infty} S_n(x), \text{ where } S_n(x) = \sum_{k=1}^{n} g_k(x)$$

$$\Rightarrow g_1(x) = S_1(x),\ g_k(x) = S_k(x) - S_{k-1}(x) \text{ when } k>1.$$

Examples ① Define $S_n : [0,1] \to \mathbb{R}$ by $\underbrace{S_n(x) = x^n}_{\text{continuous}}$.

$S_n(x)$ converges pointwise on $[0,1]$ to



$$S(x) = \lim_{n\to\infty} S_n(x) = \underbrace{\begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}}_{\text{discontinuous}}$$

Let $g_1(x) = S_1(x)$, $g_k(x) = S_k(x) - S_{k-1}(x)$ for $k>1$.

$$\lim_{x\to 1}(g_1(x)+g_2(x)+\cdots) = \lim_{x\to 1}\sum_{k=1}^{\infty} g_k(x) = \lim_{x\to 1}\lim_{n\to\infty} S_n(x) = \lim_{x\to 1} S(x) = \underline{\underline{0}}$$

But

$$\lim_{x\to 1} g_1(x) + \lim_{x\to 1} g_2(x) + \cdots = \sum_{k=1}^{\infty} \lim_{x\to 1} g_k(x) = \lim_{n\to\infty}\sum_{k=1}^{n} \lim_{x\to 1} g_k(x)$$

$$= \lim_{n\to\infty}\lim_{x\to 1} S_n(x)$$

$$= \lim_{n\to\infty} 1 = \underline{\underline{1}}$$

$$\therefore \lim_{x\to 1}(g_1(x)+g_2(x)+\cdots) \ne \lim_{x\to 1} g_1(x) + \lim_{x\to 1} g_2(x) + \cdots$$

$$\lim_{x\to 1}\sum_{k=1}^{\infty} g_k(x) \ne \sum_{k=1}^{\infty}\lim_{x\to 1} g_k(x)$$

$$\lim_{x\to 1}\lim_{n\to\infty} S_n(x) \ne \lim_{n\to\infty}\lim_{x\to 1} S_n(x)$$

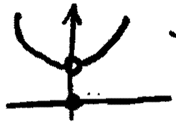
② $g_k(x) = \dfrac{x^2}{(1+x^2)^k}$, $S(x) = \sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \underbrace{\dfrac{x^2}{(1+x^2)^k}}_{\text{Continuous}}$

$S(0) = 0$. For $x \ne 0$, $S(x) = x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + \cdots$

$$= x^2\Big(\underbrace{1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \cdots}_{\text{Geometric Series}}\Big)$$

$$= \frac{x^2}{1 - \frac{1}{1+x^2}} = 1 + x^2$$

$$S(x) = \underbrace{\begin{cases} 0 & \text{if } x = 0 \\ 1+x^2 & \text{if } x \ne 0 \end{cases}}_{\text{discontinuous}}$$



$$g_k'(0) = \left.\frac{2x - 2(k-1)x^3}{(1+x^2)^{k+1}}\right|_{x=0} = 0 \qquad S_n(x) = \sum_{k=0}^{n} g_k(x)$$

$$\Rightarrow S_n'(0) = 0$$

At $\underline{\underline{x=0}}$,

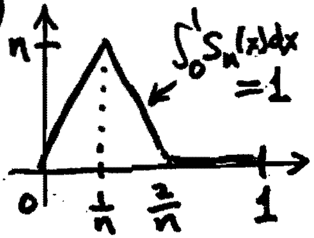
$$\frac{d}{dx}\sum_{k=0}^{\infty} g_k(x) = \frac{d}{dx}\lim_{n\to\infty} S_n(x) = \frac{d}{dx} S(x) \text{ doesn't exist.}$$

$$\sum_{k=0}^{\infty}\frac{d}{dx} g_k(x) = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{d}{dx} g_k(x) = \lim_{n\to\infty}\underbrace{\frac{d}{dx} S_n(x)}_{=0} = 0.$$

$$\frac{d}{dx}\sum_{k=0}^{\infty} g_k(x) \ne \sum_{k=0}^{\infty}\frac{d}{dx} g_k(x)$$

$$\frac{d}{dx}(g_1(x)+g_2(x)+\cdots) \ne \frac{d}{dx}g_1(x) + \frac{d}{dx}g_2(x) + \cdots$$

$$\frac{d}{dx}\lim_{n\to\infty} S_n(x) \ne \lim_{n\to\infty}\frac{d}{dx} S_n(x)$$

③ 

$$S_n(x) = \begin{cases} n^2 x & \text{if } 0 \le x < \frac{1}{n} \\ -n^2 x + 2n & \text{if } \frac{1}{n} \le x < \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \le x \le 1 \end{cases}$$

$S_n(x)$ converges pointwise on $[0,1]$ to $S(x) = \lim_{n\to\infty} S_n(x) = 0$.

$$\boxed{\begin{array}{l} S_n(0)=0 \Rightarrow S(0)=0 \\ \text{For } x>0,\ \exists n \in \mathbb{N}\ \ n > \frac{2}{x} \\ \quad \text{then } x > \frac{2}{n} \Rightarrow S_n(x) = 0 \Rightarrow S(x) = 0 \end{array}}$$

Let $g_1(x) = S_1(x)$, $g_k(x) = S_k(x) - S_{k-1}(x)$ for $k > 1$.

$$\int_0^1 \sum_{k=1}^{\infty} g_k(x)\,dx = \int_0^1 \lim_{n\to\infty} S_n(x) = \int_0^1 0\,dx = \underline{\underline{0}},$$

but

$$\sum_{k=1}^{\infty} \int_0^1 g_k(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} \int_0^1 g_k(x)dx = \lim_{n\to\infty} \underbrace{\int_0^1 S_n(x)dx}_{=1}$$
$$= \underline{\underline{1}}$$

$$\therefore \quad \int_0^1 \sum_{k=1}^{\infty} g_k(x)dx \ne \sum_{k=0}^{\infty} \int_0^1 g_k(x)\,dx$$

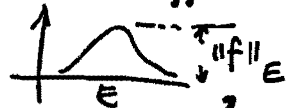
$$\int_0^1 (g_1(x) + g_2(x) + \cdots)dx \ne \int_0^1 g_1(x)dx + \int_0^1 g_2(x)dx + \cdots$$

$$\int_0^1 \lim_{n\to\infty} S_n(x)\,dx \ne \lim_{n\to\infty} \int_0^1 S_n(x)\,dx$$

<u>Question</u>: Are there conditions which ensure an interchange of limit operations is correct?

<u>Answer</u>: "uniform convergence" is sufficient.

<u>Notation</u>: For $f: E \to \mathbb{R}$,



$$\|f\|_E = \|f\|_\infty = \sup\{|f(x)| : x \in E\}$$

← sup-norm of f on E

<u>Definitions</u> ① Sequence $S_n : E \to \mathbb{R}$ <u>converges uniformly</u> on E to function $S : E \to \mathbb{R}$ iff

$$\lim_{n\to\infty} \|S_n - S\|_E = \lim_{n\to\infty} \Big(\sup\underbrace{\{|S_n(x) - S(x)| : x \in E\}}_{\text{furthest distance between } S_n(x) \text{ and } S(x) \text{ on } E} \Big) = 0$$

i.e. $\forall \varepsilon > 0\ \exists K \in \mathbb{N}$ (K depends only on ε) such that $n \ge K \Rightarrow \forall x \in E,\ |S_n(x) - S(x)| < \varepsilon$

② Series $\sum_{k=1}^{\infty} g_k(x)$ <u>converges uniformly</u> on E to function $S(x)$ iff partial sum sequence $S_n(x) = \sum_{k=1}^{n} g_k(x)$ converges uniformly on E to $S(x)$.

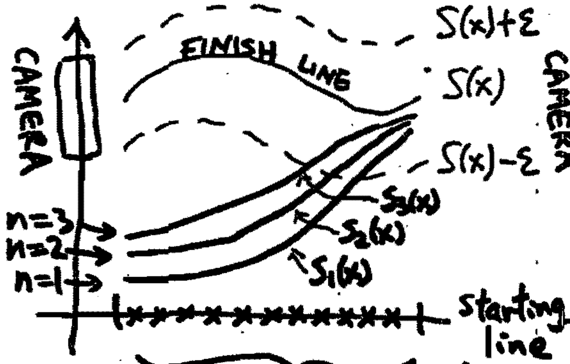
<u>Notations</u> "$S_n(x) \to S(x)$ on E" denote $S_n(x)$ converges pointwise on E to $S(x)$

"$S_n(x) \rightrightarrows S(x)$ on E" denote $S_n(x)$ converges uniformly on E to $S(x)$.

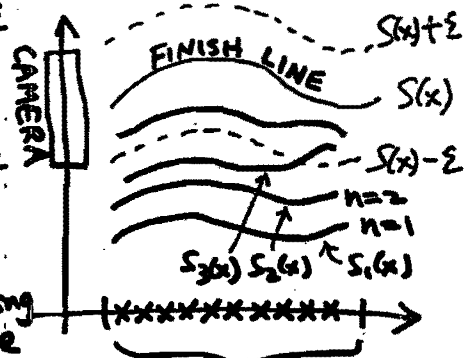
Pointwise Convergence vs uniform convergence

Pointwise Convergence	uniform convergence
$S_n(x) \to S(x)$ on E	$S_n(x) \rightrightarrows S(x)$ on E
$\lim\limits_{n\to\infty} S_n(x) = S(x)$ for all $x \in E$	$\lim\limits_{n\to\infty} \lVert S_n - S \rVert_E = 0.$
$\forall x\in E, \forall \varepsilon>0$ $\exists K \in \mathbb{N}$ (K depends on x and ε) $n \ge K \Rightarrow \lvert S_n(x)-S(x)\rvert<\varepsilon$	$\forall \varepsilon>0$ $\exists K\in\mathbb{N}$ (K depends on ε only) $n\ge K \Rightarrow \forall x\in E$ $\lvert S_n(x)-S(x)\rvert<\varepsilon$

$$\lvert S_n(x)-S(x)\rvert<\varepsilon \iff S(x)-\varepsilon < S_n(x) < S(x)+\varepsilon$$



camera time E



E

x_1	$K=1$	May not be any K common to all x
x_2	$K=2$	
x_3	$K=3$	
$\vdots$		

Example Marathon

$x_1, x_2, x_3, \cdots$ camera time $K=4$ a common K for all x's

Example 100 meter dash

Theorem $S_n(x) \rightrightarrows S(x)$ on $E \Rightarrow S_n(x) \to S(x)$ on E.

$\sum\limits_{k=1}^{\infty} g_k(x) \rightrightarrows S(x)$ on $E \Rightarrow \sum\limits_{k=0}^{\infty} g_k(x) \to S(x)$ on E.

Reason $\forall x\in E,\ \lvert S_n(x)-S(x)\rvert \le \lVert S_n - S\rVert_E$.

$$S_n(x) \rightrightarrows S(x) \text{ on } E \iff \lim_{n\to\infty}\lVert S_n - S\rVert_E = 0$$
$$\Rightarrow \forall x\in E,\ \lim_{n\to\infty}\lvert S_n(x)-S(x)\rvert = 0$$
$$\iff \forall x\in E,\ \lim_{n\to\infty} S_n(x) = S(x)$$
$$\iff S_n(x)\to S(x) \text{ on } E.$$

Continuity Theorem (for uniform convergence)

If ① every $S_n(x)$ is continuous at $c\in E$,
and ② $S_n(x) \rightrightarrows S(x)$ on E,
then $\underbrace{S(x) \text{ is continuous at } c\in E}$

$$\lim_{x\to c}\lim_{n\to\infty} S_n(x) \underset{\text{by ②}}{=} \lim_{x\to c} S(x) = S(c) \underset{\text{by ②}}{=} \lim_{n\to\infty} S_n(c) \underset{\text{by ①}}{=} \lim_{n\to\infty}\lim_{x\to c} S_n(x)$$

Similarly,
if ① every $g_k(x)$ is continuous at $c\in E$,
and ② $\sum\limits_{k=1}^{\infty} g_k(x) \rightrightarrows S(x)$ on E,
then $\underbrace{\sum\limits_{k=1}^{\infty} g_k(x) \text{ is continuous at } c \in E}$

$$\lim_{x\to c}\sum_{k=1}^{\infty} g_k(x) = \sum_{k=1}^{\infty} g_k(c) \underset{\text{by ①}}{=} \sum_{k=1}^{\infty}\lim_{x\to c} g_k(x)$$

Integration Theorem Let $[a,b]$ be a closed, bounded interval.

If ① every $S_n(x)$ is integrable on $[a,b]$

and ② $S_n(x) \rightrightarrows S(x)$ on $[a,b]$,

then $S(x)$ is integrable on $[a,b]$ and

$$\int_a^b \lim_{n\to\infty} S_n(x)\,dx \underset{\text{by ②}}{=} \int_a^b S(x)\,dx = \lim_{n\to\infty}\int_a^b S_n(x)\,dx.$$

Similarly,

if ① every $g_k(x)$ is integrable on $[a,b]$

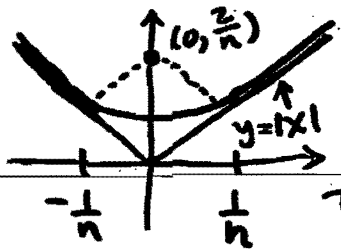
and ② $\sum_{k=1}^{\infty} g_k(x) \rightrightarrows S(x)$ on $[a,b]$,

then $\sum_{k=1}^{\infty} g_k(x)$ is integrable on $[a,b]$ and

$$\int_a^b \sum_{k=1}^{\infty} g_k(x)\,dx = \sum_{k=1}^{\infty}\int_a^b g_k(x)\,dx.$$

Question: How about differentation under unif. conv.?

Examples ① Let $S_n(x)$ be $|x|$ for $x \notin [-\frac{1}{n}, \frac{1}{n}]$.



On $[-\frac{1}{n}, \frac{1}{n}]$, let graph of $S_n(x)$ be a quarter circle centered at $(0, \frac{2}{n})$ and radius $\frac{1}{n}\sqrt{2}$.

Then

$$\sup\{|S_n(x)-|x|| : x\in\mathbb{R}\} = |S_n(0)| = \frac{2-\sqrt{2}}{n} \to 0.$$

$\therefore S_n(x) \rightrightarrows |x|$ on $\mathbb{R}$

Every $S_n(x)$ is differentiable on $\mathbb{R}$,

but $|x|$ is not differentiable.

② Let $S_n(x) = \dfrac{x}{1+nx^2}$, then $S(x) = \lim_{n\to\infty} S_n(x) = 0$.

$$S_n'(x) - S'(x) = \frac{1-nx^2}{(1+nx^2)^2} \Rightarrow \|S_n - S\|_{\mathbb{R}} = |S_n(\tfrac{1}{\sqrt{n}}) - 0| = \frac{1}{2\sqrt{n}} \to 0$$

$= 0$ if $x = \pm\frac{1}{\sqrt{n}}$ critical point of $S_n(x) - S(x)$

$\therefore S_n(x) \rightrightarrows S(x) = 0$ on $\mathbb{R}$. Now $S_n'(0) = 1$.

At $x=0$, $\frac{d}{dx}\lim_{n\to\infty} S_n(x) = \frac{d}{dx}0 = 0$,

but $\lim_{n\to\infty}\frac{d}{dx}S_n(x) = \lim_{n\to\infty} 1 = 1$

Differentiation Theorem (for uniform convergence)

If ① every $S_n(x)$ is differentiable on (a,b),

② $S_n'(x) \rightrightarrows T(x)$ on (a,b) for some function T

③ $\lim_{n\to\infty} S_n(x_0)$ exists for some $x_0 \in (a,b)$,

then $S_n(x) \rightrightarrows S(x)$ on (a,b) for some function $S(x)$

and $\frac{d}{dx}\lim_{n\to\infty} S_n(x) = \frac{d}{dx}S(x) = T(x) = \lim_{n\to\infty}\frac{d}{dx}S_n(x)$.

Similarly, if ① every $g_k(x)$ is diff. on (a,b),

② $\sum_{k=1}^{\infty} g_k'(x)$ converges unif. on (a,b)

③ $\sum_{k=1}^{\infty} g_k(x_0)$ converges for some $x_0 \in (a,b)$,

then $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly on (a,b) and

$$\frac{d}{dx}\sum_{k=1}^{\infty} g_k(x) = \sum_{k=1}^{\infty}\frac{d}{dx}g_k(x).$$

Tests for checking uniform convergence

L-Test (for sequence of functions)

If ① $S_n(x) \to S(x)$ on E

② $\forall n \in \mathbb{N}$, $\exists$ constant L_n such that

$$|S_n(x) - S(x)| \le L_n \quad \forall x \in E$$

③ $\lim\limits_{n\to\infty} L_n = 0$,

then $S_n(x) \rightrightarrows S(x)$ on E.

Proof ② implies $\|S_n - S\|_E = \sup\{|S_n(x) - S(x)| : x \in E\}$
$\le L_n$

③ and sandwich theorem imply $\lim\limits_{n\to\infty} \|S_n - S\|_E = 0$.

Examples ① Consider $S_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ on $\mathbb{R}$.

① $S(x) = \lim\limits_{n\to\infty} \dfrac{\sin nx}{\sqrt{n}} = 0$, ② $|S_n(x) - S(x)| = \left|\dfrac{\sin nx}{\sqrt{n}}\right| \le \underbrace{\dfrac{1}{\sqrt{n}}}_{L_n}$

③ $\lim\limits_{n\to\infty} L_n = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}} = 0$. $\therefore S_n(x) \rightrightarrows 0$ on $\mathbb{R}$.

② Consider $S_n(x) = e^{-\cos^2(1/x)/n}$ on $(0,1)$.

① $S(x) = \lim\limits_{n\to\infty} e^{-\cos^2(1/x)/n} = 1$

② Mean value theorem and $w > 0$ $\Rightarrow |e^{-w} - 1| = |(-e^{-c})(w-0)| \le |w|$ (c between w and 0)

$\therefore |S_n(x) - S(x)| = |e^{-\cos^2(1/x)/n} - 1| \le \left|\dfrac{\cos^2(1/x)}{n}\right| \le \underbrace{\dfrac{1}{n}}_{L_n}$

③ $\lim\limits_{n\to\infty} L_n = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. $\therefore S_n(x) \rightrightarrows 1$ on $(0,1)$.

③ Show $S_n(x) = x^n \rightrightarrows 0$ on $[0,t]$, where $t<1$, but $S_n(x)$ does not converge uniformly on $[0,1]$.

① $\forall x \in [0,t]$, $S(x) = \lim\limits_{n\to\infty} S_n(x) = \lim\limits_{n\to\infty} x^n = 0$.

② $\forall x \in [0,t]$, $|S_n(x) - S(x)| = |x^n| \le t^n = L_n$

③ $\lim\limits_{n\to\infty} L_n = \lim\limits_{n\to\infty} t^n = 0$ ($t<1$). $\therefore S_n(x) \rightrightarrows 0$ on $[0,t]$.

For $[0,1]$, $S(x) = \lim\limits_{n\to\infty} S_n(x) = \lim\limits_{n\to\infty} x^n = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$ discont.

Note $S_n(x)$ are continuous on $[0,1]$.
Assume $S_n(x) \rightrightarrows S(x)$ on $[0,1]$. By continuity theorem, $S(x)$ is continuous on $[0,1]$, contradiction.
$\therefore S_n(x)$ does not converge uniformly on $[0,1]$.

Weierstrass M-Test (for series of functions)

If ① $\forall k = 1,2,3,\dots$ $\exists$ constant M_k such that

$$|g_k(x)| \le M_k \quad \forall x \in E$$

② $\sum\limits_{k=1}^{\infty} M_k$ converges,

then $\sum\limits_{k=1}^{\infty} g_k(x)$ converges uniformly on E.

Proof. By ① and ②, $S_n(x) = \sum\limits_{k=1}^{n} g_k(x) \to S(x) = \sum\limits_{k=1}^{\infty} g_k(x)$.

$\forall x \in E$, $|S_n(x) - S(x)| \le \sum\limits_{k=n+1}^{\infty} |g_k(x)| \le \sum\limits_{k=n+1}^{\infty} M_k = L_n$

$\lim\limits_{n\to\infty} L_n = \lim\limits_{n\to\infty} \left(\sum\limits_{k=1}^{\infty} M_k - \sum\limits_{k=1}^{n} M_k\right) = 0$. $\therefore S_n(x) \rightrightarrows S(x)$ on E.

Examples ① Consider $\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$ on $\mathbb{R}$.

① $\forall x \in \mathbb{R}$, $\left|\frac{\sin kx}{k^2}\right| \le \frac{1}{k^2} = M_k$

② $\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p-test.

$\therefore \sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$ converges uniformly on $\mathbb{R}$.

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② Consider $\sum_{k=1}^{\infty} \left(\frac{\ln x}{x}\right)^k$ on $[1, \infty)$.

① $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1 - \ln x}{x^2} = 0 \Rightarrow x = e$, $\frac{\ln 1}{1} = 0$, $\frac{\ln e}{e} = \frac{1}{e}$

$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$. $\forall x \in [1, \infty)$, $\left|\frac{\ln x}{x}\right|^k \le \left(\frac{1}{e}\right)^k = M_k$

② $\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \left(\frac{1}{e}\right)^k$ converges by geometric series test.

$\therefore \sum_{k=1}^{\infty} \left(\frac{\ln x}{x}\right)^k$ converges uniformly on $[1, \infty)$.

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③ Find $\int_0^1 x^x \, dx$ to 5 decimal places.

Step 1 (Proper or Improper Integral?)

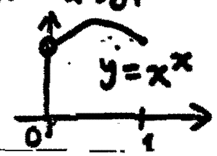
Recall for $a > 0$, $a^b = (e^{\ln a})^b = e^{b \ln a}$

So $\forall x \in (0, 1]$, $x^x = e^{x \ln x}$

$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0$ May set $0 \ln 0 = 0$

$\therefore \lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x \ln x} = e^0 = 1$.

$\therefore x^x$ is integrable on $[0, 1]$



Step 2 (Expand function into Series, check unif. conv.)

Recall $e^w = \sum_{k=0}^{\infty} \frac{w^k}{k!}$ $\forall w \in \mathbb{R}$. $\therefore x^x = e^{x \ln x} = \sum_{k=0}^{\infty} \frac{(x \ln x)^k}{k!}$

① On $[0,1]$, $\frac{d}{dx}(x \ln x) = \ln x + 1 = 0 \Rightarrow x = \frac{1}{e}$.

$\lim_{x\to 0^+} x \ln x = 0$, $\frac{1}{e} \ln \frac{1}{e} = -\frac{1}{e}$, $1 \ln 1 = 0$

$\therefore \forall x \in [0, 1]$, $\left|\frac{(x \ln x)^k}{k!}\right| \le \frac{(1/e)^k}{k!} = M_k$

② $\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} \frac{(1/e)^k}{k!}$ converges by ratio test or using $e^w = \sum_{k=0}^{\infty} \frac{w^k}{k!}$.

$\therefore \sum_{k=0}^{\infty} \frac{(x \ln x)^k}{k!} \rightrightarrows x^x$ on $[0,1]$.

Step 3 (Integrate term-by-term to get answer)

By integration theorem for uniform convergence,

$$\int_0^1 x^x \, dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(x \ln x)^k}{k!} = \sum_{k=0}^{\infty} \int_0^1 \frac{(x \ln x)^k}{k!} \, dx$$

$$\int_0^1 (x \ln x)^k \, dx \underset{t = \ln x}{=} \int_{-\infty}^{0} t^k e^{(k+1)t} \, dt \underset{u = -(k+1)t}{=} \frac{(-1)^k}{k!(k+1)^{k+1}} \int_0^{\infty} u^k e^{-u} \, du$$

$$\int_0^{\infty} u^k e^{-u} \, du = u^k(-e^{-u})\Big|_0^{\infty} + k\int_0^{\infty} u^{k-1} e^{-u} \, du = k \int_0^{\infty} u^{k-1} e^{-u} \, du$$

$$= \cdots = k!$$

$$\therefore \int_0^1 x^x \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)^{k+1}} k! = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{k+1}}$$

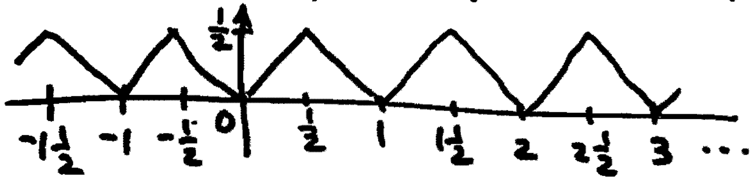
taking 10 terms ≈ 0.78343

Three Applications of Uniform Convergence

① **Theorem** (Due to Weierstrass in 1872)

There exists a continuous function $f: \mathbb{R} \to \mathbb{R}$ which is not differentiable at every $x \in \mathbb{R}$.

Outline For $x \in [0,1)$, define $f_0(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{2}) \\ 1-x & \text{if } x \in [\frac{1}{2}, 1) \end{cases}$ and extend it to a periodic function on $\mathbb{R}$.



Define $f(x) = \sum_{k=0}^{\infty} \frac{f_0(2^k x)}{2^k}$.

Continuity f_0 is continuous on $\mathbb{R}$

$|f_0(x)| \le 1 \ \forall x \in \mathbb{R} \Rightarrow \left|\frac{f_0(2^k x)}{2^k}\right| \le \frac{1}{2^k} = M_k$ on $\mathbb{R}$, $\sum_{k=0}^{\infty} M_k = \sum_{k=0}^{\infty} \frac{1}{2^k}$ converges $\Big\} \Rightarrow$ Series converges uniformly on $\mathbb{R}$

$\therefore f$ is continuous on $\mathbb{R}$ by integration theorem.

Nondifferentiability

$\forall x \in \mathbb{R}$, for $n = 0, 1, 2, 3, \cdots$, $x \in [\underbrace{\frac{m}{2^n}}_{a_n}, \underbrace{\frac{m+1}{2^n}}_{b_n})$, where $m = [2^n x]$

$f'(x) = \lim_{n\to\infty} \frac{f(b_n) - f(a_n)}{b_n - a_n} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \pm 1 = \sum_{k=0}^{\infty} \pm 1$

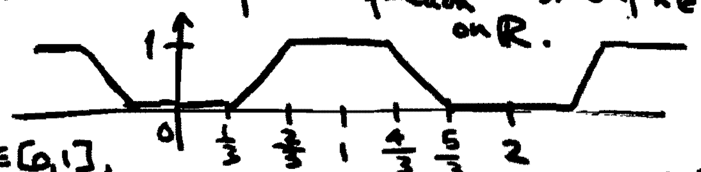
Practice exercise #68 $\qquad$ doesn't converge.

② A point has zero length and zero area in $\mathbb{R}^2$. A continuous curve can have positive length, but can it have positive area?

Theorem (Due to Peano in 1890)

There exists a surjective continuous function $f: [0,1] \to [0,1] \times [0,1]$. So its graph fills the unit square. Such a curve is called a space-filling curve.

Outline For $x \in [0,2]$, define $g(x) = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{3}] \cup [\frac{5}{3}, 2] \\ 3x-1 & \text{if } x \in [\frac{1}{3}, \frac{2}{3}] \\ 1 & \text{if } x \in [\frac{2}{3}, \frac{4}{3}] \\ -3x+5 & \text{if } x \in [\frac{4}{3}, \frac{5}{3}] \end{cases}$ and extend it to a periodic function on $\mathbb{R}$.



$\forall t \in [0,1]$, define $f(t) = (x(t), y(t))$, where $x(t) = \sum_{n=1}^{\infty} \frac{g(3^{2n-2} t)}{2^n}$ and $y(t) = \sum_{n=1}^{\infty} \frac{g(3^{2n-1} t)}{2^n}$. Weierstrass M-test, integration theorem $\Rightarrow x(t), y(t)$ are continuous

So f is continuous.

$\forall (a,b) \in [0,1] \times [0,1]$, write $a = (0.a_1 a_2 a_3 \cdots)_2$, $b = (0.b_1 b_2 b_3 \cdots)_2$ where $a_i, b_i = 0$ or 1 for all i.

Define $c = 2(0.a_1 b_1 a_2 b_2 a_3 b_3 \cdots)_3$.

Then $f(c) = (a,b)$.

③ Weierstrass Approximation Theorem (1886)

Let $f: [0,1] \to \mathbb{R}$ be continuous. For every $\varepsilon > 0$, there exists a polynomial $P(x)$ such that $|f(x) - P(x)| < \varepsilon$ for all $x \in [0,1]$. Taking $\varepsilon = 1/n$, we get a sequence of polynomial $P_n(x)$ converging uniformly on $[0,1]$ to $f(x)$ since $\|f - P_n\|_{[0,1]} \leq \frac{1}{n}$.

Remark The theorem is also true for every closed and bounded interval $[a,b]$.

Outline For $n = 1, 2, 3, \cdots$, define the n-th Bernstein polynomial of $f(x)$ by

$$f_n(x) = \sum_{k=0}^{n} \underbrace{f\left(\tfrac{k}{n}\right)\binom{n}{k}}_{\text{Constant}} \underbrace{x^k (1-x)^{n-k}}_{\text{polynomial}} \qquad \deg f_n \leq n$$

Let $\|f\|_{[0,1]} = M$. For every $\varepsilon > 0$, since f is uniformly continuous on $[0,1]$, there exists $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2.$$

Choose $n > \frac{M}{\varepsilon \delta^2}$. Then $|f(x) - f_n(x)| < \varepsilon \quad \forall x \in [0,1]$

Remarks If we take f to be a continuous, nowhere differentiable function on $[0,1]$, then the theorem provides a sequence of polynomials $P_n(x) \rightrightarrows f(x)$ on $[0,1]$. Even though all $P_n(x)$ are differentiable everywhere on $[0,1]$, their uniform limit function $f(x)$ is very bad in terms of differentiability.