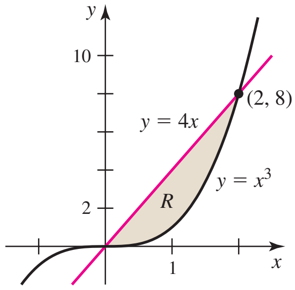
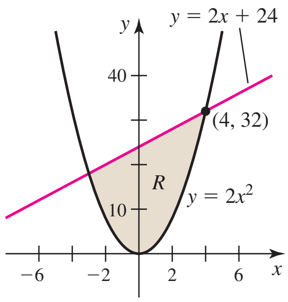
MATH 2023 • Multivariable Calculus
Problem Set #5 • Double Integrals

1. (★) Set-up the lower and upper bounds of each double integral below using **both** $dxdy$ and $dydx$ orders. Compute the integral using **both** orders and verify that they give the same value.

   (a) $\iint_R 2xy\,dA$ where R is the region as shown below:



   (b) $\iint_R 1\,dA$ where R is the region bounded between $y = 2x + 24$ and $y = 2x^2$:



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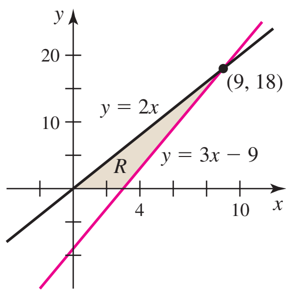
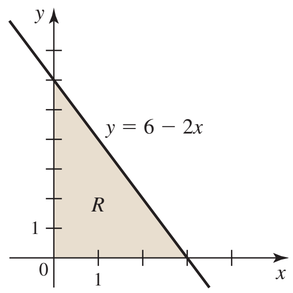


2. (★) Evaluate the integral $\iint_T \sqrt{a^2 - y^2}\,dA$ where T is the triangle with vertices $(0, 0)$, $(0, a)$ and (a, a). Set-up the integral in both $dxdy$ and $dydx$ orders, and choose the *easier* one to compute.

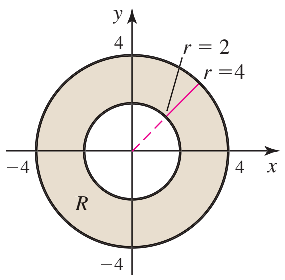
3. (★★) Consider the integral $\int_0^1 \int_x^{x^{1/3}} \sqrt{1 - y^4}\,dydx$. It is almost impossible to compute the inner integral. Try to switch the order of integration to evaluate it. [Hint: You should first sketch the region of integration.]

4. (★★) Evaluate the integrals $\iint_R \frac{1}{3-x}\,dA$ and $\iint_R \frac{1}{y-6}\,dA$. Try to avoid *integration-by-parts* if possible.



5. (★) Evaluate $\iint_R (x+y)\,dA$ using polar coordinates where R is the region in the first quadrant lying inside the disk $x^2+y^2 \leq a^2$ and under the line $y = \sqrt{3}x$.

6. (★) Consider the annular region R below. Express the integral $\iint_R (x^2+y^2)\,dA$ in **both** rectangular and polar coordinates. Choose the *easier* system to compute the integral.



7. (★★) Evaluate each of the following integrals:

   (a) $\int_0^{2\pi}\int_0^1 e^{-x^2}\sin y\,dxdy$

   (b) $\int_{-1}^{0}\int_0^{\sqrt{y+1}} \left(x-\frac{x^3}{3}\right)^{5/2}\,dxdy$

   (c) $\iint_Q \frac{1}{(1+2x^2+2y^2)^3}\,dA$ where Q is the entire first quadrant of the xy-plane

   (d) $\iint_Q (x^2-y+1)\,dA$ where Q is the region $\{(x,y) \mid x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 4\}$

   (e) $\iint_{\mathbb{R}^2} (x^2+y^2)e^{-(x^4+2x^2y^2+y^4)}\,dA$

8. (★★) Some single-variable integrals are "notoriously" difficult to compute. One example is $\int e^{-x^2}dx$ despite the fact that this integral is of central importance in mathematics (pure/applied), physics, statistics and engineering. However, some of these difficult integrals can be evaluated via double integral methods.

   This problem investigates another well-known integral which has no closed-form antiderivative:

$$\int \frac{\log(1-x)}{x}\,dx.$$

The goal of this problem is to show that this integral over $0 \le x \le 1$ can be written as an infinite series.

Consider the function

$$f(x,y) = \frac{1}{1-xy}, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

It is defined almost everywhere on the rectangle $0 \le x \le 1$ and $0 \le y \le 1$ (we say 'almost' because it's undefined only at $(x,y) = (1,1)$, but this single point is negligible).

(a) Show that: $\displaystyle\int_0^1 \frac{1}{1-xy}\,dy = -\frac{\log(1-x)}{x}$.

(b) Note that $|xy| < 1$ except for the negligible point $(x,y) = (1,1)$, so the function $f(x,y)$ can be expressed as a geometric series:

$$\frac{1}{1-xy} = 1 + (xy) + (xy)^2 + (xy)^3 + \ldots$$

Using this geometric series, show that

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\,dy\,dx = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

(c) Using (a) and (b), show that

$$-\int_0^1 \frac{\log(1-x)}{x}\,dx = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

(d) Using the above approach, *mutatis mutandis*, show that for any $0 \le z \le 1$, we have:

$$-\int_0^z \frac{\log(1-x)}{x}\,dx = \sum_{k=1}^{\infty} \frac{z^k}{k^2}.$$

[Remark: *Mutatis mutandis* is a Latin phrase meaning "changing only those things which need to be changed".]

9. (★★★) The purpose of this problem is to use double integrals to derive a somewhat surprising result in electrostatics, that is the electric force exerted on a charged particle by an infinite sheet of uniformly distributed charges is *independent* of how far the particle and the sheet are apart from each other.

The paragraphs below describe the physical set-up of the problem. Although it may be possible to proceed to the problem without knowing the physics background, it is strongly recommend to read through the paragraphs below so as to understand the motivation of this problem.

According to the Coulomb's Law, the electric force $\mathbf{F}$ exerted **on** a point particle with charge Q located at (x_0, y_0, z_0), **by** a point particle with charge q located at (x,y,z), is given by:

$$\mathbf{F} = \frac{qQ}{4\pi\varepsilon_0} \frac{(x_0 - x)\mathbf{i} + (y_0 - y)\mathbf{j} + (z_0 - z)\mathbf{k}}{\left((x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2\right)^{3/2}}$$

where ε_0 is positive constant (depending on the medium).

The Coulomb's Law is also called the Inverse Square Law because one can easily verify that the magnitude of the force satisfies:

$$|\mathbf{F}| = \frac{qQ}{4\pi\varepsilon_0 d^2}$$

where $d = \sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}$ is the distance between the two particles.

If there is a sequence of *discrete* charged particles located at (x_1, y_1, z_1), (x_2, y_2, z_2), $\ldots$, each with charge q, then the resultant electric force exerted on a particle with charge Q located at (x_0, y_0, z_0), is given by the vector sum of all forces:

$$\mathbf{F} = \sum_{i=1}^{\infty} \frac{qQ}{4\pi\varepsilon_0} \frac{(x_0 - x_i)\mathbf{i} + (y_0 - y_i)\mathbf{j} + (z_0 - z_i)\mathbf{k}}{\left((x_0 - x_i)^2 + (y_0 - y_i)^2 + (z_0 - z_i)^2\right)^{3/2}}.$$

This is called the Principle of Superposition by physicists.

Now given there is an infinite sheet of uniformly distributed charges on the xy-plane, and for each small area element dA on the xy-plane, the amount of charges is given by $\sigma\, dA$, where σ is a constant that represents the area density of charges. Suppose there is a particle with charge Q located above the xy-plane at $(0, 0, z_0)$, i.e. $z_0 > 0$. For simplicity, call this the Q-particle.

Now regard a small area element located at $(x, y, 0)$ on the xy-plane as a charged "particle" with charge $q = \sigma\, dA$, then the force exerted on the Q-particle by this area element is given by substituting $(x, y, z) = (x, y, 0)$ and $(x_0, y_0, z_0) = (0, 0, z_0)$:

$$\frac{Q(\sigma\, dA)}{4\pi\varepsilon_0} \frac{(0 - x)\mathbf{i} + (0 - y)\mathbf{j} + (z_0 - 0)\mathbf{k}}{\left((0 - x)^2 + (0 - y)^2 + (z_0 - 0)^2\right)^{3/2}} = \frac{Q\sigma}{4\pi\varepsilon_0} \frac{-x\mathbf{i} - y\mathbf{j} + z_0\mathbf{k}}{\left(x^2 + y^2 + z_0^2\right)^{3/2}}\, dA.$$

Therefore, by the Principle of Superposition, the resultant electric force exerted on the Q-particle by the sheet of charges is given by this double integral over the entire xy-plane (i.e. $\mathbb{R}^2$):

$$\mathbf{F}_{\text{resultant}} = \iint_{\mathbb{R}^2} \frac{Q\sigma}{4\pi\varepsilon_0} \frac{-x\mathbf{i} - y\mathbf{j} + z_0\mathbf{k}}{\left(x^2 + y^2 + z_0^2\right)^{3/2}}\, dA.$$

Here integrating a vector simply means integrating each component of the vector treating $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as "constants".

(a) Show that $\mathbf{i}$ and $\mathbf{j}$-components of $\mathbf{F}_{\text{resultant}}$ are zero.

(b) Derive that:

$$\mathbf{F}_{\text{resultant}} = \frac{Q\sigma}{2\varepsilon_0}\mathbf{k}.$$

[Remark: The result in (b) asserts that the resultant force on the Q-particle does *not* depend on how far it is from the infinite sheet! Believe it or not?]

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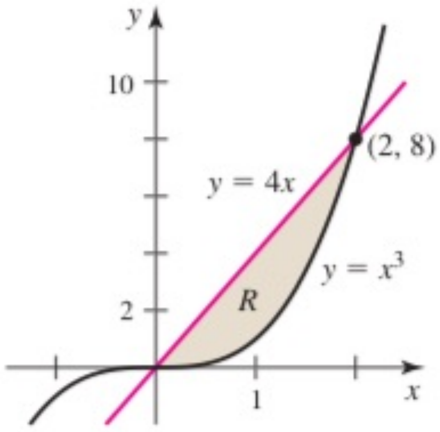
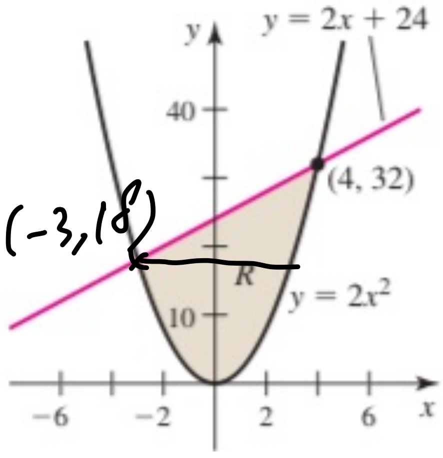
   (a) $\iint_R 2xy\,dA$ where R is the region as shown below:



$$\int_0^8 \int_{\frac{y}{4}}^{y^{\frac{1}{3}}} 2xy \; dx\,dy$$

$$\int_0^2 \int_{x^3}^{4x} 2xy \; dy\,dx$$

(b) $\iint_R 1\,dA$ where R is the region bounded between $y = 2x + 24$ and $y = 2x^2$:



$2x+24 = 2x^2$

$0 = 2x^2 - 2x - 24$

$x^2 - x - 12 = 0$

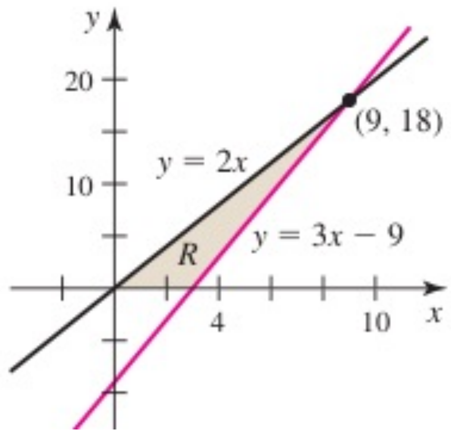
$x \quad +3$

$x \quad -4$

$$\int_0^{18} \int_{-\sqrt{\frac{y}{2}}}^{\sqrt{\frac{y}{2}}} dx\,dy + \int_{18}^{32} \int_{\frac{y-24}{2}}^{\sqrt{\frac{y}{2}}} dx\,dy$$

$$\int_{-3}^{4} \int_{2x^2}^{2x+24} dy\,dx$$

(c) $\iint_R x^2\,dA$ where R is the region bounded between $y = 2x$, $y = 3x - 9$ and the x-axis:

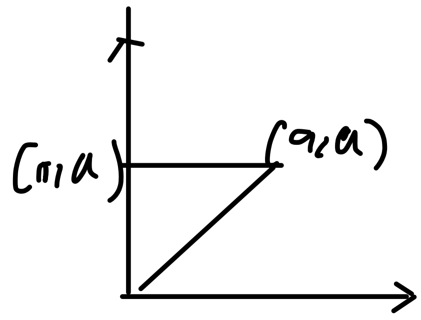


$$\int_0^{18}\int_{\frac{y}{2}}^{\frac{y+9}{3}} x^2 \quad dx\,dy$$

$$\int_0^{3}\int_0^{2x} x^2 \quad dy\,dx \;+$$

$$\int_3^{9}\int_{3x-9}^{2x} x^2 \quad dy\,dx$$

2. (★) Evaluate the integral $\iint_T \sqrt{a^2 - y^2}\, dA$ where T is the triangle with vertices $(0,0)$, $(0,a)$ and (a,a). Set-up the integral in both $dxdy$ and $dydx$ orders, and choose the *easier* one to compute.

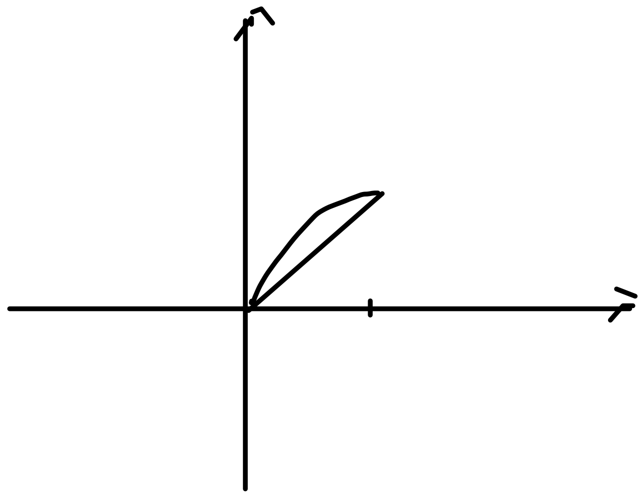


$$\int_0^a \int_0^y \sqrt{a^2 - y^2}\, dx\, dy$$

$$\int_0^a \int_x^a \sqrt{a^2 - y^2}\, dy\, dx$$

3. (★★) Consider the integral $\int_0^1 \int_x^{x^{1/3}} \sqrt{1-y^4}\, dy dx$. It is almost impossible to compute the inner integral. Try to switch the order of integration to evaluate it. [Hint: You should first sketch the region of integration.]

$$\int_0^1 \int_{y^3}^{y} \sqrt{1-y^4}\, dx\, dy$$



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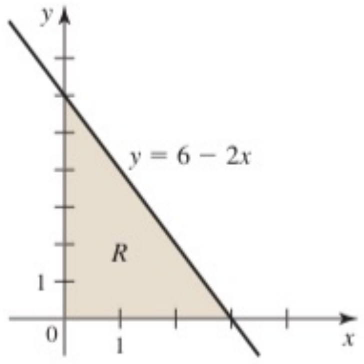
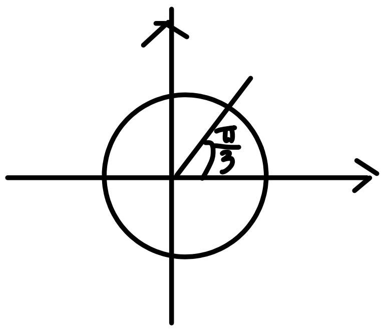


$$\int_0^3 \int_0^{6-2x} \frac{1}{3-x}\,dy\,dx$$

$$\int_0^6 \int_0^{\frac{6-y}{2}} \frac{1}{y-6}\,dx\,dy$$

5. (★) Evaluate $\iint_R (x+y)\, dA$ using polar coordinates where R is the region in the first quadrant lying inside the disk $x^2+y^2 \le a^2$ and under the line $y = \sqrt{3}x$.

$$\int_0^{\frac{\pi}{3}} \int_0^a (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$



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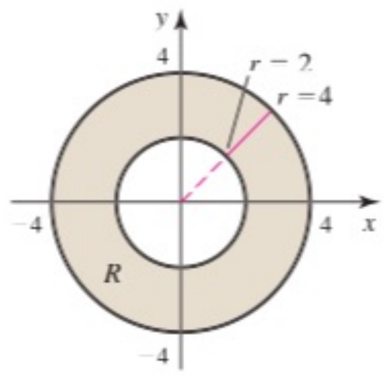


$$\int_0^{2\pi}\int_2^4 r^2 \; r\,dr\,d\theta$$

7. (★★) Evaluate each of the following integrals:

(a) $\int_0^{2\pi} \int_0^1 e^{-x^2} \sin y \, dx dy$

? (b) $\int_{-1}^{0} \int_0^{\sqrt{y+1}} \left(x - \frac{x^3}{3}\right)^{5/2} \, dx dy$

(c) $\iint_Q \frac{1}{(1+2x^2+2y^2)^3} \, dA$ where Q is the entire first quadrant of the xy-plane

(d) $\iint_Q (x^2 - y + 1) \, dA$ where Q is the region $\{(x,y) \mid x \geq 0, \ y \geq 0, \ x^2 + y^2 \leq 4\}$

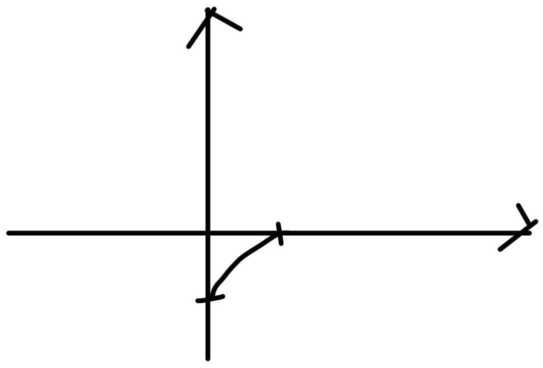
(e) $\iint_{\mathbb{R}^2} (x^2+y^2) e^{-(x^4+2x^2y^2+y^4)} \, dA$

a). $\left(\int_0^{2\pi} \sin y \, dy\right)\left(\int_0^1 e^{-x^2} dx\right)$

$x = \sqrt{y+1}$

$\left([-\cos y]_0^{2\pi}\right) \left(\int_0^1 e^{-x^2} dx\right) = 0.$

$-1 - (-1)$

b). 

$\int_0^1 \int_{x^2-1}^{0} \left(x - \frac{x^3}{3}\right)^{\frac{5}{2}} dy \, dx$

c). $\int_0^{\frac{\pi}{2}} \int_0^{\infty} \frac{1}{(1+2r^2)^3} \, r dr \, d\theta$

d). $\int_0^{\frac{\pi}{2}} \int_0^{2} (r^2 \cos^2\theta - r\sin\theta + 1) r \, dr d\theta$

e). $\int_0^{2\pi} \int_0^{\infty} r^3 e^{-(r^2)} \, dr d\theta$

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The goal of this problem is to show that this integral over $0 \le x \le 1$ can be written as an infinite series.

Consider the function

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It is defined almost everywhere on the rectangle $0 \le x \le 1$ and $0 \le y \le 1$ (we say 'almost' because it's undefined only at $(x,y) = (1,1)$, but this single point is negligible).

(a) Show that: $\displaystyle\int_0^1 \frac{1}{1-xy}dy = -\frac{\log(1-x)}{x}$.

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$$\frac{1}{1-xy} = 1 + (xy) + (xy)^2 + (xy)^3 + \dots$$

Using this geometric series, show that

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(c) Using (a) and (b), show that

$$-\int_0^1 \frac{\log(1-x)}{x}dx = \sum_{k=1}^{\infty}\frac{1}{k^2}.$$

(d) Using the above approach, *mutatis mutandis*, show that for any $0 \le z \le 1$, we have:

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[Remark: *Mutatis mutandis* is a Latin phrase meaning "changing only those things which need to be changed".]

a). $\displaystyle\int_0^1 \frac{1}{1-xy}dy$

Let $u = 1-xy$, $du = -x\,dy$

$\displaystyle -\frac{1}{x}\int_1^{1-x} \frac{1}{u}du$

$\displaystyle = \frac{-\ln|1-x|}{x}$

$\displaystyle = \frac{-\log(1-x)}{x}$.

b). $\displaystyle\int_0^1 \frac{1}{1-xy}dy$

$\displaystyle = [y]_0^1 + \left[\frac{xy^2}{2}\right]_0^1 + \left[\frac{x^2y^3}{3}\right]_0^1 + \dots$

$\displaystyle = 1 + \frac{x}{2} + \frac{x^2}{3} + \dots$

$\displaystyle = \int_0^1\int_0^1 \frac{1}{1-xy}dy\,dx$

$\displaystyle = [x]_0^1 + \left[\frac{x^2}{2^2}\right]_0^1 + \left[\frac{x^3}{3^2}\right]_0^1 + \dots$

$\displaystyle = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

c). $\displaystyle\int_0^1 \frac{\log(1-x)}{x}dx = \sum_{k=1}^{\infty}\frac{1}{k^2}$

9. (★★★) The purpose of this problem is to use double integrals to derive a somewhat surprising result in electrostatics, that is the electric force exerted on a charged particle by an infinite sheet of uniformly distributed charges is *independent* of how far the particle and the sheet are apart from each other.

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where ε_0 is positive constant (depending on the medium).

The Coulomb's Law is also called the Inverse Square Law because one can easily verify that the magnitude of the force satisfies:

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where $d = \sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}$ is the distance between the two particles.

If there is a sequence of *discrete* charged particles located at (x_1, y_1, z_1), (x_2, y_2, z_2), $\ldots$, each with charge q, then the resultant electric force exerted on a particle with charge Q located at (x_0, y_0, z_0), is given by the vector sum of all forces:

$$\mathbf{F} = \sum_{i=1}^{\infty} \frac{qQ}{4\pi\varepsilon_0} \frac{(x_0 - x_i)\mathbf{i} + (y_0 - y_i)\mathbf{j} + (z_0 - z_i)\mathbf{k}}{\left((x_0 - x_i)^2 + (y_0 - y_i)^2 + (z_0 - z_i)^2\right)^{3/2}}.$$

This is called the Principle of Superposition by physicists.

Now given there is an infinite sheet of uniformly distributed charges on the xy-plane, and for each small area element dA on the xy-plane, the amount of charges is given by $\sigma\, dA$, where σ is a constant that represents the area density of charges. Suppose there is a particle with charge Q located above the xy-plane at $(0, 0, z_0)$, i.e. $z_0 > 0$. For simplicity, call this the Q-particle.

Now regard a small area element located at $(x, y, 0)$ on the xy-plane as a charged "particle" with charge $q = \sigma\, dA$, then the force exerted on the Q-particle by this area element is given by substituting $(x, y, z) = (x, y, 0)$ and $(x_0, y_0, z_0) = (0, 0, z_0)$:

$$\frac{Q(\sigma\, dA)}{4\pi\varepsilon_0} \frac{(0 - x)\mathbf{i} + (0 - y)\mathbf{j} + (z_0 - 0)\mathbf{k}}{\left((0 - x)^2 + (0 - y)^2 + (z_0 - 0)^2\right)^{3/2}} = \frac{Q\sigma}{4\pi\varepsilon_0} \frac{-x\mathbf{i} - y\mathbf{j} + z_0\mathbf{k}}{\left(x^2 + y^2 + z_0^2\right)^{3/2}}\, dA.$$

Therefore, by the Principle of Superposition, the resultant electric force exerted on the Q-particle by the sheet of charges is given by this double integral over the entire xy-plane (i.e. $\mathbb{R}^2$):

$$\mathbf{F}_{\text{resultant}} = \iint_{\mathbb{R}^2} \frac{Q\sigma}{4\pi\varepsilon_0} \frac{-x\mathbf{i} - y\mathbf{j} + z_0\mathbf{k}}{\left(x^2 + y^2 + z_0^2\right)^{3/2}}\, dA.$$

Here integrating a vector simply means integrating each component of the vector treating $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as "constants".

(a) Show that $\mathbf{i}$ and $\mathbf{j}$-components of $\mathbf{F}_{\text{resultant}}$ are zero.

(b) Derive that:

$$\mathbf{F}_{\text{resultant}} = \frac{Q\sigma}{2\varepsilon_0}\mathbf{k}.$$

[Remark: The result in (b) asserts that the resultant force on the Q-particle does *not* depend on how far it is from the infinite sheet! Believe it or not?]