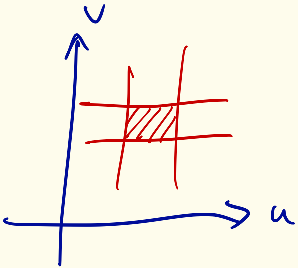
Last Time

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \underbrace{(\nabla\times\vec{F})\cdot\vec{k}}_{\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\vec{k}}\, dA$$

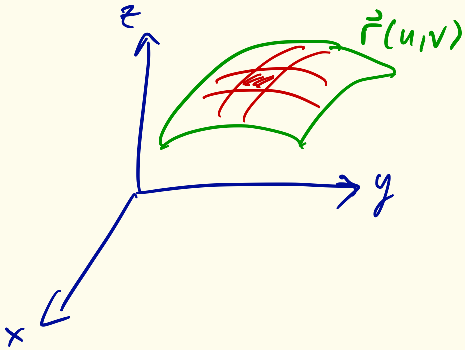
generalize surface integral in 3D

$\Longrightarrow$ parametric surfaces.

$$\vec{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle$$



input

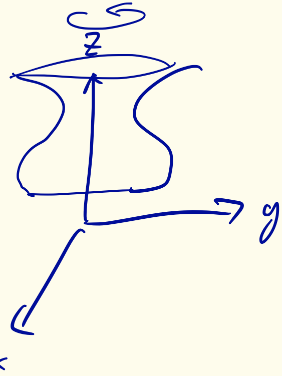


output

Ex **Graphs** $z = f(x,y)$

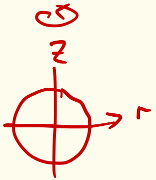
$$\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$$

Ex **Surface of Revolutions**



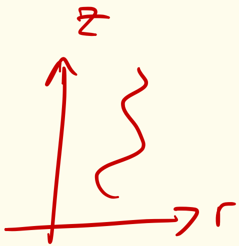
$$\vec{r}(u,v) = \langle f(u)\cos v, f(u)\sin v, g(u) \rangle$$

rotating about z-axis.



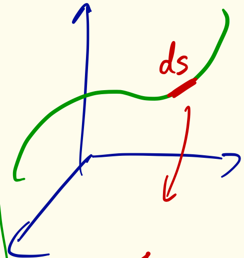
$$\begin{cases} r = f(u) \\ z = g(u) \end{cases}$$

shape you are rotating.



Ex **Sphere of radius ρ**: $\vec{r}(\phi,\theta) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi \rangle$

Arc Length: $\int_C ds = \int_a^b |r'(t)|\,dt$



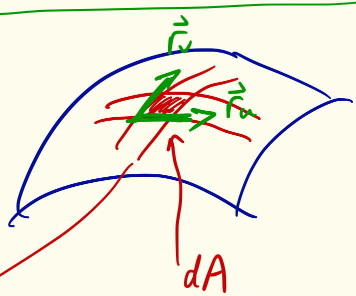
$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

$$= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}\,\Delta t$$

$$\approx \sqrt{(x')^2 + (y')^2 + (z')^2}\,dt$$

$$= |r'(t)|\,dt$$

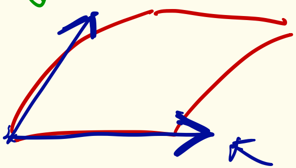
Surface Area: $\iint_D dA$



normal vector of tangent plane: $\vec r_u \times \vec r_v$

$\vec r(u, v+\Delta v) - \vec r(u,v)$

$\approx \vec r_v \cdot \Delta v$



$\leftarrow \vec r(u+\Delta u, v) - \vec r(u,v)$

$$= \frac{\vec r(u+\Delta u, v) - \vec r(u,v)}{\Delta u}\,\Delta u$$

$\Delta u \to 0: \quad \vec r_u \cdot \Delta u$

$\Delta A = \text{Area}(▱) = |\vec r_u \times \vec r_v|\,\Delta u\,\Delta v$

Area element: $dA = |\vec{r}_u \times \vec{r}_v| \, du\, dv.$

<u>Surface Area</u> : $\iint_D dA = \iint_D |\vec{r}_u \times \vec{r}_v| \, du\, dv$

appropriate domain for u, v.

<u>Ex</u> Sphere of radius ρ. $(4\pi\rho^2)$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$\vec{r}(\phi, \theta) = \langle \rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi \rangle$

$$\left| \begin{array}{lccc} & i & j & k \\ \vec{r}_\phi = & \langle \rho\cos\phi\cos\theta, & \rho\cos\phi\sin\theta, & -\rho\sin\phi \rangle \\ \vec{r}_\theta = & \langle -\rho\sin\phi\sin\theta, & \rho\sin\phi\cos\theta, & 0 \rangle \end{array} \right|$$

$$\vec{r}_\phi \times \vec{r}_\theta = \langle \rho^2 \sin^2\phi \cos\theta, \rho^2\sin^2\phi\sin\theta, \rho^2\cos\phi\sin\phi\,\cancel{(\cos^2\theta + \sin^2\theta)} \rangle$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{\rho^4\sin^4\phi\cos^2\theta + \rho^4\sin^4\phi\sin^2\theta + \rho^4\cos^2\phi\sin^2\phi} = \rho^2\sin\phi$$

Surface area of Sphere :

$$\int_0^{2\pi}\int_0^{\pi} \rho^2 \sin\phi \, d\phi \, d\theta$$

$$2\pi\rho^2 \underbrace{\int_0^{\pi} \sin\phi \, d\phi}_{=2}$$

$$= 4\pi\rho^2$$

$z = f(x,y)$: $\vec{r}(u,v) = \langle u, v, f(u,v) \rangle$

$$\begin{array}{ll} & \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \end{array} \\ \vec{r}_u = & \langle 1, 0, f_u \rangle \\ \vec{r}_v = & \langle 0, 1, f_v \rangle \end{array}$$

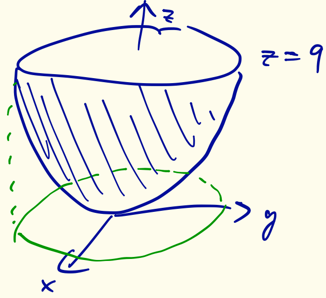
$\vec{r}_u \times \vec{r}_v = \langle -f_u, -f_v, 1 \rangle$ ← upward pointing normal.

$|\vec{r}_u \times \vec{r}_v| = \sqrt{f_u^2 + f_v^2 + 1}$

Surface area of a graph $z = f(x,y)$ is

$$A(S) = \iint_D |\vec{r}_x \times \vec{r}_y| \, dA = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dx\, dy.$$

Ex $\quad z = x^2+y^2 \;,\; 0 \le z \le 9$



$D : x^2+y^2 \le 9$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1+(2x)^2+(2y)^2}$$

$$\iint_D \sqrt{1+4x^2+4y^2}\, dA \overset{\text{polar}}{=} \int_0^3 \int_0^{2\pi} \sqrt{1+4r^2}\, r\, dr\, d\theta$$

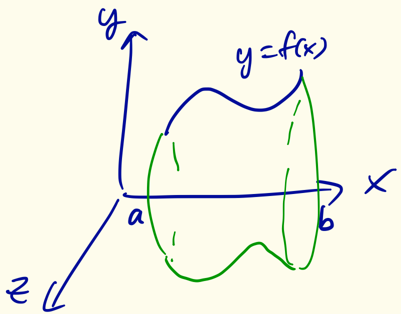
$$= 2\pi \int_0^3 \sqrt{1+4r^2}\, r\, dr \qquad \text{let } u = r^2 \,,\; du = 2r\,dr$$

$$= \int_0^9 \sqrt{1+4u}\, \frac{du}{2} = (1+4u)^{3/2} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{4} \Big|_0^9$$

$$= \ldots$$

Made with Goodnotes

Ex Surface of Revolution: around x-axis



$$\vec{r}(x,\theta) = \langle x,\ f(x)\cos\theta,\ f(x)\sin\theta \rangle$$

$$\begin{array}{ccc} & i & j & k \\ \vec{r}_x = & \langle 1, & f'(x)\cos\theta, & f'(x)\sin\theta \rangle \\ \vec{r}_\theta = & \langle 0, & -f(x)\sin\theta, & f(x)\cos\theta \rangle \end{array}$$

$$\vec{r}_x \times \vec{r}_\theta = \langle f'(x)f(x),\ -f(x)\cos\theta,\ -f(x)\sin\theta \rangle$$

$$|\vec{r}_x \times \vec{r}_\theta| = \sqrt{(f'(x)f(x))^2 + f(x)^2} = f(x)\sqrt{1+f'(x)^2}$$

$$A(S) = \iint f(x)\sqrt{1+(f'(x))^2}\,dx\,d\theta = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$$

Def Surface Integral of $f(x,y,z)$:

$$\iint f(x,y,z) \times (\text{area of patch})$$

$$= \iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA.$$

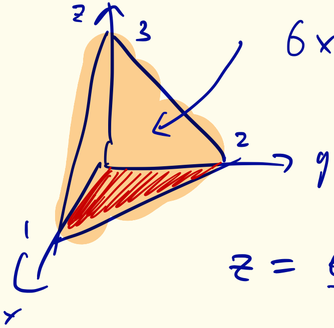
Special Case : $f(x,y,z) = 1$: Surface Area.

If f is "thickness" $\Rightarrow$ $\iint f\, dS$ = Volume of the shell.

Special Case $z = z(x,y)$

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,z(x,y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

Ex $\iint_S x\, dS$, S

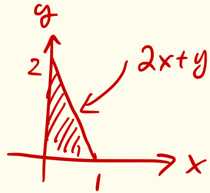


$6x+3y+2z=6.$

$z = \dfrac{6-6x-3y}{2}$

$$\iint_D x\sqrt{1+(-3)^2+\left(-\tfrac{3}{2}\right)^2}\, dA$$

D :



$$= \int_0^1 \int_0^{2-2x} x\sqrt{\frac{49}{4}}\, dy\, dx$$

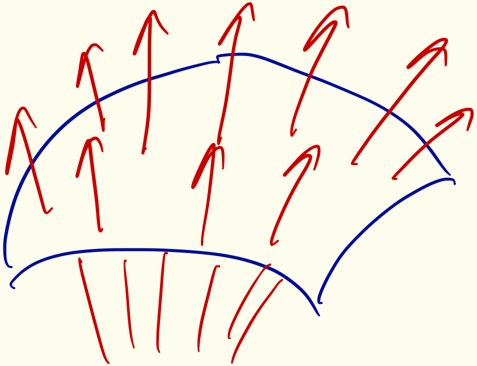
$$= \int_0^1 (2-2x)\, x\, \frac{7}{2}\, dx$$

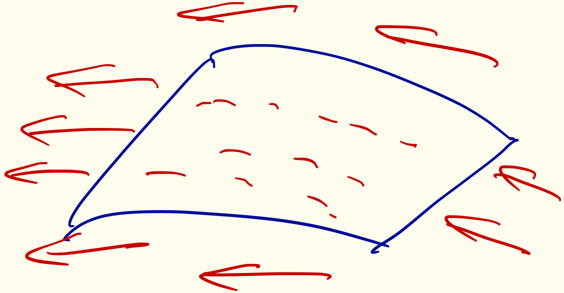
$= \cdots$

Surface Integral for Vector Fields

= measure the "flux" through the surface

= rate of flow (mass/time)





no flow across surface!

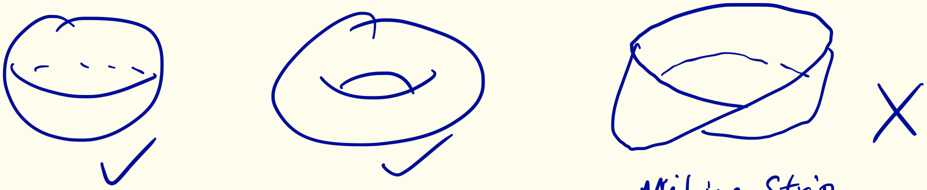
"Add up the component of vector $\perp$ surface"

Def $$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S \vec{F}\cdot d\vec{S}$$

unit normal to S.

up or down ?

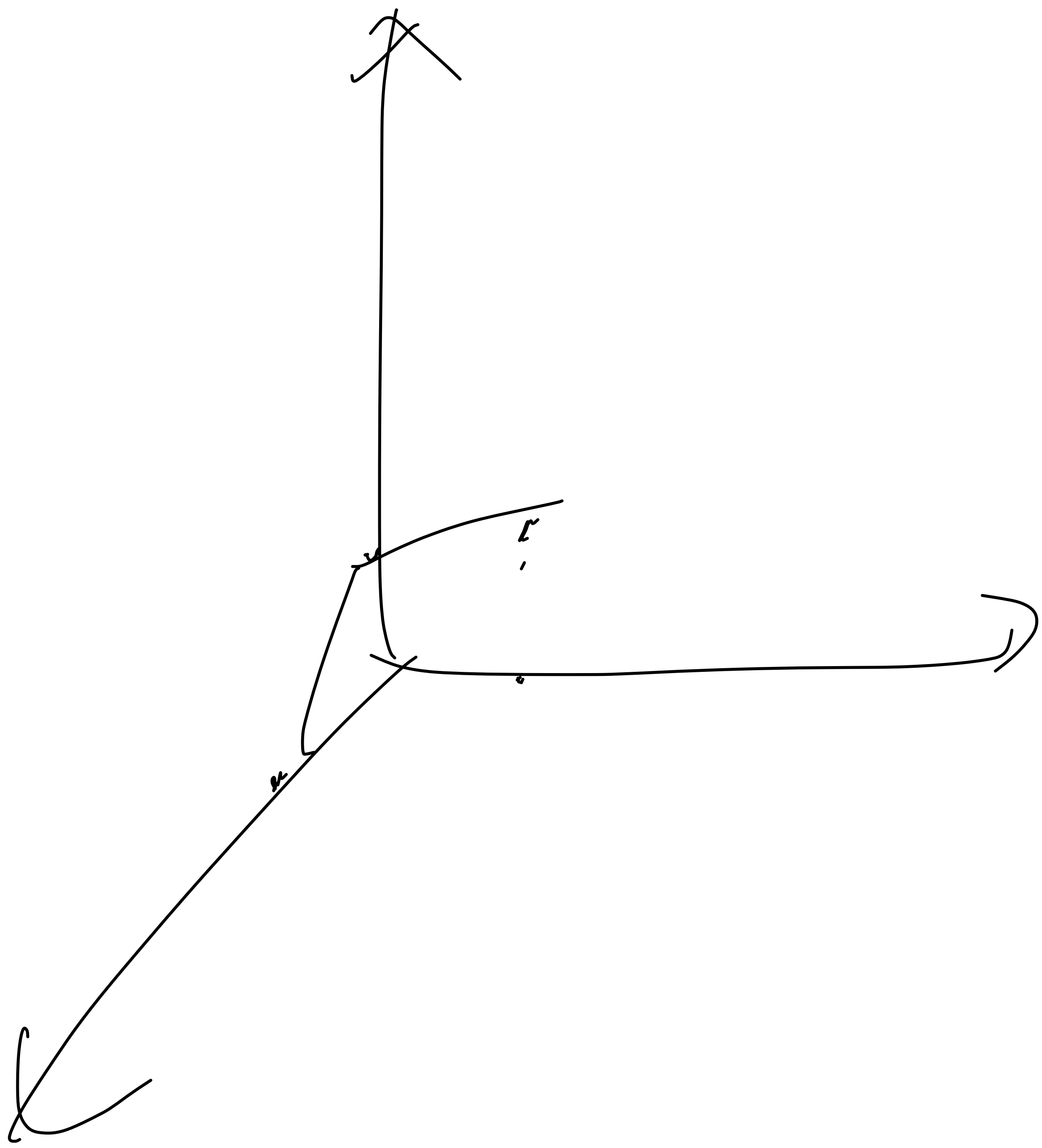
Rem Oriented Surface : has 2 sides.



"can choose unit normal vector continuously"

$$\iint_S \vec{F} \cdot \vec{n}\, dS$$

$$3y^2 + 3x^2 + 3z^2$$



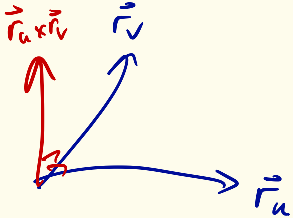
$r u$

Convention for $\vec{n}$ (positive orientation).

Graph: upward orientation

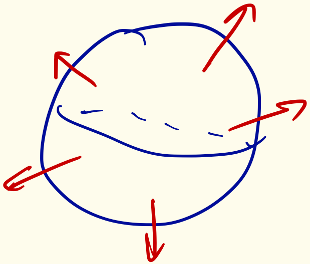
$z = f(x,y)$ $\quad\quad$ $\vec{n} = \langle -f_x, -f_y, 1 \rangle$

Parametric Equation: $\vec{r}_u \times \vec{r}_v$



Priority

Closed Surface: Outward Orientation



If $S = \vec{r}(u,v)$, $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \cancel{|\vec{r}_u \times \vec{r}_v|}\, dS$$

$$= \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, \underbrace{dA}_{du\,dv}$$

<u>Special Case</u> $S : z = f(x,y)$

$$\iint \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \langle -f_x, -f_y, 1 \rangle\, dA.$$

Ex Find $\iint_S \vec{F}\cdot d\vec{S}$ on unit sphere.

$$\vec{F} = \langle z, y, x\rangle$$

$$\vec{r}(\phi,\theta) = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, \cos\phi\rangle$$

$$\begin{array}{ccccc} & i & j & k & \\ \vec{r}_\phi = & \langle 3\cos\phi\cos\theta, & 3\cos\phi\sin\theta, & -\sin\phi\rangle & \\ \vec{r}_\theta = & \langle -3\sin\phi\sin\theta, & 3\sin\phi\cos\theta, & 0\rangle & \end{array}$$

$$9\sin\phi\cos\phi\cos^2\theta + 9$$

$$\vec{r}_\phi\times\vec{r}_\theta = \langle 3\sin^2\phi\cos\theta, 3\sin^2\phi\sin\theta, 9\sin\phi\cos\phi\rangle \quad \text{outward pointing.}$$

$$\vec{F} = \langle \cancel{\cos\phi}, \sin\phi\sin\theta, \sin\phi\cos\theta\rangle$$

$$81\sin^4\phi\cos\theta\sin^2\theta, 81\sin^4\phi\cos^2\theta\sin\theta, \cos^3\phi$$

$$\int_0^{2\pi}\int_0^{\pi} \vec{F}\cdot(\vec{r}_\phi\times\vec{r}_\theta)\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\sin^2\theta\,d\phi\,d\theta. = \cdots$$

Made with Goodnotes

$\cancel{\vec{r}(\varphi, \theta)} =$

$$\int_0^{9} \int_0^{10} \int_0^{4} (2x-1)\, dz\, dy\, dx$$

$$= 9 \times 10 \; \cancel{7 \times 6} \times \left(x^2 - x\right)_0^4$$

$$= 90 \times 12$$

$$40 \times (81 - 9)$$

$$\int_0^{2\pi}\int_0^{\pi} \sin^3\phi \, \sin^2\theta \; d\phi \, d\theta.$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \pi$$

$$\int_0^{\pi} \sin^3\phi \, d\phi = \int_0^{\pi} \sin\phi \,(1-\cos^2\phi)\, d\phi = -\cos\phi + \frac{\cos^3\phi}{3}\Bigg|_0^{\pi} = \frac{4}{3}.$$

$$\int_0^{2\pi}\int_0^{\pi} \sin^3\phi \, \sin^2\theta \; d\phi \, d\theta .$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \pi$$

$$\int_0^{\pi} \sin^3\phi \, d\phi = \int_0^{\pi} \sin\phi \, (1-\cos^2\phi) \, d\phi = -\cos\phi + \frac{\cos^3\phi}{3} \Big|_0^{\pi} = \frac{4}{3} .$$