Scalars, vectors, matrices

① transpose A^T $A = [a_{ij}]$ $B = A^T$, $b_{ij} = a_{ji}$.

② Hermitian matrix $A^* = A$, $B = A^*$, $b_{ij} = \overline{a_{ji}}$

$A \in \mathbb{R}^{m\times n}$, $A^* = A^T = A$

$A \in \mathbb{C}^{m\times n}$, $A^* = A$

- Addition / subtraction $C = A \pm B$, $c_{ij} = a_{ij} \pm b_{ij}$
- Scalar product $B = cA$, $b_{ij} = ca_{ij}$
- Inner product $\langle a, b \rangle = \sum_{i=1}^{n} a_i b_i$ $a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$
- Matrix - Vector product: $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

$b = Ax$, $b_i = \sum_{j=1}^{n} A_{ij} x_j$

$\langle a, b \rangle = a^T b = \sum_{i=1}^{n} a_i b_i$

$[\;] \in \mathbb{R}^m$ $\in \mathbb{R}^{m\times 1}$

- Matrix - Matrix product: $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, $B = [b_{ij}] \in \mathbb{R}^{n\times p}$

$C = AB$ $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

- the range of A is $Ran(A) = \{Ax \mid x \in \mathbb{R}^n\}$ $A \in \mathbb{R}^{m\times n}$, $a_{(i)} \in \mathbb{R}^m$ $A = [a_{(1)}, \cdots, a_{(n)}]$

$= \{a_{(1)} x_1 + \cdots + a_{(n)} x_n \mid x \in \mathbb{R}^n\} \subset \mathbb{R}^m$.

$Ran(A)$ column space of A. $Col(A)$.

- the null space of A is $Null(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} \subset \mathbb{R}^n$
- the rank of A is $Rank(A) = dim(Ran(A))$

$Rank(A) \leq \min\{m, n\}$

1. $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 4 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix}$

1) $\operatorname{rank}(A)$

2) $\dim(\operatorname{Ran}(A))$

3) $\operatorname{Null}(A)$, $\dim(\operatorname{Null}(A))$

Elementary Row/ Column permutation.

Solution $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 4 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 3 \\ 2 & -6 & 1 \\ 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -6 & -17 \\ 1 & -1 & -4 \end{bmatrix}$

$\times 3$ $\quad a_{(1)}\ a_{(2)}\ a_{(3)}$

$\operatorname{Ran}(A) = \{ a_{(1)} x_1 + a_{(2)} x_2 + a_{(3)} x_3 \mid x \in \mathbb{R}^3 \} \subset \mathbb{R}^4$

$\dim(\operatorname{Ran}(A)) = 3.$ $\quad \operatorname{rank}(A) = 3.$

3) $\operatorname{Null}(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \} \subset \mathbb{R}^n$

$Ax = 0.$ $\quad \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 4 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$

$\operatorname{Null}(A) = \{ \vec{0} \} \subset \mathbb{R}^3$ $\quad \operatorname{Null}(A) = \vec{0}$ X

$\dim(\operatorname{Null}(A)) = 0$

Definition of norm $\|\cdot\|$

① $\|x\| \geq 0$, $\forall x \in \mathbb{R}^n$, $\|x\| = 0 \Leftrightarrow x = 0$

② $\|\alpha x\| = |\alpha| \|x\|$, $\forall \alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$

③ $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in \mathbb{R}^n$. triangle Ineq.

- Some frequently used norms of vectors

① $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$ $\quad (p \geq 1)$ $\quad \forall x \in \mathbb{R}^n$

② $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/p}$

③ $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

④ $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

$x = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -8 \\ 5 \end{bmatrix}$ $\quad$ compute $\|x\|_1$, $\|x\|_2$, $\|x\|_\infty$

$$\|x\|_1 = \sum_{i=1}^{5} |x_i| = 1+2+0+8+5 = 16$$

$$\|x\|_2 = \left(\sum_{i=1}^{5} |x_i|^2\right)^{1/2} = \sqrt{1+4+0+64+25} = \sqrt{94}$$

$$\sqrt{12} = 2\sqrt{3}$$

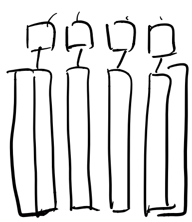
$$\|x\|_\infty = \max_{1 \le i \le 5} \{1, 2, 0, 8, 5\} = 8$$

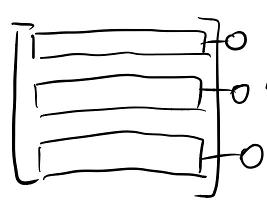
- Some frequently used norms of matrices

① Frobenius norm $\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^{2}\right)^{1/2}$ $\forall A \in \mathbb{R}^{m \times n}$ (p, $1/p$)

② matrix operator p-norm. $\|A\|_p = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p = 1} \|Ax\|_p$

- 2-norm $\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 = (\text{max eigenvalue of } A^TA)^{1/2}$
- 1-norm $\|A\|_1 = \max_{\|x\|_1 = 1} \|Ax\|_1 = \max_{1 \le j \le n} \|a_{(j)}\|_1$
- ∞-norm $\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty = \max_{1 \le i \le m} \|a^{(i)}\|_1$

$A =$ 

$A =$  $A \in \mathbb{R}^{3\times4}$

$\|A\|_\infty$.

Exercise $\|A\|_1$

$$A = \begin{array}{cc} \begin{array}{cccc} 5 & 7 & 5 & 3 \end{array} & \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 3 & 2 & 0 & 2 \\ 1 & 4 & 3 & 0 \end{bmatrix} & \begin{array}{c} 5 \\ 7 \\ \to 8 \end{array} \end{array} \in \mathbb{R}^{3\times4}$$

A^TA.

$\|A\|_2$, $\|A\|_1$, $\|A\|_\infty$, $\|A\|_F$

$$\|A\|_F = \left(\sum_{i=1}^{3}\sum_{j=1}^{4} |a_{ij}|^2\right)^{1/2} = \sqrt{50} = 5\sqrt{2}$$

$$\|A\|_1 = 7$$

$$\|A\|_\infty = 8$$

$\|A\|_2$ A^TA. (eigenvalues of A^TA)

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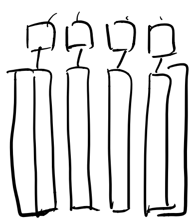
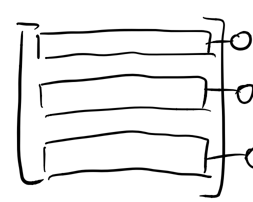
$A =$ 

$A =$  $A \in \mathbb{R}^{3\times 4}$

$\|A\|_1$ $\|A\|_\infty$.

Exercise

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A^TA.

$\|A\|_2$, $\|A\|_1$, $\|A\|_\infty$, $\|A\|_F$

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$\|A\|_2$ A^TA. (eigenvalues of A^TA)