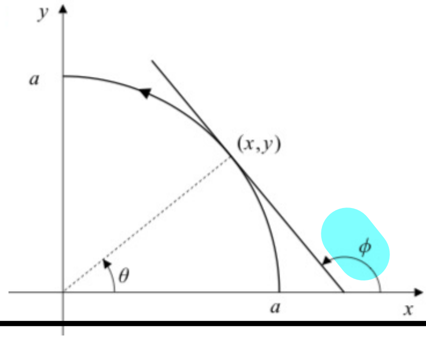
3. In terms of the angle between the tangent line and the positive x-axis, oriented counterclockwise

Remark: this angle means:
(ϕ)

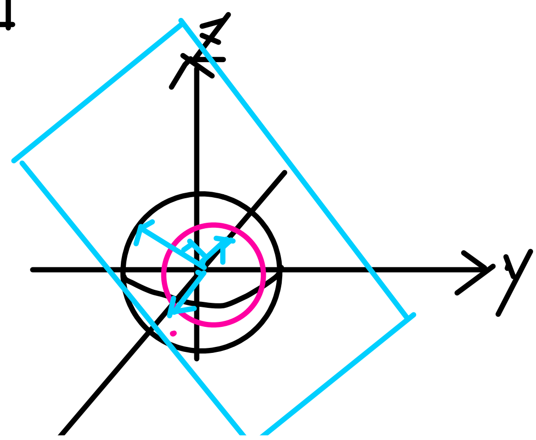


11. The plane $z = 1 + x$ intersects the cone $z^2 = x^2 + y^2$ in a parabola. Try to parametrize the parabola using as parameter:

(a) $t = x$, (b) $t = y$, and (c) $t = z$.
Which of these choices for t leads to a single parametrization that represents the whole parabola? What is that parametrization? What happens with the other two choices?

Remark: 如果計算過程當中出現 $x^2/y^2/z^2 = f(t)$，那麼便需要 $x = \pm\sqrt{f(t)}$，因此需要 2 different parametrization.

12. The plane $x + y + z = 1$ intersects the sphere $x^2 + y^2 + z^2 = 1$ in a circle $\mathscr{C}$. Find the centre $\mathbf{r}_0$ and radius r of $\mathscr{C}$. Also find two perpendicular unit vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ parallel to the plane of $\mathscr{C}$. (*Hint:* To be specific, show that $\hat{\mathbf{v}}_1 = (\mathbf{i} - \mathbf{j})/\sqrt{2}$ is one such vector; then find a second that is perpendicular to $\hat{\mathbf{v}}_1$.) Use your results to construct a parametrization of $\mathscr{C}$.



Solving strategies: centre:(1/3,1/3,1/3)
radius就求其sub一點計與centre的distance
Find v1(v1$\cdot\vec{n}=0$) ,v2= v1$\times\vec{n}$, C= r_{centre} + radius $(\cos t\ \vec{v_1} + \sin t\ \vec{v_2})$

17. Find the length of the conical helix given by the parametrization $\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t\,\mathbf{k}$, $(0 \le t \le 2\pi)$. Why is the curve called a conical helix?

17. $\mathbf{r} = t\cos t\mathbf{i} + t\sin t\mathbf{j} + t\mathbf{k}, \qquad 0 \le t \le 2\pi$

$\mathbf{v} = (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} + \mathbf{k}$

$v = |\mathbf{v}| = \sqrt{(1+t^2)+1} = \sqrt{2+t^2}$.

The length of the curve is

$$L = \int_0^{2\pi} \sqrt{2+t^2}\,dt \quad \text{Let } t = \sqrt{2}\tan\theta$$
$$dt = \sqrt{2}\sec^2\theta\,d\theta$$
$$= 2\int_{t=0}^{t=2\pi} \sec^3\theta\,d\theta$$
$$= \left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_{t=0}^{t=2\pi}$$
$$= \frac{t\sqrt{2+t^2}}{2} + \ln\left(\frac{\sqrt{2+t^2}}{\sqrt{2}} + \frac{t}{\sqrt{2}}\right)\Bigg|_0^{2\pi}$$
$$= \pi\sqrt{2+4\pi^2} + \ln\left(\sqrt{1+2\pi^2} + \sqrt{2}\pi\right) \text{ units.}$$

Integration Technique: trigo sub中不一定馬上change upper and lower base, 可以化trigo function做原本variable再sub

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18. Describe the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the elliptic cylinder $x^2 + 2z^2 = 1$. Find the total length of this intersection curve.

Parametrization Technique: For ellipse: $x^2 + \frac{z^2}{(\frac{1}{\sqrt{2}})^2} = 1$

$x = \cos t, \quad z = \frac{1}{\sqrt{2}}\sin t$

---

20. Find the length of the piecewise smooth curve $\mathbf{r} = t^3\mathbf{i} + t^2\mathbf{j}$, $(-1 \le t \le 2)$.

Remark: 要分 $\int_{-1}^{0} (-t)\sqrt{9t^2+4}\,dt + \int_0^2 t\sqrt{9t^2+4}\,dt$

21. Describe the piecewise smooth curve $\mathscr{C} = \mathscr{C}_1 + \mathscr{C}_2$, where $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j}$, $(0 \le t \le 1)$, and $\mathbf{r}_2(t) = (1-t)\mathbf{i} + (1+t)\mathbf{j}$, $(0 \le t \le 1)$.

Remark: C是兩段不同 line segment 加起，從 (0,0) 去 (1,1) 再到 (0,2)

---

27. Let $\mathbf{r} = \mathbf{r}_1(t)$, $(a \le t \le b)$, and $\mathbf{r} = \mathbf{r}_2(u)$, $(c \le u \le d)$, be two parametrizations of the same curve $\mathscr{C}$, each one-to-one on its domain and each giving $\mathscr{C}$ the same orientation (so that $\mathbf{r}_1(a) = \mathbf{r}_2(c)$ and $\mathbf{r}_1(b) = \mathbf{r}_2(d)$). Then for each t in $[a, b]$ there is a unique $u = u(t)$ such that $\mathbf{r}_2(u(t)) = \mathbf{r}_1(t)$. Show that

$$\int_a^b \left| \frac{d}{dt}\mathbf{r}_1(t) \right| dt = \int_c^d \left| \frac{d}{du}\mathbf{r}_2(u) \right| du,$$

and thus that the length of $\mathscr{C}$ is independent of parametrization.

Solving Technique: Chain Rule: $\frac{d}{dt} r_1(t) = \frac{d}{du} r_2(u) \frac{du}{dt}$

Integrating both dist. $\Rightarrow \int \left| \frac{d}{dt} r_1(t) \right| dt = \int \left| \frac{d}{du} r_2(u) \frac{du}{dt} \right| dt$

---

28. If the curve $\mathbf{r} = \mathbf{r}(t)$ has continuous, nonvanishing velocity $\mathbf{v}(t)$ on the interval $[a, b]$, and if t_0 is some point in $[a, b]$, show that the function

$$s = g(t) = \int_{t_0}^{t} |\mathbf{v}(u)|\, du$$

is an increasing function on $[a, b]$ and so has an inverse:

$$t = g^{-1}(s) \iff s = g(t).$$

Hence, show that the curve can be parametrized in terms of arc length measured from $\mathbf{r}(t_0)$.

Solution: increasing function: just $\frac{ds}{dt} = g'(t) = |v(t)| > 0$.

$$\vec{r} = \vec{r_2}(s) = \vec{r}(g^{-1}(s))$$

Made with Goodnotes

EXERCISES 11.3

In Exercises 1–4, find the required parametrization of the first quadrant part of the circular arc $x^2 + y^2 = a^2$.

1. In terms of the y-coordinate, oriented counterclockwise
2. In terms of the x-coordinate, oriented clockwise
3. In terms of the angle between the tangent line and the positive x-axis, oriented counterclockwise

??

4. In terms of arc length measured from $(0, a)$, oriented clockwise
5. The cylinders $z = x^2$ and $z = 4y^2$ intersect in two curves, one of which passes through the point $(2, -1, 4)$. Find a parametrization of that curve using $t = y$ as parameter.
6. The plane $x + y + z = 1$ intersects the cylinder $z = x^2$ in a parabola. Parametrize the parabola using $t = x$ as parameter.

In Exercises 7–10, parametrize the curve of intersection of the given surfaces. *Note*: The answers are not unique.

7. $x^2 + y^2 = 9$ and $z = x + y$
8. $z = \sqrt{1 - x^2 - y^2}$ and $x + y = 1$
9. $z = x^2 + y^2$ and $2x - 4y - z - 1 = 0$
10. $yz + x = 1$ and $xz - x = 1$
11. The plane $z = 1 + x$ intersects the cone $z^2 = x^2 + y^2$ in a parabola. Try to parametrize the parabola using as parameter: (a) $t = x$, (b) $t = y$, and (c) $t = z$. Which of these choices for t leads to a single parametrization that represents the whole parabola? What is that parametrization? What happens with the other two choices?
12. The plane $x + y + z = 1$ intersects the sphere $x^2 + y^2 + z^2 = 1$ in a circle $\mathcal{C}$. Find the centre $\mathbf{r}_0$ and radius r of $\mathcal{C}$. Also find two perpendicular unit vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ parallel to the plane of $\mathcal{C}$. (*Hint:* To be specific, show that $\hat{\mathbf{v}}_1 = (\mathbf{i} - \mathbf{j})/\sqrt{2}$ is one such vector; then find a second that is perpendicular to $\hat{\mathbf{v}}_1$.) Use your results to construct a parametrization of $\mathcal{C}$.
13. Find the length of the curve $\mathbf{r} = t^2\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from $t = 0$ to $t = 1$.
14. For what values of the parameter λ is the length $s(T)$ of the curve $\mathbf{r} = t\mathbf{i} + \lambda t^2\mathbf{j} + t^3\mathbf{k}$, $(0 \le t \le T)$ given by $s(T) = T + T^3$?
15. Express the length of the curve $\mathbf{r} = at^2\,\mathbf{i} + bt\,\mathbf{j} + c\ln t\,\mathbf{k}$, $(1 \le t \le T)$, as a definite integral. Evaluate the integral if $b^2 = 4ac$.
16. Describe the parametric curve $\mathcal{C}$ given by

$$x = a\cos t\sin t, \qquad y = a\sin^2 t, \qquad z = bt.$$

What is the length of $\mathcal{C}$ between $t = 0$ and $t = T > 0$?

17. Find the length of the conical helix given by the parametrization $\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t\,\mathbf{k}$, $(0 \le t \le 2\pi)$. Why is the curve called a conical helix?
18. Describe the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the elliptic cylinder $x^2 + 2z^2 = 1$. Find the total length of this intersection curve.
19. Let $\mathcal{C}$ be the curve $x = e^t\cos t$, $y = e^t\sin t$, $z = t$ between $t = 0$ and $t = 2\pi$. Find the length of $\mathcal{C}$.
20. Find the length of the piecewise smooth curve $\mathbf{r} = t^3\mathbf{i} + t^2\mathbf{j}$, $(-1 \le t \le 2)$.
21. Describe the piecewise smooth curve $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$, where $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j}$, $(0 \le t \le 1)$, and $\mathbf{r}_2(t) = (1 - t)\mathbf{i} + (1 + t)\mathbf{j}$, $(0 \le t \le 1)$.
22. A cable of length L and circular cross-section of radius a is wound around a cylindrical spool of radius b with no overlapping and with the adjacent windings touching one another. What length of the spool is covered by the cable?

In Exercises 23–26, reparametrize the given curve in the same orientation in terms of arc length measured from the point where $t = 0$.

23. $\mathbf{r} = At\mathbf{i} + Bt\mathbf{j} + Ct\mathbf{k}, \qquad (A^2 + B^2 + C^2 > 0)$
24. $\mathbf{r} = e^t\mathbf{i} + \sqrt{2}t\mathbf{j} - e^{-t}\mathbf{k}$
25. $\mathbf{r} = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j} + b\cos 2t\,\mathbf{k}, \qquad (0 \le t \le \frac{\pi}{2})$
26. $\mathbf{r} = 3t\cos t\,\mathbf{i} + 3t\sin t\,\mathbf{j} + 2\sqrt{2}t^{3/2}\mathbf{k}$
27. Let $\mathbf{r} = \mathbf{r}_1(t)$, $(a \le t \le b)$, and $\mathbf{r} = \mathbf{r}_2(u)$, $(c \le u \le d)$, be two parametrizations of the same curve $\mathcal{C}$, each one-to-one on its domain and each giving $\mathcal{C}$ the same orientation (so that $\mathbf{r}_1(a) = \mathbf{r}_2(c)$ and $\mathbf{r}_1(b) = \mathbf{r}_2(d)$). Then for each t in $[a, b]$ there is a unique $u = u(t)$ such that $\mathbf{r}_2(u(t)) = \mathbf{r}_1(t)$. Show that

$$\int_a^b \left|\frac{d}{dt}\mathbf{r}_1(t)\right| dt = \int_c^d \left|\frac{d}{du}\mathbf{r}_2(u)\right| du,$$

and thus that the length of $\mathcal{C}$ is independent of parametrization.

28. If the curve $\mathbf{r} = \mathbf{r}(t)$ has continuous, nonvanishing velocity $\mathbf{v}(t)$ on the interval $[a, b]$, and if t_0 is some point in $[a, b]$, show that the function

$$s = g(t) = \int_{t_0}^t |\mathbf{v}(u)|\, du$$

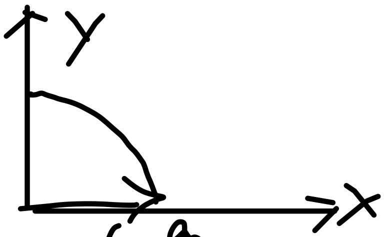
is an increasing function on $[a, b]$ and so has an inverse:

$$t = g^{-1}(s) \iff s = g(t).$$

Hence, show that the curve can be parametrized in terms of arc length measured from $\mathbf{r}(t_0)$.

In Exercises 1–4, find the required parametrization of the first quadrant part of the circular arc $x^2 + y^2 = a^2$.

1. In terms of the y-coordinate, oriented counterclockwise
2. In terms of the x-coordinate, oriented clockwise



$t=0.$

$y=t,$

$x^2=a^2-t^2$

$x=\sqrt{a^2-t^2}$

$\langle \sqrt{a^2-t^2}, t \rangle$

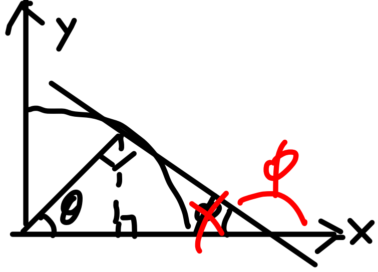
$r(y) = \sqrt{a^2-y^2}\, i + y j$ $\quad 0 \le y \le a.$

2. $y^2 = a^2 - x^2$

$r(x) = x i + \sqrt{a^2-x^2}\, j$ $\quad 0 \le x \le a.$

3. In terms of the angle between the tangent line and the positive x-axis, oriented counterclockwise

4. In terms of arc length measured from $(0, a)$, oriented clockwise

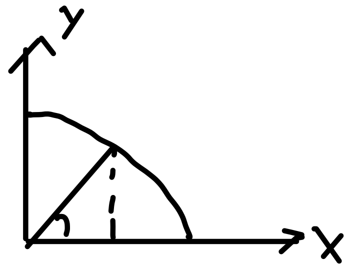


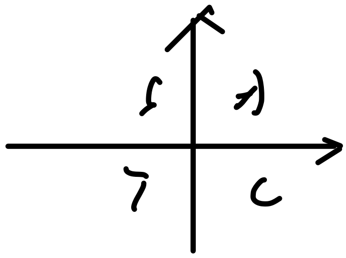
$$r(\theta) = a\cos\theta\, i + a\sin\theta\, j$$

$$= a\cos(90^\circ - \phi)\, i + a\sin(90^\circ - \phi)\, j$$

$$= a\sin\phi\, i \cancel{+} a\cos\phi\, j$$

$$a\sin\phi\, i - a\cos\phi\, j$$





4. $$\text{arc length} = \int_0^t \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$x^2 + y^2 = a^2$$

$$x = \sqrt{a^2 - y^2}$$

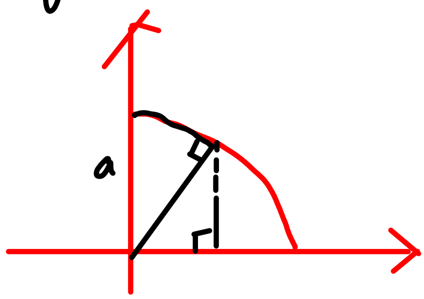
$$\frac{dx}{dy} = \frac{1}{2}(a^2 - y^2)^{-\frac{1}{2}}(-2y)$$

$$= \frac{-y}{\sqrt{a^2 - y^2}}$$

$$= \int_0^t \sqrt{1 + \frac{y^2}{a^2 - y^2}}\, dy$$

$$= \int_0^t \sqrt{\frac{a^2}{a^2 - y^2}}\, dy$$

$$= \int_0^t \frac{a}{\sqrt{a^2 - y^2}}\, dy$$



$a\int_0^t \frac{1}{\sqrt{a^2-y^2}}\,dy$ let $y=a\sin\theta$, $dy=a\cos\theta$

$\int_0^{\sin^{-1}\frac{t}{a}} dy$

$s=a\sin^{-1}\left(\frac{t}{a}\right)y$

$\frac{s}{a}\frac{1}{\sin^{-1}\left(\frac{t}{a}\right)}$

Made with Goodnotes

5. The cylinders $z = x^2$ and $z = 4y^2$ intersect in two curves, one of which passes through the point $(2, -1, 4)$. Find a parametrization of that curve using $t = y$ as parameter.

Let $t = y$, $z = 4t^2$, $4t^2 = x^2$, $x = \pm 2t$

When the curve $\vec{r}(t)$ pass through $(2, -1, 4)$,

$t = -1$.

$\therefore x = -2t$ in this case.

$\vec{r}(t) = -2t\,i + y\,j + 4t^2 k$

$t\,j$

6. The plane $x + y + z = 1$ intersects the cylinder $z = x^2$ in a parabola. Parametrize the parabola using $t = x$ as parameter.

$x + y + x^2 = 1$

when $t = x$, $z = t^2$

$t + y + t^2 = 1$

$y = 1 - t - t^2$

parabola: $\vec{r}(t) = (t\,i + (1 - t - t^2)j + t^2 k)$

Made with Goodnotes

In Exercises 7–10, parametrize the curve of intersection of the given surfaces. *Note*: The answers are not unique.

7. $x^2 + y^2 = 9$ and $z = x + y$

8. $z = \sqrt{1 - x^2 - y^2}$ and $x + y = 1$

Let $x = 3\cos\theta$, $y = 3\sin\theta$, $z = 3(\sin\theta + \cos\theta)$

$$\vec{r}(\theta) = 3\cos\theta\, i + 3\sin\theta\, j + 3(\sin\theta + \cos\theta)\, k$$

8. $x = 1 - y$ $\quad z = \sqrt{1-(1-y)^2-y^2}$

$$= \sqrt{1-(1-2y+y^2)-y^2}$$

$$= \sqrt{1-1+2y-y^2-y^2}$$

$$= \sqrt{2y-2y^2}$$

$$= \sqrt{(2y)(1-y)}$$

$$\vec{r}(t) = (1-t)\, i + t\, j + \sqrt{2t(1-t)}\, k$$

9. $z = x^2 + y^2$ and $2x - 4y - z - 1 = 0$

10. $yz + x = 1$ and $xz - x = 1$

$$2x - 4y - x^2 - y^2 - 1 = 0$$

$$-x^2 + 2x - 1 - y^2 - 4y = 0$$

$$-(x^2 - 2x + 1) - (y^2 + 4y) = 0$$

$$-(x-1)^2 - (y^2 + 4y + 4) + 4 = 0$$

$$-(x-1)^2 - (y+2)^2 + 4 = 0$$

$$(x-1)^2 + (y+2)^2 = 4$$

$$x = 2\cos\theta + 1$$

$$y = 2\sin\theta - 2$$

$$z = (2\cos\theta + 1)^2 + (2\sin\theta - 2)^2$$

$$z = 4\cos^2\theta + 4\cos\theta + 1 + 4\sin^2\theta - 8\sin\theta + 4$$

$$z = 4\cos\theta - 8\sin\theta + 9.$$

$$\vec{r}(\theta) = (2\cos\theta + 1)i + (2\sin\theta - 2)j + (4\cos\theta - 8\sin\theta + 9)k$$

10. $yz + x = 1$ and $xz - x = 1$

$$x = 1 - yz$$
$$x(z-1) = 1$$
$$z = \frac{1}{x} + 1$$
$$x = 1 - y\left(\frac{1}{x} + 1\right)$$
$$x = 1 - \frac{y}{x} - y$$
$$x - 1 = -\frac{y}{x} - y$$
$$1 - x = y\left(\frac{1}{x} + 1\right)$$
$$\frac{1-x}{\frac{1}{x}+1} = y$$
$$y = \frac{1-x}{\frac{1+x}{x}}$$
$$y = \frac{x - x^2}{1+x}$$
$$\vec{r}(t) = t\,i + \frac{t-t^2}{1+t}\,j + \left(\frac{1}{t} + 1\right)k$$

11. The plane $z = 1 + x$ intersects the cone $z^2 = x^2 + y^2$ in a parabola. Try to parametrize the parabola using as parameter: (a) $t = x$, (b) $t = y$ (c) $t = z$

(a) $z = 1 + t$

$$(1+t)^2 = t^2 + y^2$$
$$t^2 + 2t + 1 = t^2 + y^2 \longrightarrow y^2 = 2t + 1$$
$$\cancel{\sqrt{2t+1} = y} \qquad y = \pm\sqrt{2t+1}$$
$$\vec{r}(t) = t\,i + \sqrt{2t+1}\,j + (1+t)k$$

two parametrization required

11. The plane $z = 1 + x$ intersects the cone $z^2 = x^2 + y^2$ in a parabola. Try to parametrize the parabola using as parameter:

b). $t = y$:

$$z^2 = x^2 + t^2$$
$$(1+x)^2 = x^2 + t^2$$
$$x^2 + 2x + 1 = x^2 + t^2$$
$$2x + 1 = t^2$$
$$x = \frac{t^2 - 1}{2}$$

$$z = \frac{t^2-1}{2} + 1$$
$$z = \frac{t^2+1}{2}$$

$$\vec{r}(t) = \frac{t^2-1}{2}\,i + t\,j + \frac{t^2+1}{2}\,k$$

c). $t = z$:

$$t = 1 + x$$
$$x = t - 1$$

$$t^2 = (t-1)^2 + y^2$$
$$t^2 = t^2 - 2t + 1 + y^2$$
$$0 = -2t + 1 + y^2$$
$$2t - 1 = y^2$$
$$y = \pm\sqrt{2t-1}$$

$$\vec{r}(t) = (t-1)\,i + \sqrt{2t-1}\,j + t\,k$$

11. (a) $t = x$, (b) $t = y$, and (c) $t = z$.
Which of these choices for t leads to a single parametrization that represents the whole parabola? What is that parametrization? What happens with the other two choices? ??

12. The plane $x + y + z = 1$ intersects the sphere $x^2 + y^2 + z^2 = 1$ in a circle $\mathscr{C}$. Find the centre $\mathbf{r}_0$ and radius r of $\mathscr{C}$. Also find two perpendicular unit vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ parallel to the plane of $\mathscr{C}$. (*Hint:* To be specific, show that $\hat{\mathbf{v}}_1 = (\mathbf{i} - \mathbf{j})/\sqrt{2}$ is one such vector; then find a second that is perpendicular to $\hat{\mathbf{v}}_1$.) Use your results to construct a parametrization of $\mathscr{C}$.

centre $= \frac{1}{3}(i + j + k)$

radius $\sqrt{(\frac{1}{3} - 0)^2 + (\frac{1}{3} - 0)^2 + (\frac{1}{3} - 1)^2}$

13. Find the length of the curve $\mathbf{r} = t^2\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from $t = 0$ to $t = 1$.

$$r' = 2t\,i + 2t\,j + 3t^2\,k$$
$$|r'| = \sqrt{4t^2 + 4t^2 + 9t^4}$$
$$|r'| = \sqrt{8t^2 + 9t^4}$$

$$S = \int_0^1 \sqrt{8t^2 + 9t^4}\,dt$$
$$= \int_0^1 t\sqrt{8 + 9t^2}\,dt.$$

Let $u = 8 + 9t^2$, $du = 18t\,dt$, $dt = \frac{1}{18t}du$.

$$\frac{1}{18}\int_8^{17} \sqrt{u}\,du$$
$$= \frac{1}{18}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_8^{17}$$
$$= \frac{1}{27}\left(17^{\frac{3}{2}} - 8^{\frac{3}{2}}\right)$$
$$= \frac{1}{27}\left(\sqrt{17}^3 - \sqrt{8}^3\right)$$

er.

14. For what values of the parameter λ is the length $s(T)$ of the curve $\mathbf{r} = t\mathbf{i} + \lambda t^2\mathbf{j} + t^3\mathbf{k}$, $(0 \le t \le T)$ given by $s(T) = T + T^3$?

?? ???

15. Express the length of the curve $\mathbf{r} = at^2\,\mathbf{i} + bt\,\mathbf{j} + c\ln t\,\mathbf{k}$, $(1 \le t \le T)$, as a definite integral. Evaluate the integral if $b^2 = 4ac$.

$$r' = 2at\,i + bj + \frac{c}{t}k$$

$$|r'| = \sqrt{4a^2t^2 + b^2 + \frac{c^2}{t^2}}$$

$$s = \int_1^T \sqrt{4a^2t^2 + b^2 + \frac{c^2}{t^2}}\,dt$$

$$s = \int_1^T \sqrt{4a^2t^2 + 4ac + \frac{c^2}{t^2}}\,dt$$

4.4

$$s = \int_1^T \sqrt{\left(\frac{c}{t} + 2at\right)^2}\,dt$$

$$= \int_1^T \frac{c}{t} + 2at\,dt$$

$$= \left[c\ln t + at^2\right]_1^T$$

$$= c\ln T + aT^2 - a$$

16. Describe the parametric curve $\mathscr{C}$ given by

$$x = a\cos t\sin t, \qquad y = a\sin^2 t, \qquad z = bt.$$

helix., radius. a

What is the length of $\mathscr{C}$ between $t = 0$ and $t = T > 0$?

$$r' = a(\cos^2 t - \sin^2 t)\,i + 2a\sin t\cos t\,j + bk$$

$$r' = a(\cos 2t)\,i + a(\sin 2t)\,j + bk$$

$$|r'| = \sqrt{a^2\cos^2(2t) + a^2\sin^2(2t) + b^2}$$

$$= \sqrt{a^2+b^2}$$

$$s = \int_0^T \sqrt{a^2+b^2}\,dt$$

$$s = \left[\sqrt{a^2+b^2}\,t\right]_0^T$$

$$s = \sqrt{a^2+b^2}\,T$$

17. Find the length of the conical helix given by the parametrization $\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t\,\mathbf{k}$, $(0 \le t \le 2\pi)$. Why is the curve called a conical helix?

$$r' = (-t\sin t + \cos t)i + (t\cos t + \sin t)j + k$$

$$r' = (\cos t - t\sin t)i + (\sin t + t\cos t)j + k$$

$$|r'| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1}$$

$$|r'| = \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 1}$$

$$|r'| = \sqrt{1 + 1 + t^2}$$

$$|r'| = \sqrt{t^2+2}$$

$$s = \int_0^{2\pi} \sqrt{t^2+2}\,dt$$

$\sin^2\theta + \cos^2\theta = 1$

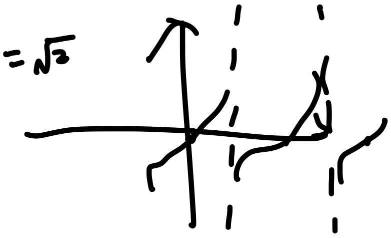
$\tan^2\theta + 1 = \sec^2\theta$

$1 + \tan^2\theta =$

Let $t = \sqrt{2}\tan\theta$, $dt = \sqrt{2}\sec^2\theta$

$$\int_0^{2\pi} \sqrt{2}\,\cancel{\sec^2\theta}\ \sec\theta \qquad \sqrt{2}\sec^2\theta$$

$t = \sqrt{2}$



$$2\int_0^{2\pi} \cancel{\sec^4\theta\,d\theta}\ \sec^3\theta$$

$\Rightarrow$ reduction formula:

$$\int^{2\pi}$$

18. Describe the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the elliptic cylinder $x^2 + 2z^2 = 1$. Find the total length of this intersection curve.

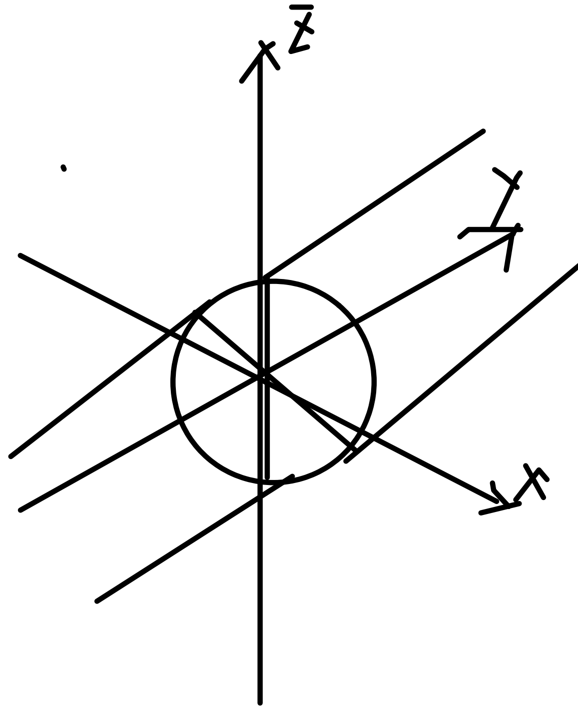
$$\begin{cases} x^2+y^2+z^2=1 \\ x^2+2z^2=1 \end{cases}$$

$$y^2-z^2=0$$

$$y^2=z^2$$

$$x^2+2y^2=1$$

$$\frac{x^2}{1}+\frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2}=1$$



$$x^2+2y^2=1$$

$$=$$

19. Let $\mathscr{C}$ be the curve $x = e^t \cos t$, $y = e^t \sin t$, $z = t$ between $t = 0$ and $t = 2\pi$. Find the length of $\mathscr{C}$.

20. Find the length of the piecewise smooth curve $\mathbf{r} = t^3\mathbf{i} + t^2\mathbf{j}$, $(-1 \le t \le 2)$.

$$\vec{r}'(t) = (-e^t\sin t + e^t\cos t)\,i + (e^t\cos t + e^t\sin t)\,j + k$$

$$|\vec{r}'| = \sqrt{(-e^t\sin t + e^t\cos t)^2 + (e^t\cos t + e^t\sin t)^2 + 1}$$

$$= \sqrt{e^{2t}\sin^2 t - 2e^{2t}\sin t\cos t + e^{2t}\cos^2 t + e^{2t}\cos^2 t + 2e^{2t}\sin t\cos t + e^{2t}\sin^2 t + 1}$$

$$= \sqrt{e^{2t} + e^{2t} + 1}$$

$$= \sqrt{2e^{2t} + 1}$$

$$S = \int_0^{2\pi} \sqrt{2e^{2t}+1}\,dt$$

let $u = 2e^{2t}+1$, $du = 4e^{2t}\,dt$, $dt = \frac{1}{2(u-1)}\,du$

$u - 1 = 2e^{2t}$

$$S = \int_0^{2\pi} \sqrt{2e^{2t}+1}\, dt$$

Let $u = \sqrt{2e^{2t}+1}$, $du = 4e^{2t} dt$, $dt = \frac{1}{4e^{2t}} du$

$u - 1 = 2e^{2t}$

$$= \int_0^{2\pi} \frac{1}{2(u-1)} \sqrt{u}\, du.$$

$$= \frac{\sqrt{u}}{2u-2}$$

Let $u = \sqrt{2e^{2t}+1}$, $du = \frac{1}{2}(2e^{2t}+1)^{-\frac{1}{2}}(4e^{2t})$

$$= \frac{2e^{2t}}{\sqrt{2e^{2t}+1}} dt$$

$$dt = \frac{u}{u^2-1} du$$

$$\int_0^{2\pi} \frac{u^2}{u^2-1} du$$

$$= \frac{u^2-1+1}{u^2-1}$$

Made with Goodnotes

20. Find the length of the piecewise smooth curve $\mathbf{r} = t^3\mathbf{i} + t^2\mathbf{j}$, $(-1 \le t \le 2)$.

$$r' = 3t^2 i + 2t j$$

$$|r'| = \sqrt{9t^4 + 4t^2}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$s = \int_{-1}^{2} \sqrt{9t^4 + 4t^2}\, dt$$

$$s = \int_{-1}^{2} t\sqrt{9t^2+4}\, dt$$

$$s = \int_{-1}^{2} 3t\sqrt{t^2 + \frac{4}{9}}$$

let $t = \frac{2}{3}\tan\theta$, $dt = \frac{2}{3}\sec^2\theta\, d\theta$

$$\int_{\tan^{-1}(-\frac{3}{2})}^{\tan^{-1}3} 2\left(\frac{2}{3}\tan\theta\right)\left(\frac{2}{3}\right)\left(\sec^2\theta\right)\left(\frac{2}{3}\sec^2\theta\right) d\theta$$

$$= \int_{\tan^{-1}(-\frac{3}{2})}^{\tan^{-1}3} \frac{16}{27} \sec^4\theta \tan\theta$$

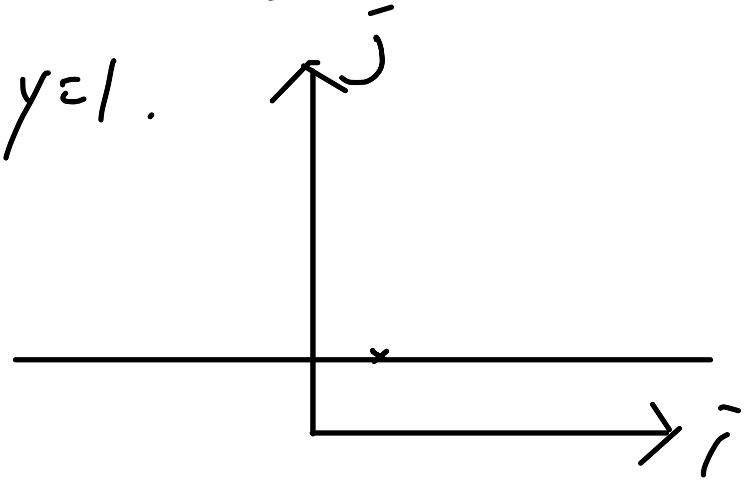
Made with Goodnotes

21. Describe the piecewise smooth curve $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2$, where $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j}$, $(0 \le t \le 1)$, and $\mathbf{r}_2(t) = (1-t)\mathbf{i} + (1+t)\mathbf{j}$, $(0 \le t \le 1)$.

$$\vec{r_1}(t) + \vec{r_2}(t) = t\vec{i} + t\vec{j} + (1-t)\vec{i} + (1+t)\vec{j}$$

$$= \vec{i} + (1+2t)\vec{j}$$

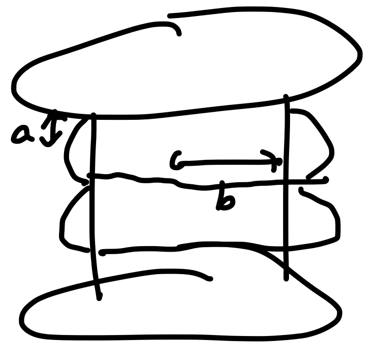
It is a straight line with $y=1$.



Made with Goodnotes

22. A cable of length L and circular cross-section of radius a is wound around a cylindrical spool of radius b with no overlapping and with the adjacent windings touching one another. What length of the spool is covered by the cable?

??



$$\vec{r}(t) \quad b\cos t\, \vec{i} + b\sin t\, \vec{j} + a t\, k$$

$$\vec{r}'(t) = -b\sin t\, \vec{i} + b\cos t\, \vec{j} + a\, k$$

$$\left|\vec{r}'(t)\right| = \sqrt{b^2 + a^2}$$

$$\int_0^L \sqrt{b^2+a^2}\, dt$$

$$= \boxed{L\sqrt{b^2+a^2}}$$

In Exercises 23–26, reparametrize the given curve in the same orientation in terms of arc length measured from the point where $t = 0$.

23. $\mathbf{r} = At\mathbf{i} + Bt\mathbf{j} + Ct\mathbf{k}, \qquad (A^2 + B^2 + C^2 > 0)$

24. $\mathbf{r} = e^t\mathbf{i} + \sqrt{2}t\mathbf{j} - e^{-t}\mathbf{k}$

23. $r' = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$

$|r'| = \sqrt{A^2 + B^2 + C^2}$

$$s = \int_0^t \sqrt{A^2+B^2+C^2}\, d\tau$$

$$s = t\sqrt{A^2+B^2+C^2}$$

$$t = \frac{s}{\sqrt{A^2+B^2+C^2}}$$

$$\vec{r} = \frac{s(A\mathbf{i} + B\mathbf{j} + C\mathbf{k})}{\sqrt{A^2+B^2+C^2}}$$

24. $\mathbf{r} = e^t\mathbf{i} + \sqrt{2}t\mathbf{j} - e^{-t}\mathbf{k}$

$$\vec{r}' = e^t i + \sqrt{2} j + e^{-t} k$$

$$|\vec{r}'| = \sqrt{e^{2t} + 2 + e^{-2t}}$$

$$|\vec{r}'| = (e^t + e^{-t})$$

sinh(t)

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|r'(t)|^3}$$

$$\kappa = \frac{\sqrt{2}e^t + \sqrt{2}e^{-t}}{(e^t + e^{-t})^3}$$

$$\kappa = \frac{\sqrt{2}}{(e^t + e^{-t})^2}$$

$$r'' = e^t i - e^{-t} k$$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} i & j & k \\ e^t & \sqrt{2} & e^{-t} \\ e^t & 0 & -e^{-t} \end{vmatrix}$$

$$= -\sqrt{2}e^{-t} i + 2j + \sqrt{2}e^t k$$

$$|r'(t) \times r''(t)| = \sqrt{2e^{-2t} + 4 + 2e^{2t}}$$

$$= (\sqrt{2}e^t + \sqrt{2}e^{-t})$$

25. $\mathbf{r} = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j} + b\cos 2t\,\mathbf{k}, \qquad (0 \le t \le \frac{\pi}{2})$

$$r' = -3a\cos^2 t\sin t\; i + 3a\sin^2 t\cos t\; j - 2b\sin 2t\; k$$

$$|r'| = \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t + 4b^2\sin^2 2t}$$

$$= \sqrt{\sin^2 t\cos^2 t\,(9a^2\cos^2 t + 9a^2\sin^2 t) + 8b^2\sin^2 t\cos^2 t}$$

$$= \sqrt{\sin^2 t\cos^2 t\,(9a^2 + 8b^2)}$$

$$S = \int_0^t \sqrt{\sin^2 \tau\cos^2 \tau\,(9a^2+8b^2)}\,d\tau$$

$$S = \sqrt{9a^2+8b^2}\int_0^t \sin\tau\cos\tau\,d\tau$$

Let $u = \sin\tau$, $du = \cos\tau\,d\tau$, $d\tau = \frac{1}{\cos\tau}du$

$$S = \sqrt{9a^2+8b^2}\int_0^{\sin t} u\,du$$

$$S = \sqrt{9a^2+8b^2}\left[\frac{u^2}{2}\right]_0^{\sin t}$$

$$S = \sqrt{9a^2+8b^2}\left(\frac{\sin^2 t}{2}\right)$$

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$$\frac{2s}{\sqrt{9a^2+8b^2}} = \sin^2 t$$

$$\sqrt{\frac{2s}{\sqrt{9a^2+8b^2}}} = \sin t$$

$$\sin^{-1}\left(\sqrt{\frac{2s}{\sqrt{9a^2+8b^2}}}\right) = t$$

Sub 2 .

26. $\mathbf{r} = 3t\cos t\,\mathbf{i} + 3t\sin t\,\mathbf{j} + 2\sqrt{2}t^{3/2}\mathbf{k}$

$$\vec{r}' = 3(-t\sin t + \cos t)\,i + 3(t\cos t + \sin t)\,j$$

$$+ \; 3\sqrt{2}\; t^{\frac{1}{2}}\; k$$

$$|\vec{r}'| = \sqrt{9(-t\sin t + \cos t)^2 + 9(t\cos t + \sin t)^2 + 18t}$$

$$= \sqrt{9(t^2\sin^2 t - 2t\sin t\cos t + \cos^2 t) + 9(t^2\cos^2 t + 2t\cos t\sin t + \sin^2 t) + 18t}$$

$$= \sqrt{9t^2 + 18t + 9}$$

$$= 3(t+1)$$

$$\vec{r} = 0$$

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$$s = 3\int_0^t \tau + 1 \, d\tau$$

$$s = 3\left[\frac{\tau^2}{2} + \tau\right]_0^t$$

$$s = 3\left(\frac{t^2}{2} + t\right)$$

$$0 = \frac{3}{2}t^2 + 3t - s$$

$$t = \frac{-3 + \sqrt{9 - 4\left(\frac{3}{2}\right)(-s)}}{2\left(\frac{3}{2}\right)}$$

$$t = \frac{-3 + \sqrt{9 + 6s}}{3}$$

$$t = \frac{-3 + 3\sqrt{1 + \frac{2}{3}s}}{3}$$

$$t = \sqrt{1 + \tfrac{2}{3}s} - 1$$

Sub $\vec{r}$ ..

27. Let $\mathbf{r} = \mathbf{r}_1(t)$, $(a \le t \le b)$, and $\mathbf{r} = \mathbf{r}_2(u)$, $(c \le u \le d)$, be two parametrizations of the same curve $\mathcal{C}$, each one-to-one on its domain and each giving $\mathcal{C}$ the same orientation (so that $\mathbf{r}_1(a) = \mathbf{r}_2(c)$ and $\mathbf{r}_1(b) = \mathbf{r}_2(d)$). Then for each t in $[a, b]$ there is a unique $u = u(t)$ such that $\mathbf{r}_2(u(t)) = \mathbf{r}_1(t)$. Show that

$$\int_a^b \left| \frac{d}{dt}\mathbf{r}_1(t) \right| dt = \int_c^d \left| \frac{d}{du}\mathbf{r}_2(u) \right| du,$$

and thus that the length of $\mathcal{C}$ is independent of parametrization.