MATH 2023 – Multivariable Calculus

Lecture #09 Worksheet ♯ March 7, 2019

Problem 1. Find the point on the sphere $\underbrace{x^2 + y^2 + z^2}_{g(x,y,z)} = 4$ that are closest and farthest from the point $(3, 1, -1)$.

$$d(x,y,z) = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$$

$$f(x,y,z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \iff \begin{cases} 2(x-3) = 2\lambda x \\ 2(y-1) = 2\lambda y \\ 2(z+1) = 2\lambda z \\ x^2+y^2+z^2 = 4 \end{cases}$$

$$2x - 6 = 2\lambda x$$
$$x(2-2\lambda) = 6$$
$$x = \frac{3}{1-\lambda}$$

$$2y - 2 = 2\lambda y$$
$$y(2-2\lambda) = 2$$
$$y = \frac{1}{1-\lambda}$$

$$2z + 2 = 2\lambda z$$
$$z(2-2\lambda) = -2$$
$$z = -\frac{1}{1-\lambda}$$

$$\left(\frac{3}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{-1}{1-\lambda}\right)^2 = 4$$

$$11 = 4(1-\lambda)^2$$

$$\lambda = 1 \pm \frac{\sqrt{11}}{2}$$

$$x = \frac{6}{\sqrt{11}} \qquad -\frac{6}{\sqrt{11}}$$
$$y = \frac{2}{\sqrt{11}} \quad \text{or} \quad -\frac{2}{\sqrt{11}}$$
$$z = -\frac{2}{\sqrt{11}} \qquad \frac{2}{\sqrt{11}}$$

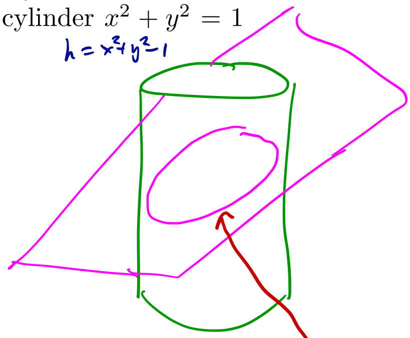
Estimate:

$$f\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) < f\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

closest. farthest

Problem 2. Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$ using

$g = x - y + z - 1$ $\qquad$ $h = x^2 + y^2 - 1$



max. f on

- Lagrange multipliers

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

$$\Longleftrightarrow \begin{cases} 1 = \lambda + 2x\mu \\ 2 = -\lambda + 2y\mu \\ 3 = \lambda \end{cases} \qquad \begin{aligned} -2 &= 2x\mu \quad \text{--- } (1) \\ 5 &= 2y\mu \quad \text{--- } (2) \end{aligned}$$

multiply by x and y on RHS

$$(1)y = 2xy\mu = -2y \Longrightarrow x = -\tfrac{2}{5}y$$
$$(2)x = 2xy\mu = 5x$$

$$x^2 + y^2 = 1 \Longleftrightarrow \frac{4}{25}y^2 + y^2 = 1 \Longrightarrow \frac{29}{25}y^2 = 1 \quad y^2 = \frac{25}{29}, \quad y = \pm\frac{5}{\sqrt{29}} \quad x = \mp\frac{2}{\sqrt{29}}$$

$$z = 1 - x + y = 1 \pm \frac{2}{\sqrt{29}} \pm \frac{5}{\sqrt{29}} = 1 \pm \frac{7}{\sqrt{29}}$$

$$f(x,y,z) = \mp\frac{2}{\sqrt{29}} \pm \frac{10}{\sqrt{29}} + 3 \pm \frac{21}{\sqrt{29}} = 3 \pm \sqrt{29}$$

max : $3 + \sqrt{29}$

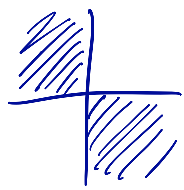
min : $3 - \sqrt{29}$.

- parametric curve

$$x = \cos t, \quad y = \sin t, \qquad z = 1 - x + y = 1 - \cos t + \sin t.$$

$$f(\vec{r}(t)) = \cos t + 2\sin t + (3 - 3\cos t + 3\sin t)$$
$$= 3 - 2\cos t + 5\sin t.$$

$$f'(t) = 2\sin t + 5\cos t = 0 \Longrightarrow \tan t = -\frac{5}{2}$$



$$y = \sin t = \frac{5}{\sqrt{29}} \qquad \text{or} \qquad \sin t = -\frac{5}{\sqrt{29}}$$
$$x = \cos t = -\frac{2}{\sqrt{29}} \qquad\qquad \cos t = +\frac{2}{\sqrt{29}}$$

arithmetic mean

geometric mean.

Problem 3. Let $x_1, ..., x_n \geq 0$. Prove the **AM-GM** inequality

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}$$

by finding the maximum value of

$$f(x_1, ..., x_n) = x_1 x_2 \cdots x_n$$

subject to the condition $x_1 + x_2 + \cdots + x_n = S$ where S is a constant.
Find the conditions in which equality holds.

$\nabla f = \lambda \nabla g$ $\qquad$ $g = x_1 + \cdots + x_n - S$

$$\Longleftrightarrow \text{n eq.} \begin{cases} x_2 \cdots x_n = \lambda \\ x_1 x_3 \cdots x_n = \lambda \\ \vdots \\ x_1 x_2 \cdots x_{n-1} = \lambda \end{cases} \Longleftrightarrow \lambda x_1 = \lambda x_2 = \cdots = \lambda x_n = x_1 x_2 \cdots x_n$$

$\lambda \neq 0$ (otherwise $x_i = 0 \Rightarrow f = 0$)

$\Rightarrow x_1 = x_2 = \cdots = x_n$

$f(x_1, \ldots, x_n) \leq x_1^n = \left(\frac{S}{n}\right)^n$ $\left(n x_1 = S \Rightarrow x_1 = \frac{S}{n} \right)$

$= \left(\frac{x_1 + \cdots + x_n}{n}\right)^n$

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n} \quad //$$

equality : $x_1 = x_2 = \cdots = x_n$