MATH 2023 – Multivariable Calculus

Lecture #15 Worksheet April 4, 2019

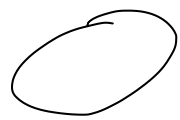
Problem 1. Use Green's Theorem to show that the area of a simple region can be expressed as

$$A = \oint_C x\,dy = -\oint y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Hence find the area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \oint_C Q\,dy + P\,dx$$

let $P = 0,\ Q = x$ =



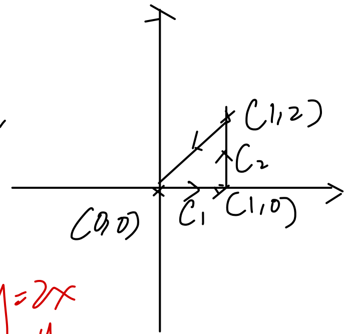
$$A = \frac{1}{2}\oint_C x\,dy - y\,dx$$

$a\cos t,\ b\sin t.$

$$\frac{1}{2}\oint_0^{2\pi} a\cos t\,(b\cos t)\,dt + b\sin t\, a\sin t\, dt$$

Problem 2. Find $\int_C e^x\,dx + (xy + \cos y^{2023})\,dy$ where C consists of the straight line segments from $(1,0)$ to $(1,2)$ to $(0,0)$.

$$\int_C + \int_{C_1} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA.$$



$y = 2x$

$x = \frac{y}{2}$

$$\frac{y-2}{x-1} = \frac{-2}{-1}$$

$$-(y-2) = -2(x-1)$$

$$y-2 = 2x-2$$

$$0 = 2x - y$$

$$= \iint_D y\,dA$$

$$= \int_0^{2x}\int_{\cancel{0}}^{1} y\,dx\,dy \qquad \int_0^{2}\int_{\frac{y}{2}}^{1} y\,dx\,dy$$

$$= \int_0^{2x} y\,dy$$

$$= \left[\frac{y^2}{2}\right]_0^{2x}$$

$$= \frac{4x^2}{2} = 2x^2.$$

$0 < t < 1$

$\langle t, 0 \rangle$

$$\int_{C_1} \to \int_0^1 e^t\,dt$$

$$= e - 1$$

Problem 3. Evaluate

$$\oint_C y^2 dx + 3xy dy$$

where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$\iint_D 3y - 2y \, dA$$

$$= \iint y \, dA$$

$$= \int_0^{\pi} \int_1^2 r^2 \sin\theta \, dr \, d\theta$$

Problem 4. Consider the change of variables

$$x = g(u,v), \qquad y = h(u,v)$$

Use Green's Theorem to prove the change of variables formula

$$\iint_R dxdy = \iint_S \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

where $\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$ is called the **Jacobian**, given by the determinant

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\iint_R dxdy = \oint_C$$