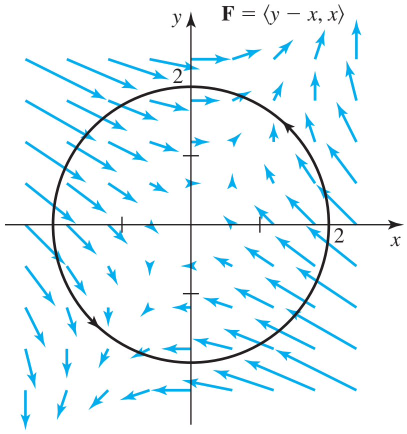
MATH 2023 • Multivariable Calculus
Problem Set #7 • Line Integrals, Conservative Vector Fields, Curl Operator

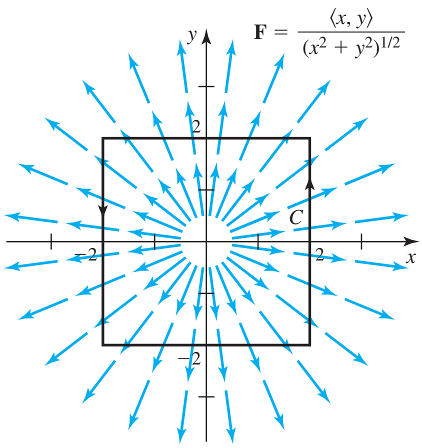
Do not use the Green's Theorem in any problem in this set.

1. (★) Let $\mathbf{F} = (y - x)\mathbf{i} + x\mathbf{j}$ on $\mathbb{R}^2$, and C be the counter-clockwise circular path with radius 2 centered at the origin. See the figure below:



   (a) On the above figure, highlight the portion of the path C at which $\mathbf{F} \cdot \mathbf{r}' > 0$.
   (b) On the above figure, highlight (with another color) the portion of the path C at which $\mathbf{F} \cdot \ \mathbf{r}' < 0$.
   (c) Calculate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ from the definition. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?
   (d) Calculate $\nabla \times \mathbf{F}$, i.e. the curl of $\mathbf{F}$. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?
   (e) Find a potential function f such that $\mathbf{F} = \nabla f$, or show that such an f does not exist. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

2. (★) Let $\mathbf{F} = \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\mathbf{j}$, and C be the counter-clockwise square path with vertices $(2,-2)$, $(2,2)$, $(-2,2)$ and $(-2,-2)$. See the figure below:



Do (a)-(e) of Problem #1 with this $\mathbf{F}$ and C instead.

3. (★) Let C be the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = y$.
   (a) Sketch the cylinder, the plane and the curve C in the same diagram.
   (b) Let $\mathbf{F} = y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$. Calculate the line integral $\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ where Γ is a portion of C from $(-1, 0, 0)$ to $(1, 0, 0)$. There are two possible such Γ's. Do both.
   Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?
   (c) Find a potential function f such that $\mathbf{F} = \nabla f$, or show that such an f does not exist.
   Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

4. (★) Determine whether or not each of the following vector fields is conservative or not. If so, find its potential function f such that $\mathbf{F} = \nabla f$.
   (a) $\mathbf{F} = (e^{-y} - ze^{-x})\mathbf{i} + (e^{-z} - xe^{-y})\mathbf{j} + (e^{-x} - ye^{-z})\mathbf{k}$
   (b) $\mathbf{F} = (x^2 - xy)\mathbf{i} + (y^2 - xy)\mathbf{j}$

5. (★) Determine the values of A and B for which the vector field below is conservative:

$$\mathbf{F}(x, y, z) = Ax \ln z \,\mathbf{i} + By^2 z \,\mathbf{j} + \left(\frac{x^2}{z} + y^3\right)\mathbf{k},$$

   where the domain of $\mathbf{F}$ is the upper-half space $\{(x, y, z) : z > 0\}$.
   For each such pair of A and B, find the potential function f for the vector field.

6. (★★) Consider the path C:

$$\mathbf{r}(t) = \left(\cos^{2M} t\right)\mathbf{i} + \left(\sin^N t\right)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi.$$

   Here M is the age of the Earth, and N is the age of the Universe. Assume both M and N are positive finite integers.
   Evaluate the line integral:

$$\int_C \left(e^{-y} - ze^{-x}\right) dx + \left(e^{-z} - xe^{-y}\right) dy + \left(e^{-x} - ye^{-z}\right) dz$$

   Provide **TWO** different solutions to this problem.

7. (★★) Given a conservative vector field $\mathbf{F}$ in $\mathbb{R}^3$, the potential *energy* of $\mathbf{F}$ is a scalar-valued function $V(x, y, z)$ such that $\mathbf{F} = -\nabla V$. Suppose $\mathbf{r}(t)$ is the path of a particle with mass m traveling in accordance to the Newton's Second Law $\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$. Then its kinetic energy is defined to be:

$$\text{KE} = \frac{1}{2}m\left|\mathbf{r}'(t)\right|^2.$$

   The total (kinetic + potential) energy of the particle at time t is therefore given by:

$$E(t) := \frac{1}{2}m\left|\mathbf{r}'(t)\right|^2 + V(\mathbf{r}(t)).$$

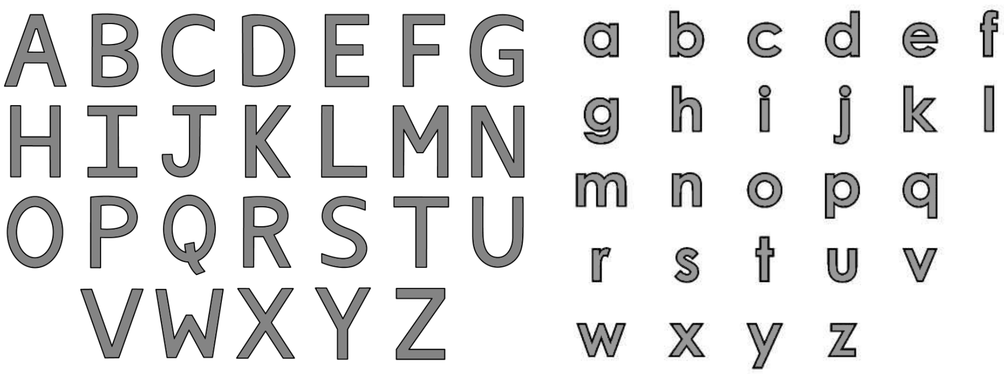
   Show that the total energy is conserved, i.e. $E'(t) = 0$ for all time t.
   [Hint: the only fact you need to know about Physics is the Newton's Second Law given above. It is purely a math problem.]

8. (★★) Denote $\mathbf{e}_\rho = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$ and $\mathbf{e}_r = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2+y^2}}$, which are the unit radial vector fields in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

   (a) Show that if $\mathbf{F}(x,y,z) = f(\rho)\mathbf{e}_\rho$ where f is a function depending only on $\rho = \sqrt{x^2+y^2+z^2}$, then $\nabla \times \mathbf{F} = \mathbf{0}$ on the domain of $\mathbf{F}$. Is this result alone sufficient to claim that $\mathbf{F}$ is conservative?

   (b) Show that if $\mathbf{G}(x,y) = g(r)\mathbf{e}_r$ where g is a function depending only on $r = \sqrt{x^2+y^2}$, then $\nabla \times \mathbf{G} = \mathbf{0}$ on the domain of $\mathbf{G}$. Is this result alone sufficient to claim that $\mathbf{G}$ is conservative?

9. (★) Regard each English letter as a solid region in $\mathbb{R}^2$. Which capital letters are simply-connected? Which small letters are simply-connected?

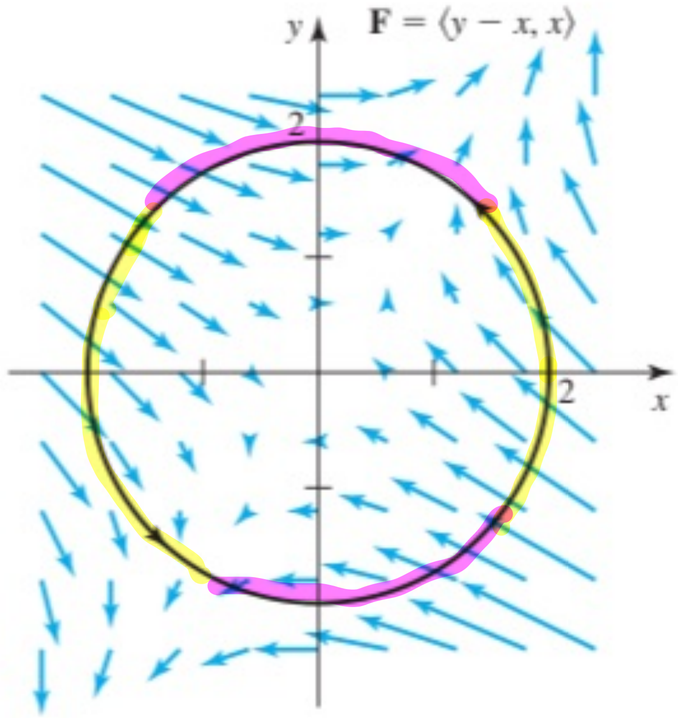


10. (★★) The notation $\mathbb{R}^3\backslash X$ means the xyz-space $\mathbb{R}^3$ with the set X removed. Determine whether $\mathbb{R}^3\backslash X$ is simply-connected when X is each of the following:

    (a) X is the origin
    (b) X is the entire y-axis
    (c) X is the positive y-axis
    (d) X is the solid sphere $x^2+y^2+z^2 \le 1$
    (e) X is the surface sphere $x^2+y^2+z^2 = 1$
    (f) X is the solid cylinder $x^2+y^2 \le 1$
    (g) X is the solid half-cylinder $x^2+y^2 \le 1$ and $z \ge 0$.
    (h) X is the surface cylinder $x^2+y^2 = 1$
    (i) X is the surface half-cylinder $x^2+y^2 = 1$ and $z \ge 0$
    (j) X is a solid torus
    (k) X is a surface torus
    (l) X is a simple closed curve

    Give an example of a proper subset X of $\mathbb{R}^3$ such that both X and $\mathbb{R}^3\backslash X$ are simply-connected. [Note: "proper" means X cannot be empty, and cannot be the whole $\mathbb{R}^3$.]

1. (★) Let $\mathbf{F} = (y - x)\mathbf{i} + x\mathbf{j}$ on $\mathbb{R}^2$, and C be the counter-clockwise circular path with radius 2 centered at the origin. See the figure below:



(a) On the above figure, highlight the portion of the path C at which $\mathbf{F} \cdot \mathbf{r}' > 0$.

(b) On the above figure, highlight (with another color) the portion of the path C at which $\mathbf{F} \cdot \mathbf{r}' < 0$.

(c) Calculate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ from the definition. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

(d) Calculate $\nabla \times \mathbf{F}$, i.e. the curl of $\mathbf{F}$. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

(e) Find a potential function f such that $\mathbf{F} = \nabla f$, or show that such an f does not exist. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

$C(t) = \langle 2\cos t, 2\sin t \rangle$

$(y-x)(-2\sin t)\,dt + x(2\cos t)\,dt$

$= \int_0^{2\pi} (2\sin t - 2\cos t)(-2\sin t)\,dt + 4\cos^2 t\,dt$

$= \int_0^{2\pi} -4\sin^2 t + 4\sin t\cos t + 4\cos^2 t\,dt$

$= \int_0^{2\pi} -4 + 4\sin t\cos t + 8\cos^2 t\,dt$

(a) On the above figure, highlight the portion of the path C at which $\mathbf{F} \cdot \mathbf{r}' > 0$.

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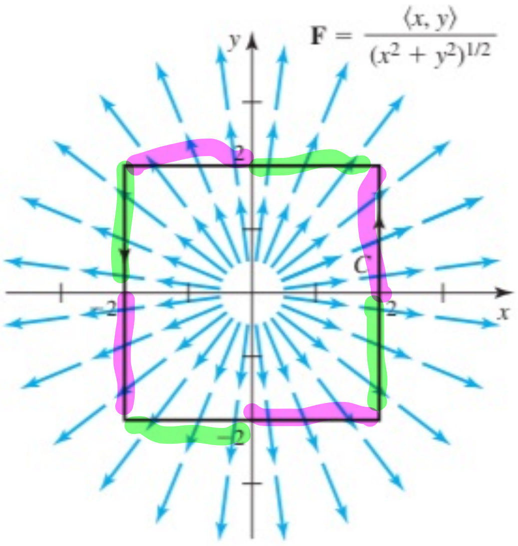

c). x

d) x

e) x

Do (a)-(e) of Problem #1 with this **F** and C instead.

3. (★) Let C be the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = y$.

(a) Sketch the cylinder, the plane and the curve C in the same diagram.

(b) Let $\mathbf{F} = y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$. Calculate the line integral $\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ where Γ is a portion of C from $(-1, 0, 0)$ to $(1, 0, 0)$. There are two possible such Γ's. Do both.
Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not? ✓

(c) Find a potential function f such that $\mathbf{F} = \nabla f$, or show that such an f does not exist.
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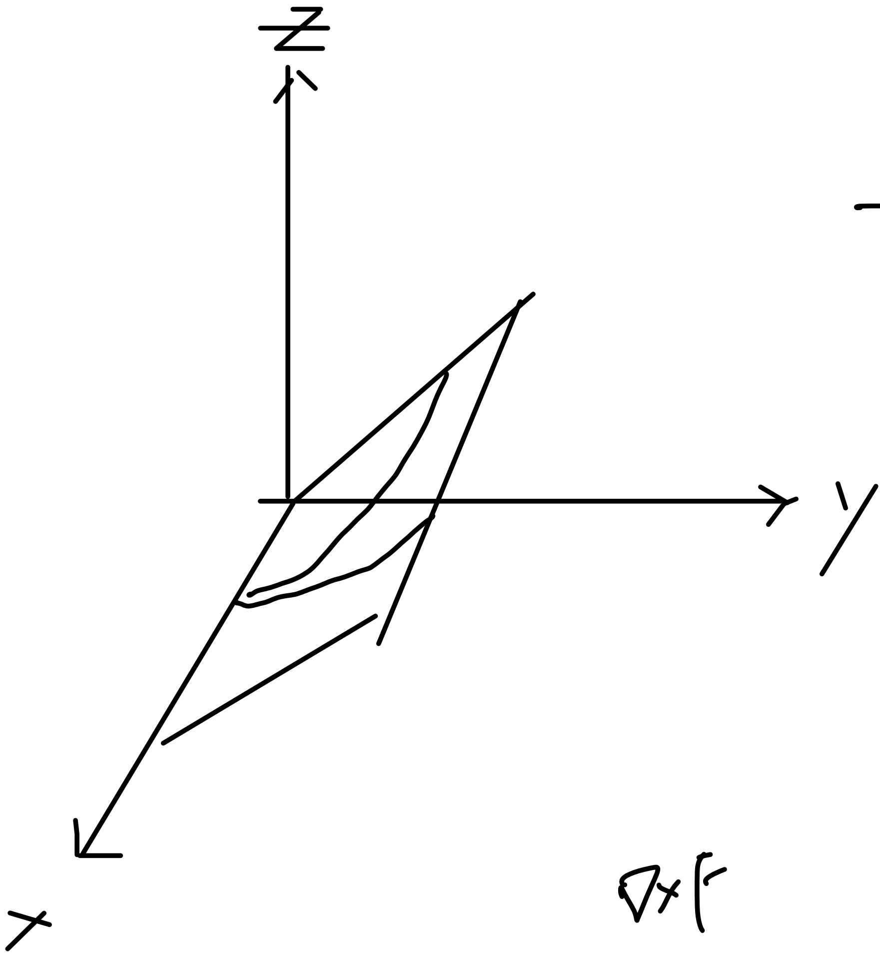
b) $(-1,0,0) \to (1,0,0)$

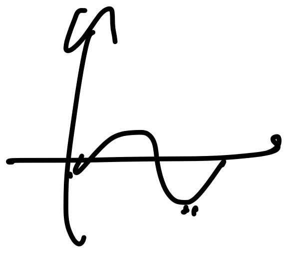
$\langle \cos\theta, \sin\theta, \sin\theta \rangle$

$-\pi \le \theta \le 0$

$\langle \sin\theta \quad \cos\theta, \cos\theta \rangle$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$





$$\nabla \times F \quad \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & -x \end{vmatrix}$$

$$= \langle -1, 1, -1 \rangle$$

4. (★) Determine whether or not each of the following vector fields is conservative or not. If so, find its potential function f such that $\mathbf{F} = \nabla f$.

(a) $\mathbf{F} = (e^{-y} - ze^{-x})\mathbf{i} + (e^{-z} - xe^{-y})\mathbf{j} + (e^{-x} - ye^{-z})\mathbf{k}$

(b) $\mathbf{F} = (x^2 - xy)\mathbf{i} + (y^2 - xy)\mathbf{j}$

4(a) - $e^{-y}x + e^{-z}y + e^{-x}z$

b) $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -y - (-x) = -y + x \neq 0.$

Made with Goodnotes

5. (★) Determine the values of A and B for which the vector field below is conservative:

$$\mathbf{F}(x,y,z) = Ax\ln z\,\mathbf{i} + By^2z\,\mathbf{j} + \left(\frac{x^2}{z} + y^3\right)\mathbf{k},$$

where the domain of $\mathbf{F}$ is the upper-half space $\{(x,y,z) : z > 0\}$.
For each such pair of A and B, find the potential function f for the vector field.

$A = 2$

$B = 3$

$x^2 \ln z + y^3 z$

6. (★★) Consider the path C:

$$\mathbf{r}(t) = \left(\cos^{2M} t\right)\mathbf{i} + \left(\sin^{N} t\right)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi.$$

Here M is the age of the Earth, and N is the age of the Universe. Assume both M and N are positive finite integers.

Evaluate the line integral:

$$\int_C \left(e^{-y} - ze^{-x}\right)\, dx + \left(e^{-z} - xe^{-y}\right)\, dy + \left(e^{-x} - ye^{-z}\right)\, dz$$??

Provide **TWO** different solutions to this problem. ?

$$e^{-y}x + e^{-z}y + e^{-x}z$$

$$f(\vec{r}(\pi)) - f(\vec{r}(0))$$

$$= f(\langle 1, 0, \pi\rangle) - f(\langle 1, 0, 0\rangle)$$

$$= 1 + e^{-z}(0) + \pi e^{-1} - (1 + 0 + 0)$$

$$= \pi e^{-1}$$

7. (★★) Given a conservative vector field $\mathbf{F}$ in $\mathbb{R}^3$, the potential *energy* of $\mathbf{F}$ is a scalar-valued function $V(x, y, z)$ such that $\mathbf{F} = -\nabla V$. Suppose $\mathbf{r}(t)$ is the path of a particle with mass m traveling in accordance to the Newton's Second Law $\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$. Then its kinetic energy is defined to be:

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The total (kinetic + potential) energy of the particle at time t is therefore given by:

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??

Show that the total energy is conserved, i.e. $E'(t) = 0$ for all time t.

[Hint: the only fact you need to know about Physics is the Newton's Second Law given above. It is purely a math problem.]

8. (★★) Denote $\mathbf{e}_\rho = \dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$ and $\mathbf{e}_r = \dfrac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$, which are the unit radial vector fields in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

(a) Show that if $\mathbf{F}(x,y,z) = f(\rho)\mathbf{e}_\rho$ where f is a function depending only on $\rho = \sqrt{x^2+y^2+z^2}$, then $\nabla \times \mathbf{F} = \mathbf{0}$ on the domain of $\mathbf{F}$. Is this result alone sufficient to claim that $\mathbf{F}$ is conservative? yes

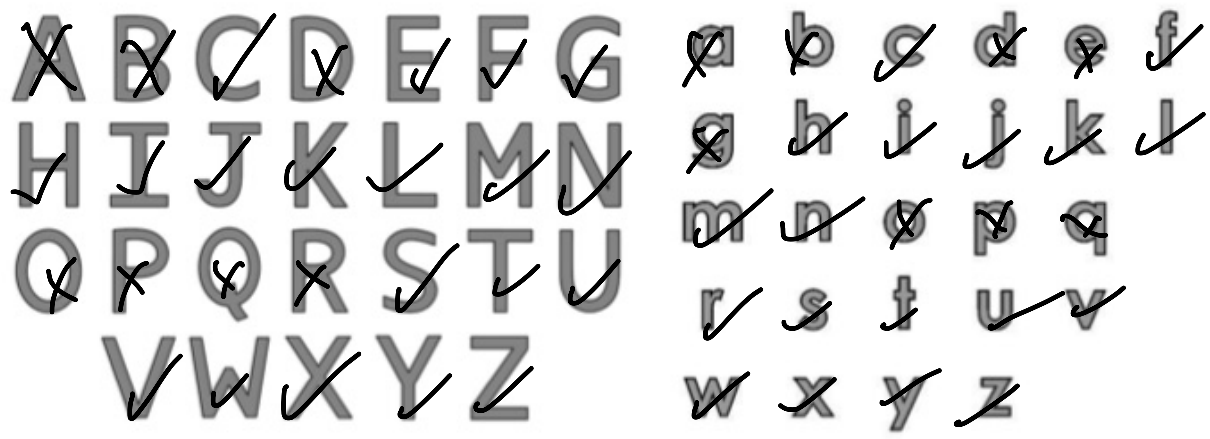
(b) Show that if $\mathbf{G}(x,y) = g(r)\mathbf{e}_r$ where g is a function depending only on $r = \sqrt{x^2+y^2}$, then $\nabla \times \mathbf{G} = \mathbf{0}$ on the domain of $\mathbf{G}$. Is this result alone sufficient to claim that $\mathbf{G}$ is conservative?

$$f(\rho)e_\rho = f(\rho)\frac{\langle x,y,z\rangle}{\rho}$$

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{f(\rho)x}{\rho} & \frac{f(\rho)y}{\rho} & \frac{f(\rho)z}{\rho} \end{vmatrix} = \vec{0}.$$

b). No.

9. (★) Regard each English letter as a solid region in $\mathbb{R}^2$. Which capital letters are simply-connected? Which small letters are simply-connected?



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(d) X is the solid sphere $x^2 + y^2 + z^2 \leq 1$ X ?

(e) X is the surface sphere $x^2 + y^2 + z^2 = 1$

(f) X is the solid cylinder $x^2 + y^2 \leq 1$ X

(g) X is the solid half-cylinder $x^2 + y^2 \leq 1$ and $z \geq 0$. X

(h) X is the surface cylinder $x^2 + y^2 = 1$ X

(i) X is the surface half-cylinder $x^2 + y^2 = 1$ and $z \geq 0$ X

(j) X is a solid torus X

(k) X is a surface torus X

(l) X is a simple closed curve ✓

Give an example of a proper subset X of $\mathbb{R}^3$ such that both X and $\mathbb{R}^3 \backslash X$ are simply-connected. [Note: "proper" means X cannot be empty, and cannot be the whole $\mathbb{R}^3$.] ?