MATH2023
Tutorial 6
March 19, 2019 Tutorial Notes by Ivan So

1 Review

- The **double integral** is defined as $\int\int_R f(\mathbf{x})dA := \lim_{n\to\infty}\sum_{i,j=1}^n f(\mathbf{x}_i^*)\Delta A_i$.
- Midpoint rule? Average value?
- **Fubini's Theorem**: If f is (1) discontinuous on finitely many number of points and (2) bounded over the rectangle $R = \{(x,y)|(x,y)\in[a,b]\times[c,d]\}\subset\mathbb{R}^2$, then

$$\int_a^b\int_c^d f(x,y)dydx = \int_c^d\int_a^b f(x,y)dxdy.$$

- Consider function of two variables. A region D is said to be of **type I** (**type II**) if $D = \{(x,y)|a\le x\le b \text{ and } g_1(x)\le y\le g_2(x)\}$ ($D = \{(x,y)|c\le y\le d \text{ and } h_1(y)\le x\le h_2(y)\}$) where g_1 and g_2 are continuous functions.
- Integration for function over type I or type II region is well-defined. But when changing order of integration, one would have to beware of the integration limits.
- If $R = R_1\sqcup R_2$, then $\int\int_R f(x,y)dA = \int\int_{R_1} f(x,y)dA + \int\int_{R_2} f(x,y)dA$.
- Recall in **polar coordinates**, $r^2 = x^2+y^2$, $\tan\theta = y/x$. In other words, $x = r\cos\theta$ and $y = r\sin\theta$. Integration of two variable function can be done with polar coordinates, with $dA = rdrd\theta$.

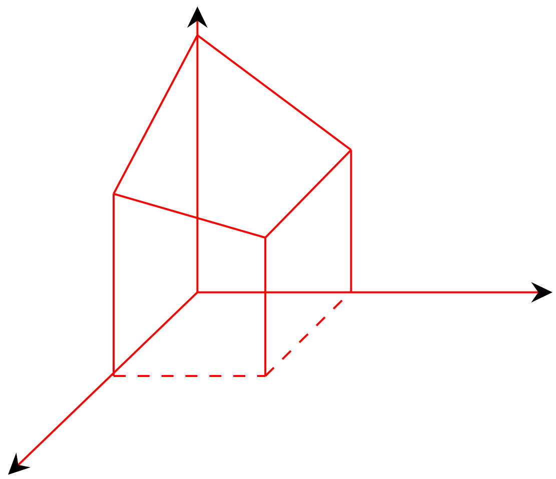
2 Problems

1. True or False

   (a) $\int_1^2\int_3^4 x^2e^ydydx = \int_1^2 x^2dx\int_3^4 e^ydy$.
   True. Can check from direct evaluation.

   (b) $\int_0^1\int_0^x\sqrt{x+y^2}dydx = \int_0^x\int_0^1\sqrt{x+y^2}dxdy$.
   False. The second integral is not well-defined (you will find it depends on x) after integration.

2. Sketch the solid bounded by the constraints $0\le x,y\le 1$ and $0\le z\le 4-x-2y$. Evaluate its volume.
   Solution: Sketch of the solid:



Volume:

$$V = \int_0^1 \int_0^1 (4 - x - 2y) dx dy = \int_0^1 \left(4 - \frac{1}{2} - 2y\right) dy = \frac{5}{2}.$$

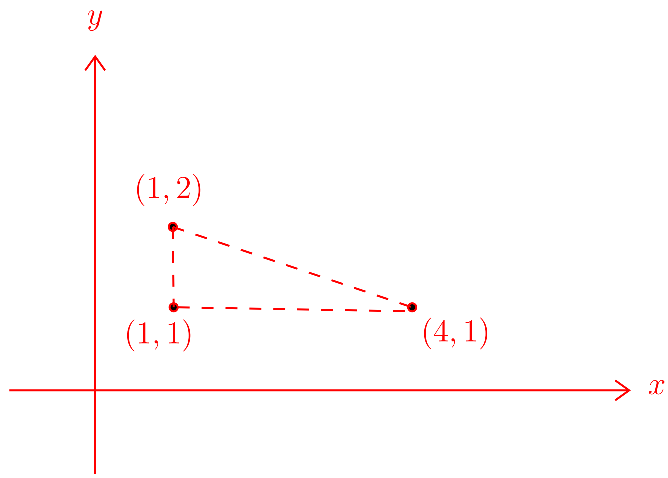
3. Show that $0 \leq \int \int_R \sin \pi x \cos \pi x dA \leq \frac{1}{32}$ for $R = [0, 1/4] \times [1/4, 1/2]$.
Solution: On $[0, 1/4] \times [1/4, 1/2]$, $0 \leq \sin \pi x \cos \pi x \leq 1$, therefore in the concerned domain

$$0 \leq \sin \pi x \cos \pi x = \frac{1}{2} \sin 2\pi x \leq \frac{1}{2} \Rightarrow 0 \int \int \sin \pi x \cos \pi x dA \leq \frac{1}{2} \int \int dA = \frac{1}{32}.$$

4. Find the average value of $f(x, y) = x^2 y$ over the rectangle with vertices $(-1, 0), (-1, 5), (1, 5), (1, 0)$.
Solution: Recall when we have discrete data, mean $\bar{x} = \sum_{i=1}^{n} x_i / n$. Using integration, the concerned mean μ is

$$\mu = \frac{\int_0^5 \int_{-1}^1 x^2 y dx dy}{\int_0^5 \int_{-1}^1 dx dy}$$

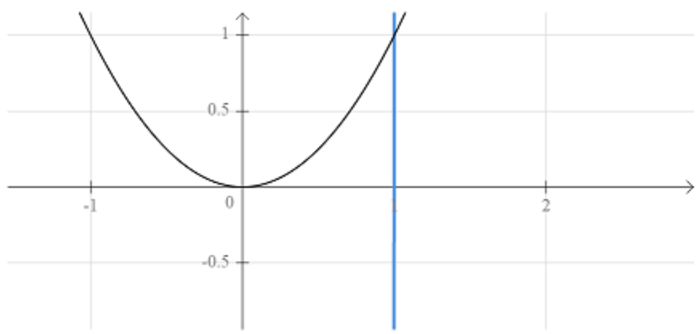
5. Write the volume integral of the solid bounded by $z = xy$ above a triangle with vertices $(1, 1), (4, 1)$ and $(1, 2)$.
Solution: The domain of integration is



In the domain of integration, $z > 0$, so the concerned volume is

$$V = \int_1^4 \int_1^{-\frac{1}{4}x+\frac{9}{4}} xy\, dydx.$$

6. Evaluate $\int\int_D x\cos y dA$ over where D is the region bounded by $y = 0,\ y = x^2,\ x = 1$.
Solution: The domain of integration:



So the concerned integral is:

$$I = \int_0^1 \int_0^{x^2} x\cos y dydx = \int_0^1 x\sin x^2 = \frac{1}{2}(1-\cos 1)$$

Additional (Change of order of integration):

$$I = \int_0^1 \int_{\sqrt{y}}^1 x\cos y dxdy = \frac{1}{2}\int_0^1 (1-y)\cos y dy = \frac{1}{2}(1-\cos 1).$$

7. Prove that if $m \le f(x,y) \le M$ for all (x,y) in D, then

$$mA(D) \le \int\int_D f(x,y)dA \le MA(D).$$

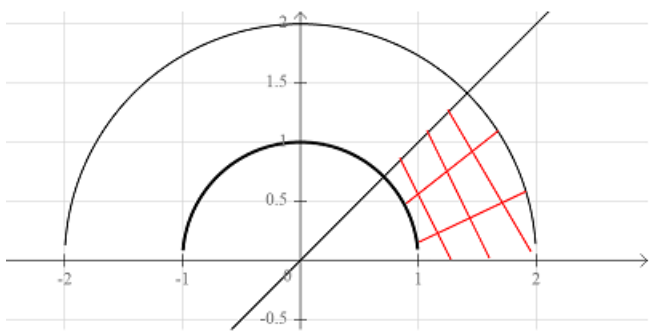
Solution: Recall that integration is the generalization of summation, so

$$m \le f(x,y) \le M \Rightarrow mA(D) \le \int\int f(x,y)dA \le MA(D).$$

8. Use polar coordinates to combine and evaluate the sum

$$\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xydydx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xydydx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xydydx$$

Solution: The integration area is given by the following:



By $x = r\cos\theta,\ y = r\sin\theta$, we have

$$\begin{aligned}
&\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xydydx + \int_{1}^{\sqrt{2}}\int_{0}^{x} xydydx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xydydx \\
&= \int_{0}^{\pi/4}\int_{1}^{2} (r\cos\theta)(r\sin\theta)rdrd\theta \\
&= \frac{15}{8}\int_{0}^{\pi/4} \sin 2\theta d\theta = \frac{15}{16}
\end{aligned}$$

9. Evaluate $\int_0^\infty e^{-x^2}dx$.

Solution: Single variable method will not work. However, let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$, then

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \quad \text{(dummy variables)} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dy dx \\ &= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r dr d\theta \\ &= \frac{1}{2} \int_{0}^{2\pi} d\theta = \pi \\ \Rightarrow I &= \sqrt{\pi} \end{aligned}$$

I has an even integrand, therefore $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.