MATH 2023 • Multivariable Calculus
Problem Set #3 • Chain Rule, Directional Derivatives, Gradients

1. (★) Suppose $w = f(x, y, z)$ where $x = g(s)$, $y = h(s, t)$ and $z = k(t)$. Assume all functions involved are C^1. Draw the tree diagram to showcase the relations between w, x, y, z, s and t. Hence, write down the chain rule for calculating the partial derivatives: $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$. Use the symbols ∂ and d appropriately.

2. (★) Recall that the rectangular-polar coordinates conversion rules are given as follows:
$$x = r\cos\theta$$
$$y = r\sin\theta$$
A function $f(x, y)$ is said to be **rotationally/radially symmetric** if $\frac{\partial f}{\partial \theta} = 0$, i.e. when regarded as a function of (r, θ), it depends only the radial variable r but not the angular variable θ. For instance, $f(x, y) = x^2 + y^2$ is rotationally symmetric since $f(r, \theta) = r^2$. Using the chain rule, show that f is rotationally symmetric if and only if:
$$y\frac{\partial f}{\partial x} = x\frac{\partial f}{\partial y}.$$

3. (★) Suppose $f(u, v)$ is a C^2 function, and $u = s^2 - t$ and $v = s + t^2$. Express the second partial derivative $\dfrac{\partial^2 f}{\partial s \partial t}$ in terms of f_{uu}, f_{uv}, f_{vv}, s and t.

4. (★) Let $f(x, y, z)$ be a C^1 function of three variables, and z be a C^1 function of (x, y) such that
$$f(x, y, z(x, y)) = 0.$$
Using the chain rule, show that:
$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}.$$

5. (★★) Let $f(x, y)$ be a C^1 function. Consider two parametric curves $\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j}$ and $\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j}$ which satisfy:
$$\mathbf{r}_1(0) = \mathbf{r}_2(0) \quad \text{and} \quad \mathbf{r}_1'(0) = \mathbf{r}_2'(0).$$
   (a) Show that
$$\left.\frac{d}{dt}\right|_{t=0} f(x_1(t), y_1(t)) = \left.\frac{d}{dt}\right|_{t=0} f(x_2(t), y_2(t)).$$
   (b) Give a geometric interpretation of the above result.

6. (★★) The wave equation is an important partial differential equation which governs the propagation of waves. Let $u(x, y, z, t)$ be the displacement of the wave at position (x, y, z) at time t. It can be shown by several physical laws (such as the Hooke's Law) that u satisfies:
$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$
where c is a constant (which is the wave speed).

In one (spatial) dimension, the wave equation can be stated as:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

It turns out that the chain rule of several variables has a nice application on solving the one dimensional wave equation. The following exercise guides you to show that if $u(x,t)$ is a solution to the one dimensional wave equation, then it must take the form $u(x,t) = F(x-ct) + G(x+ct)$ where F and G are arbitrary differentiable functions of single variable.

Let $u(x,t)$ solve the one dimensional wave equation (2).

(a) Define $\xi = x - ct$ and $\eta = x + ct$. Regard u as a function of ξ and η, and ξ and η are functions of x and t. Using the chain rule of multivariable functions, show that:

$$u_t = c(u_\eta - u_\xi) \quad \text{and} \quad u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).$$

(b) Using the chain rule agin, show that

$$u_x = u_\xi + u_\eta \quad \text{and} \quad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

(c) Combining the results of (a), (b) and the wave equation, show that $u_{\xi\eta} = 0$.

(d) Finally, deduce that u, as a function of ξ and η, must be in the form of:

$$u(\xi,\eta) = F(\xi) + G(\eta)$$

where F and G are arbitrary functions. Hence, in terms of the original variables x and t, u must take the form $u(x,t) = F(x-ct) + G(x+ct)$.

7. (★★★) In many physics, geometry and engineering applications, it is often more convenient to use polar or sphereical coordinates since many physical quantities are rotationally symmetric.

The conversion rule of rectangular and polar coordinates is given by:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Let u be a function of x and y. Since (x,y) can be converted into (r,θ), we can also regard u as a function of (r,θ). The chain rule can be used to derive some conversion formulae between u_x, u_y and u_r, u_θ.

An important operator in physics, geometry and engineering is called the **Laplacian**. In two dimensions, it is defined as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}.$$

In this exercise, we will show that $\nabla^2 u$ can be expressed in polar form as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

The polar form of the Laplacian is often used when dealing with rotationally symmetric functions, i.e. a function u which does not depend on θ but only on r. For such functions, their Laplacian is simply:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r.$$

(a) Use the fact that $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$, show that:

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}.$$

(b) Regard u as a function of (r,θ), and (r,θ) are functions of (x,y). Sketch a tree diagram to showcase these relations. Using the chain rule, show that:

$$\begin{aligned} u_x &= \frac{xu_r}{r} - \frac{yu_\theta}{r^2}, \\ u_y &= \frac{yu_r}{r} + \frac{xu_\theta}{r^2}. \end{aligned}$$

(c) Using quotient and product rules, show that:

$$\begin{aligned} u_{xx} &= \frac{u_r}{r} + \frac{xu_{rx}}{r} - \frac{x^2u_r}{r^3} - \frac{yu_{\theta x}}{r^2} + \frac{2xyu_\theta}{r^4} \\ u_{yy} &= \frac{u_r}{r} + \frac{yu_{ry}}{r} - \frac{y^2u_r}{r^3} + \frac{xu_{\theta y}}{r^2} - \frac{2xyu_\theta}{r^4} \end{aligned}$$

(d) Since u_r and u_θ are functions of (r,θ), and (r,θ) are functions of (x,y), they share the same tree diagram as u in part (b), and hence we have

$$u_{rx} = \frac{\partial u_r}{\partial x} = \frac{\partial u_r}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u_r}{\partial \theta}\frac{\partial \theta}{\partial x}$$

and similar for other second derivatives u_{ry}, $u_{\theta x}$ and $u_{\theta y}$. Show that:

$$\begin{aligned} xu_{rx} + yu_{ry} &= ru_{rr} \\ xu_{\theta y} - yu_{\theta x} &= u_{\theta\theta} \end{aligned}$$

(e) Combining the results proved in previous parts, show that:

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

8. (★) Compute the directional derivative of the following functions at the given point P in the direction of the given vector $\mathbf{v}$. Moreover, find the unit direction $\mathbf{u}$ along which the function increases most rapidly.

(a) $f(x,y) = x^2 - y^2$, $P(-1,-3)$, $\mathbf{v} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$.

(b) $g(x,y) = e^{-x-y}$, $P(\ln 2, \ln 3)$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

(c) $h(x,y) = e^{xy}$, $P(1,0)$, $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j}$.

(d) $F(x,y,z) = xy + yz + zx + 4$, $P(2,-2,1)$, $\mathbf{v} = -\mathbf{j} - \mathbf{k}$.

(e) $G(x,y,z) = e^{xyz} - 1$, $P(0,1,-1)$, $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

9. (★) For each surface and the given point P, find the value a such that P lies on the surface, and then find an equation of the tangent plane to the surface at the point P:

(a) $x^2 + y + z = 3$, $P(2,0,a)$

(b) $xy \sin z = 1$, $P(a, 2, \pi/6)$

(c) $yze^{xz} = 8$, $P(0,a,4)$

(d) $z = e^{xy}$, $P(1,0,a)$

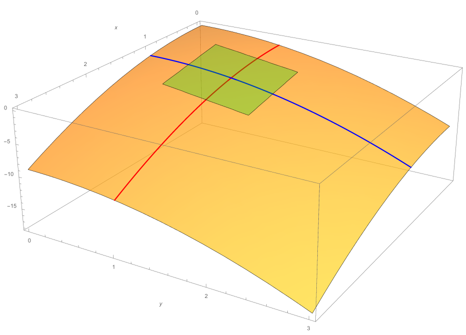
(e) $z = \ln(1+xy)$, $P(1,2,a)$.

10. (★) Let
$$V(x,y,z) = -\frac{GMm}{\sqrt{x^2+y^2+z^2}}$$
where G, M and m are constants. Define $\mathbf{F}(x,y,z) = -\nabla V(x,y,z)$.
   (a) Verify that:
   $$\mathbf{F}(x,y,z) = -GMm\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}.$$
   (b) Show that $|\mathbf{F}(x,y,z)|$ is inversely proportional to the squared distance from (x,y,z) to the origin in $\mathbb{R}^3$.

11. (★★) Consider the function
$$f(x,y) = \cos(x+y)$$
as well as the plane Π given by the equation
$$x - y = 0\ .$$
The intersection of the graph of f with Π is a curve C. Find the slope of the tangent line to C at the point (π,π) using directional derivatives. [Hint: First sketch a diagram of the graph, the plane and the curve.]

12. (★★) One approach for finding the normal vector of the tangent plane at a given point (x_0,y_0) to a graph $z = f(x,y)$ is by writing the graph equation as a level surface $z - f(x,y) = 0$ of a three-variable function $g(x,y,z) := z - f(x,y)$. Then, the gradient $\nabla g = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right\rangle$ at point $(x_0,y_0,f(x_0,y_0))$ is perpendicular to the level surface $\{g=0\}$, and so we can take it to be a normal vector of the tangent plane as long as $\nabla g \neq \mathbf{0}$ at $(x_0,y_0,f(x_0,y_0))$.

In fact, it is also possible to show the normal vector is $\left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right\rangle$ using a *purely* two-variable argument instead of *going up one higher dimension*.



   (a) Consider a given function $f(x,y)$, and a given point (x_0,y_0). Find a parametrization:
   $$\mathbf{r}_1(t) = ?\mathbf{i} + ?\mathbf{j} + ?\mathbf{k}$$
   of the curve on the graph $z = f(x,y)$ travelling in the x-direction while keeping y fixed at y_0 (i.e. the red curve in the diagram). Hence, find the tangent vector of the curve $\mathbf{r}_1(t)$ at the point $(x_0,y_0,f(x_0,y_0))$. Label this tangent vector by $\mathbf{T}_1$.
   (b) Find a parametrization $\mathbf{r}_2(t)$ of the curve on the graph $z = f(x,y)$ travelling in the y-direction while keeping x fixed at x_0 (i.e. the blue curve in the diagram). Hence, find the tangent vector of $\mathbf{r}_2(t)$ at the point $(x_0,y_0,f(x_0,y_0))$. Label this tangent vector by $\mathbf{T}_2$.

(c) Since both $\mathbf{T}_1$ and $\mathbf{T}_2$ are tangent vectors to the graph, they are parallel to the tangent plane. Therefore, the normal vector to the tangent plane must be perpendicular to both $\mathbf{T}_1$ and $\mathbf{T}_2$. Using this fact, show that the normal vector to the tangent plane is given by

$$\left\langle -\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1 \right\rangle.$$

Optional

13. The spherical coordinates (ρ, θ, ϕ) is another important coordinate system in $\mathbb{R}^3$. We will learn that in later chapters. The conversion rules between spherical and rectangular coordinates are given by:

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}$$

Given a C^2 function $f(x, y, z)$, it can be regarded as a function of (ρ, θ, ϕ) as well under the above conversion rule. Show that the Laplacian $\nabla^2 f := f_{xx} + f_{yy} + f_{zz}$ can be expressed in spherical coordinates as:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}.$$

[Note: It is a very time consuming exercise. It took me 4 hours to do it when I was an undergraduate.]