Lecture 3

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Review: basics of set theory:

Example: ϕ, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

Set relations: $A \subseteq B \iff x \in A \Rightarrow x \in B$

Power set: $P(S) = 2^S$: the set of all subsets of S.

Set operations: $\bigcup_{i \in \Lambda} A_i$ [For example: $\Lambda = \mathbb{N}$, $\Lambda = \{1, 2, 3, \dots m\}$]

Λ → index set

$\bigcap_{i \in \Lambda} A_i$, $\bigtimes_{i \in \Lambda} A_i$

Function $f: A \to B$. A rule to assign every $x \in A$ an unique element in B, denoted by $f(x)$.

A ↓ domain, B ↓ Codomain

$f(A) = \{ f(x) : x \in A \}$: the range (image) of f.

Type of Functions :

① $I_S : S \to S$. $I_S(x) = x \quad \forall x \in S$. identity function

② $f: A \to B$, $g: B \to C$, $f \circ g : A \to C$ Composite function

③ $f: A \to B$, $C \subseteq A$, $f|_C : C \to B$. restriction function.

④ $f: A \to B$ is surjective (or onto) if $f(A) = B$;

⑤ $f: A \to B$ is injective (or 1-to-1) if $f(x) = f(y) \Rightarrow x = y$.

⑥ For injective $f: A \to B$. Define

$$f^{-1} : f(A) \longrightarrow A \quad \text{by} \quad f^{-1}(f(x)) = x$$

↑
the inverse of f

⑦ $f: A \to B$ is bijective (or one-to-one correspondence) $\Leftrightarrow$ f is injective and surjective.

Remark : If $f: A \to B$ is injective, then the function $\tilde{f}$ (or still denoted by f for simplicity) : $A \to f(A)$ given by $\tilde{f}(x) = f(x)$, $\forall x \in A$ is a bijective.

Example:

Show that $f: [0,1] \to [3,4]$ defined by $f(x) = x^3 + 3$ is a bijective.

Proof: Step 1, we show that f is injective.

If $f(x) = f(y)$ for $x, y \in [0,1]$,

then $x^3 + 3 = y^3 + 3$, $x^3 = y^3$.

Since $x, y \geq 0 \Rightarrow x = y$. This shows that f is injective.

Step 2. We show that f is surjective.

$\forall\ y \in [3,4]$, Let $f(x) = x^3 + 3 = y$. then

$x^3 = y - 3$, $x = \sqrt[3]{y-3} \in [0,1]$. This show that f is surjective

Exercise : Prove the following theorem

① $f: A \to B$ is a bijective

iff $\exists\, g: B \to A$ such that

$g \circ f = I_A$ and $f \circ g = I_B$.

② $f: A \to B$ and $g: B \to C$ are

bijections, then $g \circ f: A \to C$ is a bijection.

③ Let $A, B \subseteq \mathbb{R}$ and $f: A \to B$ be a function.

If $\forall\, b \in B$, the horizontal line $y = b$ intersects

the graph of f exactly once, then f is a

bijection.

Equivalent relations

Def: A relation on a set S is any subset of $S \times S$.

A relation R on a set S is an equivalence relation iff

① $\forall x \in S$, $(x, x) \in R$ (Reflexive)

② $(x, y) \in R \Rightarrow (y, x) \in R$ (Symmetry)

③ $(x, y) \in R$, $(y, z) \in R \Rightarrow (x, z) \in R$. (Transitive)

Remark: 1. We write $x \sim y$ when $(x, y) \in R$. Then ①, ②, ③ above can be intepreted as: ①' $x \sim x$ (reflexive)

②' $x \sim y \Rightarrow y \sim x$ (symmetry) ③' $x \sim y$, $y \sim z \Rightarrow x \sim z$ (transitive).

2. $\forall x \in S$, we write $[x] = \{ y : x \sim y \}$

↑

Called the equivalence class containing x.

Some facts about equivalence relations

① Since $x \sim x$, $x \in [x]$,

$$S = \bigcup_{x \in S} [x]$$

② $x \sim y \Rightarrow [x] = [y]$. Since

$$z \in [x] \Leftrightarrow x \sim z \Leftrightarrow z \sim y \Leftrightarrow z \in [y].$$

not equivalent to

③ $x \not\sim y \Rightarrow [x] \cap [y] = \phi$

(otherwise $\exists\ z \in [x] \cap [y]$, $\Rightarrow z \sim x$, $z \sim y$ $\Rightarrow x \sim y$)

Remark 1. An equivalence relation on S gives a natural partition of the set S.

2. If we can define "$\sim$" for any two elements in S which satisfies the reflexive, symmetry and transitive properties, then one can define an equivalence relation R on S accordingly.

Examples

① S = the set of all triangles. Define a relation R on S by

$$R = \{ (T_1, T_2) : T_1 \text{ is "similar" to } T_2 \}$$

One can check that R is an equivalence relation.

Let T_1, T_2 be two triangles,

$T_1 \sim T_2 \Leftrightarrow (T_1, T_2) \in R \Leftrightarrow T_1$ is similar to T_2

Let T be a given triangle, then

$[T]$ is the set of all triangles similar to T.

② $S = \mathbb{Z}$. Introduce a relation R on $\mathbb{Z}$

by let $R = \{ (m,n) : m-n \text{ is even} \}$.

then one can $\boxed{\text{check}}$ that R is an equivalence relation.

$\forall\ m, n \in \mathbb{Z}$

$m \sim n \iff (m,n) \in R \iff m-n$ is even.

$[0] = \{ m : m-0 \text{ is even integer} \}$

$= \{ m : m \text{ is an even integer} \}$

$[1] = \{ m : m-1 \text{ is even integer} \}$

$= \{ m : m \text{ is an odd integer} \}$

$\mathbb{Z} = [0] \cup [1]$

③ Let $S = \{0, 1\}$ and $R = \{(1,1)\}$.

then R is a relation on S. However, R

is not an equivalence relation R is not

reflexive $(0,0) \bar{\in} R$.

④ For sets S_1 and S_2, define an equivalence relation

as follows: $S_1 \sim S_2$ iff

$\exists$ bijective $f : S_1 \to S_2$.

We can check that the above indeed defines an equivalence

relation [① $S \sim S$ $\forall S$. ② $S \sim T \Rightarrow T \sim S$

③ $S \sim T$, $T \sim W \Rightarrow S \sim W$]

In other word, let $\mathcal{S}$ = the set of all sets. then

$R = \{(S_1, S_2) : \exists$ bijective $f : S_1 \to S_2\}$ defines an equivalence relation on $\mathcal{S}$.

We say that S_1 and S_2 have the same cardinality if

$$S_1 \sim S_2.$$

We call the number of elements in S the cardinal number of S.

and denote it by $\text{Card } S$ or $|S|$.

Example: ① $\text{Card}\{1, 2, \cdots n\} = n$, for positive integer n.

② $\text{Card } \phi = 0$.

③ $\text{Card}\{1, 2, 3, \cdots\} = \text{Card } \mathbb{N} = \aleph_0$ (alpha-null)

④ $\text{Card } \mathbb{R} = c$. (Cardinality of the continuum)

Chapte 3 Countability

(a property that distinguishes some infinite sets)

Def: ① A set S is countably infinite iff $\exists$ bijection

$f: \mathbb{N} \to S$ (i.e Card S = Card $\mathbb{N}$ = $\aleph_0$)

② A set S is countable iff S is finite

or Countably infinite

Observation : $\exists$ bijection $f: \mathbb{N} \to S \Rightarrow S = \{ f(1), f(2), f(3) \cdots \}$

a listing of elements of S with no repetition nor omission.

On the other hand, given a listing of $S = \{ s_1, s_2, s_3, \cdots \}$ with no repetition and omission, then $f: \mathbb{N} \to S$ defined by

$f(n) = s_n$ is a bijection.

Bijection theorem: Let $g: S \to T$ be a bijection,

S is countable $\boxed{\Leftrightarrow}$ T is countable.

[The contrapositive is :

Let $g: S \to T$ be a bijection. then S is uncountable $\Leftrightarrow$ T is uncountable.]

Proof: $g: S \to T$ is a bijection, $\Leftrightarrow$ $S \sim T$

or $(\text{Card } S = \text{Card } T)$.

Now, S is countable $\Leftrightarrow$ S is finite or S is

countably infinite $\Leftrightarrow$ $S \sim \{1, 2, \cdots m\}$ for some $m \in \mathbb{N}$.

or $S \sim \mathbb{N}$ $\Leftrightarrow$ $T \sim \{1, 2, \cdots m\}$ for some $m \in \mathbb{N}$

or $T \sim \mathbb{N}$. $\Leftrightarrow$ T is finite or T is countably infinite.

$\Leftrightarrow$ T is countable.

Basic Examples

① $\mathbb{N}$ is countably infinite as $I_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ is a bijection.

② $\mathbb{Z}$ is countably infinite because

$$\begin{array}{lccccccccc} \mathbb{N} & = & \{1, & 2, & 3, & 4, & 5, & 6, & \cdots & \} \\ f\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \mathbb{Z} & = & \{0, & 1, & -1, & 2, & -2, & 3, & \cdots & \} \end{array}$$

Or more precisely, $f(n) = \begin{cases} \frac{n}{2}, & n \text{ even} \\ -\frac{n-1}{2}, & n \text{ odd} \end{cases}$

is a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

Example 3. $\mathbb{N}\times\mathbb{N} = \{(m,n) : m, n \in \mathbb{N}\}$ is countably infinite.

[use diagonal counting scheme]

$$\begin{array}{ccccc}
(1,1)^{f(1)} & (1,2)^{f(3)} & (1,3)_{f(6)} & (1,4)_{f(10)} & \cdots \\
(2,1)^{f(2)} & (2,2)_{f(5)} & (2,3)_{f(9)} & (2,4) & \cdots \\
(3,1)_{f(4)} & (3,2)\, f(8) & (3,3) & (3,4) & \cdots \\
(4,1)^{f(7)} & (4,2) & (4,3) & (4,4) & \cdots \\
(5,1)_{f(11)} & \cdots & & &
\end{array}$$

One can check that $f : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ is bijective,

[In fact $(m,n) = f\left(\sum_{k=0}^{m+n-2} k + n\right)$, for $(n,m) \in \mathbb{N}\times\mathbb{N}$, so f is surjective]

or equivalently, f provides a counting (or listing) of elements in $\mathbb{N}\times\mathbb{N}$ with no repetition and omission.

Example 4 : Open interval $(0,1) = \{ x : x \in \mathbb{R},\ 0 < x < 1 \}$

is uncountable.

[More precisely, if "countable" is defined by Def ①. ②,

then $(0,1)$ is not countable]

Proof : By contradiction, assume that $(0,1)$ is countable,

then $\exists$ bijection $f : \mathbb{N} \to (0,1)$. So

$$f(1) = 0.\underline{a_{11}}\, a_{12}\, a_{13}\, a_{14} \cdots = \sum_{n=1}^{\infty} a_{1n} \times 10^{-n}$$

$$f(2) = 0.a_{21}\, \underline{a_{22}}\, a_{23}\, a_{24} \cdots = \sum_{n=1}^{\infty} a_{2n} \times 10^{-n}$$

$$\vdots$$

$$f(m) = 0.a_{m1}\, a_{m2}\, a_{m3}\, a_{m4} \cdots = \sum_{n=1}^{\infty} a_{mn} \times 10^{-n}.$$

Consider $x = 0.b_1 b_2 b_3 \cdots = \sum_{n=1}^{\infty} b_n \times 10^{-n}$

where $b_n = \begin{cases} 9 & \text{if } a_{nn} < 5 \\ 1 & \text{if } a_{nn} \geq 5 \end{cases}$.

Then $x \in (0,1)$. However $x \neq f(n)$ for all n.

Since $|a_{nn} - b_n| > 2$.

the n-th digit of x differs from the n-th digit of $f(n)$ by at least 2.

This contradicts the surjectivity of f.

$\therefore$ $(0,1)$ is not countable.

Remark : $0.19999\cdots = 0.2000\cdots$