MATH 2023 • Multivariable Calculus
Problem Set #2 • Multivariable Functions, Partial Derivatives

1. (★) Let $f(x,y) = \sqrt{y - x^2}$
   (a) What is the (largest possible) domain of f?
   (b) Sketch the level sets $f = 0$, $f = 1$ and $f = 2$ in the same diagram.

2. (★) Let
$$f(x,y) = \frac{1}{\sqrt{x^2 + y^2 - 1}}$$
   (a) What is the (largest possible) domain of f?
   (b) Sketch the level sets $f = 1$, $f = 2$ and $f = 3$ in the same diagram.
   (c) Repeat (a) and (b) for the function $g(x,y) = \dfrac{1}{\sqrt{1 - x^2 - y^2}}$.

3. (★) Compute all the first and second partial derivatives of the following functions. For the second partials f_{xy} and f_{yx}, compute both and verify that they are indeed the same.
   (a) $f(x,y) = y^{2015} + 2x^2 + 2xy$
   (b) $f(x,y) = e^{x^2 y}$
   (c) $f(x,y) = \frac{x}{x^2+y^2}$
   (d) $f(x,y) = x\ln(x^2 + y^2)$

4. (★★) Compute the first partial derivative $\frac{\partial f}{\partial x}$ of the following functions (where $x, y > 0$).
   (a) $f(x,y) = e^{x^y}$
   (b) $f(x,y) = e^{y^x}$
   (c) $f(x,y) = x^{e^y}$
   (d) $f(x,y) = y^{e^x}$
   (e) $f(x,y) = x^{y^e}$
   (f) $f(x,y) = y^{x^e}$

5. (★) Compute both the third-order derivatives h_{xyy} and h_{yyx} of the following function, and verify that they are indeed the same.
$$h(x,y,z) = \cos(x^2 + y^3 z).$$

6. (★★) Find the second derivative $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ of each function $f(x,y)$ below. [Hint: There is a smart way to compute each of them.]
   (a)
$$f(x,y) = \sin(x+y)\ \cos(x-y)$$
   (b)
$$f(x,y) = \cos(xy) + \left(\frac{\sin^{2016} y + \cos^{2014} y}{\sin^2 \log(y^4+1) + 2015}\right)^{\frac{1}{2015}}.$$
   (c)
$$f(x,y) = \frac{e^{x+y} + e^{x-y}}{e^{x+y} - e^{x-y}}$$

7. (★★) Suppose that $f(x,y)$ is a function such that $\dfrac{\partial^2 f}{\partial x \partial y} \equiv 0$. Show that f can be decomposed into the form:
$$f(x,y) = F(x) + G(y)$$
where $F(x)$ and $G(y)$ are some single-variable functions.

8. (★★) Let $u(x,y,z,t)$ be the temperature at the point (x,y,z) at the time t. Combining with several important laws in thermodynamics, including the Fourier's Law and conservation of energy, it can be derived (detail omitted) that the temperature function $u(x,y,z,t)$ satisfies the following equation:
$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$
where k is a positive constant depending only on the medium. This equation is known as the **heat equation**.

The study of the heat equation is an important topic in physics, engineering and mathematics (both pure and applied). Through solving the heat equation with an initial condition $u(x,y,z,0) = g(x,y,z)$, it predicts how heat diffuses for a given an initial heat profile $g(x,y,z)$ at time $t = 0$.

Your task in this problem is to verify that the following given function is a solution to the heat equation:
$$\varphi(x,y,z,t) = \frac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(-\frac{x^2+y^2+z^2}{4kt}\right).$$

This particular solution φ represents the heat diffusion with highly concentrated heat source at the origin $(0,0,0)$ at time $t = 0$. As time goes by, the temperature profile becomes more and more uniformly distributed. (In physics, this solution is also closely related to the *Dirac delta function*.)

By following the outline below, show that φ satisfies the heat equation:

(a) Show that:
$$\ln \varphi(x,y,z,t) = -\ln(4\pi k)^{\frac{3}{2}} - \frac{3}{2}\ln t - \frac{x^2+y^2+z^2}{4kt}.$$

(b) Using (a), show that:
$$\frac{\partial \varphi}{\partial t} = \left(\frac{x^2+y^2+z^2}{4kt^2} - \frac{3}{2t}\right)\varphi.$$

(c) Using (a) again, show that:
$$\frac{\partial \varphi}{\partial x} = -\frac{x\varphi}{2kt} \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial x^2} = \frac{1}{2kt}\left(\frac{x^2}{2kt} - 1\right)\varphi.$$

(d) Hence, verify that φ satisfies the heat equation: $\varphi_t = k(\varphi_{xx} + \varphi_{yy} + \varphi_{zz})$.

(e) (Optional) Show that
$$\lim_{t\to 0^+} \varphi(x,y,z,t) = \begin{cases} \infty & \text{if } (x,y,z) = (0,0,0) \\ 0 & \text{if } (x,y,z) \neq (0,0,0) \end{cases}$$