MATH 2023 • Multivariable Calculus
Problem Set #8 • Green's Theorem

1. (★) Use the Green's Theorem to evaluate

$$\oint_C (4y^2 + e^{x^2})\,dx - (2x + e^{y^2})\,dy$$

where C is each of the following (assume C is counter-clockwise oriented):

(a) the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$

Solution: The line integral is associated with the vector field $\mathbf{F} = (4y^2 + e^{x^2})\mathbf{i} - (2x + e^{y^2})\mathbf{j}$. By direct computation, we get:

$$\nabla \times \mathbf{F} = -2(1+4y)\mathbf{k} \Longrightarrow (\nabla \times \mathbf{F}) \cdot \mathbf{k} = -2(1+4y).$$

By Green's Theorem:

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \int_0^1 (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dxdy \\ &= \int_0^1 \int_0^1 -2(1+4y)\,dxdy \\ &= -6\end{aligned}$$

(b) the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$

Solution: Denote the solid square by R, then by Green's Theorem:

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA \\ &= \iint_R -2(1+4y)\,dA = -\iint_R 2\,dA - \iint_R 8y\,dA \\ &= -2\,\text{Area}(R) - \iint_R 8y\,dA \\ &= -4 - \iint_R 8y\,dA\end{aligned}$$

Since y is an odd function and the region R is symmetric about the x-axis, we have $\iint_R 8y\,dA = 0$. Hence $\oint_C \mathbf{F} \cdot d\mathbf{r} = -4$.

(c) the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$

Solution: The hypotenuse of the triangle is given by equation $y = 1 - x$. Using the Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dydx = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} -2(1+4y)\,dydx = -\frac{7}{3}.$$

(d) the unit circle $x^2 + y^2 = 1$

Solution: The enclosed region is a unit circle. It is best to use polar coordinates.

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \int_0^1 (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 -2(1+4y)\, r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 -2(1+4r\sin\theta) r\, dr d\theta \\ &= -2\pi.\end{aligned}$$

2. (★★) The purpose of this problem is to explore a line integral for computing areas.

(a) Let C be a simple closed curve in $\mathbb{R}^2$ and the area enclosed by C is denoted by A. Show that:

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy$$

Solution:

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\oint_C \langle -y, x, 0\rangle \cdot d\mathbf{r} = \frac{1}{2}\iint_R (\nabla \times \langle -y, x, 0\rangle) \cdot \mathbf{k}\, dA.$$

Here R is the region enclosed by C. By direct computations, we get:

$$\nabla \times \langle -y, x, 0\rangle = 2\mathbf{k}.$$

Hence

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\iint_R 2\,dA = \text{Area of } R.$$

(b) Let E be the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $a, b > 0$. Find the area bounded by E using the result of (a).

Solution: The ellipse can be parametrized by:

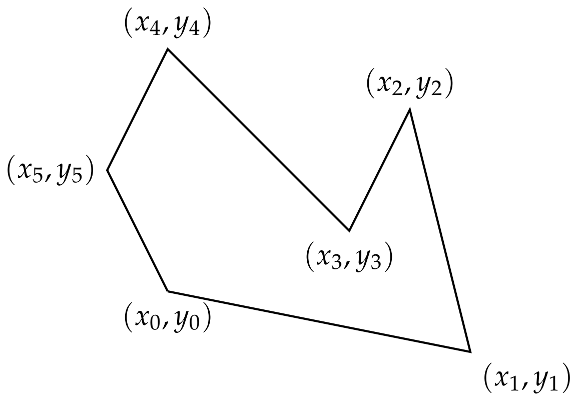
$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (b\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

In other words, $x = a\cos t$ and $y = b\sin t$.

$$\begin{aligned}A &= \frac{1}{2}\oint_C -y\,dx + x\,dy \\ &= \frac{1}{2}\int_{t=0}^{t=2\pi} -\underbrace{(b\sin t)}_{y}\, d(\underbrace{a\cos t}_{x}) + \underbrace{(a\cos t)}_{x}\, d(\underbrace{b\sin t}_{y}) \\ &= \frac{1}{2}\int_{t=0}^{t=2\pi} ab\sin^2 t\,dt + ab\cos^2 t\,dt \\ &= \frac{1}{2}\int_{t=0}^{t=2\pi} ab\,dt = ab\pi.\end{aligned}$$

(c) Let P be a n-sided polygon with vertices (x_0, y_0), (x_1, y_1), $\ldots$, (x_{n-1}, y_{n-1}). See the figure below for an example when $n = 6$. For convenience, we denote $(x_n, y_n) = (x_0, y_0)$. Using (a), show that the area $A(P)$ bounded by the polygon P is given by:

$$A(P) = \frac{1}{2}\sum_{i=1}^{n}\left(x_{i-1}y_i - x_i y_{i-1}\right).$$



Solution: Denote L_i to be the straight-line from (x_{i-1}, y_{i-1}) to (x_i, y_i), which is parametrized by:

$$\mathbf{r}_i(t) = \underbrace{\langle x_{i-1}, y_{i-1}\rangle}_{\text{starting point}} + t\underbrace{\langle x_i - x_{i-1}, y_i - y_{i-1}\rangle}_{\text{direction}}, \quad 0 \le t \le 1.$$

Then on L_i, we have $x = x_{i-1} + t(x_i - x_{i-1})$ and $y = y_{i-1} + t(y_i - y_{i-1})$, and so:

$$dx = (x_i - x_{i-1})\,dt \quad \text{and} \quad dy = (y_i - y_{i-1})\,dt.$$

The polygon can then be represented as the directed path $L_1 + L_2 + \ldots + L_n$, or simply $\sum_{i=1}^{n} L_i$. By (a), we have:

$$\begin{aligned}
A(P) &= \frac{1}{2}\oint_{\sum_{i=1}^{n} L_i} -y\,dx + x\,dy = \frac{1}{2}\sum_{i=1}^{n}\int_{L_i} -y\,dx + x\,dy \\
&= \frac{1}{2}\sum_{i=1}^{n}\int_{t=0}^{t=1} -\underbrace{\left(y_{i-1} + t(y_i - y_{i-1})\right)(x_i - x_{i-1})\,dt}_{y\,dx} \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\int_{t=0}^{t=1}\underbrace{\left(x_{i-1} + t(x_i - x_{i-1})\right)(y_i - y_{i-1})\,dt}_{x\,dy} \\
&= \frac{1}{2}\sum_{i=1}^{n}\left[-(x_i - x_{i-1})\left(y_{i-1}t + \frac{(y_i - y_{i-1})t^2}{2}\right)\right]_{t=0}^{t=1} \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\left[(y_i - y_{i-1})\left(x_{i-1}t + \frac{(x_i - x_{i-1})t^2}{2}\right)\right]_{t=0}^{t=1} \\
&= -\frac{1}{2}\sum_{i=1}^{n}(x_i - x_{i-1})\cdot\frac{y_i + y_{i-1}}{2} + \frac{1}{2}\sum_{i=1}^{n}(y_i - y_{i-1})\cdot\frac{x_i + x_{i-1}}{2}
\end{aligned}$$

which yields the desired result after simplifications.

3. (★★) Consider the following system of differential equations:

$$\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y)$$

where f and g are C^1 on $\mathbb{R}^2$. Given that $\dfrac{\partial f}{\partial x} + \dfrac{\partial g}{\partial y} > 0$ on $\mathbb{R}^2$, show that the system cannot have a non-constant periodic solution. We say a solution $(x(t), y(t))$ is periodic if there exists $T > 0$ such that $(x(0), y(0)) = (x(T), y(T))$.

Hint: Proof by contradiction. Apply Green's Theorem on $\mathbf{F} = -g(x,y)\mathbf{i} + f(x,y)\mathbf{j}$.

Solution: Suppose the system has a non-constant periodic solution $(x(t), y(t))$. Let $T > 0$ be the first time such that $(x(T), y(T)) = (x(0), y(0))$, then the curve:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \quad 0 \le t \le T$$

is a simple closed curve in $\mathbb{R}^2$. Denote this simple closed curve by C and let R be the region enclosed by C. Apply Green's Theorem on the vector field $\mathbf{F} = -g(x,y)\mathbf{i} + f(x,y)\mathbf{j}$ over C:

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA \\
&= \iint_R \left(\frac{\partial f}{\partial x} - \left(-\frac{\partial g}{\partial y} \right) \right) \mathbf{k} \cdot \mathbf{k}\, dA \\
&= \iint_R \underbrace{\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right)}_{\text{given}>0} dA > 0
\end{aligned}$$

On the other hand, the line integral can be shown to be zero:

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_{t=0}^{t=T} \underbrace{\langle -g(x,y), f(x,y) \rangle}_{\mathbf{F}} \cdot \underbrace{\langle x'(t), y'(t) \rangle}_{\mathbf{r}'(t)}\, dt \\
&= \int_{t=0}^{t=T} -g(x,y)\, x'(t) + f(x,y)\, y'(t)\, dt.
\end{aligned}$$

From the given differential equations, we have

$$-g(x,y)\, x'(t) + f(x,y)\, y'(t) = -g(x,y)\, f(x,y) + f(x,y)\, g(x,y) = 0.$$

Therefore, we have:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

which contradicts to the previous result, so the system cannot have non-constant periodic solution.

[FYI: This result is called the Bendixson-Dulac's Theorem. First established in 1901 by Ivar Bendixson. This short proof using Green's Theorem is later discovered by Henri Dulac in 1933.]

4. (★★) Consider the vector field $\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ which is defined at every point on $\mathbb{R}^2$ except the origin.

(a) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$ at every point in $\mathbb{R}^2$ except the origin.

> **Solution:** Straight-forward.

(b) Show, by direct computation, that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is non-zero where C is the unit circle, counter-clockwise oriented, with centered at the origin.

> **Solution:** The unit circle is parametrized by $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \leq t \leq 2\pi$. Hence,
>
> $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{t=2\pi} \underbrace{\left(-\frac{\sin t}{\cos^2 t + \sin^2 t}\mathbf{i} + \frac{\cos t}{\cos^2 t + \sin^2 t}\mathbf{j}\right)}_{\mathbf{F}} \cdot \underbrace{(-(\sin t)\mathbf{i} + (\cos t)\mathbf{j})}_{\mathbf{r}'(t)}\, dt$$
> $$= \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\cos^2 t + \sin^2 t}\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

(c) The following students are confused about the above vector field $\mathbf{F}$ in relation to some facts and theorems stated in class. Pretend that you are a teaching assistant of this course, point out their misconceptions.

i. Student A said, "Given that $\nabla \times \mathbf{F} = \mathbf{0}$, the Curl Test asserts that $\mathbf{F}$ is conservative and so the closed-path line integral in (b) should be zero. How come the answer for (b) is non-zero???!!!"

> **Solution:** The domain of $\mathbf{F}$ is $\mathbb{R}^2 \backslash \{(0,0)\}$ which is NOT simply-connected. The curl test cannot be used here.

ii. Student B said, "Given that $\nabla \times \mathbf{F} = \mathbf{0}$, the Green's Theorem asserts that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_R \mathbf{0} \cdot \mathbf{k}\, dA = 0$$

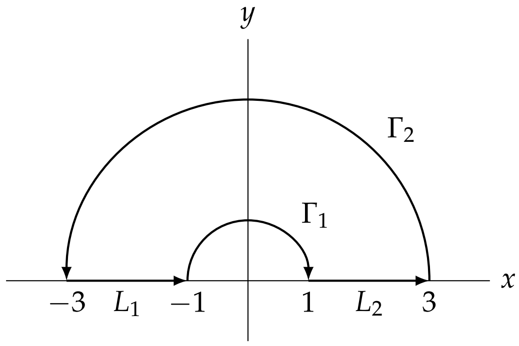
for any closed-path C. Why can the answer in (b) be non-zero???!!!" "

> **Solution:** The unit circle C encloses the origin at which $\mathbf{F}$ is not defined. Green's Theorem cannot be used for this curve C.

iii. Student C said, "It can be verified that $\mathbf{F} = \nabla\left(\tan^{-1}\frac{y}{x}\right)$ and so $\mathbf{F}$ is conservative with potential function $f(x,y) = \tan^{-1}\frac{y}{x}$. Any line integral of a conservative vector field over a closed curve must be zero. How come can the closed-path integral in (b) be non-zero???!!!"

> **Solution:** The domain of the potential function f needs to be the same as that of a vector field $\mathbf{F}$. In our case, the domain of $\mathbf{F}$ is $\mathbb{R}^2 \backslash \{(0,0)\}$ whereas the domain of $\tan^{-1}\frac{y}{x}$ is $\mathbb{R}^2 \backslash \{\text{y-axis}\}$. Hence, $\tan^{-1}\frac{y}{x}$ cannot be regarded as the (global) potential function of $\mathbf{F}$. We cannot show $\mathbf{F}$ is conservative in this way.

5. (★★) In the figure shown below, Γ_1 and Γ_2 are circular arcs centered at the origin. L_1 and L_2 are straight-lines. Consider the closed path $C = L_1 + \Gamma_1 + L_2 + \Gamma_2$.



Compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ of each vector field below using the Green's Theorem in an *appropriate* way:

(a) $\mathbf{F} = y^3\mathbf{i} - x^3\mathbf{j}$

Solution: The vector field is C^1 everywhere in $\mathbb{R}^2$. No problem to apply Green's Theorem. Direct computations show:

$$\nabla \times \mathbf{F} = -3(x^2+y^2)\mathbf{k} \Longrightarrow (\nabla \times \mathbf{F}) \cdot \mathbf{k} = -3(x^2+y^2).$$

By Green's Theorem:

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA \\ &= \int_0^{\pi} \int_1^3 -3(x^2+y^2)\, r\, dr d\theta \\ &= -3 \int_0^{\pi} \int_1^3 r^3\, dr d\theta = -60\pi.\end{aligned}$$

(b) $\mathbf{F} = -\dfrac{y-3}{(x-3)^2+(y-3)^2}\mathbf{i} + \dfrac{x-3}{(x-3)^2+(y-3)^2}\mathbf{j}$

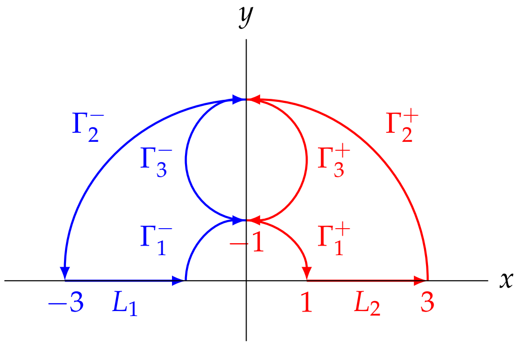
Solution: By somewhat lengthy computations, one can verify that $\nabla \times \mathbf{F} = \mathbf{0}$. The domain of $\mathbf{F}$ is $\mathbb{R}^2 \backslash \{(3,3)\}$. Fortunately, the closed path above does not enclose $(3,3)$ – no problem to use Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \underbrace{(\nabla \times \mathbf{F})}_{=\mathbf{0}} \cdot \mathbf{k}\, dA = 0$$

(c) $\mathbf{F} = -\dfrac{y-2}{x^2+(y-2)^2}\mathbf{i} + \dfrac{x}{x^2+(y-2)^2}\mathbf{j}$

Solution: By another somewhat lengthy computations, we get $\nabla \times \mathbf{F} = \mathbf{0}$. The domain of $\mathbf{F}$ is $\mathbb{R}^2 \backslash \{(0,2)\}$. However, the closed path C encloses this bad point $(0,2)$ – we can't use Green's Theorem directly.

To handle this path, we construct a "hole" with radius 1 centered at $(0, 2)$. Denote the boundary of the hole by $\Gamma_3 = \Gamma_3^+ + \Gamma_3^-$ as shown in the figure below. Note that Γ_3 is a **clockwise** circle.



Consider the red and blue paths individually. Each of the red and blue path does not enclose the bad point $(0, 2)$, we can apply Green's Theorem without problem:

$$\int_{\Gamma_2^+} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_3^+} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_1^+} \mathbf{F}\cdot d\mathbf{r} + \int_{L_2} \mathbf{F}\cdot d\mathbf{r} = \iint_{R^+} \underbrace{(\nabla\times\mathbf{F})}_{=\mathbf{0}}\cdot\mathbf{k}\,dA = 0$$

$$\int_{\Gamma_2^-} \mathbf{F}\cdot d\mathbf{r} + \int_{L_1} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_1^-} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_3^-} \mathbf{F}\cdot d\mathbf{r} = \iint_{R^-} \underbrace{(\nabla\times\mathbf{F})}_{=\mathbf{0}}\cdot\mathbf{k}\,dA = 0$$

Summing up and use the fact that $\Gamma_i = \Gamma_i^- + \Gamma_i^+$ (for $i = 1, 2, 3$), we get:

$$\underbrace{\int_{\Gamma_2} \mathbf{F}\cdot d\mathbf{r} + \int_{L_1} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_1} \mathbf{F}\cdot d\mathbf{r} + \int_{L_2} \mathbf{F}\cdot d\mathbf{r}}_{C=\Gamma_2+L_1+\Gamma_1+L_2} + \oint_{\Gamma_3} \mathbf{F}\cdot d\mathbf{r} = 0.$$

Hence,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} + \oint_{\Gamma_3} \mathbf{F}\cdot d\mathbf{r} = 0$$

To find $\oint_C \mathbf{F}\cdot d\mathbf{r}$, it suffices to find $\oint_{\Gamma_3} \mathbf{F}\cdot d\mathbf{r}$. Note that Γ_3 is **clockwise**, it is parametrized by:

$$\mathbf{r}(t) = (0 + \cos(-t))\mathbf{i} + (2 + \sin(-t))\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Then, $x = \cos t$, $y = 2 - \sin t$, and so $x^2 + (y-2)^2 = 1$.

$$\begin{aligned}\oint_{\Gamma_3} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \left(-\frac{(2-\sin t) - 2}{1}\mathbf{i} + \frac{\cos t}{1}\mathbf{j}\right)\cdot((-\sin t)\mathbf{i} - (\cos t)\mathbf{j})\ dt \\ &= \int_0^{2\pi} (-\sin^2 - \cos^2 t)\,dt = -2\pi.\end{aligned}$$

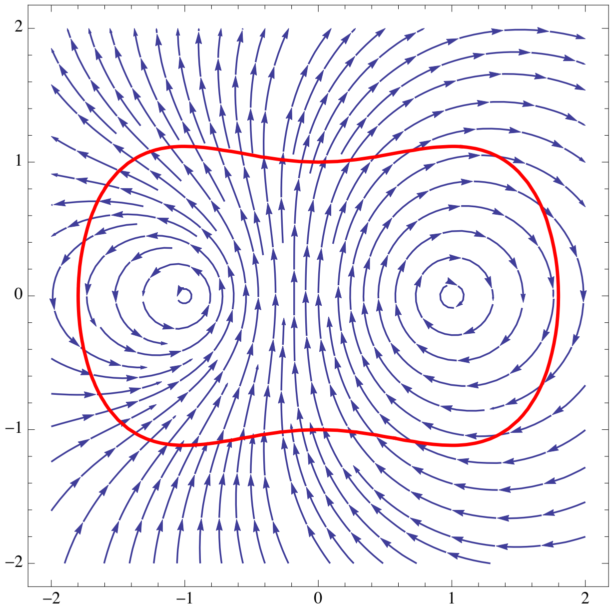
Therefore,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = -\int_{\Gamma_3} \mathbf{F}\cdot d\mathbf{r} = 2\pi.$$

6. (★★) Consider the flow of fluid (shown in blue in the figure below) which is represented by the vector field:

$$\mathbf{F} = \left(-\frac{y}{(x+1)^2+y^2} + \frac{2y}{(x-1)^2+y^2}\right)\mathbf{i} + \left(\frac{x+1}{(x+1)^2+y^2} - \frac{2(x-1)}{(x-1)^2+y^2}\right)\mathbf{j}$$

C is an arbitrary simple closed curve (red in the figure) which encloses all points at which $\mathbf{F}$ is not defined.



(a) At which point(s) the vector field $\mathbf{F}$ is/are *not* defined? Is the domain of $\mathbf{F}$ simply-connected?

Solution: $\mathbf{F}$ is NOT defined at $(-1, 0)$ and $(1, 0)$. The domain of $\mathbf{F}$ is

$$\mathbb{R}^2 \backslash \{(-1, 0), (1, 0)\}$$

which is NOT simply-connected.

(b) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$ at every point in $\mathbb{R}^2$ where $\mathbf{F}$ is defined.

Solution: Straight-forward, but quite lengthy.

(c) Show that from the definition of line integrals:

i. $\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for any counter-clockwise circle Γ centered at $(-1, 0)$ with radius less than 2.

Solution: Γ is parametrized by:

$$\mathbf{r}(t) = \langle -1 + \varepsilon \cos t, \varepsilon \sin t \rangle, \quad 0 \leq t \leq 2\pi.$$

On this path, the vector field is given by:

$$\mathbf{F} = \left\langle -\frac{\varepsilon \sin t}{\varepsilon^2} + \frac{2\varepsilon \sin t}{(\varepsilon \cos t - 2)^2 + \varepsilon^2 \sin^2 t}, \frac{\varepsilon \cos t}{\varepsilon^2} - \frac{2(\varepsilon \cos t - 2)}{(\varepsilon \cos t - 2)^2 + \varepsilon^2 \sin^2 t} \right\rangle.$$

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -\varepsilon \sin t,\, \varepsilon \cos t\rangle \\
\mathbf{F}\cdot\mathbf{r}'(t) &= 1 - \frac{2\varepsilon^2 \sin^2 t}{(\varepsilon\cos t - 2)^2 + \varepsilon^2 \sin^2 t} - \frac{2\varepsilon \cos t(\varepsilon\cos t - 2)}{(\varepsilon\cos t - 2)^2 + \varepsilon^2\sin^2 t} \\
&= 1 - \frac{2\varepsilon^2 - 4\varepsilon\cos t}{(\varepsilon\cos t - 2)^2 + \varepsilon^2\sin^2 t} \\
&= 1 - \frac{2\varepsilon^2 - 4\varepsilon\cos t}{\varepsilon^2 - 4\varepsilon\cos t + 4} \\
\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} 1 - \frac{2\varepsilon^2 - 4\varepsilon\cos t}{\varepsilon^2 - 4\varepsilon\cos t + 4}\, dt \\
&= 2\pi.
\end{aligned}$$

Mathematica was used to compute this difficult integral.

ii. $\oint_\gamma \mathbf{F}\cdot d\mathbf{r} = -4\pi$ for any counter-clockwise circle γ centered at $(1, 0)$ with radius less than 2.

Solution: Similar to (i). Parametrize the path by $\mathbf{r}(t) = \langle 1 + \varepsilon\cos t, \varepsilon \sin t\rangle$.

(d) Using the above results, show that:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = -2\pi$$

for any simple closed curve C in $\mathbb{R}^2$ that encloses all points at which $\mathbf{F}$ is not defined.

Solution: C encloses points at which $\mathbf{F}$ is not defined. We need to drill two circular holes centered at $(-1, 0)$ and $(1, 0)$. Then, apply Green's Theorem on the closed path $C + L_1 - \Gamma - L_1 + L_2 - \gamma - L_2$, which does not enclose $(-1, 0)$ and $(1, 0)$, we get:

$$\begin{aligned}
&\oint_{C+L_1-\Gamma-L_1+L_2-\gamma-L_2} \mathbf{F}\cdot d\mathbf{r} \\
&= \oint_C \mathbf{F}\cdot d\mathbf{r} + \int_{L_1}\mathbf{F}\cdot d\mathbf{r} - \oint_\gamma \mathbf{F}\cdot d\mathbf{r} - \int_{L_1}\mathbf{F}\cdot d\mathbf{r} + \int_{L_2}\mathbf{F}\cdot d\mathbf{r} - \oint_\gamma \mathbf{F}\cdot d\mathbf{r} - \int_{L_2}\mathbf{F}\cdot d\mathbf{r} \\
&= \iint_R \underbrace{(\nabla\times\mathbf{F})}_{=\mathbf{0}}\cdot\mathbf{k}\, dA = 0
\end{aligned}$$

After cancellations, we get:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} - \oint_\gamma \mathbf{F}\cdot d\mathbf{r} - \oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$$

From (c), we conclude that:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_\gamma \mathbf{F}\cdot d\mathbf{r} + \oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = -4\pi + 2\pi = -2\pi.$$

$$
\begin{aligned}
\mathbf{r}'(t) &= \langle -\varepsilon \sin t,\, \varepsilon \cos t \rangle \\
\mathbf{F} \cdot \mathbf{r}'(t) &= 1 - \frac{2\varepsilon^2 \sin^2 t}{(\varepsilon \cos t - 2)^2 + \varepsilon^2 \sin^2 t} - \frac{2\varepsilon \cos t(\varepsilon \cos t - 2)}{(\varepsilon \cos t - 2)^2 + \varepsilon^2 \sin^2 t} \\
&= 1 - \frac{2\varepsilon^2 - 4\varepsilon \cos t}{(\varepsilon \cos t - 2)^2 + \varepsilon^2 \sin^2 t} \\
&= 1 - \frac{2\varepsilon^2 - 4\varepsilon \cos t}{\varepsilon^2 - 4\varepsilon \cos t + 4} \\
\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} 1 - \frac{2\varepsilon^2 - 4\varepsilon \cos t}{\varepsilon^2 - 4\varepsilon \cos t + 4}\, dt \\
&= 2\pi.
\end{aligned}
$$

Mathematica was used to compute this difficult integral.

ii. $\oint_\gamma \mathbf{F} \cdot d\mathbf{r} = -4\pi$ for any counter-clockwise circle γ centered at $(1, 0)$ with radius less than 2.

Solution: Similar to (i). Parametrize the path by $\mathbf{r}(t) = \langle 1 + \varepsilon \cos t, \varepsilon \sin t \rangle$.

(d) Using the above results, show that:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = -2\pi$$

for any simple closed curve C in $\mathbb{R}^2$ that encloses all points at which $\mathbf{F}$ is not defined.

Solution: C encloses points at which $\mathbf{F}$ is not defined. We need to drill two circular holes centered at $(-1, 0)$ and $(1, 0)$. Then, apply Green's Theorem on the closed path $C + L_1 - \Gamma - L_1 + L_2 - \gamma - L_2$, which does not enclose $(-1, 0)$ and $(1, 0)$, we get:

$$
\begin{aligned}
&\oint_{C+L_1-\Gamma-L_1+L_2-\gamma-L_2} \mathbf{F} \cdot d\mathbf{r} \\
&= \oint_C \mathbf{F} \cdot d\mathbf{r} + \int_{L_1} \mathbf{F} \cdot d\mathbf{r} - \oint_\gamma \mathbf{F} \cdot d\mathbf{r} - \int_{L_1} \mathbf{F} \cdot d\mathbf{r} + \int_{L_2} \mathbf{F} \cdot d\mathbf{r} - \oint_\gamma \mathbf{F} \cdot d\mathbf{r} - \int_{L_2} \mathbf{F} \cdot d\mathbf{r} \\
&= \iint_R \underbrace{(\nabla \times \mathbf{F})}_{=\mathbf{0}} \cdot \mathbf{k}\, dA = 0
\end{aligned}
$$

After cancellations, we get:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} - \oint_\gamma \mathbf{F} \cdot d\mathbf{r} - \oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = 0$$

From (c), we conclude that:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_\gamma \mathbf{F} \cdot d\mathbf{r} + \oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = -4\pi + 2\pi = -2\pi.$$