Last Time

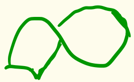
Max / Min Problems for $z = f(x, y)$ ← nice functions! ($f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ exist + cont.)

① $\nabla f = \vec{0}$ (critical points)

$\langle f_x, f_y \rangle$

② $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$

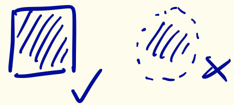
$D > 0$	$f_{xx} > 0$	local min.
$D > 0$	$f_{xx} < 0$	local max.
$D < 0$		saddle point
$D = 0$		no conclusion.



$\Longleftrightarrow$ ☆ { not local min nor local max.

Extremal Value Thm

: $f(x, y, z)$ have an absolute max & absolute min on a bounded closed set.



③ Critical Points restricted to boundary

④ Compare the values : largest = absolute max
smallest = absolute min.

$D=0$: No conclusion ?!

$f(x)=x^3$

$f'(x)=0 \Rightarrow x=0$

$f''(0)=0$

Ex $f(x,y)=x^3-3xy^2$ $\quad (=r^2\cos 3\theta)$

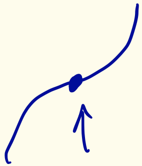
$\nabla f = \langle 3x^2-3y^2, -6xy \rangle$

$\Rightarrow x=y=0$.

$$D=\begin{vmatrix} 6x & -6y \\ -6y & -6x \end{vmatrix} = 0$$

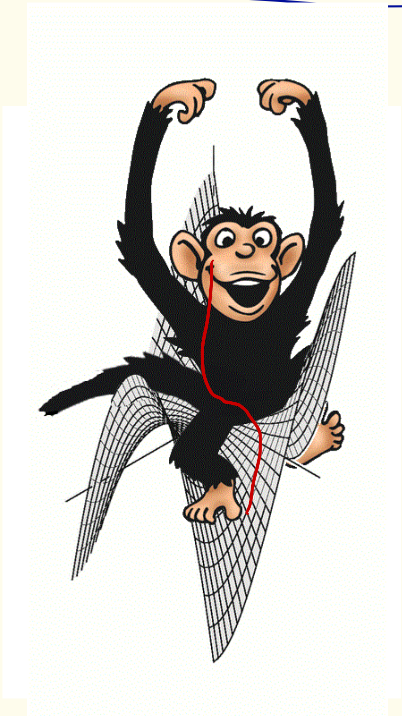
Restrict to $y=0$

$f(x,0)=x^3$:



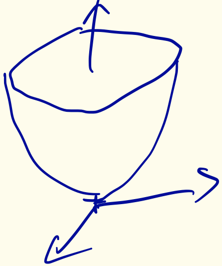
$(0,0)$ is not local max/min

$\Leftrightarrow$ $(0,0)$ is saddle point!



Monkey Saddle.

Ex $f(x,y) = x^4 + y^4$



$\nabla f = (4x^3, 4y^3) = \vec{0}$

$\Rightarrow x = y = 0$

$D = \begin{vmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{vmatrix} = 0$ at $(0,0)$

(absolute)
(local)

$(0,0)$ is min because $f(x,y) \geq 0$.

$f(0,0) = 0$.

smallest possible!

Ex $f(x,y) = 2x^4 - 3x^2y + y^2 = (y-2x^2)(y-x^2)$

$$\nabla f = \langle 8x^3 - 6xy,\ -3x^2+2y \rangle = \langle 0,0 \rangle$$

$$\iff x = y = 0$$

$$D = \begin{vmatrix} 24x^2 - 6y & -6x \\ -6x & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} = 0 \text{ at } (0,0).$$

$x=0$: y^2

$y=0$: $2x^4$

$y=mx$: $2x^4 - 3mx^3 + m^2x^2$

$\frac{d}{dx}$: $8x^3 - 9mx^2 + 2m^2x$

$\frac{d^2}{dx^2}$: $24x^2 - 18mx + 2m^2$

$= 2m^2$ at $x=0$

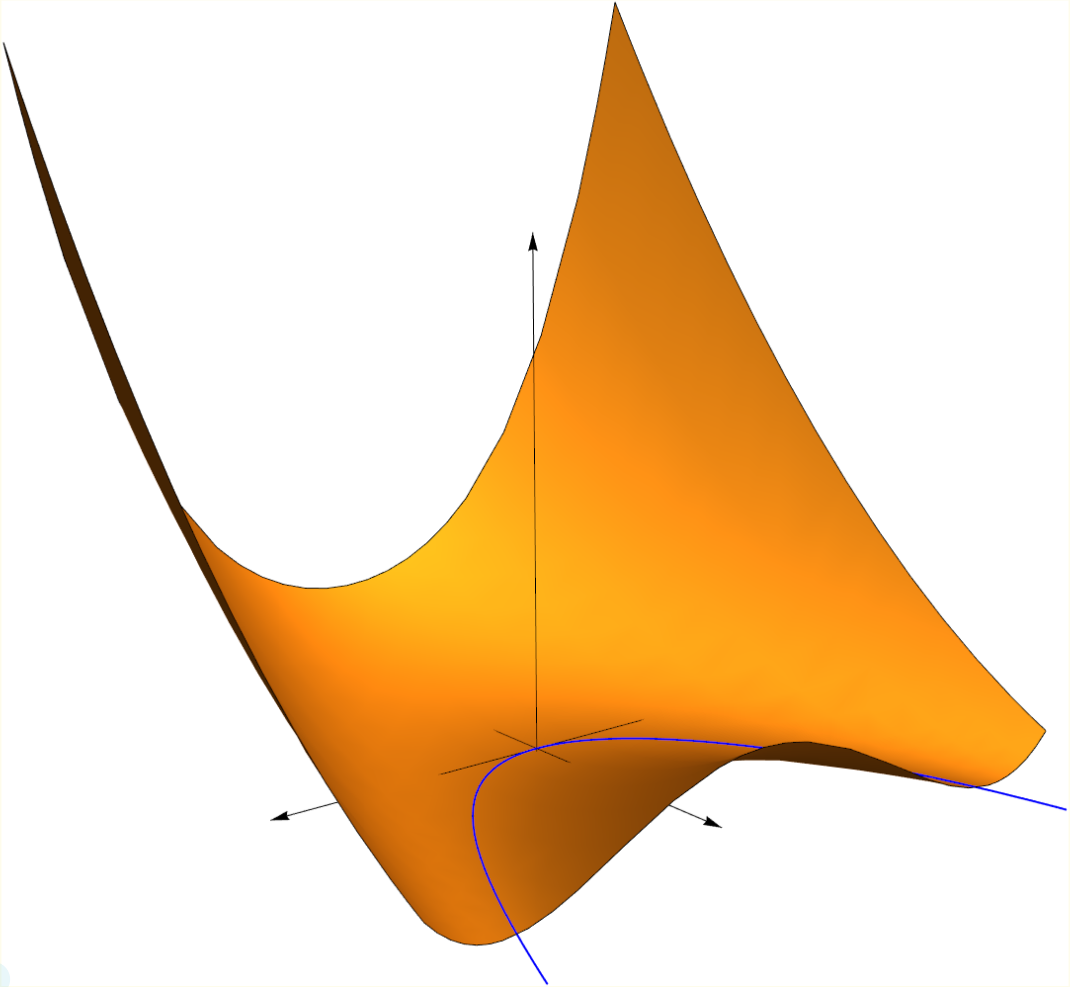
> 0

Along $y = \frac{3}{2}x^2$

$$f = \left(-\frac{1}{2}x^2\right)\left(\frac{1}{2}x^2\right) = -\frac{1}{4}x^2.$$

No matter how close to $(0,0)$, you can find $f(\varepsilon, \frac{3}{2}\varepsilon^2) < 0$

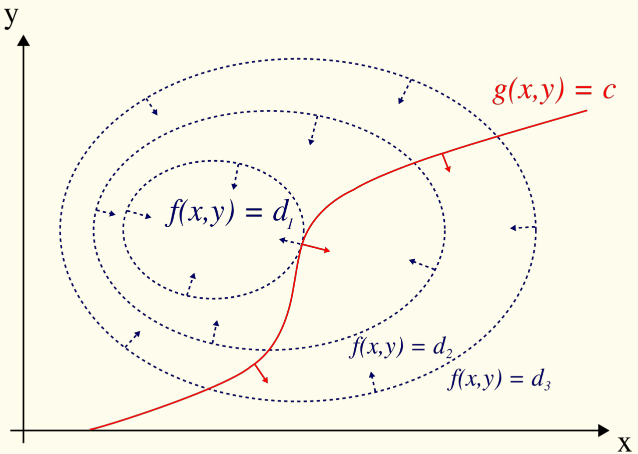
Saddle Point!



<u>Goal</u> Optimization Problem with constraints.

Maximize $f(x,y,z)$ subject to the condition $g(x,y,z) = k$

$f(x_1, \dots, x_n)$ $g(x_1, \dots, x_n) \leq k$



<u>Ex</u> Max volume of a box subject to given surface area $= 12\,cm^2$

<u>Ex</u> Max. utility function ☺ subject to budget $ constraint.

$3 apple x
$4 banana y
$5 chocolate z

$x^2 + e^{yz^2}$ ⇙ utility function.

$: $3x + 4y + 5z = 100$
↑
constraint

If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ satisfying the constraint

$$g(x,y,z) = k.$$

chain rule $\Longleftrightarrow$ $\frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} + \frac{\partial g}{\partial z}\frac{dz}{dt} = 0$ or $\star$ $\nabla g \cdot \vec{r}'(t) = 0.$

normal to tangent plane

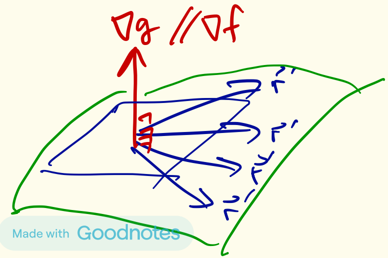
lies on tangent plane

Optimize $f(x,y,z)$ along $\vec{r}(t)$

$$\Longrightarrow F(t) = f(x(t), y(t), z(t))$$

solve $F'(t) = \nabla f \cdot \vec{r}'(t) = 0$ $\star\star$

$\star$ & $\star\star$ are true for all possible curve $\vec{r}(t)$ satisfying g.



$$\Longrightarrow \nabla f \parallel \nabla g$$

Lagrange multiplier.

$$\nabla f = \lambda \nabla g \quad \text{for some } \lambda$$

Method of Lagrange Multiplier

assume $\nabla g \neq 0$

To find max / min of $f(x,y,z)$ subject to constraint $g(x,y,z)=0$

① Solve (x,y,z,λ) such that

$$(*)\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y,z) = 0 \end{cases}$$

4 eq. in 4 variables!

② If extremal value exists, and g describes a bounded, closed set,

then $\begin{cases} \max = \text{largest value} \\ \min = \text{smallest value} \end{cases}$ at the solutions.

$$\mathcal{L}(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y,z)$$

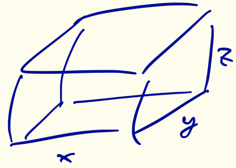
Lagragian

$$(*) \Longleftrightarrow \nabla \mathcal{L} = 0$$

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \lambda} \right\rangle$$

Ex Box without lid, with surface area $=12$. Max Volume.

$V = xyz$ subject to $xy + 2xz + 2yz = 12$



$f(x,y,z) = xyz$

$g(x,y,z) = xy + 2xz + 2yz - 12 = 0$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \iff \begin{cases} yz = \lambda(y + 2z) \\ xz = \lambda(x + 2z) \\ xy = \lambda(2x + 2y) \\ xy + 2xz + 2yz = 12. \end{cases}$$

$$\Longrightarrow \begin{cases} xyz = \lambda(xy + 2xz) \\ xyz = \lambda(xy + 2yz) \\ xyz = \lambda(2xz + 2yz) \end{cases}$$

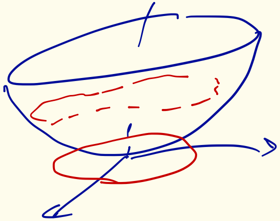
$\lambda(xy + 2xz) = \lambda(xy + 2yz)$

$2\lambda xz = 2\lambda yz \Longrightarrow x = y$

$\Longrightarrow y = 2z$

$4z^2 + 4z^2 + 4z^2 = 12 \Longrightarrow z = 1, \; x = y = 2.$

Ex Maximize x^2+2y^2 subject to $x^2+y^2=1$.



$f(x,y) = x^2+2y^2$

$g(x,y) = x^2+y^2-1 = 0$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Rightarrow \begin{cases} 2x = \lambda 2x \\ 4y = \lambda 2y \\ x^2+y^2=1 \end{cases}$$

$2x = \lambda 2x \iff x=0$ or $\lambda = 1$

$x=0$	$\lambda=1$
$y=\pm 1$	$4y=2y$
$4y = 2\lambda y$	$\Rightarrow y=0$
$\lambda = \pm 2$	$x=\pm 1$
$(0,1)$	$(1,0)$
$(0,-1)$	$(-1,0)$

max

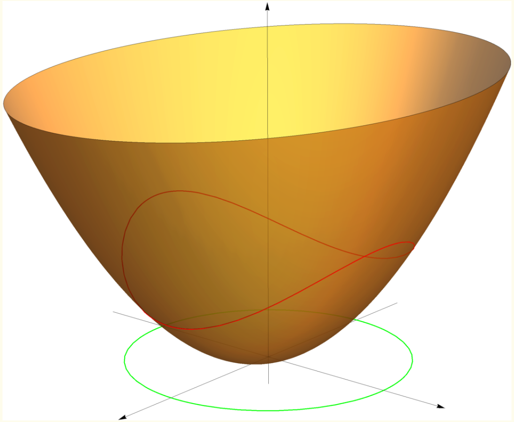
$f(0,1) = 2$

$f(0,-1) = 2$

min

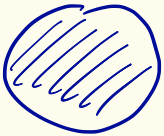
$f(1,0) = 1$

$f(-1,0) = 1$



Made with Goodnotes

<u>Ex</u> x^2+2y^2 subject to $x^2+y^2 \le 1$.



$$\nabla f = \langle 2x, 4y \rangle = \vec{0}$$

$$\Rightarrow x = y = 0.$$

$$f(0,0) = 0.$$

absolute minimum.

Ex x^2y subject to $x^2+y^2=3$.

$$\nabla f = \lambda \nabla g \iff \begin{cases} 2xy = \lambda 2x \\ x^2 = \lambda 2y \\ x^2+y^2=3. \end{cases}$$

$2x(y-\lambda)=0$

$\Rightarrow x=0$ or $y=\lambda$

$\downarrow$

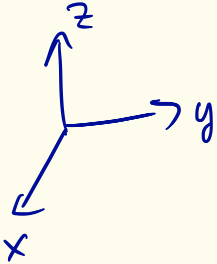
$y=\pm\sqrt{3}$

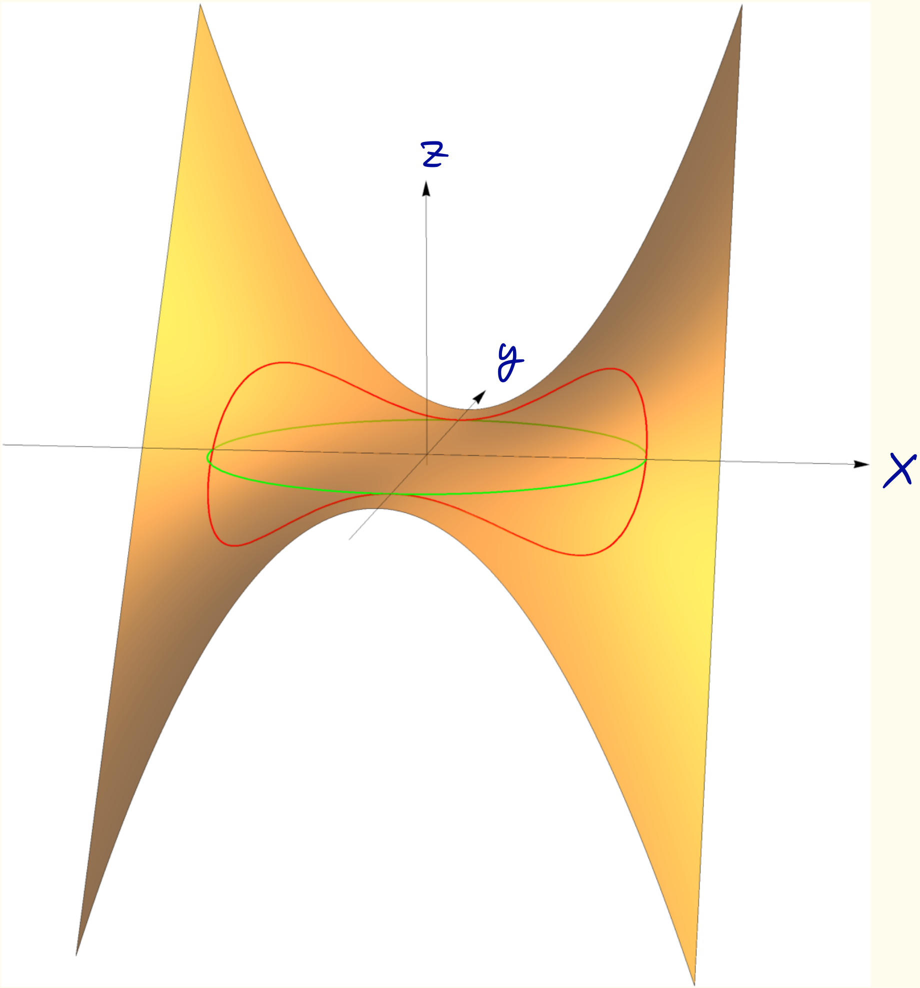
$\begin{cases} x^2=2y^2 \\ x^2+y^2=3 \end{cases} \Rightarrow \begin{array}{l} y=\pm 1 \\ x=\pm\sqrt{2} \end{array}$

$f(0,\sqrt{3})=0$ $\quad f(1,\sqrt{2})=\sqrt{2}$ $\quad f(1,-\sqrt{2})=-\sqrt{2}$

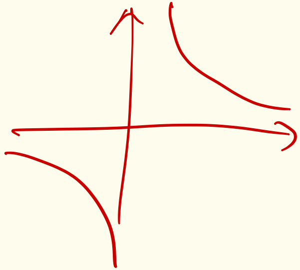
$f(0,-\sqrt{3})=0$ $\quad f(-1,\sqrt{2})=\sqrt{2}$ $\quad f(-1,-\sqrt{2})=-\sqrt{2}$

max $\quad$ min


z
y
x


z
y
X

Ex x^2+y^2 subject to $xy=1$.



$x^2+\frac{1}{x^2} \Longrightarrow$ max / min.
(NO) $(x=\pm1,\ f=2)$

$$\nabla f=\lambda\nabla g \iff \begin{cases} 2x=\lambda y \\ 2y=\lambda x \\ xy=1 \end{cases} \iff \begin{cases} x=y=\pm1. \\ \lambda=2 \end{cases}$$

$$f(x,y)=f(\pm1,\pm1)=2.$$

is min.

but cannot use Lagrange multiplier method to conclude