MATH 2023

Multivariable Calculus

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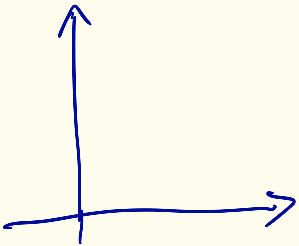
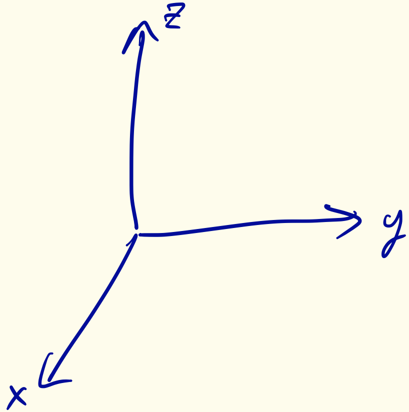
Multivariable

≤ 3

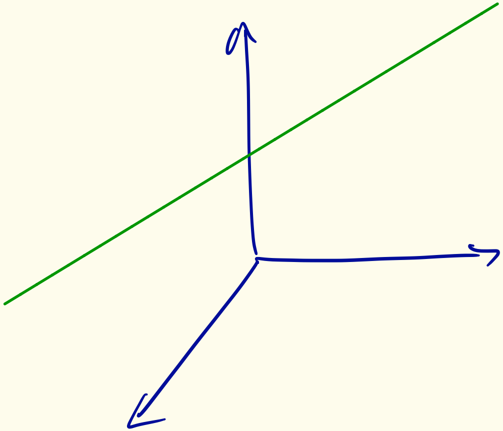
everything in 3D.



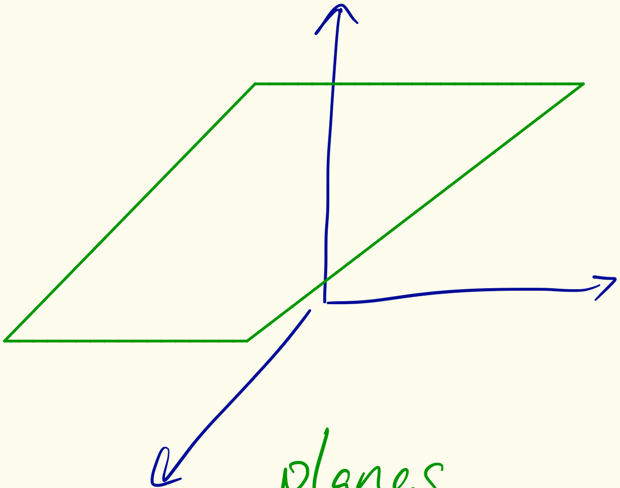
Single.



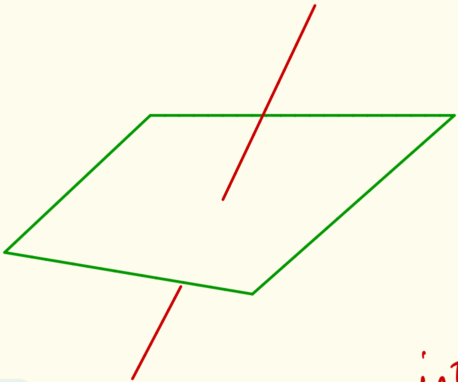
1st Goal: 3D Geometry.

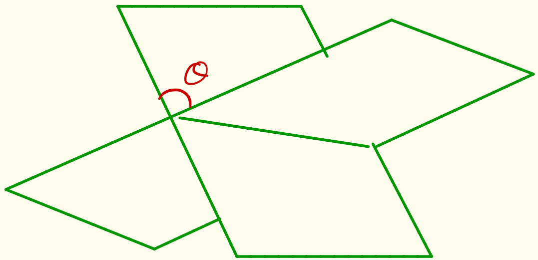


lines

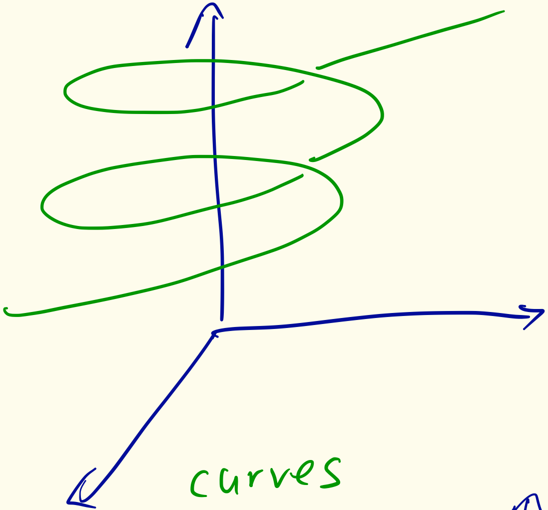


planes.

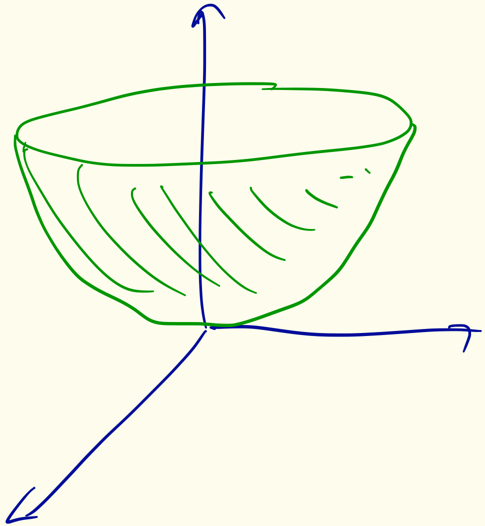




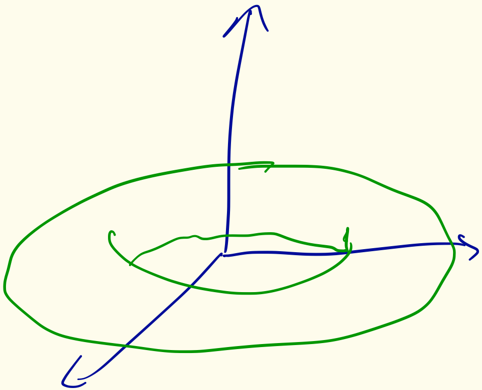
intersections



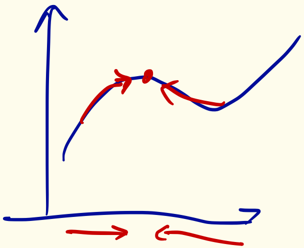
curves

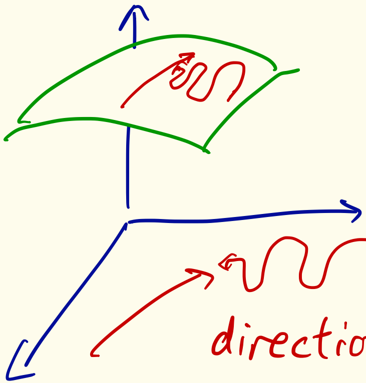


surfaces

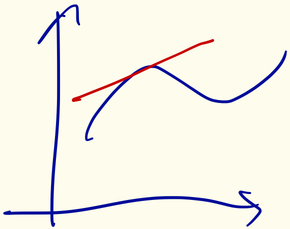


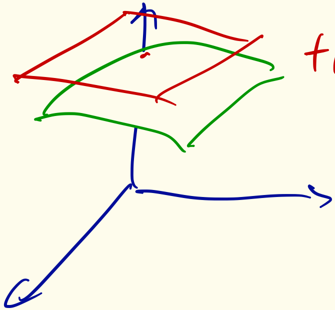
Calculus — differentiation, integration.





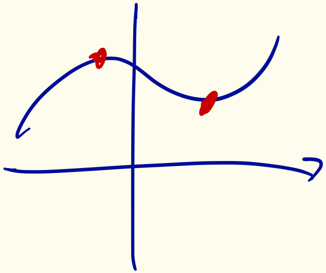
directional derivatives.

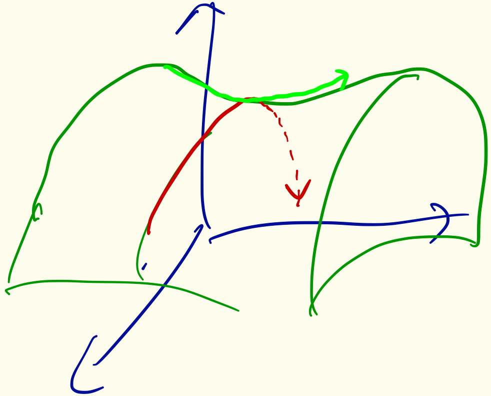




tangent planes

2nd Goal: Apply multi-theory to max/min problem





3rd Goal: Fundamental Thm of Calculus

in Higher Dimension

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

curl, div, grad ∇

$$\int_a^b F(x)\,dx = F(b) - F(a)$$

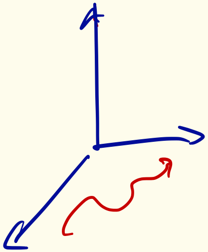
functions

Vector Fields

line integral

surface integral

triple integral



3 Main Thm:

Green's Thm

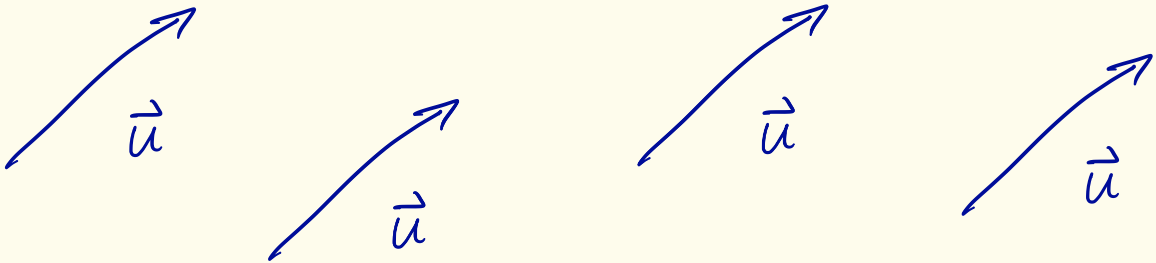
Stokes' Thm

Divergence Thm

Vectors

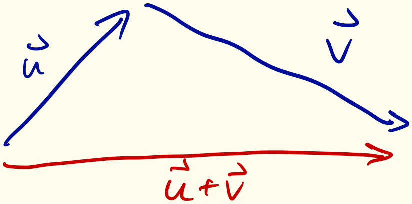
- directions ($\vec{0}$ no direction)
- magnitude (length)

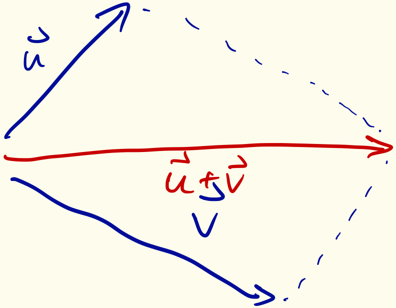
(does not depend on position!)



scalings $\quad c \cdot \vec{v}$, $c \in \mathbb{R}$ scalars (numbers).

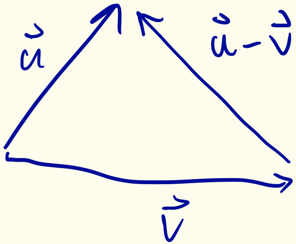
addition :





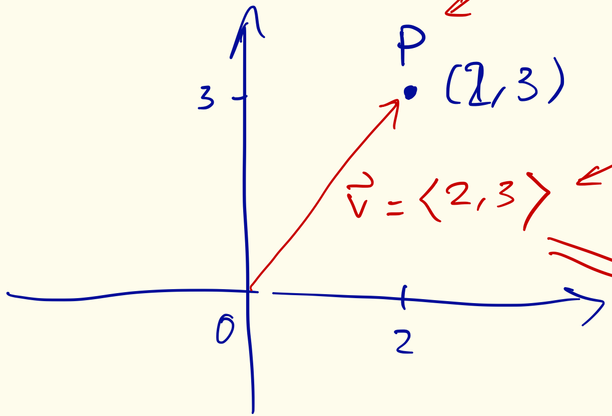
$$\vec{u}-\vec{v} = \vec{u}+(-\vec{v}).$$

$$\vec{v}+(-\vec{v})=\vec{0}$$



$\vec{u}$, $\underline{u}$, $\dot{u}$

$\mathbf{u}$ ← books.



positions

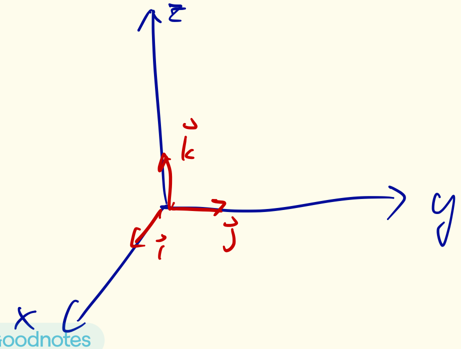
direction
length.

$$\vec{v} = \langle 2, 3 \rangle = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$\Rightarrow$ $\vec{v}$ is position vector of P.
$= \overrightarrow{OP}$

$$\vec{v} = \langle a, b, c \rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\vec{i} + b\vec{j} + c\vec{k}$$

standard vectors



$\vec{i} = \langle 1, 0, 0 \rangle$
$\vec{j} = \langle 0, 1, 0 \rangle$
$\vec{k} = \langle 0, 0, 1 \rangle$

$$|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$$

Unit Vector : length = 1

$$\vec{e} = \frac{\vec{v}}{|\vec{v}|}$$

Ex $\vec{v} = \langle 1, 2, 3 \rangle$, $\vec{e} = \dfrac{\langle 1,2,3\rangle}{\sqrt{1^2+2^2+3^2}} = \left\langle \dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}} \right\rangle$

is a unit vector.

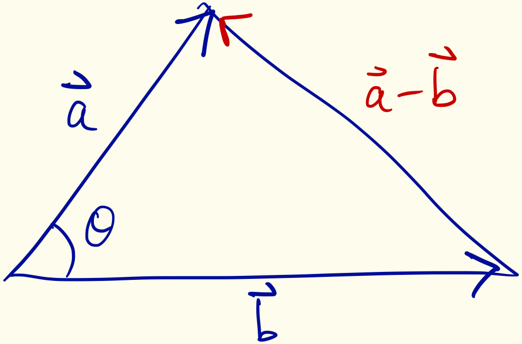
Dot Product (Scalar Product) ⟺ "angle between vectors"

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

$= a_1 b_1 + a_2 b_2 + a_3 b_3 \in \mathbb{R}$.

Made with Goodnotes



Cosine Rule

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

$$\vec{a}\cdot\vec{a} = |\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2.$$

$$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \cancel{\vec{a}\cdot\vec{a}} + \cancel{\vec{b}\cdot\vec{b}} - 2|\vec{a}||\vec{b}|\cos\theta$$

$$\parallel$$

$$\cancel{\vec{a}\cdot\vec{a}} - 2\vec{a}\cdot\vec{b} + \cancel{\vec{b}\cdot\vec{b}}$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

<u>Ex</u> $\vec{a}\cdot\vec{b} = 0$

$\Rightarrow \theta = \frac{\pi}{2}$

$\Rightarrow \vec{a} \perp \vec{b}$

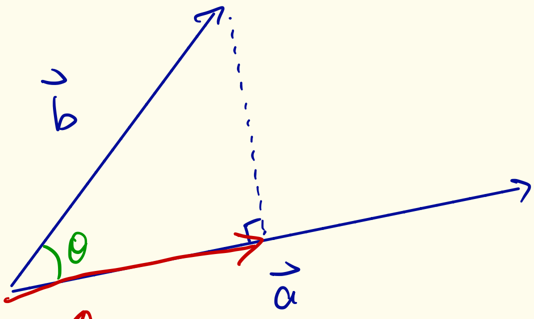
<u>Ex</u> $\vec{i}\cdot\vec{j} = 0$

$\vec{j}\cdot\vec{k} = 0$

$\vec{i}\cdot\vec{k} = 0.$

Made with Goodnotes

Projections



$\mathrm{Proj}_{\vec{a}}\vec{b}$

length : $|\vec{b}|\cos\theta$

$= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

direction : $\dfrac{\vec{a}}{|\vec{a}|}$

$\Longrightarrow \mathrm{Proj}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\cdot\vec{a}$

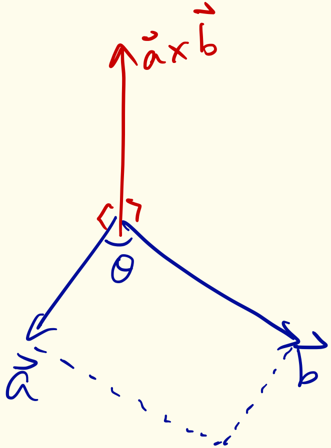
$\vec{e} = \dfrac{\vec{a}}{|\vec{a}|}$

$(\vec{b}\cdot\vec{e})\vec{e}$

Cross Product (Vector Product) (3D).

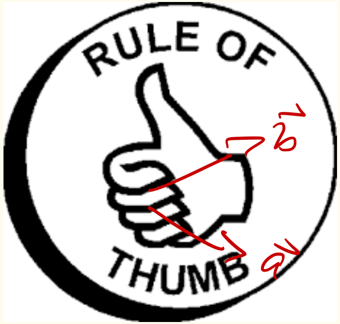
$\vec{a} \times \vec{b}$: direction • $\perp$ both $\vec{a}$ and $\vec{b}$

• right-hand rule.

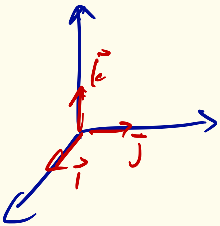


length : area of //gram spanned by $\vec{a}$ and $\vec{b}$

$= |\vec{a}||\vec{b}| \sin\theta$.



$\vec{a} \times \vec{b} = 0$ if $\vec{a} // \vec{b}$



$\vec{i} \times \vec{j} = \vec{k}$ $\qquad$ $\vec{j} \times \vec{i} = -\vec{k}$

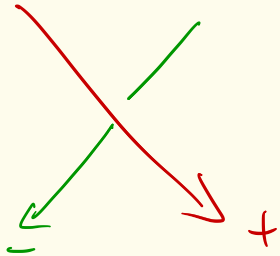
$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

<u>Ex</u> $\langle 1,2,3 \rangle \times \langle 4,5,6 \rangle$

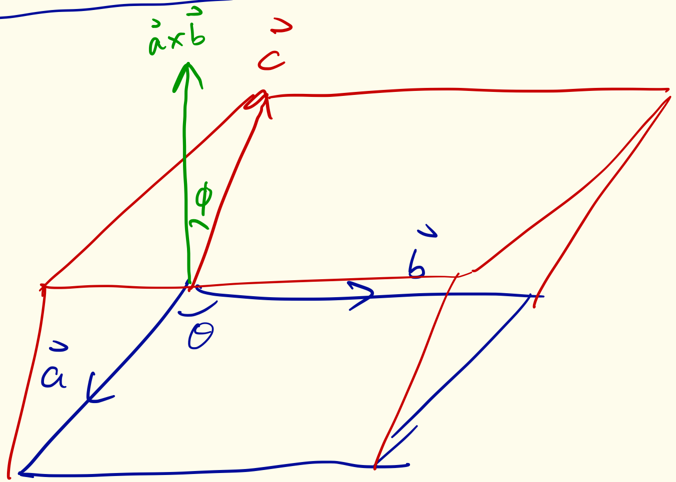
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$



$$\vec{i}\cdot 2\cdot 6 + \vec{j}\cdot 3\cdot 4 + \vec{k}\cdot 1\cdot 5$$

$$- \vec{i}\cdot 3\cdot 5 - \vec{j}\cdot 1\cdot 6 - \vec{k}\cdot 2\cdot 4 \Rightarrow -3\vec{i} + 6\vec{j} - 3\vec{k}$$

Triple Product



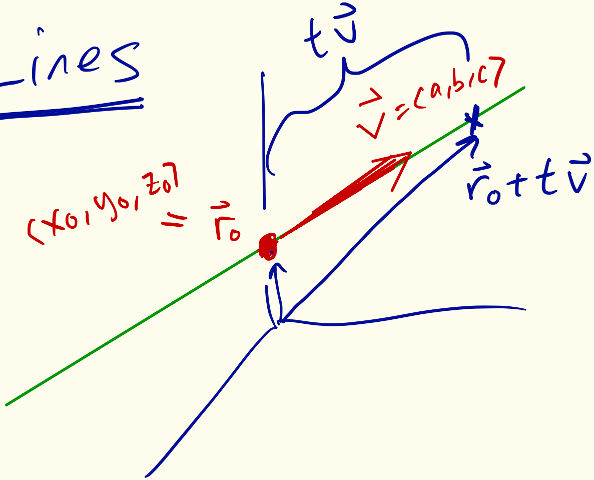
Volume of parallelopiped.

$$\text{Base} = \text{Area of } \square = |\vec{a}\times\vec{b}|$$

$$\text{Height} : |\vec{c}|\cos\phi.$$

$$\text{Volume} = |\vec{a}\times\vec{b}||\vec{c}|\cos\phi \implies |\vec{c}\cdot(\vec{a}\times\vec{b})|.$$

Lines



lines = direction + point.

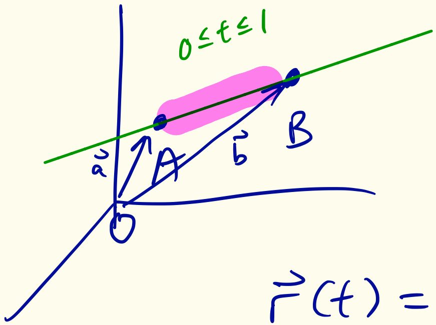
Vector Equation: $\vec{r}(t) = \vec{r}_0 + t\vec{v} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Parametric Equation: $\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$

Ex line passing through $(1,2,3)$ and $\parallel$ to $\langle 4,5,6\rangle$

$$\vec{r}(t) = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}4\\5\\6\end{pmatrix} \quad \text{or} \quad \begin{cases} x = 1+4t \\ y = 2+5t \\ z = 3+6t \end{cases}$$

Ex Line passing through A, B



direction : $\overrightarrow{AB} = \vec{b} - \vec{a}$

point : $A = \vec{a}$

$\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$

$$\vec{r}(t) = \vec{a} + t(\vec{b} - \vec{a})$$
$$= (1-t)\vec{a} + t\vec{b}.$$

line segments : $0 \le t \le 1$ $\quad \overrightarrow{AB}$.

Made with Goodnotes

Ex When line intersects xy-plane?

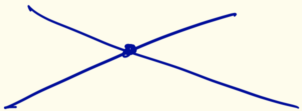
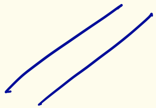
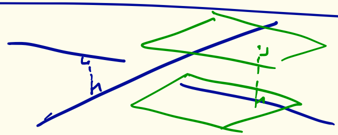
$$\vec{r}(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$z = 0$

$$z = 1 + 2t = 0 \Rightarrow t = -\frac{1}{2}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \end{pmatrix}$$ in x-y plane.

2 Lines?

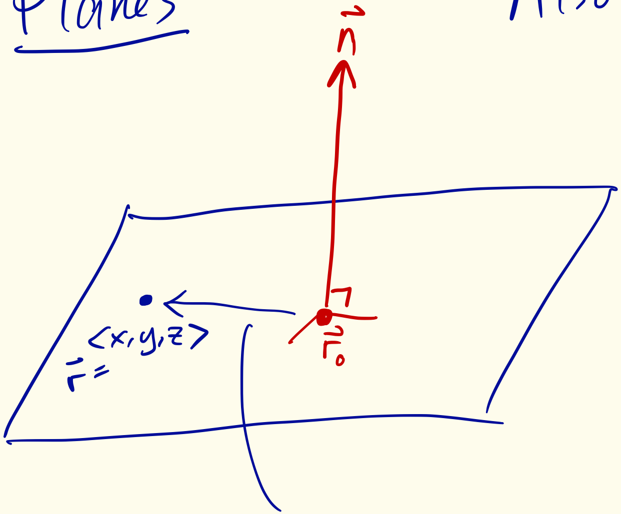
	parallel	not parallel
intersect	same line.	
not intersect.		 skew lines

Planes

Also by point + direction!

$\vec{r}_0$ $\langle x_0, y_0, z_0 \rangle$

$\vec{n}$ normal vector $\langle a, b, c \rangle$



$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$$

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\boxed{ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{\text{given numbers}} = d}$$

__Ex__ $\vec{r}_0 = (2, 4, -1)$, $\vec{n} = \langle 1, 2, 3 \rangle$.

$$\Rightarrow \quad x + 2y + 3z = \cancel{d}\ 7$$

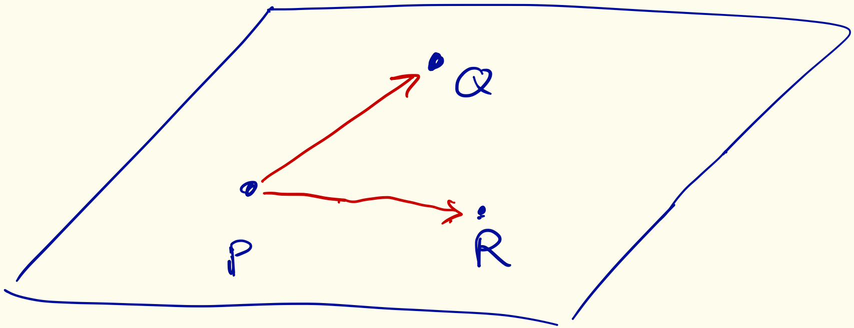
$$2 + 2(4) + 3(-1) = 7$$

$$\underline{a}x + \underline{b}y + \underline{c}z = d$$

$\vec{n}$

substitute $\vec{r}_0$

Ex Plane containing 3 points P, Q, R ?



point : P

direction : $\vec{PR} \times \vec{PQ}$

Ex $P = (1,1,1)$

$Q = (1,2,3)$

$R = (0,1,1)$

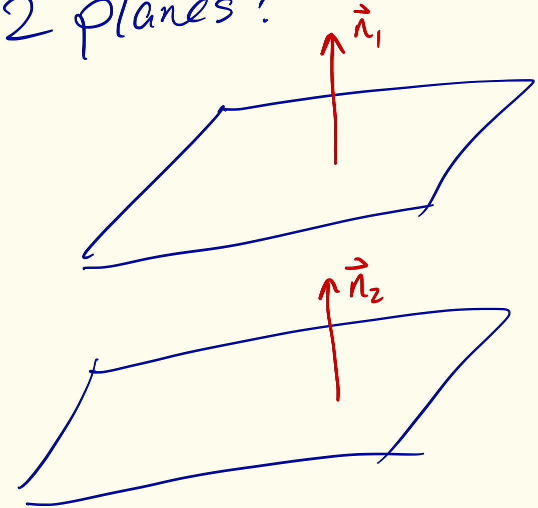
$\vec{PR} = \langle -1, 0, 0 \rangle$

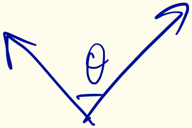
$\vec{PQ} = \langle 0, 1, 2 \rangle$

$$\vec{PR} \times \vec{PQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 2\vec{j} - \vec{k} = \underbrace{\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}}_{\vec{n}}$$

$2y - z = 1$

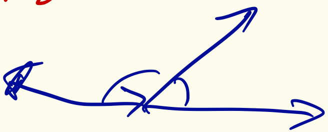
2 planes?

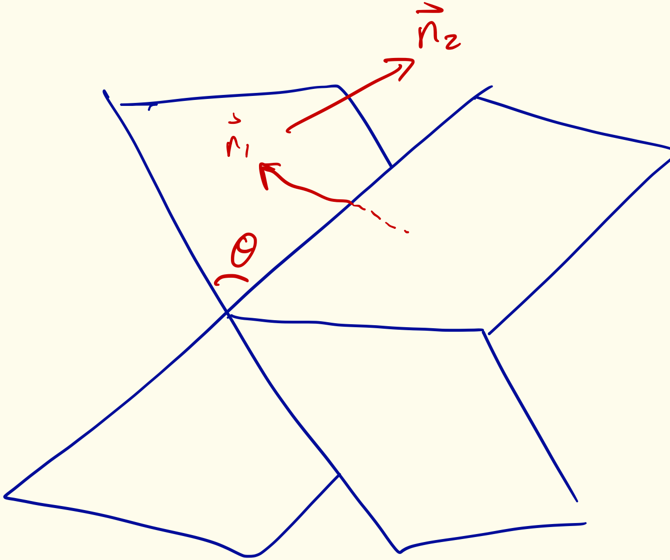




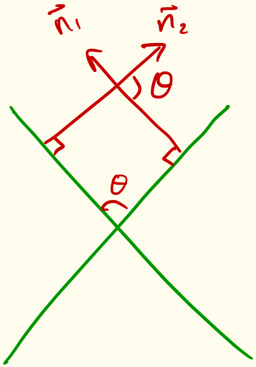
parallel.

$\vec{n}_1 \,//\, \vec{n}_2$





intersect



angle between planes

= angle between $\vec{n}_1, \vec{n}_2$!