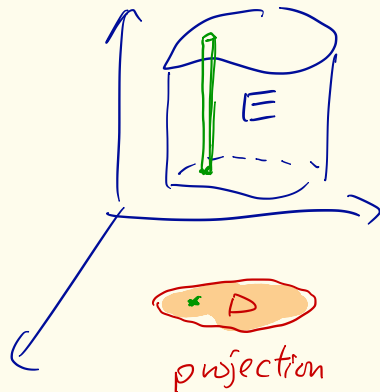


# Last Time : Triple Integrations

$$\iiint_E f(x,y,z) dV$$



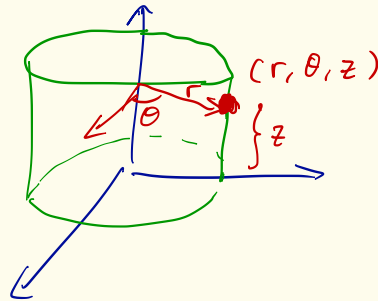
$$\iint_D \left( \int dz \right) dA$$

↑  
base  
projection

# Cylindrical Coordinate

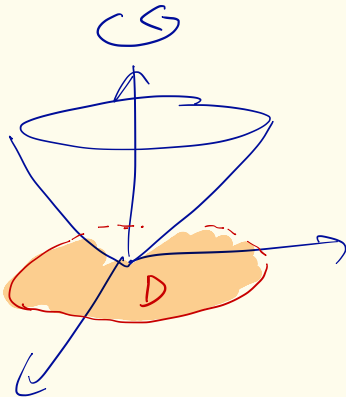
= Using polar coordinate in x-y.

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$



$$dx dy dz \Rightarrow \underline{r dz} \underline{dr d\theta}$$

use  $r dr d\theta$  as base



$$\iint_D \left( \int dz \right) dA$$

↓

$$\iint_D \left( \int dz \right) r dr d\theta$$

Ex  $E$  = solid bounded by

cylinder  $x^2 + y^2 = 1$

$z \leq 4$

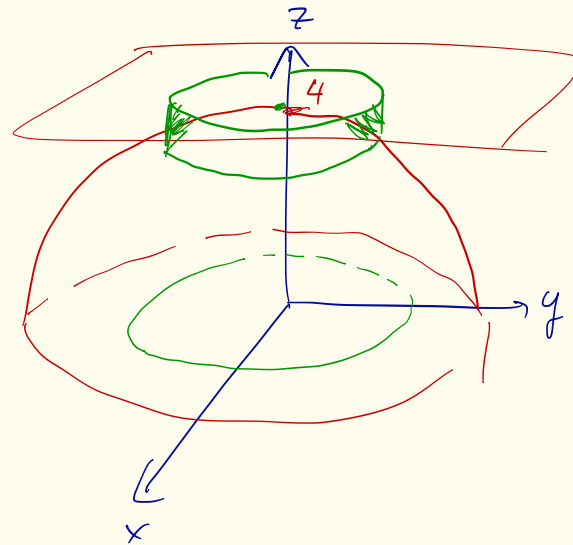
(above  $z = 4 - x^2 - y^2$ )

Find  $\iiint_E \sqrt{x^2 + y^2} dV$ .

$$\iint_{x^2 + y^2 \leq 1} \left( \int_{4 - x^2 - y^2}^4 \sqrt{x^2 + y^2} dz \right) dA$$

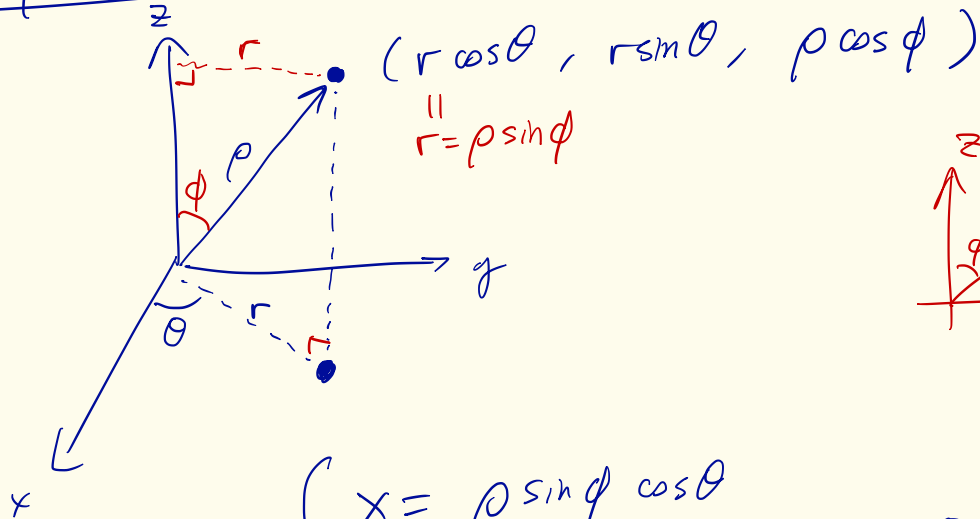
By cylindrical coord:

$$\int_0^{2\pi} \int_0^1 \left( \int_{4-r^2}^4 r dz \right) r dr d\theta = \int_0^{2\pi} \int_0^1 \int_{4-r^2}^4 r^2 \underline{dz} dr d\theta$$

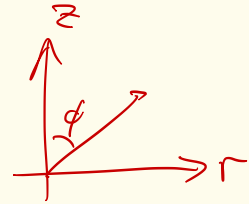


$$r^2 (4 - (4 - r^2)) = r^4$$

# Spherical Coordinate



$$r = \rho \sin \phi$$



polar here again!

$$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases}$$

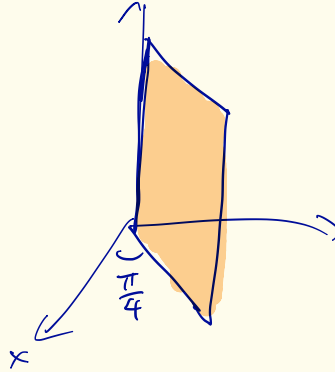
$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \pi$$

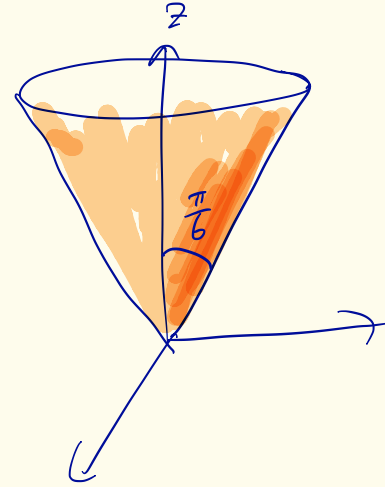
Ex  $\rho = 3$



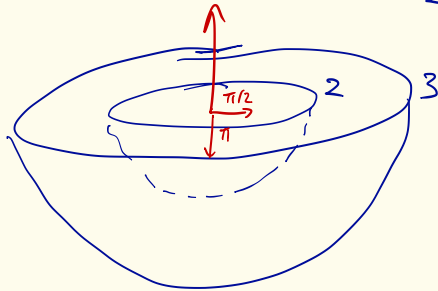
Ex  $\theta = \frac{\pi}{4}$



Ex  $\phi = \frac{\pi}{6}$



Ex  $2 \leq \rho \leq 3$  and  $\frac{\pi}{2} \leq \phi \leq \pi$



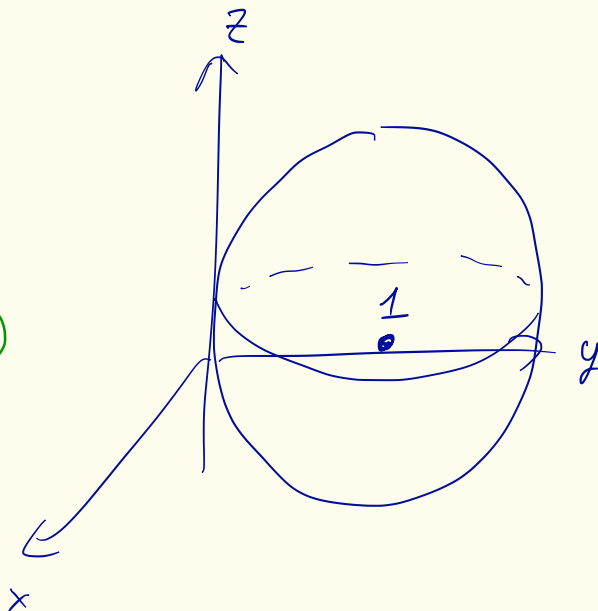
$$\underline{\text{Ex}} \quad \rho = 2 \underbrace{\sin \theta \sin \phi}_{y/\rho} \quad (0 \leq \theta \leq \pi)$$

$$\Leftrightarrow \rho^2 = 2y$$

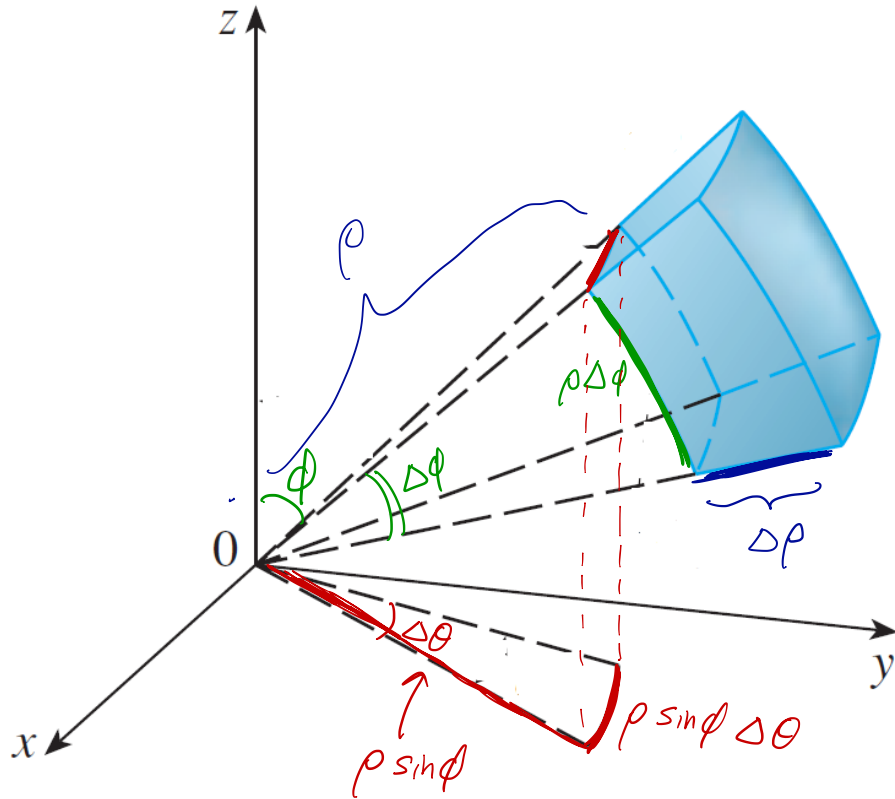
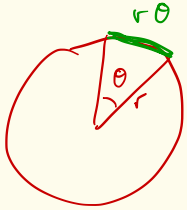
$$\Leftrightarrow x^2 + y^2 + z^2 = 2y$$

$$\Leftrightarrow x^2 + (y-1)^2 + z^2 = 1$$

(-ve  $\rho \Rightarrow$  going in opposite direction)



Volume Element  $dx dy dz \Rightarrow ?$



$$\Delta V = \rho^2 \sin \phi \Delta \rho \Delta \phi \Delta \theta \rightarrow dV = \rho^2 \sin \phi d\rho d\phi d\theta //$$

$$\underline{\text{Ex}} \quad \iiint_B e^{(x^2+y^2+z^2)^{3/2}} dV$$

$B = \text{unit ball.}$

$$\int_0^{2\pi} \int_0^\pi \int_0^1 e^{\rho^3}$$

$$\underbrace{\rho^2 \sin \phi}_{dV} d\rho d\phi d\theta$$

$$(x^2+y^2+z^2 = \rho^2)$$

$$\int e^{\rho^3} \rho^2 d\rho = \frac{e^{\rho^3}}{3}$$

$$= (2\pi)(2) \left[ \frac{e^{\rho^3}}{3} \right]_0^1$$

$$= 4\pi \left( \frac{e}{3} - \frac{1}{3} \right)$$



## Change of Variables

$$\int_a^b f(x) dx \quad x = g(u)$$

$$\Leftrightarrow \int_c^d f(g(u)) g'(u) du$$

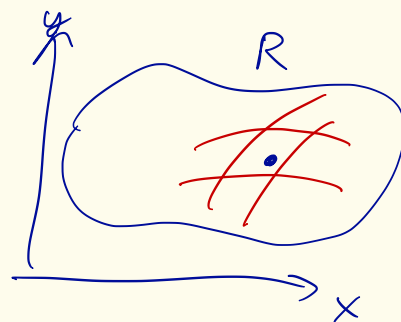
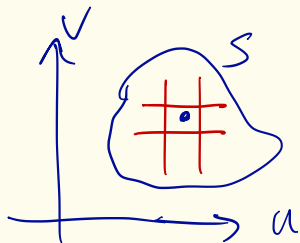
$$\begin{cases} g(c) = a \\ g(d) = b \end{cases}$$

In double integrals

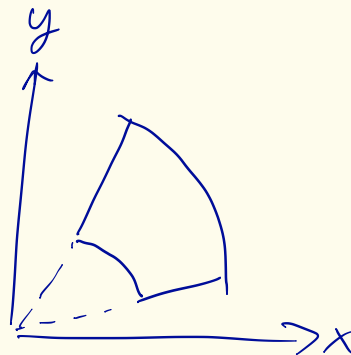
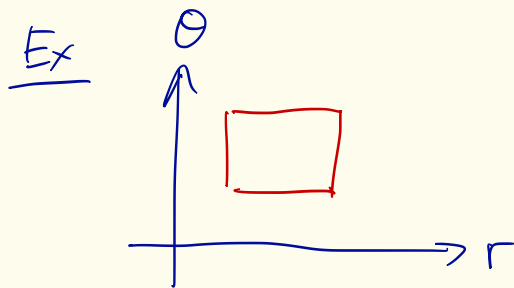
$$x = x(r, \theta) = r \cos \theta$$

$$y = y(r, \theta) = r \sin \theta$$

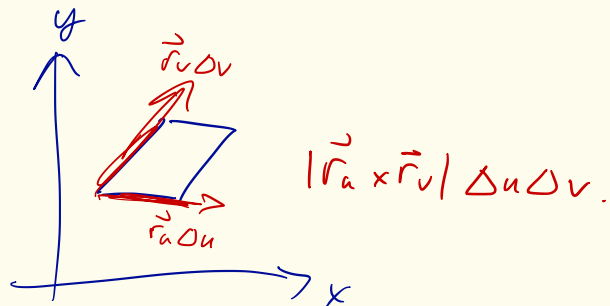
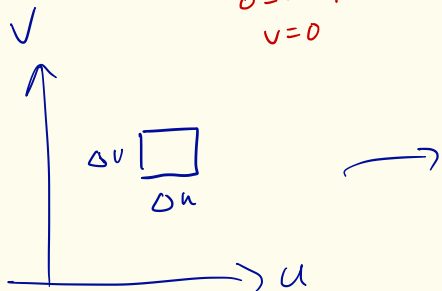
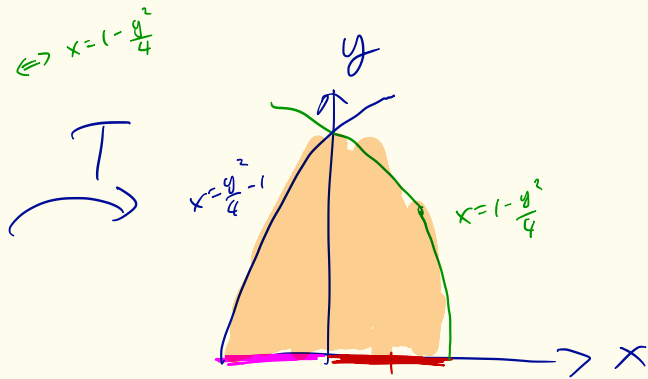
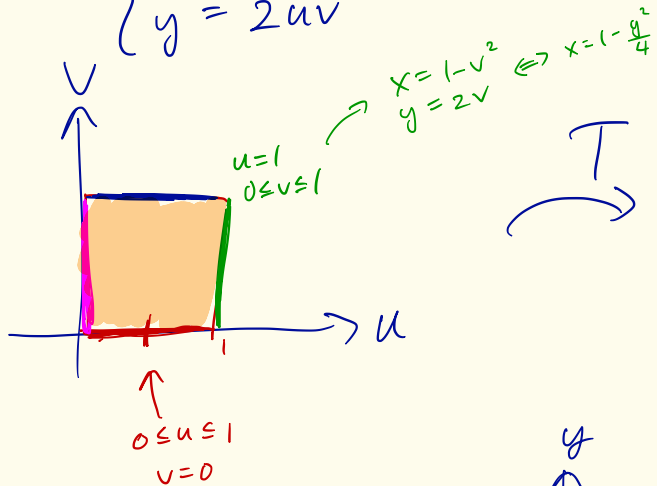
$$dx dy \Rightarrow r dr d\theta$$



$$\vec{F}(u,v) = \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = 0 \end{cases}$$



Ex  $\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$



$$|\vec{r}_u \times \vec{r}_v| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \quad \text{Jacobian}$$

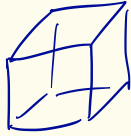
$$\underline{\text{Thm}} \quad \iint_R f(x, y) dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$



$$\underline{\text{Ex}} \quad \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \Leftrightarrow \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

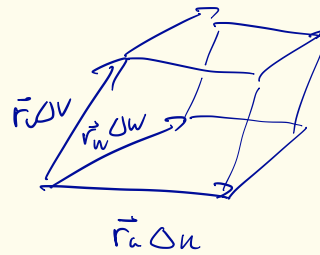
$$dx dy = r dr d\theta //$$

$$\underline{\text{Ex}} \quad \begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases} : \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2u & 2v \\ -2v & 2u \end{vmatrix} = 4(u^2 + v^2) \Leftrightarrow \int_{\Delta} dx dy = \int_{\square} 4(u^2 + v^2) du dv$$



$\Delta x \Delta y \Delta z$

$$\begin{aligned} x &= x(u, v, w) \\ y &= y(u, v, w) \\ z &= z(u, v, w) \end{aligned}$$



Volume  $dV = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} dx dy dz$

$\parallel$   
 $\frac{\partial(x, y, z)}{\partial(u, v, w)}$  Jacobian //

Ex  $\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \theta \end{cases}$

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin \phi$$