HKUST **MATH 102**

Midterm Examination

Multivariable and Vector Calculus

21 Dec 2005

Answer ALL 8 questions

Time allowed – 180 minutes

Directions – This is a closed book examination. No talking or whispering are allowed. Work must be shown to receive points. An answer alone is not enough. Please write neatly. Answers which are illegible for the grader cannot be given credit.

Note that you can work on *both* sides of the paper and do not detach pages from this exam packet or unstaple the packet.

Student Name: ____________________

Student Number: ____________________

Tutorial Session: ____________________

Question No.	Marks
1	/20
2	/20
3	/20
4	/20
5	/20
6	/20
7	/20
8	/20
Total	/160

Problem 1 (20 points) Your Score:

(a) Assume $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three dimensional vectors and if

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b} + \beta\mathbf{c}.$$

Use suffix notation to find λ, μ and β in terms of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Can you say something about the direction of the vector $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

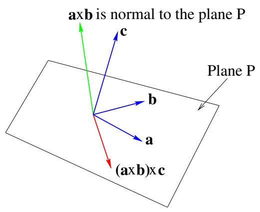
(b) Let $\mathbf{a}$ be a constant vector and $\mathbf{r} = (x, y, z)$, use suffix notation to evaluate

(i) $\nabla\cdot\mathbf{r}$, (ii) $\nabla\cdot(\mathbf{a}\times\mathbf{r})$, (iii) $\nabla\times(\mathbf{a}\times\mathbf{r})$.

Solution:

(a)

$$\begin{aligned}
[(\mathbf{a}\times\mathbf{b})\times\mathbf{c}]_i &= \epsilon_{ijk}(\mathbf{a}\times\mathbf{b})_j\, c_k \\
&= \epsilon_{ijk}\epsilon_{jpq}\, a_p b_q c_k \\
&= \epsilon_{jki}\epsilon_{jpq}\, a_p b_q c_k \\
&= (\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip}) a_p b_q c_k \\
&= a_k b_i c_k - a_i b_k c_k
\end{aligned}$$



i.e. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$, i.e. $\mu = \mathbf{a}\cdot\mathbf{c}$, $\lambda = -\mathbf{b}\cdot\mathbf{c}$ and $\beta = 0$.

The resulting vector of $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is a linear combination of the vectors $\mathbf{a}$ and $\mathbf{b}$, hence it lies on the plane containing the vectors $\mathbf{a}$ and $\mathbf{b}$.

(b) (i) $\nabla\cdot\mathbf{r} = \partial_i r_i = 3$.

(ii)

$$\begin{aligned}
\nabla\cdot(\mathbf{a}\times\mathbf{r}) &= \partial_i(\mathbf{a}\times\mathbf{r})_i \\
&= \partial_i\varepsilon_{ijk}\, a_j r_k \\
&= \varepsilon_{ijk}\, a_j \partial_i r_k \\
&= \varepsilon_{ijk}\, a_j \delta_{ik} \\
&= \varepsilon_{iji}\, a_j = 0
\end{aligned}$$

(iii)

$$\begin{aligned}
[\nabla\times(\mathbf{a}\times\mathbf{r})]_i &= \varepsilon_{ijk}\,\partial_j(\mathbf{a}\times\mathbf{r})_k \\
&= \varepsilon_{ijk}\,\partial_j\varepsilon_{kpq} a_p r_q \\
&= \varepsilon_{kij}\varepsilon_{kpq}\, a_p\delta_{jq} \\
&= \varepsilon_{kij}\varepsilon_{kpj}\, a_p \\
&= (\delta_{ip}\delta_{jj} - \delta_{ij}\delta_{jp})\, a_p \\
&= 3a_i - a_i = 2a_i
\end{aligned}$$

$\therefore\quad \nabla\times(\mathbf{a}\times\mathbf{r}) = 2\mathbf{a}$.

Problem 2 (20 points) Your Score:

(a) Sketch and describe the parametric curve C

$$\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + (2\pi - t)\,\mathbf{k}, \qquad 0 \leqslant t \leqslant 2\pi.$$

Show the direction of increasing t. Find the project curve C onto the yz-plane.

(b) Find a change of parameter $t = g(\tau)$ for the semicircle

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \qquad 0 \leqslant t \leqslant \pi$$

such that (i) the semicircle is traced counterclockwise as τ varies over the interval [0, 1],

(ii) the semicircle is traced clockwise as τ varies over the interval [0, 0.5].

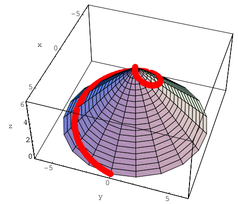
Solution:

(a)

$$\begin{aligned} x &= t\cos t \\ y &= t\sin t \\ z &= 2\pi - t \end{aligned} \quad\Rightarrow\quad x^2 + y^2 = t^2 \quad\Rightarrow\quad t = \sqrt{x^2+y^2} \quad \text{since} \quad t \geqslant 0.$$

i.e. $z = 2\pi - \sqrt{x^2+y^2}$.

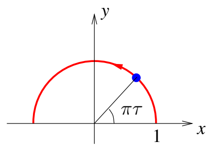
$\therefore$ $\mathbf{r}$ is a conical helix wound around the cone $z = 2\pi - \sqrt{x^2+y^2}$ starting at the vertex $(0, 0, 2\pi)$ and completing one revolution to end up at $(2\pi, 0, 0)$.



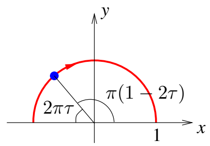
The projection curve can be obtained by considering

$$\begin{aligned} y &= t\sin t \\ z &= 2\pi - t \\ \therefore\quad \mathbf{r}(t) &= t\sin t\,\mathbf{j} + (2\pi - t)\,\mathbf{k} \end{aligned}$$

(b) (i) $\mathbf{r}(\tau) = \cos\pi\tau\,\mathbf{i} + \sin\pi\tau\,\mathbf{j}, \qquad 0 \leqslant \tau \leqslant 1$



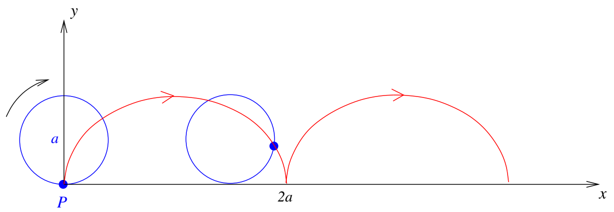
(ii) $$\begin{aligned}\mathbf{r}(\tau) &= \cos\pi(1-2\tau)\,\mathbf{i} + \sin\pi(1-2\tau)\,\mathbf{j}, && 0 \leqslant \tau \leqslant 0.5 \\ &= -\cos(2\pi\tau)\,\mathbf{i} + \sin(2\pi\tau)\,\mathbf{j}, && 0 \leqslant \tau \leqslant 0.5.\end{aligned}$$

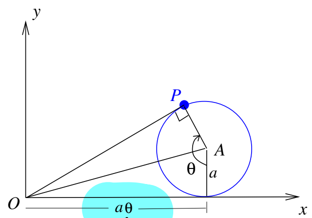


Problem 3 (20 points) Your Score:

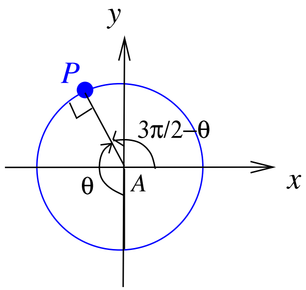
If a wheel with radius a rolls along a flat surface without slipping, a point P on the rim of the wheel traces a curve C, find the parametric equation of the point P. Suppose that the point P on the wheel is initially at the origin. Find also the arc length of the curve C if the wheel makes one complete turn (no need to carry out the integration).

Solution:

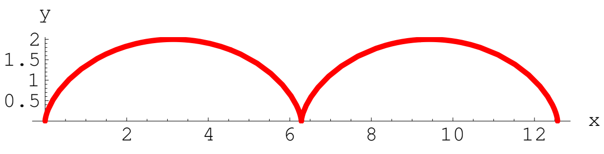




why?



$$\begin{aligned}
\mathbf{OA} &= a\theta\,\mathbf{i} + a\,\mathbf{j} \\
\mathbf{AP} &= a\cos\left(\frac{3\pi}{2} - \theta\right)\mathbf{i} + a\sin\left(\frac{3\pi}{2} - \theta\right)\mathbf{j} \\
&= -a\sin\theta\,\mathbf{i} - a\cos\theta\,\mathbf{j} \\
\therefore \quad \mathbf{OP} &= \mathbf{OA} + \mathbf{AP} \\
&= (a\theta\,\mathbf{i} + a\,\mathbf{j}) + (-a\sin\theta\,\mathbf{i} - a\cos\theta\,\mathbf{j}) \\
&= a(\theta - \sin\theta)\,\mathbf{i} + a(1 - \cos\theta)\,\mathbf{j}.
\end{aligned}$$



Problem 3 (20 points) Your Score:

Since the parametric representation of the curve is

$$\begin{aligned}
\mathbf{r}(\theta) &= a(\theta - \sin\theta)\,\mathbf{i} + a(1 - \cos\theta)\,\mathbf{j} \\
\mathbf{r}'(\theta) &= a(1 - \cos\theta)\,\mathbf{i} + a\sin\theta\,\mathbf{j} \\
\|\mathbf{r}'(\theta)\|^2 &= a^2(2 - 2\cos\theta)
\end{aligned}$$

Therefore, the arc length is given as

$$\begin{aligned}
s &= \int_0^{2\pi} \|\mathbf{r}'(\theta)\|\,d\theta \\
&= a\int_0^{2\pi} (2 - 2\cos\theta)^{\frac{1}{2}}\,d\theta \\
&= 8a.
\end{aligned}$$

Problem 4 (20 points) Your Score:

(a) Verify the formula for the arc length element is cylindrical coordinates,

$$ds = \sqrt{\left(\frac{dr}{dt}\right)^2 + (r(t))^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

(b) Find a similar formula as in (a) for the arc length element in spherical coordinates.

(c) Use part (b) or otherwise, find the arc length of the curve in spherical coordinates: $\rho = 2t$, $\theta = \ln t$, $\phi = \pi/6$; $1 \leqslant t \leqslant 5$.

Solution:

(a) In cylindrical coord.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} = \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt} \\ \frac{dy}{dt} = \sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt} \\ \frac{dz}{dt} = \frac{dz}{dt} \end{cases}$$

$$\begin{aligned}\|\mathbf{r}'(t)\|^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \\ &= \cos^2\theta\left(\frac{dr}{dt}\right)^2 - 2r\cos\theta\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} + r^2\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 \\ &\quad + \sin^2\theta\left(\frac{dr}{dt}\right)^2 + 2r\cos\theta\sin\theta\frac{dr}{dt}\frac{d\theta}{dt} + r^2\cos^2\theta\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \\ &= \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\end{aligned}$$

Hence the answer.

Problem 4 (20 points) Your Score:

(b) In spherical coord.

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi$$

therefore,

$$\begin{aligned}\frac{dx}{dt} &= \frac{d\rho}{dt}\sin\phi\cos\theta + \rho\cos\phi\cos\theta\frac{d\phi}{dt} - \rho\sin\phi\sin\theta\frac{d\theta}{dt} \\ \frac{dy}{dt} &= \frac{d\rho}{dt}\sin\phi\sin\theta + \rho\cos\phi\sin\theta\frac{d\phi}{dt} + \rho\sin\phi\cos\theta\frac{d\theta}{dt} \\ \frac{dz}{dt} &= \frac{d\rho}{dt}\cos\phi - \rho\sin\phi\frac{d\phi}{dt} \\ \|\mathbf{r}'(t)\|^2 &= \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \\ &= \left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\phi}{dt}\right)^2 + \rho^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2\end{aligned}$$

Hence

$$ds = \sqrt{\left(\frac{d\rho}{dt}\right)^2 + (\rho(t))^2\left(\frac{d\phi}{dt}\right)^2 + (\rho(t)\sin\phi(t))^2\left(\frac{d\theta}{dt}\right)^2}\, dt.$$

(c)

$$\begin{aligned}&\left(\frac{d\rho}{dt}\right)^2 + \rho^2\sin^2\phi\left(\frac{d\theta}{dt}\right)^2 + \rho^2\left(\frac{d\phi}{dt}\right)^2 \\ &= 4 + 4t^2\frac{1}{4}\frac{1}{t^2} + 0 = 5 \\ \therefore \quad L &= \int_1^5 \sqrt{5}\, dt = 4\sqrt{5}.\end{aligned}$$

Problem 5 (20 points) Your Score:

Let $f(x,y) = \begin{cases} \dfrac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) Is the function continuous at $(0,0)$?

(b) Calculate $f_x(x,y)$, $f_y(x,y)$, $f_{xy}(x,y)$ and $f_{yx}(x,y)$ at point $(x,y) \neq (0,0)$. Also calculate these derivatives at $(0,0)$.

(c) Is $f_{yx}(x,y)$ continuous at $(0,0)$?

(d) Explain why $f_{yx}(0,0) \neq f_{xy}(0,0)$.

Solution:

(a) Examine $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

Along $x=0$, $y \neq 0$ $\quad \lim\limits_{y\to 0} f(0,y) = \dfrac{0}{y^2} = 0$

or along $x \neq 0$, $y = 0$ $\quad \lim\limits_{x\to 0} f(x,0) = \dfrac{0}{x^2} = 0.$

Let $x = r\cos\theta(r)$, $\quad y = r\sin\theta(r)$, then

$$\begin{aligned} f(r\cos\theta(r), r\sin\theta(r)) &= \frac{2r^2\cos\theta\sin\theta \cdot r^2(\cos^2\theta - \sin^2\theta)}{r^2} \\ &= 2r^2\cos\theta\sin\theta(\cos^2\theta-\sin^2\theta) \leqslant 2r^2 \end{aligned}$$

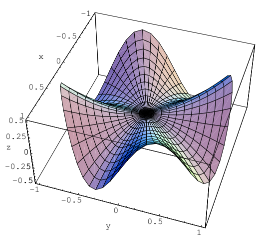
As $(x,y)\to(0,0) \quad \Rightarrow \quad r \to 0^+$, i.e. $f(x,y)\to 0$.

$$\lim_{(x,y) to (0,0)} f(x,y) = 0 = f(0,0)$$

$\therefore f(x,y)$ is continuous at $(0,0)$.

(b) Let $f(x,y) = (x^2-y^2)\dfrac{2xy}{x^2+y^2}$, then

$$\begin{aligned} f_x &= \frac{4x^2y}{x^2+y^2} - \frac{2y(y^2-x^2)^2}{(x^2+y^2)^2} \\ f_y &= -\frac{4xy^2}{x^2+y^2} + \frac{2x(x^2-y^2)^2}{(x^2+y^2)^2} \\ f_{xy} &= \frac{2(x^6+9x^4y^2-9x^2y^4-y^6)}{(x^2+y^2)^3} = f_{yx}. \end{aligned}$$



Surface of $f(x,y)$

Problem 5 (20 points) Your Score:

For the value at $(0,0)$, we use

$$\begin{aligned} f_x(0,0) &= \lim_{\Delta x\to 0} \frac{f(\Delta x,0)-f(0,0)}{\Delta x} = 0 = f_y(0,0) \\ f_{xy}(0,0) &= \lim_{\Delta y\to 0} \frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{-2\Delta y(\Delta y)^4}{\Delta y(\Delta y)^4} = -2 \\ f_{yx}(0,0) &= \lim_{\Delta x\to 0} \frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{2\Delta x(\Delta x)^4}{\Delta x(\Delta x)^4} = 2. \end{aligned}$$

(c) Along then line $y = x$, i.e.

$$\lim_{x\to o} f(x,x) = \frac{0}{x^6} = 0.$$

Along $y=0$, $x\neq 0$, then

$$\lim_{x\to 0} f(x,0) = 2.$$

$\therefore$ Different paths with different limits, hence, the limit does not exist.

(d) Observe that $f_{yx}(0,0) = 2$ and $f_{xy}(0,0) = -2$. This is because the partials f_{xy} and f_{yx} are not continuous at $(0,0)$.

(For instance, $f_{xy}(x,x) = 0$ while $f_{xy}(x,0) = 2$ for $x \neq 0$).

Problem 6 (20 points) Your Score:

Find the distance from the origin to the plane $x + 2y + 2z = 3$,

(a) using a geometric argument (no calculus),

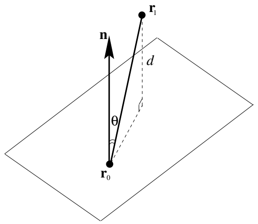
(b) by reducing the problem to an unconstrained problem in two variables, and

(c) using the method of Lagrange multipliers.

Solution:

(a) The point $\mathbf{r}_0 = (3, 0, 0)$ is on the given plane

$$\begin{aligned} d &= \|\mathbf{r} - \mathbf{r}_0\| \, |\cos\theta| \cdot \|\widehat{\mathbf{n}}\| \\ &= |(\mathbf{r} - \mathbf{r}_0) \cdot \widehat{\mathbf{n}}| \\ &= \left|(3, 0, 0) \cdot \frac{1}{3}(1, 2, 2)\right| \\ &= 1. \end{aligned}$$



Alternatively, let (x, y, z) be the point on the given plane closest to $(0, 0, 0)$. The vector (1,2,2) is normal to the plane, so must be parallel to the vector (x, y, z) from $\mathbf{0}$ to (x, y, z). Thus

$$(x, y, z) = \lambda(1, 2, 2) \quad \text{for some scalar } \lambda.$$

Since the point (x, y, z) is on the given plane, i.e.

$$t + 4t + 4t = 3 \quad \Rightarrow \quad t = \frac{1}{3}$$

$$\therefore (x, y, z) = \frac{1}{3}(1, 2, 2)$$

$$\therefore d = \frac{1}{3}\sqrt{1 + 4 + 4} = 1.$$

Problem 6 (20 points) Your Score:

(b) Let (x, y, z) be the point on the given plane closest to $\mathbf{0}$, so the problem becomes: minimize

$$s(x, y, z) = x^2 + y^2 + z^2.$$

Since $x + 2y + 2z = 3$, we have $x = 3 - 2y - 2z$

$$\therefore \ s = s(y, z) = (3 - 2y - 2z)^2 + y^2 + z^2$$

For critical points, $s_y = s_z = 0$

$$\begin{aligned} s_y &= -12 + 10y + 8z = 0 \\ s_z &= -12 + 8y + 10z = 0 \\ \Rightarrow \quad y &= z = \frac{2}{3}, \quad x = \frac{1}{3} \end{aligned}$$

$\therefore$ The distance is 1 unit as in part (a).

(c) Same as in part (b), but now the problem becomes:

Minimize $s = x^2 + y^2 + z^2$ subject to $x + 2y + 2z = 3 = g(x, y, z)$

Using Lagrangian multipliers, to find the critical points, we have

$$\begin{cases} \nabla s = \lambda \nabla g \\ g(x, y, z) = 3 \end{cases}$$

$$\therefore \begin{cases} 2x = \lambda \\ 2y = 2\lambda \\ 2z = 2\lambda \\ x + 2y + 2z = 3 \end{cases} \quad \Rightarrow \quad \begin{cases} y = z = \lambda \\ x = \dfrac{\lambda}{2} \\ \lambda = \dfrac{2}{3} \end{cases}$$

So the critical point is once again $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$, whose distance from the origin is 1 unit.

Problem 7 (20 points) Your Score:

(a) What condition must the constants a, b, and c satisfy to guarantee that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{ax^2+bxy+cy^2}$$

exists. Prove your answer.

(b) Find $\dfrac{\partial^2}{\partial y \partial x} f(y^2,\, xy,\, -x^2)$ in terms of partial derivatives of the function f.

Solution:

(a) Suppose $(x,y)\to(0,0)$ along the ray $y=kx$, then

$$f(x,y) = \frac{xy}{ax^2+bxy+cy^2} = \frac{k}{a+bk+ck^2}.$$

Thus $f(x,y)$ has different constant values along different rays from the origin unless $a=c=0$ and $b\neq 0$. If this condition is satisfied, then $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{b}$ does exist. If this condition is NOT satisfied, then $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Alternatively by using polar, let $x = r\cos\theta(r)$ and $x = r\sin\theta(r)$, then

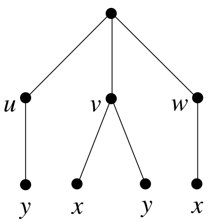
$$f(x,y) = \frac{\cos\theta\sin\theta}{a\cos^2\theta + b\cos\theta\sin\theta + c\sin^2\theta}.$$

Then similar arguement as above and we should end up with the some conclusion.

(b) Let $u=u(y)=y^2$, $v=v(x,y)=xy$, $w=w(x)=-x^2$; then $f=f(u,v,w)$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{dw}{dx} \\
&= f_v y + f_w(-2x) \\
&= y f_v - 2x f_w
\end{aligned}$$

$f,\ f_v$ and f_w



$$\begin{aligned}
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}(y f_v - 2x f_w) \\
&= f_v + y\left(\frac{\partial}{\partial y} f_v\right) - 2x\left(\frac{\partial}{\partial y} f_w\right) \\
&= f_v + y\left[\frac{\partial f_v}{\partial u}\frac{du}{dy} + \frac{\partial f_v}{\partial v}\frac{\partial v}{\partial y}\right] - 2x\left[\frac{\partial f_w}{\partial u}\frac{du}{dy} + \frac{\partial f_w}{\partial v}\frac{\partial v}{\partial y}\right] \\
&= f_v + y[f_{vu}(2y) + f_{vv}x] - 2x[f_{wu}(2y) + f_{wv}x] \\
&= f_v + 2y^2 f_{vu} + xy f_{vv} - 4xy f_{wu} - 2x^2 f_{wv},
\end{aligned}$$

where all partials are evaluated at $(y^2, xy, -x^2)$.

Problem 8 (20 points) Your Score:

(a) Find the equation of the tangent plane at the point $(-1,1,0)$ to the surface

$$x^2 - 2y^2 + z^3 = -e^{-z}.$$

(b) The temperature at a point (x,y) on a metal plate in xy-plane is $T(x,y) = x^2+y^3$ degrees Celsius.

(i) Find the rate of change of temperature at $(1,1)$ in the direction of $\mathbf{a} = 2\,\mathbf{i} + \mathbf{j}$.

(ii) An ant at $(1,1)$ wants to walk in the direction in which the temperature decreases most rapidly. Find a unit vector in that direction.

(c) Let C be the curve $x^{2/3}+y^{2/3}=a^{2/3}$ on the xy-plane, find the parametric equation of the curve C. Hence find the tangent line to the curve C at $(a,0)$.

Solution:

(a) Let $F(x,y,z) = x^2-2y^2+z^3+e^{-z} = 0$. This is a level surface in 3D and

$$\nabla F = (2x,\ -4y,\ 3z^2 - e^{-z}).$$

At $(-1,1,0)$, $\nabla F = (-2,-4,-1)$. This vector is normal to the level surface at the point $(-1,1,0)$.

$\therefore$ Tangent required plane is

$$\begin{aligned}
(x+1, y-1, z-0)\cdot(-2,-4,-1) &= 0 \\
-2(x+1) - 4(y-1) - z &= 0 \\
2x + 4y + z &= 2.
\end{aligned}$$

(b) (i)

$$\begin{aligned}
&\nabla T(x,y) = (2x, 3y^2) \\
&\nabla T(1,1) = (2,3), \qquad \text{and} \qquad \hat{\mathbf{a}} = \frac{1}{\sqrt{5}}(2\,\mathbf{i}+\mathbf{j}). \\
\therefore \quad & D_{\mathbf{a}}T = \hat{\mathbf{a}}\cdot\nabla T = \frac{1}{\sqrt{5}}(2\,\mathbf{i}+\mathbf{j})\cdot(2\,\mathbf{i}+3\,\mathbf{j}) = \frac{7}{\sqrt{5}}
\end{aligned}$$

(ii)

$$\hat{\mathbf{u}} = -\frac{1}{\sqrt{13}}(2\,\mathbf{i}+3\,\mathbf{j}), \quad \text{opposite to} \quad \nabla T(1,1).$$

becasue $\min(D_{\mathbf{u}}T) = \min(\|\hat{\mathbf{u}}\|\,\|\nabla T\|\cos\theta) = -\|\nabla T\|$ when $\theta = \pi$.

(c) Let $x = a\cos^3\theta$, $y = a\sin^3\theta$, then

$$\mathbf{r}(\theta) = a\cos^3\theta\,\mathbf{i} + a\sin^3\theta\,\mathbf{j}$$

then

$$\mathbf{r}'(\theta) = -3a\cos^2\theta\sin\theta\,\mathbf{i} + 3a\sin^2\theta\cos\theta\,\mathbf{j}.$$

At $(a,0)$, $\theta = 0$ and $\mathbf{r}'(0) = \mathbf{0}$.

$\therefore$ The curve is not differentiable at $(a,0)$, hence, there is no tangent line.