MATH2023
Tutorial 5
March 6, 2019 Tutorial Notes by Ivan So

1 Review

- A function has a **local maximum** (respectively, **local minimum**) at $\mathbf{x}_0$ if there exist $\delta > 0$ such that for any $\mathbf{x}$ satisfying $\|\mathbf{x} - \mathbf{x}_0\| < \delta$, then $f(\mathbf{x}_0) \geq f(\mathbf{x})$ (respectively, $f(\mathbf{x}_0) \leq f(\mathbf{x})$).

- A function has a **absolute maximum** (respectively, **absolute minimum**) at $\mathbf{x}_0$ if for any $\mathbf{x}$ in the domain D, $f(\mathbf{x}_0) \geq f(\mathbf{x})$ (respectively, $f(\mathbf{x}_0) \leq f(\mathbf{x})$).

- "f has local maximum/minimum at $\mathbf{x}_0 \implies f_x(\mathbf{x}_0) = f_y(\mathbf{x}_0) = 0$". Notice that the converse is in general **NOT** true. i.e. $f_x(\mathbf{x}_0) = f_y(\mathbf{x}_0) = 0 \not\Longrightarrow f$ has local maximum/minimum at $\mathbf{x}_0$ in general.

- The **Hessian matrix** of a function f is defined as

$$H(f)(\mathbf{x}_0) := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

  For function two variables, if (1) $f_x(a,b) = f_y(a,b) = 0$ and in addition (2) the second derivatives are continuous the Hessian matrix enable us to have the **second derivative test** for two variables functions. We have the following cases splitting:

  (a) In case $\det H(f)(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is *local minimum*.
  (b) In case $\det H(f)(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is *local maximum*.
  (c) In case $\det H(f)(a,b) < 0$ then $f(a,b)$ is *neither local maximum* nor *local minimum*, but a *saddle point* (think of a Pringles potato chip cut).
  (d) In case $\det H(f)(a,b) = 0$, the second derivative test is **inconclusive**.

- The absolute extrema of a function over a given domain is either the point of *local extrema* or on the *boundary*.

- Motivation for *Lagrange multiplier*: You have a function $f(\mathbf{x})$. What will be the extrema of $f(\mathbf{x})$ subject to the constraint $g(\mathbf{x}) = k$?

- The **Lagrange multiplier** λ_i's for a function f with respect to the constraint $g_i(\mathbf{x}) = k_i$ are the constant in which $\nabla f(\mathbf{x}_0) = \sum_{i=1}^m \lambda_i \nabla g_i(\mathbf{x}_0)$ for some $\mathbf{x}_0$ in the domain of f.

- The **method of Lagrange multiplier** in extrema evaluation is given as follows:

  1. Find all values of $\mathbf{x}$ and λ_i such that $\nabla f(\mathbf{x}) = \sum_{i=1}^m \lambda_i \nabla g_i(\mathbf{x})$ and $g_i(\mathbf{x}) = k_i$.
  2. Evaluate f all the $\mathbf{x}$'s obtained above. The largest and smallest give the extrema.

2 Problems

1. True or False

   (a) If $f(x, y)$ has two local maxima, then it must have a local minimum.

   (b) If $f(2, 1)$ is a critical point and $f_{xx}(2, 1)f_{yy}(2, 1) < [f_{xy}(2, 1)]^2$, then f has a saddle point at $(2, 1)$.

   (c) If f has a local minimum at (a, b) and differentiable at (a, b), then $\nabla f(a, b) = \mathbf{0}$.

2. Find the absolute maximum and minimum for $f(x, y) = x^2 + y^2 + x^2y + 4$ over the $[-1, 1] \times [-1, 1]$.

3. Find three positive numbers whose sum is 100 and whose product is a maximum.

4. Find an equation of the plane that passes through the point $(1, 2, 3)$ and cuts off the smallest volume in the first octant.

5. Find the local maximum and minimum values and saddle points of the function $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 10$.

6. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 + y^2$ subject to the constraint $xy = 1$.

7. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.

8. (a) Find the maximum value of $f(\mathbf{x}) = \sqrt[n]{x_1 \cdots x_n}$ with the constraints $x_1, \cdots, x_n > 0$ and $x_1 + \cdots + x_n =$ constant.

   (b) Prove that if $x_1, \cdots, x_n > 0$, then

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}.$$

   Deduce under what circumstances will give rise to equality.

9. (a) Maximize $\mathbf{x} \cdot \mathbf{y}$ subject to the constraint $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2 = 1$.

   (b) Put $\mathbf{x} = \frac{\mathbf{a}}{\|a\|}$ and $\mathbf{y} = \frac{\mathbf{b}}{\|b\|}$. Prove the Cauchy-Schwarz inequality

$$\mathbf{a} \cdot \mathbf{b} \leq \|\mathbf{a}\| \, \|\mathbf{b}\| \, .$$