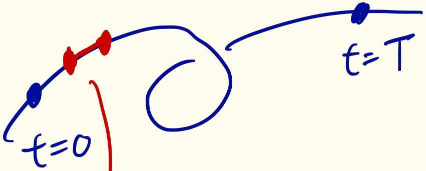
Last Time : $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$\vec{r}\,'(t)$: velocity (tangent vector)

$\vec{r}\,''(t)$: acceleration.

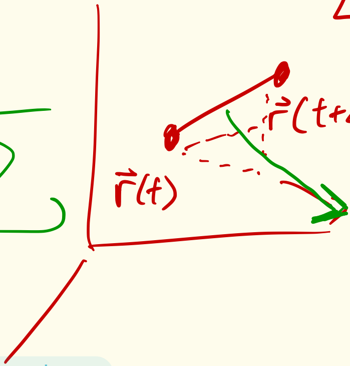
Arc length



$\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle$

$$\Rightarrow \boxed{\int_0^T |\vec{r}\,'(t)|\, dt.}$$

$$\sum \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}\ \Delta t$$



$$\sum \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

Ex Find the length of $\vec{r}(t) = \langle \sqrt{2}\,t, e^{t}, e^{-t} \rangle$ from $0 \le t \le 1$

$$\vec{r}'(t) = \langle \sqrt{2}, e^{t}, -e^{-t} \rangle$$

$$|\vec{r}'(t)| = \sqrt{\underbrace{2 + e^{2t} + e^{-2t}}_{= (e^{t}+e^{-t})^{2}}} = e^{t} + e^{-t}$$

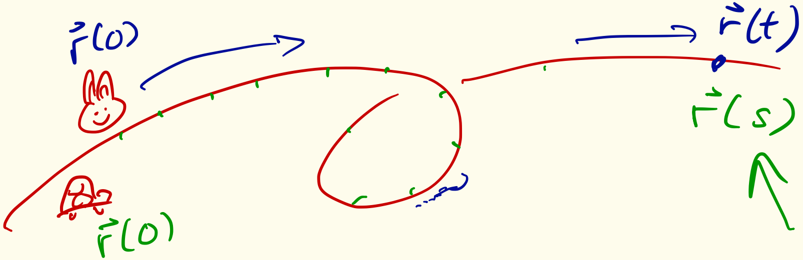
$$\int_0^1 (e^{t} + e^{-t})\,dt = e - \frac{1}{e}$$

$$\vec{r}(t) = \langle \sqrt{2}\, t^3, e^{t^3}, e^{-t^3} \rangle$$

$$= \langle \sqrt{2} \ln t, t, t^{-1} \rangle$$

different parametrization of <u>same curve</u>

<u>Arc Length Parametrization</u> : standardized parametrization.



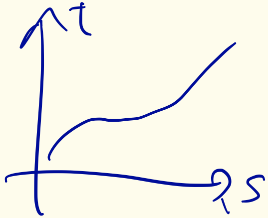
distance walked $= s$.

constant speed $|\vec{r}\,'(s)| = 1$

Q Given $\vec{r}(t)$, find $\vec{r}_{arc}(s)$ such that $|\vec{r}_{arc}\,'(s)| = 1$

$$s = s(t) = \int_0^t |\vec{r}\,'(\tau)|\, d\tau$$

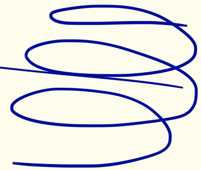
Arc length function



$\Leftrightarrow$ Rewrite t in terms of s

$$t = t(s).$$

$$\vec{r}_{arc}(s) = \vec{r}(t(s)).$$



<u>Ex</u> $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ $t=0$

$(1,0,0)$

$$s = s(t) = \int_0^t |\vec{r}\,'(\tau)|\, d\tau$$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$

$|\vec{r}\,'(t)| = \sqrt{2}$

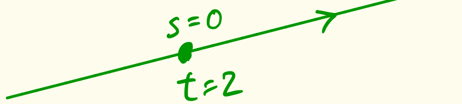
$$= \int_0^t \sqrt{2}\, d\tau = \sqrt{2}\, t$$

$$\Rightarrow t = \frac{s}{\sqrt{2}}$$

$$\boxed{\vec{r}_{arc}(s) = \left\langle \cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right\rangle}$$

<u>Ex</u> $\vec{r}(t) = \langle 4t,\ 2t+3,\ 5t \rangle$ $\qquad t=2$

$$= \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}$$



$$\vec{r}\,'(t) = \langle 4, 2, 5 \rangle, \quad |\vec{r}\,'(t)| = \sqrt{16+4+25} = \sqrt{45}$$

$$s = \int_2^t \sqrt{45}\, d\tau = \sqrt{45}\,(t-2).$$

$$t = \frac{s}{\sqrt{45}} + 2$$

$$\vec{r}_{arc}(s) = \left\langle 4\left(\frac{s}{\sqrt{45}}+2\right),\ 2\left(\frac{s}{\sqrt{45}}+2\right)+3,\ 5\left(\frac{s}{\sqrt{45}}+2\right) \right\rangle$$

$$= \langle \ \cdot\ ,\ \cdot\ ,\ \cdot\ \rangle$$

Proof $\vec{r}_{arc}(s) = \vec{r}(t(s))$

$$s = \int_0^t |\vec{r}\,'(\tau)|\,d\tau$$

$$\vec{r}\,'_{arc}(s) = \left(\vec{r}(t(s))\right)'$$

$$= \vec{r}\,'(t) \cdot \frac{dt}{ds}$$

$$\frac{ds}{dt} = |\vec{r}\,'(t)|$$

$$= \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

$$\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}} = \frac{1}{|\vec{r}\,'(t)|}$$

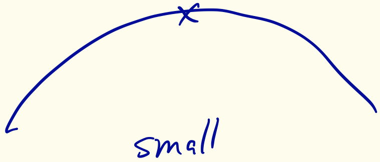
is a unit vector!

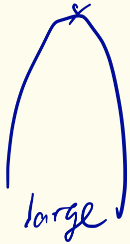
$$|\vec{r}\,'_{arc}(s)| = 1 \quad \checkmark$$

Made with Goodnotes

Curvature

Measures how "curved" the curve is.





$\Downarrow$

$\vec{F} \sim \vec{a}$

Def Curvature of $\vec{r}(t)$ is given by $\overset{K(s)}{=} |\vec{r}_{arc}''(s)|$

Ex Circle $\vec{r}(t) = \langle R\cos t, R\sin t \rangle$

$\vec{r}'(t) = \langle -R\sin t, R\cos t \rangle$, $\quad |\vec{r}'(t)| = R$

$s = \int_0^t |\vec{r}'(\tau)|\,d\tau = \int_0^t R\,d\tau = Rt \implies t = \frac{s}{R}$

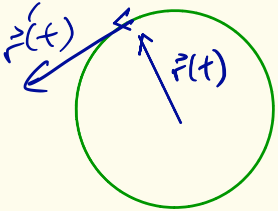
$\vec{r}_{arc}(s) = \langle R\cos\frac{s}{R}, R\sin\frac{s}{R} \rangle$

$\vec{r}'(s) = \langle -\sin\frac{s}{R}, \cos\frac{s}{R} \rangle$

$\vec{r}''(s) = \langle -\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R} \rangle$

$K(s) = |\vec{r}''(s)| = \frac{1}{R}$.

Thm $|\vec{r}(t)| = c$ const. $\Longrightarrow \vec{r}(t) \perp \vec{r}\,'(t)$



$$\Longrightarrow \vec{r}(t) \cdot \vec{r}(t) = c^2$$

$$\underbrace{\vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t)}_{2\,\vec{r}(t) \cdot \vec{r}\,'(t)} = 0$$

$$2\,\vec{r}(t) \cdot \vec{r}\,'(t) = 0$$

$\vec{r}_{arc}(s)$: $|\vec{r}_{arc}\,'(s)| = 1$

$$\Longrightarrow \vec{r}_{arc}\,'(s) \perp \vec{r}_{arc}\,''(s)$$

Thm Curvature without arc length parametrization

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$$\boxed{\text{Ignore } t:}$$ ← because I am Lazy!

$\kappa = \kappa(t)$

$\vec{r}\,' = \vec{r}\,'(t)$

$\vec{T} = \vec{T}(t)$

$\vdots$

$' = \frac{d}{dt}$

$$\kappa = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|r'|^3}$$

Pf $\kappa = |\vec{r}\,''(s(t))|$

$$s = \int_0^t |\vec{r}\,'(\tau)|\, d\tau$$

$\rightsquigarrow \frac{ds}{dt} = |\vec{r}\,'(t)|$

$$\vec{r}\,'(s) = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \vec{T} \quad \text{unit tangent vector.}$$

$$T' = \underbrace{\vec{r}\,''(s)}_{\text{“}\kappa\text{”}} \cdot \underbrace{\frac{ds}{dt}}_{= |\vec{r}\,'|} \Longrightarrow \kappa = \frac{|T'|}{|\vec{r}\,'|}$$

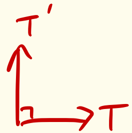
$$T = \frac{\vec{r}\,'}{|\vec{r}\,'|} \implies \vec{r}\,' = |\vec{r}\,'|\, T = \frac{ds}{dt}\cdot T$$

$$\vec{r}\,'' = \frac{d^2s}{dt^2}\, T + \frac{ds}{dt}\cdot T'$$

$$\vec{r}\,' \times \vec{r}\,'' = \cancel{T \times T}\ \left(\frac{ds}{dt}\right)^2 \cdot T \times T'$$

$|T| = 1$

constant length $\implies T \perp T'$



$|T \times T'| = |T|\,|T'|$

$$|\vec{r}\,' \times \vec{r}\,''| = \left(\frac{ds}{dt}\right)^2 \cdot |T \times T'| = \underbrace{\left(\frac{ds}{dt}\right)^2}_{= |\vec{r}\,'|^2} \underbrace{|T|}_{= 1}\, |T'|$$

$$\implies \kappa = \frac{|T'|}{|\vec{r}\,'|} = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} \quad //$$

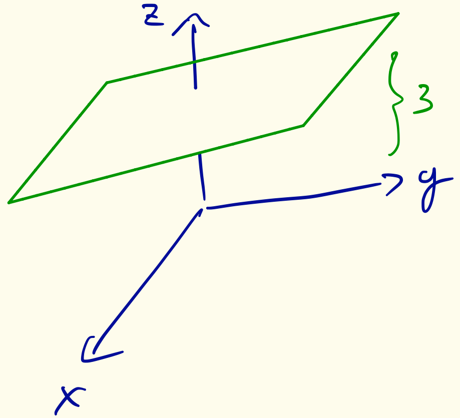
Functions of $\underbrace{\text{several}}_{2}$ variables

Input : 2 numbers (x,y)

Output : 1 number $z = f(x,y)$

height

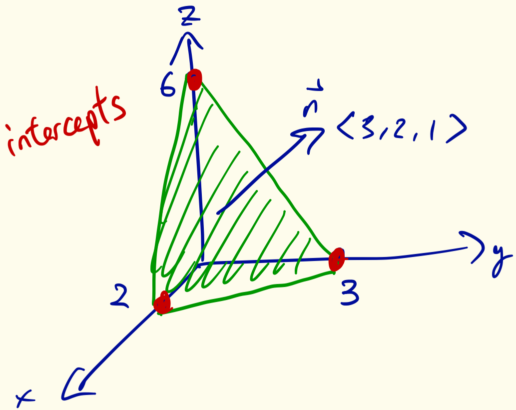
<u>Ex</u> $f(x,y) = 3$



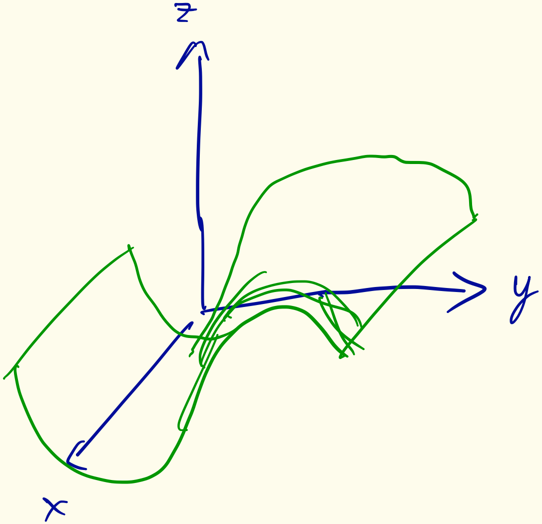
<u>Ex</u> $z = f(x,y) = 6 - 3x - 2y$

plane : $\vec{n} = \langle 3,2,1 \rangle$

$\vec{r}_0 = \langle 0,0,6 \rangle$

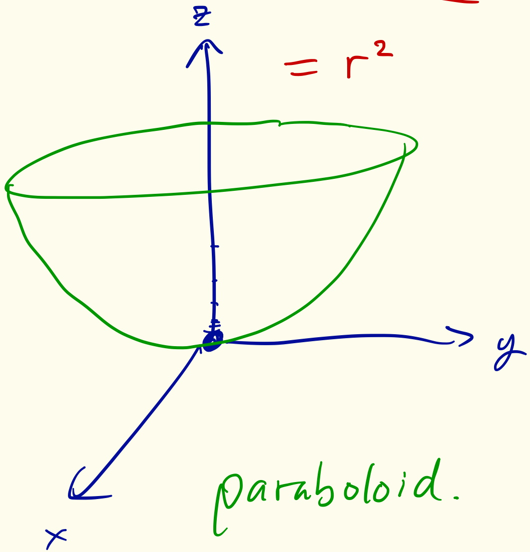


Ex $f(x,y) = \sin y$ (does not depend on x)



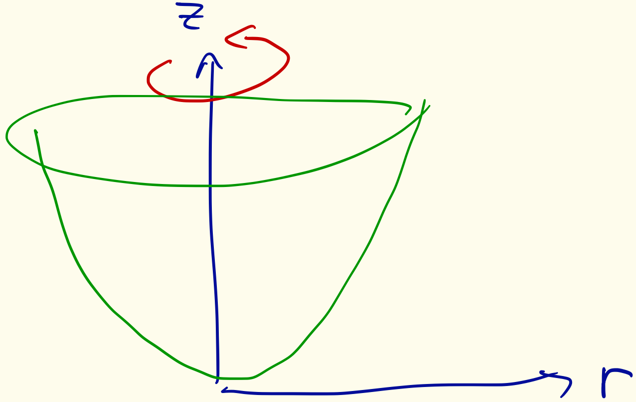
Ex $f(x,y) = \underline{\underline{x^2+y^2}}$

$= r^2$



paraboloid.

polar coordinate $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$



$$f(x,y) = x^2 + y^2$$

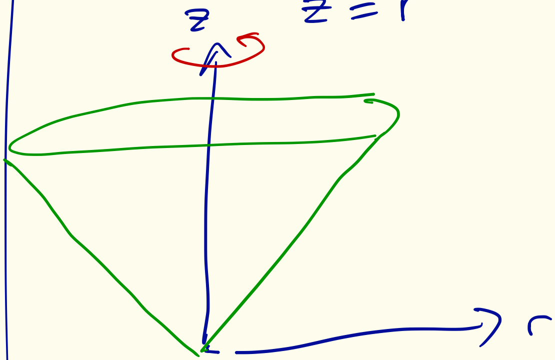
$$z = r^2$$

↓

rotational symmetric

Ex $f(x,y) = \sqrt{x^2+y^2}$

$z = r$



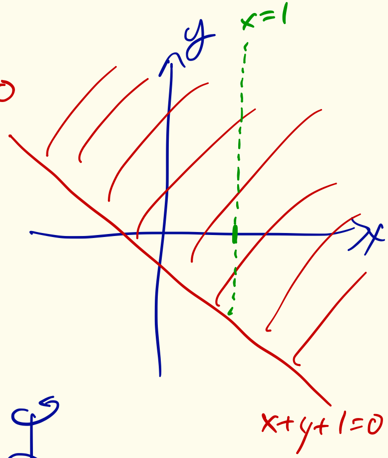
cone.

Domain : Set of (x,y) that $f(x,y)$ is defined.

Ex $f(x,y)=\dfrac{\sqrt{x+y+1}}{x-1}$

$x+y+1\geq 0$

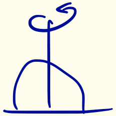
$x\neq 1$



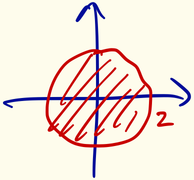
Ex $f(x,y)=\sqrt{4-x^2-y^2}$

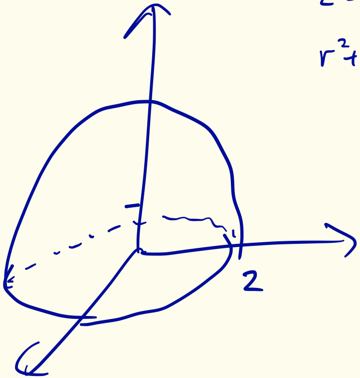
$z=\sqrt{4-r^2}$

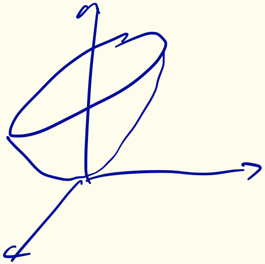
$r^2+z^2=4$



$x^2+y^2<4$



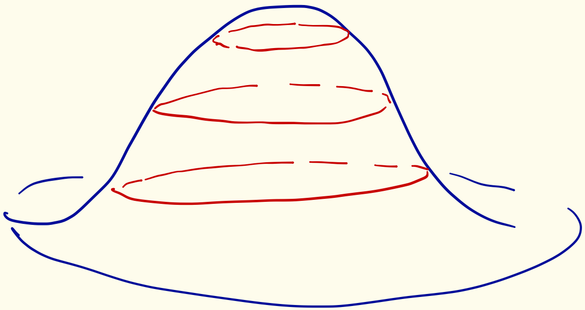


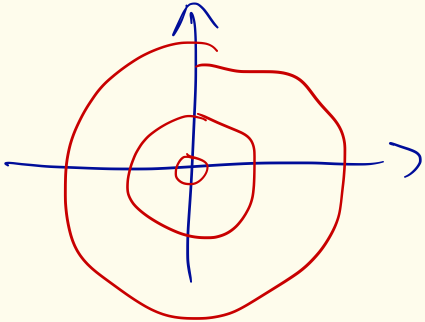


Ex $f(x,y)=x^2+4y^2$ $\quad (x,y)\in\mathbb{R}^2$

Level Curves "Contour Lines" $\Longrightarrow$ x-y plane

= fixed height

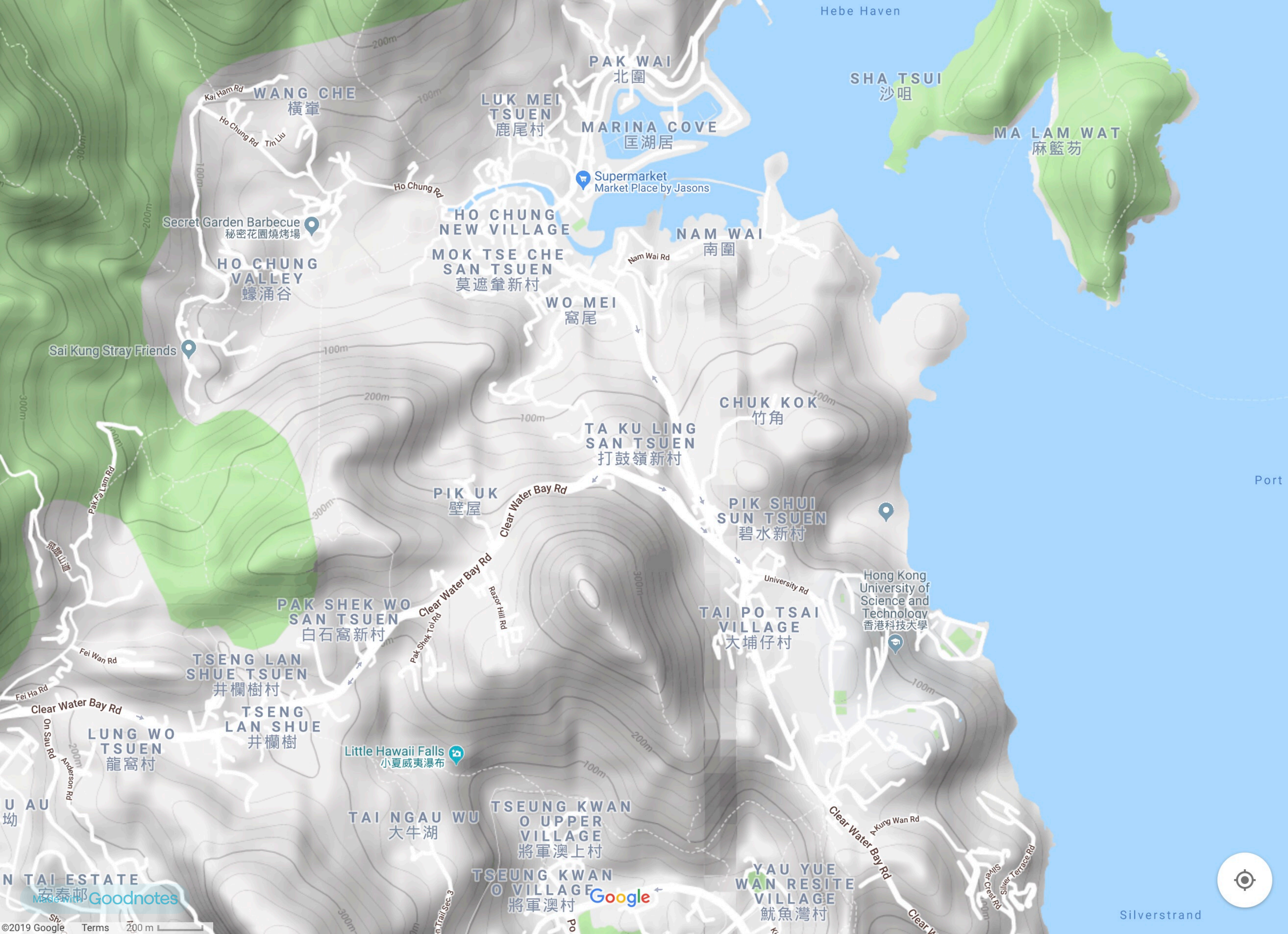




Def $f(x,y) = k$

constant

(choose many $\underline{k}$ on the same diagram)


Hebe Haven
PAK WAI
北圍
SHA TSUI
沙咀
MA LAM WAT
麻藍笏
WANG CHE
橫輋
LUK MEI
TSUEN
鹿尾村
MARINA COVE
匡湖居
Kai Ham Rd
Ho Chung Rd
Tin Liu
Ho Chung Rd
Supermarket
Market Place by Jasons
Secret Garden Barbecue
秘密花園燒烤場
HO CHUNG
NEW VILLAGE
NAM WAI
南圍
Nam Wai Rd
MOK TSE CHE
SAN TSUEN
莫遮輋新村
HO CHUNG
VALLEY
蠔涌谷
WO MEI
窩尾
Sai Kung Stray Friends
CHUK KOK
竹角
TA KU LING
SAN TSUEN
打鼓嶺新村
Port
PIK UK
壁屋
Clear Water Bay Rd
PIK SHUI
SUN TSUEN
碧水新村
Pak Fa Lam Rd
飛鵝山道
University Rd
Hong Kong
University of
Science and
Technology
香港科技大學
Clear Water Bay Rd
Razor Hill Rd
PAK SHEK WO
SAN TSUEN
白石窩新村
TAI PO TSAI
VILLAGE
大埔仔村
Pak Shek Toi Rd
Fei Wan Rd
TSENG LAN
SHUE TSUEN
井欄樹村
Fei Ha Rd
Clear Water Bay Rd
TSENG
LAN SHUE
井欄樹
LUNG WO
TSUEN
龍窩村
On Sau Rd
Anderson Rd
Little Hawaii Falls
小夏威夷瀑布
U AU
坳
TAI NGAU WU
大牛湖
TSEUNG KWAN
O UPPER
VILLAGE
將軍澳上村
Clear Water Bay Rd
Kung Wan Rd
Silver Terrace Rd
Silver Crest Rd
N TAI ESTATE
安泰邨
TSEUNG KWAN
O VILLAGE
將軍澳村
YAU YUE
WAN RESITE
VILLAGE
魷魚灣村
Silverstrand
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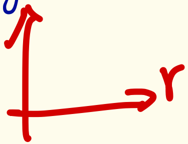
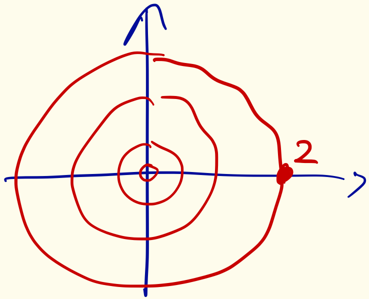
Ex $f(x,y) = \sqrt{4-x^2-y^2}$ z

$z^2 = 4 - r^2$

$z = \sqrt{4-r^2}$

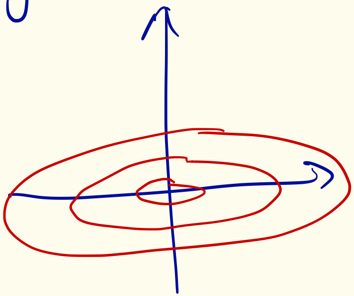




Ex $f(x,y) = x^2 + 4y^2$

$x^2 + 4y^2 = k$

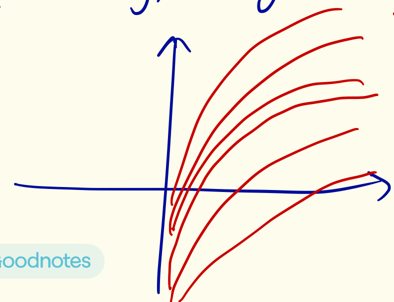
for different k:



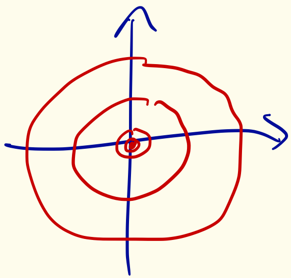
Ex $f(x,y) = y - \ln x$

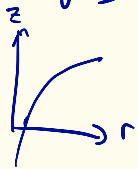
$\rightarrow$ $y - \ln x = k$

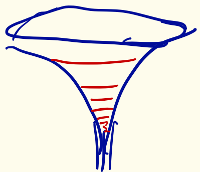
$y = \ln x + k$



Ex $f(x,y) = \ln(x^2+y^2)$







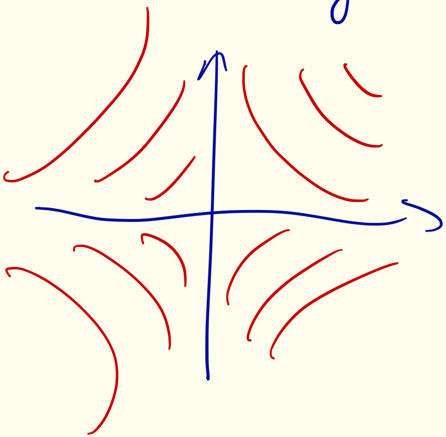
Ex $f(x,y) = \ln x^2 + \ln y^2$

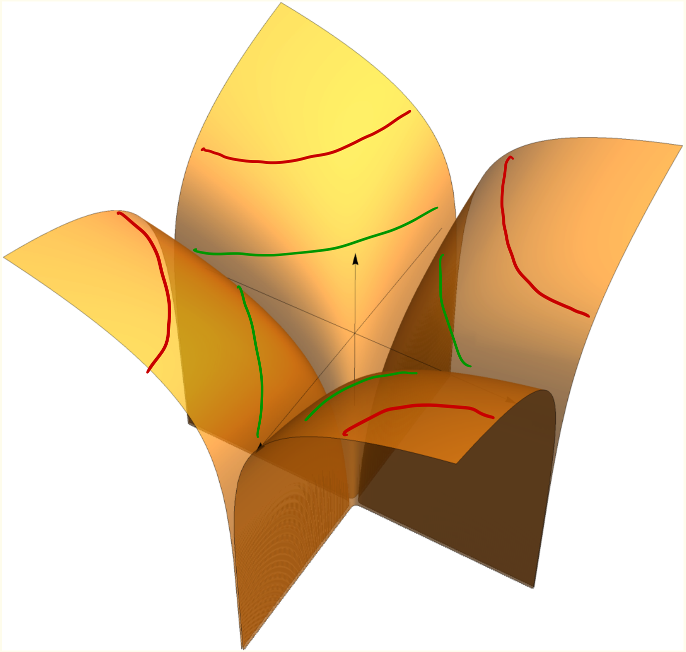
$\ln x^2 + \ln y^2 = k$

$2\ln x + 2\ln y$

$\cancel{2}\,\cancel{\ln}\, xy = e^{k/2}$

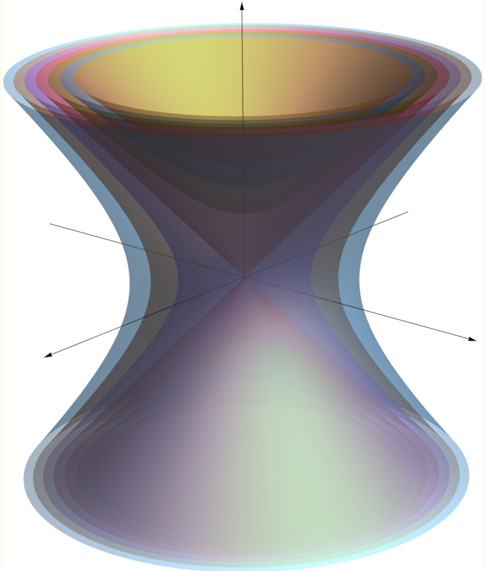
$\Rightarrow \quad xy = \text{const.}$





$f(x,y,z) = k$

Level surfaces.

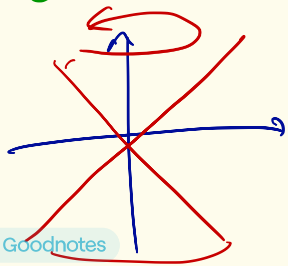


<u>Ex</u> $x^2+y^2-z^2$

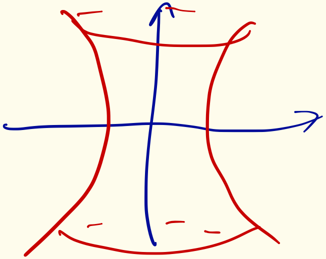
$$x^2+y^2-z^2=k$$

$$r^2-z^2=k$$

$k=0: \ r^2=z^2$



$k=1 \quad r^2-z^2=1$



$k=-1 \quad r^2-z^2=-1$