BasicLinearAlgebra

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Basic Linear Algebra and Matrix Operations

Definitions

Vector (column) = ordered field (over real numbers).

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

x_i = the ith entry of x (scalar).

$x_{i:j}$ = the vector $\begin{bmatrix} x_i & x_{i+1} & \cdots & x_j \end{bmatrix}$.

$x_{:j}$ = the vector $\begin{bmatrix} x_1 & x_2 & \cdots & x_j \end{bmatrix}$.

$x_{i:}$ = the vector $\begin{bmatrix} x_i & x_{i+1} & \cdots & x_n \end{bmatrix}$.

Matrix = ordered list of vectors (rows).

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}$$

x_{ij} = the (i,j)th entry of x (scalar).

$x_{i:j,k:l}$ = the matrix $\begin{bmatrix} x_{ik} & x_{i(k+1)} & \cdots & x_{il} \\ x_{(i+1)k} & x_{(i+1)(k+1)} & \cdots & x_{(i+1)l} \\ \vdots & \vdots & \ddots & \vdots \\ x_{jk} & x_{j(k+1)} & \cdots & x_{jl} \end{bmatrix}$.

$x_{:j,k:l}$ = the matrix $\begin{bmatrix} x_{1k} & x_{1(k+1)} & \cdots & x_{1l} \\ x_{2k} & x_{2(k+1)} & \cdots & x_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ x_{jk} & x_{j(k+1)} & \cdots & x_{jl} \end{bmatrix}$.

$x_{i:,k:l}$ = the matrix $\begin{bmatrix} x_{ik} & x_{i(k+1)} & \cdots & x_{il} \\ x_{(i+1)k} & x_{(i+1)(k+1)} & \cdots & x_{(i+1)l} \\ \vdots & \vdots & \ddots & \vdots \\ x_{nk} & x_{n(k+1)} & \cdots & x_{nl} \end{bmatrix}$.

$x_{:,k:l}$ = the matrix $\begin{bmatrix} x_{1k} & x_{1(k+1)} & \cdots & x_{1l} \\ x_{2k} & x_{2(k+1)} & \cdots & x_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ x_{nk} & x_{n(k+1)} & \cdots & x_{nl} \end{bmatrix}$.

Tensor = ordered list of matrices (or higher tensors).

Special vectors:

e_i = the vector $\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$ (1 in ith position).

$\mathbf{1}$ = the vector $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$.

$\mathbf{0}$ = the vector $\begin{bmatrix} 0 & 0 & \cdots & 0 \end{bmatrix}$.

Special matrix:

I = the matrix $\begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Matrix shape:

$x \in \mathbb{F}^n$ = x is a vector of length n.

$x \in \mathbb{F}^{n \times m}$ = x is a matrix of shape $n \times m$.

$x \in \mathbb{F}^{n \times m \times l}$ = x is a tensor of shape $n \times m \times l$.

Notation: $\mathbb{F}^{n \times m \times l} \to \mathbb{F}^{n \times m}$ = x is a function that maps a tensor of shape $n \times m \times l$ to a matrix of shape $n \times m$.

Transpose:

$x^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$ = the transpose of x (column vector).

$x^T = \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{m1} \\ x_{12} & x_{22} & \cdots & x_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{mn} \end{bmatrix}$ = the transpose of x (matrix).

Notation: $\mathbb{F}^{n \times m \times l} \to \mathbb{F}^{m \times n}$ = x is a function that maps a tensor of shape $n \times m \times l$ to a matrix of shape $m \times n$.

Symmetric matrix:

x = symmetric if $x = x^T$.

x = symmetric if $x_{ij} = x_{ji}$.

x = symmetric if $x_{i:j,i:j} = x_{i:j,i:j}^T$.

x = symmetric if $x_{i:j,i:j}$ = symmetric.

x = symmetric if $x_{i:j,i:j}$ = diagonal.

x = symmetric if $x_{i:j,i:j} = I$ = identity.

Vector Transposes, Products and Sums

Vector transpose:

Given two matrices $x \in \mathbb{F}^{n \times m}$, $y \in \mathbb{F}^{m \times l}$, then $(xy)^T = y^T x^T$ (shape $l \times n$).

Proof: $(xy)^T = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{bmatrix} \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1l} \\ y_{21} & y_{22} & \cdots & y_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{ml} \end{bmatrix}$

Dot product:

$x \cdot y = x^T y$ = the dot product of x and y (scalar).

Outer product:

$x \cdot y^T = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_n \end{bmatrix}$ = the outer product of x and y (matrix).

Inner product:

$\langle x, y \rangle = x^T y$ = the inner product of x and y (scalar).

Trace:

$\mathrm{tr}(x) = \sum_{i=1}^{n} x_{ii}$ = the trace of x (scalar).

Sum:

$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$ = the sum of the entries of x (scalar).

$\sum_{i=1}^{n} x_{ii} = \mathrm{tr}(x)$ = the sum of the diagonal entries of x (scalar).

$\sum_{i=1}^{n} x_{i:j,i:j}$ = the sum of the diagonal blocks of x (matrix).

$\sum_{i=1}^{n} x_{i:j,i:j} = x$ = the sum of the diagonal blocks of x is equal to x.

$\sum_{i=1}^{n} x_{i:j,i:j} = I$ = the sum of the diagonal blocks of x is the identity.

$\sum_{i=1}^{n} x_{i:j,i:j} = \mathbf{0}$ = the sum of the diagonal blocks of x is the zero matrix.

$\sum_{i=1}^{n} x_{i:j,i:j}$ = symmetric = the sum of the diagonal blocks of x is symmetric.

$\sum_{i=1}^{n} x_{i:j,i:j}$ = diagonal = the sum of the diagonal blocks of x is diagonal.

$\sum_{i=1}^{n} x_{i:j,i:j} = I$ = the sum of the diagonal blocks of x is the identity.

$\sum_{i=1}^{n} x_{i:j,i:j} = \mathbf{0}$ = the sum of the diagonal blocks of x is the zero matrix.

Product:

$\prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdot \cdots \cdot x_n$ = the product of the entries of x (scalar).

$\prod_{i=1}^{n} x_{ii} = \det(x)$ = the product of the diagonal entries of x (scalar).

$\prod_{i=1}^{n} x_{i:j,i:j}$ = the product of the diagonal blocks of x (matrix).

$\prod_{i=1}^{n} x_{i:j,i:j} = x$ = the product of the diagonal blocks of x is equal to x.

$\prod_{i=1}^{n} x_{i:j,i:j} = I$ = the product of the diagonal blocks of x is the identity.

$\prod_{i=1}^{n} x_{i:j,i:j} = \mathbf{0}$ = the product of the diagonal blocks of x is the zero matrix.

$\prod_{i=1}^{n} x_{i:j,i:j}$ = symmetric = the product of the diagonal blocks of x is symmetric.

$\prod_{i=1}^{n} x_{i:j,i:j}$ = diagonal = the product of the diagonal blocks of x is diagonal.

$\prod_{i=1}^{n} x_{i:j,i:j} = I$ = the product of the diagonal blocks of x is the identity.

$\prod_{i=1}^{n} x_{i:j,i:j} = \mathbf{0}$ = the product of the diagonal blocks of x is the zero matrix.

Vector Norms and Matrix Norms

Vector norms:

$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ = the p-norm of x (scalar).

$\|x\|_1 = \sum_{i=1}^{n} |x_i|$ = the 1-norm of x (scalar).

$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$ = the 2-norm of x (scalar).

$\|x\|_\infty = \max_i |x_i|$ = the ∞-norm of x (scalar).

$\|x\|_0 = \sum_{i=1}^{n} \mathbb{1}(x_i \neq 0)$ = the 0-norm of x (scalar).

$\|x\|_0 = \sum_{i=1}^{n} \mathbb{1}(x_i \neq 1)$ = the 1-norm of x (scalar).

$\|x\|_0 = \sum_{i=1}^{n} \mathbb{1}(x_i \neq -1)$ = the 2-norm of x (scalar).

$\|x\|_0 = \sum_{i=1}^{n} \mathbb{1}(x_i \neq \infty)$ = the ∞-norm of x (scalar).

Matrix norms:

$\|x\|_p = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} |x_{ij}|^p\right)^{1/p}$ = the p-norm of x (scalar).

$\|x\|_1 = \sum_{i=1}^{n} \sum_{j=1}^{m} |x_{ij}|$ = the 1-norm of x (scalar).

$\|x\|_2 = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} |x_{ij}|^2}$ = the 2-norm of x (scalar).

$\|x\|_\infty = \max_{i,j} |x_{ij}|$ = the ∞-norm of x (scalar).

$\|x\|_0 = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{1}(x_{ij} \neq 0)$ = the 0-norm of x (scalar).

$\|x\|_0 = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{1}(x_{ij} \neq 1)$ = the 1-norm of x (scalar).

$\|x\|_0 = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{1}(x_{ij} \neq -1)$ = the 2-norm of x (scalar).

$\|x\|_0 = \sum_{i=1}^{n} \sum_{j=1}^{m} \mathbb{1}(x_{ij} \neq \infty)$ = the ∞-norm of x (scalar).

Vector Products

Hadamard product:

$x \circ y = \begin{bmatrix} x_1 y_1 & x_2 y_2 & \cdots & x_n y_n \end{bmatrix}$ = the Hadamard product of x and y (vector).

$x \circ y = \begin{bmatrix} x_{11} y_{11} & x_{12} y_{12} & \cdots & x_{1m} y_{1m} \\ x_{21} y_{21} & x_{22} y_{22} & \cdots & x_{2m} y_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} y_{n1} & x_{n2} y_{n2} & \cdots & x_{nm} y_{nm} \end{bmatrix}$ = the Hadamard product of x and y (matrix).

Kronecker product:

$x \otimes y = \begin{bmatrix} x_1 y & x_2 y & \cdots & x_n y \end{bmatrix}$ = the Kronecker product of x and y (vector).

$x \otimes y = \begin{bmatrix} x_{11} y & x_{12} y & \cdots & x_{1m} y \\ x_{21} y & x_{22} y & \cdots & x_{2m} y \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} y & x_{n2} y & \cdots & x_{nm} y \end{bmatrix}$ = the Kronecker product of x and y (matrix).

Cross product:

$x \times y = \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \end{bmatrix}$ = the cross product of x and y (vector).

$x \times y = \begin{bmatrix} x_{11} y_{11} & x_{12} y_{12} & \cdots & x_{1m} y_{1m} \\ x_{21} y_{21} & x_{22} y_{22} & \cdots & x_{2m} y_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} y_{n1} & x_{n2} y_{n2} & \cdots & x_{nm} y_{nm} \end{bmatrix}$ = the cross product of x and y (matrix).

Vector Inverses and Matrix Inverses

x is invertible if there exists x^{-1} such that $x x^{-1} = I$.

x is invertible if x is square.

x is invertible if x is symmetric.

x is invertible if x is diagonal.

x is invertible if x is the identity.

x is invertible if x is the zero matrix.

x is invertible if x is symmetric.

x is invertible if x is symmetric.

x is invertible if x is symmetric