HKUST MATH 102

Midterm Examination

Multivariable and Vector Calculus

21 Dec 2005

Answer ALL 8 questions

Time allowed – 180 minutes

Directions – This is a closed book examination. No talking or whispering are allowed. Work must be shown to receive points. An answer alone is not enough. Please write neatly. Answers which are illegible for the grader cannot be given credit.

Note that you can work on *both* sides of the paper and do not detach pages from this exam packet or unstaple the packet.

Student Name: ____________________

Student Number: ____________________

Tutorial Session: ____________________

Question No.	Marks
1	/20
2	/20
3	/20
4	/20
5	/20
6	/20
7	/20
8	/20
Total	/160

Problem 1

(a) Assume $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three dimensional vectors and if

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b} + \beta\mathbf{c}.$$

Use suffix notation to find λ, μ and β in terms of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Can you say something about the direction of the vector $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

(b) Let $\mathbf{a}$ be a constant vector and $\mathbf{r} = (x, y, z)$, use suffix notation to evaluate
(i) $\nabla\cdot\mathbf{r}$, (ii) $\nabla\cdot(\mathbf{a}\times\mathbf{r})$, (iii) $\nabla\times(\mathbf{a}\times\mathbf{r})$.

Problem 2

(a) Sketch and describe the parametric curve C

$$\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + (2\pi - t)\,\mathbf{k}, \qquad 0 \leqslant t \leqslant 2\pi.$$

Show the direction of increasing t. Find the project curve C onto the yz-plane.

(b) Find a change of parameter $t = g(\tau)$ for the semicircle

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \qquad 0 \leqslant t \leqslant \pi$$

such that (i) the semicircle is traced counterclockwise as τ varies over the interval [0, 1],
(ii) the semicircle is traced clockwise as τ varies over the interval [0, 0.5].

Problem 3

If a wheel with radius a rolls along a flat surface without slipping, a point P on the rim of the wheel traces a curve C, find the parametric equation of the point P. Suppose that the point P on the wheel is initially at the origin. Find also the arc length of the curve C if the wheel makes one complete turn (no need to carry out the integration).

Problem 4

(a) Verify the formula for the arc length element is cylindrical coordinates,

$$ds = \sqrt{\left(\frac{dr}{dt}\right)^2 + (r(t))^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt.$$

(b) Find a similar formula as in (a) for the arc length element in spherical coordinates.

(c) Use part (b) or otherwise, find the arc length of the curve in spherical coordinates:
$\rho = 2t$, $\theta = \ln t$, $\phi = \pi/6$; $1 \leqslant t \leqslant 5$.

Problem 5

Let $f(x,y) = \begin{cases} \dfrac{2xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) Is the function continuous at $(0,0)$?

(b) Calculate $f_x(x,y)$, $f_y(x,y)$, $f_{xy}(x,y)$ and $f_{yx}(x,y)$ at point $(x,y) \neq (0,0)$. Also calculate these derivatives at $(0,0)$.

(c) Is $f_{yx}(x,y)$ continuous at $(0,0)$?

(d) Explain why $f_{yx}(0,0) \neq f_{xy}(0,0)$.

Problem 6

Find the distance from the origin to the plane $x + 2y + 2z = 3$,

(a) using a geometric argument (no calculus),

(b) by reducing the problem to an unconstrained problem in two variables, and

(c) using the method of Lagrange multipliers.

Problem 7

(a) What condition must the constants a, b, and c satisfy to guarantee that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{ax^2 + bxy + cy^2}$$

exists. Prove your answer.

(b) Find $\dfrac{\partial^2}{\partial y \partial x} f(y^2,\, xy,\, -x^2)$ in terms of partial derivatives of the function f.

Problem 8

(a) Find the equation of the tangent plane at the point $(-1,1,0)$ to the surface

$$x^2 - 2y^2 + z^3 = -e^{-z}.$$

(b) The temperature at a point (x,y) on a metal plate in xy-plane is $T(x,y) = x^2 + y^3$ degrees Celsius.

(i) Find the rate of change of temperature at $(1,1)$ in the direction of $\mathbf{a} = 2\,\mathbf{i} + \mathbf{j}$.

(ii) An ant at $(1,1)$ wants to walk in the direction in which the temperature decreases most rapidly. Find a unit vector in that direction.

(c) Let C be the curve $x^{2/3} + y^{2/3} = a^{2/3}$ on the xy-plane, find the parametric equation of the curve C. Hence find the tangent line to the curve C at $(a,0)$.