MATH 2023 – Multivariable Calculus

Lecture #04 Worksheet ♣ February 19, 2019

Problem 1. Let

$$f(x,y) = \begin{cases} \dfrac{3x^2y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Is this function continuous on $\mathbb{R}^2$?

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$: $x^2 \le x^2+y^2$

$$|f(x,y)| \le \frac{3(x^2+y^2)|y|}{x^2+y^2} = 3|y| \xrightarrow{y\to 0} 0$$

- By Squeeze Thm, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0. = f(0,0)$
  - $f(x,y)$ is continuous at $(0,0)$.
  - $f(x,y)$ = rational function $\Rightarrow$ continuous on $\mathbb{R}^2$.
- By Polar Coordinate: $f(x,y) = \dfrac{3r^3\cos^2\theta\sin\theta}{r^2} = 3r\cos^2\theta\sin\theta$

$$|f(x,y)| \le 3r \xrightarrow[r\to 0^+]{} 0.$$

Problem 2. Let

$$f(x,y) = x^{\left(y^{(x^y)}\right)}.$$

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

product rule

$$\frac{\partial}{\partial x}\left(x^{f(x)}\right) = \frac{\partial}{\partial x} e^{f(x)\ln x} = \cancel{e^{f(x)\ln x}}\, x^{f(x)} \cdot \left(f'(x)\ln x + \frac{f(x)}{x}\right)$$

$$\frac{\partial}{\partial x}\left(y^{f(x)}\right) = \frac{\partial}{\partial x}\, e^{f(x)\ln y} = \cancel{e^{f(x)\ln y}}\, y^{f(x)} \cdot \ln y \cdot f'(x).$$

$$\frac{\partial}{\partial x}\, x^{(y^{x^y})} = x^{y^{x^y}} \cdot \left((y^{x^y})' \ln x + \frac{y^{x^y}}{x}\right)$$

$$(y^{x^y})' = y^{x^y} \cdot \ln y \cdot (x^y)', \qquad (x^y)' = y x^{y-1}$$

$$= x^{y^{x^y}}\left(y^{x^y} \cdot \ln y \cdot y x^{y-1} \cdot \ln x + \frac{y^{x^y}}{x}\right)$$

$$\frac{\partial}{\partial y}\, x^{y^{x^y}} = ?$$

Problem 3. Find $f(x, y)$ such that

$$\begin{cases} \dfrac{\partial f}{\partial x} = 4x - y \\ \dfrac{\partial f}{\partial y} = -x + 6y^2 \end{cases}$$

$$f(x,y) \rightsquigarrow \frac{\partial f}{\partial x} = 4x - y$$

some expression in y.

$$f(x,y) = 2x^2 - xy + g(y).$$

$$\frac{\partial f}{\partial y} = 0 - x + g'(y)$$

$$g'(y) = 6y^2$$

$$\Rightarrow g(y) = 2y^3 + C$$

$$f(x,y) = 2x^2 - xy + 2y^3 + C$$