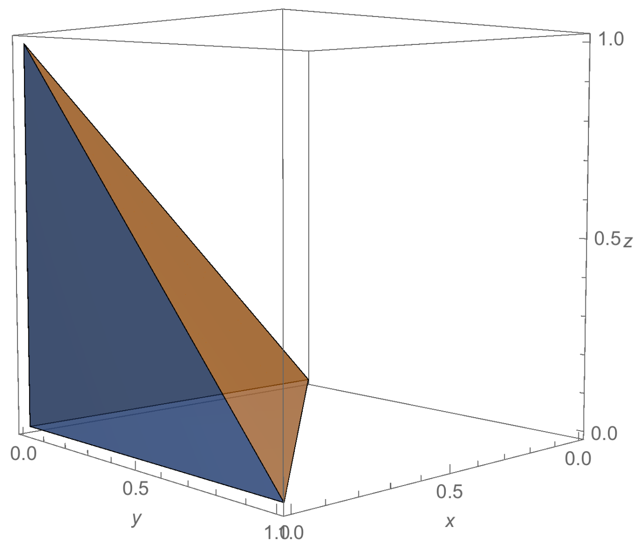
MATH 2023 • Multivariable Calculus
Problem Set #6 • Triple Integrals

1. (★) Consider the triple integral:

$$\int_0^1 \int_z^1 \int_0^{x-z} f(x,y,z)\, dydxdz.$$

(a) Sketch the solid described by the integral.



(b) Express the integral using each of the other five orders, i.e. $dydzdx$, $dxdydz$, $dxdzdy$, $dzdxdy$ and $dzdydx$.

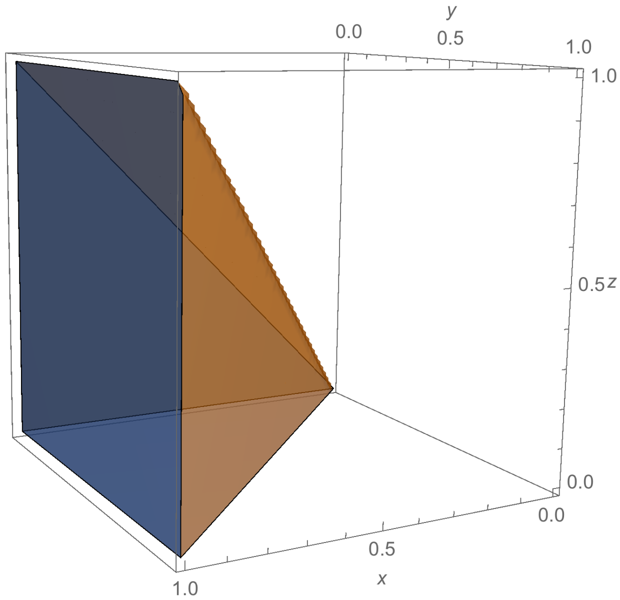
Solution: (Answer only)

$$\begin{aligned}\int_0^1 \int_z^1 \int_0^{x-z} f(x,y,z)\, dydxdz &= \int_0^1 \int_0^x \int_0^{x-z} f(x,y,z)\, dydzdx \\ &= \int_0^1 \int_0^{1-z} \int_{y+z}^1 f(x,y,z)\, dxdydz \\ &= \int_0^1 \int_0^{1-y} \int_{y+z}^1 f(x,y,z)\, dxdzdy \\ &= \int_0^1 \int_y^1 \int_0^{x-y} f(x,y,z)\, dzdxdy \\ &= \int_0^1 \int_0^x \int_0^{x-y} f(x,y,z)\, dzdydx\end{aligned}$$

2. (★★) Consider the triple integral:

$$\int_0^1 \int_z^1 \int_0^x e^{x^3}\, dydxdz.$$

(a) Sketch the solid described by the integral.

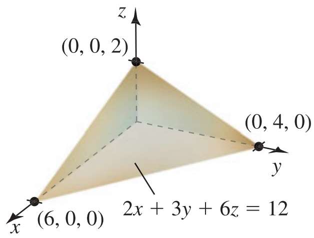


(b) Pick a *good* order of integration and compute the integral *by hand*.

Solution: We use $dzdydx$-order since the integrand e^{x^3} depends only on x. That way we should be able to compute the inner- and middle integral easily. [Note: $dydzdx$-order should works as well.]
Use z as the "pillar" variable, so that (x, y) are the base variables. The triple integral can be rewritten as:

$$\begin{aligned}\int_0^1 \int_0^x \int_0^x e^{x^3}\, dzdydx &= \int_0^1 \int_0^x xe^{x^3}\, dydx \\ &= \int_0^1 x^2 e^{x^3}\, dx \\ &= \left[\frac{1}{3}e^{x^3}\right]_{x=0}^{x=1} \\ &= \frac{1}{3}\left(e-1\right).\end{aligned}$$

3. (★★) Consider the right tetrahedron solid T in the first octant bounded by the xy-, yz-, xz-planes and the plane Π with vertices $(6,0,0)$, $(0,4,0)$ and $(0,0,2)$.



(a) Show that the equation of the plane Π is given by $2x+3y+6z=12$.

Solution: Straight-forward.

(b) Evaluate the following triple integral:

$$\iiint_T \left(\frac{1}{12-3y-6z}+\frac{1}{12-2x-6z}+\frac{1}{12-2x-3y}\right)\,dV.$$

Please do the computations *by hand*. Pick carefully the orders of integration to simplify your computations.

Solution: Denote T_{yz} the projection of T on the yz-plane. Similar for T_{xy} and T_{xz}. We split the integral into three and use different order of integration for each of them:

$$\begin{aligned}\iiint_T \frac{1}{12-3y-6z}\,dV &= \iint_{T_{yz}}\int_{x=0}^{x=\frac{1}{2}(12-3y-6z)} \frac{1}{12-3y-6z}\,dxdydz\\ &= \iint_{T_{yz}} \frac{1}{2}\,dA = \frac{1}{2}\text{ area of }T_{yz} = \frac{1}{2}\,\frac{4\times 2}{2} = 2\end{aligned}$$

$$\begin{aligned}\iiint_T \frac{1}{12-2x-6z}\,dV &= \iint_{T_{xz}}\int_{y=0}^{y=\frac{1}{3}(12-2x-6z)} \frac{1}{12-2x-6z}\,dydxdz\\ &= \iint_{T_{xz}} \frac{1}{3}\,dxdz = \frac{1}{3}\,\frac{6\times 2}{2} = 2\end{aligned}$$

$$\begin{aligned}\iiint_T \frac{1}{12-2x-3y}\,dV &= \iint_{T_{xy}}\int_{z=0}^{z=\frac{1}{6}(12-2x-3y)} \frac{1}{12-2x-3y}\,dzdxdy\\ &= \iint_{T_{xy}} \frac{1}{6}\,dxdy = \frac{1}{6}\,\frac{6\times 4}{2} = 2\end{aligned}$$

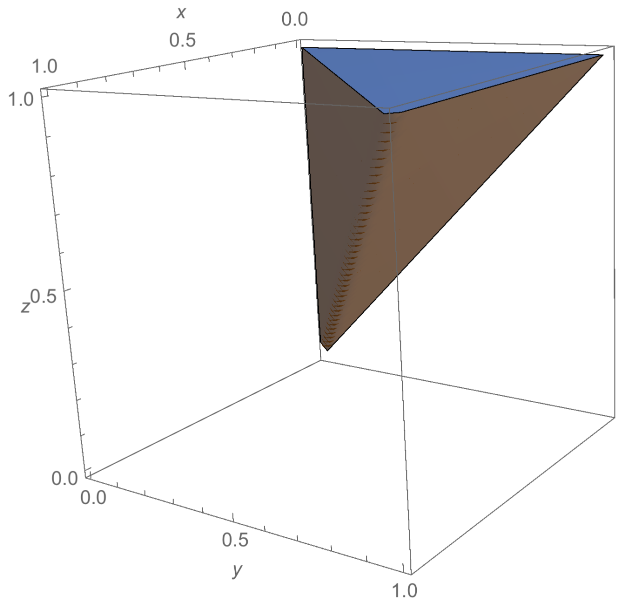
Therefore,

$$\iiint_T \left(\frac{1}{12-3y-6z}+\frac{1}{12-2x-6z}+\frac{1}{12-2x-3y}\right)\,dV = 2+2+2=6.$$

4. (★★) Let a be a positive constant. Given that $f(x)$ is a continuous function of x, show that:

$$\int_0^a \int_0^z \int_0^y f(x)\,dxdydz = \int_0^a \frac{(a-x)^2}{2} f(x)\,dx$$

Solution: The triple integral represents the following solid:



Note that the integrand $f(x)$ depends only on x. We switch the order of integration to: $dzdydx$ (so that one can compute the inner and middle integrals without any problem):

$$\begin{aligned}\int_0^a \int_0^z \int_0^y f(x)\,dxdydz &= \int_0^a \int_x^a \int_y^a f(x)\,dzdydx \\ &= \int_0^a \int_x^a f(x)\,(a-y)\,dydx \\ &= \int_0^a \left[-f(x)\cdot\frac{(a-y)^2}{2}\right]_{y=x}^{y=a} dx \\ &= \int_0^a f(x)\cdot\frac{(a-x)^2}{2}\,dx\end{aligned}$$

5. (★) Evaluate $\iiint_D (x^2+y^2)\,dV$ over the solid D which lies above the cone $z = c\sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 = a^2$.

Solution: The cone can be expressed in spherical coordinates as:

$$\underbrace{\rho\cos\phi}_{z} = \underbrace{c\rho\sin\phi}_{c\sqrt{x^2+y^2}} \Longrightarrow \phi = \tan^{-1}\frac{1}{c}.$$

Hence, the solid D can be expressed in spherical coordinates as:

$$0 \le \rho \le a, \quad 0 \le \phi \le \tan^{-1}\frac{1}{c}, \quad 0 \le \theta \le 2\pi.$$

Therefore, we have:

$$\begin{aligned}
\iiint_D (x^2+y^2)\,dV &= \int_0^{2\pi}\int_0^{\tan^{-1}\frac{1}{c}}\int_0^a \underbrace{\rho^2\sin^2\phi}_{x^2+y^2}\cdot\underbrace{\rho^2\sin\phi\,d\rho d\phi d\theta}_{dV} \\
&= \int_0^{2\pi}\int_0^{\tan^{-1}\frac{1}{c}}\int_0^a \rho^4\sin^3\phi\,d\rho d\phi d\theta \\
&= \frac{2\pi a^5}{5}\int_0^{\tan^{-1}\frac{1}{c}} \sin^3\phi\,d\phi \\
&= \frac{2\pi a^5}{5}\int_0^{\tan^{-1}\frac{1}{c}} \left(\cos^2\phi - 1\right) d(\cos\phi) \\
&= \frac{2\pi a^5}{5}\left[\frac{\cos^3\phi}{3} - \cos\phi\right]_0^{\tan^{-1}\frac{1}{c}} \\
&= \frac{2\pi a^5}{5}\left(\frac{1}{3}\cos^3\tan^{-1}\frac{1}{c} - \cos\tan^{-1}\frac{1}{c} + \frac{2}{3}\right)
\end{aligned}$$

You may use the fact that $\cos\tan^{-1}x = \frac{1}{\sqrt{1+x^2}}$ to simplify the final answer, but it is not necessary.

6. (★) Find the volume of the solid bounded by the xy-plane, the cone $z = 2a - \sqrt{x^2+y^2}$ and the cylinder $x^2+y^2 = 2ay$.

Solution: (Sketch only) In cylindrical coordinates, the cone is given by $z = 2a - r$ and the cylinder is $r^2 = 2ar\sin\theta$, or equivalently, $r = 2a\sin\theta$. Therefore,

$$\text{volume} = \iiint_D 1\,dV = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2a\sin\theta}\int_{z=0}^{z=2a-r} 1\,r\,dzdrd\theta = \frac{2}{9}(9\pi-16)a^3$$

Here we presented the case when $a > 0$. The other case is similar (yet different).

7. (★★) Let $\phi(x,y,z) = \dfrac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(-\dfrac{x^2+y^2+z^2}{4kt}\right)$ where $t > 0$. Show that for each fixed $t > 0$, we have:

$$\iiint_{\mathbb{R}^3} \phi(x,y,z)\, dV = 1.$$

Solution: The appearance of the term $x^2+y^2+z^2$ suggests it may be best to use spherical coordinates, since $x^2+y^2+z^2 = \rho^2$. The bounds for the whole space $\mathbb{R}^3$ is:

$$0 \le \rho < \infty, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

$$\begin{aligned}
\iiint_{\mathbb{R}^3} \phi(x,y,z)\, dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{(4\pi kt)^{3/2}} \exp\left(-\frac{\rho^2}{4kt}\right) \rho^2 \sin\phi \, d\rho d\phi d\theta \\
&= \frac{1}{(4\pi kt)^{3/2}} \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_0^{\infty} \rho^2 e^{-\rho^2/4kt}\, d\rho\right) \\
&= \frac{1}{(4\pi kt)^{3/2}} \cdot 2\pi \cdot 2 \cdot \left(\int_0^{\infty} \rho^2 e^{-\rho^2/4kt}\, d\rho\right)
\end{aligned}$$

It comes down to computing $\displaystyle\int_0^{\infty} \rho^2 e^{-\rho^2/4kt}\, d\rho$. Note that

$$\frac{d}{d\rho}\left(e^{-\rho^2/4kt}\right) = e^{-\rho^2/4kt} \cdot \left(-\frac{2\rho}{4kt}\right) = -\frac{\rho}{2kt} \cdot e^{-\rho^2/4kt}.$$

$$\begin{aligned}
\int_0^{\infty} \rho^2 e^{-\rho^2/4kt}\, d\rho &= -2kt \int_0^{\infty} \rho \cdot \underbrace{\left(-\frac{\rho}{2kt} e^{-\rho^2/4kt}\right) d\rho}_{d\left(e^{-\rho^2/4kt}\right)} \\
&= -2kt \left\{ \left[\rho e^{-\rho^2/4kt}\right]_{\rho=0}^{\rho\to\infty} - \int_0^{\infty} e^{-\rho^2/4kt} d\rho \right\} \\
&= -2kt \left\{ [0-0] - \sqrt{4kt} \int_0^{\infty} e^{-(\rho/\sqrt{4kt})^2}\, d\left(\frac{\rho}{\sqrt{4kt}}\right) \right\} \\
&= 2kt\sqrt{4kt} \int_0^{\infty} e^{-u^2}\, du = 4k^{3/2}t^{3/2} \cdot \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

The last two steps follows from integration by substitutions (let $u = \rho/\sqrt{4kt}$) and Lecture Notes P.64.

Combining these results, we get:

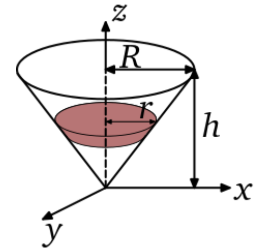
$$\iiint_{\mathbb{R}^3} \phi(x,y,z)\, dV = \frac{1}{(4\pi kt)^{3/2}} \cdot 2\pi \cdot 2 \cdot 4k^{3/2}t^{3/2} \cdot \frac{\sqrt{\pi}}{2} = 1.$$

Alternatively, one can also breakdown $\phi(x,y,z)$ into:

$$\frac{1}{(4\pi kt)^{3/2}} e^{-x^2/4kt} e^{-y^2/4kt} e^{-z^2/4kt}$$

and set up the integral using rectangular coordinates.

8. (★★) Consider a right circular solid cone (denoted by K) with radius R, height h, mass m and uniform density δ.



The moment of inertia about the z-axis of the solid is defined to be:

$$I_z := \iiint_K D_z(x,y,z)^2\,\delta dV$$

where $D_z(x,y,z)$ is the perpendicular distance between the point (x,y,z) and the z-axis.

(a) *Set up*, but do not evaluate, the integral I_z using each of the following coordinates:

i. rectangular coordinates

ii. cylindrical coordinates

iii. spherical coordinates

Solution: $D_z(x,y,z)$ is the distance from (x,y,z) to the z-axis, which is also the distance from (x,y,z) to $(0,0,z)$ – draw a picture to convince yourself on that! Therefore, $D_z(x,y,z) = \sqrt{x^2+y^2}$.

The equation of the cone is given by:

$$\begin{aligned} z &= \frac{h}{R}r && \text{(cylindrical)} \\ z &= \frac{h}{R}\sqrt{x^2+y^2} && \text{(rectangular)} \\ \varphi &= \tan^{-1}\frac{R}{h} && \text{(spherical)} \end{aligned}$$

The equation of the flat top of the cone is given by:

$$\begin{aligned} z &= h && \text{(both cylindrical and rectangular)} \\ \rho &= h\sec\theta && \text{(spherical)} \end{aligned}$$

$$I_z = \int_{y=-R}^{y=R}\int_{x=-\sqrt{R^2-y^2}}^{x=\sqrt{R^2-y^2}}\int_{z=\frac{h}{R}\sqrt{x^2+y^2}}^{z=h} \delta(x^2+y^2)\,dzdxdy$$

$$\begin{aligned} I_z &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R}\int_{z=\frac{h}{R}r}^{z=h} \delta r^2\, rdzdrd\theta \\ &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R}\int_{z=\frac{h}{R}r}^{z=h} \delta r^3\, dzdrd\theta \end{aligned}$$

$$\begin{aligned} I_z &= \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\tan^{-1}\frac{R}{h}}\int_{\rho=0}^{\rho=h\sec\theta} \delta(\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta)\,\rho^2\sin\varphi d\rho d\varphi d\theta \\ &= \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\tan^{-1}\frac{R}{h}}\int_{\rho=0}^{\rho=h\sec\theta} \delta\rho^4\sin^3\varphi\, d\rho d\varphi d\theta \end{aligned}$$

(b) Rank the ease of computations of the above coordinate systems for evaluating the integral I_z, then compute I_z using the easiest coordinate system. Express your final answer in terms of the mass m, not the density δ.

Solution: From the easiest to the hardest: cylindrical, spherical, rectangular. Using rectangular coordinates would involve some difficult trig substitution. Using spherical coordinates will amount to integrating $\sec^5\theta$

$$\begin{aligned} I_z &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R}\int_{z=\frac{h}{R}r}^{z=h} \delta r^3 \, dz dr d\theta \\ &= \frac{\delta\pi R^4 h}{10} \\ &= \frac{\pi R^4 h}{10}\,\frac{m}{\frac{1}{3}\pi R^2 h} \\ &= \frac{3}{10} m R^2 \end{aligned}$$

9. (★★) Given a solid T with mass m and uniform density δ, the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is defined to be:

$$\bar{x} = \frac{\iiint_T x \, \delta dV}{\iiint_T \delta dV}, \quad \bar{y} = \frac{\iiint_T y \, \delta dV}{\iiint_T \delta dV}, \quad \bar{z} = \frac{\iiint_T z \, \delta dV}{\iiint_T \delta dV}$$

The moment of inertia of T about the z-axis is defined as:

$$I_z := \iiint_T D_z(x,y,z)^2 \, \delta dV$$

where $D_z(x,y,z)$ is the perpendicular distance between the point (x,y,z) and the z-axis. Now consider the axis L passing through the center of mass $(\bar{x}, \bar{y}, \bar{z})$ and parallel to the z-axis. The moment of inertia of the solid about the axis L is defined as:

$$I_{\text{cm}} := \iiint_T D_L(x,y,z)^2 \, \delta dV$$

where $D_L(x,y,z)$ is the perpendicular distance between the point (x,y,z) and the axis L. Prove the following result (which is called the Parallel Axis Theorem):

$$I_z = I_{\text{cm}} + md^2$$

where d is the distance between the z-axis and the axis L.

Solution: As in Problem 2, $D_z(x,y,z)^2 = x^2 + y^2$. Therefore,

$$I_z = \iiint_T \delta(x^2 + y^2) \, dV$$

$D_L(x,y,z)$ is the distance from (x,y,z) to the axis L. Since L is a vertical line passing through $(\bar{x}, \bar{y}, \bar{z})$, the x- and y-coordinates of every point on L must be $\bar{x}$ and $\bar{y}$. The distance $D_L(x,y,z)$ is measured between the points (x,y,z) and $(\bar{x}, \bar{y}, z)$, i.e. the perpendicular distance. Therefore, $D_L(x,y,z)^2 = (x-\bar{x})^2 + (y-\bar{y})^2$.

The distance d between the two vertical axes (z-axis and L) is the distance between any two points at the same altitude. In other words, $d^2 = \bar{x}^2 + \bar{y}^2$.

Consider $I_{cm} + md^2$:

$$\begin{aligned}
I_{cm} + md^2 &= \iiint_T \delta\left((x-\bar{x})^2 + (y-\bar{y})^2\right)\, dV + m(\bar{x}^2+\bar{y}^2) \\
&= \iiint_T \delta\left(x^2 - 2\bar{x}x + \bar{x}^2 + y^2 - 2\bar{y}y + \bar{y}^2\right)\, dV + m(\bar{x}^2+\bar{y}^2) \\
&= \iiint_T \delta(x^2+y^2)\, dV - 2\iiint_T \delta\left(\bar{x}x + \bar{y}y\right)\, dV + \iiint_T \delta\left(\bar{x}^2+\bar{y}^2\right)\, dV + m(\bar{x}^2+\bar{y}^2)
\end{aligned}$$

Note that $\bar{x}$ and $\bar{y}$ are constants, we get:

$$\begin{aligned}
I_{cm} + md^2 &= I_z - 2\bar{x}\iiint_T \delta x\, dV - 2\bar{y}\iiint_T \delta y\, dV + \left(\bar{x}^2+\bar{y}^2\right)\iiint_T \delta dV + m(\bar{x}^2+\bar{y}^2) \\
&= I_z - 2\bar{x}\cdot m\bar{x} - 2\bar{y}\cdot m\bar{y} + m(\bar{x}^2+\bar{y}^2) + m(\bar{x}^2+\bar{y}^2) \\
&= I_z - 2m(\bar{x}^2+\bar{y}^2) + m(\bar{x}^2+\bar{y}^2) + m(\bar{x}^2+\bar{y}^2) = I_z.
\end{aligned}$$

Here we have used the fact that $m = \iiint_T \delta\, dV$ and the definition of $\bar{x}$ and $\bar{y}$.

10. (★) The change-of-variable formula for the volume element dV is given by:

$$dxdydz = \left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw. \qquad (*)$$

(a) Using (*), verify that:

$$dxdydz = \rho^2 \sin\phi \, d\rho d\phi d\theta.$$

Solution: It suffices to show:

$$\left|\det \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \rho^2 \sin\phi.$$

Using the conversion rules $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$ (here we used MATH convention), we get:

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{bmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \end{bmatrix} \\
&= \begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}
\end{aligned}$$

Then by direct computations, we get:

$$\begin{aligned}
\det \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \rho^2\cos^2\phi\sin\phi\cos^2\theta + \rho^2\sin^3\phi\sin^2\theta \\
&\quad - \left(-\rho^2\cos^2\phi\sin\phi\sin^2\theta - \rho^2\sin^3\phi\cos^2\theta\right) \\
&= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi = \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi.
\end{aligned}$$

(b) Let $u = 2x$, $v = 3y$ and $w = 5z$. Using (*), express $dxdydz$ in terms of $dudvdw$.

Solution: By rearrangement, we get $x = \frac{u}{2}$, $y = \frac{v}{3}$ and $z = \frac{w}{5}$

$$\det \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

$$= \det \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} = \frac{1}{2 \times 3 \times 5} = \frac{1}{30}.$$

Therefore,

$$dxdydz = \left| \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw = \frac{1}{30} dudvdw.$$

11. (★★★) Consider a solid sphere with radius R centered at the origin in $\mathbb{R}^3$ which carries a uniform distribution of charges with density δ. Each volume element dV in the sphere can be regarded as a particle with charge $\delta\, dV$.

Fix a particle with charge q at $(0, 0, z_0)$ where $z_0 > R$, i.e. outside the sphere, and call it the q-particle. As in the previous Problem Set, the electric force exerted on the q-particle by a charged element $\delta\, dV$ at (x, y, z) in the solid sphere is given by the Coulomb's Law (in vector form):

$$d\mathbf{F} = \frac{q\,\delta\, dV}{4\pi\varepsilon_0} \frac{(0-x)\mathbf{i} + (0-y)\mathbf{j} + (z_0 - z)\mathbf{k}}{\left((0-x)^2 + (0-y)^2 + (z_0-z)^2\right)^{3/2}}$$

Similar to the previous Problem Set, the Principle of Superposition asserts that the resultant force exerted on the q-particle by the whole sphere is given by "summing-up", i.e. integrating, each the force element $d\mathbf{F}$ over the sphere:

$$\mathbf{F}_{\text{resultant}} = \iiint_{\text{sphere}} d\mathbf{F}.$$

(a) Show that:

$$\mathbf{F}_{\text{resultant}} = \left(\int_0^{2\pi} \int_0^{\pi} \int_0^R \frac{q\delta}{4\pi\varepsilon_0} \frac{\rho^2 \sin\varphi \cdot (z_0 - \rho\cos\varphi)}{\left(\rho^2 - 2\rho z_0 \cos\varphi + z_0^2\right)^{3/2}}\, d\rho d\varphi d\theta \right) \mathbf{k}$$

Solution: Use spherical coordinates:

$$\begin{aligned}
&\mathbf{F}_{\text{resultant}} \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^R d\mathbf{F} \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^R \frac{q\,\delta\, dV}{4\pi\varepsilon_0} \frac{-x\mathbf{i} - y\mathbf{j} - (z - z_0)\mathbf{k}}{\left(x^2 + y^2 + (z - z_0)^2\right)^{3/2}} \\
&= \frac{q\,\delta}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_0^R \frac{-\rho\sin\phi\cos\theta\,\mathbf{i} - \rho\sin\phi\sin\theta\,\mathbf{j} - (\rho\cos\phi - z_0)\,\mathbf{k}}{(\rho^2\sin^2\phi + (\rho\cos\phi - z_0)^2)^{3/2}} \cdot \rho^2 \sin\phi\, d\rho d\phi d\theta
\end{aligned}$$

Note that the **i** and **j** components are zero since:

$$\int_0^{2\pi} \sin\theta \, d\theta = \int_0^{2\pi} \cos\theta \, d\theta = 0.$$

After simplification of the **k**-component, one can obtain the required result.

(b) Try to compute the above integral, either by software or by hand, and show that:

$$\mathbf{F}_{\text{resultant}} = \frac{q\delta R^3}{3\varepsilon_0 z_0^2}\mathbf{k} = \frac{qQ}{4\pi\varepsilon_0 z_0^2}\mathbf{k}$$

where Q is the total amount of charges in the sphere.
[Remark 1: This result shows that the resultant force exerted on the q-particle by the charged sphere will be the same if one replaces it by a particle at the origin with the same amount of charges.]
[Remark 2: Using the Gauss's Law for Electricity, the above result can be obtained easily by considering the surface flux of $\mathbf{F}_{\text{resultant}}$. We will discuss that later, and will derive the Gauss's Law using the Divergence Theorem (assuming Coulomb's Law).]

Solution: Compute the integral in (a). Being a human being in the 21th Century, you should do it using Mathematica or WolframAlpha. Don't waste your time doing it by hand (unless you are required to in later E&M course).