4028

Math 2023 Midterm Examination Spring 2013

Instructions:

Do not read the inside of this exam book until you are told to do so

1. Do all problems
2. Total Time: 75 minutes
3. Can use both sides of each page
4. Only HKEA approved calculators allowed
5. Must show steps and/or arguments to get credits
6. Put the final answer of each problem on the page where the problem appears

Student Name:

Student ID number:

Soln

Total 40

Marks:

Problem 1 (20%) ....

Problem 2 (20%) ....

Problem 3 (20%) ....

Problem 4 (20%) ....

Problem 5 (20%) ....

TOTAL ....

1. Find the minimal distance between the point $(1, 1, 0)$ and points on the sphere $x^2 + y^2 + z^2 - 2x - 4y = 4$.

$(x-1)^2 + (y-2)^2 + z^2 = 9$

$(z-0)^2$

3

center of sphere at $(1, 2, 0)$

3

distance between point and center

$= \sqrt{(1-1)^2 + (1-2)^2 + 0^2} = 1$

radius of sphere $= 3$

2

$\therefore$ distance from pt. to surface $= 3 - 1 = 2$

2. Find a vector equation for the intersection of the two planes $x+y+z=1$ and $x+2y+z=3$.

① ②

② − ① : $y = 2$

$\therefore x + z = -1$

let $z = t$

$x = -1 - z$

2 points

$\therefore$ A parameterization for the line is

Solve equations

4 points

$$\begin{cases} x = -1 - t \\ y = 2 \\ z = t \end{cases}$$

2

$\therefore$ Vector equation is

$$\vec{r}(t) = (-1, 2, 0) + t(-1, 0, 1)$$

3. Sketch the surface $z = 1 + xy$.

$z - 1 = xy$

2 points $z = 1$ for x or $y = 0$

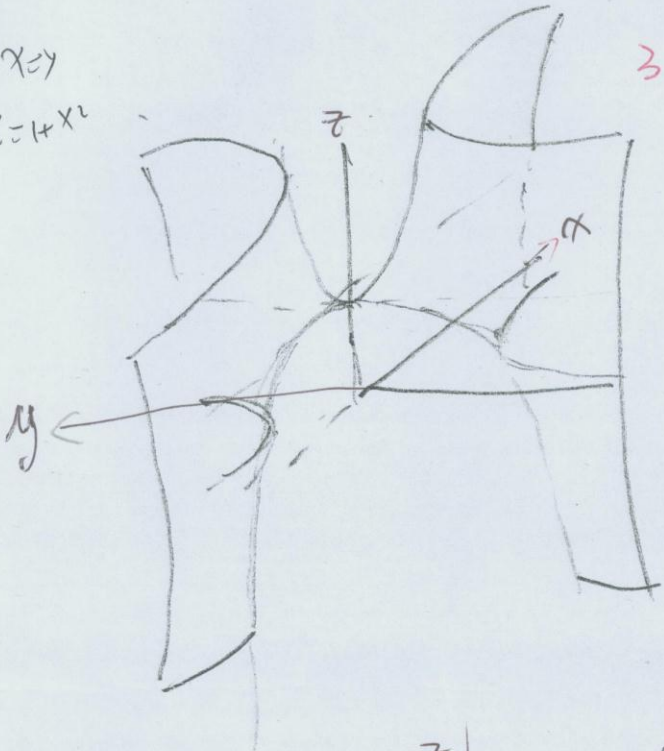
3 points on xy-plane, $z = 0$

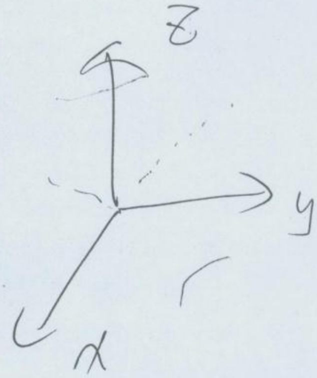
$xy = -1$ hyperbola

graph 3 points

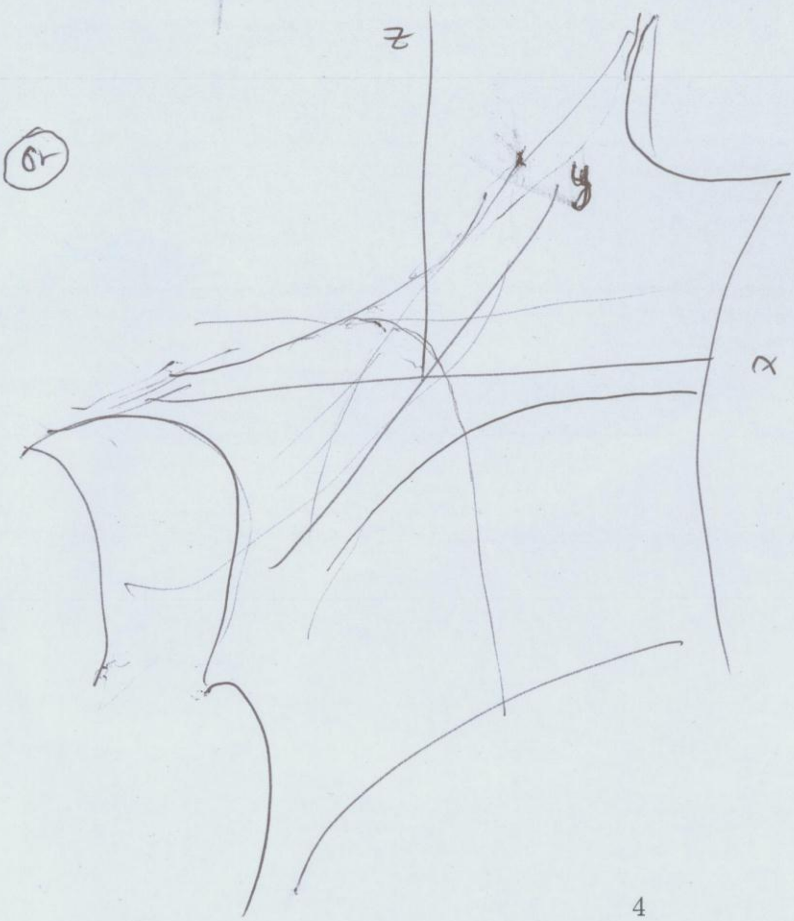
$x = y$

$z = 1 + x^2$





or



4. Let $f(x, y) = \ln(x^2 + y^2 - 2)$, $g(x) = \sqrt{1 - x^2}$. Find the domain of the composite function $g \circ f$.

(i) Domain of f satisfies

3

$x^2+y^2-2>0$

$x^2+y^2>2$ $\therefore D_f = D_{\sqrt{2}}(0,0)$

An open disk centered at $(0,0)$ with radius $\sqrt{2}$.

(ii) Domain of g satisfies

2

$1-x^2 \geq 0$ $x^2 \leq 1$ or $|x| \leq 1$

~~$D_g = [-1, 1]$~~

3 $f(x,y)$ need to be in D_g

$\therefore \ln(x^2+y^2-2) \in [-1, 1]$

$\Leftrightarrow e^{-1} \leq x^2+y^2-2 \leq e^1$

$2+e^{-1} \leq x^2+y^2 \leq 2+e^{+1}$

As both $2+e^{-1}$, $2+e^{+1} > 2$, (i) is satisfied

$\therefore D_{g\circ f} = \{(x,y) \mid 2+e^{-1} \leq x^2+y^2 \leq 2+e\}$

5. Find the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2y + xy^2 + x^2 + y^2 + 2xy}{1 - \cos\sqrt{x^2+y^2}}$ if it exists.

2

Use $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{x^2y + xy^2 + x^2 + y^2 + 2xy}{1-\cos\sqrt{x^2+y^2}}$$

$$= \frac{r^3\cos\theta\sin\theta(\cos\theta+\sin\theta) + r^2 + 2r^2\cos\theta\sin\theta}{1-\cos r}$$

↑ if one gets this form without using $\lim_{r\to 0}\frac{r^2}{1-\cos r}$, one can get 3 points

3 $$\lim_{r\to 0}\frac{r^2}{1-\cos r} = \lim_{r\to 0}\frac{2r}{\sin r} = 2$$

2 $$\lim_{r\to 0}\frac{r^3\cos\theta\sin\theta(\cos\theta+\sin\theta)}{1-\cos r}$$

$$= \underbrace{\lim_{r\to 0}\frac{r^2}{(1-\cos r)}}_{=2}\cdot\underbrace{\lim_{r\to 0} r\left[\cos\theta\sin\theta(\cos\theta+\sin\theta)\right]}_{=0}$$

$$= 0$$

1 However $\lim_{r\to 0}\frac{2r^2\cos\theta\sin\theta}{1-\cos r}$

$= 4\cos\theta\sin\theta$ for fixed θ

and depends on θ.

~~The limit 不存在.~~

See back of previous page

if $\theta = \pi/4$ i.e. along $(x,y) = t(1,1)$
then the limit is $2+2 = 4$. If ~~$\theta = 0$, the l~~ $(x,y) = t(1,0)$, limit is 2. ∴ limit ~~does~~ not exist.

Made with Goodnotes

Along $(x,y) = \cancel{(t,t)}\ (t,0)$

$$[\ ] = \frac{t^2}{1-\cos t}$$

4 $$\lim_{t\to 0^+} \frac{t^2}{1-\cos t} = \lim_{t\to 0^+} \frac{2t}{\sin t} = 2$$

Along $(x,y) = (t,t)$

3 $$[\ ] = \frac{2t^3+2t^2+2t^2}{1-\cos\sqrt{2}t} = \frac{2t^3+4t^2}{1-\cos\sqrt{2}t}$$

$$\lim_{t\to 0^+} \frac{2t^3+4t^2}{1-\cos\sqrt{2}t} = \lim_{t\to 0} \frac{6t^2+8t}{\sqrt{2}\sin\sqrt{2}t} = 4$$

1 The ~~t~~ limits along the two lines do not agree

$\therefore$ no limit.

Along $(x,y) = (t,0)$

$$[\ ] = \frac{t^2}{1-\cos t}$$

4 $$\lim_{t\to 0^+} \frac{t^2}{1-\cos t} = \lim_{t\to 0^+} \frac{2t}{\sin t} = 2$$

Along $(x,y) = (t,t)$

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$$\lim_{t\to 0^+} \frac{2t^3+4t^2}{1-\cos\sqrt{2}t} = \lim_{t\to 0} \frac{6t^2+8t}{\sqrt{2}\sin\sqrt{2}t} = 4$$

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