Ex.1: If $f: \mathbb{R} \to \mathbb{R}$ is a function such that

① $f(x) = 0$ holds for all $x \in \mathbb{Q}$ and

② f is continuous

Show that $f(x) = 0$, $\forall x \in \mathbb{R}$.

Proof: We need only show that

$$f(x_0) = \lim_{x \to x_0} f(x) = 0 \text{ for all } x_0 \notin \mathbb{Q}.$$

By f is continuous

By f continuous again, we know for every sequence (x_n) s.t. $x_n \to x_0$, we have $\lim_n f(x_n) = f(x_0)$.

By the density of $\mathbb{Q}$ in $\mathbb{R}$, we can always find a sequence $(x_n) \in \mathbb{Q}$ s.t. $|x_n - x_0| \le \frac{1}{n}$. As a result, $x_n \to x_0$, then

$$f(x_0) = \lim_n f(x_n) = \lim_n 0 = 0.$$

Ex.2 Show that the sequence $x_n := (1+\frac{1}{n})^n$ converges to e as $n \to +\infty$ by the continuity of exp function and log function.

Proof:

why do this? ⇒ we don't want the variable appear in ~~both~~ the base and index

⇓

Rewrite the sequence as $x_n = (1+\frac{1}{n})^n = (e^{\log(1+\frac{1}{n})})^n = e^{n\log(1+\frac{1}{n})}$. then the problem can be translated to:

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{n\to\infty} \exp\{n\log(1+\frac{1}{n})\}$$

$$\overset{\text{change variable } t=\frac{1}{n}}{=} \lim_{t\to 0} \exp\{\frac{\log(1+t)}{t}\}$$

using the continuity of exp-function →

$$= \exp\{\lim_{t\to 0}\frac{\log(1+t)}{t}\}.$$

Besides, using the L'hospital rule, we know

$$\lim_{t\to 0}\frac{\log(1+t)}{t} = \lim_{t\to 0}\frac{(\log(1+t))'}{t'} = \lim_{t\to 0}\frac{1}{1+t} = 1$$

So we conclude that $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

Ex. 3: Show that if $f: [0,1] \to [0,1]$ is an increasing function, then there must exist a point $x_0 \in [0,1]$ s.t. $f(x_0) = x_0$.

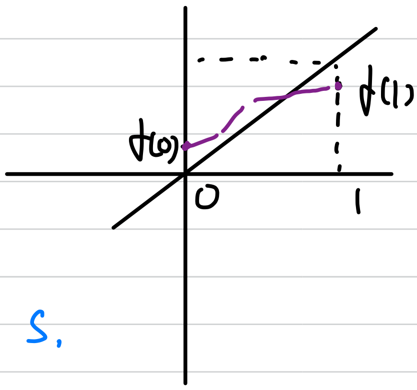
Proof:

If $f(0)=0$ or $f(1)=1$: Then the desired x_0 can be 0 or 1.

Otherwise: $f(0)>0$ and $f(1)<1$:

Consider the set

$S := \{ t \in [0,1] : t < f(t) \}$.



we know that $0 \in S$ and $1 \notin S$. 1 is an upper bound of S.

now consider $x_0' := \sup S$. then by the def of supremum.

① $\forall t > x_0'$, we have $t \geq f(t)$.

② exists a sequence t_n s.t. $t_n < f(t_n)$. $t_n \to x_0'$.

thus $x_0' = \lim_n t_n \leq \lim_{n\to\infty} f(t_n) \leq f(x_0')$
increasing prop.

1.1: If exists some $t_0 \geq x_0'$ s.t. $t_0 = f(t_0)$,
then we can set $x_0 = t_0$.

1.2: Else if $f(x_0') = x_0'$, we can set $x_0 = x_0'$.

1.3: Otherwise.

We have $\begin{cases} \forall t > x_0', \; t > f(t) \\ x_0' < f(x_0'). \end{cases}$

Now consider a sequence $x_n > x_0'$ s.t. $x_n \to x_0'$ decreasely. then

$f(x_0') > x_0' = \lim_n x_n \geq \lim_n f(x_n)$ ← why this limit exists?
Monotone seq thm.

i.e. $f(x_0') > \lim_n f(x_n)$, so exists some N s.t.

$f(x_0') > f(x_N)$. Notice that $x_N > x_0'$, so that leads to a contradiction! So we must have 1.1 or 1.2 happens.

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Ex.4: Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|x-y| \leq |f(x)-f(y)|, \forall x,y \in \mathbb{R}.$$

Show that f is a bijective.

Proof:

① Injective: For every $x_1 \neq x_2$. we have

$$|f(x_1)-f(x_2)| \geq |x_1-x_2| > 0$$

$\Rightarrow$ $f(x_1) \neq f(x_2)$. #.

② Surjective: $\forall u \in \mathbb{R}$. we need to find some v s.t. $f(v)=u$.

Idea: For every fixed u, find some a, b s.t. $f(a) < u < f(b)$, then using intermediate thm.

Now for a fixed u s.t. $-C < u < C$ for some $C > 0$. we can select C large enough s.t. $C > |f(0)|+1$, then,

$\forall M > 0$, set $w_1 := f(0) + M$, then we have

$$|f(w_1)| + |f(0)| \geq |f(w_1) - f(0)| \geq |w_1| \geq M - |f(0)|$$

$$\Rightarrow \quad |f(w_1)| \geq M - 2|f(0)|.$$

On the other hand, set $w_2 = f(0) - M$, we have

$$|f(w_2)| + |f(0)| \geq |f(w_2) - f(0)| \geq |w_2| \geq M - |f(0)|.$$

$$\Rightarrow |f(w_2)| \geq M - 2|f(0)|.$$

Claim: If $M \geq 3|f(0)| + C$, then we must have

either $f(w_1) \geq C$, $f(w_2) \leq -C$

or $f(w_1) \leq -C$. $f(w_2) \geq C$.

Proof of the claim: By the selection of M, we have

both $|f(w_1)| \geq C$, $|f(w_2)| \geq C$, so we need only show

that $f(w_1) \cdot f(w_2) < 0$: Otherwise $f(w_1) \cdot f(w_2) > 0$, W.L.O.G.

assume $f(w_1) > 0$. $f(w_2) > 0$, then by

$$\begin{aligned} f(w_i) - f(0) &= |f(w_i)| - f(0) \\ &\geq |f(w_i)| - |f(0)| \\ &\geq C, \quad i = 1, 2. \end{aligned}$$

we have by f is continuous, for every point

$z \in [|f(0)|+1, C]$, there exists some

$x_1 \in [0, w_1]$, $x_2 \in [w_2, 0]$ s.t. $f(x_1) = f(x_2) = z$.

notice that $x_1, x_2 \neq 0$. thus $x_1 \neq x_2$. that contradicts to f is an injection

why?

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So the claim holds, as a result, we can

find a, b ($a = w_1$, $b = w_2$ or $a = w_2$, $b = w_1$) s.t.

$$f(a) \leq -C \leq u \leq C \leq f(b).$$

then by the intermediate theorem, there must

exists some v between a and b s.t.

$f(v) = u$. that shows f is a surjection.

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Ex.5 If $f:[a,b]\to\mathbb{R}$ is continouous, and $a\le x_1<x_2<\cdots<x_n\le b$, show that exists some $\xi\in[x_1,x_n]$ such that

$$f(\xi)=\frac{1}{n}\sum_{i=1}^{n}f(x_i)$$

Proof:

From the mean value Inequality. we know

$$\min_i f(x_i)\le\frac{1}{n}\sum_{i=1}^{n}f(x_i)\le\max_i f(x_i)$$

So let $c\triangleq\frac{1}{n}\sum_{i=1}^{n}f(x_i)$. consider $g:=f-c$. Since f is continuous. g is continuous as well. Besides. select

$$\begin{cases} f(x_p)=\max\limits_{1\le i\le n}f(x_i) \\ f(x_q)=\min\limits_{1\le i\le n}f(x_i) \end{cases}$$

Set $x_p<x_q$. WLOG (if $x_p=x_q$. then $f(x_1)=\cdots=f(x_n)=\frac{1}{n}\sum_i f(x_i)$. directly let $\xi=x_1$ and we're done). Then we find

$$\begin{cases} g(x_p)=f(x_p)-c\ge 0 \\ g(x_q)=f(x_q)-c\le 0 \end{cases}$$

So apply the intermediate value theorem. we conclude that here $\exists$ some $\xi\in[x_p,x_q]\subset[a,b]$ such that $g(\xi)=0\iff f(\xi)=c=\frac{1}{n}\sum_i f(x_i)$

$\square$