Solution of Homework 3

① Sketch $|b_n - b_m| = \left|\sqrt{\frac{a_n}{a_n+3}+5} - \sqrt{\frac{a_m}{a_m+3}+5}\right| \le \sqrt{\left|\left(\frac{a_n}{a_n+3}+5\right)-\left(\frac{a_m}{a_m+3}+5\right)\right|} = \sqrt{\left|\frac{a_n}{a_n+3}-\frac{a_m}{a_m+3}\right|}$

$= \sqrt{\left|\frac{3a_n-3a_m}{\underbrace{(a_n+3)}_{>0}\underbrace{(a_m+3)}_{>0}}\right|} \le \sqrt{\frac{3|a_n-a_m|}{3\times 3}} = \sqrt{\frac{|a_n-a_m|}{3}} < \varepsilon \iff |a_n-a_m| < 3\varepsilon^2.$

Solution $\forall \varepsilon > 0$, Since $\{a_n\}$ is Cauchy, $\exists K \in \mathbb{N}$ such that $m, n \ge K \Rightarrow |a_n - a_m| < 3\varepsilon^2$
$\Rightarrow |b_n - b_m| < \varepsilon$.

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② Suppose $f(a) = f(b)$. Then $1-f(a) = 1-f(b)$, $1-a^9 = f(1-f(a)) = f(1-f(b)) = 1-b^9$. So $a^9 = b^9$. Taking 9^{th} root, $a=b$. Hence f is injective. Since f is differentiable (hence continuous), by the continuous injection theorem, f is strictly monotone. Since $f'(1)<0$, f is strictly decreasing. Let $g(x) = f(x) - x^{2013}$. Then $g(1) = f(1) - 1^{2013} < 0$. Since f is strictly decreasing, $0 < 1 \Rightarrow f(0) > f(1) = 0 \Rightarrow g(0) = f(0) - 0^{2013} = f(0) > 0$.
By the intermediate value theorem, $\exists r \in \mathbb{R}$, $g(r) = 0$. $\therefore f(r) = r^{2013}$.

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③ By Taylor's theorem, $\exists\, \theta_0$ between 0 and x and $\exists\, \theta_1$ between $-x$ and 0 such that $f(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{6}(x-0)^3 + \frac{f^{(4)}(\theta_0)}{24}(x-0)^4$ and $f(-x) = f(0) + f'(0)(-x-0) + \frac{f''(0)}{2}(-x-0)^2 + \frac{f'''(0)}{6}(-x-0)^3 + \frac{f^{(4)}(\theta_1)}{24}(-x-0)^4$. Adding these, we get $f(x) + f(-x) = 2f(0) + f''(0)x^2 + (f^{(4)}(\theta_0) + f^{(4)}(\theta_1))\frac{x^4}{24}$. Solving for $f''(0)$, we get $f''(0) = \frac{f(x) - 2f(0) + f(-x)}{x^2} - \frac{(f^{(4)}(\theta_0) + f^{(4)}(\theta_1))x^2}{24}$. Therefore,

$$\left|f''(0) - \frac{f(x) - 2f(0) + f(-x)}{x^2}\right| \le \frac{|f^{(4)}(\theta_0)| + |f^{(4)}(\theta_1)|}{24}x^2 \le \frac{1+1}{24}x^2 = \frac{x^2}{12}.$$

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④ For every $\varepsilon > 0$, Since S is of measure 0, $\exists (a_1, b_1), (a_2, b_2), \ldots$ such that $S \subseteq \bigcup_{i=1}^{\infty}(a_i, b_i)$ and $\sum_{i=1}^{\infty}|a_i - b_i| < \varepsilon/2$, then $T \subseteq \bigcup_{i=1}^{\infty}(2a_i, 2b_i)$ and $\sum_{i=1}^{\infty}|2a_i - 2b_i| < \varepsilon$. Therefore, T is of measure 0.

Since f is Riemann integrable, by Lebesgue's theorem, S_f is of measure 0. Observe that if $f(x)$ is continuous at w, then $g(x) = f(\frac{x}{2})$ is continuous at $2w$. Taking contrapositive, if g is discontinuous at $2w$, then f is discontinuous at w. (This means if $2w \in S_g$, then $w \in S_f$.) So we have $S_g \subseteq \{2w : w \in S_f\}$. By the first part of this problem, we have $\{2w : w \in S_f\}$ is of measure 0. Then S_g is of measure 0. By Lebesgue's theorem, g is Riemann integrable.