prescribes the *relative error* in $\hat{x}$. Relative error in the $\infty$-norm can be translated into a statement about the number of correct significant digits in $\hat{x}$. In particular, if

$$\frac{\|\hat{x} - x\|_\infty}{\|x\|_\infty} \approx 10^{-p},$$

then the largest component of $\hat{x}$ has approximately $p$ correct significant digits. For example, if $x = [1.234 \;\; .05674]^T$ and $\hat{x} = [1.235 \;\; .05128]^T$, then $\|\hat{x} - x\|_\infty / \|x\|_\infty \approx .0043 \approx 10^{-3}$. Note than $\hat{x}_1$ has about three significant digits that are correct while only one significant digit in $\hat{x}_2$ is correct.

### 2.2.4 Convergence

We say that a sequence $\{x^{(k)}\}$ of $n$-vectors *converges* to $x$ if

$$\lim_{k\to\infty} \|x^{(k)} - x\| = 0.$$

Because of (2.2.4), convergence in *any* particular norm implies convergence in *all* norms.

#### Problems

**P2.2.1** Show that if $x \in \mathbb{R}^n$, then $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

**P2.2.2** By considering the inequality $0 \le (ax + by)^T(ax + by)$ for suitable scalars $a$ and $b$, prove (2.2.3).

**P2.2.3** Verify that $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ are vector norms.

**P2.2.4** Verify (2.2.5)-(2.2.7). When is equality achieved in each result?

**P2.2.5** Show that in $\mathbb{R}^n$, $x^{(i)} \to x$ if and only if $x_k^{(i)} \to x_k$ for $k = 1{:}n$.

**P2.2.6** Show that for any vector norm on $\mathbb{R}^n$ that $|\,\|x\| - \|y\|\,| \le \|x - y\|$.

**P2.2.7** Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^m$ and assume $A \in \mathbb{R}^{m\times n}$. Show that if $\operatorname{rank}(A) = n$, then $\|x\|_A = \|Ax\|$ is a vector norm on $\mathbb{R}^n$.

**P2.2.8** Let $x$ and $y$ be in $\mathbb{R}^n$ and define $\psi{:}\mathbb{R} \to \mathbb{R}$ by $\psi(\alpha) = \|x - \alpha y\|_2$. Show that $\psi$ is minimized if $\alpha = x^Ty/y^Ty$.

**P2.2.9** Prove or disprove:

$$v \in \mathbb{R}^n \;\Rightarrow\; \|v\|_1 \|v\|_\infty \le \frac{1+\sqrt{n}}{2}\|v\|_2^2.$$

**P2.2.10** If $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$ then it can be shown that $|x^Ty| = \|x\|_2 \|y\|_2 |\cos(\theta)|$ where $\theta$ is the angle between $x$ and $y$. An analogous result exists for the *cross product* defined by

$$x \times y = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix}.$$

In particular, $\|x \times y\|_2 = \|x\|_2 \|y\|_2 |\sin(\theta)|$. Prove this.

**P2.2.11** Suppose $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Show that

$$\|x \otimes y\|_p = \|x\|_p \|y\|_p$$

for $p = 1, 2,$ and $\infty$.

#### Notes and References for §2.2

Although a vector norm is "just" a generalization of the absolute value concept, there are some noteworthy subtleties:

J.D. Pryce (1984). "A New Measure of Relative Error for Vectors," *SIAM J. Numer. Anal.* 21, 202–221.

norms are the *p-norms* defined by

$$\| x \|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}, \qquad p \geq 1\,. \tag{2.2.1}$$

The $1-$, $2-$, and $\infty-$ norms are the most important:

$$\begin{aligned}
\| x \|_1 &= |x_1| + \cdots + |x_n|, \\
\| x \|_2 &= (|x_1|^2 + \cdots + |x_n|^2)^{\frac{1}{2}} = (x^T x)^{\frac{1}{2}}, \\
\| x \|_\infty &= \max_{1 \leq i \leq n} |x_i|.
\end{aligned}$$

A *unit vector* with respect to the norm $\| \cdot \|$ is a vector $x$ that satisfies $\| x \| = 1$.

### 2.2.2 Some Vector Norm Properties

A classic result concerning $p$-norms is the *Hölder inequality*:

$$|x^T y| \leq \| x \|_p \| y \|_q \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{2.2.2}$$

A very important special case of this is the *Cauchy-Schwarz inequality*:

$$|x^T y| \leq \| x \|_2 \| y \|_2. \tag{2.2.3}$$

All norms on $\mathbb{R}^n$ are *equivalent*, i.e., if $\| \cdot \|_\alpha$ and $\| \cdot \|_\beta$ are norms on $\mathbb{R}^n$, then there exist positive constants $c_1$ and $c_2$ such that

$$c_1 \| x \|_\alpha \leq \| x \|_\beta \leq c_2 \| x \|_\alpha \tag{2.2.4}$$

for all $x \in \mathbb{R}^n$. For example, if $x \in \mathbb{R}^n$, then

$$\| x \|_2 \leq \| x \|_1 \leq \sqrt{n} \| x \|_2, \tag{2.2.5}$$

$$\| x \|_\infty \leq \| x \|_2 \leq \sqrt{n} \| x \|_\infty, \tag{2.2.6}$$

$$\| x \|_\infty \leq \| x \|_1 \leq n \| x \|_\infty. \tag{2.2.7}$$

Finally, we mention that the 2-norm is preserved under orthogonal transformation. Indeed, if $Q \in \mathbb{R}^{n \times n}$ is orthogonal and $x \in \mathbb{R}^n$, then

$$\| Qx \|_2^2 = (Qx)^T (Qx) = (x^T Q^T)(Qx) = x^T (Q^T Q) x = x^T x = \| x \|_2^2.$$

### 2.2.3 Absolute and Relative Errors

Suppose $\hat{x} \in \mathbb{R}^n$ is an approximation to $x \in \mathbb{R}^n$. For a given vector norm $\| \cdot \|$ we say that

$$\epsilon_{\text{abs}} = \| \hat{x} - x \|$$

is the *absolute error* in $\hat{x}$. If $x \neq 0$, then

$$\epsilon_{\text{rel}} = \frac{\| \hat{x} - x \|}{\| x \|}$$

**P2.2.2** By considering the inequality $0 \leq (ax + by)^T(ax + by)$ for suitable scalars $a$ and $b$, prove (2.2.3).

$$|x^T y| \leq \|x\|_2 \|y\|_2$$

$$(ax+by)^T(ax+by) = \|ax+by\|_2^2 \geq 0$$

$$= \sum_{i=1}^{n}(ax_i + by_i)(ax_i + by_i) \geq 0$$

$$= \sum_{i=1}^{n}\left( ax_i^2 + 2ax_i by_i + by_i^2 \right) \geq 0$$

$$= a\|x\|_2^2 + 2ab\sum_{i=1}^{n} x_i y_i + b\|y\|_2^2 \geq 0$$

$$p(t) = \|t\vec{y} - \vec{x}\|^2 \geq 0$$

$$p(t) = (\vec{y}\cdot\vec{y})\,t^2 - 2(\vec{x}\cdot\vec{y})\,t + \vec{x}\cdot\vec{x} \geq 0$$

$$p(t) = at^2 - bt + c \geq 0$$

$$p\left(\frac{b}{2a}\right) = a\frac{b^2}{4a^2} - b\left(\frac{b}{2a}\right) + c \geq 0$$

$$= \frac{b^2}{4a} - \frac{2b^2}{4a} + c \geq 0$$

$$= \frac{-b^2}{4a} + c \geq 0$$

$$c \geq \frac{b^2}{4a}$$

$$4ac \geq b^2$$

$$4(\|\vec{y}\|_2^2 \|\vec{x}\|_2^2) \geq (-2(\vec{x}\cdot\vec{y}))^2$$

$$4\|\vec{x}\|_2^2 \|y\|_2^2 \geq 4(\vec{x}\cdot\vec{y})^2$$

$$\|\vec{x}\|_2 \|y\|_2 \geq |x^T y|$$