EXERCISES 12.8

In Exercises 1–12, calculate the indicated derivative from the given equation(s). What condition on the variables will guarantee the existence of a solution that has the indicated derivative? Assume that any general functions F, G, and H have continuous first partial derivatives.

1. $\dfrac{dx}{dy}$ if $xy^3 + x^4 y = 2$

2. $\dfrac{\partial x}{\partial y}$ if $xy^3 = y - z$

3. $\dfrac{\partial z}{\partial y}$ if $z^2 + xy^3 = \dfrac{xz}{y}$

4. $\dfrac{\partial y}{\partial z}$ if $e^{yz} - x^2 z \ln y = \pi$

5. $\dfrac{\partial x}{\partial w}$ if $x^2y^2 + y^2z^2 + z^2t^2 + t^2w^2 - xw = 0$

6. $\dfrac{dy}{dx}$ if $F(x, y, x^2 - y^2) = 0$

7. $\dfrac{\partial u}{\partial x}$ if $G(x, y, z, u, v) = 0$

8. $\dfrac{\partial z}{\partial x}$ if $F(x^2 - z^2, y^2 + xz) = 0$

9. $\dfrac{\partial w}{\partial t}$ if $H(u^2 w, v^2 t, wt) = 0$

10. $\left(\dfrac{\partial y}{\partial x}\right)_u$ if $xyuv = 1$ and $x + y + u + v = 0$

11. $\left(\dfrac{\partial x}{\partial y}\right)_z$ if $x^2 + y^2 + z^2 + w^2 = 1$, and $x + 2y + 3z + 4w = 2$

12. $\dfrac{du}{dx}$ if $x^2 y + y^2 u - u^3 = 0$ and $x^2 + yu = 1$

13. If $x = u^3 + v^3$ and $y = uv - v^2$ are solved for u and v in terms of x and y, evaluate

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial(u, v)}{\partial(x, y)}$$

at the point where $u = 1$ and $v = 1$.

14. Near what points (r, s) can the transformation

$$x = r^2 + 2s, \qquad y = s^2 - 2r$$

be solved for r and s as functions of x and y? Calculate the values of the first partial derivatives of the solution at the origin.

15. Evaluate the Jacobian $\partial(x, y)/\partial(r, \theta)$ for the transformation to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. Near what points (r, θ) is the transformation one-to-one and therefore invertible to give r and θ as functions of x and y?

16. Evaluate the Jacobian $\partial(x, y, z)/\partial(R, \phi, \theta)$, where

$$x = R\sin\phi\cos\theta, \quad y = R\sin\phi\sin\theta, \quad \text{and } z = R\cos\phi.$$

This is the transformation from Cartesian to spherical coordinates in 3-space that we discussed in Section 10.6. Near what points is the transformation one-to-one and hence invertible to give R, ϕ, and θ as functions of x, y, and z?

17. Show that the equations

$$\begin{cases} xy^2 + zu + v^2 = 3 \\ x^3 z + 2y - uv = 2 \\ xu + yv - xyz = 1 \end{cases}$$

can be solved for x, y, and z as functions of u and v near the point P_0 where $(x, y, z, u, v) = (1, 1, 1, 1, 1)$, and find $(\partial y/\partial u)_v$ at $(u, v) = (1, 1)$.

18. Show that the equations $\begin{cases} xe^y + uz - \cos v = 2 \\ u\cos y + x^2 v - yz^2 = 1 \end{cases}$ can be solved for u and v as functions of x, y, and z near the point P_0 where $(x, y, z) = (2, 0, 1)$ and $(u, v) = (1, 0)$, and find $(\partial u/\partial z)_{x,y}$ at $(x, y, z) = (2, 0, 1)$.

19. Find dx/dy from the system

$$F(x, y, z, w) = 0, \; G(x, y, z, w) = 0, \; H(x, y, z, w) = 0.$$

20. Given the system

$$\begin{aligned} F(x, y, z, u, v) &= 0 \\ G(x, y, z, u, v) &= 0 \\ H(x, y, z, u, v) &= 0, \end{aligned}$$

how many possible interpretations are there for $\partial x/\partial y$? Evaluate them.

21. Given the system

$$\begin{aligned} F(x_1, x_2, \ldots, x_8) &= 0 \\ G(x_1, x_2, \ldots, x_8) &= 0 \\ H(x_1, x_2, \ldots, x_8) &= 0, \end{aligned}$$

how many possible interpretations are there for the partial $\dfrac{\partial x_1}{\partial x_2}$? Evaluate $\left(\dfrac{\partial x_1}{\partial x_2}\right)_{x_4, x_6, x_7, x_8}$.

22. If $F(x, y, z) = 0$ determines z as a function of x and y, calculate $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x\partial y$, and $\partial^2 z/\partial y^2$ in terms of the partial derivatives of F.

23. If $x = u + v$, $y = uv$, and $z = u^2 + v^2$ define z as a function of x and y, find $\partial z/\partial x$, $\partial z/\partial y$, and $\partial^2 z/\partial x\partial y$.

24. A certain gas satisfies the law $pV = T - \dfrac{4p}{T^2}$,

where p = pressure, V = volume, and T = temperature.

(a) Calculate $\partial T/\partial p$ and $\partial T/\partial V$ at the point where $p = V = 1$ and $T = 2$.

(b) If measurements of p and V yield the values $p = 1 \pm 0.001$ and $V = 1 \pm 0.002$, find the approximate maximum error in the calculated value $T = 2$.

25. If $F(x, y, z) = 0$, show that $\left(\dfrac{\partial x}{\partial y}\right)_z \left(\dfrac{\partial y}{\partial z}\right)_x \left(\dfrac{\partial z}{\partial x}\right)_y = -1.$

Derive analogous results for $F(x, y, z, u) = 0$ and for $F(x, y, z, u, v) = 0$. What is the general case?

! 26. If the equations $F(x, y, u, v) = 0$ and $G(x, y, u, v) = 0$ are solved for x and y as functions of u and v, show that

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial(F, G)}{\partial(u, v)} \Bigg/ \frac{\partial(F, G)}{\partial(x, y)}.$$

27. If the equations $x = f(u, v)$, $y = g(u, v)$ can be solved for u and v in terms of x and y, show that

$$\frac{\partial(u, v)}{\partial(x, y)} = 1 \Bigg/ \frac{\partial(x, y)}{\partial(u, v)}.$$

Hint: Use the result of Exercise 26.

28. If $x = f(u, v)$, $y = g(u, v)$, $u = h(r, s)$, and $v = k(r, s)$, then x and y can be expressed as functions of r and s. Verify by direct calculation that

$$\frac{\partial(x, y)}{\partial(r, s)} = \frac{\partial(x, y)}{\partial(u, v)} \frac{\partial(u, v)}{\partial(r, s)}.$$

This is a special case of the Chain Rule for Jacobians.

29. Two functions, $f(x, y)$ and $g(x, y)$, are said to be functionally dependent if one is a function of the other; that is, if there exists a single-variable function $k(t)$ such that $f(x, y) = k\Big(g(x, y)\Big)$ for all x and y. Show that in this case $\partial(f, g)/\partial(x, y)$ vanishes identically. Assume that all necessary derivatives exist.

30. Prove the converse of Exercise 29 as follows: Let $u = f(x, y)$ and $v = g(x, y)$, and suppose that $\partial(u, v)/\partial(x, y) = \partial(f, g)/\partial(x, y)$ is identically zero for all x and y. Show that $(\partial u/\partial x)_v$ is identically zero. Hence u, considered as a function of x and v, is independent of x; that is, $u = k(v)$ for some function k of one variable. Why does this imply that f and g are functionally dependent?

Thermodynamics Problems

31. Use the different versions of the equation of state, presented in this section, to determine explicit functions u and v such that $S = u(E, V, N)$ and $S = v(T, V, N)$.

In Exercises 32–34, verify the given Maxwell relation by using a suitable Legendre transformation (see the Thermodynamics subsection of Section 12.6) to involve the appropriate set of independent variables.

32. $\left(\dfrac{\partial P}{\partial T}\right)_{V,N} = \left(\dfrac{\partial S}{\partial V}\right)_{T,N}$

33. $\left(\dfrac{\partial V}{\partial S}\right)_{P,N} = \left(\dfrac{\partial T}{\partial P}\right)_{S,N}$

34. $\left(\dfrac{\partial S}{\partial P}\right)_{T,N} = -\left(\dfrac{\partial V}{\partial T}\right)_{P,N}$

In Exercises 1–12, calculate the indicated derivative from the given equation(s). What condition on the variables will guarantee the existence of a solution that has the indicated derivative? Assume that any general functions F, G, and H have continuous first partial derivatives.

1. $\dfrac{dx}{dy}$ if $xy^3 + x^4y = 2$

2. $\dfrac{\partial x}{\partial y}$ if $xy^3 = y - z$

1. $\dfrac{dx}{dy} = \dfrac{dx}{dy} \cdot \dfrac{dy}{dy}$

$$\frac{dx}{dy} = x^3 y.$$