MATH 2023 • Multivariable Calculus
Problem Set #8 • Green's Theorem

1. (★) Use the Green's Theorem to evaluate

$$\oint_C (4y^2 + e^{x^2})\,dx - (2x + e^{y^2})\,dy$$

where C is each of the following (assume C is counter-clockwise oriented):

(a) the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$

(b) the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$

(c) the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$

(d) the unit circle $x^2 + y^2 = 1$

2. (★★) The purpose of this problem is to explore a line integral for computing areas.

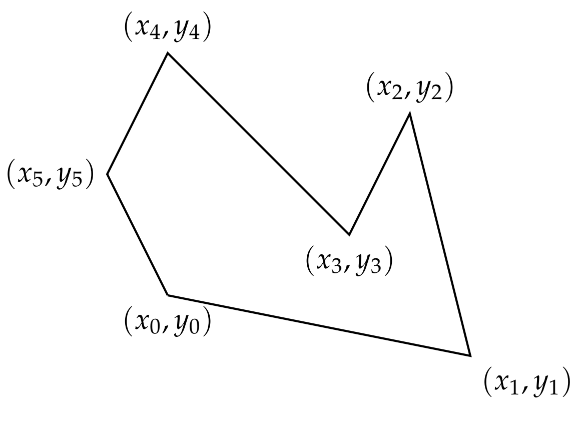
(a) Let C be a simple closed curve in $\mathbb{R}^2$ and the area enclosed by C is denoted by A. Show that:

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy$$

(b) Let E be the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $a, b > 0$. Find the area bounded by E using the result of (a).

(c) Let P be a n-sided polygon with vertices (x_0, y_0), (x_1, y_1), $\ldots$, (x_{n-1}, y_{n-1}). See the figure below for an example when $n = 6$. For convenience, we denote $(x_n, y_n) = (x_0, y_0)$. Using (a), show that the area $A(P)$ bounded by the polygon P is given by:

$$A(P) = \frac{1}{2}\sum_{i=1}^{n}(x_{i-1}y_i - x_i y_{i-1}).$$



3. (★★) Consider the following system of differential equations:

$$\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y)$$

where f and g are C^1 on $\mathbb{R}^2$. Given that $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} > 0$ on $\mathbb{R}^2$, show that the system cannot have a non-constant periodic solution. We say a solution $(x(t), y(t))$ is periodic if there exists $T > 0$ such that $(x(0), y(0)) = (x(T), y(T))$.

Hint: Proof by contradiction. Apply Green's Theorem on $\mathbf{F} = -g(x,y)\mathbf{i} + f(x,y)\mathbf{j}$.

4. (★★) Consider the vector field $\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ which is defined at every point on $\mathbb{R}^2$ except the origin.

   (a) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$ at every point in $\mathbb{R}^2$ except the origin.

   (b) Show, by direct computation, that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is non-zero where C is the unit circle, counter-clockwise oriented, with centered at the origin.

   (c) The following students are confused about the above vector field $\mathbf{F}$ in relation to some facts and theorems stated in class. Pretend that you are a teaching assistant of this course, point out their misconceptions.

      i. Student A said, "Given that $\nabla \times \mathbf{F} = \mathbf{0}$, the Curl Test asserts that $\mathbf{F}$ is conservative and so the closed-path line integral in (b) should be zero. How come the answer for (b) is non-zero???!!!"

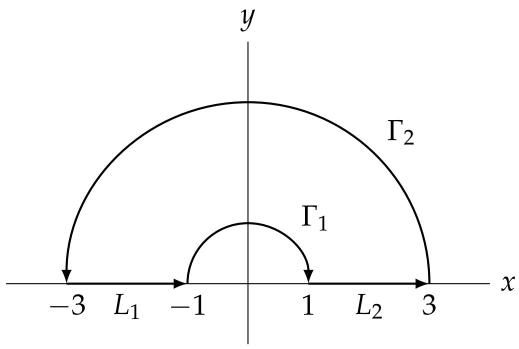
      ii. Student B said, "Given that $\nabla \times \mathbf{F} = \mathbf{0}$, the Green's Theorem asserts that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_R \mathbf{0} \cdot \mathbf{k}\, dA = 0$$

      for any closed-path C. Why can the answer in (b) be non-zero???!!!" "

      iii. Student C said, "It can be verified that $\mathbf{F} = \nabla\left(\tan^{-1}\frac{y}{x}\right)$ and so $\mathbf{F}$ is conservative with potential function $f(x,y) = \tan^{-1}\frac{y}{x}$. Any line integral of a conservative vector field over a closed curve must be zero. How come can the closed-path integral in (b) be non-zero???!!!"

5. (★★) In the figure shown below, Γ_1 and Γ_2 are circular arcs centered at the origin. L_1 and L_2 are straight-lines. Consider the closed path $C = L_1 + \Gamma_1 + L_2 + \Gamma_2$.



Compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ of each vector field below using the Green's Theorem in an *appropriate* way:

   (a) $\mathbf{F} = y^3\mathbf{i} - x^3\mathbf{j}$

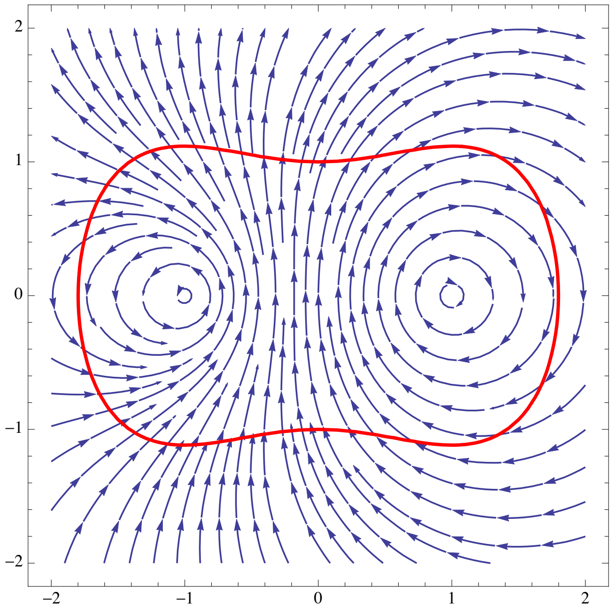
   (b) $\mathbf{F} = -\frac{y-3}{(x-3)^2+(y-3)^2}\mathbf{i} + \frac{x-3}{(x-3)^2+(y-3)^2}\mathbf{j}$

   (c) $\mathbf{F} = -\frac{y-2}{x^2+(y-2)^2}\mathbf{i} + \frac{x}{x^2+(y-2)^2}\mathbf{j}$

6. (★★) Consider the flow of fluid (shown in blue in the figure below) which is represented by the vector field:

$$\mathbf{F} = \left(-\frac{y}{(x+1)^2+y^2} + \frac{2y}{(x-1)^2+y^2}\right)\mathbf{i} + \left(\frac{x+1}{(x+1)^2+y^2} - \frac{2(x-1)}{(x-1)^2+y^2}\right)\mathbf{j}$$

C is an arbitrary simple closed curve (red in the figure) which encloses all points at which $\mathbf{F}$ is not defined.



(a) At which point(s) the vector field $\mathbf{F}$ is/are *not* defined? Is the domain of $\mathbf{F}$ simply-connected?

(b) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$ at every point in $\mathbb{R}^2$ where $\mathbf{F}$ is defined.

(c) Show that from the definition of line integrals:

i. $\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for any counter-clockwise circle Γ centered at $(-1, 0)$ with radius less than 2.

ii. $\oint_\gamma \mathbf{F} \cdot d\mathbf{r} = -4\pi$ for any counter-clockwise circle γ centered at $(1, 0)$ with radius less than 2.

(d) Using the above results, show that:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = -2\pi$$

for any simple closed curve C in $\mathbb{R}^2$ that encloses all points at which $\mathbf{F}$ is not defined.

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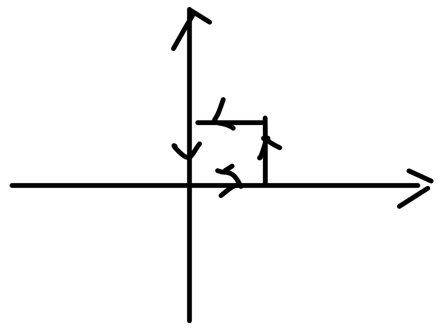
a).



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\, dS$$

$$\nabla\times F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2-(8y)$$

$$\int_0^1\int_0^1 -2-8y\, dx\, dy$$

$$= \int_0^1 -2-8y\, dy$$

$$= -2-4$$

$$= -6$$

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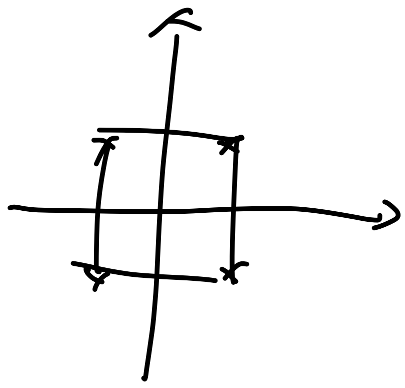

b).
$$\int_{-1}^{1}\int_{-1}^{1} -2-8y \, dx\,dy$$

$$= \int_{-1}^{1} -4-16y \, dy$$

$$= -8 - 16\left[\frac{y^2}{2}\right]_{-1}^{1}$$

$$= -8 - 16\left(\frac{1}{2} - \frac{1}{2}\right)$$

$$= -8$$

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$$\int_0^1 \int_0^{1-y} -2 - 8y \, dx\, dy$$

$$= \int_0^1 (-2-8y)(1-y)\, dy$$

$$= \int_0^1 -2 + 2y - 8y + 8y^2 \, dy$$

$$= \int_0^1 -2 - 6y + 8y^2 \, dy$$

$$= -2 - 6\left[\frac{y^2}{2}\right]_0^1 + 8\left[\frac{y^3}{3}\right]_0^1$$

$$= -6 - 6\left(\frac{1}{2}\right) + 8\left(\frac{1}{3}\right)$$

$$= -6 - 3 + \frac{8}{3}$$

$$= -9 + \frac{8}{3}$$

$$=$$

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$$\iint -2 - 8y \, dA$$

$$= \int_0^{2\pi}\int_0^1 (-2 - 8r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 -2r - 8r^2\sin\theta \, dr\, d\theta$$

$$= \int_0^{2\pi} -1 - \frac{8}{3}\sin\theta \, d\theta$$

$$= -2\pi - \frac{8}{3}\left[-\cos\theta\right]_0^{2\pi}$$

$$= -2\pi$$

2. (★★) The purpose of this problem is to explore a line integral for computing areas.

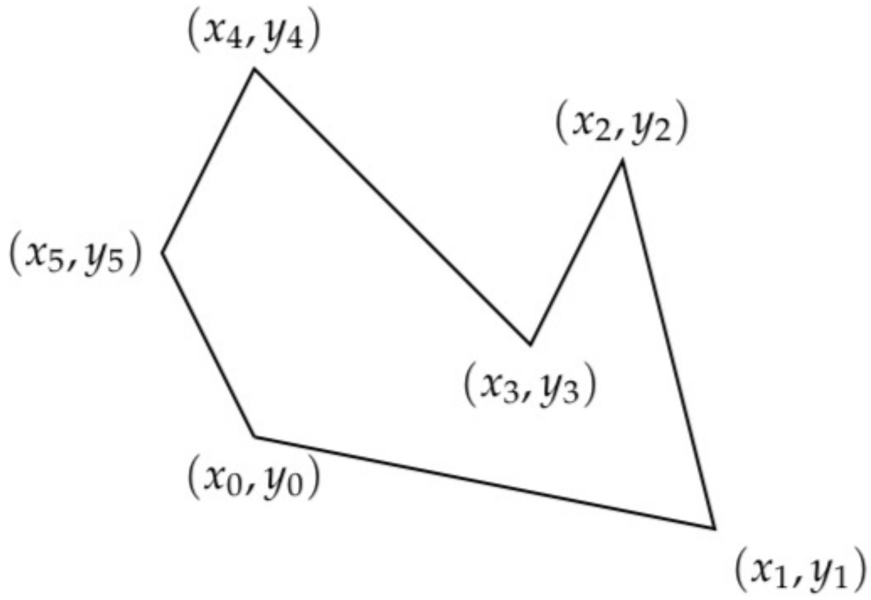
(a) Let C be a simple closed curve in $\mathbb{R}^2$ and the area enclosed by C is denoted by A. Show that:

$$A = \frac{1}{2} \oint_C -y\,dx + x\,dy$$

(b) Let E be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a, b > 0$. Find the area bounded by E using the result of (a).

(c) Let P be a n-sided polygon with vertices (x_0, y_0), (x_1, y_1), $\ldots$, (x_{n-1}, y_{n-1}). See the figure below for an example when $n = 6$. For convenience, we denote $(x_n, y_n) = (x_0, y_0)$. Using (a), show that the area $A(P)$ bounded by the polygon P is given by:

$$A(P) = \frac{1}{2} \sum_{i=1}^{n} (x_{i-1} y_i - x_i y_{i-1}).$$



a). $\iint 1\,dA = \oint_C x\,dy$ Let $F\langle 0, x \rangle$

$= \oint_C -y\,dx$

$2A = \oint_C x\,dy - \oint_C y\,dx$

$= \frac{1}{2} \oint -y\,dx + \oint_C x\,dy.$

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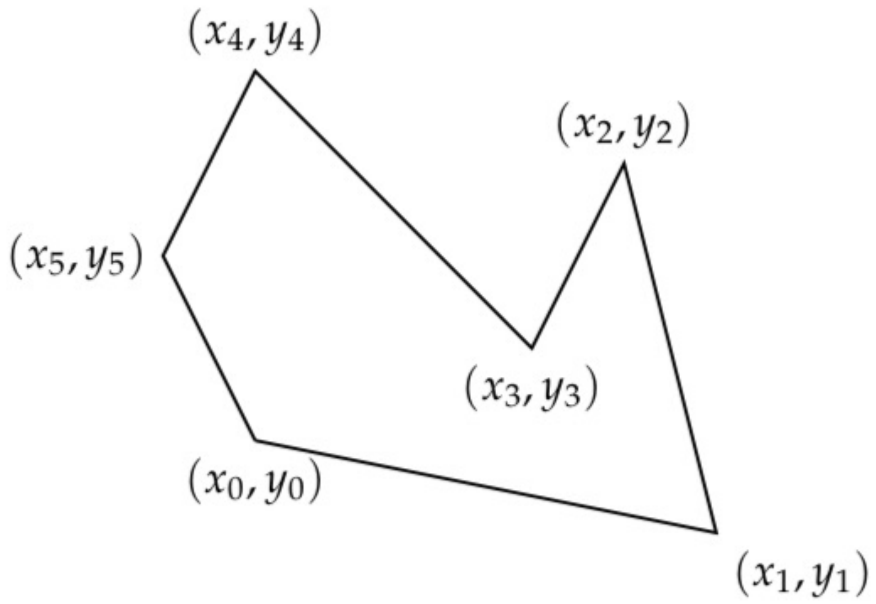

b). $\vec{r}(t) = \langle a\cos t,\ b\sin t \rangle$

$$A = \frac{1}{2}\int_0^{2\pi} -b\sin t\,(-a\sin t)\,dt + a\cos t\,(b\cos t)\,dt$$

$$A = \frac{1}{2}(2\pi \cdot ab)$$

$$A = \pi ab$$

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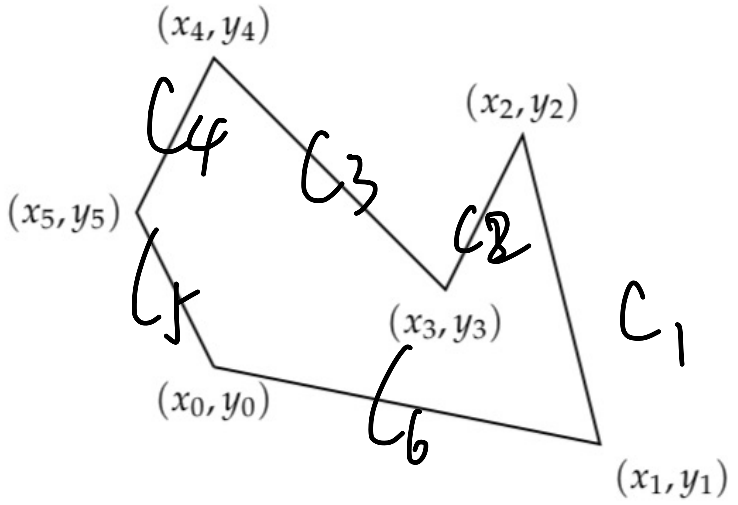

c). $A(P) = \frac{1}{2}\left(\oint_C -y\,dx + x\,dy\right)$

$= \frac{1}{2}\left(\int_{C_1+C_2+\cdots+C_6} -y\,dx + x\,dy\right)$

$= \frac{1}{2}\left(-y_2 \cdot (x_2 - x_1) + x_2(y_2 - y_1)\right)$

$= \frac{1}{2}\left(x_1 y_2 - x_2 y_1 + \cdots\right)$

$= \frac{1}{2}\sum_{i=1}^{n}(x_{i-1} y_i - x_i y_{i-1})$

3. (★★) Consider the following system of differential equations:

$$\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y)$$

where f and g are C^1 on $\mathbb{R}^2$. Given that $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} > 0$ on $\mathbb{R}^2$, show that the system cannot have a non-constant periodic solution. We say a solution $(x(t), y(t))$ is periodic if there exists $T > 0$ such that $(x(0), y(0)) = (x(T), y(T))$.

Hint: Proof by contradiction. Apply Green's Theorem on $\mathbf{F} = -g(x,y)\mathbf{i} + f(x,y)\mathbf{j}$.

$$\int_0^T -g(x,y)\, x'(t) + f(x,y)\, y'(t)\, dt$$

$$= \int_0^T -g(x,y)\, f(x,y) + f(x,y)\, g(x,y)\, dt = 0$$

$$0 = \int_{y(0)}^{y(T)} \int_{x(0)}^{x(T)} \frac{\partial f(x,y)}{\partial x} + \frac{\partial g(x,y)}{\partial y}\, dx\,dy$$

$\cancel{\#}$ Since $\dfrac{\partial f(x,y)}{\partial x} + \dfrac{\partial g(x,y)}{\partial y} > 0$,

$$\int_{x(0)}^{x(T)} \frac{\partial f(x,y)}{\partial x} + \frac{\partial g(x,y)}{\partial y}\, dx > 0$$

$$x(T) \neq x(0)$$

Made with Goodnotes

4. (★★) Consider the vector field $\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ which is defined at every point on $\mathbb{R}^2$ except the origin.

(a) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$ at every point in $\mathbb{R}^2$ except the origin.

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i. Student A said, "Given that $\nabla \times \mathbf{F} = \mathbf{0}$, the Curl Test asserts that $\mathbf{F}$ is conservative and so the closed-path line integral in (b) should be zero. How come the answer for (b) is non-zero???!!!"

ii. Student B said, "Given that $\nabla \times \mathbf{F} = \mathbf{0}$, the Green's Theorem asserts that

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for any closed-path C. Why can the answer in (b) be non-zero???!!!" "

iii. Student C said, "It can be verified that $\mathbf{F} = \nabla\left(\tan^{-1}\frac{y}{x}\right)$ and so $\mathbf{F}$ is conservative with potential function $f(x,y) = \tan^{-1}\frac{y}{x}$. Any line integral of a conservative vector field over a closed curve must be zero. How come can the closed-path integral in (b) be non-zero???!!!"

4a). $\nabla \times F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} - \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = \vec{0}$$

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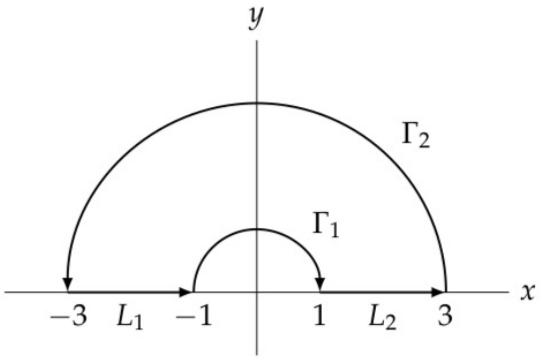
$\vec{r}(t) = \langle \cos t, \sin t \rangle \quad 0 \le t \le 2\pi$

b). $\displaystyle\int_0^{2\pi} -\frac{y}{x^2+y^2}(-\sin t)\,dt + \frac{x}{x^2+y^2}(\cos t)\,dt$

$= \displaystyle\int_0^{2\pi} \sin^2 t + \cos^2 t \, dt$

$= 2\pi$

5. (★★) In the figure shown below, Γ_1 and Γ_2 are circular arcs centered at the origin. L_1 and L_2 are straight-lines. Consider the closed path $C = L_1 + \Gamma_1 + L_2 + \Gamma_2$.



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a). $\iint \nabla \times F \cdot \hat{k} \, dA$

$\iint -3x^2 - 3y^2 \, dA$

$= \int_0^{\pi}\int_1^3 -3r^2 (r) dr d\theta$

$= \pi \left(-3 \left[\frac{r^4}{4}\right]_1^3 \right)$

$= \pi \left(-3 \left(\frac{3^4}{4} - \frac{1}{4} \right) \right)$

b). $\dfrac{(y-3)^2 - (x-3)^2}{((x-3)^2 + (y-3)^2)^2} - \dfrac{(y-3)^2 - (x-3)^2}{((x-3)^2 + (y-3)^2)^2}$

$= 0$

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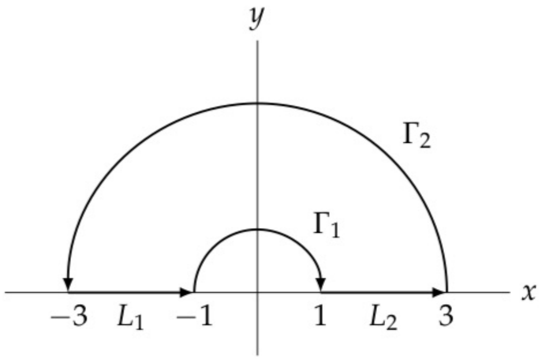

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(c) $\mathbf{F} = -\dfrac{y-2}{x^2+(y-2)^2}\mathbf{i} + \dfrac{x}{x^2+(y-2)^2}\mathbf{j}$ $\quad \langle -3+2t, 0 \rangle$

c). $\displaystyle\int_0^1 -\frac{(y-2)}{x^2+(y-2)^2}\, dt$

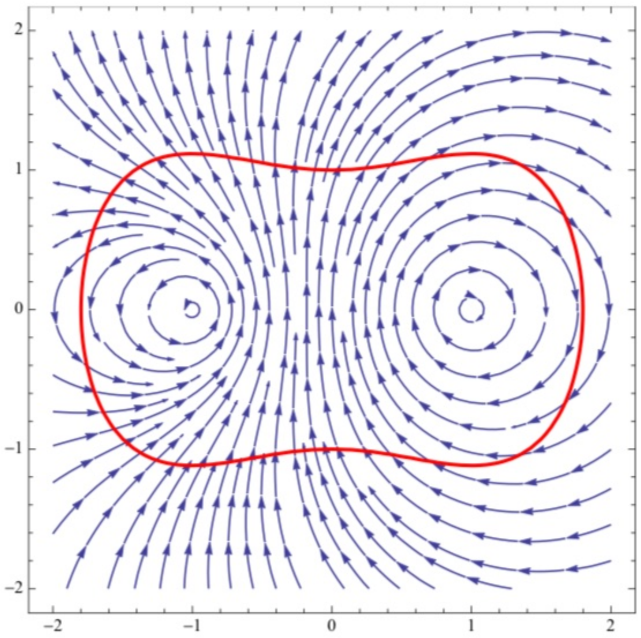
$$= \int_0^1 \frac{2}{(-3+2t)^2+4}\, dt$$

$$= \int_0^1 \frac{2}{4t^2-12t-5}\, dt$$

6. (★★) Consider the flow of fluid (shown in blue in the figure below) which is represented by the vector field:

$$\mathbf{F} = \left(-\frac{y}{(x+1)^2 + y^2} + \frac{2y}{(x-1)^2 + y^2} \right) \mathbf{i} + \left(\frac{x+1}{(x+1)^2 + y^2} - \frac{2(x-1)}{(x-1)^2 + y^2} \right) \mathbf{j}$$

C is an arbitrary simple closed curve (red in the figure) which encloses all points at which $\mathbf{F}$ is not defined.



$\langle -1, 0 \rangle \quad \langle 1, 0 \rangle$

(a) At which point(s) the vector field $\mathbf{F}$ is/are *not* defined? Is the domain of $\mathbf{F}$ simply-connected?

(b) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$ at every point in $\mathbb{R}^2$ where $\mathbf{F}$ is defined.

(c) Show that from the definition of line integrals:

i. $\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for any counter-clockwise circle Γ centered at $(-1, 0)$ with radius less than 2.

ii. $\oint_\gamma \mathbf{F} \cdot d\mathbf{r} = -4\pi$ for any counter-clockwise circle γ centered at $(1, 0)$ with radius less than 2.

(d) Using the above results, show that:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = -2\pi$$

for any simple closed curve C in $\mathbb{R}^2$ that encloses all points at which $\mathbf{F}$ is not defined.