Review: Set unions of sets; intersection of sets; complement of sets;
Cartesian product

Problem Set 2.

Problem 1: A, B : any two sets. The symmetric difference of A and B (denoted by $A \Delta B$). is defined by

set theory, real analysis, measure theory, $\longrightarrow$ $A \Delta B = (A \setminus B) \cup (B \setminus A)$

(a) show that $A \Delta B = (A \cup B) \setminus (A \cap B)$

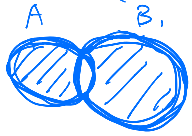
(b) show that $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

for any sets A, B, C.

Proof:

$A \Delta B = (A \setminus B) \cup (B \setminus A)$.

$(A \cup B) \setminus (A \cap B)$



Proof $A = B$

① $A \subseteq B$

② $B \subseteq A$,

(a).

- $A \Delta B \subseteq (A \cup B) \setminus (A \cap B)$

For all $x \in A \Delta B = (A \setminus B) \cup (B \setminus A)$

$\Rightarrow x \in A \setminus B$ or $x \in B \setminus A$

$\Rightarrow x \in A,\ x \notin B$ or $x \in B,\ x \notin A$

$\Rightarrow x \in A \cup B,\ x \notin A \cap B$

$\Rightarrow x \in (A \cup B) \setminus (A \cap B)$

$\Rightarrow A \Delta B \subseteq (A \cup B) \setminus (A \cap B)$

- $(A \cup B) \setminus (A \cap B) \subseteq A \Delta B$

For all $x \in (A \cup B) \setminus (A \cap B)$

$\Rightarrow x \in A \cup B,\ x \notin A \cap B$

$\Rightarrow x \in A,\ x \notin B$ or $x \in B,\ x \notin A$

$\Rightarrow x \in A \setminus B$ or $x \in B \setminus A$

$\Rightarrow x \in (A \setminus B) \cup (B \setminus A)$

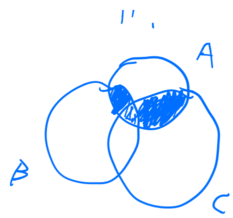
$\Rightarrow x \in A \Delta B$

$\Rightarrow (A \cup B) \setminus (A \cap B) \subseteq A \Delta B$

Conclusion: $A \Delta B = (A \cup B) \setminus (A \cap B)$.

$A \cap (B \Delta C)$.

$||$



(b) $x \in A \cap (B \Delta C)$

$(A\cap B)\Delta(A\cap C)$

$\Leftrightarrow x \in A,\ x \in B \Delta C.$

$\Leftrightarrow x \in A,\ x \in B,\ x \notin C$ OR $x \in A,\ x \notin B,\ x \in C.$

$\Leftrightarrow x \in A\cap B,\ x \notin A\cap C$ OR $x \in A \cap C,\ x \notin A \cap B$

$\Leftrightarrow x \in (A\cap B)\Delta(A\cap C).$

Problem 2: We let A, B, C be 3 sets, prove that

(a) $A\setminus(B\setminus A) = A$

(b) $(A\cap B)\setminus C = (A\setminus C)\cap(B\setminus C).$

Proof:

(a).

- $A\setminus(B\setminus A) \subseteq A$

For all $x \in A\setminus(B\setminus A)$

$\Rightarrow x\in A,\ x \notin B\setminus A.$

$\Rightarrow x \in A$

$\Rightarrow A\setminus(B\setminus A) \subseteq A$

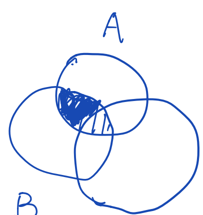
- $A \subseteq A\setminus(B\setminus A)$

For all $x \in A$

$\Rightarrow x \notin B\setminus A$

$\Rightarrow x \in A\setminus(B\setminus A)$

$\Rightarrow A \subseteq A\setminus(B\setminus A).$



$(A\cap B)\setminus C.$

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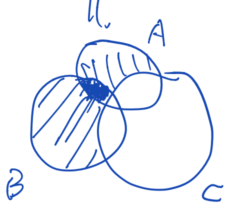


$(A\setminus C)\cap(B\setminus C).$

(b). $x \in (A\cap B)\setminus C$

$\Leftrightarrow x \in A\cap B,\ x\notin C$

$\Leftrightarrow x\in A,\ x\in B,\ x\notin C$

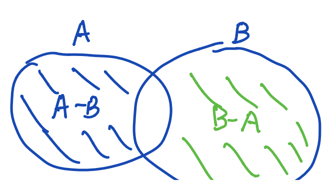
$\Leftrightarrow x \in A\setminus C,\ x\in B\setminus C.$

$\Leftrightarrow x \in (A\setminus C)\cap(B\setminus C)$

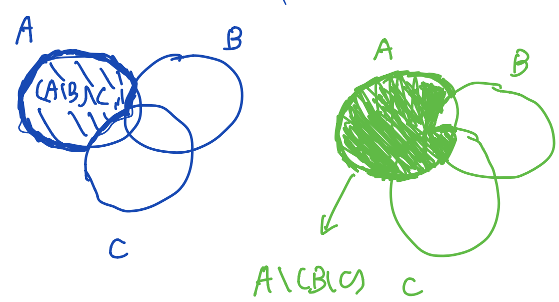
Problem 3.

(a)

$(A-B)\cap(B-A)=\phi.$



(b)



" — "

$(A\setminus B)\setminus C$

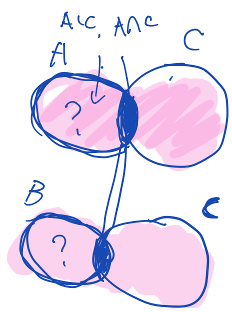
$\neq$ $A\setminus(B\setminus C)$

Problem 4: We let A, B be two subsets of a set Ω.

If there exists a subset $C\subseteq\Omega$ such that

$\rightarrow$ $B\cap C = A\cap C$ and $B\cup C = A\cup C$

Show that $A=B$.



$A = (A\cup C)\setminus(A\setminus C)$.

Proof:

$$A = \underset{\overset{\parallel}{B\cap C}}{(A\cap C)} \cup \underset{\overset{\downarrow}{?}}{(A\setminus C)}$$

$A\setminus C = (A\cup C)\setminus C$ $\rightarrow$ Proof:

$= (B\cup C)\setminus C$

$= B\setminus C$.

$A\setminus C = (A\cup C)\setminus C$.

- $A\setminus C \subseteq (A\cup C)\setminus C$

  For all $x\in A\setminus C$

  $x\in A$, $x\notin C$

  $\Rightarrow x\in A\cup C$, $x\notin C$

  $\Rightarrow x\in(A\cup C)\setminus C$

- $(A\cup C)\setminus C \subseteq A\setminus C$

  For all $x\in(A\cup C)\setminus C$

  $\Rightarrow x\in A\cup C$, $x\notin C$.

  $\Rightarrow x\in A$, $x\notin C$

  $\Rightarrow x\in A\setminus C$.

  $\Rightarrow (A\cup C)\setminus C\subseteq A\setminus C$

$A = (A\cap C)\cup(A\setminus C)$

$= (B\cap C)\cup(B\setminus C)$

$= B$. #

Problem 5) $f:\mathbb{R}\to\mathbb{R}$. Suppose f is strictly increasing

($f(x_1)<f(x_2)$ for any x_1, x_2 which $x_1<x_2$).

(a) Show that $f(x)$ is injective

(b) Determine if $f(x)$ is always surjective.

Proof:

$f(x_1)=f(x_2) \Leftrightarrow x_1=x_2$

if $x_1 < x_2$

$f(x_1) < f(x_2)$

if $x_1 > x_2$

$f(x_1) > f(x_2)$.

(a). if $x_1, x_2 \in \mathbb{R}$, and,

$f(x_1)=f(x_2)$.

① $x_1<x_2 \Rightarrow f(x_1)<f(x_2)$.

$\Rightarrow \underline{x_1<x_2}$ is impossible.

②. $x_1>x_2 \Rightarrow f(x_1)>f(x_2)$

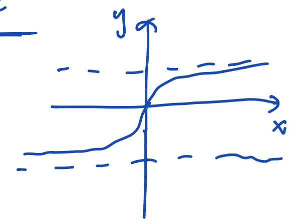
$\Rightarrow \underline{x_1>x_2}$ is impossible

$\Rightarrow \underline{x_1=x_2}$.

$\Rightarrow$ f is injective

(b) $f(x)=\arctan(x)$.

$f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$. machine learning.



not continuous

$$f(x)=\begin{cases} x+1 & x\geq 0 \\ x & x<0. \end{cases}$$

$[0,1)$, X

Not surjective!

Problem 6: $f(x)=\sin x$.

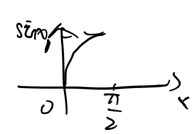
①. range: $[-1, 1]$.

②. injective: not.

$f(x)=f(x+2k\pi)$ for all $k \in \mathbb{Z}$

surjective: $(1,+\infty)\cup(-\infty,-1)$ is not covered.

③ Sure,



Problem 7:

$$|f(x_1) - f(x_2)| = |x_1 - x_2|.$$

Proof: Suppose $x_1, x_2 \in \mathbb{R}$ and $f(x_1) = f(x_2)$

$\Rightarrow |f(x_1) - f(x_2)| = 0$

$= |x_1 - x_2|.$

$\Rightarrow x_1 = x_2.$

Problem 9:

$$A \xrightarrow{f_1, f_2} B \xrightarrow{g} C \xrightarrow{h_1, h_2} D.$$

① $g(f_1(x)) = g(f_2(x))$ for all $x \in A$.

g: injective.

$g(?) = g(A) \Rightarrow ? = A$

$f_1(x) = f_2(x)$. for all $x \in A$.

② $h_1(g(x)) = h_2(g(x))$ for all $x \in B$.

g: surjective.

$\Downarrow$

For any $y \in C$, there exist $x \in B$ such that $g(x) = y$

$\Downarrow$ $h_1(y) = h_1(g(x)) = h_2(g(x)) = h_1(y)$.

$\Rightarrow h_1(y) = h_2(y)$. for all $y \in C$.