HKUST **MATH 101**

Midterm Examination

Multivariable Calculus

16 October 2003

Answer ALL 6 questions

Time allowed – 120 minutes

Problem 1

True (T) or False (F) questions: Write T or F at the bottom table for your answer. No justifications are needed.

(a) At a local maximum (x_0, y_0) of $f(x, y)$, one has $f_{yy}(x_0, y_0) \geqslant 0$. F

(b) The gradient $(2x, 2y)$ is perpendicular to the surface $z = x^2 + y^2$. T × F

(c) The equation $f(x, y) = k$ implicitly defines x as a function of y and $\dfrac{dx}{dy} = \dfrac{\partial f}{\partial y} \Big/ \dfrac{\partial f}{\partial x}$. F

(d) $f(x, y) = \sqrt{16 - x^2 - y^2}$ has both an absolute maximum and an absolute minimum on its domain of definition. F × T

(e) If (x_0, y_0) is a critical point of $f(x, y)$ under the constraint $g(x, y) = 0$, and $f_{xy}(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point. T × F

(f) The vector $\mathbf{r}_u(u, v)$ of a parameterized surface $(u, v) \quad \Rightarrow \quad \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ is normal to the surface. ? T × F

(g) $f(x, y)$ and $g(x, y) = f(x^2, y^2)$ have the same critical points. F

(h) At a saddle point, the directional derivative is zero for two different vectors $\mathbf{u}$, $\mathbf{v}$. F ? × T

(i) The value of the function $f(x, y) = e^x y$ at $(0.001, -0.001)$ can by linear approximation be estimated as -0.001. ? × T

(j) The maximum of $f(x, y)$ under the constraint $g(x, y) = 0$ is the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$. F

Problem 2

(a) Find the distance from the origin to the line $x + y + z = 0$, $2x - y - 5z = 1$.

(b) Use suffix notation to prove the Lagrange's identity: $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$.

Problem 3

Consider a particle which moves on a circular helix in $\mathbb{R}^3$ with position vector given by (all scalars are non-zero):

$$\mathbf{r}(t) = (a \cos \omega t,\ a \sin \omega t,\ b\omega t).$$

(a) Show that the speed of the particle is a constant.

(b) Show that the velocity vector makes a constant non-zero angle with the z-axis.

(c) If $t_1 = 0$ and $t_2 = \dfrac{2\pi}{\omega}$, notice that $\mathbf{r}(t_1) = (a, 0, 0)$ and $\mathbf{r}(t_2) = (a, 0, 2\pi b)$, so the vector $\mathbf{r}(t_2) - \mathbf{r}(t_1)$ is vertical. Conclude that the equation

$$\mathbf{r}(t_2) - \mathbf{r}(t_1) = (t_2 - t_1)\mathbf{r}'(\tau)$$

cannot hold for any $\tau \in (t_1, t_2)$. Thus the Mean Value Theorem does not hold for vector-valued functions.

Problem 4

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \dfrac{x^2 y}{x^2 + y^2}$ unless $x = y = 0$ and $f(0, 0) = 0$.

(a) Show that $D_{\mathbf{v}} f(0, 0)$ exists for all $\mathbf{v} \in \mathbb{R}^2$ by direct computation.

(b) Show that f satisfies the homogeneous relation $f(t\mathbf{v}) = tf(\mathbf{v})$ for all $t \in \mathbb{R}$ and all $\mathbf{v} \in \mathbb{R}^2$.

(c) Show that any differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ satisfying the homogeneous relation $g(t\mathbf{v}) = tg(\mathbf{v})$, $\forall\, t \in \mathbb{R}$, $\forall\, \mathbf{v} \in \mathbb{R}^n$ and $g(\mathbf{0}) = 0$ also satisfies the relation

$$g(\mathbf{v}) = \nabla g(\mathbf{0}) \cdot \mathbf{v} \quad \text{for all} \quad \mathbf{v} \in \mathbb{R}^n$$

and hence must be *linear*.

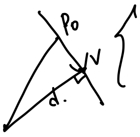
(d) Conclude that f possesses directional derivatives in all directions at $(0, 0)$, but that f is *not* differentiable at $(0, 0)$.

Problem 2

(a) Find the distance from the origin to the line $x + y + z = 0$, $2x - y - 5z = 1$. ?

(b) Use suffix notation to prove the Lagrange's identity: $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$.

$\|u \times v\|^2$??



$$\begin{cases} x + y + z = 0 \\ 2x - y - 5z = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & -5 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -7 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{3} & -\frac{1}{3} \\ 0 & -3 & -7 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 0 & -4 & -1 \\ 0 & -3 & -7 & 1 \end{array}\right]$$

$$3x - 4z = -1$$
$$-3y - 7z = 1$$

$z = t$, $\frac{-1+4t}{3} = x$, $\frac{1+7t}{-3} = y$

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(a) Show that the speed of the particle is a constant.

(b) Show that the velocity vector makes a constant non-zero angle with the z-axis.

$v \cdot (\lambda k) = |v|\,|k| \cos\theta$

(c) If $t_1 = 0$ and $t_2 = \frac{2\pi}{\omega}$, notice that $\mathbf{r}(t_1) = (a, 0, 0)$ and $\mathbf{r}(t_2) = (a, 0, 2\pi b)$, so the vector $\mathbf{r}(t_2) - \mathbf{r}(t_1)$ is vertical. Conclude that the equation

$$\mathbf{r}(t_2) - \mathbf{r}(t_1) = (t_2 - t_1)\mathbf{r}'(\tau)$$

cannot hold for any $\tau \in (t_1, t_2)$. Thus the Mean Value Theorem does not hold for vector-valued functions.

a). $r'(t) = \langle -a\omega\sin\omega t,\ a\omega\cos\omega t,\ b\omega \rangle$

$|r'(t)| = \sqrt{a^2\omega^2 + b^2\omega^2} = C$

b). $v \cdot (\lambda k) = |C|\lambda \cos\theta$ with $\lambda \neq 0$.

$b\omega\lambda = C\lambda \cos\theta$

$\cos\theta = \dfrac{b\omega\lambda}{C\lambda}$

$\cos\theta = \dfrac{b\omega}{C}$

c).

Made with Goodnotes

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(c) Show that any differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ satisfying the homogeneous relation $g(t\mathbf{v}) = tg(\mathbf{v})$, $\forall\, t \in \mathbb{R}$, $\forall\, \mathbf{v} \in \mathbb{R}^n$ and $g(\mathbf{0}) = 0$ also satisfies the relation

$$g(\mathbf{v}) = \nabla g(\mathbf{0}) \cdot \mathbf{v} \quad \text{for all} \quad \mathbf{v} \in \mathbb{R}^n$$

and hence must be *linear*.

(d) Conclude that f possesses directional derivatives in all directions at $(0, 0)$, but that f is *not* differentiable at $(0, 0)$.

$v = (ai, bj)$

$$\nabla f = \left\langle \frac{(x^2+y^2)(2xy) - x^2y(2x)}{(x^2+y^2)^2}, \frac{(x^2+y^2)(x^2) - x^2y(2y)}{(x^2+y^2)^2} \right\rangle$$

$$= \left\langle \frac{2x^3y + 2xy^3 - 2x^3y,\ x^4 + x^2y^2 - 2x^2y^2}{(x^2+y^2)^2} \right\rangle$$

$$= \left\langle \frac{2xy^3,\ x^4 - x^2y^2}{(x^2+y^2)^2} \right\rangle$$

$$D_v f(0,0) = \frac{2xy^3}{(x^2+y^2)^2}\, a + \frac{x^4 - x^2y^2}{(x^2+y^2)^2}\, b$$

$=$

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(d) Conclude that f possesses directional derivatives in all directions at $(0,0)$, but that f is *not* differentiable at $(0,0)$.

4a). $$D_v f(0,0) = \lim_{t\to 0} \frac{f(x+ta,\, y+tb) - f(0,0)}{t}$$

$$= \lim_{t\to 0} \frac{(x^2+2tax+t^2a^2)(y+tb)}{(x^2+2tax+t^2a^2) + (y^2+2tay+t^2b^2)} \left(\frac{1}{t}\right)$$

=

Problem 5

(a) Find the equation of the level curve of the function $z = g(x, y) = xf(xy)$ at the point (x_0, y_0), where both f and g are differentiable. Show that $\nabla g(x_0, y_0)$ is normal to the tangent line to the level curve at (x_0, y_0).

?

(b) Show that, in terms of polar coordinates (r, θ) (where $x = r\cos\theta$, and $y = r\sin\theta$), the gradient of a function $f(r, \theta)$ is given by

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}},$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction of the position vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\hat{\boldsymbol{\theta}}$ is a unit vector at right angles to $\hat{\mathbf{r}}$ in the direction of increasing θ.

[a). $\nabla f = \langle f_r, f_\theta)$

$\nabla f = (f_x X_r + f_y Y_r,\ f_x X_\theta + f_y y_\theta)$

$= \langle \cos\theta f_x + \sin\theta f_y,\ -r\sin\theta f_x + r\cos\theta f_y \rangle$

$=$

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Problem 6

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P\ (2, 0)$ in the direction from P to $Q\left(\frac{1}{2}, 2\right)$. In what direction does f have the maximum rate of change?

(b) Find a single equation of the form $Ax + By + Cz = D$ that describes the plane given parametrically as

$$x = 3s - t + 2$$
$$y = 4s + t$$
$$z = s + 5t + 3.$$

(c) Locate all relative maxima, relative minima and saddle points of the function $f(x, y) = x^4 - y^3$.

a). $V = \langle -\frac{3}{2}, 2\rangle$

$\hat{V} = \dfrac{\langle -\frac{3}{2}, 2\rangle}{\sqrt{\frac{9}{4} + 4}} = \dfrac{\langle -\frac{3}{2}, 2\rangle}{\frac{5}{2}} = \langle -\frac{3}{5}, \frac{4}{5}\rangle$

$\nabla f = \langle e^y, xe^y\rangle$ $\quad \langle 1, 2\rangle \cdot \langle -\frac{3}{5}, \frac{4}{5}\rangle$

$\nabla f(2,0) = \langle 1, 2\rangle$ $\quad = 1$ $\quad \nabla f = \langle 1, 2\rangle$

(b) Find a single equation of the form $Ax + By + Cz = D$ that describes the plane given parametrically as

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(c) Locate all relative maxima, relative minima and saddle points of the function $f(x, y) = x^4 - y^3$.

$$\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} s + \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} t + \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$n = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$$

$$= \begin{vmatrix} i & j & k \\ 3 & 4 & 1 \\ -1 & 1 & 5 \end{vmatrix}$$

$$n = 19i - 16j + 7k$$

$$19x - 16y + 7z = 59.$$

c. $f_x = 4x^3$, $f_y = -3y^2$

$f_{xx} = 12x^2$, $f_{yy} = -6y$

$4x^3 = -3y^2 = 0$

$f_{xy} = 0$ $(0,0)$.

$D = -72x^2y$

$= 0.$

walk along $x=0$, it is saddle.

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