In Exercises 1–6, find all the second partial derivatives of the given function.

1. $z = x^2(1 + y^2)$
2. $f(x, y) = x^2 + y^2$
3. $w = x^3y^3z^3$
4. $z = \sqrt{3x^2 + y^2}$
5. $z = x\,e^y - y\,e^x$
6. $f(x, y) = \ln\big(1 + \sin(xy)\big)$
7. How many mixed partial derivatives of order 3 can a function of three variables have? If they are all continuous, how many different values can they have at one point? Find the mixed partials of order 3 for $f(x, y, z) = x\,e^{xy}\cos(xz)$ that involve two differentiations with respect to z and one with respect to x.

1. $f_x = 2(1+y^2)x$ $\quad f_y = 2x^2y$ $\quad f_z = -1$

$f_{xx} = 2(1+y^2)$ $\quad f_{yy} = 2x^2$ $\quad f_{zz} = 0$

$f_{xy} = 4y$

$f_{yx} = 4x$

2. $f_x = 2x$ $\quad f_y = 2y$

$f_{xx} = 2$ $\quad f_{yy} = 2$ $\quad f_{xy} = 0$

$f_{yx} = 0$

3. $f_x = 3y^3z^3x^2$ $\quad f_y = 3x^3z^3y^2$ $\quad f_z = 3x^3z^2y^3$

$f_{xx} = 6y^3z^3x$ $\quad f_{yy} = 6x^3z^3y$ $\quad f_{zz} = 6x^3y^3z$

$f_{xy} = 9y^2x^2z^3$ $\quad f_{yx} = 9x^2y^2z^3$ $\quad f_{zy} = 9x^3z^2y^2$

$\vdots$

4. $f_x = \frac{1}{2}(3x^2+y^2)^{-\frac{1}{2}}(6x)$

$$f_x = \frac{3x}{\sqrt{3x^2+y^2}}$$

$$f_{xx} = \frac{\sqrt{3x^2+y^2}\,(3) - (3x)\left(\frac{1}{2}\right)\left(\frac{6x}{\sqrt{3x^2+y^2}}\right)}{3x^2+y^2}$$

$$f_{xx} = \frac{3\sqrt{3x^2+y^2} - \frac{9x^2}{\sqrt{3x^2+y^2}}}{3x^2+y^2}$$

$$= \frac{9x^2+3y^2-9x^2}{(3x^2+y^2)^{\frac{3}{2}}}$$

$$= \frac{3y^2}{(3x^2+y^2)^{\frac{3}{2}}}$$

Made with Goodnotes

7. How many mixed partial derivatives of order 3 can a function of three variables have? If they are all continuous, how many different values can they have at one point? Find the mixed partials of order 3 for $f(x, y, z) = x\,e^{xy}\cos(xz)$ that involve two differentiations with respect to z and one with respect to x.

$3^3 - 3$

f_{xx}

$f_{xy} = f_{yx}$

$f_{xyz} = f_{yxz}$

$f_{xyx} = f_{yxx}$

$f_{xyy} = f_{yxy}$

$f_{yz} = f_{zy}$

$f_{yzx} = f_{zyx}$

$f_{yzy} = f_{zyy}$

$f_{y(xz)} = f_{(xz)y}$

xz

f_{xxy}

f_{xxz}

f_{yyx}

f_{yyz}

f_{zzx}

f_{zzy}.

f_{xyz}.

7.

EXERCISES 12.4

In Exercises 1–6, find all the second partial derivatives of the given function.

1. $z = x^2(1 + y^2)$

2. $f(x, y) = x^2 + y^2$

3. $w = x^3y^3z^3$

4. $z = \sqrt{3x^2 + y^2}$

5. $z = x\,e^y - y\,e^x$

6. $f(x, y) = \ln\big(1 + \sin(xy)\big)$

7. How many mixed partial derivatives of order 3 can a function of three variables have? If they are all continuous, how many different values can they have at one point? Find the mixed partials of order 3 for $f(x, y, z) = x\,e^{xy}\cos(xz)$ that involve two differentiations with respect to z and one with respect to x.

Show that the functions in Exercises 8–12 are harmonic in the plane regions indicated.

8. $f(x, y) = A(x^2 - y^2) + Bxy$ in the whole plane (A and B are constants.)

9. $f(x, y) = 3x^2y - y^3$ in the whole plane (Can you think of another polynomial of degree 3 in x and y that is also harmonic?)

10. $f(x, y) = \dfrac{x}{x^2 + y^2}$ everywhere except at the origin

11. $f(x, y) = \ln(x^2 + y^2)$ everywhere except at the origin

12. $\tan^{-1}(y/x)$ except at points on the y-axis

13. Show that $w = e^{3x+4y}\sin(5z)$ is harmonic in all of $\mathbb{R}^3$, that is, it satisfies everywhere the 3-dimensional Laplace equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0.$$

14. Assume that $f(x, y)$ is harmonic in the xy-plane. Show that each of the functions $z\,f(x, y)$, $x\,f(y, z)$, and $y\,f(z, x)$ is harmonic in the whole of $\mathbb{R}^3$. What condition should the constants a, b, and c satisfy to ensure that $f(ax + by, cz)$ is harmonic in $\mathbb{R}^3$?

15. Let the functions $u(x, y)$ and $v(x, y)$ have continuous second partial derivatives and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Show that u and v are both harmonic.

16. Let $F(x, y) = \begin{cases} \dfrac{2xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$

Calculate $F_1(x, y)$, $F_2(x, y)$, $F_{12}(x, y)$, and $F_{21}(x, y)$ at points $(x, y) \neq (0, 0)$. Also calculate these derivatives at $(0, 0)$. Observe that $F_{21}(0, 0) = 2$ and $F_{12}(0, 0) = -2$. Does this result contradict Theorem 1? Explain why.

The heat (diffusion) equation

17. Show that the function $u(x, t) = t^{-1/2}\,e^{-x^2/4t}$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

This equation is called the **one-dimensional heat equation** because it models heat diffusion in an insulated rod (with $u(x, t)$ representing the temperature at position x at time t) and other similar phenomena.

18. Show that the function $u(x, y, t) = t^{-1}\,e^{-(x^2+y^2)/4t}$ satisfies the **two-dimensional heat equation**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

19. By comparing the results of Exercises 17 and 18, guess a solution to the **three-dimensional heat equation**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Verify your guess. (If you're feeling lazy, use Maple.)

Biharmonic functions

A function $u(x, y)$ with continuous partials of fourth order is **biharmonic** if $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ is a harmonic function.

20. Show that $u(x, y)$ is biharmonic if and only if it satisfies the biharmonic equation

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$$

21. Verify that $u(x, y) = x^4 - 3x^2y^2$ is biharmonic.

22. Show that if $u(x, y)$ is harmonic, then $v(x, y) = xu(x, y)$ and $w(x, y) = yu(x, y)$ are biharmonic.

Use the result of Exercise 22 to show that the functions in Exercises 23–25 are biharmonic.

23. $x\,e^x \sin y$

24. $y\ln(x^2 + y^2)$

25. $\dfrac{xy}{x^2 + y^2}$

26. Propose a definition of a biharmonic function of three variables, and prove results analogous to those of Exercises 20 and 22 for biharmonic functions $u(x, y, z)$.

27. Use Maple to verify directly that the function of Exercise 25 is biharmonic.