MATH 2023 – Multivariable Calculus

Lecture #01 Worksheet ♠ January 31, 2019

Problem 1. Let

$$A = (1,2,3), \qquad B = (3,4,5), \qquad C = (1,0,-1), \qquad D = (3,2,1)$$

be four points in $\mathbb{R}^3$. ABDC

(a) Show that $ABCD$ is a parallelogram

(b) Find the area of this parallelogram.

$\vec{AC} = \vec{BD}$:

$\vec{AC} = \vec{OC} - \vec{OA}$

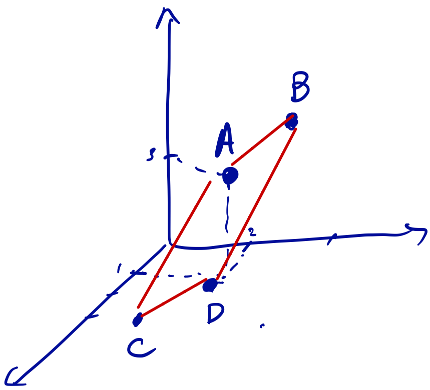
$= \langle 1,0,-1\rangle - \langle 1,2,3\rangle$

$= \langle 0,-2,-4\rangle$

$\vec{BD} = \langle 3,2,1\rangle - \langle 3,4,5\rangle$

$= \langle 0,-2,-4\rangle$

$\vec{AB} = \vec{CD}$



$$|\vec{AC} \times \vec{CD}| = \langle 0,-2,-4\rangle \times \langle 2,2,2\rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & -4 \\ 2 & 2 & 2 \end{vmatrix} = |4\vec{i} - 8\vec{j} + 4\vec{k}|$$

$$= \sqrt{16+64+16} = \sqrt{96} = 4\sqrt{6}$$

Problem 2. Describe the four different relationships between the line L

$$L = \begin{cases} x = 1 + 4s \\ y = 2 + 5s \\ z = 3 + 6s \end{cases} \qquad \vec{v} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

and the lines

$$\ell_1 = \begin{cases} x = 9 - 8t \\ y = 12 - 10t \\ z = 15 - 12t \end{cases} \leftarrow \vec{v}_1 = \begin{pmatrix} -8 \\ -10 \\ -12 \end{pmatrix} = -2\vec{v}. \ //$$

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ in. ℓ_1? $t = 1$ ✓.

$L = \ell_1$ same line!

$$\ell_2 = \begin{cases} x = 12t \\ y = 3 + 15t \\ z = 5 + 18t \end{cases} \qquad \vec{v}_2 = \begin{pmatrix} 12 \\ 15 \\ 18 \end{pmatrix} = 3\vec{v} \quad //$$

$$\begin{cases} 1 + 4s = 12t \\ 2 + 5s = 3 + 15t \\ 3 + 6s = 5 + 18t \end{cases} \Longrightarrow \text{no solution.}$$

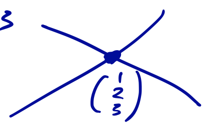
$L \parallel \ell_2$ parallel lines not intersecting.

$$\ell_3 = \begin{cases} x = -2 + 3t \\ y = 4 - 2t \\ z = -1 + 4t \end{cases} \qquad \vec{v}_3 = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \quad \vec{v}_3 \nparallel \vec{v}$$

$s = 0$

$$\begin{cases} 1 + 4s = -2 + 3t \\ 2 + 5s = 4 - 2t \\ 3 + 6s = -1 + 4t \end{cases} \quad t = 1 \longrightarrow \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

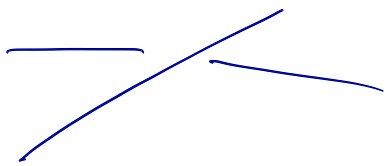
L intersects ℓ_3



$$\ell_4 = \begin{cases} x = -1 + t \\ y = t \\ z = 2 + t \end{cases} \qquad \vec{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{v}_4 \nparallel \vec{v}$$

$$\begin{cases} 1 + 4s = -1 + t \\ 2 + 5s = t \\ 3 + 6s = 2 + t \end{cases} \longrightarrow \text{no solution!}$$

skew lines.



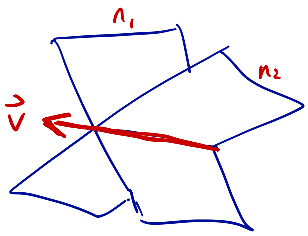
Problem 3. Find the angle between the planes and their line of intersection

$$\begin{cases} x+y+z=1 \\ x-2y+3z=1 \end{cases}$$

$\vec{n}_1 = \langle 1, 1, 1 \rangle$

$\vec{n}_2 = \langle 1, -2, 3 \rangle$

$\vec{n}_1, \vec{n}_2$: $\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \dfrac{2}{\sqrt{3}\sqrt{14}} = \dfrac{2}{\sqrt{42}}$



$\theta \simeq 72° \ldots$

$\vec{v} \perp \vec{n}_1$ & $\vec{v} \perp \vec{n}_2$

$\longrightarrow \vec{v} \parallel \vec{n}_1 \times \vec{n}_2$

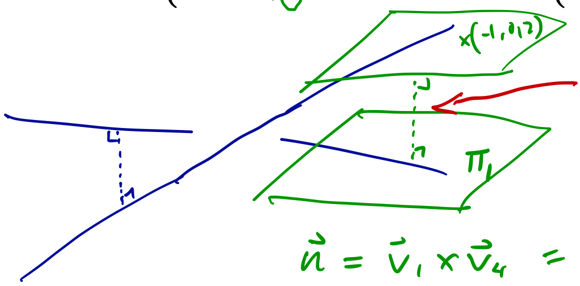
$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\vec{i} - 2\vec{j} - 3\vec{k}$.

point? $\begin{cases} x=1 \\ y=0 \\ z=0 \end{cases} \Longrightarrow \vec{r}(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ -2 \\ -3 \end{pmatrix}$ ✓.

Problem 4. Find the shortest distance between the skew lines

$$L = \begin{cases} x = 1 + 4s \\ y = 2 + 5s \\ z = 3 + 6s \end{cases} \qquad \text{and} \qquad \ell_4 = \begin{cases} x = -1 + t \\ y = 0 + t \\ z = 2 + t \end{cases}$$

$\vec{v}_1$ $\qquad \vec{v}_4 = \langle 1, 1, 1 \rangle$



= distance between 2 // planes.

$$\vec{n} = \vec{v}_1 \times \vec{v}_4 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 5 & 6 \\ 1 & 1 & 1 \end{vmatrix} = -\vec{i} + 2\vec{j} - \vec{k}$$

$\vec{r}_0 = \langle 1, 2, 3 \rangle$, $\vec{n} = \langle -1, 2, -1 \rangle$

$\Pi_1 \Longrightarrow -x + 2y - z = 0$

by distance formula: $(-1, 0, 2)$ and $-x + 2y - z = 0$ d

$$= \left| \frac{1 + 0 - 2}{\sqrt{1^2 + 2^2 + 1^2}} \right| = \frac{1}{\sqrt{6}}$$