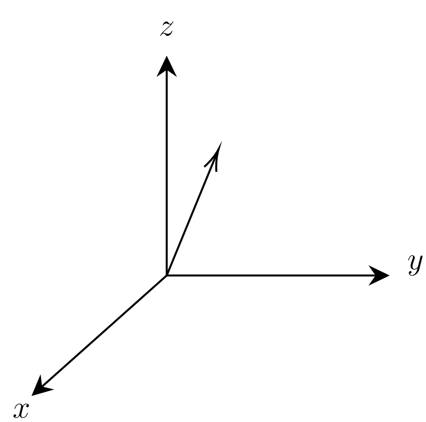
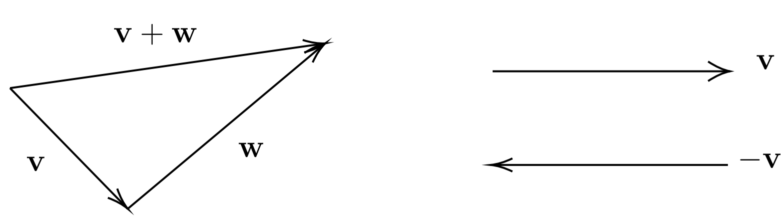
1 Review

In the following we will assume V to be a 3-dimensional real vector space (A rank 3 free $\mathbb{R}$-module :D).

- **Scalar**:
  - Is an *one*-entry object belongs to $\mathbb{R}$.
  - Represent a quantity.
  - *Ordered*.
- **Vector**:
  - Is a *three*-entry object represented by $\mathbf{x} = (x_1, x_2, x_3) = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, which represent an "arrow" in 3D space.



  - Addition of vector: follows the head to tail rules.



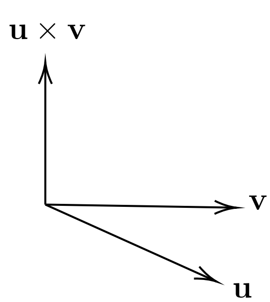
  - The **norm** $\|\cdot\| : V \to \mathbb{R}$ is a function which measures the *length* of the arrow. It is defined by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ (in our consideration).
  - Property: $\|\mathbf{v}\| = 0 \Leftrightarrow \mathbf{v} = \mathbf{0} = (0, 0, 0)$
  - **Unit Vector** is a vector with norm 1.
  - $\mathbf{v}_1$ and $\mathbf{v}_2$ are **linearly dependent** if $\mathbf{v}_1 = \alpha\mathbf{v}_2$ for some $\alpha \in \mathbb{R}$.

- Two vectors are said to be **orthogonal** if the angle in between them is $\pi/2$.
- **NOT** *ordered.*

- **Determinant**
  - for 2×2 matrix, $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
  - for 3×3 matrix, $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$

- **Dot product**:
  -
  $$\begin{aligned} ``\cdot" : V \times V &\to \mathbb{R} \\ (\mathbf{v}_1, \mathbf{v}_2) &\mapsto \mathbf{v}_1 \cdot \mathbf{v}_2 := v_{1x}v_{2x} + v_{1y}v_{2y} + v_{1z}v_{2z} = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos\theta \end{aligned}$$
  - Note that $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$.
  - Dot product of *orthogonal* vectors is 0.
  - It represent the length of projection of $\mathbf{v}_1$ on $\mathbf{v}_2$.

- **Cross product**:
  - $`` \times " : (\mathbf{v}_1, \mathbf{v}_2) \in V \times V \mapsto \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_{1x} & v_{1y} & v_{1z} \\ v_{2x} & v_{2y} & v_{2z} \end{bmatrix} \in V.$
  - Cross product gives a vector which is *perpendicular* to both of the given vectors.
  - $\|\mathbf{v}_1 \times \mathbf{v}_2\| = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \sin\theta$
  - Cross product of *linearly dependent* vectors is 0.
  - A handy convention concerning the direction of cross product can be found by the **right-hand grip rule**.



Procedures:

1. Point your index finger in the direction of the *first* vector of the cross product.

2. Curl your index finger (in the natural direction) towards the direction of the *second* vector.
3. The direction of your thumb in the process give you the direction of the cross product.

Remember Remember Remember (so important that is has to be mentioned three times) that "like " has to be given with **right hand** instead of left hand.

- The way to find a equation of **line** passing through *two* points:
  1. Given points A, B, we can find the vector $\overrightarrow{AB}$.
  2. The equation of line $\overline{AB}$ is given by $\overrightarrow{OA} + t\overrightarrow{AB}$ ($\overrightarrow{AB}$ is known as the **tangent vector**).
- The way to find a equation of **plane** passing through *three* points:
  1. Given points A, B and C, we can find vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
  2. The normal vector of the plane is given by $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC}$.
  3. If $P = (x, y, z)$ is a point on the plane, then $\overrightarrow{AB} \perp \mathbf{n}$, so $\overrightarrow{AB} \cdot \mathbf{n} = 0$, which gives the equation of plane.

2 Problems

1. True or False

   (a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
   True. From the definition of dot product, $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta = \|\mathbf{v}\| \|\mathbf{u}\| \cos\theta = \mathbf{v} \cdot \mathbf{u}$.

   (b) If $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
   True. Without loss of generality, we can assume $\|\mathbf{u}\| \neq 0$ (otherwise there will be no space for discussion). $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta = 0$ implies there will be 2 possibilities:

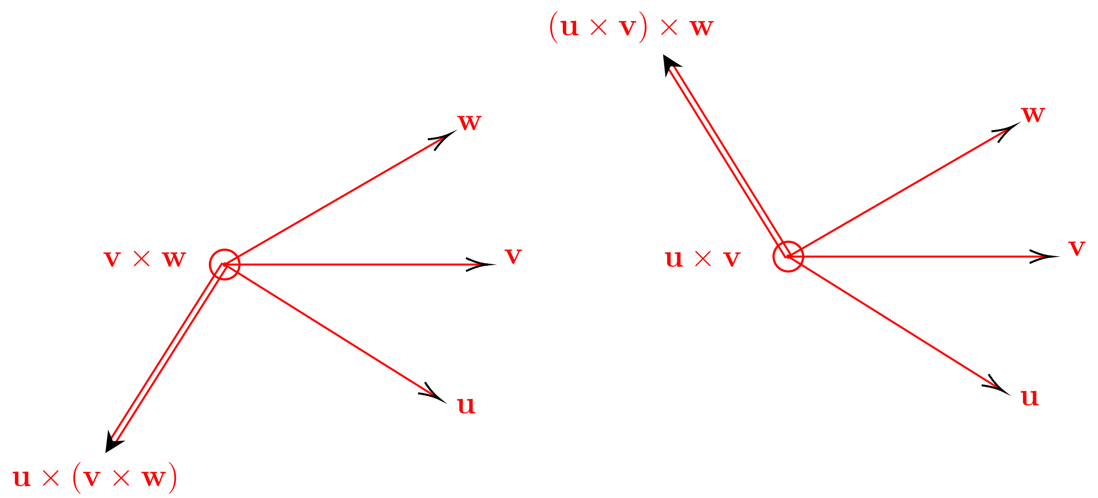
   1. $\|\mathbf{v}\| = 0$, then we are done.
   2. $\cos\theta = 0$. Let see what will happen if $\|\mathbf{v}\| \neq 0$. $\cos\theta = 0$ implies $\theta = \pi/2$ or $3\pi/2$. For either value of θ, $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \neq 0$, in contradiction with the given fact $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

   Therefore $\mathbf{v}$ has to be $\mathbf{0}$.

   (c) If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
   False. Counterexample: cross product of two linearly dependent non-zero vectors.

   (d) If $\mathbf{u} \cdot \mathbf{v} = 0$ then $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
   False. Counterexample: dot product of of orthogonal vectors.

(e) For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.
False. One can verify the components are not the same in general. Pictorial counterexample: Consider three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ lying on the same plane (on the paper),



(The circle represent a vector pointing out of the page)

2. Compute the angle between $\mathbf{v}_1 = (6, 2, 3)$ and $\mathbf{v}_2 = (5, -1, 4)$.
Solution: $\|\mathbf{v}_1\| = \sqrt{6^2 + 2^2 + 3^2} = 7$, $\|\mathbf{v}_2\| = \sqrt{5^2 + (-1)^2 + 4^2} = \sqrt{42}$

$$\|\mathbf{v}_1\| \, \|\mathbf{v}_2\| \cos\theta = \mathbf{v}_1 \cdot \mathbf{v}_2 = 6 \cdot 5 + 2 \cdot (-1) + 3 \cdot 4 = \sqrt{40}$$
$$\Rightarrow \theta = \underline{\arccos\left(\frac{\sqrt{40}}{7\sqrt{42}}\right)}$$

3. Compute the cross product of $\mathbf{v}_1 = (6, 2, 3)$ and $\mathbf{v}_2 = (5, -1, 4)$.
Solution:

$$\mathbf{v}_1 \times \mathbf{v}_2 = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & 3 \\ 5 & -1 & 4 \end{bmatrix} = \det \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \mathbf{i} - \det \begin{bmatrix} 6 & 3 \\ 5 & 4 \end{bmatrix} \mathbf{j} + \det \begin{bmatrix} 6 & 2 \\ 5 & -1 \end{bmatrix} \mathbf{k} = \underline{11\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}}$$

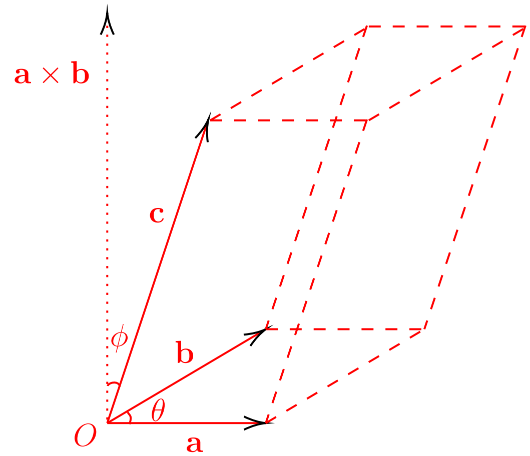
4. Compute the determinant of the matrix $\begin{bmatrix} 6 & 2 & 3 \\ 5 & -1 & 4 \\ 1 & 2 & 3 \end{bmatrix}$.

Solution:

$$\det \begin{bmatrix} 6 & 2 & 3 \\ 5 & -1 & 4 \\ 1 & 2 & 3 \end{bmatrix} = 6 \det \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix} - 2 \det \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix} + 3 \det \begin{bmatrix} 5 & -1 \\ 1 & 2 \end{bmatrix} = \underline{-55}.$$

5. Express the volume of the parallelepiped with vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as the three edges sharing the same vertex.

Solution: Pictorially, the parallelepiped is the following:



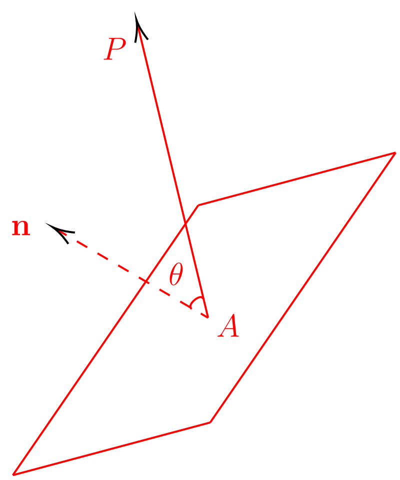
So it's volume is given by

$$V = (\|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta) \, \|\mathbf{c}\| \cos\phi = \|\mathbf{a} \times \mathbf{b}\| \, \|\mathbf{c}\| \cos\phi = \underline{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}}$$

assuming $0 \le \theta \le \pi$ and $0 \le \phi \le \pi/2$.

6. Which of the points $P = (6,2,3), Q = (5,-1,4)$ and $R = (0,3,8)$, is closest to the plane $x + 4y - 3z = 1$? Which point lies in the yz-plane?

Solution: The unit normal vector of the plane $x + 4y - 3z = 1$ is $\mathbf{n} = \frac{1}{\sqrt{26}}(1,4,-3)$ and the point $A = (1,0,0)$ is on the plane. Then we have the vectors $\overrightarrow{AP} = (5,2,3)$, $\overrightarrow{AQ} = (4,-1,4)$, $\overrightarrow{AR} = (-1,3,8)$. So



$$\text{seperation of } P \text{ with plane} = \left\|\text{Proj}_{\mathbf{n}} \overrightarrow{AP}\right\| = \left|\overrightarrow{AP} \cdot \mathbf{n}\right| = \frac{4}{\sqrt{26}}$$

$$\text{seperation of } Q \text{ with plane} = \left\|\text{Proj}_{\mathbf{n}} \overrightarrow{AQ}\right\| = \left|\overrightarrow{AQ} \cdot \mathbf{n}\right| = \frac{12}{\sqrt{26}}$$

$$\text{seperation of } R \text{ with plane} = \left\|\text{Proj}_{\mathbf{n}} \overrightarrow{AR}\right\| = \left|\overrightarrow{AR} \cdot \mathbf{n}\right| = \frac{13}{\sqrt{26}}$$

So $\underline{P \text{ is closest to the plane } x + 4y - 3z = 1}$.

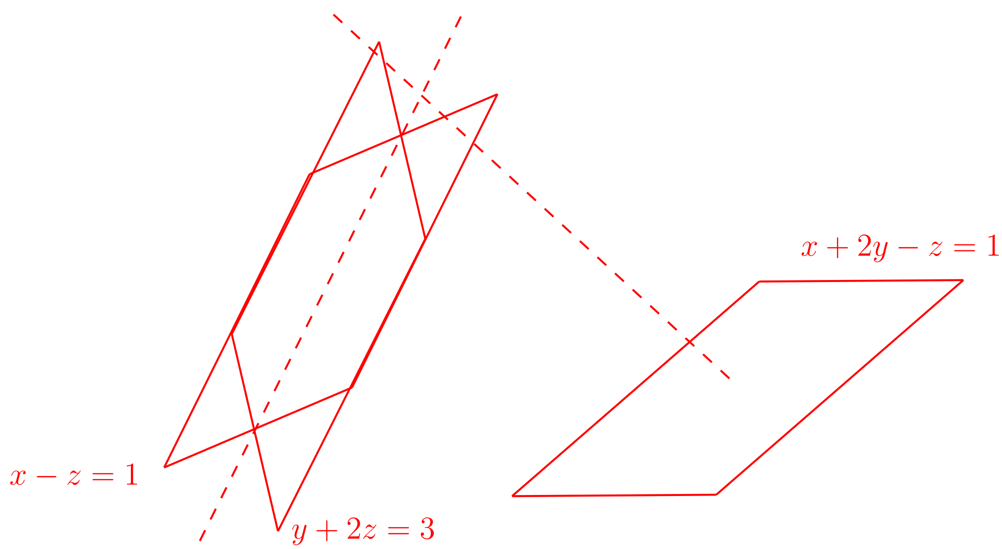
On yz-plane, $x = 0$. So R is the point lies in the yz-plane.

7. Find the parametric equation line through $(4, 1, -2)$ and $(1, 2, 5)$.
Solution: Let $A = (4, 1, -2)$ and $B = (1, 2, 5)$, then $\overline{AB} = (-3, 1, 7)$. The equatiom of line is therefore given by $\overrightarrow{OA} + t\overrightarrow{AB} = (4 - 3t, 1 + t, -2 + 7t)$.

8. Find the plane through $(2, 1, 0)$ and parallel to $x + 4y - 3z = 1$.

Solution:Two planes are parallel implies they **share the same normal vector**.
$x + 4y - 3z = 1 \Leftrightarrow (x - 1, y, z) \cdot (1, 4, -3) = 0$, so the shared normal vector is (1,4,-3).
For any point $P = (x, y, z)$ in the plane passing through $A = (2, 1, 0)$ and parallel to $x + 4y - 3z = 1$, it satisfies $\overrightarrow{AP} \cdot (1, 4, -3) = 0$. So the desired equation of plane is given by
$$(x - 2, y - 1, z) \cdot (1, 4, -3) = 0 \Leftrightarrow \underline{x + 4y - 3z = 6}$$

9. Find an equation of the plane through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and perpendicular to the plane $x + y - 2z = 1$.
Solution: The normal vector of $P_1 : x - z = 1$ is $\mathbf{n}_1 = (1, 0, -1)$ and the normal vector of $P_2 : y + 2z = 3$ is $\mathbf{n}_2 = (0, 1, 2)$. The vector $\mathbf{n}_1 \times \mathbf{n}_2 = (1, -2, 1)$ lies in the concerned plane P.
Since the concerned plane P is perpendicular to the plane $x + y - 2z = 1$, its normal vector $\mathbf{n}_3 = (1, 1, -2)$ lies in the concerned plane.
Notice that the point $(1, 3, 0)$ lies in the intersection equation (by trial and error or solving system of linear equations) So the equation of plane is given by
$$(x - 1, y - 3, z) \cdot (3, 3, 3) = 0 \Leftrightarrow \underline{x + y + z = 4}.$$



The dotted lines span the concerned plane.

10. Find a vector perpendicular to the plane through the points $A = (1,0,0)$, $B = (2,0,-1)$, $C = (1,4,3)$. Find the area of the triangle ABC.
Solution: $\overrightarrow{AB} = (1,0,-1)$ and $\overrightarrow{AC} = (0,4,3)$ are two edges of triangle ABC. Vectors perpendicular to the triangle is the scalar multiple of the vector $\underline{\overrightarrow{AB} \times \overrightarrow{AC} = (4,-3,4)}$. From trigonometry, the area of triangle ABC is

$$\frac{1}{2}|AB||AC|\sin\angle BAC = \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \underline{\frac{\sqrt{41}}{2}}.$$

11. Find the point in which the line with parametric equations $x = 2-t$, $y = 1+3t$, $z = 4t$ intersects the plane $2x - y + z = 2$.
Solution: Through direct substitution of the components of the equation of line,

$$2(2-t) - (1+3t) + 4t = 2 \Leftrightarrow t = 1$$

So the point of intersection corresponds to $t = 1$ in the equation of line, which is $\underline{(1,4,4)}$.

12. Determine wheter the following pair of lines are parallel, skew, or intersecting. If intersect, find the point of intersection.

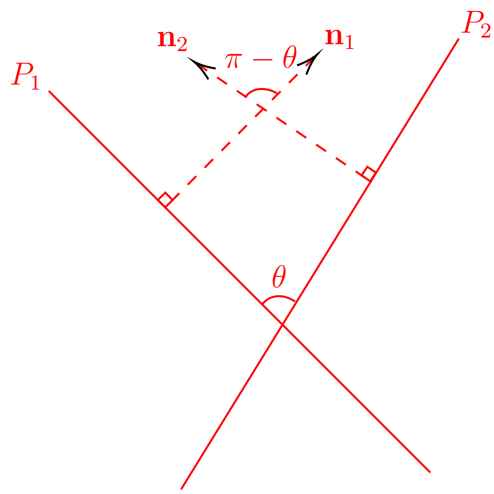
   (a) $L_1 : x = -6t,\ y = 1 + 9t,\ z = -3t,$
   $L_2 : x = 1 + 2s,\ y = 4 - 3s,\ z = s.$
   Solution: The tangent vector for L_1 from taking derivative is $(-6,9,-3)$ and the tangent vector for L_2 is $(2,-3,1) = -\frac{1}{3}(-6,9,-3)$, which implies the two lines are **parallel**.

   (b) $L_1 : \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$
   $L_2 : \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$
   Solution: The equation of L_1 can be rewritten as $(t, 2t+1, 3t+2)$ and L_2 can be rewritten as $(-4s+3, -3s+2, 2s+1)$. From taking derivative of their parameters, the tangents are $(1,2,3)$ and $(-4,-3,2)$, which shows two lines are **NOT** parallel. If two lines intersect, there will be solution for the system of linear equations

$$\begin{cases} t = -4s + 3 \\ 2t + 1 = -3s + 2 \\ 3t + 2 = 2s + 1 \end{cases}$$

   solving for s, t from the last two equations, we have $s = 3/5$ and $t = -3/5$, but substituting such s, t into the first equation **does not** give equality ($-3/5 \neq -4(3/5) + 3 = 3/5$). Therefore the above linear system does not have solution and the L_1, L_2 are **skew**.

13. Find the angle between two planes.
Solution: We can always find a "viewpoint" in which the two planes and be represented by two lines.



From the analysis at such viewpoint, the dot product of normal vectors calculate the angle between planes by:

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \|\mathbf{n}_1\| \, \|\mathbf{n}_2\| \cos(\pi - \theta) \Leftrightarrow \theta = \underline{\arccos\left(\frac{-\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \, \|\mathbf{n}_2\|}\right)}.$$