HKUST **MATH 102**

Midterm Examination

Multivariable and Vector Calculus

6 Nov 2006

Answer ALL 5 questions

Time allowed – 120 minutes

Directions – This is a closed book examination. No talking or whispering are allowed. Work must be shown to receive points. An answer alone is not enough. Please write neatly. Answers which are illegible for the grader cannot be given credit.

Note that you can work on *both* sides of the paper and do not detach pages from this exam packet or unstaple the packet.

Student Name: ______________________

Student Number: ______________________

Tutorial Session: ______________________

Question No.	Marks
1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100

Problem 1 (20 points) Your Score:

(a) Assume **a**, **b** and **c** are three dimensional vectors and if

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b} + \beta\mathbf{c}.$$

Use suffix notation to find λ, μ and β in terms of the vectors **a**, **b** and **c**. Can you say something about the direction of the vector $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

(b) (i) Find the distance (in terms of $\mathbf{n}$, $\mathbf{r}_0$ and $\mathbf{r}_1$ only) from the point $\mathbf{r}_1$ to the plane $(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{n} = 0$.

(ii) Use (i) or otherwise, find the distance d between the two parallel planes determined by the equations $Ax + By + Cz = D_1$ and $Ax + By + Cz = D_2$.

(iii) Use (ii) or otherwise, find equations for the planes that are parallel to $x + 3y - 5z = 2$ and lie three units from it.

Solution:

(a)

$$\begin{aligned}
[(\mathbf{a}\times\mathbf{b})\times\mathbf{c}]_i &= \epsilon_{ijk}(\mathbf{a}\times\mathbf{b})_j\, c_k \\
&= \epsilon_{ijk}\epsilon_{jpq}\, a_p b_q c_k \\
&= \epsilon_{jki}\epsilon_{jpq}\, a_p b_q c_k \\
&= (\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip}) a_p b_q c_k \\
&= a_k b_i c_k - a_i b_k c_k
\end{aligned}$$

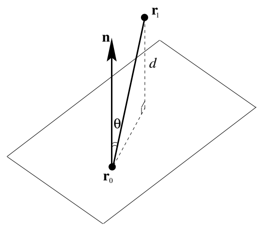
i.e. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$, i.e. $\mu = \mathbf{a}\cdot\mathbf{c}$, $\lambda = -\mathbf{b}\cdot\mathbf{c}$ and $\beta = 0$.

The resulting vector of $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is a linear combination of the vectors **a** and **b**, hence it lies on the plane containing the vectors **a** and **b**.

Problem 1 (20 points) Your Score:

(b) (i)

$$\begin{aligned}
d &= \|\mathbf{r}_1 - \mathbf{r}_0\|\ |\cos\theta| \\
&= \|\mathbf{r}_1 - \mathbf{r}_0\|\cdot\|\widehat{\mathbf{n}}\|\ |\cos\theta| \\
&= |(\mathbf{r}_1 - \mathbf{r}_0)\cdot\widehat{\mathbf{n}}|
\end{aligned}$$



(ii) First locate a point on P_1, i.e. we can let $y = z = 0$, then $x = D_1/A$, i.e. $\mathbf{r}_1 = (D_1/A, 0, 0)$ is a point on P_1. Similarly, $\mathbf{r}_2 = (D_2/A, 0, 0)$ is a point on P_2. Then $\mathbf{r}_1 - \mathbf{r}_2 = \dfrac{D_1 - D_2}{A}\,\mathbf{i}$ and $\widehat{\mathbf{n}} = \dfrac{(A, B, C)}{(A^2 + B^2 + C^2)^{1/2}}$. Therefore, from (i),

$$\begin{aligned}
d &= \frac{|A\cdot(D_1 - D_2)/A|}{(A^2 + B^2 + C^2)^{1/2}} \\
&= \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}.
\end{aligned}$$

(iii) Use (ii), here $A = 1$, $B = 3$, $C = -5$ and $D_1 = 2$. The above equation becomes:

$$3 = \frac{|2 - D_2|}{\sqrt{1^2 + 3^2 + (-5)^2}} = \frac{|2 - D_2|}{\sqrt{35}}.$$

So

$$3\sqrt{35} = |2 - D_2| \quad \text{or} \quad 2 - D_2 = \pm 3\sqrt{35}.$$

So $D_2 = 2 \pm 3\sqrt{35}$ and the equation of the two planes are:

$$x + 3y - 5z = 2 \pm 3\sqrt{35}.$$

Problem 2 (20 points) Your Score:

Let $f(x,y) = \begin{cases} xy\left(\dfrac{x^2-y^2}{x^2+y^2}\right) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$

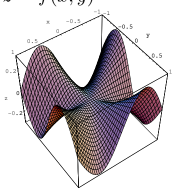
(i) Find $f_x(x,y)$ and $f_y(x,y)$ for $(x,y) \neq (0,0)$.

(ii) Find the partial derivatives $f_x(0,y)$ and $f_y(x,0)$.

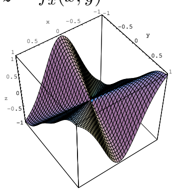
(iii) Find the values of $f_{xy}(0,0)$ and $f_{yx}(0,0)$. Are the mixed partials equal (i.e. $f_{xy}(0,0) = f_{yx}(0,0)$)? Why?

Solution:

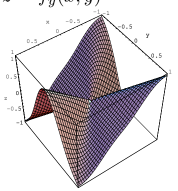
$z = f(x,y)$



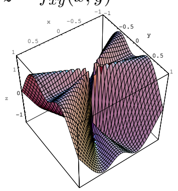
$z = f_x(x,y)$



$z = f_y(x,y)$



$z = f_{xy}(x,y)$



(i) For $(x,y) \neq (0,0)$, compute the partial derivatives:

$$f_x(x,y) = y\left(\frac{x^2-y^2}{x^2+y^2}\right) + xy\left(\frac{[x^2+y^2](2x) - [x^2-y^2](2x)}{(x^2+y^2)^2}\right)$$

$$= \frac{y(x^2-y^2)(x^2+y^2) + xy(4xy^2)}{(x^2+y^2)^2} = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$$

and similarly $\quad f_y(x,y) = \dfrac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}.$

(ii) We use part (i):

$$f_x(0,y) = \frac{y(-y^4)}{(y^2)^2} = -y \quad \text{for} \quad y \neq 0,$$

$$\text{and} \quad f_y(x,0) = x \quad \text{for} \quad x \neq 0.$$

(iii) From part (ii), $f_{xy}(0,y) = -1$ while $f_{yx}(x,0) = 1$ and $f_x(0,y)$ and $f_y(x,0)$ are continuous at the origin so you can conclude that $f_{xy}(0,0) = -1$ while $f_{yx}(0,0) = 1$.

Why aren't the mixed partials equal? The answer is that the second partials are not continuous at the origin. We can see this calculation

$$f_{xy}(x,y) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}.$$

$$\text{Therefore, along } y=0 \quad f_{xy}(x,0) = 1 \quad \Rightarrow \quad \lim_{(x,y)\to(0,0)} f(x,y) = 1$$

$$\text{and along } x=0 \quad f_{xy}(0,y) = -1 \quad \Rightarrow \quad \lim_{(x,y)\to(0,0)} f(x,y) = -1.$$

Hence $\displaystyle\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ does not exist. In other words, f_{xy} is not continuous at the origin. This is from the theorem in the Lecture Notes: Chapter 12, page 21.

Problem 3 (20 points) Your Score:

(i) Let $z = f(x,y) = \big|\,|x| - |y|\,\big| - |x| - |y|$, use the fundamental theorem of partial differentiation to find $f_x(0,0)$ and $f_y(0,0)$.

(ii) Is the function $f(x,y)$ in (i) differentiable at $(0,0)$. Why?

Hint: You may use the following theorem to do part (ii), the function $f(x,y)$ is differentiable at the point (a,b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - hf_x(a,b) - kf_y(a,b)}{\sqrt{h^2+k^2}} = 0.$$

Solution:

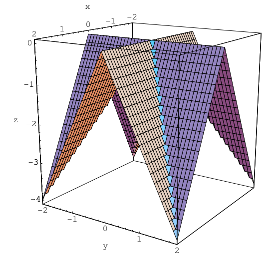
(i) The partial derivatives of f at the origin may be calculated from:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\big|\,|h-0|\,\big| - |h|}{h} = \lim_{h\to 0} 0 = 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\big|\,0-|h|\,\big| - |h|}{h} = \lim_{h\to 0} 0 = 0.$$



(ii) We already know that the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ exist and equal zero. Thus at $(0,0)$, we have

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - 0 - 0}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\big|\,|h|-|k|\,\big| - |h| - |k|}{\sqrt{h^2+k^2}}.$$

However, it is not hard to see that the limit in question fails to exist. Along the line $k=0$ we have

$$\frac{f(h,0)}{\|(h,0)\|} = \frac{\big|\,|h|-0\big| - |h| - |0|}{\sqrt{h^2}} = \frac{0}{|h|} = 0,$$

but along the line $h = k$,

$$\frac{f(h,k)}{\|(h,k)\|} = \frac{\big|\,|h|-|h|\,\big| - |h| - |h|}{\sqrt{h^2+h^2}} = \frac{-2|h|}{\sqrt{2}|h|} = -\sqrt{2}.$$

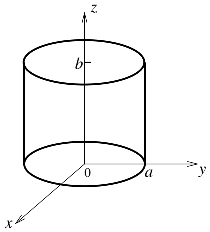
Hence f fails to be differentiable at $(0,0)$.

Problem 4 (20 points) Your Score:

(a) The solid cylinder is positioned such that the center of the bottom disk is at the origin and the z-axis is the axis of the cylinder as shown.

(i) Describe this solid, using cylindrical coordinates.

(ii) Describe this solid, using spherical coordinates.



(b) Show that if a differentiable path lies on a sphere centered at the origin, then its position vector is always perpendicular to its velocity vector.

Solution:

(a) **(i)** In cylindrical coordinates: θ is free to take on any values between 0 and 2π. The z coordinate is bounded by 0 and b, and $0 \leqslant r \leqslant a$. To sum up:

$$\{(r,\theta,z)\,|\,0 \leqslant r \leqslant a, \quad 0 \leqslant z \leqslant b, \quad 0 \leqslant \theta < 2\pi\}.$$

(ii) In spherical coordinates: For $z = b$, in spherical,coord. $\rho\cos\phi = b \quad \Rightarrow \quad \rho = b/\cos\phi$. For cylinder with radius a, i.e. $r = a$, in spherical coordinates

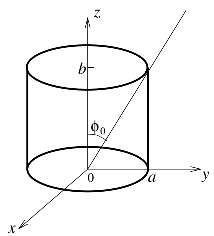
$$\rho\sin\phi = a$$

$$\therefore \quad \rho = a/\sin\phi.$$

$\therefore$ the solid cylinder is

$$\{(\rho,\theta,\phi)\,|\,0 \leqslant \rho \leqslant b/\cos\phi, \quad 0 \leqslant \phi \leqslant \phi_0, \quad 0 \leqslant \theta < 2\pi\}$$
$$\cup\,\{(\rho,\theta,\phi)\,|\,0 \leqslant \rho \leqslant a/\sin\phi, \quad \phi_0 \leqslant \phi \leqslant \pi/2, \quad 0 \leqslant \theta < 2\pi\}$$

where $\phi_0 = \tan^{-1}(a/b)$.



(b) Let $\mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$ be any path on a sphere centered at the origin with radius a. Then

$$\|\mathbf{r}\|^2 = \mathbf{r}\cdot\mathbf{r} = a^2.$$

Differentiate wrt t, we have

$$2\mathbf{r}'\cdot\mathbf{r} = 0.$$

Therefore $\mathbf{r}' \perp \mathbf{r}$, i.e. its position vector is always perpendicular to its velocity vector.

Problem 5 (20 points) Your Score:

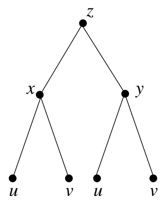
(a) Let $z = f(x,y)$, where $x(u,v) = u+v$, $y(u,v) = u-v$ and f is a differentiable function. Show that $\dfrac{\partial^2 z}{\partial v\partial u} = a\dfrac{\partial^2 z}{\partial x^2} + b\dfrac{\partial^2 z}{\partial x\partial y} + c\dfrac{\partial^2 z}{\partial y^2}$. Find a, b and c.

(b) Suppose that $z = x^2 + y^3$, where $x = 2st$ and y is a function of s and t. Suppose further that when $(s,t) = (2,1)$, $\dfrac{\partial y}{\partial t} = 0$. Determine $\left.\dfrac{\partial z}{\partial t}\right|_{(2,1)}$.

Solution:

(a)

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \\
\frac{\partial^2 z}{\partial v\partial u} &= \frac{\partial z_u}{\partial v} \\
&= \frac{\partial z_u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z_u}{\partial y}\frac{\partial y}{\partial v} \\
&= \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\partial y}\right)\frac{\partial x}{\partial v} + \left(\frac{\partial^2 z}{\partial y\partial x} + \frac{\partial^2 z}{\partial y^2}\right)\frac{\partial y}{\partial v} \\
&= \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y\partial x} + \frac{\partial^2 z}{\partial x\partial y} - \frac{\partial^2 z}{\partial y^2} \\
&= \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}.
\end{aligned}$$



Therefore, $a = 1$, $b = 0$ and $c = -1$.

(b) We can think of $z = z(s,t) = [x(s,t)]^2 + [y(s,t)]^3$. So

$$\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\times\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\times\frac{\partial y}{\partial t} \\
&= 2x \times 2s + 0 \\
&= 8s^2 t \\
\therefore \left.\frac{\partial z}{\partial t}\right|_{(2,1)} &= 32.
\end{aligned}$$