Marking Scheme

MATH2033 Mathematical Analysis (2021 Spring)
Suggested Solution of Final Examination

Problem 1 (18 marks)

We consider a function $f: [-1,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^m & if\ x = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{4}, \dots\dots \\ 0 & if\ otherwise \end{cases},$$

where m is a positive integer.

(a) (9 marks) Find the value(s) of m such that $f(x)$ is continuous at $x = 0$.

(b) (9 marks) Find the value(s) of m such that $f(x)$ is differentiable at $x = 0$.

☺Solution

(a) We shall argue that $\lim_{x\to 0} f(x) = f(0) = 0$ for any $m \geq 1$.

Since $f(x) = 0$ or x^m, it follows that $|f(x)| \leq |x^m| = |x|^m$ and

$$-|x|^m \leq -|f(x)| \leq f(x) \leq |f(x)| \leq |x|^m.$$

Note that $\lim_{x\to 0}|x|^m = 0$, it follows from sandwich theorem that

$$\lim_{x\to 0} f(x) = 0 = f(0).$$

7 marks

So $f(x)$ is continuous at $x = 0$ for any $m \geq 1$.

2 marks (for conclusion)

(b) For any positive integer $m \geq 2$, we note that for $x \neq 0$

$$|x|^{m-1} \leq -\left|\frac{f(x)}{x}\right| \leq \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} \leq \left|\frac{f(x)}{x}\right| \overset{|f(x)| \leq |x|^m}{\leq} |x|^{m-1}.$$

Note that $\lim_{x\to 0}|x|^{m-1} = 0$ for $m2$, it follows from sandwich theorem that

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

So $f(x)$ is differentiable at $x = 0$ for $m \geq 2$.

4 marks

For $m = 1$, we consider two sequences $\{x_n\}$ and $\{y_n\}$ defined by $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n\sqrt{2}}$.

Then we have $f(x_n) = x_n$ and $f(y_n) = 0$ so that

$$\lim_{n\to\infty} \frac{f(x_n) - f(0)}{x_n - 0} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} 1 = 1.$$

$$\lim_{n\to\infty} \frac{f(y_n) - f(0)}{y_n - 0} = \lim_{n\to\infty} \frac{0}{y_n} = \lim_{n\to\infty} 0 = 0.$$

Argument 3 marks

Since $\lim_{n\to\infty} \frac{f(x_n)-f(0)}{x_n-0} \neq \lim_{n\to\infty} \frac{f(y_n)-f(0)}{y_n-0}$, then the limits $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ does not exist for $m =$ 1. So $f(x)$ is not differentiable at $x = 0$ for $m = 1$.

Reason and conclusion (2 marks)

Problem 2 (10 marks)
We let $f: [0,2] \to \mathbb{R}$ be a continuous function. Show that there exists $c \in [0,1]$ such that
$$f(c+1) - f(c) = \frac{1}{2}(f(2) - f(0)).$$

☺Solution
We consider a function $g: [0,1] \to \mathbb{R}$ defined by
$$g(x) = f(x+1) - f(x).$$
3 marks
Since $f(x)$ is continuous on $[0,2]$, it follows that $g(x)$ is also continuous on $[0,1]$. Then the statement is equivalent to
$$\underbrace{g(c) = \frac{1}{2}(g(1) + g(0))}_{denoted\ by\ K} \quad for\ some\ c \in [0,1].$$
If $g(0) = K$ or $g(1) = K$, then the above equation holds for $c = 0$ (if $g(0) = K$) or $c = 1$ (if $g(1) = K$).
If $g(0) \neq K$ and $g(1) \neq K$, since $g(0) < K = \frac{g(0)+g(1)}{2} < g(1)$, it follows from intermediate value theorem that there exists $c \in (0,1)$ such that
$$g(c) = K = \frac{1}{2}(g(1) + g(0)).$$

7 marks
(3 mark for highlighting $g(0) \le \frac{g(0)+g(1)}{2} \le g(1)$ (assume $g(0) \le g(1)$.)
+4 marks for intermediate value theorem)

Problem 3 (18 marks)

(a) **(8 marks)** We let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathbb{R}$ such that $|f'(x)| \le C$ for all $x \in \mathbb{R}$, where C is a positive constant. We let $\{x_n\}$ be a Cauchy sequence. Show that the sequence $\{y_n\}$ defined by $y_n = f(x_n)$ is also a Cauchy sequence.

(b) **(10 marks)** We let $f: (a,b) \to \mathbb{R}$ be 4-times differentiable function on (a,b) such that $|f^{(4)}(x)| \le M$ for all $x \in (a,b)$. Show that
$$\left|\frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} - f''(x_0)\right| \le \frac{M}{12}h^2$$
for any x_0 and h satisfying $a < x_0 - h < x_0 < x_0 + h < b$.

☺Solution

(a) For any $\varepsilon > 0$, we note that

- Since $\{x_n\}$ is Cauchy sequence, then there exists $K \in \mathbb{N}$ such that for $m, n \ge K$,
$$|x_m - x_n| < \frac{\varepsilon}{C}.$$
2 points
- For any $x_m \neq x_n$, we apply mean value theorem on $f(x)$ over the interval $[x_m, x_n]$ and deduce that there exists $c \in (x_m, x_n)$ such that
$$\frac{f(x_m) - f(x_n)}{x_m - x_n} = f'(c) \Rightarrow f(x_m) - f(x_n) = f'(c)(x_m - x_n).$$
2 points

Then it follows that for any $m, n \ge K$
$$|y_m - y_n| = |f(x_m) - f(x_n)| = |f'(c)(x_m - x_n)| \le C|x_m - x_n| < C\left(\frac{\varepsilon}{C}\right) = \varepsilon.$$
4 points
(*Note: Although the above inequality requires that $x_m \neq x_n$, it can be see that the inequality also holds for $x_m = x_n$ since $|y_m - y_n| = 0 < \varepsilon$ in this case.

("*Note" is not required)

So it follows that $\{y_n\}$ is Cauchy sequence by definition.

(b) For any $x \in (a,b)$ and $x_0 \in (a,b)$, we apply Taylor theorem on $f(x)$ and deduce that there exists $c_x \in (x_0, x)$ such that

$$\left.\begin{aligned} f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 \\ + \frac{f^{(4)}(c_x)}{4!}(x-x_0)^4. \end{aligned}\right\} \text{3 marks}$$

By taking $x = x_0 + h$ and $x = x_0 - h$, we have

$c_1 \in (x_0, x_0+h)$

$$\left.\begin{aligned} f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f^{(3)}(x_0)}{3!}h^3 + \frac{f^{(4)}(c_1)}{4!}h^4 \;\ldots(*) \\ f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2!}h^2 - \frac{f^{(3)}(x_0)}{3!}h^3 + \frac{f^{(4)}(c_2)}{4!}h^4 \;\ldots(**) \end{aligned}\right\} \text{4 marks}$$

$c_2 \in (x_0-h, x_0)$

Then it follow that

$$\left|\frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} - f''(x_0)\right|$$

$$= \left|\frac{f''(x_0)h^2 + \frac{f^{(4)}(c_1)}{4!}h^4 + \frac{f^{(4)}(c_2)}{4!}h^4}{h^2} - f''(x_0)\right|$$

$$= \left|\frac{f^{(4)}(c_1)}{24}h^2 + \frac{f^{(4)}(c_2)}{24}h^2\right| \overset{f^{(4)}(x)\le M}{\le} \frac{M}{24}h^4 + \frac{M}{24}h^4 = \frac{M}{12}h^2.$$

3 marks

Problem 4 (18 marks)

(a) **(8 marks)** We let $f: (a,b) \to \mathbb{R}$ be n-times differentiable function and suppose that $f^{(n)}(x) > 0$ for all $x \in (a,b)$. Show that $f(x) = 0$ has at most n solutions in the interval (a,b).

(b) **(10 marks)** We consider the equation $4x^2 - 8x + 5 = 2^x$.

(i) Show that the equation has at least one solution over $(0,1)$.

(ii) Show that the equation has exactly two solutions over $(0,2)$.

☺Solution

1 marks

(a) Suppose that $f(x) = 0$ has at least $(n+1)$ solutions over (a,b). We let $x_1 < x_2 < \cdots < x_{n+1}$ be some solutions (may not all) of $f(x) = 0$.

For any $k = 1,2,\ldots,n$, we apply Rolle's theorem on $f(x)$ over $[x_k, x_{k+1}]$ and deduce that there exists $c_k \in (x_k, x_{k+1})$ such that

$$f'(c_k) = 0.$$

4 marks

So $f'(x) = 0$ has at least n solutions.

By applying Rolle's theorem on $f'(x)$ over $[c_k, c_{k+1}]$ for $k = 1,2,\ldots,n-1$, we deduce that there exists $d_k \in (x_k, x_{k+1})$ such that

$$f''(d_k) = (f')'(d_k) = 0.$$

2 marks (Repeated use of Rolle's thm)

Then $f''(x) = 0$ has at least $n-1$ solutions.

By repeating this process, we deduce that $f^{(3)}(x) = 0$ has at least $n-2$ solutions, $f^{(4)}(x) = 0$ has at least $n-3$ solutions and so on.

2 marks Finally, we deduce that $f^{(n)}(x) = 0$ has at least 1 solution over (a, b) which contradicts to the assumption that $f^{(n)}(x) > 0$ for all $x \in (a, b)$.
Therefore we conclude that $f(x) = 0$ has at most n solutions.

(b) (i) We let $f(x) = 4x^2 - 8x + 5 - 2^x$ which is continuous over $\mathbb{R}$. Note that
- ✓ $f(0) = 0 - 0 + 5 - 1 = 4 > 0$ and
- ✓ $f(1) = 4 - 8 + 5 - 2 = -1 < 0$

3 marks

It follows from intermediate value theorem that there exists $c \in (0,1)$ such that $f(c) = 0$ so that $f(x) = 0$ (equivalent to $4x^2 - 8x + 5 = 2^x$) has at least one solution over $(0,1)$. 1 mark.

(ii) On the other hand, we see that $f(2) = 16 - 16 + 5 - 4 = 1 > 0$. It follows from intermediate value theorem that there exists $d \in (1,2)$ (and $d \neq c$) such that $f(d) = 0$. So $f(x) = 0$ has at least two solutions over $(0,2)$. 3 marks
Furthermore, we note that
$$f''(x) = 8 - 2^x(\ln 2)^2 > 8 - 2^2 \underbrace{(\ln 2)^2}_{<1} > 0 \quad for \ x \in (0,2).$$
It follows from the result of (a) that the equation $f(x) = 0$ has at most two solutions over $(0,2)$. Combining the earlier result, we deduce that the equation $f(x) = 0$ has exactly two solutions. 3 marks

Problem 5 (20 marks)

(a) (10 marks) We let $[a, b]$ (where $a < b$) be an closed interval. For any closed interval $[c, d] \subseteq [a, b]$ (where $a < c < d < b$), we define a function $g: [a, b] \to \mathbb{R}$ as
$$g(x) = \begin{cases} 1 & if \ x \in [c, d] \\ 0 & if \ otherwise' \end{cases}$$
Using integral criterion or the definition of integrability, determine if $g(x)$ is integrable.

(b) (10 marks) We let $f: [a, b] \to \mathbb{R}$ be a bounded Riemann integrable function and let $g: [a, b] \to \mathbb{R}$ be another bounded function such that the set $\{x \in [a, b] | f(x) \neq g(x)\} = \{x_1, x_2, \dots, x_n\}$ where $a < x_1 < x_2 < \cdots < x_n < b$.

(i) Show that $g(x)$ is integrable. (☺Hint: Consider the function $h(x) = g(x) - f(x)$)

(ii) Show that
$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

☺Solution

(a) For any $\varepsilon > 0$, we consider the following partition
$$\mathcal{P} = \left\{ \underbrace{a}_{x_0}, \underbrace{c - \delta}_{x_1}, \underbrace{c + \delta}_{x_2}, \underbrace{d - \delta}_{x_3}, \underbrace{d + \delta}_{x_4}, \underbrace{b}_{x_5} \right\}$$
4 points for correct choice of partition.

where $\delta > 0$ is some positive constant (the value will be determined later).
Under this partition, we have

$$U(\mathcal{P},g) - L(\mathcal{P},g) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1})$$
$$= (0-0)(x_1 - x_0) + (1-0)(x_2 - x_1) + (1-1)(x_3 - x_2) + (1-0)(x_4 - x_3) + (0-0)(x_5 - x_4)$$
$$= 2\delta + 2\delta = 4\delta.$$

Calculation of $U-L$ (4 points)

By taking $\delta < \frac{\varepsilon}{4}$, we deduce that

$$U(\mathcal{P},g) - L(\mathcal{P},g) < 4\left(\frac{\varepsilon}{4}\right) = \varepsilon.$$

So $g(x)$ integrable on $[a,b]$ by integral criterion.

2 points

(b) (i) We consider the function $h(x) = g(x) - f(x)$. Note that

$$h(x) = \begin{cases} g(x_k) - f(x_k) & if\ x = x_1, x_2, \dots, x_n \\ 0 & if\ otherwise \end{cases}.$$

2 marks. (correct formula of $h(x)$)

One can see that $h(x)$ is not continuous at $x = x_k$ if $g(x_k) - f(x_k) \neq 0$. Thus, the number of discontinuity points of $h(x)$ is *at most* n and therefore finite. So it follows that $h(x)$ is integrable over $[a,b]$.

2 marks

Therefore, we conclude that the function

$$g(x) = \underbrace{f(x)}_{integrable} + \underbrace{(g(x) - f(x))}_{integrable}.$$

is integrable over $[a,b]$.

(ii) We shall prove that $\int_a^b h(x)dx = 0$. To facilitate the analysis, we let

$$M = \sup\{h(x_1), h(x_2), \dots, h(x_n)\} \quad and \quad m = \inf\{h(x_1), h(x_2), \dots, h(x_n)\}.$$

For any $\varepsilon > 0$, we consider the partition

$$\mathcal{P} = \{a, x_1 - \delta, x_1 + \delta, x_2 - \delta, x_2 + \delta, \dots, x_n - \delta, x_n + \delta, b\},$$

where $\delta = \frac{\varepsilon}{2nM}$.

Then we have

$$\int_a^b h(x)dx \leq U(\mathcal{P},h) \leq \sum_{k=1}^{n} M\big(x_k + \delta - (x_k - \delta)\big) = 2nM\delta \overset{\delta = \frac{\varepsilon}{2nM}}{=} \varepsilon.$$

By taking $\varepsilon \to 0^+$, we have $\int_a^b h(x)dx \leq 0$.

On the other hand,

$$\int_a^b h(x)dx \geq L(\mathcal{P},h) \geq \sum_{k=1}^{n} m\big(x_k + \delta - (x_k - \delta)\big) = 2nm\delta = \frac{m}{M}\varepsilon.$$

By taking $\varepsilon \to 0^+$, we have $\int_a^b h(x)dx \geq 0$.

So it follows that $\int_a^b h(x)dx = 0$.

4 marks for arguing $\int_a^b h(x)dx = 0$

Hence, it follows from property of integral that

$$\int_a^b g(x)dx = \int_a^b f(x)dx + \int_a^b \underbrace{(g(x) - f(x))}_{h(x)} dx = \int_a^b f(x)dx.$$

2 marks.

Problem 6 (16 marks)

We let $f: \mathbb{R} \to \mathbb{R}$ be a function.

(a) (12 marks) We let L be a real number. Show that $\lim_{x\to+\infty} f(x) = L$ if and only if $\lim_{n\to\infty} f(x_n) = L$ for any sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = +\infty$.

(b) (4 marks) Does the limits $\lim_{x\to\infty} \frac{\sin x}{2+\cos x}$ converge to a real number? Explain your answer.

☺Solution

(a) ("⇒" part)

We let $\{x_n\}$ be a sequence which $\lim_{n\to\infty} x_n = +\infty$. We shall argue that $\lim_{n\to\infty} f(x_n) = L$ using the definition of limits.

For any $\varepsilon > 0$,

3 marks

- ✓ Since $\lim_{x\to+\infty} f(x) = L$, then there exists $M > 0$ such that $|f(x) - L| < \varepsilon$ when $x > M$.
- ✓ Since $\lim_{n\to\infty} x_n = +\infty$, then there exists $K \in \mathbb{N}$ such that $x_n > M$ for $n \geq K$.

3 marks

For this integer K, it follows that when $n \geq K$,

$$|f(x_n) - L| \overset{\substack{x_n > M \\ for\ n \geq K}}{<} \varepsilon.$$

So $\lim_{n\to\infty} f(x_n) = L$ using the definition of limits.

("⇐" part)

2 marks

Suppose that $\lim_{x\to+\infty} f(x) \neq L$, then there exists $\varepsilon_0 > 0$ such that for any $M > 0$, there exists x_0 such that $x_0 > M$ and $|f(x_0) - L| \geq \varepsilon_0$.

By taking $M = n$ (where $n \in \mathbb{N}$), we deduce that there exists x_n satisfying

$$x_n > M = n \quad and \quad |f(x_n) - L| \geq \varepsilon_0 \ldots (*)$$

2 marks for correct $\{x_n\}$

By repeating the process for all positive integer n, we obtain a sequence $\{x_n\}$ such that each x_n satisfies the inequalities $(*)$.

2 marks for arriving successful contradiction

Note that $\lim_{n\to\infty} x_n = +\infty$, it follows that $\lim_{n\to\infty} f(x_n) = L$, then for $\varepsilon = \varepsilon_0$, there exists $K \in \mathbb{N}$ such that

$$|f(x_n) - L| < \varepsilon = \varepsilon_0 \quad for\ n \geq K.$$

This contradicts to the inequality $(*)$ since the inequality $(*)$ is supposed to be valid for all positive integer n. Hence, we conclude that $\lim_{x\to+\infty} f(x) = L$.

(b) We consider two sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_n = 2n\pi \quad and \quad y_n = 2n\pi + \frac{\pi}{2}.$$

2 marks each for each of $\{x_n\}$ and $\{y_n\}$

We observe that $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} y_n = +\infty$. On the other hand, we deduce that

$$\lim_{n\to\infty} \frac{\sin x_n}{2 + \cos x_n} = \lim_{n\to\infty} \frac{0}{2+1} = 0$$

$$\lim_{n\to\infty} \frac{\sin y_n}{2 + \cos y_n} = \lim_{n\to\infty} \frac{1}{2+0} = \frac{1}{2}.$$

Since $\lim_{n\to\infty} \frac{\sin x_n}{2+\cos x_n} \neq \lim_{n\to\infty} \frac{\sin y_n}{2+\cos y_n}$, so the limits $\lim_{x\to\infty} \frac{\sin x}{2+\cos x}$ does not exist by the result of (a).