Solutions to Practice Exercises

① $\sim((x>0 \text{ and } x<1) \text{ or } x=-1) = \sim(x>0 \text{ and } x<1) \text{ and } x\neq -1$

$= (x\leq 0 \text{ or } x\geq 1) \text{ and } x\neq -1$

② $\sim(x>0 \text{ and } (x<1 \text{ or } x=-1)) = x\leq 0 \text{ or } \sim(x<1 \text{ or } x=-1)$

$= x\leq 0 \text{ or } (x\geq 1 \text{ and } x\neq -1)$

$= x\leq 0 \text{ or } x\geq 1$

③ $\sim(\forall \triangle ABC, \angle A+\angle B+\angle C=180^\circ) = \exists \triangle ABC$ such that $\angle A+\angle B+\angle C\neq 180^\circ$

(There is a triangle ABC such that $\angle A+\angle B+\angle C\neq 180^\circ$.)

④ $\sim(\exists$ man such that man does not have wife$) = \forall$ man, man has a wife

(Every man has a wife.)

⑤ $\sim(\forall x.\ \exists y$ such that $x+y=0) = \exists x\ \forall y,\ x+y\neq 0$

(There is an x such that for every y, $x+y\neq 0$.)

⑥ $\sim(\exists\alpha\ \forall\beta\ \exists\gamma$ such that $|\alpha-\beta|<\gamma) = \forall\alpha\ \exists\beta\ \forall\gamma,\ |\alpha-\beta|\geq\gamma.$

⑦ $\sim($If $(x>0)$ and $(y>0)$, then $x+y>0) = (x>0)$ and $(y>0)$ and $(x+y\leq 0)$

⑧ (a) If $\angle B\neq\angle C$ in $\triangle ABC$, then $AB\neq AC$ in $\triangle ABC$.

(b) If a function is not continuous, then it is not differentiable.

(c) If $\lim\limits_{x\to 0}(f(x)+g(x))\neq a+b$, then $\lim\limits_{x\to 0} f(x)\neq a$ or $\lim\limits_{x\to 0} g(x)\neq b$.

(d) If $x\neq \frac{-b+\sqrt{b^2-4c}}{2}$ and $x\neq\frac{-b-\sqrt{b^2-4c}}{2}$, then $x^2+bx+c\neq 0$.

⑨ (a) $(\{x,y,z\}\cup\{w,z\})\setminus\{u,v,w\} = \{w,x,y,z\}\setminus\{u,v,w\} = \{x,y,z\}$.

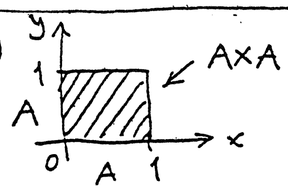
(b) $\{1,2\}\times\{3,4\}\times\{5\} = \{(1,3,5),(1,4,5),(2,3,5),(2,4,5)\}$.

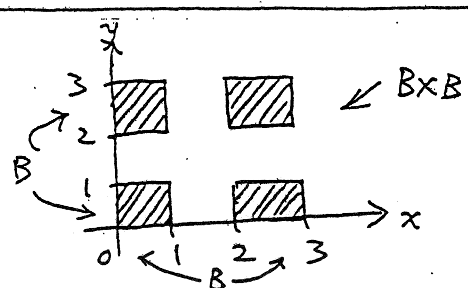
(c) $\mathbb{Z}\cap[0,10]\cap\{n^2+1 : n\in\mathbb{N}\} = \{0,1,2,\cdots,10\}\cap\{2,5,10,\cdots\} = \{2,5,10\}$.

(d) $\{n\in\mathbb{N}: 5<n<9\}\setminus\{2m : m\in\mathbb{N}\} = \{6,7,8\}\setminus\{2,4,6,8,10,\cdots\} = \{7\}$.

(e) $([0,2]\setminus[1,3])\cup([1,3]\setminus[0,2]) = [0,1)\cup(2,3]$.

⑩ (i)





(ii) $A=B$ (Reason: For every $a\in A$, $b\in B$, we have $(a,b)\in A\times B = B\times A$. By the definition of Cartesian product, this means $a\in B$, $b\in A$. So $A\subseteq B$ and $B\subseteq A$.)

(11) (a) If $x\in A\cup B$, then $x\in A$ or $x\in B$, which implies $x\in A$ or $x\in C$ (because $B\subseteq C$ and $x\in B$ will yield $x\in C$). So $x\in A\cup C$. So every element of $A\cup B$ is also an element of $A\cup C$. Therefore, $A\cup B\subseteq A\cup C$.

(b) If $x\in (X\setminus Y)\setminus Z$, then $x\in X\setminus Y$ and $x\notin Z$. So $x\in X$ and $x\notin Y$ and $x\notin Z$. Then $x\in X$ and $x\notin Z$ and $x\notin Y$. Hence, $x\in X\setminus Z$ and $x\notin Y$. Therefore, $x\in (X\setminus Z)\setminus Y$. We get $(X\setminus Y)\setminus Z\subseteq (X\setminus Z)\setminus Y$.

Interchanging Y, Z everywhere in the last paragraph, we also get $(X\setminus Z)\setminus Y\subseteq (X\setminus Y)\setminus Z$. Therefore, $(X\setminus Y)\setminus Z=(X\setminus Z)\setminus Y$.

---

(12) (i) False. For example, $A=\mathbb{R}\setminus\mathbb{Q}$, $B=\mathbb{Q}=C$, then $(A\cup B)\cap C=\mathbb{R}\cap\mathbb{Q}=\mathbb{Q}$, $A\cup(B\cap C)=(\mathbb{R}\setminus\mathbb{Q})\cup\mathbb{Q}=\mathbb{R}$

(ii) False. For example, $A=\mathbb{R}=B$, $C=\mathbb{Q}$, then $A\cup B=\mathbb{R}=A\cup C$, but $B\neq C$.

(iii) True. (Reason: For every $x\in A\setminus(B\cup C)$, we have $x\in A$ and $x\notin B\cup C$. Now $\boxed{x\notin B\cup C=\sim(x\in B\cup C)=\sim((x\in B)\text{ or }(x\in C))=x\notin B\text{ and }x\notin C}$. So $x\in A\setminus B$ and $x\in A\setminus C$. We get $x\in(A\setminus B)\cap(A\setminus C)$. $\therefore$ $A\setminus(B\cup C)\subseteq(A\setminus B)\cap(A\setminus C)$.

Next we reverse steps. For every $x\in(A\setminus B)\cap(A\setminus C)$, we have $x\in A\setminus B$ and $x\in A\setminus C$. So $x\in A$ and $x\notin B$ and $x\notin C$. By the box above, we get $x\in A$ and $x\notin B\cup C$. So $x\in A\setminus(B\cup C)$. $\therefore$ $(A\setminus B)\cap(A\setminus C)\subseteq A\setminus(B\cup C)$.)

---

(13) (i) For every $x\in A\cup C$, we have $x\in A$ or $x\in C$. If $x\in A$, then $A\subseteq B$ implies $x\in B$. If $x\in C$, then $C\subseteq D$ implies $x\in D$. So $x\in B$ or $x\in D$, which implies $x\in B\cup D$.

(ii) False. For example, let $A=\{0\}$, $C=\{1\}$, $B=\{0,1\}=D$, then $A\cup C=\{0,1\}=B\cup D$.

(iii) Yes. (Reason: Since $(\frac{1}{n},2)_{\mathbb{Q}}\subseteq[\frac{1}{n},2)_{\mathbb{Q}}$ for each n, so as in part (i),

$$\bigcup_{n=1}^{\infty}(\tfrac{1}{n},2)_{\mathbb{Q}}=(1,2)_{\mathbb{Q}}\cup(\tfrac{1}{2},2)_{\mathbb{Q}}\cup(\tfrac{1}{3},2)_{\mathbb{Q}}\cup\cdots\subseteq[1,2)_{\mathbb{Q}}\cup[\tfrac{1}{2},2)_{\mathbb{Q}}\cup[\tfrac{1}{3},2)_{\mathbb{Q}}\cup\cdots=\bigcup_{n=1}^{\infty}[\tfrac{1}{n},2)_{\mathbb{Q}}$$

For the reverse inclusion, Since $[\frac{1}{n},2)_{\mathbb{Q}}\subseteq(\frac{1}{n+1},2)_{\mathbb{Q}}$ for each n, we have

$$\bigcup_{n=1}^{\infty}[\tfrac{1}{n},2)_{\mathbb{Q}}=[1,2)_{\mathbb{Q}}\cup[\tfrac{1}{2},2)_{\mathbb{Q}}\cup[\tfrac{1}{3},2)_{\mathbb{Q}}\cup\cdots\subseteq(\tfrac{1}{2},2)_{\mathbb{Q}}\cup(\tfrac{1}{3},2)_{\mathbb{Q}}\cup(\tfrac{1}{4},2)_{\mathbb{Q}}\cup\cdots=\bigcup_{n=1}^{\infty}(\tfrac{1}{n},2)_{\mathbb{Q}})$$

because $(\frac{1}{2},2)_{\mathbb{Q}}=(1,2)_{\mathbb{Q}}\cup(\frac{1}{2},2)_{\mathbb{Q}}$.

$\boxed{\text{Actually, } \bigcup_{n=1}^{\infty}[\tfrac{1}{n},2)_{\mathbb{Q}}=(0,2)_{\mathbb{Q}}=\bigcup_{n=1}^{\infty}(\tfrac{1}{n},2)_{\mathbb{Q}} \text{ but this is less rigorous.}}$

---

(14) f is not injective because $f(1)=0=f(2)$. f is not surjective because $f(\mathbb{R})=\{0,1\}\neq\mathbb{R}$. g is injective because $g(x)=g(y)\Leftrightarrow 1-2x=1-2y$ implies $x=y$. g is surjective because for every $y\in\mathbb{R}$, $y=g(\frac{1-y}{2})$ and so $g(\mathbb{R})=\mathbb{R}$.

$f\circ g:\mathbb{R}\to\mathbb{R}$ is given by $(f\circ g)(x)=f(g(x))=f(1-2x)=\begin{cases}0 & \text{if } 1-2x>0\\ 1 & \text{if } 1-2x\leq 0\end{cases}$

$=\begin{cases}0 & \text{if } \frac{1}{2}>x\\ 1 & \text{if } \frac{1}{2}\leq x\end{cases}$. $g\circ f:\mathbb{R}\to\mathbb{R}$ is given by $(g\circ f)(x)=g(f(x))=\begin{cases}1=g(0) & \text{if } x>0\\ -1=g(1) & \text{if } x\leq 0\end{cases}$.

(15) (i) To show f is injective, let $f(x)=f(y)$. Then $x=(g\circ f)(x)=g(f(x))=g(f(y))$ $=(g\circ f)(y)=y$. Next we will show f is surjective. For every $b\in B$, since $b=(f\circ g)(b)=f(g(b))$, we see that $b\in f(A)$. $\therefore$ $f(A)=B$.

(ii) To show $h\circ f$ is injective, let $(h\circ f)(x)=(h\circ f)(y)$. Then $h(f(x))=h(f(y))$. Since h is injective, we get $f(x)=f(y)$. Since f is injective, we get $x=y$. Next we will show $h\circ f$ is surjective. For every $c\in C$, since h is surjective, $C=h(B)$, which implies $c=h(b)$ for some $b\in B$. Since f is surjective, $B=f(A)$, which implies $b=f(a)$ for some $a\in A$. Then $c=h(b)=h(f(a))$ $=(h\circ f)(a)\in(h\circ f)(A)$. $\therefore$ $(h\circ f)(A)=C$.

---

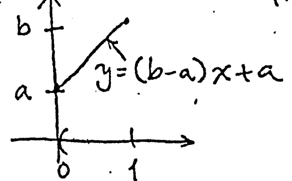
(16) For the 'at most once' case, to show f is injective, let $f(x_0)=f(y_0)$. Using the choice $b=f(x_0)$, we see that the line $y=b$ intersects the graph of f at the point $(x_0,f(x_0))$ and at the point $(y_0,f(y_0))$. Since the intersection is at most one point, we have $(x_0,f(x_0))=(y_0,f(y_0))$, which implies $x_0=y_0$.

For the 'at least once' case, we can conclude f is surjective. (Reason: For every $b\in B$, the line $y=b$ intersects the graph of f at least once. This implies there is a point (a,b) on the graph of f. Then $b=f(a)\in f(A)$. $\therefore$ $f(A)=B$.)

(Comments: Combining the two cases, we see that if for every $b\in B$, the horizontal line $y=b$ intersects the graph of f exactly once, then f is a bijection. This "horizontal line test" is useful to check bijections by inspecting the graphs.)

---

(17)



The function $f:(0,1)\to(a,b)$ defined by $f(x)=(b-a)x+a$ is a bijection. (This is clear from the graph. As x varies from a to b, $f(x)$ takes each of the values between a and b exactly once.) Since $(0,1)$ is uncountable, by the bijection theorem we see that (a,b) is uncountable. Since $(a,b)\subseteq[a,b]$, by the countable subset theorem, $[a,b]$ is uncountable.

---

(18) Let $S=\{(0,y): y\in\mathbb{R}\setminus\mathbb{Q}\}$. The function $f:\mathbb{R}\setminus\mathbb{Q}\to S$ defined by $f(y)=(0,y)$ is a bijection. Since $\mathbb{R}\setminus\mathbb{Q}$ is uncountable, by the remarks, S is uncountable. Since $S\subseteq\mathbb{Q}\times(\mathbb{R}\setminus\mathbb{Q})$, by the countable subset theorem, $\mathbb{Q}\times(\mathbb{R}\setminus\mathbb{Q})$ is uncountable.

---

(19) For $n,m\in\mathbb{Z}$, $\frac{1}{2^n}+\frac{1}{3^m}\in\mathbb{Q}$. So $A\subseteq\mathbb{Q}$. Since $\mathbb{Q}$ is countable, by the countable subset theorem, A is countable.

---

(20) For $x\in\mathbb{N}$, let $B_x=\{x+\sqrt{2}y : y\in\mathbb{N}\}$. The function $f:\mathbb{N}\to B_x$ defined by $f(y)=x+\sqrt{2}y$ is a bijection. So B_x is countable. Now $B=\bigcup_{x\in\mathbb{N}} B_x$, $\mathbb{N}$ is countable, each B_x is countable for $x\in\mathbb{N}$, so by the countable union theorem, B is countable.

---

(21) Let $S=\{L_m : L_m$ is the line with equation $y=mx$, $m\in\mathbb{R}\}$. The function $f:\mathbb{R}\to S$ defined by $f(m)=L_m$ is a bijection. Since $\mathbb{R}$ is uncountable, by the remarks, S is uncountable. Since $S\subseteq C$, by the countable subset theorem, C is uncountable.

C contains vertical line, not in S,

---

(22) For $r\in\mathbb{Q}$, $D_r=\{x\in\mathbb{R}\mid x^5+x+2=r\}$ has at most 5 elements, so D_r is countable. Now $D=\bigcup_{r\in\mathbb{Q}} D_r$, $\mathbb{Q}$ is countable and each D_r is countable for $r\in\mathbb{Q}$, so by the countable union theorem, D is countable.

---

(23) Let $\mathbb{Q}^+$ be the positive rational numbers. Since $\mathbb{Q}^+\subseteq\mathbb{Q}$ and $\mathbb{Q}$ is countable, by the countable subset theorem, $\mathbb{Q}^+$ is countable. Now the function $f:\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}^+\to E$ defined by letting $f(x,y,r)$ be the circle centered at (x,y) and radius r is a bijection. Since $\mathbb{Q}$ and $\mathbb{Q}^+$ are countable, by the product theorem, $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}^+$ is countable. By the bijection theorem, E is countable.

---

(24) Suppose $x^4+ax-5=0$ has a rational root r. (If $r=0$, then $r^4+ar-5\neq 0$.) We get $r\neq 0$ and $r^4+ar-5=0 \Rightarrow a=\frac{5-r^4}{r}\in\mathbb{Q}$. So $F\subseteq\mathbb{Q}$. Therefore, F is countable.

---

(25) Since X is nonempty, let $a_0\in X$. Consider the subset $G'=\{a_0^3+b^3 : b\in Y\}$ of G. The function $f:Y\to G'$ defined by $f(b)=a_0^3+b^3$ is a bijection (From $w=a_0^3+b^3 \Leftrightarrow b=\sqrt[3]{w-a_0^3}$, we see $g:G'\to Y$ defined by $g(w)=\sqrt[3]{w-a_0^3}$ is the inverse of f.) Since Y is uncountable, so G' is uncountable. Since $G'\subseteq G$, so G is also uncountable.

---

(26) We will show $Y\setminus X$ is uncountable first. Suppose $Y\setminus X$ is countable. Since X is countable and $X\cap Y\subseteq X$, we get $X\cap Y$ countable by the countable subset theorem. Then $Y=(Y\setminus X)\cup(X\cap Y)$ is countable by the countable union theorem, a contradiction. $\therefore$ $Y\setminus X$ is uncountable. Since $Y\setminus X\subseteq(X\setminus Y)\cup(Y\setminus X)$, $H=(X\setminus Y)\cup(Y\setminus X)$ is uncountable by the countable subset theorem.

Solutions to Presentation Exercises (Week 6)

(27) Solution 1

For $k=0,1,2,\cdots$, let S_k be the set of all subsets of $\mathbb{N}$ having exactly k elements. Then $S_0=\{\emptyset\}$ has one element and so S_0 is countable. For $k\in\mathbb{N}$, the function $f_k: S_k \to \underbrace{\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}}_{k\ \mathbb{N}'s}$ defined by $f(\{n_1,n_2,\cdots,n_k\}) = \underbrace{(n_1,n_2,\cdots,n_k)}_{\text{increasing order}}$ is an injective function. Since $\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}$ is countable by the product theorem, we can use the bijection theorem to conclude that S_k is countable. Then $F = S_0 \cup \left(\bigcup_{k=1}^{\infty} S_k\right)$ is countable by the countable union theorem.

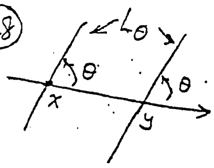
Solution 2 Define $g: F \to \mathbb{N}\cup\{0\}$ by assigning to each finite subset S of $\mathbb{N}$ the nonnegative integer n having base 2 representation $n=(\cdots d_3d_2d_1)_2$, where $d_j=1$ if and only if $j\in S$. (For example, $S=\{1,2,4\} \to n=(1011)_2=8+2+1=11$.) Note g has the inverse $g^{-1}: \mathbb{N}\cup\{0\}\to F$ by assigning $n=(\cdots d_3d_2d_1)_2$ the subset $\{j: d_j=1\}$. It follows g is a bijection. As $\mathbb{N}\cup\{0\}$ is countable, so F is countable.

Solution 3 Let $F_0=\{\emptyset\}$. Let F_k be the set of all subsets of $\{1,2,\cdots,k\}$ for $k\in\mathbb{N}$. By math induction, we can check that F_k has 2^k elements. (This was discussed in the power set examples.) Clearly, $F_k\subseteq F$ because every element of F_k is a finite subset of $\mathbb{N}$. So $\bigcup_{k=0}^{\infty} F_k \subseteq F$ by the definitions of union and subset. For every $S\in F$, S is a finite subset of $\mathbb{N}$. If $S\neq\emptyset$, then S has a maximum element k. Then $S\in F_k$. Hence $F\subseteq \bigcup_{k=0}^{\infty} F_k$.

$\therefore$ $F=\bigcup_{k=0}^{\infty} F_k$ is a countable union of finite sets, hence countable by the countable union theorem.

---

(28)



For $\theta\in(0,\pi)$, let L_θ be the pair of lines through x and y respectively making an angle θ with the axis from x to y. Let $T=\{L_\theta: \theta\in(0,\pi)\}$. The function $f:(0,1)\to T$ defined by $f(x)=L_{\pi x}$ has the inverse $g: T\to(0,1)$ given by $g(L_\theta)=\frac{\theta}{\pi}$. So f is a bijection. Since $(0,1)$ is uncountable, T is uncountable.

Next observe that for every $z\in S$, there are at most two θ's such that z is on one of the lines of L_θ, namely when $\overleftrightarrow{xz}$ or $\overleftrightarrow{yz}$ is one of the lines of L_θ. So $V=\{L_\theta: L_\theta \text{ contains some } z\in S\} = \bigcup_{\underset{\text{countable}}{z\in S}} \underbrace{\{L_\theta: L_\theta \text{ contains } z\}}_{\text{at most 2 elements}}$ is countable.

Since T is uncountable and V is countable, so $T\setminus V$ is uncountable. In particular, taking two distinct L_θ's in $T\setminus V$, the parallelogram determined by them is in $\mathbb{R}^2\setminus S$ and has x,y as opposite vertices.

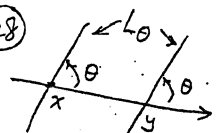
---

(27) Solution 1

For $k=0,1,2,\cdots$, let S_k be the set of all subsets of $\mathbb{N}$ having exactly k elements. Then $S_0=\{\emptyset\}$ has one element and so S_0 is countable. For $k\in\mathbb{N}$, the function $f_k: S_k \to \underbrace{\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}}_{k\ \mathbb{N}'s}$ defined by $f(\{n_1,n_2,\cdots,n_k\}) = \underbrace{(n_1,n_2,\cdots,n_k)}_{\text{increasing order}}$ is an injective function. Since $\mathbb{N}\times\mathbb{N}\times\cdots\times\mathbb{N}$ is countable by the product theorem, we can use the bijection theorem to conclude that S_k is countable. Then $F=S_0\cup\left(\bigcup_{k=1}^{\infty}S_k\right)$ is countable by the countable union theorem.

Solution 2 Define $g: F\to\mathbb{N}\cup\{0\}$ by assigning to each finite subset S of $\mathbb{N}$ the nonnegative integer n having base 2 representation $n=(\cdots d_3d_2d_1)_2$, where $d_j=1$ if and only if $j\in S$. (For example, $S=\{1,2,4\}\to n=(1011)_2=8+2+1=11$.) Note g has the inverse $g^{-1}: \mathbb{N}\cup\{0\}\to F$ by assigning $n=(\cdots d_3d_2d_1)_2$ the subset $\{j: d_j=1\}$. It follows g is a bijection. As $\mathbb{N}\cup\{0\}$ is countable, so F is countable.

---

(28) 

For $\theta\in(0,\pi)$, let L_θ be the pair of lines through x and y respectively making an angle θ with the axis from x to y. Let $T=\{L_\theta:\theta\in(0,\pi)\}$. The function $f:(0,1)\to T$ defined by $f(x)=L_{\pi x}$ has the inverse $g: T\to(0,1)$ given by $g(L_\theta)=\frac{\theta}{\pi}$. So f is a bijection. Since $(0,1)$ is uncountable, T is uncountable.

Next observe that for every $z\in S$, there are at most two θ's such that z is on one of the lines of L_θ, namely when $\overleftrightarrow{xz}$ or $\overleftrightarrow{yz}$ is one of the lines of L_θ. So $V=\{L_\theta: L_\theta \text{ contains some } z\in S\}=\underbrace{\bigcup_{z\in S}}_{\text{countable}}\underbrace{\{L_\theta: L_\theta \text{ contains } z\}}_{\text{at most 2 elements}}$ is countable.

Since T is uncountable and V is countable, so $T\setminus V$ is uncountable. In particular, taking two distinct L_θ's in $T\setminus V$, the parallelogram determined by them is in $\mathbb{R}^2\setminus S$ and has x,y as opposite vertices.

---

(29) For $x\in[0,1]$, let $x \doteq (0.a_1a_2a_3\cdots)_3$. Observe that $(\frac13,\frac23)=\{x: a_1=1\}$ where we take $\frac13=(0.022\cdots)_3$. So $K_1=[0,1]\setminus(\frac13,\frac23)=\{x: a_1\neq 1\}$.

Also, $(\frac19,\frac29)=\{x: a_1=0, a_2=1\}$ where we take $\frac19=(0.0022\cdots)_3$ and $(\frac79,\frac89)=\{x: a_1=2, a_2=1\}$ where we take $\frac79=(0.2022\cdots)_3$.

So $K_2=K_1\setminus\left((\frac19,\frac29)\cup(\frac79,\frac89)\right)=\{x: a_1\neq1, a_2\neq 1\}$. Similarly, we will get $K_n=\{x: a_1\neq1, a_2\neq1,\cdots,a_n\neq1\}$. Therefore $K=\{x: \text{all } a_i\neq 1\}$ $=\{x: \text{all } a_i=0 \text{ or } 2\}$. Define $f:\{0,1\}\times\{0,1\}\times\cdots\to K$ by $f((b_1,b_2,\cdots))=x$ where $a_i=2b_i$ for $i=1,2,3,\cdots$. This function has the inverse $g: K\to\{0,1\}\times\{0,1\}\times\cdots$ defined by $g(x)=(b_1,b_2,\cdots)$, where $b_i=\frac{a_i}{2}$ for $i=1,2,3,\cdots$. So f is a bijection. Since $\{0,1\}\times\{0,1\}\times\cdots$ is uncountable, K is uncountable.

Remarks In the above solution, when we wrote K_1, K_2, K_n, K as sets of x with $a_i\neq 1$, we mean "x has at least one base 3 representation, where the a_i's $\neq 1$."

---

30 (a) Since $\lim_{k\to\infty} \cos(\sin\frac{1}{k}) = \cos(\sin 0) = \cos 0 = 1 \neq 0$, by term test, the series diverges

---

(b) For k large, $\frac{1}{\sqrt{k(k+1)(k+2)}} \approx \frac{1}{\sqrt{k\cdot k\cdot k}} = \frac{1}{k^{3/2}}$. So we apply the limit comparison test. Since $\lim_{k\to\infty} \frac{\frac{1}{\sqrt{k(k+1)(k+2)}}}{\frac{1}{\sqrt{k\cdot k\cdot k}}} = \lim_{k\to\infty}\sqrt{\frac{k}{k}\frac{k}{k+1}\frac{k}{k+2}} = \sqrt{1\cdot 1\cdot 1} = 1$ and $\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}$ (p series, $p=3/2>1$) converges by p-test, so $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k(k+1)(k+2)}}$ converges.

(c). $\sum_{k=1}^{\infty}\ln(1+\frac{1}{k}) = \sum_{k=1}^{\infty}\ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{\infty}(\ln(k+1)-\ln k) = \lim_{n\to\infty}(\ln(n+1)-\underbrace{\ln 1}_{0}) = +\infty$,
↑ telescoping series
the series diverges (to $+\infty$).

(d) Solution 1 By root test, $\lim_{k\to\infty}\sqrt[k]{\left(\frac{1}{2}+\frac{1}{k}\right)^k} = \lim_{k\to\infty}\left(\frac{1}{2}+\frac{1}{k}\right) = \frac{1}{2} < 1 \Rightarrow$ Series Converges.

Solution 2 For $k \geq 3$, $0 \leq \left(\frac{1}{2}+\frac{1}{k}\right)^k \leq \left(\frac{1}{2}+\frac{1}{3}\right)^k = \left(\frac{5}{6}\right)^k$ and $\sum_{k=3}^{\infty}\left(\frac{5}{6}\right)^k$ (geometric series, $r=\frac{5}{6}$) converges by geometric series test, so $\sum_{k=1}^{\infty}\left(\frac{1}{2}+\frac{1}{k}\right)^k = \frac{3}{2}+1+\sum_{k=3}^{\infty}\left(\frac{1}{2}+\frac{1}{k}\right)^k$ converges.

(e) Solution 1 Since $\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2} < 0$ for $x > e$ and $\lim_{k\to\infty}\frac{\ln k}{k} \left(= \lim_{k\to\infty}\frac{\frac{1}{k}}{1} = 0\right)$, so $\frac{\ln x}{x}$ decreases to 0 for $x > e$. Now $\int_3^{\infty}\frac{\ln x}{x}dx = \frac{1}{2}(\ln x)^2\Big|_3^{\infty} = \infty$. By the integral test, $\sum_{k=2}^{\infty}\frac{\ln k}{k} = \frac{\ln 2}{2} + \sum_{k=3}^{\infty}\frac{\ln k}{k}$ diverges.

Solution 2 For $k \geq 3$; $\frac{1}{k} \leq \frac{\ln k}{k}$. Since $\sum_{k=3}^{\infty}\frac{1}{k}$ (p-series, $p=1$) diverges by p-test, so $\sum_{k=2}^{\infty}\frac{\ln k}{k}$ diverges by the comparison test.

(f) Since $\left|\frac{\cos 2^k}{k^2}\right| \leq \frac{1}{k^2}$ and $\sum_{k=1}^{\infty}\frac{1}{k^2}$ (p-series, $p=2$) converges by p-test, so $\sum_{k=1}^{\infty}\left|\frac{\cos 2^k}{k^2}\right|$ converges by the comparison test. Then $\sum_{k=1}^{\infty}\frac{\cos 2^k}{k^2}$ converges by the absolute convergence test.

(g) Solution 1 (Limit Comparison Test) Since $\lim_{k\to\infty}\frac{\left(\frac{k+2}{k+1}\right)\left(\frac{2}{3}\right)^k}{\left(\frac{2}{3}\right)^k} = \lim_{k\to\infty}\frac{k+2}{k+1} = 1$ and $\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k$ (geometric, $r=\frac{2}{3}<1$) converges, so $\sum_{k=1}^{\infty}\frac{k+2}{k+1}\left(\frac{2}{3}\right)^k$ converges.

Solution 2 (Ratio test) $\lim_{k\to\infty}\frac{\frac{k+3}{k+2}\left(\frac{2}{3}\right)^{k+1}}{\frac{k+2}{k+1}\left(\frac{2}{3}\right)^k} = \lim_{k\to\infty}\frac{(k+3)(k+1)}{(k+2)^2}\frac{2}{3} = \frac{2}{3} < 1$, Series Converges.

(h) Since $\cos k\pi = (-1)^k$ and ($k \nearrow \infty \Rightarrow \sqrt{k} \nearrow \infty \Rightarrow \frac{1}{\sqrt{k}} \searrow 0$), so $\sum_{k=1}^{\infty}\frac{\cos k\pi}{\sqrt{k}}$ converges by the alternating series test.

8 (i) Solution 1 Since $\frac{d}{dx}(xe^{-x^2}) = e^{-x^2}(1-2x^2) < 0$ for $x \ge 1$ and $\lim_{x\to\infty}\frac{x}{e^{x^2}} = \lim_{x\to\infty}\frac{1}{2xe^{x^2}} = 0$, xe^{-x^2} decreases to 0 as $x \to \infty$. Now $\int_1^\infty xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}\Big|_0^\infty = 0-(-\frac{1}{2}e^{-1}) < \infty$. By the integral test, $\sum_{k=1}^{\infty} ke^{-k^2}$ converges.

Solution 2 (Ratio test) $\lim_{k\to\infty}\frac{(k+1)e^{-(k+1)^2}}{k\,e^{-k^2}} = \lim_{k\to\infty}\frac{k+1}{k}e^{-2k-1} = 0 < 1 \Rightarrow \sum_{k=1}^{\infty} ke^{-k^2}$ converges.

(j) Solution 1 (Ratio test) $\lim_{k\to\infty}\frac{\frac{k+1}{(k+2)!}}{\frac{k}{(k+1)!}} = \lim_{k\to\infty}\frac{k+1}{k}\frac{(k+1)!}{(k+2)!} = \lim_{k\to\infty}\frac{k+1}{k}\lim_{k\to\infty}\frac{1}{k+2} = 1\cdot 0 = 0 < 1$, Series Converges.

Solution 2 $\sum_{k=1}^{\infty}\frac{k}{(k+1)!} = \sum_{k=1}^{\infty}\frac{(k+1)-1}{(k+1)!} = \sum_{k=1}^{\infty}\left(\frac{1}{k!}-\frac{1}{(k+1)!}\right) = \lim_{n\to\infty}\left(\frac{1}{1!}-\frac{1}{(n+1)!}\right) = 1.$

telescoping series

(k) Solution 1 Since $0 \le \frac{\text{Arctan}\,k}{k^2+1} \le \frac{\frac{\pi}{2}}{k^2}$ and $\sum_{k=1}^{\infty}\frac{\frac{\pi}{2}}{k^2} = \frac{\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by p-test, so $\sum_{k=1}^{\infty}\frac{\text{Arctan}\,k}{k^2+1}$ converges by the comparison test.

Solution 2 $0 \le \frac{\text{Arctan}\,k}{k^2+1} \le \frac{\frac{\pi}{2}}{k^2+1} \xRightarrow{\text{sandwich thm}} \lim_{k\to\infty}\frac{\text{Arctan}\,k}{k^2+1} = 0$. Now $\frac{d}{dx}\left(\frac{\text{Arctan}\,x}{x^2+1}\right) = \frac{1-2x\,\text{Arctan}\,x}{(x^2+1)^2}$

Since x, $\text{Arctan}\,x$ are increasing, $1-2x\,\text{Arctan}\,x \le 1-2\,\text{Arctan}\,1 < 0$ for $x \ge 1$.
So $\frac{\text{Arctan}\,x}{x^2+1}$ decreases to 0 as $x\to\infty$. Now $\int_1^\infty\frac{\text{Arctan}\,x}{x^2+1}dx = \frac{1}{2}(\text{Arctan}\,x)^2\Big|_1^\infty = \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 < \infty$. By the integral test, $\sum_{k=1}^{\infty}\frac{\text{Arctan}\,k}{k^2+1}$ converges.

(l) Since $\lim_{k\to\infty}\frac{\frac{1}{k^{1+1/k}}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{1}{k^{1/k}} = 1$ and $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges by p-test, so $\sum_{k=1}^{\infty}\frac{1}{k^{1+1/k}}$ diverges by the limit comparison test.

(m) By the root test, $\lim_{k\to\infty}\sqrt[k]{\left|\tan^k\left(\frac{k+1}{k}\right)\right|} = \lim_{k\to\infty}\left|\tan\left(\frac{k+1}{k}\right)\right| = \tan 1 > \tan\frac{\pi}{4} = 1$
$\Rightarrow$ Series diverges

(n) $\lim_{k\to\infty}\frac{1-\cos\frac{1}{k}}{\frac{1}{k^p}} \underset{\theta=\frac{1}{k}}{=} \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^p} = \lim_{\theta\to 0}\frac{\sin\theta}{p\theta^{p-1}} = \frac{1}{2}$ (if we set $p=2$). Since $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by p-test, so $\sum_{k=1}^{\infty}(1-\cos\frac{1}{k})$ converges by the limit comparison test.

(o) $\lim_{k\to\infty}\frac{k^2\sin^p(\frac{1}{k})}{k^2(\frac{1}{k})^p} = \lim_{k\to\infty}\left(\frac{\sin(\frac{1}{k})}{\frac{1}{k}}\right)^p \underset{\theta=\frac{1}{k}}{=} \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^p = 1$. Since $\sum_{k=1}^{\infty}k^2\left(\frac{1}{k}\right)^p = \sum_{k=1}^{\infty}\frac{1}{k^{p-2}}$ converges iff $p-2>1$ by the p-test, so $\sum_{k=1}^{\infty}k^2\sin^p(\frac{1}{k})$ converges iff $p>3$.

(p) $\sqrt{k+1}-\sqrt{k} = (\sqrt{k+1}-\sqrt{k})\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}+\sqrt{k}} = \frac{1}{\sqrt{k+1}+\sqrt{k}}$. As $k\nearrow\infty$, $\sqrt{k+1}+\sqrt{k}\nearrow\infty$, $\sqrt{k+1}-\sqrt{k} = \frac{1}{\sqrt{k+1}+\sqrt{k}} \searrow 0$.
So $\sum_{k=1}^{\infty}(-1)^{k+1}(\sqrt{k+1}-\sqrt{k})$ converges by the alternating series test.

(31) Let $s_n = a_1 + a_2 + \cdots + a_n$ and $t_k = 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}$. By definition, $\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$ and $\sum_{k=1}^{\infty} 2^k a_{2^k} = \lim_{k\to\infty} t_k$. Since $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$, s_n and t_k's are increasing. Their limits are either numbers or $+\infty$. Now

$$S_{2^k-1} = a_1 + (a_2+a_3) + (a_4 + \cdots + a_7) + \cdots + (a_{2^{k-1}} + \cdots + a_{2^k-1})$$
$$\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^{k-1} a_{2^{k-1}} = a_1 + t_{k-1}$$

So if $\lim_{k\to\infty} t_k < \infty$, then $\lim_{n\to\infty} S_n \underset{\uparrow}{=} \lim_{k\to\infty} S_{2^k-1} \leq a_1 + \lim_{k\to\infty} t_k < \infty$.
(S_n increasing)

Conversely, $t_k = 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = 2(a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k}) \leq 2(a_2 + (a_3+a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}))$
$= 2(S_{2^k} - a_1)$.

So if $\lim_{k\to\infty} S_n = \lim_{k\to\infty} S_{2^k} < \infty$, then $\lim_{k\to\infty} t_k \leq 2(\lim_{k\to\infty} S_{2^k} - a_1) < \infty$.

For the second part, $\sum_{k=3}^{\infty} 2^k \frac{1}{2^k \ln 2^k \ln(\ln 2^k)} = \sum_{k=3}^{\infty} \frac{1}{k \ln 2 (\ln k + \ln\ln 2)}$. We compare this with $\sum_{k=3}^{\infty} \frac{1}{k \ln k}$. Since $\lim_{k\to\infty} \frac{\frac{1}{k\ln 2(\ln k + \ln\ln 2)}}{\frac{1}{k\ln k}} = \lim_{k\to\infty} \frac{1}{\ln 2 + \frac{\ln\ln 2}{\ln k}} = \frac{1}{\ln 2}$ and $\sum_{k=3}^{\infty} \frac{1}{k\ln k}$ diverges by example of integral test; so $\sum_{k=3}^{\infty} 2^k \frac{1}{2^k \ln 2^k \ln(\ln 2^k)}$ diverges by the limit comparison test. By first part, $\sum_{k=3}^{\infty} \frac{1}{k \ln k \ln(\ln k)}$ diverges.

---

(32) Since $\lim_{k\to\infty} \frac{k+1}{2^k} \Big/ \frac{k}{2^{k-1}} = \lim_{k\to\infty} \frac{k+1}{k} \frac{1}{2} = \frac{1}{2} < 1$, the series converges by the ratio test.

Now $S = \sum_{k=2}^{\infty} \frac{k}{2^{k-1}} = \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \cdots$

$\frac{1}{2} S = \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots$

$\frac{1}{2} S = S - \frac{1}{2} S = \frac{2}{2} + \left(\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\right) = \frac{2}{2} + \frac{\frac{1}{2^2}}{1 - \frac{1}{2}} = \frac{3}{2}$

Therefore, $S = 3$.

---

(33). Suppose $\sum_{k=1}^{\infty} \frac{1}{p_k} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ converges to s. Then $s_n = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p_n}$ has limit s as $n\to\infty$, i.e. $\lim_{n\to\infty}(s - s_n) = 0$. So for some n, $s - s_n = \frac{1}{p_{n+1}} + \frac{1}{p_{n+2}} + \cdots = \sum_{k=n+1}^{\infty} \frac{1}{p_k} < \frac{1}{2}$.

Let $Q = p_1 p_2 \cdots p_n$, then the numbers $1 + mQ$ cannot be divisible by $p_1, p_2, \cdots, p_n$.

So $1 + mQ = p_{n+1}^{e_{n+1}} p_{n+2}^{e_{n+2}} \cdots$, where the exponents e_k are nonnegative integers.

Let $j = e_{n+1} + e_{n+2} + \cdots$ (only finitely many $e_k \neq 0$), then

$$\frac{1}{1+mQ} = \frac{1}{p_{n+1}^{e_{n+1}}} \cdot \frac{1}{p_{n+2}^{e_{n+2}}} \cdots \text{ is a term in } \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^{e_{n+1}} \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^{e_{n+2}} \cdots = \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^j$$

So the numbers $\frac{1}{1+Q}, \frac{1}{1+2Q}, \cdots, \frac{1}{1+NQ}$ will correspond to N terms of $\sum_{j=1}^{\infty} \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^j$, which is less than $\sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = 1$. Then

$$\sum_{m=1}^{N} \frac{1}{1+mQ} \leq \sum_{j=1}^{\infty} \left(\sum_{k=n+1}^{\infty} \frac{1}{p_k}\right)^j < \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = 1 \text{ for every positive integer } N.$$

Since $\frac{1}{2mQ} \leq \frac{1}{1+mQ}$ and $\sum_{m=1}^{\infty} \frac{1}{2mQ} = \frac{1}{2Q} \sum_{m=1}^{\infty} \frac{1}{m}$ diverges to $+\infty$ by p-test, so $\sum_{m=1}^{\infty} \frac{1}{1+mQ}$ diverges to $+\infty$ by the comparison test. Therefore $\sum_{m=1}^{N} \frac{1}{1+mQ} < 1$ cannot hold for all positive integer N, a contradiction.

(34) (a) $A = \{\sqrt{1}+\sqrt{1}, \sqrt{2}+\sqrt{1}, \sqrt{1}+\sqrt{2}, \dots\}$ is not bounded above. However, A has 2 as a lower bound because $\sqrt{m}+\sqrt{n} \geq \sqrt{1}+\sqrt{1} = 2$ for every $m, n \in \mathbb{N}$. In fact, $\inf A = 2$ because 2 is a lower bound and every lower bound $b \leq \sqrt{1}+\sqrt{1} \in A$.

(b) $B = (-\infty, \pi] \cup \{3, 3\frac{1}{2}, 3\frac{2}{3}, \dots\}$ is not bounded below. However, B has 4 as an upper bound because $\pi \leq 4$ and $4 - \frac{1}{n} \leq 4$ for all $n \in \mathbb{N}$. (Note $4 \notin B$.) We will show $\sup B = 4$. Assume there is an upper bound $t < 4$. By the Archimedian principle, there is $n \in \mathbb{N}$ such that $n > \frac{1}{4-t}$. Then $4 - \frac{1}{n} > t$ and $4 - \frac{1}{n} \in B$, which contradicts t being an upper bound.

(c) For $n, m \in \mathbb{N}$, $0 < \frac{1}{n} + \frac{1}{2^m} \leq \frac{1}{1} + \frac{1}{2^1} = \frac{3}{2}$. So C has 0 as a lower bound and $\frac{3}{2}$ as an upper bound. In fact, $\sup C = \frac{3}{2}$ because $\frac{1}{1} + \frac{1}{2^1} = \frac{3}{2} \in C$ and every upper bound $M \geq \frac{1}{1} + \frac{1}{2^1}$. Also, we can show $\inf C = 0$ as follow. Assume there is a lower bound $t > 0$. By Archimedean principle, there is $k \in \mathbb{N}$ such that $k > \frac{1}{t}$. Then taking $m = n = 2k$, we have $t > \frac{1}{k} = \frac{1}{2k} + \frac{1}{2k} \geq \underset{n}{\frac{1}{n}} + \underset{m}{\frac{1}{2^m}} \in C$, contradicting t being a lower bound.

(d) For $x \in D$, $0 < x < \sqrt{2}$. So D has 0 as a lower bound and $\sqrt{2}$ as an upper bound. In fact, $\sup D = \sqrt{2}$ because if there is an upper bound $t < \sqrt{2}$, then by density of rationals, there will be $\frac{m}{n} \in \mathbb{Q}$ such that $\max(t, 0) < \frac{m}{n} < \sqrt{2}$, which means $t < \frac{m}{n} \in D$, contradicting t being an upper bound.

Next, $\inf D = 0$ because if there is a lower bound $s > 0$, then by the density of rationals, there will be $\frac{p}{q} \in \mathbb{Q}$ such that $0 < \frac{p}{q} < \min(s, \sqrt{2})$, which means $\frac{p}{q} \in D$ and $\frac{p}{q} < s$, contradicting s being a lower bound.

Remarks If supremum limit theorem and infimum limit theorem are allowed, then the proofs by contradiction above can be avoided.

For (b), taking $w_n = 4 - \frac{1}{n} \in B$, we have $\lim_{n\to\infty} w_n = 4$. Since 4 is an upper bound, $\sup B = 4$ by the supremum limit theorem.

For (c), taking $w_n = \frac{1}{n} + \frac{1}{2^n} \in C$, we have $\lim_{n\to\infty} w_n = 0$. Since 0 is a lower bound, $\inf C = 0$ by the infimum limit theorem.

For (d), taking $w_n = \frac{1}{n} \in D$ and $z_n = \frac{[10^n\sqrt{2}]}{10^n} \in D$, we have $\lim_{n\to\infty} w_n = 0$ and $\lim z_n = \sqrt{2}$. Since 0 is a lower bound and $\sqrt{2}$ is an upper bound, so $\inf D = 0$ and $\sup D = \sqrt{2}$ by the infimum limit theorem and the supremum limit theorem.

---

(35) Let $A = (-\infty, 0) = B$, then both A and B are bounded above by 0, but $S = (0, +\infty)$ is not bounded above, $T = (-\infty, \infty)$ is not bounded above.

---

(36) For every $x \in A$, $y \in B$, we have $x \le \sup A$ and $y \le \sup B$. So $x+y \le \sup A + \sup B$.
$\therefore$ C is bounded above by $\sup A + \sup B$. As $\sup A + \sup B$ is an upper bound of C, we have $\sup C \le \sup A + \sup B$. Assume $\sup C < \sup A + \sup B$. Let $\varepsilon = \frac{\sup A + \sup B - \sup C}{2} > 0$. By the supremum property, $\exists\, x \in A$ such that $\sup A - \varepsilon < x \le \sup A$ and $\exists\, y \in B$ such that $\sup B - \varepsilon < y \le \sup B$. Adding these, we get $\sup C = \sup A + \sup B - 2\varepsilon < x+y \in C$, a contradiction. Therefore, $\sup C = \sup A + \sup B$.

Another Solution: As in the first solution, $\sup C \le \sup A + \sup B$.
Conversely, for every $x \in A$, $y \in B$, $x+y \le \sup C$, so $x \le \sup C - y$.
Then $\sup C - y$ is an upper bound of A. So $\sup A \le \sup C - y$. Then $y \le \sup C - \sup A$. This implies $\sup C - \sup A$ is an upper bound of B. So $\sup B \le \sup C - \sup A$. Then $\sup A + \sup B \le \sup C$. $\therefore \sup C = \sup A + \sup B$.

Alternate Solution (using Supremum Limit theorem): As above, we have $\sup C \le \sup A + \sup B$. By supremum limit theorem, $\exists\, a_n \in A$ with $\lim_{n\to\infty} a_n = \sup A$ and $\exists\, b_n \in B$ with $\lim_{n\to\infty} b_n = \sup B$. Then $a_n + b_n \in C$ and $\lim_{n\to\infty}(a_n+b_n) = C$. By the supremum limit theorem, the upper bound $\sup A + \sup B$ of set C is the supremum of C.

---

(37) Given $\varepsilon > 0$. (Consider the inequalities $\frac{4}{n^2} < \frac{\varepsilon}{2}$ and $\frac{5}{n^3} < \frac{\varepsilon}{2}$. If n satisfies these, then $\frac{4n+5}{n^3} = \frac{4}{n^2} + \frac{5}{n^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.) So let $K = \left[\max\left(\sqrt{\frac{8}{\varepsilon}}, \sqrt[3]{\frac{10}{\varepsilon}}\right)\right] + 1$, then
$n \ge K \Rightarrow n > \sqrt{\frac{8}{\varepsilon}}$ and $n > \sqrt[3]{\frac{10}{\varepsilon}} \Rightarrow \left|\frac{4n+5}{n^3} - 0\right| = \frac{4}{n^2} + \frac{5}{n^3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.
For $\varepsilon = 0.1$, we can choose $K = \left\lceil \max\left(\sqrt{\frac{8}{0.1}}, \sqrt[3]{\frac{10}{0.1}}\right)\right\rceil$, e.g. $K = 9$ will do.

---

(38) We have $y - 1 < [y] \le y$. So $\frac{(x-1)+(2x-1)+\cdots+(nx-1)}{n^2} < a_n \le \frac{x+2x+\cdots+nx}{n^2}$,
i.e. $\frac{\frac{n(n+1)}{2}x - n}{n^2} = \frac{(n+1)x}{2n} - \frac{1}{n} < a_n \le \frac{\frac{n(n+1)}{2}x}{n^2} = \frac{(n+1)x}{2n}$.
Since $\lim_{n\to\infty}\left(\frac{(n+1)x}{2n} - \frac{1}{n}\right) = \frac{x}{2} = \lim_{n\to\infty}\frac{(n+1)x}{2n}$, by squeeze limit theorem, $\lim_{n\to\infty} a_n = \frac{x}{2}$.

---

(39) Let $x \in \mathbb{R}$. For every $n \in \mathbb{N}$, by the density of rational numbers, there is $r_n \in \mathbb{Q}$ such that $x - \frac{1}{n} < r_n < x$. Since $\lim_{n\to\infty}(x - \frac{1}{n}) = x = \lim_{n\to\infty} x$, by the squeeze limit theorem or Sandwich theorem, $\lim_{n\to\infty} r_n = x$.

---

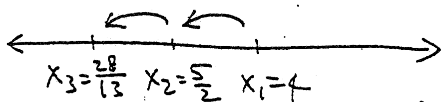
(40) Let $r = |x-y|$. By triangle inequality, $|x| = |(x-y)+y| \le |x-y| + |y| = r + |y|$ and so $|x| - |y| \le r$. Also $|y| = |(y-x)+x| \le |y-x| + |x| = r + |x|$ and so $-r \le |x| - |y|$. Then $-r \le |x| - |y| \le r$. Therefore, $\big||x| - |y|\big| \le r = |x-y|$.
Next we will show if $\lim_{n\to\infty} a_n = A$, then $\lim_{n\to\infty}|a_n| = |A|$. For $\varepsilon > 0$, since $\lim_{n\to\infty} a_n = A$, by definition of convergence, there is $K \in \mathbb{N}$ such that $n \ge K \Rightarrow |a_n - A| < \varepsilon$. Then $n \ge K \Rightarrow \big||a_n| - |A|\big| \le |a_n - A| < \varepsilon$.
(Alternatively, $\lim_{n\to\infty} a_n = A \Leftrightarrow \lim_{n\to\infty}|a_n - A| = 0$. Since $0 \le \big||a_n| - |A|\big| \le |a_n - A|$ and $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty}|a_n - A|$, by squeeze limit theorem, $\lim_{n\to\infty}\big||a_n| - |A|\big| = 0 \Leftrightarrow \lim_{n\to\infty}|a_n| = |A|$.) The converse is false. Take $a_n = (-1)^n$. Then $\lim_{n\to\infty}|a_n| = 1$, but $\lim_{n\to\infty} a_n$ doesn't exist.

(41) For $\varepsilon>0$, since $\lim_{n\to\infty} a_n = A$, by definition of convergence, there is $K\in\mathbb{N}$ such that $n\ge K \Rightarrow |a_n-A|<\varepsilon$. Then

$$n\ge K \Rightarrow n+1\ge K \Rightarrow \left|\frac{a_n+a_{n+1}}{2}-A\right| = \left|\frac{a_n-A}{2}+\frac{a_{n+1}-A}{2}\right| \le \left|\frac{a_n-A}{2}\right|+\left|\frac{a_{n+1}-A}{2}\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

---

(42) $x_1=4,\ x_2=\frac{5}{2},\ x_3=\frac{28}{13}$



(We suspect $\{x_n\}$ is decreasing. If $\lim_{n\to\infty} x_n = x$, then $x=\lim_{n\to\infty} x_{n+1} = \lim \frac{4(1+x_n)}{4+x_n} = \frac{4(1+x)}{4+x}$. Solving this, we get $x=\pm 2$. Since $x_n>0$, $x=2$.)

<u>We will first show $x_n\ge 2$ for all $n\in\mathbb{N}$ by mathematical induction.</u> For $n=1$, $x_1=4\ge 2$. Assume $x_n\ge 2$, then $2x_n\ge 4 \Rightarrow 4+4x_n\ge 8+2x_n \Rightarrow x_{n+1}=\frac{4(1+x_n)}{4+x_n}\ge 2$

<u>Next we will show $x_n\ge x_{n+1}$ for all $n\in\mathbb{N}$ by mathematical induction.</u> For $n=1$, $x_1=4\ge x_2=\frac{5}{2}$. Assume $x_n\ge x_{n+1}$. Since $x_{n+1}\ge 2$, so $4x_{n+1}+x_{n+1}^2\ge 4x_{n+1}+4$

$\Rightarrow x_{n+1}\ge \frac{4(1+x_{n+1})}{4+x_{n+1}} = x_{n+2}$. $\because x_{n+1}^2\ge 4$

By the monotone sequence theorem, $\{x_n\}$ converges. (In fact, we saw above that $\lim_{n\to\infty} x_n = 2$)

---

(43) By AM-GM inequality, $1+\frac{1}{n+1} = \frac{\overbrace{(1+\frac{1}{n})+\cdots+(1+\frac{1}{n})}^{n \text{ times}}+1}{n+1} \ge \sqrt[n+1]{(1+\frac{1}{n})^n\cdot 1}$. Taking $(n+1)$-st power of both sides, we get $\left(1+\frac{1}{n+1}\right)^{n+1}\ge \left(1+\frac{1}{n}\right)^n$. So $\{(1+\frac{1}{n})^n\}$ is increasing. Next, by binomial theorem,

$$\left(1+\frac{1}{n}\right)^n = 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3+\cdots+\left(\frac{1}{n}\right)^n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$$

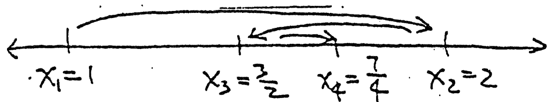
$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots = 3.$$

---

(44) (a) Since $\{x_n\}$ is bounded, $\exists$ upper bound u, lower bound v for $\{x_n\}$. Then $v\le x_n\le u$ for all n. Then $v\le m_n\le M_n\le u$ for all n, i.e. $\{M_n\}$ and $\{m_n\}$ are bounded.
Now M_n is an upper bound of $\{x_{n+1},x_{n+2},\cdots\}$ and m_n is a lower bound of $\{x_{n+1},x_{n+2},\cdots\}$ imply $M_{n+1}\le M_n$ and $m_n\le m_{n+1}$. So $\{M_n\}$ is decreasing, $\{m_n\}$ is increasing.
By the monotone limit theorem, both $\{M_n\}$ and $\{m_n\}$ converge.

(b) Since $m_n\le x_n\le M_n$, so $\lim_{n\to\infty} M_n = x = \lim_{n\to\infty} m_n \Rightarrow \lim_{n\to\infty} x_n = x$ by Sandwich theorem.
Conversely, if $\lim_{n\to\infty} x_n = x$, then $\forall\varepsilon>0\ \exists K$ such that $n\ge K\Rightarrow |x_n-x|<\varepsilon_0=\varepsilon/2$
$\Rightarrow x_K, x_{K+1}, x_{K+2},\cdots \in(x-\varepsilon_0, x+\varepsilon_0) \Rightarrow M_K, M_{K+1}, M_{K+2},\cdots\in[x-\varepsilon_0,x+\varepsilon_0]\subseteq(x-\varepsilon,x+\varepsilon)$
$m_K, m_{K+1}, m_{K+2},\cdots\in[x-\varepsilon_0,x+\varepsilon_0]\subseteq(x-\varepsilon,x+\varepsilon)$.
So $n\ge K\Rightarrow |M_n-x|<\varepsilon$ and $|m_n-x|<\varepsilon$. $\therefore \lim_{n\to\infty} M_n = x = \lim_{n\to\infty} m_n$.

(45) $x_1=1,\ x_2=2,\ x_3=\frac{3}{2},\ x_4=\frac{7}{4}$



Let $I_n=[x_{2n-1},x_{2n}]$. We will show $I_n \supseteq I_{n+1}$ (i.e. $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$) by mathematical induction. For $n=1$, $x_1=1\le x_3=\frac{3}{2}\le x_4=\frac{7}{4}\le x_2=2$.
Assume $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$. Then $x_{2n+1}\le \frac{x_{2n+1}+x_{2n+2}}{2}(=x_{2n+3})\le x_{2n+2}$ and $x_{2n+3}\le \frac{x_{2n+3}+x_{2n+2}}{2}(=x_{2n+4})\le x_{2n+2}$. So $x_{2n+1}\le x_{2n+3}\le x_{2n+4}\le x_{2n+2}$.

Now $I_1\supseteq I_2\supseteq I_3\supseteq\cdots$ implies $\lim\limits_{n\to\infty}x_{2n-1}=x$ and $\lim\limits_{n\to\infty}x_{2n}=x'$. We will show $x=x'$. (By the intertwining sequence theorem, this will imply $\{x_n\}$ converges.)

Method I. Since x_{k+1} is the midpoint of x_k and x_{k-1}, so $x_{k+1}-x_k=\frac{x_{k-1}-x_k}{2}$.
Then $x_{2n-1}-x_{2n}=\frac{-(x_{2n-2}-x_{2n-1})}{2}=\frac{x_{2n-3}-x_{2n-2}}{2^2}=\cdots=\frac{x_1-x_2}{2^{2n-2}}=\frac{-1}{2^{2n-2}}$. So $\lim\limits_{n\to\infty}|x_{2n-1}-x_{2n}|=0$. By the nested interval theorem, $\bigcap\limits_{n=1}^{\infty}I_n=\{x\}$. So $x=x'$.

Method II. $x=\lim\limits_{n\to\infty}x_{2n+1}=\lim\limits_{n\to\infty}\frac{x_{2n}+x_{2n-1}}{2}=\frac{x'+x}{2}\Rightarrow x=x'$.

(Remarks We can find $\lim\limits_{n\to\infty}x_n$ as follow: $x_n=x_1+(x_2-x_1)+(x_3-x_2)+\cdots+(x_n-x_{n-1})$
$=1+1-\frac{1}{2}+\frac{1}{4}-\cdots+(-\frac{1}{2})^{n-1}$. So $\lim\limits_{n\to\infty}x_n=1+(1-\frac{1}{2}+\frac{1}{4}-\cdots)=1+\frac{1}{1-(-\frac{1}{2})}=\frac{5}{3}$.)

---

(46) Let $s_n=\sum\limits_{k=2}^{n}|x_k-x_{k-1}|$ and $s=\sum\limits_{k=2}^{\infty}|x_k-x_{k-1}|$. For every $\varepsilon>0$, since $\sum\limits_{k=2}^{\infty}|x_k-x_{k-1}|$ converges $\Leftrightarrow \lim\limits_{n\to\infty}s_n=s$, so $\exists K$ such that $n\ge K\Rightarrow |s_n-s|=\sum\limits_{k=n+1}^{\infty}|x_k-x_{k-1}|<\varepsilon$.
Then for $m,n\ge K$, say $m\ge n$, we have

$$|x_m-x_n|\le|x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n|\le\sum_{k=n+1}^{\infty}|x_k-x_{k-1}|<\varepsilon.$$

Therefore, $\{x_n\}$ is a Cauchy sequence.

---

(47) Claim: $|\sqrt{x}-\sqrt{y}|\le\sqrt{|x-y|}$ for every $x,y\ge 0$.
Proof. Let $u=\max(x,y)$ and $v=\min(x,y)$. Then $|\sqrt{x}-\sqrt{y}|=\sqrt{u}-\sqrt{v}$ and $|x-y|=u-v$.
Now $\sqrt{u}-\sqrt{v}\le\sqrt{u-v}\Leftrightarrow\sqrt{u}\le\sqrt{v}+\sqrt{u-v}\Leftrightarrow u\le \underbrace{v+2\sqrt{v(u-v)}+(u-v)}_{u+2\sqrt{v(u-v)}}$, which is true.

converge $\Rightarrow$ Cauchy

If $a_n\ge 0$ for all $n\in\mathbb{N}$ and $\lim\limits_{n\to\infty}a_n=a$, then for every $\varepsilon>0$, there exists $K\in\mathbb{N}$ such that $m,n\ge K\Rightarrow |a_m-a_n|<\varepsilon^2$. So $m,n\ge K\Rightarrow|\sqrt{a_m}-\sqrt{a_n}|\le\sqrt{|a_m-a_n|}<\sqrt{\varepsilon^2}=\varepsilon$.

---

(48) If $x_2=x_1$, then $|x_{n+1}-x_n|\le k|x_n-x_{n-1}|$ implies all $x_n=x_1$. In this case, for every $\varepsilon>0$, take $K=1$ and $m,n\ge K\Rightarrow|x_m-x_n|=0<\varepsilon$. The sequence $\{x_n\}$ is Cauchy.
If $x_2\ne x_1$, then $\forall\varepsilon>0$, let $K>\log_k\frac{(1-k)\varepsilon}{|x_2-x_1|}$ so that $|x_2-x_1|\frac{k^N}{1-k}<\varepsilon$. We have $m,n\ge K$, say $m>n$, implies $|x_m-x_n|\le|x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n|$

$$\le|x_2-x_1|(k^{m-2}+k^{m-3}+\cdots+k^{n-1})$$
$$\le|x_2-x_1|(k^K+k^{K+1}+\cdots)=|x_2-x_1|\frac{k^K}{1-k}<\varepsilon.$$

So the sequence $\{x_n\}$ is Cauchy.

(49) Let $b_n = a_n - A$ and $\beta_n = \frac{b_1+b_2+\cdots+b_n}{n}$, then $\lim_{n\to\infty}\alpha_n = A \Leftrightarrow \lim_{n\to\infty}(\alpha_n - A) = 0 \Leftrightarrow$
$\lim_{n\to\infty}\frac{(a_1-A)+\cdots+(a_n-A)}{n} = 0 \Leftrightarrow \lim_{n\to\infty}\beta_n = 0$, which is to be shown.

Since $\lim_{n\to\infty} b_n = \lim_{n\to\infty}(a_n - A) = 0$, $\{b_n\}$ is bounded, say $|b_n| \le M$ for all $n \in \mathbb{N}$.
For $\varepsilon > 0$, there is $K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |b_n - 0| < \frac{\varepsilon}{2}$. Let $K = \lceil \max(K_1, \frac{2(K_1-1)M}{\varepsilon})$.
Then $n \ge K \Rightarrow |\beta_n - 0| = \left|\frac{b_1+b_2+\cdots+b_{K_1-1}}{n} + \frac{b_{K_1}+\cdots+b_n}{n}\right| < \frac{(K_1-1)M}{n} + \frac{(n-K_1+1)\frac{\varepsilon}{2}}{n} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$0 < n - K_1 + 1 \le n$

Therefore, $\lim_{n\to\infty}\beta_n = 0$.

To see the converse is false, take $a_n = (-1)^n$. Then $\alpha_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$. So $\lim_{n\to\infty}\alpha_n = 0$ and $\{a_n\}$ doesn't converge.

---

(50) Assume $\lim_{n\to\infty} x_n \ne x$. Then $\sim(\forall \varepsilon > 0\ \exists K$, such that $n \ge K \Rightarrow |x_n - x| < \varepsilon)$
$= \exists \varepsilon > 0\ \forall K\ \exists n \ge K$ and $|x_n - x| \ge \varepsilon$. So $\exists \varepsilon > 0$ such that
for $K = 1$, $\exists\ n_1 \ge 1$ and $|x_{n_1} - x| \ge \varepsilon$,
for $K = n_1 + 1$, $\exists\ n_2 \ge n_1 + 1$ and $|x_{n_2} - x| \ge \varepsilon$,
for $K = n_2 + 1$, $\exists\ n_3 \ge n_2 + 1$ and $|x_{n_3} - x| \ge \varepsilon$, ...

Then $n_1 < n_2 < n_3 < \cdots$ and subsequence $\{x_{n_j}\}$ satisfies $|x_{n_j} - x| \ge \varepsilon$ for all j.
Since $\{x_{n_j}\}$ is bounded, by Bolzano-Weierstrass theorem, it has a Convergence subsequence $\{x_{n_{j_k}}\}$. Then $\lim_{k\to\infty} x_{n_{j_k}} = x$ and $0 = \lim_{k\to\infty}|x_{n_{j_k}} - x| \ge \varepsilon$ leads to a contradiction. Therefore, $\lim_{n\to\infty} x_n = x$.

---

(51) $\forall \varepsilon > 0$, by the Archimedean principle, $\exists m \in \mathbb{N}$ such that $m > \log_2 \frac{1}{\varepsilon}$ $(\Leftrightarrow 2^{-m} < \varepsilon)$.
Since f is injective, the set, $T = \{n \in \mathbb{N} : f(n) = 2^{-1}$ or 2^{-2} or $\cdots$ or $2^{-(m-1)}\}$ has at most $m-1$ elements.
If the set is empty, then let $K = 1$, otherwise let K be larger than the maximum of T.
Then $n \ge K \Rightarrow n \notin T \Rightarrow |f(n) - 0| = f(n) \le 2^{-m} < \varepsilon$. Therefore, $\lim_{n\to\infty} f(n) = 0$.

---

(52) Solution 1 Since $\lim_{n\to\infty}(a_{n+1} - \frac{a_n}{2}) = 0$, so $\forall \varepsilon > 0\ \exists K \in \mathbb{N}$ such that $n \ge K \Rightarrow$
$|a_{n+1} - \frac{a_n}{2}| < \varepsilon_0 = \frac{\varepsilon}{3}$. Let $K' \in \mathbb{N}$ be such that $\frac{|a_K|}{2^{K'}} < \varepsilon_0$. Then
$$m \ge K + K' \Rightarrow |a_m| < \frac{1}{2}|a_{m-1}| + \varepsilon_0 < \frac{1}{2^2}|a_{m-2}| + \frac{\varepsilon_0}{2} + \varepsilon_0 < \cdots$$
$$< \frac{1}{2^{m-K}}|a_K| + \frac{\varepsilon_0}{2^{m-K+1}} + \cdots + \frac{\varepsilon_0}{2} + \varepsilon_0 < \frac{1}{2^{K'}}|a_K| + 2\varepsilon_0 < 3\varepsilon_0 = \varepsilon,$$
$\therefore \lim_{n\to\infty} a_n = 0$.

(52) Solution 2 Let $b_n = a_{n+1} - \frac{1}{2}a_n$. Define $c_1 = c_2 = 0$ and for $k=1,2,3,\ldots$ define $c_{2^k+1} = \cdots = c_{2^{k+1}} = b_k$. Then $b_n \to 0$ implies $c_n \to 0$, which implies (by exercise 49) that we have $\lim_{n\to\infty} \frac{c_1+c_2+\cdots+c_n}{n} = 0$.

So, $2\left(\frac{c_1+c_2+\cdots+c_{2^{n+1}}}{2^{n+1}}\right) = \frac{2b_1+4b_2+\cdots+2^n b_n}{2^n} = a_n - \frac{1}{2^n}a_1 \to 0$. $\therefore \lim_{n\to\infty} a_n = 0$.

---

(53) For $\varepsilon>0$, there is $K\in\mathbb{N}$ such that $n\ge K \Rightarrow |x_n - x_{n-2}| < \varepsilon/2$. For $n \ge K$,

$x_n - x_{n-1} = (x_n - x_{n-2}) - (x_{n-1} - x_{n-3}) + (x_{n-2} - x_{n-4}) - \cdots \pm (x_{K+1} - x_{K-1}) \mp (x_K - x_{K-1})$.

So $|x_n - x_{n-1}| \le (n-K)\frac{\varepsilon}{2} + |x_K - x_{K-1}|$. For $n \ge \frac{2}{\varepsilon}|x_K - x_{K-1}|$, $\left|\frac{x_n - x_{n-1}}{n}\right| \le \underbrace{\frac{n-K}{n}}_{<1}\frac{\varepsilon}{2} + \frac{|x_K - x_{K-1}|}{n} < 1\cdot\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$\therefore \lim_{n\to\infty} \frac{x_n - x_{n-1}}{n} = 0$.

---

(54) Let $y_n \to y$ and let $x = y/3$. We will show $\lim_{n\to\infty} x_n = x$. $\forall \varepsilon>0, \exists K\in\mathbb{N}$ such that $n\ge K \Rightarrow |y_n - y| < \frac{\varepsilon}{2}$. Now $\frac{\varepsilon}{2} > |y_n - y| = |x_{n-1} + 2x_n - 3x| = |2(x_n - x) + (x_{n-1} - x)|$ $\ge 2|x_n - x| - |x_{n-1} - x|$. So $|x_n - x| < \frac{\varepsilon}{4} + \frac{1}{2}|x_{n-1} - x|$. Repeat usage of this lead to $|x_{n+m} - x| < \varepsilon/4\,(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^m}) + \frac{1}{2^{m+1}}|x_{n-1} - x| < \frac{\varepsilon}{2} + \frac{1}{2^{m+1}}|x_{n-1} - x|$.

Next choose M large so that $\frac{1}{2^{M+1}}|x_{K-1} - x| < \frac{\varepsilon}{2}$. Then $j \ge K+M$ implies $|x_j - x| < \frac{\varepsilon}{2} + \frac{1}{2^{j-K+1}}|x_{K-1} - x| \le \frac{\varepsilon}{2} + \frac{1}{2^{M+1}}|x_{K-1} - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\therefore \lim_{n\to\infty} x_n = x$.

---

(55) (a) If $a_n = 1 - \frac{2}{n(n+1)} = \frac{n^2+n-2}{n(n+1)} = \frac{(n-1)(n+2)}{n(n+1)}$, then

$\prod_{n=2}^{\infty}\left(1 - \frac{2}{n(n+1)}\right) = \lim_{k\to\infty} a_2 a_3 \cdots a_k = \lim_{k\to\infty}\left(\frac{1\cdot\cancel{4}}{\cancel{2}\cdot 3}\,\frac{\cancel{2}\cdot\cancel{5}}{3\cdot\cancel{4}}\cdots\frac{(\cancel{k-1})(k+2)}{k(\cancel{k+1})}\right) = \lim_{k\to\infty}\frac{k+2}{3k} = \frac{1}{3}$.

(b) If $a_n = 1 - \frac{1}{n^2} = \frac{n^2-1}{n^2} = \frac{(n-1)(n+1)}{n^2}$, then

$\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right) = \lim_{k\to\infty} a_2 a_3 \cdots a_k = \lim_{k\to\infty}\left(\frac{1\cdot 3}{2^2}\,\frac{\cancel{2}\cdot 4}{3^2}\,\frac{\cancel{3}\cdot 5}{\cancel{4^2}}\cdots\frac{(\cancel{k-1})(k+1)}{k^2}\right) = \lim_{k\to\infty}\frac{k+1}{2k} = \frac{1}{2}$.

(c) Note $\frac{n^3-1}{n^3+1} = \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)}$ and $n^2+n+1 = (n+1)^2 - (n+1) + 1$. So

$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1} = \lim_{k\to\infty}\left(\frac{1\cdot\cancel{7}}{3\cdot\cancel{3}}\,\frac{2\cdot\cancel{13}}{4\cdot\cancel{7}}\,\frac{\cancel{3}\cdot\cancel{21}}{5\cdot\cancel{13}}\cdots\frac{(\cancel{k-1})(k^2+k+1)}{(k+1)(\cancel{k^2-k+1})}\right) = \lim_{k\to\infty}\frac{2(k^2+k+1)}{3k(k+1)} = \frac{2}{3}$.

(d) Note $(1-z)(1+z)(1+z^2)\cdots(1+z^{2^k}) = (1-z^2)(1+z^2)\cdots(1+z^{2^k}) = \cdots$
$= (1-z^{2^k})(1+z^{2^k}) = 1 - z^{2^{k+1}}$. So $(1+z)(1+z^2)\cdots(1+z^{2^k}) = \frac{1-z^{2^{k+1}}}{1-z}$.

Therefore, $\prod_{n=0}^{\infty}(1+z^{2^n}) = \lim_{k\to\infty}\frac{1-z^{2^{k+1}}}{1-z} = \frac{1}{1-z}$ as $|z|<1 \Rightarrow \lim_{k\to\infty} z^{2^{k+1}} = 0$.

(56) Let S be a bounded infinite subset of $\mathbb{R}$. Then we choose $x_1 \in S$. Since S is infinite, there $\exists x_2 \in S$, $x_2 \neq x_1$, $\cdots$, $\exists x_n \in S$, $x_n \neq x_1, \cdots, x_{n-1}$. So the sequence $\{x_n\}$ consists of distinct terms in S. Since S is bounded, $\{x_n\}$ is bounded. By Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergence subsequence, say $\lim_{j\to\infty} x_{n_j} = x_0$. If $x_0 = x_{n_k}$ for some k, then x_0 is the limit of $x_{n_{k+1}}, x_{n_{k+2}}, \cdots$ in $S \setminus \{x_0\}$. So S has x_0 as an accumulation point.

---

(57) (Note: $S=(0,+\infty)$, so $x \in S \Rightarrow x+1 > 1$) For every $\varepsilon > 0$, let $\delta = 2\varepsilon$, then for every $x \in S = (0,\infty)$, $0 < |x-1| < \delta = 2\varepsilon \Rightarrow \left|\frac{x}{x+1} - \frac{1}{2}\right| = \frac{|x-1|}{2(x+1)} < \frac{2\varepsilon}{2} = \varepsilon$.

---

(58) Suppose $\lim_{x\to x_0} f(x)$ exists at x_0. By density of rational, there is $r_n \in \mathbb{Q}$ such that $x_0 - \frac{1}{n} < r_n < x_0$. By density of irrational, there is $s_n \in \mathbb{R}\setminus\mathbb{Q}$ such that $x_0 - \frac{1}{n} < s_n < x_0$. By squeeze limit theorem, $\lim_{n\to\infty} r_n = x_0 = \lim_{n\to\infty} s_n$. By the sequential limit theorem, $\lim_{x\to x_0} f(x) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} 8r_n = 8x_0$ and $\lim_{x\to x_0} f(x) = \lim_{n\to\infty} f(s_n) = \lim_{n\to\infty}(2s_n^2+8) = 2x_0^2+8$. By the uniqueness of limit, $8x_0 = 2x_0^2 + 8$. So $x_0 = 2$.

Next we show $\lim_{x\to 2} f(x)$ exists. (The limit should be $8\times 2 = 16 = 2\cdot 2^2+8$.)

We have $0 \le |f(x)-16| \le |8x-16| + |(2x^2+8)-16|$ for x rational or irrational.

Since $\lim_{x\to 2}(|8x-16| + |(2x^2+8)-16|) = 0$, by squeeze limit theorem, $\lim_{x\to 2} f(x) = 16$.

---

(59) For $w \in \mathbb{R}$, there is a sequence $\{x_n\}$ of rational numbers converging to w (by practice exercise #39 or last exercise). Since f is continuous at w, by the sequential limit theorem, $f(w) = \lim_{x\to w} f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$.

---

(60) (a) ① $f(0+0) = f(0)+f(0) \Rightarrow f(0) = 0$, ② $0 = f(x+(-x)) = f(x)+f(-x) \Rightarrow f(-x) = -f(x)$.

③ For $n \in \mathbb{N}$, $f(nx) = nf(x)$ by mathematical induction ($f(1x) = 1f(x)$. If $f(nx) = nf(x)$, then $f((n+1)x) = f(nx+x) = f(nx)+f(x) = nf(x)+f(x) = (n+1)f(x)$.)

④ Taking $x = \frac{1}{n}$ in ③ we get $f(1) = nf(\frac{1}{n}) \Rightarrow f(\frac{1}{n}) = \frac{1}{n}f(1)$.

Taking $x = \frac{1}{k}$ in ③ we get $f(\frac{n}{k}) = nf(\frac{1}{k}) = \frac{n}{k}f(1)$.

⑤ By ②, $f(-\frac{n}{k}) = -f(\frac{n}{k}) = -\frac{n}{k}f(1)$. If $c = f(1)$, then $f(r) \overset{\text{by ①,④,⑤}}{=} cr$ for $r \in \mathbb{Q}$.

Conversely, the function $f(r) = cr$ satisfies $f(x+y) = c(x+y) = cx+cy = f(x)+f(y)$ for any $c \in \mathbb{R}$.

(b) For $w \in \mathbb{R}$, by density of rational numbers, there are $r_n, s_n \in \mathbb{Q}$ such that $w - \frac{1}{n} < r_n < w < s_n < w + \frac{1}{n}$. For f strictly increasing, by part (a), $r_n f(1) = f(r_n) < f(w) < f(s_n) = s_n f(1)$.

Taking limit, we get $f(w) = wf(1)$ by squeeze limit theorem. So the functions we are looking for are $f: \mathbb{R} \to \mathbb{R}$ given by $f(w) = cw$, where $c = f(1) > 0$.

(61) For every r, $f(r)$ is the maximum or minimum of $f(x)$ on some interval (a,b) containing r. Then $a<r<b$. By density of rational numbers, there are $c, d \in \mathbb{Q}$ such that $a<c<r$ and $r<d<b$. Let $S=\{(q_0,q_1): q_0,q_1\in\mathbb{Q} \text{ and } q_0<q_1\}$, then $S\subseteq \mathbb{Q}\times\mathbb{Q}$ and so S is countable.

$$f(\mathbb{R})=\{f(r): r\in\mathbb{R}\}\subseteq\{\max_{c<x<d} f(x) : (c,d)\in S\}\cup\{\min_{c<x<d} f(x) : (c,d)\in S\}$$

$$=\underbrace{\bigcup_{(c,d)\in S}}_{\text{countable}} \underbrace{\{\max_{c<x<d} f(x), \min_{c<x<d} f(x)\}}_{\text{finite}}$$ which is countable by the countable union theorem

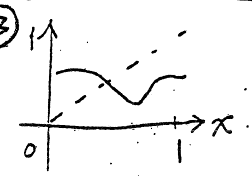
So $f(\mathbb{R})$ is countable. By the intermediate value theorem, f is constant.

---

(62) Suppose such function g exists. We first show g is injective. (If $g(a)=g(b)$, then $-a^9=g(g(a))=g(g(b))=-b^9 \Rightarrow a=b$.) Since g is continuous and injective, by the continuous injection theorem, g is strictly increasing or strictly decreasing. If g is strictly increasing, then $x<y \Rightarrow g(x)<g(y) \Rightarrow g(g(x))<g(g(y))$. If g is strictly decreasing, then $x<y \Rightarrow g(x)>g(y) \Rightarrow g(g(x))<g(g(y))$. So in both cases, $g(g(x))$ is strictly increasing, which cannot equal to the decreasing function $-x^9$, a contradiction. So no such g exists.

---

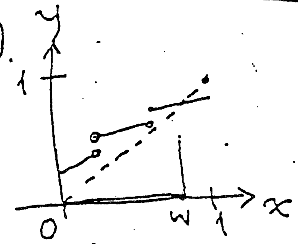
(63)



Let $g(x)=f(x)-x$, then g is continuous on $[0,1]$ because f is continuous on $[0,1]$. Since $f(0),f(1)\in[0,1]$, so $g(0)=f(0)-0\geq 0$ and $g(1)=f(1)-1\leq 0$. By the intermediate value theorem, there is at least one w between 0 and 1 such that $g(w)=0$. Then $f(w)=w$.

---

(64)



Let $S=\{t\in[0,1]: t<f(t)\}$. Since $0\in S$ and S is bounded above by 1, $\sup S = w\in[0,1]$. By the supremum limit theorem, there is a sequence $t_n\in S$ converging to w. By the monotone function theorem, $w=\lim_{n\to\infty} t_n \leq \lim_{n\to\infty} f(t_n) \overset{\text{seq. limit thm}}{=} f(w-) \overset{\text{monotone function theorem}}{\leq} f(w)$. In particular, $w\neq 1$. So $w<1$. Let $\{s_n\}$ be a strictly decreasing sequence in $[0,1]$ converging to w. Since $s_n>w$, $s_n\notin S$ and so $w=\lim_{n\to\infty} s_n \underset{s_n\notin S}{\geq} \lim_{n\to\infty} f(s_n) \underset{\text{seq. limit theorem}}{=} f(w+) \underset{\text{monotone function theorem}}{\geq} f(w)$.

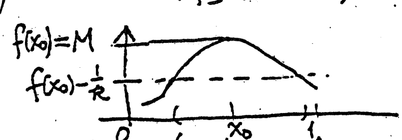
Therefore, $w=f(w)$.

---

(65) f is injective because $f(a)=f(b) \Rightarrow 0=|f(a)-f(b)|\geq|a-b| \Rightarrow a=b$. Next, since f is continuous and injective, f is strictly monotone by the continuous injection theorem. To show f is surjective, let $w\in\mathbb{R}$ and $M=|w-f(0)|$. The given inequality implies $|f(M)-f(0)|\geq|M-0|=M=|w-f(0)|$ and $|f(0)-f(-M)|\geq|0-(-M)|=M=|w-f(0)|$. Since f is strictly monotone, $f(0)$ is between $f(-M)$ and $f(M)$. The inequalities above imply w is closer to $f(0)$ than $f(M)$ and $f(-M)$. So w is between $f(-M)$ and $f(M)$. The intermediate value theorem implies $w=f(x)$ for some x between $-M$ and M. So f is surjective. Therefore, f is bijective.

(66) Since $M=\sup_{x\in[0,1]} f(x)$, $\left(\int_0^1 f(x)^n dx\right)^{\frac1n} \le \left(\int_0^1 M^n dx\right)^{\frac1n} = M$ for all $n\in\mathbb{N}$. By the extreme value theorem, $M=f(x_0)$ for some $x_0\in[0,1]$. For every $k\in\mathbb{N}$, we consider $g(x)=f(x)-(f(x_0)-\frac1k)$ on $[0,1]$. Since g is continuous and $g(x_0)=\frac1k>0$, by the sign preserving property, there is $\delta>0$ such that $g(x)>0$ ($\Leftrightarrow f(x)>M-\frac1k$) on the interval $(x_0-\delta,x_0+\delta)\cap[0,1]$. Let a, b be the endpoints of the interval with $a<b$. Since $f(x)>0$, $\left(\int_a^b (M-\frac1k)^n dx\right)^{\frac1n} < \left(\int_a^b f(x)^n dx\right)^{\frac1n} \le \left(\int_0^1 f(x)^n dx\right)^{\frac1n}$.

So $\boxed{(M-\frac1k)(b-a)^{\frac1n} \le \left(\int_0^1 f(x)^n dx\right)^{\frac1n} \le M}$. Since $\lim_{n\to\infty}(b-a)^{\frac1n}=1$, we have $M-\frac1k \le \lim_{n\to\infty}\left(\int_0^1 f(x)^n dx\right)^{\frac1n} \le M$ for every $k\in\mathbb{N}$. As $k\to\infty$, we get by sandwich theorem that $\lim_{n\to\infty}\left(\int_0^1 f(x)^n dx\right)^{\frac1n}=M$.



Comments: In fact, the limit must exist. From the box above we have

$$\left|\left(\int_0^1 f(x)^n dx\right)^{\frac1n}-M\right| \le M-(M-\tfrac1k)(b-a)^{\frac1n} = (M-\tfrac1k)\left(1-(b-a)^{\frac1n}\right)+\tfrac1k.$$

For every $\varepsilon>0$, by the Archimedean Principle, there is $k\in\mathbb{N}$ such that $\frac1k<\frac{\varepsilon}{2}$ and $\frac1k<M$. With one such k, since $\lim_{n\to\infty}(b-a)^{\frac1n}=1$, there is $K\in\mathbb{N}$ such that

$$n\ge K \Rightarrow \left|(b-a)^{\frac1n}-1\right| < \frac{\varepsilon}{2(M-\frac1k)}.$$ Then

$$n\ge K \Rightarrow \left|\left(\int_0^1 f(x)^n dx\right)^{\frac1n}-M\right| \le M-(M-\tfrac1k)(b-a)^{\frac1n}$$
$$= (M-\tfrac1k)(1-(b-a)^{\frac1n})+\tfrac1k < \tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2}=\varepsilon.$$

---

(67) Since $f(0)=0=0^2$, so $f(x)=x^2$ for all $x\in\mathbb{R}$. Then for every $x_0\in\mathbb{R}$,

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0}\frac{x^2-x_0^2}{x-x_0}=\lim_{x\to x_0} x+x_0 = 2x_0.$$

Remarks We have $f(x)=2x=\begin{cases}2x & \text{if } x\neq0\\ 0 & \text{if } x=0\end{cases} \neq \begin{cases}\frac{d}{dx}(x^2) & \text{if } x\neq0\\ \frac{d}{dx}(x) & \text{if } x=0.\end{cases}$ This is to illustrate that if $f(x)=\begin{cases}h_0(x) & \text{if } x\in S\\ h_1(x) & \text{if } x\notin S\end{cases}$, then in general, $f'(x)\neq\begin{cases}h_0'(x) & \text{if } x\in S\\ h_1'(x) & \text{if } x\notin S.\end{cases}$

For $g(x)=|\cos x|$, let $r(x)=|x|$ and $s(x)=\cos x$, then $r'(x)=\begin{cases}1 & \text{if } x>0\\ -1 & \text{if } x<0\\ \text{not exist} & \text{if } x=0\end{cases}$ and $s'(x)=-\sin x$. By chain rule, if $\cos x>0$ ($\Leftrightarrow x\in\bigcup_{n\in\mathbb{Z}}\left((2n-\frac12)\pi,(2n+\frac12)\pi\right)$), then $g'(x)=(r\circ s)'(x)=r'(s(x))\cdot s'(x)=-\sin x$; if $\cos x<0$ ($\Leftrightarrow x\in\bigcup_{n\in\mathbb{Z}}\left((2n+\frac12)\pi,(2n+\frac32)\pi\right)$), then $g'(x)=(r\circ s)'(x)=r'(s(x))\cdot s'(x)=\sin x$. If $\cos x=0$ ($\Leftrightarrow x=(2n\pm\frac12)\pi$, $n\in\mathbb{Z}$), then $\lim_{t\to x^+}\frac{|\cos t|-|\cos x|}{t-x}=\left|\lim_{t\to x^+}\frac{\cos t}{t-x}\right|=1$, but $\lim_{t\to x^-}\frac{|\cos t|-|\cos x|}{t-x}=-\left|\lim_{t\to x^+}\frac{\cos t}{t-x}\right|=-1$, so $g'(x)$ doesn't exist.

Remarks Even $r'(0)$ doesn't exist, $(s\circ r)(x)=\cos|x|=\cos x$ has derivative $-\sin x$ everywhere!

(68) $$\frac{f(b_n)-f(a_n)}{b_n-a_n}-f'(c) = \left(\frac{f(b_n)-f(c)}{b_n-a_n}+\frac{f(c)-f(a_n)}{b_n-a_n}\right)-f'(c)\left(\frac{b_n-c}{b_n-a_n}+\frac{c-a_n}{b_n-a_n}\right)$$
$$= \frac{f(b_n)-f(c)}{b_n-c}\,\frac{b_n-c}{b_n-a_n}+\frac{f(c)-f(a_n)}{c-a_n}\,\frac{c-a_n}{b_n-a_n}-f'(c)\frac{b_n-c}{b_n-a_n}-f'(c)\frac{c-a_n}{b_n-a_n}$$
$$= \left(\frac{f(b_n)-f(c)}{b_n-c}-f'(c)\right)\underbrace{\frac{b_n-c}{b_n-a_n}}_{\le 1}+\left(\frac{f(c)-f(a_n)}{c-a_n}-f'(c)\right)\underbrace{\frac{c-a_n}{b_n-a_n}}_{\le 1}.$$

So $$\left|\frac{f(b_n)-f(a_n)}{b_n-a_n}-f'(c)\right| \le \underbrace{\left|\frac{f(b_n)-f(c)}{b_n-c}-f'(c)\right|}_{\to 0 \text{ as } n\to\infty}\cdot 1+\underbrace{\left|\frac{f(c)-f(a_n)}{c-a_n}-f'(c)\right|}_{\to 0 \text{ as } n\to\infty}\cdot 1 \to 0$$

$\therefore \lim_{n\to\infty}\frac{f(b_n)-f(a_n)}{b_n-a_n}=f'(c).$

---

(69) $$f(x)=\begin{cases}x^3 & \text{if } x\ge 0\\ -x^3 & \text{if } x<0\end{cases} \Rightarrow f'(x)=\begin{cases}3x^2 & \text{if } x>0\\ -3x^2 & \text{if } x<0\\ 0 & \text{if } x=0\end{cases} \Rightarrow f''(x)=\begin{cases}6x & \text{if } x>0\\ -6x & \text{if } x<0\\ 0 & \text{if } x=0\end{cases} = 6|x| \text{ is continuous}$$

$|f'(0)|=\left|\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}\right|=\lim|x|^2=0$ $\quad |f''(0)|=\left|\lim_{x\to 0}\frac{f'(x)-f'(0)}{x-0}\right|=\lim_{x\to 0}3|x|=0 \Rightarrow f\in C^2(\mathbb{R})$.

$f'''(0)=\lim_{x\to 0}\frac{f''(x)-f''(0)}{x-0}=6\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

---

(70) $|f'(b)|=\left|\lim_{x\to b}\frac{f(x)-f(b)}{x-b}\right|=\lim_{x\to b}\left|\frac{f(x)-f(b)}{x-b}\right|\le\lim_{x\to b}|x-b|=0$ for every $b\in\mathbb{R}$. So $f'\equiv 0$. Therefore, f is a constant function. The same is true if 2 is replaced by $n>1$ because $\left|\frac{f(x)-f(b)}{x-b}\right|\le|x-b|^{n-1}\to 0$ as $x\to b$. However if 2 is replaced by 1, then it is not true as can be seen by taking $f(x)=x$, then $|f(a)-f(b)|=|a-b|$ and f is not constant.

---

(71) Since f has roots at ± 1 with multiplicities n, so $f(\pm1)=f'(\pm1)=\cdots=f^{(n-1)}(\pm1)=0$. Since $f(-1)=f(1)=0$, by Rolle's theorem, there is $x_0\in(-1,1)$ such that $f'(x_0)=0$. Then f' has at least three distinct roots $-1, x_0, 1$. By Rolle's theorem, f'' will have at least four distinct roots. Repeating this until the $(n-1)^{st}$ derivative, we see that $f^{(n-1)}$ will have at least $n+1$ distinct roots. So by Rolle's theorem, $f^{(n)}$ will have at least n distinct roots. Since $\deg f^{(n)}=n$, $f^{(n)}$ has exactly n distinct roots.

---

(72) Let $g(x)=e^{-x}f(x)$, then $g'(x)=-e^{-x}f(x)+e^{-x}f'(x)=e^{-x}(f'(x)-f(x))\le 0$. So $g(x)$ is decreasing on $[0,\infty)$. Then $g(x)\le g(0)=f(0)=0$ for $x\in[0,\infty)$. So $f(x)=e^{x}g(x)\le 0$ for $x\in[0,\infty)$.

(73) We first show $x_n = f(\frac{1}{n})$ is a Cauchy sequence. For every $\varepsilon > 0$, let $K \in \mathbb{N}$ such that $K > \frac{2}{\varepsilon}$ (by Archimedian principle). Then $m, n \geq K \Rightarrow |x_m - x_n| = |f(\frac{1}{m}) - f(\frac{1}{n})| \underset{\text{mean-value theorem}}{=} |f'(x_0)||\frac{1}{m} - \frac{1}{n}|$ $\leq 2|\frac{1}{m} - \frac{1}{n}| \leq 2(\frac{1}{K} - 0) = \frac{2}{K} < \varepsilon$. $\Rightarrow 0 < \frac{1}{m}, \frac{1}{n} \leq \frac{1}{K}$

Next, to show $\lim_{x \to 0^+} f(x)$ exists, it is enough to show $\lim_{n \to \infty} f(t_n)$ exists for every $t_n \to 0$ in $(0, +\infty)$ by the remark following the sequential limit theorem. For every $t_n \to 0$ in $(0, +\infty)$, $\{t_n\}$ is a Cauchy sequence by Cauchy's theorem. We will show $\lim_{n \to \infty} f(t_n)$ exists by showing $\{f(t_n)\}$ is a Cauchy sequence. For every $\varepsilon > 0$, since $\{t_n\}$ is Cauchy, $\exists K_1 \in \mathbb{N}$ such that $m, n \geq K_1 \Rightarrow |t_m - t_n| < \frac{\varepsilon}{2} \Rightarrow |f(t_m) - f(t_n)| = |f'(y_0)(t_m - t_n)|$ $\leq 2|t_m - t_n| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon$. So $\{f(t_n)\}$ is Cauchy, $\lim_{n \to \infty} f(t_n)$ exists by Cauchy's theorem.

---

(74) For $0 < x < \frac{\pi}{2}$, consider the function $f: [0, x] \to \mathbb{R}$ defined by $f(t) = \ln(\cos t)$. Now f is continuous on $[0, x]$ and differentiable on $(0, x)$. By mean-value theorem, $|\ln(\cos x)| = |f(x) - f(0)| = |f'(t_0)(x - 0)| = |(-\tan t_0)x|$ for some t_0 on $(0, x)$. Now $\tan$ is strictly increasing on $(0, \frac{\pi}{2})$, $\tan t_0 < \tan x$. $\therefore |\ln(\cos x)| \leq (\tan t_0)x < x \tan x$.

---

(75) Let $|f|$ has maximum value M on $[0, \frac{1}{2}]$. Since $|f|$ is continuous on $[0, \frac{1}{2}]$, so by extreme value theorem, $M = |f(w)|$ for some $w \in [0, \frac{1}{2}]$. By mean value theorem, there is $x_0 \in (0, w)$ such that $f(w) - f(0) = f'(x_0)(w - 0)$. Then $M = |f(w) - f(0)| \leq |f'(x_0)||w|$ $\leq |f(x_0)| \frac{1}{2} \leq \frac{M}{2}$. Since $0 \leq M \leq \frac{M}{2}$, we get $M = 0$. Then $f(x) = 0$ for all $x \in [0, \frac{1}{2}]$. Similarly, replacing $[0, \frac{1}{2}]$ by $[\frac{1}{2}, 1]$ and using $f(\frac{1}{2}) = 0$ instead of $f(0) = 0$, the argument above shows $f(x) = 0$ for all $x \in [\frac{1}{2}, 1]$. Keep on going, we will get $f(x) = 0$ for all $x \geq 0$.

---

(76) Since $\lim_{h \to 0} f(x_0 + h) + f(x_0 - h) - 2f(x_0) = 0$ and $\lim_{h \to 0} h^2 = 0$, we consider using l'Hopital's rule

$$\lim_{h \to 0} \frac{f'(x_0 + h) - f'(x_0 - h)}{2h} = \frac{1}{2} \lim_{h \to 0} \left(\frac{f'(x_0 + h) - f'(x_0)}{h} + \frac{f'(x_0 - h) - f'(x_0)}{-h} \right) = \frac{1}{2}(f''(x_0) + f''(x_0)) = f''(x_0).$$

By l'Hopital's rule, $\lim_{h \to 0} \frac{f(x_0 + h) + f(x_0 - h) - 2f(x_0)}{h^2} = f''(x_0)$.

---

(77) Since $\frac{d^4}{d\theta^4} \cos\theta = \cos\theta$, by Taylor's theorem, there is $\theta_0 \in (0, \theta)$ such that

$\cos\theta = 1 + 0(\theta - 0) - \frac{1}{2!}(\theta - 0)^2 + \frac{0}{3!}(\theta - 0)^3 + \frac{\cos\theta_0}{4!}(\theta - 0)^4$. Since $0 \leq \theta_0 \leq \theta \leq \frac{\pi}{2}$, so $0 \leq \cos\theta_0 \leq 1$.

Therefore, $1 - \frac{\theta^2}{2} \leq \cos\theta \leq 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$.

---

(78) Let $h>0$ and $x=c+2h$. By Taylor's theorem, there is $x_0\in(c,x)$ such that
$$f(x)=f(c)+f'(c)\underbrace{(x-c)}_{2h}+\frac{f''(x_0)}{2}\underbrace{(x-c)^2}_{4h^2} \Rightarrow f'(c)=\frac{f(x)-f(c)}{2h}-f''(x_0)h$$
$$\Rightarrow |f'(c)|\le \frac{1}{2h}(|f(x)|+|f(c)|)+|f''(x_0)|h$$
$$\le \frac{M_0}{h}+M_2h \text{ for every } h>0.$$
By calculus, $\frac{M_0}{h}+M_2h$ has minimum value $2\sqrt{M_0M_2}$ when $h=\sqrt{\frac{M_0}{M_2}}$, so $|f'(c)|\le 2\sqrt{M_0M_2}$ for every $c\in\mathbb{R}$. Then $M_1\le 2\sqrt{M_0M_2}$, i.e. $M_1^2\le 4M_0M_2$.

---

(79) (a) For every $\varepsilon>0$, let $\delta=\frac{\varepsilon}{2}$, then $|x-t|<\delta \Rightarrow |f(x)-f(t)| \underset{\text{mean-value theorem}}{=} |f'(c_0)(x-t)|\le 2|x-t| < 2\delta=\varepsilon$. $\therefore$ f is uniformly continuous.

(b) Suppose $f(x)=\sin\frac{1}{x}$ is uniformly continuous on $(0,\infty)$. Then for every $\varepsilon>0$ (in particular $\varepsilon=1$), there is $\delta>0$ such that $\forall x,t\in(0,\infty)$, $|x-t|<\delta \Rightarrow |f(x)-f(t)|<\varepsilon=1$. By Archimedian principle, $\exists\, n\in\mathbb{N}$ such that $n>\sqrt{\frac{1}{\pi\delta}}$. Now let $x=\frac{1}{n\pi}$ and $t=\frac{1}{(n+\frac12)\pi}$, then $|x-t|=\left|\frac{1}{n\pi}-\frac{1}{(n+\frac12)\pi}\right|=\frac{1}{2n(n+\frac12)\pi}<\frac{1}{n^2\pi}<\delta$, but $|f(x)-f(t)|=|\sin n\pi-\sin(n+\frac12)\pi|=1$, a contradiction.

---

(80) (a) Suppose the statement is false. Let $m_1=(a+b)/2$. Then one of $[a,m_1]$ or $[m_1,b]$ is not contained in the union of finitely many of these open intervals, call that interval I_1. Again, we divide I_1 into two using its midpoint. Then one of these two, call it I_2, is not contained in the union of finitely many of these open intervals. Continuing this process, we get closed intervals $[a,b]\supseteq I_1\supseteq I_2\supseteq\cdots$ and length of I_n goes to 0. So by nested interval theorem, $\bigcap_{n=1}^{\infty} I_n=\{x\}$. Since $x\in[a,b]$, one of the open intervals will contain x. Since length of I_n goes to 0, this open interval containing x will contain some I_n, contradicting the definition of I_n. Therefore, the statement must be true.

(b). If $f:[a,b]\to\mathbb{R}$ is continuous, then $\forall\varepsilon>0$, $\forall t\in[a,b]$, $\underset{f\text{ continuous at } t \text{ implies}}{\Rightarrow}$ $\exists\,\delta_t>0$ such that $x\in(t-\delta_t,t+\delta_t)\Rightarrow |f(x)-f(t)|<\frac{\varepsilon}{2}$. Since $[a,b]\subseteq\bigcup_{t\in[a,b]}(t-\frac{\delta_t}{2},t+\frac{\delta_t}{2})$, by part (a) $\exists\, t_1,\dots,t_n\in[a,b]$ such that $[a,b]\subseteq(t_1-\frac{\delta_{t_1}}{2},t_1+\frac{\delta_{t_1}}{2})\cup\cdots\cup(t_n-\frac{\delta_{t_n}}{2},t_n+\frac{\delta_{t_n}}{2})$.
Let $\delta=\frac12\min\{\delta_{t_1},\dots,\delta_{t_n}\}>0$. Now for every $x,y\in[a,b]$ with $|x-y|<\delta$, we have $x\in(t_i-\frac{\delta_{t_i}}{2},t_i+\frac{\delta_{t_i}}{2})$ for some i. So $|x-t_i|<\frac{\delta_{t_i}}{2}<\delta_{t_i}$ and $|y-t_i|\le|y-x|+|x-t_i|<\delta+\frac{\delta_{t_i}}{2}\le\frac{\delta_{t_i}}{2}+\frac{\delta_{t_i}}{2}=\delta_{t_i}$. Then
$|f(x)-f(y)|\le|f(x)-f(t_i)|+|f(t_i)-f(y)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.
Therefore, f is uniformly continuous on $[a,b]$.

(81) Solution 1 Assume $f(x_0)>0$ for some $x_0\in[a,b]$. Since f is continuous at x_0, for $\varepsilon=\frac{f(x_0)}{2}$, there is a $\delta>0$ such that $x\in[a,b]\cap(x_0-\delta,x_0+\delta)$ implies $|f(x)-f(x_0)|<\varepsilon=\frac{f(x_0)}{2}$. Then $-\frac{f(x_0)}{2}<f(x)-f(x_0)$ so that $f(x)>\frac{f(x_0)}{2}>0$. Now $[a,b]\cap(x_0-\delta,x_0+\delta)$ contains a closed interval $[c,d]$ of positive length. Then $0<\int_c^d \frac{f(x_0)}{2}\,dx<\int_c^d f(x)\,dx\le\int_a^b f(x)\,dx=0$, contradiction. So $f(x)\equiv 0$ $\forall x\in[a,b]$.

Solution 2 Define $g(t)=\int_a^t f(x)\,dx$. Since f is continuous on $[a,b]$, by the fundamental theorem of calculus, $g'(x)=f(x)\ge 0$ for all $x\in[a,b]$. So g is increasing on $[a,b]$. Since $0=g(a)\le g(t)\le g(b)=\int_a^b f(x)\,dx=0$, we must have $g(x)=0$ for all $x\in[a,b]$. Then $f(x)=g'(x)=0$ for all $x\in[a,b]$.

---

(82) (i) For $\varepsilon>0$, since f is integrable on $[a,b]$ and $[b,c]$, by the integral criterion, there are partition P_1 of $[a,b]$ such that $U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{2}$ and partition P_2 of $[b,c]$ such that $U(f,P_2)-L(f,P_2)<\frac{\varepsilon}{2}$. Then $P=P_1\cup P_2$ is a partition of $[a,c]$ and

$$\begin{aligned}U(f,P)-L(f,P)&=(U(f,P_1)+U(f,P_2))-(L(f,P_1)+L(f,P_2))\\&=(U(f,P_1)-L(f,P_1))+(U(f,P_2)-L(f,P_2))\\&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

So by the integral criterion, f is integrable on $[a,c]$.

(ii) For $\varepsilon>0$, since f is integrable on $[a,d]$, by the integral criterion, there is a partition P_1 of $[a,d]$ such that $U(f,P_1)-L(f,P_1)<\varepsilon$. Then $P_2=P_1\cup\{b,c\}$ is finer partition of P_1 so that $L(f,P_1)\le L(f,P_2)\le U(f,P_2)\le U(f,P_1)$. Then $U(f,P_2)-L(f,P_2)\le U(f,P_1)-L(f,P_1)<\varepsilon$. Now $P=P_2\cap[b,c]$ is a partition of $[b,c]$ and $\underbrace{U(f,P)-L(f,P)}\le U(f,P_2)-L(f,P_2)<\varepsilon$.

Only the terms of $U(f,P_2)-L(f,P_2)=\sum_{i=1}^{n}\underbrace{(M_i-m_i)}_{\ge 0}\Delta x_i$ in $[b,c]$ are used to compute $U(f,P)-L(f,P)$

So by the integral criterion, f is integrable on $[b,c]$.

---

(83) Consider the subintervals $[a, x_1], [x_1, \frac{x_1+x_2}{2}], [\frac{x_1+x_2}{2}, x_2], \cdots, [\frac{x_{n-1}+x_n}{2}, x_n], [x_n, b]$. By exercise 82(i), it is enough to show f is integrable on each of these intervals. (If $a=x_1$, then ignore $[a,x_1]$. If $x_n=b$, then ignore $[x_n,b]$.) In each of the subinterval $[u,v]$ above, <u>either</u> f is discontinuous only at u <u>or</u> f is discontinuous only at v. <u>In the former case</u>, since f is bounded on $[a,b]$, there is $K>0$ such that $|f(x)| \le K$ for every $x \in [a,b]$. For $\varepsilon>0$, choose $w \in (u,v)$ such that $2K(w-u) < \frac{\varepsilon}{2}$ ($\Leftrightarrow w < u + \frac{\varepsilon}{4K}$). Since f is continuous on $[w,v]$, f is integrable on $[w,v]$. By the integral criterion, there is a partition P_1 of $[w,v]$ such that $U(f,P_1)-L(f,P_1) < \frac{\varepsilon}{2}$. Let $P=\{u\} \cup P_1$, then P is a partition of $[u,v]$ and $U(f,P)-L(f,P) = (M_1-m_1)(w-u) + U(f,P_1)-L(f,P_1) \le 2K(w-u)+\frac{\varepsilon}{2} < \varepsilon$. So by the integral criterion, f is integrable on $[u,v]$. <u>The latter case</u> when f is discontinuous only at v is similar.

---

(84) (i) Since $\inf_{x\in[x_{i-1},x_i]} f(x) + \inf_{x\in[x_{i-1},x_i]} g(x)$ is a lower bound of $\{f(x)+g(x) : x \in [x_{i-1},x_i]\}$, we get

$$\inf_{x\in[x_{i-1},x_i]} f(x) + \inf_{x\in[x_{i-1},x_i]} g(x) \le \inf_{x\in[x_{i-1},x_i]} (f(x)+g(x)). \text{ So } L(f,P)+L(g,P) \le L(f+g,P).$$

Similarly, $U(f+g,P) \le U(f,P)+U(g,P)$ since $\sup_{x\in[x_{i-1},x_i]}(f(x)+g(x)) \le \sup_{x\in[x_{i-1},x_i]} f(x) + \sup_{x\in[x_{i-1},x_i]} g(x)$.

(ii) For $\varepsilon>0$, since $\int_a^b f(x)\,dx = \sup\{L(f,P) : P \text{ partition of } [a,b]\}$, by the supremum property, there is a partition P_1 such that $\int_a^b f(x)\,dx - \frac{\varepsilon}{2} < L(f,P_1) \le \int_a^b f(x)\,dx$.
Similarly, there is a partition P_2 such that $\int_a^b g(x)\,dx - \frac{\varepsilon}{2} < L(g,P_2) \le \int_a^b g(x)\,dx$.
Letting $P = P_1 \cup P_2$, then $P_1, P_2 \subseteq P$. So

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon < L(f,P_1)+L(g,P_2) \le L(f,P)+L(g,P) \overset{\text{by part (i)}}{\le} L(f+g,P) \le (L)\int_a^b (f(x)+g(x))\,dx.$$

By the infinitesimal principle, $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \le (L)\int_a^b (f(x)+g(x))\,dx$.
Similarly, the inequality $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \ge (U)\int_a^b (f(x)+g(x))\,dx$ can be obtained by using the infimum property. Combining, we get

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \le (L)\int_a^b (f(x)+g(x))\,dx \le (U)\int_a^b (f(x)+g(x))\,dx \le \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Therefore, equality must hold throughout, i.e. $f+g$ is integrable and $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(85) (a) $\int_0^\infty \frac{dx}{\sqrt{e^x}} = \int_0^\infty e^{-\frac{1}{2}x}dx = \lim_{d\to+\infty}\int_0^d e^{-\frac{1}{2}x}dx = \lim_{d\to+\infty}\left(-\frac{1}{2}e^{-\frac{1}{2}x}\Big|_0^d\right) = \lim_{d\to+\infty}\left(-2e^{-\frac{1}{2}d}+2\right) = 2$

Integral exists

(b) $\int_0^\infty \sin x\,dx = \lim_{d\to+\infty}\int_0^d \sin x\,dx = \lim_{d\to+\infty}\left(-\cos x\Big|_0^d\right) = \lim_{d\to+\infty}(-\cos d+1)$ does not exist.

(c) Note $0 \le \frac{1}{6x} \le \frac{1}{x^2+5x}$ for $x\in(0,1]$. $\int_0^1 \frac{1}{6x}dx = \lim_{c\to0^+}\int_c^1 \frac{1}{6x}dx = \lim_{c\to0^+}\left(\frac{1}{6}\ln x\Big|_c^1\right)$

$= \lim_{c\to0^+}\left(-\frac{1}{6}\ln c\right)$ does not exists. By comparison test, $\int_0^1 \frac{dx}{x^2+5x}$ does not exist.

(d) $\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = \int_{-1}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{d\to0^-}\int_{-1}^d \frac{dx}{\sqrt[3]{x}} + \lim_{c\to0^+}\int_c^1 \frac{dx}{\sqrt[3]{x}} = \lim_{d\to0^-}\left(\frac{3}{2}x^{\frac{2}{3}}\Big|_{-1}^d\right) + \lim_{c\to0^+}\left(\frac{3}{2}x^{\frac{2}{3}}\Big|_c^1\right)$

$= \lim_{d\to0^-}\left(\frac{3}{2}d^{\frac{2}{3}}-\frac{3}{2}\right) + \lim_{c\to0^+}\left(\frac{3}{2}-\frac{3}{2}c^{\frac{2}{3}}\right) = -\frac{3}{2}+\frac{3}{2} = 0$. Integral exists.

(e) $\int_0^1 \frac{dx}{x(x-1)} = \int_0^{\frac{1}{2}} \frac{dx}{x(x-1)} + \int_{\frac{1}{2}}^1 \frac{dx}{x(x-1)}$. $\int_0^{\frac{1}{2}} \frac{dx}{x(x-1)} = \lim_{c\to0^+}\int_c^{\frac{1}{2}} \frac{dx}{x(x-1)} = \lim_{c\to0^+}\int_c^{\frac{1}{2}}\left(\frac{1}{x-1}-\frac{1}{x}\right)dx$

$= \lim_{c\to0^+}\left(\ln|x-1| - \ln|x|\right)\Big|_0^{\frac{1}{2}} = \lim_{c\to0^+}\left(-\ln|c-1| + \ln|c|\right) = 0-\infty$ does n't exist (as a number). So $\int_0^1 \frac{dx}{x(x-1)}$ does not exist.

(f) For $x\in(0,+\infty)$, $\left|\frac{\cos x}{1+x^2}\right| \le \frac{1}{1+x^2}$. Since $\int_0^{+\infty}\frac{1}{1+x^2}dx = \lim_{b\to+\infty}\int_0^b \frac{1}{1+x^2}dx = \lim_{b\to+\infty}\tan^{-1}b = \frac{\pi}{2}$,

$\int_0^{+\infty}\left|\frac{\cos x}{1+x^2}\right|dx$ exists by the comparison test. Then $\int_0^{+\infty}\frac{\cos x}{1+x^2}dx$ exists by the absolute convergence test.

(86) (a) P.V. $\int_{-\infty}^\infty \frac{x}{e^{x^2}}dx = \lim_{b\to+\infty}\int_{-b}^b xe^{-x^2}dx = \lim_{b\to+\infty}\left(-\frac{1}{2}e^{-x^2}\Big|_{-b}^b\right) = \lim_{b\to+\infty}\underbrace{\left(-\frac{1}{2}e^{-b^2}+\frac{1}{2}e^{-b^2}\right)}_{=0} = 0$

(b) P.V. $\int_0^2 \frac{dx}{x^2-1} = \lim_{\varepsilon\to0^+}\left(\int_0^{1-\varepsilon}\frac{dx}{x^2-1} + \int_{1+\varepsilon}^2 \frac{dx}{x^2-1}\right) = \lim_{\varepsilon\to0^+}\left(\int_0^{1-\varepsilon}\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx + \int_{1+\varepsilon}^2 \frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx\right)$

$= \lim_{\varepsilon\to0^+}\left(\cancel{\frac{1}{2}\ln\varepsilon} - \frac{1}{2}\ln(2-\varepsilon) - \cancel{\frac{1}{2}\ln\varepsilon} - \frac{1}{2}\ln 3 + \frac{1}{2}\ln(2+\varepsilon)\right) = -\frac{1}{2}\ln 3$.

(87) We have $\int_0^\infty t^{x-1}e^{-t}dt = \int_0^1 t^{x-1}e^{-t}dt + \int_1^\infty t^{x-1}e^{-t}dt$.

For $\int_0^1 t^{x-1}e^{-t}dt$, since $\lim_{t\to0^+}\frac{t^{x-1}e^{-t}}{t^{x-1}} = \lim_{t\to0^+}e^{-t} = 1$, by the limit comparison tes

$\int_0^1 t^{x-1}e^{-t}dt$ converges $\Leftrightarrow \int_0^1 t^{x-1}dt = \int_0^1 \frac{1}{t^{1-x}}dt$ converges $\overset{\text{p-test}}{\Longleftrightarrow} 1-x<1 \Leftrightarrow x>$

For $\int_1^\infty t^{x-1}e^{-t}dt$, note that $\lim_{t\to+\infty}\frac{t^{x-1}e^{-t}}{\frac{1}{t^2}} = \lim_{t\to+\infty}\frac{t^{x+1}}{e^t} = 0$ by example 1 on p.39.

Since $\int_1^\infty \frac{1}{t^2}dt$ converges by p-test, so by the limit comparison test, $\int_1^\infty t^{x-1}e^{-t}dt$ converges. Therefore, $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ converges for $x>0$.

(88) (a) S is a <u>countably infinite</u> set iff there exists a bijection $f: \mathbb{N} \to S$.

(b) S is a <u>countable</u> set iff S is a finite set or a countably infinite set.

(c) A series $\sum_{k=1}^{\infty} a_k$ <u>converges</u> to a number S iff $\lim_{n\to\infty}(a_1+a_2+\cdots+a_n)=S$.

(d) A nonempty subset S of $\mathbb{R}$ is <u>bounded above</u> iff there exists some $M\in\mathbb{R}$ such that $x\le M$ for all $x\in S$.

(e) $\tilde{M}$ is the <u>supremum</u> of a subset S of $\mathbb{R}$ that is bounded above iff $\tilde{M}$ is an upper bound of S and $\tilde{M}\le M$ for all upper bounds M of S.

(f) A sequence $\{x_n\}$ <u>converges</u> to a number x iff for every $\varepsilon>0$, there exists $K\in\mathbb{N}$ such that $n\ge K$ implies $|x_n - x|<\varepsilon$.

(g) A sequence $\{x_n\}$ is a <u>Cauchy</u> sequence iff for every $\varepsilon>0$, there exists $K\in\mathbb{N}$ such that $m,n\ge K$ implies $|x_m-x_n|<\varepsilon$.

(h) x is an <u>accumulation point</u> of a set S iff there exists a sequence $\{x_n\}$ in S such that $x_n\ne x$ for all n and $\lim_{n\to\infty} x_n = x$.

(i) $f: S\to\mathbb{R}$ has a <u>limit</u> L at x_0 iff for every $\varepsilon>0$, there exists a $\delta>0$ such that $x\in S$ and $0<|x-x_0|<\delta$ imply $|f(x)-L|<\varepsilon$.

(j) $f: S\to\mathbb{R}$ is <u>continuous</u> at $x_0\in S$ iff for every $\varepsilon>0$, there exists a $\delta>0$ such that $x\in S$ and $|x-x_0|<\delta$ imply $|f(x)-f(x_0)|<\varepsilon$.

---

(89) (a) For a fixed $m\in\mathbb{Z}$, the curves $y=\pi x$ and $y=x^3+x+m$ intersect in at most 3 points (because $\pi x = x^3+x+m \Rightarrow x^3+(1-\pi)x+m=0$.) Now $S=\underbrace{\bigcup_{m\in\mathbb{Z}}}_{\text{countable}}\underbrace{\{(x,y): y=\pi x, y=x^3+x+m\}}_{\text{at most 3 points hence countable}}$ is countable by the countable union theorem.

(b) For a fixed $m\in\mathbb{Z}$, the curves $y=x^3+x+1$ and $y=mx$ intersects in at most 3 points (because $mx=x^3+x+1 \Rightarrow x^3+(1-m)x+1=0$). Now $S=\underbrace{\bigcup_{m\in\mathbb{Z}}}_{\text{countable}}\underbrace{\{(x,y): y=x^3+x+1, y=mx\}}_{\text{at most 3 points hence countable}}$ is countable by the countable union theorem.

(c) For a fixed $m\in\mathbb{N}$, the curves $x^2+y^2=1$ and $xy=\frac{1}{m}$ intersect in at most 4 points (because $x^2+(\frac{1}{mx})^2=1 \Rightarrow x^4-x^2+\frac{1}{m^2}=0$.) Now $S=\underbrace{\bigcup_{m\in\mathbb{N}}}_{\text{countable}}\underbrace{\{(x,y): x^2+y^2=1, xy=\frac{1}{m}\}}_{\text{at most 4 points hence countable}}$ is countable by the countable union theorem.

(d) Taking $b=0$, we see that $S\supseteq M$. Since M is uncountable, so S is uncountable by the countable subset theorem.

(e) Note if $x=|a|$, then $a=x$ or $-x$. So

$$S=\{a+b: |a|\in M, b\in\mathbb{Q}\}=\{x+b: x\in M, b\in\mathbb{Q}\}\cup\{-x+b: x\in M, b\in\mathbb{Q}\}$$
$$=\underbrace{\bigcup_{(x,b)\in M\times\mathbb{Q}}}_{\text{countable}}\underbrace{\{x+b,-x+b\}}_{\text{2 elements, countable}}$$

is countable by the countable union theorem.

(89)(f) The set $S_0 = \{a+b\sqrt{2} : a,b \in \mathbb{Q}\} = \bigcup_{\underbrace{(a,b)\in \mathbb{Q}\times\mathbb{Q}}_{\text{Countable}}} \underbrace{\{a+b\sqrt{2}\}}_{\substack{\text{1 element} \\ \text{finite, countable}}}$ is countable.

The set $\{c+d\sqrt{2} : c,d \in \mathbb{Q},\ c+d\sqrt{2} \neq 0\} = S_0 \setminus \{0\}$ is also countable by Countable Subset theorem.

$\therefore S = \mathbb{Q}(\sqrt{2}) = \{\frac{x}{y} : x \in S_0,\ y \in S_0\setminus\{0\}\} = \bigcup_{\underbrace{(x,y)\in S_0\times(S_0\setminus\{0\})}_{\text{countable}}} \underbrace{\{\frac{x}{y}\}}_{\substack{\text{1 element} \\ \text{finite, countable}}}$ is countable

---

(g) Since $A\cap B \subseteq A$, $\mathbb{Q}\cap A \subseteq \mathbb{Q}$, $B\cap\mathbb{Q} \subseteq \mathbb{Q}$ and A, $\mathbb{Q}$ are countable, so by the Countable subset theorem, $A\cap B$, $\mathbb{Q}\cap A$, $B\cap\mathbb{Q}$ are countable. For $x\in A\cap B$, $y\in\mathbb{Q}\cap A$ and $z\in B\cap\mathbb{Q}$, let $S_{x,y,z} = \{x^2+y^2+z^2\}$. Then $S_{x,y,z}$ is a one element set. So $S_{x,y,z}$ is countable.

Finally, $S = \bigcup_{\underbrace{(x,y,z)\in(A\cap B)\times(\mathbb{Q}\cap A)\times(B\cap\mathbb{Q})}_{\text{countable by product theorem}}} \underbrace{S_{x,y,z}}_{\text{Countable}}$ is countable by the countable union theorem

---

(h) Let $y_0 \in A$ and $T = \{x - y_0 : x\in A\}$. Then $T \subseteq S$. Now $f: A\to T$ defined by $f(x) = x - y_0$ is a bijection (with $f^{-1}(t) = t + y_0$). By bijection theorem, A uncountable $\Rightarrow$ T uncountable. Finally, since $T\subseteq S$, S must also be uncountable by contrapositive of countable subset theorem

---

(i) Solution 1 For $x\in A$, let $S_x = \{x^2+y^2 : y\in A\} = \bigcup_{\underbrace{y\in A}_{\text{countable}}} \underbrace{\{x^2+y^2\}}_{\substack{\text{1 element} \\ \text{so countable}}}$, then S_x is countable by countable union theorem. Then $S = \bigcup_{x\in A} S_x$ is countable by countable union theorem.

Solution 2 A countable $\underset{\text{product theorem}}{\Rightarrow}$ $A\times A$ countable $\underset{\text{Countable union theorem}}{\Rightarrow}$ $S = \bigcup_{(x,y)\in A\times A}\{x^2+y^2\}$ is countable.

Solution 3 The function $f: A\times A \to S$ defined by $f(x,y) = x^2+y^2$ is surjective. Since A is countable, $A\times A$ is countable by product theorem. Then S is countable by the surjection theorem.

(j) Since A is countable, $\mathbb{R}\setminus A$ must be uncountable. Taking $y=0$, we have $S \supseteq \mathbb{R}\setminus A$. By the Countable subset theorem, S is uncountable.

(k) Since A is countable, $\mathbb{R}-A$ must be uncountable. Let $a\in A$, then S contains the subset $S_a = \{(a,y) : y\in\mathbb{R}\setminus A\}$. The function $f: \mathbb{R}\setminus A \to S_a$ defined by $f(y) = (a,y)$ is a bijection with $f^{-1}(a,y) = y$. Since $\mathbb{R}\setminus A$ is uncountable, so S_a is uncountable. Then S is uncountable by the Countable subset theorem.

(l) $S = \bigcup_{x\in\mathbb{Z}} S_x$, where $S_x = \{x + y\sqrt{2} : y\in A\}$. The function $f: A\to S_x$ defined by $f(y) = x + y\sqrt{2}$ is a bijection with $f^{-1}(x+y\sqrt{2}) = y$. Since A is countable, each S_x is countable, then $S = \bigcup_{x\in\mathbb{Z}} S_x$ is countable by the countable union theorem.

89 (m) Since $f: \mathbb{Q} \to T$ defined by $f(r) = r\pi$ is a bijection with $f^{-1}(t) = \frac{t}{\pi}$, so T is countable. The set
$$U = \{a+b\sqrt{2}-c\sqrt{3} : a,b,c \in T\} = \bigcup_{\underbrace{(a,b,c)\in T\times T\times T}_{\substack{\text{Countable} \\ \text{by product theorem}}}} \underbrace{\{a+b\sqrt{2}-c\sqrt{3}\}}_{\substack{\text{1 element set} \\ \Rightarrow \text{Countable}}}$$
is Countable by the countable union theorem. Then $S = \mathbb{R} \setminus U$ is uncountable.

(n) $$\{\sqrt{m}+\sqrt{n} : m,n \in \mathbb{N}\} = \bigcup_{\underbrace{(m,n)\in \mathbb{N}\times\mathbb{N}}_{\substack{\text{Countable} \\ \text{by product theorem}}}} \underbrace{\{\sqrt{m}+\sqrt{n}\}}_{\substack{\text{one element} \\ \Rightarrow \text{Countable}}}$$ is countable.

Since $\mathbb{R} \setminus (T\cap U) = (\mathbb{R}\setminus T) \cup (\mathbb{R}\setminus U) = \mathbb{Q} \cup \{\sqrt{m}+\sqrt{n} : m,n\in\mathbb{N}\}$ is countable, so $S = T \cap U = \mathbb{R} \setminus (\mathbb{R}\setminus(T\cap U))$ is uncountable.

(o) Consider the subset of S of squares having the unit circle at the origin as circumcircle. This subset is uncountable because for every $\alpha \in [0, \frac{\pi}{2})$, there is a unique square having $(\cos\alpha, \sin\alpha)$ as a vertex and $[0, \frac{\pi}{2})$ is uncountable. So S is uncountable.

(p). $$G = \bigcup_{\underbrace{(a,b)\in\mathbb{Z}\times\mathbb{Z}}_{\text{Countable}}} \underbrace{\{a+bi\}}_{\substack{\text{1 element} \\ \Rightarrow \text{Countable}}}$$ is Countable by countable union theorem. Let S_n be the degree n polynomials. So
$$S_n = \bigcup_{\underbrace{(a_0,a_1,\dots,a_n)\in G\times G\times\cdots\times G\setminus\{0\}}_{\text{Countable}}} \underbrace{\{a_0+a_1x+\cdots+a_nx^n\}}_{\text{1 element} \Rightarrow \text{Countable}}$$
is Countable. Then $S = \bigcup_{n\in\mathbb{N}} S_n$ is countable by countable union theorem.

90 (a) Alternating Series Test $\sum_{k=1}^{\infty} \frac{\cos k\pi}{k^2+2^k} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2+2^k}$. As $k \uparrow \infty$, $k^2+2^k \uparrow \infty$ and $\frac{1}{k^2+2^k} \downarrow 0$. So $\sum_{k=1}^{\infty} \frac{\cos k\pi}{k^2+2^k}$ Converges.

Comparison Test Since $\frac{e^{\sqrt{k}}}{\sqrt{k}} > \frac{1}{\sqrt{k}} = \frac{1}{k^{1/2}}$ and $\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ diverges by p-test, so $\sum_{k=1}^{\infty} \frac{e^{\sqrt{k}}}{\sqrt{k}}$ diverges.

(b) Ratio test $$\lim_{k\to\infty} \frac{(2(k+1))!}{3^{k+1}(k+1)^4} \frac{3^k k^4}{(2k)!} = \lim_{k\to\infty} \frac{(2k+2)(2k+1)}{3}\left(\frac{k}{k+1}\right)^4 = \infty \Rightarrow \sum_{k=1}^{\infty} \frac{(2k)!}{3^k k^4} \text{ diverges.}$$

Absolute Convergence Test and Comparison Test $\left|\frac{(\cos k)(\sin 2k)}{2^k}\right| \le \left(\frac{1}{2}\right)^k$ and $\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k$ Converges
$$\Rightarrow \sum_{k=1}^{\infty} \left|\frac{(\cos k)(\cos 2k)}{2^k}\right| \text{ Converges} \Rightarrow \sum_{k=1}^{\infty} \frac{(\cos k)(\sin 2k)}{2^k} \text{ Converges.}$$

(c) Term test $$\lim_{k\to\infty} \frac{1}{2}\left(\cos\frac{1}{k}+\sin\frac{1}{k}\right) = \frac{1}{2}(1+0) = \frac{1}{2} \neq 0 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{2}\left(\cos\frac{1}{k}+\sin\frac{1}{k}\right) \text{ diverges.}$$

Limit Comparison Test $$\lim_{k\to\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) = 0 \Rightarrow \lim_{k\to\infty} \frac{\sin\left(\frac{1}{k}-\frac{1}{k+1}\right)}{\frac{1}{k}-\frac{1}{k+1}} \overset{\theta = \frac{1}{k}-\frac{1}{k+1}}{=} \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$
Since $\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right) = 1 - \lim_{k\to\infty}\frac{1}{k+1} = 1$ (by telescoping series test), $\sum_{k=1}^{\infty} \sin\left(\frac{1}{k}-\frac{1}{k+1}\right)$ converges

(90) (d) Since $0 \le \frac{2^k+3^k}{1^k+4^k} \le \frac{3^k+3^k}{4^k} = 2\left(\frac{3}{4}\right)^k$ and $\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^k$ converges by geometric series test, so $\sum_{k=1}^{\infty}\frac{2^k+3^k}{1^k+4^k}$ converges by comparison test.

Since $\lim_{k\to\infty}\cos\left(\sin\frac{1}{k}\right) = \cos(\sin 0) = \cos 0 = 1 \ne 0$, $\sum_{k=1}^{\infty}\cos\left(\sin\frac{1}{k}\right)$ diverges by term test.

(e) $\lim_{k\to\infty}\frac{2^{1/k}+3^{1/k}}{1^{1/k}+4^{1/k}} = \frac{2^0+3^0}{1^0+4^0} = 1 \ne 0 \Rightarrow \sum_{k=1}^{\infty}\frac{2^{1/k}+3^{1/k}}{1^{1/k}+4^{1/k}}$ diverges by term test.

Since $\cos k\pi = (-1)^k$ and $k \nearrow \infty \Rightarrow \frac{1}{k\pi} \searrow 0 \Rightarrow \sin\frac{1}{k\pi} \searrow 0$, by the alternating series test, $\sum_{k=1}^{\infty}(\cos k\pi)\left(\sin\frac{1}{k\pi}\right)$ converges

sin is increasing on $[0, \frac{1}{\pi}]$.

(f) Since $\lim_{k\to\infty}\frac{(k+1)!^2}{((k+1)^2)!}\Big/\frac{(k!)^2}{(k^2)!} = \lim_{k\to\infty}\frac{(k+1)!^2\,(k^2)!}{(k!)^2((k+1)^2)!} = \lim_{k\to\infty}\frac{1}{(k^2+1)(k^2+2)\cdots(k^2+2k)} = 0 < 1$,

by ratio test, $\sum_{k=1}^{\infty}\frac{(k!)^2}{(k^2)!}$ converges.

Note $0 < \left(\cos\frac{1}{k}\right)\left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right) \le \left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right)$. Since $\lim_{k\to\infty}\frac{\left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right)}{\frac{1}{k^2}} \underset{\theta=\frac{1}{k}}{=} \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\frac{\tan\theta}{\theta} = 1$ and $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges by p-test, so $\sum_{k=1}^{\infty}\left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right)$ converges by limit comparison test. Therefore $\sum_{k=1}^{\infty}\left(\cos\frac{1}{k}\right)\left(\sin\frac{1}{k}\right)\left(\tan\frac{1}{k}\right)$ converges by comparison test.

(g) $\left|\frac{2^k\cos k}{(k-1)!}\right| \le \frac{2^k}{(k-1)!}$. Now $\lim_{k\to\infty}\frac{2^{k+1}}{k!}\Big/\frac{2^k}{(k-1)!} = \lim_{k\to\infty}\frac{2}{k} = 0 < 1$. So by the ratio test, $\sum_{k=2}^{\infty}\frac{2^k}{(k-1)!}$ converge. By the comparison test, $\sum_{k=2}^{\infty}\left|\frac{2^k\cos k}{(k-1)!}\right|$ converges. By the absolute convergence test, $\sum_{k=2}^{\infty}\frac{2^k\cos k}{(k-1)!}$ converges.

$\lim_{k\to\infty}\frac{\sin(\frac{1}{k})}{\ln k}\Big/\frac{\frac{1}{k}}{\ln k} = \lim_{k\to\infty}\frac{\sin(\frac{1}{k})}{1/k} \underset{\theta=\frac{1}{k}}{=} \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$. Now as $k\to\infty$, $k\ln k$ increases to ∞, $\frac{1}{k\ln k}$ decreases to 0. Since $\int_2^{\infty}\frac{1}{x\ln x}dx = \ln(\ln x)\Big|_2^{\infty} = \infty$, by the integral test, $\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ diverges. So by the limit comparison test, $\sum_{k=2}^{\infty}\frac{\sin(\frac{1}{k})}{\ln k}$ diverges.

(h) (When k is large, $\frac{k^{\pi}+\cos k\pi}{3+k^4} \sim \frac{k^{\pi}}{k^4}$.) $\lim_{k\to\infty}\frac{\frac{k^{\pi}+\cos k\pi}{3+k^4}}{\frac{k^{\pi}}{k^4}} = \lim_{k\to\infty}\frac{1+\frac{\cos k\pi}{k^{\pi}}}{\frac{3}{k^4}+1} = 1$. Since $\sum_{k=1}^{\infty}\frac{k^{\pi}}{k^4} = \sum_{k=1}^{\infty}\frac{1}{k^{4-\pi}}$ diverges by p-test (because $4-\pi \le 1$), so $\sum_{k=1}^{\infty}\frac{k^{\pi}+\cos k\pi}{3+k^4}$ diverges.

Next $\sum_{k=1}^{\infty}\frac{k^{\pi}\cos k\pi}{3k^4} = \sum_{k=1}^{\infty}\frac{(-1)^k}{3k^{4-\pi}}$ and $a_k = \frac{1}{3k^{4-\pi}}$ decreases to 0, so by the alternating series test, $\sum_{k=1}^{\infty}\frac{k^{\pi}\cos k\pi}{3k^4}$ converges.

(90)(i) $\lim\limits_{k\to\infty} \frac{(2k+2)!}{(k+2)!\,k!} \cdot \frac{(k+1)!\,(k-1)!}{(2k)!} = \lim\limits_{k\to\infty} \frac{(2k+2)(2k+1)}{(k+2)(k)} = 4 > 1 \overset{\text{ratio test}}{\Longrightarrow} \sum\limits_{k=2}^{\infty} \frac{(2k)!}{(k+1)!(k-1)!}$ diverges

$\lim\limits_{k\to\infty} k\cos\left(\frac{1}{k^2}\right) = \infty\cdot\cos 0 = \infty\cdot 1 = \infty \neq 0 \overset{\text{term test}}{\Longrightarrow} \sum\limits_{k=1}^{\infty} k\cos\left(\frac{1}{k^2}\right)$ diverges;

(j) $\lim\limits_{k\to\infty} \frac{(3(k+1))!}{(k+1)!\,(2(k+1))!} \Big/ \frac{3k!}{k!\,(2k)!} = \lim\limits_{k\to\infty} \frac{(3k+3)(3k+2)(3k+1)}{(k+1)(2k+2)(2k+1)} = \frac{27}{4} > 1 \Rightarrow \sum\limits_{k=1}^{\infty} \frac{(3k)!}{k!(2k)!}$ diverges by ratio test.

$0 \le \frac{\cos(1/k)}{k^2-1} \le \frac{1}{k^2-1} < \frac{2}{k^2}$ for $k \ge 2$. Since $\sum\limits_{k=2}^{\infty} \frac{2}{k^2} = 2\sum\limits_{k=2}^{\infty}\frac{1}{k^2}$ converges by p-test,

so $\sum\limits_{k=2}^{\infty} \frac{\cos(1/k)}{k^2-1}$ converges by comparison test.

(k) Ratio Test $\lim\limits_{k\to\infty} \frac{(k+1)!}{(2(k+1)-1)!} \Big/ \frac{k!}{(2k-1)!} = \lim\limits_{k\to\infty} \frac{k+1}{(2k+1)2k} = 0 < 1 \Rightarrow \sum\limits_{k=1}^{\infty} \frac{k!}{(2k-1)!}$ converges.

Alternating Series Test $\sum\limits_{k=1}^{\infty} \frac{\cos k\pi}{\sqrt{k}+1} = \sum\limits_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k}+1}$, the sequence $\left\{\frac{1}{\sqrt{k}+1}\right\}$ decrease to 0 because $k > k' \Rightarrow \sqrt{k} > \sqrt{k'} \Rightarrow \frac{1}{\sqrt{k}+1} < \frac{1}{\sqrt{k'}+1}$ and $\lim\limits_{k\to\infty}\frac{1}{\sqrt{k}+1} = 0$.

So $\sum\limits_{k=1}^{\infty} \frac{\cos k\pi}{\sqrt{k}+1}$ converges.

(l) Ratio Test $\lim\limits_{k\to\infty} \frac{2^{k+1}(k+1)^2}{(k+1)!} \Big/ \frac{2^k k^2}{k!} = \lim\limits_{k\to\infty} \frac{2(k+1)}{k^2} = 0 < 1 \Rightarrow \sum\limits_{k=1}^{\infty} \frac{2^k k^2}{k!}$ converges.

Limit Comparison Test $\lim\limits_{k\to\infty} \frac{\frac{1}{\sqrt{k}}\sin\left(\frac{1}{\sqrt{k}}\right)}{\frac{1}{\sqrt{k}}\frac{1}{\sqrt{k}}} = \lim\limits_{k\to\infty} \frac{\sin\frac{1}{\sqrt{k}}}{\frac{1}{\sqrt{k}}} = \lim\limits_{\substack{\theta\to 0 \\ \theta = \frac{1}{\sqrt{k}}}} \frac{\sin\theta}{\theta} = 1$. Since $\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k}}\frac{1}{\sqrt{k}} = \sum\limits_{k=1}^{\infty}\frac{1}{k}$ diverges by p-test, so $\sum\limits_{k=1}^{\infty} \frac{1}{\sqrt{k}}\sin\left(\frac{1}{\sqrt{k}}\right)$ diverges.

(m) Since $\frac{1}{k}$ decreases to 0 as $k \nearrow \infty$, by alternating series test, $\sum\limits_{k=1}^{\infty} \frac{1}{k\cos k\pi} = \sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges.

Since $0 \le \frac{k^2\sin(1/k)}{(2k+1)!} \le \frac{k^2}{(2k+1)!}$ and $\lim\limits_{k\to\infty} \frac{(k+1)^2}{(2k+3)!} \Big/ \frac{k^2}{(2k+1)!} = \lim\limits_{k\to\infty} \left(\frac{k+1}{k}\right)^2 \frac{1}{(2k+3)(2k+2)} = 0$, by ratio test, $\sum\limits_{k=1}^{\infty} \frac{k^2}{(2k+1)!}$ converges. By comparison test, $\sum\limits_{k=1}^{\infty} \frac{k^2\sin(1/k)}{(2k+1)!}$ converges.

(n) By root test, $\lim\limits_{k\to\infty} \cos\left(1+\frac{1}{k}\right) = \cos 1 < 1 \Rightarrow \sum\limits_{k=1}^{\infty} \cos^k\left(1+\frac{1}{k}\right)$ converges.

By term test, $\lim\limits_{k\to\infty} \frac{\cos(\sin(\frac{1}{k}))}{\sin(\cos(\frac{1}{k}))} = \frac{1}{\sin 1} \neq 0 \Rightarrow \sum\limits_{k=1}^{\infty} \frac{\cos(\sin(1/k))}{\sin(\cos(1/k))}$ diverges.

---

(91)(a) For $m, n \in \mathbb{N}$, $0 < \frac{1}{m} + \frac{1}{n}$ and $\frac{1}{1} + \frac{1}{1} = \frac{2}{1} \notin S$. So $S \subseteq (0, 1+\frac{1}{2}]$. Then S has lower bound 0 and upper bound $\frac{3}{2}$. Let $x_k = \frac{1}{k} + \frac{1}{k+1}$, then $x_k \in S$. (Note $\frac{2}{k+1} < x_k < \frac{2}{k}$.) Since $\lim\limits_{k\to\infty} x_k = 0 + 0 = 0$, by the infimum limit theorem, $\inf S = 0$. Next, every upper bound $M \ge \frac{1}{1} + \frac{1}{2} = \frac{3}{2} \in S$. So $\sup S = \frac{3}{2}$.

(31) (b) For $x, y \in [\frac{1}{2}, 1)$, $1 = \frac{1}{2} + \frac{1}{2} \le x+y < 1+1 = 2$. So $S \subseteq [1,2)$. Then S has lower bound 1 and upper bound 2. Take $x = y = \frac{1}{2} + \frac{1}{2\sqrt{2}k} \in [\frac{1}{2}, 1)$. Then $x_k = x+y \in S$. (Note x_k is irrational, so $x_k \ne 2 - \frac{1}{n}$ for all $n \in \mathbb{N}$.) Since $\lim_{k\to\infty} x_k = \frac{1}{2} + \frac{1}{2} = 1$, by the infimum limit theorem, $\inf S = 1$. Next, take $x = y = 1 - \frac{1}{\sqrt{2}k}$. Then $w_k = x+y \in S$ and $\lim_{k\to\infty} w_k = 1+1 = 2$. By the supremum limit theorem, $\sup S = 2$.

(c) For $x \in [0,1] \cap \mathbb{Q}$, $n \in \mathbb{N}$, $-1 = 0 - \frac{1}{1} \le x - \frac{1}{n} < 1 - 0 = 1$. So $S \subseteq [\frac{1}{2}, 1)$. Then $\frac{1}{2}$ is a lower bound of S and 1 is an upper bound of S. Now every lower bound $m \le \frac{1}{2} = 1 - \frac{1}{2} \in S$, so $\inf S = \frac{1}{2}$. Also let $x_n = 1 - \frac{1}{n+1} \in S$, then $\lim_{n\to\infty} x_n = 1$. By supremum limit theorem, $\sup S = 1$.

(d) (When $x \to \pi$, $\frac{x-\pi}{x+\pi} \to 0$ and when $x \to \infty$, $\frac{x-\pi}{x+\pi} \to 1$.) We will show that $\inf S = 0$ and $\sup S = 1$. For $x \in (\mathbb{R}\setminus\mathbb{Q}) \cap [\pi, \infty)$, $0 \le \frac{x-\pi}{x+\pi} \Leftrightarrow \pi \le x$, which is true. So 0 is a lower bound of S. Also $0 = \frac{\pi-\pi}{\pi+\pi} \in S$. So every lower bound $t \le \frac{\pi-\pi}{\pi+\pi} = 0$. $\therefore \inf S = 0$.

For $x \in (\mathbb{R}\setminus\mathbb{Q}) \cap [\pi, \infty)$, $\frac{x-\pi}{x+\pi} \le 1 \Leftrightarrow x - \pi \le x + \pi$, which is true. So 1 is an upper bound of S. Now $w_n = \frac{n\pi - \pi}{n\pi + \pi} \in S$ for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty} w_n = 1$, so by the supremum limit theorem, $\sup S = 1$.

(e) (When $x \to 0$, $\frac{x-\pi}{x+\pi} \to -1$ and when $x \to \infty$, $\frac{x-\pi}{x+\pi} \to 1$.) We will show that $\inf S = -1$ and $\sup S = 1$. For $x \in \mathbb{Q} \cap [0, \infty)$, $-1 \le \frac{x-\pi}{x+\pi} \Leftrightarrow -x - \pi \le x - \pi$ $\Leftrightarrow 0 \le x$, which is true. So -1 is a lower bound of S. Also $-1 = \frac{0-\pi}{0+\pi} \in S$. So every lower bound $t \le \frac{0-\pi}{0+\pi} = -1$. $\therefore \inf S = -1$.

For $x \in \mathbb{Q} \cap [0, \infty)$, $\frac{x-\pi}{x+\pi} \le 1 \Leftrightarrow x - \pi \le x + \pi$, which is true. So 1 is an upper bound of S. Now $w_n = \frac{n-\pi}{n+\pi} \in S$ for every $n \in \mathbb{N}$. Since $\lim_{n\to\infty} w_n = 1$, so by the supremum limit theorem, $\sup S = 1$.

(f) For $x \in \mathbb{Q} \cap [0,1]$, $y \in (-1, 0]$, $-1 = 0^3 + (-1)^3 < x^3 + y^3 \le 1^3 + 0^3 = 1$. So -1 is a lower bound of S and 1 is an upper bound of S. Note $1 = 1^3 + 0^3 \in S$. So for every upper bound M of S, $M \ge 1$. Therefore, $\sup S = 1$.

Next for every $n \in \mathbb{N}$, $w_n = 0^3 + (-\frac{n}{n+1})^3 \in S$ and $\lim_{n\to\infty} w_n = -1$. So by the infimum limit theorem, $\inf S = -1$.

---

(9) (g) Since $0 < \frac{\sqrt{2}}{m+n} + \frac{1}{k\sqrt{2}} \le \frac{\sqrt{2}}{1+1} + \frac{1}{1\sqrt{2}} = \frac{\sqrt{2}}{2} + \frac{1}{\sqrt{2}} = \sqrt{2}$, S is bounded below by 0 and above by $\sqrt{2}$. Now every upper bound $M \ge \sqrt{2} \in S$, so $\sup S = \sqrt{2}$. Next considering $a_n = \frac{\sqrt{2}}{n+n} + \frac{1}{n\sqrt{2}} \in S$, we have $\lim_{n\to\infty} a_n = 0$, which is a lower bound. So by the infimum limit theorem, $\inf S = 0$.

(h) $S = [0, \frac{1}{2}) \cup [\frac{2}{3}, \frac{3}{4}) \cup [\frac{4}{5}, \frac{5}{6}) \cup \cdots$. Since $0 \le 1 - \frac{1}{2k-1}$ and $1 - \frac{1}{2k} < 1$ for $k = 1, 2, 3, \cdots$, so $0 \le x < 1$ for all $x \in S$. So S is bounded below by 0 and above by 1. Since every lower bound $m \le 0 \in S$, so $\inf S = 0$. Next since $1 - \frac{1}{2k-1} \in S$ and $\lim_{k\to\infty} (1 - \frac{1}{2k-1}) = 1$, so by the supremum limit theorem, $\sup S = 1$.

(i) For $x, y \in (0,1] \cap \mathbb{Q}$, $0 \le \sqrt{x} + y^2 \le \sqrt{1} + 1^2 = 2$. So 0 is a lower bound and 2 is an upper bound. Now let $w_n = \sqrt{\frac{1}{n}} + (\frac{1}{n})^2$ for $n = 1, 2, 3, \cdots$, then $w_n \in S$ and $\lim_{n\to\infty} w_n = 0$. So by infimum limit theorem, $\inf S = 0$. Next, $2 = \sqrt{1} + 1^2 \in S$ and so every upper bound of S is greater than or equal to 2. Therefore, $\sup S = 2$.

(j) Since $0 \le \frac{1}{n} + x \le 2$ for $x \in [0,1] \cap \mathbb{Q}$, $n = 1, 2, 3, \cdots$, the set S is bounded below by 0 and bounded above by 2. We will show $\inf S = 0$ and $\sup S = 2$. Since 0 is a lower bound, $0 \le \inf S$. For $n = 1, 2, 3, \cdots$, $\frac{1}{n} = \frac{1}{n} + 0 \in S$ and so $\inf S \le \frac{1}{n}$. Then $\inf S \le \lim_{n\to\infty} \frac{1}{n} = 0$. So $\inf S = 0$.
Since 2 is an upper bound, $\sup S \le 2$. If m is an upper bound of S, then $m \ge \frac{1}{1} + 1 = 2 \in S$. So $\sup S = 2$.

(k) Since $0 \le x + y \le 2$ for $x \in [0,1] \cap \mathbb{Q}$, $y \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$, S is bounded below by 0 and bounded above by 2. We will show $\inf S = 0$ and $\sup S = 2$.
Let $w_n = \frac{1}{n} + \frac{1}{n\sqrt{2}}$, then $w_n \in S$ and $\lim_{n\to\infty} w_n = 0$. So by infimum limit theorem, $\inf S = 0$.
Let $v_n = \frac{n}{n+1} + \frac{1}{n+1\sqrt{2}}$, then $v_n \in S$ and $\lim v_n = 2$, So by supremum limit theorem, $\sup S = 2$.

(l) Note $x(x+1) \le 0 \Leftrightarrow x \in [-1, 0]$. So $S = [-1, 0] \cap (\mathbb{R} \setminus \mathbb{Q})$. Hence S is bounded below by -1 and above by 0. We will show $\inf S = -1$ and $\sup S = 0$.
Let $w_n = -\frac{1}{n+1\sqrt{2}}$, then $w_n \in S$ and $\lim_{n\to\infty} w_n = -1$. So by infimum limit theorem, $\inf S = -1$.
Let $v_n = -\frac{1}{n\sqrt{2}}$, then $v_n \in S$ and $\lim_{n\to\infty} v_n = 0$. So by supremum limit theorem, $\sup S = 0$.

(m) $\forall \frac{k}{n!} \in S$, $0 < \frac{k}{n!} < \sqrt{2}$. So S has lower bound 0 and upper bound $\sqrt{2}$. Will show $\inf S = 0$, $\sup S = \sqrt{2}$.
If $\sup S < \sqrt{2}$, then by density of rational, there is $\frac{m}{n} \in \mathbb{Q}$ such that $\sup S < \frac{m}{n} < \sqrt{2}$. However, $\frac{m}{n} = \frac{m(n-1)!}{n!} \in S$, contradicting $\sup S$ is an upper bound of S. $\therefore \sup S = \sqrt{2}$.
If $\inf S > 0$, then by density of rational, there is $\frac{p}{q} \in \mathbb{Q}$ such that $0 < \frac{p}{q} < \inf S$. However, $\frac{p}{q} = \frac{p(q-1)!}{q!} \in S$, contradicting $\inf S$ is a lower bound of S. $\therefore \inf S = 0$.

91 (n) Note $S=\bigcup_{n=1}^{10}\left([\frac{1}{n\sqrt2}, 2-\frac1n]\setminus\mathbb{Q}\right) = [\frac{1}{10\sqrt2}, 1.9]\setminus\mathbb{Q}$. So S is bounded below by $\frac{1}{10\sqrt2}$ and above by 1.9. We will show $\inf S=\frac{1}{10\sqrt2}$ and $\sup S = 1.9$.
Since $\frac{1}{10\sqrt2}\in S$, every lower bound $m\le\frac{1}{10\sqrt2}$, so $\inf S=\frac{1}{10\sqrt2}$.
Next, let $w_n = 1.9 - \frac{1}{n\sqrt2}$, then $\frac{1}{10\sqrt2}<1<1.9-\frac{1}{\sqrt2}\le w_n<1.9$. So $w_n\in S$. Since $\lim_{n\to\infty} w_n = 1.9$, by the supremum limit theorem, $\sup S=1.9$.

(o) $0\le x^2+y^3+z^4\le 1+1+1=3$ for $x\in(-1,0)\setminus\mathbb{Q}$, $y\in(0,1)\cap\mathbb{Q}$, $z\in(-1,1)$. So 0 is a lower bound and 3 is an upper bound of S. Since $(-\frac{1}{n\sqrt2})^2+(\frac{1}{n+1})^3+(\frac{1}{n+1})^4$ is in S and has limit 0, so $\inf S=0$. Since $(-1+\frac{1}{n\sqrt2})^2+(1-\frac{1}{n+1})^3+((1-\frac{1}{n+1}))^4$ is in S and has limit 3, so $\sup S=3$.

---

92 (a) (Note $x_1=1<x_2=\frac12+\sqrt1=\frac32<x_3=\frac34+\sqrt{\frac32}=\frac{3+2\sqrt6}{4}$. Also $x=\frac{x}{2}+\sqrt{x}\Rightarrow x=0$ or 4.)
We will show $x_n\le x_{n+1}\le 4$ by induction. For $n=1$, $1\le\frac32\le4$. Next suppose $x_n\le x_{n+1}\le4$. Then $\frac{x_n}{2}\le\frac{x_{n+1}}{2}\le2$ and $\sqrt{x_n}\le\sqrt{x_{n+1}}\le\sqrt4\Rightarrow x_{n+1}=\frac{x_n}{2}+\sqrt{x_n}\le x_{n+2}=\frac{x_{n+1}}{2}+\sqrt{x_{n+1}}$ $\le 2+\sqrt4=4$. Therefore, $\{x_n\}$ is increasing and bounded above. By the monotone sequence theorem, $\lim_{n\to\infty}x_n=x$ exists. Then $x=\frac{x}{2}+\sqrt{x}\Rightarrow x=0$ or 4. Since $x_1>1$, $\lim_{n\to\infty}x_n=x=4$.

(b) (Note $x_1=1<x_2=2<x_3=\sqrt2+\sqrt1=\sqrt2+1$, so we suspect $\{x_n\}$ is increasing.)
We will show $x_n\le x_{n+1}$ for all $n\in\mathbb{N}$ by induction. The cases $n=1, 2$ are true as shown above. Assume the cases $n<k$ are true. For the case $n=k$, we have $x_k\le x_{k+1}\Leftrightarrow\sqrt{x_{k-1}}+\sqrt{x_{k-2}}\le\sqrt{x_k}+\sqrt{x_{k-1}}\Leftrightarrow x_{k-2}\le x_k$, which is true by cases $n=k-2$ $(x_{k-2}\le x_{k-1})$ and $n=k-1$ $(x_{k-1}\le x_k)$. So $\{x_n\}$ is increasing.
Next we will show $x_n\le4$ for all $n\in\mathbb{N}$. For $n=1,2$, this is clear. Assume the cases $n<k$ are true, then $x_k=\sqrt{x_{k-1}}+\sqrt{x_{k-2}}\le\sqrt4+\sqrt4=4$. So by induction, $x_n\le4$ for all $n\in\mathbb{N}$. By the monotone sequence theorem, $\{x_n\}$ converges. Let $x=\lim_{n\to\infty}x_n$, then $x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}(\sqrt{x_n}+\sqrt{x_{n-1}})=2\sqrt{x}\Rightarrow x=0$ or 4. Since $1=x_1\le x$, $x=4$.

(c) $x_2=\frac14<x_4=\frac{19}{46}<x_3=\frac{7}{13}<x_1=1$. Assume $x_{2n}<x_{2n+2}<x_{2n+1}<x_{2n-1}$. Now $x_{k+1}=\frac{2-x_k}{3+x_k}=\frac{5}{3+x_k}-1$. So $x_{2n+1}=\frac{5}{3+x_{2n}}-1>x_{2n+3}=\frac{5}{3+x_{2n+2}}-1>x_{2n+2}=\frac{5}{3+x_{2n+1}}-1>x_{2n}=\frac{5}{3+x_{2n-1}}-1$. Repeating this once more, we get $x_{2n+2}=\frac{5}{3+x_{2n+1}}-1<x_{2n+4}=\frac{5}{3+x_{2n+3}}-1<x_{2n+3}=\frac{5}{3+x_{2n+2}}-1<x_{2n+1}=\frac{5}{3+x_{2n}}-1$. Therefore, $x_{2k}<x_{2k+2}<x_{2k+1}<x_{2k-1}$ for all k by mathematical induction.

(c). (Cont.) Now $|x_m - x_{m-1}| = \left|\frac{2-x_{m-1}}{3+x_{m-1}} - \frac{2-x_{m-2}}{3+x_{m-2}}\right| = \frac{5|x_{m-1}-x_{m-2}|}{(3+x_{m-1})(3+x_{m-2})} \le \frac{5|x_{m-1}-x_{m-2}|}{(3\frac{1}{4})(3\frac{1}{4})} = \frac{80}{169}|x_{m-1}-x_{m-2}|$

$\le \cdots \le \left(\frac{80}{169}\right)^{m-2}|x_2-x_1|$. Since $\lim_{m\to\infty}\left(\frac{80}{169}\right)^{m-2}|x_2-x_1| = 0$, $\lim_{m\to\infty}|x_m - x_{m-1}| = 0$ and $\lim_{k\to\infty}|x_{2k}-x_{2k-1}| = 0$. By the nested interval theorem and intertwining sequence theorem, $\{x_n\}$ converges to some x. Now $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\frac{2-x_n}{3+x_n} = \frac{2-x}{3+x}$. Solving, we find $x = -2 \pm \sqrt{6}$. Since $x_n \ge x_2 = \frac{1}{4}$, $x = -2+\sqrt{6}$.

Alternatively, after we showed $x_{2k} < x_{2k+2} < x_{2k+1} < x_{2k-1}$ for all k, we can argue as follow. Since $\{x_{2n}\}$ is increasing and bounded above by x_1, $\{x_{2n}\}$ must converge to some a by the monotone sequence theorem. Also $\{x_{2n+1}\}$ is decreasing and bounded below by x_2, so $\{x_{2n+1}\}$ must converge to some b by the monotone sequence theorem. Then

$$\left.\begin{aligned} b &= \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty}\frac{2-x_{2n}}{3+x_{2n}} = \frac{2-a}{3+a} \\ a &= \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty}\frac{2-x_{2n-1}}{3+x_{2n-1}} = \frac{2-b}{3+b} \end{aligned}\right\} \Rightarrow \begin{aligned} 3b+ab &= 2-a \\ 3a+ab &= 2-b \end{aligned}$$

$$\overset{\text{subtracting}}{\Rightarrow} 3(b-a) = b-a \Rightarrow b = a.$$

By the intertwining sequence theorem, $\{x_n\}$ converges. Then the limit of $\{x_n\}$ is found as above.

(d) $x_3 = \frac{1}{2}$, $x_2 = \frac{3}{4}$, $x_1 = \frac{15}{16}$

$x = 1 - \sqrt{1-x} \Leftrightarrow \sqrt{1-x} = 1-x \Leftrightarrow 1-x = (1-x)^2$
$\Leftrightarrow (1-x)x = 0 \Leftrightarrow x = 0$ or 1

We will prove $0 < x_{n+1} < x_n$ for $n = 1, 2, \cdots$ by induction. For $n=1$, $0 < x_2 = \frac{3}{4} < x_1 = \frac{15}{16}$.
Assume $0 < x_{n+1} < x_n$. Then $1 > 1 - x_{n+1} > 1 - x_n \Rightarrow 1 > \sqrt{1-x_{n+1}} > \sqrt{1-x_n}$
$\Rightarrow 0 < 1 - \sqrt{1-x_{n+1}} = x_{n+2} < 1 - \sqrt{1-x_n} = x_{n+1}$,
completing the induction. Therefore $\{x_n\}$ is decreasing and bounded below. By the monotone sequence theorem, $\{x_n\}$ converges to some limit x. Then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}(1-\sqrt{1-x_n}) = 1-\sqrt{1-x}$.
So $x = 0$ or 1. Since $x_n < 1$ and $\{x_n\}$ is decreasing, $x = 0$.
Now $\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\frac{1-\sqrt{1-x_n}}{x_n}\frac{(1+\sqrt{1-x_n})}{(1+\sqrt{1-x_n})} = \lim_{n\to\infty}\frac{1-(1-x_n)}{x_n(1+\sqrt{1-x_n})} = \lim_{n\to\infty}\frac{1}{1+\sqrt{1-x_n}} = \frac{1}{2}$.

(e) $x_2 = \frac{3}{2}$ $x_4 = \frac{8}{5}$ $x_3 = \frac{5}{3}$ $x_1 = 2$

Let $I_n = [x_{2n}, x_{2n-1}]$. We will show $x_{2n} \le x_{2n+2} \le x_{2n+1} \le x_{2n-1}$. The case $n=1$ is shown on the left. Suppose the case $n=k$ is true, i.e. $x_{2k} \le x_{2k+2} \le x_{2k+1} \le x_{2k-1}$. Since $x_n = \frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{1}{x_{n-1}}$, so

$$\underbrace{1+\frac{1}{x_{2k}}}_{= x_{2k+1}} \ge \underbrace{1+\frac{1}{x_{2k+2}}}_{= x_{2k+3}} \ge \underbrace{1+\frac{1}{x_{2k+1}}}_{= x_{2k+2}} \ge \underbrace{1+\frac{1}{x_{2k-1}}}_{= x_{2k}}$$

and

$$\underbrace{1+\frac{1}{x_{2k+1}}}_{= x_{2k+2}} \le \underbrace{1+\frac{1}{x_{2k+3}}}_{= x_{2k+4}} \le \underbrace{1+\frac{1}{x_{2k+2}}}_{= x_{2k+3}} \le \underbrace{1+\frac{1}{x_{2k}}}_{= x_{2k+1}} \Rightarrow \left\{ n = k+1 \text{ is true} \right.$$

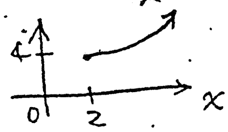
This implies $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$. Next we will show $\lim_{n\to\infty}(x_{2n-1} - x_{2n}) = 0$. Note that $|x_m - x_{m+1}| = \left|\left(1+\frac{1}{x_{m-1}}\right) - \left(1+\frac{1}{x_m}\right)\right| = \frac{|x_{m-1}-x_m|}{x_{m-1}x_m} \le \frac{4}{9}|x_{m-1}-x_m| \Rightarrow |x_{2n-1}-x_{2n}| \le \left(\frac{4}{9}\right)^{2n-2}\left(\frac{1}{2}\right)$
Since $\lim_{n\to\infty}\left(\frac{4}{9}\right)^{2n-2}\left(\frac{1}{2}\right) = 0$, by the squeeze limit theorem, $\lim_{n\to\infty}(x_{2n-1}-x_{2n}) = 0$. By the nested interval theorem and the intertwining sequence theorem, $\{x_n\}$ converges, say to x.
Then $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(1+\frac{1}{x_{n-1}}\right) = 1 + \frac{1}{x} \Rightarrow x = \frac{1\pm\sqrt{5}}{2}$. Since $x_n > 0$, so $x = \frac{1+\sqrt{5}}{2}$

(92) (f) (

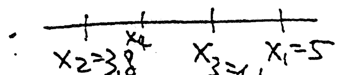
$x = 1-\frac{1}{4x} \Rightarrow 4x^2 = 4x-1 \Rightarrow 0 = 4x^2-4x+1 = (2x-1)^2 \Rightarrow x = \frac{1}{2}$.)

We will show $x_n \geq x_{n+1} \geq \frac{1}{2}$ for $n = 1, 2, 3, \cdots$ by induction. We have $x_1 = 1 \geq x_2 = \frac{3}{4} \geq \frac{1}{2}$.
Assume $x_n \geq x_{n+1} \geq \frac{1}{2}$. Then $\frac{1}{4x_n} \leq \frac{1}{4x_{n+1}} \leq \frac{1}{2}$ and $x_{n+1} = 1-\frac{1}{4x_n} \geq x_{n+2} = 1-\frac{1}{4x_{n+1}} \geq 1-\frac{1}{4(\frac{1}{2})} = \frac{1}{2}$,
completing the induction. So $\{x_n\}$ is decreasing and bounded below by $\frac{1}{2}$.
By monotone sequence theorem, $\{x_n\}$ converges to some x. Then $x = \lim_{n\to\infty} x_{n+1}$
$= \lim_{n\to\infty}(1-\frac{1}{4x_n}) = 1-\frac{1}{4x}$. So $x = 1-\frac{1}{4x}$. As above, $x = \frac{1}{2}$.

(g) Since $f'(x) = 1-\frac{4}{x^2} \geq 0$ for $x \geq 2$ and $\lim_{x\to\infty}(x+\frac{4}{x}) = \infty$, $f(x)$ is increasing to ∞.



$x_1 = 4$, $x_2 = \frac{5}{2} = 2.5$, $x_3 = \frac{1}{2}(2.5+1.6) = 2.05$. We suspect $\{x_n\}$ is decreasing.
(If $\{x_n\}$ converges to x, then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\frac{1}{2}(x_n+\frac{4}{x_n}) = \frac{1}{2}(x+\frac{4}{x})$, which implies $x = \pm 2$. Since $x_n > 0$ by induction, $x = 2$.)
We will show $2 \leq x_{n+1} \leq x_n$ for $n = 1, 2, \cdots$. (This implies $\{x_n\}$ is decreasing and bounded below by 2. By the monotone sequence theorem, we get $\{x_n\}$ converges.)
For $n=1$, $2 \leq x_2 = 2.5 \leq x_1 = 4$. Suppose $2 \leq x_{n+1} \leq x_n$. Then since $f(x) = x+\frac{4}{x}$ is increasing for $x \geq 2$, we get $2 = \frac{1}{2}f(2) \leq x_{n+2} = \frac{1}{2}f(x_{n+1}) \leq x_{n+1} = \frac{1}{2}f(x_n)$,
completing the induction.

(h) $x_1 = 5$, $x_2 = 3\frac{4}{5} = 3.8$, $x_3 = 4\frac{1}{19}$, $x_4 = 3+\frac{4}{x_3} > 3+\frac{4}{5} = x_2$.


Define $I_n = [x_{2n}, x_{2n-1}]$, we will show $x_{2n} \leq x_{2n+2} \leq x_{2n+1} \leq x_{2n-1}$, i.e. $I_{n+1} \subseteq I_n$.
The case $n=1$ is done above. Suppose the case n is true. Then $\frac{4}{x_{2n}} \geq \frac{4}{x_{2n+2}} \geq \frac{4}{x_{2n+1}} \geq \frac{4}{x_{2n-1}}$
$\Rightarrow x_{2n+1} \geq x_{2n+3} \geq x_{2n+2} \geq x_{2n} \Rightarrow \frac{4}{x_{2n+1}} \leq \frac{4}{x_{2n+3}} \leq \frac{4}{x_{2n+2}} \leq \frac{4}{x_{2n}} \Rightarrow x_{2n+2} \leq x_{2n+4} \leq x_{2n+3} \leq x_{2n+1}$,
completing the induction.
*Next observe that $|x_{m+1}-x_m| = |\frac{4}{x_m}-\frac{4}{x_{m-1}}| = \frac{4|x_m-x_{m-1}|}{x_m x_{m-1}} \leq \frac{4}{(3.8)^2}|x_m-x_{m-1}|$. So
$|x_{2n-1}-x_{2n}| \leq \frac{4}{(3.8)^2}|x_{2n-2}-x_{2n-1}| \leq \cdots \leq \left(\frac{4}{(3.8)^2}\right)^{2n-3}|x_2-x_1|$. Since $\frac{4}{(3.8)^2} < 1$, $\lim_{n\to\infty}\left(\frac{4}{(3.8)^2}\right)^{2n-3}|x_2-x_1| = 0$
and $\lim_{n\to\infty}|x_{2n-1}-x_{2n}| = 0$. Hence $\bigcap_{n=1}^{\infty} I_n = \{x\}$ and $\lim_{n\to\infty} x_{2n} = x = \lim_{n\to\infty} x_{2n-1}$. So by the intertwining sequence theorem, $\{x_n\}$ converges to x.
Taking limit of $x_{n+1} = 3+\frac{4}{x_n}$, we get $x = 3+\frac{4}{x} \Rightarrow x^2-3x-4 = 0 \Rightarrow x = -1$ or 4.
Since $x \in I_1 = [3.8, 5]$, so $x = 4$.
* Alternatively, the 2nd paragraph can be replaced by the following argument. Let $x_{2n} \to a$, $x_{2n-1} \to b$,
then ($x_{2n+1} = 3+\frac{4}{x_{2n}} \Rightarrow b = 3+\frac{4}{a}$) and ($x_{2n} = 3+\frac{4}{x_{2n-1}} \Rightarrow a = 3+\frac{4}{b}$). So $(b-3)a = 4 = (a-3)b$.
$ba-3a = ab-3b \Rightarrow a = b$. Therefore, $\{x_n\}$ converges to $x = a = b$.

(92)(i) $x_1=2,\ x_2=\frac{3}{2}=1.5,\ x_3=\frac{4}{3}=1.33\cdots$. We suspect $\{x_n\}$ is decreasing.

(If $\{x_n\}$ converges to x, then $x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}(2-\frac{1}{x_n})=2-\frac{1}{x}$, which implies $x=1$ by algebra.)

We will show $1\le x_{n+1}\le x_n$ for $n=1,2,\cdots$; (this implies $\{x_n\}$ is decreasing and bounded below by 1. By the monotone sequence theorem, we get $\{x_n\}$ converges.)

For $n=1$, we have $1\le x_2=1.5\le x_1=2$. Suppose $1\le x_{n+1}\le x_n$, then $\frac{1}{1}\ge\frac{1}{x_{n+1}}\ge\frac{1}{x_n}$ and $1=2-\frac{1}{1}\le x_{n+2}=2-\frac{1}{x_{n+1}}\le x_{n+1}=2-\frac{1}{x_n}$, completing M. I.

(j) ($x_1=0<x_2=\frac{0^2+4}{5}=\frac{4}{5}<x_3=\frac{(\frac{4}{5})^2+4}{5}$ $\quad x=\frac{x^2+4}{5}\iff x^2-5x+4=(x-1)(x-4)=0,$ $x=1\quad x=4$)

We will show $x_n\le x_{n+1}\le 1$ by math induction. For $n=1$, $x_1=0\le x_2=\frac{4}{5}\le 1$. Suppose $x_n\le x_{n+1}\le 1$. Then $x_n^2+4\le x_{n+1}^2+4\le 1^2+4$. Dividing by 5, we get $x_{n+1}\le x_{n+2}\le 1$ completing the induction. This shows $\{x_n\}$ is increasing and bounded above. By monotone sequence theorem, $\{x_n\}$ converges to some $x\in\mathbb{R}$. Now

$x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}\frac{x_n^2+4}{5}=\frac{x^2+4}{5}\Rightarrow x^2-5x+4=0\Rightarrow x=1$ or $\cancel{4}$ $\leftarrow$ rejected as $x_n\le 1$

(k) (Note $x_1=1>x_2=\sqrt{1}-\frac{1}{4}=\frac{3}{4}>x_3=\sqrt{\frac{3}{4}}-\frac{1}{4}=\frac{2\sqrt{3}-1}{4}$ and $x=\sqrt{x}-\frac{1}{4}\Rightarrow x=\frac{1}{4}$.)

We will show $x_n\ge x_{n+1}\ge\frac{1}{4}$ by induction. For $n=1$, $1\ge\frac{3}{4}\ge\frac{1}{4}$. Suppose $x_n\ge x_{n+1}\ge\frac{1}{4}$. Then $x_{n+1}=\sqrt{x_n}-\frac{1}{4}\ge x_{n+2}=\sqrt{x_{n+1}}-\frac{1}{4}\ge\frac{1}{4}=\sqrt{\frac{1}{4}}-\frac{1}{4}$. Therefore, $\{x_n\}$ is decreasing and bounded below. By the monotone sequence theorem, $\lim_{n\to\infty}x_n=x$ exists. Then $x=\sqrt{x}-\frac{1}{4}\Rightarrow x=\frac{1}{4}$. So $\lim_{n\to\infty}x_n=\frac{1}{4}$.

(l) (Note $x_1=3>x_2=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}>x_3=\sqrt{1-\frac{2}{2+\sqrt{3}}}=\sqrt{\frac{3}{2+\sqrt{3}}}$ and $x=\sqrt{1-\frac{1}{x+1}}=\sqrt{\frac{x}{x+1}}\Rightarrow x=0$ or $x(x+1)-1=0$ $\Rightarrow x=0$ or $\frac{-1\pm\sqrt{5}}{2}$.) We will show $x_n\ge x_{n+1}\ge\frac{-1+\sqrt{5}}{2}$. For $n=1$, $3>\frac{\sqrt{3}}{2}\approx\frac{1.73}{2}>\frac{-1+\sqrt{5}}{2}\approx\frac{1.2}{2}$.

Suppose $x_n\ge x_{n+1}\ge\frac{-1+\sqrt{5}}{2}$. Then $x_n+1\ge x_{n+1}+1\ge\frac{1+\sqrt{5}}{2}\Rightarrow\frac{1}{x_n+1}\le\frac{1}{x_{n+1}+1}\le\frac{2}{1+\sqrt{5}}\frac{(1-\sqrt{5})}{(1-\sqrt{5})}=\frac{\sqrt{5}-1}{2}$

and so $\sqrt{1-\frac{1}{x_n+1}}\ge\sqrt{1-\frac{1}{x_{n+1}+1}}\ge\sqrt{1-(\frac{\sqrt{5}-1}{2})}=\sqrt{\frac{3-\sqrt{5}}{2}}=\frac{-1+\sqrt{5}}{2}$ (as $(\frac{-1+\sqrt{5}}{2})^2=\frac{6-2\sqrt{5}}{4}=\frac{3-\sqrt{5}}{2}$)

So $x_{n+1}\ge x_{n+2}\ge\frac{-1+\sqrt{5}}{2}$. Therefore, $\{x_n\}$ is decreasing and bounded below. By the monotone sequence theorem, $\lim_{n\to\infty}x_n=x$ exists. Then $x=\sqrt{1-\frac{1}{x+1}}=\sqrt{\frac{x}{x+1}}\Rightarrow$ $x=0$ or $\frac{-1+\sqrt{5}}{2}$. Since $x_n\ge\frac{-1+\sqrt{5}}{2}>0>\frac{-1-\sqrt{5}}{2}$, $\lim_{n\to\infty}x_n=x=\frac{-1+\sqrt{5}}{2}$.

(m) We claim that $0<x_n<1$ for $n=1,2,3,\cdots$. The case $n=1$ is given. Suppose $0<x_n<1$, then $0<x_{n+1}=\frac{x_n^3+6}{7}<\frac{1+6}{7}=1$, completing the induction. Next, $x_{n+1}-x_n=\frac{x_n^3+6}{7}-x_n=\frac{x_n^3-7x_n+6}{7}=\frac{(x_n-1)(x_n-2)(x_n+3)}{7}>0$ implies $\{x_n\}$ is increasing. Since it is bounded above by 1, $\{x_n\}$ converges to some x by monotone sequence theorem. We have $7x=\lim_{n\to\infty}7x_{n+1}=\lim_{n\to\infty}x_n^3+6=x^3+6\Rightarrow x^3-7x+6=0\Rightarrow x=1,2$ or -3. Since $0<x_n<1$, $x=1$.

(92)(n) ($x=\sqrt{3x-2} \Rightarrow x^2-3x+2=0 \Rightarrow x=1$ or 2.) If $x_1=1$ and $x_n=1$, then $x_{n+1}=\sqrt{3\cdot 1-2}=1$ and so $\lim_{n\to\infty} x_n = 1$ in that case. If $x_1 \in (1,2]$, then we claim $1<x_n \le x_{n+1} \le 2$. For $x_n>1$ $x_n \le x_{n+1} \Leftrightarrow x_n^2 \le 3x_n-2 \Leftrightarrow x_n^2-3x_n+2=(x_n-1)(x_n-2)\le 0 \Leftrightarrow 1\le x_n \le 2$. Since $1<x_1\le 2$, so if $1<x_n\le 2$, then $1<x_n\le x_{n+1}=\sqrt{3x_n-2}\le\sqrt{3\cdot 2-2}=2$, completing induction. So $\lim_{n\to\infty} x_n$ exists in this case. It is a root of $x=\sqrt{3x-2}$ in $(1,2]$. So $\lim_{n\to\infty} x_n=2$ in this case. If $x_1\in(2,\infty)$, then we claim $x_n \ge x_{n+1} \ge 2$. For $x_n>2$, $x_{n+1}\le x_n \Leftrightarrow 3x_n-2\le x_n^2$ $\Leftrightarrow x_n^2-3x_n+2=(x_n-1)(x_n-2)\ge 0 \Leftrightarrow x_n \ge 2$. Since $x_1>2$, so if $x_n>2$, then $x_n\ge x_{n+1}=\sqrt{3x_n-2}\ge\sqrt{3\cdot 2-2}=2$, completing induction. So $\lim_{n\to\infty} x_n$ exists in this case. It is a root of $x=\sqrt{3x-2}$ in $[2,\infty)$. $\therefore \lim_{n\to\infty} x_n = 2$ in this case.

(o) ($x_1 \le x_2$. If $x_2=0$, then $x_3=\frac{1}{3}$, so suspect $\{x_n\}$ is increasing. The equation $x=\frac{1}{3}(1+x+x^3)$ has $x=1$ as a root. So $x=\frac{1}{3}(1+x+x^3) \Leftrightarrow x^3-2x+1=0$ can be solved by factoring $x-1$. The roots are $1, \frac{-1+\sqrt5}{2}, \frac{-1-\sqrt5}{2}$. Note $a_1\le a_2<\frac{-1+\sqrt5}{2}$.)

Claim: $x_n \le x_{n+1} \le \frac{-1+\sqrt5}{2}$. (Note $\frac{-1-\sqrt5}{2}<\frac{-1+\sqrt5}{2}<1$.)

Case $n=1$ is true as $x_1\le x_2 \le \frac{-1+\sqrt5}{2}$. Case $n=2$ is true because $x_2\le\frac12 \Leftrightarrow$ $x_2 \le \frac13(1+x_2)=x_3 \le \frac13(1+\frac{-1+\sqrt5}{2})=\frac13(\frac{1+\sqrt5}{2})<\frac{-1+\sqrt5}{2}$. Assume cases $n-1$ and n, we have $x_{n-1}\le x_n \le \frac{-1+\sqrt5}{2}$ and $x_n \le x_{n+1} \le \frac{-1+\sqrt5}{2}$. So

$$x_{n+1}=\frac13(1+x_n+x_{n-1}^3)\le\frac13(1+x_{n+1}+x_n^3)=x_{n+2}\le\frac13\left(1+\frac{-1+\sqrt5}{2}+\left(\frac{-1+\sqrt5}{2}\right)^3\right)=\frac{-1+\sqrt5}{2}.$$

Completing induction.

By monotone sequence theorem, $\lim_{n\to\infty} x_n$ exists. It is a root of $x=\frac13(1+x+x^3)$ in $[0,\frac{-1+\sqrt5}{2}]$, so $\lim_{n\to\infty} x_n=\frac{-1+\sqrt5}{2}$.

---

(93) From $x_2=a_1-a_2\le x_4=a_1-a_2+a_3-a_4\le x_3=a_1-a_2+a_3\le x_1=a_1$, we define $I_n=[x_{2n},x_{2n-1}]$. We claim $I_1\supseteq I_2\supseteq I_3\supseteq I_4\supseteq\cdots$. For this, we have to check $I_n=[x_{2n},x_{2n-1}]\supseteq I_{n+1}=[x_{2n+2},x_{2n+1}]$. (Since $\{a_n\}$ is decreasing, $x_{2n}\le x_{2n+2}=x_{2n}+a_{2n+1}-a_{2n+2}\le x_{2n+1}=x_{2n}+a_{2n+1}=x_{2n-1}-a_{2n}+a_{2n+1}\le x_{2n-1}$.) Finally since $\lim_{n\to\infty}|x_{2n}-x_{2n-1}|=\lim_{n\to\infty} a_{2n}=0$, we have $\bigcap_{n=1}^{\infty} I_n=\{x\}$, $\lim_{n\to\infty} x_{2n}=x=\lim_{n\to\infty} x_{2n-1}$. So $\lim_{n\to\infty} x_n=x$.

Alternative Solution Applying summation by parts, we get $x_n=S_na_n-\sum_{k=1}^{n-1}S_k(a_{k+1}-a_k)$, where $S_j=\sum_{k=1}^{j}(-1)^{k+1}=\begin{cases}0 & \text{if } j \text{ is even}\\ 1 & \text{if } j \text{ is odd}\end{cases}$. Since $\{a_n\}$ is a decreasing sequence with limit 0 and $0\le S_n\le 1$, we have $\lim_{n\to\infty} S_na_n=0$. Also, $-S_k(a_{k+1}-a_k)\ge 0$ so that $y_n=-\sum_{k=1}^{n-1}S_k(a_{k+1}-a_k)$ is increasing. Since $y_n\le\sum_{k=1}^{n-1}1(a_k-a_{k+1})=a_1-a_n\le a_1$, by monotone sequence theorem, $\lim_{n\to\infty} y_n$ exists. Then $\lim_{n\to\infty} x_n=\lim_{n\to\infty} S_na_n+\lim_{n\to\infty} y_n$ exists.

(84) Observe $a_1=a\le a_2=\frac{a+b}{2}=\sqrt{\frac{a^2+2ab+b^2}{4}}\le b_2=\sqrt{\frac{2a^2+2b^2}{4}}\le b_1=b$. We will try to show $a_n\le a_{n+1}\le b_{n+1}\le b_n$ by mathematical induction. Case $n=1$ is done above. Suppose case $n=k$ is true. For case $n=k+1$, since $a_{k+1}\le b_{k+1}$,

$$a_{k+1}\le a_{k+2}=\frac{a_{k+1}+b_{k+1}}{2}=\sqrt{\frac{a_{k+1}^2+2a_{k+1}b_{k+1}+b_{k+1}^2}{4}}\le b_{k+2}=\sqrt{\frac{2a_{k+1}^2+2b_{k+1}^2}{4}}\le b_{k+1}.$$

So $\{a_n\}$ is increasing and bounded above by $b_1=b$, hence converges to some A. Also $\{b_n\}$ is decreasing and bounded below by $a_1=a$, hence converges to some B. Since $a_{n+1}=\frac{a_n+b_n}{2}$, so $A=\lim_{n\to\infty}a_{n+1}=\lim_{n\to\infty}\frac{a_n+b_n}{2}=\frac{A+B}{2}\Rightarrow A=B$.

---

(85) (i) If $a\le b$ and $0<t<1$, then $a=ta+(1-t)a\le ta+(1-t)b\le tb+(1-t)b=b$.

(ii) (Note $x_1=1$, $x_2=2$, $x_3=\frac13 2+\frac23 1=\frac43$, $x_4=\frac13\frac43+\frac23 2=\frac{16}{9}$ $1\ \ \frac43\ \ \frac{16}{9}\ \ 2$)

Let $I_n=[x_{2n-1},x_{2n}]$, then we will show $I_n\supseteq I_{n+1}$, i.e. $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$ for all $n\in\mathbb{N}$. Now $x_{2n-1}\le x_{2n+1}=\frac13 x_{2n}+\frac23 x_{2n-1}=\frac23 x_{2n-1}+\frac13 x_{2n}\le x_{2n}$ by part (i). Also $x_{2n+1}\le x_{2n+2}=\frac13 x_{2n+1}+\frac23 x_{2n}\le x_{2n}$ by part (i) again. So we get $x_{2n-1}\le x_{2n+1}\le x_{2n+2}\le x_{2n}$ for every $n\in\mathbb{N}$. Note $1\le x_{2n-1}\le x_{2n}\le 2$. By the monotone sequence theorem, $\{x_{2n-1}\}$ converges to a and $\{x_{2n}\}$ converges to b for some $a,b\in\mathbb{R}$. Since $x_{2n+1}=\frac13 x_{2n}+\frac23 x_{2n-1}$, let $n\to\infty$, we get $a=\frac13 b+\frac23 a$ $\Rightarrow a=b$. By the intertwining sequence theorem, $\{x_n\}$ converges.

---

(96) ($x_0=0$, $x_1=1$, $x_2=\sqrt{\frac14 1+\frac34 0}=\frac12$, $x_3=\sqrt{\frac14\frac14+\frac34 1}=\sqrt{\frac{13}{16}}$ $0\ \ \frac12\ \ \sqrt{\frac{13}{16}}\ \ 1$)

If $x_n\le x_{n-1}$, then $x_n=\sqrt{\frac14x_n^2+\frac34x_n^2}\le x_{n+1}=\sqrt{\frac14x_n^2+\frac34x_{n-1}^2}\le x_{n-1}=\sqrt{\frac14x_{n-1}^2+\frac34x_{n-1}^2}$

If $x_{n-1}<x_n$, then $x_{n-1}=\sqrt{\frac14x_{n-1}^2+\frac34x_{n-1}^2}\le x_{n+1}=\sqrt{\frac14x_n^2+\frac34x_{n-1}^2}\le x_n=\sqrt{\frac14x_n^2+\frac34x_n^2}$.

So x_{n+1} is always between x_{n-1} and x_n. Define $I_n=[x_{2n},x_{2n+1}]$ for $n=0,1,2,\cdots$ Then $x_{2n}\le x_{2n+2}\le x_{2n+3}\le x_{2n+1}$ for $n=0,1,2,\cdots$; So $[0,1]=I_0\supseteq I_1\supseteq I_2\supseteq\cdots$.

By nested interval theorem, $\lim_{n\to\infty}x_{2n}=a$ and $\lim_{n\to\infty}x_{2n+1}=b$ exist. Taking limit of $x_{2n+1}=\sqrt{\frac14x_{2n}^2+\frac34x_{2n-1}^2}$, we get $b=\sqrt{\frac14a^2+\frac34b^2}\Rightarrow a=b$. By intertwining sequence theorem, x_n converges to some limit x.

To find x, write

$$\cancel{x_2^2}=\frac14x_1^2+\frac34x_0^2$$
$$\cancel{x_3^2}=\frac14\cancel{x_2^2}+\frac34x_1^2$$
$$\vdots$$
$$\frac34x_k^2\quad \cancel{x_k^2}=\frac14\cancel{x_{k-1}^2}+\frac34\cancel{x_{k-2}^2}$$
$$x_{k+1}^2=\frac14\cancel{x_k^2}+\frac34\cancel{x_{k-1}^2}$$

Adding these equations and cancelling common terms on both sides, we get $x_{k+1}^2+\frac34x_k^2=x_1^2+\frac34x_0^2=1$. Taking limit, we get $\frac74x^2=1$. So $x=\sqrt{\frac47}$.

(96) $x_{n+1} = \sqrt{\frac{1}{4}x_n^2 + \frac{3}{4}x_{n-1}^2} \Rightarrow x_{n+1}^2 = \frac{1}{4}x_n^2 + \frac{3}{4}x_{n-1}^2 \Rightarrow 4x_{n+1}^2 = x_n^2 + 3x_{n-1}^2$

$\Rightarrow 4x_{n+1}^2 - 4x_n^2 = -3x_n^2 + 3x_{n-1}^2 \Rightarrow x_{n+1}^2 - x_n^2 = -\frac{3}{4}(x_n^2 - x_{n-1}^2) = \cdots = \left(-\frac{3}{4}\right)^n \underbrace{(x_1^2 - x_0^2)}_{=1}$

$\Rightarrow x_{n+1}^2 = \underbrace{\sum_{k=0}^{n}(x_{k+1}^2 - x_k^2)}_{\text{telescoping series}} = \sum_{k=0}^{n}\left(-\frac{3}{4}\right)^k \Rightarrow x_n = \sqrt{\sum_{k=0}^{n-1}\left(-\frac{3}{4}\right)^k} \Rightarrow \lim_{n\to\infty} x_n = \sqrt{\sum_{k=0}^{\infty}\left(-\frac{3}{4}\right)^k} = \sqrt{\frac{1}{1-\left(-\frac{3}{4}\right)}} = \sqrt{\frac{4}{7}}.$

Geometric series

(97) Assume S is unbounded. Then for every $n \in \mathbb{N}$, there is $x_n \in S$ outside $[-n, n]$, i.e. $|x_n| > n$. We are given that $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Then $\{x_{n_k}\}$ is bounded. Since $|x_{n_k}| > n_k \geq k$ can be arbitrarily large, $\{x_{n_k}\}$ cannot be bounded, a contradiction. Therefore S is bounded.

---

(98) We have $x \in A, y \in A \Rightarrow x^2 + y^2 \leq (\sup A)^2 + (\sup A)^2 = 2(\sup A)^2$. So $2(\sup A)^2$ is an upper bound for B.

By supremum limit theorem, there is a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = \sup A$. Then $\{x_n^2 + x_n^2\}$ is a sequence in B and $\lim_{n\to\infty}(x_n^2 + x_n^2) = 2(\sup A)^2$. So by the Supremum limit theorem, $\sup B = 2(\sup A)^2$.

---

(99) For $x \in \bigcup_{n=1}^{10} A_n$, $x \in A_n$ for some $n \Rightarrow x \leq x_n = \sup A_n \leq \max(x_1, \cdots, x_{10})$.

So $\max(x_1, \cdots, x_{10})$ is an upper bound of $\bigcup_{n=1}^{10} A_n$. Let $x_i = \max(x_1, \cdots, x_{10})$, then since $x_i = \sup A_i$, there is $\{a_n\}$ in A_i such that $\lim_{n\to\infty} a_n = x_i$. Since $\{a_n\} \in \bigcup_{n=1}^{10} A_n$, so $x_i = \sup(\bigcup_{i=1}^{10} A_i)$. $\therefore \sup(\bigcup_{i=1}^{10} A_i) = \max(x_1, \cdots, x_{10})$.

Alternative Solution

As in first solution, $x_i = \max(x_1, \cdots, x_{10})$ is an upper bound of $\bigcup_{n=1}^{10} A_n$. For any upper bound M of $\bigcup_{n=1}^{10} A_n$, $M \geq x$ for every $x \in \bigcup_{n=1}^{10} A_n$. Since $A_i \subseteq \bigcup_{n=1}^{10} A_n$, $M \geq x$ for every $x \in A_i$. So M is an upper bound of A_i, too. Then $M \geq x_i$. So $x_i = \max(x_1, \cdots, x_{10})$ is the least upper bound of $\bigcup_{n=1}^{10} A_n$.

---

(100) Since $f(x,y) \in [0,1]$, all inf and sup expressions exist by completeness axiom.

For every $x_0 \in \mathbb{R}$, $h(y) = \inf\{f(x,y) : x \in \mathbb{R}\} \leq f(x_0, y) \leq g(x_0) = \sup\{f(x_0, y) : y \in \mathbb{R}\}$.

So $g(x_0)$ is an upper bound of $\{h(y) : y \in \mathbb{R}\}$. Then $\sup\{h(y) : y \in \mathbb{R}\} \leq g(x_0)$.

So $\sup\{h(y) : y \in \mathbb{R}\}$ is a lower bound of $\{g(x_0) : x_0 \in \mathbb{R}\}$. Therefore, $\sup\{h(y) : y \in \mathbb{R}\} \leq \inf\{g(x_0) : x_0 \in \mathbb{R}\}$.

---

(101) Let $x \in \mathbb{R}$. By the density of irrational numbers, there is $x_1 \in \mathbb{R} \setminus \mathbb{Q}$ such that $x - 1 < x_1 < x$. Suppose $x_n < x$ has been chosen, then we use density of irrational numbers to choose $x_{n+1} \in \mathbb{R} \setminus \mathbb{Q}$ such that $\max(x_n, x - \frac{1}{n+1}) < x_{n+1} < x$. Then $x_n < x_{n+1}$ and $x - \frac{1}{k} < x_k < x$ implies $\lim_{k\to\infty} x_k = x$ by the squeeze limit theorem.

---

(102) (Note $\frac{1}{n^2} < \frac{\varepsilon}{2} \Leftrightarrow \sqrt{\frac{2}{\varepsilon}} < n$ and $\frac{\sqrt{2}}{n^3} < \frac{\varepsilon}{2} \Leftrightarrow \sqrt[3]{\frac{2\sqrt{2}}{\varepsilon}} < n$.) For every $\varepsilon > 0$, by the Archimedean principle, there exists $K \in \mathbb{N}$ such that $K > \max\left(\sqrt{\frac{2}{\varepsilon}}, \sqrt[3]{\frac{2\sqrt{2}}{\varepsilon}}\right)$.

Then $n \geq K \Rightarrow \left|\left(\frac{1}{n^2} - \frac{\sqrt{2}}{n^3}\right) - 0\right| \leq \frac{1}{n^2} + \frac{\sqrt{2}}{n^3} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So $\lim_{n\to\infty}\left(\frac{1}{n^2} - \frac{\sqrt{2}}{n^3}\right) = 0$ by definition.

(103) (Note $\frac{2}{n+1}<\frac{\varepsilon}{2} \Leftrightarrow \frac{4}{\varepsilon}-1<n$ and $\frac{1}{n^2}<\frac{\varepsilon}{2} \Leftrightarrow \sqrt{\frac{2}{\varepsilon}}<n$.) For every $\varepsilon>0$, by the Archimedean principle, there is $K\in\mathbb{N}$ such that $K>\max(\frac{4}{\varepsilon}-1, \sqrt{\frac{2}{\varepsilon}})$. Then $n\geq K \Rightarrow |(\frac{2}{n+1}-\frac{1}{n^2})-0| \leq \frac{2}{n+1}+\frac{1}{n^2} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. So $\lim_{n\to\infty}(\frac{2}{n+1}-\frac{1}{n^2})=0$ by definition.

---

(104) For every $\varepsilon>0$, since $\lim_{n\to\infty} x_n=0$, there is $K_1\in\mathbb{N}$ such that $n\geq K_1 \Rightarrow |x_n-0|<\frac{\varepsilon}{2}$. By the Archimedean principle, there is $K_2\in\mathbb{N}$ such that $K_2>\frac{2}{\varepsilon}$. Let $K=\max(K_1,K_2)$ then $n\geq K \Rightarrow |(x_n+\frac{1}{n})-0| \leq |x_n-0|+\frac{1}{n} < \frac{\varepsilon}{2}+\frac{1}{K_2} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. Therefore, $\lim_{n\to\infty}(x_n+\frac{1}{n})=0$ by definition.

$n\geq K_1$ $n\geq K_2 \Rightarrow \frac{1}{n}\leq\frac{1}{K_2}<\frac{\varepsilon}{2}$

---

(105) Since $\lim_{n\to\infty} x_n=\frac{1}{2}$, so for $\varepsilon_0=\frac{1}{3}$, there is $K_1\in\mathbb{N}$ such that $n\geq K_1 \Rightarrow |x_n-\frac{1}{2}|<\varepsilon_0=\frac{1}{3}$ $\Rightarrow -\frac{1}{3}<x_n-\frac{1}{2}<\frac{1}{3} \Rightarrow \frac{1}{6}<x_n<\frac{5}{6} \Rightarrow |x_n^n-0|<(\frac{5}{6})^n$. So for every $\varepsilon>0$, let $K=\max(K_1, \lceil\frac{\ln 1/\varepsilon}{\ln 6/5}\rceil)$, then $n\geq K \Rightarrow |x_n^n-0|<(\frac{5}{6})^n\leq\varepsilon$.

---

(106) Since $\lim_{n\to\infty} x_n=8$, so for $\varepsilon_0=8$, there is $K_1\in\mathbb{N}$ such that $n\geq K_1 \Rightarrow |x_n-8|<\varepsilon_0=8$ $\Rightarrow -8<x_n-8 \Rightarrow x_n>0$. For $\varepsilon>0$, there is $K_2\in\mathbb{N}$ such that $n\geq K_2 \Rightarrow |x_n-8|<4\varepsilon$. Let $K=\max(K_1,K_2)$, then $n\geq K \Rightarrow n\geq K_1$ and $n\geq K_2$. Since $x_n>0$ for $n\geq K$ and $|x_n-8|=|\sqrt[3]{x_n}-2||(\sqrt[3]{x_n})^2+2\sqrt[3]{x_n}+4|>|\sqrt[3]{x_n}-2|4$, so $|\sqrt[3]{x_n}-2|<\frac{1}{4}|x_n-8|<\frac{1}{4}4\varepsilon=\varepsilon$.

Alternative Solution: Claim: $|\sqrt[3]{x}-\sqrt[3]{y}|\leq\sqrt[3]{|x-y|}$ for $x,y\geq 0$. Let $u=\max(x,y)$ and $v=\min(x,y)$, then we have to show $\sqrt[3]{u}-\sqrt[3]{v}\leq\sqrt[3]{u-v}$ ($\Leftrightarrow \sqrt[3]{u}\leq\sqrt[3]{v}+\sqrt[3]{u-v} \Leftrightarrow$ $u\leq v+3v^{2/3}(u-v)^{1/3}+3v^{1/3}(u-v)^{2/3}+(u-v)=u+3v^{2/3}(u-v)^{1/3}+3v^{1/3}(u-v)^{2/3}$) which is true. For the problem, since $\lim_{n\to\infty} x_n=8$, so for $\varepsilon_0=8$, there is $K_1\in\mathbb{N}$ such that $n\geq K_1 \Rightarrow |x_n-8|<\varepsilon_0=8 \Rightarrow -8<x_n-8 \Rightarrow x_n>0$. For $\varepsilon>0$, there is $K_2\in\mathbb{N}$ such that $|x_n-8|<\varepsilon^3$. Then for $n\geq K=\max(K_1,K_2)$, $|\sqrt[3]{x_n}-2|\leq\sqrt[3]{|x_n-8|}<\sqrt[3]{\varepsilon^3}=\varepsilon$.

---

(107) Let $\varepsilon>0$. Since $\{x_n\}$ and $\{y_n\}$ converge to A, so by definition, there are $K_1, K_2\in\mathbb{N}$ such that $n\geq K_1$ implies $|x_n-A|<\varepsilon$, and $n\geq K_2$ implies $|y_n-A|<\varepsilon$. Let $K=\max(K_1,K_2)$, then $n\geq K \Rightarrow n\geq K_1$ and $n\geq K_2 \Rightarrow |x_n-A|<\varepsilon$ and $|y_n-A|<\varepsilon$ $\Rightarrow |z_n-A|<\varepsilon$ (because $z_n=x_n$ or y_n.)

---

(108) For every $\varepsilon>0$, by Archimedean principle, there is integer $K>\frac{1}{\varepsilon}$.

Then $m,n\geq K$, $m>n$ $\Rightarrow$

$$\begin{aligned}|x_m-x_n| &= |(x_m-x_{m-1})+(x_{m-1}-x_{m-2})+\cdots+(x_{n+1}-x_n)| \\ &\leq |x_m-x_{m-1}|+|x_{m-1}-x_{m-2}|+\cdots+|x_{n+1}-x_n| \\ &< \frac{1}{2^{m-1}}+\frac{1}{2^{m-2}}+\cdots+\frac{1}{2^n} < \sum_{j=n}^{\infty}\frac{1}{2^j} = \frac{1}{2^{n-1}} \leq \frac{1}{2^{K-1}} \leq \frac{1}{K} < \varepsilon\end{aligned}$$

The case $m<n$ is similar. The case $m=n$ leads to $|x_m-x_n|=0<\varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence.

$K\leq 2^{K-1}$ can be proved by mathematical induction.

(109) (a) $f(x)$ converges to L (or has limit L) as x tends to x_0 in S iff for every $\varepsilon>0$, there exists a $\delta>0$ such that $x\in S$ and $0<|x-x_0|<\delta$ imply $|f(x)-L|<\varepsilon$.

(b) For every $\varepsilon>0$, take $\delta=\frac{2}{11}\varepsilon>0$. If $0<|x-2|<\delta$ and $x\in(1,3)$, then

$$|f(x)-\tfrac{9}{2}|=|(x^2+\tfrac{1}{x})-\tfrac{9}{2}|=|(x^2-4)+(\tfrac{1}{x}-\tfrac{1}{2})|\le|x^2-4|+|\tfrac{1}{x}-\tfrac{1}{2}|=|x+2||x-2|+\frac{|x-2|}{2|x|}$$
$$\le 5|x-2|+\tfrac{1}{2}|x-2|=\tfrac{11}{2}|x-2|<\tfrac{11}{2}\delta=\varepsilon.$$

(c) Solution 1 For every $\varepsilon>0$, take $\delta=\frac{\varepsilon}{6}>0$. If $0<|x-2|<\delta$ and $x\in(1,4)$, then

$$|f(x)-5|=||x^2-9|-|-5||\le|x^2-4|=|x-2||x+2|\le 6|x-2|<6\delta=\varepsilon.$$

by exercise 40, $||a|-|b||\le|a-b|$ $a=x^2-9$, $b=-5$

Solution 2 (Note that for $x\in[1,3]$, $x^2-9\le 0\Rightarrow f(x)=9-x^2$.)

For every $\varepsilon>0$, take $\delta=\min(1,\frac{\varepsilon}{5})>0$. If $0<|x-2|<\delta$, then $|x-2|<1\Rightarrow$ $x\in(1,3)\Rightarrow|f(x)-5|=|(9-x^2)-5|=|4-x^2|=|2-x||2+x|\le 5|x-2|<5\delta\le\varepsilon$.

---

(110) Since $\max(a,b)+\min(a,b)=a+b$ and $\max(a,b)-\min(a,b)=|a-b|$, so adding the two equation and dividing by 2, we get $\max(a,b)=\frac{a+b+|a-b|}{2}$.

Let S_f, S_g, S_h be the set of jumps of f,g,h, respectively. If f,g are continuous at x, then $h=\frac{f+g+|f-g|}{2}$ will also be continuous at x. Taking contrapositive, if $x\in S_h$, then $x\in S_f\cup S_g$. So $S_h\subseteq S_f\cup S_g$. By the monotone function theorem, S_f, S_g are countable. By the countable union theorem, $S_f\cup S_g$ is countable. By the countable subset theorem, S_h is countable.

---

(111) Define $f(x)=\begin{cases}\sin\pi x & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$. For every $m\in\mathbb{Z}$, $|f(x)|\le|\sin\pi x|\to 0$ as $x\to m$. So $\lim_{x\to m}f(x)=0=f(m)$. So f is continuous at every $m\in\mathbb{Z}$. For $x_0\notin\mathbb{Z}$, let $r_n\in\mathbb{Q}$ and $s_n\notin\mathbb{Q}$ such that $\lim_{n\to\infty}r_n=x_0=\lim_{n\to\infty}s_n$. Then $\lim_{n\to\infty}f(r_n)=\sin\pi x_0\ne 0=\lim_{n\to\infty}f(s_n)$. So f is not continuous at x_0 by the sequential continuity theorem.

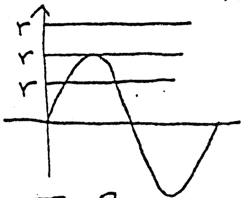
---

(112) (a) If $f:[a,b]\to\mathbb{R}$ is continuous and y_0 is between $f(a)$ and $f(b)$, then there is (at least one) $x_0\in[a,b]$ such that $f(x_0)=y_0$.

(b) Define $g:[0,1]\to\mathbb{R}$ by $g(x)=f(x)-f(x+1)$. Note $g(0)=f(0)-f(1)$ and $g(1)=f(1)-f(2)=f(1)-f(0)=-g(0)$. So $g(1)$ and $g(0)$ are of opposite sign. Since g is continuous on $[0,1]$, by the intermediate value theorem, $\exists c\in[0,1]$ such that $0=g(c)=f(c)-f(c+1)$. Then $f(c)=f(c+1)$.

(c) Observe that $|t|^r+|2t|^r+|3t|^r=|4t|^r+|5t|^r$ for every $t\in\mathbb{R}$ is equivalent to $1+2^r+3^r=4^r+5^r$. We will show this equation has a solution. Let $f(r)=1+2^r+3^r-4^r-5^r$, which is continuous. Since $f(0)=1$, $f(1)=-3$, by the intermediate value theorem, there is $r\in(0,1)$ such that $f(r)=0$. For this r, let $g(t)=|t|^r$, then $g(t)+g(2t)+g(3t)=g(4t)+g(5t)$. $\forall t\in\mathbb{R}$

(113) (a) For a fixed rational r, $\{x : \sin x = r\} = \bigcup_{k \text{ even integer}} \{x : \sin x = r, x \in [k\pi, k\pi + 2\pi)\}$



Since $\sin x = r$ on $[k\pi, k\pi+2\pi)$ has at most 2 solutions,

$$\{x : \sin x = r\} = \underbrace{\bigcup_{k \text{ even integer}}}_{\text{Countable}} \underbrace{\{x : \sin x = r, x \in [k\pi, k\pi+2\pi)\}}_{\text{Countable}} \text{ is Countable.}$$

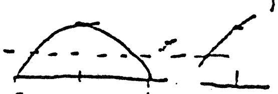
So $T = \{x : \sin x \in \mathbb{Q}\} = \underbrace{\bigcup_{r \in \mathbb{Q}}}_{\text{Countable}} \underbrace{\{x : \sin x = r\}}_{\text{Countable}}$ is Countable.

(b) For every $x \in [0,1]$, $\sin f(x) \in \mathbb{Q}$ implies $f(x) \in T$. So $f([0,1]) = \{f(x) : x \in [0,1]\} \subseteq T$. By (a), T is countable, so $f([0,1])$ is Countable.

Assume f is not a constant function, then $f([0,1])$ contains an interval (of positive length) by the intermediate value theorem. Then $f([0,1])$ is uncountable, a contradiction. Therefore, f is a constant function.

---

(114) Suppose such a function exists. Let a, b be the solutions of $f(x) = 0$ with $a < b$.



Case 1 $\max_{x \in [a,b]} f(x) = f(x_0) \neq 0$. Let y_0 be the other solution of $f(x) = f(x_0)$. If $y_0 \notin [a,b]$, then by the intermediate value theorem, there will be 3 solutions of $f(x) = \frac{1}{2} f(x_0)$, one on (a, x_0), one on (x_0, b) and one between y_0 and the closer endpoint of $[a,b]$ to y_0.

If $y_0 \in [a,b]$, then let $f(z_0) = \min_{x \in [x_0, y_0]} f(x)$ with $z_0 \in [x_0, y_0]$. Let $w = \max\{f(z_0), 0\}$, then by the intermediate value theorem, there are at least 3 solutions of $f(z) = w$, one on (a, x_0), one on (x_0, y_0), one on (y_0, b).

Thus, whether $y_0 \notin [a,b]$ or $y_0 \in [a,b]$ will lead to a contradiction.

Case 2 $\min_{x \in [a,b]} f(x) \neq 0$. This case is similar to case 1. (Turn figures upside down.)

Case 3 $\max_{x \in [a,b]} f(x) = 0 = \min_{x \in [a,b]} f(x)$. Then $f(x) \equiv 0$ on $[a,b]$, a contradiction.

---

(115) (a) $f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$. $-\frac{x^4}{|x|} \le f(x) \le \frac{x^4}{|x|} \Rightarrow \lim_{x \to 0} f(x) \overset{\swarrow}{=} 0 = f(0)$ (Sandwich Theorem)

So f is continuous at 0.

(b) $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} [f(x - x_0) + f(x_0)] = \lim_{x \to x_0} f(x - x_0) + f(x_0) = \lim_{y \to 0} f(y) + f(x_0) = 0 + f(x_0) = f(x_0)$

So f is continuous at every $x_0 \in \mathbb{R}$.

(c) $f(x) \equiv 0$ satisfies $f(x+y) = f(x) + f(y)$ and $|f(x)| \le x^4/|x|$ for $x \neq 0$.

---

(116) Since $\{g(x) : x \in [1,2]\} = [3,4]$, so there are $x_0, x_1 \in [1,2]$ such that $g(x_0) = 3$ and $g(x_1) = 4$. On the closed interval I with endpoints x_0 and x_1, since $f : I \to [3,4]$, $(f-g)(x_0) = f(x_0) - 3 \ge 0$ and $(f-g)(x_1) = f(x_1) - 4 \le 0$, $f - g$ is continuous on I, so by intermediate value theorem, there is $c \in I \subseteq [1,2]$ such that $(f-g)(c) = 0$. So $f(c) = g(c)$.

(117) (a) Observe that $|x_{k+1}-x_k| = |f(x_k)-f(x_{k-1})| \le \frac{1}{2}|x_k - x_{k-1}|$. Repeating this, we get $|x_{k+1}-x_k| \le \frac{1}{2}|x_k - x_{k-1}| \le (\frac{1}{2})^2 |x_{k-1}-x_{k-2}| \le \cdots \le (\frac{1}{2})^{k-1}|x_2 - x_1|$. So for $m > n$, we have

$$\begin{aligned}|x_m - x_n| &= |(x_m - x_{m-1}) + (x_{m-1}-x_{m-2}) + \cdots + (x_{n+1}-x_n)| \\ &\le |x_m - x_{m-1}| + |x_{m-1}-x_{m-2}| + \cdots + |x_{n+1}-x_n| \\ &\le \left(\left(\tfrac{1}{2}\right)^{m-2} + \left(\tfrac{1}{2}\right)^{m-3} + \cdots + \left(\tfrac{1}{2}\right)^{n-1}\right)|x_2 - x_1| \le \overbrace{\left(\tfrac{1}{2}\right)^{n-2}}^{=\sum_{k=n-1}^{\infty}\left(\frac{1}{2}\right)^k} |x_2 - x_1|.\end{aligned}$$

If $x_1 = x_2$, then $x_m = x_n$ for all m, n and $\{x_n\}$ is a constant sequence. Hence $\{x_n\}$ converges and is a Cauchy sequence. If $x_1 \ne x_2$, then for every $\varepsilon > 0$, by the Archimedean principle, there is $K \in \mathbb{N}$ such that $K > 2 - \log_2 \frac{\varepsilon}{|x_2 - x_1|}$, which implies $(\frac{1}{2})^{K-2}|x_2 - x_1| < \varepsilon$. So $m, n \ge K \Rightarrow |x_m - x_n| \le (\frac{1}{2})^{K-2}|x_2 - x_1| < \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence.

(b) Let $w \in \mathbb{R}$. Define $\{x_n\}$ as in (a). Then $\{x_n\}$ is a Cauchy sequence by (a). By Cauchy's theorem, $\{x_n\}$ converges to some $x \in \mathbb{R}$. We have

$$x \underset{\text{Subsequence theorem}}{=} \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) \underset{\text{Sequential continuity theorem}}{=} f(\lim_{n\to\infty} x_n) = f(x).$$

---

(118) Define $f(x) = \begin{cases} (x-1)^2 \sin\frac{1}{x-1} & \text{if } x \in (0,1)\cup(1,2) \\ 0 & \text{if } x = 1 \end{cases}$. For $x \in (0,1)\cup(1,2)$, by product rule, $f'(x) = 2(x-1)\sin\frac{1}{x-1} - \cos\frac{1}{x-1}$. For $x = 1$, $f'(1) = \lim_{x\to1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1}(x-1)\sin\frac{1}{x-1}$ $= 0$ as $|(x-1)\sin\frac{1}{x-1}| \le |x-1| \to 0$ as $x \to 1$. So f is differentiable on $(0,2)$. However, $\lim_{x\to1} f'(x) = -\lim_{x\to1}\cos\frac{1}{x-1}$ doesn't exist. So $f'(x)$ is not continuous at 1.

---

(119) We have $\left|\frac{f(a)-f(b)}{a-b}\right| \le \frac{\sin^2|a-b|}{|a-b|}$ for $a \ne b$, a, b in $(0,\pi)$. Since $\lim_{a\to b}\frac{\sin^2|a-b|}{|a-b|}$ $= \lim_{\theta\to0}\frac{\sin^2\theta}{\theta} = \lim_{\theta\to0}\frac{\sin\theta}{\theta}\lim_{\theta\to0}\sin\theta = 1\cdot0 = 0$, we have $f'(b) = \lim_{a\to b}\frac{f(a)-f(b)}{a-b} = 0$ for every b. Therefore, f is a constant function.

---

(120) (a) Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $x_0 \in (a,b)$ such that $f(b) - f(a) = f'(x_0)(b-a)$.

(b) By the mean value theorem, $|\sin b - \sin a| = |(\cos x_0)(b-a)| \le 1|b-a|$. If there is a K such that $|f(b)-f(a)| \le K|b-a|$ for every $a, b \in \mathbb{R}$, then $|f'(a)| = \lim_{b\to a}\left|\frac{f(b)-f(a)}{b-a}\right| \le K$ for every $a \in \mathbb{R}$. Since $f'(0) = \cos 0 = 1$, so $K \ge 1$. Therefore, the smallest K is 1.

---

(121) Since $\lim_{x\to0}\frac{f'(x)}{1} = \lim_{x\to0} f'(x)$ exists, by l'Hopital's rule, we have $\lim_{x\to0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to0}\frac{f'(x)}{1}$ $= \lim_{x\to0} f'(x)$ exists in $\mathbb{R}$. Therefore, $f'(0) \underset{\text{by definition of } f'(0)}{=} \lim_{x\to0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to0} f'(x)$, i.e. f' is continuous at 0.

(122) By the mean value theorem, $|\sin 5b - \sin 5a| = |(5\cos 5x_0)(b-a)| \le 5|b-a|$.
So for every $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{5} > 0$. With this δ, we have
for every $a, b \in \mathbb{R}$, $|b-a| < \delta \Rightarrow |\sin 5b - \sin 5a| \le 5|b-a| < 5\delta = \varepsilon$.

---

(123) For every $\varepsilon > 0$, since f is uniformly continuous, so $\exists \delta > 0$ such that $\forall x, y \in \mathbb{R}$
$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon^2$. Then $|x-y| < \delta \Rightarrow |\sqrt{f(x)} - \sqrt{f(y)}| \le \sqrt{|f(x) - f(y)|} < \varepsilon$,
(where we used $|\sqrt{a} - \sqrt{b}| \le \sqrt{|a-b|}$ ~~as in homework 2, #236(b)~~) Therefore,
$\sqrt{f(x)}$ is also uniformly continuous.

---

(124) (b) Solution 1. (using Lebesgue's Theorem) $\Rightarrow f, g$ bounded on $[0,2]$ $\Rightarrow h$ bounded on $[0,2]$.
Since f, g are Riemann integrable on $[0,2]$, so $S_f = \{x \in [0,2] : f$ is discontinuous at $x\}$ and $S_g = \{x \in [0,2] : g$ is discontinuous at $x\}$ are of measure 0.
Now for $x \in [0,1)$, h is discontinuous at x if and only if f is discontinuous at x.
Also for $x \in (1,2]$, h is discontinuous at x if and only if g is discontinuous at x.
(These are because $h = f$ on $[0,1)$ and $h = g$ on $(1,2]$.) So

$$S_h = \{x \in [0,2] : h \text{ is discontinuous at } x\} \subseteq (S_f \cap [0,1)) \cup (S_g \cap (1,2]) \cup \{1\}$$
$$\subseteq S_f \cup S_g \cup \{1\}.$$

Since $S_f, S_g, \{1\}$ are of measure 0, we have $S_f \cup S_g \cup \{1\}$ is of measure 0.
Then S_h is also of measure 0. Therefore, h is Riemann integrable by Lebesgue's Theorem.

Solution 2 (using integral criterion)
Since f and g are integrable on $[0,2]$, they are bounded on $[0,2]$. So there are $m, M \in \mathbb{R}$ such that $m \le f(x), g(x) \le M$ for all $x \in [0,2]$.
If $m = M$, then $h(x)$ is a constant function, hence h is integrable on $[0,2]$.
If $m < M$, then for every $\varepsilon > 0$, let $d \in (0,1)$ such that $1 - d < \frac{\varepsilon}{3(M-m)}$.
By 82(ii), f is integrable on $[0,d]$ and g is integrable on $[1,2]$.
By integral criterion, there are partitions P_1 of $[0,d]$ and P_2 of $[1,2]$ such that $U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{3}$ and $U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{3}$.
Now $P = P_1 \cup P_2$ is a partition of $[0,2]$ and

$$U(h,P) - L(h,P) \le U(f,P_1) - L(f,P_1) + (M-m)(1-d) + U(g,P_2) - L(g,P_2)$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

(125) Note $\max(f,g)+\min(f,g)=f+g$ and $\max(f,g)-\min(f,g)=|f-g|$. Subtracting, then dividing by 2, we have $h=\min(f,g)=\frac{f+g-|f-g|}{2}$. If f,g are integrable, then $f+g$, $f-g$ are integrable. Since $|x|$ is continuous, so $|f-g|$ is also integrable. Therefore, $h=\frac{f+g-|f-g|}{2}$ is integrable.

---

(126) Since $\mathbb{Q}\cap[0,1]$ is countable, let $r_1,r_2,r_3,\dots$ be a listing of the elements of $\mathbb{Q}\cap[0,1]$ without repetition nor omission. Define $f_n(x)=\begin{cases}1 & \text{if } x=r_1 \text{ or } r_2 \text{ or} \dots \text{or } r_n\\ 0 & \text{: otherwise}\end{cases}$. Then on $[0,1]$, f_n is discontinuous exactly at $r_1,r_2,\dots,r_n$. Since $\{r_1,r_2,\dots,r_n\}$ is countable, hence of measure 0, f_n is Riemann integrable by Lebesgue's theorem.

Now $f(x)=\lim_{n\to\infty}f_n(x)=\begin{cases}1 & \text{if } x=r_i \text{ for } i=1,2,3,\dots\\ 0 & \text{otherwise}\end{cases}=\begin{cases}1 & \text{if } x\in\mathbb{Q}\cap[0,1]\\ 0 & \text{otherwise}\end{cases}$.

On $[0,1]$, f is discontinuous everywhere as is shown in class. Since $[0,1]$ is not of measure 0, f is not Riemann integrable by Lebesgue's theorem.

---

(127) (a) Since $\left|\frac{\cos 3x}{1+x^2}\right|=\frac{|\cos 3x|}{1+x^2}\le\frac{1}{1+x^2}$ and $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\lim_{c\to-\infty}\int_c^0\frac{1}{1+x^2}dx+\lim_{d\to+\infty}\int_0^d\frac{1}{1+x^2}dx$

$=\lim_{c\to-\infty}(-\operatorname{Arctan} c)+\lim_{d\to+\infty}(\operatorname{Arctan} d)=\frac{\pi}{2}+\frac{\pi}{2}=\pi$, so $\frac{1}{1+x^2}$ is improper integrable on $(-\infty,\infty)$.

By comparison test, $\left|\frac{\cos 3x}{1+x^2}\right|$ is improper integrable on $(-\infty,\infty)$. By the absolute convergence test, $\frac{\cos 3x}{1+x^2}$ is improper integrable on $(-\infty,\infty)$. So $\int_{-\infty}^{\infty}\frac{\cos 3x}{1+x^2}dx$ exists.

(b) Since $\int_{-\infty}^{\infty}\frac{\cos 3x}{1+x^2}dx$ is improper integrable on $(-\infty,\infty)$, so P.V. $\int_{-\infty}^{\infty}\frac{\cos 3x}{1+x^2}dx$ exists.

(c) $\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}dx=\lim_{c\to0^-}\int_{-1}^{c}\frac{1}{\sqrt[3]{x}}dx+\lim_{d\to0^+}\int_d^1\frac{1}{\sqrt[3]{x}}dx=\lim_{c\to0^-}\left(\frac{3}{2}x^{2/3}\right)\Big|_{-1}^{c}+\lim_{d\to0^+}\left(\frac{3}{2}x^{2/3}\right)\Big|_d^1=-\frac{3}{2}+\frac{3}{2}=0$

(d) Since $\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}dx$ exists as an improper integral, so PV $\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}dx$ also exist.

Alternatively, PV $\int_{-1}^{1}\frac{1}{\sqrt[3]{x}}dx=\lim_{\varepsilon\to0^+}\left(\int_{-1}^{-\varepsilon}\frac{1}{\sqrt[3]{x}}dx+\int_{\varepsilon}^{1}\frac{1}{\sqrt[3]{x}}dx\right)=\lim_{\varepsilon\to0^+}0=0$

$1/\sqrt[3]{x}$ is an odd function.

(e) $\int_0^{\infty}\sin x\,dx=\lim_{c\to+\infty}\int_0^c\sin x\,dx=\lim_{c\to+\infty}(-\cos x)\Big|_0^c=\lim_{c\to+\infty}(-\cos c+1)$ doesn't exist.

So $\int_{-\infty}^{\infty}\sin x\,dx$ doesn't exist.

(f) PV $\int_{-\infty}^{\infty}\sin x\,dx=\lim_{c\to+\infty}\int_{-c}^{c}\sin x\,dx=\lim_{c\to+\infty}(-\cos x)\Big|_{-c}^{c}=\lim_{c\to+\infty}0=0$.

Solutions to Math 202 Past Exam Problems (Part I)

(128) Since $0+2 \le w+z \le 1+3$, we have $7 = 2^2+3 \le f(w+z) \le 4^2+3 = 19$. So 7 is a lower bound of S and 19 is an upper bound of S.
Let $w_n = 0$ and $z_n = 2 + \frac{1}{\sqrt{2}n}$, then $f(w_n+z_n) \in S$ and $f(w_n+z_n) = (w_n+z_n)^2+3$ converges to 7. $\therefore \inf S = 7$ by infimum limit theorem. Next let $w_n = 1$ and $z_n = 3 - \frac{1}{\sqrt{2}n}$, then $f(w_n+z_n) \in S$ and $f(w_n+z_n) = (w_n+z_n)^2+3$ converges to 19. $\therefore \sup S = 19$ by supremum limit theorem.

---

(129) $\lim\limits_{k\to\infty} \frac{2^{k+1}\sqrt{k+1}}{(2k+2)!}\frac{(2k)!}{2^k\sqrt{k}} = \lim\limits_{k\to\infty}\frac{1}{(2k+1)\sqrt{k(k+1)}} = 0 < 1 \Rightarrow \sum\limits_{k=1}^{\infty}\frac{2^k\sqrt{k}}{(2k)!}$ Converges by ratio test.

$|(\cos k)(\sin\frac{1}{k^2})| \le \sin(\frac{1}{k^2})$; $\lim\limits_{k\to\infty}\frac{\sin(\frac{1}{k^2})}{\frac{1}{k^2}} \underset{\theta=\frac{1}{k^2}}{=} \lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, $\sum\limits_{k=1}^{\infty}\frac{1}{k^2}$ converges by p-test $\underset{p=2>1}{}$

$\Rightarrow \sum\limits_{k=1}^{\infty}\sin(\frac{1}{k^2})$ Converges by limit comparison test $\Rightarrow \sum\limits_{k=1}^{\infty}(\cos k)(\sin\frac{1}{k^2})$ Converges by the Comparison test and the absolute convergence test.

---

(130) Let $W = \{(a,b,c,d) : a,b,c,d \in \mathbb{Q},\ (a,b)\neq(c,d)\}$. Then $W \subseteq \underbrace{\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}}_{\text{Countable by product theorem}}$ implies W is countable. For $(a,b,c,d)\in W$, let $L_{(a,b,c,d)}$ be the line through (a,b) and (c,d).
Then $S = \bigcup\limits_{(a,b,c,d)\in W}\underbrace{\{L_{(a,b,c,d)}\}}_{1 \text{ element} \Rightarrow \text{Countable}}$ is countable by countable union theorem.
Let $V = \{(L,L') : L, L' \in S,\ L \neq L'\}$. Then $V \subseteq \underbrace{S\times S}_{\text{Countable by product theorem}}$ implies V is countable.
Now $T = \bigcup\limits_{(L,L')\in V}\underbrace{(L\cap L')}_{0 \text{ or } 1 \text{ element} \Rightarrow \text{Countable}}$ is countable by countable union theorem.

Alternatively, each L in S has equation of the form $ax+by=c$ for some $a,b,c\in\mathbb{Q}$. The intersection point of two such lines (if they intersect) is in $\mathbb{Q}\times\mathbb{Q}$ (by algebra or Cramer's rule). So $T \subseteq \mathbb{Q}\times\mathbb{Q}$. Hence T is countable.

---

(131) For every $\varepsilon>0$, since $\lim\limits_{n\to\infty}x_n = 0$, $\exists K_1$ such that $n\ge K_1 \Rightarrow |x_n - 0| < \frac{1}{2} \Leftrightarrow x_n \in (-\frac{1}{2},\frac{1}{2})$ $\Leftrightarrow 1+x_n \in (\frac{1}{2},\frac{3}{2}) \Leftrightarrow \frac{1}{1+x_n}\in(\frac{2}{3},2)$ and $\exists K_2$ such that $n\ge K_2 \Rightarrow |x_n - 0| < \frac{\varepsilon}{2}$. Let $K = \max(K_1,K_2)$. Then $n \ge K \Rightarrow n\ge K_1$ and $n \ge K_2 \Rightarrow \left|\frac{x_n}{1+x_n} - 0\right| = \frac{|x_n|}{1+x_n} \le 2|x_n| < \varepsilon$.

---

(132) For every $\varepsilon > 0$, since $\lim\limits_{n\to\infty}x_n = w$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |x_n - w| < \varepsilon$.
In particular, for $n = 2K_1 \ge K_1$, we have $x_{2K_1} < w$ and $|x_{2K_1} - w| = w - x_{2K_1} < \varepsilon$.
Let $\underline{K = 2K_1 - 1}$. Then for $n \ge K = 2K_1 - 1$, we have $K_1 \le \frac{n+1}{2}$, which implies
$x_{2K_1} \in \{x_{2k} : k\in\mathbb{N},\ k \le \frac{n+1}{2}\}$. So $x_{2K_1} \le y_n < w \Rightarrow \underbrace{|y_n - w|}_{= w - y_n} \le \underbrace{|x_{2K_1} - w|}_{= w - x_{2K_1}} < \varepsilon$.
finite set $\uparrow$ $\uparrow$ bounded above by w
$\Rightarrow y_n < w$

(133) $w \in (\frac{\pi}{4}, \frac{\pi}{3}) \setminus \mathbb{Q} \Rightarrow \frac{\sqrt{2}}{2} \le f(w) \le \frac{\sqrt{3}}{2}$. For $n \in \mathbb{N}$, $\frac{\sqrt{2}}{2} - 1 \le f(w) - \frac{1}{n} \le \frac{\sqrt{3}}{2}$. So $\frac{\sqrt{2}}{2} - 1$ is a lower bound of S and $\frac{\sqrt{3}}{2}$ is an upper bound of S.

Let $w_k = \frac{\pi}{4} + \frac{\pi}{k+12}$ and $n = 1$, then $f(w_k) - \frac{1}{1} \in S$ and $f(w_k) - \frac{1}{1} \to \frac{\sqrt{2}}{2} - 1$
$\therefore \frac{\sqrt{2}}{2} - 1 = \inf S$ by infimum limit theorem. Next let $w_k = \frac{\pi}{3} - \frac{\pi}{k+12}$ and $n = k$; then $f(w_k) - \frac{1}{k} \in S$ and $f(w_k) - \frac{1}{k} \to \frac{\sqrt{3}}{2} - 0 = \frac{\sqrt{3}}{2}$. $\therefore \frac{\sqrt{3}}{2} = \sup S$ by supremum limit theorem.

---

(134) $\lim_{k\to\infty} \frac{(2k+3)^5}{(k+1)!} \frac{k!}{(2k+1)^5} = \lim_{k\to\infty} \left(\frac{2k+3}{2k+1}\right)^5 \frac{1}{k+1} = 0 < 1 \Rightarrow \sum_{k=1}^{\infty} \frac{(2k+1)^5}{k!}$ converges by ratio test.

$\left|\frac{\cos k}{k^4+k+1}\right| \le \frac{1}{k^4+k+1} \le \frac{1}{k^4}$ and $\sum_{k=1}^{\infty} \frac{1}{k^4}$ converges by p-test $_{p=4>1}$ $\Rightarrow \sum_{k=1}^{\infty} \left|\frac{\cos k}{k^4+k+1}\right|$ converges by comparison test $\Rightarrow \sum_{k=1}^{\infty} \frac{\cos k}{k^4+k+1}$ converges by absolute convergence test.

---

(135) In the case $x = 0$, S will contain all circles passing through $(1,1)$ and $(0,0)$. For every point (other than $(\frac{1}{2}, \frac{1}{2})$) on the perpendicular bisector of the segment joining $(0,0)$ to $(1,1)$, it can be used as a center for a unique such circle as it is equal distance from $(0,0)$ and $(1,1)$. As the perpendicular bisector minus $(\frac{1}{2}, \frac{1}{2})$ contains uncountably many points, S will contain uncountably many circles. So S is uncountable.

---

(136) For every $\varepsilon > 0$, since $\lim_{n\to\infty} x_n = 0$, $\exists K_1$ such that $n \ge K_1 \Rightarrow |x_n - 0| < \frac{1}{2} \Leftrightarrow x_n \in (-\frac{1}{2}, \frac{1}{2})$ $\Leftrightarrow x_n - 1 \in (-\frac{3}{2}, -\frac{1}{2}) \Leftrightarrow \left|\frac{1}{x_n - 1}\right| = \frac{1}{1-x_n} \in (\frac{2}{3}, 2)$ and $\exists K_2$ such that $n \ge K_2 \Rightarrow |x_n - 0| < \frac{\varepsilon}{2}$.
Let $K = \max(K_1, K_2)$. Then $n \ge K \Rightarrow n \ge K_1$ and $n \ge K_2 \Rightarrow \left|\frac{x_n}{x_n - 1} - 0\right| = \frac{|x_n|}{1 - x_n} \le 2|x_n| < \varepsilon$.

---

(137) For every $\varepsilon > 0$, since $\lim_{n\to\infty} x_n = w$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |x_n - w| < \varepsilon$.
In particular, for $n = 2K_1 \ge K_1$, we have $x_{2K_1} > w$ and $|x_{2K_1} - w| = x_{2K_1} - w < \varepsilon$.
Let $\underline{\underline{K = 2K_1 - 1}}$. Then for $n \ge K = 2K_1 - 1$, we have $K_1 \le \frac{n+1}{2}$, which implies $x_{2K_1} \in \{x_{2k} : k \in \mathbb{N}, k \le \frac{n+1}{2}\}$. So $x_{2K_1} \ge y_n > w \Rightarrow |y_n - w| \le |x_{2K_1} - w| < \varepsilon$

finite set; bounded below by w $\Rightarrow y_n > w$

$|y_n - w| = y_n - w$, $|x_{2K_1} - w| = x_{2K_1} - w$

(138) For $n \in \mathbb{N}$, $1+(-1)^n = 0$ or 2. For $w \in [1,\sqrt{2}) \setminus \mathbb{Q}$, $1 \le w \le \sqrt{2}$. So we have $1 = 0^2 + 1^2 \le f(1+(-1)^n, w) \le 2^2 + (\sqrt{2})^2 = 6$. So 1 is a lower bound of S and 6 is an upper bound of S.

Let $n=1$, $w_k = 1 + \frac{\sqrt{2}-1}{k+1}$, then $f(1+(-1)^n, w_k) = f(0, w_k) \in S$ and $f(0, w_k) = w_k^2 \to 1$. $\therefore 1 = \inf S$ by infimum limit theorem. Next, let $n=2$, $w_k = \sqrt{2} - \frac{1}{k+2}$, then $f(1+(-1)^n, w_k) = f(2, w_k) \in S$ and $f(2, w_k) = 2^2 + w_k^2 \to 2^2 + (\sqrt{2})^2 = 6$. $\therefore 6 = \sup S$ by supremum limit theorem.

---

(139) $\lim_{k\to\infty} \frac{2^{k+1}}{(k+1)^3(3k+3)!} \frac{k^3(3k)!}{2^k} = \lim_{k\to\infty} 2\left(\frac{k}{k+1}\right)^3 \frac{1}{(3k+3)(3k+2)(3k+1)} = 0 < 1 \Rightarrow \sum_{k=1}^{\infty} \frac{2^k}{k!(3k)!}$ converges by ratio test.

$\left|\left(\frac{1}{e}+\frac{1}{k}\right)^k \sin k\right| \le \left(\frac{1}{e}+\frac{1}{k}\right)^k$ and $\lim_{k\to\infty} \sqrt[k]{\left(\frac{1}{e}+\frac{1}{k}\right)^k} = \lim_{k\to\infty}\left(\frac{1}{e}+\frac{1}{k}\right) = \frac{1}{e} < 1 \Rightarrow \sum_{k=1}^{\infty}\left(\frac{1}{e}+\frac{1}{k}\right)^k$ converges by root test $\Rightarrow \sum_{k=1}^{\infty} \left|\left(\frac{1}{e}+\frac{1}{k}\right)^k \sin k\right|$ converges by comparison test $\Rightarrow \sum_{k=1}^{\infty}\left(\frac{1}{e}+\frac{1}{k}\right)^k \sin k$ converges by absolute convergence test.

---

(140) $x_1 = 1 < x_2 = \frac{\sqrt{1^2+4}}{2} = \frac{\sqrt{5}}{2} < x_3 = \frac{\sqrt{\frac{5}{4}+2\sqrt{5}}}{2}$ $\qquad x = \frac{\sqrt{x^2+4x}}{2} \Rightarrow 4x^2 = x^2+4x \Rightarrow x = 0, \frac{4}{3}$.

$\frac{5}{4} + 2\sqrt{5} > 1 + 2\sqrt{4} = 5$ $\qquad 5 = (\sqrt{5})^2 < \left(\frac{8}{3}\right)^2 = \frac{64}{9}$ rejected

We will show $x_n \le x_{n+1} \le \frac{4}{3}$ by math induction. For $n=1$, $x_1 = 1 < x_2 = \frac{\sqrt{5}}{2} < \frac{4}{3}$.

Suppose $x_n \le x_{n+1} \le \frac{4}{3}$. Then $x_n^2 + 4x_n \le x_{n+1}^2 + 4x_{n+1} \le \left(\frac{4}{3}\right)^2 + 4\left(\frac{4}{3}\right)$. Taking square root and dividing by 2, we get $x_{n+1} \le x_{n+2} \le \frac{4}{3}$ completing the induction. This shows $\{x_n\}$ is increasing and bounded above. By monotone sequence theorem, $\{x_n\}$ converges to some $x \in \mathbb{R}$. Now $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{\sqrt{x_n^2+4x_n}}{2} = \frac{\sqrt{x^2+4x}}{2} \Rightarrow 4x^2 = x^2+4x \Rightarrow x = \frac{4}{3}$ or 0

rejected as $x_1 > 0$.

---

(141) Let T be the set of all circles on the coordinate plane with center $(x,y) \in \mathbb{Q}\times\mathbb{Q}$ and radius $r \in \underset{\mathbb{Q}\cap(0,\infty)}{\overset{\shortparallel}{\mathbb{Q}^+}}$. Then $T = \bigcup_{(x,y,r)\in \underbrace{\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}^+}_{\text{countable}}} \{C_{(x,y,r)}\}$ is countable.

$C_{(x,y,r)}$: one circle with center (x,y), radius r

Now $S \subseteq \underbrace{(\mathbb{Q}\times\mathbb{Q})\times T}_{\text{countable by product theorem}}$ implies S is countable.

---

(142) For every $\varepsilon > 0$, since $\lim_{n\to\infty} x_n = 1$, $\exists K_1$ such that $n \ge K_1 \Rightarrow |x_n - 1| < 1 \Leftrightarrow x_n \in (0,2)$ and $\exists K_2$ such that $n \ge K_2 \Rightarrow |x_n - 1| < \frac{\varepsilon}{3}$. Let $K = \max(K_1, K_2)$. Then $n \ge K \Rightarrow n \ge K_1$ and $n \ge K_2 \Rightarrow |x_n^2 - 1| = |x_n - 1|(x_n + 1) \le 3|x_n - 1| < \varepsilon$. $\quad x_n + 1 \in (1,3)$

---

(143) (a) Since $\{x_n\}$ is bounded, $\exists M$ such that $|x_n| \le M \Leftrightarrow -M \le x_n \le M$ for all n. Then $-M-1 \in S$ and S is bounded above by M.

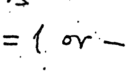
(b) By Supremum limit theorem, $\exists\, w_j \in S$ and $w_j \to s = \sup S$.
$\forall j \in \mathbb{N}$; $s+\frac{1}{j} > \sup S \Rightarrow s+\frac{1}{j} \notin S \Leftrightarrow$ there are only finitely many $n \in \mathbb{N}$ such that $s+\frac{1}{j} < x_n$.
Since $w_j \in S$, there are infinitely many $n \in \mathbb{N}$ such that $w_j < x_n$ and of these x_n's, only finitely many satisfy $s+\frac{1}{j} < x_n$. So there are infinitely many $x_n \in (w_j, s+\frac{1}{j}]$ for every j. Now pick $x_{n_1} \in (w_1, s+1]$. Once x_{n_j} is picked in $(w_j, s+\frac{1}{j}]$, since there are infinitely many x_n's in $(w_{j+1}, s+\frac{1}{j+1}]$ we can pick a $x_{n_{j+1}}$ with $n_{j+1} > n_j$ in $(w_{j+1}, s+\frac{1}{j+1}]$. So $\{x_{n_j}\}$ is a subsequence of x_n. Now $w_j \to s$, $s+\frac{1}{j} \to s$ and $w_j < x_{n_j} < s+\frac{1}{j}$ imply $x_{n_j} \to s$ by Sandwich theorem.

---

(144) For positive b, since $\lim_{k\to\infty} \sqrt[k]{(b+\frac{1}{k})^k} = \lim_{k\to\infty}(b+\frac{1}{k}) = b$, by root test, the series will converge if $b<1$ and will diverge if $b>1$. When $b=1$, $(b+\frac{1}{k})^k \ge 1^k = 1$ so that $\lim_{k\to\infty}(b+\frac{1}{k})^k \ne 0$. By term test, the series will diverge if $b=1$.
The answer is $0<b<1$.

---

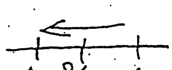
(145) For $a \in \mathbb{R}$, $S_a = \{\theta : \theta \in \mathbb{R}, \sin\theta = a\} = \underset{\substack{n\in\mathbb{Z} \\ \uparrow \\ \text{countable}}}{\bigcup} \underbrace{\{\theta : \theta \in [2n\pi, 2(n+1)\pi), \sin\theta = a\}}_{\text{at most 2 elements hence countable}}$ is countable by union theorem.
So $S = \bigcup_{a\in A} \underset{\substack{\uparrow \\ \text{countable}}}{S_a}$ is countable by union theorem.



Next $T = \mathbb{R} \setminus S$ is uncountable since $\mathbb{R}$ is uncountable and S is countable.

---

(146) $\left(x_1 = 6,\ x_2 = \frac{40}{15} = \frac{8}{3},\ x_3 = \frac{\frac{64}{9}+4}{\frac{16}{3}+3} = \frac{64+36}{48+27} = \frac{100}{75} = \frac{4}{3}\right.$



Suspect $\{x_n\}$ is decreasing.
$x = \frac{x^2+4}{2x+3} \Rightarrow 2x^2+3x = x^2+4 \Rightarrow x^2+3x-4 = 0 \Rightarrow x = 1$ or -4.)
We claim $x_n \ge 1$ and $x_n \ge x_{n+1}$ for $n = 1, 2, \cdots$. Note $x_1 = 6 \ge 1$ and suppose $x_n \ge 1$, then
$x_{n+1} = \frac{x_n^2+4}{2x_n+3} \ge 1 \overset{x_n\ge 1}{\Longleftrightarrow} x_n^2+4 \ge 2x_n+3 \Leftrightarrow x_n^2-2x_n+1 = (x_n-1)^2 \ge 0$, which is true.
So by induction $x_n \ge 1$ for $n = 1, 2, \cdots$. Next, for $n = 1, 2, \cdots$
$$x_n - x_{n+1} = x_n - \frac{x_n^2+4}{2x_n+3} = \frac{2x_n^2+3x_n-(x_n^2+4)}{2x_n+3} = \frac{x_n^2+3x_n-4}{2x_n+3} = \frac{(x_n+4)(x_n-1)}{2x_n+3} \overset{x_n \ge 1}{\ge} 0$$
So $x_n \ge x_{n+1}$. By monotone sequence theorem, since $\{x_n\}$ is decreasing and bounded below by 1, we see $\{x_n\}$ converges to some $x \in \mathbb{R}$. Then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n^2+4}{2x_n+3} = \frac{x^2+4}{2x+3}$ implies $x = 1$ or -4. Since $x_n \ge 1$, we have $x = 1$.

(147) Solution 1

Let $y_n = \frac{x_n}{x_n+1}$, then by algebra, $x_n = \frac{y_n}{1-y_n}$ and $x_n - 1 = \frac{2y_n - 1}{1-y_n} = \frac{2(y_n-\frac{1}{2})}{1-y_n}$.

Note if $|y_n - \frac{1}{2}| < \frac{1}{3}$, then $y_n \in (\frac{1}{6}, \frac{5}{6})$ so that $1-y_n \in (\frac{1}{6}, \frac{5}{6})$, $|x_n - 1| \le \frac{2|y_n-\frac{1}{2}|}{1/6}$.

For every $\varepsilon > 0$, since $\lim_{n\to\infty} y_n = \frac{1}{2}$, there is K_1 such that $n \ge K_1 \Rightarrow |y_n - \frac{1}{2}| < \frac{1}{3}$ and there is K_2 such that $n \ge K_2 \Rightarrow |y_n - \frac{1}{2}| < \frac{\varepsilon}{12}$. Let $K = \max(K_1, K_2)$, We have $n \ge K \Rightarrow n \ge K_1$ and $n \ge K_2 \Rightarrow |x_n - 1| \le \frac{2|y_n - \frac{1}{2}|}{1/6} = 12|y_n - \frac{1}{2}| < \varepsilon$.

Solution 2 Since $\lim_{n\to\infty} \frac{x_n}{x_n+1} = \frac{1}{2}$, $\exists K_1 \in \mathbb{N}$ such that $|\frac{x_n}{x_n+1} - \frac{1}{2}| < \frac{1}{4}$. Then

$\frac{x_n}{x_n+1} - \frac{1}{2} < \frac{1}{4} \Rightarrow \frac{x_n}{x_n+1} < \frac{3}{4} \Rightarrow \frac{1}{x_n+1} > 1 - \frac{3}{4} = \frac{1}{4} \Rightarrow 0 < x_n + 1 < 4$. Next,

$\exists K_2 \in \mathbb{N}$ such that $|\frac{x_n}{x_n+1} - \frac{1}{2}| = |\frac{x_n - 1}{2(x_n+1)}| < \frac{\varepsilon}{8}$. Let $K = \max(K_1, K_2)$.

Then $n \ge K \Rightarrow n \ge K_1$ and $n \ge K_2 \Rightarrow |x_n - 1| = |\frac{x_n-1}{2(x_n+1)}| \cdot 2|x_n+1| < \frac{\varepsilon}{8} \cdot 2 \cdot 4 = \varepsilon$.

---

(148) Let $V = \{\cos c : c \in S\}$ Since $S \subseteq [0, \frac{\pi}{2}]$, $U, V \subseteq [0,1]$, $x = \sup V$ and $y = \inf U$ exist in $\mathbb{R}$.
By supremum limit theorem, $\exists \cos c_n \in V$ (with $c_n \in S$) such that $\lim_{n\to\infty} \cos c_n = x$. For $a, b \in S$, $\cos^2 a + \cos^2 b \le 2x^2$ and $\lim_{n\to\infty} (\cos^2 c_n + \cos^2 c_n) = 2x^2$ imply $2x^2 = \sup T = \frac{1}{2}$. Hence $x = \frac{1}{2}$.
For $c \in S$, $\sin c = \sqrt{1-\cos^2 c} \ge \sqrt{1-(\frac{1}{2})^2} = \frac{\sqrt{3}}{2}$ Since $\lim_{n\to\infty} \sin c_n = \lim_{n\to\infty} \sqrt{1-\cos^2 c_n} = \frac{\sqrt{3}}{2}$ We have $\inf U = \frac{\sqrt{3}}{2}$.

---

(149) For $m \in \mathbb{Z} \setminus \{-1, 1\}$, let $S_m = \{(x,y) : x \in \mathbb{R}, y \in \mathbb{R}, x^2 + my^2 = 1$ and $mx^2 + y^2 = 1\}$.
Then S_m has at most 4 elements because $y^2 = 1 - mx^2 \Rightarrow \underbrace{x^2 + m(1-mx^2) = 1}_{(1-m^2)x^2 + m = 1}$ has at most 2 solutions for x.

So $S = \overbrace{\bigcup_{\underbrace{m \in \mathbb{Z} \setminus \{-1,1\}}_{\text{Countable}}} S_m}^{\text{Countable}}$ is countable.

---

(150) ($x_1 = 9$, $x_2 = \frac{\sqrt{9} + 2\cdot 9}{3} = 7$, $x_3 = \frac{\sqrt{7} + 2\cdot 7}{3} < \frac{3 + 2\cdot 7}{3} = 5\frac{2}{3}$. Suspect $\{x_n\}$ decreasing

$x = \frac{\sqrt{x} + 2x}{3} \Rightarrow x = \sqrt{x} \Rightarrow x = 0$ or 1.)

We claim: $x_n > x_{n+1} > 1$. For $n = 1$, $x_1 = 9 > x_2 = 7 > 1$. Suppose $x_n > x_{n+1} > 1$.
Then $\sqrt{x_n} > \sqrt{x_{n+1}} > 1$. So $\frac{\sqrt{x_n} + 2x_n}{3} > \frac{\sqrt{x_{n+1}} + 2x_{n+1}}{3} > \frac{\sqrt{1} + 2}{3}$, i.e. $x_{n+1} > x_{n+2} > 1$.
By the monotone sequence theorem, $\{x_n\}$ converges to some x. Then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{\sqrt{x_n} + 2x_n}{3} = \frac{\sqrt{x} + 2x}{3} \Rightarrow x = \sqrt{x} \Rightarrow x = 0$ or 1. Since $x_n > 1$, $x \ge 1$. Therefore the answer is 1.

---

(151) $\lim_{k\to\infty} \sqrt[k]{\frac{(b+a_k)^k}{k}} = \lim_{k\to\infty} \frac{b + a_k}{\sqrt[k]{k}} = b$ (because $\sum a_k$ converges $\Rightarrow \lim_{k\to\infty} a_k = 0$ and $\lim_{k\to\infty} \sqrt[k]{k} = 1$.)

So if $0 < b < 1$, then $\sum_{k=1}^{\infty} \frac{(b+a_k)^k}{k}$ converges. If $b > 1$, then $\sum_{k=1}^{\infty} \frac{(b+a_k)^k}{k}$ diverges.
If $b = 1$, then $\frac{(b+a_k)^k}{k} = \frac{(1+a_k)^k}{k} > \frac{1}{k}$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by p-test, $p=1$ so $\sum_{k=1}^{\infty} \frac{(b+a_k)^k}{k}$ diverges.

(152) $\lim_{k\to\infty} x_{2k}=0.5 \Rightarrow$ for $\varepsilon_0=0.2$, $\exists K_0\in\mathbb{N}$, $k\ge K_0 \Rightarrow |x_{2k}-0.5|<\varepsilon_0 \Rightarrow x_{2k}\in(0.3,0.7)$

$\lim_{k\to\infty} x_{2k+1}=0.6 \Rightarrow$ for $\varepsilon_1=0.1$, $\exists K_1\in\mathbb{N}$, $k\ge K_1 \Rightarrow |x_{2k+1}-0.6|<\varepsilon_1 \Rightarrow x_{2k+1}\in(0.4,0.7)$.

$\forall \varepsilon>0$, $(0.7)^n\le\varepsilon \Leftrightarrow n\ge\left\lceil\frac{\ln\varepsilon}{\ln 0.7}\right\rceil$. Let $K=\max\left(2K_0, 2K_1+1, \left\lceil\frac{\ln\varepsilon}{\ln 0.7}\right\rceil\right)$.

Then $n\ge K \Rightarrow n\ge 2K_0$ and $n\ge 2K_1+1$ and $n\ge\left\lceil\frac{\ln\varepsilon}{\ln 0.7}\right\rceil$

<u>Case n is even</u> $n=2k\ge 2K_0 \Rightarrow k\ge K_0 \Rightarrow |x_n^n-0|=x_{2k}^{2k}<(0.7)^n\le\varepsilon$

<u>Case n is odd</u> $n=2k+1\ge 2K_1+1 \Rightarrow k\ge K_1 \Rightarrow |x_n^n-0|=x_{2k+1}^{2k+1}<(0.7)^n\le\varepsilon$.

So $n\ge K \Rightarrow |x_n^n-0|<\varepsilon$.

---

(153) $\forall t\in I$, $A_t\subseteq A \Rightarrow x_t=\sup A_t\le\sup A \Rightarrow \sup\{x_t: t\in I\}\le\sup A$. Conversely, by supremum limit theorem, $\exists\, c_n\in A$ such that $\lim_{n\to\infty} c_n=\sup A$. $\forall n$, $c_n\in A=\bigcup_{t\in I} A_t$ $\Rightarrow c_n\in A_t$ for some $t\in I \Rightarrow c_n\le\sup A_t=x_t\le\sup\{x_t:t\in I\} \Rightarrow \sup A=\lim_{n\to\infty} c_n\le\sup\{x_t:t\in I\}$.

$\therefore \sup A=\sup\{x_t:t\in I\}$.

---

(154) Since $\left|\frac{\cos 3x}{\sqrt{x}}\right|\le\frac{1}{\sqrt{x}}$ and $\int_0^1\frac{1}{\sqrt{x}}dx=\lim_{a\to 0^+}\int_a^1\frac{1}{\sqrt{x}}dx=\lim_{a\to 0^+}2\sqrt{x}\Big|_a^1=2$, so by the comparison test, $\int_0^1\left|\frac{\cos 3x}{\sqrt{x}}\right|dx$ exists. By the absolute convergence theorem, $\int_0^1\frac{\cos 3x}{\sqrt{x}}$ exists.

---

(155) Assume such continuous function f exists. If $f(a)=f(b)$, then $-a=f(f(a))=f(f(b))$ $=-b \Rightarrow a=b$. So f is injective. By the continuous injection theorem, f is strictly monotone on $\mathbb{R}$. If f is strictly increasing, then $x<y\Rightarrow f(x)<f(y)\Rightarrow f(f(x))<f(f(y))$. If f is strictly decreasing, then $x<y\Rightarrow f(x)>f(y)\Rightarrow f(f(x))<f(f(y))$. In both cases, we have $f(f(x))$ is strictly increasing. However, $f(f(x))=-x$ is not strictly increasing, a contradiction. Therefore no such continuous function f exists.

---

(156) Since $\{x_n\}$ is Cauchy, for every $\varepsilon>0$, there exists K such that $m,n\ge K$ implies $|x_m-x_n|<\frac{\varepsilon}{5}$. Then $|\sin 5x_m-\sin 5x_n| \underset{\substack{\uparrow\\ \text{mean-value theorem}}}{=} |5\cos t\,(x_m-x_n)|\le 5|x_m-x_n|<\varepsilon$.

---

(157) Since f, g are Riemann integrable on $[0,2]$, $\max(f,g) = \frac{f+g+|f-g|}{2}$, $\min(f,g) = \frac{f+g-|f-g|}{2}$ are also integrable by theorem and remark in notes. By Lebesgue's theorem,

$S_{max} = \{x \in [0,2] : \max(f,g) \text{ discontinuous at } x\}$
$S_{min} = \{x \in [0,2] : \min(f,g) \text{ discontinuous at } x\}$ are both of measure 0.

Then $S_{max} \cap [0,1)$ and $S_{min} \cap (1,2]$ are both of measure 0. Since

$$S_h = \{x \in [0,2] : h \text{ is discontinuous at } x\} \subseteq \underbrace{(S_{max} \cap [0,1)) \cup (S_{min} \cap (1,2]) \cup \{1\}}_{\text{of measure } 0},$$

we see S_h is of measure 0. By Lebesgue's theorem, h is Riemann integrable on $[0,2]$.

Remarks

f, g Riemann integrable on $[0,2] \Rightarrow f, g$ bounded on $[0,2] \Rightarrow h$ bounded on $[0,2]$.

---

(158) $\sum_{k=1}^{\infty} \sin^k(1+\frac{1}{k})$ Root Test $\lim_{k\to\infty} \sqrt[k]{|\sin^k(1+\frac{1}{k})|} = \lim \sin(1+\frac{1}{k}) = \sin 1 < 1 \Rightarrow$ Series converges.

Comparison Test Since $0 \le \sin^k(1+\frac{1}{k}) \le \sin^k(1\frac{1}{2})$ and $0 < \sin(1\frac{1}{2}) < 1$, so $\sum_{k=1}^{\infty} \sin^k(1\frac{1}{2})$ converges by geometric series test. Hence $\sum_{k=1}^{\infty} \sin^k(1+\frac{1}{k})$ converges by comparison test.

$\sum_{k=1}^{\infty} \frac{1-\cos(1/k)}{1/k^2}$ Term Test $\lim_{k\to\infty} \frac{1-\cos(1/k)}{1/k^2} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \lim \frac{\sin\theta}{2\theta} = \frac{1}{2} \ne 0 \Rightarrow$ Series diverges

---

(159) $\inf A = 0$ and $\sup A = 1$ imply $A \subseteq [0,1]$. On $[0,1]$, $\frac{d}{dx}(x^3-4x+1) = 3x^2-4 = 3(x^2-\frac{4}{3}) < 0$.
So $f(x) = x^3-4x+1$ is strictly decreasing on $[0,1]$. So for every $a \in A$,
$f(1) = -2 \le f(a) = a^3-4a+1 \le f(0) = 1$. By infimum limit theorem and supremum limit theorem, there are sequences $\{r_n\}, \{s_n\}$ such that $\lim_{n\to\infty} r_n = 0$, $\lim_{n\to\infty} s_n = 1$ due to $\inf A = 0$ and $\sup A = 1$. Then $\{f(r_n)\}, \{f(s_n)\}$ are sequences in S such that $\lim_{n\to\infty} f(r_n) = 1$ and $\lim_{n\to\infty} f(s_n) = -2$. By infimum limit theorem and supremum limit theorem, $\inf S = -2$ and $\sup S = 1$.

---

(160) The set $S = \{\sqrt{x^2+y^2} : (x,y) \in P\} = \bigcup_{\underset{\text{countable}}{(x,y)\in P}} \underbrace{\{\sqrt{x^2+y^2}\}}_{\text{1 element, hence countable}}$ is countable by countable union theorem.

Then $(0,\infty) \setminus S$ is uncountable; in particular, nonempty.

Let $r \in \underbrace{(0,\infty) \setminus S}_{\text{uncountable}}$. The circle C with the origin as center and radius $r > 0$ contains no point in P as every point (x,y) in P has distance $\sqrt{x^2+y^2} \ne r$ from origin.

(161) (a) Let $A_r=\{f(t): 0<|t-w|<r\}$. Note $0<r_1\le r_2 \Rightarrow A_{r_1}\subseteq A_{r_2}$, which implies

$$u\le \underset{m(r_2)}{\underset{\|}{\inf A_{r_2}}} \le \underset{m(r_1)}{\underset{\|}{\inf A_{r_1}}} \le \underset{M(r_1)}{\underset{\|}{\sup A_{r_1}}} \le \underset{M(r_2)}{\underset{\|}{\sup A_{r_2}}} \le v.$$

Hence $M(r)$ is increasing and $m(r)$ is decreasing on $(0,\infty)$. By the monotone function theorem*, $\lim_{r\to 0+} m(r)=m(0+)$ and $\lim_{r\to 0+} M(r)=M(0+)$ exist.

*<u>Remark</u> For $r\le 0$, we may define $M(r)=u$ and $m(r)=v$ so that $M(r)$ is increasing and $m(r)$ is decreasing on $(-\infty,\infty)$.

(b) If $\lim_{r\to 0+} m(r)=L=\lim_{r\to 0+} M(r)$, then for $x\ne w$, let $r=2|x-w|$ to get $0<|x-w|<r$ and $m(r)\le f(x)\le M(r)$. Now $x\to w \Rightarrow r\to 0^+ \overset{\text{Sandwich theorem}}{\Longrightarrow} f(x)\to L$.

Conversely, if $\lim_{x\to w} f(x)=L$, then $\forall \varepsilon>0\ \exists \delta>0$ such that $0<|x-w|<\delta \Rightarrow |f(x)-L|<\frac{\varepsilon}{2}$. $f(x)\in(L-\frac{\varepsilon}{2},L+\frac{\varepsilon}{2})$

This means $A_\delta\subseteq(L-\frac{\varepsilon}{2},L+\frac{\varepsilon}{2})$. So $0<r<\delta \Rightarrow$ $\Rightarrow x\in A_\delta=(w-\delta,w+\delta)$

$$L-\varepsilon<L-\tfrac{\varepsilon}{2}\le m(\delta)\le m(r)\le M(r)\le M(\delta)\le L+\tfrac{\varepsilon}{2}<L-\varepsilon,$$

i.e. $0<|r-0|<\delta \Rightarrow |m(r)-L|<\varepsilon$ and $|M(r)-L|<\varepsilon$ $\therefore \lim_{r\to 0+} m(r)=L=\lim_{r\to 0+} M(r)$.

One student presented the following alternative proof of the converse part:
By (a) and sequential limit theorem, $\lim_{r\to 0+} M(r)=\lim_{n\to\infty} M(\frac1n)$. By the Supremum Property, $\exists t_n$ such that $0<|t_n-w|<\frac1n$ and $M(\frac1n)-\frac1n<f(t_n)\le M(\frac1n)$. Since $0<|t_n-w|<\frac1n$, by Sandwich theorem, $\lim_{n\to\infty} t_n=w$. By the sequential limit theorem, $\lim_{x\to w} f(x)=L \Rightarrow \lim_{n\to\infty} f(t_n)=L$. Since $f(t_n)\le M(\frac1n)\le f(t_n)+\frac1n$, by Sandwich theorem, $\lim_{n\to\infty} M(\frac1n)=L$. So $\lim_{r\to 0+} M(r)=L$. Similarly, $\lim_{r\to 0+} m(r)=L$.

---

(162) <u>Solution 1</u> $\forall\varepsilon>0$, Since $f:\mathbb{R}\to\mathbb{R}$ is uniformly continuous, $\exists\delta>0$ such that $|x-t|<\delta \Rightarrow |f(x)-f(t)|<\varepsilon$. Now $|x-t|<\delta \Rightarrow$

$$\left|\frac{1}{1+f(x)^2}-\frac{1}{1+f(t)^2}\right|=\left|\frac{f(x)^2-f(t)^2}{(1+f(x)^2)(1+f(t)^2)}\right|\le\frac{|f(x)|+|f(t)|}{(1+f(x)^2)(1+f(t)^2)}|f(x)-f(t)|$$

$$=\left(\frac{|f(x)|}{1+f(x)^2}\,\frac{1}{1+f(t)^2}+\frac{1}{1+f(x)^2}\,\frac{|f(t)|}{1+f(t)^2}\right)|f(x)-f(t)|<\left(\tfrac12\cdot1+1\cdot\tfrac12\right)\varepsilon=\varepsilon.$$

<u>Note</u> $\frac{r}{1+r^2}\le\frac12 \Leftrightarrow 0\le \underset{(1-r)^2}{\underset{\|}{1-2r+r^2}}$ is true.

<u>Remarks</u> If we let $a=|f(x)|$, $b=|f(t)|$, then the key step is to show $\frac{a+b}{(1+a^2)(1+b^2)}\le 1$. Multiplying by denominator and transferring terms, this is equivalent to showing
$0\le(1+a^2)(1+b^2)-(a+b)=1+a^2+b^2+a^2b^2-a-b=\frac12+(a^2-a+\frac14)+(b^2-b+\frac14)+a^2b^2$
$=\frac12+(a-\frac12)^2+(b-\frac12)^2+a^2b^2$, which is clear.

Solution 2 Let $h(x)=\frac{1}{1+x^2}$, then $|h'(x)|=\left|\frac{2x}{(1+x^2)^2}\right|\le\frac{1+x^2}{(1+x^2)^2}=\frac{1}{1+x^2}\le 1$ $\forall x\in\mathbb{R}$.
By mean-value theorem, $\forall a,b\in\mathbb{R}$, $|h(a)-h(b)|=|h'(c)(a-b)|\le|a-b|$.
$\forall\varepsilon>0$, Since $f:\mathbb{R}\to\mathbb{R}$ is uniformly continuous, $\exists\delta>0$ such that $|x-t|<\delta\Rightarrow$ $|f(x)-f(t)|<\varepsilon$. Then $\left|\frac{1}{1+f(x)^2}-\frac{1}{1-f(t)^2}\right|=|h(f(x))-h(f(t))|\le|f(x)-f(t)|<\varepsilon$.
Therefore, $g(x)=\frac{1}{1+f(x)^2}$ is also uniformly continuous.

---

(163) Since f is Riemann integrable, for every $\varepsilon>0$, $\exists$ partition $P_1=\{0=x_0<x_1<\cdots<x_n=1\}$ Such that $U(f,P_1)-L(f,P_1)<\frac{\varepsilon}{3}$. Choose $r\in(0,x_1)$ and $r<\frac{\varepsilon}{6}$. Also choose $s\in(x_{n-1},1)$ and $1-s<\frac{\varepsilon}{6}$. Let $P=P_1\cup\{r,s\}$, then $U(f,P)-L(f,P)<\frac{\varepsilon}{3}$ by the refinement theorem. Note $-1\le g(x)\le 1$ $\forall x\in[0,1]$. Then

$$U(g,P)-L(g,P)\le\left(\max_{x\in[0,r]}g(x)-\min_{x\in[0,r]}g(x)\right)r+\left(U(f,P)-L(f,P)\right)+\left(\max_{x\in[s,1]}g(x)-\min_{x\in[s,1]}g(x)\right)(1-s)$$
$$<2\cdot\frac{\varepsilon}{6}+\frac{\varepsilon}{3}+2\cdot\frac{\varepsilon}{6}=\varepsilon.$$

By the integral Criterion, g is Riemann integrable.

---

(164) Since $\lim_{x\to0^+}\frac{(\sin x)/x^{3/2}}{1/x^{1/2}}=\lim_{x\to0^+}\frac{\sin x}{x}=1$, so $\int_0^1\frac{\sin x}{x^{3/2}}dx<\infty\iff\int_0^1\frac{1}{x^{1/2}}dx<\infty$ by the limit Comparison test. Now $\int_0^1\frac{1}{x^{1/2}}dx=2x^{1/2}\Big|_0^1=2$, so $\int_0^1\frac{\sin x}{x^{3/2}}dx<\infty$.
For $\int_1^\infty\frac{\sin x}{x^{3/2}}dx$, we have $\left|\frac{\sin x}{x^{3/2}}\right|\le\frac{1}{x^{3/2}}$ and $\int_1^\infty\frac{1}{x^{3/2}}dx<\infty$ by $\underset{p=3/2>1}{\text{p-test}}$, So by the Comparison test, $\int_1^\infty\left|\frac{\sin x}{x^{3/2}}\right|dx<\infty$. By the absolute convergence test, $\int_1^\infty\frac{\sin x}{x^{3/2}}dx<\infty$; therefore, $\int_0^\infty\frac{\sin x}{x^{3/2}}dx$ converges.

---

(165) Let $S_f=\{x: f \text{ is discontinuous at } x\}$ and similarly for S_g and S_{fg}. By the monotone function theorem, S_f and S_g are countable sets. If f and g are continuous at x, then fg is continuous at x. Taking contrapositive, if fg is discontinuous at x, then f is discontinuous at x or g is discontinuous at x. So $S_{fg}\subseteq S_f\cup S_g$.
Since S_f, S_g countable $\underset{\text{Countable union theorem}}{\Longrightarrow}$ $S_f\cup S_g$ countable $\underset{\text{Countable subset theorem}}{\Longrightarrow}$ S_{fg} countable, we are done.

(166) We have shown in class that $h(x)=\begin{cases}1 & \text{if } x\in[0,1]\cap\mathbb{Q}\\ 0 & \text{if } x\in[0,1]\setminus\mathbb{Q}\end{cases}$ is not Riemann integrable on $[0,1]$. So it suffices to define $g(x)=\begin{cases}1 & \text{if } x=\frac{1}{n},\ n=1,2,3,\dots\\ 0 & \text{if } x\in[0,1]\setminus\{1,\frac{1}{2},\frac{1}{3},\dots\}\end{cases}$

Then $g\circ f(x)=\begin{cases}g(\frac{1}{n})=1 & \text{if } x\in[0,1]\cap\mathbb{Q}\\ g(0)=0 & \text{if } x\in[0,1]\setminus\mathbb{Q}\end{cases}=h(x)$ is not Riemann integrable on $[0,1]$.

Now g is discontinuous only on $S_g=\{1,\frac{1}{2},\frac{1}{3},\dots\}$, which is countable, hence of measure 0. So the bounded function g is Riemann integrable on $[0,1]$ by Lebesgue theorem.

Another solution, define $g(x)=\begin{cases}0 & \text{if } x=0\\ 1 & \text{if } 0<x\le 1.\end{cases}$ $S_g=\{0\}\Rightarrow$ g is Riemann integrable on $[0,1]$, $g\circ f(x)=h(x)$ is not Riemann integrable on $[0,1]$

---

(167) By mean value theorem, $|\sin a-\sin b|=|(\cos c)(a-b)|\le|a-b|$ for all $a,b\in\mathbb{R}$. So $|f(x)-f(y)|\le|x^2-y^2|$. For every $\varepsilon>0$, by Archimedean principle, there exist $K>\frac{1}{\sqrt{\varepsilon}}$. If $m,n\ge K$, then

$$|x_m-x_n|=|f(\tfrac{1}{m})-f(\tfrac{1}{n})|\le\left|\frac{1}{m^2}-\frac{1}{n^2}\right|\le\frac{1}{K^2}<\varepsilon.\ \therefore \{x_n\} \text{ is a Cauchy sequence.}$$

---

(168) $\forall x\in(2,3]\setminus\mathbb{Q}$ and $n=1,2,3,\dots$, $\frac{1}{x}+\frac{1}{n\sqrt{2}}<\frac{1}{2}+\frac{1}{\sqrt{2}}$. So $\frac{1}{2}+\frac{1}{\sqrt{2}}$ is an upper bound of S. Let $x_n=\frac{1}{2+\frac{1}{n\sqrt{2}}}+\frac{1}{\sqrt{2}}$, then $x_n\in S$ and $\lim_{n\to\infty}x_n=\frac{1}{2}+\frac{1}{\sqrt{2}}$. By the supremum limit theorem, $\sup S=\frac{1}{2}+\frac{1}{\sqrt{2}}$.

---

(169) (a) $f:S\to\mathbb{R}$ is continuous at $x_0\in S$ iff $\lim_{x\to x_0}f(x)=f(x_0)$, i.e. $\forall\varepsilon>0\ \exists\delta>0$ such that $\forall x\in S$, $|x-x_0|<\delta\Rightarrow|f(x)-f(x_0)|<\varepsilon$.

(b) $f(2)=1$. $|f(x)-1|=\left|\frac{2+3x}{x^2+4}-1\right|=\frac{|x^2-3x+2|}{x^2+4}=\frac{|x-2||x-1|}{x^2+4}$. If $x\in(1,3)$, then $x-1\in(0,2)$.

For every $\varepsilon>0$, let $\delta=\min(1,2\varepsilon)>0$. $\forall x\in S$, $|x-2|<\delta\Rightarrow|x-2|<1$, $|x-2|<2\varepsilon$ $\Rightarrow x\in(1,3)$, $|x-2|<2\varepsilon\Rightarrow|f(x)-1|=\frac{|x-2||x-1|}{x^2+4}\le\frac{|x-2|2}{4}<\varepsilon$.

---

(170) By the mean value theorem, if $a,b>0$, then $|e^{-a}-e^{-b}|=|(-e^{-c})(a-b)|\le e^{-0}|a-b|$.
c between a and b
$\forall\varepsilon>0$, since $\{x_n\}$ is Cauchy, $\exists K$ such that $m,n\ge K\Rightarrow|x_m-x_n|<\varepsilon$. Then $|e^{-x_m}-e^{-x_n}|\le|x_m-x_n|<\varepsilon$.

---

(171) (a) Since $S\setminus\{0,1\}\subseteq S$, $S\setminus\{0,1\}$ is of measure 0. For every $\varepsilon>0$, $\exists$ intervals (a_1,b_1), (a_2,b_2), (a_3,b_3), ... such that $S\setminus\{0,1\}\subseteq\bigcup_{n=1}^{\infty}(a_n,b_n)$ and $\sum_{n=1}^{\infty}|a_n-b_n|<\frac{\varepsilon}{2}$. Let $(c_n,d_n)=(a_n,b_n)\cap(0,1)$, then $S\setminus\{0,1\}\subseteq\bigcup_{n=1}^{\infty}(c_n,d_n)$ and $\sum_{n=1}^{\infty}|c_n-d_n|\le\sum_{n=1}^{\infty}|a_n-b_n|<\frac{\varepsilon}{2}$. Then $T\setminus\{0,1\}\subseteq\bigcup_{n=1}^{\infty}(c_n^2,d_n^2)$ and $\sum_{n=1}^{\infty}|c_n^2-d_n^2|=\sum_{n=1}^{\infty}|c_n+d_n||c_n-d_n|\le 2\sum_{n=1}^{\infty}|c_n-d_n|<\varepsilon$. So $T\setminus\{0,1\}$ is of measure 0. Since $T\subseteq\underbrace{(T\setminus\{0,1\})\cup\{0,1\}}_{\text{both are of measure 0}}$, T is of measure 0.

(b) f and h are given to be bounded. $g(t)=\sqrt{t}$ is continuous at x^2. So if f is continuous at $g(x^2)=\sqrt{x^2}=x$, then $h=f\circ g$ is continuous at x^2. Taking contrapositive, if h is discontinuous at x^2, then f is discontinuous at x. So $S_h=\{x^2\in[0,1]: h$ is discontinuous at $x^2\}\subseteq\{x^2: x\in S_f\}$.

$\therefore$ f integrable $\overset{\text{by Leb. Thm}}{\Longleftrightarrow}$ S_f is of measure 0 $\overset{\text{by (a)}}{\Longrightarrow}$ $\{x^2: x\in S_f\}$ is of measure 0

$\Rightarrow S_h$ is of measure 0 $\overset{\text{by Leb. Thm}}{\Longleftrightarrow}$ h integrable.

---

(172) (a) $$\lim_{k\to\infty}\frac{3^{k+1}}{(2(k+1))!\,(k+1)!}\cdot\frac{(2k)!\,k!}{3^k}=\lim_{k\to\infty}\frac{3}{(2k+2)(2k+1)(k+1)}=0<1\Rightarrow\sum_{k=1}^{\infty}\frac{3^k}{(2k)!\,k!}\text{ converges}$$ by ratio test

(b) By term test, $\sum_{k=1}^{\infty}|a_k|$ converges $\Rightarrow\lim_{k\to\infty}a_k=0$. Now $\lim_{k\to\infty}\dfrac{\left|\frac{a_k}{1+a_k}\right|}{|a_k|}=\lim_{k\to\infty}\frac{1}{|1+a_k|}=1$ and $\sum_{k=1}^{\infty}|a_k|$ converges $\Rightarrow\sum_{k=1}^{\infty}\left|\frac{a_k}{1+a_k}\right|$ converges $\Rightarrow\sum_{k=1}^{\infty}\frac{a_k}{1+a_k}$ converges.

---

(173) $x_1=4>x_2=4-\frac{4}{4}=3>x_3=4-\frac{4}{3}=8/3=2\frac{2}{3}$ $\quad x=4-\frac{4}{x}\Rightarrow x^2-4x+4=0$ $\Rightarrow x=2$.

Claim: $x_n>x_{n+1}>2$. For $n=1$, $x_1=4>x_2=3>x_3=2\frac{2}{3}$ is true. Then $x_n>x_{n+1}>2\Rightarrow\frac{4}{x_n}<\frac{4}{x_{n+1}}<\frac{4}{2}\Rightarrow 4-\frac{4}{x_n}>4-\frac{4}{x_{n+1}}>4-\frac{4}{2}$. Assume case n $\Rightarrow x_{n+1}>x_{n+2}>2$. $\therefore$ the claim is true for all n. By the monotone sequence theorem, $\lim_{n\to\infty}x_n=x$ exists. Then $x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}(4-\frac{4}{x_n})=4-\frac{4}{x}\Rightarrow x=2$.

---

(174) We have $0-2\sqrt{2}\le x-y\sqrt{2}\le 1-(-2)\sqrt{2}=1+2\sqrt{2}$. Since $\sup A=1$ and $\inf A=0$, by the supremum limit theorem and the infimum limit theorem, $\exists\, a_n\in A$ such that $\lim_{n\to\infty}a_n=1$ and $\exists\, a_n'\in A$ such that $\lim_{n\to\infty}a_n'=0$. Then $a_n'-(2-\frac{1}{n})\sqrt{2}\in S$ and $\lim_{n\to\infty}(a_n'-(2-\frac{1}{n})\sqrt{2})=0-2\sqrt{2}=-2\sqrt{2}$ and $a_n-(-2)\sqrt{2}\in S$ and $\lim_{n\to\infty}(a_n-(-2)\sqrt{2})=1+2\sqrt{2}$. So $\inf S=-2\sqrt{2}$ and $\sup S=1+2\sqrt{2}$ by the supremum limit theorem (by infimum limit theorem.

(175) Since $\lim_{n\to\infty} b_n = 3$, for $\varepsilon_0 = 1$, $\exists K_0 \in \mathbb{N}$ such that $n \ge K_0 \Rightarrow |b_n - 3| < 1$ $\Rightarrow b_n \in (2,4) \Rightarrow b_n + 2 \in (4,6)$. Now for $\varepsilon > 0$, $\exists K_1 \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |a_n - 2| < 2\varepsilon$ and $\exists K_2 \in \mathbb{N}$ such that $n \ge K_2 \Rightarrow |b_n - 3| < 2\varepsilon$. Let $K = \max(K_0, K_1, K_2)$. Then $n \ge K \Rightarrow n \ge K_1, K_2, K_3 \Rightarrow$

$$\left|\frac{a_n+3}{b_n+2} - 1\right| = \left|\frac{a_n+1-b_n}{b_n+2}\right| = \left|\frac{(a_n-2)-(b_n-3)}{b_n+2}\right| \le \frac{|a_n-2|+|b_n-3|}{4} < \frac{2\varepsilon+2\varepsilon}{4} = \varepsilon.$$

---

(176) (a) $\lim_{k\to\infty} \frac{(3(k+1))!}{(k+1)!\,2^{k+1}} \frac{k!\,2^k}{(3k)!} = \lim_{k\to\infty} \frac{\cancel{(3k+3)}^{\,3}(3k+2)(3k+1)}{\cancel{(k+1)}\,2} = \infty > 1 \Rightarrow \sum_{k=1}^{\infty} \frac{(3k)!}{k!\,2^k}$ diverges, by ratio test

(b) By term test, $\sum_{k=1}^{\infty} |a_k|$ converges $\Rightarrow \lim_{k\to\infty} a_k = 0$. Now $\lim_{k\to\infty} \frac{|a_k/\cos a_k|}{|a_k|} = \lim_{k\to\infty} \frac{1}{|\cos a_k|} = 1$ and $\sum_{k=1}^{\infty} |a_k|$ converges $\Rightarrow \sum_{k=1}^{\infty} |a_k/\cos a_k|$ converges $\Rightarrow \sum_{k=1}^{\infty} a_k/\cos a_k$ converges.

---

(177) $x_1 = 1 < x_2 = \frac{1^2+15}{8} = 2 < x_3 = \frac{2^2+15}{8} = 2\frac{3}{8}$ $\quad x = \frac{x^2+15}{8} \Rightarrow x^2 - 8x + 15 = 0 \Rightarrow x = 3$ or 5

Claim: $x_n < x_{n+1} < 3$. For $n=1$, $x_1 = 1 < x_2 = 2 < x_3 = 2\frac{3}{8}$. Assume case n is true. Then $x_n < x_{n+1} < 3 \Rightarrow \frac{x_n^2+15}{8} < \frac{x_{n+1}^2+15}{8} < \frac{3^2+15}{8} \Leftrightarrow x_{n+1} < x_{n+2} < 3$. $\therefore$ the claim is true for all n. By monotone sequence theorem, $\lim_{n\to\infty} x_n = x$ exists. Then $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n^2+15}{8} = \frac{x^2+15}{8} \Rightarrow x = 3$ or 5. Since $x_n < 3$, $x = 3$.

---

(178) Note $\frac{1}{2} \le y < 1 \Rightarrow 2 \ge \frac{1}{y} > 1$ and $-\frac{1}{2} \ge -y > -1 \Rightarrow 2 - \frac{1}{2} \ge \frac{1}{y} - y \ge 1 - 1 = 0$

For $x \in A$, $9 = 6\left(2 - \frac{1}{2}\right) \ge x\left(\frac{1}{y} - y\right) = \frac{x}{y} - xy > 2 \cdot 0 = 0$. By the supremum limit theorem, $\exists x_n \in A$ such that $\lim_{n\to\infty} x_n = 6$; then $\frac{x_n}{1/2} - x_n\left(\frac{1}{2}\right)$ has limit $6\left(2 - \frac{1}{2}\right) = 9$ and $\frac{x_n}{1-\frac{1}{n+2}} - x_n\left(1 - \frac{1}{n+2}\right)$ has limit $\frac{6}{1} - 6 = 0$. By the supremum limit theorem, $\sup S = 9$. By the infimum limit theorem, $\inf S = 0$.

---

(179) Since $\lim_{n\to\infty} a_n = 1$, for $\varepsilon_0 = 1$, $\exists K_0 \in \mathbb{N}$ such that $n \ge K_0 \Rightarrow |a_n - 1| < 1$ $\Rightarrow a_n \in (0,2)$. Let $K = \max\left(K_0, \left\lceil \frac{6}{\varepsilon} + 2 \right\rceil\right)$. Then $n \ge K \Rightarrow$ $n \ge K_0$ and $n \ge \frac{6}{\varepsilon} + 2 \Rightarrow \left|\frac{a_n^2 + n}{n - a_n} - 1\right| = \left|\frac{a_n^2 + a_n}{n - a_n}\right| < \frac{2^2+2}{n-2} = \frac{6}{n-2} \le \varepsilon.$

(180) (a) $f(x)$ converges to L as x tends to x_0 iff for every $\varepsilon>0$, there exists $\delta>0$ such that for every $x\in S$, $0<|x-x_0|<\delta$ implies $|f(x)-L|<\varepsilon$.

(b) Solution 1 $\forall\varepsilon>0$, set $\delta=\frac{\varepsilon}{\sqrt{2}}>0$

$\forall x\in(0.5,+\infty)$

Used $|\sqrt{a}-\sqrt{b}|\le\sqrt{|a-b|}$; $x>0$

$$0<|x-1|<\delta \Rightarrow \left|\sqrt{x+\tfrac{1}{x}}-\sqrt{2}\right| \le \sqrt{\left|x+\tfrac{1}{x}-2\right|} = \sqrt{\left|\frac{x^2-2x+1}{x}\right|} = \sqrt{\frac{(x-1)^2}{x}}$$

$$= \frac{|x-1|}{\sqrt{x}} \underset{x>0.5}{<} \sqrt{2}\,|x-1| < \varepsilon$$

need $\delta=\frac{\varepsilon}{\sqrt{2}}$

Solution 2 $\forall\varepsilon>0$, set $\delta=\sqrt{\frac{\varepsilon}{\sqrt{2}}}$

$\forall x\in(0.5,+\infty)$

$|\sqrt{a}-\sqrt{b}|=\frac{|a-b|}{\sqrt{a}+\sqrt{b}}$; $\sqrt{x+\frac{1}{x}}\ge 0$; $x>0.5$

$$0<|x-1|<\delta \Rightarrow \left|\sqrt{x+\tfrac{1}{x}}-\sqrt{2}\right| = \frac{\left|x+\frac{1}{x}-2\right|}{\sqrt{x+\frac{1}{x}}+\sqrt{2}} \le \frac{(x-1)^2/x}{\sqrt{2}} < \frac{2}{\sqrt{2}}(x-1)^2 < \varepsilon$$

need $\delta=\sqrt{\varepsilon/\sqrt{2}}$

(181) Solution 1 Let $b_n=na_n$. Then $b_{n+1}=(n+1)a_{n+1}=na_n+\frac{\cos n}{(n+1)^2}=b_n+\frac{\cos n}{(n+1)^2}$

To show $\lim_{n\to\infty} na_n=\lim_{n\to\infty} b_n$ exists, by Cauchy's theorem, it is the same as showing $\{b_n\}$ is a Cauchy sequence.

$\forall\varepsilon>0$, take $K\in\mathbb{N}$ such that $K>\frac{1}{\varepsilon}$.

$m,n\ge K \Rightarrow |b_m-b_n|=|(b_m-b_{m-1})+(b_{m-1}-b_{m-2})+\cdots+(b_{n+1}-b_n)|$

Say $m\ge n$

$$\le \left|\frac{\cos(m-1)}{m^2}+\frac{\cos(m-2)}{(m-1)^2}+\cdots+\frac{\cos n}{(n+1)^2}\right|$$

$$\le \frac{1}{m^2}+\frac{1}{(m-1)^2}+\cdots+\frac{1}{(n+1)^2}$$

$$< \frac{1}{m(m-1)}+\frac{1}{(m-1)(m-2)}+\cdots+\frac{1}{(n+1)n}$$

$$= \left(\frac{1}{m-1}-\frac{1}{m}\right)+\left(\frac{1}{m-2}-\frac{1}{m-1}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

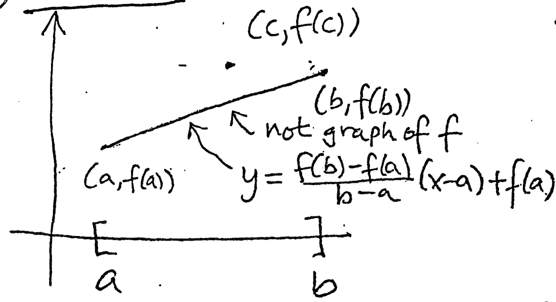
$$= \frac{1}{n}-\frac{1}{m} < \frac{1}{n} \le \frac{1}{K} < \varepsilon$$

need $K>1/\varepsilon$

Solution 2 Let $b_n=na_n$. Then $b_{n+1}=(n+1)a_{n+1}=na_n+\frac{\cos n}{(n+1)^2}=b_n+\frac{\cos n}{(n+1)^2}$.
Then $b_{n+1}-b_n=\frac{\cos n}{(n+1)^2}$. So $b_n=b_1+\sum_{k=2}^{n}(b_k-b_{k-1})=1+\sum_{k=2}^{n}\frac{\cos(k-1)}{k^2}$. Hence, $\lim_{n\to\infty} b_n$ exists $\Leftrightarrow \lim_{n\to\infty}\left(1+\sum_{k=2}^{n}\frac{\cos(k-1)}{k^2}\right)=1+\sum_{k=2}^{\infty}\frac{\cos(k-1)}{k^2}$ converges.

Since $\left|\frac{\cos(k-1)}{k^2}\right|\le\frac{1}{k^2}$ and $\sum_{k=2}^{\infty}\frac{1}{k^2}$ converges by p-test ($p=2>1$), by comparison test and absolute convergence test, $\sum_{k=2}^{\infty}\frac{\cos(k-1)}{k^2}$ converges. So $\lim_{n\to\infty} b_n$ exists.

(182) Solution 1



Since the graph of f is not a line segment, there exists $c \in (a,b)$ such that $(c, f(c))$ is either above or below the line segment joining $(a, f(a))$ and $(b, f(b))$. So either $f(c) > \frac{f(b)-f(a)}{b-a}(c-a)+f(a)$ or $f(c) < \frac{f(b)-f(a)}{b-a}(c-a)+f(a)$.

In the former case, solving for $\frac{f(b)-f(a)}{b-a}$, we get $\frac{f(c)-f(a)}{c-a} > \frac{f(b)-f(a)}{b-a}$.

By mean-value theorem, $\exists\, x_2 \in (a,c) \subseteq (a,b)$ such that $f'(x_2) = \frac{f(c)-f(a)}{c-a}$.

Then $\frac{f(b)-f(a)}{b-a} < f'(x_2)$. Next, we should consider $\frac{f(c)-f(b)}{c-b}$ in view of the expression $\frac{f(c)-f(a)}{c-a}$. We have

$$f(c)-f(b) > \frac{f(b)-f(a)}{b-a}(c-a)+f(a)-f(b) = (f(b)-f(a))\Big(\overbrace{\frac{c-a}{b-a}-1}^{=\frac{c-b}{b-a}}\Big).$$

Since $c-b<0$, we get $\frac{f(c)-f(b)}{c-b} < \frac{f(b)-f(a)}{b-a}$. By the mean-value theorem, $\exists\, x_1 \in (b,c) \subseteq (a,b)$ such that $f'(x_1) = \frac{f(c)-f(b)}{c-b}$.

Therefore, $f'(x_1) < \frac{f(b)-f(a)}{b-a} < f'(x_2)$. The latter case is similar.

Solution 2 Let $L(x) = \frac{f(b)-f(a)}{b-a}(x-a)+f(a)$. Assume there is no $x \in (a,b)$ such that $f'(x) < \frac{f(b)-f(a)}{b-a}$. Then $f'(x) \geq \frac{f(b)-f(a)}{b-a}$ for all $x \in (a,b)$. So $(f(x)-L(x))' = f'(x) - \frac{f(b)-f(a)}{b-a} \geq 0$ for all $x \in (a,b)$. This implies $f(x)-L(x)$ is increasing.

Since $f(a)-L(a) = 0 = f(b)-L(b)$, so $f(x)-L(x) = 0\ \forall x \in [a,b]$.

Then $f(x) = L(x)$, contradicting the graph of f is not a line segment.

$\therefore\ \exists\, x_1 \in (a,b)$ such that $f'(x_1) < \frac{f(b)-f(a)}{b-a}$. Similarly, $\exists\, x_2 \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} < f'(x_2)$.

(183) Solution 1

Since f, g are continuous on $[0,1]$, $h(x)=g(x)-f(x)$ is also continuous on $[0,1]$. By the extreme value theorem, $\exists\, u, v \in [0,1]$ such that $h(u)=\min_{x\in[0,1]} h(x)$ and $h(v)=\max_{x\in[0,1]} h(x)$. So $\forall x\in[0,1]$, $h(u)\le h(x)\le h(v)$. Then

$$h(u)\le h(x_n)=g(x_n)-f(x_n)=f(x_{n+1})-f(x_n)\le h(v).$$

(Now $f(x_{n+1})-f(x_n)$ suggests mean-value theorem or telescoping series. Since f is not known to be differentiable, we consider telescoping series.)

We have $h(x_1)+\cdots+h(x_n)=f(x_{n+1})-f(x_1)$.

Since $h(u)\le h(x_1),\cdots,h(x_n)\le h(v)$, so $h(u)\le c_n=\dfrac{h(x_1)+\cdots+h(x_n)}{n}\le h(v)$.

By the intermediate value theorem, $\exists\, w_n\in[u,v]\subseteq[0,1]$ such that $h(w_n)=c_n$.

Now $|c_n|=\left|\dfrac{f(x_{n+1})-f(x_1)}{n}\right|\le \dfrac{2\max_{x\in[0,1]} f(x)}{n}\to 0$ as $n\to\infty$. By the Bolzano-Weierstrass theorem, $\exists\, w_{n_i}\to w\in[0,1]$. Then $h(w)=\lim_{i\to\infty} h(w_{n_i})=\lim_{i\to\infty} c_{n_i}=0$. So $g(w)=f(w)$.

Solution 2 Define $h:[0,1]\to\mathbb{R}$ by $h(x)=g(x)-f(x)$. Assume $h(x)\ne 0$ for all $x\in[0,1]$. By the intermediate value theorem, either $h(x)>0$ for all $x\in[0,1]$ or $h(x)<0$ for all $x\in[0,1]$.

In the former case, $h(x_n)=g(x_n)-f(x_n)=f(x_{n+1})-f(x_n)>0$. Then $f(x_{n+1})>f(x_n)$. So $\{f(x_n)\}$ is strictly increasing. Since $f(x_n)\le\max_{x\in[0,1]} f(x)$, $\{f(x_n)\}$ is bounded above.

So $f(x_n)\to c\in[0,1]$. Then $g(x_n)=f(x_{n+1})\to c\in[0,1]$ by subsequence theorem. So $h(x_n)=g(x_n)-f(x_n)\to c-c=0$.

By Bolzano-Weierstrass theorem, $\exists\, x_{n_i}\to x\in[0,1]$.

Then $h(x)=\lim_{i\to\infty} h(x_{n_i})=0$, a contradiction.

The latter case is similar.

Therefore, there must exist some $x\in[0,1]$ such that $h(x)=0$, then $g(x)=f(x)$.