Lecture 11 14-03-2019

Review :

① Big question : given an sequence $\{x_n\}$, how can we know it is convergent without knowing the limit?

② Subsequence thm : $\lim_{n\to\infty} x_n = x \Rightarrow \lim_{j\to\infty} x_{n_j} = x$

for any subsequence $\{x_{n_j}\} \subset \{x_n\}$

Remark : the converse is also true. (Exercise)

③ Monotone convergece thm :

$$\begin{cases} x_n \nearrow + \{x_n\} \text{ bounded above} \Rightarrow \lim x_n = \sup\{x_n : n\geq 1\} \\ x_n \searrow + \{x_n\} \text{ bounded below} \Rightarrow \lim x_n = \inf\{x_n : n\geq 1\} \end{cases}$$

④ Interwining Sequence thm : $\lim x_{2n-1} = \lim x_{2n} = x$

$\Rightarrow \lim x_n = x$

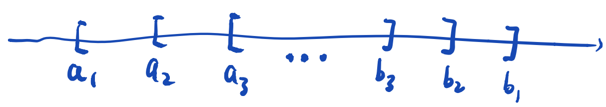
Nested Interval Theorem

THM: If $\forall n \in \mathbb{N}$, $I_n = [a_n, b_n]$ where $a_n \le b_n$ s.t

$I_1 \supseteq I_2 \supseteq I_3 \cdots$ Then $\bigcap_{n=1}^{\infty} I_n = [a, b]$, where

$a = \lim_{n\to\infty} a_n$, $b = \lim_{n\to\infty} b_n$.

If $\lim_{n\to\infty} b_n - a_n = 0$, Then $\bigcap_{n=1}^{\infty} I_n = \{a\}$.



proof: $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ implies that

$a_1 \le a_2 \le a_3 \le \cdots \le b_1$, $b_1 \ge b_2 \ge b_3 \ge \cdots \ge a_1$

By monotone sequence theorem, $\lim_{n\to\infty} a_n = \sup\{a_1, a_2, \cdots\} = a$

$\lim_{n\to\infty} b_n = \inf\{b_1, b_2, \cdots\} = b$

Since $a_n \le b_n$ $\forall n \in \mathbb{N}$,

Taking the limit $n \to \infty$, we obtain $a \le b$.

We now show that $\bigcap_{n \in \mathbb{N}} I_n = [a, b]$

Note that $a_n \le a$, $b_n \ge b$ $\Rightarrow$ $[a, b] \subseteq [a_n, b_n] = I_n$ for all $n \ge 1$

$\Rightarrow$ $[a, b] \subseteq \bigcap_{n \in \mathbb{N}} I_n$

On the other hand, $\forall$ $x \in \bigcap_{n \in \mathbb{N}} I_n$, then $x \in I_n$ for $n \ge 1$

$\Rightarrow$ $a_n \le x \le b_n$ $\forall n$. let $n \to \infty$, we get by limit inequality $a \le x \le b$ $\Rightarrow$ $x \in [a, b]$

$\Rightarrow$ $\bigcap_{n \in \mathbb{N}} I_n \subseteq [a, b]$. Therefore $\bigcap_{n \in \mathbb{N}} I_n = [a, b]$.

Finally, if $0 = \lim_{n \to \infty} b_n - a_n = 0$ then $0 = b - a$ $\Rightarrow$ $a = b$

$\Rightarrow$ $[a, b] = \{a\} = \bigcap_{n \in \mathbb{N}} I_n$.

R.K.1. Nested interval theorem $\Leftrightarrow$ Monotone sequence theorem

R.K.2 The nested interval thm is not true if we replace the closed intervals $[a_n, b_n]$ by open intervals (a_n, b_n).

For instance, $I_n = (0, \frac{1}{n})$, then $I_n \supset I_{n+1} \supset \cdots$

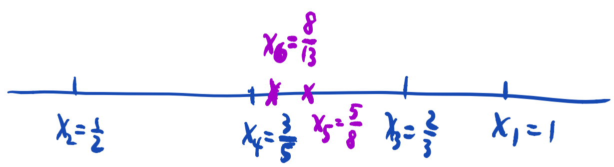
But $\bigcap_{n \in \mathbb{N}} I_n = \phi$.

Example: Does $\dfrac{1}{1+\frac{1}{1+\cdots}}$ represent a number? More precisely

let $x_1 = 1$, $x_{n+1} = \dfrac{1}{1+x_n}$ for $n \geq 1$. Does $\{x_n\}$ converges to some number?

Solution: Observe that $x_1 = 1$, $x_2 = \frac{1}{2}$, $x_3 = \dfrac{1}{1+\frac{1}{2}} = \dfrac{2}{3}$,

$$x_4 = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}, \qquad x_5 = \frac{1}{1+\frac{3}{5}} = \frac{5}{8}$$

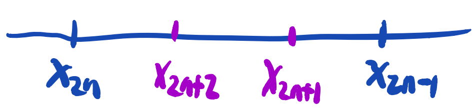


$$[x_2, x_1] \supset [x_4, x_3] \supset [x_6, x_5] \supset \cdots$$

We claim that we have nested intervals $I_n = [x_{2n}, x_{2n-1}]$ for $n \geq 1$, or equivalently

$$I_n \supset I_{n+1} \iff x_{2n} < x_{2n+2} < x_{2n+1} < x_{2n-1}$$

For $n \geq 1$.



We prove by induction. For $n=1$, we have.

$$x_2 < x_4 < x_3 < x_1$$

Now, assume that $x_{2n} < x_{2n+2} < x_{2n+1} < x_{2n-1}$ for n.

$$\Rightarrow 1+x_{2n} < 1+x_{2n+2} < 1+x_{2n+1} < 1+x_{2n-1}$$

$$\begin{array}{lccccccc}
\Rightarrow & \dfrac{1}{1+x_{2n}} & > & \dfrac{1}{1+x_{2n+2}} & > & \dfrac{1}{1+x_{2n+1}} & > & \dfrac{1}{1+x_{2n-1}} \\
 & \parallel & & \parallel & & \parallel & & \parallel \\
 & x_{2n+1} & > & x_{2n+3} & > & x_{2n+2} & > & x_{2n}
\end{array}$$

$$\Rightarrow 1+x_{2n+1} > 1+x_{2n+3} > 1+x_{2n+2} > 1+x_{2n}$$

$$\begin{array}{lccccccc}
\Rightarrow & \dfrac{1}{1+x_{2n+1}} & < & \dfrac{1}{1+x_{2n+3}} & < & \dfrac{1}{1+x_{2n+2}} & < & \dfrac{1}{1+x_{2n}} \\
 & \parallel & & \parallel & & & & \\
 & x_{2n+2} & < & x_{2n+4} & < & x_{2n+3} & < & x_{2n+1} \\
 & \parallel & & \parallel & & \parallel & & \parallel \\
 & x_{2(n+1)} & < & x_{2(n+1)+2} & < & x_{2(n+1)+1} & < & x_{2(n+1)-1}
\end{array}$$

As a result, $x_{2n} < x_{2n+2} < x_{2n+1} < x_{2n-1}$

is true for all $n \geq 1$.

$\Rightarrow$ Conditions for the nested interval holds

$\Rightarrow \bigcap_{n \in \mathbb{N}} I_n = [a, b]$ where $a = \lim x_{2n}$

$b = \lim x_{2n+1}$

Note that
$$\left\{ \begin{array}{lll} x_{2n} = \dfrac{1}{1 + x_{2n-1}} & \Rightarrow & a = \dfrac{1}{1+b} \\ x_{2n+1} = \dfrac{1}{1 + x_{2n}} & \Rightarrow & b = \dfrac{1}{1+a} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} a + ab = 1 \\ b + ab = 1 \end{array} \right. \Rightarrow a = b, \quad a + a^2 = 1$$

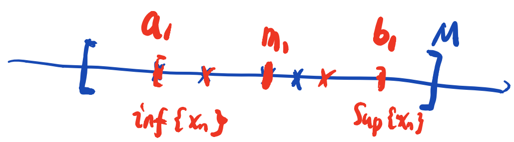
$$\Rightarrow a = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $a \in I_1 = [\frac{1}{2}, 1] \Rightarrow a \geq \frac{1}{2}$

$\Rightarrow a = \frac{-1 + \sqrt{5}}{2} \Rightarrow \lim x_n = \frac{-1 + \sqrt{5}}{2}$

Bolzano-Weierstrass Theorem

THM: If $\{x_1, x_2, \cdots\}$ is bounded, then $\exists$ subsequence $\{x_{n_k}\}$ which converges.



Proof: let $a_1 = \inf\{x_1, x_2, \cdots\}$

$b_1 = \sup\{x_1, x_2, \cdots\}$.

If $a_1 = b_1$, then $x_1 = x_2 = \cdots = x_n = a_1$ $\lim x_n = a$

If $a_1 < b_1$, then let $m_1 = \frac{a_1+b_1}{2}$, If $[a_1, m_1]$ has ∞ terms of $\{x_n\}$, then let $a_2 = a_1$, $b_2 = m_1$ and $I_2 = [a_2, b_2]$. If $[a_1, m_1]$ has finite terms of $\{x_n\}$, then $[m_1, b_1]$ must have ∞ terms of $\{x_n\}$. then we let $a_2 = m_1$, $b_2 = b_1$ and $I_2 = [a_2, b_2]$.

Keep repeating, we get $I_1 \supset I_2 \supset I_3 \supset \cdots$. Since

We have $b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \cdots (\frac{1}{2})^{n-1}(b_1 - a_1)$

$\lim_{n\to\infty}(b_n - a_n) = 0$. By the nested interval theorem,

$\exists \; a = \lim_{n\to\infty} a_n$ s.t $\bigcap_{n\in\mathbb{N}} I_n = \{a\}$

Now, take $x_{n_1} \in I_1$, Since $\exists$ ∞ terms in I_2

$\exists$ x_{n_2} with $n_2 > n_1$ s.t $x_{n_2} \in I_2$. Similarly,

for I_3, $\exists$ $x_{n_3} \in I_3$ s.t $n_3 > n_2$. Keeping doing this,

we get $x_{n_1}, x_{n_2}, x_{n_3}, \cdots x_{n_j}, \cdots$ s.t

$x_{n_j} \in I_j$ and $n_1 < n_2 < n_3 < \cdots$ for $j \geq 1$

We claim that $\lim x_{n_j} = a$.

Since $x_{n_j} \in I_j$, $a \in I_j$ $0 \leq |x_{n_j} - a| <$ length of $I_j = (\frac{1}{2})^{j-1}(b_1 - a_1)$

by the sandwich theorem $\lim_{j\to\infty} |x_{n_j} - a| = 0$ or $\lim_{j\to\infty} x_{n_j} = a$.

Cauchy Sequence

Question: How can we prove a sequence converges without knowing the limit?

In 19th, Cauchy introduced the following definition.

Def: $\{x_n\}$ is a Cauchy sequence iff $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t

$m, n \geq K \Rightarrow |x_n - x_m| < \varepsilon$.

[intuitively, this means that the terms are as close as desired when the indices are sufficiently large]

Example: Let $x_n = \frac{1}{n}$, show that $\{x_n\}$ is Cauchy.

Solution: $\forall \varepsilon > 0$, need to find K s.t

$|x_n - x_m| < \varepsilon$ for all $m, n \geq K$.

Note that $|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \max\left\{\frac{1}{n}, \frac{1}{m}\right\}$

Let $K = [\frac{1}{\varepsilon}] + 1$. then for $m, n \geq \frac{1}{K}$

$\frac{1}{n} < \varepsilon$, $\frac{1}{m} < \varepsilon$ $\Rightarrow |x_n - x_m| < \varepsilon$.

boundedness of Cauchy sequence

Lemma : If $\{x_n\}$ is Cauchy, then

$\{x_1, x_2, \cdots\}$ is bounded.

Proof : Since $\{x_n\}$ is cauchy, for $\varepsilon=1$,

$\exists\ K \in \mathbb{N}$ s.t $|x_n - x_m| < 1$ for all $m, n \geq K$.

Especially $|x_n - x_K| < 1$ for all $n \geq K$.

$$\Rightarrow |x_n| < |x_K| + |x_n - x_K| < |x_K| + 1$$

for all $n \geq K$

Let $M = \operatorname{Max}\{|x_1|, |x_2|, \cdots, |x_K|, |x_K|+1\}$

then $|x_n| < M$ for all $n \geq 1$

So $\{x_n\}$ is bounded

Cauchy's Theorem

THM: $\{x_n\}$ converges $\Leftrightarrow$ $\{x_n\}$ is Cauchy.

Proof: $(\Rightarrow)$. We need to show that $\forall \varepsilon > 0$, $\exists K$ s.t

$$|x_n - x_m| < \varepsilon \quad \text{for all } m, n \geq K.$$

Since $\{x_n\}$ converges to some x. $\exists K_0 \in \mathbb{N}$ s.t

$$|x_n - x| < \varepsilon/2 \quad \text{for all } n \geq K_0.$$

We take $K = K_0$. Then for $m, n \geq K_0$,

$$|x_n - x_m| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$(\Leftarrow)$. Given $\{x_n\}$ is Cauchy, we need to show that $\{x_n\}$ converges. First, we note that $\{x_1, x_2, \cdots\}$ is bounded by the Lemma proved previously. Then Bolzano-Weierstrass Theorem implies that $\exists$ subsequence $\{x_{n_k}\}$ which converges to some $x \in \mathbb{R}$. We show next that $\{x_n\}$ converges x as well.

$\forall \varepsilon > 0$, we need to find K s.t

$$|x_n - x| < \varepsilon \quad \text{for all } n \geq K.$$

Since $\{x_n\}$ is Cauchy, $\exists\, K_1 \in \mathbb{N}$ s.t

$$|x_m - x_n| < \varepsilon/2 \quad \forall\, m, n \geq K_1 .$$

Since $\{x_{n_k}\}$ converges to x, $\exists\, K_2 \in \mathbb{N}$ s.t

$$|x_{n_j} - x| < \varepsilon/2 \quad \forall\, j \geq K_2 .$$

Let $K = \max\{K_1, K_2\}$. Then for $n \geq K$,

$n_K \geq K \geq K_1 \Rightarrow |x_n - x_{n_K}| < \frac{\varepsilon}{2}$, and $K \geq K_2 \Rightarrow |x_{n_K} - x| < \frac{\varepsilon}{2}$

So $|x_n - x| < |x_n - x_{n_K}| + |x_{n_K} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Therefore $\{x_n\}$ converges to x.