supplemental exercise 2.

1. $f(x)$ is defined on $(-\infty,+\infty)$ for $\forall x\in(-\infty,+\infty)$ $f(x^2)=f(x)$

$f(x)$ is continuous ~~on~~ at $x=0$, $x=1$

prove: $f(x)$ is a constant on $(-\infty,+\infty)$

2. $f(x)$ is continuous on $[a,b]$, for $\forall x\in[a,b]$ $\exists y\in[a,b]$ and $|f(y)|\le\frac{1}{2}|f(x)|$

prove: $\exists\, \xi\in[a,b]$ $f(\xi)=0$

3. $f(x)$ is continuous on $[a,b]$. $M(x)=\sup\limits_{a\le t\le x} f(t)$ $m(x)=\inf\limits_{a\le t\le x} f(t)$

prove: $M(x)$ $m(x)$ are continuous on $[a,b]$.

4. $f(x)$ is continuous on $[a,b]$. $f(x)>0$ for $\forall x\in[a,b]$.

prove by definition: $\frac{1}{f(x)}$ is continuous on $[a,b]$

5. $f(x)$ is cont on $[0,1]$ $f(0)=f(1)$.

prove: for $\forall n\ge 2$, $n\in\mathbb{N}$, $\exists\, \xi_n\,[0,1]$, $f(\xi_n+\frac{1}{n})=f(\xi_n)$

6. $f_n(x)=x+x^2+\dots+x^n$ $(n=2,3,\dots)$

prove: (1). $f_n(x)=1$ has only one real solution on $[0,+\infty)$.

(2). denote the solution as x_n, find $\lim\limits_{n\to\infty} x_n$.

7. $f(x)$ is differentiable in (a,b). prove: for $\forall x_0\in(a,b)$

$\exists\, x_n\in(a,b)$ $n=1,2,3\dots$ $\lim\limits_{n\to\infty}x_n=x_0$ and $\lim\limits_{n\to\infty}f'(x_n)=f'(x_0)$

8. $f(x)$. cont on $[a,b]$ diff in (a,b). $a>0$. prove that:

(1). $\exists\, \xi\in(a,b)$. $f(b)-f(a)=\xi f'(\xi)\ln\frac{b}{a}$.

(2). $\lim\limits_{n\to\infty} n(~~x~~\, x^{\frac{1}{n}}-1)=\ln x$ $(x>0)$.

9. $f(x)$ cont on $[0,1]$, diff in $(0,1)$. $f(0)=f(1)=0$. $f(\frac{1}{2})=1$
prove : $\exists\ \xi \in (0,1)$ $f'(\xi)=1$.

10. $f(x)$ cont on $[0,1]$, diff in $(0,1)$. $f(0)=f(1)=0$.
prove: for $\forall\ x_0 \in (0,1)$. $\exists\ \xi \in (0,1)$. $f'(\xi)=f(x_0)$.

11. $f(x)$. cont on $[0,1]$ diff in $(0,1)$ $|f'(x)|<1$ $f(0)=f(1)$
prove: $\forall$ ~~x_0~~ $x_1, x_2 \in (0,1)$ $|f(x_1)-f(x_2)|<\frac{1}{2}$.

12. $f(x)$ $g(x)$. cont on $[a,b]$, diff in (a,b). $g(a)=0$ $f(b)=0$
$f(x), g(x) \neq 0$ for $\forall\ x \in (a,b)$. prove: $\exists\ \xi \in (a,b)$. s.t.
$$\frac{f'(\xi)}{f(\xi)} = -\frac{g'(\xi)}{g(\xi)}$$

13. $f(x)$. diff on $[a,b]$ $f'(a)=f'(b)=0$. prove $\exists\ c \in (a,b)$.
$f(c)-f(a)=(c-a)f'(c)$.

14. $f(x)$ diff on $[a,b]$ twice diff in (a,b). $f(a)=f(b)=0$. $f'(a)\cdot f'(b)>0$.
prove. $\exists\ \xi \in (a,b)$ $f(\xi)=0$, $\exists\ \eta \in (a,b)$ $f''(\eta)=f(\eta)$.

15. $f(x)$ cont on $[a,b]$, diff in (a,b). $f(a)=f(b)$ $f(x) \not\equiv$ constant.
prove. $\exists\ \xi \in (a,b)$ $f'(\xi)>0$.

16. $f(x)$ cont on $[0,1]$, diff $(0,1)$ $f(0)=0$, ~~prove~~. $f(x) \neq 0$. in $(0,1)$
prove. $\exists\ \xi \in (0,1)$. $f(\xi)\cdot f'(\xi)>0$

17. $f(x)$. diff in (a,b). $f'(x)$ monotone. prove ~~$f(x)$ cont~~ $f'(x)$ is cont in (a,b)

18. $f(x)$ bdd in $[a,+\infty)$ $f'(x)$ exists $\lim_{x\to+\infty} f'(x)=b$ prove $b=0$

19. $f(x)$ diff. $f''(x_0)$ exists prove:

$$\lim_{h\to 0} \frac{f(x_0+2h) - 2f(x_0+h) + f(x_0)}{h^2} = f''(x_0).$$

20. $f(x)$ twice diff on $[0,1]$. $|f(x)| \le a$, $|f''(x)| \le b$.
~~$a \neq 0, b \neq 0$~~. $a \ge 0$, $b \ge 0$. prove: for $\forall\, c \in (0,1)$.
$|f'(c)| \le 2a + \frac{b}{2}$.

21. $f(x)$ diff at $x = x_0$. $\alpha_n < x_0 < \beta_n$. $(n = 1, 2, \ldots)$. $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = x_0$
prove $\lim_{n\to\infty} \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n} = f'(x_0)$

22. $f(x)$ cont at $x=0$, $\lim_{x\to 0} \frac{f(2x) - f(x)}{x} = A$. prove $f'(0)$ exists
and $f'(0) = A$

23. $f(x)$ defined in a neighbourhood of x_0. $f(x)$ diff at x_0.
prove. $\lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h)}{2h} = f'(x_0)$

If the limit exists. can you prove $f(x)$ is diff at x_0?
If not, give a counter-example.

24. $f(x)$ diff on $[0, +\infty)$ $f(0) = 0$. $\exists A > 0$. $|f'(x)| \le A|f(x)|$
prove $f(x) \equiv 0$ for $x \in [0, +\infty)$.

25. prove. Darboux Theorem f diff on $[a, b]$, $f'(a) < f'(b)$
for $\forall c$: $f'(a) < c < f'(b)$ $\exists \xi \in (a,b)$ $f'(\xi) = c$.

26. $f(x)$ cont on $[a, b]$, diff in $[a, b]$, $f(a) < 0$. $f(b) < 0$.
$\exists c \in (a, b)$ $f(c) > 0$. prove $\exists \xi \in (a, b)$ $f(\xi) + f'(\xi) = 0$

27. $f(x)$ diff in (a,b). prove (1): If $f'(x)$ bdd in (a,b). Then $f(x)$ bdd in (a,b).

(2) If $f(x)$ is not bdd in (a,b). then $f'(x)$ is not bdd in (a,b).

28. $f(x)$ cont in $[0,+\infty)$. $f(0)=0$. prove that:

(1) If $f'(x)\uparrow$, then $\frac{f(x)}{x}\uparrow$ in $(0,+\infty)$

(2) If $f'(x)\downarrow$, then $\frac{f(x)}{x}\downarrow$ in $(0,+\infty)$

29. $f(x)=\begin{cases} |x| & x\neq 0 \\ 1 & x=0 \end{cases}$ prove $f(x)$ can't be a derivative function.

30. $f(x)$ twice diff on $[a,b]$, $f'(a)=f'(b)=0$. prove $\exists\, \xi\in(a,b)$

$|f''(\xi)|\geq \frac{4}{(b-a)^2}|f(b)-f(a)|$

31. $f(x)$ twice diff. $f(0)=f(1)=0$. $\max\limits_{0\leq x\leq 1} f(x)=2$. prove.

$\min\limits_{0\leq x\leq 1} f''(x)\leq -16$.

32. $f(x)$ twice diff on $[0,1]$, $0\leq x\leq 1$, $|f(x)|\leq 1$ $|f''(x)|\leq 2$.

prove $|f'(x)|\leq 3$ when $0\leq x\leq 1$.

33. $f(x)$ twice diff on $[0,+\infty)$ If $\lim\limits_{x\to+\infty} f(x)$ exists.

$f''(x)$ bdd on $[0,+\infty)$.

prove. $\lim\limits_{x\to+\infty} f'(x)=0$.

34. $f(x)$ twice diff on $[0,1]$, $f(0)=f(1)=0$, $\min\limits_{0\leq x\leq 1} f(x)=-1$

prove $\max\limits_{0\leq x\leq 1} f''(x)\geq 8$

35. $f(x)$ twice diff on $[0,1]$. $f(1)=f(0)$. $|f''(x)| \le M$. $M>0$
prove. $|f'(x)| \le \frac{M}{2}$.

36. $f(x)$ twice diff on $[0,1]$ $f(1)=f(0)=f(\frac{1}{2})=0$. $|f''(x)| \le M$.
prove. $|f'(x)| < \frac{M}{2}$.

37. $f(0)=0$ $f'(0)$ exists. $X_n = f(\frac{1}{n^2}) + f(\frac{2}{n^2}) + \dots + f(\frac{n}{n^2})$
find $\lim\limits_{n\to\infty} X_n$ $\lim\limits_{n\to\infty} [\sin\frac{1}{n^2} + \sin\frac{2}{n^2} + \dots + \sin\frac{n}{n^2}]$

$\lim\limits_{n\to\infty} [(1+\frac{1}{n^2})(1+\frac{2}{n^2}) \cdots (1+\frac{n}{n^2})]$

38. $f(x)$ diff in $(-\infty,+\infty)$ There is no point satisfies $f(x)=f'(x)=0$
prove. $\{x\in[0,1] \mid f(x)=0\}$ is finite

39. $f(x)$ diff in $(0,+\infty)$. $\lim\limits_{x\to+\infty} f'(x)=+\infty$, prove. $f(x)$ is not uniformly continuous in $(0,+\infty)$

40. $f(x)$ cont in $(0,a]$, $\lim\limits_{x\to 0^+} \sqrt{x} f'(x)$ exists. prove. $f(x)$ is uniformly continuous in $(0,a]$

41. $f(x)$ cont on $[a,b]$, twice diff in (a,b). $f(a)=f(b)=0$.
$\exists c\in(a,b)$. $f(c)<0$, prove $\exists \xi\in(a,b)$. $f''(\xi)>0$.

42. $f(x)$ twice diff in R. $M_k = \sup\limits_{x\in R} |f^{(k)}(x)| < +\infty$ $k=0,1,2$.
prove $M_1 \le M_2 M_0$

43 find min β and max α so that $\forall n\in\mathbb{N}$. $(1+\frac{1}{n})^{n+\alpha} \le e \le (1+\frac{1}{n})^{n+\beta}$

44. $f(x)$ ~~[illegible]~~ cont on $[a,b]$ $\max f(x)=M$ $\min f(x)=m$. prove. $\exists [\alpha,\beta]\subset[a,b]$,
so that (1). $f(\alpha)=m$ $f(\beta)=M$ or $f(\alpha)=M$ $f(\beta)=m$.
(2). $m<f(x)<M$ $\forall x\in(\alpha,\beta)$

45. $f(x)$ is not bdd on $[a,b]$. prove $\exists\ c\in[a,b]$, for $\forall\ \delta>0$
$f(x)$ is not bdd on. $(c-\delta, c+\delta)\cap[a,b]$.

46. $f.\ g$ diff in $(a,+\infty)$ $|g'(x)|<f'(x)$ prove. if $\lim_{x\to+\infty} f(x)$ exists, then. $\lim_{x\to+\infty} g(x)$ exists.

47. $f(x)$ diff in. $(a,+\infty)$. $|f'(x)|\downarrow$. $\lim_{x\to+\infty} f(x)$ exists.
prove. $\lim_{x\to+\infty} xf'(x)=0$.

48. $f(x)$ cont on $[a,b]$, has 且 only one max point x_0
$\{x_n\}\subset[a,b]$ $\lim_{n\to\infty} f(x_n)=f(x_0)$ prove. $\lim_{n\to\infty} x_n=x_0$

49. $f(x)$ diff on $[0,c]$, $f'(x)\downarrow$. $f(0)=0$. prove. $\forall\ 0\le a\le b\le a+b\le c$.
we have. $f(a+b)\le f(a)+f(b)$

50. $f(x)$. twice diff in a neighbourhood of x_0.
$f(x_0+\Delta x)=f(x_0)+f'(x_0+\theta\Delta x)\Delta x$. $f''(x_0)\neq 0$.
$f''(x)$ cont at $x=x_0$. prove. $\lim_{\Delta x\to 0}\theta=\frac{1}{2}$

51. $f(x)$ diff in $(0,+\infty)$. $\lim_{x\to+\infty}(f(x)+f'(x))=0$ prove. $\lim_{x\to+\infty} f(x)=0$.

52. $f(x)$ diff on $[0,1]$, $f(\lambda x)=f'(x)$ $x\in[0,1)$. $0<\lambda\le 1$.
$f(0)=0$. prove $f(x)\equiv 0$.

53. $f(x)$ cont on $[0,1]$ $f(x)>0$. $M(x)=\max_{0\le t\le x} f(t)$ $(0<x\le 1)$.
$\alpha(x)=\lim_{n\to\infty}\left(\frac{f(x)}{M(x)}\right)^n$
prove $\alpha(x)$ cont $\iff f(x)\uparrow$

54. $f(x)$ uniformly cont on $[a, +\infty)$. $\varphi(x)$ cont on $[a, +\infty)$.
$\lim_{x\to+\infty} (f(x) - \varphi(x)) = 0$. prove $\varphi(x)$ uniformly cont

55. prove cauchy theorem. for function.
$\lim_{x\to x_0} f(x)$ exists $\Leftrightarrow$. $\forall \varepsilon>0$. $\exists \delta>0$. when. $0<|x'-x_0|<\delta$,
$0<|x''-x_0|<\delta$, we have. $|f(x')-f(x'')|<\varepsilon$.

56. $f(x)$ cont on $[a, b]$, $f(a)<0$ $f(b)>0$. prove $\exists \xi \in (a,b)$.
$f(\xi)=0$ and $f(x)>0$ for $x\in(\xi, b]$

57. $f(x)$ is bdd and diff on R. $|f(x)+f'(x)| \le 1$.
prove $|f(x)| \le 1$.

58. $f(x)$ twice diff on $[a,b]$ $f''(x)$ cont on $[a,b]$. $f(a)=f(b)=0$.
prove (1) $\max_{a\le x\le b} |f(x)| \le \frac{1}{8}(b-a)^2 \cdot \max_{a\le x\le b} |f''(x)|$
(2). $\max_{a\le x\le b} |f'(x)| \le \frac{1}{2}(b-a) \max_{a\le x\le b} |f''(x)|$.

59. $f(x)$. twice diff on R. and bdd prove $\exists x_0 \in R$. $f''(x_0)=0$.

60. $f(x)$ is defined in $(0,1)$. $\lim_{x\to 0} f(x)=0$. $\lim_{x\to 0} \frac{f(x)-f(\frac{x}{2})}{x}=0$
prove. $\lim_{x\to 0} \frac{f(x)}{x}=0$.