MATH 2031 Introduction to Real Analysis

March 12, 2013

Tutorial Note 15

Landau's Big-Oh and Little-Oh Notation (Big-O, Small-O)

(I) **Definition:** Let $c \in \mathbb{R}$ or $c = +\infty$ or $c = -\infty$
Let I be an interval containing c or with c as an endpoint.
Let $f(x)$ and $g(x)$ be functions on I, $g(x) \neq 0$ on $I \setminus \{c\}$

(1) We write $f(x) = O(g(x))$ iff $\exists A \in \mathbb{R}$ such that $\forall x \in I,\ |f(x)| \leq A|g(x)|$ $\left(\text{or } \left|\dfrac{f(x)}{g(x)}\right| \text{ is bounded on } I\right)$.

(2) We write $f(x) = o(g(x))$ as $x \to c$ iff $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = 0$.

(3) We write $f(x) = O^*(g(x))$ as $x \to c$ iff $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = k$ where k is nonzero.

(4) We write $f(x) \sim g(x)$ as $x \to c$ iff $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = 1$.

(II) **Properties:**

(i) $O(g(x)) \pm O(g(x)) = O(g(x))$

(ii) $o(g(x)) \pm o(g(x)) = o(g(x))$

(iii) $O(g_1(x))O(g_2(x)) = O(g_1(x)g_2(x))$

(iv) $\forall p > 0,\ O(g(x))^p = O(g(x)^p)$

(v) $O^*(g_1(x))O^*(g_2(x)) = O^*(g_1(x)g_2(x))$

(vi) $\forall r \in \mathbb{R},\ O^*(g(x))^r = O^*(g(x)^r)$

(vii) $\left.\begin{array}{l} o(g_1(x))o(g_2(x)) \\ o(g_1(x))O^*(g_2(x)) \\ o(g_1(x))O(g_2(x)) \\ o(g_1(x))g_2(x) \end{array}\right\} = o(g_1(x)g_2(x))$

(viii) $o(o(g(x)) = o(g(x))$; $O^*(o(g(x)) = o(g(x))$; $o(O^*(g(x)) = o(g(x))$

(ix) $a < b \Rightarrow O^*(x^a) \pm O^*(x^b) = O^*(x^a)$ as $x \to 0$
$o(x^n) \pm o(x^m) = o(x^{\min\{n,m\}})$ as $x \to 0$

(III) **Useful examples:** As $x \to 0$ we have the following:

(a) $1 - e^x \sim -x$

(b) $\sin x \sim x$

(c) $\ln(1 + x) \sim x$

(d) $\tan x \sim x$

(e) $\arctan x \sim x$

(f) $\arcsin x \sim x$

(IV) **Stolz' Theorem**

Let $b_1, b_2, b_3, \cdots$ be a strictly monotone sequence.

$$\begin{array}{ll} \text{If either} & \lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n \\ \text{or} & \lim_{n\to\infty} b_n = \pm\infty \end{array}$$

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$$

provided that limit on RHS exists as a number or $\pm\infty$.

Problem 1 For a fixed $k \in \mathbb{N}$, prove that

$$1^k + 2^k + \cdots + n^k = \frac{1}{k+1} n^{k+1} + O(n^k).$$

Solution:

Denote $1^k + 2^k + \cdots + n^k$ as $S(n)$.

Consider each term m^k of $S(n)$ as the area of the rectangle with base 1 and height m^k for $m = 1, 2, \cdots, n$.

Then $S(n)$ is the sum of area of all such rectangles, if we compare it with the curve $y = x^k$. We get

$$\begin{pmatrix}\text{Area under the curve } y = x^k \\ \text{for } 0 \le x \le n\end{pmatrix} < S(n) < \begin{pmatrix}\text{Area under the curve } y = x^k \\ \text{for } 1 \le x \le n+1\end{pmatrix}$$

Then

$$\begin{aligned}\begin{pmatrix}\text{Area under the curve } y = x^k \\ \text{for } 0 \le x \le n\end{pmatrix} &= \int_0^n x^k dx \\ &= \left.\frac{x^{k+1}}{k+1}\right|_0^n \\ &= \frac{n^{k+1}}{k+1}\end{aligned}$$

Similarly,

$$\begin{aligned}\begin{pmatrix}\text{Area under the curve } y = x^k \\ \text{for } 1 \le x \le n+1\end{pmatrix} &= \int_1^{n+1} x^k dx \\ &= \left.\frac{x^{k+1}}{k+1}\right|_1^{n+1} \\ &= \frac{(n+1)^{k+1}-1}{k+1} \\ &< \frac{(n+1)^{k+1}}{k+1}\end{aligned}$$

Then the above inequality becomes

$$\frac{n^{k+1}}{k+1} < S(n) < \frac{(n+1)^{k+1}}{k+1}$$

Subtracting $\frac{n^{k+1}}{k+1}$ we get

$$0 < S(n) - \frac{n^{k+1}}{k+1} < \frac{(n+1)^{k+1} - n^{k+1}}{k+1} \tag{1}$$

Next, binomial expansion gives

$$(n+1)^{k+1} = n^{k+1} + \binom{k+1}{1} n^k + \binom{k+1}{2} n^{k-2} + \cdots + \binom{k+1}{n} n + 1$$

so

$$
\begin{aligned}
(n+1)^{k+1} - n^{k+1} &= \binom{k+1}{1} n^k + \binom{k+1}{2} n^{k-2} + \cdots + \binom{k+1}{n} n + 1 \\
&\leq \binom{k+1}{1} n^k + \binom{k+1}{2} n^k + \cdots + \binom{k+1}{n} n^k + n^k \\
&= M n^k \qquad \text{where } M \text{ is a constant depending on } k \text{ only}
\end{aligned}
$$

Thus equation (1) becomes

$$0 < S(n) - \frac{n^{k+1}}{k+1} < \frac{M}{k+1} n^k$$

Therefore, $S(n) = 1^k + 2^k + \cdots + n^k = \dfrac{1}{k+1} n^{k+1} + O(n^k)$.

Problem 2 Check which of the following functions is/are $o(1)$ as $x \to \infty$

$$f(x) = \frac{1}{\sqrt{x}} \qquad g(x) = \frac{1}{\sin x} \qquad h(x) = 2^{\frac{\ln x}{x}}$$

Solution:

(i) Since $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{x}} = 0$, $f(x) = o(1)$.

(ii) As $x \to \infty$, $\sin x$ is oscillating and so is $g(x) = \frac{1}{\sin x}$.
Thus $\lim\limits_{x\to\infty} g(x)$ does not exist, and therefore $g(x)$ is not $o(1)$.

(iii) Consider $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} = \lim\limits_{x\to\infty} \dfrac{1}{x} = 0$, so $\lim\limits_{x\to\infty} h(x) = \lim\limits_{x\to\infty} 2^{\frac{\ln x}{x}} = 2^0 = 1$. Thus $h(x)$ is not $o(1)$. ($h(x) \sim 1$)

Problem 3 Determine the value of the limit $\lim\limits_{n\to\infty} \dfrac{1 - e^{\frac{1}{n}}}{\sin\left(\frac{1}{n}\right)}$.

Solution:
Since for $0 < x < \frac{\pi}{2}$, $\sin x$ is strictly monotone, then $\sin\left(\frac{1}{n}\right)$ is a strictly monotone sequence. Also,

$$\lim_{n\to\infty} 1 - e^{\frac{1}{n}} = 1 - 1 = 0 \qquad \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 0,$$

so we may apply the Stolz' theorem,

$$\lim_{n\to\infty} \frac{1 - e^{\frac{1}{n}}}{\sin\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{e^{\frac{1}{n}} - e^{\frac{1}{n+1}}}{\sin\left(\frac{1}{n+1}\right) - \sin\left(\frac{1}{n}\right)}$$

Since both $\sin x$ and e^x are differentiable, by mean value theorem, there exist $c, d \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$ such that

$$e^{\frac{1}{n}} - e^{\frac{1}{n+1}} = e^c \left(\frac{1}{n} - \frac{1}{n+1}\right) \qquad \sin\frac{1}{n+1} - \sin\frac{1}{n} = \cos d \left(\frac{1}{n+1} - \frac{1}{n}\right).$$

Then we get

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1 - e^{\frac{1}{n}}}{\sin\left(\frac{1}{n}\right)} &= \lim_{n\to\infty} \frac{e^{\frac{1}{n}} - e^{\frac{1}{n+1}}}{\sin\left(\frac{1}{n+1}\right) - \sin\left(\frac{1}{n}\right)} \\
&= -\lim_{n\to\infty} \frac{e^c}{\cos d}
\end{aligned}
$$

As both $\cos x$ and e^x are positive on $(0,1)$, we have $e^{\frac{1}{n+1}} \leq e^c \leq e^{\frac{1}{n}}$ and $\dfrac{1}{\cos\frac{1}{n+1}} \leq \dfrac{1}{\cos d} \leq \dfrac{1}{\cos\frac{1}{n}}$.

Then

$$1 = \lim_{n\to\infty} \frac{e^{\frac{1}{n+1}}}{\cos\frac{1}{n+1}} \leq \lim_{n\to\infty} \frac{e^c}{\cos d} \leq \lim_{n\to\infty} \frac{e^{\frac{1}{n}}}{\cos\frac{1}{n}} = 1$$

By sandwich theorem, we have $\lim\limits_{n\to\infty} \dfrac{1 - e^{\frac{1}{n}}}{\sin\left(\frac{1}{n}\right)} = \lim\limits_{n\to\infty} \dfrac{e^c}{\cos d} = 1$.

Problem 4 For a fixed $k \in \mathbb{N}$, compute the following limit.

$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}}$$

Solution 1: With the result from Problem 1, we have

$$\frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} = \frac{1}{k+1} + \frac{O(n^k)}{n^{k+1}} = \frac{1}{k+1} + \frac{O(1)}{n}$$

Then taking limit on both sides, as $n \to \infty$, we get

$$\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} = \frac{1}{k+1}$$

Solution 2: Without the help of Problem 1, we may consider Stolz' theorem.
As n^{k+1} is strictly monotone and approaching ∞, as $n \to \infty$, by Stolz' theorem

$$\begin{aligned}
\lim_{n\to\infty} \frac{1^k + 2^k + \cdots + n^k}{n^{k+1}} &= \lim_{n\to\infty} \frac{(1^k + 2^k + \cdots + n^k + (n+1)^k) - (1^k + 2^k + \cdots + n^k)}{(n+1)^{k+1} - n^{k+1}} \\
&= \lim_{n\to\infty} \frac{(n+1)^k}{(n+1)^{k+1} - n^{k+1}} \\
&= \lim_{n\to\infty} \frac{1}{(n+1) - n^{k+1}(n+1)^{-k}} \\
&= \lim_{n\to\infty} \frac{n^{-1}}{1 + n^{-1} - (1+n^{-1})^{-k}} \\
&= \lim_{n\to\infty} \frac{-n^{-2}}{-n^{-2} - k(1+n^{-1})^{-k-1} - n^{-2}} \qquad \text{(L'Hôpital's rule)} \\
&= \lim_{n\to\infty} \frac{1}{1 + k(1+n^{-1})^{-k-1}} \\
&= \frac{1}{k+1}
\end{aligned}$$