MATH2023
Tutorial 4
March 6, 2019 Tutorial Notes by Ivan So

1 Review

- The **tangent plane** is the analogy of tangent line in single variable calculus, explicitly it is the first order approximation by partial derivatives given by:

$$P(\mathbf{x}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} f_{x_i}(\mathbf{x}_0)\Delta x_i.$$

  The idea of **total differential** ($df = \sum_{i=1}^{n} f_{x_i}\Delta x_i$) is derived based on linear approximation.

  - **Theorem**: Normal vector of the surface defined by $x_{n+1} = f(\mathbf{x})$ is $(f_{x_1}, \cdots, f_{x_n}, -1)$.

- FYI: In analytical aspects, a function $f : \mathbb{R}^n \to \mathbb{R}$ is **first order differentiable** if for any $\epsilon > 0$, there exist δ_ϵ such that $\|\mathbf{x} - \mathbf{x}_0\| < \delta_\epsilon \Longrightarrow |f(\mathbf{x}) - P(\mathbf{x})| < \epsilon \|\mathbf{x} - \mathbf{x}_0\|$.

- The **directional derivative** of $f(\mathbf{x})$ in the direction of $\hat{\mathbf{v}}$ by definition is

$$D_{\hat{\mathbf{v}}} f(\mathbf{x}) := \lim_{t \to 0} \frac{f(\mathbf{x} + t\hat{\mathbf{v}}) - f(\mathbf{x})}{t}$$

  It represent the derivative of the curve of cross section if we"cut" the surface from above by the line passing through the origin and in the direction of $\hat{\mathbf{v}}$.

- The **gradient operator** is an operator which maps a function into a vector by

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right).$$

  Indeed, the directional derivative can be rewritten as:

$$D_{\hat{\mathbf{v}}} f(\mathbf{x}) = \nabla f \cdot \hat{\mathbf{v}}$$

- Suppose $\mathbf{x} \in \mathbb{R}^n$ are set of variables which depends on $\mathbf{t} \in \mathbb{R}^m$, then the **chain rule** in multivariable case is given by

$$\frac{\partial f}{\partial t_i} = \nabla f \cdot \frac{\partial \mathbf{x}}{\partial t_i}.$$

  we can draw *tree diagram* for the chain relation.

- Given the relation $F(\mathbf{x}) = C$, we can find the dependence of x_j on x_i by **implicit differentiation**. The process of implicit differentiation is carried as follows:

1. Take the partial derivative $F(\mathbf{x}) = C$ with respect to x_i, then we obtain the relation $\nabla F \cdot \frac{\partial \mathbf{x}}{\partial x_i} = 0$.
2. Find the expression $\nabla F \cdot \frac{\partial \mathbf{x}}{\partial x_i} = 0$ with $\frac{\partial x_j}{\partial x_i}$ on left hand side.
3. Integrate the expression of $\frac{\partial x_j}{\partial x_i}$ with respect to x_i.

2 Problems

1. True or False

   (a) True or False. If $f(x, y) = \ln y$, then $\nabla f(x, y) = 1/y$.

   (b) Give the rationale for ∇f being the direction of steepest ascent/descent.

2. Find $\frac{\partial f(x/y)}{\partial y}$ and $\frac{\partial f(x/y)}{\partial x}$.

$f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)$ $\qquad$ $f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)$

3. Find the tangent plane of the surface $z = f(x, y) = \frac{1}{x^2+y^2+1}$ at $(1, 1, 1/3)$.

$\nabla f = \left\langle \frac{-2x}{(x^2+y^2+1)^2}, \frac{-2y}{(x^2+y^2+1)^2}, -1 \right\rangle$

$\nabla f = \left\langle \frac{-2}{9}, \frac{-2}{9}, -1 \right\rangle$

$-\frac{2}{9}x - \frac{2}{9}y - z = -\frac{7}{9}$

4. Find the directional derivative of $f(x, y) = \frac{1}{x^2+y^2+1}$ in the direction of $(1, 1)$ at (1,1).

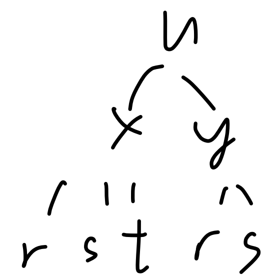
$\nabla f(1,1) = \left\langle \frac{-2}{9}, -\frac{2}{9} \right\rangle$

$\hat{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$

$-\frac{2}{9} \times \frac{\sqrt{2}}{2} - \frac{2}{9} \times \frac{\sqrt{2}}{2}$

$= -\frac{2\sqrt{2}}{9}$

5. Draw the tree diagram for $u = f(x, y)$, where $x = x(r, s, t)$, $y = y(r, s)$.



$f(x,y,z) = \cos(x+y+z) - xyz$

6. Find $\frac{\partial z}{\partial x}$ for z satisfying $xyz = \cos(x + y + z)$.

7. If $z = f(x - y)$, show that $z_x + z_y = 0$.

$Z_x = f'(x-y)$

$Z_y = -f'(x-y)$