# Spring 2009 Final Exam — Math 202

① Find the domain of convergence of $f(x)=\sum_{k=1}^{\infty}\frac{(-1)^k}{k^4}(3x-2)^k$

**Solution** $\lim_{k\to\infty}\left|\frac{(-1)^{k+1}(3x-2)^{k+1}}{(k+1)^4}\frac{k^4}{(-1)^k(3x-2)^k}\right|=\lim_{k\to\infty}\left(\frac{k}{k+1}\right)^4|3x-2| = |3x-2|$

If $|3x-2|<1$, then $|x-\frac{2}{3}|<\frac{1}{3}\Leftrightarrow x\in(\frac{1}{3},1)$, series converges

If $|3x-2|=1$, then $x=\frac{1}{3}$ or $1$.

For $x=\frac{1}{3}$, $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^4}(3x-2)^k=\sum_{k=1}^{\infty}\frac{1}{k^4}$ converges by p-test

For $x=1$, $\sum\frac{(-1)^k}{k^4}(3x-2)^k=\sum_{k=1}^{\infty}\frac{(-1)^k}{k^4}$ converges by alternating series test.

$\therefore$ domain $=[\frac{1}{3},1]$.

② Determine if $\int_{-1}^{1}\frac{\sin x}{x^2\cos^2 x}dx$ converges.

Determine if $PV\int_{-1}^{1}\frac{\sin x}{x^2\cos^2 x}dx$ converges.

**Solution** $\int_{-1}^{1}\frac{\sin x}{x^2\cos^2x}dx=\int_{-1}^{0}\frac{\sin x}{x^2\cos^2x}dx+\int_{0}^{1}\frac{\sin x}{x^2\cos^2x}dx$.

(Note as $x\to0^+$, $\frac{\sin x}{x^2\cos^2x}\sim\frac{x}{x^2\cdot1}=\frac{1}{x}$). $\lim_{x\to0^+}\frac{\sin x}{x^2\cos^2x}\Big/\frac{1}{x}=\lim_{x\to0^+}\frac{\sin x}{x}\frac{1}{\cos^2x}=1$

Since $\int_0^1\frac{1}{x}dx=\ln|x|\Big|_0^1=+\infty$, so $\int_0^1\frac{\sin x}{x^2\cos^2x}dx$ diverges by limit comparison test. $\therefore\int_{-1}^{1}\frac{\sin x}{x^2\cos^2x}dx$ diverges.

$PV\int_{-1}^{1}\frac{\sin x}{x^2\cos^2x}dx=\lim_{\varepsilon\to0^+}\underbrace{\left(\int_{-1}^{-\varepsilon}\frac{\sin x}{x^2\cos^2x}dx+\int_{\varepsilon}^{1}\frac{\sin x}{x^2\sin x}dx\right)}_{=0\text{ since }\frac{\sin x}{x^2\cos^2x}\text{ is odd}}=0$

③ Prove that $\sum_{k=1}^{\infty}k\left(\frac{x^2}{1+x^3}\right)^k$ converges uniformly on $[0,+\infty)$.

**Solution** $\frac{d}{dx}\left(\frac{x^2}{1+x^3}\right)=\frac{2x(1+x^3)-x^2(3x^2)}{(1+x^3)^2}=\frac{2x-x^4}{(1+x^3)^2}=0\Leftrightarrow x=0,\sqrt[3]{2}$.

At $x=0$, $\frac{x^2}{1+x^3}=0$, At $x=\sqrt[3]{2}$, $\frac{x^2}{1+x^3}=\frac{2^{2/3}}{3}$. As $x\to\infty$, $\frac{x^2}{1+x^3}\to0$.

So on $[0,+\infty)$, $\left|k\left(\frac{x^2}{1+x^3}\right)^k\right|\le k\left(\frac{2^{2/3}}{3}\right)^k=M_k$

$\lim_{k\to\infty}\sqrt[k]{M_k}=\lim\sqrt[k]{k\left(\frac{2^{2/3}}{3}\right)^k}=\frac{2^{2/3}}{3}\lim_{k\to\infty}\sqrt[k]{k}=\frac{2^{2/3}}{3}<\frac{2}{3}<1$

By root test, $\sum_{k=1}^{\infty}M_k<\infty$. By Weierstrass M-test, series $\sum_{k=1}^{\infty}k\left(\frac{x^2}{1+x^3}\right)^k$ converges uniformly on $[0,+\infty)$.

④ Find $\lim_{n\to\infty}\frac{1^{1/n}+2^{1/n}+\cdots+n^{1/n}}{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}$ and $\lim_{x\to0^+}\frac{\sin(x^2)-x^2\cos(\sqrt{x})}{e^{x^3}-1}$.

**Solution** $0\le\frac{1^{1/n}+2^{1/n}+\cdots+n^{1/n}}{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}\le\frac{1^{1/1}+2^{1/2}+\cdots+n^{1/n}}{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}$

$\lim_{n\to\infty}\frac{1^{1/1}+2^{1/2}+\cdots+n^{1/n}}{\underbrace{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}_{\nearrow+\infty}}\underset{\text{Stolz' theorem}}{=}\lim_{n\to\infty}\frac{(n+1)^{1/(n+1)}}{\sqrt{n+1}}=\frac{1}{\infty}=0$

By sandwich theorem, $\lim_{n\to\infty}\frac{1^{1/n}+2^{1/n}+\cdots+n^{1/n}}{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}=0$.

$\cos w=1-\frac{w^2}{2}+o(w^2)$, $\sin w=w-\frac{w^3}{6}+o(w^3)$, $e^w=1+w+o(w)$ as $w\to0$.

$\lim_{x\to0^+}\frac{\sin(x^2)-x^2\cos\sqrt{x}}{e^{x^3}-1}=\lim_{x\to0^+}\frac{(x^2-\frac{x^6}{6}+o(x^6))-x^2(1-\frac{x}{2}+o(x))}{x^3+o(x)}$

As $x\to0^+$, $\frac{o(x^6)}{x^3}=\frac{o(x^6)}{x^6}x^3\to0\cdot0=0\Rightarrow o(x^6)=o(x^3)$, $x^2o(x)=o(x^3)$

$=\lim_{x\to0^+}\frac{x^3/2+o(x^3)}{x^3+o(x^3)}=\lim_{x\to0^+}\frac{\frac{1}{2}+\frac{o(x^3)}{x^3}}{1+\frac{o(x^3)}{x^3}}=\frac{\frac{1}{2}+0}{1+0}=\frac{1}{2}$

(5) Let $\{x_n\}$ be Cauchy in $\mathbb{R}$ and $y_n = x_{n+1} + x_n^2 + \cos(x_n)$ for $n=1,2,3,\dots$. Prove $\{y_n\}$ is Cauchy by checking the definition of Cauchy sequence.

Solution Since $\{x_n\}$ is Cauchy, it is bounded. So $\exists M$ such that $\forall n \in \mathbb{N}$, $|x_n| < M$.

Since $\{x_n\}$ is Cauchy, $\forall \varepsilon > 0$, $\exists K_1 \in \mathbb{N}$ such that $m,n \geq K_1 \Rightarrow |x_m - x_n| < \frac{\varepsilon}{3}$. Also $\exists K_2 \in \mathbb{N}$ such that $m,n \geq K_2 \Rightarrow |x_m - x_n| < \frac{\varepsilon}{6M}$. Let $K = \max\{K_1, K_2\}$.

Then $m,n \geq K \Rightarrow m,n \geq K_1$ and $m,n \geq K_2$, $m+1, n+1 \geq K_1$

$$\begin{aligned}\Rightarrow |y_m - y_n| &\leq |x_{n+1} - x_{m+1}| + |x_n^2 - x_m^2| + |\cos x_n - \cos x_m| \\ &\leq \varepsilon/3 + |x_n + x_m||x_n - x_m| + |x_n - x_m| \\ &< \varepsilon/3 + 2M \cdot \varepsilon/6M + \varepsilon/3 = \varepsilon.\end{aligned}$$

(6) (a) State Lebesgue's Theorem

Solution A bounded function $f:[a,b] \to \mathbb{R}$ is Riemann integrable iff $f$ is continuous almost everywhere on $[a,b]$, i.e. $S_f = \{x \in [a,b] : f \text{ is discontinuous at } x\}$ is a set of measure 0.

(b) Let $S$ be a set of measure 0. Prove that $T = \{2x : x \in S\}$ is also a set of measure 0. Let $f:[0,1] \to [0,1]$ be Riemann integrable. Prove that $g:[0,2] \to [0,1]$ defined by $g(x) = f(\frac{x}{2})$ is Riemann integrable on $[0,1]$.

Solution $\forall \varepsilon > 0$, since $S$ is of measure 0, $\exists (a_1,b_1), (a_2,b_2), \dots$ such that $S \subseteq \bigcup_{i=1}^{\infty} (a_i, b_i)$ and $\sum_{i=1}^{\infty} |a_i - b_i| < \varepsilon/2$, then $T \subseteq \bigcup_{i=1}^{\infty} (2a_i, 2b_i)$ and $\sum_{i=1}^{\infty} |2a_i - 2b_i| < \varepsilon$. $\therefore T$ is of measure 0.

Since $f$ is Riemann integrable, by Lebesgue's Theorem, $S_f$ is of measure 0. Then $S_g = \{2x : x \in S_f\}$ is also of measure 0. $\therefore g$ is Riemann integrable on $[0,2]$ and $[0,1]$.

(7) Let $f:\mathbb{R} \to \mathbb{R}$ be 7-times differentiable such that $\forall x \in \mathbb{R}$, $f^{(7)}(x) + f(x) = 0$ and $f(0) = f'(0) = \dots = f^{(7)}(0) = 0$. Prove that $f$ is $n$-times differentiable for every integer $n > 7$. Prove that $f(t) = 0$ for all $t \in \mathbb{R}$.

Solution $\forall x \in \mathbb{R}$, $f^{(7)}(x) = -f(x)$ implies for $k = 1,2,\dots,7$, $f^{(7+k)}(x) = \frac{d^k}{dx^k} f^{(7)}(x) = \frac{d^k}{dx^k}(-f(x)) = -f^{(k)}(x)$, i.e. $f$ is $(7+k)$-times differentiable. $\therefore f$ is $n$-times differentiable for all integer $n > 7$.

Since $f^{(7+k)}(0) = -f^{(k)}(0)$ and $f(0) = f'(0) = \dots = f^{(7)}(0) = 0$, it follows $f^{(n)}(0) = 0$ for every $n$. In particular, $f(0) = 0$.

For every $t \in \mathbb{R} - \{0\}$, let $I$ be the closed interval with $0$ and $t$ as endpoints. By Taylor's theorem, $\forall n = 1,2,3,\dots$, $\exists \theta_n$ between $0$ and $t$ such that

$$f(t) = f(0) + \frac{f'(0)}{1!}(t-0) + \dots + \frac{f^{(n)}(\theta_n)}{n!}(t-0)^n = \frac{f^{(n)}(\theta_n)}{n!} t^n.$$

Since $f^{(7+k)}(x) = -f^{(k)}(x)$, so $f^{(n)}(x) = \pm f(x), \pm f'(x), \cdots, \pm f^{(6)}(x)$.
Let $M$ be an upper bound of $|f(x)|, |f'(x)|, \cdots, |f^{(6)}(x)|$ for all $x \in I$ ($M$ exists because $|f(x)|, |f'(x)|, \cdots, |f^{(6)}(x)|$ are continuous on closed and bounded interval $I$).

Then $|f(t)| = \left| \dfrac{f^{(n)}(\theta_n)}{n!} t^n \right| \le \dfrac{M|t|^n}{n!}$.

Since $\lim\limits_{n\to\infty} \dfrac{M|t|^{n+1}}{(n+1)!} \Big/ \dfrac{M|t|^n}{n!} = \lim\limits_{n\to\infty} \dfrac{|t|}{n+1} = 0 < 1$,
by ratio test, $\sum\limits_{n=0}^{\infty} \dfrac{M|t|^n}{n!}$ converges. By term test,
$\lim\limits_{n\to\infty} \dfrac{M|t|^n}{n!} = 0$. $\therefore$ $f(t) = 0$ by Sandwich theorem.

# Alternative Solutions.

(4) The following are 3 more solutions to $\lim\limits_{n\to\infty} \dfrac{1^{1/n}+2^{1/n}+\cdots+n^{1/n}}{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}$.

$$0 \le \frac{1^{1/n}+2^{1/n}+\cdots+n^{1/n}}{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}} = \frac{n}{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}} \cdot \frac{1^{1/n}+2^{1/n}+\cdots+n^{1/n}}{n}$$

$$\lim_{n\to\infty}\frac{n}{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}} \underset{\text{Stolz' theorem}}{=} \lim_{n\to\infty}\frac{1}{\sqrt{n+1}} = 0,\quad 1=\frac{n}{n} \le \frac{1^{1/n}+2^{1/n}+\cdots+n^{1/n}}{n} \le \frac{n\cdot n^{1/n}}{n} = n^{1/n} \to 1.$$

$$\therefore \lim_{n\to\infty}\frac{1^{1/n}+2^{1/n}+\cdots+n^{1/n}}{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}} = 0.$$

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Let $x_n = 1^{1/n}+2^{1/n}+\cdots+n^{1/n}$. Then $x_{n+1}-x_n = (1^{\frac{1}{n+1}}-1^{\frac{1}{n}})+(2^{\frac{1}{n+1}}-2^{\frac{1}{n}})+\cdots+(n^{\frac{1}{n+1}}-n^{\frac{1}{n}})+(n+1)^{\frac{1}{n+1}}$

For $k=1,2,\cdots,n$, $\left|k^{\frac{1}{n+1}}-k^{\frac{1}{n}}\right| \overset{\text{mean value theorem}}{=} \left|(k^{\theta}\ln k)\left(\frac{1}{n+1}-\frac{1}{n}\right)\right| \le (n^{\frac{1}{n}}\ln n)\frac{1}{n(n+1)}$.

Then $\left|(1^{\frac{1}{n+1}}-1^{\frac{1}{n}})+(2^{\frac{1}{n+1}}-2^{\frac{1}{n}})+\cdots+(n^{\frac{1}{n+1}}-n^{\frac{1}{n}})\right| \le n^{\frac{1}{n}}\left(\frac{\ln n}{n+1}\right) \to 0$ as $n\to+\infty$.

$$\therefore \text{Then } \lim_{n\to\infty}\frac{1^{\frac1n}+2^{\frac1n}+\cdots+n^{1/n}}{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}} \underset{\text{Stolz' theorem}}{=} \lim_{n\to\infty}\frac{x_{n+1}-x_n}{\sqrt{n+1}} = \frac{0+1}{\infty} = 0.$$

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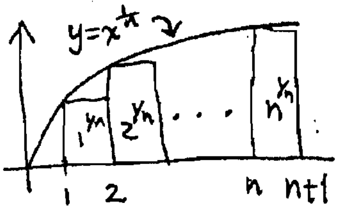
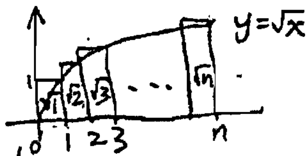



$$1^{1/n}+2^{1/n}+\cdots+n^{1/n} < \int_1^{n+1} x^{\frac1n}\,dx = \frac{n}{n+1}x^{\frac{n+1}{n}}\Big|_1^{n+1} = n\left((n+1)^{\frac1n}-\frac{1}{n+1}\right)$$




$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} > \int_0^n \sqrt{x}\,dx = \frac23 x^{3/2}\Big|_0^n = \frac23 n^{3/2}$$

$$0 \le \frac{1^{1/n}+2^{1/n}+\cdots+n^{1/n}}{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}} < \frac{n\left((n+1)^{\frac1n}-\frac{1}{n+1}\right)}{\frac23 n^{3/2}} = \frac32\,\frac{(n+1)^{\frac1n}-\frac{1}{n+1}}{n^{1/2}} \underset{\text{as } n\to\infty}{\to} \frac32\,\frac{(1-0)}{\infty} = 0$$

$$\therefore \lim_{n\to\infty}\frac{1^{1/n}+2^{1/n}+\cdots+n^{1/n}}{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}} = 0. \qquad (n+1)^{\frac1n} = \left((n+1)^{\frac{1}{n+1}}\right)^{\frac{n+1}{n}} \to 1^1 = 1$$

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(6) To show $g(x)=f(x/2)$ is Riemann integrable on $[0,2]$ (hence also on $[0,1]$), we can check by integral criterion:

$\forall \varepsilon>0$, since $f$ is Riemann integrable on $[0,1]$, $\exists$ partition $P=\{0=x_0<x_1<\cdots<x_n=1\}$ such that $U(f,P)-L(f,P)<\frac{\varepsilon}{2}$. Let $P'=\{0=x_0'<x_1'=2x_1<x_2'=2x_2<\cdots<x_n'=2\}$.

Since $x_{i+1}'-x_i'=2(x_{i+1}-x_i)$, $\sup\{g(t): t\in[x_{i+1}',x_i']\}=\sup\{f(\tfrac{t}{2}): \tfrac{t}{2}\in[x_{i+1},x_i]\}=M_i$ and $\inf\{g(t): t\in[x_{i+1}',x_i']\}=\inf\{f(\tfrac{t}{2}): \tfrac{t}{2}\in[x_{i+1},x_i]\}=m_i$, so

$$U(g,P')-L(g,P') = \sum_{i=0}^{n-1}(M_i-m_i)\underbrace{\Delta x_i'}_{=2\Delta x_i} = 2\sum_{i=1}^{n-1}(M_i-m_i)\Delta x_i = U(f,P)-L(f,P)<\varepsilon$$

$\therefore$ $g$ is Riemann integrable on $[0,2]$ (as well as on $[0,1]$).