Problem 1

(a) Assume $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three dimensional vectors and if

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b} + \beta \mathbf{c}.$$

Use suffix notation to find λ, μ and β in terms of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Can you say something about the direction of the vector $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. ?

(b) Let $\mathbf{a}$ be a constant vector and $\mathbf{r} = (x, y, z)$, use suffix notation to evaluate ?

(i) $\nabla \cdot \mathbf{r}$, (ii) $\nabla \cdot (\mathbf{a} \times \mathbf{r})$, (iii) $\nabla \times (\mathbf{a} \times \mathbf{r})$.

Problem 2

(a) Sketch and describe the parametric curve C

$$\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + (2\pi - t)\,\mathbf{k}, \qquad 0 \leqslant t \leqslant 2\pi.$$

Show the direction of increasing t. Find the project curve C onto the yz-plane.

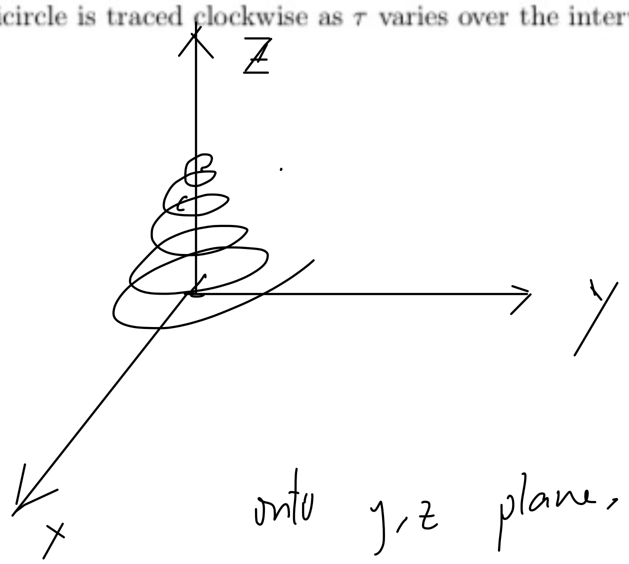
(b) Find a change of parameter $t = g(\tau)$ for the semicircle

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \qquad 0 \leqslant t \leqslant \pi$$

such that (i) the semicircle is traced counterclockwise as τ varies over the interval $[0, 1]$,

(ii) the semicircle is traced clockwise as τ varies over the interval $[0, 0.5]$.

a).



onto y, z plane, $x = 0$:

$$\vec{r} = t\sin t\, j + (2\pi - t)k$$

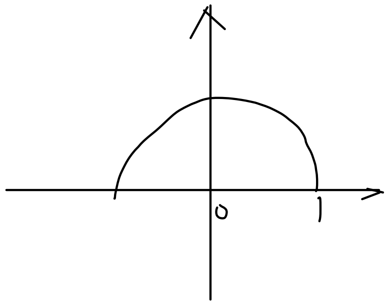
(b) Find a change of parameter $t = g(\tau)$ for the semicircle

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \qquad 0 \leqslant t \leqslant \pi$$

such that (i) the semicircle is traced counterclockwise as τ varies over the interval $[0,\ 1]$,

(ii) the semicircle is traced clockwise as τ varies over the interval $[0,\ 0.5]$.

??



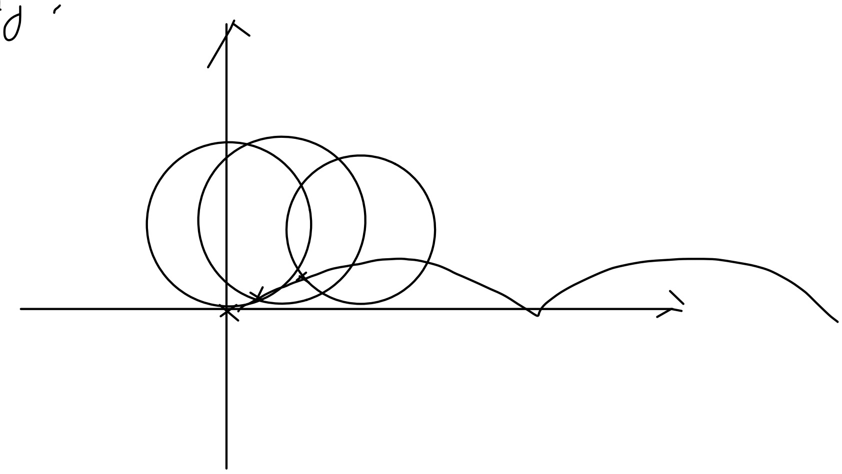
Problem 3

If a wheel with radius a rolls along a flat surface without slipping, a point P on the rim of the wheel traces a curve C, find the parametric equation of the point P. Suppose that the point P on the wheel is initially at the origin. Find also the arc length of the curve C if the wheel makes one complete turn (no need to carry out the integration).

??!!

$\boxed{\vec{r}} : a\cos\theta\, i + a\sin\theta\, j.$

$\boxed{\vec{r}}' = -a\sin\theta\, i + a\cos\theta\, j.$



Problem 4

(a) Verify the formula for the arc length element is cylindrical coordinates,

$$ds = \sqrt{\left(\frac{dr}{dt}\right)^2 + (r(t))^2 \left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

(b) Find a similar formula as in (a) for the arc length element in spherical coordinates.

(c) Use part (b) or otherwise, find the arc length of the curve in spherical coordinates:
$\rho = 2t$, $\theta = \ln t$, $\phi = \pi/6$; $1 \leqslant t \leqslant 5$.

Problem 5

Let $f(x,y) = \begin{cases} \dfrac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) Is the function continuous at $(0,0)$?

(b) Calculate $f_x(x,y)$, $f_y(x,y)$, $f_{xy}(x,y)$ and $f_{yx}(x,y)$ at point $(x,y) \neq (0,0)$. Also calculate these derivatives at $(0,0)$.

(c) Is $f_{yx}(x,y)$ continuous at $(0,0)$?

(d) Explain why $f_{yx}(0,0) \neq f_{xy}(0,0)$.

a).
$$\lim_{(x,y)\to(0,0)} \frac{2xy(x^2-y^2)}{x^2+y^2}$$

$$= \lim_{r\to 0} \frac{\dfrac{2r^2\cos\theta\sin\theta(r^2\cos\theta - r^2\sin\theta)}{r^2}}{r}$$

$$= \lim_{r\to 0} 2r\cos^2\theta\sin\theta - r\sin\theta$$

$$= 0$$

$\therefore$ continuous.

Problem 5

Let $f(x,y) = \begin{cases} \dfrac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) Is the function continuous at $(0,0)$?

(b) Calculate $f_x(x,y)$, $f_y(x,y)$, $f_{xy}(x,y)$ and $f_{yx}(x,y)$ at point $(x,y) \neq (0,0)$. Also calculate these derivatives at $(0,0)$.

(c) Is $f_{yx}(x,y)$ continuous at $(0,0)$?

(d) Explain why $f_{yx}(0,0) \neq f_{xy}(0,0)$.

b). $$f_x = \frac{(x^2+y^2)\frac{\partial}{\partial x}(2xy(x^2-y^2)) - 2xy(x^2-y^2)(2x)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(2xy(2x)+(x^2-y^2)(2y)) - 4x^2y(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(4x^2y+2x^2y-2y^3) - 4x^4y+4x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{6x^4y - 2x^2y^3 + 6x^2y^3 - 2y^5 - 4x^4y + 4x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{2x^4y + 8x^2y^3 - 2y^5}{(x^2+y^2)^2}$$

$$= 2y(x^4 + 4x^2y^2 - y^4)$$

$$f_y = \frac{(x^2+y^2)(2xy(-2y)+(x^2-y^2)(2x)) - 2xy(x^2-y^2)(2y)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(-4xy^2+2x^3-2xy^2) - 4xy^2(x^2-y^2)}{(x^2+y^2)^2}$$

$$= \frac{-6x^3y^2 - 6xy^4 + 2x^5 + 2x^3y^2 - 4x^3y^2 + 4xy^4}{(x^2+y^2)^2}$$

$$= \frac{-8x^3y^2 - 2xy^4 + 2x^5}{(x^2+y^2)^2}$$

Problem 6

Find the distance from the origin to the plane $x + 2y + 2z = 3$,

(a) using a geometric argument (no calculus),

(b) by reducing the problem to an unconstrained problem in two variables, and

(c) using the method of Lagrange multipliers.

a).

$$\left| \frac{-3}{\sqrt{9}} \right| = |-1| = 1.$$

b).

$$D = x^2+y^2+z^2$$

$$x+2y+2z=3$$

$$x=3-2y-2z$$

$$= (3-2y-2z)^2+y^2+z^2$$

$$= (3-2y-2z)(3-2y-2z)+y^2+z^2$$

$$= (9-6y-6z-6y+4y^2+4yz-6z+4yz+4z^2)+y^2+z^2$$

$$D = 9-12y-12z+8yz+5y^2+5z^2$$

$D_y = -12 + 8z + 10y$

$D_z = -12 + 8y + 10z$

$$\begin{cases} 8z + 10y = 12 \\ 10z + 8y = 12 \end{cases}$$

$2z - 2y = 0$

$z = y$

$y = \frac{12}{18} = \frac{2}{3}$

$x = 3 - 4y$

$= 3 - 4\left(\frac{2}{3}\right)$

$= 3 - \frac{8}{3}$

$= \frac{1}{3}$

$\therefore (x, y, z) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$D = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$

$= 1$

Minimize $D = x^2+y^2+z^2$ subject to $x+2y+2z=3$.

$$\nabla D = \langle 2x, 2y, 2z \rangle$$

$$\nabla g = \langle 1, 2, 2 \rangle$$

$$\begin{cases} 2x = \lambda \\ 2y = 2\lambda \\ 2z = 2\lambda \\ x+2y+2z=3 \end{cases}$$

$x = \frac{\lambda}{2}$, $y = \lambda$, $z = \lambda$.

$$\frac{\lambda}{2} + 2\lambda + 2\lambda = 3$$

$$\frac{9}{2}\lambda = 3$$

$\lambda = \frac{2}{3}$. , $x = \frac{1}{3}$, $y = \frac{2}{3}$, $z = \frac{2}{3}$.

$$D: \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1$$

Problem 7

(a) What condition must the constants a, b, and c satisfy to guarantee that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{ax^2+bxy+cy^2}$$

exists. Prove your answer.

(b) Find $\frac{\partial^2}{\partial y \partial x} f(y^2, xy, -x^2)$ in terms of partial derivatives of the function f.

a). along $x=0$, if $c \neq 0$, lim $\to 0$.

along $x=y$, $\frac{x^2}{(a+c+b)x^2} = \frac{1}{a+b+c}$

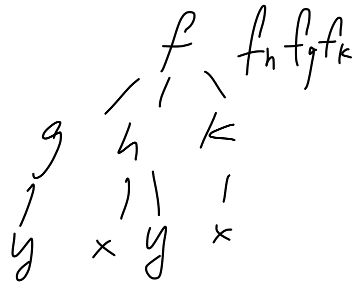
$\Rightarrow$ impossible for $\frac{1}{a+b+c}$ to be 0.

$\therefore a=c=0$, $\lim_{(x,y)\to(0,0)} \frac{xy}{bxy} = \frac{1}{b}$

$\therefore$ conditions $=$ $a=c=0$, $b \neq 0$.

(b) Find $\frac{\partial^2}{\partial y \partial x} f(y^2, xy, -x^2)$ in terms of partial derivatives of the function f.

Let $g(y)=y^2$, $h(x,y)=xy$, $k(x)=-x^2$



$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial h}\frac{\partial h}{\partial x} + \frac{\partial f}{\partial k}\frac{\partial k}{\partial x}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial (h)}(y) + \frac{\partial f}{\partial (k)}(-2x)$$

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial (h)} + y(f_{hg}g_y + f_{hh}h_y) - 2x(f_{kg}g_y + f_{kh}h_y))$$

$$= f_h + y(f_{hg}(2y) + f_{hh}\cdot x) - 2x(f_{kg}\cdot 2y + f_{kh}\cdot x)$$

$$= f_h + 2f_{hg}y^2 + f_{hh}xy - 4xy f_{kg} - 2f_{kh}x^2$$

Problem 8

(a) Find the equation of the tangent plane at the point $(-1, 1, 0)$ to the surface

$$x^2 - 2y^2 + z^3 = -e^{-z}.$$

(b) The temperature at a point (x, y) on a metal plate in xy-plane is $T(x, y) = x^2 + y^3$ degrees Celsius.

(i) Find the rate of change of temperature at $(1, 1)$ in the direction of $\mathbf{a} = 2\,\mathbf{i} + \mathbf{j}$.

(ii) An ant at $(1, 1)$ wants to walk in the direction in which the temperature decreases most rapidly. Find a unit vector in that direction.

(c) Let C be the curve $x^{2/3} + y^{2/3} = a^{2/3}$ on the xy-plane, find the parametric equation of the curve C. Hence find the tangent line to the curve C at $(a, 0)$.

a) let $f(x,y,z) = x^2 - 2y^2 + z^3 + e^{-z} = 0$

$\nabla f = \langle 2x, -4y, 3z - e^{-z} \rangle$

$\nabla f \langle -1, 1, 0 \rangle = \langle -2, -4, -1 \rangle$

Eqt: $-2x - 4y - z = -2$

$2x + 4y + z = 2.$

Made with Goodnotes

(b) The temperature at a point (x, y) on a metal plate in xy-plane is $T(x, y) = x^2 + y^3$ degrees Celsius.

(i) Find the rate of change of temperature at $(1, 1)$ in the direction of $\mathbf{a} = 2\,\mathbf{i} + \mathbf{j}$.

(ii) An ant at $(1, 1)$ wants to walk in the direction in which the temperature decreases most rapidly. Find a unit vector in that direction.

(c) Let C be the curve $x^{2/3} + y^{2/3} = a^{2/3}$ on the xy-plane, find the parametric equation of the curve C. Hence find the tangent line to the curve C at $(a, 0)$.

b) i). $\hat{a} = \frac{2}{\sqrt{5}} i + \frac{1}{\sqrt{5}} j$.

$\nabla T = \langle 2x, 3y^2 \rangle$

$\nabla T(1,1) = \langle 2, 3 \rangle$

$\nabla T \cdot \hat{a} = \frac{4}{\sqrt{5}} + \frac{3}{\sqrt{5}} = \frac{7}{\sqrt{5}}$.

ii). $-\nabla T = \langle -2x, -3y^2 \rangle$

$-\nabla T(1,1) = \langle -2, -3 \rangle$

unit vector $= \langle -\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}} \rangle$

Made with Goodnotes

(c) Let C be the curve $x^{2/3} + y^{2/3} = a^{2/3}$ on the xy-plane, find the parametric equation of the curve C. Hence find the tangent line to the curve C at $(a, 0)$.

Let $x = t$, $t^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

$y^{\frac{2}{3}} = a^{\frac{2}{3}} - t^{\frac{2}{3}}$

$y = \left(a^{\frac{2}{3}} - t^{\frac{2}{3}}\right)^{\frac{3}{2}}$