Last Time

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \qquad \vec{F} = \langle P, Q, R \rangle$$

$P(x,y,z)$ etc.

$$\operatorname{grad} f = \nabla f = \langle f_x, f_y, f_z \rangle$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = P_x + Q_y + R_z$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\vec{i} + \ldots\ldots$$

Special Case: $\vec{F} = \langle P(x,y), Q(x,y), 0 \rangle$

$$\operatorname{curl} \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

Green's Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\operatorname{curl} \vec{F}) \cdot \vec{k} \, dA.$$

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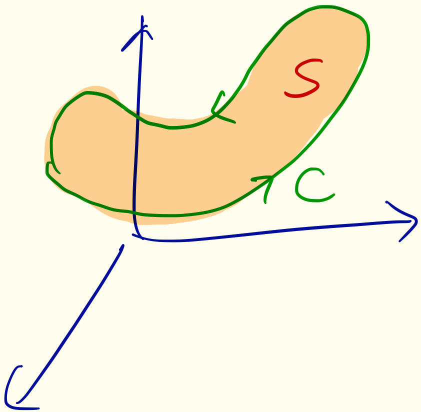
We want C in 3D!

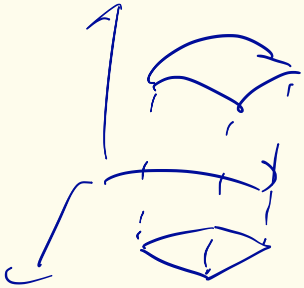


$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dS$$

To Do surface Integration

Need new way to describe surface.

Surface



① Graph: $z = f(x,y)$

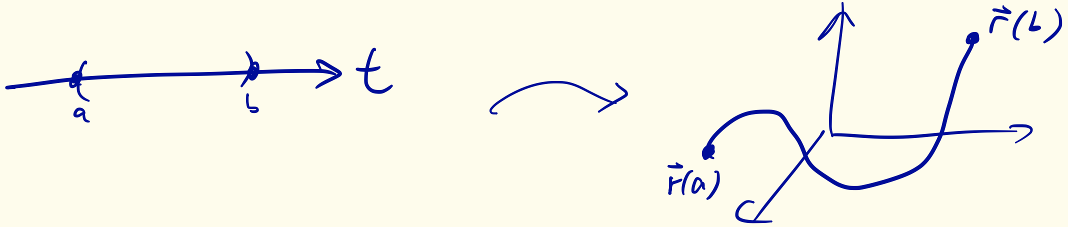
② Level surfaces: $F(x,y,z)=0$

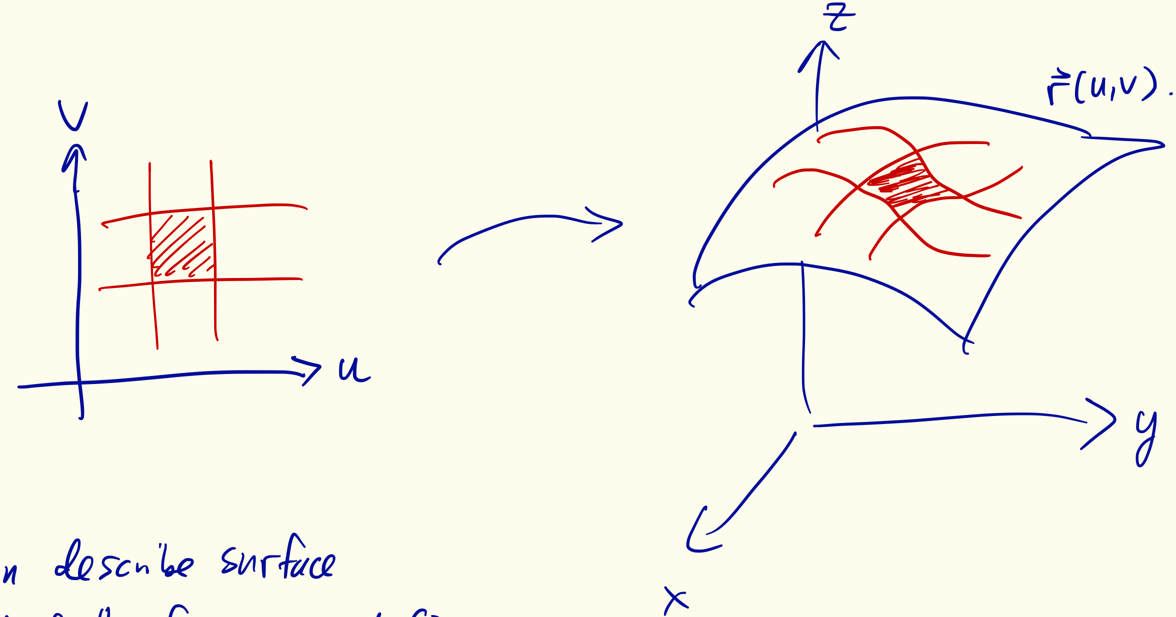
③ Parametric Surface:

parametric curve: $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

parametric surface: $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$.

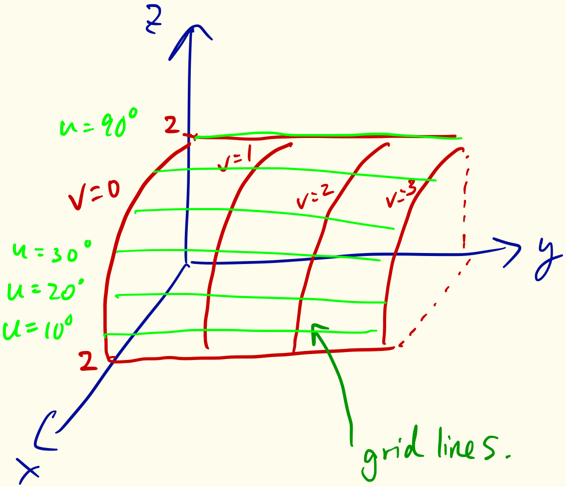
$= x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$.





Can describe surface
not of the form ① and ②

Ex $\vec{r}(u,v) = \langle 2\cos u,\ v,\ 2\sin u \rangle$ $\quad 0 \le v \le 3$, $\ 0 \le u \le \frac{\pi}{2}$.



$v=0 : \langle 2\cos u,\ 0,\ 2\sin u \rangle$ quarter circles.

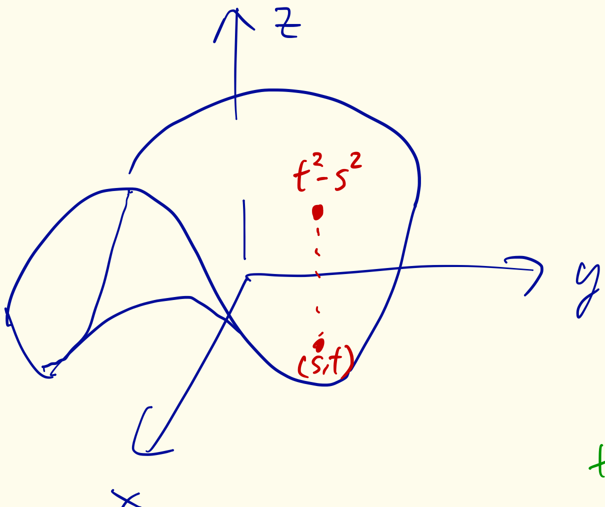
$u=45° : \langle \sqrt{2},\ v,\ \sqrt{2} \rangle$ horizontal lines

Ex $\vec{r}(s,t) = \langle s, t, t^2 - s^2 \rangle$

$\uparrow \quad \uparrow \qquad\qquad \uparrow$

x, y-coord. $\qquad z$

this is same as the graph $z = y^2 - x^2$.



saddle

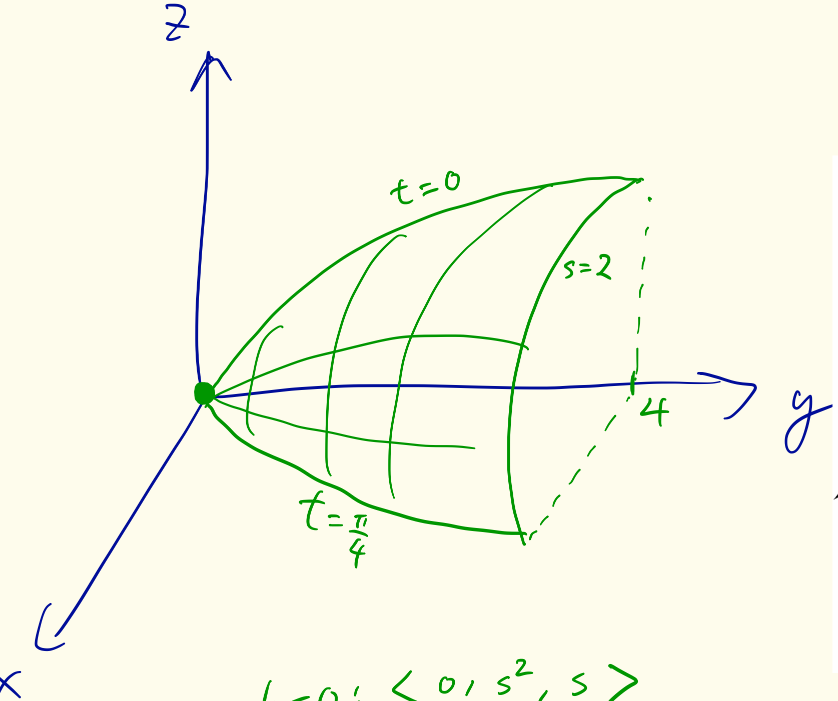
the parametric eq. for graphs.

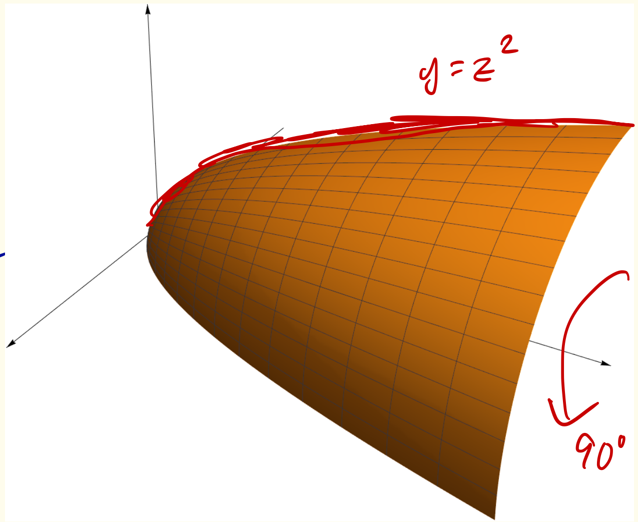
Fact For $z = f(x,y)$, $\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$

Ex $\vec{r}(s,t) = \langle s\sin 2t,\ s^2,\ s\cos 2t \rangle.$

$0 \le s \le 2, \quad 0 \le t \le \frac{\pi}{4}$

rotation! around y-axis. (polar in x-z plane)



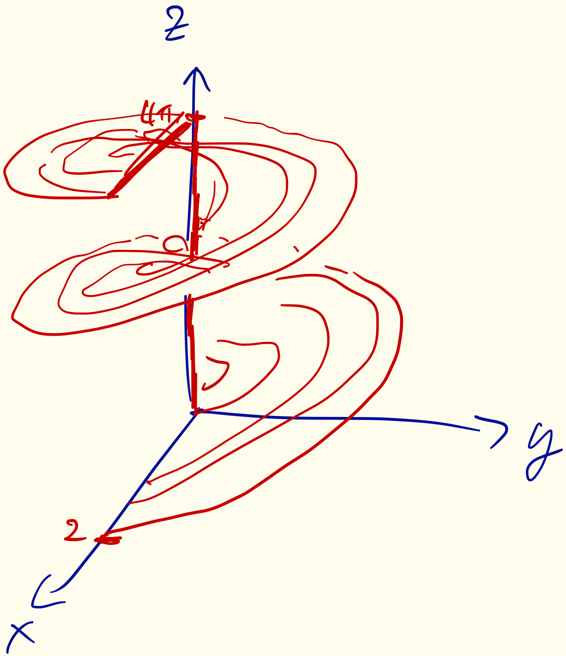


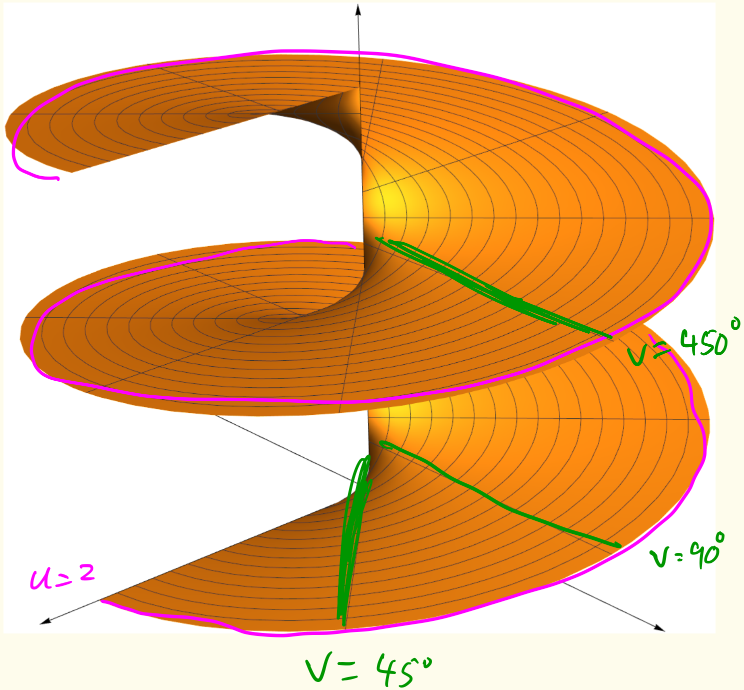
$t=0: \langle 0, s^2, s \rangle$

$\hookrightarrow y = z^2$

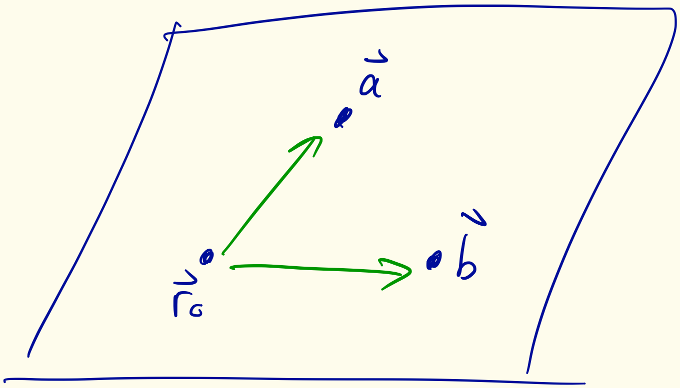
Ex $\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ v \rangle$

$0 \le u \le 2$
$0 \le v \le 4\pi$





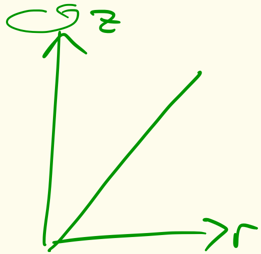
Ex Plane passing through $\vec{r}_0$ and containing $\vec{a}, \vec{b}$.

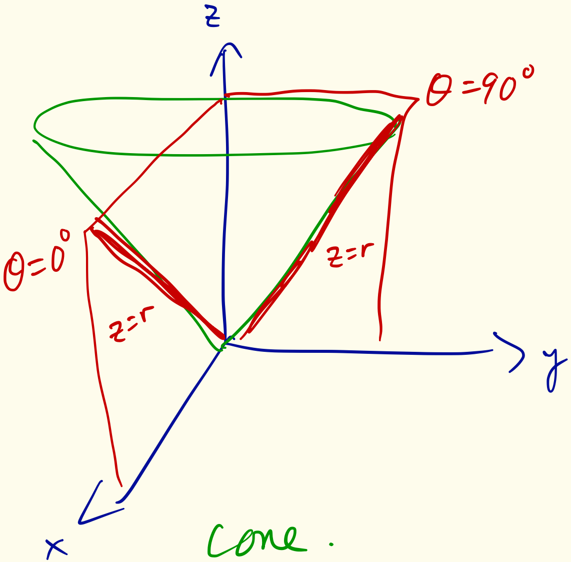


$$\vec{r}(\lambda, \mu) = \vec{r}_0 + \lambda(\vec{a} - \vec{r}_0) + \mu(\vec{b} - \vec{r}_0).$$

$$(\lambda, \mu \in \mathbb{R}).$$

Ex $z = \sqrt{x^2+y^2}$ $\longrightarrow$ polar : $z = r$





Graph :

$$\vec{r}(s,t) = \langle s, t, \sqrt{s^2+t^2} \rangle$$

$(s, t \in \mathbb{R})$.

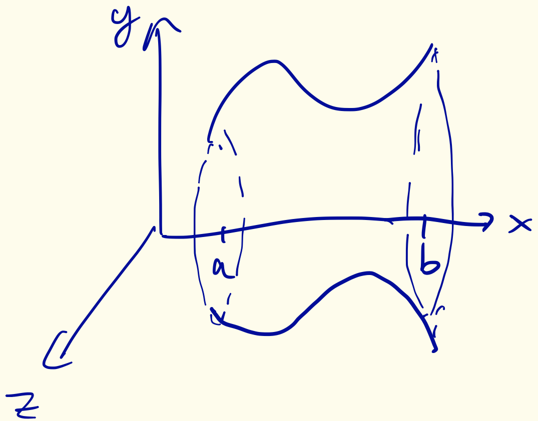
Polar :

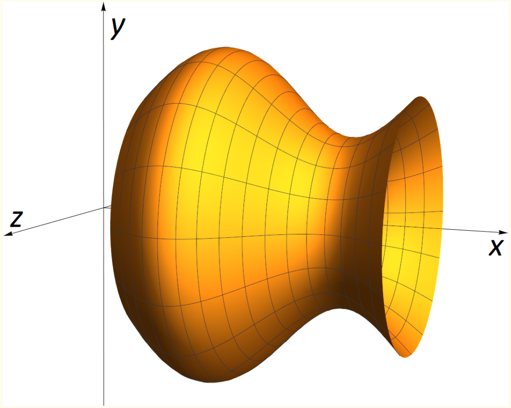
$$\vec{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle.$$

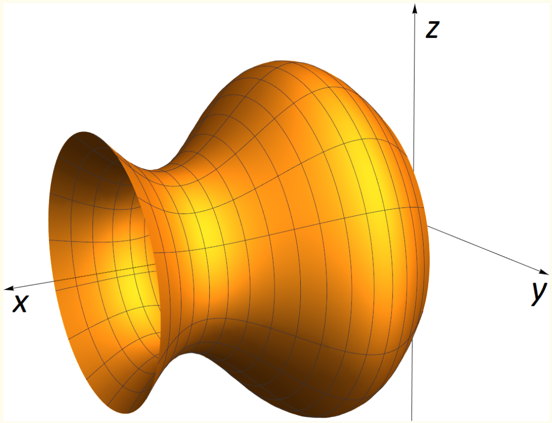
$$\left(\begin{array}{c} 0 \le \theta \le 2\pi \\ r \ge 0 \end{array}\right)$$

Ex Surface of Revolution

$y = f(x)$ rotate about x-axis.

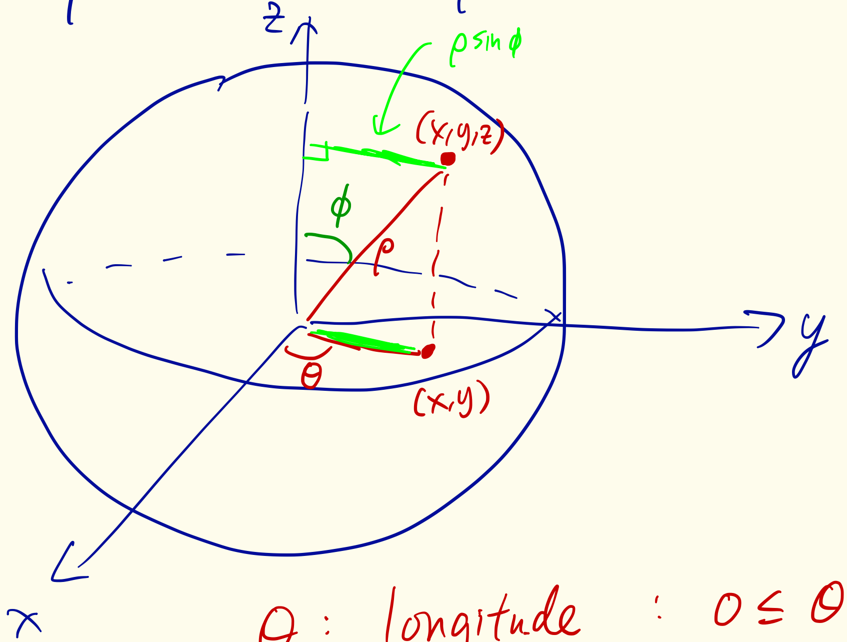






$$\vec{r}(x,\theta) = \langle x, \underbrace{f(x)\cos\theta, f(x)\sin\theta}_{r = f(x)} \rangle$$

Ex Sphere of radius ρ ("rho")



θ : longitude : $0 \le \theta \le 2\pi$

ϕ : latitude : $0 \le \phi \le \pi$

$$\vec{r}(\theta,\phi) = \langle \underline{\rho\sin\phi}\cos\theta, \underline{\rho\sin\phi}\sin\theta, \rho\cos\phi \rangle$$

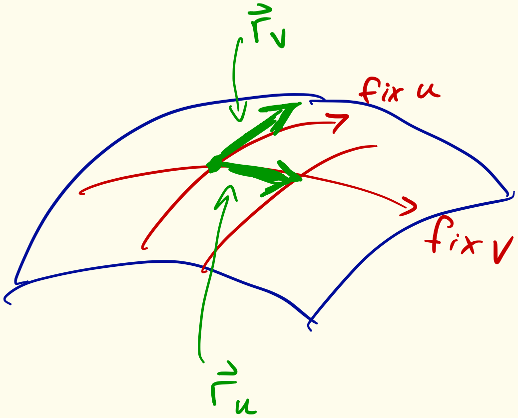
"rotating r^2+z^2 around z-axis"

Surface : Tangent Plane :

$z = f(x,y)$: $\langle -f_x, -f_y, 1 \rangle$

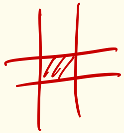
$F(x,y,z) = 0$: $\langle F_x, F_y, F_z \rangle$.

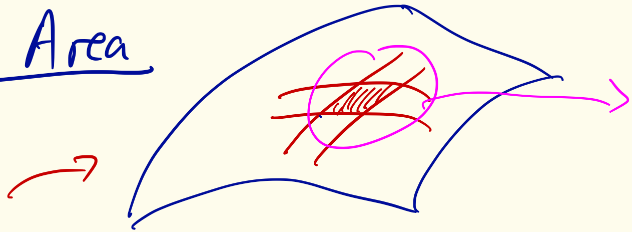
$\vec{r}(u,v)$:

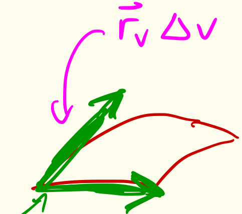


normal vector $\vec{n}$ of tangent plane $= \vec{r}_u \times \vec{r}_v$.

Surface Area







$$\frac{\vec{r}(u+\Delta u, v) - \vec{r}(u,v)}{\Delta u}\Delta u \approx \vec{r}_u \Delta u$$

$$\text{Area element} = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$$

$$A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA$$

$du\,dv$ where $(u,v) \in D$.

Ex **Sphere** $\vec{r}(\theta,\phi) = \langle \rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi \rangle$

$$\begin{array}{cccc} & i & j & k \\ \vec{r}_\theta = & \langle -\rho \sin\phi \sin\theta, & \rho \sin\phi \cos\theta, & 0 \rangle \\ \vec{r}_\phi = & \langle \rho \cos\phi \cos\theta, & \rho \cos\phi \sin\theta, & -\rho \sin\phi \rangle \end{array}$$

$$\vec{r}_\theta \times \vec{r}_\phi = \left\langle -\rho^2 \sin^2\phi \cos\theta, -\rho^2 \sin^2\phi \sin\theta, \begin{array}{l} -\rho^2 \sin^2\theta \cos\phi \sin\phi \\ -\rho^2 \cos^2\theta \cos\phi \sin\phi \\ = -\rho^2 \cos\phi \sin\phi \end{array} \right\rangle$$

$$|\vec{r}_\theta \times \vec{r}_\phi| = \sqrt{\rho^4 \sin^4\phi \cos^2\theta + \rho^4 \sin^4\phi \sin^2\theta + \rho^4 \cos^2\phi \sin^2\phi}$$

$$= \sqrt{\rho^4 (\sin^4\phi + \cos^2\phi \sin^2\phi)} = \sqrt{\rho^4 \sin^2\phi} = \rho^2 \sin\phi .$$

Made with Goodnotes

$$\iint_D \rho^2 \sin\phi \, dA$$

$$= \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin\phi \, d\phi \, d\theta$$

2π

$$\int_0^{\pi} \sin\phi \, d\phi = 2.$$

$$= 4\pi\rho^2$$