MATH 2023 – Multivariable Calculus

Lecture #16 Worksheet April 9, 2019

Problem 1. Let f be a scalar field, $\mathbf{F}$ be a vector field. Rewrite them using ∇, and state whether each expression is meaningful.

(a) curl f $\quad \nabla \times f$ ✗

(b) grad f $\quad \nabla f$ ✓

(c) div $\mathbf{F}$ $\quad \nabla \cdot \vec{F}$ ✓

(d) grad $\mathbf{F}$ $\quad \nabla \vec{F}$ ✗

(e) curl(grad f) $\quad \nabla \times (\nabla f)$ ✓

(f) div(grad f) $\quad \nabla \cdot (\nabla f)$ ✓

(g) grad(div $\mathbf{F}$) $\quad \nabla(\overbrace{\nabla \cdot \vec{F}}^{\text{scalar}})$ ✓

(h) grad(div f) $\quad \nabla(\nabla \cdot \overbrace{f}^{\text{scalar}})$ ✗

(j) curl(curl(curl $\mathbf{F}$)) $\quad \nabla \times (\nabla \times (\overbrace{\nabla \times \vec{F}}^{\text{vector}}))$ ✓

(i) div(div(div $\mathbf{F}$)) $\quad \nabla \cdot (\underline{\nabla \cdot (\overbrace{\nabla \cdot \vec{F}}^{\text{scalar}})})$ ✗

(k) (grad f) $\times$ (curl $\mathbf{F}$) $\quad (\nabla f) \times (\nabla \times \vec{F})$ ✓

(l) div(curl(grad f)) $\quad \nabla \cdot (\underbrace{\nabla \times \underbrace{\nabla f}_{\text{vector}}}_{\text{vector}})$ ✓

Thm C/D.

Problem 2. All vector fields of the form $\mathbf{F} = \nabla g$ satisfies $\nabla \times \mathbf{F} = \mathbf{0}$.

All vector fields of the form $\mathbf{F} = \nabla \times \mathbf{G}$ satisfies $\nabla \cdot \mathbf{F} = 0$. $(\nabla \cdot (\nabla \times G) = 0)$

Are there any equations that all functions of the form $f = \nabla \cdot \mathbf{G}$ must satisfy?

NO! Any function can be divergence of some vector field!

$$f(x,y,z) = \nabla \cdot \vec{G}$$

$$\vec{G} = \left\langle \int_0^x f(t,y,z)\,dt,\ 0,\ 0 \right\rangle$$

$$\nabla \cdot \vec{G} = \underbrace{\frac{\partial}{\partial x} \int_0^x f(t,y,z)\,dt}_{= f(x,y,z)} + 0 + 0$$

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

Probelm 3. Prove the following identities:

$F = \langle P, Q, R \rangle$

(a) $\nabla \cdot (f\mathbf{F}) = (\nabla f) \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F})$

(b) $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$

(c) $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

$\vec{F} = \langle P, Q, R \rangle$
$\vec{G} = \langle U, V, W \rangle$

(a) $\nabla \cdot (f\vec{F}) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle fP, fQ, fR \rangle$

$= f_x P + f P_x + f_y Q + f Q_y + f_z R + f R_z$

$= (\nabla f) \cdot \vec{F} + f(\nabla \cdot \vec{F})$

(b) $\vec{F} \times \vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ P & Q & R \\ U & V & W \end{vmatrix} = \langle (QW - RV), (RU - PW), (PV - QU) \rangle$

$\nabla \cdot (\vec{F} \times \vec{G}) = Q_x W + Q W_x - R_x V - R V_x$
$+ R_y U + R U_y - P_y W - P W_y$
$+ P_z V + P V_z - Q_z U - Q U_z$

$(\nabla \times \vec{F}) \cdot \vec{G} - \vec{F} \cdot (\nabla \times \vec{G})$

(c) LHS : $\nabla \times \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle (R_y - Q_z), (P_z - R_x), (Q_x - P_y) \rangle$

$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ R_y - Q_z & P_z - R_x & Q_x - P_y \end{vmatrix} = (Q_{xy} - P_{yy} - P_{zz} + R_{xz})\ \vec{i}$

$(P_{yz} - Q_{zz} - Q_{xx} + P_{yx})\ \vec{j}$

$(R_{zx} - R_{xx} - R_{yy} + Q_{zy})\ \vec{k}$

RHS : $\nabla(\nabla \cdot \vec{F}) = \nabla(P_x + Q_y + R_z)$

$= \langle P_{xx} + Q_{yx} + R_{zx},\ P_{xy} + Q_{yy} + R_{zy},\ P_{xz} + Q_{yz} + R_{zz} \rangle$

$\nabla^2 \vec{F} = \langle \nabla^2 P, \nabla^2 Q, \nabla^2 R \rangle$

$= \langle P_{xx} + P_{yy} + P_{zz},\ Q_{xx} + Q_{yy} + Q_{zz},\ R_{xx} + R_{yy} + R_{zz} \rangle$

LHS = RHS //

Problem 4. Let $f(x, y), g(x, y)$ have continuous partial derivatives, and C, D as in Green's Theorem. Recall that $\mathbf{n}$ is the **unit normal vector** of C away from D.

(a) Use the second form of Green's Theorem to prove the **Green's first identity**:

$$\iint_D f\nabla^2 g dA = \oint_C f(\nabla g)\cdot \mathbf{n} ds - \iint_D \nabla f \cdot \nabla g dA$$

(b) Use this to prove **Green's second identity**

$$\iint_D (f\nabla^2 g - g\nabla^2 f) dA = \oint_C (f\nabla g - g\nabla f)\cdot \mathbf{n} ds$$

(c) If g is **harmonic function**, show that

$$\nabla^2 g = 0$$

$$\oint_C (\nabla g)\cdot \mathbf{n} ds = 0$$

(a) $$\oint_C f(\nabla g)\cdot \vec{n}\, ds \underset{\text{second form}}{=} \iint_D \nabla\cdot(f\nabla g)\, dA$$

$$\underset{\text{by Q3(a)}}{=} \iint_D (\nabla f\cdot\nabla g + f\nabla^2 g)\, dA$$

(b) $$\oint g(\nabla f)\cdot\vec{n}\, ds = \iint_D (\nabla g\cdot\nabla f + g\nabla^2 f)\, dA$$

(c) By part (a), set $f = 1$,

$$\iint \cancel{\nabla^2 g}^{\,0}\, dA = \underbrace{\oint \nabla g\cdot\vec{n}\, ds}_{=0} - \iint (\cancel{\nabla 1}^{\,0}\cdot\nabla g)\, dA$$