(a) Assume **a**, **b** and **c** are three dimensional vectors and if

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b} + \beta \mathbf{c}.$$

Use suffix notation to find λ, μ and β in terms of the vectors **a**, **b** and **c**. Can you say something about the direction of the vector $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

(b) (i) Find the distance (in terms of $\mathbf{n}$, $\mathbf{r}_0$ and $\mathbf{r}_1$ only) from the point $\mathbf{r}_1$ to the plane $(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$.

(ii) Use (i) or otherwise, find the distance d between the two parallel planes determined by the equations $Ax + By + Cz = D_1$ and $Ax + By + Cz = D_2$.

(iii) Use (ii) or otherwise, find equations for the planes that are parallel to $x + 3y - 5z = 2$ and lie three units from it.

b) i).

$$\frac{r_1 \cdot n - r_0 \cdot n}{|n|}$$

ii). $\left(\frac{D_1}{A}, 0, 0\right)$ is on the plane 1.

$$\frac{\left(\frac{D_1}{A}, 0, 0\right) \cdot (A, B, C) - \left(\frac{D_2}{A}, 0, 0\right) \cdot (A, B, C)}{\sqrt{A^2 + B^2 + C^2}}$$

$$= \left| \frac{D_1 - D_2}{\sqrt{A^2 + B^2 + C^2}} \right|$$

iii). $D_1 = 2$,

$$\frac{2 - D_2}{\sqrt{1 + 9 + 25}} = 3$$

$$2 - D_2 = 3\sqrt{35}$$

$D_2 = 3\sqrt{35} - 2$ or $D_2 = 2 - 3\sqrt{35}$

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Problem 2 (20 points) Your Score:

Let $f(x,y) = \begin{cases} xy\left(\dfrac{x^2-y^2}{x^2+y^2}\right) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$

(i) Find $f_x(x,y)$ and $f_y(x,y)$ for $(x,y) \neq (0,0)$.

(ii) Find the partial derivatives $f_x(0,y)$ and $f_y(x,0)$.

(iii) Find the values of $f_{xy}(0,0)$ and $f_{yx}(0,0)$. Are the mixed partials equal (i.e. $f_{xy}(0,0) = f_{yx}(0,0)$)? Why?

I). $f(x,y) = \dfrac{x^3y}{x^2+y^2} - \dfrac{xy^3}{x^2+y^2}$

$$f_x : \frac{(x^2+y^2)(3x^2y) - x^3y(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2)y^3 - xy^3(2x)}{(x^2+y^2)^2}$$

$$= \frac{3x^4y + 3x^2y^3 - 2x^4y - x^2y^3 - y^5 + 2x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

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$$f(x,y) = \frac{x^3y}{(x^2+y^2)} - \frac{xy^3}{(x^2+y^2)}$$

$$f_y = \frac{(x^2+y^2)(x^3) - (x^3y)(2y)}{(x^2+y^2)^2} - \frac{(x^2+y^2)(3xy^2) - xy^3(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^5 + x^3y^2 - 2x^3y^2 - 3x^3y^2 - 3xy^4 + 2xy^4}{(x^2+y^2)^2}$$

$$= \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

Problem 2 (20 points) Your Score:

Let $f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$

(i) Find $f_x(x,y)$ and $f_y(x,y)$ for $(x,y) \neq (0,0)$.

(ii) Find the partial derivatives $f_x(0,y)$ and $f_y(x,0)$.

(iii) Find the values of $f_{xy}(0,0)$ and $f_{yx}(0,0)$. Are the mixed partials equal (i.e. $f_{xy}(0,0) = f_{yx}(0,0)$)? Why?

No, not continuous.

(ii). $f(x,y) = \frac{x^3y}{x^2+y^2} - \frac{xy^3}{x^2+y^2}$

$f_x(0,y) = \frac{-y^5}{y^4} = -y.$

$f_y(x,0) = x.$

(iii). $f_x = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$

$f_y = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$

$f_{xy} = \lim_{\Delta y \to 0} \frac{\frac{-\Delta y^5}{\Delta y^4} - f_x(0,0)}{\Delta y} = -1$

$f_{yx} = \lim_{\Delta x \to 0} \frac{\frac{\Delta x^5}{\Delta x^4}}{\Delta x} = 1$

For $f_x(0,0)$, $\lim_{\Delta x \to 0} \dfrac{\frac{0}{\Delta x^4}}{\Delta x} = 0.$

Problem 3 (20 points) Your Score:

(i) Let $z = f(x, y) = \big|\, |x| - |y| \,\big| - |x| - |y|$, use the fundamental theorem of partial differentiation to find $f_x(0,0)$ and $f_y(0,0)$.

(ii) Is the function $f(x, y)$ in (i) differentiable at $(0,0)$. Why?

Hint: You may use the following theorem to do part (ii), the function $f(x,y)$ is differentiable at the point (a, b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a,b) - hf_x(a,b) - kf_y(a,b)}{\sqrt{h^2+k^2}} = 0.$$

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{|\Delta x| - |\Delta x|}{\Delta x} = 0$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{|-|\Delta y|| - |\Delta y|}{\Delta y} = 0$$

(ii) : ??

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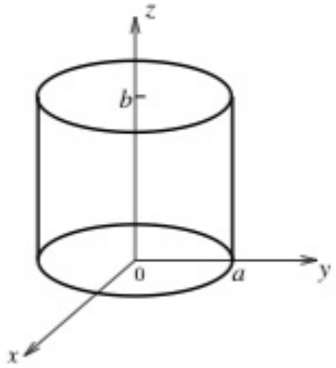
Problem 4 (20 points) Your Score:

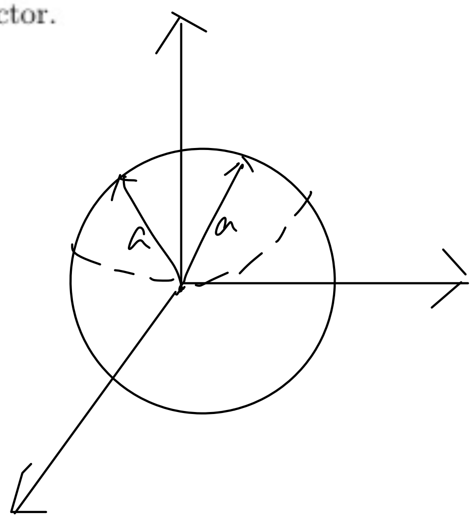
(a) The solid cylinder is positioned such that the center of the bottom disk is at the origin and the z-axis is the axis of the cylinder as shown.

(i) Describe this solid, using cylindrical coordinates.

(ii) Describe this solid, using spherical coordinates.



(b) Show that if a differentiable path lies on a sphere centered at the origin, then its position vector is always perpendicular to its velocity vector.



b). Let. radius $= a$

$|r(t)| = a$

$r(t) \cdot r(t) = a^2$

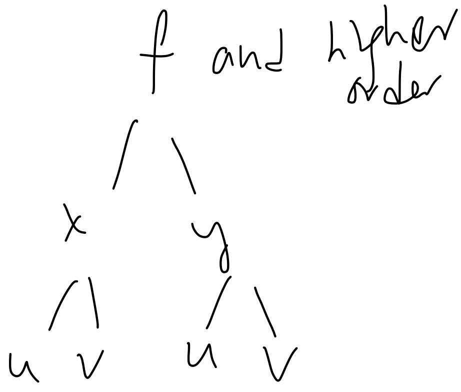
$2 r'(t) \cdot r(t) = 0$

$r(t) \cdot r'(t) = 0$

Problem 5 (20 points) Your Score:

(a) Let $z = f(x, y)$, where $x(u, v) = u + v$, $y(u, v) = u - v$ and f is a differentiable function. Show that $\dfrac{\partial^2 z}{\partial v \partial u} = a\dfrac{\partial^2 z}{\partial x^2} + b\dfrac{\partial^2 z}{\partial x \partial y} + c\dfrac{\partial^2 z}{\partial y^2}$. Find a, b and c.

(b) Suppose that $z = x^2 + y^3$, where $x = 2st$ and y is a function of s and t. Suppose further that when $(s, t) = (2, 1)$, $\dfrac{\partial y}{\partial t} = 0$. Determine $\left.\dfrac{\partial z}{\partial t}\right|_{(2,1)}$.



$\sqrt{}$ a). $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial u}$

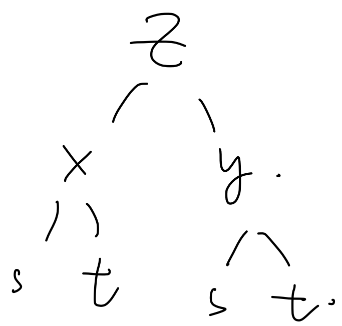
$z_u = z_x + z_y$

$z_{uv} = z_{xx} x_v + z_{xy} y_v + z_{yx} x_v + z_{yy} y_v$

$= z_{xx} + z_{xy}(-1) + z_{yx}(1) + z_{yy}(-1)$

$= z_{xx} - z_{yy}$

$a = 1, \ b = 0, \ c = -1$



b). $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$

$\dfrac{\partial z}{\partial t} = (2x)(2s) + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$

when $(s, t) = (2, 1)$,

$\left.\dfrac{\partial z}{\partial t}\right|_{(s,t)=(2,1)} = 2(4) \times 2(2) + 0 = 12$ //