MATH 2023 • Multivariable Calculus
Problem Set #6 • Triple Integrals

1. (★) Consider the triple integral:

$$\int_0^1 \int_z^1 \int_0^{x-z} f(x,y,z)\, dydxdz.$$

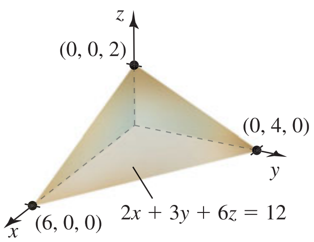
   (a) Sketch the solid described by the integral.
   (b) Express the integral using each of the other five orders, i.e. $dydzdx$, $dxdydz$, $dxdzdy$, $dzdxdy$ and $dzdydx$.

2. (★★) Consider the triple integral:

$$\int_0^1 \int_z^1 \int_0^{x} e^{x^3}\, dydxdz.$$

   (a) Sketch the solid described by the integral.
   (b) Pick a *good* order of integration and compute the integral *by hand*.

3. (★★) Consider the right tetrahedron solid T in the first octant bounded by the xy-, yz-, xz-planes and the plane Π with vertices $(6,0,0)$, $(0,4,0)$ and $(0,0,2)$.



   (a) Show that the equation of the plane Π is given by $2x + 3y + 6z = 12$.
   (b) Evaluate the following triple integral:

$$\iiint_T \left(\frac{1}{12 - 3y - 6z} + \frac{1}{12 - 2x - 6z} + \frac{1}{12 - 2x - 3y} \right) dV.$$

   Please do the computations *by hand*. Pick carefully the orders of integration to simplify your computations.

4. (★★) Let a be a positive constant. Given that $f(x)$ is a continuous function of x, show that:

$$\int_0^a \int_0^z \int_0^y f(x)\, dxdydz = \int_0^a \frac{(a-x)^2}{2} f(x)\, dx$$

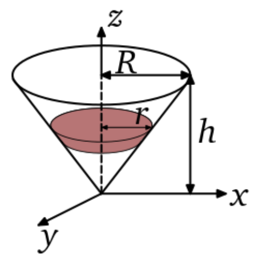
5. (★) Evaluate $\iiint_D (x^2 + y^2)\, dV$ over the solid D which lies above the cone $z = c\sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = a^2$.

6. (★) Find the volume of the solid bounded by the xy-plane, the cone $z = 2a - \sqrt{x^2 + y^2}$ and the cylinder $x^2 + y^2 = 2ay$.

7. (★★) Let $\phi(x,y,z) = \dfrac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(-\dfrac{x^2+y^2+z^2}{4kt}\right)$ where $t > 0$. Show that for each fixed $t > 0$, we have:

$$\iiint_{\mathbb{R}^3} \phi(x,y,z)\,dV = 1.$$

8. (★★) Consider a right circular solid cone (denoted by K) with radius R, height h, mass m and uniform density δ.



The moment of inertia about the z-axis of the solid is defined to be:

$$I_z := \iiint_K D_z(x,y,z)^2\,\delta dV$$

where $D_z(x,y,z)$ is the perpendicular distance between the point (x,y,z) and the z-axis.

(a) *Set up*, but do not evaluate, the integral I_z using each of the following coordinates:
   i. rectangular coordinates
   ii. cylindrical coordinates
   iii. spherical coordinates

(b) Rank the ease of computations of the above coordinate systems for evaluating the integral I_z, then compute I_z using the easiest coordinate system. Express your final answer in terms of the mass m, not the density δ.

9. (★★) Given a solid T with mass m and uniform density δ, the center of mass $(\bar{x},\bar{y},\bar{z})$ is defined to be:

$$\bar{x} = \frac{\iiint_T x\,\delta dV}{\iiint_T \delta dV}, \quad \bar{y} = \frac{\iiint_T y\,\delta dV}{\iiint_T \delta dV}, \quad \bar{z} = \frac{\iiint_T z\,\delta dV}{\iiint_T \delta dV}$$

The moment of inertia of T about the z-axis is defined as:

$$I_z := \iiint_T D_z(x,y,z)^2\,\delta dV$$

where $D_z(x,y,z)$ is the perpendicular distance between the point (x,y,z) and the z-axis.

Now consider the axis L passing through the center of mass $(\bar{x},\bar{y},\bar{z})$ and parallel to the z-axis. The moment of inertia of the solid about the axis L is defined as:

$$I_{\text{cm}} := \iiint_T D_L(x,y,z)^2\,\delta dV$$

where $D_L(x,y,z)$ is the perpendicular distance between the point (x,y,z) and the axis L.

Prove the following result (which is called the Parallel Axis Theorem):

$$I_z = I_{\text{cm}} + md^2$$

where d is the distance between the z-axis and the axis L.

10. (★) The change-of-variable formula for the volume element dV is given by:

$$dxdydz = \left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw. \tag{*}$$

(a) Using (*), verify that:

$$dxdydz = \rho^2 \sin\phi \, d\rho d\phi d\theta.$$

(b) Let $u = 2x$, $v = 3y$ and $w = 5z$. Using (*), express $dxdydz$ in terms of $dudvdw$.

11. (★★★) Consider a solid sphere with radius R centered at the origin in $\mathbb{R}^3$ which carries a uniform distribution of charges with density δ. Each volume element dV in the sphere can be regarded as a particle with charge $\delta\, dV$.

Fix a particle with charge q at $(0,0,z_0)$ where $z_0 > R$, i.e. outside the sphere, and call it the q-particle. As in the previous Problem Set, the electric force exerted on the q-particle by a charged element $\delta\, dV$ at (x,y,z) in the solid sphere is given by the Coulomb's Law (in vector form):

$$d\mathbf{F} = \frac{q\,\delta\, dV}{4\pi\varepsilon_0} \frac{(0-x)\mathbf{i} + (0-y)\mathbf{j} + (z_0 - z)\mathbf{k}}{\left((0-x)^2 + (0-y)^2 + (z_0-z)^2\right)^{3/2}}$$

Similar to the previous Problem Set, the Principle of Superposition asserts that the resultant force exerted on the q-particle by the whole sphere is given by "summing-up", i.e. integrating, each the force element $d\mathbf{F}$ over the sphere:

$$\mathbf{F}_{\text{resultant}} = \iiint_{\text{sphere}} d\mathbf{F}.$$

(a) Show that:

$$\mathbf{F}_{\text{resultant}} = \left(\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{q\delta}{4\pi\varepsilon_0} \frac{\rho^2 \sin\varphi \cdot (z_0 - \rho\cos\varphi)}{\left(\rho^2 - 2\rho z_0 \cos\varphi + z_0^2\right)^{3/2}} \, d\rho d\varphi d\theta\right)\mathbf{k}$$

(b) Try to compute the above integral, either by software or by hand, and show that:

$$\mathbf{F}_{\text{resultant}} = \frac{q\delta R^3}{3\varepsilon_0 z_0^2}\mathbf{k} = \frac{qQ}{4\pi\varepsilon_0 z_0^2}\mathbf{k}$$

where Q is the total amount of charges in the sphere.

[Remark 1: This result shows that the resultant force exerted on the q-particle by the charged sphere will be the same if one replaces it by a particle at the origin with the same amount of charges.]

[Remark 2: Using the Gauss's Law for Electricity, the above result can be obtained easily by considering the surface flux of $\mathbf{F}_{\text{resultant}}$. We will discuss that later, and will derive the Gauss's Law using the Divergence Theorem (assuming Coulomb's Law).]

1. (★) Consider the triple integral:

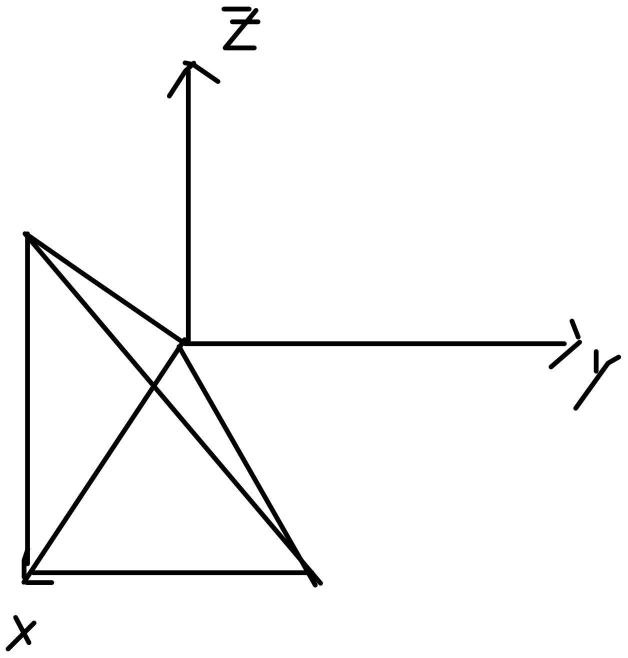
$$\int_0^1 \int_z^1 \int_0^{x-z} f(x,y,z)\, dydxdz.$$

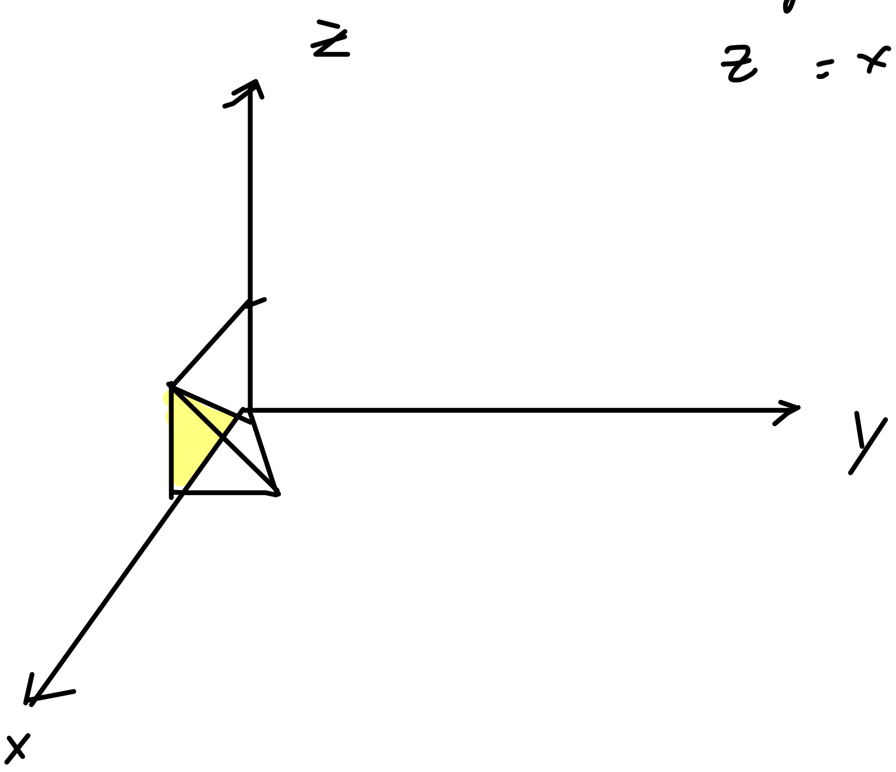
(a) Sketch the solid described by the integral.

(b) Express the integral using each of the other five orders, i.e. $dydzdx$, $dxdydz$, $dxdzdy$, $dzdxdy$ and $dzdydx$.

$y = x - z$

$z = x - y$





$$\int_0^1 \int_0^x \int_0^{x-z} \quad dy\,dz\,dx$$

$$\int_0^1 \int_0^{1-z} \int_{y+z}^1 \quad dx\,dy\,dz$$

$$\int_0^1 \int_0^{1-y} \int_{y+z}^1 \quad dx\,dz\,dy$$

$$\int_0^1 \int_y^1 \int_0^{x-y} \quad dz\,dx\,dy$$

$$\int_0^1 \int_0^x \int_0^{x-y} \quad dz\,dy\,dx$$

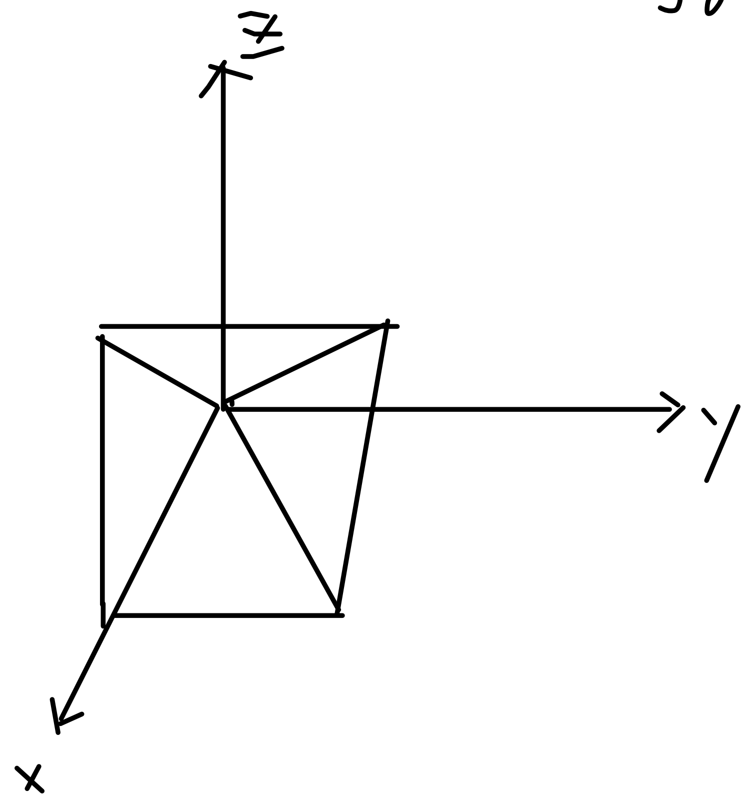
2. (★★) Consider the triple integral:

$$\int_0^1 \int_z^1 \int_0^x e^{x^3}\, dy dx dz.$$

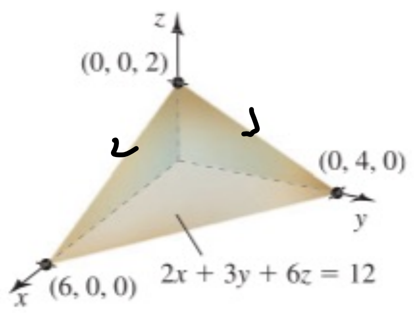
(a) Sketch the solid described by the integral.

(b) Pick a *good* order of integration and compute the integral *by hand*.

$$\int_0^1 \int_0^x \int_0^x e^{x^3}\, dz dy dx$$



3. (★★) Consider the right tetrahedron solid T in the first octant bounded by the xy-, yz-, xz-planes and the plane Π with vertices $(6,0,0)$, $(0,4,0)$ and $(0,0,2)$.



(a) Show that the equation of the plane Π is given by $2x + 3y + 6z = 12$.

(b) Evaluate the following triple integral:

$$\iiint_T \left(\frac{1}{12 - 3y - 6z} + \frac{1}{12 - 2x - 6z} + \frac{1}{12 - 2x - 3y} \right) \, dV.$$

Please do the computations *by hand*. Pick carefully the orders of integration to simplify your computations.

a). $r_1 = \langle 6, 0, -2 \rangle$ $\quad r_2 = \langle 0, 4, -2 \rangle$

$$r_1 \times r_2 = \begin{vmatrix} i & j & k \\ 6 & 0 & -2 \\ 0 & 4 & -2 \end{vmatrix} = \langle 8, 12, 24 \rangle$$

$8x + 12y + 24z$

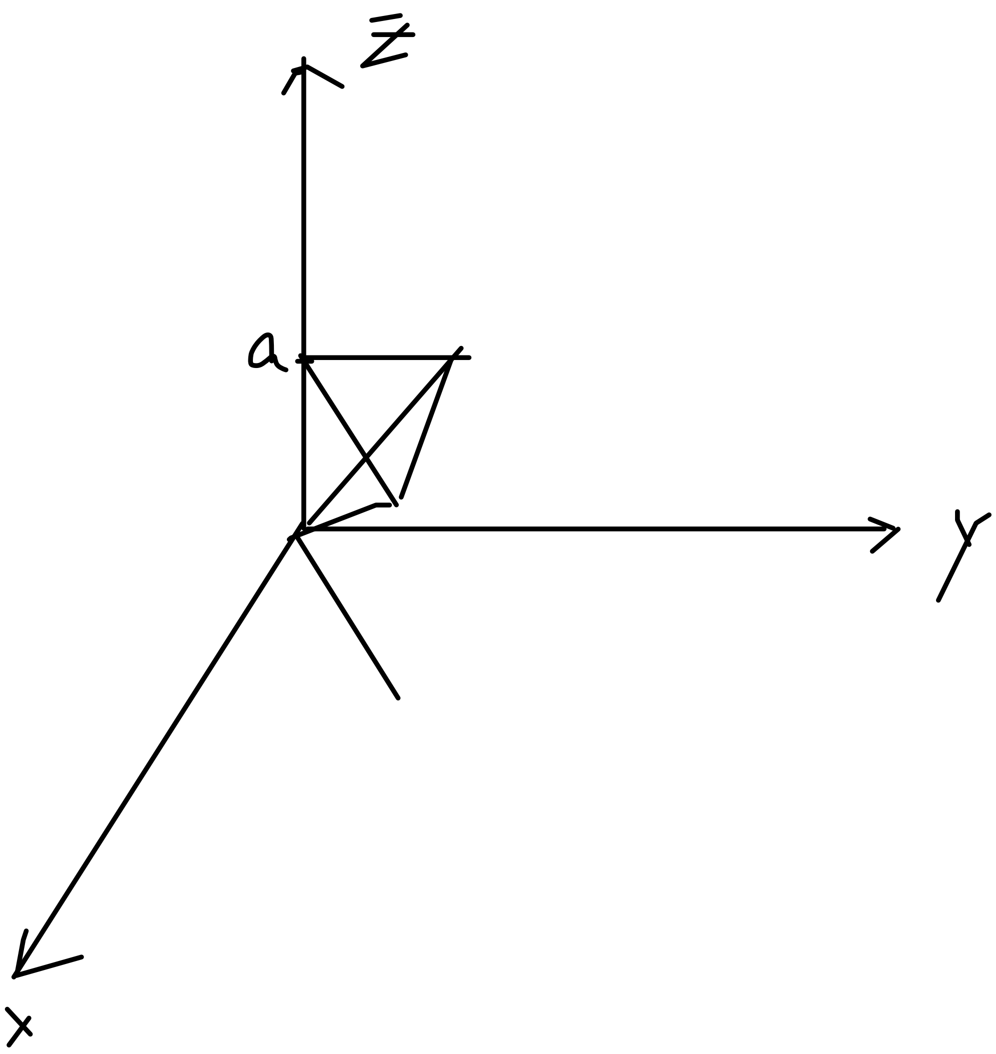
$2x + 3y + 6z$

$$\int_0^2 \int_0^{6z} \int_0^{\frac{12-3y-6z}{2}} \frac{1}{12 - 3y - 6z} \, dx \, dy \, dz$$

Made with Goodnotes

4. (★★) Let a be a positive constant. Given that $f(x)$ is a continuous function of x, show that:

$$\int_0^a \int_0^z \int_0^y f(x)\, dx dy dz = \int_0^a \frac{(a-x)^2}{2} f(x)\, dx$$



$$\int_0^a \int_x^a \int_y^a f(x) \quad dz\, dy\, dx$$

$$\int_x^a af(x) - yf(x)\, dy$$

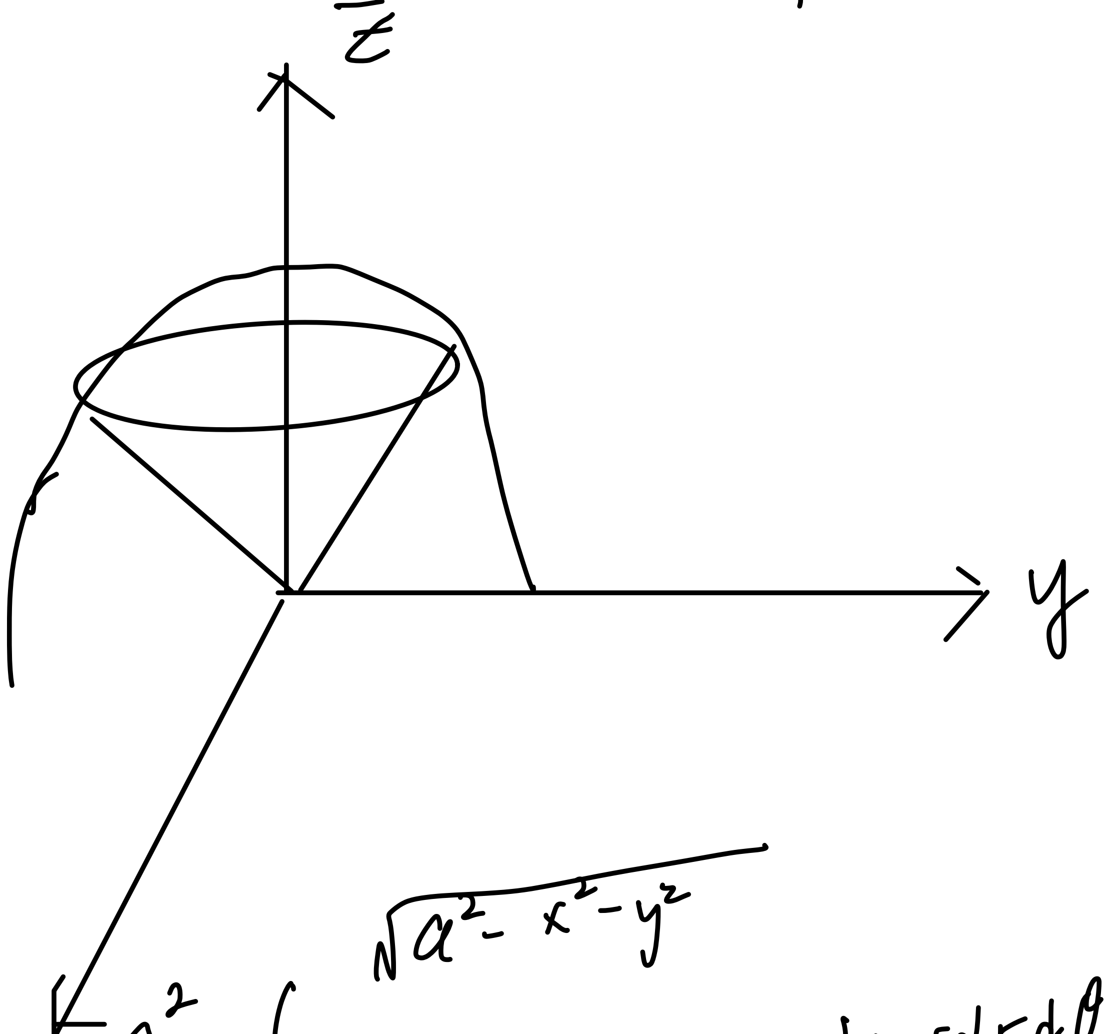
$$a^2f(x) - \left[\frac{y^2}{2}f(x)\right]_x^a - axf(x)$$

$$a^2f(x) - \frac{a^2}{2}f(x) + \frac{x^2}{2}f(x) - axf(x)$$

5. (★) Evaluate $\iiint_D (x^2+y^2)\,dV$ over the solid D which lies above the cone $z = c\sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 = a^2$.

$$x^2+y^2+c^2(x^2+y^2) = a^2$$

$$(x^2+y^2) = \frac{a^2}{1+c^2}$$



$$\int_0^{2\pi} \int_0^{\frac{a^2}{1+c^2}} \int_{c\sqrt{x^2+y^2}}^{\sqrt{a^2-x^2-y^2}} dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{a^2}{1+c^2}} \int_{cr}^{\sqrt{a^2-r^2}} r^3\, dz\, dr\, d\theta$$

6. (★) Find the volume of the solid bounded by the xy-plane, the cone $z = 2a - \sqrt{x^2 + y^2}$ and the cylinder $x^2 + y^2 = 2ay$.

$r^2 = 2r\sin\theta a$

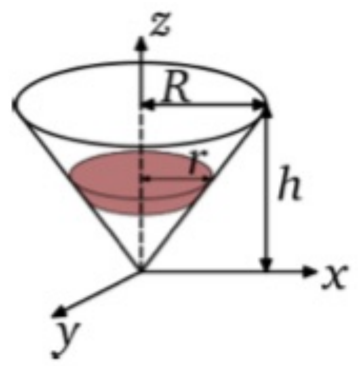
$$\int_0^{\pi} \int_0^{2a\sin\theta} \int_0^{2a-r} dz\, r\, dr\, d\theta$$

7. (★★) Let $\phi(x,y,z) = \frac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(-\frac{x^2+y^2+z^2}{4kt}\right)$ where $t > 0$. Show that for each fixed $t > 0$, we have:

$$\iiint_{\mathbb{R}^3} \phi(x,y,z)\, dV = 1.$$

$$\int \int \int \frac{1}{(4\pi kt)^{\frac{3}{2}}} e^{-\frac{\rho^2}{4kt}} (\rho^2 \sin\varphi)\, d\rho\, d\varphi\, d\theta$$

8. (★★) Consider a right circular solid cone (denoted by K) with radius R, height h, mass m and uniform density δ.



The moment of inertia about the z-axis of the solid is defined to be:

$$I_z := \iiint_K D_z(x,y,z)^2\, \delta dV$$

where $D_z(x,y,z)$ is the perpendicular distance between the point (x,y,z) and the z-axis.

(a) *Set up*, but do not evaluate, the integral I_z using each of the following coordinates:

i. rectangular coordinates

ii. cylindrical coordinates

iii. spherical coordinates

(b) Rank the ease of computations of the above coordinate systems for evaluating the integral I_z, then compute I_z using the easiest coordinate system. Express your final answer in terms of the mass m, not the density δ.

a). i. $$\int_{-R}^{R}\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}}\int_{\sqrt{x^2+y^2}}^{h} dz\,dx\,dy$$

ii. $$\int_0^{2\pi}\int_0^{R}\int_r^{h} dz\, r\, dr\, d\theta$$

iii. $$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{R\sec\varphi} \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

9. (★★) Given a solid T with mass m and uniform density δ, the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is defined to be:

$$\bar{x} = \frac{\iiint_T x \, \delta dV}{\iiint_T \delta dV}, \quad \bar{y} = \frac{\iiint_T y \, \delta dV}{\iiint_T \delta dV}, \quad \bar{z} = \frac{\iiint_T z \, \delta dV}{\iiint_T \delta dV}$$

The moment of inertia of T about the z-axis is defined as:

$$I_z := \iiint_T D_z(x, y, z)^2 \, \delta dV$$

where $D_z(x, y, z)$ is the perpendicular distance between the point (x, y, z) and the z-axis. Now consider the axis L passing through the center of mass $(\bar{x}, \bar{y}, \bar{z})$ and parallel to the z-axis. The moment of inertia of the solid about the axis L is defined as:

$$I_{\text{cm}} := \iiint_T D_L(x, y, z)^2 \, \delta dV$$

where $D_L(x, y, z)$ is the perpendicular distance between the point (x, y, z) and the axis L. Prove the following result (which is called the Parallel Axis Theorem):

$$I_z = I_{\text{cm}} + md^2$$

where d is the distance between the z-axis and the axis L.

10. (★) The change-of-variable formula for the volume element dV is given by:

$$dxdydz = \left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw. \qquad (*)$$

(a) Using (*), verify that:
$$dxdydz = \rho^2 \sin\phi \, d\rho d\phi d\theta.$$

(b) Let $u = 2x$, $v = 3y$ and $w = 5z$. Using (*), express $dxdydz$ in terms of $dudvdw$.

$$\begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{vmatrix} = \frac{1}{2}\left(\frac{1}{15}\right) = \frac{1}{30} \, du \, dv \, dw.$$

11. (★★★) Consider a solid sphere with radius R centered at the origin in $\mathbb{R}^3$ which carries a uniform distribution of charges with density δ. Each volume element dV in the sphere can be regarded as a particle with charge $\delta \, dV$.

Fix a particle with charge q at $(0, 0, z_0)$ where $z_0 > R$, i.e. outside the sphere, and call it the q-particle. As in the previous Problem Set, the electric force exerted on the q-particle by a charged element $\delta \, dV$ at (x, y, z) in the solid sphere is given by the Coulomb's Law (in vector form):

$$d\mathbf{F} = \frac{q \, \delta \, dV}{4\pi\varepsilon_0} \frac{(0-x)\mathbf{i} + (0-y)\mathbf{j} + (z_0 - z)\mathbf{k}}{((0-x)^2 + (0-y)^2 + (z_0-z)^2)^{3/2}}$$

Similar to the previous Problem Set, the Principle of Superposition asserts that the resultant force exerted on the q-particle by the whole sphere is given by "summing-up", i.e. integrating, each the force element $d\mathbf{F}$ over the sphere:

$$\mathbf{F}_{\text{resultant}} = \iiint_{\text{sphere}} d\mathbf{F}.$$

(a) Show that:

$$\mathbf{F}_{\text{resultant}} = \left(\int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{q\delta}{4\pi\varepsilon_0} \frac{\rho^2 \sin\varphi \cdot (z_0 - \rho\cos\varphi)}{(\rho^2 - 2\rho z_0 \cos\varphi + z_0^2)^{3/2}} \, d\rho d\varphi d\theta \right) \mathbf{k}$$

(b) Try to compute the above integral, either by software or by hand, and show that:

$$\mathbf{F}_{\text{resultant}} = \frac{q\delta R^3}{3\varepsilon_0 z_0^2}\mathbf{k} = \frac{qQ}{4\pi\varepsilon_0 z_0^2}\mathbf{k}$$

where Q is the total amount of charges in the sphere.

[Remark 1: This result shows that the resultant force exerted on the q-particle by the charged sphere will be the same if one replaces it by a particle at the origin with the same amount of charges.]

[Remark 2: Using the Gauss's Law for Electricity, the above result can be obtained easily by considering the surface flux of $\mathbf{F}_{\text{resultant}}$. We will discuss that later, and will derive the Gauss's Law using the Divergence Theorem (assuming Coulomb's Law).]