MATH2023
Tutorial 5
March 23, 2019 Tutorial Notes by Ivan So

1 Review

1.1 Vector

- **Scalar** is an *one*-entry object belongs to $\mathbb{R}$.
- **Vector** is a *three*-entry object represented by $\mathbf{x} = (x_1, x_2, x_3) = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, which represent an "arrow" in 3D space.
- Addition of vector: follows the head to tail rules.
- The **norm** $\|\cdot\| : V \to \mathbb{R}$ is a function which measures the *length* of the arrow. Defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ (in our consideration).
- **Dot Product**:

$$\begin{aligned} "\cdot" &: V \times V \to \mathbb{R} \\ (\mathbf{v}_1, \mathbf{v}_2) &\mapsto \mathbf{v}_1 \cdot \mathbf{v}_2 := v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z} = \|\mathbf{v}_1\| \, \|\mathbf{v}_2\| \cos\theta \end{aligned}$$

- **Cross Product**: $" \times " : (\mathbf{v}_1, \mathbf{v}_2) \in V \times V \mapsto \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_{1x} & v_{1y} & v_{1z} \\ v_{2x} & v_{2y} & v_{2z} \end{bmatrix} \in V.$
- You are reminded the concept of **linearly independent**, **unit vector**, **orthogonal** and **determinant**.
- The way to find a equation of **line** passing through *two* points:
  1. Given points A, B, find $\overrightarrow{AB}$.
  2. Equation of line:$\overrightarrow{OA} + t\overrightarrow{AB}$ $t \in \mathbb{R}$.
- The way to find a equation of **plane** passing through *three* points:
  1. Given points A, B and C, find vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
  2. Find normal of plane $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.
  3. Let $P = (x, y, z)$, then equation of plane: $\overrightarrow{AP} \cdot \mathbf{n} = 0$.

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1.2 Vector Valued Functions, Tangent/Normal Vectors, Curvature and Arc Length of Curve

- **Vector valued function** $V : t \in \mathbb{R} \mapsto \mathbf{r}(t) = (r_1(t), r_2(t), r_3(t)) \in \mathbb{R}^3$.
- **Tangent vector** $\mathbf{T}(t) := \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, **Normal vector** $\mathbf{N}(t) := \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$.
- **Arc length parametrization**: parametrization of curve in which $|\mathbf{r}'(s)| = 1$.
- **Curvature** $\kappa := \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.
- **Arc length** calculation: The arc length of the curve defined by $\mathbf{r}(t)$ for $a \le t \le b$ is given by $L = \int_a^b \sqrt{[r_1'(t)]^2 + [r_2'(t)]^2 + [r_3'(t)]^2}dt$.

1.3 Multivariable Functions, Limits and Differentiation

- **Multivariable function** is defined as the map $f : \mathbb{R}^n \to \mathbb{R}$.
- **Domain** D: Subset $D \subset \mathbb{R}^n$ on which the function is defined. **Range**: The set $\{f(\mathbf{x})|\mathbf{x} \in D\}$.
- **Graph**: The set $\{(x, y, f(x, y))|(x, y) \in D\}$ for the domain D of f.
- **Level curve**: The curve with satisfying $f(x, y) = k$.
- **Limit**: The value that f "approach" as (x, y) approach (a, b)). Notationally,

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = L.$$

  - If the limit exist, then **MUST** be unique.
  - Evaluation can be done with *squeeze theorem* or *polar coordinates*.
- **Continuouity**: f is continuous at $\mathbf{x}_0$ if it satisfies $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$.
- **Partial derivative** $\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}_0} := \lim_{h\to 0} \frac{f(\mathbf{x}_0+h\mathbf{e}_i)-f(\mathbf{x}_0)}{h}$. Other notations: f_{x_i}. One can regard *unrelated variables* as constants in taking partial derivative.
- **Theorem** (Clairaut): f_{xy} and f_{yx} are continuous $\Rightarrow f_{xy} = f_{yx}$.

1.4 Tangent Plane, Directional Derivatives and Implicit Differentiation

- **Tangent plane** of a multivariable function $P(\mathbf{x}) := f(\mathbf{x}_0) + \sum_{i=1}^n f_{x_i}(\mathbf{x}_0)\Delta x_i$. **Total differential** $df := \sum_{i=1}^n f_{x_i}\Delta x_i$.
- **Theorem**: Normal vector of the surface defined by $x_{n+1} = f(\mathbf{x})$ is $(f_{x_1}, \cdots, f_{x_n}, -1)$.

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- **Gradient operator** maps a function into a vector by $\nabla f := \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right)$.

- **Directional derivative** in the direction of $\hat{\mathbf{v}}$ $D_{\hat{\mathbf{v}}} f(\mathbf{x}) := \lim_{t \to 0} \frac{f(\mathbf{x}+t\hat{\mathbf{v}})-f(\mathbf{x})}{t} = \nabla f \cdot \hat{\mathbf{v}}$.
  * Always the best to work with unit vector.

- Suppose $\mathbf{x}(t) \in \mathbb{R}^n$ are set of variables which depends on $\mathbf{t} \in \mathbb{R}^m$, then the **chain rule** in multivariable case is given by $\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \mathbf{x}}{\partial t_i}$. we can draw *tree diagram* for the chain relation.

- If $F(\mathbf{x}) = C$, we can find $\frac{\partial x_j}{\partial x_i}$ by **implicit differentiation**. Procedures:

  1. Take the partial derivative $F(\mathbf{x}) = C$ with respect to x_i, then we obtain the relation $\nabla F \cdot \frac{\partial \mathbf{x}}{\partial x_i} = 0$.
  2. Find the expression $\nabla F \cdot \frac{\partial \mathbf{x}}{\partial x_i} = 0$ with $\frac{\partial x_j}{\partial x_i}$ on left hand side.

1.5 Optimization Problem

- You are reminded the concepts of **local min/max** and **absolute min/max**.

- f has local min/max or saddle point at $\mathbf{x}_0 \Rightarrow f_{x_i} = 0$.

- The **Hessian matrix** for two-variable function f is defined as

$$H(f)(\mathbf{x}_0) := \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

  If (1) $f_x(a,b) = f_y(a,b) = 0$ and (2) the second derivatives are continuous, then:

  (a) In case $\det H(f)(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is *local minimum.*
  (b) In case $\det H(f)(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is *local maximum.*
  (c) In case $\det H(f)(a,b) < 0$ then $f(a,b)$ is *neither local maximum* nor *local minimum*, but a *saddle point* (think of a Pringles potato chip cut).
  (d) In case $\det H(f)(a,b) = 0$, the second derivative test is **inconclusive**.

- The absolute extrema of a function over a given domain is either the point of *local extrema* or on the *boundary.*

- The **method of Lagrange multiplier** in extrema evaluation is given as follows:

  1. Find all values of x_i's and λ_i's such that $\nabla f(\mathbf{x}) = \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x})$ given the constraints $g_i(\mathbf{x}) = k_i$.
  2. Evaluate f at all the $\mathbf{x}$'s obtained above. The largest and smallest give the extrema.

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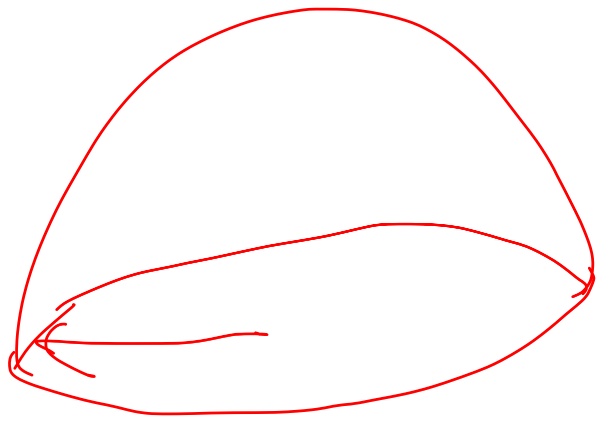
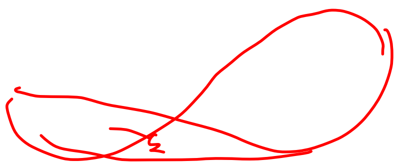
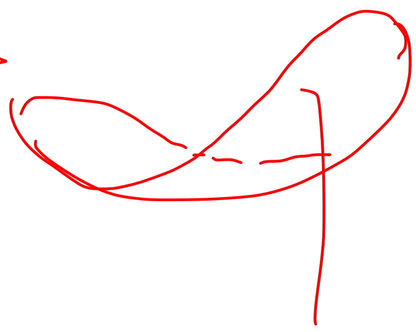




$\langle 1, 1, 1 \rangle$



$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}$$

$\nabla \times \vec{F}$

2 Problems

2.1 Vectors

1. (a) Find the equation of P_1, P_2 in which P_1 satisfies: $(1/3, 1/3, 1/3) \in P_1$ and perpendicular to the vector $\langle 1, 1, 1 \rangle$. P_2 satisfies: $(1, 1, 0) \in P_2$ and perpendicular to the vector $\langle 1, 2, 1 \rangle$.

   (b) Find the equation of line of intersection of P_1 and P_2.

2. Show that two lines $\mathbf{r}_1(t) = \mathbf{a} + \mathbf{v}t$ and $\mathbf{r}_2(t) = \mathbf{b} + \mathbf{u}t$ will intersect if $\mathbf{a} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{b} \cdot (\mathbf{u} \times \mathbf{v})$.

2.2 Vector Valued Function

1. Find the curvature expression for the curve $r = 4\cos 2\theta$ for $0 \leq \theta < 2\pi$.

$r = 4\cos 2\theta$

$$\begin{cases} x = 4\cos 2\theta \cos\theta \\ y = 4\cos 2\theta \sin\theta \end{cases}$$

$\vec{F}(r) = \langle 4\cos 2t \cos t, 4\cos 2t \sin t, 0 \rangle$

Need to do: compute $\vec{r}\,'(t)$, $\vec{r}\,''(t)$, $K(s) = \dfrac{r'(t) \times r''(t)}{|r'(t)|^3}$

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2.3 Limit

1. (a) Can the function $f(x,y) = \frac{\sin x \sin^3 y}{1-\cos(x^2+y^2)}$ be defined at $(0,0)$ such that it is continuous at $(0,0)$? If so, how?

   (b) Let $f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$.
   Calculate each of the following derivatives or explain why they do not exists:
   (i) $f_x(0,0)$ (ii) $f_y(0,0)$ (iii) $f_{xy}(0,0)$ (iv) $f_{yx}(0,0)$ (v) $f_{xx}(0,0)$.

2. Find the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y+xy^2+x^2+y^2+2xy}{1-\cos\sqrt{x^2+y^2}}$.

2.4 Derivatives

1. (a) Show that if f is differentiable and $z = xf\left(\frac{x}{y}\right)$, then all tangent planes to the graph of this equation pass through a common point. Find the common point.

   (b) Find the equation of the level curve of the function $z = g(x,y) = xf\left(\frac{x}{y}\right)$ at the point (x_0, y_0). Show that $\nabla g(x_0, y_0)$ is normal to the tangent line to the level curve at (x_0, y_0).

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1(a). $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin x \sin^3 y}{1-\cos(x^2+y^2)}$

$= \lim\limits_{r\to 0} \dfrac{\sin(r\cos\theta)\sin^3(r\sin\theta)}{1-\cos(r^2)}$

$= \lim\limits_{r\to 0} r^2 \sin^3\theta \cos^2\theta \cdot \lim\limits_{r\to 0} \dfrac{r^2}{1-\cos r^2}$

$= \lim\limits_{r\to 0} \dfrac{\sin(r\cos\theta)}{r\cos\theta} \cdot \lim\limits_{r\to 0} \dfrac{\sin^3(r\sin\theta)}{(r\sin\theta)^3}$

$= 0$

define $f(0,0) = 0$.

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$$\frac{x^3}{x^2+y^2}$$

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{\frac{\Delta x^3}{\Delta x^2} - 0}{\Delta x} = 1$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{\frac{0}{\Delta y^2} - 0}{\Delta y} = 0$$

$$f_x = \frac{(x^2+y^2)(3x^2) - (x^3)(2x)}{(x^2+y^2)^2}$$

$$= \frac{3x^4 + 3x^2y^2 - 2x^4}{(x^2+y^2)^2}$$

$$= \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2}$$

$$f_{xy} = \lim_{\Delta y \to 0} \frac{0-1}{(\Delta y)^2} = \text{does not exist.}$$

$$f_y = \frac{(x^2+y^2)(0) - x^3(2y)}{(x^2+y^2)^2} = \frac{-2yx^3}{(x^2+y^2)^2}$$

$$\lim_{\Delta x \to 0} \frac{\frac{0}{\Delta x^4} - 0}{\Delta x} = 0.$$

$$f_{xx} = \lim_{\Delta x \to 0} \frac{\frac{\Delta x^4}{\Delta x^4} - 1}{\Delta x} = 0.$$

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2. Find the limit $\lim_{(x,y)\to(0,0)} \frac{x^2y+xy^2+x^2+y^2+2xy}{1-\cos\sqrt{x^2+y^2}}$.

$$\frac{r^3\cos^2\theta\sin\theta + r^3\cos\theta\sin^2\theta + r^2 + 2r^2\sin\theta\cos\theta}{1-\cos r}$$

$$= \frac{r^3\sin 2\theta}{2(1-\cos r)} + \frac{r^2}{1-\cos r} + \frac{r^2\sin 2\theta}{1-\cos r}$$

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Sol.
$$z_x = f\left(\frac{x}{y}\right) + \frac{x}{y} f'\left(\frac{x}{y}\right)$$
$$z_y = -\frac{x^2}{y^2} f'\left(\frac{x}{y}\right)$$

$$P_{(x_0,y_0)}(x,y) = z_0 + z_x(x_0,y_0)(x-x_0) + z_y(x_0,y_0)(y-y_0)$$

Try:
$$P_{(x_0,y_0)}\big((x,y)=(0,0)\big) = \cancel{x_0 f\left(\frac{x_0}{y_0}\right)} + \cancel{\left(f\left(\frac{x_0}{y_0}\right) + \frac{x_0}{y_0} f'\left(\frac{x_0}{y_0}\right)\right)(-x_0)}$$
$$\cancel{\left(-\frac{x_0^2}{y_0^2} f'\left(\frac{x_0}{y_0}\right)(-y_0)\right)}$$

$= 0$

$(0,0,0)$ lies inside $P_{(x_0,y_0)}(x,y)$ $\forall (x_0, y_0)$

All tangent plane ~~share~~ the origin.

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(b) Find the equation of the level curve of the function $z = g(x, y) = xf\left(\frac{x}{y}\right)$ at the point (x_0, y_0). Show that $\nabla g(x_0, y_0)$ is normal to the tangent line to the level curve at (x_0, y_0).

Eq. of level curve: $xf\left(\frac{x}{y}\right) = x_0 f\left(\frac{x_0}{y_0}\right)$

We define $\langle x(t), y(t) \rangle$ to be the level curve $(*)$

$$\frac{d}{dt} g((x(t), y(t))) = \frac{d}{dt}\left(x_0 f\left(\frac{x_0}{y_0}\right)\right) = 0$$

$$\Rightarrow g_x x'(t) + g_y y'(t) = 0$$

$$\nabla g \cdot \langle x', y' \rangle = 0$$

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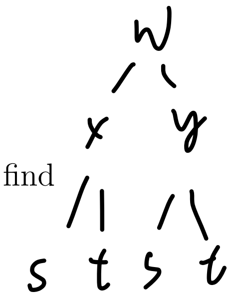

2. If $w = f(x, y)$ is differentiable. Suppose $x = s^2 + t^2$, $y = s^2 - t^2$. Use chain rule to find (i) w_s (ii) w_{st} (iii) w_{stt}.

i). $W_s = W_x X_s + W_y Y_s$

$= W_x(2s) + W_y(2s)$

$= 2s(W_x + W_y)$

ii). $W_{st} =$

2.5 Optimization Problems

1. Find the point on the surface $z^2 = -\frac{1}{2}x^2 + 2y^2 + xy$ which is closest to the point $\left(-\frac{1}{2}, -3, 0\right)$ through:

   (a) reducing the problem into an unconstrained problem of 2-variables.

   (b) the method of Lagrange multiplier.

$(0, -1, \pm\sqrt{2})$

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   (b) the method of Lagrange multiplier.

a. $f(x,y,z) = \sqrt{(x+\frac{1}{2})^2 + (y+3)^2 + z^2}$

$f(x,y,z(x,y)) = \sqrt{(x+\frac{1}{2})^2 + (y+3)^2 - \frac{1}{2}x^2 + 2y^2 + xy}$

$\Rightarrow$ Take partial d, solve critical point, check D test

b). $g(x,y,z) = (x+\frac{1}{2})^2 + (y+3)^2 + z^2$, subject to

$-\frac{1}{2}x^2 + 2y^2 + xy - z^2 = 0$

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2. Find the maximum and minimum of the function $f(x, y, z) = xy + yz$ subject to the constraints $x + 2y - 6 = 0$ and $x - 3z = 0$ using the method of Lagrange multiplier. Also find the values of the Lagrange multipliers. $\square$

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