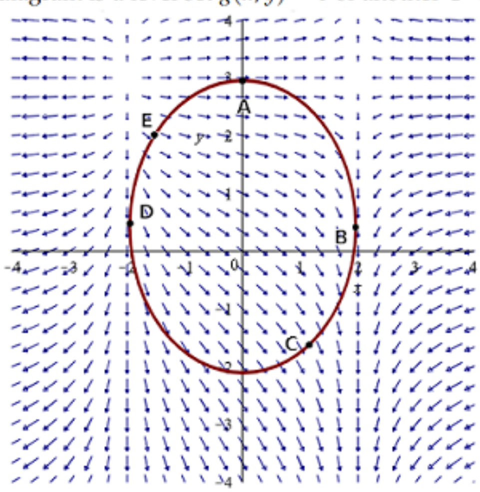
MATH2023
Tutorial 8
April 7, 2019 Tutorial Notes by Ivan So

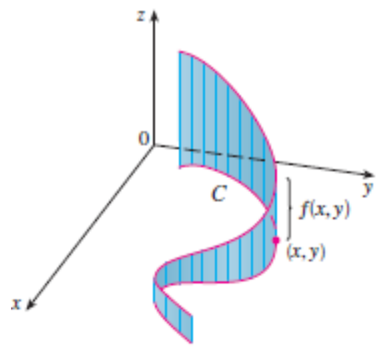
1 Review

- A **vector field** $F : \mathbb{R}^m \to \mathbb{R}^n$ which assign every point in the concerned space a *vector*.
- **Conservative vector field**:
  - The *gradient operator* assigns a function into a vector field. I.e. for a function $\phi : \mathbb{R}^n \to \mathbb{R}$, the gradient operator map the function into an n-dimensional vector field.
  - In such a case, we say the field is **conservative** and the function ϕ is the **potential function**.
  - e.g. Q1(g) from sample midterm

(g) Let $f(x, y)$ be a C^1 function. The diagram below is the plot of the vector field ∇f. The ellipse in the diagram is a level set $g(x, y) = c$ of another C^1 function g.



- The **line integral of function** along a curve C $\int_C f(\mathbf{x})ds := \lim_{n\to\infty} \sum_{i=1}^n f(\mathbf{x}_i^*)\Delta s_i$ for sampling points $\mathbf{x}_i^* \in C$. Explicitly, if C is parametrized by $\mathbf{r}(t)$ with $a \leq t \leq b$, then
$$\int_C f(\mathbf{x})ds = \int_a^b f(\mathbf{r}(t)) \left\| \frac{d\mathbf{r}}{dt} \right\| dt.$$
Pictorially, its meaning



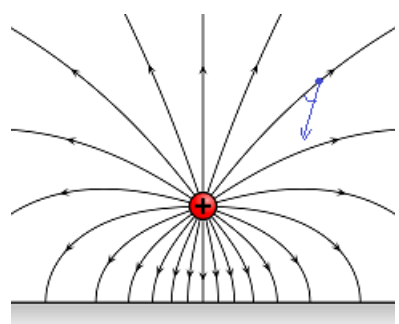
Analogy:

- Single variable calculus: definite integral = area under curve.
- Multivariable calculus: line integral of function = area under the function along the curve.

- The **line integral of a vector field F** along a curve C (parametrized by $\mathbf{r}(t)$ with $a \le t \le b$) is defined as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \sum_{i=1}^{n} (\mathbf{F})_{x_i} dx_i = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

Physical realization: Work done of moving a charge in electric field



- **Theorem**(Fundamental Theorem for Line Integral): If the concerned vector field is conservative, given by ∇f, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

if the parameter t of the path satisfies $a \le t \le b$.

  - **Corollary 1**: The path integral of a conservative field is path independent.
  - **Corollary 2** : The path integral of a conservative field over a closed path (a path in which starting point is the end point) is zero.

- *Notation*:
  - If C is a curve obtained by connecting C_1 and C_2, then $\int_{C_1} + \int_{C_2} = \int_C$.
  - Path integral is direction sensitive. $\int_{-C}$ represent the integral of the same path go in the opposite direction (i.e. from t evaluate from b to a instead of a to b).
  - The integral over a closed path is denoted by $\oint_C$.
  - The *positive orientation* of a closed path is the path going in *counterclockwise* direction.
- Some useful theorems:
  1. (Does not matter on domain) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path $C \Leftrightarrow \int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.
  2. On open connected domain, path independence $\Leftrightarrow$ conservative.
  3. If $\mathbf{F} = \langle P, Q \rangle$, then conservative $\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.
- **Theorem** (Green's Theorem): Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be any vector field (not necessarily conservative) with P, Q having continuous derivatives, then

$$\oint_{C=\partial D} \mathbf{F} \cdot d\mathbf{r} = \int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

  - **Corollary**: (Partial inverse of 3, Theorem D in lecture) On open simply connected region [no hole], if $\mathbf{F} = \langle P, Q \rangle$, then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \Rightarrow$ conservative.

  Comment:
  1. Line integral is not always easy to compute, this theorem provides an alternative, the area integral which could be easier to compute in some instances.
  2. This is the specialization of the later introduced **Stoke's Theorem**.

2 Problems

1. True or False.

   (a) $\int_C f(x, y) ds = - \int_{-C} f(x, y) ds$. True.

   (b) $\mathbf{F} = \langle P, Q \rangle$, then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \Rightarrow$ conservative. False

(c) The set $\{(x, y) : x, y \geq 0\}$ is a open connected subset of $\mathbb{R}^2$.

True

(d) The set $\{(x, y) : x \neq 0\} \cup \{(0, 0)\}$ is a simply connected subset of $\mathbb{R}^2$.

False

2. Write $\int_C (2x + 9z) ds$ with C parametrized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$ in terms of t.

3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \sin x\mathbf{i} + \cos y\mathbf{j} + xz\mathbf{k}$ and C is parametrized by $\mathbf{r}(t) = \langle t^3, -t^2, t \rangle$ with $0 \leq t \leq 1$.

4. If the domain is the whole $\mathbb{R}^2$, determine whether the field $\mathbf{F}(x, y) = (2x - 3y)\mathbf{i} + (-3x + 4y - 8)\mathbf{j}$ is conservative.
   If conservative, evaluate the path integral over the path C parametrized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$ with the Fundamental Theorem of Line Integral.

5. If the domain is the whole $\mathbb{R}^2$, determine whether the field $\mathbf{F}(x, y) = e^x \sin y\mathbf{i} + e^{-x} \cos y\mathbf{j}$ is conservative.
   If conservative, evaluate the path integral over the path C parametrized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$ with the Fundamental Theorem of Line Integral.

6. Evaluate $\oint_C xydx + x^2y^3dy$ with C being the triangle with vertices $(0,0),(1,0)$ and $(1,2)$ using Green's theorem. Check your answer with the classical method.

7. Use Green's theorem to prove the 2-dimensional change of variable formula (which will be useful later)

$$\int\int_R dxdy = \int\int_S \det\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

where R is the domain of integration in terms of x,y-variables. S is for u,v. (Assume the interchangeability of second order derivative of x,y with respect to u,v)

2. Write $\int_C (2x + 9z)ds$ with C parametrized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \le t \le 1$ in terms of t.

$$\int_0^1 2t + 9t^3 \sqrt{t^2 + t^4 + t^6} \quad dt$$

3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = \sin x\mathbf{i} + \cos y\mathbf{j} + xz\mathbf{k}$ and C is parametrized by $\mathbf{r}(t) = \langle t^3, -t^2, t\rangle$ with $0 \le t \le 1$.

$$\int_0^1 \sin x\,(3t^2)dt + \cos y\,(-2t)\,dt + xz\,dt$$

$$= \int_0^1 3\sin(t^3)t^2\,dt - \int_0^1 2t\cos(-t^2)\,dt + \int_0^1 t^4\,dt$$

let $u = t^3$, $du = 3t^2$

let $u = -t^2$, $du = -2t$

$$= \int_0^1 \sin u\,du + \int_0^{-1} \cos u\,du + \frac{1}{5}$$

$$= \left[-\cos u\right]_0^1 + \left[\sin u\right]_0^{-1}$$

$$-\cos 1 + 1 + \sin(-1) + \frac{1}{5}$$

4. If the domain is the whole $\mathbb{R}^2$, determine whether the field $\mathbf{F}(x, y) = (2x - 3y)\mathbf{i} + (-3x + 4y - 8)\mathbf{j}$ is conservative.
If conservative, evaluate the path integral over the path C parametrized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$ with the Fundamental Theorem of Line Integral.

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -3 - (-3) = 0$ yes.

$f(x,y) = x^2 - 3xy + 2y^2 - 8y$

$f(1,1,1) - f(0,0,0) = 1 - 3 + 2 - 8 - 0 = -8$

5. If the domain is the whole $\mathbb{R}^2$, determine whether the field $\mathbf{F}(x, y) = e^x \sin y\mathbf{i} + e^{-x} \cos y\mathbf{j}$ is conservative.

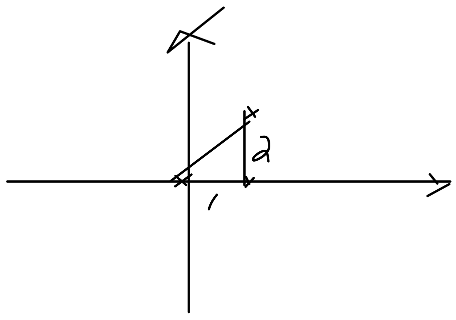
If conservative, evaluate the path integral over the path C parametrized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$ with the Fundamental Theorem of Line Integral.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -e^{-x}\cos y - e^{x}\cos y$$

6. Evaluate $\oint_C xy\,dx + x^2y^3\,dy$ with C being the triangle with vertices $(0,0), (1,0)$ and $(1,2)$ using Green's theorem. Check your answer with the classical method.

$$\iint 2xy^3 - x \; dA$$

y=2x



$$= \int_0^2 \int_{\frac{y}{2}}^1 2xy^3 - x \quad dx\,dy$$

$$= \int_0^2 \left[x^2y^3 - \frac{x^2}{2} \right]_{\frac{y}{2}}^1 dy$$

$$= \int_0^2 y^3 - \frac{1}{2} - \left(\frac{y^2}{4} \cdot y^3 - \frac{y^2}{8} \right) dy$$

$$= \left[\frac{y^4}{4} - \frac{1}{2}y - \frac{y^6}{24} - \frac{y^3}{24} \right]_0^2$$

$$= 0$$

7. Use Green's theorem to prove the 2-dimensional change of variable formula (which will be useful later)

$$\int\int_R dxdy = \int\int_S \det\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv.$$

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