Solution to Presentation Exercises

501 Since $\{a_n\}$ is Cauchy, $\exists M>0$ such that $\forall n\in\mathbb{N}$, $|a_n|\le M$.
For every $\varepsilon>0$, $\exists K_1\in\mathbb{N}$ such that $n,m\ge K_1 \Rightarrow |a_n-a_m|<\frac{\varepsilon}{4M}$ and
$\exists K_2\in\mathbb{N}$ such that $n,m\ge K_2 \Rightarrow |a_n-a_m|<\frac{\varepsilon^3}{56}$.
Let $K=\max(K_1,K_2)$. Then

$$\begin{aligned}
n,m\ge K \Rightarrow |b_n-b_m| &= |\sin(a_n^2)-\sin(a_m^2)+\sqrt[3]{7a_n}-\sqrt[3]{7a_m}|\\
&\le |\sin(a_n^2)-\sin(a_m^2)|+|\sqrt[3]{7a_n}-\sqrt[3]{7a_m}|\\
&\le |a_n^2-a_m^2|+\sqrt[3]{|7a_n-7a_m|}\\
&\le |a_n+a_m||a_n-a_m|+\sqrt[3]{7|a_n-a_m|}\\
&< 2M\frac{\varepsilon}{4M}+\sqrt[3]{7\frac{\varepsilon^3}{56}}=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

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503 <u>Solution.</u> Note $\{a_n\}$ is Cauchy, so $\exists M>0$ such that $|a_n|\le M$.

$$\begin{aligned}
|c_n-c_m| &= |a_n^2-a_m^2+\sqrt{b_n}-\sqrt{b_m}+\sin(a_n+b_n)-\sin(a_m+b_m)|\\
&\le |a_n+a_m||a_n-a_m|+\sqrt{|b_n-b_m|}+|(a_n+b_n)-(a_m+b_m)|\\
&\le (|a_n|+|a_m|)|a_n-a_m|+\sqrt{|b_n-b_m|}+|a_n-a_m|+|b_n-b_m|.\\
&\le (2M+1)|a_n-a_m|+\sqrt{|b_n-b_m|}+|b_n-b_m|
\end{aligned}$$

$\forall\varepsilon>0$, since $\{a_n\}$ is Cauchy, $\exists K_1\in\mathbb{N}$, $m,n\ge K$
$\Rightarrow |a_n-a_m|<\frac{\varepsilon}{2(2M+1)}$. Since $\{b_n\}$ is Cauchy,
$\exists K_2\in\mathbb{N}$, $m,n\ge K_2 \Rightarrow |b_n-b_m|<\varepsilon/4$
$\exists K_3\in\mathbb{N}$, $m,n\ge K_3 \Rightarrow |b_n-b_m|<\varepsilon^2/16$.
Let $K=\max\{K_1,K_2,K_3\}$. Then
$m,n\ge K \Rightarrow |c_n-c_m|<\frac{\varepsilon}{2}+\sqrt{\frac{\varepsilon^2}{16}}+\frac{\varepsilon}{4}=\varepsilon.$

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602 <u>Solution 1</u>

Scratch: $\left|\frac{x+8}{x^2+3}-\frac{9}{4}\right|=\frac{|-9x^2+4x+5|}{4(x^2+3)}\le\frac{|9x+5||x-1|}{12}<\frac{23|x-1|}{12}<\varepsilon$

$|x-1|<1 \Leftrightarrow x\in(0,2)$

$9x+5\in(5,23)$

$|x-1|<\frac{12}{23}\varepsilon$

$\forall\varepsilon>0$, let $\delta=\min\{1,\frac{12}{23}\varepsilon\}$, then

$$\begin{aligned}
0<|x-1|<\delta &\Rightarrow |x-1|<1 \text{ and } |x-1|<\tfrac{12}{23}\varepsilon\\
&\Rightarrow x\in(0,2) \text{ and } |x-1|<\tfrac{12}{23}\varepsilon \qquad 9x+5\in(5,23)\\
&\Rightarrow \left|\frac{x+8}{x^2+3}-\frac{9}{4}\right|=\frac{|-9x^2+4x+5|}{4(x^2+3)}\le\frac{|9x+5||x-1|}{12}<\frac{23}{12}|x-1|<\varepsilon
\end{aligned}$$

602 Solution 2 By SLT, for every $x_n \to 1$, we need to show $\lim_{n\to\infty} \frac{x_n+8}{x_n^2+3} = \frac{9}{4}$.

Since $x_n \to 1$, $\exists K \in \mathbb{N}$ such that $n \ge K_1 \Rightarrow |x_n - 1| < 1 \Rightarrow x_n \in (0,2)$

$\Rightarrow 9x_n + 5 \in (5, 23)$

$\forall \varepsilon > 0, \exists K_2 \in \mathbb{N}$ such that $n \ge K_2 \Rightarrow |x_n - 1| < \frac{12}{23}\varepsilon$.

Let $K = \max\{K_1, K_2\}$.

Then $n \ge K \Rightarrow n \ge K_1$ and $n \ge K_2$

$$\Rightarrow \left|\frac{x_n+8}{x_n^2+3} - \frac{9}{4}\right| = \frac{|-9x_n^2+4x_n+5|}{4(x_n^2+3)} \le \frac{|9x_n+5||x_n-1|}{12} < \frac{23}{12}|x_n-1| < \varepsilon.$$

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603 Scratch Work

$$\left|f(x) - \tfrac{1}{2}\right| = \left|\sqrt[4]{\frac{1}{x^2+6x}} - \sqrt[4]{\frac{1}{16}}\right| \le \sqrt[4]{\left|\frac{1}{x^2+6x} - \frac{1}{16}\right|}$$

$$= \sqrt[4]{\frac{|x^2+6x-16|}{16(x^2+6x)}} = \sqrt[4]{\frac{|x+8||x-2|}{16(x^2+6x)}} \underset{1 \le x \le 3}{\le} \sqrt[4]{\frac{11|x-2|}{112}}$$

Solution $\forall \varepsilon > 0$, let $\delta = \frac{112}{11}\varepsilon^4$, then

$\forall x \in [1,3]$, $0 < |x-2| < \delta \Rightarrow |f(x) - \frac{1}{2}| \le \sqrt[4]{\frac{11|x-2|}{112}}$

$< \sqrt[4]{\frac{11\delta}{112}} = \varepsilon.$

Variation

$$\left|f(x) - \tfrac{1}{2}\right| = \left|\frac{1}{\sqrt[4]{x^2+6x}} - \frac{1}{2}\right| = \left|\frac{2 - \sqrt[4]{x^2+6x}}{2\sqrt[4]{x^2+6x}}\right| \times \left|\frac{2+\sqrt[4]{x^2+6x}}{2+\sqrt[4]{x^2+6x}}\right|$$

$$= \frac{|4 - \sqrt{x^2+6x}|}{2\sqrt[4]{x^2+6x}\,(2+\sqrt[4]{x^2+6x})} \times \left|\frac{4+\sqrt{x^2+6x}}{4+\sqrt{x^2+6x}}\right|$$

$$= \frac{|16-(x^2+6x)|}{2\sqrt[4]{x^2+6x}\,(2+\sqrt[4]{x^2+6x})(4+\sqrt{x^2+6x})} \underset{x\in[1,3]}{\le} \frac{|x+8||x-2|}{2\cdot 1(2)(4)}$$

$\le \frac{11}{16}|x-2| < \varepsilon$ if $|x-2| < \frac{16}{11}\varepsilon$.

(6.04) **Solution** $\forall \varepsilon > 0$, let $\delta = \varepsilon^4 > 0$. Then $\forall x \in [0, +\infty)$, $0 < |x-1| < \delta$ implies

$$\left|\sin^2\left(\frac{1}{1+\sqrt[4]{x}}\right) - \sin^2\frac{1}{2}\right| = \left|\sin\left(\frac{1}{1+\sqrt[4]{x}}\right) + \sin\frac{1}{2}\right|\left|\sin\left(\frac{1}{1+\sqrt[4]{x}}\right) - \sin\frac{1}{2}\right|$$

$$\leq 2\left|\frac{1}{1+\sqrt[4]{x}} - \frac{1}{2}\right| = 2\left|\frac{1-\sqrt[4]{x}}{2(1+\sqrt[4]{x})}\right|$$

$$= 2\,\frac{|1-\sqrt[4]{x}|}{2(1+\sqrt[4]{x})} \leq \frac{|1-\sqrt[4]{x}|}{1+0} \leq \sqrt[4]{|1-x|} < \sqrt[4]{\delta} = \varepsilon.$$

switch

Variation

$$\frac{|1-\sqrt[4]{x}|}{1+\sqrt[4]{x}} = \frac{|1-\sqrt[4]{x}|(1+\sqrt[4]{x})}{(1+\sqrt[4]{x})(1+\sqrt[4]{x})} = \frac{|1-\sqrt{x}|}{(1+\sqrt[4]{x})^2}\,\frac{(1+\sqrt{x})}{(1+\sqrt{x})}$$

$$= \frac{|1-x|}{(1+\sqrt[4]{x})^2(1+\sqrt{x})} \leq \frac{|1-x|}{(1+0)^2(1+0)} = |1-x| < \delta$$

$\forall \varepsilon > 0$, let $\delta = \varepsilon$ in this case.

(7.01) (This is similar to $g(g(x)) = -x^9$ problem #62)

$f : [0,1] \to [0,1]$ continuous

$f(0) = 0$, $f(1) = 1$

$f(f(x)) = x \quad \forall x \in [0,1]$

Prove $f(x) = x \quad \forall x \in [0,1]$

f is injective since $f(a) = f(b) \Rightarrow \underset{a}{\underset{\shortparallel}{f(f(a))}} = \underset{b}{\underset{\shortparallel}{f(f(b))}}$

By continuous injection theorem, f is strictly monotone.

Since $f(0) = 0$, $f(1) = 1$, f is strictly increasing.

$x \leq f(x) \Rightarrow f(x) \leq f(f(x)) = x \Rightarrow x = f(x)$

$f(x) \leq x \Rightarrow x = f(f(x)) \leq f(x) \Rightarrow f(x) = x.$

$\therefore \forall x \in [0,1]$, $f(x) = x$.

(702) By the extreme value theorem, $\sup\{f(x): x\in[0,2\pi]\}=M$, $\inf\{f(x):x\in[0,2\pi]\}=m$ exist in $\mathbb{R}$. Since $f(x+2\pi)=f(x)$, so $\forall x\in\mathbb{R}$, $m\le f(x)\le M$.
Now $g(t)=f(t)-t$ is continuous on $\mathbb{R}$. We have
$g(M)=f(M)-M\le 0$ and $g(m)=f(m)-m\ge 0$.
By the intermediate value theorem, $\exists\, x_0\in[m,M]$ such that $g(x_0)=0$.
Then $f(x_0)=x_0$.

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(703) We claim $f(x)=x$ for all $x\in[0,1]$ (Then $f(f(f(x)))+f(x)=x+x=2x$.)
Suppose $f(x)$ is a continuous function $f:[0,1]\to[0,1]$ such that
$f(f(f(x)))+f(x)=2x$ for all $x\in[0,1]$.
We check f is injective:
$f(a)=f(b)\Rightarrow f(f(f(a)))+f(a)=f(f(f(b)))+f(b)\Rightarrow 2a=2b\Rightarrow a=b$.
Since f is continuous and injective, by the continuous injection theorem, either ① f is strictly decreasing or ② f is strictly increasing.
In case ①, $a<b\Rightarrow f(a)>f(b)\Rightarrow f(f(a))<f(f(b))\Rightarrow f(f(f(a)))>f(f(f(b)))$
$\Rightarrow 2a=f(f(f(a)))+f(a)>f(f(f(b)))+f(b)=2b\Rightarrow a>b$ contradiction.
In case ②, we show $f(x)>x$ and $f(x)<x$ lead to contradictions
If $\underbrace{f(x)>x}_{(*)}$, then $f(f(x))\underset{\text{by ② and }(*)}{>}f(x)\underset{\text{by }(*)}{\Rightarrow}f(f(x))>x\underset{\text{by ②}}{\Rightarrow}f(f(f(x)))>f(x)\underset{\text{by }(*)}{\Rightarrow}f(f(f(x)))>x$
$\Rightarrow 2x=f(f(f(x)))+f(x)>x+x=2x$, contradiction. Similarly, $f(x)<x$ is false.

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(706) Let $g(x)=f(x)-x^2$. Then g is continuous on $[0,2]$
Since f is continuous on $[0,2]$.
$g(2)=f(2)-2^2=0-4<0$.
Next, $f(x)-2=\dfrac{f(x)-2}{\sqrt{x}-1}(\sqrt{x}-1)$ for $x\ne 1$.
Then
$$f(1)-2=\lim_{x\to1}(f(x)-2)=\lim_{x\to1}\frac{f(x)-2}{\sqrt{x}-1}\cdot\lim_{x\to1}(\sqrt{x}-1)$$
$$=1\cdot 0=0$$
$\therefore f(1)=2$
Then $g(1)=f(1)-1^2=2-1>0$.
By intermediate value theorem, $\exists\, x\in[1,2]$ such that $g(x)=0$, so $f(x)=x^2$.

(801) By Taylor's Theorem, $f(x) = \underbrace{f(0)}_{=2} + \underbrace{f'(0)}_{0} x + \frac{f''(\theta_x)}{2} x^2$ for some θ_x between x and 0. Letting $x = -1$ and 1, we get

$$0 = f(-1) = 2 + \frac{f''(\theta_{-1})}{2} \quad \text{and} \quad 5 = f(1) = 2 + \frac{f''(\theta_1)}{2}.$$

This implies $f''(\theta_{-1}) = -4$ and $f''(\theta_1) = 6$. Since $f''(x)$ is continuous and $-4 < \sqrt{2} < 6$, by the intermediate value theorem, there exists $c \in \mathbb{R}$ such that $f''(c) = \sqrt{2}$.

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(806) By Taylor's theorem, $f(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(\theta_x)}{6}(x-1)^3$,

where θ_x is some number between x and 1

Then $f(2) = f(1) + f'(1) + \frac{f''(1)}{2} + \frac{f'''(\theta_2)}{6}$

and $f(0) = f(1) - f'(1) + \frac{f''(1)}{2} - \frac{f'''(\theta_0)}{6}$.

Subtracting these, we get $2 = 2f'(1) + \frac{f'''(\theta_2) - f'''(\theta_0)}{6}$. Solving for $f'(1)$,

$f'(1) = 1 - \frac{f'''(\theta_2) - f'''(\theta_0)}{12}$. So $|f'(1)| \le 1 + \frac{1}{12}(|f'''(\theta_2)| + |f'''(\theta_0)|) \le 1 + \frac{1}{12}(6+6) = 2$.
