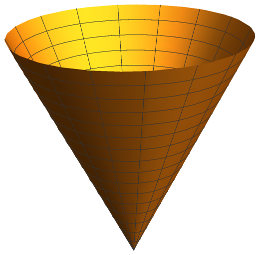
MATH 2023 • Multivariable Calculus
Problem Set #9 • Surface Integrals, Stokes' Theorem

1. (★) Consider the right circular cone surface (just the *shell*, and the flat top is *not* included) with base radius R and height h, and with z-axis as the central axis and the origin as the vertex. See the figure below):



Suppose the cone has uniform surface density σ and its total mass is m.

(a) Write down a parametrization $\mathbf{r}(u, v)$ of the cone, and indicate the range of the parameters. It is OK to use other letters for the pair of parameters.

(b) Find the surface area of the cone.

(c) Find the moment of inertia $I_z := \iint_S (x^2 + y^2)\sigma \, dS$. about the z-axis. Express your final answer in terms of the mass m.

(d) Compute the surface flux of the vector field $\mathbf{F} = \mathbf{i}$ through the cone. Choose $\hat{\mathbf{n}}$ to be the upward unit normal. Do not use Stokes' Theorem in this problem.

Solution: There are at least two ways to parametrize the surface cone.

Using Cylindrical Coordinates: The cone surface is represented by $z = \frac{h}{R}r$ (use similar triangles to figure out the ratio of sides)

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \frac{h}{R}r\mathbf{k}, \quad 0 \le r \le R,\ 0 \le \theta \le 2\pi.$$

By direct computations (omitted here), one can get:

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \left(-\frac{h}{R}r\cos\theta\right)\mathbf{i} + \left(-\frac{h}{R}r\sin\theta\right)\mathbf{j} + r\mathbf{k}$$

$$\left|\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\right| = \sqrt{\frac{h^2r^2}{R^2} + r^2} = r\sqrt{1 + \frac{h^2}{R^2}}$$

Therefore,

$$\begin{aligned}\text{surface area} &= \iint_S dS = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R}\left|\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\right| \, dr d\theta \\ &= \int_0^{2\pi}\int_0^R r\sqrt{1+\frac{h^2}{R^2}} \, dr d\theta = \pi R^2\sqrt{1+\frac{h^2}{R^2}}\end{aligned}$$

$$\begin{aligned}
I_z = \iint_S \delta r^2 dS &= \int_0^{2\pi} \int_{r=0}^{r=R} \delta r^3 \sqrt{1+\frac{h^2}{R^2}}\, dr d\theta \\
&= \delta \cdot \frac{\pi R^4}{2}\sqrt{1+\frac{h^2}{R^2}} \\
&= \frac{m}{\pi R^2 \sqrt{1+\frac{h^2}{R^2}}} \cdot \frac{\pi R^4}{2} \quad \sqrt{1+\frac{h^2}{R^2}} \\
&= \frac{mR^2}{2}
\end{aligned}$$

$\delta = \dfrac{m}{\text{surface area}}$

$$\begin{aligned}
\iint_S \mathbf{i}\cdot\hat{\mathbf{n}}\, dS &= \int_0^{2\pi}\int_{r=0}^{r=R} \mathbf{i}\cdot\left(\frac{\partial \mathbf{r}}{\partial r}\times\frac{\partial \mathbf{r}}{\partial \theta}\right) dr d\theta \\
&= \int_0^{2\pi}\int_0^R -\frac{h}{R} r\cos\theta\, dr d\theta \\
&= -\frac{h}{R}\left(\int_0^{2\pi}\cos\theta\, d\theta\right)\left(\int_0^R r\, dr\right) = 0
\end{aligned}$$

Using Spherical Coordinates: The surface cone is represented by the equation $\varphi = \tan^{-1}\frac{R}{h}$. For simplicity, denote $\alpha = \tan^{-1}\frac{R}{h}$, then the cone is represented by $\varphi = \alpha$ and so:

$$\mathbf{r}(\rho,\theta) = (\rho\sin\alpha\cos\theta)\mathbf{i} + (\rho\sin\alpha\sin\theta)\mathbf{j} + (\rho\cos\alpha)\mathbf{k}, \quad 0 \le \rho \le \sqrt{h^2+R^2},\ 0 \le \theta \le 2\pi.$$

Direct computations give:

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \rho}\times\frac{\partial \mathbf{r}}{\partial \theta} &= (-\rho\sin\alpha\cos\alpha\cos\theta)\mathbf{i} + (-\rho\sin\alpha\cos\alpha\sin\theta)\mathbf{j} + (\rho\sin^2\alpha)\mathbf{k} \\
\left|\frac{\partial \mathbf{r}}{\partial \rho}\times\frac{\partial \mathbf{r}}{\partial \theta}\right| &= \rho\sin\alpha
\end{aligned}$$

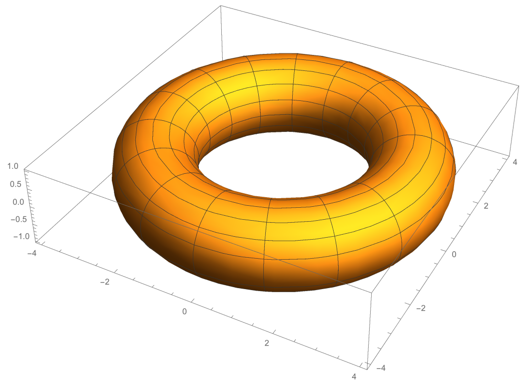
$$\begin{aligned}
\text{surface area} &= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{h^2+R^2}} \rho\sin\alpha\, d\rho d\theta \\
&= \pi(h^2+R^2)\sin\alpha = \pi(h^2+R^2)\sin(\tan^{-1}\frac{R}{h}) \\
&= \pi(h^2+R^2)\cdot\frac{R}{\sqrt{h^2+R^2}} = \pi R\sqrt{h^2+R^2} \\
I_z &= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{h^2+R^2}} \delta\rho^2\sin^2\alpha\cdot\rho\sin\alpha\, d\rho d\theta \\
&= \delta\cdot\frac{(h^2+R^2)^2}{4}\cdot 2\pi\sin^3\alpha \\
&= \frac{m}{\pi R\sqrt{h^2+R^2}}\cdot\frac{(h^2+R^2)^2}{4}\cdot 2\pi\cdot\left(\frac{R}{\sqrt{h^2+R^2}}\right)^3 = \frac{mR^2}{2}
\end{aligned}$$

$$\begin{aligned}\iint_S \mathbf{i}\cdot\hat{\mathbf{n}}\,dS &= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{h^2+R^2}} \mathbf{i}\cdot\left(\frac{\partial \mathbf{r}}{\partial \rho}\times\frac{\partial \mathbf{r}}{\partial \theta}\right)\,d\rho d\theta \\ &= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{h^2+R^2}} (-\rho\sin\alpha\cos\alpha\cos\theta)\,d\rho d\theta \\ &= -\left(\int_0^{2\pi}\cos\theta\,d\theta\right)\left(\int_0^{\sqrt{h^2+R^2}}\rho\,d\rho\right)\cdot\sin\alpha\cos\alpha = 0.\end{aligned}$$

2. (★) Consider the parametrization of a torus (i.e. donut):

$$\mathbf{r}(u,v) = ((R+a\cos u)\cos v)\ \mathbf{i} + ((R+a\cos u)\sin v)\ \mathbf{j} + (a\sin u)\ \mathbf{k}$$

where $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$. Here R and a are constants such that $R > a > 0$.



Suppose the torus has uniform surface density σ and its total mass is m.

(a) Find the surface area of the torus.

(b) Find the moment of inertia $I_z := \oiint_S (x^2+y^2)\sigma\,dS$ about the z-axis. Express your final answer in terms of m.

(c) Compute the surface flux of the vector field $\mathbf{F} = \mathbf{k}$ through the torus. Choose $\hat{\mathbf{n}}$ to be the outward unit normal. Do not use Stokes' Theorem in this problem.

Solution:

$$\begin{aligned}&\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v} \\ &= (-a(R+a\cos u)\cos u\cos v)\mathbf{i} + (-a(R+a\cos u)\cos u\sin v)\mathbf{j} \\ &\quad + (-a(R+a\cos u)\sin u)\mathbf{k}\end{aligned}$$

$$\left|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right| = a(R+a\cos u)$$

$$\begin{aligned}
\text{surface area} &= \int_0^{2\pi}\int_0^{2\pi}\left|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right|\,dudv \\
&= \int_0^{2\pi}\int_0^{2\pi} a(R+a\cos u)\,dudv = 4\pi^2 aR \\
I_z &= \int_0^{2\pi}\int_0^{2\pi} \delta(R+a\cos u)^2\cdot a(R+a\cos u)\,dudv \\
&= \delta\cdot 2\pi^2 aR(3a^2+2R^2) \\
&= \frac{m}{4\pi^2 aR}\cdot 2\pi^2 aR(3a^2+2R^2) \\
&= \frac{m}{2}(3a^2+2R^2) \\
\iint_T \mathbf{k}\cdot\hat{\mathbf{n}}\,dS &= \int_0^{2\pi}\int_0^{2\pi}\mathbf{k}\cdot\left(\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right)\,dudv \\
&= \int_0^{2\pi}\int_0^{2\pi} -a(R+a\cos u)\sin u\,dudv = 0.
\end{aligned}$$

3. (★★) In Chapter 2, we claimed without proof that $\nabla f(P)$ is perpendicular to the level surface $f = c$ at P (we proved the case of level *curves* only). In this problem, we are going to complete the proof for level surfaces.

Let $f(x, y, z)$ be a C^1 function, and S be the level surface $f(x, y, z) = c$. Consider a parametrization $\mathbf{r}(u, v)$ for S, then if one can show $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are both perpendicular to ∇f, then we are done because the normal vector $\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}$ to the surface S will then be parallel to ∇f. By considering $f(\mathbf{r}(u, v)) = c$, show that $\nabla f\cdot\frac{\partial \mathbf{r}}{\partial u} = 0$.

[The fact that $\nabla f\cdot\frac{\partial \mathbf{r}}{\partial v} = 0$ can be shown in a similar way.]

Solution: Since $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ represents a point on the level surface $f(x, y, z) = c$, we have

$$f(x(u,v), y(u,v), z(u,v)) = c \quad \text{or in short} \quad f(\mathbf{r}(u,v)) = c.$$

By chain rule,

$$\begin{aligned}
f(\mathbf{r}(u,v)) &= c \\
\implies \frac{\partial}{\partial u} f(\mathbf{r}(u,v)) &= \frac{\partial c}{\partial u} \\
\implies \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} &= 0 \\
\implies \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\right) &= 0 \\
\implies \nabla f\cdot\frac{\partial \mathbf{r}}{\partial u} &= 0
\end{aligned}$$

Therefore, ∇f is perpendicular to the tangent vector $\frac{\partial \mathbf{r}}{\partial u}$. Similarly, one can show ∇f is perpendicular to the tangent vector $\frac{\partial \mathbf{r}}{\partial v}$. It concludes that ∇f is a normal vector to the level surface $f = c$.

4. (★) Suppose S is a level surface $f(x,y,z)=c$ of a C^1 function f. Show that:

$$\iint_S \nabla f \cdot \hat{\mathbf{n}}\, dS = \pm \iint_S |\nabla f|\; dS$$

where $\pm$ depends on the choice of unit normal $\hat{\mathbf{n}}$.

Solution: Since S is a level surface $f=c$, its unit normal vector is given by:

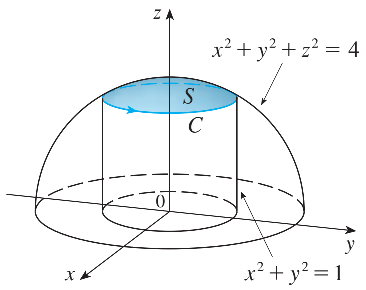
$$\hat{\mathbf{n}} = \pm\frac{\nabla f}{|\nabla f|}.$$

Therefore, we have:

$$\iint_S \nabla f \cdot \hat{\mathbf{n}}\, dS = \pm\iint_S \nabla f \cdot \frac{\nabla f}{|\nabla f|}\, dS = \pm\iint_S \frac{|\nabla f|^2}{|\nabla f|}\, dS = \pm\iint_S |\nabla f|\; dS.$$

Here we have used the fact that $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$.

5. (★) Let S be the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane. Denote C to be the boundary of S with orientation indicated in the diagram below:



(a) Write down the parametrizations of both the surface S and the curve C. For the surface S, choose a *suitable* coordinate system so that the parameters have constant bounds.

Solution:

$$\mathbf{r}(\phi,\theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k};\quad 0\le\phi\le\frac{\pi}{6},\ 0\le\theta\le 2\pi$$

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \sqrt{3}\mathbf{k};\quad 0\le t\le 2\pi$$

(b) Consider the vector field $\mathbf{F}(x,y,z) = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$. Compute both $\oint_C \mathbf{F}\cdot d\mathbf{r}$ and $\iint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\, dS$ directly. Verify that they are equal.

Solution: Along C, we have $x = \cos t$, $y = \sin t$ and $z = \sqrt{3}$.

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^{2\pi} \left(\sqrt{3}\cos t\,\mathbf{i} + \sqrt{3}\sin t\,\mathbf{j} + \cos t\, \sin t\,\mathbf{k}\right) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\, dt \\ &= \int_0^{2\pi} 0\, dt = 0.\end{aligned}$$

To compute $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$, we first compute:

$$\nabla \times \mathbf{F} = (x-y)\mathbf{i} + (x-y)\mathbf{j}.$$

Using the parametrization $\mathbf{r}(\phi, \theta)$, we get:

$$\begin{aligned}\hat{\mathbf{n}}\, dS &= \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}\, d\phi d\theta \\ &= \left\{ \left(4\sin^2\phi\cos\theta\right)\mathbf{i} + \left(4\sin^2\phi\sin\theta\right)\mathbf{j} + \left(4\sin\theta\cos\theta\right)\mathbf{k} \right\} d\phi d\theta\end{aligned}$$

Therefore,

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS &= \int_0^{2\pi}\int_0^{\pi} (x-y)(4\sin^2\phi)(\cos\theta + \sin\theta)\, d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} \underbrace{2\sin\phi(\cos\theta - \sin\theta)}_{x-y}\; 4\sin^2\phi\; (\cos\theta + \sin\theta)\, d\phi d\theta \\ &= 8\left(\int_0^{2\pi} (\cos^2\theta - \sin^2\theta)\, d\theta\right)\left(\int_0^{\pi} \sin^3\phi\, d\phi\right) \\ &= 0\end{aligned}$$

6. (★) Let C be the simple closed curve given parametrized by:

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}, \quad 0 \le t \le 2\pi.$$

(a) Show that the curve lies on the surface $z = 2xy$.

Solution: Use the identity $\sin 2t = 2\sin t\, \cos t$.

(b) Use the Stokes' Theorem to evaluate the line integral:

$$\oint_C e^{x^2} dx + yzdy + \frac{x^2}{2} dz.$$

[Why is it difficult to compute this line integral *directly*?]

Solution: The line integral can be written as $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F} = e^{x^2}\mathbf{i} + yz\mathbf{j} + \frac{x^2}{2}\mathbf{k}$$

By straight-forward computations, we get:

$$\nabla \times \mathbf{F} = -y\mathbf{i} - x\mathbf{j} + 0\mathbf{k}.$$

Let Σ be the surface $z = 2xy$ enclosed by C. It can be parametrized by cylindrical coordinates:

$$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \underbrace{(2r^2\sin\theta\cos\theta)}_{z=2xy}\mathbf{k}, \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

Using Stokes' Theorem, we have:

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_\Sigma (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS \\
&= \int_0^{2\pi}\int_0^1 \underbrace{(-y\mathbf{i} - x\mathbf{j})}_{\nabla\times\mathbf{F}} \cdot \underbrace{\frac{\nabla(z-2xy)}{|\nabla(z-2xy)|}}_{\hat{\mathbf{n}}} \underbrace{\left|\frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta}\right| dr d\theta}_{dS} \\
&= \int_0^{2\pi}\int_0^1 (-y\mathbf{i} - x\mathbf{j})\cdot\frac{-2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}}{\sqrt{1+4x^2+4y^2}}\, r\sqrt{1+4r^2}\,dr d\theta \\
&= \int_0^{2\pi}\int_0^1 2(x^2+y^2)\,\frac{1}{\sqrt{1+4r^2}}\cdot r\cdot\sqrt{1+4r^2}\,dr d\theta \\
&= \int_0^{2\pi}\int_0^1 2r^3\,dr d\theta = \pi.
\end{aligned}$$

7. (★) Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$. Show that the line integral

$$\oint_C 2y dx + 3z dy - x dz$$

depends only on the area of the region enclosed by C on the above given plane and the orientation of C, but not on the position or shape of C.

Solution: The line integral can be written as $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where

$$\mathbf{F} = 2y\mathbf{i} + 3z\mathbf{j} - x\mathbf{k}.$$

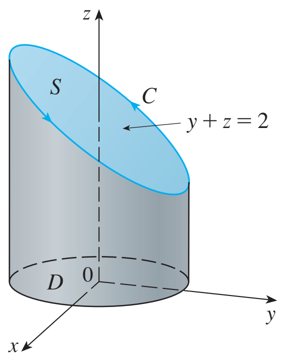
By calculation, we get: $\nabla\times\mathbf{F} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Let Σ be the region enclosed by C on the plane $2x + 2y + z = 2$, then the unit normal to the plane is given by:

$$\hat{\mathbf{n}} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{9}}$$

Stokes' Theorem asserts that:

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \pm\iint_\Sigma (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS \qquad (\pm \text{ depends on orientation of } C) \\
&= \pm\iint_\Sigma (-3\mathbf{i} + \mathbf{j} - 2\mathbf{k})\cdot\frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{9}}\,dS \\
&= \mp\frac{6}{\sqrt{9}}\iint_\Sigma 1\,dS = \mp 2 \times \text{area of } \Sigma
\end{aligned}$$

8. (★★) Consider the curve of intersection C of the plane $y+z=2$ and the cylinder $x^2+y^2=1$, with orientation shown in the diagram below. The surface S is the planar region enclosed by C, and its projection onto the xy-plane is denoted by D.



(a) Using a suitable coordinate system, write down a parametrization of S such that the parameters have constant bounds.

> **Solution:** Use cylindrical coordinates:
>
> $$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \underbrace{(2 - r\sin\theta)}_{z=2-y}\mathbf{k}$$
>
> where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

(b) Evaluate the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}$ is given by:

$$\mathbf{F}(x,y,z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$$

> **Solution:**
>
> $$\nabla\times\mathbf{F} = (1+2y)\mathbf{k}$$
>
> Using Stokes' Theorem (note that $\mathbf{F}$ is defined and C^1 everywhere in $\mathbb{R}^3$), we get:
>
> $$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS \\ &= \int_0^{2\pi}\int_0^1 (1+2y)\mathbf{k}\cdot\frac{\mathbf{j}+\mathbf{k}}{\sqrt{2}}\left|\frac{\partial\mathbf{r}}{\partial r}\times\frac{\partial\mathbf{r}}{\partial\theta}\right| dr d\theta \\ &= \int_0^{2\pi}\int_0^1 \frac{1+2r\sin\theta}{\sqrt{2}}\,|r\mathbf{j}+r\mathbf{k}|\,dr d\theta \\ &= \int_0^{2\pi}\int_0^1 \frac{1+2r\sin\theta}{\sqrt{2}}\,\sqrt{2}r\,dr d\theta \\ &= \int_0^{2\pi}\int_0^1 (r+2r^2\sin\theta)\,dr d\theta \\ &= \pi\end{aligned}$$

(c) Let $\mathbf{G}(x,y,z) = -\dfrac{y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j} + \dfrac{1}{z+1}\mathbf{k}$.

i. Verify that $\nabla \times \mathbf{G} = \mathbf{0}$ at every point in the domain of $\mathbf{G}$. Does this result determine that $\mathbf{G}$ is conservative or not?

> **Solution:** Showing $\nabla \times \mathbf{G} = \mathbf{0}$ is straight-forward. Note that the domain of $\mathbf{G}$ is $\mathbb{R}^3$ removing the z-axis and the horizontal plane $z = -1$. It is not a simply-connected domain. We cannot conclude that $\mathbf{G}$ is conservative or not based the result $\nabla \times \mathbf{G} = \mathbf{0}$.

ii. Denote Γ to be the projection of C onto the xy-plane. Using the Stokes' Theorem in an *appropriate* way, show that:

$$\oint_C \mathbf{G} \cdot d\mathbf{r} = \oint_\Gamma \mathbf{G} \cdot d\mathbf{r}.$$

> **Solution:** Similar to Worksheet #22, Q3.

iii. Evaluate $\oint_C \mathbf{G} \cdot d\mathbf{r}$ using the above result.

> **Solution:** Γ can be parametrized by $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0\mathbf{k}$, where $0 \le t \le 2\pi$. We can compute the line integral $\oint_\Gamma \mathbf{G} \cdot d\mathbf{r}$ directly.
>
> $$\begin{aligned}\oint_\Gamma \mathbf{G} \cdot d\mathbf{r} &= \int_0^{2\pi} \left(-\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} + \frac{1}{z+1}\mathbf{k}\right) \cdot \underbrace{\left((-\sin t)\mathbf{i} + (\cos t)\mathbf{j}\right)}_{\mathbf{r}'(t)} \, dt \\ &= \int_0^{2\pi} \left(-\frac{\sin t}{1}\mathbf{i} + \frac{\cos t}{1}\mathbf{j} + \frac{1}{0+1}\mathbf{k}\right) \cdot \left(-(\sin t)\mathbf{i} + (\cos t)\mathbf{j}\right) dt \\ &= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = 2\pi.\end{aligned}$$

9. (★) Two of the four Maxwell's Equations (Faraday's and Ampère's Laws) assert that:

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\end{aligned}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the current, and c, μ_0 and ε_0 are positive constants. Using Stokes' Theorem, show that for any (stationary) simply-connected orientable surface S with boundary C, we have:

$$\begin{aligned}\oint_C \mathbf{E} \cdot d\mathbf{r} &= -\frac{1}{c}\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot \hat{\mathbf{n}} \, dS \\ \oint_C \mathbf{B} \cdot d\mathbf{r} &= \mu_0 \iint_S \mathbf{J} \cdot \hat{\mathbf{n}} \, dS + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \iint_S \mathbf{E} \cdot \hat{\mathbf{n}} \, dS\end{aligned}$$

[You don't need to know any physics to do this problem.]

Solution:

$$\begin{aligned} -\frac{1}{c}\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot\hat{\mathbf{n}}\,dS &= -\frac{1}{c}\iint_S \frac{\partial}{\partial t}\mathbf{B}\cdot\hat{\mathbf{n}}\,dS \\ &= \iint_S (\nabla\times\mathbf{E})\cdot\hat{\mathbf{n}}\,dS && \text{(Given)} \\ &= \oint_C \mathbf{E}\cdot d\mathbf{r} && \text{(Stokes' Theorem)} \end{aligned}$$

Similarly,

$$\begin{aligned} \mu_0\iint_S \mathbf{J}\cdot\hat{\mathbf{n}}\,dS + \mu_0\varepsilon_0\frac{\partial}{\partial t}\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,dS &= \iint_S\left(\mu_0\mathbf{J}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\cdot\hat{\mathbf{n}}\,dS \\ &= \iint_S (\nabla\times\mathbf{B})\cdot\hat{\mathbf{n}}\,dS \\ &= \oint_C \mathbf{B}\cdot d\mathbf{r} \end{aligned}$$

10. (★★) Let $\mathbf{F}(x,y,z) = \langle 0, -\frac{z}{2}, \frac{y}{2}\rangle$.

(a) Show that $\nabla\times\mathbf{F} = \mathbf{i}$.

Solution: Straight-forward

(b) Find vector fields $\mathbf{G}$ and $\mathbf{H}$ such that $\nabla\times\mathbf{G} = \mathbf{j}$ and $\nabla\times\mathbf{H} = \mathbf{k}$.

Solution: Mimic the vector field $\mathbf{F}$ in part (a), and some trial-and-errors:

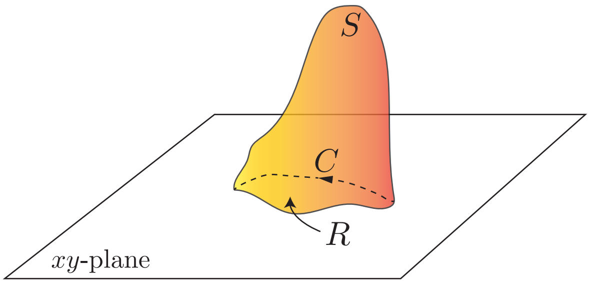
$$\begin{aligned} \nabla\times\left\langle \frac{z}{2}, 0, -\frac{x}{2}\right\rangle &= \mathbf{j} \\ \nabla\times\left\langle -\frac{y}{2}, \frac{x}{2}, 0\right\rangle &= \mathbf{k} \end{aligned}$$

Let $\mathbf{G} = \langle \frac{z}{2}, 0, -\frac{x}{2}\rangle$ and $\mathbf{H} = \langle -\frac{y}{2}, \frac{x}{2}, 0\rangle$.
[Note that the problem only requires us to find one such $\mathbf{G}$ and one such $\mathbf{H}$. In fact it can be shown that all other possible $\mathbf{G}$'s are given by:

$$\left\langle \frac{z}{2}, 0, -\frac{x}{2}\right\rangle + \nabla f$$

where f is any C^2 scalar functions defined everywhere in $\mathbb{R}^3$. Similar for $\mathbf{H}$.]

(c) Let C be an arbitrary simple closed curve on the xy-plane in the three dimensional space, and S is any surface *above* the xy-plane with boundary curve C. See the figure below.



Show that:

$$\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = c \times \text{area of the region on the } xy\text{-plane enclosed by } C.$$

Here a, b and c are all constants.

Solution: Using (a) and (b), we get:

$$a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = a(\nabla \times \mathbf{F}) + b(\nabla \times \mathbf{G}) + c(\nabla \times \mathbf{H}) = \nabla \times (a\mathbf{F} + b\mathbf{G} + c\mathbf{H}).$$

Using Stokes' Theorem, we get:

$$\begin{aligned}
\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS &= \iint_S (\nabla \times (a\mathbf{F} + b\mathbf{G} + c\mathbf{H})) \cdot \hat{\mathbf{n}}\, dS \\
&= \oint_C (a\mathbf{F} + b\mathbf{G} + c\mathbf{H}) \cdot d\mathbf{r} && \text{(Stokes')} \\
&= \iint_R (\nabla \times (a\mathbf{F} + b\mathbf{G} + c\mathbf{H})) \cdot \mathbf{k}\, dA && \text{(Green's)} \\
&= \iint_R (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \mathbf{k}\, dA \\
&= \iint_R c\, dA = c \times \text{area of } R
\end{aligned}$$

(d) Using the results of (a), (b), and the Stokes' Theorem, redo Problems #1(d) and #2(c).

Solution: For #1(d), use the fact that $\nabla \times \left\langle 0, -\frac{z}{2}, \frac{y}{2} \right\rangle = \mathbf{i}$ and Stokes' Theorem:

$$\begin{aligned}
\iint_{\text{cone}} \mathbf{i} \cdot \hat{\mathbf{n}}\, dS &= \iint_{\text{cone}} \left(\nabla \times \left\langle 0, -\frac{z}{2}, \frac{y}{2} \right\rangle\right) \cdot \hat{\mathbf{n}}\, dS \\
&= \oint_{\text{boundary circle}} \left\langle 0, -\frac{z}{2}, \frac{y}{2} \right\rangle \cdot d\mathbf{r} && \text{(Stokes')}
\end{aligned}$$

The boundary circle of the cone is parametrized by:

$$\mathbf{r}(t) = (R\cos t)\mathbf{i} + (R\sin t)\mathbf{j} + h\mathbf{k}, \quad 0 \le t \le 2\pi.$$

Therefore,

$$\oint_{\text{boundary circle}} \left\langle 0, -\frac{z}{2}, \frac{y}{2} \right\rangle \cdot d\mathbf{r} = \int_0^{2\pi} \left\langle 0, -\frac{h}{2}, \frac{R}{2}\sin t \right\rangle \cdot \langle -R\sin t, R\cos t, 0\rangle \, dt$$
$$= \int_0^{2\pi} -\frac{hR}{2}\cos t \, dt = 0.$$

#2(c) is to calculate the flux of $\mathbf{k}$ through the surface torus. From part (c) we know that $\mathbf{k} = \nabla \times \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle$. Therefore, $\mathbf{k}$ is a solenoidal vector field. From Worksheet #22 Q2 we can use the Stokes' Theorem (after cutting the torus along two circles so that it becomes simply-connected) to conclude that the surface flux is zero.

Optional – about the Gauss-Bonnet's Theorem

11. Given a oriented surface S with parametrization $\mathbf{r}(u, v)$, we denote:

$$E = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u} \qquad F = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \qquad G = \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial v}$$
$$e = \frac{\partial^2 \mathbf{r}}{\partial u^2} \cdot \hat{\mathbf{n}} \qquad f = \frac{\partial^2 \mathbf{r}}{\partial u \partial v} \cdot \hat{\mathbf{n}} \qquad g = \frac{\partial^2 \mathbf{r}}{\partial v^2} \cdot \hat{\mathbf{n}}$$

The *Gauss curvature* at the point $\mathbf{r}(u, v)$ is defined to be:

$$K(u, v) := \frac{eg - f^2}{EG - F^2}.$$

The geometric intuition behind the Gauss curvature *may* be covered in MATH 4223. In Differential Geometry, there is a beautiful theorem – the Gauss-Bonnet's Theorem – which asserts that if S is closed, oriented and smooth (without corners), then:

$$\oiint_S K \, dS = 4\pi(1 - \text{ number of holes of } S)$$

Therefore, if S is a sphere, the above surface integral should be 4π as there is no hole. If S is a torus (which has one hole), the above surface integral should be 0. Verify this theorem for the sphere and torus, by parametrizing them and compute the above integral directly over the sphere and the torus.

As an optional problem, you may use computer softwares to ease your calculations.