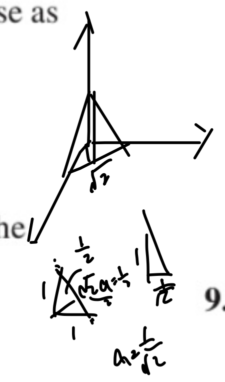
EXERCISES 16.5

1. Evaluate $\oint_{\mathcal{C}} xy\,dx + yz\,dy + zx\,dz$ around the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented clockwise as seen from the point $(1, 1, 1)$.

2. Evaluate $\oint_{\mathcal{C}} y\,dx - x\,dy + z^2\,dz$ around the curve $\mathcal{C}$ of intersection of the cylinders $z = y^2$ and $x^2 + y^2 = 4$, oriented counterclockwise as seen from a point high on the z-axis.



3. Evaluate $\iint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS$, where $\mathcal{S}$ is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$ with outward normal, and $\mathbf{F} = 3y\mathbf{i} - 2xz\mathbf{j} + (x^2 - y^2)\mathbf{k}$.

$a = \frac{1}{\sqrt{2}}$

$\sqrt{1 + \frac{1}{2}} = a$

$\sqrt{\frac{3}{2}} =$

4. Evaluate $\iint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS$, where $\mathcal{S}$ is the surface $x^2 + y^2 + 2(z - 1)^2 = 6$, $z \ge 0$, $\hat{\mathbf{N}}$ is the unit outward (away from the origin) normal on $\mathcal{S}$, and

$$\mathbf{F} = (xz - y^3 \cos z)\mathbf{i} + x^3 e^z \mathbf{j} + xyz\,e^{x^2+y^2+z^2}\mathbf{k}.$$

5. Use Stokes's Theorem to show that

$$\oint_{\mathcal{C}} y\,dx + z\,dy + x\,dz = \sqrt{3}\,\pi a^2,$$

where $\mathcal{C}$ is the suitably oriented intersection of the surfaces $x^2 + y^2 + z^2 = a^2$ and $x + y + z = 0$.

6. Evaluate $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$ around the curve

$$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sin 2t\,\mathbf{k}, \quad (0 \le t \le 2\pi),$$

where

$$\mathbf{F} = (e^x - y^3)\mathbf{i} + (e^y + x^3)\mathbf{j} + e^z\mathbf{k}.$$

Hint: Show that $\mathcal{C}$ lies on the surface $z = 2xy$.

7. Find the circulation of $\mathbf{F} = -y\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$ around the oriented boundary of the part of the paraboloid $z = 9 - x^2 - y^2$ lying above the xy-plane and having normal field pointing upward.

8. Evaluate $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$, where

$$\mathbf{F} = ye^x\mathbf{i} + (x^2 + e^x)\mathbf{j} + z^2e^z\mathbf{k},$$

and $\mathcal{C}$ is the curve

$$\mathbf{r}(t) = (1 + \cos t)\mathbf{i} + (1 + \sin t)\mathbf{j} + (1 - \cos t - \sin t)\mathbf{k}$$

for $0 \le t \le 2\pi$. *Hint:* Use Stokes's Theorem, observing that $\mathcal{C}$ lies in a certain plane and has a circle as its projection onto the xy-plane. The integral can also be evaluated by using the techniques of Section 15.4.

9. Let $\mathcal{C}_1$ be the straight line joining $(-1, 0, 0)$ to $(1, 0, 0)$, and let $\mathcal{C}_2$ be the semicircle $x^2 + y^2 = 1$, $z = 0$, $y \ge 0$. Let $\mathcal{S}$ be a smooth surface joining $\mathcal{C}_1$ to $\mathcal{C}_2$ having upward normal, and let

$$\mathbf{F} = (\alpha x^2 - z)\mathbf{i} + (xy + y^3 + z)\mathbf{j} + \beta y^2(z + 1)\mathbf{k}.$$

Find the values of α and β for which $I = \iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S}$ is independent of the choice of $\mathcal{S}$, and find the value of I for these values of α and β.

10. Let $\mathcal{C}$ be the curve $(x - 1)^2 + 4y^2 = 16$, $2x + y + z = 3$, oriented counterclockwise when viewed from high on the z-axis. Let

$$\mathbf{F} = (z^2 + y^2 + \sin x^2)\mathbf{i} + (2xy + z)\mathbf{j}) + (xz + 2yz)\mathbf{k}.$$

Evaluate $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$.

11. If $\mathcal{C}$ is the oriented boundary of surface $\mathcal{S}$, and ϕ and ψ are arbitrary smooth scalar fields, show that

$$\oint_{\mathcal{C}} \phi\nabla\psi \bullet d\mathbf{r} = -\oint_{\mathcal{C}} \psi\nabla\phi \bullet d\mathbf{r} = \iint_{\mathcal{S}} (\nabla\phi \times \nabla\psi) \bullet \hat{\mathbf{N}}\,dS.$$

Is $\nabla\phi \times \nabla\psi$ solenoidal? Find a vector potential for it.

12. Let $\mathcal{C}$ be a piecewise smooth, simple closed plane curve in $\mathbb{R}^3$, which lies in a plane with unit normal $\hat{\mathbf{N}} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and has orientation inherited from that of the plane. Show that the plane area enclosed by $\mathcal{C}$ is

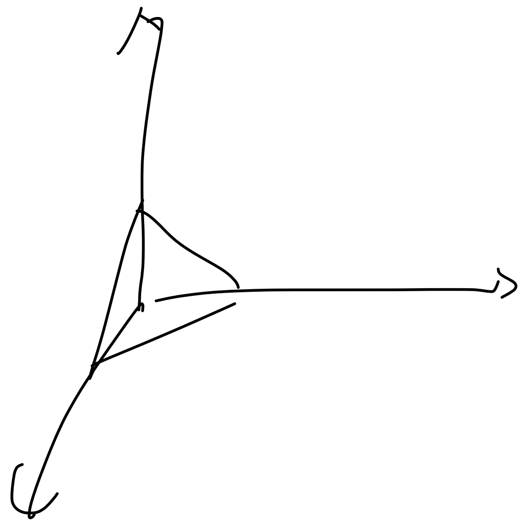
$$\frac{1}{2}\oint_{\mathcal{C}} (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz.$$

13. Use Stokes's Theorem to prove Theorem 2 of Section 16.1.

1. Evaluate $\oint_{\mathcal{C}} xy\,dx + yz\,dy + zx\,dz$ around the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, oriented clockwise as seen from the point $(1, 1, 1)$.

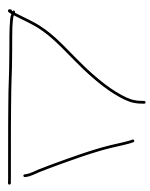
$$\iint_S \nabla\times\vec{F}\cdot dS$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix}$$



$$\nabla\times\vec{F} = \langle -y, -z, -x \rangle$$

$x+y+z=1$



$z = 1-x-y$

$$-\iint_S -y-z-x \; dS$$

$0 \le u \le 1$

$$\vec{r}(u,v) = \langle u, v, 1-u-v \rangle$$

$0 \le v \le 1$

$$-\int_0^1\int_0^{1-v} -v-(1-u-v)-u \; du\,dv$$

$$= -\int_0^1\int_0^{1-v} -v-1+u+v-u \; du\,dv$$

$$= \int_0^1\int_0^{1-v} du\,dv$$

$$= \int_0^1 1-v\,dv = \left[v-\frac{v^2}{2}\right]_0^1 = \frac{1}{2}$$

2. Evaluate $\oint_{\mathcal{C}} y\,dx - x\,dy + z^2\,dz$ around the curve $\mathcal{C}$ of intersection of the cylinders $z = y^2$ and $x^2 + y^2 = 4$, oriented counterclockwise as seen from a point high on the z-axis.

$x^2 + z = 4$

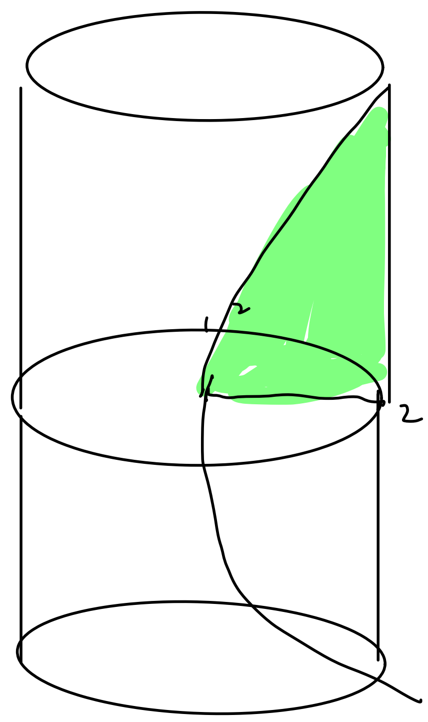
$z = 4 - x^2$

$$\iint_S \nabla \times \vec{F} \cdot dS$$

$\hat{n} = \langle 2x, 0, 1 \rangle$

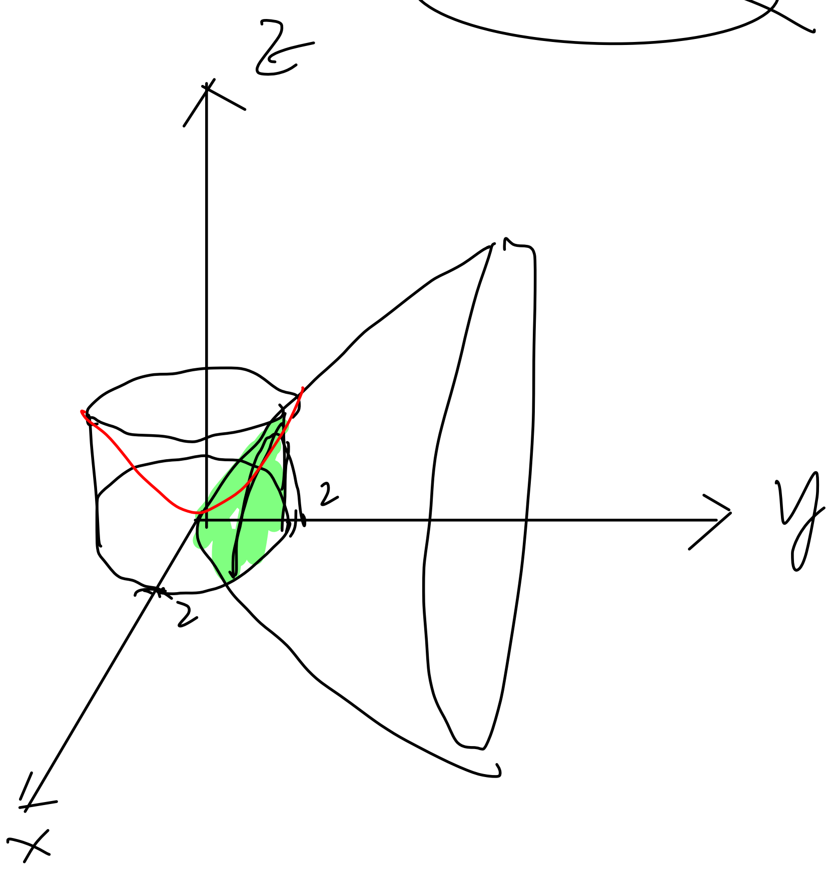
$-2 \le x \le 2$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z^2 \end{vmatrix}$$



$= \langle 0, 0, -1 -1 \rangle$

$= \langle 0, 0, -2 \rangle$

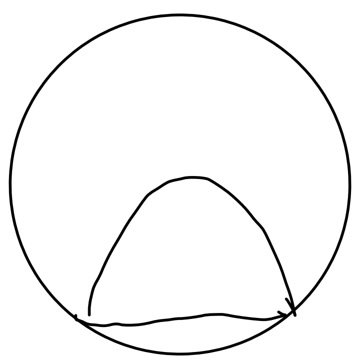


$$\iint_S -2\, dS$$

$= - \iint_S 1\, dS$

$= - \left(\frac{\pi \cdot 2^2}{2} \right)$

$= -2\pi.$



3. Evaluate $\iint_{\mathcal{S}} \mathbf{curl\,F} \bullet \hat{\mathbf{N}}\, dS$, where $\mathcal{S}$ is the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$ with outward normal, and $\mathbf{F} = 3y\mathbf{i} - 2xz\mathbf{j} + (x^2 - y^2)\mathbf{k}$.

$$\iint_S \nabla \times F \cdot \hat{N}\, dS = \oint_C \vec{F} \cdot d\vec{r}$$

$$\vec{r}(\theta) = \langle a\cos\theta, a\sin\theta, 0\rangle$$

$$\oint 3y \cdot (a - \sin\theta)\, d\theta$$

$$= -\int_0^{2\pi} 3a\sin^2\theta\, d\theta$$

$$= -3a \int_0^{2\pi} \frac{1}{2} - \frac{\cos 2\theta}{2}\, d\theta$$

$$= -3a(\pi) + 3a\left[\frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= -3a\pi.$$

4. Evaluate $\iint_{S} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS$, where S is the surface $x^2+y^2+2(z-1)^2=6$, $z \ge 0$, $\hat{\mathbf{N}}$ is the unit outward (away from the origin) normal on S, and

$$\mathbf{F} = (xz - y^3\cos z)\mathbf{i} + x^3 e^z\mathbf{j} + xyz\,e^{x^2+y^2+z^2}\mathbf{k}.$$

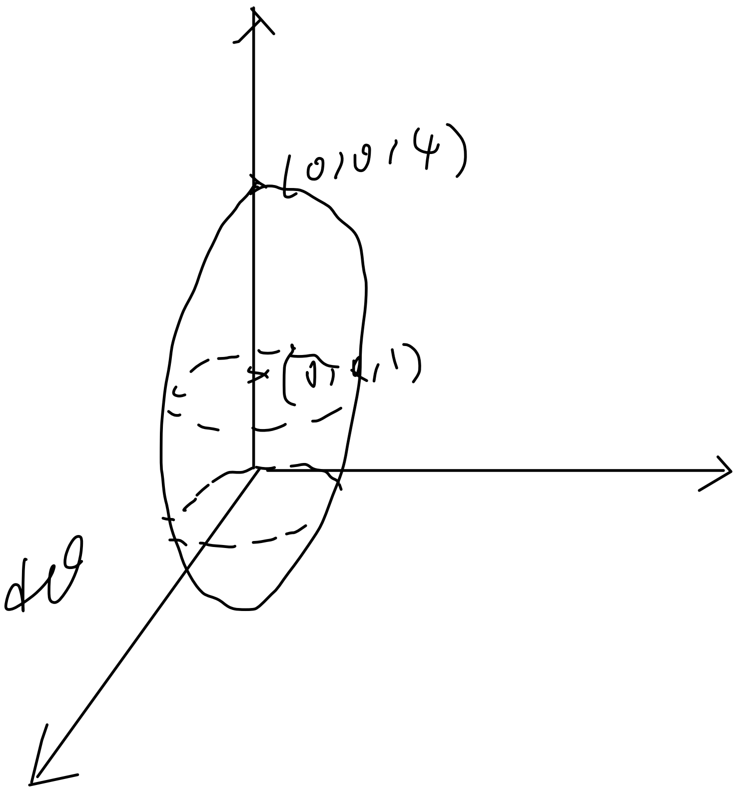
$x^2+y^2+2=6$

$x^2+y^2=4$

$$\frac{x^2}{2}+\frac{y^2}{2}+(z-1)^2=3$$

$\vec{r}(\theta)=\langle 2\cos\theta,\ 2\sin\theta,\ 0\rangle$

$$\oint_C \vec{F}\cdot d\vec{r}$$



$$= \int_0^{2\pi} -y^3(-2\sin\theta)\,d\theta +$$

$$= \int_0^{2\pi} -(8\sin^3\theta)(-2\sin\theta)\,d\theta$$

$$= 16\int_0^{2\pi} \sin^4\theta\,d\theta$$

$$16\left(\frac{3}{8}\cdot 2\pi\right) = 12\pi.$$

$$= 16\int_0^{2\pi} \left(\frac{1}{2}-\frac{\cos 2\theta}{2}\right)^2 d\theta$$

$$= 16\int_0^{2\pi} \frac{1}{4}-\frac{\cos 2\theta}{2}+\frac{\cos^2 2\theta}{4}\,d\theta$$

$$= 16\int_0^{2\pi} \frac{1}{4}-\frac{\cos 2\theta}{2}+\frac{1}{8}+\frac{\cos 4\theta}{8}\,d\theta$$

$$= 16\int_0^{2\pi} \frac{3}{8}-\frac{\cos 2\theta}{2}+\frac{\cos 4\theta}{8}\,d\theta$$

0

5. Use Stokes's Theorem to show that

$$\oint_{\mathcal{C}} y\,dx + z\,dy + x\,dz = \sqrt{3}\,\pi a^2,$$

$z = -x - y$.

where $\mathcal{C}$ is the suitably oriented intersection of the surfaces $x^2 + y^2 + z^2 = a^2$ and $x + y + z = 0$.

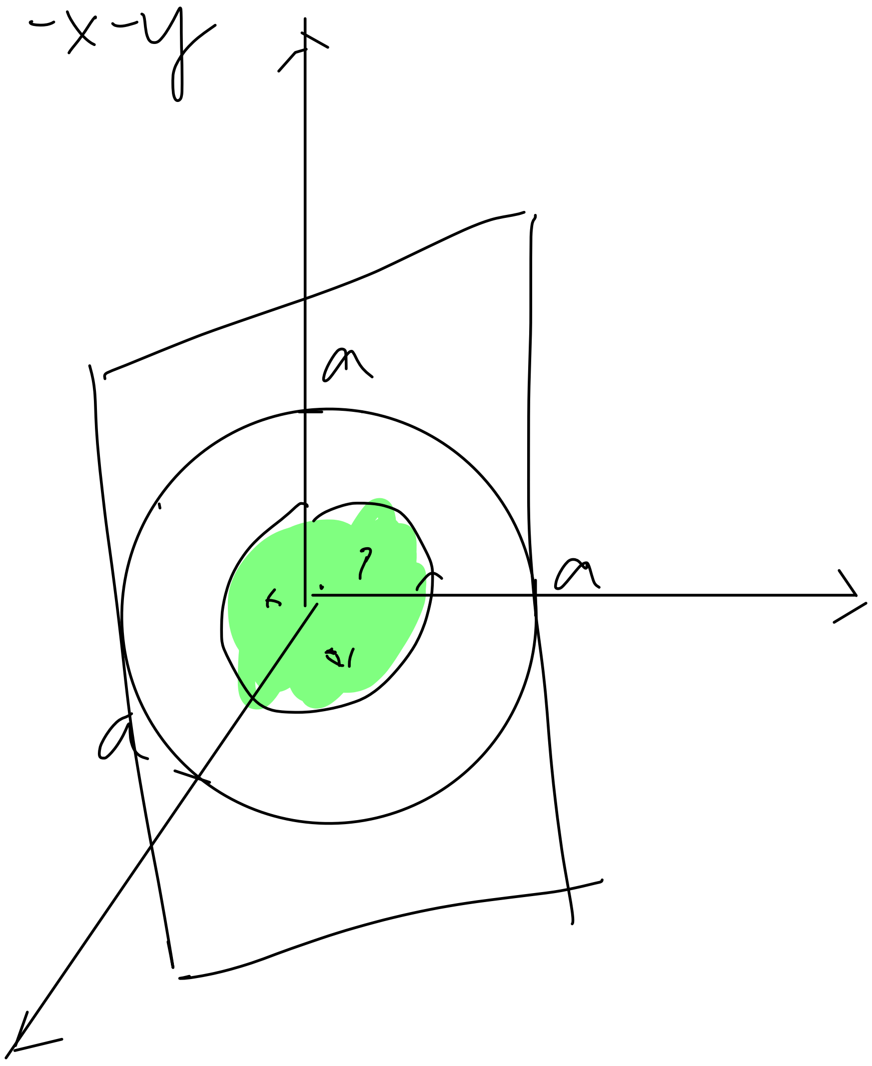
$z^2 = a^2 - x^2 - y^2$

$x^2 + y^2 + (-x-y)^2 = a^2$ $\quad z = -x - y$

$x^2 + y^2 + x^2 + 2xy + y^2 = a^2$

$2x^2 + 2xy + 2y^2 = a^2$

$=$



$\iint_S \vec{F} \cdot \hat{n}\, dS$

$= \iint_S \langle y, z, x \rangle \cdot \langle 2x, 2y, 1 \rangle\, dS$

$\vec{r}(u,v) = \langle a \sin u \cos v,\ u \sin u \sin v,\ a \cos v \rangle$

$= \iint_S 2xy + 2yz + x\, dS$

$= \iint \left(2a^2 \sin^2 u \sin v \cos v + 2a^2 \sin u \sin v \cos v + a \sin u \cos v \right) a^2 \sin u\, du\, dv$

6. Evaluate $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$ around the curve

$$\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sin 2t\,\mathbf{k}, \quad (0 \le t \le 2\pi),$$

where

$$\mathbf{F} = (e^x - y^3)\mathbf{i} + (e^y + x^3)\mathbf{j} + e^z\mathbf{k}.$$

$\vec{r}(x,y) = \langle x, y, 2xy \rangle$ $-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$z = 2xy$

$\langle -2x, -2y, 1 \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x - y^3 & e^y + x^3 & e^z \end{vmatrix}$$

$$= \langle 0, 0, 3x^2 + 3y^2 \rangle$$

$$\iint_S 3x^2 + 3y^2 \, dS$$

$$= \int_0^{2\pi}\int_0^1 3r^2 \cdot r \, dr \, d\theta$$

$$= \frac{3}{4} \cdot 2\pi$$

$$= \frac{3}{2}\pi$$

Made with Goodnotes

7. Find the circulation of $\mathbf{F} = -y\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$ around the oriented boundary of the part of the paraboloid $z = 9 - x^2 - y^2$ lying above the xy-plane and having normal field pointing upward.

$$7.\ \iint_S \nabla \times F \cdot \hat{n}\, dS$$

$\vec{r}(\theta) = \langle 3\cos\theta, 3\sin\theta, 0 \rangle$

$$= \oint_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} -y\,(-3\sin\theta) + x^2\,(3\cos\theta)\, d\theta$$

$$= \int_0^{2\pi} 9\sin^2\theta + 27\cos^3\theta\, d\theta$$

$$= \int_0^{2\pi} 9 + 18\cos^3\theta\, d\theta$$

$$= 18\pi + 18\int_0^{2\pi} (1-\sin^2\theta)\cos\theta\, d\theta$$

$$= 18\pi.$$

8. Evaluate $\oint_{\mathcal{C}} \mathbf{F} \bullet d\mathbf{r}$, where

$$\mathbf{F} = ye^x\mathbf{i} + (x^2 + e^x)\mathbf{j} + z^2e^z\mathbf{k},$$

and $\mathcal{C}$ is the curve

$$\mathbf{r}(t) = (1 + \cos t)\mathbf{i} + (1 + \sin t)\mathbf{j} + (1 - \cos t - \sin t)\mathbf{k}$$

for $0 \le t \le 2\pi$. *Hint:* Use Stokes's Theorem, observing that $\mathcal{C}$ lies in a certain plane and has a circle as its projection onto the xy-plane. The integral can also be evaluated by using the techniques of Section 15.4.

$a - 1 - \cos t - 1 - \sin t$

$a - 2 = 1$

$a = 3$

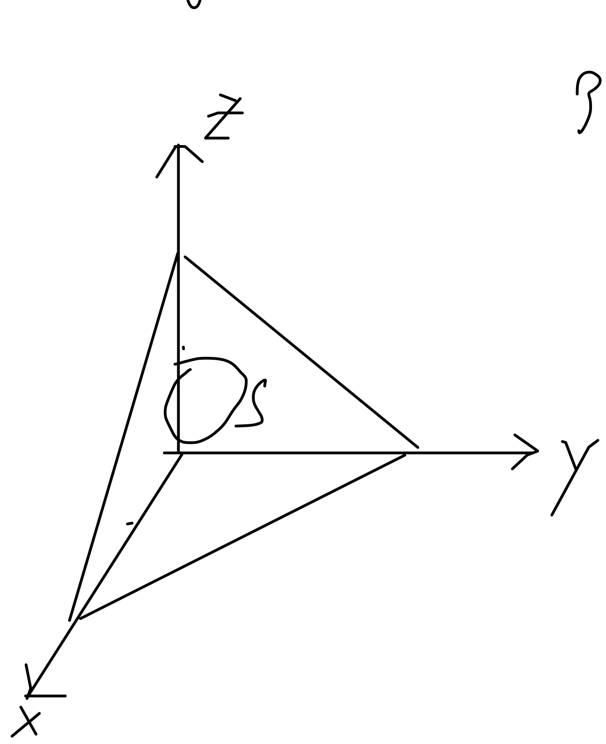
$z = 3 - x - y.$

$3 - x - y.$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^x & x^2+e^x & z^2e^z \end{vmatrix}$$

$$= \langle 0, 0, 2x + e^x - e^x \rangle$$

$$= \langle 0, 0, 2x \rangle$$



$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS$$

$$\iint_S 2x\, dS$$

!

9. Let $\mathscr{C}_1$ be the straight line joining $(-1, 0, 0)$ to $(1, 0, 0)$, and let $\mathscr{C}_2$ be the semicircle $x^2 + y^2 = 1$, $z = 0$, $y \geq 0$. Let $\mathcal{S}$ be a smooth surface joining $\mathscr{C}_1$ to $\mathscr{C}_2$ having upward normal, and let

$$\mathbf{F} = (\alpha x^2 - z)\mathbf{i} + (xy + y^3 + z)\mathbf{j} + \beta y^2(z + 1)\mathbf{k}.$$

Find the values of α and β for which $I = \iint_{\mathcal{S}} \mathbf{F} \bullet d\mathbf{S}$ is independent of the choice of $\mathcal{S}$, and find the value of I for these values of α and β.

$\iint$