Problem 1 (20 points) Your Score:

(a) Find the distance (in terms of $\mathbf{n}$, $\mathbf{r}_0$ and $\mathbf{r}_1$ only) from the point $\mathbf{r}_1$ to the plane $(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{n}=0$.

(b) A rigid body rotates about an axis through point O with angular velocity $\boldsymbol{\omega}$.

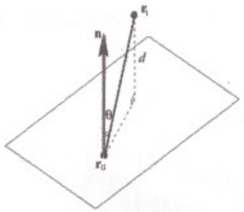
(i) Find the linear velocity $\mathbf{v}$ of a point P of the body with position vector $\mathbf{r}$.

(ii) Show that the vector $-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ is directed away from the axis of rotation and lies on the plane containing the vector $\boldsymbol{\omega}$ and $\mathbf{r}$.

Solution:

(a)

$$\begin{aligned} d &= \|\mathbf{r}_1-\mathbf{r}_0\|\ |\cos\theta| \\ &= \|\mathbf{r}_1-\mathbf{r}_0\|\cdot\|\hat{\mathbf{n}}\|\ |\cos\theta| \\ &= |(\mathbf{r}_1-\mathbf{r}_0)\cdot\hat{\mathbf{n}}| \end{aligned}$$



(b) (i) Since P travels in a circle of radius $r\sin\theta$, the magnitude of the linear velocity $\mathbf{v}$ is $\omega(r\sin\theta)=\|\boldsymbol{\omega}\times\mathbf{r}\|$. Also, $\mathbf{v}$ must be perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{r}$ and is such that $\mathbf{r}$, $\boldsymbol{\omega}$ and $\mathbf{v}$ form a right-handed system, i.e.

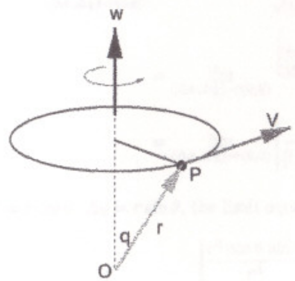
$$\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$$

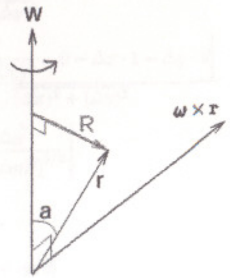
(ii)

$$\begin{aligned} [\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})]_i &= \varepsilon_{ijk}\,\omega_j(\boldsymbol{\omega}\times\mathbf{r})_k \\ &= \varepsilon_{kij}\varepsilon_{kpq}\,\omega_j\omega_p r_q \\ &= (\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp})\omega_j\omega_p r_q \\ &= \omega_j\omega_i r_j-\omega_j\omega_j r_i \\ &= (\boldsymbol{\omega}\cdot\mathbf{r})\omega_i-(\boldsymbol{\omega}\cdot\boldsymbol{\omega})r_i \\ \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) &= (\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega}-(\boldsymbol{\omega}\cdot\boldsymbol{\omega})\mathbf{r} \end{aligned}$$

i.e. the vector $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ is on the plane containing $\boldsymbol{\omega}$ and $\mathbf{r}$ since $\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ is a linear combination of $\boldsymbol{\omega}$ and $\mathbf{r}$.

From the right-hand system, we can see from the figure $\mathrm{R}=-\omega\times(\omega\times r)$ is directed away from the axis of rotation.





The direction of the centripetal force

Problem 2 (20 points) Your Score:

(a) Can the function $f(x,y)=\dfrac{\sin x\sin^3 y}{1-\cos(x^2+y^2)}$ be defined at $(0,0)$ in such a way that it becomes continuous there? If so, how?

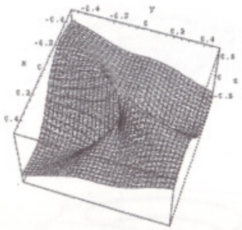
(b) Let $f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2}, & \text{if } (x,y)\neq(0,0),\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$

Calculate each of the following partial derivatives or explain why it does not exist:

(i) $f_x(0,0)$, (ii) $f_y(0,0)$, (iii) $f_{yx}(0,0)$, (iv) $f_{xy}(0,0)$ and (v) $f_{xx}(0,0)$.

Is the function $f(x,y)$ differentiable at $(0,0)$? Explain.

Solution:



(a) Along the x-axis, i.e. $y=0$, then

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{x\to 0} f(x,0)=0,$$

so the limit must be 0 if it exists at all. However, along the straight line $y=mx$, then

$$f(x,mx)=\frac{\sin x\sin^3 mx}{1-\cos(m^2+1)x^2}.$$

In particular, if $m=1$, we have

$$f(x,x)=\frac{\sin^4 x}{1-\cos(2x^2)}=\frac{\sin^4 x}{2\sin^2(x^2)}$$

$$\lim_{x\to 0} f(x,x)=\lim_{x\to 0}\frac{4\sin^3 x\cos x}{2\cdot 2\sin(x^2)\cdot 2x}=\lim_{x\to 0}\left[\frac{1}{2}\cdot\frac{\sin^3 x}{x^3}\cdot\frac{x^2}{\sin(x^2)}\cdot\cos x\right]=\frac{1}{2}$$

$\therefore\ \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist since difference paths end up different limits.

$\therefore\ f(x,y)$ cannot be defined at $(0,0)$.

(b) If $(x,y)\neq(0,0)$, we have

$$f_x(x,y)=\frac{(x^2+y^2)3x^2-x^3\cdot(2x)}{(x^2+y^2)^2}=\frac{x^4+3x^2y^2}{(x^2+y^2)^2}$$

$$f_y(x,y)=x^3(-1)(x^2+y^2)^{-2}\cdot 2y=-\frac{2x^3y}{(x^2+y^2)^2}.$$

If $(x,y)=(0,0)$, then

(i) $$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{(\Delta x)^3/[(\Delta x)^2-0]}{\Delta x}=1$$

(ii) $$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-0}{\Delta y}=0$$

(iii) $$f_{xy}(0,0)=\lim_{\Delta y\to 0}\frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0-1}{\Delta y}\ \text{does not exist}$$

(iv) $$f_{yx}(0,0)=\lim_{\Delta x\to 0}\frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0-0}{\Delta x}=0$$

(v) $$f_{xx}(0,0)=\lim_{\Delta x\to 0}\frac{f_x(\Delta x,0)-f_x(0,0)}{\Delta x}=\lim_{\Delta y\to 0}\frac{1-1}{\Delta x}=0.$$

Note that

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{|f(\Delta x,\Delta y)-f(0,0)-\Delta x f_x(0,0)-\Delta y f_y(0,0)|}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

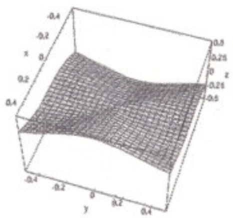
$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\left|\dfrac{(\Delta x)^3}{(\Delta x)^2+(Dy)^2}-0-\Delta x\cdot 1-\Delta y\cdot 0\right|}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\left|\frac{-\Delta x\Delta y^2}{[(\Delta x)^2+(\Delta y)^2]^{3/2}}\right|.$$

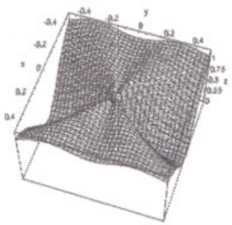
Let $\Delta x=r\cos\theta$, $\Delta y=r\sin\theta$, the limit equals

$$\left|\frac{r^3\cos\theta\sin^2\theta}{r^3}\right|=|\cos\theta\sin^2\theta|,$$

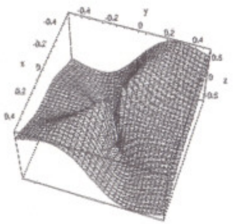
which depends on θ. So the limit does not exist and hence f is not differentiable at $(0,0)$.



$z=f(x,y)$



$z=f_x(x,y)$



$z=f_y(x,y)$

Problem 3 (20 points) Your Score: [ ]

(a) Show that the curve $\mathbf{r}(t) = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t\,\mathbf{k}$, $t \geqslant 0$, lies on the surface of the form $z = f(x,y)$. Find $f(x,y)$. Describe (or sketch) the curve.

(b) Find a vector equation of the line tangent to the graph of

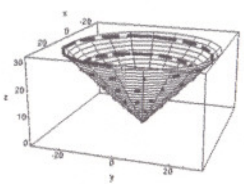
$$\mathbf{r}(t) = t^2\,\mathbf{i} - \frac{1}{t+1}\mathbf{j} + (4-t^2)\,\mathbf{k}$$

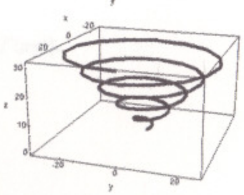
at the point $(4,1,0)$ on the curve. Find also the arc length of the curve $\mathbf{r}(t)$ from point $(4,1,0)$ to point $(0,-1,4)$.

Solution:

(a) Let

$$\begin{aligned} x &= t\cos t \\ y &= t\sin t \\ z &= t \\ \therefore\ x^2+y^2 &= t^2 = z^2 \end{aligned}$$





i.e., $z = f(x,y) = \sqrt{x^2+y^2}$

(only take +ve root since $z = t > 0$).

The curve lies on the cone $z = \sqrt{x^2+y^2}$ $(\phi = \pi/4)$.

(b)

$$\mathbf{r}'(t) = 2t\,\mathbf{i} + \frac{1}{(t+1)^2}\mathbf{j} - 2t\,\mathbf{k}.$$

At $(4,1,0)$, $t=-2$, so $\mathbf{r}'(-2) = -4\,\mathbf{i} + \mathbf{j} + 4\,\mathbf{k}$. Therefore

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_0 + t\mathbf{v} \\ &= (4,1,0) + t(-4,1,4). \end{aligned}$$

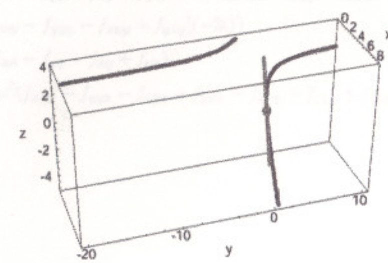
At $(0,-1,4)$, $t=0$, the required arc length is with starting point at $t=-2$ to the end point $t=0$. Note that the graph does not define at $t=-1$, hence the arc length from $t=-2$ to $t=0$ cannot be defined. Even if you don't realize that, from one variable calculus, you have some problem too in the integrand, because

$$\begin{aligned} \mathbf{r}'(t) &= 2t\,\mathbf{i} - (-1)(t+1)^{-2}\mathbf{j} + (-2t)\,\mathbf{k} \\ \|\mathbf{r}'\|^2 &= 4t^2 + \frac{1}{(t+1)^4} + 4t^2 \end{aligned}$$

and

$$s = \int_{-2}^{0} \|\mathbf{r}'(t)\|\,dt.$$

The integrand yet is not defined at $t=-1$.



Problem 4 (20 points) Your Score:

(a) Find the equation of the level curve of the function $z = g(x,y) = xf(xy)$ at the point (x_0, y_0), where both f and g are differentiable. Show that $\nabla g(x_0, y_0)$ is normal to the tangent line to the level curve at (x_0, y_0).

(b) If $w = f(x,y)$ (assume f is differentiable) and $x = s^2 + t^2$, $y = s^2 - t^2$, use the chain rule to find (i) w_s, (ii) w_{st} and (iii) w_{stt}.

Solution:

(a) The equation of the level curve at the point (x_0, y_0) is

$$xf(xy) = x_0 f(x_0 y_0).$$

Differentiation both sides of the equation wrt x, then

$$xf'(xy)\left[x\frac{dy}{dx} + y\right] + f(xy) = 0$$

$$\frac{dy}{dx} = -\frac{xyf'(xy) + f(xy)}{x^2 f'(xy)}.$$

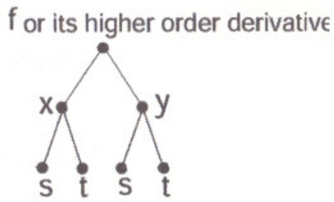
Also

$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = (xyf'(xy) + f(xy),\ x^2 f'(xy)).$$

Therefore $\left.\frac{dy}{dx}\right|_{(x_0,y_0)} \times [\text{slope of } \nabla g(x_0,y_0)] = -1,$

i.e. they must be normal to each other.

(b)



(i)
$$\begin{aligned} w_s &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial f}{\partial s} \\ &= f_x(2s) + f_y(2s) \\ &= 2s(f_x + f_y) \end{aligned}$$

(ii)
$$\begin{aligned} w_{st} &= 2s\frac{\partial}{\partial t}(f_x + f_y) \\ &= 2s\left[\frac{\partial}{\partial x}(f_x + f_y)\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}(f_x + f_y)\frac{\partial y}{\partial t}\right] \\ &= 2s[(f_{xx} + f_{yx})(2t) + (f_{xy} + f_{yy})(-2t)] \\ &= 4st[f_{xx} - f_{yy} - f_{xy} + f_{yx}] = 4stG \end{aligned}$$

where $G = f_{xx} - f_{yy} - f_{xy} + f_{yx}$.

(iii)
$$\begin{aligned} w_{stt} &= 4s(f_{xx} - f_{yy} - f_{xy} + f_{yx}) + 4st\left[\frac{\partial G}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial t}\right] \\ &= 4s(f_{xx} - f_{yy} - f_{xy} + f_{yx}) + 4st\,\{[f_{xxx} - f_{yyx} - f_{xyx} + f_{yxx}](2t) \\ &\quad + [f_{xxy} - f_{yyy} - f_{xyy} + f_{yxy}](-2t)\} \\ &= 4s(f_{xx} - f_{yy} - f_{xy} + f_{yx}) \\ &\quad + 8st^2(f_{xxx} - f_{yyx} - f_{xyx} + f_{yxx} - f_{xxy} + f_{yyy} + f_{xyy} - f_{yxy}). \end{aligned}$$

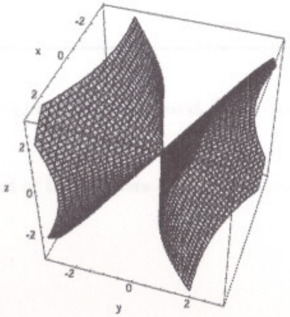
Problem 5 (20 points) Your Score:

Find the point(s) on the surface $z^2 = -\frac{1}{2}x^2 + 2y^2 + xy$ that are closest to the point $\left(-\frac{1}{2}, -3, 0\right)$

(a) by reducing the problem to an unconstrained problem in two variables, and

(b) using the method of Lagrange multipliers.

Solution:



(a)

$$\boxed{D_s = d^2 = \left(x+\frac{1}{2}\right)^2 + (y+3)^2 + z^2}$$
$$= x^2 + x + \frac{1}{4} + y^2 + 6y + 9 - \frac{1}{2}x^2 + 2y^2 + xy$$
$$= \frac{1}{2}x^2 + 3y^2 + xy + x + 6y + 9\frac{1}{4}.$$
$$\frac{\partial D_s}{\partial x} = x + y + 1$$
$$\frac{\partial D_s}{\partial y} = 6y + x + 6.$$

For critical point, $\frac{\partial D_s}{\partial x} = \frac{\partial D_s}{\partial y} = 0 \quad \Rightarrow \quad x = 0,\ y = -1.$

$$\frac{\partial^2 D_s}{\partial x^2} = 1, \quad \frac{\partial^2 D_s}{\partial y^2} = 6, \quad \frac{\partial^2 D_s}{\partial x \partial y} = 1.$$

so $D = 1 \times 6 - 1 = 5 > 0$ and $\frac{\partial^2 D_s}{\partial x^2} > 0$, therefore $(0,-1)$ is a min point.

When $x = 0$, $y = -1$, $z = \pm\sqrt{2}$, therefore the required point are $(0, -1, \pm\sqrt{2})$.

(b) Alternatively, minimize

$$D_s = f(x,y,z) = \left(x+\frac{1}{2}\right)^2 + (y+3)^2 + z^2$$

subject to

$$g(x,y,z) = z^2 + \frac{1}{2}x^2 - 2y^2 - xy = 0.$$

Then

$$\nabla f = \left(2\left(x+\frac{1}{2}\right), 2(y+3), 2z\right)$$
$$\nabla g = (x - y, -4y - x, 2z)$$

from Lagrange multipliers, $\nabla f = \lambda \nabla g$, we have

$$2\left(x+\frac{1}{2}\right) = \lambda(x-y) \tag{1}$$
$$2(y+3) = \lambda(4y - x) \tag{2}$$
$$2z = \lambda 2z \tag{3}$$

from (3), $\lambda = 1$, then from (1) and (2)

$$\begin{cases} 2x + 1 = x - y \\ 2y + 6 = -4y - x \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -1 \end{cases}.$$

From the constant equation

$$z = \pm\sqrt{2}.$$

$\therefore$ The required points are $(0, -1, \pm\sqrt{2})$.

If $z = 0$ and $\lambda \neq 1$, then we need to solve these two equations $-0.5x^2 + 2y^2 + xy = 0$ and $2x^2 - 2y^2 - 6xy - 10y + 7x = 0$. By using some numerical methods!!!, you should obtain three points $(x, y) = (0,0)$, $(-2.721, 2, 201)$ and $(86.721, 26.798)$ to satisfy the above two equations. Compute the distance from $(-0.5, -3)$ to these three points and we find that they are not the shortest distance.