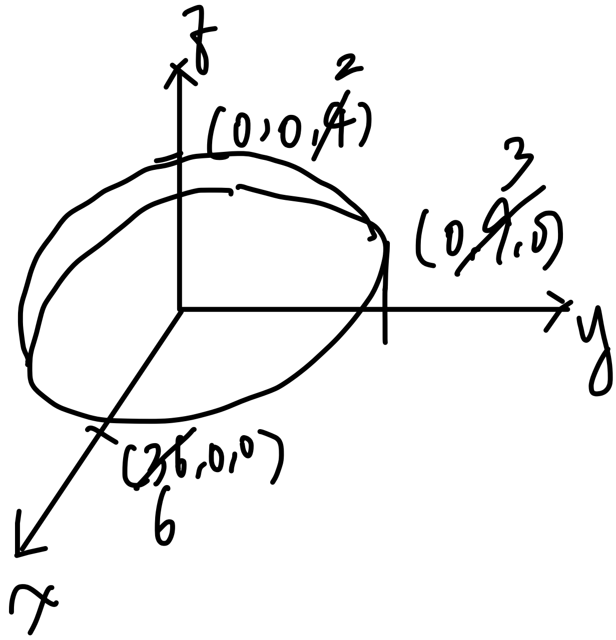
Identify the surfaces represented by the equations in Exercises 1–16 and sketch their graphs.

1. $x^2 + 4y^2 + 9z^2 = 36$ **2.** $x^2 + y^2 + 4z^2 = 4$

1. ellisoid

2. ellipsoid



3. $2x^2 + 2y^2 + 2z^2 - 4x + 8y - 12z + 27 = 0$

4. $x^2 + 4y^2 + 9z^2 + 4x - 8y = 8$

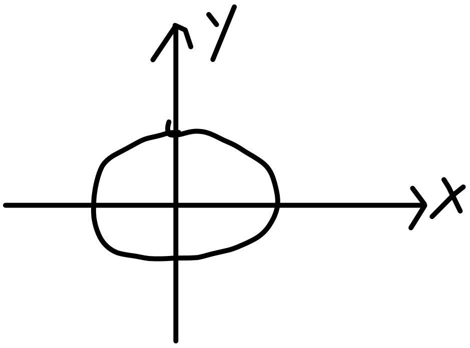
3. ??

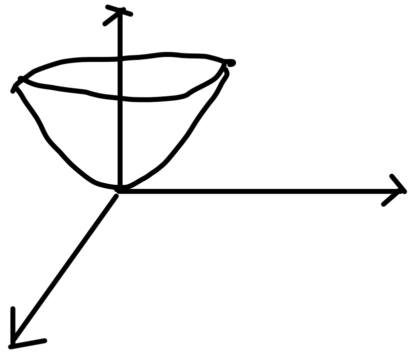
4. ??

5. $z = x^2 + 2y^2$ **6.** $z = x^2 - 2y^2$

7. $x^2 - y^2 - z^2 = 4$ **8.** $-x^2 + y^2 + z^2 = 4$

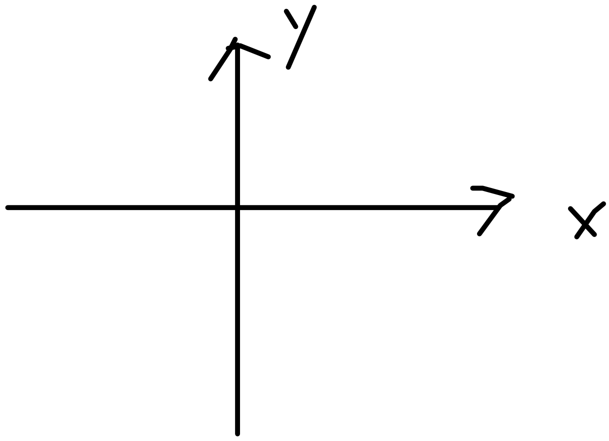
5.

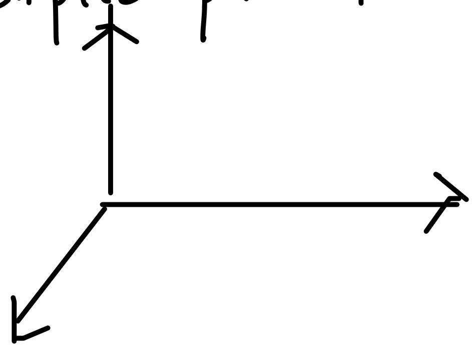




elliptic paraboloid

6.





hyperbolic paraboloid

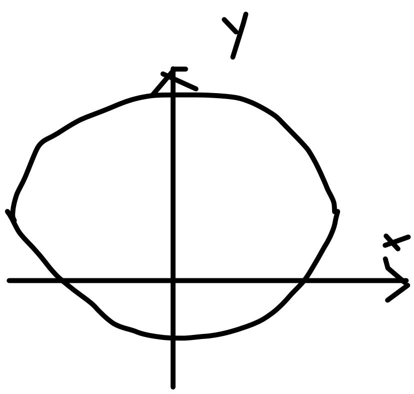
7.

$$-z^2 = 4 - x^2 + y^2$$

$$z^2 = -4 + x^2 - y^2$$

$$k^2 = x^2 - y^2 - 4$$

$$y^2 = x^2 - 4 - k^2$$



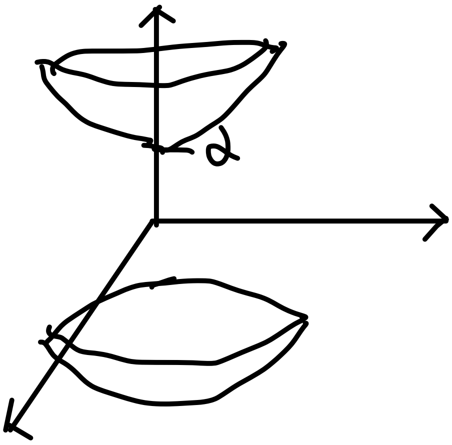
$$z = \sqrt{x^2 - y^2 - 4}$$

hyperboloid two sheet

$$x^2 - y^2 \geq 4$$

$$x^2 \geq 4 + y^2$$

7.



8. $-x^2+y^2+z^2=4$

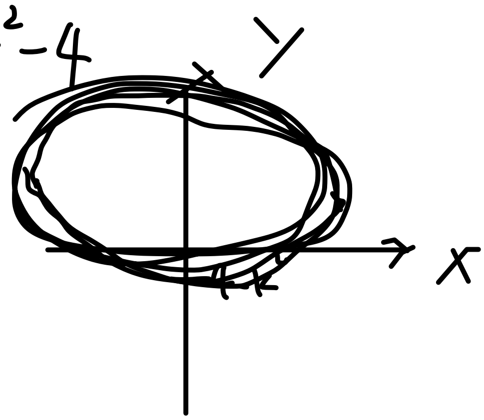
$z^2=4+x^2-y^2$

$z^2=x^2-y^2+4$

$z=\sqrt{x^2-y^2+4}$

$x^2-y^2+4\geq 0$

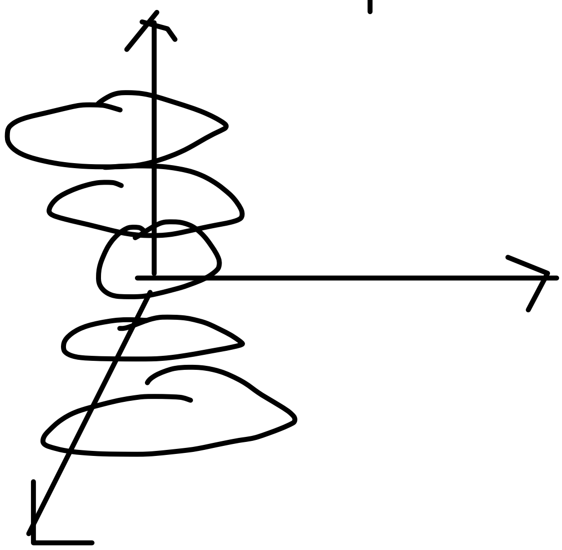
$x^2\geq y^2-4$



$z\geq 0$

$k^2=x^2-y^2+4$

$y^2=x^2+4-k^2$



EXERCISES 10.5

Identify the surfaces represented by the equations in Exercises 1–16 and sketch their graphs.

1. $x^2 + 4y^2 + 9z^2 = 36$
2. $x^2 + y^2 + 4z^2 = 4$
3. $2x^2 + 2y^2 + 2z^2 - 4x + 8y - 12z + 27 = 0$
4. $x^2 + 4y^2 + 9z^2 + 4x - 8y = 8$
5. $z = x^2 + 2y^2$
6. $z = x^2 - 2y^2$
7. $x^2 - y^2 - z^2 = 4$
8. $-x^2 + y^2 + z^2 = 4$
9. $z = xy$
10. $x^2 + 4z^2 = 4$
11. $x^2 - 4z^2 = 4$
12. $y = z^2$
13. $x = z^2 + z$
14. $x^2 = y^2 + 2z^2$
15. $(z-1)^2 = (x-2)^2 + (y-3)^2$
16. $(z-1)^2 = (x-2)^2 + (y-3)^2 + 4$

Describe and sketch the geometric objects represented by the systems of equations in Exercises 17–20.

17. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x + y + z = 1 \end{cases}$
18. $\begin{cases} x^2 + y^2 = 1 \\ z = x + y \end{cases}$
19. $\begin{cases} z^2 = x^2 + y^2 \\ z = 1 + x \end{cases}$
20. $\begin{cases} x^2 + 2y^2 + 3z^2 = 6 \\ y = 1 \end{cases}$

21. Find two one-parameter families of straight lines that lie on the hyperboloid of one sheet

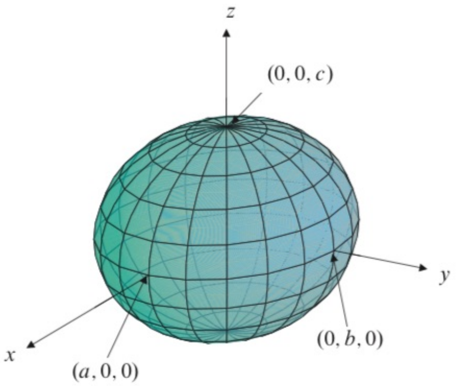
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

22. Find two one-parameter families of straight lines that lie on the hyperbolic paraboloid $z = xy$.
23. The equation $2x^2 + y^2 = 1$ represents a cylinder with elliptical cross-sections in planes perpendicular to the z-axis. Find a vector **a** perpendicular to which the cylinder has circular cross-sections.
24. The equation $z^2 = 2x^2 + y^2$ represents a cone with elliptical cross-sections in planes perpendicular to the z-axis. Find a vector **a** perpendicular to which the cone has circular cross-sections. *Hint:* Do Exercise 23 first and use its result.

how to find centre & radius

1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

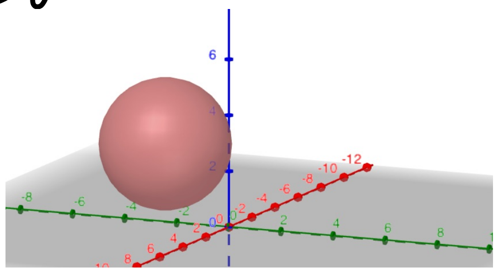
$\Rightarrow$ ellipsoid 橢球



2. $2x^2+2y^2+2z^2-4x+8y-12z+17=0$

$\rightarrow$ 可以 complete squares 成:

$(x-1)^2+(y+2)^2+(z-3)^2=\frac{1}{2}$

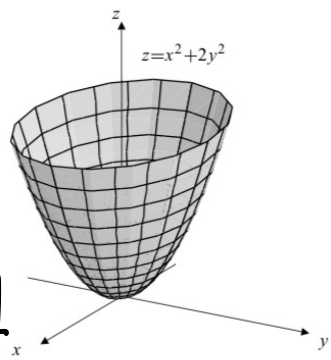


$\Rightarrow$ sphere 球體, centre $(1,-2,3)$, radius $\frac{1}{\sqrt{2}}$

3. $z=x^2+2y^2$

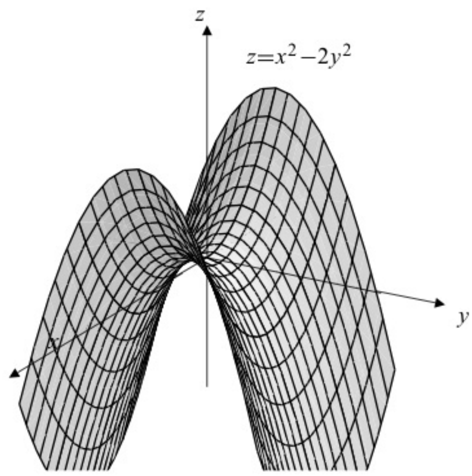
$\Rightarrow$ elliptic paraboloid 橢圓拋物面

Cross-section 是 semi-axises 為$\sqrt{k}$, $\sqrt{\frac{k}{2}}$的橢圓



4. $z=x^2-2y^2$

$\Rightarrow$ hyperbolic paraboloid 雙曲拋物面



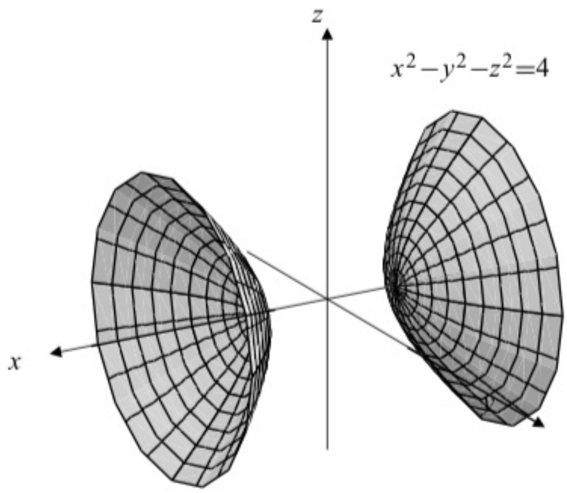
5. $x^2 - y^2 - z^2 = 4$

$$x^2 = 4 + y^2 + z^2$$

$$x = \sqrt{4 + y^2 + z^2}$$

$\Rightarrow$ hyperboloid of two sheets, vertex為 ± 2 (Sub $y, z = 0$),

cross section = 圓形, radius ≥ 2, 垂直於 x-axis



6. $-x^2+y^2+z^2=4$

$$-x^2 = 4-y^2-z^2$$

$$x^2 = -4+y^2+z^2$$

$$x = \sqrt{y^2+z^2-4}$$

Section 10.5 Quadric Surfaces (page 598)

1. $x^2 + 4y^2 + 9z^2 = 36$

$$\frac{x^2}{6^2} + \frac{y^2}{3^2} + \frac{z^2}{2^2} = 1$$

This is an ellipsoid with centre at the origin and semi-axes 6, 3, and 2.

2. $x^2 + y^2 + 4z^2 = 4$ represents an oblate spheroid, that is, an ellipsoid with its two longer semi-axes equal. In this case the longer semi-axes have length 2, and the shorter one (in the z direction) has length 1. Cross-sections in planes perpendicular to the z-axis between $z = -1$ and $z = 1$ are circles.

3. $2x^2 + 2y^2 + 2z^2 - 4x + 8y - 12z + 27 = 0$

$$2(x^2 - 2x + 1) + 2(y^2 + 4y + 4) + 2(z^2 - 6z + 9)$$
$$= -27 + 2 + 8 + 18$$
$$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = \frac{1}{2}$$

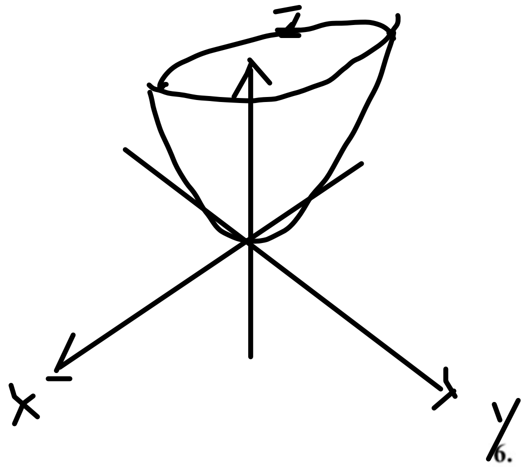
This is a sphere with radius $1/\sqrt{2}$ and centre $(1, -2, 3)$.

4. $x^2 + 4y^2 + 9z^2 + 4x - 8y = 8$

$$(x + 2)^2 + 4(y - 1)^2 + 9z^2 = 8 + 8 = 16$$
$$\frac{(x + 2)^2}{4^2} + \frac{(y - 1)^2}{2^2} + \frac{z^2}{(4/3)^2} = 1$$

This is an ellipsoid with centre $(-2, 1, 0)$ and semi-axes 4, 2, and 4/3.

5. $z = x^2 + 2y^2$ represents an elliptic paraboloid with vertex at the origin and axis along the positive z-axis. Cross-sections in planes $z = k > 0$ are ellipses with semi-axes $\sqrt{k}$ and $\sqrt{k/2}$.



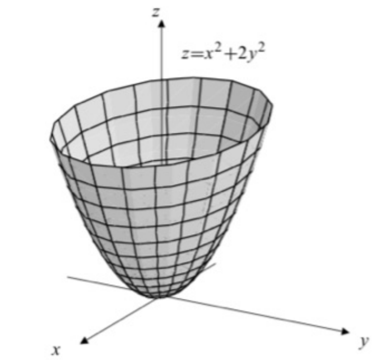


Fig. 10.5.5

6. $z = x^2 - 2y^2$ represents a hyperbolic paraboloid.

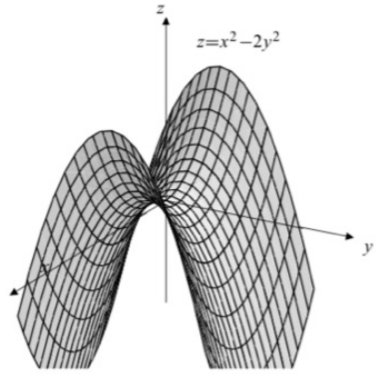
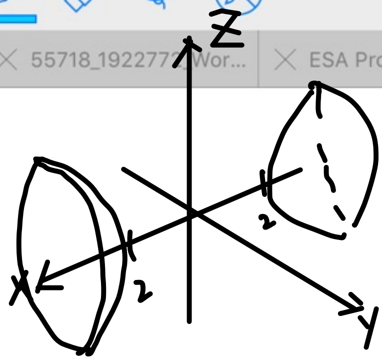


Fig. 10.5.6

7. $x^2 - y^2 - z^2 = 4$ represents a hyperboloid of two sheets with vertices at $(\pm 2, 0, 0)$ and circular cross-sections in planes $x = k$, where $|k| > 2$.

$$-x^2 + y^2 + z^2 = 4$$
$$y^2 + z^2 = 4 + x^2$$

$$x^2 = 4 + y^2 + z^2$$
$$k^2 = 4 + y^2 + z^2$$
$$k = \sqrt{4 + y^2 + z^2}$$



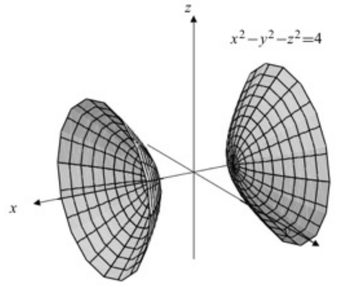


Fig. 10.5.7

8. $-x^2+y^2+z^2=4$ represents a hyperboloid of one sheet, with circular cross-sections in all planes perpendicular to the x-axis.

$-x^2=4-y^2-z^2$

$x^2=-4+y^2+z^2$

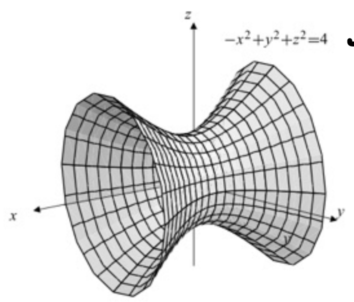


Fig. 10.5.8

9. $z=xy$ represents a hyperbolic paraboloid containing the x- and y-axes.

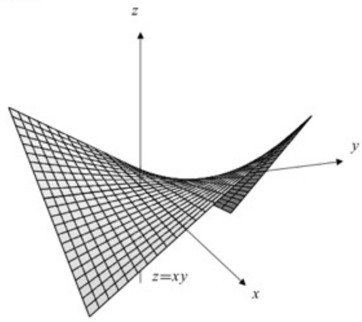


Fig. 10.5.9

10. $x^2+4z^2=4$ represents an elliptic cylinder with axis along the y-axis.

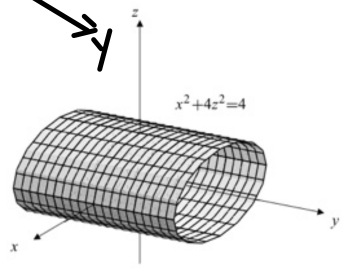


Fig. 10.5.10

11. $x^2-4z^2=4$ represents a hyperbolic cylinder with axis along the y-axis.

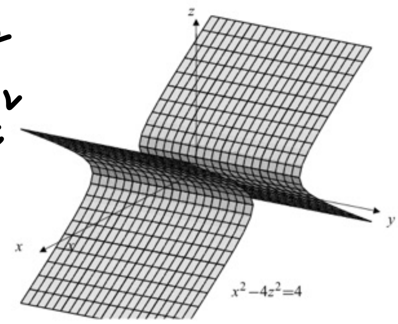


Fig. 10.5.11

12. $y=z^2$ represents a parabolic cylinder with vertex line along the x-axis.

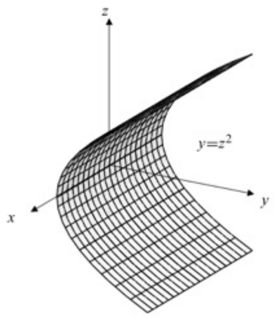
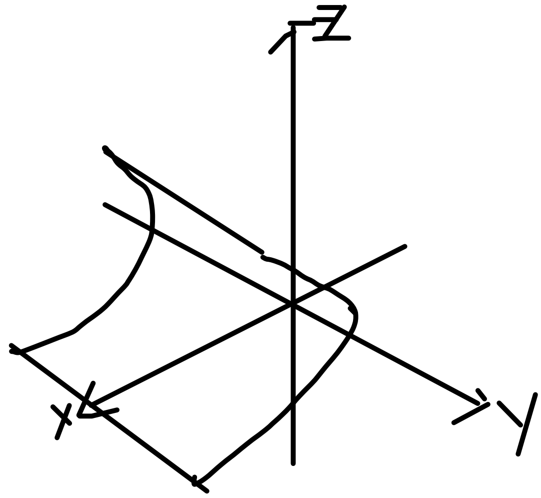


Fig. 10.5.12

13. $x=z^2+z=\left(z+\frac{1}{2}\right)^2-\frac{1}{4}$ represents a parabolic cylinder with vertex line along the line $z=-1/2$, $x=-1/4$.



$$x^2 = y^2 + 2z^2$$

$$k^2 = y^2 + 2z^2$$

$$k = \sqrt{y^2 + 2z^2}$$

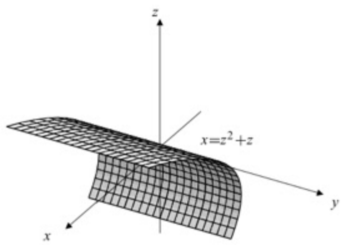


Fig. 10.5.13

14. $x^2 = y^2 + 2z^2$ represents an elliptic cone with vertex at the origin and axis along the x-axis.

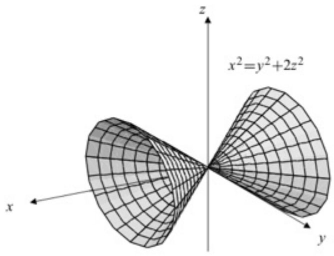


Fig. 10.5.14

15. $(z-1)^2 = (x-2)^2 + (y-3)^2$ represents a circular cone with axis along the line $x = 2$, $y = 3$, and vertex at $(2, 3, 1)$

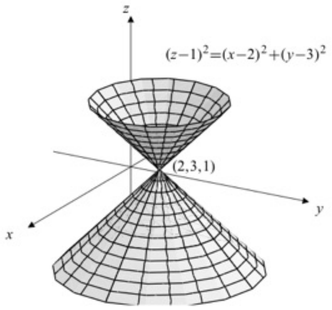


Fig. 10.5.15

16. $(z-1)^2 = (x-2)^2 + (y-3)^2 + 4$ represents a hyperboloid of two sheets with centre at $(2, 3, 1)$, axis along the line $x = 2$, $y = 3$, and vertices at $(2, 3, -1)$ and $(2, 3, 3)$.

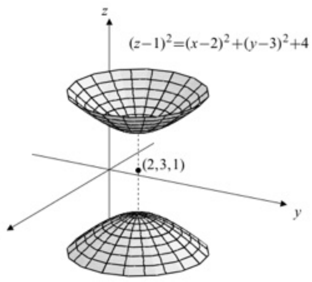


Fig. 10.5.16

17. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x + y + z = 1 \end{cases}$ represents the circle of intersection of a sphere and a plane. The circle lies in the plane $x + y + z = 1$, and has centre $(1/3, 1/3, 1/3)$ and radius $\sqrt{4 - (3/9)} = \sqrt{11/3}$.

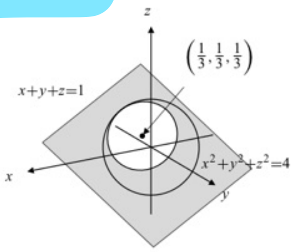


Fig. 10.5.17

18. $\begin{cases} x^2 + y^2 = 1 \\ z = x + y \end{cases}$ is the ellipse of intersection of the plane $z = x + y$ and the circular cylinder $x^2 + y^2 = 1$. The centre of the ellipse is at the origin, and the ends of the major axis are $\pm(1/\sqrt{2}, 1/\sqrt{2}, \sqrt{2})$.

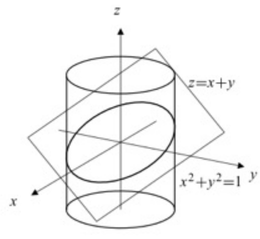


Fig. 10.5.18

19. $\begin{cases} z^2 = x^2 + y^2 \\ z = 1 + x \end{cases}$ is the parabola in which the plane $z = 1 + x$ intersects the circular cone $z^2 = x^2 + y^2$. (It is a parabola because the plane is parallel to a generator of the cone, namely the line $z = x$, $y = 0$.) The vertex of the parabola is $(-1/2, 0, 1/2)$, and its axis is along the line $y = 0$, $z = 1 + x$.

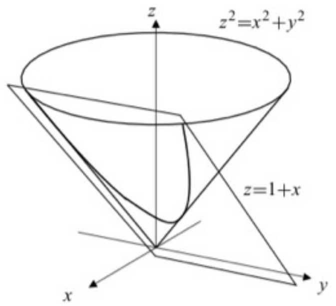


Fig. 10.5.19

20. $\begin{cases} x^2 + 2y^2 + 3z^2 = 6 \\ y = 1 \end{cases}$ is an ellipse in the plane $y = 1$. Its projection onto the xz-plane is the ellipse $x^2 + 3z^2 = 4$. One quarter of the ellipse is shown in the figure.

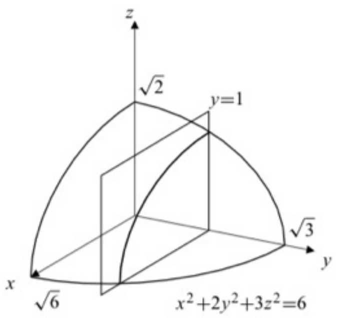


Fig. 10.5.20

21.
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$
$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}$$
$$\left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right)$$

$$\text{Family 1: } \begin{cases} \dfrac{x}{a} + \dfrac{z}{c} = \lambda\left(1 + \dfrac{y}{b}\right) \\ \lambda\left(\dfrac{x}{a} - \dfrac{z}{c}\right) = 1 - \dfrac{y}{b}. \end{cases}$$

$$\text{Family 2: } \begin{cases} \dfrac{x}{a} + \dfrac{z}{c} = \mu\left(1 - \dfrac{y}{b}\right) \\ \mu\left(\dfrac{x}{a} - \dfrac{z}{c}\right) = 1 + \dfrac{y}{b}. \end{cases}$$

22. $z = xy$

$$\text{Family 1: } \begin{cases} z = \lambda x \\ \lambda = y. \end{cases}$$

$$\text{Family 2: } \begin{cases} z = \mu y \\ \mu = x. \end{cases}$$

23. The cylinder $2x^2 + y^2 = 1$ intersects horizontal planes in ellipses with semi-axes 1 in the y direction and $1/\sqrt{2}$ in the x direction. Tilting the plane in the x direction will cause the shorter semi-axis to increase in length. The plane $z = cx$ intersects the cylinder in an ellipse with principal axes through the points $(0, \pm 1, 0)$ and $(\pm 1/\sqrt{2}, 0, \pm c/\sqrt{2})$. The semi-axes will be equal (and the ellipse will be a circle) if $(1/2) + (c^2/2) = 1$, that is, if $c = \pm 1$. Thus cross-sections of the cylinder perpendicular to the vectors $\mathbf{a} = \mathbf{i} \pm \mathbf{k}$ are circular.

24. The plane $z = cx + k$ intersects the elliptic cone $z^2 = 2x^2 + y^2$ on the cylinder

$$c^2x^2 + 2ckx + k^2 = 2x^2 + y^2$$
$$(2 - c^2)x^2 - 2ckx + y^2 = k^2$$
$$(2 - c^2)\left(x - \frac{ck}{2 - c^2}\right)^2 + y^2 = k^2 + \frac{c^2k^2}{2 - c^2} = \frac{2k^2}{2 - c^2}$$
$$\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $x_0 = \dfrac{ck}{2 - c^2}$, $a^2 = \dfrac{2k^2}{(2 - c^2)^2}$, and $b^2 = \dfrac{2k^2}{2 - c^2}$.

As in the previous exercise, $z = cx + k$ intersects the cylinder (and hence the cone) in an ellipse with principal axes joining the points

$$(x_0 - a, 0, c(x_0 - a) + k) \quad \text{to} \quad (x_0 + a, 0, c(x_0 + a) + k),$$
$$\text{and } (x_0, -b, cx_0 + k) \quad \text{to} \quad (x_0, b, cx_0 + k).$$

The centre of this ellipse is $(x_0, 0, cx_0 + k)$. The ellipse is a circle if its two semi-axes have equal lengths, that is, if

$$a^2 + c^2a^2 = b^2,$$

that is,

$$(1 + c^2)\frac{2k^2}{(2 - c^2)^2} = \frac{2k^2}{2 - c^2},$$

or $1 + c^2 = 2 - c^2$. Thus $c = \pm 1/\sqrt{2}$. A vector normal to the plane $z = \pm(x/\sqrt{2}) + k$ is $\mathbf{a} = \mathbf{i} \pm \sqrt{2}\mathbf{k}$.

Section 10.6 Cylindrical and Spherical Coordinates (page 602)

1. Cartesian: $(2, -2, 1)$;
Cylindrical: $[2\sqrt{2}, -\pi/4, 1]$;
Spherical: $[3, \cos^{-1}(1/3), -\pi/4]$.

2. Cylindrical: $[2, \pi/6, -2]$;
Cartesian: $(\sqrt{3}, 1, -2]$; Spherical: $[2\sqrt{2}, 3\pi/4, \pi/6]$.

3. Spherical: $[4, \pi/3, 2\pi/3]$;
Cartesian: $(-\sqrt{3}, 3, -2)$; Cylindrical: $[2\sqrt{3}, 2\pi/3, 2]$.