1. (15 points) .

1.1 Negating the following statements.

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall x, 0 < |x - x_0| < \delta \implies |\frac{f(x) - f(x_0)}{x - x_0} - L| < \epsilon$$

1.2 Find $\cap_{n\in\mathbb{N}}(0, \frac{1}{n})$ and justify your answer.

Four symbols are all right get 5 points

Solution:

1.1

such that

$$\exists \epsilon > 0, \forall \delta > 0, \exists x, 0 < |x - x_0| < \delta, |\frac{f(x) - f(x_0)}{x - x_0} - L| \geq \epsilon$$

1' 1' 1' 1'

1.2

$$\cap_{n\in\mathbb{N}}(0, \frac{1}{n}) = \emptyset.$$

For $x \leq 0, x \geq 1, x \notin (0, \frac{1}{n}), \forall n \in \mathbb{N}$, thus $x \notin \cap_{n\in\mathbb{N}}(0, \frac{1}{n})$ for $x \leq 0$ and $x \geq 1$. 2' For $0 < x < 1$, if $n > [\frac{1}{x}] + 1$, then $x \notin (0, \frac{1}{n})$. This implies $x \notin \cap_{n\in\mathbb{N}}(0, \frac{1}{n})$. Therefore,

4' 1'

$$\cap_{n\in\mathbb{N}}(0, \frac{1}{n}) = \emptyset.$$

3'

Second method:

By contradiction, assume $\bigcap_{n\in\mathbb{N}}(0, \frac{1}{n}) \neq \emptyset$ 1'

then $\exists\, x \in \mathbb{R}$ s.t $x \in \bigcap_{n\in\mathbb{N}}(0, \frac{1}{n})$ 1'

$\Rightarrow x \in (0, \frac{1}{n})$ for all $n \in \mathbb{N}$ 1'

$\Rightarrow 0 < x < \frac{1}{n}$ for all $n \in \mathbb{N}$

$\Rightarrow \frac{1}{x} > n$ for all $n \in \mathbb{N}$

contradicts to Archimede's principle 4'

So $\bigcap_{n\in\mathbb{N}}(0, \frac{1}{n}) = \emptyset$ 3'

2. (25 points) .

2.1 Write down the definition of infimum. State and prove the infimum property.

2.2 Determine if the following set A has an infimum. If it exists, find it and justify your answer.

$$A = \{x + y^2 : x \in [0,1] \cap \mathbb{Q}, y \in [0,1] \setminus \mathbb{Q}\}$$

Solution:

2.1 Definition: An infimum of S, denoted by $InfS$ is a lower bound such that $K \leq InfS$ for all bounds K of S.

2' 3'

Theorem:(Infimum property) then

If $InfS$ is an infimum of S in $\mathbb{R}$ ~~if and only if $InfS$ is a lower bound of S and.~~

2'

$$\forall \epsilon > 0, \exists x \in S, \text{ s.t } InfS \leq x \leq InfS + \epsilon. \quad 3'$$

Proof " $\Longrightarrow$ " If $InfS$ is an infimum of S in $\mathbb{R}$, by definition of infimum, we have $InfS$ is a lower bound of S. Since $InfS$ is the largest lower bound of S, for all $\epsilon > 0$, $InfS + \epsilon$ cannot be a lower bound of S. Hence, for all $\epsilon > 0$, we can find $x \in S$ such that

$$2' \quad InfS \leq x \leq InfS + \epsilon. \quad 3'$$

" ~~$\Longleftarrow$ " We only need to show $InfS$ is the largest bound. Otherwise, there is another lower bound m such that $m > InfS$. Then $\epsilon = m - InfS > 0$, for this ϵ, we can find $x \in S$ such that~~

$$\cancel{x < InfS + \epsilon = m.}$$

~~This contradicts m is a lower bound of S. So $InfS$ is the largest lower bound and the infimum.~~

2.2 Since $x \in [0,1] \cap \mathbb{Q}, y \in [0,1] \setminus \mathbb{Q}$, we can see $0 \leq x + y$ and thus 0 is a lower bound of A. To show 0 is an infimum of A, we note that $x_n = \frac{1}{n} \in [0,1] \cap \mathbb{Q}, y_n = \frac{1}{\sqrt{2}n} \in [0,1] \setminus \mathbb{Q}$ and

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = 0. \quad 3'$$

Hence, we find a sequence $\{x_n + y_n^2\} \subset A$ such that 2'

$$\lim_{n \to \infty} x_n + y_n^2 = 0.$$

By Infimum-limit theorem, we conclude that $InfA = 0$.

3' $\exists$ 2'

Second way:
1. decreasing bounded 6'

2. $a_n^2 = a_n$ 4'
$\Rightarrow \lim_{n\to+\infty} C^{\frac{1}{n}} = 1$

3. (25 points) .
3.1 Let $C \geq 1$, prove that
$$\lim_{n\to+\infty} C^{\frac{1}{n}} = 1.$$
3.2 Determine whether the sequence x_n defined by
$$x_1 = 3, x_{n+1} = 3 + \frac{4}{x_n} \quad \text{for } n \geq 1,$$
converges or not. If it converges, prove its convergence and find the limit.

Solution:
3.1 $\boxed{\forall \epsilon > 0}$, take $\boxed{K = [\frac{C-1}{\epsilon}] + 1} \in \mathbb{N}$. We can see, $\forall n > K, n \in \mathbb{N}, C < 1 + n\epsilon < (1+\epsilon)^n$. Therefore, for any $\epsilon > 0, n > K, n \in \mathbb{N}$, we have
$$\boxed{C^{\frac{1}{n}} < 1 + \epsilon.}$$

ϵ-N language framework 5'
expression right 5'

On the other hand, $C^{\frac{1}{n}} \geq 1^{\frac{1}{n}} = 1$. Then we have
$$\boxed{|C^{\frac{1}{n}} - 1| < \epsilon, \forall n > K.}$$
That is $\lim_{n\to\infty} C^{\frac{1}{n}} = 1$.

3.2 The sequence converges. If $3 \leq x_n \leq 5$, then by $x_{n+1} = 3 + \frac{4}{x_n}$, we have

2' if written
$$3 \leq 3 + \frac{4}{5} \leq x_{n+1} \leq 3 + \frac{4}{3} \leq 5.$$
By mathematical induction principle, we have
$$\boxed{3 \leq x_n \leq 5, \forall n \in \mathbb{N}.}$$ 3'

Then by
$$\begin{aligned}
x_{n+2} - x_n &= \frac{4}{x_{n+1}} - \frac{4}{x_{n-1}} \\
&= \frac{4}{x_{n+1}x_{n-1}}(x_{n-1} - x_{n+1}) \\
&= \frac{4}{x_{n+1}x_{n-1}}(\frac{4}{x_{n-2}} - \frac{4}{x_n}) \\
&= \frac{4}{x_{n+1}x_{n-1}}\frac{4}{x_n x_{n-2}}(x_n - x_{n-2}),
\end{aligned}$$
3'

because $\frac{4}{x_{n+1}x_{n-1}}\frac{4}{x_n x_{n-2}} > 0$, then
$$x_{n+2} > x_n \text{ if } x_n > x_{n-2}$$
and
$$x_{n+2} < x_n \text{ if } x_n < x_{n-2}.$$
Because $x_1 = 3, x_2 = \frac{13}{3}, x_3 = \frac{51}{13}, x_4 = \frac{205}{51}$, then
$$\{x_{2n+1}, \in\in \mathbb{N}\} \text{ increasing}$$
and
$$\{x_{2n}, n \in \mathbb{N}\} \text{ dereasing.}$$
4'

Suppose $x_{2n+1} \to a$ and $x_{2n} \to b$ as $n \to \infty$. Then

$$a = 3 + \frac{4}{b} \quad b = 3 + \frac{4}{a}.$$

The solution and the constraint $3 \le a, b \le 5$ gives

$$a = b = 4.$$

3'

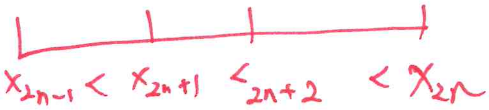
By interwining sequence theorem, the sequence $\{x_n\}$ converges and

$$\lim_{n\to+\infty} x_n = 4.$$

2'

Second way:

1.



10'

2.

$$a = 3 + \frac{4}{b} \quad b = 3 + \frac{4}{a}$$

3'

$$\lim_{x\to+\infty} x_n = 4$$

2'

4. (35 points) .
4.1 Write down the definition of Cauchy sequence.
4.2 Let $\{x_n\}$, $\{y_n\}$ be Cauchy sequence, show that both $\{x_n - y_n\}$ and $\{x_n y_n\}$ are Cauchy sequences.
4.3 Let S be the set of all Cauchy sequences in $\mathbb{Q}$. More precisely,

$$S = \{\{x_n\} : \{x_n\} \text{ is a Cauchy sequence s.t } x_n \in \mathbb{Q} \text{ for all } n \in \mathbb{N}\}.$$

Determine if S is countable and justify your answer.
4.4 Let S be defined above, let $\{x_n\} \in S$, we say that $\{x_n\}$ is positive iff there exists $\delta > 0, \delta \in \mathbb{Q}$ and $k \in \mathbb{N}$ s.t $x_n > \delta$ for all $n \geq k$. We say that $\{x_n\} < \{y_n\}$ iff $\{y_n - x_n\}$ is positive. Show that

$$\forall \{x_n\}, \{y_n\}, \{z_n\} \in S,$$

if $\{x_n\} < \{y_n\}$, and $\{z_n\}$ is positive, then

$$\{x_n z_n\} < \{y_n z_n\}.$$

Solution:
4.1 Definition: A sequence $\{x_n\}$ is a Cauchy sequence iff $\forall \epsilon > 0, \exists K \in \mathbb{N}$ such that $m, n > K$,

$$|x_m - x_n| < \epsilon.$$

5'

4.2 By definition, $\forall \epsilon > 0, \exists K_1, K_2 \in \mathbb{N}$ such that $m, n > K_1, m', n' > K_2$

$$|x_m - x_n| < \frac{\epsilon}{2}$$
$$|y_{m'} - y_{n'}| < \frac{\epsilon}{2}$$

2'

1'

Therefore, taking $K = \max(K_1, K_2)$, we have $p, q > K$ implies

$$|(x_p - y_p) - (x_q - y_q)| \leq |x_p - x_q| + |y_p - y_q| < \epsilon$$

2'

which means $\{x_n - y_n\}$ is a Cauchy sequence.
Since $\{x_n\}$, $\{y_n\}$ are Cauchy sequences, $\exists M > 0$ such that $|x_n|, |y_n| \leq M$ for all $n \in \mathbb{N}$. By definition of Cauchy sequence, $\forall \epsilon > 0, \exists K_1, K_2 \in \mathbb{N}$ such that $m, n > K_1, m', n' > K_2$

$$|x_m - x_n| < \frac{\epsilon}{2M}$$
$$|y_{m'} - y_{n'}| < \frac{\epsilon}{2M}$$

2'

1'

Therefore, taking $K = \max(K_1, K_2)$, we have $m, n > K$ implies

$$|x_n y_n - x_m y_m| = |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \leq |x_n||y_n - y_m| + |y_m||x_n - x_m| < M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon$$

2'

which means $\{x_n y_n\}$ is a Cauchy sequence.

4.3 We can construct a surjective mapping $f : S \to \mathbb{R}$ given by

$$f(\{x_n\}) = \lim_{n \to \infty} x_n.$$

2'

2'

This mapping is well defined because $\{x_n\} \subset \mathbb{Q} \subset \mathbb{R}$ is a Cauchy sequence in $\mathbb{R}$ and by Cauchy theorem, the $\lim_{n \to \infty} x_n$ exists in $\mathbb{R}$. Next, we show this map is surjective. For $\forall x \in \mathbb{R}$, we can construct a sequence $\{x_n\}$ as follow:

$$x_n \in (x - \frac{1}{n}, x + \frac{1}{n})$$

4'

where $x_n \in \mathbb{Q}$. Clearly, it is a Cauchy sequence since $\forall \epsilon > 0, \exists K = [\frac{2}{\epsilon}] + 1$, for $m, n > K$,

$$|x_m - x_n| \leq |x_m - x| + |x_n - x| \leq \frac{1}{n} + \frac{1}{m} < \epsilon.$$

Finally, by surjective theorem, S is uncountable since $\mathbb{R}$ is uncountable. 2'

4.4 Since $\{x_n\} < \{y_n\}$, there exist $\delta_1 > 0, \delta_1 \in \mathbb{Q}$ and $K_1 \in \mathbb{N}$ such that

$$y_n - x_n > \delta_1, \quad \forall n \geq K_1.$$ 3'

Similarly, since $\{z_n\}$ positive, there exists $\delta_2 > 0, \delta_2 \in \mathbb{Q}$ and $K_2 \in \mathbb{N}$ such that

$$z_n > \delta_2, \quad \forall n \geq K_2.$$ 3'

Take $K = \max\{K_1, K_2\}, \delta = \delta_1\delta_2 > 0, \delta \in \mathbb{Q}$ and for all $n > K$, we have

$$z_n y_n - z_n x_n = z_n(y_n - x_n) > \delta$$ 4'

which means

$$\{x_n z_n\} < \{y_n z_n\}.$$