(c) unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively,

(d) the dot product $\mathbf{u} \bullet \mathbf{v}$,

(e) the angle between $\mathbf{u}$ and $\mathbf{v}$,

(f) the scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$,

(g) the vector projection of $\mathbf{v}$ along $\mathbf{u}$.

2. $\mathbf{u} = \mathbf{i} - \mathbf{j}$ and $\mathbf{v} = \mathbf{j} + 2\mathbf{k}$

3. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$

4. Use vectors to show that the triangle with vertices $(-1, 1)$, $(2, 5)$, and $(10, -1)$ is right-angled.

In Exercises 5–8, prove the stated geometric result using vectors.

5. The line segment joining the midpoints of two sides of a triangle is parallel to and half as long as the third side.

6. If P, Q, R, and S are midpoints of sides AB, BC, CD, and DA, respectively, of quadrilateral $ABCD$, then $PQRS$ is a parallelogram.

7. The diagonals of any parallelogram bisect each other.

8. The medians of any triangle meet in a common point. (A median is a line joining one vertex to the midpoint of the opposite side. The common point is the *centroid* of the triangle.)

9. A weather vane mounted on the top of a car moving due north at 50 km/h indicates that the wind is coming from the west. When the car doubles its speed, the weather vane indicates that the wind is coming from the northwest. From what direction is the wind coming, and what is its speed?

10. A straight river 500 m wide flows due east at a constant speed of 3 km/h. If you can row your boat at a speed of 5 km/h in still water, in what direction should you head if you wish to row from point A on the south shore to point B on the north shore directly north of A? How long will the trip take?

11. In what direction should you head to cross the river in Exercise 10 if you can only row at 2 km/h, and you wish to row from A to point C on the north shore, k km downstream from B? For what values of k is the trip not possible?

12. A certain aircraft flies with an airspeed of 750 km/h. In what direction should it head in order to make progress in a true easterly direction if the wind is from the northeast at 100 km/h? How long will it take to complete a trip to a city 1,500 km from its starting point?

13. For what value of t is the vector $2t\mathbf{i} + 4\mathbf{j} - (10 + t)\mathbf{k}$ perpendicular to the vector $\mathbf{i} + t\mathbf{j} + \mathbf{k}$?

14. Find the angle between a diagonal of a cube and one of the edges of the cube.

15. Find the angle between a diagonal of a cube and a diagonal of one of the faces of the cube. Give all possible answers. T-J??

16. **(Direction cosines)** If a vector $\mathbf{u}$ in $\mathbb{R}^3$ makes angles α, β, and γ with the coordinate axes, show that

$$\hat{\mathbf{u}} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

is a unit vector in the direction of $\mathbf{u}$, so $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. The numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the *direction cosines* of $\mathbf{u}$.

17. Find a unit vector that makes equal angles with the three coordinate axes.

18. Find the three angles of the triangle with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 3)$.

19. If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of two points, P_1 and P_2, and λ is a real number, show that

$$\mathbf{r} = (1 - \lambda)\mathbf{r}_1 + \lambda\mathbf{r}_2$$

is the position vector of a point P on the straight line joining P_1 and P_2. Where is P if $\lambda = 1/2$? if $\lambda = 2/3$? if $\lambda = -1$? if $\lambda = 2$?

20. Let $\mathbf{a}$ be a nonzero vector. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a} \bullet \mathbf{r} = 0$.

21. Let $\mathbf{a}$ be a nonzero vector, and let b be any real number. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a} \bullet \mathbf{r} = b$.

In Exercises 22–24, $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$, and $\mathbf{w} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

22. Find two unit vectors each of which is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

23. Find a vector $\mathbf{x}$ satisfying the system of equations $\mathbf{x} \bullet \mathbf{u} = 9$, $\mathbf{x} \bullet \mathbf{v} = 4$, $\mathbf{x} \bullet \mathbf{w} = 6$.

24. Find two unit vectors each of which makes equal angles with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

25. Find a unit vector that bisects the angle between any two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$.

26. Given two nonparallel vectors $\mathbf{u}$ and $\mathbf{v}$, describe the set of all points whose position vectors $\mathbf{r}$ are of the form $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$, where λ and μ are arbitrary real numbers.

27. **(The triangle inequality)** Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors.

(a) Show that $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + 2\mathbf{u} \bullet \mathbf{v} + |\mathbf{v}|^2$.

(b) Show that $\mathbf{u} \bullet \mathbf{v} \le |\mathbf{u}||\mathbf{v}|$.

(c) Deduce from (a) and (b) that $|\mathbf{u} + \mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$.

28. (a) Why is the inequality in Exercise 27(c) called a triangle inequality?

(b) What conditions on $\mathbf{u}$ and $\mathbf{v}$ imply that $|\mathbf{u} + \mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$?

29. **(Orthonormal bases)** Let $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$, $\mathbf{v} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$, and $\mathbf{w} = \mathbf{k}$.

(a) Show that $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1$ and $\mathbf{u} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{w} = \mathbf{v} \bullet \mathbf{w} = 0$. The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular unit vectors and as such are said to constitute an **orthonormal basis** for $\mathbb{R}^3$.

(b) If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, show by direct calculation that

$$\mathbf{r} = (\mathbf{r} \bullet \mathbf{u})\mathbf{u} + (\mathbf{r} \bullet \mathbf{v})\mathbf{v} + (\mathbf{r} \bullet \mathbf{w})\mathbf{w}.$$

30. Show that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any three mutually perpendicular unit vectors in $\mathbb{R}^3$ and $\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, then $a = \mathbf{r} \bullet \mathbf{u}$, $b = \mathbf{r} \bullet \mathbf{v}$, and $c = \mathbf{r} \bullet \mathbf{w}$.

31. **(Resolving a vector in perpendicular directions)** If $\mathbf{a}$ is a nonzero vector and $\mathbf{w}$ is any vector, find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, $\mathbf{u}$ is parallel to $\mathbf{a}$, and $\mathbf{v}$ is perpendicular to $\mathbf{a}$.

32. **(Expressing a vector as a linear combination of two other vectors with which it is coplanar)** Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{r}$ are position vectors of points U, V, and P, respectively, that $\mathbf{u}$ is not parallel to $\mathbf{v}$, and that P lies in the plane containing the origin, U, and V. Show that there exist numbers λ and μ such that $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$. *Hint:* Resolve both $\mathbf{v}$ and $\mathbf{r}$ as sums of vectors parallel and perpendicular to $\mathbf{u}$ as suggested in Exercise 31.

33. Given constants r, s, and t, with $r \neq 0$ and $s \neq 0$, and given a vector $\mathbf{a}$ satisfying $|\mathbf{a}|^2 > 4rst$, solve the system of equations

$$\begin{cases} r\mathbf{x} + s\mathbf{y} = \mathbf{a} \\ \mathbf{x} \bullet \mathbf{y} = t \end{cases}$$

for the unknown vectors $\mathbf{x}$ and $\mathbf{y}$.

Hanging cables

34. **(A suspension bridge)** If a hanging cable is supporting weight with constant horizontal line density (so that the weight supported by the arc LP in Figure 10.19 is $\delta g x$ rather than $\delta g s$), show that the cable assumes the shape of a parabola rather than a catenary. Such is likely to be the case for the cables of a suspension bridge.

35. At a point P, 10 m away horizontally from its lowest point L, a cable makes an angle 55° with the horizontal. Find the length of the cable between L and P.

36. Calculate the length s of the arc LP of the hanging cable in Figure 10.19 using the equation $y = (1/a)\cosh(ax)$ obtained for the cable. Hence, verify that the magnitude $T = |\mathbf{T}|$ of the tension in the cable at any point $P = (x, y)$ is $T = \delta g y$.

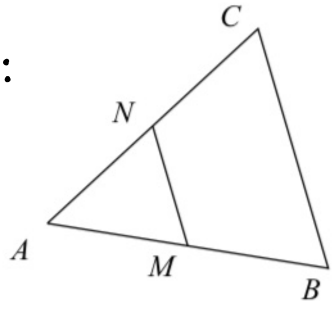
37. A cable 100 m long hangs between two towers 90 m apart so that its ends are attached at the same height on the two towers. How far below that height is the lowest point on the cable?

Q5:

Proof parallel and half 只需證明 $\frac{1}{2}\overrightarrow{MN} = \overrightarrow{BC}$ 即可.

方法: 用 $\vec{v} - \vec{u}$.

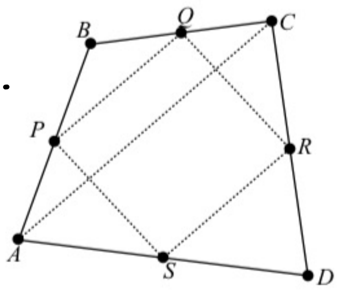
Q5:



5. The line segment joining the midpoints of two sides of a triangle is parallel to and half as long as the third side.

Fig. 10.2.5

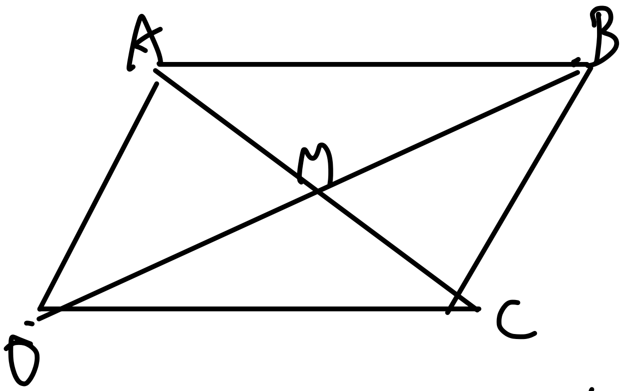
Q6.



6. If P, Q, R, and S are midpoints of sides AB, BC, CD, and DA, respectively, of quadrilateral $ABCD$, then $PQRS$ is a parallelogram.

this question is tricky. 要加對角線AC並利用AB+BC=AC,推出PQ=1/2AC 然後SR同理

Q7.



$\overrightarrow{AB} = \overrightarrow{CD}$

$\overrightarrow{AC} = \overrightarrow{BD}$

Approach: Proof Position vector of mid point of OB = midpoint of AC.

7. The diagonals of any parallelogram bisect each other.

Q8

8. The medians of any triangle meet in a common point. (A median is a line joining one vertex to the midpoint of the opposite side. The common point is the *centroid* of the triangle.)

8. Let X be the point of intersection of the medians AQ and BP as shown. We must show that CX meets AB in the midpoint of AB. Note that $\overrightarrow{PX} = \alpha\overrightarrow{PB}$ and $\overrightarrow{QX} = \beta\overrightarrow{QA}$ for certain real numbers α and β. Then

$$\begin{aligned}
\overrightarrow{CX} &= \frac{1}{2}\overrightarrow{CB} + \beta\overrightarrow{QA} = \frac{1}{2}\overrightarrow{CB} + \beta\left(\frac{1}{2}\overrightarrow{CB} + \overrightarrow{BA}\right) \\
&= \frac{1+\beta}{2}\overrightarrow{CB} + \beta\overrightarrow{BA}; \\
\overrightarrow{CX} &= \frac{1}{2}\overrightarrow{CA} + \alpha\overrightarrow{PB} = \frac{1}{2}\overrightarrow{CA} + \alpha\left(\frac{1}{2}\overrightarrow{CA} + \overrightarrow{AB}\right) \\
&= \frac{1+\alpha}{2}\overrightarrow{CA} + \alpha\overrightarrow{AB}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1+\beta}{2}\overrightarrow{CB} + \beta\overrightarrow{BA} &= \frac{1+\alpha}{2}\overrightarrow{CA} + \alpha\overrightarrow{AB} \\
(\beta+\alpha)\overrightarrow{BA} &= \frac{1+\alpha}{2}\overrightarrow{CA} - \frac{1+\beta}{2}\overrightarrow{CB} \\
(\beta+\alpha)(\overrightarrow{CA} - \overrightarrow{CB}) &= \frac{1+\alpha}{2}\overrightarrow{CA} - \frac{1+\beta}{2}\overrightarrow{CB} \\
\left(\beta+\alpha-\frac{1+\alpha}{2}\right)\overrightarrow{CA} &= \left(\beta+\alpha-\frac{1+\beta}{2}\right)\overrightarrow{CB}.
\end{aligned}$$

Since $\overrightarrow{CA}$ is not parallel to $\overrightarrow{CB}$,

$$\begin{aligned}
&\beta+\alpha-\frac{1+\alpha}{2} = \beta+\alpha-\frac{1+\beta}{2} = 0 \\
&\Rightarrow \alpha = \beta = \frac{1}{3}.
\end{aligned}$$

Since $\alpha = \beta$, x divides AQ and BP in the same ratio. By symmetry, the third median CM must also divide the other two in this ratio, and so must pass through X and $MX = \frac{1}{3}MC$.

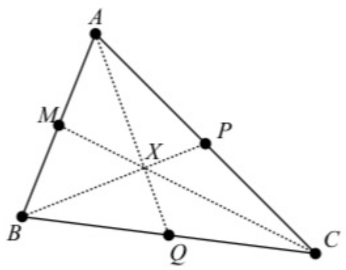


Fig. 10.2.8

Made with Goodnotes

9. A weather vane mounted on the top of a car moving due north at 50 km/h indicates that the wind is coming from the west. When the car doubles its speed, the weather vane indicates that the wind is coming from the northwest. From what direction is the wind coming, and what is its speed?

9.

9. Let $\mathbf{i}$ point east and $\mathbf{j}$ point north. Let the wind velocity be

$$\mathbf{v}_{\text{wind}} = a\mathbf{i} + b\mathbf{j}.$$

Now $\mathbf{v}_{\text{wind}} = \mathbf{v}_{\text{wind rel car}} + \mathbf{v}_{\text{car}}$.
When $\mathbf{v}_{\text{car}} = 50\mathbf{j}$, the wind appears to come from the west, so $\mathbf{v}_{\text{wind rel car}} = \lambda\mathbf{i}$. Thus

= the wind vector is pointing to i

$$a\mathbf{i} + b\mathbf{j} = \lambda\mathbf{i} + 50\mathbf{j},$$

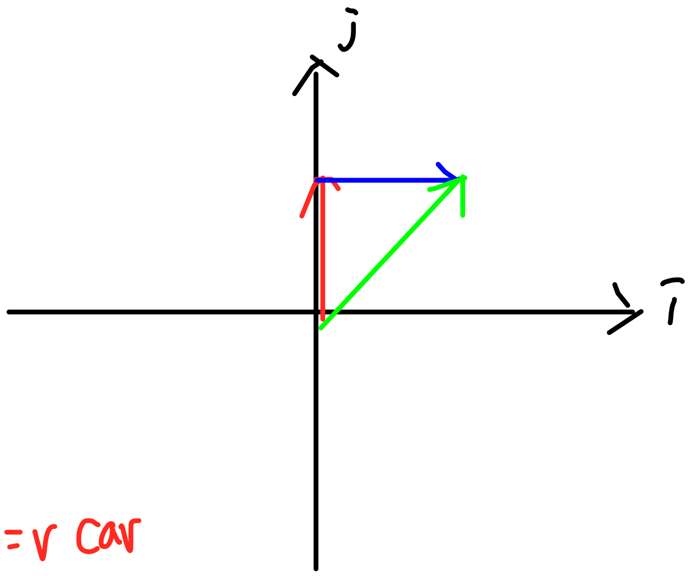
so $a = \lambda$ and $b = 50$.
When $\mathbf{v}_{\text{car}} = 100\mathbf{j}$, the wind appears to come from the northwest, so $\mathbf{v}_{\text{wind rel car}} = \mu(\mathbf{i}\text{-}\mathbf{j})$. Thus

it is i-j

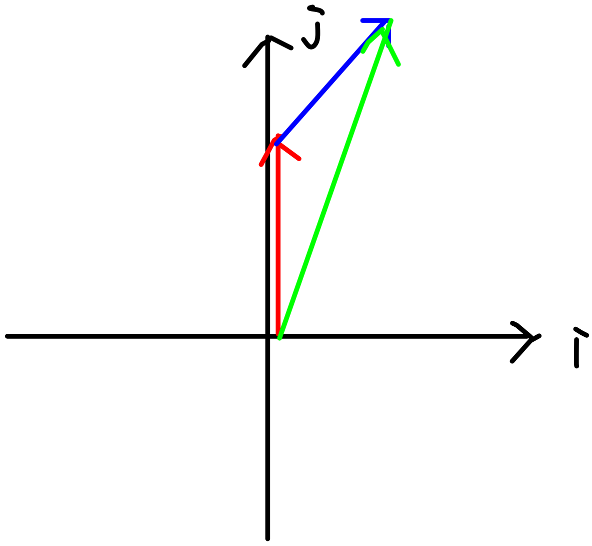
$$a\mathbf{i} + b\mathbf{j} = \mu(\mathbf{i} - \mathbf{j}) + 100\mathbf{j},$$

so $a = \mu$ and $b = 100 - \mu$.
Hence $50 = 100 - \mu$, so $\mu = 50$. Thus $a = b = 50$. The wind is from the southwest at $50\sqrt{2}$ km/h.

$V_{car} = 50j$



$V_{car} = 100j$



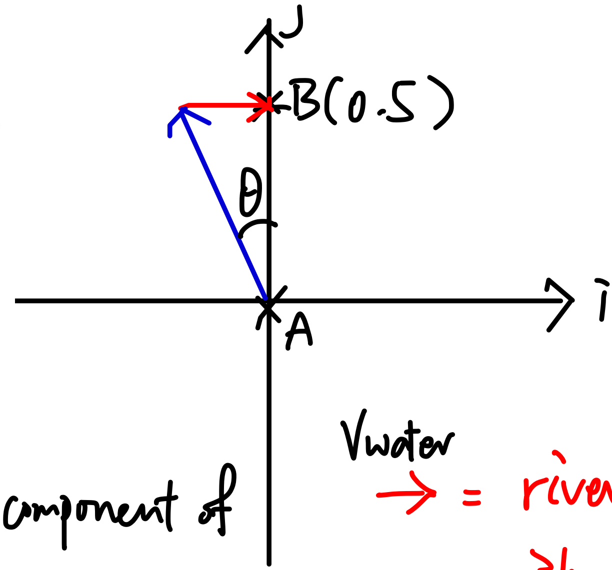
→ = v car

→ = Vwind relative to car

→ = Vwind

10. A straight river 500 m wide flows due east at a constant speed of 3 km/h. If you can row your boat at a speed of 5 km/h in still water, in what direction should you head if you wish to row from point A on the south shore to point B on the north shore directly north of A? How long will the trip take?

let $\vec{v} = ai + bj$ = boat direction



10. Let the x-axis point east and the y-axis north. The velocity of the water is

$$\mathbf{v}_{\text{water}} = 3\mathbf{i}.$$

If you row through the water with speed 5 in the direction making angle θ west of north, then your velocity relative to the water will be

$$\mathbf{v}_{\text{boat rel water}} = -5\sin\theta\mathbf{i} + 5\cos\theta\mathbf{j}.$$

Therefore, your velocity relative to the land will be

$$\begin{aligned}\mathbf{v}_{\text{boat rel land}} &= \mathbf{v}_{\text{boat rel water}} + \mathbf{v}_{\text{water}} \\ &= (3 - 5\sin\theta)\mathbf{i} + 5\cos\theta\mathbf{j}.\end{aligned}$$

To make progress in the direction $\mathbf{j}$, choose θ so that $3 = 5\sin\theta$. Thus $\theta = \sin^{-1}(3/5) \approx 36.87°$. In this case, your actual speed relative to the land will be

← component of $i = 0$

$$5\cos\theta = \frac{4}{5} \times 5 = 4 \text{ km/h}.$$

To row from A to B, head in the direction 36.87° west of north. The 1/2 km crossing will take $(1/2)/4 = 1/8$ of an hour, or about $7\frac{1}{2}$ minutes.

$\vec{V_{water}}$ = river flow 3km/h

$\vec{V_{boat}}$ = $\vec{v}$, 5 km/h

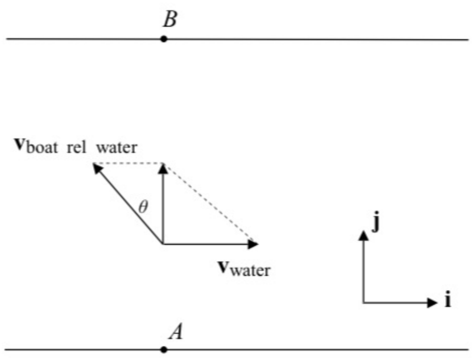


Fig. 10.2.10

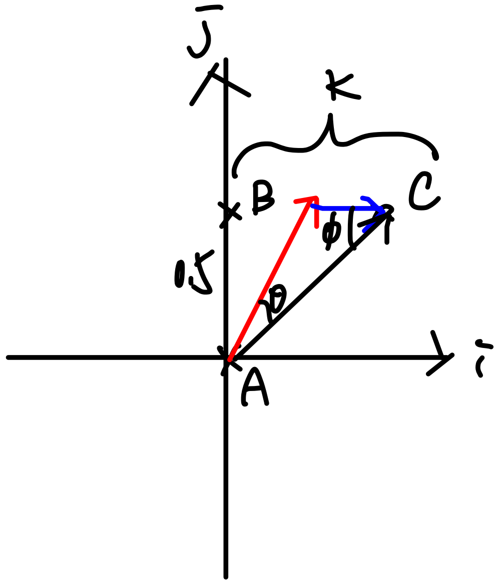
一定都用 velocity vector 計

Made with Goodnotes

11. In what direction should you head to cross the river in Exercise 10 if you can only row at 2 km/h, and you wish to row from A to point C on the north shore, k km downstream from B? For what values of k is the trip not possible?

$V_{river} = 3\mathbf{i}$

$\angle BCA = \tan^{-1}\left(\frac{0.5}{k}\right)$



$V_{black} = k\mathbf{i} + \frac{1}{2}\mathbf{j}$

11. We use the notations of the solution to Exercise 4. You now want to make progress in the direction $k\mathbf{i} + \frac{1}{2}\mathbf{j}$, that is, in the direction making angle

$$\phi = \tan^{-1}\frac{1}{2k}$$

with vector $\mathbf{i}$. Head at angle θ upstream of this direction. Since your rowing speed is 2, the triangle with angles θ and ϕ has sides 2 and 3 as shown in the figure. By the Sine Law, $\frac{3}{\sin\theta} = \frac{2}{\sin\phi}$, so

$$\sin\theta = \frac{3}{2}\sin\phi = \frac{3}{2}\,\frac{1}{2\sqrt{k^2+\frac{1}{4}}} = \frac{3}{2\sqrt{4k^2+1}}.$$

This is only possible if $\frac{3}{2\sqrt{4k^2+1}} \le 1$, that is, if $k \ge \frac{\sqrt{5}}{4}$.

Head in the direction $\theta = \sin^{-1}\frac{3}{2\sqrt{4k^2+1}}$ upstream of the direction of AC, as shown in the figure. The trip is not possible if $k < \sqrt{5}/4$.

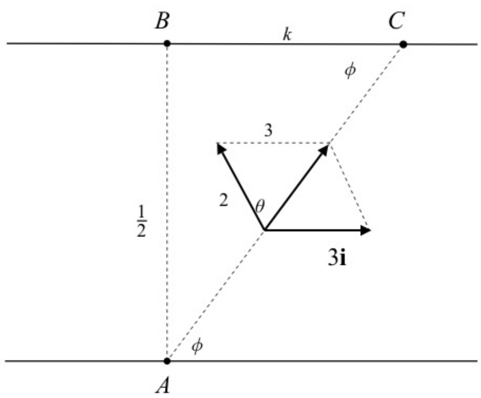


Fig. 10.2.11