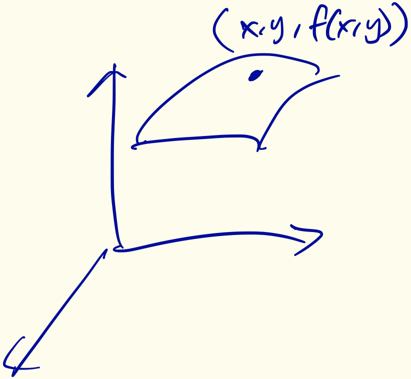
Last Time

Arc Length: $\vec{r}(t) \rightsquigarrow \int |\vec{r}\,'(t)|\, dt$

parametrization by arc length: $\vec{r}_{arc}(s)$, $|\vec{r}_{arc}\,'(s)| = 1$.

$$\vec{r}_{arc}(s) = \vec{r}(t(s)) \quad \text{where } s(t) = \int_0^t |\vec{r}\,'(u)|\, du.$$

$z = f(x,y)$

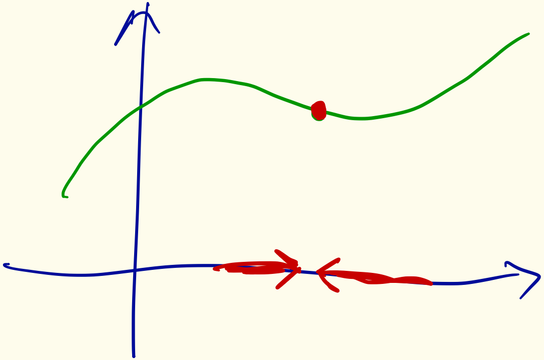


Level curves:

$$f(x,y) = k.$$

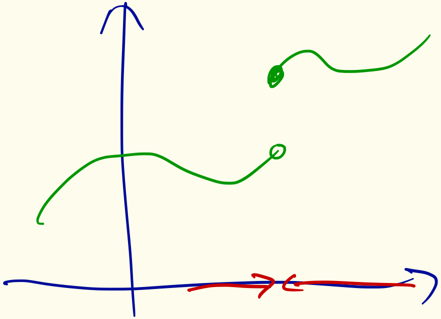
constant.

Single Variable



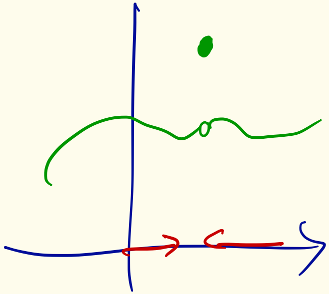
limit exists ✓

continuous



limit does not exist.

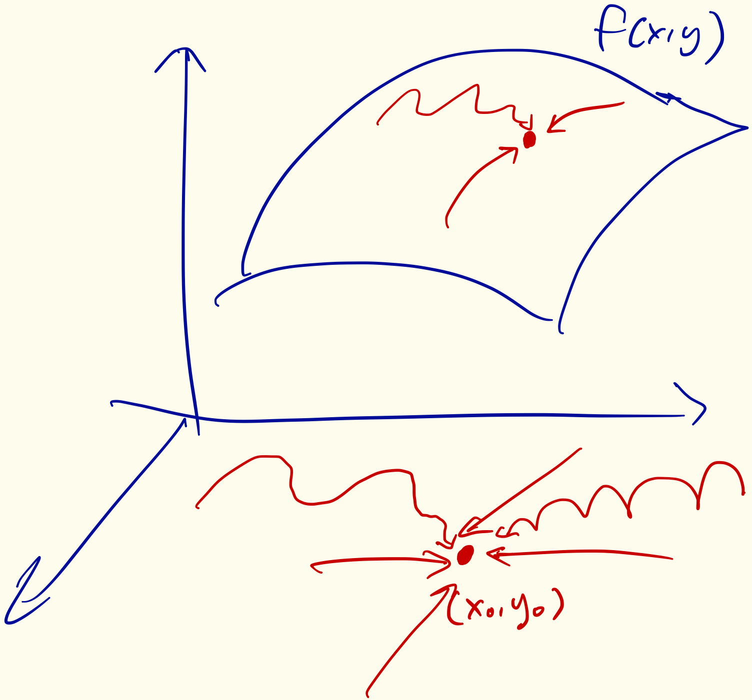
not continuous



limit exists ✓

not continuous

Multivariable Calculus:



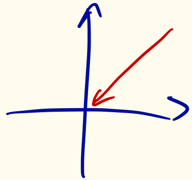
Limit exists

$\Longleftrightarrow$ Limit exist on "all" <u>possible</u> paths on the surface.

$$\underline{\text{Ex}} \quad f(x,y) = \frac{x^2-y^2}{x^2+y^2}$$

along $x=y$

$$f(x,x) = 0$$

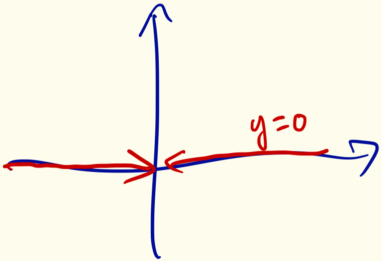


$$\lim_{(x,y)\to(0,0)} f(x,y)$$

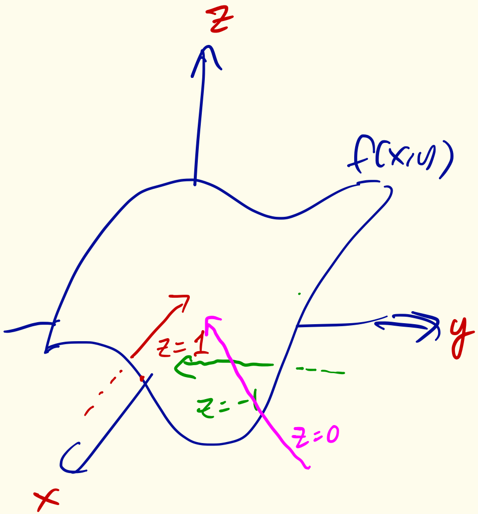
$$\boxed{\text{Limit does not exist}}$$

Test different paths:

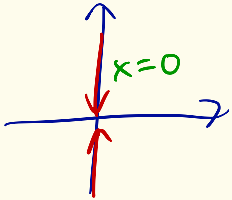
along x-axis



$$f(x,0) = \frac{x^2}{x^2} = 1$$

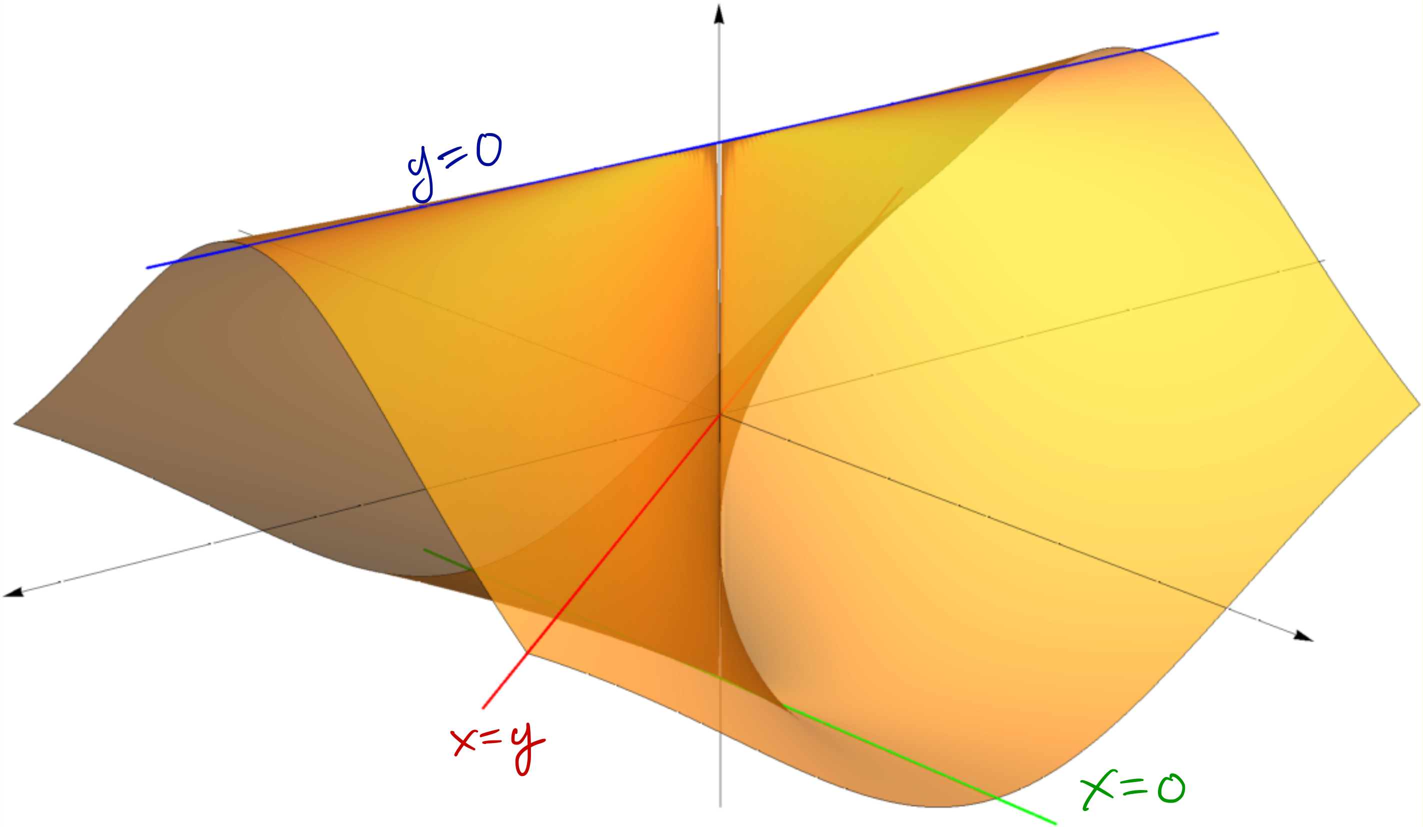


along y-axis



$$f(0,y) = -\frac{y^2}{y^2} = -1$$

Made with Goodnotes



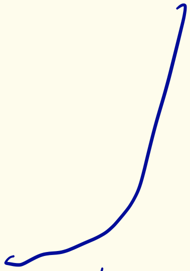
y=0
x=y
x=0

Ex $f(x,y) = \dfrac{x+y}{x^2+y^2}$

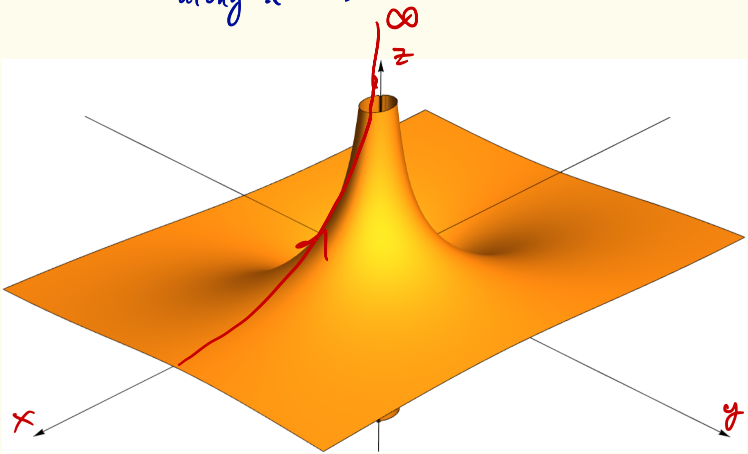
$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$

along x-axis: $y=0$

$$f(x,0) = \frac{x}{x^2} = \frac{1}{x}$$



along x-axis.



limit does not exist

Ex $f(x,y) = \dfrac{xy}{x^2+y^2}$ $\lim_{(x,y)\to(0,0)} f(x,y) = ?$

$$\left.\begin{aligned} &x\text{-axis}: f(x,0) = 0 \\ &y\text{-axis}: f(0,y) = 0 \\ &x=y : f(x,x) = \frac{x^2}{2x^2} = \frac{1}{2} \end{aligned}\right\}$$ limit does not exist.

Ex $f(x,y) = \dfrac{x^4+\sin^2 y}{2x^4+y^2}$

$$\left.\begin{aligned} &x\text{-axis}: f(x,0) = \frac{1}{2} \\ &y\text{-axis}: f(0,y) = \frac{\sin^2 y}{y^2}, \quad \lim_{y\to 0}\left(\frac{\sin y}{y}\right)^2 = 1. \end{aligned}\right\}$$ does not exist.

Ex $f(x,y) = \dfrac{xy^2}{x^2+y^4}$.

x-axis : $f(x,0) = 0$

y-axis : $f(0,y) = 0$

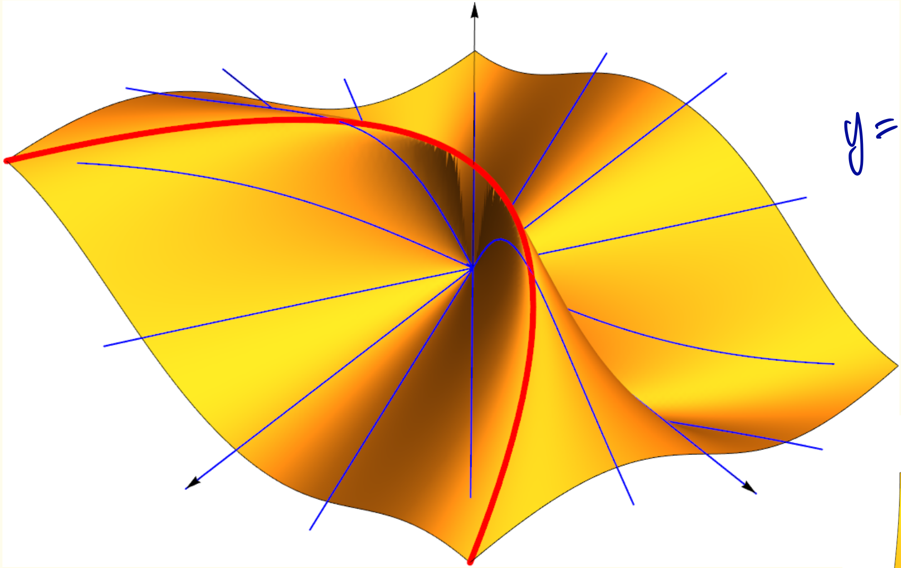
$x=y$: $f(x,x) = \dfrac{x^3}{x^2+x^4} \xrightarrow[x\to 0]{\lim} \dfrac{x}{1+x^2} = 0$

$y=mx$: $f(x,mx) = \dfrac{m^2x^3}{x^2+m^4x^4} = \dfrac{m^2x}{1+m^4x^2} \xrightarrow[x\to 0]{} 0$

along:

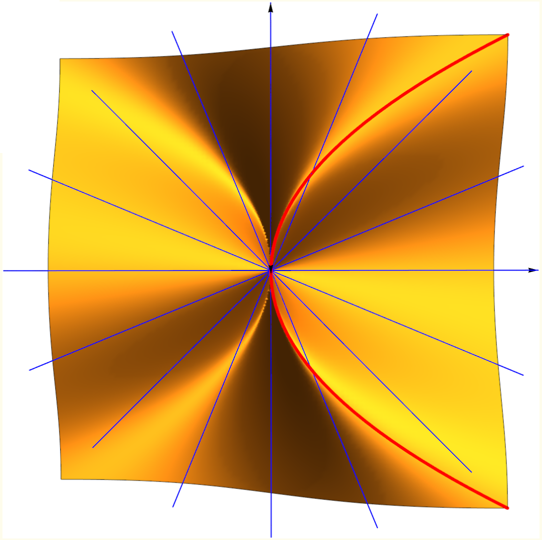
$x=y^2$: $f(y^2,y) = \dfrac{y^2y^2}{y^4+y^4} = \dfrac{1}{2}$.

limit does not exist!



$y = mx$

$x = y^2$



Prove existence : Test all paths ?

Techniques #1: Squeeze Theorem.

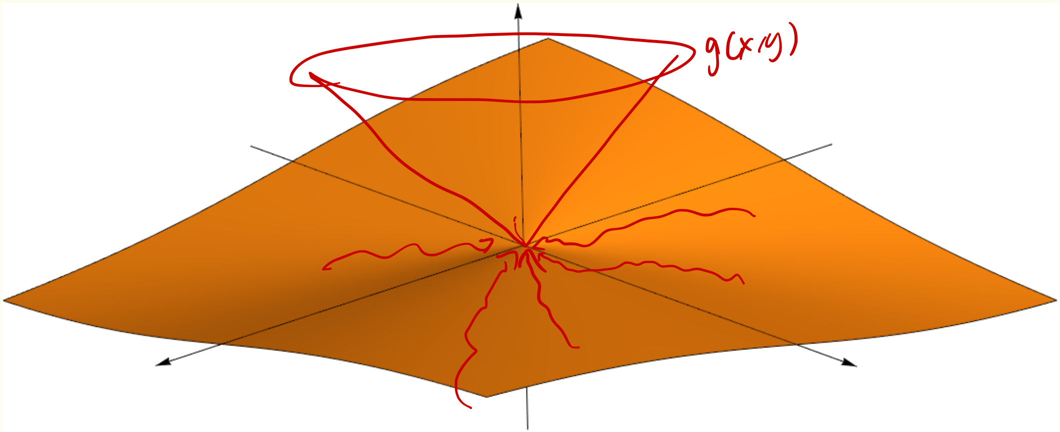
Thm If $0 \le |f(x,y)| \le g(x,y)$ and

if $\lim_{(x,y)\to(0,0)} g(x,y) = 0 \implies \lim_{(x,y)\to(0,0)} f(x,y) = 0$.

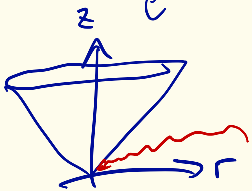
Ex $f(x,y) = \dfrac{2xy}{\sqrt{x^2+y^2}}$, NOTE $|2xy| \le x^2+y^2$. $\left(\begin{array}{c} x^2 \pm 2xy + y^2 \\ \shortparallel \\ (x\pm y)^2 \ge 0 \end{array} \right)$

$|f(x,y)| \le \dfrac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} \longrightarrow 0$ in the limit $(x,y)\to(0,0)$.

$\implies$ By Squeeze Thm, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.



$g(x,y) = \sqrt{x^2+y^2}$:



cone

Technique #2 Use Polar Coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Longrightarrow x^2+y^2 = r^2$$

$$\lim_{(x,y)\to(0,0)} = \lim_{r\to 0^+}$$

Ex $$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0^+} \frac{\sin r^2}{r^2} = 1$$

Ex $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\ln(x^2+y^2)} = \lim_{r\to 0^+} \frac{r^2}{\ln r^2} = \text{“}\frac{0}{\infty}\text{''} = 0.$$

Continuity:

Def $f(x,y)$ is continuous at (a,b) if

$\lim_{(x,y)\to(a,b)} f(x,y)$ exists and equal $f(a,b)$.

Ex Polynomials. $e^{\cdots}$, $\sin$, $\cos$ etc.

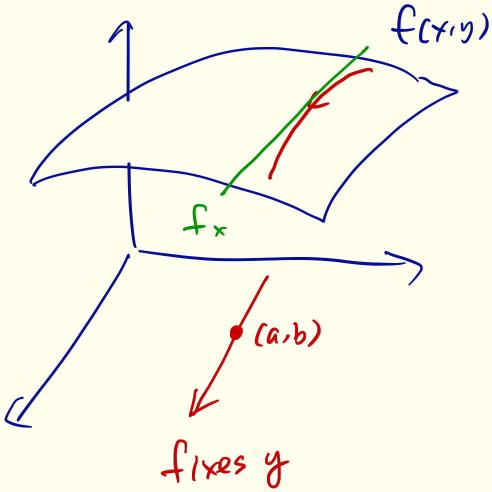
$$\frac{f(x,y)}{g(x,y)} \leftarrow \neq 0.$$

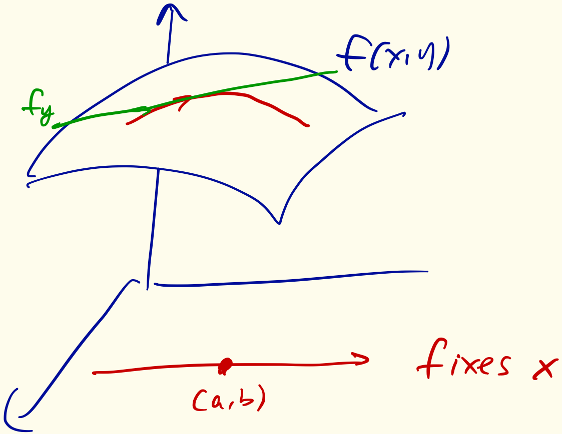
Ex $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} \\ 0 \quad \text{at } (0,0) \end{cases}$ not continuous at $(0,0)$

domain $=\mathbb{R}^2$

but continuous everywhere else. $(x^2+y^2\neq 0)$

Partial Derivative





$$f_x(a,b)$$

$$= \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$= \frac{\partial f}{\partial x}(a,b)$$

↑ "partial"

$$f_y(a,b)$$

$$= \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

$$= \frac{\partial f}{\partial y}(a,b).$$

Ex $f(x,y) = x^2y^3 + 3x^2y - 2y^5$

$$\frac{\partial f}{\partial x} = 2xy^3 + 6xy$$

$$\frac{\partial f}{\partial y} = x^2 3y^2 + 3x^2 - 10y^4$$

$$\frac{\partial f}{\partial x}(2,1) = 4\cdot 1 + 6\cdot 2\cdot 1 = 16 \quad \cdots$$

$$\frac{\partial f}{\partial y}(2,1) = 4\cdot 3\cdot 1 + 3\cdot 4 - 10 = 14 \ .$$

Made with Goodnotes

Ex $f(x,y) = \sin(x^2 y)$

$$\frac{\partial f}{\partial x} = \cos(x^2 y) \cdot 2xy$$

$$\frac{\partial f}{\partial y} = \cos(x^2 y) \cdot x^2 \cdots$$

Ex $f(x,y) = \tan^{-1}(x+y^2)$

$$\frac{\partial f}{\partial x} = \frac{1}{1+(x+y^2)^2} \cdot 1$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+(x+y^2)^2} \cdot 2y$$

Ex $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

not continuous!

$(x=0 \quad \text{or} \quad x=y)$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0.$$

$$f_x(x,y) = \frac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2}$$

$$= \frac{y(y^2-x^2)}{(x^2+y^2)^2} \quad (x,y) \neq (0,0)$$

Second Partial Derivatives

$f, f', f'', \ldots$

$f, f_x, f_{xx} \ldots \quad (f_x)_y \quad (f_y)_x \quad f_{yy}$

$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, $(f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, $(f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$, $f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

Ex $f(x,y) = \sin(x^2 y)$

$$f_x = \cos(x^2 y) \cdot 2xy$$

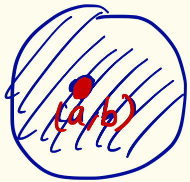
$$f_y = \cos(x^2 y) \cdot x^2$$

$$f_{xy} = \underbrace{-\sin(x^2 y) \cdot x^2 \cdot 2xy}_{-2\sin(x^2 y)\, x^3 y} + \cos(x^2 y) \cdot 2x$$

$$\|$$

$$f_{yx} = -\sin(x^2 y) \cdot 2xy \cdot x^2 + \cos(x^2 y) \cdot 2x$$

Clairaut's Theorem
(Mixed Partials Theorem)



<u>Thm</u> If $f(x,y)$ is defined on a disk containing (a,b) and f_{xy} exists and continuous at (a,b)

then f_{yx} exists and

$$f_{xy}(a,b) = f_{yx}(a,b).$$

<u>Pf</u> MATH 3033 //