MATH2033 Mathematical Analysis
Suggested Solution of Problem Set 7

Problem 1

We consider a function $\mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^3 & if \;\; x \in \mathbb{Q} \\ 0 & if \; x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}.$$

(a) Determine if $f(x)$ is differentiable at $x = 0$.

(b) Determine if $f(x)$ is differentiable at $x \neq 0$.

(c) Determine if $f(x)$ is twice differentiable at $x = 0$.

😊Solution

(a) We shall argue that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$ using the definition of limits.

For any $\varepsilon > 0$, we pick $\delta = \sqrt{\varepsilon}$. Then for any $0 < |x - 0| < \delta$, we have

$$\left|\frac{f(x) - f(0)}{x - 0} - 0\right| = \left|\frac{f(x)}{x}\right| \leq \left|\frac{x^3}{x}\right| = |x^2| < \left(\sqrt{\varepsilon}\right)^2 = \varepsilon.$$

So $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$ by definition of limits and $f(x)$ is differentiable at $x = 0$.

(b) For any $x_0 \neq 0$, there exists a sequence of rational numbers $\{q_n\}$ (where $q_n \in \mathbb{Q}$) and a sequence of irrational numbers $\{r_n\}$ (where $r_n \in \mathbb{R}\backslash\mathbb{Q}$) such that $\lim_{n\to\infty} q_n = \lim_{n\to\infty} r_n = x_0$ (see Lecture Note 6). Then we get

$$\lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} (q_n)^3 = x_0^3 \quad and$$
$$\lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 0 = 0.$$

As $\lim_{n\to\infty} f(q_n) \neq \lim_{n\to\infty} f(r_n)$, it follows from sequential limit theorem that the limits $\lim_{x\to x_0} f(x)$ does not exist. Thus, $f(x)$ is not continuous at $x = x_0$ and hence is not differentiable at $x = x_0$.

(c) As $f'(x)$ does not exists for $x \neq 0$ (as shown in (b)), thus the limits $\lim_{x\to 0} \frac{f'(x)-f'(0)}{x-0}$ (or $f''(0)$) does not exist. So $f(x)$ is not twice differentiable at $x = 0$.

Problem 2

Suppose that $f: \mathbb{R} \to \mathbb{R}$ is differentiable at $x = c$ and $f(c) = 0$. Show that $g(x) = |f(x)|$ is differentiable at $x = c$ if and only if $f'(c) = 0$.

😊Solution

"⇒" part

We shall prove it by contradiction. Suppose that $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = L \neq 0$, we let $L > 0$ (the case for $L < 0$ can be established in a similar way).

We pick $\varepsilon = L$, then there exists $\delta > 0$ such that for $0 < |x - c| < \delta$

$$\left|\frac{f(x)-f(c)}{x-c}-L\right|<\varepsilon=L\Rightarrow 0<\frac{f(x)-f(c)}{x-c}=\frac{f(x)}{x-c}<2L.$$

This implies $f(x)<0$ for $x<c$ and $f(x)>0$ for $x>c$.
Then we deduce that

$$\lim_{x\to c^+}\frac{|f(x)|-|f(c)|}{x-c}=\lim_{x\to c^+}\frac{f(x)}{x-c}=L \quad and$$

$$\lim_{x\to c^-}\frac{|f(x)|-|f(c)|}{x-c}=\lim_{x\to c^-}\frac{-f(x)}{x-c}=-L$$

As $L\neq 0$, we have $\lim_{x\to c^+}\frac{|f(x)|-|f(c)|}{x-c}\neq\lim_{x\to c^-}\frac{|f(x)|-|f(c)|}{x-c}$. So the limits $\lim_{x\to c}\frac{|f(x)|-|f(c)|}{x-c}$ does not exist and $|f(x)|$ is not differentiable and there is contradiction. So it follows that $f'(c)=0$.

"⇐" part
If $f'(c)=0$, then we have

$$\lim_{x\to c}\frac{f(x)}{x-c}=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=f'(c)=0.$$

It follows that

$$-\left|\frac{f(x)}{x-c}\right|\leq\frac{|f(x)|-|f(c)|}{x-c}=\frac{|f(x)|}{x-c}\leq\left|\frac{f(x)}{x-c}\right|$$

As $|x|$ is continuous, $\lim_{x\to c}\left|\frac{f(x)}{x-c}\right|=\left|\lim_{x\to c}\frac{f(x)}{x-c}\right|=0$. It follows from sandwich theorem that

$$\underbrace{-\lim_{x\to c}\left|\frac{f(x)}{x-c}\right|}_{=0}\leq\lim_{x\to c}\frac{|f(x)|-|f(c)|}{x-c}=\lim_{x\to c}\frac{|f(x)|}{x-c}\leq\underbrace{\lim_{x\to c}\left|\frac{f(x)}{x-c}\right|}_{=0}$$

$$\Rightarrow\lim_{x\to c}\frac{|f(x)|-|f(c)|}{x-c}=0.$$

So that $g(x)=|f(x)|$ is differentiable at $x=c$.

Problem 3 (Harder)

A function $f(x)$ is continuous on (a,b) and has finite derivative $f'(x)$ at every $x\in(a,b)\backslash\{c\}$. Suppose that $\lim_{x\to c}f'(x)=A$, show that f is also differentiable at $x=c$ and $f'(c)=A$.

😊Solution

Our goal is to show the limits $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists. To do so, we consider the one-sided limits:

- We take $x<c$. By applying mean value theorem on $f(x)$ over the interval $[x,c]$ (note that $f(x)$ is continuous on $[x,c]$ and is differentiable over (x,c)), we deduce that there exists $c_x\in(x,c)$ such that

$$\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}=\lim_{x\to c^-}f'(c_x)\overset{\substack{as\ x<c_x<c^-\\ so\ c_x\to c}}{=}A.$$

- We take $x > c$. By applying mean value theorem on $f(x)$ over the interval $[c, x]$ (note that $f(x)$ is continuous on $[c, x]$ and is differentiable over (c, x)), we deduce that there exists $d_x \in (x, c)$ such that

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} f'(d_x) \overset{\substack{as\ c^+ < d_x < x \\ so\ d_x \to c}}{\cong} A.$$

Since $\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = A$, we conclude that $\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = A$ and $f(x)$ is differentiable at $x = c$ with $f'(c) = A$.

Problem 4

(a) We consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 2x + 1$. Show that the inverse function f^{-1} exists and is differentiable at any $x_0 \in \mathbb{R}$.

(b) We let $g(x) = \tan x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Show that the inverse function $g^{-1}(y) = \tan^{-1} y$ exists and is differentiable at any $y \in \mathbb{R}$. Find $\frac{d}{dy} g^{-1}(y)$.

😊Solution

(a) Note that $f(x)$ is differentiable and $f'(x) = 3x^2 + 2 \geq 2 > 0$, it follows that f is strictly increasing. Then f is injective so that f^{-1} exists.
Since $f'(x_0) = 3x_0^2 + 2 \neq 0$ for all $x_0 \in \mathbb{R}$, it follows from inverse function theorem that f^{-1} is differentiable at any $x_0 \in \mathbb{R}$.

(b) Since $\tan x$ is strictly increasing over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so the inverse function $g^{-1}(y) = \tan^{-1} y$ exists. Since $\tan x$ is also differentiable over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\frac{d}{dx} \tan x = \sec^2 x \neq 0$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, it follows from inverse function theorem that $g^{-1}(y) = \tan^{-1} y$ is differentiable and

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{g'\left(g^{-1}(y)\right)} = \frac{1}{\sec^2(\tan^{-1} y)} = \frac{1}{1 + (\tan(\tan^{-1} y))^2}$$
$$= \frac{1}{1 + y^2}.$$

Problem 5

(a) We let $f(x)$, $g(x)$ be two differentiable functions on $\mathbb{R}$ such that $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for all $x \geq 0$, show that $f(x) \leq g(x)$ for all $x \geq 0$.

(b) Show that for any $a > b > 0$, we have $a^{\frac{1}{n}} - b^{\frac{1}{n}} < (a - b)^{\frac{1}{n}}$ for all positive integer $n \geq 2$. (😊Hint: Consider the function $f(x) = x^{\frac{1}{n}} - (x - 1)^{\frac{1}{n}}$ for $x \geq 1$)

😊Solution

(a) We take $h(x) = f(x) - g(x)$. For any $x > 0$, one can apply mean value theorem on $h(x)$ over the interval $[0, x]$ and deduce that there exists $c \in (0, x)$ such that

$$\frac{h(x)-h(0)}{x-0} = h'(c) = f'(c) - g'(c) \le 0 \Rightarrow h(x) \le h(0).$$

This implies that

$$f(x) - g(x) \le f(0) - g(0) = 0 \Rightarrow f(x) \le g(x).$$

(b) By applying mean value theorem on $f(x) = x^{\frac{1}{n}} - (x-1)^{\frac{1}{n}}$ over the interval $[1, x]$ (where $x > 1$), we deduce that there exists $c \in (1, x)$ such that

$$\frac{f(x)-f(1)}{x-1} = f'(c) = \frac{1}{n}\left[c^{\frac{1}{n}-1} - (c-1)^{\frac{1}{n}-1}\right] \overset{\frac{1}{n}-1<0}{<} 0$$

$$\Rightarrow x^{\frac{1}{n}} - (x-1)^{\frac{1}{n}} = f(x) < f(1) = 1.$$

By taking $x = \frac{a}{b} > 1$, we deduce that

$$\left(\frac{a}{b}\right)^{\frac{1}{n}} - \left(\frac{a}{b} - 1\right)^{\frac{1}{n}} < 1 \Rightarrow a^{\frac{1}{n}} - (a-b)^{\frac{1}{n}} < b^{\frac{1}{n}} \Rightarrow a^{\frac{1}{n}} - b^{\frac{1}{n}} < (a-b)^{\frac{1}{n}}.$$

Problem 6

It is given that a function $f(x)$ is continuous on $[a, b]$ and is differentiable on (a, b). Suppose that $f(a) = f(b) = 0$, show that for any $\lambda \in \mathbb{R}$, there exists $c \in (a, b)$ such that $f'(c) = \lambda f(c)$. (😊Hint: Apply Rolle's theorem to $g(x)f(x)$, where $g(x)$ is some function depending on λ.)

😊Solution

We let $g(x) = e^{-\lambda x}$ and consider the function $h(x) = g(x)f(x) = e^{-\lambda x}f(x)$.
Note that

- $h(a) = e^{-\lambda a}f(a) = 0$ and $h(b) = e^{-\lambda b}f(b) = 0$.
- $h'(x) = e^{-\lambda x}f'(x) - \lambda e^{-\lambda x}f(x) = e^{-\lambda x}\big(f'(x) - \lambda f(x)\big)$
- Since both $e^{-\lambda x}$ and $f(x)$ are continuous on $[a, b]$ and is differentiable on (a, b), so does $h(x)$.

It follows Rolle's theorem that there exists $c \in (a, b)$ such that

$$h'(c) = 0 \Leftrightarrow e^{-\lambda c}\big(f'(c) - \lambda f(c)\big) = 0 \overset{e^{-\lambda c}\neq 0}{\Leftrightarrow} f'(c) = \lambda f(c).$$

Problem 7

We let $f(x)$ be a continuous function on $[0,1]$ which $f(0) = 0$ and is differentiable at any $x \in (0,1)$. Prove that if $f'(x)$ is increasing, then a function defined by $g(x) = \frac{f(x)}{x}$ is also increasing.

😊Solution

Firstly, we deduce from quotient rule that

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} = \frac{f'(x) - \frac{f(x)}{x}}{x}, \qquad for\ x \in (0,1).$$

On the other hand, we apply mean value theorem on $f(x)$ over $[0, x]$ and deduce that there exists $c \in (0, x)$ such that

$$\frac{f(x)}{x} \overset{f(0)=0}{\widehat{=}} \frac{f(x)-f(0)}{x-0} = f'(c) \overset{\substack{f'(x)\ is \\ increasing}}{\widehat{\leq}} f'(x).$$

It follows that

$$g'(x) = \frac{\overbrace{f'(x) - \frac{f(x)}{x}}^{>0}}{\underbrace{x}_{>0}} \geq 0.$$

So $g(x)$ is increasing.

Problem 8

Suppose that $f(x)$ is differentiable over the interval $(0, \infty)$ and that $\lim_{x\to\infty} f'(x) = 0$. We let $a > 0$ be a positive number and define $g(x) = f(x+a) - f(x)$. Show that $\lim_{x\to\infty} g(x) = 0$.

😊Solution

We shall prove the statement using the definition of limits.

For any $\varepsilon > 0$

- Since $\lim_{x\to\infty} f'(x) = 0$, there exists $M > 0$ such that
$$|f'(x)| < \frac{\varepsilon}{a} \quad for \ \ x > M.$$
- On the other hand, we apply mean value theorem on $f(x)$ over $[x, x+a]$ and deduce that there exists $c_x \in (x, x+a)$ such that
$$\frac{f(x+a)-f(x)}{(x+a)-x} = f'(c_x) \Rightarrow \underbrace{f(x+a)-f(x)}_{g(x)} = af'(c_x)$$
- With this value of M, we deduce that for any $x > M$,
$$|g(x)| = a|f'(c_x)| \overset{c_x > x > M}{\widehat{<}} a\left(\frac{\varepsilon}{a}\right) = \varepsilon.$$

So $\lim_{x\to\infty} g(x) = 0$ by the definition of limits.

Problem 9

It is given that a function $f: [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and is differentiable on (a, b). Suppose that $|f'(x)| < 1$ for all $x \in (a, b)$, prove that $f(x) = x$ has at most one solution. (☺Hint: What will happen if there are two or more solutions?)

☺Solution

Suppose that the equation $f(x) = x$ has at least two solutions. We let x_1, x_2 (with $x_1 \neq x_2$) be two of the solutions.

We define a function $g(x) = f(x) - x$ which is continuous on $[a, b]$ and is differentiable on (a, b).

- Since $g(x_1) = f(x_1) - x_1 = 0$ and $g(x_2) = f(x_2) - x_2 = 0$, it follows from Rolle's theorem that there exists $c \in (x_1, x_2)$ such that
$$g'(c) = 0 \Rightarrow f'(c) - 1 = 0 \Rightarrow f'(c) = 1,$$

This contradicts to the assumption that $|f'(x)| < 1$. So we conclude that $f(x) = x$ has at most one solution.

Problem 10

Show that $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$ for all $x > 0$.

☺Solution

We let $f(x) = \sqrt{1+x}$. By direct differentiation, we get
$$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}, \qquad f''(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}, \qquad f'''(x) = \frac{3}{8}(1+x)^{-\frac{5}{2}}$$

- By applying Taylor theorem with $n = 1$, we have
$$\sqrt{1+x} = f(x) = f(0) + f'(0)x + \frac{f''(c_1)}{2!}x^2 = 1 + \frac{x}{2} - \frac{1}{4(1+c_1)^{\frac{3}{2}}}x^2,$$
$$where \;\; c_1 \in (0, x)$$
As $x > 0$ and $c_1 > 0$, we have
$$\sqrt{1+x} < 1 + \frac{1}{2}x.$$
- By applying Taylor theorem with $n = 2$. We have
$$\sqrt{1+x} = f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(c_2)}{3!}x^3$$
$$= 1 + \frac{x}{2} - \frac{1}{8}x^2 + \frac{1}{16(1+c_2)^{\frac{5}{2}}}x^3, \qquad c_2 \in (0, x).$$
As $x > 0$ and $c_2 > 0$, we have
$$e^{-x} > 1 + \frac{x}{2} - \frac{x^2}{8}.$$

Problem 11

We let f be a twice differentiable function on (a, b) which $f''(x) \geq 0$ for all $x \in (a, b)$. For any $c \in (a, b)$, show that the graph of $f(x)$ is never below the tangent line to the graph at $(c, f(c))$.

☺Solution

Recall that the equation of tangent line to the graph $y = f(x)$ at $(c, f(c))$ is
$$\frac{y - f(c)}{x - c} = f'(c) \Rightarrow y = f(c) + f'(c)(x - c).$$

By applying Taylor theorem, we deduce that for any $x \in (a,b)$, there exists $c_0 \in (c,x) \subseteq [a,b]$ such that

$$\underbrace{f(x)}_{graph} = f(c) + f'(c)(x-c) + \frac{f''(c_0)}{2!}(x-c)^2 \overbrace{\geq}^{f''(c_0)\geq 0} \underbrace{f(c) + f'(c)(x-c)}_{tangent\ line}.$$

So the graph $y = f(x)$ is always above the tangent line $y = f(c) + f'(c)(x-c)$.