MATH 2023 – Multivariable Calculus

Lecture #08 Worksheet ♯ March 5, 2019

Problem 1. Find the maximum, minimum and saddle points of the following functions:

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$$f(x,y) = x^2 + y^2 + x^{-2}y^{-2}$$

$$f(x,y) = x^2 y e^{-x^2-y^2}$$

Problem 1. Find the maximum, minimum and saddle points of the following functions:

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$\nabla f < 4x^3 - 4y, \; 4y^3 - 4x >$

$(x,y) = (0,0), (-1,-1), (1,1)$

$f_{xx} = 12x^2 \qquad f_{xy} = -4$

$f_{yy} = 12y^2$

$$D = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

For $(0,0)$, $D = -16$, saddle.

For $(-1,-1)$, $D = 8$, $f_{xx} > 0$, min.

For $(1,1)$, $D = 8$, $f_{xx} > 0$, min.

$$f(x,y) = x^2 + y^2 + x^{-2}y^{-2}$$

$$\nabla f = \left\langle 2x - \frac{2}{y^2x^3},\ 2y - \frac{2}{x^2y^3} \right\rangle$$

$$2x - \frac{2}{y^2x^3} = 0$$

$$2x = \frac{2}{y^2x^3}$$

$$y^2x^4 = 1 \qquad x^2y^4 = 1.$$

$y^6 = 1$

$y = \pm 1$

$x = \pm 1.$

$$x^2y^4 = y^2x^4$$

$$xy^2 = \pm yx^2$$

$$y^2 = \pm yx$$

$$y = \pm x.$$

$$f_{xx} = 2 + \frac{6}{y^2x^4}$$

$$f_{yy} = 2 + \frac{6}{x^2y^4}$$

$$f_{xy} = \frac{4}{y^3x^3}$$

For $(1,1)$

$D = 64 - 16 > 0,$

$f_{xx} > 0$, min.

For $(-1,-1)$.

$D = 64 - 16 > 0$

$f_{xx} > 0$ min.

$$f(x,y) = x^2 y e^{-x^2-y^2}$$

$$\nabla f = \langle\ yx^2 e^{-x^2-y^2}(-2x) + e^{-x^2-y^2}(2xy),$$
$$x^2 y e^{-x^2-y^2}(-2y) + x^2 e^{-x^2-y^2}\ \rangle$$

$$\begin{cases} e^{-x^2-y^2}(-2yx^3+2xy) = 0 \\ e^{-x^2-y^2}(-2x^2y^2+x^2) = 0 \end{cases}$$

$$\begin{cases} -2yx^3+2xy=0 \\ -2x^2y^2+x^2=0 \end{cases}$$

$2xy(1-x^2)=0$

$x^2(1-2y^2)=0$

From ①, $xy=0$ or $-2x^2+2=0$

$x^2=1$

$x=\pm 1$.

$(x=0$ or $y=0.)$

$\cancel{(1,0)}$

$(0,0)$

$\cancel{(-1,0)}$

From ②, $x^2=0$ or $-2y^2+1=0$, $y=\pm\frac{\sqrt{2}}{2}$.

$(0,\frac{\sqrt{2}}{2})$, $(0,-\frac{\sqrt{2}}{2})$ $(0,0)$

Made with Goodnotes

Problem 2. Find the shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ using calculus. Verify the result using the distance formula.

$$D = \left| \frac{1-2-4}{\sqrt{1+4+1}} \right| =$$

$$2.04124$$

$$D = (x-1)^2 + y^2 + (z+2)^2$$

$$x+2y+z=4.$$

$$z = 4-x-2y.$$

$$D = (x-1)^2 + y^2 + (6-x-2y)^2$$

$$\nabla D = \langle 2(x-1) + 2(6-x-2y)(-1),$$
$$2y + 2(6-x-2y)(-2) \rangle$$

$$= \langle 2x-2 - 2(6-x-2y),$$
$$2y - 4(6-x-2y) \rangle$$

$$= \langle 2x-2-12+2x+4y,$$
$$2y-24+4x+8y \rangle$$

$$= \langle 4x+4y-14,\ 4x+10y-24 \rangle$$

$$\begin{cases} 4x+4y-14=0 \\ 4x+10y-24=0 \end{cases}$$

$x+y=\frac{14}{4}=\frac{7}{2}$

$x=\frac{7}{2}-y.$

$4x+10y=24$

$2x+5y=12.$

$2(\frac{7}{2}-y)+5y=12$

$7-2y+5y=12$

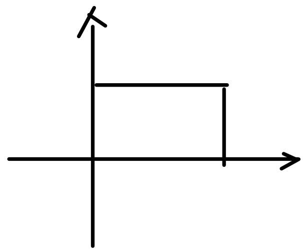
$3y=5$

$y=\frac{5}{3},\ x=\frac{7}{2}-\frac{5}{3}=\frac{11}{6}.$

$D=2.04124$

Problem 3. Find the maximum and minimum of $f(x, y) = x^2 - 2xy + 2y$ on

- the rectangle $R = \{(x, y) : 0 \leq x \leq 3, 0 \leq y \leq 2\}$.



- the closed triangle T bounded by $(0, 0), (0, 2), (3, 0)$

- the unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$

Problem 3. Find the maximum and minimum of $f(x,y)=x^2-2xy+2y$ on

- the rectangle $R=\{(x,y):0\le x\le 3, 0\le y\le 2\}$.

$$\nabla f = \langle 2x-2y, -2x+2\rangle$$

Solve $\begin{cases} 2x-2y=0 \\ -2x+2=0 \end{cases}$ $\quad x=1, y=1.$

$f_{xx}=2$

$f_{yy}=0$

$f_{xy}=-2$

At end point:

$(0,0)$ $f(0,0)=0$

$(3,2)$ $f(3,2)=1$

$(0,2)$ $f(0,2)=4$

$(3,0)$ $f(3,0)=9.$

$(1,1)$ $f(1,1)=1$

$x^2-6x+9-9+6$

$=(x-3)^2-3$

$x=0$ $\quad 2y.$

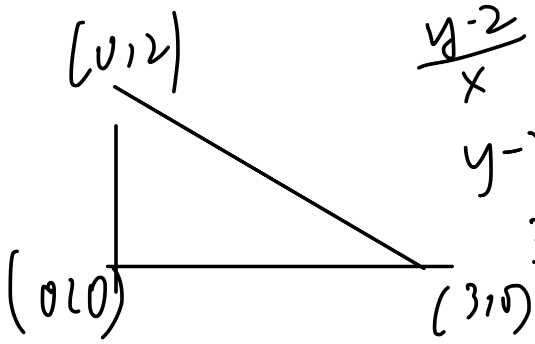
$y=0$ $\quad x^2$

$x=3$ $\quad 9-4y$

$y=3$ $\quad x^2-6x+6$

Max	when	Min	when
6	$y=3$	0	$y=0$
9	$x=3$	0	$x=0$
-3	$y=3$	5	$y=0$
-3	$x=3$	6	$x=0$

- the closed triangle T bounded by $(0,0), (0,2), (3,0)$



$\frac{y-2}{x} = \frac{-2}{3}$

$y-2 = -\frac{2}{3}x$

$3y-6 = -2x$

$2x+3y-6=0$

$3y = 6-2x$

$y = \frac{6-2x}{3}$

Max	Min
6	0
9	0

$x=0$ $\quad 2y$

$y=0$ $\quad x^2$

$y=\frac{6-2x}{3}$ $\quad x^2-2x\left(\frac{6-2x}{3}\right)+2\left(\frac{6-2x}{3}\right)$

$= x^2 + \frac{-12x+4x^2}{3} + \frac{12-4x}{3}$

$= x^2 - 4x + \frac{4}{3}x^2 + 4 - \frac{4}{3}x$

$= \frac{7}{3}x^2 - \frac{16}{3}x + 4$

$\frac{7}{3}\left(x^2 - \frac{16}{7}x + \left(\frac{8}{7}\right)^2 - \left(\frac{8}{7}\right)^2\right) + 4$

$= \frac{7}{3}\left(x-\frac{8}{7}\right)^2 - \frac{7}{3}\cdot\frac{64}{49} + 4$

$= \frac{7}{3}\left(x-\frac{8}{7}\right)^2 - \frac{64}{21} + 4$

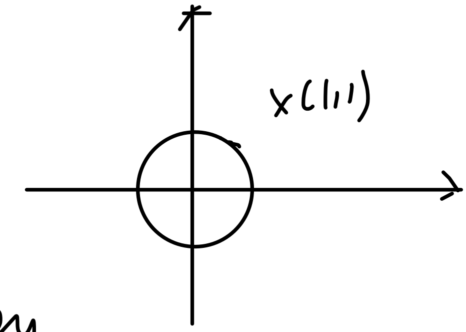
$\frac{20}{21}$, $x = \frac{8}{7}$.

9. Max.

- the unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$

only test boundary:

$x = \cos\theta$, $y = \sin\theta$.



$x^2 - 2xy + 2y$.