MATH2023
Tutorial 4
March 4, 2019 Tutorial Notes by Ivan So

1 Review

- The **tangent plane** is the analogy of tangent line in single variable calculus, explicitly it is the first order approximation by partial derivatives given by:

$$P(\mathbf{x}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} f_{x_i}(\mathbf{x}_0)\Delta x_i.$$

  The idea of **total differential** ($df = \sum_{i=1}^{n} f_{x_i}\Delta x_i$) is derived based on linear approximation.

  - **Theorem**: Normal vector of the surface defined by $x_{n+1} = f(\mathbf{x})$ is $(f_{x_1}, \cdots, f_{x_n}, -1)$.

- FYI: In analytical aspects, a function $f : \mathbb{R}^n \to \mathbb{R}$ is **first order differentiable** if for any $\epsilon > 0$, there exist δ_ϵ such that $\|\mathbf{x} - \mathbf{x}_0\| < \delta_\epsilon \implies |f(\mathbf{x}) - f(\mathbf{x}_0) - P(\mathbf{x})| < \epsilon \|\mathbf{x} - \mathbf{x}_0\|$.

- The **directional derivative** of $f(\mathbf{x})$ in the direction of $\hat{\mathbf{v}}$ by definition is

$$D_{\hat{\mathbf{v}}} f(\mathbf{x}) := \lim_{t\to 0} \frac{f(\mathbf{x} + t\hat{\mathbf{v}}) - f(\mathbf{x})}{t}$$

  It represent the derivative of the curve of cross section if we"cut" the surface from above by the line passing through the origin and in the direction of $\hat{\mathbf{v}}$.

- The **gradient operator** is an operator which maps a function into a vector by

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right).$$

  Indeed, the directional derivative can be rewritten as:

$$D_{\hat{\mathbf{v}}} f(\mathbf{x}) = \nabla f \cdot \hat{\mathbf{v}}$$

- Suppose $\mathbf{x} \in \mathbb{R}^n$ are set of variables which depends on $\mathbf{t} \in \mathbb{R}^m$, then the **chain rule** in multivariable case is given by

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \mathbf{x}}{\partial t_i}.$$

  we can draw *tree diagram* for the chain relation.

- Given the relation $F(\mathbf{x}) = C$, we can find the dependence of x_j on x_i by **implicit differentiation**. The process of implicit differentiation is carried as follows:

$$p(n) = f(\vec{x_0}) + \sum_{i=1}^{n} f_{x_i}(\vec{x_0})\,\Delta x_i$$

$$\underbrace{\overbrace{p(n)}^{z} - \underbrace{f(\vec{x_0})}_{z_0}}_{\Delta z} = \sum_{i=1}^{n} f_{x_i}(\vec{x_0})\,\Delta x_i$$

$$\Delta z = \sum_{i=1}^{n} f_{x_i}(\vec{x_0})\,\Delta x_i$$

$$0 = -\Delta z + \sum_{i=1}^{n} f_{x_i}(\vec{x_0})\,\Delta x_i$$

$$0 = \left(f_{x_1}(\vec{x_0}), f_{x_2}(\vec{x_0}), \dots, f_{x_n}(\vec{x_0}), -1\right) \cdot \left(\Delta x_1, \Delta x_2, \dots, \Delta x_n, \Delta z\right)$$

$f(y(x))$, then $\dfrac{d f(y(x))}{dx} = f'(y(x)) \cdot \dfrac{dy}{dx}$

i-th component of $\vec{x}$ depends on $\vec{t} = (t_1, \cdots, t_m)$

f: depends on $\vec{x}$, then $\dfrac{\partial f}{\partial t_i} = \sum_{j=1}^{n} \dfrac{\partial f}{\partial x_j} \dfrac{\partial x_j}{\partial t_i}$

$$= \nabla f \cdot \frac{\partial \vec{x}}{\partial t_i}$$

Implicit Differentiation:

if $\underbrace{F(\vec{x}) = c}_{\text{constraint}}$, $\vec{x}$ variable in $\mathbb{R}^n$, then

$\Rightarrow x_n$ depends on $x_1, \dots, x_{n-1}$

$$\frac{\partial}{\partial x_1} F(\vec{x}) = 0$$

$$\sum_{j=1}^{n} \frac{\partial F(\vec{x})}{\partial x_j} \frac{\partial x_j}{\partial x_1} = 0$$

$$\Rightarrow \sum_{j=1}^{n-1} \frac{\partial F(\vec{x})}{\partial x_j} + \frac{\partial F(\vec{x})}{\partial x_n} \frac{\partial x_n}{\partial x_1} = 0$$

1. Take the partial derivative $F(\mathbf{x}) = C$ with respect to x_i, then we obtain the relation $\nabla F \cdot \frac{\partial \mathbf{x}}{\partial x_i} = 0$.
2. Find the expression $\nabla F \cdot \frac{\partial \mathbf{x}}{\partial x_i} = 0$ with $\frac{\partial x_j}{\partial x_i}$ on left hand side.
3. Integrate the expression of $\frac{\partial x_j}{\partial x_i}$ with respect to x_i.

2 Problems

1. True or False

   (a) True or False. If $f(x,y) = \ln y$, then $\nabla f(x,y) = 1/y$.

   False. $\because$ Gradient operator maps a function to a vector

   (b) Give the rationale for ∇f being the direction of steepest ascent/descent.

   $D_{\hat{v}} f(\vec{x}) = \nabla f \cdot \hat{v}$

   $D_{\hat{v}} f(\vec{x}) = |\nabla f| |\hat{v}| \cdot \cos\theta \Rightarrow$ max/min $\theta = 0$ or π. $\Rightarrow \nabla f = \hat{v}$.

2. Find $\frac{\partial f(x/y)}{\partial y}$ and $\frac{\partial f(x/y)}{\partial x}$.

   $f(\frac{x}{y}) := f(u)$

   $\Rightarrow \frac{\partial f(u)}{\partial y} = f'(u) \cdot u_y = \frac{-x}{y^2} f'(\frac{x}{y})$

3. Find the tangent plane of the surface $f(x,y) = \frac{1}{x^2+y^2+1}$ at $(1,1,1/3)$.

   $\vec{n} = (f_x, f_y, -1)$

   $= \left(-\frac{2x}{(x^2+y^2+1)^2}, \frac{-2y}{(x^2+y^2+1)^2}, -1\right) = \left(-\frac{2}{9}, -\frac{2}{9}, -1\right)$

   $2x + 2y + 9z = 7$

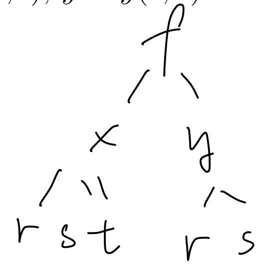
4. Find the directional derivative of $f(x,y) = \frac{1}{x^2+y^2+1}$ in the direction of $(1,1)$ at $(1,1)$.

   $\nabla f(1,1) = \left(-\frac{2}{9}, -\frac{2}{9}\right)$

   $\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

   $D_u f(1,1) = \left(-\frac{2}{9}, -\frac{2}{9}\right) \cdot \frac{1}{\sqrt{2}}(1,1) = -\frac{2\sqrt{2}}{9}$

5. Draw the tree diagram for $u = f(x, y)$, where $x = x(r, s, t)$, $y = y(r, s)$.



6. Find $\frac{\partial z}{\partial x}$ for z satisfying $xyz = \cos(x + y + z)$.

$$\frac{\partial}{\partial x}(xyz) = \frac{\partial}{\partial x}\cos(x+y+z)$$

$$yz + xyz_x = -\sin(x+y+z)\cdot(1+z_x)$$

$$xyz_x + \sin(x+y+z)z_x = -\sin(x+y+z) - yz$$

7. If $z = f(x - y)$, show that $z_x + z_y = 0$.

$$z_x\left(xy + \sin(x+y+z)\right) = -yz - \sin(x+y+z)$$

$$z_x = -\frac{yz - \sin(x+y+z)}{xy + \sin(x+y+z)}$$

7. If $z = f(x-y)$, show $z_x + z_y = 0$.

$$z = f(x-y) := f(u)$$

$$z_x + z_y = f'(u)\frac{\partial}{\partial x}(x-y) + f'(u)\frac{\partial}{\partial y}(x-y)$$

$$= f'(u) - f'(u)$$

$$= 0$$