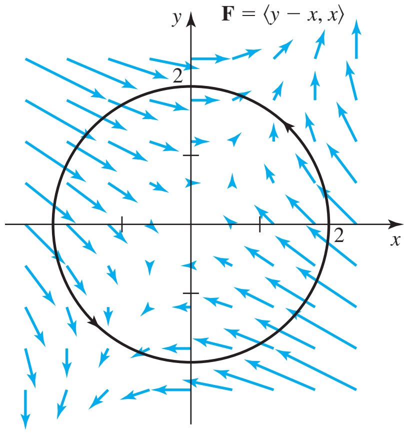
MATH 2023 • Multivariable Calculus
Problem Set #7 • Line Integrals, Conservative Vector Fields, Curl Operator

Do not use the Green's Theorem in any problem in this set.

1. (★) Let $\mathbf{F} = (y - x)\mathbf{i} + x\mathbf{j}$ on $\mathbb{R}^2$, and C be the counter-clockwise circular path with radius 2 centered at the origin. See the figure below:



   (a) On the above figure, highlight the portion of the path C at which $\mathbf{F} \cdot \mathbf{r}' > 0$.
   (b) On the above figure, highlight (with another color) the portion of the path C at which $\mathbf{F} \cdot \mathbf{r}' < 0$.
   (c) Calculate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ from the definition. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

Solution: First parametrize the path:

$$\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

That is, we have $x = 2\cos t$ and $y = 2\sin t$.

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} ((y - x)\mathbf{i} + x\mathbf{j}) \cdot ((-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j})\ dt \\ &= \int_0^{2\pi} -2(y - x)\sin t + 2x\cos t\, dt \\ &= \int_0^{2\pi} -2(2\sin t - 2\cos t)\sin t + 2(2\cos t) \cdot \cos t\, dt \\ &= \int_0^{2\pi} -4\sin^2 t + 4\sin t\cos t + 4\cos^2 t\, dt\end{aligned}$$

Recall from single-variable calculus that $\int_0^{2\pi} \sin^2 t\, dt = \int_0^{2\pi} \cos^2 t\, dt$, so we have:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 4\sin t\cos t\, dt = \int_0^{2\pi} 2\sin 2t\, dt = [-\cos 2t]_0^{2\pi} = 0.$$

We cannot argue whether or not $\mathbf{F}$ is conservative by just showing $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for ONE closed curve – we need $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for ALL closed curves.

(d) Calculate $\nabla \times \mathbf{F}$, i.e. the curl of $\mathbf{F}$. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

Solution:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-x & x & 0 \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y-x & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y-x & x \end{vmatrix} \mathbf{k} \\
&= 0\mathbf{i} - 0\mathbf{j} + \left(\frac{\partial}{\partial x} x - \frac{\partial}{\partial y}(y-x) \right) \mathbf{k} \\
&= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.
\end{aligned}$$

Since $\mathbf{F}$ is defined everywhere, the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is simply-connected. By Curl Test, we conclude that $\mathbf{F}$ is conservative.

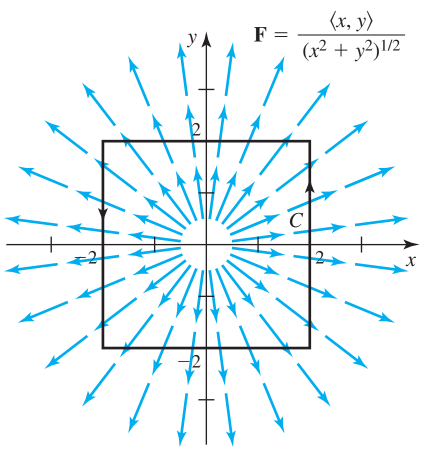
(e) Find a potential function f such that $\mathbf{F} = \nabla f$, or show that such an f does not exist. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

Solution: By inspection, it is not difficult to see that:

$$\mathbf{F} = \nabla \left(xy - \frac{x^2}{2} \right).$$

Therefore, $f(x, y)$ can be taken to be $xy - \frac{x^2}{2}$, and so $\mathbf{F}$ is conservative (by definition).

2. (★) Let $\mathbf{F} = \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\mathbf{j}$, and C be the counter-clockwise square path with vertices $(2, -2)$, $(2, 2)$, $(-2, 2)$ and $(-2, -2)$. See the figure below:



Do (a)-(e) of Problem #1 with this $\mathbf{F}$ and C instead.

Solution: Part (c): To calculate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ from the definition, we split the path C into four segments:

From $(2,2)$ to $(-2,2)$: $\mathbf{r}_1(t) = \langle 2,2\rangle + t(\langle -2,2\rangle - \langle 2,2\rangle) = \langle 2-4t, 2\rangle, \qquad 0 \le t \le 1.$

$$\begin{aligned}
\int_0^1 \mathbf{F}\cdot\mathbf{r}_1'(t)\,dt &= \int_0^1 \frac{(2-4t)\mathbf{i}+2\mathbf{j}}{\sqrt{(2-4t)^2+2^2}}\cdot(-4\mathbf{i}+0\mathbf{j})\,dt \\
&= \int_0^1 \frac{-4(2-4t)}{\sqrt{(2-4t)^2+2^2}}\,dt \\
&= \int_{-2}^2 \frac{u}{\sqrt{u^2+4}}\,dt && \text{(Let } u = 2-4t\text{)} \\
&= 0 && \text{(Odd function!)}
\end{aligned}$$

From $(-2,2)$ to $(-2,-2)$: $\mathbf{r}_2(t) = \langle -2, 2-4t\rangle, \quad 0 \le t \le 1.$

$$\begin{aligned}
\int_0^1 \mathbf{F}\cdot\mathbf{r}_2'(t)\,dt &= \int_0^1 \frac{-2\mathbf{i}+(2-4t)\mathbf{j}}{\sqrt{2^2+(2-4t)^2}}\cdot(0\mathbf{i}-4\mathbf{j})\,dt \\
&= \int_0^1 \frac{-4(2-4t)}{\sqrt{2^2+(2-4t)^2}}\,dt \\
&= 0
\end{aligned}$$

Note the integral is the same as the previous one.

From $(-2,-2)$ to $(2,-2)$: $\mathbf{r}_3(t) = \langle -2+4t, -2\rangle, \quad 0 \le t \le 1.$

$$\begin{aligned}
\int_0^1 \mathbf{F}\cdot\mathbf{r}_3'(t)\,dt &= \int_0^1 \frac{(-2+4t)\mathbf{i}-2\mathbf{j}}{\sqrt{(-2+4t)^2+2^2}}\cdot(4\mathbf{i}+0\mathbf{j})\,dt \\
&= \int_0^1 \frac{4(-2+4t)}{\sqrt{(-2+4t)^2+2^2}}\,dt \\
&= \int_0^1 \frac{-4(2-4t)}{\sqrt{2^2+(2-4t)^2}}\,dt \\
&= 0
\end{aligned}$$

From $(2,-2)$ to $(2,2)$: $\mathbf{r}_4(t) = \langle 2, -2+4t\rangle, \quad 0 \le t \le 1.$

$$\begin{aligned}
\int_0^1 \mathbf{F}\cdot\mathbf{r}_4'(t)\,dt &= \int_0^1 \frac{2\mathbf{i}+(-2+4t)\mathbf{j}}{\sqrt{2^2+(-2+4t)^2}}\cdot(0\mathbf{i}+4\mathbf{j})\,dt \\
&= \int_0^1 \frac{4(-2+4t)}{\sqrt{2^2+(-2+4t)^2}}\,dt \\
&= 0
\end{aligned}$$

Finally, adding up the above line segments, we get:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0+0+0+0 = 0.$$

Note that we can't use this result alone to argue if $\mathbf{F}$ is conservative, as we have just shown that $\oint_C \mathbf{F}\cdot d\mathbf{r}$ for ONE closed curve C (but not for ALL closed curves C).

For Part (d):

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix} \mathbf{k} \\
&= 0\mathbf{i} - 0\mathbf{j} + \left(\frac{\partial}{\partial x} \frac{y}{\sqrt{x^2+y^2}} - \frac{\partial}{\partial y} \frac{x}{\sqrt{x^2+y^2}} \right) \mathbf{k} \\
&= \left\{ \left(-\frac{y}{2}(x^2+y^2)^{-3/2} \cdot 2x \right) - \left(-\frac{x}{2}(x^2+y^2)^{-3/2} \cdot 2y \right) \right\} \mathbf{k} \\
&= 0\mathbf{k} = \mathbf{0}
\end{aligned}
$$

Although $\nabla \times \mathbf{F} = \mathbf{0}$, the domain of $\mathbf{F}$ is $\mathbb{R}^2 \backslash \{(0,0)\}$ which is NOT simply-connected. The Curl Test cannot be used here, and so this result alone cannot conclude on whether $\mathbf{F}$ is conservative.

For Part (e): We can verify that:

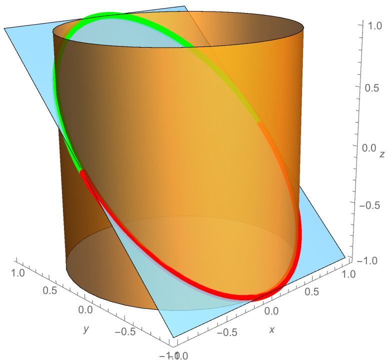
$$
\mathbf{F}(x,y) = \nabla \left(\sqrt{x^2+y^2} \right).
$$

Therefore, one can take the potential function $f(x,y) = \sqrt{x^2+y^2}$, and so $\mathbf{F}$ is conservative from definition.

3. ($\bigstar$) Let C be the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $z=y$.

(a) Sketch the cylinder, the plane and the curve C in the same diagram.

Solution:



(b) Let $\mathbf{F} = y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$. Calculate the line integral $\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ where Γ is a portion of C from $(-1,0,0)$ to $(1,0,0)$. There are two possible such Γ's. Do both.

Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

Solution: We need to first parametrize the path Γ. There are two such possible paths, namely the counter-clockwise path and clockwise path (when looking from the top).

For the counter-clockwise path, the curve lies on the cylinder $x^2 + y^2 = 1$ and therefore projects down to the unit circle centered at the origin on the xy-plane. This unit circle is parametrized by $x = \cos t$ and $y = \sin t$. Therefore, the red path also has x and y coordinates given by $x = \cos t$ and $y = \sin t$. Furthermore, the red path lies on the plane $z = y$, and so we have $z = \sin t$. To sum up, the parametric equation for the red path is:

$$\mathbf{r}_1(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin t)\mathbf{k}, \quad \pi \leq t \leq 2\pi.$$

The bounds for t are chosen so that $\mathbf{r}_1(\pi) = \langle -1, 0, 0\rangle$ and $\mathbf{r}_1(2\pi) = \langle 1, 0, 0\rangle$, which are the coordinates of the starting and ending points of the red path.

$$\begin{aligned}
&\int_{\text{red path}} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{\pi}^{2\pi} (y\mathbf{i} + z\mathbf{j} - x\mathbf{k}) \cdot \mathbf{r}_1'(t)\, dt \\
&= \int_{\pi}^{2\pi} ((\sin t)\mathbf{i} + (\sin t)\mathbf{j} - (\cos t)\mathbf{k}) \cdot ((-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\cos t)\mathbf{k})\, dt \\
&= \int_{\pi}^{2\pi} (-\sin^2 t + \sin t \cos t - \cos^2 t)\, dt = \int_{\pi}^{2\pi} (-1 + \sin t \cos t)\, dt \\
&= \left[-t - \frac{1}{2}\cos^2 t\right]_{\pi}^{2\pi} = -\pi
\end{aligned}$$

For the clockwise path, we can parametrize it by replacing all t's in $\mathbf{r}_1(t)$ by $-t$, i.e.:

$$\begin{aligned}
\mathbf{r}_2(t) &= (\cos(-t))\mathbf{i} + (\sin(-t))\mathbf{j} + (\sin(-t))\mathbf{k} \\
&= (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - (\sin t)\mathbf{k}
\end{aligned}$$

In order to give starting point $(-1, 0, 0)$ and ending point $(1, 0, 0)$, we can set the bounds for t to be $\pi \leq 2\pi$, then $\mathbf{r}_2(\pi) = \langle -1, 0, 0\rangle$ and $\mathbf{r}_2(2\pi) = \langle 1, 0, 0\rangle$.

$$\begin{aligned}
&\int_{\text{green path}} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{\pi}^{2\pi} ((-\sin t)\mathbf{i} + (-\sin t)\mathbf{j} - (\cos t)\mathbf{k}) \cdot ((-\sin t)\mathbf{i} + (-\cos t)\mathbf{j} + (-\cos t)\mathbf{k})\, dt \\
&= \int_{\pi}^{2\pi} (1 + \sin t \cos t)\, dt = \left[t - \frac{1}{2}\cos^2 t\right]_{\pi}^{2\pi} = \pi
\end{aligned}$$

Since we can find two different paths with the same starting and ending points so that the line integral of $\mathbf{F}$ over them are not equal, we conclude that $\mathbf{F}$ is not conservative.

(c) Find a potential function f such that $\mathbf{F} = \nabla f$, or show that such an f does not exist. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

Solution: Set up:

$$\frac{\partial f}{\partial x} = y$$
$$\frac{\partial f}{\partial y} = z$$
$$\frac{\partial f}{\partial z} = -x$$

Integrating the first equation gives:

$$f(x, y, z) = xy + g(y, z)$$

Then:

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}(y, z)$$

Combining with the second equation, we get:

$$z = x + \frac{\partial g}{\partial y}(y, z) \implies z - \frac{\partial g}{\partial y}(y, z) = x.$$

Now that the RHS is a function of x while the LHS is a function of y and z. It is a contradiction. Therefore, such an f does not exist and so $\mathbf{F}$ is not conservative by definition.

Alternatively, we can also check that:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & -x \end{vmatrix} = -\mathbf{i} + \mathbf{j} - \mathbf{k} \neq \mathbf{0}$$

Conservative vector field must have zero curl. Now that curl of $\mathbf{F}$ is non-zero, the vector field $\mathbf{F}$ is not conservative.

4. (★) Determine whether or not each of the following vector fields is conservative or not. If so, find its potential function f such that $\mathbf{F} = \nabla f$.

(a) $\mathbf{F} = (e^{-y} - ze^{-x})\mathbf{i} + (e^{-z} - xe^{-y})\mathbf{j} + (e^{-x} - ye^{-z})\mathbf{k}$

Solution: Set up:

$$\frac{\partial f}{\partial x} = e^{-y} - ze^{-x}$$
$$\frac{\partial f}{\partial y} = e^{-z} - xe^{-y}$$
$$\frac{\partial f}{\partial z} = e^{-x} - ye^{-z}$$

By integrating the first equation, we get:

$$f(x,y,z) = xe^{-y} + ze^{-x} + g(y,z) \quad \text{where } g(y,z) \text{ is an arbitrary function}$$

Then, by differentiation we get $\frac{\partial f}{\partial y} = -xe^{-y} + \frac{\partial g}{\partial y}$, and combined with the second equation, we must have:

$$\frac{\partial g}{\partial y} = e^{-z}.$$

Integrating this, we get $g(y,z) = ye^{-z} + h(z)$ for some arbitrary function $h(z)$, and so

$$f(x,y,z) = xe^{-y} + ze^{-x} + ye^{-z} + h(z).$$

Again by differentiating, we get: $\frac{\partial f}{\partial z} = e^{-x} - ye^{-z} + h'(z)$. Combine with the third equation, we get $h'(z) = 0$ and so h is a constant.

It can be easily verified that $\nabla\left(xe^{-y} + ze^{-x} + ye^{-z} + C\right) = \mathbf{F}$, so $\mathbf{F}$ is conservative with potential funciton $f(x,y,z) = xe^{-y} + ze^{-x} + ye^{-z} + C$.

(b) $\mathbf{F} = (x^2 - xy)\mathbf{i} + (y^2 - xy)\mathbf{j}$

Solution: Set up:

$$\begin{aligned} \frac{\partial f}{\partial x} &= x^2 - xy \\ \frac{\partial f}{\partial y} &= y^2 - xy \end{aligned}$$

Integrating the first equation we get:

$$f(x,y) = \frac{x^3}{3} - \frac{x^2 y}{2} + g(y)$$

By differentiation, we get $\frac{\partial f}{\partial y} = -\frac{x^2}{2} + g'(y)$. Combining with the second equation, we must have

$$y^2 - xy = -\frac{x^2}{2} + g'(y).$$

However, that would imply

$$\frac{x^2}{2} = g'(y) - y^2 + xy.$$

LHS is a function of x only, while RHS depends on both x and y. Therefore, such a function f cannot exist and therefore $\mathbf{F}$ is not conservative.

Alternatively, one can show $\mathbf{F}$ is not conservative by showing:

$$\nabla \times \mathbf{F} = (x - y)\mathbf{k}.$$

Therefore, $\nabla \times \mathbf{F}$ is non-zero, and so $\mathbf{F}$ is not conservative.

5. (★) Determine the values of A and B for which the vector field below is conservative:

$$\mathbf{F}(x,y,z) = Ax\ln z\,\mathbf{i} + By^2z\,\mathbf{j} + \left(\frac{x^2}{z} + y^3\right)\mathbf{k},$$

where the domain of $\mathbf{F}$ is the upper-half space $\{(x,y,z) : z > 0\}$.

For each such pair of A and B, find the potential function f for the vector field.

Solution: Note that the domain of $\mathbf{F}$ is the upper-half space, which *is* simply-connected! Therefore, we have:

$$\mathbf{F} \text{ is conservative } \iff \nabla \times \mathbf{F} = \mathbf{0}$$

By straight-forward computations (omitted here), we get:

$$\nabla \times \mathbf{F} = (3-B)y^2\mathbf{i} + \frac{(A-2)x}{z}\mathbf{j} + 0\mathbf{k}$$

Therefore, $\mathbf{F}$ is conservative if and only if $A = 2$ and $B = 3$.

For this pair of A and B, we solve the equation $\mathbf{F} = \nabla f$ for f:

$$\frac{\partial f}{\partial x} = 2x\ln z$$
$$\frac{\partial f}{\partial y} = 3y^2z$$
$$\frac{\partial f}{\partial z} = \frac{x^2}{z} + y^3$$

Integrating the first equation, we get:

$$f(x,y,z) = x^2\ln z + g(y,z).$$

Then, we have $\frac{\partial f}{\partial z} = \frac{x^2}{z} + \frac{\partial g}{\partial z}$, and by comparison with the third equation, we get $\frac{\partial g}{\partial z} = y^3$, and so $g(y,z) = y^3z + h(y)$. Substitute back into f, it comes down to solving h:

$$f(x,y,z) = x^2\ln z + y^3z + h(y)$$

By considering $\frac{\partial f}{\partial y} = 3y^2z + h'(y)$ and the second equation, we conclude $h'(y) = 0$ and so $h(y) = C$. The potential function for the vector field is: $f(x,y,z) = x^2\ln z + y^3z + C$ where C is any real constant.

6. (★★) Consider the path C:

$$\mathbf{r}(t) = \left(\cos^{2M} t\right)\mathbf{i} + \left(\sin^N t\right)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi.$$

Here M is the age of the Earth, and N is the age of the Universe. Assume both M and N are positive finite integers.

Evaluate the line integral:

$$\int_C \left(e^{-y} - ze^{-x}\right)\,dx + \left(e^{-z} - xe^{-y}\right)\,dy + \left(e^{-x} - ye^{-z}\right)\,dz$$

Solution: The vector field represented by this line integral is:

$$\mathbf{F} = (e^{-y} - ze^{-x})\mathbf{i} + (e^{-z} - xe^{-y})\mathbf{j} + (e^{-x} - ye^{-z})\mathbf{k}$$

It appeared in Problem #4(a) in which we showed it is conservative with a potential function $f(x, y, z) = xe^{-y} + ze^{-x} + ye^{-z}$. Therefore, to compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, we simply need to find the starting and ending points of the path C:

$$\begin{aligned} \mathbf{r}(0) &= \mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \langle 1, 0, 0 \rangle \\ \mathbf{r}(\pi) &= \mathbf{i} + 0\mathbf{j} + \pi\mathbf{k} = \langle 1, 0, \pi \rangle \end{aligned}$$

By Fundamental Theorem of Line Integrals, we get:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 0, \pi) - f(1, 0, 0) = (e^0 + \pi e^{-1} + 0) - (e^0 + 0 + 0) = \pi e^{-1}$$

Alternatively, given that $\mathbf{F}$ is a conservative vector field, one can calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ by choosing an easier path joining the same endpoints as C. Clearly, a straight-line path L is an easier path.

From above, the starting and ending points of C are $(1, 0, 0)$ and $(1, 0, \pi)$ respectively. The straight-line L can be parametrized by:

$$\mathbf{r}_L(t) = \langle 1, 0, 0 \rangle + t(\langle 1, 0, \pi \rangle - \langle 1, 0, 0 \rangle) = \langle 1, 0, t\pi \rangle, \qquad 0 \le t \le 1.$$

$$\begin{aligned} &\int_L \mathbf{F} \cdot d\mathbf{r} = \int_L \mathbf{F} \cdot \mathbf{r}_L'(t)\, dt \\ &= \int_0^1 \left((e^{-y} - ze^{-x})\mathbf{i} + (e^{-z} - xe^{-y})\mathbf{j} + (e^{-x} - ye^{-z})\mathbf{k}\right) \cdot \mathbf{r}_L'(t)\, dt \\ &= \int_0^1 \left((e^{-y} - ze^{-x})\mathbf{i} + (e^{-z} - xe^{-y})\mathbf{j} + (e^{-x} - ye^{-z})\mathbf{k}\right) \cdot (1\mathbf{i} + 0\mathbf{j} + \pi\mathbf{k})\, dt \\ &= \int_0^1 \pi(e^{-x} - ye^{-z})\, dt = \int_0^1 \pi(e^{-1} - 0)\, dt = \pi e^{-1} \end{aligned}$$

Note that along the straight-line L, we have $x = 1$, $y = 0$ and $z = t\pi$.

Finally, since $\mathbf{F}$ is conservative by Problem #4(a), we have $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_L \mathbf{F} \cdot d\mathbf{r} = \pi e^{-1}$.

7. (★★) Given a conservative vector field $\mathbf{F}$ in $\mathbb{R}^3$, the potential *energy* of $\mathbf{F}$ is a scalar-valued function $V(x, y, z)$ such that $\mathbf{F} = -\nabla V$. Suppose $\mathbf{r}(t)$ is the path of a particle with mass m traveling in accordance to the Newton's Second Law $\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$. Then its kinetic energy is defined to be:

$$\text{KE} = \frac{1}{2}m\left|\mathbf{r}'(t)\right|^2.$$

The total (kinetic + potential) energy of the particle at time t is therefore given by:

$$E(t) := \frac{1}{2}m\left|\mathbf{r}'(t)\right|^2 + V(\mathbf{r}(t)).$$

Show that the total energy is conserved, i.e. $E'(t) = 0$ for all time t.

[Hint: the only fact you need to know about Physics is the Newton's Second Law given above. It is purely a math problem.]

Solution: The key idea is to write $|\mathbf{r}'(t)|^2$ as $\mathbf{r}'(t) \cdot \mathbf{r}'(t)$. Also, it's essential to observe that along the path, the potential energy V is first of all a function of x, y and z, and (x, y, z) are functions of t. Therefore, one can use the chain rule to find $\frac{dV}{dt}$.

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\left(\frac{1}{2}m\mathbf{r}'(t)\cdot\mathbf{r}'(t)\right) + \frac{dV}{dt} \\
&= \frac{1}{2}m\underbrace{\left(\mathbf{r}''(t)\cdot\mathbf{r}'(t) + \mathbf{r}'(t)\cdot\mathbf{r}''(t)\right)}_{\text{product rule}} + \underbrace{\left(\frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}\right)}_{\text{chain rule}} \\
&= m\mathbf{r}''(t)\cdot\mathbf{r}'(t) + \underbrace{\left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right)\cdot\left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right)}_{\text{a good trick to learn}} \\
&= \mathbf{F}\cdot\mathbf{r}'(t) + \nabla V\cdot\mathbf{r}'(t) \\
&= -\nabla V\cdot\mathbf{r}'(t) + \nabla V\cdot\mathbf{r}'(t) \\
&= 0.
\end{aligned}$$

8. (★★) Denote $\mathbf{e}_\rho = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$ and $\mathbf{e}_r = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$, which are the unit radial vector fields in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

(a) Show that if $\mathbf{F}(x, y, z) = f(\rho)\mathbf{e}_\rho$ where f is a function depending only on $\rho = \sqrt{x^2+y^2+z^2}$, then $\nabla \times \mathbf{F} = \mathbf{0}$ on the domain of $\mathbf{F}$. Is this result alone sufficient to claim that $\mathbf{F}$ is conservative?

Solution: Using $\rho^2 = x^2 + y^2 + z^2$, one can differentiate both sides by x, y and z individually and show:

$$\frac{\partial \rho}{\partial x} = \frac{x}{\rho}, \quad \frac{\partial \rho}{\partial y} = \frac{y}{\rho}, \quad \frac{\partial \rho}{\partial z} = \frac{z}{\rho}.$$

Now consider the vector field $\mathbf{F} = f(\rho)\mathbf{e}_\rho$ whose components are given by:

$$\mathbf{F} = f(\rho)\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho} = \frac{f(\rho)}{\rho}x\mathbf{i} + \frac{f(\rho)}{\rho}y\mathbf{j} + \frac{f(\rho)}{\rho}z\mathbf{k}$$

Then,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{f(\rho)}{\rho}x & \frac{f(\rho)}{\rho}y & \frac{f(\rho)}{\rho}z \end{vmatrix}$$

We are going to compute the $\mathbf{i}$-component as an example (the $\mathbf{j}$- and $\mathbf{k}$-components are similar). The **i**-component is given by:

$$\begin{aligned}
\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{f(\rho)}{\rho}y & \frac{f(\rho)}{\rho}z \end{vmatrix} \mathbf{i} &= \left\{ \frac{\partial}{\partial y}\left(\frac{f(\rho)}{\rho}z\right) - \frac{\partial}{\partial z}\left(\frac{f(\rho)}{\rho}y\right) \right\} \mathbf{i} \\
&= \left\{ \frac{d}{d\rho}\left(\frac{f(\rho)}{\rho}\right)\frac{\partial \rho}{\partial y}\cdot z - \frac{d}{d\rho}\left(\frac{f(\rho)}{\rho}\right)\frac{\partial \rho}{\partial z}\cdot y \right\} \mathbf{i}.
\end{aligned}$$

Here we have used the chain rule on $\frac{\partial}{\partial y}\left(\frac{f(\rho)}{\rho}\right)$ and $\frac{\partial}{\partial z}\left(\frac{f(\rho)}{\rho}\right)$, as $\frac{f(\rho)}{\rho}$ is a function of ρ, and ρ is a function of (x, y, z). Note that:

$$\begin{aligned}
&\frac{d}{d\rho}\left(\frac{f(\rho)}{\rho}\right)\frac{\partial \rho}{\partial y}\cdot z - \frac{d}{d\rho}\left(\frac{f(\rho)}{\rho}\right)\frac{\partial \rho}{\partial z}\cdot y \\
&= \frac{d}{d\rho}\left(\frac{f(\rho)}{\rho}\right)\cdot\frac{y}{\rho}\cdot z - \frac{d}{d\rho}\left(\frac{f(\rho)}{\rho}\right)\cdot\frac{z}{\rho}\cdot y \\
&= 0.
\end{aligned}$$

Therefore, the **i**-component is zero. Similarly one can also show that the **j**- and **k**-components are zero.

Assuming f is C^1 – it should have been stated in the problem, the domain of $\mathbf{F}$ is $\mathbb{R}^3\backslash\{(0,0,0)\}$ since $\mathbf{e}_\rho$ is undefined only at the origin. Therefore, the domain of $\mathbf{F}$ is simply-connected, and now that we showed $\nabla \times \mathbf{F} = \mathbf{0}$. By curl test, we conclude that $\mathbf{F}$ is conservative.

(b) Show that if $\mathbf{G}(x, y) = g(r)\mathbf{e}_r$ where g is a function depending only on $r = \sqrt{x^2 + y^2}$, then $\nabla \times \mathbf{G} = \mathbf{0}$ on the domain of $\mathbf{G}$. Is this result alone sufficient to claim that $\mathbf{G}$ is conservative?

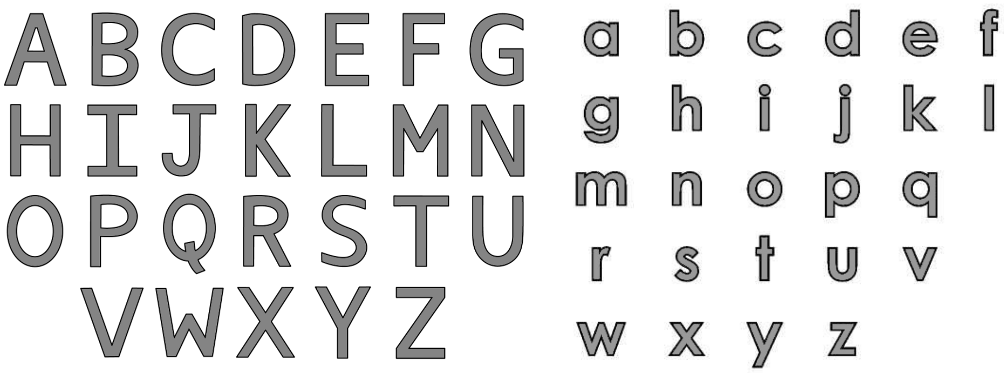
Solution: The way to show $\nabla \times \mathbf{G} = \mathbf{0}$ is very similar to part (a). Here we need to know:

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \text{and} \quad \mathbf{G} = \frac{g(r)}{r}(x\mathbf{i} + y\mathbf{j})$$

One can then calculate the curl of $\mathbf{G}$ using the chain rule.

However, this result alone cannot conclude whether $\mathbf{G}$ is conservative. Since $\mathbf{e}_r$ is undefined at $(0, 0)$ but $\mathbb{R}^2\backslash\{(0,0)\}$ is NOT simply-connected. The curl test does not apply here.

9. (★) Regard each English letter as a solid region in $\mathbb{R}^2$. Which capital letters are simply-connected? Which small letters are simply-connected?



Solution:

Simply-connected capital letters:	C E F G H I J K L M N S T U V W X Y Z
Simply-connected small letters:	c f h k l m n r s t u v w x y z

Note that the small i and j are not connected, and hence not simply-connected.

10. (★★) The notation $\mathbb{R}^3 \backslash X$ means the xyz-space $\mathbb{R}^3$ with the set X removed. Determine whether $\mathbb{R}^3 \backslash X$ is simply-connected when X is each of the following:

(a) X is the origin

(b) X is the entire y-axis

(c) X is the positive y-axis

(d) X is the solid sphere $x^2 + y^2 + z^2 \leq 1$

(e) X is the surface sphere $x^2 + y^2 + z^2 = 1$

(f) X is the solid cylinder $x^2 + y^2 \leq 1$

(g) X is the solid half-cylinder $x^2 + y^2 \leq 1$ and $z \geq 0$.

(h) X is the surface cylinder $x^2 + y^2 = 1$

(i) X is the surface half-cylinder $x^2 + y^2 = 1$ and $z \geq 0$

(j) X is a solid torus

(k) X is a surface torus

(l) X is a simple closed curve

Give an example of a proper subset X of $\mathbb{R}^3$ such that both X and $\mathbb{R}^3 \backslash X$ are simply-connected. [Note: "proper" means X cannot be empty, and cannot be the whole $\mathbb{R}^3$.]

Solution: $\mathbb{R}^3 \backslash X$ is simply-connected for those X's in: (a)(c)(d)(g)(i), whereas $\mathbb{R}^3 \backslash X$ is not simply-connected for those X's in: (b)(e)(f)(h)(j)(k)(l).

Here is one of many examples of X so that both X and $\mathbb{R}^3 \backslash X$ are both simply-connected: When X is the upper-half space $\{(x, y, z) : z > 0\}$. Then $\mathbb{R}^3 \backslash X$ is the lower-half space.