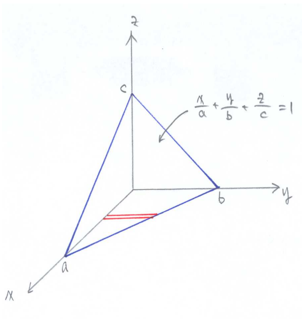
Exercise 14.5

Qu. 4

$$
\begin{aligned}
\iiint\limits_R x\,dV &= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} x\,dz \\
&= c\int_0^a \int_0^{b(1-\frac{x}{a})} x(1-\frac{x}{a}-\frac{y}{b})\,dy \\
&= c\int_0^a x\left[b(1-\frac{x}{a})^2 - \frac{b^2}{2b}(1-\frac{x}{a})^2\right]dx \\
&= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 x\,dx \\
&= \frac{bc}{2}\int_0^a \left(1-2\frac{x}{a}+\frac{x^2}{a^2}\right) x\,dx \\
&= \frac{1}{24}a^2bc.
\end{aligned}
$$



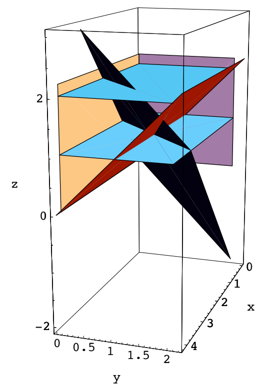
Homework 7 **MATH2023**

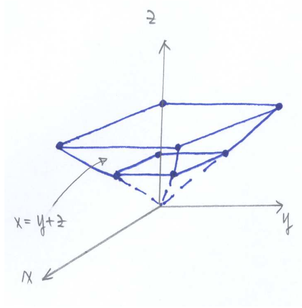
Qu. 11 Note that the region is

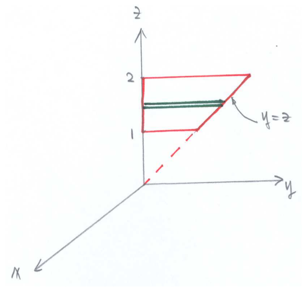
from $x = 0$ to $x = y + z$,

from $y = 0$ to $y = z$,

from $z = 1$ to $z = 2$.





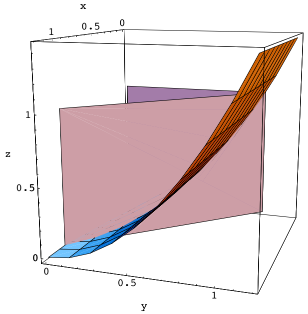


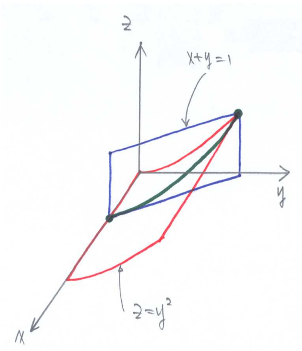
Note also that for the plane $z = x - y$, when

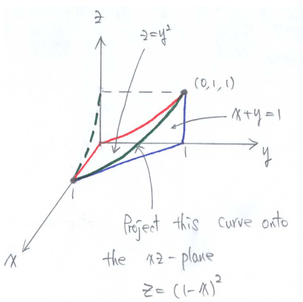
$$
\begin{aligned}
x = y = 0, &\quad z = 0 \\
y = 0, &\quad z = x \\
\text{and} \quad x = 0, &\quad z = -y
\end{aligned}
$$

$$
\begin{aligned}
\therefore \iiint\limits_R \frac{1}{(x+y+z)^3}\,dV &= \int_1^2\int_0^z\int_0^{y+z} \frac{1}{(x+y+z)^3}\,dx\,dy\,dz \\
&= \int_1^2\int_0^z \left.\frac{-1}{2(x+y+z)^2}\right|_0^{y+z} dy\,dz \\
&= \frac{3}{8}\int_1^2\int_0^z \frac{1}{(y+z)^2}\,dy\,dz \\
&= \frac{3}{8}\int_1^2 \left.\frac{-1}{(y+z)}\right|_0^z dz \\
&= \frac{3}{16}\int_1^2 \frac{1}{z}\,dz \\
&= \frac{3}{16}\ln 2.
\end{aligned}
$$

Qu. 16







$$\iiint_R f(x,y,z)\,dV = \int_0^1\int_0^{1-x}\int_0^{y^2} f(x,y,z)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1\int_{\sqrt{z}}^1\int_0^{1-y} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^{(1-x)^2}\int_{\sqrt{z}}^{1-x} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1\int_0^{1-\sqrt{z}}\int_{\sqrt{z}}^{1-x} f(x,y,z)\,dy\,dx\,dz.$$

Qu. 19 Note that the integration region is

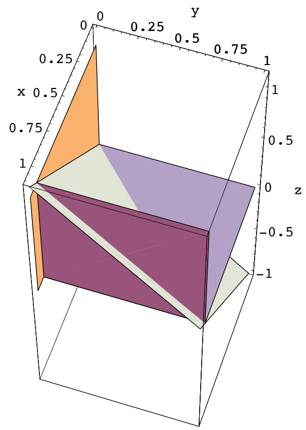
from $y=0$ to $y=x-z$,

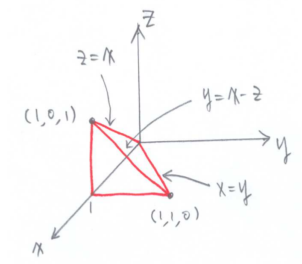
from $x=z$ to $x=1$,

from $z=0$ to $z=1$.

Note also that for the plane $y=x-z$, we have

$$\begin{aligned} x=z=0, &\quad y=0\\ x=0, &\quad y=z\\ y=0, &\quad z=x\\ z=0, &\quad y=x \end{aligned}$$





$$\therefore \int_0^1\int_z^1\int_0^{x-z} f(x,y,z)\,dy\,dx\,dz = \iiint_R f(x,y,z)\,dV \quad (R \text{ is the tetrahedron in the figure})$$

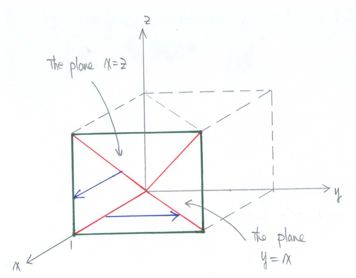
$$= \int_0^1\int_0^x\int_0^{x-y} f(x,y,z)\,dz\,dy\,dx.$$

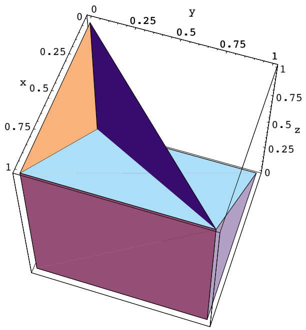
Qu. 27 Note that the integration region is

from $y=0$ to $y=x$,

from $x=z$ to $x=1$,

from $z=0$ to $z=1$.





$$\int_0^1\int_z^1\int_0^x e^{x^3}\,dy\,dx\,dz$$

$$=\iiint_R e^{x^3}\,dV \quad (R \text{ is the pyramid in the figure with vertex } (0,0,0), \text{ rectangle base } x=1)$$

$$=\int_0^1\int_0^x\int_0^x e^{x^3}\,dz\,dy\,dx$$

$$=\int_0^1 x^2e^{x^3}\,dx$$

$$=\frac{1}{3}(e-1).$$

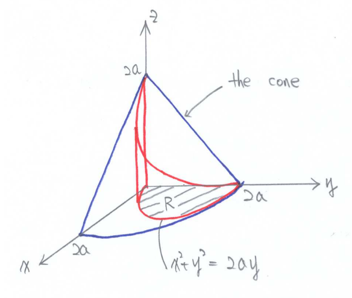
Exercise 14.6

Qu. 19 One half of the required volume V lies in the first octant, inside the cylinder

$$x^2+y^2=2ay \quad\Rightarrow\quad x^2+(y-a)^2=a^2$$

In polar,

$$r=2a\,\sin\theta$$



$$V=2\iint_R\int_0^{2a-\sqrt{x^2+y^2}}dz\,dA$$

$$=2\iint_R(2a-\sqrt{x^2+y^2})\,dA$$

$$=2\int_0^{\pi/2}\int_0^{2a\sin\theta}(2a-r)r\,dr\,d\theta \qquad \text{(in polar)}$$

$$=2a\int_0^{\pi/2}4a^2\sin^2\theta\,d\theta-\frac{2}{3}\int_0^{\pi/2}8a^3\sin^3\theta\,d\theta$$

$$=4a^3\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta+\frac{16}{3}a^3\int_0^{\pi/2}(1-\cos^2\theta)\,d(\cos\theta)$$

$$=2\pi a^3-\frac{32}{9}a^3.$$

Qu. 25

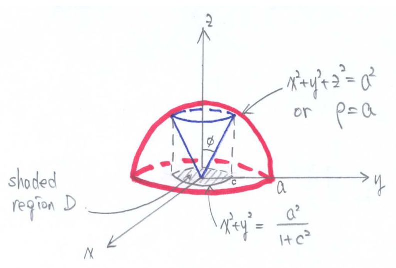
$$\iiint_B(x^2+y^2)\,dV=\int_0^{2\pi}\int_0^{\pi}\int_0^{a}(\rho^2\sin^2\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=2\pi\int_0^{\pi}\sin^3\phi\,d\phi\times\int_0^a\rho^4\,d\rho$$

$$=2\pi\left(\frac{4}{3}\right)\frac{a^5}{5}$$

$$=\frac{8}{15}\pi a^5.$$

Qu. 30 In spherical coord



$$\iiint_R (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^{\tan^{-1}(\frac{1}{c})}\int_0^a (\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{2\pi a^5}{5}\int_0^{\tan^{-1}(\frac{1}{c})} \sin\phi(1-\cos^2\phi)\,d\phi$$

$$= \frac{2\pi a^5}{5}\int_{\frac{c}{\sqrt{c^2+1}}}^1 (1-u^2)\,du, \qquad \text{let} \quad u=\cos\phi, \quad \text{then} \quad du=-\sin\phi\,d\phi$$

$$= \frac{2\pi a^5}{5}\left(u-\frac{u^3}{3}\right)\Big|_{\frac{c}{\sqrt{c^2+1}}}^1$$

$$= \frac{2\pi a^5}{5}\left[\frac{2}{3}-\frac{c}{\sqrt{c^2+1}}+\frac{c^3}{3(c^2+1)^{3/2}}\right].$$

In cylindrical coord:

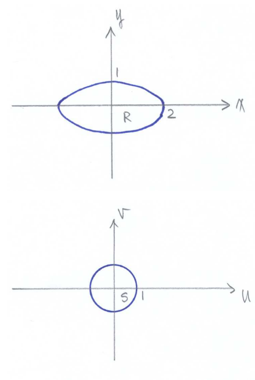
$$\iiint_R (x^2+y^2)\,dV = \iiint_R\int_{c\sqrt{x^2+y^2}}^{\sqrt{a^2-x^2-y^2}} dz\,dA$$

$$= \iint_R (\sqrt{a^2-x^2-y^2}-c\sqrt{x^2+y^2})\,dA$$

$$= \int_0^{2\pi}\int_0^{\frac{a}{\sqrt{1+c^2}}}\left[\sqrt{a^2-r^2}-cr\right] r\,dr\,d\theta = \text{Ans.}$$

Exercise 14.7

Qu. 2



$$z=\frac{3x-4y}{5}, \quad \frac{\partial z}{\partial x}=\frac{3}{5}, \quad \frac{\partial z}{\partial y}=\frac{4}{5}$$

$$dS = \sqrt{1+(z_x)^2+(z_y)^2}\,dA$$

$$= \sqrt{1+\frac{3^2+4^2}{5^2}}\,dA = \sqrt{2}\,dA.$$

$$S = \iint_R \sqrt{2}\,dA$$

Let $x=2u, \quad y=v, \quad$ then $\quad u^2+v^2=1$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 1\end{vmatrix} = 2.$$

$$\therefore \quad \sqrt{2}\iint_R dA = \sqrt{2}\iint_S 2\,dA, \quad \text{where } S \text{ is a circle with radius 1 in } uv\text{-plane}$$

$$= 2\sqrt{2}\pi.$$

Qu. 6

$$z=1-x^2-y^2, \quad \frac{\partial z}{\partial x}=-2x, \quad \frac{\partial z}{\partial y}=-2y$$

$$dS = \sqrt{1+(z_x)^2+(z_y)^2}\,dA = \sqrt{1+4x^2+4y^2}\,dA$$

$$\therefore S = \iint_R \sqrt{1+4(x^2+y^2)}\,dA$$

$$= \int_0^{\pi/2}\int_0^1 \sqrt{1+4r^2}\,r\,dr\,d\theta$$

$$= \frac{\pi}{2}\frac{1}{8}\frac{2(1+4r^2)^{3/2}}{3}\Big|_0^1$$

$$= \frac{\pi(5\sqrt{5}-1)}{24}.$$

Qu. 10 The area elements on $z = 2xy$ and $z = x^2 + y^2$ are

$$dS_1 = \sqrt{1 + (2y)^2 + (2x)^2}\, dA = \sqrt{1 + 4x^2 + 4y^2}\, dA$$
$$dS_2 = \sqrt{1 + (2x)^2 + (2y)^2}\, dA = \sqrt{1 + 4x^2 + 4y^2}\, dA$$

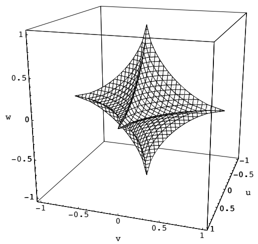
Since these elements are equal, the area of the parts of both surfaces defined over any region of the xy-plane will be equal.

Qu. Let $x = u^3$, $y = v^3$, $z = w^3$, then the region R bounded by the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ gets mapped to the ball B bounded by $u^2 + v^2 + w^2 = a^{2/3}$. Assume that $a > 0$. Since

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = 27u^2v^2w^2,$$

the volume of R is

$$V = 27 \iiint\limits_B u^2v^2w^2\, du\, dv\, dw.$$



In spherical coord,

$$dudvdw = (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho \cos\phi),$$

we have

$$\begin{aligned}
V &= 27 \int_0^{2\pi}\int_0^{\pi}\int_0^{a^{1/3}} \cos^2\theta \sin^2\theta \sin^5\phi \cos^2\phi\, \rho^8\, d\rho\, d\phi\, d\theta \\
&= 3a^3 \int_0^{2\pi} \frac{\sin^2(2\theta)}{4} \int_0^{\pi} (1 - \cos^2\phi)^2 \cos^2\phi \sin\phi\, d\phi\, d\theta \\
&= 3a^3 \int_0^{2\pi} \frac{1 - \cos(4\theta)}{8}\, d\theta \cdot \int_{-1}^{1} (1 - t^2)^2 t^2\, dt, \qquad \text{let} \quad t = \cos\phi \\
&= \frac{3a^3}{8}(2\pi)2 \int_0^1 (t^2 - 2t^4 + t^6)\, dt \\
&= \frac{4}{35}\pi a^3.
\end{aligned}$$