Multivariable Calculus

Lecture Notes for
MATH 2023
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Jacobian

change of variable in triple integrals

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div curl

surface integral, flux

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1 — Three-Dimensional Space

"There is no royal road to geometry"

Euclid

1.1 Rectangular Coordinates in $\mathbb{R}^3$

Throughout the course, we will use an ordered triple (x, y, z) to represent a point in the three dimensional space. The real numbers x, y and z in an ordered triple (x, y, z) are respectively the x-, y- and z-coordinates which, by convention, are defined according to the following diagram:

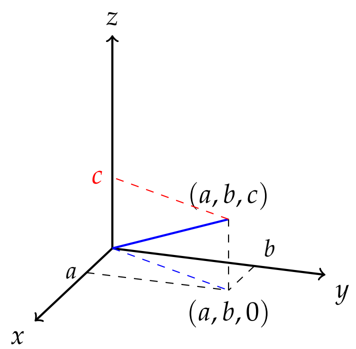

Figure 1.1: Rectangular coordinates system in 3-space

Notation We will use the notation $\mathbb{R}^3$ to denote the entire three dimensional space.

Any point on the x-axis has the form $(x, 0, 0)$, i.e. $y = 0$ and $z = 0$. Similarly, points on the y-axis are of the form $(0, y, 0)$, and points on the z-axis are of the form $(0, 0, z)$. The three coordinate axes meet at a point with coordinates $(0, 0, 0)$ which is called the **origin**.

A vector in $\mathbb{R}^3$ is an arrow which is based at one point and is pointing at another point. If a vector $\mathbf{v}$ is based at (x_0, y_0, z_0) and points toward (x_1, y_1, z_1), then the vector is written as:

$$\mathbf{v} = (x_1 - x_0)\mathbf{i} + (y_1 - y_0)\mathbf{j} + (z_1 - z_0)\mathbf{k}.$$

For example, the vector based at $(3, 2, -1)$ pointing at $(5, 2, 0)$ is expressed as

$$(5 - 3)\mathbf{i} + (2 - 2)\mathbf{j} + (0 - (-1))\mathbf{k} = 2\mathbf{i} + \mathbf{k}.$$

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Consequently, any two vectors that are **pointing at the same direction** and have **the same length** are considered to be **equal**, even though they may have different base points. For instance, a vector $\mathbf{v}$ based at $(1,2,3)$ pointing at $(4,3,2)$, i.e.

$$\mathbf{v} = (4-1)\mathbf{i} + (3-2)\mathbf{j} + (2-3)\mathbf{k} = 3\mathbf{i} + \mathbf{j} - \mathbf{k},$$

is considered to be equal to the vector $\mathbf{w}$ based at $(0,0,0)$ pointing at $(3,1,-1)$. In other words, we can write $\mathbf{w} = \mathbf{v}$.

An alternative notation for a vector is the angular bracket $\langle a,b,c\rangle$. We will sometimes write a vector this way to save the hassle of writing down $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

Notation $\langle a,b,c\rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

In this course, we make very little conceptual distinction between a *point* (x,y,z) and a *vector based at* $(0,0,0)$ *pointing at the point* (x,y,z). However, speaking of notations, one should use $\langle x,y,z\rangle$ or $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ to denote a vector and (x,y,z) to denote a point so as to avoid confusion.

Vector additions can scalar multiplications are defined as follows:

Definition 1.1 — Vector Additions and Scalar Multiplications. Let $\mathbf{a} = \langle a_1,a_2,a_3\rangle$ and $\mathbf{b} = \langle b_1,b_2,b_3\rangle$ be two vectors in $\mathbb{R}^3$, and c be a real scalar, then:

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle \qquad \text{(vector addition)}$$
$$c\mathbf{a} = \langle ca_1, ca_2, ca_3\rangle \qquad \text{(scalar multiplication)}$$

The negative of a vector is defined as: $-\mathbf{a} = (-1)\mathbf{a}$. The difference between vectors is defined as $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

Geometrically, these vector operations can be represented by the following diagrams:

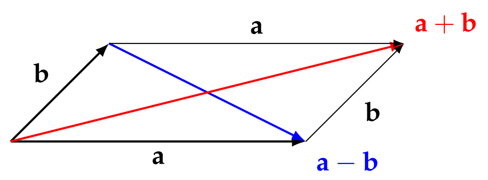

Figure 1.2: Geometric representations of various vector operations

Property Vector additions and scalar multiplications have the following algebraic properties

1. commutative rule: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. associatative rule: $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$
3. distributive rules: $(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$ and $\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$

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1.2 Dot Product

There are two types of products for vectors in $\mathbb{R}^3$, namely the *dot product* and the *cross product*. The former outputs a scalar whereas the latter outputs a vector. In this section, We first talk about the dot product.

Definition 1.2 — Dot Product. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then dot product between the vectors $\mathbf{a}$ and $\mathbf{b}$ are defined as:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

It is important to note that the dot product between a vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and itself is given by:

$$\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2$$

which is incidently the **square of the length** of the vector $\mathbf{a}$ (by the Pythagoreas' Theorem in $\mathbb{R}^3$).

Notation Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$. We denote the length of a vector $\mathbf{a}$ by $|\mathbf{a}|$, which is given by:

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

It is important to note that $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$.

The *length* of a vector is sometimes called the *norm*, or the *magnitude*, of the vector.

Property It can easily be verified that the dot product satisfies the following algebraic properties:

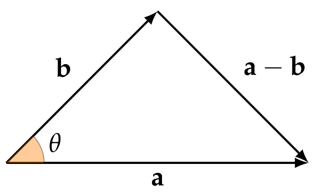
1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.
2. $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$.
3. $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda (\mathbf{a} \cdot \mathbf{b})$
4. $\mathbf{0} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{0} = 0$.

The following theorem gives the geometric meaning of the dot product:

Theorem 1.1 Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ be two vectors in $\mathbb{R}^3$, and θ be the angle between these two vectors. Then we have:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta. \tag{1.1}$$

Proof. The proof uses the Law of Cosines. Consider the triangle in the diagram below:



The side opposite to the angle is represented by the vector $\mathbf{a} - \mathbf{b}$. Using the Law of Cosines:

$$\begin{aligned}
|\mathbf{a} - \mathbf{b}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}| \cos\theta \\
(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) &= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\,|\mathbf{a}|\,|\mathbf{b}| \cos\theta \\
\mathbf{a} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} &= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\,|\mathbf{a}|\,|\mathbf{b}| \cos\theta \\
\cancel{\mathbf{a} \cdot \mathbf{a}} - 2\mathbf{a} \cdot \mathbf{b} + \cancel{\mathbf{b} \cdot \mathbf{b}} &= \cancel{\mathbf{a} \cdot \mathbf{a}} + \cancel{\mathbf{b} \cdot \mathbf{b}} - 2\,|\mathbf{a}|\,|\mathbf{b}| \cos\theta \\
-2\mathbf{a} \cdot \mathbf{b} &= -2\,|\mathbf{a}|\,|\mathbf{b}| \cos\theta \\
\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}| \cos\theta.
\end{aligned}$$

■

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One immediate consequence of Equation (1.1) is that it allows us to use the dot product to find the angle between two vectors. Precisely, the angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by:

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\right).$$

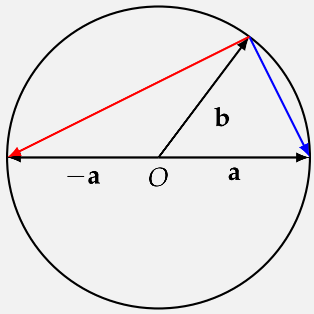
The most important case is that the two vectors $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular**, also known as **orthogonal**. The angle between the vectors is $\frac{\pi}{2}$ and so we have the following important fact:

Corollary 1.2 Two non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a}\cdot\mathbf{b} = 0$.

This corollary is particularly useful to determine whether two vectors are perpendicular.

■ **Example 1.1** Show that any triangle which is inscribed in a circle and has one of its side coincides the diameter of the circle must be a right-angled triangle.

■ **Solution** Let O be the center of the circle. Define vectors $\mathbf{a}$ and $\mathbf{b}$ as in the diagram below.



We would like to show the vectors in red and blue are orthogonal to each other. By basic vector additions and subtractions:

$$\text{Red vector} = -\mathbf{a} - \mathbf{b} \qquad \text{Blue vector} = \mathbf{a} - \mathbf{b}.$$

Their dot product equals to

$$\begin{aligned}(-\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) &= -\mathbf{a}\cdot\mathbf{a} + \cancel{\mathbf{a}\cdot\mathbf{b}} - \cancel{\mathbf{b}\cdot\mathbf{a}} + \mathbf{b}\cdot\mathbf{b} \\ &= -|\mathbf{a}|^2 + |\mathbf{b}|^2 \qquad \text{(recall } \mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2\text{)}\end{aligned}$$

Since both $\mathbf{a}$ and $\mathbf{b}$ represent the radii of the circle, they have the same magnitude. Therefore $|\mathbf{a}| = |\mathbf{b}|$ and we have $(-\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = 0$. This shows the red and blue vectors are orthogonal.

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1.3 Cross Product

The cross product is another important vector operation. In contrast to the dot product, the cross product gives a **vector** instead of a scalar. A vector is characterized by its length and direction, we define the cross product by declaring these two attributes:

Definition 1.3 — Cross Product. Given two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ in $\mathbb{R}^3$ with angle θ between them, the cross product between $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \times \mathbf{b}$, is defined as the vector such that:

- the length is given by $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, i.e. the area of the parallelogram formed by vectors $\mathbf{a}$ and $\mathbf{b}$;
- the cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$;
- the direction of $\mathbf{a} \times \mathbf{b}$ is determined by the right-hand grab rule illustrated by the Figure 1.3.

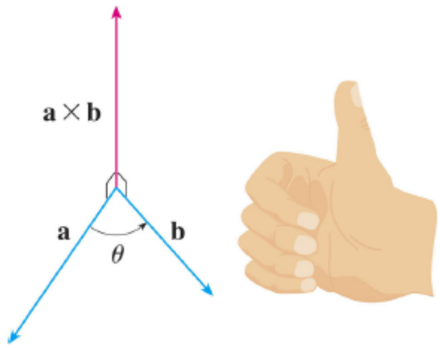
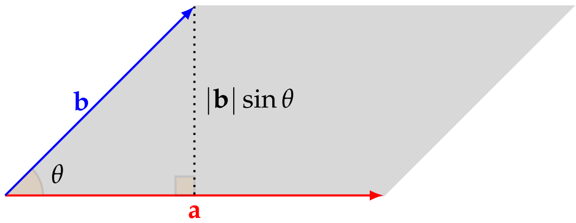


Figure 1.3: right-hand grab rule

From the right-hand grab rule, we can clearly see that $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ are vectors with the same length but in **opposite** direction, i.e. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$. The magnitude of $|\mathbf{a} \times \mathbf{b}|$, which is defined to be $|\mathbf{a}|\,|\mathbf{b}| \sin\theta$, is the **area of the parallelogram** formed by $\mathbf{a}$ and $\mathbf{b}$:



The following are some useful algebraic properties of the cross products. Based on the definition of cross products we presented above, the proofs are purely geometric and are omitted here.

Property The cross product satisfies:

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
3. $\mathbf{a} \times \mathbf{0} = \mathbf{0}$
4. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$

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For simple vectors such as $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, their cross products can be easily found from the definition:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

For more complicated vectors, the cross product can be computed using the following determinant formula:

Theorem 1.3 — Determinant Formula of Cross Product. Given two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, their cross product is given by:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \end{aligned} \tag{1.2}$$

Proof. The proof follows from expanding:

$$\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

using the algebraic properties of the cross product. It is left as an exercise for readers. ■

The cross product can be used to find a vector which is orthogonal to a plane.

■ **Example 1.2** Given three points in the xyz-space:

$$A(0,2,-1), \quad B(4,0,-1), \quad C(7,-3,0)$$

Find a vector $\mathbf{n}$ which is orthogonal to the plane passing through A, B and C. Moreover, find the area of the triangle $\triangle ABC$.

■ **Solution** A vector $\mathbf{n}$ is orthogonal to the plane if and only if it is orthogonal to any two (non-parallel) vectors on the plane. We will first find two vectors on the plane and then take the cross product. The outcome will give a vector orthogonal to these two vectors (hence orthogonal to the plane as well).

The following two vectors lie on the plane:

$$\begin{aligned} \overrightarrow{AB} &= \langle 4,0,-1\rangle - \langle 0,2,-1\rangle = \langle 4,-2,0\rangle \\ \overrightarrow{AC} &= \langle 7,-3,0\rangle - \langle 0,2,-1\rangle = \langle 7,-5,1\rangle \end{aligned}$$

Taking the cross product: $\overrightarrow{AB} \times \overrightarrow{AC} = \langle -2,-4,-6\rangle$.

Therefore, the required vector $\mathbf{n}$ can be taken to be any scalar multiple of $\langle -2,-4,-6\rangle$, such as $\langle 1,2,3\rangle$ or $\langle 2,4,6\rangle$.

The length of the cross product $\overrightarrow{AB} \times \overrightarrow{AC}$ is equal to the area of the parallelogram formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The area of the triangle $\triangle ABC$ is $\frac{1}{2}$ of the area of this parallelogram. Therefore,

$$\text{Area of } \triangle ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BC}\right| = \frac{1}{2}\sqrt{(-2)^2 + (-4)^2 + (-6)^2} = \sqrt{14}.$$

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1.4 Lines and Planes

1.4.1 Parametric Equations of Lines

In the three dimensional space, lines are no longer represented by an equation like $x + 2y = 1$ in the two dimensional plane. In order to represent a straight-line (and a curve as well), we need to introduce the **time variable** t, and think of a straight-line or a curve as the **path of a particle** travelling as t varies.

Suppose the line L passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ (see Figure 1.4). For any variable point $P(x, y, z)$, the vector $\overrightarrow{P_0P}$ is parallel to the vector $\mathbf{v}$, meaning that $\overrightarrow{P_0P} = t\mathbf{v}$ for some real number t. Therefore,

$$\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle = t\mathbf{v}$$
$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle v_1, v_2, v_3 \rangle$$

Therefore, we have:

$$x = x_0 + tv_1$$
$$y = y_0 + tv_2$$
$$z = z_0 + tv_3$$

which is called the **parametric equation** of the line L. It is called this way because the variable t is called the **parameter** of the line.

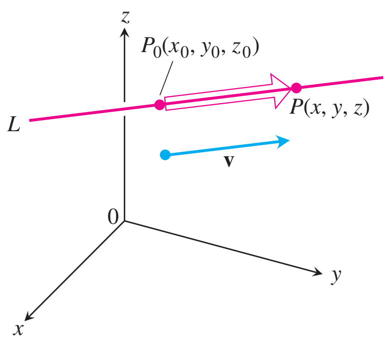

Figure 1.4: a straight line L passing through P_0 and parallel to $\mathbf{v}$

Notation In this course, the vector $\mathbf{r}$ is "reserved" to denote the position vector $\langle x, y, z \rangle$.

Using this notation, we can also write the parametric equation of the line L in vector form:

$$\mathbf{r}(t) = \overrightarrow{OP_0} + t\mathbf{v}.$$

The t-variable in $\mathbf{r}(t)$ emphasizes the fact that the position vector $\mathbf{r}$ depends on t. It can be omitted if it is clear that t is the parameter letter.

■ **Example 1.3** Find parametric equation of the line L passing through both $A(3, -2, 0)$ and $B(1, 0, 1)$. Express your answer in both equation form and vector form.

■ **Solution** In order to write down the parametric equation of a straight-line, we need two "ingredients":

1. a given point P_0 on the line, and
2. the direction $\mathbf{v}$ of the line.

In this case, we can take P_0 to be $A(3, -2, 0)$. The direction of the line L is the vector $\overrightarrow{AB}$,

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which is given by:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 1,0,1\rangle - \langle 3,-2,0\rangle = \langle -2,2,1\rangle\,.$$

With $P_0(3,-2,0)$ and $\mathbf{v} = \langle -2,2,1\rangle$, the parametric equation of the line L is given by:

$$\begin{aligned} x &= 3-2t \\ y &= -2+2t \\ z &= 0+t \end{aligned}$$

or equivalently,

$$\mathbf{r}(t) = \langle 3,-2,0\rangle + t\,\langle -2,2,1\rangle\,.$$

In the above example, one may also take P_0 to be $B(1,0,1)$ and keeping $\mathbf{v}$ to be $\langle -2,2,1\rangle$, then the parametric equation of L is given by:

$$\mathbf{r}(t) = \langle 1,0,1\rangle + t\,\langle -2,2,1\rangle\,.$$

Although it gives a different $\mathbf{r}(t)$, this parametric equation represents the same straight line L. Every straight line can be represented by many different parametric equations!

1.4.2 Equation of Planes

In three dimensions, equation of a plane can be represented in the form of $Ax+By+Cz=D$. For a plane through a given point $P_0(x_0,y_0,z_0)$ with a normal vector $\mathbf{n} = \langle A,B,C\rangle$, the equation of the plane is given by:

$$Ax+By+Cz = Ax_0+By_0+Cz_0. \tag{1.3}$$

Equation (1.3) can be proved by considering a variable point $P(x,y,z)$. As illustrated in Figure 1.5, the vector $\overrightarrow{P_0P}$ lies on the plane and therefore is orthogonal to the normal vector $\mathbf{n}$. Therefore, we have:

$$\begin{aligned} \mathbf{n}\cdot\overrightarrow{P_0P} &= 0 \\ \langle A,B,C\rangle\cdot(\langle x,y,z\rangle - \langle x_0,y_0,z_0\rangle) &= 0 \\ \langle A,B,C\rangle\cdot\langle x,y,z\rangle - \langle A,B,C\rangle\cdot\langle x_0,y_0,z_0\rangle &= 0 \\ Ax+By+Cz &= Ax_0+By_0+Cz_0. \end{aligned}$$

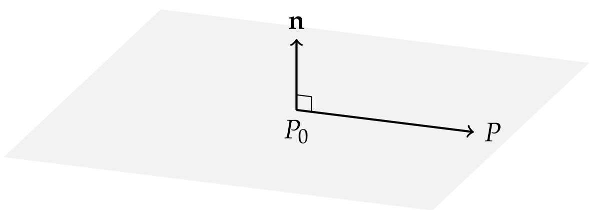

Figure 1.5: equation of a plane

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In order to find **n**, one may use the cross product. Here is an example:

■ **Example 1.4** Find an equation of the plane in $\mathbb{R}^3$ passing through the following three points.

$$A(0,2,-1), \quad B(4,0,-1), \quad C(7,-3,0).$$

■ **Solution** The two "ingredients" of finding the equation of a plane are

1. a given point on the plane; and
2. a normal vector to the plane.

In order to find the normal vector the plane through A, B and C, we take the cross product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \langle 4,0,-1\rangle - \langle 0,2,-1\rangle = \langle 4,-2,0\rangle$$
$$\overrightarrow{AC} = \langle 7,-3,0\rangle - \langle 0,2,-1\rangle = \langle 7,-5,1\rangle$$

Taking the cross product: $\overrightarrow{AB} \times \overrightarrow{AC} = \langle -2,-4,-6\rangle$. Any non-zero vector parallel to this cross product is a normal vector to the plane. For simplicity, we can take:

$$\mathbf{n} = \langle 1,2,3\rangle .$$

Take $A(0,2,-1)$ to be the given point P_0, then the equation of the plane through A, B and C is given by:

$$\underbrace{1x + 2y + 3z = 1(0) + 2(2) + 3(-1)}_{(x_0,y_0,z_0)=(0,2,-1) \text{ and } \mathbf{n}=\langle 1,2,3\rangle}$$

After simplification: $x + 2y + 3z = 1$.

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1.5 Parametric Curves

1.5.1 Parametric Equations of Curves

In two dimensions, there are two ways to represent a curve, namely in the form of $x^2 + y^2 = 1$, or of the form

$$x = \cos t$$
$$y = \sin t.$$

The former is called the **Cartesian equation** and the second one is called the **parametric equation**.

However, in three dimensions, a single Cartesian equation such as $x^2 + y^2 + z^2 = 1$ represents a **surface** instead. Therefore, we will only use the parametric equations to present curves in three dimensions.

Definition 1.4 — Parametric Equation of a Curve. The parametric equation of a curve is of the form:

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

where $f(t)$, $g(t)$ and $h(t)$ are differentiable functions of t. In vector notations, the parametric equation of this curve is written as:

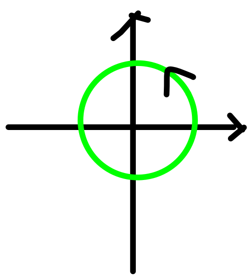
$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

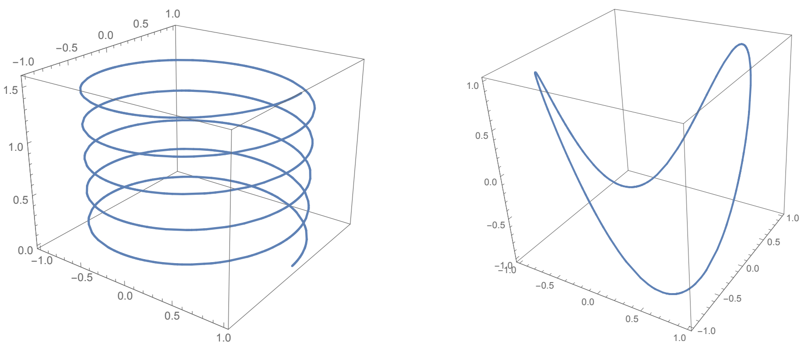
The parametric equation of a straight-line is a special case of parametric equation of a "curve". An interesting example of a parametric curve is the **helix**:

$$\mathbf{r}_1(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \frac{t}{20}\mathbf{k}.$$

It is a curve that goes around the circle but the altitude is constantly increasing. See Figure 1.6a for the computer sketch. Here is another example of a parametric curve. See Figure 1.6b for the sketch.

$$\mathbf{r}_2(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin 2t)\mathbf{k}.$$





(a) sketch of $\mathbf{r}_1(t)$, a helix (b) sketch of $\mathbf{r}_2(t)$

Figure 1.6: Sketches of two parametric curves

domain of r is domain of t such that all component functions are defined

Eg. $\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$

$t \in \mathbb{R}$ $\quad$ $3-t>0$, $t<3$ $\quad$ $t \geq 0$ $\quad$ $\Big\}\ 0 \leq t < 3$

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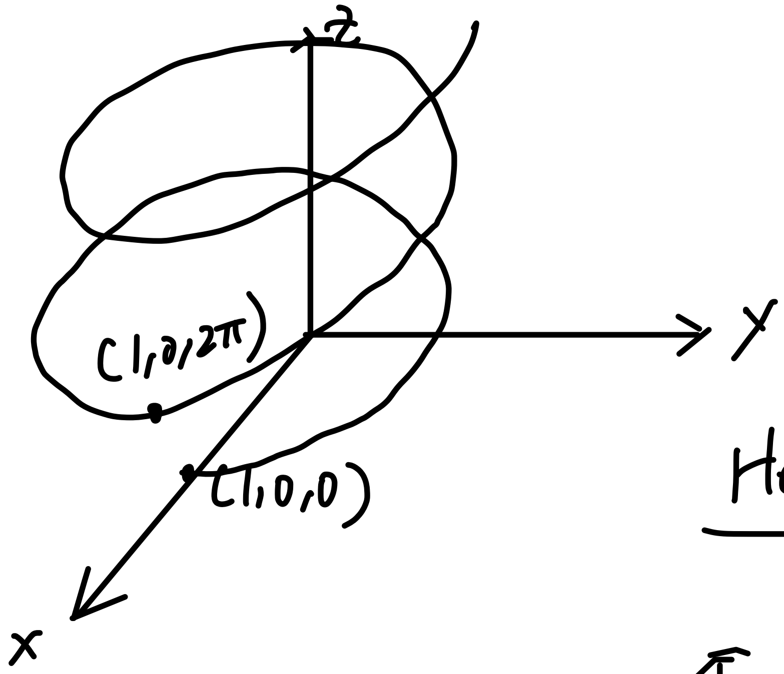
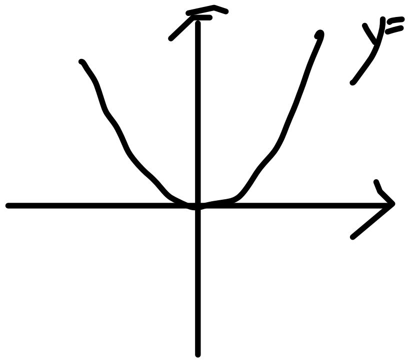
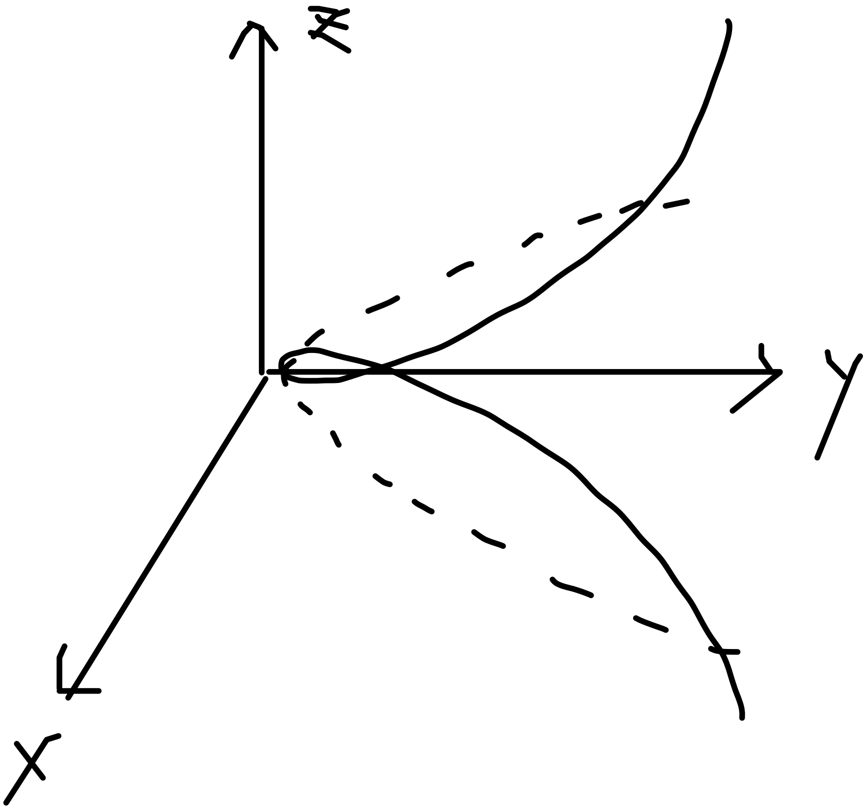


Helix.

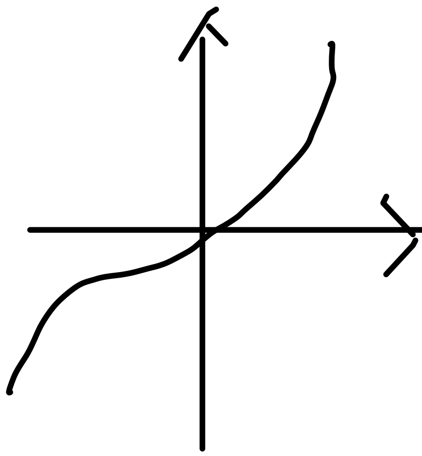
$y = x^2$



Eg. $\vec{r}(t) = \langle \underbrace{t, t^2}_{\text{parabola}}, t^3 \rangle$



$z = x^3$



Limit:

$$\lim_{t \to a} \vec{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

Continuity:

$$\lim_{t \to a} \vec{r}(t) = \vec{r}(a) \quad \text{(exist + equal)}$$

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$\vec{r}'(t) = \frac{d\vec{r}}{dt} = \langle f'(t), g'(t), h'(t) \rangle$

$\lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$

1.5.2 Derivatives of Parametric Curves

One reason for using vector notations for a parametric curve is that their derivatives carry various physical and geometric meanings.

Given a parametric curve $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, regarded as the position vector of a particle at time t, then:

- the first derivative, $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$, represents the **velocity vector** of the particle,
- geometrically, the first derivative $\mathbf{r}'(t)$ (if non-zero) is a **tangent vector** of the curve,
- the length $|\mathbf{r}'(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$ represents the **speed** of the particle, and
- the second derivative, $\mathbf{r}''(t) = f''(t)\mathbf{i} + g''(t)\mathbf{j} + h''(t)\mathbf{k}$, represents the **acceleration vector** of the particle.

Example 1.5 Find the velocity, speed and acceleration of the particle whose path is:

$$\mathbf{r}(t) = (\sin t)\,\mathbf{i} + \left(t^2 - \cos t\right)\mathbf{j} + e^t\mathbf{k}.$$

Solution

$$\begin{aligned}
\text{velocity} &= \mathbf{r}'(t) = (\sin t)'\,\mathbf{i} + \left(t^2 - \cos t\right)'\mathbf{j} + \left(e^t\right)'\mathbf{k} \\
&= (\cos t)\,\mathbf{i} + (2t + \sin t)\,\mathbf{k} + e^t\mathbf{k} \\
\text{speed} &= \left|\mathbf{r}'(t)\right| = \sqrt{(\cos t)^2 + (2t + \sin t)^2 + \left(e^t\right)^2} \\
&= \sqrt{\cos^2 t + 4t^2 + 4t\sin t + \sin^2 t + e^{2t}} \\
&= \sqrt{1 + 4t^2 + 4t\sin t + e^{2t}} \\
\text{acceleration} &= \mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) \\
&= (\cos t)'\,\mathbf{i} + (2t + \sin t)'\,\mathbf{j} + \left(e^t\right)'\mathbf{k} \\
&= (-\sin t)\mathbf{i} + (2 + \cos t)\,\mathbf{j} + e^t\mathbf{k}.
\end{aligned}$$

Conservation of Angular Momentum

In physics, given a particle with mass m travelling along the $\mathbf{r}(t)$, the following vector is defined to be the **angular momentum** about the origin of the particle:

$$\mathbf{L}(t) = \mathbf{r}(t) \times m\mathbf{r}'(t).$$

When $\mathbf{L}(t)$ is a non-zero constant vector (independent of t), we say that the angular momentum is *conserved*. The conservation of angular momentum implies that the path of the particle is contained in a plane. It can be explained as follows:

By the definition of cross product, the angular momentum $\mathbf{L}(t)$ is always orthogonal to $\mathbf{r}(t)$ (and to $\mathbf{r}'(t)$ too, but we do not need this). Therefore, at any time t, we have:

$$\mathbf{L}(t) \cdot \mathbf{r}(t) = 0.$$

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If $\mathbf{L}(t)$ is a constant vector, it can be expressed as $\mathbf{L} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ where A, B and C are fixed numbers. Then:

$$\begin{aligned}
(A\mathbf{i} + B\mathbf{j} + C\mathbf{k}) \cdot (x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}) &= 0 \\
Ax(t) + By(t) + Cz(t) &= 0.
\end{aligned}$$

Therefore, the point $(x(t), y(t), z(t))$ lies on the plane $Ax + By + Cz = 0$, which is a plane with normal vector $\mathbf{L}$ passing through the origin. In other words, the path of the particle is confined in this plane.

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$$\text{speed} = \frac{\text{distance}}{\text{time}} \rightarrow v = \frac{ds(t)}{dt} = |\vec{v}|$$

↑ magnitude

$$\text{velocity} = \frac{\text{displacement}}{\text{time}} \rightsquigarrow \vec{v} = \frac{d\vec{r}}{dt} = \vec{r}\,'(t)$$

size of tangent vector tells you the speed, how fast you are going.

in terms of classical mechanics:

displacement $\vec{r}(t)$

velocity $\vec{r}\,'(t) = \vec{v}(t)$ speed $= |\vec{v}(t)|$

acceleration $\vec{r}\,''(t) = \vec{a}(t)$

example:

$$\vec{r}(t) = \sqrt{1+t}\,\vec{i} + (t^3+4)\,\vec{j} + e^{\sin t}\,\vec{k}$$

$$\vec{r}\,'(t) = \left\langle \frac{1}{2\sqrt{1+t}},\ 3t^2,\ e^{\sin t}\cos t \right\rangle$$

at $(1,4,1)$ $(t=0)$ velocity: $\vec{r}\,'(0) = \langle \frac{1}{2}, 0, 1 \rangle$, speed $= \sqrt{\frac{1}{4}+0+1} = \frac{\sqrt{5}}{2}$

acceleration: $r''(t) =$ d 多次., at $(1,4,1)$,

$$r''(t) = \langle -\tfrac{1}{4}, 0, 1 \rangle$$

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tangent line at $t=0$

$$\vec{r}(t) = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + t \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

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Newton's Law of Motion

$$\vec{F}(t) = m\vec{a}(t)$$

Example: Rotation

$$\vec{r}(t) = \langle \rho\cos kt, \rho\sin kt \rangle$$

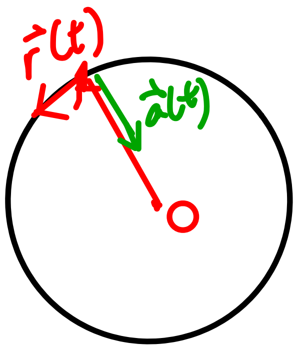
radius

freq.

$$\vec{v}(t) = \langle -\rho k\sin kt, \rho k\cos kt \rangle$$

$$\vec{a}(t) = \langle -\rho k^2\cos kt, -\rho k^2\sin kt \rangle$$

$$= -k^2\,\vec{r}(t)$$



$\vec{F}(t) \parallel \vec{a}(t)$

centripetal force 向心力

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Differentiation Rules

$$\frac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \vec{u}'(t) + \vec{v}'(t)$$

$$\frac{d}{dt}\left(c\,\vec{u}(t)\right) = c\,\vec{u}'(t) \qquad c \in \mathbb{R}$$

Product Rule

$$\frac{d}{dt}\big(\underbrace{f(t)}_{\text{scalar function}}\,\underbrace{\vec{v}(t)}_{\text{vector function}}\big) = f'(t)\,\vec{v}(t) + f(t)\,\vec{v}'(t)$$

$$\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$$

$$\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \vec{u}'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}'(t)$$

Chain Rule

$$\frac{d}{dt}\left(\vec{v}(f(t))\right) = \vec{v}'(f(t))\,.\,f'(t)$$

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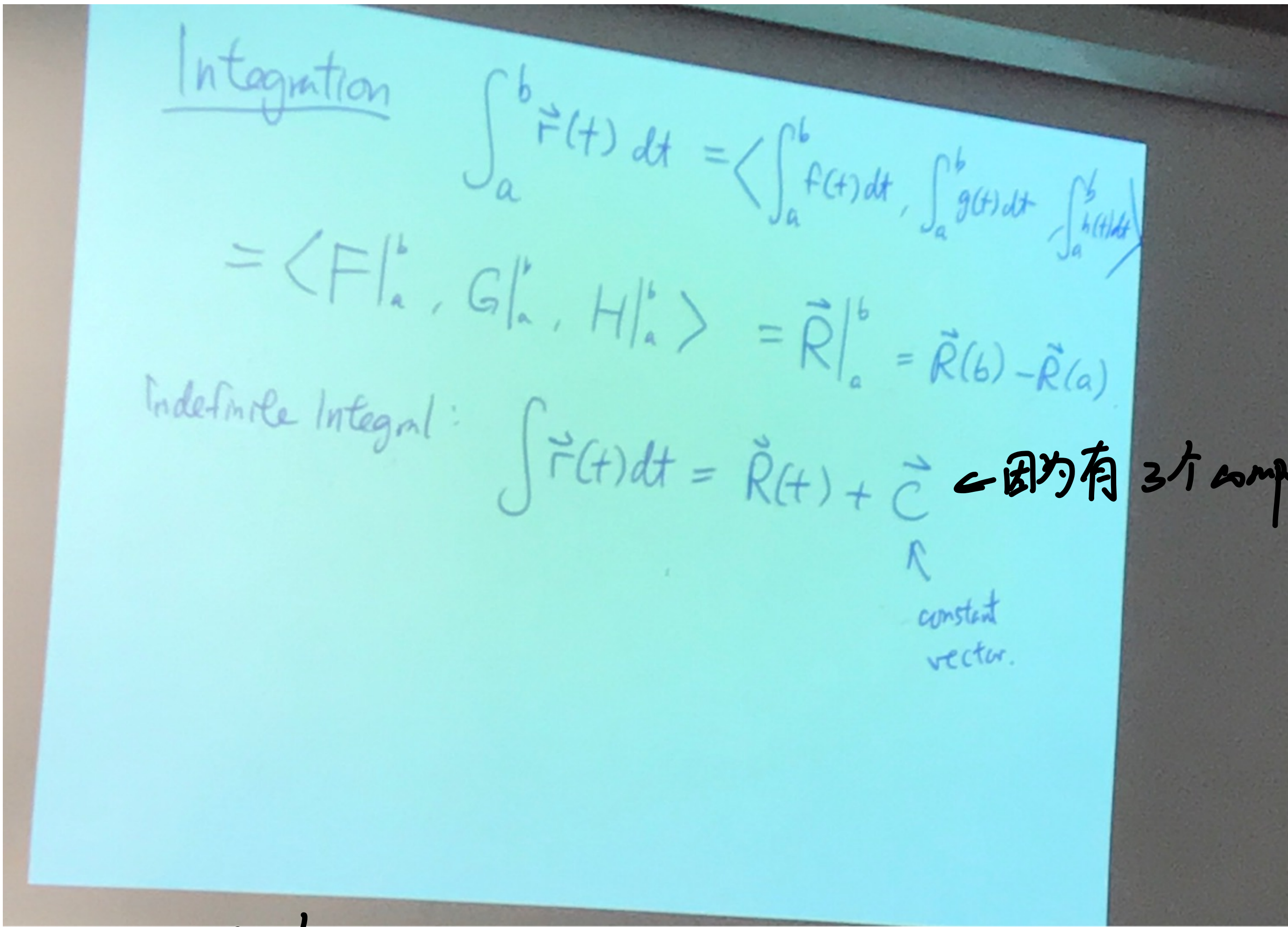
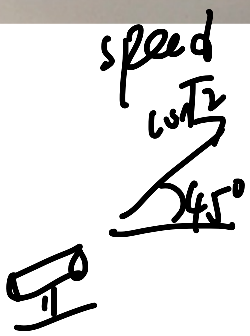


Eg.



Ask when does it hit the ground

Find $\vec{r}(t)$ and solve when $y=0$

$\boxed{\text{Ans: } \therefore t=2}$

$\vec{a}(t) = \langle 0, -10 \rangle$

$\vec{v}(0) = \langle 10, 10 \rangle$

$\vec{r}(0) = \langle 0, 0 \rangle$

$\Rightarrow \vec{v}(t) = \int \vec{a}(t) = \langle C_1, -10t + C_2 \rangle$ plug $t=0$, const $= 10, 10$

$\langle 10, 10 - 10t \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle 10t + C_1, 10t - 5t^2 + C_2 \rangle$

plug $t=0$, $C_1, C_2 = 0, 0$,

$\vec{r}(t) = \langle 10t, 10t - 5t^2 \rangle$

When $y=0$, $t=2$.

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Product Rules of Differentiating Curves

When differentiating the dot or cross product of two curves $\mathbf{u}(t)$ and $\mathbf{v}(t)$, you may apply the product rule as like in single variable calculus.

Property Given two curves $\mathbf{u}(t)$ and $\mathbf{v}(t)$, and a scalar function $f(t)$, we have:

1. $\frac{d}{dt}(f(t)\mathbf{u}(t)) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
2. $\frac{d}{dt}(\mathbf{u}(t) \cdot \mathbf{v}(t)) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
3. $\frac{d}{dt}(\mathbf{u}(t) \times \mathbf{v}(t)) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.

Here is a good example on the use of one of the above product rules.

■ **Example 1.6** Given $\mathbf{r}(t)$ represents a particle travelling at uniform speed C. Show that its velocity and acceleration vectors are always orthogonal.

■ **Solution** The particle is travelling at uniform speed C. Therefore, $|\mathbf{r}'(t)| \equiv C$. We want to show that $\mathbf{r}'(t) \cdot \mathbf{r}''(t) \equiv 0$, so it is natural to differentiate $|\mathbf{r}'(t)|$ with respect to t so that the RHS vanishes and the LHS perhaps may be related to $\mathbf{r}''(t)$.

However, since $|\mathbf{r}'(t)|$ is the form of a square root so it is cumbersome to differentiate it. Instead, we differentiate $|\mathbf{r}'(t)|^2 = C^2$ using the fact that $|\mathbf{r}'(t)|^2 = \mathbf{r}'(t) \cdot \mathbf{r}'(t)$:

$$\begin{aligned} |\mathbf{r}'(t)|^2 &= C^2 \\ \mathbf{r}'(t) \cdot \mathbf{r}'(t) &= C^2 \\ \frac{d}{dt}\left(\mathbf{r}'(t) \cdot \mathbf{r}'(t)\right) &= 0 \\ \mathbf{r}''(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}''(t) &= 0 \\ 2\mathbf{r}'(t) \cdot \mathbf{r}''(t) &= 0. \end{aligned}$$

Therefore, we have $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 0$, which is desired.

1.5.3 Arc Lengths of Curves

For a particle travelling at uniform speed, the distance (i.e. arc length) travelled is simply:

$$\text{distance} = \text{speed} \times \text{time lapsed}$$

As compared to the area of a rectangle: height × width. However, if one is asked to calculate the area under a curve $y = f(x)$, $a \leq x \leq b$, one should consider the integral

$$\int_a^b y\,dx = \int_a^b f(x)\,dx$$

under the rationale of Riemann sum: $\int_a^b y\,dx \approx \sum_i y_i \Delta x_i$.

At the same token, if the particle is not travelling at uniform speed, one should calculate the distance by integration:

$$\text{distance} = \int (\text{speed}) \times d(\text{time}),$$

as compared to area $= \int \text{height} \times d(\text{width})$. Precisely, we have:

Theorem 1.4 Given a parametric curve $\mathbf{r}(t)$. The arc length of the curve from the point at $t = a$ to the point at $t = b$ is given by:

$$\text{arc length} = \int_a^b |\mathbf{r}'(t)|\,dt. \tag{1.4}$$

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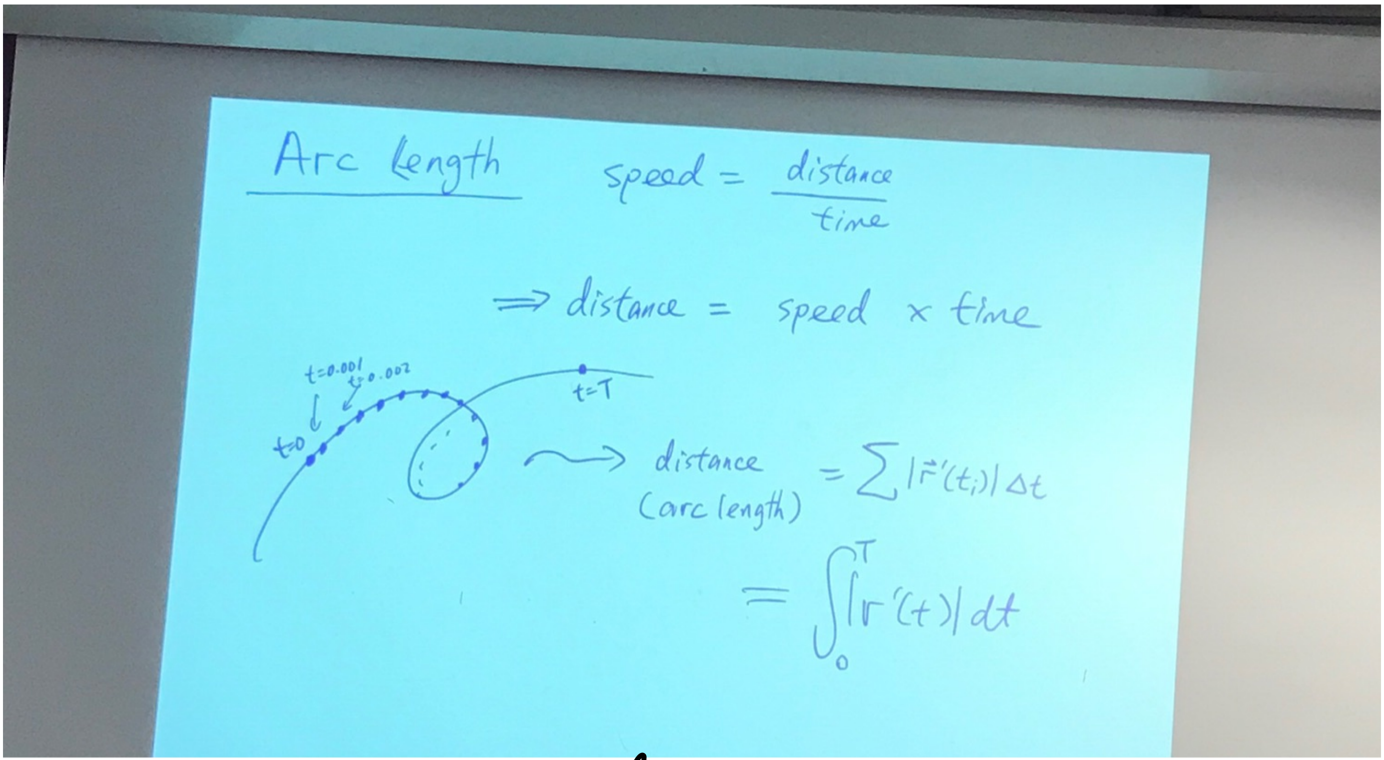

Eg. Find the length of $\vec{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$ from $0 \le t \le 1$

$$\vec{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$$

$$|\vec{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = e^t + e^{-t}$$

$$\int_0^1 e^t + e^{-t}\, dt = \left[e^t - e^{-t} \right]_0^1 = e - e^{-1}$$

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$\vec{r}(t) = \langle \sqrt{2}t^3, e^{t^3}, e^{-t^3} \rangle$

$\langle \sqrt{2}\ln t, t, t^{-1} \rangle$

same curve with before, but different speed

We want to find different parametrization of same curve

Arc length parametrization : standard parametrization

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■ **Example 1.7** Find the arc length of the curve:

$$\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} + \frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\mathbf{j} + t\mathbf{k}$$

from $(0,0,0)$ to $(2, \frac{8}{3}, 2)$.

■ **Solution** It is simple to verify that the initial point corresponds to $t = 0$ since $\mathbf{r}(0) = \langle 0,0,0 \rangle$, whereas the final point corresponds to $t = 2$ since $\mathbf{r}(2) = \langle 2, \frac{8}{3}, 2 \rangle$.

$$\begin{aligned}
\mathbf{r}'(t) &= \left\langle t, \sqrt{2}t^{\frac{1}{2}}, 1 \right\rangle \\
\left|\mathbf{r}'(t)\right| &= \sqrt{t^2 + 2t + 1} \\
&= \sqrt{(t+1)^2} \\
&= t + 1 \qquad \text{(note that } t + 1 > 0 \text{ in our case)}
\end{aligned}$$

The desired arc length is given by:

$$\begin{aligned}
\int_0^2 \left|\mathbf{r}'(t)\right| dt &= \int_0^2 (t+1)dt \\
&= \frac{t^2}{2} + t\Big|_0^2 \\
&= 10.
\end{aligned}$$

1.5.4 Arc-Length Parametrization

Let's begin our discussion by considering the three curves:

$$\begin{aligned}
\mathbf{r}_1(t) &= (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 2\pi \\
\mathbf{r}_2(t) &= (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j} + 2t\mathbf{k}, \quad 0 \le t \le \pi
\end{aligned}$$

If you plot them using Mathematica, you should find out that these two curves are the same, although their speeds are different. The curve $\mathbf{r}_2(t)$ is obtained by replacing every t in $\mathbf{r}_1(t)$ by $2t$. The initial and final times are adjusted so that the end-points of both $\mathbf{r}_1$ and $\mathbf{r}_2$ are $(0,0,0)$ and $(0,0,2\pi)$. We say $\mathbf{r}_2$ is a **reparametrization** of $\mathbf{r}_1$.

If $\mathbf{r}(s)$ is a parametric curve such that $|\mathbf{r}'(s)| = 1$ for any s, we say the curve is **parametrized by arc-length**. For such a parametrization, it is conventional to use s to denote the parameter. Given a parametric curve $\mathbf{r}(t)$, *in theory* one can reparametrize the curve by arc-length, such that with the new parameter s, the curve $\mathbf{r}(s)$ travels at unit speed. To find the arc-length parametrization, you may follow the procedure:

1. Given a curve $\mathbf{r}(t) : [a,b] \to \mathbb{R}^3$, compute the following integral:

$$s = \int_a^t \left|\mathbf{r}'(\tau)\right| d\tau.$$

2. Since the upper limit of the above integral is t, the function s should be a function of t. Express t in terms of s whenever it is possible, so that t is a function of s, i.e. $t = t(s)$.
3. Finally, replace all t's by this function of s in the curve $\mathbf{r}(t)$.

The new parametrization $\mathbf{r}(s)$ will be arc-length parametrized. Let's see some examples before we learn why it works:

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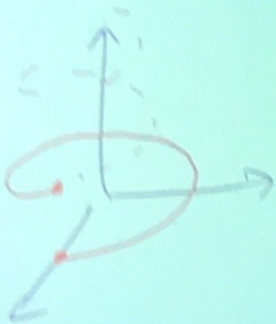
Ex Length of Helix:

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

from $t=0$ to $t=2\pi$



$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 + \cos^2 + 1} = \sqrt{2}$$

$$\int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}$$

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■ **Example 1.8** Find the arc-length parametrization of the curve:

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k},\ t \in [0, 2\pi].$$

■ **Solution** By straight-forward computations, we get

$$|\mathbf{r}'(t)| = \sqrt{2}.$$

Therefore,

$$s(t) = \int_0^t |\mathbf{r}'(\tau)|\, d\tau = \int_0^t \sqrt{2} d\tau = \sqrt{2}t.$$

Express t in terms of s, we get $t = \frac{s}{\sqrt{2}}$. Replace all t's in $\mathbf{r}(t)$ by $\frac{s}{\sqrt{2}}$, we get an arc-length parametrization:

$$\mathbf{r}(s) = \left(\cos \frac{s}{\sqrt{2}}\right)\mathbf{i} + \left(\sin \frac{s}{\sqrt{2}}\right)\mathbf{j} + \frac{s}{\sqrt{2}}\mathbf{k}.$$

■ **Example 1.9** Find the arc-length parametrization of the curve:

$$\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} + \frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\mathbf{j} + t\mathbf{k}, \quad t \geq 0.$$

■ **Solution** By straight-forward computations (and simplification), we get:

$$|\mathbf{r}'(t)| = t + 1.$$

Consider:

$$s = \int_0^t |\mathbf{r}'(\tau)|\, d\tau = \int_0^t (\tau + 1) d\tau = \frac{t^2}{2} + t.$$

To solve t in terms of s, we use the quadratic equation. One should get:

$$t = \frac{-2 + \sqrt{4 + 8s}}{2} = -1 + \sqrt{1 + 2s}.$$

Finally, replace all t's in $\mathbf{r}(t)$ by this function of s, we get an arc-length parametrization:

$$\mathbf{r}(s) = \frac{1}{2}\left(-1 + \sqrt{1 + 2s}\right)^2\mathbf{i} + \frac{2\sqrt{2}}{3}\left(-1 + \sqrt{1 + 2s}\right)^{3/2}\mathbf{j} + \left(-1 + \sqrt{1 + 2s}\right)\mathbf{k}.$$

To see why this procedure gives an arc-length parametrization, we need to show $|\mathbf{r}'(s)| = 1$. We first use chain rule:

$$\begin{aligned} |\mathbf{r}'(s)| &= \left|\frac{d\mathbf{r}}{ds}\right| = \left|\frac{d\mathbf{r}}{dt}\frac{dt}{ds}\right| \\ &= |\mathbf{r}'(t)| \left|\frac{dt}{ds}\right|. \end{aligned}$$

Recall that s is defined to be $s = \displaystyle\int_a^t |\mathbf{r}'(\tau)|\, d\tau$. The Fundamental Theorem of Calculus tells us that $\dfrac{ds}{dt} = |\mathbf{r}'(t)|$ and so,

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{|\mathbf{r}'(t)|}.$$

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Therefore, we have:

$$|\mathbf{r}'(s)| = |\mathbf{r}'(t)| \cdot \frac{1}{|\mathbf{r}'(t)|} = 1.$$

The parametrization $\mathbf{r}(s)$ has unit speed, and hence is an arc-length parametrization.

Although the above procedure of finding arc-length parametrization works in the two examples we have seen, in general it may be hard to find an arc-length parametrization. Since both steps – integration and solving t in terms of s – can be difficult if the given curve $\mathbf{r}(t)$ is not *nice*.

下一版是讲有些 technique 可以用

1.5.5 Curvature

Curvature is quantity that measures the sharpness of a curve, and is closely related to the acceleration. Imagine you are driving a car along a curved road. On a sharp turn, the force exerted on your body is proportional to the acceleration according to the Newton's Second Law. Therefore, given a parametric curve $\mathbf{r}(t)$, the magnitude of the acceleration $|\mathbf{r}''(t)|$ *somewhat* reflects the sharpness of the path – the sharper the turn, the larger the $|\mathbf{r}''(t)|$.

However, the magnitude $|\mathbf{r}''(t)|$ is not *only* affected by the sharpness of the curve, but also on how *fast* you drive. In order to give a *fair* and *standardized* measurement of sharpness, we need to get an arc-length parametrization $\mathbf{r}(s)$ so that the "car" travels at unit speed.

Definition 1.5 — Curvature. Given a curve γ in $\mathbb{R}^2$ or $\mathbb{R}^3$ which can be arc-length parametrized by $\mathbf{r}(s)$, then it's curvature is a function of s defined as:

$$\kappa(s) := |\mathbf{r}''(s)|.$$

■ **Example 1.10** Find the curvature of the circle of radius R centered at the origin $(0,0)$ in $\mathbb{R}^2$.

■ **Solution** The circle of radius R centered at the origin $(0,0)$ on the xy-plane can be parametrized by:

$$\mathbf{r}(t) = (R\cos t,\ R\sin t).$$

It can be easily verified that $|\mathbf{r}'(t)| = R$ and so $\mathbf{r}(t)$ is not an arc-length parametrization.

To find an arc-length parametrization, we let:

$$s(t) = \int_0^t |\mathbf{x}'(\tau)|\ d\tau = \int_0^t R\ d\tau = Rt.$$

Therefore, $t(s) = \frac{s}{R}$ as a function of s and so an arc-length parametrization of the circle is:

$$\mathbf{r}(s) = \left(R\cos\frac{s}{R},\ R\sin\frac{s}{R}\right).$$

To find its curvature, we compute:

$$\begin{aligned}
\mathbf{r}'(s) &= \frac{d}{ds}\left(R\cos\frac{s}{R},\ R\sin\frac{s}{R}\right) \\
&= \left(-\sin\frac{s}{R},\ \cos\frac{s}{R}\right) \\
\mathbf{r}''(s) &= \left(-\frac{1}{R}\cos\frac{s}{R},\ -\frac{1}{R}\sin\frac{s}{R}\right) \\
\kappa(s) &= |\mathbf{r}''(s)| = \frac{1}{R}.
\end{aligned}$$

Thus the curvature of the circle is given by $\frac{1}{R}$, i.e. the larger the circle, the smaller the curvature.

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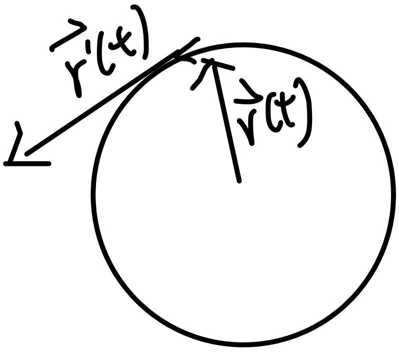
Thm $|\vec{r}(t)| = c \Rightarrow \vec{r}(t) \perp \vec{r}'(t)$

Proof

$\vec{r}(t) \cdot \vec{r}(t) = c^2$

d兩边: $\vec{r}(t) \cdot \vec{r}'(t) + \vec{r}'(t) \cdot \vec{r}(t) = 0$



$2\vec{r}(t)\,\vec{r}'(t) = 0$

所以 $r'(t) \perp \vec{r}'(t)$

$\vec{r}_{arc}(s)$: $|\vec{r}_{arc}'(s)| = 1 \Leftarrow$ Def.

$\Rightarrow \vec{r}_{arc}'(s) \perp \vec{r}_{arc}''(s)$ (用上面個 Thm)

Thm Curvature without arc length parametrization

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|r'(t)|^3}$$

→ No integration!

為求簡化，下面的 proof 將会省略了.

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Proof $\quad K = \left|\vec{r}''(s(t))\right| \Leftarrow$ original definition

$$s = \int_0^t |\vec{r}'(t)|\,dt$$

$$\vec{r}'(s) = \frac{\vec{r}'}{|\vec{r}'|} \Leftarrow \text{unit vector} = \vec{T}$$

($\Rightarrow$ call it to be $\vec{T}$) (unit tangent vectr)

$$T' = \underbrace{\vec{r}''(s)}_{\text{“}K\text{”}} \cdot \underbrace{\frac{ds}{dt}}_{\text{“}r'\text{”}}$$

$$K = \frac{|T'|}{|\vec{r}'|}$$

$$T = \frac{\vec{r}'}{|\vec{r}'|} \Rightarrow \vec{r}' = |\vec{r}'|\,T$$

$$= \frac{ds}{dt} \cdot T$$

$$\vec{r}'' = \frac{d^2 s}{dt^2} T + \frac{ds}{dt} T'$$

劇告一段落

$$\vec{r}' \times \vec{r}'' = \cancel{T \times T} \quad \left(\frac{ds}{dt}\right)^2 T \times T'$$

$\uparrow$ constant length $\Rightarrow T \perp T'$ (Thm 1)

$$|\vec{r}' \times \vec{r}''| = \left(\frac{ds}{dt}\right)^2 |T \times T'| = \underbrace{\left(\frac{ds}{dt}\right)^2}_{|\vec{r}'|^2} \underbrace{|T|}_{1}\,|T'|$$

$$\Rightarrow K = \frac{|T'|}{|r'|} = \frac{|\vec{r}' \times \vec{r}''|}{|r'|^3}$$

Proof 参考

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Polar Coordinates

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2 — Partial Differentiations

> "As you will find in multivariable calculus, there is often a number of solutions for any given problem."
>
> *John Nash*

2.1 Functions of Several Variables

As the name implies, a function of several variables (or a multivariable function) is a function which depends on several quantities. Examples of which include:

$$\text{volume of a cylinder} = \pi r^2 h$$

where r is the radius of the cylinder and h is the height.

Symbolically, we denote the function for the volume of a cylinder by $V(r,h)$, which indicates V depends on both r and h. We can write:

$$V(r,h) = \pi r^2 h.$$

In this chapter, we will extend theory and applications of single-variable differentiation to multivariable differentiation. Multivariable integration will be discussed in the next chapter.

2.1.1 Domains of Functions

An input of a function $f(x,y)$ of two variables involves two quantities x and y. Each input is then represented by a **point** (x,y) in $\mathbb{R}^2$. As in single-variable calculus, the domain of a function is the **set of allowable inputs**. For instance, the function $f(x,y) = \sqrt{y - x^2}$ is defined only when $y \geq x^2$. The domain of this function is given by:

$$D = \{(x,y) : y \geq x^2\}$$

which is the region above the parabola $y = x^2$ in $\mathbb{R}^2$ (the parabola is included).

The function $g(x,y) = \dfrac{1}{x^2 + y^2}$ is defined everywhere on $\mathbb{R}^2$ except the origin $(0,0)$. Therefore the domain of g is given by $\{(x,y) : x \neq 0 \text{ and } y \neq 0\}$. In short, we can write this set as $\mathbb{R}^2 \backslash \{(0,0)\}$, meaning $\mathbb{R}^2$ with the origin removed.

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The function $h = \dfrac{1}{xy}$ is undefined when $xy = 0$, or equivalently when at least one of the x and y is zero. We can write its domain as $\{(x, y) : x \neq 0 \text{ or } y \neq 0\}$. Geometrically, it is $\mathbb{R}^2$ with both x- and y-axes removed.

2.1.2 Graphs of Two-Variable Functions

In single-variable calculus, we visualize a function $y = f(x)$ through its graph. The horizontal x-axis stands for the inputs, and the height of the graph above x represents the output $f(x)$. Many concepts in single-variable calculus, such as derivatives, integrals, critical points, etc. are introduced using the graph of a function.

For functions of two variables, i.e. $f(x, y)$, the graph is no longer a *curve* in $\mathbb{R}^2$, but a *surface* in $\mathbb{R}^3$. The inputs involve two variables x and y, and are represented by points on the xy-plane. The value of the function $f(x, y)$ is now represented by the height z of the surface above the point (x, y).

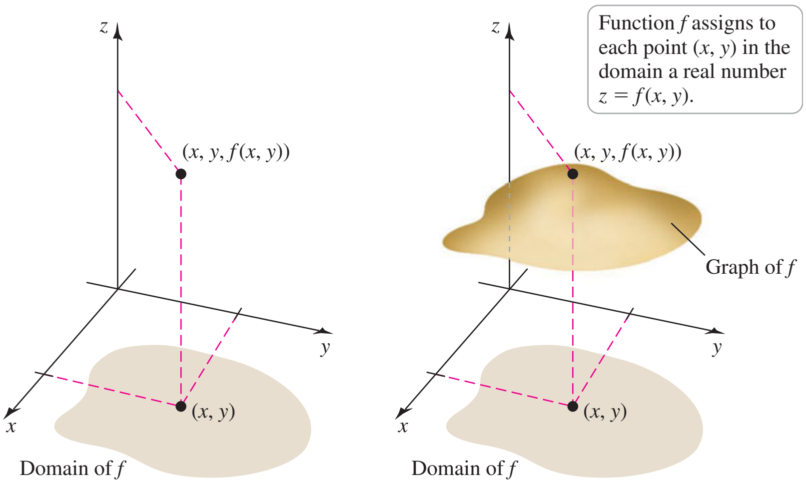
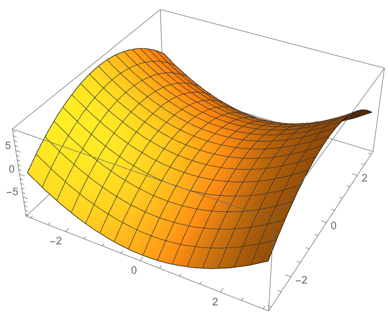
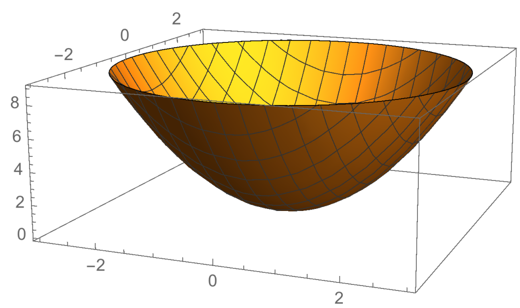


Figure 2.1: value of $f(x_0, y_0)$ is the height of the surface above the point $(x_0, y_0, 0)$



(a) the graph of $z = x^2 - y^2$



(b) the graph of $z = x^2 + y^2$

Figure 2.2: graphs of two-variables functions

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2.1.3 Level Set Diagrams

Another common way to visualize a two-variable function is through its level sets:

Definition 2.1 — Level Sets. Given a function $f(x_1, \dots, x_n) : \mathbb{R}^n \to \mathbb{R}$, a level set of the function f is a subset of $\mathbb{R}^n$ of the form:

$$f(x_1, \dots, x_n) = c$$

where c is a constant.

Given $f(x,y) = x^2 + y^2$, which is a function from $\mathbb{R}^2$ to $\mathbb{R}$. An example of a level set of f is $\underbrace{x^2 + y^2}_{f(x,y)} = 1$, which is a unit circle on $\mathbb{R}^2$ centered at the origin. By taking c to be different values, we get several level sets on the plane. They are circles centered at the origin with varying radii depending on the value of c chosen:

$$\begin{array}{lll} c = 0 & x^2 + y^2 = 0 & \text{the origin only} \\ c = 1 & x^2 + y^2 = 1 & \text{radius} = 1 \\ c = 2 & x^2 + y^2 = 2 & \text{radius} = \sqrt{2} \\ c = 3 & x^2 + y^2 = 3 & \text{radius} = \sqrt{3} \end{array}$$

The **level set diagram** of the two-variable function $f(x,y)$ consists of some representative level sets of the function on $\mathbb{R}^2$. See Figure 2.3.

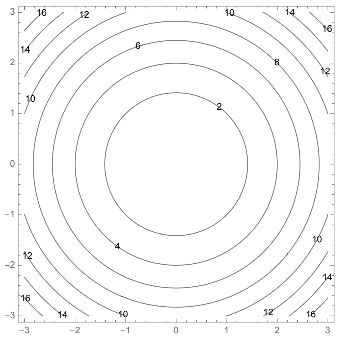

Figure 2.3: level set diagram of $f(x,y) = x^2 + y^2$

For three-variable functions $f(x,y,z)$, we do not attempt to visualize its **graph**, but we can visualize its level set diagram. The former requires the fourth dimension while the latter can be visualized in $\mathbb{R}^3$. A *generic* level set of a three-variable function $f(x,y,z)$ is a **surface** in $\mathbb{R}^3$ (see Figures 2.4ab).

To summarize, a the graph and level set of a function on several variables are:

Functions	Graph	Level Sets
$f(x)$	$y = f(x)$ is a curve in $\mathbb{R}^2$	$f(x) = c$ is *generically* a point on $\mathbb{R}$
$f(x,y)$	$z = f(x,y)$ is a surface in $\mathbb{R}^3$	$f(x,y) = c$ is *generically* a curve on $\mathbb{R}^2$
$f(x,y,z)$	cannot visualize	$f(x,y,z) = c$ is *generically* a surface in $\mathbb{R}^3$
$f(x,y,z,w)$	cannot visualize	cannot visualize

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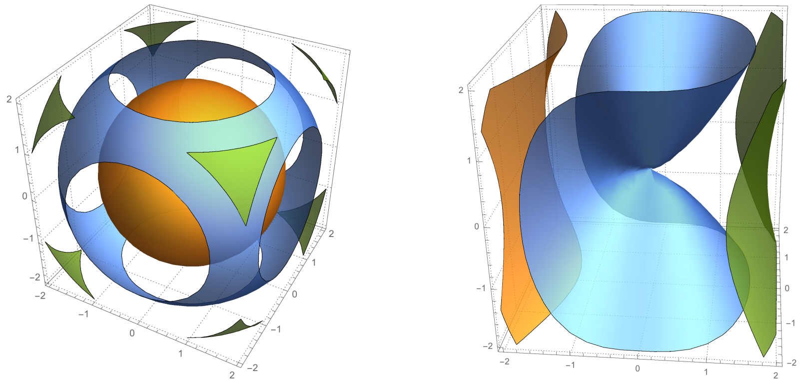

(a) some level sets of $x^2 + y^2 + z^2$ (b) some level sets of $x^3 + y^2 - z^2$

Figure 2.4: level sets of three-variable functions

2.1.4 Continuous Functions

The concept of continuity plays an important role in single variable calculus since many theorems require continuity as one of the conditions. For multivariable functions, the notion of continuity is formally defined as follows: a function $f(x, y)$ is continuous at (x_0, y_0) if:

1. (x_0, y_0) is in the domain of $f(x, y)$; and
2. for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever (x, y) is in the domain of f and that

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta,$$

we have $|f(x, y) - f(x_0, y_0)| < \varepsilon$.

Don't panic if you cannot understand this definition at this moment! We will *not* deal with this definition in this course, but instead we learn continuity through some examples. The rigorous approach of dealing with continuity will be covered systematically in MATH 2033 and MATH 3033. In this course, we will use the following facts about continuity without proof:

1. Any polynomial such as $f(x, y) = x^2 + xy + y^5$ is continuous at every (x_0, y_0) in $\mathbb{R}^2$ (we can also say it is continuous on $\mathbb{R}^2$).
2. $\sin x$, $\cos x$, e^x, $|x|$ are all continuous everywhere.
3. $\ln x$, $\tan x$, $\sqrt{x}$ are continuous on their domains.
4. The sum, difference and product of two continuous functions are all continuous.
5. The quotient $\frac{f(x,y)}{g(x,y)}$ of two continuous functions $f(x, y)$ and $g(x, y)$ is continuous at (x_0, y_0) whenever $g(x_0, y_0) \neq 0$. For instance, the function

$$\frac{x^2 - y^2}{x^2 + y^2}$$

is continuous on $\mathbb{R}^2 \backslash \{(0, 0)\}$.
6. The composition $f \circ g$ of two continuous functions $f(t)$ and $g(x, y)$ is continuous on the domain of $f \circ g$. For example, the functions

$$e^{x-y}, \quad \cos \frac{x}{1 + x^2 + y^2}$$

are continuous on $\mathbb{R}^2$, whereas the function

$$\sqrt{y - x^2}, \quad \frac{1}{xy}$$

are continuous on their domains.

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2.2 Partial Derivatives

2.2.1 First Derivatives

Given a multivariable function such as $f(x,y)$, one can talk about derivatives with respect to both variables x and y. Taking partial derivatives means differentiating $f(x,y)$ with respect to one of the variables while keeping the other variables fixed.

Definition 2.2 — Partial Derivatives. Given a multivariable function $f(x,y)$, we define

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &:= \text{ the derivative of } f(x,y) \text{ with respect to } x \text{ regarding } y \text{ constant}\\ &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h};\\ \frac{\partial f}{\partial y}(x,y) &:= \text{ the derivative of } f(x,y) \text{ with respect to } y \text{ regarding } x \text{ constant}\\ &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}.\end{aligned}$$

Notation Alternatively, we sometimes denote $\frac{\partial f}{\partial x}$ by f_x, and $\frac{\partial f}{\partial y}$ by f_y. Note also that we do not use $f'(x,y)$ for multivariable functions, since it is ambigious to whether it means f_x or f_y.

Computations of partial derivatives are as easy as single-variable derivatives, as illustrated in the following example:

■ **Example 2.1** Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the function:

$$f(x,y) = x^2\sin(xy).$$

■ **Solution** To calculate $\frac{\partial f}{\partial x}$, we regard y as a constant:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(x^2\sin(xy)\right) && \text{(regarding } y \text{ constant)}\\ &= \frac{\partial x^2}{\partial x}\sin(xy) + x^2\frac{\partial}{\partial x}\sin(xy) && \text{(product rule)}\\ &= 2x\sin(xy) + x^2\cdot\cos(xy)\frac{\partial}{\partial x}xy\\ &= 2x\sin(xy) + x^2 y\cos(xy).\end{aligned}$$

Similarly, to calculate $\frac{\partial f}{\partial y}$, regard x as a constant:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x^2\sin(xy)\right)\\ &= x^2\frac{\partial}{\partial y}\sin(xy) && \text{(here } x^2 \text{ is regarded as a constant)}\\ &= x^2\cos(xy)\cdot\frac{\partial}{\partial y}xy\\ &= x^2\cos(xy)\cdot x\\ &= x^3\cos(xy).\end{aligned}$$

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Similar to single-variable calculus, to evaluate f_x at a fixed point say $(x, y) = (1, \pi)$, one should perform the differentiation *first*, and *then* substitute $(x, y) = (1, \pi)$ into the derivative. Not the other way round! For example,

$$f_x(1, \pi) = 2x\sin(xy) + x^2 y\cos(xy)\Big|_{(x,y)=(1,\pi)} = 2 \cdot 1 \sin\pi + 1^2 \cdot \pi\cos(\pi) = -\pi.$$

Geometric interpretation of partial derivatives

The geometric meaning of the partial derivative f_x is illustrated in Figure 2.5. By keeping y constant (say we let $y = b$) and let x varies, the path traced on the surface is the curve of intersection between the plane $y = b$ and the graph of the function. This curve is sometimes called an x-curve. The partial derivative $f_x(a, b)$ is the slope of the tangent to this x-curve at (a, b):

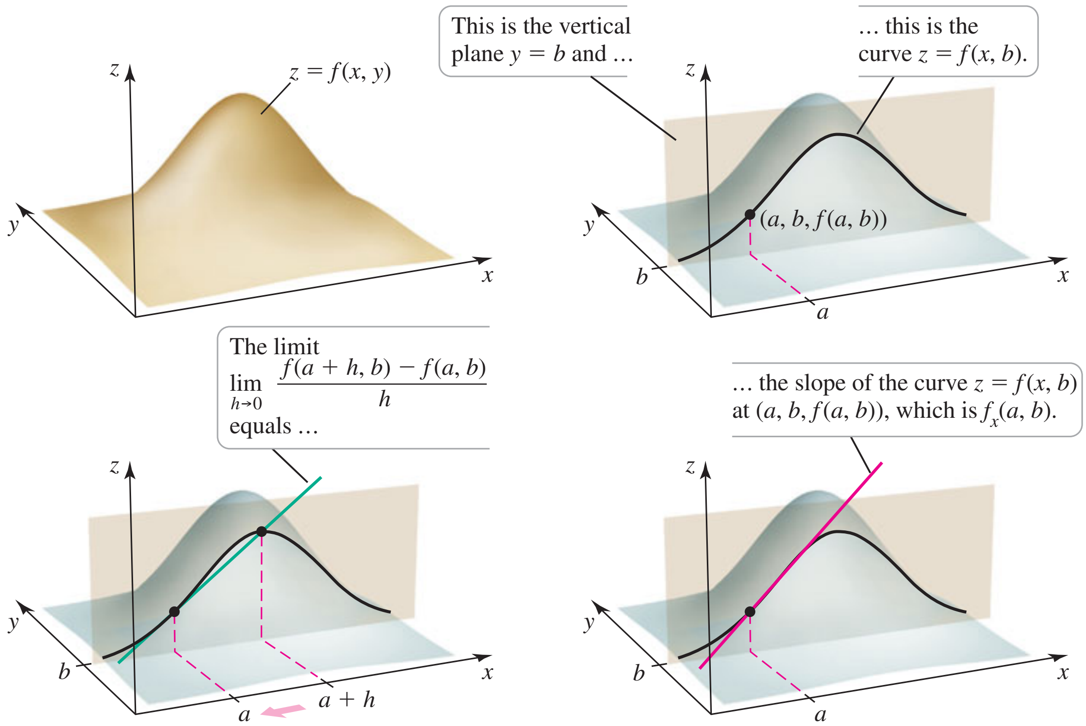

Figure 2.5: geometric interpretation of $\frac{\partial f}{\partial x}$

Partial derivatives of function with more than two variables

Partial derivatives of functions with more than two variables, say $f(x, y, z)$, are defined in an analogous way. For instance, $\frac{\partial f}{\partial x}$ is the derivative with respect to x regarding all other variables, i.e. y and z, constant. Although it is not easy to interpret $\frac{\partial f}{\partial x}$ geometrically since the graph of $f(x, y, z)$ sits inside a 4-dimensional space, the way to compute $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are exactly the same as two-variable functions.

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■ **Example 2.2** Given $f(x,y,z) = e^{x^2+y^3+xyz}$. Compute $\frac{\partial f}{\partial z}$.

■ **Solution**

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z} e^{x^2+y^3+xyz} \\ &= e^{x^2+y^3+xyz} \cdot \frac{\partial (x^2+y^3+xyz)}{\partial z} && \text{(chain rule)} \\ &= e^{x^2+y^3+xyz} \cdot (0+0+xy) \\ &= xye^{x^2+y^3+xyz}.\end{aligned}$$

2.2.2 Second Derivatives

As in single-variable calculus, one can also talk about second derivatives for multivariable functions. Given a two-variable function $f(x,y)$, its first partial derivatives f_x and f_y are also functions of x and y. Therefore, we can further differentiate them with respect to either x or y.

■ **Example 2.3** Let $f(x,y) = 3x^4y - 2xy + 5xy^3$. Compute all first and second partial derivatives.

■ **Solution** It is easy to see that

$$\begin{aligned}\frac{\partial f}{\partial x} &= 12x^3y - 2y + 5y^3 \\ \frac{\partial f}{\partial y} &= 3x^4 - 2x + 15xy^2\end{aligned}$$

Then, the second derivatives are:

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial}{\partial x}(12x^3y - 2y + 5y^3) = 36x^2y \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial}{\partial y}(12x^3y - 2y + 5y^3) = 12x^3 - 2 + 15y^2 \\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial}{\partial x}(3x^4 - 2x + 15xy^2) = 12x^3 - 2 + 15y^2 \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial}{\partial y}(3x^4 - 2x + 15xy^2) = 30xy.\end{aligned}$$

Notation Since it is a bit clumsy to write $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ every time, we can use the following short-hand:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

Similarly for the subscript notations, we write $f_{xx} = (f_x)_x$, and $f_{xy} = (f_x)_y$. The later means to differentiate by x first and then by y. Therefore, it is related to the fraction notation by:

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}.$$

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The above remark *seems* to suggest that we should be very careful when converting $\frac{\partial^2 f}{\partial y \partial x}$ into the subscript notation f_{xy}. The order of x and y needs to be switched in the conversion.

However, thanks to the following important theorem, we don't need to worry about this too much, since in *many cases*, we have $f_{xy} = f_{yx}$.

Clairant's Theorem

Theorem 2.1 — Mixed Partials Theorem. Consider the function $f(x, y)$, if at least one of the second partials f_{xy} and f_{yx} exists and is continuous, then we must have $f_{xy} = f_{yx}$.

Proof. Beyond the scope of this course. To be covered in MATH 3033. ■

Although the theorem requires that f_{xy} or f_{yx} needs to be continuous, most functions we will encounter in this course are continuous and so this theorem applies. In Example 2.3, you may have already noticed that f_{xy} and f_{yx} are equal. The Mixed Partials Theorem tells us that it is not a coincident!

■ **Example 2.4** Consider the function:

$$f(x, y) = \sqrt{\frac{e^{\sin x}}{x^{2014} + \sqrt{x^{2012}+1}}} + \cos(xy).$$

Find the second partial derivative $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$.

■ **Solution** Needless to say, it is very tedious and time-consuming to compute $\frac{\partial f}{\partial x}$. However, by the Mixed Partials Theorem, we can try to find $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$, and if it is continuous, then $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$.

It is much easier to compute $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ since the "monster" term is gone after differentiating the function by y:

$$\begin{aligned} \frac{\partial f}{\partial y} &= 0 - \sin(xy) \cdot \frac{\partial}{\partial y} xy \\ &= -x \sin(xy) \\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= -\frac{\partial}{\partial x}(x \sin(xy)) \\ &= -\sin(xy) - xy\cos(xy), \end{aligned}$$

which is a continuous function. Therefore, $f_{yx} = f_{xy}$ and so

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin(xy) - xy\cos(xy).$$

All examples of multivariable functions we have seen so far are "nice", in a sense that partial derivatives exist and are continuous at every point in their domains. In this course, we will use the following terminology:

Definition 2.3 — C^k functions. A multivariable function $f(x, y)$ is said to be C^0 on its domain D if it is continuous at every point (x_0, y_0) in D. Moreover, a function $f(x, y)$ is said to be C^k on its domain D if all partial derivatives up to and including order k exist and are continuous at every point (x_0, y_0) in D.

ⓘ In this course, we will not discuss the difficult notion of *differentiable functions*, which will be covered in MATH 3033. Meanwhile, please note that a function being C^1 is not the same as saying it is differentiable!

Omitted : Tangent-plane approximation

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2.3 Chain Rule

In this section, we assume that the partial derivatives of all functions involved exist and are continuous (i.e. C^k for any k), so that we do not need to worry about whether we can differentiate the functions.

2.3.1 Multivariable Chain Rule

In single-variable calculus, we apply the chain rule when there is a chain of relations between variables. For example, if $f(x)$ is a function of x, and x is in turn a function of t, then f is ultimately a function of t. The derivative $\frac{df}{dt}$ can be calculated by:

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}.$$

One may represent this chain of relations by the schematic diagram:

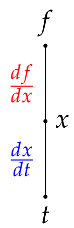

Figure 2.6: the schematic diagram for chain rule formula $\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$.

For multivariable functions, the relation between variables can be more complicated. For example, let the function $u(x,y,z)$ be the temperature at the point (x,y,z) in the space. Suppose a particle moves along the path

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

Then, the coordinates x, y and z all depend on t, and so u is ultimately a function of t. See Figure 2.7 for the tree diagram of the variables.

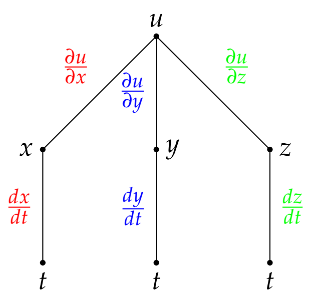

Figure 2.7: tree diagram of variables for $u(x,y,z)$.

The derivative $\frac{du}{dt}$ is the rate of change of the temperature that the particle "feels" as it travels. The multivariable chain rule can be read off from the tree diagram 2.7:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Precisely, to write down the chain rule for $\frac{du}{dt}$, we find all possible paths from u to t in the tree diagram. Each path consists of some segments. A segment, say, from u to x represents the derivative $\frac{\partial u}{\partial x}$. To write down the chain rule, we "multiply" all segments of each path, and "add" up all the paths.

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■ **Example 2.5** Let $u(x,y,z) = x^2 + y^2 - 2z^2$ and $\langle x,y,z\rangle = \langle \cos t, \sin t, t\rangle$. Compute $\frac{du}{dt}$.

■ **Solution** Although it can be done by substituting $x = \cos t$, $y = \sin t$ and $z = t$ into $u(x,y,z) = x^2 + y^2 - 2z^2$ and then compute $\frac{du}{dt}$ directly, let's try to use the chain rule to do it. From the tree diagram (Figure 2.7), we have:

$$\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \\
&= \underbrace{2x}_{\frac{\partial u}{\partial x}} \cdot \underbrace{(-\sin t)}_{\frac{dx}{dt}} + \underbrace{2y}_{\frac{\partial u}{\partial y}} \cdot \underbrace{\cos t}_{\frac{dy}{dt}} + \underbrace{(-2z)}_{\frac{\partial z}{\partial t}} \cdot \underbrace{1}_{\frac{dz}{dt}} && \text{(calculate each derivative)} \\
&= -2\cos t \sin t + 2\sin t \cos t - 2t && \text{(write } x, y \text{ and } z \text{ in terms of } t\text{)} \\
&= -2t.
\end{aligned}$$

As illustrated in the above example, once the chain rule formula is correctly written according to the tree diagram, the remaining computations are straight-forward. From now on, we will investigate how to write down the chain rule under various configuration of variables. The computations will usually be skipped and are left as exercises for readers.

Examples with more diversed variable configurations

Suppose now that the temperature distribution is changing over time as well, i.e. the temperature $u(x,y,z,t)$ is a function of four variables. Again, a particle travels along a path $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then, the tree diagram of variables in this case can be drawn as in Figure 2.8.

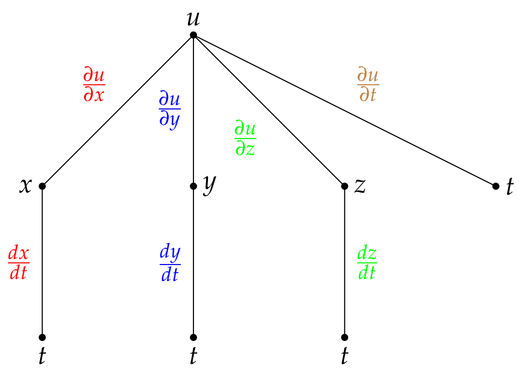

Figure 2.8: tree diagram of variables for $u(x,y,z,t)$.

There are four paths from u to t, so we expect the chain rule formula for $\frac{du}{dt}$ consists of four terms:

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}.$$

Note that $\frac{du}{dt}$ is different from $\frac{\partial u}{\partial t}$. As the particle travels along its path, the temperature the particle "feels" is a ultimately a function of t, and so the rate of change of temperature is represented by $\frac{du}{dt}$ (using d instead of the partial ∂).

On the other hand, the partial $\frac{\partial u}{\partial t}$ is the time derivative of u regarding x, y and z constant! Therefore, it is the rate of change of temperature when the position is **fixed**! It is not the rate of change of temperature for the **moving** particle!

Now consider a slightly more complicated example. Let w be a function of x, y and z, and each of x, y and z is a function of s and t, as illustrated in Figure 2.9. The function w is ultimately a function of s and t.

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There are three paths from w to s, so the chain rule for $\frac{\partial w}{\partial s}$ is given by the following:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

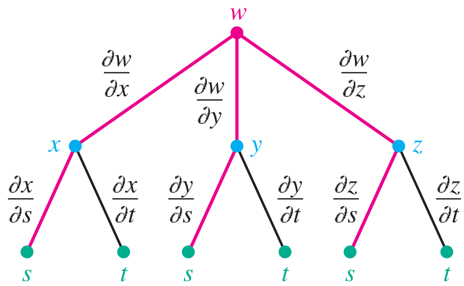

Figure 2.9: tree diagram

■ **Example 2.6** Express $\frac{\partial w}{\partial s}$ in terms of s and r where:

$$w = x + 2y + z^2$$

$$x = \frac{r}{s} \qquad y = r^2 + \ln s \qquad z = 2r$$

■ **Solution** According to the tree diagram Figure 2.9, the chain rule for $\frac{\partial w}{\partial s}$ is given by:

$$\begin{aligned}
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= \underbrace{1}_{w_x} \cdot \underbrace{\left(-\frac{r}{s^2}\right)}_{x_s} + \underbrace{2}_{w_y} \cdot \underbrace{\frac{1}{s}}_{y_s} + \underbrace{2z}_{w_z} \cdot \underbrace{0}_{z_s} \\
&= -\frac{r}{s^2} + \frac{2}{s}.
\end{aligned}$$

Now suppose w is a function of z only, z is a function of x and y, and both x and y are functions of t, as illustrated in Figure 2.10. Ultimately, w is a function t. There are two paths from w to t in the tree diagram. Each path consists of three segments. The chain rule for $\frac{dw}{dt}$ is given by:

$$\frac{dw}{dt} = \frac{dw}{dz}\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{dw}{dz}\frac{\partial z}{\partial y}\frac{dy}{dt}.$$

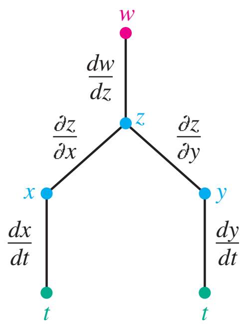

Figure 2.10: tree diagram

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2.3.2 Implicit Differentiation: revisited

Given an implicit equation such as:

$$x^2 + y^3 + \sin^2 y = 1,$$

it is very difficult (possibly impossible) to express y in terms of x. In single-variable calculus, we learned how to find $\frac{dy}{dx}$ using implicit differentiation – regard y as a function of x, and differentiate both sides by x then solve for $\frac{dy}{dx}$.

The multivariable chain rule offers an alternative approach to implicit differentiation.

Define $f(x,y) = x^2 + y^3 + \sin^2 y$, then the above implicit equation can be written as $f(x,y) = 1$. Regarding y as a function of x, $f(x,y)$ is ultimately a function of x. Figure 2.11 shows the tree diagram for the variables.

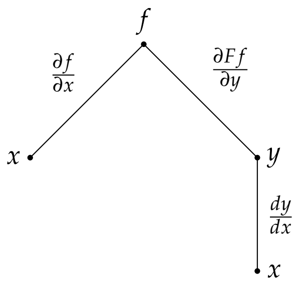

Figure 2.11: tree diagram for implicit differentiation

Therefore, by the chain rule, we have

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}.$$

Recall that $f(x,y) = 1$ is a constant, so $\frac{df}{dx} = 0$, which yields:

$$0 = f_x + f_y \frac{dy}{dx}, \quad \text{and so:} \quad \frac{dy}{dx} = -\frac{f_x}{f_y}.$$

It is a much straight-forward formula than the implicit differentiation method learned in single-variable calculus. When $f(x,y) = x^2 + y^3 + \sin^2 y$, we have:

$$\frac{dy}{dx} = -\frac{2x}{3y^2 + 2\sin y \cos y}.$$

2.3.3 Chain Rule on Second Derivatives

Suppose $u(x,y)$ is a function of x and y. The rectangular-polar coordinates conversion rule is given by:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Therefore, both x and y can be regarded as functions of r and θ, and so u can be regarded as a function of r and θ as well. By the chain rule, we know:

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= u_x \cos\theta + u_y \sin\theta. \end{aligned}$$

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To find $u_{r\theta}$, one can differentiate the above expression by θ:

$$\begin{aligned}\frac{\partial^2 u}{\partial\theta\partial r} &= \frac{\partial}{\partial\theta}\left(u_x\cos\theta + u_y\sin\theta\right)\\ &= \frac{\partial u_x}{\partial\theta}\cos\theta - u_x\sin\theta + \frac{\partial u_y}{\partial\theta}\sin\theta + u_y\cos\theta.\end{aligned}$$

Next we would like to express $\frac{\partial u_x}{\partial\theta}$ and $\frac{\partial u_y}{\partial\theta}$ as partial derivatives with respect to x and y only. The reason of doing so is because u_{xx} is much easier to compute than $u_{x\theta}$. For instance, if $u(x,y) = x^2 + y$, then $u_x = 2x$, and so $u_{xx} = 2$ and $u_{xy} = 0$. However, to find $u_{x\theta}$ one needs to first express u_x as $2r\cos\theta$.

Since u_x and u_y are both functions of x and y, and (x,y) are functions of (r,θ). Therefore, u_x and u_y will have the same tree diagram as the function u. The chain rule for them is given by:

$$\begin{aligned}\frac{\partial u_x}{\partial\theta} &= \frac{\partial u_x}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial u_x}{\partial y}\frac{\partial y}{\partial\theta}\\ &= u_{xx}\left(-r\sin\theta\right) + u_{xy}\left(r\cos\theta\right)\\ \frac{\partial u_y}{\partial\theta} &= \frac{\partial u_y}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial u_y}{\partial y}\frac{\partial y}{\partial\theta}\\ &= u_{yx}\left(-r\sin\theta\right) + u_{yy}\left(r\cos\theta\right)\end{aligned}$$

Substitute them back in, we get:

$$\begin{aligned}\frac{\partial^2 u}{\partial\theta\partial r} &= \frac{\partial u_x}{\partial\theta}\cos\theta - u_x\sin\theta + \frac{\partial u_y}{\partial\theta}\sin\theta + u_y\cos\theta\\ &= \left(-u_{xx}r\sin\theta + u_{xy}r\cos\theta\right)\cos\theta - u_x\sin\theta\\ &\quad + \left(-u_{yx}r\sin\theta + u_{yy}r\cos\theta\right)\sin\theta + u_y\cos\theta\\ &= -u_{xx}r\sin\theta\cos\theta + u_{xy}r\left(\cos^2\theta - \sin^2\theta\right) + u_{yy}r\sin\theta\cos\theta\\ &\quad - u_x\sin\theta + u_y\cos\theta\end{aligned}$$

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2.4 Directional Derivatives

Recall that the physical meaning of the partial derivative $\frac{\partial f}{\partial x}$ is the rate of change of f in the direction of x. The geometric interpretation was discussed in Figure 2.5. Similarly, $\frac{\partial f}{\partial y}$ is the rate of change of f in the direction of y. In this section, we introduce the rate of change of f in any other directions:

Definition 2.4 — Directional Derivative. Given a **unit** direction $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and a two-variable function $f(x,y)$, the directional derivative of f in the direction of $\mathbf{u}$ at point (x,y) is denoted and defined to be:

$$D_{\mathbf{u}}f(x,y) = \frac{d}{dt}f(x+tu_1, y+tu_2)\bigg|_{t=0}.$$

When $\mathbf{u} = \mathbf{i}$, then $u_1 = 1$ and $u_2 = 0$ and so

$$D_{\mathbf{i}}f(x,y) = \frac{d}{dt}f(x+t,y)\bigg|_{t=0} = \lim_{t\to 0}\frac{f(x+t,y)-f(x,y)}{t} = \frac{\partial f}{\partial x}(x,y).$$

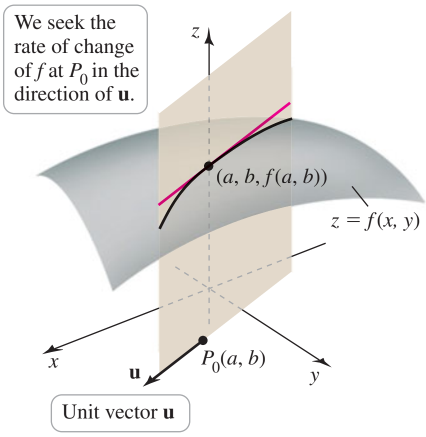

Figure 2.12: directional derivative

In practice, we do not need to compute $D_{\mathbf{u}}f$ from the definition, since Theorem 2.2 to be introduced will come in handy help us. In order to introduce this theorem, we first define:

Definition 2.5 — Gradient Vector. Given a two-variable function $f(x,y)$ which is C^1 on its domain, the gradient vector of f at (x,y) is denoted and defined as:

$$\nabla f(x,y) = \frac{\partial f}{\partial x}(x,y)\mathbf{i} + \frac{\partial f}{\partial y}(x,y)\mathbf{j}.$$

As an example, let $f(x,y) = x^2y + x^3$, then

$$\frac{\partial f}{\partial x}(x,y) = 2xy + 3x^2$$
$$\frac{\partial f}{\partial y}(x,y) = x^2$$

Therefore,

$$\nabla f(x,y) = (2xy+3x^2)\mathbf{i} + x^2\mathbf{j}.$$

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The vector $\nabla f(x,y)$ depends on (x,y). By taking different values of (x,y), a different gradient vector is produced. For instance,

$$\nabla f(1,1) = 5\mathbf{i} + \mathbf{j},$$
$$\nabla f(1,0) = 3\mathbf{i} + \mathbf{j}.$$

Theorem 2.2 Given a two-variable function $f(x,y)$ which is C^1 on its domain, the directional derivative of f at (x,y) in the **unit direction** $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is given by:

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}.$$

By applying this theorem in the special case $\mathbf{u} = \mathbf{i}$, we can see

$$\nabla f(x,y) \cdot \mathbf{u} = \left(\frac{\partial f}{\partial x}(x,y)\mathbf{i} + \frac{\partial f}{\partial y}(x,y)\mathbf{j}\right) \cdot \mathbf{i} = \frac{\partial f}{\partial x}(x,y) = D_{\mathbf{i}} f(x,y)$$

as expected. Similarly, we can see $\nabla f(x,y) \cdot \mathbf{j} = \frac{\partial f}{\partial y}(x,y) = D_{\mathbf{j}} f(x,y)$ as expected. Let's see the proof of the general case:

Proof of Theorem 2.2. The key idea is to use the chain rule. The directional derivative $D_{\mathbf{u}} f(x_0, y_0)$ at the point (x_0, y_0) is the rate of change of f along the path $\mathbf{r}(t) = \langle x_0, y_0 \rangle + t\langle u_1, u_2 \rangle$, i.e. $x = x_0 + u_1 t$ and $y = y_0 + u_2 t$. By definition of directional derivative, $D_{\mathbf{u}} f(x_0, y_0)$ is the derivative $\frac{d}{dt} f(x_0 + u_1 t, y_0 + u_2 t)$ at $t = 0$. Therefore, f is a function of (x,y), and (x,y) are functions of t. By the chain rule, we have:

$$\begin{aligned}
D_{\mathbf{u}} f &= \frac{df}{dt} \\
&= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial f}{\partial x}\frac{d}{dt}(x_0 + tu_1) + \frac{\partial f}{\partial y}\frac{d}{dt}(y_0 + tu_2) \\
&= \frac{\partial f}{\partial x} \cdot u_1 + \frac{\partial f}{\partial y} \cdot u_2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nabla f \cdot \mathbf{u} &= \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) \\
&= \frac{\partial f}{\partial x} \cdot u_1 + \frac{\partial f}{\partial y} \cdot u_2.
\end{aligned}$$

Therefore, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$. ■

This theorem tells us that the computation of directional derivatives amounts to computing the gradient vector and a dot product. Easy enough? As an example, given $f(x,y) = x^2 y + x^3$. We worked out in the previous example that $\nabla f(1,1) = 5\mathbf{i} + \mathbf{j}$, and so the directional derivative of f at $(1,1)$ along $\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$ is given by:

$$D_{\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}} f(1,1) = \nabla f(1,1) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) = (5\mathbf{i} + \mathbf{j}) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) = \frac{6}{\sqrt{2}}.$$

2.4.1 Geometric Interpretation of Gradient Vectors

Theorem 2.2 not only tells us how to compute the directional derivative $D_{\mathbf{u}} f(x,y)$, but also tells us in what direction $\mathbf{u}$ the derivative $D_{\mathbf{u}} f(x,y)$ is the greatest and the smallest.

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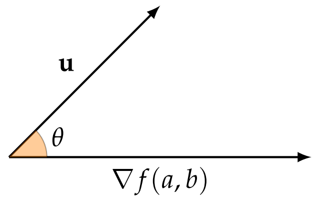

Figure 2.13: $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|u| \cos\theta$

Given a function $f(x,y)$, fix a point (a,b). From the dot product formula (1.1), we know:

$$D_{\mathbf{u}} f(a,b) = \nabla f(a,b) \cdot \mathbf{u} = |\nabla f(a,b)| \underbrace{|\mathbf{u}|}_{=1} \cos\theta = |\nabla f(a,b)| \cos\theta$$

where θ is the angle between $\nabla f(a,b)$ and $\mathbf{u}$.

Since $\cos\theta$ is the largest when $\theta = 0$ at which $\cos\theta = 1$, the directional derivative $D_{\mathbf{u}} f(a,b)$ is **maximized** when $\nabla f(a,b)$ is parallel to $\mathbf{u}$. **Therefore, $\nabla f(a,b)$ is pointing in the direction at which f increases most rapidly from** (a,b). It is quite intuitive that in order to increase the value of f most rapidly, one should go along the direction **perpendicular to the level curve**. It is indeed true. Let's state it as a theorem:

Theorem 2.3 Let $f(x,y)$ be a two-variable function which is C^1 on its domain, and (a,b) be a point on the level curve $f(x,y) = c$. Then the gradient vector $\nabla f(a,b)$ is orthogonal to the level curve $f(x,y) = c$ at the point (a,b).

Proof. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ be a parametrization of the level curve $f(x,y) = c$. In other words, we have $f(x(t), y(t)) = c$ for any t, and so

$$\frac{d}{dt} f(x(t), y(t)) = 0.$$

By the chain rule, we get:

$$\begin{aligned} \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} &= 0 \\ \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right) &= 0 \\ \nabla f \cdot \mathbf{r}'(t) &= 0. \end{aligned}$$

Therefore, the gradient vector ∇f is orthogonal to $\mathbf{r}'(t)$ which is the tangent vector of the level curve $\mathbf{r}(t)$. It completes the proof. ■

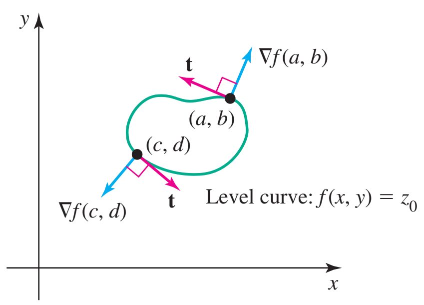

Figure 2.14: $\nabla f(a,b)$ is orthogonal to the level curve of f at (a,b)

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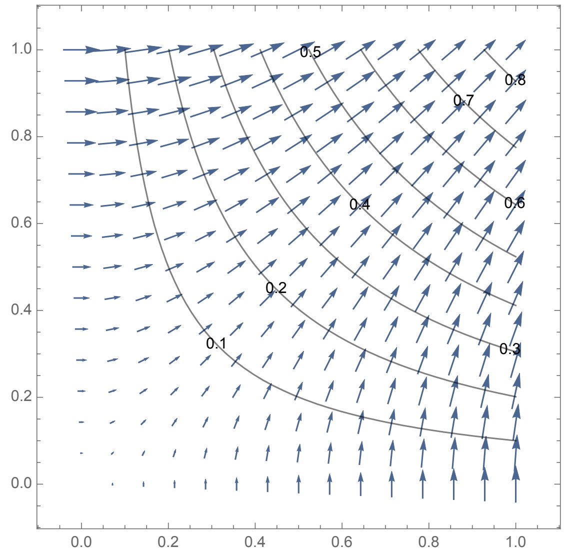

Figure 2.15: a plot of the field $\nabla(\sin xy)$ and the level curves of $\sin xy$.

2.4.2 Directional Derivative of Three-Variable Functions

The directional derivative and the gradient vector for functions $f(x, y, z)$ of three variables is defined in an analogous way as for two-variable functions. Precisely, we have:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Given a unit vector $\mathbf{u}$, the directional derivative of $f(x, y, z)$ in the direction of $\mathbf{u}$ is given by:

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}.$$

Since the level set of a three-variable function is typically a surface, the gradient vector ∇f at any given point is orthogonal to the level surface $f(x, y, z) = c$ at that point. See Figure 2.16.

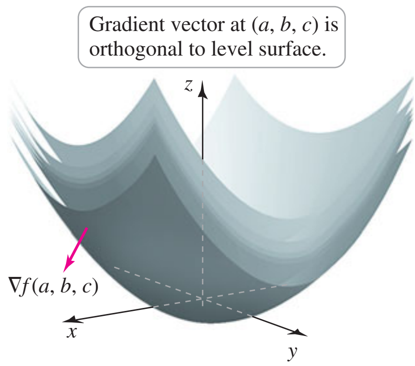

Figure 2.16: $\nabla f(a, b, c)$ is orthogonal to the level surface of f at (a, b, c)

In the next section, we will use this fact to find the equation of the tangent plane to a surface.

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2.5 Tangent Planes

In single-variable calculus, the tangent line to the curve $y = f(x)$ at a point $(x_0, f(x_0))$ has slope equal to $f'(x_0)$. Using the slope, one can easily write down the equation of the tangent line as:

$$y = f(x_0) + f'(x_0)(x - x_0).$$

In multivariable calculus, the graph of a two-variable function $z = f(x, y)$ is no longer a curve but a surface. Therefore, there are infinitely many tangent lines passing through any given point on a surface. However, there is only *one* tangent plane at any given point. In this section, we want to find an equation for the tangent plane.

Recall that the two ingredients of finding an equation of a plane are:

- a normal vector to the plane; and
- any given point on the plane.

Naturally, the point can be taken to be the contact point of the surface (x_0, y_0, z_0). To find the normal vector, we will use the **gradient vector**.

In the previous section, we see that given a three-variable function $g(x, y, z)$, the gradient vector $\nabla g(x, y, z)$ is perpendicular to the level surface $g(x, y, z) = c$. In other words, ∇g is a **normal vector** of the level surface $g(x, y, z) = c$.

▪ **Example 2.7** Find the tangent plane to the surface

$$x^2 + y^2 = z^2 + 3$$

at the point $(x, y, z) = (2, 0, -1)$.

▪ **Solution** First we need to write the equation of the surface in a level set form, i.e.

$$x^2 + y^2 - z^2 = 3$$

Define $g(x, y, z) = x^2 + y^2 - z^2$, then the given surface is a level set $g(x, y, z) = 3$. By direct computations,

$$\begin{aligned} \nabla g(x, y, z) &= 2x\mathbf{i} + 2y\mathbf{j} + (-2z)\mathbf{k} \\ \nabla g(2, 0, -1) &= 4\mathbf{i} + 2\mathbf{k}. \end{aligned}$$

Then, $\mathbf{n} := 4\mathbf{i} + 2\mathbf{k}$ is a normal vector of the surface at $(2, 0, -1)$. The equation of the tangent plane at $(2, 0, -1)$ is given by:

$$\begin{aligned} 4x + 0y + 2z &= 4(2) + 0(0) + 2(-1), \\ 4x + 2z &= 6. \end{aligned}$$

Given a two-variable function $f(x, y)$, the graph $z = f(x, y)$ is a surface. In order to find the tangent plane at a given point, one can rewrite the graph equation $z = f(x, y)$ as:

$$z - f(x, y) = 0.$$

Then, one can define $g(x, y, z) = z - f(x, y)$ so that the **graph** of the two-variable function $f(x, y)$ becomes a **level set** of a three-variable function $g(x, y, z)$. Let's look at an example:

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■ **Example 2.8** Given the function $f(x,y) = x\cos y - ye^x$, find the tangent plane at $(0,0,0)$ to the graph $z = x\cos y - ye^x$.

■ **Solution** First rearrange the terms so that the graph equation becomes a level set:

$$z - x\cos y + ye^x = 0.$$

Define $g(x,y,z) = z - x\cos y + ye^x$, then the surface under consideration is the level set $g(x,y,z) = 0$.

$$\nabla g(x,y,z) = (-\cos y + ye^x)\mathbf{i} + (x\sin y + e^x)\mathbf{j} + \mathbf{k}.$$

The normal vector at $(0,0,0)$ is given by:

$$\mathbf{n} = \nabla g(0,0,0) = -\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

The equation of the tangent plane at $(0,0,0)$ to the graph is:

$$-x + y + z = 0.$$

Generally, one can derive a formula for finding the tangent plane of any graph of a two-variable function $f(x,y)$:

Theorem 2.4 Given a function $f(x,y)$ which is C^1 on its domain. The equation of the tangent plane for the graph $z = f(x,y)$ at the point $(x_0, y_0, f(x_0,y_0))$ is given by:

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y - y_0).$$

Proof. Write the graph equation $z = f(x,y)$ as:

$$z - f(x,y) = 0$$

and define $g(x,y,z) = z - f(x,y)$, then the graph can be regarded as a level set $g(x,y,z) = 0$ of the three-variable function g.

$$\nabla g(x,y,z) = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}.$$

At the point $(x_0, y_0, f(x_0,y_0))$, the normal vector to the surface is therefore given by:

$$\mathbf{n} = \nabla g(x_0, y_0, f(x_0,y_0)) = -\frac{\partial f}{\partial x}(x_0,y_0)\mathbf{i} - \frac{\partial f}{\partial y}(x_0,y_0)\mathbf{j} + \mathbf{k}.$$

The equation of the tangent plane at $(x_0, y_0, f(x_0,y_0))$ is:

$$\left(-\frac{\partial f}{\partial x}(x_0,y_0)\right)x + \left(-\frac{\partial f}{\partial y}(x_0,y_0)\right)y + z = \left(-\frac{\partial f}{\partial x}(x_0,y_0)\right)x_0 + \left(-\frac{\partial f}{\partial y}(x_0,y_0)\right)y_0 + f(x_0,y_0)$$

By rearrangement, we get:

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y - y_0),$$

as desired. ■

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2.6 Local Extrema

For a single-variable function $f(x)$, if $f'(c) = 0$, we say $(c, f(c))$ is a critical point. It is a *candidate* for local maximum or minimum. The second derivative may be used to determine whether the critical point is a local maximum or a local minimum.

In this section, we will extend the concept of critical points to two-variable functions, and introduce the second derivative test for these functions.

2.6.1 Critical Points

Recall that the equation of the tangent plane to the graph $z = f(x,y)$ at a point (a,b) is given by:

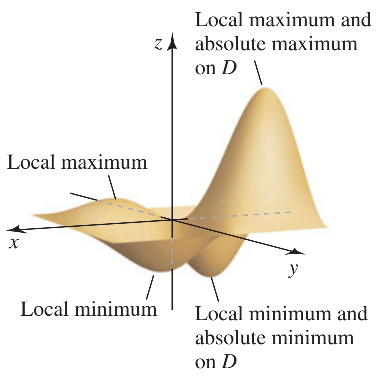
$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

This plane is horizontal if and only if $f_x(a,b) = f_y(a,b) = 0$. This motivates the definition of critical points for two-variable functions:

Definition 2.6 — Critical Points. Given a C^1 function $f(x,y)$. A point (a,b) is said to be a critical point if the tangent plane at (a,b) to the graph $z = f(x,y)$ is **horizontal**.

Therefore, (a,b) is a critical point $\Leftrightarrow \frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0 \Leftrightarrow \nabla f(a,b) = \mathbf{0}$.

As in single-variable calculus, the critical points are just *candidates* of maximum/minimum. Further investigation is needed to determine whether it is a local maximum or minimum, or neither.



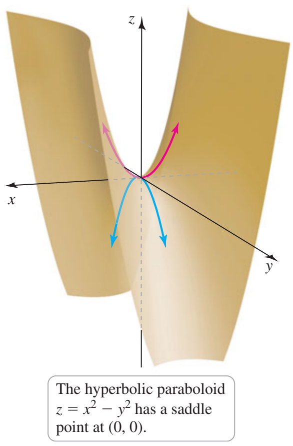


Figure 2.17: the tangent plane at a critical point of a C^1 function is horizontal. However, a critical point is not always a local maximum or minimum. It can be a **saddle** like the origin of $z = x^2 - y^2$, which is a local maximum in the y-direction but is a local minimum in the x-direction.

Example 2.9 Find all critical point(s) of the function $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.

Solution We compute $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and then set them to zero and solve for (x,y):

$$0 = \frac{\partial f}{\partial x} = y - 2x - 2$$
$$0 = \frac{\partial f}{\partial y} = x - 2y - 2.$$

It is a system of equations with unknowns x and y. The first equation gives $y = 2x + 2$, and

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substitute it into the second equation, we get:

$$x - 2(2x + 2) - 2 = 0 \Rightarrow x = -2.$$

When $x = -2$, we have $y = -2$, and so $(x, y) = (-2, -2)$ is a critical point of $f(x, y)$. See Figure 2.18a for its graph.

■ **Example 2.10** Find all critical point(s) of the function $f(x, y) = \sin x \sin y$

■ **Solution** Consider:

$$\begin{aligned} \frac{\partial f}{\partial x} = 0 \quad &\text{and} \quad \frac{\partial f}{\partial y} = 0 \\ \cos x \sin y = 0 \quad &\text{and} \quad \sin x \cos y = 0 \\ (\cos x = 0 \text{ or } \sin y = 0) \quad &\text{and} \quad (\sin x = 0 \text{ or } \cos y = 0) \\ (x = \frac{\pi}{2} + k\pi \text{ or } y = m\pi) \quad &\text{and} \quad (x = n\pi \text{ or } y = \frac{\pi}{2} + p\pi). \end{aligned}$$

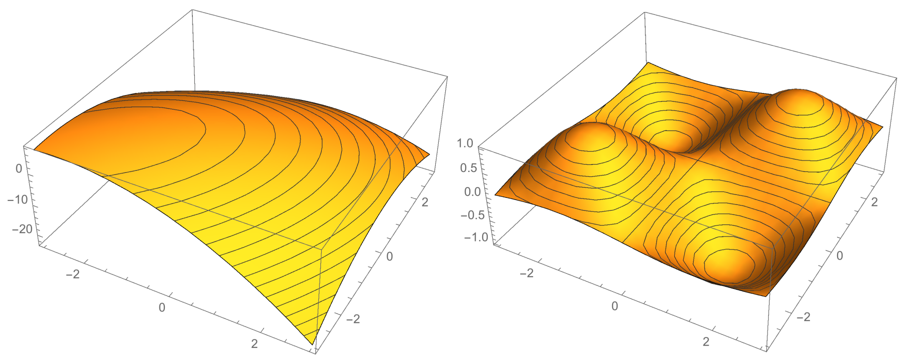
Here m, n, k, p are any integers. Some logical deductions show that these imply the following:

$$\begin{aligned} x = \frac{\pi}{2} + k\pi \quad &\text{and} \quad y = \frac{\pi}{2} + p\pi \\ \text{or:} \quad y = m\pi \quad &\text{and} \quad x = n\pi. \end{aligned}$$

Therefore, there are infinitely many critical points:

$$\left(\frac{\pi}{2} + k\pi, \frac{\pi}{2} + p\pi\right), \quad (m\pi, n\pi)$$

where m, n, k, p are any integers. See Figure 2.18b for its graph.



(a) graph of $z = xy - x^2 - y^2 - 2x - 2y + 4$ (b) graph of $z = \sin x \sin y$

Figure 2.18: graphs of functions in Examples 2.9 and 2.10

2.6.2 Second Derivative Test of Multivariable Functions

In single-variable calculus, to determine the nature of a critical point x_0 of a function $f(x)$, we look at its second derivative at x_0. If $f''(x_0) > 0$, the graph $y = f(x)$ is concave up around x_0 and so $(x_0, f(x_0))$ is a local minimum point. On the other hand, if $f''(x_0) < 0$, the graph is concave down near x_0 and so $(x_0, f(x_0))$ is a local maximum point.

In multivariable calculus, however, to determine whether a critical point (x_0, y_0) of a two-variable function $f(x, y)$ is not as simple as in single-variable calculus. Take the following

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function as an example:

$$f(x,y) = x^2 + 4xy + y^2.$$

One can easily verify that $\nabla f(0,0) = \mathbf{0}$ and so $(0,0)$ is a critical point. For the second derivatives, we find that:

$$f_{xx}(0,0) = 2$$
$$f_{yy}(0,0) = 2$$

for every (x,y) on the $\mathbb{R}^2$ plane. Both are positive numbers. You may be tempted to conclude that $(0,0)$ is a local maximum point. However, if one plots the graph of this function (see Figure 2.19), one can see easily that $(0,0)$ is neither a local maximum or a local minimum.

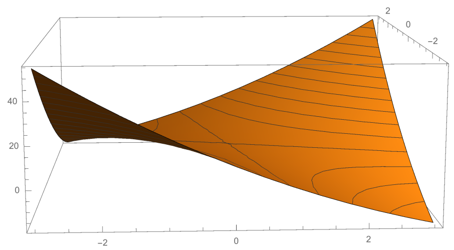

Figure 2.19: $(0,0)$ is neither a maximum or minimum

Around $(0,0)$, the graph is a concave up in some directions but concave down in other directions. We call this $(0,0)$ a **saddle**.

This example shows the signs of f_{xx} and f_{yy} alone could not conclude the nature of the critical point. In fact, the second derivative test for two-variable functions is slightly more complicated than that in single-variable calculus:

Theorem 2.5 — Second Derivative Test for Two-Variable Functions. Let $f(x,y)$ be a C^2 function and (x_0,y_0) is a critical point of f, i.e. $\nabla f(x_0,y_0) = \mathbf{0}$. Then the nature of this critical point (x_0,y_0) is determined by the following table:

$\left(f_{xx}f_{yy} - f_{xy}^2\right)\Big\|_{(x_0,y_0)}$	$f_{xx}(x_0,y_0)$	(x_0,y_0) is a:
> 0	> 0	local minimum
> 0	< 0	local maximum
< 0	anything	saddle

Any other cases are inconclusive.

For the function $f(x,y) = x^2 + 4xy + y^2$ in the above example, to determine the nature of $(0,0)$ we also need $f_{xy}(0,0)$, which can be found as equal to 4.

Therefore, we have:

$$\left(f_{xx}f_{yy} - f_{xy}^2\right)\Big|_{(0,0)} = 2 \times 2 - 4^2 < 0,$$
$$f_{xx}(0,0) = 2 > 0.$$

From the table in Theorem 2.5, we conclude $(0,0)$ is a saddle, as expected from the plot of the its graph. Let's look at one more example before we learn the proof of the Second Derivative Test.

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■ **Example 2.11** Let $f(x,y) = 3y^2 - 2y^3 - 3x^2 + 6xy$. Find all critical points and determine the nature of each of them.

■ **Solution** To find all critical points, we set:

$$\frac{\partial f}{\partial x} = -6x + 6y = 0,$$
$$\frac{\partial f}{\partial y} = 6y - 6y^2 + 6x = 0.$$

From the first equation, we get $y = x$. Substitute this into the second equation, we yield:

$$6x - 6x^2 + 6x = 0, \text{ or equivalently } 2x - x^2 = 0.$$

By factorization, we get $x(2-x) = 0$. Therefore

$$x = 0 \text{ or } x = 2.$$

By noting that $y = x$, we have two critical points: $(0,0)$ and $(2,2)$.

Next we compute the second derivatives of f:

$$f_{xx} = -6 \qquad f_{xy} = 6$$
$$f_{yx} = 6 \qquad f_{yy} = 6 - 12y$$

Critical point P	$f_{xx}(P)$	$f_{yy}(P)$	$f_{xy}(P)$	$\left(f_{xx}f_{yy} - f_{xy}^2\right)(P)$	Nature of P
$(0,0)$	-6	6	6	-72	saddle
$(2,2)$	-6	-18	6	72	local maximum

Explanation of the Second Derivative Test

In single-variable, the second derivative test can be explained using convexity of the graph $y = f(x)$. However, this approach cannot be generalized to higher dimensions.

Before we explain why the above second derivative test works for two-variable functions $f(x,y)$, we first seek an alternative explanation of the single-variable second derivative test using Taylor's series.

Recall that the Taylor's series of a given function $f(x)$ about $x = a$ is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

If $f(x)$ has a critical point at $x = a$, then $f'(a) = 0$. Also, when x is very close to a, the higher-order terms $(x-a)^3$, $(x-a)^4$, etc. are significantly smaller than the quadratic term $(x-a)^2$. Therefore, the function $f(x)$ is approximately given by:

$$f(x) \simeq f(a) + \frac{f''(a)}{2!}(x-a)^2 \quad \text{when } x \text{ is near } a.$$

The right-hand side $f(a) + \frac{f''(a)}{2!}(x-a)^2$ is a quadratic function. If $f''(a) > 0$, then the graph $y = f(a) + \frac{f''(a)}{2!}(x-a)^2$ is a concave up parabola and so $f(a) + \frac{f''(a)}{2!}(x-a)^2 \geq f(a)$. Therefore, $f(x)$, which is approximately $f(a) + \frac{f''(a)}{2!}(x-a)^2$, is also $\geq f(a)$ when x is near a. This explains $f(x)$ has a local minimum at $x = a$.

On the other hand, if $f''(a) < 0$, then the graph $y = f(a) + \frac{f''(a)}{2!}(x-a)^2$ is a concave down parabola. Similar argument as above shows $f(x)$ has a local maximum at $x = a$.

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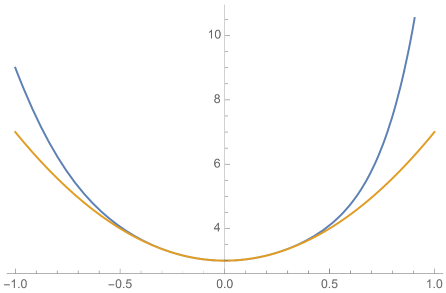

Figure 2.20: blue graph shows $y = f(x)$ where $f'(0) = 0$; yellow graph shows $y = f(0) + \frac{f''(0)}{2!}x^2$ where $f''(0) > 0$

Back to multivariable calculus, we now explain the second derivative test using the Taylor's series approach. Given a function $f(x,y)$, the multivariable Taylor's series about $(x,y) = (x_0,y_0)$ is given by:

$$\begin{aligned} f(x,y) = f(x_0,y_0) &+ f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{f_{xx}(x_0,y_0)}{2!}(x-x_0)^2 + \frac{2f_{xy}(x_0,y_0)}{2!}(x-x_0)(y-y_0) + \frac{f_{yy}(x_0,y_0)}{2!}(y-y_0)^2 \\ &+ \text{higher-order terms} \end{aligned}$$

The proof is beyond the scope of the course. If (x_0,y_0) is a critical point of $f(x,y)$, then $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$. For simplicity, denote $P = (x_0,y_0)$, then when (x,y) is near P, we have:

$$f(x,y) \simeq f(x_0,y_0) + \frac{1}{2}\left(f_{xx}(P)(x-x_0)^2 + 2f_{xy}(P)(x-x_0)(y-y_0) + f_{yy}(P)(y-y_0)^2\right).$$

Therefore, to determine whether (x_0,y_0) is a local maximum/minimum or a saddle of $f(x,y)$, one should determine whether the quadratic function:

$$f_{xx}(P)(x-x_0)^2 + 2f_{xy}(P)(x-x_0)(y-y_0) + f_{yy}(P)(y-y_0)^2$$

is positive/negative or neither.

For simplicity, denote

$$A = f_{xx}(P), \quad B = f_{xy}(P), \quad C = f_{yy}(P),$$
$$X = x - x_0, \quad Y = y - y_0.$$

Then, the quadratic expression can be simplified as:

$$f_{xx}(P)(x-x_0)^2 + 2f_{xy}(P)(x-x_0)(y-y_0) + f_{yy}(P)(y-y_0)^2 = AX^2 + 2BXY + CY^2.$$

To determine whether $AX^2 + 2BXY + CY^2$ is always positive/negative or neither, one looks the discriminant $\Delta = (2B)^2 - 4AC = 4(B^2 - AC)$:

$AC - B^2$	$\Delta = 4(B^2 - AC)$	A	$AX^2 + 2BXY + CY^2$	near P, $f(x,y)$ is
> 0	< 0	> 0	≥ 0	$\geq f(P)$
> 0	< 0	< 0	≤ 0	$\leq f(P)$
< 0	> 0	anything	$+$ or $-$	$\geq f(P)$ or $\leq f(P)$

Translate back to previous notations, we can conclude:

$\left(f_{xx}f_{yy} - f_{xy}^2\right)\Big\vert_{(x_0,y_0)}$	$f_{xx}(x_0,y_0)$	(x_0,y_0) is a:
> 0	> 0	local minimum
> 0	< 0	local maximum
< 0	anything	saddle

This explains the second derivative test for two-variable functions!

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2.7 Lagrange's Multiplier

In the previous section, we learned how to find critical points in the **interior** of a domain, namely by solving $\nabla f = \mathbf{0}$. These critical points are candidates of the maximum or minimum of the function. We also learn how to determine the local nature of the critical points. However, to determine the maximum/minimum on the boundary of a domain, the gradient method does not work as the tangent plane at the maximum/minimum needs not be horizontal (see Figure 2.21).

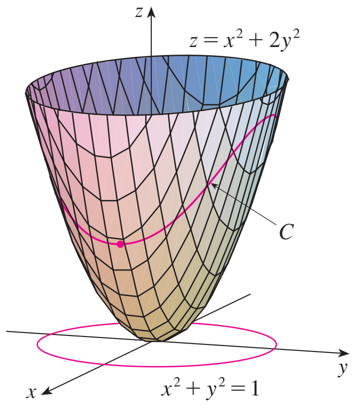

Figure 2.21: At the maximum and minimum points of the function $x^2 + 2y^2$ when (x, y) is restricted the unit circle $x^2 + y^2 = 1$, the tangent plane may not be horizontal. Therefore, solving $\nabla f = 0$ does not give the maximum or minimum points of the function.

When the domain of a function $f(x, y)$ is restricted on a level set such as $x^2 + y^2 = 1$, which is a unit circle, we use a method called the **Lagrange's Multiplier**. We first state the method, then look at a few examples, and finally explain why it works.

Given a function $f(x, y)$ which we want to maximize or minimize, and the variables (x, y) are restricted by the constraint $g(x, y) = c$. Then, to determine all possible candidates of maximum/minimum point on the constraint, we:

1. Solve the system of equations

$$\begin{aligned} \nabla f(x, y) &= \lambda \nabla g(x, y) \\ g(x, y) &= c \end{aligned}$$

   Here the unknowns are x, y and λ.
2. The solutions of (x, y) are the possible candidates of the maximum or minimum points of the function $f(x, y)$. We call these **boundary critical points**.
3. Finally, evaluate $f(x, y)$ at each boundary critical point found. The point giving the largest value of $f(x, y)$ is the maximum point on the boundary, and that giving the smallest value of $f(x, y)$ is the minimum.

ⓘ We call this the Lagrange's Multiplier method because the scalar λ is called the Lagrange's Multiplier.

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■ **Example 2.12** Let $f(x,y) = x^2 + y^2 + 2x + 2y$, find the maximum and minimum values of f when (x,y) is restricted on the constraint $x^2 + y^2 = 1$.

■ **Solution** $f(x,y)$ is the function we want to maximize and minimize. Let $g(x,y) = x^2 + y^2$ so that the level set $g(x,y) = 1$ is our constraint. First we compute:

$$\nabla f = \langle 2x+2, 2y+2 \rangle$$
$$\nabla g = \langle 2x, 2y \rangle.$$

Therefore, the vector equation $\nabla f(x,y) = \lambda \nabla g(x,y)$ is equivalent to the two equations $2x + 2 = 2\lambda x$ and $2y + 2 = 2\lambda y$. Combining with the constraint equation $x^2 + y^2 = 1$, we get a system of three equations with three unknowns x, y and λ:

$$2x + 2 = 2\lambda x \quad (1)$$
$$2y + 2 = 2\lambda y \quad (2)$$
$$x^2 + y^2 = 1 \quad (3)$$

Since we are interested in solving for (x,y) and finding λ is optional, we divide (1) by (2) so that the λ can be canceled. However, we may worry that whether (2) is zero! Therefore, we split into two cases.

Case 1: $2y + 2 \neq 0$ (and so $2\lambda y \neq 0$ too)

(1) $\div$ (2) gives:

$$\frac{2x+2}{2y+2} = \frac{2\lambda x}{2\lambda y}.$$

After cancellations, we get:

$$\frac{x+1}{y+1} = \frac{x}{y}.$$

By cross multiplication:

$$y(x+1) = x(y+1) \Rightarrow xy + y = xy + x \Rightarrow y = x.$$

Substitute $y = x$ into (3), we have $2x^2 = 1$, and so $x = \frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$. Since $y = x$, the solutions for (x,y) **in this case** are:

$$(x,y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Case 2: $2y + 2 = 0$

In this case, $y = -1$. Substitute this into (3), we get $x = 0$. However, putting $x = 0$ into (1) yields $2 = 0$ which is absurd! Therefore, there is no solution in this case.

To sum up, the boundary critical points are: $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Evaluate $f(x,y)$ at each point gives:

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 1 + 2\sqrt{2}$$
$$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = 1 - 2\sqrt{2}$$

Therefore, subject to the constraint $x^2 + y^2 = 1$, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is the maximum point of f with value $1 + 2\sqrt{2}$, and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is the minimum point of f with value $1 - 2\sqrt{2}$. See Figure 2.22 (the blue circle is the constraint).

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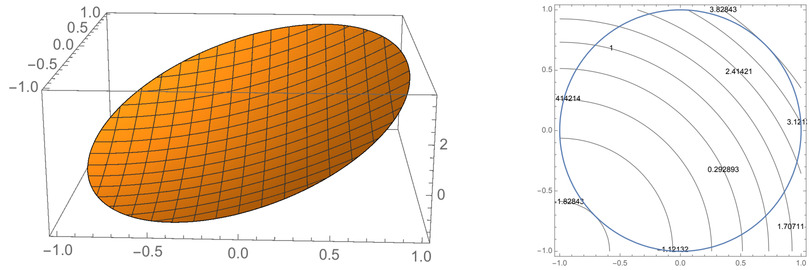


(a) the graph (b) the level set diagram

Figure 2.22: $f(x,y) = x^2 + y^2 + 2x + 2y$ in Example 2.12

■ **Example 2.13** Let $f(x,y) = x^2 - 4x + y^2 + 9$. Find the maximum and minimum points and values of $f(x,y)$ subject to the constraint $4x^2 + 9y^2 = 36$.

■ **Solution** Define $g(x,y) = 4x^2 + 9y^2$, then the constraint is the level set $g(x,y) = 36$. Set-up the Lagrange's Multiplier system:

$$\begin{aligned} \nabla f(x,y) &= \lambda \nabla g(x,y) \\ g(x,y) &= 36 \end{aligned}$$

By computing ∇f and ∇g, the above is equivalent to a system of three equations:

$$\begin{aligned} 2x - 4 &= 8\lambda x && 1 \\ 2y &= 18\lambda y && 2 \\ 4x^2 + 9y^2 &= 36 && 3 \end{aligned}$$

Case 1: 2 $\neq 0$

By 1 $\div$ 2, we get:

$$\begin{aligned} \frac{2x-4}{2y} &= \frac{8\lambda x}{18\lambda y} \\ \frac{x-2}{y} &= \frac{4x}{9y} && \text{(cancel } \lambda) \\ 9y(x-2) &= 4xy && \text{(cross multiplication)} \\ 9(x-2) &= 4x && \text{(cancel } y \neq 0) \\ x &= \frac{18}{5} \end{aligned}$$

However, substitute $x = \frac{18}{5}$ into 3, we get:

$$9y^2 = 36 - 4\left(\frac{18}{5}\right)^2$$

which is a negative number, but $9y^2$ must be positive (or zero)! Therefore, there is no solution in this case.

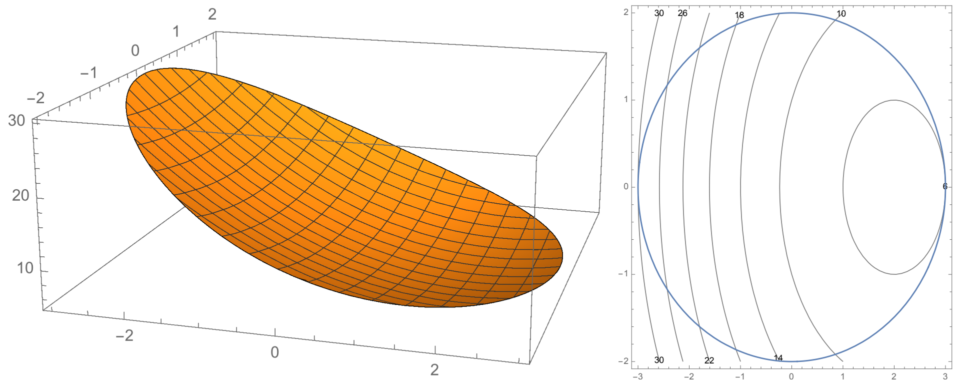
Case 2: ② $= 0$

In this case, we have $2y = 18\lambda y = 0$, so $y = 0$. From ③, we have $4x^2 = 36$ and so $x = 3$ or $x = -3$. Therefore, $(x, y) = (3, 0)$ and $(x, y) = (-3, 0)$ are the solutions in this case.

Finally, we evaluate f at each boundary critical point:

$$f(3,0) = 6$$
$$f(-3,0) = 30$$

Therefore, minimum point is $(3, 0)$ with value 6; maximum point is $(-3, 0)$ with value 30. See Figure 2.23



(a) the graph (b) the level set diagram

Figure 2.23: $f(x,y) = x^2 - 4x + y^2 + 9$ in Example 2.13

Next we explain how the Lagrange's Multiplier method works. Given a function $f(x, y)$ subject to the constraint $g(x, y) = c$. At the point (a, b) on the constraint where the maximum or minimum of $f(x, y)$ is achieved, **the level set of** $f(x, y)$ **at** (a, b) **is tangent to the constraint** $g(x, y) = c$. Consequently, the gradient vector $\nabla f(a, b)$, which is perpendicular to the level set of f at (a, b), must be parallel to the gradient vector $\nabla g(a, b)$, which is perpendicular to the constraint $g = c$. See Figure 2.24 for an illustration. Therefore, at such a point, we must have:

$$\nabla f(a, b) = \lambda \nabla g(a, b)$$

where λ is a scalar.

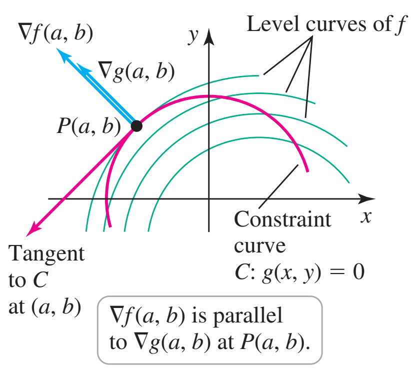


Figure 2.24: ∇f and ∇g are parallel at the boundary critical point (a, b)

The Lagrange's Multiplier method also works for three-variable functions, yet the system of equations may be more complicated. Let's look at the example:

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■ **Example 2.14** Find the distance from $(0,0,0)$ to the plane $2x+3y+4z=29$ using Lagrange's Multiplier.

■ **Solution** The distance from a point P to a plane is defined to be the shortest possible distance between the given point P and any point Q on the plane. Let's first formulate this problem in a mathematical way. We want to:

$$\begin{aligned} \text{minimize} \quad & \sqrt{x^2+y^2+z^2} \\ \text{subject to constraint} \quad & 2x+3y+4z=29 \end{aligned}$$

However, to minimize $\sqrt{x^2+y^2+z^2}$ amounts to calculating $\nabla\left(\sqrt{x^2+y^2+z^2}\right)$. As you can imagine, it would be messy. It is useful to observe that minimizing $\sqrt{x^2+y^2+z^2}$ is equivalent to minimizing $x^2+y^2+z^2$, i.e. the square of the distance from the origin. The latter is much easier to handle. Let:

$$\begin{aligned} f(x,y,z) &= x^2+y^2+z^2 \\ g(x,y,z) &= 2x+3y+4z, \end{aligned}$$

then the constraint is the level set $g=29$. Set-up the Lagrange's Multiplier system $\nabla f=\lambda\nabla g$ as in previous examples:

$$\begin{aligned} 2x &= 2\lambda \\ 2y &= 3\lambda \\ 2z &= 4\lambda \\ 2x+3y+4z &= 29 \end{aligned}$$

Then $x=\lambda$, $y=\frac{3\lambda}{2}$ and $z=2\lambda$. Substitute them into the constraint equation, we get:

$$2\lambda+\frac{9\lambda}{2}+8\lambda=29.$$

It is easy to see that $\lambda=2$, and therefore $(x,y,z)=(2,3,4)$. It gives the unique critical point. It is intuitive that the minimum point must exist in this problem (and there is no maximum point), so this unique critical point must give the minimum point. Since $f(2,3,4)=29$, the distance from $(0,0,0)$ to the plane is $\sqrt{29}$.

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2.8 Optimizations

In this section we will learn some examples of optimization using the gradient and/or Lagrange's Multiplier methods.

■ **Example 2.15** Many airlines require that the sum of length, width and height of a checked baggage cannot exceed 62 inches. Find the dimensions of the rectangular baggage that has the greatest possible volume under this regulation.

■ **Solution** Denote l, w, h to be the length, width and height respectively. We need to maximize the volume of the baggage, which is given by:

$$V(l, w, h) = lwh \text{ (cubic inches)}.$$

The constraint is $l + w + h \leq 62$ (inches), but it is intuitively clear that in order to maximize the volume, the sum $l + w + h$ has better be at maximum possible. Define $g(l, w, h) = l + w + h$, then the constraint can be regarded as the level set $g = 62$. Set up the Lagrange's Multiplier system:

$$\begin{aligned} \nabla V &= \lambda \nabla g \\ g(l, w, h) &= 62 \end{aligned}$$

which is equivalent to

$$\begin{aligned} wh &= \lambda \\ lh &= \lambda \\ lw &= \lambda \\ l + w + h &= 62 \end{aligned}$$

Although it is not too difficult to solve them by hand, let's type the following command on Mathematica to solve them:

```
Solve[{w h == L, l h == L, l w == L, l + w + h == 62}, {l, w, h, L}]
```

These are all critical points:

$$(l, w, h) = \left(\frac{62}{3}, \frac{62}{3}, \frac{62}{3}\right), \quad (0, 0, 62), \quad (0, 62, , 0), \quad (62, 0, 0).$$

Only the first one is physically relevant. Therefore, the rectangular baggage with the largest volume under this restriction is the square cube!

■ **Example 2.16** Three cities A, B and C are located at $(5, 2)$, $(-4, 4)$ and $(-1, -3)$ respectively on the (x, y)-plane. There is a railtrack whose equation is $y = x^3 + 1$, and a station is going to be built on the track so that the sum of squares of the distances from each city to the station is minimized. Find the coordinates of the station.

■ **Solution** Quantity to be minimized is

$$f(x, y, z) = \underbrace{(x-5)^2 + (y-2)^2}_{\text{distance}^2 \text{ from station to city A}} + \underbrace{(x+4)^2 + (y-4)^2}_{\text{distance}^2 \text{ from station to city B}} + \underbrace{(x+1)^2 + (y+3)^2}_{\text{distance}^2 \text{ from station to city C}}.$$

The constraint is that the station has to be on the track, i.e. $y = x^3 + 1$. Define $g(x, y) = y - x^3$, then the constraint can be written as $g(x, y) = 1$. Set up the Lagrange's Multiplier system

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$\nabla f = \lambda \nabla g$ and $g(x,y) = 1$:

$$\begin{aligned} 2(x-5)+2(x+4)+2(x+1) &= -3\lambda x^2 \\ 2(y-2)+2(y-4)+2(y+3) &= \lambda \\ y - x^3 &= 1 \end{aligned}$$

Solving the system, we get $(x,y) = (0,1)$. Therefore, the station should be located at $(0,1)$ in order to minimize the sum of squares of the distances.

■ **Example 2.17 — Least Square Approximation.** Given a set of data points:

$$(x_1, y_1), \ldots, (x_N, y_N)$$

on the xy-plane. Find the straight-line $y = mx + c$ such that the sum of squares of distances between each (x_i, y_i) and $(x_i, mx_i + c)$ is minimized.

■ **Solution** The quantity to be minimized is:

$$f(m,c) = \sum_{i=1}^{N} (y_i - mx_i - c)^2.$$

Note that (x_i, y_i)'s are given so they should be regarded as constants. The variables are m and c. Note that there is no constraint for m and c, so we can simply solve $\nabla f(m,c) = \mathbf{0}$ for critical points.

$$\begin{aligned} \frac{\partial f}{\partial m} &= -2\sum_{i=1}^{N}(y_i - mx_i - c)x_i = -2\left(\sum_{i=1}^{N} x_i y_i - m\sum_{i=1}^{N} x_i^2 - c\sum_{i=1}^{N} x_i\right) \\ \frac{\partial f}{\partial c} &= -2\sum_{i=1}^{N}(y_i - mx_i - c) = -2\left(\sum_{i=1}^{N} y_i - m\sum_{i=1}^{N} x_i - cN\right) \end{aligned}$$

Set $\frac{\partial f}{\partial m} = \frac{\partial f}{\partial c} = 0$, regarding all x_i's and y_i's to be constants, then:

$$\begin{aligned} Am + Bc &= E \\ Bm + Nc &= F \end{aligned}$$

where $A = \sum_{i=1}^{N} x_i^2$, $B = \sum_{i=1}^{n} x_i$, $E = \sum_{i=1}^{N} x_i y_i$ and $F = \sum_{i=1}^{N} y_i$. By solving the system carefully, one should get:

$$\begin{aligned} m &= \frac{BF - EN}{B^2 - AN} = \frac{\left(\sum x_i\right)\left(\sum y_i\right) - N\left(\sum x_i y_i\right)}{\left(\sum x_i\right)^2 - N\left(\sum x_i^2\right)} \\ c &= \frac{BE - AF}{B^2 - AN} = \frac{\left(\sum x_i\right)\left(\sum x_i y_i\right) - \left(\sum x_i^2\right)\left(\sum y_i\right)}{\left(\sum x_i\right)^2 - N\left(\sum x_i^2\right)} \end{aligned}$$

It is quite intuitive that this pair of m and c should minimize f since $f \geq 0$ and so a minimum must exist.

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■ **Example 2.18** Let $f(x,y) = x^2 - 4x + y^2 + 9$ (which was considered in Example 2.13 in the previous section). Find the absolute maximum and absolute minimum of f restricted to the domain $4x^2 + 9y^2 \leq 36$.

■ **Solution** The Lagrange's Multiplier method finds us the boundary critical points on $4x^2 + 9y^2 = 36$. For the interior $4x^2 + 9y^2 < 36$, the critical points are simply solutions to $\nabla f = \mathbf{0}$. The general procedure of an optimization problem with a solid domain is that:

1. Find all interior critical points by solving $\nabla f = \mathbf{0}$;
2. Find all boundary critical points using Lagrange's Multiplier;
3. Evaluate f at each critical points found, and look for the point that gives greatest/lowest value of f.

Interior: Set $\nabla f = \mathbf{0}$, we get:

$$\begin{aligned} 2x - 4 &= 0 \\ 2y &= 0 \end{aligned}$$

Therefore, the only interior critical point is $(2,0)$, which can be checked easily that it is in the given region.

Boundary: The boundary is the ellipse $4x^2 + 9y^2 = 36$. We have already done in Example 2.13, using Lagrange's Multiplier, that the boundary critical points are $(3,0)$ and $(-3,0)$.

Finally, evaluate f at each critical point found:

$$\begin{aligned} f(2,0) &= 5 \\ f(3,0) &= 6 \\ f(-3,0) &= 30 \end{aligned}$$

Therefore, the absolute minimum is 5 (attained at $(2,0)$), and the absolute maximum is 30 (attained at $(-3,0)$).

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3 — Multiple Integrations

> "Mathematics is not about numbers, equations, computations, or algorithms: it is about understanding"
>
> *William Thurston*

3.1 Double Integrals in Rectangular Coordinates

In single-variable calculus, integration is used to find the area of the graph of a function $f(x)$. In this chapter, we will generalize the concept of integrations to multivariable functions. There are many applications of multiple integrals in sciences, including deriving moments of inertia, probability, and in later part of the course: finding surface area and surface flux.

Computations of multivariable integrals are not much different from those in single-variable integrals, but *setting up* a multivariable integral involves a lot more geometric intuitions. Let's first look at some computations first before we explain the geometry of these integrals.

Example 3.1 Compute the following double integral:

$$\int_{y=1}^{y=2}\int_{x=0}^{x=1}(4-x-y^2x)dxdy.$$

Solution A double integral consists of an inner integral and an outer integral:

$$\overbrace{\int_{y=1}^{y=2}\underbrace{\int_{x=0}^{x=1}(4-x-y^2x)dx}_{\text{inner}}\,dy}^{\text{outer}}.$$

When computing the inner integral (which is respect to x in this example), we regard all

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other variable(s) (i.e. y) to be constant(s):

$$\begin{aligned}\int_{y=1}^{y=2}\int_{x=0}^{x=1}(4-x-y^2x)dxdy &= \int_{y=1}^{y=2}\left[4x-\frac{x^2}{2}-\frac{y^2x^2}{2}\right]_{x=0}^{x=1}dy\\ &= \int_{y=1}^{y=2}\left(\left(4-\frac{1}{2}-\frac{y^2}{2}\right)-0\right)dy\\ &= \int_{y=1}^{y=2}\left(\frac{7}{2}-\frac{y^2}{2}\right)dy = \left[\frac{7y}{2}-\frac{y^3}{6}\right]_{y=1}^{y=2} = \frac{7}{3}.\end{aligned}$$

It is worthwhile to note that if we switch the inner and outer integrals, the final answer is the same!

$$\begin{aligned}\int_{x=0}^{x=1}\int_{y=1}^{y=2}(4-x-y^2x)dydx &= \int_{y=1}^{y=2}\left[4y-xy-\frac{y^3x}{3}\right]_{y=1}^{y=2}dx\\ &= \int_{x=0}^{x=1}\left(8-2x-\frac{8x}{3}\right)-\left(4-x-\frac{x}{3}\right)dx\\ &= \int_{x=0}^{x=1}\left(4-\frac{10x}{3}\right)dx = \frac{7}{3}.\end{aligned}$$

It is not a coincident! Let's explain why it is true by learning the geometric meaning of double integrals. Consider the integral:

$$\int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dydx.$$

The inner integral:

$$A(x) := \int_{y=c}^{y=d} f(x,y)\,dy$$

is an integral with respect to y keeping x fixed. This quantity represents the **area under the curve obtained by moving along the y-direction from $y=c$ to $y=d$ on the surface $z=f(x,y)$, while keeping x unchanged**. See Figure 3.1.

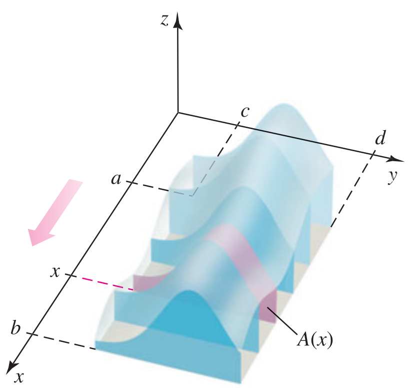

Figure 3.1: geometric meaning of a double integral

The outer integral integrates the inner integral $A(x)$ from $x=a$ to $x=b$, i.e.

$$\int_{x=a}^{x=b}\int_{y=c}^{y=d} f(x,y)\,dydx = \int_{x=a}^{x=b} A(x)\,dx.$$

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Since $A(x)\ dx$ can be thought as the volume of a solid slice with width dx and cross-section area $A(x)$, by integrating $A(x)\ dx$ it means adding up the volume of these thin slices and so the double integral

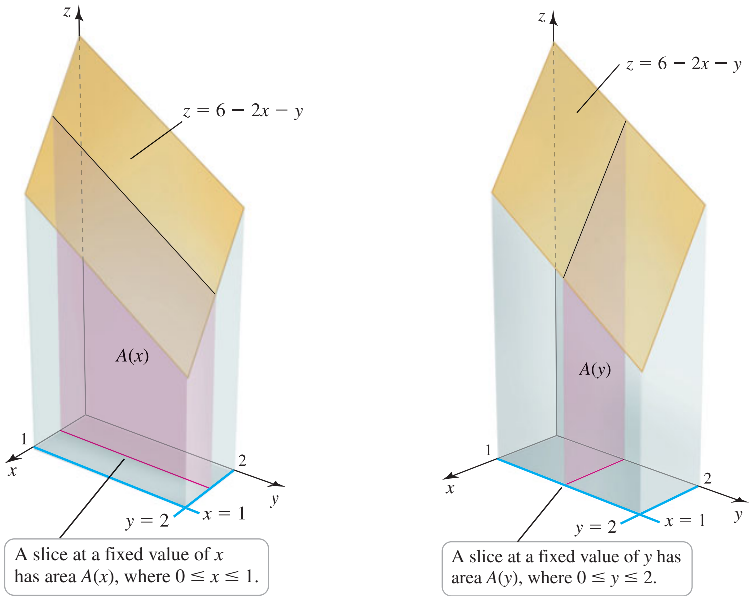
$$\int_{x=a}^{x=b} A(x)\ dx$$

is the **volume under the graph** $z = f(x, y)$ **over the base rectangle bounded by** $x = a$**,** $x = b$**,** $y = c$ **and** $y = d$. It is important to understand the geometric meanings of the inner and outer integrals in order to set-up a double integral correctly.

As a double integral represents the volume of a solid, one should not expect there is any difference if we slice the solid in a different way. For instance, to find the volume under the graph $z = 6 - 2x - y$ over the rectangular region $0 \le x \le 1$ and $0 \le y \le 2$, one can set up the double integral in either way:

$$\int_{x=0}^{x=1} \underbrace{\int_{y=0}^{y=2} (6 - 2x - y)\ dy\, dx}_{A(x)} \qquad \text{see Figure 3.2a}$$

$$\int_{y=0}^{y=2} \underbrace{\int_{x=0}^{x=1} (6 - 2x - y)\ dx\, dy}_{A(y)} \qquad \text{see Figure 3.2b}$$



(a) slices with fixed x (b) slices with fixed y

Figure 3.2: volume under the same graph

Readers should verify that the above integrals indeed give the same value (the answer is 8). In general, the following Fubini's Theorem asserts that switching dx and dy (and the corresponding integral signs) give the same double integral. Although the statement of the theorem is geometrically intuitive, the proof is not easy and is beyond the scope of this course.

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Theorem 3.1 — Fubini's Theorem for Rectangular Regions. Let $f(x,y)$ be a continuous function over a rectangular region $a \leq x \leq b$ and $c \leq y \leq d$, then:

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y)\, dxdy = \int_{x=a}^{x=b} \int_{y=c}^{y=d} f(x,y)\, dydx.$$

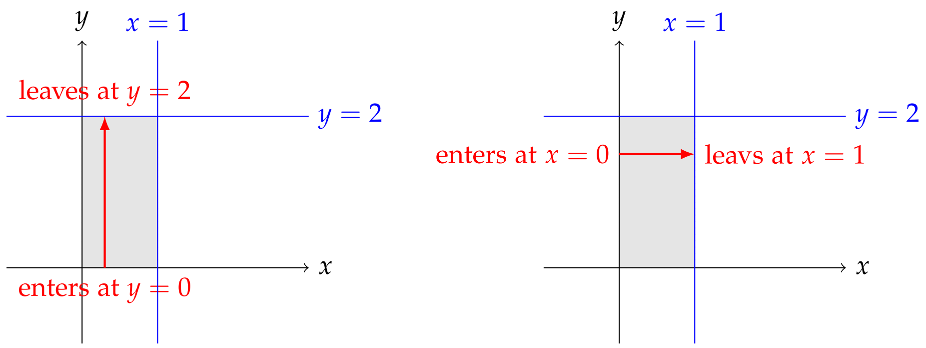
Since the order of integration (i.e. $dxdy$ or $dydx$) determines the order of the integral signs, the above two double integrals can simply be written as:

$$\int_c^d \int_a^b f(x,y)\, dxdy = \int_a^b \int_c^d f(x,y)\, dydx.$$

Even simpler, one may denote $dA := dxdy$ or $dydx$ and the rectangular region by R. Then, we can write the integral as:

$$\iint_R f(x,y)\, dA.$$

When setting up a double integral to find the volume of the solid under a graph $z = f(x,y)$, it is worthwhile to observe that the lower and upper limits of the integral are not affected by the function $f(x,y)$. Therefore, in order to interpret a double integral in a geometric way, one may simply draw the base region (or in other words, the top-down view) instead of drawing the solid in the three-dimensional space.



(a) top-down view of Figure 3.2a (b) top-down view of Figure 3.2b

Figure 3.3: the red arrows represent the cross-section slices in Figures 3.2a and 3.2b.

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3.2 Fubini's Theorem for General Regions

In this section, we demonstrate some examples of double integrals whose base regions are general regions such as triangles.

■ **Example 3.2** Find the volume of the solid under the plane $z = 3 - x - y$ over the triangle region R bounded by the x-axis, $x = 1$ and $y = x$.

■ **Solution** First we choose an order of integration, say $dydx$. The inner integral should calculate the area of slices with y varies and x fixed. Since the height $3 - x - y$ of the solid does not affect how we set-up the upper/lower limits, we consider the top-down view of the solid (see Figure 3.4).

The red strip in Figure 3.4b represents a sample slice with fixed x. The strip **enters** at $y = 0$ and **leaves** at $y = x$. Hence, the area of this slice is:

$$\int_{y=0}^{y=x} \underbrace{(3 - x - y)}_{\text{height}} \, dy.$$

"Summing up" the area of these slices, we integrate by dx over the range of x: $0 \leq x \leq 1$ as shown in Figure 3.4b, i.e.

$$\int_{x=0}^{x=1} \underbrace{\int_{y=0}^{y=x} (3 - x - y) \, dy \, dx}_{\text{inner integral}}.$$

It will give the volume of the solid as required in this problem. The rest of the task is to compute the integral:

$$\begin{aligned}
\int_{x=0}^{x=1} \int_{y=0}^{y=x} (3 - x - y) \, dydx &= \int_{x=0}^{x=1} \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx \\
&= \int_{x=0}^{x=1} \left(3x - x^2 - \frac{x^2}{2} \right) dx \\
&= \int_0^1 \left(3x - \frac{3x^2}{2} \right) \, dx \\
&= \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_0^1 \\
&= 1.
\end{aligned}$$

Alernatively, we can also integrate first by dx then by dy. Then, the inner integral is represented by the red strip in Figure 3.4c. It **enters** at $x = y$ and **leaves** at $x = 1$. The double integral is therefore:

$$\int_{y=0}^{y=1} \int_{x=y}^{x=1} (3 - x - y) \, dxdy.$$

Readers should compute as an exercise that the answer is again 1.

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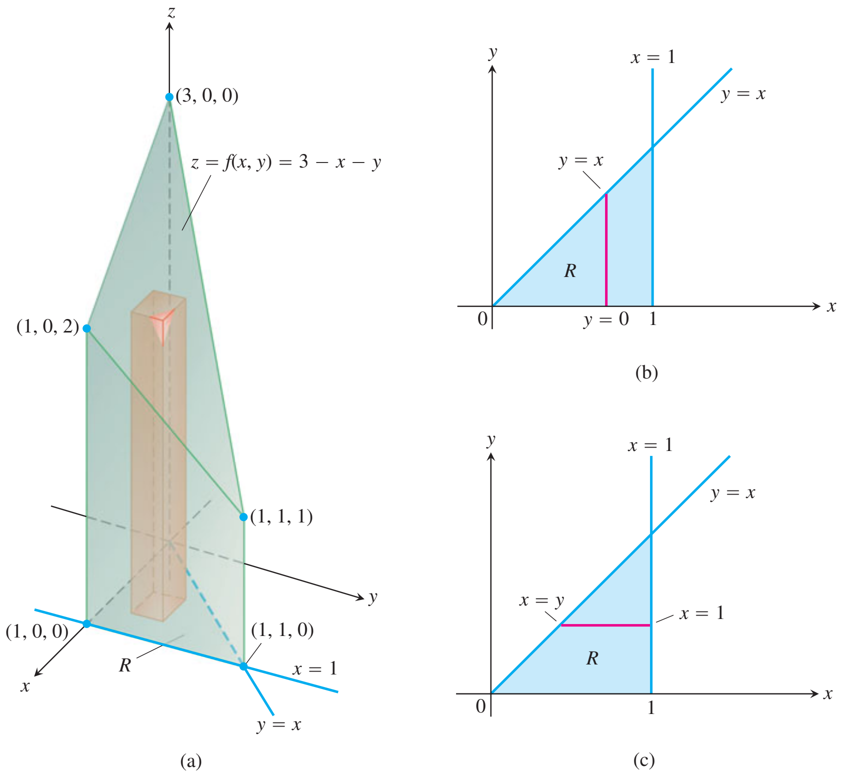

Figure 3.4: the graph and the top-down views of the function in Example 3.2

The fact that we have the freedom to choose our order of integration is guaranteed by the Fubini's Theorem, whose proof is again beyond the scope of this course.

Theorem 3.2 — Fubini's Theorem for General Regions. Let R be a region on the xy-plane and $f(x,y)$ is a continuous function on R, then

$$\iint_R f(x,y)dxdy = \iint_R f(x,y)dydx$$

where the lower/upper limits of each integral are set up according to the region R.

Notation As there is no difference between $dxdy$ and $dydx$ as far as the upper and lower limits are set according to the same region, we may simply write:

$$dA = dxdy \text{ or } dydx$$

Although choosing the $dxdy$-order will yield the same result as the $dydx$-order, it happens often that one order is easier while the other one is harder. Let's look at the following example:

■ **Example 3.3** Let R be the region in the first quadrant of the xy-plane bounded by the unit circle $x^2 + y^2 = 1$ and the straight-line $x + y = 1$. Evaluate the integral

$$\iint_R \sqrt{1 - x^2}dA.$$

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■ **Solution** First choose the order of integration. However, it seems like integrating $\sqrt{1-x^2}$ by dx involves trig substitutions that we want to avoid if possible. Let's try integrating first by dy then by dx (to see if there is any luck).

Set-up the double integral according the top-down view of the solid (Figure 3.5):

$$\int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} \sqrt{1-x^2}\, dydx.$$

As x is regarded as a constant when dealing with the inner integral, we can easily see that:

$$\begin{aligned}
\int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} \sqrt{1-x^2}\, dydx &= \int_{x=0}^{x=1} \left[y\sqrt{1-x^2}\right]_{y=1-x}^{y=\sqrt{1-x^2}} dx \\
&= \int_{x=0}^{x=1} (1-x^2) - (1-x)\sqrt{1-x^2}\, dx \\
&= \int_0^1 (1-x^2 - \sqrt{1-x^2} + x\sqrt{1-x^2})\, dx
\end{aligned}$$

The only two difficult parts are

$$\int_0^1 \sqrt{1-x^2}\, dx \text{ and } \int_0^1 x\sqrt{1-x^2}\, dx.$$

The former can be evaluated by substitution $x = \sin\theta$, while the latter can be done by substituting $u = 1-x^2$. Readers should complete the rest of computations as an exercise. The final answer should be $1 - \frac{\pi}{4}$.

We need a trig substitution anyway, but it is easier than doing a substitution for the inner integral.

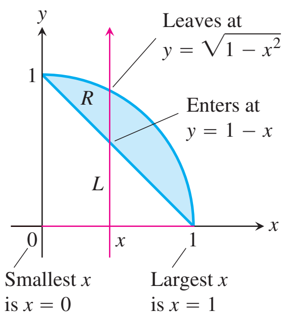

Figure 3.5: top-down view of the region in Example 3.3

In the previous example, we see that although the Fubini's Theorem tells that *in theory* we can choose our favorite the order of integration, *in practice* we sometimes have to make a smart choice. In the next example, let's demonstrate an example that one order gives an integral which is impossible to compute, while another is extremely easy.

■ **Example 3.4** Evaluate the following double integral:

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dxdy.$$

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■ **Solution** The integrand $\frac{\sin x}{x}$ does not have a simple antiderivative when integrating by dx! Let's switch the order of integration first.

The region corresponds to the double integral is formed by strips entering at $x = y$ and leaving at $x = 1$. The range of y is from 0 to 1. A sketch of the diagram can be found in Figure 3.6.

Switching the order of integration, the strip for each x enters at $y = 0$ and leaves at $y = x$. Therefore, Fubini's Theorem says:

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dxdy = \int_0^1 \int_0^x \frac{\sin x}{x}\, dydx.$$

The RHS is much easier to compute:

$$\begin{aligned}\int_0^1 \int_0^x \frac{\sin x}{x}\, dydx &= \int_0^1 \left[y\frac{\sin x}{x} \right]_{y=0}^{y=x} dx \\ &= \int_0^1 \left(x\frac{\sin x}{x} - 0 \right) dx \\ &= \int_0^1 \sin x\, dx = -\cos 1.\end{aligned}$$

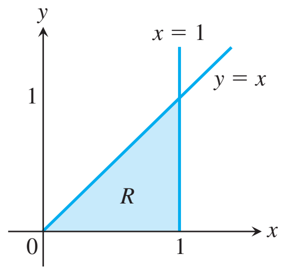
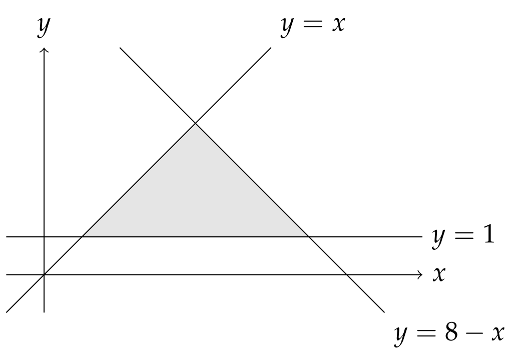


Figure 3.6: the region of integration in Example 3.4

If the region of integration is the shaded triangle below, which order of integration is better?

$$\iint_R f(x,y)\, dxdy \quad \text{or} \quad \iint_R f(x,y)\, dydx,$$

where $f(x,y)$ is not very complicated, say $f(x,y) = x^2y$.



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3.3 Double Integrals in Polar Coordinates

When the region of integration is *circular* in shape, or the integrand is rotationally symmetric, it is often more convenient to use *polar coordinates* to set-up the integral instead of the using the rectangular coordinates. We will see in some examples in this section that some tedious trig substitution can be avoid if polar coordinates are used.

Recall that the polar coordinates (r, θ) and the rectangular coordinates (x, y) are related by the following rules:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Here r is the distance from the point to the origin, and θ is the angle made with the positive x-axis.

In rectangular coordinates, the region defined by inequalities like $a \le x \le b$ and $c \le y \le d$, where a, b, c and d are constants, describe a rectangle. We have seen that it is very easy to set-up a double integral when the region is a rectangle.

In polar coordinates, regions defined by inequalities like $a \le r \le b$ and $\alpha \le \theta \le \beta$, where a, b, α and β are constants, describe a fan shape or a circular sector (see Figure 3.7). It is wise to use polar coordinates instead of rectangular coordinates when the region of integration is given by one of these circular shapes.

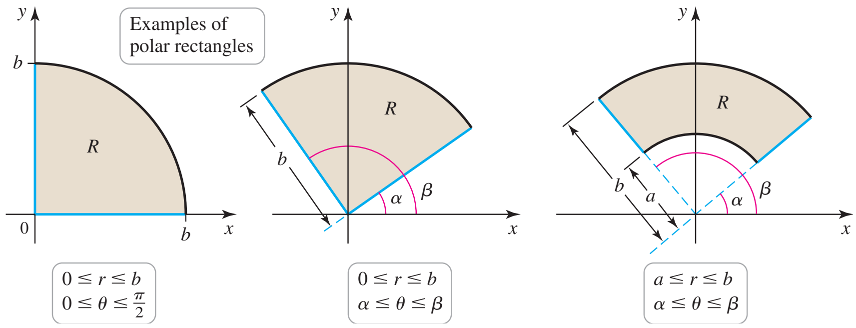

Figure 3.7: examples of regions good for polar coordinates

To set-up a double integral of a region $a \le r \le b$ and $\alpha \le \theta \le \beta$ using polar coordinates, the upper/lower limits are simply:

$$\int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{r=b} \quad \text{or} \quad \int_{r=a}^{r=b}\int_{\theta=\alpha}^{\theta=\beta}$$

depending on the order of integration $drd\theta$ or $d\theta dr$. However, one should be very cautious that while $dA = dxdy$ in rectangular coordinates, it is instead:

$$dA = rdrd\theta \quad \text{or} \quad rd\theta dr$$

in polar coordinates. We will explain why it is so after learning a few examples.

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■ **Example 3.5** Evaluate the integral:

$$\iint_R (x^2+y^2)dA$$

where R is the semicircular region bounded by the x-axis and the curve $y=\sqrt{1-x^2}$. See Figure 3.8.

■ **Solution** The problem is extremely difficult to do in rectangular coordinates. If one attempts to set it up using xy-coordinates, one will get the following double integral:

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}(x^2+y^2)dydx.$$

After computing the inner integral, one should get:

$$\int_{-1}^{1}\left(x^2\sqrt{1-x^2}+\frac{(1-x^2)^{3/2}}{3}\right)dx.$$

However, things go much better if we switch to polar coordinates, since the region is in the form $a\le r\le b$ and $\alpha\le\theta\le\beta$. From Figure 3.8, the region is defined by:

$$0\le r\le 1,\quad 0\le\theta\le\pi.$$

Keeping in mind that $dA=rdrd\theta$, the required integral is:

$$\begin{aligned}\iint_R (x^2+y^2)dA &= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1}\underbrace{\left((r\cos\theta)^2+(r\sin\theta)^2\right)}_{x^2+y^2}rdrd\theta\\ &= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1} r^2(\cos^2\theta+\sin^2\theta)rdrd\theta\\ &= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1} r^3drd\theta\\ &= \int_{\theta=0}^{\theta=\pi}\left[\frac{r^4}{4}\right]_{r=0}^{r=1}d\theta\\ &= \int_{\theta=0}^{\theta=\pi}\frac{1}{4}d\theta=\frac{\pi}{4}.\end{aligned}$$

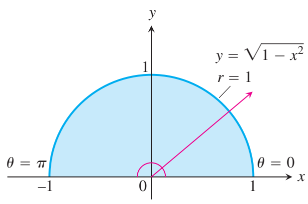

Figure 3.8: region of integration for Example 3.5

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Annular regions, to be discussed in the next example, are extremely clumsy using rectangular coordinates but relatively easy using polar coordinates.

■ **Example 3.6** Evaluate the following integral:

$$\iint_R x \, dA$$

where R is an annular region shown in Figure 3.9.

■ **Solution** The annular region R is defined by inequalities:

$$2 \leq r \leq 4, \quad 0 \leq \theta \leq 2\pi.$$

Therefore,

$$\begin{aligned}
\iint_R x \, dA &= \int_0^{2\pi} \int_2^4 r\cos\theta \cdot r dr d\theta \\
&= \int_0^{2\pi} \int_2^4 r^2 \cos\theta \, dr d\theta \\
&= \int_0^{2\pi} \left[\frac{r^3}{3}\right]_{r=2}^{r=4} \cdot \cos\theta \, d\theta \\
&= \int_0^{2\pi} \frac{56}{3} \cos\theta \, d\theta \\
&= \left[\frac{56}{3}\sin\theta\right]_0^{2\pi} \\
&= 0
\end{aligned}$$

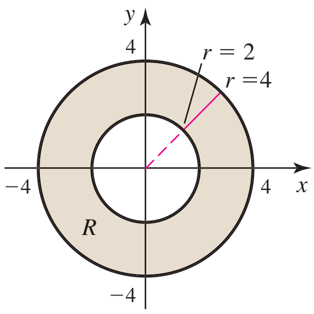

Figure 3.9: region of integration for Example 3.6

A Notoriously Difficult Single-Variable Integral

Next we present an application of evaluating a notoriously difficult single-variable integral using a double integral in polar coordinates.

It is famous (or infamous) that the integral:

$$\int e^{-x^2} \, dx$$

does not have an easy explicit expression. Nonetheless, this integral is extremely important in probability, heat flow and many physics and engineering subjects. While this integral is generally hard to compute, it can be *magically* done, with the help of a double integral, when

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the upper/lower limits are:

$$\int_0^\infty e^{-x^2}\,dx.$$

Consider the double integral:

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dxdy.$$

Observing that $e^{-x^2-y^2} = e^{-x^2}e^{-y^2}$, we can regard e^{-y^2} as a constant in the inner dx-integral:

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dxdy = \int_0^\infty \int_0^\infty e^{-x^2}e^{-y^2}\,dxdy = \int_0^\infty e^{-y^2}\left(\int_0^\infty e^{-x^2}\,dx\right)\,dy.$$

Now that $\int_0^\infty e^{-x^2}\,dx$ is a constant, and in particular, independent of y. Therefore, with respect to the outer dy-integral, one can factor it out and yield:

$$\int_0^\infty e^{-y^2}\left(\int_0^\infty e^{-x^2}\,dx\right)\,dy = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right).$$

Now the double integral are split as a product of two single-variable integrals. Note that the two single-variable integrals are the same as x and y are merely dummy variables! Therefore, combining everything we have shown above, we have:

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dxdy = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \left(\int_0^\infty e^{-x^2}\,dx\right)^2.$$

Consequently, in order to evaluate the single-variable integral $\int_0^\infty e^{-x^2}\,dx$, one can first evaluate the double integral

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dxdy$$

and then the square root of the double integral will give the value of $\int_0^\infty e^{-x^2}\,dx$.

In contrast to the single-variable integral, the double integral is relatively easy to find using polar coordinates. The region of integration is the entire first quadrant which, in polar coordinates, can be described as:

$$0 \le r \le \infty, \quad 0 \le \theta \le \frac{\pi}{2}.$$

Therefore,

$$\begin{aligned}
\int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dxdy &= \int_0^{\pi/2}\int_0^\infty e^{-r^2}r\,drd\theta \\
&= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{r=\infty} d\theta \qquad \text{note that } \frac{d}{dr}e^{-r^2} = e^{-r^2}\cdot(-2r) \\
&= \int_0^{\pi/2}\left(-0+\frac{1}{2}\right)\,d\theta \\
&= \frac{\pi}{4}.
\end{aligned}$$

Therefore,

$$\int_0^\infty e^{-x^2}\,dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}.$$

Furthermore, e^{-x^2} is an even function, we also have:

$$\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}.$$

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However, this trick does not work well for any similar definite integral such as:

$$\int_0^1 e^{-x^2}\,dx.$$

Although it is still true (from a similar argument as above) that we have

$$\left(\int_0^1 e^{-x^2}\,dx\right)^2 = \int_0^1 \int_0^1 e^{-x^2-y^2}\,dxdy,$$

the double integral is not "polar-friendly" since the region of integration is a square!

Explanation of the Polar Method

We end this section by explaining why $dA = rdrd\theta$ but not simply $dA = drd\theta$.

In rectangular coordinates, the area of a region is calculated by chopping off the region into tiny rectangular pieces with length and width denoted by the changes Δx and Δy. The area of each rectangular piece is given by $\Delta A = \Delta x \cdot \Delta y$. When these rectangular pieces get smaller and smaller, Δx and Δy become dx and dy, and so the area of these rectangular pieces becomes $dA = dxdy$.

However, things are a bit different in polar coordinates. Instead of chopping off a given region into tiny rectangles, the region is chopped into "fan-shaped" pieces (see Figure 3.10. Given a pair of Δr and $\Delta\theta$, the area ΔA of the piece is *not* simply $\Delta r \cdot \Delta\theta$, but is getting proportionally larger when the piece is further from the origin!

Therefore, one needs to multiply r to $\Delta r \cdot \Delta\theta$ in order to accurately reflect the size of ΔA. Precisely, ΔA can be calculated as the difference between the area of an outer sector (with radius $r + \Delta r$) and an inner section (with radius r):

$$\begin{aligned}\Delta A &= \underbrace{\frac{1}{2}(r+\Delta r)^2\,\Delta\theta}_{\text{area of outer sector}} - \underbrace{\frac{1}{2}r^2\Delta\theta}_{\text{area of inner sector}} \\ &= \frac{1}{2}\left(r^2 + 2r\Delta r + (\Delta r)^2 - r^2\right)\Delta\theta \\ &= r\Delta r \cdot \Delta\theta + \frac{1}{2}(\Delta r)^2\Delta\theta.\end{aligned}$$

When Δr and $\Delta\theta$ get smaller and smaller, the last term $(\Delta r)^2\Delta\theta$ is negligible when compared to the other term $r\Delta r\Delta\theta$. Therefore, as the region is chopped into infinitesimal pieces, we have:

$$dA = rdrd\theta.$$

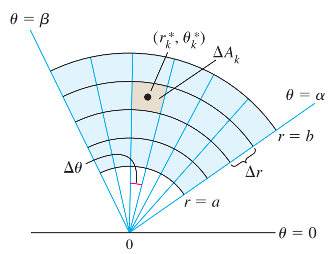

Figure 3.10: area of ΔA gets larger when r gets larger.

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3.4 Triple Integrals in Rectangular Coordinates

We now move one dimension up and talk about triple integrals, whose integrands are functions of three variables such as $f(x,y,z)$. An example of such an integral is:

$$\int_0^1 \int_0^y \int_z^y f(x,y,z)dxdzdy.$$

Some triple integrals have important physical meanings. For instance, if $f(x,y,z)$ is the density function, then the triple integral

$$\int_0^1 \int_0^y \int_z^y f(x,y,z)\, dxdzdy$$

represents the mass of the solid (whose shape is determined by the upper/lower limits of the integral). It is because $dxdzdy$ represents (infinitesimal) volume, and

$$\text{density} \times \text{volume} = \text{mass}.$$

In physics, some calculations such as the center of mass of an object and the moment of inertia about an axis also involve evaluations of triple integrals.

Pillar-Base Approach

Before we see some practical applications of triple integrals, let's learn how a triple integral is set-up. One helpful technique is so-called the **pillar-base** approach or **pillar-shadow** approach. Let's explain this through an example:

■ **Example 3.7** Let D be the tetrahedral solid bounded by the plane $z = y - x$, the plane $y = 1$, the xy-plane and the yz-plane (see Figure 3.11). Evaluate the triple integral:

$$\iiint_D x^2 dzdydx.$$

■ **Solution** We demonstrate the pillar-base approach in the solution. The so-called *pillar* is the orange ray labeled M in Figure 3.11, and it determines how the inner-most integral is set-up:

$$\int_{z=?}^{z=?} x^2 dz.$$

Its lower and upper limits of z are determined by, respectively, where the pillar **enters** the solid and where the pillar **leaves** the solid. According to the diagram, it enters at $z = 0$, i.e. the xy-plane; and it leaves at $z = y - x$. Therefore, the inner-most integral should be:

$$\int_{z=0}^{z=y-x} x^2 dz.$$

Note that again the integrand x^2 does not affect how we set-up this integral.

Next we call the other two variables x and y to be *base* variables or *shadow* variables. Suppose light is coming in along the direction parallel to the pillar (i.e. z-direction), the shadow of the solid appears on the base xy-plane as a triangle. To way to set-up the middle and outer-most integrals are just the same as what we did for double integrals.

Since we picked the $dydx$-order, we draw a sample strip L on the base in the direction of y. This strip enters at $y = x$ and leaves at $y = 1$, and therefore the middle and outer-most integrals should be set-up as:

$$\underbrace{\int_{x=0}^{x=1} \int_{y=x}^{y=1}}_{\text{base}} \underbrace{\int_{z=0}^{z=y-x}}_{\text{pillar}} x^2\, dzdydx.$$

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Finally, the easiest step is to evaluate the integral. This is as straight-forward as in double integrals:

$$\begin{aligned}
\int_0^1 \int_x^1 \int_0^{y-x} x^2 \, dzdydx &= \int_0^1 \int_x^1 [x^2 z]_{z=0}^{z=y-x} \, dydx \\
&= \int_0^1 \int_x^1 x^2 (y-x) dydx \\
&= \int_0^1 \left[\frac{x^2 y^2}{2} - x^3 y \right]_{y=x}^{y=1} dx \\
&= \int_0^1 \left(\frac{x^2}{2} - x^3 - \frac{x^4}{2} + x^4 \right) dx \\
&= \int_0^1 \left(\frac{x^2}{2} - x^3 + \frac{x^4}{2} \right) dx \\
&= \left[\frac{x^3}{6} - \frac{x^4}{4} + \frac{x^5}{10} \right]_0^1 \\
&= \frac{1}{6} - \frac{1}{4} + \frac{1}{10} = \frac{1}{60}.
\end{aligned}$$

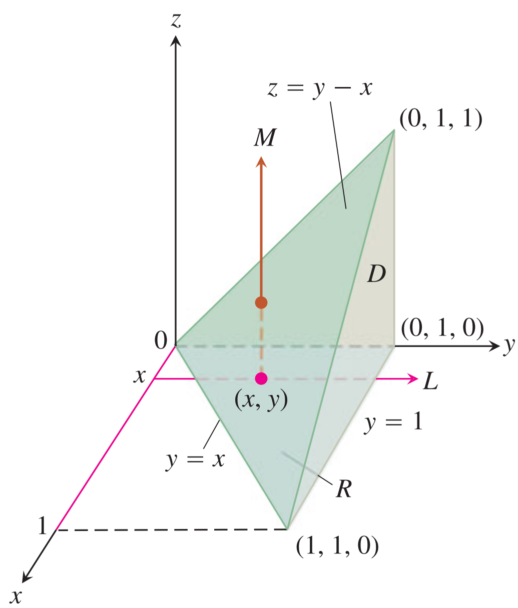

Figure 3.11: solid D in Example 3.7.

The Fubini's Theorem also holds for triple integrals, i.e. we can evaluate the integral by different order of integration, say $dxdydz$ or $dydxdz$ (there are 6 possible orders), we should get the same value provided that the upper and lower limits are adjusted to present the same solid.

Example 3.8 Let D be the tetrahedral solid in Example 3.7. Evaluate the triple integral

$$\iiint_D x^2 dydzdx$$

using the $dydzdx$-order.

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■ **Solution** Now the inner-most integral is with respect to dy, meaning the *pillar* is pointing along the positive y-axis. It is the ray labeled as M in Figure 3.12. It enters the solid through the plane $z = y - x$, or equivalently $y = x + z$, and it leaves at $y = 1$. Therefore, the inner-most integral should be:

$$\int_{y=x+z}^{y=1} x^2\,dy.$$

The middle and the outer-most variables are z and x, so the *base* is the shadow of the solid on the xz-plane, which is the triangle labeled R in Figure 3.12.

Here the order of integration is $dzdx$, so we draw a sample strip (labeled L) along the z-axis direction. It enters the region through $z = 0$ and leaves at the line $x + z = 1$, or equivalently, $z = 1 - x$. Therefore, the whole triple integral should be set as:

$$\int_{x=0}^{x=1}\int_{z=0}^{z=1-x}\int_{y=x+z}^{y=1} x^2\,dydzdx.$$

It can be verified by straight-forward computations that the answer should be the same as we got in Example 3.7:

$$\begin{aligned}\int_{x=0}^{x=1}\int_{z=0}^{z=1-x}\int_{y=x+z}^{y=1} x^2\,dydzdx &= \int_0^1\int_0^{1-x}\left[x^2y\right]_{y=x+z}^{y=1} dzdx \\ &= \int_0^1\int_0^{1-x}\left(x^2 - x^2(x+z)\right) dzdx \\ &= \int_0^1 \left[(x^2-x^3)z - \frac{x^2z^2}{2}\right]_{z=0}^{z=1-x} dx \\ &= \int_0^1 \left((x^2-x^3)(1-x) - \frac{x^2(1-x)^2}{2}\right) dx \\ &= \int_0^1 \left(\frac{x^2}{2} - x^3 + \frac{x^4}{2}\right) dx = \frac{1}{60} \quad \text{(same!)}\end{aligned}$$

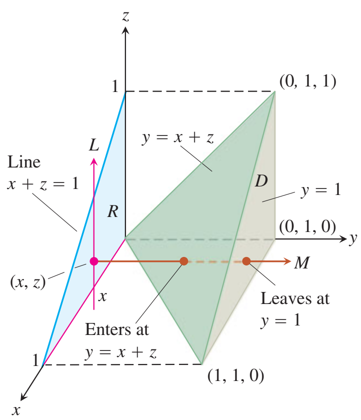

Figure 3.12: the pillar-base diagram for the triple integral in Example 3.8

The Fubini's Theorem tells us that no matter what order of integration we choose, we always get the same answer. Therefore, we can sometimes us the notation dV to denote the volume element $dxdydz$ (or any other order). A generic triple integral can be written as:

$$\iiint_D f(x,y,z)\,dV.$$

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Although Fubini's Theorem allows us to switch the order of integral, we sometimes need to choose a *smart* choice of pillar and base variables to ease our computation. Let's look at the next example:

■ **Example 3.9** Let D be the solid bounded by the paraboloid $y = x^2 + z^2$ and $y = 16 - 3x^2 - z^2$ (see Figure 3.13). Find the volume of the solid, i.e. evaluate the integral:

$$\iiint_D 1 dV.$$

■ **Solution** After taking a careful look at the diagram, one should see that it would be a bad choice if we chose either x or z to be the pillar variable. The solid can be decomposed into two parts as shown in the diagram. If z were chosen to be the pillar direction, then the pillar would enter the yellow part in a way different from it does in the blue part. The yellow part would have $z = \pm\sqrt{16 - 3x^2 - y}$ as the lower/upper limits, while the blue part have $z = \pm\sqrt{y - x^2}$. Even worse, the part near the intersection of the blue and the yellow surfaces is geometrically complicated – it is not easy to set up the inner integral for that part.

However, life is much easier if we choose y as the pillar variable. Since then the y-pillar will enter through the blue surface and leave through the yellow surface. The shadow is an ellipse on the xz-plane.

To set-up the inner integral, we note that the y-pillar enters at $y = x^2 + z^2$ and leaves at $y = 16 - 3x^2 - z^2$:

$$\int_{y=x^2+z^2}^{y=16-3x^2-z^2} dy.$$

The shadow (or base) is an ellipse, whose equation can be obtained by setting $y = x^2 + z^2 = 16 - 3x^2 - z^2$, which gives

$$2x^2 + z^2 = 8.$$

For the base integral, we can either choose $dzdx$ or $dxdz$. For the former, the sample z-strip enters the ellipse at $-\sqrt{8 - 2x^2}$ and leaves at $\sqrt{8 - 2x^2}$. The min/max values of x are -2 and 2 for the ellipse. Therefore, the middle and outer integrals should be:

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{8-2x^2}}^{z=\sqrt{8-2x^2}} \int_{y=x^2+z^2}^{y=16-3x^2-z^2} dydzdx.$$

Evaluation of this integral is straight-forward (but be careful). It is left as an exercise for readers. The answer should be $32\pi\sqrt{2}$.

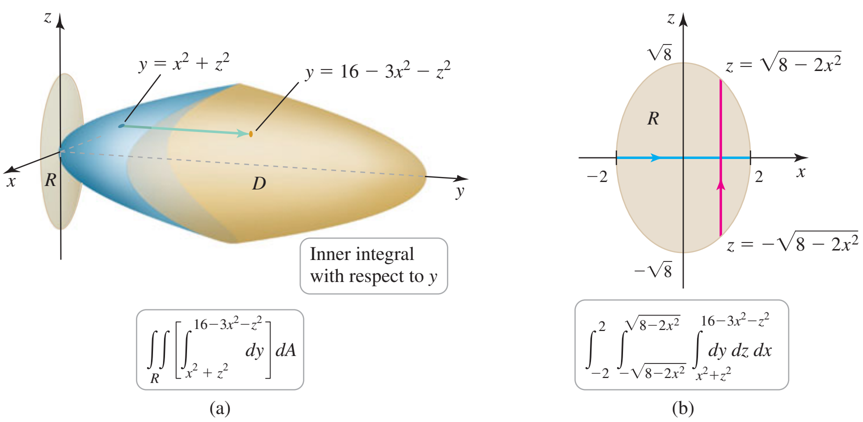

Figure 3.13: the pillar-base diagram for the triple integral in Example 3.9

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3.5 Triple Integrals in Cylindrical Coordinates

In order to set-up or compute a double integral of a *circular* region, it is often more convenient to convert the problem into polar coordinates. For triple integrals, there are also some solids that are not so "compatible" with rectangular coordinates but are easily to handle if one convert the problem into cylindrical or spherical coordinates. These solids include cylinders, cones, spheres, etc.

Cylindrical coordinates in $\mathbb{R}^3$ is simply combining the polar coordinates for the (x, y)-directions, and keeping the z coordinates. The conversion rule is given by:

$$
\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z
\end{aligned}
$$

where $r \geq 0$, $0 \leq \theta \leq 2\pi$ and z can be any real number. Just like polar coordinates, it is good to keep in mind that $x^2 + y^2 = r^2$, which will sometimes simplify your calculations. Figure 3.14 explains the geometry of these conversion rules.

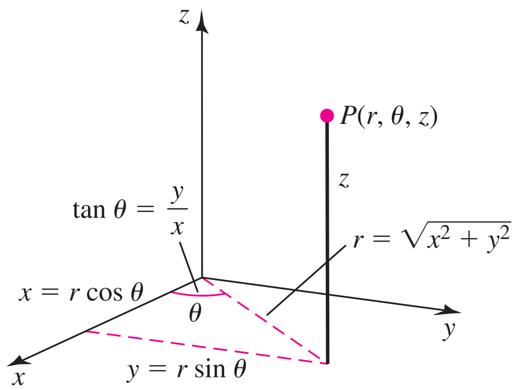

Figure 3.14: cylindrical coordinates

If one sets $r =$ constant, then it describes an infinite cylinder in $\mathbb{R}^3$ with z-axis as the central axis. Therefore, if the solid is cylindrical in shape, it is usually easier to set-up a triple integral using cylindrical coordinates. Analogous to polar coordinates, the volume element dV is:

Theorem 3.3 Under cylindrical coordinates (r, θ, z), we have:

$$dV = r dz dr d\theta.$$

Let's look at an example:

■ **Example 3.10** The volume of the solid bounded by two surfaces $z = 4 - 4(x^2 + y^2)$ and $z = (x^2 + y^2)^2 - 1$, as shown in Figure 3.15.

■ **Solution** When using cylindrical coordinates, one often (though not always) set z as the pillar variable, then r and θ become the base variable.

The z-pillar enters the solid at $z = (x^2 + y^2)^2 - 1$, which is $z = r^4 - 1$ in cylindrical coordinates, and leaves the solid at $z = 4 - 4(x^2 + y^2)$, which is $z = 4 - 4r^2$ in cylindrical coordinates. These determine the lower and upper limit of the inner integral.

The shadow is a circle centered at the origin. To find its radius, we solve:

$$r^4 - 1 = z = 4 - 4r^2$$

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which gives $r = 1$. Therefore, the outer and middle integrals have upper and lower limits given by:

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} .$$

Combining all of the above, the volume of the solid is given by:

$$\begin{aligned}
\iiint_D dV &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r^4-1}^{z=4-4r^2} 1 \, r dz dr d\theta \\
&= \int_0^{2\pi} \int_0^1 [z]_{r^4-1}^{z=4-4r^2} \, r dr d\theta \\
&= \int_0^{2\pi} \int_0^1 (4 - 4r^2 - r^4 + 1) r \, dr d\theta \\
&= \int_0^{2\pi} \int_0^1 (5r - 4r^3 - r^5) \, dr d\theta \\
&= \int_0^{2\pi} \left[-\frac{5r^2}{2} - r^4 - \frac{r^6}{6} \right]_{r=0}^{r=1} d\theta \\
&= \int_0^{2\pi} \left(\frac{5}{2} - 1 - \frac{1}{6} \right) d\theta = \frac{8\pi}{3}.
\end{aligned}$$

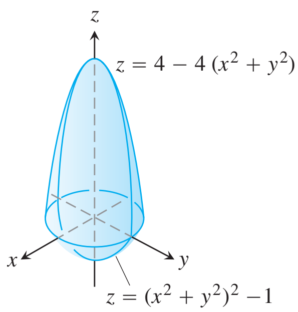

Figure 3.15: the solid in Example 3.10.

■ **Example 3.11** Let D be a solid cylinder of radius a with z-axis as the central axis, is bounded by planes $z = z_0$ and $z = z_0 + h$, where z_0 and h are constants. Therefore, the cylinder has height h. Suppose the solid has uniform density δ. Evaluate the following triple integral:

$$I_z := \iiint_D \delta(x^2 + y^2) dV$$

which is the **moment of inertia** of the solid about the z-axis.

■ **Solution** We choose the order of integration $dz dr d\theta$. The z-pillar enters the solid at $z = z_0$ and leaves at $z = z_0 + h$. The shadow is the circle with radius a centered at the origin.

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Therefore,

$$
\begin{aligned}
I_z &= \int_0^{2\pi} \int_0^a \int_{z_0}^{z_0+h} \delta \underbrace{(x^2+y^2)}_{r^2} \cdot r dz dr d\theta \\
&= \int_0^{2\pi} \int_0^a \int_{z_0}^{z_0+h} \delta r^3 dz dr d\theta \\
&= \int_0^{2\pi} \int_0^a \delta r^3 h dr d\theta \\
&= \int_0^{2\pi} \left[\frac{\delta h r^4}{4} \right]_{r=0}^{r=a} d\theta \\
&= \int_0^{2\pi} \frac{\delta h a^4}{4} d\theta \\
&= \frac{\delta h \pi a^4}{2}.
\end{aligned}
$$

Most physics/engineering textbook expresses the moment of inertia in terms of the total mass m rather than the density δ. To rewrite the above answer in terms of m, we note that:

$$\delta = \frac{m}{V}$$

where V is the total volume of the solid, which is $\pi a^2 h$. Therefore, combining with the above calculation, we have:

$$I_z = \frac{m}{\pi a^2 h} \cdot \frac{h \pi a^4}{2} = \frac{m a^2}{2},$$

which is exactly what you can find in physics or engineering textbooks.

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3.6 Triple Integrals in Spherical Coordinates

Another common coordinate system is the spherical coordinates. A point in $\mathbb{R}^3$ can be represented by three numbers (ρ, θ, ϕ) which have the following geometric meaning:

$$\begin{aligned} \rho &= \text{distance from the point to the origin} \\ \theta &= \text{the projected angle on the } xy\text{-plane counting from the positive } x\text{-axis} \\ \phi &= \text{the angle counting from the positive } z\text{-axis} \end{aligned}$$

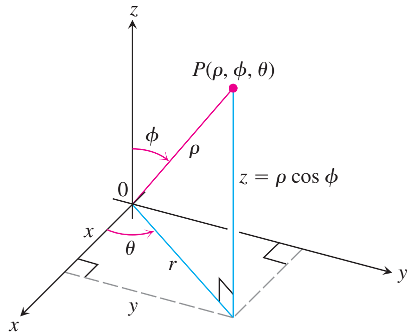

Figure 3.16: spherical coordinates

The ranges of (ρ, θ, ϕ) are:

$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

From standard trigonometry, one can figure out the following conversion rules:

$$\begin{aligned} x &= \rho \sin \phi \cos \theta \\ y &= \rho \sin \phi \sin \theta \\ z &= \rho \cos \phi \end{aligned}$$

Similar to polar and cylindrical coordinates, it is good to keep in mind that $\rho^2 = x^2 + y^2 + z^2$.

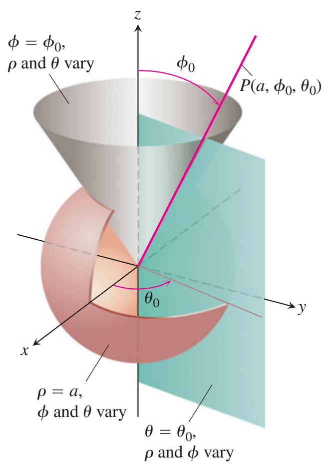

Figure 3.17: coordinate planes

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The equation $\rho = \rho_0$, where ρ_0 is a positive constant, represents the sphere of radius ρ_0 centered at the origin. The equation $\phi = \phi_0$ represents a cone with angle ϕ_0 with the origin as its vertex. Therefore, the spherical coordinates are particular useful when handling spherical or conical objects (see Figure 3.17).

To integrate using spherical coordinates, one should note that:

Theorem 3.4 Under the spherical coordinates (ρ, θ, ϕ), the volume element is given by:

$$dV = \rho^2 \sin\phi \, d\rho d\phi d\theta.$$

Infinitesimally, $\rho^2 \sin\phi \, d\rho d\phi d\theta$ is the volume of the little cube in red in Figure 3.18. The dimensions of the little cube can be found using trigonometry.

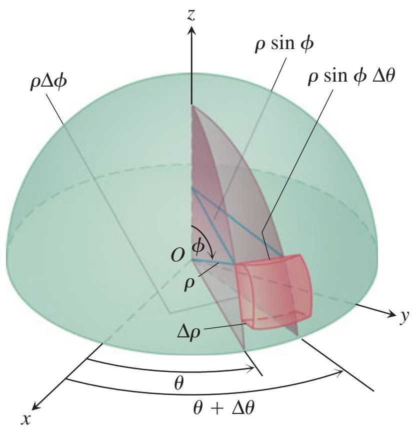

Figure 3.18: geometric meaning of $dV = \rho^2 \sin\phi \, d\rho d\phi d\theta$.

There is a general way to derive a formula for dV for any kind of coordinate systems. Suppose each coordinate of (x, y, z) is a function of (u, v, w). The *Jacobian matrix* is defined to be:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}.$$

There is a general conversion formula (whose proof is beyond the scope of the course):

$$dxdydz = \left| \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw. \tag{3.1}$$

Let's take cylindrical coordinates as an example. Since (x, y, z) are related to (r, θ, z) by the conversion rules:

$$x = r\cos\theta, \; y = r\sin\theta, \; z = z.$$

Therefore, the Jacobian matrix is given by:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{bmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial \theta} r\cos\theta & \frac{\partial}{\partial z} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial \theta} r\sin\theta & \frac{\partial}{\partial z} r\sin\theta \\ \frac{\partial}{\partial r} z & \frac{\partial}{\partial \theta} z & \frac{\partial}{\partial z} z \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is straight-forward to compute that $\det \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r\cos^2\theta - (-r\sin^2\theta) = r$. Therefore, (3.1) shows:

$$dxdydz = rdrd\theta dz.$$

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■ **Example 3.12** Consider a solid sphere S of radius r centered at the origin. Suppose it has uniform density δ. Derive the moment of inertia about z-axis, which is given by:

$$I_z := \iiint_S \delta(x^2+y^2)dV$$

■ **Solution** The solid sphere S can be written as $0 \le \rho \le r$ in spherical coordinates. As a full sphere, the ranges for the angles are:

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

Therefore, the set-up of the integral is:

$$I_z = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=r} \delta(x^2+y^2)\rho^2 \sin\phi \, d\rho d\phi d\theta.$$

To compute the integral, we need to express x^2+y^2 using spherical coordinates:

$$x^2+y^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = \rho^2\sin^2\phi.$$

Therefore,

$$\begin{aligned}
I_z &= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \delta\cdot\rho^2\sin^2\phi\cdot\rho^2\sin\phi \, d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \delta\rho^4\sin^3\phi \, d\rho d\phi d\theta \\
&= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin^3\phi \, d\phi\right)\left(\int_0^{r}\delta\rho^4 \, d\rho\right) \\
&= 2\pi\cdot\left(\int_0^{\pi}(1-\cos^2\phi)\sin\phi \, d\phi\right)\cdot\frac{\delta r^5}{5} \\
&= 2\pi\cdot\left[-\cos\phi+\frac{\cos^3\phi}{3}\right]_0^{\pi}\cdot\frac{\delta r^5}{5} \\
&= 2\pi\cdot\left(1-\frac{1}{3}+1-\frac{1}{3}\right)\cdot\frac{\delta r^5}{5}. \\
&= \frac{8\pi\delta r^5}{15}.
\end{aligned}$$

Although it is a perfectly acceptable answer, the moment of inertia in many physics and engineering books is expressed in terms of the total mass m rather than of the density. Since the density in this problem is uniform, one can easily derive the moment of inertia formula in terms of m:

$$I_z = \frac{8\pi r^5}{15}\cdot\frac{m}{\frac{4\pi r^3}{3}} = \frac{2mr^2}{5}$$

which is what one can find in physics or engineering books.

Conical objects are another type of solids which are compatible with spherical coordinates. The inequality $0 \le \phi \le \frac{\pi}{3}$ represents an infinite cone above the xy-plane with cone angle $\frac{\pi}{3}$ counting from the positive z-axis. If one further impose the inequality $0 \le \rho \le 1$ which represents the sphere with radius 1 centered at the origin, then the combined inequalities:

$$0 \le \rho \le 1, \quad 0 \le \phi \le \frac{\pi}{3}$$

represents the common part of the sphere and the cone, which is in an ice-cream shape.

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■ **Example 3.13** Find the volume of the "ice-cream cone" D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \frac{\pi}{3}$, as shown in Figure 3.19.

■ **Solution** The volume is given by $\iiint_D \rho^2 \sin\phi \, d\rho d\phi d\theta$. The limits of the inner-most integral are determine by where the ρ-ray, labeled M in the diagram, enters the solid and where it leaves the solid. Evidently, it enters from the origin $\rho = 0$. When it leaves, it always leaves from the sphere part and never the cone part, so the upper limit should be $\rho = 1$.

The ϕ-angle run from 0 to $\frac{\pi}{3}$, and the θ-ray, labeled L in the diagram, sweeps over the shadow from 0 to 2π.

Combining all these limits, the volume is given by the integral:

$$\begin{aligned}
\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/3} \int_{\rho=0}^{\rho=1} \rho^2 \sin\phi \, d\rho d\phi d\theta &= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/3} \sin\phi \, d\phi\right)\left(\int_0^1 \rho^2 \, d\rho\right) \\
&= 2\pi \cdot [-\cos\phi]_0^{\pi/3} \cdot \frac{1}{3} \\
&= 2\pi \cdot \left(-\frac{1}{2} + 1\right) \cdot \frac{1}{3} = \frac{\pi}{3}.
\end{aligned}$$

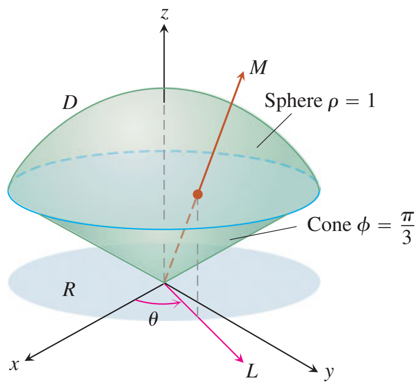
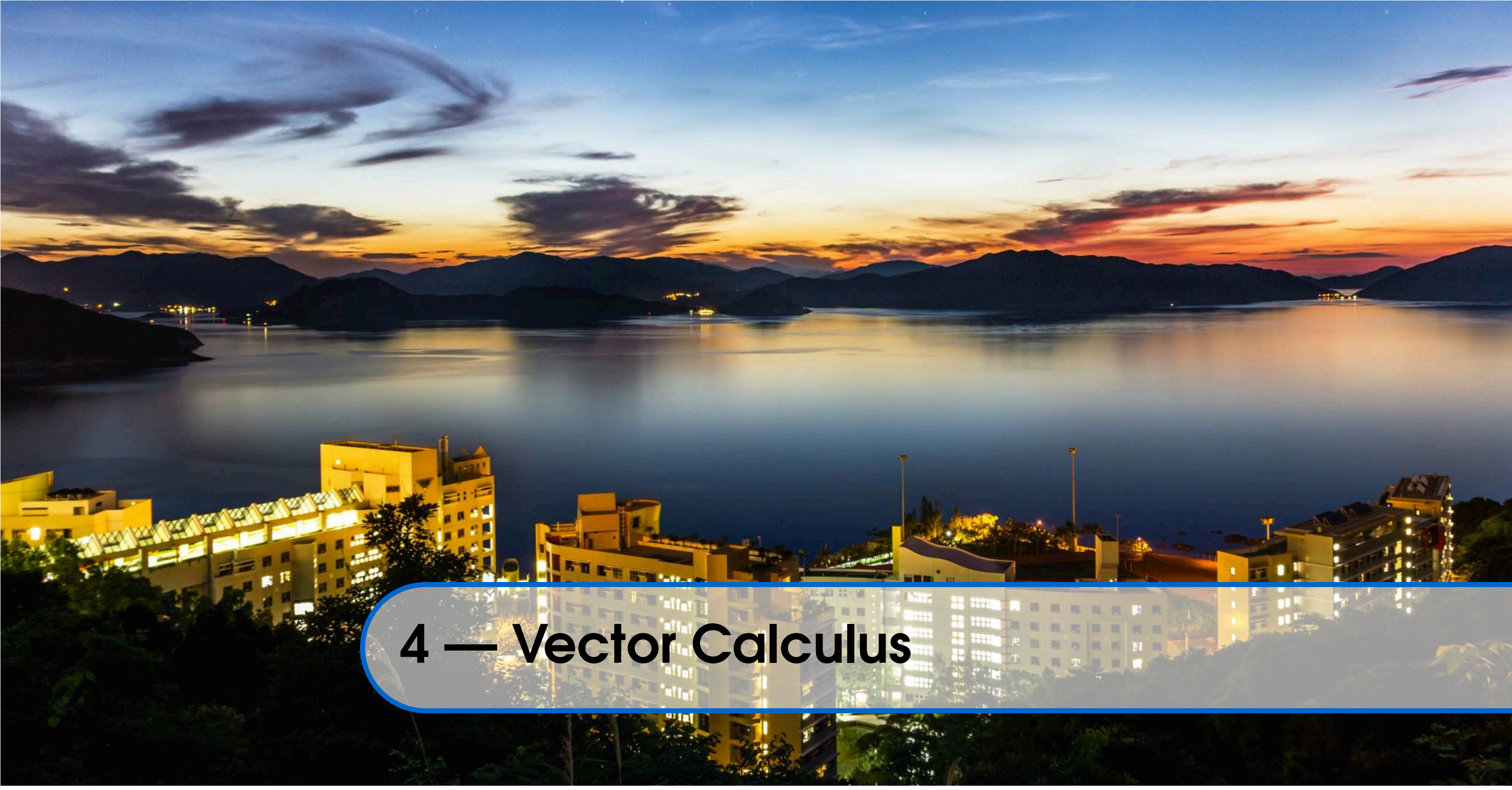


Figure 3.19: ice cream cone



4 — Vector Calculus

> "Equations are just the boring part of mathematics. I attempt to see things in terms of geometry"
>
> *Stephen Hawking*

4.1 Vector Fields on $\mathbb{R}^2$ and $\mathbb{R}^3$

4.1.1 Examples of Vector Fields

Vector Calculus is an important tool in physics and engineering. Many concepts in physics, such as gravitational and electrostatic forces, fluid flow, heat flow, etc. are described using a mathematical concept called vector fields. Gravitational, magnetic and electric forces are not just one single vector. Both directions and magnitudes may vary from place to place and even changing over time. A vector field consists of a collection of vectors that are denoted by a vector-valued function $\mathbf{F}(x,y,z) : \mathbb{R}^3 \to \mathbb{R}^3$, which assigns to each point (x,y,z) a vector labeled $\mathbf{F}(x,y,z)$. For instance, the gravitational force field due to the Sun (whose center is at the origin) is given by:

$$\mathbf{F}(x,y,z) = -GMm\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}},$$

where G, M and m are constants. That is to say, at the point $(1,0,0)$, the gravitational force is:

$$\mathbf{F}(1,0,0) = -GMm\,\mathbf{i},$$

whereas at the point $(1,0,1)$, the gravitation force is:

$$\mathbf{F}(1,0,1) = -GMm\left(\frac{1}{2^{3/2}}\mathbf{i} + \frac{1}{2^{3/2}}\mathbf{k}\right).$$

With the help of computer software (such as Mathematica), one can visualize a vector field easily. A plot of the above gravitational field can be found in Figure 4.1.

The general form of a vector field in $\mathbb{R}^3$ is given by:

$$\mathbf{F}(x,y,z) = F_x(x,y,z)\mathbf{i} + F_y(x,y,z)\mathbf{j} + F_z(x,y,z)\mathbf{k}$$

where each of F_x, F_y and F_z is a scalar-valued function.

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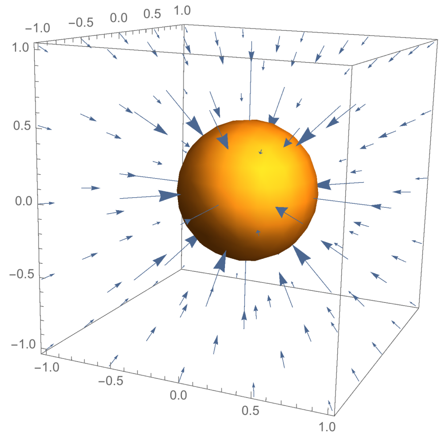

Figure 4.1: plot of the gravitational force $\mathbf{F}$

Do NOT confuse F_x with the partial derivative $\frac{\partial F}{\partial x}$! Although we used the subscript notation F_x to denote a partial derivative before, we should avoid using it in this chapter. Here F_x means the x-component of the vector field $\mathbf{F}$.

Many vector fields in physics are three-dimensional. To begin with, we will also study two-dimensional vector fields. A two-dimensional vector field is a collection of vectors in $\mathbb{R}^2$ that are denoted by a vector-valued function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, which assigns to each point (x, y) a vector $\mathbf{F}(x, y)$. The general form of a two-dimensional vector field is given by:

$$\mathbf{F}(x, y) = F_x(x, y)\mathbf{i} + F_y(x, y)\mathbf{j}.$$

Figure 4.2 shows the plot of two examples of two-dimensional vector fields:

$$\mathbf{F}(x, y) = (1 - y^2)\mathbf{i} \quad \text{and} \quad \mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}.$$

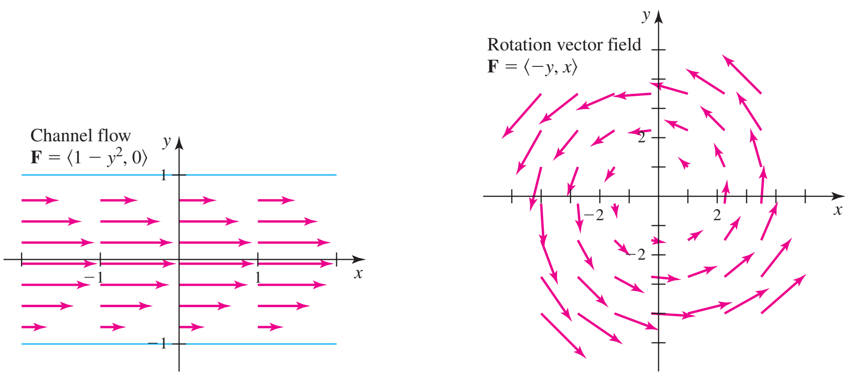

Figure 4.2: two examples of vector fields in $\mathbb{R}^2$

For brevity, we sometimes write down a vector field $\mathbf{F}$ in the form of $\langle F_x, F_y \rangle$ in $\mathbb{R}^2$, or $\langle F_x, F_y, F_z \rangle$ in $\mathbb{R}^3$. For instance,

$$(1 - y^2)\mathbf{i} = \langle 1 - y^2,\ 0 \rangle$$
$$-y\mathbf{i} + x\mathbf{j} = \langle -y,\ x \rangle$$
$$-GMm\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}} = -GMm\frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$$

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4.2 Line Integrals of Vector Fields

4.2.1 Definition of Line Integrals

Work is defined to be: force $\times$ displacement. Precisely, if $\mathbf{F}$ is a **constant** force, and $\Delta\mathbf{r} := \mathbf{r}_2 - \mathbf{r}_1$ is the displacement vector which denotes the change of position from $\mathbf{r}_1$ to $\mathbf{r}_2$, then:

$$\text{Work} = \mathbf{F} \cdot \Delta\mathbf{r}.$$

However, if the force is not a constant, meaning that either the direction or the magnitude is not uniform, the work done by the force is not as simple as stated above. Likewise, if the path is not a straight-path but a curved one so that $\Delta\mathbf{r}$ is changing over time, the work done by the force is again a bit more complicated. **Line integrals** of vector fields are introduced to handle these more complicated scenarios.

Definition 4.1 — Line Integrals of Vector Fields. Given a continuous vector field $\mathbf{F}(x, y, z)$ and a path C which is parametrized by $\mathbf{r}(t)$, $a \le t \le b$, the line integral of $\mathbf{F}$ over C is defined to be:

$$\int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt.$$

Notation In the spirit of $\mathbf{r}'(t)\, dt = \frac{d\mathbf{r}}{dt}\, dt = d\mathbf{r}$, we may denote the above line integral in a concise way as:

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is the given path.

Let's first look at some computational examples before we explain the physical and geometric meanings of line integrals.

■ **Example 4.1** Let $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ and C be the counter-clockwise path along the circular arc from $(1, 0)$ to $(0, 1)$. Find the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

■ **Solution** First we parametrize C:

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le \frac{\pi}{2}.$$

By the definition of line integrals,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{t=0}^{t=\pi/2} \mathbf{F} \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^{\pi/2} (-y\mathbf{i} + x\mathbf{j}) \cdot ((-\sin t)\mathbf{i} + (\cos t)\mathbf{j})\, dt \\
&= \int_0^{\pi/2} (y \sin t + x \cos t)\, dt.
\end{aligned}$$

Along the path C, we have $x = \cos t$ and $y = \sin t$ according to the parametrization, so we have:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} \left(\sin^2 t + \cos^2 t\right)\, dt \\
&= \int_0^{\pi/2} 1\, dt = \frac{\pi}{2}.
\end{aligned}$$

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Line integrals in three dimension can be computed in an exactly the same way as in two dimension. Let's see one example:

■ **Example 4.2** Let $\mathbf{F}(x,y,z) = (y-x^2)\mathbf{i} + (z-y^2)\mathbf{j} + (x-z^2)\mathbf{k}$ and C be a path parametrized by: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ where $0 \le t \le 1$. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

■ **Solution** We are given the parametrization in the problem so we can proceed to the computation of the line integral:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \left\langle y-x^2, z-y^2, x-z^2 \right\rangle \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^1 \left\langle y-x^2, z-y^2, x-z^2 \right\rangle \cdot \left\langle 1, 2t, 3t^2 \right\rangle\, dt \\ &= \int_0^1 \left(y-x^2+2t(z-y^2)+3t^2(x-z^2) \right)\, dt.\end{aligned}$$

Along the path C, we have $x = t$, $y = t^2$ and $z = t^3$, so:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \left(t^2 - t^2 + 2t(t^3-t^4) + 3t^2(t-t^6) \right)\, dt \\ &= \int_0^1 \left(3t^3 + 2t^4 - 2t^5 - 3t^8 \right)\, dt = \frac{29}{60}.\end{aligned}$$

It is often that a path can be broken into several segments as shown in Figure 4.3. If C denotes the combined path, then the line integral $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ of any continuous vector field $\mathbf{F}$ can be calculated by breaking it down into:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} + \int_{C_5} \mathbf{F} \cdot d\mathbf{r}.$$

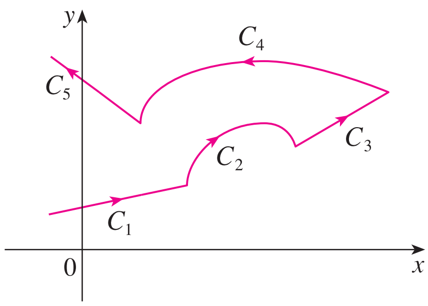

Figure 4.3: piecewise path

In this spirit, we often denote the path C by:

$$C = C_1 + C_2 + C_3 + C_4 + C_5.$$

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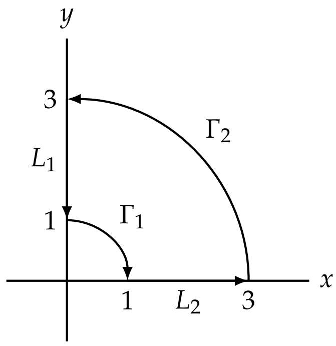

Figure 4.4: the path in Example 4.3

■ **Example 4.3** Consider the directed path $C = L_1 + \Gamma_1 + L_2 + \Gamma_2$ starting from $(0,3)$, first along the y-axis to the point $(0,1)$, then along the circular arc Γ_1 to $(1,0)$, then along the x-axis to $(3,0)$, and finally back to the point $(0,3)$ along the circular arc Γ_2. See Figure 4.4.

Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}$ is given by:

$$\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}.$$

■ **Solution** Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{L_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r} + \int_{L_2} \mathbf{F} \cdot d\mathbf{r} + \int_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r},$$

we compute the line integral of each segment individually.

To compute any line integral, we need to parametrize the path. The line segment L_1 connects the points $(0,3)$ and $(0,1)$. From Chapter 1, we learned that its parametrization of the straight-line path from $\mathbf{a}$ to $\mathbf{b}$ is given by:

$$\mathbf{r}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a}), \quad 0 \le t \le 1.$$

Therefore, L_1 is parametrized by:

$$L_1: \qquad \mathbf{r}_1(t) = \langle 0,3\rangle + t\left(\langle 0,1\rangle - \langle 0,3\rangle\right) = \langle 0, 3-2t\rangle, \quad 0 \le t \le 1.$$

Hence,

$$\begin{aligned}
\int_{L_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F} \cdot \mathbf{r}_1'(t)\, dt \\
&= \int_0^1 \langle -y, x\rangle \cdot \langle 0, -2\rangle\, dt \\
&= \int_0^1 \langle -(3-2t), 0\rangle \cdot \langle 0, -2\rangle\, dt \\
&= \int_0^1 0\, dt = 0
\end{aligned}$$

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Next we consider Γ_1, which is a clockwise circular arc of unit radius. The reverse path, commonly denoted as $-\Gamma_1$, is a counter-clockwise circular arc of unit radius which can be parametrized by:

$$-\Gamma_1 : \qquad \mathbf{r}_2(t) = \langle \cos t,\ \sin t \rangle, \quad 0 \le t \le \frac{\pi}{2}.$$

We will compute $\int_{-\Gamma_1} \mathbf{F} \cdot d\mathbf{r}$, then the line integral over the original path Γ_1 is given by:

$$\int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r} = -\int_{-\Gamma_1} \mathbf{F} \cdot d\mathbf{r}.$$

$$\begin{aligned}
\int_{-\Gamma_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} \mathbf{F} \cdot \mathbf{r}_2'(t)\ dt \\
&= \int_0^{\pi/2} \langle -y,\ x \rangle \cdot \langle -\sin t, \cos t \rangle\ dt \\
&= \int_0^{\pi/2} \langle -\sin t,\ \cos t \rangle \cdot \langle -\sin t,\ \cos t \rangle\ dt \\
&= \int_0^{\pi/2} (\sin^2 t + \cos^2 t)\ dt = \int_0^{\pi/2} 1\ dt = \frac{\pi}{2}.
\end{aligned}$$

Therefore,

$$\int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r} = -\int_{-\Gamma_1} \mathbf{F} \cdot d\mathbf{r} = -\frac{\pi}{2}.$$

Similarly, we parametrize L_2:

$$L_2 : \qquad \mathbf{r}_3(t) = \langle 1 + 2t,\ 0 \rangle, \quad 0 \le t \le 1.$$

$$\begin{aligned}
\int_{L_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \langle -y,\ x \rangle \cdot \mathbf{r}_3'(t)\ dt \\
&= \int_0^1 \langle 0,\ 1 + 2t \rangle \cdot \langle 2,\ 0 \rangle\ dt \\
&= \int_0^1 0\ dt = 0.
\end{aligned}$$

For the circular arc Γ_2 with radius 3, the parametrization is given by:

$$\Gamma_2 : \qquad \mathbf{r}_4(t) = \langle 3\cos t,\ 3\sin t \rangle, \quad 0 \le t \le \frac{\pi}{2}.$$

$$\begin{aligned}
\int_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} \langle -y,\ x \rangle \cdot \langle -3\sin t,\ 3\cos t \rangle\ dt \\
&= \int_0^{\pi/2} \langle -3\sin t,\ 3\cos t \rangle \cdot \langle -3\sin t,\ 3\cos t \rangle\ dt \\
&= \int_0^{\pi/2} (9\sin^2 t + 9\cos^2 t)\ dt = \int_0^{\pi/2} 9\ dt = \frac{9\pi}{2}.
\end{aligned}$$

Finally, we sum up these results to find the value of the desired line integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{L_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r} + \int_{L_2} \mathbf{F} \cdot d\mathbf{r} + \int_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r} = 0 - \frac{\pi}{2} + 0 + \frac{9\pi}{2} = 4\pi.$$

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4.2.2 Physical Meaning of Line Integrals

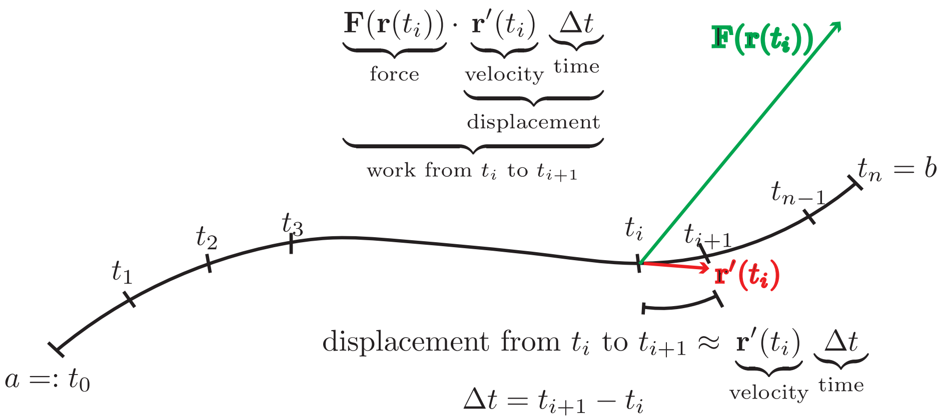
We introduce line integrals here because $\int_C \mathbf{F}\cdot d\mathbf{r}$ is exactly equal to the **work done by the force F to move a particle from the starting point of** C **to the ending point of** C.

By the virtue of Riemann sums, the line integral can be thought as:

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt \simeq \sum_i \mathbf{F}\cdot\mathbf{r}'(t_i)\Delta t_i$$

where we divide the time interval $[a,b]$ into small subdivisions:

$$a =: t_0 < t_1 < \ldots < t_i < \ldots < t_n := b.$$



Assume each subdivision is very small that $\mathbf{F}$ and $\mathbf{r}'$ are roughly constants in each subdivision. By definition of derivatives, we have:

$$\mathbf{r}'(t_i) \simeq \frac{\Delta\mathbf{r}(t_i)}{\Delta t_i}.$$

Therefore,

$$\int_a^b \mathbf{F}\cdot\mathbf{r}'(t)\,dt \simeq \sum_i \mathbf{F}\cdot\Delta\mathbf{r}(t_i)$$

Since $\mathbf{F}\cdot\Delta\mathbf{r}(t_i)$ is the work done by $\mathbf{F}$ with displacement $\Delta\mathbf{r}(t_i)$ (as they are roughly constants), summing them up gives the approximated total work done by the force over the whole path C. As $n\to\infty$, the subdivisions become infinitesimal and $\sum_i \mathbf{F}\cdot\Delta\mathbf{r}(t_i)$ becomes more accurate and approaches to the total work done by the force.

4.2.3 Geometric meaning of line integrals

The line integral of a given vector field $\mathbf{F}$ over a path C can indicate whether the path C is overall *flowing along* the vector field. Recall that the line integral is given by:

$$\int_C \mathbf{F}\cdot\mathbf{r}'(t)\,dt.$$

The value of the integral is determined by many factors, including the length of C, the magnitude of $\mathbf{F}$ and the velocity of $\mathbf{r}(t)$. However, the sign of $\mathbf{F}\cdot\mathbf{r}'(t)$ is solely determined by the angle θ between $\mathbf{F}$ and the tangent vector $\mathbf{r}'(t)$ of the curve C, since:

$$\mathbf{F}\cdot\mathbf{r}'(t) = |\mathbf{F}|\,|\mathbf{r}'(t)|\cos\theta.$$

It is *positive* when $\mathbf{F}$ and $\mathbf{r}'(t)$ make an acute angle, and is *negative* when they make an obtuse angle. Therefore, the sign of the line integral can reveal whether the path C is overall *along*

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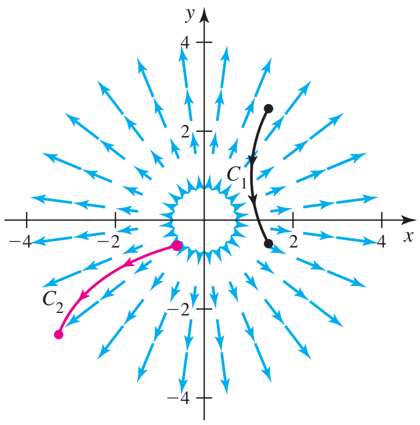

Figure 4.5: geometric meaning of line integrals

or *against* the direction of the vector field $\mathbf{F}$. The more positive is the value, the more often the path is traveling along the vector field on average. Let's illustrate this point through the following example:

Consider the vector field as shown in Figure 4.5 and the two paths C_1 and C_2 with directions indicated in the diagram. Along the path C_1, the velocity vector is pointing *against* the vector field at the beginning of the path. Then, it turns slightly *along* the vector field near the very end of the path. Therefore, $\mathbf{F} \cdot \mathbf{r}'$ is *negative* for most of the time, and so we should expect that:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} < 0.$$

On the other hand, the path C_2 is along the vector field at *all* time. The integrand $\mathbf{F} \cdot \mathbf{r}'$ is *positive* throughout the path C_2. Therefore, it is certain that:

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} > 0.$$

4.2.4 Independence of parametrization

Recall that when computing a line integral, we first pick a parametrization of the path. There is one logical gap we need to fill in, namely if we choose two different parametrizations of the same path, will we get the same answer for the line integral?

The answer is positive, as we can show it is true using the chain rule:

Suppose $\mathbf{r}(\tau)$, $a \leq \tau \leq b$, and $\mathbf{r}(t)$, $c \leq t \leq d$, are two parametrizations of a path C. Therefore, their endpoints must match, i.e. $\tau = a$ if and only if $t = c$, and $\tau = b$ if and only if $t = d$.

Then, using the τ-parametrization, the line integral is given by:

$$\int_{\tau=a}^{\tau=b} \mathbf{F} \cdot \mathbf{r}'(\tau)\, d\tau.$$

By chain rule, we have:

$$\mathbf{r}'(\tau) = \frac{d\mathbf{r}}{d\tau} = \frac{d\mathbf{r}}{dt}\frac{dt}{d\tau} = \mathbf{r}'(t)\,\frac{dt}{d\tau},$$

and so

$$\int_{\tau=a}^{\tau=b} \mathbf{F} \cdot \mathbf{r}'(\tau)\, d\tau = \int_{\tau=a}^{\tau=b} \mathbf{F} \cdot \mathbf{r}'(t)\,\frac{dt}{d\tau}\, d\tau.$$

By change of variables, we have $dt = \frac{dt}{d\tau}\, d\tau$ and so:

$$\int_{\tau=a}^{\tau=b} \mathbf{F} \cdot \mathbf{r}'(\tau)\, d\tau = \int_{t=c}^{t=d} \mathbf{F} \cdot \mathbf{r}'(t)\, dt$$

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which is exactly how we defined line integral using t as the parameter.

The above shows that no matter how we parametrize the path, as far as the endpoints of the path are kept unchanged, the line integral must be the same. We call this **independence of parametrization**. Physically speaking, this tells us as far as a particle travels along a fixed path C, **the work done by the force does not depend on how fast the particle travels**.

4.2.5 Alternative notations for line integrals

In some textbooks, a line integral is denoted using the arc-length parametrization. Recall that a path $\mathbf{r}(s)$ is said to be arc-length parametrized if $|\mathbf{r}'(s)| = 1$ for any s, i.e. unit speed. For such a parametrization, it is a convention to use s as a parameter.

Since the line integral of a given vector field $\mathbf{F}$ over a path C is independent of parametrization, we can also denote the line integral using the s parameter:

$$\int_C \mathbf{F} \cdot \mathbf{r}'(s)\, ds.$$

The value is the same using any other parametrizations.

Since $\mathbf{r}'(s)$ is a unit vector, conventionally it is often denoted by $\hat{\mathbf{T}}$ inside a line integral, meaning that it is a unit tangent vector of the path C. Therefore, you may see that occasionally the line integral is denoted by:

$$\int_C \mathbf{F} \cdot \hat{\mathbf{T}}\, ds.$$

Another common notation for line integrals is so-called the *differential form* notation. Recall that $\mathbf{r}$ is the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and so symbolically $d\mathbf{r}$ can be regarded as:

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}.$$

Let $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$, then (again symbolically)

$$\mathbf{F} \cdot d\mathbf{r} = F_x\, dx + F_y\, dy + F_z\, dz.$$

Therefore, another common notation for line integrals is:

$$\int_C F_x\, dx + F_y\, dy + F_z\, dz.$$

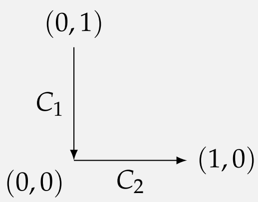
While the ds-notation does not have much practical use, there are some practical advantages for using the differential form notations if the path C is parallel to one of the coordinate axes. Let's illustrate this through an example:

■ **Example 4.4** Compute the line integral

$$\int_C -y\, dx + x\, dy$$

where C is the path from $(0,1)$ down to $(0,0)$ along the y-axis, then to $(1,0)$ along the x-axis.

■ **Solution** Note that the path C has two segments. Break C into two segments C_1 and C_2 where C_1 is the path from $(0,1)$ to $(0,0)$ along the y-axis, and C_2 is the path from $(0,0)$ to $(1,0)$ along the x-axis.



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The line integral is broken down into two:

$$\int_C -y\,dx + x\,dy = \int_{C_1} -y\,dx + x\,dy + \int_{C_2} -y\,dx + x\,dy.$$

Along C_1, we have $x = 0$, and so $dx = 0$ too. From this we can immediately tell that

$$\int_{C_1} -y\,dx + x\,dy = \int_{C_1} -y\,d(0) + 0\,dy = 0.$$

Similarly, along C_2, we have $y = 0$, and so:

$$\int_{C_2} -y\,dx + x\,dy = \int_{C_2} -0\,dx + x\,d(0) = 0.$$

Combining the two results, we know:

$$\int_C -y\,dx + x\,dy = 0.$$

To conclude, given a path C parametrized by $\mathbf{r}(t)$ and a vector field $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$, then the following are all equivalent notations for line integrals:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{r}'(t)\,dt = \int_C \mathbf{F} \cdot \hat{\mathbf{T}}\,ds = \int_C F_x\,dx + F_y\,dy + F_z\,dz.$$

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4.3 Conservative Vector Fields

4.3.1 Definition and Consequences

This section discusses a special type of fields called *conservative vector fields*. The term *conservative* is rooted from physics, not politics! We will first study its definition, and then investigate the features that make conservative vector fields distinguished from generic ones.

Definition 4.2 — Conservative Vector Field. A vector field $\mathbf{F}$ is called a conservative vector field if and only if it is in the form of $\mathbf{F} = \nabla f$ where f is a scalar function. The scalar function f is called a **potential function** of the vector field $\mathbf{F}$.

Recall that ∇f denotes the gradient vector of f, defined by:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The most preliminary method to determine whether a given vector field is conservative is to solve for the scalar potential f, as illustrated by the following example:

Example 4.5 Consider the vector field

$$\mathbf{F}(x,y,z) = (2x+y)\mathbf{i} + (x+z^3)\mathbf{j} + (3yz^2+1)\mathbf{k}.$$

Determine whether or not $\mathbf{F}$ is a conservative vector field. If so, find its potential function f such that $\mathbf{F} = \nabla f$.

Solution $\mathbf{F}$ is conservative if and only if $\mathbf{F} = \nabla f$ for some scalar function f, or equivalently, the following equations hold simultaneously:

$$\begin{aligned} &① \quad \frac{\partial f}{\partial x} = 2x+y \\ &② \quad \frac{\partial f}{\partial y} = x+z^3 \\ &③ \quad \frac{\partial f}{\partial z} = 3yz^2+1 \end{aligned}$$

From ①, one can find $f(x,y,z)$ by integrating $2x+y$ by x, regarding y to be a constant. In single variable calculus, an integration constant will be added after the integration. However, we are now considering partial derivatives, and not only constants but also y or z will vanish after differentiating by x! In other words, the "integration constant" is no longer a constant but instead a function of y and z. Precisely, we have:

$$④ \quad f(x,y,z) = x^2 + yx + g(y,z)$$

where $g(y,z)$ is some function of y and z. We will figure out $g(y,z)$ in the remaining steps.

By differentiating both sides of ④ with respect to y, we get:

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}.$$

Compare this with ②, one need to have

$$\frac{\partial g}{\partial y} = z^3.$$

An integration by y yields:

$$g(y,z) = yz^3 + h(z).$$

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Note that from similar principle discussed above for f, the integration "constant" is no longer just a constant but a function not depending on y and hence a function of z only.

By differentiating both sides with respect to z, we get:

$$\frac{\partial g}{\partial z} = 3yz^2 + h'(z).$$

Finally, by comparing this result with ③, one must have $h'(z) = 1$, and so clearly $h(z) = z + C$, where C is geniunely a constant this time!

Combining all results above, we have

$$f(x, y, z) = x^2 + yx + yz^3 + z + C$$

where C is any real constant.

It can be easily checked that $\mathbf{F} = \nabla f$. Therefore, $\mathbf{F}$ is a conservative vector field with potential functions given by the above f's.

It is important to keep in mind that not all vector fields are conservative! Here is one which such an f does not exist:

■ **Example 4.6** Let $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$. Determine whether or not $\mathbf{F}$ is a conservative vector field. If so, find its potential function f such that $\mathbf{F} = \nabla f$.

■ **Solution** If f is a scalar function such that $\mathbf{F} = \nabla f$, then we have:

$$\begin{aligned} &① \quad \frac{\partial f}{\partial x} = -y \\ &② \quad \frac{\partial f}{\partial y} = x \end{aligned}$$

Solving ① for f by integration, we get $f(x, y) = -xy + g(y)$ for some function $g(y)$. However, by differentiating this result with respect to y, we get:

$$\frac{\partial f}{\partial y} = -x + g'(y).$$

In order to be consistent with ②, we would require

$$-x + g'(y) = x, \text{ or equivalently, } g'(y) = 2x.$$

However, $g'(y)$ is a function of y, not of x! Since it leads to inconsistency, such an f cannot exist and so $\mathbf{F}$ is not conservative.

One important feature of conservative vector fields is the **path independence** of line integral, meaning that the line integral depends only on the end-points of the path. Precisely, we have:

Theorem 4.1 Given a conservative vector field $\mathbf{F} = \nabla f$, where f is a potential function, then along any path C connecting from point $P_0(x_0, y_0, z_0)$ to point $P_1(x_1, y_1, z_1)$, then the line integral is given by:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(x_1, y_1, z_1) - f(x_0, y_0, z_0).$$

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Proof. The proof is a consequence of the multivariable chain rule and the Fundamental Theorem of Calculus. Suppose $\mathbf{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ and the path C is parametrized by:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \qquad a \leq t \leq b.$$

Then,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \\
&= \int_C \nabla f \cdot \frac{d\mathbf{r}}{dt} dt \\
&= \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt \qquad \text{(chain rule)} \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \qquad \text{(Fundamental Theorem of Calculus)}
\end{aligned}$$

If (x_0, y_0, z_0) and (x_1, y_1, z_1) are the initial and final positions respectively, then $\mathbf{r}(a) = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{r}(b) = \langle x_1, y_1, z_1 \rangle$. Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(x_1, y_1, z_1) - f(x_0, y_0, z_0).$$

■

The significance of this theorem is that the RHS depends only on the initial and final points of the curve C, but not the intermediate path. In other words, if C_1 and C_2 are two paths with the same initial and final points, then we have $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. Moreover, if C is closed path whose initial and final positions are the same, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.

Notation If C is a closed path, meaning that the two endpoints are the same, it is a convention to use denote the line integral as:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

To summarize, we have:

Corollary 4.2 For a conservative vector field $\mathbf{F}$, if C_1 and C_2 are two paths with the same initial and final positions, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Moreover, if C is a closed path, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

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■ **Example 4.7** Consider the vector field $\mathbf{F}(x,y,z) = (2x+y)\mathbf{i} + (x+z^3)\mathbf{j} + (3yz^2+1)\mathbf{k}$ which appeared in Example 4.5, and the path C given by the parametric equation

$$\mathbf{r}(t) = (e^t \sin^{2012} t)\mathbf{i} + (\cos^{2013} t)\mathbf{j} + \frac{t}{\pi}\mathbf{k}, \qquad 0 \le t \le 2\pi.$$

Find the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

■ **Solution** It is clear that direct computation of this line integral is extremely laborious (and may be impossible). Fortunately, it was shown in Example 4.5 that $\mathbf{F}$ is a conservative vector field with potential function

$$f(x,y,z) = x^2 + yx + yz^3 + z + C.$$

The initial and final positions of the given path are respectively:

$$\mathbf{r}(0) = \langle 0,1,0\rangle$$
$$\mathbf{r}(2\pi) = \langle 0,1,2\rangle.$$

By Theorem 4.1, the line integral in question is simply given by:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(0,1,2) - f(0,1,0) = (10+C) - C = 10.$$

Alternatively, one can also find this line integral using the path independence property of conservative vector fields. Let L be the straight path connecting from $(0,1,0)$ to $(0,1,2)$ which are the initial and final positions respectively. Since $\mathbf{F}$ is conservative, we must have:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_L \mathbf{F}\cdot d\mathbf{r}.$$

The latter is much easier to figure out: line L is parametrized by:

$$\mathbf{r}_L(t) = \langle 0,1,0\rangle + t\left(\langle 0,1,2\rangle - \langle 0,1,0\rangle\right) = \langle 0,1,2t\rangle, \quad 0 \le t \le 1.$$

Therefore,

$$\begin{aligned}
\int_L \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \left\langle 2x+y, x+z^3, 3yz^2+1\right\rangle \cdot \mathbf{r}_L'(t)\, dt \\
&= \int_0^1 \left\langle 2(0)+1, 0+8t^3, 3(4t^2)+1\right\rangle \cdot \langle 0,0,2\rangle \; dt \\
&= \int_0^1 (24t^2+2)\, dt \\
&= 8t^3 + 2t\Big|_0^1 = 10.
\end{aligned}$$

By path independence, $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_L \mathbf{F}\cdot d\mathbf{r} = 10.$

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4.3.2 Curl Test

It is possible to test whether a given vector field is conservative or not without attempting to find its potential function. This test is commonly called the **curl test**. The use of term *curl* comes from the fact that it involves calculations of the curl of a vector field.

Definition 4.3 — Curl of a Vector Field. Given a vector field

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k},$$

the curl of the vector field $\mathbf{F}$, denoted either by $\mathrm{curl}(\mathbf{F})$ or $\nabla \times \mathbf{F}$, is defined as:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times \left(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}\right) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\
&= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}.
\end{aligned}$$

ⓘ The symbol $\nabla := \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ should be considered as an operator rather than a vector. It carries no physical or geometric meaning. "Multiplying" $\frac{\partial}{\partial x}$ by a function P gives the partial derivative $\frac{\partial P}{\partial x}$ as the product.

ⓘ If $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$ is a two dimensional vector field, the curl $\nabla \times \mathbf{F}$ can also be defined by regarding the $\mathbf{k}$-component to be zero, i.e.

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + 0\mathbf{k}.$$

It can be easily verified that

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}.$$

ⓘ $\nabla \times \mathbf{F}$ is called the *curl* of $\mathbf{F}$ because it measures how circular the vector field $\mathbf{F}$ is, as a consequence of the Green's and Stokes' Theorems which we will learn very soon.

■ **Example 4.8** Compute the curl of the vector field:

$$\mathbf{F} = (2x + y)\mathbf{i} + (x + z^3)\mathbf{j} + (3yz^2 + 1)\mathbf{k}.$$

■ **Solution**

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + y & x + z^3 & 3yz^2 + 1 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(3yz^2 + 1) - \frac{\partial}{\partial z}(x + z^3)\right)\mathbf{i} + \left(\frac{\partial}{\partial z}(2x + y) - \frac{\partial}{\partial x}(3yz^2 + 1)\right)\mathbf{j} \\
&\quad + \left(\frac{\partial}{\partial x}(x + z^3) - \frac{\partial}{\partial y}(2x + y)\right)\mathbf{k} \\
&= (3z^2 - 3z^2)\mathbf{i} + (0 - 0)\mathbf{j} + (1 - 1)\mathbf{k} = \mathbf{0}.
\end{aligned}$$

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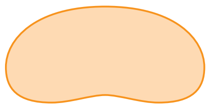
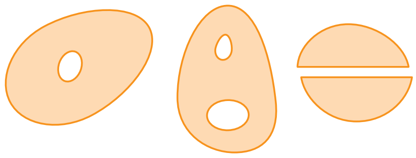
In order to introduce the curl test, we first need to introduce a topological concept about a region, namely *simply-connectedness*. A *connected* region means the region is in one piece, and a *simply-connected* region is defined as follows:

Definition 4.4 — Simply-Connected Regions. A region Ω is simply-connected if Ω is connected and every closed loop in Ω can be contracted to a point continuously without leaving the region Ω.



simply-connected region



regions that are not simply-connected

The set $\mathbb{R}^2$ with the origin removed, is not simply-connected, as the loops that go around the origin cannot be contracted to a point without "touching" the origin. However, the set $\mathbb{R}^3$ with the origin removed is simply-connected – draw a picture to convince yourself on that!

The curl test to be introduced is a very straight-forward test to check whether a given vector field $\mathbf{F}$ is conservative, without the need to solve for the potential function f. There is one crucial condition on the domain of the vector field when applying the curl test.

Theorem 4.3 — Curl Test. Given a vector field $\mathbf{F}$ is defined and C^1 on a region Ω, then:

1. If $\mathbf{F} = \nabla f$ for some scalar function f defined on Ω, then $\nabla \times \mathbf{F} = \mathbf{0}$ on Ω.
2. If $\nabla \times \mathbf{F} = \mathbf{0}$ and Ω is simply-connected, then $\mathbf{F} = \nabla f$ for some scalar function f defined on Ω.

Proof. The theorem has two parts. Part (1) is easy, while the other part is very technical (and hence the proof is omitted).

Part (1) is a consequence of the Mixed Partial Theorem. Suppose $\mathbf{F} = \nabla f$, then $\mathbf{F} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$, and so

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.
\end{aligned}$$

■

Therefore, to check *whether or not* $\mathbf{F}$ is conservative assuming it is defined and smooth on a simply-connected region, it is not necessary to solve for the potential function f. All is needed is to find $\nabla \times \mathbf{F}$ which only involves *differentiation* but not *integration*. However, the curl test only tells you whether or not the vector field is conservative. It fails to tell you what the potential function is. However, knowing that $\mathbf{F}$ is conservative without knowing the potential f can still be helpful since one can then pick an easier path when computing a line integral. Let's consider the following example:

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■ **Example 4.9** Let C be the path in $\mathbb{R}^2$ parametrized by:

$$\mathbf{r}(t) = \left(\cos^{24601} t\right)\mathbf{i} + \frac{2t}{\pi}\mathbf{j}, \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

Find the line integral:

$$\int_C \left(2xe^{xy} + x^2ye^{xy}\right)\, dx + \left(x^3e^{xy} + 2y\right)\, dy$$

■ **Solution** Recall that the required line integral can be equivalently written as $\int_C \mathbf{F} \cdot d\mathbf{r}$ where the vector field $\mathbf{F}$ is given by:

$$\mathbf{F}(x,y) = \left(2xe^{xy} + x^2ye^{xy}\right)\mathbf{i} + \left(x^3e^{xy} + 2y\right)\mathbf{j}.$$

If we compute the line integral directly from the definition, then one would have to first compute $\mathbf{F} \cdot \mathbf{r}'(t)$, which is given by:

$$\underbrace{\left(2\cos^{24601} t \cdot e^{\cos^{24601} t \cdot \frac{2t}{\pi}} + \cos^{2\times 24601} t \cdot \frac{2t}{\pi} e^{\cos^{24601} t \cdot \frac{2t}{\pi}}\right)}_{2xe^{xy}+x^2ye^{xy}} \cdot \underbrace{24601 \cos^{24600} t \cdot (-\sin t)}_{\left(\cos^{24601} t\right)'} + \ldots$$

Needless to say, it is overly complicated and almost impossible to integrate it by hand.

However, if we try to find the curl $\nabla \times \mathbf{F}$, we see that:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xe^{xy} + x^2ye^{xy} & x^3e^{xy} + 2y & 0 \end{vmatrix} \\
&= 0\mathbf{i} - 0\mathbf{j} + \left(\frac{\partial(x^3e^{xy} + 2y)}{\partial x} - \frac{\partial(2xe^{xy} + x^2ye^{xy})}{\partial y}\right)\mathbf{k} \\
&= \left(3x^2e^{xy} + x^3ye^{xy} - 2x^2e^{xy} - x^2e^{xy} - x^3ye^{xy}\right)\mathbf{k} \\
&= 0\mathbf{k} = \mathbf{0}.
\end{aligned}$$

Note that $\mathbf{F}$ is defined and C^1 everywhere in $\mathbb{R}^2$. The curl test applies and therefore it shows $\mathbf{F}$ is conservative. Even though the curl test does not tell us what the potential function is, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is now shown to be path-independent. One may simply join the end-points of C by a straight-line path L, and we can compute the line integral $\int_L \mathbf{F} \cdot d\mathbf{r}$ instead.

The end-points of C are given by:

$$\begin{aligned}
\mathbf{r}(-\pi/2) &= (\cos(-\pi/2))^{24601}\,\mathbf{i} + \frac{2}{\pi}\left(-\frac{\pi}{2}\right)\mathbf{j} = 0\mathbf{i} - \mathbf{j} \\
\mathbf{r}(\pi/2) &= (\cos(\pi/2))^{24601}\,\mathbf{i} + \frac{2}{\pi}\left(\frac{\pi}{2}\right)\mathbf{j} = 0\mathbf{i} + \mathbf{j}
\end{aligned}$$

The straight-line path L from $(0,-1)$ to $(0,1)$ is parametrized by:

$$\mathbf{r}_L(t) = \langle 0,-1\rangle + t\left(\langle 0,1\rangle - \langle 0,-1\rangle\right) = \langle 0, 2t-1\rangle, \quad 0 \leq t \leq 1.$$

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Since $\mathbf{F}$ is conservative, by the path-independence property of line integrals, we get:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_L \mathbf{F}\cdot d\mathbf{r}\\ &= \int_0^1 \mathbf{F}\cdot \mathbf{r}_L'(t)\,dt\\ &= \int_0^1 \langle 2xe^{xy}+x^2ye^{xy}, x^3e^{xy}+2y\rangle\cdot\langle 0,2\rangle\,dt\\ &= \int_0^1 2\left(x^3e^{xy}+2y\right)\,dt\\ &= \int_0^1 2\left(0+2(2t-1)\right)\,dt\\ &= 4\left[t^2-t\right]_0^1 = 0.\end{aligned}$$

It is worthwhile to note the condition that $\mathbf{F}$ is defined and C^1 on a simply-connected domain is crucial when applying the curl test to show $\mathbf{F}$ is conservative. The following example tells us why:

Let

$$\mathbf{H}(x,y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$

This vector field is not defined when $(x,y)\neq(0,0)$. It can be easily verified (left as an exercise) that $\nabla\times\mathbf{H}=\mathbf{0}$ for any $(x,y)\neq(0,0)$. However, when integrating along the unit circle $C:\mathbf{r}(t)=(\cos t)\mathbf{i}+(\sin t)\mathbf{j}$ where $0\leq t\leq 2\pi$:

$$\begin{aligned}\oint_C \mathbf{H}\cdot d\mathbf{r} &= \oint_C \frac{-ydx+xdy}{x^2+y^2}\\ &= \int_0^{2\pi}\frac{-\sin t\,d(\cos t)+\cos t\,d(\sin t)}{\sin^2 t+\cos^2 t}\\ &= \int_0^{2\pi}\frac{(\sin^2 t+\cos^2 t)dt}{\sin^2+\cos^2 t}\\ &= \int_0^{2\pi} 1dt = 2\pi\neq 0.\end{aligned}$$

C is a closed path, so $\mathbf{H}$ cannot be conservative, even thought the curl of $\mathbf{H}$ is zero! The curl test fails here because the domain of $\mathbf{H}$ is not simply-connected.

The gravitational vector field

$$\mathbf{F}(x,y,z) = -GMm\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$$

is not defined on $(0,0,0)$ but is defined and C^1 on everywhere *else* in $\mathbb{R}^3$. With some straightforward (although lengthy) computations, one can verify that $\nabla\times\mathbf{F}=\mathbf{0}$ for any $(x,y,z)\neq(0,0,0)$. Since the domain $\mathbb{R}^3$ with origin removed is simply-connected, the curl test applies to this vector field! It concludes that the gravitational vector field $\mathbf{F}$ is conservative.

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4.4 Green's Theorem

4.4.1 The Theorem and its Uses

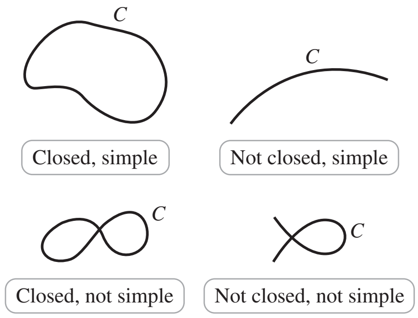
We will exclusively deal with two-dimensional vector fields in this section. In the previous section, we see that if $\mathbf{F}$ is a two-dimensional vector field which is defined and C^1 on a simply-connected region Ω in $\mathbb{R}^2$ (such as the entire $\mathbb{R}^2$ plane), then the curl test says $\nabla \times \mathbf{F} = \mathbf{0}$ if and only if $\mathbf{F}$ is conservative, and so for such a vector field, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for any closed curve C in $\mathbb{R}^2$.

It is natural to ask if there is any hidden relation between $\nabla \times \mathbf{F}$ and $\oint_C \mathbf{F} \cdot d\mathbf{r}$, given that the former being a zero vector implies the latter is zero. The Green's Theorem gives the relationship between them. Before introducing the Green's Theorem, we need to understand:

Definition 4.5 — Simple Closed Curves. A curve C is called a simple closed curve if the two endpoints coincide and it does not intersect itself at any point (other than the endpoints).



Theorem 4.4 — Green's Theorem. Let C be a simple closed curve in $\mathbb{R}^2$ which is counter-clockwise oriented. Suppose the curve C encloses region R. Let $\mathbf{F}(x, y)$ be a vector field which is defined and C^1 at every point in R, then:

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{line integral}} = \underbrace{\iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA}_{\text{double integral}}$$

ⓘ In terms of components of $\mathbf{F}$ in rectangular coordinates, i.e. $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$, one can easily verify that $\nabla \times \mathbf{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$, so the above Green's Theorem can be stated as:

$$\oint_C F_x dx + F_y dy = \iint_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dA.$$

ⓘ In particular, if $\nabla \times \mathbf{F} = \mathbf{0}$ and $\mathbf{F}$ is defined on all of $\mathbb{R}^2$, then the Green's Theorem tells us that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \mathbf{0} \cdot \mathbf{k} \, dA = 0$$

for any simple closed curve C. This is exactly what the curl test tells us!

The proof of the Green's Theorem is quite technical if the curve C is complicated. The proof of the theorem in some special cases can be found in some reference textbooks listed in the syllabus. Let's first look at some examples.

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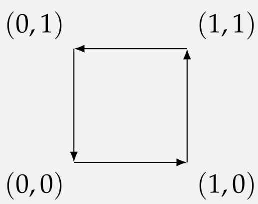
■ **Example 4.10** Use the Green's Theorem to evaluate the line integral:

$$\oint_C -ydx + xdy$$

where C is a rectangular loop $(0,0) \to (1,0) \to (1,1) \to (0,1) \to (0,0)$.

■ **Solution** The vector field corresponding to this line integral is $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, which is defined and is smooth everywhere on $\mathbb{R}^2$. The given path C encloses the rectangle R with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.



By straight-forward calculation, the curl of $\mathbf{F}$ is given by:

$$\nabla \times \mathbf{F} = 2\mathbf{k}.$$

By the Green's Theorem, we have:

$$\begin{aligned}
\oint_C -ydx + xdy &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\
&= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA \\
&= \iint_R 2\mathbf{k} \cdot \mathbf{k}\, dA \\
&= \iint_R 2\, dA \\
&= 2 \times \text{area of } R \\
&= 2.
\end{aligned}$$

If a two-dimensional C^1 vector field $\mathbf{F}$ has a property that $(\nabla \times \mathbf{F}) \cdot \mathbf{k} = c$ where c is a constant, it is best to apply the Green's Theorem when finding the line integral over a simple-closed curve C. It is because we then have:

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA \\
&= \iint_R c\, dA \\
&= c \times \text{area of } R
\end{aligned}$$

When C is a closed path going around a rectangle or a square, it is often a good idea to apply the Green's Theorem because $\iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA$ is a double integral over a rectangle or a square, which is very easy to set up. On the other hand, computing $\oint_C \mathbf{F} \cdot d\mathbf{r}$ would require breaking it down into four segments and computing the line integral of each of them.

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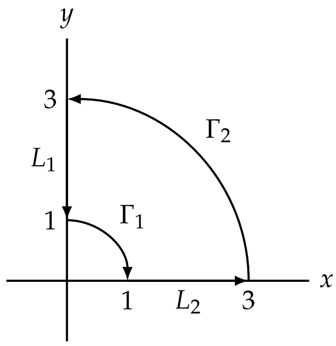

Figure 4.6: the path in Example 4.11

■ **Example 4.11** Let $\mathbf{F} = -y^2\mathbf{i} + xy\mathbf{j}$ and C is the multi-segment path appeared in Example 4.3. For easy reference, see Figure 4.6. Find the line integral using Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

■ **Solution** The closed path C consists of four segments. If we attempt to find the line integral directly, we would have to break it down into four integrals, one for each segment. However, this closed path C encloses a fan-shape region (denoted by R). The double integral over R is easy to set up if one converts it into polar coordinates. This suggests that the Green's Theorem may come in handy here.

Since $\mathbf{F}$ is defined and is C^1 everywhere, we can apply the Green's Theorem without any issues. First we compute its curl:

$$\nabla \times \mathbf{F} = 3y\mathbf{k}.$$

By the Green's Theorem

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA \\ &= \iint_R 3y\mathbf{k} \cdot \mathbf{k}\, dA \\ &= \iint_R 3y\, dA.\end{aligned}$$

The region R can be represented in polar coordinates as:

$$1 \le r \le 3, \quad 0 \le \theta \le \frac{\pi}{2}.$$

Recall that $y = r\sin\theta$ and $dA = rdrd\theta$, we have:

$$\begin{aligned}\iint_R 3y\, dA &= \int_0^{\pi/2} \int_1^3 3r\sin\theta\, rdrd\theta \\ &= \int_0^{\pi/2} \left[r^3\right]_1^3 \cdot \sin\theta\, d\theta \\ &= 26\left[-\cos\theta\right]_0^{\pi/2} \\ &= 26\end{aligned}$$

Therefore,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 26.$$

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4.4.2 Significance of the Green's Theorem

The significance of the Green's Theorem is far beyond than making computations of line integrals easier. One important geometric or physical significance is that it gives an interpretation of the curl of a two-dimensional vector field.

Consider a vector field $\mathbf{F}$ in $\mathbb{R}^2$ defined and being C^1 everywhere. Suppose C is a very tiny simple closed curve enclosing a tiny region R. Then, the quantity $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$ can be regarded as a constant inside the region R.

By the Green's Theorem, we have:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA \simeq (\nabla \times \mathbf{F}) \cdot \mathbf{k}\ (\text{area of } R).$$

In other words,

$$(\nabla \times \mathbf{F}) \cdot \mathbf{k} \simeq \frac{1}{\text{area of } R} \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

The line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is larger if $\mathbf{F} \cdot \mathbf{r}'$ is large for most of the time. It will happen when $\mathbf{F}$ is *circular* around the curve C. In other words, the closed-loop line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ indicates how *circular* the vector field $\mathbf{F}$ is around the curve C. The above result tells us that the quantity $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$ at a point is roughly proportional to the circulation of $\mathbf{F}$ around that point. That's why we call $\nabla \times \mathbf{F}$ to be curl of $\mathbf{F}$ because it is an indicator of curliness of a vector field!

As an example, you can verify that the curls of the vector fields $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ and $\mathbf{G} = x\mathbf{i} + y\mathbf{j}$ are respectively given by:

$$\begin{aligned}(\nabla \times \mathbf{F}) \cdot \mathbf{k} &= 2 \\ (\nabla \times \mathbf{G}) \cdot \mathbf{k} &= 0\end{aligned}$$

It suggests that $\mathbf{F}$ is more circular than $\mathbf{G}$. By plotting them in Mathematica, we can see that vectors in $\mathbf{F}$ are circling around the origin, while vectors in $\mathbf{G}$ are diverging from the origin.

4.4.3 Limitations of the Green's Theorem

We have stated in the Green's Theorem that the vector field $\mathbf{F}$ needs to be defined at every point in the region R enclosed by the simple closed curve C. It is a crucial condition!

Let's consider the vector field

$$\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$

It is not defined at the origin $(0,0)$. If we let C be the (counter-clockwise) unit circle centered at the origin, then C can be parametrized by:

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

By computing the line integral over C directly, one should get:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \oint_C \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt = \oint_C 1\, dt = 2\pi.$$

However, one can check from direct computations that $\nabla \times \mathbf{F} = \mathbf{0}$ at every point except the origin, and is undefined at the origin. Therefore, the double integral

$$\iint_R \underbrace{(\nabla \times \mathbf{F}) \cdot \mathbf{k}}_{=0}\, dA = 0.$$

Therefore, in this case the Green's Theorem does not hold as we have:

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{=2\pi} \ne \underbrace{\iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA}_{=0}.$$

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To summarize, one cannot directly apply the Green's Theorem if the given curve encloses a point at which the vector field is not defined. In such cases, we should either compute the line integral directly by parametrization, or use some other tools to compute it. We will compute this integral by so-called the *hole-drilling technique* to be presented in the next subsection.

If the curve does not enclose the origin (at which $\mathbf{F}$ is undefined), then Green's Theorem applies to $\mathbf{F}$ and such a curve without any issues. For instance, if Γ is a circle centered at $(3,0)$ with radius 2, then it does not enclose the origin (which is the only point at which $\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ is undefined). The Green's Theorem does show that

$$\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = \iint_R \underbrace{(\nabla \times \mathbf{F}) \cdot \mathbf{k}}_{=0 \text{ at every point in R}} \, dA = \iint_R 0 \, dA = 0.$$

Here R is the region enclosed by Γ.

4.4.4 Winding Number of a Closed Curve

The following vector field (discussed in the previous part)

$$\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$

is a famous one which concerns about the winding number of a closed curve. As discussed before, it is not defined at the origin, and $\nabla \times \mathbf{F} = \mathbf{0}$ at every point except the origin. If C is a closed curve (not necessarily simple closed) in $\mathbb{R}^2$ not passing through the origin, there is a celebrated result in topology that says:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 2\pi \times \text{ number of times } C \text{ travels around the origin counter-clockwisely.}$$

Consequently, the quantity $\frac{1}{2\pi}\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ is often called the *winding number* of the curve C. This number is not only important in pure mathematics but also in physics. Furthermore, the computation of surface flux of a vector field satisfying the inverse square law, as we will see later, are also in the same spirit as the computation of the winding number.

Our goal here is to use the Green's Theorem to explain why this line integral gives the winding number.

Circles Centered at Origin

We start with the simplest case where the curve is a circle with radius ε centered at the origin, which from now on denoted by Γ_ε. The path is parametrized by:

$$\Gamma_\varepsilon : \quad \mathbf{r}(t) = (\varepsilon \cos t)\mathbf{i} + (\varepsilon \sin t)\mathbf{j}, \quad 0 \leq t \leq 2\pi.$$

Then, by straight-forward calculations, we get:

$$\begin{aligned}
\oint_{\Gamma_\varepsilon} -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy &= \int_{t=0}^{t=2\pi} -\frac{\varepsilon \sin t}{\varepsilon^2} d(\varepsilon \cos t) + \frac{\varepsilon \cos t}{\varepsilon^2} d(\varepsilon \sin t) \\
&= \int_{t=0}^{t=2\pi} \left(\sin^2 t + \cos^2 t\right) \, dt \\
&= \int_{t=0}^{t=2\pi} 1 \, dt = 2\pi.
\end{aligned}$$

Simple Closed Curves Enclosing the Origin

Next we see how to apply the Green's Theorem on a more arbitrary curve, say the curve C in Figure 4.7a. We are going to show that the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$, is equal to 2π. First note that $\mathbf{F}$ is not defined at the origin which is enclosed by the curve C, so we can't directly apply the Green's Theorem on this curve C.

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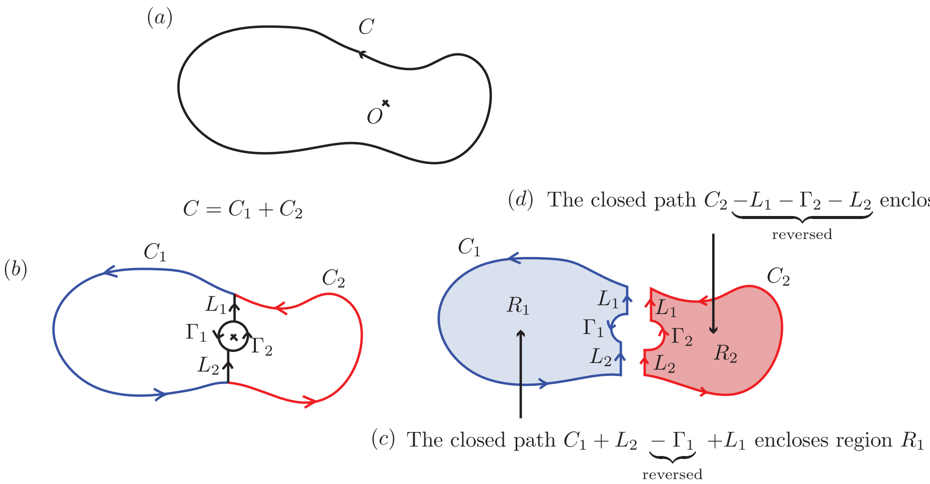

Figure 4.7: a curve circling around the origin once

To handle this issue, we drill a hole near the origin, i.e. we remove a tiny ball with radius ε centered at the origin from the region (see Figure 4.7b), and further split the punctured region into two parts R_1 and R_2 by cutting it through line segments L_1 and L_2 (see Figure 4.7cd). We label each segment of the boundaries by C_1, C_2, L_1, L_2, Γ_1 and Γ_2 with directions indicated in the diagram. Then, according to the directions of C_1, C_2 and C, the line integral in question can be expressed as:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_{C_1+C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}.$$

Likewise, according to the directions of Γ_1, Γ_2 and Γ_ε (recall that Γ_ε is the counter-clockwise circle with radius ε centered at the origin), we have:

$$\oint_{\Gamma_\varepsilon} \mathbf{F}\cdot d\mathbf{r} = \oint_{\Gamma_1+\Gamma_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\Gamma_1} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_2} \mathbf{F}\cdot d\mathbf{r}.$$

Now our goal is to show that $\oint_C \mathbf{F}\cdot d\mathbf{r}$ and $\oint_{\Gamma_\varepsilon} \mathbf{F}\cdot d\mathbf{r}$ are equal. Note that we have already computed the latter, which is 2π. The above result will show that the line integral over C is also 2π.

In order to show this, we consider each of the regions R_1 and R_2 shown in Figure 4.7cd. Since R_1 does not enclose the origin and $\mathbf{F}$ is defined everywhere in R_1, the Green's Theorem can be applied to the region R_1. According to the indicated directions of the boundary curves, we see that:

$$\text{boundary of } R_1 = C_1 + L_2 - \Gamma_1 + L_1.$$

Therefore, by the Green's Theorem, we have:

$$\underbrace{\int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{L_2} \mathbf{F}\cdot d\mathbf{r} - \int_{\Gamma_1} \mathbf{F}\cdot d\mathbf{r} + \int_{L_1} \mathbf{F}\cdot d\mathbf{r}}_{\oint_{\text{boundary of } R_1} \mathbf{F}\cdot d\mathbf{r}} = \iint_{R_1} \underbrace{(\nabla\times\mathbf{F})\cdot\mathbf{k}}_{=0}\, dA = 0. \tag{4.1}$$

Similarly, R_2 does not enclose the origin so the Green's Theorem can be applied to R_2:

$$\underbrace{\int_{C_2} \mathbf{F}\cdot d\mathbf{r} - \int_{L_1} \mathbf{F}\cdot d\mathbf{r} - \int_{\Gamma_2} \mathbf{F}\cdot d\mathbf{r} - \int_{L_2} \mathbf{F}\cdot d\mathbf{r}}_{\oint_{\text{boundary of } R_2} \mathbf{F}\cdot d\mathbf{r}} = \iint_{R_2} \underbrace{(\nabla\times\mathbf{F})\cdot\mathbf{k}}_{=0}\, dA = 0. \tag{4.2}$$

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By summing up (4.1) and (4.2) and canceling out the line integrals over L_1 and L_2, we get:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} - \int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r} - \int_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r} = 0.$$

Since $C = C_1 + C_2$ and $\Gamma_\varepsilon = \Gamma_1 + \Gamma_2$ according to their directions in the diagram, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} - \oint_{\Gamma_\varepsilon} \mathbf{F} \cdot d\mathbf{r} = 0$$

and hence:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{\Gamma_\varepsilon} \mathbf{F} \cdot d\mathbf{r} = 2\pi.$$

Self-Intersecting Curves

Next, let's use the above result to deal with some intersecting curves. Consider the curve C in Figure 4.8.

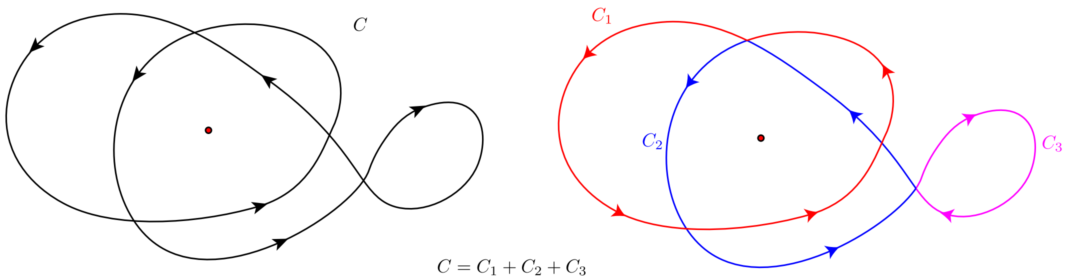

Figure 4.8: a self-intersecting curve circling around the origin twice

Since we already know how to deal with any simple closed curve enclosing the origin, we are going to split the curve C into simple closed curves C_1, C_2 and C_3 according to Figure 4.8.

As both C_1 and C_2 are simple closed curves enclosing the origin, by our previous discussion we know:

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 2\pi.$$

For C_3, it is a simple closed curve not enclosing the origin. Therefore, the Green's Theorem can be applied on C_3 without any issues. Hence, we have

$$\oint_{-C_3} \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = 0$$

where R is the region enclosed by C_3. Here we have again used the fact that $\nabla \times \mathbf{F} = \mathbf{0}$ inside the region R.

According the directions indicated on the diagram, we have:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_3} \mathbf{F} \cdot d\mathbf{r} = 2\pi + 2\pi + 0 = 4\pi.$$

Therefore, the winding number of this curve C is equal to 2. Similar technique can be applied to more complicated curves which go around the origin a lot of times.

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4.5 Parametric Surfaces

The goal of this and the next sections is to generalize the Green's Theorem for vector fields on the xy-plane to the Stokes' Theorem for vector fields on the xyz-space. While the Green's Theorem relates the line integral of a closed curve to the double integral over the enclosed region, the Stokes' Theorem relates the line integral to an integral over a *surface* whose boundary is the a closed curve. For that we need to define and make sense of *surface integrals*.

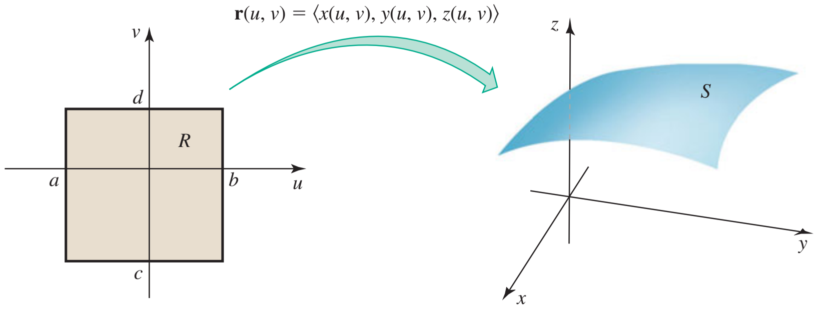
4.5.1 Surface Parametrizations

To begin with, we need to know how to describe a surface in the xyz-space. Recall that a curve in space is represented in parametric form $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, and is thought as the path of a particle. The values of $x(t)$, $y(t)$ and $z(t)$ represent the coordinates of the particle at time t.

To present a surface, we need *two* parameters, say u and v. The general form of a parametric equation of a surface is:

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}.$$

Instead of regarding u and v as time variables, we regard them as the coordinates on a uv-plane, and the vector $\mathbf{r}(u,v)$ is a function that associates each point (u,v) on the uv-plane to a point $(x(u,v), y(u,v), z(u,v))$ in the xyz-space. Since there are two parameters, the image of the function is a surface in the xyz-space. In other words, the function $\mathbf{r}(u,v)$ can be thought as a transformation that "wraps" the uv-paper onto the surface.



The function $\mathbf{r}(u,v)$ is called a **parametrization** of the surface, and by what we mean "parametrizing a surface" is to give a parametrization of the surface. As we will see later, parametrizing a surface is often the first step of computing a surface integral.

Although we use (u,v) to denote the parameters, you can use whatever pair of variables for the parameters, provided that there is no confusion.

Parametrization via Coordinate Systems

Let's look at several elementary examples such as cylinders, spheres and cones.

Example 4.12 Find a parametrization for the cylinder with radius r_0 and with z-axis as the central axis.

Solution If, under a certain coordinate system, the surface has *one* of the coordinates being constant, then giving a parametrization to that surface is fairly easy: simply take the other *two* coordinates as parameters, and define $\mathbf{r}$ according to the conversion rules between this coordinate system and the rectangular coordinate system.

The cylinder described in the problem can be presented by equation $r = r_0$ under cylindrical coordinates (r, θ, z). The conversion rule between cylindrical and rectangular

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coordinates is given by:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Fix $r = r_0$, then x, y and z are functions of (θ, z). Simply set:

$$x(\theta,z) = r_0\cos\theta, \quad y(\theta,z) = r_0\sin\theta, \quad z(\theta,z) = z,$$

then the parametrization is given by:

$$\mathbf{r}(\theta,z) = (r_0\cos\theta)\mathbf{i} + (r_0\sin\theta)\mathbf{j} + z\mathbf{k}, \quad 0 \le \theta \le 2\pi, \ -\infty < z < \infty.$$

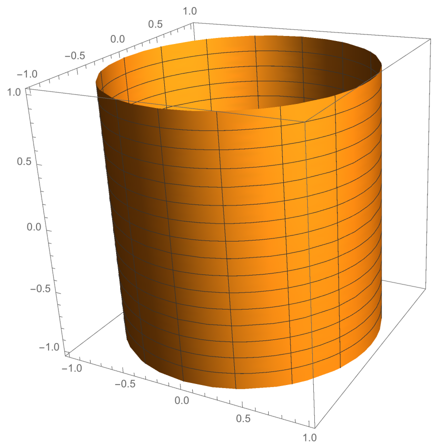

Figure 4.9: parametric plot of a cylinder

One can also specify the range of z so that the parametrization gives a finite cylinder. For instance,

$$\mathbf{r}(\theta,z) = (r_0\cos\theta)\mathbf{i} + (r_0\sin\theta)\mathbf{j} + z\mathbf{k}, \quad 0 \le \theta \le 2\pi, \ 0 \le z \le 1$$

gives the finite cylinder with unit height (from $z = 0$ to $z = 1$).

Similarly, a cone making $\frac{\pi}{4}$ angle with the z-axis can be represented by $z = r$ in cylindrical coordinates, or $\phi = \frac{\pi}{4}$ in spherical coordinates. Therefore, it is can be parametrized by two different ways:

$$\begin{aligned}
\mathbf{r}_1(r,\theta) &= (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \quad 0 \le r < \infty, \ 0 \le \theta \le 2\pi \\
\mathbf{r}_2(\rho,\theta) &= \left(\rho\sin\frac{\pi}{4}\cos\theta\right)\mathbf{i} + \left(\rho\cos\frac{\pi}{4}\sin\theta\right)\mathbf{j} + \left(\rho\cos\frac{\pi}{4}\right)\mathbf{k} \\
&= \left(\frac{\rho\cos\theta}{\sqrt{2}}\right)\mathbf{i} + \left(\frac{\rho\sin\theta}{\sqrt{2}}\right)\mathbf{j} + \frac{\rho}{\sqrt{2}}\mathbf{k}, \quad 0 \le \rho < \infty, \ 0 \le \theta \le 2\pi
\end{aligned}$$

A sphere with radius 3 can be presented by $\rho = 3$ in spherical coordinates, and so it can be parametrized by:

$$\mathbf{r}_3(\theta,\phi) = (3\sin\phi\cos\theta)\mathbf{i} + (3\sin\phi\sin\theta)\mathbf{j} + (3\sin\phi)\mathbf{k}, \quad 0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi.$$

Parametrization of Graphs

If a surface can be represented by a Cartesian equation (i.e. level-set form) such as $x^2 + y - z^3 = 1$ and that you can write one of the variables as a function of the other two variables (such as $y = z^3 - x^2 + 1$ in this example), then we can use *the other two variables* as parameters. For instance, the surface $y = z^3 - x^2 + 1$ can be parametrized as:

$$\mathbf{r}(x,z) = x\mathbf{i} + \underbrace{(z^3 - x^2 + 1)}_{y}\mathbf{j} + z\mathbf{k}.$$

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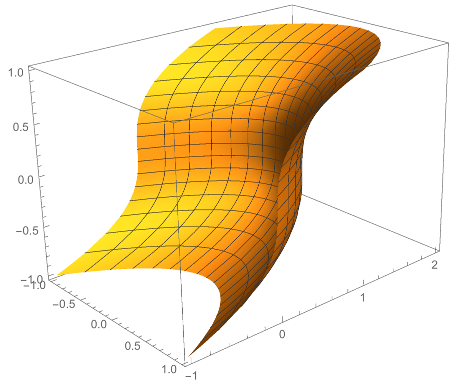


However, x cannot be written as a function of y and z for the surface $y = z^3 - x^2 + 1$, since $x = \pm\sqrt{z^3 - y + 1}$ and the $\pm$ makes x *not* be a function of y and z. On the other hand, z can be written as a function of x and y, since $z^3 = x^2 + y - 1$ and there is exactly one cubic root for $x^2 + y - 1$. However, the resulting parametrization

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + \left(x^2 + y - 1\right)^{1/3}\mathbf{k}$$

is not easy to work with.

Some Interesting Surfaces

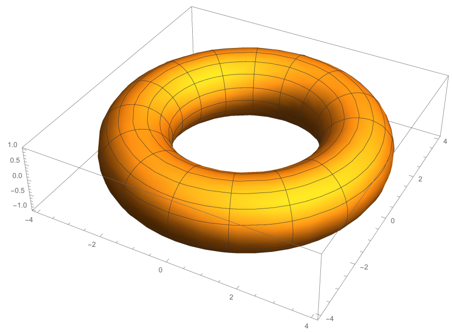
Although surfaces such as cones, spheres and cylinders can be easily parametrized, it is not the case for many interesting surfaces.

Below are some interesting examples of surfaces. It is not easy to write down (or even to explain) the parametrization since it demands quite a lot of geometric intuitions.

A torus (i.e. donut), for example, has a complicated parametrization given by:

$$\mathbf{r}(u,v) = ((3 + \cos u)\cos v)\,\mathbf{i} + ((3 + \cos u)\sin v)\,\mathbf{j} + (\sin u)\,\mathbf{k},$$

with $0 \le u \le 2\pi,\ 0 \le v \le 2\pi$.



The Möbius strip, a famous object in topology, has the following parametrization

$$\mathbf{r}(u,v) = \left(\left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\cos u\right)\mathbf{i} + \left(\left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\sin u\right)\mathbf{j} + \left(\frac{v}{2}\sin\frac{u}{2}\right)\mathbf{k}$$

where $0 \le u \le 2\pi$ and $-1 \le v \le 1$.

Since $\mathbf{r}$ denotes the position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we can also write down a parametrization in an equation form (especially when the expression of the parametrization is too long to fit in one line). For instance, the Möbius strip parametrization can be equivalently written as:

$$\begin{aligned} x &= \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\cos u \\ y &= \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\sin u \\ z &= \frac{v}{2}\sin\frac{u}{2} \end{aligned}$$

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where $0 \le u \le 2\pi$ and $-1 \le v \le 1$.

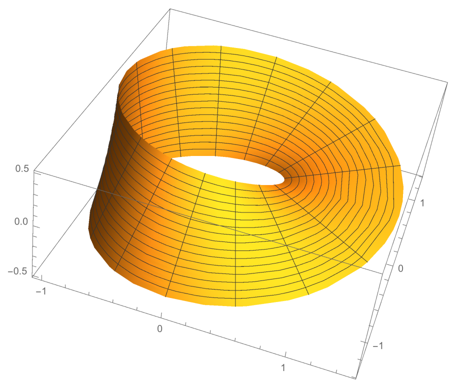

Figure 4.10: Möbius Strip

The following parametrization, which looks intimidating, describes a very beautiful surface called the Klein bottle (see Figure 4.11):

$$
\begin{aligned}
x &= -\frac{2}{15}\cos u\left(3\cos v - 30\sin u + 90\cos^4 u\sin u - 60\cos^6 u\sin u + 5\cos u\sin u\cos v\right)\\
y &= -\frac{1}{15}\sin u(3\cos v - 3\cos^2 u\cos v - 48\cos^4 u\cos v + 48\cos^6 u\cos v\\
&\qquad\qquad\qquad\qquad - 60\sin u + 5\cos u\sin u\cos v - 5\cos^3 u\cos v\sin u\\
&\qquad\qquad\qquad\qquad - 80\cos^5 u\sin u\cos v + 80\cos^7 u\sin u\cos v)\\
z &= \frac{2}{15}(3 + 5\cos u\sin u)\sin v
\end{aligned}
$$

for $0 \le u \le \pi$ and $0 \le v \le 2\pi$.

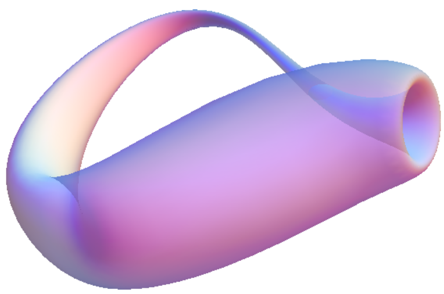

Figure 4.11: the Klein Bottle

Note that the Klein bottle is self-intersecting as depicted in the diagram.

Geometry of Parametric Surfaces

Consider a parametrization $\mathbf{r}(u, v)$ of a surface. Keeping $v = v_0$ fixed and letting u vary, the parametrization function $\mathbf{r}(u, v_0)$ depends only on u and it will give a curve on the surface. This curve is often called a u-curve for u being varied. The tangent vector to any u-curve can be computed by taking the u-derivative of $\mathbf{r}$, i.e.

$$\frac{\partial \mathbf{r}}{\partial u} = \text{tangent vector to the } u\text{-curve}$$

Likewise, when $u = u_0$ is fixed while v varies, the function $\mathbf{r}(u_0, v)$ depends only on v. The curve traced out by this function when v varies is called a v-curve, and

$$\frac{\partial \mathbf{r}}{\partial v} = \text{tangent vector to the } v\text{-curve}$$

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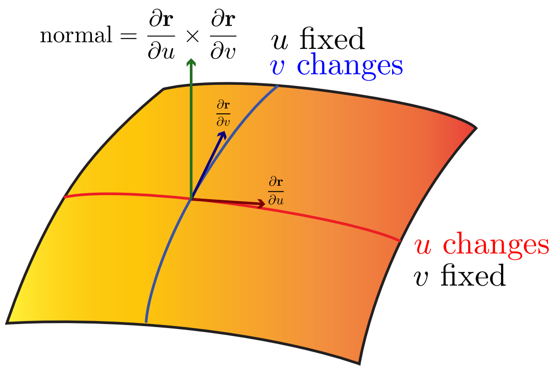
Both $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are *tangent* to the surface, hence their cross product is *normal* to the surface. A unit normal at each point is therefore given by: $\hat{\mathbf{n}} = \dfrac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|}$.



4.5.2 Surface Integrals

We are going to introduce surface integrals in this subsection. Double integrals such as

$$\iint_R f(x,y)\, dA$$

is a special type of surface integral where the region of integration R is on the flat xy-plane. A *surface integral* is one that the region of integration can be a curved surface. Many geometric and physical quantities, such as surface area, surface flux, and moment of inertia for some shell objects, can be computed using surface integrals. The Stokes' Theorem to be introduced in the next section also involves surface integrals.

We first state the definition of surface integrals, compute some examples and then explain its geometric and physical meaning.

Definition 4.6 — Surface Integrals. Given a surface S parametrized by $\mathbf{r}(u,v)$ with $a \le u \le b$ and $c \le v \le d$, and a continuous, scaled-valued function $f(x,y,z)$, the surface integral of f over the surface S is denoted and defined to be:

$$\iint_S f\, dS = \int_{v=c}^{v=d} \int_{u=a}^{u=b} f(\mathbf{r}(u,v)) \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dudv.$$

The line integral of a vector field over a curve C is independent of how we parametrize C. It is also true that the surface integral over a surface S is also independent of the surface parametrization $\mathbf{r}(u,v)$. The proof involves change of variables technique in multivariable calculus and is omitted here.

Notation If a surface is closed (meaning it has no boundaries), it is conventional to denote the integral sign by:

$$\oiint_S$$

Examples of closed surfaces include spheres and torus, while a hemisphere (only the spherical part, the flat part is not included) is not closed since it has a circle as its boundary.

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Example 4.13 Let S be the sphere of radius a centered at the origin. Evaluate the surface integral

$$\oiint_S \left(x^2 + y^2\right)\, dS$$

Solution We first parametrize the surface. Using spherical coordinates, the sphere is presented by $\rho = a$. Take (θ, ϕ) as parameters, then a parametrization of the sphere is given by:

$$\mathbf{r}(\theta,\phi) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.

According to the definition of surface integrals, we need to compute $\left|\frac{\partial r}{\partial\theta} \times \frac{\partial r}{\partial\phi}\right|$. It is straight-forward, although quite tedious:

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \theta} &= (-a\sin\phi\sin\theta)\mathbf{i} + a\sin\phi\cos\theta)\mathbf{j} + 0\mathbf{k} \\
\frac{\partial \mathbf{r}}{\partial \phi} &= (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} + (-a\sin\phi)\mathbf{k} \\
\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \end{vmatrix} \\
&= (-a^2\sin^2\phi\cos\theta)\mathbf{i} + (-a^2\sin^2\phi\sin\theta)\mathbf{j} + (-a^2\sin\phi\cos\phi)\mathbf{k} \\
\left|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi}\right| &= \sqrt{a^4\sin^4\phi(\cos^2\theta + \sin^2\theta) + a^4\sin^2\phi\cos^2\phi} \\
&= \sqrt{a^4\sin^2\phi(\sin^2\phi + \cos^2\phi)} \\
&= a^2\sin\phi.
\end{aligned}$$

Next we compute the surface integral. When (x, y, z) is on the sphere S, we have: $x = a\sin\phi\cos\theta$ and $y = a\sin\phi\sin\theta$. Therefore, the integrand is given by:

$$x^2 + y^2 = a^2\sin^2\phi(\cos^2\theta + \sin^2\theta) = a^2\sin^2\phi.$$

According to the bounds of θ and ϕ in the parametrization, the surface integral of $x^2 + y^2$ over S is given by:

$$\begin{aligned}
\oiint_S \left(x^2 + y^2\right)\, dS &= \int_{\phi=0}^{\phi=\pi}\int_{\theta=0}^{\theta=2\pi} \left(x^2 + y^2\right) \left|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi}\right|\, d\theta d\phi \\
&= \int_{\phi=0}^{\phi=\pi}\int_{\theta=0}^{\theta=2\pi} \underbrace{a^2\sin^2\phi}_{x^2+y^2} \cdot \underbrace{a^2\sin\phi}_{\left|\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi}\right|}\, d\theta d\phi \\
&= \int_{\phi=0}^{\phi=\pi}\int_{\theta=0}^{\theta=2\pi} a^4\sin^3\phi\, d\theta d\phi \\
&= \int_{\phi=0}^{\phi=\pi} 2\pi a^4\sin^3\phi\, d\phi \\
&= 2\pi a^4 \cdot \frac{4}{3} = \frac{8\pi a^4}{3}.
\end{aligned}$$

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Here we used the fact from single-variable calculus that:

$$\int_0^{\pi} \sin^3 \phi \, d\phi = \frac{4}{3}.$$

We leave this part as an exercise.

■ **Example 4.14** Let S be the plane $3x + 2y + z = 1$ defined over the region $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Compute the surface integral:

$$\iint_S (x + y + z) \, dS.$$

■ **Solution** The equation of the given plane can be written as a graph $z = 1 - 3x - 2y$ over the given region $0 \leq x \leq 1$ and $0 \leq y \leq 1$. We can take x and y as parameters and so a parametrization of the plane is given by:

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (1 - 3x - 2y)\mathbf{k}$$

where $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

Next we compute all the ingredients of the surface integral:

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial x} &= \mathbf{i} - 3\mathbf{k} \\
\frac{\partial \mathbf{r}}{\partial y} &= \mathbf{j} - 2\mathbf{k} \\
\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} &= 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} \\
\left| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right| &= \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14} \\
x + y + z &= x + y + (1 - 3x - 2y) = 1 - 2x - y
\end{aligned}$$

Finally we compute the surface integral:

$$\begin{aligned}
\iint_S (x + y + z) \, dS &= \int_0^1 \int_0^1 (1 - 2x - y) \cdot \sqrt{14} \, dx dy \\
&= \sqrt{14} \int_0^1 \left[x - x^2 - xy \right]_{x=0}^{x=1} dy \\
&= \sqrt{14} \int_0^1 -y \, dy \\
&= \left[-\frac{y^2 \sqrt{14}}{2} \right]_0^1 \\
&= -\frac{\sqrt{14}}{2}.
\end{aligned}$$

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Surface Element

Now we explain the geometric and physical meaning of surface integrals. Given a parametric surface S with parametrization $\mathbf{r}(u,v)$, $a \le u \le b$ and $c \le v \le d$, from now on we denote:

> **Notation** $dS = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| \, dudv$

It is a called the *surface element* of the integral. If we subdivide the domain in the uv-plane into small rectangular pieces with area $\Delta u \, \Delta v$, the parametrization $\mathbf{r}(u,v)$ transform them into small pieces ΔS on the surface (see Figure 4.12).

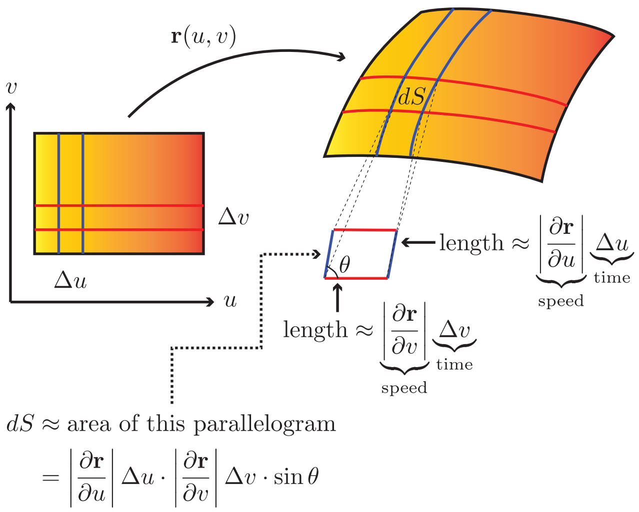

Figure 4.12: geometric meaning of the surface element

If the number of subdivisions is very large so that each ΔS is very small, then ΔS is approximately a parallelogram. The red side of the parallelogram has length approximately equal to $\left|\frac{\partial \mathbf{r}}{\partial u}\right| \Delta u$ (to see this, regard u is the time then the distance traveled after Δu unit time is approximately the speed $\times$ time. Similarly, the blue side of the parallelogram has length approximately equal to $\left|\frac{\partial \mathbf{r}}{\partial v}\right| \Delta v$. Suppose the angle of the parallelogram is θ, then the area of ΔS is approximately:

$$\left|\frac{\partial \mathbf{r}}{\partial u}\right| \Delta u \cdot \left|\frac{\partial \mathbf{r}}{\partial v}\right| \Delta v \cdot \sin\theta = \left|\frac{\partial \mathbf{r}}{\partial u}\right| \left|\frac{\partial \mathbf{r}}{\partial v}\right| \sin\theta \cdot \Delta u \, \Delta v = \underbrace{\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right|}_{\left|\frac{\partial \mathbf{r}}{\partial u}\right|\left|\frac{\partial \mathbf{r}}{\partial v}\right| \sin\theta} \Delta u \, \Delta v.$$

Infinitesimally, the ΔS becomes the surface element dS, and $\Delta u \, \Delta v$ becomes $dudv$. In other words, the surface element dS presents the area of a very tiny piece of subdivision on the surface. Summing up the area of all tiny subdivisions, we get the **surface area** of S, i.e.

$$\iint_S dS = \text{surface area of } S.$$

Different geometric or physical meanings of the integrand f give different meanings to the surface integral of f over S. For instance,

If f is	each element $f dS$ means	$\iint_S f dS$ means
1	area of dS	total surface area
surface density	mass of dS	total mass of the surface
density $\times (x^2 + y^2)$	moment of inertia of dS	moment of inertia of S

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Let S be the sphere of radius a centered at the origin. We computed in Example 4.13 that

$$\oiint_S \left(x^2+y^2\right)\, dS = \frac{8\pi a^4}{3}.$$

Suppose this spherical shell has a uniform surface density δ, then the integral $\oiint_S \delta\left(x^2+y^2\right)\, dS$ is the moment of inertia I_z about z-axis. Since its formula in many physics textbooks is written in terms of the total mass m of the sphere, let's rewrite it in terms of m.

$$\begin{aligned} I_z &= \oiint_S \delta\left(x^2+y^2\right)\, dS \\ &= \delta\left(\frac{8\pi a^4}{3}\right) \\ &= \frac{m}{4\pi a^2}\cdot\frac{8\pi a^4}{3} \\ &= \frac{2}{3}ma^2. \end{aligned}$$

4.5.3 Surface Flux

Surface flux is an important type of surface integrals in both mathematics and physics. It will appear in the statement of the Stokes' Theorem, and also plays an important in electricity and magnetism. In a nutshell, the surface flux of a vector field $\mathbf{F}$ through a surface S measures the "amount" of vectors passing through S.

Let's begin our discussion with the simplest case where the vector field $\mathbf{F}$ is constant and the surface is simply a flat plane.

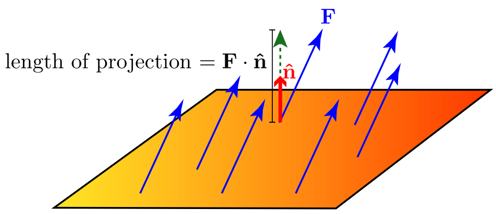

Figure 4.13: surface flux for uniform vector field through a plane

The force $\mathbf{F}$ can be decomposed into two components, one perpendicular to the plane, another parallel to the plane. As the flux counts only the amount of vectors *through* the plane, only those perpendicular to the plane should be counted. Suppose the force $\mathbf{F}$ makes an angle θ with the unit normal vector $\hat{\mathbf{n}}$, then the perpendicular component of the force has length $\mathbf{F}\cdot\hat{\mathbf{n}} = |\mathbf{F}|\cos\theta$.

Furthermore, there are vectors at every point on the plane. The larger the plane, the more vectors passing through it. Therefore, the flux of $\mathbf{F}$ through the plane should be defined as:

$$(\mathbf{F}\cdot\hat{\mathbf{n}})(\text{area of the plane}).$$

Now consider a curved surface (so that $\hat{\mathbf{n}}$ is no longer constant) and $\mathbf{F}$ is no longer a constant vector field. Infinitesimally, each area element can be regarded as a tiny flat parallelogram, and both $\mathbf{F}$ and $\hat{\mathbf{n}}$ can be regarded as constant vectors over each of tiny parallelogram. Recall that dS is the area of this element. Therefore, the flux of $\mathbf{F}$ through this tiny bit of surface is given by:

$$\mathbf{F}\cdot\hat{\mathbf{n}}\, dS$$

Summing up, the total flux of $\mathbf{F}$ through the entire surface is given by a surface integral as stated in the definition below:

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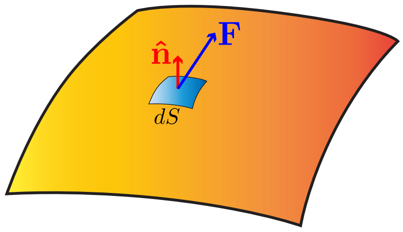

Figure 4.14: flux through a tiny surface element

Definition 4.7 — Surface Flux. Given a vector field $\mathbf{F}$ and a surface S, the surface flux of $\mathbf{F}$ through S is defined to be the surface integral:

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

where $\hat{\mathbf{n}}$ denotes unit normal vector to S at each point.

Note that there are often two choice of $\hat{\mathbf{n}}$, so the sign of the surface flux depends on which direction of $\hat{\mathbf{n}}$ is chosen. For a closed surface such as a sphere, it is a convention to choose the **outward** unit normal.

There are some surfaces which you cannot "choose" a unit normal convention. For instance, if you pick a normal vector on the Möbius strip and let it vary continuously over the strip, the normal vector may end up pointing at opposite direction when it returns to the original position! We call these **non-orientable** surfaces. We do not define surface flux on non-orientable surfaces.

Generally speaking, the surface flux can be computed by parametrizing the surface. Given a parametrization $\mathbf{r}(u, v)$, with $a \le u \le b$ and $c \le v \le d$, of a surface S, the unit normal vector $\mathbf{n}$ is given by:

$$\hat{\mathbf{n}} = \pm \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|}$$

where $\pm$ depends on the convention chosen.

It looks complicated to compute the normal vector and the surface flux. However, as a coincidence, the area element dS is given by:

$$dS = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dudv.$$

Therefore, when one multiplies $\mathbf{F} \cdot \hat{\mathbf{n}}$ by dS, the term $\left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right|$ got canceled! Precisely, we have:

Theorem 4.5 Let $\mathbf{r}(u, v)$, with $a \le u \le b$ and $c \le v \le d$, be a parametrization of a surface S. The surface flux of a vector field $\mathbf{F}$ through S can computed by:

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \pm \int_c^d \int_a^b \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dudv$$

where $\pm$ depends on the chosen convention of $\hat{\mathbf{n}}$.

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Proof. From the given parametrization $\mathbf{r}(u,v)$, we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \int_c^d\int_a^b \mathbf{F}\cdot\hat{\mathbf{n}}\left|\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right|dudv\\ &= \pm\int_c^d\int_a^b \mathbf{F}\cdot\frac{\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}}{\left|\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right|}\left|\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right|dudv\\ &= \pm\int_c^d\int_a^b \mathbf{F}\cdot\left(\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right)dudv.\end{aligned}$$

■

■ **Example 4.15** Consider the vector field:

$$\mathbf{F} = -GMm\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}.$$

Let S be part of the horizontal plane $z=1$ over the region $x^2+y^2\le 1$. Compute the surface flux of $\mathbf{F}$ through S, i.e.

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS.$$

Here $\hat{\mathbf{n}}$ is chosen to be the upward normal.

■ **Solution** First we parametrize the surface S. Since the plane $z=1$ is a graph over the xy-plane, it seems the easiest way is to use x and y as parameters. However, the base region $x^2+y^2\le 1$ is not a rectangle so it may be difficult to set up the ranges of x and y for the parametrization. Since the base region is a solid circle, we can use cylindrical coordinates too. Let

$$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}+\mathbf{k}\quad (\text{since } z=1),$$

where $0\le r\le 1$ and $0\le\theta\le 2\pi$. Next we compute all the ingredients:

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial r} &= (\cos\theta)\mathbf{i}+(\sin\theta)\mathbf{j}\\ \frac{\partial\mathbf{r}}{\partial\theta} &= (-r\sin\theta)\mathbf{i}+(r\cos\theta)\mathbf{j}\\ \frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & \sin\theta & 0\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix}\\ &= (r\cos^2\theta + r\sin^2\theta)\mathbf{k}\\ &= r\mathbf{k}\quad(\text{which is upward})\\ \mathbf{F} &= -GMm\frac{(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}+z\mathbf{k}}{(r^2\cos^2\theta+r^2\sin^2\theta+z^2)^{3/2}}\\ &= -GMm\frac{(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}+\mathbf{k}}{(r^2+1)^{3/2}} \qquad (\text{since } z=1)\\ \mathbf{F}\cdot\left(\frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta}\right) &= -\frac{GMmr}{(r^2+1)^{3/2}}.\end{aligned}$$

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Therefore, by Theorem 4.5, the surface flux is given by

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \int_0^{2\pi}\int_0^1 \mathbf{F}\cdot\left(\frac{\partial \mathbf{r}}{\partial r}\times\frac{\partial \mathbf{r}}{\partial \theta}\right)\,dr d\theta\\
&= -\int_0^{2\pi}\int_0^1 \frac{GMmr}{(r^2+1)^{3/2}} dr d\theta\\
&= \int_0^{2\pi}\left[\frac{GMm}{\sqrt{1+r^2}}\right]_{r=0}^{r=1} d\theta && \text{(by inspection)}\\
&= GMm\int_0^{2\pi}\left(\frac{1}{\sqrt{2}}-1\right)\,d\theta\\
&= 2\pi GMm\left(\frac{1}{\sqrt{2}}-1\right).
\end{aligned}$$

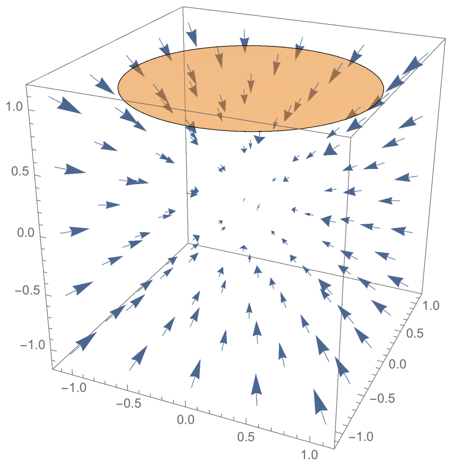

Figure 4.15: **F** and S in Example 4.15

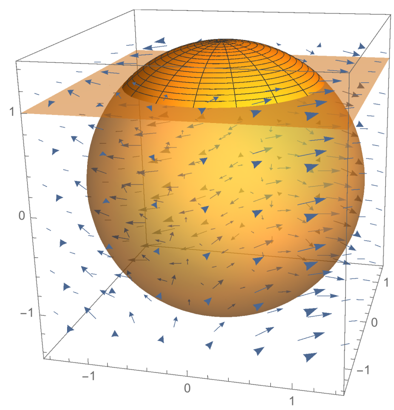

Figure 4.16: **F** and S in Example 4.16

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■ **Example 4.16** Let $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$. Find the upward flux of $\mathbf{F}$ over S which is the upper part of the sphere with radius $\sqrt{2}$ centered at the origin cut out by the plane $z = 1$.

■ **Solution** Since the surface is spherical, it is usually the best to use spherical coordinates to parametrize it. Under spherical coordinates, the sphere is represented by $\rho = \sqrt{2}$, so we use θ and ϕ for the parameters. Let

$$\mathbf{r}(\theta,\phi) = \left\langle \sqrt{2}\sin\phi\cos\theta, \sqrt{2}\sin\phi\sin\theta, \sqrt{2}\cos\phi \right\rangle.$$

The domain of the parameters are $0 \le \theta \le 2\pi$ and $0 \le \phi \le \frac{\pi}{4}$.

As in the previous example, we first compute all the ingredients. Since they are all straight-forward computations, some detail will be omitted here.

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \theta} &= \left\langle -\sqrt{2}\sin\phi\sin\theta, \sqrt{2}\sin\phi\cos\theta, 0 \right\rangle \\
\frac{\partial \mathbf{r}}{\partial \phi} &= \left\langle \sqrt{2}\cos\phi\cos\theta, \sqrt{2}\cos\phi\sin\theta, -\sqrt{2}\sin\phi \right\rangle \\
\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} &= \left\langle -2\cos\phi\sin^2\theta,\ -2\sin\phi\sin^2\theta,\ -2\cos\phi\sin\phi \right\rangle \\
\mathbf{F} = x\mathbf{i} - y\mathbf{j} &= \left\langle \sqrt{2}\sin\phi\cos\theta,\ -\sqrt{2}\sin\phi\sin\theta,\ 0 \right\rangle \\
\mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} \right) &= -2\sqrt{2}\cos 2\theta \sin^3\phi.
\end{aligned}$$

Note that $\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi}$ obtained above is a downward normal since the $\mathbf{k}$-component is negative. The upward flux is given by:

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \int_0^{2\pi} \int_0^{\pi/4} \mathbf{F} \cdot \left(-\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} \right) d\phi d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} 2\sqrt{2}\cos 2\theta \sin^3\phi\ d\phi d\theta \\
&= 2\sqrt{2} \underbrace{\left(\int_0^{2\pi} \cos 2\theta\ d\theta \right)}_{=0} \left(\int_0^{\pi/4} \sin^3\phi\ d\phi \right) = 0.
\end{aligned}$$

Physical Interpretations of Surface Flux

Generally speaking, the flux of a vector field $\mathbf{F}$ through a surface S measures the net "amount" of vectors $\mathbf{F}$ passing through S along a chosen direction of normal vector $\hat{\mathbf{n}}$. The unit of the "amount" depends on the physical meaning of the vector field $\mathbf{F}$.

For instance, if $\mathbf{u}$ is the velocity vector field of fluid (in ms^{-1}), and the surface area of S has unit m^2, then the surface flux

$$\iint_S \mathbf{u} \cdot \hat{\mathbf{n}}\, dS$$

has unit m^3s^{-1} and so it measures the net volume of fluid through the surface S in the direction of $\hat{\mathbf{n}}$. If the flux is positive, then there is more fluid flowing along the direction $\hat{\mathbf{n}}$ than against it. On the other hand, if the flux is negative, there is more fluid flowing against $\hat{\mathbf{n}}$ than along it.

If S is a closed surface (such as a sphere, a cube, a torus), it is a convention to take $\hat{\mathbf{n}}$ as the outward unit normal. The surface flux

$$\oiint_S \mathbf{u} \cdot \hat{\mathbf{n}}\, dS$$

over this closed surface measures the net volume of fluid flowing in the direction of $\hat{\mathbf{n}}$ through

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the surface, or in other words, the net volume of fluid flowing **out** from region D enclosed by S.

If one denotes ϱ as the uniform density of the fluid, then

$$\oiint_S \varrho\mathbf{u} \cdot \hat{\mathbf{n}}\, dS$$

measures the net mass of the fluid flowing out from the enclosed region D through its boundary S per unit time. Assuming there is no sink or source inside D, by the conservation of mass, the rate of change of the total mass of fluid enclosed by S is related to the surface flux by:

$$\frac{\partial}{\partial t} \underbrace{\iiint_D \varrho\, dV}_{\text{total mass in } D} = -\oiint_S \varrho\mathbf{u} \cdot \hat{\mathbf{n}}\, dS.$$

Later we can apply the Divergence Theorem on the above relation to derive an important equation to the fluid flow.

Now suppose the vector field $\mathbf{J}$ represents the transfer of heat energy in unit Joule per second. Note that while energy is a scalar, the transfer of heat at different point may have a different direction and so it is a vector quantity. Again, take S to be a closed surface enclosing a solid region D, then the surface flux of $\mathbf{j}$ through S:

$$\oiint_S \mathbf{J} \cdot \hat{\mathbf{n}}\, dS$$

measures the amount of heat energy flowing out from the region D through S. This flux integral is commonly called the *heat flux* through S by physicists.

If $\mathbf{E}$ is a electric field and, again, S is a closed surface, then the flux integral

$$\oiint_S \mathbf{E} \cdot \hat{\mathbf{n}}\, dS$$

is commonly called the *electric flux* through S. A result by Gauss claims that this flux integral is proportional to the total amount of charges enclosed by the surface. If $\mathbf{B}$ is a magnetic field and, again, S is a closed surface, then the flux integral

$$\oiint_S \mathbf{B} \cdot \hat{\mathbf{n}}\, dS$$

is commonly called the *magnetic flux* through S. Gauss's Law for Magnetism asserts that it must be zero.

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4.6 Stokes' Theorem

This section was written during the author's first flight of Boeing 787 from Boston to Tokyo. The author is grateful that Japan Airlines offer in-seat power for his laptop throughout the entire 14-hour journey.

4.6.1 Stokes' Theorem for Simply-Connected Surfaces

Recall that the Green's Theorem relates the line integral of a closed curve to a certain double integral over the region enclosed by the curve. The Stokes' Theorem is its generalization to the three dimenions, which allows the region to be a curved surface in $\mathbb{R}^3$.

Theorem 4.6 — Stokes' Theorem. Let S be an orientable, simply-connected surface in $\mathbb{R}^3$, and C be the boundary curve of the surface S. Suppose $\mathbf{F}$ is a vector field which is defined and is C^1 on the surface S, then we have:

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{line integral}} = \underbrace{\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS}_{\text{surface integral}}$$

where $\hat{\mathbf{n}}$ is the unit normal vector to S, with direction determined by the right-hand rule (see Figure 4.17).

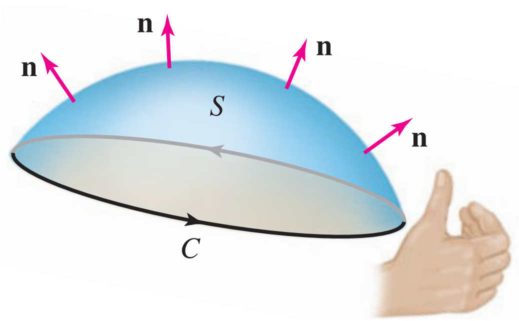

Figure 4.17: right-hand rule

By comparing the statements of the Green's and Stokes' Theorems, one can easily see that the Green's Theorem is a special case of the Stokes' Theorem, in a sense that the former applies to plane curve and the flat region enclosed by the curve on the xy-plane. The unit normal vector $\hat{\mathbf{n}}$ for the plane region is obviously given by $\mathbf{k}$ if the plane curve is traveling in the counter-clockwise orientation.

For closed curves in the three-dimensional space, one cannot say whether they are counter-clockwise or clockwise as it depends on the direction of observations. Therefore, the counter-clockwise convention of the Green's Theorem is generalized to the right-hand rule condition in the statement of the Stokes' Theorem.

The Stokes' Theorem applies only on orientable surfaces. That says, it may not hold for surfaces such as the Möbius strip. Also, the condition where the vector field $\mathbf{F}$ needs to be defined and is C^1 on the surface S is crucial. However, we will mostly deal with vector fields that satisfy this condition.

The condition that S has to be simply-connected is also crucial, but we will later learn how to modify the Stokes' Theorem so as to allow non-simply-connected surface S.

The proof of the Stokes' Theorem is omitted here. Interested readers may consult a reference textbook for a proof of one special case. Let's look at some examples.

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■ **Example 4.17** Let S be the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$ above the the xy-plane, and C be its boundary curve oriented counter-clockwise on the xy-plane. Given $\mathbf{F} = (z - y)\mathbf{i} + x\mathbf{j} - x\mathbf{k}$. Determine the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

using the Stokes' Theorem.

■ **Solution** The Stokes' Theorem asserts that:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS.$$

Since the RHS is a surface integral, we need to parametrize it first in order to compute it.

Using spherical coordinates, the parametrization of S is given by:

$$\mathbf{r}(\theta, \phi) = (2 \sin\phi \cos\theta)\mathbf{i} + (2 \sin\phi \sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{2}$. Then,

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \theta} &= (-2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\cos\theta)\mathbf{j} + 0\mathbf{k} \\
\frac{\partial \mathbf{r}}{\partial \phi} &= (2\cos\phi\cos\theta)\mathbf{i} + 2\cos\phi\sin\theta)\mathbf{j} + (-2\sin\phi)\mathbf{k} \\
\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi} &= (-4\sin^2\phi\cos\theta)\mathbf{i} + (-4\sin^2\phi\sin\theta)\mathbf{j} + (-4\sin\phi\cos\phi)\mathbf{k} \\
\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} &= (4\sin^2\phi\cos\theta)\mathbf{i} + (4\sin^2\phi\sin\theta)\mathbf{j} + (4\sin\phi\cos\phi)\mathbf{k}
\end{aligned}$$

Note that $\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi}$ is pointing downward, so we use $\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}$ instead.

Next we need to compute $\nabla \times \mathbf{F}$:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - y & x & -x \end{vmatrix} \\
&= 2\mathbf{j} + 2\mathbf{k}.
\end{aligned}$$

Therefore, using the Stokes' Theorem, we have

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS = \iint_S (2\mathbf{j} + 2\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS \\
&= \int_0^{\pi/2} \int_0^{2\pi} (2\mathbf{j} + 2\mathbf{k}) \cdot \left(\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right) d\theta d\phi \qquad \text{(using Theorem 4.5)} \\
&= \int_0^{\pi/2} \int_0^{2\pi} (8\sin^2\phi\sin\theta + 8\sin\phi\cos\phi)\, d\theta d\phi \\
&= 8 \left(\int_0^{\pi/2} \sin^2\phi\, d\phi \right) \underbrace{\left(\int_0^{2\pi} \sin\theta\, d\theta \right)}_{=0} + 2\pi \int_0^{\pi/2} 4\sin 2\phi\, d\phi \\
&= 8\pi \left[-\frac{1}{2}\cos 2\phi \right]_0^{\pi/2} = 8\pi.
\end{aligned}$$

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Although the Stokes' Theorem was used in the previous example (as required by the problem), it is actually easier to compute the line integral directly by parametrizing the curve:
Let $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + 0\mathbf{k}$ where $0 \le t \le 2\pi$. Then

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C \mathbf{F} \cdot \mathbf{r}'(t)\ dt \\
&= \int_0^{2\pi} ((z-y)\mathbf{i} + x\mathbf{j} - x\mathbf{k}) \cdot ((-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 0\mathbf{k})\ dt \\
&= \int_0^{2\pi} (-2(z-y)\sin t + 2x\cos t)\ dt \\
&= \int_0^{2\pi} -2(-2\sin t)\sin t + 2(2\cos t)\cos t\ dt \\
&= \int_0^{2\pi} 4(\sin^2 t + \cos^2 t)\ dt = 8\pi.
\end{aligned}$$

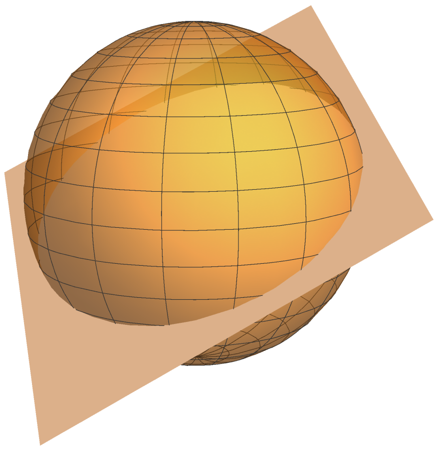

Figure 4.18: the sphere and the plane in Example 4.18

The purpose of the previous example is simply to illustrate how to use the Stokes' Theorem, although it is not necessary to use it. The line integral in the next example, however, would be extremely difficult to compute without the Stokes' Theorem.

Example 4.18 Let C be the curve of intersection of the plane $Ax + By + Cz = 0$ and the sphere $x^2 + y^2 + z^2 = a^2$. See Figure 4.18. Show that

$$\oint_C ydx + zdy + xdz = \pm\frac{\pi a^2(A+B+C)}{\sqrt{A^2+B^2+C^2}}$$

where $\pm$ is determined by the orientation of C.

Solution The plane $Ax + By + Cz = 0$ passes through the origin. Therefore, the curve C is a great circle on the sphere. However, this great circle is neither horizontal or vertical, so it is difficult to parametrize C to compute the line integral.

Let's use the Stokes' Theorem to see if there is any luck! When using the Stokes' Theorem, one needs to pick a surface S whose boundary curve is C. There are three choices:

1. the disk region enclosed by C on the plane
2. the hemisphere above the plane
3. the hemisphere below the plane

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All of the above choice should give the same answer. However, let's pick the region of the plane enclosed by C to be the surface S and we will explain why it is the smartest choice among all three.

Now the given line integral is associated to the vector field $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$, i.e.

$$\oint_C y\,dx + z\,dy + x\,dz = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

In order to apply the Stokes' Theorem, we need to compute the curl:

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} \\ &= -\mathbf{i} - \mathbf{j} - \mathbf{k}. \end{aligned}$$

We also need the unit normal vector $\hat{\mathbf{n}}$, but since the region S is a plane whose equation is $Ax + By + Cz = 0$. The unit normal is a constant vector given by:

$$\hat{\mathbf{n}} = \pm \frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}}$$

where $\pm$ is determined by the orientation of C.

Therefore,

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} = \pm(-\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot \left(\frac{A\mathbf{i} + B\mathbf{j} + C\mathbf{k}}{\sqrt{A^2 + B^2 + C^2}} \right) = \pm \frac{A + B + C}{\sqrt{A^2 + B^2 + C^2}}.$$

Next, we apply the Stokes' Theorem on these C and S:

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS \\ &= \pm \iint_S \frac{A + B + C}{\sqrt{A^2 + B^2 + C^2}}\, dS \\ &= \pm \frac{A + B + C}{\sqrt{A^2 + B^2 + C^2}} \underbrace{\iint_S dS}_{\text{surface area}} \\ &= \pm \frac{\pi a^2 (A + B + C)}{\sqrt{A^2 + B^2 + C^2}}. \end{aligned}$$

Note that the surface area of the region S on the plane is πa^2, since its boundary is a circle with radius a.

There are two major reasons why the part of the plane enclosed by C is a smarter choice for S than the hemispheres. For one thing, both $\nabla \times \mathbf{F}$ and $\hat{\mathbf{n}}$ are constant vector field if S is chosen to be a planar region, so that computing its surface integral is very easy – no parametrization! For another, if any of the hemispheres were chosen to be S, then the surface integral needs to be computed by parametrization – which can be tedious. It is also very difficult to determine the range of values of ϕ and θ since the plane cutting the sphere is not a horizontal one.

Occasionally, the Stokes' Theorem can be applied to evaluate a surface integral over an arbitrary or complicated surface, which is not easy to be parametrized. If the given vector field $\mathbf{G}$ can be expressed in the form of $\mathbf{F} = \nabla \times \mathbf{G}$ for another vector field $\mathbf{G}$, by the (backward) Stokes' Theorem asserts that:

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_S (\nabla \times \mathbf{G}) \cdot \hat{\mathbf{n}}\, dS = \oint_C \mathbf{G} \cdot d\mathbf{r}$$

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where C is the boundary curve of the surface S. Very often, the line integral is easier to compute than the surface integral.

> Note that the above discussion holds only when the given vector field $\mathbf{F}$ is of the form $\mathbf{F} = \nabla \times \mathbf{G}$. If such an $\mathbf{F}$ is not in this form, there is no easy way to apply the Stokes' Theorem backward.

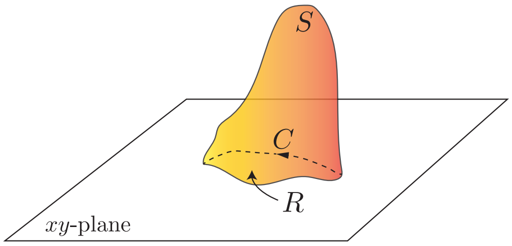

Figure 4.19: the curve and surfaces in Example 4.19

Example 4.19 Let C be an arbitrary simple closed curve on the xy-plane in the xyz-space, and S be an arbitrary surface above the xy-plane with boundary curve C. See Figure 4.19.

1. Verify that $\mathbf{i} = \nabla \times \left(-\frac{z}{2}\mathbf{j} + \frac{y}{2}\mathbf{k}\right)$.
2. Show that:

$$\iint_S \mathbf{i} \cdot \hat{\mathbf{n}}\, dS = 0.$$

Solution Part (1) is straight-forward:

$$\begin{aligned}
\nabla \times \left(-\frac{z}{2}\mathbf{j} + \frac{y}{2}\mathbf{k}\right) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -\frac{z}{2} & \frac{y}{z} \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}\left(\frac{y}{2}\right) - \frac{\partial}{\partial z}\left(-\frac{z}{2}\right)\right)\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} \\
&= \mathbf{i}
\end{aligned}$$

For part (2), we denote $\mathbf{G} = -\frac{z}{2}\mathbf{j} + \frac{y}{2}\mathbf{k}$ for simplicity. Then $\mathbf{i} = \nabla \times \mathbf{G}$. Applying the Stokes' Theorem backward, we get:

$$\begin{aligned}
\iint_S \mathbf{i} \cdot \hat{\mathbf{n}}\, dS &= \iint_S (\nabla \times \mathbf{G}) \cdot \hat{\mathbf{n}}\, dS \\
&= \oint_C \mathbf{G} \cdot d\mathbf{r}.
\end{aligned}$$

Since the curve C is arbitrary in nature, there is no way to parametrize C. However, it is given that C is on the xy-plane! Therefore, one can use the Green's Theorem to evaluate the above line integral. Denote R to be the region on the xy-plane enclosed by the curve C, then the Green's Theorem asserts that:

$$\begin{aligned}
\oint_C \mathbf{G} \cdot d\mathbf{r} &= \iint_R (\nabla \times \mathbf{G}) \cdot \mathbf{k}\, dA \\
&= \iint_R \mathbf{i} \cdot \mathbf{k}\, dA = \iint_R 0\, dA = 0.
\end{aligned}$$

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4.6.2 Significance of the Stokes' Theorem

Interpretation of Curl

Using the Stokes' Theorem, one can give a geometric interpretation of $\nabla \times \mathbf{F}$. Consider a tiny surface S with boundary curve C. Denote $\hat{\mathbf{n}}$ to be the unit normal vector of S with direction determined by the right-hand rule. Since the surface S is very small, one can regard the quantity $(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}$ is nearly a constant over the surface S. By the Stokes' Theorem, we have:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS \simeq (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \left(\iint_S dS \right).$$

Therefore,

$$(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \simeq \frac{\oint_C \mathbf{F} \cdot d\mathbf{r}.}{\text{Surface area of } S}.$$

This quantity is large when the vector field $\mathbf{F}$ is circular about the normal vector $\hat{\mathbf{n}}$. In other words, the quantity $(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}$ measures the circulation density around any given point. That's why $\nabla \times \mathbf{F}$ is often called the *curl* of $\mathbf{F}$.

Conservative Vector Fields

Recall that a vector field $\mathbf{F}$ is conservative if there exists a scalar function f such that $\mathbf{F} = \nabla f$. If $\mathbf{F}$ is defined and C^1 everywhere (or on a simply-connected domain), then the curl test asserts that $\mathbf{F}$ is conservative if and only if $\nabla \times \mathbf{F} = \mathbf{0}$. Using the Stokes' Theorem, if $\mathbf{F}$ is conservative, then:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \underbrace{(\nabla \times \mathbf{F})}_{=\mathbf{0}} \cdot \hat{\mathbf{n}}\, dS = 0,$$

which recovers the result we proved before in Theorem 4.1. Of course, the Stokes' Theorem asserts more than Theorem 4.1 does because it applies to any vector field, not just the conservative ones.

4.6.3 Surfaces with Multiple Boundaries

When applying the Stokes' Theorem on surfaces with multiple boundaries (i.e. not simply-connected), such as the one in Figure 4.20, one needs to be very careful when dealing with the inner boundary.

The form of the Stokes' Theorem, as stated in Theorem 4.6, applies only to simply-connected regions. However, using a similar technique illustrated in the subsection about the winding number, one can extend the Stokes' Theorem so that it can be applied to surfaces with *holes* as well. Take the region in Figure 4.20 as an example. One can subdivide the surface by cutting along arcs that connect the outer boundary and the inner boundaries.

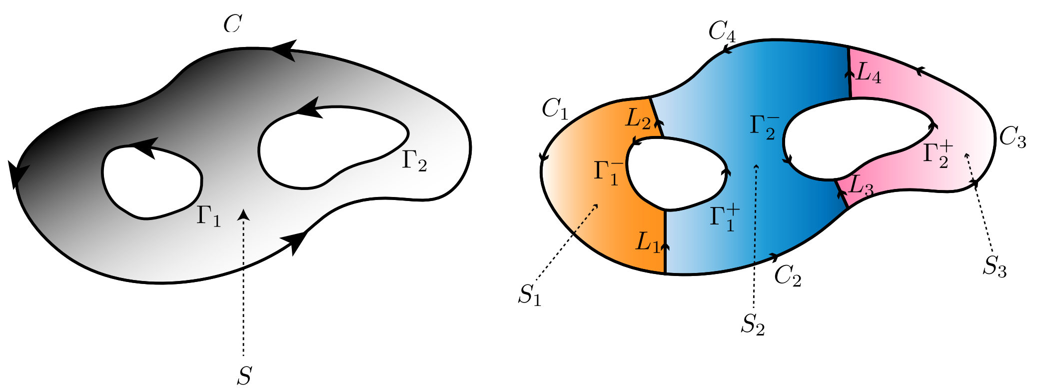

Figure 4.20: apply Stokes' Theorem for surfaces with holes

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Each sub-surface of S_1, S_2 and S_3 now becomes simply-connected and so the Stokes' Theorem applies to each sub-surface:

$$\underbrace{\left(\int_{C_1} + \int_{L_1} - \int_{\Gamma_1^-} + \int_{L_2}\right)}_{\text{form the boundary of } S_1} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$$

$$\underbrace{\left(\int_{C_2} + \int_{L_3} - \int_{\Gamma_2^-} + \int_{L_4} + \int_{C_4} - \int_{L_2} - \int_{\Gamma_1^+} - \int_{L_1}\right)}_{\text{form the boundary of } S_2} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$$

$$\underbrace{\left(\int_{C_3} - \int_{L_4} - \int_{\Gamma_2^+} - \int_{L_3}\right)}_{\text{form the boundary of } S_3} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_3} (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS.$$

Summing up all three equations and cancelling out the L_i's terms, we get:

$$\left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} - \int_{\Gamma_1^-} - \int_{\Gamma_1^+} - \int_{\Gamma_2^-} - \int_{\Gamma_2^+}\right) \mathbf{F} \cdot d\mathbf{r}$$
$$= \left(\iint_{S_1} + \iint_{S_2} + \iint_{S_3}\right) (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$$

Combining all C_i's, Γ_i's and S_i's, we yield:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} - \oint_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r} - \oint_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS.$$

While the surface integral (RHS) is in the same form as in the usual Stokes' Theorem, the LHS is not *summing* up the boundary line integrals, but rather the outer boundary has a plus sign in front and the inner boundaries each has a minus sign in front.

For more complicated surfaces (with many, but finitely many, holes), one can apply the technique illustrated above to establish:

Theorem 4.7 — Stokes' Theorem for Higher Genus[a] Surfaces. Let S be an orientable surface in $\mathbb{R}^3$ with n holes. Denote C to be its outer boundary, and $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ to be its inner boundaries. Suppose $\mathbf{F}$ is a vector field is defined and C^1 on the surface S, then:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} - \sum_{i=1}^{n} \oint_{\Gamma_i} \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS.$$

Here $\hat{\mathbf{n}}$ is the unit normal vector to S with orientation determined by the right-hand rule applied to the outer boundary C.

[a] The word *genus* is the mathematical term for number of "holes" inside the surface.

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4.7 Divergence Theorem

The Green's and Stokes' Theorems relate the line integral of a vector field over a simple closed curve with a double/surface integral of the curl of the vector field. In this section, we are going to learn the Divergence Theorem, which relates the surface integral over a closed surface with a triple integral over the solid region enclosed by the surface.

4.7.1 Divergence Operator

In order to state the Divergence Theorem, we need to define:

Definition 4.8 — Divergence Operator. Given a C^1 vector field $\mathbf{F}$ in $\mathbb{R}^3$ whose components in rectangular coordinates are:

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k},$$

the divergence of $\mathbf{F}$, denoted by $\nabla \cdot \mathbf{F}$, is defined as:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

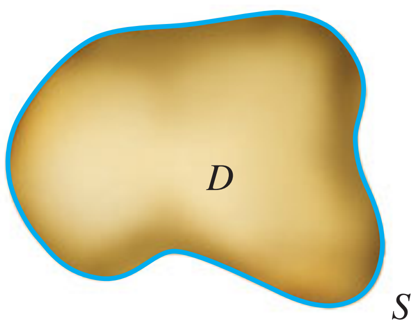
ⓘ We will see the geometric interpretation of $\nabla \cdot \mathbf{F}$ after we state the Divergence Theorem. Essentially, it measures how *diverging* the vector field is.

ⓘ The symbol ∇ can be regarded as $\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ so that we can regard the divergence of $\mathbf{F}$ as a dot product:

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}).$$

However, the "vector" $\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ is regarded as an *operator* and has no physical or geometric meaning. Note that the divergence of $\mathbf{F}$ is a *scalar* function, in contrast to the curl $\nabla \times \mathbf{F}$ which is a vector field.

4.7.2 Divergence Theorem for Solids without Holes



Theorem 4.8 — Divergence Theorem. Let S be a closed orientable surface enclosing a simply-connected solid region D. Suppose $\mathbf{F}$ is a vector field defined and being C^1 in and near the region D, then we have:

$$\underbrace{\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS}_{\text{surface integral}} = \underbrace{\iiint_D \nabla \cdot \mathbf{F}\, dV}_{\text{triple integral}}.$$

Here $\hat{\mathbf{n}}$ is the outward normal of S.

The Divergence Theorem is particularly useful for computing the flux over a closed surface, as the theorem says we do not need to parametrize the surface and compute the normal vector. Let's look at some examples.

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■ **Example 4.20** Consider the vector field $\mathbf{F} = 3x\mathbf{i} + 4y\mathbf{j} - 5z\mathbf{k}$. Let S be the sphere with radius a centered at the origin. Evaluate the flux integral $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$ with outward unit normal $\hat{\mathbf{n}}$.

■ **Solution** You can imagine the computation would be quite tedious if we computed this flux integral directly by parametrizing the sphere. However, since S is a closed surface, one can try to use the Divergence Theorem:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(-5z) = 3 + 4 - 5 = 2.$$

Denote D to be the solid sphere with radius a centered at the origin, i.e. the solid region enclosed by S. By the Divergence Theorem, we have:

$$\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 2\, dV = 2 \cdot \frac{4}{3}\pi a^3 = \frac{8}{3}\pi a^3.$$

■ **Example 4.21** Let $\mathbf{F} = x^2\mathbf{i} + 4xyz\mathbf{j} + ze^x\mathbf{k}$, D be the rectangular box defined by $0 \le x \le 3$, $0 \le y \le 2$ and $0 \le z \le 1$ and S be the boundary surface of the D (i.e. S is the shell of the box). Evaluate the flux integral:

$$\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

where $\hat{\mathbf{n}}$ is the outward normal.

■ **Solution** The closed surface S has six faces! If one attempts to compute the flux integral directly, one needs to split it into six integrals, corresponding to each of its six faces.

However, if one applies the Divergence Theorem, the difficult surface integral becomes a triple integral over a rectangular region which is very easy to set-up.

We first compute:

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(4xyz) + \frac{\partial}{\partial z}(ze^x) = 2x + 4xz + e^x.$$

By the Divergence Theorem, we get:

$$\begin{aligned} \oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \iiint_D \nabla \cdot \mathbf{F}\, dV \\ &= \int_{z=0}^{z=1} \int_{y=0}^{y=2} \int_{x=0}^{x=3} (2x + 4xz + e^x)\ dxdydz \\ &= 34 + 2e^3. \end{aligned}$$

We omit the computational detail of the triple integral above, which is a very straight-forward.

One should note that the surface integral stated in the Divergence Theorem is

$$\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

but not of $(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}$. In fact, using the Divergence Theorem, one can show that

$$\oiint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS = 0$$

for any C^1 vector field $\mathbf{F}$.

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It is because:

$$\oiint_S \underbrace{(\nabla \times \mathbf{F})}_{=:\mathbf{G}} \cdot \hat{\mathbf{n}}\, dS = \iiint_D \nabla \cdot \mathbf{G}\, dV$$
$$= \iiint_D \nabla \cdot (\nabla \times \mathbf{F})\, dV.$$

We next show that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any C^1 vector field $\mathbf{F}$.

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$
$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$
$$= \frac{\partial^2 F_z}{\partial x \partial y} - \frac{\partial^2 F_y}{\partial x \partial z} + \frac{\partial^2 F_x}{\partial y \partial z} - \frac{\partial^2 F_z}{\partial y \partial x} + \frac{\partial^2 F_y}{\partial z \partial x} - \frac{\partial^2 F_x}{\partial z \partial y}.$$

Using the Mixed Partials Theorem, all of the above second derivatives are canceled out, and so $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

Therefore, we get:

$$\oiint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS = 0.$$

4.7.3 Interpretation of Divergence Operator

By taking a tiny solid region D with boundary surface S, then the Divergence Theorem applied to a vector field $\mathbf{F}$ asserts that:

$$\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

When the region D is very small, one can regard $\nabla \times \mathbf{F}$ is nearly a constant, and so we have:

$$\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}} \simeq (\nabla \cdot \mathbf{F}) \times \text{volume of } D.$$

This result gives a geometric interpretation of $\nabla \cdot \mathbf{F}$:

$$\nabla \cdot \mathbf{F} \simeq \frac{1}{\text{volume of } D} \oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS.$$

In other words, $\nabla \times \mathbf{F}$ measures the *flux density* near a point. The more diverging $\mathbf{F}$ is around a point, the higher the flux over a tiny closed surface around that point, resulting in greater value of $\nabla \cdot \mathbf{F}$. This justifies the use of name *divergence* for $\nabla \cdot \mathbf{F}$.

4.7.4 Limitations of Divergence Theorem

The condition that $\mathbf{F}$ has to be defined in the region D is crucial. Consider the gravitational force field:

$$\mathbf{F} = -GMm\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} = -\frac{GMm}{x^2+y^2+z^2} \cdot \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}.$$

Under the spherical coordinates (ρ, θ, ϕ), we have $\rho^2 = x^2 + y^2 + z^2$. The vector field $\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$ is the **unit** radial vector field. For simplicity, we denote $\mathbf{e}_\rho = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$, then the gravitational vector field can be expressed as:

$$\mathbf{F} = -\frac{GMm}{\rho^2}\mathbf{e}_\rho.$$

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If S is the sphere with radius a centered at the origin, then its outward unit normal is also radial and so we have $\hat{\mathbf{n}} = \mathbf{e}_\rho$, and $\rho = a$ on S. Therefore, we have:

$$\begin{aligned}
\mathbf{F} \cdot \hat{\mathbf{n}} &= -\frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \mathbf{e}_\rho \\
&= -\frac{GMm}{\rho^2}\left|\mathbf{e}_\rho\right|^2 \\
&= -\frac{GMm}{\rho^2}.
\end{aligned}$$

Hence, the outward flux is given by:

$$\begin{aligned}
\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \oiint_S -\frac{GMm}{\rho^2}\, dS \\
&= \oiint_S -\frac{GMm}{a^2}\, dS && (\text{since } \rho = a \text{ on } S) \\
&= -\frac{GMm}{a^2} \underbrace{\oiint_S dS}_{\text{surface area of } S} \\
&= -\frac{GMm}{a^2} \cdot 4\pi a^2 = -4\pi GMm.
\end{aligned}$$

On the other hand, by direct computation (left as an exercise), one can verify that:

$$\nabla \cdot \mathbf{F} = \nabla \cdot \left(-GMm \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} \right) = 0$$

whenever $(x, y, z) \neq (0, 0, 0)$.

Then, the triple integral would be:

$$\iiint_D \underbrace{\nabla \cdot \mathbf{F}}_{=0 \text{ except origin}} dV = 0.$$

Here D is the solid sphere enclosed by S.

Therefore, in this case we have:

$$\underbrace{\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS}_{=-4\pi GMm} \neq \underbrace{\iiint_D \nabla \cdot \mathbf{F}\, dV}_{=0}.$$

The Divergence Theorem does *not* hold in this case. The reason is that the vector field $\mathbf{F}$ is not defined at the origin and the surface S *encloses* the origin!

To conclude, one needs to be very careful when applying the Divergence Theorem if the region D contains some points at which the vector field is not defined. In this next subsection, we will learn how to apply the Divergence Theorem (in a modified way) when the surface encloses some points at which the vector field in not defined.

4.7.5 Gauss's Law for Gravity

The purpose of this subsection is to give a proof of the *Gauss's Law for Gravity* (assuming the inverse-square law), which says that the gravitational flux:

$$\oiint_S \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}\, dS$$

is given by $4\pi GMm$ for **any** closed surface S enclosing the origin.

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We have shown that it is so when S is a sphere centered at the origin, and we are going to use the Divergence Theorem to show that it is always true for any closed surface S enclosing the origin. However, we need to be very careful when applying the Divergence Theorem since the gravitational field is undefined at the origin.

We will adopt the "hole-drilling" technique which was previously used in computing the winding number integral. Given a solid D containing the origin, we first construct a small sphere B with radius a centered at the origin. Then, the solid $D \backslash B$ (i.e. the solid D with B removed) is a solid not enclosing the "bad" point origin.

Next, we cut this solid into two parts by the horizontal plane $z = 0$. Label each side of the resulting solids by S_i, Π and Σ_i as shown in the Figure 4.21. Note that Π is the common side.

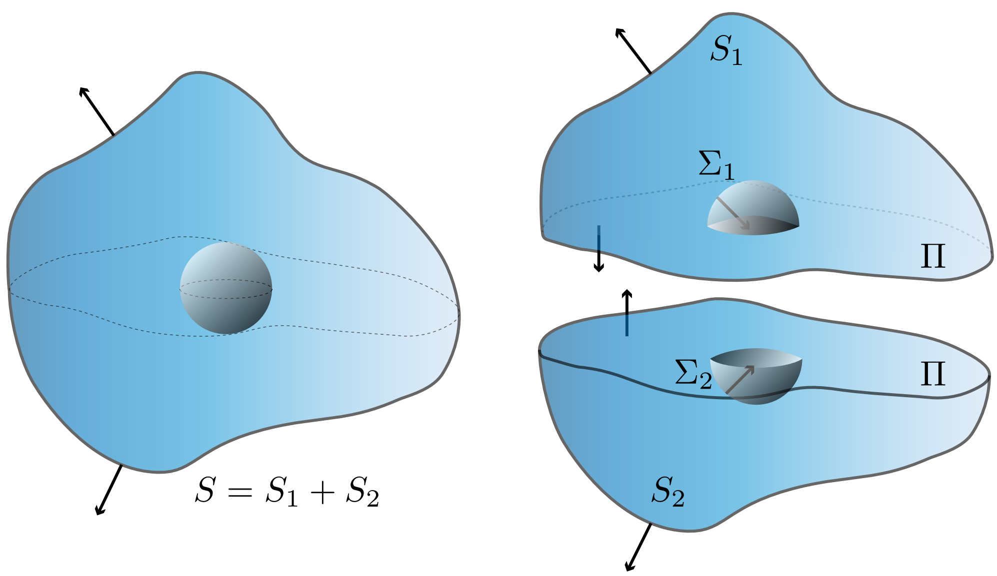

Figure 4.21: applying Divergence Theorem on the gravitational force field

Gluing S_1, Π and Σ_1 together gives a closed surface not enclosing the origin. Denote D_1 to be the solid enclosed by this closed surface. Hence, one can apply the Divergence Theorem without any issue:

$$\left(\iint_{S_1} + \iint_{\Pi} + \iint_{\Sigma_1}\right) \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}\, dS = \iiint_{D_1} \underbrace{\nabla \cdot \left(\frac{GMm}{\rho^2}\mathbf{e}_\rho\right)}_{=0}\, dV = 0$$

where $\hat{\mathbf{n}}$ is the **outward** unit normal of the boundary surface of D_1. Denote $\hat{\mathbf{n}}_{\text{up}}$ and $\hat{\mathbf{n}}_{\text{down}}$ to be the upward and downward normal vector respectively. The above integrals can be expressed as:

$$\iint_{S_1} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{up}}\, dS + \iint_{\Pi} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{down}}\, dS + \iint_{\Sigma_1} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{down}}\, dS = 0. \tag{4.3}$$

Similarly, gluing S_2, Π and Σ_2 together gives a closed surface not enclosing the origin. By the Divergence Theorem applied to this surface, we get:

$$\iint_{S_2} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{down}}\, dS + \iint_{\Pi} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{up}}\, dS + \iint_{\Sigma_2} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{up}}\, dS = 0. \tag{4.4}$$

We then add up the above two equations. First note that S_1 and S_2 can glue together to form the closed surface S. Both $\hat{\mathbf{n}}_{\text{up}}$ of S_1, and $\hat{\mathbf{n}}_{\text{down}}$ of S_2 become the **outward** normal of S.

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Therefore,

$$\iint_{S_1} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{down}}\, dS + \iint_{S_2} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{up}}\, dS = \iint_{S} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{outward}}\, dS.$$

For the planar surface Π, the downward normal $\hat{\mathbf{n}}_{\text{down}}$ is in the opposite direction of the upward normal $\hat{\mathbf{n}}_{\text{up}}$, i.e. $\hat{\mathbf{n}}_{\text{down}} = -\hat{\mathbf{n}}_{\text{up}}$. Therefore,

$$\iint_{\Pi} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{down}}\, dS + \iint_{\Pi} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{up}}\, dS = 0.$$

Finally, the surfaces Σ_1 and Σ_2 glue together to form the closed sphere Σ. The normal vectors $\hat{\mathbf{n}}_{\text{down}}$ of Σ_1, and $\hat{\mathbf{n}}_{\text{up}}$ of Σ_2, are the **inward** unit normal of Σ. Therefore,

$$\iint_{\Sigma_1} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{down}}\, dS + \iint_{\Sigma_2} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{up}}\, dS = \iint_{\Sigma} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{inward}}\, dS.$$

Summing up (4.3) and (4.4), we get:

$$\oiint_{S} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{outward}}\, dS + 0 + \oiint_{\Sigma} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{inward}}\, dS = 0.$$

Therefore,

$$\begin{aligned}
\oiint_{S} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{outward}}\, dS &= -\iint_{\Sigma} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{inward}}\, dS \\
&= \oiint_{\Sigma} \frac{GMm}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}_{\text{outward}}\, dS \\
&= 4\pi GMm \qquad \text{(computed before)}
\end{aligned}$$

This holds true for **any closed surface** S enclosing the origin. This proves the Gauss's Law for Gravity (assuming the inverse-square law).

However, if S **does not** enclose the origin, then one can apply the Divergence Theorem on the gravitational vector field without any issue.

To conclude, for any closed surface S not passing through the origin, we have:

$$\oiint_{S} GMm\frac{1}{\rho^2}\mathbf{e}_\rho \cdot \hat{\mathbf{n}}\, dS = \begin{cases} 4\pi GMm & \text{if } S \text{ encloses the origin;} \\ 0 & \text{otherwise.} \end{cases}$$

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4.8 Heat Diffusion (Optional)

In this section, we discuss an important differential equation in both mathematics and physics, the heat equation. Let $u(x,y,z,t)$ be the temperature at point (x,y,z) at time t. The heat equation:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

governs the diffusion of heat.

4.8.1 Derivation of Heat Equation

To begin with, we will use several fundamental laws in physics and the Divergence Theorem to derive the heat equation.

Heat diffusion is caused by displacement of heat energy. Fourier's Law in physics asserts that heat energy transfers according to the following rule:

$$\mathbf{J} = -a\nabla u$$

where $\mathbf{J}$ is a vector field (in energy per second) representing the flow of heat energy, and a is a positive constant depending on the medium. In other words, heat energy diffuses from higher temperature regions to lower ones, and the rate of diffusion is proportional to the magnitude of ∇u.

Let D be an arbitrary solid region with boundary surface S. Denote ϱ to be the energy density function (in energy per volume), which equals to bu for some positive constant b whose value depends on the medium. Then the triple integral:

$$\iiint_D \varrho \, dV$$

is the **total** amount of heat energy contained in the region D.

On the other hand, the outward flux

$$\oiint_S \mathbf{J} \cdot \hat{\mathbf{n}} \, dS$$

measures the amount of heat loss through the closed surface S. By the *conservation of heat energy*, heat energy must escape through the surface S. In mathematical terms, it is stated as:

$$\frac{\partial}{\partial t} \iiint_D \varrho \, dV = -\oiint_S \mathbf{J} \cdot \hat{\mathbf{n}} \, dS.$$

The negative sign appears on the RHS because of the outward convention of $\hat{\mathbf{n}}$.

Applying the above physical laws, we get:

$$\begin{aligned}
\iiint_D b\frac{\partial u}{\partial t} \, dV &= -\oiint_S (-a\nabla u) \cdot \hat{\mathbf{n}} \, dS \\
\iiint_D b\frac{\partial u}{\partial t} \, dV &= \oiint_S a\nabla u \cdot \hat{\mathbf{n}} \, dS \\
\iiint_D b\frac{\partial u}{\partial t} \, dV &= \iiint_D \nabla \cdot (a\nabla u) \, dV && \text{(Divergence Theorem)}
\end{aligned}$$

Since D is arbitrary, we must have:

$$b\frac{\partial u}{\partial t} = \nabla \cdot (a\nabla u) = a\nabla \cdot \nabla u.$$

We leave it as an exercise for readers to verify that:

$$\nabla \cdot \nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

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Therefore, we can conclude that:

$$\frac{\partial u}{\partial t} = \frac{b}{a}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

which is exactly the heat equation by defining $k = \frac{b}{a}$.

Very often, $\nabla \cdot \nabla u$ is denoted by $\nabla^2 u$ or Δu. As such, the heat equation can be written as:

$$\frac{\partial u}{\partial t} = k\Delta u.$$

4.8.2 Fundamental Solution

It can be verified that the following function satisfies the heat equation:

$$\Phi(x, y, z, t) = \frac{1}{(4\pi kt)^{3/2}} \exp\left(-\frac{x^2 + y^2 + z^2}{4kt}\right).$$

At a point $(x, y, z) = (0, 0, 0)$, we have $\Phi(0, 0, 0, t) = \frac{1}{(4\pi kt)^{3/2}}$, and so:

$$\lim_{t\to 0} \Phi(0, 0, 0, t) = \lim_{t\to 0} \frac{1}{(4\pi kt)^{3/2}} = \infty.$$

In contrast, if $(x, y, z) \neq (0, 0, 0)$, both $\exp\left(-\frac{x^2+y^2+z^2}{4kt}\right)$ and $(4\pi kt)^{3/2}$ go to 0 as $t \to 0$. However, the exponential term goes to 0 *faster* than the $t^{3/2}$ term, so

$$\lim_{t\to 0} \Phi(x, y, z, t) = \lim_{t\to 0} \frac{1}{(4\pi kt)^{3/2}} \exp\left(-\frac{x^2 + y^2 + z^2}{4kt}\right) = 0 \quad \text{when } (x, y, z) \neq (0, 0, 0).$$

Therefore, the function $\Phi(x, y, z, t)$ represents the heat diffusion starting from a highly concentrated heat source at $t = 0$. As time goes, the temperature distribution becomes more and more uniform.

In general, if the initial temperature distribution is given by the function $g(x, y, z)$, it can be shown (proof beyond the scope of the course) that the following function

$$u(x, y, z, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(x - u, y - v, z - w, t)g(u, v, w)\, dudvdw$$

satisfies the heat equation $\frac{\partial u}{\partial t} = k\Delta u$ with initial condition $g(x, y, z)$, meaning that

$$\lim_{t\to 0} u(x, y, z, t) = g(x, y, z).$$

In other words, the function $u(x, y, z, t)$ predicts how heat diffuses when given an initial temperature profile $g(x, y, z)$. However, the triple integral involved is generally difficult to be found explicitly.

4.8.3 Steady State

A temperature distribution $u(x, y, z, t)$ is said to be at steady state if it is independent of the time t, i.e. $\frac{\partial u}{\partial t} = 0$. For such a temperature distribution, the heat equation implies that

$$\Delta u = 0.$$

The above equation is often called the Laplace Equation.

Now given a closed surface S which encloses a solid region D. Using the Divergence Theorem, one can show that at a steady state if the temperature *on* the surface S is constant, then the temperature *inside* the surface S is also a constant. To argue this, we denote $u(x, y, z)$ to be a steady state temperature distribution (i.e. $\Delta u = 0$), and that:

$$u(x, y, z) = C \quad \text{for any } (x, y, z) \text{ on } S.$$

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Next we consider the vector field $(u-C)\nabla u$. We leave it as an exercise for readers to verify from the definition that:

$$\nabla \cdot ((u-C)\nabla u) = |\nabla u|^2 + (u-C)\Delta u.$$

At steady state, we have $\Delta u = 0$ and so $\nabla \cdot ((u-C)\nabla u) = |\nabla u|^2$. Next we integrate this result over D:

$$\iiint_D \nabla \cdot ((u-C)\nabla u)\ dV = \iiint_D |\nabla u|^2\ dV.$$

Applying the Divergence Theorem on the LHS, we get:

$$\oiint_S (u-C)\nabla u \cdot \hat{\mathbf{n}}\, dS = \iiint_D |\nabla u|^2\ dV.$$

From our assumption, we have $u = C$ for any point on S. Therefore, the integrand $(u-C)\nabla u \cdot \hat{\mathbf{n}}$ of the flux integral on LHS is zero, and so:

$$0 = \iiint_D |\nabla u|^2\ dV.$$

Since the integrand $|\nabla u|^2$ is non-negative, the only scenario for the above to happen is that

$$\nabla u(x,y,z) = 0 \quad \text{for any } (x,y,z) \text{ in } D.$$

Therefore, u must be a constant in the region D, and by continuity, this constant must be C.

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