

## SAMPLE MIDTERM

**Course Code:** MATH 2023  
**Course Title:** Multivariable Calculus  
**Time Limit:** 2 Hours

### Instructions

- Do **NOT** open the exam until instructed to do so.
- This is a **CLOSED BOOK, CLOSED NOTES** exam.
- All mobile phones and communication devices should be switched **OFF**.
- Only calculators approved by HKEAA can be used.
- Answer **ALL** eight problems.
- You must **SHOW YOUR WORK** to receive credits in all problems except Problem #1. Answers alone (whether correct or not) will not receive any credit.
- Some problems are structured into several parts. You can quote the results stated in the preceding parts to do the next part, regardless of whether you can complete the preceding parts or not. However, different parts in the same problem are not necessarily co-related.

### About this practice test

The purpose of this sample midterm is to let students get a rough idea of the style of problems and the format of the exam. Do **NOT** expect the actual exam is simply a minor variation of this sample exam. The level of difficulties, the point allocation of each problem, and the choice of topics may be different from the actual test. For better preparation of the test, students should extensively review the course materials covered in class, in the lecture notes and in the lecture worksheets, and should have worked seriously on the Problem Sets and WebWorks.

| Problem | 1  | 2  | 3 | 4  | 5  | 6  | 7  | 8  | Total |
|---------|----|----|---|----|----|----|----|----|-------|
| Max     | 25 | 10 | 8 | 10 | 10 | 15 | 12 | 10 | 100   |
| Score   |    |    |   |    |    |    |    |    |       |

## FORMULAE TABLE

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\sin(\theta \pm \phi) = \sin \theta \cos \phi \pm \cos \theta \sin \phi$$

$$\cos(\theta \pm \phi) = \cos \theta \cos \phi \mp \sin \theta \sin \phi$$

$$\tan(\theta \pm \phi) = \frac{\tan(\theta) \pm \tan(\phi)}{1 \mp \tan \theta \tan \phi}$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$$

$$= 1 - 2 \sin^2 \theta$$

$$= 2 \cos^2 \theta - 1$$

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

---

For any  $C^2$  function  $f(x, y)$  and  $z = f(a + tu_1, b + tu_2)$ , we have:

$$\begin{aligned} \frac{d^2 z}{dt^2} &= f_{xx} u_1^2 + 2f_{xy} u_1 u_2 + f_{yy} u_2^2 \\ &= f_{xx} \left[ \left( u_1 + \frac{f_{xy}}{f_{xx}} u_2 \right)^2 + \left( \frac{f_{xx} f_{yy} - f_{xy}^2}{f_{xx}^2} \right) u_2^2 \right] \quad \text{if } f_{xx} \neq 0 \end{aligned}$$

1. Answer the following conceptual questions. No justification is needed.

- (a) Let  $\Pi$  be the plane in  $\mathbb{R}^3$  whose equation is given by  $x + 2y + 3z = 1$ . Write down a three-variable function  $g(x, y, z)$  such that  $\Pi$  is a level set of  $g$ . /1

ANSWER:

$$g(x, y, z) = x + 2y + 3z$$

- (b) Let  $\Pi$  be the plane in  $\mathbb{R}^3$  whose equation is given by  $x + 2y + 3z = 1$ . Write down a two-variable function  $f(x, y)$  such that the plane  $\Pi$  is the graph of this function  $f$ . /2

ANSWER:

$$f(x, y) = \frac{1 - x - 2y}{3}$$

- (c) Consider the parametric curve  $\mathbf{r}(t) = (\cos 2t) \mathbf{i} + (\sin 2t) \mathbf{j}$ . Write down an arc-length parametrization  $\mathbf{r}(s)$  of this curve. /2

ANSWER:

$$\mathbf{r}(s) = \cos(s) \mathbf{i} + \sin(s) \mathbf{j}$$

- (d) Let  $f(x, y)$  be a  $C^1$  function such that  $\nabla f(0, 0) = 3\mathbf{i} + 4\mathbf{j}$ . Write down a unit vector  $\mathbf{u}$  such that  $D_{\mathbf{u}}f(0, 0)$  is maximized. /2

ANSWER:

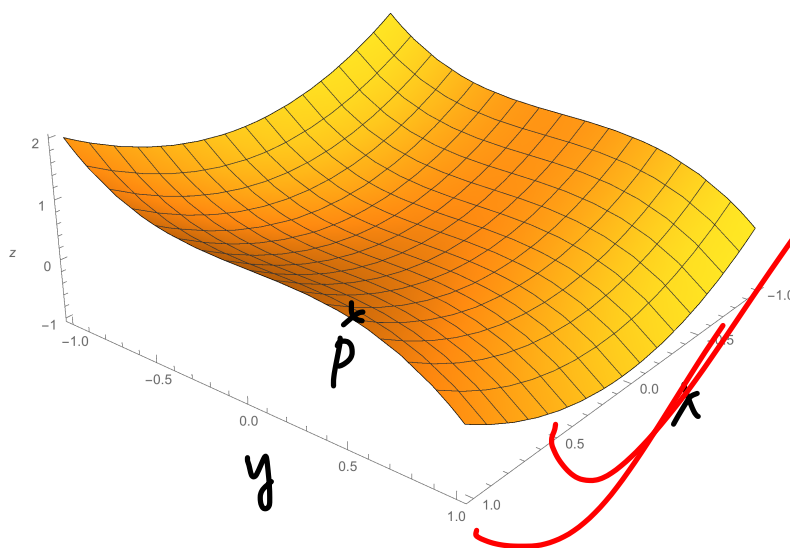
$$\frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j}$$

- (e) Let  $f(x, y)$  be a  $C^1$  function. Suppose  $\Pi$  is the tangent plane to the graph  $z = f(x, y)$  at a point  $(x_0, y_0, f(x_0, y_0))$ . Write down an upward-pointing normal  $\mathbf{n}$  to the plane  $\Pi$ . /2

ANSWER:

$$\frac{1}{n} = \left\langle -\frac{\partial f}{\partial x} \Big|_{x=x_0}, -\frac{\partial f}{\partial y} \Big|_{y=y_0}, 1 \right\rangle$$

- (f) The following diagram shows the graph of the function  $f(x, y) = x^2 - y^3$ . The positive  $z$ -axis is pointing up. However, the labels for the  $x$ - and  $y$ -axes are missing. /4



- i. On the diagram above, label the  $x$ - and  $y$ -axes.  
 ii. Mark a point  $P$  on the graph at which  $f_x(P) > 0$  and  $f_{yy}(P) < 0$ .

down

Problem #1 continues on next page...

$$\vec{r}(t) = \cos(2t) \mathbf{i} + (\sin 2t) \mathbf{j}$$

$$\vec{r}'(t) = -2 \sin(2t) \mathbf{i} + 2 \cos 2t \mathbf{j}$$

$$|\vec{r}'(t)| = \sqrt{4 \sin^2(2t) + 4 \cos^2(2t)}$$

$$= 2$$

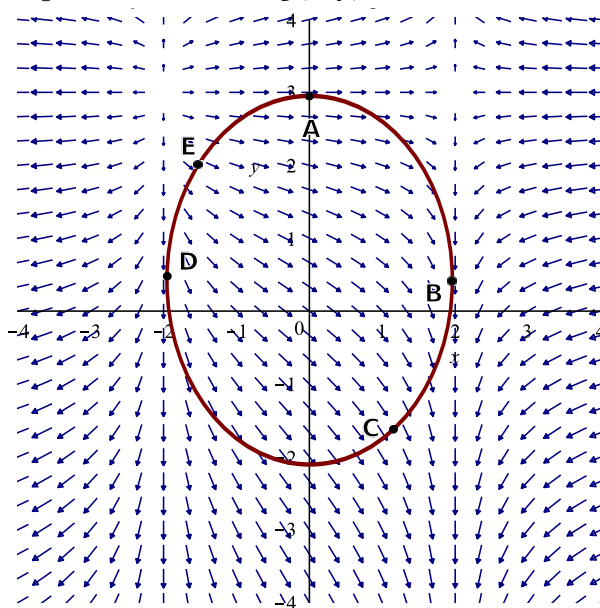
$$s = \int_0^t 2 \, dz$$

$$s = 2t$$

$$t = \frac{s}{2}$$

- (g) Let  $f(x, y)$  be a  $C^1$  function. The diagram below is the plot of the vector field  $\nabla f$ . The ellipse in the diagram is a level set  $g(x, y) = c$  of another  $C^1$  function  $g$ .

/8



- i. What is  $\frac{\partial f}{\partial x}(a, b)$  when  $a = 2$ ?

i. 0

- ii. How many critical point(s) of  $f$  is/are there in the region  $-4 \leq x \leq 0$  and  $-4 \leq y \leq 4$ ?

ii. 1

- iii. Which of the point A, B, C, D and E has/have  $(x, y)$ -coordinates solving the following Lagrange's Multiplier system? State all such point(s).

$$\nabla f(x, y) = \lambda \nabla g(x, y), \quad g(x, y) = c$$

iii. A, B, D

- iv. Suppose  $f(x, y)$  represents the temperature at point  $(x, y)$ . Now you are located at point A and want to cool down as quickly as possible. Which direction will you go along?

iv. to negative x direction

- (h) Let  $f(x, y)$  be a  $C^2$  function such that  $(x_0, y_0)$  is its critical point. Suppose we know:

/2

$$\begin{matrix} <0 & <0 \\ f_{xx}f_{yy} & > f_{xy}^2 \end{matrix}$$

$$\left( f_{xx}f_{yy} - f_{xy}^2 \right) (x_0, y_0) > 0 \quad \text{and} \quad f_{yy}(x_0, y_0) < 0$$

What is the nature of the critical point  $(x_0, y_0)$ ? Circle the correct answer:

local minimum    local maximum    saddle    not enough data

- (i) Let  $f(x, y)$  be a  $C^2$  function such that  $(x_0, y_0)$  is its critical point. Suppose we know:

/2

$$f_{xx}(x_0, y_0) = 25, \quad f_{xy}(x_0, y_0) = 5, \quad f_{yy}(x_0, y_0) = 1.$$

What is the nature of the critical point  $(x_0, y_0)$ ? Circle the correct answer:

local minimum    local maximum    saddle    not enough data

2. Suppose  $u(x, y)$  and  $v(x, y)$  are  $C^2$  functions defined on  $\mathbb{R}^2$  which satisfy the relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (*)$$

[FYI: It is so-called the Cauchy-Riemann relations in Complex Analysis.]

$$\begin{array}{c} \text{V. U} \\ \diagdown \quad \diagup \\ \text{x} \quad \text{y} \end{array}$$

- (a) Show that  $v(x, y)$  satisfies the Laplace equation:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$$\frac{\partial}{\partial x} \frac{\partial v}{\partial x} = \frac{\partial}{\partial x} - \frac{\partial u}{\partial y}$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial x} \left( -\frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \frac{\partial u}{\partial x}$$

$$= -\frac{\partial u}{\partial xy} + \frac{\partial u}{\partial yx}$$

$$= 0.$$

- (b) Suppose we define  $v(x, y) = x^3 - 3xy^2 + 10$ . Find a function  $u(x, y)$  such that  $u$  and  $v$  satisfy the relation (\*) above.

2. Suppose  $u(x, y)$  and  $v(x, y)$  are  $C^2$  functions defined on  $\mathbb{R}^2$  which satisfy the relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (*)$$

[FYI: It is so-called the Cauchy-Riemann relations in Complex Analysis.]

(a) Show that  $v(x, y)$  satisfies the Laplace equation:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$\begin{matrix} v & u \\ x & y \end{matrix}$

(b) Suppose we define  $v(x, y) = x^3 - 3xy^2 + 10$ . Find a function  $u(x, y)$  such that  $u$  and  $v$  satisfy the relation (\*) above.

/6

$$\frac{\partial v}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial v}{\partial y} = -6xy$$

$$\frac{\partial u}{\partial y} = 3y^2 - 3x^2, \quad \frac{\partial u}{\partial x} = -6xy.$$

$$u(x, y) = y^3 - 3x^2y + C, \text{ which } C \text{ is constant.}$$

3. Consider the function  $f(x, y) = 1 - x^2 - y^2$  and the point  $P = (\sqrt{a}, 0, 0)$ .

(a) Find the value of  $a$  such that the point  $P$  is on the graph of  $f$ .

$$a=1$$

$$\begin{aligned} 1 - a - 0 &= 0 \\ 1 - a &= 0 \\ a &= 1 \end{aligned}$$

/1

(b) Find a downward-pointing normal vector  $\mathbf{n}$  to the graph of  $f$  at  $P$ .

$$\nabla f = \langle -2x, -2y, -1 \rangle$$

$$\nabla f(1, 0, 0) = \langle -2, 0, -1 \rangle$$

/4

(c) Find an equation of the tangent plane to the graph of  $f$  at  $P$ .

$$-2x - z = -2$$

$$2x + z = 2$$

/3



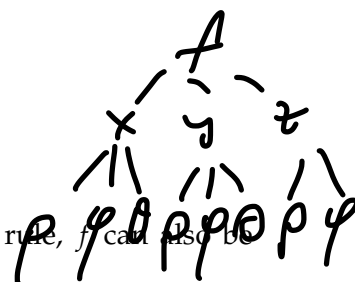
4. The spherical coordinates  $(\rho, \theta, \varphi)$  in  $\mathbb{R}^3$  are related to the rectangular coordinates  $(x, y, z)$  by the following conversion rule:

$$x = \rho \sin \varphi \cos \theta$$

$$y = \rho \sin \varphi \sin \theta$$

$$z = \rho \cos \varphi$$

Let  $f(x, y, z)$  be a three-variable function. By the above conversion rule,  $f$  can also be regarded as a function of  $(\rho, \theta, \varphi)$ .



- (a) Using the chain rule, show that:

/6

$$\begin{aligned} \frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} \\ &= \frac{\partial f}{\partial x} (\rho \sin \varphi \sin \theta) + \frac{\partial f}{\partial y} (\rho \sin \varphi \cos \theta) \\ &= \frac{\partial f}{\partial x} (-y) + x \frac{\partial f}{\partial y} \end{aligned}$$

- (b) Suppose  $\nabla f(x, y, z)$  is orthogonal to the vector  $-y\mathbf{i} + x\mathbf{j}$  for any  $(x, y, z)$  in  $\mathbb{R}^3$ . Show that  $f$  does not depend on  $\theta$ .

/4

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \langle -y, x \rangle = 0$$

$$-y \left( \frac{\partial f}{\partial x} \right) + x \left( \frac{\partial f}{\partial y} \right) = 0$$

$$\frac{\partial f}{\partial \theta} = 0$$

$\therefore f$  does not depend on  $\theta$

5. Consider the function  $f(x, y) = e^{-(x^2+y^2)}$ .

(a) Find all critical point(s) of  $f(x, y)$ .

/4

$$f_x = e^{-(x^2+y^2)} (-2x)$$

$$f_y = e^{-(x^2+y^2)} (-2y)$$

Solving  $f_x = f_y = 0$ , we have  $x = y = 0$

(b) For each critical point of  $f$  found in (a), determine whether it is a local maximum, a local minimum or a saddle point.

/6

$$\frac{d}{dr} e^{-r^2} = -2re^{-r^2}, \quad \frac{d^2}{dr^2} -2re^{-r^2}$$

$$= -2(r(-2re^{-r^2}) + e^{-r^2}) = -2(-2r^2 + 1)(e^{-r^2})$$

$$\text{For } r=0, \text{ we have } \frac{d^2 f}{dr^2} = -2(1)(1) = -2 < 0$$

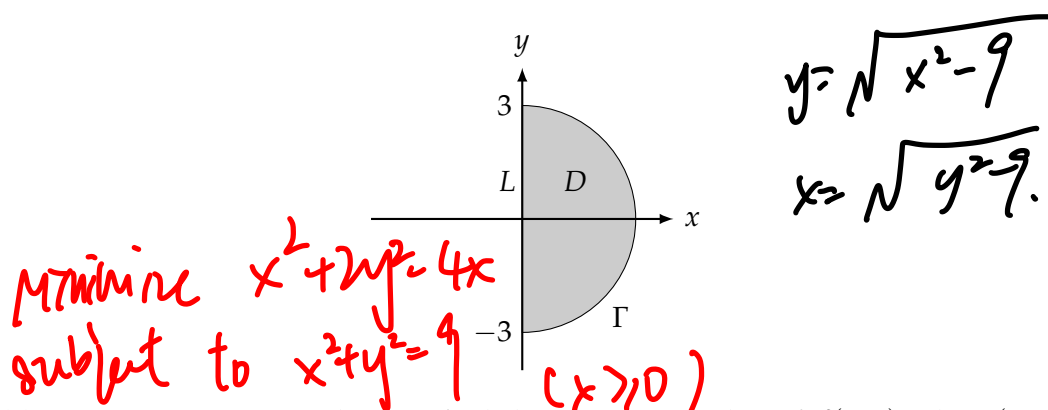
it is local maximum,

6. Consider the function  $f(x, y) = x^2 + 2y^2 - 4x$ . Let  $D$  be the semi-disk region (shaded in the diagram below) defined by:

$$x \geq 0 \quad \text{and} \quad x^2 + y^2 \leq 9.$$

The boundary of  $D$  has two components:

- $L$  is the line segment joining  $(0, 3)$  and  $(0, -3)$ ; and
- $\Gamma$  is the circular arc joining  $(0, 3)$  and  $(0, -3)$ .



- (a) Using Lagrange's Multiplier, find the minimum value of  $f(x, y)$  when  $(x, y)$  is restricted to the curve  $\Gamma$ .

/8

$$\Gamma = \sqrt{y^2 - 9} - x.$$

$$\left(\frac{2}{3}, \pm \frac{\sqrt{5}}{3}\right)$$

$$\nabla \Gamma = \left\langle -1, \frac{y}{\sqrt{y^2 - 9}} \right\rangle$$

Minimum of  $f$  =

$$\nabla f = \langle 2x - 4, 4y \rangle$$

$$\left(\frac{2}{3}\right)^2 + 2\left(\frac{\sqrt{5}}{3}\right)^2 - 4\left(\frac{2}{3}\right)$$

$$\begin{cases} 2x - 4 = \lambda(-1) \\ 4y = \frac{y}{\sqrt{y^2 - 9}} (\lambda) \\ x = \sqrt{y^2 - 9} \end{cases}$$

$$= \frac{4}{9} + \frac{10}{9} - \frac{8}{3} = \frac{10}{9} - \frac{8}{3} = \frac{10 - 24}{9} = -\frac{14}{9}$$

$$\begin{aligned} 2x - 4 &= -\lambda \\ 4 - 2x &= \lambda \end{aligned}$$

$$y = 0 \Rightarrow 4 = \frac{1}{x} (4 - 2x)$$

$$4x = 4 - 2x$$

$$6x = 4$$

$$x = \frac{2}{3}$$

test  $(0, 3)$   $(0, -3)$

$$\text{When } x = \frac{2}{3}, \frac{4}{9} = y^2$$

$$\frac{4}{9} = y^2 \Rightarrow y = \pm \frac{\sqrt{85}}{3}$$

Problem #6 continues on next page...

For easy reference:  $f(x, y) = x^2 + 2y^2 - 4x$ .

- (b) Find the minimum value of  $f(x, y)$  when  $(x, y)$  is restricted to the line  $L$ .

/3

When  $x=0$ ,  $f(x, y)$  obtain its minimum at  $2y^2=0$ ,

$$y=0,$$

$$\therefore f(x, y) = 0.$$

- (c) Determine the minimum value of  $f(x, y)$  over the solid region  $D$ .

/4

$$\nabla f = \langle 2x-4, 4y \rangle$$

Solving  $\nabla f = \vec{0}$ , we have  $\{x=2, y=0\}$ .

$$f(2, 0) = 4 + 0 - 4 = 0. -4$$

$\therefore$  Minimum value = 0.

7. Let  $\mathbf{c}(t)$  be parametric curve in  $\mathbb{R}^3$  such that  $|\mathbf{c}(t)| = 1$  at all time  $t$ . Define another parametric curve  $\mathbf{r}(t)$  by:

$$\mathbf{r}(t) = f(t)\mathbf{c}(t).$$

- (a) Show that  $\mathbf{c}(t)$  and  $\mathbf{c}'(t)$  are orthogonal at all time  $t$ .

/3

$$\begin{aligned} |\mathbf{c}(t)| &= 1 \\ \mathbf{c}(t) \cdot \mathbf{c}(t) &= 1^2 \\ \frac{d}{dt} \mathbf{c}(t) \cdot \mathbf{c}(t) &= 0 \end{aligned}$$

$$(\mathbf{c}(t)^2 + \mathbf{c}'(t)^2)$$

- (b) Using (a), or otherwise, show that  $|\mathbf{r}'(t)| = \sqrt{f(t)^2 |\mathbf{c}'(t)|^2 + f'(t)^2}$ .

/4

$$\begin{aligned} \mathbf{r}'(t) &= f(t)\mathbf{c}'(t) + f'(t)\mathbf{c}(t) \\ |\mathbf{r}'(t)| &= \sqrt{f(t)^2 \mathbf{c}'(t)^2 + (f'(t))^2 \mathbf{c}(t)^2} \end{aligned}$$

$$= \sqrt{f(t)^2}$$



/5

$$\mathbf{q}(t) = \frac{3}{5}e^{-t} [(\sin t) \mathbf{i} + (\cos t) \mathbf{k}]$$

[Typo in previous version]

This image shows a single sheet of white paper with horizontal ruling lines. The lines are evenly spaced and run across the width of the page. There are no margins, text, or other markings on the paper.

8. Given a fixed point  $(x_0, y_0, z_0)$  in  $\mathbb{R}^3$ , we denote its position vector by  $\mathbf{p} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ . Let  $\mathbf{r}(t)$  be the path of a particle in  $\mathbb{R}^3$ . Given that  $\mathbf{r}(t) - \mathbf{p}$  is parallel to  $\mathbf{r}''(t)$  at all time  $t$ .

Define:

$$\mathbf{L}_p(t) := (\mathbf{r}(t) - \mathbf{p}) \times \mathbf{r}'(t).$$

- (a) Show that  $\mathbf{L}_p(t)$  does not depend on  $t$ .

$$(\vec{r}(t) - \mathbf{p}) = \lambda \mathbf{r}''(t) \quad /5$$

$$\begin{aligned} \mathbf{L}'_p(t) &:= (\mathbf{r}(t) - \mathbf{p})' \times \mathbf{r}'(t) + (\mathbf{r}(t) - \mathbf{p}) \times \mathbf{r}''(t) \\ &= (\mathbf{r}'(t)) \times \mathbf{r}'(t) \\ &= \mathbf{0}. \end{aligned}$$

$\therefore \mathbf{L}'_p(t)$  does not depend on  $p$ .

- (b) Using (a), show that the path of the particle is contained in a single plane. /5

[Hint: consider the dot product  $\mathbf{L}_p \cdot (\mathbf{r}(t) - \mathbf{p})$ .]

$$\begin{aligned} 2\mathbf{p} \cdot (\mathbf{r}(t) - \mathbf{p}) &= \mathbf{L}_p \cdot \mathbf{r}(t) - \mathbf{L}_p \cdot \mathbf{p} \\ &= \end{aligned}$$

8. Given a fixed point  $(x_0, y_0, z_0)$  in  $\mathbb{R}^3$ , we denote its position vector by  $\mathbf{p} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ . Let  $\mathbf{r}(t)$  be the path of a particle in  $\mathbb{R}^3$ . Given that  $\mathbf{r}(t) - \mathbf{p}$  is parallel to  $\mathbf{r}''(t)$  at all time  $t$ .

Define:

$$\mathbf{L}_p(t) := (\mathbf{r}(t) - \mathbf{p}) \times \mathbf{r}'(t).$$

- (a) Show that  $\mathbf{L}_p(t)$  does not depend on  $t$ .

/5

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

- (b) Using (a), show that the path of the particle is contained in a single plane.  
[Hint: consider the dot product  $\mathbf{L}_p \cdot (\mathbf{r}(t) - \mathbf{p})$ .]

/5

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

---

End of Exam