Solution of Math 2033 Homework 1

① (a) To show $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$, let $x\in A\cap(B\cup C)$. Then $x\in A$ and ($x\in B$ or $x\in C$). If $x\in B$, then $x\in A\cap B$. If $x\in C$, then $x\in A\cap C$. Hence $x\in A\cap B$ or $x\in A\cap C$. So we get $x\in(A\cap B)\cup(A\cap C)$. This means $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.

Next, to show $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$, let $x\in(A\cap B)\cup(A\cap C)$. Then $x\in A\cap B$ or $x\in A\cap C$. In both cases, $x\in A$. In both cases, $x\in B$ or $x\in C$, which gives $x\in B\cup C$. So we get $x\in A\cap(B\cup C)$. This means $(A\cap B)\cup(A\cap C)\subseteq A\cap(B\cup C)$.

Combining the conclusions of the two paragraphs, $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$.

(b) To show $X\setminus(Y\cup Z)\subseteq(X\setminus Y)\cap(X\setminus Z)$, let $x\in X\setminus(Y\cup Z)$. Then $x\in X$ and $x\notin(Y\cup Z)$. So $x\in X$. Also,

$$\boxed{x\notin(Y\cup Z)\Leftrightarrow\sim(x\in Y\cup Z)\Leftrightarrow\sim(x\in Y\text{ or }x\in Z)\Leftrightarrow x\notin Y\text{ and }x\notin Z.}$$

Then $x\in X$ and $x\notin Y$ and $x\notin Z$. Hence, $x\in X\setminus Y$ and $x\in X\setminus Z$. So we get $x\in(X\setminus Y)\cap(X\setminus Z)$. This means $X\setminus(Y\cup Z)\subseteq(X\setminus Y)\cap(X\setminus Z)$.

Next to show $(X\setminus Y)\cap(X\setminus Z)\subseteq X\setminus(Y\cup Z)$, let $x\in(X\setminus Y)\cap(X\setminus Z)$. Then $x\in X\setminus Y$ and $x\in X\setminus Z$. In both cases, $x\in X$. Also $x\notin Y$ and $x\notin Z$. By the box above, $x\notin(Y\cup Z)$. So we get $x\in X\setminus(Y\cup Z)$. This means $(X\setminus Y)\cap(X\setminus Z)\subseteq X\setminus(Y\cup Z)$.

Combinining, we get $X\setminus(Y\cup Z)=(X\setminus Y)\cap(X\setminus Z)$.

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② (a) We will prove the statement for every $n\in\mathbb{N}=\{1,2,3,\cdots\}$,

$P(n)\rightarrow$ (S contains an element a_n such that $\forall\, m\in\mathbb{N}$ and $m<n$, we have $a_n\neq a_m$)

Case $n=1$. S infinite $\Rightarrow S\neq\emptyset$ (empty set has 0 elements, $0<\infty$)

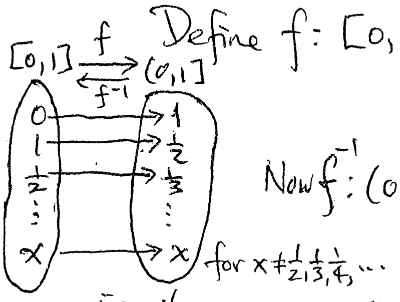
Then there exists an element a_1 in S. There is no $m\in\mathbb{N}$ and $m<1$.

Suppose case n is true. Then $A_n=\{a_m: m\in\mathbb{N}\text{ and }m\le n\}$ has n elements. Since $n<\infty$, A_n is a finite subset of S. Then $A_n\subset S$. We get $S\setminus A_n\neq\emptyset$. Then there exists an element a_{n+1} in $S\setminus A_n$. For every $m\in\mathbb{N}$ and $m<n+1$, $a_m\in A_n$ and $a_{n+1}\in S\setminus A_n$. So $a_{n+1}\neq a_m$.

The function $f:\mathbb{N}\to\{a_m: m\in\mathbb{N}\}$ defined by $f(n)=a_n$ for $n\in\mathbb{N}$ is surjective ($a_n=f(n)$) and injective ($m\neq n\Rightarrow f(m)=a_m\neq a_n=f(n)$).

$\therefore$ S contains the countably infinite set $\{a_m: m\in\mathbb{N}\}$.

② (b) Let $X=\{0,1,\frac{1}{2},\frac{1}{3},\cdots\}$. Then $[0,1]\setminus X=(0,1)\setminus\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots\}=(0,1]\setminus X$.

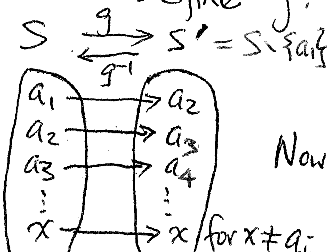


Define $f:[0,1]\to(0,1]$ by
$$f(x)=\begin{cases} x & \text{if } x\in(0,1)\setminus\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots\}\\ 1 & \text{if } x=0\\ \frac{1}{n+1} & \text{if } x=\frac{1}{n},\ n=1,2,3,\cdots \end{cases}$$

Now $f^{-1}:(0,1]\to[0,1]$ given by
$$f^{-1}(x)=\begin{cases} x & \text{if } x\in(0,1)\setminus\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\cdots\}\\ 0 & \text{if } x=1\\ \frac{1}{n-1} & \text{if } x=\frac{1}{n},\ n=2,3,4,\cdots \end{cases}$$
is the inverse of f. $\therefore f$ is a bijection.

(c) By part (a), S contains a countably infinite subset $\{a_m : m\in\mathbb{N}\}$

Let $S'=S\setminus\{a_1\}$



Define $g:S\to S'$ by
$$g(x)=\begin{cases} x & \text{if } x\in S\setminus\{a_1,a_2,a_3,\cdots\}\\ a_{n+1} & \text{if } x=a_n,\ n=1,2,3,\cdots \end{cases}$$

Now $g^{-1}:S'\to S$ given by
$$g^{-1}(x)=\begin{cases} x & \text{if } x\in S\setminus\{a_1,a_2,a_3,\cdots\}\\ a_{n-1} & \text{if } x=a_n,\ n=2,3,4,\cdots \end{cases}$$
is the inverse of g. $\therefore g$ is a bijection.

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③ For every $(m,r)\in\mathbb{Z}\times\mathbb{Q}$, $\begin{cases} y=mx\\ x^2+y^2=r^2\end{cases} \Rightarrow \begin{cases} x^2+(mx)^2=r^2\\ y=mx\end{cases} \Rightarrow \begin{cases} x^2=\frac{r^2}{1+m^2}\\ y=mx\end{cases}$

has at most 2 solutions since $x^2=r^2/(1+m^2)$ has at most 2 solutions and at most 1 y for each solution of x.

Let $S_{(m,r)}$ be the solutions of the system $\begin{cases} y=mx\\ x^2+y^2=r^2\end{cases}$ for $(m,r)\in\mathbb{Z}\times\mathbb{Q}$.

Then $S=\bigcup\limits_{\underbrace{(m,r)\in\mathbb{Z}\times\mathbb{Q}}_{\text{Countable}}}\underbrace{S_{(m,r)}}_{\text{finite, hence countable}}$ is countable by countable union theorem.

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④ For every $(x,y,z)\in\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$, let $S_{(x,y,z)}=\{2^x+3^y+5^z\}$, ← 1 element set.

then $S=\{2^x+3^y+5^z : (x,y,z)\in\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}\}=\bigcup\limits_{(x,y,z)\in\underbrace{\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}}_{\text{Countable}}}\underbrace{S_{(x,y,z)}}_{\text{finite, hence countable}}$

is countable by countable union theorem.

By example 6 (on transparencies page 21) $(0,+\infty)\setminus S$ is uncountable.

Hence $(0,+\infty)\setminus S$ is not a finite set (finite sets are countable).

So $(0,+\infty)\setminus S$ is infinite. $\therefore$ there are infinitely many $r\in(0,\infty)$ and r is not in S.

⑤ Note $(0,1) = \bigcup_{K=1}^{\infty} [\frac{1}{K+1}, \frac{1}{K})$ (See solution of practice exercise #13 (iii) as a reference)

So every element of $T \subseteq (0,1)$ is in at least one of the interval $[\frac{1}{k+1}, \frac{1}{k})$.

If $x_1, x_2, \cdots, x_n$ are in T and $[\frac{1}{k+1}, \frac{1}{k})$, then $1 > x_1^2 + \cdots + x_n^2 \geq \underbrace{\frac{1}{(k+1)^2} + \cdots + \frac{1}{(k+1)^2}}_{n \text{ times}}$.

From this we get $1 > \frac{n}{(k+1)^2}$. So $n < (k+1)^2$.

Let $T_k = T \cap [\frac{1}{k+1}, \frac{1}{k})$, then T_k has less than $(k+1)^2$ elements.

So T_k is a finite set. $\therefore$ $T = \bigcup_{k=1}^{\infty} \underbrace{T_k}_{\text{finite, hence countable}}$ is countable by the Countable union theorem.