MATH 2023 – Multivariable Calculus

Lecture #05 Worksheet ♣ February 21, 2019

Problem 1. Find $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ where

$$f(x, y) = \frac{e^{2019x^2}}{\ln\sqrt{x^2 + 2023}} + \sin(xy)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(x\cos(xy)\right)$$

$$= -xy\sin(xy) + \cos(xy)$$

is continuous

(ln, e, x^2 are cont. in their domain)

By Mixed Partial Theorem,

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Problem 2. Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

① ②

Show that f is continuous, f_x, f_y continuous, but

③ $$f_{xy}(0,0) \neq f_{yx}(0,0)$$

Why does this violate the Mixed Partial Theorem?

f, f_x, f_y, f_{xy} etc are all continuous on $\mathbb{R}^2 \setminus (0,0)$

① Polar Coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{r^2\cos\theta\sin\theta(r^2\cos^2\theta - r^2\sin^2\theta)}{r^2}$$

$\frac{1}{2}\sin 2\theta$ $\qquad$ $\cos 2\theta$

$$= \frac{r^2}{2}\sin 2\theta\cos 2\theta$$

$$= \frac{r^2}{4}\sin 4\theta$$

$|f(x,y)| \le \frac{r^2}{4} \to 0$ Limit $(x,y) \to (0,0)$

exists and equal $(0,0)$

② f_x : $\dfrac{(x^2+y^2)(y(x^2-x^2)+xy(2x)) - xy(x^2-y^2)(2x)}{(x^2+y^2)^2}$

$$= \frac{y(x^2+4xy-y^2)}{(x^2+y^2)} \quad (\text{for } (x,y) \neq (0,0))$$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \frac{0-0}{h} = 0$$

Check $\lim_{(x,y) \to (0,0)} f_x = 0$ (use polar coord.)

f_x = cont.

$$f_y = \frac{-x(y^2+4xy-x^2)}{x^2+y^2}$$

$f_y \to$ continuous.

3 $f_{xy}(0,0) = \lim_{h \to 0} \dfrac{f_x(0,h) - f_x(0,0)}{}$

Problem 3. (a) Show that

$$u(x,t) = \sin(x - at)$$

is a solution to the **wave equation**

$$u_{tt} = a^2 u_{xx}$$

(b) Show that

$$u(x,y,z) = e^{3x+4y} \sin 5z$$

is a solution to the **Laplace's equation**

$$u_{xx} + u_{yy} + u_{zz} = 0$$

Problem 4. Let $z = f(x, y) = x^2 + 3xy - y^2$.

(a) Find the differential dz

(b) Find the tangent plane of $f(x, y)$ at $(2, 3)$

(c) Compare the values of Δz and dz when x changes from 2 to 2.05 and y changes from 3 to 2.96.