Remark: $r(t) \times F(t) = 0$

$\Rightarrow r(t)$ parallel to $F(t)$.

Remark: arc-length parametrization 當中，

$r(t)$ 如果是 ti，$|r'(t)|$ 要變為 $|t|i$.

7. (★★) Suppose that $f(x,y)$ is a function such that $\dfrac{\partial^2 f}{\partial x \partial y} \equiv 0$. Show that f can be decomposed into the form:

$$f(x,y) = F(x) + G(y)$$

where $F(x)$ and $G(y)$ are some single-variable functions.

Technique:

Solution: Given that:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = 0,$$

we know $\frac{\partial f}{\partial y}$ is independent of x, and depends only on y. Therefore, it can be written as:

$$\frac{\partial f}{\partial y} = g(y)$$

for an arbitrary differentiable function $g(y)$.

Now, we obtained that the y-derivative of f is $g(y)$. To find the function f, we can integrate $g(y)$ with respect to y:

$$f(x,y) = \int g(y)\, dy + F(x).$$

在 In 翻 partial y 的時候，個 c 唔係 c，而是 $F(x)$.

As $\frac{\partial}{\partial y}$ is a partial derivative, the integration "constant" is not really a constant but is a quantity not depending on y. In other words, the integration "constant" is a function $F(x)$ of x.

Since $\int g(y)\, dy$ is also an arbitrary function of y, for simplicity we relabel it as $G(y)$. Therefore, we get $f(x,y) = F(x) + G(y)$.

5. (★★) Let $f(x, y)$ be a C^1 function. Consider two parametric curves $\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j}$ and $\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j}$ which satisfy:

$$\mathbf{r}_1(0) = \mathbf{r}_2(0) \quad \text{and} \quad \mathbf{r}_1'(0) = \mathbf{r}_2'(0).$$

(a) Show that

$$\left.\frac{d}{dt}\right|_{t=0} f(x_1(t), y_1(t)) = \left.\frac{d}{dt}\right|_{t=0} f(x_2(t), y_2(t)).$$

Solution:

Using the chain rule, we get:

partial
d左先.

$$\frac{d}{dt} f(x_1(t), y_1(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$
$$= \frac{\partial f}{\partial x} x_1'(t) + \frac{\partial f}{\partial y} y_1'(t) \quad (\text{since } x = x_1 \text{ and } y = y_1 \text{ in this case})$$

Sub $t=0$

At $t = 0$, we get:

$$\left.\frac{d}{dt}\right|_{t=0} f(x_1(t), y_1(t)) = \left.\frac{\partial f}{\partial x}\right|_{(x_1(0), y_1(0))} x_1'(0) + \left.\frac{\partial f}{\partial y}\right|_{(x_1(0), y_1(0))} y_1'(0)$$

是在呢點時找的value,
非找 $|x_1(0)$

Similarly, one can also show:

$$\left.\frac{d}{dt}\right|_{t=0} f(x_2(t), y_2(t)) = \left.\frac{\partial f}{\partial x}\right|_{(x_2(0), y_2(0))} x_2'(0) + \left.\frac{\partial f}{\partial y}\right|_{(x_2(0), y_2(0))} y_2'(0)$$

Sub點去
指出相同

It is given from the problem that:

$$\mathbf{r}_1(0) = \mathbf{r}_2(0) \quad \text{and} \quad \mathbf{r}_1'(0) = \mathbf{r}_2'(0).$$

Therefore, we have:

$$x_1(0) = x_2(0) \qquad x_1'(0) = x_2'(0)$$
$$y_1(0) = y_2(0) \qquad y_1'(0) = y_2'(0)$$

and so:

$$\left.\frac{d}{dt}\right|_{t=0} f(x_1(t), y_1(t)) = \left.\frac{d}{dt}\right|_{t=0} f(x_2(t), y_2(t)).$$

8. (★★) Let $u(x,y,z,t)$ be the temperature at the point (x,y,z) at the time t. Combining with several important laws in thermodynamics, including the Fourier's Law and conservation of energy, it can be derived (detail omitted) that the temperature function $u(x,y,z,t)$ satisfies the following equation:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

where k is a positive constant depending only on the medium. This equation is known as the **heat equation**.

The study of the heat equation is an important topic in physics, engineering and mathematics (both pure and applied). Through solving the heat equation with an initial condition $u(x,y,z,0) = g(x,y,z)$, it predicts how heat diffuses for a given an initial heat profile $g(x,y,z)$ at time $t = 0$.

Your task in this problem is to verify that the following given function is a solution to the heat equation:

$$\varphi(x,y,z,t) = \frac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(-\frac{x^2+y^2+z^2}{4kt}\right). \qquad (*)$$

This particular solution φ represents the heat diffusion with highly concentrated heat source at the origin $(0,0,0)$ at time $t = 0$. As time goes by, the temperature profile becomes more and more uniformly distributed. (In physics, this solution is also closely related to the *Dirac delta function*.)

By following the outline below, show that φ satisfies the heat equation:

(a) Show that:

$$\ln\varphi(x,y,z,t) = -\ln(4\pi k)^{\frac{3}{2}} - \frac{3}{2}\ln t - \frac{x^2+y^2+z^2}{4kt}. \qquad (⊛)$$

(b) Using (a), show that:

$$\frac{\partial \varphi}{\partial t} = \left(\frac{x^2+y^2+z^2}{4kt^2} - \frac{3}{2t}\right)\varphi.$$

(c) Using (a) again, show that:

$$\frac{\partial \varphi}{\partial x} = -\frac{x\varphi}{2kt} \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{2kt}\left(\frac{x^2}{2kt} - 1\right)\varphi.$$

(d) Hence, verify that φ satisfies the heat equation: $\varphi_t = k(\varphi_{xx} + \varphi_{yy} + \varphi_{zz})$.

(e) (Optional) Show that

$$\lim_{t\to 0^+} \varphi(x,y,z,t) = \begin{cases} \infty & \text{if } (x,y,z) = (0,0,0) \\ 0 & \text{if } (x,y,z) \neq (0,0,0) \end{cases}$$

Procedure: 呢題跟住題目枕得，目的是要 proof

$\varphi_t = k(\varphi_{xx} + \varphi_{yy} + \varphi_{zz})$ for equation $(*)$.

第一步：兩边一齊 take log，得出 (⊛)

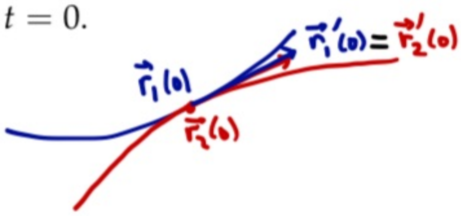
第二步：$\frac{\partial \varphi}{\partial t}$，用 Implicit differentiation 方法.

第三步：$\frac{\partial \varphi}{\partial x}$，$\frac{\partial^2 \varphi}{\partial x^2}$，再計 $k(\varphi_{xx} + \varphi_{yy} + \varphi_{zz})$

承上題:

(b) Give a geometric interpretation of the above result.

Solution: This result shows that any two parametric curves with the same position and velocity at $t = 0$ will give the same rate of change of a function f along these two curves at $t = 0$.



6. (★★) The wave equation is an important partial differential equation which governs the propagation of waves. Let $u(x, y, z, t)$ be the displacement of the wave at position (x, y, z) at time t. It can be shown by several physical laws (such as the Hooke's Law) that u satisfies:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$

where c is a constant (which is the wave speed).

In one (spatial) dimension, the wave equation can be stated as:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

It turns out that the chain rule of several variables has a nice application on solving the one dimensional wave equation. The following exercise guides you to show that if $u(x, t)$ is a solution to the one dimensional wave equation, then it must take the form $u(x, t) = F(x - ct) + G(x + ct)$ where F and G are arbitrary differentiable functions of single variable.

Let $u(x, t)$ solve the one dimensional wave equation (2).

(a) Define $\xi = x - ct$ and $\eta = x + ct$. Regard u as a function of ξ and η, and ξ and η are functions of x and t. Using the chain rule of multivariable functions, show that:

$$u_t = c(u_\eta - u_\xi) \quad \text{and} \quad u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).$$

(b) Using the chain rule agin, show that

$$u_x = u_\xi + u_\eta \quad \text{and} \quad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

(c) Combining the results of (a), (b) and the wave equation, show that $u_{\xi\eta} = 0$.

(d) Finally, deduce that u, as a function of ξ and η, must be in the form of:

$$u(\xi, \eta) = F(\xi) + G(\eta)$$

where F and G are arbitrary functions. Hence, in terms of the original variables x and t, u must take the form $u(x, t) = F(x - ct) + G(x + ct)$.

呢題的 a, b, c 都是廢話, 而(d)的推法就是:

$u_{\xi\eta} = 0 \Rightarrow \frac{\partial}{\partial \eta} u_\xi = 0 \Rightarrow u_\xi$ is independent of η, only function of $\xi \Rightarrow \frac{\partial u}{\partial \xi} = f(\xi) \Rightarrow$ 兩邊一齊 $\int \Rightarrow$

$u = \int f(\xi)\,d\xi + C(?) \Rightarrow C(?)$ 不是 numerical constant 咁簡單, 而是 function of η. $\therefore u = \int f(\xi)\,d\xi + g(\eta)$

Remark: Geographical meaning of directional derivatives =

11. (★★) Consider the function

$$f(x,y) = \cos(x+y)$$

as well as the plane Π given by the equation

$$x - y = 0\,.$$

The intersection of the graph of f with Π is a curve C. Find the slope of the tangent line to C at the point (π, π) using directional derivatives. [Hint: First sketch a diagram of the graph, the plane and the curve.]

此題解法: $\hat{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$\nabla f = \langle -\sin(x+y), -\sin(x+y) \rangle$

$\nabla f_{(\pi,\pi)} \cdot \hat{u}$ 即是此 slope at (π,π).

Local Extrema: $f(x,y) = \sin x \cos y$.

經過一輪計算後計出下列情況, 但要留意, solving $\sin x = 0$, x 係 $\frac{\pi}{2} + k\pi$, 之後用 D test 都要分翻 D arbitary constant 是 odd / even;

$\sin x\ \sin y = 0$ then tells us that $\sin y = 0$, and so $y = m\pi$ where m is any integer. Critical points obtained in this case are:

$$(x,y) = \left(\frac{\pi}{2} + k\pi,\ m\pi\right), \quad \text{where } m \text{ and } k \text{ are any integers.}$$

Case B: From $\cos y = 0$, we get $y = \frac{\pi}{2} + n\pi$ where n is any integer, then substitute it into the second equation, we get $\sin x \cdot (\pm 1) = 0$ where $\pm$ depends on whether k is even or odd. Therefore, $x = q\pi$ where q is any integer. Critical points obtained in this case are:

$$(x,y) = \left(q\pi,\ \frac{\pi}{2} + n\pi\right) \quad \text{where } n \text{ and } q \text{ are any integers.}$$

To apply the Second Derivative Test, we first compute:

$$\begin{aligned} f_{xx} &= -\sin x\ \cos y \\ f_{xy} &= -\cos x\ \sin y \\ f_{yy} &= -\sin x\ \cos y \\ f_{xx}f_{yy} - f_{xy}^2 &= \sin^2 x\ \cos^2 y - \cos^2 x\ \sin^2 y \end{aligned}$$

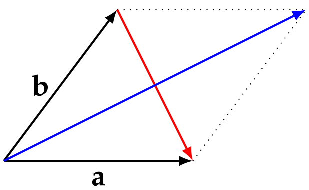
P		f_{xx}	f_{yy}	f_{xy}	$\left(f_{xx}f_{yy} - f_{xy}^2\right)$	P is:
$(\frac{\pi}{2} + k\pi, m\pi)$	k odd, m odd	-1	-1	0	1	local max
$(\frac{\pi}{2} + k\pi, m\pi)$	k odd, m even	1	1	0	1	local min
$(\frac{\pi}{2} + k\pi, m\pi)$	k even, m odd	1	1	0	1	local min
$(\frac{\pi}{2} + k\pi, m\pi)$	k even, m even	-1	-1	0	1	local max
$(q\pi, \frac{\pi}{2} + n\pi)$	any integers q, n	0	0	± 1	-1	saddle

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #0 • Dot and Cross Products (Review)

1. (★) Given three points in $\mathbb{R}^3$:
$$A(1,2,3),\ B(4,0,5)\ \text{ and }\ C(x,6,4)$$
Determine the number of possible value(s) of x such that the triangle ABC has a right angle.

2. (★★) Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.
   (a) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal (i.e. $\mathbf{u} \perp \mathbf{v}$, $\mathbf{v} \perp \mathbf{w}$ and $\mathbf{w} \perp \mathbf{u}$).
   (b) Given any vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in $\mathbb{R}^3$, show that:
$$\mathbf{r} = \frac{\mathbf{r}\cdot\mathbf{u}}{|\mathbf{u}|^2}\mathbf{u} + \frac{\mathbf{r}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} + \frac{\mathbf{r}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}.$$
   [Hint: You may use the fact that since $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal and non-zero, the vector $\mathbf{r}$ can be expressed as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, i.e.
$$\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$
   Solve for the scalars a, b and c.]
   (c) Express the vector $\mathbf{i}$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

3. (★) The figure below shows two vectors $\mathbf{a}$ and $\mathbf{b}$ which span a parallelogram. The vectors in blue and red represent the two diagonals of the parallelogram.



   (a) Express the red and the blue vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.
   (b) By considering the dot product, show that: $|\mathbf{a}| = |\mathbf{b}|$ if and only if the diagonals of the parallelogram are orthogonal to each other.

4. (★) Let $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be a variable **unit** vector in $\mathbb{R}^3$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
   (a) Find x, y and z such that $\mathbf{u}\cdot\mathbf{v}$ is the maximum possible. Explain your answer.
   (b) Find x, y and z such that $|\mathbf{u}\times\mathbf{v}|$ is the maximum possible. Explain your answer.

5. (★★) Given two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, prove the following:
   (a) Cauchy-Schwarz's Inequality: $|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}|$
   (b) Triangle Inequality: $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$
   (c) If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, show that $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$.

6. (★) Let A, B and C be the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively in the three dimensional space, and O be the origin $(0,0,0)$. Denote $[ABC]$ the area of the triangle with vertices A, B and C (analogously for $[OAB]$, $[OBC]$, etc.). Show that:
$$[ABC]^2 = [OAB]^2 + [OBC]^2 + [OCA]^2.$$
With the help of a diagram, explain why this result can be regarded as the *three-dimensional analogue of the Pythagoreas' Theorem*.

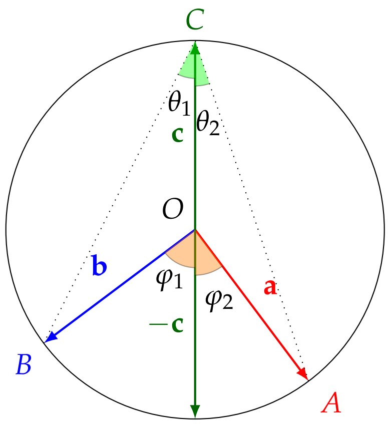
7. (★★) Given three non-zero vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, provide a *geometric explanation* to each of the following facts:

   (a) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

   (b) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$

   (c) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is a vector on the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

8. (★★★) The diagram below shows a circle with radius r centered at O. Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$. The purpose of the problem is to use dot products to show that the angle at the center of a circle is twice the corresponding angle at the circumference. Precisely, with the notations in the diagram below, we want to show $\angle BOA = 2\angle BCA$. We will prove this by showing $\varphi_1 = 2\theta_1$, and $\varphi_2 = 2\theta_2$ can be proved in a similar way. Follow the steps structured below:



   (a) Show that $\cos\varphi_1 = -\dfrac{\mathbf{b} \cdot \mathbf{c}}{r^2}$. Recall that r is the radius of the circle.

   (b) Show that $\cos\theta_1 = \dfrac{r^2 - \mathbf{b} \cdot \mathbf{c}}{|\mathbf{b} - \mathbf{c}|\,|\mathbf{c}|}$.

   (c) Showing that $|\mathbf{b} - \mathbf{c}|^2 = 2(r^2 - \mathbf{b} \cdot \mathbf{c})$.

   (d) Using the result proved in the previous parts, show that $\cos^2\theta_1 = \dfrac{r^2 - \mathbf{b} \cdot \mathbf{c}}{2r^2}$.

   (e) Finally, find a relation between $\cos^2\theta_1$ and $\cos\varphi_1$, and conclude that $\varphi_1 = 2\theta_1$.
   [Hint: Double angle formula for cos.]

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #1 • Lines, Planes and Curves

1. (★) Consider the two straight-lines:

$$L_1: \mathbf{r}_1(t) = \langle 1,2,3\rangle + t\langle 1,-1,-1\rangle$$
$$L_2: \mathbf{r}_2(t) = \langle 2+t, 3-3t, -2+3t\rangle$$

   (a) Show that L_1 and L_2 intersects each other. Find the coordinates of the intersection point.
   (b) Find an equation of the plane containing both L_1 and L_2.

2. (★) Consider the following four points in three-dimensional space:

$$A(0,2,-1),\ B(4,0,-1),\ C(7,-3,0) \text{ and } D(\tfrac{1}{3},\tfrac{1}{6},\tfrac{1}{9})$$

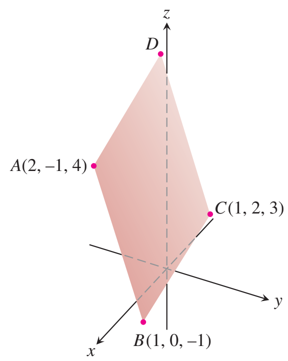
   Determine whether or not these four points are coplanar (i.e. contained in a single plane).

3. (★) A parallelogram in $\mathbb{R}^3$ has vertices:

$$A(2,-1,4),\quad B(1,0,-1),\quad C(1,2,3),\quad D(x_0,y_0,z_0)$$

   as shown in the figure below. Answer the following questions:

   (a) Find the coordinates of D.
   (b) Find the area of the parallelogram $ABCD$.
   (c) Find an equation of the plane containing the parallelogram $ABCD$.
   (d) Project the parallelogram $ABCD$ orthogonally onto the plane $z = -1$. Find the coordinates the projection of each vertices, then find the area of the *projected* parallelogram.



4. (★) Consider a particle whose path is represented by:

$$\mathbf{r}(t) = \left(\ln(t^2+1)\right)\mathbf{i} + \left(\tan^{-1} t\right)\mathbf{j} + \sqrt{t^2+1}\mathbf{k}$$

   Find the velocity, speed and acceleration of the particle at $t = 0$.

5. (★★) Consider a plane through the point $P_0(x_0,y_0,z_0)$ with normal vector $\mathbf{n} = \langle A,B,C\rangle$. Prove that the perpendicular distance d from a given point $Q(x_1,y_1,z_1)$ to the plane is given by:

$$d = \frac{\left|\overrightarrow{P_0Q}\cdot\mathbf{n}\right|}{|\mathbf{n}|} = \left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right|$$

   where $D = Ax_0 + By_0 + Cz_0$.

6. (★) Suppose $\mathbf{r}(t)$ represents the path of a particle traveling on a sphere centered at the origin. Show that the position vector $\mathbf{r}(t)$ and the velocity $\mathbf{r}'(t)$ are orthogonal to each other at any time.

7. (★★) Suppose that the path of a particle at time t is given by $\mathbf{r}(t)$ and the force exerted on the particle at time t is $\mathbf{F}(t)$. By Newton's Second Law, $\mathbf{F}(t)$ and $\mathbf{r}(t)$ are related by:

$$\mathbf{F}(t) = m\mathbf{r}''(t),$$

where m is the mass of the particle. The angular momentum $\mathbf{L}(t)$ about the origin of the particle at time t is defined to be:

$$\mathbf{L}(t) := \mathbf{r}(t) \times m\mathbf{r}'(t)$$

   (a) Show that

$$\frac{d}{dt}\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{F}(t).$$

   (b) When $\mathbf{L}(t)$ is a constant vector, we say that the angular momentum is *conserved*. According to the result in (a), under what condition on $\mathbf{r}(t)$ and $\mathbf{F}(t)$ will the angular momentum be conserved? Also, give one example in physics that this condition is satisfied.

8. (★★) Consider two point particles with masses m_1 and m_2, and their trajectories are $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ respectively. Denote $\mathbf{F}(t)$ to be the force exerted on the m_1-particle by the m_2-particle at time t. By Newton's Third Law, the force exerted on the m_2-particle by the m_1-particle at time t (i.e. the reverse force) is given by $-\mathbf{F}(t)$. Assume there are no other forces exerted on any of these particles.

   (a) Consider the following vector:

$$\mathbf{C}(t) := \frac{m_1\mathbf{r}_1(t) + m_2\mathbf{r}_2(t)}{m_1 + m_2}.$$

   In physics, this vector is pointing at the center of mass of the two particles. Show that $\mathbf{C}''(t) = \mathbf{0}$ for any t using Newton's Second and Third Laws.

   (b) Hence, show that there exist two constant vectors $\mathbf{r}_0$ and $\mathbf{v}$ such that

$$\frac{m_1\mathbf{r}_1(t) + m_2\mathbf{r}_2(t)}{m_1 + m_2} = \mathbf{r}_0 + t\mathbf{v}.$$

   [Question: What is the physical significance of this result?]

9. (★) For each of the following curves, first reparametrize it by arc-length and then compute its curvature function $\kappa(s)$:

   (a) $\mathbf{r}_1(t) = (R\cos\omega t)\,\mathbf{i} + (R\sin\omega t)\,\mathbf{j}, \quad 0 \le t \le \frac{2\pi}{\omega}.$

   (b) $\mathbf{r}_2(t) = \langle 1, 2, 3\rangle + (\ln t)\,\langle 1, 0, -1\rangle, \quad 0 < t < \infty$

   (c) $\mathbf{r}_3(t) = (\cos^3 t)\,\mathbf{i} + (\sin^3 t)\,\mathbf{j}, \quad 0 \le t \le \frac{\pi}{2}.$

   Give an example of a path whose arc-length parametrization cannot be explicitly found even with computer softwares.

10. (★★) Suppose

$$\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} + \frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\mathbf{j} + t\mathbf{k}$$

represents the path of a race-car climbing up a hill from $(0, 0, 0)$ at $t = 0$. A truck, on the other hand, drives slowly in unit speed from $(0, 0, 0)$ at time $t = 0$ along the same path and direction as the race-car. Find a parametrization which represents the path of the truck.

11. (★★★) We define the curvature of a path by $\kappa(s) = |\mathbf{r}''(s)|$ where $\mathbf{r}(s)$ is the arc-length parametrization of the path. However, the arc-length parametrization $\mathbf{r}(s)$ is often difficult to find explicitly. The purpose of this exercise is to derive an equivalent formula for the curvature which does not require finding an arc-length parametrization.

Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that s and t are related by:

$$s = \int_0^t |\mathbf{r}'(\tau)| \, d\tau.$$

(a) Show, using the chain rule, that:

$$\mathbf{r}'(t) = \mathbf{r}'(s)\frac{ds}{dt}$$
$$\mathbf{r}''(t) = \mathbf{r}''(s)\left(\frac{ds}{dt}\right)^2 + \mathbf{r}'(s)\frac{d^2s}{dt^2}$$

(b) Show that:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^3 \mathbf{r}'(s) \times \mathbf{r}''(s)$$

(c) Using (a) and (b), show that the curvature, which is defined as $\kappa(s) := |\mathbf{r}''(s)|$, can be expressed in terms of t as:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Although it looks more complicated, this formula does not require the procedure of finding arc-length parametrization.

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #2 • Multivariable Functions, Partial Derivatives

1. (★) Let $f(x,y) = \sqrt{y - x^2}$
   (a) What is the (largest possible) domain of f?
   (b) Sketch the level sets $f = 0$, $f = 1$ and $f = 2$ in the same diagram.

2. (★) Let
$$f(x,y) = \frac{1}{\sqrt{x^2 + y^2 - 1}}$$
   (a) What is the (largest possible) domain of f?
   (b) Sketch the level sets $f = 1$, $f = 2$ and $f = 3$ in the same diagram.
   (c) Repeat (a) and (b) for the function $g(x,y) = \dfrac{1}{\sqrt{1 - x^2 - y^2}}$.

3. (★) Compute all the first and second partial derivatives of the following functions. For the second partials f_{xy} and f_{yx}, compute both and verify that they are indeed the same.
   (a) $f(x,y) = y^{2015} + 2x^2 + 2xy$
   (b) $f(x,y) = e^{x^2 y}$
   (c) $f(x,y) = \frac{x}{x^2+y^2}$
   (d) $f(x,y) = x \ln(x^2 + y^2)$

4. (★★) Compute the first partial derivative $\frac{\partial f}{\partial x}$ of the following functions (where $x, y > 0$).
   (a) $f(x,y) = e^{x^y}$
   (b) $f(x,y) = e^{y^x}$
   (c) $f(x,y) = x^{e^y}$
   (d) $f(x,y) = y^{e^x}$
   (e) $f(x,y) = x^{y^e}$
   (f) $f(x,y) = y^{x^e}$

5. (★) Compute both the third-order derivatives h_{xyy} and h_{yyx} of the following function, and verify that they are indeed the same.
$$h(x,y,z) = \cos(x^2 + y^3 z).$$

6. (★★) Find the second derivative $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ of each function $f(x,y)$ below. [Hint: There is a smart way to compute each of them.]
   (a)
$$f(x,y) = \sin(x+y)\ \cos(x-y)$$
   (b)
$$f(x,y) = \cos(xy) + \left(\frac{\sin^{2016} y + \cos^{2014} y}{\sin^2 \log(y^4+1) + 2015}\right)^{\frac{1}{2015}}.$$
   (c)
$$f(x,y) = \frac{e^{x+y} + e^{x-y}}{e^{x+y} - e^{x-y}}$$

7. (★★) Suppose that $f(x,y)$ is a function such that $\dfrac{\partial^2 f}{\partial x \partial y} \equiv 0$. Show that f can be decomposed into the form:
$$f(x,y) = F(x) + G(y)$$
where $F(x)$ and $G(y)$ are some single-variable functions.

8. (★★) Let $u(x,y,z,t)$ be the temperature at the point (x,y,z) at the time t. Combining with several important laws in thermodynamics, including the Fourier's Law and conservation of energy, it can be derived (detail omitted) that the temperature function $u(x,y,z,t)$ satisfies the following equation:
$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$
where k is a positive constant depending only on the medium. This equation is known as the **heat equation**.

The study of the heat equation is an important topic in physics, engineering and mathematics (both pure and applied). Through solving the heat equation with an initial condition $u(x,y,z,0) = g(x,y,z)$, it predicts how heat diffuses for a given an initial heat profile $g(x,y,z)$ at time $t = 0$.

Your task in this problem is to verify that the following given function is a solution to the heat equation:
$$\varphi(x,y,z,t) = \frac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(-\frac{x^2+y^2+z^2}{4kt}\right).$$

This particular solution φ represents the heat diffusion with highly concentrated heat source at the origin $(0,0,0)$ at time $t = 0$. As time goes by, the temperature profile becomes more and more uniformly distributed. (In physics, this solution is also closely related to the *Dirac delta function*.)

By following the outline below, show that φ satisfies the heat equation:

(a) Show that:
$$\ln \varphi(x,y,z,t) = -\ln(4\pi k)^{\frac{3}{2}} - \frac{3}{2}\ln t - \frac{x^2+y^2+z^2}{4kt}.$$

(b) Using (a), show that:
$$\frac{\partial \varphi}{\partial t} = \left(\frac{x^2+y^2+z^2}{4kt^2} - \frac{3}{2t}\right)\varphi.$$

(c) Using (a) again, show that:
$$\frac{\partial \varphi}{\partial x} = -\frac{x\varphi}{2kt} \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{2kt}\left(\frac{x^2}{2kt} - 1\right)\varphi.$$

(d) Hence, verify that φ satisfies the heat equation: $\varphi_t = k(\varphi_{xx} + \varphi_{yy} + \varphi_{zz})$.

(e) (Optional) Show that
$$\lim_{t\to 0^+} \varphi(x,y,z,t) = \begin{cases} \infty & \text{if } (x,y,z) = (0,0,0) \\ 0 & \text{if } (x,y,z) \neq (0,0,0) \end{cases}$$

1. (★) Let $f(x, y) = \sqrt{y - x^2}$
   (a) What is the (largest possible) domain of f?
   (b) Sketch the level sets $f = 0$, $f = 1$ and $f = 2$ in the same diagram.

a). $y - x^2 \geq 0$

$y \geq x^2$

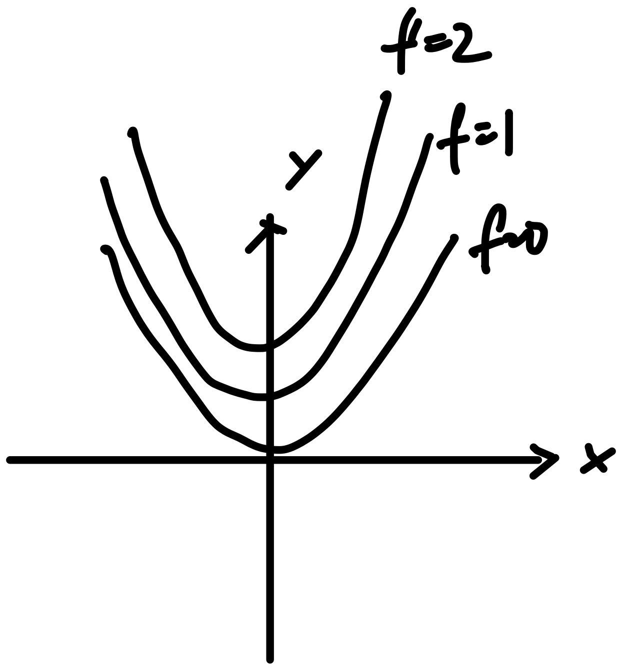
b). $\sqrt{y-x^2} = 0$

$y - x^2 = 0$

$y = x^2$

$y - x^2 = 1$

$y = x^2 + 1$



2. (★) Let

$$f(x,y) = \frac{1}{\sqrt{x^2+y^2-1}}$$

(a) What is the (largest possible) domain of f?

(b) Sketch the level sets $f = 1$, $f = 2$ and $f = 3$ in the same diagram.

(c) Repeat (a) and (b) for the function $g(x,y) = \frac{1}{\sqrt{1-x^2-y^2}}$.

2a). $x^2+y^2-1 > 0$

$x^2+y^2 > 1$

b). $1 = x^2+y^2-1$

$2 = x^2+y^2$

$1 = 2\sqrt{x^2+y^2-1}$

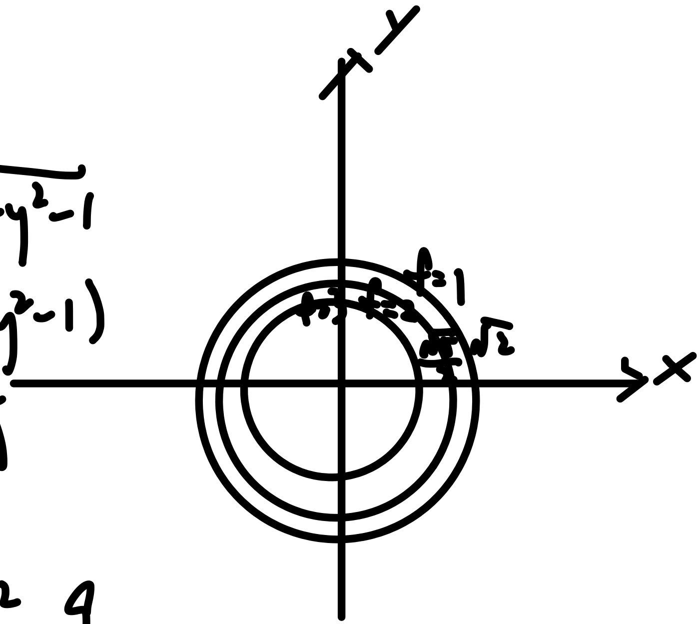
$1 = 4(x^2+y^2-1)$

$5 = 4x^2+4y^2$

$\frac{5}{4} = x^2+y^2$

$1 = 9x^2+9y^2-9$

$10 = 9x^2+9y^2$



(c) Repeat (a) and (b) for the function $g(x, y) = \frac{1}{\sqrt{1-x^2-y^2}}$.

c) a). $1-x^2-y^2 > 0$

$1 > x^2+y^2$

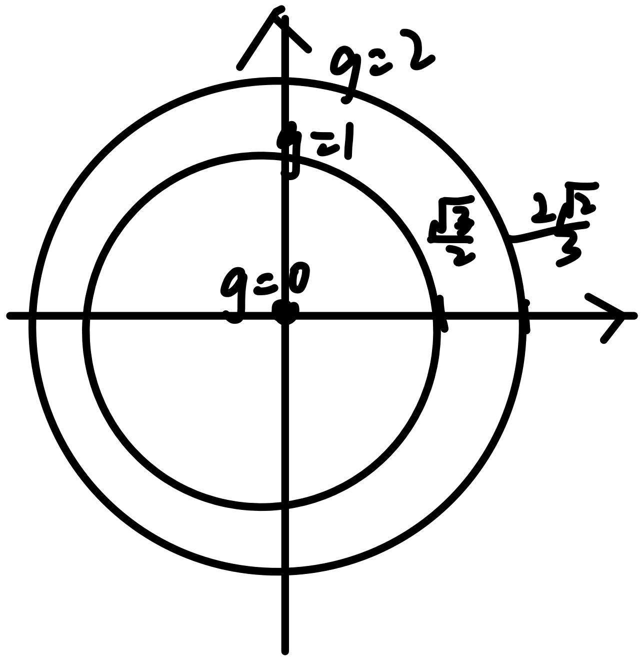
$x^2+y^2 < 1$

b). $g(x,y)=1$

$1 = \sqrt{1-x^2-y^2}$

$1 = 1-x^2-y^2$

$x^2+y^2=0$



$g(x,y)=2.$

$1 = 2\sqrt{1-x^2-y^2}$

$1 = 4(1-x^2-y^2)$

$\frac{1}{4} = 1-x^2-y^2$

$x^2+y^2 = \frac{3}{4}$

$g(x,y)=3$

$1 = 3\sqrt{1-x^2-y^2}$

$1 = 9(1-x^2-y^2)$

$\frac{1}{9} = 1-x^2-y^2$

$x^2+y^2 = \frac{8}{9}$

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3. (★) Compute all the first and second partial derivatives of the following functions. For the second partials f_{xy} and f_{yx}, compute both and verify that they are indeed the same.

(a) $f(x,y) = y^{2015} + 2x^2 + 2xy$

(b) $f(x,y) = e^{x^2y}$

(c) $f(x,y) = \frac{x}{x^2+y^2}$

(d) $f(x,y) = x\ln(x^2+y^2)$

a). $f_x = 4x + 2y.$

$f_y = 2015\,y^{2014} + 2x$

$f_{xx} = 4$

$f_{xy} = 2$

$f_{yx} = 2$

$f_{yy} = (2015 \cdot 2014)\,y^{2013}.$

b). $f_x = 2x^3y^2 \; e^{x^2y}$

$f_y = (x^2y)e^{x^2y}\,(x^2)$

$f_{xx} = 2y^2$

7. (★★) Suppose that $f(x, y)$ is a function such that $\frac{\partial^2 f}{\partial x \partial y} \equiv 0$. Show that f can be decomposed into the form:

$$f(x, y) = F(x) + G(y)$$

where $F(x)$ and $G(y)$ are some single-variable functions.

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0 \cdot$$

f

x y.

x.

$$\frac{\partial}{\partial x} f = c_1$$

$$f = c_1 x + c_2$$

$$f = c_3 y + c_4.$$

8. (★★) Let $u(x,y,z,t)$ be the temperature at the point (x,y,z) at the time t. Combining with several important laws in thermodynamics, including the Fourier's Law and conservation of energy, it can be derived (detail omitted) that the temperature function $u(x,y,z,t)$ satisfies the following equation:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

where k is a positive constant depending only on the medium. This equation is known as the **heat equation**.

The study of the heat equation is an important topic in physics, engineering and mathematics (both pure and applied). Through solving the heat equation with an initial condition $u(x,y,z,0) = g(x,y,z)$, it predicts how heat diffuses for a given an initial heat profile $g(x,y,z)$ at time $t = 0$.

Your task in this problem is to verify that the following given function is a solution to the heat equation:

$$\varphi(x,y,z,t) = \frac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(-\frac{x^2+y^2+z^2}{4kt}\right).$$

This particular solution φ represents the heat diffusion with highly concentrated heat source at the origin $(0,0,0)$ at time $t = 0$. As time goes by, the temperature profile becomes more and more uniformly distributed. (In physics, this solution is also closely related to the *Dirac delta function*.)

By following the outline below, show that φ satisfies the heat equation:

(a) Show that:

$$\ln\varphi(x,y,z,t) = -\ln(4\pi k)^{\frac{3}{2}} - \frac{3}{2}\ln t - \frac{x^2+y^2+z^2}{4kt}.$$

(b) Using (a), show that:

$$\frac{\partial\varphi}{\partial t} = \left(\frac{x^2+y^2+z^2}{4kt^2} - \frac{3}{2t}\right)\varphi.$$

(c) Using (a) again, show that:

$$\frac{\partial\varphi}{\partial x} = -\frac{x\varphi}{2kt} \quad\text{and}\quad \frac{\partial^2\varphi}{\partial x^2} = \frac{1}{2kt}\left(\frac{x^2}{2kt} - 1\right)\varphi.$$

(d) Hence, verify that φ satisfies the heat equation: $\varphi_t = k(\varphi_{xx} + \varphi_{yy} + \varphi_{zz})$.

(e) (Optional) Show that

$$\lim_{t\to 0^+} \varphi(x,y,z,t) = \begin{cases} \infty & \text{if } (x,y,z) = (0,0,0) \\ 0 & \text{if } (x,y,z) \neq (0,0,0) \end{cases}$$

a).

$$\ln\varphi(x,y,z,t) = -\frac{3}{2}\ln(4\pi kt) - \frac{x^2+y^2+z^2}{4kt}$$

$$-\frac{3}{2}\ln(4\pi k) - \frac{3}{2}\ln t$$

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MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #3 • Chain Rule, Directional Derivatives, Gradients

1. (★) Suppose $w = f(x, y, z)$ where $x = g(s)$, $y = h(s, t)$ and $z = k(t)$. Assume all functions involved are C^1. Draw the tree diagram to showcase the relations between w, x, y, z, s and t. Hence, write down the chain rule for calculating the partial derivatives: $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$. Use the symbols ∂ and d appropriately.

2. (★) Recall that the rectangular-polar coordinates conversion rules are given as follows:
$$x = r\cos\theta$$
$$y = r\sin\theta$$
A function $f(x, y)$ is said to be **rotationally/radially symmetric** if $\frac{\partial f}{\partial \theta} = 0$, i.e. when regarded as a function of (r, θ), it depends only the radial variable r but not the angular variable θ. For instance, $f(x, y) = x^2 + y^2$ is rotationally symmetric since $f(r, \theta) = r^2$. Using the chain rule, show that f is rotationally symmetric if and only if:
$$y\frac{\partial f}{\partial x} = x\frac{\partial f}{\partial y}.$$

3. (★) Suppose $f(u, v)$ is a C^2 function, and $u = s^2 - t$ and $v = s + t^2$. Express the second partial derivative $\dfrac{\partial^2 f}{\partial s \partial t}$ in terms of f_{uu}, f_{uv}, f_{vv}, s and t.

4. (★) Let $f(x, y, z)$ be a C^1 function of three variables, and z be a C^1 function of (x, y) such that
$$f(x, y, z(x, y)) = 0.$$
Using the chain rule, show that:
$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}.$$

5. (★★) Let $f(x, y)$ be a C^1 function. Consider two parametric curves $\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j}$ and $\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j}$ which satisfy:
$$\mathbf{r}_1(0) = \mathbf{r}_2(0) \quad \text{and} \quad \mathbf{r}_1'(0) = \mathbf{r}_2'(0).$$
   (a) Show that
$$\left.\frac{d}{dt}\right|_{t=0} f(x_1(t), y_1(t)) = \left.\frac{d}{dt}\right|_{t=0} f(x_2(t), y_2(t)).$$
   (b) Give a geometric interpretation of the above result.

6. (★★) The wave equation is an important partial differential equation which governs the propagation of waves. Let $u(x, y, z, t)$ be the displacement of the wave at position (x, y, z) at time t. It can be shown by several physical laws (such as the Hooke's Law) that u satisfies:
$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$
where c is a constant (which is the wave speed).

In one (spatial) dimension, the wave equation can be stated as:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

It turns out that the chain rule of several variables has a nice application on solving the one dimensional wave equation. The following exercise guides you to show that if $u(x,t)$ is a solution to the one dimensional wave equation, then it must take the form $u(x,t) = F(x-ct) + G(x+ct)$ where F and G are arbitrary differentiable functions of single variable.

Let $u(x,t)$ solve the one dimensional wave equation (2).

(a) Define $\xi = x - ct$ and $\eta = x + ct$. Regard u as a function of ξ and η, and ξ and η are functions of x and t. Using the chain rule of multivariable functions, show that:

$$u_t = c(u_\eta - u_\xi) \quad \text{and} \quad u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).$$

(b) Using the chain rule agin, show that

$$u_x = u_\xi + u_\eta \quad \text{and} \quad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

(c) Combining the results of (a), (b) and the wave equation, show that $u_{\xi\eta} = 0$.

(d) Finally, deduce that u, as a function of ξ and η, must be in the form of:

$$u(\xi,\eta) = F(\xi) + G(\eta)$$

where F and G are arbitrary functions. Hence, in terms of the original variables x and t, u must take the form $u(x,t) = F(x-ct) + G(x+ct)$.

7. (★★★) In many physics, geometry and engineering applications, it is often more convenient to use polar or sphereical coordinates since many physical quantities are rotationally symmetric.

The conversion rule of rectangular and polar coordinates is given by:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Let u be a function of x and y. Since (x,y) can be converted into (r,θ), we can also regard u as a function of (r,θ). The chain rule can be used to derive some conversion formulae between u_x, u_y and u_r, u_θ.

An important operator in physics, geometry and engineering is called the **Laplacian**. In two dimensions, it is defined as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}.$$

In this exercise, we will show that $\nabla^2 u$ can be expressed in polar form as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

The polar form of the Laplacian is often used when dealing with rotationally symmetric functions, i.e. a function u which does not depend on θ but only on r. For such functions, their Laplacian is simply:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r.$$

(a) Use the fact that $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$, show that:

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}.$$

(b) Regard u as a function of (r,θ), and (r,θ) are functions of (x,y). Sketch a tree diagram to showcase these relations. Using the chain rule, show that:

$$u_x = \frac{xu_r}{r} - \frac{yu_\theta}{r^2},$$
$$u_y = \frac{yu_r}{r} + \frac{xu_\theta}{r^2}.$$

(c) Using quotient and product rules, show that:

$$u_{xx} = \frac{u_r}{r} + \frac{xu_{rx}}{r} - \frac{x^2u_r}{r^3} - \frac{yu_{\theta x}}{r^2} + \frac{2xyu_\theta}{r^4}$$
$$u_{yy} = \frac{u_r}{r} + \frac{yu_{ry}}{r} - \frac{y^2u_r}{r^3} + \frac{xu_{\theta y}}{r^2} - \frac{2xyu_\theta}{r^4}$$

(d) Since u_r and u_θ are functions of (r,θ), and (r,θ) are functions of (x,y), they share the same tree diagram as u in part (b), and hence we have

$$u_{rx} = \frac{\partial u_r}{\partial x} = \frac{\partial u_r}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u_r}{\partial \theta}\frac{\partial \theta}{\partial x}$$

and similar for other second derivatives u_{ry}, $u_{\theta x}$ and $u_{\theta y}$. Show that:

$$xu_{rx} + yu_{ry} = ru_{rr}$$
$$xu_{\theta y} - yu_{\theta x} = u_{\theta\theta}$$

(e) Combining the results proved in previous parts, show that:

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

8. (★) Compute the directional derivative of the following functions at the given point P in the direction of the given vector $\mathbf{v}$. Moreover, find the unit direction $\mathbf{u}$ along which the function increases most rapidly.

(a) $f(x,y) = x^2 - y^2$, $P(-1,-3)$, $\mathbf{v} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$.

(b) $g(x,y) = e^{-x-y}$, $P(\ln 2, \ln 3)$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

(c) $h(x,y) = e^{xy}$, $P(1,0)$, $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j}$.

(d) $F(x,y,z) = xy + yz + zx + 4$, $P(2,-2,1)$, $\mathbf{v} = -\mathbf{j} - \mathbf{k}$.

(e) $G(x,y,z) = e^{xyz} - 1$, $P(0,1,-1)$, $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

9. (★) For each surface and the given point P, find the value a such that P lies on the surface, and then find an equation of the tangent plane to the surface at the point P:

(a) $x^2 + y + z = 3$, $P(2,0,a)$

(b) $xy\sin z = 1$, $P(a,2,\pi/6)$

(c) $yze^{xz} = 8$, $P(0,a,4)$

(d) $z = e^{xy}$, $P(1,0,a)$

(e) $z = \ln(1+xy)$, $P(1,2,a)$.

10. (★) Let

$$V(x,y,z) = -\frac{GMm}{\sqrt{x^2+y^2+z^2}}$$

where G, M and m are constants. Define $\mathbf{F}(x,y,z) = -\nabla V(x,y,z)$.

(a) Verify that:

$$\mathbf{F}(x,y,z) = -GMm\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}.$$

(b) Show that $|\mathbf{F}(x,y,z)|$ is inversely proportional to the squared distance from (x,y,z) to the origin in $\mathbb{R}^3$.

11. (★★) Consider the function

$$f(x,y) = \cos(x+y)$$

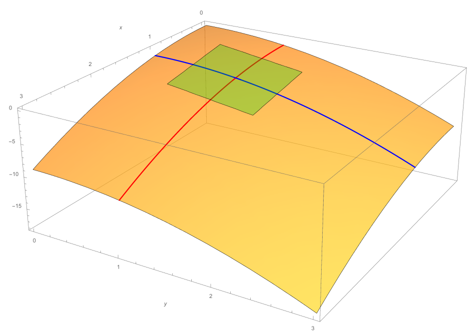
as well as the plane Π given by the equation

$$x - y = 0\ .$$

The intersection of the graph of f with Π is a curve C. Find the slope of the tangent line to C at the point (π,π) using directional derivatives. [Hint: First sketch a diagram of the graph, the plane and the curve.]

12. (★★) One approach for finding the normal vector of the tangent plane at a given point (x_0,y_0) to a graph $z = f(x,y)$ is by writing the graph equation as a level surface $z - f(x,y) = 0$ of a three-variable function $g(x,y,z) := z - f(x,y)$. Then, the gradient $\nabla g = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right\rangle$ at point $(x_0,y_0,f(x_0,y_0))$ is perpendicular to the level surface $\{g = 0\}$, and so we can take it to be a normal vector of the tangent plane as long as $\nabla g \neq \mathbf{0}$ at $(x_0,y_0,f(x_0,y_0))$.

In fact, it is also possible to show the normal vector is $\left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right\rangle$ using a *purely* two-variable argument instead of *going up one higher dimension*.



(a) Consider a given function $f(x,y)$, and a given point (x_0,y_0). Find a parametrization:

$$\mathbf{r}_1(t) = ?\mathbf{i} + ?\mathbf{j} + ?\mathbf{k}$$

of the curve on the graph $z = f(x,y)$ travelling in the x-direction while keeping y fixed at y_0 (i.e. the red curve in the diagram). Hence, find the tangent vector of the curve $\mathbf{r}_1(t)$ at the point $(x_0,y_0,f(x_0,y_0))$. Label this tangent vector by $\mathbf{T}_1$.

(b) Find a parametrization $\mathbf{r}_2(t)$ of the curve on the graph $z = f(x,y)$ travelling in the y-direction while keeping x fixed at x_0 (i.e. the blue curve in the diagram). Hence, find the tangent vector of $\mathbf{r}_2(t)$ at the point $(x_0,y_0,f(x_0,y_0))$. Label this tangent vector by $\mathbf{T}_2$.

(c) Since both $\mathbf{T}_1$ and $\mathbf{T}_2$ are tangent vectors to the graph, they are parallel to the tangent plane. Therefore, the normal vector to the tangent plane must be perpendicular to both $\mathbf{T}_1$ and $\mathbf{T}_2$. Using this fact, show that the normal vector to the tangent plane is given by

$$\left\langle -\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1 \right\rangle.$$

Optional

13. The spherical coordinates (ρ, θ, ϕ) is another important coordinate system in $\mathbb{R}^3$. We will learn that in later chapters. The conversion rules between spherical and rectangular coordinates are given by:

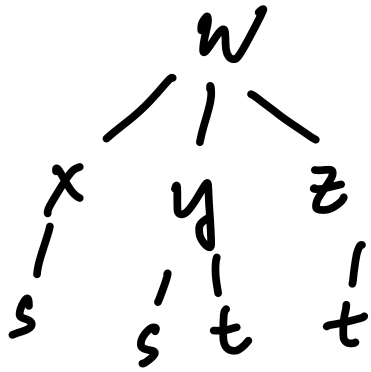
$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}$$

Given a C^2 function $f(x, y, z)$, it can be regarded as a function of (ρ, θ, ϕ) as well under the above conversion rule. Show that the Laplacian $\nabla^2 f := f_{xx} + f_{yy} + f_{zz}$ can be expressed in spherical coordinates as:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}.$$

[Note: It is a very time consuming exercise. It took me 4 hours to do it when I was an undergraduate.]

1. (★) Suppose $w = f(x, y, z)$ where $x = g(s)$, $y = h(s, t)$ and $z = k(t)$. Assume all functions involved are C^1. Draw the tree diagram to showcase the relations between w, x, y, z, s and t. Hence, write down the chain rule for calculating the partial derivatives: $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$. Use the symbols ∂ and d appropriately.



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

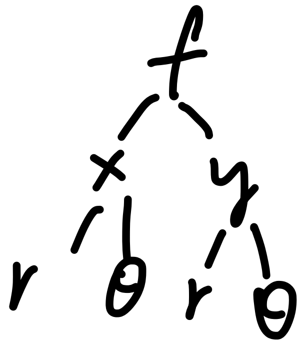
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{dz}{\partial t}$$

2. (★) Recall that the rectangular-polar coordinates conversion rules are given as follows:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

A function $f(x, y)$ is said to be **rotationally/radially symmetric** if $\frac{\partial f}{\partial \theta} = 0$, i.e. when regarded as a function of (r, θ), it depends only the radial variable r but not the angular variable θ. For instance, $f(x, y) = x^2 + y^2$ is rotationally symmetric since $f(r, \theta) = r^2$. Using the chain rule, show that f is rotationally symmetric if and only if:

$$y\frac{\partial f}{\partial x} = x\frac{\partial f}{\partial y}.$$



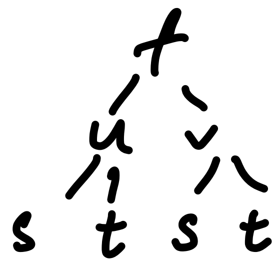
$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$$0 = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$$

$$\frac{\partial f}{\partial x} r\sin\theta = \frac{\partial f}{\partial y}(r\cos\theta)$$

$$\frac{\partial f}{\partial x} y = \frac{\partial f}{\partial y} x$$

3. (★) Suppose $f(u,v)$ is a C^2 function, and $u = s^2 - t$ and $v = s + t^2$. Express the second partial derivative $\dfrac{\partial^2 f}{\partial s \partial t}$ in terms of f_{uu}, f_{uv}, f_{vv}, s and t.



$$\frac{\partial}{\partial s}\frac{\partial f}{\partial t}$$

$$= \frac{\partial}{\partial s}\left(f_u u_t + f_v v_t\right)$$

$$= \frac{\partial}{\partial s}\left(f_u(-1) + f_v(2t)\right)$$

$$= \frac{\partial}{\partial s}\left(2 f_v t - f_u\right)$$

$$= 2t\left(f_{vu} u_s + f_{vv} v_s\right) - f_{uu} u_s - f_{uv} v_s$$

$$= 2t\left(f_{uv}(2s) + f_{vv}(1)\right) - f_{uu}(2s) - f_{uv}(1)$$

$$= 4st f_{uv} + 2t f_{vv} - 2s f_{uu} - f_{uv}$$

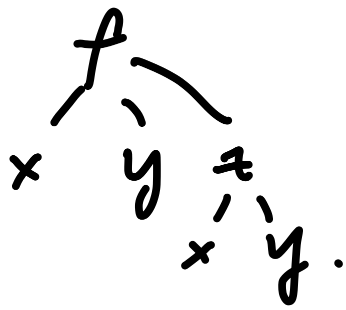
$$= -2s f_{uu} + (4st-1) f_{uv} + 2t f_{vv}$$

4. (★) Let $f(x,y,z)$ be a C^1 function of three variables, and z be a C^1 function of (x,y) such that

$$f(x,y,z(x,y)) = 0.$$

Using the chain rule, show that:

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}.$$



$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$0 = f_x + f_z \cdot \frac{\partial z}{\partial x}$$

$$\frac{-f_x}{f_z} = \frac{\partial z}{\partial x}.$$

$$\frac{df}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y}$$

$$0 = f_y + f_z \cdot \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{-f_y}{f_z}$$

5. (★★) Let $f(x,y)$ be a C^1 function. Consider two parametric curves $\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j}$ and $\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j}$ which satisfy:

$$\mathbf{r}_1(0) = \mathbf{r}_2(0) \quad \text{and} \quad \mathbf{r}_1'(0) = \mathbf{r}_2'(0).$$

(a) Show that

$$\left.\frac{d}{dt}\right|_{t=0} f(x_1(t), y_1(t)) = \left.\frac{d}{dt}\right|_{t=0} f(x_2(t), y_2(t)).$$

(b) Give a geometric interpretation of the above result.

$$\vec{r}_1'(0) = \vec{r}_2'(0)$$

$$x_1'(0) + y_1'(0) = x_2'(0) + y_2'(0)$$

$$x_1(0) + y_1(0) = x_2(0) + y_2(0).$$

$$\begin{array}{cccc} & f & & \\ x_2 & x_1 & y_1 & y_2 \\ t & t & t & t \end{array}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial f}{\partial y_2}\frac{dy_2}{dt}$$

$=$

×

6. (★★) The wave equation is an important partial differential equation which governs the propagation of waves. Let $u(x, y, z, t)$ be the displacement of the wave at position (x, y, z) at time t. It can be shown by several physical laws (such as the Hooke's Law) that u satisfies:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{1}$$

where c is a constant (which is the wave speed).

In one (spatial) dimension, the wave equation can be stated as:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

It turns out that the chain rule of several variables has a nice application on solving the one dimensional wave equation. The following exercise guides you to show that if $u(x, t)$ is a solution to the one dimensional wave equation, then it must take the form $u(x, t) = F(x - ct) + G(x + ct)$ where F and G are arbitrary differentiable functions of single variable.

Let $u(x, t)$ solve the one dimensional wave equation (2).

(a) Define $\xi = x - ct$ and $\eta = x + ct$. Regard u as a function of ξ and η, and ξ and η are functions of x and t. Using the chain rule of multivariable functions, show that:

$$u_t = c(u_\eta - u_\xi) \quad \text{and} \quad u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).$$

(b) Using the chain rule agin, show that

$$u_x = u_\xi + u_\eta \quad \text{and} \quad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

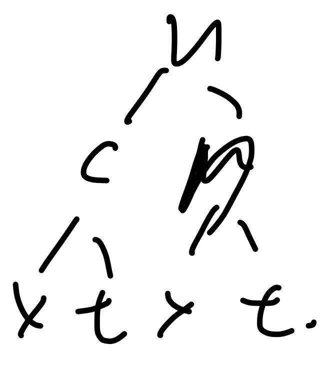
(c) Combining the results of (a), (b) and the wave equation, show that $u_{\xi\eta} = 0$.

(d) Finally, deduce that u, as a function of ξ and η, must be in the form of:

$$u(\xi, \eta) = F(\xi) + G(\eta)$$

where F and G are arbitrary functions. Hence, in terms of the original variables x and t, u must take the form $u(x, t) = F(x - ct) + G(x + ct)$.

$$u_t = u_\xi \xi_t + u_\eta \eta_t.$$
$$= u_\xi(-c) + u_\eta(c)$$
$$= c(u_\eta - u_\xi)$$



(a) Define $\xi = x - ct$ and $\eta = x + ct$. Regard u as a function of ξ and η, and ξ and η are functions of x and t. Using the chain rule of multivariable functions, show that:

$$u_t = c(u_\eta - u_\xi) \quad \text{and} \quad u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).$$

(b) Using the chain rule agin, show that

$$u_x = u_\xi + u_\eta \quad \text{and} \quad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

$\xi = x - ct$

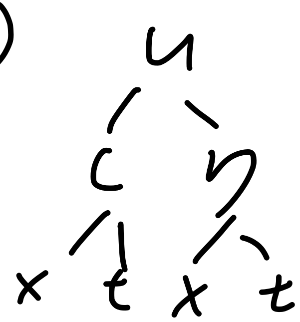
$\eta = x + ct$

$$u_{tt} = \frac{\partial}{\partial t}\left(c(u_\eta - u_\xi)\right)$$

$$= c(u_{\eta\xi}\xi_t + u_{\eta\eta}\eta_t - u_{\xi\eta}\eta_t - u_{\xi\xi}\xi_t)$$

$$= c((-c)u_{\eta\xi} + cu_{\eta\eta} - cu_{\eta\xi} + cu_{\xi\xi})$$

$$= c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$



b/. $u_x = u_\xi \xi_x + u_\eta \eta_x$

$u_x = u_\xi + u_\eta$

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x$

$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

In one (spatial) dimension, the wave equation can be stated as:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

It turns out that the chain rule of several variables has a nice application on solving the one dimensional wave equation. The following exercise guides you to show that if $u(x,t)$ is a solution to the one dimensional wave equation, then it must take the form $u(x,t) = F(x-ct) + G(x+ct)$ where F and G are arbitrary differentiable functions of single variable.

Let $u(x,t)$ solve the one dimensional wave equation (2).

(a) Define $\xi = x - ct$ and $\eta = x + ct$. Regard u as a function of ξ and η, and ξ and η are functions of x and t. Using the chain rule of multivariable functions, show that:

$$u_t = c(u_\eta - u_\xi) \quad \text{and} \quad u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).$$

(b) Using the chain rule agin, show that

$$u_x = u_\xi + u_\eta \quad \text{and} \quad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

$$u_{tt} = c^2 u_{xx}$$

$$c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) = c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$$0 = 4u_{\xi\eta}$$

$$u_{\xi\eta} = 0.$$

d/.

7. (★★★) In many physics, geometry and engineering applications, it is often more convenient to use polar or sphereical coordinates since many physical quantities are rotationally symmetric.

The conversion rule of rectangular and polar coordinates is given by:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Let u be a function of x and y. Since (x, y) can be converted into (r, θ), we can also regard u as a function of (r, θ). The chain rule can be used to derive some conversion formulae between u_x, u_y and u_r, u_θ.

An important operator in physics, geometry and engineering is called the **Laplacian**. In two dimensions, it is defined as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}.$$

In this exercise, we will show that $\nabla^2 u$ can be expressed in polar form as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

The polar form of the Laplacian is often used when dealing with rotationally symmetric functions, i.e. a function u which does not depend on θ but only on r. For such functions, their Laplacian is simply:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r.$$

(a) Use the fact that $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$, show that:

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}.$$

(b) Regard u as a function of (r, θ), and (r, θ) are functions of (x, y). Sketch a tree diagram to showcase these relations. Using the chain rule, show that:

$$u_x = \frac{xu_r}{r} - \frac{yu_\theta}{r^2},$$
$$u_y = \frac{yu_r}{r} + \frac{xu_\theta}{r^2}.$$

(c) Using quotient and product rules, show that:

$$u_{xx} = \frac{u_r}{r} + \frac{xu_{rx}}{r} - \frac{x^2u_r}{r^3} - \frac{yu_{\theta x}}{r^2} + \frac{2xyu_\theta}{r^4}$$
$$u_{yy} = \frac{u_r}{r} + \frac{yu_{ry}}{r} - \frac{y^2u_r}{r^3} + \frac{xu_{\theta y}}{r^2} - \frac{2xyu_\theta}{r^4}$$

(d) Since u_r and u_θ are functions of (r, θ), and (r, θ) are functions of (x, y), they share the same tree diagram as u in part (b), and hence we have

$$u_{rx} = \frac{\partial u_r}{\partial x} = \frac{\partial u_r}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u_r}{\partial \theta}\frac{\partial \theta}{\partial x}$$

and similar for other second derivatives u_{ry}, $u_{\theta x}$ and $u_{\theta y}$. Show that:

$$xu_{rx} + yu_{ry} = ru_{rr}$$
$$xu_{\theta y} - yu_{\theta x} = u_{\theta\theta}$$

(e) Combining the results proved in previous parts, show that:

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

8. (★) Compute the directional derivative of the following functions at the given point P in the direction of the given vector $\mathbf{v}$. Moreover, find the unit direction $\mathbf{u}$ along which the function increases most rapidly.

(a) $f(x,y) = x^2 - y^2$, $P(-1,-3)$, $\mathbf{v} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$.

(b) $g(x,y) = e^{-x-y}$, $P(\ln 2, \ln 3)$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

(c) $h(x,y) = e^{xy}$, $P(1,0)$, $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j}$.

(d) $F(x,y,z) = xy + yz + zx + 4$, $P(2,-2,1)$, $\mathbf{v} = -\mathbf{j} - \mathbf{k}$.

(e) $G(x,y,z) = e^{xyz} - 1$, $P(0,1,-1)$, $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

(a). $|v| = 1.$

$$\nabla f = \langle 2x, -2y \rangle$$

$$\nabla f(-1,-3) = \langle -2, 6 \rangle$$

$$\nabla f \cdot v = -2\left(\tfrac{3}{5}\right) - 6\left(\tfrac{4}{5}\right) = -\tfrac{6}{5} - \tfrac{24}{5} = -6.$$

b). $$\nabla f = \langle \overset{-e^{-x-y}}{\cancel{(x+y)e^{-x-y}}}, \overset{-e^{-x-y}}{\cancel{(x+y)e^{-x-y}}} \rangle$$

$$\nabla f(\ln 2, \ln 3) = \langle \ln 6\, e^{-\ln 2 - \ln 3}, \dots \rangle$$

$$= (\ln 6)\, e^{\ln \frac{1}{6}}$$

$$= \left(\tfrac{1}{6} \ln 6\right)$$

$$\hat{v} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$$

$$\nabla f \cdot \hat{v} = \frac{2\sqrt{2}}{3} \ln 6$$

$h(x,y) = e^{xy}$ $\quad P(1,0)$. $\quad v = \langle 5, 12 \rangle$

$\nabla h = \langle ye^{xy}, xe^{xy} \rangle$

$\nabla h(1,0) = \cancel{\langle 1, 0 \rangle}$ $\langle 0, 1 \rangle$

$\hat{v} = \langle \frac{5}{13}, \frac{12}{13} \rangle$

$\nabla h \cdot \hat{v} = \cancel{\frac{5}{13}} \cdot \frac{12}{13}$

(d) $F(x,y,z) = xy + yz + zx + 4$, $P(2,-2,1)$, $\mathbf{v} = -\mathbf{j} - \mathbf{k}$.

(e) $G(x,y,z) = e^{xyz} - 1$, $P(0,1,-1)$, $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

$$\nabla F = \langle y+z, x+z, x+y \rangle$$

$$\nabla F(2,-2,1) = \langle -1, 3, 0 \rangle$$

$$\hat{v} = \langle 0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$$

$$\nabla F \cdot \hat{v} = \frac{-3}{\sqrt{2}}.$$

e).

$$\nabla G = \langle yze^{xyz}, xze^{xyz}, xye^{xyz} \rangle$$

$$\nabla G(0,1,-1) = \langle -1, 0, 0 \rangle$$

$$\hat{v} = \langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \rangle$$

$$\nabla G \cdot \hat{v} = \frac{2}{3}.$$

9. (★) For each surface and the given point P, find the value a such that P lies on the surface, and then find an equation of the tangent plane to the surface at the point P:

(a) $x^2 + y + z = 3$, $P(2, 0, a)$

(b) $xy \sin z = 1$, $P(a, 2, \pi/6)$

(c) $yze^{xz} = 8$, $P(0, a, 4)$

(d) $z = e^{xy}$, $P(1, 0, a)$

(e) $z = \ln(1 + xy)$, $P(1, 2, a)$.

a). $(2)^2 + a = 3$

$a = -1$

$P(2, 0, -1)$

$\nabla F = \langle 2x, 1, 1 \rangle$

$\nabla F(2, 0, -1) = \langle 4, 1, 1 \rangle$

$4x + y + z = 7.$

$\nabla F = \langle y\sin z, x\sin z, xy\cos z \rangle$

$\nabla F(\cancel{\tfrac{\sqrt{2}}{2}}\ 1, 2, \tfrac{\pi}{6}) = \times$

$\langle \sqrt{2}, \tfrac{1}{2}, \sqrt{3} \rangle$

$\sqrt{2}x + \tfrac{1}{2}y + \sqrt{3}z =$

$2 + \frac{\sqrt{3}\pi}{6}.$

$0\ \frac{\pi}{6}\ \frac{\pi}{4}\ \frac{\pi}{3}$

b). $2a\sin\frac{\pi}{6} = 1$

$\sin\frac{\pi}{6} = \frac{1}{2a}$

$\sqrt{\frac{1}{2}} = \frac{1}{2a}$

$\frac{1}{\sqrt{2}} = \frac{1}{2a}$

$2a = \sqrt{2}$

$a = \frac{\sqrt{2}}{2}.$

$x\ 2\sin\frac{\pi}{6} = 1$

$x\sin\frac{\pi}{6} = \frac{1}{2}$

$x\,\frac{1}{2} = \frac{1}{2}$

$x = 1.$

9. (★) For each surface and the given point P, find the value a such that P lies on the surface, and then find an equation of the tangent plane to the surface at the point P:

(a) $x^2 + y + z = 3, \; P(2, 0, a)$

(b) $xy \sin z = 1, \; P(a, 2, \pi/6)$

(c) $yze^{xz} = 8, \; P(0, a, 4)$

(d) $z = e^{xy}, \; P(1, 0, a)$

(e) $z = \ln(1 + xy), \; P(1, 2, a)$.

c). $\nabla F = \langle yz^2 e^{xz}, \; ze^{xz}, \; yxze^{xz} + ye^{xz} \rangle$

$a(4) = 8$

$a = 2$

$\nabla F(0,2,4) = \langle 32, 4, 2 \rangle$

$32x + 4y + 2z = 16$

$16x + 2y + z = 8$

d) $\nabla F = \frac{\partial}{\partial x} \; e^{xy} - z$

$= \langle ye^{xy}, \; xe^{xy}, -1 \rangle$

$P = (1, 0, 1)$

$\nabla F(1,0,1) = \langle 0, 1, -1 \rangle$

$y - z = -1$

$z = y + 1$

Made with Goodnotes

(d) $z = \dots$, $P(1, 0, a)$

(e) $z = \ln(1 + xy)$, $P(1, 2, a)$.

$\ln(1+2) = \ln 3.$

$\nabla F = \left\langle \frac{y}{(1+xy)}, \frac{x}{(1+xy)}, -1 \right\rangle$

$\nabla F(1, 2, \ln 3) = \left\langle \frac{2}{3}, \frac{1}{3}, -1 \right\rangle$

$\frac{2}{3}x + \frac{1}{3}y - z = \frac{4}{3} - \ln 3.$

$2x + y - 3z = 4 - 3\ln 3.$

10. (★) Let

$$V(x,y,z) = -\frac{GMm}{\sqrt{x^2+y^2+z^2}}$$

where G, M and m are constants. Define $\mathbf{F}(x,y,z) = -\nabla V(x,y,z)$.

(a) Verify that:

$$\mathbf{F}(x,y,z) = -GMm\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}.$$

(b) Show that $|\mathbf{F}(x,y,z)|$ is inversely proportional to the squared distance from (x,y,z) to the origin in $\mathbb{R}^3$.

a). $\frac{\partial V}{\partial x} = \frac{GMm\left(\frac{1}{2}\right)(x^2+y^2+z^2)^{-\frac{1}{2}}(2x)}{(x^2+y^2+z^2)}$

$\frac{\partial V}{\partial x} = \frac{GMm}{(x^2+y^2+z^2)^{\frac{3}{2}}}x$

b). $|F(x,y,z)| =$

$$\sqrt{G^2M^2m^2\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^3}}$$

$$= \sqrt{G^2M^2m^2\left(\frac{1}{x^2+y^2+z^2}\right)^2}$$

$$= GMm\frac{1}{x^2+y^2+z^2}$$

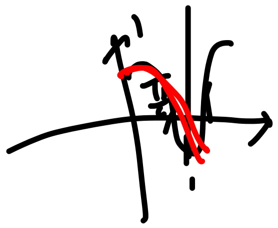
11. (★★) Consider the function

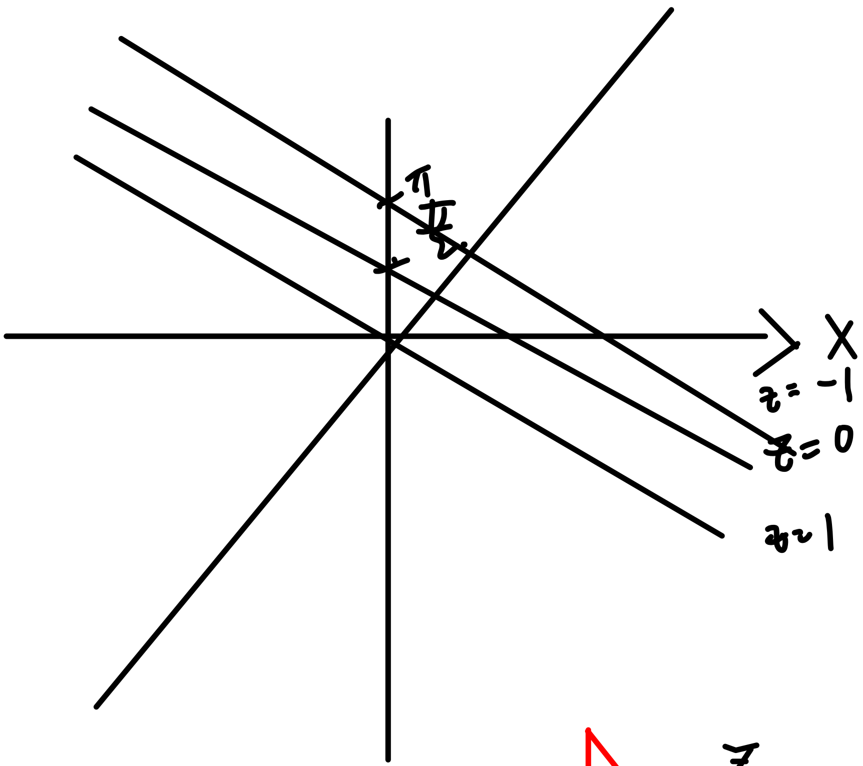
$$f(x,y) = \cos(x+y)$$

as well as the plane Π given by the equation

$$x - y = 0\,.$$

The intersection of the graph of f with Π is a curve C. Find the slope of the tangent line to C at the point (π, π) using directional derivatives. [Hint: First sketch a diagram of the graph, the plane and the curve.]





$\cos(x+y) = 0.$

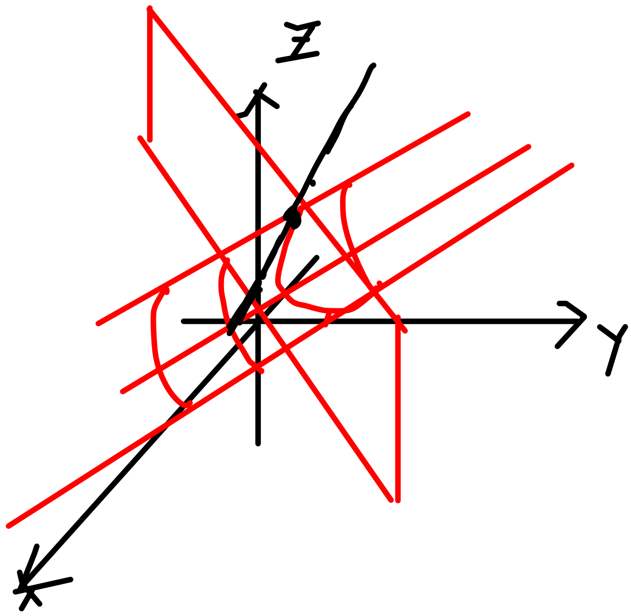
$x+y = \frac{\pi}{2}.$

$y = -x + \frac{\pi}{2}.$

$\cos(x+y) = 1$

$x+y = 0$

$y = -x$



$\cos(x+y) = -1$

$x+y = \pi$

$z = \cos(x+y)$, $x - y = 0$, $(\pi, \pi, 1)$

$$\hat{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$$

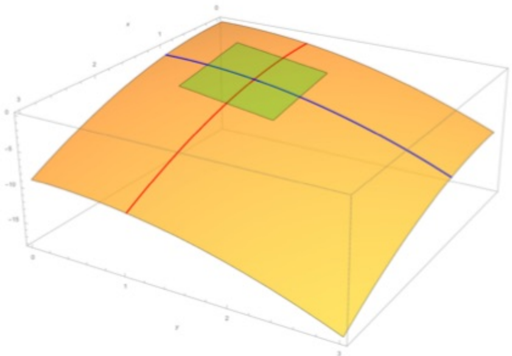
$$\nabla f = \langle -\sin(x+y), -\sin(x+y), -1 \rangle$$

$$\nabla f = \langle 0, 0, -1 \rangle$$

$$\nabla f \cdot \hat{v} = 0.$$

12. (★★) One approach for finding the normal vector of the tangent plane at a given point (x_0, y_0) to a graph $z = f(x, y)$ is by writing the graph equation as a level surface $z - f(x, y) = 0$ of a three-variable function $g(x, y, z) := z - f(x, y)$. Then, the gradient $\nabla g = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$ at point $(x_0, y_0, f(x_0, y_0))$ is perpendicular to the level surface $\{g = 0\}$, and so we can take it to be a normal vector of the tangent plane as long as $\nabla g \neq \mathbf{0}$ at $(x_0, y_0, f(x_0, y_0))$.

In fact, it is also possible to show the normal vector is $\left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$ using a *purely* two-variable argument instead of *going up one higher dimension*.



(a) Consider a given function $f(x, y)$, and a given point (x_0, y_0). Find a parametrization:

$$\mathbf{r}_1(t) = ?\mathbf{i} + ?\mathbf{j} + ?\mathbf{k}$$

of the curve on the graph $z = f(x, y)$ travelling in the x-direction while keeping y fixed at y_0 (i.e. the red curve in the diagram). Hence, find the tangent vector of the curve $\mathbf{r}_1(t)$ at the point $(x_0, y_0, f(x_0, y_0))$. Label this tangent vector by $\mathbf{T}_1$.

(b) Find a parametrization $\mathbf{r}_2(t)$ of the curve on the graph $z = f(x, y)$ travelling in the y-direction while keeping x fixed at x_0 (i.e. the blue curve in the diagram). Hence, find the tangent vector of $\mathbf{r}_2(t)$ at the point $(x_0, y_0, f(x_0, y_0))$. Label this tangent vector by $\mathbf{T}_2$.

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #4 • Critical Points, Lagrange's Multiplier

1. (★) Find all local extrema (a.k.a. critical points) of the following functions $f(x,y)$. Determine the nature (a local minimum, a local maximum or a saddle) of each of them using the Second Derivative Test whenever possible. If the Second Derivative Test is inconclusive, use some other methods to determine its nature.
   (a) $f(x,y) = 4 + x^3 + y^3 - 3xy$
   (b) $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$
   (c) $f(x,y) = \sin x \cos y$
   (d) $f(x,y) = x^4 + y^4$

2. (★★) Give an example of a C^2 function $f(x,y)$ such that:
   (a) $(0,0)$ is a saddle point and $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(0,0)$.
   (b) $(0,0)$ is a local minimum, $f_{xx}f_{yy} - f_{xy}^2 = 0$, $f_{xx} > 0$ and $f_{yy} = 0$ at $(0,0)$.
   (c) $(0,0)$ is a local maximum, $f_{xx}f_{yy} - f_{xy}^2 = 0$, $f_{xx} = 0$ and $f_{yy} < 0$ at $(0,0)$.

3. (★) Using Lagrange's Multipliers, find the maximum and minimum values of the given function subject to the given constraint:
   (a) $f(x,y) = x + 2y$ subject to $x^2 + y^2 = 4$
   (b) $f(x,y) = x - y$ subject to $x^2 + y^2 = 20 + 3xy$
   (c) $f(x,y,z) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$
   (d) $f(x_1, x_2, \ldots, x_n) = x_1 + 2x_2 + \ldots + nx_n$ subject to $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$

4. (★★) The rationale behind Lagrange's Multipliers method is that if a function f achieves its maximum or minimum on the constraint $g = c$ at points P, then the level sets of f and g at P must be tangent to each other, and so $\nabla f(P)$ is parallel to $\nabla g(P)$. Therefore, we solve the system:
$$\nabla f(P) = \lambda \nabla g(P) \quad \text{and} \quad g(P) = c$$
   to locate all such P's.

   However, another way to determine whether $\nabla f(P)$ and $\nabla g(P)$ are parallel is by their cross product:
$$\nabla f(P) \parallel \nabla g(P) \quad \text{if and only if} \quad \nabla f(P) \times \nabla g(P) = \mathbf{0}.$$
   By solving the vector equation $\nabla f(P) \times \nabla g(P) = \mathbf{0}$ for P (instead of using Lagrange's Multiplier), try to redo Problems #3(a)(b)(c).

   What is the limitation of this method when compared to Lagrange's Multiplier?

5. (★) A closed rectangular water tank is to be made with three different materials. The top part will be made by a thin material which costs \$1 per cm^2. The four sides will use stronger material which costs \$2 per cm^2. To support the weight of water, the bottom of the tank has to be made with very durable and strengthened material which costs \$5 per cm^2.

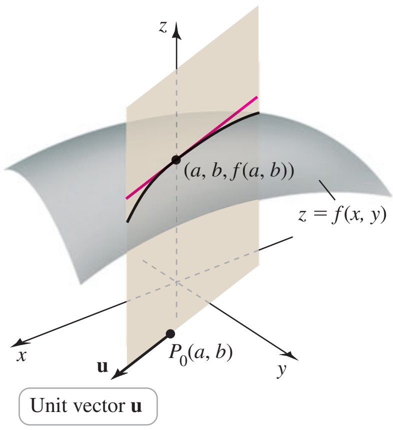
   Suppose the volume of the water tank is to be 96cm^3 . What dimensions of the tank will *minimize* the cost of construction?

6. (★) Find the point(s) on the cone:
$$z = \sqrt{x^2 + y^2}$$
that is/are closest to the point $(1, 3, 1)$.

7. (★★) Consider the surface given by the equation
$$x^2y^2z^2 = 1.$$
   (a) Show that for any point (a, b, c) on the surface, its tangent plane does not contain the origin.
   (b) Find all points (a, b, c) on the surface such that the tangent plane at these points are closest to the origin.

8. (★★) Suppose
$$f(x, y) = 2x^2 + xy - 8x - y + 6$$
Let T be the triangular region (boundary included) in the xy-plane with vertices $(0, 0)$, $(0, 3)$ and $(3, 0)$.
   (a) Find all the interior critical point(s) of f in the region T.
   (b) Find the maximum and minimum values of $f(x, y)$ when (x, y) is restricted on the vertical side of T, i.e. the line segment joining $(0, 0)$ and $(0, 3)$.
   (c) Find the maximum and minimum values of $f(x, y)$ when (x, y) is restricted on the horizontal side of T, i.e. the line segment joining $(0, 0)$ and $(3, 0)$.
   (d) Write the equation of the line joining the vertices $(0, 3)$ and $(3, 0)$. Hence find the maximum and minimum values of $f(x, y)$ when (x, y) is restricted to the hypotenuse of T.
   (e) Determine the absolute maximum and minimum of $f(x, y)$ over the domain T.

9. (★★) This purpose of this problem is to explain why the Second Derivative Test works for two-variable functions. Let $f(x, y)$ be a C^2 function with a critical point $P_0(a, b)$. Given any unit direction $\hat{\mathbf{u}} = u_1\mathbf{i} + u_2\mathbf{j}$, the path:
$$x = a + tu_1, \quad y = b + tu_2$$
is a straight-line passing through (a, b) in the direction of $\hat{\mathbf{u}}$. As such, the intersection curve of the vertical plane shown in the diagram below can be expressed as:
$$z = f(a + tu_1, b + tu_2)$$



Therefore, the first derivative $\left.\frac{dz}{dt}\right|_{t=0}$ measures the slope of tangent at $(a, b, f(a, b))$, whereas the second derivative $\left.\frac{d^2z}{dt^2}\right|_{t=0}$ indicates whether the curve is concave up or down around the point $(a, b, f(a, b))$.

(a) Using the chain rule, show that:

$$\frac{dz}{dt} = f_x\, u_1 + f_y\, u_2$$
$$\frac{d^2z}{dt^2} = f_{xx}u_1^2 + 2f_{xy}u_1u_2 + f_{yy}u_2^2$$

(b) By *completing-the-square*, show further that:

$$\frac{d^2z}{dt^2} = \begin{cases} f_{xx}\left[\left(u_1 + \frac{f_{xy}}{f_{xx}}u_2\right)^2 + \left(\frac{f_{xx}f_{yy}-f_{xy}^2}{f_{xx}^2}\right)u_2^2\right] & \text{if } f_{xx} \neq 0 \\ f_{yy}\left[\left(\frac{f_{xy}}{f_{yy}}u_1 + u_2\right)^2 + \left(\frac{f_{xx}f_{yy}-f_{xy}^2}{f_{yy}^2}\right)u_1^2\right] & \text{if } f_{yy} \neq 0 \\ 2f_{xy}u_1u_2 & \text{if } f_{xx} = f_{yy} = 0 \end{cases}$$

(c) The nature of the point $(a, b, f(a, b))$ can be determined by the following argument:

i. If $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$, what can you say about $\frac{d^2z}{dt^2}$? Using this observation, conclude the nature of the point $(a, b, f(a, b))$.

ii. How about if $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$?

iii. How about if $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} = 0$?

iv. What if $f_{xx}f_{yy} - f_{xy}^2 < 0$?

1. (★) Find all local extrema (a.k.a. critical points) of the following functions $f(x, y)$. Determine the nature (a local minimum, a local maximum or a saddle) of each of them using the Second Derivative Test whenever possible. If the Second Derivative Test is inconclusive, use some other methods to determine its nature.

(a) $f(x, y) = 4 + x^3 + y^3 - 3xy$

(b) $f(x, y) = x^2 + 4y^2 - 2x^2y + 4$

(c) $f(x, y) = \sin x \cos y$

(d) $f(x, y) = x^4 + y^4$

$f_x = 3x^2 - 3y = 0$

$f_y = 3y^2 - 3x = 0.$

$x^2 - y = 0$

$y^2 - x = 0$

$x^4 - x = 0$

$x = 0$ or $(x^3 - 1) = 0$

$(x-1) = 0$ or $x^2 + x + 1 = 0$ (neg.)

$x = 0$ or $x = 1$

$y = 0$ or $y = 1.$

$f_{xx} = 6x$, $f_{yy} = 6y$. $f_{xy} = -3.$

$$D = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix}$$

At $(0,0)$, $D = -9$. (saddle). At $(1,1)$, $D = 27$, $f_{xx} > 0$.

Local min.

A.

x	y	
0	0	saddle
1	1	min.

1. (★) Find all local extrema (a.k.a. critical points) of the following functions $f(x,y)$. Determine the nature (a local minimum, a local maximum or a saddle) of each of them using the Second Derivative Test whenever possible. If the Second Derivative Test is inconclusive, use some other methods to determine its nature.

   (a) $f(x,y) = 4 + x^3 + y^3 - 3xy$
   (b) $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$
   (c) $f(x,y) = \sin x \cos y$
   (d) $f(x,y) = x^4 + y^4$

b). $f_x = 2x - 4xy$

$f_y = 8y - 2x^2$

$-2x^2 = -8y$

$x^2 = 4y$

$y = \frac{x^2}{4}$.

$2x - 4x\left(\frac{x^2}{4}\right) = 0$

$2x - x^3 = 0$

$x = 0$ or

$2 - x^2 = 0$

$x = \pm\sqrt{2}$!

$(0, 0)$

$(\sqrt{2}, \frac{1}{2})$

$f_{xx} = 2$

$f_{xy} = -4x$

$f_{yy} = 8$

x	y	
0	0	min
$\sqrt{2}$	$\frac{1}{2}$	~~min.~~ saddle.

$-\sqrt{2}$ $\frac{1}{2}$

$$D = \begin{vmatrix} 2 & -4x \\ -4x & 8 \end{vmatrix}$$

For $(0, 0)$, $D = 16 - 0 = 16$ Local min

For $(\sqrt{2}, \frac{1}{2})$, $D = 16 - 16(\cancel{\sqrt{2}})(2) > 0$

$f(x) = \sin x \cos y$.

$f_x = \cos x \cos y$

$f_y = -\sin x \sin y$

$$\begin{cases} \cos x \cos y = 0 \\ \sin x \sin y = 0 \end{cases}$$

$\cos x = 0$ or $\cos y = 0$

$$\begin{pmatrix} x = \frac{\pi}{2} + 2\pi n \\ y = 2\pi n \end{pmatrix} \text{ or } \begin{pmatrix} y = \frac{\pi}{2} + 2\pi n \\ x = 2\pi n \end{pmatrix}$$

$\sin x = 0$ or $\sin y = 0$

$f_{xx} = -\cos y \sin x$

$f_{xy} = -\sin y \cos x$

$f_{yy} = -\sin x \cos y$

$$D = \begin{vmatrix} -\cos y \sin x & -\sin y \cos x \\ -\sin y \cos x & -\sin x \cos y \end{vmatrix}$$

at $(\frac{\pi}{2}, 0)$, $D = 1 - 0 = 1$, $f_{xx} = -1$, Local max.

at $(0, \frac{\pi}{2})$, $D = 0 - 1 = -1$, saddle.

$f(x,y) = x^4 + y^4$

$f_x = 4x^3$

$f_y = 4y^3$ $\qquad (x,y) = (0,0)$

$f_{xx} = 12x^2$

$f_{xy} = 0$

$f_{yy} = 12y^2$

$$D = \begin{vmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{vmatrix} = 0$$

it is min.

Made with Goodnotes

2. (★★) Give an example of a C^2 function $f(x,y)$ such that:

(a) $(0,0)$ is a saddle point and $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(0,0)$.

(b) $(0,0)$ is a local minimum, $f_{xx}f_{yy} - f_{xy}^2 = 0$, $f_{xx} > 0$ and $f_{yy} = 0$ at $(0,0)$.

(c) $(0,0)$ is a local maximum, $f_{xx}f_{yy} - f_{xy}^2 = 0$, $f_{xx} = 0$ and $f_{yy} < 0$ at $(0,0)$.

a). $x^3 - 3x^2y$.

$f_x = 3x^2 - 6xy$.

$f_{xx} = 6x - 6y$

$f_{xy} = -6x$

$f_y = -3x^2$

$f_{yy} = 0$.

$$D = \begin{vmatrix} 6x-6y & -6x \\ -6x & 0 \end{vmatrix} = 0.$$

But along $y=0$, $(0,0)$ is not local max/min.

b).

3. (★) Using Lagrange's Multipliers, find the maximum and minimum values of the given function subject to the given constraint:

(a) $f(x,y) = x + 2y$ subject to $x^2 + y^2 = 4$

(b) $f(x,y) = x - y$ subject to $x^2 + y^2 = 20 + 3xy$

(c) $f(x,y,z) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$

(d) $f(x_1, x_2, \ldots, x_n) = x_1 + 2x_2 + \ldots + nx_n$ subject to $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$

a). $g(x,y) = x^2 + y^2$

$\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \langle 1, 2 \rangle$

$$\begin{cases} 1 = \lambda(2x) \\ 2 = \lambda(2y) \\ x^2 + y^2 = 4. \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = \frac{1}{\lambda} \\ x^2 + y^2 = 4. \end{cases}$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = 4$$

$$\frac{5}{4\lambda^2} = 4$$

$$5 = 16\lambda^2$$

$$\lambda = \pm\sqrt{\frac{5}{16}}$$

$x = \pm \frac{1}{2\sqrt{\frac{5}{16}}} = \pm \frac{2}{\sqrt{5}}$

$y = \pm \frac{4}{\sqrt{5}}$

Max : $\frac{2}{\sqrt{5}} + 2\left(\frac{4}{\sqrt{5}}\right) = \frac{10}{\sqrt{5}} = 2\sqrt{5}$

Min = $-2\sqrt{5}$.

Made with Goodnotes

3. (★) Using Lagrange's Multipliers, find the maximum and minimum values of the given function subject to the given constraint:

(a) $f(x,y) = x + 2y$ subject to $x^2 + y^2 = 4$

(b) $f(x,y) = x - y$ subject to $x^2 + y^2 = 20 + 3xy$

(c) $f(x,y,z) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$

(d) $f(x_1, x_2, \ldots, x_n) = x_1 + 2x_2 + \ldots + nx_n$ subject to $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$

3b. $\nabla f = \langle 1, -1 \rangle$

$g(x) = x^2 + y^2 - 3xy.$

$\nabla g = \langle 2x - 3y, 2y - 3x \rangle$

$$\begin{cases} 1 = \lambda(2x - 3y) \\ 1 = \lambda(3x - 2y) \\ x^2 + y^2 - 3xy = 20 \end{cases}$$

$2\lambda x - 3\lambda y = 1$

$x = \frac{1 + 3\lambda y}{2\lambda}$

$1 = \lambda\left(3\left(\frac{1+3\lambda y}{2\lambda}\right) - 2y\right)$

$1 = 3\lambda\left(\frac{1+3\lambda y}{2\lambda}\right) - 2\lambda y$

$1 = \frac{3}{2}(1 + 3\lambda y) - 2\lambda y$

$1 = \frac{3}{2} + \frac{9}{2}\lambda y - 2\lambda y$

$-\frac{1}{2} = \frac{5}{2}\lambda y$

$\lambda y = -\frac{1}{5}$

$x = \frac{1 - \frac{3}{5}}{2\lambda} = \frac{\frac{2}{5}}{2\lambda} = \frac{1}{5\lambda}$

$1 = \lambda\left(\frac{2}{5\lambda} - 3y\right)$

$1 = \frac{2}{5} - 3\lambda y$

$-\frac{3}{5} = 3\lambda y$

$-\frac{1}{5} = \lambda y$

$y = \frac{-1}{5\lambda}$

$y = -x.$

$2x^2 + 3x^2 = 20$

$x = \pm 2.$

$y = \mp 2.$

For $(2, -2)$, $x - y = 4$

$\Rightarrow$ Max.

For $(-2, 2)$ $x - y = -4$

$\Rightarrow$ Min.

Made with Goodnotes

3. (★) Using Lagrange's Multipliers, find the maximum and minimum values of the given function subject to the given constraint:

(a) $f(x,y) = x + 2y$ subject to $x^2 + y^2 = 4$

(b) $f(x,y) = x - y$ subject to $x^2 + y^2 = 20 + 3xy$

(c) $f(x,y,z) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$

(d) $f(x_1, x_2, \ldots, x_n) = x_1 + 2x_2 + \ldots + nx_n$ subject to $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$

$$\nabla f = \langle yz, xz, xy \rangle$$

$$\nabla g = \langle 2x, 4y, 8z \rangle$$

$$\begin{cases} yz = 2\lambda x \\ xz = 4\lambda y \\ xy = 8\lambda z \\ x^2 + 2y^2 + 4z^2 = 9 \end{cases}$$

$$\frac{2\lambda x}{y} = \frac{4\lambda y}{x}$$

$$2x^2 = 4y^2$$

$$x^2 = 2y^2$$

$$\frac{8\lambda z}{x} = \frac{2\lambda x}{z}$$

$$8\lambda z^2 = 2\lambda x^2$$

$$2y^2 = 4z^2 = x^2$$

$$3x^2 = 9$$

$$x = \pm 1 \quad 3$$

$$\cancel{z = \pm \frac{1}{2}} \qquad z = \pm \frac{3}{2}$$

$$\cancel{y = \pm \frac{1}{\sqrt{2}}} \qquad y = \pm \frac{3}{\sqrt{2}}$$

$$f\langle 1, \tfrac{1}{\sqrt{2}}, \tfrac{1}{2} \rangle = \frac{\sqrt{2}}{4}. \quad \text{Max}$$

$$f\langle -1, -\tfrac{1}{\sqrt{2}}, -\tfrac{1}{2} \rangle = -\frac{\sqrt{2}}{4}. \quad \text{Min.}$$

3. (★) Using Lagrange's Multipliers, find the maximum and minimum values of the given function subject to the given constraint:

(a) $f(x,y) = x + 2y$ subject to $x^2 + y^2 = 4$

(b) $f(x,y) = x - y$ subject to $x^2 + y^2 = 20 + 3xy$

(c) $f(x,y,z) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$

(d) $f(x_1, x_2, \ldots, x_n) = x_1 + 2x_2 + \ldots + nx_n$ subject to $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$

$$\nabla f = \langle yz, xz, xy \rangle$$

$$\nabla g = \langle 2x, 4y, 8z \rangle$$

漏左3个root，

$$yz = 2\lambda x$$
$$xz = 4\lambda y$$
$$xy = 8\lambda z$$

$$\frac{3}{2} = y^2$$
$$y = \pm\frac{\sqrt{3}}{\sqrt{2}}$$

$$2\lambda x^2 = 4\lambda y^2 = 8\lambda z^2$$
$$2x^2 = 4y^2 = 8z^2$$

$$\frac{3}{4} = z^2$$
$$z = \frac{\sqrt{3}}{2}.$$

$$x^2 = 2y^2 = 4z^2 \Leftarrow x = 0 \text{ or } y = 0 \text{ or } z = 0$$

係其中三root!!

$$x^2 + x^2 + x^2 = 9$$
$$x^2 = 3$$
$$x = \pm\sqrt{3}$$

$$\frac{9}{2} = y^2$$
$$y = \pm\frac{3}{\sqrt{2}}, \quad z = \pm\frac{3}{2}.$$

3. (★) Using Lagrange's Multipliers, find the maximum and minimum values of the given function subject to the given constraint:

(a) $f(x,y) = x + 2y$ subject to $x^2 + y^2 = 4$

(b) $f(x,y) = x - y$ subject to $x^2 + y^2 = 20 + 3xy$

(c) $f(x,y,z) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$

(d) $f(x_1, x_2, \ldots, x_n) = x_1 + 2x_2 + \ldots + nx_n$ subject to $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$

$$\nabla f = \langle 1, 2, \cdots, n \rangle$$

$$\nabla g = \langle 2x_1, 2x_2, \cdots, 2x_n \rangle$$

$$\begin{cases} 1 = 2x_1\lambda \\ 2 = 2x_2\lambda \\ 3 = 2x_3\lambda \\ \vdots \\ n = 2x_n\lambda \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 1 \end{cases}$$

$x_1 = \frac{1}{2\lambda}$ $\quad x_1^2 = \frac{1}{4\lambda^2}$

$x_2 = \frac{1}{\lambda}$ $\quad x_2^2 = \frac{1}{\lambda^2}$

$x_3 = \frac{3}{2\lambda}$ $\quad x_3^2 = \frac{9}{4\lambda^2}$

$x_4 = \frac{2}{\lambda}$ $\quad x_4^2 = \frac{4}{\lambda^2}$

$$\langle x_1, x_2, \cdots, x_n \rangle \cdot \langle x_1, x_2, \cdots, x_n \rangle = 1$$

$$|\langle x_1, x_2, \cdots, x_n \rangle|^2 = 1$$

Made with Goodnotes

3. (★) Using Lagrange's Multipliers, find the maximum and minimum values of the given function subject to the given constraint:

(a) $f(x,y) = x + 2y$ subject to $x^2 + y^2 = 4$

(b) $f(x,y) = x - y$ subject to $x^2 + y^2 = 20 + 3xy$

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(d) $f(x_1, x_2, \ldots, x_n) = x_1 + 2x_2 + \ldots + nx_n$ subject to $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$

d). $\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases}$

$$\begin{cases} 1 = 2x_1(\lambda) \\ 2 = 2x_2(\lambda) \\ \vdots \\ n = 2x_n(\lambda) \end{cases}$$

$\frac{1}{2}\left(\frac{1}{\lambda}\right) = x_1$

$\frac{2}{2}\left(\frac{1}{\lambda}\right) = x_2$

$\frac{3}{2}\left(\frac{1}{\lambda}\right) = x_3$

$\vdots$

$\frac{n}{2}\left(\frac{1}{\lambda}\right) = x_n$

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1$$

$$\frac{1}{\lambda^2}\left(\frac{1^2}{2^2} + \frac{2^2}{2^2} + \frac{3^2}{2^2} + \cdots + \frac{n^2}{2^2}\right) = 1$$

$$\frac{1}{4}\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right) = \lambda^2$$

$$\sum_{i=1}^{n} i^2 = 4\lambda^2$$

$$\frac{n(n+1)(2n+1)}{6} = 4\lambda^2$$

$$\pm\sqrt{\frac{n(n+1)(2n+1)}{24}} = \lambda$$

$$\frac{2}{1}x_1 = \frac{2}{2}x_2 = \frac{2}{3}x_3 = \cdots = \frac{2}{n}x_n$$

$$x_n = \frac{n}{2}\left(\pm\frac{\sqrt{24}}{\sqrt{n(2n+1)(n+1)}}\right)$$

$$2x_1 = \frac{2}{n}x_n \qquad \frac{2}{3}x_3 = \frac{2}{n}x_n$$

$$x_1 = \pm\left(\frac{1}{2}\right)\frac{\sqrt{24}}{\sqrt{n(n+1)(2n+1)}} \qquad \left(\frac{3}{2}\right)$$

$$x_2 = \pm\left(\frac{2}{2}\right)\frac{\sqrt{24}}{\sqrt{n(n+1)(2n+1)}}$$

$$x_3 = \pm\left(\frac{3}{2}\right)\frac{\sqrt{24}}{\sqrt{n(n+1)(2n+1)}}$$

for $x_1 = \left(\frac{1}{2}\right)\frac{\sqrt{24}}{\sqrt{n(n+1)(2n+1)}} = k\left(\frac{1}{2}\right).$

$$f = \left(\frac{1}{2} + 2\left(\frac{2}{2}\right) + 3\left(\frac{3}{2}\right) + 4\left(\frac{4}{2}\right) + \cdots + n\left(\frac{n}{2}\right)\right)k$$

$$= \frac{1}{2}\left(1+2^2+3^2+\cdots n^2\right)k$$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}\right)\frac{\sqrt{24}}{\sqrt{n(n+1)(2n+1)}}$$

$$= \pm\frac{\sqrt{n(n+1)(2n+1)\times 6}}{6}$$

4. (★★) The rationale behind Lagrange's Multipliers method is that if a function f achieves its maximum or minimum on the constraint $g = c$ at points P, then the level sets of f and g at P must be tangent to each other, and so $\nabla f(P)$ is parallel to $\nabla g(P)$. Therefore, we solve the system:

$$\nabla f(P) = \lambda \nabla g(P) \quad \text{and} \quad g(P) = c$$

to locate all such P's.

However, another way to determine whether $\nabla f(P)$ and $\nabla g(P)$ are parallel is by their cross product:

$$\nabla f(P) \parallel \nabla g(P) \quad \text{if and only if} \quad \nabla f(P) \times \nabla g(P) = \mathbf{0}.$$

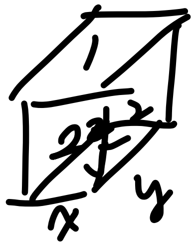
By solving the vector equation $\nabla f(P) \times \nabla g(P) = \mathbf{0}$ for P (instead of using Lagrange's Multiplier), try to redo Problems #3(a)(b)(c).

What is the limitation of this method when compared to Lagrange's Multiplier?

5. (★) A closed rectangular water tank is to be made with three different materials. The top part will be made by a thin material which costs \$1 per cm^2. The four sides will use stronger material which costs \$2 per cm^2. To support the weight of water, the bottom of the tank has to be made with very durable and strengthened material which costs \$5 per cm^2.

Suppose the volume of the water tank is to be 96cm^3 . What dimensions of the tank will *minimize* the cost of construction?

5.



$C = 6xy + 4xz + 4yz$ subject to

$xyz = 96.$

$$\begin{cases} \nabla C = \lambda \nabla g \\ g = 96 \end{cases}$$

$$\begin{cases} 6y + 4z = \lambda yz \\ 6x + 4z = \lambda xz \\ 4x + 4y = \lambda xy \\ xyz = 96 \end{cases}$$

$4x^2 = \lambda x^2$

$\lambda = 4$

$(6y+4z)x = (6x+4z)y$

$6xy + 4xz = 6xy + 4yz$

$4xz = 4yz$

$4x = 4y$

$\boxed{x = y}$.

$$4x + 4y = \lambda xy$$
$$x + y = xy$$
$$(x+y)z = 96$$
$$z = \frac{96}{x^2}$$

$$(6x+4z)y = (4x+4y)z$$
$$6xy = 4xz$$
$$6y = 4z$$
$$\frac{3y}{2} = z$$

$$x^2\left(\frac{3x}{2}\right) = 96$$
$$x^3 = 64$$

Dimension $\begin{cases} x = 4 \\ y = 4 \\ z = 6 \end{cases}$

Made with Goodnotes

6. (★) Find the point(s) on the cone:

$$z = \sqrt{x^2 + y^2}$$

that is/are closest to the point $(1, 3, 1)$.

Can rewrite as $z^2 - x^2 - y^2 = 0$

$$D = \sqrt{(x-1)^2 + (y-3)^2 + (z-1)^2}$$ subject

to $z = \sqrt{x^2+y^2}$

$$\begin{cases} \nabla D^2 = \lambda \nabla g \\ g = 0. \end{cases}$$

$$\begin{cases} 2(x-1) = \dfrac{x}{\sqrt{x^2+y^2}}(\lambda) \\ 2(y-3) = \dfrac{y}{\sqrt{x^2+y^2}}(\lambda) \\ 2(z-1) = -\lambda \\ \sqrt{x^2+y^2} = z \end{cases}$$

$$(2x-2)y = (2y-6)x$$

$$2xy - 2y = 2xy - 6x$$

$$2y = 6x$$

$$y = 3x$$

$$z = \sqrt{10}\,x$$

$$2\sqrt{10}\,x - 2 = -\lambda$$

$$2 - 2\sqrt{10}\,x = \lambda$$

$$2x - 1 = \frac{x}{\sqrt{10}\,x} \qquad (2 - 2\sqrt{10}\,x)$$

$$2x-1 = \frac{2}{\sqrt{10}} - 2x$$

$$4x = \frac{2}{\sqrt{10}} + 1$$

$$x = \frac{1}{2\sqrt{10}} + \frac{1}{4}$$

$$y = \frac{3}{2\sqrt{10}} + \frac{3}{4}$$

$$z = \frac{1}{2} + \frac{1}{4}\sqrt{10}$$

$$\frac{2+\sqrt{10}}{4\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}}$$

$$= \frac{2\sqrt{10}+10}{40}$$

$$= \frac{\sqrt{10}+5}{20}$$

7. (★★) Consider the surface given by the equation

$$x^2y^2z^2 = 1.$$

(a) Show that for any point (a, b, c) on the surface, its tangent plane does not contain the origin.

(b) Find all points (a, b, c) on the surface such that the tangent plane at these points are closest to the origin.

a).
$$\nabla f = \langle 2xy^2z^2, 2x^2yz^2, 2x^2y^2z \rangle$$

$a^2b^2c^2 = 1$
$$= \langle 2ab^2c^2, 2a^2bc^2, 2a^2b^2c \rangle$$

$$\left\langle \frac{2}{a}, \frac{2}{b}, \frac{2}{c} \right\rangle$$

$$\frac{2}{a}x + \frac{2}{b}y + \frac{2}{c}z = 6$$
$$\frac{1}{a}x + \frac{1}{b}y + \frac{1}{c}z = 3$$

b).
$$g(x,y,z) = x^2y^2z^2 - 1$$
$$h(x,y,z) = \frac{1}{a}x + \frac{1}{b}y + \frac{1}{c}z - 3$$
$$\cancel{f(x,y,z) = x^2 + y^2 + z^2}$$

用 $d = \dfrac{3}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$

→ 再通分一下 用 $g(x,y,z)$ 得出 $d = \dfrac{3}{\sqrt{\frac{b^2c^2 + a^2c^2 +}{a^2b^2}}}$

8. (★★) Suppose

$$f(x,y) = 2x^2 + xy - 8x - y + 6$$

Let T be the triangular region (boundary included) in the xy-plane with vertices $(0,0)$, $(0,3)$ and $(3,0)$.

(a) Find all the interior critical point(s) of f in the region T.

(b) Find the maximum and minimum values of $f(x,y)$ when (x,y) is restricted on the vertical side of T, i.e. the line segment joining $(0,0)$ and $(0,3)$.

(c) Find the maximum and minimum values of $f(x,y)$ when (x,y) is restricted on the horizontal side of T, i.e. the line segment joining $(0,0)$ and $(3,0)$.

(d) Write the equation of the line joining the vertices $(0,3)$ and $(3,0)$. Hence find the maximum and minimum values of $f(x,y)$ when (x,y) is restricted to the hypotenuse of T.

(e) Determine the absolute maximum and minimum of $f(x,y)$ over the domain T.

a). $f_x = 4x + y - 8 = 0$

$f_y = x - 1 = 0$

$\begin{cases} x = 1 \\ y = 4 \end{cases}$ No interior critical pt.

b). $f(0,y) = -y+6$ Max when $y=0$, 6

Min when $y=3$, 3

c). $f(x,0) = 2x^2 - 8x + 6$ Min when $x=2$, $\boxed{-2}$

$= 2(x^2-4x)+6$ Max when $x=0$ $\boxed{6}$

$= 2(x^2-4x+4-4)+6$

$= 2(x-2)^2 - 8 + 6$

$= 2(x-2)^2 - 2$

d). $\frac{y-3}{x} = \frac{-3}{3}$

$y-3 = -x$ $y = 3-x$

$x+y=3$

$$f(x,3-x) = 2x^2 + x(3-x) - 8x - (3-x) + 6$$

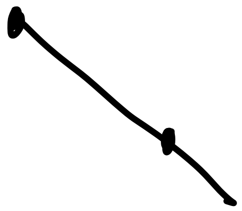
$$= 2x^2 + 3x - x^2 - 8x - 3 + x + 6$$

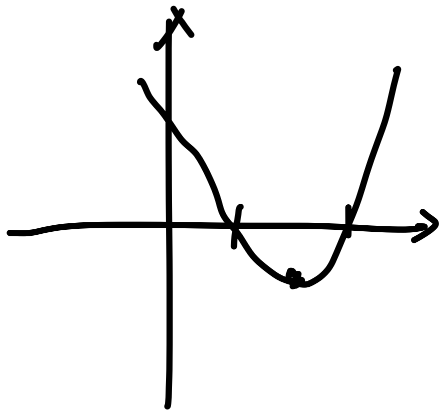
$$= x^2 - 4x + 3$$

$$(x-2)^2 - 1$$

Min $= -1$ when $x = 2$.

Max $= 3$ when $x = 0$





D). Absolute Max : 6 $(0,0)$

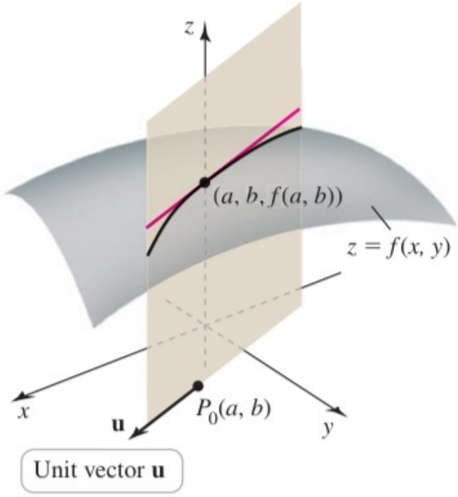
Min : -2 $(2,0)$

9. (★★) This purpose of this problem is to explain why the Second Derivative Test works for two-variable functions. Let $f(x, y)$ be a C^2 function with a critical point $P_0(a, b)$. Given any unit direction $\hat{\mathbf{u}} = u_1\mathbf{i} + u_2\mathbf{j}$, the path:

$$x = a + tu_1, \quad y = b + tu_2$$

is a straight-line passing through (a, b) in the direction of $\hat{\mathbf{u}}$. As such, the intersection curve of the vertical plane shown in the diagram below can be expressed as:

$$z = f(a + tu_1, b + tu_2)$$



Therefore, the first derivative $\left.\frac{dz}{dt}\right|_{t=0}$ measures the slope of tangent at $(a, b, f(a, b))$, whereas the second derivative $\left.\frac{d^2z}{dt^2}\right|_{t=0}$ indicates whether the curve is concave up or down around the point $(a, b, f(a, b))$.

(a) Using the chain rule, show that:

$$\frac{dz}{dt} = f_x\, u_1 + f_y\, u_2$$
$$\frac{d^2z}{dt^2} = f_{xx}u_1^2 + 2f_{xy}u_1u_2 + f_{yy}u_2^2$$

(b) By *completing-the-square*, show further that:

$$\frac{d^2z}{dt^2} = \begin{cases} f_{xx}\left[\left(u_1 + \frac{f_{xy}}{f_{xx}}u_2\right)^2 + \left(\frac{f_{xx}f_{yy}-f_{xy}^2}{f_{xx}^2}\right)u_2^2\right] & \text{if } f_{xx} \neq 0 \\ f_{yy}\left[\left(\frac{f_{xy}}{f_{yy}}u_1 + u_2\right)^2 + \left(\frac{f_{xx}f_{yy}-f_{xy}^2}{f_{yy}^2}\right)u_1^2\right] & \text{if } f_{yy} \neq 0 \\ 2f_{xy}u_1u_2 & \text{if } f_{xx} = f_{yy} = 0 \end{cases}$$

(c) The nature of the point $(a, b, f(a, b))$ can be determined by the following argument:

i. If $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$, what can you say about $\frac{d^2z}{dt^2}$? Using this observation, conclude the nature of the point $(a, b, f(a, b))$.

ii. How about if $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$?

iii. How about if $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} = 0$?

iv. What if $f_{xx}f_{yy} - f_{xy}^2 < 0$?

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= f_x u_1 + f_y u_2$$

$f_x = u_1$

$f_y = u_2$

$$\frac{d^2 z}{dt^2} = f_{xx} x_t (u_1) + f_{xy}(u_1) + f_{yx}(u_2) y_t + f_{yy} y_t (u_2)$$

f, f_x, f_y

$x \quad y$

$t \quad t$

$$= f_{xx}(u_1)^2 + 2(u_1)(u_2) f_{xy} + f_{yy}(u_2)^2$$

$$f_{xx}\left(u_1^2 + 2u_1 u_2 \frac{f_{xy}}{f_{xx}} \right) + f_{yy}(u_2)^2$$

$$f_{xx}\left(u_1^2 + 2u_1 u_2 \frac{f_{xy}}{f_{xx}} + \left(u_2 \frac{f_{xy}}{f_{xx}}\right)^2 - \left(u_2 \frac{f_{xy}}{f_{xx}}\right)^2 \right) + f_{yy}(u_2)^2$$

$$= f_{xx}\left(u_1 + u_2 \frac{f_{xy}}{f_{xx}} \right)^2 - u_2^2 \frac{(f_{xy})^2 f_{xx}}{f_{xx}^2} + f_{yy} u_2^2$$

$$\frac{f_{xx} f_{yy} - \cancel{f_{xx}}(f_{xy})^2}{f_{xx}^2}$$

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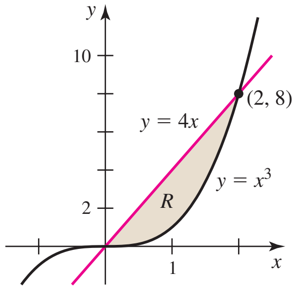
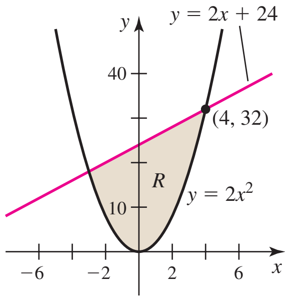
Problem Set #5 • Double Integrals

1. (★) Set-up the lower and upper bounds of each double integral below using **both** $dxdy$ and $dydx$ orders. Compute the integral using **both** orders and verify that they give the same value.

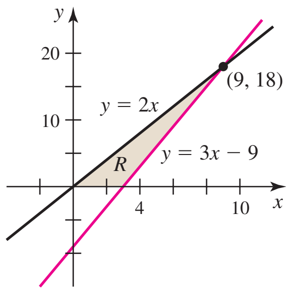
   (a) $\iint_R 2xy\,dA$ where R is the region as shown below:



   (b) $\iint_R 1\,dA$ where R is the region bounded between $y = 2x + 24$ and $y = 2x^2$:



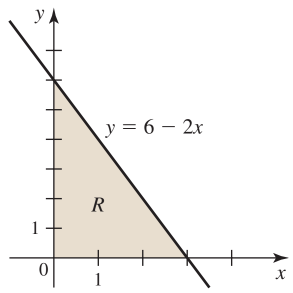
   (c) $\iint_R x^2\,dA$ where R is the region bounded between $y = 2x$, $y = 3x - 9$ and the x-axis:



2. (★) Evaluate the integral $\iint_T \sqrt{a^2 - y^2}\,dA$ where T is the triangle with vertices $(0,0)$, $(0,a)$ and (a,a). Set-up the integral in both $dxdy$ and $dydx$ orders, and choose the *easier* one to compute.

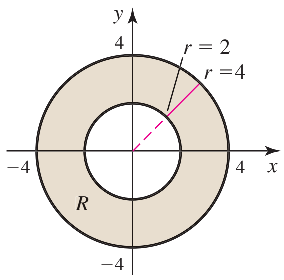
3. (★★) Consider the integral $\int_0^1 \int_x^{x^{1/3}} \sqrt{1-y^4}\,dydx$. It is almost impossible to compute the inner integral. Try to switch the order of integration to evaluate it. [Hint: You should first sketch the region of integration.]

4. (★★) Evaluate the integrals $\iint_R \frac{1}{3-x}\,dA$ and $\iint_R \frac{1}{y-6}\,dA$. Try to avoid *integration-by-parts* if possible.



5. (★) Evaluate $\iint_R (x+y)\,dA$ using polar coordinates where R is the region in the first quadrant lying inside the disk $x^2+y^2 \le a^2$ and under the line $y=\sqrt{3}x$.

6. (★) Consider the annular region R below. Express the integral $\iint_R (x^2+y^2)\,dA$ in **both** rectangular and polar coordinates. Choose the *easier* system to compute the integral.



7. (★★) Evaluate each of the following integrals:

   (a) $\int_0^{2\pi}\int_0^1 e^{-x^2}\sin y\,dxdy$

   (b) $\int_{-1}^{0}\int_0^{\sqrt{y+1}} \left(x-\frac{x^3}{3}\right)^{5/2} dxdy$

   (c) $\iint_Q \frac{1}{(1+2x^2+2y^2)^3}\,dA$ where Q is the entire first quadrant of the xy-plane

   (d) $\iint_Q (x^2-y+1)\,dA$ where Q is the region $\{(x,y) \mid x \ge 0,\ y \ge 0,\ x^2+y^2 \le 4\}$

   (e) $\iint_{\mathbb{R}^2} (x^2+y^2)e^{-(x^4+2x^2y^2+y^4)}\,dA$

8. (★★) Some single-variable integrals are "notoriously" difficult to compute. One example is $\int e^{-x^2}dx$ despite the fact that this integral is of central importance in mathematics (pure/applied), physics, statistics and engineering. However, some of these difficult integrals can be evaluated via double integral methods.

   This problem investigates another well-known integral which has no closed-form antiderivative:

$$\int \frac{\log(1-x)}{x}dx.$$

The goal of this problem is to show that this integral over $0 \le x \le 1$ can be written as an infinite series.

Consider the function

$$f(x,y) = \frac{1}{1-xy}, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

It is defined almost everywhere on the rectangle $0 \le x \le 1$ and $0 \le y \le 1$ (we say 'almost' because it's undefined only at $(x,y) = (1,1)$, but this single point is negligible).

(a) Show that: $\displaystyle\int_0^1 \frac{1}{1-xy}\,dy = -\frac{\log(1-x)}{x}$.

(b) Note that $|xy| < 1$ except for the negligible point $(x,y) = (1,1)$, so the function $f(x,y)$ can be expressed as a geometric series:

$$\frac{1}{1-xy} = 1 + (xy) + (xy)^2 + (xy)^3 + \ldots$$

Using this geometric series, show that

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\,dydx = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

(c) Using (a) and (b), show that

$$-\int_0^1 \frac{\log(1-x)}{x}\,dx = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

(d) Using the above approach, *mutatis mutandis*, show that for any $0 \le z \le 1$, we have:

$$-\int_0^z \frac{\log(1-x)}{x}\,dx = \sum_{k=1}^{\infty} \frac{z^k}{k^2}.$$

[Remark: *Mutatis mutandis* is a Latin phrase meaning "changing only those things which need to be changed".]

9. (★★★) The purpose of this problem is to use double integrals to derive a somewhat surprising result in electrostatics, that is the electric force exerted on a charged particle by an infinite sheet of uniformly distributed charges is *independent* of how far the particle and the sheet are apart from each other.

The paragraphs below describe the physical set-up of the problem. Although it may be possible to proceed to the problem without knowing the physics background, it is strongly recommend to read through the paragraphs below so as to understand the motivation of this problem.

According to the Coulomb's Law, the electric force $\mathbf{F}$ exerted **on** a point particle with charge Q located at (x_0, y_0, z_0), **by** a point particle with charge q located at (x,y,z), is given by:

$$\mathbf{F} = \frac{qQ}{4\pi\varepsilon_0} \frac{(x_0-x)\mathbf{i} + (y_0-y)\mathbf{j} + (z_0-z)\mathbf{k}}{\left((x_0-x)^2 + (y_0-y)^2 + (z_0-z)^2\right)^{3/2}}$$

where ε_0 is positive constant (depending on the medium).

The Coulomb's Law is also called the Inverse Square Law because one can easily verify that the magnitude of the force satisfies:

$$|\mathbf{F}| = \frac{qQ}{4\pi\varepsilon_0 d^2}$$

where $d = \sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}$ is the distance between the two particles.

If there is a sequence of *discrete* charged particles located at (x_1, y_1, z_1), (x_2, y_2, z_2), $\ldots$, each with charge q, then the resultant electric force exerted on a particle with charge Q located at (x_0, y_0, z_0), is given by the vector sum of all forces:

$$\mathbf{F} = \sum_{i=1}^{\infty} \frac{qQ}{4\pi\varepsilon_0} \frac{(x_0 - x_i)\mathbf{i} + (y_0 - y_i)\mathbf{j} + (z_0 - z_i)\mathbf{k}}{\left((x_0 - x_i)^2 + (y_0 - y_i)^2 + (z_0 - z_i)^2\right)^{3/2}}.$$

This is called the Principle of Superposition by physicists.

Now given there is an infinite sheet of uniformly distributed charges on the xy-plane, and for each small area element dA on the xy-plane, the amount of charges is given by $\sigma\, dA$, where σ is a constant that represents the area density of charges. Suppose there is a particle with charge Q located above the xy-plane at $(0, 0, z_0)$, i.e. $z_0 > 0$. For simplicity, call this the Q-particle.

Now regard a small area element located at $(x, y, 0)$ on the xy-plane as a charged "particle" with charge $q = \sigma\, dA$, then the force exerted on the Q-particle by this area element is given by substituting $(x, y, z) = (x, y, 0)$ and $(x_0, y_0, z_0) = (0, 0, z_0)$:

$$\frac{Q(\sigma\, dA)}{4\pi\varepsilon_0} \frac{(0 - x)\mathbf{i} + (0 - y)\mathbf{j} + (z_0 - 0)\mathbf{k}}{\left((0 - x)^2 + (0 - y)^2 + (z_0 - 0)^2\right)^{3/2}} = \frac{Q\sigma}{4\pi\varepsilon_0} \frac{-x\mathbf{i} - y\mathbf{j} + z_0\mathbf{k}}{\left(x^2 + y^2 + z_0^2\right)^{3/2}}\, dA.$$

Therefore, by the Principle of Superposition, the resultant electric force exerted on the Q-particle by the sheet of charges is given by this double integral over the entire xy-plane (i.e. $\mathbb{R}^2$):

$$\mathbf{F}_{\text{resultant}} = \iint_{\mathbb{R}^2} \frac{Q\sigma}{4\pi\varepsilon_0} \frac{-x\mathbf{i} - y\mathbf{j} + z_0\mathbf{k}}{\left(x^2 + y^2 + z_0^2\right)^{3/2}}\, dA.$$

Here integrating a vector simply means integrating each component of the vector treating $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as "constants".

(a) Show that $\mathbf{i}$ and $\mathbf{j}$-components of $\mathbf{F}_{\text{resultant}}$ are zero.

(b) Derive that:

$$\mathbf{F}_{\text{resultant}} = \frac{Q\sigma}{2\varepsilon_0}\mathbf{k}.$$

[Remark: The result in (b) asserts that the resultant force on the Q-particle does *not* depend on how far it is from the infinite sheet! Believe it or not?]

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Problem Set #6 • Triple Integrals

1. (★) Consider the triple integral:

$$\int_0^1 \int_z^1 \int_0^{x-z} f(x,y,z)\, dydxdz.$$

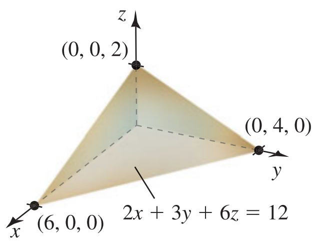
   (a) Sketch the solid described by the integral.
   (b) Express the integral using each of the other five orders, i.e. $dydzdx$, $dxdydz$, $dxdzdy$, $dzdxdy$ and $dzdydx$.

2. (★★) Consider the triple integral:

$$\int_0^1 \int_z^1 \int_0^{x} e^{x^3}\, dydxdz.$$

   (a) Sketch the solid described by the integral.
   (b) Pick a *good* order of integration and compute the integral *by hand*.

3. (★★) Consider the right tetrahedron solid T in the first octant bounded by the xy-, yz-, xz-planes and the plane Π with vertices $(6,0,0)$, $(0,4,0)$ and $(0,0,2)$.



   (a) Show that the equation of the plane Π is given by $2x + 3y + 6z = 12$.
   (b) Evaluate the following triple integral:

$$\iiint_T \left(\frac{1}{12-3y-6z} + \frac{1}{12-2x-6z} + \frac{1}{12-2x-3y} \right) dV.$$

   Please do the computations *by hand*. Pick carefully the orders of integration to simplify your computations.

4. (★★) Let a be a positive constant. Given that $f(x)$ is a continuous function of x, show that:

$$\int_0^a \int_0^z \int_0^y f(x)\, dxdydz = \int_0^a \frac{(a-x)^2}{2} f(x)\, dx$$

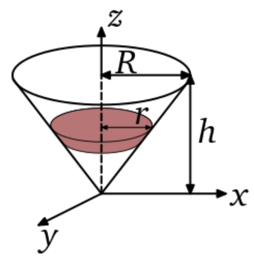
5. (★) Evaluate $\iiint_D (x^2+y^2)\, dV$ over the solid D which lies above the cone $z = c\sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 = a^2$.

6. (★) Find the volume of the solid bounded by the xy-plane, the cone $z = 2a - \sqrt{x^2+y^2}$ and the cylinder $x^2+y^2 = 2ay$.

7. (★★) Let $\phi(x, y, z) = \frac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(-\frac{x^2+y^2+z^2}{4kt}\right)$ where $t > 0$. Show that for each fixed $t > 0$, we have:

$$\iiint_{\mathbb{R}^3} \phi(x, y, z)\, dV = 1.$$

8. (★★) Consider a right circular solid cone (denoted by K) with radius R, height h, mass m and uniform density δ.



The moment of inertia about the z-axis of the solid is defined to be:

$$I_z := \iiint_K D_z(x, y, z)^2\, \delta dV$$

where $D_z(x, y, z)$ is the perpendicular distance between the point (x, y, z) and the z-axis.

(a) *Set up*, but do not evaluate, the integral I_z using each of the following coordinates:
   i. rectangular coordinates
   ii. cylindrical coordinates
   iii. spherical coordinates

(b) Rank the ease of computations of the above coordinate systems for evaluating the integral I_z, then compute I_z using the easiest coordinate system. Express your final answer in terms of the mass m, not the density δ.

9. (★★) Given a solid T with mass m and uniform density δ, the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is defined to be:

$$\bar{x} = \frac{\iiint_T x\, \delta dV}{\iiint_T \delta dV}, \quad \bar{y} = \frac{\iiint_T y\, \delta dV}{\iiint_T \delta dV}, \quad \bar{z} = \frac{\iiint_T z\, \delta dV}{\iiint_T \delta dV}$$

The moment of inertia of T about the z-axis is defined as:

$$I_z := \iiint_T D_z(x, y, z)^2\, \delta dV$$

where $D_z(x, y, z)$ is the perpendicular distance between the point (x, y, z) and the z-axis.

Now consider the axis L passing through the center of mass $(\bar{x}, \bar{y}, \bar{z})$ and parallel to the z-axis. The moment of inertia of the solid about the axis L is defined as:

$$I_{\text{cm}} := \iiint_T D_L(x, y, z)^2\, \delta dV$$

where $D_L(x, y, z)$ is the perpendicular distance between the point (x, y, z) and the axis L.

Prove the following result (which is called the Parallel Axis Theorem):

$$I_z = I_{\text{cm}} + md^2$$

where d is the distance between the z-axis and the axis L.

10. (★) The change-of-variable formula for the volume element dV is given by:

$$dxdydz = \left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw. \tag{*}$$

   (a) Using (*), verify that:

$$dxdydz = \rho^2 \sin\phi \, d\rho d\phi d\theta.$$

   (b) Let $u = 2x$, $v = 3y$ and $w = 5z$. Using (*), express $dxdydz$ in terms of $dudvdw$.

11. (★★★) Consider a solid sphere with radius R centered at the origin in $\mathbb{R}^3$ which carries a uniform distribution of charges with density δ. Each volume element dV in the sphere can be regarded as a particle with charge $\delta\, dV$.

Fix a particle with charge q at $(0,0,z_0)$ where $z_0 > R$, i.e. outside the sphere, and call it the q-particle. As in the previous Problem Set, the electric force exerted on the q-particle by a charged element $\delta\, dV$ at (x,y,z) in the solid sphere is given by the Coulomb's Law (in vector form):

$$d\mathbf{F} = \frac{q\,\delta\, dV}{4\pi\varepsilon_0} \frac{(0-x)\mathbf{i} + (0-y)\mathbf{j} + (z_0 - z)\mathbf{k}}{\left((0-x)^2 + (0-y)^2 + (z_0-z)^2\right)^{3/2}}$$

Similar to the previous Problem Set, the Principle of Superposition asserts that the resultant force exerted on the q-particle by the whole sphere is given by "summing-up", i.e. integrating, each the force element $d\mathbf{F}$ over the sphere:

$$\mathbf{F}_{\text{resultant}} = \iiint_{\text{sphere}} d\mathbf{F}.$$

   (a) Show that:

$$\mathbf{F}_{\text{resultant}} = \left(\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{q\delta}{4\pi\varepsilon_0} \frac{\rho^2\sin\varphi\cdot(z_0 - \rho\cos\varphi)}{\left(\rho^2 - 2\rho z_0\cos\varphi + z_0^2\right)^{3/2}}\, d\rho d\varphi d\theta\right)\mathbf{k}$$

   (b) Try to compute the above integral, either by software or by hand, and show that:

$$\mathbf{F}_{\text{resultant}} = \frac{q\delta R^3}{3\varepsilon_0 z_0^2}\mathbf{k} = \frac{qQ}{4\pi\varepsilon_0 z_0^2}\mathbf{k}$$

   where Q is the total amount of charges in the sphere.

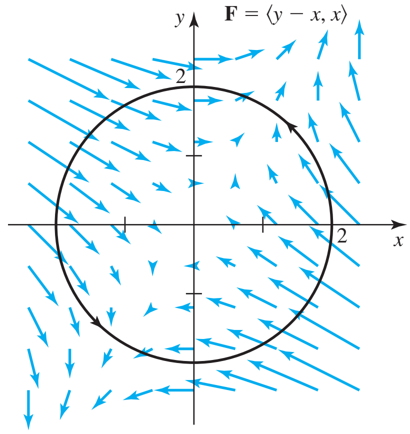
   [Remark 1: This result shows that the resultant force exerted on the q-particle by the charged sphere will be the same if one replaces it by a particle at the origin with the same amount of charges.]

   [Remark 2: Using the Gauss's Law for Electricity, the above result can be obtained easily by considering the surface flux of $\mathbf{F}_{\text{resultant}}$. We will discuss that later, and will derive the Gauss's Law using the Divergence Theorem (assuming Coulomb's Law).]

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #7 • Line Integrals, Conservative Vector Fields, Curl Operator

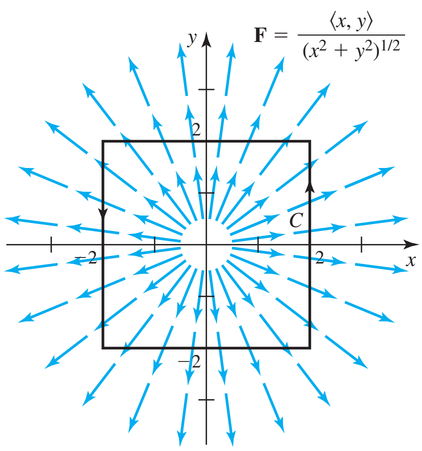
Do not use the Green's Theorem in any problem in this set.

1. (★) Let $\mathbf{F} = (y - x)\mathbf{i} + x\mathbf{j}$ on $\mathbb{R}^2$, and C be the counter-clockwise circular path with radius 2 centered at the origin. See the figure below:



   (a) On the above figure, highlight the portion of the path C at which $\mathbf{F} \cdot \mathbf{r}' > 0$.
   (b) On the above figure, highlight (with another color) the portion of the path C at which $\mathbf{F} \cdot \mathbf{r}' < 0$.
   (c) Calculate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ from the definition. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?
   (d) Calculate $\nabla \times \mathbf{F}$, i.e. the curl of $\mathbf{F}$. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?
   (e) Find a potential function f such that $\mathbf{F} = \nabla f$, or show that such an f does not exist. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

2. (★) Let $\mathbf{F} = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$, and C be the counter-clockwise square path with vertices $(2, -2)$, $(2, 2)$, $(-2, 2)$ and $(-2, -2)$. See the figure below:



Do (a)-(e) of Problem #1 with this $\mathbf{F}$ and C instead.

3. (★) Let C be the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = y$.
   (a) Sketch the cylinder, the plane and the curve C in the same diagram.
   (b) Let $\mathbf{F} = y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$. Calculate the line integral $\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ where Γ is a portion of C from $(-1,0,0)$ to $(1,0,0)$. There are two possible such Γ's. Do both.
   Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?
   (c) Find a potential function f such that $\mathbf{F} = \nabla f$, or show that such an f does not exist.
   Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

4. (★) Determine whether or not each of the following vector fields is conservative or not. If so, find its potential function f such that $\mathbf{F} = \nabla f$.
   (a) $\mathbf{F} = (e^{-y} - ze^{-x})\mathbf{i} + (e^{-z} - xe^{-y})\mathbf{j} + (e^{-x} - ye^{-z})\mathbf{k}$
   (b) $\mathbf{F} = (x^2 - xy)\mathbf{i} + (y^2 - xy)\mathbf{j}$

5. (★) Determine the values of A and B for which the vector field below is conservative:

$$\mathbf{F}(x,y,z) = Ax \ln z \ \mathbf{i} + By^2 z \ \mathbf{j} + \left(\frac{x^2}{z} + y^3\right) \mathbf{k},$$

   where the domain of $\mathbf{F}$ is the upper-half space $\{(x,y,z) : z > 0\}$.
   For each such pair of A and B, find the potential function f for the vector field.

6. (★★) Consider the path C:

$$\mathbf{r}(t) = \left(\cos^{2M} t\right) \mathbf{i} + \left(\sin^N t\right) \mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi.$$

   Here M is the age of the Earth, and N is the age of the Universe. Assume both M and N are positive finite integers.
   Evaluate the line integral:

$$\int_C \left(e^{-y} - ze^{-x}\right) \, dx + \left(e^{-z} - xe^{-y}\right) \, dy + \left(e^{-x} - ye^{-z}\right) \, dz$$

   Provide **TWO** different solutions to this problem.

7. (★★) Given a conservative vector field $\mathbf{F}$ in $\mathbb{R}^3$, the potential *energy* of $\mathbf{F}$ is a scalar-valued function $V(x,y,z)$ such that $\mathbf{F} = -\nabla V$. Suppose $\mathbf{r}(t)$ is the path of a particle with mass m traveling in accordance to the Newton's Second Law $\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$. Then its kinetic energy is defined to be:

$$\text{KE} = \frac{1}{2} m \left|\mathbf{r}'(t)\right|^2.$$

   The total (kinetic + potential) energy of the particle at time t is therefore given by:

$$E(t) := \frac{1}{2} m \left|\mathbf{r}'(t)\right|^2 + V(\mathbf{r}(t)).$$

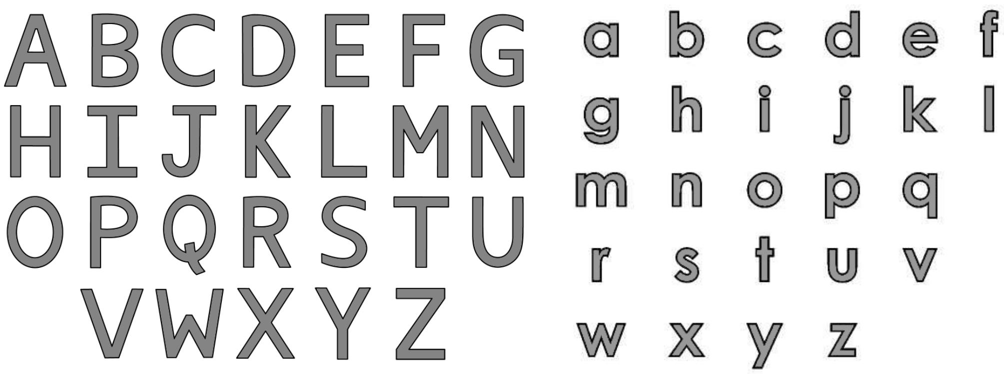
   Show that the total energy is conserved, i.e. $E'(t) = 0$ for all time t.
   [Hint: the only fact you need to know about Physics is the Newton's Second Law given above. It is purely a math problem.]

8. (★★) Denote $\mathbf{e}_\rho = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$ and $\mathbf{e}_r = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$, which are the unit radial vector fields in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

   (a) Show that if $\mathbf{F}(x,y,z) = f(\rho)\mathbf{e}_\rho$ where f is a function depending only on $\rho = \sqrt{x^2+y^2+z^2}$, then $\nabla \times \mathbf{F} = \mathbf{0}$ on the domain of $\mathbf{F}$. Is this result alone sufficient to claim that $\mathbf{F}$ is conservative?

   (b) Show that if $\mathbf{G}(x,y) = g(r)\mathbf{e}_r$ where g is a function depending only on $r = \sqrt{x^2+y^2}$, then $\nabla \times \mathbf{G} = \mathbf{0}$ on the domain of $\mathbf{G}$. Is this result alone sufficient to claim that $\mathbf{G}$ is conservative?

9. (★) Regard each English letter as a solid region in $\mathbb{R}^2$. Which capital letters are simply-connected? Which small letters are simply-connected?



10. (★★) The notation $\mathbb{R}^3 \backslash X$ means the xyz-space $\mathbb{R}^3$ with the set X removed. Determine whether $\mathbb{R}^3 \backslash X$ is simply-connected when X is each of the following:

    (a) X is the origin

    (b) X is the entire y-axis

    (c) X is the positive y-axis

    (d) X is the solid sphere $x^2 + y^2 + z^2 \le 1$

    (e) X is the surface sphere $x^2 + y^2 + z^2 = 1$

    (f) X is the solid cylinder $x^2 + y^2 \le 1$

    (g) X is the solid half-cylinder $x^2 + y^2 \le 1$ and $z \ge 0$.

    (h) X is the surface cylinder $x^2 + y^2 = 1$

    (i) X is the surface half-cylinder $x^2 + y^2 = 1$ and $z \ge 0$

    (j) X is a solid torus

    (k) X is a surface torus

    (l) X is a simple closed curve

    Give an example of a proper subset X of $\mathbb{R}^3$ such that both X and $\mathbb{R}^3 \backslash X$ are simply-connected. [Note: "proper" means X cannot be empty, and cannot be the whole $\mathbb{R}^3$.]

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #8 • Green's Theorem

1. (★) Use the Green's Theorem to evaluate

$$\oint_C (4y^2 + e^{x^2})\,dx - (2x + e^{y^2})\,dy$$

where C is each of the following (assume C is counter-clockwise oriented):

(a) the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$

(b) the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$

(c) the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$

(d) the unit circle $x^2 + y^2 = 1$

2. (★★) The purpose of this problem is to explore a line integral for computing areas.

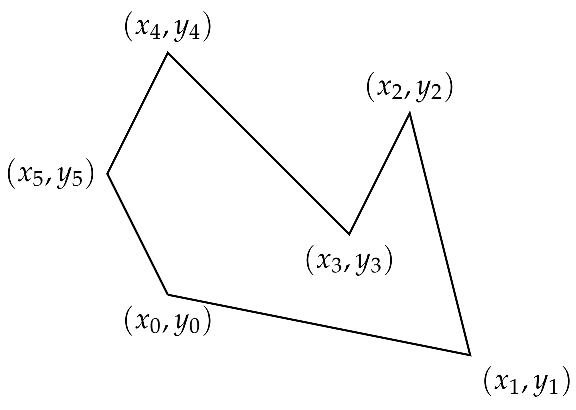
(a) Let C be a simple closed curve in $\mathbb{R}^2$ and the area enclosed by C is denoted by A. Show that:

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy$$

(b) Let E be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a, b > 0$. Find the area bounded by E using the result of (a).

(c) Let P be a n-sided polygon with vertices (x_0, y_0), (x_1, y_1), $\ldots$, (x_{n-1}, y_{n-1}). See the figure below for an example when $n = 6$. For convenience, we denote $(x_n, y_n) = (x_0, y_0)$. Using (a), show that the area $A(P)$ bounded by the polygon P is given by:

$$A(P) = \frac{1}{2}\sum_{i=1}^{n}(x_{i-1}y_i - x_i y_{i-1})\,.$$



3. (★★) Consider the following system of differential equations:

$$\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y)$$

where f and g are C^1 on $\mathbb{R}^2$. Given that $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} > 0$ on $\mathbb{R}^2$, show that the system cannot have a non-constant periodic solution. We say a solution $(x(t), y(t))$ is periodic if there exists $T > 0$ such that $(x(0), y(0)) = (x(T), y(T))$.

Hint: Proof by contradiction. Apply Green's Theorem on $\mathbf{F} = -g(x,y)\mathbf{i} + f(x,y)\mathbf{j}$.

4. (★★) Consider the vector field $\mathbf{F} = -\dfrac{y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j}$ which is defined at every point on $\mathbb{R}^2$ except the origin.

   (a) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$ at every point in $\mathbb{R}^2$ except the origin.

   (b) Show, by direct computation, that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is non-zero where C is the unit circle, counter-clockwise oriented, with centered at the origin.

   (c) The following students are confused about the above vector field $\mathbf{F}$ in relation to some facts and theorems stated in class. Pretend that you are a teaching assistant of this course, point out their misconceptions.

      i. Student A said, "Given that $\nabla \times \mathbf{F} = \mathbf{0}$, the Curl Test asserts that $\mathbf{F}$ is conservative and so the closed-path line integral in (b) should be zero. How come the answer for (b) is non-zero???!!!"

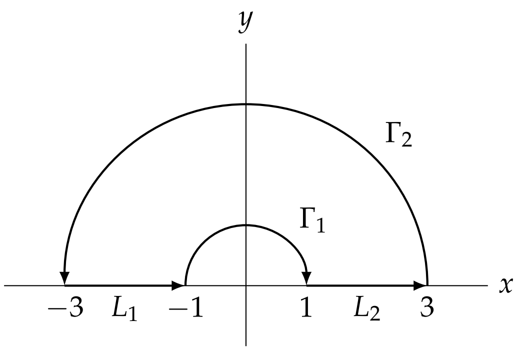
      ii. Student B said, "Given that $\nabla \times \mathbf{F} = \mathbf{0}$, the Green's Theorem asserts that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_R \mathbf{0} \cdot \mathbf{k}\, dA = 0$$

      for any closed-path C. Why can the answer in (b) be non-zero???!!!" "

      iii. Student C said, "It can be verified that $\mathbf{F} = \nabla \left(\tan^{-1} \dfrac{y}{x}\right)$ and so $\mathbf{F}$ is conservative with potential function $f(x,y) = \tan^{-1} \dfrac{y}{x}$. Any line integral of a conservative vector field over a closed curve must be zero. How come can the closed-path integral in (b) be non-zero???!!!"

5. (★★) In the figure shown below, Γ_1 and Γ_2 are circular arcs centered at the origin. L_1 and L_2 are straight-lines. Consider the closed path $C = L_1 + \Gamma_1 + L_2 + \Gamma_2$.



Compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ of each vector field below using the Green's Theorem in an *appropriate* way:

   (a) $\mathbf{F} = y^3\mathbf{i} - x^3\mathbf{j}$

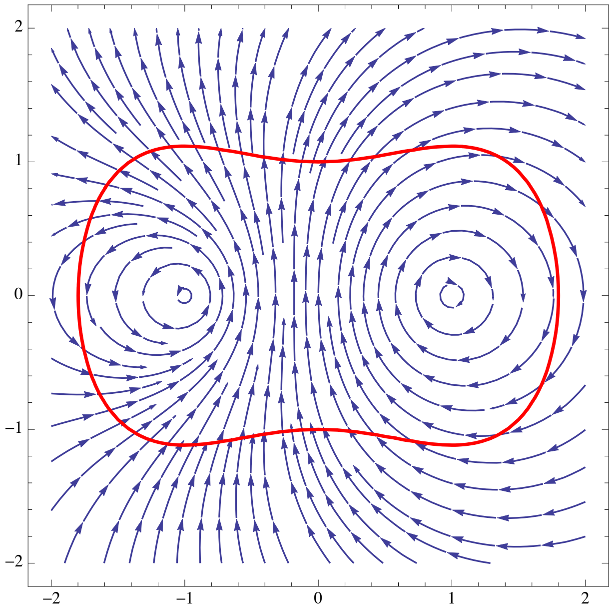
   (b) $\mathbf{F} = -\dfrac{y-3}{(x-3)^2+(y-3)^2}\mathbf{i} + \dfrac{x-3}{(x-3)^2+(y-3)^2}\mathbf{j}$

   (c) $\mathbf{F} = -\dfrac{y-2}{x^2+(y-2)^2}\mathbf{i} + \dfrac{x}{x^2+(y-2)^2}\mathbf{j}$

6. (★★) Consider the flow of fluid (shown in blue in the figure below) which is represented by the vector field:

$$\mathbf{F} = \left(-\frac{y}{(x+1)^2+y^2} + \frac{2y}{(x-1)^2+y^2}\right)\mathbf{i} + \left(\frac{x+1}{(x+1)^2+y^2} - \frac{2(x-1)}{(x-1)^2+y^2}\right)\mathbf{j}$$

C is an arbitrary simple closed curve (red in the figure) which encloses all points at which $\mathbf{F}$ is not defined.



(a) At which point(s) the vector field $\mathbf{F}$ is/are *not* defined? Is the domain of $\mathbf{F}$ simply-connected?

(b) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$ at every point in $\mathbb{R}^2$ where $\mathbf{F}$ is defined.

(c) Show that from the definition of line integrals:

i. $\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for any counter-clockwise circle Γ centered at $(-1, 0)$ with radius less than 2.

ii. $\oint_\gamma \mathbf{F} \cdot d\mathbf{r} = -4\pi$ for any counter-clockwise circle γ centered at $(1, 0)$ with radius less than 2.

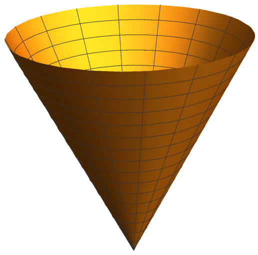
(d) Using the above results, show that:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = -2\pi$$

for any simple closed curve C in $\mathbb{R}^2$ that encloses all points at which $\mathbf{F}$ is not defined.

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #9 • Surface Integrals, Stokes' Theorem

1. (★) Consider the right circular cone surface (just the *shell*, and the flat top is *not* included) with base radius R and height h, and with z-axis as the central axis and the origin as the vertex. See the figure below):



Suppose the cone has uniform surface density σ and its total mass is m.

(a) Write down a parametrization $\mathbf{r}(u, v)$ of the cone, and indicate the range of the parameters. It is OK to use other letters for the pair of parameters.

(b) Find the surface area of the cone.

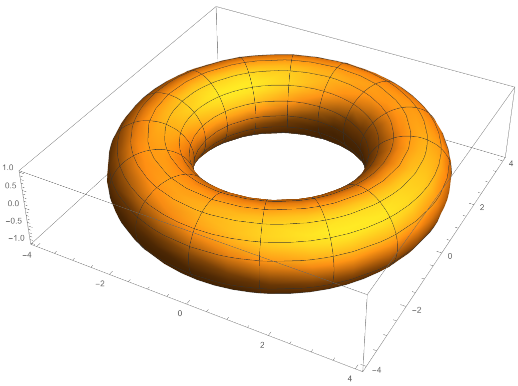
(c) Find the moment of inertia $I_z := \iint_S (x^2 + y^2)\sigma\, dS$. about the z-axis. Express your final answer in terms of the mass m.

(d) Compute the surface flux of the vector field $\mathbf{F} = \mathbf{i}$ through the cone. Choose $\hat{\mathbf{n}}$ to be the upward unit normal. Do not use Stokes' Theorem in this problem.

2. (★) Consider the parametrization of a torus (i.e. donut):

$$\mathbf{r}(u, v) = ((R + a\cos u)\cos v)\ \mathbf{i} + ((R + a\cos u)\sin v)\ \mathbf{j} + (a\sin u)\ \mathbf{k}$$

where $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$. Here R and a are constants such that $R > a > 0$.



Suppose the torus has uniform surface density σ and its total mass is m.

(a) Find the surface area of the torus.

(b) Find the moment of inertia $I_z := \oiint_S (x^2 + y^2)\sigma\, dS$ about the z-axis. Express your final answer in terms of m.

(c) Compute the surface flux of the vector field $\mathbf{F} = \mathbf{k}$ through the torus. Choose $\hat{\mathbf{n}}$ to be the outward unit normal. Do not use Stokes' Theorem in this problem.

3. (★★) In Chapter 2, we claimed without proof that $\nabla f(P)$ is perpendicular to the level surface $f = c$ at P (we proved the case of level *curves* only). In this problem, we are going to complete the proof for level surfaces.

   Let $f(x, y, z)$ be a C^1 function, and S be the level surface $f(x, y, z) = c$. Consider a parametrization $\mathbf{r}(u, v)$ for S, then if one can show $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are both perpendicular to ∇f, then we are done because the normal vector $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ to the surface S will then be parallel to ∇f. By considering $f(\mathbf{r}(u, v)) = c$, show that $\nabla f \cdot \frac{\partial \mathbf{r}}{\partial u} = 0$.

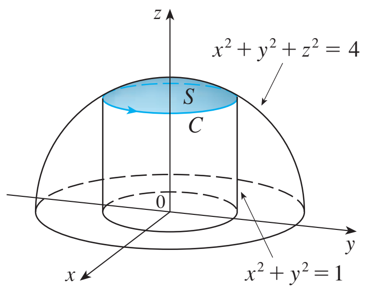
   [The fact that $\nabla f \cdot \frac{\partial \mathbf{r}}{\partial v} = 0$ can be shown in a similar way.]

4. (★) Suppose S is a level surface $f(x, y, z) = c$ of a C^1 function f. Show that:

$$\iint_S \nabla f \cdot \hat{\mathbf{n}}\, dS = \pm \iint_S |\nabla f|\, dS$$

   where $\pm$ depends on the choice of unit normal $\hat{\mathbf{n}}$.

5. (★) Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. Denote C to be the boundary of S with orientation indicated in the diagram below:



   (a) Write down the parametrizations of both the surface S and the curve C. For the surface S, choose a *suitable* coordinate system so that the parameters have constant bounds.

   (b) Consider the vector field $\mathbf{F}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$. Compute both $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$ directly. Verify that they are equal.

6. (★) Let C be the simple closed curve given parametrized by:

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}, \quad 0 \le t \le 2\pi.$$

   (a) Show that the curve lies on the surface $z = 2xy$.

   (b) Use the Stokes' Theorem to evaluate the line integral:

$$\oint_C e^{x^2} dx + yzdy + \frac{x^2}{2} dz.$$

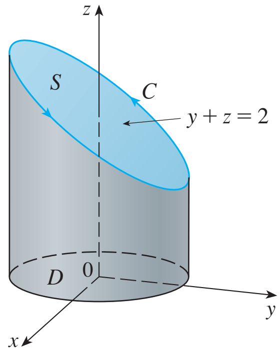
   [Why is it difficult to compute this line integral *directly*?]

7. (★) Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$. Show that the line integral

$$\oint_C 2ydx + 3zdy - xdz$$

   depends only on the area of the region enclosed by C on the above given plane and the orientation of C, but not on the position or shape of C.

8. (★★) Consider the curve of intersection C of the plane $y+z=2$ and the cylinder $x^2+y^2=1$, with orientation shown in the diagram below. The surface S is the planar region enclosed by C, and its projection onto the xy-plane is denoted by D.



(a) Using a suitable coordinate system, write down a parametrization of S such that the parameters have constant bounds.

(b) Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}$ is given by:

$$\mathbf{F}(x, y, z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$$

(c) Let $\mathbf{G}(x, y, z) = -\dfrac{y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j} + \dfrac{1}{z+1}\mathbf{k}$.

i. Verify that $\nabla \times \mathbf{G} = \mathbf{0}$ at every point in the domain of $\mathbf{G}$. Does this result determine that $\mathbf{G}$ is conservative or not?

ii. Denote Γ to be the projection of C onto the xy-plane. Using the Stokes' Theorem in an *appropriate* way, show that:

$$\oint_C \mathbf{G} \cdot d\mathbf{r} = \oint_\Gamma \mathbf{G} \cdot d\mathbf{r}.$$

iii. Evaluate $\oint_C \mathbf{G} \cdot d\mathbf{r}$ using the above result.

9. (★) Two of the four Maxwell's Equations (Faraday's and Ampère's Laws) assert that:

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the current, and c, μ_0 and ε_0 are positive constants. Using Stokes' Theorem, show that for any (stationary) orientable surface S with boundary C, we have:

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{1}{c}\frac{\partial}{\partial t}\iint_S \mathbf{B} \cdot \hat{\mathbf{n}}\, dS$$
$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \iint_S \mathbf{J} \cdot \hat{\mathbf{n}}\, dS + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\iint_S \mathbf{E} \cdot \hat{\mathbf{n}}\, dS$$

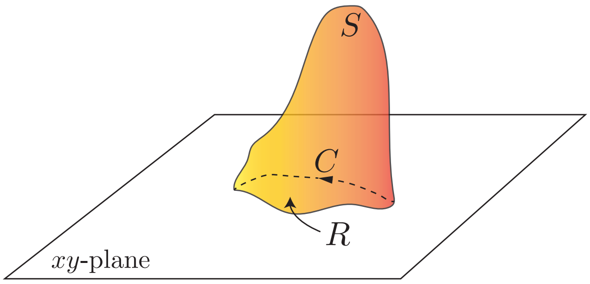
[You don't need to know any physics to do this problem.]

10. (★★) Let $\mathbf{F}(x,y,z) = \langle 0, -\frac{z}{2}, \frac{y}{2} \rangle$.

   (a) Show that $\nabla \times \mathbf{F} = \mathbf{i}$.

   (b) Find vector fields $\mathbf{G}$ and $\mathbf{H}$ such that $\nabla \times \mathbf{G} = \mathbf{j}$ and $\nabla \times \mathbf{H} = \mathbf{k}$.

   (c) Let C be an arbitrary simple closed curve on the xy-plane in the three dimensional space, and S is any surface *above* the xy-plane with boundary curve C. See the figure below.



   Show that:

$$\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = c \times \text{area of the region on the } xy\text{-plane enclosed by } C.$$

   Here a, b and c are all constants.

   (d) Using the results of (a), (b), and the Stokes' Theorem, redo Problems #1(d) and #2(c).

Optional – about the Gauss-Bonnet's Theorem

11. Given a oriented surface S with parametrization $\mathbf{r}(u,v)$, we denote:

$$E = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u} \qquad F = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \qquad G = \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial v}$$

$$e = \frac{\partial^2 \mathbf{r}}{\partial u^2} \cdot \hat{\mathbf{n}} \qquad f = \frac{\partial^2 \mathbf{r}}{\partial u \partial v} \cdot \hat{\mathbf{n}} \qquad g = \frac{\partial^2 \mathbf{r}}{\partial v^2} \cdot \hat{\mathbf{n}}$$

The *Gauss curvature* at the point $\mathbf{r}(u,v)$ is defined to be:

$$K(u,v) := \frac{eg - f^2}{EG - F^2}.$$

The geometric intuition behind the Gauss curvature *may* be covered in MATH 4223. In Differential Geometry, there is a beautiful theorem – the Gauss-Bonnet's Theorem – which asserts that if S is closed, oriented and smooth (without corners), then:

$$\oiint_S K\, dS = 4\pi(1 - \text{ number of holes of } S)$$

Therefore, if S is a sphere, the above surface integral should be 4π as there is no hole. If S is a torus (which has one hole), the above surface integral should be 0. Verify this theorem for the sphere and torus, by parametrizing them and compute the above integral directly over the sphere and the torus.

As an optional problem, you may use computer softwares to ease your calculations.

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #10 • Divergence Theorem

1. (★) Use the Divergence Theorem to find the outward flux $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}}\, dS$ for each of the following $\mathbf{F}$ and S:

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the surface of any square cube of length b.

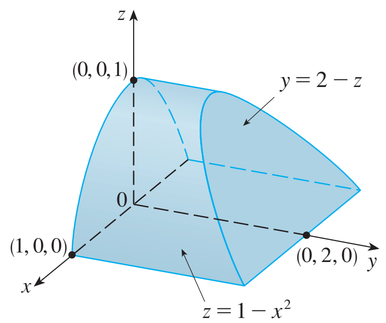
   (b) $\mathbf{F} = x^3\mathbf{i} + 3yz^2\mathbf{j} + (3y^2z + x^2)\mathbf{k}$ and S is the sphere with radius $a > 0$ centered at the origin.

   (c) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S is the boundary surface of the cylinder D defined by $x^2 + y^2 \leq 1$ and $0 \leq z \leq 4$.

2. (★) Evaluate $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}}\, dS$ where

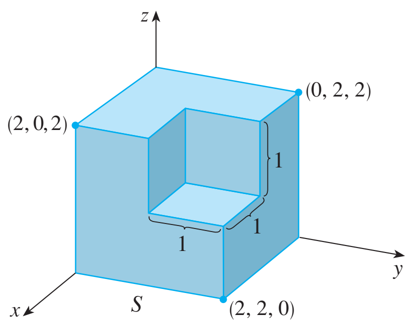
$$\mathbf{F} = xy\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\mathbf{k}$$

and S is the surface boundary of the region D defined by $z \leq 1 - x^2$, $z \geq 0$, $y \geq 0$ and $y \leq 2 - z$. See the figure below:



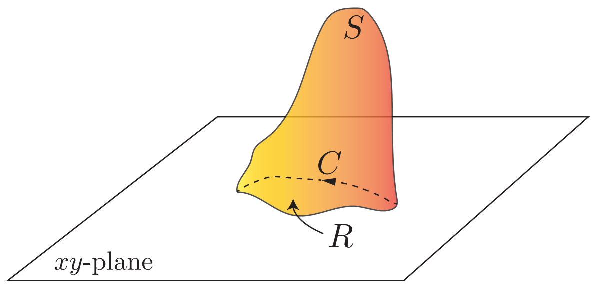
Comment on why it is preferable to use the Divergence Theorem instead of computing the surface flux directly.

3. (★) Let D be the solid square cube of length 2 with one corner unit cube removed. See the figure below.



Evaluate the outward flux $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}}\, dS$ where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Comment on why it is preferable to use the Divergence Theorem instead of computing the flux directly.

4. (★★) Let C be an arbitrary simple closed curve on the xy-plane in the three dimensional space, and S is any surface *above* the xy-plane with boundary curve C. See the figure below.



Using the Divergence Theorem, show that:

$$\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = c \times \text{area of the region on the } xy\text{-plane enclosed by } C.$$

Here a, b and c are all constants.

5. (★★) Suppose $f(x, y, z)$ is a C^2 function on $\mathbb{R}^3$ such that $\nabla^2 f(x, y, z) = 0$ on $\mathbb{R}^3$. Here $\nabla^2 f$ means the Laplacian of f, i.e. $\nabla^2 f = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}$.
   (a) Show that:
   $$\oiint_S f\nabla f \cdot \hat{\mathbf{n}}\, dS = \iiint_D |\nabla f|^2\, dV$$
   for any closed oriented surface S enclosing the solid region D.
   (b) If, furthermore, assume that $f(x, y, z) = 0$ for any (x, y, z) on S, what can you say about $f(x, y, z)$ for any (x, y, z) in D?

6. (★★) Suppose S is a closed oriented level surface $f(x, y, z) = c$ of a C^2 function f. Denote D to be the solid enclosed by S. Show that:
   $$\oiint_S |\nabla f|\, dS = \pm \iiint_D \nabla^2 f\, dV$$
   where $\pm$ depends on whether ∇f points inward or outward on the surface S.

7. (★★) Given two C^2 functions $u(x, y, z)$ and $v(x, y, z)$ defined on $\mathbb{R}^3$. Let S be a closed oriented surface and D is the solid enclosed by S.
   (a) Rewrite $\nabla \cdot (u\nabla v - v\nabla u)$ using **curl**, **grad** and **div**.
   (b) Show that
   $$\oiint_S (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}}\, dS = \iiint_D \left(u\nabla^2 v - v\nabla^2 u\right) dV$$
   (c) Assume further that $\nabla u(x, y, z) \cdot \hat{\mathbf{n}} = \nabla v(x, y, z) \cdot \hat{\mathbf{n}} = 0$ for any (x, y, z) on S, show that
   $$\iiint_D u\nabla^2 v\, dV = \iiint_D v\nabla^2 u\, dV.$$
   [FYI: Using the language in Linear Algebra or Functional Analysis, this result asserts that the Laplace operator ∇^2 is *self-adjoint* with respect to the L^2-inner product on C^2 functions under the boundary condition $D_{\hat{\mathbf{n}}} = 0$.]