MATH 2023 – Multivariable Calculus

Lecture #06 Worksheet ◇ February 26, 2019

Problem 1. Let $u = x^4y + y^2z$ where $u(x,y,z)$

$x = rse^t$ $x(r,s,t)$

$y = s^2e^{-tr}$ $y(r,s,t)$

$z = rt$ $z(r,t)$

$r = st^2$ $r(s,t)$

Find $\frac{\partial u}{\partial s}$ in terms of s, t

$$= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r}\frac{\partial r}{\partial s} = 4x^3 y\, se^t t^2$$

$s^2e^{-trst^2}$

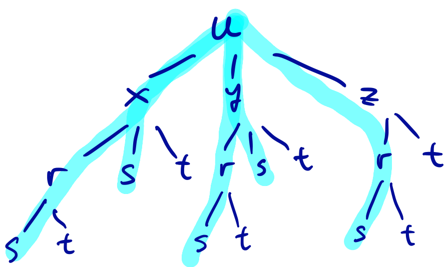
rse^t, $r = st^2$

$$= 4st^2se^ts^2e^{-tst^2}se^tt^2$$

$$= 4s^5t^4e^{2t}e^{-st^3}.$$

$$+ \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r}\frac{\partial r}{\partial s}$$

$\ldots$



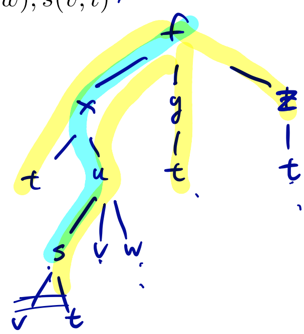
Let $f(x, y, z)$ be a function, where we have the dependence of variables:

$$x(t,u), y(t), u(s,v,w), s(v,t), z(t)$$

Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}\frac{\partial u}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}\frac{\partial u}{\partial s}\frac{\partial s}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

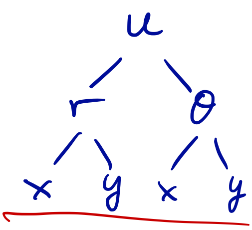


$$x = r\cos\theta \qquad y = r\sin\theta \iff r = \sqrt{x^2+y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$$

Problem 2. Let $u(r,\theta)$ be a function in polar coordinates. Express the Laplace equation

$$u_{xx} + u_{yy} = 0$$

in terms of r and θ.



$$u_x = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = u_r\cos\theta - u_\theta\frac{\sin\theta}{r}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot -\frac{y}{x^2} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$$

$$u_{xx} = (u_x)_x = \left(u_r\cos\theta - u_\theta\frac{\sin\theta}{r}\right)_r\cos\theta - \left(u_r\cos\theta - u_\theta\frac{\sin\theta}{r}\right)_\theta\frac{\sin\theta}{r}$$

$$= \left(u_{rr}\cos\theta + u_\theta\frac{\sin\theta}{r^2} - u_{\theta r}\frac{\sin\theta}{r}\right)\cos\theta - \left(-u_r\sin\theta + u_{r\theta}\cos\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta\frac{\cos\theta}{r}\right)\frac{\sin\theta}{r}$$

$$= u_{rr}\cos^2\theta + \cancel{2u_\theta\frac{\sin\theta\cos\theta}{r^2}} + u_r\frac{\sin^2\theta}{r} + u_{\theta\theta}\frac{\sin^2\theta}{r^2} + u_\theta\frac{\sin\theta\cos\theta}{r^2} - u_{r\theta}\frac{\sin\theta\cos\theta}{r}$$
$$\cancel{-2u_{\theta r}\frac{\sin\theta\cos\theta}{r}}$$

$$\frac{\partial r}{\partial y} = \sin\theta \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

$$u_y = u_r\sin\theta + u_\theta\frac{\cos\theta}{r}$$

$$u_{yy} = u_{rr}\sin^2\theta + \cancel{2u_{\theta r}\frac{\sin\theta\cos\theta}{r}} + u_r\frac{\cos^2\theta}{r} + u_{\theta\theta}\frac{\cos^2\theta}{r^2} \cancel{- 2u_\theta\frac{\cos\theta\sin\theta}{r}}$$

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$