Final Review Session (3 hours)

Midterm : ($\sim 20\%$) (curves, normal vectors, chain rules, $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$, Lagrange ...)

Integration

- functions
- vector fields

Integration:
- line
- surface
- solid

Coordinate system

- polar
- cylindrical
- spherical

4 Main Theorems

- FTL.
- Green
- Stokes
- Divergence

∇ operators

- grad
- curl
- div.
- ∇^2 : Laplacian.

- <u>NO</u> PHYSICS Concept

- <u>No</u> proofs of 4 Main Thm

- Graphing ^o^

- All f, $\vec{F}$ will have continuous partial derivatives.

$C : \vec{r}(t)$ curve $\quad a \le t \le b$

$$\int_C f(x,y,z)\,\underline{\underline{ds}} := \int_a^b f(\underbrace{\vec{r}(t)}_{\langle x(t),\, y(t),\, z(t)\rangle})\, |\vec{r}\,'(t)|\, dt$$

← does not depend on orientation.

$$\int_C \vec{F} \cdot d\vec{r} := \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

← depends on orientation of C

$$\|$$

$$\int_C P\,dx + Q\,dy + R\,dz = \int_a^b P\,x'(t) + Q\,y'(t) + R\,z'(t)\, dt$$

$$\int_C = -\int_{-C}$$

$$\underbrace{\vec{F}}_{\vec{F}(x,y,z)} = \langle \underbrace{P}_{P(x,y,z)}, \underbrace{Q}_{Q(x,y,z)}, \underbrace{R}_{R(x,y,z)} \rangle$$

2D

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx$$

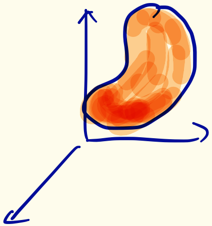
Polar coordinate : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad dy\,dx \Longrightarrow r\,dr\,d\theta$.

$$= \iint f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta .$$

3D

$$\iint_S f(x,y,z)\,dS := \iint_D f(\underline{\vec{r}(u,v)})\,|\vec{r}_u \times \vec{r}_v|\,dA$$

$D \leftarrow$ appropriate range of u,v.



$S = \vec{r}(u,v)$?

Parametric Surface

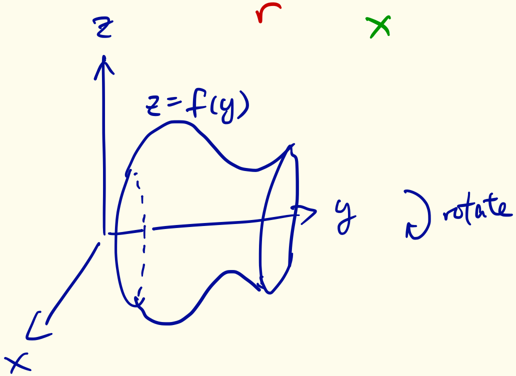
$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$

to visualize: fix u, v to get grid lines.

Important Surface of Revolution

$$\vec{r}(u,v) = \langle \underbrace{f(u)}_{r}\underbrace{\cos v}_{x},\ u,\ \underbrace{f(u)}_{r}\underbrace{\sin v}_{z} \rangle$$

$\Rightarrow$ rotating around y-axis



Tangent Plane: $\vec{n} = \pm\, \vec{r}_u \times \vec{r}_v$

= normal vector

Area: $\iint_D |\vec{r}_u \times \vec{r}_v|\, dA$.

Graph: $z = z(x,y)$

$$\vec{r}(u,v) = \langle u, v, z(u,v) \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle -z_x, -z_y, 1 \rangle$$

"FLUX through S" $\displaystyle\iint_S \vec{F}\cdot d\vec{S} = \iint_S (\vec{F}\cdot\vec{n})\,dS$

unit normal

orientation of $\vec{n}$ is ALWAYS given!

★ S closed: $\vec{n}$ is outward.

★ depends on boundary C

★ explicitly given (e.g. downward)

z component of $\vec{n}$ is negative.

If $S = \vec{r}(u,v)$:

$$\iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

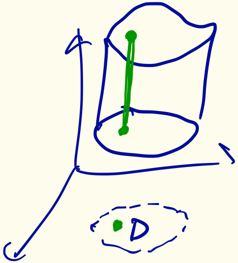
If $S = z(x,y)$

$$\iint_D \vec{F}\cdot\langle -z_x, -z_y, 1\rangle\,dA$$

(1 means upward)

$$\iiint_E f(x,y,z)\,dV = \iiint f(x,y,z)\,dz\,dy\,dx$$

6 different orders.



$$= \iint_D \left(\int dz \right) dA$$

height

projection onto x-y plane

Cylindrical (polar)

$$= \iiint_D r\,dz\,dr\,d\theta$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Spherical :

$$\iiint \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \qquad \begin{cases} r = \rho\sin\phi \\ z = \rho\cos\phi \end{cases}$$

$$\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{cases}$$

Change of Variables

$$x = x(u,v)$$
$$y = y(u,v)$$

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

ugly region in xy

nice region in uv

$$\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = x_u y_v - y_u x_v$$

$$\iiint f \; dx\,dy\,dz = \iiint \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

$$\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix}$$

∇ operators

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$\nabla f = \text{grad } f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \qquad \text{vector}$$

$$\nabla \times \vec{F} = \text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \qquad \text{vector}$$

$$\nabla \cdot \vec{F} = \text{div } \vec{F} = P_x + Q_y + R_z \qquad \text{scalar.}$$

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz}. \qquad \text{Scalar.}$$

Identities

$$\nabla \times \nabla f = \vec{0}$$

$$\nabla \cdot (\nabla \times \vec{F}) = 0$$

Made with Goodnotes

☆ Important

$\nabla \times \vec{F} = \vec{0}$ irrotational

how rotate the field is

$\nabla \cdot \vec{F} = 0$ solenoidal

source of field

$\nabla^2 f = 0$ harmonic

heat eq.

$f_{xx} + f_{yy} = 0$

$f_{xx} + f_{yy} + f_{zz} = 0$

$\vec{F} = \nabla f$ conservative

$\vec{F} = \nabla f$ conservative vector fields.

$\Longrightarrow \quad \nabla \times \vec{F} = \vec{0}$.

☆ If $\nabla \times \vec{F} = \vec{0}$ and $\vec{F}$ is defined on <u>simply connected domain D</u>

<u>Thm</u> (Curl Test)

then $\vec{F} = \nabla f$ is conservative.

☆ $D = \mathbb{R}^2$ or $\mathbb{R}^3$

Made with Goodnotes

Fund Thm of Line Integral

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

(If $\vec{F}$ is conservative $\Rightarrow$ $\oint_C \vec{F}\cdot d\vec{r} = 0$ $\forall$ closed loop C.

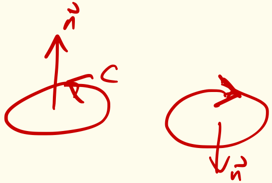
$\Longleftrightarrow$ Independence of path

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$)

Stokes' Thm

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$$

$\vec{n}$ is determined by C



Green's Thm

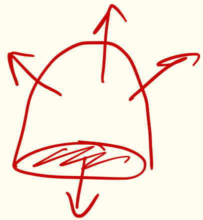
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C P\,dx + Q\,dy$$

2D, xy domain!

Divergence Thm

$$\iiint_E \nabla \cdot \vec{F}\, dV = \iint_S \vec{F}\cdot d\vec{S}$$

$\vec{n}$ outward



☆ Important Field   Curl Field

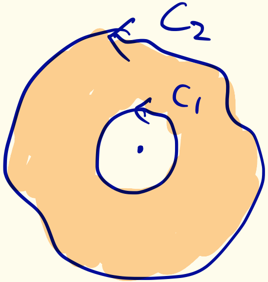
$$\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

$$\text{curl } \vec{F} = 0 \quad (\text{not conservative!})$$

☆ **Thm** $\oint_{C_1} \vec{F} \cdot d\vec{r} = \oint_{C_2} \vec{F} \cdot d\vec{r}$ for any closed curve surrounding $(0,0)$.

(counterclockwise simple)

$$= 2\pi$$



$$0 = \iint_{O} \text{curl } \vec{F} \cdot d\vec{A} = \oint_{C_2} \vec{F} \cdot d\vec{r} - \oint_{C_1} \vec{F} \cdot d\vec{r}$$

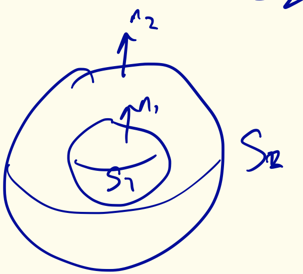
3D $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3}$ gravitational field.

$$\operatorname{curl} \vec{F} = \vec{0} \quad , \operatorname{div} \vec{F} = 0$$

but it is also conservative!

$$V = -\frac{1}{|\vec{r}|}$$

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} = 4\pi$$



$\forall$ closed surface surrounding $(0,0,0)$.

3D $\quad \vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$ gravitational field.

$$\operatorname{curl} \vec{F} = \vec{0} \quad , \operatorname{div} \vec{F} = 0$$

but it is also conservative!

$$V = -\frac{1}{|\vec{r}|}$$

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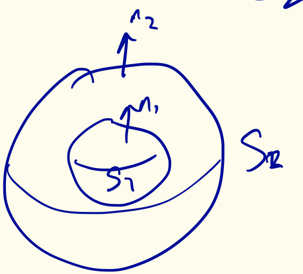

$\forall$ closed surface surrounding $(0,0,0)$.