# MATH2033 Mathematical Analysis

## Lecture Note 5 (Part 1)

## Limits of sequence

## Introduction

Roughly speaking, an (infinite) sequence of real number is a list of elements (denoted by $\{x_1, x_2, x_3, \ldots\}$ or $\{x_n\}$) of $\mathbb{R}$ in a specific order. Some examples of the sequence are:

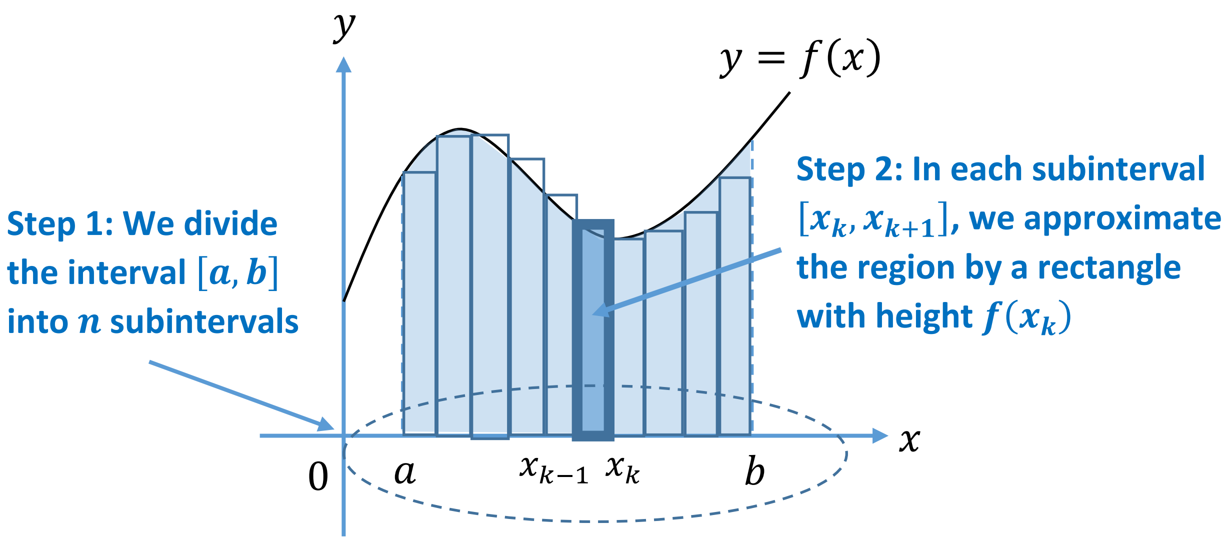
1. **Arithmetic sequence**: $\{x_1, x_2, x_3, x_4 \ldots\} = \{1, 4, 7, 10, \ldots\}$ (or $x_n = 3n - 2$ for $n = 1,2,3, \ldots$)
2. **Geometric sequence**: $\{x_1, x_2, x_3, x_4, \ldots\} = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots\right\}$ (or $x_n = \frac{1}{2^n}$ for $n = 1,2,3, \ldots$)
3. **Sequence defined by recurrence relation**: $x_1 = 2$, $x_2 = 3$ and $x_{n+2} = 2x_{n+1} - x_n$ for $n = 1,2,3, \ldots$ (This implies $x_3 = 4$, $x_4 = 5$, $x_5 = 6$, …)

Given a sequence of real numbers, one important issue is to examine the value of $x_n$ when $n$ is very large?

Mathematically, the limits of $x_n$, denoted by $\lim_{n\to\infty} x_n$, describes the *limiting behavior* of $x_n$. It has a wide application in different aspects.

## Application 1: Estimating the area of irregular region

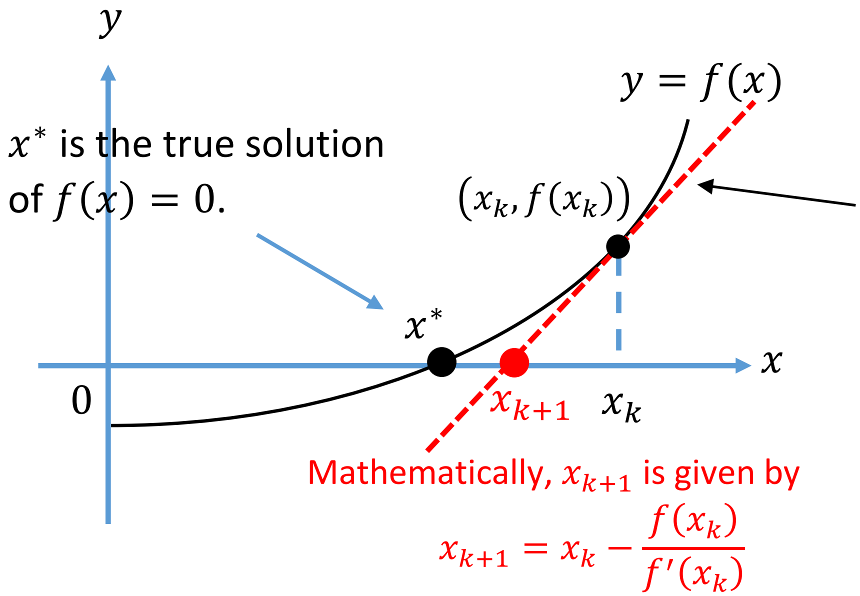
We would like to find the area under the curve $y = f(x)$ over the interval $[a, b]$. Then one can estimate the area as follows:




- The area of the region is approximated by $I_n = \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1})$.
- Then the area of the region can be obtained by considering the limit $\lim_{n\to\infty} I_n$. For example, if $f(x) = x$ and $x_{k-1} = a + k\left(\frac{b-a}{n}\right)$. Then $\lim_{n\to\infty} I_n = \lim_{n\to\infty} \sum_{k=1}^{n} \left(a + (k-1)\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right) = \cdots = \lim_{n\to\infty}\left[a(b-a) + \frac{(b-a)^2(n-1)}{2n}\right] = \frac{b^2-a^2}{2}$.

**Application 2: Finding the approximation of the solution of equation – Newton's method.**

As we have learnt in MATH1013, Newton's method is a numerical algorithm that try to obtain numerical solution of the equation $f(x) = 0$ using graphical method.




Given $x_k$, the method generate a better guess by considering the tangent line at $x = x_k$. The tangent line passes through the $x$-axis at $x_{k+1}$ which is a better guess (closer to $x^*$)

Given the initial guess $x_0$, the method generates a sequence of iterates $\{x_1, x_2, x_3, \dots\}$ which are getting closer to $x^*$ when $n \to \infty$. Thus, we expect that $\lim_{n\to\infty} x_n = x^*$ and $x_n$ (with large $n$) can approximate the solution of $x^*$.

## A formal definition of limits

We let $\{x_n\}$ be a sequence of real numbers. Intuitively, we say $x_n$ converges to a real number $L \in \mathbb{R}$ (i.e. $\lim_{n\to\infty} x_n = L$) if and only if $x_n$ is getting closed to $L$ when $n$ is sufficiently large.

However, this definition is not satisfactory since the words "close" and "sufficiently large" are not clearly defined. Hence, it is necessary to specify the meaning of "close" and "large":

- We let $\varepsilon > 0$ be a positive number, we say $x_n$ is close to $L$ if $|x_n - L| < \varepsilon$.
- We let $K \in \mathbb{N}$ be a positive integer, we say $n$ is sufficiently large if $n \geq K$.

Next, we use these "definitions" to rephrase the definition of limits.

As an example, we consider the sequence defined by $x_n = \frac{1}{n^2}, n = 1,2,3, \ldots$.

- It is clear that the value of $\frac{1}{n^2}$ is getting close to $0$ when $n$ is getting large (or $n \to \infty$). So we expect that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$.

- For any $\varepsilon > 0$, we note that $\underbrace{|x_n - 0| < \varepsilon}_{x_n \text{ is close to } 0} \Leftrightarrow \frac{1}{n^2} < \varepsilon \Leftrightarrow n > \frac{1}{\sqrt{\varepsilon}}$. By picking $K$ to be the smallest positive integer which $K > \frac{1}{\sqrt{\varepsilon}}$ (guaranteed by Archimedean property and well-ordering property), we see that $x_n$ is close to $0$ (i.e. $|x_n - 0| < \varepsilon$) if $n$ is sufficiently large in the sense that $n \geq K$.

As inspired by the above example, we can obtain the mathematical definition of limits as follows:

**Definition (Limits of sequence)**

A sequence $\{x_n\}$ converges to a real number $x \in \mathbb{R}$ ($\lim_{n\to\infty} x_n = x$) if and only if for any $\varepsilon > 0$, there exists a positive integer $K \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ for all $n \geq K$.

*Remark about the definition*

- Since we expect that $x_n$ should be arbitrary close to $x$ when $n$ is large, so the statement should hold for <u>all</u> $\varepsilon > 0$.
- In general, <u>the value of $K$ depends on the value of $\varepsilon$ chosen</u> (i.e. $K = K(\varepsilon)$). From the above example, we see $K = 11$ if $\varepsilon = 0.01$ and $K = 51$ if $\varepsilon = 0.0004$.

**Example 1**

We let $x_n = 2^{\frac{2n+3}{n}}$ for $n \in \mathbb{N}$, show that $\lim_{n\to\infty} x_n = 4$ using definition of limits.

😊Solution

For any $\varepsilon > 0$, we note that

$$|x_n - 4| < \varepsilon \overbrace{\Leftrightarrow}^{x_n \geq 4} 2^{\frac{2n+3}{n}} - 4 < \varepsilon \Leftrightarrow 4\left(2^{\frac{3}{n}} - 1\right) < \varepsilon \Leftrightarrow n > \frac{3\ln 2}{\ln\left(1 + \frac{\varepsilon}{4}\right)}.$$

We pick $K = \left\lfloor \frac{3\ln 2}{\ln\left(1+\frac{\varepsilon}{4}\right)} \right\rfloor + 1$, then we see from the above inequality that

$$n \geq K > \frac{3\ln 2}{\ln\left(1 + \frac{\varepsilon}{4}\right)} \Rightarrow |x_n - 4| < \varepsilon.$$

So we conclude that $\lim_{n\to\infty} x_n = 4$ by definition of limits.

**Example 2 (Harder)**

**(a)** Show that $\lim\limits_{n\to\infty} \frac{\cos n}{n} = 0$ using definition of limits.

**(b)** We let $\{x_n\}$ be a sequence which $\lim\limits_{n\to\infty} x_n = 2$. Show that $\lim\limits_{n\to\infty} x_n^2 = 4$ using the definition of limits.

Solution

**(a)** For any $\varepsilon > 0$, we pick $K = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1$, then for any $n \geq K > \frac{1}{\varepsilon}$, we have

$$\left|\frac{\cos n}{n} - 0\right| \overset{|\cos n| \leq 1}{\leq} \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon.$$

So $\lim\limits_{n\to\infty} \frac{\cos n}{n} = 0$ by definition.

**(b)** For any $\varepsilon > 0$ and since $\lim\limits_{n\to\infty} x_n = 2$, there exists $K_1 \in \mathbb{N}$ and $K_2 \in \mathbb{N}$ such that

- $|x_n - 2| < 1 \Leftrightarrow 1 < x_n < 3$ for all $n \geq K_1$ ... ... $(*)$
- $|x_n - 2| < \frac{\varepsilon}{5}$ for all $n \geq K_2$ ... ... $(**)$

We pick $K = \max(K_1, K_2)$, then for any $n \geq K$ (which $(*)$ and $(**)$ hold),

$$|x_n^2 - 4| = |x_n + 2||x_n - 2| \overset{(*)}{<} 5|x_n - 2| \overset{(**)}{<} 5\left(\frac{\varepsilon}{5}\right) = \varepsilon.$$

Thus $\lim\limits_{n\to\infty} x_n^2 = 4$ by definition.

**Example 3**

We let $x_n = e^n$ for $n \in \mathbb{N}$, show that $\{x_n\}$ does not converge to any real number.

😊Solution

One can prove this by contradiction. Suppose that $\lim\limits_{n\to\infty} x_n = x$ for some $x \in \mathbb{R}$, then for a fixed $\varepsilon_0 > 0$, there exists $K \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon_0 \Leftrightarrow x - \varepsilon_0 < x_n < x + \varepsilon_0 \quad for \;\; n \geq K \ldots (*)$$

On the other hand, we deduce from Archimedean property that there exists $K_1 \in \mathbb{N}$ such that $K_1 > \ln(x + \varepsilon_0)$. This implies

$$x_n = e^n \geq e^{K_1} > x + \varepsilon_0 \quad for \;\; all \; n \geq K_1 \ldots (**)$$

By choosing $n \geq \max(K, K_1)$, we deduce from the equation inequalities that

$$x_n \quad \overbrace{<}^{from\,(*)} \quad x + \varepsilon_0 \quad and \quad x_n \quad \overbrace{>}^{from\,(**)} \quad x + \varepsilon_0.$$

This leads to contradiction and the sequence $\{x_n\}_{n\in\mathbb{N}}$ does not converge to any real number.

*Remark of Example 3*

- Alternatively, one can prove the statement by considering the *negation* of the definition of limits. That is,
  "$x_n$ *does not* converge to $x$ if and only if there exists $\varepsilon > 0$ such that for any positive integer $K$, there exists $n \geq K$ such that $|x_n - x| > \varepsilon$."
- In this example, we observe the value of $x_n$ approaches to $\infty$ when $n$ is getting larger. We can still define $\lim_{n\to\infty} x_n = +\infty$ although we cannot describe its limits by real number.
- The formal definition of $\lim_{n\to\infty} x_n = +\infty$ is given as follows. The limits $\lim_{n\to\infty} x_n = -\infty$ can be defined in a similar manner.

**Definition (Limits to infinity)**

A sequence $\{x_n\}$ diverges to a positive infinity ($\lim_{n\to\infty} x_n = +\infty$) if and only if for any $M > 0$, there exists a positive integer $K \in \mathbb{N}$ such that $x_n > M$ for any $n \geq K$.

**Example 4**

Show that the sequence $\{x_n\}$ defined by $x_n = (-1)^n$ for $n \in \mathbb{N}$ does not converge to any real number.

😊Solution

Suppose that the sequence converges to $x \in \mathbb{R}$ which $\lim_{n\to\infty} x_n = x$.

We pick $\varepsilon = 0.5$ (or any number less than 1), there exists $K \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon = 0.5 \quad for\ any\ n \geq K.$$

It follows that for any $n \geq K$, we have

$$|x_{n+1} - x_n| \leq |x_n - x| + |x - x_{n+1}| < 0.5 + 0.5 = 1.$$

However, one can deduce from direct calculation that

$$|x_{n+1} - x_n| = |(-1)^n[(-1) - 1]| = 2$$

which contradicts to the fact that $|x_{n+1} - x_n| < 1$. Hence, we conclude that the sequence $\{x_n\}$ does not converge.

**Some basic properties of convergent sequence**

In this section, we present several properties of convergent sequence which are useful for computation and doing proof.

**Property 1: Uniqueness of limits**

We let $\{x_n\}$ be a sequence. Suppose that $\{x_n\}$ converges to $x$ and $\{x_n\}$ converges to $y$, then $x = y$.

(*In other words, the limits of convergent sequence must be unique.)

*Proof*

For any $\varepsilon > 0$, since $\{x_n\}$ converges to $x$ and $\{x_n\}$ converges to $y$, there exists $K_1 \in \mathbb{N}$ and $K_2 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{2} \ \ for\ all\ n \geq K_1 \ \ and \ \ |x_n - y| < \frac{\varepsilon}{2} \ \ for\ all\ n \geq K_2.$$

By taking $n = \max(K_1, K_2)$, we deduce that

$$|x - y| \leq |x - x_n| + |x_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since the inequality holds for any $\varepsilon > 0$, it follows from infinitesimal property that $|x - y| \leq 0 \Rightarrow |x - y| = 0 \Rightarrow x = y$.

**Example 5 (Another example on the use of infinitesimal property)**

We let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences which $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Suppose that $\lim_{n\to\infty}(x_n - y_n) = 0$, prove that $x = y$.

😊Solution

Our goal is to show $|x - y| = 0$ using infinitesimal property.

For any $\varepsilon > 0$, note that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty}(x_n - y_n) = 0$, there exists $K_1 \in \mathbb{N}$, $K_2 \in \mathbb{N}$ and $K_3 \in \mathbb{N}$ such that

- $|x_n - x| < \frac{\varepsilon}{3}$ for $n \geq K_1$;
- $|y_n - y| < \frac{\varepsilon}{3}$ for $n \geq K_2$ and
- $|x_n - y_n| < \frac{\varepsilon}{3}$ for $n \geq K_3$

By choosing $n$ such that $n \geq \max(K_1, K_2, K_3)$ and using triangle inequality, we have

$$\begin{aligned}|x - y| &= |(x - x_n) + (x_n - y_n) + (y_n - y)| \leq |x - x_n| + |x_n - y_n| + |y_n - y| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Since $|x - y| < \varepsilon$ for all $\varepsilon > 0$, it follows from infinitesimal property that $|x - y| = 0 \Rightarrow x = y$.

**Property 2: Boundness theorem (Useful fact in doing proof)**

If the sequence $\{x_n\}$ converges to some $x \in \mathbb{R}$, then $\{x_n\}$ is bounded.

*Proof of property 2*

Since $\{x_n\}$ is convergent, we let $\lim_{n\to\infty} x_n = x$.

We pick a small positive number $\varepsilon_0 > 0$. Then there is $K \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon_0 \Leftrightarrow x - \varepsilon_0 < x_n < x + \varepsilon_0 \quad for \ \ all\, n \geq K.$$

Next, we take $L = \min(x_1, x_2, \dots, x_{K-1}, x - \varepsilon_0)$ and $U = \max(x_1, x_2, \dots, x_{K-1}, x + \varepsilon_0)$.
One can see that $L \leq x_n \leq U \ \ for \ \ all\, n \in \mathbb{N}$.

So $\{x_n\}_{n\in\mathbb{N}}$ is both bounded above and bounded below and hence is bounded.

*Remark*

- By taking sufficiently value of $M$ such that $[L, U] \subseteq [-M, M]$, one can see that $-M \leq x_n \leq M \Rightarrow |x_n| \leq M$.
- By taking the contrapositive of the statement, we have "if $\{x_n\}$ is unbounded (not bounded from above or not bounded from below), then $\{x_n\}$ does not converge."
- The converse of this property is *false*. For example, $x_n = (-1)^n$.

**Property 3: Computational formula for limits (Useful fact in doing proof)**

We let $\{x_n\}$ and $\{y_n\}$ be two sequences which $\{x_n\}$ converges to $x \in \mathbb{R}$ and $\{y_n\}$ converges to $y \in \mathbb{R}$. Then

**(1)** $\{x_n \pm y_n\}$ converges and $\lim_{n\to\infty}(x_n \pm y_n) = \lim_{n\to\infty} x_n \pm \lim_{n\to\infty} y_n = x \pm y$.

**(2)** $\{x_n y_n\}$ converges and $\lim_{n\to\infty} x_n y_n = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right) = xy$.

**(3)** Provided that $y_n \neq 0$ for all $n \in \mathbb{N}$ and $y \neq 0$, then $\left\{\frac{x_n}{y_n}\right\}$ converges and

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n} = \frac{x}{y}.$$

*Proof of property 3*

**(1)** We prove the case for $\{x_n + y_n\}$. The case for $\{x_n - y_n\}$ can be proved in a similar manner.

For any $\varepsilon > 0$, note that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, there exists $K_1 \in \mathbb{N}$ and $K_2 \in \mathbb{N}$ such that

- $|x_n - x| < \frac{\varepsilon}{2}$ for any $n \geq K_1$.
- $|y_n - y| < \frac{\varepsilon}{2}$ for any $n \geq K_2$.

By taking $K = \max(K_1, K_2)$, then for any $n \geq K$, we have

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\lim_{n\to\infty} (x_n + y_n) = x + y$ by the definition of limits.

**(2)** Since both $\{x_n\}$ and $\{y_n\}$ are convergent, so both $\{x_n\}$ and $\{y_n\}$ are bounded. We can write $|x_n| \leq M_x$ and $|y_n| \leq M_y$ for some positive constants $M_x$ and $M_y$. For any $\varepsilon > 0$, note that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, there exists $K_1 \in \mathbb{N}$ and $K_2 \in \mathbb{N}$ such that

- $|x_n - x| < \frac{\varepsilon}{2\max(|y|, M_y)}$ for any $n \geq K_1$ (provided that $|y| \neq 0$).
- $|y_n - y| < \frac{\varepsilon}{2M_x}$ for any $n \geq K_2$.

By picking $K = \max(K_1, K_2)$, then for any $n \geq K$, we get

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| = |x_n||y_n - y| + |y||x_n - x| \\ &\leq M_x |y_n - y| + \max(|y|, M_y)\, |x_n - x| \\ &< M_x \left(\frac{\varepsilon}{2M_x}\right) + \max(|y|, M_y) \left(\frac{\varepsilon}{2\max(|y|, M_y)}\right) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $\lim_{n\to\infty} x_n y_n = xy$.

**(3)** It suffices to prove the sequence $\left\{\frac{1}{y_n}\right\}$ converges to $\frac{1}{y}$. Then if follows from (2) that $\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty} x_n\left(\frac{1}{y_n}\right) = x\left(\frac{1}{y}\right) = \frac{x}{y}$.

- Since $\lim_{n\to\infty} y_n = y \neq 0$ (so that $|y| > 0$), then there exists $K_1 \in \mathbb{N}$ such that
$$|y_n - y| < \varepsilon_0 = \frac{|y|}{2} \Rightarrow^{(*)} |y| - |y_n| < \frac{|y|}{2} \Rightarrow |y_n| > \frac{|y|}{2} \quad for\ any\ n \geq K_1.$$
(*Note: This step follows from the fact that
$$|y| \leq |y - y_n| + |y_n| \Rightarrow |y| - |y_n| \leq |y - y_n|.)$$
- On the other hand, there exists $K_2 \in \mathbb{N}$ such that
$$|y_n - y| < \frac{|y|^2}{2}\varepsilon.$$
- By picking $K = \max(K_1, K_2)$, then for any $n \geq K$, we have
$$\left|\frac{1}{y_n} - \frac{1}{y}\right| = \frac{|y - y_n|}{|y_n||y|} < \frac{|y - y_n|}{\left(\frac{|y|}{2}\right)|y|} < \frac{2}{|y|^2}\left(\frac{|y|^2}{2}\varepsilon\right) = \varepsilon.$$

So we conclude that $\lim_{n\to\infty}\frac{1}{y_n} = \frac{1}{y}$.

**Property 4: Sandwich Theorem (or squeeze theorem)**

We let $\{x_n\}, \{y_n\}, \{z_n\}$ be three sequences which $x_n \leq z_n \leq y_n$ for all $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = z$, then the sequence $\{z_n\}$ converges and $\lim_{n\to\infty} z_n = z$

Proof:

Note $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = z$, then for any $\varepsilon > 0$, there exists $K_1 \in \mathbb{N}$ and $K_2 \in \mathbb{N}$ such that

- $|x_n - z| < \varepsilon \Rightarrow x_n > z - \varepsilon$ for all $n \geq K_1$ and
- $|y_n - z| < \varepsilon \Rightarrow y_n < z + \varepsilon$ for all $n \geq K_2$.

By picking $K = \max(K_1, K_2)$, we have

$$z - \varepsilon < x_n \leq z_n \leq y_n < z + \varepsilon \Rightarrow |z_n - z| < \varepsilon \;\; for \;\; all \; n \geq K$$

So we conclude that $\lim_{n\to\infty} z_n = z$ by definition.

**Example 6 (A quick example on the application of sandwich theorem)**

We consider a sequence $\{x_n\}$ defined by

$$x_n = \frac{\sin n}{n^2}, \qquad for\ n \in \mathbb{N}.$$

Show that $\{x_n\}$ converges and find its limits.

😊Solution

Since $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$, it follows that

$$\underbrace{\frac{-1}{n^2}}_{y_n} \leq x_n = \frac{\sin n}{n^2} \leq \underbrace{\frac{1}{n^2}}_{z_n}.$$

One can show that $\lim_{n\to\infty}\left(-\frac{1}{n^2}\right) = \lim_{n\to\infty}\frac{1}{n^2} = 0$ by definition (I omit the details here). Then it follows from sandwich theorem that $\frac{\sin n}{n^2}$ converges and $\lim_{n\to\infty}\frac{\sin n}{n^2} = 0.$

**Property 5: Limit inequality**

We let $\{x_n\}$ and $\{y_n\}$ be two sequences which $x_n \le y_n$ for all $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $x \le y$.

*Remark*

- As a corollary, the above property implies that if a convergent sequence $\{x_n\}$ satisfies $a \le x_n \le b$ for all $n \in \mathbb{N}$, then $a \le \lim_{n\to\infty} x_n \le b$.
- One has to be careful that the above property may not hold if all inequality sign ($\le$) becomes strict inequality sign ($<$). For example, we take $x_n = -\frac{1}{n}$ and $y_n = \frac{1}{n}$ for $n \in \mathbb{N}$. We see that $x_n < y_n$ but $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.

*Proof of property 5*

We shall prove this by contradiction. Suppose that $x > y$, we take $\varepsilon = \frac{x-y}{2}$, then there exists $K_1 \in \mathbb{N}$ and $K_2 \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon = \frac{x-y}{2} \Rightarrow x_n > x - \frac{x-y}{2} = \frac{x+y}{2} \quad for\ n \ge K_1.$$

$$|y_n - y| < \varepsilon = \frac{x-y}{2} \Rightarrow y_n < y + \frac{x-y}{2} + \frac{x+y}{2} \quad for\ \ n \ge K_2.$$

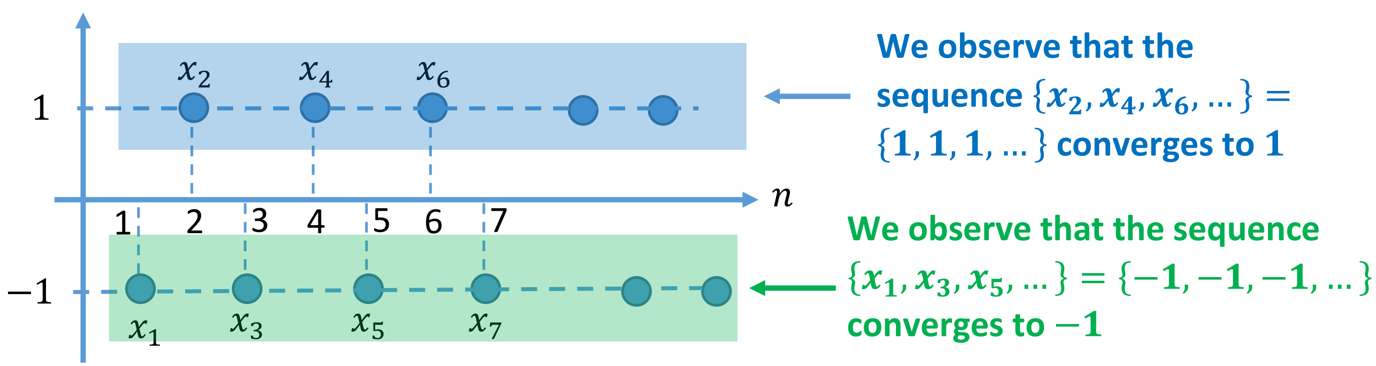
This implies that $x_n > \frac{x+y}{2} > y_n$ for $n \ge \max(K_1, K_2)$ and this leads to contradiction.

**Subsequence and its application – A quick introduction**

Practically, the limits $\lim_{n\to\infty} x_n$ can describe the limiting behavior of $x_n$ in the sense that it reveals the approximated value of $x_n$ when $n$ is large. However, it is not necessary that $\lim_{n\to\infty} x_n$ exists (either as number and $\pm\infty$) for every sequence $\{x_n\}$.

As an example, we take $x_n = (-1)^n$ for $n \in \mathbb{N}$.

- Similar to Example 4, one can show that $x_n$ does not converge to any number so that $\lim_{n\to\infty} x_n$ does not exist.
- By plotting $x_n$ against $n$, we observe that



- It appears that we can still investigate the limiting behavior of the sequence $\{x_n\}$ by observing the limits of convergent *subsequences* of $\{x_n\}$ (i.e. $\{x_{2n}\}$ and $\{x_{2n-1}\}$).

**Definition (Subsequence)**

We let $\{x_n\}$ be a sequence. A subsequence of $\{x_n\}$ defined as a sequence $\left\{x_{n_j}\right\} = \{x_{n_1}, x_{n_2}, x_{n_3}, \dots\}$, where $n_j \in \mathbb{N}$ and $1 \leq n_1 < n_2 < n_3 < \cdots$.

*Subsequence of convergent sequence* $\{x_n\}$

**Theorem 1**

A sequence $\{x_n\}$ converges to $x$ (i.e. $\lim\limits_{n\to\infty} x_n = x$) if and only if every subsequence $\left\{x_{n_j}\right\}$ converges to $x$ (i.e. $\lim\limits_{j\to\infty} x_{n_j} = x$)

*Remark of theorem 1*

Practically, this theorem is useful in proving a sequence is not convergent.

*Proof of theorem 1*

The " $\Leftarrow$ " is obvious since one can take $\{x_{n_j}\} = \{x_1, x_2, x_3, \dots\}$ (i.e. $n_j = j$)

For "$\Rightarrow$" direction, we first note that $n_j \geq j$ for all $j \in \mathbb{N}$.

For any $\varepsilon > 0$, since $\lim_{n\to\infty} x_n = x$, there exists $K \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon \quad for \;\; n \geq K.$$

This implies that for any $j \geq K$ (so that $n_j \geq j \geq K$), we have $\left|x_{n_j} - x\right| < \varepsilon$. So $\lim_{n\to\infty} x_{n_j} = x$ by definition of limits.

**Example 7**

Show that the sequence defined by $x_n = \sin\frac{n\pi}{4}$ for $n \in \mathbb{N}$ is not convergent.

😊Solution

We consider two subsequences $\{x_{4n}\} = \{x_4, x_8, x_{12}, \dots\} = \{0, 0, 0, \dots\}$ and $\{x_{8n-6}\} = \{x_2, x_{10}, x_{18}, \dots\} = \{1, 1, 1, \dots\}$.

Since $\lim_{n\to\infty} x_{4n} = 0$ and $\lim_{n\to\infty} x_{8n-6} = 1$, so it follows from the contrapositive of the theorem that $\{x_n\}$ is not convergent.

*Application of subsequence – Limit superior and limit inferior*

As mentioned earlier, the limits of subsequences are useful for examining the limiting behavior of a sequence $\{x_n\}$, which may not be convergent.

Given a sequence $\{x_n\}$, we define $L$ be a set which collects the limits of all convergent subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$. That is,

$$L = \left\{x \middle| \lim_{j\to\infty} x_{n_j} = x \;\; for \;\; some \; subsequence \; \left\{x_{n_j}\right\}\right\}.$$

For example, if $x_n = (-1)^n$ for $n \in \mathbb{N}$, we have $L = \{-1, 1\}$.

In some cases, one would like to study the *range* of $x_n$ when $n \to \infty$, then we can study them using limit superior and limit inferior which are defined by

$$\limsup_{n\to\infty} x_n = \sup L\,, \qquad \liminf_{n\to\infty} x_n = \inf L.$$

In the earlier example which $L = \{-1, 1\}$, we have $\limsup\limits_{n\to\infty} x_n = \sup L = 1$ and $\liminf\limits_{n\to\infty} x_n = \inf L = -1$. So we can deduce that $x_n$ is roughly between $-1$ and $1$ when $n \to \infty$.

*Existence of convergent subsequence*

The only technical problem is that whether the set $L$ is always non-empty in the sense that there always exists a convergent subsequence.

The Bolzano-Weierstrass theorem confirms that convergent subsequence always exists for bounded sequence $\{x_n\}$.

**Theorem 4: Bolzano-Weietstrass theorem**

If the sequence $\{x_n\}$ is bounded (i.e. $|x_n| \le M$), there exists a convergent subsequence $\left\{x_{n_j}\right\}$.

*Remark*

For the case when $\{x_n\}$ is unbounded, one can show that

- if the sequence $\{x_n\}$ is not bounded from above (no upper bound), then there exists a subsequence $\left\{x_{n_j}\right\}$ which $\lim_{n\to\infty} x_{n_j} = +\infty$ and
- if the sequence $\{x_n\}$ is not bounded from below (no lower bound), then there exists a subsequence $\left\{x_{n_j}\right\}$ which $\lim_{n\to\infty} x_{n_j} = -\infty$.

So we can conclude from the theorem 4 and the remarks that $L$ is always non-empty so that $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ are well-defined.

*Proof of Bolzano-Weietstrass theorem*

The proof is divided into three key steps.

*Step 1: Construct the required subsequence* $\left\{x_{n_j}\right\}$

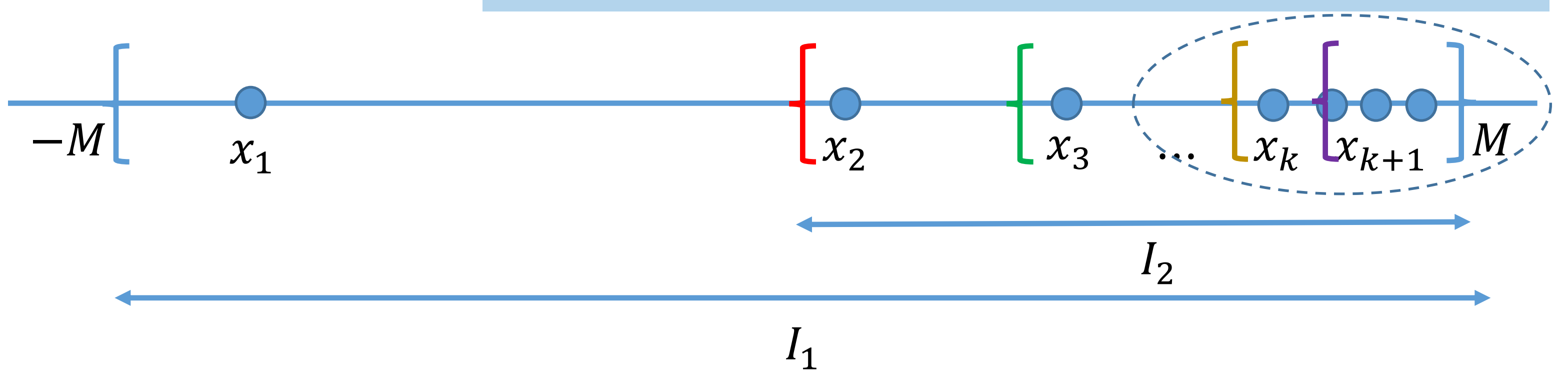
Since $\{x_n\}$ is convergent and hence bounded, so we have $-M \leq x_n \leq M$ for $n \in \mathbb{N}$.

Next, we will construct a convergent sequence as follows:

- **Step 1**: We let $I_1 = [-M, M]$. Then divide the interval equally into two subintervals (i.e. $[-M, 0]$ and $[0, M]$). Since the entire sequence $x_n$ lies in $I_1$, it follows that there are infinitely many $x_n s$ that lies in one of the intervals $[-M, 0]$ and $[0, M]$. Then we choose the interval with infinitely many $x_n$s and denoted it by $I_2$.
- **Step 2**: Suppose that the interval $I_k = [a_k, b_k]$ contains infinitely many $x_n s$, then we divide $I_k$ equally into two subintervals, $\left[a_k, \frac{a_k+b_k}{2}\right]$ and $\left[\frac{a_k+b_k}{2}, b_k\right]$ and pick the one with infinitely many $x_n s$ and denoted by $I_{k+1}$ (similar to step 1).
- **Step 3**: By repeating Step 1, we obtain a sequence of subintervals $I_1, I_2, I_3, \ldots$ which
  - $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_k \supseteq I_{k+1} \supseteq \cdots$ and
  - Length of $I_k$ is $b_k - a_k = \frac{2M}{2^{k-1}} = \frac{M}{2^{k-2}}$ for $k = 1,2,3, \ldots$.
  - Each $I_k$ contains infinitely many $x_n s$.

- **Step 4**: We pick a subsequence $\left\{x_{n_j}\right\}$ as follows:
  - For $j = 1$, we pick $x_{n_1} = x_1 \in I_1$.
  - For $j = 2$, we pick a number $x_{n_2} \in I_2$, where $n_2 > 1$.
    (*Note: This step is to avoid of picking the same $x_1$. It is feasible since there are infinitely many $x_n$ in the interval $I_2$)
  - For $j = 3$, we pick a number $x_{n_3} \in I_{n_3}$ with $n_3 > n_2$ and so on.
- Then we obtain a sequence $\left\{x_{n_j}\right\}$ which $x_{n_j} \in I_j$ for all $n \in \mathbb{N}$ and $1 = n_1 < n_2 < \cdots < n_j < n_{j+1} < \cdots$. Based on this construction, one would expect that this sequence converges.

**Since $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ and the length of $I_k$ shrinks to $0$ whe $k \to \infty$, we observe that $x_{n_k}$ is close to each other when $k$ is large. This gives a signal that the sequence converges to some point.**



*Step 2: Find the limits of the subsequence*

It remains to find the limits of this subsequence *in order to prove the convergence using the definition*. Since the intervals $I_k = [a_k, b_k]$ satisfy $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, it follows from nested interval theorem that $\bigcap_{k=1}^{\infty} I_k = [a, b]$, where $a = \sup\{a_k | k \in \mathbb{N}\} = \lim_{k\to\infty} a_k$ and $b = \inf\{b_k | k \in \mathbb{N}\} = \lim_{k\to\infty} b_k$.

(*The last equality follows from monotone sequence theorem).

Since $b - a = \lim_{k\to\infty}(b_k - a_k) = \lim_{k\to\infty} \frac{M}{2^{k-2}} = 0$, so that $a = b = x$ and $\bigcap_{k=1}^{\infty} I_k = \{x\}$. Then we conjecture that $x$ will be the limits of the subsequence $\left\{x_{n_j}\right\}$.

*Step 3: Prove the convergence using the definition*

For any $\varepsilon > 0$, we deduce from Archimedean property that there exists $K \in \mathbb{N}$ such that

$$K > 2 + \frac{\ln \frac{M}{\varepsilon}}{\ln 2} \Leftrightarrow \frac{M}{2^{K-2}} < \varepsilon.$$

Then for any $j \geq K$, we have $x_{n_j} \in I_j$ and $x \in \bigcap_{k=1}^{\infty} I_k \subseteq I_j$. Then we have

$$\left|x_{n_j} - x\right| \leq |b_j - a_j| = \frac{M}{2^{j-2}} \leq \frac{M}{2^{K-2}} < \varepsilon.$$

So we conclude that $\lim_{j\to\infty} x_{n_j} = x$ by definition and the proof is completed.

*How to prove a sequence is convergent without knowing the limits?*

Sometimes, we may not able to figure out the limits of sequence (especially that defined by recurrence relation) so that we cannot check the convergence using the definition.

In this section, we shall examine some approaches to resolve this problem:

**Approach 1 – Argue that the sequence $\{x_n\}$ is some special sequence such as monotone sequence, intertwining sequence etc.**

*1. Monotonic sequence*

We say a sequence $\{x_n\}$ is increasing if and only if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *strictly increasing* if the strictly inequality holds (i.e. $x_n < x_{n+1}$)

We say a sequence $\{x_n\}$ is decreasing if and only if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *strictly decreasing* if the strictly inequality holds (i.e. $x_n > x_{n+1}$)

The following theorem gives the criterion for the convergence of monotone sequence.

**Theorem 2 (Monotone sequence theorem)**

If a sequence $\{x_n\}$ is increasing and bounded from above, then $\lim_{n\to\infty} x_n = \sup\{x_n | n \in \mathbb{N}\}$.

If a sequence $\{x_n\}$ is decreasing and bounded from below, then $\lim_{n\to\infty} x_n = \inf\{x_n | n \in \mathbb{N}\}$.

*Proof of theorem 2*

We shall prove the case when $\{x_n\}$ is increasing and bounded from above. The proof consists of two key steps:

**Step 1: Identify the limits of $\{x_n\}$**

Since $\{x_n\}$ is bounded above, it follows that $x = \sup\{x_n | n \in \mathbb{N}\}$ exists. Since $x_n$ is increasing, so the "maximum" of $\{x_n\}$ will occur when $n \to \infty$, we expect that $x = \sup\{x_n | n \in \mathbb{N}\}$ will be the limits of $\{x_n\}$

**Step 2: Prove $\lim\limits_{n\to\infty} x_n = x$ using definition**

For any $\varepsilon > 0$, we note that $x - \varepsilon$ is not upper bound of $\{x_n\}$. It follows that there exists $x_K$ such that $x - \varepsilon < x_K$.

Since $x_n$ is increasing and $x_n \leq x = \sup\{x_n | n \in \mathbb{N}\}$, it follows that for $n \geq K$,

$$x - \varepsilon < x_K \leq x_n \leq x \Rightarrow |x_n - x| < \varepsilon.$$

So $\{x_n\}$ converges to $x = \sup\{x_n | n \in \mathbb{N}\}$ by definition.

**Example 8**

We let $\{x_n\}$ be a sequence defined by

$$x_1 = 1, \qquad x_{n+1} = \sqrt{6 + x_n}.$$

Show that the sequence $\{x_n\}$ converges.

😊Solution

We shall prove the convergence of the sequence by proving the followings:

- $\{x_n\}$ is increasing and
- $\{x_n\}$ is bounded above by 3

(Question: How do we make these conjectures?)

If the above conjectures are correct, then the convergence is guaranteed by monotonic sequence theorem.

Step 1: Show that $1 \leq x_n \leq 3$ for all $n \in \mathbb{N}$.

One can prove this using mathematical induction.

For $n = 1$, it is clear that $1 \leq x_1 = 1 \leq 3$ so the that statement is true for $n = 1$.

Assuming that $1 \le x_k \le 3$ where $k$ is positive integer, then for $n = k + 1$

$$1 < \sqrt{6+1} \le x_{k+1} = \sqrt{6 + x_k} \le \sqrt{6+3} = 3.$$

So by mathematical induction, we confirm that $1 \le x_n \le 3$ for all $n \in \mathbb{N}$.

<u>Step 2: Show that $x_n$ is monotonic increasing. That is, $x_n \le x_{n+1}$ for all $n \in \mathbb{N}$</u>

One can prove this using mathematical induction.

For $n = 1$, one can see that $x_2 = \sqrt{6 + x_1} = \sqrt{7} > 1 = x_1$.

Assuming that $x_k \le x_{k+1}$ for some $k \in \mathbb{N}$, then for $n = k + 1$, we consider

$$x_{k+2} - x_{k+1} = \sqrt{6 + x_{k+1}} - \sqrt{6 + x_k} = \frac{(6 + x_{k+1}) - (6 + x_k)}{\sqrt{6 + x_{k+1}} + \sqrt{6 + x_k}}$$

$$= \frac{x_{k+1} - x_k}{\sqrt{6 + x_{k+1}} + \sqrt{6 + x_k}} \overset{\substack{x_n \ge 1 > 0 \\ x_k+1 \ge x_k}}{\overbrace{\ge}} 0.$$

So it follows from mathematical induction that $x_{n+1} \ge x_n$ for all $n \in \mathbb{N}$ so that $\{x_n\}$ is increasing. Therefore, the sequence $\{x_n\}$ converges by monotonic sequence theorem. (Here, one can show that $\lim_{n\to\infty} x_n = 3$)

**Example 9 (Harder)**

We let $\{x_n\}$ be a convergent sequence and let $\{y_n\}$ be a sequence defined by

$$y_n = \min(x_1, x_2, \dots, x_n), \qquad n \in \mathbb{N}.$$

Prove that $\{y_n\}$ converges.

😊Solution

Firstly, one can show that $\{y_n\}$ is decreasing as

$$y_{n+1} = \min(x_1, x_2, \dots, x_n, x_{n+1}) \le \min(x_1, x_2, \dots, x_n) = y_n.$$

It remains to show $\{y_n\}$ is bounded from below.

- Since $\{x_n\}$ is convergent, so $\{x_n\}$ must be bounded and $|x_n| \le M \Rightarrow -M \le x_n \le M$ for any $n \in \mathbb{N}$. Here, $M$ is some positive number.
- Then it follows that
$$y_n = \min(x_1, x_2, \dots, x_n) \ge \min(-M, -M, \dots, -M) = -M.$$

So $y_n$ is bounded below by $-M$.

Thus it follows from monotone sequence theorem that $\{y_n\}$ converges.

## *2. Intertwining sequence*

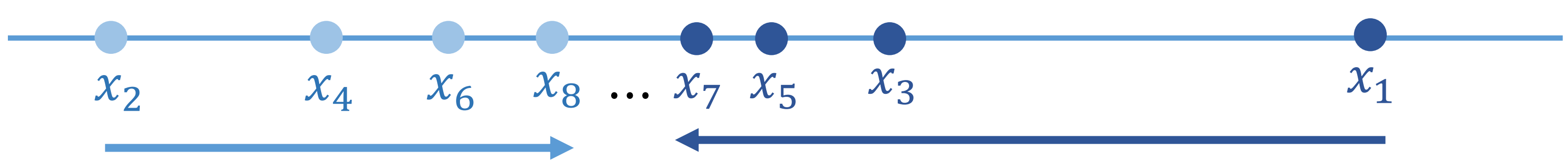
To motivate this concept, we consider the following sequence $\{x_n\}$ defined by

$$x_1 = 1 \quad and \quad x_{n+1} = \frac{1}{1+x_n} \quad for\ n \in \mathbb{N}.$$

By computing the first few terms, we observe that

$$x_2 = \frac{1}{2} = 0.5, \quad x_3 = \frac{2}{3} \approx 0.667, \quad x_4 = \frac{3}{5} = 0.6, \quad x_5 = \frac{5}{8} = 0.625, \quad x_6 \approx 0.615, \quad x_7 = 0.619, \quad x_8 = 0.618, \ldots\ldots$$

We observe that the sequence is not monotonic. On the other hand, one can plot the points $x_n$ on the real number line as follows:




We observe that the sequence consists of two subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ which $\{x_{2n}\}$ is increasing and $\{x_{2n+1}\}$ is decreasing.

In general, we let $\{a_n\}$ and $\{b_n\}$ be two sequences. The intertwining sequence $\{x_n\}$ is a sequence which takes the following form:

$$\{x_1, x_2, x_3, x_4, x_5, x_6, \dots\} = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$$

where $x_{2n-1} = a_n$ and $x_{2n} = b_n$ for all $n \in \mathbb{N}$. In other words, the odd term consists of the sequence $\{a_n\}$ and the even term consists of the sequence $\{b_n\}$.

*Convergence of intertwining sequence*

**Theorem 3 (Intertwining sequence theorem)**

If $\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} x_{2n} = x$ (or $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$), then $\lim_{n\to\infty} x_n = x$.

*Proof of theorem 3*

For any $\varepsilon > 0$, there exists $K_1, K_2$ such that

$$|x_{2n-1} - x| < \varepsilon \ \ for \ \ n \geq K_1 \ \ and \ \ |x_{2n} - x| < \varepsilon \ \ for \ \ n \geq K_2.$$

By picking $K = \max(2K_1 - 1, 2K_2)$, then for $n \geq K$, we get

$$|x_n - x| = \begin{cases} |x_{2m-1} - x| & if \ \ n = 2m - 1 \\ |x_{2m} - x| & if \ \ n = 2m \end{cases} < \varepsilon.$$

So we get $\lim_{n\to\infty} x_n = x$ by definition.

As an example, we revisit the sequence introduced earlier.

By some algebra, one can write the recurrence relation as

$$x_{n+2} = \frac{1}{1+x_{n+1}} = \frac{1}{1+\frac{1}{1+x_n}} = \frac{x_n+1}{x_n+2} = 1 - \frac{1}{x_n+2}, \qquad for\ n \in \mathbb{N}$$

The convergence of $\{x_n\}$ can be proved as follows:

We first argue that the subsequence $\{x_{2n-1}\}$ converges, one can use mathematical induction and prove that

- $\frac{1}{2} \leq x_{2n-1} \leq 1$ for all $n \in \mathbb{N}$
  - ✓ For $n = 1$, one can see that $\frac{1}{2} < x_1 = 1 \leq 1$.
  - ✓ Assuming that $\frac{1}{2} \leq x_{2k-1} \leq 1$, then for $n = k+1$

$$\frac{1}{2} < \frac{3}{5} = 1 - \frac{1}{\frac{1}{2}+2} \leq x_{2k+1} = 1 - \frac{1}{x_n+2} \leq 1 - \frac{1}{1+2} = \frac{2}{3} < 1.$$

So we conclude from mathematical induction that $\frac{1}{2} \leq x_{2n-1} \leq 1$ for all $n \in \mathbb{N}$.

- $\{x_{2n-1}\}$ is decreasing. That is, $x_{2n+1} \le x_{2n-1}$ for all $n \in \mathbb{N}$.
  - ✓ For $n = 1$, we have $x_3 = 1 - \frac{1}{x_1+2} = \frac{2}{3} < x_1$.
  - ✓ Assuming that $x_{2k+1} \le x_{2k-1}$, then for $n = k+1$, we consider

$$x_{2k+3} - x_{2k+1} = 1 - \frac{1}{x_{2k+1}+2} - \left(1 - \frac{1}{x_{2k-1}+2}\right)$$

$$= \frac{x_{2k+1} - x_{2k-1}}{(x_{2k+1}+2)(x_{2k-1}+2)} \overset{As\ x_{2k-1} \ge 0.5 > 0}{\overbrace{\le}} 0.$$

This implies that $x_{2k+3} \le x_{2k+1}$ and the statement is true for $n = k+1$. Thus we conclude from mathematical induction that $\{x_{2k-1}\}$ is decreasing

So we conclude that $\{x_{2k-1}\}$ converges by monotonic sequence theorem since the sequence is decreasing and bounded from below.

To compute the limits, we let $\lim_{k\to\infty} x_{2k-1} = x$. Then we deduce that

$$\lim_{k\to\infty} x_{2k+1} = \lim_{k\to\infty}\left(1 - \frac{1}{x_{2k-1}+2}\right) \Rightarrow x = 1 - \frac{1}{x+2} \Rightarrow x^2 + x - 1 = 0$$

$$\Rightarrow x = \frac{-1+\sqrt{5}}{2} \quad or \quad x = \frac{-1-\sqrt{5}}{2} \quad \left(rejected \ \ since \ \ x \ge \frac{1}{2}\right).$$

Next, we consider the subsequence $\{x_{2n}\}$. Using mathematical induction, one can show that

- $\frac{1}{2} \leq x_{2n} \leq 1$ for all $n \in \mathbb{N}$.
- $\{x_{2n}\}$ is increasing. That is, $x_{2n+2} \geq x_{2n}$ for all $n \in \mathbb{N}$.

Since the sequence is increasing and bounded from above, then $\{x_{2n}\}$ converges by monotone sequence theorem.

We let $\lim_{k\to\infty} x_{2k} = y$, one can show that

$$\lim_{k\to\infty} x_{2k+2} = \lim_{k\to\infty}\left(1 - \frac{1}{x_{2k}+2}\right) \Rightarrow y = 1 - \frac{1}{y+2} \Rightarrow y^2 + y - 1 = 0$$

$$\Rightarrow y = \frac{-1+\sqrt{5}}{2} \quad or \quad y = \frac{-1-\sqrt{5}}{2} \left(rejected \;\; since \;\; y \geq \frac{1}{2}\right).$$

Since $\lim_{k\to\infty} x_{2k-1} = \lim_{k\to\infty} x_{2k} = \frac{-1+\sqrt{5}}{2}$, so it follows from intertwining sequence theorem that the sequence $\{x_n\}$ converges and $\lim_{n\to\infty} x_n = \frac{-1+\sqrt{5}}{2}$.

## Cauchy sequence

Roughly speaking, Cauchy sequence provides an alternative way to describe the convergence of a sequence *without specifying its limits*. The idea is that when $\{x_n\}$ converges to $x$, then all $x_n$ should be closed to $x$ when $n$ is large. Intuitively, one would expect that the $x_n s$ should be closed to each other when $n$ is large.

**Definition (Cauchy Sequence)**

A sequence $\{x_n\}$ is said to be Cauchy sequence if and only if for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for any $m, n \geq K$.

**Example 10**

Show that the sequence $\{x_n\}$ defined by $x_n = \frac{1}{n^2}$ is Cauchy sequence.

😊Solution

For any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $\frac{1}{K^2} < \varepsilon$.

Then for any $m, n \geq K$ (assume $m \geq n$), we have

$$|x_m - x_n| = \left|\frac{1}{m^2} - \frac{1}{n^2}\right| = \frac{1}{n^2} - \frac{1}{m^2} < \frac{1}{n^2} < \varepsilon.$$

So $\{x_n\}$ is Cauchy sequence.

**Example 11**

Suppose that a sequence $\{x_n\}$ satisfy $|x_{n+2} - x_{n+1}| \leq \frac{1}{2}|x_{n+1} - x_n| \quad for\ \ any\ \ n \in \mathbb{N}$.

Show that $\{x_n\}$ is Cauchy.

😊Solution

Applying the above inequality repeatedly, one can deduce that for any $n \in \mathbb{N}$.

$$|x_{n+1} - x_n| \leq \frac{1}{2}|x_n - x_{n-1}| \leq \frac{1}{2^2}|x_{n-1} - x_{n-2}| \leq \cdots \leq \frac{1}{2^{n-1}}|x_2 - x_1|.$$

For any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $K > 2 + \frac{\ln\frac{|x_2 - x_1|}{\varepsilon}}{\ln 2} \Rightarrow \frac{|x_2 - x_1|}{2^{K-2}} < \varepsilon$.

Then for any $m, n \geq K$ (assume $m > n$), we have

$|x_m - x_n| = |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)|$

$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots |x_{n+1} - x_n|$

$$\leq \frac{1}{2^{m-2}}|x_2 - x_1| + \frac{1}{2^{m-3}}|x_2 - x_1| + \cdots + \frac{1}{2^{n-1}}|x_2 - x_1| < |x_2 - x_1|\frac{1}{2^{n-1}}\left(\frac{1 - \frac{1}{2^{m-n}}}{1 - \frac{1}{2}}\right)$$

$$< \frac{|x_2 - x_1|}{2^{n-1}\left(1 - \frac{1}{2}\right)} = \frac{|x_2 - x_1|}{2^{n-2}} < \varepsilon.$$

Thus $\{x_n\}$ is Cauchy.

## Relationship between Cauchy sequence and convergent sequence

Given a convergent sequence $\{x_n\}$ with limits of $x$, one would expect that it is also Cauchy since $x_m$ and $x_n$ will be close to $x$ when $m, n$ are large so that $x_m$, $x_n$ should also be closed to each other. It is confirmed by the following theorem.

**Theorem 4**

If a sequence $\{x_n\}$ converges to $x$, then $\{x_n\}$ is also Cauchy.

*Proof of theorem 4*

For any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$ for all $n \geq K$.

It follows that for all $m > n \geq K$,

$$|x_m - x_n| \leq |x_m - x| + |x - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $\{x_n\}$ is Cauchy by definition.

Conversely, suppose that we have a Cauchy sequence $\{x_n\}$, one can also prove that the sequence also converges. One challenge in proving this is that one has to "determine" the limits of $x$ in order to prove the convergence using definition.

**Theorem 5**

If $\{x_n\}$ is also Cauchy, then $\{x_n\}$ converges to $x \in \mathbb{R}$.

*Proof of theorem 5*

Step 1: Obtain the limits of $\{x_n\}$

This can be done using Bolzano-Weierstrass theorem. To use it, one has to argue that $\{x_n\}$ is bounded.

- Since $\{x_n\}$ is Cauchy, we pick $\varepsilon = \varepsilon_0 > 0$ and deduce that there exists $K \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon_0$ any $m > n \geq K$.
- By taking $n = K$, we deduce that
$$|x_m - x_K| < \varepsilon_0 \Leftrightarrow x_K - \varepsilon_0 < x_m < x_K + \varepsilon_0 \quad for \;\; m > K.$$
- We take $m^* = \min(x_1, x_2, \dots, x_K, x_K - \varepsilon_0)$ and $M^* = \max(x_1, x_2, \dots, x_K, x_K + \varepsilon_0)$. Then we observe that
$$m^* \leq x_n \leq M^* \quad for \;\; all\, n \in \mathbb{N}.$$
Thus, $x_n$ is bounded.

Then by Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{x_{n_k}\}$ with limits $x$. Since $\{x_n\}$ is Cauchy, we expect that $\{x_n\}$ has limits $x$.

Step 2: Prove that $\{x_n\}$ converges to $x$ using definition of limits

For any $\varepsilon > 0$

- Note that $\lim_{j\to\infty} x_{n_j} = x$. There exists $K \in \mathbb{N}$ such that $\left|x_{n_j} - x\right| < \frac{\varepsilon}{2}$ for $j \geq K$.
- Since $\{x_n\}$ is Cauchy, then there exists $K_1 \in \mathbb{N}$ such that

$$|x_m - x_n| < \frac{\varepsilon}{2} \quad for \;\; m > n \geq K_1.$$

We take $j^* \in \mathbb{N}$ which $n_{j^*} \geq K_1$ and take $K^* = \max\left(n_{j^*}, K_1\right)$, then for any $n \geq K^*$, we have

$$|x_n - x| \leq \left|x_n - x_{n_{j^*}}\right| + \left|x_{n_{j^*}} - x\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus we conclude that $\lim_{n\to\infty} x_n = x$ by definition.