MATH2033 Mathematical Analysis (2021 Spring)
Final Examination

Time allowed: 120 minutes (12:30p.m.- 2:30p.m.)

Instructions: Answer ALL problems. Full details must be clearly shown to receive full credits. Please submit your work via the submission system in canvas before 2:45p.m.. Late submission will not be accepted.

Your submission must be

- **100% handwritten (typed solution will not be accepted)**
- **In a single pdf. files (other file format will not be accepted)**
- **With your full name (as shown in your student ID card), student ID number and your signature on the cover page of your submission.**

Problem 1 (18 marks)

We consider a function $f: [-1,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^m & if\ \ x = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \dots\dots \\ 0 & if\ otherwise \end{cases},$$

where m is a positive integer.

(a) **(9 marks)** Find the value(s) of m such that $f(x)$ is continuous at $x = 0$.

(b) **(9 marks)** Find the value(s) of m such that $f(x)$ is differentiable at $x = 0$.

Problem 2 (10 marks)

We let $f: [0,2] \to \mathbb{R}$ be a continuous function. Show that there exists $c \in [0,1]$ such that

$$f(c+1) - f(c) = \frac{1}{2}\big(f(2) - f(0)\big).$$

Problem 3 (18 marks)

(a) **(8 marks)** We let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathbb{R}$ such that $|f'(x)| \le C$ for all $x \in \mathbb{R}$, where C is a positive constant. We let $\{x_n\}$ be a Cauchy sequence. Show that the sequence $\{y_n\}$ defined by $y_n = f(x_n)$ is also a Cauchy sequence.

(b) **(10 marks)** We let $f: (a,b) \to \mathbb{R}$ be 4-times differentiable function on (a,b) such that $\left|f^{(4)}(x)\right| \le M$ for all $x \in (a,b)$. Show that

$$\left|\frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} - f''(x_0)\right| \le \frac{M}{12}h^2$$

for any x_0 and h satisfying $a < x_0 - h < x_0 < x_0 + h < b$.

Problem 4 (18 marks)

(a) **(8 marks)** We let $f: (a,b) \to \mathbb{R}$ be n-times differentiable function and suppose that $f^{(n)}(x) > 0$ for all $x \in (a,b)$. Show that $f(x) = 0$ has at most n solutions in the interval (a,b).

(b) **(10 marks)** We consider the equation $4x^2 - 8x + 5 = 2^x$.

(i) Show that the equation has at least one solution over $(0,1)$.

(ii) Show that the equation has exactly two solutions over $(0,2)$.

Problem 5 (20 marks)

(a) **(10 marks)** We let $[a, b]$ (where $a < b$) be an closed interval. For any closed interval $[c, d] \subseteq [a, b]$ (where $a < c < d < b$), we define a function $g: [a, b] \to \mathbb{R}$ as

$$g(x) = \begin{cases} 1 & if \ \ x \in [c, d] \\ 0 & if \ \ otherwise \end{cases},$$

Using integral criterion or the definition of integrability, determine if $g(x)$ is integrable.

(b) **(10 marks)** We let $f: [a, b] \to \mathbb{R}$ be a bounded Riemann integrable function and let $g: [a, b] \to \mathbb{R}$ be another bounded function such that the set $\{x \in [a, b] | f(x) \neq g(x)\} = \{x_1, x_2, \dots, x_n\}$ where $a < x_1 < x_2 < \cdots < x_n < b$.

(i) Show that $g(x)$ is integrable. (☺Hint: Consider the function $h(x) = g(x) - f(x)$)

(ii) Show that

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

Problem 6 (16 marks)

We let $f: \mathbb{R} \to \mathbb{R}$ be a function.

(a) **(12 marks)** We let L be a real number. Show that $\lim_{x \to +\infty} f(x) = L$ if and only if $\lim_{n \to \infty} f(x_n) = L$ for any sequence $\{x_n\}$ with $\lim_{n \to \infty} x_n = +\infty$.

(b) **(4 marks)** Does the limits $\lim_{x \to \infty} \frac{\sin x}{2 + \cos x}$ converge to a real number? Explain your answer.

End of Paper