MATH 2023 • Multivariable Calculus
Problem Set #1 • Lines, Planes and Curves

1. (★) Consider the two straight-lines:

$$\begin{aligned} L_1 &: \mathbf{r}_1(t) = \langle 1, 2, 3 \rangle + t \langle 1, -1, -1 \rangle \\ L_2 &: \mathbf{r}_2(t) = \langle 2 + t, 3 - 3t, -2 + 3t \rangle \end{aligned}$$

   (a) Show that L_1 and L_2 intersects each other. Find the coordinates of the intersection point.
   (b) Find an equation of the plane containing both L_1 and L_2.

2. (★) Consider the following four points in three-dimensional space:

$$A(0, 2, -1), \ B(4, 0, -1), \ C(7, -3, 0) \ \text{ and } \ D(\tfrac{1}{3}, \tfrac{1}{6}, \tfrac{1}{9})$$

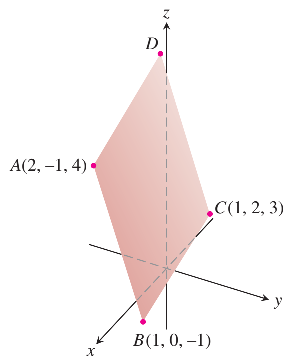
   Determine whether or not these four points are coplanar (i.e. contained in a single plane).

3. (★) A parallelogram in $\mathbb{R}^3$ has vertices:

$$A(2, -1, 4), \quad B(1, 0, -1), \quad C(1, 2, 3), \quad D(x_0, y_0, z_0)$$

   as shown in the figure below. Answer the following questions:

   (a) Find the coordinates of D.
   (b) Find the area of the parallelogram $ABCD$.
   (c) Find an equation of the plane containing the parallelogram $ABCD$.
   (d) Project the parallelogram $ABCD$ orthogonally onto the plane $z = -1$. Find the coordinates the projection of each vertices, then find the area of the *projected* parallelogram.



4. (★) Consider a particle whose path is represented by:

$$\mathbf{r}(t) = \left(\ln(t^2 + 1)\right) \mathbf{i} + \left(\tan^{-1} t\right) \mathbf{j} + \sqrt{t^2 + 1}\mathbf{k}$$

   Find the velocity, speed and acceleration of the particle at $t = 0$.

5. (★★) Consider a plane through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle A, B, C \rangle$. Prove that the perpendicular distance d from a given point $Q(x_1, y_1, z_1)$ to the plane is given by:

$$d = \frac{\left| \overrightarrow{P_0 Q} \cdot \mathbf{n} \right|}{|\mathbf{n}|} = \left| \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}} \right|$$

   where $D = Ax_0 + By_0 + Cz_0$.

6. (★) Suppose $\mathbf{r}(t)$ represents the path of a particle traveling on a sphere centered at the origin. Show that the position vector $\mathbf{r}(t)$ and the velocity $\mathbf{r}'(t)$ are orthogonal to each other at any time.

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7. (★★) Suppose that the path of a particle at time t is given by $\mathbf{r}(t)$ and the force exerted on the particle at time t is $\mathbf{F}(t)$. By Newton's Second Law, $\mathbf{F}(t)$ and $\mathbf{r}(t)$ are related by:

$$\mathbf{F}(t) = m\mathbf{r}''(t),$$

where m is the mass of the particle. The angular momentum $\mathbf{L}(t)$ about the origin of the particle at time t is defined to be:

$$\mathbf{L}(t) := \mathbf{r}(t) \times m\mathbf{r}'(t)$$

(a) Show that

$$\frac{d}{dt}\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{F}(t).$$

(b) When $\mathbf{L}(t)$ is a constant vector, we say that the angular momentum is *conserved*. According to the result in (a), under what condition on $\mathbf{r}(t)$ and $\mathbf{F}(t)$ will the angular momentum be conserved? Also, give one example in physics that this condition is satisfied.

8. (★★) Consider two point particles with masses m_1 and m_2, and their trajectories are $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ respectively. Denote $\mathbf{F}(t)$ to be the force exerted on the m_1-particle by the m_2-particle at time t. By Newton's Third Law, the force exerted on the m_2-particle by the m_1-particle at time t (i.e. the reverse force) is given by $-\mathbf{F}(t)$. Assume there are no other forces exerted on any of these particles.

(a) Consider the following vector:

$$\mathbf{C}(t) := \frac{m_1\mathbf{r}_1(t) + m_2\mathbf{r}_2(t)}{m_1 + m_2}.$$

In physics, this vector is pointing at the center of mass of the two particles. Show that $\mathbf{C}''(t) = \mathbf{0}$ for any t using Newton's Second and Third Laws.

(b) Hence, show that there exist two constant vectors $\mathbf{r}_0$ and $\mathbf{v}$ such that

$$\frac{m_1\mathbf{r}_1(t) + m_2\mathbf{r}_2(t)}{m_1 + m_2} = \mathbf{r}_0 + t\mathbf{v}.$$

[Question: What is the physical significance of this result?]

9. (★) For each of the following curves, first reparametrize it by arc-length and then compute its curvature function $\kappa(s)$:

(a) $\mathbf{r}_1(t) = (R\cos\omega t)\,\mathbf{i} + (R\sin\omega t)\,\mathbf{j}, \quad 0 \le t \le \frac{2\pi}{\omega}.$

(b) $\mathbf{r}_2(t) = \langle 1,2,3\rangle + (\ln t)\,\langle 1,0,-1\rangle, \quad 0 < t < \infty$

(c) $\mathbf{r}_3(t) = (\cos^3 t)\,\mathbf{i} + (\sin^3 t)\,\mathbf{j}, \quad 0 \le t \le \frac{\pi}{2}.$

Give an example of a path whose arc-length parametrization cannot be explicitly found even with computer softwares.

10. (★★) Suppose

$$\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} + \frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\mathbf{j} + t\mathbf{k}$$

represents the path of a race-car climbing up a hill from $(0,0,0)$ at $t = 0$. A truck, on the other hand, drives slowly in unit speed from $(0,0,0)$ at time $t = 0$ along the same path and direction as the race-car. Find a parametrization which represents the path of the truck.

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11. (★★★) We define the curvature of a path by $\kappa(s) = |\mathbf{r}''(s)|$ where $\mathbf{r}(s)$ is the arc-length parametrization of the path. However, the arc-length parametrization $\mathbf{r}(s)$ is often difficult to find explicitly. The purpose of this exercise is to derive an equivalent formula for the curvature which does not require finding an arc-length parametrization.

    Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that s and t are related by:

$$s = \int_0^t \left|\mathbf{r}'(\tau)\right| \, d\tau.$$

    (a) Show, using the chain rule, that:

$$\begin{aligned}\mathbf{r}'(t) &= \mathbf{r}'(s)\,\frac{ds}{dt} \\ \mathbf{r}''(t) &= \mathbf{r}''(s)\left(\frac{ds}{dt}\right)^2 + \mathbf{r}'(s)\,\frac{d^2s}{dt^2}\end{aligned}$$

    (b) Show that:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^3 \mathbf{r}'(s) \times \mathbf{r}''(s)$$

    (c) Using (a) and (b), show that the curvature, which is defined as $\kappa(s) := |\mathbf{r}''(s)|$, can be expressed in terms of t as:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

    Although it looks more complicated, this formula does not require the procedure of finding arc-length parametrization.

Made with Goodnotes

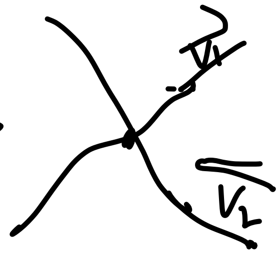
1. (★) Consider the two straight-lines:

$$L_1 : \mathbf{r}_1(t) = \langle 1, 2, 3 \rangle + t\,\langle 1, -1, -1 \rangle$$
$$L_2 : \mathbf{r}_2(t) = \langle 2 + t, 3 - 3t, -2 + 3t \rangle$$

(a) Show that L_1 and L_2 intersects each other. Find the coordinates of the intersection point.

(b) Find an equation of the plane containing both L_1 and L_2.

$\langle 1,-1,-1\rangle^{t} \neq \langle 1,-3,3\rangle^{t}$



$\langle 3, 0, 1 \rangle$

$$x = 1 + t_1 = 2 + t_2$$
$$y = 2 - t_1 = 3 - 3t_2$$
$$z = 3 - t_1 = -2 + 3t_2$$

$t_1 = 2,$
$t_2 = 1.$

$$\cancel{t_1}^{2} = 1 + \cancel{t_2}^{1}$$
$$\cancel{t_1}^{2} = -1 + 3\cancel{t_2}^{1}$$
$$\cancel{t_1}^{2} = 5 - 3\cancel{t_2}^{1}$$

$$2t_1 = 4$$
$$t_1 = 2$$

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$\langle 1,-1,-1\rangle \times \langle 1,-3,3\rangle$

$$= \begin{vmatrix} i & j & k \\ 1 & -1 & -1 \\ 1 & -3 & 3 \end{vmatrix} = \langle -6, -4, -2\rangle$$

$$= \langle 3, 2, 1\rangle$$

$$3x + 2y + z = 10$$

Made with Goodnotes

2. (★) Consider the following four points in three-dimensional space:

$$A(0,2,-1),\ B(4,0,-1),\ C(7,-3,0) \text{ and } D\left(\tfrac{1}{3},\tfrac{1}{6},\tfrac{1}{9}\right)$$

Determine whether or not these four points are coplanar (i.e. contained in a single plane).

$$\vec{AB} = \langle 4, -2, 0 \rangle$$

$$\vec{AC} = \langle 7, -5, 1 \rangle$$

$$\vec{AD} = \left\langle \frac{1}{3}, \frac{-11}{6}, \frac{10}{9} \right\rangle$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 4 & -2 & 0 \\ 7 & -5 & 1 \end{vmatrix} =$$

$$\langle -2, -4, -6 \rangle = -2\langle 1, 2, 3 \rangle$$

$$\langle 1, 2, 3 \rangle \cdot \left\langle \frac{1}{3}, \frac{-11}{6}, \frac{10}{9} \right\rangle$$

$$= \frac{1}{3} - \frac{11}{3} + \frac{10}{3} = 0.$$

$\therefore$ They are coplanar

Made with Goodnotes

3. (★) A parallelogram in $\mathbb{R}^3$ has vertices:

$$A(2,-1,4), \quad B(1,0,-1), \quad C(1,2,3), \quad D(x_0,y_0,z_0)$$

as shown in the figure below. Answer the following questions:

(a) Find the coordinates of D.

(b) Find the area of the parallelogram $ABCD$.

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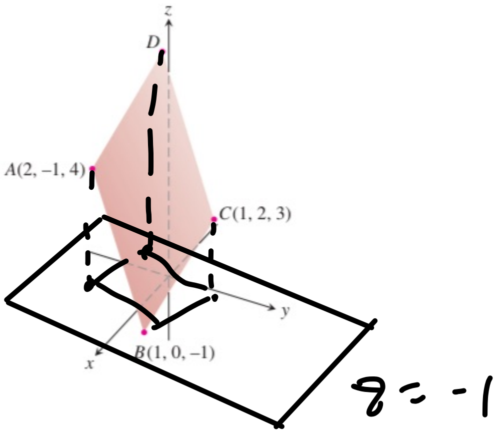

$$\vec{BC} = \langle 0, 2, 4 \rangle$$

$$\vec{AD} = \vec{BC} = \langle 0, 2, 4 \rangle$$

$$\langle x_0 - 2,\ y_0 + 1,\ z_0 - 4 \rangle = \langle 0, 2, 4 \rangle$$

$$\langle x_0, y_0, z_0 \rangle = \langle 2, 1, 8 \rangle$$

b) $\vec{BA} = \langle 1, -1, 5 \rangle$

$$\vec{BA} \times \vec{BC} = \begin{vmatrix} i & j & k \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= \langle -14, -4, 2 \rangle$$

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$$|\langle -14, -4, 2\rangle|$$

$$= \sqrt{14^2 + 4^2 + 2^2}$$

$$= \sqrt{196 + 16 + 4}$$

$$= \sqrt{216}$$

$(1, 0, -1)$

$$-14x - 4y + 2z = -16$$

$$7x + 2y - z = 8$$

(d). $A: (2, -1)$

$B: (1, 0)$

$C: (1, 2)$

$D: (2, 1)$

$\overrightarrow{AB} = \langle -1, 1\rangle$

$\overrightarrow{AC} = \langle -1, 3\rangle$

$\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0, 0, -2\rangle$

$$\begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 3 & 0 \end{vmatrix}$$

Area = 2

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4. (★) Consider a particle whose path is represented by:

$$\mathbf{r}(t) = \left(\ln(t^2+1)\right)\mathbf{i} + \left(\tan^{-1} t\right)\mathbf{j} + \sqrt{t^2+1}\mathbf{k}$$

Find the velocity, speed and acceleration of the particle at $t = 0$.

$$\vec{r}'(t) = \frac{2t}{t^2+1} i + \frac{1}{1+t^2} j + \frac{1}{2}(t^2+1)^{-\frac{1}{2}}(2t)k$$

at $t=0$: $\langle 0, 1, 0 \rangle$

$$= \frac{2t}{t^2+1} \bar{i} + \frac{1}{1+t^2} j + \frac{t}{\sqrt{t^2+1}} k$$

$$\text{speed} = |\vec{r}'(t)| = \sqrt{\frac{4t^2+t^2+1}{(t^2+1)^2}}$$

$$|\vec{r}'(t)| = \sqrt{\frac{5t^2+1}{(t^2+1)^2}}$$

$$|\vec{r}'(t)| = 1$$

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$$\vec{a}(t) = \frac{2(t^2+1) - (2t)(2t)}{(t^2+1)^2} i +$$

$$\frac{-(t^2+1)}{(t^2+1)^2} j +$$

$$\frac{(t^2+1) - t(2t)}{(t^2+1)^2} k$$

$$= \frac{2-2t^2}{(t^2+1)^2} i +$$

$$\frac{1}{t^2+1} j +$$

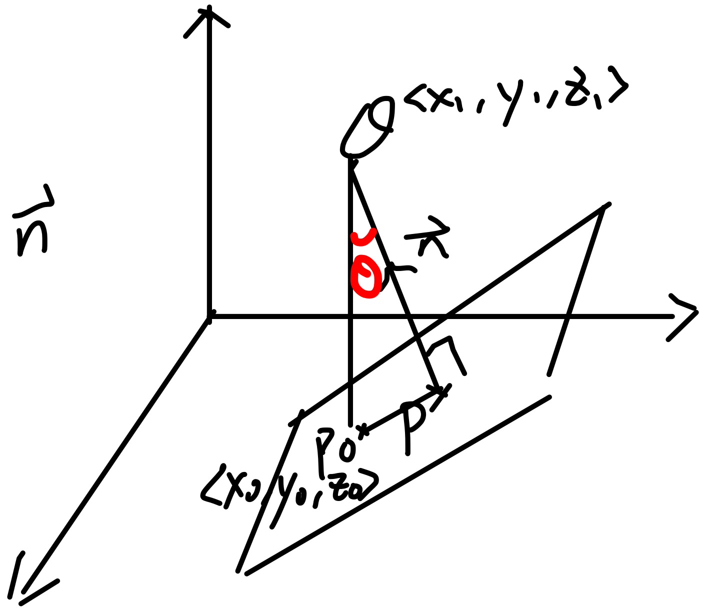
$$\frac{1-t^2}{(t^2+1)^2} k$$

At $t=0$, $\vec{a}(t) = \langle 2, 1, 1 \rangle$

5. (★★) Consider a plane through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle A, B, C \rangle$. Prove that the perpendicular distance d from a given point $Q(x_1, y_1, z_1)$ to the plane is given by:

$$d = \frac{\left|\overrightarrow{P_0Q} \cdot \mathbf{n}\right|}{|\mathbf{n}|} = \left|\frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

where $D = Ax_0 + By_0 + Cz_0$.



distance = $\overrightarrow{P_0Q}$ project on $\vec{n}$

$$\frac{\overrightarrow{P_0Q} \cdot \vec{n}}{|\vec{n}|}$$

$$= \frac{\langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle \cdot \langle A, B, C\rangle}{\sqrt{A^2 + B^2 + C^2}}$$

Made with Goodnotes

6. (★) Suppose $\mathbf{r}(t)$ represents the path of a particle traveling on a sphere centered at the origin. Show that the position vector $\mathbf{r}(t)$ and the velocity $\mathbf{r}'(t)$ are orthogonal to each other at any time.

$$r(t) \cdot r(t) = |r(t)|^2$$

$$2r(t) \cdot r'(t) = 0$$

$$|r(t)|^2 = c^2$$

$$\frac{d}{dt} c^2 = 0.$$

Made with Goodnotes

7. (★★) Suppose that the path of a particle at time t is given by $\mathbf{r}(t)$ and the force exerted on the particle at time t is $\mathbf{F}(t)$. By Newton's Second Law, $\mathbf{F}(t)$ and $\mathbf{r}(t)$ are related by:

$$\mathbf{F}(t) = m\mathbf{r}''(t),$$

where m is the mass of the particle. The angular momentum $\mathbf{L}(t)$ about the origin of the particle at time t is defined to be:

$$\mathbf{L}(t) := \mathbf{r}(t) \times m\mathbf{r}'(t)$$

(a) Show that

$$\frac{d}{dt}\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{F}(t).$$

(b) When $\mathbf{L}(t)$ is a constant vector, we say that the angular momentum is *conserved*. According to the result in (a), under what condition on $\mathbf{r}(t)$ and $\mathbf{F}(t)$ will the angular momentum be conserved? Also, give one example in physics that this condition is satisfied.

$$\frac{d}{dt} L(t) = r(t) \times \frac{d}{dt} mr'(t) + r'(t) \times mr'(t)$$

$$= r(t) \times m''(t) + r'(t) \times mr'(t)$$

$$= r(t) \times F(t) + 0$$

$$= r(t) \times F(t).$$

b). When $L(t) =$ constant, $\frac{d}{dt} L(t) = 0$

$r(t) \times F(t) = 0$

$r(t) = F(\cancel{t})$.

parallel.

反作用力

奈何.

$\rightarrow$ straight ~~line~~ motion.

Made with Goodnotes

8. (★★) Consider two point particles with masses m_1 and m_2, and their trajectories are $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ respectively. Denote $\mathbf{F}(t)$ to be the force exerted on the m_1-particle by the m_2-particle at time t. By Newton's Third Law, the force exerted on the m_2-particle by the m_1-particle at time t (i.e. the reverse force) is given by $-\mathbf{F}(t)$. Assume there are no other forces exerted on any of these particles.

(a) Consider the following vector:

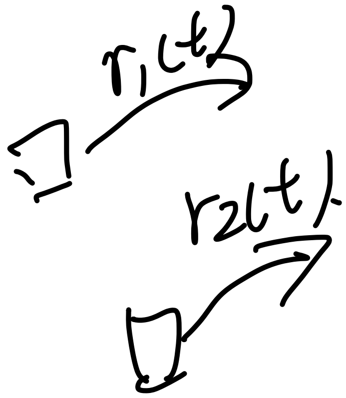
$$\mathbf{C}(t) := \frac{m_1\mathbf{r}_1(t) + m_2\mathbf{r}_2(t)}{m_1 + m_2}.$$

In physics, this vector is pointing at the center of mass of the two particles. Show that $\mathbf{C}''(t) = \mathbf{0}$ for any t using Newton's Second and Third Laws.

(b) Hence, show that there exist two constant vectors $\mathbf{r}_0$ and $\mathbf{v}$ such that

$$\frac{m_1\mathbf{r}_1(t) + m_2\mathbf{r}_2(t)}{m_1 + m_2} = \mathbf{r}_0 + t\mathbf{v}.$$

[Question: What is the physical significance of this result?]



Made with Goodnotes

9. (★) For each of the following curves, first reparametrize it by arc-length and then compute its curvature function $\kappa(s)$:

(a) $\mathbf{r}_1(t) = (R\cos\omega t)\,\mathbf{i} + (R\sin\omega t)\,\mathbf{j}, \quad 0 \le t \le \frac{2\pi}{\omega}$.

(b) $\mathbf{r}_2(t) = \langle 1,2,3\rangle + (\ln t)\,\langle 1,0,-1\rangle, \quad 0 < t < \infty$

(c) $\mathbf{r}_3(t) = (\cos^3 t)\,\mathbf{i} + (\sin^3 t)\,\mathbf{j}, \quad 0 \le t \le \frac{\pi}{2}$.

Give an example of a path whose arc-length parametrization cannot be explicitly found even with computer softwares.

a). $r'(t) = (-R\omega\sin\omega t)\,i + (R\omega\cos\omega t)\,j$

$|r'(t)| = |R|\omega$

$s = \int_0^t |R|\omega \, dt$

$s = R\omega t$

$t = \frac{s}{R\omega}$

$r(s) = R\cos\left(\frac{s}{R}\right) i + R\sin\left(\frac{s}{R}\right) j$

Made with Goodnotes

b)- (b) $\mathbf{r}_2(t) = \langle 1,2,3\rangle + (\ln t)\langle 1,0,-1\rangle, \quad 0 < t < \infty$

$$r_2'(t) = \langle \frac{1}{t}, 0, -\frac{1}{t} \rangle$$

$$|r_2'(t)| = \sqrt{\frac{2}{t^2}} = \frac{\sqrt{2}}{t}$$

$$s = \int_1^t \frac{\sqrt{2}}{t}\,dt$$

$$s = [\sqrt{2}\ln t]_1^t$$

$$s = \sqrt{2}\ln t$$

$$\frac{s}{\sqrt{2}} = \ln t$$

$$t = e^{\frac{s}{\sqrt{2}}}$$

$$r_2(t) = \langle 1,2,3\rangle + (\ln e^{\frac{s}{\sqrt{2}}})\langle 1,0,-1\rangle$$

$$r_2(t) = \langle 1,2,3\rangle + \frac{s}{\sqrt{2}}\langle 1,0,-1\rangle$$

$|r''(s)|$ = curvature.

(c) $\mathbf{r}_3(t) = (\cos^3 t)\,\mathbf{i} + (\sin^3 t)\,\mathbf{j}, \quad 0 \le t \le \frac{\pi}{2}.$

$$r_3'(t) = (-3\cos^2 t \sin t)\,i + (3\sin^2 t \cos t)\,j$$

$$|r_3'(t)| = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$$

$$|r_3'(t)| = 3.$$

$$s = \int_0^t 3 \, d\tau$$

$$s = 3t$$

$$t = \frac{s}{3}$$

$$\vec{r_3}(t) = \cos^3\left(\frac{s}{3}\right) i + \sin^3\left(\frac{s}{3}\right) j$$

Made with Goodnotes

10. (★★) Suppose

$$\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} + \frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\mathbf{j} + t\mathbf{k}$$

represents the path of a race-car climbing up a hill from $(0,0,0)$ at $t = 0$. A truck, on the other hand, drives slowly in unit speed from $(0,0,0)$ at time $t = 0$ along the same path and direction as the race-car. Find a parametrization which represents the path of the truck.

$$r'(t) = t\,i + \sqrt{2}\,t^{\frac{1}{2}} + k$$

$$\|r'(t)\| = t+1$$

$$s = \int_0^t \tau + 1 \, d\tau$$

$$s = \left[\frac{t^2}{2} + t\right]_0^t$$

$$s = \frac{t^2}{2} + t$$

Made with Goodnotes