MATH202 Introduction to Analysis (2007 Fall and 2008 Spring)

Tutorial Note #14

Part I: Cauchy Sequence

Definition (Cauchy Sequence): A sequence of real number $\{x_n\}$ is Cauchy if and only if for any $\varepsilon > 0$, there is positive integer K such that $m, n > K \rightarrow |x_m - x_n| < \varepsilon$
Theorem: (Cauchy Theorem) The sequence is Cauchy if and only if $\{x_n\}$ converges to some real number L.

In last semester, we have seen some examples about it (See Tutorial Note #13). Try to have a look on that. Here we try to show more technique.

One useful technique is using **mean value theorem**, we state the theorem here (where the proof will be discussed in Chapter 8-Differentation)

Theorem (Mean Value Theorem)

Let $f: [a, b] \rightarrow \mathbf{R}$ be continuous on [a,b] and differentiable on (a,b). Then

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

For some $c \in (a, b)$

Example 1

If $a_n \geq 0$ for $n = 1,2,3, \ldots$ and $\{a_n\}$ is Cauchy Sequence. Show that the sequence $\{b_n\}$ defined by $b_n = \ln(1 + a_n)$ Cauchy by checking the definition.

IDEA: Let $f(x) = \ln|1 + x| \rightarrow f'(x) = 1/(1 + x)$

$$|b_m - b_n| = \left|\ln(1 + a_m) - \ln(1 + a_n)\right| = \frac{1}{1 + c}\left|a_m - a_n\right|$$

$$\leq \frac{1}{1}|a_m - a_n| < \varepsilon$$

So we require $|a_m - a_n| < \varepsilon$ (this can be done as $\{a_n\}$ is Cauchy Sequence)

<u>Solution:</u>

For any $\varepsilon > 0$, since a_n is Cauchy, then there exists K_1 such that for $m, n > K_1$, we have $|a_m - a_n| < \varepsilon$

Pick $K = K_1$, then for $m, n > K_1$, from the previous work, we get

$|b_m - b_n| < \varepsilon$

So $\{b_n\}$ is Cauchy

Example 2 (Practice Exercise #47)
Let $a_n \geq 0$ for $n = 1,2,3,\ldots$ and $\{a_n\}$ is Cauchy. Show that $\{\sqrt{a_n}\}$ is Cauchy by checking the definition.

IDEA: If $\{a_n\}$ is Cauchy, then a_n converges by Cauchy Theorem

Case i) If $\lim_{n\to\infty} a_n > 0$, then say $\lim_{n\to\infty} a_n = a$. Consider $f(x) = \sqrt{x}$,

$$\left|\sqrt{a_m} - \sqrt{a_n}\right| = \frac{1}{2\sqrt{c}}|a_m - a_n| < \frac{1}{2\sqrt{\frac{a}{2}}}|a_m - a_n| = \frac{1}{\sqrt{2a}}|a_m - a_n| < \varepsilon$$

We require $|a_n - a| < \frac{a}{2}$ and $|a_m - a_n| < \sqrt{2a}\varepsilon$

Case ii) If $\lim_{n\to\infty} a_n = 0$, then

$$\left|\sqrt{a_m} - \sqrt{a_n}\right| \leq \left|\sqrt{a_m}\right| + \left|\sqrt{a_n}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

We require $|a_n - 0| < \frac{\varepsilon^2}{4}$

<u>Solution</u>:

Case i) If $\lim_{n\to\infty} a_n > 0$, then say $\lim_{n\to\infty} a_n = a$,

There exists K_1 such that for $n > K_1$, $|a_n - a| < \frac{a}{2}$

There exists K_2 such that $m, n > K_2$, $|a_m - a_n| < \sqrt{2a}\varepsilon$
Pick $K = \max\{K_1, K_2\}$, then for $m, n > K$, from the arguments above, we get
$\left|\sqrt{a_m} - \sqrt{a_n}\right| < \varepsilon$

Case ii) If $\lim_{n\to\infty} a_n = 0$

There exists K such that for $n > K$, $|a_n - 0| < \frac{\varepsilon^2}{4}$

Then for any $m, n > K$, we get

$$\left|\sqrt{a_m} - \sqrt{a_n}\right| \leq \left|\sqrt{a_m}\right| + \left|\sqrt{a_n}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Combining two cases, we complete the proof.

Example 3 (Modified from Rudin P.82 #23)
Suppose $\{p_n\}, \{q_n\}$ are Cauchy Sequences in $\mathbf{R}$, show that the distance $d_n = |p_n - q_n|$ is a Cauchy Sequence.

IDEA:
Applying triangle inequality and assume $d_m \geq d_n$ we get
$|d_m - d_n| = |p_m - q_m| - |p_n - q_n|$

$= |(p_m - p_n) + (p_n - q_n) + (q_n - q_m)| - |p_n - q_n|$
$\leq |p_m - p_n| + |p_n - q_n| + |q_n - q_m| - |p_n - q_n|$
$= |p_m - p_n| + |q_m - q_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
We require $|p_m - p_n| < \varepsilon/2$ and $|q_m - q_n| < \varepsilon/2$

Solution:
For any $\varepsilon > 0$, since $\{p_n\}, \{q_n\}$ are both Cauchy Sequence,

There exist K_1, such that $m, n > K_1$, $|p_m - p_n| < \frac{\varepsilon}{2}$

There exist K_2, such that $m, n > K_2$, $|q_m - q_n| < \frac{\varepsilon}{2}$

Pick $K = \max\{K_1, K_2\}$, then for $m, n > K$, from the steps above, we get
$|d_m - d_n| < \varepsilon$
Hence $\{d_n\}$ is Cauchy.

Example 4
Let $\{x_n\}$ be the Cauchy such that $x_n \in \mathbf{N}$ for $n = 1,2,3,..$ (i.e. x_n is positive integers for every n). Show that there exists K such that for $n > K$, x_n is constant (i.e.x_n will become constant when n is large)

IDEA: Since x_n is positive integer and two different integers must have distance at least 1 (For example: 1,2). Now x_n is Cauchy, for large n, the distance between x_m, x_n will be very close (distance less than one). It will force all x_n need to be same.

Solution:
Since x_n is Cauchy, pick $\varepsilon = 0.5$ (it can be any number less than 1), there exists K such that for $m, n > K$, $|x_m - x_n| < 0.5$.....(*)

Next we claim x_n is constant for $n > K$, we prove by contradiction, suppose there are x_m, x_n $(m, n > K)$ such that $x_m \neq x_n$. From (*), we get $|x_m - x_n| < 0.5$, but since both x_m, x_n are both positive integers, then $|x_m - x_n| \geq 1$. Contradiction

Hence x_n must be constant for $n > K$.

Part 2: Limit of Function

Definition: (Limit of Function)
Given a function $f(x)$, we say $\lim_{x\to x_0} f(x) = L$ if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $|x - x_0| < \delta$, we have $|f(x) - L| < \varepsilon$

Roughly speaking, the definition means if x is sufficiently close to $x_0(|x - x_0| < \delta)$, Then $f(x)$ should be very close to L. Technically, when we apply the definition to show $\lim_{x\to x_0} f(x) = L$, similar as the one in sequence, we need to find the δ so that $|f(x) - L| < \varepsilon$

Example 5
Using the definition of limit, show that

$$\lim_{x\to 2}\frac{1}{x} = \frac{1}{2}$$

IDEA: From the definition, we need to find the δ such that $\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \left|\frac{2-x}{2x}\right| < \frac{|2-x|}{2(1.5)} \quad (\text{We hope } |2-x| < 0.5 \to 1.5 < x < 2.5)$$

$$= \frac{|2-x|}{3} < \varepsilon \quad (We\ hope\ |2-x| < 3\varepsilon)$$

Overall, we need $|2 - x| < min\{0.5, 3\varepsilon\}$

Solution:
For any $\varepsilon > 0$, pick $\delta = \min\{0.5, 3\varepsilon\}$, then for $|x - 2| < \delta$, we get

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$$

Example 6
Using the definition of limit, show that

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x} = \frac{1}{4}$$

IDEA: From the definition, we need to find the δ such that $\left|\frac{\sqrt{x+4}-2}{x} - \frac{1}{4}\right| < \varepsilon$

Note that

$$\left|\frac{\sqrt{x+4}-2}{x} - \frac{1}{4}\right| = \left|\frac{4\sqrt{x+4}-8-x}{4x}\right| = \left|\frac{4\sqrt{x+4}-(8+x)}{4x}\right|$$

$$= \left|\frac{\left(4\sqrt{x+4}-(8+x)\right)\left(4\sqrt{\mathbf{x+4}}+(\mathbf{8}+\mathbf{x})\right)}{4x\left(4\sqrt{\mathbf{x+4}}+(\mathbf{8}+\mathbf{x})\right)}\right| = \left|\frac{16(x+4)-(8+x)^2}{4x\left(4\sqrt{x+4}+(8+x)\right)}\right|$$

$$= \left|\frac{-x^2}{4x\left(4\sqrt{x+4}+(8+x)\right)}\right| = \left|\frac{x}{\left(16\sqrt{x+4}+4(8+x)\right)}\right|$$

$$< \frac{|x|}{16\sqrt{(4-3)}+4(8-3)} = \left|\frac{x}{36}\right| < \varepsilon$$

We require $|x-0| < 3$ (so that $-3 < x < 3$) and $|x-0\} < 36\varepsilon$

Solution:

For any $\varepsilon > 0$, pick our $\delta = \min\{3, 36\varepsilon\}$, then for $|x-0| < \delta$, from the above steps, we get $\left|\frac{\sqrt{x+4}-2}{x} - \frac{1}{4}\right| < \varepsilon$. We completes the proof.

Example 7 Show by definition that if $\lim_{x\to 0}\frac{f(x)}{x} = L \in \boldsymbol{R}$ and $a \neq 0$, then $\lim_{x\to 0}\frac{f(ax)}{x} = aL$ What is the case when $a = 0$?

IDEA: Note that when $x \to 0$, then $ax \to 0$

$$\left|\frac{f(ax)}{x} - aL\right| = \left|a\frac{f(ax)}{ax} - aL\right| = |a|\left|\frac{f(ax)}{ax} - L\right| < \varepsilon$$

So we need $\left|\frac{f(ax)}{ax} - L\right| < \frac{\varepsilon}{|a|}$

Solution:

For any $\varepsilon > 0$, since $\lim_{y\to 0}\frac{f(y)}{y} = L$, then there exists $\delta' > 0$ such that for

$|y-0| < \delta'$, we get $\left|\frac{f(y)}{y} - L\right| < \frac{\varepsilon}{|a|}$

Pick $\delta = \frac{\delta'}{|a|}$, then for $|x-0| < \delta = \frac{\delta'}{|a|} \to |ax-0| < a\delta = \delta'$

$$\left|\frac{f(ax)}{x} - aL\right| = \left|a\frac{f(ax)}{ax} - aL\right| = |a|\left|\frac{f(ax)}{ax} - L\right| = \frac{|a|\varepsilon}{|a|} < \varepsilon$$

We complete the proof.

Besides the definition, there is one useful theorem in limit.

Theorem: (Sequential Limit Theorem) $\lim_{x\to x_0} f(x) = L$ if and only if for every sequence $x_n \to x_0$ and $x_n \neq x_0$, we have $\lim_{n\to\infty} f(x_n) = L$

One application of this theorem is to show the limit of some functions DO NOT exist

Example 8

Show that $\lim_{x\to 1} x-[x]$ does not exist

(where $[x]$ denotes the greatest integer less than or equal to x)

Solution:

Consider two sequences which

$x_n = 1-\frac{1}{n+1}$ and $y_n = 1+\frac{1}{n+1}$

Then

$$\lim_{n\to\infty} x_n - [x_n] = \lim_{n\to\infty} 1-\frac{1}{n+1}-0=1$$

$$\lim_{n\to\infty} y_n - [y_n] = \lim_{n\to\infty} 1+\frac{1}{n+1}-1=0$$

Hence $\lim_{n\to\infty} x_n \neq \lim_{n\to\infty} y_n$, by sequential limit theorem, the limit does not exist

Example 9

Define $f(x) = \begin{cases} 1 \text{ if x is rational} \\ 0 \text{ if x is irrational} \end{cases}$, show that $\lim_{x\to x_0} f(x)$ does not exist.

Solution:

For each x_0, we pick

If x_0 is rational, pick $x_n = x_0\left(1-\frac{1}{n}\right)$ and $y_n = x_0\left(1-\frac{1}{\sqrt{2}n}\right)$

We get $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 0 = 0$

If x_0 is irrational, pick $x_n = \frac{[10^n x_0]}{10^n}$ and $y_n = x_0\left(1-\frac{1}{n}\right)$,

We get $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 0 = 0$

Hence the limit does not exists for every x_0

Try to do the following exercises, you may submit your work to me so that I can give some comments to your work.

☺Exercise 1 (Practice Exercise #156, #170)

Given $\{x_n\}$ is a Cauchy Sequence, show that both $\{e^{-x_n}\}$ and $\{\sin 5x_n\}$ are Cauchy Sequence by checking the definition.

☺Exercise 2

Suppose $\{y_n\}$ is Cauchy Sequence. Show $\{\sqrt[3]{y_n}\}$ is also Cauchy

(Hint: The method is similar to Example 2)

☺Exercise 3 (Practice Exercise #167)

Let $f: (0,\infty) \to \mathbf{R}$ satisfy $|f(x) - f(y)| \le |\sin(x^2) - \sin(y^2)|$ for all $x, y > 0$,

show that the sequence $x_1, x_2, x_3, \ldots.$ defined by $x_n = f\left(\frac{1}{n}\right)$ is a Cauchy Sequence.

(Hint: From the mean value theorem, we get $|\sin a - \sin b| \le |\cos c||a - b|$. Apply this result to R.H.S. of the inequality)

☺Exercise 4

Let $\{x_n\}$ and $\{y_n\}$ be two Cauchy Sequence in $\mathbf{R}$. Show that the product $\{x_n y_n\}$ is also Cauchy.

(Hint: Apply the similar trick from Example 3 on $|x_m y_m - x_n y_n|$) and use the fact that if $\{x_n\}$ is Cauchy→ $\{x_n\}$ converges→ $\{x_n\}$ is bounded (i.e. $|x_n| \le M$))

☺Exercise 5

Show by definition of limit that

a) $\lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}$ (Practice Exercise #57)

b) $\lim_{x\to 2} |x^2 - 9| = 5$ (Practice Exercise #109c)

c) $\lim_{x\to 6} \frac{\sqrt{x-2}-2}{x-6} = \frac{1}{4}$

d) $\lim_{x\to a} \tan^{-1}\frac{1}{x} = \tan^{-1}\frac{1}{a}$ for $a \ne 0$ ***(☹Difficult!!)***

☺Exercise 6

Consider $f(x) = \begin{cases} x - 1 \text{ if } x \le 1 \\ x^3 \quad \text{if } x > 1 \end{cases}$, show that $\lim_{x\to 1} f(x)$ does not exist.

(Hint: Try to plot the graph and get the idea, then prove it property)

☺Exercise 7

Consider $f(x) = \begin{cases} 2x & \text{if x is rational} \\ 1 - 2x & \text{if x is irrational} \end{cases}$. Determine with proof whether

$\lim_{x\to\frac{1}{2}} f(x)$ and $\lim_{x\to\frac{1}{4}} f(x)$ exist or not.

☺Exercise 8

Suppose $\lim_{x\to a} f(x) = A > 0$, show that there exists $\delta > 0$ such that for $0 < |x - a| < \delta$, we have $f(x) > 0$.

(Hint: If $f(x)$ has a positive limit at a, then it implies that if x is close enough to a, f(x) will be very close to a, then f(x) will be eventually positive.)