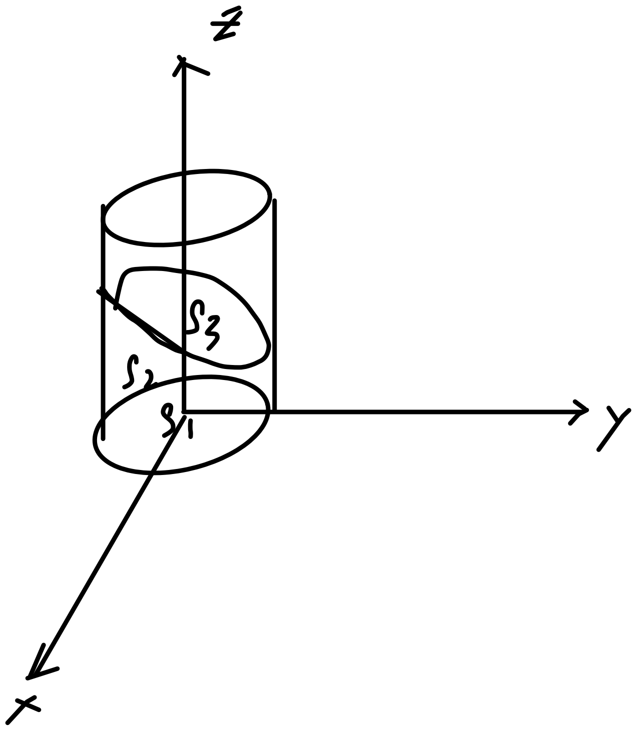
MATH 2023 – Multivariable Calculus

Lecture #18 Worksheet   April 16, 2019

Problem 1. Find the surface integral

$$\iint_S z\,dS$$

where S is the surface of the solid bounded by the cylinder $x^2 + y^2 = 1$, the disk $x^2 + y^2 \leq 1$ and under the plane $z = x + 1$.



S_1: $\vec{n}$: $\langle 0, 0, -1 \rangle$, $S_1 = 0$.

S_2: $r(u,v) = \langle \cos u, \sin u, v \rangle$

$r(u,v) = \langle \cos u, \sin u, v \rangle$

$r_u = \langle -\sin u, \cos u, 0 \rangle$

$r_v = \langle 0, 0, 1 \rangle$

$r_u \times r_v = \langle \cos u, \sin u, 0 \rangle$

$$\iint z \, dA$$

$$= \int_0^{2\pi} \int_0^{\cos u + 1} v \, dv \, du$$

$$= \int_0^{2\pi} \frac{(\cos u + 1)^2}{2} \, du.$$

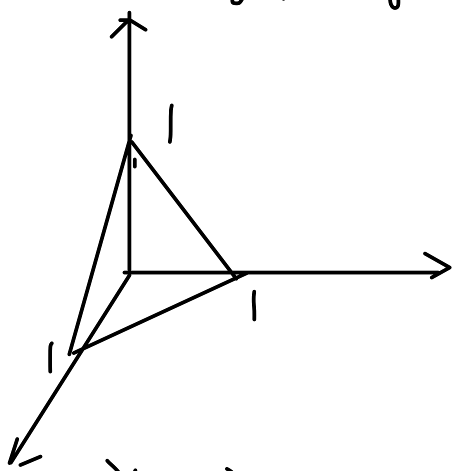
For S_3, $\vec{}_3 = \langle -1, 0, 1 \rangle$

$r(u,v) = \langle u, v, u+1 \rangle$

$$\int_{-1}^{1} \int (u+1)(\sqrt{2}) \, du \, dv$$

Problem 2. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = y\mathbf{i} + (z-y)\mathbf{j} + x\mathbf{k}$ and S is the surface of the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

$z = 1 - x - y$.



$\vec{n} = \langle 1, 1, 1 \rangle$

$y + z - y + x \quad dA$.

$\iint x + z \; dA$

$\vec{r}(u, v) = \langle u, v, 1 - u - v \rangle$

$\iint u + 1 - u - v \; dA$

$\int_0^1 \int_0^{1-v} 1 - v \; du\,dv$

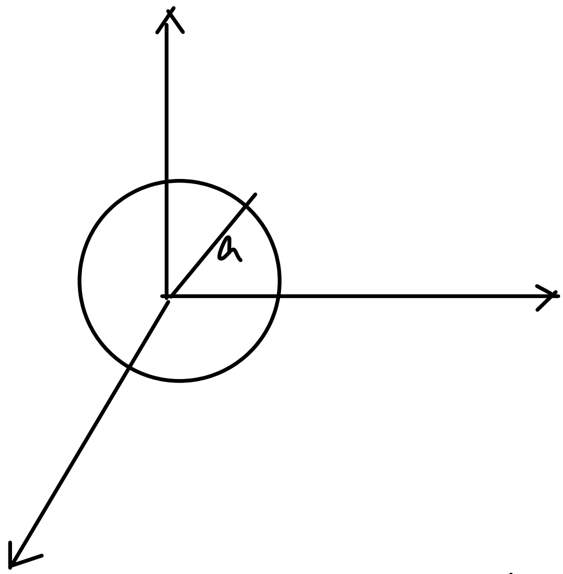
$= \int_0^1 (1-v)(1-v)\, dv$

$= \int_0^1 1 - 2v + v^2 \, dv$

$= \left[v - v^2 + \frac{v^3}{3} \right]_0^1$

$= \frac{1}{3}$.

Problem 3. Let $\mathbf{G} = \dfrac{\mathbf{r}}{|\mathbf{r}|^3}$ be the gravitational field, where $\mathbf{r} = \langle x, y, z \rangle$.
Show that the flux of $\mathbf{G}$ across a sphere S with center at the origin is indepndent of the radius of S.



$$\vec{r}(u,v) = \langle a\cos u \cos v, a\sin u \cos v, a\sin v \rangle$$

$$r_u = \langle -a\sin u \cos v, a\cos u \cos v, 0 \rangle$$

$$r_v = \langle -a\cos u \sin v, -a\sin u \sin v, a\cos v \rangle$$

$$r_u \times r_v = \langle a^2 \cos u \cos^2 v, -a^2 \sin u \cos^2 v, a^2 \sin v \cos v \rangle$$

$$\iint \frac{r}{|r|^3} \cdot r_u \times r_v \, dA$$

$$= \frac{x}{|\vec{r}|^3}(a^2 \cos u \cos^2 v) - \frac{y}{|\vec{r}|^3} \cdot a^2 \sin u \cos^2 v + \frac{z}{|\vec{r}|^3} \cdot a^2 \sin u \cos v$$

$$\left(\frac{1}{|\vec{r}|^3}\right) \frac{a^3 \cos^2 u \cos^3 v - a^3 \sin^2 u \cos^3 v + a^3 \sin u \cos v \sin v}{a^3}.$$