MATH 2023 • Multivariable Calculus
Problem Set #10 • Divergence Theorem

1. (★) Use the Divergence Theorem to find the outward flux $\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}_{\text{out}}\,dS$ for each of the following $\mathbf{F}$ and S:

(a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the surface of any square cube of length b.

Solution: It is easy to see that $\nabla\cdot\mathbf{F} = 3$. The solid D enclosed by S is the solid square cube of length b. Divergence Theorem shows:

$$\begin{aligned}\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}_{\text{out}}\,dS &= \iiint_D \nabla\cdot\mathbf{F}\,dV\\ &= \iiint_D 3\,dV = 3\times\text{volume of } D\\ &= 3b^2.\end{aligned}$$

(b) $\mathbf{F} = x^3\mathbf{i} + 3yz^2\mathbf{j} + (3y^2z + x^2)\mathbf{k}$ and S is the sphere with radius $a > 0$ centered at the origin.

Solution: $\nabla\cdot\mathbf{F} = 3(x^2+y^2+z^2) = 3\rho^2$. The solid D enclosed by S is the solid sphere with radius a centered at the origin, i.e. $D = \{\rho \le a\}$.

$$\begin{aligned}\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \iiint_D \nabla\cdot\mathbf{F}\,dV\\ &= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 3\rho^2\cdot\rho^2\sin\phi\,d\rho d\phi d\theta\\ &= \frac{12\pi a^5}{5}.\end{aligned}$$

(c) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S is the boundary surface of the cylinder D defined by $x^2 + y^2 \le 1$ and $0 \le z \le 4$.

Solution: $\nabla\cdot\mathbf{F} = 2(x+y+z)$. The solid D is described by inequalities $0 \le r \le 1$, $0 \le \theta \le 2\pi$ and $0 \le z \le 4$ in cylindrical coordinates:

$$\begin{aligned}\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \int_0^{2\pi}\int_0^{1}\int_0^{4} \nabla\cdot\mathbf{F}\,r dz dr d\theta\\ &= \int_0^{2\pi}\int_0^{1}\int_0^{4} 2(r\cos\theta + r\sin\theta + z)\,r dz dr d\theta\\ &= 16\pi\end{aligned}$$

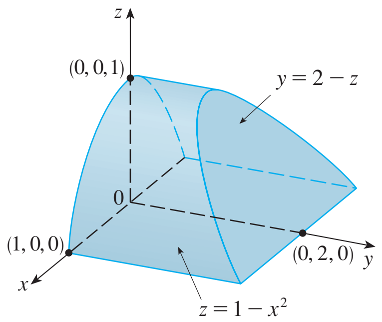
Remark: To simplify the computations, it is good to keep in mind that:

$$\int_0^{2\pi}\cos\theta\,d\theta = \int_0^{2\pi}\sin\theta\,d\theta = 0.$$

2. (★) Evaluate $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}}\, dS$ where

$$\mathbf{F} = xy\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\mathbf{k}$$

and S is the surface boundary of the region D defined by $z \leq 1 - x^2$, $z \geq 0$, $y \geq 0$ and $y \leq 2 - z$. See the figure below:



Comment on why it is preferable to use the Divergence Theorem instead of computing the surface flux directly.

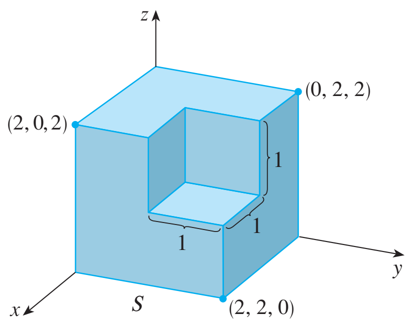
Solution:

$$\nabla \cdot \mathbf{F} = 3y$$

$$\begin{aligned}
\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \iiint_D \nabla \cdot \mathbf{F}\, dV \\
&= \int_{x=-1}^{x=1} \int_{z=0}^{z=1-x^2} \int_{y=0}^{y=2-z} 3y\, dydzdx \\
&= \frac{184}{35}
\end{aligned}$$

Easier to use Divergence Theorem as the surface S has 4 faces. To compute the surface flux directly we would need to split the surface flux into 4 parts and parametrize them individually.

3. (★) Let D be the solid square cube of length 2 with one corner unit cube removed. See the figure below.



Evaluate the outward flux $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}}\, dS$ where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Comment on why it is preferable to use the Divergence Theorem instead of computing the flux directly.

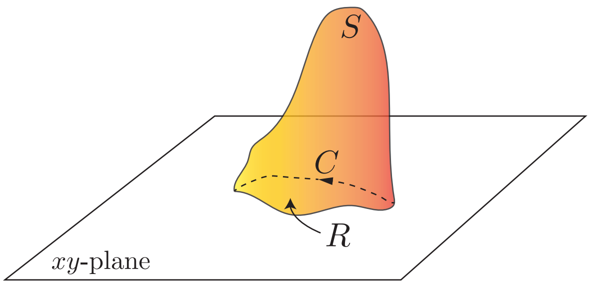
Solution:

$$\nabla \cdot \mathbf{F} = 3$$

$$\begin{aligned}\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \iiint_D \nabla \cdot \mathbf{F}\, dV \\ &= \iiint_D 3\, dV = 3 \times \text{volume of } D \\ &= 3(2^3 - 1^3) = 21.\end{aligned}$$

The surface S has 9 faces!!! Without the Divergence Theorem, we will need to compute the surface flux by split it into 9 parts!

4. (★★) Let C be an arbitrary simple closed curve on the xy-plane in the three dimensional space, and S is any surface *above* the xy-plane with boundary curve C. See the figure below.



Using the Divergence Theorem, show that:

$$\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\; dS = c \times \text{area of the region on the } xy\text{-plane enclosed by } C.$$

Here a, b and c are all constants.

Solution:

$$\nabla \cdot (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} = 0 + 0 + 0 = 0.$$

However, note that S is not a closed surface, but $S \cup R$ is closed. Apply the Divergence Theorem on $S \cup R$ instead:

$$\oiint_{S \cup R} (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = \iiint_{\text{solid enclosed}} \underbrace{\nabla \cdot \mathbf{F}}_{0}\, dV = 0.$$

$$\iint_R (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = \iint_R (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (-\mathbf{k})\, dS = -\iiint_R c\, dS = -c \times \text{area}(R)$$

Since:

$$\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS + \iint_R (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S \cup R} (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = 0$$

we conclude that:

$$\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = -\iint_R (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = c \times \text{area of } R.$$

5. (★★) Suppose $f(x, y, z)$ is a C^2 function on $\mathbb{R}^3$ such that $\nabla^2 f(x, y, z) = 0$ on $\mathbb{R}^3$. Here $\nabla^2 f$ means the Laplacian of f, i.e. $\nabla^2 f = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}$.

(a) Show that:

$$\oiint_S f \nabla f \cdot \hat{\mathbf{n}}\, dS = \iiint_D |\nabla f|^2\, dV$$

for any closed oriented surface S enclosing the solid region D.

Solution:

$$\begin{aligned}
\oiint_S f \nabla f \cdot \hat{\mathbf{n}}\, dS &= \iiint_D \nabla \cdot (f \nabla f)\, dV \\
&= \iiint_D (\nabla f \cdot \nabla f) + f \nabla \cdot \nabla f\, dV \\
&= \iiint_D |\nabla f|^2 + f \underbrace{\nabla^2 f}_{=0}\, dV \\
&= \iiint_D |\nabla f|^2\, dV
\end{aligned}$$

(b) If, furthermore, assume that $f(x, y, z) = 0$ for any (x, y, z) on S, what can you say about $f(x, y, z)$ for any (x, y, z) in D?

Solution: If $f = 0$ on S, then the surface integral:

$$\oiint_S f \nabla f \cdot \hat{\mathbf{n}}\, dS = \oiint_S 0\, \nabla f \cdot \hat{\mathbf{n}}\, dS = 0.$$

Then from (a), we get:

$$\iiint_D |\nabla f|^2\, dV = 0$$

Since $|\nabla f|^2 \geq 0$, the only chance that the above integral is zero is that $\nabla f = \mathbf{0}$ at every point in D. This means f is a constant function is D. By continuity, this constant must match with the value of f on the boundary S, hence $f \equiv 0$ in D.

6. (★★) Suppose S is a closed oriented level surface $f(x,y,z)=c$ of a C^2 function f. Denote D to be the solid enclosed by S. Show that:

$$\oiint_S |\nabla f|\; dS = \pm \iiint_D \nabla^2 f\, dV$$

where $\pm$ depends on whether ∇f points inward or outward on the surface S.

Solution: Note that S is the level surface $f=c$. Hence $\hat{\mathbf{n}} = \pm\frac{\nabla f}{|\nabla f|}$.

$$\begin{aligned}\iiint_D \nabla^2 f\, dV &= \iiint_D \nabla\cdot\nabla f\, dV\\ &= \oiint_S \nabla f\cdot\hat{\mathbf{n}}\, dS\\ &= \pm\oiint_S \nabla f\cdot\frac{\nabla f}{|\nabla f|}\, dS\\ &= \pm\oiint_S \frac{|\nabla f|^2}{|\nabla f|}\, dS = \pm\oiint_S |\nabla f|\; dS\end{aligned}$$

7. (★★) Given two C^2 functions $u(x,y,z)$ and $v(x,y,z)$ defined on $\mathbb{R}^3$. Let S be a closed oriented surface and D is the solid enclosed by S.

(a) Rewrite $\nabla\cdot(u\nabla v - v\nabla u)$ using **curl**, **grad** and **div**.

Solution:

$$\operatorname{div}(u\operatorname{grad}(v) - v\operatorname{grad}(u)).$$

(b) Show that

$$\oiint_S (u\nabla v - v\nabla u)\cdot\hat{\mathbf{n}}\, dS = \iiint_D \left(u\nabla^2 v - v\nabla^2 u\right)\; dV$$

Solution:

$$\begin{aligned}\oiint_S (u\nabla v - v\nabla u)\cdot\hat{\mathbf{n}}\, dS &= \iiint_D \nabla\cdot(u\nabla v - v\nabla u)\; dV\\ &= \iiint_D (\nabla u\cdot\nabla v + u\nabla\cdot\nabla v - \nabla v\cdot\nabla u - v\nabla\cdot\nabla u)\, dV\\ &= \iiint_D (u\nabla^2 v - v\nabla^2 u)\, dV\end{aligned}$$

(c) Assume further that $\nabla u(x,y,z)\cdot\hat{\mathbf{n}} = \nabla v(x,y,z)\cdot\hat{\mathbf{n}} = 0$ for any (x,y,z) on S, show that

$$\iiint_D u\nabla^2 v\, dV = \iiint_D v\nabla^2 u\, dV.$$

Solution: Simply apply the result of (b) using the given conditions that $\nabla u\cdot\hat{\mathbf{n}} = \nabla v\cdot\hat{\mathbf{n}} = 0$ on S.