MATH 2023 – Multivariable Calculus

Lecture #16 Worksheet   April 9, 2019

Problem 1. Let f be a scalar field, $\mathbf{F}$ be a vector field. Rewrite them using ∇, and state whether each expression is meaningful.

(a) curl f

(b) grad f

(c) div $\mathbf{F}$

(d) grad $\mathbf{F}$

(e) curl(grad f)

(f) div(grad f)

(g) grad(div $\mathbf{F}$)

(h) grad(div f)

(j) curl(curl(curl $\mathbf{F}$))

(i) div(div(div $\mathbf{F}$))

(k) (grad f) $\times$ (curl $\mathbf{F}$)

(l) div(curl(grad f))

Problem 2. All vector fields of the form $\mathbf{F} = \nabla g$ satisfies $\nabla \times \mathbf{F} = \mathbf{0}$.

All vector fields of the form $\mathbf{F} = \nabla \times \mathbf{G}$ satisfies $\nabla \cdot \mathbf{F} = 0$.

Are there any equations that all functions of the form $f = \nabla \mathbf{G}$ must satisfy?

Probelm 3. Prove the following identities:

(a) $\nabla \cdot (f\mathbf{F}) = (\nabla f) \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F})$

(b) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

(c) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

Problem 4. Let $f(x, y), g(x, y)$ have continuous partial derivatives, and C, D as in Green's Theorem. Recall that $\mathbf{n}$ is the **unit normal vector** of C away from D.

(a) Use the second form of Green's Theorem to prove the **Green's first identity**:

$$\iint_D f\nabla^2 g dA = \oint_C f(\nabla g) \cdot \mathbf{n} ds - \iint_D \nabla f \cdot \nabla g dA$$

(b) Use this to prove **Green's second identity**

$$\iint_D (f\nabla^2 g - g\nabla^2 f) dA = \oint_C (f\nabla g - g\nabla f) \cdot \mathbf{n} ds$$

(c) If g is **harmonic function**, show that

$$\oint_C (\nabla g) \cdot \mathbf{n} ds = 0$$