MATH2023
Tutorial 11
May 9, 2019 Tutorial Notes by Ivan So

1 Review

- The **triple integral** is defined as

$$\int\int\int_D f(x,y,z)dV := \lim_{n\to\infty}\sum_{i,j,k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_i.$$

  Interpretation: See triple integration as the calculation of "mass" of an object. Think of the function f as the "density" function, then $f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_i$ is the mass of part of the object. Then $\sum_{i,j,k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_i$ will approximate the "mass" of the whole object.
  Other examples:

  1. See f as the charge density of an object, then $\int\int\int_D f dV$ calculate the amount of charge of the object D.
  2. See f as the function representing the density of air molecules, then $\int\int\int_D f dV$ calculate the number of molecules in region D.

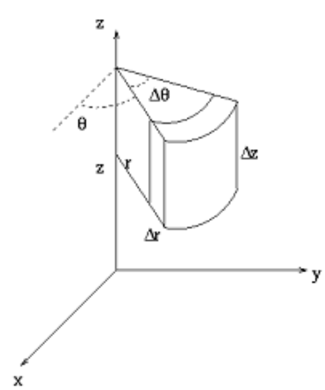
- The **Fubini's theorem** applies in 3D (less technically, the order of integration can be changed).

- Alternative coordinate systems:

  – **Cylindrical Coordinates**: We redefine our coordinate system by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

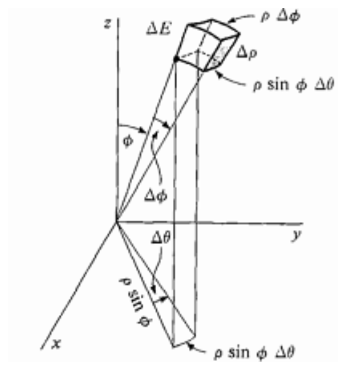
  The *differential volume* is given by $dV = rdrd\theta dz$. Pictorially, the following represent the object with volume dV:



– **Spherical Coordinates**: We redefine our coordinate system by

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

The *differential volume* is given by $dV = \rho^2 \sin\phi d\rho d\phi d\theta$. Pictorially, the following represent the object with volume dV:



– Change of variable in general: Change of variable can be done by inserting **Jacaobian** in the integration. Explicitly,

$$\int \cdots \int_R f(\mathbf{x}) dR = \int \cdots \int_D f(\mathbf{x}(\mathbf{y})) \det \frac{\partial \mathbf{x}}{\partial \mathbf{y}} d^n \mathbf{y}.$$

Simply, for our consideration:

* In 2D: We can think of change of variable as "no longer integrating over a flat surface", i.e. a surface integral:

$$\int\int_{S_{x,y}} f(x,y) dS = \int\int_{R_{u,v}} f(x(u,v), y(u,v)) \underbrace{|\mathbf{r}_u \times \mathbf{r}_v|}_{\text{Jacobian}} dudv.$$

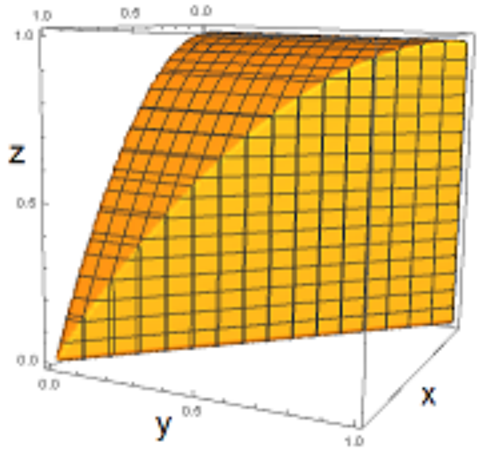
* In 3D: We approximate the differential volume with **parallelepiped** (recall tutor 1). Given the approximation process.

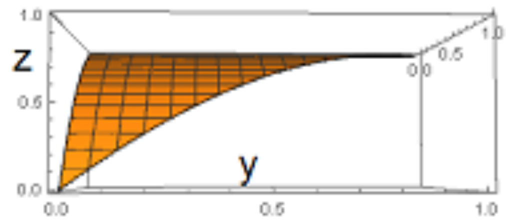
$$\begin{aligned} &\int\int\int_{D_{x,y,z}} f(x,y,z) dD \\ = &\int\int\int_{R_{u,v,w}} f(x(u,v,w), y(u,v,w), z(u,v,w)) \underbrace{|(\mathbf{r}_u \times \mathbf{r}_v) \cdot \mathbf{r}_w|}_{\text{Jacobian}} dudvdw \end{aligned}$$

Remark: The absolute sign of Jacobian guarantee it is always non-negative.

2 Problems

1. Rewrite the integral
$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dydzdx$$
in other five different orders.
Solution: The region of integration is as follows:





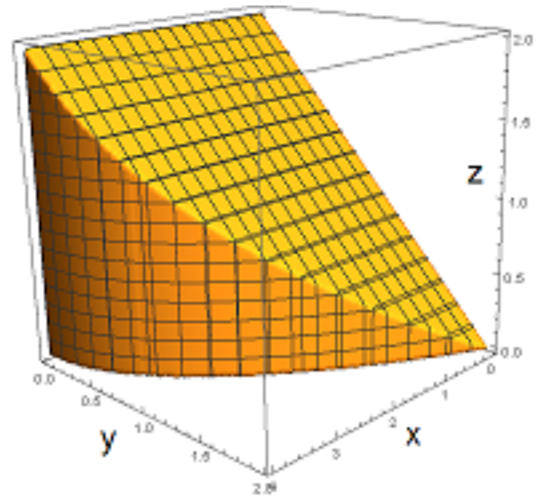
First of all lets rewrite the known constraints. $y = 1-x \leftrightarrow x = 1-y$, $1-x^2 = z \leftrightarrow x = \sqrt{1-z}$. The three easier orders are
$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f\,dydxdz,\quad \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f\,dzdydx,\quad \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f\,dzdxdy.$$
The remaining integration orders are $dxdydz$ and $dxdzdy$. For these orders, we will need to divide the yz-region by the curve $y = 1-\sqrt{1-z}$ (or $z = 1-(1-y)^2$). Then the remaining orders are
$$\int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{\sqrt{1-z}} f\,dxdydz + \int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{1-y} f\,dxdydz$$
$$\int_0^1 \int_{1-(1-y)^2}^1 \int_0^{\sqrt{1-z}} f\,dxdzdy + \int_0^1 \int_0^{1-(1-y)^2} \int_0^{1-y} f\,dxdzdy$$

2. Find the center of mass of the solid defined by $0 \le x, y, z \le a$ where the density is $\rho(x,y,z) = x^2+y^2+z^2$.
Solution: The x coordinate of the center of mass is given by
$$\overline{x} = \int_0^a \int_0^a \int_0^a x\rho\,dxdydz \Big/ \int_0^a \int_0^a \int_0^a \rho\,dxdydz \quad \text{(I forgot to divide the mass in class!)}$$
$$= \frac{7a^6}{12} \Big/ a^5 = \frac{7a}{12}$$
The y, z-coordinates are calculated in the same spirit, therefore the center of mass is: $\left(\frac{7a}{12}, \frac{7a}{12}, \frac{7a}{12}\right)$.

3. Sketch the solid whose volume is given by the integral

$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx dz dy.$$

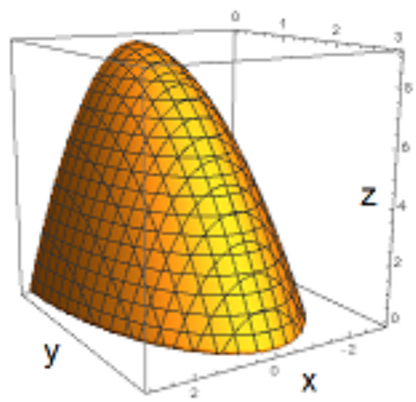
Solution:



4. Evaluate the integral

$$\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2+y^2} dz dy dx.$$

Solution: The region of integration is:



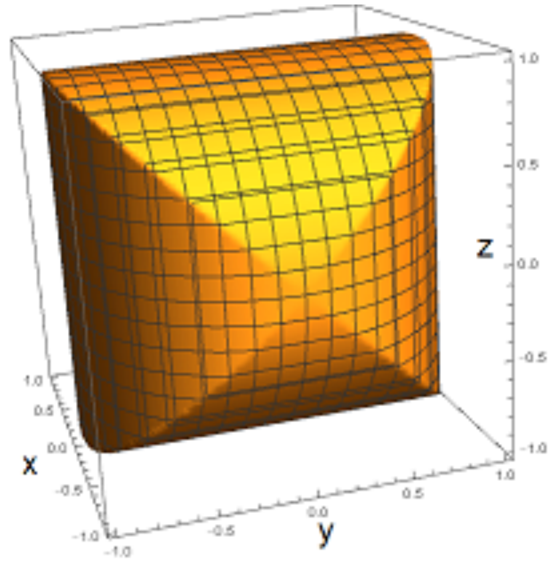
From the expressions like $x^2 + y^2$, evaluation with cylindrical coordinates will come handy. So we have the evaluation

$$\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2+y^2} dz dy dx = \int_0^{\pi} \int_0^{3} \int_0^{9-r^2} r dz dr d\theta$$

$$\pi \int_0^3 r(9-r^2) dr$$

$$= \frac{81}{4}\pi$$

5. Find the volume of the intersection solid of perpendicular cylinders.
Solution: The concerned solid is as in the following diagram.



Though it is intersection of two cylinders, it is not necessary to use cylindrical coordinates. One would see that

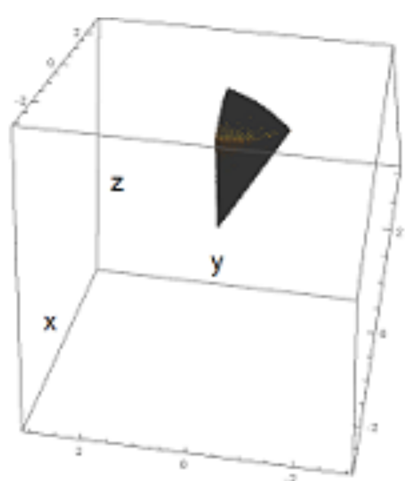
$$\int_{-r}^{r}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dzdydx = \frac{16}{3}r^3$$

is a correct integral of volume. One can check that the boundary surface are indeed those we desired for the intersection solid.

6. Sketch the solid whose volume is given by the integral

$$\int_0^{\pi/3}\int_0^{\pi/6}\int_0^{3} \rho^2 \sin\phi d\rho d\theta d\phi.$$

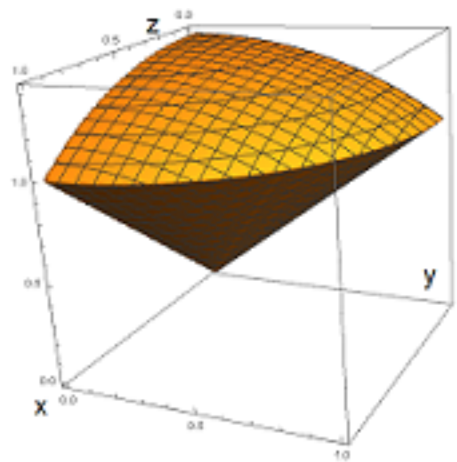
Solution:



7. Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xydzdydx.$$

Solution: The region of integration is:



The use of spherical coordinates might come handy in this case. One can check that the region is represented by the set $\{(\rho, \theta, \phi) | 0 \le \rho \le \sqrt{2}, 0 \le \theta \le \pi/4, 0 \le \phi \le \pi/2\}$. So the we can rewrite the integral as

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xydzdydx = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{2}} (\rho \sin\theta \cos\phi)(\rho \sin\theta \sin\phi)[\rho^2 \sin\theta] d\rho d\theta d\phi$$
$$= 2^{5/2} \cdot \frac{1}{2} \cdot 1 = 2^{3/2}$$

8. Rewrite the integral $\int\int\int_R xydV$ under the change of coordinates $x = v + w^2$, $y = w + u^2$, $z = u + v^3$.

Solution: We first compute the Jacobian

$$\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \left| \det \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} \right| = |12uv^2w - 1|.$$

So, in terms of the variables u, v, w,

$$\int\int\int_{R_{x,y,z}} xydV_{x,y,z} = \int\int\int_{D_{u,v,w}} (v^+w^2)(w+u^2)|12uv^2w - 1| dV_{u,v,w}.$$

The subscripts are noting which set of variables the region and the differential volume are respecting. **You are reminded the importance of absolute sign towards the Jacobian**.

9. Evaluate $\int\int_R \sin(9x^2+4y^2)dA$, where R is the region with lying inside the ellipse $9x^2+4y^2=1$ in the first quadrant.
Solution:
Motivation in changing coordinates: the argument of sin is not easy to integrate. Change coordinates in order for it to look nicer. We change our coordinates into $x=\frac{1}{3}r\cos\theta$, $y=\frac{1}{2}r\sin\theta$, then $9x^2+4y^2=r^2$. Meanwhile,

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix}\right| = \frac{1}{6}r.$$

In the first quadrant, $0\le\theta\le\frac{\pi}{2}$. Therefore

$$\begin{aligned}\int\int_R \sin(9x^2+4y^2)dA &= \int_0^{\pi/2}\int_0^1 \sin(r^2)\left(\frac{1}{6}r\right)drd\theta \\ &= \frac{\pi}{24}(1-\cos 1)\end{aligned}$$