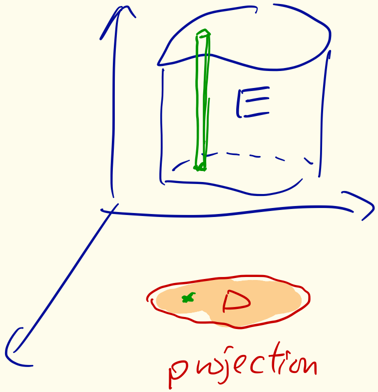
Last Time : Triple Integrations

$$\iiint_E f(x,y,z)\, dV$$

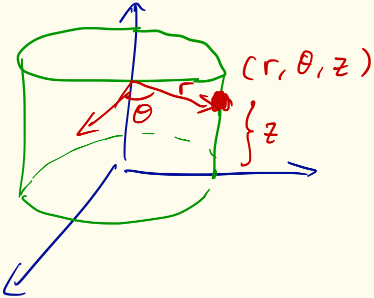


$$\iint_D \left(\int dz \right) dA$$

base projection

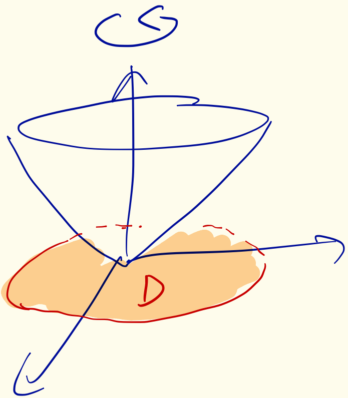
Cylindrical Coordinate = Using polar coordinate in x-y.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



$$dx\,dy\,dz \Longrightarrow r\,dz\,dr\,d\theta$$

use $r\,dr\,d\theta$ as base



$$\iint_D \left(\int dz\right) dA$$

$$\Downarrow$$

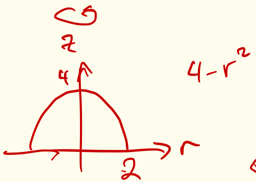
$$\iint \left(\int dz\right) r\,dr\,d\theta$$

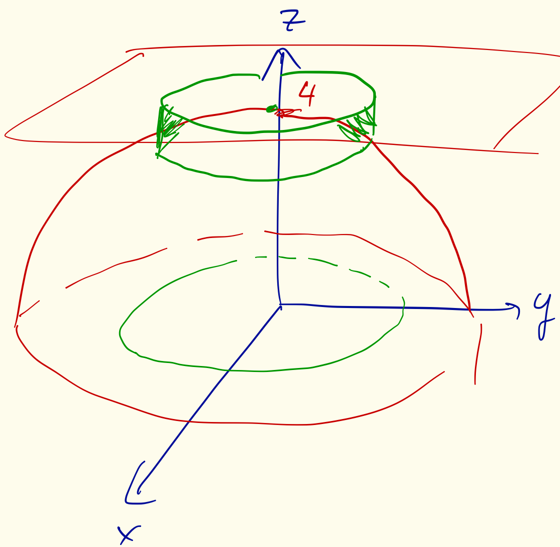
Ex E = solid bounded by

cylinder $x^2+y^2=1$

$z \le 4$

above $z = 4-x^2-y^2$





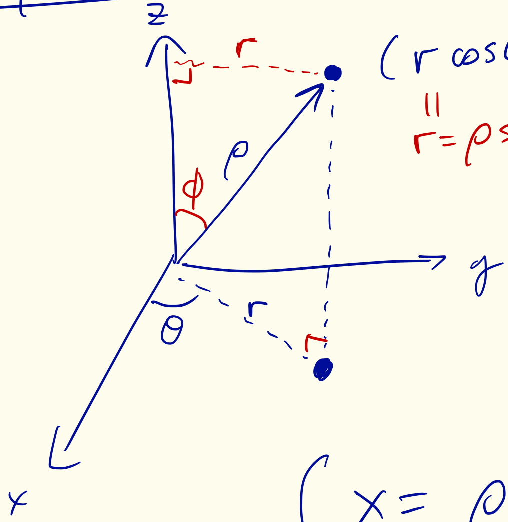
Find $\iiint_E \sqrt{x^2+y^2}\, dV$.

$$= \iint_{x^2+y^2\le 1} \left(\int_{4-x^2-y^2}^{4} \sqrt{x^2+y^2}\, dz \right) dA$$

By cylndrical coord:

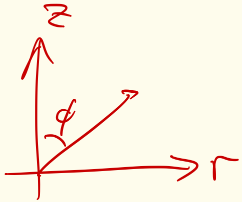
$$\int_0^{2\pi}\int_0^1 \left(\int_{4-r^2}^{4} r\, dz \right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \underbrace{\int_{4-r^2}^{4} r^2\, dz}_{= \; r^2(4-(4-r^2)) = r^4}\, dr\, d\theta$$

Spherical Coordinate



$(r\cos\theta,\ r\sin\theta,\ \rho\cos\phi)$

$$r = \rho\sin\phi$$



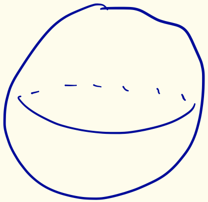
polar here again!

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

$$0 \le \theta \le 2\pi$$

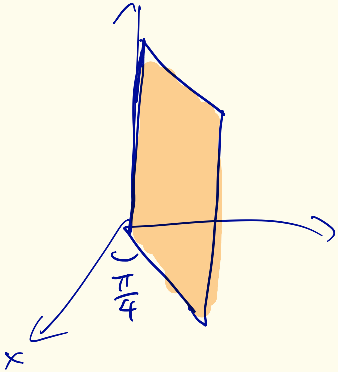
$$0 \le \phi \le \pi$$

Ex $\rho = 3$

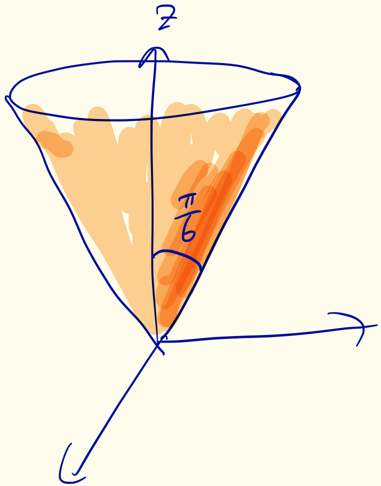


radius 3

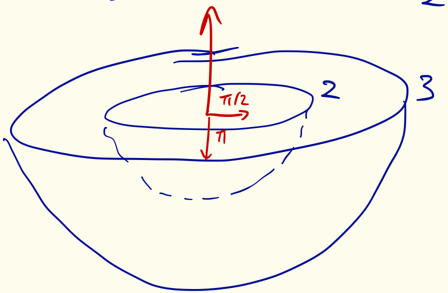
Ex $\theta = \frac{\pi}{4}$



Ex $\phi = \frac{\pi}{6}$



Ex $2 \le \rho \le 3$ and $\frac{\pi}{2} \le \phi \le \pi$



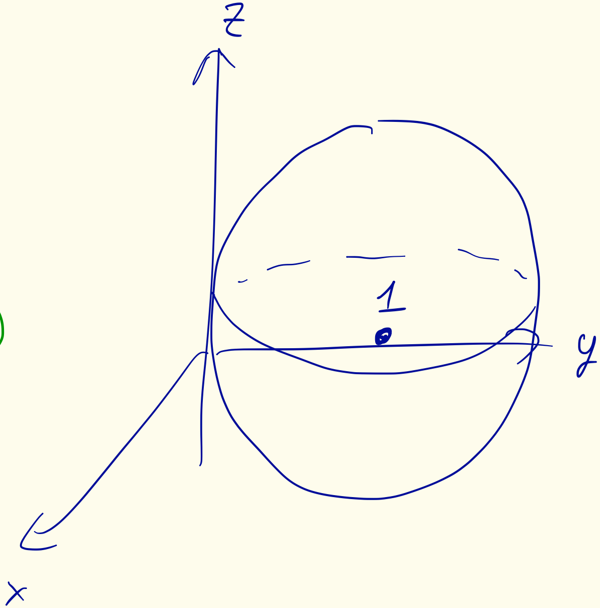
Ex $\rho = 2 \underbrace{\sin\theta \sin\phi}_{y/\rho}$ $\quad (0 \le \theta \le \pi)$

$$\iff \rho^2 = 2y$$

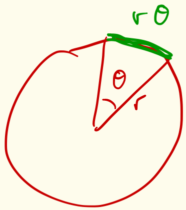
$$\iff x^2 + y^2 + z^2 = 2y$$

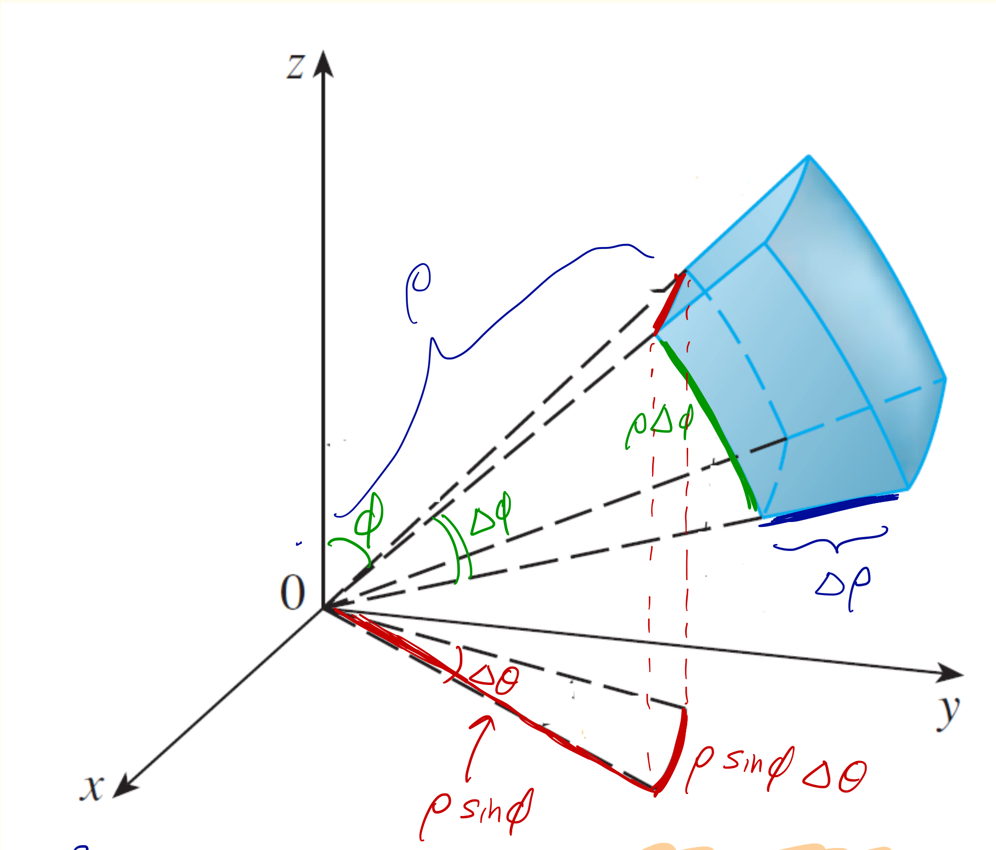
$$\iff x^2 + (y-1)^2 + z^2 = 1$$

(−ve $\rho \Rightarrow$ going in opposite direction)



Volume Element $dx\,dy\,dz \Rightarrow ?$





$$\Delta V = \rho^2 \sin\phi \,\Delta\rho\,\Delta\phi\,\Delta\theta \rightarrow \boxed{dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$$ //

__Ex__ $$\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV \qquad B = \text{unit ball}.$$

$$(x^2+y^2+z^2=\rho^2)$$

$$\boxed{\int_0^{2\pi}}\boxed{\int_0^{\pi}}\int_0^1 e^{\rho^3}\ \underbrace{\rho^2\boxed{\sin\phi}\,d\rho\,d\phi\,\boxed{d\theta}}_{dV}$$

$$\int e^{\rho^3}\rho^2\,d\rho = \frac{e^{\rho^3}}{3}$$

$$= (2\pi)(2)\left[\frac{e^{\rho^3}}{3}\right]_0^1$$

$$= 4\pi\left(\frac{e}{3}-\frac{1}{3}\right)$$

Change of Variables

$$\int_a^b f(x)\,dx \qquad x = g(u)$$

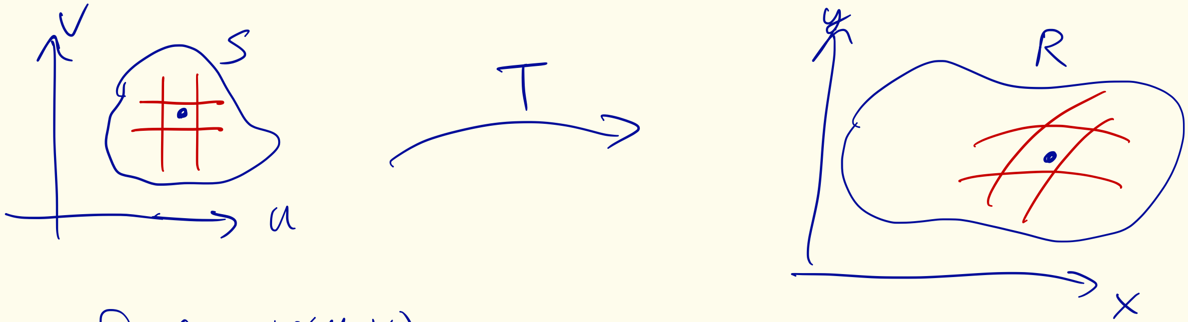
$$\Rightarrow \int_c^d f(g(u))\, g'(u)\, du$$

$$\begin{cases} g(c) = a \\ g(d) = b \end{cases}$$

In double integrals

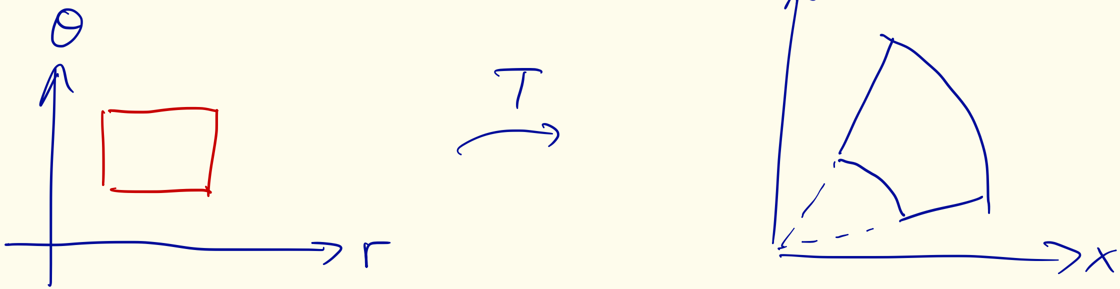
$$x = x(r,\theta) = r\cos\theta$$
$$y = y(r,\theta) = r\sin\theta$$

$$dx\,dy \Longrightarrow r\,dr\,d\theta$$



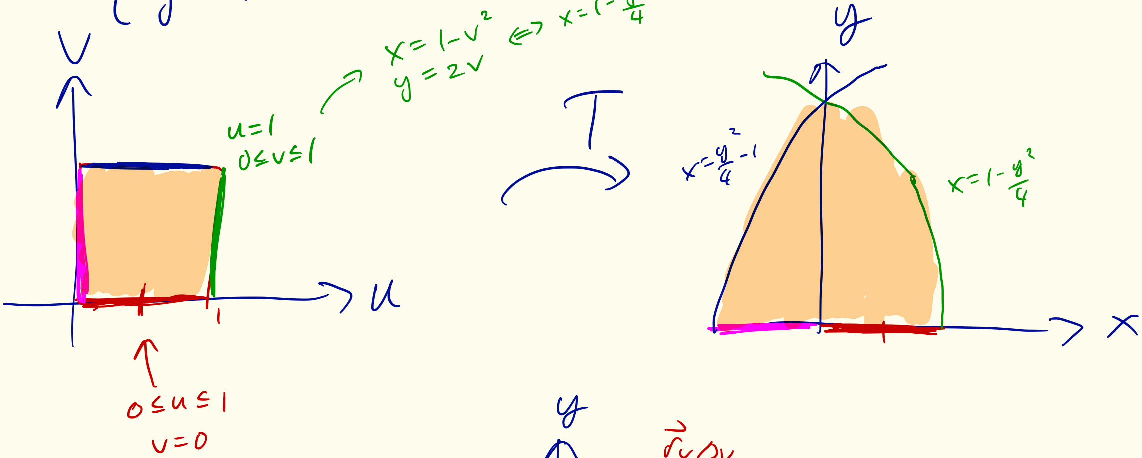
$$\vec{r}(u,v) = \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = 0 \end{cases}$$

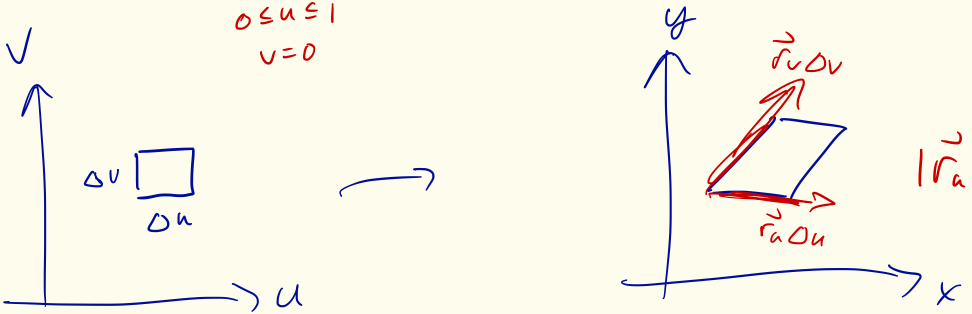
<u>Ex</u>



Ex

$$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$$





$|\vec{r}_u \times \vec{r}_v| \, \Delta u \Delta v.$

$$|\vec{r}_u \times \vec{r}_v| = \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{matrix} \right| = \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{matrix} \right| \quad \text{Jacobian}$$

Thm $$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

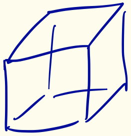
$$\Updownarrow$$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Ex $$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \iff \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

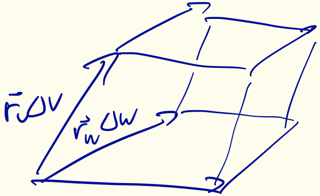
$$dx\,dy = r\,dr\,d\theta$$

Ex $$\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases} : \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & 2v \\ -2v & 2u \end{vmatrix} = 4(u^2+v^2) \iff \iint_{\triangle} dx\,dy = \iint_{\square} 4(u^2+v^2)\,du\,dv$$



$\Delta x \Delta y \Delta z$

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$$



$\vec{r}_u \Delta u$

Volume $\quad dV = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} dx\,dy\,dz$

$$\| \qquad$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \qquad \text{Jacobian}$$

Ex $\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\theta \end{cases} \qquad \dfrac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \rho^2 \sin\phi$