

# 1 Review

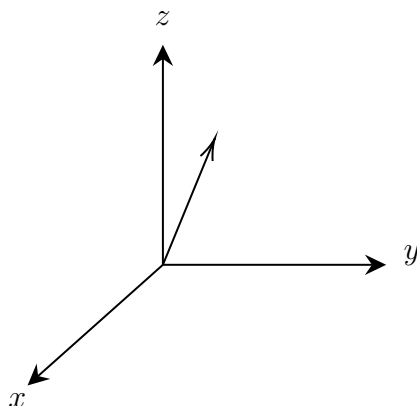
In the following we will assume  $V$  to be a 3-dimensional real vector space (A rank 3 free  $\mathbb{R}$ -module :D).

- **Scalar:**

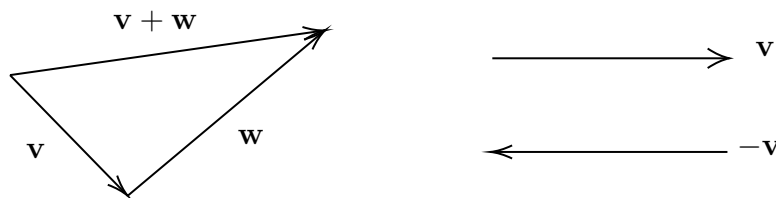
- Is an *one*-entry object belongs to  $\mathbb{R}$ .
- Represent a quantity.
- *Ordered*.

- **Vector:**

- Is a *three*-entry object represented by  $\mathbf{x} = (x_1, x_2, x_3) = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ , which represent an "arrow" in 3D space.



- Addition of vector: follows the head to tail rules.



- The **norm**  $\|\cdot\| : V \rightarrow \mathbb{R}$  is a function which measures the *length* of the arrow. It is defined by  $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$  (in our consideration).
- Property:  $\|\mathbf{v}\| = 0 \Leftrightarrow \mathbf{v} = \mathbf{0} = (0, 0, 0)$
- **Unit Vector** is a vector with norm 1.
- $\mathbf{v}_1$  and  $\mathbf{v}_2$  are **linearly dependent** if  $\mathbf{v}_1 = \alpha\mathbf{v}_2$  for some  $\alpha \in \mathbb{R}$ .

- Two vectors are said to be **orthogonal** if the angle in between them is  $\pi/2$ .
- **NOT** ordered.

- **Determinant**

- for  $2 \times 2$  matrix,  $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
- for  $3 \times 3$  matrix,  $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$

- **Dot product:**

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$$" \cdot " : V \times V \rightarrow \mathbb{R}$$

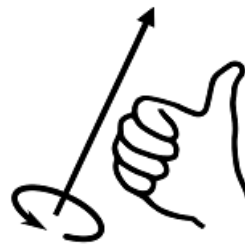
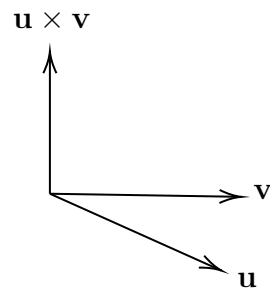
$$(\mathbf{v}_1, \mathbf{v}_2) \mapsto \mathbf{v}_1 \cdot \mathbf{v}_2 := v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z} = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos \theta$$

- Note that  $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$ .
- Dot product of *orthogonal* vectors is 0.
- It represent the length of projection of  $\mathbf{v}_1$  on  $\mathbf{v}_2$ .

- **Cross product:**

$$" \times " : (\mathbf{v}_1, \mathbf{v}_2) \in V \times V \mapsto \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_{1x} & v_{1y} & v_{1z} \\ v_{2x} & v_{2y} & v_{2z} \end{bmatrix} \in V.$$

- Cross product gives a vector which is *perpendicular* to both of the given vectors.
- $\|\mathbf{v}_1 \times \mathbf{v}_2\| = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \sin \theta$
- Cross product of *linearly dependent* vectors is 0.
- A handy convention concerning the direction of cross product can be found by the **right-hand grip rule**.



Procedures:

1. Point your index finger in the direction of the *first* vector of the cross product.

2. Curl your index finger (in the natural direction) towards the direction of the *second* vector.
3. The direction of your thumb in the process give you the direction of the cross product.

Remember Remember Remember (so important that it has to be mentioned three times) that “like ” has to be given with **right hand** instead of left hand.

- The way to find a equation of **line** passing through *two* points:
  1. Given points  $A, B$ , we can find the vector  $\overrightarrow{AB}$ .
  2. The equation of line  $\overline{AB}$  is given by  $\overrightarrow{OA} + t\overrightarrow{AB}$ .
- The way to find a equation of **plane** passing through *three* points:
  1. Given points  $A, B$  and  $C$ , we can find vectors  $\overrightarrow{AB}$  and  $\overrightarrow{AC}$ .
  2. The normal vector of the plane is given by  $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$ .
  3. If  $P = (x, y, z)$  is a point on the plane, then  $\overrightarrow{AP} \perp \mathbf{n}$ , so  $\overrightarrow{AP} \cdot \mathbf{n} = 0$ , which gives the equation of plane.

## 2 Problems

1. True or False

(a)  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ .

(b) If  $\mathbf{u} \cdot \mathbf{v} = 0$  and  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$  then  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{v} = \mathbf{0}$ .

(c) If  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$  then  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{v} = \mathbf{0}$ .

(d) If  $\mathbf{u} \cdot \mathbf{v} = 0$  then  $\mathbf{u} = \mathbf{0}$  or  $\mathbf{v} = \mathbf{0}$ .

(e) For any vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$ ,  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ .

2. Compute the angle between  $\mathbf{v}_1 = (6, 2, 3)$  and  $\mathbf{v}_2 = (5, -1, 4)$ .

3. Compute the cross product of  $\mathbf{v}_1 = (6, 2, 3)$  and  $\mathbf{v}_2 = (5, -1, 4)$ .

4. Compute the determinant of the matrix  $\begin{bmatrix} 6 & 2 & 3 \\ 5 & -1 & 4 \\ 1 & 2 & 3 \end{bmatrix}$ .

5. Express the volume of the parallelepiped with vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  as the three edges sharing the same vertex.

6. Which of the points  $P = (6, 2, 3)$ ,  $Q = (5, -1, 4)$  and  $R = (0, 3, 8)$ , is closest to the plane  $x + 4y - 3z = 1$ ? Which point lies in the  $yz$ -plane?

7. Find the parametric equation line through  $(4, 1, -2)$  and  $(1, 2, 5)$ .
8. Find the plane through  $(2, 1, 0)$  and parallel to  $x + 4y - 3z = 1$ .
9. Find an equation of the plane through the line of intersection of the planes  $x - z = 1$  and  $y + 2z = 3$  and perpendicular to the plane  $x + y - 2z = 1$ .
10. Find a vector perpendicular to the plane through the points  $A = (1, 0, 0)$ ,  $B = (2, 0, -1)$ ,  $C = (1, 4, 3)$ . Find the area of the triangle  $ABC$ .
11. Find the point in which the line with parametric equations  $x = 2 - t$ ,  $y = 1 + 3t$ ,  $z = 4t$  intersects the plane  $2x - y + z = 2$ .
12. Determine whether the following pair of lines are parallel, skew, or intersecting. If intersect, find the point of intersection.
- (a)  $L_1 : x = -6t, y = 1 + 9t, z = -3t$ ,  
 $L_2 : x = 1 + 2s, y = 4 - 3s, z = s$ .

$$\begin{aligned} \text{(b)} \quad L_1 : \frac{x}{1} &= \frac{y-1}{2} = \frac{z-2}{3} \\ L_2 : \frac{x-3}{-4} &= \frac{y-2}{-3} = \frac{z-1}{2} \end{aligned}$$

13. Find the angle between two planes.