Last Time

Chain Rule: $f(x, y, z)$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

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y is independent of x:

when x changes, y does NOT change.

y is dependent of x:

when x changes, y will change.

$f(x, y, z)$ (can change freely)

independent variables.

Implicit Differentiation: (n variables function)

$$f(x, y, z) = 0$$ Relationship between x, y, z.

$\begin{cases} 1 \text{ dependent variable} \\ n-1 \text{ independent variables} \end{cases}$

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$: z - dep.
x, y - indep.

$\frac{\partial y}{\partial x}, \frac{\partial y}{\partial z}$: y - dep.
x, z - indep.

$\frac{\partial x}{\partial y}, \frac{\partial x}{\partial z}$: x - dep.
y, z - indep.

Partial Derivatives

$\dfrac{\partial F}{\partial x}$: Want to know how F changes when x changes "freely" $\Longleftrightarrow$ independently.

always independent!

"Coordinates" : Information to specify location of a point.

$(x, y) \in \mathbb{R}^2 \quad \Longleftrightarrow \quad (r, \theta)$

independent | independent

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$x(r,\theta) \qquad \Longleftrightarrow \qquad r(x,y)$

$y(r,\theta) \qquad\qquad\qquad \theta(x,y)$

Implicit Differentiation DOES NOT WORK!!

$$\frac{\partial r}{\partial x} \neq \frac{1}{\cos\theta}$$

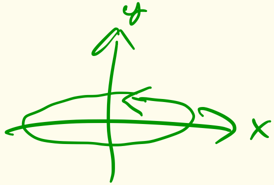
Chain Rule

$$f(x, y, t) = \sin(xy + t)$$

What is $\frac{\partial f}{\partial t} = ?$ change of height with respect to t

with **fixed** (x, y)

If $x(t) = \cos t \qquad y(t) = 2\sin t$



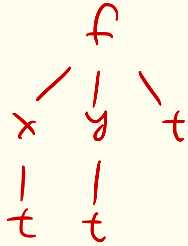
What is $\cancel{\frac{\partial f}{\partial t}} = ?$ $\qquad \frac{df}{dt}$

Bad notation $\qquad$ (Physics books

Bad notation!)

Better : $F(t) = f(x(t), y(t), t)$. $\qquad \frac{dF}{dt} = ?$

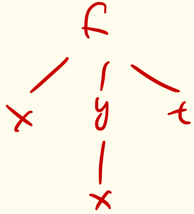
$$\frac{dF}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}.$$



Ex If $y = x^2$ What is $\cancel{\dfrac{\partial f}{\partial x}}$?

Bad Notation!

Better $F(x,t) = f(x, x^2, t) = \sin(x^3+t)$



then $$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

indep.

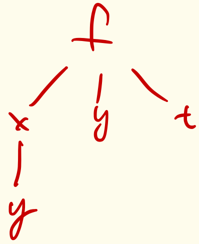
$$= y\cos(xy+t) + x\cos(xy+t)(2x)$$

$$= x^2\cos(x^3+t) + 2x^2\cos(x^3+t) \,//$$

$\cos(x^3+t)\cdot 3x^2$ =

What is $\cancel{\dfrac{\partial f}{\partial y}}$? $\quad \dfrac{\partial F}{\partial y}$?

indep.



$$F(y,t) = f(\sqrt{y}, y, t).$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial x}\frac{dx}{dy} + \frac{\partial f}{\partial y} \quad //$$

Gradient Vector

$\nabla f = \langle f_x , f_y \rangle$

Direction derivatives: $D_{\vec{u}} f = \nabla f \cdot \vec{u}$

$|\vec{u}| = 1$

unit vector.

$|D_{\vec{u}} f|$ is maximum when $\vec{u} \parallel \nabla f$

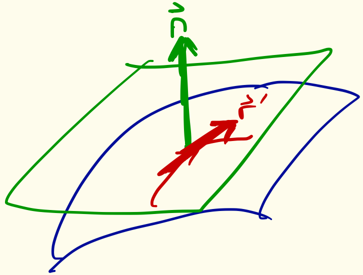
∇f is the direction of greatest ascent.

$\nabla f \perp$ level curves.

Tangent Plane to Surfaces

$$F(x, y, z) = 0$$

$\vec{n} = \nabla F$



$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

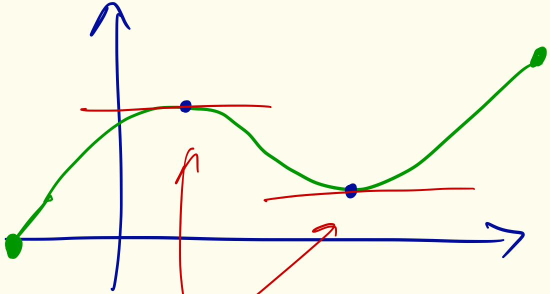
$$F(x(t), y(t), z(t)) = 0$$

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

$$\nabla F \cdot \vec{r}\,'(t) = 0$$

Normal vector of tangent plane at (x_0, y_0, z_0) of $F(x, y, z) = 0$ is $\nabla F(x_0, y_0, z_0)$

Max / Min Problem



local max/min. $\longrightarrow f'(x) = 0$

global (absolute) max/min

$$\begin{cases} f''(x) < 0 & : \text{max} \\ f''(x) > 0 & : \text{min} \end{cases}$$

2nd derivative test.

Def (a,b) is a local maximum (minimum) for $f(x,y)$ if

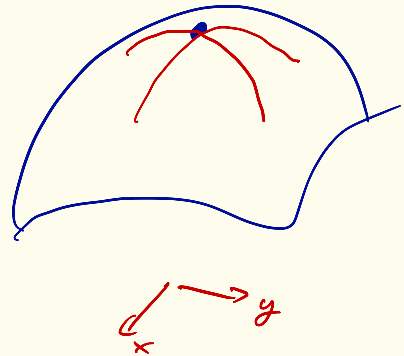
$f(x,y) \le f(a,b)$ for (x,y) near (a,b).

$(\geq)$ $\rightarrow$ maximum value

Thm If $f(x,y)$ has a local max or min at (a,b), and f_x, f_y exist, then

$$f_x(a,b) = f_y(a,b) = 0.$$

or $\nabla f(a,b) = 0$



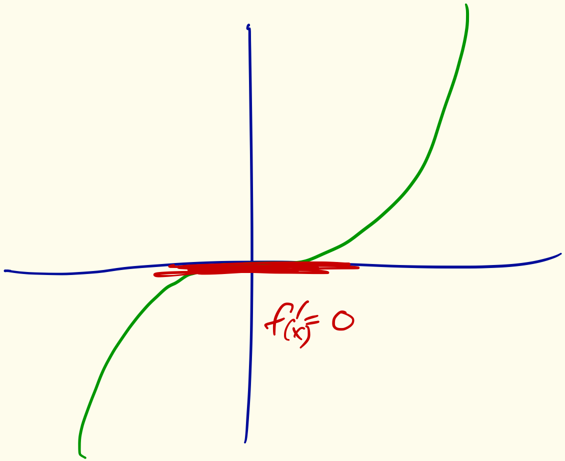
Def (a,b) is called a critical point of $f(x,y)$ if

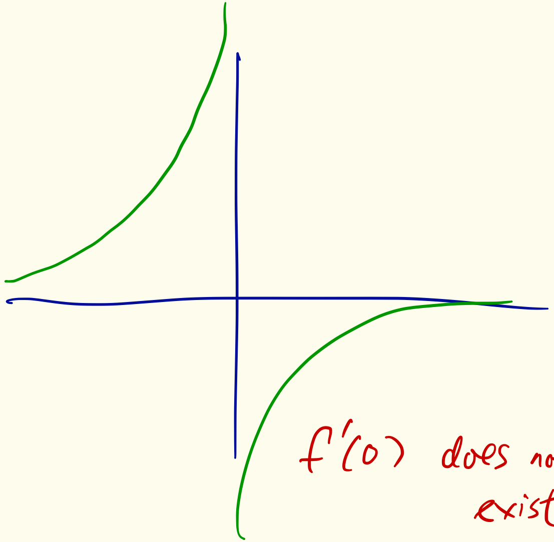
$\nabla f(a,b) = 0$ or f_x or f_y does not exist at (a,b)

local max/min $\Longrightarrow$ critical point

$\not\Longleftarrow$

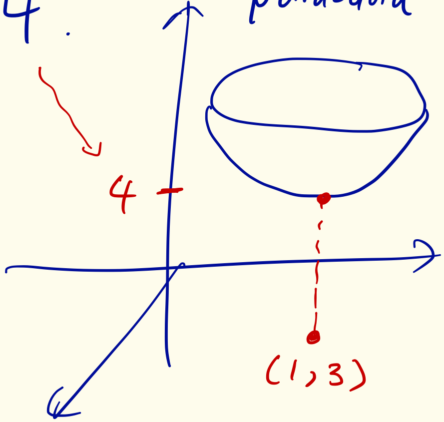
Made with Goodnotes


$f'(x) = 0$


$f'(0)$ does not exist

<u>Ex</u> $f(x,y) = x^2 + y^2 - 2x - 6y + 14.$

$$= (x-1)^2 + (y-3)^2 + 4.$$



$\Rightarrow$ It has one (global) local minimum $(1,3)$ with value 4.

$$\nabla f = \langle 2x-2,\ 2y-6 \rangle = \vec{0}$$

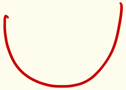
$$\Rightarrow (x,y) = (1,3)$$

Ex $f(x,y) = x^2 - y^2$

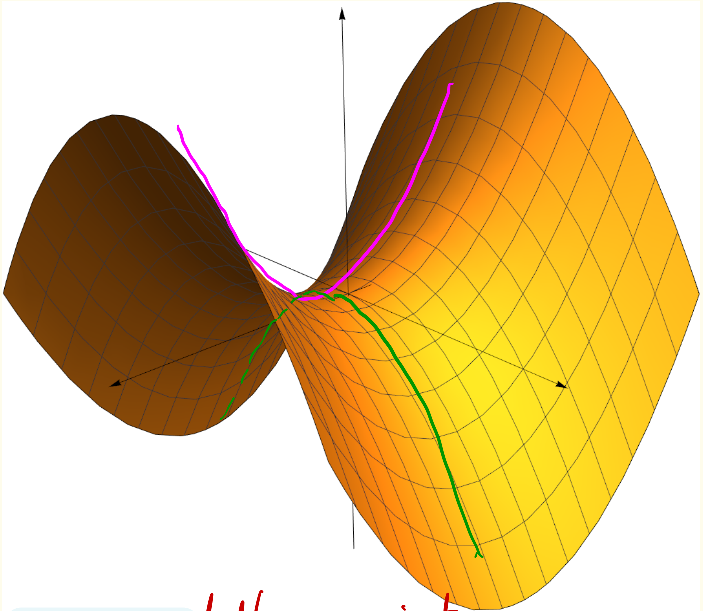
$\nabla f = \langle 2x, -2y \rangle = \vec{0}$

$\Rightarrow (x,y) = (0,0)$

along x-axis: $(y=0)$: x^2



along y-axis: $(x=0)$: $-y^2$



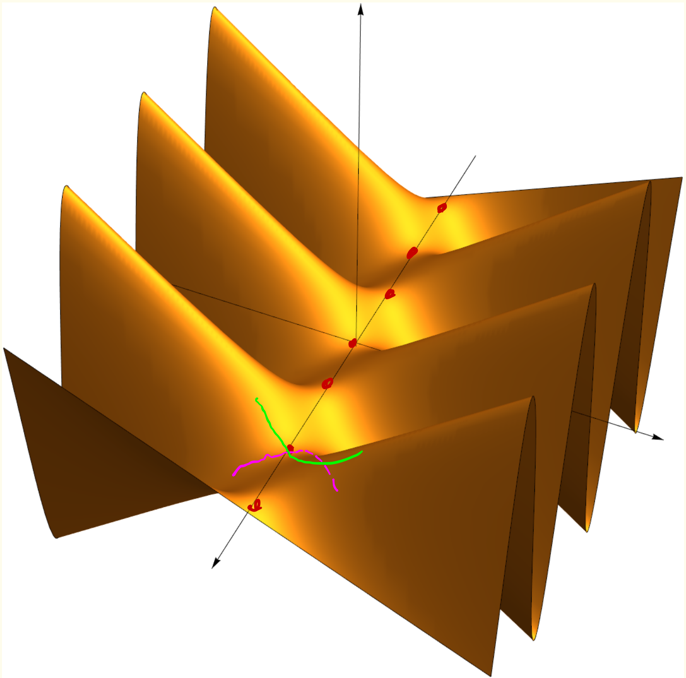
saddle point

<u>Ex</u> $f(x,y) = y \sin x$

$$\nabla f = \langle y\cos x, \sin x \rangle = \vec{0}$$

$$\Rightarrow \begin{cases} x = k\pi \\ y = 0. \end{cases}, \quad k \in \mathbb{Z}.$$

Every Critical Points are saddle points.



<u>Ex</u> $f(x,y) = e^x \cos y$

$$\nabla f = \langle e^x \cos y, -e^x \sin y \rangle = \langle 0,0 \rangle$$

No solutions!

$\Leftrightarrow$ No critical points.