HKUST MATH 102

Second Midterm Examination

Multivariable and Vector Calculus

15 Dec 2006

Problem 1

(a) $x + 2y - z = 10$

(b) $x^2 + (y-1)^2 + z^2 = 1$

which is the equation of a sphere with center $(0, 1, 0)$ and radius 1.

(c) The distance between two planes is

$$d = \left| (\mathbf{a} - \mathbf{b}) \cdot \frac{(\mathbf{u} \times \mathbf{v})}{\|\mathbf{u} \times \mathbf{v}\|} \right|.$$

If they meet, then $d = 0$, i.e.

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{u} \times \mathbf{v}) = 0.$$

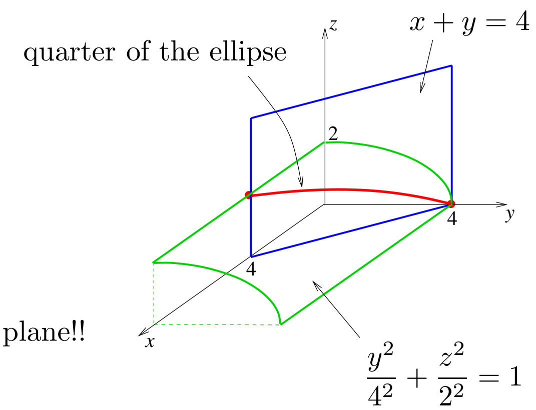
Problem 2

(a) $\mathbf{r}_1$ describes the equation of a parabola in the plane $x = -2$. [vertex at $(-2, 0, -1)$, opening upward]. The required tangent line is

$$\mathbf{r}(t) = -2\,\mathbf{i} + \mathbf{j} + t(\mathbf{j} + 2\,\mathbf{k}).$$

(b) From $\frac{y^2}{4^2} + \frac{z^2}{2^2} = 1$, we know that this is an elliptic cylinder with its axis equal to the x-axis, therefore we let

$$y = 4\cos\theta$$
$$z = 2\sin\theta$$



and from $x + y = 4$, we have

$$x = 4 - 4\cos\theta$$

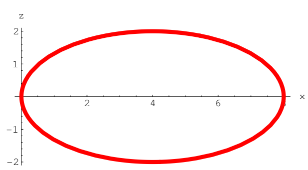
where $0 \leqslant \theta \leqslant 2\pi$.

The curve of intersection is an ellipse on the $x + y = 4$ plane!!

The required projection curve C onto the xz-plane is

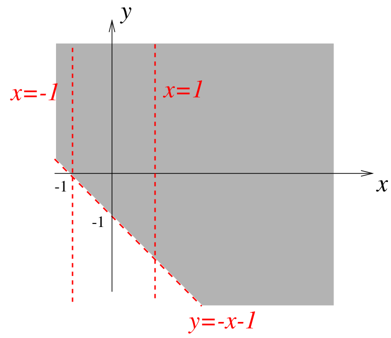
$$\frac{(x-4)^2}{16} + \frac{z^2}{4} = 1$$

which is an ellipse centered at $(4, 0)$ on the xz-plane.



Problem 3

(a)



(b) f is continuous throughout $\mathbb{R}^2$.

(c) The level surface are $(x-2)^2 + y^2 = k$, where $k \geqslant 0$.

These form a family of concentric cylinders with radius $\sqrt{k}$

and their axis centered at $(2,0,0)$ and parallel with the z-axis.

Problem 4

(a)

$$\left.\frac{\partial z}{\partial x}\right|_{(4,2)} = \frac{3}{8}$$

$$\left.\frac{\partial z}{\partial y}\right|_{(4,2)} = \frac{1}{4}.$$

(b) Note that the domain of $g(x,y)$ is $D = \{(x,y)\,|\,x, y \in \mathbb{R}\}$

$$g_x(x,y) = \frac{2}{3}(x^2+y^3)^{-\frac{1}{3}} \times 2x = \frac{4x}{3(x^2+y^3)^{\frac{1}{3}}}, \quad \text{in } D \text{ and } (x,y) \neq (0,0)$$

$$g_x(0,0) = \left.\frac{d}{dx} g(x,0)\right|_{x=0} = \left.\frac{d}{dx} x^{\frac{4}{3}}\right|_{x=0} = \left.\frac{4}{3} x^{\frac{1}{3}}\right|_{x=0} = 0.$$

(c) $2sf_{uuu} + (1-8st)f_{uuv} + 4t(2st-1)f_{uvv} + 4t^2 f_{vvv} + 4sf_{uv} + 2f_{vv}.$

Problem 5

(a) $\nabla f = \mathbf{B} \times (\mathbf{r} \times \mathbf{A}) + \mathbf{A} \times (\mathbf{r} \times \mathbf{B})$,

i.e. $\mathbf{P} = \mathbf{B}$ and $\mathbf{Q} = \mathbf{A}$.

(b) (i) $\mathbf{C} = -\mathbf{A} \cdot \mathbf{r}$. (ii) $\mathbf{D} = 3(\mathbf{A} \cdot \mathbf{r})(\mathbf{B} \cdot \mathbf{r})$.

Problem 6

(i) $f(x) = x^2$. (ii) The common points are $\mathbf{r}(t) = (t, t^2, -t)$, where $t \in \mathbb{R}$

Problem 7

(b) At an arbitrary point (x, y) the gradient vector is

$$\nabla f(x, y) = (2x - 3y)\,\mathbf{i} - 3x\,\mathbf{j}.$$

At the point $(1, 2)$ we have $\nabla f(1, 2) = -4\,\mathbf{i} - 3\,\mathbf{j}$. For the representation of C the unit tangent vector $\mathbf{T}(1)$ is $(\mathbf{i} + \mathbf{j})/\sqrt{2}$ and the required directional derivative is $\nabla f(1, 2) \cdot \mathbf{T}(1) = -7/\sqrt{2}$.

Problem 8

$x = 3$ and $y = 5$.

$$\lambda = 160 \times 100 \,\frac{4}{(3+4)^2} - 1000$$
$$= 306.122 \quad \text{(for each 1000)}.$$

Since the change in this promotion/development is \$100, the corresponding change in profit is \$30.61.