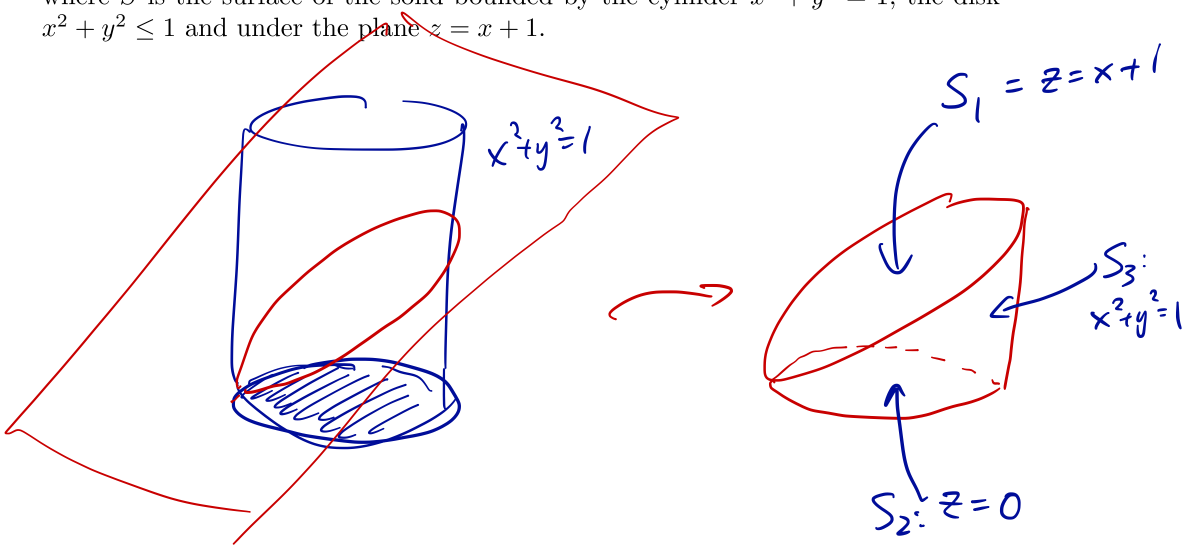
MATH 2023 – Multivariable Calculus

Lecture #18 Worksheet April 16, 2019

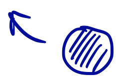
Problem 1. Find the surface integral

$$\iint_S z\,dS$$

where S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$, the disk $x^2 + y^2 \leq 1$ and under the plane $z = x + 1$.



$$S_1 : \iint_S z\,dS = \iint_D (x+1)\sqrt{1+1+0}\,dA = \int_0^{2\pi}\int_0^1 (r\cos\theta + 1)\sqrt{2}\,r\,dr\,d\theta$$



$$\int_0^{2\pi} \cos\theta = 0$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{2}\,r\,dr\,d\theta$$

$$= \sqrt{2}\,\pi$$

$$S_2 : z = 0 : \iint_S z\,dS = 0.$$

$$S_3 : \vec{r}(u,\theta) = \langle \cos\theta, \sin\theta, u \rangle$$

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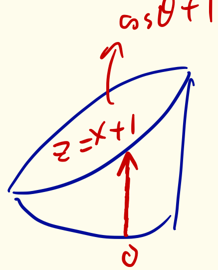
$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ \end{array}$$

$$\begin{aligned} \vec{r}_u &= \langle 0, 0, 1 \rangle \\ \vec{r}_\theta &= \langle -\sin\theta, \cos\theta, 0 \rangle \end{aligned}$$

$$\vec{r}_u \times \vec{r}_\theta = \langle -\cos\theta, -\sin\theta, 0 \rangle$$

$$|\vec{r}_u \times \vec{r}_\theta| = 1.$$



$$\iint_{S_3} z\,dS = \int_0^{2\pi}\int_0^{\cos\theta+1} u\,du\,d\theta$$

$$= \int_0^{2\pi} \frac{u^2}{2}\Big|_0^{\cos\theta+1} d\theta$$

$$= \int_0^{2\pi} \frac{(\cos\theta+1)^2}{2} d\theta$$

$$= \frac{3}{2}\pi \;//$$

$$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS = \sqrt{2}\pi + 0 + \frac{3}{2}\pi$$

Problem 2. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = y\mathbf{i} + (z-y)\mathbf{j} + x\mathbf{k}$ and S is the surface of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

Problem 3. Let $\mathbf{G} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$ be the gravitational field, where $\mathbf{r} = \langle x, y, z \rangle$. Show that the flux of $\mathbf{G}$ across a sphere S with center at the origin is indepndent of the radius of S.