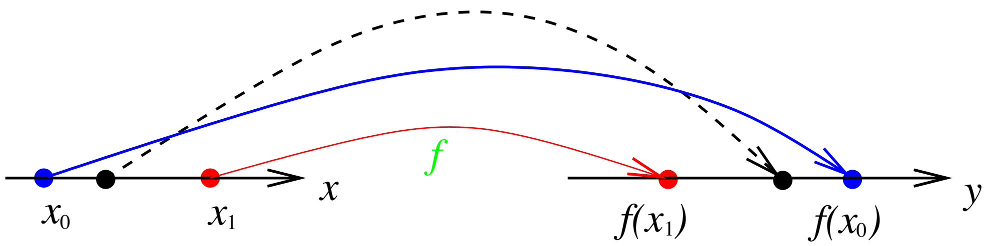
Chapter 12
Partial differentiation

Contents

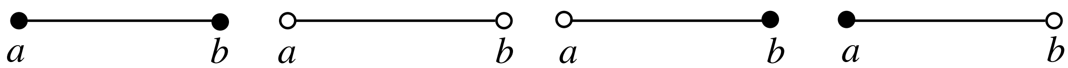
Review

Function of one variable: $f : \mathbb{R} \to \mathbb{R}$, $y = f(x)$. This is a curve in a plane.



domain of f is the set of allowable values for independent variable x

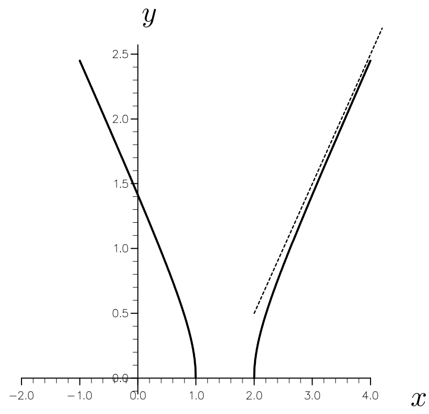
range: the set of all possible values of f



$[a, b]$ is a *closed* interval, (a, b) is an *open* interval, $(a, b]$ or $[a, b)$ are neither open nor closed.

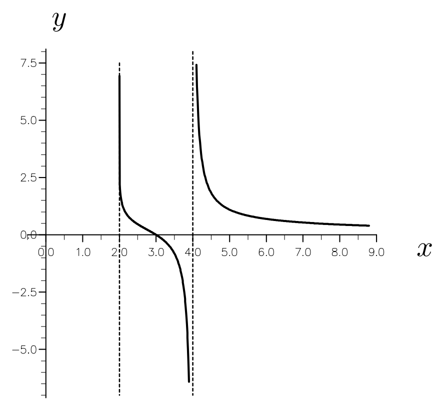
Ex. 1 Find the domain and the range of the function

$$f(x) = \sqrt{x^2 - 3x + 2}.$$



Ex. 2 Find the domain and the range of the function

$$f(x) = \frac{\ln(x-2)}{x-4}.$$

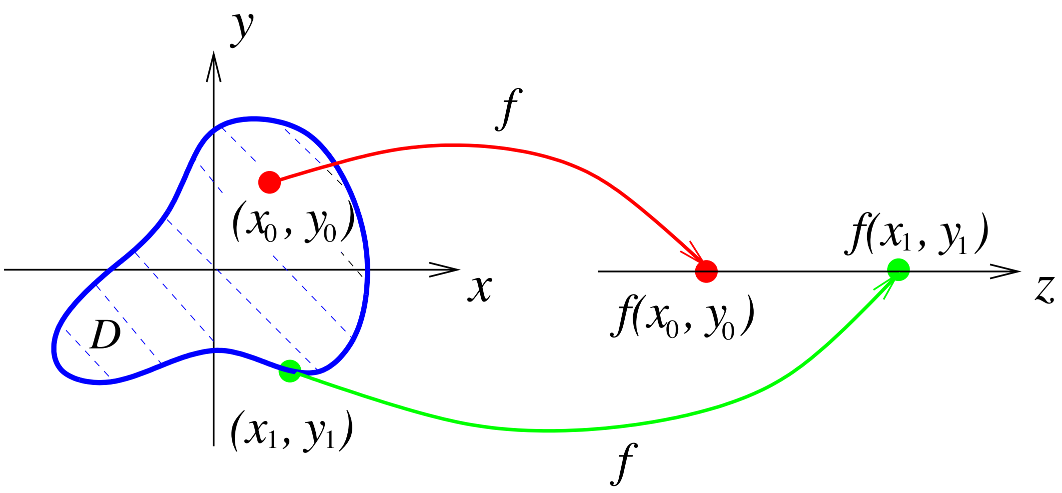


12.1 Function of two variables

$f : \mathbb{R}^2 \to \mathbb{R}$, $z = f(x, y)$. This is a surface in 3D-space.

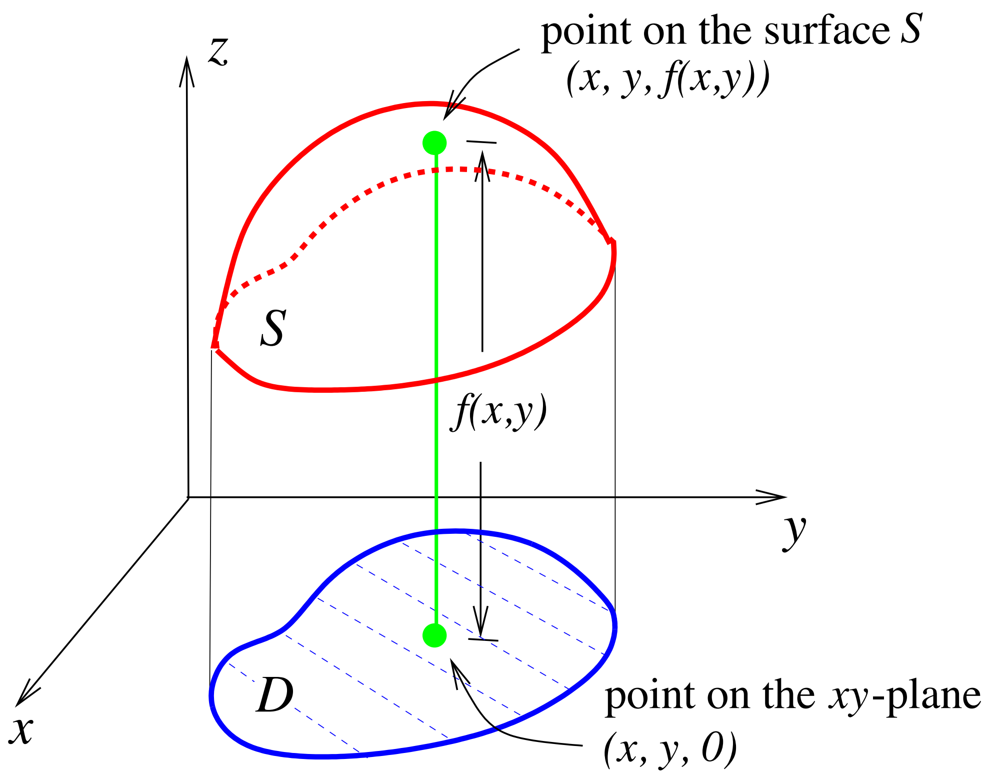
(x_0, y_0) – interior point

(x_1, y_1) – boundary point



A function of two (or n) variables is a rule that assigns to each ordered pair (x, y) (or $(x_1, x_2, \cdots, x_n)$) in D a *unique* real number $z = f(x, y)$ (or $z = f(x_1, x_2, \cdots, x_n)$).

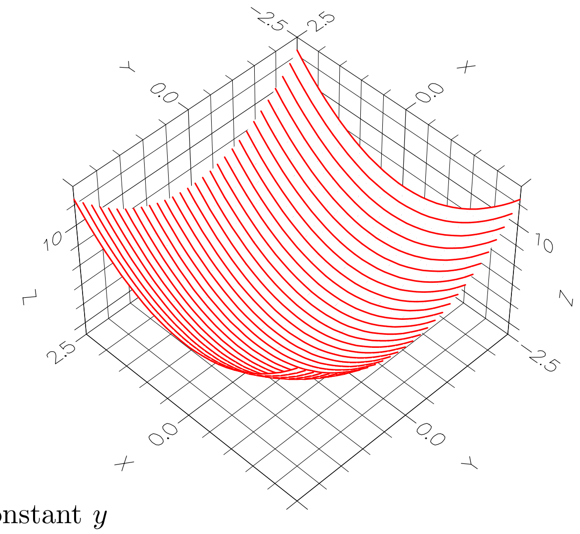
The *domain* of a function of n variables $f : \mathbb{R}^n \to \mathbb{R}$, $z = f(x_1, x_2, \cdots, x_n)$ is the set of points $(x_1, x_2, \cdots, x_n)$ such that the formula given by f can be evaluated at $(x_1, x_2, \cdots, x_n)$.



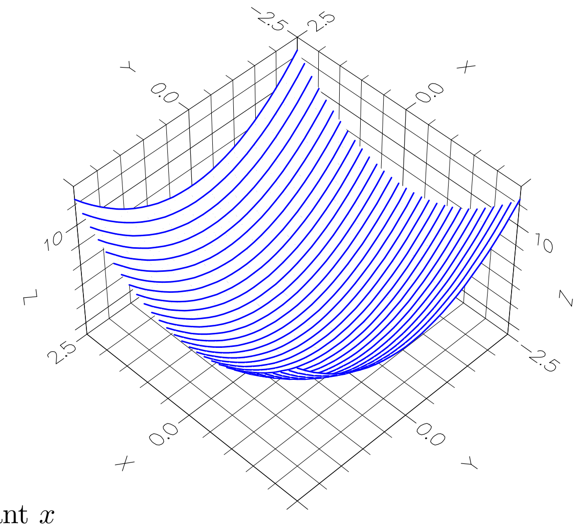
In general, the graph of a function of n variables is an n-dimensional surface in $\mathbb{R}^{n+1}$. We will not attempt to *draw* graphs of functions of more than two variables.

How to draw surfaces?

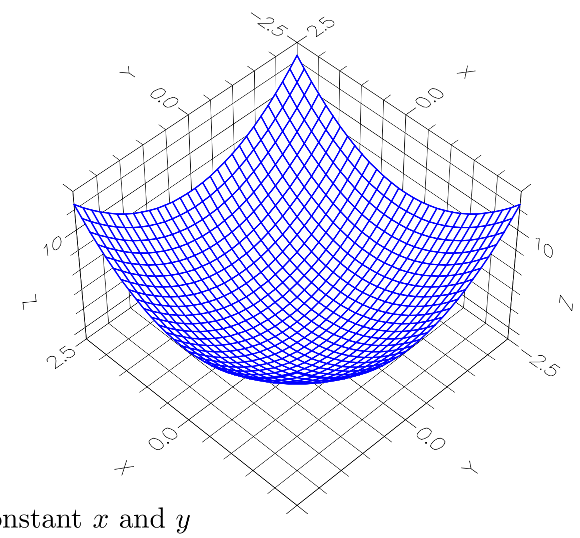
For example, $z = f(x, y) = x^2 + y^2$



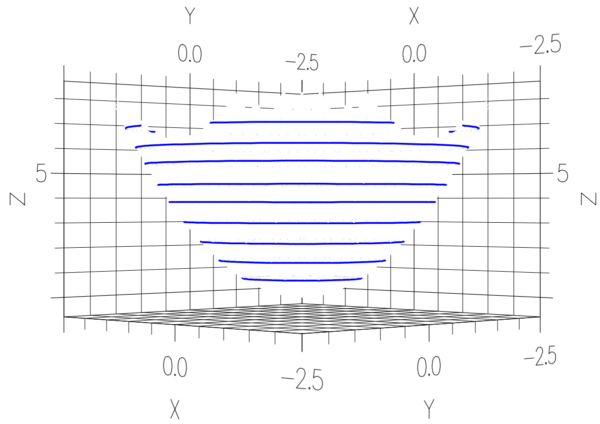
constant y



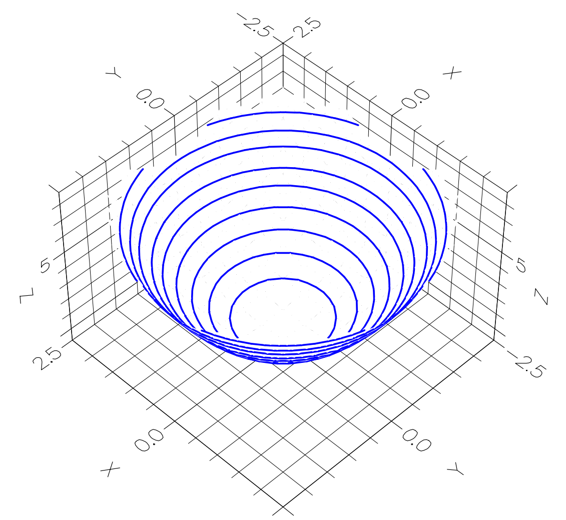
constant x

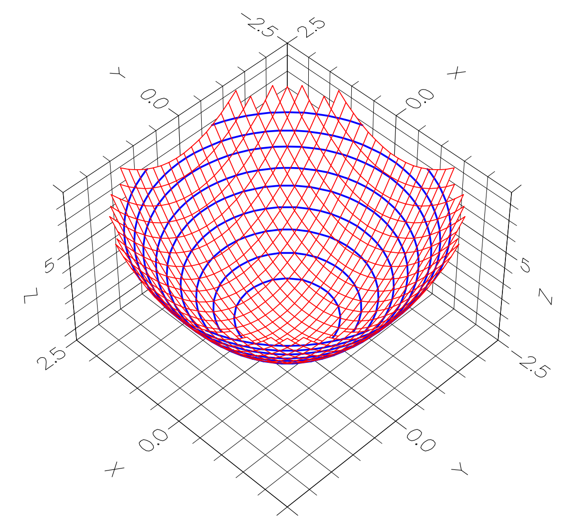


constant x and y



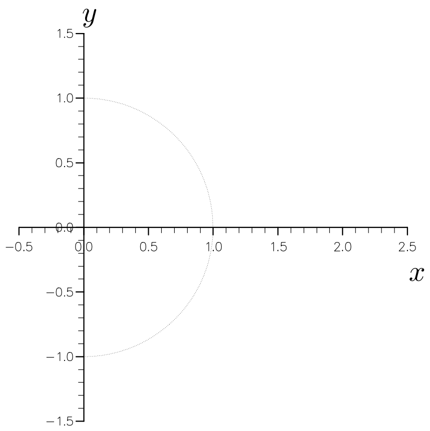
constant z

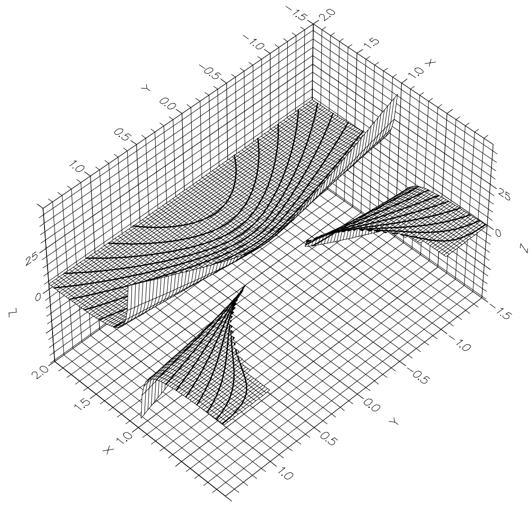




constant z (view at different angle)

Ex. 1.1 Find the domain and the range of the function $f(x, y) = \dfrac{\sqrt{x^2 + y^2 - 1}}{\ln x}$.



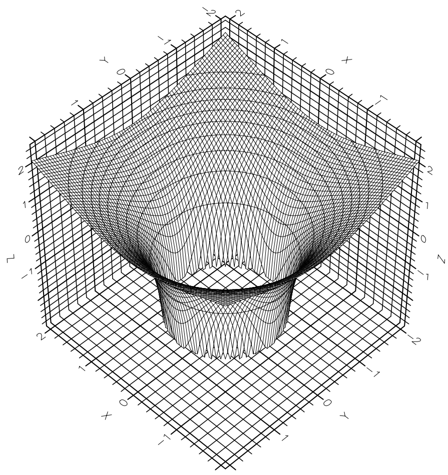


Ex. 1.2 Find the domain and the range of the function

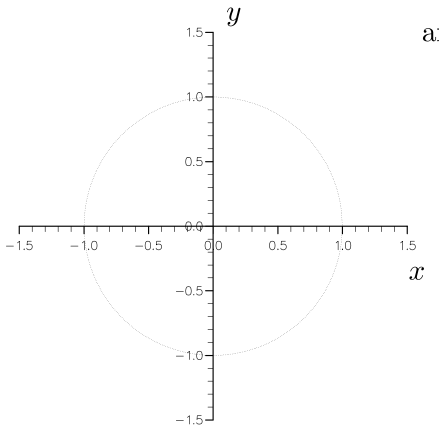
$$f(x, y) = x^2 + y^2.$$

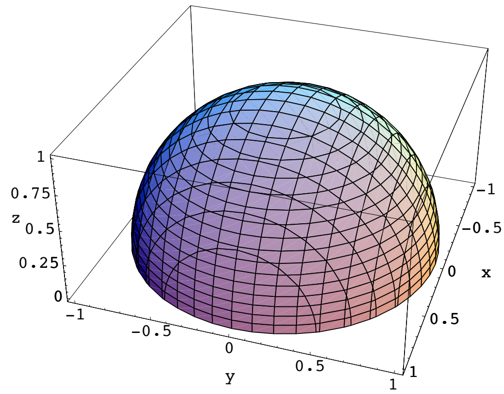
Ex. 1.3 Find the domain and the range of the function

$$f(x, y) = \ln(x^2 + y^2 - 1).$$

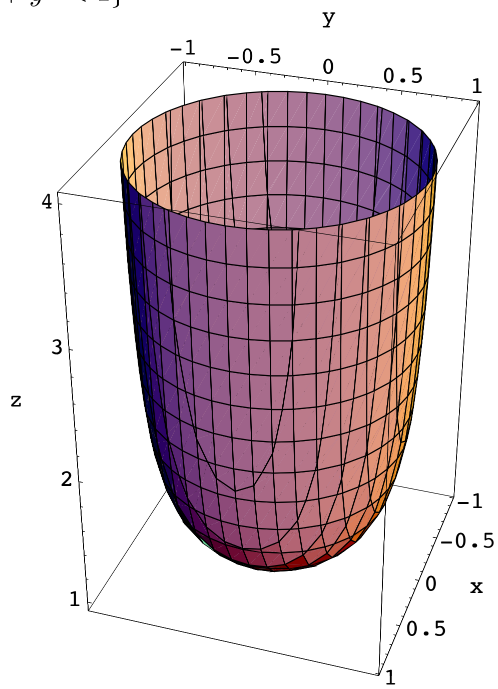


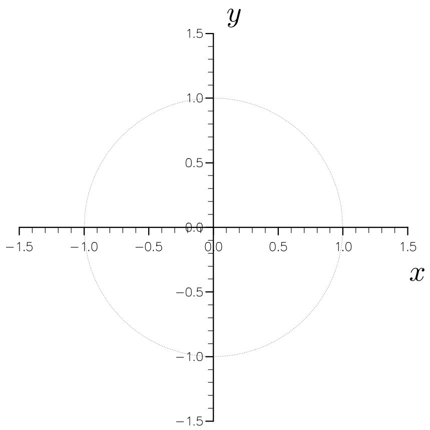
Ex. 1.4 $z = g(x,y) = \sqrt{1-x^2-y^2}$ has domain $\{(x,y)\,|\,x^2+y^2 \leqslant 1\}$ and range $z \in [0,1]$.



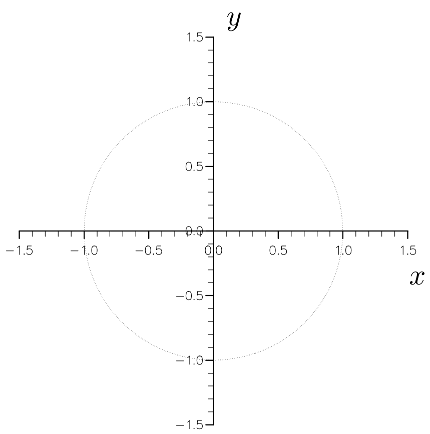


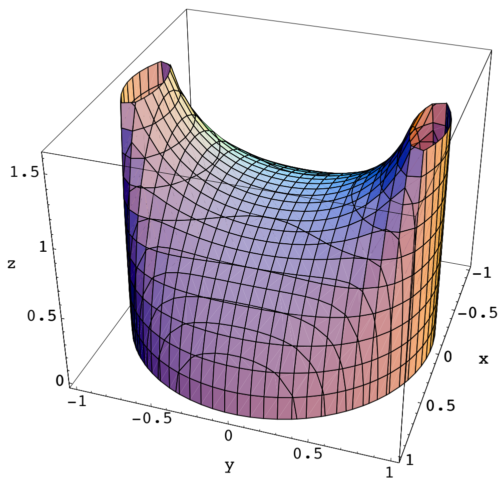
Ex. 1.5 $z = f(x,y) = \dfrac{1}{\sqrt{1-x^2-y^2}}$ has domain $\{(x,y)\,|\,x^2+y^2 < 1\}$ and range $z \in [1,\infty)$.





Ex. 1.6 $z = h(x,y) = \dfrac{\sqrt{1-x^2-y^2}}{1-y^2}$ has domain $\{(x,y)\,|\,x^2+y^2 \leqslant 1,\ y \neq \pm 1\}$ and range $z \in [0,\infty)$.





Definition

A set which has no boundary point is called *an open set* (e.g. domain of f).

A set has all boundary points is called *a closed set* (e.g. domain of g).

A set which has some, but not all, boundary point(s) is neither open nor closed (e.g. domain of h misses two boundary points $(0, \pm 1)$).

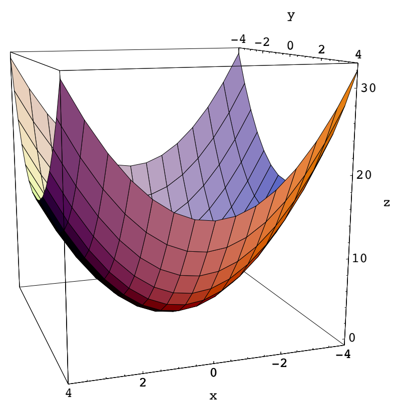
Level curves (surfaces)

The *level set* with constant c for a function $f : \mathbb{R}^n \to \mathbb{R}$ is $\{(x_1, x_2, \cdots, x_n) \,|\, f(x_1, x_2, \cdots, x_n) = c\}$.

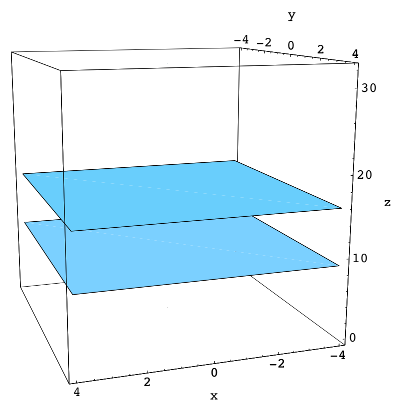
When $n = 2$: a level set is also called a *level curve* (e.g. $f(x, y) = x^2 + y^2 = c$), it shows where the graph of f has constant height c.

When $n = 3$: a level set is also called a *level surface* (e.g. $f(x, y, z) = x^2 + y^2 + z^2 = c$).

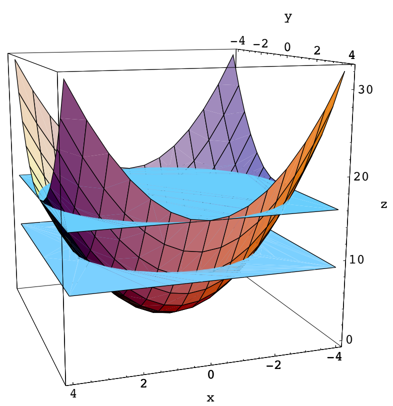
Ex. 1.7 Consider $f(x, y) = x^2 + y^2$. Show the level curves at 0, 1, 4, 9 and 16.

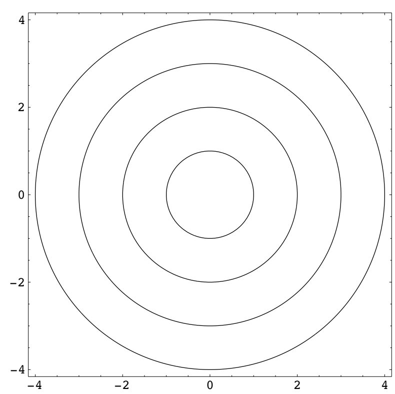


$z = f(x, y) = x^2 + y^2$

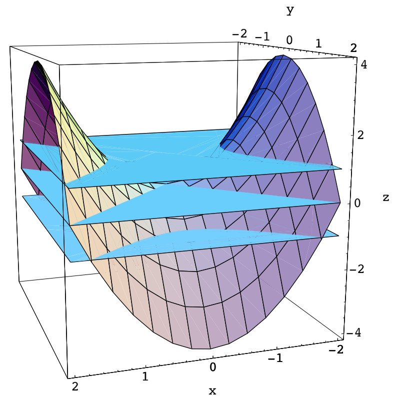


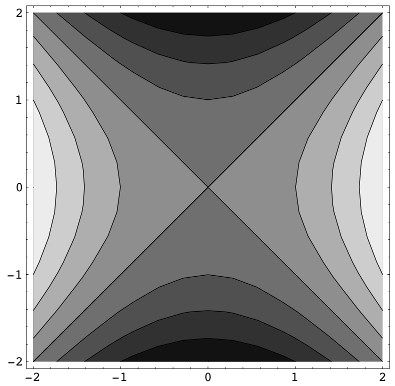
The planes $z = 9$ and $z = 16$





Ex. 1.8 Consider $f(x, y) = x^2 - y^2$. Show the level curves (the contours at levels) at -1, 0 and 1.





The level curves are all hyperbolas, with the exception of the level curve at height 0, which is a pair of intersecting lines.

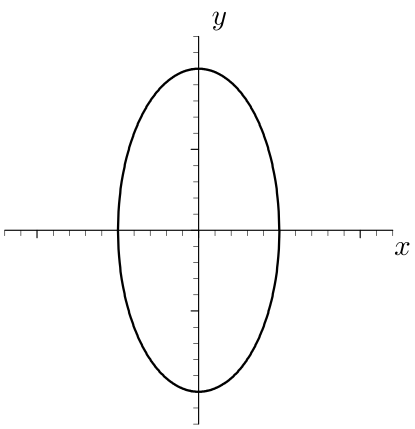
Ex. 1.9 Sketch some level curves of the function $f(x, y) = 4x^2 + y^2$.

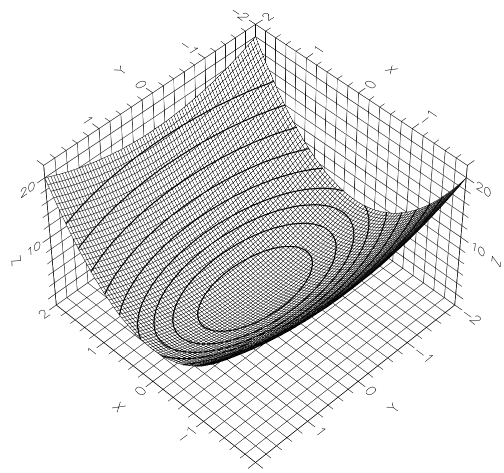
The level curves are $4x^2 + y^2 = k$ or $\dfrac{x^2}{k/4} + \dfrac{y^2}{k} = 1$.

For $k > 0$, describes a family of ellipses with semiaxes $\sqrt{k}/2$ and $\sqrt{k}$.

For $k = 0$,

For $k < 0$,





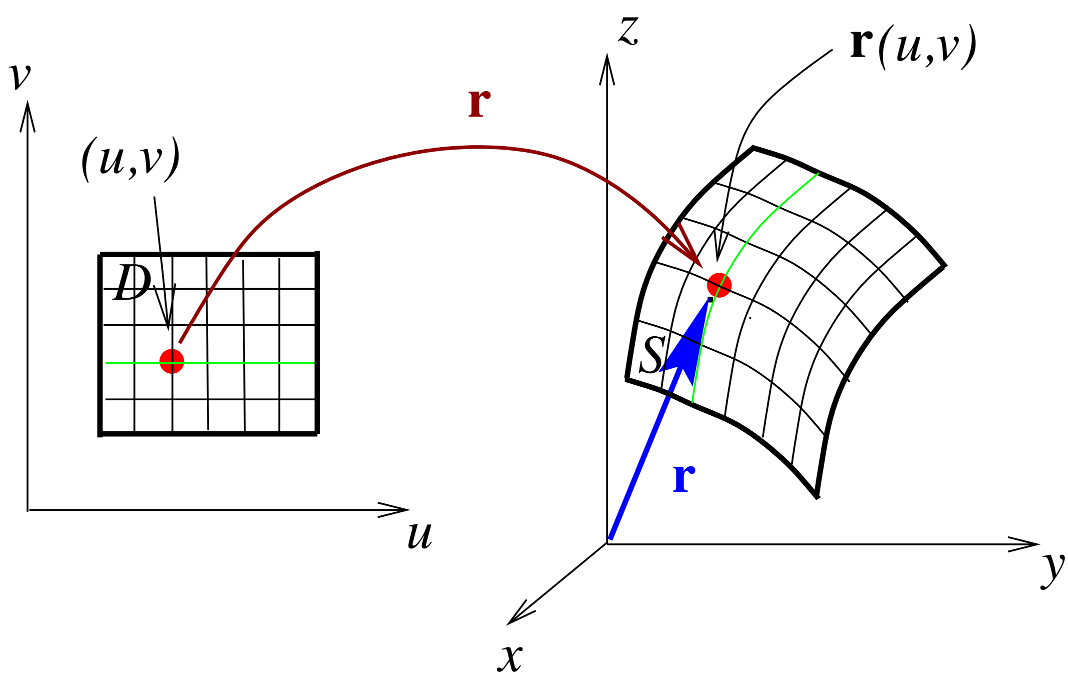
$z = f(x, y) = 4x^2 + y^2$

Ex. 1.10 Find the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2$.

Parametric representation of surfaces

A parametric surface in 3d-space is a continuous function $\mathbf{r}$ ($\mathbb{R}^2 \to \mathbb{R}^3$) defined on some region D in the uv-plane, and have values in 3d-space:

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad \text{where} \quad x, y, z : \mathbb{R}^2 \to \mathbb{R}.$$



Ex. 1.11 If the surface $z(x,y) = \dfrac{1}{1+x^2+y^2}$, write this surface in terms of (r, θ).

Ex. 1.12 Describe the parametric surface $\mathbf{r}(\theta, z) = 3\sin\theta\,\mathbf{i} + 2\cos\theta\,\mathbf{j} + 2z\,\mathbf{k}$, where $\theta \in [0, 2\pi]$, $z \in [1, 2]$.

Ex. 1.13 Describe the parametric surface

$$\mathbf{r}(s,t) = s\mathbf{a} + t\mathbf{b} + \mathbf{c}$$

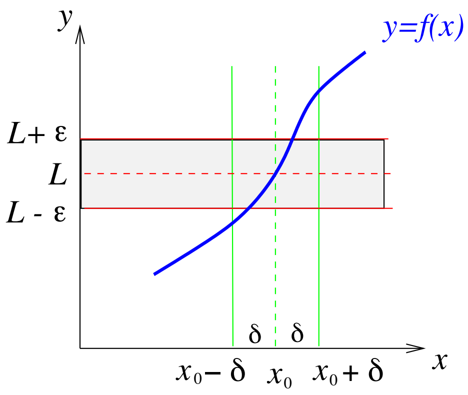
where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are constant vectors and $\mathbf{a}$ is not parallel to $\mathbf{b}$.

Exercise for students: Find the parametric equation of the cone $z^2 = x^2 + y^2$.

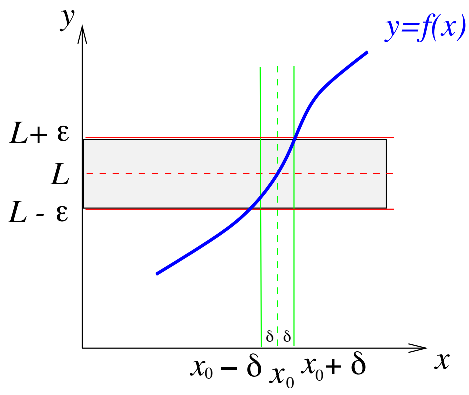
12.2 Limits and continuity

Recall function of *one variable*, there are only two directions from which x can approach x_0, $\lim_{x_0^-} f(x)$ or $\lim_{x_0^+} f(x)$. The function is *continuous* at $x = x_0$ if $\lim_{x_0^-} f(x) = \lim_{x_0^+} f(x)$ and $\lim_{x \to x_0} f(x) = f(x_0)$. $f(x)$ can be made as close as we like to L (says with ε units of L) by restricting x to lie within some sufficiently small interval δ.

$$|f(x) - L| < \varepsilon \quad \text{provided that} \quad |x - x_0| < \delta.$$



δ is not small enough



δ is small enough

Exercises for students

(a) $\displaystyle\lim_{x \to 0^+} \frac{1}{x}$

(b) $\displaystyle\lim_{x \to 0} \frac{\sin x}{x}$

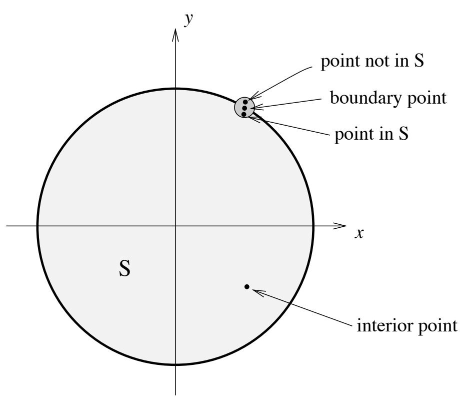
(c) $\displaystyle\lim_{x \to 1} \frac{\sqrt{x+3} - 2}{x - 1}$

(d) $\displaystyle\lim_{x \to \infty} \frac{e^{ax}}{x^n}$, where $a > 0$, $n > 0$.

(e) $\displaystyle\lim_{x \to 0^+} x^x$

(f) $\displaystyle\lim_{x \to \infty} \frac{(\log x)^n}{x}$

Let S be a set of points in the xy-plane.



Functions of two or three variables

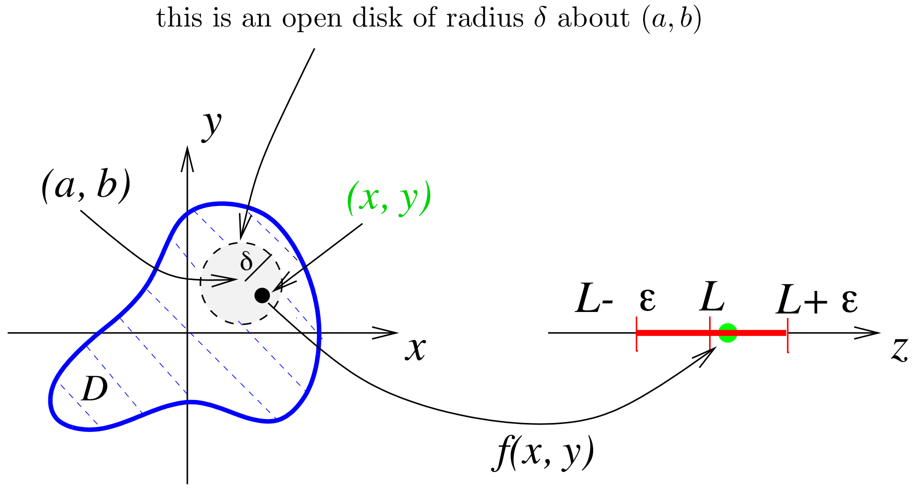
We write

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = L.$$

i.e. the *distance* between $f(x,y)$ and L can be made arbitrary small by making the *distance* from $\mathbf{x}$ to $\mathbf{x}^0$ sufficiently small (but not 0).

For $n=2$, $\lim_{(x,y)\to(a,b)} f(x,y) = L$, i.e., given every ε, there is a δ such that

$$|f(x,y)-L| < \varepsilon \quad \text{whenever} \quad \sqrt{(x-a)^2+(y-b)^2} < \delta.$$



If the limit *exists*, then $f(x,y)$ must approach the same limit *no matter how* (x,y) approaches (a,b).

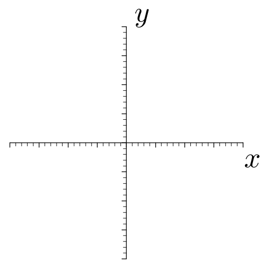
Let $f:\mathbb{R}^n \to \mathbb{R}$ $(n=2,3,\cdots)$ has a limit L as $(x_1,x_2,\cdots,x_n)\to(x_1^0,x_2^0,\cdots,x_n^0)$ if given every ε, there is a δ such that

$$|f(x_1,x_2,\cdots,x_n)-L| < \varepsilon \quad \text{whenever} \quad \|\mathbf{x}-\mathbf{x}^0\| < \delta.$$

Definition: If $f:\mathbb{R}^n\to\mathbb{R}$, then $f(\mathbf{x})$ is continuous at $\mathbf{x}^0$ if

$$\lim_{\mathbf{x}\to\mathbf{x}^0} f(\mathbf{x}) = f(\mathbf{x}^0).$$

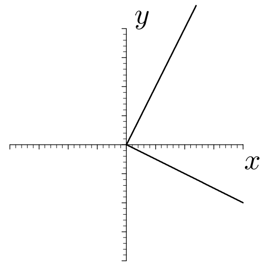
Ex. 2.1 If $f(x,y)=\dfrac{xy^2}{x^2+y^4}$, does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist?



As $(x,y)\to(0,0)$ along x-axis $(y=0)$ or along y-axis $(x=0)$,

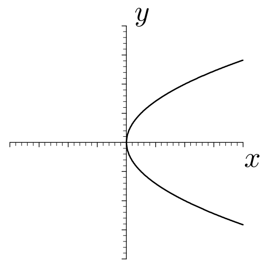
$$\frac{xy^2}{x^2+y^4}=0.$$

As $(x,y)\to(0,0)$ along the line $y=mx$ with $m\neq 0$, then



$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{\substack{x\to 0\\ m\neq 0}}\frac{xm^2x^2}{x^2+m^4x^4}=m^2\lim_{\substack{x\to 0\\ m\neq 0}}\frac{x}{1+m^4x^2}=0.$$

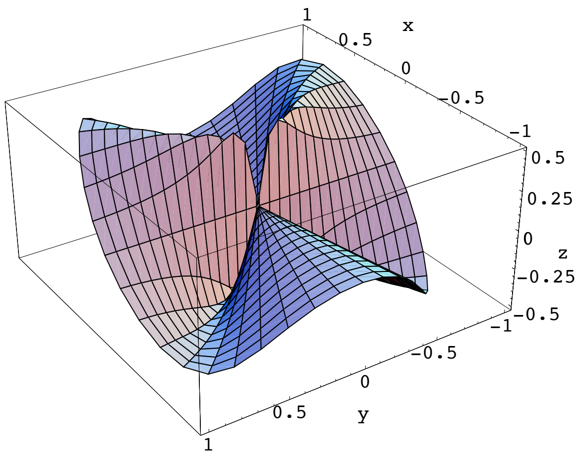
However, as $(x,y)\to(0,0)$ along the parabola $x=y^2$, we have

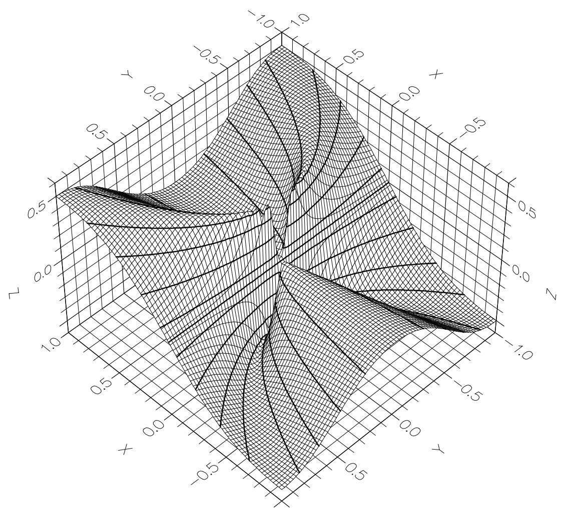


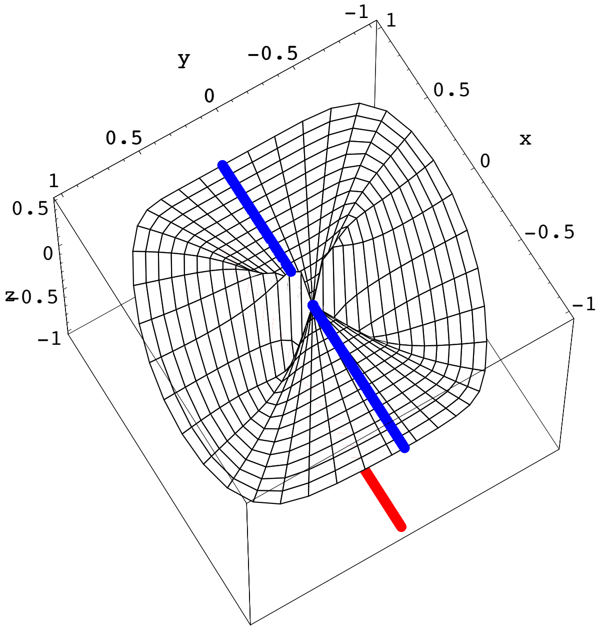
$$f(x,y)=f(y^2,y)=\frac{y^4}{2y^4}=\frac{1}{2},$$

$$\text{so}\quad f(x,y)\to\frac{1}{2}\quad\text{as}\quad(x,y)\to(0,0)\quad\text{along}\quad x=y^2.$$

So we obtained *different limits along different paths*, the limit doesn't exist.



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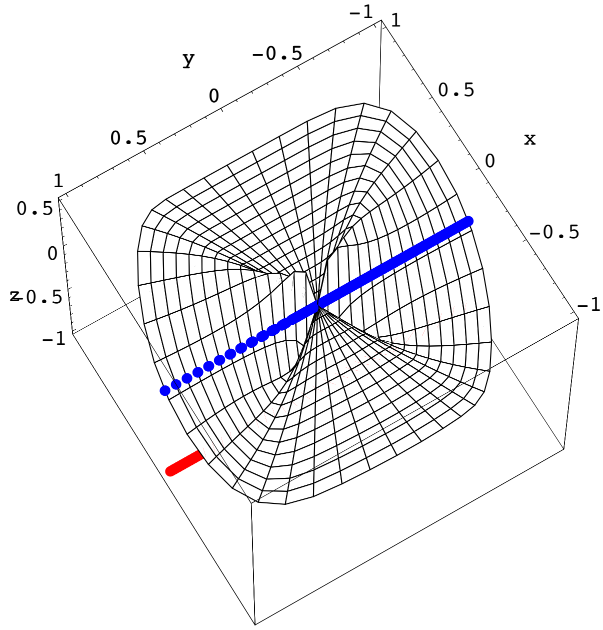
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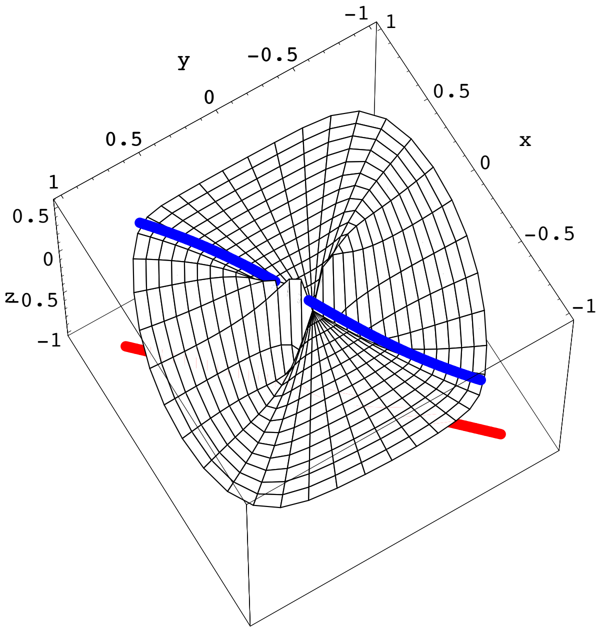
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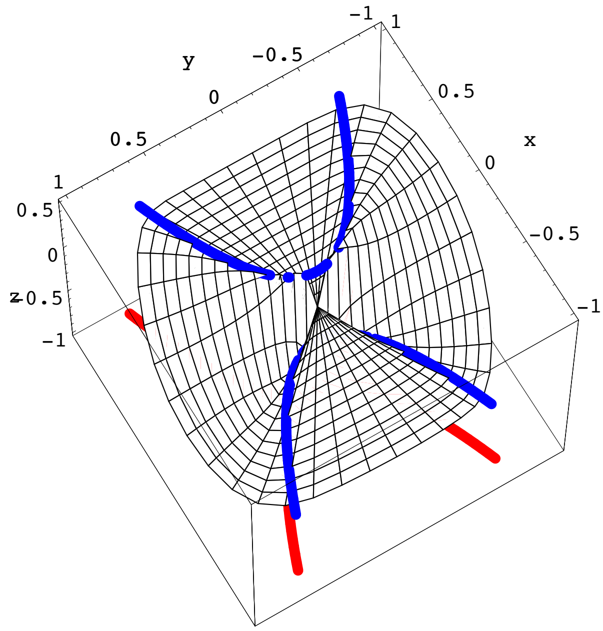
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Ex. 2.2 Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

As $(x,y) \to (0,0)$ along the line $y = mx$, then

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \frac{3m}{1+m^2} \lim_{x\to 0} x = 0,$$

but the limits along the parabola $y = x^2$ or $x = y^2$ also turn out to be 0, so we begin to suspect that the limit does exist.

Let $\varepsilon > 0$, find $\delta > 0$ such that

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{x^2+y^2} < \delta. \quad (\because \ |f(x,y) - L| < \varepsilon \quad \text{whenever} \quad \|\mathbf{x} - \mathbf{x}_0\| < \delta)$$

Note $\dfrac{3x^2|y|}{x^2+y^2} \leqslant 3|y| = 3\sqrt{y^2} \leqslant 3\sqrt{x^2+y^2}$,

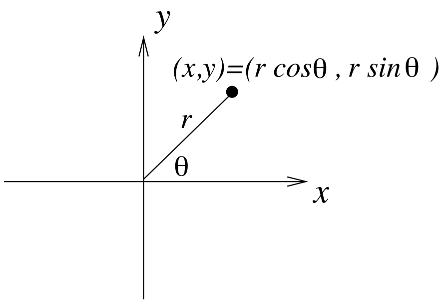
$\therefore$ if $\delta = \varepsilon/3$, and let $0 < \sqrt{x^2+y^2} < \delta$, then $\left|\dfrac{3x^2y}{x^2+y^2} - 0\right| \leqslant 3\delta = \varepsilon$.

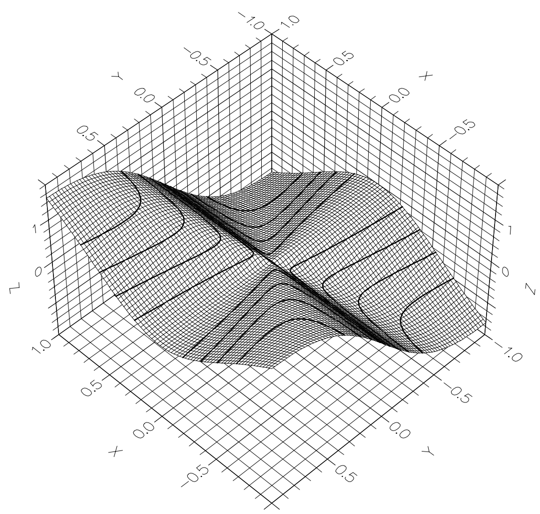
Here by definition, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

Alternatively, try *arbitrary* approach of $(x,\ y) \to (0,\ 0)$ in polar coord., $r = \sqrt{x^2+y^2} \to 0^+$

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| = \left|\frac{3r^3\cos^2\theta\sin\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right| = \left|3r\cos^2\theta\sin\theta\right| \leqslant$$

$\therefore$ the limit exists.





OR, try to examine the quantity

$$0 < \left|\frac{3x^2y}{x^2+y^2}\right|$$

Thus, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

Continuity

For function of single variable:

If $\lim_{x \to x_0} f(x) = L$ and $f(x_0) = L$, then $f(x)$ is *continuous* at $x = x_0$.

For function of n variables:

If $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L$ and $f(\mathbf{x}_0) = L$, then $f(\mathbf{x})$ is *continuous* at $\mathbf{x} = \mathbf{x}_0$.

Note also that if $f(x)$ and $g(x)$ are both *continuous* functions, then the composite function $f[g(x)]$ is also *continuous*.

If $f(x)$ is a *continuous* function of one variable and $g(x, y)$ is a *continuous* function of two variables, then

$$h(x, y) = f[g(x, y)]$$

is a *continuous* function of x and y.

From **Ex. 2.1**, define

$$g(x, y) = \begin{cases} f(x, y) = \dfrac{xy^2}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Here g is defined at $(0, 0)$, but g is still *discontinuous* at 0 because $\lim_{(x,y) \to (0,0)} g(x, y)$ does *not* exist.

From **Ex. 2.2**, define

$$h(x, y) = \begin{cases} f(x, y) = \dfrac{3x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

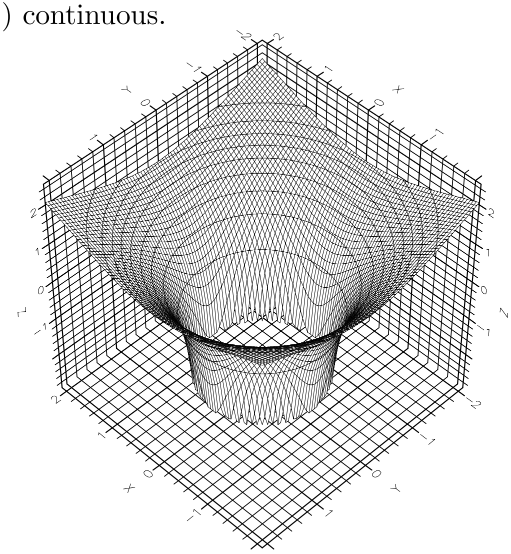
From Ex. 2.2, $\lim_{(x,y) \to (0,0)} h(x, y) = 0 = h(0, 0)$, therefore h is continuous at $(0, 0)$.

Also, we know f is continuous for $(x, y) \neq (0, 0)$ since it is equal to a rational function there, so $h(x, y)$ is *continuous* on $\mathbb{R}^2$.

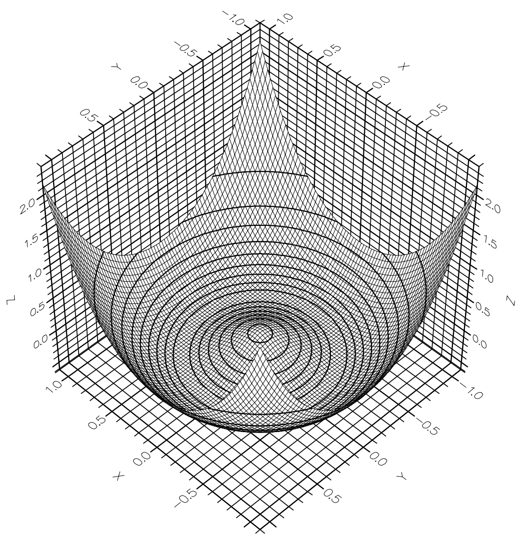
Theorem Let D be a closed, bounded set in the plane, and let f be continuous on D. Then $f(x, y)$ has both a maximum and a minimum on D.

Function of one variable	**Function of two variables**
$z = f(x)$	$z = f(x_1, x_2)$
$\lim\limits_{x \to x_0} f(x)$ exists if	$\lim\limits_{(x,x_0) \to (a,b)} f(x_1, x_2)$ exists if
$\lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x)$	as $(x_1, x_2) \to (a, b)$ [**independent** of path].
Then	Then
$\lim\limits_{x \to x_0} f(x) = L$ exists.	$\lim\limits_{(x_1,x_2) \to (a,b)} f(x_1, x_2) = L$ exists.
or	**or**
$\lvert f(x) - L \rvert < \varepsilon$ provided that	$\lvert f(x_1, x_2) - L \rvert < \varepsilon$ provided that
$\lvert x - x_0 \rvert < \delta$.	$\lVert \mathbf{x} - \mathbf{x}_0 \rVert < \delta$, where $\mathbf{x} = (x_1, x_2)$, $\mathbf{x}_0 = (a, b)$.
If $\lim\limits_{x \to x_0^-} f(x) = L^-$,	If $\lim\limits_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L_1$ along path 1
$\lim\limits_{x \to x_0^+} f(x) = L^+$,	$\lim\limits_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L_2$ along path 2,
and $L^- \neq L^+$, then	and $L_1 \neq L_2$, then
$\lim\limits_{x \to x_0} f(x)$ does not exist.	$\lim\limits_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$ does not exist.

Ex. 2.3 On what set is the function $h(x,y) = \ln(x^2+y^2-1)$ continuous.



Ex. 2.4 Find $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$.



Ex. 2.5 Let $f(x,y) = xy\ln(x^2+y^2)$. Is it possible to define $f(0,0)$ so that f will be continuous at $(0,0)$.

Ex. 2.6 Find $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$ (using spherical polar coordinates).

12.3 Partial derivatives

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant, is called the *partial derivative* of the function w.r.t. the variable. Partial derivative of $f(x, y)$ w.r.t. x and y are denoted by

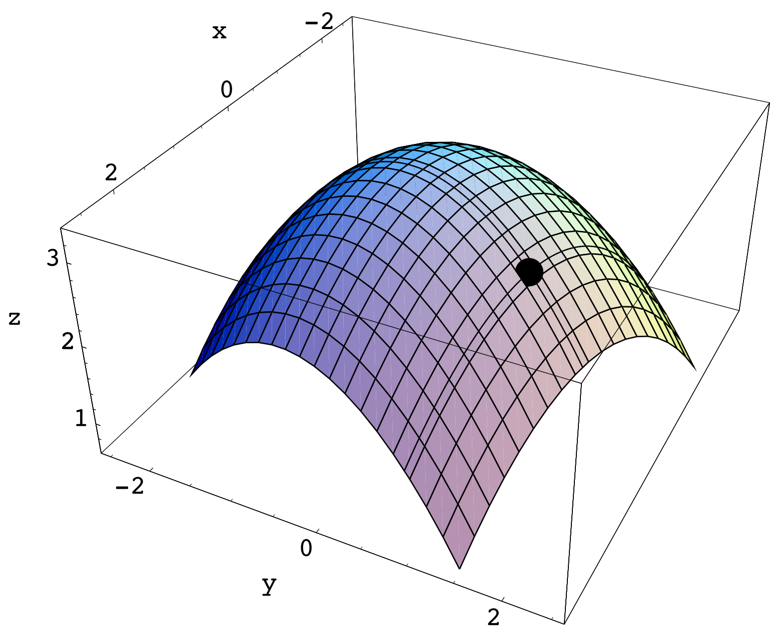
$$\frac{\partial f}{\partial x} \quad \text{or} \quad f_x \quad \text{or} \quad D_1 f; \qquad\qquad \frac{\partial f}{\partial y} \quad \text{or} \quad f_y \quad \text{or} \quad D_2 f.$$

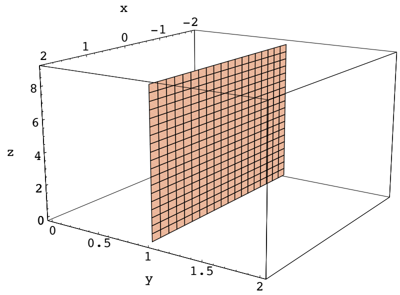
Definition: $f : \mathbb{R}^n \to \mathbb{R}$,

$$\boxed{\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, \cdots, x_i + \Delta x_i, \cdots, x_n) - f(x_1, x_2, \cdots, x_i, \cdots, x_n)}{\Delta x_i}}$$

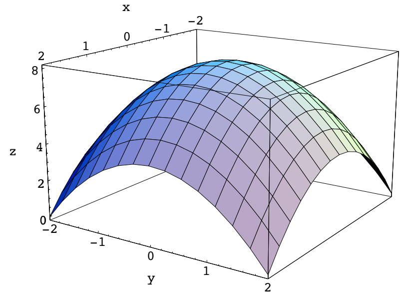
For example: If $f : \mathbb{R}^2 \to \mathbb{R}$, then to find f_x, regard y as a constant and differentiate $f(x, y)$ w.r.t. x.

Geometric interpretation of partial derivatives

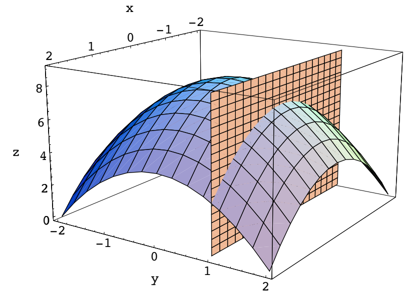




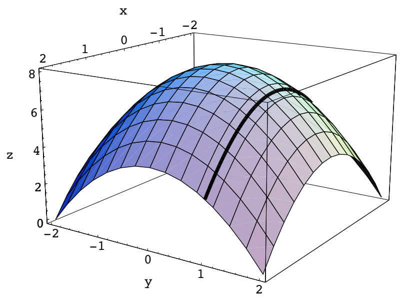
$y = a$



$z = f(x, y)$

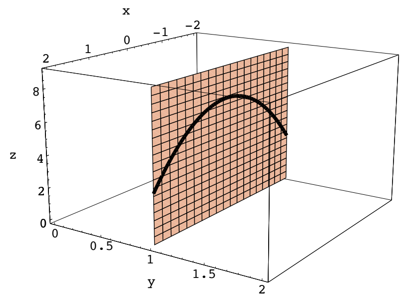


Two surfaces intersect



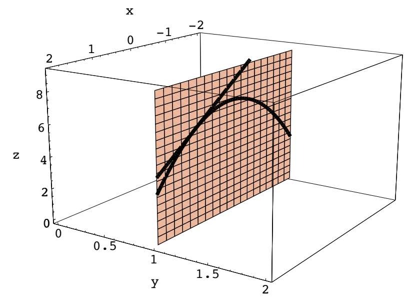
curve of intersection

lies on z



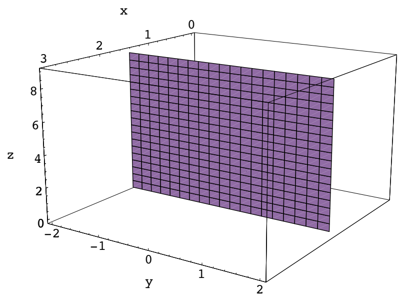
curve of intersection

lies on $y = a$

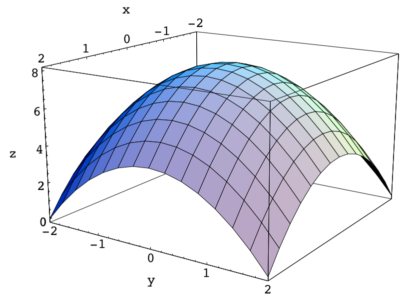


The slope of the tangent line

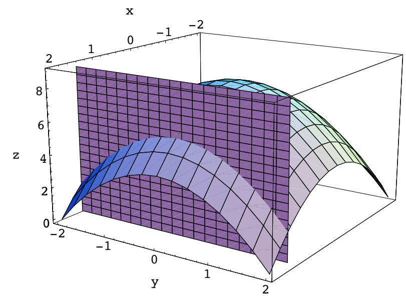
at (x_0, a) is $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,a)}$



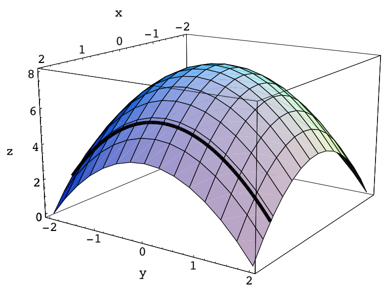
$x = b$



$z = f(x, y)$

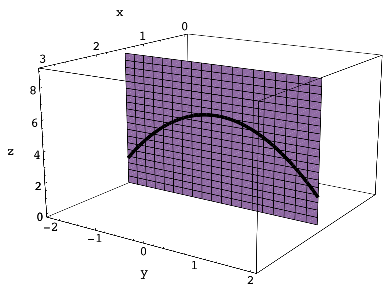


Two surfaces intersect



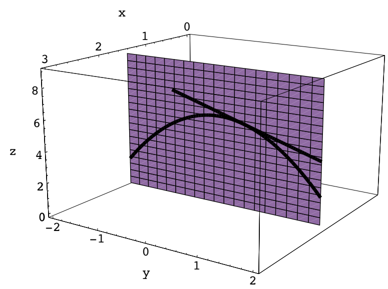
curve of intersection

lies on z



curve of intersection

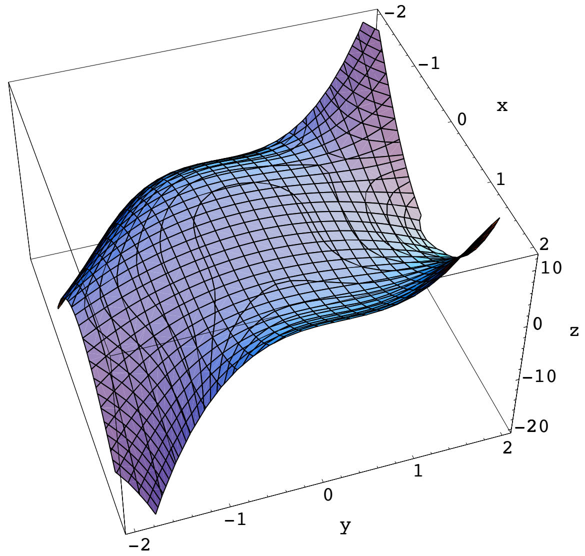
lies on $x = b$



The slope of the tangent line

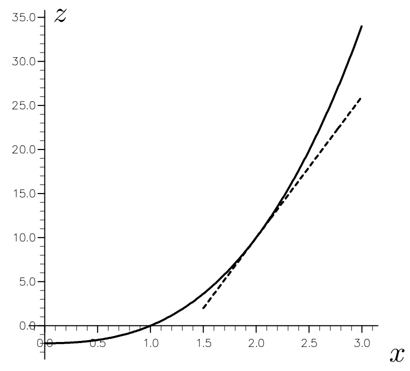
at (b, y_0) is $\left.\dfrac{\partial f}{\partial y}\right|_{(b,y_0)}$

Ex. 3.1 If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.



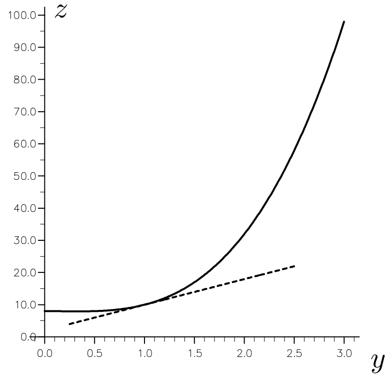
Geometric interpretation

In the plane $y = 1$



$f(x,1) = x^3 + x^2 - 2$

In the plane $x = 2$



$f(2,y) = 8 + 4y^3 - 2y^2$

Ex. 3.2 If $f(x,y,z) = e^{xy}\ln z$, find f_x, f_y and f_z.

Ex. 3.3 If $f(x,y) = (x^3+y^2)^{\frac{1}{3}}$, find $f_x(0,0)$.

12.4 Higher derivatives

Since f_x and f_y are functions of x and y, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$ and $(f_y)_y$, which are called the second partial derivatives. For example

$$(f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} \quad \text{and} \quad (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}.$$

Theorem

Let f be a function of two variables, if f_x, f_y, f_{xy} and f_{yx} are continuous on an open set (in a neighbour of (x,y)), then $f_{xy} = f_{yx}$ at each point of the set, i.e., the order of differentiation is immaterial (note that the proof is not required).

12.5 Differentiability and chain rules

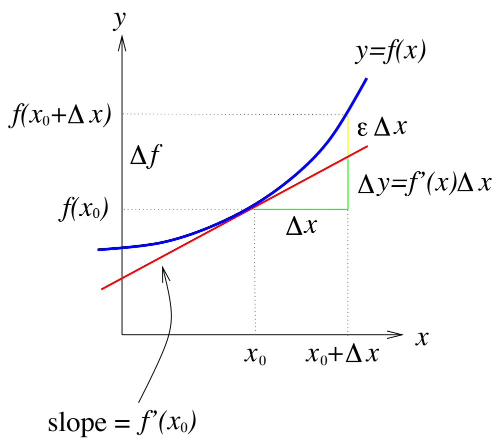
Recall function of *one* variable

$$\boxed{\Delta f = f(x_0+\Delta x) - f(x_0) = f'(x_0)\Delta x + \varepsilon \Delta x}$$

where ε is function of Δx, i.e., $\varepsilon = \varepsilon(\Delta x)$.

Definition: f is differentiable at x_0 if there exists a number $f'(x_0)$ such that

$$\Delta f = f'(x)\cdot \Delta x + \varepsilon \Delta x, \quad \text{where} \quad \varepsilon \to 0 \quad \text{as} \quad \Delta x \to 0.$$



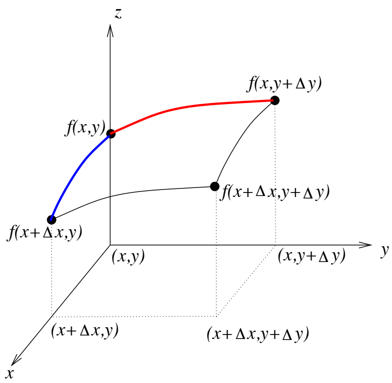
Similarly, $f : \mathbb{R}^2 \to \mathbb{R}$, $z = f(x, y)$ is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist, then the change

$$\begin{aligned}\Delta f &= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)\\ &= [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)] + [f(x_0, y_0 + \Delta y) - f(x_0, y_0)]\end{aligned}$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ satisfies an equation of the form

$$\boxed{\Delta f = (f_x(x_0, y_0) + \varepsilon_1)\Delta x + (f_y(x_0, y_0) + \varepsilon_2)\Delta y}$$

where $\varepsilon_1(\Delta x, \Delta y) \to 0$ and $\varepsilon_2(\Delta x, \Delta y) \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.



Alternatively, we say that the function $f(x, y)$ is **differentiable** at the point (a, b) if the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ exist and the function

$$h(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is a good linear approximation of f near (a, b). That is, if

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - h(x, y)}{\|(x, y) - (a, b)\|} = 0.$$

Moveover, if f is differentiable at (a, b), then the equation $z = h(x, y)$ defines the **tangent plane** to the graph of f at the point $(a, b, f(a, b))$. If f is differentiable at all points of its domain, then we simply say that f is **differentiable**.

It is not difficult now to see how to generalize to three (or more) variables: For a scalar-valued function of three variables to be differentiable at a point (a, b, c), we must have

(i) that the three partial derivatives exist at (a, b, c) and

(ii) that the function $h : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$h(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

is a good linear approximation to f near (a, b, c). In other words, (ii) means that

$$\lim_{(x,y,z)\to(a,b,c)} \frac{f(x, y, z) - h(x, y, z)}{\|(x, y, z) - (a, b, c)\|} = 0.$$

The passage from three variables to arbitrarily many is now straightforward.

Theorem: If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Theorem: If the first-order partial derivatives of f **exist** and are **continuous** on an open set (in a neighbour of (x, y)), then $f(x, y)$ is differentiable at (x, y) (see page 744).

Chain rule (Case 1)

If $x(t)$, $y(t)$ are differentiable at t_0, and $f(x, y)$ is differentiable at $(x_0, y_0) = (x(t_0), y(t_0))$, then

$$\begin{aligned}\frac{d}{dt}f(x(t), y(t))\Big|_{t=t_0} &= \lim_{\Delta t\to 0}\frac{\Delta f}{\Delta t} = \lim_{\Delta t\to 0}\frac{f(x(t_0+\Delta t), y(t_0+\Delta t)) - f(x(t_0), y(t_0))}{\Delta t}\\ &= \lim_{\Delta t\to 0}\left[f_x(x_0, y_0)\frac{\Delta x}{\Delta t} + f_y(x_0, y_0)\frac{\Delta y}{\Delta t} + \varepsilon_1\frac{\Delta x}{\Delta t} + \varepsilon_2\frac{\Delta y}{\Delta t}\right]\\ &= f_x(x_0, y_0)\frac{dx}{dt}\Big|_{t=t_0} + f_y(x_0, y_0)\frac{dy}{dt}\Big|_{t=t_0} \qquad \because\ \varepsilon_1 \text{ and } \varepsilon_2 \to 0 \text{ as } \Delta t \to 0.\end{aligned}$$

i.e. $\boxed{\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}}$

Chain rule (Case 2)

Suppose $z = f(x, y)$, $x = x(s, t)$, $y = y(s, t)$ are differentiable, then

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}} \quad \text{and} \quad \boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}$$

If $f(x_1, x_2, \cdots, x_n)$ and $x_i = x_i(t_1, t_2, \cdots, t_m)$ are differentiable, then

$$\frac{\partial f}{\partial t_j} = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t_j}, \quad \text{where} \quad j = 1, 2, 3, \ldots, m.$$

Ex. 5.1 Suppose that $z = x^2 y$, $x = t^2$, $y = t^3$. Find dz/dt.

Ex. 5.2 If $z = e^{xy}$, $x = 2u + v$ and $y = u/v$. Find z_u and z_v.

Ex. 5.3 If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation $tg_s + sg_t = 0$.

Let $x(s,t) = s^2 - t^2$ and $y(s,t) = t^2 - s^2$, then $g(s,t) = f(x,y)$

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} =$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} =$$

$$\therefore \quad tg_s + sg_t = 0.$$

Higher-order derivatives

Ex. 5.4 Calculate $\dfrac{\partial^2}{\partial x \partial y} f(x^2 - y^2, xy)$ in terms of partial derivatives of the function f. Assume that the partials of f are continuous.

Let $u(x, y) = x^2 - y^2, v(x, y) = xy$, then $f = f(u, v)$.

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \\ &= f_u(-2y) + f_v(x) = -2yf_u + xf_v\end{aligned}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) =$$

Ex. 5.5 Show that any function of the form $z(x, t) = f(x + at) + g(x - at)$ is a solution of the wave equation $\dfrac{\partial^2 z}{\partial t^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$.

Hint: Let $u(x, t) = x + at$, $v(x, t) = x - at$, then $z = f(u) + g(v)$.

12.6 Total differentials

If $z = f(x, y)$ and let Δx and Δy be increments given to x and y, then the increment in z (total differentials) is

$$\begin{aligned}\Delta z &= f(x+\Delta x, y+\Delta y) - f(x,y) \\ &= \frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\cdot \Delta x + \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}\cdot \Delta y\end{aligned}$$

In the limit $\Delta x \to 0$, $\Delta y \to 0$, we have

$$\boxed{\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y}$$

If $z = f(x_1, x_2, x_3, \cdots, x_n)$, then $\Delta z = \displaystyle\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Delta x_i$.

Ex. 6.1 Find the approximate value of the function $f(x, y) = x^2y^3$ at $(3.1, 0.9)$.

$$\begin{aligned}f(x+\Delta x, y+\Delta y) &\simeq f(x,y) + \Delta x f_x + \Delta y f_y \\ &= x^2y^3 + \Delta x(2xy^3) + \Delta y(x^2\cdot 3y^2).\end{aligned}$$

when $x = 3$, $y = 1$, $\Delta x = 0.1$, $\Delta y = -0.1$, then

$$f(3.1, 0.9) \simeq 9 + 0.6 - 2.7 = 6.9.$$

$$f(3.1, 0.9) = 7.00569. \qquad \text{(exact value)}.$$

Ex. 6.2 If the radius of a right-circular cone is changed from 10cm to 10.1cm and the height is changed from 1m to 0.99m, use differential to approximate the change in its volume.

12.7 Gradients of functions and directional derivatives

Recall that

1. If $y = f(x)$, then the *gradient* of the function at $x = x_0$ is

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f'(x_0).$$

2. Suppose that C is the graph of a vector-valued function $\mathbf{r}(t)$ in 2D or 3D, then the derivative $\mathbf{r}'(t_0)$ is tangent to C (the *gradient* of the curve C at $t = t_0$) and points in the direction of increasing parameter t.

3. If $z = f(x, y)$, then the partial derivative $f_x(x_0, y_0)$ can be interpreted as the *gradient* (change in z per unit increase in x) of the tangent line to the curve C at the point (x_0, y_0), where

$$C: \qquad z = f(x, y_0) = g(x).$$

Similarly for $f_y(x_0, y_0)$.

Gradient of the function f

The definition of the gradient of a function of n variables is

$$\text{grad}\, f = \nabla f(x_1, x_2, \cdots, x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right),$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \cdots, \frac{\partial}{\partial x_n}\right)$.

∇f is a vector, i.e. $(f_{x_1}, f_{x_2}, \ldots, f_{x_n})$ are the components of ∇f in the $x_1, x_2, \ldots, x_n$ directions.

Ex. 7.1 Let $z = f(x, y) = x^2 + y^2$, find the gradient of f at $(1, 1)$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Ex. 7.2 Let $f(x, y, z) = x^2y + yz^2$. Find ∇f.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

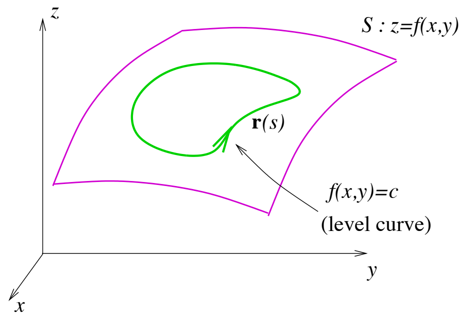
Theorem:

If f is a function of two variables, then $\nabla f(x_0, y_0)$ is *normal* to the *level curve* of f through (x_0, y_0).

Proof:

Suppose $z = f(x, y) = c$ (level curve since c is a constant).

Let $\mathbf{r}(s) = (x(s), y(s))$ be a parametric curve of the level curve, then



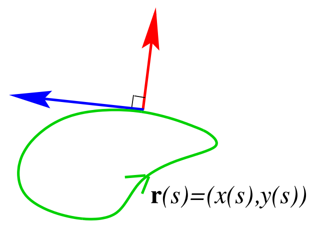
$$f(x(s),\, y(s)) = c.$$

Differentiate it w.r.t. s, we have

$$\frac{df}{ds} = \frac{dc}{ds} = 0.$$

By the Chain rule

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = 0$$



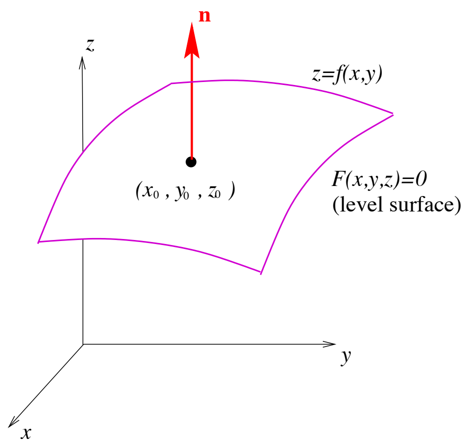
Similarly, if f is a function of three variables, then $\nabla f(x_0, y_0, z_0)$ is *normal* to the *level surface* of f through (x_0, y_0, z_0).

Normal vector to the surface $z = f(x, y)$

If $z = f(x, y)$, then redefine $F(x, y, z) = f(x, y) - z = 0$, then

$$\mathbf{n} = \nabla F(x_0, y_0, z_0)$$

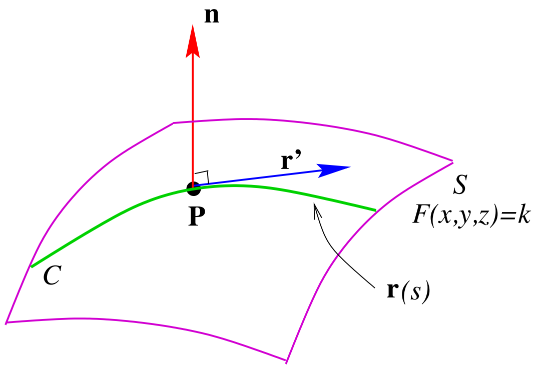
is a vector which is normal to the surface of $z = f(x, y)$ at (x_0, y_0, z_0).



In general, if f is differentiable at $(x_1,\, x_2,\, x_3,\, \ldots,\, x_n)$, then $\nabla f(x_1^0,\, x_2^0,\, x_3^0,\, \ldots,\, x_n^0)$ is **normal** to the level surface of $f(x_1,\, x_2,\, x_3,\, \ldots,\, x_n)$, through $(x_1^0,\, x_2^0,\, x_3^0,\, \ldots,\, x_n^0)$. (*Proof*: By using Chain rule.)

Tangent plane

Suppose S is a surface with equation $F(x, y, z) = k$ (e.g. $z = f(x, y) \Rightarrow F(x, y, z) = f(x, y) - z = 0$), and let $\mathbf{P} = (x_0, y_0, z_0)$ be a point on S. Let C be *any* curve that lies on the surface S and passes through the point $\mathbf{P}$.



Parametric curve C: $\quad \mathbf{r}(s) = x(s)\,\mathbf{i} + y(s)\,\mathbf{j} + z(s)\,\mathbf{k}$.

At point $\mathbf{P}$: $\quad \mathbf{r}(s_0) = x(s_0)\,\mathbf{i} + y(s_0)\,\mathbf{j} + z(s_0)\,\mathbf{k} = x_0\,\mathbf{i} + y_0\,\mathbf{j} + z_0\,\mathbf{k}$,

also $F(x(s),\, y(s),\, z(s)) = F(\mathbf{r}) \;= k$ since C lies on S.

Chain rule $\dfrac{\partial F}{\partial x}\dfrac{dx}{ds} + \dfrac{\partial F}{\partial y}\dfrac{dy}{ds} + \dfrac{\partial F}{\partial z}\dfrac{dz}{ds} = \dfrac{d}{ds}k = 0.$

$$\boldsymbol{\nabla} F\cdot \mathbf{r}'(s) = 0, \quad \text{where} \quad \mathbf{r}'(s) = (x'(s), y'(s), z'(s)) \quad \text{and} \quad \boldsymbol{\nabla} = \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

In particular, when $s = s_0$, we have

$\nabla F\cdot \mathbf{r}'(s_0) = 0$, this implies that the tangent vector $\mathbf{r}'(s_0)$ to *any* curve C on S that passes through $\mathbf{P}$ is perpendicular to

$$\boldsymbol{\nabla} F(\mathbf{r}(s_0)).$$

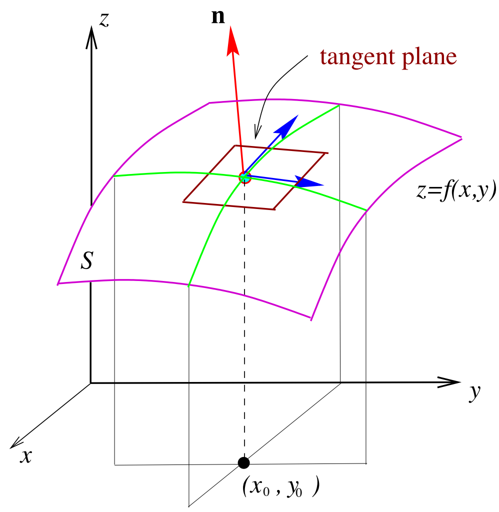
Since the tangent plane to $F(x, y, z) = k$ of $\mathbf{P}$ is the plane that passes through $\mathbf{P}$ and has normal vector $\nabla F(\mathbf{r}(s_0))$,

$$\therefore \quad \nabla F(r(s_0))\cdot (\mathbf{r} - \mathbf{r}(s_0)) = 0.$$

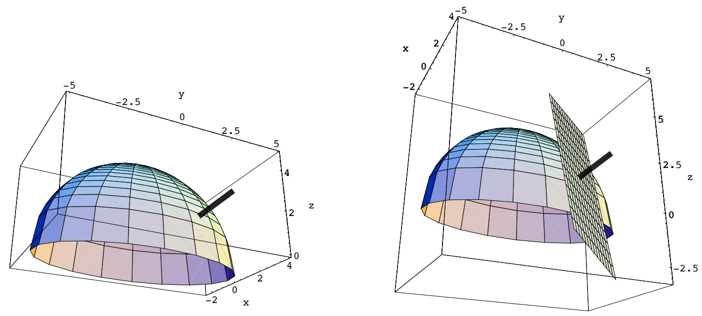
In the case $F(x, y, z) = f(x, y) - z = 0$, the equation of the tangent plane is

$$(f_x,\, f_y,\, -1)\cdot (x - x_0,\, y - y_0,\, z - z_0) = 0$$

$$\therefore \qquad z - z_0 = f_x(x_0,\, y_0)(x - x_0) + f_y(x_0,\, y_0)(y - y_0).$$



Ex. 7.3 Find the equation of the tangent plane $4x^2 + y^2 + z^2 = 24$ at $(2, 2, 2)$.



$$z = +\sqrt{24 - 4x^2 - y^2}$$

Exercise for students:

If $h(t) = f(\mathbf{r}(t))$, where $\mathbf{r}(t) = (x_1(t), x_2(t), \cdots, x_n(t))$, show that $\dfrac{dh}{dt} = \nabla f \cdot \mathbf{r}'(t)$.

[Note that $\nabla = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \cdots, \dfrac{\partial}{\partial x_n}\right)$]

Ex. 7.4 Find the normal vector to the surface $z = x^2 + y^2$ at (a) (1, 1, 2) and (b) (0, 0, 0).

Redefine the surface as $F(x, y, z) = x^2 + y^2 - z = 0$, then $\nabla F = (2x, 2y, -1)$.

(a) $\nabla F(1, 1, 2) =$

(b) $\nabla F(0, 0, 0) =$

Ex. 7.5 Find the normal to the following curve or surfaces:

(a) $ax + by + cz = d$ (a plane)

Rewrite it as

$$f(x, y, z) = ax + by + cz = d$$

(b) $x^2 + y^2 = a^2$ (a circle)

Rewrite it as

$$f(x, y) = x^2 + y^2 = a^2$$

(c) $x^2 + y^2 + z^2 = a^2$ (a sphere)

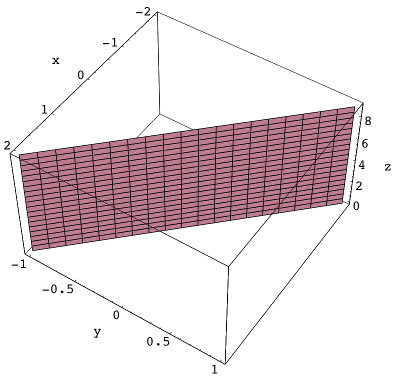
Rewrite it as

$$f(x, y, z) = x^2 + y^2 + z^2 = a^2$$

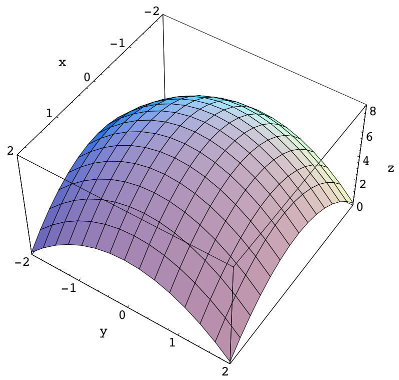
$\nabla f(\mathbf{r}_0)$ is perpendicular to any tangent vector of level surface $f(\mathbf{r}_0) = k$ at $\mathbf{r}_0$.

Note that $f_x(x, y)$ and $f_y(x, y)$ represent the rate of change of $f(x, y)$ in the direction *parallel* to the x- and y-axes.

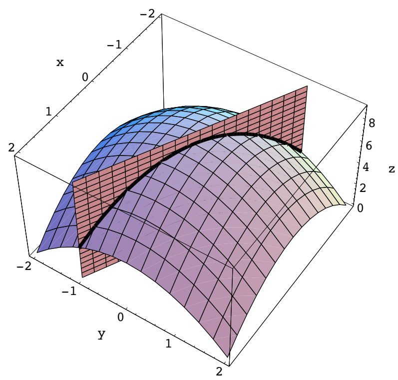
Directional derivative



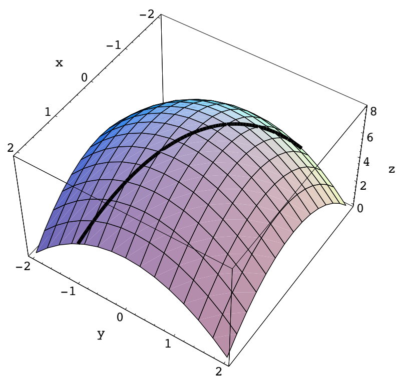
$y = ax + b$



$z = f(x, y)$

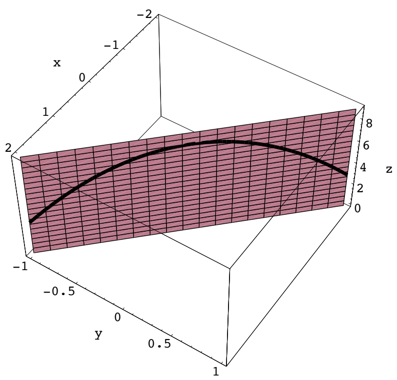


Two surfaces intersect



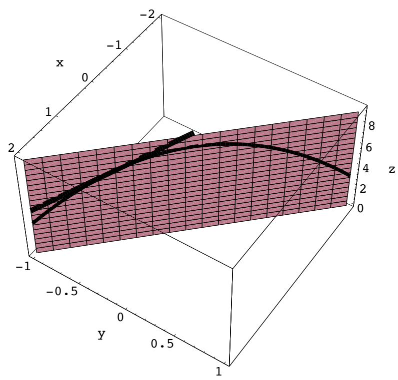
curve of intersection

lies on z



curve of intersection

lies on $y = ax + b$



The slope of the tangent line

in the direction of $\mathbf{u}$ at

(x_0, y_0) is $\nabla f(x_0, y_0) \cdot \widehat{\mathbf{u}}$

If f is differentiable at (x_0, y_0), then the directional derivative in the direction $\widehat{\mathbf{u}}$ of the function f is

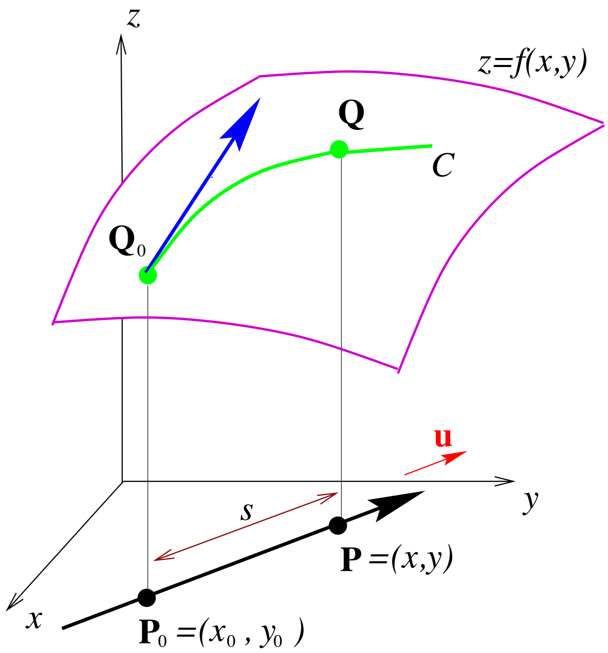
$$\boxed{D_{\mathbf{u}}\, f = \widehat{\mathbf{u}} \cdot \nabla f = (\widehat{\mathbf{u}} \cdot \nabla) f}$$

This measures the rate of change of the function $f(x, y)$ in the direction of $\widehat{\mathbf{u}}$.

Proof:

Let $\widehat{\mathbf{u}} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ and

$$z = f(x_0 + su_1, y_0 + su_2) = f(x, y), \quad \text{where} \quad \begin{aligned} x(s) &= x_0 + su_1 \\ y(s) &= y_0 + su_2 \end{aligned}$$



Then

$$D_{\mathbf{u}} f = \frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} = \widehat{\mathbf{u}}.\nabla f$$

Note: Maximum $D_{\mathbf{u}} f$ occurs when $\mathbf{u}$ is in the direction of ∇f and $\max(D_{\mathbf{u}} f) = |\nabla f|$.

Minimum $D_{\mathbf{u}} f$ occurs when $\mathbf{u}$ is in the direction of $-\nabla f$ and $\min(D_{\mathbf{u}} f) = -|\nabla f|$.

Alternative definition

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0_+} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}.$$

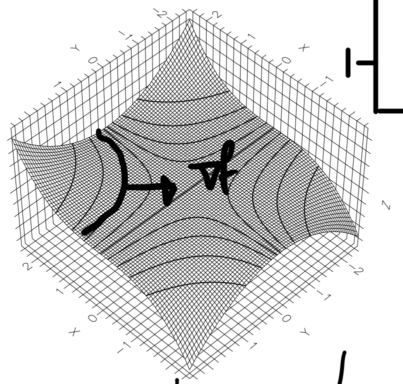
Ex. 7.6 Find the gradient of $f(x,y) = 4x^3y^2$ at the point $(2,1)$ and use it to calculate the directional derivative of f at (2, 1) in the direction of the vector $\mathbf{u} = 4\,\mathbf{i} - 3\,\mathbf{j}$.

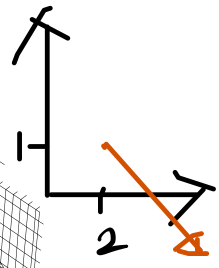
$$\nabla f = 12x^2y^2\,\mathbf{i} + 8x^3y\,\mathbf{j} = f_x \mathbf{i} + f_y \mathbf{j}$$

$$\nabla f(2,1) = 48\,\mathbf{i} + 64\,\mathbf{j}$$

$$\widehat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{5}(4\,\mathbf{i} - 3\,\mathbf{j})$$

$$D_{\mathbf{u}}f = \widehat{\mathbf{u}}\cdot\nabla f = \frac{4}{5}\times 48 - \frac{3}{5}\times 64 = 0.$$





directional derivative 係你向上望,
發現 height change = 0, 因為佢 move along level set 個方向

Ex. 7.7 Find the directional derivative of $f(x,y,z) = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $\mathbf{u} = 2\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}$.

$$\nabla f = (2xyz + 4z^2)\,\mathbf{i} + x^2z\,\mathbf{j} + (x^2y + 8xz)\,\mathbf{k}$$

$$= 8\,\mathbf{i} - \mathbf{j} - 10\,\mathbf{k} \quad \text{at} \quad (1,-2,-1).$$

unit vector $\hat{u} = \frac{2}{3}i - \frac{1}{3}j - \frac{2}{3}k$

$\therefore D_u f = \hat{u}\cdot\nabla f = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$

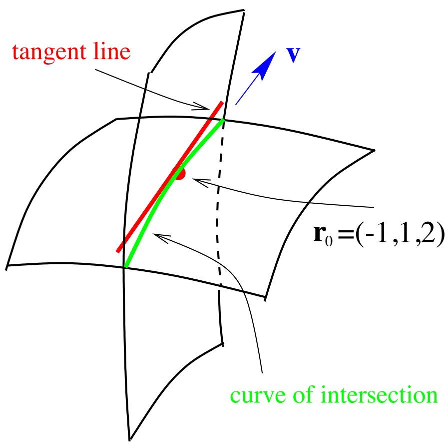
Ex. 7.8 Find vector equation of the tangent line at the point $(-2,2,4)$ to the curve of intersection of the surface $z = 2x^2 - y^2$ and the plane $z = 4$.

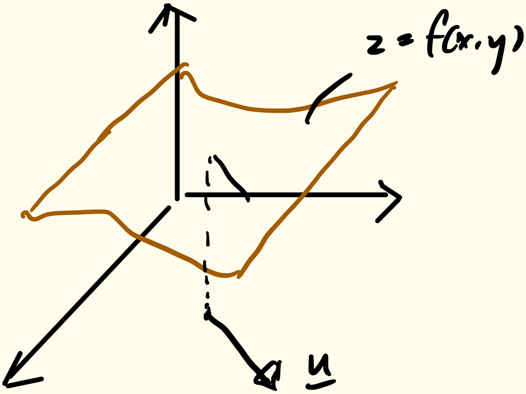
Ex. 7.9 Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1,1,2)$.

Hint: The vector equation of the line is

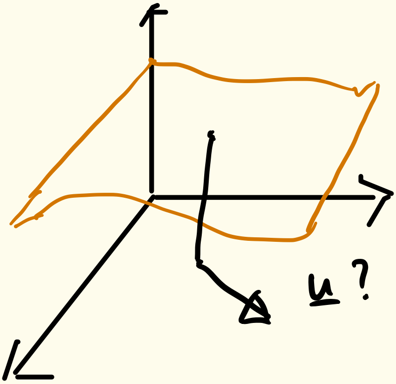
$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}t,$$

where $\mathbf{r}_0 = (-1,1,2)$, so all you need to do is to find $\mathbf{v}$.





Q_1 Given $\underline{u}$, find rate of change of f in the direction of $\underline{u}$



Q_2 Find $\underline{u}$ s.t. $D_{\underline{u}}f$ has the max. rate of increase (decrease)

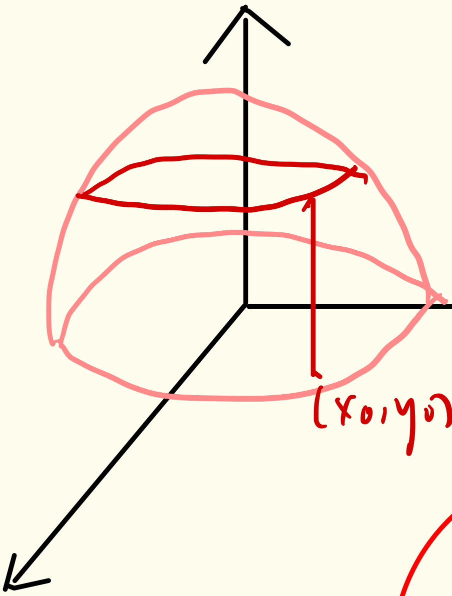
$$D_{\underline{u}}f = \underline{u} \cdot \nabla f = \|\hat{u}\| \, \|\nabla f\| \cos\theta$$

θ is the angle between $\hat{u}$ and ∇f

$\text{Max}(D_{\underline{u}} f) = 1 \cdot \|\nabla f\| \, \text{Max}(\cos\theta) = \nabla f$

when $\theta = 0^\circ$ ($\hat{u} \parallel \nabla f$)

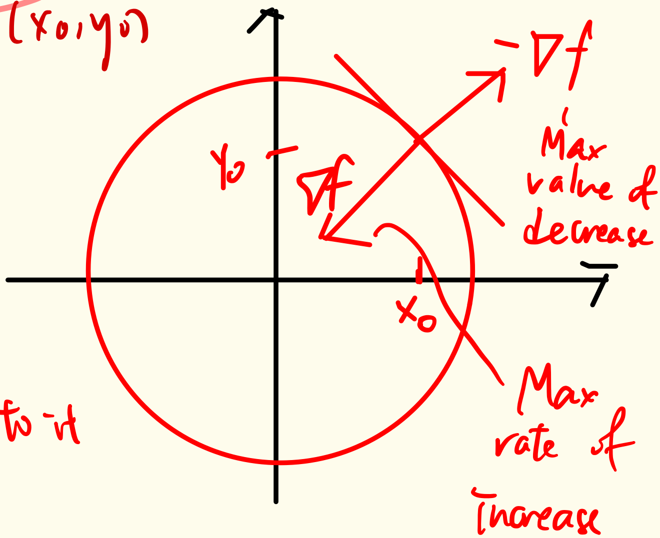
Max rate of decrease = $\text{Max}(D_{\underline{u}} f)$ when $\theta = 180^\circ$, $\hat{u} \parallel -\nabla f$



$z = a - x^2 - y^2 = f(x,y)$

$\nabla f = (-2x, -2y)$

$= -2(x, y)$

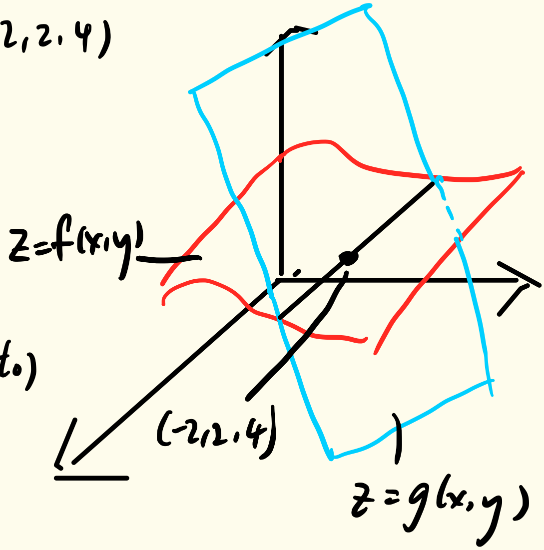


像 contour line more perpendicular to it = shortest path.

Ex7.8

$z = f(x,y) = 2x^2 - y^2 \qquad z = g(x,y) = 4$

Insect at the point $(-2,2,4)$



Find the intersection curve

$\underline{r}(t) = (x(t), y(t), z(t))$

$\underline{r_0} = (-2,2,4)$, $\underline{v} = \underline{r}'(t_0)$

For intersection curve,

$$2x^2 - y^2 = 4$$

let $x = t$,

$y = \sqrt{2t^2-4}$ (呢刻取正數因为 $(-2,\underset{\uparrow}{2},4)$ 2 is positive)

$\underline{r}(t) = (t, \sqrt{2t^2-4}, 4)$

$\underline{r}'(t) = (1, \frac{1}{2}(2t^2-4)^{-\frac{1}{2}}(4t), 0)$

At $(-2,2,4) = t = -2$

$\underline{r}'(-2) = (1,-2,0) = \underline{v}$

$\underline{r} = \underline{r_0} + \underline{v}t = (-2,2,4) + (1,-2,0)t$

Ex 7.8 (Another approach)

let $F(x,y,z) = z = f(x,y) = 0$

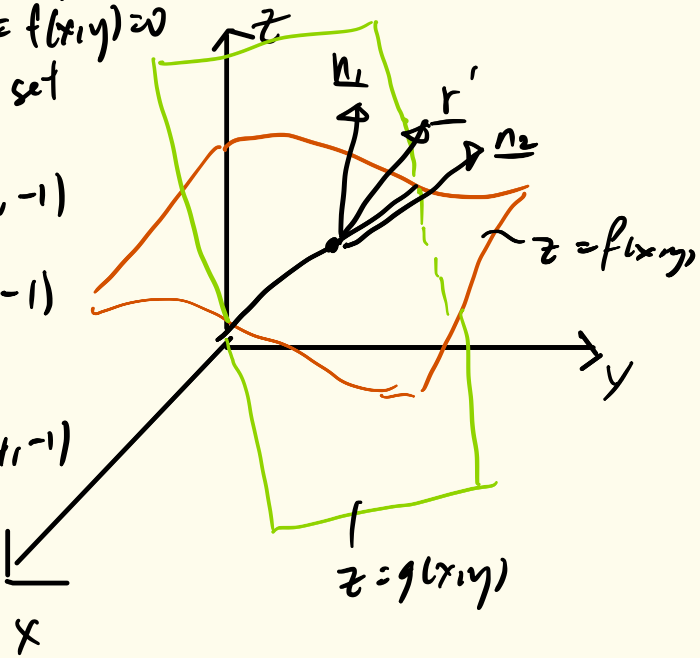
this is a level set

$\nabla F = (-f_x, -f_y, -1)$

$= (-4x, 2y, -1)$

$\nabla F(-2,2,4) = (8, 4, -1)$

$= \underline{n_1}$



let $G(x,y,z) = z - g(x,y) = 0$

$\nabla G = (-g_x, -g_y, -1)$

$= (0, 0, -1) = n_2$

At $(-2,2,4)$ $\quad \underline{r}' = \underline{v} = \underline{n}_1 \times \underline{n_2} = \begin{vmatrix} i & j & k \\ 8 & 4 & -1 \\ 0 & 0 & -1 \end{vmatrix}$

Find vector equation of the tangent line at the point (2, 2, 4) to the curve of intersection of
the surface z = 2x
2
y
2
and the plane z = 4.

H.W.4 Q1. $T(x,y)$ at pts of xy plane is given by $T(x,y) = x^2 - 2y^2$

a). Draw contour diagram T showing some isothermal

$\therefore x^2 - 2y^2 = c$

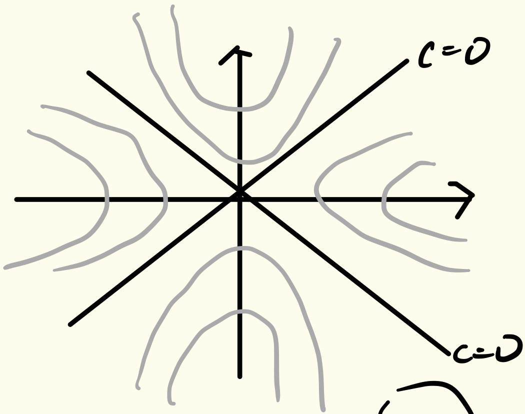
$c = 0,\ x^2 - 2y^2 = 0$

$x = \pm\sqrt{2}\,y \Rightarrow y = \pm\frac{1}{\sqrt{2}}x$

$c > 0\quad x^2 - 2y^2 = 1$

(hyperbola)

$c < 0\quad x^2 - 2y^2 = -1$



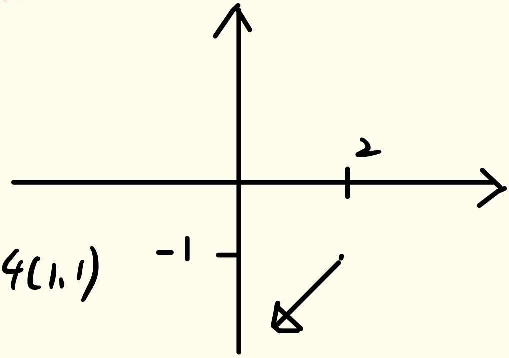
3

b). In what direction should an ant at position (2, −1) move if it wishes to cool off as quickly as possible?

At $(2,-1)$

$\nabla T = (2x, -4y)$

$\nabla T(2,-1) = (4,4) = 4(1,1)$



To cool off (max) $= -\nabla T(2,-1)$
(max rate of decrease)

3

i.e. in the direction of $(-1,-1)$

c). If an ant moves in that direction at speed k (units distance per unit time), at what rate does it experience the decrease of temperature?

$$D_{\underline{u}} f = \hat{\underline{u}} \cdot \nabla f$$

$$\text{Max}(D_{\underline{u}} f) = -\|\nabla f\| = \frac{df}{ds}$$

$$dT = \|\nabla T\| \, ds = (4\sqrt{2})k \text{ - speed}$$

4

d). At what rate would the ant experience the decrease of temperature if it moves from (2, –1) at speed k in the direction of the vector –i-2j ?

$$\frac{dT}{ds} = \hat{\underline{u}} \cdot \nabla T$$

$$dT = (\hat{\underline{u}} \cdot \nabla T) \times ds = \frac{1}{\sqrt{5}}(-1,-2) \cdot 4(1,1) \cdot k$$

$$= \frac{12}{\sqrt{5}} k$$

4

e). Along what curve through (2, –1) should the ant move in order to continue to experience maximum rate of cooling?

At any point (x,y), the direction the ant should move $-\nabla T(x,y) \parallel \underline{v}$

If the position vector of the ant is $\underline{r_0}(x,y)$

$$\frac{d\underline{r}}{dt} = \underline{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \lambda \nabla T \qquad \lambda \in \mathbb{R}$$

$$\frac{dx}{dt} = \lambda \frac{dT}{dx} = \lambda(2x)$$

$$\frac{dy}{dt} = \lambda \frac{dT}{dy} = \lambda(-4y)$$

$$\frac{1}{2} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\lambda(-4y)}{\lambda(2x)} = \frac{-2y}{x}$$

$$\frac{dy}{dx} = \frac{-2y}{x}$$

$$\int \frac{1}{y}\, dy = -2\int \frac{1}{x}\, dx$$

$$yx^2 = c$$

At $(2,-1)$ $\quad -1(2)^2 = c \Rightarrow c = -4$

$$\therefore y = -\frac{4}{x^2}$$

6

Exercises for students

(1a) Find and sketch the domain of the function

$$f(x,y) = \frac{\ln(x+y+1)}{x-1}.$$

(1b) Determine the largest set on which the function

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

is continuous.

(2a) Describe the surface

$$\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{i} + u\,\mathbf{k}$$

for $0 \leqslant u \leqslant 2$ and $0 \leqslant v < 2\pi$.

(2b) Let $f(x,y) = xy\ln(x^2+y^2)$. Is it possible to define $f(0,0)$ so that f will be continuous at $(0,0)$.

(3) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \dfrac{x^2y}{x^2+y^2}$ unless $x = y = 0$ and $f(0,0) = 0$.

(a) Show that $D_{\mathbf{v}}f(0,0)$ exists for all $\mathbf{v} \in \mathbb{R}^2$ by direct computation.

(b) Show that f satisfies the homogeneous relation $f(t\mathbf{v}) = tf(\mathbf{v})$ for all $t \in \mathbb{R}$ and all $\mathbf{v} \in \mathbb{R}^2$.

(c) Show that any differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ satisfying the homogeneous relation $g(t\mathbf{v}) = tg(\mathbf{v})$, $\forall\, t \in \mathbb{R}$, $\forall\, \mathbf{v} \in \mathbb{R}^n$ and $g(\mathbf{0}) = 0$ also satisfies the relation

$$g(\mathbf{v}) = \nabla g(\mathbf{0}) \cdot \mathbf{v} \quad \text{for all} \quad \mathbf{v} \in \mathbb{R}^n$$

and hence must be *linear*.

(d) Conclude that f possesses directional derivatives in all directions at $(0,0)$, but that f is *not* differentiable at $(0,0)$.

(e) (Not required) Graph f using Mathematica.

(4) Suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping; that is,

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x}, \quad \text{where} \quad \mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$$

and A is an $m \times n$ matrix. Calculate $\partial f_i/\partial x_j$ (or $D\mathbf{f}(\mathbf{x})$) and relate your result to the derivative of the one-variable linear function $f(x) = ax$.

(5) Let $z = g(x, y)$ be a function of class C^2, and let $x = e^r \cos\theta, y = e^r \sin\theta$.

(a) Use the chain rule to find $\partial z/\partial r$ and $\partial z/\partial\theta$ in terms of $\partial z/\partial x$ and $\partial z/\partial y$. Use your results to solve for $\partial z/\partial x$ and $\partial z/\partial y$ in terms of $\partial z/\partial r$ and $\partial z/\partial\theta$.

(b) Use part (a) and the product rule to show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 z}{\partial r^2} + \frac{\partial^2 z}{\partial\theta^2}\right).$$

(6) Prove: If f, f_x and f_y are continuous on a circular region containing points $A\,(x_0, y_0)$ and $B\ (x_1, y_1)$, then there is a point (x^*, y^*) on the line segment joining A and B such that

$$f(x_1, y_1) - f(x_0, y_0) = f_x(x^*, y^*)(x_1 - x_0) + f_y(x^*, y^*)(y_1 - y_0).$$

This result is the two-dimensional version of the Mean-Value Theorem. [*Hint*: Express the line segment joining A and B in parametric form and use the Mean-Value Theorem for functions of one variable.]

(7) Prove: If $f_x(x, y) = 0$ and $f_y(x, y) = 0$ throughout a circular region, then $f(x, y)$ is constant on that region. [*Hint*: Use the result of Exercise 6.]

(8) Show that

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)\,dt\right] = f(b(x))b'(x) - f(a(x))a'(x)$$

[*Hint*: Let $u = a(x)$, $v = b(x)$, and $F(u, v) = \displaystyle\int_u^v f(t)\,dt$.

(9) Show that if $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = (x_0, y_0)$, then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0)\cdot(\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = 0$$

[*Hint*: Use the definition of the lecture note on differentiability.]

(10) (a) Let $f : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}^n$ be two differentiable curves, with $f'(t) \neq 0$ and $g'(t) \neq 0$ for all $t \in \mathbb{R}$. Suppose that $\mathbf{p} = f(s_0)$ and $\mathbf{q} = g(t_0)$ are closer than any other pair of points on the two curves. Prove that the vector $\mathbf{p} - \mathbf{q}$ is orthogonal to both velocity vectors $f'(s_0)$ and $g'(t_0)$.

(*Hint*: The point (s_0, t_0) must be a critical point for the function $\rho : \mathbb{R}^2 \to \mathbb{R}$ defined by $\rho(s, t) = \|f(s) - g(t)\|^2$)

(b) Apply the result of part (a) to find the closest pair of points of the "skew" straight lines in $\mathbb{R}^3$ defined by $f(s) = (s, 2s, -s)$ and $g(t) = (t + 1, t - 2, 2t + 3)$.