MATH 2023 – Multivariable Calculus

Lecture #11 Worksheet ♮ March 14, 2019

Problem 1. Sketch the following curves in polar coordinates :

$\theta = 0 : r = -\infty$

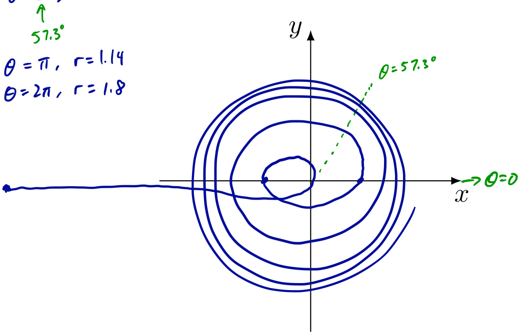
$\theta = 1$, $r = 0$

$57.3°$

$\theta = \pi$, $r = 1.14$

$\theta = 2\pi$, $r = 1.8$

(a) $r = \ln\theta$



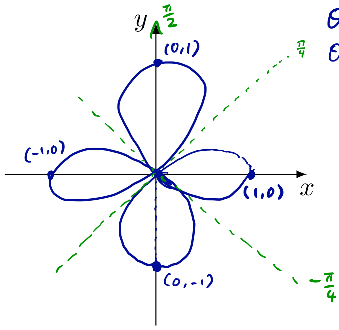
(b) $r = \cos 2\theta$ ← periodic in $0 \le \theta \le \pi$

$\theta = 0$, $r = 1$

$\theta = \frac{\pi}{4}$, $r = 0$

$\theta = \frac{\pi}{2}$, $r = -1$

⋮



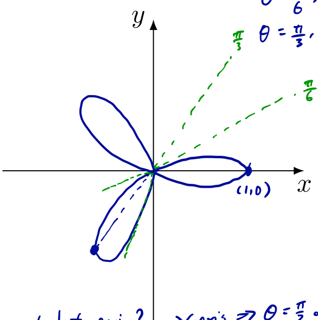
→ $\theta = 0$

(c) $r = \cos 3\theta$

$\theta = 0$, $r = 1$

$\theta = \frac{\pi}{6}$, $r = 0$

$\theta = \frac{\pi}{3}$, $r = -1$



where it hit axis? y-axis ⇒ $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, $r = 0$

x-axis ⇒ $\theta = 0$ or π, $r = 1$ -1

(d) $r = \cos\dfrac{\theta}{2}$

$\theta = 0$: $r = 1$

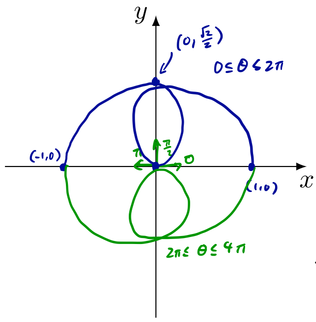
$\theta = \frac{\pi}{2}$: $r = \frac{\sqrt{2}}{2}$

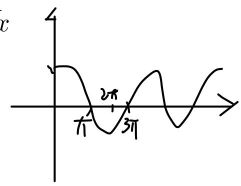
$\theta = \pi$: $r = 0$

⋮

$\frac{3\pi}{2}$: $r = -\frac{\sqrt{2}}{2}$

2π : $r = -1$





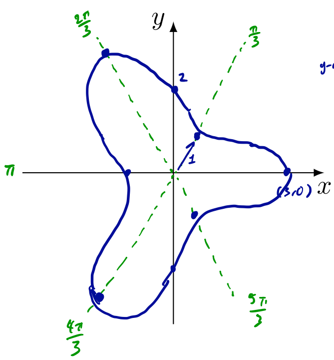
(e) $r = 2 + \cos 3\theta$

$\theta = 0$, $r = 3$

y-axis: $\theta = \frac{\pi}{2}$, $r = 2$

r is biggest when $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, 0$

r is smallest when $\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$



(f) $r = 1 + \sin\theta$

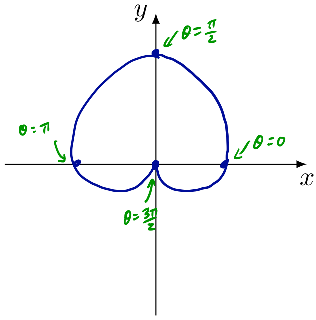
$\theta = 0$, $r = 1$

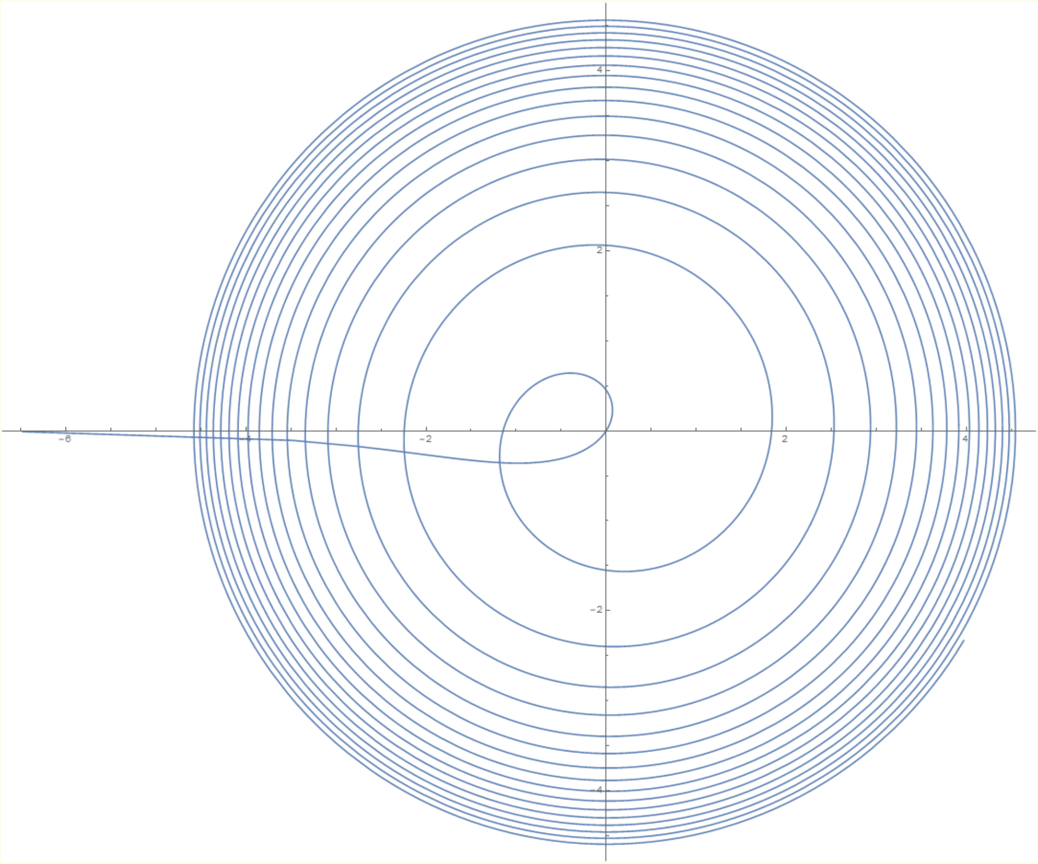
$\theta = \frac{\pi}{2}$, $r = 2$

$\theta = \pi$, $r = 1$

$\theta = \frac{3\pi}{2}$, $r = 0$

$\theta = 2\pi$, $r = 1$

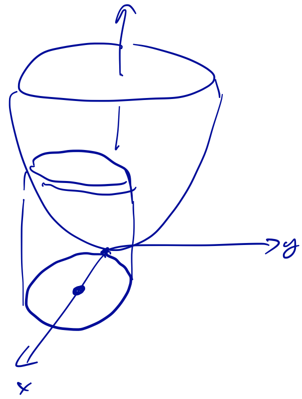




4
2
-2
-4
-6
-2
2
4

$(x-1)^2+y^2=1$

$x^2-2x+1+y^2=1$



Problem 2. Find the volume:

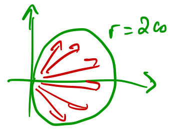
(a) Under $z = x^2 + y^2$ and inside the cyliner $x^2 + y^2 = 2x$

$$\iint_R x^2+y^2\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\, r\, dr\, \theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(2\cos\theta)^4}{4}\, d\theta$$

$x^2-2x+y^2=0$

$r^2-2r\cos\theta=0$

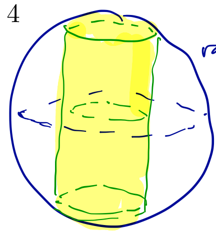
$r=2\cos\theta$



$r=2\cos\theta$

θ range from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

(b) Under the sphere $x^2 + y^2 + z^2 = 9$ and inside the cylinder $x^2 + y^2 = 4$



radius = 3

$= 2\times$(Volume above xy plane)

$$=2\iint_R \sqrt{9-x^2-y^2}\, dA = 2\int_0^{2\pi}\int_0^2 \sqrt{9-r^2}\, r\, dr\, d\theta$$

circle of radius 2

$$= 4\pi \int_0^2 \sqrt{9-r^2}\, r\, dr$$

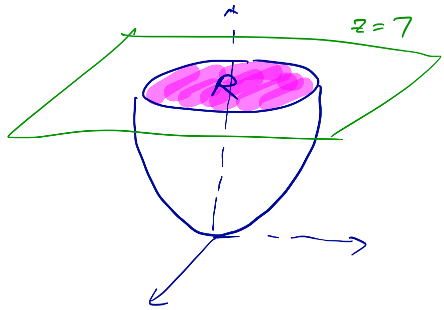
$u=r^2$ . . . .

(c) Bounded by $z = x^2 + y^2$ and $z = 7$

$$\iint_R 7-(x^2+y^2)\, dA = \int_0^{2\pi}\int_0^{\sqrt{7}} (7-r^2)\, r\, dr\, d\theta$$

$x^2+y^2=7$

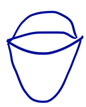
difference in height.

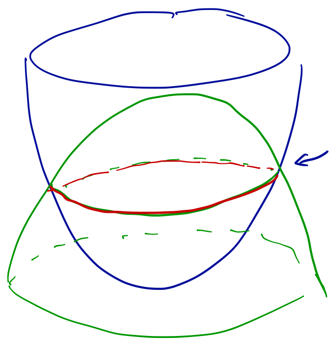


(d) Bounded by $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$

$3x^2+3y^2=4-x^2-y^2$

$\Rightarrow x^2+y^2=1$





$x^2+y^2=1$

$z=3$

$$\iint_R (4-x^2-y^2)-(3x^2+3y^2)\, dA$$

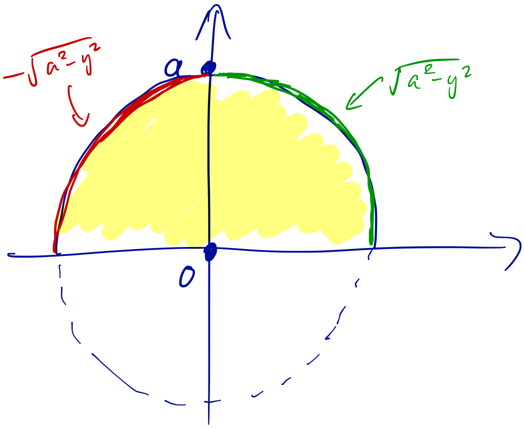
$x^2+y^2=1$

$$= \int_0^{2\pi}\int_0^1 (4-4r^2)\, r\, dr\, d\theta.$$

$x = \sqrt{a^2-y^2} \Rightarrow x^2+y^2=a^2$

Problem 3. Convert the following integral into polar coordinates:

(a) $\displaystyle\int_0^a \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} dxdy$

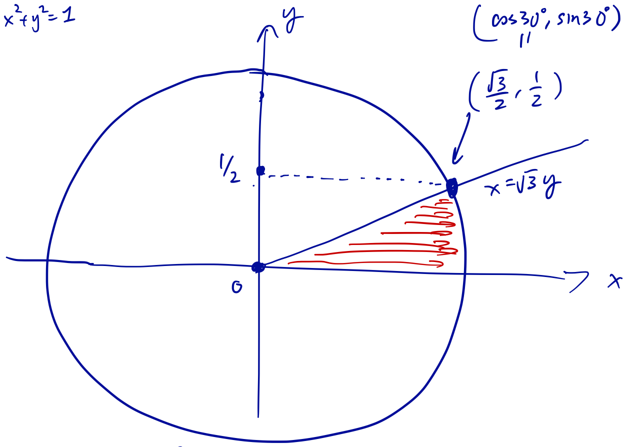


$$\int_0^{\pi}\int_0^{a} r\,dr\,d\theta.$$

(b) $\displaystyle\int_0^{\frac{1}{2}} \int_{\sqrt{3}y}^{\sqrt{1-y^2}} dxdy$

$x=\sqrt{1-y^2} \Rightarrow x^2+y^2=1$

$x=\sqrt{3}y$



$$\int_0^{\frac{\pi}{6}}\int_0^{1} r\,dr\,d\theta$$

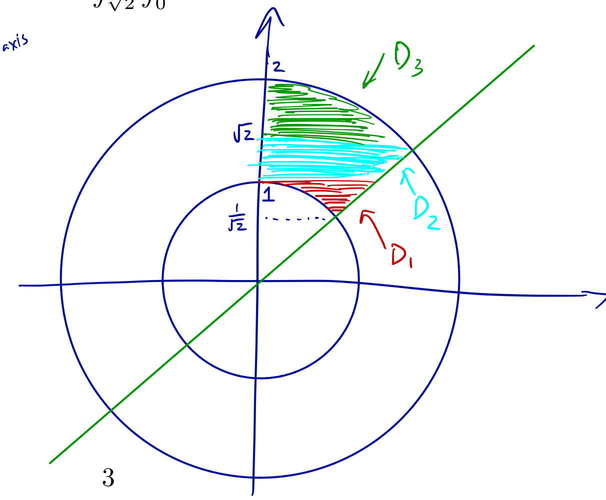
(c) $\displaystyle\int_{\frac{1}{\sqrt{2}}}^1 \int_{\sqrt{1-\cancel{x}^2}\;y}^{\cancel{x}\;y} dxdy + \int_1^{\sqrt{2}} \int_0^{\cancel{x}\;y} dxdy + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-\cancel{x}^2}\;y} dxdy$

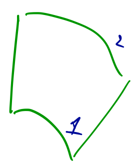
$y=x$

$x^2+y^2=1$

y-axis

$x^2+y^2=4$





$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_1^{2} r\,dr\,d\theta$$