HKUST MATH 101

Midterm Examination

Multivariable Calculus

14 October 2004

Answer ALL 5 questions

Time allowed – 120 minutes

Directions – This is a closed book examination. No talking or whispering are allowed. Work must be shown to receive points. An answer alone is not enough. Please write neatly. Answers which are illegible for the grader cannot be given credit.

Note that you can work on *both* sides of the paper and do not detach pages from this exam packet or unstaple the packet.

Student Name: ______________________

Student Number: ______________________

Tutorial Session: ______________________

Question No.	Marks
1	/20
2	/20
3	/20
4	/20
5	/20
Total	/100

Problem 1

(a) If $\mathbf{e}$ is any unit vector and $\mathbf{a}$ an arbitrary vector show that

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{e})\mathbf{e} + \mathbf{e}\times(\mathbf{a}\times\mathbf{e}).$$

This shows that $\mathbf{a}$ can be resolved into a component parallel to and one perpendicular to an arbitrary direction $\mathbf{e}$.

(b) Show that the two lines

$$\mathbf{r} = \mathbf{a} + \mathbf{v}t, \qquad \mathbf{r} = \mathbf{b} + \mathbf{u}t$$

where t is a parameter and $\mathbf{u}$ and $\mathbf{v}$ are two unit vectors, will intersect if

$$\mathbf{a}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{b}\cdot(\mathbf{u}\times\mathbf{v}).$$

Problem 2

(a) Describe (sketch) the intersection curve C of the sphere $x^2+y^2+z^2=1$ and the elliptic cylinder $y^2+2z^2=1$ in the first octant.

(b) Find the parametric equation of the curve C in the first octant.

(c) Find the vector equation of the tangent line L of C at the point $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$.

(d) Find the equation of the plane through the point $(1,1,1)$ and parallel with the tangent line L obtained in part (iii).

Problem 3

Let $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & \text{if} \quad (x,y)\neq(0,0) \\ 0, & \text{if} \quad (x,y)=(0,0). \end{cases}$

Calculate $f_x(x,y)$ and $f_y(x,y)$ at all points (x,y) (*include* the point $(0,0)$) in the xy-plane. Are f_x and f_y continuous at $(0,0)$ (why)? Is f continuous at $(0,0)$ (why)?

Problem 4

(a) Show that

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)\,dt\right] = f(b(x))b'(x) - f(a(x))a'(x)$$

[*Hint*: Let $u=a(x)$, $v=b(x)$, and $F(u,v)=\displaystyle\int_u^v f(t)\,dt$.

(b) Show that if $z=f(x,y)$ is differentiable at $\mathbf{x}_0=(x_0,y_0)$, then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x})-f(\mathbf{x}_0)-\nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{\|\mathbf{x}-\mathbf{x}_0\|} = 0$$

[*Hint*: Use the definition on differentiability.]

Problem 5

(a) The temperature $T(x, y)$ at points of the xy-plane is given by $T(x, y) = x^2 - 2y^2$.

(i) Draw a contour diagram for T showing some isotherms (curves of constant temperature).

(ii) In what direction should an ant at position $(2, -1)$ move if it wishes to cool off as quickly as possible?

(iii) If an ant moves in that direction at speed k (units distance per unit time), at what rate does it experience the decrease of temperature?

(iv) At what rate would the ant experience the decrease of temperature if it moves from $(2, -1)$ at speed k in the direction of the vector $-\mathbf{i} - 2\mathbf{j}$?

(v) Along what curve through $(2, -1)$ should the ant move in order to continue to experience maximum rate of cooling?

(b) Find and classify the critical points of the given function $f(x, y) = x^4 + y^4 - 4xy$.