MATH2023
Tutorial 6
March 18, 2019 Tutorial Notes by Ivan So

1 Review

- The **double integral** is defined as $\int\int_R f(\mathbf{x})dA := \lim_{n\to\infty}\sum_{i,j=1}^n f(\mathbf{x}_i^*)\Delta A_i$.
- **Fubini's Theorem**: If f is (1) discontinuous on finitely many number of points and (2) bounded over the rectangle $R = \{(x,y)|(x,y) \in [a,b]\times[c,d]\} \subset \mathbb{R}^2$, then

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy.$$

- Consider function of two variables. A region D is said to be of **type I** (**type II**) if $D = \{(x,y)|a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x)\}$ ($D = \{(x,y)|c \leq y \leq d \text{ and } h_1(x) \leq y \leq h_2(x)\}$) where g_1 and g_2 are continuous functions.
- Integration for function over type I or type II region is well-defined. But when changing order of integration, one would have to beware of the integration limits.
- If $R = R_1 \sqcup R_2$, then $\int\int_R f(x,y)dA = \int\int_{R_1} f(x,y)dA + \int\int_{R_2} f(x,y)dA$.
- Recall in **polar coordinates**, $r^2 = x^2+y^2$, $\tan\theta = y/x$. In other words, $x = r\cos\theta$ and $y = r\sin\theta$. Integration of two variable function can be done with polar coordinates, with $dA = rdrd\theta$.

2 Problems

1. True or False

   (a) $\int_1^2\int_3^4 x^2e^y dydx = \int_1^2 x^2dx\int_3^4 e^y dy$. True.

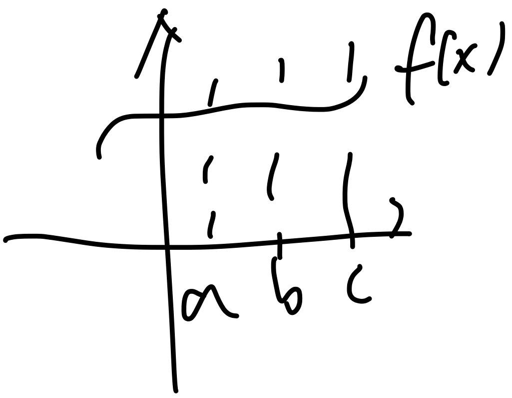
   (b) $\int_0^1\int_0^x \sqrt{x+y^2}dydx = \int_0^x\int_0^1 \sqrt{x+y^2}dxdy$.

2. Sketch the solid bounded by the constraints $0 \leq x, y \leq 1$ and $0 \leq z \leq 4 - x - 2y$. Evaluate its volume.

$$\int_a^b \int_{g_2(x)}^{g_1(x)} f(x,y)\,dy\,dx$$

$$= \int_{a'}^{b'} \int_{g_2^{-1}(y)}^{g_1^{-1}(y)} f(x,y)\,dx\,dy.$$

be careful of the domain

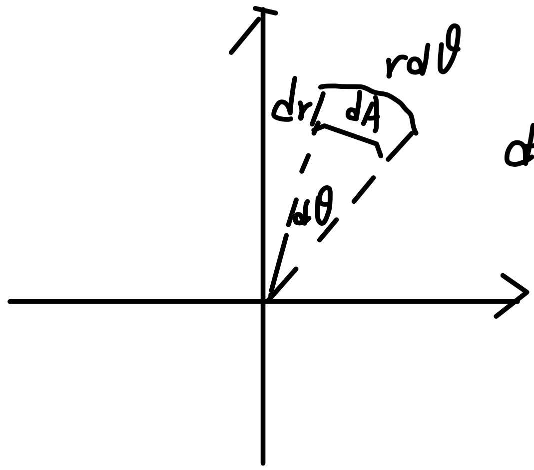


$[a,b]\,[b,c]$

Disjoint.

= area 分开加埋

推 $dA = r\,dr\,d\theta$



$$dA = dr \cdot r\,d\theta$$

3. Show that $0 \leq \int\int_R \sin \pi x \cos \pi x dA \leq \frac{1}{32}$ for $R = [0, 1/4] \times [1/4, 1/2]$.

4. Find the average value of $f(x, y) = x^2 y$ over the rectangle with vertices $(-1, 0), (-1, 5), (1, 5), (1, 0)$.

5. Write the volume integral of the solid bounded by $z = xy$ above a triangle with vertices $(1, 1), (4, 1)$ and $(1, 2)$.

6. Evaluate $\int\int_D x \cos y dA$ over where D is the region bounded by $y = 0,\ y = x^2,\ x = 1$.

7. Prove that if $m \leq f(x, y) \leq M$ for all (x, y) in D, then

$$mA(D) \leq \int\int_D f(x, y) dA \leq MA(D).$$

8. Use polar coordinates to combine and evaluate the sum

$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xy dy dx + \int_{1}^{\sqrt{2}} \int_{0}^{x} xy dy dx + \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{4-x^2}} xy dy dx$$

2 Problems

1. True or False

   (a) $\int_1^2 \int_3^4 x^2 e^y dy dx = \int_1^2 x^2 dx \int_3^4 e^y dy$. True.

1a. True ∵ Check the value. of Integration

Remark: most of the time, if boundary of integrals are constants ⇒ Interchangable (order)

(b) $\int_0^1 \int_0^x \sqrt{x+y^2} dy dx = \int_0^x \int_0^1 \sqrt{x+y^2} dx dy$.

False.

∵ 2nd Integral is absurd.

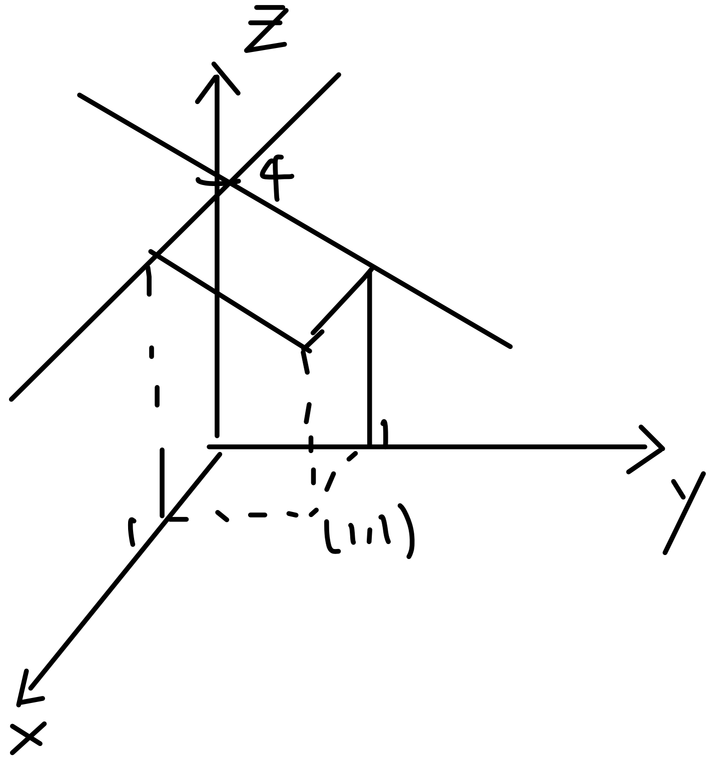
∵ x appears after Integration

but not for LHS.

2. Sketch the solid bounded by the constraints $0 \leq x, y \leq 1$ and $0 \leq z \leq 4 - x - 2y$. Evaluate its volume.

$z = 4 - x - 2y$

$x + 2y + z = 4.$



$$\int_0^1 \int_0^1 4 - x - 2y \; dy . dx$$

$$= \int_0^1 4x - \frac{x^2}{2} - 2xy \Big|_0^1 \; dy$$

$$= \int_0^1 4 - \frac{1}{2} - 2y \; dy$$

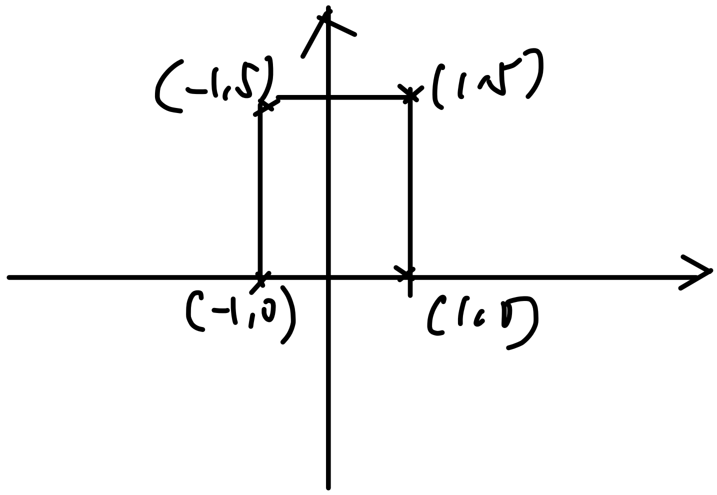
$$= \left[\frac{7}{2} y - y^2 \right]_0^1$$

$$= \frac{7}{2} - 1 = \frac{5}{2}.$$

4. Find the average value of $f(x, y) = x^2y$ over the rectangle with vertices $(-1, 0), (-1, 5), (1, 5), (1, 0)$.

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

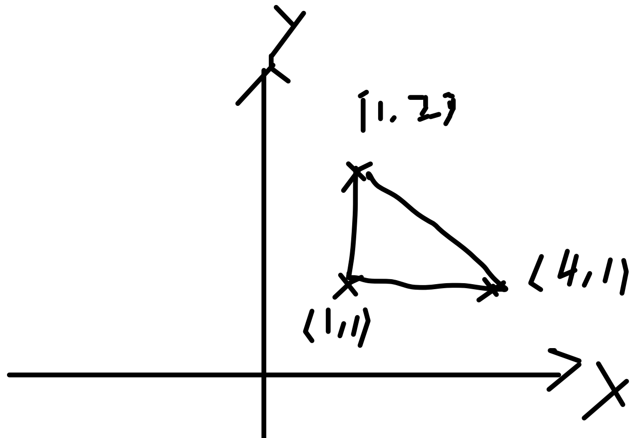
$$\bar{f} = \frac{\int f\,dx}{\int dx}$$ (in one-dimension)



$$\bar{f} = \frac{\int_{-1}^{1}\int_{0}^{5} x^2y\,dy\,dx}{\int_{-1}^{1}\int_{0}^{5} dy\,dx}$$

Made with Goodnotes

5. Write the volume integral of the solid bounded by $z = xy$ above a triangle with vertices $(1, 1), (4, 1)$ and $(1, 2)$.



$y-2 = -\frac{1}{3}(x-1)$

$y = -\frac{1}{3}x + \frac{7}{3}$

$y - \frac{7}{3} = -\frac{1}{3}x$

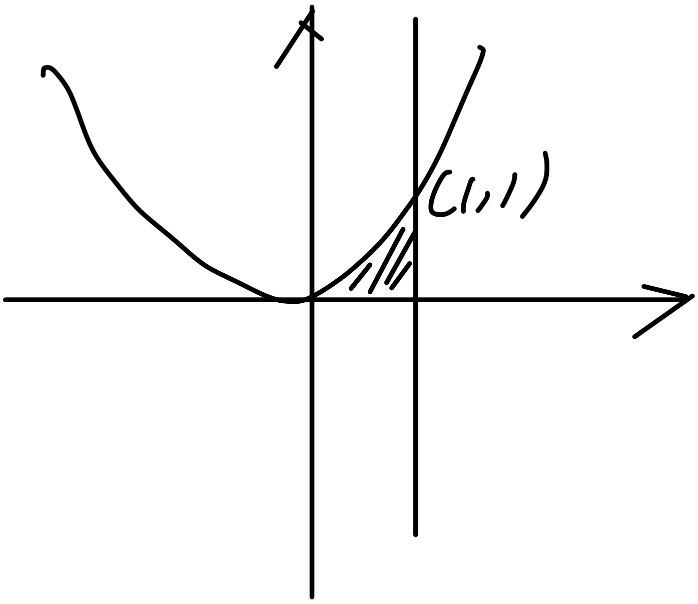
$-3y + 7 = x$

$x = 7 - 3y$

$$V = \int_1^4 \int_1^{-\frac{1}{3}x + \frac{7}{3}} xy \, dy \, dx$$

$$V = \int_1^2 \int_1^{7-3y} xy \, dx \, dy$$

6. Evaluate $\int\int_D x\cos y\,dA$ over where D is the region bounded by $y=0,\ y=x^2,\ x=1$.



$$\int_0^1\int_0^{x^2} x\cos y\,dy\,dx$$

$$=\int_0^1\int_{\sqrt{y}}^1 x\cos y\,dx\,dy$$

7. Prove that if $m \leq f(x,y) \leq M$ for all (x,y) in D, then

$$mA(D) \leq \int\int_D f(x,y)dA \leq MA(D).$$

Recall if $x_i < y_i < z_i$

$$\sum x_i < \sum y_i < \sum z_i$$

$\Leftrightarrow$ if $f < g < h$

then $\int f dA < \int g dA < \int h dA$

$$m \leq f \leq M$$

$$\iint_D m\,dA \leq \iint_D f\,dA \leq \iint_D M\,dA$$

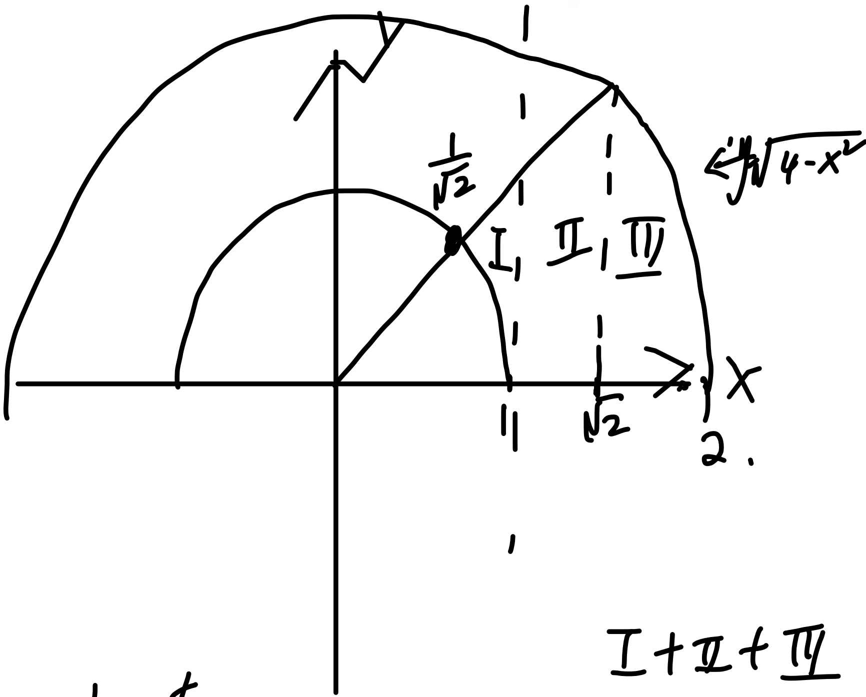
$$mA(D) \leq \iint_D f\,dA \leq MA(D)$$

just const, pull out

$= \iint_D dA$

8. Use polar coordinates to combine and evaluate the sum

$$\overset{\mathrm{I}}{\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xydydx} + \int_{1}^{\sqrt{2}}\int_{0}^{x} xydydx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} xydydx$$



$\mathrm{I}+\mathrm{II}+\mathrm{III} =$

Using polar coordinates:

$x = r\cos\theta$, $y = r\sin\theta$

$$\mathrm{I} = \int_0^{\frac{\pi}{4}}\int_1^2 \underbrace{r\sin\theta\, r\cos\theta}_{(xy)}\ \underbrace{r\,dr\,d\theta}_{(dA)}$$

Made with Goodnotes

9. Evaluate $\int_0^\infty e^{-x^2}\,dx$.

Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$

$$I^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2} dy\right)$$

Since they are independent variable.

$$I^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(e^{-r^2}\right)_0^{\infty} d\theta$$

even function

$$= \pi$$

So

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = I = \sqrt{\pi}$$

$$\parallel$$

$$2\int_0^{+\infty} e^{-x^2} dx = I$$

$$\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$