MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #27

Uniform Convergence and Consequence (A brief preview)

Previously, we know how to check the uniform convergence of $\{f_n(x)\}$ by checking the definition directly. (namely, find the limit function $f(x)$, then compute $\sup_{x\in E}|f_n(x)-f(x)|$ and show $\lim_{n\to\infty}(\sup_{x\in E}|f_n(x)-f(x)|)=0$).

When the functions are too complicated, if we use the definition to do checking, the calculation can be tedious. Hence to simplify the checking process, we need some theorems to help us. They are L-test (for sequences of functions) and Weierstrass's M-test (for series of functions).

<u>L-test (for sequence of functions)</u> Let $f_n: E\to R$ be sequences of functions on set E, suppose 1) $\lim_{n\to\infty} f_n(x)=f(x)$ (Pointwise Limit) 2) For each $n=1,2,3,\ldots$ there is constant L_n such that $\|f_n(x)-f(x)\|\le L_n$ for all $x\in E$ 3) $\lim_{n\to\infty} L_n=0$ Then $f_n(x)$ converges uniformly to $f(x)$
<u>M-test (for series of functions)</u> Let $g_k: E\to R$ be sequences of functions of E, suppose 1) For each $k=1,2,3,..$, there is constant M_k such that $\|g_k(x)\|\le M_k$ 2) $\sum_{k=1}^{\infty} M_k$ converges Then $\sum_{k=1}^{\infty} g_k(x)$ converges uniformly

Example 1 Show that the sequence of functions $f_n(x)=e^{-n}\sin(n^2x)$ converges uniformly on $\mathbf{R}$

Solution:

(Step 1: Find the limit function) For each $x\in\mathbf{R}$

$$f(x)=\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} e^{-n}\sin(n^2x)=\lim_{n\to\infty}\frac{\sin(n^2x)}{e^n}=0$$

(Step 2)

Since $|f_n(x)-f(x)|=|e^{-n}\sin(n^2x)-0|\le e^{-n}$ $\quad(\sin(n^2x)\le 1)$

Clearly $\lim\limits_{n\to\infty} e^{-n}=0$

Hence by L-test, $f_n(x)$ converges uniformly on $\mathbf{R}$

Example 2

Show that the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^x}$$

Converges uniformly on $[p,\infty)$ where $p>1$

Solution:

(Step 1) For $x\in[p,\infty)$

$$\left|\frac{(-1)^n}{n^x}\right| = \frac{1}{n^x} \le \frac{1}{n^p} \quad (Since\ \ x\ge p)$$

(Step 2)

Since $p>1$, then by p-test, the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges.

Hence by M-test, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^x}$ converges uniformly.

Example 3 (Useful Techniques)

Show that the series

$$\sum_{k=1}^{\infty}\frac{1}{k^2(e^{kx}+e^{-kx})}$$

converges uniformly on $\mathbf{R}$

Solution:

(Step 1) Note that for any $x\in\mathbf{R}$, $e^{kx}+e^{-kx}\ge 2\sqrt{(e^{kx})(e^{-kx})}=2$ by AM-GM inequality. Apply this, we have

$$\left|\frac{1}{k^2(e^{kx}+e^{-kx})}\right| \le \frac{1}{2k^2}$$

(Step 2) Clearly, by p-test, $\sum_{k=1}^{\infty}\frac{1}{2k^2}$ converges (in fact, it equals to $\frac{\pi^2}{12}$). So by M-test,

the series $\sum_{k=1}^{\infty}\frac{1}{k^2(e^{kx}+e^{-kx})}$ converges uniformly on $\mathbf{R}$

Example 4 (Another Techniques: Art of Calculus)

Show that the series

$$\sum_{n=1}^{\infty}\frac{x}{n^{\alpha}(1+nx^2)}$$

Converges uniformly on $\mathbf{R}$ where $\alpha>\frac{1}{2}$

Solution:

(Step 1)

Let $f(x)=\frac{x}{1+nx^2}\to f'(x)=\frac{(1+nx^2)-x(2nx)}{(1+nx^2)^2}=\frac{1-nx^2}{(1+nx^2)^2}$,

Set $f'(x)=0\to 1-nx^2=0\to x=\pm\sqrt{\frac{1}{n}}$

$$f''(x)=\frac{-2nx}{(1+nx^2)^2}-2\frac{(1-nx^2)(2nx)}{(1+nx^2)^3}=\frac{-2nx(3-nx^2)}{(1+nx^2)^3}$$

We see $f''(\sqrt{\frac{1}{n}})<0$ and $f''(-\sqrt{\frac{1}{n}})>0$, $f(x)$ attains maximum at $x=\sqrt{\frac{1}{n}}$ and minimum at $x=-\sqrt{\frac{1}{n}}$

Furthermore when $x\to\pm\infty$,$\lim_{x\to+\infty}f(x)=0$ and $\lim_{x\to-\infty}f(x)=0$

(The reason why we need to check the $\lim_{x\to+\infty}f(x)$ and $\lim_{x\to-\infty}f(x)$ is in first derivative test, we can only find the "relative" maximum/ minimum, but it may happen that the absolute maximum/minimum appears at $x=\pm\infty$! <u>Therefore, when you apply derivative to find max/min, remember to check this</u>)

So $-\frac{1}{2\sqrt{n}}=\frac{-\sqrt{\frac{1}{n}}}{1+n\left(-\sqrt{\frac{1}{n}}\right)^2}=f\left(-\sqrt{\frac{1}{n}}\right)\le f(x)\le f\left(\sqrt{\frac{1}{n}}\right)=\frac{\sqrt{\frac{1}{n}}}{1+n\left(\sqrt{\frac{1}{n}}\right)^2}=\frac{1}{2\sqrt{n}}$

therefore $|f(x)|\le\frac{1}{2\sqrt{n}}$

$$\left|\frac{x}{n^\alpha(1+nx^2)}\right|=\left|\frac{1}{n^\alpha}\left(\frac{x}{(1+nx^2)}\right)\right|\le\frac{1}{2n^{\alpha+\frac{1}{2}}}$$

(Step 2) Since $\alpha>\frac{1}{2}\to\alpha+\frac{1}{2}>1$, by p-test, the series $\sum_{n=1}^{\infty}\frac{1}{2n^{\alpha+\frac{1}{2}}}$ converges.

Therefore by M-test, the series $\sum_{n=1}^{\infty}\frac{x}{n^\alpha(1+nx^2)}$ converges uniformly.

<u>Consequence of Uniform Convergence</u>

Sometimes when we come across some calculations which involves interchange of limit, integration and differentiation, namely

$$\lim_{x\to a}\lim_{n\to\infty}f_n(x)\overset{?}{=}\lim_{n\to\infty}\lim_{x\to a}f_n(x)\qquad\int_a^b\lim_{n\to\infty}f_n(x)\,dx\overset{?}{=}\lim_{n\to\infty}\int_a^b f_n(x)dx$$

$$\frac{d}{dx}\lim_{n\to\infty}f_n(x)\overset{?}{=}\lim_{n\to\infty}\frac{d}{dx}f_n(x)$$

However, such actions are not allowed in general

Some simple examples show that it is not true in general

Counter-Example 1:

$$\left(\lim_{x\to a}\lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty}\lim_{x\to a} f_n(x)\right)$$

$$f_n(x) = \frac{x}{x+n}$$

We can see that

$$\lim_{x\to\infty}\lim_{n\to\infty} f_n(x) = \lim_{x\to\infty}\lim_{n\to\infty}\frac{x}{x+n} = \lim_{x\to\infty} 0 = 0$$

But

$$\lim_{n\to\infty}\lim_{x\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to\infty}\frac{x}{x+n} = \lim_{n\to\infty} 1 = 1$$

Counter Example 2:

$$\left(\int_a^b \lim_{n\to\infty} f_n(x)\,dx \neq \lim_{n\to\infty}\int_a^b f_n(x)dx\right)$$

Pick $f_n(x) = n^2x(1-x^2)^n$ for $0 \le x \le 1$

One can show

$$\lim_{n\to\infty} f_n(x) = 0 \ \ for\ 0 \le x \le 1$$

Now

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0dx = 0$$

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx = \lim_{n\to\infty}\int_0^1 n^2x(1-x^2)^n dx = \lim_{n\to\infty} -\frac{n^2(1-x^2)^{n+1}}{2(n+1)}\Big|_0^1$$

$$= \lim_{n\to\infty}\frac{n^2}{2(n+1)} = +\infty$$

Counter Example 3:

$$\left(\frac{d}{dx}\lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty}\frac{d}{dx} f_n(x)\right)$$

Pick $f_n(x) = \frac{\sin nx}{\sqrt{n}}$

$$f(x) = \lim_{n\to\infty} f_n(x) = 0 \to \frac{d}{dx}\lim_{n\to\infty} f_n(x) = f'(x) = 0$$

and $f_n'(x) = \sqrt{n}\cos nx$

Consider $x = 0$

$f'(0) = \frac{d}{dx}\lim_{n\to\infty} f_n(x) = 0$ but $\lim_{n\to\infty} f_n'(0) = \lim_{n\to\infty}\sqrt{n} \to \infty$

(More Examples can be found in Kin Li's Note in MATH301 Chapter 10)

But with uniform convergence, one can interchange the above operations freely (except differentiation as it needs more conditions)

Continuity Theorem (Interchange of Limit and Limit or Limit and Summation)
Let $f_n(x)$ converges uniformly to $f(x)$ on E and if $f_n(x)$ is continuous at $x = c$
Then $f(x) = \lim_{n\to\infty} f_n(x)$ is also continuous at $x = c$

$$\lim_{x\to c}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to c} f_n(x)$$

In case for series, if $\sum g_k(x)$ converges uniformly, then we have

$$\lim_{x\to c}\sum_{k=1}^{\infty} g_k(x) = \sum_{k=1}^{\infty}\lim_{x\to c} g_k(x)$$

Integration Theorem:
If $f_n(x)$ converges uniformly on $[a,b]$ and integrable, then we have

$$\int_a^b \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty}\int_a^b f_n(x)dx$$

In case of series, if $\sum g_k(x)$ converges uniformly and $g_k(x)$ is integrable, then

$$\int_a^b \sum_{k=1}^{\infty} g_k(x)\,dx = \sum_{k=1}^{\infty}\int_a^b g_k(x)dx$$

Example 5
Evaluate the integral

$$\int_1^2 \sum_{n=1}^{\infty} ne^{-nx}\,dx$$

(Step 0: Clearly $g_n(x) = ne^{-nx}$ is continuous and therefore integrable)
(Step 1: Check the uniform convergence)
Note that $|ne^{-nx}| = ne^{-nx} \le ne^{-(1)n} = ne^{-n}$ for $1 \le x \le 2$
By using root test, the series $\sum_{n=1}^{\infty} ne^{-nx}$ converges
(Since $\lim_{n\to\infty}\sqrt[n]{ne^{-n}} = \lim_{n\to\infty}\sqrt[n]{n}e^{-1} = e^{-1} < 1$)

So by M-test, the series $\sum_{n=1}^{\infty} ne^{-nx}$ converges uniformly on $[1,2]$

(Step 2: Do integration)

$$\int_1^2 \sum_{n=1}^{\infty} ne^{-nx}\,dx = \sum_{n=1}^{\infty}\int_1^2 ne^{-nx}dx = \sum_{n=1}^{\infty} -e^{-nx}|_1^2 = \sum_{n=1}^{\infty}(e^{-n} - e^{-2n})$$

$$=^{(*)} \sum_{n=1}^{\infty} e^{-n} - \sum_{n=1}^{\infty} e^{-2n} = \frac{e^{-1}}{1-e^{-1}} - \frac{e^{-2}}{1-e^{-2}} = \frac{1}{e-1} - \frac{1}{e^2-1} = \frac{e}{e^2-1}$$

(*Note: $\sum_{n=1}^{\infty}(e^{-n} - e^{-2n}) = \sum_{n=1}^{\infty} e^{-n}(1-e^{-n}) \le \sum_{n=1}^{\infty} e^{-n}$, since RHS converges and LHS converges by comparison test, so we can rearrange the terms in this way.)

Example 6 (MATH301 2000 Final)
Compute the integral

$$\int_1^e e^x \ln x dx$$

(Step 0: Clearly $e^x \ln x$ is continuous on $[1, e]$ and therefore integrable)
(Step 1: Note that

$$\int_1^e e^x \ln x dx = \int_1^e \sum_{n=0}^{\infty} \frac{x^n}{n!} \ln x \, dx$$

In order to interchange the operations, we need to check the series is uniformly convergence on $[1, e]$

Note $\left|\frac{x^n \ln x}{n!}\right| \leq \frac{e^n \ln e}{n!} = \frac{e^n}{n!}$,

Applying ratio test, we see the series $\sum_{n=0}^{\infty} \frac{e^n}{n!}$ converges

(Since $\lim_{n\to\infty} \frac{\frac{e^{n+1}}{(n+1)!}}{\frac{e^n}{n!}} = \lim_{n\to\infty} \frac{e}{n+1} = 0 < 1$)

So by M-test, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!} \ln x$ converges uniformly.

(Step 2: Do integration)

$$\int_1^e e^x \ln x dx = \int_1^e \sum_{n=0}^{\infty} \frac{x^n}{n!} \ln x \, dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_1^e x^n \ln x dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \int_1^e \ln x d(x^{n+1})$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(x^{n+1} \ln x |_1^e - \int_1^e x^{n+1} dx \right) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(e^{n+1} - \int_1^e x^n dx \right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{(n+1)!} \left(e^{n+1} - \frac{e^{n+1}}{n+1} + \frac{1}{n+1} \right) = \sum_{n=0}^{\infty} \frac{ne^{n+1} + 1}{((n+1)!)(n+1)}$$

The Consequence of uniform convergence on differentiation is quite different from Continuity and Integration. Here we state the theorem first

Differentiation Theorem
Let $\{f_n\}$ be a sequence of differentiable functions on $[a, b]$, suppose $f_n{}'$ converges uniformly to some $T(x)$ and $\lim_{n\to\infty} f(x_0)$ exists for some x_0.
Then f_n converges uniformly to $f(x)$ and

$$\lim_{n\to\infty} \frac{d}{dx} f_n(x) = \frac{d}{dx} \lim_{n\to\infty} f_n(x)$$

In case of series,

If $g_k(x)$ is differentiable and $\sum g_k{}'$ converges uniformly to some $T(x)$ and $\sum g_k(x_0)$ converges for some x_0, then $\sum g(x_k)$ converges uniformly and

$$\sum_{k=1}^{\infty}\frac{d}{dx}g_k(x)=\frac{d}{dx}\sum_{k=1}^{\infty}g_k(x)$$

One interesting application of this theorem is to find out the sum of special series. Here comes an example.

Example 7

Given $\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$ for $-1<x<1$, compute $\sum_{n=1}^{\infty}nx^n$

IDEA: Let us do the casual calculation first

Note that $\frac{d}{dx}x^n=nx^{n-1}\rightarrow x\left(\frac{d}{dx}x^n\right)=nx^n$

So we expect

$\sum_{n=1}^{\infty}nx^n=x\sum_{n=1}^{\infty}nx^{n-1}=x\sum_{n=0}^{\infty}nx^{n-1}=x\sum_{n=0}^{\infty}\frac{d}{dx}x^n\overset{?}{=}x\frac{d}{dx}\sum_{n=0}^{\infty}x^n=$

$x\left(\frac{d}{dx}\left(\frac{1}{1-x}\right)\right)=x\left(\frac{1}{(1-x)^2}\right)=\frac{x}{(1-x)^2}$

So we need to justify the step (?)
Let $g_n(x)=x^n$ and it is differentiable,
Next $g_n'(x)=nx^{n-1}$ and $\sum_{n=1}^{\infty}nx^{n-1}$ converges uniformly on $(-1,1)$ (The actual reason is out of scope of MATH202, you will know it in MATH301)

(Reason: $\lim_{n\to\infty}\sqrt[n]{|nx^{n-1}|}=\lim_{n\to\infty}\sqrt[n]{n}|x|^{\frac{n-1}{n}}=|x|<1$)

And $\sum_{n=0}^{\infty}g_n(x)=\sum_{n=0}^{\infty}x^n$ converges (by geometric series test, since $-1<x<1$)
Hence by differentiation theorem, the step (?) is justified.

Try to work on the following exercises. You are welcome to submit your solutions to me for comments. Anyway, good luck in your final.

☺Exercise 1

Show that the following sequences of functions converges uniformly on indicated intervals.

a) $f_n(x)=\frac{e^{-nx}\cos nx}{n^3}$ on $x\in[0,\infty)$

b) $f_n(x)=\frac{x}{1+(nx^3)^{\frac{1}{3}}+(nx^3)^{\frac{2}{3}}}$ on $x\in[1,\infty)$

(Hint: Use AM-GM somewhere)

c) $f_n(x) = \frac{n^2 x}{1+n^3x^2}$ on $x \in \mathbf{R}$

(Hint: Use Calculus to find the maximum!!)

☺Exercise 2

Show that the following series of functions converges uniformly on indicated intervals

a) $\sum_{n=1}^{\infty} n^2 x^n$ on $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

b) $\sum_{n=1}^{\infty} \frac{\cos nx}{5^n}$ on $x \in \mathbf{R}$

c) $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ on $x \in [0,1]$

(Use Calculus to help you)

d) $\sum_{k=1}^{\infty} \frac{kx}{e^{kx}}$ on $[r, \infty)$, where $r > 0$

(Use Calculus to help you)

e) $\sum_{k=1}^{\infty} k^2 x^k$ on $[-r, r]$ where $0 < r < 1$

☺Exercise 3

Calculate the integral

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{n\sin(nx)}{e^n} dx$$

(The answer is $\frac{2e}{e^2-1}$)

☺Exercise 4

If $\sum |a_n|$ converges, prove that

$$\int_0^1 \sum_{n=1}^{\infty} a_n x^n \, dx = \sum_{n=1}^{\infty} \frac{a_n}{n+1}$$

☺Exercise 5

Compute the integral

$$\int_a^b \sum_{n=1}^{\infty} \sin(nx)\, e^{-xn} \, dx$$

Where $0 < a < b$

☺Exercise 6 (Another application of uniform convergence)

Show that $f_n(x) = \sin^n x$ does not converge uniformly on $[0, \pi]$ by using Continuity theorem

(Hint: Clearly $f_n(x)$ is continuous, suppose $f_n(x)$ converges uniformly, then the limit $f(x) = \lim_{n\to\infty} f_n(x)$ is also continuous by continuity theorem.)

☺Exercise 7

Differentiate $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}$ for $x \in \mathbf{R}$