MATH2023
Tutorial 11
April 29, 2019 Tutorial Notes by Ivan So

1 Review

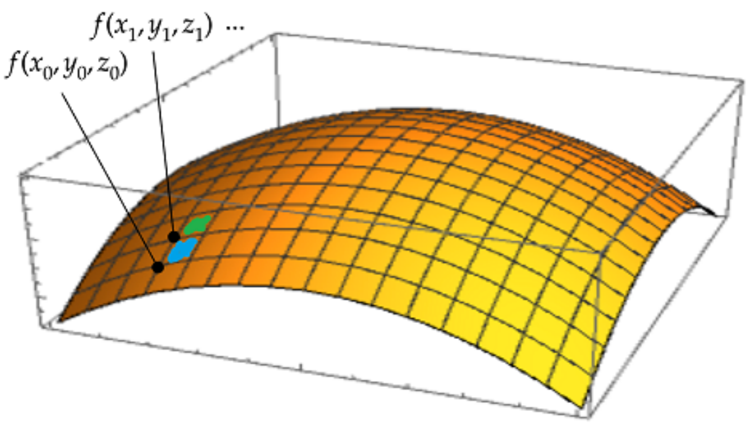
- The **surface area** of S parametrized by $\mathbf{r}(u, v)$ is given by

$$A(S) = \int\int_D |\mathbf{r}_u \times \mathbf{r}_v| dA.$$

- The **surface integral for function** f over S is defined by

$$\int_S f dS = \int\int_D f(x(u, v), y(u, v), z(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dvdu.$$

Interpretation: We are summing the value of function a point multiplied by a differential area of a small patch.

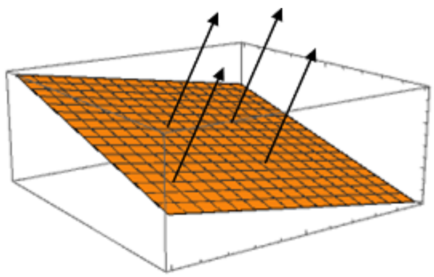


- The **surface integral** for *vector field* over S is defined by

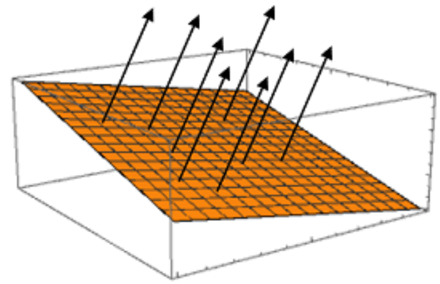
$$\int_S \mathbf{F} \cdot \mathbf{n} dS = \int\int_D \langle P(u, v), Q(u, v), R(u, v)\rangle \cdot \langle n_1(u, v), n_2(u, v), n_3(u, v)\rangle dudv$$

for normal vector of S $\mathbf{n}$ (**very important**: $\mathbf{n}$ necessary to be unit in length).

Interpretation: The measure of flux of vector field (component of the vector field parallel to the normal vector) through a surface (you may non-rigorously think of it as counting the number of vector field line passing through the surface).



fewer flux



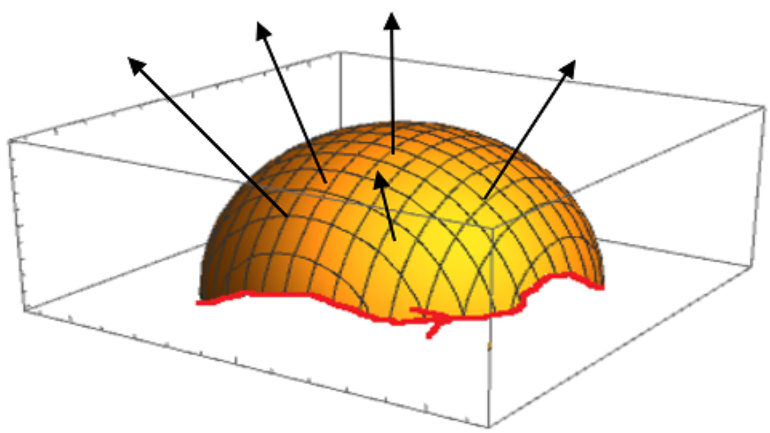
more flux

- **Stoke's Theorem**: The line integral over a closed curve is:

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

where S is a surface with ∂S as boundary and $\mathbf{n}$ is the unit normal vector of the surface of obeying the *positive orientation* (satisfying the right hand grip rule).

Interpretation: Stoke's theorem said the loop integral of a vector field can be calculated by measuring the flux of the *curl* through the surface with the concerned curve as the boundary.



The arrows are the vector field lines of the curl of the vector field.

2 Problems

1. **True or False**

(a) The constant vector field $\mathbf{F}(x, y, z) = \langle 1, -1/2, -1/2 \rangle$ has a non-vanishing flux through the surface $x + y + z = 1$.

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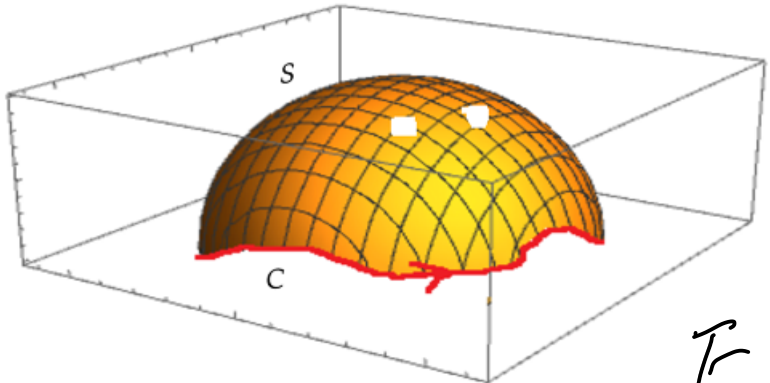
(b) In the following diagram, $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$ is still satisfied, where S is a surface with holes.



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2. Evaluate $\int\int_S (x^2 z + y^2 z) dS$, where S is the upper hemisphere with radius 2.

3. Let $\mathbf{F}(x, y, z) = \mathbf{r}/|\mathbf{r}|^3$, where $\mathbf{r} = \langle x, y, z \rangle$. Show that the flux through the surface of the sphere is independent of the radius.

4. Find the center of mass of a hemisphere shell assuming uniform density.

5. Use Stoke's Theorem to evaluate the close loop integral $\oint_F \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle x + y^2, y + z^2, z + x^2 \rangle$ over C, where C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, where the loop is running counterclockwise if we view from infinitely far away in the positive quadrant.

6. Let C be the closed simple curve lies in the plane $x + y + z = 1$. Show that the line integral

$$\oint_C z dx - 2x dy + 3y dz$$

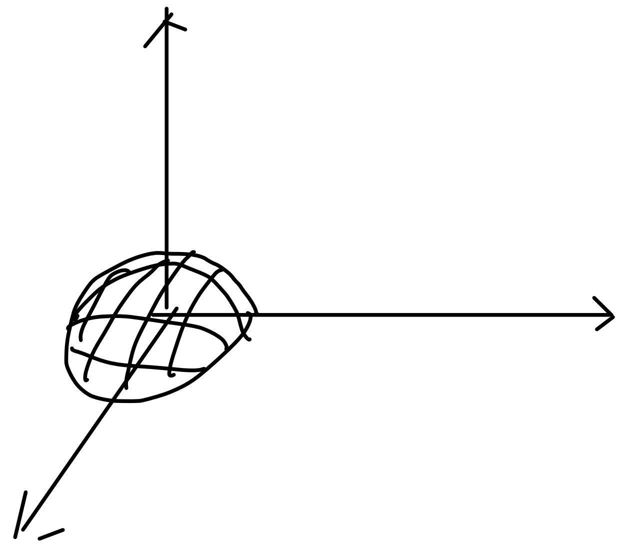
depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

7. Evaluate

$$\oint_C (y + \sin x) dx + (z^2 + \cos y) dy + x^3 dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$ for $0 \leq t \leq 2\pi$.

2. Evaluate $\int\int_S (x^2z + y^2z)dS$, where S is the upper hemisphere with radius 2.



$$\vec{r}(\varphi,\theta) = \langle 2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta,\ 2\cos\varphi \rangle$$

$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \frac{\pi}{2}$$

$$r_\varphi = \langle 2\cos\varphi\cos\theta,\ 2\cos\varphi\sin\theta,\ -2\sin\varphi \rangle$$

$$r_\theta = \langle -2\sin\varphi\sin\theta,\ 2\sin\varphi\cos\theta,\ 0 \rangle$$

$$r_\varphi \times r_\theta = \langle 4\sin^2\varphi\cos\theta,\ 4\sin^2\varphi\sin\theta,\ 4\sin\varphi\cos\varphi \rangle$$

$$16\sin^4\varphi\cos^2\theta + 16\sin^4\varphi\sin^2\theta + 16\sin^2\varphi\cos^2\varphi$$

$$= 16\sin^4\varphi + 16\sin^2\varphi\cos^2\varphi$$

$$= 16\sin^2\varphi$$

$$|r_\varphi \times r_\theta| = 4\sin\varphi$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(8\sin^2\varphi\cos^2\theta\cos\varphi + 8\sin^2\varphi\sin^2\theta\cos\varphi\right) 4\sin\varphi \, d\varphi \, d\theta$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left(8\sin^2\varphi\cos^2\theta\cos\varphi + 8\sin^2\varphi\sin^2\theta\cos\varphi\right)4\sin\varphi \, d\varphi \, d\theta$$

$$= 32\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin^3\varphi\cos\varphi \, d\varphi \, d\theta$$

3. Let $\mathbf{F}(x, y, z) = \mathbf{r}/|\mathbf{r}|^3$, where $\mathbf{r} = \langle x, y, z\rangle$. Show that the flux through the surface of the sphere is independent of the radius. a.

$$\vec{r}(\varphi, \theta) = \langle a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi \rangle$$

$$r_\varphi \times r_\theta = \langle a^2\sin^2\varphi\cos\theta,\ a^2\sin^2\varphi\sin\theta,\ a^2\sin\varphi\cos\varphi \rangle$$

$$\int_0^{2\pi}\int_0^{\pi} \frac{x}{|r|^3}(a^2\sin^2\varphi\cos\theta) + \frac{y}{|r|^3}(a^2\sin^2\varphi\sin\theta) + \frac{z}{|r|^3}(a^2\sin\varphi\cos\varphi)\, d\varphi\, d\theta$$

$$= \frac{a^3\sin^3\varphi\cos^2\theta + a^3\sin^3\varphi\sin^2\theta + a^3\sin\varphi\cos^2\varphi}{a^3}\, d\varphi\, d\theta.$$

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4. Find the center of mass of a hemisphere shell assuming uniform density.

$\rho(x,y,z) = 1.$