Chapter 16
Vector Calculus

Contents

16.3 The Divergence Theorem

Let G be a simple solid whose boundary surface S has positive (outward) orientation and let $\mathbf{F}(x, y, z) = f(x, y, z)\,\mathbf{i} + g(x, y, z)\,\mathbf{j} + h(x, y, z)\,\mathbf{k}$ be a smooth vector field with f, g and h have continuous partial derivatives on an open region that contain G. Then

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \oiint_S \mathbf{F}\cdot \widehat{\mathbf{n}}\, dS = \iiint_G \nabla\cdot \mathbf{F}\, dV.$$

Note that $\widehat{\mathbf{n}}$ is a unit normal field pointing out of G. The volume integral of the divergence of a vector field, taken throughout a bounded domain G, equals the surface integral of the normal component of the vector field taken over the boundary of G $(= S)$. In other words, **the total divergence within G equals the net flux emerging from G.**

(Proof: see the textbook p434)

A two-dimensional Divergence Theorem

Let R be a domain in the xy-plane with piecewise smooth boundary curve C. If

$$\mathbf{F} = F_1(x, y)\,\mathbf{i} + F_2(x, y)\,\mathbf{j}$$

is a smooth vector field, then

$$\oint_C \mathbf{F}\cdot \widehat{\mathbf{n}}\, ds = \iint_R \nabla\cdot \mathbf{F}\, dA$$

where $\widehat{\mathbf{n}}$ is the unit outward normal on C.

(Proof: see the textbook p439)

Prove the 3-Dimensional Divergence Theorem.

Let the smooth vector field be $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$, then the 3D Divergence Theorem stated that

$$\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_G \nabla\cdot\mathbf{F}\,dV$$

$$\oiint_S (F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k})\cdot\hat{\mathbf{n}}\,dS = \iiint_G \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV,$$

where G is the volume enclosed by the closed surface S. Prove

$$\oiint_S F_1\,\mathbf{i}\cdot\hat{\mathbf{n}}\,dS = \iiint_G \frac{\partial F_1}{\partial x}\,dV \qquad (1)$$

$$\oiint_S F_2\,\mathbf{j}\cdot\hat{\mathbf{n}}\,dS = \iiint_G \frac{\partial F_2}{\partial y}\,dV \qquad (2)$$

$$\oiint_S F_3\,\mathbf{k}\cdot\hat{\mathbf{n}}\,dS = \iiint_G \frac{\partial F_3}{\partial z}\,dV \qquad (3)$$

Since the proofs of (1) – (3) are similar, we need to prove (3) only.

$dV = dz\,dA$ where $dA = dx\,dy$

$$\text{RHS} = \iiint_G \frac{\partial F_3}{\partial z}\,dV$$

$$= \oiint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} \frac{\partial F_3}{\partial z}\,dz\right] dA = \oiint_R F_3(x,y,z)\Big|_{z=g_1}^{z=g_2}\,dA$$

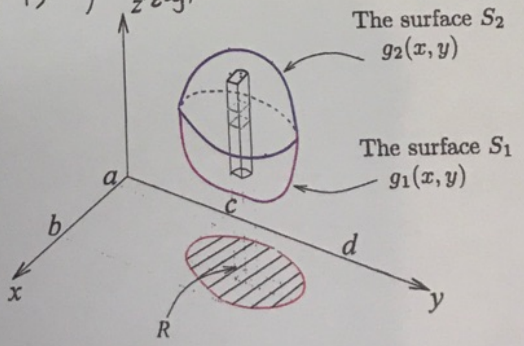
$$= \oiint_R \left[F_3(x,y,g_2) - F_3(x,y,g_1)\right] dA$$

$$\text{LHS} = \oiint_S F_3\,\mathbf{k}\cdot\hat{\mathbf{n}}\,dS$$

$$= \iint_{S_1} F_3\,\mathbf{k}\cdot\hat{\mathbf{n}}\,dS + \iint_{S_2} F_3\,\mathbf{k}\cdot\hat{\mathbf{n}}\,dS$$

$$= \mathrm{I}_1 + \mathrm{I}_2$$

兩個$\hat{n}$不同，
各自垂直于S_1及S_2.



$S = S_1 + S_2$

$\mathrm{I}_1 = \int_{S_1}$ …

$=$

$=$

$\mathrm{I}_2 =$

$=$

$\therefore\ \mathrm{I}_1 + \mathrm{I}_2$ …

ce Theorem

(1)

(2)

(3)

= dzdA where dA = dxdy

e surface S_2
$g_2(x,y)$

surface S_1
(x,y)

y

projection

$$I_1 = \iint_{S_1} F_3\,\mathbf{k}\cdot\hat{\mathbf{n}}\,dS = \iint_R F_3(x,y,g_1)\,\mathbf{k}\cdot\hat{\mathbf{n}}\sqrt{1+(g_{1x})^2+g_{1y})^2}\,dA_{xy}$$

downward normal

$$= \iint_R F_3(x,y,g_1)\,\frac{\mathbf{k}\cdot(g_{1x},g_{1y},-1)}{\sqrt{1+(g_{1x})^2+g_{1y})^2}}\sqrt{1+(g_{1x})^2+g_{1y})^2}\,dA_{xy}$$

$$= -\iint_R F_3(x,y,g_1)\,dA_{xy}$$

$$I_2 = \iint_{S_2} F_3\,\mathbf{k}\cdot\hat{\mathbf{n}}\,dS = \iint_R F_3(x,y,g_2)\,\mathbf{k}\cdot\hat{\mathbf{n}}\sqrt{1+(g_{2x})^2+g_{2y})^2}\,dA_{xy}$$

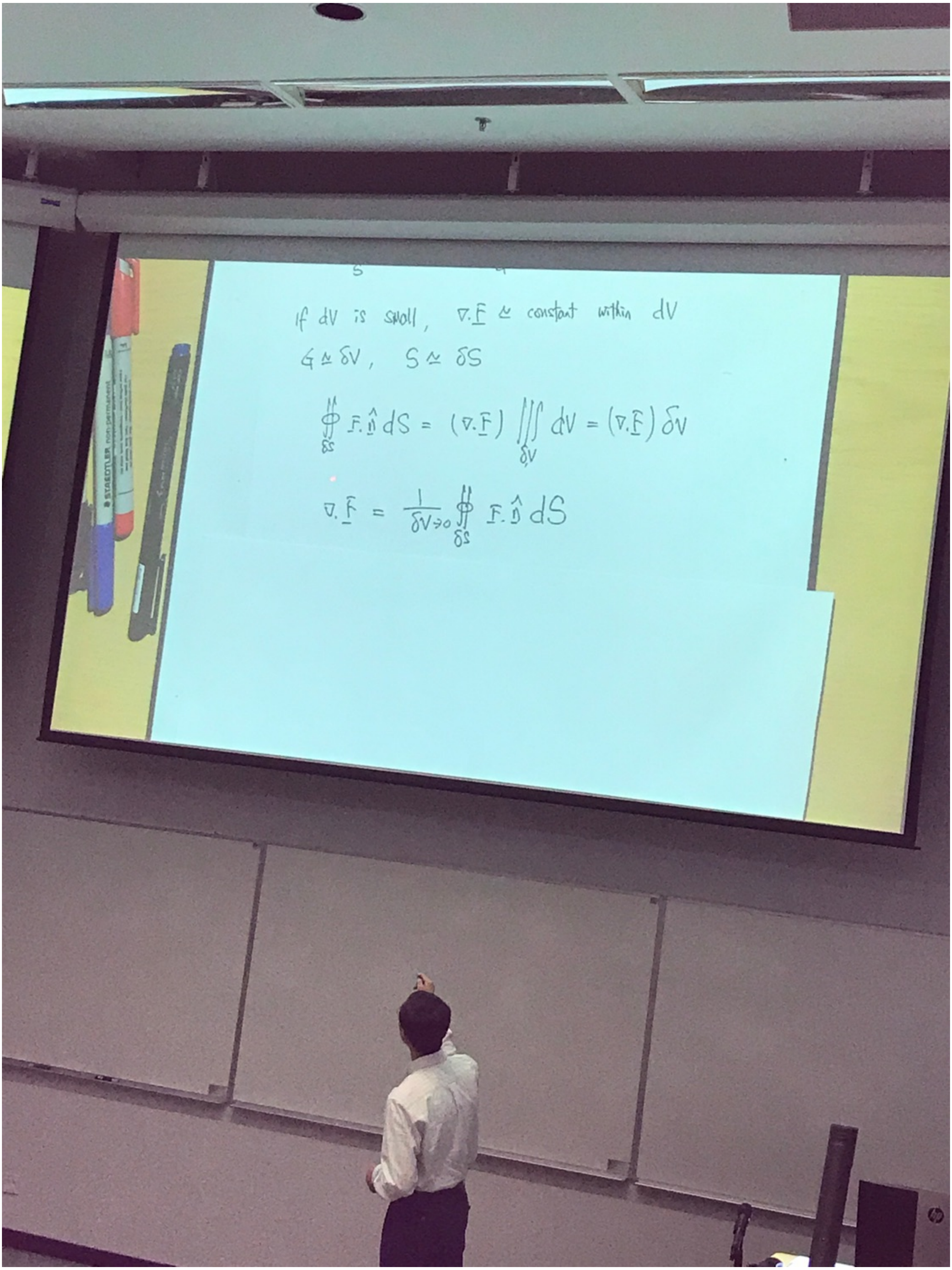
upper normal

$$= \iint_R F_3(x,y,g_2)\,\frac{\mathbf{k}\cdot(-g_{2x},-g_{2y},1)}{\sqrt{1+(g_{2x})^2+g_{2y})^2}}\sqrt{1+(g_{2x})^2+g_{2y})^2}\,dA_{xy}$$

$$= \iint_R F_3(x,y,g_2)\,dA_{xy}$$

$$\therefore\quad I_1+I_2 = \iint_R [F_3(x,y,g_2)-F_3(x,y,g_1)]\,dA_{xy}$$

$$= \text{RHS.}$$


if dV is small, $\nabla.\underline{F} \simeq$ constant within dV
$G \simeq \delta V$, $S \simeq \delta S$
$\oiint_{\delta S} \underline{F}.\hat{\underline{n}}\, dS = (\nabla.\underline{F}) \iiint_{\delta V} dV = (\nabla.\underline{F})\, \delta V$
$\nabla.\underline{F} = \frac{1}{\delta V \to 0} \oiint_{\delta S} \underline{F}.\hat{\underline{n}}\, dS$

Ex. 3.1 A fluid with density 1500 flows with velocity $\mathbf{v} = -y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k}$. Find the rate of flow outward through the sphere $x^2 + y^2 + z^2 = 5^2$, i.e., find the flux of the vector field $\mathbf{F}(x, y, z) = 1500(-y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k})$ over the sphere $x^2 + y^2 + z^2 = 5^2$.

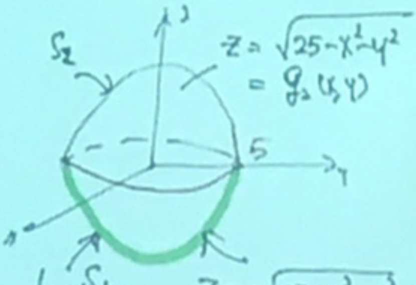
$$\nabla \cdot \underline{F} = 1500\left(\frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial(2z)}{\partial z}\right) = 3000$$

$\underline{F}(x,y,z) = 1500\,(-y,\ x,\ 2z)$

$$\oiint \underline{F}\cdot\hat{\underline{n}}\,dS = \iint_{S_1} \underline{F}\cdot\hat{\underline{n}}\,dS + \iint_{S_2} \underline{F}\cdot\hat{\underline{n}}\,dS$$

$$= \iiint_G \nabla\cdot\underline{F}\,dV$$ where G is the volume enclosed by S

$S = S_1 + S_2$

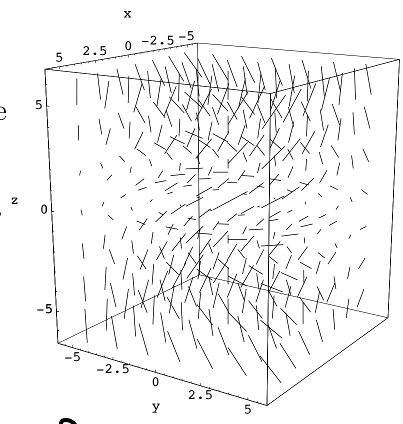


$$= \iiint_G 3000\,dV$$

$$= 3000 \iiint_G dV = 3000\left(\frac{4}{3}\pi(5)^3\right)$$

Ex. 3.1 A fluid with density 1500 flows with velocity $\mathbf{v} = -y\,\mathbf{i} + \mathbf{j} + z\,\mathbf{k}$. Find the rate of flow outward through the sphere $x^2+y^2+z^2 = 5^2$, i.e., find the flux of the vector field $\mathbf{F}(x,y,z) = 1500(-y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k})$ over the sphere $x^2+y^2+z^2 = 5^2$.

$$\nabla\cdot\mathbf{F} = 1500 \times 2 = 3000$$



Using the divergence theorem because the sphere is a *closed* surface

$$\oiint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS = \iiint_G 3000\,dV = 3000 \times \frac{4}{3}\pi 5^3 = 500,000\pi.$$

Using the method of surface integral: the surface S is given by

$$\mathbf{r}(\phi,\theta) = 5\sin\phi\cos\theta\,\mathbf{i} + 5\sin\phi\sin\theta\,\mathbf{j} + 5\cos\phi\,\mathbf{k},$$

where $\phi \in [0,\pi]$, $\theta \in [0, 2\pi]$,

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$$\therefore\ \mathbf{r}_\phi\times\mathbf{r}_\theta = 25\sin^2\phi\cos\theta\,\mathbf{i} + 25\sin^2\phi\sin\theta\,\mathbf{j} + 25\sin\phi\cos\phi\,\mathbf{k}.$$

$$\|\mathbf{r}_\phi\times\mathbf{r}_\theta\| = 25\sin\phi$$

$$\widehat{\mathbf{n}} = \frac{\mathbf{r}_\phi\times\mathbf{r}_\theta}{\|\mathbf{r}_\phi\times\mathbf{r}_\theta\|} = \frac{1}{5}\mathbf{r}(\phi,\theta)$$

$$\therefore \iint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS = 1500\int_0^{2\pi}\int_0^{\pi}(-125\sin^3\phi\sin\theta\cos\theta + 125\sin^3\phi\sin\theta\cos\theta + 250\sin\phi\cos^2\phi)\,d\phi\,d\theta$$

$$= 3000\pi(250)\left[-\frac{1}{3}\cos^3\phi\right]_0^{\pi} = 500,000\pi.$$

Ex. 3.2 Find $\iint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS$ if $\mathbf{F}(x,y,z) = (x^3+yz)\,\mathbf{i} + x^2y\,\mathbf{j} + xy^2\,\mathbf{k}$, S is the surface of the solid bounded by the sphere $x^2+y^2+z^2 = 4$ and $x^2+y^2+z^2 = 9$.

$$\nabla\cdot\mathbf{F} = 3x^2 + x^2 + 0 = 4x^2$$

Using the divergence theorem

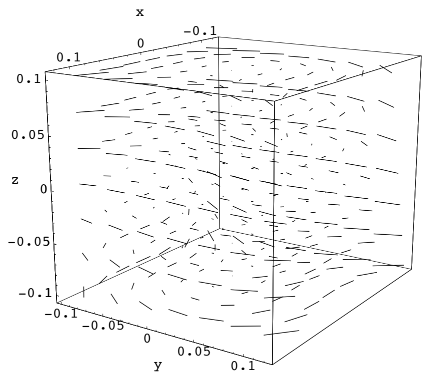
spherical coordinate

$$\oiint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS = \iiint_G \nabla\cdot\mathbf{F}\,dV$$

$$= \iiint_G 4x^2\,dV \qquad (x = \rho\sin\phi\cos\theta)$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_2^3 (4\rho^2\sin^2\phi\cos^2\theta)\rho^2\sin\phi\,d\rho d\phi d\theta$$

$$= \frac{3376}{15}\pi.$$



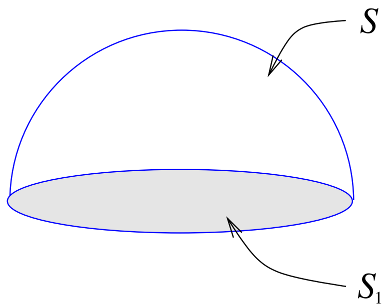
Ex. 3.3 $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F}(x,y,z) = z^2x\,\mathbf{i} + \left(\frac{1}{3}y^3 + \tan z\right)\mathbf{j} + (x^2z + y^2)\,\mathbf{k}$ and S is the top half of the sphere $x^2+y^2+z^2 = 1$. (Note that S is *not* a **closed surface**).

Let S_1 be the disk $x^2+y^2 \leqslant 1$ and $S_2 = S_1 + S$ (S_2 is a *closed* surface).

For S_1, $\widehat{\mathbf{n}} = -\mathbf{k}$, so $\mathbf{F}\cdot\widehat{\mathbf{n}} = -x^2z - y^2 = -y^2$ (since $z=0$ on S_1)

$$\iint_{S_1} \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS = \iint_{S_1} -y^2\,dA$$

$$= -\int_0^{2\pi}\int_0^1 r^2\sin^2\theta\, r\,dr\,d\theta = -\frac{1}{4}\pi.$$



Note also that $\nabla\cdot\mathbf{F} = z^2+y^2+x^2$, therefore

$$\oiint_{S_2} \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS = \iiint_G (x^2+y^2+z^2)\,dV$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{2}{5}\pi.$$

Since the surface $S = S_2 - S_1$,

$$\therefore \iint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS = \iint_{S_2} \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS - \iint_{S_1} \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS = \frac{2}{5}\pi + \frac{1}{4}\pi = \frac{13}{20}\pi.$$

Ex. 3.4 Show that $\nabla\cdot(f\nabla g) = (\nabla g)\cdot(\nabla f) + f\nabla^2 g$.

Hence show that $\displaystyle\oiint_S (f\nabla g - g\nabla f)\cdot\widehat{\mathbf{n}}\,dS = \iiint_G (f\nabla^2 g - g\nabla^2 f)\,dV$.

Ex. 3.5 Let G be the region $x^2+y^2+z^2 \leqslant 4a^2$, $x^2+y^2 \geqslant a^2$. The surface S of G consists of cylindrical part, S_1 and a spherical part, S_2. Evaluate the flux of

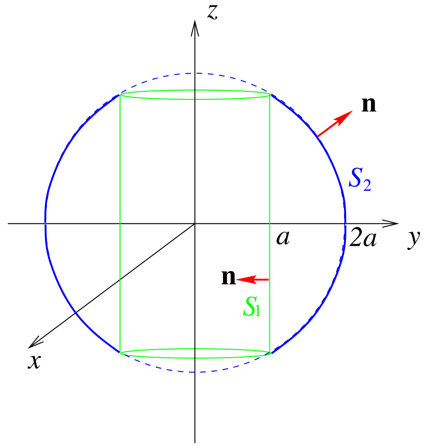
$$\mathbf{F} = (x+yz)\,\mathbf{i} + (y-xz)\,\mathbf{j} + (z-e^x \sin y)\,\mathbf{k}$$

out of G through

(a) the whole surface S,

(b) the surface S_1, and

(c) the surface S_2.



$$\begin{aligned}\mathbf{F} &= (x+yz)\,\mathbf{i} + (y-xz)\,\mathbf{j} + (z-e^x \sin y)\,\mathbf{k} \\ \operatorname{div}\mathbf{F} &= 1+1+1 = 3.\end{aligned}$$

(a) The flux of $\mathbf{F}$ out of G through $S = S_1 \cup S_2$ is

$$\begin{aligned}\oint\!\!\!\oint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iiint_G \nabla\cdot\mathbf{F}\,dV \\ &= 2\int_0^{2\pi}\int_a^{2a}\int_0^{\sqrt{4a^2-r^2}} 3\,r\,dzdrd\theta \\ &= 12\pi\int_a^{2a} r\sqrt{4a^2-r^2}\,dr \\ &= 12\sqrt{3}\pi a^3.\end{aligned}$$

(b) On S_1, $\widehat{\mathbf{n}} = -\dfrac{x\,\mathbf{i}+y\,\mathbf{j}}{a}$, $dS = a\ d\theta dz$. The flux of $\mathbf{F}$ out of G through S_1 is

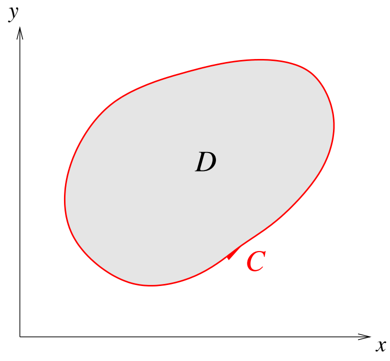
$$\begin{aligned}\iint_{S_1} \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS &= \iint_{S_1} \frac{-x^2-xyz-y^2+xyz}{a} a\,d\theta\ dz \\ &= -a^2\int_0^{2\pi}\int_{-\sqrt{3}a}^{\sqrt{3}a} dzd\theta = -4\sqrt{3}\pi a^3.\end{aligned}$$

(c) The flux of $\mathbf{F}$ out of G through the spherical part S_2 is

$$\begin{aligned}\iint_{S_2} \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS &= \oint\!\!\!\oint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS - \iint_{S_1} \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS \\ &= 12\sqrt{3}\pi a^3 + 4\sqrt{3}\pi a^3 = 16\sqrt{3}\pi a^3.\end{aligned}$$

16.4 Green's Theorem (2D) and Stoke's Theorem (3D)

Green's Theorem gives the relationship between a line integral around a simple, closed, piecewise smooth curve C oriented counter-clockwise and a double integral over the plane region D bounded by C.



Counter-clockwise
– positive orientation

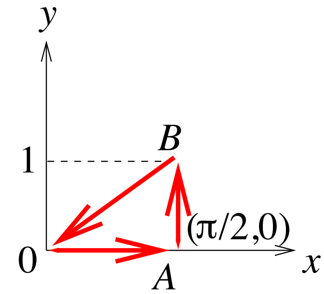
<u>Green's Theorem</u>: Let C be a positive oriented, piecewise-smooth simple closed curve in the xy-plane and D be the region bounded by C. If $\mathbf{F}(\mathbf{r}) = P(\mathbf{r})\,\mathbf{i} + Q(\mathbf{r})\,\mathbf{j}$, where P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D (\nabla\times\mathbf{F})\cdot \mathbf{k}\,dA \qquad \text{where} \qquad \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{i} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ P & Q & 0 \end{vmatrix} = (Q_x - P_y)\,\mathbf{k}.$$

Note that $\mathbf{k}$ is normal to the xy-plane, hence it is normal to the region D.

Ex. 4.1 Evaluate $\oint_C (y - \sin x)\,dx + \cos x\,dy$, where C is the triangle of the figure. (a) directly, (b) by using Green's theorem in the plane.



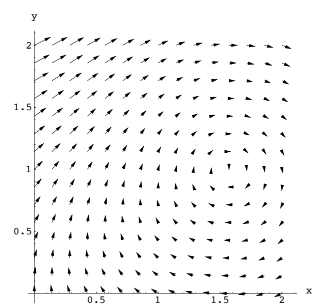
(a) Along **OA**, $\mathbf{r} = (1-t)(0,0) + t(\frac{\pi}{2},0) = (\frac{\pi}{2}t, 0), \quad 0 \leqslant t \leqslant 1$

$$\therefore \int_0^1 \left(0 - \sin\frac{\pi}{2}t\right)\left(\frac{\pi}{2}dt\right) + \cos\frac{\pi}{2}t\cdot(0) = -1$$

Along **AB**, $\mathbf{r} = (1-t)(\frac{\pi}{2},0) + t(\frac{\pi}{2},1) = (\frac{\pi}{2}, t), \quad 0 \leqslant t \leqslant 1$

$$\therefore \int_0^1 (t-1)\cdot(0) + (0)\cdot dt = 0$$

Along **BO**, $\mathbf{r} = (1-t)(\frac{\pi}{2},1) + t(0,0) = (\frac{\pi}{2} - \frac{\pi}{2}t, 1-t), \quad 0 \leqslant t \leqslant 1$



$\mathbf{F}(\mathbf{r}) = (y - \sin x)\,\mathbf{i} + \cos x\,\mathbf{j}$

$$\therefore \int_0^1 \left[1 - t - \cos\frac{\pi}{2}t\right]\left(-\frac{\pi}{2}dt\right) + \sin\frac{\pi}{2}t\,(-dt)$$

$$= -\frac{\pi}{2}\left[t - \frac{t^2}{2}\right]\Bigg|_0^1 + \sin\frac{\pi}{2}t\Bigg|_0^1 + \frac{2}{\pi}\cos\frac{\pi}{2}t\Bigg|_0^1$$

$$= 1 - \frac{\pi}{4} - \frac{2}{\pi}$$

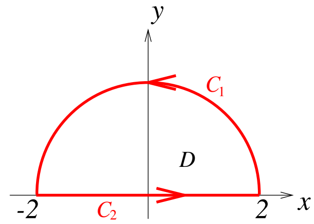
(b) Let $P = y - \sin x$ and $Q = \cos x$, then

$$\oint_C P\,dx + Q\,dy =$$

Ex. 4.2 $\displaystyle\int_C xy\,dx + 2x^2dy$, C consists of the segment from $(-2,0)$ to $(2,0)$ and top half of the circle $x^2+y^2=4$.

C_1: $\mathbf{r}(t) = (1-t)(-2,0) + t(2,0) = (4t-2, 0) \quad 0 \leqslant t \leqslant 1$

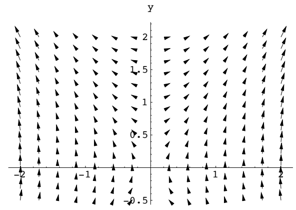
C_2: $\mathbf{r}(t) = (2\cos t, 2\sin t) \quad 0 \leqslant t \leqslant \pi$



$$\int_{C_1} xy\,dx + 2x^2\,dy = \int_0^1 (4t-2)\cdot 0\cdot 4\,dt + 2(4t-2)^2\cdot(0) = 0$$

$$\int_{C_2} xy\,dx + 2x^2\,dy = \int_0^\pi (2\cos t)(2\sin t)(-2\sin t)\,dt + 2(2\cos t)^2(2\cos t)\,dt$$

$$= 8\int_0^\pi (-\cos t\sin^2 t + \cos^3 t)\,dt = 0$$



$\mathbf{F}(\mathbf{r}) = xy\,\mathbf{i} + 2x^2\,\mathbf{j}$

or using Green's theorem

$$\oint_C xy\,dx + 2x^2dy =$$

Computing Area:

If $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$, then $A = \displaystyle\iint_D dA = \oint_C P\,dx + Q\,dy$.

There are several possibilities:

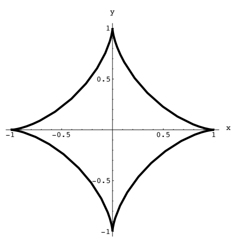
(i) $P = 0, \quad Q = x$

(ii) $P = -y, \quad Q = 0$

(iii) $P = -y/2 \quad Q = x/2$

$$\therefore A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Example Find the area of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$.



Let $x = a\cos^3\theta$, $y = a\sin^3\theta$, where $0 \leqslant \theta \leqslant 2\pi$.

$$A = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi}(a\cos^3\theta \times 3a\sin^2\theta\cos\theta\,d\theta + a\sin^3\theta\times 3a\cos^2\theta\sin\theta\,d\theta)$$

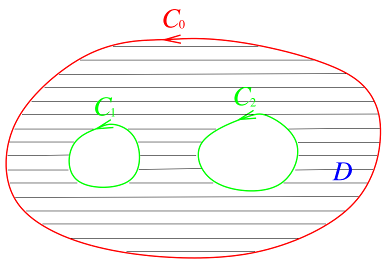
$$= \frac{3}{2}a^2\int_0^{2\pi}(\cos^4\sin^2\theta + \sin^4\theta\cos^2\theta)\,d\theta$$

$$= \frac{3}{2}a^2\int_0^{2\pi}\frac{1}{4}\sin^2 2\theta\,d\theta$$

$$= \frac{3}{8}a^2\int_0^{2\pi}\left(\frac{1-\cos 4\theta}{2}\right)d\theta = \frac{3\pi}{8}a^2.$$

General version of Green's Theorem*

Green's Theorem can be extended to apply to region with holes. Let $P(x,y)$, $Q(x,y)$, P_y, Q_x be continuous on a closed set D whose boundary consists of closed curves, C_0, C_1, $C_2, \cdots, C_n$ oriented counter clockwisely with C_0 enclosing the others. Then



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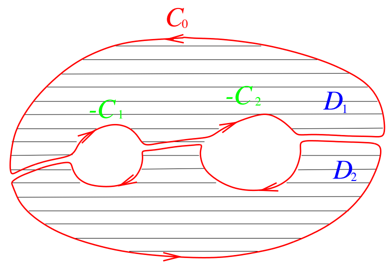


where $D = D_1 + D_2$ and D_i is bounded by C_i'.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_{D_1} + \iint_{D_2} = \int_{C_1'} + \int_{C_2'}$$

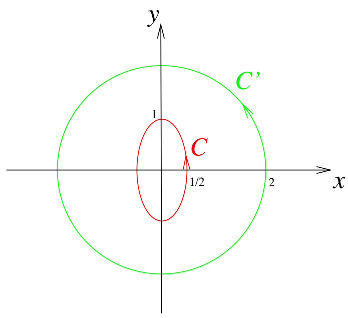
$$= \int_{C_0} (P\,dx + Q\,dy) - \sum_{j=1}^{n} \int_{C_j} (P\,dx + Q\,dy).$$

(The purpose of this general version of Green's theorem is for dealing with the cases when P or Q is not defined somewhere insider a closed curve **<u>or</u>** when changing curves to simplify the computation.)

Example $\displaystyle\oint_C \frac{-x^2 y\,dx + x^3\,dy}{(x^2+y^2)^2}$, where C is the ellipse $4x^2 + y^2 = 1$.

If C' is the circle $x^2 + y^2 = 4$, then C is interior to C', and everywhere except at $(0,0)$. Note also that

$$\frac{\partial}{\partial x}\left[\frac{x^3}{(x^2+y^2)^2}\right] = \frac{\partial}{\partial y}\left[\frac{-x^2 y}{(x^2+y^2)^2}\right]$$

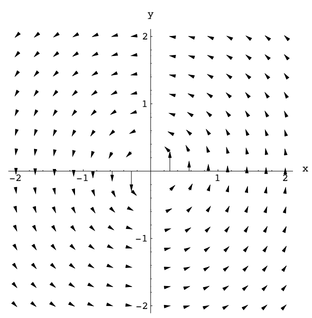


$$\therefore\ I = \oint_C \frac{-x^2 y\,dx + x^2\,dy}{(x^2+y^2)^2} = \oint_{C'} \frac{-x^2 y\,dx + x^3\,dy}{(x^2+y^2)^2}$$

On C', let $x = 2\cos\theta$, $y = 2\sin\theta$, where $0 \leqslant \theta \leqslant 2\pi$, then

$$I = \int_0^{2\pi} \frac{-4\cos^2\theta\ 2\sin\theta(-2\sin\theta)\,d\theta + (2\cos\theta)^2\ 2\cos\theta\,d\theta}{16}$$

$$= \int_0^{2\pi} \cos^2\theta\,d\theta = \int_0^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \pi.$$

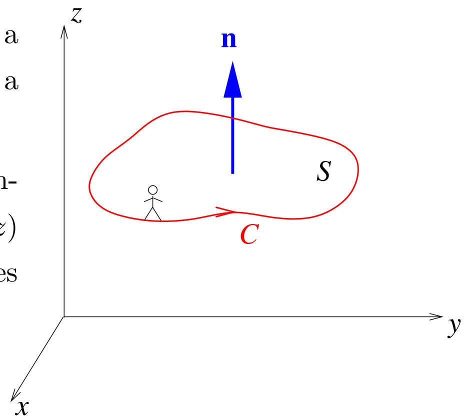


$$\mathbf{F}(\mathbf{r}) = \frac{-x^2 y\,\mathbf{i} + x^3\,\mathbf{j}}{(x^2+y^2)^2}$$

* Will NOT be tested.

Stokes' Theorem

Stokes' Theorem can be regarded as a higher dimensional version of Green's Theorem. Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve).



positive direction around C means the surface will always be on your left, then your head pointing in the direction of $\mathbf{n}$

Stokes' Theorem Let S be a non-closed surface, whose boundary consists of a closed smooth curve C with positive orientation. Let $\mathbf{F}(x, y, z)$ be a vector field whose components have continuous partial derivatives on S. Then

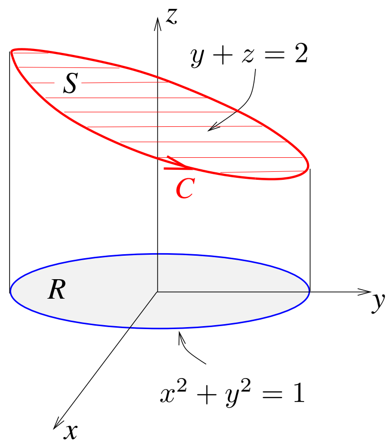
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\widehat{\mathbf{n}}\, dS$$

Note $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \mathbf{F}\cdot\mathbf{T}\, ds$.

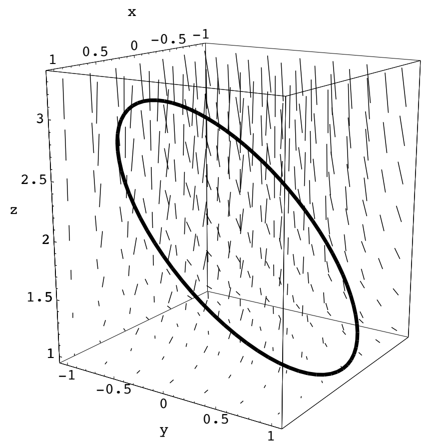
Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$, taken over a bounded surface S.

Ex. 4.3 Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y)\,\mathbf{k}$$



Let $g(x, y, z) = y + z - 2 = 0$. This is a level surface, hence the gradient of g is $\nabla g = (0, 1, 1)$, i.e. $\mathbf{n} = (0, 1, 1)$.



Ex. 4.4 Verify Stokes' Theorem: $\mathbf{F}(x,y,z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$, S is the part of the plane $x+y+z=1$ that lies in the first octant, oriented upward $C = C_1 + C_2 + C_3$.

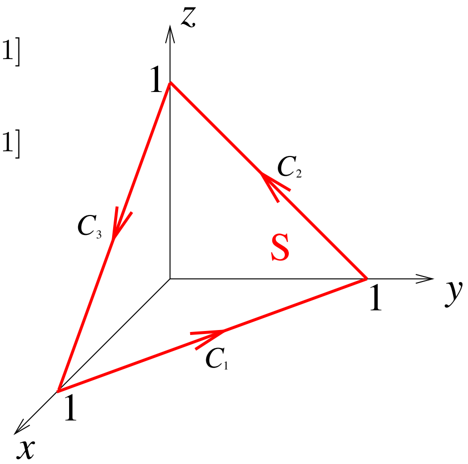
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} (y\,dx + z\,dy + x\,dz) \quad x = 1-t,\ y = t,\ z = 0 \qquad t \in [0,1]$$

$$+ \int_{C_2} (y\,dx + z\,dy + x\,dz) \quad x = 0,\ y = t,\ z = -t \qquad t \in [0,1]$$

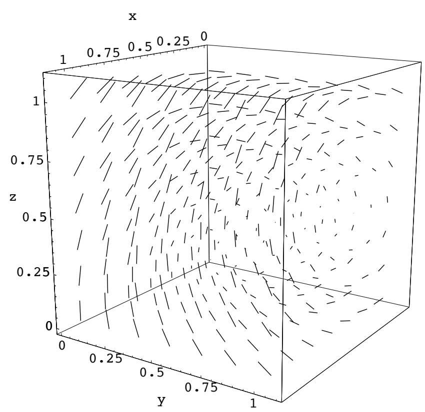
$$+ \int_{C_3} (y\,dx + z\,dy + x\,dz) \quad x = t,\ y = 0,\ z = -t \qquad t \in [0,1]$$

$$= -\int_0^1 t\,dt + \int_0^1 (-t)\,dt + \int_0^1 t(-dt)$$

$$= -3\int_0^1 t\,dt = -3/2.$$

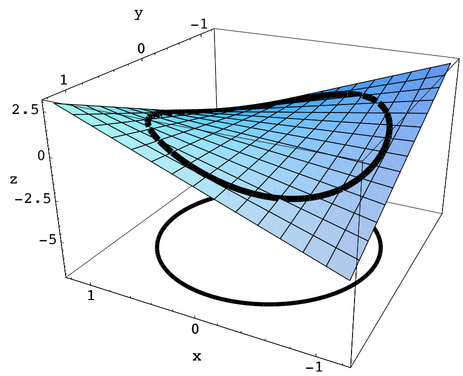


Alternatively,



Ex. 4.5 Evaluate $\displaystyle\int_C (y+\sin x)\,dx + (z^2+\cos y)\,dy + x^3\,dz$, where C is the curve

$\mathbf{r}(t) = (\sin t,\, \cos t,\, \sin 2t), \quad 0 \leqslant t \leqslant 2\pi.$



Note that C is a closed space curve.

$$\therefore \oint_C (y+\sin x)\,dx + (z^2+\cos y)\,dy + x^3\,dz = \oint_C \mathbf{F}\cdot d\mathbf{r},$$

where $\mathbf{F}(x,y,z) = (y+\sin x,\, z^2+\cos y,\, x^3)$.

$\nabla\times\mathbf{F} = (-2z,\, -3x^2,\, -1)$, since $\sin 2t = 2\sin t\cos t \;\Rightarrow\; z = 2xy$, i.e. C lies on the surface $z = 2xy$. Let S be the part of this surface that is bounded by C. Then the projection of S onto xy-plane is the unit disk D $(x^2+y^2 \leqslant 1)$.

Normal vector of the level surface $f(x,y,z) = z - 2xy = 0$ is

$\mathbf{n} = \nabla f = (-2y, -2x, 1).$ (Note that $\mathbf{n}$ points upward, Why?)

Therefore

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \nabla\times\mathbf{F}\cdot\widehat{\mathbf{n}}\,dS \\
&= \iint_D \nabla\times\mathbf{F}\cdot\mathbf{n}\,dA = \iint_D (-4xy, -3x^2, -1)\cdot(-2y,-2x,1)\,dA \\
&= \iint_D (8xy^2 + 6x^3 - 1)\,dA \\
&= \int_0^{2\pi}\int_0^1 (8r^3\cos\theta\sin^2\theta + 6r^3\cos^3\theta - 1)\,r dr\,d\theta \\
&= -\pi.
\end{aligned}$$

Ex. 4.6 Show that $\displaystyle\oint_C (f\nabla g)\cdot d\mathbf{r} = \iint_S (\nabla f\times\nabla g)\cdot\widehat{\mathbf{n}}\,dS$