MATH2023
Tutorial 6
March 19, 2019 Tutorial Notes by Ivan So

1 Review

- The **double integral** is defined as $\int\int_R f(\mathbf{x})dA := \lim_{n\to\infty}\sum_{i,j=1}^n f(\mathbf{x}_i^*)\Delta A_i$.
- **Fubini's Theorem**: If f is (1) discontinuous on finitely many number of points and (2) bounded over the rectangle $R = \{(x,y)|(x,y) \in [a,b]\times[c,d]\} \subset \mathbb{R}^2$, then

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

- Consider function of two variables. A region D is said to be of **type I** (**type II**) if $D = \{(x,y)|a \le x \le b \text{ and } g_1(x) \le y \le g_2(x)\}$ ($D = \{(x,y)|c \le y \le d \text{ and } h_1(y) \le x \le h_2(y)\}$) where g_1 and g_2 are continuous functions.
- Integration for function over type I or type II region is well-defined. But when changing order of integration, one would have to beware of the integration limits.
- If $R = R_1 \sqcup R_2$, then $\int\int_R f(x,y)dA = \int\int_{R_1} f(x,y)dA + \int\int_{R_2} f(x,y)dA$.
- Recall in **polar coordinates**, $r^2 = x^2 + y^2$, $\tan\theta = y/x$. In other words, $x = r\cos\theta$ and $y = r\sin\theta$. Integration of two variable function can be done with polar coordinates, with $dA = rdrd\theta$.

2 Problems

1. True or False

   (a) $\int_1^2\int_3^4 x^2e^ydydx = \int_1^2 x^2dx\int_3^4 e^ydy$.

   (b) $\int_0^1\int_0^x \sqrt{x+y^2}dydx = \int_0^x\int_0^1 \sqrt{x+y^2}dxdy$.

2. Sketch the solid bounded by the constraints $0 \le x, y \le 1$ and $0 \le z \le 4 - x - 2y$. Evaluate its volume.

3. Show that $0 \leq \int\int_R \sin \pi x \cos \pi x dA \leq \frac{1}{32}$ for $R = [0, 1/4] \times [1/4, 1/2]$.

4. Find the average value of $f(x, y) = x^2y$ over the rectangle with vertices $(-1, 0), (-1, 5), (1, 5), (1, 0)$.

5. Write the volume integral of the solid bounded by $z = xy$ above a triangle with vertices $(1, 1), (4, 1)$ and $(1, 2)$.

6. Evaluate $\int\int_D x \cos y dA$ over where D is the region bounded by $y = 0,\ y = x^2,\ x = 1$.

7. Prove that if $m \leq f(x, y) \leq M$ for all (x, y) in D, then

$$mA(D) \leq \int\int_D f(x, y)dA \leq MA(D).$$

8. Use polar coordinates to combine and evaluate the sum

$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xydydx + \int_{1}^{\sqrt{2}} \int_{0}^{x} xydydx + \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{4-x^2}} xydydx$$

9. Evaluate $\int_0^\infty e^{-x^2}\, dx$.