Lecture 8 05-03-2019

Review :

① Four Axioms for the theory of real numbers

I: Field Axiom ; II: Order Axiom ; III: Well-ordering Axiom ;

IV : Completeness Axiom

② Supremum / Infimum property : $\forall \varepsilon > 0$, $\exists\ x \in S$ s.t

$$\operatorname{Sup} S - \varepsilon < x \leq \operatorname{Sup} S \quad (\text{or } \operatorname{Inf} S \leq x < \operatorname{Inf} S + \varepsilon)$$

③ Theorems derived from the Four Axioms

(1) Archimedean Principal : $\forall\ x \in \mathbb{R}$, $\exists\ n \in \mathbb{N}$ s.t $x < n$

(2) A Lemma : $\forall\ x \in \mathbb{R}$, $\exists$ integer number $[x]$ s.t

$$[x] \leq x < [x] + 1$$

(3) Density of $\mathbb{Q}$: $\forall\ x < y$, $\exists\ q \in \mathbb{Q}$ s.t $x < q < y$

(4) Density of $\mathbb{R} \setminus \mathbb{Q}$, $\forall\ x < y$, $\exists\ r \in \mathbb{R} \setminus \mathbb{Q}$ s.t $x < r < y$.

A historic note

1. The idea of using Axiom system to represent math theory dates back to the "Elements" by Euclid 300BC. It has proven instrumental in the development of logic and Mordern science. Its logic rigor was not surpassed until the 19th century.

2. Mathematicians and philosophers, such as Thomas Hubbes, Baruch Spinoza, Bertrand Russell, have attempted to creat their own foundamental "Elements" for their respective disciplines.

3. In 1899, D. Hilbert "moderized" the axioms of Euclid, and proposed a form set of "Hilbert's axioms" in the book "Foundations of Geometry". (his axioms were modified several times. This leads to his grand axiomatic program to treat mathematics, which was unfortunately turned down by Gödel in 1931.

Chapter 5: Limit (ε-δ)

Def: We say a sequence $x_1, x_2, \cdots$ is in S iff every term $x_1, x_2, \cdots$ is an element of the set S.

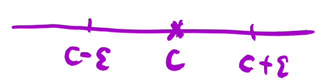
A sequence $x_1, x_2, \cdots$ in $\mathbb{R}$ is bounded above / below iff the set $\{x_1, x_2, x_3, \cdots\}$ is bounded above / below in $\mathbb{R}$.

Notation: we denote the sequence $x_1, x_2, \cdots$ by $\{x_n\}$.

Warning: the sequence $\{x_n\} \neq$ the set $\{x_1, x_2, \cdots\} = \{x_n : n \in \mathbb{N}\}$

Def: $\forall\, x, y \in \mathbb{R}$, We define the distance between x and y, denoted by $d(x,y)$, to be: $d(x,y) = |x-y|$.

Let $c \in \mathbb{R}$, $\forall\, \varepsilon > 0$, a ε-neighborhood of c is defined to be the set



$$\{x : x \in \mathbb{R},\ d(x,c) < \varepsilon\} = \{x : |x-c| < \varepsilon,\ x \in \mathbb{R}\} = (c-\varepsilon,\ c+\varepsilon)$$

Intuitive meaning of limit of sequence :

" $\lim_{n\to\infty} x_n = c$ " means : ① x_n 's are closer and closer to c as n gets larger and larger

② the distance $d(x_n, c)$ goes to zero as n tends to infinity.

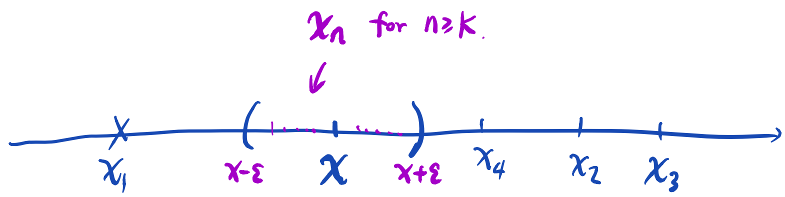
③ using idea of infinitesimal principal, $d(x_n, c)$ is arbitarily small when $\boxed{n \text{ is sufficient large}}$. More precisely,

$\forall \varepsilon > 0$, $d(x_n, c) < \varepsilon$, for n sufficiently large,

[or $\boxed{n \geq K \text{ for some } K \in \mathbb{N}}$.]

This leads to the following definition

Def : We say that a sequence $x_1, x_2, x_3, \ldots$ converges to a number x (or has a limit x), denoted by $\lim_{n\to\infty} x_n = x$, iff $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ (depending on ε) such that $|x_n - x| < \varepsilon$ for all $n \geq K$. [or equivalently, $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$, s.t $n \geq K \Rightarrow |x_n - x| < \varepsilon$]



R.K. ① $|x_n - x| < \varepsilon$ for all $n \geq K$ $\Leftrightarrow$

$x_n \in (x - \varepsilon, x + \varepsilon)$ for all $n \geq K$.

② For a given ε, it may be not easy to compute K explicitly. However, we only need to show the existence of K.

Example ①. We check the definition of limit on the sequence $x_n = \frac{1}{n}$, $n=1, 2, 3, \cdots$.

Intuitively, we should have $\lim x_n = 0$

$\forall\ \varepsilon > 0$, we need to find $K \in \mathbb{N}$ s.t

$|x_n - 0| < \varepsilon$ for all $n \geq K$.

i.e $\frac{1}{n} < \varepsilon$ for all $n \geq K$

or equivalently $n > \frac{1}{\varepsilon}$ for $n \geq K$.

We may take $K = [\frac{1}{\varepsilon}] + 1$. Therefore $\forall\ \varepsilon > 0$

$\exists\ K = [\frac{1}{\varepsilon}] + 1$ s.t $n \geq K \Rightarrow |\frac{1}{n} - 0| < \varepsilon$. So $\lim_{n\to\infty} \frac{1}{n} = 0$.

R.K : $\varepsilon = 0.1$, we may take $K = 10$

$\varepsilon = 0.01$, we take $K = 100$

$\varepsilon = 10^{-10}$, we take $K = 10^{10}$.

Example ② Let $x_n = c$, show that $\lim_{n\to\infty} x_n = c$.

proof: $\forall\, \varepsilon > 0$, let $K = 1$. Then $\forall\, n \geq K$

$$|x_n - c| = 0 < \varepsilon.$$

$$\therefore \quad \lim_{n\to\infty} x_n = c$$

Example ③ $x_n = \dfrac{n}{\cos n - n}$. Show that $\lim_{n\to\infty} x_n = -1$

proof: $\forall\, \varepsilon > 0$, we need to find $K \in \mathbb{N}$ s.t

$$|x_n - (-1)| < \varepsilon \quad \text{for all } n \geq K.$$

$$|x_n - (-1)| = \left|\frac{n}{\cos n - n} + 1\right| = \left|\frac{\cos n}{\cos n - n}\right| \overset{?}{<} \varepsilon$$

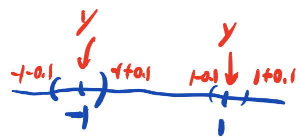
Note that $\left|\dfrac{\cos n}{\cos n - n}\right| < \dfrac{1}{n-1}$ for $n \geq 2$

We only need to have $\dfrac{1}{n-1} < \varepsilon$. or $n-1 > \dfrac{1}{\varepsilon}$ or $n \geq 1 + [\frac{1}{\varepsilon}] + 1$

Then, $\forall\, \varepsilon > 0$, let $K = [\frac{1}{\varepsilon}] + 2$, then for $n \geq K$,

$$|x_n - (-1)| = \left|\frac{\cos n}{\cos n - n}\right| < \frac{1}{n-1} < \varepsilon. \quad \text{So } \lim_{n\to\infty} x_n = -1.$$

Example ④ $y_n = (-1)^n$. Show that $\{y_n\}$ does not converge.



Proof. By contradiction, assume that $\{y_n\}$ converges to y for some $y \in \mathbb{R}$.

Then for $\varepsilon = 0.1$, $\exists\ K \in \mathbb{N}$ s.t $|y_n - y| < \varepsilon = 0.1$ for all $n \geq K$.

i.e $|(-1)^n - y| < 0.1$ for all $n \geq K$.

For n even and $n \geq K$, we have $|1-y| < 0.1$, i.e $y \in (0.9, 1.1)$

For n odd and $n \geq K$, we have $|-1-y| < 0.1$, i.e $y \in (-1.1, -0.9)$

Such y does not exist since $(-1.1, -0.9) \cap (0.9, 1.1) = \phi$.

So $\{y_n\}$ does not converge.

A useful remark.

The following are equivalent:

① $\{x_n\}$ converges to x

② $\forall\, \overset{M\varepsilon}{\boxed{\varepsilon}} > 0$, $\exists\, K \in \mathbb{N}$, s.t $n \geq K$

$\Rightarrow |x_n - x| < \varepsilon$.

③ $\forall\, \varepsilon > 0$, the number of x_n's such that

$|x_n - x| \geq \varepsilon$ is finite. [Exercise]

④ $\exists\, M > 0$, $\forall\, \varepsilon > 0$, $\exists\, K \in \mathbb{N}$ s.t $n \geq K$

$\Rightarrow |x_n - x| < M\varepsilon$ [Exercise]

Uniqueness of limit

THM: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$.

then $x = y$. [This justifies that the notation

$\lim_{n\to\infty} x_n$ is a well-defined number]

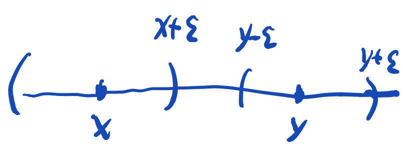
Proof: By contradiction, assume that $x \neq y$.

Since $\lim_{n\to\infty} x_n = x$, for $\varepsilon = \dfrac{|x-y|}{3}$, $\exists\ K_1 \in \mathbb{N}$ s.t

$|x_n - x| < \varepsilon$ for $n \geq K_1$

Since $\lim_{n\to\infty} x_n = y$, for $\varepsilon = \dfrac{|x-y|}{3}$, $\exists\ K_2 \in \mathbb{N}$ s.t $|x_n - x| < \varepsilon$.

for $n \geq K_2$. Then for $n \geq \operatorname{Max}(K_1, K_2)$, $|x_n - x| < \varepsilon$, $|x_n - y| < \varepsilon$.



Since $\varepsilon = \dfrac{|x-y|}{3}$,

$(x-\varepsilon, x+\varepsilon) \cap (y-\varepsilon, y+\varepsilon) = \phi$. There exists no common element

in both the ε-neighborhood of x and y. This contradicts to the conclusion that

$|x_n - x| < \varepsilon$ and $|x_n - y| < \varepsilon$ for all $n \geq K$.

Equivalence of limit

Remark : The following are equivalent

① $\{x_n\}$ Converges to x $\left(\lim_{n\to\infty} x_n = x\right)$

$\forall \varepsilon > 0,\ \exists K \in \mathbb{N}$ s.t $n \geq K \Rightarrow |x_n - x| < \varepsilon$.

② $\{x_n - x\}$ Converges to 0 $\left(\lim_{n\to\infty} (x_n - x) = 0\right)$

$\forall \varepsilon > 0,\ \exists K \in \mathbb{N}$ s.t $n \geq K \Rightarrow \underbrace{|x_n - x - 0|}_{\Downarrow \; |x_n - x|} < \varepsilon$

③ $\{|x_n - x|\}$ Converges to 0 $\left(\lim_{n\to\infty} |x_n - x| = 0\right)$

$\forall \varepsilon > 0,\ \exists K \in \mathbb{N}$ s.t $n \geq K \Rightarrow \underbrace{\big||x_n - x| - 0\big|}_{\parallel \; |x_n - x|} < \varepsilon$

Boundedness Theorem

THM: If $\{x_n\}$ converges, then the set $\{x_1, x_2, \cdots\}$ is bounded

Proof: Let $x = \lim_{n \to \infty} x_n$. For $\varepsilon = 1$, $\exists\ K \in \mathbb{N}$, s.t

$n \geq K \Rightarrow |x_n - x| < 1$

$\Rightarrow |x_n| < |x| + |x_n - x| < |x| + 1$

Let $M = \text{Max}(|x_1|, |x_2|, \cdots |x_{K-1}|, 1+|x|)$

Then $|x_n| \leq M$ for $n \leq K-1$. and

$|x_n| < 1+|x| \leq M$ for $n \geq K$.

As a result $|x_n| \leq M$ for all $n \geq 1$. $\Rightarrow$ $\{x_1, x_2, \cdots\}$ is bounded.

R.K: The converse of boundedness theorem is FAlSE.

Counterexample: $x_n = (-1)^n$.