Problem 1

(a) Find an equation of the plane through $(-1, 4, -3)$ and perpendicular to the line

$$x = t + 2, \quad y = 2t - 3, \quad z = -t.$$

(b) Find a rectangular equation for the surface whose spherical equation is $\rho = 2\sin\theta\sin\phi$. Describe the surface.

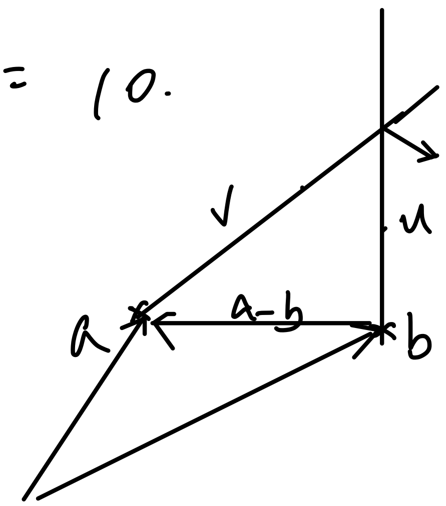
(c) Show that the two lines $\mathbf{r} = \mathbf{a} + \mathbf{v}t$ and $\mathbf{r} = \mathbf{b} + \mathbf{u}t$, where t is a parameter and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{u}$ and $\mathbf{v}$ are constant vectors, will intersect if $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

1a). $\vec{n} = \langle 1, 2, -1 \rangle$

$x + 2y - z = 10.$

b). X.

c). ??



$a + ? = b$

?

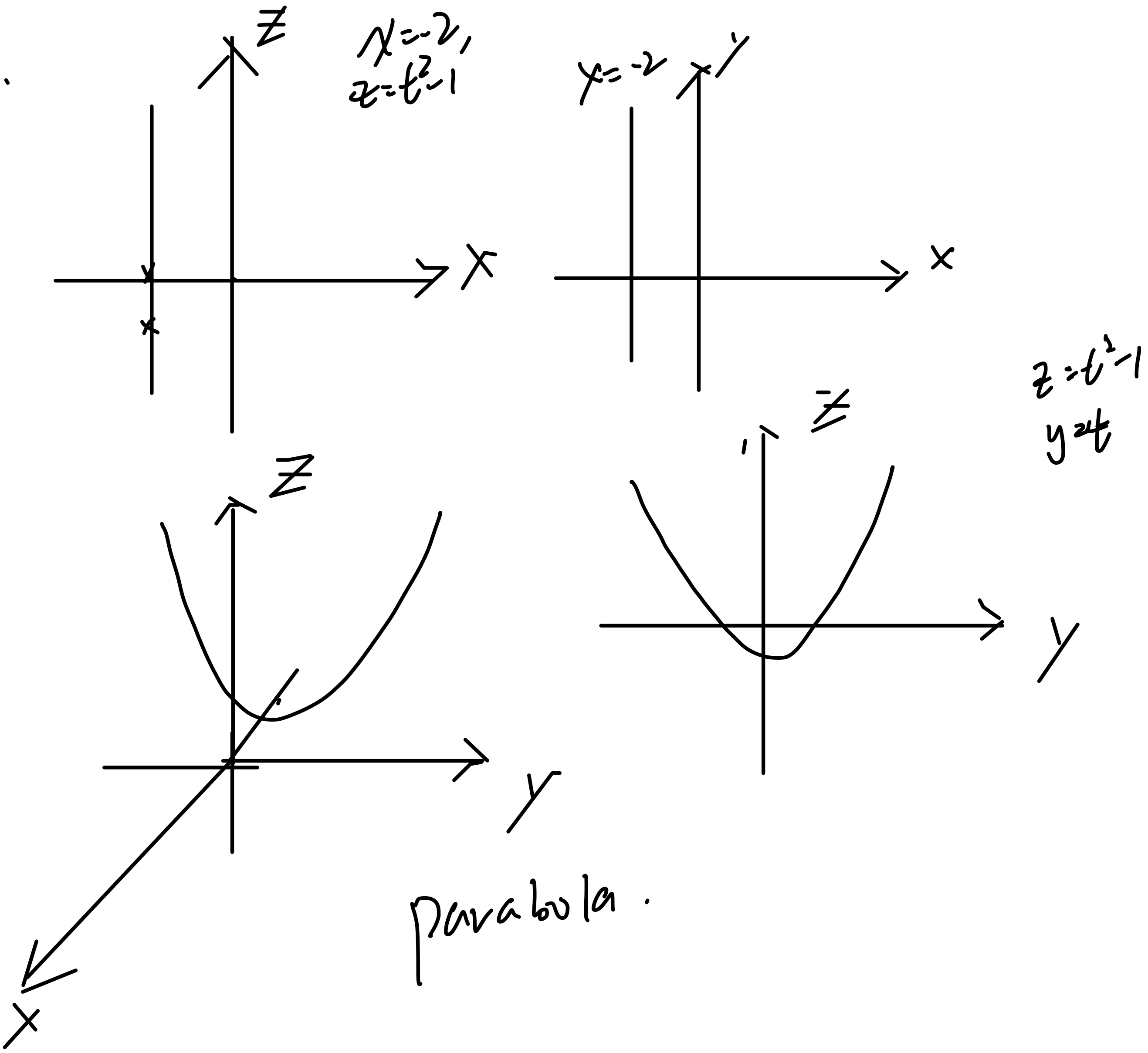
Problem2

(a) Describe the graph of the equation $\mathbf{r}_1(t) = -2\,\mathbf{i} + t\,\mathbf{j} + (t^2 - 1)\,\mathbf{k}$.

Find also the vector equation of the tangent line to the curve $\mathbf{r}_1(t)$ such that it is parallel to the line $\mathbf{r}_2(t) = \mathbf{i} + (2 + 2t)\,\mathbf{j} + (3 + 4t)\,\mathbf{k}$.

(b) Sketch the surfaces $x + y = 4$ and $\dfrac{y^2}{4^2} + \dfrac{z^2}{2^2} = 1$ in the *first* octant. Find the parametric equations of the curve C of intersection of the two surfaces above. Find the parametric equation of the projection curve C onto the xz-plane. Describe the projection curve.

2a).



parabola.

Problem2

(a) Describe the graph of the equation $\mathbf{r}_1(t) = -2\,\mathbf{i} + t\,\mathbf{j} + (t^2 - 1)\,\mathbf{k}$.

Find also the vector equation of the tangent line to the curve $\mathbf{r}_1(t)$ such that it is parallel to the line $\mathbf{r}_2(t) = \mathbf{i} + (2 + 2t)\,\mathbf{j} + (3 + 4t)\,\mathbf{k}$.

(b) Sketch the surfaces $x + y = 4$ and $\dfrac{y^2}{4^2} + \dfrac{z^2}{2^2} = 1$ in the *first* octant. Find the parametric equations of the curve C of intersection of the two surfaces above. Find the parametric equation of the projection curve C onto the xz-plane. Describe the projection curve.

parallel to $v = \langle 0, 2, 4 \rangle$

$r_1'(t) = \langle 0, 1, 2t \rangle$

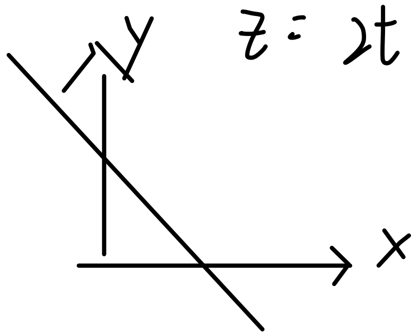
when $t=1$, it is parallel to v. $\vec{r}_1(t) = \langle -2, 1, 0 \rangle$

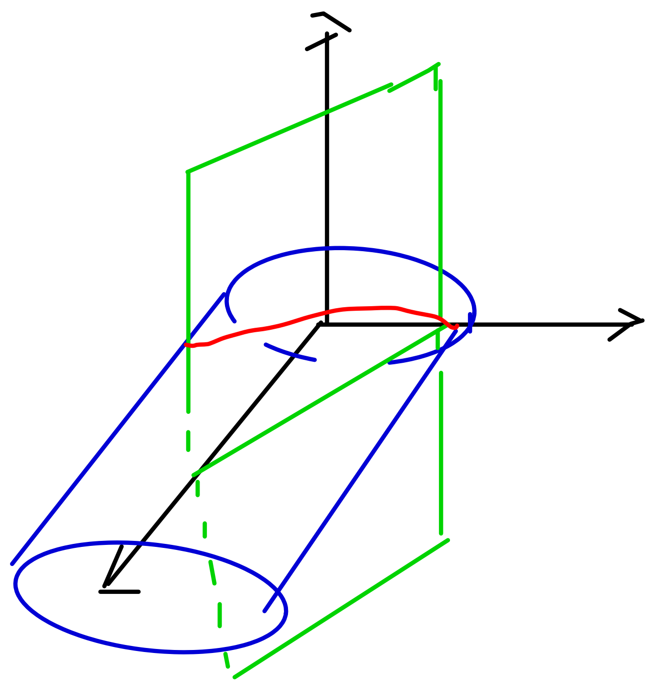
$x = -2$

$y = \cancel{2}\,1 + t$

$z = 2t$

b).



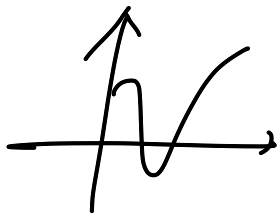


$x+y=4$

$\frac{y^2}{4^2}+\frac{z^2}{2^2}=1$

let $y=4\cos t$, $z=2\sin t$, $x=4-4\cos t$

$\vec{r}(t)=\langle 4-4\cos t, 4\cos t, 2\sin t\rangle$



set $y=0$, $4\cos t=0$, $t=\frac{\pi}{2}$,

$\vec{r}(\frac{\pi}{2})=\langle 4,0,0\rangle$

Problem 3

(a) Sketch the domain of the function $f(x, y) = \dfrac{\ln(x + y + 1)}{x^2 - 1}$.

(b) Determine the largest set on which the function

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

is continuous.

(c) Describe the level surfaces of the function $f(x, y, z) = (x - 2)^2 + y^2$.

$x^2 \neq 1$

$x+y+1 > 0$

$x+y > -1$

$y > -1-x$

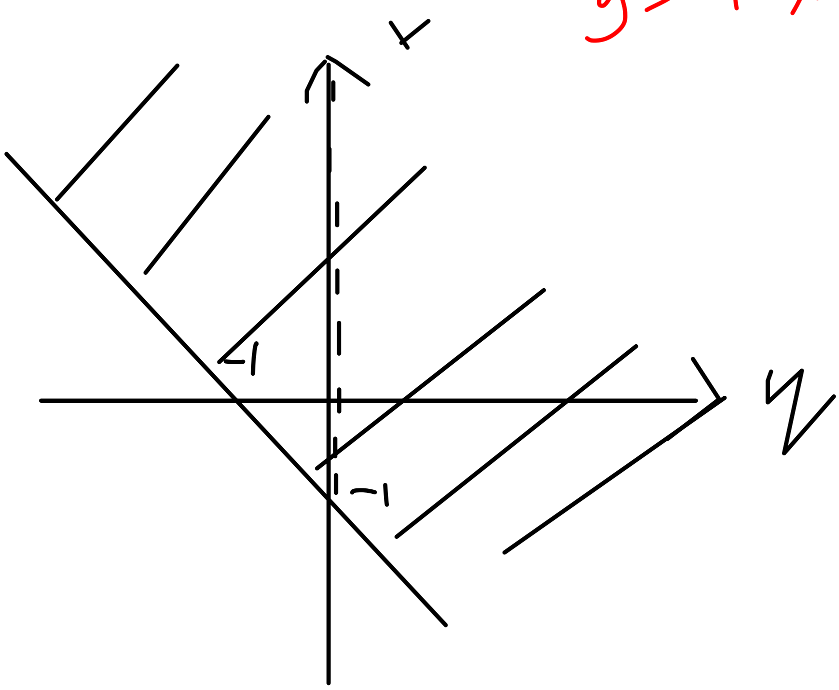
$x+y+1 > 0$

$x^2 - 1 \neq 0$

$x^2 \neq 0$

$x \neq 0$

$x+y > -1$



(c) Describe the level surfaces of the function $f(x, y, z) = (x-2)^2 + y^2$.

b). Let $x = r\cos\theta$, $y = r\sin\theta$,

$$\lim_{r\to 0} \frac{\frac{r^3\cos^2\theta\sin\theta}{r^2} - 0}{r}$$

$$= \lim_{r\to 0} \cos^2\theta\sin\theta$$

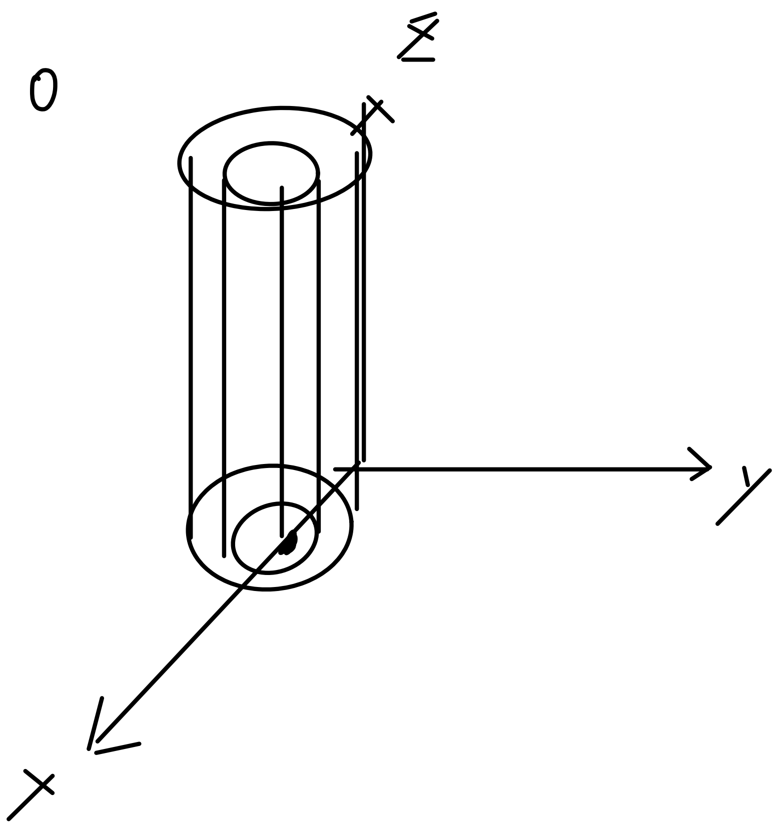
which depends on θ, does not cont.

c). $f(x,y,z) = 0$

$(x-2)^2 + y^2 = 0$.

$x = 2$ and $y = 0$.

$(x-2)^2 + y^2 = 1$



Problem 4

(a) Let $f(x,y)=\sqrt{3x+2y}$.

(i) Find the slope of the surface $z=f(x,y)$ in the x-direction at the point $(4,2)$.

(ii) Find the slope of the surface $z=f(x,y)$ in the y-direction at the point $(4,2)$.

(b) Let $g(x,y)=(x^2+y^3)^{\frac{2}{3}}$. Find $g_x(x,y)$, at all points (x,y) in the xy-plane (*include* the point $(0,0)$).

(c) Find $\dfrac{\partial^3}{\partial t^2\partial s}f(s^2-t,\ s+t^2)$ in terms of partial derivatives of f. Assume that f has continuous partial derivatives of all orders.

4a) i). $\dfrac{\partial z}{\partial x}=\dfrac{3}{2\sqrt{3x+2y}}$

$\left.\dfrac{\partial z}{\partial x}\right|_{(x,y)=(4,2)}=\dfrac{3}{2\sqrt{3(4)+2(2)}}=\dfrac{3}{8}$

ii). $\dfrac{\partial z}{\partial y}=\dfrac{1}{\sqrt{3x+2y}}=\dfrac{1}{4}.$

Problem 4

(a) Let $f(x,y)=\sqrt{3x+2y}$.

(i) Find the slope of the surface $z=f(x,y)$ in the x-direction at the point $(4,2)$.

(ii) Find the slope of the surface $z=f(x,y)$ in the y-direction at the point $(4,2)$.

(b) Let $g(x,y)=(x^2+y^3)^{\frac{2}{3}}$. Find $g_x(x,y)$, at all points (x,y) in the xy-plane (*include* the point $(0,0)$).

(c) Find $\dfrac{\partial^3}{\partial t^2 \partial s} f(s^2-t,\ s+t^2)$ in terms of partial derivatives of f. Assume that f has continuous partial derivatives of all orders.

b).

$$g_x = \frac{2}{3}(x^2+y^3)^{-\frac{1}{3}}(2x)$$

$$g_x = \frac{4x}{3(x^2+y^3)^{\frac{1}{3}}}$$

$$g_x(0,0) = \lim_{\Delta x \to 0} \frac{(\Delta x^2)^{\frac{2}{3}}}{\Delta x} = \frac{\Delta x^{\frac{4}{3}}}{\Delta x} = \Delta x^{\frac{1}{3}} = 0$$

c).

Problem 5

(a) If $f(x,y,z) = (\mathbf{r}\times\mathbf{A})\cdot(\mathbf{r}\times\mathbf{B})$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and $\mathbf{A}$ and $\mathbf{B}$ are constant vectors, show that $\nabla f(x,y,z) = \mathbf{P}\times(\mathbf{r}\times\mathbf{A}) + \mathbf{Q}\times(\mathbf{r}\times\mathbf{B})$. Find $\mathbf{P}$ and $\mathbf{Q}$ in terms of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{r}$.

i j k
1 0 7
3 6 9

(b) Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and let $r = \|\mathbf{r}\|$. If $\mathbf{A}$ and $\mathbf{B}$ are constant vectors, show that:

(i) $\mathbf{A}\cdot\nabla\left(\dfrac{1}{r}\right) = \dfrac{\mathbf{C}}{r^3}$. Find $\mathbf{C}$ in terms of $\mathbf{A}$ and $\mathbf{r}$.

(ii) $\mathbf{B}\cdot\nabla\left(\mathbf{A}\cdot\nabla\left(\dfrac{1}{r}\right)\right) = \dfrac{\mathbf{D}}{r^5} - \dfrac{\mathbf{A}\cdot\mathbf{B}}{r^3}$. Find $\mathbf{D}$ in terms of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{r}$.

a).
$$f_x = (r\times A)\cdot(r\times B)_x + (r\times A)_x\cdot(r\times B)$$
$$= (r\times A)\cdot(r_x\times B + B_x\times r_x) + (r_x\times A + r\times A_x)\cdot(r\times B)$$
$$= (r\times A)\cdot(i\times B) + (r\times B)\cdot(i\times A)$$
$$=$$

(b) Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and let $r = \|\mathbf{r}\|$. If **A** and **B** are constant vectors, show that:

(i) $\mathbf{A} \cdot \nabla\left(\frac{1}{r}\right) = \frac{\mathbf{C}}{r^3}$. Find **C** in terms of **A** and **r**.

(ii) $\mathbf{B} \cdot \nabla\left(\mathbf{A} \cdot \nabla\left(\frac{1}{r}\right)\right) = \frac{\mathbf{D}}{r^5} - \frac{\mathbf{A} \cdot \mathbf{B}}{r^3}$. Find **D** in terms of **A**, **B** and **r**.

$$\frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$\left(\frac{1}{r}\right)_x = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x)$$

$$= -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{-x}{r^3}$$

$$A \cdot \nabla\left(\frac{1}{r}\right) =$$

$$-\frac{x}{r^3}A_x - \frac{y}{r^3}A_y - A_z\frac{z}{r^3}$$

$$= \frac{-(xA_x + yA_y + zA_z)}{r^3}$$

$$= \frac{-(\vec{r} \cdot A)}{r^3} \qquad C = -(\vec{r} \cdot A)$$

Made with Goodnotes

$$= \frac{-(xA_x + yA_y + zA_z)}{r^3}$$

$$A \cdot \nabla\left(\frac{1}{r}\right) = \frac{-(\vec{r} \cdot A)}{r^3} \qquad C = -(\vec{r} \cdot A)$$

$$\nabla\left(A \cdot \nabla\left(\frac{1}{r}\right)\right) = \frac{r^3 \frac{\partial}{\partial x}(-xA_x) + (\vec{r} \cdot A)(3r^2)(r_x)}{r^6}$$

$$= \frac{r^3(-A_x) + 3r^2(\vec{r} \cdot A)}{r^6}$$

$$= \frac{-A_x}{r^3} + \frac{3(\vec{r} \cdot A)}{r^4}$$

$$B \cdot \nabla\left(A \cdot \nabla\left(\frac{1}{r}\right)\right) = B_x\left(\frac{-A_x}{r^3} + \frac{3(\vec{r} \cdot A)}{r^4}\right) + B_y\left(-\frac{A_y}{r^3} + \frac{3(\vec{r} \cdot A)}{r^4}\right) + B_z\left(-\frac{A_z}{r^3} + \frac{3(\vec{r} \cdot A)}{r^4}\right)$$

$$= \frac{-A \cdot B}{r^3} + \frac{(3(\vec{r} \cdot A) \cdot B)}{r^5}$$

$$\left(\frac{-\vec{r}\cdot A}{r^3}\right)_x = \frac{r^3\frac{\partial}{\partial x}\left((-\vec{r})\cdot A\right) - (-\vec{r}\cdot A)\frac{\partial}{\partial x}r^3}{r^6}$$

$$= \frac{r^3\left(-\vec{r}_x\cdot A + (-\vec{r})\cdot A_x\right) + (\vec{r}\cdot A)(3r^2)\,r_x}{r^6}$$

$$= \frac{r^3(-A_x) + (\vec{r}\cdot A)(3r^2)}{r^6}$$

$$= \frac{-rA_x + 3(\vec{r}\cdot A)}{r^4}$$

Problem 6

A three dimensional surface whose equation is $y = f(x)$ is tangent to the surface $z^2 + 2xz + y = 0$ at all points common to the two surfaces. (i) Find $f(x)$. (ii) Find all common points.

??

let $g(x,y,z) = z^2 + 2xz + y$

$\nabla g = \langle 2z, 1, 2z \rangle$

gradient $\perp$ level/curve

gradient $\perp$ tangent

$\therefore$ at all point

$\nabla g \cdot f(x) = 0$

$2zx + y + 2z^2 = 0$

$y = -2z^2 - 2zx$

$2z f(x) - 1 = 0$

$2z f(x) = 1$

$f(x) = \frac{1}{2z}$.

$$\begin{cases} y = -2z^2 - 2zx \\ z^2 + 2xz + y = 0 \end{cases}$$

$f(x) = \frac{1}{x}$

$z^2 + 2xz - 2z^2 - 2zx = 0$

$-z^2 = 0$

$z = 0$,

$y = 0$,

$x = 0$.

Made with Goodnotes

Problem 7

(a) Let f be a nonconstant scalar field, differentiable everywhere in the plane, and let c be a constant. Assume the Cartesian equation $f(x, y) = c$ describes a curve C having a tangent at each of its points. Prove that f has the following properties at each point of C:

(i) The gradient vector ∇f is normal to C.

(ii) The directional derivative of f is zero along C.

(iii) The directional derivative of f has its largest value in a direction normal to C.

(b) Find the directional derivative of the scalar field $f(x, y) = x^2 - 3xy$ along the parabola $y = x^2 - x + 2$ at the point $(1, 2)$.

7a). $f(x,y)=c$: is tangent ∇C.

~~形式~~ $\nabla f \perp \nabla C$. , given $f(x,y)-c = \nabla C$

$\nabla f \cdot \nabla c$ $\quad \nabla f = \langle f_x, f_y \rangle$

$C = \int f(x,y)\, \partial$

$f(x,y) \parallel C$

$\nabla f \perp f(x,y)$

$\nabla f \perp C$

ii). $D_u f$: $D_{tangent} f = f \cdot \nabla f = 0$.

(iii) largest $= \nabla f$, $\nabla f \perp C$.

Made with Goodnotes

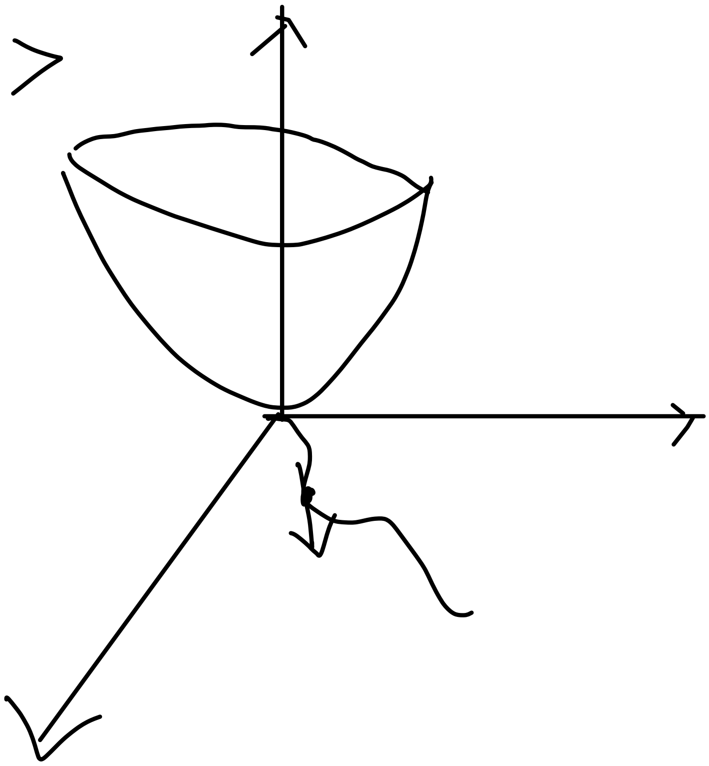
(b) Find the directional derivative of the scalar field $f(x, y) = x^2 - 3xy$ along the parabola $y = x^2 - x + 2$ at the point $(1, 2)$.

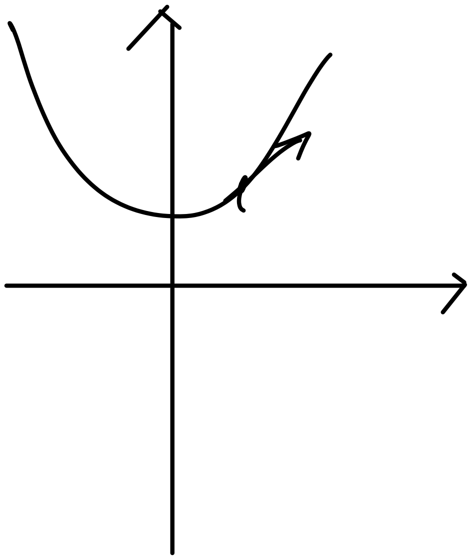
b). $g_x = 2x - 1 = 1$, $g_y = 1$

$\hat{u} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$.

$\nabla f = \langle 2x - 3y, -3x \rangle$

$\nabla f(1,2) = \langle -4, -3 \rangle$





$\nabla f \cdot \hat{u} = -\frac{4}{\sqrt{2}} - \frac{3}{\sqrt{2}} = -\frac{7}{\sqrt{2}}$.

Problem 8

A manufacturer is planning to sell a new product at the price of \$150 per unit and estimates that if x thousand dollars is spent on development and y thousand dollars is spent on promotion, approximately $\frac{320y}{y+2} + \frac{160x}{x+4}$ units of the product will be sold. The cost of manufacturing the product is \$50 per unit. If the manufacturer has a total \$8,000 to spend on development and promotion, how should this money be allocated to generate the largest possible profit?

Suppose the manufacturer in the above exercise decides to spend \$8,100 instead of \$8,000 on the development and promotion of the new product. Use the Lagrange multiplier λ to estimate how this change will affect the maximum possible profit.

$$P = \left(\frac{320y}{y+2} + \frac{160x}{x+4}\right) 150 \cdot \; -x - y$$

$$8000 = 50\left(\frac{320y}{y+2} + \frac{160x}{x+4}\right)$$

$$\nabla P = \left(150\left(\frac{(x+4)(160)-(160x)}{(x+4)^2}\right), \frac{(y+2)(320)-320y}{(y+2)^2}\right)$$

$$\nabla P = \left(\frac{96000}{(x+4)^2} - 1, \frac{96000}{(y+2)^2} - 1\right)$$

$$\nabla g = \left(\frac{50\cdot(160)(4)}{(x+4)^2}, \frac{50\cdot 320\cdot 2}{(y+2)^2}\right)$$

$$\nabla g = \left(\frac{32000}{(x+4)^2}, \frac{32000}{(y+2)^2}\right)$$

$$\frac{96000}{(x+4)^2} - 1 = \lambda \frac{32000}{(x+4)^2} \qquad \lambda = 3.$$

$$6000 = 50\left(\frac{320y}{y+2} + \frac{160x}{x+4}\right)$$

$$\frac{96000 - (x+4)^2}{(x+4)^2} = \lambda \frac{(32000)}{(x+4)^2}$$

$$96000 - (x+4)^2 = 32000\lambda$$

$$96000 - x^2 - 8x - 16 = 32000\lambda$$

$$96000 - y^2 - 4x - 4 = 32000\lambda$$

$$(x+4)^2 = (y+2)^2$$

$$x+4 = \pm(y+2)$$

$$x+4 = y+2 \quad \text{or} \quad x+4 = -(y+2)$$

$$x = y-2 \quad \text{or} \quad x+4 = -y-2$$

$$x = -y-6$$

For $x = y - 2$,

$$8000 = 50\left(\frac{320y}{y+2} + \frac{160(y-2)}{y+2}\right)$$

$$8000 = \frac{(50)480y - 320(50)}{y+2}$$

$$8000(y+2) = 24000y - 16000$$

$$8000y + 16000 = 24000y - 16000$$

$$32000 = 16000y$$

$$y = 2$$

$$x = 0,$$

$$(24000)$$

Made with Goodnotes