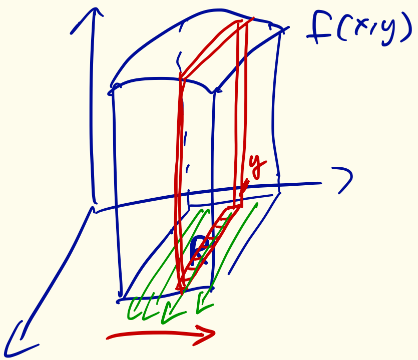
Last Time   Double Integration



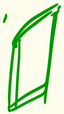
$$\iint_R f\, dA = \text{Volume under the graph.}$$

Fubini's Theorem: If f is continuous, and

$$R = [a,b] \times [c,d]$$

then

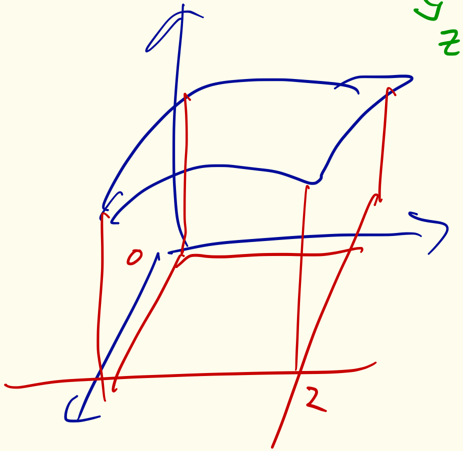
"Volume of the slices"



$$\iint_R f\, dA = \int_c^d \underbrace{\int_a^b f(x,y)\,dx}_{\text{treating } y \text{ as const.}}\,dy = \int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{\text{treating } x \text{ as const.}}\,dx.$$

Ex Volume of $x^3+2y^2+z=16$ bounded by $x=2$, $y=2$ and coordinate plane.

$$z = 16 - x^3 - 2y^2$$



$$V = \int_0^2 \int_0^2 (16 - x^3 - 2y^2)\, dx\, dy.$$

$$= \int_0^2 \left(16x - \frac{x^4}{4} - 2y^2 x\right)\Big|_0^2 \, dy$$

$$= \int_0^2 (28 - 4y^2)\, dy$$

$$= 28y - \frac{4y^3}{3}\Big|_0^2$$

$$= 56 - \frac{32}{3}$$

<u>In general</u>

$$\int_a^b \int_c^d f(x)\, g(y)\, dy\, dx$$

const

$$= \int_a^b f(x) \underbrace{\left(\int_c^d g(y)\, dy \right)}_{\text{just number}} dx$$

$$= \left(\int_a^b f(x)\, dx \right) \left(\int_c^d g(y)\, dy \right)$$

<u>Ex</u> $\int_0^1 \int_0^{\pi} (\sin x) e^y \, dx\, dy$

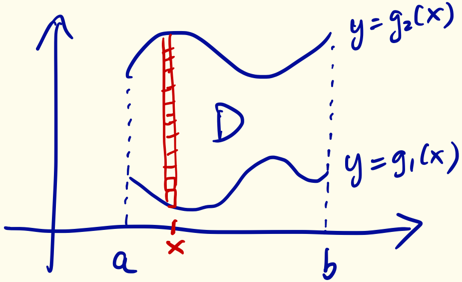
$$= \left(\int_0^{\pi} \sin x\, dx \right) \left(\int_0^1 e^y\, dy \right)$$

$$= 2(e-1)$$

General Region

$$\iint_D f(x,y)\,dA \qquad D \subset \mathbb{R}^2$$

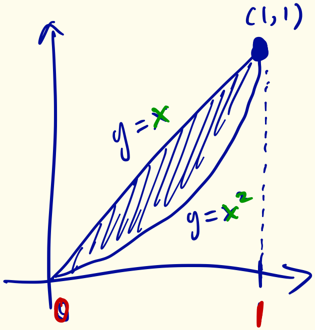
$\boxed{\text{2 types}}$ Type I



$$\| $$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Ex

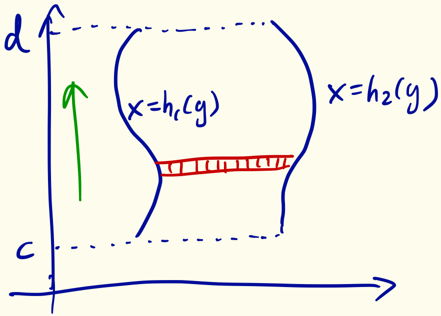


$$\text{Area} = \int_0^1 \int_{x^2}^{x} 1\,dy\,dx$$

$$= \int_0^1 y\Big|_{x^2}^{x}\,dx$$

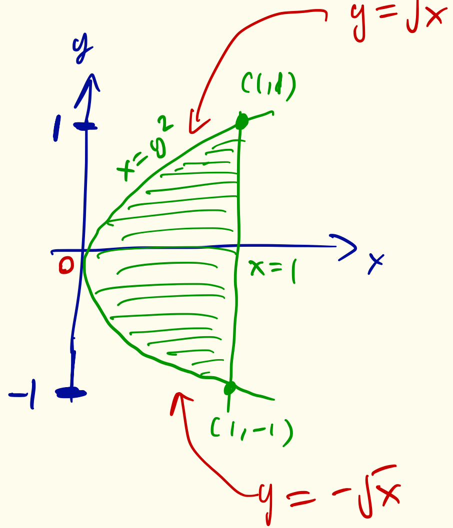
$$= \int_0^1 x - x^2\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^1 = \frac{1}{6}.$$

Type II



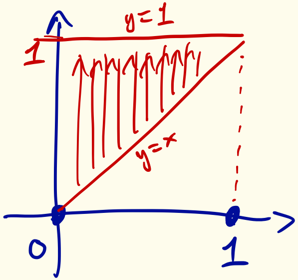
$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

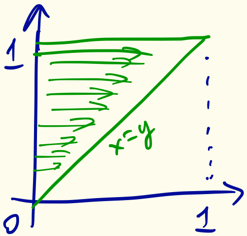
Ex $\displaystyle\int_{-1}^{1}\int_{y^2}^{1} dx\,dy \iff$



$$= \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} dy\,dx$$

Ex $\displaystyle\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$ $\qquad \displaystyle\int \sin(y^2)\,dy = ???$





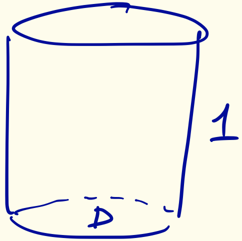
$$\int_0^1 \int_0^y \underbrace{\sin(y^2)}_{\text{is constant in } \int dx}\,dx\,dy$$

$$= \int_0^1 x \sin y^2 \Big|_0^y \,dy = \int_0^1 y \sin y^2\,dy$$

$$\left(\text{Let } u = y^2,\ du = 2y\,dy \right.$$

$$\left. = \int_0^1 \sin u \frac{du}{2} = -\frac{\cos u}{2}\Big|_0^1 = \frac{1}{2} - \frac{\cos 1}{2}. \right)$$

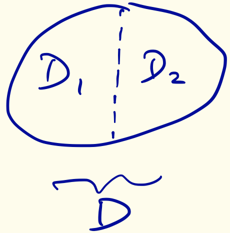
Fact $\iint_D f\,dA$ = Volume of f over D.



- $\iint_D 1\,dA$ = Area of D $(A(D))$

- $$\iint_{D_1} f\,dA + \iint_{D_2} f\,dA = \iint_{D_1 \sqcup D_2} f\,dA.$$

interior are disjoint



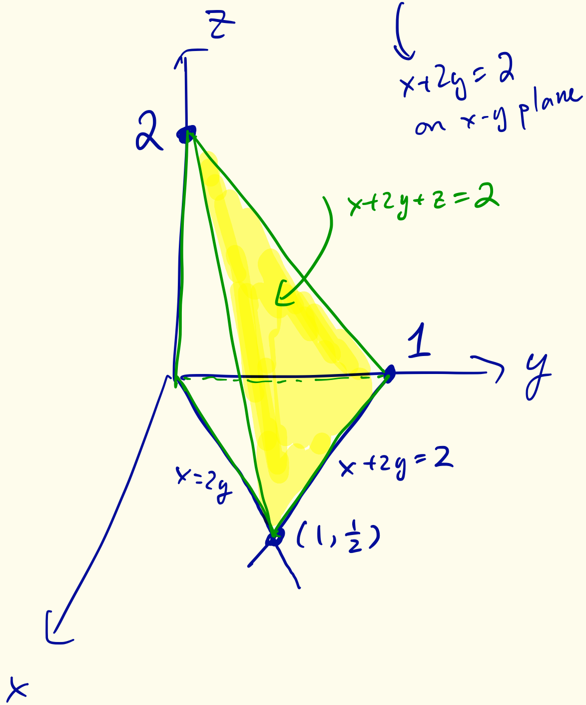
- $m \le f(x,y) \le M$

$$m\,A(D) \le \iint_D f(x,y)\,dA \le M\,A(D).$$

Ex Volume of Tetrahedron

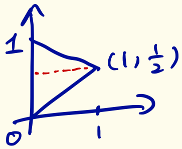
bounded by $\cdot\, x = 2y$, $\cdot\, x = 0$, $\cdot\, z = 0$, $\cdot\, x + 2y + z = 2$

vertical "wall"



With integration

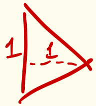
$$\iint_D 2-x-2y \, dA$$



$$\int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} (2-x-2y) \, dy \, dx = \cdots$$

$$\int_0^{\frac{1}{2}} \int_0^{2y} + \int_{\frac{1}{2}}^{1} \int_0^{2-2y} (2-x-2y) \, dx \, dy = \cdots$$

Easiest : By geometry



$$= \frac{\text{Base} \times \text{Height}}{3} = \frac{\left(\frac{1 \times 1}{2}\right) \times 2}{3} = \frac{1}{3}.$$