1. $Ab = \begin{pmatrix} 2 & -1 & 1 & -4 & 0 \\ 1 & -2 & 0 & 2 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ -7 \end{pmatrix}$

2. $\|Ab\|_\infty = \max_{1 \le i \le 2} |(Ab)_i| = 7.$

3. $\|A\|_1 = \max\{3, 3, 1, 6, 3\}$

$= 6$

4. $\|A\|_2 = (\max \text{ eigenvalues of } A^TA)^{\frac{1}{2}}.$

$A^TA = \begin{pmatrix} 2 & -1 & 1 & -4 & 0 \\ 1 & -2 & 0 & 2 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -2 \\ 1 & 0 \\ -4 & 2 \\ 0 & -3 \end{pmatrix}$

$A^TA = \begin{pmatrix} 22 & -4 \\ -4 & 18 \end{pmatrix}$

$$|A^TA - \lambda I| = \begin{vmatrix} 22-\lambda & -4 \\ -4 & 18-\lambda \end{vmatrix}$$

$$= (22-\lambda)(18-\lambda) - 16$$

$$= 396 - 22\lambda - 18\lambda + \lambda^2 - 16$$

$$= \lambda^2 - 40\lambda + 380.$$

$$\lambda = \frac{40 \pm \sqrt{40^2 - 4(1)(380)}}{2(1)}$$

$$\lambda = \frac{40 \pm \sqrt{80}}{2}$$

$$\lambda = \frac{40 \pm 4\sqrt{5}}{2}$$

$$\lambda = 20 \pm 2\sqrt{5}$$

$$\therefore \|A\|_2 = \sqrt{20 + 2\sqrt{5}}$$

2.

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$$

$$\|x\|_\infty^2 = \max_{1\le i\le n} |x_i|^2 \le \sum_{i=1}^{n} |X_i|^2 = \|x\|_2^2 \quad -①$$

$$\|x\|_2^2 = \sum_{i=1}^{n} |x_i|^2 \le \sum_{i=1}^{n}\left(\max_{1\le i\le n} |x_i|^2\right) = n\cdot\|x\|_\infty^2 \quad -②$$

Taking square root on both side of ①, we have

$$\|x\|_\infty \le \|x\|_2$$

similarly,

$$\|x\|_2 \le \sqrt{n}\,\|x\|_\infty$$

$$\therefore \quad \|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty$$

2.2).

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_\infty = \max_{1\le i\le n} |x_i| \le \sum_{i=1}^{n} |x_i| = \|x\|_1$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| \le \sum_{i=1}^{n} \left(\max_{1\le i\le n} |x_i| \right) = n \cdot \|x\|_\infty$$

$\therefore$ we have

$$\|x\|_\infty \le \|x\|_1 \le n \cdot \|x\|_\infty$$

3.
$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4/3 \end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 4/3 \end{array}\right)$$

Interchange ① & ②, $L_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 2 & -1 & 1 & 3 \\ 0 & 1 & 1 & 4/3 \end{array}\right)$$

① $\times(-2)$ + ②, $L_2 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 1 & 1 & 4/3 \end{array}\right)$$

② $\times(-\frac{1}{3})$ + ③, $L_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{pmatrix}$

$$= \left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2/3 & 1 \end{array}\right)$$

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_2 + x_3 = 1 \\ \frac{2}{3}x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{2}{3} \\ x_2 = -\frac{1}{6} \\ x_3 = 3/2 \end{cases}$$

LU decomposition of matrix A:
(using algorithm learnt in class)

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 \\ 1/2 & -2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 2 & -1 & 1 \\ 1/2 & -3/2 & -1/2 \\ 0 & -2/3 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 \\ 1/2 & -3/2 & -1/2 \\ 0 & -2/3 & 2/3 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & -2/3 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 2 & -1 & 1 \\ 0 & -3/2 & -1/2 \\ 0 & 0 & 2/3 \end{pmatrix}$$