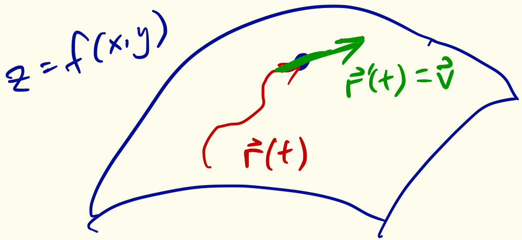
Last time: Max / Min Problems:

Critical Points: $\nabla f = 0$ or does not exist.

$\nabla f = \langle f_x, f_y \rangle$

$$\Uparrow$$

local max/min.



local max should look like $\frown$ in one-dimension along every path.

$F(t) = f(x(t), y(t))$ and we want $F'(t) = 0$.

$$F'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

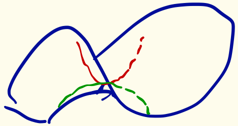
$$= \nabla f \cdot \vec{r}\,'(t) = \nabla f \cdot \vec{v}$$

$\Longrightarrow$ only need to concern x, y direction

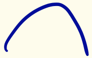
$\Longleftrightarrow \nabla f = 0.$

Saddle Points

Ex $f(x,y) = x^2 - y^2$:



↳ Idea

you may find one direction : 

another direction : 

In general

Saddle Point = Not local max nor local min.

Ex $z = \dfrac{x^4 - 6x^2y^2 + y^4}{x^2+y^2}$ $\rightsquigarrow$ Polar Coord: $x = r\cos\theta$, $y = r\sin\theta$

$$= \frac{(x^2+y^2)^2 - 8x^2y^2}{r^2} = \frac{r^4 - 8r^4\cos^2\theta\sin^2\theta}{r^2} = r^2(1 - 8\cos^2\theta\sin^2\theta)$$

$$= r^2(1 - 2\sin^2 2\theta)$$

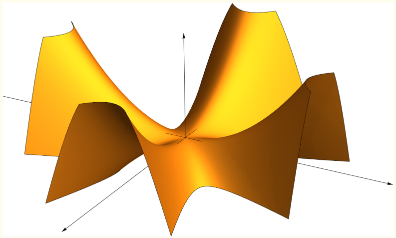
$$z = r^2\cos 4\theta.$$

along $\theta = 0$: looks like r^2

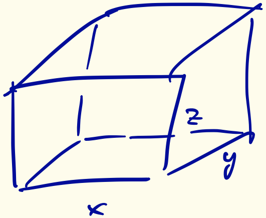
$\theta = \frac{\pi}{4}$: looks like $-r^2$

$\theta = \frac{\pi}{2}$: looks like r^2

⋮



Ex Max. volume of a box without cover with surface area $12\ \text{cm}^2$



: $V = xyz$.

"Constrained Problem"

Condition : $xy + 2xz + 2yz = 12$.

$$\hookrightarrow \quad z = \frac{12 - xy}{2x+2y}$$

$$V = \frac{xy(12-xy)}{2x+2y} = \frac{12xy - x^2y^2}{2x+2y}$$

$$\frac{\partial V}{\partial x} = \frac{(2x+2y)(12y - 2xy^2) - (12xy - x^2y^2)2}{(2x+2y)^2} = \frac{\cancel{24xy} + 24y^2 - \cancel{2}x^2y^2 - 4xy^3 - \cancel{24xy} + \cancel{2x^2y^2}}{} = \frac{y^2(24 - 2x^2 - 4xy)}{(2x+2y)^2} = 0$$

$$\Longleftrightarrow \quad 12 - x^2 - 2xy = 0$$

By Symmetry:

$$\frac{\partial V}{\partial y} = 0 \iff 12 - y^2 - 2xy = 0$$

$$\begin{cases} 12 - x^2 - 2xy = 0 \\ 12 - y^2 - 2xy = 0 \end{cases}$$

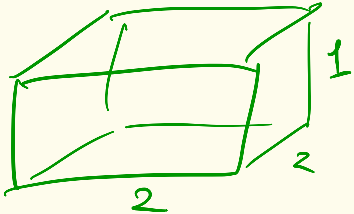
$$2xy = 12 - x^2$$
$$\parallel$$
$$2xy = 12 - y^2 \Longrightarrow x = y$$

$$12 - x^2 - 2x^2 = 0$$

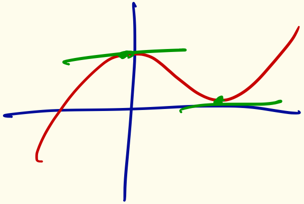
$$x = 2 \,/\!/$$

$$y = 2 \,/\!/$$

$$z = \frac{12 - xy}{2x + 2y} = 1.$$



2^{nd} Derivative Test



$f'(x) = 0$,

2^{nd} dev. test:

$f''(x) < 0$ max

$f''(x) > 0$ min.

$= 0$ no conclusion.

$\langle f_x\ f_y \rangle = \nabla f = 0$

f_{xx}, f_{yy}, $f_{xy} = f_{yx}$

Thm Assume 2^{nd} partial derivatives of f are continuous near (a,b), and (a,b) is critical point of f.

Let $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$ at (a,b)

Then

Then

☆ $D > 0$ and $f_{xx} > 0$ $\Rightarrow$ $f(a,b)$ is local min.

$D > 0$ and $f_{xx} < 0$ $\Rightarrow$ $f(a,b)$ is local max.

$D < 0$ $\Rightarrow$ saddle point

$D = 0$ $\Rightarrow$ no conclusion.

Proof : Local min : Idea is to look at every direction $\vec{v}$ should look like 

$\vec{v} = \langle h, k \rangle$ (unit direction)

$$\begin{cases} D_{\vec{u}} f = 0 \iff h f_x + k f_y = \nabla f \cdot \vec{u} = 0 \\ (D_{\vec{u}})^2 f > 0 \end{cases}$$

$$(D_{\vec{u}})^2 f = h(h f_x + k f_y)_x + k(h f_x + k f_y)_y$$

$$= h^2 f_{xx} + 2hk f_{xy} \cancel{+ kh f_{xy}} + k^2 f_{yy}$$

completing square :

$$f_{xx}\left(h + \frac{f_{xy}}{f_{xx}} k\right)^2 + \frac{k^2}{f_{xx}} \underbrace{\left(f_{xx} f_{yy} - f_{xy}^2\right)}_{D}.$$

So, if $D > 0$, $f_{xx} > 0 \Rightarrow D_{\vec{u}}^2 f > 0 \Rightarrow$ local min !

$\quad\quad\quad\quad >0 \quad\quad$ " $<0 \quad\quad$ " $<0 \quad\quad\quad$ max

Ex $xye^{-2x^2-2y^2}$: $(xe^{-2x^2})(ye^{-2y^2})$

$\downarrow d_x$

$e^{-2x^2} - 4x^2e^{-2x^2}$

$e^{-2x^2}(1-4x^2)$

$\downarrow d_x$

$-4x(1-4x^2)e^{-2x^2} - 8xe^{-2x^2}$

$= e^{-2x^2}(16x^3 - 12x)$

$= 4e^{-2x^2}x(4x^2-3)$

$\downarrow d_y$

$e^{-2y^2}(1-4y^2)$

$\downarrow d_y$

$4e^{-2y^2}y(4y^2-3)$.

$$\nabla f = \left\langle e^{-2x^2-2y^2}(1-4x^2)y \; , \; e^{-2x^2-y^2}x(1-4y^2) \right\rangle = \langle 0, 0 \rangle$$

$$\begin{cases} (1-4x^2)y = 0 \\ ((1-4y^2)x = 0 \end{cases} \Rightarrow (x,y) = (0,0), \left(\tfrac{1}{2}, \tfrac{1}{2}\right), \left(-\tfrac{1}{2}, \tfrac{1}{2}\right), \left(\tfrac{1}{2}, -\tfrac{1}{2}\right), \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right).$$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4xy(4x^2-3)e^{-2x^2-2y^2} & e^{-2x^2-2y^2}(1-4x^2)(1-4y^2) \\ e^{-2x^2-2y^2}(1-4x^2)(1-4y^2) & 4xy(4y^2-3)e^{-2x^2-2y^2} \end{vmatrix}$$

D at $(0,0)$: $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \Rightarrow$ saddle point.

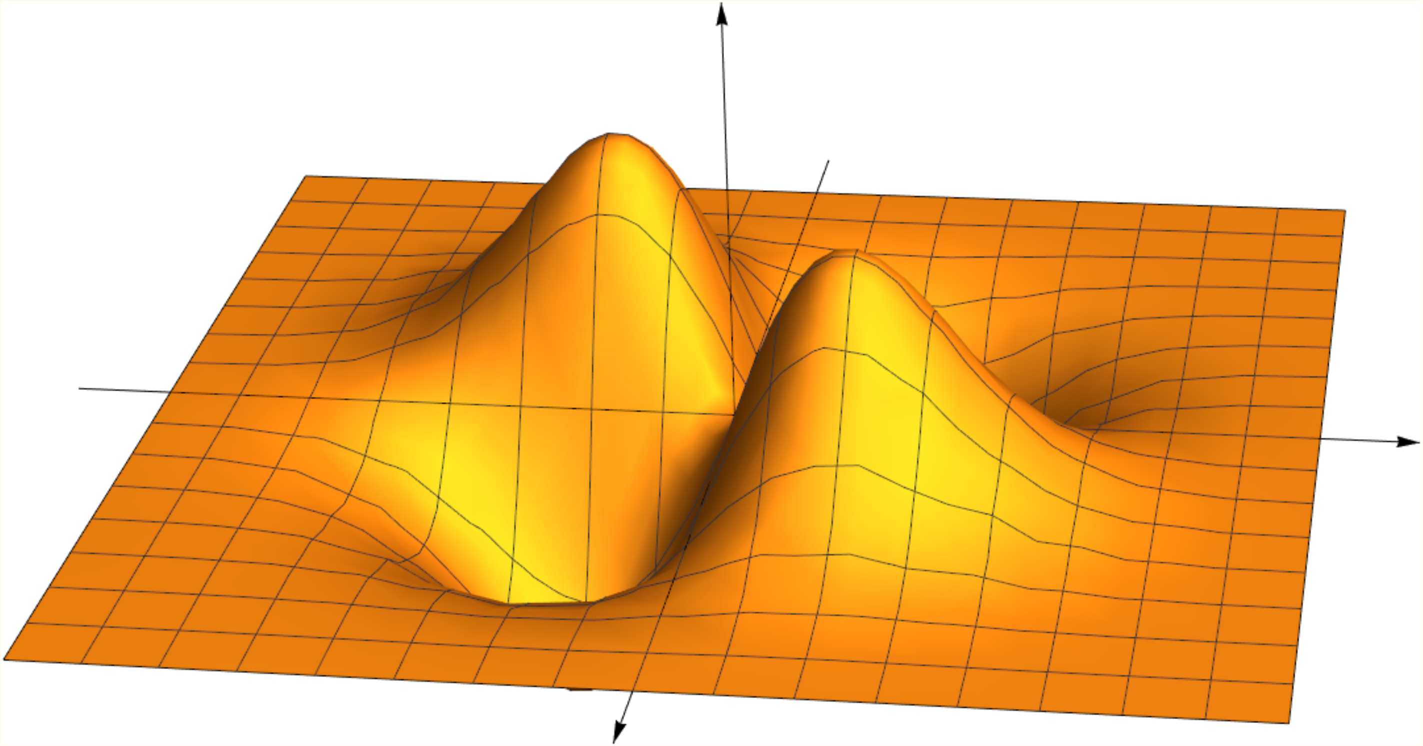
D at $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$: $\begin{vmatrix} \approx -xy & 0 \\ 0 & -xy \end{vmatrix} > 0$

$f_{xx} \approx -xy > 0$

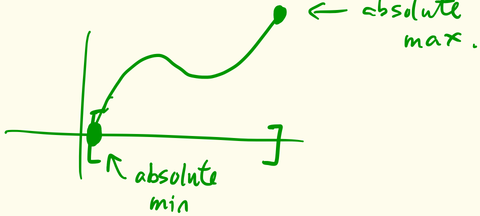
$\left(\frac{1}{2}, -\frac{1}{2}\right)$
$\left(-\frac{1}{2}, \frac{1}{2}\right)$ $\Rightarrow$ local min

$f_{xx} \approx -xy < 0$

$\left(\frac{1}{2}, \frac{1}{2}\right)$
$\left(-\frac{1}{2}, -\frac{1}{2}\right)$ $\Rightarrow$ local max

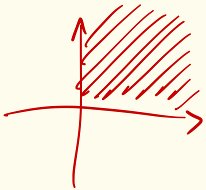


Absolute Max/Min. (Global) $\Leftarrow$ Biggest / Smallest in the domain of f



Extreme Value Theorem

If f is continuous on a bounded closed subset $D \subseteq \mathbb{R}^2$ then it has an absolute max and absolute min.

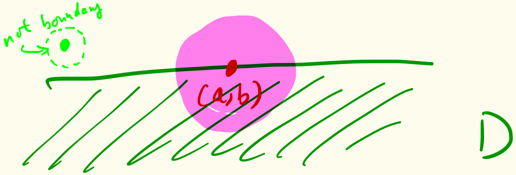


not bounded

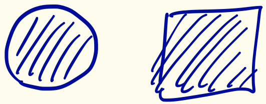
bounded

"closed set = contains all boundary points"

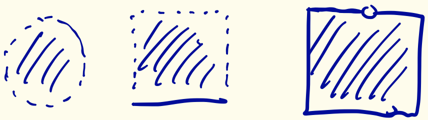
(a,b) is boundary point $\Leftrightarrow$ any disk with (a,b) at center will contain points inside D & points outside D



Closed



Not Closed



Find absolute max/min

① Find values of f at critical points in D

② Find extreme values of f at the boundary of D

∂D

③ Compare largest/smallest value.