18.02 Practice Final 3hrs.

Problem 1. Given the points $P : (1,1,-1), Q : (1,2,0), R : (-2,2,2)$ find

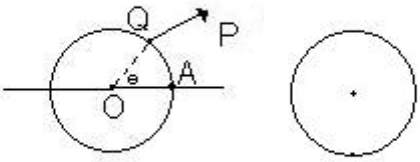
a)$PQ \times PR$ b) a plane $ax + by + cz = d$ trough P, Q and R

Problem 2. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & c \\ 2 & c & 1 \\ 1 & -1 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{A}^{-1} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \times & \cdot \end{pmatrix}$.

a) For what valu(s) of constant c will $\mathbf{A}\mathbf{x} = 0$ have a non-zero solution?

b) Take $c = 2$, and tell what entry the inverse matrix has in the position market $\times$

Problem 3. The roll of Scotch tape has outer radius a and is fixed in position (i.e., does not turn). Its end P is originally at the point A; the tape is then pulled from the roll so the free portion makes a 45-degree



angle with the horizontal.

Write the parametric equation $x = x(\theta)$ $y = y(\theta)$ for the curve C traced out by the point P as it moves. (Use vectore methods; θ is the angle shown)

Sketch the curve on the second picture, showing its behavior at its endpoints.

Problem 4. The position vectore of point P is $r =< 3\cos t, 5\sin t, 4\cos t >$.

a) Show its speed is constant.

b) At what point $A : (a, b, c)$ does P pass through the yz-plane?

Problem 5. Let $\omega = x^2y - xy^3$, and $P = (2, 1)$

a)Find the directional derivative $\frac{d\omega}{ds}$ at P in the diraction of $\mathbf{A} = 3i + 4j$.

b) If you start at P and go a distance .01 in the diraction of $\mathbf{A}$, by approximately how much will ω change? (Give a decimal with one significant digit.)

Problem 6. a) Find the tangent plane at (1,1,1) to the surface $z^2 + 2y^2 + 2z^2 = 5$; give the equation in the form $az + by + cz = d$ and simplify the coefficients.

b) What dihedral angle does the tangent plane make with the xy-plane? (Hint: consider the normal vectors of the two planes.)

Problem 7. Find the point on the plane $2z + y - z = 6$ which is closest to the origin, by using Lagrange multipliers. (Minimize the square of the distance. Only 10 points if you use some other method)

Problem 8. Let $\omega = f(x, y, z)$ with the constraint $g(x, y, z) = 3$.

At the point $P : (0, 0, 0)$, we have $\nabla f =< 1, 1, 2 >$ and $\nabla g =< 2, -1, -1 >$, Find the value at P of the two quantities (show work): a) $\left(\frac{\partial z}{\partial x}\right)_y$ b) $\left(\frac{\partial \omega}{\partial x}\right)_y$

Problem 9. Evaluate by changing the order of integration: $\int_0^3 \int_{z^3}^9 xe^{-y^2} dydz$.

Problem 10. A plane region R is bounded by four semicircles of radius 1. having ends at $(1,1),(1,=1),(-1,1),(-1,-1)$ and centerpoints at $(2,0),(-2,0),(0,2),(0,-2)$.

Set up an iterated integral in polar coordinates for the moment of inertia of R about the origin; take the density $\delta = 1$. Supply integrand and limits, but *do not evaluate* the integral.

Use symmetry to simplify the limits of integration.

Problem 11. a) In the xy-plane, let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Give in terms of P and Q the line integral representing the flux $\mathbf{F}$ across a simple closed curve C, with outward-pointing normal.

b) Let $\mathbf{F} = ax\mathbf{i} + by\mathbf{j}$. How should the constants a and b be related if the flux of $\mathbf{F}$ over any simple closed curve C is equal to the area inside C?

Problem 12. A solid hemisphere of radius 1 has its lower flat base on the xy-plane and center at the origin. Its density function is $\delta = z$. Find the force of gravitational attraction it exerts on a unit mass at the origin.

Problem 13. Evaluate $\int_C (y-x)dz + (y-z)dz$ over the line segment C from $P:(1,1,1)$ to $Q:(2,4,8)$.

Problem 14. a) Let $\mathbf{F} = ay^2\mathbf{i} + 2y(x+z)\mathbf{j} + (by^2+z^2)\mathbf{k}$. For what values of the constants a and b will F be conservative? Show work.

b) Using these values, find a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.

c) Using these values, give the equation of a surface S having the property : $\int_P^Q \mathbf{F} \cdot dr = 0$ for any two points P and Q on the surface S.

Problem 15. Let S be the closed surface whose bottom face B is the unit disc in the xy-plane and whose upper surface is the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across U by using the divergence theorem.

Problem 16. Using the data of the preceding problem, calculate the flux of $\mathbf{F}$ across U directly, by setting up the surface integral for the flux and evaluating the resulting double integral in the xy-plane.

Problem 17. An xz-cylinder in 3-space is a surface given by an equation $f(x,z) = 0$ in x and z alone; its section by any plane $y = c$ perpendicular to the y-axis is always the same xz-curve.

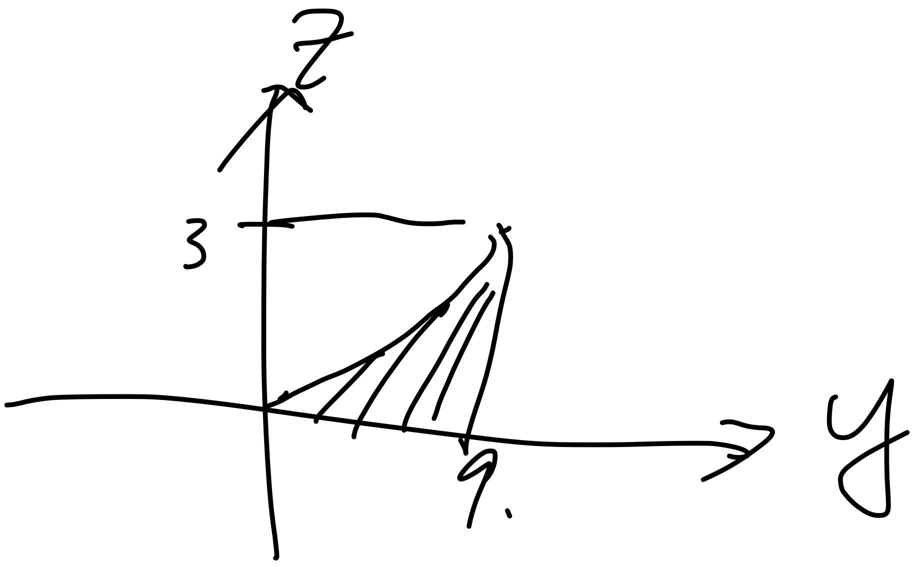
Show that if $\mathbf{F} = z^2\mathbf{i} + y^2\mathbf{j} + xz\mathbf{k}$ then $\oint_C \mathbf{F} \cdot dr = 0$ for any simple closed curve C lying on an xz-cylinder. (Use Stokes' theorem)

Problem 18. $\int e^{-x^2} dx$ is not elementary but $I = \int_0^\infty e^{-x^2} dx$ can still be evaluated.

a) Evaluate the iterated integral $\int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} dydx$, in terms of I.

b) Then evaluate the integral in (a) by switching to polar coordinates. Comparing the two evaluations, what value do you get for I?

Problem 9. Evaluate by changing the order of integration: $\int_0^3 \int_{z^3}^9 xe^{-y^2}\,dy\,dz$.



$$\int_0^9 \int_0^{y^{\frac{1}{3}}} xe^{-y^2}\,dz\,dy$$

$$\int_0^9 \left(xe^{-y^2}\right) y^{\frac{1}{3}}\,dy$$

Let $u = -y^2$, $du = -2y\,dy$,

$$-\frac{1}{2}\int_0^9 x\, e^u \quad y^{-\frac{2}{3}}\,dy$$

$$= \frac{1}{2}\int_0^{-81} x\, e^u u^{-\frac{1}{3}}\,du$$

$J0 \quad J z^3$

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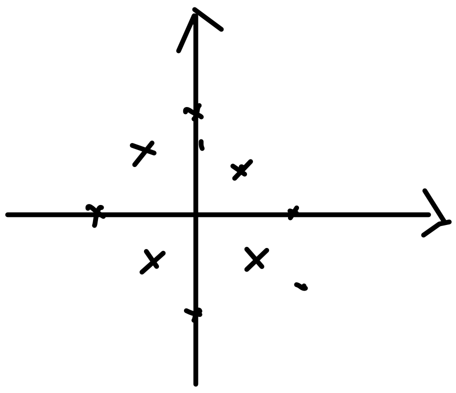
Set up an iterated integral in polar coordinates for the moment of inertia of R about the origin; take the density $\delta = 1$. Supply integrand and limits, but *do not evaluate* the integral.

Use symmetry to simplify the limits of integration.



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$C(t) = \langle 1+t,\ 1+3t,\ 1+7t \rangle$

$0 \le t \le 1$

$\int_0^1 (1+3t-(1+t))(7)\,dt + \int_0^1 (1+3t-(1+7t))(7)\,dt$

$= 7\int_0^1 2t - 4t\,dt$

$= 7\int_0^1 -2t\,dt$

$= 7\left[-t^2\right]_0^1$

$= -7.$

Problem 14. a) Let $\mathbf{F} = ay^2\mathbf{i} + 2y(x+z)\mathbf{j} + (by^2+z^2)\mathbf{k}$. For what values of the constants a and b will F be conservative? Show work.

b) Using these values, find a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.

c) Using these values, give the equation of a surface S having the property : $\int_P^Q \mathbf{F}\cdot dr = 0$ for any two points P and Q on the surface S.

a). $\nabla \times F = 0$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay^2 & 2y(x+z) & by^2+z^2 \end{vmatrix}$$

$\langle 2by - 2y, 0, 2y - 2ay \rangle$

$a = b = 1$.

b). $\vec{F} = \langle y^2, 2y(x+z), y^2+z^2 \rangle$

$f(x,y,z) = y^2x + y^2z + \frac{z^3}{3} + C$

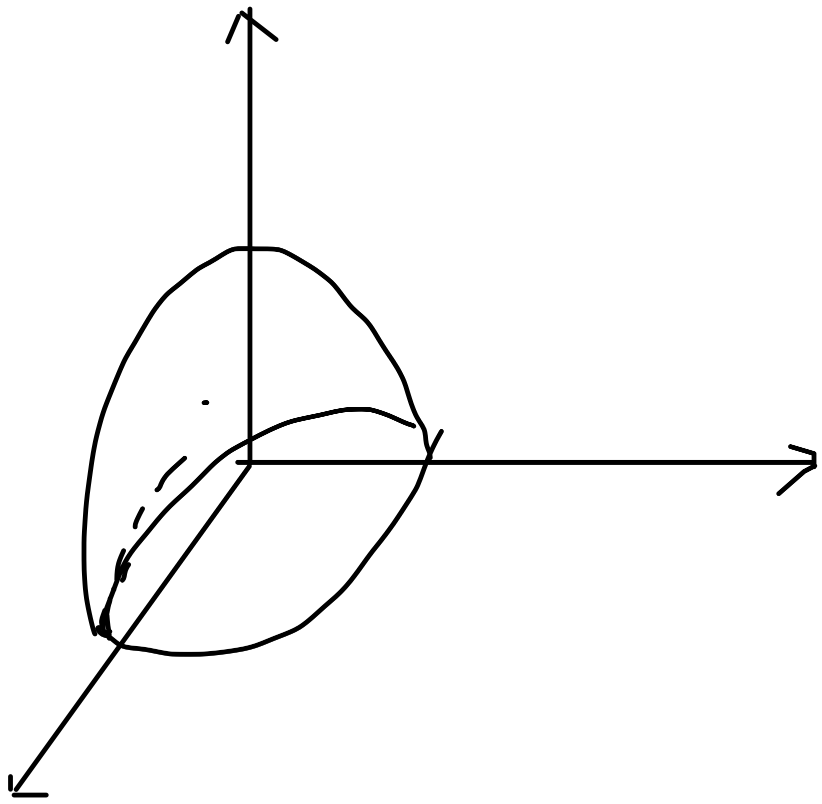
c). $\oint_C \vec{F}\cdot d\vec{r} = 0$

For close surface S,

such as $\vec{r}(u,v) = \langle \cos u \sin v, \sin u \sin v, \cos v \rangle$

Made with Goodnotes

Problem 15. Let S be the closed surface whose bottom face B is the unit disc in the xy-plane and whose upper surface is the paraboloid $z = 1 - x^2 - y^2$, $z \geq 0$. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across U by using the divergence theorem.



$$\iiint_U 3\,dV =$$

$$3 \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta$$

$$= 3 \cdot 2\pi \cdot \int_0^1 r - r^3\,dr$$

$$= 6\pi \cdot \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1$$

$$= \frac{3}{2}\pi.$$

Problem 16. Using the data of the preceding problem, calculate the flux of **F** across U directly, by setting up the surface integral for the flux and evaluating the resulting double integral in the xy-plane.

$$r(u,v) = \langle u\cos v,\ u\sin v,\ 1-u^2 \rangle$$

$$r_u = \langle \cos v,\ \sin v,\ -2u \rangle$$

$$r_v = \langle u\sin v,\ u\cos v,\ 0 \rangle$$

$$r_u \times r_v = \langle 2u^2\cos v,\ 2u^2\sin v,\ u \rangle$$

$$\int_0^{2\pi}\int_0^1 x\cdot 2u^2\cos v + y\cdot 2u^2\sin v + z(u)\ du\,dv$$

$$= \int_0^{2\pi}\int_0^1 2u^3 + (1-u^2)u\ du\,dv$$

$$= 2\pi\left(\int_0^1 u^3 + u\ du\right)$$

$$= 2\pi\left[\frac{u^4}{4} + \frac{u^2}{2}\right]_0^1$$

$$= \frac{3}{2}\pi$$

$$r(\theta) = \langle \cos\theta, \sin\theta, 0 \rangle$$

$$\oint_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} -x\sin\theta + y\cos\theta + 0\ d\theta$$

$$= 0$$

Made with Goodnotes

Problem 17. An xz-cylinder in 3-space is a surface given by an equation $f(x,z)=0$ in x and z alone; its section by any plane $y=c$ perpendicular to the y-axis is always the same xz-curve.
Show that if $\mathbf{F}=z^2\mathbf{i}+y^2\mathbf{j}+xz\mathbf{k}$ then $\oint_C \mathbf{F}\cdot dr=0$ for any simple closed curve C lying on an xz-cylinder.
(Use Stokes' theorem)

$$\nabla\times F=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & xz \end{vmatrix}$$

$$=\langle 0,\ 2z-z,\ 0\rangle$$

$$=\langle 0,\ z,\ 0\rangle$$

For C lying on xz-cylinder, $\hat{n}=\langle 0,1,0\rangle$

$$\iint \nabla\times F\cdot\hat{n}\,dS$$

$$=\iint_A z\,dx\,dz$$

$$=\int_0^{2\pi}\int_0^{f(r)} r^2\sin\theta\,dr\,d\theta$$

$$=\int_0^{2\pi}\frac{f(r)^3}{3}\sin\theta\,d\theta=\frac{f(r)^3}{3}\left[-\cos\theta\right]_0^{2\pi}=0.$$

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a). $\int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,dr\,d\theta$