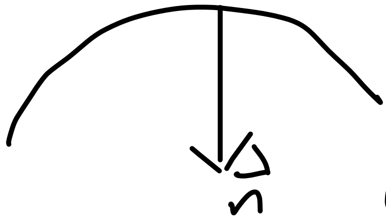
MATH2023
Tutorial 2
February 17, 2019 Tutorial Notes by Ivan So

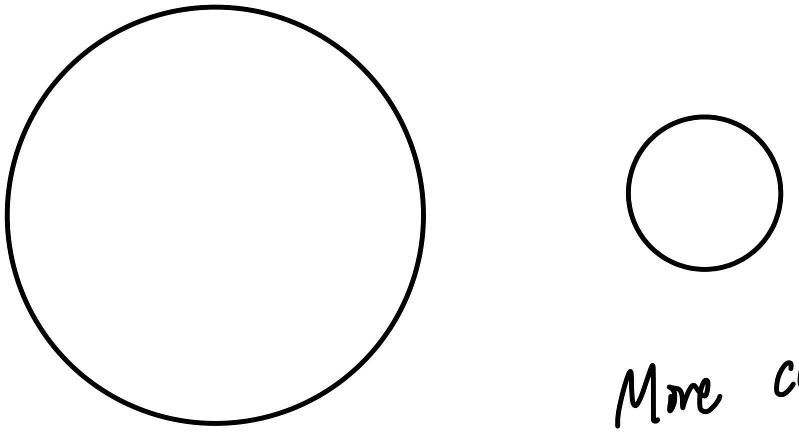
1 Review

In the following we will assume V to be a 3-dimensional vector space.

- **Vector valued function** is a map from subset of $\mathbb{R}$ to V. In general, they can be expressed as $\mathbf{r}(t) = (r_1(t), r_2(t), r_3(t))$
- **Limit** for vector valued function is evaluated componentwise, concepts in single variable calculus generalize.
- The **tangent vector** is defined by $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, which is the vector pointing *along* the curve.
- The **normal vector** for a curve is defined by $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$, which is *perpendicular* to the tangent vector.
- **Integrals** for vector function is defined by integration *componentwise*.
- **Arc length**: The length of a curve over the given interval of t. We can derive the equation as follows:
  1. Classically, separation of two points in space is given by the Pythagoras formula $\Delta s = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$.
  2. The length over a curve is approximated by $\sum_{i=1}^{n} \Delta s_i = \sum_{i=1}^{n} \frac{\Delta s_i}{\Delta t} \Delta t$.
  3. Taking limit,
  $$L = \int_a^b \sqrt{[r_1'(t)]^2 + [r_2'(t)]^2 + [r_3'(t)]^2} dt.$$
- The **curvature** of the curve is defined by $\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$, where s is the arc length function of the curve.
  An alternative expression for curvature is
  $$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$
- The **arc length parametrization** is a parametrization of a curve in which $|\vec{r}'(s)| = 1$.
- Definition for vector value function provided a way for us to apply Newton's second law in 3D, namely $\mathbf{F}(t) = m\mathbf{r}''(t))$.

Normal vector def:





$$\frac{d\vec{r}(t)}{ds} = \frac{d\vec{r}(t)}{dt} \Big/ \frac{ds}{dt}$$

$$\left|\frac{d\vec{r}(t)}{ds}\right| = \left|\frac{d\vec{r}(t)}{dt}\right| \Big/ \frac{ds}{dt}$$

speed = speed

[illegible] = |

$F = ma$

$\vec{F} = m\vec{r}''(t)$

2 Problems

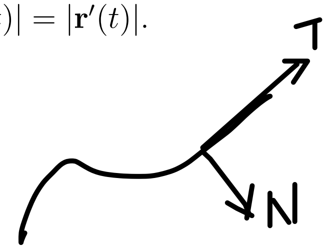
1. True or False

   (a) If $|\mathbf{r}(t)| = 1$ for all t, then $|\mathbf{r}'(t)|$ is a constant.

   (b) If $|\mathbf{r}(t)| = 1$ for all t, then $\mathbf{r}'(t)$ is a orthogonal to $\mathbf{r}(t)$ for all t.

   (c) If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable vector-valued function, then $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.

   (d) If $\mathbf{r}(t)$ is differentiable, then $\frac{d}{dt}|\mathbf{r}(t)| = |\mathbf{r}'(t)|$.



2. Prove that $\mathbf{T} \perp \mathbf{N}$.

3. Show that the curvature κ is related to the tangent and normal vectors by the equation $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$.

4. Find the length of the curve $\mathbf{r}(t) = (2t^{3/2}, \cos 2t, \sin 2t)$ for $0 \le t \le 1$.

5. Show that if $|\mathbf{r}'(t)| = C$, then $\mathbf{r}'(t) \perp \mathbf{r}''(t)$.

1a).

(a) If $|\mathbf{r}(t)| = 1$ for all t, then $|\mathbf{r}'(t)|$ is a constant.

False.

Counter example: $\vec{r}(t) = \left(\frac{1}{\sqrt{1+t^2}}, 0, \frac{t}{\sqrt{1+t^2}} \right)$

$$|\vec{r}(t)| = 1$$

$$\vec{r}\,'(t) = \left(-\frac{t}{(1+t^2)^{3/2}}, 0, \frac{1}{(1+t^2)^{3/2}} \right)$$

$$|\vec{r}\,'(t)| = \frac{1}{\sqrt{1+t^2}}$$

(b) If $|\mathbf{r}(t)| = 1$ for all t, then $\mathbf{r}'(t)$ is a orthogonal to $\mathbf{r}(t)$ for all t.

True.

$$\frac{d}{dt}\, \vec{r}(t) \cdot \vec{r}(t) = \frac{d}{dt}(1) = 0$$

$$\vec{r}(t) \cdot \vec{r}\,'(t) + \vec{r}\,'(t) \cdot \vec{r}(t) = 0$$

$$2\, \vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$$\vec{r}(t) \cdot \vec{r}\,'(t) = 0.$$

(c) If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are differentiable vector-valued function, then $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.

True. $[\vec{u}(t) \times \vec{v}(t)]_x = u_y v_z - u_z v_y$

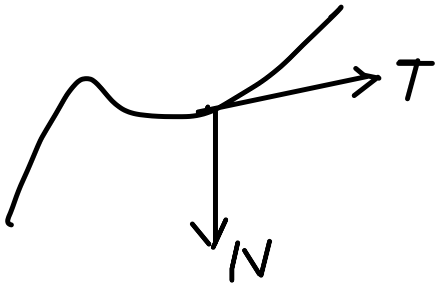
$$\frac{d}{dt}[u_y v_z - u_z v_y] = u_y' v_z - u_z' v_y + u_y v_z' - u_z v_y'$$

$$= [u'(t) \times v(t) + u(t) \times v'(t)]_x$$

(d) If $\mathbf{r}(t)$ is differentiable, then $\frac{d}{dt}|\mathbf{r}(t)| = |\mathbf{r}'(t)|$.

Counter-example: part (a)

2. Prove that $\mathbf{T} \perp \mathbf{N}$.



$$\vec{T} = \frac{r'(t)}{|r'(t)|} \Rightarrow |\vec{T}| = 1$$

$$\vec{T} \cdot \vec{T} = 1$$

$$\vec{T} \cdot \frac{d\vec{T}}{dt} = 0 \Rightarrow \vec{T} \cdot \vec{T}' = 0 \Rightarrow \vec{T} \cdot \frac{\vec{T}'}{|\vec{T}'|} = 0 \Rightarrow \vec{T} \cdot \vec{N} = 0$$

def. of $\vec{N} = \dfrac{\vec{T}'}{|\vec{T}'|}$

Made with Goodnotes

3. Show that the curvature κ is related to the tangent and normal vectors by the equation $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$.

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt} \Big/ \frac{ds}{dt} = \vec{T}' \Big/ \frac{ds}{dt} \Rightarrow \text{parallel to } \vec{N}$$

chain rule

by def.

4. Find the length of the curve $\mathbf{r}(t) = (2t^{3/2}, \cos 2t, \sin 2t)$ for $0 \leq t \leq 1$.

$$\vec{r}'(t) = (3t^{\frac{1}{2}}, -2\sin 2t, 2\cos 2t)$$

$$\text{length} = \int_0^1 \sqrt{9t + (-2\sin 2t)^2 + (2\cos 2t)^2}\, dt$$

$$= \int_0^1 \sqrt{9t + 4}\, dt$$

let $u = 9t+4$, $du = 9dt$, $dt = \frac{1}{9}du$

$$\frac{1}{9}\int_4^{13} \sqrt{u}\, du$$

$$= \frac{1}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_4^{13}$$

$$= \frac{2}{27}\left[(13)^{3/2} - 8\right]$$

6. Find the tangential and normal component of $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$.

$$\vec{r}'(t) = (-\sin t, \cos t, 1)$$

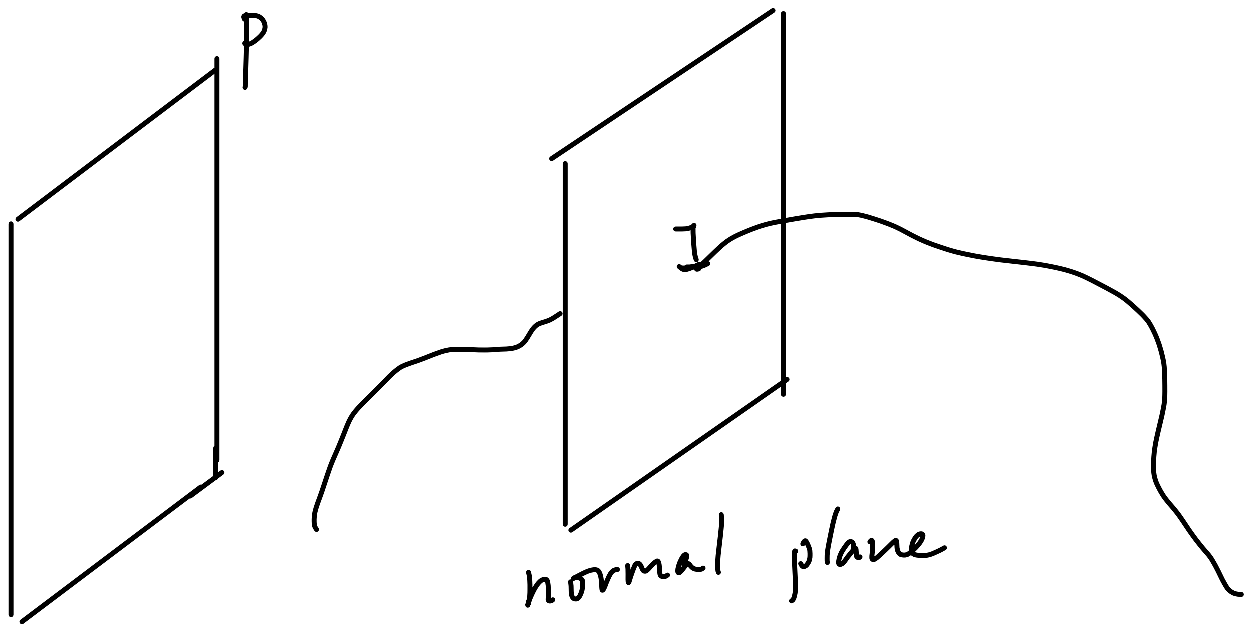
$$\vec{r}''(t) = (-\cos t, -\sin t, 0)$$

$$\vec{T} = \frac{(-\sin t, \cos t, 1)}{\sqrt{2}}$$

$$\vec{N} = \frac{(-\cos t, -\sin t, 0)/\sqrt{2}}{1/\sqrt{2}}$$

$$= (-\cos t, -\sin t, 0)$$

7. At what point on a curve C is the normal plane parallel to the plane P?



Given P, we know normal vector $\vec{n}$. what to find:

Solve $\vec{T}(t) = \vec{n}$, $\vec{n} \parallel r'(t)$

8. Show that $\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t)$.

$$\frac{d}{dt}|r(t)| = \frac{d}{dt}\sqrt{x(t)^2 + y(t)^2 + z(t)^2}$$

$$= \frac{x(t)\,x'(t) + y(t)\,y'(t) + z(t)\,z'(t)}{|\vec{r}(t)|}$$

$$= \frac{\vec{r}(t) \cdot \vec{r}'(t)}{|\vec{r}(t)|}$$

9. Find the arc length parametrization of the curve $C : (2t^{3/2}, \cos 2t, \sin 2t)$.

Goal: Find t in term of s:

$$s(t) = L(t) = \int_0^t \sqrt{9t+4}\,dt$$

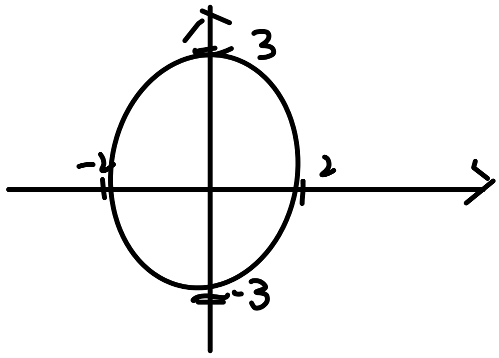
$$s(t) = \frac{2}{27}(9t+4)^{3/2} - 4$$

$$\Rightarrow t = \frac{1}{9}\left[\left(\frac{27}{2}s+4\right)^{2/3} - 4\right]$$

$$\frac{dl}{ds} = 1.$$

10. Find the curvature expression of the curve $C: \frac{x^2}{4} + \frac{y^2}{9} = 1$.

→先 parametrization



$$\vec{r}(t) = (2\cos t, 3\sin t, 0)$$

$$s(t) = \int_0^t \sqrt{(-2\sin t)^2 + (3\cos t)^2}\, dt$$

$$= \int_0^t \sqrt{5 + 4\sin^2 t}\, dt$$ ← 無得 in, 所以 要用另一條 formula

$$\vec{r}(t) = (2\cos t, 3\sin t, 0)$$

$$\vec{r}'(t) = (-2\sin t, 3\cos t, 0)$$

$$\vec{r}''(t) = (-2\cos t, -3\sin t, 0)$$

$$\frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} \quad \frac{\|(0, 0, 6\sin^2 t + 6\cos^2 t)\|}{\left(\sqrt{4\sin^2 t + 9\cos^2 t}\right)^3}$$

$$= \frac{6}{(4 + 5\cos^2 t)^{3/2}}$$

所以係 $t=0$ 是 less curved, $t=\frac{\pi}{2}$ 最 curved

Made with Goodnotes

6. Find the tangential and normal component of $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$.

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