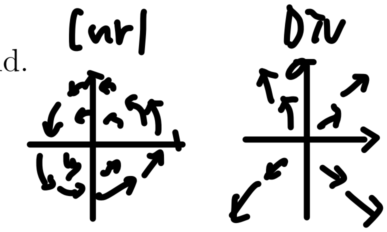
MATH2023
Tutorial 9
April 13, 2019 Tutorial Notes by Ivan So

1 Review

- The **gradient operator** can be realized as a *vector*, explicitly

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

Based on such realization, there are essential quantities derived from the gradient operator:



1. **Gradient Field** of a function. In the spirit of a conservative field.

contrast.

2. **Curl** of a vector field $\mathbf{F}$ is defined as $\nabla \times \mathbf{F}$.
   Intuitively it measures the *rotation* directed by the vector field.
3. **Divergence** of a vector field $\mathbf{F}$ is defined as $\nabla \cdot \mathbf{F}$.
   Intuitively it measures the amount of *source* of the vector field.
4. **Laplacian** of a function f, which is the *second order derivative*

$$\nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- Some important theorems concerning divergence and curl.

1. $\nabla \cdot (\nabla \times \mathbf{F}) = 0$
2. $\nabla \times \mathbf{F} = 0$ does **NOT** necessarily implies $\mathbf{F}$ is conservative.
3. $\nabla \times (\nabla f) = 0$. Consequently, curl of conservative field is *zero.*
4. If the domain is simply connected, then $\nabla \times \mathbf{F} = \mathbf{0} \Rightarrow \mathbf{F}$ is conservative.

- **Green's Theorem** (flux form): If $\mathbf{F} = \langle P, Q \rangle$, then

measure flux in D.

$$\int_{\partial D} \mathbf{F} \cdot \mathbf{n} ds = \int\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy.$$

- A surface can be *parametrized* by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,z) \rangle$. Remark:

  - For a graph of function, it can be parametrized by $\mathbf{r}(u,v) = \langle u, v, f(u,v) \rangle$.
  - $\frac{\partial \mathbf{r}}{\partial u}$, $\frac{\partial \mathbf{r}}{\partial v}$ represent two tangent vectors on the surface. This implies their cross product gives a normal vector of the surface.
  - Notice that

1. $\left|\frac{\Delta_u \mathbf{r}}{\Delta u} \times \frac{\Delta_v \mathbf{r}}{\Delta v}\right| \Delta u \Delta v = |\Delta_u \mathbf{r}||\Delta_v \mathbf{r}||\sin\theta|$ (assuming $\Delta u, \Delta v \geq 0$), which is the area of the parallelogram with $\Delta_u \mathbf{r}$ and $\Delta_v \mathbf{r}$ as sides.
2. Therefore the sum

$$\sum_{i,j} \left|\frac{\Delta_u \mathbf{r}}{\Delta u_i}(u_i^*, v_j^*) \times \frac{\Delta_v \mathbf{r}}{\Delta v_j}(u_i^*, v_j^*)\right| \Delta u_i \Delta v_j$$

approximate the area of the surface over the domain of u, v. Therefore

$$\int\int_D \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| dudv = \text{area of the surface over the domain } D \text{ of } u, v.$$

2 Problems

1. True or False

(a) Given a vector field $\mathbf{F}$, $\nabla \cdot \mathbf{F}$ is a vector.

False. $\vec{F} = \langle P, Q, R \rangle$

$\nabla \vec{F} = P_x + Q_y + R_z$

$\Rightarrow$ scalar

(b) If $\mathbf{F}$ is a constant vector field, then $\oint_C \mathbf{F} \cdot \mathbf{n} ds = 0$.

$\oint_{\partial D} F \cdot n ds = \iint_D (P_x + Q_y) dxdy$ True.

$= 0$

(c) There is a vector field such that $\mathbf{F} = \langle x, y, z \rangle$.

$\nabla \times \vec{F} = \langle x, y, z \rangle$

$\nabla \cdot (\nabla \times \vec{F}) = 1 + 1 + 1 = 3 \neq 0$ $\therefore$ False.

2. Find the curl and divergence of $\mathbf{F} = \langle 2xy, xz^2, y^2 \rangle$.

3. Find the area of $3x + 2y + z = 6$ in the first octant.

2. Find the curl and divergence of $\mathbf{F} = \langle 2xy, xz^2, y^2 \rangle$.

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & xz^2 & y^2 \end{vmatrix} = \langle 2y - 2xz,\ 0,\ z^2 - 2x \rangle$$

$$\nabla \cdot \vec{F} = 2y + 0 + 0 = 2y.$$

3. Find the area of $3x + 2y + z = 6$ in the first octant.

Need parametrization

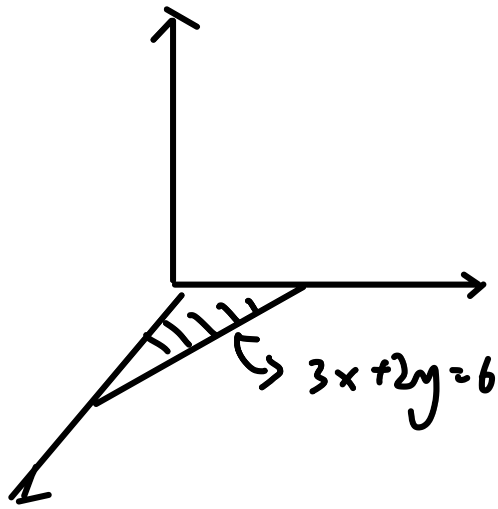
$$\vec{r}(u,v) = \langle u, v, 6 - 3u - 2v \rangle$$

$$\vec{r}_u = \langle 1, 0, -3 \rangle$$

$$\vec{r}_v = \langle 0, 1, -2 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 3, 2, 1 \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{14}$$



$$\text{area} = \int_0^2 \int_0^{\frac{1}{2}(6-3u)} \sqrt{14}\, dv\, du$$

4. Use the flux form Green's theorem to prove the identity

$$\int\int_D f\nabla^2 g = \oint_{\partial D} f(\nabla g)\cdot \mathbf{n} ds - \int\int_D \nabla f \cdot \nabla g dA.$$

5. Given the Maxwell's equations

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$$

prove that (a) $\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial^2 \mathbf{B}}{\partial t^2}$, (b) $\nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$.

6. Verify that $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ and $\langle z_x, z_y, -1\rangle$ give the same normal vector up to a multiple at a point on the upper unit hemisphere.

4. Use the flux form Green's theorem to prove the identity

$$\int\int_D f\nabla^2 g = \oint_{\partial D} f(\nabla g)\cdot \mathbf{n} ds - \int\int_D \nabla f \cdot \nabla g dA.$$

$$\oint_{\partial D} f(\nabla g)\cdot \vec{n}\, ds$$

$$\overset{\text{Green's thm (flux)}}{=} \iint_D \nabla\cdot (f(\nabla g))\, dA$$

$$= \iint_D \left((\nabla f \cdot \nabla g) + (f \nabla^2 g) \right) dA$$

移項即可

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5. Given the Maxwell's equations

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$$

prove that (a) $\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial^2 \mathbf{B}}{\partial t^2}$, (b) $\nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial t^2}$.

無中生有: $\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 E$

Proof: $\left[\nabla \times (\nabla \times \vec{E})\right]_i = \partial_j (\vec{\nabla} \times \vec{E})_k - \partial_k (\vec{\nabla} \times \vec{E})_j$

$$= \partial_j (\partial_i F_j - \partial_j F_j) - \partial_k(\partial_k E_i - \partial_i E_k)$$

$$= \partial_j \partial_i E_j - \partial_j^2 E_i - \partial_k^2 E_i - \partial_i \partial_k E_k$$

再無中生有，一加一减

$$= \partial_j \partial_i E_j - \partial_j^2 E_i - \partial_k^2 E_i - \partial_i \partial_k E_k - \partial_i^2 E_i + \partial_i^2 E_i$$

$$= \partial_i (\partial_j E_j - \partial_k E_k - \partial_i E_i) - (\nabla^2 E)$$

$$= \partial_i (\nabla \cdot E) - (\nabla^2 \vec{E})_i$$

b). $\nabla \times (\nabla \times \vec{E})$

$= \nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right)$

$= -\frac{\partial}{\partial t} (\nabla \times \vec{B})$

$= -\frac{\partial}{\partial t} \left(\frac{\partial \vec{E}}{\partial t} \right) = -\frac{\partial^2 \vec{E}}{\partial t^2}$

5. a). $\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$

(b). $\nabla \times (\nabla \times \vec{E}) = -\frac{\partial^2 \vec{E}}{\partial t^2}$

$\underbrace{\nabla (\nabla \cdot \vec{E})}_{= 0} - \nabla^2 E = -\frac{\partial^2 E}{\partial t}$

$\Rightarrow \nabla^2 E = -\frac{\partial^2 E}{\partial t^2}$

FYI: Solution is like $A \sin(kx - \omega t)$

measure wave.

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6. Verify that $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ and $\langle z_x, z_y, -1 \rangle$ give the same normal vector up to a multiple at a point on the upper unit hemisphere.

$$z = \sqrt{1-x^2-y^2}$$

$$z_x = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$z_y = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$$\underset{(\vec{n}_1)}{z_x \times z_y} = \left\langle \frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}}, -1 \right\rangle$$

$$\propto \langle -x, -y, -z \rangle$$

$$\propto \langle x\sqrt{x^2+y^2}, y\sqrt{x^2+y^2}, z\sqrt{x^2+y^2} \rangle$$

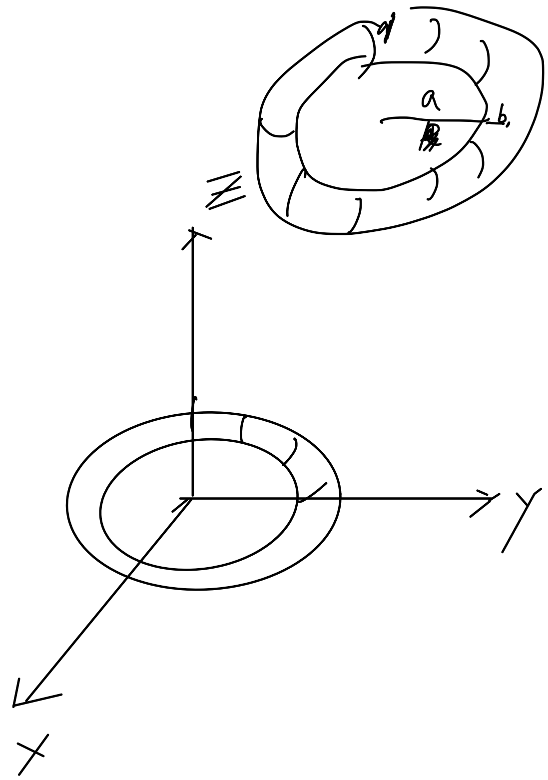
$$\vec{r}(u,v) = \langle \sin u \cos v, \sin u \sin v, \cos u \rangle$$

$$\vec{r}_u = \langle \cos u \cos v, \cos u \sin v, -\sin u \rangle$$

$$\vec{r}_v = \langle -\sin u \sin v, \sin u \cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle x\sqrt{x^2+y^2}, y\sqrt{x^2+y^2}, z\sqrt{x^2+y^2} \rangle$$

7. Find the surface area of a ring torus.



$$\vec{r}(\theta, \alpha) = \langle \cos\theta(b + a\cos\alpha), \sin\theta(b + a\cos\alpha), a\sin\alpha \rangle$$