Problem 1

(a) If $\mathbf{e}$ is any unit vector and $\mathbf{a}$ an arbitrary vector show that

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{e})\mathbf{e} + \mathbf{e}\times(\mathbf{a}\times\mathbf{e}).$$

This shows that $\mathbf{a}$ can be resolved into a component parallel to and one perpendicular to an arbitrary direction $\mathbf{e}$.

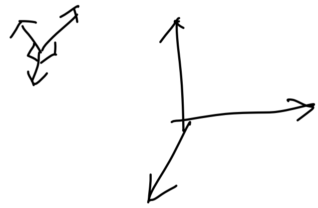
?

(b) Show that the two lines

$$\mathbf{r} = \mathbf{a} + \mathbf{v}t, \qquad \mathbf{r} = \mathbf{b} + \mathbf{u}t$$

where t is a parameter and $\mathbf{u}$ and $\mathbf{v}$ are two unit vectors, will intersect if

$$\mathbf{a}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{b}\cdot(\mathbf{u}\times\mathbf{v}).$$



Let $a = \alpha i + \beta j + \gamma k$

$e = xi + yj + zk, \quad x^2 + y^2 + z^2 = 1$

$$a\times e = \begin{vmatrix} i & j & k \\ \alpha & \beta & \gamma \\ x & y & z \end{vmatrix} = \langle \beta z - y\gamma,\ x\gamma - \alpha z,\ \alpha y - x\beta \rangle$$

$e\times(a\times e) = \langle y(\alpha y - x\beta),$

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$$\mathbf{a} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{b} \cdot (\mathbf{u} \times \mathbf{v}).$$

b).

Problem 2

(a) Describe (sketch) the intersection curve C of the sphere $x^2 + y^2 + z^2 = 1$ and the elliptic cylinder $y^2 + 2z^2 = 1$ in the first octant.

?

(b) Find the parametric equation of the curve C in the first octant.

(c) Find the vector equation of the tangent line L of C at the point $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$.

(d) Find the equation of the plane through the point $(1, 1, 1)$ and parallel with the tangent line L obtained in part (iii).

$$\begin{cases} x^2+y^2+z^2=1 \\ y^2+2z^2=1 \end{cases}$$

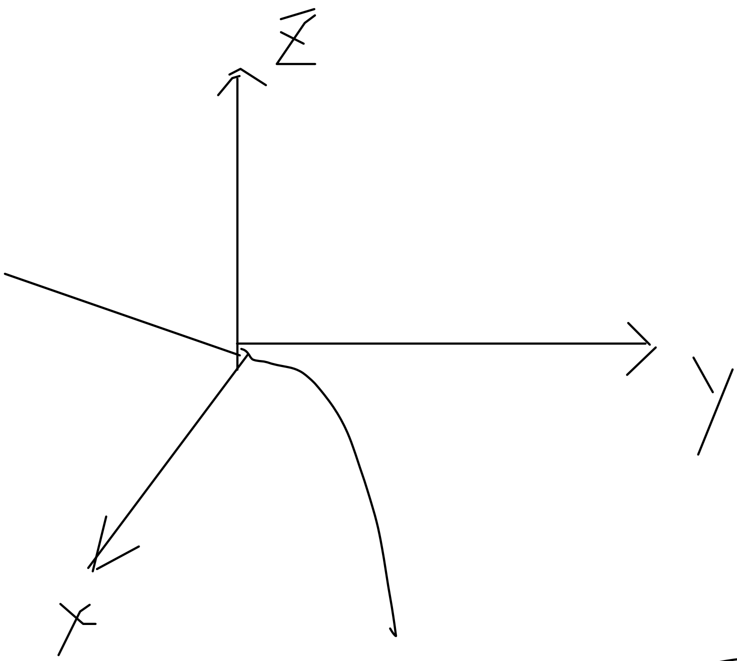
$$x^2 - z^2 = 0$$

$$x = \pm z$$

$$2x^2+y^2 = 1$$

$$y^2 = 1-2x^2$$

$$y = \sqrt{1-2x^2}$$



b). $x=t$, $z=t$, $y=\sqrt{1-2t^2}$

$$\vec{r}(t) = \langle t, \sqrt{1-2t^2}, t \rangle$$

Problem 2

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c). $t = \frac{1}{2}$. , $\vec{r}'(t) = \langle 1, \frac{-4t}{2\sqrt{1-2t^2}}, 1 \rangle$

$$\vec{r}'(t) = \langle 1, \frac{-2t}{\sqrt{1-2t^2}}, 1 \rangle$$

$$\vec{r}'(\tfrac{1}{2}) = \langle 1, \frac{-2(\frac{1}{2})}{\sqrt{1-2(\frac{1}{4})}}, 1 \rangle$$

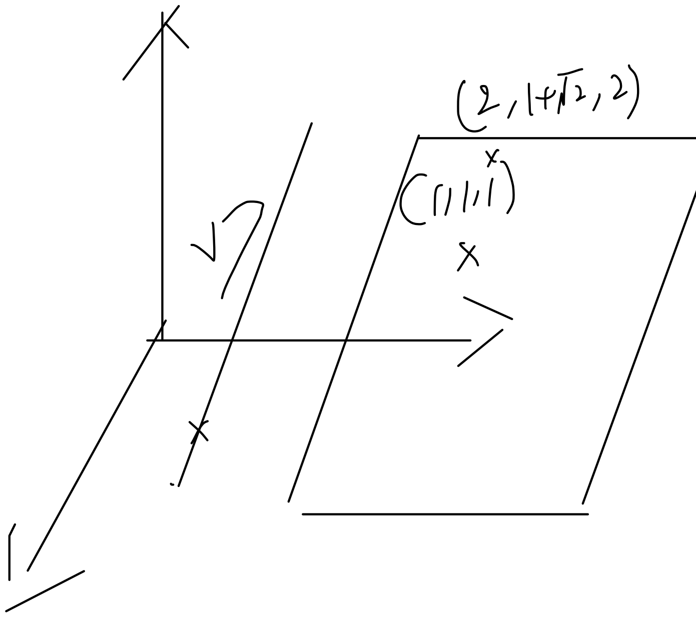
$$= \langle 1, \frac{-1}{\frac{\sqrt{2}}{2}}, 1 \rangle$$

$$= \langle 1, \sqrt{2}, 1 \rangle$$

$$\vec{v} = \langle \tfrac{1}{2} + t, \tfrac{\sqrt{2}}{2} + \sqrt{2}t, \tfrac{1}{2} + t \rangle$$

Made with Goodnotes

parallel to $\langle 1, \sqrt{2}, 1 \rangle$



Problem 3

Let $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2}, & \text{if} \quad (x,y) \neq (0,0) \\ 0, & \text{if} \quad (x,y) = (0,0). \end{cases}$

Calculate $f_x(x,y)$ and $f_y(x,y)$ at all points (x,y) (*include* the point $(0,0)$) in the xy-plane. Are f_x and f_y continuous at $(0,0)$ (why)? Is f continuous at $(0,0)$ (why)?

$$f_x = \frac{(x^2+y^2)(3x^2) - (x^3-y^3)(2x)}{(x^2+y^2)^2}$$

$$f_x = \frac{3x^4 + 3x^2y^2 - 2x^4 + 2xy^3}{(x^2+y^2)^2}$$

$$f_x = \frac{x^4 + 3x^2y^2 + 2xy^3}{(x^2+y^2)^2}$$

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{\frac{\Delta x^3}{\Delta x^2} - 0}{\Delta x} = 1$$

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + 3x^2y^2 + 2xy^3}{(x^2+y^2)^2} = \frac{r^4\cos^4\theta + 3r^4\cos^2\theta\sin^2\theta + 2r^4\cos\theta\sin^3\theta}{r^4}$$

$$= \cos^4\theta + 3\cos^2\theta\sin^2\theta + 2\cos\theta\sin^3\theta$$

depends on θ, many possibilities, no limit

not continuous at $(0,0)$.

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$$f_y = \frac{(x^2+y^2)(-3y^2) - (x^3-y^3)(2y)}{(x^2+y^2)^2}$$

$x = r\cos\theta$

$y = r\sin\theta$

$$= \frac{-3x^2y^2 - 3y^4 - 2yx^3 + 2y^4}{(x^2+y^2)^2}$$

$$= \frac{-3x^2y^2 - y^4 - 2yx^3}{(x^2+y^2)^2}$$

$$\lim_{\Delta y \to 0} \frac{\frac{-\Delta y^3}{\Delta y^2}}{\Delta y} = -1$$

$$\lim_{(x,y)\to(0,0)} f_y = \frac{-3r^4\cos^2\theta\sin^2\theta - r^4\sin^4\theta - 2r^4\sin\theta\cos^3\theta}{r^4}$$

$$= -3\cos^2\theta\sin^2\theta - \sin^4\theta - 2\sin\theta\cos^3\theta$$

depends on θ, limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$$

$$= \quad 0 \le \frac{x^3-y^3}{x^2+y^2} \le \frac{x^3}{x^2+y^2} \le x$$

$$\lim_{x\to 0} x \to 0$$

f is continuous.

Problem 4

(a) Show that

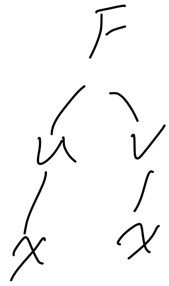
$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)\,dt\right] = f(b(x))b'(x) - f(a(x))a'(x)$$

[*Hint*: Let $u = a(x)$, $v = b(x)$, and $F(u,v) = \displaystyle\int_u^v f(t)\,dt$.

(b) Show that if $z = f(x,y)$ is differentiable at $\mathbf{x}_0 = (x_0, y_0)$, then

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{\|\mathbf{x}-\mathbf{x}_0\|} = 0$$

[*Hint*: Use the definition on differentiability.]



a). Let $u = a(x)$, $v = b(x)$, $F(u,v) = \int_u^v f(t)\,dt$

$$\frac{d}{dx}F(u,v) = \frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial \int_u^v f(t)\,dt}{\partial u}\cdot a'(x) + \frac{\partial \int_u^v f(t)\,dt}{\partial v}\cdot b'(x)$$

$$= -f(a(x))\,a'(x) + f(b(x))\,b'(x)$$

(b) Show that if $z = f(x, y)$ is differentiable at $\mathbf{x}_0 = (x_0, y_0)$, then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{\|\mathbf{x} - \mathbf{x}_0\|} = 0$$??

[*Hint*: Use the definition on differentiability.]

b)-