MATH 2023 – Multivariable Calculus

Lecture #15 Worksheet   April 4, 2019

Problem 1. Use Green's Theorem to show that the area of a simple region can be expressed as

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Hence find the area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$A = \iint_D 1\,dA \qquad (\text{If } P=0,\ Q=x,\ \vec{F}=\langle 0,x\rangle)$$

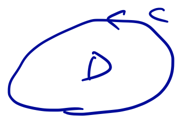
$$= \oint_C x\,dy \quad \text{by Green's Thm.}$$

$$(\text{If } P=-y,\ Q=0,\ \vec{F}=\langle -y,0\rangle)$$

$$= \oint_C -y\,dx$$

$$2A = A + A = \oint_C x\,dy - \oint_C y\,dx \iff A = \frac{1}{2}\oint x\,dy - y\,dx.$$

Area of ellipse: $C = \vec{r}(t) = \langle a\cos t, b\sin t\rangle$



$dx = -a\sin t\,dt$

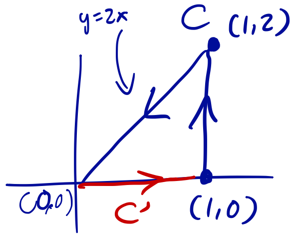
$dy = b\cos t\,dt$

$$\Rightarrow \frac{1}{2}\int_0^{2\pi} a\cos t\, b\cos t\,dt - b\sin t(-a\sin t)\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} \cos^2 + \sin^2\,dt = \pi ab$$

P Q

Problem 2. Find $\int_C e^x\,dx + (xy + \cos y^{2023})\,dy$ where C consists of the straight line segments from $(1,0)$ to $(1,2)$ to $(0,0)$.



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$\parallel$$

$$\int_{C} + \int_{C'}$$

$$\iint_D y\, dA = \int_0^2 \int_{\frac{y}{2}}^1 y\, dx\, dy = \int_0^2 \left(1 - \frac{y}{2}\right) y\, dy = \frac{y^2}{2} - \frac{y^3}{6}\Big|_0^2$$

$$= \cancel{2 - \frac{8}{6}}\ \frac{2}{3}.$$

$$\int_{C'} \vec{F}\cdot d\vec{r} = \int_0^1 e^x\, dx = e - 1$$

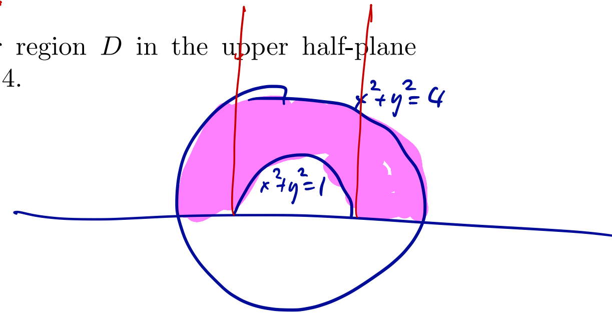
y=0!

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA - \int_{C'} \vec{F}\cdot d\vec{r} = \frac{3}{2} - (e-1).$$

Problem 3. Evaluate

$$\oint_C \overset{P}{\underline{y^2}}\,dx + \overset{Q}{\underline{3xy}}\,dy$$

where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.



Green's Thm is OK!

$$= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= \iint_D 3y - 2y \, dA$$

$$= \iint_D \underline{y}\, \underline{dA}$$

$$= \int_0^{\pi} \int_1^2 \underline{r\sin\theta}\; \underline{r\,dr\,d\theta}$$

Problem 4. Consider the change of variables

$$x = g(u,v), \qquad y = h(u,v)$$

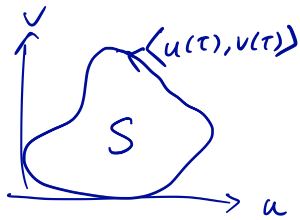
Use Green's Theorem to prove the change of variables formula

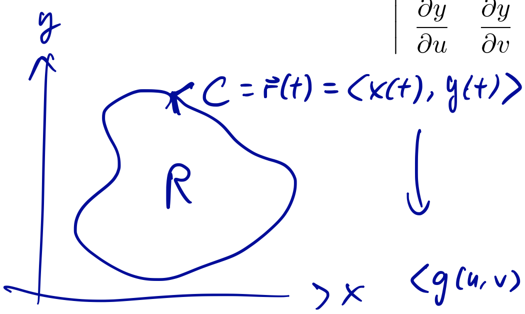
$$\iint_R dxdy = \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv$$

(R, S simple region)

where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ is called the **Jacobian**, given by the determinant

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$





$x = g(u,v)$

$y = h(u,v)$

$\langle g(u,v), h(u,v)\rangle$

$u = u(\tau)$

$v = v(\tau)$

$$\text{Area}(R) = \oint_C x\,dy = \oint_{C'} g(u,v)\underbrace{\left(\frac{\partial h}{\partial u}\frac{du}{d\tau} + \frac{\partial h}{\partial v}\frac{dv}{d\tau}\right) d\tau}_{dy = y'(\tau)d\tau}, \quad y = h(u(\tau), v(\tau)).$$

C': $\langle u(\tau), v(\tau)\rangle$

$$\oint_{C'} g(u,v)\, \nabla h \cdot d\vec{r} \qquad \text{(in } (u,v) \text{ coordinate)}$$

$P : g \cdot h_u \quad , Q : g \cdot h_v$

$$\iint_S \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) dA$$

$$\|$$

$$g_u \cdot h_v + \cancel{g \cdot h_{vu}} - (g_v h_u + \cancel{g \cdot h_{uv}})$$

$$= g_u h_v - g_v h_u = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$