HKUST **MATH 101**

Midterm Examination

Multivariable Calculus

30 October 2002

Answer ALL 5 questions

Time allowed – 120 minutes

Directions – This is a closed book examination. No talking or whispering are allowed. Work must be shown to receive points. An answer alone is not enough.

Note that you can work on *both* sides of the paper and no part of these papers is to be torn out.

Student Name: ____________________

Student Number: ____________________

Tutorial Session: ____________________

(1) **(a)** Find the distance (in terms of $\mathbf{n}$, $\mathbf{r}_0$ and $\mathbf{r}_1$ only) from the point $\mathbf{r}_1$ to the plane $(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{n} = 0$.

(b) A rigid body rotates about an axis through point O with angular velocity $\boldsymbol{\omega}$.

(i) Find the linear velocity $\mathbf{v}$ of a point P of the body with position vector $\mathbf{r}$.

(ii) Show that the vector $-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ is directed away from the axis of rotation and lies on the plane containing the vector $\boldsymbol{\omega}$ and $\mathbf{r}$.

(2) **(a)** Can the function $f(x,y) = \dfrac{\sin x \sin^3 y}{1-\cos(x^2+y^2)}$ be defined at $(0,0)$ in such a way that it becomes continuous there? If so, how?

(b) Let $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}, & \text{if } (x,y)\neq(0,0), \\ 0, & \text{if } (x,y)=(0,0). \end{cases}$

Calculate each of the following partial derivatives or explain why it does not exist:

(i) $f_x(0,0)$, (ii) $f_y(0,0)$, (iii) $f_{yx}(0,0)$, (iv) $f_{xy}(0,0)$ and (v) $f_{xx}(0,0)$.

Is the function $f(x,y)$ differentiable at $(0,0)$? Explain.

(3) **(a)** Show that the curve $\mathbf{r}(t) = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t\,\mathbf{k}$, $t\geqslant 0$, lies on the surface of the form $z = f(x,y)$. Find $f(x,y)$. Describe (or sketch) the curve.

(b) Find a vector equation of the line tangent to the graph of

$$\mathbf{r}(t) = t^2\,\mathbf{i} - \frac{1}{t+1}\,\mathbf{j} + (4-t^2)\,\mathbf{k}$$

at the point $(4,1,0)$ on the curve. Find also the arc length of the curve $\mathbf{r}(t)$ from point $(4,1,0)$ to point $(0,-1,4)$.

(4) **(a)** Find the equation of the level curve of the function $z = g(x,y) = xf(xy)$ at the point (x_0,y_0), where both f and g are differentiable. Show that $\nabla g(x_0,y_0)$ is normal to the tangent line to the level curve at (x_0,y_0).

(b) If $w = f(x,y)$ (assume f is differentiable) and $x = s^2+t^2$, $y = s^2-t^2$, use the chain rule to find (i) w_s, (ii) w_{st} and (iii) w_{stt}.

(5) Find the point(s) on the surface $z^2 = -\frac{1}{2}x^2+2y^2+xy$ that are closest to the point $\left(-\frac{1}{2},-3,0\right)$

(a) by reducing the problem to an unconstrained problem in two variables, and

(b) using the method of Lagrange multipliers.