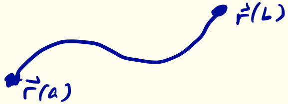
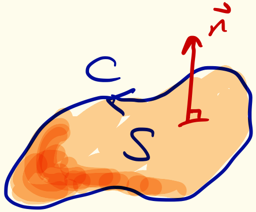
Fund Thm

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$



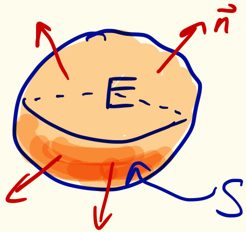
Stokes'

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$



Divergence Thm

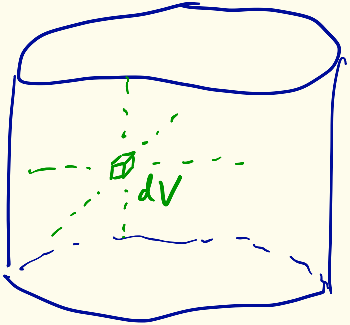
$$\iiint_E (\nabla \cdot \vec{F})\, dV = \iint_S \vec{F} \cdot d\vec{S}$$



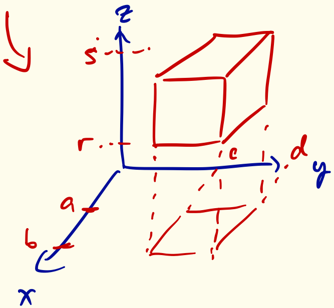
how to calculate triple integral?

Triple Integrations

$$\iiint_E \underbrace{\underbrace{f(x,y,z)}_{\text{density (mass/volume)}}\, dV}_{\text{mass.}}$$



$$\iiint_{[a,b]\times[c,d]\times[r,s]} f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$



Note

$$\iiint_E 1\,dV = \text{Volume}(E).$$

Ex $\iiint_E x^2yz\,dV$, $E=[1,2]\times[-1,0]\times[0,3]$.

$$= \int_0^3\int_{-1}^0\int_1^2 x^2yz \; dx\,dy\,dz$$

Separation of Variables.

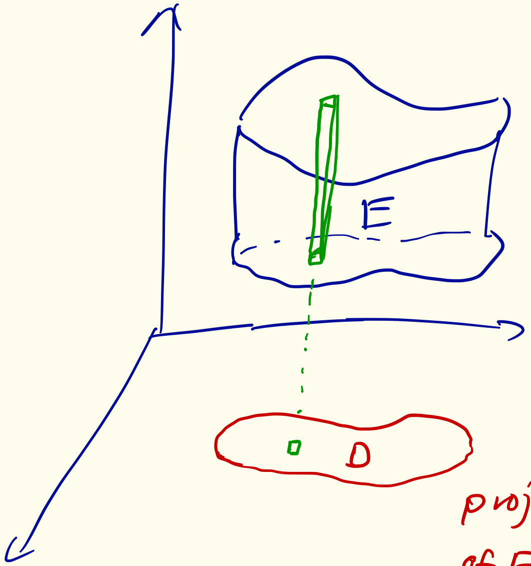
$$= \left(\int_0^3 z\,dz\right)\left(\int_{-1}^0 y\,dy\right)\left(\int_1^2 x^2\,dx\right)$$

$$= \left(\frac{9}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{4}{3}-\frac{1}{3}\right) = -\frac{9}{4},$$

General Strategy

Build a house from a base

$$\iiint dV = \iint_D \left(\int dz \right) dA$$



projection of E on plane

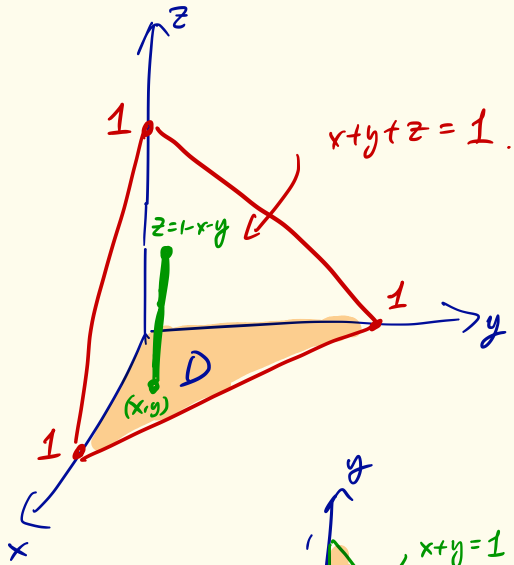
Ex $\displaystyle\iiint_E z\,dV$ where E is tetrahedron bounded by $x=y=z=0$ and $x+y+z=1$.

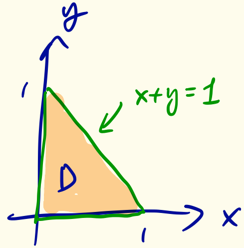
$$= \iint_D \left(\int_0^{1-x-y} z\,dz \right) dA$$

$$= \int_0^1 \int_0^{1-x} \underbrace{\int_0^{1-x-y} z\,dz}_{\frac{z^2}{2}\Big|_0^{1-x-y}} \,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \frac{(1-x-y)^2}{2}\,dy\,dx$$

$= \ldots$

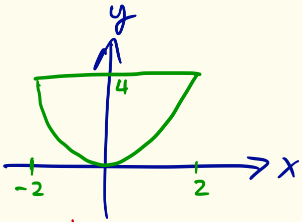




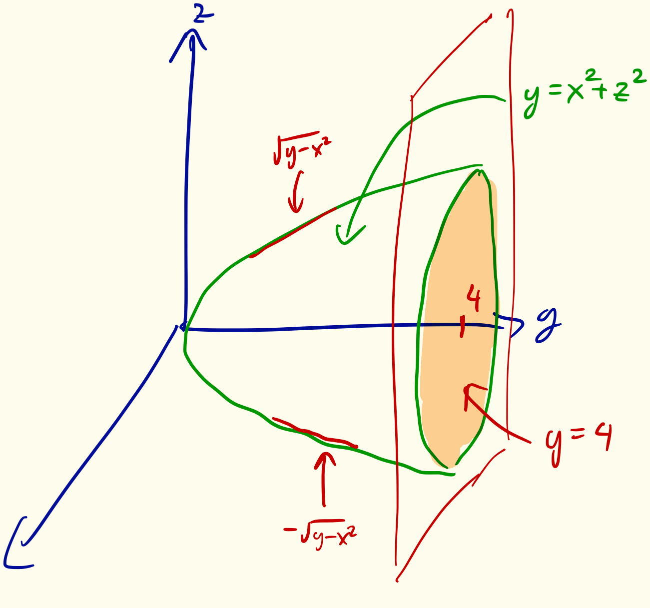
Ex Find $\iiint_E \sqrt{x^2+z^2}\,dV$, E bounded by paraboloid $y=x^2+z^2$ and $y=4$.

$y=r^2$

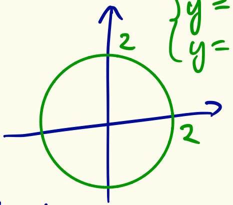
Project to x-y plane



$$\int_{-2}^{2}\int_{x^2}^{4}\left(\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}}\sqrt{x^2+z^2}\,dz\right)dy\,dx$$



Project to x-z plane



$$\begin{cases} y=r^2 \\ y=4 \end{cases}$$

$$\int_0^{2\pi}\int_0^{2}\left(\int_{x^2+z^2}^{4}\sqrt{x^2+z^2}\,dy\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r^2\,dy\,dr\,d\theta = \cdots$$

$r=\sqrt{x^2+z^2}$

Remark $\rho(x,y,z)$ density (mass/volume)

$$\text{Mass } M = \iiint_E \rho(x,y,z)\, dV$$

Center of Mass $(\bar{x}, \bar{y}, \bar{z})$

$$\bar{x} = \frac{\iiint_E x\rho(x,y,z)\, dV}{M} \qquad \text{(Moment)}$$

$$\bar{y} = \frac{\iiint y\rho(x,y,z)\, dV}{M}, \qquad \bar{z} = \frac{\iiint z\rho(x,y,z)\, dV}{M}.$$

If ρ is constant : $(\bar{x}, \bar{y}, \bar{z})$ centroid.

Ex Find center of mass of a cube $0 \le x, y, z \le a$ with density $\rho(x,y,z) = x^2+y^2+z^2$.

$$M = \iiint_E \rho(x,y,z)\, dV = \int_0^a \int_0^a \int_0^a (x^2+y^2+z^2)\, dx\, dy\, dz$$

$$= \int_0^a \int_0^a \left(\frac{x^3}{3} + y^2 x + z^2 x \right)\Big|_0^a \, dy\, dz$$

$$= \int_0^a \int_0^a \left(\frac{a^3}{3} + a(y^2+z^2) \right) dy\, dz$$

$$= \int_0^a \left(\frac{a^3}{3} y + \frac{a y^3}{3} + a z^2 y \right)\Big|_0^a \, dz$$

$$= \int_0^a \left(\frac{2a^4}{3} + a^2 z^2 \right) dz$$

$$= \frac{2a^4}{3} z + \frac{a^2 z^3}{3} \Big|_0^a = a^5$$

$$\bar{x} = \int_0^a \int_0^a \int_0^a x(x^2+y^2+z^2)\, dx\, dy\, dz \Big/ M$$

$$= \cdots \frac{7}{12} a^6 / a^5 = \frac{7}{12} a$$

$$\bar{y} = \bar{z} = \frac{7}{12} a$$

$\therefore$ center of mass $= \left(\frac{7}{12} a, \frac{7}{12} a, \frac{7}{12} a \right)$