MATH2023 Multivariable Calculus 2013

From the textbook Calculus - Several Variables (5th) by R. Adams, Addison/Wesley/Longman.

Homework 3

(*Total: 21 questions*)

Ex. 10.5

4 Identify the surface represented by the equation and sketch the graph

$$x^2 + 4y^2 + 9z^2 + 4x - 8y = 8.$$

10 Identify the surface represented by the equation and sketch the graph $x^2 + 4z^2 = 4$.

Ex. 12.1

4 Specify the domain of the function $f(x, y) = \dfrac{xy}{x^2 - y^2}$.

10 Specify the domain of the function $f(x, y, z) = \dfrac{e^{xyz}}{\sqrt{xyz}}$.

14 Sketch the graph of the function $f(x, y) = 4 - x^2 - y^2$, $\quad (x^2 + y^2 \leqslant 4,\ x \geqslant 0,\ y \geqslant 0)$.

24 Sketch some of the level curves of the function $f(x, y) = \dfrac{y}{x^2 + y^2}$.

36 Find $f(x, y, z)$ if for each constant C the level surface $f(x, y, z) = C$ is a plane having intercepts C^3, $2C^3$, and $3C^3$ on the x-axis, the y-axis and the z-axis respectively.

42 Describe the "level hypersurfaces" of the function $f(x, y, z, t) = x^2 + y^2 + z^2 + t^2$.

Ex. 12.2

6 Evaluate $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^2 + (y-1)^2}$, or explain why it does not exist.

12 Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4 + y^4}$, or explain why it does not exist

13 How can the function $f(x, y) = \dfrac{x^2 + y^2 - x^3y^3}{x^2 + y^2}$, $\quad (x, y) \neq (0, 0)$, be defined at the origin so that it becomes continuous at all points of the xy-plane?

17 Let $\mathbf{u} = u\,\mathbf{i} + v\,\mathbf{j}$ be a unit vector, and let

$$f_{\mathbf{u}}(t) = f(a + tu,\, b + tv)$$

be the single-variable function obtained by restricting the domain of $f(x, y)$ to points of the straight line through (a, b) parallel to $\mathbf{u}$, If $f_{\mathbf{u}}(t)$ is continuous at $t = 0$ for every unit vector $\mathbf{u}$, does it follow that f is continuous at (a, b)? Conversely, does the continuity of f at (a, b) guarantee the continuity of $f_{\mathbf{u}}(t)$ at $t = 0$? Justify your answers.

18 What condition must the nonnegative integers m, n, and p satisfy to guarantee that

$$\lim_{(x,y)\to(0,0)} \frac{x^m y^n}{(x^2 + y^2)^p}$$

exists? Prove your answer.

Ex. 12.3

2 Find all the first partial derivatives of the function specified and evaluate them at the given point

$$f(x, y) = xy + x^2, \qquad (2, 0).$$

8 Find all the first partial derivatives of the function specified and evaluate them at the given point

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}, \quad (-3, 4).$$

9 Find all the first partial derivatives of the function specified and evaluate them at the given point

$$w = x^{(y \ln z)}, \qquad (e, 2, e).$$

12 Calculate the first partial derivatives of the given function at $(0, 0)$. You will have to use the definition of *first partial derivatives*.

$$f(x, y) = \begin{cases} \dfrac{x^2 - 2y^2}{x - y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

28 Show that the given function satisfies the given partial differential equation

$$w = x^2 + yz, \quad x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 2w.$$

36 Let $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

Note that f is not continuous at $(0,0)$. Therefore its graph is not smooth there. Show, however, that $f_x(0,0)$ and $f_y(0,0)$ both exist. Hence the existence of partial derivatives does not imply that a function of several variables is continuous. This is in contrast to the single-variable case.

Lecture Note (Exercises for students - (p9) - just after Ex. 1.13), Ex. 2.6 (p16)

Homework 4

(*Total: 18 questions*)

Ex. 12.4

4 Find all the second partial derivatives of the function $z = \sqrt{3x^2+y^2}$.

16 Let $f(x,y) = \begin{cases} \dfrac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

Calculate $f_x(x,y)$, $f_y(x,y)$, $f_{xy}(x,y)$ and $f_{yx}(x,y)$ at point $(x,y) \neq (0,0)$. Also calculate these derivatives at $(0,0)$. Observe that $f_{yx}(0,0) = 2$ and $f_{xy}(0,0) = -2$. Does this results contradict Theorem 1? Explain why.

18 Show that the function $u(x,y,t) = t^{-1}e^{-(x^2+y^2)/4t}$ satisfies the two-dimensional heat equation

$$u_t = u_{xx} + u_{yy}.$$

Ex. 12.5

2 Write appropriate versions of the Chain Rule for the indicated derivatives.

$$\partial w/\partial t \quad \text{if} \quad w = f(x,y,z), \quad \text{where} \quad x = g(s),\ y = h(s,t), \quad \text{and} \quad z = k(t).$$

12 Find the indicated derivative, assuming that the function $f(x,y)$ has continuous first partial derivatives

$$\frac{\partial}{\partial y} f(yf(x,t),\ f(y,t))$$

20 Find $\dfrac{\partial^3}{\partial t^2 \partial s} f(s^2-t, s+t^2)$ in terms of partial derivatives of f.

21 Assume that f has continuous partial derivatives of all orders and suppose that $u(x,y)$ and $v(x,y)$ have continuous second partial derivatives and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Suppose also that $f(u,v)$ is a harmonic function of u and v. Show that $f(u(x,y),\ v(x,y))$ is a harmonic function of x and y.

Ex. 12.6

6 Use suitable linearization to find approximate value for the given function at the points indicated.

$$f(x,y) = xe^{y+x^2} \quad \text{at} \quad (2.05, -3.92).$$

12 By approximately what percentage will the value of $w = x^2y^3/z^4$ increase or decrease if x increases by 1%, y increases by 2%, and z and increases by 3%?

17 Prove that if $f(x,y)$ is differentiable at (a,b), then $f(x,y)$ is continuous at (a,b).

18 Prove the following version of the Mean-Value Theorem: if $f(x,y)$ has first partial derivatives continuous near every point of the straight line segment joining the points (a,b) and $(a+h, b+k)$, then there exists a number θ satisfying $0 < \theta < 1$ such that

$$f(a+h, b+k) = f(a,b) + hf_x(a+\theta h, b+\theta k) + kf_y(a+\theta h, b+\theta k).$$

(*Hint:* apply the single-variable Mean-Value Theorem to $g(t) = f(a+th, b+tk)$.)

Ex. 12.7

6 Let $f(x,y) = \dfrac{2xy}{x^2+y^2}$. Find

(a) the gradient of the given function at the point $(0,2)$,

(b) an equation of the plane tangent to the graph of the given function at the point $(0,2)$ whose x and y coordinates are given, and

(c) an equation of the straight line tangent, at the point $(0,2)$, to the level curve of the given function passing through that point.

10 Find the rate of change of the given function at the given point in the specified direction.
$f(x, y) = 3x - 4y$ at $(0, 2)$ in the direction of the vector $-2\,\mathbf{i}$.

14 Let $f(x, y) = \ln \|\mathbf{r}\|$ where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{i}$. Show that $\nabla f = \dfrac{\mathbf{r}}{\|\mathbf{r}\|^2}$.

16 Show that, in terms of polar coordinates (r, θ) (where $x = r\cos\theta$, and $y = r\sin\theta$), the gradient of a function $f(r, \theta)$ is given by

$$\boldsymbol{\nabla} f = \frac{\partial f}{\partial r}\,\widehat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\widehat{\boldsymbol{\theta}},$$

where $\widehat{\mathbf{r}}$ is a unit vector in the direction of the position vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\widehat{\boldsymbol{\theta}}$ is a unit vector at right angles to $\widehat{\mathbf{r}}$ in the direction of increasing θ.

18 In what direction at the point (a, b, c) does the function $f(x, y, z) = x^2 + y^2 - z^2$ increase at half of its maximal rate at that point?

21 The temperature $T(x, y)$ at points of the xy-plane is given by $T(x, y) = x^2 - 2y^2$.

(a) Draw a contour diagram for T showing some isotherms (curves of constant temperature).

(b) In what direction should an ant at position $(2, -1)$ move if it wishes to cool off as quickly as possible?

(c) If an ant moves in that direction at speed k (units distance per unit time), at what rate does it experience the decrease of temperature?

(d) At what rate would the ant experience the decrease of temperature if it moves from $(2, -1)$ at speed k in the direction of the vector $-\,\mathbf{i} - 2\,\mathbf{j}$?

(e) Along what curve through $(2, -1)$ should the ant move in order to continue to experience maximum rate of cooling?

26 Find a vector tangent to the curve of intersection of the two cylinders $x^2+y^2 = 2$ and $y^2+z^2 = 2$ at the point $(1, -1, 1)$.

Homework 5

(*Total: 7 questions*)

Ex. 13.1

4 Find and classify the critical points of the given function $f(x, y) = x^4 + y^4 - 4xy$.

20 Find the absolute minimum value of $f(x, y) = x + 8y + \dfrac{1}{xy}$ in the first quadrant $x > 0$, $y > 0$. How do you know that an absolute minimum exists?

27 Let $f(x, y) = (y - x^2)(y - 3x^2)$. Show that the origin is a critical point of f and that the restriction of f to every straight line through the origin has a local minimum value at the origin. (That is, show that $f(x, kx)$ has a local minimum value at $x = 0$ for every k, and that $f(x, y)$ has a local minimum value at $y = 0$.) Does $f(x, y)$ have a local minimum value at the origin? What happens to f on the curve $y = 2x^2$? What does the second derivative test say about this situation?

Ex. 13.3

3 Find the distance from the origin to the plane $x + 2y + 2z = 3$,

(a) using a geometric argument (no calculus),

(b) by reducing the problem to an unconstrained problem in two variables, and

(c) using the method of Lagrange multipliers.

12 Find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ on the ellipse formed by the intersection of the cone $z^2 = x^2 + y^2$ and the plane $x - 2z = 3$.

22 Find the maximum and minimum values of $xy + z^2$ on the ball $x^2 + y^2 + z^2 \leqslant 1$. Use Lagrange multipliers to treat the boundary case.

26 What is the shortest distance from the point $(0, -1)$ to the curve $y = \sqrt{1 - x^2}$? Can this problem be solved by the Lagrange multiplier method? Why?