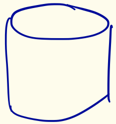
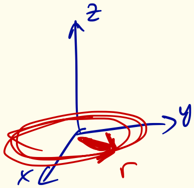
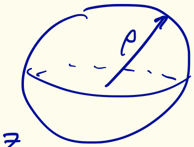
Last Time



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\iiint dV = \iiint \underline{r}\, dz\, \underline{dr\, d\theta}$$



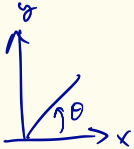


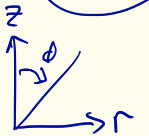
$$\begin{cases} x = \rho\sin\phi\, \underline{\cos\theta} \\ y = \rho\sin\phi\, \underline{\sin\theta} \\ z = \rho\cos\phi \end{cases}$$

$r = \rho\, \underline{\sin\phi}$

$z = \rho\, \underline{\cos\phi}$

$$\begin{pmatrix} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{pmatrix}$$





$$\iiint dV = \iiint \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$(\rho^2 = x^2 + y^2 + z^2)$$

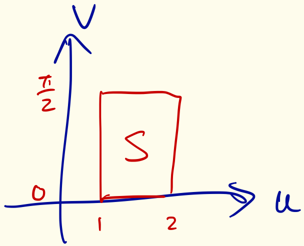
Change of Variables

absolute value

$$\iint_R dA = \iint_S \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

Jacobian

$$= \left\| \begin{matrix} x_u & y_u \\ x_v & y_v \end{matrix} \right\|$$



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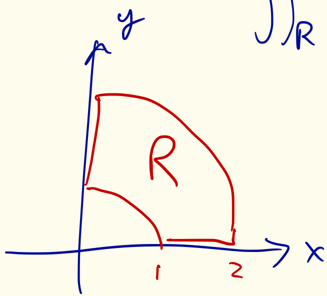


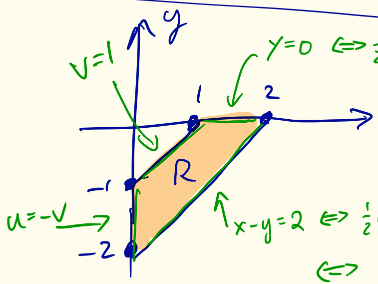
$$\begin{cases} x = u\cos v \\ y = u\sin v \end{cases} \longrightarrow u\, du\, dv$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{vmatrix} x_r & y_r \\ x_\theta & y_\theta \end{vmatrix} = r$$

$$dA = r\,dr\,d\theta$$

<u>Ex</u> $\displaystyle\iint_R e^{\frac{x+y}{x-y}}\,dA$

R :



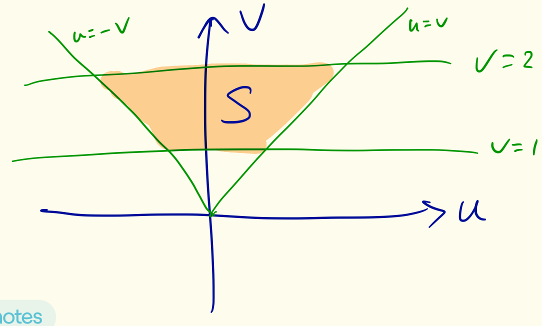
$y=0 \iff \frac{1}{2}(u-v)=0 \iff u=v$

$x-y=2 \iff \frac{1}{2}(u+v)-\frac{1}{2}(u-v)=2 \iff v=2$

$$\begin{cases} u = x+y \\ v = x-y \end{cases} \iff \boxed{\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}, \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2}.$$

S :



$$\boxed{dA = \frac{1}{2}\,du\,dv}$$

$$\int_1^2\int_{-v}^{v} e^{\frac{u}{v}}\,\frac{1}{2}\,du\,dv = \frac{1}{2}\int_1^2 v e^{u/v}\Big|_{u=-v}^{u=v}$$

$$= \frac{1}{2}\int_1^2 (e-e^{-1})\,v\,dv$$

$$= \frac{3}{4}(e-e^{-1})$$

3D

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \\ x_w & y_w & z_w \end{vmatrix}$$ Jacobian

Ex

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho\sin\phi \sin\theta \\ z = \rho\cos\phi \end{cases} \Longrightarrow \frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \rho^2 \sin\phi$$

$$\frac{dy}{\partial x} = \frac{1}{\frac{\partial x}{\partial y}}$$

(Inverse function Thm)

$$\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$$

$$(T \cdot T^{-1} = 1.)$$

- - - -

Divergence Theorem

E closed solid

S boundary of E, oriented outward

$\vec{F}$ vector field.

$$\iiint_E \nabla \cdot \vec{F} \, dV = \iint_S \vec{F} \cdot d\vec{S}$$

" amount of field generated inside S = flux across S

Ex $\vec{F} = \langle x, y, z \rangle$.

Find the flux through the unit sphere.

$$\iint_S \vec{F} \cdot dS = \iiint_E \nabla \cdot \vec{F} \, dV$$

S: sphere

E: solid ball

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 3.$$

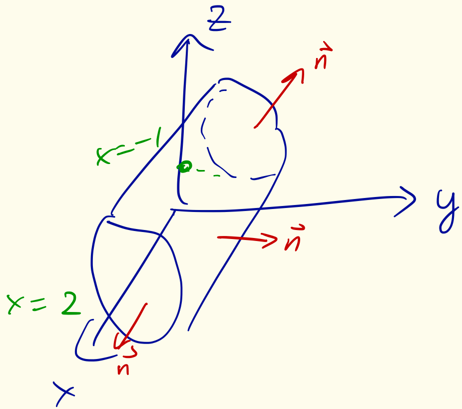
$$= 3 \iiint_E dV$$

$$= 3 \, \text{Volume}(E) = 4\pi$$

$$= 3 \left(\frac{4}{3} \pi (1)^3 \right)$$

Ex $\vec{F} = \langle 3xy^2, xe^z, z^3 \rangle$ $\qquad S = \begin{cases} y^2+z^2=1 \\ x=-1 \\ x=2 \end{cases}$

Find $\iint_S \vec{F} \cdot d\vec{S}$



Div. Thm

$$\iiint (\nabla \cdot \vec{F}) \cdot dV$$

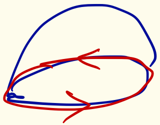
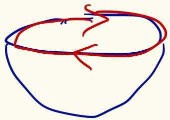
$$= \iiint 3y^2 + 0 + 3z^2 \, dV$$

$$= \int_{-1}^{2} \int_0^{2\pi} \int_0^1 (3r^2)\, r\,dr\,d\theta\,dx = 3(2\pi)\frac{3}{4} = \frac{9}{2}\pi.$$

Rem

$$0 \overset{\text{stokes'}}{=} \iint_S \nabla \times \vec{F} \cdot d\vec{S} \overset{\text{div Thm}}{=} \iiint_E \overbrace{\nabla \cdot (\nabla \times \vec{F})}^{0 \,=} \, dV$$

closed

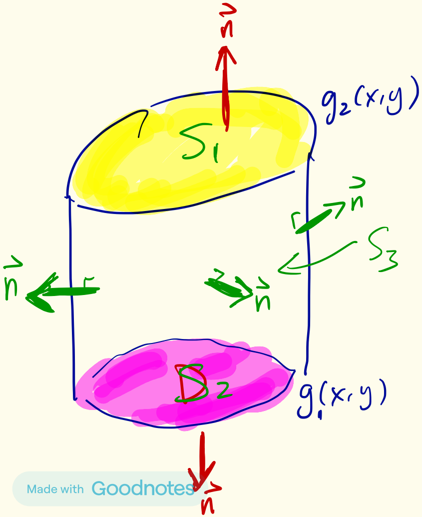
$S =$  $+$ 

Proof

$$\vec{F} = \langle P, Q, R \rangle = P\vec{i} + Q\vec{j} + R\vec{k}$$

$$LHS = \iiint_E \nabla \cdot \vec{F}\, dV = \iiint_E \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\, dV$$

$$RHS = \iint_S \vec{F} \cdot \underbrace{d\vec{S}}_{\vec{n}\, dS} = \iint_S P(\vec{i} \cdot \vec{n}) + Q(\vec{j} \cdot \vec{n}) + R(\vec{k} \cdot \vec{n})\, dS$$



$$\iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D \int_{g_1(x,y)}^{g_2(x,y)} \frac{\partial R}{\partial z}\, dz\, dA$$

$$\overset{\text{Fund. Thm.}}{=} \iint_D R(x,y,g_2(x,y)) - R(x,y,g_1(x,y))\, dA$$

$$RHS: \iint_S R(\vec{k} \cdot \vec{n})\, dS \quad : \quad \iint_{S_3} = 0 \quad (\vec{k} \cdot \vec{n} = 0)$$

$$\iint_{S_3} R(\vec{k}\cdot\vec{n})\, dS = 0 \qquad (\vec{k}\cdot\vec{n} = 0)$$

$$\iint_{S_1} R\,\vec{k}\cdot d\vec{S}$$

$S_1 : \langle x, y, g_2(x,y) \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = \langle -g_x, -g_y, 1 \rangle$$

$$\vec{n} = \langle -g_x, -g_y, 1 \rangle$$

$(0,0,1)$

$$\iint_D R \cdot \vec{k}\cdot\vec{n}\; dx\,dy$$

$$= \iint_D R(x, y, g_2(x,y))\, dA$$

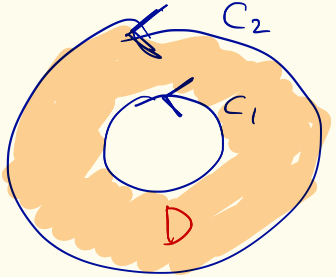
$$\iint_{S_2} R\,\vec{k}\cdot d\vec{S}$$

$$= \dots$$

$$= -\iint_D R(x, y, g_1(x,y))\, dA$$

Green's Thm

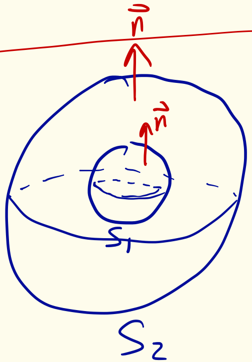
$$\vec F = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle, \quad \nabla\times\vec F = \vec 0$$



$$\iint_D (\nabla\times\vec F)\cdot\vec k\, dA = \oint_{C_2} \vec F\cdot d\vec r - \oint_{C_1} \vec F\cdot d\vec r$$

$$\iint_D (\nabla\times\vec F)\cdot\vec k\, dA = 0$$

$$\Longleftrightarrow \oint_{C_1} \vec F\cdot d\vec r = \oint_{C_2} \vec F\cdot d\vec r \quad (= 2\pi)$$



$$\vec F = \frac{\vec r}{|\vec r|^3} \qquad \vec r = \langle x, y, z\rangle \qquad \nabla\cdot\vec F = 0$$

$$0 = \iiint_E \nabla\cdot\vec F = \iint_{S_2} \vec F\cdot d\vec S - \iint_{S_1} \vec F\cdot d\vec S$$

E ← solid between S_1 & S_2

$$\Longleftrightarrow \iint_{S_1} \vec F\cdot d\vec S = \iint_{S_2} \vec F\cdot d\vec S$$

<u>Ex</u> Electric field : $\vec{E} = \varepsilon Q \dfrac{\vec{r}}{|\vec{r}|^3}$ for point charge.

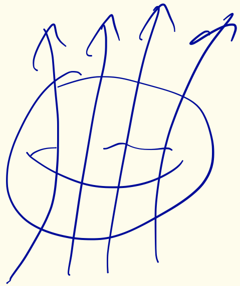
$\uparrow$ (charge)

$$\iint_S \vec{E} \cdot d\vec{S}$$ is the same for all closed surface containing $(0,0,0)$

$$= 4\pi \varepsilon Q$$ Gauss' Law.

<u>Ex</u> $\vec{B}$ magnetic field.

$$\nabla \cdot \vec{B} = 0 \xRightarrow{\text{div Thm}} \iint_S \vec{B} \cdot d\vec{S} = 0$$



"no net magnetic flux"

Made with Goodnotes

Heat Equation

T temperature.

Heat flow $\vec{F} = -k \nabla T$

thermal conductivity.

Energy $\iiint_E \rho T \, dV$

Conservation of Energy

$$\frac{d}{dt} \iiint_E \rho T \, dV = - \iint_S \vec{F} \cdot d\vec{S}$$

$$= - \iiint_E \nabla \cdot \vec{F} \, dV$$

$\nabla^2 = \nabla \cdot \nabla$

$$\frac{\partial T}{\partial t} = - c \nabla^2 T$$

Summary

Fund. Thm of Line Integral $\int_C \nabla \vec{F}(\vec{r}(t))\,dt = \vec{F}(\vec{r}(b)) - \vec{F}(\vec{r}(a))$

Cauchy
~~Green's~~
~~Stokes'~~
Lord Kelvin
$$\iint_S (\nabla \times F) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

Divergence
(Gauss).
$$\iiint (\nabla \cdot \vec{F})\, dV = \iint_S \vec{F} \cdot d\vec{S}$$

Side Quest Suffix Notation

Hidden Boss:

Generalized ~~Stokes'~~ Thm
(Cartan)
$$\int_S dw = \int_{\partial S} w$$

Summary

Fund. Thm of Line Integral $\displaystyle\int_C \nabla \vec{F}(\vec{r}(t))\,dt = \vec{F}(\vec{r}(b)) - \vec{F}(\vec{r}(a))$

Cauchy

~~Green's~~

~~Stokes'~~ $\displaystyle\iint_S (\nabla \times F)\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$

Lord Kelvin

Divergence $\displaystyle\iiint (\nabla \cdot \vec{F})\,dV = \iint_S \vec{F}\cdot d\vec{S}$

(Gauss).

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Generalized ~~Stokes'~~ Thm $\displaystyle\int_S dw = \int_{\partial S} w$

(Cartan)