MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #9

Midterm Review (Part I) (Countable)

Example 1

Determine whether the following are countable or not

a) $A = \{x \in \mathbf{R}: 9\cos^9 x + 3\cos^3 x + 1 = 0\}$

b) $B = \{\pi x^3 + 2y - 3\sqrt{z}: x \in \mathbf{R}\backslash\mathbf{Q}, y \in \mathbf{R}, z \in \mathbf{Q}\}$

c) $C = \{y^2 - 5x: y \in A \text{ and } x \in \mathbf{R}\backslash A\}$ where A is countable set

d) $D = \{(x, y) \in \mathbf{R}^2: x^2 + y^2 = 2 \text{ and } dx^3 - y = 3 \text{ for some integer } d\}$

Solution:

a) Let $S = \{y \in \mathbf{R}: 9y^9 + 3y^9 + 1 = 0\}$

Since the polynomial has degree 9, therefore it has at most 9 real roots, therefore S is countable.

Now we see that $A = \bigcup_{y\in S}\{x \in \mathbf{R}: \cos x = y\}$

For $\{x \in \mathbf{R}: \cos x = y\}$,

If $y > 1 \; or \; y < -1$, then $\{x \in \mathbf{R}: \cos x = y\}$ is an empty set and so countable

If $-1 \leq y \leq 1$, since $\cos x = y \rightarrow x = 2n\pi \pm \cos^{-1} y$, hence

$$\{x \in \mathbf{R}: \cos x = y\} = \{x = 2n\pi \pm \cos^{-1} y : n \in \mathbf{Z}\} = \bigcup_{n\in\mathbf{Z}}\{x = 2n\pi \pm \cos^{-1} y\}$$

which is countable by countable union theorem.

Therefore since $\{x \in \mathbf{R}: \cos x = y\}$ is countable and S is countable, then A is countable by countable union theorem.

b) By intuition, we guess B is uncountable since variable x and y lies in uncountable set ($\mathbf{R}\backslash\mathbf{Q}$ and $\mathbf{R}$)

To show this, we first fix x and z and let $B_y = \{\pi x^3 + 2y - 3\sqrt{z}: y \in \mathbf{R}\}$

we consider a map,

$$f: B_y \rightarrow \mathbf{R}$$

want to show where y lies (Trouble!!)

where $f(\pi x^3 + 2y - 3\sqrt{z}) = y$.

We see that f is bijective because we can find an inverse map $g(y) = \pi x^3 + 2y - 3\sqrt{z}$. So by bijection theorem, B_y is uncountable.

Finally, $B \supseteq B_y$, so by countable subset theorem, B is uncountable

c) **<u>Case i) If A is empty</u>**, then we cannot assign any value to y. Therefore C will be an empty set and therefore countable. **(☹You may lose points if you fail to consider this case!!!!!!!)**

<u>Case ii) If A is non-empty,</u> note that $\mathbf{R}$ is uncountable and A is countable, therefore $\mathbf{R}\backslash \mathrm{A}$ is uncountable. (If $\mathbf{R}\backslash \mathrm{A}$ is countable, then $(\mathbf{R}\backslash \mathrm{A}) \cup \mathrm{A} = \mathbf{R}$ will imply **R** is countable which is a contradiction)

Since variable x lies in $\mathbf{R}\backslash \mathrm{A}$ which is uncountable, we suspect C is uncountable (One can use similar argument in b) to show C is uncountable)

d) We first rewrite

$\mathrm{D} = \{(\mathrm{x},\mathrm{y}) \in \mathbf{R}^2 : \mathrm{x}^2 + \mathrm{y}^2 = 2 \text{ and } \mathrm{d}\mathrm{x}^3 - \mathrm{y} = 3 \text{ for some integer } \mathrm{d}\}$

$= \{(\mathrm{x},\mathrm{y}) \in \mathbf{R}^2 : \mathrm{x}^2 + \mathrm{y}^2 = 2 \text{ and } \mathrm{d}\mathrm{x}^3 - \mathrm{y} = 3 \ \mathrm{d} \in \mathrm{Z}\}$

$= \bigcup_{\mathrm{d}\in\mathrm{Z}} \{(\mathrm{x},\mathrm{y}) \in \mathbf{R}^2 : \mathrm{x}^2 + \mathrm{y}^2 = 2 \text{ and } \mathrm{d}\mathrm{x}^3 - \mathrm{y} = 3\}$

For the set $\{(\mathrm{x},\mathrm{y}) \in \mathbf{R}^2 : \mathrm{x}^2 + \mathrm{y}^2 = 2 \text{ and } \mathrm{d}\mathrm{x}^3 - \mathrm{y} = 3\}$

Note that $\mathrm{d}\mathrm{x}^3 - \mathrm{y} = 3 \rightarrow \mathrm{y} = \mathrm{d}\mathrm{x}^3 - 3 = \mathrm{y}$, substitute it into 1st equation

$\mathrm{x}^2 + \mathrm{y}^2 = 2 \rightarrow \mathrm{x}^2 + (\mathrm{d}\mathrm{x}^3 - 3)^2 = 2 \rightarrow \mathrm{d}^2\mathrm{x}^6 - 6\mathrm{d}\mathrm{x}^3 + \mathrm{x}^2 + 7 = 0$

It has at most 6 real roots for x, therefore (x,y) has at most 6 elements and so the set is countable.

Since $\mathbf{Z}$ is countable, therefore D is countable by countable union theorem.

Example 2 (2004 Midterm)

Let A be a non-empty countable subset of **R.** Let

$$S = \{\theta \in \mathbf{R}: \sin\theta \in A\} \text{ and } T = \{\theta \in \mathbf{R}: \sin\theta \notin A\}$$

Determine (with proof) if each of the sets S and T is countable or uncountable.

(For Set S)

$$S = \{\theta \in \mathbf{R}, \sin\theta \in A\} = \{\theta \in \mathbf{R}: \sin\theta = a, a \in A\} = \bigcup_{a \in A} \{\theta \in \mathbf{R}: \sin\theta = a\}$$

Note that $\sin\theta = a \rightarrow \theta = n\pi + (-1)^n \sin^{-1} a$ for $n \in \mathbf{Z}$

So $\{\theta \in \mathbf{R}: \sin\theta = a\} = \{\theta = n\pi + (-1)^n \sin^{-1} a : n \in \mathbf{Z}\}$

$= \cup_{n \in \mathbf{Z}} \{\theta = n\pi + (-1)^n \sin^{-1} a\}$ which is countable.

Hence S is countable by countable union theorem.

(For Set T)

Since the description of Set T is just the opposite of that of set S. Therefore, $T = \mathbf{R} \backslash S$

Note since R is uncountable and S is countable, then $T = \mathbf{R} \backslash S$ is uncountable.

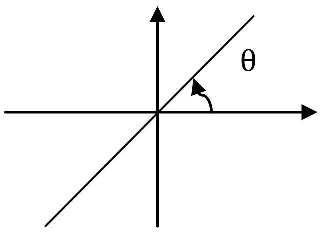
(Otherwise, if $\mathbf{R} \backslash S$ is countable, then $(\mathbf{R} \backslash S) \cup S = \mathbf{R}$ will imply $\mathbf{R}$ is uncountable which is a contradiction)

Example 3

Let P be a countable set of points in R^2. Prove that there exists a line L passing through origin such that every point of the line L is not in P.

Solution:

Let L_θ be a line pass through origin and make an angle θ with x-axis. (See figure)



Let $A = \{L_\theta : 0 \leq \theta < \pi\} = \{L_\theta : \theta \in [0, \pi)\}$ (Collection of all such lines)

We can show **<u>A is uncountable</u>**

Consider a map $f: A \rightarrow [0, \pi)$, where $f(L_\theta) = \theta$

f is bijective because we can find an inverse map $g(\theta) = L_\theta$

Therefore since $[0, \pi)$ is uncountable, then A is uncountable by bijection theorem.

Let $B = \{L_\theta : \text{It passes through some } p \in P\}$ (Collection of all lines which pass through some points in P)

Since P is countable, we can show B is countable by writing

$$B = \bigcup_{p \in P} \{L_\theta : \text{It passes through } p\}$$

And for each p, there is only 1 lines pass through it, so $\{L_\theta : \text{It passes through } p\}$ is countable and **<u>B is countable</u>**.

Then $A \backslash B = \{L_\theta : \text{It does not pass through any point in P}\}$ is uncountable and therefore non-empty, so there exist a line (in $A \backslash B$) such that it does not pass through any points in P.

<u>☺Exercise for Countability</u>

Exercise 1 (Basic Question)

Determine (with proof) the following sets are countable or not.

a) $A = \{x \in \mathbf{R} : e^{3x} + 3e^x - 9 = 0\}$

b) $B = \{7x^2 - 6y + \pi^z : x \in \mathbf{Q} \cap A, y, z \in \mathbf{Q}\}$ where A is an uncountable subset of $\mathbf{R}$

c) $C = \{(x, y, z) \in \mathbf{R}^3, x^2 - z^3 + y^4 = 4, y + 2x = 6 \text{ and } x^3 + 3z = 1\}$

d) $D = \{x^2 - 5y : x \in A, y \in B\}$ where A is an uncountable subset of $\mathbf{R}$ and B is any subset of $\mathbf{R}$. (Be careful, B can be an empty set)

e) $E = \{x - y : x, y \in A\}$ where A is an uncountable set.

f) $F = \{(x, y) : [x] \in \mathbf{N} \text{ and } y \in \mathbf{N}\}$ where [x] is greatest integer less than or equal to x. (i.e. $[7] = 7$, $[7.2] = 7$, $[7.9] = 7$, $[-1] = -1$, $[-1.2] = -2$, $[-1.9] = -2$)

Exercise 2 (2004 Midterm)

Let S be the set of all intersection points $(x, y) \in \mathbf{R}^2$ of the graphs of the equations $x^2 + my^2 = 1$ and $mx^2 + y^2 = 1$, where $m \in \mathbf{Z} \backslash \{-1, 1\}$. Determine if S is countable or uncountable. Provide a proof of your answer.

Exercise 3 (2003 Final)

Let P be a countable set of points in $\mathbf{R}^2$. Prove that there exists a circle C with the origin as center and positive radius such that every point of the circle C is not in P. (Note points inside the circle do not belong to the circle.)

(Hint: If you understand Example 3, this one should be an easy question)

*Exercise 4 (2002 Midterm)

Let S be the set of all lines L on the R^2 such that L passes through 2 distinct points in $\mathbf{Q} \times \mathbf{Q}$ and T be the set of all points, each of which is the intersection of a pair of distinct lines in S. Determine if T is countable set or not.

(Hint: Try to draw some graphs to understand the problem)

Part II Series

Special Series Test

Geometric Series Test, Telescoping Test, p-test

Test for NON-NEGATIVE SERIES

i) integral test

ii) comparison test and limit comparison test

Test for GENERAL SERIES

i) alternating series test

ii) absolute convergence test

iii) root test and ratio test

Example 4

Check whether the following series converges

a) $\sum_{\mathrm{k}=1}^{\infty} \frac{\cos \mathrm{k}\,\pi}{\mathrm{k}^2+2^{\mathrm{k}}}$

b) $\sum_{\mathrm{k}=1}^{\infty} \mathrm{e}^{\sqrt{\mathrm{k}}}/\sqrt{k}$

c) $\sum_{\mathrm{k}=1}^{\infty} \frac{(2\mathrm{k})!}{3^{\mathrm{k}}\mathrm{k}^4}$

d) $\sum_{\mathrm{k}=1}^{\infty} \frac{(\cos \mathrm{k})(\sin 2\mathrm{k})}{2^{\mathrm{k}}}$

e) $\sum_{\mathrm{k}=2}^{\infty} \frac{\sin\left(\frac{1}{\mathrm{k}}\right)}{\ln \mathrm{k}}$

f) $\sum_{\mathrm{k}=1}^{\infty} \cos^{\mathrm{k}}(e^{\frac{1}{k}})$

g) $\sum_{\mathrm{k}=1}^{\infty} \frac{2^{\mathrm{k}}\cos \mathrm{k}}{(\mathrm{k}-1)!}$

Solution:

a) $\sum_{\mathrm{k}=1}^{\infty} \frac{\cos \mathrm{k}\,\pi}{\mathrm{k}^2+2^{\mathrm{k}}} = \sum_{\mathrm{k}=1}^{\infty} \frac{(-1)^{\mathrm{k}}}{\mathrm{k}^2+2^{\mathrm{k}}}$ which is an alternating series

Since $\mathrm{c}_{\mathrm{k}} = \frac{1}{\mathrm{k}^2+2^{\mathrm{k}}}$ is decreasing and $\lim_{\mathrm{k}\to\infty} c_k = 0$.

By alternating series test, $\sum_{\mathrm{k}=1}^{\infty} \frac{\cos \mathrm{k}\,\pi}{\mathrm{k}^2+2^{\mathrm{k}}}$ converges

b) $\lim_{\mathrm{k}\to\infty} \frac{e^{\sqrt{k}}}{\sqrt{k}} = \lim_{k\to\infty} \frac{\frac{1}{2\sqrt{k}}e^{\sqrt{k}}}{\frac{1}{2\sqrt{k}}} = \lim_{k\to\infty} e^{\sqrt{k}} = \infty \neq 0$

Therefore by term test, the series diverges.

(Remark, you may also use integral test to show the series diverges)

c) Since the series involves factorials, we may use ratio test,

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{\dfrac{\big(2(k+1)\big)!}{3^{k+1}(k+1)^4}}{\dfrac{(2k)!}{3^k k^4}} = \lim_{k\to\infty}\frac{k^4(2k+2)(2k+1)}{3(k+1)^4} = \infty > 1$$

By ratio test, the series diverges.

d) Since $\cos k$ and $\sin 2k$ can be negative, so we first consider

$$\sum_{k=1}^{\infty}\left|\frac{(\cos k)(\sin 2k)}{2^k}\right| \le \sum_{k=1}^{\infty}\frac{1}{2^k} \quad \text{since } \cos k, \sin 2k \le 1$$

Note that $\sum_{k=1}^{\infty}\frac{1}{2^k}$ converges by geometric series test $\frac{1}{2} < 1$.

Then $\sum_{k=1}^{\infty}\left|\frac{(\cos k)(\sin 2k)}{2^k}\right|$ converges by comparison test

Finally $\sum_{k=1}^{\infty}\frac{(\cos k)(\sin 2k)}{2^k}$ converges by absolute convergence test.

e) We can apply the limit comparison test

Note $\sin\left(\frac{1}{k}\right) \approx \frac{1}{k}$ which is large, therefore $\frac{\sin\left(\frac{1}{k}\right)}{\ln k} \approx \frac{1}{k\ln k}$ where k is large

$$\lim_{k\to\infty}\frac{\sin\left(\frac{1}{k}\right)}{\ln k}\Big/\frac{1}{k\ln k} = \lim_{k\to\infty}\frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} = 1$$

For series $\sum_{k=2}^{\infty}\frac{1}{k\ln k}$

Let $f(x) = \frac{1}{x\ln x}$ and it is clear that f is decreasing function in $[2, \infty)$ and

$\lim_{x\to\infty}\frac{1}{x\ln x} = 0.$

Then $\int_2^{\infty}\frac{1}{x\ln x}dx = \int_2^{\infty}\frac{1}{\ln x}d(\ln x) = \ln(\ln x)\,\Big|_2^{\infty} = \infty$

Hence by integral test, the series $\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ diverges.

By limit comparison test, $\sum_{k=2}^{\infty}\frac{\sin\left(\frac{1}{k}\right)}{\ln k}$

f) Since the power involves k, we can apply root test in this case

$$\lim_{k\to\infty} \sqrt[k]{\cos^k(e^{\frac{1}{k}})} = \lim_{k\to\infty} \cos\left(e^{\frac{1}{k}}\right) = \cos(e^0) = cos1 = 0.9998 < 1$$

By root test, the series converges

g) *Note: if we use the ratio test directly, then

$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{2\cos(k+1)}{kcosk}$ which may lend to great trouble.

So we should simpify it a little bit.

Consider $\sum_{k=1}^{\infty}\left|\frac{2^k \cos k}{(k-1)!}\right| \le \sum_{k=1}^{\infty}\frac{2^k}{(k-1)!}$

Apply ratio test, $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{2}{k} = 0 < 1$

By ratio test, $\sum_{k=1}^{\infty}\left|\frac{2^k \cos k}{(k-1)!}\right|$ converges

By absolute convergence test, the series $\sum_{k=1}^{\infty}\frac{2^k \cos k}{(k-1)!}$ converges.

☺Exercise 5

Check whether the following series converges or not

(Try to do as many as possible)

a) $\sum_{k=1}^{\infty}\cos\left(\frac{1}{k}\right)\sin\left(\frac{1}{k}\right)\tan\left(\frac{1}{k}\right)$ (simpify it first!!)

b) $\sum_{k=1}^{\infty}\frac{(2k+1)!2^k}{3(k+2)^2}$

c) $\sum_{k=1}^{\infty}\frac{2^k\sqrt{k}}{(2k)!}$ and $\sum_{k=1}^{\infty}(\cos k)\left(\sin\left(\frac{1}{k^2}\right)\right)$ (2002 L1 Midterm)

d) $\sum_{k=1}^{\infty}\frac{(2k+1)^5}{k!}$ and $\sum_{k=1}^{\infty}\frac{\cos k}{k^4+k+1}$ (2002 L2 Midterm)

e) $\sum_{k=1}^{\infty}\sin^k\left(1+\frac{1}{k}\right)$ and $\sum_{k=1}^{\infty}\frac{\left(1-\cos\left(\frac{1}{k}\right)\right)}{\frac{1}{k^2}}$ (2003 Final)

f) $\sum_{k=1}^{\infty}\frac{3^k}{(2k)!k!}$ (2005 Fall Exam)

g) $\sum_{k=1}^{\infty}\frac{(3k)!}{k!2^k}$ (2005 Fall Exam)