1. Prove that if a Cauchy sequence of rational numbers $(a_k)_{k\in\mathbb{N}}$ does not converge to 0, then there exists a positive integer N such that all the rational numbers $\{a_k \mid k \geq N\}$ have the same sign, that is, either every number in $\{a_k \mid k \geq N\}$ is positive, or every number in $\{a_k \mid k \geq N\}$ is negative.

2. Recall that for two Cauchy sequences of rational numbers $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$, we define the equivalence relation by

$$(a_k)_{k\in\mathbb{N}} \sim (b_k)_{k\in\mathbb{N}} \quad \text{if } (a_k - b_k)_{k\in\mathbb{N}} \text{ converges to } 0.$$

For a Cauchy sequence $(a_k)_{k\in\mathbb{N}}$, we denote $[a_k]$ as the equivalence class of $(a_k)_{k\in\mathbb{N}}$. Define the multiplication of two equivalence classes by

$$[a_k] \cdot [b_k] = [a_k b_k].$$

Prove that this multiplication is well-defined.

3. Prove that the order "$<$" on $\mathbb{R}$ which was defined in class satisfies the following property: If $x, y, z \in \mathbb{R}$ with $x < y$ and $z > 0$, then $xz < yz$.

4. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of rational numbers converging to a rational number a. Suppose that $a \neq 0$. For $k = 1, 2, \cdots$, let

$$b_k = \begin{cases} 0 & \text{if } a_k = 0 \\ \frac{1}{a_k} & \text{if } a_k \neq 0 \end{cases}$$

Prove that $\{b_n\}_{n\in\mathbb{N}}$ converges to $\frac{1}{a}$.