香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
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數學系
DEPARTMENT OF MATHEMATICS

SAMPLE MIDTERM

Course Code: MATH 2023
Course Title: Multivariable Calculus
Time Limit: 2 Hours

Instructions

- Do **NOT** open the exam until instructed to do so.
- This is a **CLOSED BOOK, CLOSED NOTES** exam.
- All mobile phones and communication devices should be switched **OFF**.
- Only calculators approved by HKEAA can be used.
- Answer **ALL** eight problems.
- You must **SHOW YOUR WORK** to receive credits in all problems except Problem #1. Answers alone (whether correct or not) will not receive any credit.
- Some problems are structured into several parts. You can quote the results stated in the preceding parts to do the next part, regardless of whether you can complete the preceding parts or not. However, different parts in the same problem are not necessarily co-related.

About this practice test

The purpose of this sample midterm is to let students get a rough idea of the style of problems and the format of the exam. Do **NOT** expect the actual exam is simply a minor variation of this sample exam. The level of difficulties, the point allocation of each problem, and the choice of topics may be different from the actual test. For better preparation of the test, students should extensively review the course materials covered in class, in the lecture notes and in the lecture worksheets, and should have worked seriously on the Problem Sets and WebWorks.

Problem	1	2	3	4	5	6	7	8	Total
Max	25	10	8	10	10	15	12	10	100
Score									

Made with Goodnotes

FORMULAE TABLE

$$
\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 \\
1 + \tan^2\theta &= \sec^2\theta \\
1 + \cot^2\theta &= \csc^2\theta \\
\tan\theta &= \frac{\sin\theta}{\cos\theta} \\
\sin(\theta \pm \phi) &= \sin\theta\cos\phi \pm \cos\theta\sin\phi \\
\cos(\theta \pm \phi) &= \cos\theta\cos\phi \mp \sin\theta\sin\phi \\
\tan(\theta \pm \phi) &= \frac{\tan(\theta) \pm \tan(\phi)}{1 \mp \tan\theta\tan\phi} \\
\sin(2\theta) &= 2\sin\theta\cos\theta \\
\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\
&= 1 - 2\sin^2\theta \\
&= 2\cos^2\theta - 1 \\
\tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta}
\end{aligned}
$$

For any C^2 function $f(x,y)$ and $z = f(a + tu_1, b + tu_2)$, we have:

$$
\begin{aligned}
\frac{d^2z}{dt^2} &= f_{xx}u_1^2 + 2f_{xy}u_1u_2 + f_{yy}u_2^2 \\
&= f_{xx}\left[\left(u_1 + \frac{f_{xy}}{f_{xx}}u_2\right)^2 + \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}^2}\right)u_2^2\right] \qquad \text{if } f_{xx} \neq 0
\end{aligned}
$$

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1. Answer the following conceptual questions. No justification is needed.

(a) Let Π be the plane in $\mathbb{R}^3$ whose equation is given by $x + 2y + 3z = 1$. Write down a three-variable function $g(x, y, z)$ such that Π is a level set of g. /1

ANSWER: $g(x,y,z) = x+2y+3z$

Acceptable answer
$x+2y+3z-1$

(b) Let Π be the plane in $\mathbb{R}^3$ whose equation is given by $x + 2y + 3z = 1$. Write down a two-variable function $f(x, y)$ such that the plane Π is the graph of this function f. /2

ANSWER: $f(x,y) = \frac{1}{3}(1-x-2y)$

$x+2y+3z=1$
$3z = 1-x-2y$
$z = \frac{1}{3}(1-x-2y)$

(c) Consider the parametric curve $\mathbf{r}(t) = (\cos 2t)\,\mathbf{i} + (\sin 2t)\,\mathbf{j}$. Write down an arc-length parametrization $\mathbf{r}(s)$ of this curve. /2

ANSWER: $\vec{r}(s) = (\cos s)\hat{\imath} + (\sin s)\hat{\jmath}$

(d) Let $f(x, y)$ be a C^1 function such that $\nabla f(0, 0) = 3\mathbf{i} + 4\mathbf{j}$. Write down a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}} f(0, 0)$ is maximized. when $\nabla f(0,0)$ and $\hat{u}$ point in the same direction /2

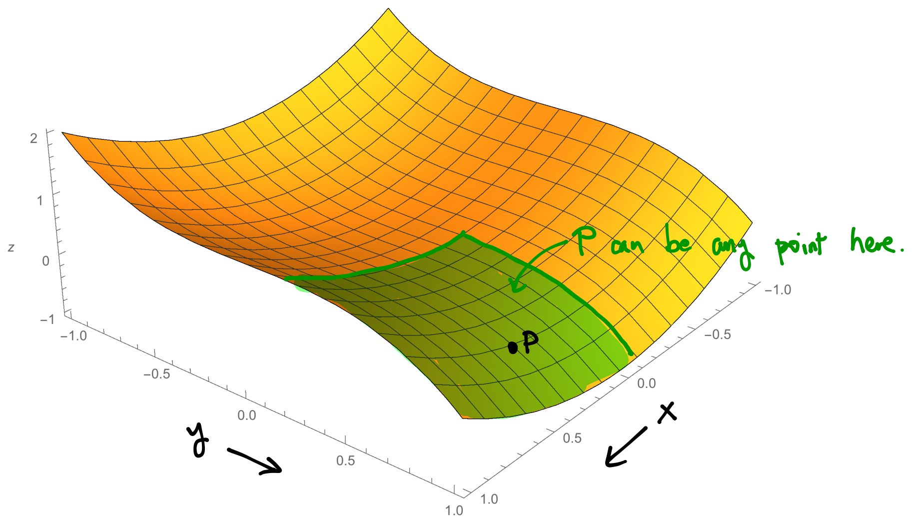
ANSWER: $\hat{u} = \frac{\nabla f(0,0)}{|\nabla f(0,0)|} = \frac{3\hat{\imath}+4\hat{\jmath}}{5}$

(e) Let $f(x, y)$ be a C^1 function. Suppose Π is the tangent plane to the graph $z = f(x, y)$ at a point $(x_0, y_0, f(x_0, y_0))$. Write down an upward-pointing unit normal $\hat{\mathbf{n}}$ to the plane Π. /2

$\nabla(z - f(x,y)) = \langle -f_x, -f_y, 1\rangle$

ANSWER: $\vec{n} = -\frac{\partial f}{\partial x}(x_0,y_0)\hat{\imath} - \frac{\partial f}{\partial y}(x_0,y_0)\hat{\jmath} + \hat{k}$

(f) The following diagram shows the graph of the function $f(x, y) = x^2 - y^3$. The positive z-axis is pointing up. However, the labels for the x- and y-axes are missing. /4



i. On the diagram above, label the x- and y-axes.

ii. Mark a point P on the graph at which $f_x(P) > 0$ and $f_{yy}(P) < 0$.

Problem #1 continues on next page. . .

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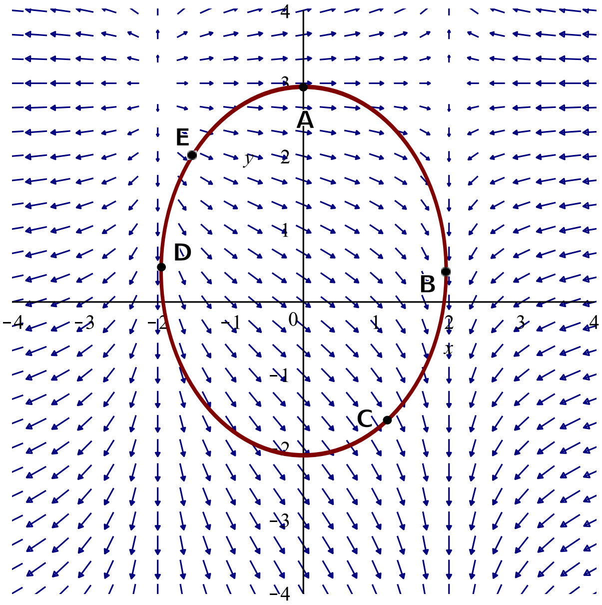
(g) Let $f(x,y)$ be a C^1 function. The diagram below is the plot of the vector field ∇f. The ellipse in the diagram is a level set $g(x,y)=c$ of another C^1 function g. /8



i. What is $\frac{\partial f}{\partial x}(a,b)$ when $a=2$?

i. 0

ii. How many critical point(s) of f is/are there in the region $-4 \le x \le 0$ and $-4 \le y \le 4$?

ii. 1

iii. Which of the point A, B, C, D and E has/have (x,y)-coordinates solving the following Lagrange's Multiplier system? State all such point(s).

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad g(x,y)=c$$

iii. C , E

iv. Suppose $f(x,y)$ represents the temperature at point (x,y). Now you are located at point A and want to cool down as quickly as possible. Which direction will you go along?

iv. $-\hat{i}$ (or : west, left)

(h) Let $f(x,y)$ be a C^2 function such that (x_0,y_0) is its critical point. Suppose we know: /2

$$\left(f_{xx}f_{yy} - f_{xy}^2\right)(x_0,y_0) > 0 \quad \text{and} \quad f_{yy}(x_0,y_0) < 0$$

$f_{xx} f_{yy} > f_{xy}^2 > 0$

What is the nature of the critical point (x_0,y_0)? Circle the correct answer:

local minimum   local maximum   saddle   not enough data

(i) Let $f(x,y)$ be a C^2 function such that (x_0,y_0) is its critical point. Suppose we know: /2

$$f_{xx}(x_0,y_0) = 25, \; f_{xy}(x_0,y_0) = 5, \; f_{yy}(x_0,y_0) = 1.$$

What is the nature of the critical point (x_0,y_0)? Circle the correct answer:

local minimum   local maximum   saddle   not enough data

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2. Suppose $u(x,y)$ and $v(x,y)$ are C^2 functions defined on $\mathbb{R}^2$ which satisfy the relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (*)$$

[FYI: It is so-called the Cauchy-Riemann relations in Complex Analysis.]

(a) Show that $v(x,y)$ satisfies the Laplace equation: **/4**

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x}$$

$= 0$ (By Mixed Partials Theorem, note that u is C^2 and so u_{xy} and u_{yx} are continuous)

(b) Suppose we define $v(x,y) = x^3 - 3xy^2 + 10$. Find a function $u(x,y)$ such that u and v satisfy the relation (*) above. **/6**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -6xy \quad \text{— ①}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -3x^2 + 3y^2 \quad \text{— ②}$$

① $\Rightarrow$ $u(x,y) = \int -6xy \, dx = -3x^2y + g(y)$ ← arbitrary function of y

Need to find $g(y)$:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(-3x^2y + g(y)\right) = -3x^2 + g'(y)$$

② $\Rightarrow$ $-3x^2 + 3y^2 = -3x^2 + g'(y) \Rightarrow g'(y) = 3y^2 \Rightarrow g(y) = y^3 + C$

$\therefore u(x,y) = -3x^2y + y^3 + C$

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3. Consider the function $f(x,y) = 1 - x^2 - y^2$ and the point $P = (\sqrt{a}, 0, 0)$.

(a) Find the value of a such that the point P is on the graph of f. **/1**

Graph of f : $z = f(x,y) = 1 - x^2 - y^2$. Sub $(x,y,z) = (\sqrt{a}, 0, 0)$

$0 = 1 - a - 0 \Rightarrow a = 1$

(b) Find a *downward-pointing* normal vector $\mathbf{n}$ to the graph of f at P. **/4**

$z - f(x,y) = 0$

Let $g(x,y,z) = z - f(x,y)$

then the graph is a level-set $g = 0$

$\nabla g = \langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \rangle = \langle 2x, 2y, 1 \rangle$

$\nabla g(P) = \langle 2, 0, 1 \rangle$

$\therefore n = \langle -2, 0, -1 \rangle$

downward-pointing

(c) Find an equation of the tangent plane to the graph of f at P. **/3**

$\vec{n} = \langle -2, 0, -1 \rangle$ $\quad P(1, 0, 0)$

Equation of the tangent plane:

$-2x + 0y - 1z = (-2)(1) + 0 \cdot 0 + (-1)(0)$

$\Rightarrow -2x - z = -2$ (or : $2x + z = 2$)

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4. The spherical coordinates (ρ, θ, φ) in $\mathbb{R}^3$ are related to the rectangular coordinates (x, y, z) by the following conversion rule:

$$\begin{aligned} x &= \rho \ \sin\varphi \ \cos\theta \\ y &= \rho \ \sin\varphi \ \sin\theta \\ z &= \rho \ \cos\varphi \end{aligned}$$

Let $f(x, y, z)$ be a three-variable function. By the above conversion rule, f can also be regarded as a function of (ρ, θ, φ).

(a) Using the chain rule, show that: **/6**

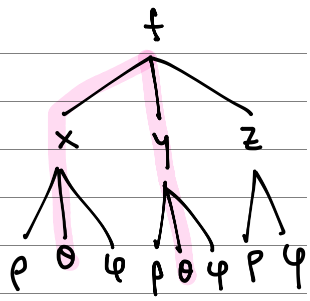
$$\frac{\partial f}{\partial \theta} = -y\,\frac{\partial f}{\partial x} + x\,\frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial f}{\partial x}\cdot\frac{\partial}{\partial \theta}(\rho\sin\varphi\cos\theta) + \frac{\partial f}{\partial y}\cdot\frac{\partial}{\partial \theta}(\rho\sin\varphi\sin\theta)$$

$$= \frac{\partial f}{\partial x}\cdot(\underbrace{-\rho\sin\varphi\sin\theta}_{=y}) + \frac{\partial f}{\partial y}\cdot(\underbrace{\rho\sin\varphi\cos\theta}_{=x})$$

$$= -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}$$



(b) Suppose $\nabla f(x, y, z)$ is orthogonal to the vector $-y\mathbf{i} + x\mathbf{j}$ for any (x, y, z) in $\mathbb{R}^3$. Show that f does not depend on θ. **/4**

Given $\nabla f \cdot (-y\hat{\imath} + x\hat{\jmath}) = 0$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot(-y\hat{\imath} + x\hat{\jmath}) = 0$$

$$\Rightarrow -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0$$

From (a), $\frac{\partial f}{\partial \theta} = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0$

$\therefore f$ does not depend on θ.

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5. Consider the function $f(x,y) = e^{-(x^2+y^2)} = e^{-x^2} \cdot e^{-y^2}$

(a) Find all critical point(s) of $f(x,y)$. /4

$\frac{\partial f}{\partial x} = -2x e^{-x^2} \cdot e^{-y^2}$ $\quad$ $\frac{\partial f}{\partial y} = -2y e^{-x^2} \cdot e^{-y^2}$

$e^{\text{any real}}$ must be >0.

Solve : $\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Rightarrow \begin{cases} -2x\, e^{-x^2-y^2} = 0 \Rightarrow x = 0 \\ -2y\, e^{-x^2-y^2} = 0 \Rightarrow y = 0 \end{cases}$

$\therefore$ The only critical point : $(0,0)$

(b) For each critical point of f found in (a)(i), determine whether it is a local maximum, a local minimum or a saddle point. /6

$f_{xx} = -2e^{-x^2-y^2} - 2x \cdot (-2x e^{-x^2-y^2})$
$= (4x^2-2)e^{-x^2-y^2}$

$f_{xy} = -2x e^{-x^2} \cdot (-2y e^{-y^2})$
$= 4xy\, e^{-x^2-y^2}$

$f_{yx} = 4xy\, e^{-x^2-y^2}$ $\quad$ $f_{yy} = (4y^2-2)e^{-x^2-y^2}$

At $(0,0)$:

$f_{xx}(0,0) = -2$ $\quad$ $f_{yy}(0,0) = -2$ $\quad$ $f_{xy} = 0$

$\therefore (f_{xx} f_{yy} - f_{xy}^2)(0,0) = 4 > 0$

$\therefore$ $(0,0)$ is a local maximum.

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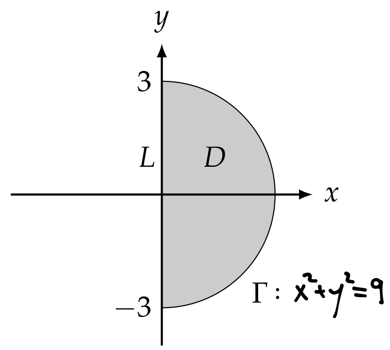
6. Consider the function $f(x, y) = x^2 + 2y^2 - 4x$. Let D be the semi-disk region (shaded in the diagram below) defined by:

$$x \geq 0 \quad \text{and} \quad x^2 + y^2 \leq 9.$$

The boundary of D has two components:

- L is the line segment joining $(0, 3)$ and $(0, -3)$; and
- Γ is the circular arc joining $(0, 3)$ and $(0, -3)$.



(a) Using Lagrange's Multiplier, find the minimum value of $f(x, y)$ when (x, y) is restricted to the curve Γ. /8

Minimize : $f(x,y) = x^2 + 2y^2 - 4x$

Subject to : $g(x,y) = x^2 + y^2 = 9$ (and $x \geq 0$)

$$\begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \\ g(x,y) = 9 \end{cases} \Rightarrow \begin{cases} 2x - 4 = 2\lambda x & \text{— ①} \\ 4y = 2\lambda y & \text{— ②} \\ x^2 + y^2 = 9 & \text{— ③} \end{cases}$$

Case a) : $4y = 2\lambda y \neq 0$

$\frac{①}{②} \Rightarrow \frac{2x-4}{4y} = \frac{2\lambda x}{2\lambda y} \Rightarrow \frac{x-2}{2y} = \frac{x}{y} \Rightarrow xy - 2y = 2xy$

$\Rightarrow x - 2 = 2x$ (as $y \neq 0$ in this case)

$\Rightarrow x = -2$ (rejected as $x \geq 0$)

Case b) : $4y = 2\lambda y = 0$, then $y = 0$

③ $\Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

$\therefore (x,y) = (3,0)$ or $(-3,0)$ ← rejected

	(x,y)	$f(x,y)$	
	$(3,0)$	-3	— min
end points of Γ	$(0,3)$	18	max
	$(0,-3)$	18	

Problem #6 continues on next page...

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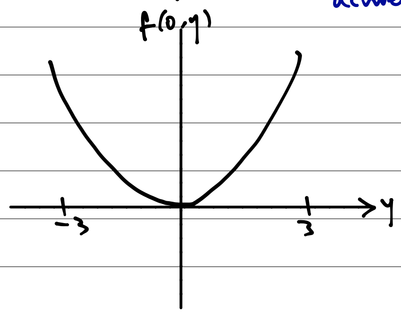
For easy reference: $f(x,y) = x^2 + 2y^2 - 4x$.

(b) Find the minimum value of $f(x,y)$ when (x,y) is restricted to the line L. /3

On L: $x=0$, $-3 \le y \le 3$

$f(0,y) = 2y^2$ ← achieve minimum when $y=0$. $f(0,0)=0$



$\therefore$ Min value of f on $L = 0$

(c) Determine the minimum value of $f(x,y)$ over the *solid* region D. /4

Left to locate interior critical points:

$\nabla f(x,y) = (2x-4)\hat{i} + 4y\,\hat{j}$

Solve for $\nabla f(x,y) = \vec{0} \Rightarrow \begin{cases} 2x-4=0 \Rightarrow x=2 \\ 4y=0 \Rightarrow y=0 \end{cases}$

$(x,y) = (2,0)$ is in the solid region D.

Compare the value of $f(x,y)$ at all candidates points:

	(x,y)	$f(x,y)$	
interior critical point →	$(2,0)$	-4	← min.
corner points	$(0,-3)$	18	
	$(0,3)$	18	
min. pt on Γ →	$(3,0)$	-3	
min. pt. on L →	$(0,0)$	0	

Minimum of f on $D = -4$.

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7. Let $\mathbf{c}(t)$ be parametric curve in $\mathbb{R}^3$ such that $|\mathbf{c}(t)| = 1$ at all time t. Define another parametric curve $\mathbf{r}(t)$ by:

$$\mathbf{r}(t) = f(t)\mathbf{c}(t).$$

(a) Show that $\mathbf{c}(t)$ and $\mathbf{c}'(t)$ are orthogonal at all time t. /3

Given: $|\vec{c}(t)| = 1 \Rightarrow |\vec{c}(t)|^2 = 1$

$\Rightarrow \vec{c}(t) \cdot \vec{c}(t) = 1$

$$\frac{d}{dt}\left[\vec{c}(t) \cdot \vec{c}(t)\right] = \frac{d}{dt}1 = 0$$

$$\frac{d}{dt}\vec{c}(t) \cdot \vec{c}(t) + \vec{c}(t) \cdot \frac{d}{dt}\vec{c}(t) = 0$$

$$\Rightarrow \underbrace{\vec{c}'(t) \cdot \vec{c}(t)} + \underbrace{\vec{c}(t) \cdot \vec{c}'(t)} = 0$$

same

$$\Rightarrow 2\vec{c}'(t) \cdot \vec{c}(t) = 0$$

$$\Rightarrow \vec{c}'(t) \cdot \vec{c}(t) = 0$$

$\therefore \vec{c}'(t) \perp \vec{c}(t)$ at all time t.

(b) Using (a), or otherwise, show that $|\mathbf{r}'(t)| = \sqrt{f(t)^2\,|\mathbf{c}'(t)|^2 + f'(t)^2}$. /4

$$\vec{r}(t) = f(t)\,\vec{c}(t) \Rightarrow \vec{r}'(t) = \frac{d}{dt}\left(f(t)\,\vec{c}(t)\right)$$

$$= \frac{d}{dt}f(t)\,\vec{c}(t) + f(t)\frac{d}{dt}\vec{c}(t)$$

$$= f'(t)\,\vec{c}(t) + f(t)\,\vec{c}'(t)$$

$$|\vec{r}'(t)|^2 = \vec{r}'(t) \cdot \vec{r}'(t)$$

$$= \left(f'(t)\,\vec{c}(t) + f(t)\,\vec{c}'(t)\right) \cdot \left(f'(t)\,\vec{c}(t) + f(t)\,\vec{c}'(t)\right)$$

$$= f'(t)^2\,\vec{c}(t) \cdot \vec{c}(t) + 2f'(t)f(t)\,\underbrace{\vec{c}(t) \cdot \vec{c}'(t)}_{=0 \text{ from (a)}} + f(t)^2\,\vec{c}'(t) \cdot \vec{c}'(t)$$

$$= f'(t)^2\,\underbrace{|\vec{c}(t)|^2}_{=1 \text{ (given)}} + f(t)^2\,|\vec{c}'(t)|^2$$

$$= f'(t)^2 + f(t)^2\,|\vec{c}'(t)|^2$$

$$\Rightarrow |\vec{r}'(t)| = \sqrt{f'(t)^2 + f(t)^2\,|\vec{c}'(t)|^2}$$

Problem #7 continues on next page...

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(c) Using (b), or otherwise, find an arc-length parametrization $\mathbf{q}(s)$ of the following curve: **/5**

$$\mathbf{q}(t) = e^{-t}\left[\left(\frac{\overset{3}{\cancel{4}}}{5}\sin t\right)\,\mathbf{i} + \left(\frac{3}{5}\cos t\right)\,\mathbf{k}\right]$$

Let $f(t) = \frac{3}{5}e^{-t}$ and $\vec{c}(t) = (\sin t)\hat{\imath} + (\cos t)\hat{k}$

then $|\vec{c}(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$ and $\vec{c}\,'(t) = (\cos t)\hat{\imath} + (-\sin t)\hat{k}$

and $\vec{q}(t) = f(t)\,\vec{c}(t)$ $\Rightarrow |\vec{c}\,'(t)| = 1$

From (a), $|\vec{q}\,'(t)| = \sqrt{f(t)^2|\vec{c}\,'(t)|^2 + f'(t)^2}$

$$= \sqrt{\frac{9}{25}e^{-2t} + \left(-\frac{3}{5}e^{-t}\right)^2}$$

$$= \sqrt{\frac{18}{25}}\, e^{-t}$$

$$\therefore\ s = \int_0^t |\vec{q}\,'(u)|\,du = \int_0^t \sqrt{\frac{18}{25}}\, e^{-u}\,du$$

$$= \sqrt{\frac{18}{25}}\left[-e^{-u}\right]_{u=0}^{u=t}$$

$$= \sqrt{\frac{18}{25}}\left(1 - e^{-t}\right)$$

$$\Rightarrow 1 - e^{-t} = \sqrt{\frac{25}{18}}\, s \Rightarrow e^{-t} = 1 - \sqrt{\frac{25}{18}}\, s \Rightarrow -t = \ln\left(1 - \sqrt{\frac{25}{18}}\, s\right)$$

$$\Rightarrow t = -\ln\left(1 - \sqrt{\frac{25}{18}}\, s\right) = \ln\frac{1}{1 - \sqrt{\frac{25}{18}}\, s}$$

$$\therefore\ \vec{q}(s) = \left(1 - \sqrt{\frac{25}{18}}\, s\right)\left[\left(\frac{3}{5}\sin \ln\frac{1}{1 - \sqrt{\frac{25}{18}}\, s}\right)\hat{\imath} + \left(\frac{3}{5}\cos \ln\frac{1}{1 - \sqrt{\frac{25}{18}}\, s}\right)\hat{k}\right]$$

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8. Given a fixed point (x_0, y_0, z_0) in $\mathbb{R}^3$, we denote its position vector by $\mathbf{p} = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$. Let $\mathbf{r}(t)$ be the path of a particle in $\mathbb{R}^3$. Given that $\mathbf{r}(t) - \mathbf{p}$ is parallel to $\mathbf{r}''(t)$ at all time t.

Define:

$$\mathbf{L_p}(t) := (\mathbf{r}(t) - \mathbf{p}) \times \mathbf{r}'(t).$$

(a) Show that $\mathbf{L_p}(t)$ does not depend on t. /5

$$\frac{d\vec{L}_p}{dt} = \frac{d}{dt}\left((\vec{r}(t)-\vec{p}) \times \vec{r}'(t)\right)$$

$$= (\vec{r}(t)-\vec{p})' \times \vec{r}'(t) + (\vec{r}(t)-\vec{p}) \times \vec{r}''(t)$$

$$= \vec{r}'(t) \times \vec{r}'(t) + (\vec{r}(t)-\vec{p}) \times \vec{r}''(t) \qquad (\text{as } \vec{p}' = \vec{0})$$

$$= \underbrace{\vec{0}}_{\text{as } \vec{a}\times\vec{a}=\vec{0}} + \vec{0} \quad \text{as } \vec{r}(t)-\vec{p} \,/\!/\, \vec{r}''(t).$$

(b) Using (a), show that the path of the particle is contained in a single plane. /5
[Hint: consider the dot product $\mathbf{L_p} \cdot (\mathbf{r}(t) - \mathbf{p})$.]

(a) $\Rightarrow$ $\vec{L}_p(t)$ is a constant vector.

Let $\vec{L}_p = A\hat{i} + B\hat{j} + C\hat{k}$ where A, B, C are constants

By definition of $\vec{L}_p := (\vec{r}(t) - \vec{p}) \times \vec{r}'(t)$, it must be orthogonal to $\vec{r}(t) - \vec{p}$.

$$\therefore \vec{L}_p \cdot (\vec{r}(t)-\vec{p}) = 0$$

$$\Rightarrow (A\hat{i}+B\hat{j}+C\hat{k}) \cdot \left[\left(\underbrace{x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}}_{\vec{r}(t)}\right) - \left(\underbrace{x_0\hat{i}+y_0\hat{j}+z_0\hat{k}}_{\vec{p}}\right)\right] = 0$$

$$\Rightarrow (A\hat{i}+B\hat{j}+C\hat{k}) \cdot \left((x(t)-x_0)\hat{i} + (y(t)-y_0)\hat{j} + (z(t)-z_0)\hat{k}\right) = 0$$

$$\Rightarrow A(x(t)-x_0) + B(y(t)-y_0) + C(z(t)-z_0) = 0$$

Hence $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is always contained in the plane

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0.$$

End of Exam

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