MATH 2023 – Multivariable Calculus

Lecture #20 Worksheet — April 30, 2019

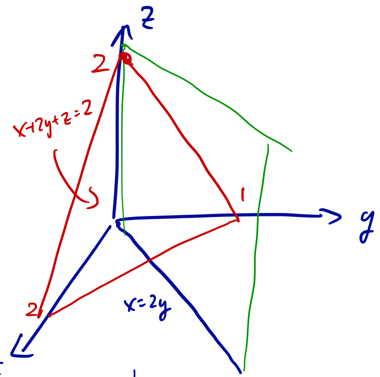
Problem 1. Set up the triple integrations over the solid E bounded by:

(a)

$$\begin{cases} x + 2y + z = 2 \\ x = 2y \\ x = 0 \\ z = 0 \end{cases}$$

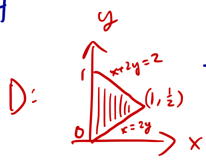
on yz-plane : $2y+z=2$

on xy-plane : $x+2y=2$

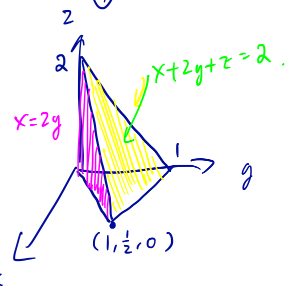


$$\iiint dV = \iint_D \left(\int dz \right) dA$$

D:



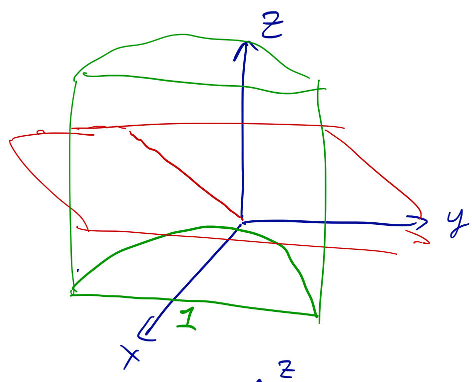
$$\int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} \left(\int_0^{2-x-2y} dz \right) dy\, dx$$

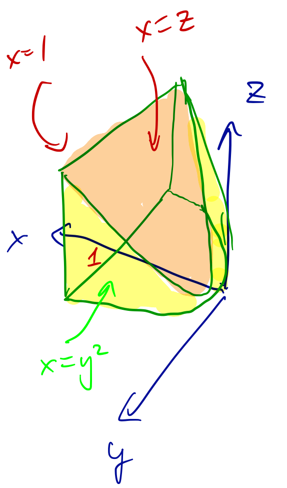


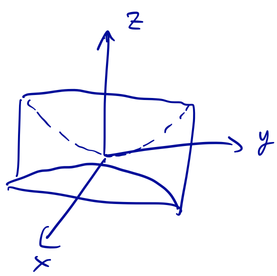
(b)

$$\begin{cases} x = y^2 \\ x = z \\ z = 0 \\ x = 1 \end{cases}$$

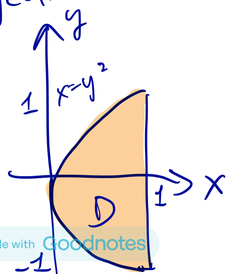
$$\int_{-1}^{1} \int_{y^2}^{1} \left(\int_0^{x} dz \right) dx\, dy$$







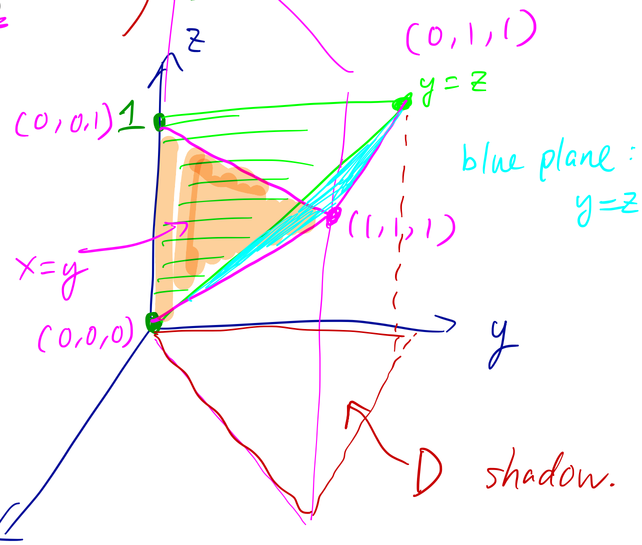
Projection



Problem 2. Change the order of integration

$$\int_0^1 \int_0^z \left(\int_0^y f(x,y,z)\,dx\right) dy\,dz$$

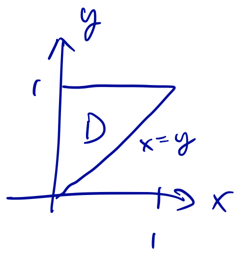
from $dxdydz$ to $dzdydx$



$$\iint_D \left(\int dz\right) dA$$

$$\parallel$$

$$\int_0^1 \int_x^1 \left(\int_y^1 dz\right) dy\,dx$$



Made with Goodnotes

Problem 3. Sketch the solid of integration, and rewrite it in the 5 other orders.

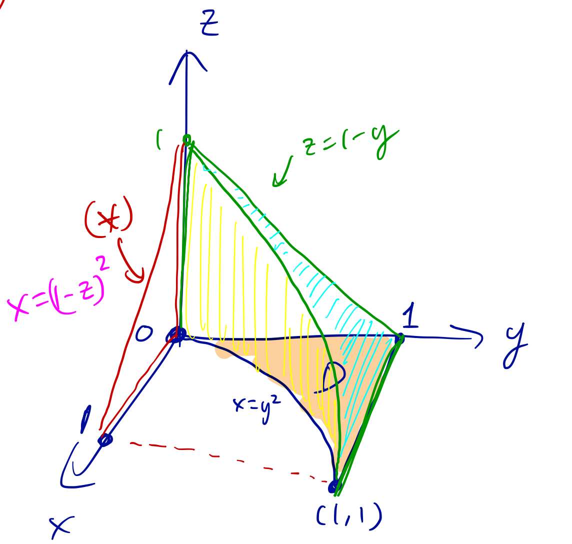
(a)

$$\int_0^1 \int_{\sqrt{x}}^1 \left(\int_0^{1-y} dz \right) dy\,dx$$

$z = 1-y$

$y = \sqrt{x}$

$x = y^2$

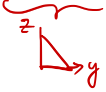


$(*)$ = Projection of $\begin{cases} z = 1-y \\ x = y^2 \end{cases}$

$\iff x = (1-z)^2$

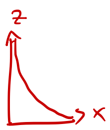
$$\int_0^1 \int_0^{y^2} \int_0^{1-y} dz\,dx\,dy$$

$$\int_0^1 \int_0^{1-z} \int_0^{y^2} dx\,dy\,dz$$



$$\int_0^1 \int_0^{1-y} \int_0^{y^2} dx\,dz\,dy$$

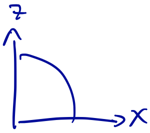
$$\int_0^1 \int_0^{(1-z)^2} \int_{\sqrt{x}}^{1-z} dy\,dx\,dz$$



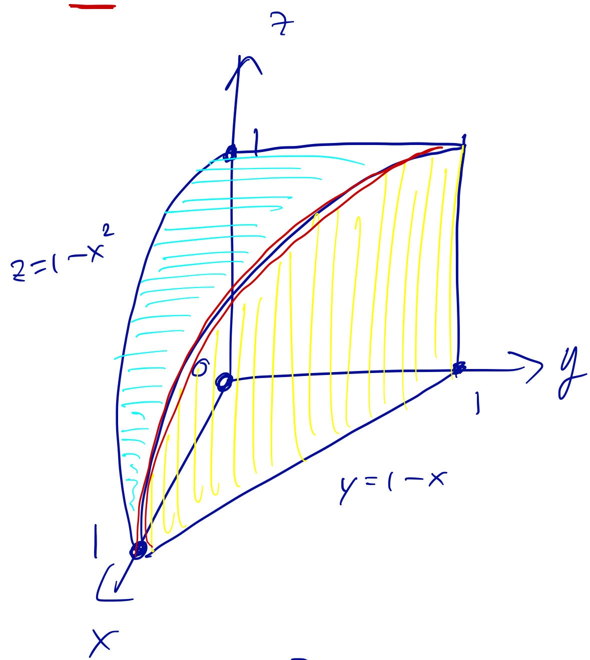
$$\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} dy\,dz\,dx$$

Sketch the solid of integration, and rewrite it in the 5 other orders.

(b)

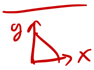


$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} dydzdx$$

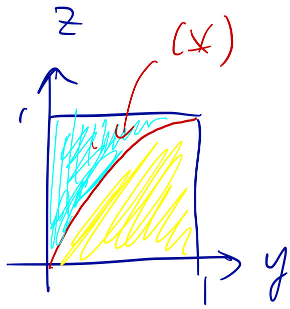


$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} dy\,dx\,dz$$

$$\int_0^1 \int_0^{1-x} \left(\int_0^{1-x^2} dz\right) dy\,dx$$



$$\int_0^1 \int_0^{1-y} \left(\int_0^{1-x^2} dz\right) dx\,dy$$



$$\iint \left(\int_0^{\sqrt{1-z}} dx\right) dA = dzdy$$

$$+ \iint \left(\int_0^{1-y} dx\right) dA$$

Projection of $\begin{cases} z=1-x^2 \\ y=1-x \end{cases}$

eliminate x variable.

$$z = 1-(1-y)^2$$

Problem 4. Evaluate

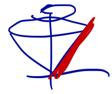
$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx$$

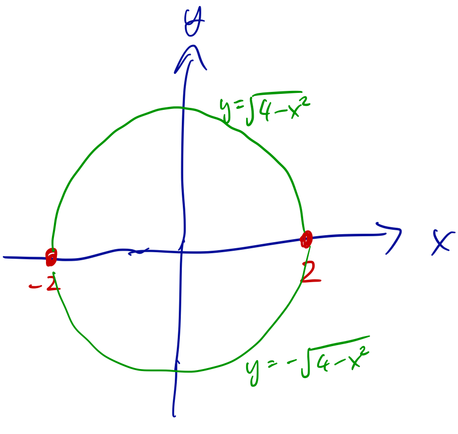
$y=\sqrt{4-x^2} \Rightarrow x^2+y^2=4$

$\boxed{z=\sqrt{x^2+y^2}}$

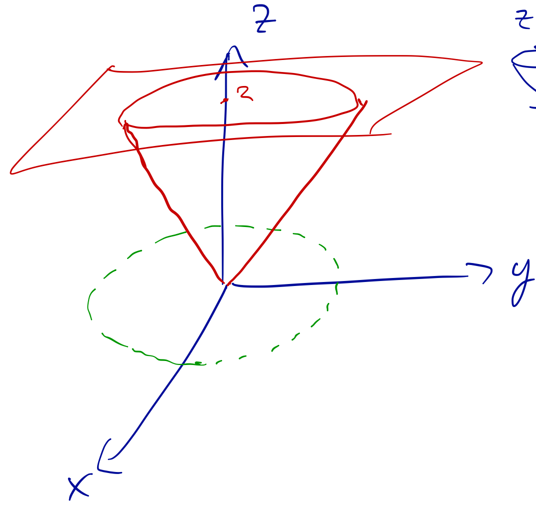
$\parallel$

$z=r$





base



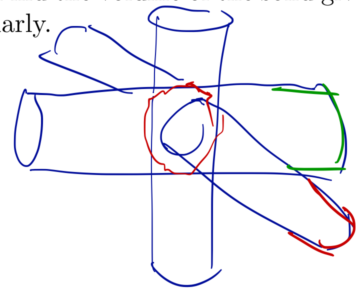
$$\int_0^{2\pi}\int_0^{2}\left(\int_r^{2} r^2\,dz\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2}\int_r^{2} r^3\,dz\,dr\,d\theta$$

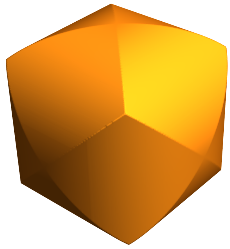
$$2\pi\int_0^{2} r^3(2-r)\,dr$$

$$=2\pi\left(\frac{2r^4}{4}-\frac{r^5}{5}\right)\Big|_0^2 = \frac{16\pi}{5}$$

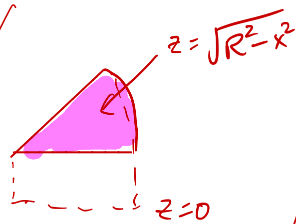
$$x \pm y \iff x^2 = y^2 \iff \begin{cases} x^2+y^2=R^2 \\ x^2+z^2=R^2 \\ y^2+z^2=R^2 \end{cases}$$

Problem 5. Find the volume of the solid given by intersecting 3 cylinders of radius R perpendicularly.

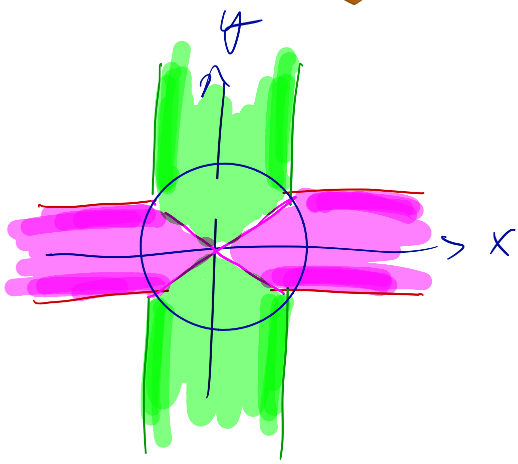




$16 \times \Big($



$\Big) \longleftarrow$

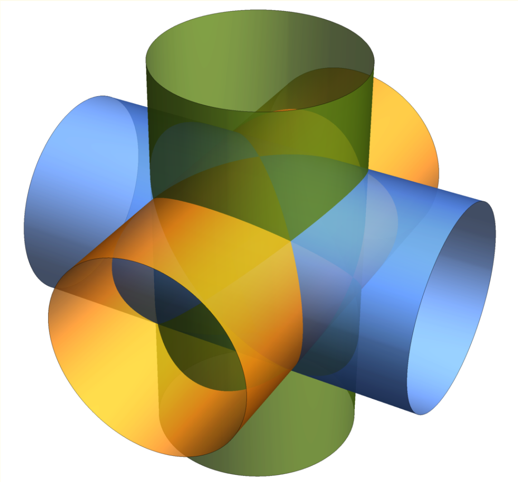


$$16 \int_0^{\frac{\pi}{4}} \int_0^R \left(\int_0^{\sqrt{R^2 - r^2\cos^2\theta}} dz \right) r\,dr\,d\theta$$

$$= 16 \int_0^{\frac{\pi}{4}} \int_0^R r\sqrt{R^2 - r^2\cos^2\theta}\; dr\,d\theta$$

$$= \ldots = 8R^3(2-\sqrt{2})$$

(MATH1014) (see next page)

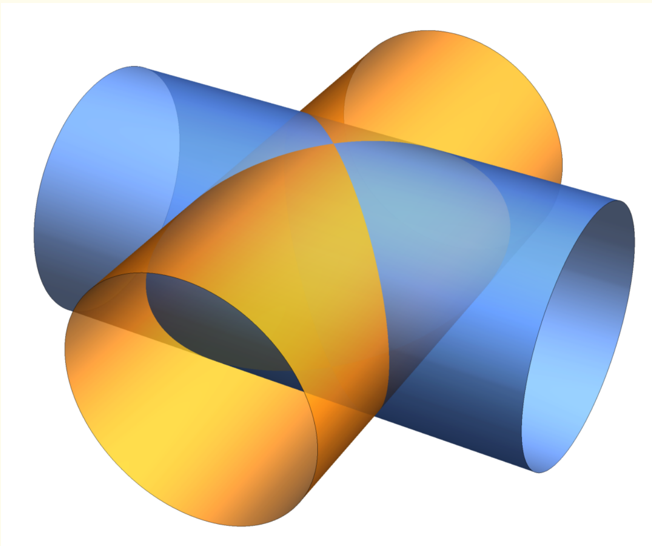


$$\int_0^{\frac{\pi}{4}}\int_0^R r\sqrt{R^2-r^2\cos^2\theta}\;dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{(R^2-r^2\cos^2\theta)^{3/2}}{-3\cos^2\theta}\Bigg|_{r=0}^{r=R} d\theta$$

$$-\frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{(R^2-R^2\cos^2\theta)^{3/2}}{\cos^2\theta} - \frac{R^3}{\cos^2\theta}\;d\theta$$

$$= -\frac{R^3}{3}\int_0^{\frac{\pi}{4}} \frac{\sin^3\theta}{\cos^2\theta} - \frac{1}{\cos^2\theta}\;d\theta$$



$$= -\frac{R^3}{3}\int_0^{\frac{\pi}{4}}\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right)d(\cos\theta) - \int_0^{\frac{\pi}{4}}\sec^2\theta\;d\theta$$

$$= -\frac{R^3}{3}\left(\left(\cos\theta+\frac{1}{\cos\theta}\right)\Bigg|_{\theta=0}^{\theta=\frac{\pi}{4}} - \tan\theta\Bigg|_0^{\frac{\pi}{4}}\right)$$

$$= -\frac{R^3}{3}\left(\frac{\sqrt{2}}{2}-1+\sqrt{2}-1-1\right)$$

$$= R^3\left(1-\frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow 16\left(R^3\left(1-\frac{\sqrt{2}}{2}\right)\right) = 8R^3(2-\sqrt{2})$$

Problem 5. Find the volume of the solid given by intersecting 3 cylinders of radius R perpendicularly.