

## Exercise 16.4

Qu. 4 If  $\mathbf{F} = x^3 \mathbf{i} + 3yz^2 \mathbf{j} + (3y^2z + x^2) \mathbf{k}$ , then

$$\begin{aligned}\nabla \cdot \mathbf{F} &= 3x^2 + 3z^2 + 3y^2, \text{ and} \\ \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS &= \iiint_G \nabla \cdot \mathbf{F} dV \quad (\text{Divergence Theorem}) \\ &= 3 \iiint_G (x^2 + y^2 + z^2) dV \\ &= 3 \int_0^{2\pi} \int_0^\pi \int_0^a \rho^2 \cdot \rho^2 \sin \phi d\rho d\phi d\theta \\ &= \frac{12}{5} \pi a^5.\end{aligned}$$

Qu. 8 Since the cylinder is a closed surface, so we can use the divergence theorem to do this question

$$\mathbf{F} = x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k}$$

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z$$

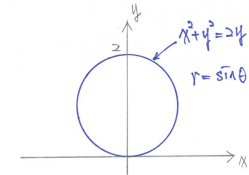
$$\begin{aligned}\therefore \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS &= \iiint_G \nabla \cdot \mathbf{F} dV \quad (\text{where } G \text{ is the cylinder}) \\ &= 2 \iiint_G (x + y + z) dV\end{aligned}$$

$$\begin{aligned}\therefore \iiint_G x dV &= \int_0^\pi \int_0^{2\sin\theta} \int_0^4 r \cos \theta \cdot r dz dr d\theta \\ &= 4 \int_0^\pi \int_0^{2\sin\theta} r^2 \cos \theta dr d\theta \\ &= 4 \times \frac{8}{3} \int_0^\pi \cos \theta \sin^3 \theta d\theta \\ &= \frac{32}{3} \sin^4 \theta \Big|_0^\pi = 0\end{aligned}$$

$$\begin{aligned}\iiint_G y dV &= \int_0^\pi \int_0^{2\sin\theta} \int_0^4 r \sin \theta \cdot r dz dr d\theta \\ &= 4 \int_0^\pi \int_0^{2\sin\theta} r^2 \sin \theta dr d\theta \\ &= \frac{32}{3} \int_0^\pi \sin^4 \theta d\theta \\ &= \frac{32}{3} \frac{1}{4} \left( \frac{3}{2} - 2 \cos 2\theta + \frac{1}{4} \cos 4\theta \right) \Big|_0^\pi = 4\pi\end{aligned}$$

$$\begin{aligned}\iiint_G z dV &= \int_0^\pi \int_0^{2\sin\theta} \int_0^4 z \cdot r dz dr d\theta \\ &= 8 \int_0^\pi \int_0^{2\sin\theta} r dr d\theta \\ &= 16 \int_0^\pi \sin^2 \theta d\theta \\ &= 16 \times \frac{1}{2} \left( \theta - \frac{1}{2} \sin 2\theta \right) \Big|_0^\pi = 8\pi\end{aligned}$$

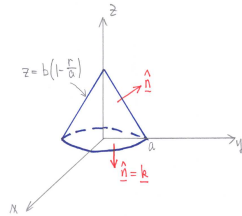
$$\therefore \iint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = 2(0 + 4\pi + 8\pi) = 24\pi.$$



**Qu. 11** This is a closed surface, we can use Divergence Theorem to do this question.

$$\mathbf{F} = (x + y^2)\mathbf{i} + (3x^2y + y^3 - x^3)\mathbf{j} + (z + 1)\mathbf{k}$$

$$\nabla \cdot \mathbf{F} = 1 + 3(x^2 + y^2) + 1 = 2 + 3(x^2 + y^2).$$



Let  $G$  be the conical domain,  $S$  its conical surface, and  $B$  its base disk, as shown in the figure.

We have

$$\begin{aligned} \iint_{S+B} \mathbf{F} \cdot \hat{\mathbf{n}} dS &= \iiint_G \nabla \cdot \mathbf{F} dV \\ &= \iiint_G [2 + 3(x^2 + y^2)] dV \\ &= \int_0^{2\pi} \int_0^a \int_0^{b(1-\frac{r}{a})} (2 + 3r^2)r dz dr d\theta \quad (\text{Cylindrical coord}) \\ &= 2\pi b \int_0^a r(2 + 3r^2)(1 - \frac{r}{a}) dr \\ &= 2\pi b \int_0^a (2r + 3r^3 - \frac{2r^2}{a} - \frac{3r^4}{a}) dr \\ &= \frac{2\pi a^2 b}{3} + \frac{3\pi a^4 b}{10}. \end{aligned}$$

**Qu. 23** Note that  $\nabla \cdot (\phi \mathbf{F}) = \phi \nabla \cdot \mathbf{F} + \nabla \phi \cdot \mathbf{F}$ , thus

$$\begin{aligned} \iiint_D \phi \nabla \cdot \mathbf{F} dV + \iiint_D \nabla \phi \cdot \mathbf{F} dV &= \iiint_D \nabla \cdot (\phi \mathbf{F}) dV \\ &= \iint_S \phi \mathbf{F} \cdot \hat{\mathbf{n}} dS \quad (\text{Divergence Theorem}) \end{aligned}$$

**Qu. 24** If  $\mathbf{F} = \nabla \phi$  in Qu. 23, then  $\nabla \cdot \mathbf{F} = \nabla^2 \phi$  and

$$\iiint_D \phi \nabla^2 \phi dV + \iint_S \|\nabla \phi\|^2 dV = \iint_S \phi \nabla \phi \cdot \hat{\mathbf{n}} dS.$$

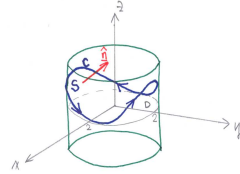
If  $\nabla^2 \phi = 0$  in  $D$  and  $\phi = 0$  on  $S$ , then

$$\iiint_D \|\nabla \phi\|^2 dV = 0.$$

Since  $\phi$  is assumed to be smooth,  $\nabla \phi = 0$  throughout  $D$ , and therefore  $\phi$  is constant on each connected component of  $D$ . Since  $\phi = 0$  on  $S$ , these constants must all be zero, and  $\phi = 0$  on  $D$ .

## Exercise 16.5

**Qu. 2** Let  $S$  be the part of the surface  $z = y^2$  lying inside the cylinder  $x^2 + y^2 = 4$ , and having upward normal  $\hat{\mathbf{n}}$ . Then  $C$  is the oriented boundary of  $S$ . Let  $D$  be the disk  $x^2 + y^2 \leq 4$ ,  $z = 0$ , that is, the projection of  $S$  onto the  $xy$ -plane. If  $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}$ , then

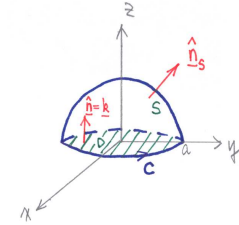


$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z^2 \end{vmatrix} = -2\mathbf{k}.$$

Let  $f(x, y, z) = z - y^2 = 0$ , this is a level surface in 3D, so  $\nabla f = (0, -1, 1) = \mathbf{n}$ , so

$$\begin{aligned} \oint_C y dx - x dy + z^2 dz &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\ &= \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} dS \quad (\text{Stoke's Theorem}) \\ &= \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA \\ &= \iint_D (-2\mathbf{k}) \cdot (-\mathbf{j} + \mathbf{k}) dA \\ &= \iint_D (-2) dA \\ &= -2\pi(2)^2 \\ &= -8\pi. \end{aligned}$$

**Qu. 3** Let  $C$  be the circle  $x^2 + y^2 = a^2$ ,  $z = 0$ , oriented counter-clockwise as seen from the positive  $z$ -axis. Let  $D$  be the disk bounded by  $C$ , with normal  $\hat{\mathbf{n}} = \mathbf{k}$ .



$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2xz & x^2 - y^2 \end{vmatrix} \\ &= 2(x - y)\mathbf{i} + 2x\mathbf{j} - (2z + 3)\mathbf{k}. \end{aligned}$$

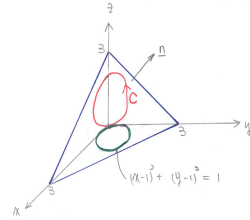
$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}_S dS &= \oint_C \mathbf{F} \cdot d\mathbf{r} \quad (\text{Stoke's Theorem}) \\ &= \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA \quad (\text{Green's Theorem}) \\ &= - \iint_D 3 dA \\ &= -3\pi a^2. \end{aligned}$$

# Homework 10

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**Qu. 8** Note that  $\mathbf{r}(0) = 2\mathbf{i} + \mathbf{j} = \mathbf{r}(2\pi)$ , therefore  $\mathbf{r}(t)$  is a closed space curve for  $0 \leq t \leq 2\pi$ . So we can use Stoke's Theorem.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^x & x + e^x & z^2 \end{vmatrix} = \mathbf{k}.$$



Note also that

$$x = 1 + \cos t$$

$$y = 1 + \sin t$$

$$z = 1 - \sin t - \cos t$$

so we have  $x + y + z = 3$  and  $(x - 1)^2 + (y - 1)^2 = 1$

i.e.  $C$  lies on the surface  $x + y + z = 3$  and  $(x - 1)^2 + (y - 1)^2 = 1$ .

In fact,  $C$  is the boundary of the elliptic disk in the plane  $x + y + z = 3$  lying inside the cylinder  $(x - 1)^2 + (y - 1)^2 = 1$ .

Let  $f(x, y, z) = x + y + z = 3$ , this is a level surface in 3D, so  $\nabla f = (1, 1, 1) = \mathbf{n}$ .

$$\begin{aligned} \text{Therefore } \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \\ &= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA \\ &= \iint_R \mathbf{k} \cdot (1, 1, 1) dA \\ &= \iint_R dA \\ &= \pi(1)^2 = \pi. \end{aligned}$$

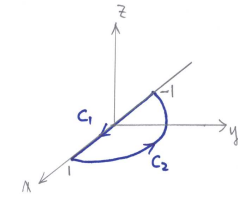
Alternatively, note that  $\mathbf{F} = ye^x \mathbf{i} + (x + e^x) \mathbf{j} + z^2 \mathbf{k} = x \mathbf{j} + \nabla \phi$ , where  $\phi = ye^x + \frac{z^3}{3}$ . The curve  $C$  is a closed curve in a simply connected domain, so

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C x \mathbf{j} \cdot d\mathbf{r} + \oint_C \nabla \phi \cdot d\mathbf{r} \\ &= \oint_C x dy \\ &= \int_0^{2\pi} (1 + \cos t) \cos t dt \\ &= \int_0^{2\pi} \cos^2 t dt = \pi. \end{aligned}$$

# Homework 10

# MATH2023

**Qu. 9** If  $S_1$  and  $S_2$  are two surfaces joining  $C_1$  to  $C_2$  each having upward normal, then the closed surface  $S_3$  consisting of  $S_1$  and  $-S_2$  (that is,  $S_2$  with downward normal) bound a region  $G$  in 3-space. Then



$$\begin{aligned} \iint_{S_1} \mathbf{F} \cdot \mathbf{n} dS - \iint_{S_2} \mathbf{F} \cdot \mathbf{n} dS &= \iint_S \mathbf{F} \cdot \mathbf{n} dS + \iint_{-S_2} \mathbf{F} \cdot \mathbf{n} dS \\ &= \iint_{S_3} \mathbf{F} \cdot \mathbf{n} dS \\ &= \pm \iiint_G \nabla \cdot \mathbf{F} dV \\ &= 0 \end{aligned}$$

provided that  $\nabla \cdot \mathbf{F} \equiv 0$ . Since

$$\mathbf{F} = (\alpha x^2 - z) \mathbf{i} + (xy + y^3 + z) \mathbf{j} + \beta y^2(z + 1) \mathbf{k}$$

we have

$$\begin{aligned} \nabla \cdot \mathbf{F} &= 2\alpha x + x + 3y^2 + \beta y^2 \\ &= 0 \quad \text{if } \alpha = -\frac{1}{2} \quad \text{and } \beta = -3. \end{aligned}$$

In this case we can evaluate  $\iint_S \mathbf{F} \cdot \mathbf{n} dS$  for any such surface  $S$  by evaluating the special case

where  $S$  is the half-disk  $H : \{x^2 + y^2 \leq 1, z = 0, y \geq 0\}$ , with upward normal  $\mathbf{n} = \mathbf{k}$ . We have

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= -3 \iint_H y^2 dA \\ &= -3 \int_0^\pi \int_0^1 r^2 \sin \theta r dr d\theta \\ &= -\frac{3\pi}{8}. \end{aligned}$$