HKUST **MATH 101**

Midterm Examination

Multivariable Calculus

16 October 2003

Answer ALL 6 questions

Time allowed – 120 minutes

Directions – This is a closed book examination. No talking or whispering are allowed. Work must be shown to receive points. An answer alone is not enough. Please write neatly. Answers which are illegible for the grader cannot be given credit.

Note that you can work on *both* sides of the paper and do not detach pages from this exam packet or unstaple the packet.

Student Name: ____________________

Student Number: ____________________

Tutorial Session: ____________________

Question No.	Marks
1	/20
2	/20
3	/20
4	/20
5	/20
6	/20
Total	/120

Problem 1 (20 points) Your Score:

True (T) or False (F) questions: Write T or F at the bottom table for your answer. No justifications are needed.

(a) At a local maximum (x_0, y_0) of $f(x, y)$, one has $f_{yy}(x_0, y_0) \geqslant 0$.

(b) The gradient $(2x, 2y)$ is perpendicular to the surface $z = x^2 + y^2$.

(c) The equation $f(x, y) = k$ implicitly defines x as a function of y and $\frac{dx}{dy} = \frac{\partial f}{\partial y} \Big/ \frac{\partial f}{\partial x}$.

(d) $f(x, y) = \sqrt{16 - x^2 - y^2}$ has both an absolute maximum and an absolute minimum on its domain of definition.

(e) If (x_0, y_0) is a critical point of $f(x, y)$ under the constraint $g(x, y) = 0$, and $f_{xy}(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.

(f) The vector $\mathbf{r}_u(u, v)$ of a parameterized surface $(u, v) \Rightarrow \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ is normal to the surface.

(g) $f(x, y)$ and $g(x, y) = f(x^2, y^2)$ have the same critical points.

(h) At a saddle point, the directional derivative is zero for two different vectors $\mathbf{u}$, $\mathbf{v}$.

(i) The value of the function $f(x, y) = e^x y$ at $(0.001, -0.001)$ can by linear approximation be estimated as -0.001.

(j) The maximum of $f(x, y)$ under the constraint $g(x, y) = 0$ is the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$.

Answer:

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)
F	**F**	**F**	**T**	**F**	**F**	**F**	**T**	**T**	**F**

(a) False. At a local maximum, $f_{yy} < 0$.

(b) False. The surface is the graph of a function $f(x, y)$. While the gradient of f is perpendicular to the level curve of f, it is only the projection of the gradient to the function $g(x, y, z) = f(x, y) - z$. The later is perpendicular to the surface.

(c) False. Almost right, the sign is wrong.

(d) True. The domain of definition is the disc $x^2 + y^2 \leqslant 16$. The maximum 4 is in the center the absolute minimum 0 at the boundary.

(e) False. The point (x_0, y_0) does not need to be a critical point of f at all.

(f) False. The vector is always tangent to the surface.

(g) False. The function g has always $(0, 0)$ as a critical point, even if f has not.

(h) True. The directional derivative can be both positive and negative at a saddle point. By the intermediate value theorem, there are two directions, where the directional derivative vanishes.

(i) True. Because the gradient at $(0, 0)$ is $(0, 1)$ and $f(0, 0) = 0$, the linear approximation is $L(x, y) = y$.

(j) False. This can not be true, because the first problem is the same if we replace $g(x, y)$ with $2g(x, y)$, but this will change the value of the maximum of g on the right hand side.

Problem 2 (20 points) Your Score:

(a) Find the distance from the origin to the line $x + y + z = 0$, $2x - y - 5z = 1$.

(b) Use suffix notation to prove the Lagrange's identity: $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$.

Solution:

(a) A line parallel to $x + y + z = 0$ and $2x - y - 5z = 1$ is parallel to the cross product of the normal vectors to these two planes, that is, to the vector

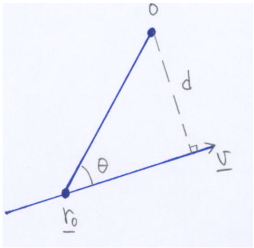
$$\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & -1 & -5 \end{vmatrix} = (-4, 7, -3).$$

We need a point on this line: set $z = 0$, then we have

$$\begin{cases} x + y = 0 \\ 2x - y = 1 \end{cases} \Rightarrow \begin{cases} x = \dfrac{1}{3} \\ y = -\dfrac{1}{3} \end{cases}$$

$\therefore \quad \mathbf{r}_0 = \left(\frac{1}{3}, -\frac{1}{3}, 0\right)$.

$\therefore$ The required distance is



$$\begin{aligned} d &= \|\mathbf{r}_0\| \sin\theta \\ &= \|\mathbf{r}_0\| \, \|\hat{\mathbf{v}}\| \sin\theta \\ &= \|\mathbf{r}_0 \times \hat{\mathbf{v}}\| \\ &= \frac{1}{\sqrt{74}} \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{3} & -\frac{1}{3} & 0 \\ -4 & 7 & -3 \end{vmatrix} \right\| \\ &= \frac{1}{\sqrt{74}} \, \|\mathbf{i} + \mathbf{j} + \mathbf{k}\| \\ &= \sqrt{\frac{3}{74}}. \end{aligned}$$

(b)

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\|^2 &= (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) \\ &= (\mathbf{u} \times \mathbf{v})_i \, (\mathbf{u} \times \mathbf{v})_i \\ &= \epsilon_{ijk} u_j v_k \, \epsilon_{ipq} u_p v_q \\ &= (\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{pk}) u_j v_k u_p v_q \\ &= u_j v_k u_j v_k - u_j v_k u_k v_j \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2. \end{aligned}$$

Problem 2 (20 points) Your Score:

Problem 3 (20 points) Your Score:

Consider a particle which moves on a circular helix in $\mathbb{R}^3$ with position vector given by (all scalars are non-zero):

$$\mathbf{r}(t) = (a\cos\omega t,\, a\sin\omega t,\, b\omega t).$$

(a) Show that the speed of the particle is a constant.

(b) Show that the velocity vector makes a constant non-zero angle with the z-axis.

(c) If $t_1 = 0$ and $t_2 = \dfrac{2\pi}{\omega}$, notice that $\mathbf{r}(t_1) = (a, 0, 0)$ and $\mathbf{r}(t_2) = (a, 0, 2\pi b)$, so the vector $\mathbf{r}(t_2) - \mathbf{r}(t_1)$ is vertical. Conclude that the equation

$$\mathbf{r}(t_2) - \mathbf{r}(t_1) = (t_2 - t_1)\mathbf{r}'(\tau)$$

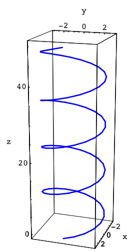
cannot hold for any $\tau \in (t_1, t_2)$. Thus the Mean Value Theorem does not hold for vector-valued functions.

Solution:

(a) Clearly $\mathbf{r}'(t) = (a^2\omega\sin\omega t,\, a\omega\cos\omega t,\, b\omega)$. Then the speed of the particle is

$$\|\mathbf{r}'(t)\| = \sqrt{a^2\omega^2\sin^2\omega t + a^2\omega^2\cos^2\omega t + b^2\omega^2} = |\omega|\sqrt{a^2+b^2},$$

which clearly in a constant.
Remark: Many people made a mistake here by missing the absolute value. At the end ω might be negative and in general $\sqrt{\omega^2} = |\omega|$.



(b) Let's consider the unit vector on the z-axis: $\mathbf{k} = (0, 0, 1)$ and let θ be the angle between $\mathbf{r}'(t)$ and $\mathbf{k}$. We need to show that θ is a non-zero constant. To do that let's consider the scalar product of $\mathbf{r}'(t)$ and $\mathbf{k}$. On one hand it is

$$\|\mathbf{r}'(t)\| \cdot \|\mathbf{k}\| \cos\theta = |\omega|\sqrt{a^2+b^2}\cos\theta$$

and on the other it equals $b\omega$. Therefore

$$\cos\theta = \frac{b\omega}{|\omega|\sqrt{a^2+b^2}},$$

which clearly is a constant. Moreover, it is non-zero since if it were zero we directly get that $a = 0$, which is not the case.

(c) The equation

$$\mathbf{r}(t_2) - \mathbf{r}(t_1) = (t_2 - t_1)\mathbf{r}'(\tau) \qquad \Leftrightarrow \qquad (0, 0, 2\pi b) = \frac{2\pi}{\omega}\mathbf{r}'(\tau)$$

cannot hold for any $\tau \in (t_1, t_2)$ since the left-hand side is vertical, i.e. makes a zero angle with the z-axis, whereas the right-hand side, as we showed in (b), makes a non-zero angle with the z-axis.

Remark: That the given equation cannot hold for any $\tau \in (t_1, t_2)$ can also be shown via a direct calculation, as many people did, but it requires some work, which is avoided in this solution.

Problem 4 (20 points) Your Score:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \dfrac{x^2 y}{x^2 + y^2}$ unless $x = y = 0$ and $f(0, 0) = 0$.

(a) Show that $D_{\mathbf{v}} f(0, 0)$ exists for all $\mathbf{v} \in \mathbb{R}^2$ by direct computation.

(b) Show that f satisfies the homogeneous relation $f(t\mathbf{v}) = tf(\mathbf{v})$ for all $t \in \mathbb{R}$ and all $\mathbf{v} \in \mathbb{R}^2$.

(c) Show that any differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ satisfying the homogeneous relation $g(t\mathbf{v}) = tg(\mathbf{v})$, $\forall\ t \in \mathbb{R}$, $\forall\ \mathbf{v} \in \mathbb{R}^n$ and $g(\mathbf{0}) = 0$ also satisfies the relation

$$g(\mathbf{v}) = \nabla g(\mathbf{0}) \cdot \mathbf{v} \quad \text{for all} \quad \mathbf{v} \in \mathbb{R}^n$$

and hence must be *linear*.

(d) Conclude that f possesses directional derivatives in all directions at $(0, 0)$, but that f is *not* differentiable at $(0, 0)$.

Solution:

(a) Let's try out the limit. First set $\mathbf{v} = (x, y)$, where neither x or $y = 0$.

$$D_{\mathbf{v}} f(0, 0) = \lim_{h\to 0} \frac{f(hx, hy) - f(0, 0)}{h} = \lim_{h\to 0} \frac{(h^2x^2)hy}{h(h^2x^2 + h^2y^2)} = \lim_{h\to 0} \frac{x^2 y}{x^2 + y^2} = f(x, y).$$

And hey, if either $x = 0$ or $y = 0$ then $D_{\mathbf{v}} f(0, 0) = 0$ for sure since the numerator goes away.

(b) Suppose we're still playing with the same $\mathbf{v}$ from up above, then

$$f(t\mathbf{v}) = \frac{t^2x^2ty}{t^2x^2 + t^2y^2} = \frac{t^3}{t^2}\frac{x^2 y}{x^2 + y^2} = tf(\mathbf{v}).$$

Again, if either $x = 0$, $y = 0$ or $t = 0$ then it looks pretty true, too.

(c) So we've already shown this basically. Just reuse parts (a) and (b) at the same time and call on the definition of the directional derivative.

$$D_{\mathbf{v}} g(\mathbf{0}) = \lim_{h\to 0} \frac{g(h\mathbf{v}) - g(\mathbf{0})}{h} = \lim_{h\to 0} \frac{hg(\mathbf{v})}{h} = g(\mathbf{v}).$$

Because g is homogeneous and $g(\mathbf{0}) = 0$, that h swings around and out of the limit. Meanwhile, back at the ranch, we know that $D_{\mathbf{v}} g(\mathbf{0}) = \nabla g(\mathbf{0}) \cdot \mathbf{v}$. Put it all together and what've you got?

$$g(\mathbf{v}) = \nabla g(\mathbf{0}) \cdot \mathbf{v}$$

(d) Well, in part (a) we showed that the directional derivatives exist. And it's not too bad to see that the partials $\partial_x f = 2xy^3/(x^2 + y^2)^2$ and $\partial_y f = (x^4 - x^2y^2)/(x^2 + y^2)^2$ are zero at $(0, 0)$ (note that $\partial_x f$ and $\partial_y f$ is not continuous at $(0, 0)$), hence it is not differentiable at $(0, 0)$. Unfortunately, if we try going in the direction of $(1, 1)$ we get something else (0.5). That means f isn't linear despite its being homogeneous. But then part (c) kicks in to tell us that f isn't differentiable at $(0, 0)$.

Problem 4 (20 points) Your Score:

Problem 5 (20 points) Your Score:

(a) Find the equation of the level curve of the function $z = g(x, y) = xf(xy)$ at the point (x_0, y_0), where both f and g are differentiable. Show that $\nabla g(x_0, y_0)$ is normal to the tangent line to the level curve at (x_0, y_0).

(b) Show that, in terms of polar coordinates (r, θ) (where $x = r\cos\theta$, and $y = r\sin\theta$), the gradient of a function $f(r, \theta)$ is given by

$$\nabla f = \frac{\partial f}{\partial r}\widehat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\widehat{\boldsymbol{\theta}},$$

where $\widehat{\mathbf{r}}$ is a unit vector in the direction of the position vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\widehat{\boldsymbol{\theta}}$ is a unit vector at right angles to $\widehat{\mathbf{r}}$ in the direction of increasing θ.

Solution:

(a) The equation of the level curve at the point (x_0, y_0) is

$$xf(xy) = x_0 f(x_0 y_0).$$

Differentiation both sides of the equation wrt x, then

$$xf'(xy)\left[x\frac{dy}{dx} + y\right] + f(xy) = 0$$
$$\frac{dy}{dx} = -\frac{xyf'(xy) + f(xy)}{x^2 f'(xy)}.$$

Also

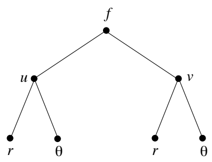
$$\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = (xyf'(xy) + f(xy),\, x^2 f'(xy)).$$

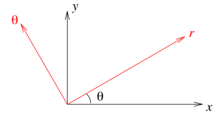
Therefore $\left.\frac{dy}{dx}\right|_{(x_0,y_0)} \times [\text{slope of } \nabla g(x_0, y_0)] = -1$,

i.e. they must be normal to each other.

(b) $x = r\cos\theta$, $y = r\sin\theta$ and $f = f(x, y) = f(r, \theta)$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$
$$= \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}$$
$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$
$$= -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}.$$





Also note that

$$\widehat{\mathbf{r}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{r} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$$
$$\widehat{\boldsymbol{\theta}} = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{r} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$$

Note that $\widehat{\mathbf{r}} \perp \widehat{\boldsymbol{\theta}}$, therefore

$$\frac{\partial f}{\partial r}\widehat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\widehat{\boldsymbol{\theta}}$$
$$= \left(\cos^2\theta\frac{\partial f}{\partial x} + \sin\theta\cos\theta\frac{\partial f}{\partial y}\right)\mathbf{i} + \left(\cos\theta\sin\theta\frac{\partial f}{\partial x} + \sin^2\theta\frac{\partial f}{\partial y}\right)\mathbf{j}$$
$$+ \left(\sin^2\theta\frac{\partial f}{\partial x} - \sin\theta\cos\theta\frac{\partial f}{\partial y}\right)\mathbf{i} + \left(-\cos\theta\sin\theta\frac{\partial f}{\partial x} + \cos^2\theta\frac{\partial f}{\partial y}\right)\mathbf{j}$$
$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$
$$= \nabla f.$$

Problem 6 (20 points) Your Score:

(a) If $f(x,y) = xe^y$, find the rate of change of f at the point $P\ (2,0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$. In what direction does f have the maximum rate of change?

(b) Find a single equation of the form $Ax + By + Cz = D$ that describes the plane given parametrically as

$$\begin{aligned} x &= 3s - t + 2 \\ y &= 4s + t \\ z &= s + 5t + 3. \end{aligned}$$

(c) Locate all relative maxima, relative minima and saddle points of the function $f(x,y) = x^4 - y^3$.

Solution:

(a)

$$\begin{aligned} \nabla f &= (f_x, f_y) = (e^y, xe^y) \\ \nabla f(2,0) &= (1,2) \end{aligned}$$

$$\mathbf{PQ} = \mathbf{OQ} - \mathbf{OP} = \left(-\frac{3}{2}, 2\right), \qquad \hat{\mathbf{u}} = \left(-\frac{3}{5}, \frac{4}{5}\right).$$

Therefore, the rate of change of f in the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}} f = \hat{\mathbf{u}} \cdot \nabla f = \left(-\frac{3}{5}, \frac{4}{5}\right) \cdot (1,2) = 1.$$

f increases fastest in the direction of the gradient vector $\nabla f(2,0) = (1,2)$.

(b) We combine the parametric equations into the single equation:

$$\mathbf{r}(s,t) = s(3,4,1) + t(-1,1,5) + (2,0,3).$$

Use the cross product to find the normal vector to the plane:

$$\mathbf{n} = (3,4,1) \times (-1,1,5) = (19,-16,7)$$

so the equation of the plane is

$$\begin{aligned} (19,-16,7) \cdot (x-2, y, z-3) &= 0 \\ 19x - 16y + 7z &= 59. \end{aligned}$$

(c)

$$\begin{aligned} f(x,y) &= x^4 - y^3 \\ f_x &= 4x^3 \\ f_y &= -3y^2 \\ f_{xx} &= 12x^2 \\ f_{yy} &= -6y \\ f_{xy} &= f_{yx} = 0 \end{aligned}$$

For critical points, $f_x = f_y = 0 \quad \Rightarrow \quad x = y = 0$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 0$ (D-test fails – no information)

Along the x-axis, $y = 0$, $f(x,0) = x^4$ (at $x = 0$ is a minimum point).

Along the y-axis, $x = 0$, $f(0,y) = y^3$ (at $y = 0$ is a point of inflection)

$\therefore$ $(0,0)$ is a saddle point.