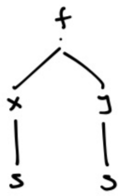
8. Let $f(x, y)$ be a function of two variables and $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$ be an **arc-length parametrized** curve in $\mathbb{R}^2$ such that /6

$$\frac{d}{ds}\mathbf{r}(s) = \nabla f(x(s), y(s)).$$

Here $\nabla f(x(s), y(s))$ means ∇f evaluated at the point $(x(s), y(s))$. Assume that both $f(x, y)$ and $\mathbf{r}(t)$ are C^2. Show that:

$$\frac{d^2}{ds^2} f(x(s), y(s)) = 0.$$



By chain rule,

$$\frac{d}{ds} f(x(s), y(s)) = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

For simplicity: we write $\frac{\partial f}{\partial x}$ in place of $\frac{\partial f}{\partial x}(x(s), y(s))$. Same for $\frac{\partial f}{\partial y}$.

$$= \left(\frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath}\right) \cdot \left(x'(s)\,\hat{\imath} + y'(s)\,\hat{\jmath}\right)$$

A "trick" we used for several places in the proofs presented in class.

★ 写成 dot product!

$$= \nabla f \cdot \vec{r}\,'(s)$$

Given that $\vec{r}\,'(s) = \nabla f$:

$$\frac{d}{ds} f(x(s), y(s)) = \vec{r}\,'(s) \cdot \vec{r}\,'(s)$$

$$= \left|\vec{r}\,'(s)\right|^2 = 1$$

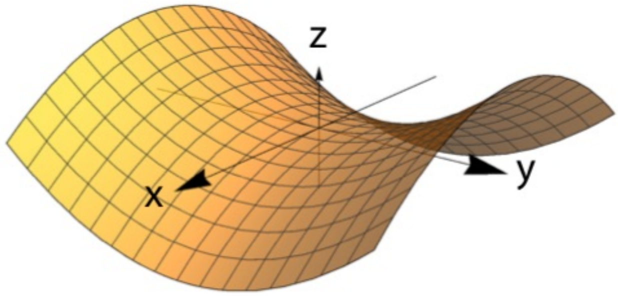
$\vec{r}(s)$ is arc-length parametrized

$$\Rightarrow \frac{d^2}{ds^2} f(x(s), y(s)) = \frac{d}{ds}\left(\frac{d}{ds} f(x(s), y(s))\right) = \frac{d}{ds} 1 = 0$$

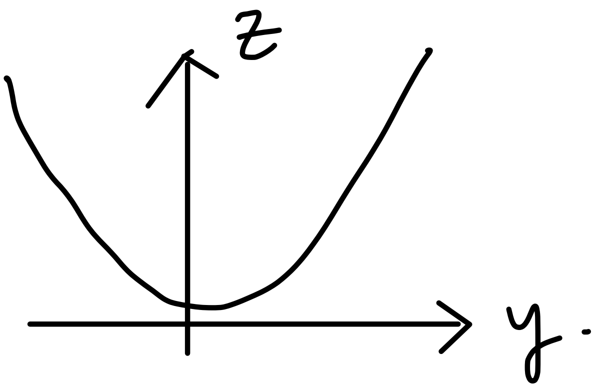
※

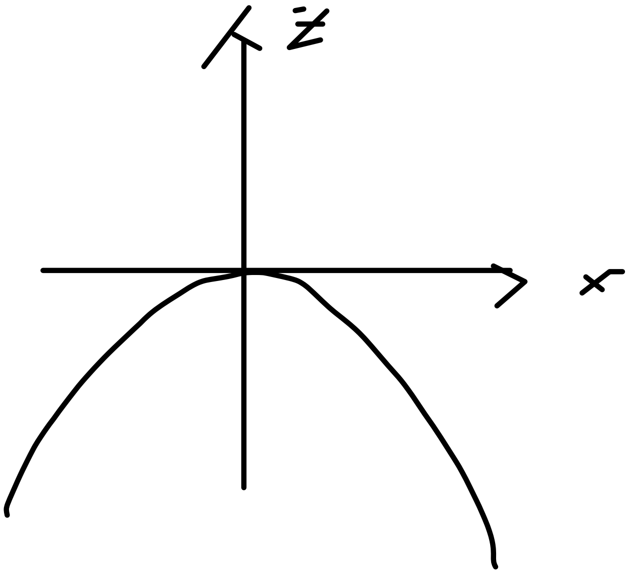
2015-16 Actual Midterm

(a) Consider the following graph of a saddle: **/2**

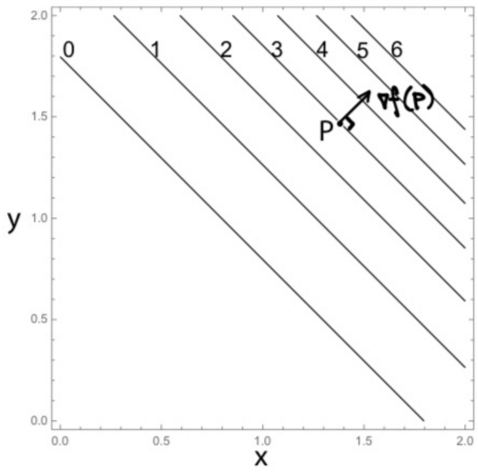


Is it the graph of $f(x,y) = x^2 - y^2$, or the graph of $g(x,y) = y^2 - x^2$? **Circle** the correct answer:





(b) Below is the level-set diagram of a C^2 function f. The contours $f(x,y) = c$ with $c = 0, 1, \ldots, 6$ are shown. **/10**



Consider the point P indicated on the diagram. Answer the following questions:

i. Determine whether each of the following quantities is positive, zero or negative. **Circle** the correct answers.

$f(P)$	positive	zero	negative
$f_y(P)$	positive	zero	negative
$f_{xx}(P)$	positive	zero	negative
$D_{\frac{-\mathrm{i}+\mathrm{j}}{\sqrt{2}}} f(P)$	positive	zero	negative

ii. Sketch the direction of $\nabla f(P)$ on the diagram.

(c) In class, we went through the proofs of several important theorems. Which of the following theorem(s) was/were proved using the chain rule in at least one of the steps? /3

Put "$\surd$" in **all** correct answer(s):

?

?

- ☑ The formula $D_{\hat{\mathbf{u}}} f = \nabla f \cdot \hat{\mathbf{u}}$
- see lecture #7 or notes P.36 → ☑ The fact that $\nabla f(P)$ is perpendicular to the level curve of f at P
- ☑ Second Derivative Test for two-variable functions $f(x, y)$ ✓

$|\vec{r}'(s)| = 1 \quad \therefore \int_{1997}^{2047} |\vec{r}'(s)|\,ds = \int_{1997}^{2047} 1\,ds = 2047 - 1997$

(d) Let $\mathbf{r}(s)$ be an arc-length parametrization of a curve. Write down the arc-length of the curve from $s = 1997$ to $s = 2047$. /2

$|\vec{r}(s)'| = 1$

arc length $= \int_{1997}^{2047} |\vec{r}'(s)|\,ds = 50$

$1x + 0y + (-2)z = 1$

(e) Write down an *upward-pointing* normal vector of the plane $x - 2z = 1$.

$$\nabla f = \langle 1, 0, -2 \rangle$$

$$-\nabla f = \langle -1, 0, 2 \rangle$$

Made with Goodnotes

(f) Give an example of a parametric curve $\mathbf{r}(t)$ in $\mathbb{R}^2$ such that:

$$\frac{d}{dt}|\mathbf{r}(t)| = 0 \quad \text{whereas} \quad \left|\frac{d}{dt}\mathbf{r}(t)\right| = 1.$$

$x(t)i + y(t)j$

$|\vec{r}(t)|$ is constant.

speed = 1

$|r'(t)| = 1$

$$\sqrt{x^2+y^2} = c$$

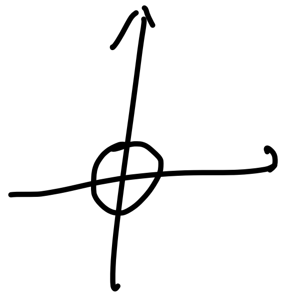
$$x^2+y^2 = c^2$$

$\frac{d}{dt}|r(t)| = 0$

distance = constant

$$\sqrt{x'(t)^2 + y'(t)^2} = 1$$

$$x'(t)^2 + y'(t)^2 = 1$$



$$\vec{r}(t) = \cos t\, i + \sin t\, j$$

(g) Let $f(x,y)$ be a C^2 function defined on $\mathbb{R}^2$ which satisfies: /6

$$\nabla f(0,0) = 0\mathbf{i} + 0\mathbf{j}$$
$$f_{xx}(0,0) = -2$$
$$\left(f_{xx}f_{yy} - f_{xy}^2\right)(0,0) = 1$$

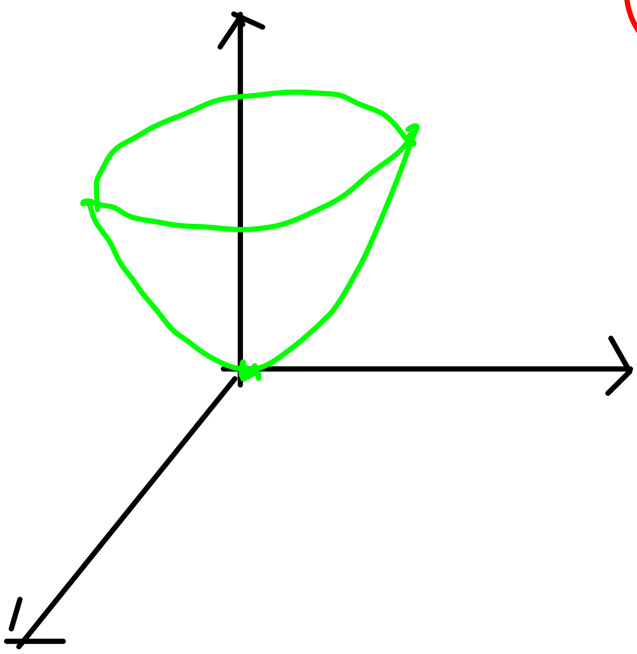
critical point

$\therefore$ Horizontal tangent plane $\Leftrightarrow \vec{n} \parallel \hat{k}$

Answer the following short questions:

i. Write down a normal vector to the graph of f at the point $(0,0,f(0,0))$.

$\vec{0}$? Why not $\vec{0}$



$f_x = 0$

$f_y = 0$

$$-2f_{yy} - f_{xy}^2 = 1$$
$$-2f_{yy} = \underbrace{1 + f_{xy}^2}_{+}$$

ii. Is $(0,0)$ a local maximum, a local minimum or a saddle point of f? **Circle** the correct answer:

local maximum   local minimum   saddle point   not enough data

iii. Is $f_{yy}(0,0)$ positive, negative or zero? **Circle** the correct answer:

$$\underbrace{f_{xx}}_{\ominus} f_{yy} = \underbrace{f_{xy}^2 + 1}_{\oplus}$$

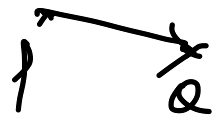
$f_{yy} < 0$

positive   zero   negative   not enough data

2. Given three points in $\mathbb{R}^3$:

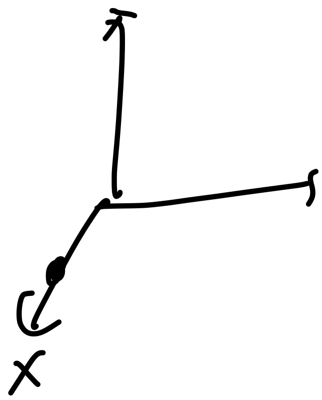
$$P(1,-2,0), \quad Q(3,1,4) \quad \text{and} \quad R(0,-1,2).$$

(a) Find an equation of the plane containing P, Q and R.



$\vec{PQ} = \langle -2, -3, -4 \rangle$

$\vec{PR} = \langle 1, -1, -2 \rangle$



$\vec{n} = \vec{PQ} \times \vec{PR}$

$= \begin{vmatrix} i & j & k \\ -2 & -3 & -4 \\ 1 & -1 & -2 \end{vmatrix}$

$= (6-4)i - 8j + 5k$

$= 2i - 8j + 5k$

Equation of plane:

$2x - 8y + 5z = 18$

$2-16$

(b) Find the point S on the x-axis such that P, Q, R and S are coplanar. /3

$(a, 0, 0)$

$2a = 18$

$a = 9$

$(9, 0, 0)$.

3. (a) Let $f(x,y) = x^{e^y}$ where $x, y > 0$. Find the partial derivatives f_x and f_y.

$$f_x = e^y x^{e^y - 1}$$

$$a^x$$

$$f_y = x^{e^y} \ln(x^{e^y})$$

$$a^x \ln a$$

$$f_y = e^y x^{e^y} \ln x$$

写 $(x^e)^y$ 咁 chain

(b) Let $g(x,y) = x^2y^3 + \boxed{e^{x^2}\tan^{-1}\left[\log\left(\frac{\sqrt{1+x^2+x^4}}{1+(x^2+1)^{x^4+1}}\right)\right]}$. ← independent of y /5

Find the second partial derivative $\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right)$.

$$\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right)$$

By mixed partial thm.

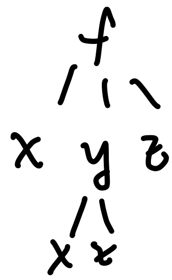
$$= \frac{\partial}{\partial x}\left(3x^2y^2\right)$$

$$= 6xy^2$$

4. (a) Let $f(x, y, z)$ be a C^1 function of three variables. Regard y to be a C^1 function of x and z such that: /6

$$f(x,\, y(x, z),\, z) = 0.$$

Using the chain rule, show that $\dfrac{\partial y}{\partial z} = -\dfrac{f_z}{f_y}$.



$$\frac{df}{dz} = \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial z} + \frac{\partial f}{\partial z}$$

$$0 = f_y \cdot \frac{\partial y}{\partial z} + f_z$$

$$-\frac{f_z}{f_y} = \frac{\partial y}{\partial z}$$

(b) Given that: **/4**

$$\sin x + \cos y + y^2 + \cos^2 z = 0.$$

Regarding y as a C^1 function of x and z, find the partial derivative $\frac{\partial y}{\partial z}$. Your final answer can be in terms of all x, y and z.

$$\frac{\partial y}{\partial z} = -\frac{f_z}{f_y}$$

$$f_z = -2\cos z \sin z = -\sin 2z$$

$$f_y = -\sin y + 2y$$

$$\frac{\partial y}{\partial z} = -\left(\frac{-\sin 2z}{-\sin y + 2y}\right)$$

5. Consider the parametric curve in $\mathbb{R}^2$:

$$\mathbf{r}(t) = \left(e^t \cos t\right)\mathbf{i} + \left(e^t \sin t\right)\mathbf{j}.$$

(a) Show that: $|\mathbf{r}'(t)| = \sqrt{2}\, e^t$ /6

$$\vec{r}'(t) = \left(-e^t \sin t + \cos t\, e^t\right)\vec{i} + \left(e^t \cos t + \sin t\, e^t\right)\vec{j}$$

$$|r'(t)| = \sqrt{\left(e^t \cos t - e^t \sin t\right)^2 + \left(e^t \cos t + \sin t\, e^t\right)^2}$$

$$= \sqrt{e^{2t}\cos^2 t - 2e^{2t}\cos t \sin t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2e \sin t \cos t + e^{2t}\sin^2 t}$$

$$= \sqrt{2e^{2t}}$$

$$= \sqrt{2}\, e^{t}$$

(b) Find an arc-length parametrization $\mathbf{r}(s)$ of the curve.

$$s = \int_0^t \sqrt{2}\, e^{z}\, dz$$

$$s = \left[\sqrt{2}\, e^{z}\right]_0^t$$

$$s = \sqrt{2}\, e^{t} - \sqrt{2}$$

$$\frac{s+\sqrt{2}}{\sqrt{2}} = e^{t}$$

$$t = \ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)$$

$$\vec{r}(s) = \left(\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\cos\left(\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\right)\vec{i} + \left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\sin\left(\ln\left(\frac{s+\sqrt{2}}{\sqrt{2}}\right)\right)\vec{j}\right)$$

6. Find the maximum and minimum of the function $f(x,y) = xy$ subject to the constraint $x^2 - xy + y^2 = 3$.

$$\nabla f = \langle y, x \rangle$$

$$\nabla g = \langle 2x - y, -x + 2y \rangle$$

$$\begin{cases} y = (2x-y)\lambda \\ x = (-x+2y)\lambda \\ x^2 - xy + y^2 = 3 \end{cases}$$

$$\frac{y}{2x-y} = \frac{x}{-x+2y}$$

$$y(-x+2y) = 2x^2 - xy$$

$$-xy + 2y^2 = 2x^2 - xy$$

$$2y^2 = 2x^2$$

$$x = \pm y.$$

When $x = +y$,

$$y^2 - y^2 + y^2 = 3$$

$$y = \pm\sqrt{3}$$

$$x = \pm\sqrt{3}$$

When $x = -y$,

$$y^2 + y^2 + y^2 = 3$$

$$y = \pm 1$$

$$x = \mp 1$$

$\therefore$ $(\sqrt{3}, \sqrt{3})$ 3
$(-\sqrt{3}, -\sqrt{3})$ 3 $\}$ Max

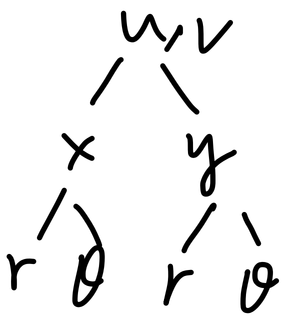
$(-1, 1)$ -1
$(1, -1)$ -1 $\}$ Min

7. Suppose $u(x,y)$ and $v(x,y)$ are C^2 functions defined on $\mathbb{R}^2$ which satisfy the relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(a) The rectangular and polar coordinates are related by $x = r\cos\theta$ and $y = r\sin\theta$. Under this relation, we can also regard u and v as functions of (r,θ). **/10**

Show that: $$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$



$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial y}(\cos\theta) - \frac{\partial v}{\partial x}(\sin\theta)$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$

$$\frac{\frac{\partial u}{\partial r}}{\frac{\partial v}{\partial \theta}} = \frac{1}{r}$$

$$\frac{\partial u}{\partial r} = \left(\frac{1}{r}\right)\left(\frac{\partial v}{\partial \theta}\right)$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial v}{\partial r} = \frac{-\partial u}{\partial y}(\cos\theta) + \frac{\partial u}{\partial x}(\sin\theta)$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta)$$

$$\frac{\frac{\partial v}{\partial r}}{\frac{\partial u}{\partial \theta}} = -\frac{1}{r}.$$

$$\frac{\partial v}{\partial r} = \left(-\frac{1}{r}\right)\left(\frac{\partial u}{\partial \theta}\right)$$

(b) Using (a), show that:

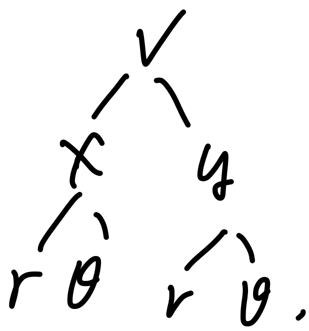
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

[Hint and remark: You can assume that the Mixed Partials Theorem holds in polar coordinates, i.e. $u_{r\theta} = u_{\theta r}$ and $v_{r\theta} = v_{\theta r}$.]

∂u

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = \left(-\frac{1}{r}\right)\left(\frac{\partial u}{\partial \theta}\right)$$



7. Suppose $u(x,y)$ and $v(x,y)$ are C^2 functions defined on $\mathbb{R}^2$ which satisfy the relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

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Show that: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

u, v

b).
$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta \partial r}$$

$$= -\frac{1}{r^2}\left(\frac{\partial v}{\partial \theta \partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial \theta \partial y}\frac{\partial y}{\partial r}\right)$$

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\left(\frac{\partial v}{\partial \theta \partial x}\cos\theta + \frac{\partial v}{\partial \theta \partial y}\sin\theta\right)$$

$$= -\frac{1}{r^2}\Big($$

8. Let $f(x,y)$ be a function of two variables and $\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j}$ be an **arc-length parametrized** curve in $\mathbb{R}^2$ such that /6

$$\frac{d}{ds}\mathbf{r}(s) = \nabla f(x(s), y(s)).$$

Here $\nabla f(x(s), y(s))$ means ∇f evaluated at the point $(x(s), y(s))$. Assume that both $f(x,y)$ and $\mathbf{r}(t)$ are C^2. Show that:

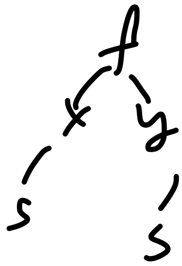
$$\frac{d^2}{ds^2} f(x(s), y(s)) = 0.$$

$$r'(s) = \langle x'(s), y'(s) \rangle$$

$$|r'(s)| = 1$$

$$\sqrt{x'(s)^2 + y'(s)^2} = 1$$

$$x'(s)^2 + y'(s)^2 = 1$$



$$\frac{d}{ds} f = \frac{\partial f}{\partial x} x'(s) + \frac{\partial f}{\partial y} y'(s)$$

$$\frac{d^2}{ds^2} f = \frac{d}{ds}\left(\frac{\partial f}{\partial x} x'(s)\right) + \frac{d}{ds}\left(\frac{\partial f}{\partial y} y'(s)\right)$$

$$= \frac{\partial f}{\partial x} x''(s) + x'(s) \frac{d}{ds}\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y''(s) + y'(s) \frac{d}{ds}\frac{\partial f}{\partial y}$$

$$= \frac{\partial f}{\partial x} x''(s) + x'(s)\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial y}{\partial s}\right) + \frac{\partial f}{\partial y} y''(s) + y(s)\left(\frac{\partial^2 f}{\partial y \partial x}\frac{\partial x}{\partial s} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial s}\right)$$