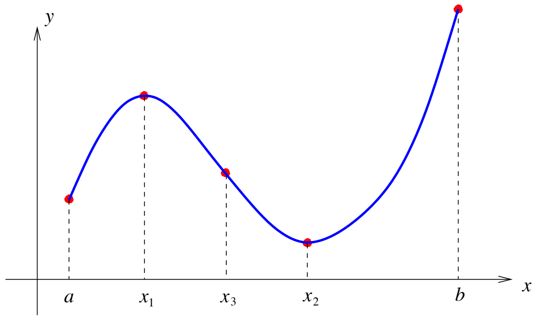
Chapter 13
Applications of partial derivatives

Contents

Review

Let $y = f(x)$ be a continuous function in a *closed* interval $[a, b]$, i.e.



x_1	is a relative maximum point, $f'(x_1) = 0$ and $f''(x_1) < 0$
x_2	is a relative minimum point, $f'(x_2) = 0$ and $f''(x_2) > 0$
x_3	is a point of inflection, $f''(x_3) = 0$
a, b	are the end points in the closed interval $[a, b]$
b	is an absolute maximum point in $[a, b]$ if $f(b) \geqslant f(x)$ for $a \leqslant x \leqslant b$
x_2	is an absolute minimum point in $[a, b]$ if $f(x_2) \leqslant f(x)$ for $a \leqslant x \leqslant b$

If $f(x)$ is continuous on a closed and bounded interval, then f must have an absolute maximum and an absolute minimum.

13.1 Extreme values

For a differentiable function $f(x_1, x_2, \cdots, x_n)$ with domain D, if

$\|\mathbf{r}-\mathbf{r}_0\| \geqslant \varepsilon$

$$f(x_1^0, x_2^0, \cdots, x_n^0) \geqslant f(x_1, x_2, \cdots, x_n)$$

$f(\mathbf{r}_0) \geqslant f(\mathbf{r})$

for all $(x_1, x_2, \cdots, x_n)$ in D, then $(x_1^0, x_2^0, \cdots, x_n^0)$ is an *absolute* maximum of f.

If $f(x_1^0, x_2^0, \cdots, x_n^0) \geqslant f(x_1, x_2, \cdots, x_n)$ for $(x_1, x_2, \cdots, x_n)$ in some ball centered at $f(x_1^0, x_2^0, \cdots, x_n^0)$ inside $S \subset D$, then $f(x_1^0, x_2^0, \cdots, x_n^0)$ is a *relative* maximum of f.

Similar definitions can be defined for absolute minimum and relative minimum. The word *extremum* means a maximum or a minimum.

Theorem

If $f(x_1, x_2, \cdots, x_n)$ is continuous on a closed (have all boundary points) and bounded (i.e. lies inside some large ball) set, then f must have an absolute maximum and an absolute minimum.

Note: Absolute *extrema* may come from relative extrema or boundary points.

Definition: Critical point

$\mathbf{r}_0 = (x_1^0, x_2^0, \cdots, x_n^0)$ is a critical point of $f(\mathbf{r}) = f(x_1, x_2, \cdots, x_n)$ if

$$\boxed{\nabla f(\mathbf{r}_0) = (f_{x_1}(\mathbf{r}_0), f_{x_2}(\mathbf{r}_0), \cdots, f_{x_n}(\mathbf{r}_0)) = \mathbf{0},} \qquad \text{where} \quad f_{x_1}(\mathbf{r}_0) = \frac{\partial f}{\partial x_1}\Big|_{\mathbf{r}_0}$$

If f has a relative extremum at a point $\mathbf{r}_0$, then $\nabla f(\mathbf{r}_0) = 0$.

($\because$ By the first derivative test of one-variable calculus, if $g(x_1) = f(x_1, x_2^0, x_3^0, \cdots, x_n^0)$ has a relative extremum at $x_1 = x_1^0$, then

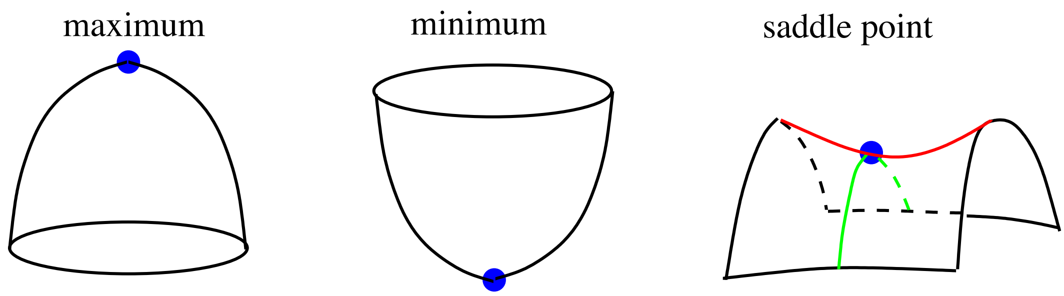
$$g'(x_1^0) = f_{x_1}(x_1^0, x_1^0, \cdots, x_n^0) = f_{x_1}(\mathbf{r}_0) = 0$$

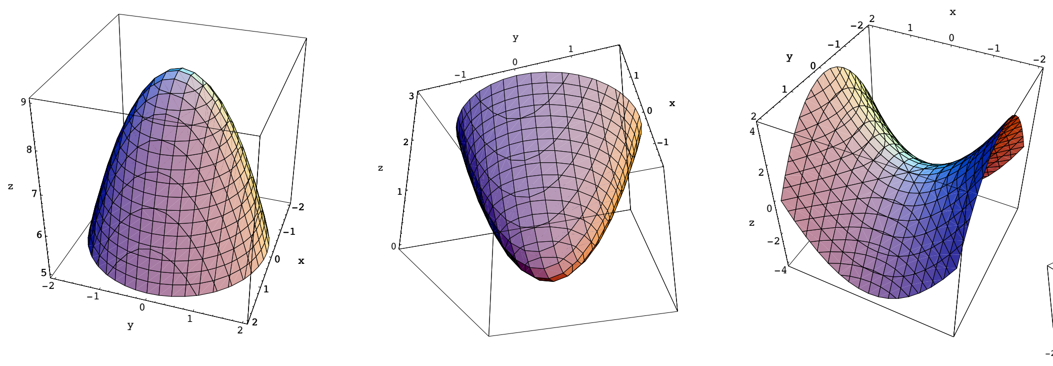
and similarly for the other coord.)

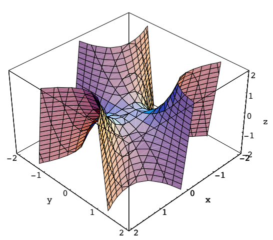
If f has a relative extremum at $\mathbf{r}_0$, then $\mathbf{r}_0$ is a critical point of f, i.e.

if $\mathbf{r}_0$ is a relative extreme	$\Rightarrow$	$\mathbf{r}_0$ is a critical point,
but if $\mathbf{r}_0$ is a critical point	$\not\Rightarrow$	$\mathbf{r}_0$ is a relative extremum.

A critical point which is not a relative extremum is a *saddle point* of $z = f(x, y)$.





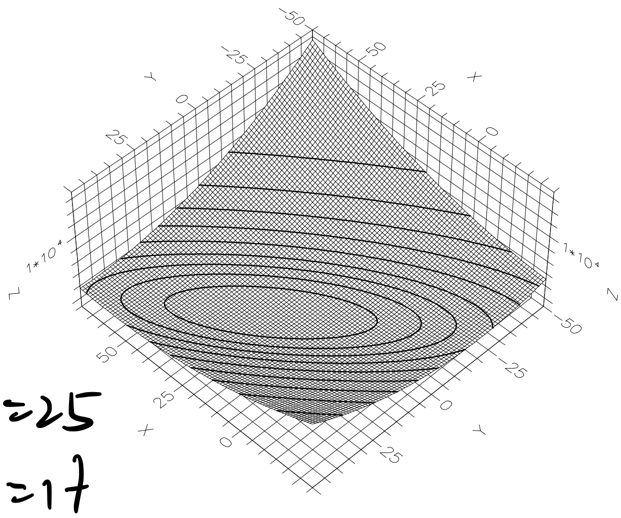


Graph of the monkey saddle

Ex. 1.1 Find the critical points of $z = f(x, y) = x^2 - 2xy + 1.5y^2 - 16x - y + 1000$.

$f_x = 2x - 2y - 16$

$f_y = -2x + 3y - 1$



For critical pt, $\nabla(f_x, f_y) = 0$

$$\begin{cases} 2x - 2y - 16 = 0 \\ -2x + 3y - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = 25 \\ y = 17 \end{cases}$$

$f_{xx} = 2 \quad f_{yy} = 3 \quad f_{xy} = -2 = f_{yx}$

$D = f_{xx} f_{yy} - (f_{xy})^2 = 6 - 4 = 2$

pt	f_{xx}	f_{yy}	f_{xy}	D
(25,17)	2	3	-2	2
	>0	>0		>0

+ve

$\boxed{\text{min}}$

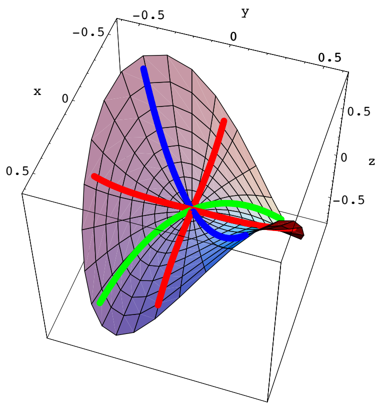
Classify the critical points - the second derivative test

Suppose $f(x, y)$ has a critical point at $\mathbf{r}_0 = (x_0, y_0)$ (i.e. $\nabla f(\mathbf{r}_0) = \mathbf{0}$) and the second partial derivative of $f(x, y)$ are continuous in a disk with center $\mathbf{r}_0 = (x_0, y_0)$. Let

$$D = \begin{vmatrix} f_{xx}(\mathbf{r}_0) & f_{xy}(\mathbf{r}_0) \\ f_{yx}(\mathbf{r}_0) & f_{yy}(\mathbf{r}_0) \end{vmatrix} = f_{xx}(\mathbf{r}_0) f_{yy}(\mathbf{r}_0) - f_{xy}^2(\mathbf{r}_0).$$

D	$f_{xx}(\mathbf{r}_0)$ or $f_{yy}(\mathbf{r}_0)$	nature of $\mathbf{r}_0$
> 0	> 0	relative minimum
> 0	< 0	relative maximum
< 0		saddle point
$= 0$		no conclusion can be drawn

‘Proof’ of D-test:



By analogy with one-variable case we see that $\partial^2 f/\partial x^2$ and $\partial^2 f/\partial y^2$ must be both positive for a minimum and both be negative for a maximum. However these are not sufficient conditions since they may also be obeyed at saddle points (see the function $z = 4xy - x^2 + y^4$: at $(0,0)$, $f_{xx} = f_{yy} = 0$). What is important for a minimum (or maximum) is that the second partial derivative must be positive (or negative) in *all* directions, not just the x- and y-direction.

If $z = f(x,y)$ and assume also that $f_x(a,b) = f_y(a,b) = 0$, i.e. (a,b) is a critical point of $f(x,y)$. We compute the second-order **directional derivative** of f in the direction of $\mathbf{u} = (h,k)$ at the critical point. The first-order derivative is given by

$$D_{\mathbf{u}} f = f_x\, h + f_y\, k$$

The second-order derivative is

$$\begin{aligned} D^2_{\mathbf{u}} f = D_{\mathbf{u}}(D_{\mathbf{u}} f) &= \frac{\partial}{\partial x}(D_{\mathbf{u}} f)\, h + \frac{\partial}{\partial y}(D_{\mathbf{u}} f)\, k \\ &= (f_{xx}\, h + f_{yx}\, k)\, h + (f_{xy}\, h + f_{yy}\, k)\, k \\ &= f_{xx}\, h^2 + 2 f_{xy}\, hk + f_{yy}\, k^2. \end{aligned}$$

If we complete the square in this expression, we obtain

$$D^2_{\mathbf{u}} f = f_{xx}\left[\left(h + \frac{f_{xy}}{f_{xx}}\, k\right)^2 + \frac{k^2}{f_{xx}^2}\left(f_{xx} f_{yy} - f_{xy}^2\right)\right] \qquad (*)$$

For a **minimum**, we require $(*)$ to be positive for all h and k, and hence

$$f_{xx} > 0, \qquad D = f_{xx} f_{yy} - (f_{xy})^2 > 0.$$

Similarly for a **maximum** we require $(*)$ to be negative, and hence

$$f_{xx} < 0, \qquad D = f_{xx} f_{yy} - f_{xy})^2 > 0.$$

For **minima** and **maxima**, symmetric requires that f_{yy} obeys the same criteria as f_{xx}.

If $D = f_{xx} f_{yy} - (f_{xy})^2 < 0$, then the expression in the square brackets $(*)$ is a difference of positive quantities.

$$[\qquad] > 0 \quad \text{when} \quad \mathbf{u} = (h,\, 0)$$

$$[\qquad] < 0 \quad \text{when} \quad \mathbf{u} = \left(-\frac{f_{xy}}{f_{xx}}\, k,\, k\right)$$

Thus f must be a **saddle** point at (a,b).

D test fail 的話
的处理方法 out

If $f_{xx} = 0$, then $D^2_{\mathbf{u}} f = 2 f_{xy}\, hk + f_{yy}\, k^2 = k\,(2 f_{xy}\, h + f_{yy}\, k)$

$$\begin{array}{llll} D^2_{\mathbf{u}} f > 0 & \text{if} \quad k > 0 & \text{and} & 2 f_{xy}\, h + f_{yy}\, k > 0 \\ D^2_{\mathbf{u}} f < 0 & \text{if} \quad k > 0 & \text{and} & 2 f_{xy}\, h + f_{yy}\, k < 0 \end{array}$$

Thus f must be a **saddle** point at (a,b).

Consider the functions

$$\begin{aligned} f(x,y) &= x^4 + y^4, \\ g(x,y) &= -x^4 - y^4, \\ h(x,y) &= x^4 - y^4. \end{aligned}$$

They all have a critical point at $(0,0)$ and at the point $(0,0)$, $D = 0$ for all three cases. However,

$(0,0)$ is a min point of f

$(0,0)$ is a max point of g

and $(0,0)$ is a saddle point of h.

i.e. $D = 0$ cannot determine the nature of the critical point.

Ex. 1.2 Find the relative minima and maxima of $f(x,y) = x^3 + y^3 - 3x - 12y + 20$.

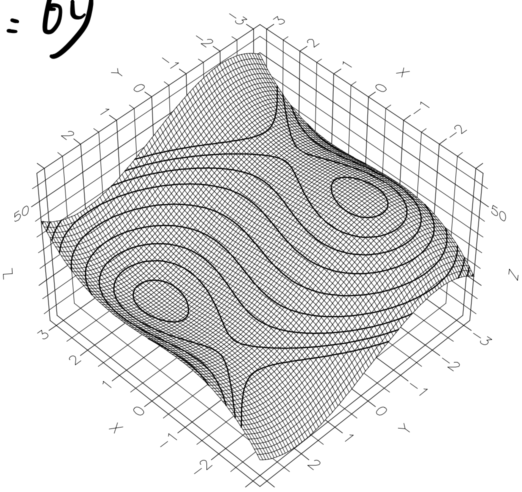
$$f_x = 3x^2 - 3 \qquad \text{and} \qquad f_y = 3y^2 - 12.$$

For critical points, $f_x = f_y = 0 \quad \Rightarrow \quad x = \pm 1,\ y = \pm 2$.

$\therefore$ $(1,2)$, $(-1,2)$, $(1,-2)$, $(-1,-2)$ are critical points.

$f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 6y$

$D = 36xy$

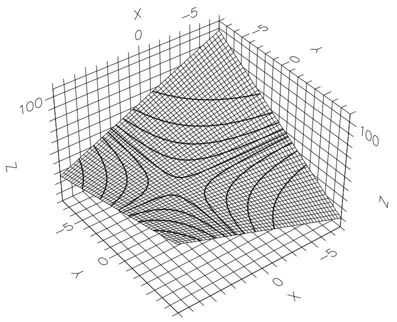


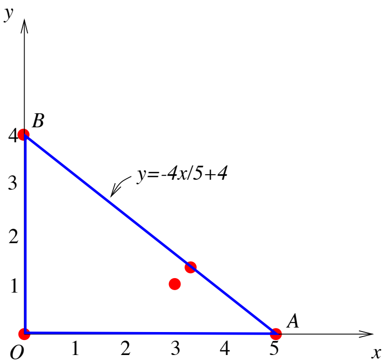
Point	f_{xx}	f_{yy}	f_{xy}	D	Type
$(1,2)$	6	12	0	>0	minimum
$(-1,2)$	-6	12	0	<0	saddle pt
$(1,-2)$	6	-12	0	<0	saddle pt
$(-1,-2)$	-6	-12	0	>0	maximum

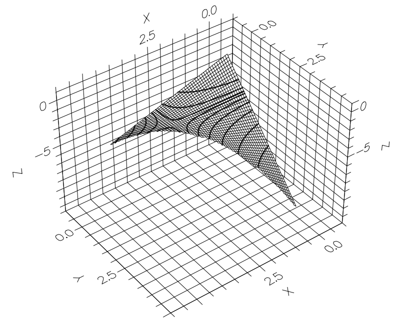
Ex. 1.3 Find the *absolute* extrema of the function

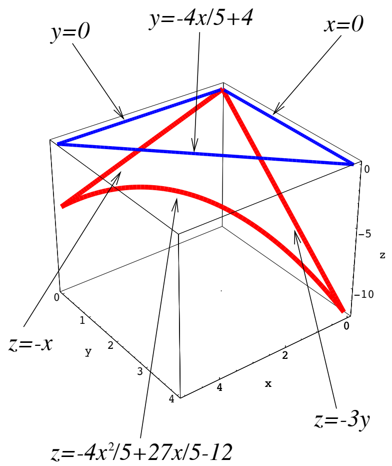
$$z = f(x, y) = xy - x - 3y$$

on the *closed* and *bounded* set R, where R is the triangular region with vertices $(0, 0)$, $(0, 4)$ and $(5, 0)$.









$$f_x = y - 1$$
$$f_y = x - 3$$
$$f_{xy} = 1 = f_{yx}$$
$$f_{xx} = 0 = f_{yy}$$
$$D = f_{xx}f_{yy} - (f_{xy})^2 = 0 - 1 = -1.$$

For critical points, $\nabla f = (f_x, f_y) = (0, 0) \quad \Rightarrow \quad x = 3,\ y = 1.$

This point is inside the domain $\left(\because \text{ when } x = 3,\ y = -\dfrac{4}{5}x + 4 = 1.6\right)$

On the boundary of domain R

Along OA, $\quad y = 0$ for $0 \leqslant x \leqslant 5$, $\quad z = f(x, 0) = -x$; $\quad$ (no critical points).

Along OB, $\quad x = 0$ for $0 \leqslant y \leqslant 4$, $\quad z = f(0, y) = -3y$; $\quad$ (no critical points).

Along AB, $\quad y = -\dfrac{4}{5}x + 4$ for $0 \leqslant x \leqslant 5$;

$$z = f\left(x,\ -\frac{4}{5}x + 4\right) = -\frac{4}{5}x^2 + \frac{27}{5}x - 12.$$

$$\frac{dz}{dx} = -\frac{8}{5}x + \frac{27}{5}.$$

For critical points, $dz/dx = 0$, $\quad x = 27/8,\ y = 13/10.$

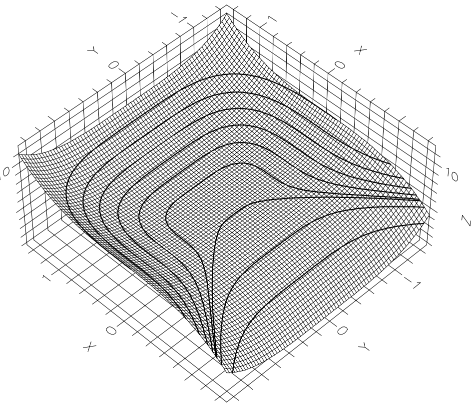
(x, y)	$f(x, y)$
$(3, 1)$	-3
$(27/8,\ 13/10)$	$-231/80$
$(0, 0)$	0
$(5, 0)$	-5
$(0, 4)$	-12

$\therefore$ Absolute maximum value is 0 which occurs at $(0, 0)$.

Absolute minimum value is -12 which occurs at $(0, 4)$.

Ex. 1.4 Find absolute minimum and maximum value of $f(x,y) = 2x^3 + y^4$ on the set $D = \{(x,y)\,|\,x^2+y^2 \leqslant 1\}$.

First, find the critical point of $f(x,y)$ on the entire x-y plane.



$f_x = 6x^2, \quad f_y = 4y^3$

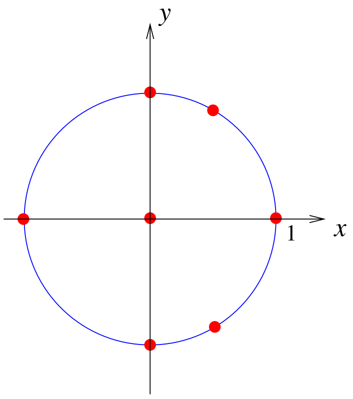
and for critical points

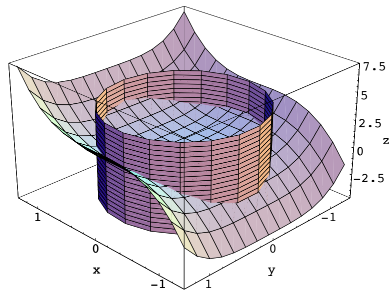
$f_x = f_y = 0 \qquad \Rightarrow \qquad x = y = 0$

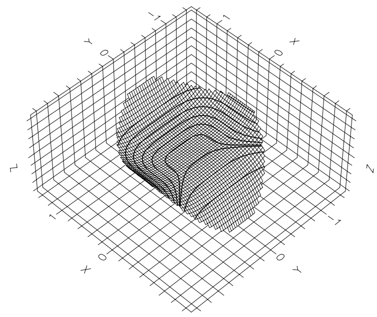
$\therefore \quad f(0,0) = 0.$

$\underline{r}(\theta) = (\cos\theta, \sin\theta) \qquad 0 \leq \theta < 2\pi$

$f(x(\theta), y(\theta)) = 2\cos^3\theta + \sin^4\theta$







Now **on** the circle $x^2+y^2=1$, let $x = \cos\theta$, $y = \sin\theta$, where $0 \leqslant \theta \leqslant 2\pi$ then

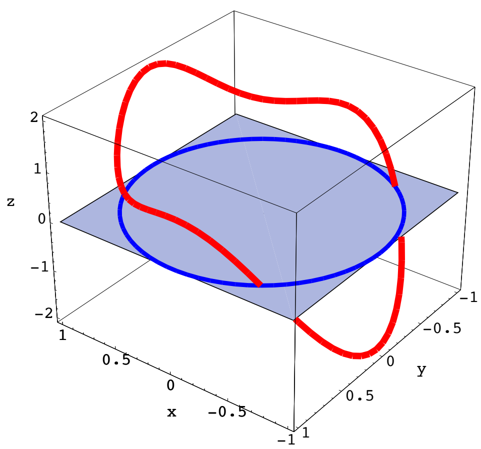
$$z = f(x,y) = f(r=1,\theta) = 2\cos^3\theta + \sin^4\theta$$

$$\begin{aligned}\frac{dz}{d\theta} &= 6\cos^2\theta(-\sin\theta) + 4\sin^3\theta\cos\theta \\ &= 2\cos\theta\sin\theta(2\sin^2\theta - 3\cos\theta) \\ &= 2\cos\theta\sin\theta(2\cos^2\theta - 3\cos\theta - 2) \\ &= 2\cos\theta\sin\theta(2\cancel{\cos^2}\theta - 1)(\cos\theta + 2)\end{aligned}$$

$2\cos\theta - 1$

For critical points, $dz/d\theta = 0$, i.e.

$\theta =$



(x,y) or θ	$f(x,y)$ or $f(1,\theta)$
$(0,0)$	
0	
$\pi/3$	
$\pi/2$	
π	
$3\pi/2$	
$5\pi/3$	

Abs. max: $f(1,0) = 2$ and abs. min: $f(-1,0) = -2$.

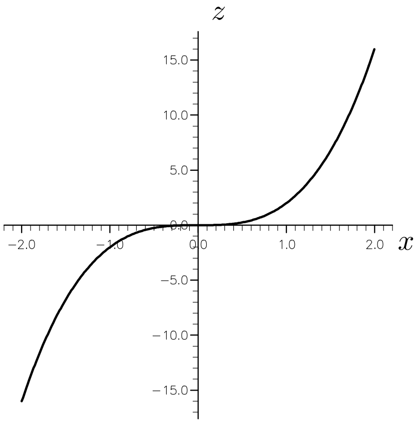
Abs max./min. need not do D test

If you are interested to the nature of the critical point at $(0,0)$, we need to do the D-test.

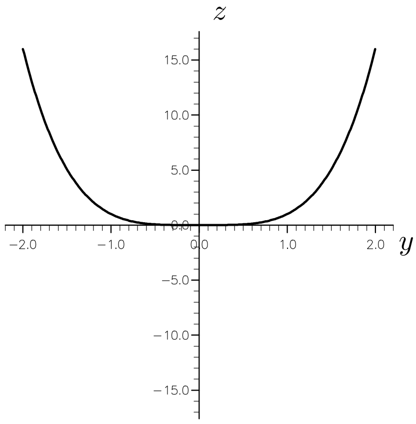
$$\begin{aligned} D &= f_{xx}f_{yy} - f_{xy}^2 = 144xy^2 \\ &= 0 \quad \text{at} \quad (0,0) \end{aligned}$$

so the D-test fails. We need to use some other means to determine the nature of the critical point – by examining the curve of intersection between the function $f(x,y)$ and the $x=0$ plane (or the $y=0$ plane).

In the plane $y=0$, $f(x,0)=2x^3$



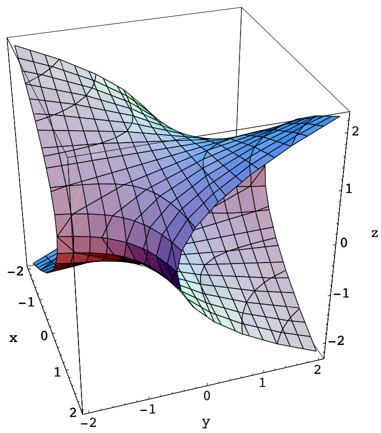
In the plane $x=0$, $f(0,y)=y^4$



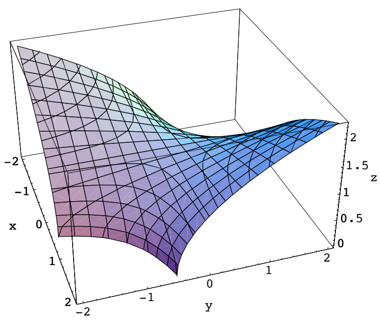
In general, if D-test fails, one needs to looks at the higher order derivatives of the function.

Ex. 1.5 Find the points on the surface $z^2 = xy+1$ that are closest to the origin.

Refer to the hangout



$z^2 = xy+1$

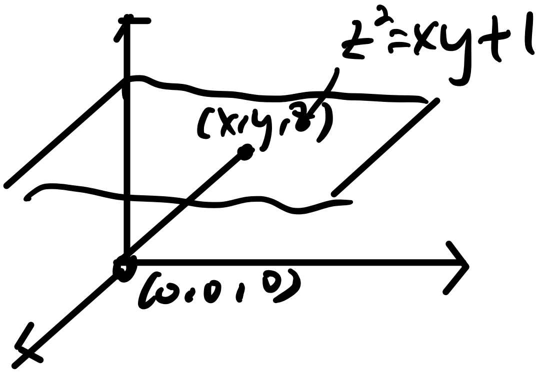


$z = +\sqrt{xy+1}$

Ans: $(0,0,\pm 1)$.

$D = d^2 = (x-0)^2 + (y-0)^2 + (z-0)^2$
$= x^2+y^2+z^2$

$D(x,y) = x^2+y^2+xy+1$ (min/max)



Ex. 1.6 If f is a function of one variable, and f is continuous on an interval I and has exactly one relative extremum on I, say at x_0, then f has an absolute extremum at x_0. This exercise shows that a similar result does not hold for functions of two variables.

(a) Show that $f(x,y) = 3xe^y - x^3 - e^{3y}$ has only one critical point and that a relative maximum occurs there.

(b) Show that f does not have an absolute maximum.

Solution

(a)

$$f_x = 3e^y - 3x^2 = 3(e^y - x^2)$$
$$f_y = 3xe^y - 3e^{3y} = 3e^y(x - e^{2y})$$

For critical points, $f_x = f_y = 0$, i.e.

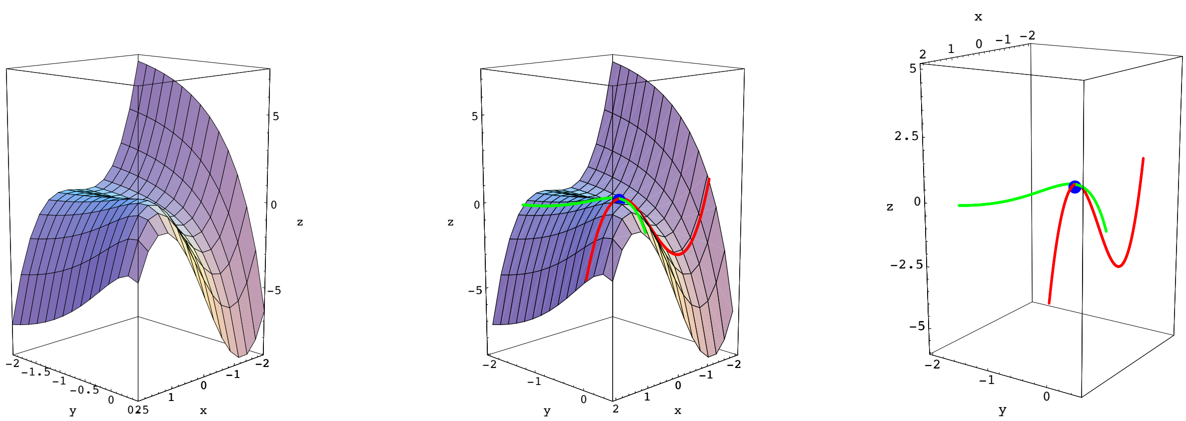
$$e^y = x^2 \quad \text{and} \quad e^{2y} = x.$$
$$\Rightarrow \quad x^4 = x$$
$$x(x^3 - 1) = 0$$
$$x = 0 \quad \text{or} \quad x = 1$$
$$y = -\infty \quad \text{or} \quad y = 0.$$

and

$$f_{xx} = -6x$$
$$f_{yy} = 3e^y(x - 3e^{2y})$$
$$f_{xy} = 3e^y = f_{yx}$$

At $(1,0)$, $f_{xx} < 0$ and $D > 0$, relative maximum.

Notice also that when $x = 0$, $f(0,y) = e^{-3y}$, this function of y has no critical point.



(b) $\lim\limits_{x\to-\infty} f(x,0) = \lim\limits_{x\to-\infty}(3x - x^3 - 1) = +\infty$ so no absolute maximum.

Ex. 1.7 Let f be a function of two variables that is continuous everywhere. One might think that if f has relative maxima at two points, then f must have another critical point because it is possible to have two mountains without some sort of valley in between. This exercise shows that this is not true. Let $f(x,y) = 4x^2e^y - 2x^4 - e^{4y}$. Show that f has exactly two critical points and that a relative maximum occurs at each one.

Solution

$$f_x = 8xe^y - 8x^3 = 8x(e^y - x^2)$$
$$f_y = 4x^2e^y - 4e^{4y} = 4e^y(x^2 - e^{3y})$$

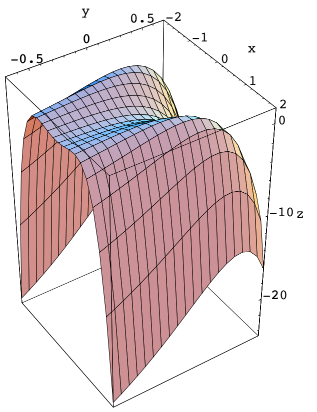
For critical points, $f_x = f_y = 0$, i.e.

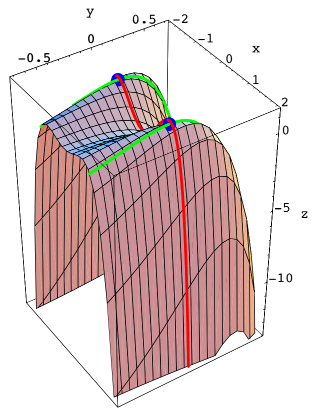
$$x^2 = e^y \quad \text{and} \quad x^2 = e^{3y}$$
$$\Rightarrow \qquad e^y = e^{3y}$$
$$e^{2y} = 1$$
$$y = 0 \quad \text{and} \quad x = \pm 1.$$

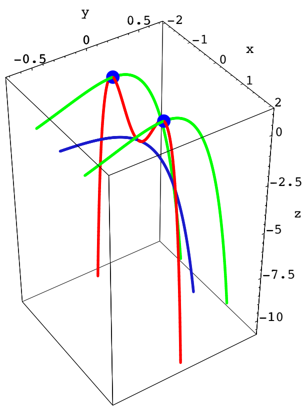
and

$$f_{xx} = 8(e^y - 3x^2)$$
$$f_{yy} = 4(x^2e^y - 4e^{4y})$$
$$f_{xy} = 8xe^y.$$

At $(\pm 1, 0)$, $f_{xx} < 0$ and $D > 0$, so both points are relative maximum points.







Ex. 1.8 A manufacturer with exclusive rights to a sophisticated new industrial machine is planning to sell a limited number of the machines to both foreign and domestic firms. The price the manufacturer can expect to receive for the machines will depend on the number of machines made available. (For example, if only a few of the machines are placed on the market, competitive bidding among prospective purchasers will tend to drive the price up.) It is estimated that if the manufacturer supplies x machines to the domestic market and y machines to the foreign market, the machines will sell for $60-\frac{x}{5}+\frac{y}{20}$ thousand dollars apiece at home and for $50-\frac{y}{10}+\frac{x}{20}$ thousand dollars apiece abroad. If the manufacturer can produce the machines at the cost of \$10,000 apiece, how many should be supplied to each market to generate the largest possible profit?

$$P_d = x\left(60-\frac{x}{5}+\frac{y}{20}-10\right) = x\left(50-\frac{x}{5}+\frac{y}{20}\right) \qquad \text{(revenue} - \text{cost)}$$
$$P_f = y\left(50-\frac{y}{10}+\frac{x}{20}-10\right) = y\left(40-\frac{y}{10}+\frac{x}{20}\right)$$

Product

Foreign

$z = y$

$p = 50 - \frac{y}{10} + \frac{x}{20}$

Domestic

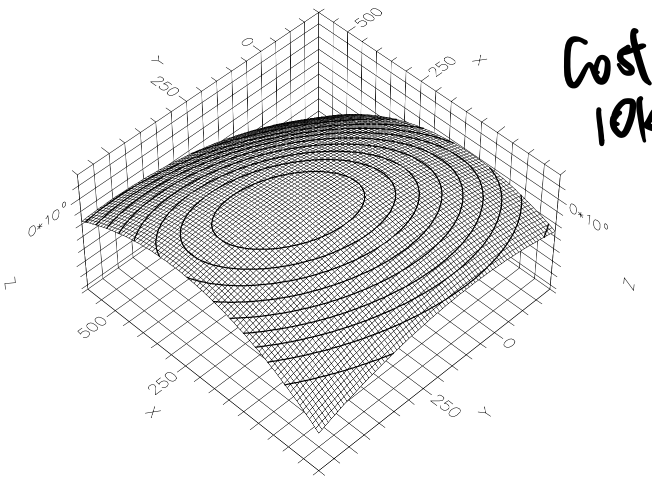
$z = x$

$p = 60 - \frac{x}{5} + \frac{y}{20}$

Cost = 10k

Ans: $x = 200$, $y = 300$.

Refer to hangout



$$P(x,y) = 50x - \frac{x^2}{5} + \frac{xy}{10} + 40y - \frac{y^2}{10}$$

Ex. 1.9 Imagine that you are selling spring water from your own spring. Since you are the only supplier in town, you have a "monopoly". Suppose that your price function is $p = 12 - 0.005x$, where p is the price at which you will sell precisely x gallons ($x \leqslant 2400$).

(a) Find the maximum revenue and the price.

(b) Duopoly: Suppose now that your neighbor opens a competing spring water business. Now both of you must share the same market. If he sells y gallons per day (and you sell x).

(i) Calculate the quantities x, and y that maximize revenue for each duopolist.

(ii) Calculate the price p and the two revenues.

(iii) Are more goods produced under a monopoly or a duopoly?

(iv) Is the price lower under a monopoly or duopoly?

(a) Revenue = price × quantity, i.e.

$$R = px = (12 - 0.005x)x = 12x - 0.005x^2$$
$$R' = 12 - 0.01x$$
$$R'' = -0.01$$

For critical points, $R' = 0 \quad \Rightarrow \quad x = 1200$. Therefore

$$R = 7200 \qquad \text{with} \qquad p = 6 \quad \text{and} \quad x = 1200.$$

b). Price function : $12 - 0.005(x+y)$

Refer to hangout.

out - c.

n-variable optimization

Relative maxima and minima (functions of more than two variables), i.e.

$$z = f(x_1,\ x_2,\ \ldots,\ x_n)$$

If the function $z = f(x_1,\ x_2,\ \ldots,\ x_n)$ has either a relative maximum or a relative minimum at

$$x_1 = a_1, \quad x_2 = a_2, \quad \ldots, \quad x_n = a_n$$

i.e. at the point $(a_1,\ a_3,\ \ldots,\ a_n)$, then

$$\begin{aligned} f_{x_1}(a_1,\ a_2,\ \ldots,\ a_n) &= 0 \\ f_{x_2}(a_1,\ a_2,\ \ldots,\ a_n) &= 0 \\ f_{x_3}(a_1,\ a_2,\ \ldots,\ a_n) &= 0 \\ &\vdots \\ f_{x_n}(a_1,\ a_2,\ \ldots,\ a_n) &= 0 \end{aligned}$$

The point $(a_1,\ a_3,\ \ldots,\ a_n)$ is called a critical point of the function

$$z = f(x_1,\ x_2,\ \ldots,\ x_n).$$

Thus to determine relative maximum or relative minimum for function of n independent variables, we first search for critical points. Once we find a critical point $(a_1,\ a_2,\ \ldots,\ a_n)$ of $z = f(x_1,\ x_2,\ \ldots,\ x_n)$, we must use *second-derivative* test for functions of n variables to determine the nature of the critical point.

We define the matrix

$$H = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} & \cdots & f_{x_2x_n} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} & \cdots & f_{x_3x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & f_{x_nx_3} & \cdots & f_{x_nx_n} \end{bmatrix}$$

The matrix H is called a Hessian matrix and *must* be evaluated at the critical point.

Note that in general, determining the nature of critical point requires a knowledge of the eigenvectors and eigenvalues of matrices.

The second-derivative test for functions of n variables also requires that we define a special set of n sub-matrices of H. These n sub-matrices are denoted by H_1, H_2, …, H_n and are defined next.

$$H_1 = [f_{x_1x_1}]$$

$$H_2 = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{bmatrix}$$

$$H_3 = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & f_{x_1x_3} \\ f_{x_2x_1} & f_{x_2x_2} & f_{x_2x_3} \\ f_{x_3x_1} & f_{x_3x_2} & f_{x_3x_3} \end{bmatrix}$$

$$\vdots$$

and

$$H_n = H.$$

The second-derivative test for function of n variables requires that the determinants of H_1, H_2, …, H_n evaluate at the critical point under consideration be computed. These determinants are denoted by $|H_1|$, $|H_2|$, …, $|H_n|$ and are called principal minors of H. We now state the second-derivative test for functions of n variables.

(1) A relative maximum exists at the critical point $(a_1, a_2, \ldots, a_n)$ if

$$|H_1| < 0, \quad |H_2| > 0, \quad |H_3| < 0, \ \ldots$$

i.e. the principal minors *alternate* in sign such that those with odd subscripts are negative and those with even subscripts are positive.

(2) A relative minimum exists at the critical point if

$$|H_1| > 0, \quad |H_2| > 0, \quad |H_3| > 0, \ \ldots, \ |H_n| > 0$$

i.e. the principal minors are all *positive*.

(3) If neither of the above two conditions occurs, then no information is given about the critical point.

Example Find any relative maximum and minimum of the function

$$z = f(x_1, x_2, x_3) = -x_1^2 - 6x_2^2 - x_3^3 + 12x_1x_3 + 12x_2 + 100.$$

Solution:

$$\begin{aligned} f_{x_1} &= -2x_1 + 12x_3 \\ f_{x_2} &= -12x_2 + 12 \\ f_{x_3} &= -3x_3^2 + 12x_1 \end{aligned}$$

For critical points, we have

$$\begin{aligned} -2x_1 + 12x_3 &= 0 \\ -12x_2 + 12 &= 0 \\ -3x_3^2 + 12x_1 &= 0 \end{aligned}$$

The above equations have solutions at $(0, 1, 0)$ and $(144, 1, 24)$.

To apply the second-derivative test, we compute

$$\begin{array}{lll} f_{x_1x_1} = -2 & f_{x_2x_2} = -12 & f_{x_3x_3} = -6x_3 \\ f_{x_1x_2} = 0 & f_{x_1x_3} = 12 & f_{x_2x_3} = 0 \end{array}$$

At $\underline{(0, 1, 0)}$

$$H = \begin{bmatrix} -2 & 0 & 12 \\ 0 & -12 & 0 \\ 12 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{ll} H_1 = [-2], & |H_1| = -2 \\ H_2 = \begin{bmatrix} -2 & 0 \\ 0 & -12 \end{bmatrix}, & |H_2| = 24 \\ H_3 = H, & |H_3| = 1728 \end{array}$$

i.e. no information is given.

At $\underline{(144, 1, 24)}$

$$H = \begin{bmatrix} -2 & 0 & 12 \\ 0 & -12 & 0 \\ 12 & 0 & -144 \end{bmatrix}$$

$$\begin{array}{ll} H_1 = [-2], & |H_1| = -2 \\ H_2 = \begin{bmatrix} -2 & 0 \\ 0 & -12 \end{bmatrix}, & |H_2| = 24 \\ H_3 = H, & |H_3| = -1728 \end{array}$$

Since $|H_1| < 0$, $|H_2| > 0$, $|H_3| < 0$, i.e. point $(144, 1, 24)$ is a maximum point and

$$z_{\max} = f(144, 1, 24) = 7018.$$

out c.

Ex. 1.10 A firm manufactures three products. The join cost function is

$$
\begin{aligned}
C &= f(x,\, y,\, y) \\
&= 10x^2 + 30y^2 + 20z^2 - 400x - 900y - 1000z + 750,000
\end{aligned}
$$

where C is the total cost (in dollars) of producing x, y, and z units of products 1, 2, and 3, respectively.

(a) Determine the quantities which result in a minimum total cost. Confirm that the critical point is a relative minimum.

(b) What is the expected minimum total cost?

13.3 Lagrange multipliers

Motivation

Ex. A manufacturer is planning to sell a new product at the price of \$150 per unit and estimates that if x thousand dollars is spent on development and y thousand dollars is spent on promotion, approximately $\dfrac{320y}{y+2} + \dfrac{160x}{x+4}$ units of the product will be sold. The cost of manufacturing the product is \$50 per unit. If the manufacturer has a total \$8,000 to spend on development and promotion, how should this money be allocated to generate the largest possible profit? [*Hint;* Profit = (number of units)(price per unit − cost per unit) − total amount spent on development and promotion.]

If unlimited funds are available, how much should the manufacturer spend on development and how much on promotion in order to generate the largest possible profit?

$x+y=8 \Rightarrow$ Constraint. Find Max $P = f(x,y)$ subject to $x+y=8$

Ex. A warehouse which will have a volume of 850,000 cubic feet is to be constructed. The warehouse is to have a rectangular foundation of dimensions x feet by y feet and height of z feet. The floor area cannot be bigger than 100,000 square feet. Construction costs are estimated based upon floor and ceiling area as well as wall area. The estimated costs are

\$6 per square foot of wall area,

\$8 per square foot of floor area,

\$9 per square foot of ceiling area.

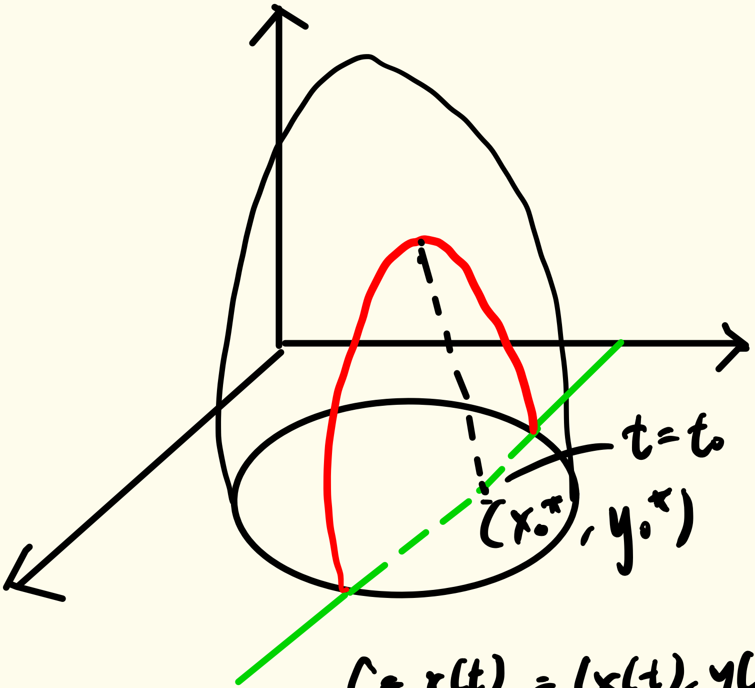
Determine the building dimensions which result in minimum construction costs.

Problems encountered will be of the form:

(a) Maximize $f(x, y)$ subject to $g(x, y) = k$.

(b) Maximize $f(x, y, z)$ subject to $g(x, y, z) = k_1$ and $h(x, y, z) = k_2$.

(c) Maximize $f(x, y)$ subject to $g(x, y) \leqslant k$.

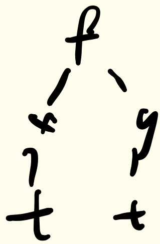


$C = \underline{r}(t) = (x(t), y(t))$

At t_0, $\underline{r}(t_0) = (x(t_0), y(t_0)) = (x_0^*, y_0^*)$

Along C,

$$z = f(\underline{r}(t))$$
$$= f(\underline{x}(t), \underline{y}(t))$$



rate of change of z wrt t,

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

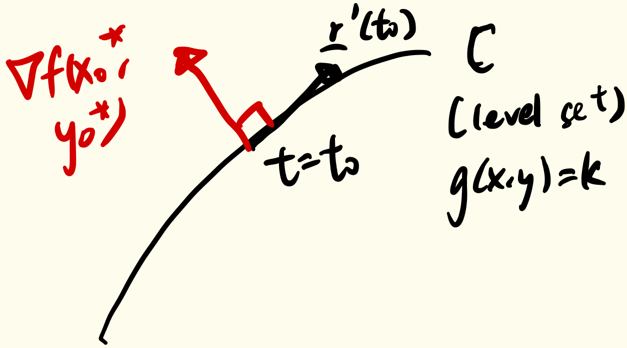
At $t = t_0$,

$$0 = \left.\frac{dz}{dt}\right|_{t=0} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

$$0 = \nabla f \cdot s'(t_0)$$

$$0 = \nabla f(x_0^*, y_0^*) \cdot \underline{r}'(t_0)$$

$$\nabla f(x_0^*, y_0^*) \perp r'(t_0)$$

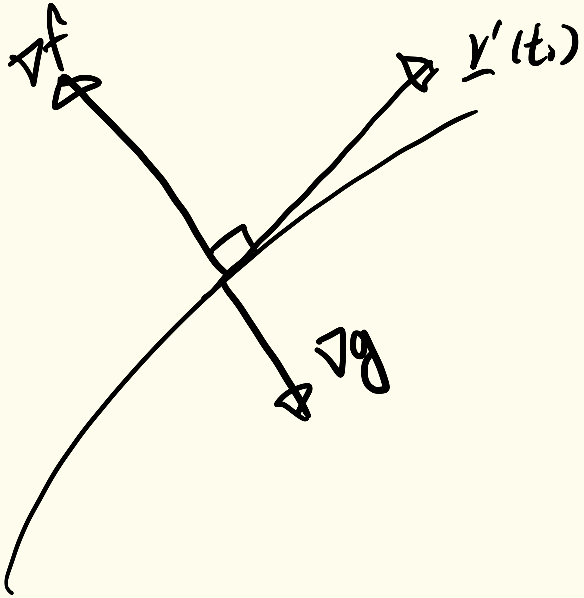


Also note that $g(x_0^*, y_0^*) \perp \underline{r}'(t_0)$

Therefore $\nabla g(x_0^*, y_0^*) \parallel \nabla f(x_0^*, y_0^*)$

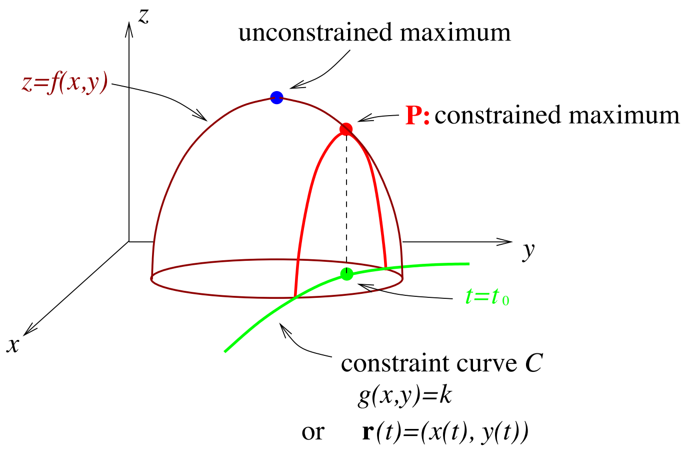
Eqt - to be solved

$$\boxed{\nabla f = \lambda \nabla g \quad \lambda \in \mathbb{R}}$$



Method for maximizing or minimizing a general function $f(x_1, x_2, \cdots, x_n)$ subject to a constraint of the form $g(x_1, x_2, \cdots, x_n) = k$. The point $(x_1, x_2, \cdots, x_n)$ is restricted to lie on the level surface S with equation $g(x_1, x_2, \cdots, x_n) = k$.

For $n = 2$, maximize (or minimize) $z = f(x, y)$ subject to the constraint $g(x, y) = k$.



Suppose z has a maximum value at a point $\mathbf{P}$, and let C be the curve with vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \quad \text{be on} \quad g(x, y) = k.$$

Assume also that at the point $\mathbf{P}$, $t = t_0$. On the constraint curve C

$$z(t) = f(x(t), y(t)),$$

then, by using the chain rule,

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \\ &= \nabla f \cdot \mathbf{r}'(t)\,. \end{aligned}$$

Then at the point $\mathbf{P}$,

$$\left.\frac{dz}{dt}\right|_{t=t_0} = \nabla f(x(t_0),\, y(t_0)) \cdot \mathbf{r}'(t_0) = 0 \qquad (\textbf{why})$$

$$\therefore \qquad \nabla f(x(t_0),\, y(t_0)) \quad \perp \quad \mathbf{r}'(t_0).$$

But

$$\begin{aligned} &\nabla g(x(t_0),\, y(t_0)) \quad \text{also} \perp \quad \mathbf{r}'(t_0) \qquad (\textbf{why}) \\ &\therefore \qquad \nabla f \quad \| \quad \nabla g \qquad (\textbf{why}) \\ &\Rightarrow \qquad \nabla f = \lambda \nabla g. \end{aligned}$$

The number λ is called a Lagrange multiplier.

How to find the maximum or minimum value

(a) Find all values of $\mathbf{r}$ and λ such that

$$\nabla f(\mathbf{r}) = \lambda \nabla g(\mathbf{r})$$

and

$$g(\mathbf{r}) = k.$$

(b) Evaluate f at all the points $\mathbf{r}$ that arise from step (a). The largest (smallest) of these values is the maximum (min) value of f.

Note that this Lagrange's method only finds critical points, it does *not* tell whether the function is maximized or minimized.

Proof for n-variable

Suppose f has an extreme value at a point $\mathbf{P}$, $\mathbf{r}_0 = (x_1^0, x_2^0, \cdots, x_n^0)$ on S and let C be a curve with vector equation $\mathbf{r}(t)$ that lies on S and passes through $\mathbf{P}$, let $\mathbf{r}(t_0) = \mathbf{r}_0$.

$\therefore h(t) = f(\mathbf{r}(t))$ the values that f takes on the curve C

$h'(t_0) = \nabla f(\mathbf{r}(t_0)) \cdot \mathbf{r}'(t_0) = 0 \quad \because \mathbf{r}(t_0)$ is extreme point $\quad \Rightarrow \quad h'(t_0) = 0$

i.e. the gradient vector $\nabla f(\mathbf{r}_0)$ is *orthogonal* to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C. The gradient vector of g, $\nabla g(\mathbf{r}_0)$ is also *orthogonal* to $\mathbf{r}'(t_0)$, i.e.

$$\nabla g(\mathbf{r}_0) \parallel \nabla f(\mathbf{r}_0)$$

$\therefore \boxed{\nabla f(\mathbf{r}_0) = \lambda \nabla g(\mathbf{r}_0)}$, the number λ is called a Lagrange multiplier.

Sufficient condition for relative extrema

To determine the behaviour of $f(x, y)$ at a critical point, we define the matrix

$$H = \begin{bmatrix} 0 & g_x & g_y \\ g_x & f_{xx} & f_{xy} \\ g_y & f_{yx} & f_{yy} \end{bmatrix}.$$

The matrix H is called a bordered Hessian matrix and must be evaluated at the critical point. The determinant of H (denote $|H|$) indicates whether $|H| > 0$ or $|H| < 0$.

Second-order condition

Given critical values $x = x_0$, $y = y_0$, $\lambda = \lambda_0$ for the Lagrangian function (auxiliary function), $|H|$ is evaluated at the critical values.

If $|H| > 0$, a relative maximum exists.

If $|H| < 0$, a relative minimum exists.

Ex. 3.1 Find the extreme values of $f(x, y) = x^2 - y^2$ subject to $x^2 + y^2 = 1$. $\leftarrow g$.

目标 = Solving $\nabla f = \lambda \nabla g$

Solution:

Find x, y and λ such that

$$\nabla f = \lambda \nabla g, \quad \text{where} \quad g = x^2 + y^2 = 1 \quad \text{and} \quad x^2 + y^2 = 1$$

$$2x\,\mathbf{i} - 2y\,\mathbf{j} = \lambda(2x\,\mathbf{i} + 2y\,\mathbf{j})$$

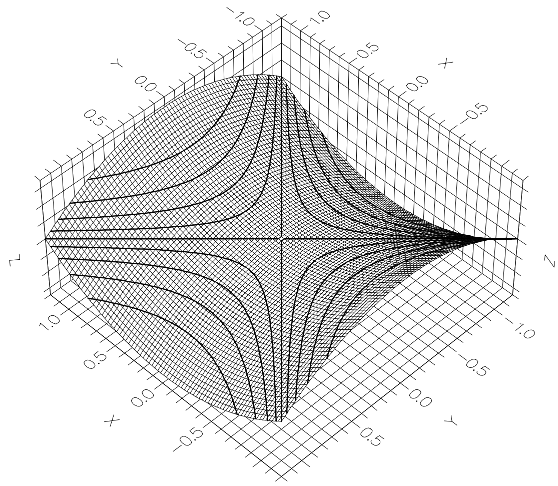
$$\Rightarrow \begin{cases} 2x = \lambda 2x \\ -2y = \lambda 2y \end{cases} \quad \Rightarrow \quad \begin{cases} \lambda = 1 & \text{or} \quad x = 0 \\ \lambda = -1 & \text{or} \quad y = 0 \end{cases}$$

From $x^2 + y^2 = 1$, we have

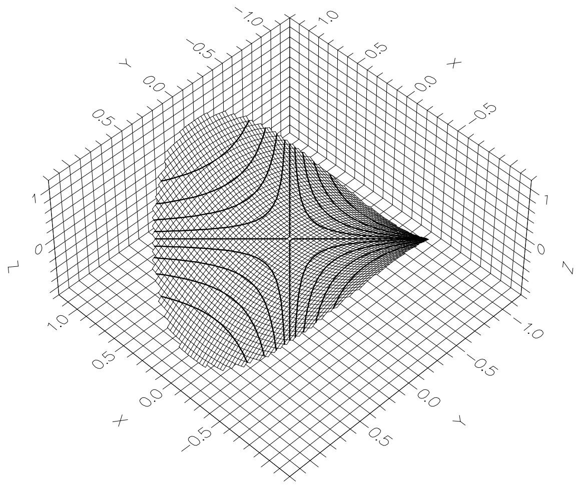
$$x = 0, \quad y = \pm 1, \quad \lambda = -1$$
$$y = 0, \quad x = \pm 1, \quad \lambda = 1.$$

Therefore, f has possible extreme values at point $(0, 1)$, $(0, -1)$, $(-1, 0)$ and $(1, 0)$. Evaluating f at these four points, we find that

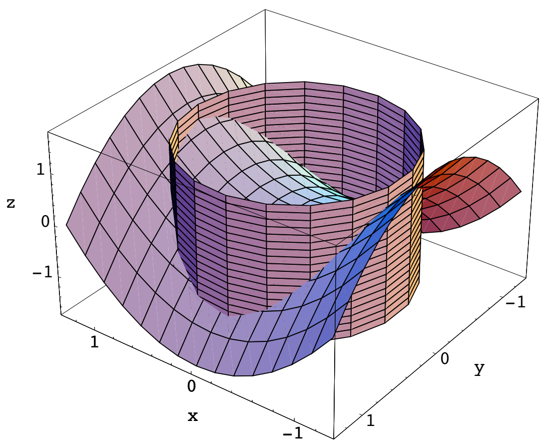
$$f(0, 1) = f(0, -1) = -1 \quad \text{(min)}$$
$$f(1, 0) = f(-1, 0) = 1 \quad \text{(max)}.$$

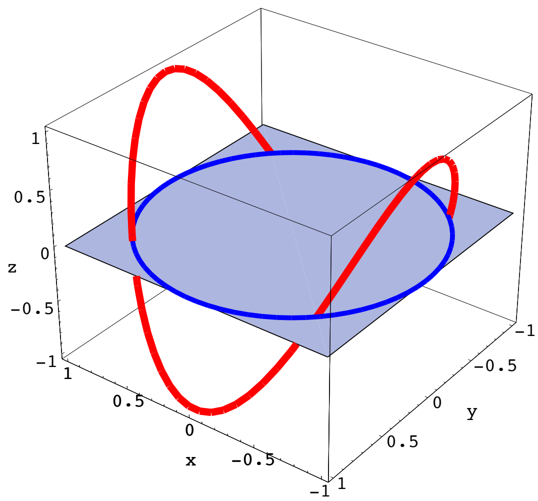


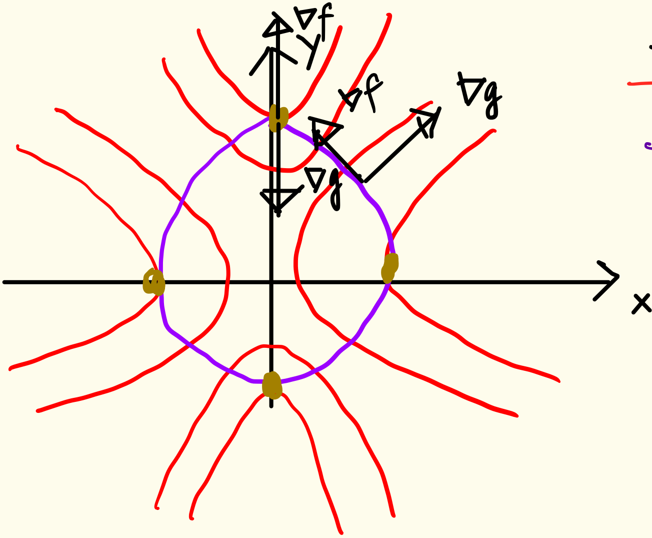
$f(x, y) = x^2 - y^2$



$f(x, y) = x^2 - y^2$ with $x^2 + y^2 \leqslant 1$







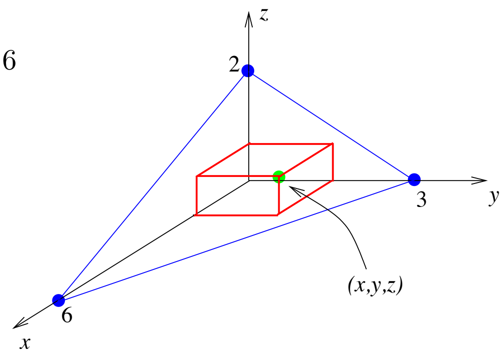
$f(x,y)=x^2-y^2$

$g(x,y)=x^2+y^2=1$

= critical pt

Ex. 3.2 Find the volume of the largest rectangular box in the first octant with three faces in the coordinates planes and are vertex in the plane $x + 2y + 3z = 6$.

Solution:



Method 1: $V = f(x, y, z) = xyz, \quad g(x, y, z) = x + 2y + 3z = 6$

constraint optimization

$$\nabla f = (yz, xz, xy) = \lambda \nabla g = (\lambda, 2\lambda, 3\lambda).$$

Then $\lambda = yz = \frac{1}{2}xz = \frac{1}{3}xy \quad \Rightarrow \quad x = 2y, \quad z = \frac{2}{3}y$

From $x + 2y + 3z = 6 \quad \Rightarrow \quad x = 2,\ y = 1,\ z = 2/3.$

$$\therefore\ V = f(2, 1, 2/3) = 4/3.$$

Method 2: Maximize $V = xyz = \frac{xy}{3}(6 - x - 2y)$

solve critical pt = 0

$$\therefore\ V_x = \frac{y}{3}(6 - 2x - 2y), \qquad V_y = \frac{x}{3}(6 - x - 4y)$$

For critical points, $V_x = V_y = 0 \quad \Rightarrow \quad x = 2,\ y = 1.$

Ex. 3.3 A metal box with no lid is to be assembled from rectangular pieces of sheet metal and is to have a volume of 1 ft^3. Suppose that material costs \$2/ft^2 for the base, \$3/ft^2 for the sides, and that it costs \$1/ft to weld each of the eight seams. Find the dimensions and cost of the most economical box.

Ans.: $x = 1.37,\ y = 1.37,\ z = 0.531$ and $C_{\min} = 20.12.$

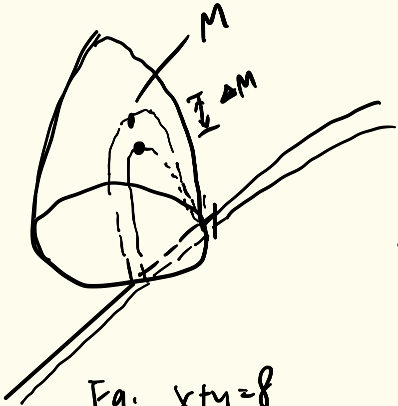
$C=\underbrace{2xy+6xz+6yz}_{\text{Area}}+\underbrace{2x+2y+4z}_{\text{Edges}}$ where xyz=1,so z=1/(xy)

=2xy+6/y+6/x+2x+2y+4/(xy)

Ex. 3.4 Find $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ subject to $x^2 + y^2 \leqslant 16$.

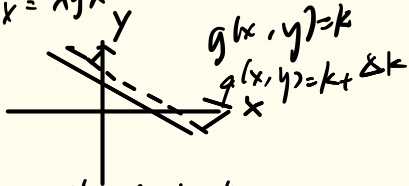
Ans.: $f(1, 0) = -7$ (min) and $f(-1, \pm 2\sqrt{3}) = 77$ (max).

$z = f(x,y)$ subject to $g(x,y) = k$



$\nabla f = \lambda \nabla g$

$f_x = \lambda g_x$



Eg. $x+y=8$

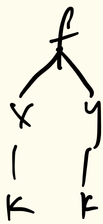
$x+y=7.8$

slightly decrease the budget

the question is about if i slightly decrease the budget, how much will the maximum pt changed.

$M = f(x,y)$

$= f(x(k), y(k))$



$$\frac{dM}{dk} = \frac{\partial f}{\partial x}\frac{dx}{dk} + \frac{\partial f}{\partial y}\frac{dy}{dk}$$

$g(x(k), y(k)) = k$

$$= \lambda g_x \frac{dx}{dk} + \lambda g_y \frac{dy}{dk}$$

$$= \lambda \left[\frac{\partial g}{\partial x}\frac{dx}{dk} + \frac{\partial g}{\partial y}\frac{dy}{dk} \right] = \lambda \frac{dg}{dk} = \lambda(1) = \lambda$$

i.e. $\Delta M \simeq \lambda \Delta k$

Interpretation of λ

Lambda is more than just an artificial creation allowing for the solution of constrained optimization problems. It has an interpretation which can be very useful.

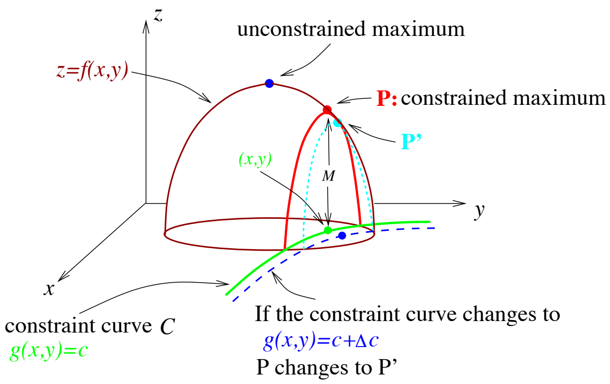
Suppose M is the optimal value of $f(x, y)$ subject to the constraint $g(x, y) = c$.

Then $f(x, y) = M$ for some ordered pair (x, y) that satisfies the three Lagrangian equations

$$\begin{aligned} f_x - \lambda g_x &= 0 \\ f_y - \lambda g_y &= 0 \\ g &= c \end{aligned}$$

Since $M = f(x, y)$

$$\begin{aligned} \frac{dM}{dc} &= \frac{\partial f}{\partial x}\frac{dx}{dc} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dc} \\ &= f_x\frac{dx}{dc} + f_y\frac{dy}{dc} \\ &= \lambda g_x\frac{dx}{dc} + \lambda g_y\frac{dy}{dc} \\ &= \lambda\left(g_x\frac{dx}{dc} + g_y\frac{dy}{dc}\right) \\ &= \lambda\frac{dg}{dc} = \lambda \end{aligned}$$



where dM/dc is evaluated at the optimal solution values. In other words, λ *measures* the sensitivity of the optimal value of f to change in c.

Ex. 3.5 Use Lagrangian multiplier to find the maximum and minimum values of the function

$$f(x, y) = 4x^3 + y^2$$

subject to the constraint $2x^2 + y^2 = 1$.

If the constraint equation changes to $2x^2 + y^2 = 0.9$, estimate how this changes will affect the maximum and minimum values of f.

Ans: $\min = -\dfrac{3\sqrt{2}}{2}$, $\quad\max = \dfrac{3\sqrt{2}}{2}$

$\Delta\min = \dfrac{3\sqrt{2}}{20}$, $\quad\Delta\max = -\dfrac{3\sqrt{2}}{20}$

refer to hangout

Ex. 3.6 A manufacturer is planning to sell a new product at the price of \$150 per unit and estimates that if x thousand dollars is spent on development and y thousand dollars is spent on promotion, approximately $\dfrac{320y}{y+2} + \dfrac{160x}{x+4}$ units of the product will be sold. The cost of manufacturing the product is \$50 per unit. If the manufacturer has a total \$8,000 to spend on development and promotion, how should this money be allocated to generate the largest possible profit? [*Hint;* Profit = (number of units)(price per unit − cost per unit) − total amount spent on development and promotion.]

Suppose the manufacturer in the above exercise decides to spend \$8,100 instead of \$8,000 on the development and promotion of the new product. Use the Lagrange multiplier λ to estimate how this change will affect the maximum possible profit.

Ex. 3.7 (a) If unlimited funds are available, how much should the manufacturer in the above exercise spend on development and how much on promotion in order to generate the largest possible profit?

(b) What is the value of the Lagrange multiplier λ that corresponds to the optimal budget in part (a)? Explain your answer in light of the interpretation of λ as $\dfrac{dM}{dk}$.

(c) Your answer to part (b) should suggest another method for solving the problem in part (a). Solve the problem using this new method.

k-constraints optimization

out c

Minimize or Maximize $f(x_1, x_2, \cdots, x_n)$ subject to

$$\begin{aligned} \phi_1(x_1, x_2, \cdots, x_n) &= c_1 \\ \phi_2(x_1, x_2, \cdots, x_n) &= c_2 \\ &\vdots \\ \phi_k(x_1, x_2, \cdots, x_n) &= c_k. \end{aligned}$$

Find x_1, x_2, $\cdots$, x_n and λ_1, λ_2, $\cdots$, λ_k such that

$$\boxed{\nabla f = \sum_{i=1}^{k} \lambda_i \nabla \phi_i}$$

and together with the constraint equations. λ_1, λ_2, $\cdots$, λ_k which are independent of x_1, x_2, $\cdots$, x_n are the Lagrangian multipliers.

Ex. 3.8 Find extrema of $f(x, y, z) = x + 2y$ subject to $x + y + z = 1$ and $y^2 + z^2 = 4$. Give the geometric interpretation of the above result, can you solve the problem again by using a different (easier!) method.

Let $\phi_1(x, y, z) = x + y + z - 1$

and $\phi_2(x, y, z) = y^2 + z^2 - 4$.

Then

$$\nabla f = (1, 2, 0) = \lambda_1 \nabla \phi_1 + \lambda_2 \nabla \phi_2 = (\lambda_1, \lambda_1, \lambda_1) + (0, 2\lambda_2 y, 2\lambda_2 z)$$

$\therefore$ $\lambda_1 = 1$ and

$$\begin{cases} 2 = \lambda_1 + 2\lambda_2 y \\ 0 = \lambda_1 + 2\lambda_2 z \end{cases} \quad \Rightarrow \quad \begin{aligned} y &= 1/(2\lambda_2) \\ z &= -1/(2\lambda_2) \end{aligned} \quad \text{i.e.} \quad y + z = 0.$$

Thus from $x + y + z = 1 \quad \Rightarrow \quad x = 1$,

and from $y^2 + z^2 = 4 \quad \Rightarrow \quad \lambda_2 = \pm\dfrac{1}{2\sqrt{2}}$

$\therefore$ The possible points are $(1, \pm\sqrt{2}, \mp\sqrt{2})$ and

$$\begin{aligned} f(1, \sqrt{2}, -\sqrt{2}) &= 1 + 2\sqrt{2} \quad \text{(max)} \\ f(1, -\sqrt{2}, \sqrt{2}) &= 1 - 2\sqrt{2} \quad \text{(min)}. \end{aligned}$$

Ex. 3.9 Find the minimum and maximum values of $x^2 + y^2 + z^2$ subject to the constraint conditions $x^2/4 + y^2/5 + z^2/25 = 1$ and $z = x + y$. Give the geometric interpretation of the above result.

Ans.: $f_{\text{min}} = 75/17$ and $f_{\text{max}} = 10$.

Exercise for students

(1) (a) Show that the maximum value of $f(x,y,z) = x^2y^2z^2$ subject to the constraint that $x^2 + y^2 + z^2 = a^2$ is

$$\frac{a^6}{27} = \left(\frac{a^2}{3}\right)^3.$$

(b) Use part (a) to show that, for all x, y and z,

$$(x^2y^2z^2)^{1/3} \leqslant \frac{x^2+y^2+z^2}{3}.$$

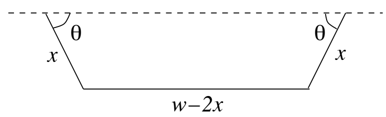
(c) Show that, for any positive numbers x_1, x_2, …, x_n,

$$(x_1x_2\cdots x_n)^{1/n} \leqslant \frac{x_1+x_2+\cdots+x_n}{n}.$$

The quantity on the right of the inequality is the *arithmetic mean* of the numbers x_1, x_2, …, x_n, and the quantity on the left is called the *geometric mean*. The inequality itself is, appropriately, called the *arithmetic-geometric inequality*.

(d) Under what conditions will equality hold in the arithmetic-geometric inequality?

(2) A long piece of galvanized sheet metal w inches wide is to be bend into a symmetric form with three straight sides to make a rain gutter A cross-section is shown in the figure.



(a) Determine the dimensions that allow the maximum possible flow; that is, find the dimensions that give the maximum possible cross-sectional area.

(b) Would it be better to bend the metal into a gutter with a semicircular cross-section than a three-side cross-section?

(3) If the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is to enclose the circle $x^2+y^2 = 2y$, what values of a and b minimize the area of the ellipse?

(4) Find the maximum and minimum values of $f(x,y,z) = xy + z^2$ on the ball $x^2+y^2+z^2 \leqslant 1$. Use Lagrange multipliers to treat the boundary case.

If the constraint equation changes to $x^2+y^2+z^2 \leqslant 0.9$, estimate how this change will affect the maximum and minimum values of f.

(5) Use Lagrangian multiplier to find the maximum and minimum values of the function

$$f(x, y) = 4x^3 + y^2$$

subject to the constraint $2x^2 + y^2 = 1$.

If the constraint equation changes to $2x^2 + y^2 = 0.9$, estimate how this changes will affect the maximum and minimum values of f.

(6) Marine biologists have determined that when a shark detects the presence of blood in the water, it will swim in the direction in which the concentration of the blood increases most rapidly. Based on certain tests in seawater, the concentration of blood (in parts per million) at a point $P(x, y)$ on the surface is approximated by

$$C(x, y) = e^{-(x^2+2y^2)/10^4}$$

where x and y are measured in meters in a rectangular coordinate system with the blood source at the origin.

(a) Identify the level curves of the concentration function and sketch several members of this family.

(b) Suppose a shark is at the point (x_0, y_0) when it first detects the presence of blood in the water. Determine the path that the shark will follow to the source (the origin).

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refer to the handout

a). Set it to be constant.