MATH 2031 Introduction to Real Analysis

December 4, 2012

Tutorial Note 7

Countability

(I) **Definition:**
A set S is **countably infinite** iff $\exists$ bijection $f : \mathbb{N} \to S$ (i.e. $Card\ S = \aleph_0$)

(II) **Definition:**
A set S is **countable** iff S is finite or S is countably infinite.

(III) **Definition:**
A set S is **uncountable** iff S is not countable.

Remark:
Notices that uncountable sets must be **infinite**.
When a problem ask you to show existing infinitely many elements with some properties then you should think of the "complement part" is a countable set and the set complement is still uncountable that ensure you have infinitely many of the required elements.

Tools related to Set operations

(a) Countable Subset Theorem:
Let $A \subseteq B$. If B is countable, then A is countable.(Contrapositive: If A is uncountable, then B is uncountable.)

Remark:
Please beware of the containment, essentially the theorem says that subsets of countable set are still countable, while a set contain an uncountable set is not countable.
(In homework some claim that the subset of uncountable set is uncountable which is not valid. We have the following countable example $\mathbb{Q} \subset \mathbb{R}$ where $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable.)

(b) Countable Union Theorem:
If $\forall n \in \mathbb{N}$, A_n is countable, then $\bigcup_{n\in\mathbb{N}} A_n$ is countable. In general, we can take union over a countable set, instead of $\mathbb{N}$.(i.e. countable union of countable sets is countable).

Remark:
When you would like to use this theorem to show certain set, say A, is countable, you should write A as a union **over a countable index** first, $A = \bigcup_{\alpha\in S} A_\alpha$ where S is countable. And check each component A_α are all countable. Then conclude by the theorem.

(c) Product Theorem:
For $n \in \mathbb{N}$, if $A_1, A_2, \cdots A_n$ are countable, then $A_1 \times A_2 \times \cdots \times A_n$ is countable.(i.e. finite product of countable sets is countable).

Remark:
Similar as above that you should check each component are countable and be sure that this theorem is for **finite** product.

Tools related to (Constructing) Functions

(i) Injection Theorem:
Let $f : A \to B$ be an injection. If B is countable, then A is countable.(contrapositive: if A is uncountable, then B is uncountable.)

(ii) Surjection Theorem:
Let $g : A \to B$ be an surjection. If A is countable, then B is countable.(contrapositive: if B is uncountable, then A is uncountable.)

(iii) Bijection Theorem:
Let $g : S \to T$ be a bijection. S is countable $\Leftrightarrow$ T is countable.

Remark:
When you would like to use the above theorem, you should clearly express you function, like the domain and range, $f : A \to B$ as well as the construction, $f(x) = \cdots$. Please make sure your function is well defined.

Then depends on which on which one you would like to use and check the corresponding properties of the functions.

For Injection Check: for any $x, y \in A$, if $f(x) = f(y)$ then $x = y$ or for any $x, y \in A$ $x \neq y$ $f(x) \neq f(y)$.

For Surjection Check: for each $b \in B$ there exist $a \in A$ such that $f(a) = b$ or simply $f(A) = B$

For Bijection Gives the inverse of the function, please be sure the inverse you give is well defined and satisfies $f \circ f^{-1} = I_B$ and $f^{-1} \circ f = I_A$

Examples

Problem 1 Let P be a countable set points of $\mathbb{R}^2 \setminus (0, 0)$.Prove that there exists a line L passing through origin such that it does not pass through any point in P.

Solution:

Following the idea in the first remark, we would like to show the collection of all line passing through origin, $\mathcal{A} = \{L | L \text{ is a line passing the origin}\}$, is uncountable.
And the "complement part", the collection of all line passing through origin and at least one point in P, $\mathcal{B} = \{L | L \text{ is a line passing the origin and at least a point in } P\}$, is countable.
Then there is infinitely many of required L in $\mathcal{A} \setminus \mathcal{B}$.

Since every line passing the origin make an unique angle in $[0, \pi)$ and also given an angle we can uniquely determine a line passing the origin, we may consider the bijection theorem.

Let $f : \mathcal{A} \to [0, \pi)$, given by $f(L) = \theta$ where θ in the angle measure from the positive x-axis anti-clockwise to the line L, with the inverse $f^{-1} : [0, \pi) \to \mathcal{A}$ given by $f^{-1}(\theta) = L$ where L is the uniquely passing the origin with the angle with the the positive x-axis is θ. Since $[0, \pi)$ is uncountable, we get $\mathcal{A}$ is also uncountable by bijection theorem.

With this bijection we may write $\mathcal{A} = \{L_\theta | \theta \in [0, \pi)\}$.

Consider $\mathcal{B}$,

$$\begin{aligned}
\mathcal{B} &= \{L | L \text{ is a line passing the origin and at least a point in } P\} \\
&= \bigcup_{p \in P} \{L | L \text{ is a line passing the origin and } p\}
\end{aligned}$$

Since any distinct 2 point in $\mathbb{R}^2$ uniquely determine a single line, the component of the above union has only one element, countable. As P is a countable set, by countable union theorem, $\mathcal{B}$ is countable.

As $\mathcal{A}$ is uncountable and $\mathcal{B}$ is countable, we get $\mathcal{A} \setminus \mathcal{B}$ is uncountable, which is infinite.

Thus there must exist a line L with the required property.

Problem 2 Determine $S = \{x^2+y^2+z^2 : x \in A\cap B,\ y \in \mathbb{Q}\cap A,\ z \in B\cap\mathbb{Q}\}$, where A is non-empty countable subset of $\mathbb{R}$ and B is an uncountable subset of $\mathbb{R}$, is countable or not.

Solution 1:

Even B is uncountable, $A\cap B$ is a subset of A which is countable, so $A\cap B$ is also countable by countable subset theorem, similar for $\mathbb{Q}\cap A$ and $B\cap\mathbb{Q}$. $\mathbb{Q}\cap A$ is a subset of A which is countable and $B\cap\mathbb{Q}$ is a subset of $\mathbb{Q}$,countable, so both sets are countable.

By product theorem, $(A\cap B)\times(\mathbb{Q}\cap A)\times(B\cap\mathbb{Q})$ is a countable, denote such set as D.

Then we get

$$\begin{aligned} S &= \{x^2+y^2+z^2 : x \in A\cap B,\ y \in \mathbb{Q}\cap A,\ z \in B\cap\mathbb{Q}\} \\ &= \bigcup_{(x,y,z)\in \underbrace{D}_{\text{countable}}} \{ \underbrace{x^2+y^2+z^2}_{\text{one element}\Rightarrow\text{countable}} \} \end{aligned}$$

Thus by countable union theorem, S is countable.

Solution 2:

Similar argument as above to show that $(A\cap B)\times(\mathbb{Q}\cap A)\times(B\cap\mathbb{Q})$ is a countable, also denote this set as D.

Define $f : D \to S$ is given by $f((x,y,z)) = x^2+y^2+z^2$. From the definition of S this f is surjective. (beware that f may not be injective)

As D is countable, by surjection theorem we get S is countable.

Problem 3 Let S be the set of non-constant polynomials with coefficients in G, where $G = \{a+bi|a,b\in\mathbb{Z}\}$, $i=\sqrt{-1}$. Determine whether S is countable or uncountable.

Solution:

The problem is mater of how you view the set and how you write it as a countable union. First, $G = \{a+bi|a,b\in\mathbb{Z}\} = \bigcup_{(a,b)\in\mathbb{Z}\times\mathbb{Z}} \{a+bi\}$. As $\mathbb{Z}$ is countable, $\mathbb{Z}\times\mathbb{Z}$ is countable by product theorem.

Since for fixed $a,b\in\mathbb{Z}$, $\{a+bi\}$ is a singleton, so countable, thus G is countable by countable union theorem.

Then we get also G^n ,denoted the product of G n-times, is also countable for all $n\in\mathbb{N}$ by product theorem.

Since a polynomial has only finite terms, we may intuitively think that S is the collection of all finite subset of G, thus come up with the following argument.

For any $n\in\mathbb{N}$, let S_n be the set of all polynomial with G-coefficient of degree n.

i.e. $S = \bigcup_{(a_0,\cdots,a_n\in G^n)} \{\underbrace{a_0+a_1x+a_2x^2+\cdots+a_nx^n}_{\text{one element, countable}}\}$.

Then we get $S = \bigcup_{n\in\mathbb{N}} S_n$ which is countable by countable union theorem.

Infinite series

(I) Definition of infinite series $\sum_{k=1}^{\infty} a_k$ and partial sum $S_n = \sum_{k=1}^{n} a_k$,
converges ($\lim_{n\to\infty} S_n \in \mathbb{R}$) and divergence ($\lim_{n\to\infty} S_n = \infty$ or $\lim_{n\to\infty} S_n$ doesn't exist)

(II) Series Tests (make sure you know **all** the conditions and conclusions for each of the tests below)
Tests for particular series

(a) Geometric Series Test
(b) Telescoping Series Test
(c) p-test
(d) Alternating Series Test

Tests for non-negative series

(a) Integral Test
(b) Comparison Test
(c) Limit Comparison Test

Tests for general series

(a) Term test (for showing diverges only)
(b) Absolute Convergence Test
(c) Ratio Test
(d) Root Test

Examples

Problem 4 Determine if each of the following series converges of diverges

(a) $\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{k^2 + 2^k}$

(b) $\sum_{k=1}^{\infty} \frac{e^{\sqrt{k}}}{\sqrt{k}}$

(c) $\sum_{k=1}^{\infty} \frac{(2k)!}{3^k k^4}$

(d) $\sum_{k=1}^{\infty} \frac{(\cos(k))(\sin(2k))}{2^k}$

(e) $\sum_{k=1}^{\infty} \frac{1}{2}\left(\cos\frac{1}{k} + \sin\frac{1}{k}\right)$

(f) $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k} - \frac{1}{k+1}\right)$

(g) $\sum_{k=1}^{\infty} \left[\left(\cos\frac{1}{k}\right)\left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right)\right]$

Solution :

(a) Since $\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{k^2 + 2^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 + 2^k}$, alternate series, as $k \nearrow \infty$, $k^2 + 2^k \nearrow \infty$ and so $\frac{1}{k^2 + 2^k} \searrow 0$. So $\sum_{k=1}^{\infty} \frac{\cos(k\pi)}{k^2 + 2^k}$ converges by alternate series test.

(b) Since $\dfrac{e^{\sqrt{k}}}{\sqrt{k}} \geq \dfrac{1}{\sqrt{k}} = \dfrac{1}{k^{\frac{1}{2}}} \geq 0$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$ diverges by p-test as $\frac{1}{2} < 1$, $\displaystyle\sum_{k=1}^{\infty} \frac{e^{\sqrt{k}}}{\sqrt{k}}$ diverges by comparison test.

(c) $\displaystyle\lim_{k\to\infty} \left| \frac{\frac{(2k+2)!}{3^{k+1}(k+1)^4}}{\frac{(2k)!}{3^k k^4}} \right| = \lim_{k\to\infty} \frac{2(k+1)(2k+1)}{3} \left(\frac{k}{k+1}\right) = \infty$. So $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{3^k k^4}$ diverges by ratio test.

(d) $\left| \dfrac{(\cos(k))(\sin(2k))}{2^k} \right| \leq \left(\dfrac{1}{2}\right)^k$ and $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k$ converges by geometric series test, $\displaystyle\sum_{k=1}^{\infty} \left| \frac{(\cos(k))(\sin(2k))}{2^k} \right|$ converges by comparison test and $\displaystyle\sum_{k=1}^{\infty} \frac{(\cos(k))(\sin(2k))}{2^k}$ converges by absolute converges test.

(e) Since $\displaystyle\lim_{k\to\infty} \frac{1}{2}\left(\cos\frac{1}{k} + \sin\frac{1}{k}\right) = \frac{1}{2}(1+0) = \frac{1}{2} \neq 0$, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2}\left(\cos\frac{1}{k} + \sin\frac{1}{k}\right)$ diverges by term test.

(f) Since for all $k \in \mathbb{N}$ $\sin\left(\dfrac{1}{k} - \dfrac{1}{k+1}\right), \left(\dfrac{1}{k} - \dfrac{1}{k+1}\right) > 0$ and $\displaystyle\lim_{k\to\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 0$, so we could consider the following

$$\lim_{k\to\infty} \frac{\sin\left(\frac{1}{k} - \frac{1}{k+1}\right)}{\left(\frac{1}{k} - \frac{1}{k+1}\right)} = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

By telescoping series test, we get $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \lim_{k\to\infty} \frac{1}{k+1} = 1$ converges.

Thus, we get $\displaystyle\sum_{k=1}^{\infty} \sin\left(\frac{1}{k} - \frac{1}{k+1}\right)$ converges by limit comparison test.

(g) Since for all $k \in \mathbb{N}$ $\left(\cos\frac{1}{k}\right)\left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right), \frac{1}{k^2} > 0$, and

$$\lim_{k\to\infty} \frac{\left(\cos\frac{1}{k}\right)\left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right)}{\frac{1}{k^2}} = \lim_{k\to\infty} \left(\cos(\frac{1}{k})\right)\left(\frac{\sin\frac{1}{k}}{\frac{1}{k}}\right)\left(\frac{\tan\frac{1}{k}}{\frac{1}{k}}\right) = 1$$

And as $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges by p-test since $2 > 1$, we get $\displaystyle\sum_{k=1}^{\infty} \left[\left(\cos\frac{1}{k}\right)\left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right)\right]$ converges by limit comparison test.