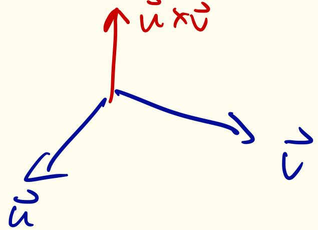
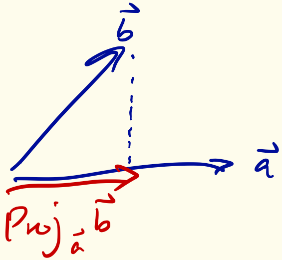
Last Time

$\vec{u}\times\vec{v}$:



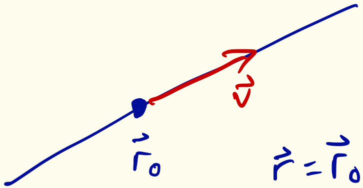
$\vec{u}\cdot\vec{v}$: $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

Projection :



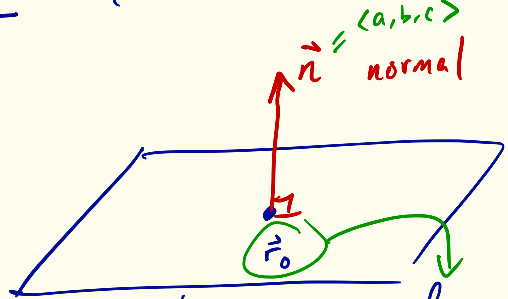
: $(\vec{b}\cdot\vec{e})\vec{e}$ $\quad \vec{e} = \frac{\vec{a}}{|\vec{a}|}$

<u>Lines</u> : point + vector



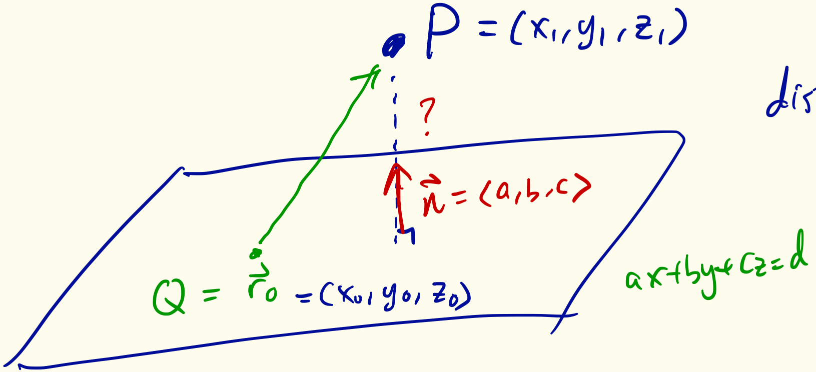
$\vec{r} = \vec{r}_0 + t\vec{v}$

<u>Planes</u> : point + vector



$ax+by+cz=d$

Distance Formula from point (P) to plane $\begin{cases}\vec{r}_0\\ \vec{n}\end{cases}$



distance = projection from $\vec{PQ}$ to $\vec{n}$.

$$\text{distance}=\left|\left(\underset{\vec{PQ}}{\vec{b}}\cdot\underset{\frac{\vec{n}}{|\vec{n}|}}{\vec{e}}\right)\right| = \left|\langle x_0-x_1,\ y_0-y_1,\ z_0-z_1\rangle\cdot\frac{\langle a,b,c\rangle}{\sqrt{a^2+b^2+c^2}}\right|$$

$$=\left|\frac{\overbrace{ax_0+by_0+cz_0}^{=d}-ax_1-by_1-cz_1}{\sqrt{a^2+b^2+c^2}}\right|$$

$$=\left|\frac{ax_1+by_1+cz_1-d}{\sqrt{a^2+b^2+c^2}}\right| \quad //$$

① plug in P to eq. of plane

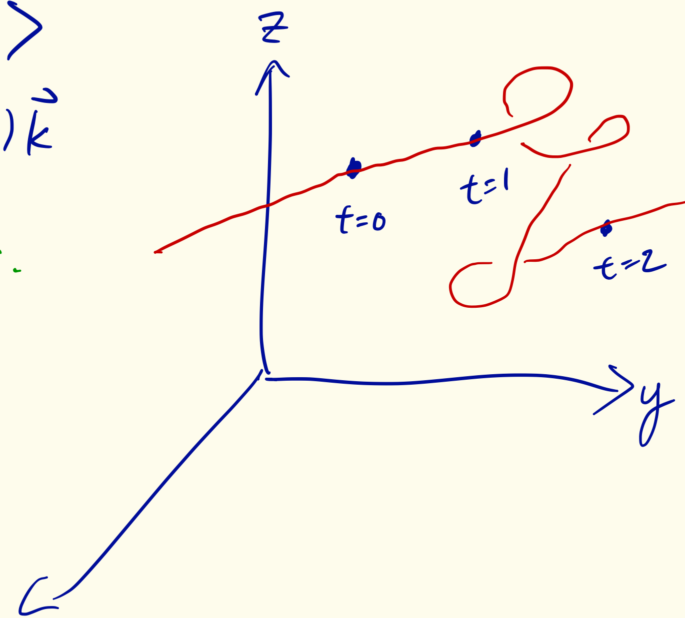
② divide by $|\vec{n}|$.

Vector Functions (Curves)

1 input $\mathbb{R}$
3 outputs

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$
$$= f(t)\,\vec{i} + g(t)\,\vec{j} + h(t)\,\vec{k}$$

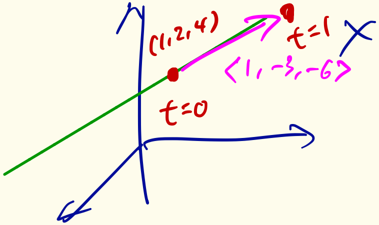
component functions.
(continuous)



$t =$ "time"

Ex $\langle 1+t, 2-3t, 4-6t \rangle$ $\langle 2,-1,-2 \rangle$

$$= \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ -6 \end{pmatrix}$$



Domain of $\vec{r}$: domain of t such that all component functions are defined.

Ex $\vec{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$

domain : $t \in \mathbb{R}$ $\quad 3-t>0$ $\quad t \geq 0$
$\quad t<3$

$\Big\} \; 0 \leq t < 3$

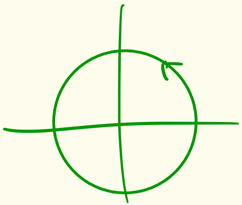
Limit $\lim\limits_{t\to a} \vec{r}(t) = \langle \lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t) \rangle$

Continuity : $\lim\limits_{t\to a} \vec{r}(t) = \vec{r}(a)$ (exists + equal)

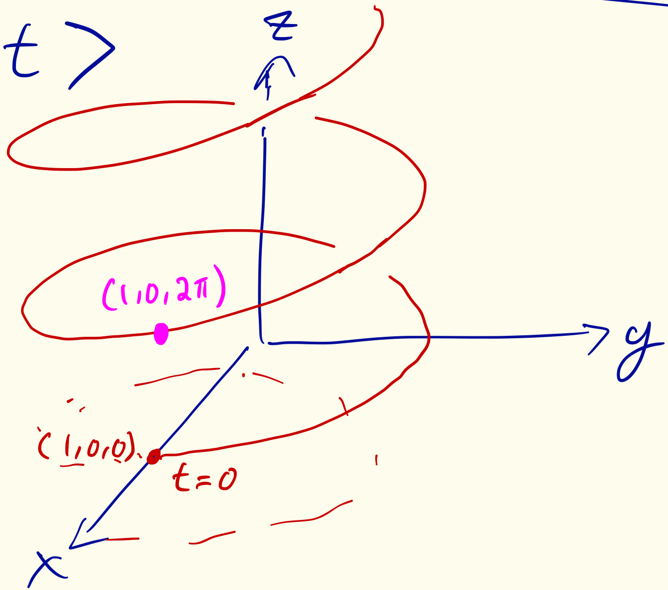
Parametric Equations $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$$

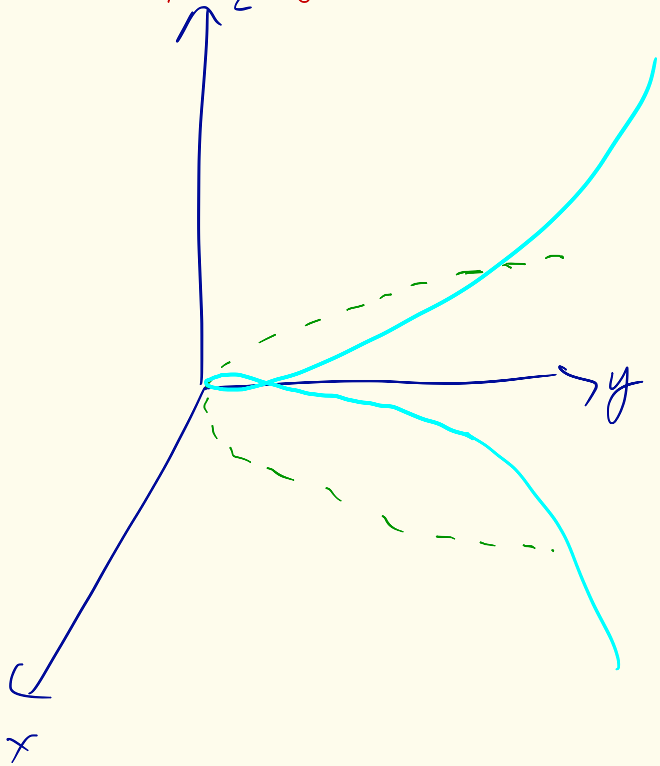
Ex $\langle \underbrace{\cos t, \sin t}_{\text{circle on } \mathbb{R}^2}, t \rangle$



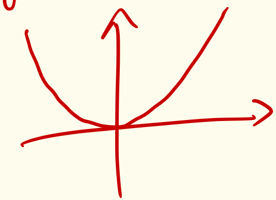
Helix



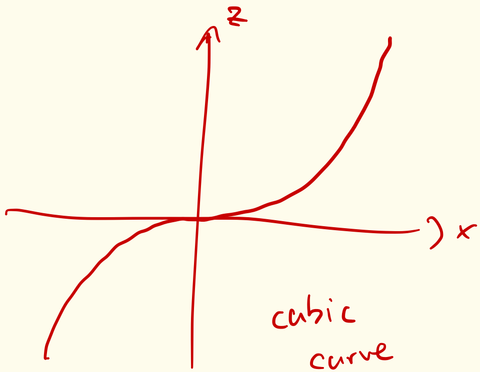
Ex $\langle \overbrace{\underset{=x}{t}, \underset{=y}{t^2}}, \underset{=z}{t^3} \rangle$



$y = x^2$



$z = x^3$

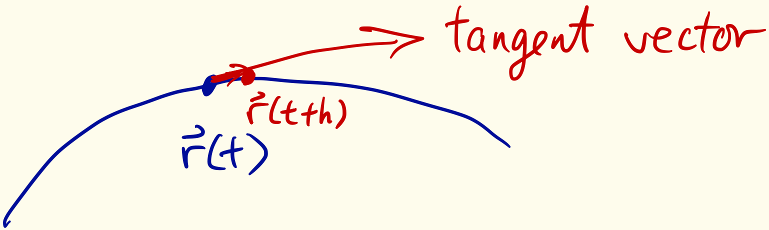


cubic curve

Derivatives

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \langle f'(t), g'(t), h'(t) \rangle$$

$$= \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$



Meaning:

$$\text{speed} = \frac{\text{distance}}{\text{time}} \rightsquigarrow v = \frac{ds(t)}{dt} = |\vec{v}|$$

magnitude

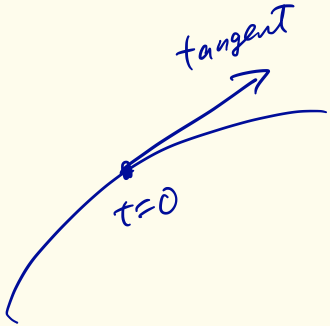
$$\text{velocity} = \frac{\text{displacement}}{\text{time}} \rightsquigarrow \vec{v} = \frac{d\vec{r}}{dt} = \vec{r}\,'(t)$$

In terms of classical mechanics:

displacement $\vec{r}(t)$

velocity $\vec{r}\,'(t) = \vec{v}(t)$ speed $= |\vec{v}(t)|$

acceleration $\vec{r}\,''(t) = \vec{a}(t)$



Ex $\vec{r}(t) = \sqrt{1+t}\ \vec{i} + (t^3+4)\vec{j} + e^{\sin t}\ \vec{k}$

$\vec{r}\,'(t) = \left\langle \frac{1}{2\sqrt{1+t}},\ 3t^2,\ e^{\sin t}\cos t \right\rangle$

at $(1,4,1)$ $\Rightarrow \vec{r}\,'(0) = \langle \frac{1}{2}, 0, 1 \rangle$, speed $= \sqrt{\frac{1}{4}+0+1} = \frac{\sqrt{5}}{2}$

$(t=0)$

acceleration: $r''(t) = \ldots = \langle -\frac{1}{4}, 0, 1 \rangle$

at $t=0$.

tangent line at $t=0$

$$\vec{r}(t) = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + t \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

Newton's Law of Motion

$$\vec{F}(t) = m\vec{a}(t).$$

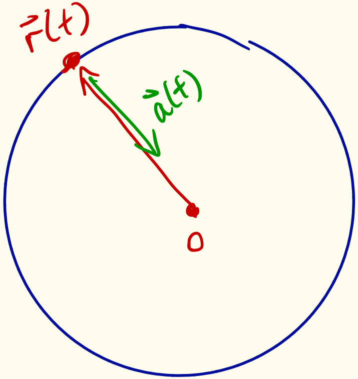
Ex Rotation

$$\vec{r}(t) = \langle \rho\cos kt, \rho\sin kt \rangle$$

$$\vec{v}(t) = \langle -\rho k\sin kt, \rho k\cos kt \rangle$$

$$\vec{a}(t) = \langle -\rho k^2\cos kt, -\rho k^2\sin kt \rangle$$

$$= -k^2\vec{r}(t).$$



$\vec{F}(t) \parallel \vec{a}(t)$

centripetal force.

Differentiation Rules

$$\frac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \vec{u}'(t) + \vec{v}'(t)$$

$$\frac{d}{dt}\left(c\,\vec{u}(t)\right) = c\,\vec{u}'(t) \qquad c \in \mathbb{R}$$

Product Rule

$$\frac{d}{dt}\left(f(t)\vec{v}(t)\right) = f'(t)\,\vec{v}(t) + f(t)\,\vec{v}'(t).$$

scalar function

vector function

$$\frac{d}{dt}\left(\vec{u}(t) \cdot \vec{v}(t)\right) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

$$\frac{d}{dt}\left(\vec{u}(t) \times \vec{v}(t)\right) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t).$$

Chain Rule

$$\frac{d}{dt}\left(\vec{v}(f(t))\right) = \vec{v}'(f(t)) \cdot f'(t)$$

Integration

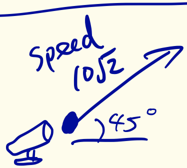
$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$$

$$= \left\langle F\Big|_a^b,\ G\Big|_a^b,\ H\Big|_a^b \right\rangle = \vec{R}\Big|_a^b = \vec{R}(b) - \vec{R}(a).$$

Indefinite Integral: $$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C}$$

constant vector.

Ex



When does it hit the ground? $t = 2$.

$\therefore$ Find $\vec{r}(t)$ and solve when $y = 0$

$\vec{a}(t) = \langle 0, -10 \rangle \qquad \Rightarrow \vec{v}(t) = \int \vec{a}(t)\,dt = \langle C_1, -10t + C_2 \rangle \xrightarrow{t=0} \langle 10, 10 - 10t \rangle$

$\vec{v}(0) = \langle 10, 10 \rangle \qquad \Rightarrow \vec{r}(t) = \int \vec{v}(t)\,dt = \langle 10t + c_1, 10t - 5t^2 + c_2 \rangle \xrightarrow{t=0}$

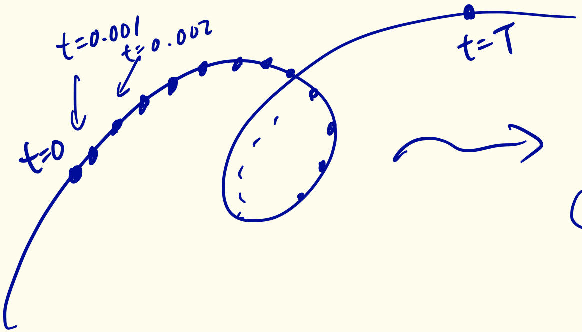
$\vec{r}(0) = \langle 0, 0 \rangle$

$\vec{r}(t) = \langle 10t, 10t - 5t^2 \rangle$

Arc Length

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

$$\Longrightarrow \text{distance} = \text{speed} \times \text{time}$$



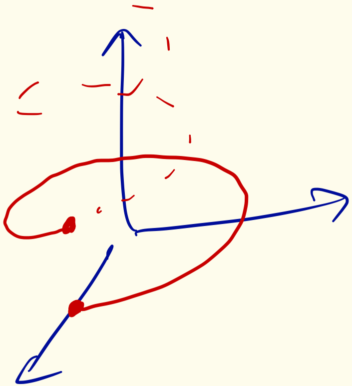
distance (arc length) $= \sum |\vec{r}\,'(t_i)| \Delta t$

$$= \int_0^T |r'(t)|\, dt$$

Ex Length of Helix:

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

from $t=0$ to $t=2\pi$



$$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$$

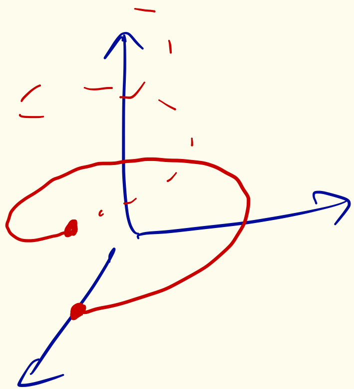
$$|\vec{r}\,'(t)| = \sqrt{\sin^2 + \cos^2 + 1} = \sqrt{2}$$

$$\int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}.$$

Ex Length of Helix:

$$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$$

from $t=0$ to $t=2\pi$



$$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{r}'(t)| = \sqrt{\sin^2 + \cos^2 + 1} = \sqrt{2}$$

$$\int_0^{2\pi} \sqrt{2}\, dt = 2\pi\sqrt{2}.$$