Solutions to Practice Exercises MATH 201 Li

- 3 DEC 2004

① $\sim((x>0 \text{ and } x<1) \text{ or } x=-1) = \sim(x>0 \text{ and } x<1) \text{ and } x\neq -1$
$= (x\leq 0 \text{ or } x\geq 1) \text{ and } x\neq -1$

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② $\sim(x>0 \text{ and } (x<1 \text{ or } x=-1)) = x\leq 0 \text{ or } \sim(x<1 \text{ or } x=-1)$
$= x\leq 0 \text{ or } (x\geq 1 \text{ and } x\neq -1)$
$= x\leq 0 \text{ or } x\geq 1$

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③ $\sim(\forall \triangle ABC,\ \angle A+\angle B+\angle C = 180^\circ) = \exists \triangle ABC$ such that $\angle A+\angle B+\angle C \neq 180^\circ$
(There is a triangle ABC such that $\angle A+\angle B+\angle C\neq 180^\circ$.)

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④ $\sim(\exists$ man such that man does not have wife$)$ = $\forall$ man, man has a wife
(Every man has a wife.)

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⑤ $\sim(\forall x.\ \exists y$ such that $x+y=0) = \exists x\ \forall y,\ x+y\neq 0$
(There is an x such that for every y, $x+y\neq 0$.)

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⑥ $\sim(\exists \alpha\ \forall \beta\ \exists \gamma$ such that $|\alpha-\beta|<\gamma) = \forall\alpha\ \exists\beta\ \forall\gamma,\ |\alpha-\beta|\geq\gamma$.

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⑦ $\sim($If $(x>0)$ and $(y>0)$, then $x+y>0) = (x>0)$ and $(y>0)$ and $(x+y\leq 0)$

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⑧ (a) If $\angle B\neq\angle C$ in $\triangle ABC$, then $\overline{AB}\neq \overline{AC}$ in $\triangle ABC$.

(b) If a function is not continuous, then it is not differentiable.

(c) If $\lim\limits_{x\to 0}(f(x)+g(x))\neq a+b$, then $\lim\limits_{x\to 0} f(x)\neq a$ or $\lim\limits_{x\to 0} g(x)\neq b$.

(d) If $x\neq \frac{-b+\sqrt{b^2-4c}}{2}$ and $x\neq\frac{-b-\sqrt{b^2-4c}}{2}$, then $x^2+bx+c\neq 0$.

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⑨ (a) $(\{x,y,z\}\cup\{w,z\})\setminus\{u,v,w\} = \{w,x,y,z\}\setminus\{u,v,w\} = \{x,y,z\}$.

(b) $\{1,2\}\times\{3,4\}\times\{5\} = \{(1,3,5),(1,4,5),(2,3,5),(2,4,5)\}$.

(c) $\mathbb{Z}\cap[0,10]\cap\{n^2+1 : n\in\mathbb{N}\} = \{0,1,2,\cdots,10\}\cap\{2,5,10,\dots\} = \{2,5,10\}$.

(d) $\{n\in\mathbb{N} : 5<n<9\}\setminus\{2m : m\in\mathbb{N}\} = \{6,7,8\}\setminus\{2,4,6,8,10,\dots\} = \{7\}$.

(e) $([0,2]\setminus[1,3])\cup([1,3]\setminus[0,2]) = [0,1)\cup(2,3]$.

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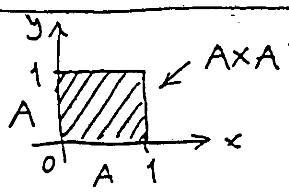
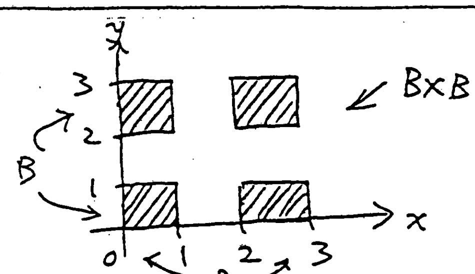
⑩ (i)





(ii) $A=B$ (Reason: For every $a\in A$, $b\in B$, we have $(a,b)\in A\times B = B\times A$.
By the definition of Cartesian product, this means $a\in B$, $b\in A$. So $A\subseteq B$ and $B\subseteq A$.)

(11) (a) If $x \in A \cup B$, then $x \in A$ or $x \in B$, which implies $x \in A$ or $x \in C$ (because $B \subseteq C$ and $x \in B$ will yield $x \in C$). So $x \in A \cup C$. So every element of $A \cup B$ is also an element of $A \cup C$. Therefore, $A \cup B \subseteq A \cup C$.

(b) If $x \in (X \setminus Y) \setminus Z$, then $x \in X \setminus Y$ and $x \notin Z$. So $x \in X$ and $x \notin Y$ and $x \notin Z$. Then $x \in X$ and $x \notin Z$ and $x \notin Y$. Hence, $x \in X \setminus Z$ and $x \notin Y$. Therefore, $x \in (X \setminus Z) \setminus Y$. We get $(X \setminus Y) \setminus Z \subseteq (X \setminus Z) \setminus Y$.

Interchanging Y, Z everywhere in the last paragraph, we also get $(X \setminus Z) \setminus Y \subseteq (X \setminus Y) \setminus Z$. Therefore, $(X \setminus Y) \setminus Z = (X \setminus Z) \setminus Y$.

(12) (i) False. For example, $A = \mathbb{R} \setminus \mathbb{Q}$, $B = \mathbb{Q} = C$, then $(A \cup B) \cap C = \mathbb{R} \cap \mathbb{Q} = \mathbb{Q}$, $A \cup (B \cap C) = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}$

(ii) False. For example, $A = \mathbb{R} = B$, $C = \mathbb{Q}$, then $A \cup B = \mathbb{R} = A \cup C$, but $B \neq C$.

(iii) True. (Reason: For every $x \in A \setminus (B \cup C)$, we have $x \in A$ and $x \notin B \cup C$. Now $\boxed{x \notin B \cup C = \sim(x \in B \cup C) = \sim((x \in B) \text{ or } (x \in C)) = x \notin B \text{ and } x \notin C}$. So $x \in A \setminus B$ and $x \in A \setminus C$. We get $x \in (A \setminus B) \cap (A \setminus C)$. $\therefore$ $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$.

Next we reverse steps. For every $x \in (A \setminus B) \cap (A \setminus C)$, we have $x \in A \setminus B$ and $x \in A \setminus C$. So $x \in A$ and $x \notin B$ and $x \notin C$. By the box above, we get $x \in A$ and $x \notin B \cup C$. So $x \in A \setminus (B \cup C)$. $\therefore$ $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$.)

(13) (i) For every $x \in A \cup C$, we have $x \in A$ or $x \in C$. If $x \in A$, then $A \subseteq B$ implies $x \in B$. If $x \in C$, then $C \subseteq D$ implies $x \in D$. So $x \in B$ or $x \in D$, which implies $x \in B \cup D$.

(ii) False. For example, let $A = \{0\}$, $C = \{1\}$, $B = \{0,1\} = D$, then $A \cup C = \{0,1\} = B \cup D$.

(iii) Yes. (Reason: Since $(\frac{1}{n}, 2)_{\mathbb{Q}} \subseteq [\frac{1}{n}, 2)_{\mathbb{Q}}$ for each n, so as in part (i),

$$\bigcup_{n=1}^{\infty} (\tfrac{1}{n}, 2)_{\mathbb{Q}} = (1,2)_{\mathbb{Q}} \cup (\tfrac{1}{2}, 2)_{\mathbb{Q}} \cup (\tfrac{1}{3}, 2)_{\mathbb{Q}} \cup \cdots \subseteq [1,2)_{\mathbb{Q}} \cup [\tfrac{1}{2}, 2)_{\mathbb{Q}} \cup [\tfrac{1}{3}, 2)_{\mathbb{Q}} \cup \cdots = \bigcup_{n=1}^{\infty} [\tfrac{1}{n}, 2)_{\mathbb{Q}}.$$

For the reverse inclusion, since $[\frac{1}{n}, 2)_{\mathbb{Q}} \subseteq (\frac{1}{n+1}, 2)_{\mathbb{Q}}$ for each n, we have

$$\bigcup_{n=1}^{\infty} [\tfrac{1}{n}, 2)_{\mathbb{Q}} = [1,2)_{\mathbb{Q}} \cup [\tfrac{1}{2}, 2)_{\mathbb{Q}} \cup [\tfrac{1}{3}, 2)_{\mathbb{Q}} \cup \cdots \subseteq (\tfrac{1}{2}, 2)_{\mathbb{Q}} \cup (\tfrac{1}{3}, 2)_{\mathbb{Q}} \cup (\tfrac{1}{4}, 2)_{\mathbb{Q}} \cup \cdots = \bigcup_{n=1}^{\infty} (\tfrac{1}{n}, 2)_{\mathbb{Q}}$$

because $(\frac{1}{2}, 2)_{\mathbb{Q}} = (1,2)_{\mathbb{Q}} \cup (\frac{1}{2}, 2)_{\mathbb{Q}}$.)

$\boxed{\text{Actually, } \bigcup_{n=1}^{\infty} [\tfrac{1}{n}, 2)_{\mathbb{Q}} = (0,2)_{\mathbb{Q}} = \bigcup_{n=1}^{\infty} (\tfrac{1}{n}, 2)_{\mathbb{Q}} \text{ but this is less rigorous.}}$

(14) f is not injective because $f(1) = 0 = f(2)$. f is not surjective because $f(\mathbb{R}) = \{0,1\} \neq \mathbb{R}$. g is injective because $g(x) = g(y) \Leftrightarrow 1 - 2x = 1 - 2y$ implies $x = y$. g is surjective because for every $y \in \mathbb{R}$, $y = g(\frac{1-y}{2})$ and so $g(\mathbb{R}) = \mathbb{R}$.

$f \circ g : \mathbb{R} \to \mathbb{R}$ is given by $(f \circ g)(x) = f(g(x)) = f(1-2x) = \begin{cases} 0 & \text{if } 1-2x > 0 \\ 1 & \text{if } 1-2x \leq 0 \end{cases}$
$= \begin{cases} 0 & \text{if } \frac{1}{2} > x \\ 1 & \text{if } \frac{1}{2} \leq x \end{cases}$. $g \circ f : \mathbb{R} \to \mathbb{R}$ is given by $(g \circ f)(x) = g(f(x)) = \begin{cases} 1 = g(0) & \text{if } x > 0 \\ -1 = g(1) & \text{if } x \leq 0 \end{cases}$.

(15) (i) To show f is injective, let $f(x)=f(y)$. Then $x=(g\circ f)(x)=g(f(x))=g(f(y))$ $=(g\circ f)(y)=y$. Next we will show f is surjective. For every $b\in B$, since $b=(f\circ g)(b)=f(g(b))$, we see that $b\in f(A)$. $\therefore f(A)=B$.

(ii) To show $h\circ f$ is injective, let $(h\circ f)(x)=(h\circ f)(y)$. Then $h(f(x))=h(f(y))$. Since h is injective, we get $f(x)=f(y)$. Since f is injective, we get $x=y$. Next we will show $h\circ f$ is surjective. For every $c\in C$, since h is surjective, $C=h(B)$, which implies $c=h(b)$ for some $b\in B$. Since f is surjective, $B=f(A)$, which implies $b=f(a)$ for some $a\in A$. Then $c=h(b)=h(f(a))$ $=(h\circ f)(a)\in(h\circ f)(A)$. $\therefore (h\circ f)(A)=C$.

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(16) For the 'at most once' case, to show f is injective, let $f(x_0)=f(y_0)$. Using the choice $b=f(x_0)$, we see that the line $y=b$ intersects the graph of f at the point $(x_0,f(x_0))$ and at the point $(y_0,f(y_0))$. Since the intersection is at most one point, we have $(x_0,f(x_0))=(y_0,f(y_0))$, which implies $x_0=y_0$.
For the 'at least once' case, we can conclude f is surjective. (<u>Reason</u>: For every $b\in B$, the line $y=b$ intersects the graph of f at least once. This implies there is a point (a,b) on the graph of f. Then $b=f(a)\in f(A)$. $\therefore f(A)=B$.)

(<u>Comments</u>: Combining the two cases, we see that if for every $b\in B$, the horizontal line $y=b$ intersects the graph of f <u>exactly once</u>, then f is a <u>bijection</u>. This "horizontal line test" is useful to check bijections by inspecting the graphs.)

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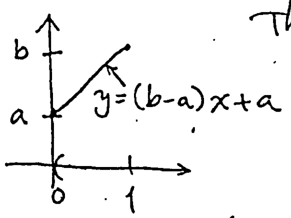
(17)



The function $f:(0,1)\to(a,b)$ defined by $f(x)=(b-a)x+a$ is a bijection. (This is clear from the graph. As x varies from a to b, $f(x)$ takes each of the values between a and b exactly once.) Since $(0,1)$ is uncountable, by the bijection theorem we see that (a,b) is uncountable. Since $(a,b)\subseteq[a,b]$, by the countable subset theorem, $[a,b]$ is uncountable.

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(18) Let $S=\{(0,y): y\in\mathbb{R}\setminus\mathbb{Q}\}$. The function $f:\mathbb{R}\setminus\mathbb{Q}\to S$ defined by $f(y)=(0,y)$ is a bijection. Since $\mathbb{R}\setminus\mathbb{Q}$ is uncountable, by the remarks, S is uncountable. Since $S\subseteq\mathbb{Q}\times(\mathbb{R}\setminus\mathbb{Q})$, by the countable subset theorem, $\mathbb{Q}\times(\mathbb{R}\setminus\mathbb{Q})$ is uncountable.

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(19) For $n,m\in\mathbb{Z}$, $\frac{1}{2^n}+\frac{1}{3^m}\in\mathbb{Q}$. So $A\subseteq\mathbb{Q}$. Since $\mathbb{Q}$ is countable, by the countable subset theorem, A is countable.

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(20) For $x \in \mathbb{N}$, let $B_x = \{x+\sqrt{2}y : y \in \mathbb{N}\}$. The function $f: \mathbb{N} \to B_x$ defined by $f(y) = x+\sqrt{2}y$ is a bijection. So B_x is countable. Now $B = \bigcup_{x\in\mathbb{N}} B_x$, $\mathbb{N}$ is countable, each B_x is countable for $x \in \mathbb{N}$, so by the countable union theorem, B is countable. 12

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(21) Let $S = \{L_m : L_m$ is the line with equation $y = mx$, $m \in \mathbb{R}\}$. The function $f: \mathbb{R} \to S$ defined by $f(m) = L_m$ is a bijection. Since $\mathbb{R}$ is uncountable, by the remarks, S is uncountable. Since $S \subsetneq C$, by the countable subset theorem, C is uncountable.

C contains vertical line, not in S,

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(22) For $r \in \mathbb{Q}$, $D_r = \{x \in \mathbb{R} \mid x^5 + x + 2 = r\}$ has at most 5 elements, so D_r is countable. Now $D = \bigcup_{r\in\mathbb{Q}} D_r$, $\mathbb{Q}$ is countable and each D_r is countable for $r \in \mathbb{Q}$, so by the countable union theorem, D is countable.

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(23) Let $\mathbb{Q}^+$ be the positive rational numbers. Since $\mathbb{Q}^+ \subseteq \mathbb{Q}$ and $\mathbb{Q}$ is countable, by the countable subset theorem, $\mathbb{Q}^+$ is countable. Now the function $f: \mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}^+ \to E$ defined by letting $f(x,y,r)$ be the circle centered at (x,y) and radius r is a bijection. Since $\mathbb{Q}$ and $\mathbb{Q}^+$ are countable, by the product theorem, $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}^+$ is countable. By the remarks, we see E is countable.

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(24) Suppose $x^4 + ax - 5 = 0$ has a rational root r. (If $r=0$, then $r^4 + ar - 5 \neq 0$.) We get $r \neq 0$ and $r^4 + ar - 5 = 0 \Rightarrow a = \frac{5-r^4}{r} \in \mathbb{Q}$. So $F \subseteq \mathbb{Q}$. Therefore, F is countable.

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(25) Since X is nonempty, let $a_0 \in X$. Consider the subset $G' = \{a_0^3 + b^3 : b \in Y\}$ of G. The function $f: Y \to G'$ defined by $f(b) = a_0^3 + b^3$ is a bijection (From $w = a_0^3 + b^3 \Leftrightarrow b = \sqrt[3]{w - a_0^3}$, we see $g: G' \to Y$ defined by $g(w) = \sqrt[3]{w - a_0^3}$ is the inverse of f.) Since Y is uncountable, so G' is uncountable. Since $G' \subseteq G$, so G is also uncountable.

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(26) We will show $Y \setminus X$ is uncountable first. Suppose $Y \setminus X$ is countable. Since X is countable and $X \cap Y \subseteq X$, we get $X \cap Y$ countable by the countable subset theorem. Then $Y = (Y\setminus X) \cup (X \cap Y)$ is countable by the countable union theorem, a contradiction. $\therefore$ $Y \setminus X$ is uncountable. Since $Y \setminus X \subseteq (X\setminus Y) \cup (Y \setminus X)$, $H = (X \setminus Y) \cup (Y\setminus X)$ is uncountable by the countable subset theorem.

(27) Solution 1. For $k=0,1,2,\cdots$, let S_k be the set of all subsets of $\mathbb{N}$ having exactly k elements. Then $S_0=\{\emptyset\}$ has one element and so S_0 is countable. For $k\in\mathbb{N}$, the function $f_k: S_k \to \underbrace{\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}}_{k\ \mathbb{N}\text{'s}}$ defined by $f(\{n_1,n_2,\cdots,n_k\}) = (n_1,n_2,\cdots,n_k)$ (increasing order) is an injective function. Since $\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}$ is countable by the product theorem, we can use the first part of the theorem on page 10 to conclude that S_k is countable. Then $F = S_0 \cup \left(\bigcup_{k=1}^{\infty} S_k\right)$ is countable by the countable union theorem.

Solution 2 Define $g: F \to \mathbb{N}\cup\{0\}$ by assigning to each finite subset S of $\mathbb{N}$ the nonnegative integer n having base 2 representation $n=(\cdots d_3d_2d_1)_2$, where $d_j=1$ if and only if $j\in S$. (For example, $S=\{1,2,4\} \to n=(1011)_2=8+2+1(=11)$) Note g has the inverse $g^{-1}: \mathbb{N}\cup\{0\}\to F$ by assigning $n=(\cdots d_3d_2d_1)_2$ the subset $\{j: d_j=1\}$. It follows g is a bijection. As $\mathbb{N}\cup\{0\}$ is countable, so F is countable.

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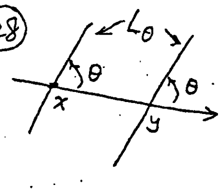
(28) 

For $\theta\in(0,\pi)$, let L_θ be the pair of lines through x and y respectively making an angle θ with the axis from x to y. Let $T=\{L_\theta : \theta\in(0,\pi)\}$. The function $f:(0,1)\to T$ defined by $f(x)=L_{\pi x}$ has the inverse $g: T\to(0,1)$ given by $g(L_\theta)=\frac{\theta}{\pi}$. So f is a bijection. Since $(0,1)$ is uncountable, T is uncountable.
Next observe that for every $z\in S$, there are at most two θ's such that z is on one of the lines of L_θ, namely when $\overleftrightarrow{xz}$ or $\overleftrightarrow{yz}$ is one of the lines of L_θ. So $V=\{L_\theta : L_\theta \text{ contains some } z\in S\} = \underbrace{\bigcup_{z\in S}}_{\text{countable}} \underbrace{\{L_\theta : L_\theta \text{ contains } z\}}_{\text{at most 2 elements}}$ is countable.
Since T is uncountable and V is countable, so $T\setminus V$ is uncountable. In particular, taking two distinct L_θ's in $T\setminus V$, the parallelogram determined by them is in $\mathbb{R}^2\setminus S$ and has x,y as opposite vertices.

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(29) For $x\in[0,1]$, let $x \doteq (0.a_1a_2a_3\cdots)_3$. Observe that $(\frac{1}{3},\frac{2}{3})=\{x: a_1=1\}$ where we take $\frac{1}{3}=(0.022\cdots)_3$. So $K_1=[0,1]\setminus(\frac{1}{3},\frac{2}{3})=\{x: a_1\neq 1\}$
Also, $(\frac{1}{9},\frac{2}{9})=\{x: a_1=0, a_2=1\}$ where we take $\frac{1}{9}=(0.0022\cdots)_3$ and
$(\frac{7}{9},\frac{8}{9})=\{x: a_1=2, a_2=1\}$ where we take $\frac{7}{9}=(0.2022\cdots)_3$.
So $K_2=K_1\setminus\left((\frac{1}{9},\frac{2}{9})\cup(\frac{7}{9},\frac{8}{9})\right)=\{x: a_1\neq 1, a_2\neq 1\}$. Similarly, we will get $K_n=\{x: a_1\neq 1, a_2\neq 1,\cdots,a_n\neq 1\}$. Therefore $K=\{x: \text{all } a_i\neq 1\}$ $=\{x: \text{all } a_i = 0 \text{ or } 2\}$. Define $f:\{0,1\}\times\{0,1\}\times\cdots\to K$ by $f((b_1,b_2,\cdots))=x$ where $a_i=2b_i$ for $i=1,2,3,\cdots$. This function has the inverse $g: K\to\{0,1\}\times\{0,1\}\times\cdots$ defined by $g(x)=(b_1,b_2,\cdots)$, where $b_i=\frac{a_i}{2}$ for $i=1,2,3,\cdots$. So f is a bijection. Since $\{0,1\}\times\{0,1\}\times\cdots$ is uncountable, K is uncountable.

Remarks In the above solution, when we wrote K_1, K_2, K_n, K as sets of x with $a_i\neq 1$, we mean "x has at least one base 3 representation, where the a_i's $\neq 1$"

(30) (a) Since $\lim_{k\to\infty} \cos(\sin \frac{1}{k}) = \cos(\sin 0) = \cos 0 = 1 \neq 0$, by term test, the series diverges.

(b) For k large, $\frac{1}{\sqrt{k(k+1)(k+2)}} \approx \frac{1}{\sqrt{k\cdot k\cdot k}} = \frac{1}{k^{3/2}}$. So we apply the limit comparison test. Since $\lim_{k\to\infty} \frac{\frac{1}{\sqrt{k(k+1)(k+2)}}}{\frac{1}{\sqrt{k\cdot k\cdot k}}} = \lim_{k\to\infty} \sqrt{\frac{k}{k}\frac{k}{k+1}\frac{k}{k+2}} = \sqrt{1\cdot 1\cdot 1} = 1$ and $\underbrace{\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}}_{p \text{ series, } p=3/2>1}$ converges by p-test, so $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)(k+2)}}$ converges.

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(c) $\sum_{k=1}^{\infty} \ln(1+\frac{1}{k}) = \sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right) = \underbrace{\sum_{k=1}^{\infty}(\ln(k+1) - \ln k)}_{\text{telescoping series}} = \lim_{n\to\infty}(\ln(n+1) - \underbrace{\ln 1}_{0}) = +\infty$, the series diverges (to $+\infty$).

(d) Solution 1 By root test, $\lim_{k\to\infty} \sqrt[k]{\left(\frac{1}{2}+\frac{1}{k}\right)^k} = \lim_{k\to\infty}\left(\frac{1}{2}+\frac{1}{k}\right) = \frac{1}{2} < 1 \Rightarrow$ series converges.

Solution 2 For $k \geq 3$, $0 \leq \left(\frac{1}{2}+\frac{1}{k}\right)^k \leq \left(\frac{1}{2}+\frac{1}{3}\right)^k = \left(\frac{5}{6}\right)^k$ and $\underbrace{\sum_{k=3}^{\infty}\left(\frac{5}{6}\right)^k}_{\text{geometric series, } r=\frac{5}{6}}$ converges by geometric series test, so $\sum_{k=1}^{\infty}\left(\frac{1}{2}+\frac{1}{k}\right)^k = \frac{3}{2}+1+\sum_{k=3}^{\infty}\left(\frac{1}{2}+\frac{1}{k}\right)^k$ converges.

(e) Solution 1 Since $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2} < 0$ for $x > e$ and $\lim_{k\to\infty} \frac{\ln k}{k} \left(= \lim_{k\to\infty} \frac{\frac{1}{k}}{1} = 0\right)$, so $\frac{\ln x}{x}$ decreases to 0 for $x > e$. Now $\int_3^{\infty} \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2\Big|_3^{\infty} = \infty$. By the integral test, $\sum_{k=2}^{\infty} \frac{\ln k}{k} = \frac{\ln 2}{2} + \sum_{k=3}^{\infty} \frac{\ln k}{k}$ diverges.

Solution 2 For $k \geq 3$, $\frac{1}{k} \leq \frac{\ln k}{k}$. Since $\underbrace{\sum_{k=3}^{\infty} \frac{1}{k}}_{p\text{-series, } p=1}$ diverges by p-test, so $\sum_{k=2}^{\infty} \frac{\ln k}{k}$ diverges by the comparison test.

(f) Since $\left|\frac{\cos 2^k}{k^2}\right| \leq \frac{1}{k^2}$ and $\underbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}_{p\text{-series, } p=2}$ converges by p-test, so $\sum_{k=1}^{\infty} \left|\frac{\cos 2^k}{k^2}\right|$ converges by the comparison test. Then $\sum_{k=1}^{\infty} \frac{\cos 2^k}{k^2}$ converges by the absolute convergence test.

(g) Solution 1 (Limit Comparison Test) Since $\lim_{k\to\infty} \frac{\left(\frac{k+2}{k+1}\right)\left(\frac{2}{3}\right)^k}{\left(\frac{2}{3}\right)^k} = \lim_{k\to\infty} \frac{k+2}{k+1} = 1$ and $\underbrace{\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k}_{\text{geometric, } r=\frac{2}{3}<1}$ converges, so $\sum_{k=1}^{\infty} \frac{k+2}{k+1}\left(\frac{2}{3}\right)^k$ converges.

Solution 2 (Ratio test) $\lim_{k\to\infty} \frac{\frac{k+3}{k+2}\left(\frac{2}{3}\right)^{k+1}}{\frac{k+2}{k+1}\left(\frac{2}{3}\right)^k} = \lim_{k\to\infty} \frac{(k+3)(k+1)}{(k+2)^2}\frac{2}{3} = \frac{2}{3} < 1$, series converges.

(h) Since $\cos k\pi = (-1)^k$ and ($k \nearrow \infty \Rightarrow \sqrt{k} \nearrow \infty \Rightarrow \frac{1}{\sqrt{k}} \searrow 0$), so $\sum_{k=1}^{\infty} \frac{\cos k\pi}{\sqrt{k}}$ converges by the alternating series test.

30 (i) Solution 1 Since $\frac{d}{dx}(xe^{-x^2}) = e^{-x^2}(1-2x^2) < 0$ for $x \ge 1$ and $\lim_{x\to\infty} \frac{x}{e^{x^2}} = \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0$, xe^{-x^2} decreases to 0 as $x \to \infty$. Now $\int_1^\infty xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}\Big|_0^\infty = 0-(-\frac{1}{2}e^{-1}) < \infty$. By the integral test, $\sum_{k=1}^{\infty} ke^{-k^2}$ converges.

Solution 2 (Ratio test) $\lim_{k\to\infty} \frac{(k+1)e^{-(k+1)^2}}{ke^{-k^2}} = \lim_{k\to\infty} \frac{k+1}{k} e^{-2k-1} = 0 < 1 \Rightarrow \sum_{k=1}^{\infty} ke^{-k^2}$ converges.

(j) Solution 1 (Ratio test) $\lim_{k\to\infty} \frac{\frac{k+1}{(k+2)!}}{\frac{k}{(k+1)!}} = \lim_{k\to\infty} \frac{k+1}{k}\frac{(k+1)!}{(k+2)!} = \lim_{k\to\infty}\frac{k+1}{k}\lim_{k\to\infty}\frac{1}{k+2} = 1\cdot 0 = 0 < 1$, Series Converges.

Solution 2 $\sum_{k=1}^{\infty}\frac{k}{(k+1)!} = \sum_{k=1}^{\infty}\frac{(k+1)-1}{(k+1)!} = \underbrace{\sum_{k=1}^{\infty}\left(\frac{1}{k!}-\frac{1}{(k+1)!}\right)}_{\text{telescoping series}} = \lim_{n\to\infty}\left(\frac{1}{1!}-\frac{1}{(n+1)!}\right) = 1$.

(k) Solution 1 Since $0 \le \frac{\text{Arctan}\, k}{k^2+1} \le \frac{\frac{\pi}{2}}{k^2}$ and $\sum_{k=1}^{\infty}\frac{\frac{\pi}{2}}{k^2} = \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by p-test, so $\sum_{k=1}^{\infty}\frac{\text{Arctan}\, k}{k^2+1}$ converges by the comparison test.

Solution 2 $0 \le \frac{\text{Arctan}\, k}{k^2+1} \le \frac{\frac{\pi}{2}}{k^2+1} \overset{\text{sandwich thm}}{\Rightarrow} \lim_{k\to\infty}\frac{\text{Arctan}\, k}{k^2+1} = 0$. Now $\frac{d}{dx}\left(\frac{\text{Arctan}\, x}{x^2+1}\right) = \frac{1-2x\,\text{Arctan}\, x}{(x^2+1)^2}$

Since x, Arctan x are increasing, $1-2x\,\text{Arctan}\, x \le 1-2\,\text{Arctan}\, 1 < 0$ for $x \ge 1$.

So $\frac{\text{Arctan}\, x}{x^2+1}$ decreases to 0 as $x\to\infty$. Now $\int_1^\infty \frac{\text{Arctan}\, x}{x^2+1}dx = \frac{1}{2}(\text{Arctan}\, x)^2\Big|_1^\infty$ $= \frac{1}{2}(\frac{\pi}{2})^2 - \frac{1}{2}(\frac{\pi}{4})^2 < \infty$. By the integral test, $\sum_{k=1}^{\infty}\frac{\text{Arctan}\, k}{k^2+1}$ converges.

(l) Since $\lim_{k\to\infty}\frac{\frac{1}{k^{1+1/k}}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{1}{k^{1/k}} = 1$ and $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges by p-test, so $\sum_{k=1}^{\infty}\frac{1}{k^{1+1/k}}$ diverges by the limit comparison test.

(m) By the root test, $\lim_{k\to\infty}\sqrt[k]{\left|\tan^k\left(\frac{k+1}{k}\right)\right|} = \lim_{k\to\infty}\left|\tan\left(\frac{k+1}{k}\right)\right| = \tan 1 > \tan\frac{\pi}{4} = 1$

$\Rightarrow$ Series diverges

(n) $\lim_{k\to\infty}\frac{1-\cos\frac{1}{k}}{\frac{1}{k^p}} \underset{\theta=\frac{1}{k}}{=} \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^p} = \lim_{\theta\to 0}\frac{\sin\theta}{p\theta^{p-1}} = \frac{1}{2}$ (if we set $p=2$). Since $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converge by p-test, so $\sum_{k=1}^{\infty}(1-\cos\frac{1}{k})$ converges by the limit comparison test.

(o) $\lim_{k\to\infty}\frac{k^2\sin^p(\frac{1}{k})}{k^2(\frac{1}{k})^p} = \lim_{k\to\infty}\left(\frac{\sin(\frac{1}{k})}{\frac{1}{k}}\right)^p \underset{\theta=\frac{1}{k}}{=} \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^p = 1$. Since $\sum_{k=1}^{\infty}k^2(\frac{1}{k})^p = \sum_{k=1}^{\infty}\frac{1}{k^{p-2}}$ converges iff $p-2>1$ by the p-test, so $\sum_{k=1}^{\infty}k^2\sin^p(\frac{1}{k})$ converges iff $p>3$.

(p) $\sqrt{k+1}-\sqrt{k} = (\sqrt{k+1}-\sqrt{k})\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}} = \frac{1}{\sqrt{k+1}+\sqrt{k}}$ decreases to 0 as $x\to\infty$.
So $\sum_{k=1}^{\infty}(-1)^{k+1}(\sqrt{k+1}-\sqrt{k})$ converges by the alternating series test.

(31) Let $S_n = a_1 + a_2 + \cdots + a_n$ and $t_k = 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$. By definition, $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n$ and $\sum_{k=1}^{\infty} 2^k a_{2^k} = \lim_{k\to\infty} t_k$. Since $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$, S_n and t_k's are increasing. Their limits are either numbers or $+\infty$. Now

$$S_{2^k-1} = a_1 + (a_2+a_3) + (a_4+\cdots+a_7) + \cdots + (a_{2^{k-1}} + \cdots + a_{2^k-1})$$
$$\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^{k-1} a_{2^{k-1}} = a_1 + t_{k-1}$$

So if $\lim_{k\to\infty} t_k < \infty$, then $\lim_{n\to\infty} S_n \underset{(S_n \text{ increasing})}{=} \lim_{k\to\infty} S_{2^k-1} \leq a_1 + \lim_{k\to\infty} t_k < \infty$.

Conversely, $t_k = 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = 2(a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k}) \leq 2(a_2 + (a_3+a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k})) = 2(S_{2^k} - a_1)$.

So if $\lim_{k\to\infty} S_n = \lim_{k\to\infty} S_{2^k} < \infty$, then $\lim_{k\to\infty} t_k \leq 2(\lim_{k\to\infty} S_{2^k} - a_1) < \infty$.

For the second part, $\sum_{k=3}^{\infty} 2^k \frac{1}{2^k \ln 2^k \ln(\ln 2^k)} = \sum_{k=3}^{\infty} \frac{1}{k \ln 2 (\ln k + \ln \ln 2)}$. We compare this with $\sum_{k=3}^{\infty} \frac{1}{k \ln k}$. Since $\lim_{k\to\infty} \frac{\frac{1}{k\ln 2(\ln k + \ln\ln 2)}}{\frac{1}{k \ln k}} = \lim_{k\to\infty} \frac{1}{\ln 2 + \frac{\ln \ln 2}{\ln k}} = \frac{1}{\ln 2}$ and $\sum_{k=3}^{\infty} \frac{1}{k \ln k}$ diverges by example of integral test, so $\sum_{k=3}^{\infty} 2^k \frac{1}{2^k \ln 2^k \ln(\ln 2^k)}$ diverges by the limit comparison test. By first part, $\sum_{k=3}^{\infty} \frac{1}{k \ln k \ln(\ln k)}$ diverges.

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(32) Since $\lim_{k\to\infty} \frac{k+1}{2^k} \Big/ \frac{k}{2^{k-1}} = \lim_{k\to\infty} \frac{k+1}{k} \frac{1}{2} = \frac{1}{2} < 1$, the series converges by the ratio test.

Now
$$S = \sum_{k=2}^{\infty} \frac{k}{2^{k-1}} = \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \cdots$$
$$\frac{1}{2}S = \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots$$
$$\frac{1}{2}S = S - \frac{1}{2}S = \frac{2}{2} + \left(\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\right) = \frac{2}{2} + \frac{\frac{1}{2^2}}{1-\frac{1}{2}} = \frac{3}{2}$$

Therefore, $S = 3$.

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(33) Suppose $\sum_{k=1}^{\infty} \frac{1}{p_k} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ converges to s. Then $s_n = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p_n}$ has limit s as $n \to \infty$, i.e. $\lim_{n\to\infty}(s - s_n) = 0$. So for some n, $s - s_n = \frac{1}{p_{n+1}} + \frac{1}{p_{n+2}} + \cdots = \sum_{k=n+1}^{\infty} \frac{1}{p_k} < \frac{1}{2}$.

Let $Q = p_1 p_2 \cdots p_n$, then the numbers $1 + mQ$ cannot be divisible by $p_1, p_2, \cdots, p_n$.

So $1 + mQ = p_{n+1}^{e_{n+1}} p_{n+2}^{e_{n+2}} \cdots$, where the exponents e_k are nonnegative integers.

Let $j = e_{n+1} + e_{n+2} + \cdots$ (only finitely many $e_k \neq 0$), then

$$\frac{1}{1+mQ} = \frac{1}{p_{n+1}^{e_{n+1}}} \cdot \frac{1}{p_{n+2}^{e_{n+2}}} \cdots \text{ is a term in } \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^{e_{n+1}} \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^{e_{n+2}} \cdots = \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^{j}$$

So the numbers $\frac{1}{1+Q}, \frac{1}{1+2Q}, \cdots, \frac{1}{1+NQ}$ will correspond to N terms of $\sum_{j=1}^{\infty} \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^j$, which is less than $\sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = 1$. Then

$$\sum_{m=1}^{N} \frac{1}{1+mQ} \leq \sum_{j=1}^{\infty} \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^j < \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = 1 \quad \text{for every positive integer } N.$$

Since $\frac{1}{2mQ} \leq \frac{1}{1+mQ}$ and $\sum_{m=1}^{\infty} \frac{1}{2mQ} = \frac{1}{2Q} \sum_{m=1}^{\infty} \frac{1}{m}$ diverges to $+\infty$ by p-test, so $\sum_{m=1}^{\infty} \frac{1}{1+mQ}$ diverges to $+\infty$ by the comparison test. Therefore $\sum_{m=1}^{N} \frac{1}{1+mQ} < 1$ cannot hold for all positive integer N, a contradiction.

(34) (a) $A = \{\sqrt{1}+\sqrt{1}, \sqrt{2}+\sqrt{1}, \sqrt{1}+\sqrt{2}, \cdots\}$ is not bounded above. However, A has 2 as a lower bound because $\sqrt{m}+\sqrt{n} \geq \sqrt{1}+\sqrt{1} = 2$ for every $m, n \in \mathbb{N}$. In fact, $\inf A = 2$ because 2 is a lower bound and every lower bound $b \leq \sqrt{1}+\sqrt{1} \in A$.

(b) $B = (-\infty, \pi] \cup \{3, 3\frac{1}{2}, 3\frac{2}{3}, \cdots\}$ is not bounded below. However, B has 4 as an upper bound because $\pi \leq 4$ and $4 - \frac{1}{n} \leq 4$ for all $n \in \mathbb{N}$. (Note $4 \notin B$) We will show $\sup B = 4$. Assume there is an upper bound $t < 4$. By the Archimedian principle, there is $n \in \mathbb{N}$ such that $n > \frac{1}{4-t}$. Then $4 - \frac{1}{n} > t$ and $4 - \frac{1}{n} \in B$, which contradicts t being an upper bound.

(c) For $n, m \in \mathbb{N}$, $0 < \frac{1}{n} + \frac{1}{2^m} \leq \frac{1}{1} + \frac{1}{2^1} = \frac{3}{2}$. So C has 0 as a lower bound and $\frac{3}{2}$ as an upper bound. In fact, $\sup C = \frac{3}{2}$ because $\frac{1}{1} + \frac{1}{2^1} = \frac{3}{2} \in C$ and every upper bound $M \geq \frac{1}{1} + \frac{1}{2^1}$. Also, we can show $\inf C = 0$ as follow. Assume there is a lower bound $t > 0$. By Archimedean principle, there is $k \in \mathbb{N}$ such that $k > \frac{1}{t}$. Then taking $m = n = 2k$, we have $t > \frac{1}{k} = \frac{1}{2k} + \frac{1}{2k} \geq \frac{1}{\underset{n}{\overset{\shortparallel}{2k}}} + \frac{1}{2^{\overset{\shortparallel}{m}}}$ — i.e. $t > \frac{1}{k} = \frac{1}{2k}+\frac{1}{2k} \geq \frac{1}{n} + \frac{1}{2^m} \in C$, contradicting t being a lower bound.

(d) For $x \in D$, $0 < x < \sqrt{2}$. So D has 0 as a lower bound and $\sqrt{2}$ as an upper bound. In fact, $\sup D = \sqrt{2}$ because if there is an upper bound $t < \sqrt{2}$, then by density of rationals, there will be $\frac{m}{n} \in \mathbb{Q}$ such that $\max(t, 0) < \frac{m}{n} < \sqrt{2}$, which means $t < \frac{m}{n} \in D$, contradicting t being an upper bound.

Next, $\inf D = 0$ because if there is a lower bound $s > 0$, then by the density of rationals, there will be $\frac{p}{q} \in \mathbb{Q}$ such that $0 < \frac{p}{q} < \min(s, \sqrt{2})$, which means $\frac{p}{q} \in D$ and $\frac{p}{q} < s$, contradicting s being a lower bound.

Remarks If supremum limit theorem and infimum limit theorem are allowed, then the proofs by contradiction above can be avoided.

For (b), taking $w_n = 4 - \frac{1}{n} \in B$, we have $\lim_{n\to\infty} w_n = 4$. Since 4 is an upper bound, $\sup B = 4$ by the supremum limit theorem.

For (c), taking $w_n = \frac{1}{n} + \frac{1}{2^n} \in C$, we have $\lim_{n\to\infty} w_n = 0$. Since 0 is a lower bound, $\inf C = 0$ by the infimum limit theorem.

For (d), taking $w_n = \frac{1}{n} \in D$ and $z_n = \frac{[10^n\sqrt{2}]}{10^n} \in D$, we have $\lim_{n\to\infty} w_n = 0$ and $\lim z_n = \sqrt{2}$. Since 0 is a lower bound and $\sqrt{2}$ is an upper bound, so $\inf D = 0$ and $\sup D = \sqrt{2}$ by the infimum limit theorem and the supremum limit theorem.

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(35) Let $A = (-\infty, 0) = B$, then both A and B are bounded above by 0, but $S = (0, +\infty)$ is not bounded above, $T = (-\infty, \infty)$ is not bounded above.

(36) For every $x \in A$, $y \in B$, we have $x \le \sup A$ and $y \le \sup B$. So $x+y \le \sup A + \sup B$. $\therefore$ C is bounded above by $\sup A + \sup B$. As $\sup A + \sup B$ is an upper bound of C, we have $\sup C \le \sup A + \sup B$. Assume $\sup C < \sup A + \sup B$. Let $\varepsilon = \frac{\sup A + \sup B - \sup C}{2} > 0$. By the supremum property, $\exists\, x \in A$ such that $\sup A - \varepsilon < x \le \sup A$ and $\exists\, y \in B$ such that $\sup B - \varepsilon < y \le \sup B$. Adding these, we get $\sup C = \sup A + \sup B - 2\varepsilon < x+y \in C$, a contradiction. Therefore, $\sup C = \sup A + \sup B$.

Another Solution As in the first solution, $\sup C \le \sup A + \sup B$. Conversely, for every $x \in A$, $y \in B$, $x+y \le \sup C$, so $x \le \sup C - y$. Then $\sup C - y$ is an upper bound of A. So $\sup A \le \sup C - y$. Then $y \le \sup C - \sup A$. This implies $\sup C - \sup A$ is an upper bound of B. So $\sup B \le \sup C - \sup A$. Then $\sup A + \sup B \le \sup C$. $\therefore \sup C = \sup A + \sup B$.

Alternate Solution (using Supremum Limit theorem) As above, we have $\sup C \le \sup A + \sup B$. By Supremum limit theorem, $\exists\, a_n \in A$ with $\lim_{n\to\infty} a_n = \sup A$ and $\exists\, b_n \in B$ with $\lim_{n\to\infty} b_n = \sup B$. Then $a_n + b_n \in C$ and $\lim_{n\to\infty}(a_n+b_n) = C$. By the Supremum limit theorem, the upper bound $\sup A + \sup B$ of set C is the supremum of C.

(37) Given $\varepsilon > 0$. (Consider the inequalities $\frac{4}{n^2} < \frac{\varepsilon}{2}$ and $\frac{5}{n^3} < \frac{\varepsilon}{2}$. If n satisfies these, then $\frac{4n+5}{n^3} = \frac{4}{n^2} + \frac{5}{n^3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.) So let $K = [\max(\sqrt{\frac{8}{\varepsilon}}, \sqrt[3]{\frac{10}{\varepsilon}})] + 1$, then $n \ge K \Rightarrow n > \sqrt{\frac{8}{\varepsilon}}$ and $n > \sqrt[3]{\frac{10}{\varepsilon}} \Rightarrow |\frac{4n+5}{n^3} - 0| = \frac{4}{n^2} + \frac{5}{n^3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

For $\varepsilon = 0.1$, we can choose $K = \lceil \max(\sqrt{\frac{8}{0.1}}, \sqrt[3]{\frac{10}{0.1}}) \rceil$, e.g. $K = 9$ will do.

(38) We have $y - 1 < [y] \le y$. So $\frac{(x-1)+(2x-1)+\cdots+(nx-1)}{n^2} < a_n \le \frac{x+2x+\cdots+nx}{n^2}$, i.e. $\frac{\frac{n(n+1)}{2}x - n}{n^2} = \frac{(n+1)x}{2n} - \frac{1}{n} < a_n \le \frac{\frac{n(n+1)}{2}x}{n^2} = \frac{(n+1)x}{2n}$.

Since $\lim_{n\to\infty}\left(\frac{(n+1)x}{2n} - \frac{1}{n}\right) = \frac{x}{2} = \lim_{n\to\infty}\frac{(n+1)x}{2n}$, by Squeeze limit theorem, $\lim_{n\to\infty} a_n = \frac{x}{2}$.

(39) Let $x \in \mathbb{R}$. For every $n \in \mathbb{N}$, by the density of rational numbers, there is $r_n \in \mathbb{Q}$ such that $x - \frac{1}{n} < r_n < x$. Since $\lim_{n\to\infty}(x - \frac{1}{n}) = x = \lim_{n\to\infty} x$, by the Squeeze limit theorem, or Sandwich theorem $\lim_{n\to\infty} r_n = x$.

(40) Let $r = |x-y|$. By triangle inequality, $|x| = |(x-y)+y| \le |x-y| + |y| = r + |y|$ and so $|x| - |y| \le r$. Also $|y| = |(y-x)+x| \le |y-x| + |x| = r + |x|$ and so $-r \le |x| - |y|$. Then $-r \le |x| - |y| \le r$. Therefore, $||x| - |y|| \le r = |x-y|$.

Next we will show if $\lim_{n\to\infty} a_n = A$, then $\lim_{n\to\infty}|a_n| = |A|$. For $\varepsilon > 0$, since $\lim_{n\to\infty} a_n = A$, by definition of convergence, there is $K \in \mathbb{N}$ such that $n \ge K \Rightarrow |a_n - A| < \varepsilon$. Then $n \ge K \Rightarrow ||a_n| - |A|| \le |a_n - A| < \varepsilon$.

(Alternatively, $\lim_{n\to\infty} a_n = A \Leftrightarrow \lim_{n\to\infty}|a_n - A| = 0$. Since $0 \le ||a_n| - |A|| \le |a_n - A|$ and $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty}|a_n - A|$, by Squeeze limit theorem, $\lim_{n\to\infty}||a_n| - |A|| = 0 \Leftrightarrow \lim_{n\to\infty}|a_n| = |A|$.) The converse is false. Take $a_n = (-1)^n$. Then $\lim_{n\to\infty}|a_n| = 1$, but $\lim_{n\to\infty} a_n$ doesn't exist.

(41) For $\varepsilon > 0$, since $\lim_{n\to\infty} a_n = A$, by definition of convergence, there is $K \in \mathbb{N}$ such that $n \geq K \Rightarrow |a_n - A| < \varepsilon$. Then

$$n \geq K \Rightarrow n+1 \geq K \Rightarrow \left|\frac{a_n + a_{n+1}}{2} - A\right| = \left|\frac{a_n - A}{2} + \frac{a_{n+1} - A}{2}\right| \leq \left|\frac{a_n - A}{2}\right| + \left|\frac{a_{n+1} - A}{2}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

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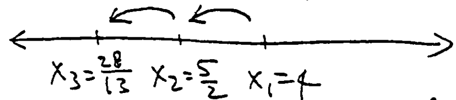
(42) $x_1 = 4,\ x_2 = \frac{5}{2},\ x_3 = \frac{28}{13}$



(We suspect $\{x_n\}$ is decreasing. If $\lim_{n\to\infty} x_n = x$, then $x = \lim_{n\to\infty} x_{n+1} = \lim \frac{4(1+x_n)}{4+x_n} = \frac{4(1+x)}{4+x}$. Solving this, we get $x = \pm 2$. Since $x_n > 0$, $x = 2$.)

We will first show $x_n \geq 2$ for all $n \in \mathbb{N}$ by mathematical induction. For $n=1$, $x_1 = 4 \geq 2$. Assume $x_n \geq 2$, then $2x_n \geq 4 \Rightarrow 4 + 4x_n \geq 8 + 2x_n \Rightarrow x_{n+1} = \frac{4(1+x_n)}{4+x_n} \geq 2$

Next we will show $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$ by mathematical induction. For $n=1$, $x_1 = 4 \geq x_2 = \frac{5}{2}$. Assume $x_n \geq x_{n+1}$. Since $x_{n+1} \geq 2$, so $4x_{n+1} + x_{n+1}^2 \geq 4x_{n+1} + 4$ ($x_{n+1}^2 \geq 4$)

$\Rightarrow x_{n+1} \geq \frac{4(1+x_{n+1})}{4+x_{n+1}} = x_{n+2}$.

By the monotone sequence theorem, $\{x_n\}$ converges. (In fact, we saw above that $\lim_{n\to\infty} x_n = 2$)

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(43) By AM-GM inequality, $1 + \frac{1}{n+1} = \frac{\overbrace{(1+\frac{1}{n}) + \cdots + (1+\frac{1}{n})}^{n \text{ times}} + 1}{n+1} \geq \sqrt[n+1]{(1+\frac{1}{n})^n \cdot 1}$. Taking $(n+1)$-st power of both sides, we get $\left(1+\frac{1}{n+1}\right)^{n+1} \geq \left(1+\frac{1}{n}\right)^n$. So $\{(1+\frac{1}{n})^n\}$ is increasing. Next, by binomial theorem,

$$\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{1}{n}\right)^n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$
$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 3.$$

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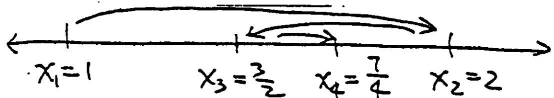
(44) (a) Since $\{x_n\}$ is bounded, $\exists$ upper bound u, lower bound v for $\{x_n\}$. Then $v \leq x_n \leq u$ for all n, Then $v \leq m_n \leq M_n \leq u$ for all n, i.e. $\{M_n\}$ and $\{m_n\}$ are bounded.
Now M_n is an upper bound of $\{x_{n+1}, x_{n+2}, \cdots\}$ and m_n is a lower bound of $\{x_{n+1}, x_{n+2}, \cdots\}$ imply $M_{n+1} \leq M_n$ and $m_n \leq m_{n+1}$. So $\{M_n\}$ is decreasing, $\{m_n\}$ is increasing.
By the monotone limit theorem, both $\{M_n\}$ and $\{m_n\}$ converge.

(b) Since $m_n \leq x_n \leq M_n$, so $\lim_{n\to\infty} M_n = x = \lim_{n\to\infty} m_n \Rightarrow \lim_{n\to\infty} x_n = x$ by Sandwich theorem.
Conversely, if $\lim_{n\to\infty} x_n = x$, then $\forall \varepsilon > 0\ \exists K$ such that $n \geq K \Rightarrow |x_n - x| < \varepsilon_0 = \varepsilon/2$
$\Rightarrow x_K, x_{K+1}, x_{K+2}, \cdots \in (x - \varepsilon_0, x + \varepsilon_0) \Rightarrow M_K, M_{K+1}, M_{K+2}, \cdots \in [x-\varepsilon_0, x+\varepsilon_0] \subseteq (x-\varepsilon, x+\varepsilon)$
$m_K, m_{K+1}, m_{K+2}, \cdots \in [x-\varepsilon_0, x+\varepsilon_0] \subseteq (x-\varepsilon, x+\varepsilon)$.
So $n \geq K \Rightarrow |M_n - x| < \varepsilon$ and $|m_n - x| < \varepsilon$. $\therefore \lim_{n\to\infty} M_n = x = \lim_{n\to\infty} m_n$.

(45) $x_1=1,\ x_2=2,\ x_3=\frac{3}{2},\ x_4=\frac{7}{4}$



Let $I_n=[x_{2n-1},x_{2n}]$. We will show $I_n \supseteq I_{n+1}$ (i.e. $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$) by mathematical induction. For $n=1$, $x_1=1\le x_3=\frac{3}{2}\le x_4=\frac{7}{4}\le x_2=2$.
Assume $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$. Then $x_{2n+1}\le \frac{x_{2n+1}+x_{2n+2}}{2}(=x_{2n+3})\le x_{2n+2}$ and $x_{2n+3}\le \frac{x_{2n+3}+x_{2n+2}}{2}(=x_{2n+4})\le x_{2n+2}$. So $x_{2n+1}\le x_{2n+3}\le x_{2n+4}\le x_{2n+2}$.

Now $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$ implies $\lim_{n\to\infty}x_{2n-1}=x$ and $\lim_{n\to\infty}x_{2n}=x'$. We will show $x=x'$. (By the intertwining sequence theorem, this will imply $\{x_n\}$ converges.)

Method I Since x_{k+1} is the midpoint of x_k and x_{k-1}, so $x_{k+1}-x_k=\frac{x_{k-1}-x_k}{2}$.
Then $x_{2n-1}-x_{2n}=\frac{-(x_{2n-2}-x_{2n-1})}{2}=\frac{x_{2n-3}-x_{2n-2}}{2^2}=\cdots=\frac{x_1-x_2}{2^{2n-2}}=\frac{-1}{2^{2n-2}}$. So $\lim_{n\to\infty}|x_{2n-1}-x_{2n}|=0$. By the nested interval theorem, $\bigcap_{n=1}^{\infty}I_n=\{x\}$. So $x=x'$.

Method II $x=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}\frac{x_{2n}+x_{2n-1}}{2}=\frac{x'+x}{2}\Rightarrow x=x'$.

(Remarks We can find $\lim_{n\to\infty}x_n$ as follow: $x_n=x_1+(x_2-x_1)+(x_3-x_2)+\cdots+(x_n-x_{n-1})$ $=1+1-\frac{1}{2}+\frac{1}{4}-\cdots+(-\frac{1}{2})^{n-1}$. So $\lim_{n\to\infty}x_n=1+(1-\frac{1}{2}+\frac{1}{4}-\cdots)=1+\frac{1}{1-(-\frac{1}{2})}=\frac{5}{3}$.)

---

(46) Let $s_n=\sum_{k=2}^{n}|x_k-x_{k-1}|$ and $s=\sum_{k=2}^{\infty}|x_k-x_{k-1}|$. For every $\varepsilon>0$, since $\sum_{k=2}^{\infty}|x_k-x_{k-1}|$ converges $\Leftrightarrow \lim_{n\to\infty}s_n=s$, so $\exists K$ such that $n\ge K\Rightarrow |s_n-s|=\sum_{k=n+1}^{\infty}|x_k-x_{k-1}|<\varepsilon$.
Then for $m,n\ge K$, say $m\ge n$, we have

$$|x_m-x_n|\le|x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n|\le\sum_{k=n+1}^{\infty}|x_k-x_{k-1}|<\varepsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence.

---

(47) Claim: $|\sqrt{x}-\sqrt{y}|\le\sqrt{|x-y|}$ for every $x,y\ge0$.
Proof. Let $u=\max(x,y)$ and $v=\min(x,y)$. Then $|\sqrt{x}-\sqrt{y}|=\sqrt{u}-\sqrt{v}$ and $|x-y|=u-v$.
Now $\sqrt{u}-\sqrt{v}\le\sqrt{u-v}\Leftrightarrow\sqrt{u}\le\sqrt{v}+\sqrt{u-v}\Leftrightarrow u\le\underbrace{v+2\sqrt{v(u-v)}+(u-v)}_{u+2\sqrt{v(u-v)}}$, which is true.

If $a_n\ge0$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty}a_n=a$, then for every $\varepsilon>0$, there exists $K\in\mathbb{N}$ such that $n\ge K\Rightarrow|a_n-a|<\varepsilon^2$. So $n\ge K\Rightarrow|\sqrt{a_n}-\sqrt{a}|\le\sqrt{|a_n-a|}<\sqrt{\varepsilon^2}=\varepsilon$.

---

(48) If $x_2=x_1$, then $|x_{n+1}-x_n|\le k|x_n-x_{n-1}|$ implies all $x_n=x_1$. In this case, for every $\varepsilon>0$, take $K=1$ and $m,n\ge K\Rightarrow|x_m-x_n|=0<\varepsilon$. The sequence $\{x_n\}$ is Cauchy.
If $x_2\ne x_1$, then $\forall\varepsilon>0$, let $K>\log_k\frac{(1-k)\varepsilon}{|x_2-x_1|}$ so that $|x_2-x_1|\frac{k^K}{1-k}<\varepsilon$. We have $m,n\ge K$, say $m>n$, implies $|x_m-x_n|\le|x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n|$

$$\le|x_2-x_1|(k^{m-2}+k^{m-3}+\cdots+k^{n-1})$$
$$\le|x_2-x_1|(k^K+k^{K+1}+\cdots)=|x_2-x_1|\frac{k^K}{1-k}<\varepsilon.$$

So the sequence $\{x_n\}$ is Cauchy.

(49) Let $b_n = a_n - A$ and $\beta_n = \frac{b_1+b_2+\cdots+b_n}{n}$, then $\lim_{n\to\infty}\alpha_n = A \Leftrightarrow \lim_{n\to\infty}(\alpha_n - A) = 0 \Leftrightarrow \lim_{n\to\infty}\frac{(a_1-A)+\cdots+(a_n-A)}{n} = 0 \Leftrightarrow \lim_{n\to\infty}\beta_n = 0$, which is to be shown.

Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n - A) = 0$, $\{b_n\}$ is bounded, say $|b_n| \le M$ for all $n \in \mathbb{N}$.
For $\varepsilon > 0$, there is $K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |b_n - 0| < \frac{\varepsilon}{2}$. Let $K = \lceil \max(K_1, \frac{2(K_1-1)M}{\varepsilon}) \rceil$.
Then $n \ge K \Rightarrow |\beta_n - 0| = \left|\frac{b_1+b_2+\cdots+b_{K_1-1}}{n} + \frac{b_{K_1}+\cdots+b_n}{n}\right| < \frac{(K_1-1)M}{n} + \frac{(n-K_1+1)\frac{\varepsilon}{2}}{n} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$0 < n - K_1 + 1 \le n$

Therefore, $\lim_{n\to\infty}\beta_n = 0$.

To see the converse is false, take $a_n = (-1)^n$. Then $\alpha_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$. So $\lim_{n\to\infty}\alpha_n = 0$ and $\{a_n\}$ doesn't converge.

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(50) Assume $\lim_{n\to\infty} x_n \ne x$. Then $\sim(\forall \varepsilon > 0\ \exists K$ such that $n \ge K \Rightarrow |x_n - x| < \varepsilon)$
$= \exists \varepsilon > 0\ \forall K\ \exists n \ge K$ and $|x_n - x| \ge \varepsilon$. So $\exists \varepsilon > 0$ such that
for $K = 1$, $\exists\ n_1 \ge 1$ and $|x_{n_1} - x| \ge \varepsilon$,
for $K = n_1 + 1$, $\exists\ n_2 \ge n_1 + 1$ and $|x_{n_2} - x| \ge \varepsilon$,
for $K = n_2 + 1$, $\exists\ n_3 \ge n_2 + 1$ and $|x_{n_3} - x| \ge \varepsilon$, ...
Then $n_1 < n_2 < n_3 < \cdots$ and subsequence $\{x_{n_j}\}$ satisfies $|x_{n_j} - x| \ge \varepsilon$ for all j.
Since $\{x_{n_j}\}$ is bounded, by Bolzano-Weierstrass theorem, it has a convergence subsequence $\{x_{n_{j_k}}\}$. Then $\lim_{k\to\infty} x_{n_{j_k}} = x$ and $0 = \lim_{k\to\infty}|x_{n_{j_k}} - x| \ge \varepsilon$ leads to a contradiction. Therefore, $\lim_{n\to\infty} x_n = x$.

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(51) $\forall \varepsilon > 0$, by the Archimedean principle, $\exists m \in \mathbb{N}$ such that $m > \log_2 \frac{1}{\varepsilon}$ ($\Leftrightarrow 2^{-m} < \varepsilon$).
Since f is injective, the set $T = \{n \in \mathbb{N} : f(n) = 2^{-1}$ or 2^{-2} or $\cdots$ or $2^{-(m-1)}\}$ has at most $m-1$ elements.
If the set is empty, then let $K = 1$, otherwise let K be larger than the maximum of T.
Then $n \ge K \Rightarrow n \notin T \Rightarrow |f(n) - 0| = f(n) \le 2^{-m} < \varepsilon$. Therefore, $\lim_{n\to\infty} f(n) = 0$.

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(52) 〔题说〕 第十一届（1977年）全苏数学奥林匹克九、十年级题2.

Solution 1
〔证〕 由 $\lim_{n\to\infty}(a_{n+1} - \frac{a_n}{2}) = 0$ 知，任给 $\varepsilon > 0$，存在 N，当正整数 $n > N$ 时，
since ... for every $\varepsilon > 0$, there exists N such that when $n > N$,

$$\left|a_{n+1} - \frac{a_n}{2}\right| < \varepsilon_0 = \frac{\varepsilon}{3}.$$

~~Let k be such~~ that $|a_N|/2^k < \varepsilon_0$. Then for $m \ge N + k$,
取正整数 k，使 $\frac{|a_N|}{2^k} < \varepsilon_0$. 则当正整数 $m \ge N + k$ 时，

$$|a_m| < \frac{1}{2}|a_{m-1}| + \varepsilon_0 < \frac{1}{2^2}|a_{m-2}| + \frac{\varepsilon_0}{2} + \varepsilon_0$$
$$< \cdots < \frac{1}{2^{m-N}}|a_N| + \frac{\varepsilon_0}{2^{m-N+1}} + \cdots + \frac{\varepsilon_0}{2^2} + \frac{\varepsilon_0}{2} + \varepsilon_0$$
$$< \frac{1}{2^k}|a_N| + 2\varepsilon_0 < 3\varepsilon_0 = \varepsilon.$$

所以

$$\lim_{n\to\infty} a_n = 0,$$

(52) Solution 2 Let $b_n = a_{n+1} - \frac{1}{2}a_n$. Define $c_1 = c_2 = 0$ and

$$c_{2^k+1} = \cdots = c_{2^{k+1}} = b_k$$

for $k = 1, 2, 3, \ldots$. Then $b_n \to 0$ implies $c_n \to 0$, which implies

by exercise 49, $$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{c_1 + c_2 + \cdots + c_n}{n} = 0.$$

Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(a_n - \frac{1}{2^n}a_1\right) = \lim_{n\to\infty} 2d_{2^{n+1}} = 0.$$

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(53) For $\epsilon > 0$, let N be sufficiently large so that $|x_n - x_{n-2}| < \epsilon$ for all $n \geqq N$. Note that for any $n > N$,

$$x_n - x_{n-1} = (x_n - x_{n-2}) - (x_{n-1} - x_{n-3}) + (x_{n-2} - x_{n-4}) - \cdots$$
$$\pm (x_{N+1} - x_{N-1}) \mp (x_N - x_{N-1}).$$

Thus $|x_n - x_{n-1}| \leqq (n-N)\epsilon + |x_N - x_{N-1}|$ and $\lim_{n\to\infty} (x_n - x_{n-1})/n = 0$.

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(54) Let $\bar{y} = \lim_{n\to\infty} y_n$ and set $\bar{x} = \bar{y}/3$. We will show that $\bar{x} = \lim_{n\to\infty} x_n$. For any $\epsilon > 0$ there is an N such that for all $n > N$, $|y_n - \bar{y}| < \epsilon/2$.

$$\epsilon/2 > |y_n - \bar{y}| = |x_{n-1} + 2x_n - 3\bar{x}| = |2(x_n - \bar{x}) + (x_{n-1} - \bar{x})|$$
$$\geqq 2|x_n - \bar{x}| - |x_{n-1} - \bar{x}|.$$

This may be rewritten as $|x_n - \bar{x}| < \epsilon/4 + \frac{1}{2}|x_{n-1} - \bar{x}|$, which can be iterated to give

$$|x_{n+m} - \bar{x}| < \epsilon/4\left(\sum_{i=0}^{m} 2^{-i}\right) + 2^{-(m+1)}|x_{n-1} - \bar{x}| < \epsilon/2 + 2^{-(m+1)}|x_{n-1} - \bar{x}|.$$

By taking m large enough, $2^{-(m+1)}|x_{n-1} - \bar{x}| < \epsilon/2$. Thus for all sufficiently large k, $|x_k - \bar{x}| < \epsilon$.

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(55) (a) If $a_n = 1 - \frac{2}{n(n+1)} = \frac{n^2+n-2}{n(n+1)} = \frac{(n-1)(n+2)}{n(n+1)}$, then

$$\prod_{n=2}^{\infty}\left(1 - \frac{2}{n(n+1)}\right) = \lim_{k\to\infty} a_2 a_3 \cdots a_k = \lim_{k\to\infty}\left(\frac{1\cdot 4}{\cancel{2}\cdot 3}\,\frac{\cancel{2}\cdot \cancel{5}}{\cancel{3}\cdot \cancel{4}} \cdots \frac{(\cancel{k-1})(k+2)}{k\,(\cancel{k+1})}\right) = \lim_{k\to\infty} \frac{k+2}{3k} = \frac{1}{3}.$$

(b) If $a_n = 1 - \frac{1}{n^2} = \frac{n^2-1}{n^2} = \frac{(n-1)(n+1)}{n^2}$, then

$$\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \lim_{k\to\infty} a_2 a_3 \cdots a_k = \lim_{k\to\infty}\left(\frac{1\cdot 3}{2^2}\,\frac{\cancel{2}\cdot 4}{\cancel{3}^2}\,\frac{\cancel{3}\cdot 5}{\cancel{4}^2} \cdots \frac{(\cancel{k-1})(k+1)}{k^2}\right) = \lim_{k\to\infty} \frac{k+1}{2k} = \frac{1}{2}.$$

(c) Note $\frac{n^3-1}{n^3+1} = \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)}$ and $n^2+n+1 = (n+1)^2 - (n+1) + 1$. So

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1} = \lim_{k\to\infty}\left(\frac{1\cdot \cancel{7}}{3\cdot \cancel{3}}\,\frac{2\cdot \cancel{13}}{4\cdot \cancel{7}}\,\frac{3\cdot \cancel{21}}{5\cdot \cancel{13}} \cdots \frac{(\cancel{k-1})(k^2+k+1)}{(k+1)(\cancel{k^2-k+1})}\right) = \lim_{k\to\infty} \frac{2(k^2+k+1)}{3k(k+1)} = \frac{2}{3}.$$

(d) Note $(1-z)(1+z)(1+z^2)\cdots(1+z^{2^k}) = (1-z^2)(1+z^2)\cdots(1+z^{2^k}) = \cdots$
$= (1-z^{2^k})(1+z^{2^k}) = 1 - z^{2^{k+1}}$. So $(1+z)(1+z^2)\cdots(1+z^{2^k}) = \frac{1-z^{2^{k+1}}}{1-z}$.

Therefore, $\prod_{n=0}^{\infty}(1+z^{2^n}) = \lim_{k\to\infty} \frac{1-z^{2^{k+1}}}{1-z} = \frac{1}{1-z}$ as $|z| < 1 \Rightarrow \lim_{k\to\infty} z^{2^{k+1}} = 0$.

56 Let S be a bounded infinite subset of $\mathbb{R}$. Then we choose $x_1 \in S$. Since S is infinite, there $\exists x_2 \in S$, $x_2 \neq x_1$, $\cdots$, $\exists x_n \in S$, $x_n \neq x_1, \cdots, x_{n-1}$. So the sequence $\{x_n\}$ consists of distinct terms in S. Since S is bounded, $\{x_n\}$ is bounded. By Bolzano-Weierstrass theorem, $\{x_{n_j}\}$ has a convergence subsequence, say $\lim_{j\to\infty} x_{n_j} = x_0$. If $x_0 = x_{n_k}$ for some k, then x_0 is the limit of $x_{n_{k+1}}, x_{n_{k+2}}, \cdots$ in $S \setminus \{x_0\}$. So S has x_0 as an accumulation point.

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57 (Note: $S=(0,+\infty)$, so $x\in S \Rightarrow x+1>1$) For every $\varepsilon>0$, let $\delta = 2\varepsilon$, then for every $x\in S=(0,\infty)$, $0<|x-1|<\delta=2\varepsilon \Rightarrow \left|\frac{x}{x+1}-\frac{1}{2}\right| = \frac{|x-1|}{2(x+1)} < \frac{2\varepsilon}{2} = \varepsilon$.

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58 Suppose $\lim_{x\to x_0} f(x)$ exists at x_0. By density of rational, there is $r_n\in\mathbb{Q}$ such that $x_0-\frac{1}{n}<r_n<x_0$. By density of irrational, there is $s_n \in \mathbb{R}\setminus\mathbb{Q}$ such that $x_0-\frac{1}{n}<s_n<x_0$. By Squeeze limit theorem, $\lim_{n\to\infty} r_n = x_0 = \lim_{n\to\infty} s_n$. By the sequential limit theorem, $\lim_{x\to x_0} f(x) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 8r_n = 8x_0$ and $\lim_{x\to x_0} f(x) = \lim_{n\to\infty} f(s_n) = \lim_{n\to\infty}(2s_n^2+8) = 2x_0^2+8$. By the uniqueness of limit, $8x_0 = 2x_0^2+8$. So $x_0=2$.

Next we show $\lim_{x\to 2} f(x)$ exists. (The limit should be $8\times 2 = 16 = 2\cdot 2^2+8$.)

We have $0\le |f(x)-16| \le |8x-16| + |(2x^2+8)-16|$ for x rational or irrational. Since $\lim_{x\to 2}(|8x-16|+|(2x^2+8)-16|)=0$, by squeeze limit theorem, $\lim_{x\to 2} f(x) = 16$.

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59 For $w\in\mathbb{R}$, there is a sequence $\{x_n\}$ of rational numbers converging to w (by practice exercise #39 or last exercise). Since f is continuous at w, by the sequential limit theorem, $f(w) = \lim_{x\to w} f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$.

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60 (a) ① $f(0+0)=f(0)+f(0) \Rightarrow f(0)=0$, ② $0 = f(x+(-x)) = f(x)+f(-x) \Rightarrow f(-x) = -f(x)$.

③ For $n\in\mathbb{N}$, $f(nx)=nf(x)$ by mathematical induction ($f(1x)=1f(x)$. If $f(nx)=nf(x)$, then $f((n+1)x) = f(nx+x) = f(nx)+f(x) = nf(x)+f(x) = (n+1)f(x)$.)

④ Taking $x=\frac{1}{n}$ in ③ we get $f(1)=nf(\frac{1}{n}) \Rightarrow f(\frac{1}{n}) = \frac{1}{n}f(1)$.

Taking $x=\frac{1}{k}$ in ③ we get $f(\frac{n}{k}) = nf(\frac{1}{k}) = \frac{n}{k}f(1)$.

⑤ By ②, $f(-\frac{n}{k}) = -f(\frac{n}{k}) = -\frac{n}{k}f(1)$. If $c=f(1)$, then $f(r) \overset{\text{by ①,④,⑤}}{=} cr$ for $r\in\mathbb{Q}$.

Conversely, the function $f(r)=cr$ satisfies $f(x+y)=c(x+y)=cx+cy=f(x)+f(y)$ for any $c\in\mathbb{R}$.

(b) For $w\in\mathbb{R}$, by density of rational numbers, there are $r_n, s_n\in\mathbb{Q}$ such that $w-\frac{1}{n}<r_n<w<s_n<w+\frac{1}{n}$. For f strictly increasing, by part (a), $r_nf(1)=f(r_n)<f(w)<f(s_n)=s_nf(1)$. Taking limit, we get $f(w)=wf(1)$ by squeeze limit theorem. So the functions we are looking for are $f:\mathbb{R}\to\mathbb{R}$ given by $f(w)=cw$, where $c=f(1)>0$.

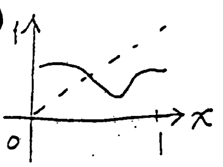
(61) For every r, $f(r)$ is the maximum or minimum of $f(x)$ on some interval (a,b) containing r. Then $a<r<b$. By density of rational numbers, there are $c, d \in \mathbb{Q}$ such that $a<c<r$ and $r<d<b$. Let $S=\{(q_0,q_1): q_0, q_1 \in \mathbb{Q}$ and $q_0<q_1\}$, then $S \subseteq \mathbb{Q}\times\mathbb{Q}$ and so S is countable.

$$f(\mathbb{R})=\{f(r): r\in\mathbb{R}\} \subseteq \{\max_{c<x<d} f(x) : (c,d)\in S\} \cup \{\min_{c<x<d} f(x) : (c,d)\in S\}$$

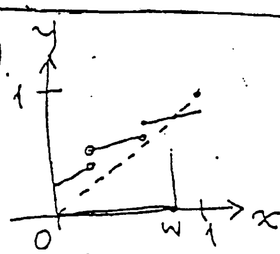
$$= \bigcup_{\substack{(c,d)\in S \\ \uparrow \\ \text{countable}}} \underbrace{\{\max_{c<x<d} f(x), \min_{c<x<d} f(x)\}}_{\text{finite}}$$ which is countable by the countable union theorem

So $f(\mathbb{R})$ is countable. By the intermediate value theorem, f is constant.

(62) Suppose such function g exists. We first show g is injective. (If $g(a)=g(b)$, then $-a^9=g(g(a))=g(g(b))=-b^9 \Rightarrow a=b$.) Since g is continuous and injective, by the continuous injection theorem, g is strictly increasing or strictly decreasing. If g is strictly increasing, then $x<y \Rightarrow g(x)<g(y) \Rightarrow g(g(x))<g(g(y))$. If g is strictly decreasing, then $x<y \Rightarrow g(x)>g(y) \Rightarrow g(g(x))<g(g(y))$. So in both cases, $g(g(x))$ is strictly increasing, which cannot equal to the decreasing function $-x^9$, a contradiction. So no such g exists.

(63) 

Let $g(x)=f(x)-x$, then g is continuous on $[0,1]$ because f is continuous on $[0,1]$. Since $f(0), f(1)\in[0,1]$, so $g(0)=f(0)-0\geq 0$ and $g(1)=f(1)-1\leq 0$. By the intermediate value theorem, there is at least one w between 0 and 1 such that $g(w)=0$. Then $f(w)=w$.

(64) 

Let $S=\{t\in[0,1]: t<f(t)\}$. Since $0\in S$ and S is bounded above by 1, $\sup S = w \in [0,1]$. By the supremum limit theorem, there is a sequence $t_n\in S$ converging to w. By the monotone function theorem, $w=\lim_{n\to\infty} t_n \leq \lim_{n\to\infty} f(t_n) \overset{\text{seq. limit thm}}{=} f(w-) \overset{\text{monotone function theorem}}{\leq} f(w)$. In particular, $w\neq 1$. So $w<1$. Let $\{s_n\}$ be a strictly decreasing sequence in $[0,1]$ converging to w. Since $s_n>w$, $s_n\notin S$ and so $w=\lim_{n\to\infty} s_n \underset{s_n\notin S}{\geq} \lim_{n\to\infty} f(s_n) \underset{\text{seq. limit theorem}}{=} f(w+) \underset{\text{monotone function theorem}}{\geq} f(w)$. Therefore, $w=f(w)$.

(65) f is injective because $f(a)=f(b) \Rightarrow 0=|f(a)-f(b)|\geq|a-b| \Rightarrow a=b$. Next, since f is continuous and injective, f is strictly monotone by the continuous injection theorem. To show f is surjective, let $w\in\mathbb{R}$ and $M=|w-f(0)|$. The given inequality implies $|f(M)-f(0)|\geq|M-0|=M=|w-f(0)|$ and $|f(0)-f(-M)|\geq|0-(-M)|=M=|w-f(0)|$. Since f is strictly monotone, $f(0)$ is between $f(-M)$ and $f(M)$. The inequalities above imply w is closer to $f(0)$ than $f(M)$ and $f(-M)$. So w is between $f(-M)$ and $f(M)$. The intermediate value theorem implies $w=f(x)$ for some x between $-M$ and M. So f is surjective. Therefore, f is bijective.

(66) Since $M=\sup_{x\in[0,1]} f(x)$, $\left(\int_0^1 f(x)^n dx\right)^{\frac1n} \le \left(\int_0^1 M^n dx\right)^{\frac1n} = M$ for all $n\in\mathbb{N}$. By the extreme value theorem, $M=f(x_0)$ for some $x_0\in[0,1]$. For every $k\in\mathbb{N}$, we consider $g(x)=f(x)-(f(x_0)-\frac1k)$ on $[0,1]$. Since g is continuous and $g(x_0)=\frac1k>0$, by the sign preserving property, there is $\delta>0$ such that $g(x)>0$ ($\Leftrightarrow f(x)>M-\frac1k$) on the interval $(x_0-\delta,x_0+\delta)\cap[0,1]$. Let a, b be the endpoints of the interval with $a<b$. Since $f(x)>0$, $\left(\int_a^b (M-\frac1k)^n dx\right)^{\frac1n} < \left(\int_a^b f(x)^n dx\right)^{\frac1n} \le \left(\int_0^1 f(x)^n dx\right)^{\frac1n}$.

So $\boxed{(M-\frac1k)(b-a)^{\frac1n} \le \left(\int_0^1 f(x)^n dx\right)^{\frac1n} \le M}$. Since $\lim_{n\to\infty}(b-a)^{\frac1n}=1$, we have $M-\frac1k \le \lim_{n\to\infty}\left(\int_0^1 f(x)^n dx\right)^{\frac1n} \le M$ for every $k\in\mathbb{N}$. As $k\to\infty$, we get by sandwich theorem that $\lim_{n\to\infty}\left(\int_0^1 f(x)^n dx\right)^{\frac1n} = M$.

Comments: In fact, the limit must exist. From the box above we have

$$\left|\left(\int_0^1 f(x)^n dx\right)^{\frac1n} - M\right| \le M-(M-\tfrac1k)(b-a)^{\frac1n} = (M-\tfrac1k)\left(1-(b-a)^{\frac1n}\right)+\tfrac1k.$$

For every $\varepsilon>0$, by the Archimedean principle, there is $k\in\mathbb{N}$ such that $\frac1k<\frac{\varepsilon}{2}$ and $\frac1k<M$. With one such k, since $\lim_{n\to\infty}(b-a)^{\frac1n}=1$, there is $K\in\mathbb{N}$ such that

$$n\ge K \Rightarrow \left|(b-a)^{\frac1n}-1\right| < \frac{\varepsilon}{2(M-\frac1k)}.$$ Then

$$n\ge K \Rightarrow \left|\left(\int_0^1 f(x)^n dx\right)^{\frac1n} - M\right| \le M-(M-\tfrac1k)(b-a)^{\frac1n} = (M-\tfrac1k)\left(1-(b-a)^{\frac1n}\right)+\tfrac1k < \tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2}=\varepsilon.$$

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(67) Since $f(0)=0=0^2$, so $f(x)=x^2$ for all $x\in\mathbb{R}$. Then for every $x_0\in\mathbb{R}$,

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{x^2-x_0^2}{x-x_0}=\lim_{x\to x_0} x+x_0 = 2x_0.$$

Remarks We have $f(x)=2x=\begin{cases}2x & \text{if } x\ne0\\ 0 & \text{if } x=0\end{cases} \ne \begin{cases}\frac{d}{dx}(x^2) & \text{if } x\ne 0\\ \frac{d}{dx}(x) & \text{if } x=0.\end{cases}$ This is to illustrate that if $f(x)=\begin{cases}h_0(x) & \text{if } x\in S\\ h_1(x) & \text{if } x\notin S\end{cases}$, then in general, $f'(x)\ne\begin{cases}h_0'(x) & \text{if } x\in S\\ h_1'(x) & \text{if } x\notin S.\end{cases}$

For $g(x)=|\cos x|$, let $r(x)=|x|$ and $s(x)=\cos x$, then $r'(x)=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0\\ \text{not exist} & \text{if } x=0\end{cases}$ and $s'(x)=-\sin x$. By chain rule, if $\cos x>0$ ($\Leftrightarrow x\in\bigcup_{n\in\mathbb{Z}}\left((2n-\frac12)\pi,(2n+\frac12)\pi\right)$), then $g'(x)=(r\circ s)'(x)=r'(s(x))\cdot s'(x)=-\sin x$; if $\cos x<0$ ($\Leftrightarrow x\in\bigcup_{n\in\mathbb{Z}}\left((2n+\frac12)\pi,(2n+\frac32)\pi\right)$), then $g'(x)=(r\circ s)'(x)=r'(s(x))\cdot s'(x)=\sin x$. If $\cos x=0$ ($\Leftrightarrow x=(2n\pm\frac12)\pi$, $n\in\mathbb{Z}$), then $\lim_{t\to x^+}\frac{|\cos t|-|\cos x|}{t-x}=\left|\lim_{t\to x^+}\frac{\cos t}{t-x}\right|=1$, but $\lim_{t\to x^-}\frac{|\cos t|-|\cos x|}{t-x}=-\left|\lim_{t\to x^+}\frac{\cos t}{t-x}\right|=-1$, so $g'(x)$ doesn't exist.

Remarks Even $r'(0)$ doesn't exist, $(s\circ r)(x)=\cos|x|=\cos x$ has derivative $-\sin x$ everywhere!

(68) $\dfrac{f(b_n)-f(a_n)}{b_n-a_n} - f'(c) = \left(\dfrac{f(b_n)-f(c)}{b_n-a_n} + \dfrac{f(c)-f(a_n)}{b_n-a_n}\right) - f'(c)\left(\dfrac{b_n-c}{b_n-a_n} + \dfrac{c-a_n}{b_n-a_n}\right)$

$$= \frac{f(b_n)-f(c)}{b_n-c}\,\frac{b_n-c}{b_n-a_n} + \frac{f(c)-f(a_n)}{c-a_n}\,\frac{c-a_n}{b_n-a_n} - f'(c)\frac{b_n-c}{b_n-a_n} - f'(c)\frac{c-a_n}{b_n-a_n}$$

$$= \left(\frac{f(b_n)-f(c)}{b_n-c} - f'(c)\right)\underbrace{\frac{b_n-c}{b_n-a_n}}_{\le 1} + \left(\frac{f(c)-f(a_n)}{c-a_n} - f'(c)\right)\underbrace{\frac{c-a_n}{b_n-a_n}}_{\le 1}.$$

So $\left|\dfrac{f(b_n)-f(a_n)}{b_n-a_n} - f'(c)\right| \le \underbrace{\left|\dfrac{f(b_n)-f(c)}{b_n-c} - f'(c)\right|}_{\to 0 \text{ as } n\to\infty}\cdot 1 + \underbrace{\left|\dfrac{f(c)-f(a_n)}{c-a_n} - f'(c)\right|}_{\to 0 \text{ as } n\to\infty}\cdot 1 \to 0$

$\therefore \lim\limits_{n\to\infty} \dfrac{f(b_n)-f(a_n)}{b_n-a_n} = f'(c)$.

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(69) $f(x) = \begin{cases} x^3 & \text{if } x \ge 0 \\ -x^3 & \text{if } x<0 \end{cases} \Rightarrow f'(x) = \begin{cases} 3x^2 & \text{if } x>0 \\ -3x^2 & \text{if } x<0 \\ 0 & \text{if } x=0 \end{cases} \Rightarrow f''(x) = \begin{cases} 6x & \text{if } x>0 \\ -6x & \text{if } x<0 \\ 0 & \text{if } x=0 \end{cases} = 6|x|$ is continuous

$|f'(0)| = \left|\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}\right| = \lim |x|^2 = 0$ $\quad |f''(0)| = \left|\lim\limits_{x\to 0}\dfrac{f'(x)-f'(0)}{x-0}\right| = \lim\limits_{x\to 0} 3|x| = 0 \Rightarrow f\in C^2(\mathbb{R})$.

$f'''(0) = \lim\limits_{x\to 0}\dfrac{f''(x)-f''(0)}{x-0} = 6\lim\limits_{x\to 0}\dfrac{|x|}{x}$ does not exist.

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(70) $|f'(b)| = \left|\lim\limits_{x\to b}\dfrac{f(x)-f(b)}{x-b}\right| = \lim\limits_{x\to b}\left|\dfrac{f(x)-f(b)}{x-b}\right| \le \lim\limits_{x\to b}|x-b| = 0$ for every $b\in\mathbb{R}$. So $f'\equiv 0$. Therefore, f is a constant function. The same is true if 2 is replaced by $n>1$ because $\left|\dfrac{f(x)-f(b)}{x-b}\right| \le |x-b|^{n-1} \to 0$ as $x\to b$. However if 2 is replaced by 1, then it is not true as can be seen by taking $f(x)=x$, then $|f(a)-f(b)| = |a-b|$ and f is not constant.

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(71) Since f has roots at ± 1 with multiplicities n, so $f(\pm 1) = f'(\pm 1) = \cdots = f^{(n-1)}(\pm 1) = 0$. Since $f(-1) = f(1) = 0$, by Rolle's theorem, there is $x_0\in(-1,1)$ such that $f'(x_0)=0$. Then f' has at least three distinct roots $-1, x_0, 1$. By Rolle's theorem, f'' will have at least four distinct roots. Repeating this until the $(n-1)^{\text{st}}$ derivative, we see that $f^{(n-1)}$ will have at least $n+1$ distinct roots. So by Rolle's theorem, $f^{(n)}$ will have at least n distinct roots. Since $\deg f^{(n)} = n$, $f^{(n)}$ has exactly n distinct roots.

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(72) Let $g(x) = e^{-x}f(x)$, then $g'(x) = -e^{-x}f(x) + e^{-x}f'(x) = e^{-x}(f'(x)-f(x)) \le 0$. So $g(x)$ is decreasing on $[0,\infty)$. Then $g(x)\le g(0) = f(0) = 0$ for $x\in[0,\infty)$. So $f(x) = e^{x}g(x) \le 0$ for $x\in[0,\infty)$.

(73) We first show $x_n = f(\frac{1}{n})$ is a Cauchy sequence. For every $\varepsilon > 0$, let $K \in \mathbb{N}$ such that $K > \frac{2}{\varepsilon}$ (by Archimedian principle). Then $m, n \geq K \Rightarrow |x_m - x_n| = |f(\frac{1}{m}) - f(\frac{1}{n})| = |f'(x_0)||\frac{1}{m} - \frac{1}{n}|$ (mean-value theorem) $\leq 2|\frac{1}{m} - \frac{1}{n}| \leq 2(\frac{1}{K} - 0) = \frac{2}{K} < \varepsilon$. ($\Rightarrow 0 < \frac{1}{m}, \frac{1}{n} \leq \frac{1}{K}$)

Next, to show $\lim_{x \to 0^+} f(x)$ exists, it is enough to show $\lim_{n\to\infty} f(t_n)$ exists for every $t_n \to 0$ in $(0, +\infty)$ by the remark following the sequential limit theorem. For every $t_n \to 0$ in $(0,+\infty)$, $\{t_n\}$ is a Cauchy sequence by Cauchy's theorem. We will show $\lim_{n\to\infty} f(t_n)$ exists by showing $\{f(t_n)\}$ is a Cauchy sequence. For every $\varepsilon > 0$, since $\{t_n\}$ is Cauchy, $\exists K_1 \in \mathbb{N}$ such that $m, n \geq K_1 \Rightarrow |t_m - t_n| < \frac{\varepsilon}{2} \Rightarrow |f(t_m) - f(t_n)| = |f'(y_0)(t_m - t_n)| \leq 2|t_m - t_n| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$. So $\{f(t_n)\}$ is Cauchy. $\lim_{n\to\infty} f(t_n)$ exists by Cauchy's theorem

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(74) For $0 < x < \frac{\pi}{2}$, consider the function $f: [0, x] \to \mathbb{R}$ defined by $f(t) = \ln(\cos t)$. Now f is continuous on $[0, x]$ and differentiable on $(0, x)$. By mean-value theorem, $|\ln(\cos x)| = |f(x) - f(0)| = |f'(t_0)(x - 0)| = |(-\tan t_0) x|$ for some t_0 on $(0, x)$. Now $\tan$ is strictly increasing on $(0, \frac{\pi}{2})$, $\tan t_0 < \tan x$. $\therefore |\ln(\cos x)| \leq (\tan t_0) x < x \tan x$.

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(75) Let $|f|$ has maximum value M on $[0, \frac{1}{2}]$. Since $|f|$ is continuous on $[0, \frac{1}{2}]$, so by extreme value theorem, $M = |f(w)|$ for some $w \in [0, \frac{1}{2}]$. By mean value theorem, there is $x_0 \in (0, w)$ such that $f(w) - f(0) = f'(x_0)(w - 0)$. Then $M = |f(w) - f(0)| \leq |f'(x_0)||w| \leq |f(x_0)| \frac{1}{2} \leq \frac{M}{2}$. Since $0 \leq M \leq \frac{M}{2}$, we get $M = 0$. Then $f(x) = 0$ for all $x \in [0, \frac{1}{2}]$. Similarly, replacing $[0, \frac{1}{2}]$ by $[\frac{1}{2}, 1]$ and using $f(\frac{1}{2}) = 0$ instead of $f(0) = 0$, the argument above shows $f(x) = 0$ for all $x \in [\frac{1}{2}, 1]$. Keep on going, we will get $f(x) = 0$ for all $x \geq 0$.

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(76) Since $\lim_{h\to 0} f(x_0+h) + f(x_0-h) - 2f(x_0) = 0$ and $\lim_{h \to 0} h^2 = 0$, we consider using l'Hopital's rule

$$\lim_{h\to 0} \frac{f'(x_0+h) - f'(x_0-h)}{2h} = \frac{1}{2} \lim_{h\to 0} \left(\frac{f'(x_0+h) - f'(x_0)}{h} + \frac{f'(x_0-h) - f'(x_0)}{-h} \right) = \frac{1}{2}(f''(x_0) + f''(x_0)) = f''(x_0).$$

By l'Hopital's rule, $\lim_{h\to 0} \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2} = f''(x_0)$.

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(77) Since $\frac{d^4}{d\theta^4} \cos\theta = \cos\theta$, by Taylor's theorem, there is $\theta_0 \in (0, \theta)$ such that

$\cos\theta = 1 + 0(\theta - 0) - \frac{1}{2!}(\theta - 0)^2 + \frac{0}{3!}(\theta - 0)^3 + \frac{\cos\theta_0}{4!}(\theta - 0)^4$. Since $0 \leq \theta_0 \leq \theta \leq \frac{\pi}{2}$, so $0 \leq \cos\theta_0 \leq 1$.

Therefore, $1 - \frac{\theta^2}{2} \leq \cos\theta \leq 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$.

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(78) Let $h>0$ and $x=c+2h$. By Taylor's theorem, there is $x_0\in(c,x)$ such that
$f(x)=f(c)+f'(c)\underbrace{(x-c)}_{2h}+\frac{f''(x_0)}{2}\underbrace{(x-c)^2}_{4h^2} \Rightarrow f'(c)=\frac{f(x)-f(c)}{2h}-f''(x_0)h$

$\Rightarrow |f'(c)|\le \frac{1}{2h}(|f(x)|+|f(c)|)+|f''(x_0)|h$

$\le \frac{M_0}{h}+M_2h$ for every $h>0$.

By calculus, $\frac{M_0}{h}+M_2h$ has minimum value $2\sqrt{M_0M_2}$ when $h=\sqrt{\frac{M_0}{M_2}}$, so $|f'(c)|\le 2\sqrt{M_0M_2}$ for every $c\in\mathbb{R}$. Then $M_1\le 2\sqrt{M_0M_2}$, i.e. $M_1^2\le 4M_0M_2$.

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(79) (a) For every $\varepsilon>0$, let $\delta=\frac{\varepsilon}{2}$, then $|x-t|<\delta \Rightarrow |f(x)-f(t)| \underset{\text{mean-value theorem}}{=} |f'(c_0)(x-t)|\le 2|x-t| < 2\delta=\varepsilon$. $\therefore$ f is uniformly continuous.

(b) Suppose $f(x)=\sin\frac{1}{x}$ is uniformly continuous on $(0,\infty)$. Then for every $\varepsilon>0$ (in particular $\varepsilon=1$), there is $\delta>0$ such that $\forall\, x,t\in(0,\infty)$, $|x-t|<\delta \Rightarrow |f(x)-f(t)|<\varepsilon=1$. By Archimedian principle, $\exists\, n\in\mathbb{N}$ such that $n>\sqrt{\frac{1}{\pi\delta}}$.
Now let $x=\frac{1}{n\pi}$ and $t=\frac{1}{(n+\frac12)\pi}$, then $|x-t|=\left|\frac{1}{n\pi}-\frac{1}{(n+\frac12)\pi}\right|=\frac{1}{2n(n+\frac12)\pi}<\frac{1}{n^2\pi}<\delta$,
but $|f(x)-f(t)|=|\sin n\pi-\sin(n+\frac12)\pi|=1$, a contradiction.

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(80) (a) Suppose the statement is false. Let $m_1=(a+b)/2$. Then one of $[a,m_1]$ or $[m_1,b]$ is not contained in the union of finitely many of these open intervals, call that interval I_1. Again, we divide I_1 into two using its midpoint. Then one of these two, call it I_2, is not contained in the union of finitely many of these open intervals. Continuing this process, we get closed intervals $[a,b]\supseteq I_1\supseteq I_2\supseteq\cdots$ and length of I_n goes to 0. So by nested interval theorem, $\bigcap_{n=1}^{\infty} I_n=\{x\}$. Since $x\in[a,b]$, one of the open intervals will contain x. Since length of I_n goes to 0, this open interval containing x will contain some I_n, contradicting the definition of I_n. Therefore, the statement must be true.

(b). If $f:[a,b]\to\mathbb{R}$ is continuous, then $\forall\,\varepsilon>0$, $\forall\, t\in[a,b]$, $\underset{f \text{ continuous at } t \text{ implies}}{\exists}\ \delta_t>0$ such that $x\in(t-\delta_t,t+\delta_t)\Rightarrow |f(x)-f(t)|<\frac{\varepsilon}{2}$. Since $[a,b]\subseteq\bigcup_{t\in[a,b]}(t-\frac{\delta_t}{2},t+\frac{\delta_t}{2})$, by part (a) $\exists\, t_1,\dots,t_n\in[a,b]$ such that $[a,b]\subseteq(t_1-\frac{\delta_{t_1}}{2},t_1+\frac{\delta_{t_1}}{2})\cup\cdots\cup(t_n-\frac{\delta_{t_n}}{2},t_n+\frac{\delta_{t_n}}{2})$.
Let $\delta=\frac12\min\{\delta_{t_1},\dots,\delta_{t_n}\}>0$. Now for every $x,y\in[a,b]$ with $|x-y|<\delta$, we have $x\in(t_i-\frac{\delta_{t_i}}{2},t_i+\frac{\delta_{t_i}}{2})$ for some i. So $|x-t_i|<\frac{\delta_{t_i}}{2}<\delta_{t_i}$ and $|y-t_i|\le|y-x|+|x-t_i|<\delta+\frac{\delta_{t_i}}{2}\le\frac{\delta_{t_i}}{2}+\frac{\delta_{t_i}}{2}=\delta_{t_i}$. Then
$|f(x)-f(y)|\le|f(x)-f(t_i)|+|f(t_i)-f(y)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.
Therefore, f is uniformly continuous on $[a,b]$.

(81) Solution 1 Assume $f(x_0)>0$ for some $x_0 \in [a,b]$. Since f is continuous at x_0, for $\varepsilon = \frac{f(x_0)}{2}$, there is a $\delta>0$ such that $x \in [a,b] \cap (x_0-\delta, x_0+\delta)$ implies $|f(x)-f(x_0)| < \varepsilon = \frac{f(x_0)}{2}$. Then $-\frac{f(x_0)}{2} < f(x)-f(x_0)$ so that $f(x) > \frac{f(x_0)}{2} > 0$.
Now $[a,b] \cap (x_0-\delta, x_0+\delta)$ contains a closed interval $[c,d]$ of positive length.
Then $0 < \int_c^d \frac{f(x_0)}{2}\,dx < \int_c^d f(x)\,dx \le \int_a^b f(x)\,dx = 0$, contradiction. So $f(x) \equiv 0$ $\forall x \in [a,b]$.

Solution 2 Define $g(t) = \int_a^t f(x)\,dx$. Since f is continuous on $[a,b]$, by the fundamental theorem of calculus, $g'(x) = f(x) \ge 0$ for all $x \in [a,b]$. So g is increasing on $[a,b]$. Since $0 = g(a) \le g(t) \le g(b) = \int_a^b f(x)\,dx = 0$, we must have $g(x) = 0$ for all $x \in [a,b]$. Then $f(x) = g'(x) = 0$ for all $x \in [a,b]$.

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(82) (i) For $\varepsilon>0$, since f is integrable on $[a,b]$ and $[b,c]$, by the integral criterion, there are partition P_1 of $[a,b]$ such that $U(f,P_1)-L(f,P_1) < \frac{\varepsilon}{2}$ and partition P_2 of $[b,c]$ such that $U(f,P_2)-L(f,P_2) < \frac{\varepsilon}{2}$. Then $P = P_1 \cup P_2$ is a partition of $[a,c]$ and

$$U(f,P)-L(f,P) = (U(f,P_1)+U(f,P_2)) - (L(f,P_1)+L(f,P_2))$$
$$= (U(f,P_1)-L(f,P_1)) + (U(f,P_2)-L(f,P_2))$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So by the integral criterion, f is integrable on $[a,c]$.

(ii) For $\varepsilon>0$, since f is integrable on $[a,d]$, by the integral criterion, there is a partition P_1 of $[a,d]$ such that $U(f,P_1)-L(f,P_1) < \varepsilon$. Then $P_2 = P_1 \cup \{b,c\}$ is finer partition of P_1 so that $L(f,P_1) \le L(f,P_2) \le U(f,P_2) \le U(f,P_1)$.
Then $U(f,P_2)-L(f,P_2) \le U(f,P_1)-L(f,P_1) < \varepsilon$. Now $P = P_2 \cap [b,c]$ is a partition of $[b,c]$ and $\underbrace{U(f,P)-L(f,P)} \le U(f,P_2)-L(f,P_2) < \varepsilon$.

Only the terms of $U(f,P_2)-L(f,P_2) = \sum_{i=1}^{n} \underbrace{(M_i-m_i)}_{\ge 0}\Delta x_i$ in $[b,c]$ are used to compute $U(f,P)-L(f,P)$

So by the integral criterion, f is integrable on $[b,c]$.

(83) Consider the subintervals $[a,x_1]$, $[x_1,\frac{x_1+x_2}{2}]$, $[\frac{x_1+x_2}{2},x_2]$, $\ldots$, $[\frac{x_{n-1}+x_n}{2},x_n]$, $[x_n,b]$. By exercise 82(i), it is enough to show f is integrable on each of these intervals. (If $a=x_1$, then ignore $[a,x_1]$. If $x_n=b$, then ignore $[x_n,b]$.) In each of the subinterval $[u,v]$ above, either f is discontinuous only at u or f is discontinuous only at v. In the former case, since f is bounded on $[a,b]$, there is $K>0$ such that $|f(x)|\le K$ for every $x\in[a,b]$. For $\varepsilon>0$, choose $w\in(u,v)$ such that $2K(w-u)<\frac{\varepsilon}{2}$ ($\Leftrightarrow w<u+\frac{\varepsilon}{4K}$). Since f is continuous on $[w,v]$, f is integrable on $[w,v]$. By the integral criterion, there is a partition P_1 of $[w,v]$ such that $U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{2}$. Let $P=\{u\}\cup P_1$, then P is a partition of $[u,v]$ and $U(f,P)-L(f,P)=(M_1-m_1)(w-u)+U(f,P_1)-L(f,P_1)\le 2K(w-u)+\frac{\varepsilon}{2}<\varepsilon$. So by the integral criterion, f is integrable on $[u,v]$. The latter case when f is discontinuous only at v is similar.

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(84) (i) Since $\inf\limits_{x\in[x_{i-1},x_i]} f(x)+\inf\limits_{x\in[x_{i-1},x_i]} g(x)$ is a lower bound of $\{f(x)+g(x): x\in[x_{i-1},x_i]\}$, we get

$$\inf_{x\in[x_{i-1},x_i]} f(x)+\inf_{x\in[x_{i-1},x_i]} g(x)\le\inf_{x\in[x_{i-1},x_i]}(f(x)+g(x)).\ \text{So}\ L(f,P)+L(g,P)\le L(f+g,P).$$

Similarly, $U(f+g,P)\le U(f,P)+U(g,P)$ since $\sup\limits_{x\in[x_{i-1},x_i]}(f(x)+g(x))\le\sup\limits_{x\in[x_{i-1},x_i]}f(x)+\sup\limits_{x\in[x_{i-1},x_i]}g(x)$.

(ii) For $\varepsilon>0$, since $\int_a^b f(x)\,dx=\sup\{L(f,P): P \text{ partition of } [a,b]\}$, by the supremum property, there is a partition P_1 such that $\int_a^b f(x)\,dx-\frac{\varepsilon}{2}<L(f,P_1)\le\int_a^b f(x)\,dx$. Similarly, there is a partition P_2 such that $\int_a^b g(x)\,dx-\frac{\varepsilon}{2}<L(g,P_2)\le\int_a^b g(x)\,dx$. Letting $P=P_1\cup P_2$, then $P_1,P_2\subseteq P$. So

$$\int_a^b f(x)\,dx+\int_a^b g(x)\,dx-\varepsilon<L(f,P_1)+L(g,P_2)\le L(f,P)+L(g,P)\overset{\text{by part (i)}}{\le} L(f+g,P)\le (L)\int_a^b (f(x)+g(x))\,dx.$$

By the infinitesimal principle, $\int_a^b f(x)\,dx+\int_a^b g(x)\,dx\le (L)\int_a^b (f(x)+g(x))\,dx$. Similarly, the inequality $\int_a^b f(x)\,dx+\int_a^b g(x)\,dx\ge (U)\int_a^b (f(x)+g(x))\,dx$ can be obtained by using the infimum property. Combining, we get

$$\int_a^b f(x)\,dx+\int_a^b g(x)\,dx\le (L)\int_a^b (f(x)+g(x))\,dx\le (U)\int_a^b (f(x)+g(x))\,dx\le\int_a^b f(x)\,dx+\int_a^b g(x)\,dx.$$

Therefore, equality must hold throughout, i.e. $f+g$ is integrable and $\int_a^b (f(x)+g(x))\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx$.

(85) (a) $\int_0^\infty \frac{dx}{\sqrt{e^x}} = \int_0^\infty e^{-\frac{1}{2}x}dx = \lim_{d\to+\infty}\int_0^d e^{-\frac{1}{2}x}dx = \lim_{d\to+\infty}\left(-\frac{1}{2}e^{-\frac{1}{2}x}\Big|_0^d\right) = \lim_{d\to+\infty}\left(-2e^{-\frac{1}{2}d}+2\right) = 2$

Integral exists

(b) $\int_0^\infty \sin x\,dx = \lim_{d\to+\infty}\int_0^d \sin x\,dx = \lim_{d\to+\infty}\left(-\cos x\Big|_0^d\right) = \lim_{d\to+\infty}(-\cos d+1)$ does not exist.

(c) Note $0 \le \frac{1}{6x} \le \frac{1}{x^2+5x}$ for $x\in(0,1]$. $\int_0^1 \frac{1}{6x}dx = \lim_{c\to0^+}\int_c^1 \frac{1}{6x}dx = \lim_{c\to0^+}\left(\frac{1}{6}\ln x\Big|_c^1\right)$
$= \lim_{c\to0^+}\left(-\frac{1}{6}\ln c\right)$ does not exists. By comparison test, $\int_0^1 \frac{dx}{x^2+5x}$ does not exist.

(d) $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = \int_{-1}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{d\to0^-}\int_{-1}^d \frac{dx}{\sqrt[3]{x}} + \lim_{c\to0^+}\int_c^1 \frac{dx}{\sqrt[3]{x}} = \lim_{d\to0^-}\left(\frac{3}{2}x^{\frac{2}{3}}\Big|_{-1}^d\right) + \lim_{c\to0^+}\left(\frac{3}{2}x^{\frac{2}{3}}\Big|_c^1\right)$
$= \lim_{d\to0^-}\left(\frac{3}{2}d^{\frac{2}{3}}-\frac{3}{2}\right) + \lim_{c\to0^+}\left(\frac{3}{2}-\frac{3}{2}c^{\frac{2}{3}}\right) = -\frac{3}{2}+\frac{3}{2} = 0$. Integral exists.

(e) $\int_0^1 \frac{dx}{x(x-1)} = \int_0^{\frac{1}{2}} \frac{dx}{x(x-1)} + \int_{\frac{1}{2}}^1 \frac{dx}{x(x-1)}$. $\int_0^{\frac{1}{2}} \frac{dx}{x(x-1)} = \lim_{c\to0^+}\int_c^{\frac{1}{2}} \frac{dx}{x(x-1)} = \lim_{c\to0^+}\int_c^{\frac{1}{2}}\left(\frac{1}{x-1}-\frac{1}{x}\right)dx$
$= \lim_{c\to0^+}\left(\ln|x-1|-\ln|x|\right)_c^{\frac{1}{2}} = \lim_{c\to0^+}\left(-\ln|c-1|+\ln|c|\right) = 0-\infty$ does n't exist (as a number). So $\int_0^1 \frac{dx}{x(x-1)}$ does not exist.

(f) For $x\in(0,+\infty)$, $\left|\frac{\cos x}{1+x^2}\right| \le \frac{1}{1+x^2}$. Since $\int_0^{+\infty}\frac{1}{1+x^2}dx = \lim_{b\to+\infty}\int_0^b \frac{1}{1+x^2}dx = \lim_{b\to+\infty}\tan^{-1}b = \frac{\pi}{2}$,
$\int_0^{+\infty}\left|\frac{\cos x}{1+x^2}\right|dx$ exists by the comparison test. Then $\int_0^{+\infty}\frac{\cos x}{1+x^2}dx$ exists by the absolute convergence test.

(86) (a) P.V. $\int_{-\infty}^\infty \frac{x}{e^{x^2}}dx = \lim_{b\to+\infty}\int_{-b}^b xe^{-x^2}dx = \lim_{b\to+\infty}\left(-\frac{1}{2}e^{-x^2}\Big|_{-b}^b\right) = \lim_{b\to+\infty}\underbrace{\left(-\frac{1}{2}e^{-b^2}+\frac{1}{2}e^{-b^2}\right)}_{=0} = 0$

(b) P.V. $\int_0^2 \frac{dx}{x^2-1} = \lim_{\varepsilon\to0^+}\left(\int_0^{1-\varepsilon}\frac{dx}{x^2-1} + \int_{1+\varepsilon}^2 \frac{dx}{x^2-1}\right) = \lim_{\varepsilon\to0^+}\left(\int_0^{1-\varepsilon}\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx + \int_{1+\varepsilon}^2\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx\right)$
$= \lim_{\varepsilon\to0^+}\left(\cancel{\frac{1}{2}\ln\varepsilon} - \frac{1}{2}\ln(2-\varepsilon) - \cancel{\frac{1}{2}\ln\varepsilon} - \frac{1}{2}\ln 3 + \frac{1}{2}\ln(2+\varepsilon)\right) = -\frac{1}{2}\ln 3$.

(87) We have $\int_0^\infty t^{x-1}e^{-t}dt = \int_0^1 t^{x-1}e^{-t}dt + \int_1^\infty t^{x-1}e^{-t}dt$.

For $\int_0^1 t^{x-1}e^{-t}dt$, since $\lim_{t\to0^+}\frac{t^{x-1}e^{-t}}{t^{x-1}} = \lim_{t\to0^+}e^{-t} = 1$, by the limit comparison tes
$\int_0^1 t^{x-1}e^{-t}dt$ converges $\Leftrightarrow \int_0^1 t^{x-1}dt = \int_0^1 \frac{1}{t^{1-x}}dt$ converges $\overset{\text{p-test}}{\Leftrightarrow} 1-x<1 \Leftrightarrow x>$

For $\int_1^\infty t^{x-1}e^{-t}dt$, note that $\lim_{t\to+\infty}\frac{t^{x-1}e^{-t}}{\frac{1}{t^2}} = \lim_{t\to+\infty}\frac{t^{x+1}}{e^t} = 0$ by example 1 on p. 38.
Since $\int_1^\infty \frac{1}{t^2}dt$ converges by p-test, so by the limit comparison test, $\int_1^\infty t^{x-1}e^{-t}dt$ converges. Therefore, $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ converges for $x>0$.

(88) (a) S is a countably infinite set iff there exists a bijection $f: \mathbb{N} \to S$.

(b) S is a countable set iff S is a finite set or a countably infinite set.

(c) A series $\sum_{k=1}^{\infty} a_k$ converges to a number S iff $\lim_{n\to\infty}(a_1+a_2+\cdots+a_n)=S$.

(d) A nonempty subset S of $\mathbb{R}$ is bounded above iff there exists some $M\in\mathbb{R}$ such that $x\le M$ for all $x\in S$.

(e) $\tilde{M}$ is the supremum of a subset S of $\mathbb{R}$ that is bounded above iff $\tilde{M}$ is an upper bound of S and $\tilde{M}\le M$ for all upper bounds M of S.

(f) A sequence $\{x_n\}$ converges to a number x iff for every $\varepsilon>0$, there exists $K\in\mathbb{N}$ such that $n\ge K$ implies $|x_n-x|<\varepsilon$.

(g) A sequence $\{x_n\}$ is a Cauchy sequence iff for every $\varepsilon>0$, there exists $K\in\mathbb{N}$ such that $m,n\ge K$ implies $|x_m-x_n|<\varepsilon$.

(h) x is an accumulation point of a set S iff there exists a sequence $\{x_n\}$ in S such that $x_n\ne x$ for all n and $\lim_{n\to\infty} x_n=x$.

(i) $f: S\to\mathbb{R}$ has a limit L at x_0 iff for every $\varepsilon>0$, there exists a $\delta>0$ such that $x\in S$ and $0<|x-x_0|<\delta$ imply $|f(x)-L|<\varepsilon$,

(j) $f: S\to\mathbb{R}$ is continuous at $x_0\in S$ iff for every $\varepsilon>0$, there exists a $\delta>0$ such that $x\in S$ and $|x-x_0|<\delta$ imply $|f(x)-f(x_0)|<\varepsilon$.

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(89)(a) For a fixed $m\in\mathbb{Z}$, the curves $y=\pi x$ and $y=x^3+x+m$ intersect in at most 3 points (because $\pi x=x^3+x+m \Rightarrow x^3+(1-\pi)x+m=0$.) Now $S=\underbrace{\bigcup_{m\in\mathbb{Z}}}_{\text{countable}}\underbrace{\{(x,y): y=\pi x, y=x^3+x+m\}}_{\text{at most 3 points hence countable}}$ is countable by the countable union theorem.

(b) For a fixed $m\in\mathbb{Z}$, the curves $y=x^3+x+1$ and $y=mx$ intersects in at most 3 points (because $mx=x^3+x+1 \Rightarrow x^3+(1-m)x+1=0$). Now $S=\underbrace{\bigcup_{m\in\mathbb{Z}}}_{\text{countable}}\underbrace{\{(x,y): y=x^3+x+1, y=mx\}}_{\text{at most 3 points hence countable}}$ is countable by the countable union theorem.

(c) For a fixed $m\in\mathbb{N}$, the curves $x^2+y^2=1$ and $xy=\frac{1}{m}$ intersect in at most 4 points (because $x^2+(\frac{1}{mx})^2=1 \Rightarrow x^4-x^2+\frac{1}{m^2}=0$.) Now $S=\underbrace{\bigcup_{m\in\mathbb{N}}}_{\text{countable}}\underbrace{\{(x,y): x^2+y^2=1, xy=\frac{1}{m}\}}_{\text{at most 4 points hence countable}}$ is countable by the countable union theorem.

(d) Taking $b=0$, we see that $S\supseteq M$. Since M is uncountable, so S is uncountable by the countable subset theorem.

(e) Note if $x=|a|$, then $a=x$ or $-x$. So

$$S=\{a+b: |a|\in M, b\in\mathbb{Q}\}=\{x+b: x\in M, b\in\mathbb{Q}\}\cup\{-x+b: x\in M, b\in\mathbb{Q}\}$$
$$=\underbrace{\bigcup_{(x,b)\in M\times\mathbb{Q}}}_{\text{countable}}\underbrace{\{x+b,-x+b\}}_{\text{2 elements, countable}}$$

is countable by the countable union theorem.

89 (f) The set $S=\{a+b\sqrt{2}: a,b\in\mathbb{Q}\}=\underbrace{\bigcup_{(a,b)\in\mathbb{Q}\times\mathbb{Q}}}_{\text{Countable}}\underbrace{\{a+b\sqrt{2}\}}_{\text{1 element finite, countable}}$ is countable.

The set $\{c+d\sqrt{2}: c,d\in\mathbb{Q},\ c+d\sqrt{2}\neq 0\}=S\setminus\{0\}$ is also countable.

Now $\mathbb{Q}(\sqrt{2})=\{\frac{x}{y}: x\in S, y\in S\setminus\{0\}\}=\underbrace{\bigcup_{(x,y)\in S\times(S\setminus\{0\})}}_{\text{countable}}\underbrace{\{\frac{x}{y}\}}_{\text{1 element finite, countable}}$ is countable

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(g) Since $A\cap B\subseteq A$, $\mathbb{Q}\cap A\subseteq\mathbb{Q}$, $B\cap\mathbb{Q}\subseteq\mathbb{Q}$ and A, $\mathbb{Q}$ are countable, so by the countable subset theorem, $A\cap B$, $\mathbb{Q}\cap A$, $B\cap\mathbb{Q}$ are countable. For $x_0\in A\cap B$, $y_0\in\mathbb{Q}\cap A$, the set $S_{x_0,y_0}=\{x_0^2+y_0^2+z^2: z\in B\cap\mathbb{Q}\}=\underbrace{\bigcup_{z\in B\cap\mathbb{Q}}}_{\text{countable}}\underbrace{\{x_0^2+y_0^2+z^2\}}_{\text{1 element, hence finite.}}$ is countable by countable union theorem.

Now $S=\underbrace{\bigcup_{x\in A\cap B}}_{\text{countable}}\Big(\underbrace{\bigcup_{y\in\mathbb{Q}\cap A}}_{\text{countable}}\underbrace{S_{x,y}}_{\text{countable}}\Big)$ is countable, again by the countable union theorem.

(h) Let $y_0\in A$ and $T=\{x-y_0: x\in A\}$. Then $T\subseteq S$. Now $f:A\to T$ defined by $f(x)=x-y_0$ is a bijection. So since A is uncountable, T must be uncountable. Finally, since $T\subseteq S$, S must also be uncountable.

(i) Solution 1 For $x\in A$, let $S_x=\{x^2+y^2: y\in A\}=\bigcup_{y\in A}\{x^2+y^2\}$, then S_x is countable by countable union theorem. (A countable; $\{x^2+y^2\}$ 1 element so countable) Then $S=\bigcup_{x\in A}S_x$ is countable by countable union theorem.

Solution 2 A countable $\underset{\text{product theorem}}{\Rightarrow}$ $A\times A$ countable $\underset{\text{countable union theorem}}{\Rightarrow}$ $S=\bigcup_{(x,y)\in A\times A}\{x^2+y^2\}$ is countable.

Solution 3 The function $f: A\times A\to S$ defined by $f(x,y)=x^2+y^2$ is surjective. Since A is countable, $A\times A$ is countable by product theorem. Then S is countable by further useful fact (2) on p. 9.

(j) Since A is countable, $\mathbb{R}\setminus A$ must be uncountable. Taking $y=0$, we have $S\supseteq\mathbb{R}\setminus A$. By the countable subset theorem, S is uncountable.

(k) Since A is countable, $\mathbb{R}-A$ must be uncountable. Let $a\in A$, then S contains the subset $S_a=\{(a,y): y\in\mathbb{R}\setminus A\}$. The function $f:\mathbb{R}-A\to S_a$ defined by $f(y)=(a,y)$ is a bijection. Since $\mathbb{R}\setminus A$ is uncountable, so S_a is uncountable. Then S is uncountable by the countable subset theorem.

(l) $S=\bigcup_{x\in\mathbb{Z}}S_x$, where $S_x=\{x+y\sqrt{2}: y\in A\}$. The function $f: A\to S_x$ defined by $f(y)=x+y\sqrt{2}$ is a bijection. Since A is countable, each S_x is countable, then $S=\bigcup_{x\in\mathbb{Z}}S_x$ is countable by the countable union theorem.

89 (m) Since $f: \mathbb{Q} \to T$ defined by $f(r) = r\pi$ is a bijection, so T is countable. The set $U = \{a+b\sqrt{2}-c\sqrt{3} : a,b,c \in T\} = \bigcup_{\underbrace{(a,b,c)\in T\times T\times T}_{\text{countable}}} \underbrace{\{a+b\sqrt{2}-c\sqrt{3}\}}_{\text{1 element set} \Rightarrow \text{countable}}$ is countable by the countable union theorem.
Then $S = \mathbb{R} \setminus U$ is uncountable.

(n) $\{\sqrt{m}+\sqrt{n} : m,n \in \mathbb{N}\} = \underbrace{\bigcup_{m=1}^{\infty}}_{\text{countable}} \underbrace{\Big(\underbrace{\bigcup_{n=1}^{\infty}}_{\text{countable}} \underbrace{\{\sqrt{m}+\sqrt{n}\}}_{\text{one element}} \Big)}_{\text{countable}}$ is countable.

Since $\mathbb{R}\setminus(T\cap U) = (\mathbb{R}\setminus T)\cup(\mathbb{R}\setminus U) = \mathbb{Q} \cup \{\sqrt{m}+\sqrt{n} : m,n\in\mathbb{N}\}$ is countable, so $S = T\cap U = \mathbb{R}\setminus(\mathbb{R}\setminus(T\cap U))$ is uncountable.

(o) Consider the subset of S of squares having the unit circle at the origin as circumcircle. This subset is uncountable because for every $\alpha \in [0, \frac{\pi}{2})$, there is a unique square having $(\cos\alpha, \sin\alpha)$ as a vertex and $[0, \frac{\pi}{2})$ is uncountable. So S is uncountable.

(p). $G = \bigcup_{\underbrace{(a,b)\in\mathbb{Z}\times\mathbb{Z}}_{\text{countable}}} \underbrace{\{a+bi\}}_{\text{1 element} \Rightarrow \text{countable}}$ is countable by countable union theorem. Let S_n be the degree n polynomials. So $S_n = \bigcup_{\underbrace{(a_0,a_1,\dots,a_n)\in G\times G\times\cdots\times G\setminus\{0\}}_{\text{countable}}} \underbrace{\{a_0+a_1x+\cdots+a_nx^n\}}_{\text{1 element} \Rightarrow \text{countable}}$ is countable. Then $S = \bigcup_{n\in\mathbb{N}} S_n$ is countable by countable union theorem.

90 (a) Alternating Series Test $\sum_{k=1}^{\infty} \frac{\cos k\pi}{k^2+2^k} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2+2^k}$. As $k \nearrow \infty$, $k^2+2^k \nearrow \infty$ and $\frac{1}{k^2+2^k} \searrow 0$. So $\sum_{k=1}^{\infty} \frac{\cos k\pi}{k^2+2^k}$ converges.

Comparison Test Since $\frac{e^{\sqrt{k}}}{\sqrt{k}} > \frac{1}{\sqrt{k}} = \frac{1}{k^{1/2}}$ and $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ diverges by p-test, so $\sum_{k=1}^{\infty} \frac{e^{\sqrt{k}}}{\sqrt{k}}$ diverges.

(b) Ratio test $\lim_{k\to\infty} \frac{(2(k+1))!}{3^{k+1}(k+1)^4} \frac{3^k k^4}{(2k)!} = \lim_{k\to\infty} \frac{(2k+2)(2k+1)}{3} \left(\frac{k}{k+1}\right)^4 = \infty \Rightarrow \sum_{k=1}^{\infty} \frac{(2k)!}{3^k k^4}$ diverges.

Absolute Convergence Test and Comparison Test $\left|\frac{(\cos k)(\sin 2k)}{2^k}\right| \le \left(\frac{1}{2}\right)^k$ and $\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ converges $\Rightarrow \sum_{k=1}^{\infty} \left|\frac{(\cos k)(\cos 2k)}{2^k}\right|$ converges $\Rightarrow \sum_{k=1}^{\infty} \frac{(\cos k)(\sin 2k)}{2^k}$ converges.

(c) Term test $\lim_{k\to\infty} \frac{1}{2}(\cos\frac{1}{k} + \sin\frac{1}{k}) = \frac{1}{2}(1+0) = \frac{1}{2} \ne 0 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{2}(\cos\frac{1}{k}+\sin\frac{1}{k})$ diverges.

Limit Comparison Test $\lim_{k\to\infty}(\frac{1}{k} - \frac{1}{k+1}) = 0 \Rightarrow \lim_{k\to\infty} \frac{\sin(\frac{1}{k}-\frac{1}{k+1})}{\frac{1}{k}-\frac{1}{k+1}} \overset{\theta = \frac{1}{k}-\frac{1}{k+1}}{=} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

Since $\sum_{k=1}^{\infty}(\frac{1}{k} - \frac{1}{k+1}) = 1 - \lim_{k\to\infty}\frac{1}{k+1} = 1$ (by telescoping series test), $\sum_{k=1}^{\infty} \sin(\frac{1}{k}-\frac{1}{k+1})$ converges

(80) (d) Since $0 \le \frac{2^k+3^k}{1^k+4^k} \le \frac{3^k+3^k}{4^k} = 2\left(\frac{3}{4}\right)^k$ and $\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^k$ converges by geometric series test, so $\sum_{k=1}^{\infty} \frac{2^k+3^k}{1^k+4^k}$ converges by comparison test.

Since $\lim_{k\to\infty} \cos\left(\sin\frac{1}{k}\right) = \cos(\sin 0) = \cos 0 = 1 \ne 0$, $\sum_{k=1}^{\infty} \cos\left(\sin\frac{1}{k}\right)$ diverges by term test.

(e) $\lim_{k\to\infty} \frac{2^{1/k}+3^{1/k}}{1^{1/k}+4^{1/k}} = \frac{2^0+3^0}{1^0+4^0} = 1 \ne 0 \Rightarrow \sum_{k=1}^{\infty} \frac{2^{1/k}+3^{1/k}}{1^{1/k}+4^{1/k}}$ diverges by term test.

Since $\cos k\pi = (-1)^k$ and $k \uparrow \infty \Rightarrow \frac{1}{k\pi} \searrow 0 \Rightarrow \sin\frac{1}{k\pi} \searrow 0$, by the alternating series test, $\sum_{k=1}^{\infty} (\cos k\pi)\left(\sin\frac{1}{k\pi}\right)$ converges

sin is increasing on $[0, \frac{1}{\pi}]$.

(f) Since $\lim_{k\to\infty} \frac{(k+1)!^2}{((k+1)^2)!} \Big/ \frac{(k!)^2}{(k^2)!} = \lim_{k\to\infty} \frac{(k+1)!^2 (k^2)!}{(k!)^2((k+1)^2)!} = \lim_{k\to\infty} \frac{1}{(k^2+1)(k^2+2)\cdots(k^2+2k)} = 0 < 1$,

by ratio test, $\sum_{k=1}^{\infty} \frac{(k!)^2}{(k^2)!}$ converges.

Note $0 < \left(\cos\frac{1}{k}\right)\left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right) \le \left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right)$. Since $\lim_{k\to\infty} \frac{\left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right)}{\frac{1}{k^2}} \overset{\theta=\frac{1}{k}}{=}$ $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\frac{\tan\theta}{\theta} = 1$ and $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by p-test, so $\sum_{k=1}^{\infty}\left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right)$ converges by limit comparison test. Therefore $\sum_{k=1}^{\infty}\left(\cos\frac{1}{k}\right)\left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right)$ converges by comparison test.

(g) $\left|\frac{2^k\cos k}{(k-1)!}\right| \le \frac{2^k}{(k-1)!}$. Now $\lim_{k\to\infty} \frac{2^{k+1}}{k!} \Big/ \frac{2^k}{(k-1)!} = \lim_{k\to\infty} \frac{2}{k} = 0 < 1$. So by the ratio test, $\sum_{k=2}^{\infty} \frac{2^k}{(k-1)!}$ converge. By the comparison test, $\sum_{k=2}^{\infty}\left|\frac{2^k\cos k}{(k-1)!}\right|$ converges. By the absolute convergence test, $\sum_{k=2}^{\infty} \frac{2^k\cos k}{(k-1)!}$ converges.

$\lim_{k\to\infty} \frac{\sin(\frac{1}{k})}{\ln k} \Big/ \frac{\frac{1}{k}}{\ln k} = \lim_{k\to\infty} \frac{\sin(\frac{1}{k})}{1/k} \overset{\theta=\frac{1}{k}}{=} \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$. Now as $k\to\infty$, $k\ln k$ increases to ∞, $\frac{1}{k\ln k}$ decreases to 0. Since $\int_2^{\infty} \frac{1}{x\ln x}dx = \ln(\ln x)\Big|_2^{\infty} = \infty$, by the integral test, $\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ diverges. So by the limit comparison test, $\sum_{k=2}^{\infty}\frac{\sin(\frac{1}{k})}{\ln k}$ diverges.

(h) (When k is large, $\frac{k^\pi+\cos k\pi}{3+k^4} \sim \frac{k^\pi}{k^4}$.) $\lim_{k\to\infty} \frac{\frac{k^\pi+\cos k\pi}{3+k^4}}{\frac{k^\pi}{k^4}} = \lim_{k\to\infty} \frac{1+\frac{\cos k\pi}{k^\pi}}{\frac{3}{k^4}+1} = 1$. Since $\sum_{k=1}^{\infty}\frac{k^\pi}{k^4} = \sum_{k=1}^{\infty}\frac{1}{k^{4-\pi}}$ diverges by p-test (because $4-\pi \le 1$), so $\sum_{k=1}^{\infty}\frac{k^\pi+\cos k\pi}{3+k^4}$ diverges.

Next $\sum_{k=1}^{\infty}\frac{k^\pi\cos k\pi}{3k^4} = \sum_{k=1}^{\infty}\frac{(-1)^k}{3k^{4-\pi}}$ and $a_k = \frac{1}{3k^{4-\pi}}$ decreases to 0, so by the alternating series test, $\sum_{k=1}^{\infty}\frac{k^\pi\cos k\pi}{3k^4}$ converges.

(90)(i) $\lim_{k\to\infty} \frac{(2k+2)!}{(k+2)!\,k!} \cdot \frac{(k+1)!\,(k-1)!}{(2k)!} = \lim_{k\to\infty} \frac{(2k+2)(2k+1)}{(k+2)(k)} = 4 > 1 \overset{\text{ratio test}}{\Longrightarrow} \sum_{k=2}^{\infty} \frac{(2k)!}{(k+1)!(k-1)!}$ diverges

$\lim_{k\to\infty} k\cos(\frac{1}{k^2}) = \infty \cdot \cos 0 = \infty \cdot 1 = \infty \neq 0 \overset{\text{Term test}}{\Longrightarrow} \sum_{k=1}^{\infty} k\cos(\frac{1}{k^2})$ diverges.

(j) $\lim_{k\to\infty} \frac{(3(k+1))!}{(k+1)!\,(2(k+1))!} \Big/ \frac{3k!}{k!(2k)!} = \lim_{k\to\infty} \frac{(3k+3)(3k+2)(3k+1)}{(k+1)(2k+2)(2k+1)} = \frac{27}{4} > 1 \Rightarrow \sum_{k=1}^{\infty} \frac{(3k)!}{k!(2k)!}$ diverges by ratio test.

$0 \le \frac{\cos(1/k)}{k^2-1} \le \frac{1}{k^2-1} < \frac{2}{k^2}$ for $k \ge 2$. Since $\sum_{k=2}^{\infty} \frac{2}{k^2} = 2\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges by p-test,

so $\sum_{k=2}^{\infty} \frac{\cos(1/k)}{k^2-1}$ converges by comparison test.

(k) Ratio Test $\lim_{k\to\infty} \frac{(k+1)!}{(2(k+1)-1)!} \Big/ \frac{k!}{(2k-1)!} = \lim_{k\to\infty} \frac{k+1}{(2k+1)2k} = 0 < 1 \Rightarrow \sum_{k=1}^{\infty} \frac{k!}{(2k-1)!}$ converges.

Alternating Series Test $\sum_{k=1}^{\infty} \frac{\cos k\pi}{\sqrt{k}+1} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k}+1}$, the sequence $\{\frac{1}{\sqrt{k}+1}\}$ decrease to 0 because $k > k' \Rightarrow \sqrt{k} > \sqrt{k'} \Rightarrow \frac{1}{\sqrt{k}+1} < \frac{1}{\sqrt{k'}+1}$ and $\lim_{k\to\infty} \frac{1}{\sqrt{k}+1} = 0$.

So $\sum_{k=1}^{\infty} \frac{\cos k\pi}{\sqrt{k}+1}$ converges.

(l) Ratio Test $\lim_{k\to\infty} \frac{2^{k+1}(k+1)^2}{(k+1)!} \Big/ \frac{2^k k^2}{k!} = \lim_{k\to\infty} \frac{2(k+1)}{k^2} = 0 < 1 \Rightarrow \sum_{k=1}^{\infty} \frac{2^k k^2}{k!}$ converges.

Limit Comparison Test $\lim_{k\to\infty} \frac{\frac{1}{\sqrt{k}}\sin(\frac{1}{\sqrt{k}})}{\frac{1}{\sqrt{k}}\frac{1}{\sqrt{k}}} = \lim_{k\to\infty} \frac{\sin\frac{1}{\sqrt{k}}}{\frac{1}{\sqrt{k}}} \underset{\theta=\frac{1}{\sqrt{k}}}{=} \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$. Since $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}\frac{1}{\sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges by p-test, so $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}\sin(\frac{1}{\sqrt{k}})$ diverges.

(m) Since $\frac{1}{k}$ decreases to 0 as $k \uparrow \infty$, by alternating series test, $\sum_{k=1}^{\infty} \frac{1}{k\cos k\pi} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges.

Since $0 \le \frac{k^2\sin(1/k)}{(2k+1)!} \le \frac{k^2}{(2k+1)!}$ and $\lim_{k\to\infty} \frac{(k+1)^2}{(2k+3)!} \Big/ \frac{k^2}{(2k+1)!} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^2 \frac{1}{(2k+3)(2k+2)} = 0$, by ratio test, $\sum_{k=1}^{\infty} \frac{k^2}{(2k+1)!}$ converges. By comparison test, $\sum_{k=1}^{\infty} \frac{k^2\sin(1/k)}{(2k+1)!}$ converges.

(n) By root test, $\lim_{k\to\infty} \cos(1+\frac{1}{k}) = \cos 1 < 1 \Rightarrow \sum_{k=1}^{\infty} \cos^k(1+\frac{1}{k})$ converges.

By term test, $\lim_{k\to\infty} \frac{\cos(\sin(\frac{1}{k}))}{\sin(\cos(\frac{1}{k}))} = \frac{1}{\sin 1} \neq 0 \Rightarrow \sum_{k=1}^{\infty} \frac{\cos(\sin(1/k))}{\sin(\cos(1/k))}$ diverges.

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(91)(a) For $m, n \in \mathbb{N}$, $0 < \frac{1}{m} + \frac{1}{n}$ and $\frac{1}{1} + \frac{1}{1} = \frac{2}{1} \notin S$. So $S \subseteq (0, 1+\frac{1}{2}]$. Then S has lower bound 0 and upper bound $\frac{3}{2}$. Let $x_k = \frac{1}{k} + \frac{1}{k+1}$, then $x_k \in S$. (Note $\frac{2}{k+1} < x_k < \frac{2}{k}$.) Since $\lim_{k\to\infty} x_k = 0 + 0 = 0$, by the infimum limit theorem, $\inf S = 0$. Next, every upper bound $M \ge \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \in S$. So $\sup S = \frac{3}{2}$.

(91)(b) For $x,y\in[\frac{1}{2},1)$, $1=\frac{1}{2}+\frac{1}{2}\le x+y<1+1=2$. So $S\subseteq[1,2)$. Then S has lower bound 1 and upper bound 2. Take $x=y=\frac{1}{2}+\frac{1}{\pi k}\in[\frac{1}{2},1)$. Then $x_k=x+y\in S$. (Note x_k is irrational, so $x_k\neq 2-\frac{1}{n}$ for all $n\in\mathbb{N}$.) Since $\lim_{k\to\infty}x_k=\frac{1}{2}+\frac{1}{2}=1$, by the infimum limit theorem, $\inf S=1$. Next, take $x=y=1-\frac{1}{\pi k}$. Then $w_k=x+y\in S$ and $\lim_{k\to\infty}w_k=1+1=2$. By the supremum limit theorem, $\sup S=2$.

(c) For $x\in[0,1]\cap\mathbb{Q}$, $n\in\mathbb{N}$, $-1=0-\frac{1}{1}\le x-\frac{1}{n}<1-0=1$. So $S\subseteq[\frac{1}{2},1)$. Then $\frac{1}{2}$ is a lower bound of S and 1 is an upper bound of S. Now every lower bound $m\le\frac{1}{2}=1-\frac{1}{2}\in S$, so $\inf S=\frac{1}{2}$. Also let $x_n=1-\frac{1}{n+1}\in S$, then $\lim_{n\to\infty}x_n=1$. By supremum limit theorem, $\sup S=1$.

(d) (When $x\to\pi$, $\frac{x-\pi}{x+\pi}\to 0$ and when $x\to\infty$, $\frac{x-\pi}{x+\pi}\to 1$.) We will show that $\inf S=0$ and $\sup S=1$. For $x\in(\mathbb{R}\setminus\mathbb{Q})\cap[\pi,\infty)$, $0\le\frac{x-\pi}{x+\pi}\iff\pi\le x$, which is true. So 0 is a lower bound of S. Also $0=\frac{\pi-\pi}{\pi+\pi}\in S$. So every lower bound $t\le\frac{\pi-\pi}{\pi+\pi}=0$. $\therefore\inf S=0$.

For $x\in(\mathbb{R}\setminus\mathbb{Q})\cap[\pi,\infty)$, $\frac{x-\pi}{x+\pi}\le 1\iff x-\pi\le x+\pi$, which is true. So 1 is an upper bound of S. Now $w_n=\frac{n\pi-\pi}{n\pi+\pi}\in S$ for every $n\in\mathbb{N}$. Since $\lim_{n\to\infty}w_n=1$, so by the supremum limit theorem, $\sup S=1$.

(e) (When $x\to 0$, $\frac{x-\pi}{x+\pi}\to -1$ and when $x\to\infty$, $\frac{x-\pi}{x+\pi}\to 1$.) We will show that $\inf S=-1$ and $\sup S=1$. For $x\in\mathbb{Q}\cap[0,\infty)$, $-1\le\frac{x-\pi}{x+\pi}\iff -x-\pi\le x-\pi$ $\iff 0\le x$, which is true. So -1 is a lower bound of S. Also $-1=\frac{0-\pi}{0+\pi}\in S$. So every lower bound $t\le\frac{0-\pi}{0+\pi}=-1$. $\therefore\inf S=-1$.

For $x\in\mathbb{Q}\cap[0,\infty)$, $\frac{x-\pi}{x+\pi}\le 1\iff x-\pi\le x+\pi$, which is true. So 1 is an upper bound of S. Now $w_n=\frac{n-\pi}{n+\pi}\in S$ for every $n\in\mathbb{N}$. Since $\lim_{n\to\infty}w_n=1$, so by the supremum limit theorem, $\sup S=1$.

(f) For $x\in\mathbb{Q}\cap[0,1]$, $y\in(-1,0]$, $-1=0^3+(-1)^3<x^3+y^3\le 1^3+0^3=1$. So -1 is a lower bound of S and 1 is an upper bound of S. Note $1=1^3+0^3\in S$. So for every upper bound M of S, $M\ge 1$. Therefore, $\sup S=1$.

Next for every $n\in\mathbb{N}$, $w_n=0^3+(-\frac{n}{n+1})^3\in S$ and $\lim_{n\to\infty}w_n=-1$. So by the infimum limit theorem, $\inf S=-1$.

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(9)(g) Since $0 < \frac{\sqrt{2}}{m+n} + \frac{1}{k\sqrt{2}} \leq \frac{\sqrt{2}}{1+1} + \frac{1}{1\sqrt{2}} = \frac{\sqrt{2}}{2} + \frac{1}{\sqrt{2}} = \sqrt{2}$, S is bounded below by 0 and above by $\sqrt{2}$. Now every upper bound $M \geq \sqrt{2} \in S$, so $\sup S = \sqrt{2}$. Next considering $a_n = \frac{\sqrt{2}}{n+n} + \frac{1}{n\sqrt{2}} \in S$, we have $\lim_{n\to\infty} a_n = 0$, which is a lower bound. So by the infimum limit theorem, $\inf S = 0$.

(h) $S = [0, \frac{1}{2}) \cup [\frac{2}{3}, \frac{3}{4}) \cup [\frac{4}{5}, \frac{5}{6}) \cup \cdots$. Since $0 \leq 1 - \frac{1}{2k-1}$ and $1 - \frac{1}{2k} < 1$ for $k = 1, 2, 3, \cdots$, so $0 \leq x < 1$ for all $x \in S$. So S is bounded below by 0 and above by 1. Since every lower bound $m \leq 0 \in S$, so $\inf S = 0$. Next since $1 - \frac{1}{2k-1} \in S$ and $\lim_{k\to\infty}(1 - \frac{1}{2k-1}) = 1$, so by the supremum limit theorem, $\sup S = 1$.

(i) For $x, y \in (0,1] \cap \mathbb{Q}$, $0 \leq \sqrt{x} + y^2 \leq \sqrt{1} + 1^2 = 2$. So 0 is a lower bound and 2 is an upper bound. Now let $w_n = \sqrt{\frac{1}{n}} + (\frac{1}{n})^2$ for $n = 1, 2, 3, \cdots$, then $w_n \in S$ and $\lim_{n\to\infty} w_n = 0$. So by infimum limit theorem, $\inf S = 0$. Next, $2 = \sqrt{1} + 1^2 \in S$ and so every upper bound of S is greater than or equal to 2. Therefore, $\sup S = 2$.

(j) Since $0 \leq \frac{1}{n} + x \leq 2$ for $x \in [0,1] \cap \mathbb{Q}$, $n = 1, 2, 3, \cdots$, the set S is bounded below by 0 and bounded above by 2. We will show $\inf S = 0$ and $\sup S = 2$.
Since 0 is a lower bound, $0 \leq \inf S$. For $n = 1, 2, 3, \cdots$, $\frac{1}{n} = \frac{1}{n} + 0 \in S$ and so $\inf S \leq \frac{1}{n}$. Then $\inf S \leq \lim_{n\to\infty} \frac{1}{n} = 0$. So $\inf S = 0$.
Since 2 is an upper bound, $\sup S \leq 2$. If m is an upper bound of S, then $m \geq \frac{1}{1} + 1 = 2 \in S$. So $\sup S = 2$.

(k) Since $0 \leq x+y \leq 2$ for $x \in [0,1] \cap \mathbb{Q}$, $y \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$, S is bounded below by 0 and bounded above by 2. We will show $\inf S = 0$ and $\sup S = 2$.
Let $w_n = \frac{1}{n} + \frac{1}{n\sqrt{2}}$, then $w_n \in S$ and $\lim_{n\to\infty} w_n = 0$. So by infimum limit theorem, $\inf S = 0$.
Let $v_n = \frac{n}{n+1} + \frac{1}{n+\sqrt{2}}$, then $v_n \in S$ and $\lim v_n = 2$. So by supremum limit theorem, $\sup S = 2$.

(l) Note $x(x+1) \leq 0 \Leftrightarrow x \in [-1, 0]$. So $S = [-1, 0] \cap (\mathbb{R} \setminus \mathbb{Q})$. Hence S is bounded below by -1 and above by 0. We will show $\inf S = -1$ and $\sup S = 0$.
Let $w_n = -\frac{1}{n+\sqrt{2}}$, then $w_n \in S$ and $\lim_{n\to\infty} w_n = -1$. So by infimum limit theorem, $\inf S = -1$.
Let $v_n = -\frac{1}{n\sqrt{2}}$, then $v_n \in S$ and $\lim_{n\to\infty} v_n = 0$. So by supremum limit theorem, $\sup S = 0$.

(m) $\forall \frac{k}{n!} \in S$, $0 < \frac{k}{n!} < \sqrt{2}$. So S has lower bound 0 and upper bound $\sqrt{2}$. Will show $\inf S = 0$, $\sup S = \sqrt{2}$.
If $\sup S < \sqrt{2}$, then by density of rational, there is $\frac{m}{n} \in \mathbb{Q}$ such that $\sup S < \frac{m}{n} < \sqrt{2}$. However, $\frac{m}{n} = \frac{m(n-1)!}{n!} \in S$, contradicting $\sup S$ is an upper bound of S. $\therefore \sup S = \sqrt{2}$.
If $\inf S > 0$, then by density of rational, there is $\frac{p}{q} \in \mathbb{Q}$ such that $0 < \frac{p}{q} < \inf S$. However, $\frac{p}{q} = \frac{p(q-1)!}{q!} \in S$, contradicting $\inf S$ is a lower bound of S. $\therefore \inf S = 0$.

(91)(n) Note $S=\bigcup_{n=1}^{10}\left(\left[\frac{1}{n\sqrt{2}}, 2-\frac{1}{n}\right]\setminus\mathbb{Q}\right)=\left[\frac{1}{10\sqrt{2}}, 1.9\right]\setminus\mathbb{Q}$. So S is bounded below by $\frac{1}{10\sqrt{2}}$ and above by 1.9. We will show $\inf S=\frac{1}{10\sqrt{2}}$ and $\sup S=1.9$.
Since $\frac{1}{10\sqrt{2}}\in S$, every lower bound $m\le\frac{1}{10\sqrt{2}}$, so $\inf S=\frac{1}{10\sqrt{2}}$.
Next, let $w_n=1.9-\frac{1}{n\sqrt{2}}$, then $\frac{1}{10\sqrt{2}}<1<1.9-\frac{1}{\sqrt{2}}\le w_n<1.9$. So $w_n\in S$. Since $\lim_{n\to\infty} w_n=1.9$, by the supremum limit theorem, $\sup S=1.9$.

(o) $0\le x^2+y^3+z^4\le 1+1+1=3$ for $x\in(-1,0)\setminus\mathbb{Q}$, $y\in(0,1)\cap\mathbb{Q}$, $z\in(-1,1)$. So 0 is a lower bound and 3 is an upper bound of S. Since $\left(-\frac{1}{n\sqrt{2}}\right)^2+\left(\frac{1}{n+1}\right)^3+\left(\frac{1}{n+1}\right)^4$ is in S and has limit 0, so $\inf S=0$. Since $\left(-1+\frac{1}{n\sqrt{2}}\right)^2+\left(1-\frac{1}{n+1}\right)^3+\left(1-\frac{1}{n+1}\right)^4$ is in S and has limit 3, so $\sup S=3$.

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(92)(a) (Note $x_1=1<x_2=\frac{1}{2}+\sqrt{1}=\frac{3}{2}<x_3=\frac{3}{4}+\sqrt{\frac{3}{2}}=\frac{3+2\sqrt{6}}{4}$. Also $x=\frac{x}{2}+\sqrt{x}\Rightarrow x=0$ or 4.)
We will show $x_n\le x_{n+1}\le 4$ by induction. For $n=1$, $1\le\frac{3}{2}\le 4$. Next suppose $x_n\le x_{n+1}\le 4$. Then $\frac{x_n}{2}\le\frac{x_{n+1}}{2}\le 2$ and $\sqrt{x_n}\le\sqrt{x_{n+1}}\le\sqrt{4}\Rightarrow x_{n+1}=\frac{x_n}{2}+\sqrt{x_n}\le x_{n+2}=\frac{x_{n+1}}{2}+\sqrt{x_{n+1}}\le 2+\sqrt{4}=4$. Therefore, $\{x_n\}$ is increasing and bounded above. By the monotone sequence theorem, $\lim_{n\to\infty}x_n=x$ exists. Then $x=\frac{x}{2}+\sqrt{x}\Rightarrow x=0$ or 4. Since $x_1>1$, $\lim_{n\to\infty}x_n=x=4$.

(b) (Note $x_1=1<x_2=2<x_3=\sqrt{2}+\sqrt{1}=\sqrt{2}+1$, so we suspect $\{x_n\}$ is increasing.)
We will show $x_n\le x_{n+1}$ for all $n\in\mathbb{N}$ by induction. The cases $n=1, 2$ are true as shown above. Assume the cases $n<k$ are true. For the case $n=k$, we have $x_k\le x_{k+1}\Leftrightarrow\sqrt{x_{k-1}}+\sqrt{x_{k-2}}\le\sqrt{x_k}+\sqrt{x_{k-1}}\Leftrightarrow x_{k-2}\le x_k$, which is true by cases $n=k-2$ $(x_{k-2}\le x_{k-1})$ and $n=k-1$ $(x_{k-1}\le x_k)$. So $\{x_n\}$ is increasing.
Next we will show $x_n\le 4$ for all $n\in\mathbb{N}$. For $n=1,2$, this is clear. Assume the cases $n<k$ are true, then $x_k=\sqrt{x_{k-1}}+\sqrt{x_{k-2}}\le\sqrt{4}+\sqrt{4}=4$. So by induction, $x_n\le 4$ for all $n\in\mathbb{N}$. By the monotone sequence theorem, $\{x_n\}$ converges. Let $x=\lim_{n\to\infty}x_n$, then $x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}(\sqrt{x_n}+\sqrt{x_{n-1}})=2\sqrt{x}\Rightarrow x=0$ or 4. Since $1=x_1\le x$, $x=4$.

(c) $x_2=\frac{1}{4}<x_4=\frac{19}{46}<x_3=\frac{7}{13}<x_1=1$. Assume $x_{2n}<x_{2n+2}<x_{2n+1}<x_{2n-1}$. Now $x_{k+1}=\frac{2-x_k}{3+x_k}=\frac{5}{3+x_k}-1$.
So $x_{2n+1}=\frac{5}{3+x_{2n}}-1>x_{2n+3}=\frac{5}{3+x_{2n+2}}-1>x_{2n+2}=\frac{5}{3+x_{2n+1}}-1>x_{2n}=\frac{5}{3+x_{2n-1}}-1$. Repeating this once more, we get $x_{2n+2}=\frac{5}{3+x_{2n+1}}-1<x_{2n+4}=\frac{5}{3+x_{2n+3}}-1<x_{2n+3}=\frac{5}{3+x_{2n+2}}-1<x_{2n+1}=\frac{5}{3+x_{2n}}-1$. Therefore, $x_{2k}<x_{2k+2}<x_{2k+1}<x_{2k-1}$ for all k by mathematical induction.

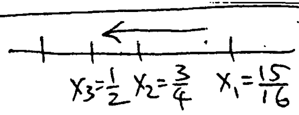
(c) (Cont.) Now $|x_m - x_{m-1}| = \left|\frac{2-x_{m-1}}{3+x_{m-1}} - \frac{2-x_{m-2}}{3+x_{m-2}}\right| = \frac{5|x_{m-1}-x_{m-2}|}{(3+x_{m-1})(3+x_{m-2})} \le \frac{5|x_{m-1}-x_{m-2}|}{(3\frac{1}{4})(3\frac{1}{4})} = \frac{80}{169}|x_{m-1}-x_{m-2}|$
$\le \cdots \le \left(\frac{80}{169}\right)^{m-2}|x_2-x_1|$. Since $\lim_{m\to\infty}\left(\frac{80}{169}\right)^{m-2}|x_2-x_1| = 0$, $\lim_{m\to\infty}|x_m - x_{m-1}| = 0$ and $\lim_{k\to\infty}|x_{2k}-x_{2k-1}| = 0$. By the nested interval theorem and intertwining sequence theorem, $\{x_n\}$ converges to some x. Now $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\frac{2-x_n}{3+x_n} = \frac{2-x}{3+x}$. Solving, we find $x = -2\pm\sqrt{6}$. Since $x_n \ge x_2 = \frac{1}{4}$, $x = -2+\sqrt{6}$.

Alternatively, after we showed $x_{2k} < x_{2k+2} < x_{2k+1} < x_{2k-1}$ for all k, we can argue as follow. Since $\{x_{2n}\}$ is increasing and bounded above by x_1, $\{x_{2n}\}$ must converge to some a by the monotone sequence theorem. Also $\{x_{2n+1}\}$ is decreasing and bounded below by x_2, so $\{x_{2n+1}\}$ must converge to some b by the monotone sequence theorem. Then

$$\left.\begin{aligned} b &= \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty}\frac{2-x_{2n}}{3+x_{2n}} = \frac{2-a}{3+a} \\ a &= \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty}\frac{2-x_{2n-1}}{3+x_{2n-1}} = \frac{2-b}{3+b} \end{aligned}\right\} \Rightarrow \begin{aligned} 3b+ab &= 2-a \\ 3a+ab &= 2-b \end{aligned}$$

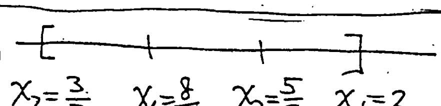
$$\xRightarrow{\text{subtracting}} 3(b-a) = b-a \Rightarrow b = a.$$

By the intertwining sequence theorem, $\{x_n\}$ converges. Then the limit of $\{x_n\}$ is found as above.

(d) 

$x = 1-\sqrt{1-x} \iff \sqrt{1-x} = 1-x \iff 1-x = (1-x)^2 \iff (1-x)x = 0 \iff x = 0$ or 1

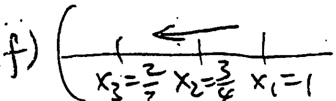
We will prove $0 < x_{n+1} < x_n$ for $n = 1, 2, \cdots$ by induction. For $n=1$, $0 < x_2 = \frac{3}{4} < x_1 = \frac{15}{16}$. Assume $0 < x_{n+1} < x_n$. Then $1 > 1-x_{n+1} > 1-x_n \Rightarrow 1 > \sqrt{1-x_{n+1}} > \sqrt{1-x_n}$
$\Rightarrow 0 < 1-\sqrt{1-x_{n+1}} = x_{n+2} < 1-\sqrt{1-x_n} = x_{n+1}$,
completing the induction. Therefore $\{x_n\}$ is decreasing and bounded below. By the monotone sequence theorem, $\{x_n\}$ converges to some limit x. Then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}(1-\sqrt{1-x_n}) = 1-\sqrt{1-x}$. So $x = 0$ or 1. Since $x_n < 1$ and $\{x_n\}$ is decreasing, $x = 0$.
Now $\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\frac{1-\sqrt{1-x_n}}{x_n}\frac{(1+\sqrt{1-x_n})}{(1+\sqrt{1-x_n})} = \lim_{n\to\infty}\frac{1-(1-x_n)}{x_n(1+\sqrt{1-x_n})} = \lim_{n\to\infty}\frac{1}{1+\sqrt{1-x_n}} = \frac{1}{2}$.

(e) 

Let $I_n = [x_{2n}, x_{2n-1}]$. We will show $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$. The case $n=1$ is shown on the left. Suppose the case $n=k$ is true, i.e. $x_{2k} \le x_{2k+2} \le x_{2k+1} \le x_{2k-1}$. Since $x_n = \frac{a_{n+1}}{a_n} = \frac{a_n+a_{n-1}}{a_n} = 1+\frac{1}{x_{n-1}}$, so

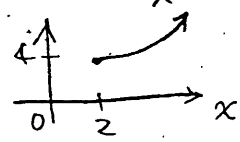
$$\underbrace{1+\frac{1}{x_{2k}}}_{x_{2k+1}} \ge \underbrace{1+\frac{1}{x_{2k+2}}}_{x_{2k+3}} \ge \underbrace{1+\frac{1}{x_{2k+1}}}_{x_{2k+2}} \ge \underbrace{1+\frac{1}{x_{2k-1}}}_{x_{2k}} \text{ and } \underbrace{1+\frac{1}{x_{2k+1}}}_{x_{2k+2}} \le \underbrace{1+\frac{1}{x_{2k+3}}}_{x_{2k+4}} \le \underbrace{1+\frac{1}{x_{2k+2}}}_{x_{2k+3}} \le \underbrace{1+\frac{1}{x_{2k}}}_{x_{2k+1}} \Rightarrow \begin{cases} n=k+1 \\ \text{is true} \end{cases}$$

This implies $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Next we will show $\lim_{n\to\infty}(x_{2n-1}-x_{2n}) = 0$. Note that $|x_m - x_{m+1}| = \left|\left(1+\frac{1}{x_{m-1}}\right)-\left(1+\frac{1}{x_m}\right)\right| = \frac{|x_{m-1}-x_m|}{x_{m-1}x_m} \le \frac{4}{9}|x_{m-1}-x_m| \Rightarrow |x_{2n-1}-x_{2n}| \le \left(\frac{4}{9}\right)^{2n-2}\left(\frac{1}{2}\right)$.
Since $\lim_{n\to\infty}\left(\frac{4}{9}\right)^{2n-2}\left(\frac{1}{2}\right) = 0$, by the squeeze limit theorem, $\lim_{n\to\infty}(x_{2n-1}-x_{2n}) = 0$. By the nested interval theorem and the intertwining sequence theorem, $\{x_n\}$ converges, say to x. Then $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(1+\frac{1}{x_{n-1}}\right) = 1+\frac{1}{x} \Rightarrow x = \frac{1\pm\sqrt{5}}{2}$. Since $x_n > 0$, so $x = \frac{1+\sqrt{5}}{2}$

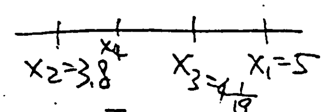
(f) ( $x=1-\frac{1}{4x} \Rightarrow 4x^2=4x-1 \Rightarrow 0=4x^2-4x+1=(2x-1)^2 \Rightarrow x=\frac{1}{2}$.)

We will show $x_n \geq x_{n+1} \geq \frac{1}{2}$ for $n=1,2,3,\cdots$ by induction. We have $x_1=1\geq x_2=\frac{3}{4}\geq\frac{1}{2}$.
Assume $x_n \geq x_{n+1} \geq \frac{1}{2}$. Then $\frac{1}{4x_n} \leq \frac{1}{4x_{n+1}} \leq \frac{1}{2}$ and $x_{n+1}=1-\frac{1}{4x_n} \geq x_{n+2}=1-\frac{1}{4x_{n+1}} \geq 1-\frac{1}{4(\frac{1}{2})}=\frac{1}{2}$,
completing the induction. So $\{x_n\}$ is decreasing and bounded below by $\frac{1}{2}$.
By monotone sequence theorem, $\{x_n\}$ converges to some x. Then $x=\lim\limits_{n\to\infty} x_{n+1}$
$=\lim\limits_{n\to\infty}(1-\frac{1}{4x_n})=1-\frac{1}{4x}$. So $x=1-\frac{1}{4x}$. As above, $x=\frac{1}{2}$.

(g) Since $f'(x)=1-\frac{4}{x^2}\geq 0$ for $x\geq 2$ and $\lim\limits_{x\to\infty}(x+\frac{4}{x})=\infty$, $f(x)$ is increasing to ∞.



$x_1=4$, $x_2=\frac{5}{2}=2.5$, $x_3=\frac{1}{2}(2.5+1.6)=2.05$. We suspect $\{x_n\}$ is decreasing.
(If $\{x_n\}$ converges to x, then $x=\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}\frac{1}{2}(x_n+\frac{4}{x_n})=\frac{1}{2}(x+\frac{4}{x})$, which implies $x=\pm 2$. Since $x_n>0$ by induction, $x=2$.)
We will show $2\leq x_{n+1}\leq x_n$ for $n=1,2,\cdots$. (This implies $\{x_n\}$ is decreasing and bounded below by 2. By the monotone sequence theorem, we get $\{x_n\}$ converges.)
For $n=1$, $2\leq x_2=2.5\leq x_1=4$. Suppose $2\leq x_{n+1}\leq x_n$. Then since $f(x)=x+\frac{4}{x}$ is increasing for $x\geq 2$, we get $2=\frac{1}{2}f(2)\leq x_{n+2}=\frac{1}{2}f(x_{n+1})\leq x_{n+1}=\frac{1}{2}f(x_n)$,
completing the induction.

(h) $x_1=5$, $x_2=3\frac{4}{5}=3.8$, $x_3=4\frac{1}{19}$, $x_4=3+\frac{4}{x_3}>3+\frac{4}{5}=x_2$. 

Define $I_n=[x_{2n},x_{2n-1}]$, we will show $x_{2n}\leq x_{2n+2}\leq x_{2n+1}\leq x_{2n-1}$, i.e. $I_{n+1}\subseteq I_n$.
The case $n=1$ is done above. Suppose the case n is true. Then $\frac{4}{x_{2n}}\geq\frac{4}{x_{2n+2}}\geq\frac{4}{x_{2n+1}}\geq\frac{4}{x_{2n-1}}$
$\Rightarrow x_{2n+1}\geq x_{2n+3}\geq x_{2n+2}\geq x_{2n} \Rightarrow \frac{4}{x_{2n+1}}\leq\frac{4}{x_{2n+3}}\leq\frac{4}{x_{2n+2}}\leq\frac{4}{x_{2n}} \Rightarrow x_{2n+2}\leq x_{2n+4}\leq x_{2n+3}\leq x_{2n+1}$,
completing the induction.
Next observe that $|x_{m+1}-x_m|=|\frac{4}{x_m}-\frac{4}{x_{m-1}}|=\frac{4|x_m-x_{m-1}|}{x_m x_{m-1}}\leq\frac{4}{(3.8)^2}|x_m-x_{m-1}|$. So
$|x_{2n-1}-x_{2n}|\leq\frac{4}{(3.8)^2}|x_{2n-2}-x_{2n-1}|\leq\cdots\leq\left(\frac{4}{(3.8)^2}\right)^{2n-3}|x_2-x_1|$. Since $\frac{4}{(3.8)^2}<1$, $\lim\limits_{n\to\infty}\left(\frac{4}{(3.8)^2}\right)^{2n-3}|x_2-x_1|=0$
and $\lim\limits_{n\to\infty}|x_{2n-1}-x_{2n}|=0$. Hence $\bigcap\limits_{n=1}^{\infty}I_n=\{x\}$ and $\lim\limits_{n\to\infty}x_{2n}=x=\lim\limits_{n\to\infty}x_{2n-1}$. So by the intertwining sequence theorem, $\{x_n\}$ converges to x.
Taking limit of $x_{n+1}=3+\frac{4}{x_n}$, we get $x=3+\frac{4}{x}\Rightarrow x^2-3x-4=0\Rightarrow x=-1$ or 4
Since $x\in I_1=[3.8,5]$, so $x=4$.

P2 (i) $x_1=2$, $x_2=\frac{3}{2}=1.5$, $x_3=\frac{4}{3}=1.33\cdots$. We suspect $\{x_n\}$ is decreasing.
(If $\{x_n\}$ converges to x, then $x=\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}\left(2-\frac{1}{x_n}\right)=2-\frac{1}{x}$, which implies $x=1$ by algebra.)
We will show $1\le x_{n+1}\le x_n$ for $n=1,2,\cdots$; (this implies $\{x_n\}$ is decreasing and bounded below by 1. By the monotone sequence theorem, we get $\{x_n\}$ converges.)
For $n=1$, we have $1\le x_2=1.5\le x_1=2$. Suppose $1\le x_{n+1}\le x_n$, then $\frac{1}{1}\ge\frac{1}{x_{n+1}}\ge\frac{1}{x_n}$ and $1=2-\frac{1}{1}\le x_{n+2}=2-\frac{1}{x_{n+1}}\le x_{n+1}=2-\frac{1}{x_n}$, completing M.I.

(j) ($x_1=0<x_2=\frac{0^2+4}{5}=\frac{4}{5}<x_3=\frac{(\frac{4}{5})^2+4}{5}$ $x=\frac{x^2+4}{5}\Leftrightarrow x^2-5x+4=(x-1)(x-4)=0$, $x=1$ $x=4$)
We will show $x_n\le x_{n+1}\le 1$ by math induction. For $n=1$, $x_1=0\le x_2=\frac{4}{5}\le 1$. Suppose $x_n\le x_{n+1}\le 1$. Then $x_n^2+4\le x_{n+1}^2+4\le 1^2+4$. Dividing by 5, we get $x_{n+1}\le x_{n+2}\le 1$ completing the induction. This shows $\{x_n\}$ is increasing and bounded above. By monotone sequence theorem, $\{x_n\}$ converges to some $x\in\mathbb{R}$. Now $x=\lim\limits_{n\to\infty}x_{n+1}=\lim\limits_{n\to\infty}\frac{x_n^2+4}{5}=\frac{x^2+4}{5}\Rightarrow x^2-5x+4=0\Rightarrow x=1$ or $\cancel{4}$ ← rejected as $x_n\le 1$

(k) (Note $x_1=1>x_2=\sqrt{1}-\frac{1}{4}=\frac{3}{4}>x_3=\sqrt{\frac{3}{4}}-\frac{1}{4}=\frac{2\sqrt{3}-1}{4}$ and $x=\sqrt{x}-\frac{1}{4}\Rightarrow x=\frac{1}{4}$.)
We will show $x_n\ge x_{n+1}\ge\frac{1}{4}$ by induction. For $n=1$, $1\ge\frac{3}{4}\ge\frac{1}{4}$. Suppose $x_n\ge x_{n+1}\ge\frac{1}{4}$. Then $x_{n+1}=\sqrt{x_n}-\frac{1}{4}\ge x_{n+2}=\sqrt{x_{n+1}}-\frac{1}{4}\ge\frac{1}{4}=\sqrt{\frac{1}{4}}-\frac{1}{4}$. Therefore, $\{x_n\}$ is decreasing and bounded below. By the monotone sequence theorem, $\lim\limits_{n\to\infty}x_n=x$ exists. Then $x=\sqrt{x}-\frac{1}{4}\Rightarrow x=\frac{1}{4}$. So $\lim\limits_{n\to\infty}x_n=\frac{1}{4}$.

(l) (Note $x_1=3>x_2=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}>x_3=\sqrt{1-\frac{2}{2+\sqrt{3}}}=\sqrt{\frac{3}{2+\sqrt{3}}}$ and $x=\sqrt{1-\frac{1}{x+1}}=\sqrt{\frac{x}{x+1}}\Rightarrow \overbrace{x(x+1)-1=0}^{x=0\text{ or}}$
$\Rightarrow x=0$ or $\frac{-1\pm\sqrt{5}}{2}$.) We will show $x_n\ge x_{n+1}\ge\frac{-1+\sqrt{5}}{2}$. For $n=1$, $3>\frac{\sqrt{3}}{2}\approx\frac{1.73}{2}>\frac{-1+\sqrt{5}}{2}\approx\frac{1.2}{2}$.
Suppose $x_n\ge x_{n+1}\ge\frac{-1+\sqrt{5}}{2}$. Then $x_n+1\ge x_{n+1}+1\ge\frac{1+\sqrt{5}}{2}\Rightarrow\frac{1}{x_n+1}\le\frac{1}{x_{n+1}+1}\le\frac{2}{1+\sqrt{5}}\frac{(1-\sqrt{5})}{(1-\sqrt{5})}=\frac{\sqrt{5}-1}{2}$
and so $\sqrt{1-\frac{1}{x_n+1}}\ge\sqrt{1-\frac{1}{x_{n+1}+1}}\ge\sqrt{1-\left(\frac{\sqrt{5}-1}{2}\right)}=\sqrt{\frac{3-\sqrt{5}}{2}}=\frac{-1+\sqrt{5}}{2}$ (as $\left(\frac{-1+\sqrt{5}}{2}\right)^2=\frac{6-2\sqrt{5}}{4}=\frac{3-\sqrt{5}}{2}$.)
So $x_{n+1}\ge x_{n+2}\ge\frac{-1+\sqrt{5}}{2}$. Therefore, $\{x_n\}$ is decreasing and bounded below. By the monotone sequence theorem, $\lim\limits_{n\to\infty}x_n=x$ exists. Then $x=\sqrt{1-\frac{1}{x+1}}=\sqrt{\frac{x}{x+1}}\Rightarrow$ $x=0$ or $\frac{-1+\sqrt{5}}{2}$. Since $x_n\ge\frac{-1+\sqrt{5}}{2}>0>\frac{-1-\sqrt{5}}{2}$, $\lim\limits_{n\to\infty}x_n=x=\frac{-1+\sqrt{5}}{2}$.

(m) We claim that $0<x_n<1$ for $n=1,2,3,\cdots$. The case $n=1$ is given. Suppose $0<x_n<1$, then $0<x_{n+1}=\frac{x_n^3+6}{7}<\frac{1+6}{7}=1$, completing the induction. Next, $x_{n+1}-x_n=\frac{x_n^3+6}{7}-x_n=\frac{x_n^3-7x_n+6}{7}=\frac{(x_n-1)(x_n-2)(x_n+3)}{7}>0$ implies $\{x_n\}$ is increasing. Since it is bounded above by 1, $\{x_n\}$ converges to some x by monotone sequence theorem. We have $7x=\lim\limits_{n\to\infty}7x_{n+1}=\lim\limits_{n\to\infty}x_n^3+6=x^3+6\Rightarrow x^3-7x+6=0\Rightarrow x=1,2$ or -3. Since $0<x_n<1$, $x=1$.

(92)(n) ($x=\sqrt{3x-2} \Rightarrow x^2-3x+2=0 \Rightarrow x=1$ or 2.) If $x_1=1$ and $x_n=1$, then $x_{n+1}=\sqrt{3\cdot 1-2}=1$ and so $\lim_{n\to\infty} x_n = 1$ in that case. If $x_1\in(1,2]$, then we claim $1<x_n\le x_{n+1}\le 2$. For $x_n>1$ $x_n\le x_{n+1} \Leftrightarrow x_n^2\le 3x_n-2 \Leftrightarrow x_n^2-3x_n+2=(x_n-1)(x_n-2)\le 0 \Leftrightarrow 1\le x_n\le 2$. Since $1<x_1\le 2$, so if $1<x_n\le 2$, then $1<x_n\le x_{n+1}=\sqrt{3x_n-2}\le\sqrt{3\cdot 2-2}=2$, completing induction. So $\lim_{n\to\infty}x_n$ exists in this case. It is a root of $x=\sqrt{3x-2}$ in $(1,2]$. So $\lim_{n\to\infty}x_n=2$ in this case. If $x_1\in(2,\infty)$, then we claim $x_n\ge x_{n+1}\ge 2$. For $x_n>2$, $x_{n+1}\le x_n \Leftrightarrow 3x_n-2\le x_n^2 \Leftrightarrow x_n^2-3x_n+2=(x_n-1)(x_n-2)\ge 0 \Leftrightarrow x_n\ge 2$. Since $x_1>2$, so if $x_n>2$, then $x_n\ge x_{n+1}=\sqrt{3x_n-2}\ge\sqrt{3\cdot 2-2}=2$, completing induction. So $\lim_{n\to\infty}x_n$ exists in this case. It is a root of $x=\sqrt{3x-2}$ in $[2,\infty)$. $\therefore \lim_{n\to\infty}x_n=2$ in this case.

(o) ($x_1\le x_2$. If $x_2=0$, then $x_3=\frac{1}{3}$, so suspect $\{x_n\}$ is increasing. The equation $x=\frac{1}{3}(1+x+x^3)$ has $x=1$ as a root. So $x=\frac{1}{3}(1+x+x^3)\Leftrightarrow x^3-2x+1=0$ can be solved by factoring $x-1$. The roots are $1, \frac{-1+\sqrt5}{2}, \frac{-1-\sqrt5}{2}$. Note $a_1\le a_2<\frac{-1+\sqrt5}{2}$.)

Claim: $x_n\le x_{n+1}\le\frac{-1+\sqrt5}{2}$. (Note $\frac{-1-\sqrt5}{2}<\frac{-1+\sqrt5}{2}<1$.)

Case $n=1$ is true as $x_1\le x_2\le\frac{-1+\sqrt5}{2}$. Case $n=2$ is true because $x_2\le\frac{1}{2}\Leftrightarrow$ $x_2\le\frac{1}{3}(1+x_2)=x_3\le\frac{1}{3}(1+\frac{-1+\sqrt5}{2})=\frac{1}{3}(\frac{1+\sqrt5}{2})<\frac{-1+\sqrt5}{2}$. Assume cases $n-1$ and n, we have $x_{n-1}\le x_n\le\frac{-1+\sqrt5}{2}$ and $x_n\le x_{n+1}\le\frac{-1+\sqrt5}{2}$. So

$$x_{n+1}=\frac{1}{3}(1+x_n+x_{n-1}^3)\le\frac{1}{3}(1+x_{n+1}+x_n^3)=x_{n+2}\le\frac{1}{3}\left(1+\frac{-1+\sqrt5}{2}+\left(\frac{-1+\sqrt5}{2}\right)^3\right)=\frac{-1+\sqrt5}{2}.$$

Completing induction.

By monotone sequence theorem, $\lim_{n\to\infty}x_n$ exists. It is a root of $x=\frac{1}{3}(1+x+x^3)$ in $[0,\frac{-1+\sqrt5}{2}]$, so $\lim_{n\to\infty}x_n=\frac{-1+\sqrt5}{2}$.

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(93) From $x_2=a_1-a_2\le x_4=a_1-a_2+a_3-a_4\le x_3=a_1-a_2+a_3\le x_1=a_1$, we define $I_n=[x_{2n},x_{2n-1}]$. We claim $I_1\supseteq I_2\supseteq I_3\supseteq I_4\supseteq\cdots$. For this, we have to check $I_n=[x_{2n},x_{2n-1}]\supseteq I_{n+1}=[x_{2n+2},x_{2n+1}]$. (Since $\{a_n\}$ is decreasing, $x_{2n}\le x_{2n+2}=x_{2n}+a_{2n+1}-a_{2n+2}\le x_{2n+1}=x_{2n}+a_{2n+1}=x_{2n-1}-a_{2n}+a_{2n+1}\le x_{2n-1}$.) Finally since $\lim_{n\to\infty}|x_{2n}-x_{2n-1}|=\lim_{n\to\infty}a_{2n}=0$, we have $\bigcap_{n=1}^{\infty}I_n=\{x\}$, $\lim_{n\to\infty}x_{2n}=x=\lim_{n\to\infty}x_{2n-1}$. So $\lim_{n\to\infty}x_n=x$.

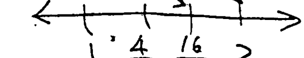
Alternative Solution Applying summation by parts, we get $x_n=S_na_n-\sum_{k=1}^{n-1}S_k(a_{k+1}-a_k)$, where $S_j=\sum_{k=1}^{j}(-1)^{k+1}=\begin{cases}0 & \text{if } j \text{ is even}\\ 1 & \text{if } j \text{ is odd}\end{cases}$. Since $\{a_n\}$ is a decreasing sequence with limit 0 and $0\le S_n\le 1$, we have $\lim_{n\to\infty}S_na_n=0$. Also, $-S_k(a_{k+1}-a_k)\ge 0$ so that $y_n=-\sum_{k=1}^{n-1}S_k(a_{k+1}-a_k)$ is increasing. Since $y_n\le\sum_{k=1}^{n-1}1(a_k-a_{k+1})=a_1-a_n\le a_1$, by monotone sequence theorem, $\lim_{n\to\infty}y_n$ exists. Then $\lim_{n\to\infty}x_n=\lim_{n\to\infty}S_na_n+\lim_{n\to\infty}y_n$ exists.

(94) Observe $a_1=a\le a_2=\frac{a+b}{2}=\sqrt{\frac{a^2+2ab+b^2}{4}}\le b_2=\sqrt{\frac{2a^2+2b^2}{4}}\le b_1=b$. We will try to show $a_n\le a_{n+1}\le b_{n+1}\le b_n$ by mathematical induction. Case $n=1$ is done above. Suppose case $n=k$ is true. For case $n=k+1$, since $a_{k+1}\le b_{k+1}$,

$$a_{k+1}\le a_{k+2}=\frac{a_{k+1}+b_{k+1}}{2}=\sqrt{\frac{a_{k+1}^2+2a_{k+1}b_{k+1}+b_{k+1}^2}{4}}\le b_{k+2}=\sqrt{\frac{2a_{k+1}^2+2b_{k+1}^2}{4}}\le b_{k+1}.$$

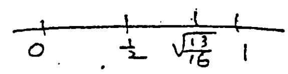
So $\{a_n\}$ is increasing and bounded above by $b_1=b$, hence converges to some A. Also $\{b_n\}$ is decreasing and bounded below by $a_1=a$, hence converges to some B. Since $a_{n+1}=\frac{a_n+b_n}{2}$, so $A=\lim_{n\to\infty}a_{n+1}=\lim_{n\to\infty}\frac{a_n+b_n}{2}=\frac{A+B}{2}\Rightarrow A=B$.

(95) (i) If $a\le b$ and $0<t<1$, then $a=ta+(1-t)a\le ta+(1-t)b\le tb+(1-t)b=b$.

(ii) (Note $x_1=1$, $x_2=2$, $x_3=\frac13 2+\frac23 1=\frac43$, $x_4=\frac13\frac43+\frac23 2=\frac{16}{9}$ 

)

Let $I_n=[x_{2n-1},x_{2n}]$, then we will show $I_n\supseteq I_{n+1}$, i.e. $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$ for all $n\in\mathbb{N}$. Now $x_{2n-1}\le x_{2n+1}=\frac13x_{2n}+\frac23x_{2n-1}=\frac23x_{2n-1}+\frac13x_{2n}\le x_{2n}$ by part (i). Also $x_{2n+1}\le x_{2n+2}=\frac13x_{2n+1}+\frac23x_{2n}\le x_{2n}$ by part (i) again. So we get $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$ for every $n\in\mathbb{N}$. Note $1\le x_{2n-1}\le x_{2n}\le 2$. By the monotone sequence theorem, $\{x_{2n-1}\}$ converges to a and $\{x_{2n}\}$ converges to b for some $a,b\in\mathbb{R}$. Since $x_{2n+1}=\frac13x_{2n}+\frac23x_{2n-1}$, let $n\to\infty$, we get $a=\frac13b+\frac23a$ $\Rightarrow a=b$. By the intertwining sequence theorem, $\{x_n\}$ converges.

(96) ($x_0=0$, $x_1=1$, $x_2=\sqrt{\frac14 1+\frac34 0}=\frac12$, $x_3=\sqrt{\frac14\frac14+\frac34 1}=\sqrt{\frac{13}{16}}$ 

)

If $x_n\le x_{n-1}$, then $x_n=\sqrt{\frac14x_n^2+\frac34x_n^2}\le x_{n+1}=\sqrt{\frac14x_n^2+\frac34x_{n-1}^2}\le x_{n-1}=\sqrt{\frac14x_{n-1}^2+\frac34x_{n-1}^2}$.

If $x_{n-1}<x_n$, then $x_{n-1}=\sqrt{\frac14x_{n-1}^2+\frac34x_{n-1}^2}\le x_{n+1}=\sqrt{\frac14x_n^2+\frac34x_{n-1}^2}\le x_n=\sqrt{\frac14x_n^2+\frac34x_n^2}$.

So x_{n+1} is always between x_{n-1} and x_n. Define $I_n=[x_{2n},x_{2n+1}]$ for $n=0,1,2,\cdots$ Then $x_{2n}\le x_{2n+2}\le x_{2n+3}\le x_{2n+1}$ for $n=0,1,2,\cdots$. So $[0,1]=I_0\supseteq I_1\supseteq I_2\supseteq\cdots$. By nested interval theorem, $\lim_{n\to\infty}x_{2n}=a$ and $\lim_{n\to\infty}x_{2n+1}=b$ exist. Taking limit of $x_{2n+1}=\sqrt{\frac14x_{2n}^2+\frac34x_{2n-1}^2}$, we get $b=\sqrt{\frac14a^2+\frac34b^2}\Rightarrow a=b$. By intertwining sequence theorem, x_n converges to some limit x.

To find x, write

$$\begin{aligned}
\cancel{x_2^2}&=\tfrac14x_1^2+\tfrac34x_0^2\\
\cancel{x_3^2}&=\tfrac14\cancel{x_2^2}+\tfrac34x_1^2\\
&\ \ \vdots\\
\tfrac34x_k^2\quad \cancel{x_k^2}&=\tfrac14\cancel{x_{k-1}^2}+\tfrac34\cancel{x_{k-2}^2}\\
x_{k+1}^2&=\tfrac14\cancel{x_k^2}+\tfrac34\cancel{x_{k-1}^2}
\end{aligned}$$

Adding these equations and cancelling common terms on both sides, we get $x_{k+1}^2+\frac34x_k^2=x_1^2+\frac34x_0^2=1$. Taking limit, we get $\frac74x^2=1$. So $x=\sqrt{\frac47}$.

(97) Assume $\overline{S}$ is unbounded. Then for every $n \in \mathbb{N}$, there is $x_n \in S$ outside $[-n, n]$, i.e. $|x_n| > n$. We are given that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Then $\{x_{n_k}\}$ is bounded. Since $|x_{n_k}| > n_k \geq k$ can be arbitrarily large, $\{x_{n_k}\}$ cannot be bounded, a contradiction. Therefore S is bounded.

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(98) We have $x \in A, y \in A \Rightarrow x^2 + y^2 \leq (\sup A)^2 + (\sup A)^2 = 2(\sup A)^2$. So $2(\sup A)^2$ is an upper bound for B.
By supremum limit theorem, there is a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = \sup A$. Then $\{x_n^2 + x_n^2\}$ is a sequence in B and $\lim_{n\to\infty}(x_n^2 + x_n^2) = 2(\sup A)^2$. So by the Supremum limit theorem, $\sup B = 2(\sup A)^2$.

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(99) For $x \in \bigcup_{n=1}^{10} A_n$, $x \in A_n$ for some $n \Rightarrow x \leq x_n = \sup A_n \leq \max(x_1, \cdots, x_{10})$.
So $\max(x_1, \cdots, x_{10})$ is an upper bound of $\bigcup_{n=1}^{10} A_n$. Let $x_i = \max(x_1, \cdots, x_{10})$, then since $x_i = \sup A_i$, there is $\{a_n\}$ in A_i such that $\lim_{n\to\infty} a_n = x_i$. Since $\{a_n\} \in \bigcup_{n=1}^{10} A_n$, so $x_i = \sup(\bigcup_{i=1}^{10} A_i)$. $\therefore \sup(\bigcup_{i=1}^{10} A_i) = \max(x_1, \cdots, x_{10})$.

Alternative Solution

As in first solution, $x_i = \max(x_1, \cdots, x_{10})$ is an upper bound of $\bigcup_{n=1}^{10} A_n$.
For any upper bound M of $\bigcup_{n=1}^{10} A_n$, $M \geq x$ for every $x \in \bigcup_{n=1}^{10} A_n$. Since $A_i \subseteq \bigcup_{n=1}^{10} A_n$, $M \geq x$ for every $x \in A_i$. So M is an upper bound of A_i, too. Then $M \geq x_i$.
So $x_i = \max(x_1, \cdots, x_{10})$ is the least upper bound of $\bigcup_{n=1}^{10} A_n$.

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(100) Since $f(x,y) \in [0,1]$, all inf and sup expressions exist by completeness axiom.
For every $x_0 \in \mathbb{R}$, $h(y) = \inf\{f(x,y) : x \in \mathbb{R}\} \leq f(x_0, y) \leq g(x_0) = \sup\{f(x_0,y) : y \in \mathbb{R}\}$.
So $g(x_0)$ is an upper bound of $\{h(y) : y \in \mathbb{R}\}$. Then $\sup\{h(y) : y \in \mathbb{R}\} \leq g(x_0)$.
So $\sup\{h(y) : y \in \mathbb{R}\}$ is a lower bound of $\{g(x_0) : x_0 \in \mathbb{R}\}$. Therefore, $\sup\{h(y) : y \in \mathbb{R}\} \leq \inf\{g(x_0) : x_0 \in \mathbb{R}\}$.

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(101) Let $x \in \mathbb{R}$. By the density of irrational numbers, there is $x_1 \in \mathbb{R} \setminus \mathbb{Q}$ such that $x - 1 < x_1 < x$. Suppose $x_n < x$ has been chosen, then we use density of irrational numbers to choose $x_{n+1} \in \mathbb{R} \setminus \mathbb{Q}$ such that $\max(x_n, x - \frac{1}{n+1}) < x_{n+1} < x$. Then $x_n < x_{n+1}$ and $x - \frac{1}{k} < x_k < x$ implies $\lim_{k\to\infty} x_k = x$ by the squeeze limit theorem.

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(102) (Note $\frac{1}{n^2} < \frac{\varepsilon}{2} \Leftrightarrow \sqrt{\frac{2}{\varepsilon}} < n$ and $\frac{\sqrt{2}}{n^3} < \frac{\varepsilon}{2} \Leftrightarrow \sqrt[3]{\frac{2\sqrt{2}}{\varepsilon}} < n$.) For every $\varepsilon > 0$, by the Archimedean principle, there exists $K \in \mathbb{N}$ such that $K > \max(\sqrt{\frac{2}{\varepsilon}}, \sqrt[3]{\frac{2\sqrt{2}}{\varepsilon}})$.
Then $n \geq K \Rightarrow |(\frac{1}{n^2} - \frac{\sqrt{2}}{n^3}) - 0| \leq \frac{1}{n^2} + \frac{\sqrt{2}}{n^3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So $\lim_{n\to\infty}(\frac{1}{n^2} - \frac{\sqrt{2}}{n^3}) = 0$ by definition.

(103) (Note $\frac{2}{n+1} < \frac{\varepsilon}{2} \Leftrightarrow \frac{4}{\varepsilon} - 1 < n$ and $\frac{1}{n^2} < \frac{\varepsilon}{2} \Leftrightarrow \sqrt{\frac{2}{\varepsilon}} < n$.) For every $\varepsilon > 0$, by the Archimedean principle, there is $K \in \mathbb{N}$ such that $K > \max(\frac{4}{\varepsilon} - 1, \sqrt{\frac{2}{\varepsilon}})$. Then $n \geq K \Rightarrow |(\frac{2}{n+1} - \frac{1}{n^2}) - 0| \leq \frac{2}{n+1} + \frac{1}{n^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So $\lim_{n\to\infty}(\frac{2}{n+1} - \frac{1}{n^2}) = 0$ by definition.

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(104) For every $\varepsilon > 0$, since $\lim_{n\to\infty} x_n = 0$, there is $K_1 \in \mathbb{N}$ such that $n \geq K_1 \Rightarrow |x_n - 0| < \frac{\varepsilon}{2}$. By the Archimedean principle, there is $K_2 \in \mathbb{N}$ such that $K_2 > \frac{2}{\varepsilon}$. Let $K = \max(K_1, K_2)$ then $n \geq K \Rightarrow |(x_n + \frac{1}{n}) - 0| \leq |x_n - 0| + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{1}{K_2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, Therefore, $\lim_{n\to\infty}(x_n + \frac{1}{n}) = 0$ by definition.

$n \geq K_1$ $\quad n \geq K_2 \Rightarrow \frac{1}{n} \leq \frac{1}{K_2} < \frac{\varepsilon}{2}$

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(105) Since $\lim_{n\to\infty} x_n = \frac{1}{2}$, so for $\varepsilon_0 = \frac{1}{3}$, there is $K_1 \in \mathbb{N}$ such that $n \geq K_1 \Rightarrow |x_n - \frac{1}{2}| < \varepsilon_0 = \frac{1}{3}$ $\Rightarrow -\frac{1}{3} < x_n - \frac{1}{2} < \frac{1}{3} \Rightarrow \frac{1}{6} < x_n < \frac{5}{6} \Rightarrow |x_n^n - 0| < (\frac{5}{6})^n$. So for every $\varepsilon > 0$, let $K = \max(K_1, \lceil \frac{\ln 1/\varepsilon}{\ln 6/5} \rceil)$, then $n \geq K \Rightarrow |x_n^n - 0| < (\frac{5}{6})^n \leq \varepsilon$.

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(106) Since $\lim_{n\to\infty} x_n = 8$, so for $\varepsilon_0 = 8$, there is $K_1 \in \mathbb{N}$ such that $n \geq K_1 \Rightarrow |x_n - 8| < \varepsilon_0 = 8$ $\Rightarrow -8 < x_n - 8 \Rightarrow x_n > 0$. For $\varepsilon > 0$, there is $K_2 \in \mathbb{N}$ such that $n \geq K_2 \Rightarrow |x_n - 8| < 4\varepsilon$. Let $K = \max(K_1, K_2)$, then $n \geq K \Rightarrow n \geq K_1$ and $n \geq K_2$. Since $x_n > 0$ for $n \geq K$ and $|x_n - 8| = |\sqrt[3]{x_n} - 2||(\sqrt[3]{x_n})^2 + 2\sqrt[3]{x_n} + 4| > |\sqrt[3]{x_n} - 2| 4$, so $|\sqrt[3]{x_n} - 2| < \frac{1}{4}|x_n - 8| < \frac{1}{4} 4\varepsilon = \varepsilon$.

<u>Alternative Solution</u>: Claim: $|\sqrt[3]{x} - \sqrt[3]{y}| \leq \sqrt[3]{|x-y|}$ for $x, y \geq 0$. Let $u = \max(x,y)$ and $v = \min(x,y)$, then we have to show $\sqrt[3]{u} - \sqrt[3]{v} \leq \sqrt[3]{u-v}$ ($\Leftrightarrow \sqrt[3]{u} \leq \sqrt[3]{v} + \sqrt[3]{u-v} \Leftrightarrow$ $u \leq v + 3v^{2/3}(u-v)^{1/3} + 3v^{1/3}(u-v)^{2/3} + (u-v) = u + 3v^{2/3}(u-v)^{1/3} + 3v^{1/3}(u-v)^{2/3}$,) which is true. For the problem, since $\lim_{n\to\infty} x_n = 8$, so for $\varepsilon_0 = 8$, there is $K_1 \in \mathbb{N}$ such that $n \geq K_1 \Rightarrow |x_n - 8| < \varepsilon_0 = 8 \Rightarrow -8 < x_n - 8 \Rightarrow x_n > 0$. For $\varepsilon > 0$, there is $K_2 \in \mathbb{N}$ such that $|x_n - 8| < \varepsilon^3$. Then for $n \geq K = \max(K_1, K_2)$, $|\sqrt[3]{x_n} - 2| \leq \sqrt[3]{|x_n - 8|} < \sqrt[3]{\varepsilon^3} = \varepsilon$.

---

(107) Let $\varepsilon > 0$. Since $\{x_n\}$ and $\{y_n\}$ converge to A, so by definition, there are $K_1, K_2 \in \mathbb{N}$ such that $n \geq K_1$ implies $|x_n - A| < \varepsilon$, and $n \geq K_2$ implies $|y_n - A| < \varepsilon$. Let $K = \max(K_1, K_2)$, then $n \geq K \Rightarrow n \geq K_1$ and $n \geq K_2 \Rightarrow |x_n - A| < \varepsilon$ and $|y_n - A| < \varepsilon$ $\Rightarrow |z_n - A| < \varepsilon$ (because $z_n = x_n$ or y_n.)

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(108) For every $\varepsilon > 0$, by Archimedean principle, there is integer $K > \frac{1}{\varepsilon}$.

Then $m, n \geq K$, $m > n$ $\Rightarrow$

$$|x_m - x_n| = |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)|$$
$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$$
$$< \frac{1}{2^{m-1}} + \frac{1}{2^{m-2}} + \cdots + \frac{1}{2^n} < \sum_{j=n}^{\infty} \frac{1}{2^j} = \frac{1}{2^{n-1}} \leq \frac{1}{2^{K-1}} \leq \frac{1}{K} < \varepsilon$$

The case $m < n$ is similar. The case $m = n$ leads to $|x_m - x_n| = 0 < \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence.

$K \leq 2^{K-1}$ can be proved by mathematical induction.

(109) (a) $f(x)$ converges to L (or has limit L) as x tends to x_0 in S iff for every $\varepsilon>0$, there exists a $\delta>0$ such that $x\in S$ and $0<|x-x_0|<\delta$ imply $|f(x)-L|<\varepsilon$.

(b) For every $\varepsilon>0$, take $\delta=\frac{2}{11}\varepsilon>0$. If $0<|x-2|<\delta$ and $x\in(1,3)$, then

$$|f(x)-\tfrac{9}{2}|=|(x^2+\tfrac{1}{x})-\tfrac{9}{2}|=|(x^2-4)+(\tfrac{1}{x}-\tfrac{1}{2})|\le|x^2-4|+|\tfrac{1}{x}-\tfrac{1}{2}|=|x+2||x-2|+\frac{|x-2|}{2|x|}$$
$$\le 5|x-2|+\tfrac{1}{2}|x-2|=\tfrac{11}{2}|x-2|<\tfrac{11}{2}\delta=\varepsilon.$$

(c) Solution 1 For every $\varepsilon>0$, take $\delta=\frac{\varepsilon}{6}>0$. If $0<|x-2|<\delta$ and $x\in(1,4)$, then

$$|f(x)-5|=\big||x^2-9|-|-5|\big|\le|x^2-4|=|x-2||x+2|\le 6|x-2|<6\delta=\varepsilon.$$

by exercise 40, $||a|-|b||\le|a-b|$ $a=x^2-9$, $b=-5$

Solution 2 (Note that for $x\in[1,3]$, $x^2-9\le 0\Rightarrow f(x)=9-x^2$.)
For every $\varepsilon>0$, take $\delta=\min(1,\frac{\varepsilon}{5})>0$. If $0<|x-2|<\delta$, then $|x-2|<1\Rightarrow$ $x\in(1,3)\Rightarrow|f(x)-5|=|(9-x^2)-5|=|4-x^2|=|2-x||2+x|\le 5|x-2|<5\delta\le\varepsilon$.

(110) Since $\max(a,b)+\min(a,b)=a+b$ and $\max(a,b)-\min(a,b)=|a-b|$, so adding the two equation and dividing by 2, we get $\max(a,b)=\frac{a+b+|a-b|}{2}$.
Let S_f, S_g, S_h be the set of jumps of f,g,h, respectively. If f,g are continuous at x, then $h=\frac{f+g+|f-g|}{2}$ will also be continuous at x. Taking contrapositive, if $x\in S_h$, then $x\in S_f\cup S_g$. So $S_h\subseteq S_f\cup S_g$. By the monotone function theorem, S_f, S_g are countable. By the countable union theorem, $S_f\cup S_g$ is countable. By the countable subset theorem, S_h is countable.

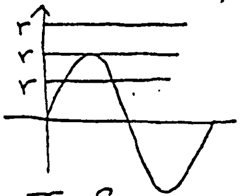
(111) Define $f(x)=\begin{cases}\sin\pi x & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$. For every $m\in\mathbb{Z}$, $|f(x)|\le|\sin\pi x|\to 0$ as $x\to m$. So $\lim_{x\to m}f(x)=0=f(m)$. So f is continuous at every $m\in\mathbb{Z}$. For $x_0\notin\mathbb{Z}$, let $r_n\in\mathbb{Q}$ and $s_n\notin\mathbb{Q}$ such that $\lim_{n\to\infty}r_n=x_0=\lim_{n\to\infty}s_n$. Then $\lim_{n\to\infty}f(r_n)=\sin\pi x_0\ne 0=\lim_{n\to\infty}f(s_n)$. So f is not continuous at x_0 by the sequential continuity theorem.

(112) (a) If $f:[a,b]\to\mathbb{R}$ is continuous and y_0 is between $f(a)$ and $f(b)$, then there is (at least one) $x_0\in[a,b]$ such that $f(x_0)=y_0$.

(b) Define $g:[0,1]\to\mathbb{R}$ by $g(x)=f(x)-f(x+1)$. Note $g(0)=f(0)-f(1)$ and $g(1)=f(1)-f(2)=f(1)-f(0)=-g(0)$. So $g(1)$ and $g(0)$ are of opposite sign. Since g is continuous on $[0,1]$, by the intermediate value theorem, $\exists c\in[0,1]$ such that $0=g(c)=f(c)-f(c+1)$. Then $f(c)=f(c+1)$.

(c) Observe that $|t|^r+|2t|^r+|3t|^r=|4t|^r+|5t|^r$ for every $t\in\mathbb{R}$ is equivalent to $1+2^r+3^r=4^r+5^r$. We will show this equation has a solution. Let $f(r)=1+2^r+3^r-4^r-5^r$, which is continuous. Since $f(0)=1$, $f(1)=-3$, by the intermediate value theorem, there is $r\in(0,1)$ such that $f(r)=0$. For this r, let $g(t)=|t|^r$, then $g(t)+g(2t)+g(3t)=g(4t)+g(5t)$, $\forall t\in\mathbb{R}$,

(113) (a) For a fixed rational r, $\{x : \sin x = r\} = \bigcup_{k \text{ even integer}} \{x : \sin x = r, x \in [k\pi, k\pi+2\pi)\}$

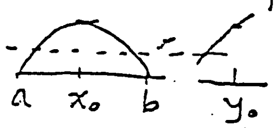


Since $\sin x = r$ on $[k\pi, k\pi+2\pi)$ has at most 2 solutions,
$\{x : \sin x = r\} = \underbrace{\bigcup_{k \text{ even integer}}}_{\text{countable}} \underbrace{\{x : \sin x = r, x \in [k\pi, k\pi+2\pi)\}}_{\text{countable}}$ is countable.

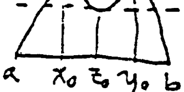
So $T = \{x : \sin x \in \mathbb{Q}\} = \underbrace{\bigcup_{r \in \mathbb{Q}}}_{\text{countable}} \underbrace{\{x : \sin x = r\}}_{\text{countable}}$ is countable.

(b) For every $x \in [0,1]$, $\sin f(x) \in \mathbb{Q}$ implies $f(x) \in T$. So $f([0,1]) = \{f(x) : x \in [0,1]\} \subseteq T$. By (a), T is countable, so $f([0,1])$ is countable.
Assume f is not a constant function, then $f([0,1])$ contains an interval (of positive length) by the intermediate value theorem. Then $f([0,1])$ is uncountable, a contradiction. Therefore, f is a constant function.

(114) Suppose such a function exists. Let a, b be the solutions of $f(x) = 0$ with $a < b$.



Case 1 $\max_{x \in [a,b]} f(x) = f(x_0) \neq 0$. Let y_0 be the other solution of $f(x) = f(x_0)$. If $y_0 \notin [a,b]$, then by the intermediate value theorem, there will be 3 solutions of $f(x) = \frac{1}{2} f(x_0)$, one on (a, x_0), one on (x_0, b) and one between y_0 and the closer endpoint of $[a,b]$ to y_0.



If $y_0 \in [a,b]$, then let $f(z_0) = \min_{x \in [x_0, y_0]} f(x)$ with $z_0 \in [x_0, y_0]$. Let $w = \max\{f(z_0), 0\}$, then by the intermediate value theorem, there are at least 3 solutions of $f(z) = w$, one on (a, x_0), one on (x_0, y_0), one on (y_0, b).
Thus, whether $y_0 \notin [a,b]$ or $y_0 \in [a,b]$ will lead to a contradiction.

Case 2 $\min_{x \in [a,b]} f(x) \neq 0$. This case is similar to case 1. (Turn figures upside down.)

Case 3 $\max_{x \in [a,b]} f(x) = 0 = \min_{x \in [a,b]} f(x)$. Then $f(x) \equiv 0$ on $[a,b]$, a contradiction.

(115) (a) $f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$. $-\frac{x^4}{|x|} \le f(x) \le \frac{x^4}{|x|} \Rightarrow \lim_{x \to 0} f(x) \overset{\text{(Sandwich Theorem)}}{=} 0 = f(0)$
So f is continuous at 0.

(b) $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} [f(x - x_0) + f(x_0)] = \lim_{x \to x_0} f(x - x_0) + f(x_0) = \lim_{y \to 0} f(y) + f(x_0) = 0 + f(x_0) = f(x_0)$
So f is continuous at every $x_0 \in \mathbb{R}$.

(c) $f(x) \equiv 0$ satisfies $f(x+y) = f(x) + f(y)$ and $|f(x)| \le x^4/|x|$ for $x \neq 0$.

(116) Since $\{g(x) : x \in [1,2]\} = [3,4]$, so there are $x_0, x_1 \in [1,2]$ such that $g(x_0) = 3$ and $g(x_1) = 4$. On the closed interval I with endpoints x_0 and x_1, since $f : I \to [3,4]$, $(f-g)(x_0) = f(x_0) - 3 \ge 0$ and $(f-g)(x_1) = f(x_1) - 4 \le 0$, $f-g$ is continuous on I, so by intermediate value theorem, there is $c \in I \subseteq [1,2]$ such that $(f-g)(c) = 0$. So $f(c) = g(c)$.

(117) (a) Observe that $|x_{k+1}-x_k| = |f(x_k)-f(x_{k-1})| \le \frac{1}{2}|x_k - x_{k-1}|$. Repeating this, we get $|x_{k+1}-x_k| \le \frac{1}{2}|x_k-x_{k-1}| \le (\frac{1}{2})^2|x_{k-1}-x_{k-2}| \le \cdots \le (\frac{1}{2})^{k-1}|x_2-x_1|$. So for $m>n$, we have

$$\begin{aligned}|x_m-x_n| &= |(x_m-x_{m-1})+(x_{m-1}-x_{m-2})+\cdots+(x_{n+1}-x_n)| \\ &\le |x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n| \\ &\le \left((\tfrac{1}{2})^{m-2}+(\tfrac{1}{2})^{m-3}+\cdots+(\tfrac{1}{2})^{n-1}\right)|x_2-x_1| \le (\tfrac{1}{2})^{n-2}|x_2-x_1|.\end{aligned}$$

$= \sum_{k=n-1}^{\infty} (\frac{1}{2})^k$

If $x_1 = x_2$, then $x_m = x_n$ for all m, n and $\{x_n\}$ is a constant sequence. Hence $\{x_n\}$ converges and is a Cauchy sequence. If $x_1 \ne x_2$, then for every $\varepsilon>0$, by the Archimedean principle, there is $K \in \mathbb{N}$ such that $K > 2 - \log_2 \frac{\varepsilon}{|x_2-x_1|}$, which implies $(\frac{1}{2})^{K-2}|x_2-x_1| < \varepsilon$. So $m,n \ge K \Rightarrow |x_m-x_n| \le (\frac{1}{2})^{K-2}|x_2-x_1| < \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence.

(b) Let $w \in \mathbb{R}$. Define $\{x_n\}$ as in (a). Then $\{x_n\}$ is a Cauchy sequence by (a). By Cauchy's theorem, $\{x_n\}$ converges to some $x \in \mathbb{R}$. We have

$$x \underset{\substack{\uparrow\\ \text{subsequence theorem}}}{=} \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) \underset{\substack{\uparrow\\ \text{sequential continuity theorem}}}{=} f(\lim_{n\to\infty} x_n) = f(x).$$

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(118) Define $f(x) = \begin{cases}(x-1)^2 \sin\frac{1}{x-1} & \text{if } x\in(0,1)\cup(1,2)\\ 0 & \text{if } x=1\end{cases}$. For $x \in (0,1)\cup(1,2)$, by product rule, $f'(x) = 2(x-1)\sin\frac{1}{x-1} - \cos\frac{1}{x-1}$. For $x=1$, $f'(1) = \lim_{x\to1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1}(x-1)\sin\frac{1}{x-1} = 0$ as $|(x-1)\sin\frac{1}{x-1}| \le |x-1| \to 0$ as $x\to1$. So f is differentiable on $(0,2)$. However, $\lim_{x\to1} f'(x) = -\lim_{x\to1}\cos\frac{1}{x-1}$ doesn't exist. So $f'(x)$ is not continuous at 1.

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(119) We have $\left|\frac{f(a)-f(b)}{a-b}\right| \le \frac{\sin^2|a-b|}{|a-b|}$ for $a\ne b$, a,b in $(0,\pi)$. Since $\lim_{a\to b}\frac{\sin^2|a-b|}{|a-b|} = \lim_{\theta\to0}\frac{\sin^2\theta}{\theta} = \lim_{\theta\to0}\frac{\sin\theta}{\theta}\lim_{\theta\to0}\sin\theta = 1\cdot0 = 0$, we have $f'(b) = \lim_{a\to b}\frac{f(a)-f(b)}{a-b} = 0$ for every b. Therefore, f is a constant function.

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(120) (a) Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $x_0 \in (a,b)$ such that $f(b)-f(a) = f'(x_0)(b-a)$.

(b) By the mean value theorem, $|\sin b - \sin a| = |(\cos x_0)(b-a)| \le 1|b-a|$. If there is a K such that $|f(b)-f(a)| \le K|b-a|$ for every $a,b\in\mathbb{R}$, then $|f'(a)| = \lim_{b\to a}\left|\frac{f(b)-f(a)}{b-a}\right| \le K$ for every $a\in\mathbb{R}$. Since $f'(0) = \cos 0 = 1$, so $K \ge 1$. Therefore, the smallest K is 1.

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(121) Since $\lim_{x\to0}\frac{f'(x)}{1} = \lim_{x\to0} f'(x)$ exists, by l'Hopital's rule, $f'(0) = \lim_{x\to0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to0}\frac{f'(x)}{1} = \lim_{x\to0} f'(x)$. Therefore, f' is continuous at 0.

(122) By the mean value theorem, $|\sin 5b - \sin 5a| = |(5\cos 5x_0)(b-a)| \le 5|b-a|$. So for every $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{5} > 0$. With this δ, we have for every $a, b \in \mathbb{R}$, $|b-a| < \delta \Rightarrow |\sin 5b - \sin 5a| \le 5|b-a| < 5\delta = \varepsilon$.

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(123) For every $\varepsilon > 0$, since f is uniformly continuous, so $\exists \delta > 0$ such that $\forall x, y \in \mathbb{R}$ $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon^2$. Then $|x-y| < \delta \Rightarrow |\sqrt{f(x)} - \sqrt{f(y)}| \le \sqrt{|f(x) - f(y)|} < \varepsilon$, (where we used $|\sqrt{a} - \sqrt{b}| \le \sqrt{|a-b|}$ as in homework 2, #236 (b)). Therefore, $\sqrt{f(x)}$ is also uniformly continuous.

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(124) (b) Solution 1. (using Lebesgue's Theorem) $\nearrow f, g$ bounded on $[0,2]$ $\Rightarrow h$ bounded on $[0,2]$.

Since f, g are Riemann integrable on $[0,2]$, so $S_f = \{x \in [0,2] : f$ is discontinuous at $x\}$ and $S_g = \{x \in [0,2] : g$ is discontinuous at $x\}$ are of measure 0.

Now for $x \in [0,1)$, h is discontinuous at x if and only if f is discontinuous at x. Also for $x \in (1,2]$, h is discontinuous at x if and only if g is discontinuous at x. (These are because $h = f$ on $[0,1)$ and $h = g$ on $(1,2]$.) So

$$S_h = \{x \in [0,2] : h \text{ is discontinuous at } x\} \subseteq (S_f \cap [0,1)) \cup (S_g \cap (1,2]) \cup \{1\} \subseteq S_f \cup S_g \cup \{1\}.$$

Since $S_f, S_g, \{1\}$ are of measure 0, we have $S_f \cup S_g \cup \{1\}$ is of measure 0. Then S_h is also of measure 0. Therefore, h is Riemann integrable by Lebesgue's Theorem.

Solution 2 (using integral criterion)

Since f and g are integrable on $[0,2]$, they are bounded on $[0,2]$. So there are $m, M \in \mathbb{R}$ such that $m \le f(x), g(x) \le M$ for all $x \in [0,2]$.

If $m = M$, then $h(x)$ is a constant function, hence h is integrable on $[0,2]$.

If $m < M$, then for every $\varepsilon > 0$, let $d \in (0,1)$ such that $1 - d < \frac{\varepsilon}{3(M-m)}$.

By 82(ii), f is integrable on $[0,d]$ and g is integrable on $[1,2]$.

By integral criterion, there are partitions P_1 of $[0,d]$ and P_2 of $[1,2]$ such that $U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{3}$ and $U(g, P_2) - L(g, P_2) < \frac{\varepsilon}{3}$.

Now $P = P_1 \cup P_2$ is a partition of $[0,2]$ and

$$U(h,P) - L(h,P) \le U(f, P_1) - L(f, P_1) + (M-m)(1-d) + U(g, P_2) - L(g, P_2)$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

(125) Note $\max(f,g)+\min(f,g)=f+g$ and $\max(f,g)-\min(f,g)=|f-g|$. Subtracting, then dividing by 2, we have $h=\min(f,g)=\frac{f+g-|f-g|}{2}$. If f,g are integrable, then $f+g$, $f-g$ are integrable. Since $|x|$ is continuous, so $|f-g|$ is also integrable. Therefore, $h=\frac{f+g-|f-g|}{2}$ is integrable.

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(126) Since $\mathbb{Q}\cap[0,1]$ is countable, let $r_1,r_2,r_3,\cdots$ be a listing of the elements of $\mathbb{Q}\cap[0,1]$ without repetition nor omission. Define $f_n(x)=\begin{cases}1 & \text{if } x=r_1 \text{ or } r_2 \text{ or} \cdots \text{or } r_n\\ 0 & \text{: otherwise}\end{cases}$. Then on $[0,1]$, f_n is discontinuous exactly at $r_1,r_2,\cdots,r_n$. Since $\{r_1,r_2,\cdots,r_n\}$ is countable, hence of measure 0, f_n is Riemann integrable by Lebesgue's theorem.

Now $f(x)=\lim_{n\to\infty}f_n(x)=\begin{cases}1 & \text{if } x=r_i \text{ for } i=1,2,3,\cdots\\ 0 & \text{otherwise}\end{cases}=\begin{cases}1 & \text{if } x\in\mathbb{Q}\cap[0,1]\\ 0 & \text{otherwise}\end{cases}$.

On $[0,1]$, f is discontinuous everywhere as is shown in class. Since $[0,1]$ is not of measure 0, f is not Riemann integrable by Lebesgue's theorem.

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(127) (a) Since $\left|\frac{\cos 3x}{1+x^2}\right|=\frac{|\cos 3x|}{1+x^2}\le\frac{1}{1+x^2}$ and $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\lim_{c\to-\infty}\int_c^0\frac{1}{1+x^2}dx+\lim_{d\to+\infty}\int_0^d\frac{1}{1+x^2}dx$ $=\lim_{c\to-\infty}(-\operatorname{Arctan} c)+\lim_{d\to+\infty}(\operatorname{Arctan} d)=\frac{\pi}{2}+\frac{\pi}{2}=\pi$, so $\frac{1}{1+x^2}$ is improper integrable on $(-\infty,\infty)$. By comparison test, $\left|\frac{\cos 3x}{1+x^2}\right|$ is improper integrable on $(-\infty,\infty)$. By the absolute convergence test, $\frac{\cos 3x}{1+x^2}$ is improper integrable on $(-\infty,\infty)$. So $\int_{-\infty}^{\infty}\frac{\cos 3x}{1+x^2}dx$ exists.

(b) Since $\int_{-\infty}^{\infty}\frac{\cos 3x}{1+x^2}dx$ is improper integrable on $(-\infty,\infty)$, so $\text{P.V.}\int_{-\infty}^{\infty}\frac{\cos 3x}{1+x^2}dx$ exists.

(c) $\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}dx=\lim_{c\to0^-}\int_{-1}^{c}\frac{1}{\sqrt[3]{x}}dx+\lim_{d\to0^+}\int_d^1\frac{1}{\sqrt[3]{x}}dx=\lim_{c\to0^-}\left(\frac{3}{2}x^{2/3}\right)\Big|_{-1}^{c}+\lim_{d\to0^+}\left(\frac{3}{2}x^{2/3}\right)\Big|_d^1=-\frac{3}{2}+\frac{3}{2}=0$

(d) Since $\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}dx$ exists as an improper integral, so $PV\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}dx$ also exist.

Alternatively, $PV\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}dx=\lim_{\varepsilon\to0^+}\left(\int_{-1}^{-\varepsilon}\frac{1}{\sqrt[3]{x}}dx+\int_{\varepsilon}^{1}\frac{1}{\sqrt[3]{x}}dx\right)=\lim_{\varepsilon\to0^+}0=0$

$1/\sqrt[3]{x}$ is an odd function.

(e) $\int_0^{\infty}\sin x\,dx=\lim_{c\to+\infty}\int_0^c\sin x\,dx=\lim_{c\to+\infty}(-\cos x)\Big|_0^c=\lim_{c\to+\infty}(-\cos c+1)$ doesn't exist.

So $\int_{-\infty}^{\infty}\sin x\,dx$ doesn't exist.

(f) $PV\int_{-\infty}^{\infty}\sin x\,dx=\lim_{c\to+\infty}\int_{-c}^{c}\sin x\,dx=\lim_{c\to+\infty}(-\cos x)\Big|_{-c}^{c}=\lim_{c\to+\infty}0=0$.