

MATH2023 Multivariable Calculus 2013

From the textbook [Calculus of Several Variables \(5th\)](#) by R. Adams, Addison Wesley.

Homework 6

(Total: 17 questions)

Ex. 14.1

17 Evaluate the given double integral by inspection

$$\iint_{x^2+y^2 \leq 1} (4x^2y^3 - x + 5) dA.$$

22 Evaluate the given double integral by inspection

$$\iint_R \sqrt{b^2 - y^2} dA,$$

where R is the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$.

Ex. 14.2

12 Calculate the iterated integral $\iint_T \sqrt{a^2 - y^2} dA$, where T is the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, a) .

18 Sketch the domain of integration and evaluate the iterated integral.

$$\int_0^1 \int_x^{x^{\frac{1}{3}}} \sqrt{1 - y^4} dy dx.$$

22 Find the volume of the solid which is under $z = 1 - y^2$ and above $z = x^2$.

30 Let $F'(x) = f(x)$ and $G'(x) = g(x)$ on the interval $a \leq x \leq b$. Let T be the triangle with vertices (a, a) , (b, a) , and (b, b) . By iterating $\iint_T f(x)g(y) dA$ in both directions, show that

$$\int_a^b f(x)G(x) dx = F(b)G(b) - F(a)G(a) - \int_a^b g(y)F(y) dy.$$

(This is an alternative derivation of the formula for integration by parts.)

Qu. $\int_{-2}^3 \int_0^1 |x| \sin \pi y dy dx.$

Ex. 14.3

4 Determine the integral converges or diverges. Try to evaluate it if it converges.

$$\iint_T \frac{1}{x\sqrt{y}} dA \text{ over the triangle } T \text{ with vertices } (0, 0), (1, 1) \text{ and } (1, 2).$$

5 Determine the integral converges or not. Try to evaluate it if it exists.

$$\iint_Q \frac{x^2 + y^2}{(1 + x^2)(1 + y^2)} dA, \text{ where } Q \text{ is the first quadrant of the } xy\text{-plane.}$$

21 Evaluate both iterations of the improper integral

$$\iint_S \frac{x - y}{(x + y)^3} dA,$$

where S is the square $0 < x < 1$, $0 < y < 1$. Show that the above improper double integral does not exist, by considering

$$\iint_T \frac{x - y}{(x + y)^3} dA,$$

where T is that part of the square S lying under the line $x = y$.

30. (**Another proof of equality of mixed partials**) Suppose that $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous in a neighbourhood of the point (a, b) . Without the equality of these mixed partial derivatives, show that

$$\iint_R f_{xy}(x, y) dA = \iint_R f_{yx}(x, y) dA,$$

where R is the rectangle with vertices (a, b) , $(a + h, b)$, $(a, b + k)$, and $(a + h, b + k)$ and $h^2 + k^2$ is sufficiently small. Now use the result of Exercise 29 to show that $f_{xy}(a, b) = f_{yx}(a, b)$. (This reproves Theorem 1 of Section 12.4 (or see below: the mean-value theorem). However, in that theorem we only assumed continuity of the mixed partials *at* (a, b) . Here, we assume the continuity *at all points sufficiently near* (a, b) .)

A mean-value theorem for double integrals

If the function $f(x, y)$ is continuous on a closed, bounded, connected set D in the xy -plane, then there exists a point (x_0, y_0) in D such that

$$\iint_D f(x, y) dA = f(x_0, y_0) \times (\text{area of } D).$$

Ex. 14.4

11 Evaluate $\iint_S (x+y) dA$, where S is the region in the first quadrant lying inside the disk $x^2 + y^2 \leq a^2$ and under the line $y = \sqrt{3}x$.

14 Evaluate $\iint_{x^2+y^2 \leq 1} \ln(x^2 + y^2) dA$.

22 Find the volume lying inside both the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$.

26 Find the volume of the region lying inside the circular cylinder $x^2 + y^2 = 2y$ and inside the parabolic cylinder $z^2 = y$.

37 (**The gamma function**) The error function, $\text{Erf}(x)$, is defined for $x \geq 0$ by

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

Show that $[\text{Erf}(x)]^2 = \frac{4}{\pi} \int_0^{\frac{\pi}{4}} (1 - e^{-x^2 / \cos^2 \theta}) d\theta$. Hence deduce that

$$\text{Erf}(x) \geq \sqrt{1 - e^{-x^2}}.$$

Qu Find the volume lying outside the cone $z^2 = x^2 + y^2$ and inside the sphere $x^2 + (y-a)^2 + z^2 = a^2$.

* At least try to do the underlined ones, the others are recommended exercises.

Ex. 14.1

17 Evaluate the given double integral by inspection

$$\iint_{x^2+y^2 \leq 1} (4x^2y^3 - x + 5) dA.$$

$$\begin{aligned} & \int_0^{2\pi} \int_0^1 (4r^5 \cos^2 \theta \sin^3 \theta - r \cos \theta + 5) r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 5r dr d\theta \\ &= \underline{5\pi} \end{aligned}$$

22 Evaluate the given double integral by inspection

$$\iint_R \sqrt{b^2 - y^2} dA,$$

where R is the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$.

$$\int_0^a \int_0^b \sqrt{b^2 - y^2} dy dx$$

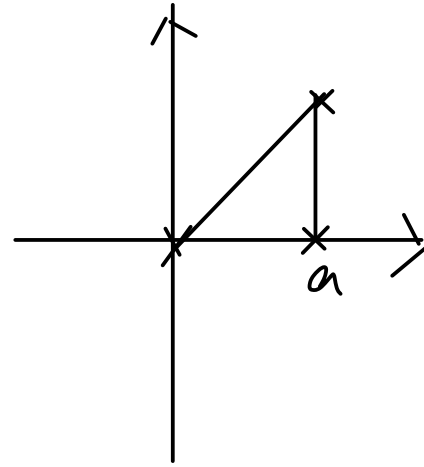
$$= \int_0^a \frac{\pi \cdot b^2}{2} dx$$

$$= \frac{\pi a b^2}{2}$$

$$\frac{\pi a b^2}{4}$$

Ex. 14.2

12 Calculate the iterated integral $\iint_T \sqrt{a^2 - y^2} dA$, where T is the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, a) .



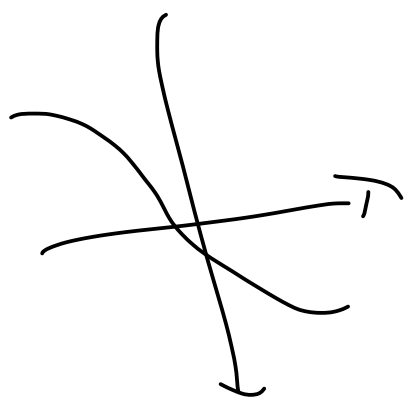
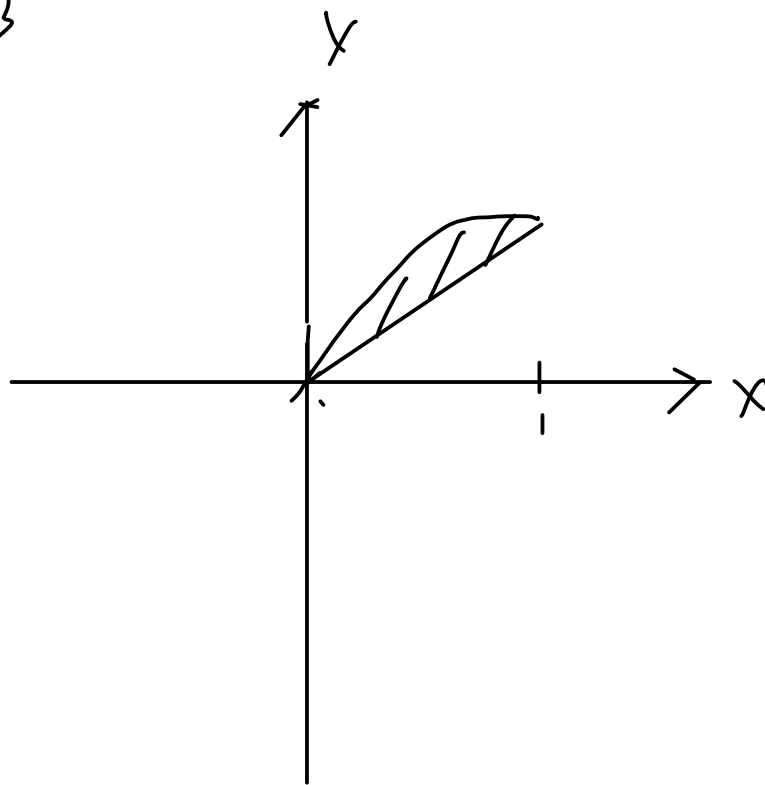
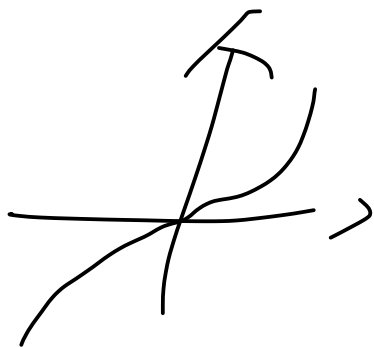
$$\int_0^a \int_0^x \sqrt{a^2 - y^2} dy dx$$

18 Sketch the domain of integration and evaluate the iterated integral.

$$\int_0^1 \int_x^{x^{\frac{1}{3}}} \sqrt{1-y^4} dy dx.$$

$$y = x^{\frac{1}{3}}$$

$$x = y^3$$



$$\int_0^1 \int_{y^3}^y \sqrt{1-y^4} dx dy$$

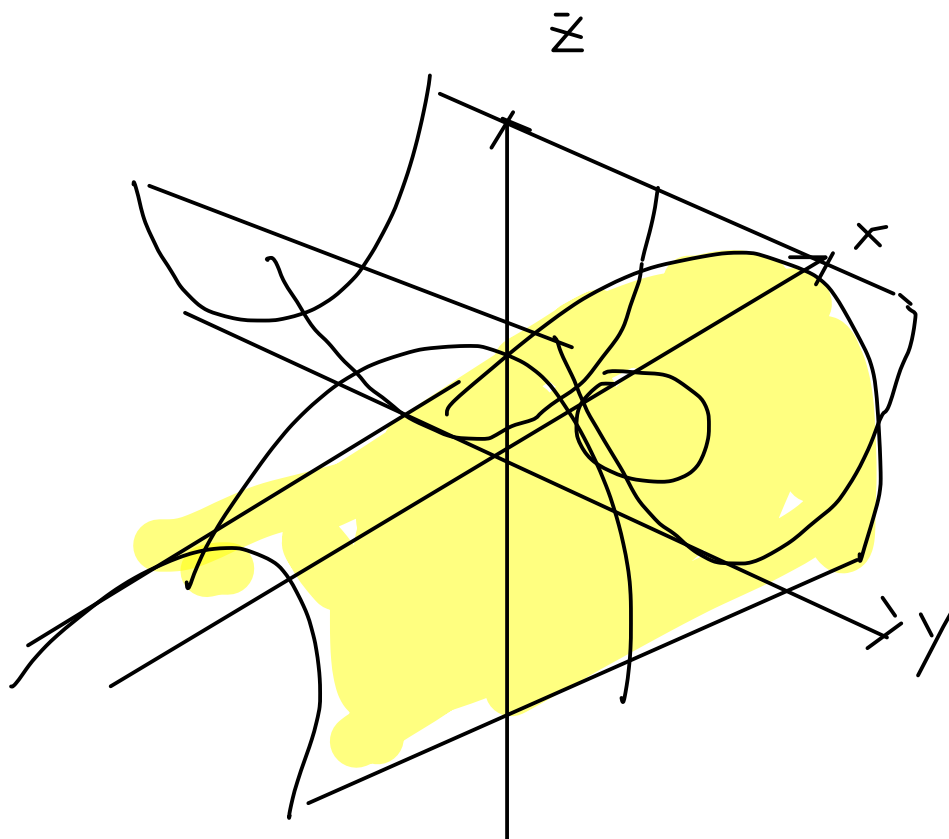
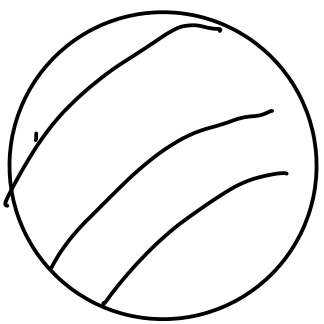
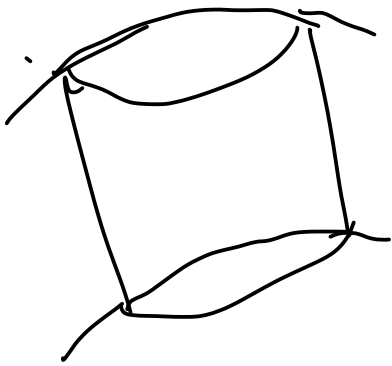
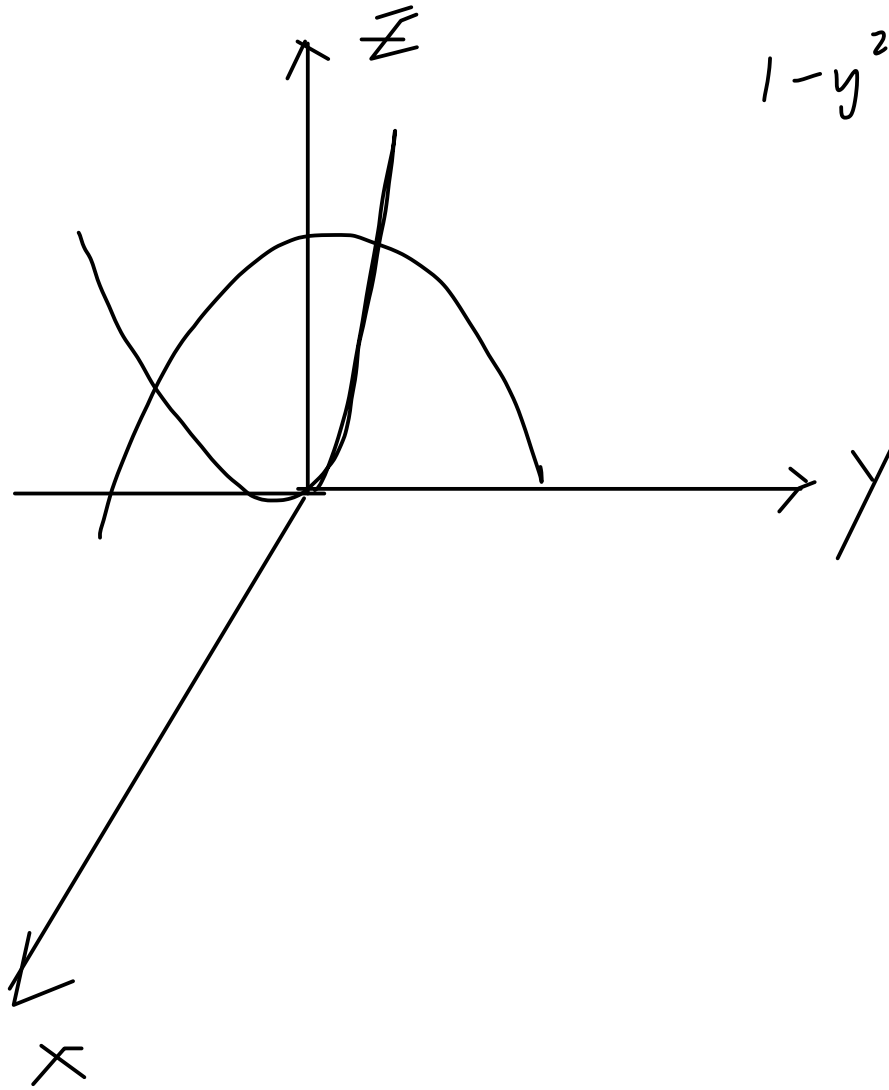
$$\int_0^1 \sqrt{1-y^4} (y - y^3) dy$$

$$\int_0^1 \underbrace{y \sqrt{1-y^4}}_{?} - y^3 \sqrt{1-y^4} dy$$

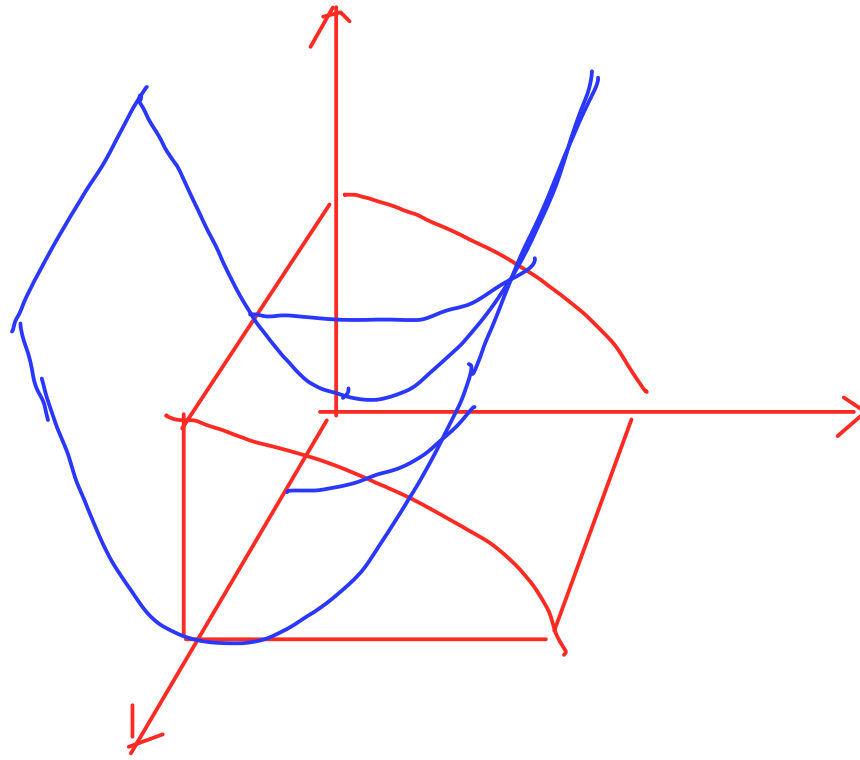
22 Find the volume of the solid which is under $z = 1 - y^2$ and above $z = x^2$.

$$1 - y^2 = x^2$$

$$1 - y^2 - x^2 = 0$$



$$z = 1 - y^2, \quad z = x^2$$

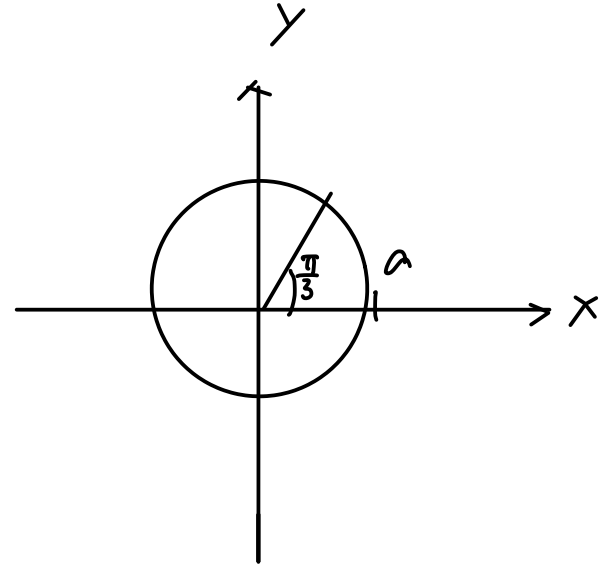


Qu. $\int_{-2}^3 \int_0^1 |x| \sin \pi y \, dy dx.$

Ex. 14.4

- 11 Evaluate $\iint_S (x + y) dA$, where S is the region in the first quadrant lying inside the disk $x^2 + y^2 \leq a^2$ and under the line $y = \sqrt{3}x$.

$$\int_0^{\frac{\pi}{3}} \int_0^a (r \cos \theta + r \sin \theta) r dr d\theta$$



14 Evaluate $\iint_{x^2+y^2 \leq 1} \ln(x^2 + y^2) dA$.

$$\int_0^{2\pi} \int_0^1 \ln(r^2) r dr d\theta$$

$$\text{let } u = r^2, du = 2r dr$$

$$2\pi \cdot \frac{1}{2} \cdot \int_0^1 \ln u du$$

$$= \pi \cdot \left([u \ln u]_0^1 - \int_0^1 1 du \right)$$

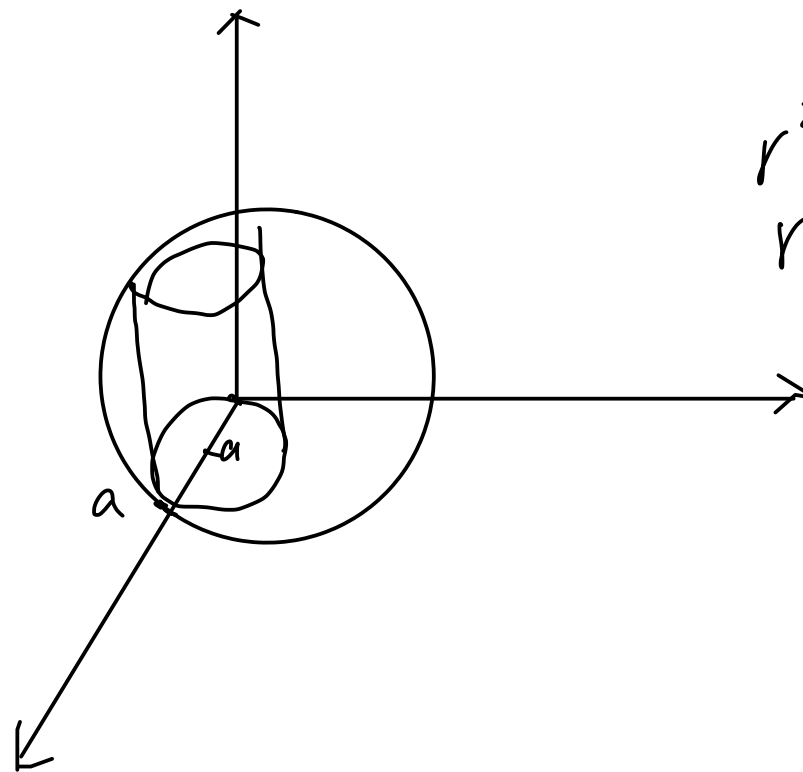
$$= -\pi.$$

22 Find the volume lying inside both the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ax$.

$$x^2 - ax + y^2 = 0$$

$$x^2 - ax + \left(\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + y^2 = 0$$

$$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$$



$$r^2 = ar \cos \theta$$

$$r = a \cos \theta$$

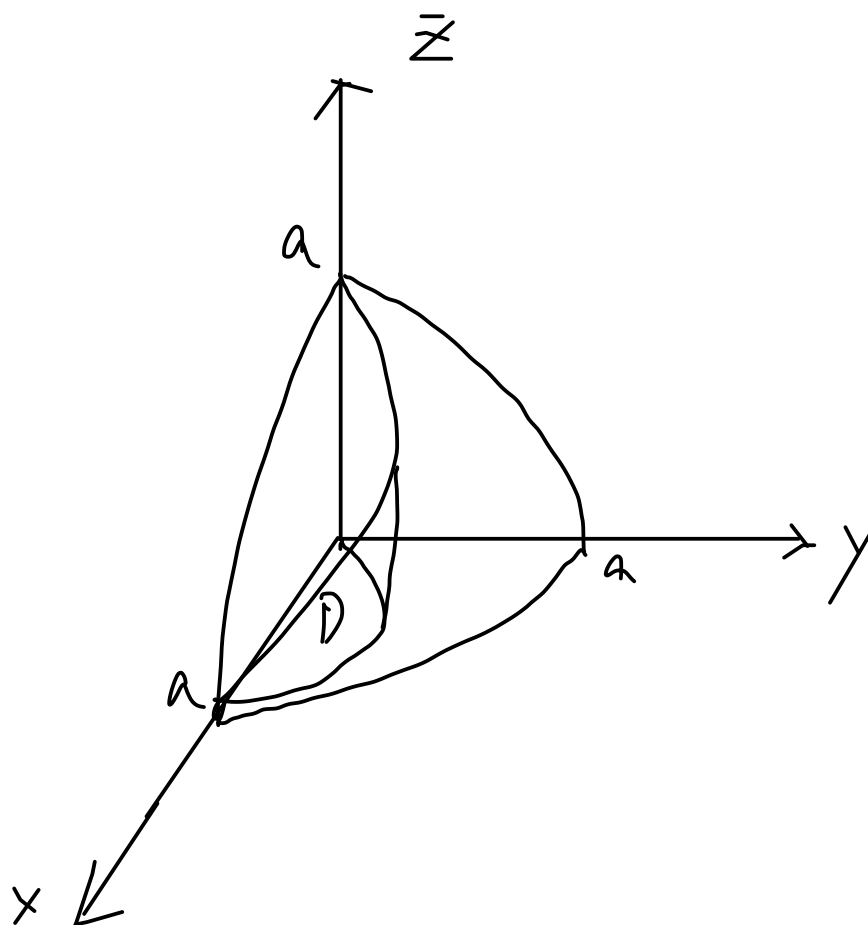
$$\rho^2 \sin \varphi = a$$

$$\rho \sin \varphi \cos \theta$$

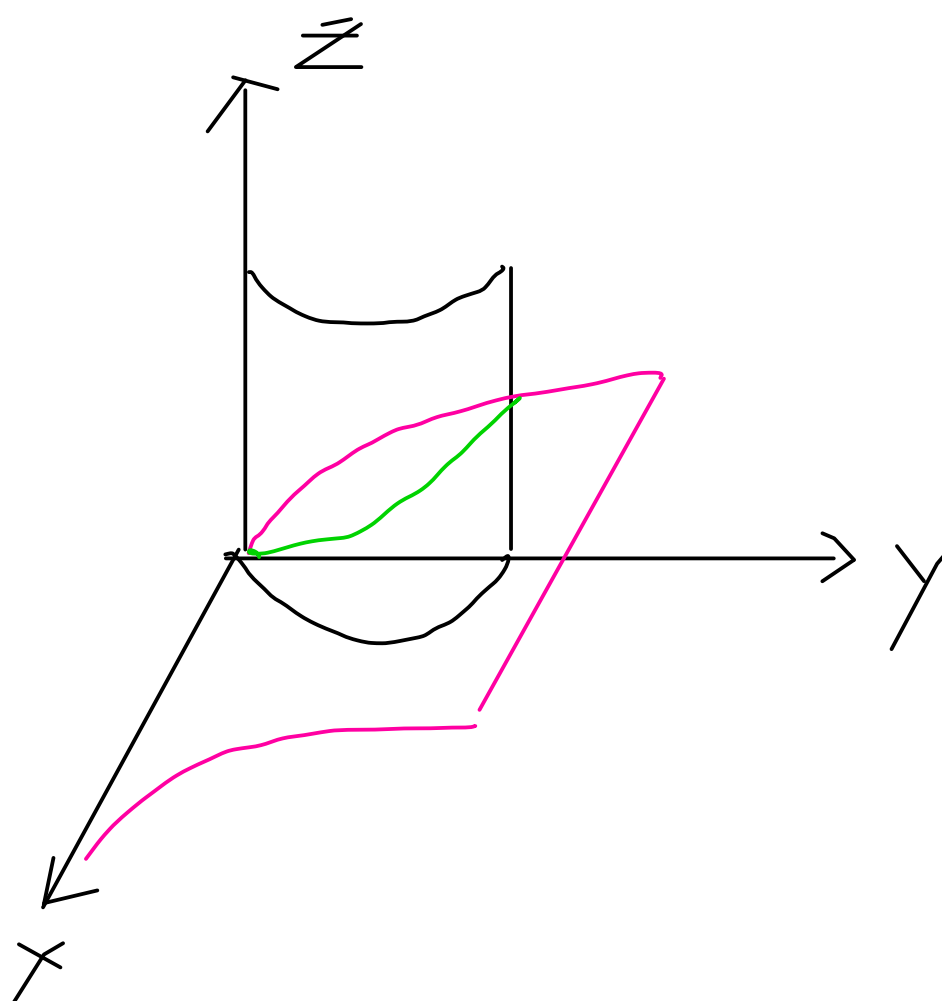
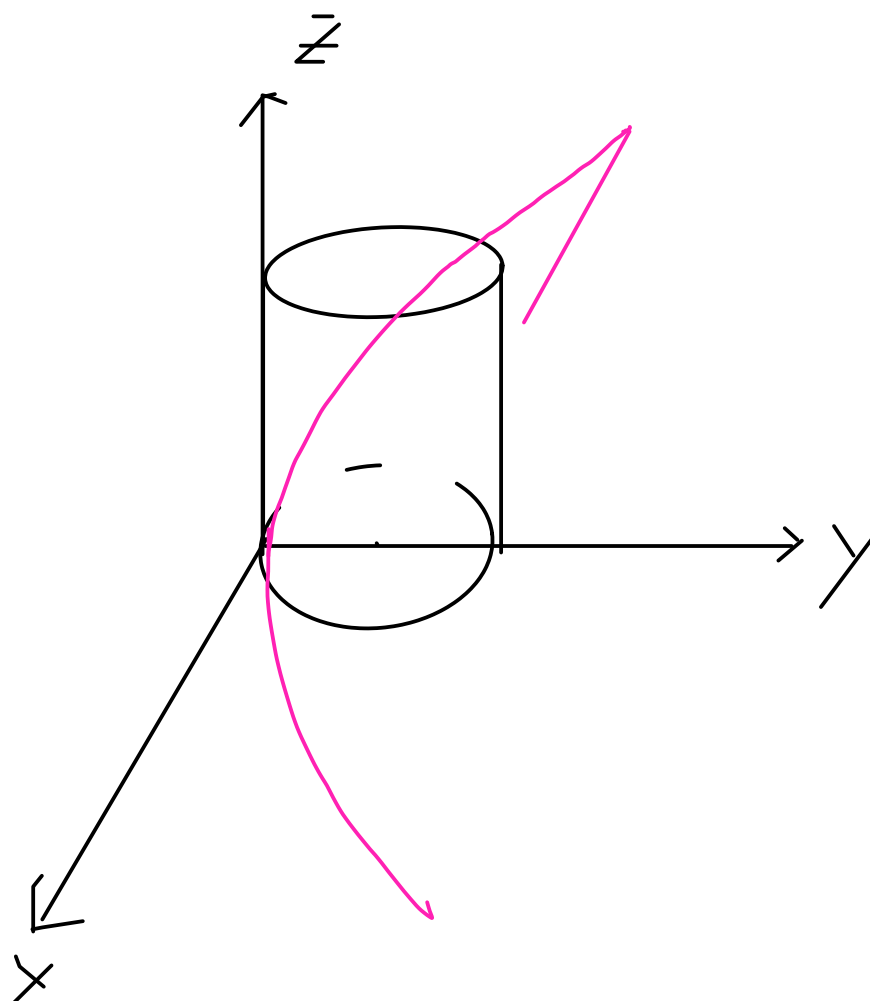
$$\rho = a \cos \theta$$

$$\int \int \int_0^{a \cos \theta}$$

$$\rho^2 \sin \varphi d\rho d\varphi d\theta$$



6 Find the volume of the region lying inside the circular cylinder $x^2 + y^2 = 2y$ and inside the parabolic cylinder $z^2 = y$.



Qu Find the volume lying outside the cone $z^2 = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + (y - a)^2 + z^2 = a^2$.

