MATH2033 Mathematical Analysis
Problem Set 2

Problem 0
Do problem #9 and problem #10 in practice exercise.

Problem 1
We let A, B be any two sets. The symmetric difference of A and B (denoted by $A \Delta B$) is defined by

$$A \Delta B = (A \backslash B) \cup (B \backslash A).$$

(a) Show that $A \Delta B = (A \cup B) \backslash (A \cap B)$
(b) Show that $A \cap (B \Delta \mathrm{C}) = (A \cap B) \Delta (A \cap C)$ for any sets A, B, C.

Problem 2
We let A, B, C be 3 sets, prove that
(a) $A \backslash (B \backslash A) = A$
(b) $(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C)$.

Problem 3
Prove that the following statements are <u>incorrect.</u>
(a) $A \backslash B = B \backslash A$ for all non-empty sets A and B
(b) $(A \backslash B) \backslash C = A \backslash (B \backslash C)$ for any non-empty sets A, B and C
(c) $(A \backslash B) \cup C = (A \cup C) \backslash (B \cup C)$ for any non-empty sets A, B and C.
(d) If $A \cap B = A \cap C$, then $B = C$. Here, A, B, C are non-empty sets.

Problem 4 (Harder)
We let A, B be two subsets of a set Ω. If there exists a subset $C \subseteq \Omega$ such that

$$B \cap C = A \cap C \quad and \quad B \cup C = A \cup C.$$

Show that $A = B$.

Problem 5
We consider a function $f: \mathbb{R} \to \mathbb{R}$. Suppose that the function is strictly increasing (i.e. $f(x_1) < f(x_2)$ for any x_1, x_2 which $x_1 < x_2$),
(a) Show that $f(x)$ is injective.
(b) Determine if $f(x)$ is always surjective. (😊Hint: The answer is false)

Problem 6
We consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$.
(a) Write down the range of $f(x)$.
(b) Determine if the function is injective. Also, determine if the function is surjective

(c) Suppose that the domain of f is changed to $\left[0, \frac{\pi}{2}\right]$, determine if the function is injective.

Problem 7

We let $f: \mathbb{R} \to \mathbb{R}$ be a function satisfying $|f(x_1) - f(x_2)| = |x_1 - x_2|$ for any $x_1, x_2 \in \mathbb{R}$. Show that f is injective.

Problem 8

We let $f: \mathbb{R} \backslash \{-1,1\} \to \mathbb{R}$ be a function defined by $f(x) = \frac{x}{1-x^2}$.

(a) Determine if the function is injective.

(b) Determine if the function is surjective.

Problem 9 (Harder)

We let $f_1: A \to B$, $f_2: A \to B$, $g: B \to C$, $h_1: C \to D$ and $h_2: C \to D$ be five functions (Here, A, B, C and D ae subsets of $\mathbb{R}$).

(a) Suppose that g is injective and $g\big(f_1(x)\big) = g\big(f_2(x)\big)$ for all $x \in A$, show that $f_1(x) = f_2(x)$ for all $x \in A$.

(b) Suppose that g is surjective and $h_1\big(g(x)\big) = h_2\big(g(x)\big)$ for all $x \in B$, show that $h_1(x) = h_2(x)$ for all $x \in C$.

Problem 10

We consider a relation $\sim$ on $\mathbb{Z}$ which

$$m \sim n \iff m + n \ \ is\ even$$

(a) Prove that $\sim$ is an equivalent relation.

(b) Determine all equivalent classes.

Problem 11

We consider a relation $\sim$ on $\mathbb{Z}$

$$m \sim n \Leftrightarrow |m - 5| = |n - 5|$$

(a) Prove that $\sim$ is an equivalent relation.

(b) Determine all equivalent classes.

Problem 12 (Harder)

We consider an non-empty set A and let $\sim_1$ and $\sim_2$ be two equivalent relations on A.

(a) We let $\sim$ be another relation on A which

$$x \sim y \Leftrightarrow x \sim_1 y \ \ and \ \ x \sim_2 y.$$

(i) Prove that $\sim$ is an equivalent relation.

(ii) We consider an element $a \in A$ and let $[a], [a]_1$ and $[a]_2$ be the equivalence class of a with respect to the equivalent relations $\sim$, $\sim_1$ and $\sim_2$ respectively. Show that $[a] = [a]_1 \cap [a]_2$.

(b) We let $\sim^*$ be another relation on A which

$$x \sim^* y \Leftrightarrow x \sim_1 y \ \ or \ \ x \sim_2 y.$$

Determine if $\sim^*$ is an equivalence relation for any A, $\sim_1, \sim_2$.

(😊Hint: If your answer is yes, provide a mathematical proof. If your answer is no, you need to provide a counter-example and you need to specify the set A and the equivalent relations $\sim_1$ and $\sim_2$.)