香港科技大學
THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

# MIDTERM EXAMINATION

**Course Code:** MATH 2023
**Course Title:** Multivariable Calculus
**Semester:** Spring 2015-16
**Date and Time:** 17 March 2016, 7pm-9pm

## Instructions

- Do **NOT** open the exam until instructed to do so.
- This is a **CLOSED BOOK, CLOSED NOTES** exam.
- All mobile phones and communication devices should be switched **OFF**.
- Only calculators approved by HKEAA can be used.
- Answer **ALL** eight problems.
- You must **SHOW YOUR WORK** to receive credits in all problems except Problem #1. Answers alone (whether correct or not) will not receive any credit.
- Some problems are structured into several parts. You can quote the results stated in the preceding parts to do the next part, regardless of whether you can complete the preceding parts or not. However, different parts in the same problem are not necessarily co-related.

## Academic Integrity

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

"I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."

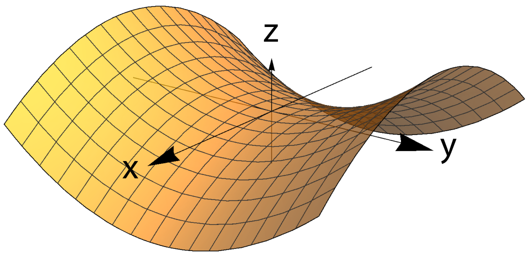
**Student's Signature:** ______________________________

**Student's Name:** ______________________ **HKUST ID:** __________

| Problem | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|---|---|---|---|---|---|---|---|---|---|
| Max | 27 | 10 | 10 | 10 | 10 | 12 | 15 | 6 | 100 |
| Score | | | | | | | | | |

1. Answer the following conceptual questions. **No justification is needed.**

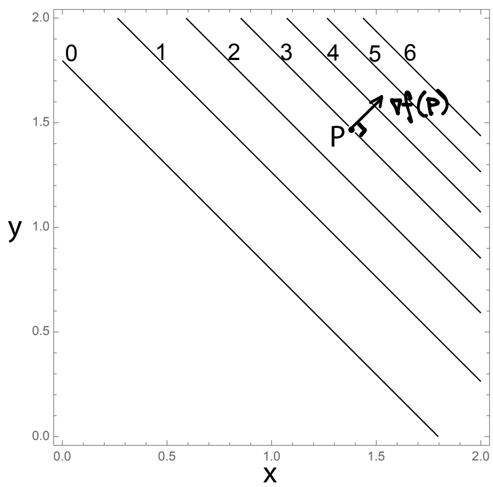
   (a) Consider the following graph of a saddle: **/2**




Is it the graph of $f(x,y) = x^2 - y^2$, or the graph of $g(x,y) = y^2 - x^2$? **Circle** the correct answer:

$f(x,y)$ $\qquad$ $g(x,y)$

   (b) Below is the level-set diagram of a $C^2$ function $f$. The contours $f(x,y) = c$ with $c = 0, 1, \ldots, 6$ are shown. **/10**




Consider the point $P$ indicated on the diagram. Answer the following questions:

i. Determine whether each of the following quantities is positive, zero or negative. **Circle** the correct answers.

| | | | |
|---|---|---|---|
| $f(P)$ | positive | zero | negative |
| $f_y(P)$ | positive | zero | negative |
| $f_{xx}(P)$ | positive | zero | negative |
| $D_{\frac{-\mathbf{i}+\mathbf{j}}{\sqrt{2}}} f(P)$ | positive | zero | negative |

ii. Sketch the direction of $\nabla f(P)$ on the diagram.

Problem #1 continues on next page...

(c) In class, we went through the proofs of several important theorems. Which of the following theorem(s) was/were proved using the chain rule in at least one of the steps? /3

Put "$\sqrt{}$" in **all** correct answer(s):

☑ The formula $D_{\hat{\mathbf{u}}} f = \nabla f \cdot \hat{\mathbf{u}}$

see lecture #7 or notes P.36 → ☑ The fact that $\nabla f(P)$ is perpendicular to the level curve of $f$ at $P$

☑ Second Derivative Test for two-variable functions $f(x, y)$

$|\vec{r}'(s)| = 1$ $\quad \therefore \int_{1997}^{2047} |\vec{r}'(s)|\, ds = \int_{1997}^{2047} 1\, ds = 2047 - 1997$

(d) Let $\mathbf{r}(s)$ be an arc-length parametrization of a curve. Write down the arc-length of the curve from $s = 1997$ to $s = 2047$. /2

50

$1x + 0y + (-2)z = 1$

(e) Write down an *upward-pointing* normal vector of the plane $x - 2z = 1$. /2

$-\hat{\imath} + 2\hat{k}$ (or $-c\hat{\imath} + 2c\hat{k}$ for any const. $c > 0$)

(f) Give an example of a parametric curve $\mathbf{r}(t)$ in $\mathbb{R}^2$ such that: /2

$$\frac{d}{dt}|\mathbf{r}(t)| = 0 \quad \text{whereas} \quad \left|\frac{d}{dt}\mathbf{r}(t)\right| = 1.$$

One possible answer:

$\vec{r}(t) = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath}$

(g) Let $f(x,y)$ be a $C^2$ function defined on $\mathbb{R}^2$ which satisfies: **/6**

$$\nabla f(0,0) = 0\mathbf{i} + 0\mathbf{j}$$
$$f_{xx}(0,0) = -2$$
$$\left(f_{xx}f_{yy} - f_{xy}^2\right)(0,0) = 1$$

critical point

$\therefore$ Horizontal tangent plane $\Leftrightarrow$ $\vec{n} \parallel \hat{k}$

Answer the following short questions:

i. Write down a normal vector to the graph of $f$ at the point $(0,0,f(0,0))$.

| |
|---|
| $\hat{k}$ (or $c\hat{k}$ for any $c \neq 0$) |

ii. Is $(0,0)$ a local maximum, a local minimum or a saddle point of $f$? **Circle** the correct answer:

local maximum  local minimum  saddle point  not enough data

iii. Is $f_{yy}(0,0)$ positive, negative or zero? **Circle** the correct answer:

$$\underbrace{f_{xx}}_{\ominus} f_{yy} = \underbrace{f_{xy}^2 + 1}_{\oplus}$$

$f_{yy} < 0$

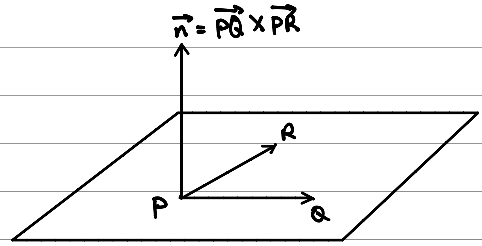
positive  zero  negative  not enough data

2. Given three points in $\mathbb{R}^3$:

$$P(1,-2,0), \quad Q(3,1,4) \quad \text{and} \quad R(0,-1,2).$$

(a) Find an equation of the plane containing $P$, $Q$ and $R$. /7

$\overrightarrow{PQ} = \langle 3,1,4 \rangle - \langle 1,-2,0 \rangle$
$= \langle 2,3,4 \rangle$




$\overrightarrow{PR} = \langle 0,-1,2 \rangle - \langle 1,-2,0 \rangle$
$= \langle -1,1,2 \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & 4 \\ -1 & 1 & 2 \end{vmatrix} = 2\hat{\imath} - 8\hat{\jmath} + 5\hat{k}$$

Take $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2,-8,5 \rangle$
The plane passes through $P(1,-2,0)$

Equation of the plane:

$$2x - 8y + 5z = (2)(1) + (-8)(-2) + (5)(0)$$
$$2x - 8y + 5z = 18$$

(b) Find the point $S$ on the $x$-axis such that $P$, $Q$, $R$ and $S$ are coplanar. /3

$S$ needs to be on the plane $2x-8y+5z=18$ found in (a).
$S$ on the x-axis $\Rightarrow S = (x,0,0) \rightsquigarrow$ Need to find $x$.
Sub $(x,0,0)$ into the plane equation:

$$2x - 8(0) + 5(0) = 18 \Rightarrow 2x = 18 \Rightarrow x = 9$$

$\therefore S(9,0,0)$

3. (a) Let $f(x,y) = x^{e^y}$ where $x, y > 0$. Find the partial derivatives $f_x$ and $f_y$. /5

$f_x = \frac{\partial}{\partial x} x^{(e^y)} = e^y x^{e^y - 1}$

$f_y = \frac{\partial}{\partial y} x^{(e^y)} = \frac{\partial}{\partial (e^y)} x^{(e^y)} \cdot \frac{\partial}{\partial y} e^y$

$= x^{(e^y)} \ln x \cdot e^y$

(b) Let $g(x,y) = x^2y^3 + \boxed{e^{x^2} \tan^{-1}\left[\log\left(\frac{\sqrt{1+x^2+x^4}}{1+(x^2+1)^{x^4+1}}\right)\right].}$ ← independent of y /5

Find the second partial derivative $\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)$.

$\frac{\partial g}{\partial y} = 3x^2y^2 + 0$

$\frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2) = 6xy^2$ ← continuous

By Mixed Partials Theorem,

$\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = 6xy^2$

4. (a) Let $f(x,y,z)$ be a $C^1$ function of three variables. Regard $y$ to be a $C^1$ function of $x$ and $z$ such that: **/6**

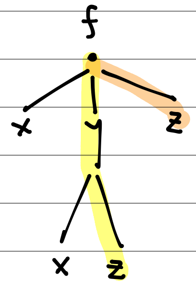
$$f(x,\, y(x,z),\, z) = 0.$$

Using the chain rule, show that $\dfrac{\partial y}{\partial z} = -\dfrac{f_z}{f_y}$.

$$f(x,\, y(x,z),\, z) = 0$$

$$\frac{\partial}{\partial z} f(x, y(x,z), z) = 0$$

$$\frac{\partial f}{\partial y}\frac{\partial y}{\partial z} + \frac{\partial f}{\partial z} = 0$$ by chain rule




$$\Rightarrow \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial z} = -\frac{\partial f}{\partial z}$$

$$\Rightarrow \frac{\partial y}{\partial z} = -\frac{\partial f}{\partial z} \Big/ \frac{\partial f}{\partial y} = -\frac{f_z}{f_y}$$

(b) Given that: **/4**

$$\sin x + \cos y + y^2 + \cos^2 z = 0.$$

Regarding $y$ as a $C^1$ function of $x$ and $z$, find the partial derivative $\dfrac{\partial y}{\partial z}$. Your final answer can be in terms of all $x$, $y$ and $z$.

Let $f(x,y,z) = \sin x + \cos y + y^2 + \cos^2 z$

Then from (a), we have:

$$\frac{\partial y}{\partial z} = \frac{-\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}} = -\frac{2\cos z\cdot(-\sin z)}{-\sin y + 2y} = \frac{2\sin z\,\cos z}{2y - \sin y}$$

5. Consider the parametric curve in $\mathbb{R}^2$:

$$\mathbf{r}(t) = \left(e^t \cos t\right)\ \mathbf{i} + \left(e^t \sin t\right)\ \mathbf{j}.$$

(a) Show that: $|\mathbf{r}'(t)| = \sqrt{2}\, e^t$ **/6**

$$\vec{r}\,'(t) = \left(e^t\cos t - e^t \sin t\right)\hat{\imath} + \left(e^t \sin t + e^t \cos t\right)\hat{\jmath}$$
$$= e^t\left[(\cos t - \sin t)\hat{\imath} + (\sin t + \cos t)\hat{\jmath}\right]$$

$$|\vec{r}\,'(t)| = e^t\left|(\cos t - \sin t)\hat{\imath} + (\sin t + \cos t)\hat{\jmath}\right|$$
$$= e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}$$
$$= e^t\sqrt{\cos^2 t - \cancel{2\cos t \sin t} + \sin^2 t + \sin^2 t + \cancel{2 \sin t \cos t} + \cos^2 t}$$
$$= e^t\sqrt{2(\cos^2 t + \sin^2 t)}$$
$$= e^t \cdot \sqrt{2}$$

(b) Find an arc-length parametrization $\mathbf{r}(s)$ of the curve. **/4**

Let $s = \int_0^t |\vec{r}\,'(u)|\,du$ or $s = \int_c^t |\vec{r}\,'(u)|\,du$, $c \leftarrow$ any const. or: $s = \int_{-\infty}^t |\vec{r}\,'(u)|\,du$

$$= \int_0^t \sqrt{2}\, e^u\, du$$

$\Downarrow$ $s = \sqrt{2}(e^t - e^c)$, $t = \ln\left(e^c + \frac{s}{\sqrt{2}}\right)$

$\Downarrow$ $s = \sqrt{2}e^t$, $t = \ln\left(\frac{s}{\sqrt{2}}\right)$

$$= \sqrt{2}e^u\Big]_0^t = \sqrt{2}\left(e^t - 1\right)$$

$$\Rightarrow e^t = 1 + \frac{s}{\sqrt{2}} \Rightarrow t = \ln\left(1 + \frac{s}{\sqrt{2}}\right)$$

$$\vec{r}(s) = \underbrace{\left(1 + \frac{s}{\sqrt{2}}\right)}_{e^t}\left[\left(\cos \ln\left(1+\frac{s}{\sqrt{2}}\right)\right)\hat{\imath} + \left(\sin \ln\left(1 + \frac{s}{\sqrt{2}}\right)\right)\hat{\jmath}\right]$$

6. Find the maximum and minimum of the function $f(x,y) = xy$ subject to the constraint $x^2 - xy + y^2 = 3$. /12

Let $g(x,y) = x^2 - xy + y^2$ Need to solve: $\nabla f = \lambda \nabla g$ and $g = 3$

$$\begin{cases} \dfrac{\partial f}{\partial x} = \lambda \dfrac{\partial g}{\partial x} \Rightarrow y = \lambda(2x-y) \text{ --- } ① \\ \dfrac{\partial f}{\partial y} = \lambda \dfrac{\partial g}{\partial y} \Rightarrow x = \lambda(-x+2y) \text{ --- } ② \\ g(x,y) = 3 \Rightarrow x^2 - xy + y^2 = 3 \text{ --- } ③ \end{cases}$$

Case a): $x = \lambda(-x+2y) \neq 0$

then $\dfrac{①}{②} \Rightarrow \dfrac{y}{x} = \dfrac{\cancel{\lambda}(2x-y)}{\cancel{\lambda}(-x+2y)} \Rightarrow y(-x+2y) = x(2x-y)$

$\Rightarrow \cancel{-xy} + 2y^2 = 2x^2 \cancel{-xy} \Rightarrow y^2 = x^2 \Rightarrow y = x$ or $-x$.

When $y = x$, ③ $\Rightarrow x^2 - x^2 + x^2 = 3 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

$\therefore (x,y) = (\sqrt{3}, \sqrt{3})$ or $(-\sqrt{3}, -\sqrt{3})$

When $y = -x$, ③ $\Rightarrow x^2 + x^2 + x^2 = 3 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = 1$ or $-1$

$\therefore (x,y) = (1,-1)$ or $(-1,1)$

Case b): $x = \lambda(-x+2y) = 0$

then we have $x = 0$. The three equations become:

$$\begin{aligned} y &= -\lambda y \\ 0 &= \lambda \cdot \cancel{2}y \end{aligned} \Rightarrow y = 0$$

$$y^2 = 3 \Rightarrow y = \sqrt{3} \text{ or } -\sqrt{3}$$

contradiction

$\therefore$ No solution in this case.

| $(x,y)$ | $f(x,y) = xy$ | |
|---|---|---|
| $(\sqrt{3}, \sqrt{3})$ | 3 | max |
| $(-\sqrt{3}, -\sqrt{3})$ | 3 | |
| $(1,-1)$ | $-1$ | min |
| $(-1,1)$ | $-1$ | |

7. Suppose $u(x, y)$ and $v(x, y)$ are $C^2$ functions defined on $\mathbb{R}^2$ which satisfy the relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(a) The rectangular and polar coordinates are related by $x = r\cos\theta$ and $y = r\sin\theta$. Under this relation, we can also regard $u$ and $v$ as functions of $(r, \theta)$. **/10**

Show that: $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial r}(r\cos\theta) + \frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial \theta}(r\sin\theta)$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$= \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial}{\partial \theta}(r\cos\theta) + \frac{\partial v}{\partial y}\frac{\partial}{\partial \theta}(r\sin\theta)$$

$$= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}r\cos\theta$$

$$= r\left(-\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta\right) = r\frac{\partial u}{\partial r}$$

$$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \#$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}\underbrace{(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)}_{\text{computed above}}$$

$$= \frac{\partial v}{\partial y}\cdot(-r\sin\theta) - \frac{\partial v}{\partial x}(r\cos\theta) = -r\left(\frac{\partial v}{\partial y}\sin\theta + \frac{\partial v}{\partial x}\cos\theta\right)$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \underbrace{\frac{\partial v}{\partial x}\cdot\cos\theta + \frac{\partial v}{\partial y}\sin\theta}_{\text{computed before}}$$

$$\therefore \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \implies \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \#$$

Problem #7 continues on next page…

(b) Using (a), show that: **/5**

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

[Hint and remark: You can assume that the Mixed Partials Theorem holds in polar coordinates, i.e. $u_{r\theta} = u_{\theta r}$ and $v_{r\theta} = v_{\theta r}$.]

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial \theta}\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(-r\frac{\partial v}{\partial r}\right) \qquad \text{(from (a))}$$

$$= \cancel{-\frac{1}{r^2}\frac{\partial v}{\partial \theta}} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} + \cancel{\frac{1}{r^2}\frac{\partial v}{\partial \theta}} + \frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial \theta \partial r}\right)$$

$$= \frac{1}{r}\left(\frac{\partial^2 v}{\partial r \partial \theta} - \frac{\partial^2 v}{\partial \theta \partial r}\right)$$

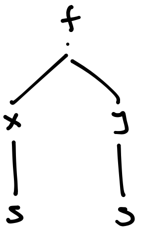
$= 0$ by Mixed Partials Theorem.

8. Let $f(x, y)$ be a function of two variables and $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$ be an **arc-length parametrized** curve in $\mathbb{R}^2$ such that /6

$$\frac{d}{ds}\mathbf{r}(s) = \nabla f(x(s), y(s)).$$

Here $\nabla f(x(s), y(s))$ means $\nabla f$ evaluated at the point $(x(s), y(s))$. Assume that both $f(x, y)$ and $\mathbf{r}(t)$ are $C^2$. Show that:

$$\frac{d^2}{ds^2} f(x(s), y(s)) = 0.$$




By chain rule,

$$\frac{d}{ds} f(x(s), y(s)) = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

For simplicity: We write $\frac{\partial f}{\partial x}$ in place of $\frac{\partial f}{\partial x}(x(s), y(s))$. Same for $\frac{\partial f}{\partial y}$.

$$= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right) \cdot \left(x'(s)\hat{i} + y'(s)\hat{j}\right)$$

A "trick" we used for several places in the proofs presented in class.

$$= \nabla f \cdot \vec{r}\,'(s)$$

Given that $\vec{r}\,'(s) = \nabla f$:

$$\frac{d}{ds} f(x(s), y(s)) = \vec{r}\,'(s) \cdot \vec{r}\,'(s)$$

$$= |\vec{r}\,'(s)|^2 = 1$$ $\vec{r}(s)$ is arc-length parametrized

$$\Rightarrow \frac{d^2}{ds^2} f(x(s), y(s)) = \frac{d}{ds}\left(\frac{d}{ds} f(x(s), y(s))\right) = \frac{d}{ds} 1 = 0$$

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