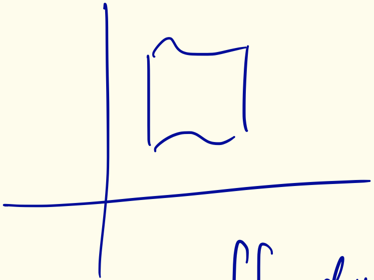
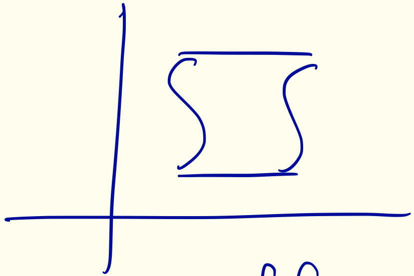
Last Time

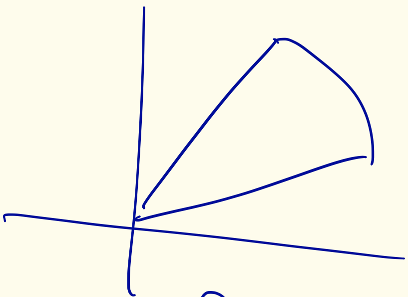
$$\iint_D f(x,y)\,dA$$



$$\iint_R dy\,dx$$



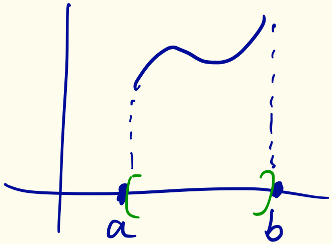
$$\iint_R dx\,dy$$



$$\iint_R r\,dr\,d\theta$$

Goal: Generalize Fundamental Theorem of Calculus

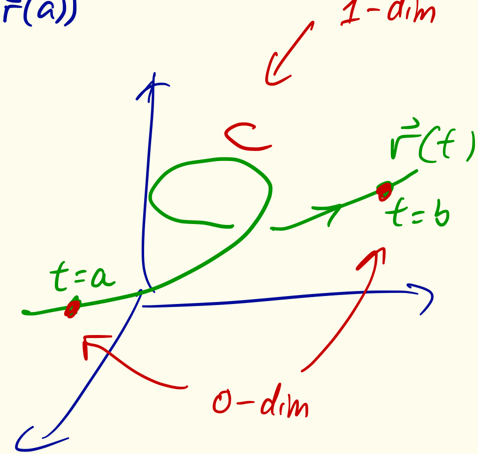
FTC: $$\int_a^b F'(x)\,dx = F(b) - F(a)$$



$\langle f_x, f_y, f_z \rangle$ $\quad$ $\langle dx, dy, dz \rangle$

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

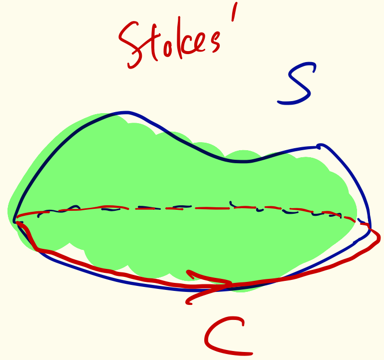
Vector fields integration

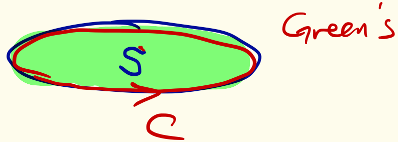


Green's Theorem / Stokes' Theorem :

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

2-dim curl 1-dim

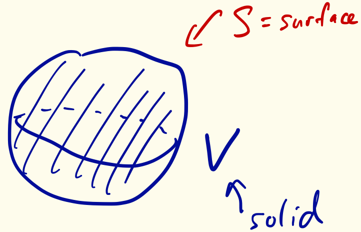




Divergence Theorem

$$\iiint_V \nabla \cdot \vec{F} \, dV = \iint_S \vec{F} \cdot d\vec{S}$$

3-dim div. 2-dim



Generalized Stokes' Theorem

$$\int_{\Omega} d\omega = \int_{\partial \Omega} \omega$$

differential forms.

n-dim (n-1)-dim

recover above formula when $n = 1, 2, 3$.

Vector Fields

(e.g. $\nabla f = \langle f_x, f_y \rangle$ gradient field)

$$\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j} \qquad (\vec{F} = \langle P, Q \rangle)$$

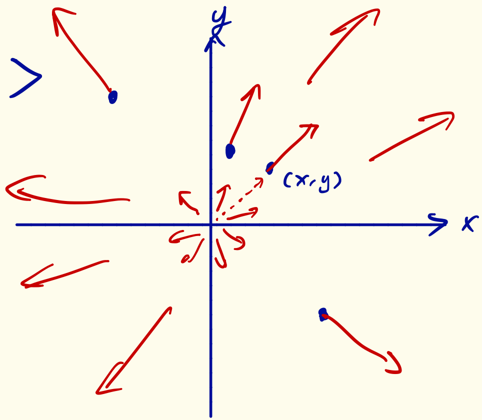
$$\vec{F}(x,y,z) = P(x,y,z)\,\vec{i} + Q(x,y,z)\,\vec{j} + R(x,y,z)\,\vec{k} \qquad (\vec{F} = \langle P, Q, R \rangle)$$

[Given a coordinate, assign a vector]

Very Important Examples

Divergence field:

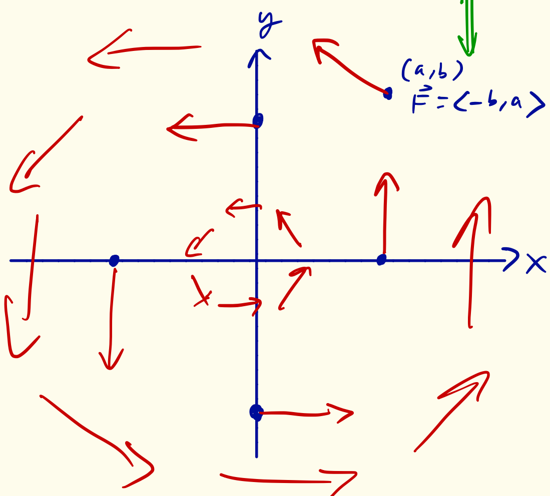
$\vec{F} = \langle x, y \rangle$



Curl Field

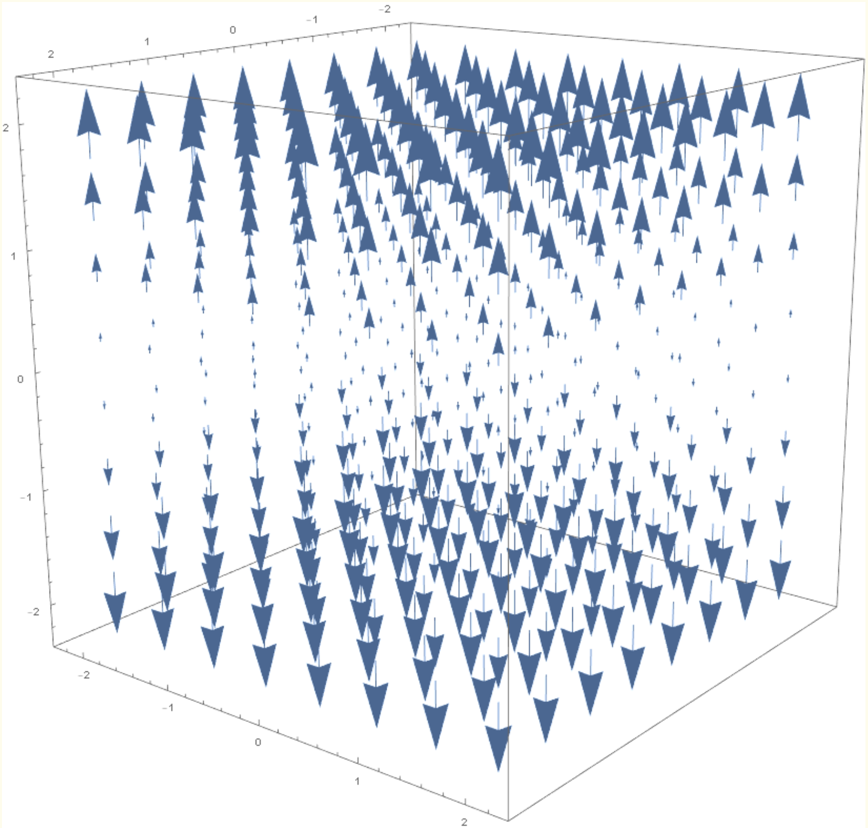
$\vec{F} = \langle -y, x \rangle$

$\perp$ to pos. vector

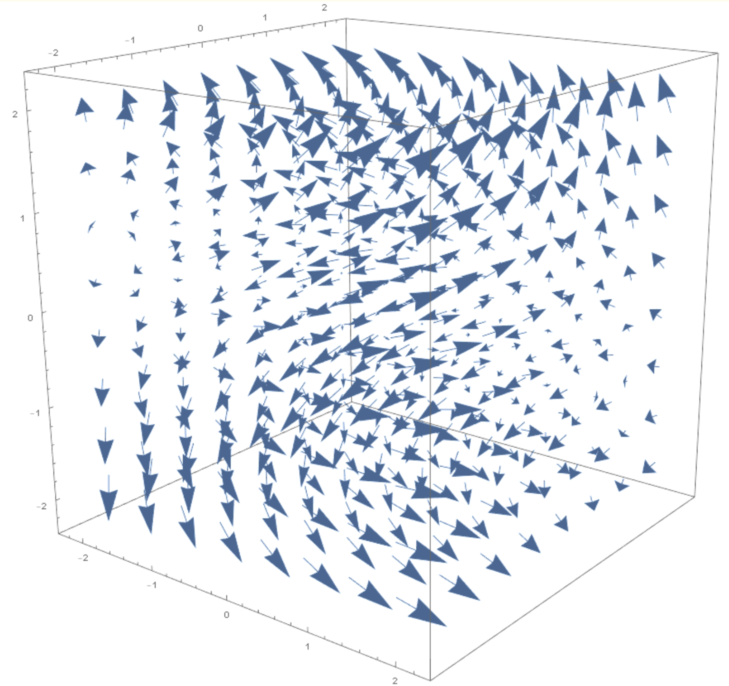


3D Fields

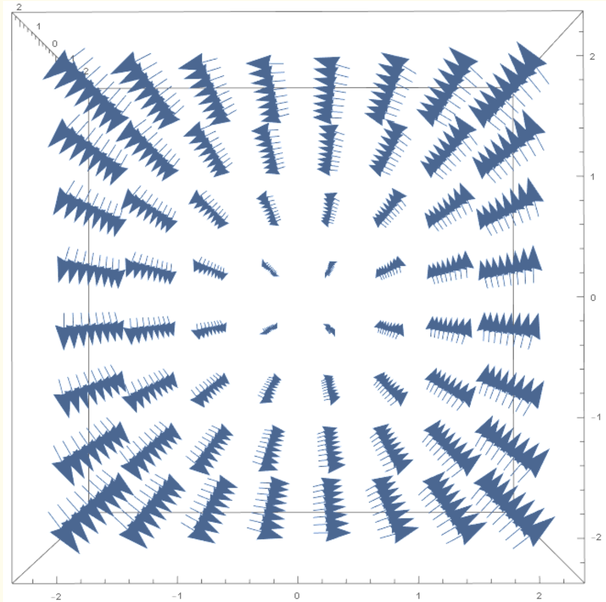
$\vec{F} = z\vec{k} = \langle 0, 0, z \rangle$



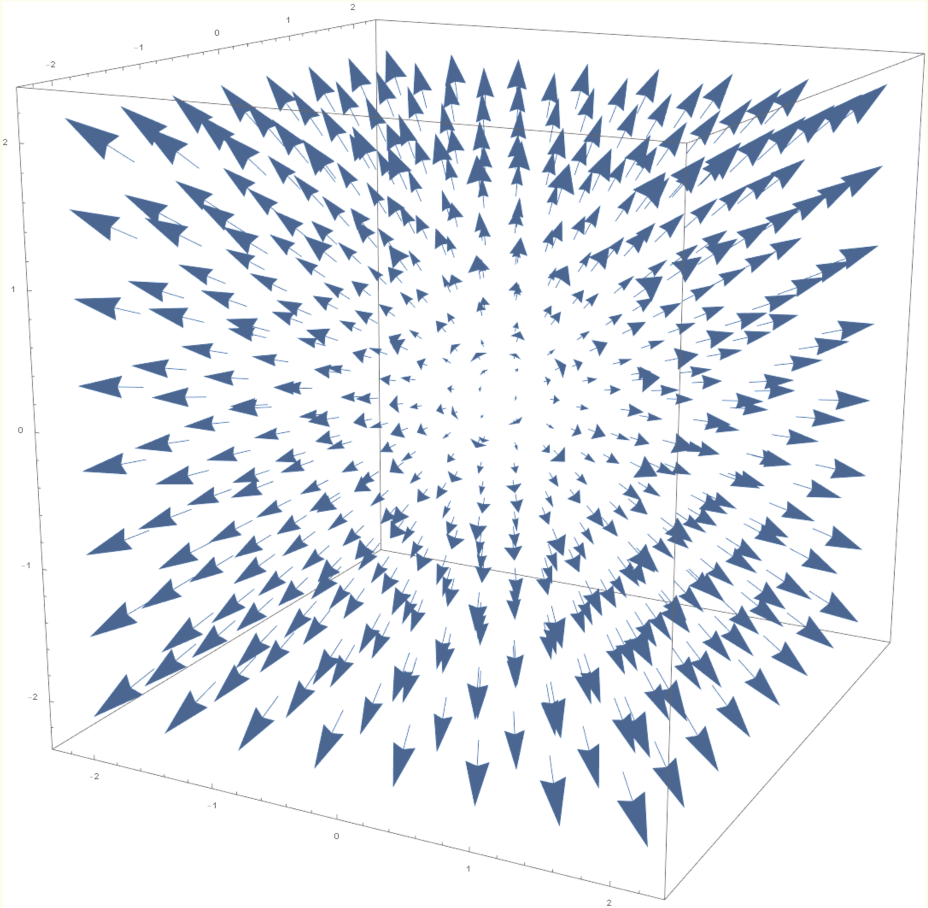
$\vec{F} = \langle -y, x, z \rangle$



from top.



$\vec{F} = \langle x, y, z \rangle$ divergence field.

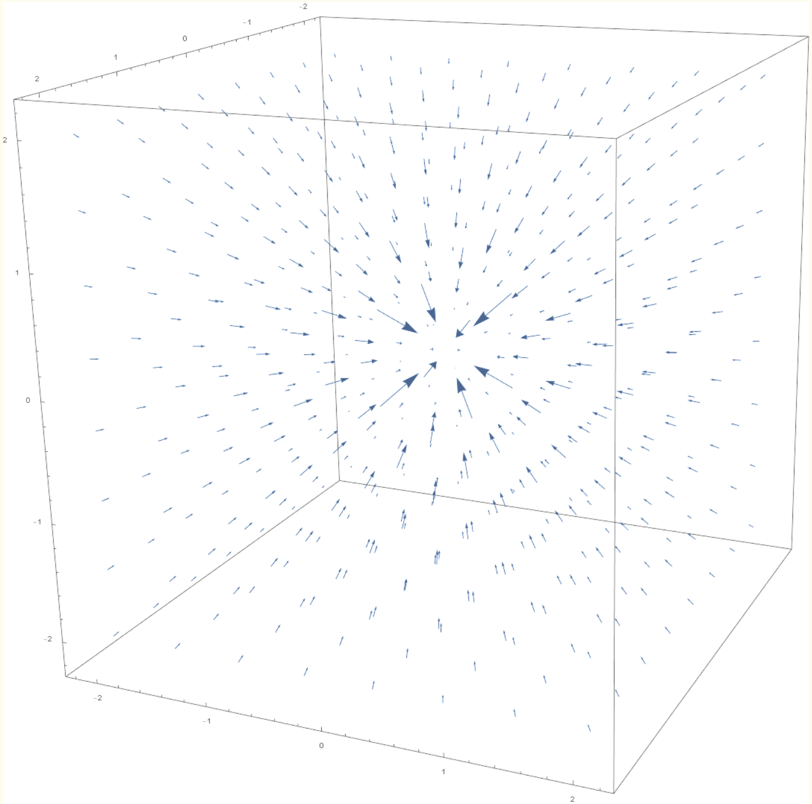


Gravitational Field

$$\vec{r} = \langle x, y, z \rangle$$

$$\vec{G} = -\frac{\vec{r}}{|\vec{r}|^3} = \frac{-\langle x, y, z \rangle}{(x^2+y^2+z^2)^{3/2}}$$

(force is proportional to $\frac{1}{r^2}$)



Question Given a vector field,

is it a gradient of some function?

IS $\vec{F} = \nabla f$?

If yes, conservative vector field.

$$\iff \oint \vec{F} \cdot d\vec{r} = 0$$

$$\iff \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \quad \text{independent of paths.}$$

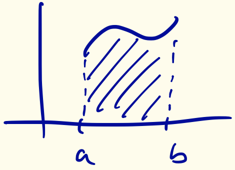
Ex $\vec{G} = -\dfrac{\vec{r}}{|\vec{r}|^3}$, $\vec{G} = \nabla V$

$$V = \frac{1}{|\vec{r}|} = \frac{1}{\sqrt{x^2+y^2+z^2}} \quad \text{gravitational potential.}$$

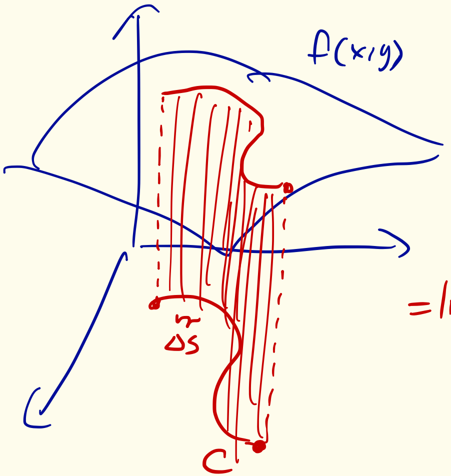
Check:

$$\frac{\partial V}{\partial x} = -\frac{1}{\cancel{2}} \frac{\cancel{2}x}{(x^2+y^2+z^2)^{3/2}} \quad \text{etc}$$

Line Integrals



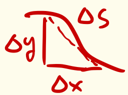
$\int_a^b f(x)\,dx$ = Area under curve.



$\int_C f(x,y)\,ds$ = Area under $f(x,y)$ along the curve C.

$= \lim \sum f(x_i, y_i)\,\Delta s$

$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$ = "arc-length"



If $C = \vec{r}(t)$

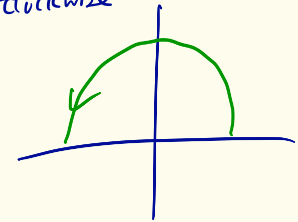
(recall: length = speed × time)

then $\Delta s \approx \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = |\vec{r}\,'(t)|\,dt$.

Def $\displaystyle\int_C f(x,y)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$.

$\underline{\text{Ex}}$ $f = 1$, $\int_C ds$ = arc length of C.

$\underline{\text{Ex}}$ $\int_C (2 + x^2 y)\, ds$, C = upper half unit circle.
counterclockwise



$C: \vec{r}(t) = \langle \cos t, \sin t \rangle \quad 0 \le t \le \pi$

$\vec{r}'(t) = \langle -\sin t, \cos t \rangle$

$|\vec{r}'(t)| = 1$

$\Rightarrow \int_0^{\pi} (2 + \cos^2 t \sin t)\, dt \quad = 2\pi + \frac{2}{3}.$

$2\pi \qquad -\frac{1}{3}\cos^3 t \Big|_0^{\pi}$

Made with Goodnotes

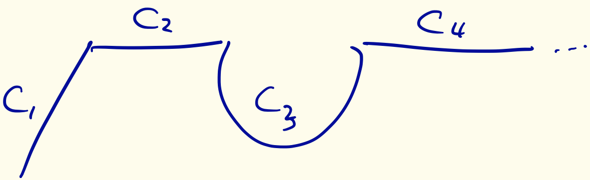
Rem Orientation of C does not matter.

$$\int_C f(x,y)\,ds$$

$$= \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

$$= \int_b^a f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

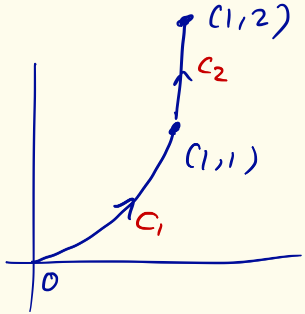
$=$ Area under curve.



$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \cdots$$

Ex $\int_C 2x\,ds$ along $y = x^2$ $(0,0)$ to $(1,1)$

followed by straight line $(1,1)$ to $(1,2)$.



$C_1 : \langle t, t^2 \rangle$

$\vec{r}\,'(t) = \langle 1, 2t \rangle$

$|\vec{r}\,'(t)| = \sqrt{1+4t^2}$

$C_2 : \langle 1, t \rangle$ $1 \le t \le 2$

$\vec{r}\,'(t) = \langle 0, 1 \rangle$

$|\vec{r}\,'(t)| = 1$

$$\int_0^1 2t\sqrt{1+4t^2}\,dt + \int_1^2 2\,dt$$

Let $u = 4t^2$

$du = 8t\,dt$

$\vdots$

Another type of line integrals (with respect to x & y)

$$\int_C f(x,y)\,dx \qquad , \qquad \int_C f(x,y)\,dy$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\int_a^b f(x(t),y(t))\,x'(t)\,dt \qquad\qquad \int_a^b f(x(t),y(t))\,y'(t)\,dt.$$

$$\frac{dx}{dt}\,dt$$

<u>Rem</u> Orientation matters.

$$\int_C = -\int_{-C}$$

Remark Also works for 3D. Same formula!

$$\int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt \qquad \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Ex $\int_C y \sin z \, ds$, $C =$ helix $\langle \cos t, \sin t, t \rangle$ $0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$

$|r'(t)| = \sqrt{2}$

$$\hookrightarrow \int_0^{2\pi} \sin t \sin t \sqrt{2} \, dt.$$

Ex $\int y \, dx + z \, dy + x \, dz$ $\quad C = (2,0,0)$ to $(3,4,5)$ straight line.

$\vec{r} = \langle 2,0,0 \rangle + t \langle 1,4,5 \rangle$

$$\hookrightarrow \int_0^1 4t \cdot 1 + 5t \cdot 4 + (2+t) \cdot 5 \; dt$$

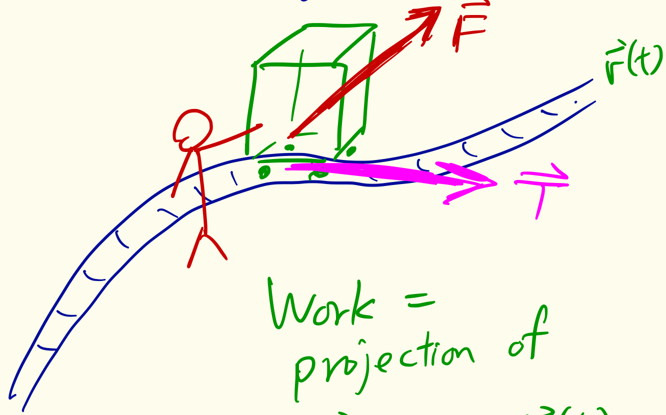
Line Integral of Vector Fields

Find total "work done" of ~~force~~ along a curve

↪ add up all the work

$$\Leftrightarrow \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}} \cancel{|\vec{r}\,'(t)|}\, dt$$



Work = projection of $\vec{F}$ onto $\vec{r}(t)$.

$= \vec{F} \cdot \vec{T}$

unit tangent vector.

Def

$(\vec{F} = \langle F_1, F_2, F_3 \rangle)$

$\vec{r}\,'(t) = \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

$$\int_C \vec{F} \cdot \underbrace{d\vec{r}}_{\langle dx,\, dy,\, dz \rangle} = \int_C F_1\, dx + F_2\, dy + F_3\, dz.$$

$$\left(= \int_C \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt \right)$$

Remark

$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_{C} \vec{F} \cdot d\vec{r}.$$