MATH 2023 – Multivariable Calculus

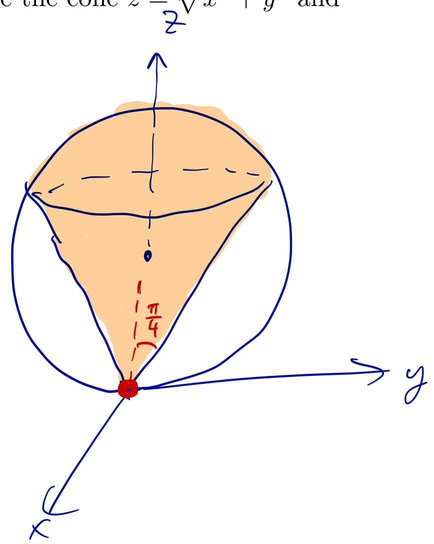
Lecture #21 Worksheet — April 30, 2019

$\phi = \frac{\pi}{4}$

Problem 1. Find the volume of the solid lying above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

$\rho^2 = \rho\cos\phi$

$\rho = \cos\phi$



$$\iiint_E dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \frac{\cos^3\phi}{3} \sin\phi \, d\phi$$

(Let $u = \cos\phi$ etc...)

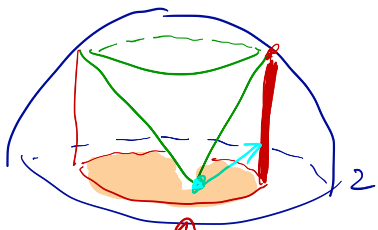
$$= 2\pi \int_0^{\frac{\pi}{4}} -\frac{\cos^3\phi}{3}\, d(\cos\phi) = 2\pi \left(-\frac{\cos^4\phi}{12} \right)\Bigg|_0^{\frac{\pi}{4}} = \frac{\pi}{8}$$

Problem 2. Find the volume of the ~~sphere~~ solid inside the sphere $x^2 + y^2 + z^2 = 4$, under $z = \sqrt{x^2 + y^2}$ and above the xy plane.

Spherical coord:

$0 \le \theta \le 2\pi$

$\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$



cylinder of radius $\sqrt{2}$

$x^2+y^2=2$

$\Leftrightarrow \rho \sin\phi = \sqrt{2}$

$\rho = \frac{\sqrt{2}}{\sin\phi}$

$$\begin{cases} x^2+y^2+z^2=4 \\ z=\sqrt{x^2+y^2} \end{cases}$$

$\Rightarrow x^2+y^2=2$

radius $= \sqrt{2}$

$$\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\frac{\sqrt{2}}{\sin\phi}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Cylndrical coord:

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\left(\int_0^{r} dz\right) r \, dr \, d\theta = \frac{4\sqrt{2}}{3}\pi$$

$$\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_0^{\sqrt{x^2+y^2}} dz \, dy \, dx$$

Problem 3. Evaluate the integrals:

$\rho = \sec\phi$

$= \frac{1}{\cos\phi}$

$\rho\cos\phi = 1$

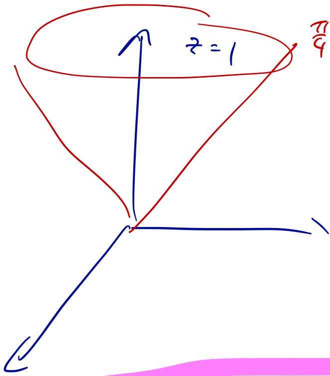
$\boxed{z = 1}$

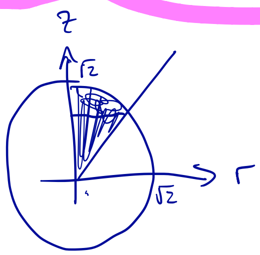
$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$d\theta$?

dV

$$= \int_0^{2\pi}\int_0^1\int_r^1 r \, dz \, dr \, d\theta .$$





$x^2+y^2=1$

$z^2 = 2 - x^2 - y^2$

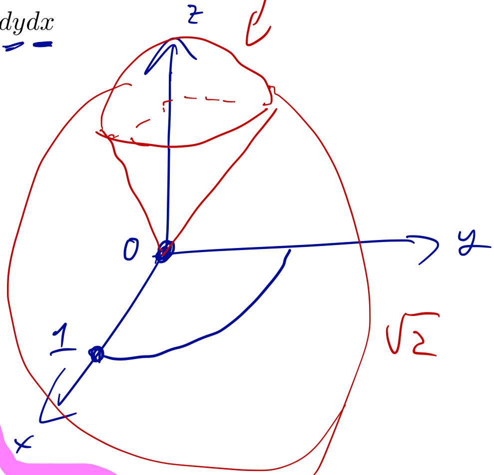
$x^2+y^2+z^2 = 2$

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy \, dz \, dy \, dx$$

cone

$z = \sqrt{x^2+y^2}$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} \rho^2\sin^2\phi\sin\theta\cos\theta \; \rho^2\sin\phi \; d\rho \, d\phi \, d\theta$$



$\rho = 2a$

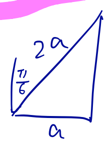
$\rho = \frac{a}{\sin\phi}$

$\Leftrightarrow \rho\sin\phi = a$

$r = a$

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_{a/\sin\phi}^{2a} \boxed{\rho^4 \sin^3\phi} \, d\rho \, d\phi \, d\theta$$

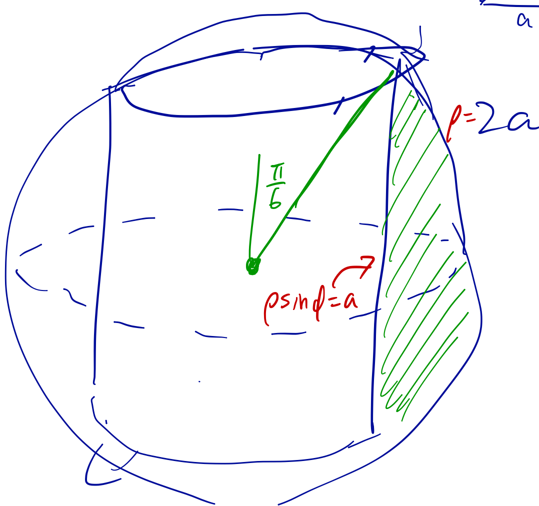
$\rho^2\sin^2\phi \;\; \rho^2\sin\phi \, d\rho \, d\phi \, d\theta$



= volume inside sphere $\rho = 2a$

outside cylinder $r = a$.

$$= \int_0^{2\pi}\int_a^{2a}\left(\int_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} r^2 \, dz\right) r \, dr \, d\theta$$

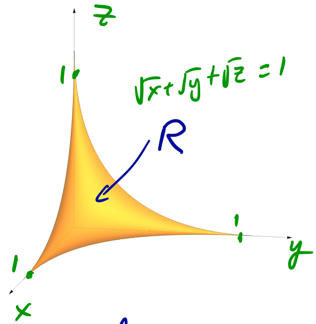


Problem 4. Find the volume of the region bounded by the surface

$$\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$$

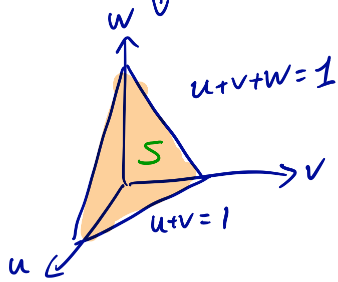
$0 \le x \le 1$

and the coordinate planes.



$$\begin{cases} x = u^2 \\ y = v^2 \\ z = w^2 \end{cases}$$

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,v)}\right| = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw$$

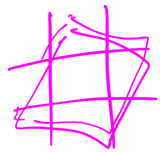


$$V = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} 8uvw \quad dw\, dv\, du$$

$$= \cdots\cdots$$

$\langle a_1, a_2, a_3 \rangle$

Problem 5. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are constant vectors, $\mathbf{r} = \langle x, y, z \rangle$ is the position vector, and E is the region bounded by (parallelepiped)

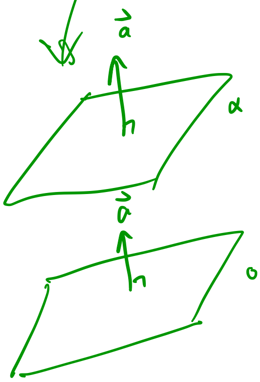


$$0 \le \mathbf{a} \cdot \mathbf{r} \le \alpha, \qquad 0 \le \mathbf{b} \cdot \mathbf{r} \le \beta, \qquad 0 \le \mathbf{c} \cdot \mathbf{r} \le \gamma$$

$0 \le a_1x + a_2y + a_3z \le \alpha$

show that

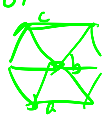
$$\iiint_E (\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})(\mathbf{c} \cdot \mathbf{r})dV = \frac{(\alpha\beta\gamma)^2}{8|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}$$



$$\begin{cases} u = a_1x + a_2y + a_3z = \vec{a} \cdot \vec{r} \\ v = b_1x + b_2y + b_3z = \vec{b} \cdot \vec{r} \\ w = \quad \cdots \quad = \vec{c} \cdot \vec{r} \end{cases}$$

Jacobian : $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ (this is volume of

$= |\vec{a} \cdot (\vec{b} \times \vec{c})|$



$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{|\vec{a} \cdot (\vec{b} \times \vec{c})|}$$

$$\int_0^\alpha \int_0^\beta \int_0^\gamma uvw \cdot \frac{dw\,dv\,du}{|\vec{a} \cdot (\vec{b} \times \vec{c})|} = \frac{\alpha^2}{2} \frac{\beta^2}{2} \frac{\gamma^2}{2} \, |\vec{a} \cdot (\vec{b} \times \vec{c})|$$