Midterm Review

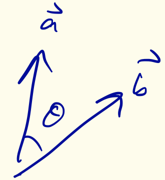
Geometry + Differentiation.

Vectors

$\vec{v}$: length, direction :

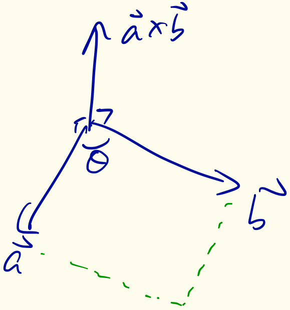
$\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}$

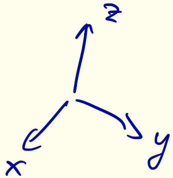
$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$



$\vec{a} \times \vec{b}$: direction $\perp$ $\vec{a}$ & $\vec{b}$

length: $\underbrace{|\vec{a}||\vec{b}| \sin\theta}_{\text{Area of //gram}}$

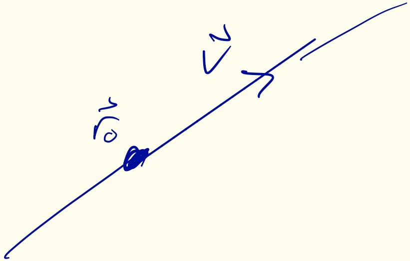




$\vec{i} \times \vec{j} = \vec{k}$

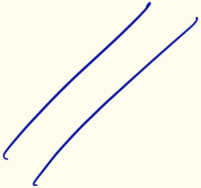
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Lines = point $\vec{r}_0$ + direction $\vec{v}$



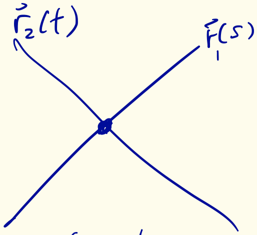
$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

velocity vector.



parallel lines

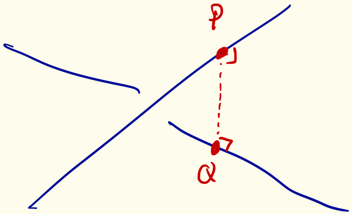
$\vec{v}_1 \parallel \vec{v}_2$



intersecting

$\vec{r}_1(s) = \vec{r}_2(t)$

has solution



skew lines

$\vec{r}_1(s) = \vec{r}_2(t)$

has NO solution.

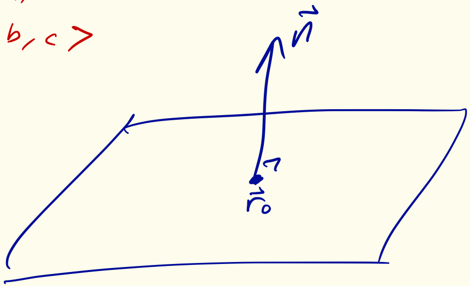
If P, Q are closest, then

$\overrightarrow{PQ} \perp \vec{v}_1$ & $\vec{v}_2$

i.e. $\overrightarrow{PQ} \parallel \vec{v}_1 \times \vec{v}_2$

Planes : point $\vec{r}_0$ + direction $\vec{n}$ (normal vector)

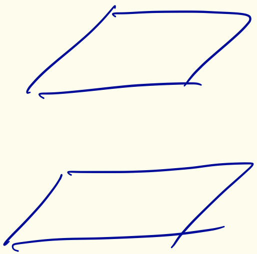
(x_0, y_0, z_0) $\qquad$ $\parallel \langle a, b, c \rangle$

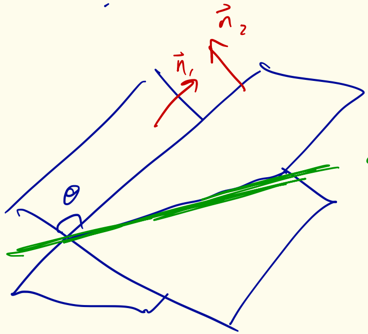


$$ax + by + cz = d$$

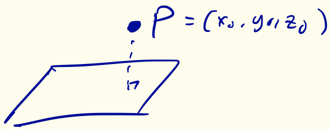
$$d = ax_0 + by_0 + cz_0 .$$

$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0 .$$





intersecting line is $\parallel$ to $\vec{n}_1 \times \vec{n}_2$



$\parallel$ planes $\Leftrightarrow \vec{n}_1 \parallel \vec{n}_2$

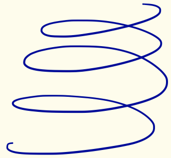
angle between planes = angle between $\vec{n}_1$ & $\vec{n}_2$

$$\text{dist} = \left| \frac{ax_0 + by_0 + cz_0 - d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Vector Functions

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

☆ **Ex** helix $\langle \cos t, \sin t, t \rangle$



Continuous: $\lim_{t \to a} \vec{r}(t) = \vec{r}(a)$.

tangent vector: $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$

$$\left.\begin{aligned} &\text{velocity} \quad \vec{v}(t) = \vec{r}\,'(t) \\ &\text{acceleration} \quad \vec{a}(t) = \vec{r}\,''(t) \end{aligned}\right\} \cancel{\vec{F} = m\vec{a}}$$

speed $= |\vec{r}\,'(t)|$

→ arc length: $\int_a^b |\vec{r}\,'(t)| \, dt$

Arc length parametrization: $|\vec{r}\,'(s)| = 1$ everywhere.

To find arc length parametrization:

① arc length function: $s(t) = \int_0^t |\vec{r}\,'(u)|\,du$

(arc length from $\vec{r}(0)$ to $\vec{r}(t)$)

② find t in terms of s.

$$t = t(s)$$

③ $\vec{r}_{arc}(s) = \vec{r}(t(s)) \iff |\vec{r}_{arc}\,'(s)| = 1$

$\frac{d}{ds}$

<u>Curvature</u>: how "curve" $\vec{r}(t)$ is

force you experience.

$\approx \vec{a}$

$$K(s) = |\vec{r}_{arc}\,''(s)| = \cancel{\frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}}$$

$\frac{d}{ds}$

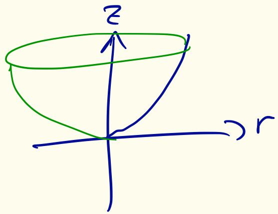
Surface

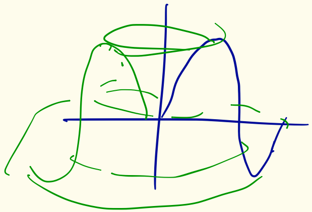
$z = f(x,y)$ graph $\qquad f(x,y) = k$ level curves

$F(x,y,z) = 0$ level surfaces.

$z = x^2+y^2$: $\begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix}$ $\qquad z = \sin(x^2+y^2)$

$\hookrightarrow \ z = r^2$





Tangent Planes

$\vec{n} = \nabla \vec{F} = \langle F_x, F_y, F_z \rangle$

downward

$\left(\text{graphs}: \ F = f(x,y) - z \ : \ \vec{n} = \langle f_x, f_y, -1 \rangle \right)$

Linear Approximation : $\underbrace{z = L(x,y)}_{\text{tangent Plane}}$.

Ex .... tangent plane $= 2x+3y+4z=5 \implies$ Linearization: $z = \dfrac{5-2x-3y}{4}$.

Limits

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

$$\left(\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(0,0)} f(x+a,\ y+b) \right)$$

Not Exists : If Limit along 2 paths have different value.

usually: $x=0$, $y=0$, $x=y$, $x=2y$, $x=-y$, $y=x^2$ ...

Exists Use squeeze theorem + Polar coordinates

$\dfrac{f(x,y)}{g(x,y)}$ order f > order g

$$0 \le |2xy| \le x^2+y^2$$

$$0 \le |\sin x| \le 1$$

$$0 \le |\cos x| \le 1$$

$$0 \le x^2 \le x^2+y^2$$

$$0 \le |f(x,y)| \le g(x,y) \to 0$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\lim_{(x,y)\to(0,0)} \Rightarrow \lim_{r\to 0^+}$$

<u>Continuity</u> : Limit exist + Equal.

$f, g = \text{poly.}$

<u>Always continuous</u> : $\dfrac{f(x,y)}{g(x,y)}$ if f, g are continuous and $g \neq 0$ on domain.

e.g. rational functions.

$$\boxed{\text{Quote this fact}}$$!!!

Partial D

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h} \quad \text{at } (a,b).$$

$$f_y = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

Mixed Partials Theorem

If f_{xy} exists & continuous, then

$$f_{xy} \;\; \text{''} \;\; (f_x)_y$$

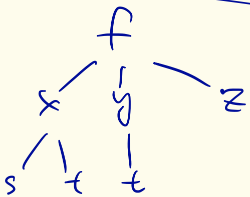
$$f_{xy} = f_{yx}$$

Chain Rule

$f(x, y, z)$

$x = g(s, t)$

$y = h(t)$



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$$

independent variables.

Implicit Differention

1 – dep

other – indep.

$$F(x, y, z, w) = 0$$ ← apply chain rule

e.g. $\dfrac{\partial z}{\partial x}$ ← dep. ↖ y, w indep.

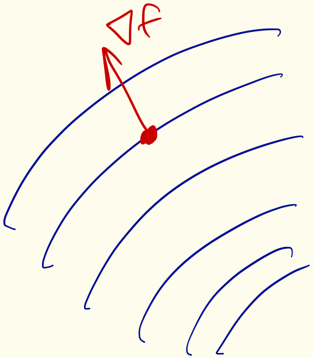
$$\frac{\partial F}{\partial x}\underbrace{\cancel{\frac{\partial x}{\partial x}}}_{1} + \frac{\partial F}{\partial y}\underbrace{\cancel{\frac{\partial y}{\partial x}}}_{y,x\ \text{indep}}^{0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial w}\underbrace{\cancel{\frac{\partial w}{\partial x}}}_{w,x\ \text{indep}}^{0} = 0$$

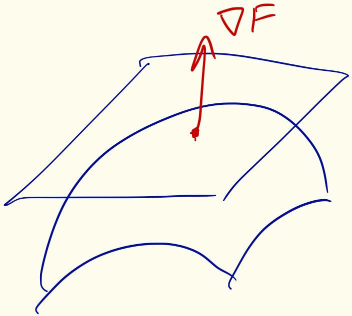
$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

they are still functions in (x, y, z, w)!

Gradient Vector $\nabla f = \langle f_x, f_y \rangle$ or $\nabla F = \langle F_x, F_y, F_z \rangle$

- ∇f is $\perp$ level curves / level surfaces.





Directional Derivatives $\vec{u} = \langle h, k \rangle$ unit vector

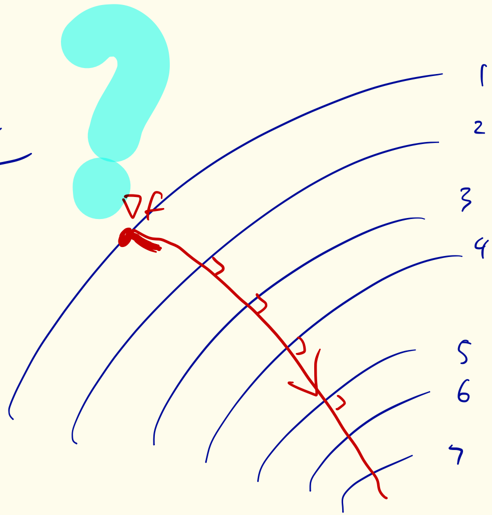
$$D_{\vec{u}} f = \nabla f \cdot \vec{u} \quad \text{"slope along } \vec{u}\text{"}$$

$$= \left.\frac{d}{dt}\right|_{t=0} f(a+th,\ b+tk) \quad \text{at } (a,b)$$

$\vec{u} = \underline{\underline{\nabla f}}$ = direction of greatest ascent

($D_{\vec{u}} f$ is largest)

$-\nabla f$ = greatest descent.



Integral curve

$\vec{r}(t) \,/\!/\, \nabla f$

everywhere.

$\Updownarrow$

solve differential equation.

Max / Min Problems

① Solve critical points $\nabla f = \vec{0}$ (or does not exist)

② 2^{nd} Derivative Test : $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$

D	f_{xx}	
>0	>0	local min
>0	<0	local max
<0		saddle
$=0$		no conclusion ⟵ use other method

internal points?

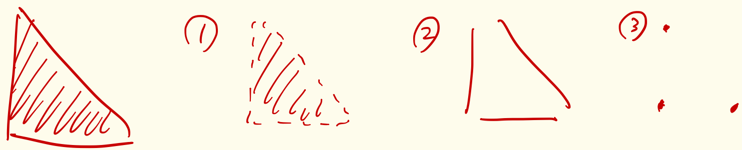
Domain with boundary

only on boundary

③ Restrict function to boundary or Lagrange multipler (boundary = constraints)
Find critical points

④ Find values at endpoints

⑤ Compare the values.



Lagrange Multiplier

$$\left.\begin{array}{l} \text{Max / Min} \quad f(x, y, z) \\ \text{Subject to} \quad g(x, y, z) = 0 \end{array}\right\} \text{ constrained system.}$$

$$\Longleftrightarrow \begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$

4 eq. in 4 unknowns (x, y, z, λ)

technique : eliminate λ by equating RHS.

$$\begin{cases} (\ \cdot\ )\,y = \lambda x\,y \\ \quad \| \\ (\ \cdot\ \cdot\ \cdot\ )\,x = \lambda y\,x \end{cases}$$

$f(w,x,y,z)$ subject to $\begin{cases} g(w,x,y,z)=0 \\ h(w,x,y,z)=0 \end{cases}$

$\Leftrightarrow$ Solve

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 0 \\ h = 0 \end{cases}$$

6 equations in 6 unknowns (w,x,y,z,λ,μ)