MATH3322 Matrix Computation
Homework 1

Due date: 15 March, Sunday

1. Let
$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
Find
$$\boldsymbol{Ax}, \quad \boldsymbol{A}^T\boldsymbol{y}, \quad \boldsymbol{AB}^T.$$

2. Show that $\|\boldsymbol{x}\|_1 := \sum_{i=1}^n |x_i|$ defines a norm on $\mathbb{R}^n$.

3. Show that $\|\boldsymbol{A}\|_\infty = \max_{1\le i\le m} \|\boldsymbol{a}_i\|_1$ for any $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $\boldsymbol{a}_i^T \in \mathbb{R}^{1\times n}$, $i = 1, \ldots, m$, being its row vectors.

4. Show that for any vector norm on $\mathbb{R}^n$ that $|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \le \|\boldsymbol{x} - \boldsymbol{y}\|$.

5. Compare performances of different implementations of the matrix-vector multiplication. An incomplete Matlab code `matvecprod.m` is attached.
   (a) Complete the code for matrix-vector multiplication.
   (b) Run the program on your computer and record the time needed for each implementation.

   Here are two remarks. 1. Different implementations may have very different performances, though mathematically they are equivalent. 2. We should call build-in functions for matrix computations if they are available, because they are optimized for your computer.

6. Use Gaussian Elimination to solve the following system of linear equations.
$$\begin{cases} 2x_1 + x_2 + x_3 = 2 \\ 4x_1 + 5x_2 + 3x_3 = 2 \\ 2x_1 - 2x_2 + 3x_3 = 7. \end{cases}$$

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1. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Find

$$\boldsymbol{Ax}, \quad \boldsymbol{A}^T\boldsymbol{y}, \quad \boldsymbol{AB}^T.$$

$$Ax = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2+3 \\ 4+5+6 \end{bmatrix} = \begin{bmatrix} 6 \\ 15 \end{bmatrix}$$

$$A^T y = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+12 \\ 4+15 \\ 6+18 \end{bmatrix} = \begin{bmatrix} 14 \\ 19 \\ 24 \end{bmatrix}$$

$$AB^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & -1+2 \\ 4-6 & -4+5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$$

2. Show that $\|\boldsymbol{x}\|_1 := \sum_{i=1}^{n} |x_i|$ defines a norm on $\mathbb{R}^n$.

First, $\|x\|_1 = \sum_{i=1}^{n} |x_i| \geqslant 0$

and $\|x\|_1 = 0 \iff \sum_{i=1}^{n} |x_i| = 0$

$\iff |x_i| = 0 \ \forall i \iff x = 0$

secondly, $\|\alpha x\|_1 = \sum_{i=1}^{n} |\alpha x_i| = \sum_{i=1}^{n} |\alpha||x_i|$

$= |\alpha| \sum_{i=1}^{n} |x_i| = |\alpha| \|x\|_1$

Thirdly, $\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} \left(|x_i| + |y_i| \right)$

$= \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|x\|_1 + \|y\|_1 \quad \forall x, y \in \mathbb{R}^n$

Thus, $\|x\|_1$ defines a norm on $\mathbb{R}^n$.

3. Show that $\|\boldsymbol{A}\|_\infty = \max_{1\le i\le m} \|\boldsymbol{a}_i\|_1$ for any $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ with $\boldsymbol{a}_i^T \in \mathbb{R}^{1\times n}$, $i = 1, \ldots, m$, being its row vectors.

$$\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\| = \max_{\|x\|_\infty=1} \max_{1\le i\le m} \left\| \sum_{j=1}^{n} a_{ij} x_j \right\|$$

$$= \max_{1\le i\le m} \max_{\|x\|_\infty=1} \left\| \sum_{i=1}^{n} a_{ij} x_j \right\| = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$$

$$= \max_{1\le i\le m} \|a_i\|_1$$

4. Show that for any vector norm on $\mathbb{R}^n$ that $|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x} - \boldsymbol{y}\|$.

$$\|x\| = \|(x-y)+y\| \leq \|x-y\| + \|y\|$$

$$\therefore \quad \|x\| - \|y\| \leq \|x-y\|$$

$$\|y\| = \|y-x+x\| \leq \|y-x\| + \|x\|$$

$$= \|x-y\| + \|x\|$$

$$\therefore \quad \|y\| - \|x\| \leq \|x-y\|$$

Thus, it proves that $|\|x\| - \|y\|| \leq \|x-y\|$

5. Compare performances of different implementations of the matrix-vector multiplication. An incomplete Matlab code `matvecprod.m` is attached.

   (a) Complete the code for matrix-vector multiplication.

   (b) Run the program on your computer and record the time needed for each implementation.

   Here are two remarks. 1. Different implementations may have very different performances, though mathematically they are equivalent. 2. We should call build-in functions for matrix computations if they are available, because they are optimized for your computer.

a).

```
c(i) = c(i) + A(i,j) * b(j);
c(i) = c(i) + A(i,j) * b(j);
```

b). ij loop, time: 16.2484 seconds.

ji loop, time: 4.625 seconds

build-in function, time: 0.12998 seconds.

6. Use Gaussian Elimination to solve the following system of linear equations.

$$\begin{cases} 2x_1 + x_2 + x_3 = 2 \\ 4x_1 + 5x_2 + 3x_3 = 2 \\ 2x_1 - 2x_2 + 3x_3 = 7. \end{cases}$$

Coefficient matrix:

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & 5 & 3 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 4 & 5 & 3 & 2 \\ 2 & -2 & 3 & 7 \end{array}\right]$$

$$\underset{\substack{① \times (-2) + ② \\ ① \times (-1) + ③}}{=} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 3 & 1 & -2 \\ 0 & -3 & 2 & 5 \end{array}\right]$$

$$\underset{② \times 1 + ③}{=} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2x_1 + x_2 + x_3 = 2 \\ 3x_2 + x_3 = -2 \\ 3x_3 = 3 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = -1 \\ x_3 = 1 \end{cases}$$

$$x = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$