MATH 2023 – Multivariable Calculus

Lecture #07 Worksheet ♢ February 28, 2019

Problem 1. Let $z = f(x, y) = e^{-x-y}$.

(a) Find ∇f at the point $P = (\ln 2, \ln 3)$

(b) Find the directional derivative $D_{\mathbf{u}} f$ where $\mathbf{u}$ is the unit vector parallel to $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$

(c) Find the unit direction such that $|D_{\mathbf{u}} f|$ is maximum.

(a) $\nabla f = \langle f_x, f_y \rangle = \langle -e^{-x-y}, -e^{-x-y} \rangle$

$\nabla f(\ln 2, \ln 3) = \langle -\frac{1}{6}, -\frac{1}{6} \rangle$ $\quad \left(-e^{-\ln 2 - \ln 3} = -e^{-\ln 6} \right)$

(b) $\vec{v} = \langle 1, 2 \rangle$, $\quad \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$D_{\vec{u}} f(\ln 2, \ln 3) = \nabla f \cdot \vec{u}$

$= -\frac{1}{6}\left(\frac{1}{\sqrt{5}}\right) - \frac{1}{6}\left(\frac{2}{\sqrt{5}}\right) = -\frac{1}{2\sqrt{5}}$

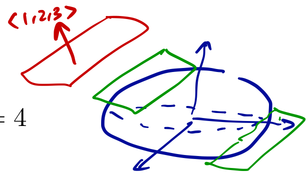
(c) $\vec{u} \parallel \nabla f = \langle -\frac{1}{6}, -\frac{1}{6} \rangle$ $\quad \Longrightarrow \vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ ← greatest descent

or $\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$. ← greatest ascent

Problem 2. At what point on the surface

$$x^2 + 2y^2 + 3z^2 = 4$$

is the tangent plane parallel to $x + 2y + 3z = 4$?



ellipsoid.

$$F = x^2 + 2y^2 + 3z^2 - 4 = 0$$

$$\nabla F = \langle 2x, 4y, 6z \rangle \parallel \langle 1, 2, 3 \rangle \ ?$$

$$\frac{\cancel{2}x}{\cancel{4}y} = \frac{1}{\cancel{2}}, \quad \frac{\cancel{2}x}{\cancel{6}z} = \frac{1}{\cancel{3}}, \quad x^2 + 2y^2 + 3z^2 = 4$$

$x = y$ $\qquad$ $x = z$

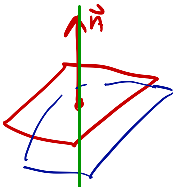
$x = y = z$ $\qquad$ $6x^2 = 4 \Rightarrow x = \pm \frac{2}{\sqrt{6}}$

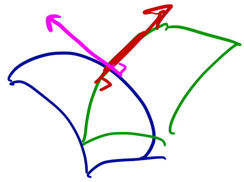
$\qquad\qquad\qquad\qquad\qquad\quad \parallel$

$\qquad\qquad\qquad\qquad\qquad\quad y = z$

The points with tangent plane // to ⟨1,2,3⟩ are

$\left(\frac{2}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ and $\left(-\frac{2}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$





Problem 3. Two surfaces are **orthogonal** at a point of intersection if their normal lines are perpendicular at that point.

(a) Show that two surfaces $F(x,y,z)=0, G(x,y,z)=0$ are orthogonal at a point P where $\nabla F \neq 0, \nabla G \neq 0$ if and only if

$$F_x G_x + F_y G_y + F_z G_z = 0 \qquad \text{at } P$$

(b) Given $r>0$. Show that the surfaces $z^2 = x^2+y^2$ and $x^2+y^2+z^2=r^2$ intersects orthogonally everywhere.

$z^2 = R^2$ $\qquad$ $R^2 + z^2 = r^2$

(c) Explain (b) without using calculus.

(a) $\nabla F \cdot \nabla G = 0$

(b) $F = x^2+y^2-z^2$ $\qquad$ $G = x^2+y^2+z^2-r^2$

$\nabla F = \langle 2x, 2y, -2z \rangle$ $\qquad$ $\nabla G = \langle 2x, 2y, 2z \rangle$

$\nabla F \cdot \nabla G = 4x^2+4y^2-4z^2 = 0$ $\quad$ ($F(x,y,z)=0$!)

in the intersection.

(c) Use Polar Coordinate!

$x = R\cos\theta$

$y = R\sin\theta$