Chapter 15

Vector Fields

Contents

Review

$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$

— vector-valued functions of a single (scalar) variable, that is, curves.

$z = f(x_1, x_2, \ldots, x_n) = f(\mathbf{r})$

— scalar valued functions of a vector variable $\mathbf{r}$, (that is, functions of several real variables). This is a scalar field.

In the next two chapters, we will look at vector-valued function $\mathbf{F}$ of a vector variable $\mathbf{r}$, i.e. $\mathbf{F}(\mathbf{r})$.

15.1 Vector Fields

Definition: A **vector field** is a function that associates a unique vector $\mathbf{F}(P)$ with each point P in a region of 2D or 3D, i.e.

$$\mathbf{F}(x, y) = F_1(x, y)\,\mathbf{i} + F_2(x, y)\,\mathbf{j} \qquad \text{(2D)}$$

or $$\mathbf{F}(\mathbf{r}) = F_1(\mathbf{r})\,\mathbf{i} + F_2(\mathbf{r})\,\mathbf{j}, \quad \text{where the position vector} \quad \mathbf{r} = (x, y).$$

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\,\mathbf{i} + F_2(x, y, z)\,\mathbf{j} + F_3(x, y, z)\,\mathbf{k} \qquad \text{(3D)}$$

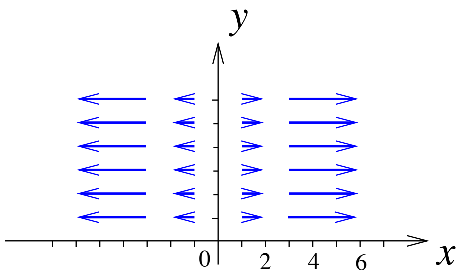
or $$\mathbf{F}(\mathbf{r}) = F_1(\mathbf{r})\,\mathbf{i} + F_2(\mathbf{r})\,\mathbf{j} + F_3(\mathbf{r})\,\mathbf{k}, \quad \text{where the position vector} \quad \mathbf{r} = (x, y, z).$$

Note that the components of a vector field are scalar fields.

A vector field is smooth when its component scalar fields have continuous partial derivatives of all orders. (For most purposes, however, second order would be sufficient.)

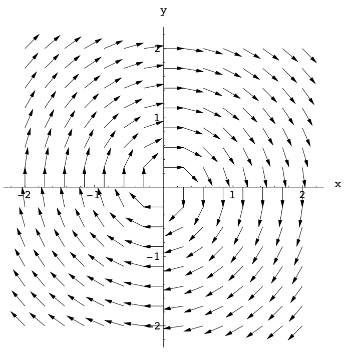
Ex. 1.1 $\mathbf{F}(x,y) = x\,\mathbf{i}$.

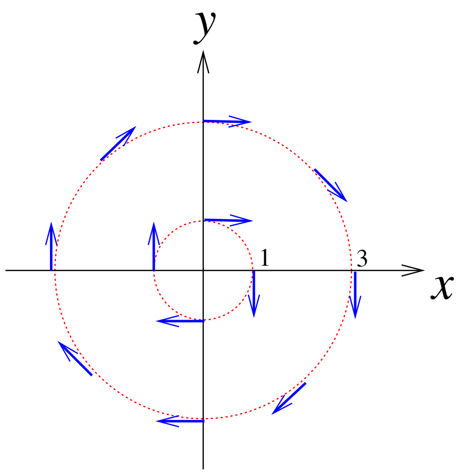
This is a 2D vector field and it is independent of y. Observe how the lengths of the vectors indicate the strength of the vector field $\|\mathbf{F}\| = \sqrt{x^2} = |x|$.



Ex. 1.2 $\mathbf{F}(x,y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2+y^2}}$

All the vectors $\mathbf{F}(x,y)$ are unit vectors tangent to circles centered at the origin with radius $\sqrt{x^2+y^2}$.

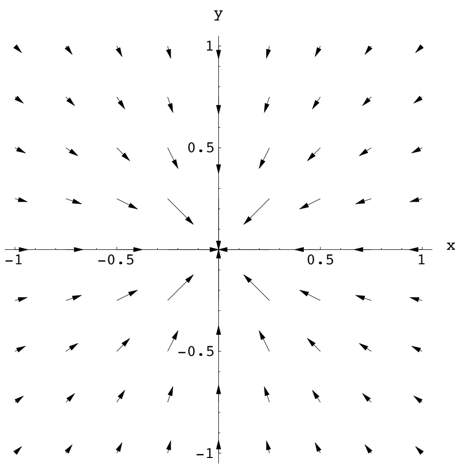




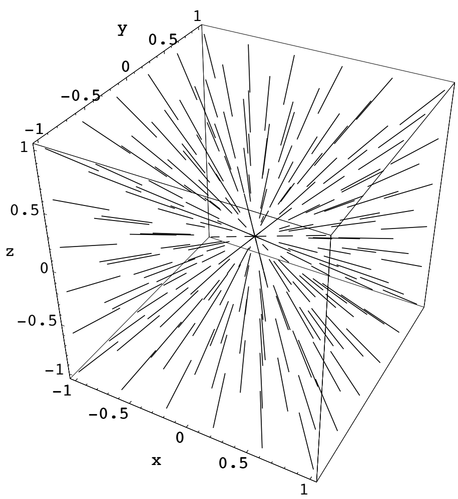
Ex. 1.3 The gravitational field of a point mass at the origin.

$$\mathbf{F}(\mathbf{r}) = -k\frac{m}{r^2}\,\widehat{\mathbf{r}},$$

where k is a constant and m is the mass.

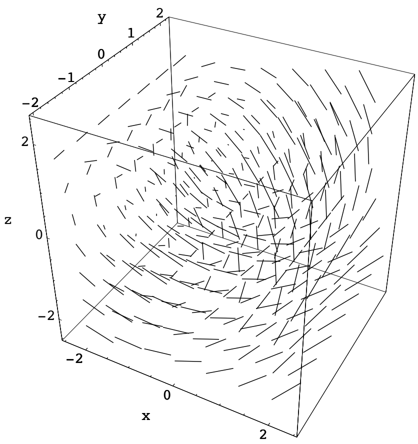


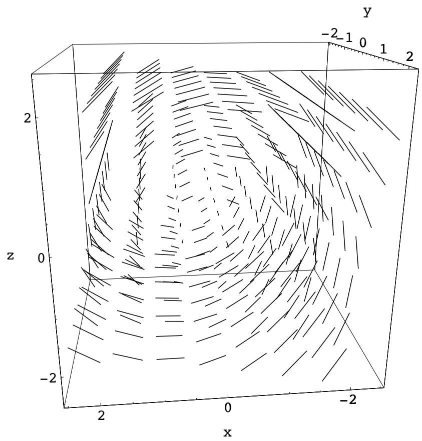
2D field



3D field

Ex. 1.4 Draw a 3D vector field : $\mathbf{F}(x, y, z) = -z\,\mathbf{i} + \mathbf{j} + x\,\mathbf{k}$.





If $f(x, y, z)$ is a scalar function of three variables, its gradient ∇f is defined by

$$\text{grad}\, f = \nabla f = f_x\,\mathbf{i} + f_y\,\mathbf{j} + f_z\,\mathbf{k}.$$

Therefore ∇f is called a *gradient vector field*.

Gradient of a scalar field f

Let $f = f(x, y, z)$, then

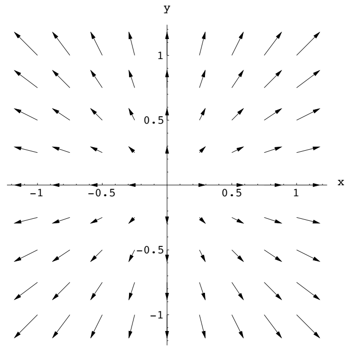
$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \nabla f \cdot d\mathbf{r} \qquad\qquad \text{where } \mathbf{r} = (x, y, z) \\ &= \nabla f \cdot \widehat{\mathbf{n}}\, ds \qquad\qquad \text{where } \frac{d\mathbf{r}}{ds} = \widehat{\mathbf{n}}, \end{aligned}$$

$\widehat{\mathbf{n}}$ is the unit normal to the level surface and s is a distance measured along the normal.

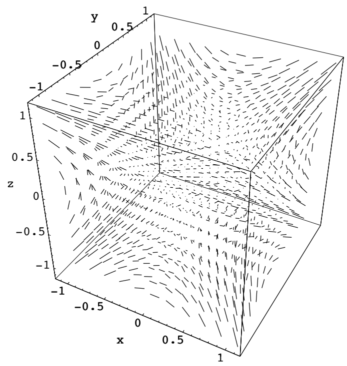
$$\frac{df}{ds} = \nabla f \cdot \widehat{\mathbf{n}} = \|\nabla f\| \qquad (\because \nabla f \parallel \widehat{\mathbf{n}}).$$

Hence the magnitude of ∇f is the rate of change of f with position along the normal, and points in the direction of the maximum upward gradient.

Ex. 1.5 If $f(x,y) = x^2 + y^2$, then $\nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

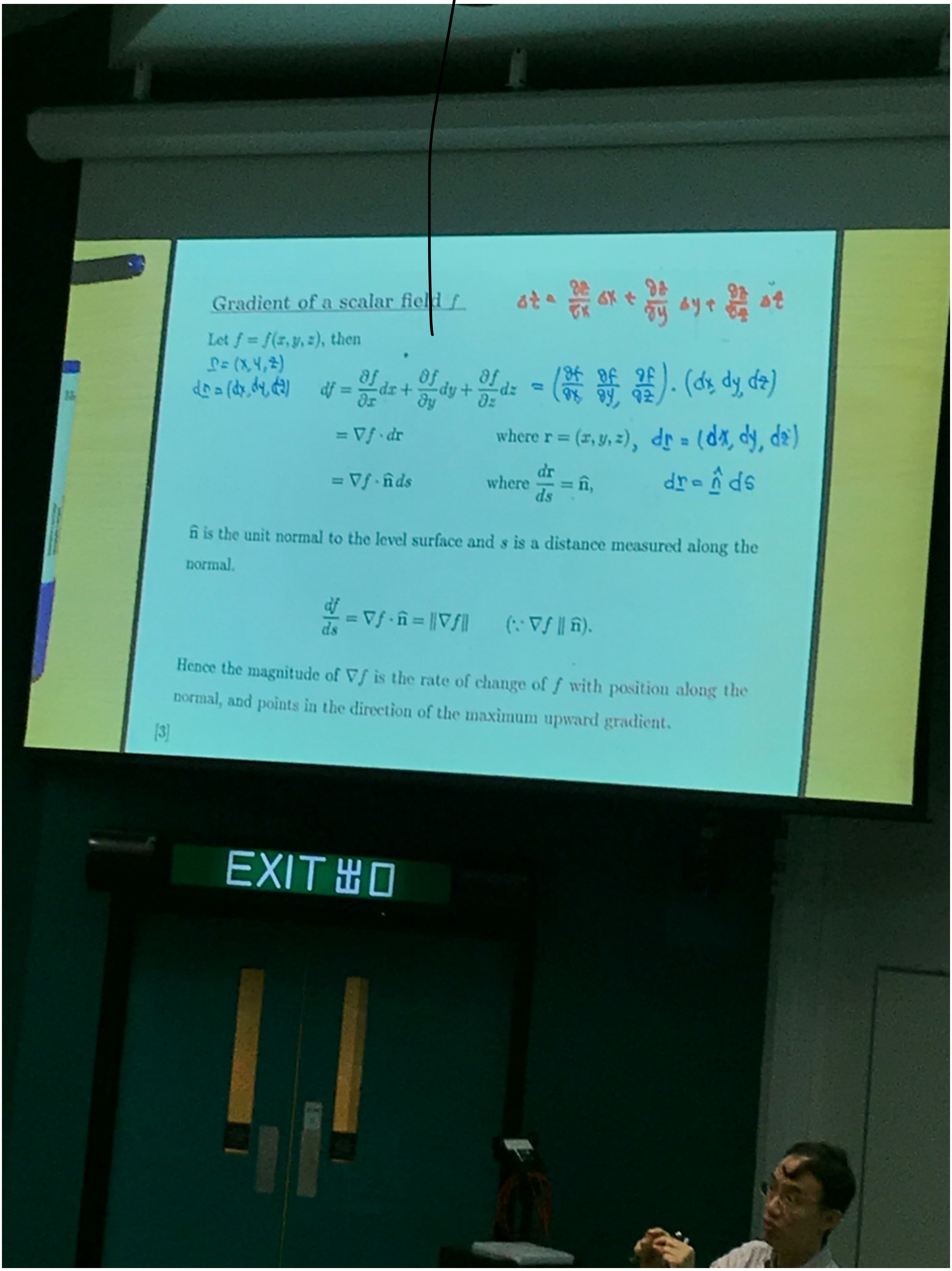


Ex. 1.6 If $f(x,y,z) = xyz$, then $\nabla f = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$.



Note that the concept of gradient applies only to scalar field. We now consider the more complicated problem of describing the rate of change of a vector field. There are two fundamental measures of the change of a vector field: the divergence and the curl.

total differential



Direction: ∇f is normal to the level set of

$f(x,y,z) = C$

<u>Magnitude:</u>

$W = f(x,y,z) = x^2+y^2+z$ (Temp)

$f(1,1,1) = 1+1+1 = 3$

$f(1.1, 1.2, 0.9)$

$d\underline{r} = (0.1, 0.2, -0.1)$

$df \sim 2.5$

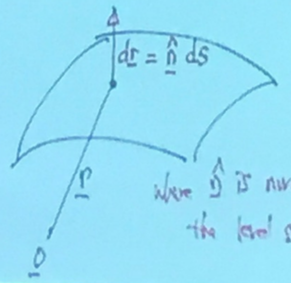
$d\underline{r}$

Temp $= 3$

$$
\begin{aligned}
df &= 2x \cdot dx + 2y\,dy + dz \\
&= 2(1)(0.1) + 2(1)0.2 - 0.1 \\
&= 0.2 + 0.4 - 0.1 \\
&= 0.5
\end{aligned}
$$

$d\underline{r} = \underline{\hat{n}}\,ds$

$W = f(x,y,z) = c$ level set



where $\underline{\hat{n}}$ is normal to the level set.

$Z = f(x,y,z)$ scalar function

$\nabla f = (f_x, f_y, f_z)$ — vector

Direction and mag.

Direction: ∇f is normal to the level set of

$f(x,y,z) = C$

Magnitude: $Z = f(x,y,z) = x^2 + y^2 + z$ (Temp)

$f(1,1,1) = 1+1+1 = 3$

$ds = (0.1, 0.2, -0.1)$ $f(1.1, 1.2, 0.9)$

Temp $= 3$ $\frac{df}{ds} \sim 2.5$

$df = 2x \cdot dx + 2y\,dy + dz$

$= 2(1)(0.1) + 2(1)0.2 - 0.1$

$= 0.2 + 0.4 - 0.1$

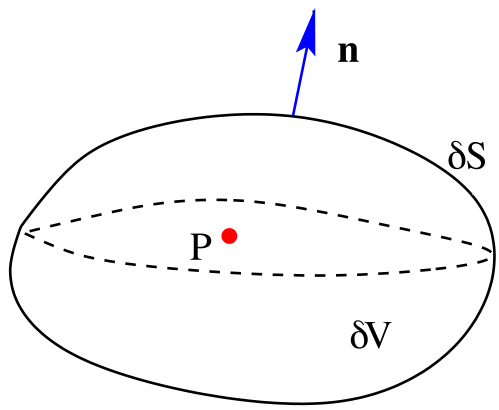
$= 0.5$

Divergence of a vector field

The divergence at any point P is defined as the limit (as the size of the region tends to zero) of the flux of $\mathbf{F}$ out of some small volume δV (has surface δS and outward normal $\widehat{\mathbf{n}}$) surrounding P, divided by δV. Thus

$$\nabla \cdot \mathbf{F} = \lim_{\delta V \to 0} \frac{1}{\delta V} \oiint_{\delta S} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$$

Hence the integration extends over closed surface surrounding the small volume. This can be written in terms of the differential operator $\nabla \cdot \mathbf{F}$.



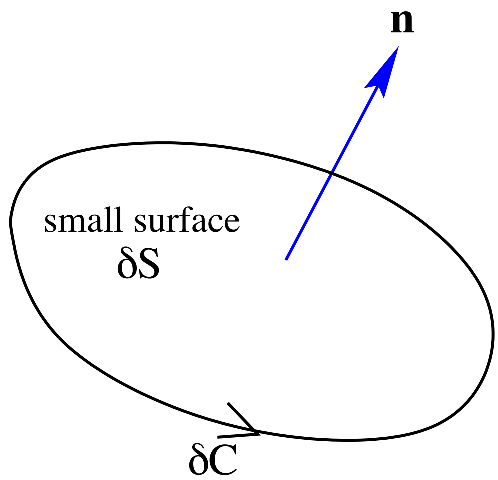
Curl of a vector field

The curl of a vector field $\mathbf{F}$ is a vector field. Its component in the direction of the unit vector $\mathbf{n}$ is

$$\widehat{\mathbf{n}} \cdot \nabla \times \mathbf{F} = \lim_{\delta S \to 0} \frac{1}{\delta S} \oint_{\delta C} \mathbf{F} \cdot d\mathbf{r}$$

where δS is a small surface element perpendicular to $\mathbf{n}$, δS is the closed curve forming the boundary of δS and δC and $\mathbf{n}$ are oriented in a right-handed sense.

The small surface δS is enclosed by the curve δC and has unit normal vector $\widehat{\mathbf{n}}$.



Definition: If $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\,\mathbf{i} + g(\mathbf{r})\,\mathbf{j} + h(\mathbf{r})\,\mathbf{k}$, then we define the **divergence** of $\mathbf{F}$, written $\operatorname{div}\mathbf{F}$ by

$$\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}, \quad \text{where} \quad \nabla = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}$$

scalar

and the curl of $\mathbf{F}$, written $\operatorname{curl}\mathbf{F}$, by

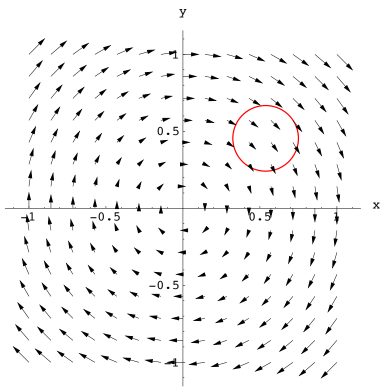
$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathbf{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\mathbf{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathbf{k}.$$

vector

Roughly speaking, the divergence of a vector field is a *scalar field* that tells us, at each point, the extent to which the field "diverges" or "spreads away" from that point. The curl of a vector field is a *vector field* that gives us at each point, an indication of how the field swirls in the vicinity of that point. However, it is possible for a field to have a positive divergence without appearing to "diverge" at all, and it is possible for field to have a nontrivial curl and yet have flow lines that do not bend at all.

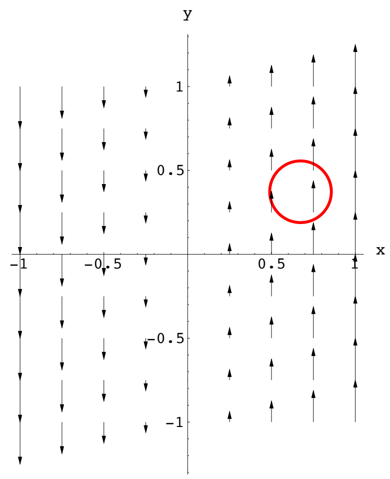
Ex. 1.7 $\mathbf{F}(\mathbf{r}) = y\,\mathbf{i} - x\,\mathbf{j}$

Note $\nabla\cdot(x\,\mathbf{j}) = 0$ and $\nabla\times(x\,\mathbf{j}) = -2\,\mathbf{k}$.



Ex. 1.8 $\mathbf{F}(\mathbf{r}) = x\,\mathbf{j}$

Note $\nabla\cdot(x\,\mathbf{j}) = 0$ and $\nabla\times(x\,\mathbf{j}) = \mathbf{k}$.



Ex. 1.9 Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$, where a, b and c are constants, show that

(a) $\nabla\cdot \mathbf{r} = 3$,

(b) $\nabla \times \mathbf{r} = \mathbf{0}$,

(c) $\nabla\cdot (\mathbf{u} \times \mathbf{r}) = 0$,

(d) $\nabla \times (\mathbf{u} \times \mathbf{r}) = 2\mathbf{u}$.

(a) $$\nabla\cdot \mathbf{r} = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right)\cdot (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

(b) $$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$$

(c) $$\mathbf{u} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\,\mathbf{i} - (az - cx)\,\mathbf{j} + (ay - bx)\,\mathbf{k}.$$

$$\therefore\ \nabla\cdot (\mathbf{u} \times \mathbf{r}) = \frac{\partial}{\partial x}(bz - cy) - \frac{\partial}{\partial y}(az - cx) + \frac{\partial}{\partial z}(ay - bx) = 0.$$

(d) $$\nabla \times (\mathbf{u} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ bz - cy & -az + cx & ay - bx \end{vmatrix} = 2(a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}) = 2\mathbf{u}.$$

Suffix Notation:

(a) $\nabla \cdot \mathbf{r} = \partial_i r_i = 3$ where $\partial_i = \dfrac{\partial}{\partial x_i}$ $i = 1,\ 2$ or 3 (ith component of ∇)

(b) $(\nabla\times\mathbf{r})_i = \varepsilon_{ijk}\,\partial_j r_k$, $\partial_j r_k$ would be non-zero if $j = k$, but when $j = k$, $\varepsilon_{ijk} = 0 \Rightarrow (\nabla\times\mathbf{r})_i = 0$.

$\therefore\ \nabla \times \mathbf{r} = \mathbf{0}$.

(c) $\nabla \cdot (\mathbf{u} \times \mathbf{r}) = \partial_i(\mathbf{u} \times \mathbf{r})_i = \partial_i\,\varepsilon_{ijk}u_j r_k = \varepsilon_{ijk}u_j\partial_i r_k = 0$.

(d) $[\nabla \times (\mathbf{u} \times \mathbf{r})]_i =$

Some identities involving Grad, Div and Curl

Let f be a scalar field and $\mathbf{F}(\mathbf{r}) = F_1(\mathbf{r})\,\mathbf{i} + F_2(\mathbf{r})\,\mathbf{j} + F_3(\mathbf{r})\,\mathbf{k}$ be a vector field, then

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k} \qquad \text{(vector field)}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial x} + \frac{\partial F_3}{\partial z} \qquad \text{(scalar field)}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \qquad \text{(vector field)}$$

Definition: Laplacian Operator

$$\begin{aligned} \nabla^2 = \nabla \cdot \nabla &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \\ &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \end{aligned}$$

∇^2 is a scalar differential operator. Note that

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 \mathbf{F} = \nabla^2 F_1\,\mathbf{i} + \nabla^2 F_2\,\mathbf{j} + \nabla^2 F_3\,\mathbf{k}.$$

Vector differential identities

Let ϕ, ψ are scalar fields and $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

(a) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(b) $\nabla \cdot (\phi\mathbf{F}) = \nabla\phi \cdot \mathbf{F} + \phi(\nabla \cdot \mathbf{F})$

(c) $\nabla \times (\phi\mathbf{F}) = \nabla\phi \times \mathbf{F} + \phi(\nabla \times \mathbf{F})$

(d) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

(e) $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

(f) $\nabla \times (\nabla\phi) = \mathbf{0}$

(g) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$

Proof:

(a)

$$[\nabla(\phi\psi)]_i = \partial_i(\phi\psi) = (\partial_i\phi)\psi + \phi(\partial_i\psi)$$
$$\text{i.e.}\quad \nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi.$$

(d)

$$\begin{aligned}\nabla\cdot(\mathbf{F}\times\mathbf{G}) &= \partial_i(\mathbf{F}\times\mathbf{G})_i\\ &= \partial_i\,\varepsilon_{ijk}F_jG_k\\ &= \varepsilon_{ijk}(\partial_iF_j)G_k + \varepsilon_{ijk}F_j(\partial_iG_k)\\ &= \varepsilon_{kij}(\partial_iF_j)G_k + \varepsilon_{jki}F_j(\partial_iG_k)\\ &= (\nabla\times\mathbf{F})_kG_k + F_j(-\nabla\times G)_j\\ &= (\nabla\times\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot(\nabla\times\mathbf{G})\end{aligned}$$

(g)

$$\begin{aligned}[\nabla\times(\nabla\times\mathbf{F})]_i &= \varepsilon_{ijk}\partial_j(\nabla\times\mathbf{F})_k\\ &= \varepsilon_{kij}\partial_j\ \varepsilon_{kpq}\partial_pF_q\\ &= (\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp})\,\partial_j\partial_pF_q\\ &= \partial_j\partial_iF_j - \partial_j\partial_jF_i\\ &= \partial_i(\nabla\cdot\mathbf{F}) - \nabla^2F_i\\ \nabla\times(\nabla\times\mathbf{F}) &= \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}\end{aligned}$$

Ex. 1.10 Verify the identity

$$\nabla\cdot(f(\nabla g\times\nabla h)) = \nabla f\cdot(\nabla g\times\nabla h)$$

for smooth scalar fields f, g and h.

Field lines

If the velocity of the particle (with position vector: $\mathbf{r}(t)$) is given by the field, then

$$\frac{d\mathbf{r}}{dt} = \mathbf{F}(\mathbf{r}).$$

The path of the particle will be a curve to which the field is tangent at every point. Such curves are called **field lines**. If we break the equation into components, then

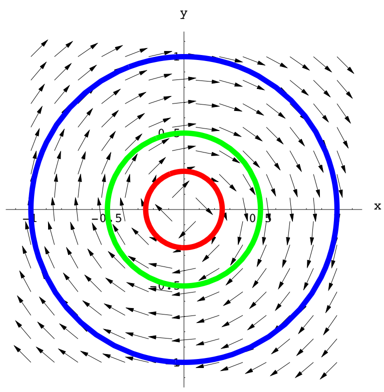
$$\frac{dx}{dt} = F_1(\mathbf{r}), \qquad \frac{dy}{dt} = F_2(\mathbf{r}), \qquad \frac{dz}{dt} = F_3(\mathbf{r}).$$

$\therefore$ The differential equation for the field lines is

$$\frac{dx}{F_1(\mathbf{r})} = \frac{dy}{F_2(\mathbf{r})} = \frac{dz}{F_3(\mathbf{r})}.$$

Note that the field lines of $\mathbf{F}$ do not depend on the magnitude of $\mathbf{F}$ at any point, but only on the direction of the field.

Ex. 1.11 Find the field lines of the velocity field $\mathbf{F}(x, y) = \dfrac{y\,\mathbf{i} - x\,\mathbf{j}}{\sqrt{x^2 + y^2}}$.



$$\begin{aligned}
\frac{dx}{y} &= \frac{dy}{-x} \\
xdx &= -ydy \\
\frac{x^2}{2} &= -\frac{y^2}{2} + \frac{c}{2} \\
x^2 + y^2 &= c.
\end{aligned}$$

$\therefore$ The field lines are circles centred at the origin in xy-plane.

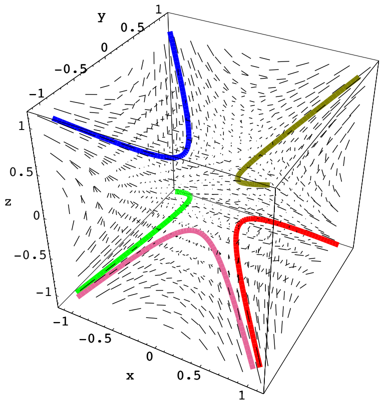
Ex. 1.12 Find the field lines of the velocity field $\mathbf{F}(x,y,z) = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$.

$$\frac{dx}{yz} = \frac{dy}{xz} = \frac{dz}{xy}$$

i.e. $$\frac{dx}{y} = \frac{dy}{x} \quad \Rightarrow \quad x\;dx = y\,dy \quad \Rightarrow \quad x^2 - y^2 = C_1$$

and

$$\frac{dy}{z} = \frac{dz}{y} \quad \rightarrow \quad y\,dy = z\,dz \quad \Rightarrow \quad y^2 - z^2 = C_2$$



Therefore the field lines have parametric equation

$$x = \sqrt{C_1 + t^2}$$
$$y = t$$
$$z = \sqrt{t^2 - C_2}$$

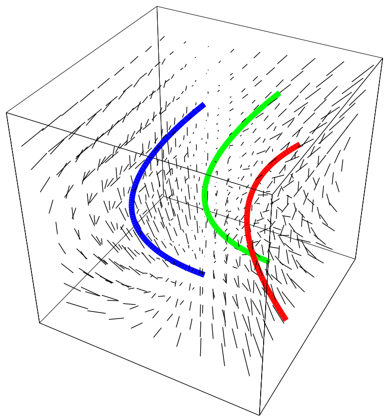
Ex. 1.13 Find the field lines of the velocity field $\mathbf{F}(x,y,z) = xz\,\mathbf{i} + yz\,\mathbf{j} + x\,\mathbf{k}$.

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{x}$$

i.e. $$\frac{dx}{x} = \frac{dy}{y} \quad \Rightarrow \quad \ln x = \ln y + C \quad \Rightarrow \quad y = C_1 x$$

and

$$\frac{dx}{z} = dz \quad \rightarrow \quad x = \frac{z^2}{2} + C_2$$



Therefore the field lines have parametric equation

$$x = \frac{t^2}{2} + C_2$$
$$y = C_1\left(\frac{t^2}{2} + C_2\right)$$
$$z = t$$

15.3 Line integrals in space

Let C be a *smooth* curve on the xy-plane with parametric equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}, \quad \text{where} \quad a = t_1 < t_2 < t_3 < \cdots < t_n = b.$$

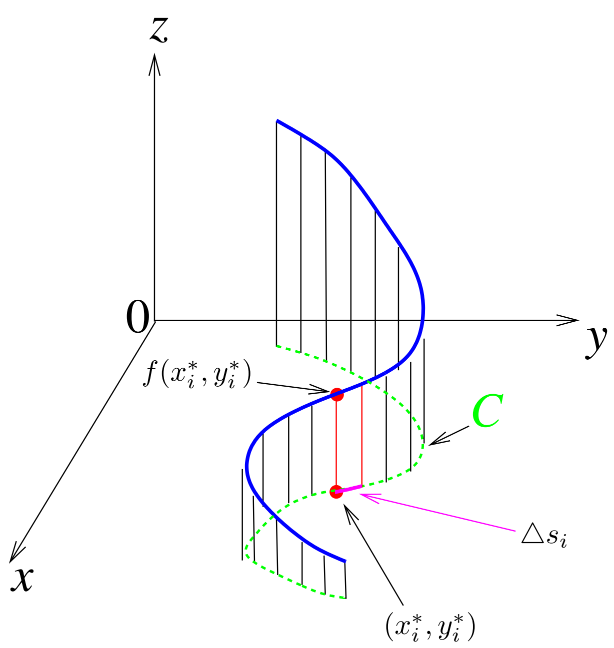
If f is any function of two variables whose domain includes C, we can evaluate f at the point (x_i^*, y_i^*), multiply by the length $\triangle s_i$ of the sub-arc and form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\triangle s_i.$$

The *line integral* of f along C is

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\triangle s_i \quad \text{as} \quad \triangle s_i \to 0,$$

if the limit exists.



It can be shown that if f is a continuous function, then the above limit always exists.

Evaluating line integral: Since $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ and let $\mathbf{r}(t) = (x(t), y(t))$, then

$$\int_C f(x,y)\,ds = \int_C f(x,y)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_C f(\mathbf{r})\,\|\mathbf{r}'(t)\|\,dt.$$

If C is a line segment from $(a,0)$ to $(b,0)$, using x as a parameter, then along C, $x = x$, $y = 0$ and $a \leqslant x \leqslant b$.

$$\int_C f(x,y)\,ds = \int_a^b f(x,0)\,dx \quad \text{(ordinary single integral).}$$

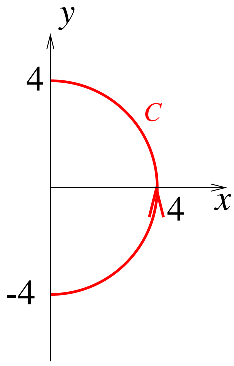
Ex. 3.1 $\int_C xy^4\,ds$, C is the right half of the circle $x^2+y^2=16$.

Let $f(\mathbf{r}) = f(x,y) = xy^4$ and

the parametric equation of the curve C is

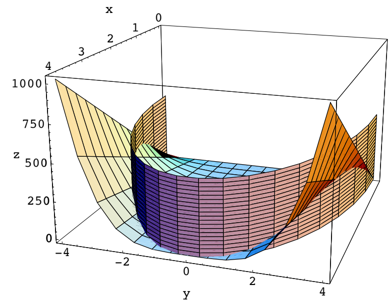
$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j}$ with $t \in [-\pi/2,\,\pi/2]$, then

$\mathbf{r}'(t) = -4\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j}$ and $\|\mathbf{r}'(t)\| = 4$,



$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} f(\mathbf{r})\,\|\mathbf{r}'(t)\|\,dt = \int_{-\pi/2}^{\pi/2} \left[4^5\cos t\sin^4 t\right](4)\,dt$$

$$= 4^6\frac{1}{5}\left[\sin^5 t\right]_{-\pi/2}^{\pi/2} = \frac{2\times 4^6}{5} = 1638.4.$$



Alternatively, if we had parameterized the curve C as

$$\mathbf{r}(t) = \sqrt{16-t^2}\,\mathbf{i} + t\,\mathbf{j} \qquad \text{where} \qquad -4 \leqslant t \leqslant 4.$$

Then

$$\mathbf{r}'(t) = -\frac{t}{\sqrt{16-t^2}}\,\mathbf{i} + \mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{\frac{16}{16-t^2}}$$

$$\therefore \quad \int_C xy^4 ds = \int_C f(\mathbf{r})\,\|\mathbf{r}'(t)\|\,dt$$

$$= \int_{-4}^{4} \sqrt{16-t^2}\times t^4 \times \sqrt{\frac{16}{16-t^2}}\,dt$$

$$= 4\int_{-4}^{4} t^4\,dt$$

$$= 4\left[\frac{t^5}{5}\right]\Bigg|_{-4}^{4} = \frac{2\times 4^6}{5}$$

Note that the line integral is *independent of parametrization* of the curve C.

Two other line integrals are obtained by replacing $\triangle s_i$ by $\triangle x_i$ and $\triangle y_i$. They are called the *line integrals* of f along C respect to x and y:

$$\int_C f(x,y)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*,y_i^*)\,\triangle x_i, \qquad \int_C f(x,y)\,dy = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*,y_i^*)\,\triangle y_i$$
$$= \int_C f(x(t),y(t))\,x'(t)\,dt \qquad\qquad = \int_C f(x(t),y(t))\,y'(t)\,dt$$

since $dx = x'(t)dt$ and $dy = y'(t)dt$.

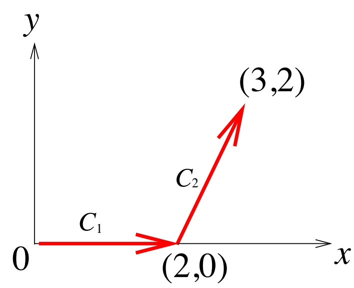
Ex. 3.2 $\displaystyle\int_C xy\,dx + (x-y)\,dy$, C consists of line segments from $(0,0)$ to $(2,0)$ and from $(2,0)$ to $(3,2)$.

Let $C = C_1 + C_2$, then

On C_1 : $x = x$, $y = 0$, $\quad 0 \leqslant x \leqslant 2$

On C_2 : $x = x$, $y = 2x-4$, $\quad 2 \leqslant x \leqslant 3$. Then

$$\int_C xy\,dx + (x-y)\,dy$$
$$= \int_{C_1} [xy\,dx + (x-y)\,dy] + \int_{C_2} [xy\,dx + (x-y)\,dy]$$
$$= \int_0^2 0\,dx + \int_2^3 (2x^2-4x)\,dx + \int_2^3 (-x+4)\,2\,dx$$
$$=$$



Orientation: $\displaystyle\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx, \qquad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$

But $\displaystyle\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds$ (independent of orientation of C).

This is because $\triangle s_i$ is always positive, whereas $\triangle x_i$, $\triangle y_i$ change sign when we reverse the orientation of C.

Let C be a *smooth* space curve with parametric equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \quad \text{where} \quad a \leqslant t \leqslant b.$$

If f is any function of three variables that is continuous on some region containing C, then

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\triangle s_i \\ &= \int_C f(x,y,z)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\ &= \int_C f(\mathbf{r})\|\mathbf{r}'(t)\|\,dt.\end{aligned}$$

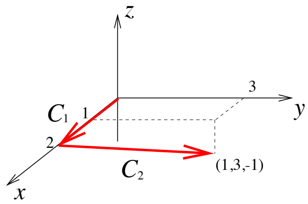
If $f(\mathbf{r}) = 1$, then the length of the curve $C = \displaystyle\int_a^b \|\mathbf{r}'(t)\|\,dt = L$.

Also line integral along C w.r.t. x, y and z can also be defined. For example

$$\int_C f(x,y,z)\,dx = \int_C f(x,y,z)x'(t)\,dt.$$

Ex. 3.3 $\mathrm{I} = \displaystyle\int_C yz\,dx + xz\,dy + xy\,dz$, C consists of line segments from $(0,0,0)$ to $(2,0,0)$, and from $(2,0,0)$ to $(1,3,-1)$.

[*Hint*: $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$, where $0 \leqslant t \leqslant 1$.]



Let $C = C_1 + C_2$, where

$C_1:\ (0,0,0)$ to $(2,0,0) \quad \Rightarrow \quad x = 2t, \quad y = z = 0, \quad$ where $\quad 0 \leqslant t \leqslant 1.$

$C_2:\ (2,0,0)$ to $(1,3,-1) \quad \Rightarrow \quad x = -t+2, \quad y = 3t, \quad z = -t, \quad$ where $\quad 0 \leqslant t \leqslant 1.$

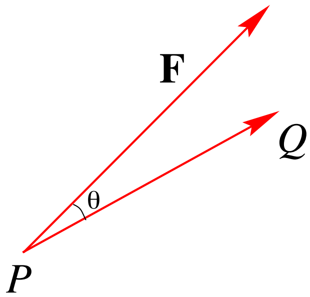
Then

$$\mathrm{I} = 0 + \int_0^1 \left[(3t^2) + 3(t^2 - 2t) - 3(2t - t^2)\right]\,dt =$$

15.4 Line integrals of vector fields

Recall that work done by a constant force f in moving a particle from a to b along x-axis is

$$W = f(b-a) \quad \text{(force} \times \text{distance).}$$

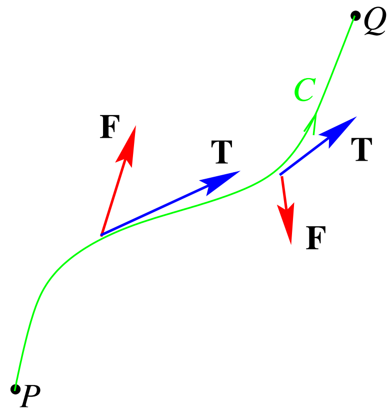


If f is a variable force, then $W = \int_a^b f(x)\,dx$.

If $\mathbf{F}$ is a constant force, moving a particle from P to Q in space, then $W = \|\mathbf{F}\| \cos\theta \cdot \|\overrightarrow{PQ}\| = \mathbf{F}\cdot \overrightarrow{PQ}$.

Now suppose that the force is a vector field, i.e. $\mathbf{F}(\mathbf{r})$, moving a particle along a curve C in space with parametric equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, where $t \in [a,b]$.

$$W = \int_C \mathbf{F}\cdot \mathbf{T}\,ds$$



This is a line integral of the *tangential component* of $\mathbf{F}$.

But $\mathbf{T} = \dfrac{d\mathbf{r}}{ds}$ (a unit vector tangent to the path),

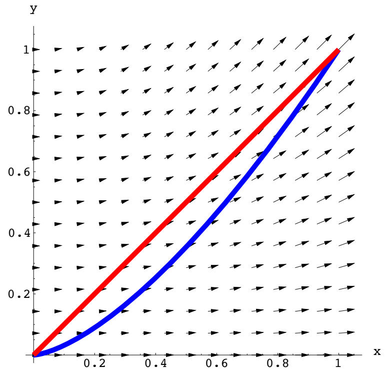
$$\therefore\; W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}\cdot \mathbf{r}'(t)\,dt. \qquad \text{(scalar)}$$

This line integral changes sign if the orientation of C is reversed, it is independent of the particular parametrization used for C.

Ex. 4.1 Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = e^{x-1}\,\mathbf{i} + xy\,\mathbf{j}$ and C is given by

(a) $\mathbf{r}(t) = t^2\,\mathbf{i} + t^3\,\mathbf{j}$; $\quad 0 \leqslant t \leqslant 1.$

(b) $\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j}$; $\quad 0 \leqslant t \leqslant 1.$



(a)
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 \mathbf{F}(t^2,t^3)\cdot\mathbf{r}'(t)\,dt\\ &= \int_0^1 \left(e^{t^2-1}\,\mathbf{i} + t^5\,\mathbf{j}\right)\cdot(2t\,\mathbf{i} + 3t^2\,\mathbf{j})\,dt\\ &= \int_0^1 \left(2te^{t^2-1} + 3t^7\right)\,dt\\ &= \left[e^{t^2-1} + \frac{3}{8}t^8\right]_0^1 = \frac{11}{8} - \frac{1}{e}\end{aligned}$$

(b)
$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 \mathbf{F}(t,t)\cdot\mathbf{r}'(t)\,dt\\ &= \int_0^1 (e^{t-1}\,\mathbf{i} + t^2\,\mathbf{j})\cdot(\mathbf{i}+\mathbf{j})\,dt\\ &= \int_0^1 (e^{t-1} + t^2)\,dt\\ &= \frac{4}{3} - \frac{1}{e}\end{aligned}$$

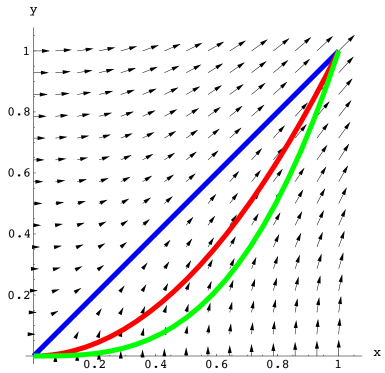
Note that the line integral depends on the path from $(0,0)$ to $(1,1)$ along which the integral is taken.

Ex. 4.2 Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = y\,\mathbf{i} + x\,\mathbf{j}$ and C is given by

(a) $\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j}$; $\quad 0 \leqslant t \leqslant 1.$

(b) $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j}$; $\quad 0 \leqslant t \leqslant 1.$

(c) $\mathbf{r}(t) = t\,\mathbf{i} + t^3\,\mathbf{j}$; $\quad 0 \leqslant t \leqslant 1.$



(a) $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (t\,\mathbf{i} + t\,\mathbf{j})\cdot(\mathbf{i}+\mathbf{j})\,dt = \int_0^1 2t\,dt = 1$

(b) $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (t^2\,\mathbf{i} + t\,\mathbf{j})\cdot(\mathbf{i}+2\,\mathbf{j})\,dt = \int_0^1 3t^2\,dt = 1$

(c) $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (t^3\,\mathbf{i} + t\,\mathbf{j})\cdot(\mathbf{i}+3t^2\,\mathbf{j})\,dt = \int_0^1 4t^3\,dt = 1$

The results in this example are not accidental; we shall soon see that the value of this line integral is the same over *all* piecewise smooth path from (0,0) to (1,1) – *independent of path.*

15.2 Conservative vector fields

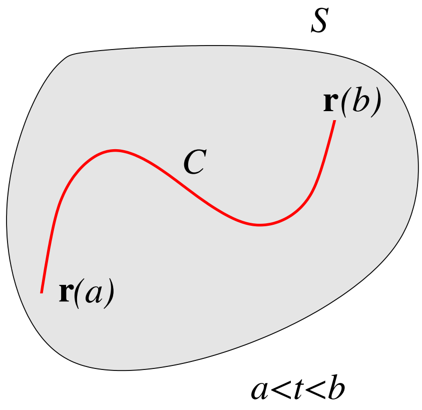
Recall that $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$, i.e. the integral only depends on the *end points*.

If $\mathbf{F}(\mathbf{r}) = f(\mathbf{r})\,\mathbf{i} + g(\mathbf{r})\,\mathbf{j} + h(\mathbf{r})\,\mathbf{k}$ is the gradient of the function $\phi(\mathbf{r})$ on S, i.e.

$$\mathbf{F}(\mathbf{r}) = \nabla\phi(\mathbf{r}) \quad \Rightarrow \quad f(\mathbf{r}) = \frac{\partial\phi}{\partial x}, \quad g(\mathbf{r}) = \frac{\partial\phi}{\partial y} \quad \text{and} \quad h(\mathbf{r}) = \frac{\partial\phi}{\partial z}.$$

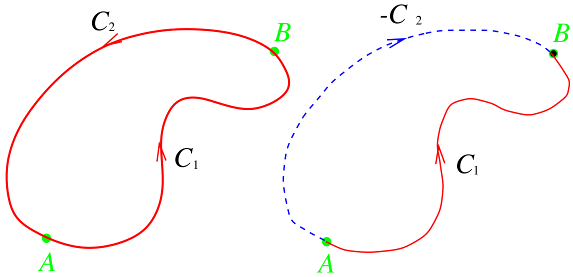
In that case, we say $\mathbf{F}(\mathbf{r})$ is a *conservative field* and ϕ is a *(scaler) potential function* of $\mathbf{F}$ on S. Then

$$\begin{aligned}\int_C \mathbf{F}(\mathbf{r})\cdot d\mathbf{r} &= \int_a^b \nabla\phi(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt \\ &= \int_a^b \left(\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\right) dt \\ &= \int_a^b \frac{d\phi}{dt}\,dt = \phi(x(b), y(b), z(b)) - \phi(x(a), y(a), z(a)) \\ &= \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)).\end{aligned}$$



This depends only on the endpoints $\mathbf{r}(b)$ and $\mathbf{r}(a)$, **not** on the curve C.

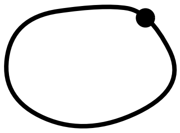
Note: $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is *independent* of path in S if and only if $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in S.



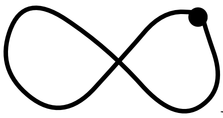
$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} \\ &= \int_{C_1} \mathbf{F}\cdot d\mathbf{r} - \int_{-C_2} \mathbf{F}\cdot d\mathbf{r} = 0,\end{aligned}$$

since C_1 and $-C_2$ have the same initial and terminal points.

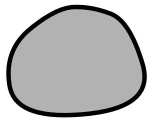
Simple - not intersect itself anywhere
between its end points



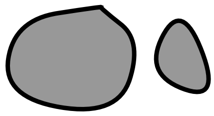
Simple



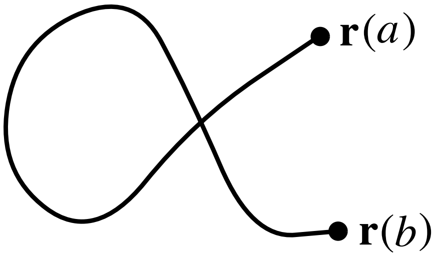
Not simple



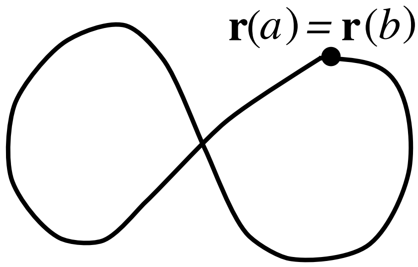
Connected



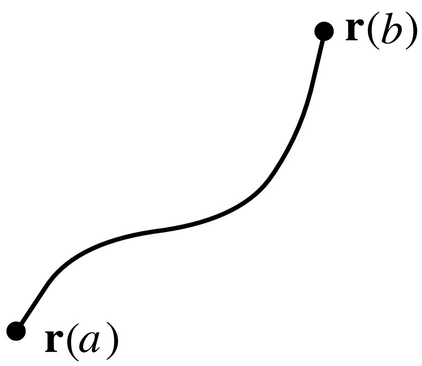
Not connected



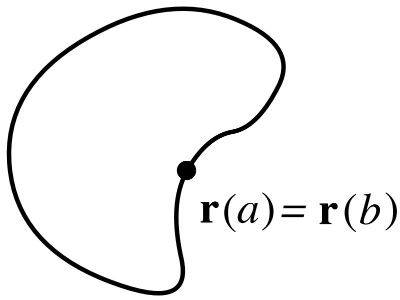
Not simple
not closed



Closed
but not simple



Simple but
not closed



Simple and
closed

Simply connected set in 2d space is connected and has no holes.

A continuously vector field $\mathbf{F}$ defined in a simply-connected domain S is conservative if, and only if, it possesses any one of the following properties.

(i) It is the gradient of a scalar function, $\mathbf{F}(\mathbf{r}) = \nabla\phi(\mathbf{r})$.

(ii) Its line integral along any regular curve extending from a point P to a point Q is independent of the path.

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \cdots$$

(iii) Its line integral around any regular closed curved is zero, i.e. $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$.

Note that **(i)** $\Rightarrow$ **(ii)** $\Rightarrow$ **(iii)**.

Converse situation: (ii) implies (i)

If $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{C_3} \mathbf{F}\cdot d\mathbf{r}\cdots$, then $\mathbf{F} = \nabla\phi$.

Proof: see the text book (Howard Anton) p941

To prove this, we need to assume that the domain S of $\mathbf{F}(\mathbf{r})$ is *open and simply-connected region.*

Theorem

Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region S. If $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in S, then $\mathbf{F}$ is a conservative vector field on S; that is, there exists a function ϕ such that $\nabla\phi = \mathbf{F}$.

From (ii), we can see that it is especially easy to evaluate the line integral $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ if $\mathbf{F}(\mathbf{r})$ is a conservative vector field. Hence the following two questions are of practical interest.

(1) How can we tell whether a given vector field $\mathbf{F}(\mathbf{r})$ is conservative in S. In other words, how can we tell whether a potential function $\phi(\mathbf{r})$ exists such that $\mathbf{F}(\mathbf{r}) = \nabla\phi(\mathbf{r})$.

(2) If a potential function $\phi(\mathbf{r})$ does exist, how can we find it?

Answer:

Exercises for students

Prove that (a) $\nabla \times (\nabla \phi) = \mathbf{0}$

(b) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(c) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

From (a), we can see that if $\mathbf{F} = \nabla \phi$, i.e. $\mathbf{F}$ is a conservative vector field on $\mathbb{R}^3$, then $\nabla \times \mathbf{F} = \mathbf{0}$. Therefore, we can add a fourth property, equivalent to any one of the other three.

(iv) $\nabla \times \mathbf{F} = \mathbf{0}$.

Ex. 2.1 Show that $\mathbf{F}(x, y, z) = y^2 z^3\, \mathbf{i} + 2xyz^3\, \mathbf{j} + 3xy^2 z^2\, \mathbf{k}$ is a conservative vector field.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix} =$$

A necessary condition for the existence of a potential function ϕ is $\nabla \times \mathbf{F} = \mathbf{0}$.

On the other hand, this condition is not sufficient; even if $\nabla \times \mathbf{F} = \mathbf{0}$ at every point in S, there may still be no potential function ϕ. In order to guarantee the existence of a potential function, we must place an additional restriction on S, namely the domain S must be open and simply connected (see example 2.3).

Necessary condition for potential function

2D: If $\mathbf{F}(x,y) = f(x,y)\,\mathbf{i} + g(x,y)\,\mathbf{j}$ (where f, g have continuous first partial derivatives on a domain S) has a potential function $\phi(x,y)$ on S, then

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad \text{on} \quad S.$$

Since $\mathbf{F}$ is conservative, so $f = \dfrac{\partial \phi}{\partial x}$, $\quad g = \dfrac{\partial \phi}{\partial y}$, $\quad$ so $\quad \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \phi}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \phi}{\partial y}\right) = \dfrac{\partial g}{\partial x}$.

i.e. $\nabla \times \mathbf{F} = \mathbf{0}$ on S.

3D: Similarly, if $\mathbf{F}(x,y,z) = f(x,y,z)\,\mathbf{i} + g(x,y,z)\,\mathbf{j} + h(x,y,z)\,\mathbf{k}$ is conservative (i.e. $\nabla\phi = \mathbf{F}$) and f, g and h have continuous first-order partial derivatives, then

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}, \qquad \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x} \qquad \text{and} \qquad \frac{\partial g}{\partial z} = \frac{\partial h}{\partial y}.$$

Since $f = \dfrac{\partial \phi}{\partial x}$, $\quad g = \dfrac{\partial \phi}{\partial y}$ $\quad$ and $\quad h = \dfrac{\partial \phi}{\partial z}$, so

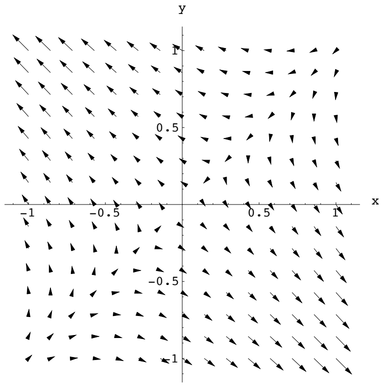
$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial g}{\partial x} \quad \text{and similarly for the other two.}$$

i.e. $\nabla \times \mathbf{F} = \mathbf{0}$ on S.

Ex. 2.2 Determine whether or not $\mathbf{F}(x,y) = (2x-3y)\,\mathbf{i} + (2y-3x)\,\mathbf{j}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

Note that

$$\nabla \times \mathbf{F} = \mathbf{0}, \quad \text{i.e.} \quad \frac{\partial}{\partial y}(2x-3y) = -3 = \frac{\partial}{\partial x}(2y-3x)$$



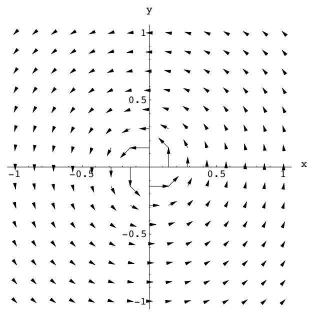
and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so $\mathbf{F}$ is conservative.

$\therefore \quad f_x = 2x - 3y \quad$ and $\quad f_y = 2y - 3x$.

$$f(x,y) = x^2 - 3xy + g(y)$$

$$f_y = -3x + g'(y) \quad \Rightarrow \quad g'(y) = 2y \quad \Rightarrow \quad g(y) = y^2 + K, \quad \text{where } K \text{ is a constant}$$

Therefore $f(x,y) = x^2 - 3xy + y^2 + K$ is the potential for $\mathbf{F}$.



Ex. 2.3 Let $\mathbf{F}(x,y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2+y^2} = f(x,y)\,\mathbf{i} + g(x,y)\,\mathbf{j}$ (see also Ex. 1.2).

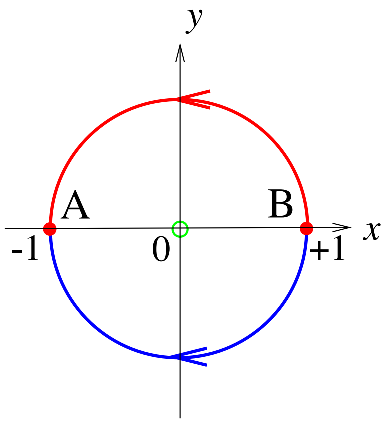
(a) Show that $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$

(b) Show that $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is not independent of path.

(a) $\dfrac{\partial f}{\partial y} = \dfrac{y^2-x^2}{(x^2+y^2)^2} = \dfrac{\partial g}{\partial x}$

(b) $C_1: \quad x = \cos\theta, \quad y = \sin\theta$

$C_2: \quad x = \cos\theta, \quad y = \sin\theta$



$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi} (-\sin\theta, \cos\theta)\cdot(-\sin\theta,\cos\theta)\,d\theta = \int_0^{\pi} d\theta = \pi$$

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{2\pi}^{\pi} d\theta = -\pi \neq \int_{C_1} \mathbf{F}\cdot d\mathbf{r}$$

$\therefore \quad \displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ is not independent of path. **Why**?

Therefore, $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$ is a **necessary condition** for potential function to exist, but not a **sufficient condition**.

Sufficient condition for potential function

If $\mathbf{F}(\mathbf{r})$ is a vector field on an **open simply-connected** (no hole) region S, and suppose that $\mathbf{F}$ satisfying the necessary condition above on S, then $\mathbf{F}$ is conservative, i.e. $\mathbf{F}$ has a potential function ϕ on S.

In **Ex. 2.3**, the set S has a "hole" at $(0,0)$. Although the necessary condition is satisfied on S, there is still no potential function on S. However, if $\widetilde{S} = \{(x,y) \mid x > 0\}$, then $\widetilde{S}$ is an open simply-connected region and the necessary condition of $\mathbf{F}$ is satisfied on $\widetilde{S}$. So there is a $\phi(x,y)$ such that $\mathbf{F} = \nabla\phi$, i.e.

$$\phi_x = -\frac{y}{x^2+y^2} \qquad \text{and} \qquad \phi_y = \frac{x}{x^2+y^2}$$

$$\therefore \quad \phi = -y\int \frac{1}{x^2+y^2}\,dx \qquad \text{let} \quad x = y\tan\theta, \quad \text{then} \quad dx = y\sec^2\theta\,d\theta$$

$$= -y\int \frac{1}{y^2\tan^2\theta + y^2}\,y\sec^2\theta\,d\theta = -\int d\theta = -\theta + g(y) = -\tan^{-1}\left(\frac{x}{y}\right) + g(y)$$

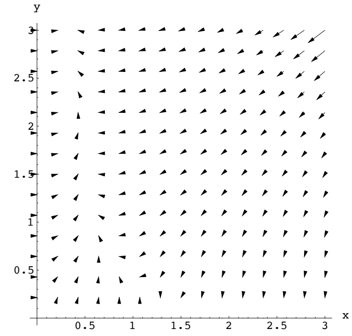
and $$\frac{\partial\phi}{\partial y} = -\frac{1}{1+\left(\frac{x}{y}\right)^2}\,\frac{d}{dy}\left(\frac{x}{y}\right) + g'(y) = \frac{x}{x^2+y^2} + g'(y) \quad \Rightarrow \quad g'(y) = 0, \quad g(y) = K.$$

$$\therefore \quad \phi = -\tan^{-1}\left(\frac{x}{y}\right) + K.$$

Ex. 2.4 Show that the following line integral is independent of path and evaluate the integral,

$$\mathrm{I} = \int_C (2y^2 - 12x^3y^3)\,dx + (4xy - 9x^4y^2)\,dy,$$

where C is any path from $(1,1)$ to $(3,2)$.



$\mathbf{F}(\mathbf{r}) = (2y^2 - 12x^3y^3)\,\mathbf{i} + (4xy - 9x^4y^2)\,\mathbf{j}$

Let $\quad \mathrm{I} = \displaystyle\int_C \mathbf{F}\cdot d\mathbf{r},$

where

$$\begin{aligned}\mathbf{F}(x,y) &= (2y^2 - 12x^3y^3)\,\mathbf{i} + (4xy - 9x^4y^2)\,\mathbf{j}\\ &= f\,\mathbf{i} + g\,\mathbf{j}\end{aligned}$$

Note that
$$\frac{\partial f}{\partial y} = 4y - 36x^3y^2 = \frac{\partial g}{\partial x},$$

$\therefore$ $\mathbf{F}$ is a conservative vector field in an open simply-connected region, i.e. $\mathbf{F} = \nabla\phi$

$$\therefore\ \phi_x = 2y^2 - 12x^3y^3 \quad (1) \qquad \text{and} \qquad \phi_y = 4xy - 9x^4y^2 \quad (2).$$

Thus the line integral is independent of path.

Method I

From (1) $\quad \phi(x,y) = 2xy^2 - 3x^4y^3 + \phi_1(y)$

$$\phi_y(x,y) = 4xy - 9x^4y^2 + \phi_1'(y)$$

Method II

From (1) $\quad \phi(x,y) = 2xy^2 - 3x^4y^3 + \phi_1(y)$

From (2) $\quad \phi(x,y) = 2xy^2 - 3x^4y^3 + \phi_2(x)$

Comparing these two equations, $\phi(x,y) = 2xy^2 - 3x^4y^3 + K$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} =$$

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For what values of b and c will

$$\mathbf{F}(x,y,z) = (y^2+2czx)\,\mathbf{i} + y(bx+cz)\,\mathbf{j} + (y^2+cx^2)\,\mathbf{k}$$

have potential functions? For each pair of these values of b and c, find a potential function for $\mathbf{F}$.

For $\mathbf{F}(\mathbf{r})$ to have a potential function, iff $\nabla\times\mathbf{F}(\mathbf{r}) = \mathbf{0}$.

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{i} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2+2czx & y(bx+cz) & y^2+cx^2 \end{vmatrix}$$

For the $\mathbf{i}$ component,

$$2y = yc \quad\Rightarrow\quad c = 2$$

For the $\mathbf{j}$ component,

$$2cx = 2cx \quad\Rightarrow\quad c \text{ cannot be determined.}$$

For the $\mathbf{k}$ component,

$$by = 2y \quad\Rightarrow\quad b = 2.$$

$\therefore b = 2$ and $c = 2$. In the case, we have

$$\mathbf{F}(\mathbf{r}) = (y^2+4zx)\,\mathbf{i} + y(2x+2z)\,\mathbf{j} + (y^2+2x^2)\,\mathbf{k} = \nabla\phi.$$

Therefore

$$\frac{\partial\phi}{\partial x} = y^2+4zx \tag{1}$$

$$\frac{\partial\phi}{\partial y} = 2xy+2yz \tag{2}$$

$$\frac{\partial\phi}{\partial z} = y^2+2x^2 \tag{3}$$

From (1), we have

$$\phi(x,y,z) = xy^2+2x^2z+p(y,z) \tag{4}$$

$$\phi_y(x,y,z) = 2xy+p_y(y,z). \tag{5}$$

Comparing (2) and (5), we have $p_y(y,z) = 2yz \quad\Rightarrow\quad p(y,z) = y^2z+q(z)$. From (4), we have

$$\phi(x,y,z) = xy^2+2x^2z+y^2z+q(z) \tag{6}$$

$$\phi_z(x,y,z) = 2x^2+y^2+q'(z). \tag{7}$$

Comparing (7) and (3), we have $q'(z) = 0 \quad\Rightarrow\quad q(z) = k$ (constant).

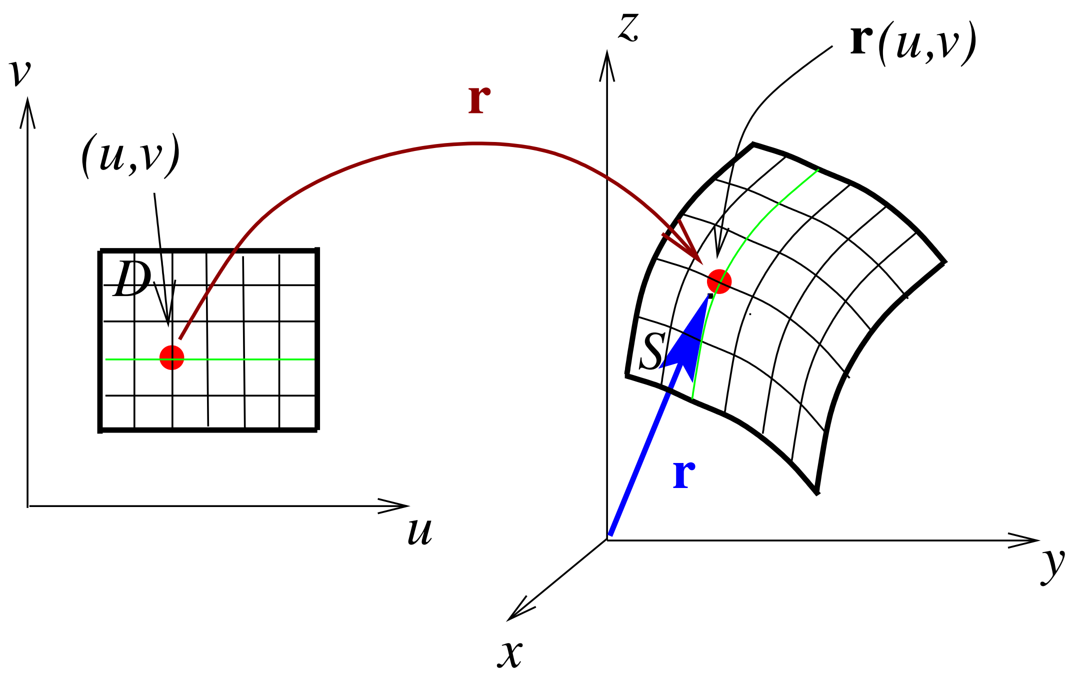
$$\therefore\quad \phi(x,y,z) = xy^2+2x^2z+y^2z+k.$$

15.5 Introduction to surface integrals

Parametric representation of surfaces

A parametric surface in 3D-space is a continuous function $\mathbf{r}$ ($\mathbb{R}^2 \to \mathbb{R}^3$) defined on some region D in the uv-plane, and have values in 3d-space:

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad \text{where} \quad x, y, z : \mathbb{R}^2 \to \mathbb{R}.$$



Ex. 5.1 If the surface $z(x,y) = \dfrac{1}{1+x^2+y^2}$, write this surface in terms of (r, θ).

Ex. 5.2 Describe the parametric surface $\mathbf{r}(\theta, z) = 3\sin\theta\ \mathbf{i} + 2\cos\theta\ \mathbf{j} + 2z\ \mathbf{k}$, where $\theta \in [0, 2\pi]$, $z \in [1,2]$.

Ex. 5.3 Describe the parametric surface

$$\mathbf{r}(s,t) = s\mathbf{a} + t\mathbf{b} + \mathbf{c}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are constant vectors and $\mathbf{a}$ is not parallel to $\mathbf{b}$.

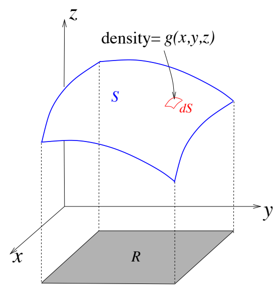
Ex. 5.4 Find the parametric equation of the cone $z^2 = x^2 + y^2$.

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length. The expression $\sqrt{f_x^2 + f_y^2 + 1}\, dA$ is approximately the surface area dS of a small patch on the surface and the total surface area is the sum (or integral) of the area of these patches.

If there is a continuous function $g(x, y, z)$ (e.g. temperature, density) defined on every point of the surface, then we might be interested in summing the values of g over the entire surface (total temperature or mass).

For a surface S given by $z = f(x, y)$, define

$$\iint_S g(x, y, z)\, dS = \iint_R g(x, y, f(x, y)) \sqrt{1 + f_x^2 + f_y^2}\, dA.$$



Note that $dS = \sqrt{1 + f_x^2 + f_y^2}\, dA$.

Ex. 5.5 $\iint_S yz\, dS$, S is the part of the plane $z = y + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.

$$\iint_S yz\, dS = \iint_R (y+3)y \sqrt{1 + z_x^2 + z_y^2}\, dA = \iint_R (y+3)y\sqrt{1+0+1}\, dA$$
$$= \sqrt{2} \iint_{x^2+y^2 \leqslant 1} (y+3)y\, dA$$

Ex. 5.6 Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2+y^2}$, $z \in [1, 4]$, if its density function is $\rho(x, y, z) = 10 - z$.

$$m = \iint_S \rho(x, y, z)\, dS = \iint_R \rho(x, y, z) \sqrt{1 + z_x^2 + z_y^2}\, dA = \iint_R (10 - \sqrt{x^2+y^2})\sqrt{2}\, dA$$
$$= \sqrt{2} \int_0^{2\pi} \int_1^4 (10 - r)\, r dr\, d\theta = 2\sqrt{2}\pi \left[5r^2 - \frac{1}{3}r^3\right]_1^4 = 108\sqrt{2}\pi$$

Ex. 5.7 Find the surface area of the surface with parametric equations

$$x = uv, \quad y = u + v \quad \text{and} \quad z = u - v \quad \text{such that} \quad u^2 + v^2 \leqslant 1$$

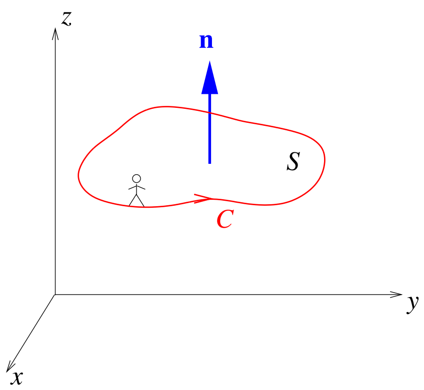
Let $\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k} = uv\,\mathbf{i} + (u+v)\,\mathbf{j} + (u - v)\,\mathbf{k}$, then $\mathbf{r}_u = (v, 1, 1)$, $\quad \mathbf{r}_v = (u, 1, -1)$ and $\mathbf{r}_u \times \mathbf{r}_v = (-2, u+v, v-u)$, therefore

$$S = \iint_R \|\mathbf{r}_u \times \mathbf{r}_v\|\, dA = \int\int_{u^2+v^2 \leqslant 1} \sqrt{4 + 2u^2 + 2v^2}\, dA = \int_0^{2\pi} \int_0^1 \sqrt{4 + 2r^2}\, r dr d\theta = \pi\left(2\sqrt{6} - \frac{8}{3}\right).$$

15.6 Surface integrals of vector fields; flux

Oriented surfaces

Let S be a non-closed surface, whose boundary consists of a closed smooth curve C with positive orientation – positive direction around C means the surface will always be on your left, then your head pointing in the direction of $\mathbf{n}$

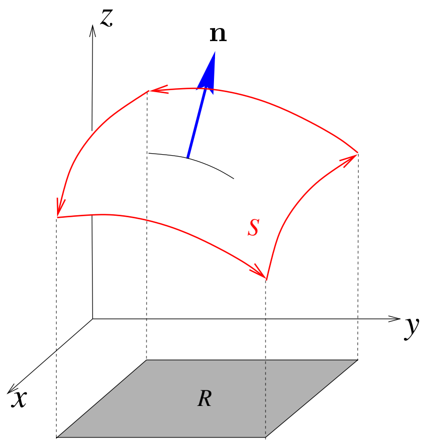


Suppose $z = f(x, y)$ is the equation for the surface S, we can define $\mathbf{n}$ by noting that S is also the level surface

$$g(x, y, z) = z - f(x, y) = 0.$$

We know that the gradient ∇g is normal to S at (x, y, z), so

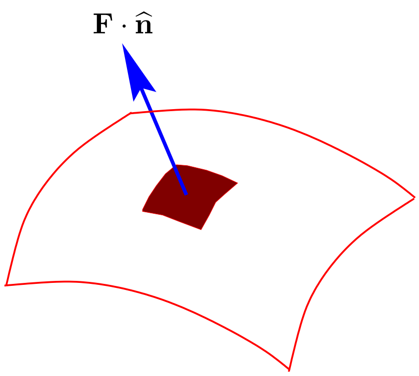
$$\widehat{\mathbf{n}} = \frac{-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}.$$



The flux of a vector field across a surface S

Suppose $\mathbf{F}(x, y, z)$ is a continuous vector field defined on a smooth, oriented surface S and $f(x, y, z)$ is the component of $\mathbf{F}$ in one of the two normal directions to S, i.e. $f = \mathbf{F}\cdot\widehat{\mathbf{n}}$

$$\iint_S f\,dS = \iint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS.$$

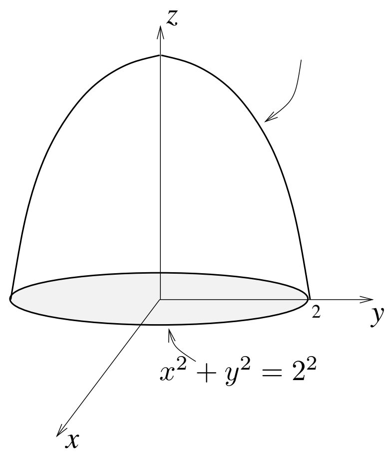


This integral is called the **flux** of $\mathbf{F}$ across S.

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS &= \iint_S \mathbf{F}\cdot\frac{-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}\,dS \\
&= \iint_R \mathbf{F}\cdot\frac{-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}\sqrt{1 + f_x^2 + f_y^2}\,dA \\
&= \iint_R \mathbf{F}\cdot\mathbf{n}\,dA.
\end{aligned}$$

Ex. 6.1 Evaluate $\displaystyle\iint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS$, where $\mathbf{F} = x^2y\,\mathbf{i} + xz\,\mathbf{j}$ and $\widehat{\mathbf{n}}$ is the upper normal to the surface

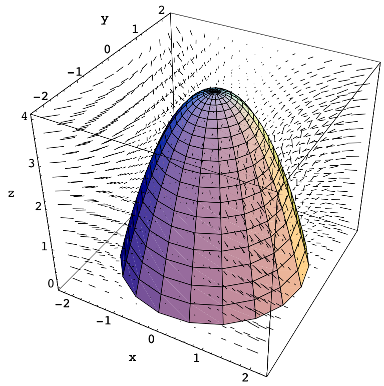
$S : z = 4 - x^2 - y^2, \quad z \geqslant 0.$



Let $g = z - 4 + x^2 + y^2$, this is a level surface in 3D,

then $\nabla g = (2x, 2y, 1) = \mathbf{n}$

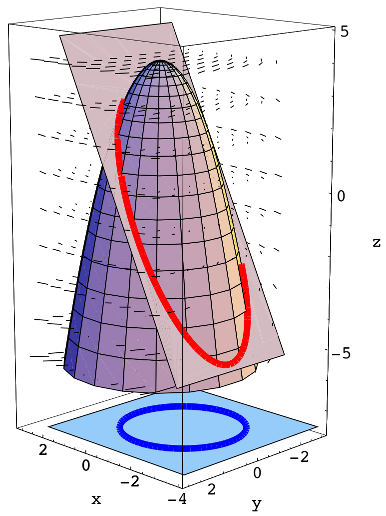
$$\begin{aligned}\therefore \iint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS &= \iint_{S_{xy}} \mathbf{F}\cdot\mathbf{n}\,dA \\ &= \iint_{S_{xy}} (x^2y, xz, 0)\cdot(2x, 2y, 1)\,dA \\ &= 2\iint_{S_{xy}} (x^3y + xyz)\,dA\end{aligned}$$



Ex. 6.2 Find the flux of $\mathbf{F} = y^3\,\mathbf{i} + z^2\,\mathbf{j} + x\,\mathbf{k}$ downward through the part of the surface $z = 4 - x^2 - y^2$ that lies above the plane $z = 2x + 1$.

The part of S of $z = 4 - x^2 - y^2$ lying above $z = 2x + 1$ has projection onto the xy-plane the disk $D = S_{xy}$ bounded by

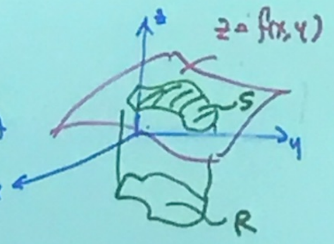
$$\begin{aligned} 2x + 1 &= 4 - x^2 - y^2 \\ (x+1)^2 + y^2 &= 4. \end{aligned}$$



Now let $g(x, y, z) = -x^2 - y^2 - z = -4$, this is a level surface in 3D, then $\nabla g = (-2x, -2y, -1) = \mathbf{n}$ which points downward.

Surface Integral

① $\iint_S dS = \iint_R \sqrt{1+(f_x)^2+(f_y)^2}\, dA$



② $\iint_S g(x,y,z)\sqrt{1+(f_x)^2+(f_y)^2}\, dA$ where $z=f(x,y)$

function of x and y.

density function

③ function of x and y.

$$\text{Flux} = \iint_S \underline{F}\cdot\hat{\underline{n}}\, dS$$

where $S: z=f(x,y)$

$\underline{n}$ is a normal vector to S.

$\underline{F}(x,y,z)$ is evaluated on the surface $z=f(x,y)$

$$dS = \sqrt{1+(f_x)^2+(f_y)^2}\, dA$$

飯碗不是

close surface

無 enclose volume

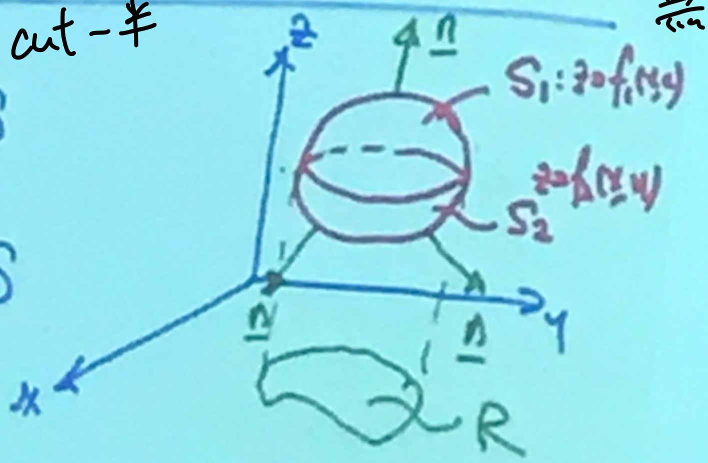
Outward flux

cut一半

closed surface → $\oiint_S \underline{F}\cdot\hat{\underline{n}}\, dS = \iint_{S_1} \underline{F}\cdot\hat{\underline{n}}\, dS + \iint_{S_2} \underline{F}\cdot\hat{\underline{n}}\, dS$

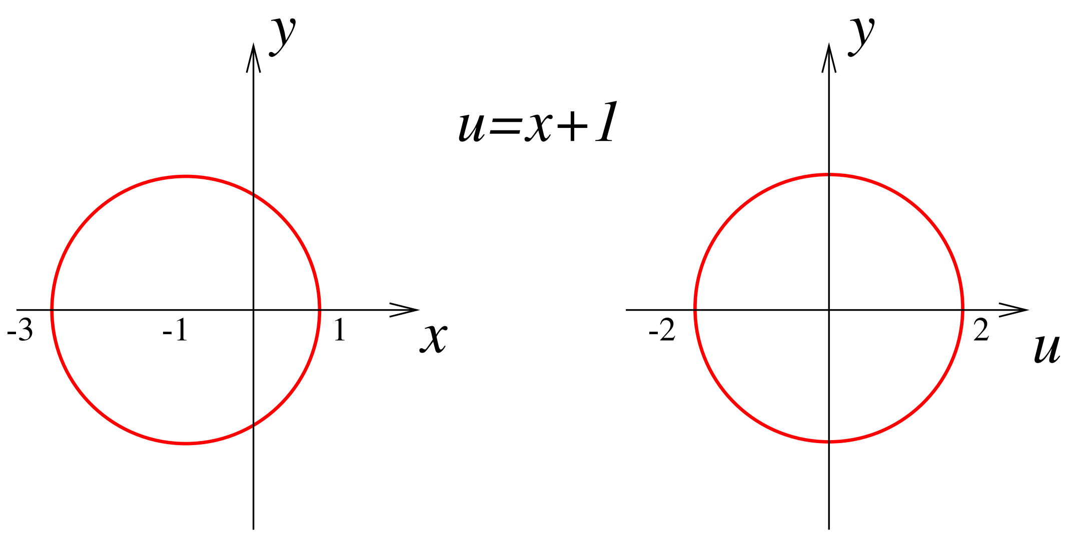
$= \iiint_V \nabla\cdot\underline{F}\, dV$

diverge thm.



$S = S_1 + S_2$

$$
\begin{aligned}
\therefore \qquad \iint_S \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS &= \iint_{S_{xy}} \mathbf{F}\cdot\mathbf{n}\,dA \\
&= \iint_{S_{xy}} (y^3, z^2, x)\cdot(-2x,-2y,-1)\,dA \\
&= -\iint_{S_{xy}} (2xy^3 + 2yz^2 + x)\,dA \\
&= -\iint_{S_{xy}} [2xy^3 + 2y(4-x^2-y^2)^2 + x]\,dA
\end{aligned}
$$



$(x-1)^2 + y^2 = 2^2$ $\qquad$ $u^2 + y^2 = 2^2$

Ex. 6.3 Evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(\mathbf{r}) = x^3z\,\mathbf{i} + (z+\cos^4 y)\,\mathbf{j} + x\,\mathbf{k}$ and C is the curve of intersection of the paraboloid $z = x^2+y^2$ and the plane $z = 2y$. (**Ans.** 2π).