Prove limit does not exist:

找不同 path limit 不相同.

Prove limit exist:

$$0 \le |f(x)| \le g(x)$$

And if $\lim_{\{x,y\}\to\{a,b\}} g(x) = 0$, then $\lim_{\{x,y\}\to\{a,b\}} f(x) = 0$

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* Remark: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Made with Goodnotes

19. What condition must the constants a, b, and c satisfy to guarantee that $\lim_{(x,y)\to(0,0)} xy/(ax^2 + bxy + cy^2)$ exists? Prove your answer.

Difficult questions: Sub $y = kx$, $\dfrac{k}{ak^2 + bk + c}$

$\therefore$ only condition: $a = c = 0$, $b \neq 0$ $\leftarrow$ in this case, no limit

EXERCISES 12.2

In Exercises 1–12, evaluate the indicated limit or explain why it does not exist.

1. $\displaystyle\lim_{(x,y)\to(2,-1)} xy + x^2$

2. $\displaystyle\lim_{(x,y)\to(0,0)} \sqrt{x^2 + y^2}$

3. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{y}$

4. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{x^2 + y^2}$

5. $\displaystyle\lim_{(x,y)\to(1,\pi)} \frac{\cos(xy)}{1 - x - \cos y}$

6. $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^2 + (y-1)^2}$

7. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2 + y^2}$

8. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x - y)}{\cos(x + y)}$

9. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2 + y^2}$

10. $\displaystyle\lim_{(x,y)\to(1,2)} \frac{2x^2 - xy}{4x^2 - y^2}$

11. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{x^2 + y^4}$

12. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y^2}{2x^4 + y^4}$

13. How can the function

$$f(x, y) = \frac{x^2 + y^2 - x^3 y^3}{x^2 + y^2}, \qquad (x, y) \neq (0, 0),$$

be defined at the origin so that it becomes continuous at all points of the xy-plane?

14. How can the function

$$f(x, y) = \frac{x^3 - y^3}{x - y}, \qquad (x \neq y),$$

be defined along the line $x = y$ so that the resulting function is continuous on the whole xy-plane?

15. What is the domain of

$$f(x, y) = \frac{x - y}{x^2 - y^2}?$$

Does $f(x, y)$ have a limit as $(x, y) \to (1, 1)$? Can the domain of f be extended so that the resulting function is continuous at $(1, 1)$? Can the domain be extended so that the resulting function is continuous everywhere in the xy-plane?

16. Given a function $f(x, y)$ and a point (a, b) in its domain, define single-variable functions g and h as follows:

$$g(x) = f(x, b), \qquad h(y) = f(a, y).$$

If g is continuous at $x = a$ and h is continuous at $y = b$, does it follow that f is continuous at (a, b)? Conversely, does the continuity of f at (a, b) guarantee the continuity of g at a and the continuity of h at b? Justify your answers.

17. Let $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ be a unit vector, and let

$$f_{\mathbf{u}}(t) = f(a + tu, b + tv)$$

be the single-variable function obtained by restricting the domain of $f(x, y)$ to points of the straight line through (a, b) parallel to $\mathbf{u}$. If $f_{\mathbf{u}}(t)$ is continuous at $t = 0$ for every unit vector $\mathbf{u}$, does it follow that f is continuous at (a, b)? Conversely, does the continuity of f at (a, b) guarantee the continuity of $f_{\mathbf{u}}(t)$ at $t = 0$? Justify your answers.

18. What condition must the nonnegative integers m, n, and p satisfy to guarantee that $\lim_{(x,y)\to(0,0)} x^m y^n/(x^2 + y^2)^p$ exists? Prove your answer.

19. What condition must the constants a, b, and c satisfy to guarantee that $\lim_{(x,y)\to(0,0)} xy/(ax^2 + bxy + cy^2)$ exists? Prove your answer.

20. Can the function $f(x, y) = \dfrac{\sin x \sin^3 y}{1 - \cos(x^2 + y^2)}$ be defined at $(0, 0)$ in such a way that it becomes continuous there? If so, how?

21. Use two- and three-dimensional mathematical graphing software to examine the graph and level curves of the function $f(x, y)$ of Example 3 on the region $-1 \le x \le 1$, $-1 \le y \le 1$, $(x, y) \neq (0, 0)$. How would you describe the behaviour of the graph near $(x, y) = (0, 0)$?

22. Use two- and three-dimensional mathematical graphing software to examine the graph and level curves of the function $f(x, y)$ of Example 4 on the region $-1 \le x \le 1$, $-1 \le y \le 1$, $(x, y) \neq (0, 0)$. How would you describe the behaviour of the graph near $(x, y) = (0, 0)$?

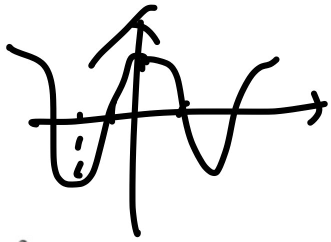
23. The graph of a single-variable function $f(x)$ that is continuous on an interval is a curve that has no *breaks* in it there and that intersects any vertical line through a point in the interval exactly once. What analogous statement can you make about the graph of a bivariate function $f(x, y)$ that is continuous on a region of the xy-plane?

24. (a) State explicitly the version of Definition 2 that applies to a function f of a single variable x.

(b) Let f be a function with domain the set of numbers $1/n$ for $n = 1, 2, 3, \ldots$ and having values given by $f(1/n) = (n - 1)/n$. According to part (a) does $\lim_{x\to 1} f(x)$ exist? What about $\lim_{x\to 0} f(x)$? Evaluate whichever of these limits does exist.

(c) Which of the two limits in (b) exist by Definition 8 in Section 1.5?

In Exercises 1–12, evaluate the indicated limit or explain why it does not exist.



1. $\lim_{(x,y)\to(2,-1)} xy + x^2$

2. $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}$

3. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{y}$

4. $\lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$

5. $\lim_{(x,y)\to(1,\pi)} \frac{\cos(xy)}{1-x-\cos y}$

6. $\lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^2+(y-1)^2}$

7. $\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2}$

8. $\lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{\cos(x+y)}$

1. $2(-1)+2^2 = 2.$ ✓

2. 0 ✓

3. Let $x = r\cos\theta$, $y = r\sin\theta$

$$\lim_{r\to 0} \frac{r^2}{r\sin\theta} = \lim_{r\to 0} \frac{r}{\sin\theta} = 0$$

✗ $x=0$, $y=x^2$

4. $$\lim_{r\to 0} \frac{r\cos\theta}{r^2} = \lim_{r\to 0} \frac{\cos\theta}{r} = \lim_{r\to 0} \cancel{\frac{\theta}{1}} = 0$$

5. $$\lim_{(x,y)\to(0,\pi)} \frac{1}{1-0-\cos\pi} = \frac{1}{2}$$ ✗ -1

$$\lim_{(x,y)\to(1,0)} = 0.$$

does not exist

6. $\displaystyle \lim_{(x,y)\to(0,1)} \frac{x^2(y-1)^2}{x^2+(y-1)^2}$

Let $x = r\cos\theta$, $y = r\sin\theta + 1$

$$\lim_{r\to 0} \frac{r^2\cos^2\theta\sin^2\theta}{r^2\cos\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} \frac{r^2\cos^2\theta\sin^2\theta}{r^2}$$

$$= 0$$

7. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2}$

Let $x = r\cos\theta$. $y = r\sin\theta$

$$\lim_{r\to 0} \frac{r^3\sin^3\theta}{r^2} = \lim_{r\to 0} r\sin^3\theta = 0$$

8. $\lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{\cos(x+y)}$ = both continuous. → 0.

9. $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$

$$\frac{-1}{x^2+y^2} \le \frac{\sin(xy)}{x^2+y^2} \le \frac{1}{x^2+y^2}$$

10. $\lim_{(x,y)\to(1,2)} \frac{2x^2 - xy}{4x^2 - y^2}$

$$\frac{x(2x-y)}{(2x+y)(2x-y)} = \frac{x}{2x+y} = \frac{1}{4}$$

11. $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2 + y^4}$

$x=y.$ $\qquad \frac{x^4}{x^2+x^4} = 1.$

$x=0.$ $\qquad \frac{y^2}{y^4} = 0.$ Does not exist.

12. $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{2x^4+y^4}$

$x=y$ $\quad \dfrac{x^4}{2x^4+x^4} = \dfrac{1}{3}.$

$x=0.$ $\quad \dfrac{y^2}{y^4} = 0.$ Does not exist.

13. How can the function

$$f(x, y) = \frac{x^2 + y^2 - x^3y^3}{x^2 + y^2}, \qquad (x, y) \neq (0, 0),$$

be defined at the origin so that it becomes continuous at all points of the xy-plane?

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2-x^3y^3}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} 1 - \frac{x^3y^3}{x^2+y^2}$$

$$\lim_{r\to 0} \frac{r^6 \sin^3\theta \cos^3\theta}{r^2} = 0.$$

$$= 1.$$

14. How can the function

$$f(x, y) = \frac{x^3 - y^3}{x - y}, \qquad (x \neq y),$$

be defined along the line $x = y$ so that the resulting function is continuous on the whole xy-plane?

$x - y = 0.$

$$\lim_{(x,y)\to(x,x)} \frac{(x-y)(x^2+xy+y^2)}{x-y}$$

$$\lim_{(x,y)\to(x,x)} x^2 + x\cancel{y} + y^2$$

$$= x^2 + \cancel{2x}\,x^2 + x^2$$

$$= \cancel{2x^2 + 2x} \quad 3x^2$$

When $x = y$, $f(x,y) = 2x^2 + 2x$

$f(x,x) = 3x^2$

15. What is the domain of

$$f(x, y) = \frac{x - y}{x^2 - y^2}?$$

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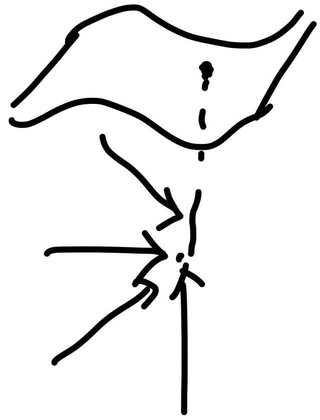
$-\frac{1}{2}$.



$$\lim_{(x,y)\to(1,1)} \frac{x-y}{x^2-y^2}$$

$$\lim_{(x,y)\to(1,1)} \frac{-1}{-1} = 1$$

no. no. no.

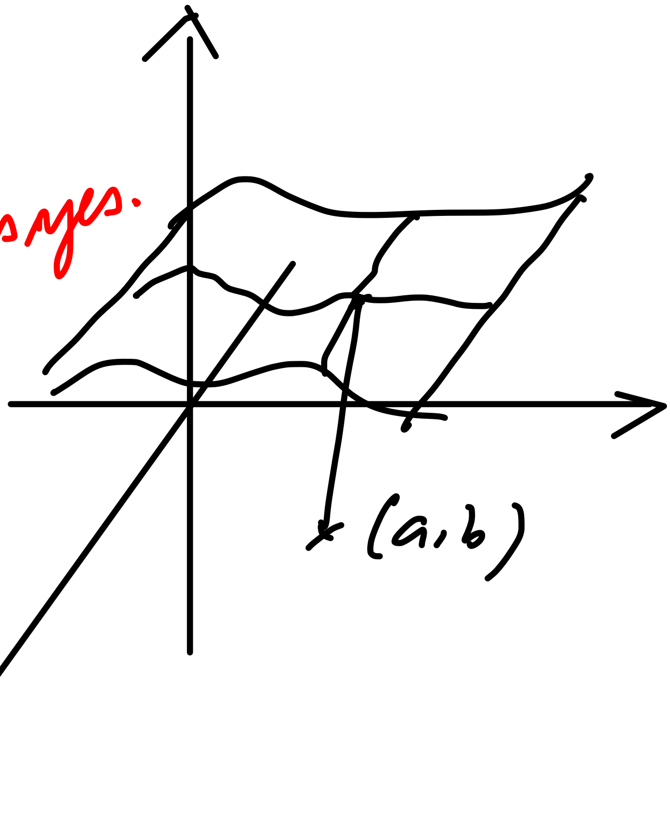
16. Given a function $f(x, y)$ and a point (a, b) in its domain, define single-variable functions g and h as follows:

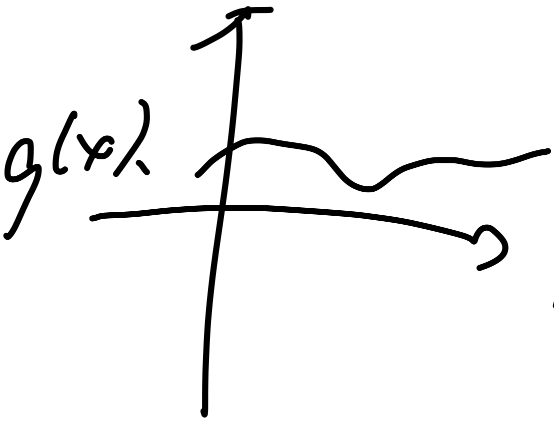
$$g(x) = f(x, b), \qquad h(y) = f(a, y).$$

If g is continuous at $x = a$ and h is continuous at $y = b$, does it follow that f is continuous at (a, b)? Conversely, does the continuity of f at (a, b) guarantee the continuity of g at a and the continuity of h at b? Justify your answers.

No, approaches from x=y.

~~No.~~ yes yes.





17. Let $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ be a unit vector, and let

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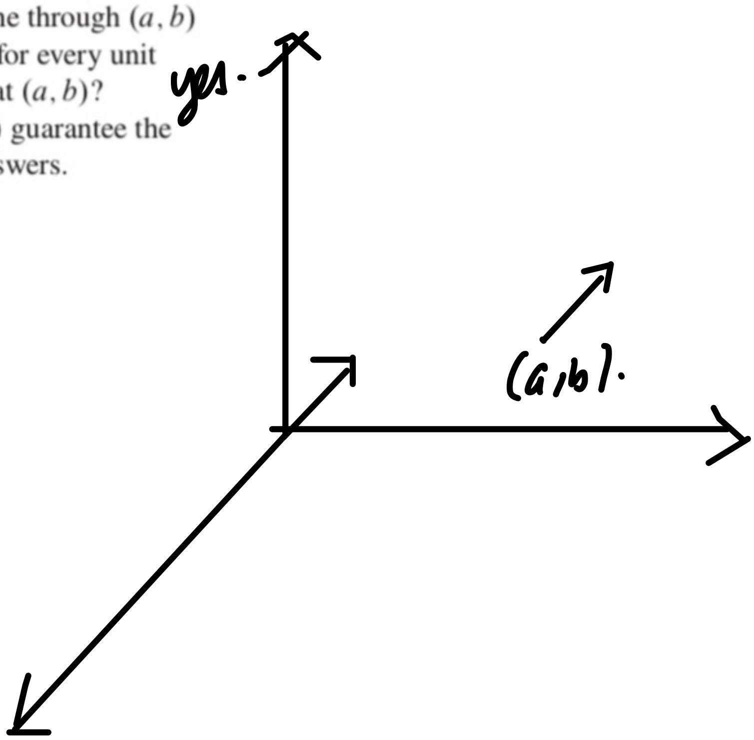
yes.

Yes.



18. What condition must the nonnegative integers m, n, and p satisfy to guarantee that $\lim_{(x,y)\to(0,0)} x^m y^n/(x^2+y^2)^p$ exists? Prove your answer.

$$\lim_{r\to 0} \frac{r^{m+n} \sin^n\theta \cos^m\theta}{r^{2p}}$$

$> 2p$.

$m+n > 2p$.

$= \quad 2p > m+n.$

$$p > \frac{m+n}{2}.$$

19. What condition must the constants a, b, and c satisfy to guarantee that $\lim_{(x,y)\to(0,0)} xy/(ax^2+bxy+cy^2)$ exists? Prove your answer.

20. Can the function $f(x, y) = \dfrac{\sin x \sin^3 y}{1 - \cos(x^2 + y^2)}$ be defined at $(0, 0)$ in such a way that it becomes continuous there? If so, how?

$\sin y(1-\cos^2 y)$

$= \sin y - \sin y \cos^2 y$

$f(x,y)$ is not continous at

$x^2+y^2 = 0 + n(2\pi)$

$\{x, y\}$.

No.

从两方向

$$-\frac{1}{1-\cos(x^2+y^2)} \le \left| \frac{\sin x \sin^3 y}{\;} \right| \le \frac{1}{1-\cos(x^2+y^2)}$$

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