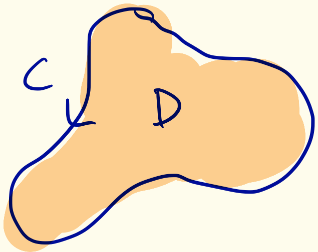
Last Time Green's Theorem.

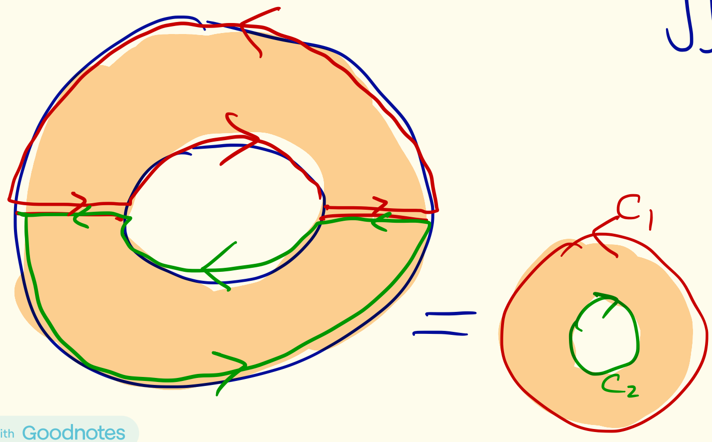
$\vec{F} = \langle P, Q \rangle$.



positive oriented.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy$$



$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \oint \vec{F}\cdot d\vec{r} + \oint \vec{F}\cdot d\vec{r}$$

$$= \oint \vec{F}\cdot d\vec{r} + \oint \vec{F}\cdot d\vec{r}$$

$$= \oint_{C_1} \vec{F}\cdot d\vec{r} + \left(\oint_{C_2} \vec{F}\cdot d\vec{r}\right)$$

$$= -\oint \vec{F}\cdot d\vec{r}$$

The "Del" operator $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$.

(nabla)

↑

"harp"

"like a vector"

Gradient: $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$ $\boxed{\text{Vector}}$

grad f

$P(x,y,z)$ $Q(x,y,z)$ $R(x,y,z)$

Divergence: $\nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$

div $\vec{F}$

$= P_x + Q_y + R_z$ $\boxed{\text{Scalar}}$

in Green's Thm!

Curl: $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}$.

curl $\vec{F}$ $\boxed{\text{Vector}}$

Laplacian: $\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

$\nabla^2 f = f_{xx} + f_{yy} + f_{zz}$

$\nabla^2 \vec{F} = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R \rangle$

Ex $\quad \vec{F} = \langle xz, xyz, -y^2 \rangle$

$$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = z + xz + 0 = z + xz.$$

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \langle (-2y - xy), x, yz \rangle$$

$$\nabla^2 \vec{F} = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R \rangle \qquad \nabla^2 f = f_{xx} + f_{yy} + f_{zz}.$$

$$= \langle 0, 0, -2 \rangle$$

Ex $\quad \vec{F} = \langle P(x,y), Q(x,y), 0 \rangle$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

Green's Thm

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\nabla \times \vec{F}) \cdot \vec{k} \, dA$$

Thm $\nabla \times (\nabla f) = \vec{0}$ (curl (grad f) $= \vec{0}$)

assume continuous second partial derivatives.

[same reason as $\vec{a} \times \vec{a} = \vec{0}$]

Pf $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = \underbrace{(f_{zy} - f_{yz})}_{=0}\vec{i} + \underbrace{(f_{xz} - f_{zx})}_{=0}\vec{j} + \underbrace{(f_{yx} - f_{xy})}_{=0}\vec{k}$

Cor If $\vec{F}$ is conservative, $\nabla \times \vec{F} = \vec{0}$ (**Thm C'**)

(If : $\nabla \times \vec{F} \neq \vec{0} \Rightarrow$ not conservative)

Ex $\vec{F} = \langle xz, xyz, -y^2 \rangle$ is not conservative ($\nabla \times \vec{F} \neq \vec{0}$)

Ex $\vec{F} = \langle P(x,y), Q(x,y), 0 \rangle$. $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Longleftrightarrow$ curl $\vec{F} = \vec{0}$

Thm D' If $\vec{F}$ has continuous partial derivatives on D in $\mathbb{R}^3$

open simply-connected.

(shrink a rubber band)

and $\text{curl}\,\vec{F} = \vec{0}$

then $\vec{F}$ is conservative. (ie. $\vec{F} = \nabla f$)

☆ If $D = \mathbb{R}^3$, can apply this theorem.

Ex $\vec{F} = \langle y^2z^3,\ 2xyz^3,\ 3xy^2z^2 \rangle$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \underbrace{(6xyz^2 - 6xyz^2)}_{=0}\vec{i} + \cdots = \vec{0}$$

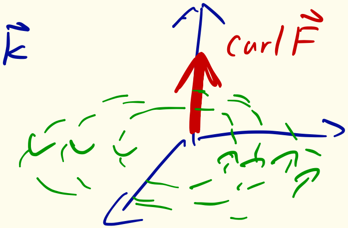
Since D is $\mathbb{R}^3$, $\vec{F}$ is conservative.

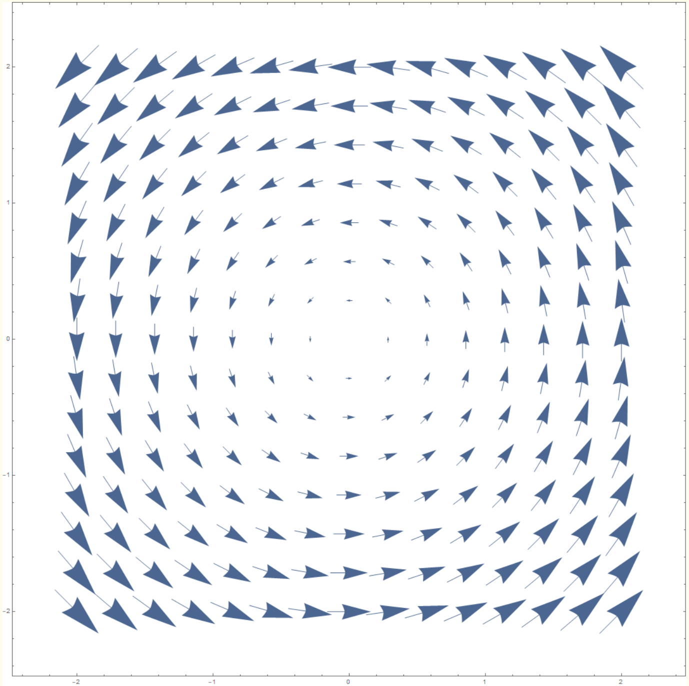
$$\vec{F} = \nabla f, \quad f(x,y,z) = xy^2z^3.$$

Recall Curl

vector field $\vec{F} = \langle -y, x, 0 \rangle$

$$\text{curl}\,\vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} = 2\vec{k}$$



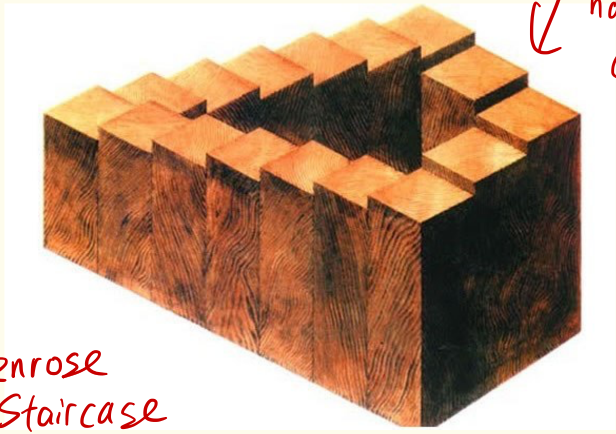


$\text{curl}\,\vec{F} = \vec{0}$

"irrotational"

$\text{curl}(\nabla f) = 0$

If ∇f has nonzero curl.



Penrose Staircase

Thm $\nabla \cdot (\nabla \times \vec{F}) = 0$ $\quad (\operatorname{div}(\operatorname{curl} \vec{F}) = 0)$

assume continuous 2nd partial derivatives

Pf $\underset{\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle}{\underset{\parallel}{\nabla}} \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$$= (R_{yx} - Q_{zx}) + (P_{zy} - R_{xy}) + (Q_{xz} - P_{yz}) = 0$$ by Mixed Partial Thm.

Ex $\vec{F} = \langle xy, xyz, -y^2 \rangle$ cannot be a curl of some vector fields

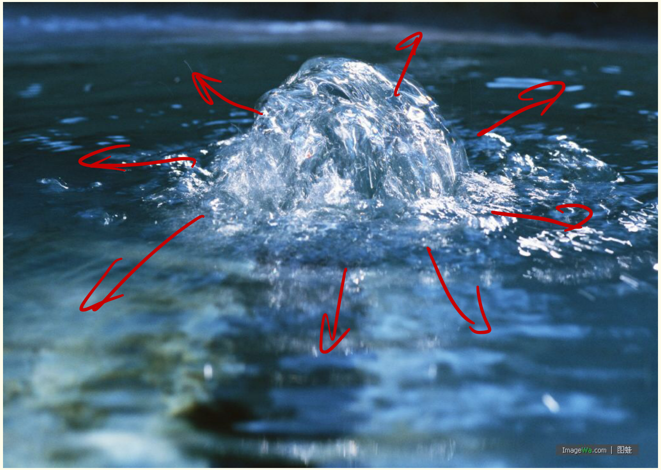
because $\nabla \cdot \vec{F} = y + xz \neq 0$.

Made with Goodnotes

Recall divergence vector field : $\langle x, y, z \rangle$.

net rate of change of mass of fluid/gas flowing from the point (x, y, z) per volume.

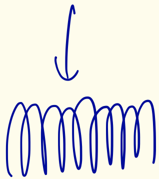
($\nabla \cdot \vec{F} > 0 \Rightarrow$ flowing out)



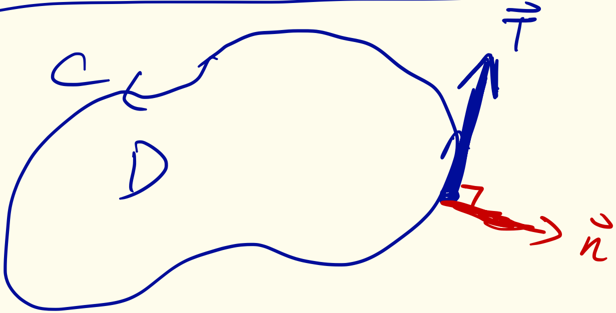
$\operatorname{div} \vec{F} = 0$

$\vec{F}$ is incompressible flow.

$\vec{F}$ is "solenoidal"



Green's Theorem : Second Form



(unit)
outward pointing normal vector.

$$C = \vec{r}(t), \qquad \vec{T} = \vec{r}\,'(t) = \langle x'(t), y'(t) \rangle$$

Take $\vec{n} = \left\langle \dfrac{y'(t), -x'(t)}{|\vec{r}\,'(t)|} \right\rangle$ Then $\vec{T} \cdot \vec{n} = 0$, $|\vec{n}| = 1$

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_D \nabla \cdot \vec{F}\, dA$$

$\left(\oint P\,dx + Q\,dy = \iint \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA \right)$

$\langle -Q, P \rangle$

$$\underline{LHS}\ \oint_C \left(\frac{P\, y'(t)}{\cancel{|\vec{r}\,'(t)|}} - \frac{Q\, x'(t)}{\cancel{|\vec{r}\,'(t)|}} \right) \cancel{|\vec{r}\,'(t)|}\, dt = \oint_C P\,dy - Q\,dx = \iint_D \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\, dA = \iint_D \nabla \cdot \vec{F}\, dA.$$

$= RHS.$

And God said:

Maxwell's Equation.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

charge density

electric field

(constant)

$$\nabla \cdot \mathbf{B} = 0$$

(no magnetic monopole)

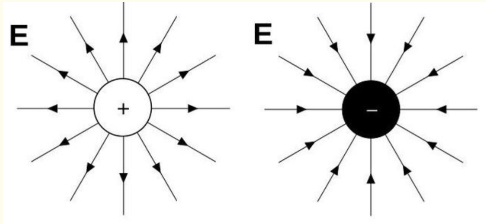
magnetic field.

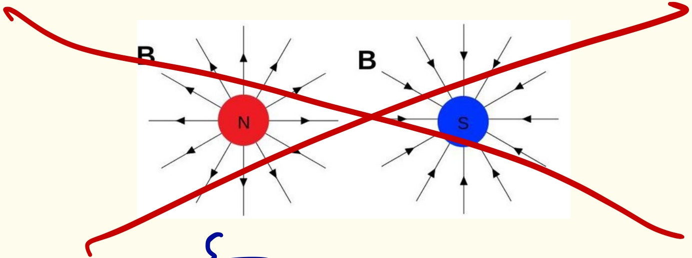
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

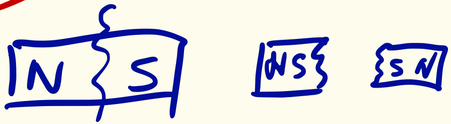
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

Faraday's Law of Induction.

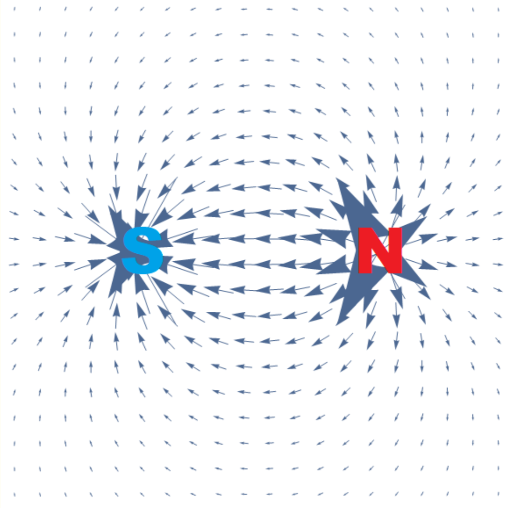
and there was light.

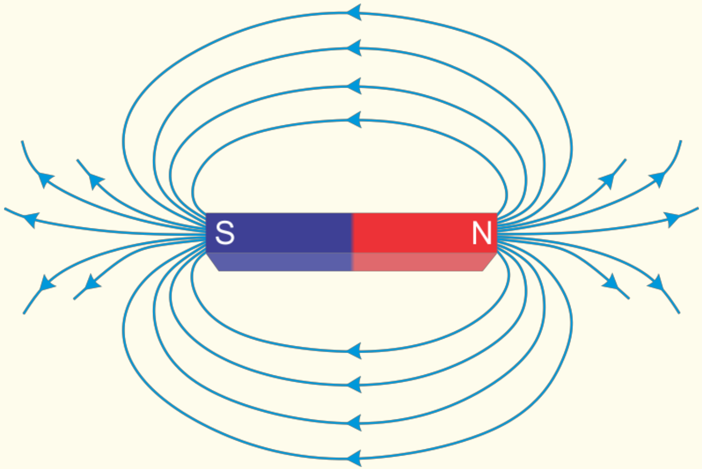






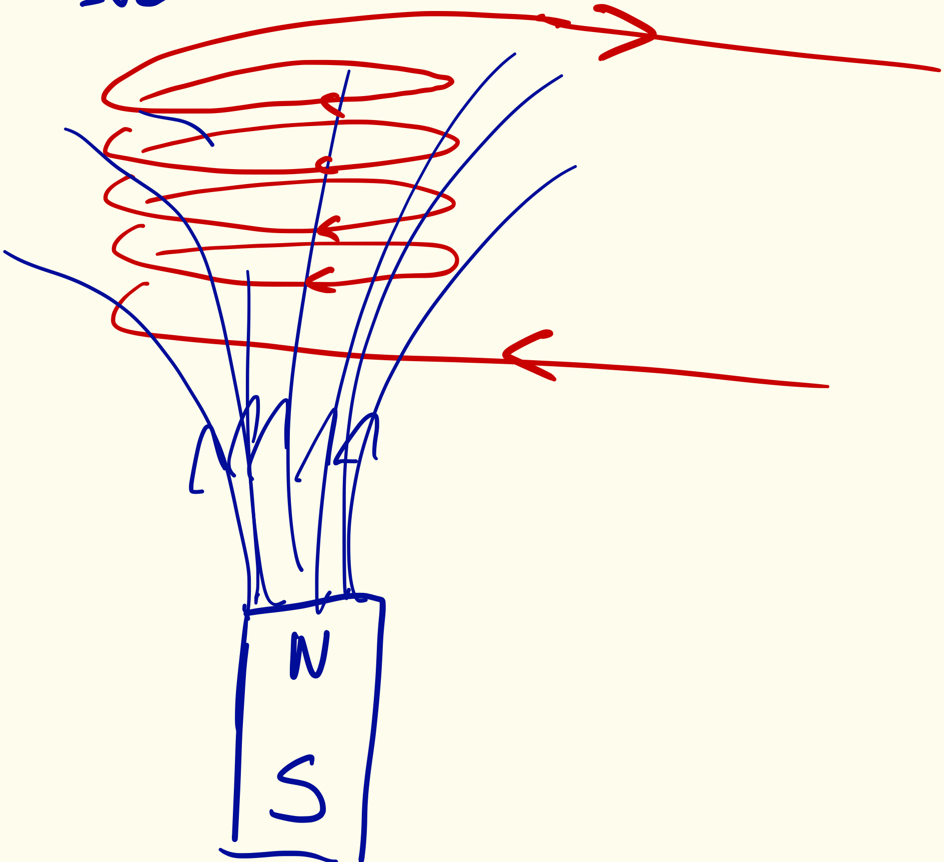
$$\operatorname{div} \vec{B} = \vec{0}$$





$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

clockwise -ve



motor