# Last Time

$$\int_C \vec{F}\cdot d\vec{r} = \int P\,dx + Q\,dy + R\,dz$$

$\vec{F} = \langle P, Q, R\rangle$ $\quad d\vec{r} = \langle dx, dy, dz\rangle$

$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ $\quad dx = x'(t)\,dt$

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P, Q has continuous partial derivatives

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Thm C $\Uparrow$ $\quad \Downarrow$ Thm D (on D open simply-connected (in 2D: "no holes"))

$$\boxed{\vec{F} \text{ is conservative}} \iff \vec{F} = \nabla f = \langle f_x, f_y\rangle$$

$\int_C \nabla f \cdot d\vec{r}$ " $f(\vec{r}(b)) - f(\vec{r}(a))$

Fund. Thm $\Downarrow$ $\quad \Uparrow$ Thm B (on D open, connected).

$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

$\Updownarrow$ (Thm A)

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$ Independent of paths

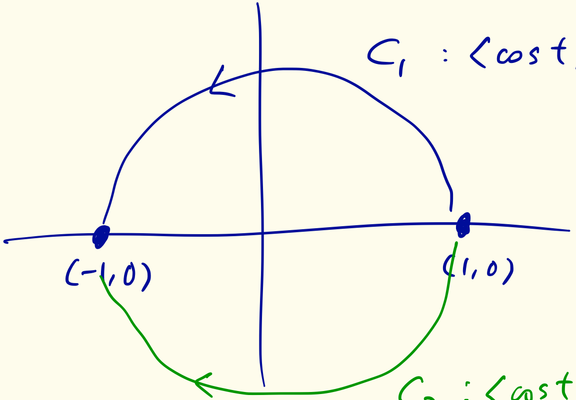
## Important Example $\vec{F} = \dfrac{-y\,\vec{i} + x\,\vec{j}}{x^2+y^2}$

**Check** $P = \dfrac{-y}{x^2+y^2}$, $Q = \dfrac{x}{x^2+y^2}$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

**Note** Not independent of path! ($\Leftrightarrow$ not conservative by Fund. Thm)

$C_1$: $dx = -\sin t\, dt$
$dy = \cos t\, dt$




$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

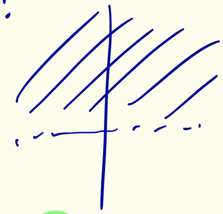
$$= \int_0^{\pi} -\underbrace{\sin t}_{1}(-\sin t\,dt) + \underbrace{\cos t}_{1}(\cos t\,dt)$$

$$= \int_0^{\pi} (\sin^2 t + \cos^2 t)\,dt = \int_0^{\pi} dt = \pi.$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = -\pi$$

# Why Thm D fails: Domain has "hole" at $(0,0)$.

What if we choose $D = \{(x,y) \mid y > 0\}$ ?

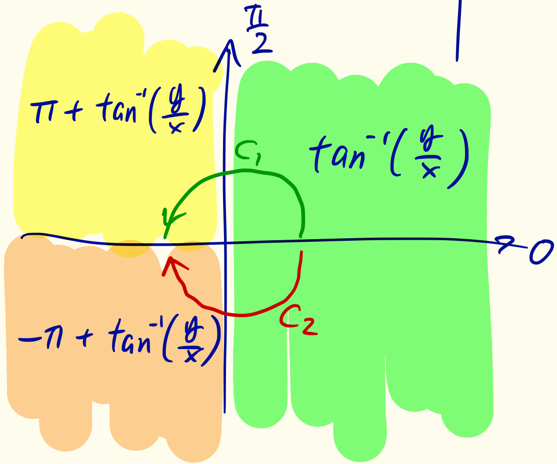


In this case Thm D applies!!

$$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right) = \theta(x,y).$$

not really good

this has a jump when $\theta = \pm\pi$.




## Then

$$\int_{C_1} \vec{F}\cdot d\vec{r} = f(-1,0) - f(1,0)$$

$$= \pi - 0 = \pi$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = f_*(-1,0) - f_*(1,0)$$

$$= -\pi - 0 = -\pi.$$

**Rem** Why $\vec{G} = \dfrac{\vec{r}}{|\vec{r}|^3}$ works?

Domain has hole!

$$\vec{G} = \nabla V \quad , \quad V = \frac{1}{|\vec{r}|} = \frac{1}{\sqrt{x^2+y^2}}$$

$\boxed{\underline{\text{Thm D}}}$ ① $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ $\Longrightarrow$ $\vec{F}$ is conservative

② D is open simply-connected

"False ⇒ Anything" is True.

i.e. no conclusion!!

**Rem** In 3D.

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0$$

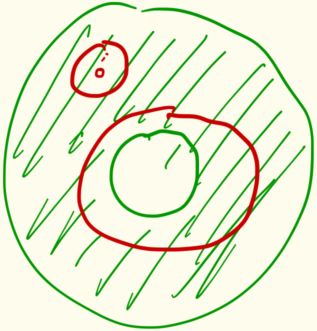
① $\iff \operatorname{curl} \vec{F} = \vec{0}$

Thm D in 3D.

② $D$ : open, simply connected.

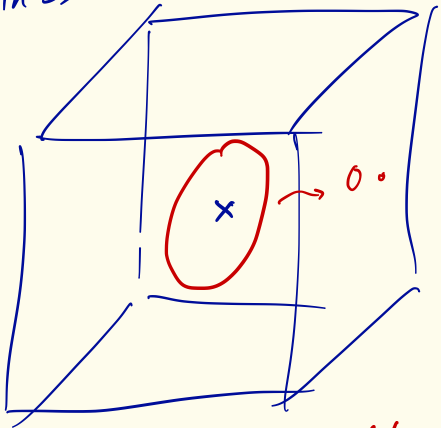
you can shrink a rubber band to a point

In 2D

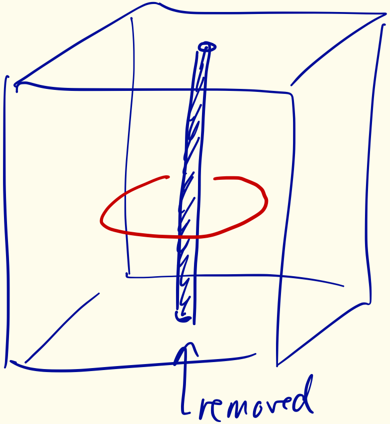


not simply connected.

In 3D



simply connected!

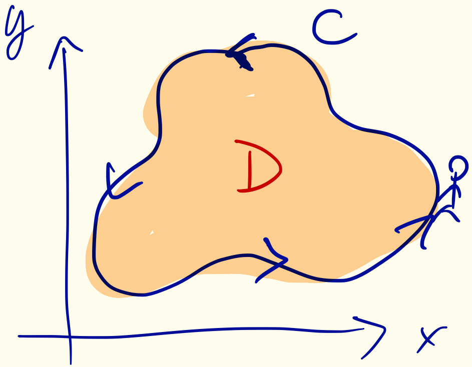



not simply connected.

# Green's Theorem

<u>Idea</u> Line integral over closed curve

$\Longleftrightarrow$ surface Integral.




<u>positive orientation</u> : counterclockwise

( surface always on the <u>LEFT</u> )

$$\partial D = C.$$

boundary.

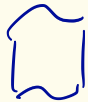
with positive orientation.

## Green's Thm

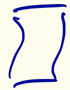
( If $P, Q$ has continuous partial derivatives on $D$ )

$$\oint_C P\,dx + Q\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

( $D$ is simple region) . $\leftarrow$ type I or type II



or

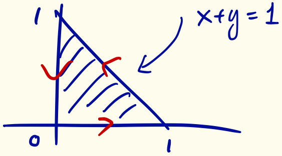


or sums.

## Special Case

$$\oint_C \vec{F} \cdot d\vec{r} = 0 \quad \text{if} \quad \vec{F} = \nabla f.$$

Ex $\oint_C \underbrace{x^4}_{P}\,dx + \underbrace{xy}_{Q}\,dy$ on




By Green's Thm

$$= \iint y\, dA = \int_0^1 \overbrace{\int_0^{1-y} y\, dx}^{= xy\big|_0^{1-y}}\, dy = \int_0^1 (1-y)y\, dy$$

$$= \frac{y^2}{2} - \frac{y^3}{3}\Big|_0^1 = \frac{1}{6}.$$

Ex $\oint_C (\underbrace{2y - \sin x^5}_{P})\, dx + (\underbrace{8x + e^{\sqrt{y^2+1}}}_{Q})\, dy$ over $x^2+y^2=9$.

counterclockwise

Green's Thm:

$$= \iint (8-2)\, dA = 6 \text{ Area(circle)} = 54\pi.$$

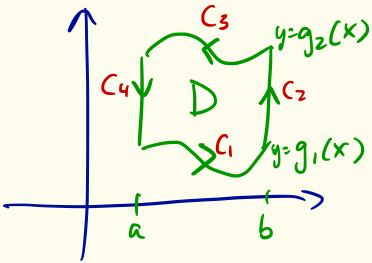
# Proof of Green's Thm

$P = P(x,y)$

We show $\displaystyle\oint_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$

$\displaystyle\oint Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA.$

Consider simple $D$:




**RHS** $\displaystyle -\iint_D \frac{\partial P}{\partial y}\,dA = -\int_a^b \underbrace{\int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy}\,dx$

$\displaystyle = -\int_a^b P(x, g_2(x)) - P(x, g_1(x))\,dx$

**LHS** $\displaystyle \int_{C_1} P(x,y)\,dx = \int_a^b P(x, g_1(x))\,dx$

$\displaystyle \int_{C_3} P(x,y)\,dx = \int_b^a P(x, g_2(x))\,dx = -\int_a^b P(x, g_2(x))\,dx$

$\displaystyle \int_{C_2} P\,dx = \int_{C_4} P\,dx = 0$ ($dx = 0$!)

LHS = RHS!

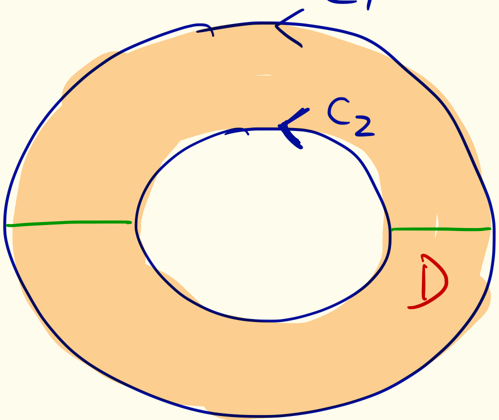
Similarly for $Q$

+

cut regions into simple ones.

# Green's Theorem : General Region

Regions with holes :




$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{C_1} + \oint_{-C_2} = \oint_{C_1} - \oint_{C_2}$$

sometimes write : $\partial D = C_1 \cup (-C_2)$

Made with Goodnotes

$$\frac{-y\hat{i} + x\hat{j}}{x^2+y^2} \qquad \langle P, Q \rangle$$

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

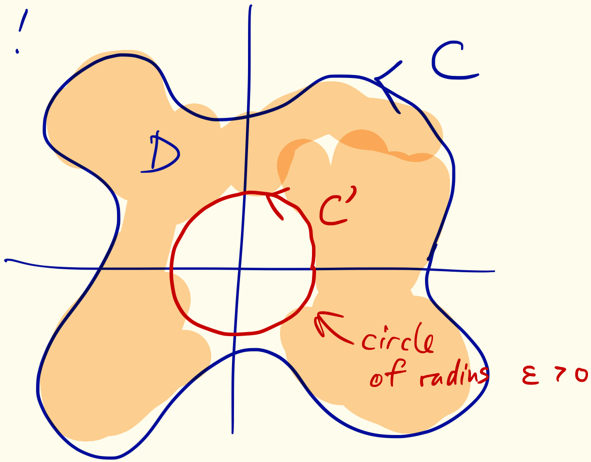
$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

**Ex** $\vec{F} = \dfrac{-y\,\vec{i} + x\,\vec{j}}{x^2+y^2}$ (D has hole!)

Then $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = 2\pi$ for every positive oriented simple closed path around origin.

**Note** $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ outside $(0,0)$!




Green's Thm

$$\iint_D \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\, dA = \oint_C \vec{F}\cdot d\vec{r} - \oint_{C'} \vec{F}\cdot d\vec{r}$$

$$\| \\ 0$$

$$\Leftrightarrow \oint_C \vec{F}\cdot d\vec{r} = \oint_{C'} \vec{F}\cdot d\vec{r} = 2\pi \qquad (C' : \langle \varepsilon\cos t, \varepsilon\sin t \rangle)$$

circle of radius $\varepsilon$.

**Pf Thm D**

① $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

② $D$ simply connected

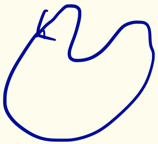
$\Longrightarrow \vec{F}$ is conservative.

$\vec{F}$ is conservative $\Longleftrightarrow \displaystyle\oint_C \vec{F}\cdot d\vec{r} = 0$ for all closed path.

By Green's $\Longleftrightarrow \displaystyle\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = 0$

by ①

If $C$ is simple
(no self intersections)



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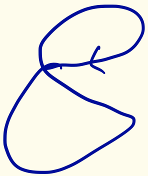

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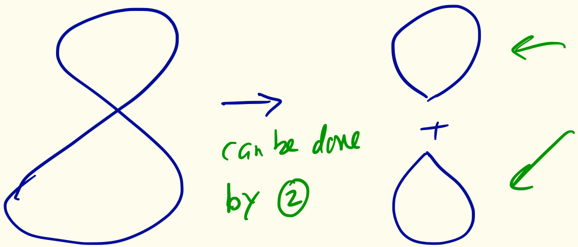
If $C$ is not simple, cut it into simple ones.




both simple.