香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

SAMPLE FINAL

Course Code: MATH 2023
Course Title: Multivariable Calculus
Time Limit: 3 Hours

Instructions

- Do **NOT** open the exam until instructed to do so.
- This is a **CLOSED BOOK, CLOSED NOTES** exam.
- All mobile phones and communication devices should be switched **OFF**.
- Only calculators approved by HKEAA can be used.
- Answer **ALL** nine problems.
- You must **SHOW YOUR WORK** to receive credits in all problems except Problem #1. Answers alone (whether correct or not) will not receive any credit.
- Some problems are structured into several parts. You can quote the results stated in the preceding parts to do the next part.

About this sample exam

The purpose of this sample final is to let students get a rough idea of the style of problems and the format of the exam. Do **NOT** expect the actual exam is simply a minor variation of this sample exam. The level of difficulties, the point allocation of each problem, and the choice of topics may be different from the actual exam. For better preparation of the final, students should extensively review the course materials covered in class, in the lecture notes and in the lecture worksheets, and should have worked seriously on the Problem Sets and WebWorks.

Problem	1	2	3	4	5	6	7	8	9	Total
Max	28	10	8	14	8	8	8	12	4	100
Score										

Made with Goodnotes

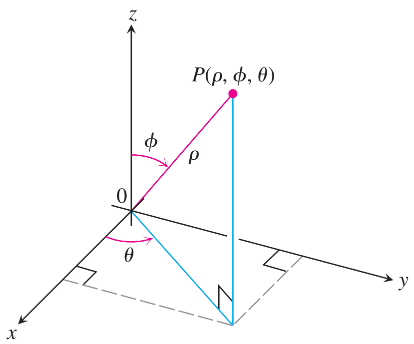
FORMULAE TABLE

$$\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 \\
1 + \tan^2\theta &= \sec^2\theta \\
1 + \cot^2\theta &= \csc^2\theta \\
\tan\theta &= \frac{\sin\theta}{\cos\theta} \\
\sin(\theta \pm \phi) &= \sin\theta\cos\phi \pm \cos\theta\sin\phi \\
\cos(\theta \pm \phi) &= \cos\theta\cos\phi \mp \sin\theta\sin\phi \\
\tan(\theta \pm \phi) &= \frac{\tan(\theta) \pm \tan(\phi)}{1 \mp \tan\theta\tan\phi} \\
\sin(2\theta) &= 2\sin\theta\cos\theta \\
\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\
&= 1 - 2\sin^2\theta \\
&= 2\cos^2\theta - 1 \\
\tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta}
\end{aligned}$$

For any C^2 function $f(x, y)$ and $z = f(a + tu_1, b + tu_2)$, we have:

$$\begin{aligned}
\frac{d^2z}{dt^2} &= f_{xx}u_1^2 + 2f_{xy}u_1u_2 + f_{yy}u_2^2 \\
&= f_{xx}\left[\left(u_1 + \frac{f_{xy}}{f_{xx}}u_2\right)^2 + \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}^2}\right)u_2^2\right] \qquad \text{if } f_{xx} \neq 0
\end{aligned}$$

Spherical coordinates (MATH version)



$$dV = \rho^2 \sin\phi \, d\rho d\phi d\theta$$

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1. Answer the following questions. Each part is independent. Justification is not required.

(a) i. Suppose $\mathbf{F}(x, y)$ is a C^1 vector field defined on $\mathbb{R}^2 \backslash \{(0,0)\}$. If there is a potential function $f(x, y)$ such that $\mathbf{F} = \nabla f$ on $\mathbb{R}^2 \backslash \{(0,0)\}$, which of the following MUST be true? Put "✓" in ALL correct answer(s). /8

- ✓ The vector field $\mathbf{F}$ is conservative on $\mathbb{R}^2 \backslash \{(0,0)\}$.
- ✓ For any closed curve C in $\mathbb{R}^2 \backslash \{(0,0)\}$, the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$.
- ✓ $\nabla \times \mathbf{F} = \mathbf{0}$ on $\mathbb{R}^2 \backslash \{(0,0)\}$.
- ○ None of the above.

ii. Suppose $\mathbf{G}(x, y)$ is a C^1 vector field defined on $\mathbb{R}^2 \backslash \{(0,0)\}$. If $\nabla \times \mathbf{G} = \mathbf{0}$ on $\mathbb{R}^2 \backslash \{(0,0)\}$, which of the following MUST be true? Put "✓" in ALL correct answer(s).

- ○ The vector field $\mathbf{G}$ is conservative on $\mathbb{R}^2 \backslash \{(0,0)\}$.
- ○ For any closed curve C in $\mathbb{R}^2 \backslash \{(0,0)\}$, the line integral $\oint_C \mathbf{G} \cdot d\mathbf{r} = 0$.
- ○ $\mathbf{G} = \nabla g$ for some potential function $g(x, y)$ defined on $\mathbb{R}^2 \backslash \{(0,0)\}$.
- ✓ None of the above.

iii. Suppose $\mathbf{H}(x, y, z)$ is a C^1 vector field defined on $\mathbb{R}^3 \backslash \{(0,0,0)\}$. If there is a potential function $h(x, y)$ such that $\mathbf{H} = \nabla h$ on $\mathbb{R}^3 \backslash \{(0,0,0)\}$, which of the following MUST be true? Put "✓" in ALL correct answer(s).

- ✓ The vector field $\mathbf{H}$ is conservative on $\mathbb{R}^3 \backslash \{(0,0,0)\}$.
- ✓ For any closed curve C in $\mathbb{R}^3 \backslash \{(0,0,0)\}$, the line integral $\oint_C \mathbf{H} \cdot d\mathbf{r} = 0$.
- ✓ $\nabla \times \mathbf{H} = \mathbf{0}$ on $\mathbb{R}^3 \backslash \{(0,0,0)\}$.
- ○ None of the above.

iv. Suppose $\mathbf{K}(x, y, z)$ is a C^1 vector field defined on $\mathbb{R}^3 \backslash \{(0,0,0)\}$. If $\nabla \times \mathbf{K} = \mathbf{0}$ on $\mathbb{R}^3 \backslash \{(0,0,0)\}$, which of the following MUST be true? Put "✓" in ALL correct answer(s).

- ✓ The vector field $\mathbf{K}$ is conservative on $\mathbb{R}^3 \backslash \{(0,0,0)\}$.
- ✓ For any closed curve C in $\mathbb{R}^3 \backslash \{(0,0,0)\}$, the line integral $\oint_C \mathbf{K} \cdot d\mathbf{r} = 0$.
- ✓ $\mathbf{K} = \nabla k$ for some potential function $k(x, y)$ defined on $\mathbb{R}^3 \backslash \{(0,0,0)\}$.
- ○ None of the above.

Problem #1 continues on next page...

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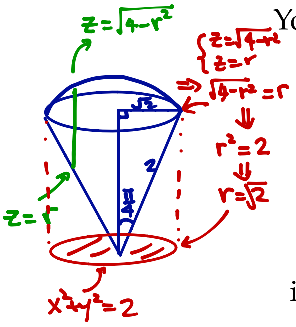
(b) A solid "ice-cream" cone D in $\mathbb{R}^3$ is bounded on top by part of the sphere $x^2 + y^2 + z^2 = 4$, and on the bottom by part of the cone $z = \sqrt{x^2 + y^2}$. The intersection of the sphere and the cone is a circle centered at the z-axis.

Set up the triple integral $\iiint_D 1\, dV$ using each of the following coordinate system. You do **not** need to evaluate the integrals.



i. Rectangular coordinates **/2**

$$\int_{x=-\sqrt{2}}^{x=\sqrt{2}} \int_{y=-\sqrt{2-x^2}}^{y=\sqrt{2-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{4-x^2-y^2}} 1 \, dz\, dy\, dx$$

ii. Cylindrical coordinates **/2**

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{2}} \int_{z=r}^{z=\sqrt{4-r^2}} r \, dz\, dr\, d\theta$$

iii. Spherical coordinates **/2**

$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\frac{\pi}{4}} \int_{\rho=0}^{\rho=2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

(c) Suppose f is a scalar function and $\mathbf{F}$ is a vector field. Both are C^2 everywhere in $\mathbb{R}^3$. Which of the following must be true? Put "✓" in ALL correct answer(s). **/2**

- ☑ $\nabla \times \nabla f = \mathbf{0}$
- ☑ $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
- ◯ $\nabla \cdot \nabla f = 0$
- ◯ $\nabla \times (\nabla \times \mathbf{F}) = \mathbf{0}$

(d) Let f be a scalar function and $\mathbf{F}$ be a vector field. Assume both are C^3 on $\mathbb{R}^3$. Determine whether each of the following quantities is a scalar, vector or undefined. Circle the correct answers. **/5**

∇f	scalar	vector	undefined
$\nabla \cdot \mathbf{F}$	scalar	vector	undefined
$\nabla \times (\nabla \cdot \mathbf{F})$	scalar	vector	undefined
$\nabla^2 (\nabla \cdot \mathbf{F})$	scalar	vector	undefined
$\nabla \cdot (\nabla^2 f)$	scalar	vector	undefined

Problem #1 continues on next page...

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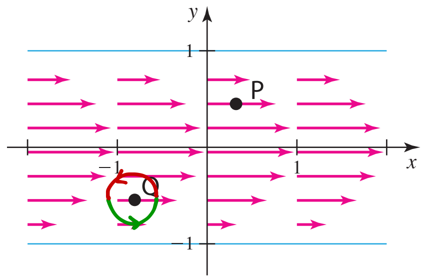
(e) The diagram below shows a C^1 vector field $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$. **/2**



At which point(s) the value of $(\nabla \times \mathbf{F}) \cdot \mathbf{k}$ is *negative*? Put "✓" in the correct answer.

◯ P only
✓ Q only ← error in previous version
◯ Both P and Q
◯ neither P nor Q

(f) Which of the following statements about simply-connectedness is/are true? Write: **/5**

✓ = true
× = false
! = does not make sense

Below is an example of a *false* statement, and an example of a *does-not-make-sense* statement:

- *false*: A sphere is not simply-connected.
- *does not make sense*: Stokes' Theorem is simply-connected.

Doesn't make sense to say a vector field is simply-connected!

i. If a vector field $\mathbf{F}$ is simply-connected, then it is conservative.

Simply-connected refers to the region! We can say ?? the domain of $\vec{F}$ is simply-connected.

i. !

ii. A vector field $\mathbf{F}$ defined on a non-simply-connected domain must not be conservative.

Counterexample: $\vec{F} = \frac{x\hat{i} + y\hat{j}}{\sqrt{x^2+y^2}} = \nabla(\sqrt{x^2+y^2})$

Domain is $\mathbb{R}^2 \setminus \{(0,0)\}$

ii. X

iii. The annular region $1 \le x^2 + y^2 \le 2$ in $\mathbb{R}^2$ is simply-connected.

iii. X

iv. If X is a simply-connected proper subset of $\mathbb{R}^3$, then $\mathbb{R}^3 \setminus X$ is not simply-connected.

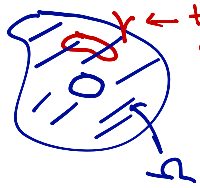
Counterexample: X = space above xy-plane
then: $\mathbb{R}^3 \setminus X$ = space below xy-plane } Both simply-connected

iv. X

v. If a closed loop γ in a region Ω can contract to a point within Ω, then ~~γ~~ Ω is simply-connected.

We need every closed loop γ contractable to a point, not just a closed loop γ.

v. X



← this γ can contract to a point within Ω, but Ω is not simply-con.

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2. (a) Evaluate the double integral $\iint_D (x^2 + y^2 + 1)\, dA$ where D is the unit disk in $\mathbb{R}^2$ given by the equation $\underbrace{x^2 + y^2 \le 1}$. **/4**

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$\iint_D (x^2+y^2+1)\, dA = \int_0^{2\pi}\int_0^1 (r^2+1)\cdot r\, dr\, d\theta$$

$$= \left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^1 (r^3+r)\, dr\right)$$

$$= 2\pi \cdot \left[\frac{r^4}{4} + \frac{r^2}{2}\right]_0^1$$

$$= 2\pi\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{3\pi}{2}$$

#

Problem #2 continues on next page...

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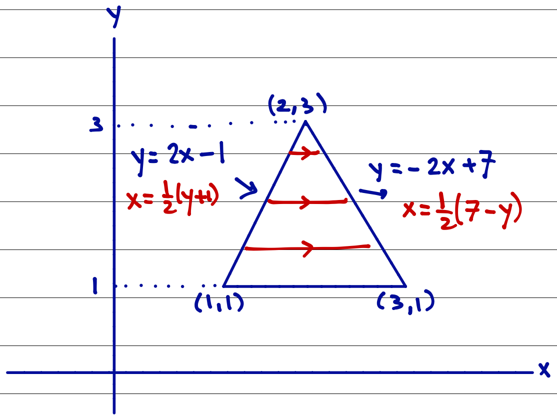
(b) Evaluate the double integral $\iint_T xe^{y^2-6y}\,dA$ where T is the triangle in $\mathbb{R}^2$ with vertices $(1,1)$, $(2,3)$ and $(3,1)$. **/6**



$$\iint_T xe^{y^2-6y}\,dA = \int_{y=1}^{y=3}\int_{x=\frac{1}{2}(y+1)}^{x=\frac{1}{2}(7-y)} xe^{y^2-6y}\,dx\,dy$$

$$= \int_1^3 \left[\frac{x^2}{2}e^{y^2-6y}\right]_{x=\frac{1}{2}(y+1)}^{x=\frac{1}{2}(7-y)} dy$$

$$= \int_1^3 \frac{1}{2}\cdot\frac{1}{4}\left((7-y)^2-(y+1)^2\right)e^{y^2-6y}\,dy$$

$$= \frac{1}{8}\int_1^3 8\,(6-2y)\;e^{y^2-6y}\,dy$$

$$= -\int_1^3 (2y-6)e^{y^2-6y}\,dy$$

$$= -\left[e^{y^2-6y}\right]_1^3 = -\left(e^{3^2-18}-e^{1-6}\right)$$

$$= e^{-5}-e^{-9}$$

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3. Consider the vector field:

$$\mathbf{F}(x,y) = \left(2xe^{xy} + x^2ye^{xy}\right)\mathbf{i} + \left(x^3e^{xy} + 2y\right)\mathbf{j}$$

(a) Calculate $\nabla \times \mathbf{F}$. /3

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xe^{xy}+x^2ye^{xy} & x^3e^{xy}+2y & 0 \end{vmatrix}$$

$$= 0\hat{i} - 0\hat{j} + \left(\left(\cancel{3x^2e^{xy}} + \cancel{x^3ye^{xy}}\right) - \left(\cancel{2x^2e^{xy}} + x^2\left(\cancel{e^{xy}} + \cancel{xye^{xy}}\right)\right)\right)\hat{k}$$

$$= \vec{0}$$

(b) Find the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ over the path C: /5

$$\mathbf{r}(t) = \left(\cos^{24601} t\right)\mathbf{i} + \frac{2t}{\pi}\mathbf{j}, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

From (a) & domain of $\vec{F}$ is $\mathbb{R}^2 \leftarrow$ simply-connected

$\Rightarrow$ $\vec{F}$ is conservative $\Rightarrow$ $\int_C \vec{F}\cdot d\vec{r}$ is independent of path!

$\vec{r}(-\frac{\pi}{2}) = 0\hat{i} - \hat{j} = \langle 0,-1\rangle$ start

$\vec{r}(\frac{\pi}{2}) = 0\hat{i} + \hat{j} = \langle 0,1\rangle$ end

Join the starting and ending points by a straight line:

L: $\vec{r}_L(t) = \langle 0,-1\rangle + t\left(\langle 0,1\rangle - \langle 0,-1\rangle\right) = \langle \underbrace{0}_{x}, \underbrace{2t-1}_{y}\rangle$, $0 \le t \le 1$

$$\int_L \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}\cdot\vec{r}_L'(t)\,dt = \int_0^1 \langle 2\cdot 0\cdot e^{0y} + 0\cdot ye^{0y},\ 0e^{0y}+2y\rangle \cdot \langle 0,2\rangle\,dt$$

sub. $x=0$, $y=2t-1$ into $\vec{F}$

$$= \int_0^1 2y\cdot 2\,dt = \int_0^1 4(2t-1)\,dt = 4\left[t^2-t\right]_0^1 = 0$$

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4. Consider the vector field:

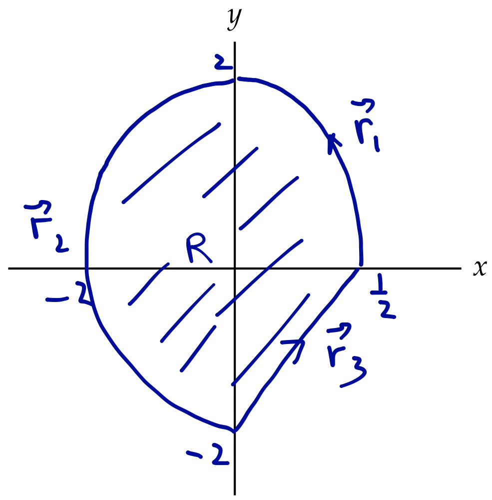
$$\mathbf{F} = \left(\frac{-y}{2} + \frac{\partial f}{\partial x}\right)\mathbf{i} + \left(\frac{x}{2} + \frac{\partial f}{\partial y}\right)\mathbf{j}$$

where f is a scalar function which is defined and $\cancel{C^1}$ C^2 everywhere on $\mathbb{R}^2$. → $f(x,y)$ indep. of z

Let C be the closed path which consists of the following segments:

- first along the ellipse $\mathbf{r}_1(t) = \left(\frac{1}{2}\cos t\right)\mathbf{i} + (2\sin t)\mathbf{j}$ from $t = 0$ to $t = \frac{\pi}{2}$;
- then along the circle $\mathbf{r}_2(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j}$ from $t = \frac{\pi}{2}$ to $t = \frac{3\pi}{2}$;
- finally along the line segment from the point $(0, -2)$ to the point $(\frac{1}{2}, 0)$.

(a) Sketch the path C on the xy-plane. Indicate all x- and y-intercepts. /3



(b) Calculate $\nabla \times \mathbf{F}$. /3

$$\nabla\times\vec F = \begin{vmatrix} \hat i & \hat j & \hat k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{2}+\frac{\partial f}{\partial x} & \frac{x}{2}+\frac{\partial f}{\partial y} & 0 \end{vmatrix}$$

$$= -\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\hat i + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)\hat j + \left(\frac12 + \cancel{\frac{\partial^2 f}{\partial x\partial y}} + \frac12 - \cancel{\frac{\partial^2 f}{\partial y\partial x}}\right)\hat k$$

Mixed Partials Theorem

since $f(x,y)$ $= \underbrace{0\hat i + 0\hat j} + \hat k = \underline{\underline{\hat k}}$

(c) Using (b), find the line integral: /3

$$\oint_C \mathbf{F}\cdot d\mathbf{r}.$$

defined, C' everywhere

Green's Theorem

$$\Rightarrow \oint_C \vec F\cdot d\vec r = \iint_R (\nabla\times\vec F)\cdot\hat k\,dA = \iint_R \hat k\cdot\hat k\,dA = \iint_R 1\,dA = \text{Area}(R)$$

$$= \underbrace{\frac14\cdot\pi\cdot\frac12\cdot 2}_{\frac14\text{ellipse}} + \underbrace{\frac12\cdot\pi\cdot 2^2}_{\frac12\text{circle}} + \underbrace{\frac12\cdot 2\cdot\frac12}_{\text{triangle}} = \frac{\pi}{4} + 2\pi + \frac12 = \frac{9\pi}{4} + \frac12 \quad *$$

error in previous version

Problem #4 continues on next page...

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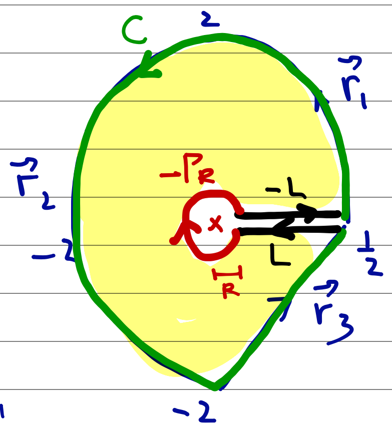
(d) Consider a vector field $\mathbf{G}$ which is defined and C^1 on $\mathbb{R}^2\backslash\{(0,0)\}$. Given that the following facts about $\mathbf{G}$: /5

- $\nabla \times \mathbf{G}(x,y) = \mathbf{0}$ for any $(x,y) \neq (0,0)$
- $\displaystyle\int_{\Gamma_R} \mathbf{G}\cdot d\mathbf{r} = \frac{\pi}{2}$ for any $R > 0$, where Γ_R is the counter-clockwise circular path centered at the origin with radius R.

Evaluate the line integral $\oint_C (\mathbf{F}+\mathbf{G})\cdot d\mathbf{r}$ where C and $\mathbf{F}$ are given in the previous page of this problem. done in (b) $\rightarrow = \oint_C \vec{F}\cdot d\vec{r} + \oint_C \vec{G}\cdot d\vec{r}$

C encloses (0,0), but $\vec{G}$ is undefined at (0,0).
Can't use Green's Theorem directly!!

Drill a hole!



Consider the closed path:
$C + L + (-\Gamma_R) + (-L)$

Γ_R for ↺
$-\Gamma_R$ for ↻

This path does not enclosed the origin!

Green's Theorem

$$\Rightarrow \oint_{C+L-\Gamma_R-L} \vec{G}\cdot d\vec{r} = \iint_{\text{Yellow region}} \underbrace{(\nabla\times\vec{G})}_{=\vec{0}\text{ given}}\cdot\hat{k}\,dA = 0$$

$$\oint_C \vec{G}\cdot d\vec{r} + \cancel{\int_L \vec{G}\cdot d\vec{r}} - \oint_{\Gamma_R} \vec{G}\cdot d\vec{r} - \cancel{\int_L \vec{G}\cdot d\vec{r}} = 0$$

$$\Rightarrow \oint_C \vec{G}\cdot d\vec{r} = \oint_{\Gamma_R} \vec{G}\cdot d\vec{r} = \frac{\pi}{2} \leftarrow \text{given!}$$

From (b),

$$\oint_C (\vec{F}+\vec{G})\cdot d\vec{r} = \oint_C \vec{F}\cdot d\vec{r} + \oint_C \vec{G}\cdot d\vec{r} = \left(\frac{9\pi}{4}+\frac{1}{2}\right) + \frac{\pi}{2} = \frac{11\pi}{4} + \frac{1}{2} \#$$

error in previous version

5. Write down a *short* proof for each fact below using the method specified in each part:

(a) Let f be a scalar function which is defined and C^2 on $\mathbb{R}^2$. Using the Fundamental Theorem of Line Integrals, show that $\oint_C \nabla f \cdot d\mathbf{r} = 0$ for any simple closed curve C in $\mathbb{R}^2$. /2

Closed curve $\Rightarrow$ start = end

$$\oint_C \nabla f \cdot d\vec{r} = f(\text{end}) - f(\text{start}) = 0$$

same

(b) Let g be a scalar function which is defined and C^2 on $\mathbb{R}^2$. Using the Green's Theorem, show that $\oint_C \nabla g \cdot d\mathbf{r} = 0$ for any simple closed curve C in $\mathbb{R}^2$. /2

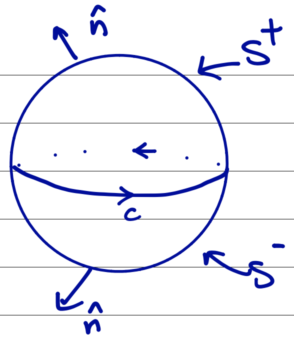
$$\oint_C \nabla g \cdot d\vec{r} = \iint_{\substack{\text{region}\\ \text{enclosed}\\ \text{by } C}} \underbrace{(\nabla \times \nabla g)}_{=\vec{0}} \cdot \hat{k}\, dA = \iint_{\substack{\text{region}\\ \text{enclosed}\\ \text{by } C}} \vec{0} \cdot \hat{k}\, dA = 0$$

(c) Let $\mathbf{F}$ be a vector field defined and C^2 everywhere in $\mathbb{R}^3$. Using the Stokes' Theorem, show that $\oiint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS = 0$ where S is the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. /2

$$\oiint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \iint_{S+} (\nabla \times \vec{F}) \cdot \hat{n}\, dS + \iint_{S_-} (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

$$= \oint_C \vec{F} \cdot d\vec{r} + \oint_{-C} \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot d\vec{r} - \oint_C \vec{F} \cdot d\vec{r} = 0$$

opposite by RH rule



(d) Let $\mathbf{G}$ be a vector field defined and C^2 everywhere in $\mathbb{R}^3$. Using the Divergence Theorem, show that $\oiint_S (\nabla \times \mathbf{G}) \cdot \hat{\mathbf{n}}\, dS = 0$ where S is the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. /2

$$\oiint_S (\nabla \times \vec{G}) \cdot \hat{n}\, dS = \iiint_{\substack{\text{solid}\\ \text{enclosed}\\ \text{by } S}} \underbrace{\nabla \cdot (\nabla \times \vec{G})}_{=0}\, dV = 0$$

Made with Goodnotes

6. Consider the vector field:

$$\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + (x\cos y)\mathbf{j} + 3y\mathbf{k}$$

whose **i**-component f is not given, but is known to be C^1 everywhere in $\mathbb{R}^3$.

(a) Verify that **/3**

$$\nabla \times \mathbf{F} = 3\mathbf{i} + \frac{\partial f}{\partial z}\mathbf{j} + \left(\cos y - \frac{\partial f}{\partial y}\right)\mathbf{k}.$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x,y,z) & x\cos y & 3y \end{vmatrix}$$

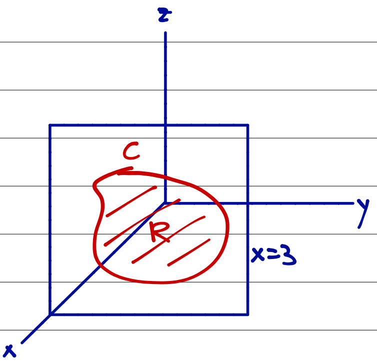
$$= \left(\frac{\partial}{\partial y}3y - \frac{\partial}{\partial z}x\cos y\right)\hat{i} - \left(\frac{\partial}{\partial x}3y - \frac{\partial f}{\partial z}\right)\hat{j} + \left(\frac{\partial}{\partial x}x\cos y - \frac{\partial f}{\partial y}\right)\hat{k}$$

$$= 3\hat{i} + \frac{\partial f}{\partial z}\hat{j} + \left(\cos y - \frac{\partial f}{\partial y}\right)\hat{k}$$

(b) Let C be an arbitrary simple closed curve on the plane $x = 3$. Using (a) and a *suitable* theorem, show that the value of $\left|\oint_C \mathbf{F}\cdot d\mathbf{r}\right|$ depends *only* on the area enclosed by C on the plane $x = 3$. **/5**

$$\oint_C \vec{F}\cdot d\vec{r} \overset{\text{Stokes}}{=} \iint_R (\nabla\times\vec{F})\cdot\hat{n}\,dS$$

Region enclosed by C on the plane x=3.



$$= \iint_R \left[3\hat{i} + \frac{\partial f}{\partial z}\hat{j} + \left(\cos y - \frac{\partial f}{\partial y}\right)\hat{k}\right]\cdot \pm\hat{i}\,dS$$

− if C

+ if C

$$= \iint_R \pm 3\,dS$$

$$= \pm 3\,\text{Area}(R)$$

$$\therefore \left|\oint_C \vec{F}\cdot d\vec{r}\right| = 3\,\text{Area}(R).$$

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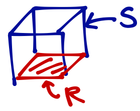
7. Let D be the solid square cube $\mathbb{R}^3$ defined by inequalities

$$1 \leq x \leq 2, \quad 2 \leq y \leq 3 \quad \text{and} \quad 0 \leq z \leq 1.$$

One of the faces is on the xy-plane. Denote this face by R, and the union of all other five faces by S. Consider the vector field:

$$\mathbf{F}(x, y, z) = (2x - xe^z)\mathbf{i} + (3y - ye^z)\mathbf{j} + 2(e^z - 1)\mathbf{k}$$



(a) Show that the surface flux $\iint_R \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$ over R is 0. /3

On R: $\hat{n} = -\hat{k}$, $z=0$

$\therefore \vec{F} = (2x - x\cdot e^0)\hat{\imath} + (3y - y\cdot e^0)\hat{\jmath} + 2(e^0 - 1)\hat{k}$

$= x\hat{\imath} + 2y\hat{\jmath} + 0\hat{k}$

$\therefore \iint_R \vec{F}\cdot\hat{n}\,dS = \iint_R (x\hat{\imath} + 2y\hat{\jmath} + 0\hat{k})\cdot(-\hat{k})\,dS = \iint_R 0\,dS = 0$

(b) Using the Divergence Theorem, find the surface flux $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$ over S. Choose $\hat{\mathbf{n}}$ to be the outward unit normal. /5

Note S is NOT closed! Can't apply Div. Theorem directly.

Glue S and R together to form a closed surface cube Σ.

Then $\oiint_\Sigma \vec{F}\cdot\hat{n}\,dS = \iiint \nabla\cdot\vec{F}\,dV = \iiint (2 - e^z + 3 - e^z + 2e^z)\,dV$

$= \iiint 5\,dV = 5\,\text{Vol}(\ ) = 5$

$\Sigma = S \cup R$

$\Rightarrow \oiint_\Sigma \vec{F}\cdot\hat{n}\,dS = \iint_S \vec{F}\cdot\hat{n}\,dS + \iint_R \vec{F}\cdot\hat{n}\,dS$

$\Rightarrow 5 = \iint_S \vec{F}\cdot\hat{n}\,dS + \underset{\text{from (a)}}{0} \Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dS = 5$

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8. Suppose $f(x,y,z)$ is a C^2 function defined on $\mathbb{R}^3$ such that $\nabla^2 f(x,y,z) = 0$. We call such a function a *harmonic function*. Recall that $\nabla^2 f := \nabla \cdot \nabla f$.

(a) Prove the following identity: **/4**

$$\nabla \cdot (f\,\nabla f) = |\nabla f|^2 \qquad (*)$$

$\nabla \cdot (f\nabla f) = \nabla \cdot \left\langle f\frac{\partial f}{\partial x}, f\frac{\partial f}{\partial y}, f\frac{\partial f}{\partial z}\right\rangle$

$= \frac{\partial}{\partial x}\left(f\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial f}{\partial z}\right)$

$= \frac{\partial f}{\partial x}\frac{\partial f}{\partial x} + f\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial f}{\partial y}\frac{\partial f}{\partial y} + f\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial f}{\partial z}\frac{\partial f}{\partial z} + f\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right)$

$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 + f\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = \left|\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right|^2 + f\,\underbrace{\cancel{\nabla^2 f}}_{0 \text{ (given)}} = |\nabla f|^2$

OR simply use product rule:

$\nabla \cdot (f\,\nabla f) = \nabla f \cdot \nabla f + f\,\nabla \cdot \nabla f$

$= |\nabla f|^2 + \cancel{f\nabla^2 f}^{\,0}$

(b) Let D be a solid bounded by a simply-connected closed oriented surface S in $\mathbb{R}^3$. Using (a), show that: **/4**

$$\oiint_S f\nabla f \cdot \hat{\mathbf{n}}\, dS = \iiint_D |\nabla f|^2\, dV$$

where $\hat{\mathbf{n}}$ is the outward unit normal of S.

$\oiint_S f\nabla f \cdot \hat{n}\, dS = \iiint_D \nabla \cdot (f\nabla f)\, dV$ (Div. Thm.)

$= \iiint_D |\nabla f|^2\, dV$ from (a)

(c) Show that if $f(x,y,z) = 0$ on the surface S, then $f(x,y,z) = 0$ in the solid D. **/4**

If $f = 0$ on S, then

$\oiint_S \underbrace{f}_{=0}\nabla f \cdot \hat{n}\, dS = 0$

$\overset{(b)}{\Rightarrow} \iiint_D \underbrace{|\nabla f|^2}_{\text{non-neg.}} dV = 0 \Rightarrow |\nabla f| = 0 \Rightarrow \nabla f = \vec{0}$

$\Rightarrow f \equiv$ const. in D. Continuity of f & $f = 0$ on S $\Rightarrow$ const. $= 0$

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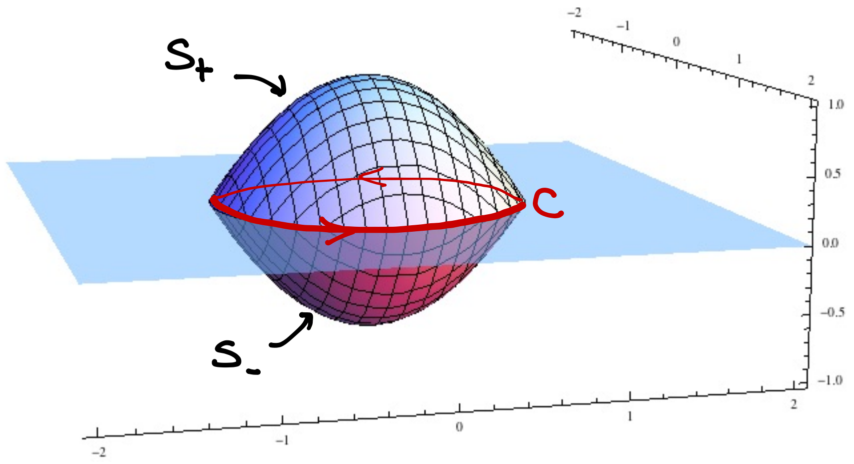
9. Let S_+ be the part of a paraboloid $z = -x^2 - y^2 + 1$ above the xy-plane, S_- be the part of a paraboloid $z = x^2 + y^2 - 1$ below the xy-plane. The intersection of the two paraboloids S_+ and S_- is the unit circle C centered at the origin on the xy-plane. Furthermore, let Σ be the union of the two paraboloids S_+ and S_-, and D be the solid region in $\mathbb{R}^3$ enclosed by Σ. See the sketch below as a reference. **/4**



A student is confused about the use of Stokes' and Divergence Theorems. First, read over the student's argument below:

"Since Σ is a closed surface, using Divergence Theorem one can say:

$$\oiint_{\Sigma} \mathbf{k} \cdot \hat{\mathbf{n}}\, dS = \iiint_D \nabla \cdot \mathbf{k}\, dV = \iiint_D 0\, dV = 0.$$

However, it seems like I get a different conclusion using Stokes' Theorem. First note that

$$\mathbf{k} = \nabla \times \left(-\frac{y}{2}\mathbf{i} + \frac{x}{2}\mathbf{j}\right).$$

By Stokes' Theorem we have:

$$\iint_{S_+} \mathbf{k} \cdot \hat{\mathbf{n}}\, dS = \iint_{S_+} \left[\nabla \times \left(-\frac{y}{2}\mathbf{i} + \frac{x}{2}\mathbf{j}\right)\right] \cdot \hat{\mathbf{n}}\, dS = \oint_C \left(-\frac{y}{2}\mathbf{i} + \frac{x}{2}\mathbf{j}\right) \cdot d\mathbf{r}.$$

Since C is the unit circle $x^2 + y^2 = 1$, by parametrizing C as $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$, where $0 \le t \le 2\pi$, we get:

$$\oint_C \left(-\frac{y}{2}\mathbf{i} + \frac{x}{2}\mathbf{j}\right) \cdot d\mathbf{r} = \int_0^{2\pi} \left(-\frac{\sin t}{2}\mathbf{i} + \frac{\cos t}{2}\mathbf{j}\right) \cdot (-\sin t\ \mathbf{i} + \cos t\ \mathbf{j})\ dt = \int_0^{2\pi} \frac{1}{2}\, dt = \pi.$$

Since both S_+ and S_- share the same boundary curve C, by applying Stokes' Theorem on S_- the ~~same way~~ as above, I can get:

$$\iint_{S_-} \mathbf{k} \cdot \hat{\mathbf{n}}\, dS = \cancel{\pi}.$$

Then, by the fact that $\Sigma = S_+ \cup S_-$, we would have:

$$\oiint_{\Sigma} \mathbf{k} \cdot \hat{\mathbf{n}}\, dS = \iint_{S_+} \mathbf{k} \cdot \hat{\mathbf{n}}\, dS + \iint_{S_-} \mathbf{k} \cdot \hat{\mathbf{n}}\, dS = \pi + \pi = 2\pi$$

How can $\oiint_{\Sigma} \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\, dS$ be both 0 and 2π? I am very confused!!!"

Problem #9 continues on next page...

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In a *short* paragraph, point out and briefly explain the fallacy of the student. You may include a diagram, or mark on the student's solution if necessary.

The student cannot apply Stokes' Theorem on S_- the same way as in S_+ !!

By the RH rule, $\hat{n}$ and orientation of C should be:

$$\iint_{S_-} \hat{k}\cdot\hat{n}\,dS = \iint_{S_-}\left[\nabla\times\left\langle -\frac{y}{2}, \frac{x}{2}, 0\right\rangle\right]\cdot\hat{n}\,dS$$

$$= \oint_{-C}\left\langle -\frac{y}{2}, \frac{x}{2}, 0\right\rangle\cdot d\vec{r}$$

↑ not C !!

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