MATH2023
Tutorial 5
March 26, 2019 Tutorial Notes by Ivan So

1 Review

1.1 Vector

- **Scalar** is an *one*-entry object belongs to $\mathbb{R}$.
- **Vector** is a *three*-entry object represented by $\mathbf{x} = (x_1, x_2, x_3) = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, which represent an "arrow" in 3D space.
- Addition of vector: follows the head to tail rules.
- The **norm** $\|\cdot\| : V \to \mathbb{R}$ is a function which measures the *length* of the arrow. Defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ (in our consideration).
- **Dot Product**:

$$\begin{aligned} "\cdot" &: V \times V \to \mathbb{R} \\ &(\mathbf{v}_1, \mathbf{v}_2) \mapsto \mathbf{v}_1 \cdot \mathbf{v}_2 := v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z} = \|\mathbf{v}_1\| \, \|\mathbf{v}_2\| \cos\theta \end{aligned}$$

- **Cross Product**: $" \times " : (\mathbf{v}_1, \mathbf{v}_2) \in V \times V \mapsto \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_{1x} & v_{1y} & v_{1z} \\ v_{2x} & v_{2y} & v_{2z} \end{bmatrix} \in V$.
- You are reminded the concept of **linearly independent**, **unit vector**, **orthogonal** and **determinant**.
- The way to find a equation of **line** passing through *two* points:
  1. Given points A, B, find $\overrightarrow{AB}$.
  2. Equation of line:$\overrightarrow{OA} + t\overrightarrow{AB}$ $t \in \mathbb{R}$.
- The way to find a equation of **plane** passing through *three* points:
  1. Given points A, B and C, find vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
  2. Find normal of plane $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.
  3. Let $P = (x, y, z)$, then equation of plane: $\overrightarrow{AP} \cdot \mathbf{n} = 0$.

1.2 Vector Valued Functions, Tangent/Normal Vectors, Curvature and Arc Length of Curve

- **Vector valued function** $V : t \in \mathbb{R} \mapsto \mathbf{r}(t) = (r_1(t), r_2(t), r_3(t)) \in \mathbb{R}^3$.
- **Tangent vector** $\mathbf{T}(t) := \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, **Normal vector** $\mathbf{N}(t) := \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$.
- **Arc length parametrization**: parametrization of curve in which $|\mathbf{r}'(s)| = 1$.
- **Curvature** $\kappa := \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.
- **Arc length** calculation: The arc length of the curve defined by $\mathbf{r}(t)$ for $a \le t \le b$ is given by $L = \int_a^b \sqrt{[r_1'(t)]^2 + [r_2'(t)]^2 + [r_3'(t)]^2} dt$.

1.3 Multivariable Functions, Limits and Differentiation

- **Multivariable function** is defined as the map $f : \mathbb{R}^n \to \mathbb{R}$.
- **Domain** D: Subset $D \subset \mathbb{R}^n$ on which the function is defined. **Range**: The set $\{f(\mathbf{x})|\mathbf{x} \in D\}$.
- **Graph**: The set $\{(x, y, f(x, y))|(x, y) \in D\}$ for the domain D of f.
- **Level curve**: The curve with satisfying $f(x, y) = k$.
- **Limit**: The value that f "approach" as (x, y) approach (a, b)). Notationally,

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = L.$$

  - If the limit exist, then **MUST** be unique.
  - Evaluation can be done with *squeeze theorem* or *polar coordinates*.
- **Continuouity**: f is continuous at $\mathbf{x}_0$ if it satisfies $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$.
- **Partial derivative** $\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}_0} := \lim_{h\to 0} \frac{f(\mathbf{x}_0 + h\mathbf{e}_i) - f(\mathbf{x}_0)}{h}$. Other notations: f_{x_i}. One can regard *unrelated variables* as constants in taking partial derivative.
- **Theorem** (Clairaut): f_{xy} and f_{yx} are continuous $\Rightarrow f_{xy} = f_{yx}$.

1.4 Tangent Plane, Directional Derivatives and Implicit Differentiation

- **Tangent plane** of a multivariable function $P(\mathbf{x}) := f(\mathbf{x}_0) + \sum_{i=1}^n f_{x_i}(\mathbf{x}_0)\Delta x_i$. **Total differential** $df := \sum_{i=1}^n f_{x_i}\Delta x_i$.
- **Theorem**: Normal vector of the surface defined by $x_{n+1} = f(\mathbf{x})$ is $(f_{x_1}, \cdots, f_{x_n}, -1)$.

- **Gradient operator** maps a function into a vector by $\nabla f := \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$.

- **Directional derivative** in the direction of $\hat{\mathbf{v}}$ $D_{\hat{\mathbf{v}}} f(\mathbf{x}) := \lim_{t\to 0} \frac{f(\mathbf{x}+t\hat{\mathbf{v}})-f(\mathbf{x})}{t} = \nabla f \cdot \hat{\mathbf{v}}$.
  * Always the best to work with unit vector.

- Suppose $\mathbf{x}(t) \in \mathbb{R}^n$ are set of variables which depends on $\mathbf{t} \in \mathbb{R}^m$, then the **chain rule** in multivariable case is given by $\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \mathbf{x}}{\partial t_i}$. we can draw *tree diagram* for the chain relation.

- If $F(\mathbf{x}) = C$, we can find $\frac{\partial x_j}{\partial x_i}$ by **implicit differentiation**. Procedures:

  1. Take the partial derivative $F(\mathbf{x}) = C$ with respect to x_i, then we obtain the relation $\nabla F \cdot \frac{\partial \mathbf{x}}{\partial x_i} = 0$.
  2. Find the expression $\nabla F \cdot \frac{\partial \mathbf{x}}{\partial x_i} = 0$ with $\frac{\partial x_j}{\partial x_i}$ on left hand side.

1.5 Optimization Problem

- You are reminded the concepts of **local min/max** and **absolute min/max**.

- f has local min/max or saddle point at $\mathbf{x}_0 \Rightarrow f_{x_i} = 0$.

- The **Hessian matrix** for two-variable function f is defined as

$$H(f)(\mathbf{x}_0) := \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

  If (1) $f_x(a,b) = f_y(a,b) = 0$ and (2) the second derivatives are continuous, then:

  (a) In case $\det H(f)(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is *local minimum.*

  (b) In case $\det H(f)(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is *local maximum.*

  (c) In case $\det H(f)(a,b) < 0$ then $f(a,b)$ is *neither local maximum* nor *local minimum*, but a *saddle point* (think of a Pringles potato chip cut).

  (d) In case $\det H(f)(a,b) = 0$, the second derivative test is **inconclusive**.

- The absolute extrema of a function over a given domain is either the point of *local extrema* or on the *boundary.*

- The **method of Lagrange multiplier** in extrema evaluation is given as follows:

  1. Find all values of x_i's and λ_i's such that $\nabla f(\mathbf{x}) = \sum_{i=1}^m \lambda_i \nabla g_i(\mathbf{x})$ given the constraints $g_i(\mathbf{x}) = k_i$.
  2. Evaluate f at all the $\mathbf{x}$'s obtained above. The largest and smallest give the extrema.

2 Problems

2.1 Vectors

1. (a) Find the equation of P_1, P_2 in which P_1 satisfies: $(1/3, 1/3, 1/3) \in P_1$ and perpendicular to the vector $\langle 1, 1, 1\rangle$. P_2 satisfies: $(1, 1, 0) \in P_2$ and perpendicular to the vector $\langle 1, 2, 1\rangle$.

   (b) Find the equation of line of intersection of P_1 and P_2.

   Solution:

   (a) Equation of P_1: $\langle x - 1/3, y - 1/3, z - 1/3\rangle \cdot \langle 1, 1, 1\rangle = 0 \Rightarrow x + y + z = 1$.
   Equation of P_2: $\langle x - 1, y - 1, z - 0\rangle \cdot \langle 1, 2, 1\rangle \Rightarrow x + 2y + z = 3$. $\square$

   (b) Direction of intersection line is perpendicular to the normal vector, therefore the tangent of the line is in the direction of

   $$\mathbf{n} = \langle 1, 1, 1\rangle \times \langle 1, 2, 1\rangle = \langle -1, 0, 1\rangle$$

   Equation of P_2 minus equation of P_1 solves $y = 2$. Substitute $y = 2$ into both equations give $x + z = -1$. So $(0, 2, -1)$ is a point of the intersection. As a result, the desired equation of line is: $\mathbf{r}(t) = \langle 0, 2, -1\rangle + t\langle -1, 0, 1\rangle$. $\square$

2. Show that two lines $\mathbf{r}_1(t) = \mathbf{a} + \mathbf{v}t$ and $\mathbf{r}_2(t) = \mathbf{b} + \mathbf{u}t$ will intersect if $\mathbf{a} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{b} \cdot (\mathbf{u} \times \mathbf{v})$.
   Solution: If the two lines intersects, then there exists t_0 such that $\mathbf{a} + \mathbf{v}t_0 = \mathbf{b} + \mathbf{u}t_0$. The equality is then immediate by taking dot product with $\mathbf{u} \times \mathbf{v}$ since $\mathbf{u} \times \mathbf{v} \perp \mathbf{u}, \mathbf{v}$ by definition. $\square$

2.2 Vector Valued Function

1. Find the curvature expression for the curve $r = 4\cos 2\theta$ for $0 \le \theta < 2\pi$.
   Solution: A parametrization of the curve is given by $\mathbf{r}(t) = \langle 4\cos 2t \cos t, 4\cos 2t \sin t, 0\rangle = \langle 2\cos 3t - 2\cos t, 2\sin 3t - 2\sin t, 0\rangle$. Then $\mathbf{r}'(t) = \langle -6\sin 3t + 2\sin t, 6\cos 3t - 2\cos t, 0\rangle$ and $\mathbf{r}''(t) = \langle -18\cos 3t + 2\cos t, -18\sin 3t + 2\sin t, 0\rangle$. So,

   $$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{48\cos 2t}{(40 - 24\cos 2t)^3} = \frac{6|\cos 2t|}{|20 - 12\cos 2t|^3}. \ \square$$

2.

2.3 Limit

1. (a) Can the function $f(x,y) = \frac{\sin x \sin^3 y}{1-\cos(x^2+y^2)}$ be defined at $(0,0)$ such that it is continuous at $(0,0)$? If so, how?

   (b) Let $f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$.
   Calculate each of the following derivatives or explain why they do not exists:
   (i) $f_x(0,0)$ (ii) $f_y(0,0)$ (iii) $f_{xy}(0,0)$ (iv) $f_{yx}(0,0)$ (v) $f_{xx}(0,0)$.

   Solution:

   (a) Using polar coordinates:

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{r\to 0} \frac{\sin(r\cos\theta)\sin^3(r\sin\theta)}{1-\cos r^2} \\ &= \lim_{r\to 0} \sin^3\theta\cos\theta \cdot \lim_{r\to 0} \frac{r^4}{1-\cos r^2} \cdot \lim_{r\to 0} \frac{\sin(r\cos\theta)}{(r\cos\theta)} \cdot \lim_{r\to 0} \frac{\sin^3(r\sin\theta)}{(r\sin\theta)^3} \\ &= 2\sin^3\theta\cos\theta\end{aligned}$$

   So the limit depends on the angle of approach, inconsistent. So it does not exist and we cannot talk about continuity. □

   (b) (i) $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 1$.
   (ii) $f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0$.
   (iii) $f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{-1}{h}$, which diverges. Therefore it does not exist.
   (iv) $f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{h}{h} = 1$.
   (v) $f_{xx}(0,0) = \lim_{h\to 0} \frac{f_x(h,0)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{1-1}{h} = 0$. □

2. Find the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y+xy^2+x^2+y^2+2xy}{1-\cos\sqrt{x^2+y^2}}$.

   Solution: In polar coordinates,

$$\begin{aligned}&\lim_{(x,y)\to(0,0)} \frac{x^2y+xy^2+x^2+y^2+2xy}{1-\cos\sqrt{x^2+y^2}} \\ &= \lim_{r\to 0}\left(\frac{r^3\sin 2\theta}{2(1-\cos r)}(\cos\theta+\sin\theta) + \frac{r^2}{1-\cos r} + \frac{r^2\sin 2\theta}{1-\cos r}\right) \\ &= 2+2\sin 2\theta\end{aligned}$$

   Therefore, the limit depends on the angle of approach, which is not unique. So the limit does not exist. □

2.4 Derivatives

1. (a) Show that if f is differentiable and $z = xf\left(\frac{x}{y}\right)$, then all tangent planes to the graph of this equation pass through a common point. Find the common point.

   (b) Find the equation of the level curve of the function $z = g(x, y) = xf\left(\frac{x}{y}\right)$ at the point (x_0, y_0). Show that $\nabla g(x_0, y_0)$ is normal to the tangent line to the level curve at (x_0, y_0).

Solution:

(a) $z_x = f\left(\frac{x}{y}\right) + \frac{x}{y}f'\left(\frac{x}{y}\right)$, $z_y = -\frac{x^2}{y^2}f'\left(\frac{x}{y}\right)$. The equation of tangent plane at the point (x_0, y_0) is then

$$\begin{aligned}
P_{(x_0,y_0)}(x,y) &= x_0 f\left(\frac{x_0}{y_0}\right) + \left[f\left(\frac{x_0}{y_0}\right) + \frac{x_0}{y_0}f'\left(\frac{x_0}{y_0}\right)\right](x - x_0) + \left[-\frac{x_0^2}{y_0^2}f'\left(\frac{x_0}{y_0}\right)\right](y - y_0) \\
P_{(x_0,y_0)}(0,0) &= x_0 f\left(\frac{x_0}{y_0}\right) + \left[f\left(\frac{x_0}{y_0}\right) + \frac{x_0}{y_0}f'\left(\frac{x_0}{y_0}\right)\right](-x_0) + \left[-\frac{x_0^2}{y_0^2}f'\left(\frac{x_0}{y_0}\right)\right](-y_0) \\
&= 0
\end{aligned}$$

The evaluation above is independent to the choice of (x_0, y_0). Therefore every plane of tangent of $z = xf\left(\frac{x}{y}\right)$ passes through the origin $(0, 0, 0)$. $\square$

(b) Equation of level curve: $xf\left(\frac{x}{y}\right) = x_0 f\left(\frac{x_0}{y_0}\right)$. Let $\langle x(t), y(t)\rangle$ be the equation of curve satisfying the level curve equation with $(x(0), y(0)) = (x_0, y_0)$, then

$$\begin{aligned}
g(x(t), y(t)) &= g(x_0, y_0) \\
\Rightarrow \frac{d}{dt}g(x(t), y(t)) &= g_x x_t + g_y y_t = 0 \\
\nabla g(x(t), y(t)) \cdot \langle x'(t), y'(t)\rangle &= 0
\end{aligned}$$

By definition $\langle x'(t), y'(t)\rangle$ is the tangent vector of the level curve. By picking $t = 0$, the claim is proved. $\square$

2. If $w = f(x, y)$ is differentiable. Suppose $x = s^2 + t^2$, $y = s^2 - t^2$. Use chain rule to find (i) w_s (ii) w_{st} (iii) w_{stt}.

Solution:

(i) $w_s = w_x x_s + w_y w_s = 2s(f_x + f_y)$. $\square$

(ii) $w_{st} = 2s(f_{xx}x_t + f_{xy}y_t + f_{yx}x_t + f_{yy}y_t) = 4st(f_{xx} - f_{xy} + f_{yx} - f_{yy})$. $\square$

(iii) $w_{stt} = 4s(f_{xx} - f_{xy} + f_{yx} - f_{yy}) + 8st^2(f_{xxx} - f_{xxy} - f_{xyx} + f_{xyy} + f_{yxx} - f_{yxy} - f_{yyx} + f_{yyy})$. $\square$

2.5 Optimization Problems

1. Find the point on the surface $z^2 = -\frac{1}{2}x^2 + 2y^2 + xy$ which is closest to the point $\left(-\frac{1}{2}, -3, 0\right)$ through:

   (a) reducing the problem into an unconstrained problem of 2-variables.

   (b) the method of Lagrange multiplier.

   Solution:

   (a) The concerned distance function is given by:

$$\begin{aligned} f(x,y,z(x,y)) &= \sqrt{(x+1/2)^2 + (y+3)^2 + z^2} \\ &= \sqrt{(x+1/2)^2 + (y+3)^2 - \frac{1}{2}x^2 + 2y^2 + xy} \\ &= \sqrt{\frac{x^2}{2} + 3y^2 + x + 6y + xy + \frac{37}{4}} \end{aligned}$$

   Solving for $\nabla f = \mathbf{0}$, we have

$$\begin{cases} f_x = (x+1+y)/2f = 0 \\ f_y = (6y+6+x)/2f = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -1 \end{cases}$$

   The second derivatives are:

$$f_{xx} = \frac{2f - 2(x+y+1)f_x}{4f^2}, f_{xy} = \frac{2f - 2(x+y+1)f_y}{4f^2},$$

$$f_{yx} = \frac{2f - 2(x+6y+6)f_x}{4f^2}, f_{yy} = \frac{12f - 2(6y+6+x)f}{4f^2}$$

   So $\det H(f)(0,-1) = \frac{5}{4(5/2)^2} > 0$. Meanwhile, $f_{xx}(0,-1) = 1/5 > 0$. Therefore $f(x,y,z(x,y))$ has a minimum for $(x,y) = (0,-1)$. As a result the point on the surface having smallest distance from the concerned point is $(0,-1,\pm\sqrt{2})$. $\square$

   (b) When $\sqrt{(x+1/2)^2 + (y+3)^2 + z^2}$ reaches maximum, $\sqrt{(x+1/2)^2 + (y+3)^2 + z^2}^2$ also reach maximum. Therefore,
   Function of concern: $f(x,y,z) = (x+1/2)^2 + (y+3)^2 + z^2$.
   Constraint: $g(x,y,z) = -z^2 - \frac{1}{2}x^2 + 2y^2 + xy = 0$.
   By the method of Lagrange multiplier, at critical point,

$$(2(x+1/2), 2(y+3), 2z) = \lambda(-x+y, 4y+x, -2z)$$

   The z-component gives two possibilities:

   - Case 1: $\lambda = -1$, then the x,y-components solve to: Give $(x,y) = (0,-1)$. Indicating $(0,-1,\pm\sqrt{2})$ are the two points having the minimum distance from $\left(-\frac{1}{2},-3,0\right)$ (instead of maximum because the surface extends indefinitely).

– Case 2: $z = 0$, in which I cannot solve and cannot give an detailed argument. Direction: For $z = 0$, the surface equation will give $(x - (1+\sqrt{5})y)(x - (1-\sqrt{5})y) = 0$. Which splits to many two cases but turns out you shall find contradiction since Lagrange multiplier gives only maximum or minimum. From (a), we already noticed there is no upper bound for the separation. □

2. Find the maximum and minimum of the function $f(x, y, z) = xy + yz$ subject to the constraints $x + 2y - 6 = 0$ and $x - 3z = 0$ using the method of Lagrange multiplier. Also find the values of the Lagrange multipliers. □
Solution: At critical point,

$$\nabla f = \langle y, x+z, y \rangle = \lambda_1 \langle 1, 2, 0 \rangle + \lambda_2 \langle 1, 0, -3 \rangle$$

So the problem reduce to solving the system of equations

$$\begin{cases} x + 2y - 6 = 0 \\ x - 3z = 0 \\ y - \lambda_1 - \lambda_2 = 0 \\ x + z - 2\lambda_1 = 0 \\ y + 3\lambda_2 = 0. \end{cases} . \square$$