HKUST MATH 102

Second Midterm Examination

Multivariable and Vector Calculus

15 Dec 2006

Answer ALL 8 questions

Time allowed – 180 minutes

Problem 1

(a) Find an equation of the plane through $(-1, 4, -3)$ and perpendicular to the line

$$x = t + 2, \quad y = 2t - 3, \quad z = -t.$$

(b) Find a rectangular equation for the surface whose spherical equation is $\rho = 2 \sin\theta \sin\phi$. Describe the surface.

(c) Show that the two lines $\mathbf{r} = \mathbf{a} + \mathbf{v}t$ and $\mathbf{r} = \mathbf{b} + \mathbf{u}t$, where t is a parameter and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$ and $\mathbf{v}$ are constant vectors, will intersect if $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

Problem2

(a) Describe the graph of the equation $\mathbf{r}_1(t) = -2\,\mathbf{i} + t\,\mathbf{j} + (t^2 - 1)\,\mathbf{k}$.

Find also the vector equation of the tangent line to the curve $\mathbf{r}_1(t)$ such that it is parallel to the line $\mathbf{r}_2(t) = \mathbf{i} + (2 + 2t)\,\mathbf{j} + (3 + 4t)\,\mathbf{k}$.

(b) Sketch the surfaces $x + y = 4$ and $\dfrac{y^2}{4^2} + \dfrac{z^2}{2^2} = 1$ in the *first* octant. Find the parametric equations of the curve C of intersection of the two surfaces above. Find the parametric equation of the projection curve C onto the xz-plane. Describe the projection curve.

Problem 3

(a) Sketch the domain of the function $f(x, y) = \dfrac{\ln(x + y + 1)}{x^2 - 1}$.

(b) Determine the largest set on which the function

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

is continuous.

(c) Describe the level surfaces of the function $f(x, y, z) = (x - 2)^2 + y^2$.

Problem 4

(a) Let $f(x,y) = \sqrt{3x+2y}$.

(i) Find the slope of the surface $z = f(x,y)$ in the x-direction at the point $(4,2)$.

(ii) Find the slope of the surface $z = f(x,y)$ in the y-direction at the point $(4,2)$.

(b) Let $g(x,y) = (x^2+y^3)^{\frac{2}{3}}$. Find $g_x(x,y)$, at all points (x,y) in the xy-plane (*include* the point $(0,0)$).

(c) Find $\dfrac{\partial^3}{\partial t^2 \partial s} f(s^2 - t,\ s + t^2)$ in terms of partial derivatives of f. Assume that f has continuous partial derivatives of all orders.

Problem 5

(a) If $f(x,y,z) = (\mathbf{r} \times \mathbf{A}) \cdot (\mathbf{r} \times \mathbf{B})$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $\mathbf{A}$ and $\mathbf{B}$ are constant vectors, show that $\nabla f(x,y,z) = \mathbf{P} \times (\mathbf{r} \times \mathbf{A}) + \mathbf{Q} \times (\mathbf{r} \times \mathbf{B})$. Find $\mathbf{P}$ and $\mathbf{Q}$ in terms of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{r}$.

(b) Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and let $r = \|\mathbf{r}\|$. If $\mathbf{A}$ and $\mathbf{B}$ are constant vectors, show that:

(i) $\mathbf{A} \cdot \nabla\left(\dfrac{1}{r}\right) = \dfrac{\mathbf{C}}{r^3}$. Find $\mathbf{C}$ in terms of $\mathbf{A}$ and $\mathbf{r}$.

(ii) $\mathbf{B} \cdot \nabla\left(\mathbf{A} \cdot \nabla\left(\dfrac{1}{r}\right)\right) = \dfrac{\mathbf{D}}{r^5} - \dfrac{\mathbf{A} \cdot \mathbf{B}}{r^3}$. Find $\mathbf{D}$ in terms of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{r}$.

Problem 6

A three dimensional surface whose equation is $y = f(x)$ is tangent to the surface $z^2 + 2xz + y = 0$ at all points common to the two surfaces. (i) Find $f(x)$. (ii) Find all common points.

Problem 7

(a) Let f be a nonconstant scalar field, differentiable everywhere in the plane, and let c be a constant. Assume the Cartesian equation $f(x,y) = c$ describes a curve C having a tangent at each of its points. Prove that f has the following properties at each point of C:

(i) The gradient vector ∇f is normal to C.

(ii) The directional derivative of f is zero along C.

(iii) The directional derivative of f has its largest value in a direction normal to C.

(b) Find the directional derivative of the scalar field $f(x,y) = x^2 - 3xy$ along the parabola $y = x^2 - x + 2$ at the point $(1,2)$.

Problem 8

A manufacturer is planning to sell a new product at the price of \$150 per unit and estimates that if x thousand dollars is spent on development and y thousand dollars is spent on promotion, approximately $\dfrac{320y}{y+2} + \dfrac{160x}{x+4}$ units of the product will be sold. The cost of manufacturing the product is \$50 per unit. If the manufacturer has a total \$8,000 to spend on development and promotion, how should this money be allocated to generate the largest possible profit?

Suppose the manufacturer in the above exercise decides to spend \$8,100 instead of \$8,000 on the development and promotion of the new product. Use the Lagrange multiplier λ to estimate how this change will affect the maximum possible profit.