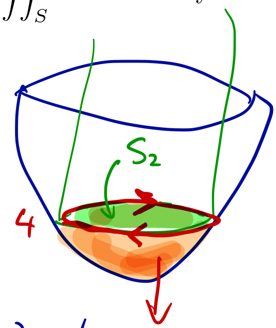
MATH 2023 – Multivariable Calculus

Lecture #19 Worksheet   April 25, 2019

Problem 1. Let $\mathbf{F} = \langle x^2z^2, y^2z^2, xyz\rangle$. Let S be the part of paraboloid $z = x^2+y^2$ inside the cylinder $x^2+y^2=4$, oriented downward. Find $\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$ by

(a) Changing to a line integral

(b) Evaluate on a different surface.



$z = x^2+y^2$
$x^2+y^2=4$

(a) $\oint_C \vec{F}\cdot d\vec{r}$

$C : \langle 2\sin t, 2\cos t, 4\rangle$

$= \int_0^{2\pi} \langle 64\sin^2 t, 64\cos^2 t, 16\cos t\sin t\rangle \cdot \langle 2\cos t, -2\sin t, 0\rangle \, dt$

$= 0$ //

(b) $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2z^2 & y^2z^2 & xyz \end{vmatrix} = \langle xz-2y^2z, 2x^2z-yz, 0\rangle$

$S_2 : \vec{n} = \langle 0,0,-1\rangle \Rightarrow \iint_{S_2} \nabla\times\vec{F}\cdot d\vec{S} = \iint \langle m, m, 0\rangle\cdot\langle 0,0,-1\rangle dS$

$= 0$ //

Problem 2. Let C be a simple closed curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\oint_C \vec{F}\cdot d\vec{r} \quad \oint_C z\,dx - 2x\,dy + 3y\,dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

Stokes' Thm :
$$\iint_S \nabla\times\vec{F}\cdot d\vec{S} = \iint_S \langle 3,1,-2\rangle\cdot\langle 1,1,1\rangle\, dS = 2\underbrace{\iint_S dS}_{=\text{ surface area.}}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ z & -2x & 3y \end{vmatrix} = \langle 3, 1, -2 \rangle$$

$$\vec{n} = \langle 1, 1, 1 \rangle$$

Problem 3. Evaluate

$$\oint \vec{F}\cdot d\vec{r} \qquad \oint_C \underline{(y+\sin x)}dx + (\underline{z^2+\cos y})dy + \underline{x^3}dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ y+\sin x & z^2+\cos y & x^3 \end{vmatrix} = \langle 2z, -3x^2, -1 \rangle$$

$$\sin 2t = 2\sin t \cos t$$

$$\Leftrightarrow C \;:\; \text{Intersection of } x^2+y^2=1 \text{ and } z=2xy.$$

S

$$\vec{n} = \langle 2y, 2x, -1 \rangle$$

$\uparrow$ downward $\because$ C is clockwise

$$\iint_{x^2+y^2\le 1} (8xy^2 - 6x^3 + 1)\, dA = \pi$$

0!

Problem 4. If $\mathbf{a}$ is a constant vector, and $\mathbf{r} = \langle x, y, z \rangle$ is the divergence vector field, show that

$$\iint_S 2\mathbf{a} \cdot d\mathbf{S} = \oint_C (\mathbf{a} \times \mathbf{r}) \cdot d\mathbf{r}$$

where the assumptions on S and C are as in Stokes' Theorem.

Just need to verify $2\vec{a} = \nabla \times (\vec{a} \times \vec{r})$.

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{r} \langle x, y, z \rangle$

$$\Rightarrow \vec{a} \times \vec{r} = \langle a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x \rangle \qquad \left| \begin{array}{ccc} \vec{i} & \vec{j} & k \\ \partial_x & \partial_y & \partial_z \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{array} \right|$$

$= 2\langle a_1, a_2, a_3 \rangle$.