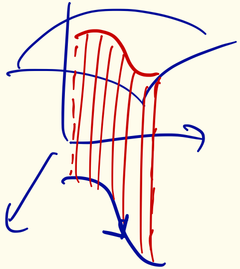
Last Time Line Integrals



$$\int_C f(x,y)\,ds = \text{area under graph above curve}$$

$$\int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt \qquad C : \vec{r}(t)$$

(does not depend on orientation of C)

$$\int_C f(x,y)\,dx = \int_a^b f(\vec{r}(t))\,x'(t)\,dt$$

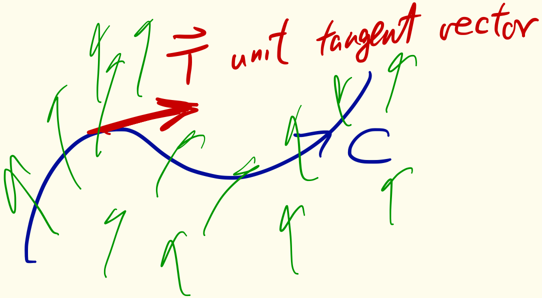
$$\int_C f(x,y)\,dy = \int_a^b f(\vec{r}(t))\,y'(t)\,dt$$

(it depends on orientation

$$\int_C = -\int_{-C}$$
)

Line Integrals of Vector Fields.

$\vec{F} = \langle P, Q, R \rangle$



Total Work Done $= \int_C (\vec{F} \cdot \vec{T})\, ds$

Work done

$$= \int_C \vec{F} \cdot \frac{\vec{r}'(t)}{\cancel{|\vec{r}'(t)|}} \cancel{|\vec{r}'(t)|}\, dt$$

$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

$$= \int_C (P x'(t) + Q y'(t) + R z'(t))\, dt$$

$\left(\int_C = - \int_{-C} \right)$

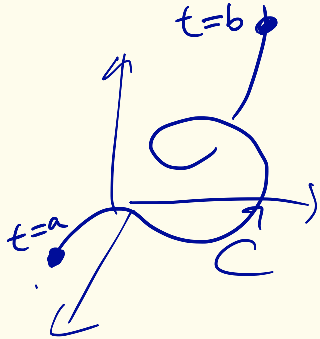
$$= \int_C P\,dx + Q\,dy + R\,dz \qquad \left(= \int_C \vec{F} \cdot d\vec{r} \right)$$

Recall : Fund. Thm. of Calculus : $\int_a^b F'(t)\,dt = F(b) - F(a)$

$C : \vec{r}(t)$ smooth curve, $a \le t \le b$, f differentiable function

Fundamental Theorem of Line Integrals :

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$



Pf $\displaystyle \int_C \nabla f \cdot d\vec{r} = \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$

$$= \int_a^b \frac{d}{dt} \underbrace{f(\vec{r}(t))}_{\text{single variable function!}} \, dt \qquad \text{(Chain Rule)}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) \qquad \text{(FTC)}$$

Ex Gravitational field $\quad \vec{G} = -\dfrac{\vec{r}}{|\vec{r}|^3} = \dfrac{\langle -x, -y, -z \rangle}{(x^2+y^2+z^2)^{3/2}}$

$\vec{G} = \nabla V, \quad V = \dfrac{1}{|\vec{r}|}$ gravitational potential.

$$\dfrac{1}{|\vec{r}|} \to \dfrac{1}{(x^2+y^2+z^2)^{1/2}}$$

Then $\underbrace{\displaystyle\int_C \vec{G} \cdot d\vec{r}}_{\text{Work done}} = \underbrace{V(\vec{r}(b)) - V(\vec{r}(a))}_{\text{potential energy}}.$ (usually: height)

Ex $f(x,y,z) = x^2y + z$

$$\nabla f = \langle 2xy, x^2, 1 \rangle$$

$$C = \langle t, t^2, t^3 \rangle \text{ where } 0 \le t \le 1$$

Then

$$\int_C \nabla f \cdot d\vec{r} = f(1,1,1) - f(0,0,0) = 2.$$

$$\|$$

$$\int_C 2xy\,dx + x^2dy + dz = \int_0^1 2t t^2 dt + t^2(2t dt) + (3t^2 dt)$$

Independence of Path

$$\int_{C_1} \nabla f \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r}$$

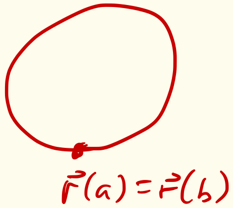
If C_1, C_2 has same endpoints.

$\vec{F} = \nabla f$

or If $\vec{F}$ is conservative vector field

Then $$\oint_C \vec{F} \cdot d\vec{r} = 0$$

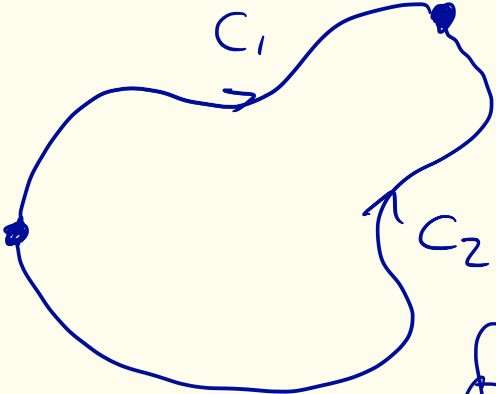
closed curve



Thm A $\int_C \vec{F}\cdot d\vec{r}$ is independent of path

$$\iff \oint_C \vec{F}\cdot d\vec{r} = 0 \text{ for all closed paths } C.$$

Pf



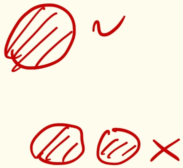
$$\int_{C_1} = \int_{C_2}$$

$$\iff$$

$$\oint_C = \int_{C_1} + \int_{-C_2} = \int_{C_1} - \int_{C_2} = 0$$

Thm B $\vec{F}$ vector field on open connected D

no boundary.



If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in D,

then $\vec{F}$ is conservative vector field.

$\left(\Rightarrow \exists f \text{ s.t. } \nabla f = \vec{F} \right)$

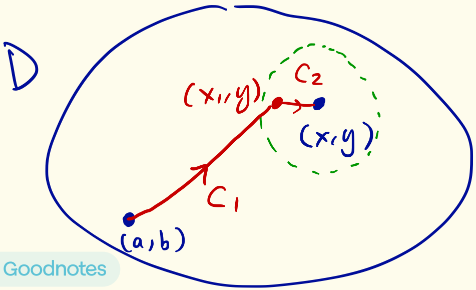
[2D Case]

Pf Given $\vec{F}$, try to find f.

$\vec{F} = \langle P, Q \rangle$

Define $f(x,y) = \int_{(a,b)}^{(x,y)} \vec{F}\cdot d\vec{r}$

Want to show $\nabla f = \langle P, Q \rangle$



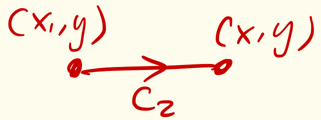
$$\int_{(a,b)}^{(x_1,y)} \vec{F}\cdot d\vec{r} + \int_{(x_1,y)}^{(x,y)} \vec{F}\cdot d\vec{r}$$

$$f(x,y) = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

does not depend on x.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{C_2} \vec{F} \cdot d\vec{r}$$

C_2 is just horizontal



$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{x_1}^{x} P(t,y)\,dt$$

$C_2 = \langle t, y \rangle$

$x_1 \le t \le x$

$$\Rightarrow (\text{FTC}) \quad \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \int_{x_1}^{x} P(t,y)\,dt = P(x,y)$$

Similarly: $\frac{\partial f}{\partial y} = Q(x,y) \quad \Rightarrow \quad \nabla f = \vec{F}$

Thm C [2D Case] If $\vec{F} = \langle P, Q \rangle$ is conservative, then

P, Q has continuous partial derivatives.

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Pf $\vec{F} = \langle f_x, f_y \rangle$, $f_x = P$, $f_y = Q$

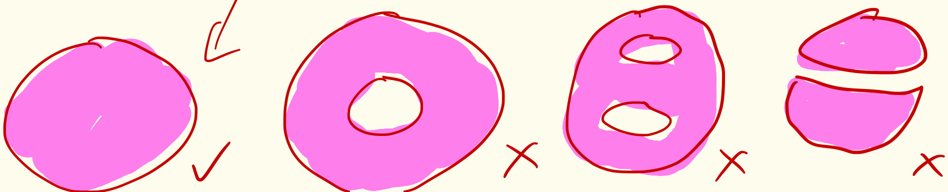
$f_{xy} = f_{yx}$

[**Most Useful**: If $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x} \Rightarrow$ NOT conservative!]

Thm D $\vec{F} = \langle P, Q \rangle$ on open simply connected region,

(no boundary) (no holes)

If $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on D, then $\vec{F}$ is conservative.



Summary

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Thm C $\Uparrow$ $\quad\quad$ $\Downarrow$ Thm D

(D open, simply connected)
$\parallel$
no holes

$$\boxed{\vec{F} \text{ is conservative}}$$

Fund. Thm. $\Downarrow$ $\quad\quad$ $\Uparrow$ Thm B.

(D open connected)

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

$\Updownarrow$ Thm A

$$\int_{C_1} = \int_{C_2}$$ independent of paths

Ex $\langle x-y \,,\, x-2 \rangle$ is not conservative.

$$\frac{\partial P}{\partial y} = -1 \,, \quad \frac{\partial Q}{\partial x} = 1$$ not equal.

By Thm C, not conservative.

Ex $\langle e^x \sin y \,,\, e^x \cos y \rangle$

$$\frac{\partial P}{\partial y} = e^x \cos y \,, \quad \frac{\partial Q}{\partial x} = e^x \cos y$$ the same.

$D = \mathbb{R}^2 \Rightarrow$ By Thm D it is conservative

$f = e^x \sin y$.

Ex $\displaystyle\int_C \tan y\, dx + x\sec^2 y\, dy$

$C: y = x^3 - 1$ from $x = 0$ to $x = 1$

$\vec{F} = \langle \tan y, \ x\sec^2 y \rangle$

$\dfrac{\partial P}{\partial y} = \sec^2 y \qquad \dfrac{\partial Q}{\partial x} = \sec^2 y \qquad \checkmark.$

$y = x^3 - 1$

$x = 0, \ y = -1$

$x = 1, \ y = 0$

$\boxed{f = x\tan y}$

By Fund. Thm. $\displaystyle\int \vec{F}\cdot d\vec{r} = f(1,0) - f(0,-1)$

$= 1\cdot\tan(0) - 0\cdot\tan(-1)$

$= 0.$

Important Example

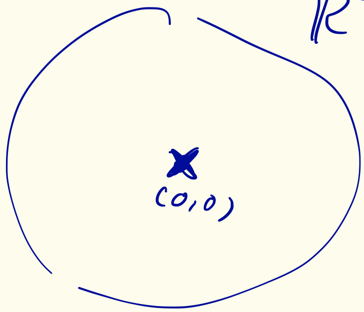
$$\vec F = \frac{-y\,\vec i + x\,\vec j}{x^2+y^2}$$

P Q

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2)(-1)-(-y)2y}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$\mathbb{R}^2$



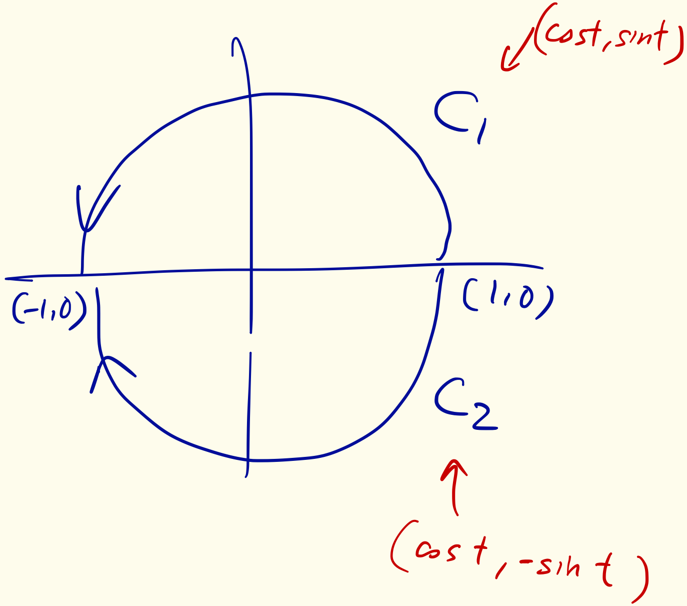
Thm D fails : D has hole :

$\vec F$ is not conservative :

$$\int_{C_1} \vec{F} \cdot d\vec{r}$$

$$= \int_0^{\pi} -\sin t \cdot (-\sin t \, dt) + \cos t (\cos t \, dt)$$

$$= \int_0^{\pi} dt = \pi$$



$$\int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_0^{\pi} \sin t (-\sin t \, dt) + \cos t (-\cos t \, dt)$$

$$= \int_0^{\pi} -1 \, dt = -\pi.$$