MATH 2023 – Multivariable Calculus

Lecture #05 Worksheet ♣ February 21, 2019

Problem 1. Find $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$ where

in x (constant)

$$f(x,y) = \frac{e^{2019x^2}}{\ln\sqrt{x^2+2023}} + \sin(xy)$$

$$\frac{\partial f}{\partial y} = x\cos(xy)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \cos(xy) + x(-\sin(xy)\cdot y)$$ * is continuous

By Mixed Partials Theorem, $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = $

Problem 2. Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Show that ① f is continuous, ② f_x, f_y continuous, but

$$③ \quad f_{xy}(0,0) \neq f_{yx}(0,0)$$

Why does this violate the Mixed Partial Theorem?

$f,\ f_x,\ f_y,\ f_{xy}$ etc. are all continuous on $\mathbb{R}^2 \setminus (0,0)$

① Polar Coord: $x = r\cos\theta,\ y = r\sin\theta$

$$\frac{\cancel{r^2}\underbrace{\cos\theta\sin\theta}_{\frac{1}{2}\sin 2\theta}\ \underbrace{r^2(\cos^2\theta - \sin^2\theta)}_{\cos 2\theta}}{\cancel{r^2}} = \frac{r^2}{2}\underbrace{\sin 2\theta\cos 2\theta}_{\frac{1}{2}\sin 4\theta}$$

$$= \frac{r^2 \sin 4\theta}{4}$$

$|f(x,y)| \leq \frac{r^2}{4} \to 0$. Limit $(x,y)\to(0,0)$ exists and equal $f(0,0)$.

② f_x : $\dfrac{(x^2+y^2)\left(y(x^2-y^2) + xy(2x)\right) - xy(x^2-y^2)(2x)}{(x^2+y^2)^2}$

$$= \frac{y(x^2+4xy-y^2)}{x^2+y^2} \quad \text{for } (x,y) \neq (0,0).$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \frac{0-0}{h} = 0.$$

Check $\lim_{(x,y)\to(0,0)} f_x = 0$ (use polar coord.)

so f_x is continuous.

$$f_y = -\frac{x(y^2+4xy-x^2)}{x^2+y^2}$$

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h \to 0} \frac{-h - 0}{h} = -1$$

$$f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1$$

not equal. <u>Why?</u>

Because f_{xy} is <u>not</u> continuous at $(0,0)$!

along x-axis :

$$f_{xy}(a,0) = \lim_{h \to 0} \frac{f_x(a,h) - f(a,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\cancel{h}(a^2 + \cancelto{0}{4ah} - \cancelto{0}{h^2})}{a^2 + \cancelto{0}{h^2}} - 0}{\cancel{h}} = \frac{a^2}{a^2} = 1$$

along y-axis

$$f_{xy}(0,b) = \lim_{h \to 0} \frac{f_x(0,b+h) - f_x(0,b)}{h}$$

NOTE: $(f_x(0,y) = -y)$

$$= \lim_{h \to 0} \frac{-(b+h) - (-b)}{h} = -1$$

NOT the same $\Rightarrow$ not <u>continuous</u>

Problem 3. (a) Show that

a is constant.

$$u(x,t) = \sin(x - at)$$

is a solution to the **wave equation**

$$u_{tt} = a^2 u_{xx}$$

$u_t = \cos(x-at) \cdot (-a)$.

$u_{tt} = -\sin(x-at) \cdot (-a)^2 = -a^2 \sin(x-at)$

$u_x = \cos(x-at)$

$u_{xx} = -\sin(x-at)$

$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} u_{tt} = a^2 u_{xx}$ ✓

(b) Show that

$$u(x,y,z) = e^{3x+4y} \sin 5z = e^{3x} e^{4y} \sin 5z$$

is a solution to the **Laplace's equation**

$$u_{xx} + u_{yy} + u_{zz} = 0$$

$u_x = 3e^{3x} e^{4y} \sin 5z$

$u_{xx} = 9e^{3x} e^{4y} \sin 5z$

$u_y = \ldots 4e^{4y} \ldots$

$u_{yy} = 16 e^{3x} e^{4y} \sin 5z$

$u_z = 5 e^{3x} e^{4y} (\cos 5z)$

$u_{zz} = -25 e^{3x} e^{4y} \sin 5z$

$u_{xx} + u_{yy} + u_{zz} = (9 + 16 - 25) e^{3x} e^{4y} \sin 5z = 0.$ ✓

Problem 4. Let $z = f(x, y) = x^2 + 3xy - y^2$.

(a) Find the differential dz

(b) Find the tangent plane of $f(x, y)$ at $(2, 3)$

(c) Compare the values of Δz and dz when x changes from 2 to 2.05 and y changes from 3 to 2.96. $\Delta y = -0.04$

$\Delta x = 0.05$

a) $dz = f_x\,dx + f_y\,dy$

$= (2x+3y)\,dx + (3x-2y)\,dy.$

b) $f_x\big|_{(2,3)} = 13 \qquad f_y\big|_{(2,3)} = 0$

$f(2,3) = 4 + 3\cdot 2\cdot 3 - 9 = 13$

$\vec{n} = \langle 13, 0, -1 \rangle \qquad \vec{r_0} = (2, 3, 13)$

$13x - z = 13$ //

$dz = 13\,dx + 0\,dy = 13\Delta x = 13(0.05) = 0.65.$

$\Delta z = f(2.05, 2.96) - f(2, 3) = 0.6449$

$f(2.05, 2.96) = 13.6449 \qquad f(2,3) = 13$

$L(2,3)$ = height of tangent plane $= 13.65.$

$(f(2,3) + dz)$

linear approximation