MATH 2023 • Multivariable Calculus
Problem Set #0 • Dot and Cross Products (Review)

1. (★) Given three points in $\mathbb{R}^3$:

$$A(1,2,3),\ B(4,0,5)\ \text{ and }\ C(x,6,4)$$

Determine the number of possible value(s) of x such that the triangle ABC has a right angle.

2. (★★) Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.

   (a) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal (i.e. $\mathbf{u} \perp \mathbf{v}$, $\mathbf{v} \perp \mathbf{w}$ and $\mathbf{w} \perp \mathbf{u}$).

   (b) Given any vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in $\mathbb{R}^3$, show that:

$$\mathbf{r} = \frac{\mathbf{r}\cdot\mathbf{u}}{|\mathbf{u}|^2}\mathbf{u} + \frac{\mathbf{r}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} + \frac{\mathbf{r}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}.$$

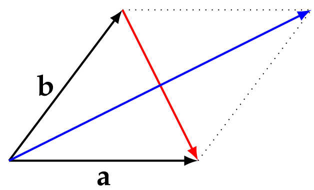
   [Hint: You may use the fact that since $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal and non-zero, the vector $\mathbf{r}$ can be expressed as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, i.e.

$$\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

   Solve for the scalars a, b and c.]

   (c) Express the vector $\mathbf{i}$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

3. (★) The figure below shows two vectors $\mathbf{a}$ and $\mathbf{b}$ which span a parallelogram. The vectors in blue and red represent the two diagonals of the parallelogram.



   (a) Express the red and the blue vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.

   (b) By considering the dot product, show that: $|\mathbf{a}| = |\mathbf{b}|$ if and only if the diagonals of the parallelogram are orthogonal to each other.

4. (★) Let $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be a variable **unit** vector in $\mathbb{R}^3$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

   (a) Find x, y and z such that $\mathbf{u}\cdot\mathbf{v}$ is the maximum possible. Explain your answer.

   (b) Find x, y and z such that $|\mathbf{u}\times\mathbf{v}|$ is the maximum possible. Explain your answer.

5. (★★) Given two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, prove the following:

   (a) Cauchy-Schwarz's Inequality: $|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}|$

   (b) Triangle Inequality: $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

   (c) If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, show that $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$.

6. (★) Let A, B and C be the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively in the three dimensional space, and O be the origin $(0,0,0)$. Denote $[ABC]$ the area of the triangle with vertices A, B and C (analogously for $[OAB]$, $[OBC]$, etc.). Show that:

$$[ABC]^2 = [OAB]^2 + [OBC]^2 + [OCA]^2.$$

With the help of a diagram, explain why this result can be regarded as the *three-dimensional analogue of the Pythagoreas' Theorem*.

1. (★) Given three points in $\mathbb{R}^3$:

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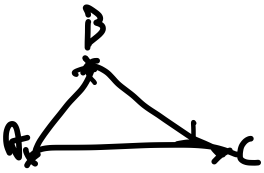
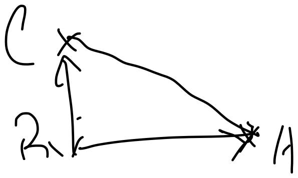
Determine the number of possible value(s) of x such that the triangle ABC has a right angle.



one.



$a = \dfrac{r \cdot u}{u \cdot u}$

2. (★★) Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.

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$$\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

Solve for the scalars a, b and c.]

(c) Express the vector $\mathbf{i}$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

$\dfrac{r \cdot u}{|u|^2} u = \dfrac{r \cdot u}{u \cdot u} u$

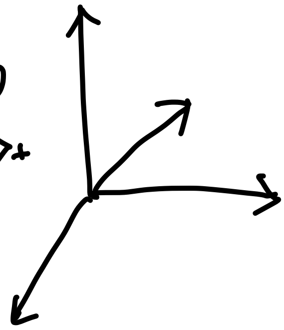
36
36
4

40

36
4
9

26
13

49



Sub $x=1, y=0, z=0$

$r = \frac{2}{3}\langle 2,1,-2\rangle +$

2a). $u \cdot v = 2 + 2 - 2 \times 2 = 0$

$v \cdot w = 0$, $w \cdot u = 0$

$w = \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{vmatrix} = \langle 6, -2, \cancel{3} \rangle$

b). $r = au + bv + cw$

$r = a(2i + j - 2k) + b(i + 2j + 2k) + c(6i - 2j + 6k)$

$r =$ $\quad r \cdot u = \quad 2x + y - 2k$

$(2a + b + 6c)i \qquad \dfrac{r \cdot u}{u \cdot u} u =$ projection of r onto u.

$x =$

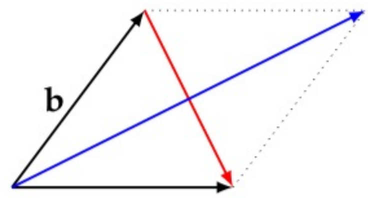
c). $\dfrac{2}{9}\langle 2,1,-2\rangle + \dfrac{1}{9}\langle 1,2,2\rangle + \dfrac{6}{49}\langle 6,-2,3\rangle$

Q2. b). $\frac{r \cdot u}{u \cdot u}$ is the projection of r onto u.

c). $r = 1i + 0j + 0k$

$r =$

3. (★) The figure below shows two vectors **a** and **b** which span a parallelogram. The vectors in blue and red represent the two diagonals of the parallelogram.



(a) Express the red and the blue vectors in terms of **a** and **b**.

(b) By considering the dot product, show that: $|\mathbf{a}| = |\mathbf{b}|$ if and only if the diagonals of the parallelogram are orthogonal to each other.

b + red = a

red = a - b

a + b = blue.

b). $(a-b)\cdot(a+b) = 0$

$a \cdot a + a \cdot b - b \cdot a - b \cdot b = 0$

$a \cdot a = b \cdot b$

$|a|^2 = |b|^2$

$|a| = |b|$ ($\because |a| > 0$)

4. (★) Let $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be a variable **unit** vector in $\mathbb{R}^3$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(a) Find x, y and z such that $\mathbf{u} \cdot \mathbf{v}$ is the maximum possible. Explain your answer.

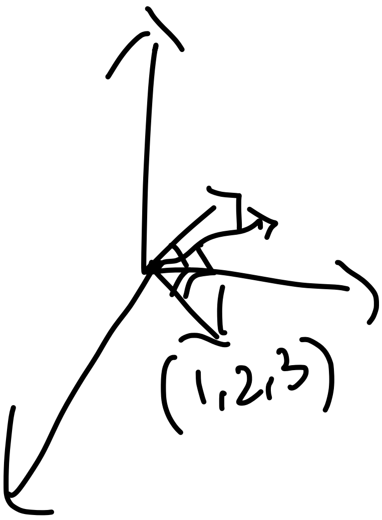
(b) Find x, y and z such that $|\mathbf{u} \times \mathbf{v}|$ is the maximum possible. Explain your answer.

$|u| = 1$ $\quad u \cdot v = 0$

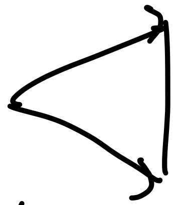
(x, y, z)

$\sqrt{x^2 + y^2 + z^2} = 1$ $\longrightarrow$ $u \cdot v = |u||v| \cos\theta$

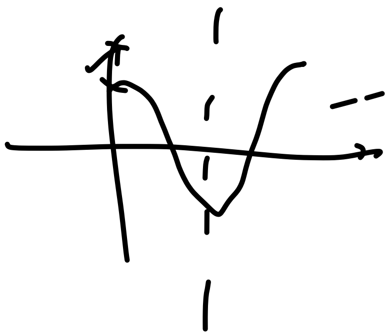
$x^2 + y^2 + z^2 = 1$



$u \cdot v = x + 2y + 3z$



$$\begin{cases} x^2 + y^2 + z^2 = 1 \\ x + 2y + 3z = u \cdot v \end{cases}$$



$u \cdot u = 1$

$\dfrac{u \cdot v}{u \cdot u} u = v$ projection

$\therefore \vec{u} = \vec{v}$,

$x = 1, y = 2, z = 3$.

$u \cdot v = |v| \cos\theta$

$|v| = \sqrt{1 + 4 + 9} = \sqrt{14}$

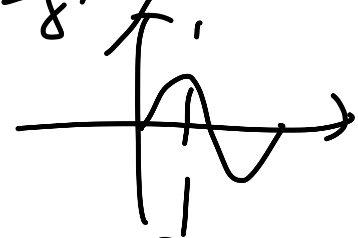
$\sqrt{14} \cos\theta$ to be maximum, $\theta = 0$,

$\cos\theta = 1$

Made with Goodnotes

$$|u\times v| = \begin{vmatrix} i & j & k \\ x & y & z \\ 1 & 2 & 3 \end{vmatrix}$$

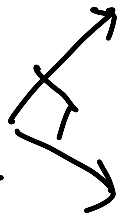
$(3y-2z)i + \frac{\pi}{2}.$



$$|u\times v| = |u|\,|v|\sin\theta$$

$$= |v|\sin\theta$$

$\sin\theta$



for $|v|\sin\theta$ to be maximized,

$$\theta = \frac{\pi}{2}.$$

$$\therefore\ u\cdot v = 0$$

$$x+2y+3z=0$$

$$x^2+y^2+z^2=1$$

$$(-2y-3z)^2+(-3z-x)^2+(-2y-x)^2=1$$

$$(2y+3z)^2+(3z+x)^2+(2y+x)^2=1$$

$$4y^2+12yz+9z^2+9z^2+6xz+x^2+4y^2+4xy+x^2=1$$

5. (★★) Given two vectors **a** and **b** in $\mathbb{R}^3$, prove the following:

(a) Cauchy-Schwarz's Inequality: $|\mathbf{a} \cdot \mathbf{b}| \le |\mathbf{a}|\,|\mathbf{b}|$

(b) Triangle Inequality: $|\mathbf{a} + \mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$

(c) If **a** and **b** are orthogonal, show that $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$.

let

$a = xi + yj + zk$

$b = \alpha i + \beta j + \gamma k$

a).

$$|a \cdot b| \le |a|\,|b|$$

$$|a \cdot b| = |x\alpha + y\beta + z\gamma|$$

$|a \cdot b|$

$$|a|\,|b|$$

$$\sqrt{x^2+y^2+z^2}\ \sqrt{\alpha^2+\beta^2+\gamma^2}$$

$$|a||b| = (x^2+y^2+z^2)(\alpha^2+\beta^2+\gamma^2)$$

$$|a||b| = x^2\alpha^2 + x^2\beta^2 + x^2\gamma^2 + y^2\alpha^2 + y^2\beta^2 + y^2\gamma^2 + z^2\alpha^2 + z^2\beta^2 + z^2\gamma^2$$

$$|a||b| \geqslant |a \cdot b|$$

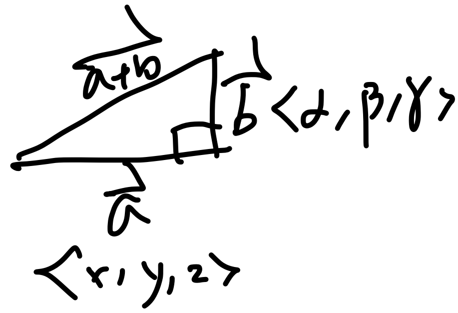
$$|a||b| \geqslant x^2\alpha^2 + y^2\beta^2 + z^2\alpha^2\ ,$$

For $x^2\alpha^2$, $x^2\alpha^2 \geqslant x\alpha$

$x \quad y \quad z$

$a \cdot b = 0$

$|a+b|^2 = |a|^2 + |b|^2$

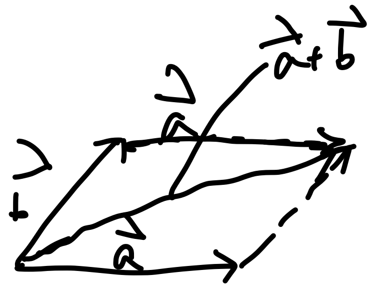


$x\alpha + y\beta + z\gamma = 0$

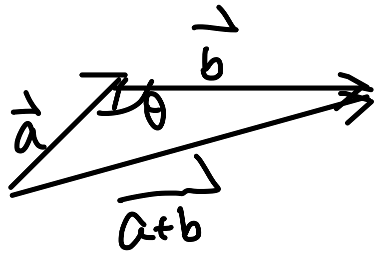
$$|a+b| \leq |a|+|b|$$

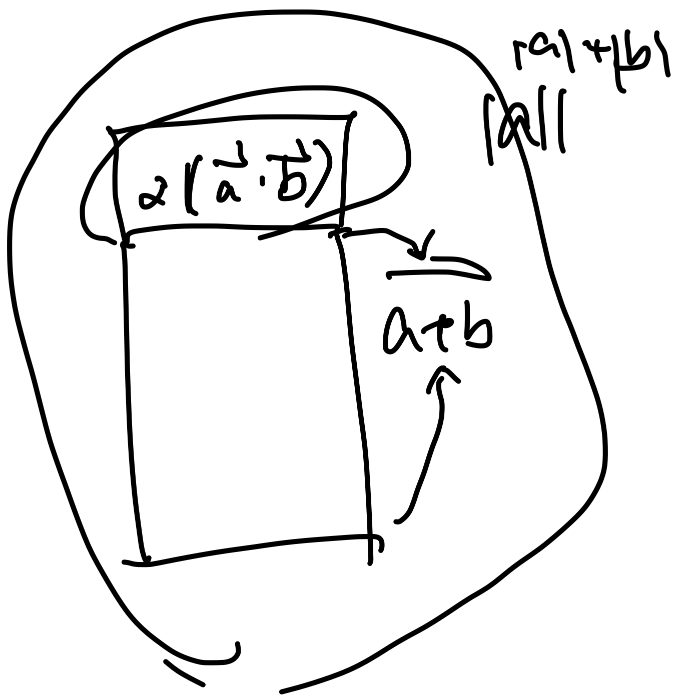
$$\sqrt{(x+\alpha)^2+(y+\beta)^2+(z+\gamma)^2} \leq \sqrt{x^2+y^2+z^2} + \sqrt{\alpha^2+\beta^2+\gamma^2}$$

$$|\overrightarrow{a+b}| = |\vec{a}|^2+|\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos\theta$$



$$2\,\vec{a}\cdot\vec{b}$$





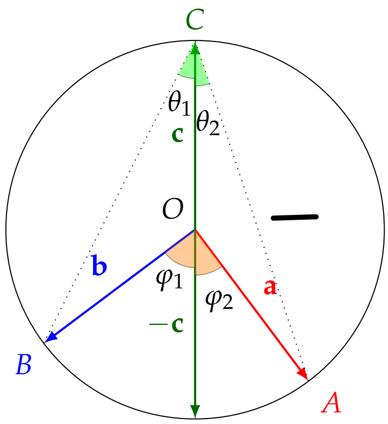
7. (★★) Given three non-zero vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, provide a *geometric explanation* to each of the following facts:

   (a) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

   (b) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$

   (c) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is a vector on the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

8. (★★★) The diagram below shows a circle with radius r centered at O. Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$. The purpose of the problem is to use dot products to show that the angle at the center of a circle is twice the corresponding angle at the circumference. Precisely, with the notations in the diagram below, we want to show $\angle BOA = 2\angle BCA$. We will prove this by showing $\varphi_1 = 2\theta_1$, and $\varphi_2 = 2\theta_2$ can be proved in a similar way. Follow the steps structured below:



   (a) Show that $\cos\varphi_1 = -\dfrac{\mathbf{b}\cdot\mathbf{c}}{r^2}$. Recall that r is the radius of the circle.

   (b) Show that $\cos\theta_1 = \dfrac{r^2 - \mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}-\mathbf{c}|\,|\mathbf{c}|}$.

   (c) Showing that $|\mathbf{b}-\mathbf{c}|^2 = 2(r^2 - \mathbf{b}\cdot\mathbf{c})$.

   (d) Using the result proved in the previous parts, show that $\cos^2\theta_1 = \dfrac{r^2 - \mathbf{b}\cdot\mathbf{c}}{2r^2}$.

   (e) Finally, find a relation between $\cos^2\theta_1$ and $\cos\varphi_1$, and conclude that $\varphi_1 = 2\theta_1$. [Hint: Double angle formula for cos.]

7. (★★) Given three non-zero vectors **u**, **v** and **w** in $\mathbb{R}^3$, provide a *geometric explanation* to each of the following facts:

(a) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

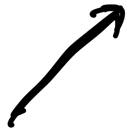
(b) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$

(c) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is a vector on the plane spanned by **u** and **v**.

a). $u \times u = |u||u| \sin\theta = 0$

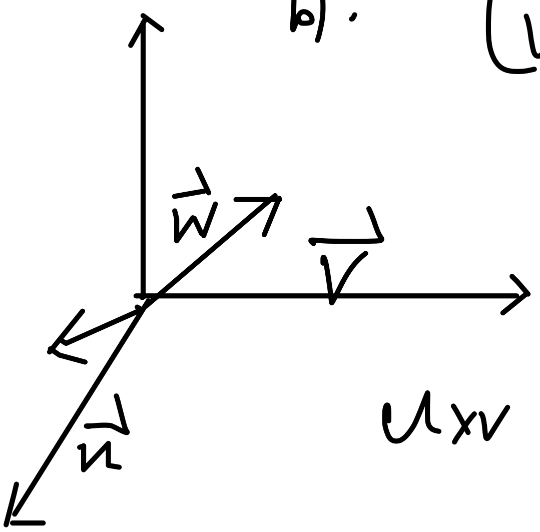
So, $|u| = 0$ (rej since u is non-zero vectors) or $\sin\theta = 0$

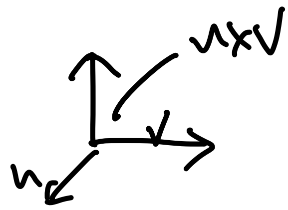
$\therefore \theta = 0$



∴ For same vectors, they don't have an area of ~~$u \times u$. so~~ parallelogram.

b). $(u \times v) \cdot u = (u \times v) \cdot v = 0$





$u \times v$ is orthogonal to u and v.

∴ $u \times v$ dot u is 0.

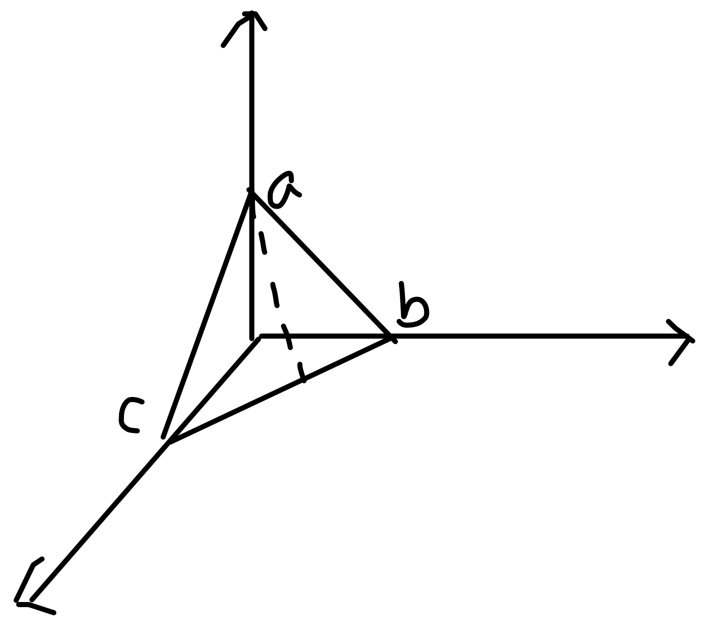
c). $(u \times v) \times w = |u \times v||w| \sin\theta$

6. (★) Let A, B and C be the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively in the three dimensional space, and O be the origin $(0,0,0)$. Denote $[ABC]$ the area of the triangle with vertices A, B and C (analogously for $[OAB]$, $[OBC]$, etc.). Show that:

$$[ABC]^2 = [OAB]^2 + [OBC]^2 + [OCA]^2.$$

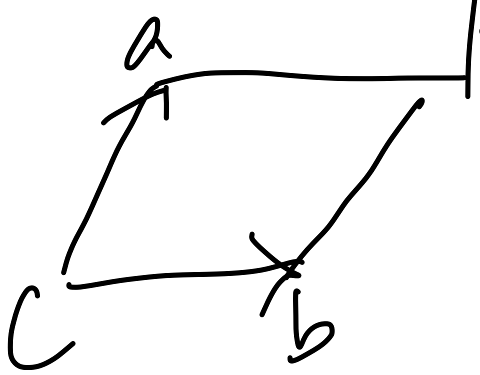
With the help of a diagram, explain why this result can be regarded as the *three-dimensional analogue of the Pythagoreas' Theorem.*

$|bc|$



$$[OAB]^2 = \left(\frac{ab}{2}\right)^2 = \frac{a^2b^2}{4}$$

$$\begin{vmatrix} i & j & k \\ -a & 0 & c \\ 0 & -b & c \end{vmatrix}$$



$$\frac{b^2c^2}{4}$$

ca

$$\frac{a^2c^2}{4}$$

$$\left(\frac{-bc-ab-ac}{2}\right)^2$$

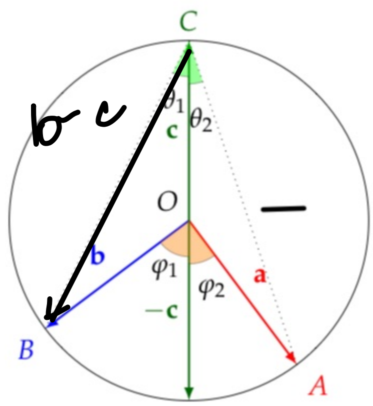
$$\frac{ca \times cb}{2}$$

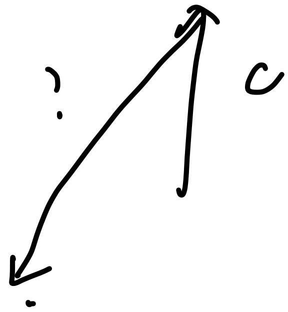
$$\frac{a^2b^2}{4} + \frac{b^2c^2}{4} + \frac{a^2c^2}{4}$$

$$\langle -a, 0, c\rangle \times \langle 0, -b, c\rangle = \frac{a^2b^2 + b^2c^2 + a^2c^2}{4}$$

Made with Goodnotes

8. (★★★) The diagram below shows a circle with radius r centered at O. Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$. The purpose of the problem is to use dot products to show that the angle at the center of a circle is twice the corresponding angle at the circumference. Precisely, with the notations in the diagram below, we want to show $\angle BOA = 2\angle BCA$. We will prove this by showing $\varphi_1 = 2\theta_1$, and $\varphi_2 = 2\theta_2$ can be proved in a similar way. Follow the steps structured below:





$c + ? = b$

$? = b - c$

$(b-c)\cdot(c)$

$c\cdot(b-c)$

$b\cdot c - c\cdot c$

(a) Show that $\cos\varphi_1 = -\dfrac{\mathbf{b}\cdot\mathbf{c}}{r^2}$. Recall that r is the radius of the circle.

(b) Show that $\cos\theta_1 = \dfrac{r^2 - \mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}-\mathbf{c}|\,|\mathbf{c}|}$.

$(\cos\theta_1)^2 = r$

(c) Showing that $|\mathbf{b}-\mathbf{c}|^2 = 2(r^2 - \mathbf{b}\cdot\mathbf{c})$.

$b-$

(d) Using the result proved in the previous parts, show that $\cos^2\theta_1 = \dfrac{r^2 - \mathbf{b}\cdot\mathbf{c}}{2r^2}$.

$\overrightarrow{CB} = b - c$

(e) Finally, find a relation between $\cos^2\theta_1$ and $\cos\varphi_1$, and conclude that $\varphi_1 = 2\theta_1$. [Hint: Double angle formula for cos.]

a). $b\cdot(-c) = |b|\,|-c| \cos\varphi_1$

$-(b\cdot c) = r^2 \cos\varphi_1$

$-b\cdot c + c\cdot c$

$r^2 - b\cdot c$

$\dfrac{-(b\cdot c)}{r^2} = \cos\varphi_1$

b). $\overrightarrow{b-c}\cdot\overrightarrow{-c} = |\vec{c}|\,|\overrightarrow{b-c}| \cos\theta_1$

$\dfrac{\overrightarrow{b-c}\cdot\overrightarrow{-c}}{|c|\,|\overrightarrow{b-c}|} = $ $b\cdot c - c\cdot c$

$\dfrac{b\cdot c - r^2}{|b-c|\,|c|} = \cos\theta_1$

$$|b-c|^2 = 2(r^2 - b\cdot c) \quad , \quad \cos\theta_1 = \frac{r^2 - b\cdot c}{|b-c|\,|c|}$$

$$(\cos\theta_1)^2 = \frac{(r^2 - b\cdot c)^2}{|b-c|^2\,|c|^2} = \frac{(r^2 - b\cdot c)^2}{2(r^2 - b\cdot c)\,|c|^2}$$

$$\cos^2\theta_1 = \frac{r^2 - b\cdot c}{2r^2} \qquad \cos^2\theta_1 = \frac{r^2 - b\cdot c}{2r^2}$$

$$\cos\varphi_1 = -\frac{b\cdot c}{r^2}$$

$$\cos^2\theta_1 = \frac{1 + \cos(2\theta_1)}{2}$$

$$\cos^2\theta_1 = \frac{1}{2} + \frac{\cos(2\theta_1)}{2}$$

$$\frac{r^2 - b\cdot c}{2r^2} = \frac{1}{2} + \frac{\cos(2\theta_1)}{2}$$

$$\frac{1}{2} - \frac{b\cdot c}{2r^2} = \frac{1}{2} + \frac{\cos(2\theta_1)}{2}$$

$$\frac{\cos\varphi_1}{2} = \frac{\cos(2\theta_1)}{2}$$

$$\cos\varphi_1 = \cos(2\theta_1) \qquad \varphi_1 = 2\theta_1$$

$$(b-c)^2 = 2(r^2 - b\cdot c)$$

$$(b-c)\cdot(b-c) = b\cdot b - cb - c\cdot b + c\cdot c$$

$$= b\cdot b - 2(c\cdot b) + c\cdot c$$

$$= r^2 + r^2 - 2(b\cdot c)$$

$$= 2(r^2 - b\cdot c)$$