Solutions to Presentation Exercises

(191) (a) Since $f(0)=0<\frac{a}{a+b}<1=f(1)$ and f is continuous on $[0,1]$, by the intermediate value theorem, there exists $x_0\in(0,1)$ such that $f(x_0)=\frac{a}{a+b}$.

(b) By the mean value theorem, there exists $x_1\in(0,x_0)$ such that $f(x_0)-f(0)=f'(x_1)(x_0-0)$ and also there exists $x_2\in(x_0,1)$ such that $f(1)-f(x_0)=f'(x_2)(1-x_0)$. These equations gives

$$\left.\begin{array}{l}\frac{a}{a+b}=f'(x_1)(x_0-0)\\ \frac{b}{a+b}=f'(x_2)(1-x_0)\end{array}\right\}\Rightarrow\left.\begin{array}{l}\frac{a}{f'(x_1)}=(a+b)x_0\\ \frac{b}{f'(x_2)}=(a+b)(1-x_0)\end{array}\right\}\Rightarrow\frac{a}{f'(x_1)}+\frac{b}{f'(x_2)}=a+b.$$

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(198) The function $f(x)=1-x+\frac{x^2}{2}-\frac{x^3}{3}+\cdots+\frac{x^{2006}}{2006}-\frac{x^{2007}}{2007}$ is continuous on $\mathbb{R}$. We have $f(0)=1$ and $f(2007)=\underbrace{(1-2007)}_{<0}+2007^2\underbrace{(\tfrac{1}{2}-\tfrac{2007}{3})}_{<0}+\cdots+2007^{2006}\underbrace{(\tfrac{1}{2006}-\tfrac{2007}{2007})}_{<0}$

So $f(2007)<0<1=f(0)$. By the intermediate value theorem, there exists $x_0\in(0,2007)$ such that $f(x_0)=0$ and $x_0>0$.

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(261) We first show if $f:\mathbb{R}\to\mathbb{R}$ is continuous and decreasing, then there exists a unique $x_0\in\mathbb{R}$ such that $f(x_0)=x_0$. (Proof: Consider $h:\mathbb{R}\to\mathbb{R}$ defined by $h(x)=f(x)-x$. Assume $h(x)\neq0$ for all $x\in\mathbb{R}$. Then either ① $\forall x\in\mathbb{R}$, $h(x)>0$ or ② $\forall x\in\mathbb{R}$, $h(x)<0$ (due to the intermediate value theorem). In case ①, $h(0)=f(0)-0>0\Rightarrow f(0)>0$. Then f decreasing $\Rightarrow f(f(0))\le f(0)\Rightarrow h(f(0))=f(f(0))-f(0)\le0$, contradicting ①. Similarly, in case ②, $h(0)=f(0)-0<0\Rightarrow f(0)<0\xRightarrow{f\text{ decreasing}} f(f(0))\ge f(0)\Rightarrow h(f(0))=f(f(0))-f(0)\ge0$, contradicting ②. $\therefore h(x_0)=0$ ($\Leftrightarrow f(x_0)=x_0$) for some $x_0\in\mathbb{R}$. For uniqueness, if $x_0>x_1$, $f(x_0)=x_0$, $f(x_1)=x_1$, then f decreasing $\Rightarrow x_0=f(x_0)\le f(x_1)=x_1$, contradicting $x_0>x_1$.)

Define $g(x)=f(f(f(x)))$ for all $x\in\mathbb{R}$. Then g is continuous and decreasing (as $x<y\Rightarrow f(x)\ge f(y)\Rightarrow f(f(x))\le f(f(y))\Rightarrow g(x)\ge g(y)$). From above, $f(x_0)=x_0\Rightarrow g(x_0)=x_0$ and such x_0 is unique. If we let $a=b=c=x_0$, then $a=f(b)$, $b=f(c)$, $c=f(a)$. Conversely, if $a=f(b)$, $b=f(c)$, $f(c)=a$, then $g(a)=f(f(f(a)))=f(f(b))=f(c)=a\Rightarrow a=x_0$. Similarly, $b=c=x_0$.

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(265) <u>Sketch</u> $|c_n-c_m|=\left|\sqrt{a_n+b_n}+\frac{a_n^2}{n}-\sqrt{a_m+b_m}-\frac{a_m^2}{m}\right|\le\left|\sqrt{a_n+b_n}-\sqrt{a_m+b_m}\right|+\left|\frac{a_n^2}{n}-\frac{a_m^2}{m}\right|$

$\le\sqrt{|a_n-a_m+b_n-b_m|}+\left|\frac{a_n^2}{n}\right|+\left|\frac{a_m^2}{n}\right|\le\sqrt{|a_n-a_m|+|b_n-b_m|}+\frac{M^2}{K}+\frac{M^2}{K}<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

$|a_n-a_m|,|b_n-b_m|<\frac{\varepsilon^2}{8}$ and $K>\frac{2M^2}{\varepsilon}$.

<u>Solution</u> Since $\{a_n\}$ is a Cauchy sequence $\Rightarrow\{a_n\}$ bounded, so $\exists M>0\ \forall a_n$, $|a_n|<M$. $\forall\varepsilon>0$, by Archimedean principle, $\exists K_1\in\mathbb{N}$ such that $K_1>\frac{2M^2}{\varepsilon}$. $\exists K_2,K_3\in\mathbb{N}$ such that $n,m\ge K_2\Rightarrow|a_n-a_m|<\frac{\varepsilon^2}{8}$, $n,m\ge K_3\Rightarrow|b_n-b_m|<\frac{\varepsilon^2}{8}$. Let $K=\max\{K_1,K_2,K_3\}$. Then $n,m\ge K\Rightarrow n,m\ge K_1,K_2,K_3\Rightarrow|c_n-c_m|<\varepsilon$ as in the sketch.

(277) By Taylor's theorem, $\exists\, a, b \in \mathbb{R}$ such that $f(0)=f(1)+f'(1)(0-1)+\frac{f''(1)}{2}(0-1)^2+\frac{f'''(a)}{6}(0-1)^3$ and $f(2)=f(1)+f'(1)(2-1)+\frac{f''(1)}{2}(2-1)^2+\frac{f'''(b)}{6}(2-1)^3$. Adding these and cancelling $f(0)+f(2)=2f(1)$, we get $0=0+0+f''(1)-\frac{f'''(a)}{6}+\frac{f'''(b)}{6}$. Rearranging terms, we have $f'''(a)-f'''(b)=6f''(c)$ with $c=1$.

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(280) By Taylor's Theorem, $5=f(0)=f(1)+f'(1)(0-1)+\frac{f''(1)}{2}(0-1)^2+\frac{f'''(x_0)}{6}(0-1)^3$ and $7=f(2)=f(1)+f'(1)(2-1)+\frac{f''(1)}{2}(2-1)^2+\frac{f'''(x_2)}{6}(2-1)^3$. Subtracting the equations, we have $2=7-5=0+2f'(1)+0+\frac{1}{6}f'''(x_2)+\frac{1}{6}f'''(x_0)$. Solving for $f'(1)$, we get $f'(1)=1-\frac{1}{12}f'''(x_2)-\frac{1}{12}f'''(x_0)$. So $|f'(1)| \le 1+\frac{1}{12}\underbrace{|f'''(x_2)|}_{\le 6}+\frac{1}{12}\underbrace{|f'''(x_0)|}_{\le 6} \le 1+\frac{1}{2}+\frac{1}{2}=2$.

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(290) By Taylor's Theorem, $\forall x \in [0,1]$, $f(x)=f(1)+f'(1)(x-1)+\frac{f''(\theta_1)}{2}(x-1)^2$ and $f(x)=f(0)+f'(0)(x-0)+\frac{f''(\theta_0)}{2}(x-0)^2$ for some θ_1 between x and 1 and some θ_0 between x and 0. Subtracting these equations, we get

$$0=f(1)-f(0)+2(-1)+\frac{f''(\theta_1)}{2}(x-1)^2-\frac{f''(\theta_0)}{2}(x-0)^2.$$

Then $f(1)-f(0)=2-\frac{f''(\theta_1)}{2}(x-1)^2-\frac{f''(\theta_0)}{2}x^2$.

Using $|f''(x)| \le 4$, we get $|f(1)-f(0)| \le 2+2(x-1)^2+2x^2 \leftarrow$ for all $x \in [0,1]$

For $x \in [0,1]$, $2(x-1)^2+2x^2=4x^2-4x+2=4(x-\frac{1}{2})^2+1$ has minimum when $x=\frac{1}{2}$. Then $|f(1)-f(0)| \le 2+2(\frac{1}{2}-1)^2+2(\frac{1}{2})^2=3$.

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(303)(c) Observe that since $|f''(x)|>1$ implies f is not the zero function (as $f''(x) \ne 0$). So $\exists\, w \in (1,3)$ such that f has a maximum value $f(w)>0$ or a minimum value $f(w)<0$ (due to $f(1)=0=f(3)$). Then $f'(w)=0$. By Taylor's theorem, $0=f(1)=f(w)+\underbrace{f'(w)}_{=0}(1-w)+\frac{f''(\theta_1)}{2}(1-w)^2$ and $0=f(3)=f(w)+\underbrace{f'(w)}_{=0}(3-w)+\frac{f''(\theta_3)}{2}(3-w)^2$ for some $\theta_1 \in [1,w]$ and some $\theta_3 \in [w,3]$. Solving for $f(w)$, we get

$$f(w)=-\frac{f''(\theta_1)}{2}(1-w)^2 \text{ and } f(w)=-\frac{f''(\theta_3)}{2}(3-w)^2.$$

Since $|f''(x)|>1$ on $[1,3]$

So $|f(w)| > \frac{1}{2}(w-1)^2$ and $\frac{1}{2}(3-w)^2$. Since $w \in (1,3)$, $\max\{w-1, 3-w\} \ge 1$,

So $|f(w)| > \frac{1}{2} \times 1^2 = \frac{1}{2}$.

(313) <u>Sketch</u> As $x\to 1$, $f(x)\to 1$, $\frac{5\pi}{4}\sqrt[3]{2f(x)+6} \to \frac{5\pi}{4}\sqrt[3]{2+6} = \frac{5\pi}{2}$, $\sin\left(\frac{5\pi}{4}\sqrt[3]{2f(x)+6}\right) \to \sin\frac{5\pi}{4} = 1$.

$$\left|\sin\left(\frac{5\pi}{4}\sqrt[3]{2f(x)+6}\right) - 1\right| = \left|\sin\left(\frac{5\pi}{4}\sqrt[3]{2f(x)+6}\right) - \sin\frac{5\pi}{2}\right| \le \left|\frac{5\pi}{4}\sqrt[3]{2f(x)+6} - \frac{5\pi}{2}\right| = \frac{5\pi}{4}\left|\sqrt[3]{2f(x)+6} - \underset{\sqrt[3]{8}}{\underset{\shortparallel}{2}}\right|$$

$$\le \frac{5\pi}{4}\sqrt[3]{|2f(x)+6-8|} = \frac{5\pi}{4}\sqrt[3]{2}\,\sqrt[3]{|f(x)-1|} < \varepsilon \Leftrightarrow |f(x)-1| < \left(\frac{4\varepsilon}{5\pi\sqrt[3]{2}}\right)^3 = \frac{32\varepsilon^3}{125\pi^3}$$

For every $\varepsilon > 0$, since $\lim_{x\to 1} f(x) = 1$, there exists $\delta > 0$ such that $0 < |x-1| < \delta$ $\Rightarrow |f(x)-1| < \frac{32\varepsilon^3}{125\pi^3} \Rightarrow \left|\sin\left(\frac{5\pi}{4}\sqrt[3]{2f(x)+6}\right) - 1\right| < \varepsilon$. $\therefore \lim_{x\to 1}\sin\left(\frac{5\pi}{4}\sqrt[3]{2f(x)+6}\right) = 1$.