Last Time Method of Lagrange Multiplier :

Solve Constrained Optimization Problem

max/min : $f(x,y,z)$

subject to $g(x,y,z)=k$

Lagrange Multiplier

① : $\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases}$ $\qquad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

4 equations in 4 unknowns (x, y, z, λ)

② If g defines a bounded closed domain

By Extreme Value Theorem $\Rightarrow$ absolute max/min exist.

Compare the values of $f(x,y,z)$ at the solutions.

Ex Maximize $x+2y+3z$ subject to $x^2+y^2+z^2=1$.

① : $\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases}$

$f(x,y,z) = x+2y+3z$

$g(x,y,z) = x^2+y^2+z^2$

$\nabla f = \langle 1, 2, 3 \rangle$

$\nabla g = \langle 2x, 2y, 2z \rangle$

$\begin{cases} 1 = 2\lambda x \\ 2 = 2\lambda y \\ 3 = 2\lambda z \end{cases} \Rightarrow \frac{1}{x} = \frac{2}{y} = \frac{3}{z} = 2\lambda$

or $\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \Rightarrow y = 2x$, $z = 3x$

$g = 1 \iff x^2+4x^2+9x^2 = 1 \Rightarrow x^2 = \frac{1}{14}$ $\quad x = \pm\frac{1}{\sqrt{14}}$

$y = \pm\frac{2}{\sqrt{14}}$, $z = \pm\frac{3}{\sqrt{14}}$

$f(x,y,z) = \pm\frac{1}{\sqrt{14}} \pm \frac{4}{\sqrt{14}} \pm \frac{9}{\sqrt{14}} = \pm\sqrt{14}$.

$\Rightarrow \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$ gives maximum.

Really is just asking which unit direction $\vec{u} = \langle x, y, z \rangle$ such that $\underset{\langle 1,2,3 \rangle}{\nabla f} \cdot \vec{u}$ is maximized?

Ex $f(x,y) = 2x + 3y$ subject to $\sqrt{x} + \sqrt{y} = 5$ $\leftarrow g(x,y)$

$$\nabla f = \langle 2, 3 \rangle$$

$$\nabla g = \left\langle \frac{1}{2\sqrt{x}}, \frac{1}{2\sqrt{y}} \right\rangle$$

$$\nabla f = \lambda \nabla g \iff \begin{cases} 2 = \dfrac{\lambda}{2\sqrt{x}} \\ 3 = \dfrac{\lambda}{2\sqrt{y}} \end{cases} \implies \begin{matrix} 4\sqrt{x} = \lambda \\ 6\sqrt{y} = \lambda \end{matrix} \implies 4\sqrt{x} = 6\sqrt{y}$$

$$16x = 36y$$

$$y = \frac{16x}{36}$$

$$g = 5 : \sqrt{x} + \sqrt{\frac{16x}{36}} = 5$$

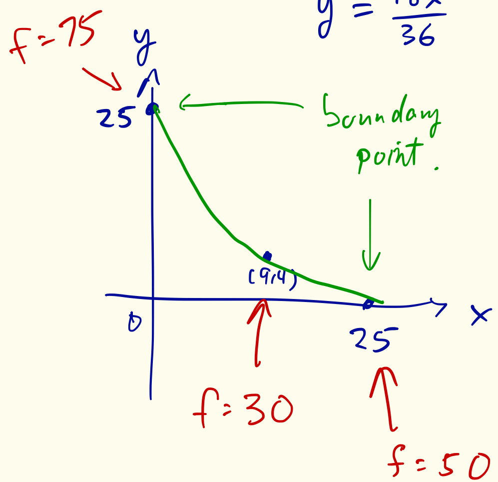
$$\sqrt{x} + \frac{2}{3}\sqrt{x} = 5$$

$$\frac{5}{3}\sqrt{x} = 5$$

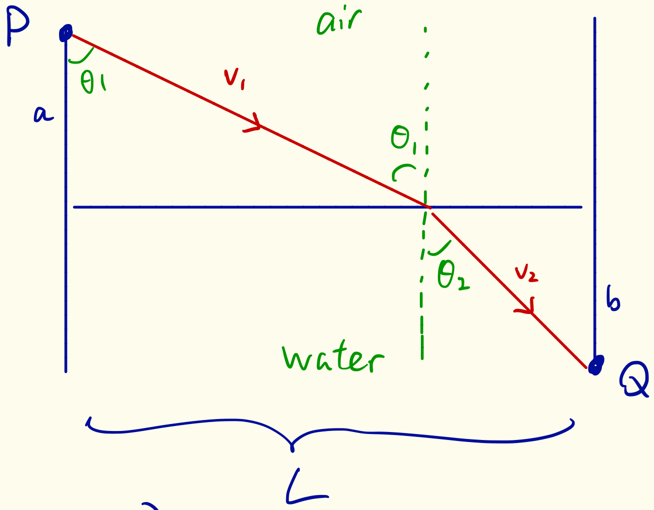
$$\sqrt{x} = 3$$

$$x = 9 \implies y = 4.$$

$(x,y) = (9,4)$. is minimum.



Snell's Law



<u>minimize</u> the time needed to travel from P to Q

$$T = \frac{\text{distance}}{\text{speed}} = \frac{a}{v_1 \cos\theta_1} + \frac{b}{v_2 \cos\theta_2} \quad \leftarrow f$$

$$L = a \tan\theta_1 + b \tan\theta_2 \quad \leftarrow g$$

$$\frac{\partial}{\partial\theta} \frac{1}{\cos\theta} = \frac{\sin\theta}{\cos^2\theta}$$

$$\left(\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}\right) \rightarrow \nabla T = \lambda \nabla L$$

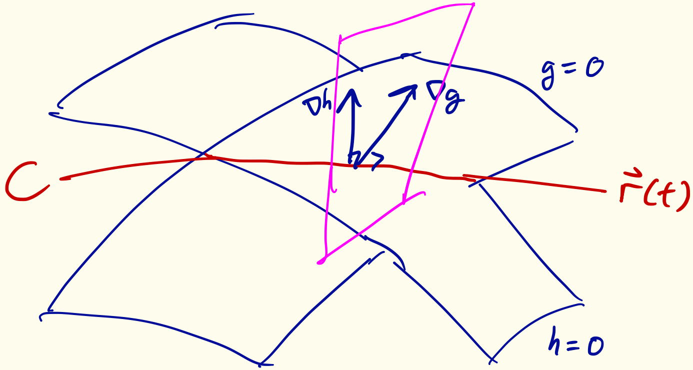
$$\begin{cases} \frac{\partial}{\partial\theta_1} : \frac{\cancel{a}}{v_1} \frac{\sin\theta_1}{\cancel{\cos^2\theta_1}} = \lambda \frac{\cancel{a}}{\cancel{\cos^2\theta_1}} \\ \frac{\partial}{\partial\theta_2} : \frac{\cancel{b}}{v_2} \frac{\sin\theta_2}{\cancel{\cos^2\theta_2}} = \frac{\lambda \cancel{b}}{\cancel{\cos^2\theta_2}} \end{cases}$$

$$\Longleftrightarrow \lambda = \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

Snell's Law

2 Constraints

Max. $f(x,y,z)$ subject to $\begin{cases} g(x,y,z)=0 \\ h(x,y,z)=0 \end{cases}$



$\vec{r}(t)$ lies on both $g=0$ & $h=0$

$\iff \vec{r}\,'(t) \perp \nabla g$ and $\vec{r}\,'(t) \perp \nabla h$

max $f(\vec{r}(t)) \Rightarrow \nabla f \cdot \vec{r}\,'(t) = 0 \iff \nabla f \perp \vec{r}\,'(t)$

$\iff$ ∇f lies in the plane spanned by ∇g and ∇h.

$\iff \nabla f = \lambda \nabla g + \mu \nabla h$

Lagrange Multipliers.

2 Constraints

Max. $f(x,y,z)$ subject to $\begin{cases} g(x,y,z)=0 \\ h(x,y,z)=0 \end{cases}$

$$① \begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 0 \\ h = 0 \end{cases}$$

5 equations in 5 unknowns (x, y, z, λ, μ)

Lagrange Multipliers Method in General

max. $f(x_1, x_2, \ldots, x_n)$

subject to $\begin{cases} g_1(x_1, \ldots, x_n) = 0 \\ g_2(x_1, \ldots, x_n) = 0 \\ \vdots \\ g_k(x_1, \ldots, x_n) = 0 \end{cases}$

$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$

$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_k \nabla g_k \\ g_1 = 0 \\ \vdots \\ g_k = 0 \end{cases}$$

$n+k$ equations in $n+k$ unknowns,

$x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k$.

Meaning of Lagrange Multiplier?

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \iff \nabla \mathcal{L} = 0 \quad \text{critical pts.}$$

$$\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda g(x,y).$$

Budget-Constraint:

max. Revenue $R(x,y)$

x — apple
y — banana.

subject to $g(x,y) = c$

Using Lagrange multiplier $\Rightarrow$ max $\iff (x^*, y^*, \lambda^*)$

Let $M^* = R(x^*, y^*)$

Treat c as variable.

If we change c, max will change!

$M^*(c)$

<u>Fact</u> $$\frac{dM^*(c)}{dc} = \lambda^*(c)$$

In <u>Economics</u> : λ^* = "shadow price"

= Change in revenue by loosening the constraint by \$1.

<u>Pf</u> $$M^*(c) = \hat{\mathcal{L}}(x^*(c), y^*(c), \lambda^*(c), c)$$

$\hat{\mathcal{L}} = f(x,y) - \lambda(g(x,y) - c)$. $\left(\frac{\partial \hat{\mathcal{L}}}{\partial c} = \lambda\right)$

$$\frac{dM^*(c)}{dc} = \underbrace{\frac{\partial \hat{\mathcal{L}}}{\partial x}}_{0} \frac{dx}{dc} + \underbrace{\frac{\partial \hat{\mathcal{L}}}{\partial y}}_{0} \frac{dy}{dc} + \underbrace{\frac{\partial \hat{\mathcal{L}}}{\partial \lambda}}_{0} \frac{d\lambda}{dc} + \frac{\partial \hat{\mathcal{L}}}{\partial c} \Longrightarrow \lambda^*(c).$$

by $\nabla \mathcal{L} = 0$

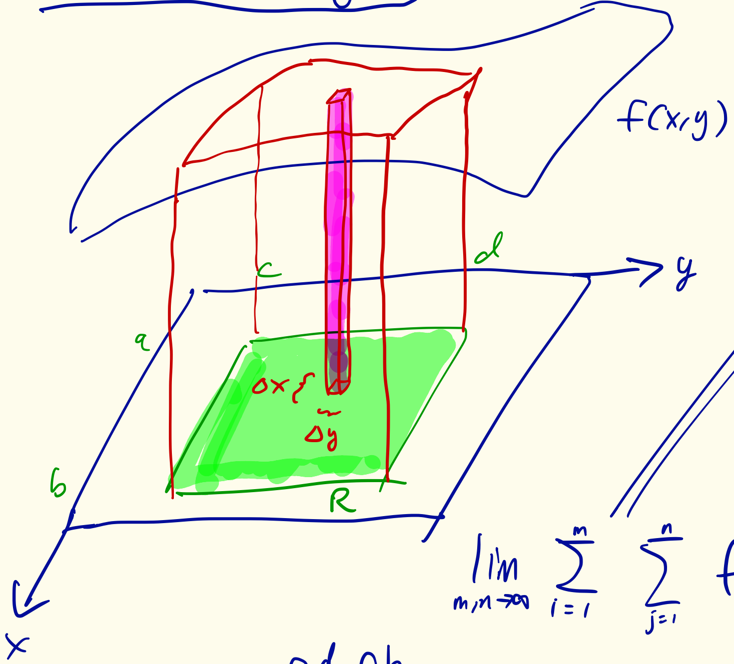
Part II Integrations

$\int_D f(x)\,dx \Rightarrow$ 

Double Integral defined on

$$R = [a,b] \times [c,d]$$

$$= \{(x,y) \mid a \le x \le b,\ c \le y \le d\}.$$



$$\iint_R f(x,y)\,dA = \text{Volume of } S$$

$$S = \left\{(x,y,z) \;\middle|\; \begin{array}{l} 0 \le z \le f(x,y) \\ (x,y) \in R \end{array}\right\}$$

$$\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\,\underset{\Delta x \Delta y}{\Delta A}$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

Fubini's Theorem (3033)

If f is continuous on $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f\, dA = \int_c^d \int_a^b f\, dx\, dy = \int_a^b \int_c^d f\, dy\, dx.$$

Properties

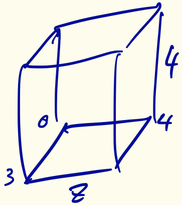
Linearity: $\iint_R (f+g)\, dA = \iint_R f\, dA + \iint_R g\, dA$

$$\iint_R cf\, dA = c \iint_R f\, dA \qquad c \in \mathbb{R}$$

If $f \ge g$, then $\iint_R f\, dA \ge \iint_R g\, dA$.

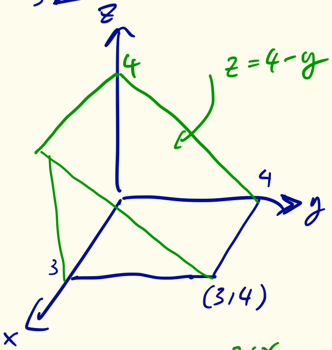
__Ex__ $R = [0,3] \times [0,4]$

$\iint_R 4\, dA =$



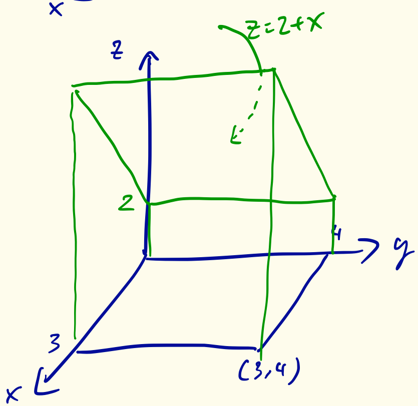
$= 3 \times 4 \times 4 = 48.$

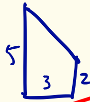
$\iint_R (4-y)\, dA$



$= \left(\frac{4 \times 4}{2}\right) \times 3 = 24.$

$\iint_R (2+x)\, dA$





$\times\, h$

2

$= \frac{7 \times 3}{2} \times 4 = 42.$

?

Ex $\iint_R (x^2-4y)\,dA \qquad R=[1,3]\times[2,4].$

$$\int_2^4\left(\int_1^3 (x^2-4y)\,dx\right)dy = \int_2^4 \left(\frac{x^3}{3}-4yx\right)\Big|_1^3 \,dy$$

y is constant here.

$$= \int_2^4 9-12y-\frac{1}{3}+4y \quad dy$$

$$= \int_2^4 \left(\frac{26}{3}-8y\right)dy.$$

$$= \frac{26}{3}y-4y^2\Big|_2^4 = -\frac{92}{3}$$

$$\int_1^3\left(\int_2^4 (x^2-4y)\,dy\right)dx = \int_1^3 \left(x^2y-2y^2\right)\Big|_2^4 \,dx$$

const

$$= \int_1^3 4x^2-32-2x^2+8 \,dx = \int_1^3 2x^2-24\,dx$$

$$= \frac{2x^3}{3}-24x\Big|_1^3 = -\frac{92}{3}$$

same.

<u>Ex</u> $\iint_R y \sin(xy)\, dA$ for $R = [1,2] \times [0,\pi]$

Choose easiest order to integrate! $\int dy$ hard

$\int dx$ easy!

$$\int_0^{\pi} \int_1^2 y \sin(xy)\, dx\, dy$$

$$= \int_0^{\pi} \cancel{y} \cdot \frac{-\cos xy}{\cancel{y}} \Bigg|_1^2 dy$$

$$= \int_0^{\pi} (\cos y - \cos 2y)\, dy$$

$$= \sin y - \frac{\sin 2y}{2} \Bigg|_0^{\pi} = 0.$$

$$\int_1^2 \int_0^{\pi} y \sin xy\, dy\, dx$$

Integration by parts!

Harder!