$V_1\ (6, 2, 3) \qquad V_2 (5, -1, 4)$

(2)

$$V_1 \cdot V_2 = |V_1||V_2| \cos\theta$$

$$30 - 2 + 12 = \sqrt{36 + 4 + 9}\ \sqrt{25 + 1 + 16}\ \cos\theta$$

$$40 = \sqrt{49}\ \sqrt{42}\ \cos\theta$$

$$40 = 7\sqrt{42} \cos\theta$$

$$\frac{40}{7\sqrt{42}} = \cos\theta$$

---

(3)

$$V_1 \times V_2 : \begin{vmatrix} i & j & k \\ 6 & 2 & 3 \\ 5 & -1 & 4 \end{vmatrix} = 11i - 9j - 16k$$

Made with Goodnotes

4. Compute the determinant of the matrix $\begin{bmatrix} 6 & 2 & 3 \\ 5 & -1 & 4 \\ 1 & 2 & 3 \end{bmatrix}$.

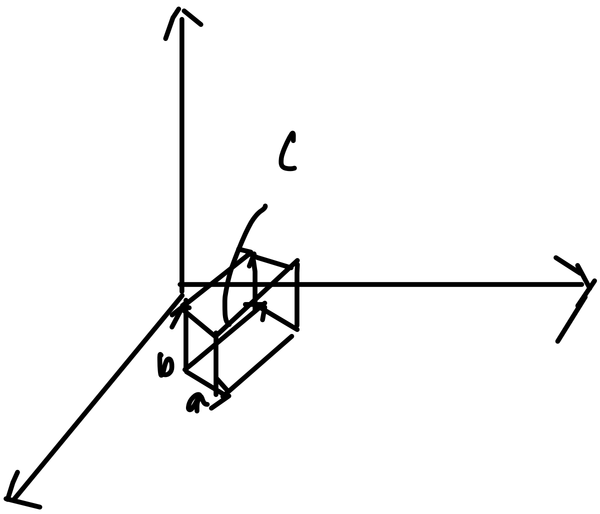
$$6(-3-8)-2(15-4)+3(10+1)$$

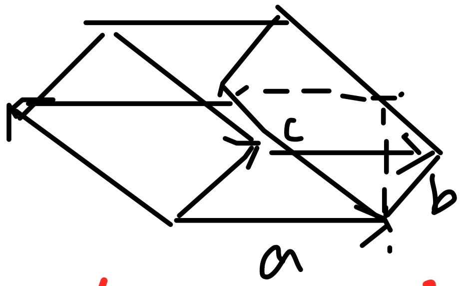
$$= 6(-11)-2(11)+3(11)$$

$$= -66-22+33$$

$$= -55$$

5. Express the volume of the parallelepiped with vectors **a**, **b**, **c** as the three edges sharing the same vertex.

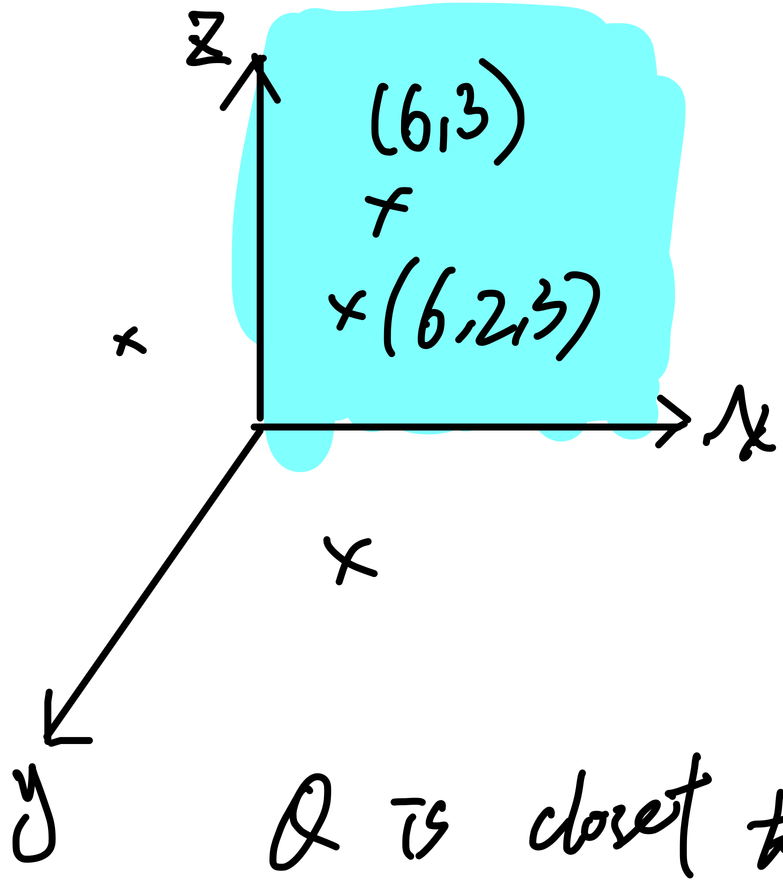




$|(a \times b) \cdot c|$

2 1 3

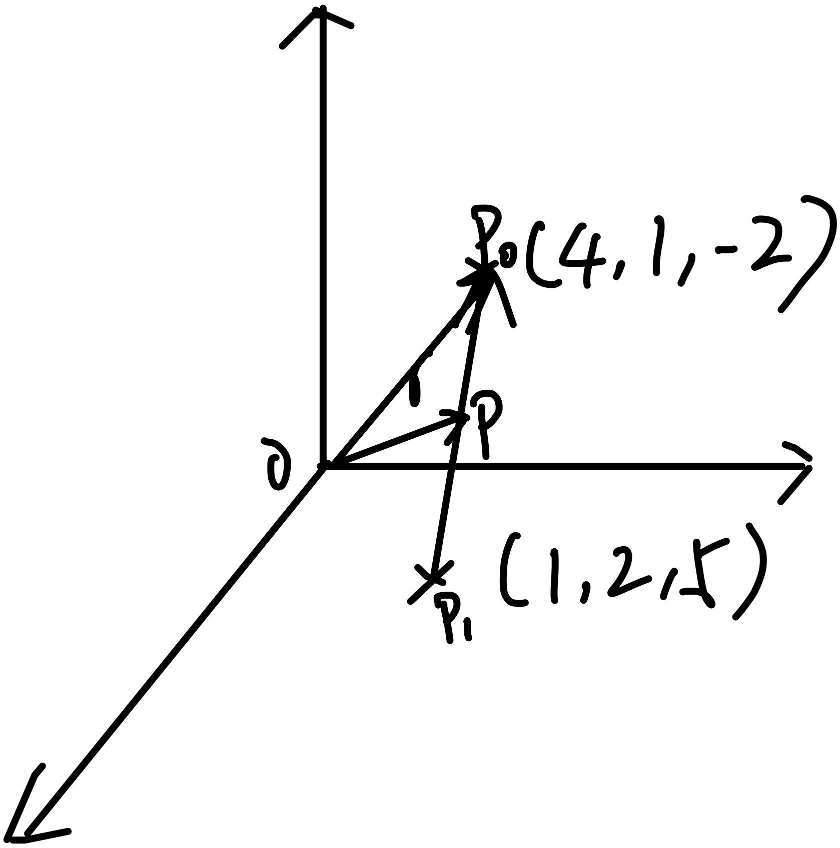
6. Which of the points $P = (6, 2, 3)$, $Q = (5, -1, 4)$ and $R = (0, 3, 8)$, is closest to the xz-plane? Which point lies in the yz-plane?



Q is closest to xz-plane.

R lies in yz-plane

7. Find the parametric equation line through $(4, 1, -2)$ and $(1, 2, 5)$.



$$\vec{OP} + \vec{PP_0} = \vec{OP_0}$$

$$\langle x,y,z\rangle + t\,\vec{P_1P_0} = \langle 4,1,-2\rangle$$

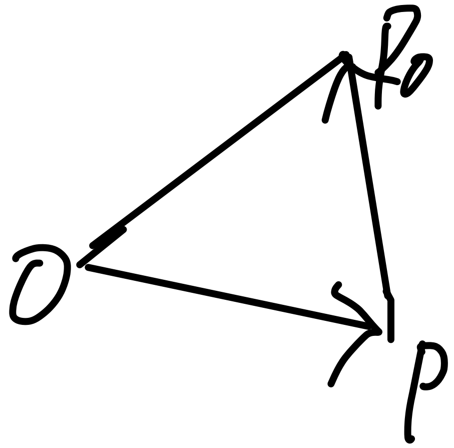
$$\langle x,y,z\rangle + t\langle 3,-1,-7\rangle = \langle 4,1,-2\rangle$$

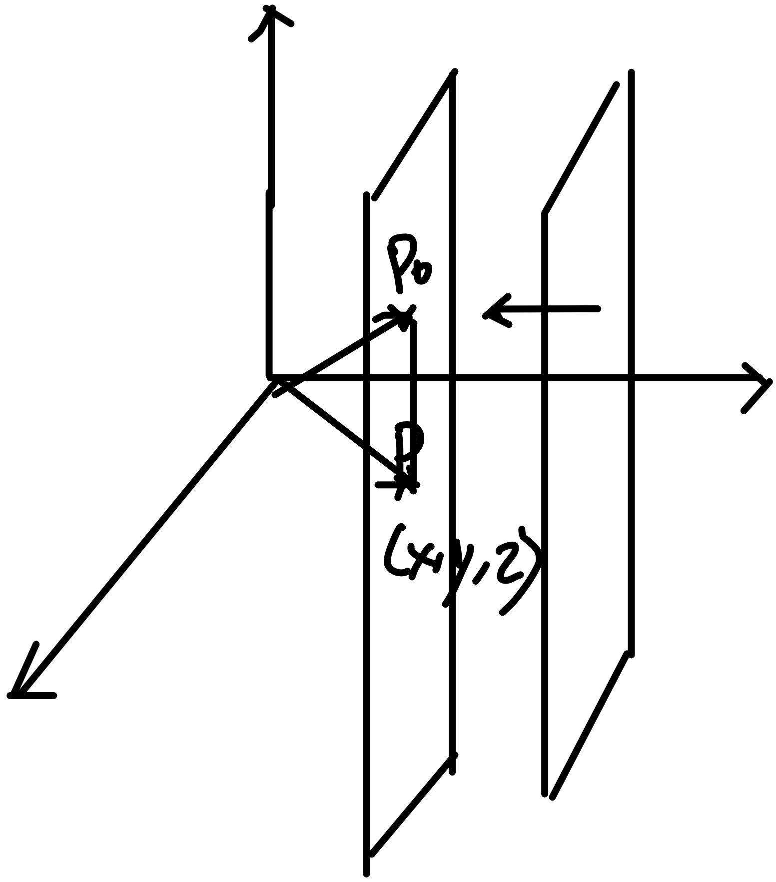
$$\langle x,y,z\rangle = \langle 4-3t,\ 1+t,\ -2+7t\rangle$$

$$\vec{r}(t) = \langle 4,1,-2\rangle - t\langle 3,-1,-7\rangle$$

$$\vec{r}(t) = \langle 4,1,-2\rangle + t\langle -3,1,7\rangle$$

8. Find the plane through $(2,1,0)$ and parallel to $x+4y-3z=1$.





$$\vec{n} = \langle 1, 4, -3 \rangle$$

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

$$\langle 1, 4, -3 \rangle \cdot \langle x-2, y-1, z \rangle = 0$$

$$x-2+4(y-1)-3z=0$$

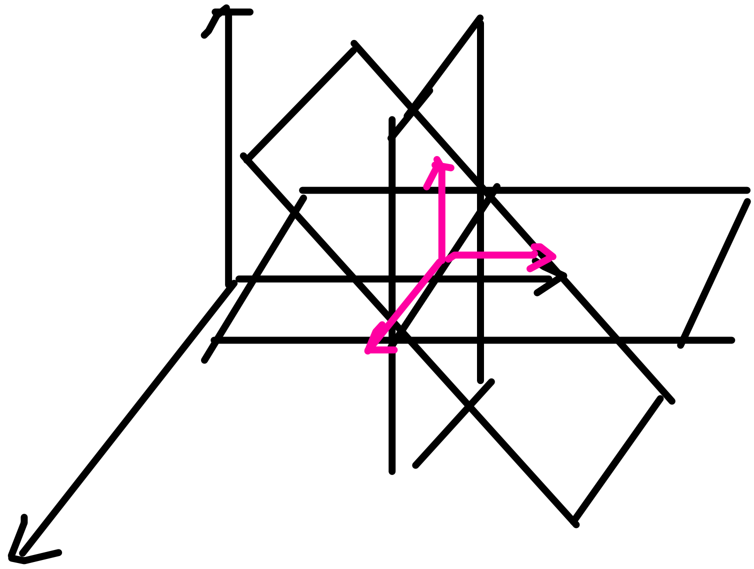
$$x-2+4y-4-3z=0$$

$$x+4y-3z-6=0$$

$$x+4y-3z=6$$

9. Find an equation of the plane through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and perpendicular to the plane $x + y - 2z = 1$.

求其係 point 上一点即可, 因此 z sub 乜都得.



$z = 0, \quad x = 1, \quad y = 3.$

$P_0 \langle 1, 3, 0 \rangle$

Parallel vector: $\langle 1, 1, -2 \rangle$

2nd Parallel vector: $\langle 1, 0, -1 \rangle \times \langle 0, 1, 2 \rangle$

$$= \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = i - 2j + k$$

$$= \langle 1, -2, 1 \rangle$$

Made with Goodnotes

$$\vec{n} = \langle 1, 1, -2 \rangle \times \langle 1, -2, 1 \rangle$$

$$= \begin{vmatrix} i & j & k \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = 5i - 3j - 3k$$

$$= \langle 5, -3, -3 \rangle$$

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

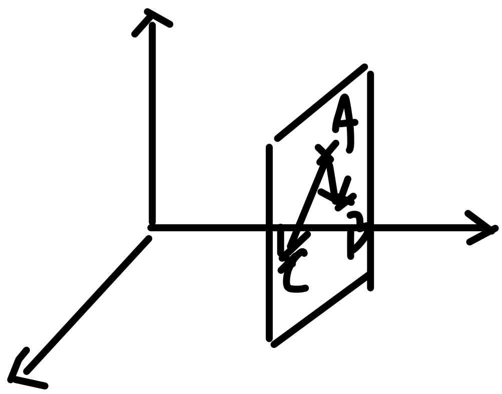
$$\langle 5, -3, -3 \rangle \cdot \langle x-1, y-3, z \rangle = 0$$

$$5x - 5 - 3(y-3) - 3z = 0$$

$$5x - 3y - 3z - 5 + 9 = 0$$

$$5x - 3y - 3z = -4$$

10. Find a vector perpendicular to the plane through the points $A = (1, 0, 0)$, $B = (2, 0, -1)$, $C = (1, 4, 3)$. Find the area of the triangle ABC.



$$\vec{AB} = \langle 1, 0, -1 \rangle$$

$$\vec{AC} = \langle 0, 4, 3 \rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 4 & 3 \end{vmatrix}$$

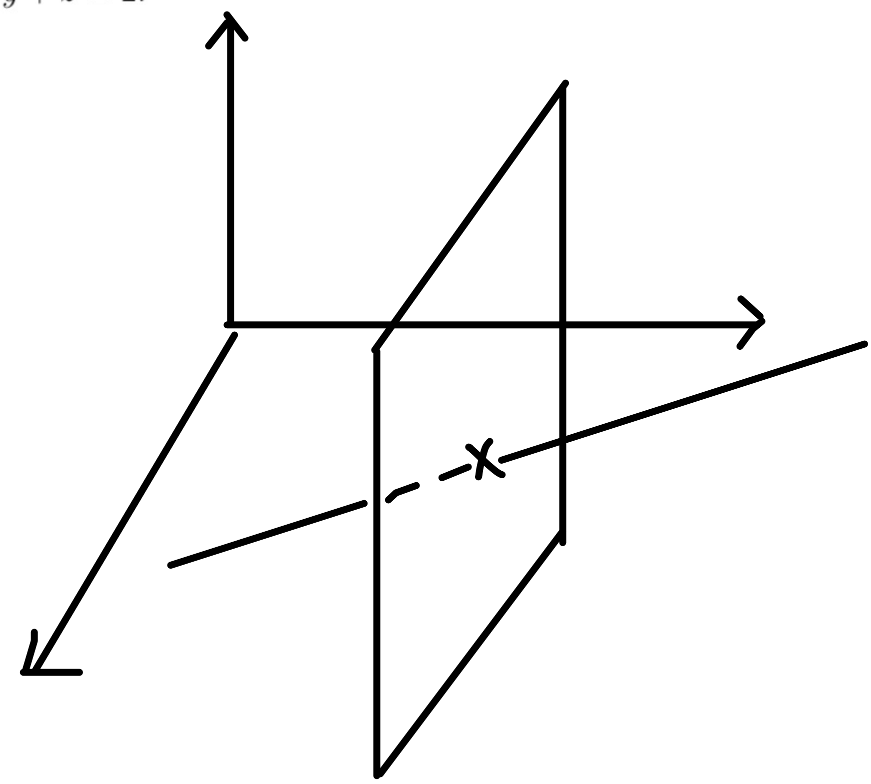
$$= 4i - 3j + 4k$$

vector required $= \langle 4, -3, 4 \rangle$

$$\text{Area} = \frac{|\vec{AB} \times \vec{AC}|}{2} = \frac{\sqrt{16 + 9 + 16}}{2}$$

$$= \frac{\sqrt{41}}{2}.$$

11. Find the point in which the line with parametric equations $x = 2 - t$, $y = 1 + 3t$, $z = 4t$ intersects the plane $2x - y + z = 2$.



$$\vec{r}(t) = \langle 2, 1, 0 \rangle + t\langle -1, 3, 4 \rangle$$

$$2(2-t) - (1-3t) + 4t = 2$$

$$4 - 2t - 1 + 3t + 4t = 2$$

$$3 + 5t = 2$$

$$5t = -1$$

$$t = -\frac{1}{5}$$

point $= \langle \frac{11}{5}, \frac{2}{5}, -\frac{4}{5} \rangle$

12. Determine wheter the following pair of lines are parallel, skew, or intersecting. If intersect, find the point of intersection.

(a) $L_1: x = -6t,\ y = 1 + 9t,\ z = -3t,$
$L_2: x = 1 + 2s,\ y = 4 - 3s,\ z = s.$

(b) $L_1: \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$
$L_2: \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$

$L1: \langle 0, 1, 0 \rangle + t \langle -6, 9, -3 \rangle$

$= \langle 0, 1, 0 \rangle - 3t \langle 2, -3, 1 \rangle$

$L2: \langle 1, 4, 0 \rangle + s \langle 2, -3, 1 \rangle$

They are parallel.

b). $6x = 3(y-1) = 2(z-2)$

$6x = 3y - 3 = 2z - 4$

$x = t$

$y = 2t + 1$

$z = 3t + 2$

$\vec{r(t)} = \langle 0, 1, 2 \rangle + \langle 1, 2, 3 \rangle$

$l_2: \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$

$-3(x-3) = -4(y-2) = 6(z-1)$

$-3x+9 = -4y+8 = 6z-6$

$x = t$

$y = \frac{3}{4}t - \frac{1}{4}$

$z = -\frac{1}{2}t + \frac{5}{2}$

$-3x+9 = -4y+8$

$-3t+1 = -4y$

$\frac{3}{4}t - \frac{1}{4} = y$

$6z-6 = -3t+9$

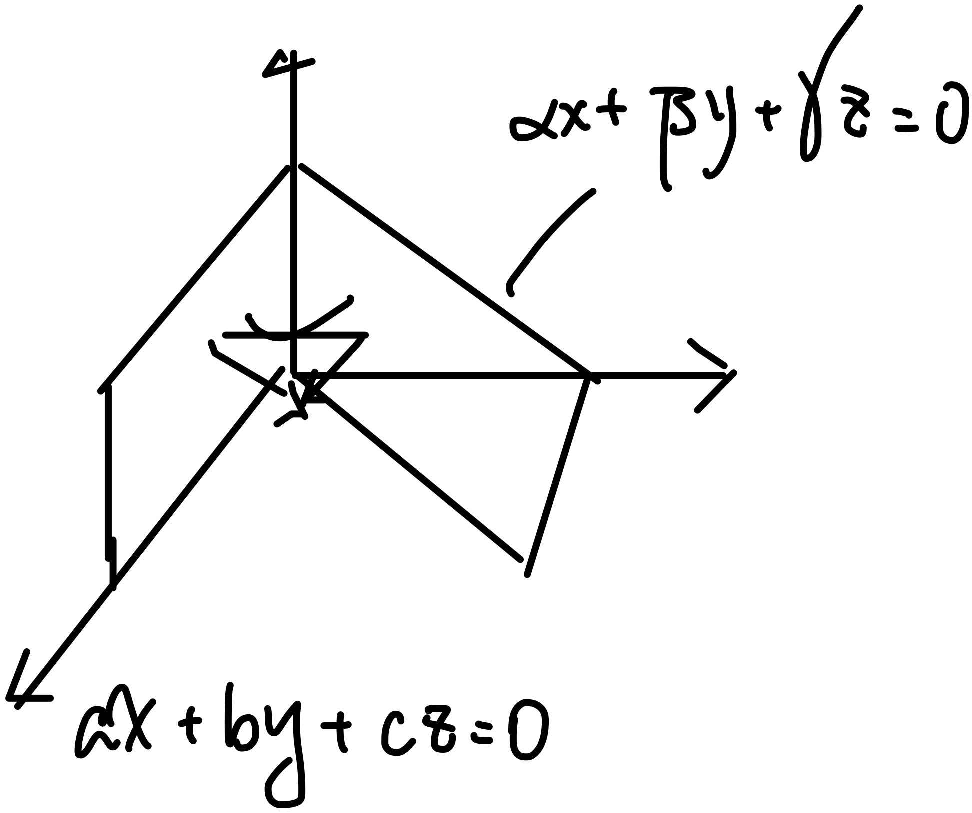
$6z = -3t+15$

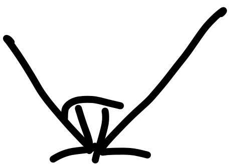
$z = -\frac{1}{2}t + \frac{5}{2}$

intersecting.

Made with Goodnotes

angle between two plane





$$-\langle \alpha, \beta, \gamma \rangle \cdot -\langle a, b, c \rangle = \sqrt{\alpha^2+\beta^2+\gamma^2}\,\sqrt{a^2+b^2+c^2}\,\cos\theta$$

MATH2023
Tutorial 0-1
January 30, 2019 Tutorial Notes by Ivan So

1 Review

In the following we will assume V to be a 3-dimensional real vector space (A rank 3 free $\mathbb{R}$-module :D).

- **Scalar**:
  - Is an *one*-entry object belongs to $\mathbb{R}$.
  - Represent a quantity.
  - *Ordered.*
- **Vector**:
  - Is a *three*-entry object represented by $\mathbf{x} = (x_1, x_2, x_3) = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, which represent an "arrow" in 3D space.
  - The **norm** $\|\cdot\| : V \to \mathbb{R}$ is a function which measures the *length* of the arrow. It is defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ (in our consideration).
  - **Unit Vector** is a vector with norm 1.
  - $\mathbf{v}_1$ and $\mathbf{v}_2$ are **linearly dependent** if $\mathbf{v}_1 = \alpha\mathbf{v}_2$ for some $\alpha \in \mathbb{R}$.
  - Two vectors are said to be **orthogonal** if the angle in between them is $\pi/2$.
  - **NOT** *ordered.*
- **Determinant**
  - for 2×2 matrix, $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
  - for 3×3 matrix, $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$
- **Dot product**:
  - $$\begin{aligned} "\cdot" &: V \times V \to \mathbb{R} \\ (\mathbf{v}_1, \mathbf{v}_2) &\mapsto \mathbf{v}_1 \cdot \mathbf{v}_2 := v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z} = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos\theta \end{aligned}$$
  - Note that $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.
  - Dot product of *orthogonal* vectors is 0.

– It represent the length of projection of $\mathbf{v}_1$ on $\mathbf{v}_2$.

- **Cross product**:
  - $" \times " : (\mathbf{v}_1, \mathbf{v}_2) \in V \times V \mapsto \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_{1x} & v_{1y} & v_{1z} \\ v_{2x} & v_{2y} & v_{2z} \end{bmatrix} \in V.$
  - $\|\mathbf{v}_1 \times \mathbf{v}_2\| = \|\mathbf{v}_1\| \, \|\mathbf{v}_2\| \sin\theta$
  - Cross product of *linearly dependent* vectors is 0.

- The way to find a equation of **line** passing through *two* points:
  1. Given points A, B, we can find the vector $\overrightarrow{AB}$.
  2. The equation of line $\overline{AB}$ is given by $\overrightarrow{OA} + t\overrightarrow{AB}$.

- The way to find a equation of **plane** passing through *three* points:
  1. Given points A, B and C, we can find vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
  2. The normal vector of the plane is given by $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.
  3. If $P = (x, y, z)$ is a point on the plane, then $\overrightarrow{AB} \perp \mathbf{n}$, so $\overrightarrow{AB} \cdot \mathbf{n} = 0$, which gives the equation of plane.

2 Problems

1. True or False
   (a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. T

   (b) If $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$. T

   (c) If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$. F

   52.42

   (d) If $\mathbf{u} \cdot \mathbf{v} = 0$ then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$. F

   (e) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$. ~~T~~ F 用 right hand rule

2. Compute the angle between $\mathbf{v}_1 = (6, 2, 3)$ and $\mathbf{v}_2 = (5, -1, 4)$.

3. Compute the cross product of $\mathbf{v}_1 = (6, 2, 3)$ and $\mathbf{v}_2 = (5, -1, 4)$.

4. Compute the determinant of the matrix $\begin{bmatrix} 6 & 2 & 3 \\ 5 & -1 & 4 \\ 1 & 2 & 3 \end{bmatrix}$.

5. Express the volume of the parallelepiped with vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as the three edges sharing the same vertex.

6. Which of the points $P = (6, 2, 3), Q = (5, -1, 4)$ and $R = (0, 3, 8)$, is closest to the xz-plane? Which point lies in the yz-plane?

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    $L_2 : \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$

13. Find the angle between two planes.