MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #0 • Dot and Cross Products (Review)

1. (★) Given three points in $\mathbb{R}^3$:

$$A(1,2,3),\ B(4,0,5)\ \text{ and }\ C(x,6,4)$$

Determine the number of possible value(s) of x such that the triangle ABC has a right angle.

> **Solution:** $\angle A$ is a right-angle if and only if $\overrightarrow{AB} \perp \overrightarrow{AC}$, if and only if:
>
> $$\begin{aligned} 0 &= \overrightarrow{AB} \cdot \overrightarrow{AC} = \langle 3,-2,2\rangle \cdot \langle x-1,4,1\rangle \\ &= 3x - 9 \end{aligned}$$
>
> Hence, $\angle A$ is a right-angle when $\boxed{x = 3}$.
>
> Similarly, $\angle B$ is a right-angle if and only if $\overrightarrow{BA} \perp \overrightarrow{BC}$, if and only if:
>
> $$\begin{aligned} 0 &= \overrightarrow{BA} \cdot \overrightarrow{BC} = \langle -3,2,-2\rangle \cdot \langle x-4,6,-1\rangle \\ &= 26 - 3x \end{aligned}$$
>
> $\angle B$ is a right-angle when $\boxed{x = 26/3}$.
>
> Finally, $\angle C$ is a right-angle if and only if $\overrightarrow{CA} \perp \overrightarrow{CB}$, if and only if:
>
> $$\begin{aligned} 0 &= \overrightarrow{CA} \cdot \overrightarrow{CB} = \langle 1-x,-4,-1\rangle \cdot \langle 4-x,-6,1\rangle \\ &= x^2 - 5x + 27 \end{aligned}$$
>
> As the quadratic equation has $\Delta = (-5)^2 - 4(27) < 0$, it has no real root.
>
> To conclude, there are $\boxed{\text{two}}$ possible values of x such that ABC has a right-angle.

2. (★★) Let $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.

   (a) Show that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal (i.e. $\mathbf{u} \perp \mathbf{v}$, $\mathbf{v} \perp \mathbf{w}$ and $\mathbf{w} \perp \mathbf{u}$).

   > **Solution:** Clearly, $\mathbf{u} \cdot \mathbf{v} = (2)(1) + (1)(2) + (-2)(2) = 0$, so $\mathbf{u} \perp \mathbf{v}$. By the definition of cross product, $\mathbf{u} \times \mathbf{v}$ is a vector which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Therefore, we have $\mathbf{w} \perp \mathbf{u}$ and $\mathbf{w} \perp \mathbf{v}$ as well.

   (b) Given any vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in $\mathbb{R}^3$, show that:

   $$\mathbf{r} = \frac{\mathbf{r} \cdot \mathbf{u}}{|\mathbf{u}|^2}\mathbf{u} + \frac{\mathbf{r} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} + \frac{\mathbf{r} \cdot \mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}.$$

   [Hint: You may use the fact that since $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are mutually orthogonal and non-zero, the vector $\mathbf{r}$ can be expressed as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, i.e.

   $$\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

   Solve for the scalars a, b and c.]

Solution: Let $\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$. Taking the dot product with $\mathbf{u}$ on both sides, we get:

$$\begin{aligned}
\mathbf{r}\cdot\mathbf{u} &= (a\mathbf{u} + b\mathbf{v} + c\mathbf{w})\cdot\mathbf{u} \\
\mathbf{r}\cdot\mathbf{u} &= a\mathbf{u}\cdot\mathbf{u} + b\mathbf{v}\cdot\mathbf{u} + c\mathbf{w}\cdot\mathbf{u} \\
\mathbf{r}\cdot\mathbf{u} &= a\,|\mathbf{u}|^2 + b(0) + c(0) \qquad \text{(since } \mathbf{v}\perp\mathbf{u} \text{ and } \mathbf{w}\perp\mathbf{u}\text{)} \\
a &= \frac{\mathbf{r}\cdot\mathbf{u}}{|\mathbf{u}|^2}
\end{aligned}$$

Similarly, taking the dot product with $\mathbf{v}$ and $\mathbf{w}$ on both sides of $\mathbf{r} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$ will give

$$b = \frac{\mathbf{r}\cdot\mathbf{v}}{|\mathbf{v}|^2} \quad \text{and} \quad c = \frac{\mathbf{r}\cdot\mathbf{w}}{|\mathbf{w}|^2}$$

respectively. This shows the desired result.

(c) Express the vector $\mathbf{i}$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

Solution: Note that the result in (b) applies to any arbitrary vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ including the vector $\mathbf{i}$ (for which we have $x = 1$, $y = z = 0$). Therefore,

$$\mathbf{i} = \frac{\mathbf{i}\cdot\mathbf{u}}{|\mathbf{u}|^2}\mathbf{u} + \frac{\mathbf{i}\cdot\mathbf{v}}{|\mathbf{v}|^2}\mathbf{v} + \frac{\mathbf{i}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}$$

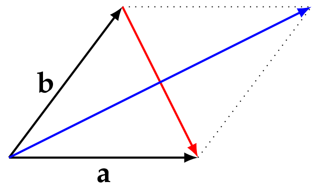
By straight-forward computations, we have:

$$\begin{aligned}
\frac{\mathbf{i}\cdot\mathbf{u}}{|\mathbf{u}|^2} &= \frac{2}{9} \\
\frac{\mathbf{i}\cdot\mathbf{v}}{|\mathbf{v}|^2} &= \frac{1}{9} \\
\mathbf{w} &= \mathbf{u}\times\mathbf{v} = 6\mathbf{i} - 6\mathbf{j} + 3\mathbf{k} \\
\frac{\mathbf{i}\cdot\mathbf{w}}{|\mathbf{w}|^2} &= \frac{6}{81} = \frac{2}{27}
\end{aligned}$$

Therefore,

$$\mathbf{i} = \frac{2}{9}\mathbf{u} + \frac{1}{9}\mathbf{v} + \frac{2}{27}\mathbf{w}.$$

3. (★) The figure below shows two vectors **a** and **b** which span a parallelogram. The vectors in blue and red represent the two diagonals of the parallelogram.



(a) Express the red and the blue vectors in terms of **a** and **b**.

Solution: Red vector is $\mathbf{a} - \mathbf{b}$; Blue vector is $\mathbf{a} + \mathbf{b}$.

(b) By considering the dot product, show that: $|\mathbf{a}| = |\mathbf{b}|$ if and only if the diagonals of the parallelogram are orthogonal to each other.

Solution: ($\Longrightarrow$) Given that $|\mathbf{a}| = |\mathbf{b}|$, we need to show the blue and the red vectors are orthogonal:

$$\begin{aligned}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b} && \text{(expansion)}\\ &= |\mathbf{a}|^2 + \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{b} - |\mathbf{b}|^2 \\ &= |\mathbf{a}|^2 - |\mathbf{b}|^2 \\ &= 0 && \text{(given } |\mathbf{a}| = |\mathbf{b}|\text{)}\end{aligned}$$

Therefore, the diagonals (which are represented by $\mathbf{a} - \mathbf{b}$ and $\mathbf{a} + \mathbf{b}$) are orthogonal to each other.
($\Longleftarrow$) Given that the diagonals $\mathbf{a} - \mathbf{b}$ and $\mathbf{a} + \mathbf{b}$ are orthogonal, we have:

$$\begin{aligned}(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) &= 0\\ \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} - \mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b} &= 0 && \text{(expansion)}\\ |\mathbf{a}|^2 + \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{b} - |\mathbf{b}|^2 &= 0\\ |\mathbf{a}|^2 - |\mathbf{b}|^2 &= 0.\end{aligned}$$

Therefore, we have $|\mathbf{a}|^2 = |\mathbf{b}|^2$, and since lengths of vectors must be non-negative, we get $|\mathbf{a}| = |\mathbf{b}|$.

4. (★) Let $\mathbf{u} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be a variable **unit** vector in $\mathbb{R}^3$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(a) Find x, y and z such that $\mathbf{u}\cdot\mathbf{v}$ is the maximum possible. Explain your answer.

Solution: $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta = |\mathbf{v}|\cos\theta$ since $\mathbf{u}$ is unit. As $\mathbf{v}$ is a fixed vector, the value of $\mathbf{u}\cdot\mathbf{v}$ is completely determined by the angle θ between $\mathbf{u}$ and $\mathbf{v}$. Since $\cos\theta$ is the maximum when $\theta = 0$ (at which we have $\cos\theta = 1$), the dot product $\mathbf{u}\cdot\mathbf{v}$ is the maximum possible when $\mathbf{u}$ is parallel to $\mathbf{v}$. Therefore,

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\sqrt{14}},$$

and so $x = \frac{1}{\sqrt{14}}$, $y = \frac{2}{\sqrt{14}}$ and $z = \frac{3}{\sqrt{14}}$.

(b) Find x, y and z such that $|\mathbf{u} \times \mathbf{v}|$ is the maximum possible. Explain your answer.

Solution: Since $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = |\mathbf{v}|\sin\theta$, it is the maximum when $\theta = \frac{\pi}{2}$ (at which $\sin\theta = 1$). Therefore, any unit vector $\mathbf{u}$ which is orthogonal to $\mathbf{v}$ will make $|\mathbf{u} \times \mathbf{v}|$ achieve the maximum possible value. As $\mathbf{u} \cdot \mathbf{v} = x + 2y + 3z$, the set of (x, y, z)'s such that $|\mathbf{u} \times \mathbf{v}|$ are those which satisfy:

$$x + 2y + 3z = 0 \quad \text{and} \quad x^2 + y^2 + z^2 = 1.$$

5. (★★) Given two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^3$, prove the following:

(a) Cauchy-Schwarz's Inequality: $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}|$

Solution:

$$|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,|\cos\theta| \leq |\mathbf{a}|\,|\mathbf{b}|\,.$$

Here θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. We have used the fact that $\cos\theta \leq 1$.

(b) Triangle Inequality: $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$

Solution: There is not much we can do with $|\mathbf{a} + \mathbf{b}|$. However, using the fact that $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$ for any vector $\mathbf{u}$, we may consider:

$$\begin{aligned}
|\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) \\
&= \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\
&= |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \\
&\leq |\mathbf{a}|^2 + 2\,|\mathbf{a} \cdot \mathbf{b}| + |\mathbf{b}|^2 && \text{(since } x \leq |x| \text{ for any real } x\text{)} \\
&\leq |\mathbf{a}|^2 + 2\,|\mathbf{a}|\,|\mathbf{b}| + |\mathbf{b}|^2 && \text{(from (a))} \\
&= (|\mathbf{a}| + |\mathbf{b}|)^2
\end{aligned}$$

As both $|\mathbf{a} + \mathbf{b}|$ and $|\mathbf{a}| + |\mathbf{b}|$ are positive, we conclude that:

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|\,.$$

(c) If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, show that $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$.

Solution: Proceed as in (b), the first three equalities give:

$$|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2\,.$$

If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, then $\mathbf{a} \cdot \mathbf{b} = 0$ and so $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$.

6. (★) Let A, B and C be the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively in the three dimensional space, and O be the origin $(0, 0, 0)$. Denote $[ABC]$ the area of the triangle with vertices A, B and C (analogously for $[OAB]$, $[OBC]$, etc.). Show that:

$$[ABC]^2 = [OAB]^2 + [OBC]^2 + [OCA]^2.$$

With the help of a diagram, explain why this result can be regarded as the *three-dimensional analogue of the Pythagoreas' Theorem*.

Solution: First we find $[ABC]$ using cross product.

$$\begin{aligned}
\overrightarrow{AB} &= \langle 0, b, 0\rangle - \langle a, 0, 0\rangle = \langle -a, b, 0\rangle \\
\overrightarrow{AC} &= \langle 0, 0, c\rangle - \langle a, 0, 0\rangle = \langle -a, 0, c\rangle \\
\overrightarrow{AB} \times \overrightarrow{AC} &= \langle bc, ac, ab\rangle \\
\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| &= \sqrt{(bc)^2 + (ac)^2 + (ab)^2} \\
[ABC] &= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{(bc)^2 + (ac)^2 + (ab)^2} \\
[ABC]^2 &= \frac{1}{4}\left((bc)^2 + (ac)^2 + (ab)^2\right)
\end{aligned}$$

The other triangles are right-angled ones whose bases and heights can be easily found from the diagram.

$$\begin{aligned}
[OAB] &= \frac{ab}{2} \\
[OAC] &= \frac{ac}{2} \\
[OBC] &= \frac{bc}{2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
[OAB]^2 + [OAC]^2 + [OBC]^2 &= \left(\frac{ab}{2}\right)^2 + \left(\frac{ac}{2}\right)^2 + \left(\frac{bc}{2}\right)^2 \\
&= \frac{1}{4}\left((bc)^2 + (ac)^2 + (ab)^2\right) \\
&= [ABC]^2,
\end{aligned}$$

as desired.

$[ABC]$ is analogous to the hypotenuse of a right-angled triangle in 2D, whiles $[OAB]$, $[OAC]$ and $[OBC]$ are analogous to the sides of the triangle. The 2D Pythagoreas' Theorem asserts the the square of the hypotenuse is the sum of squares of the *lengths* of the sides. Analogously, the squared *area* of the hypotenuse face ABC is the sum of squares of the *areas* of the other three sides.

7. (★★) Given three non-zero vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, provide a *geometric explanation* to each of the following facts:

   (a) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

   Solution: Since the "angle" between $\mathbf{u}$ and itself is 0, we have $|\mathbf{u} \times \mathbf{u}| = |\mathbf{u}|\,|\mathbf{u}| \sin 0 = 0$. The only vector with zero magnitude is the zero vector, so $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

   (b) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$

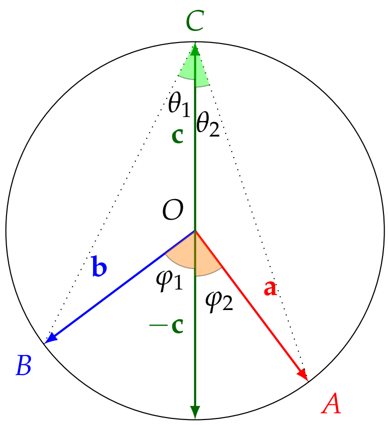
   Solution: The cross product $\mathbf{u} \times \mathbf{v}$ is geometrically defined as a vector in $\mathbb{R}^3$ which is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. Since the dot product between two perpendicular vectors are zero, so:

$$(\mathbf{u} \times \mathbf{v}) \perp \mathbf{u} \Rightarrow (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$$
$$(\mathbf{u} \times \mathbf{v}) \perp \mathbf{v} \Rightarrow (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0.$$

   (c) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is a vector on the plane spanned by $\mathbf{u}$ and $\mathbf{v}$.

   Solution: Since $(\mathbf{u} \times \mathbf{v})$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, it is a **normal vector** to the plane Π spanned by $\mathbf{u}$ and $\mathbf{v}$. Now consider $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$. No matter what $\mathbf{w}$ we pick, the product $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ must be a vector in $\mathbb{R}^3$ that is perpendicular to $(\mathbf{u} \times \mathbf{v})$. Recall that $(\mathbf{u} \times \mathbf{v})$ is a normal vector to the plane Π. The vector $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ being *perpendicular* to the *normal* vector $(\mathbf{u} \times \mathbf{v})$ must lie on the plane Π.

8. (★★★) The diagram below shows a circle with radius r centered at O. Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$. The purpose of the problem is to use dot products to show that the angle at the center of a circle is twice the corresponding angle at the circumference. Precisely, with the notations in the diagram below, we want to show $\angle BOA = 2\angle BCA$. We will prove this by showing $\varphi_1 = 2\theta_1$, and $\varphi_2 = 2\theta_2$ can be proved in a similar way. Follow the steps structured below:



(a) Show that $\cos\varphi_1 = -\dfrac{\mathbf{b}\cdot\mathbf{c}}{r^2}$. Recall that r is the radius of the circle.

Solution: According to the geometric definition of dot products (applied to vectors $\mathbf{b}$ and $-\mathbf{c}$), we have:

$$\mathbf{b}\cdot(-\mathbf{c}) = |\mathbf{b}|\,|-\mathbf{c}|\cos\varphi_1.$$

Since both $\mathbf{b}$ and $-\mathbf{c}$ represent the radii of the circle, their lengths are both r. Therefore, by rearranging the above, we get:

$$\cos\varphi_1 = \frac{\mathbf{b}\cdot(-\mathbf{c})}{|\mathbf{b}|\,|-\mathbf{c}|} = \frac{-\mathbf{b}\cdot\mathbf{c}}{r\cdot r} = -\frac{\mathbf{b}\cdot\mathbf{c}}{r^2}.$$

(b) Show that $\cos\theta_1 = \dfrac{r^2-\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}-\mathbf{c}|\,|\mathbf{c}|}$.

Solution: Similarly, apply the geometric definition of dot product on vectors $\overrightarrow{CO}$ and $\overrightarrow{CB}$. Note that $\overrightarrow{CB} = \mathbf{b}-\mathbf{c}$ and $\overrightarrow{CO} = -\mathbf{c}$, so:

$$(\mathbf{b}-\mathbf{c})\cdot(-\mathbf{c}) = |\mathbf{b}-\mathbf{c}||-\mathbf{c}|\cos\theta_1 = |\mathbf{b}-\mathbf{c}||\mathbf{c}|\cos\theta_1.$$

Expanding the LHS: we have

$$(\mathbf{b}-\mathbf{c})\cdot(-\mathbf{c}) = -\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{c} = -\mathbf{b}\cdot\mathbf{c}+|\mathbf{c}|^2 = -\mathbf{b}\cdot\mathbf{c}+r^2.$$

Therefore,

$$r^2-\mathbf{b}\cdot\mathbf{c} = |\mathbf{b}-\mathbf{c}||\mathbf{c}|\cos\theta_1,$$

which yields the desired result after rearrangement.

(c) Showing that $|\mathbf{b}-\mathbf{c}|^2 = 2(r^2-\mathbf{b}\cdot\mathbf{c})$.

Solution: Use the fact that $|\mathbf{u}|^2 = \mathbf{u}\cdot\mathbf{u}$:

$$\begin{aligned}
|\mathbf{b}-\mathbf{c}|^2 &= (\mathbf{b}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c}) \\
&= \mathbf{b}\cdot\mathbf{b}-\mathbf{b}\cdot\mathbf{c}-\mathbf{c}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{c} \\
&= |\mathbf{b}|^2-2\mathbf{b}\cdot\mathbf{c}+|\mathbf{c}|^2 \\
&= r^2-2\mathbf{b}\cdot\mathbf{c}+r^2 = 2(r^2-\mathbf{b}\cdot\mathbf{c}).
\end{aligned}$$

(d) Using the result proved in the previous parts, show that $\cos^2\theta_1 = \dfrac{r^2 - \mathbf{b}\cdot\mathbf{c}}{2r^2}$.

Solution:

$$\begin{aligned}
\cos^2\theta_1 &= \frac{(r^2-\mathbf{b}\cdot\mathbf{c})^2}{|\mathbf{b}-\mathbf{c}|^2\,|\mathbf{c}|^2} && \text{(from (b))}\\
&= \frac{(r^2-\mathbf{b}\cdot\mathbf{c})^2}{2(r^2-\mathbf{b}\cdot\mathbf{c})\,r^2} && \text{(from (c))}\\
&= \frac{r^2-\mathbf{b}\cdot\mathbf{c}}{2r^2}.
\end{aligned}$$

(e) Finally, find a relation between $\cos^2\theta_1$ and $\cos\varphi_1$, and conclude that $\varphi_1 = 2\theta_1$. [Hint: Double angle formula for cos.]

Solution:

$$\begin{aligned}
\cos^2\theta_1 &= \frac{r^2-\mathbf{b}\cdot\mathbf{c}}{2r^2} = \frac{1}{2} + \frac{1}{2}\left(-\frac{\mathbf{b}\cdot\mathbf{c}}{r^2}\right)\\
&= \frac{1}{2} + \frac{1}{2}\cos\varphi_1 && \text{(from (a))}\\
&= \cos^2\frac{\varphi_1}{2} && \text{(double/half-angle formula)}
\end{aligned}$$

Therefore, $\theta_1 = \frac{\varphi_1}{2}$, as desired.

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #1 • Lines, Planes and Curves

1. (★) Consider the two straight-lines:

$$L_1 : \mathbf{r}_1(t) = \langle 1, 2, 3\rangle + t\,\langle 1, -1, -1\rangle$$
$$L_2 : \mathbf{r}_2(t) = \langle 2 + t, 3 - 3t, -2 + 3t\rangle$$

(a) Show that L_1 and L_2 intersect each other. Find the coordinates of the intersection point.

Solution: Note that $\mathbf{r}_1(t)$ represents the position of the "particle" travelling along L_1 at time t. Similarly for $\mathbf{r}_2(t)$. Therefore, even if the lines L_1 and L_2 intersect, the two "particles" may not reach the the intersection point at the same time t. To find the intersection point, we need to find s and t such that:

$$\mathbf{r}_1(t) = \mathbf{r}_2(s)$$
$$\langle 1 + t, 2 - t, 3 - t\rangle = \langle 2 + s, 3 - 3s, -2 + 3s\rangle$$

which is equivalent to the system:

$$1 + t = 2 + s$$
$$2 - t = 3 - 3s$$
$$3 - t = -2 + 3s$$

Solving the system, we get $t = 2$ and $s = 1$.

$$\mathbf{r}_1(t = 2) = \langle 3, 0, 1\rangle$$
$$\mathbf{r}_2(s = 1) = \langle 3, 0, 1\rangle$$

Therefore, the intersection point is $\boxed{(3, 0, 1)}$.

(b) Find an equation of the plane containing both L_1 and L_2.

Solution: L_1 is parallel to the vector $\langle 1, -1, -1\rangle$. For L_2, the parametrization can be rewritten as:

$$\mathbf{r}_2(t) = \langle 2, 3, -2\rangle + t\langle 1, -3, 3\rangle$$

Therefore, L_2 is parallel to the vector $\langle 1, -3, 3\rangle$.
The required plane contains both L_1 and L_2, hence is parallel to both $\langle 1, -1, -1\rangle$ and $\langle 1, -3, 3\rangle$. The normal vector of the plane is, therefore, can be taken to be the cross-product of these two vectors:

$$\langle 1, -1, -1\rangle \times \langle 1, -3, 3\rangle = \langle -6, -4, -2\rangle = -2\langle 3, 2, 1\rangle$$

For simplicity, we take $\mathbf{n} = \langle 3, 2, 1\rangle$ which is also a normal vector to the plane.
From (a), the plane contains the point $(3, 0, 1)$. By substituting $\langle A, B, C\rangle = \langle 3, 2, 1\rangle$ and $(x_0, y_0, z_0) = (3, 0, 1)$, we find the equation of the plane is given by:

$$3x + 2y + z = 3(3) + 2(0) + 1(1)$$

After simplification, we get $\boxed{3x + 2y + z = 10}$.

2. (★) Consider the following four points in three-dimensional space:

$$A(0,2,-1),\ B(4,0,-1),\ C(7,-3,0)\ \text{ and }\ D(\tfrac{1}{3},\tfrac{1}{6},\tfrac{1}{9})$$

Determine whether or not these four points are coplanar (i.e. contained in a single plane).

Solution: First we find an equation of the plane containing A, B and C. Then, we will substitute the coordinates of D into the equation to see whether D lies on that plane.

The two "ingredients" of finding the equation of a plane are (i) a given point on the plane; and (ii) a normal vector to the plane. In order to find the normal vector the plane through A, B and C, we take the cross product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \langle 4,0,-1\rangle - \langle 0,2,-1\rangle = \langle 4,-2,0\rangle$$
$$\overrightarrow{AC} = \langle 7,-3,0\rangle - \langle 0,2,-1\rangle = \langle 7,-5,1\rangle$$

Taking the cross product: $\overrightarrow{AB}\times\overrightarrow{AC} = \langle -2,-4,-6\rangle$. Any non-zero vector parallel to this cross product is a normal vector to the plane. For simplicity, we can take:

$$\mathbf{n} = \langle 1,2,3\rangle\,.$$

Take $A(0,2,-1)$ to be the given point P_0, then the equation of the plane through A, B and C is given by:

$$\underbrace{1x+2y+3z = 1(0)+2(2)+3(-1)}_{(x_0,y_0,z_0)=(0,2,-1)\text{ and }\mathbf{n}=\langle 1,2,3\rangle}$$

After simplification: $x+2y+3z=1$.

Substitute $D(\frac{1}{3},\frac{1}{6},\frac{1}{9})$ into the equation $x+2y+3z=1$, we see:

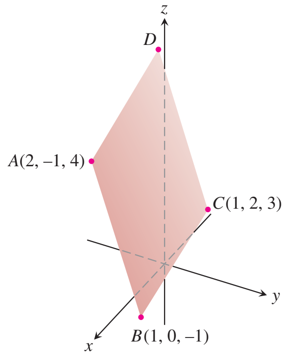
$$\text{LHS} = \frac{1}{3}+2\left(\frac{1}{6}\right)+3\left(\frac{1}{9}\right) = 1 = \text{RHS}.$$

Therefore D lies on this plane and so the four points A, B, C and D are coplanar.

3. (★) A parallelogram in $\mathbb{R}^3$ has vertices:

$$A(2,-1,4),\quad B(1,0,-1),\quad C(1,2,3),\quad D(x_0,y_0,z_0)$$

as shown in the figure below. Answer the following questions:



(a) Find the coordinates of D.

Solution: Since $ABCD$ is a parallelogram, we must have:

$$\overrightarrow{AB} = \overrightarrow{DC}$$

$$\langle -1, 1, -5 \rangle = \langle 1 - x_0, 2 - y_0, 3 - z_0 \rangle$$

Solving the equation, we get: $\boxed{(x_0, y_0, z_0) = (2, 1, 8)}$.

(b) Find the area of the parallelogram $ABCD$.

Solution:

$$\text{Area of } ABCD = \left|\overrightarrow{AB} \times \overrightarrow{AD}\right|$$

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AD} &= \langle -1, 1, -5 \rangle \times \langle 0, 2, 4 \rangle \\ &= \langle 14, 4, -2 \rangle\end{aligned}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = \sqrt{216}.$$

(c) Find an equation of the plane containing the parallelogram $ABCD$.

Solution: Similar to previous "Equation of planes" problems. Take $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AD}$. Answer: $\boxed{7x + 2y - z = 8}$

(d) Project the parallelogram $ABCD$ orthogonally onto the plane $z = -1$. Find the coordinates the projection of each vertices, then find the area of the *projected* parallelogram.

Solution: Such a projection will preserve the x- and y-coordinates, so the projection of each vertices are:

$$A'(2, -1, -1), \quad B'(1, 0, -1), \quad C'(1, 2, -1) \quad \text{and} \quad D'(2, 1, -1).$$

$$\begin{aligned}\text{Area of } A'B'C'D' &= \left|\overrightarrow{A'B'} \times \overrightarrow{A'D'}\right| \\ &= |\langle -1, 1, 0 \rangle \times \langle 0, 2, 0 \rangle| \\ &= |\langle 0, 0, -2 \rangle| = 2.\end{aligned}$$

4. (★) Consider a particle whose path is represented by:

$$\mathbf{r}(t) = \left(\ln(t^2+1)\right)\mathbf{i} + \left(\tan^{-1} t\right)\mathbf{j} + \sqrt{t^2+1}\mathbf{k}$$

Find the velocity, speed and acceleration of the particle at $t = 0$.

Solution: By straight-forward computations (omitted), we can get:

$$\begin{aligned}
\mathbf{r}'(t) &= \frac{2t}{1+t^2}\mathbf{i} + \frac{1}{1+t^2}\mathbf{j} + \frac{t}{\sqrt{t^2+1}}\mathbf{k} \\
\left|\mathbf{r}'(t)\right| &= \sqrt{\frac{4t^2}{(1+t^2)^2} + \frac{1}{(1+t^2)^2} + \frac{t^2}{t^2+1}} \\
&= \frac{\sqrt{t^4+5t^2+1}}{1+t^2} \\
\mathbf{r}''(t) &= \frac{d}{dt}\mathbf{r}'(t) \\
&= \frac{2(1-t^2)}{(1+t^2)^2}\mathbf{i} - \frac{2t}{(1+t^2)^2}\mathbf{j} + \frac{1}{(1+t^2)^{3/2}}\mathbf{k}
\end{aligned}$$

At $t = 0$, we have:

$$\begin{aligned}
\mathbf{r}'(0) &= \mathbf{j} \\
\left|\mathbf{r}'(0)\right| &= 1 \\
\mathbf{r}''(0) &= 2\mathbf{i} + \mathbf{k}
\end{aligned}$$

5. (★★) Consider a plane through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle A, B, C\rangle$. Prove that the perpendicular distance d from a given point $Q(x_1, y_1, z_1)$ to the plane is given by:

$$d = \frac{\left|\overrightarrow{P_0Q}\cdot\mathbf{n}\right|}{|\mathbf{n}|} = \left|\frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2+B^2+C^2}}\right|$$

where $D = Ax_0 + By_0 + Cz_0$.

Solution: Suppose R is the projection of the point Q onto the given plane, then we have $d = \left|\overrightarrow{QR}\right|$, which is what we need to find. Consider the triangle P_0QR and let θ be the angle $\angle P_0QR$, then we have:

$$d = \left|\overrightarrow{QR}\right| = \left|\overrightarrow{QP_0}\right|\cos\theta.$$

Since θ is also the angle between $\overrightarrow{P_0Q}$ and $\mathbf{n}$, we can deduce:

$$d = \left|\overrightarrow{QP_0}\right|\cos\theta = \underbrace{\left|\overrightarrow{QP_0}\right|\,|\mathbf{n}|\cos\theta}_{\text{dot product}}\cdot\frac{1}{|\mathbf{n}|} = \left|\overrightarrow{QP_0}\cdot\mathbf{n}\right|\cdot\frac{1}{|\mathbf{n}|}$$

as desired. The second equality follows from plugging in $\mathbf{n} = \langle A, B, C\rangle$, $P_0(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$.

6. (★) Suppose $\mathbf{r}(t)$ represents the path of a particle traveling on a sphere centered at the origin. Show that the position vector $\mathbf{r}(t)$ and the velocity $\mathbf{r}'(t)$ are orthogonal to each other at any time.

Solution: Since $\mathbf{r}(t)$ travels on a sphere centered at the origin, we have $|\mathbf{r}(t)| = C$ for some constant C. Since $\frac{d}{dt}|\mathbf{r}(t)|$ is difficult to compute, we take the square on both sides:

$$\begin{aligned}
|\mathbf{r}(t)|^2 &= C^2 \\
\mathbf{r}(t)\cdot\mathbf{r}(t) &= C^2 \\
\frac{d}{dt}\left(\mathbf{r}(t)\cdot\mathbf{r}(t)\right) &= \frac{d}{dt}C^2 = 0 \\
\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) &= 0 && \text{(product rule)} \\
2\mathbf{r}(t)\cdot\mathbf{r}'(t) &= 0 \\
\mathbf{r}(t)\cdot\mathbf{r}'(t) &= 0.
\end{aligned}$$

Therefore, $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal at any time t.

7. (★★) Suppose that the path of a particle at time t is given by $\mathbf{r}(t)$ and the force exerted on the particle at time t is $\mathbf{F}(t)$. By Newton's Second Law, $\mathbf{F}(t)$ and $\mathbf{r}(t)$ are related by:

$$\mathbf{F}(t) = m\mathbf{r}''(t),$$

where m is the mass of the particle. The angular momentum $\mathbf{L}(t)$ about the origin of the particle at time t is defined to be:

$$\mathbf{L}(t) := \mathbf{r}(t)\times m\mathbf{r}'(t)$$

(a) Show that

$$\frac{d}{dt}\mathbf{L}(t) = \mathbf{r}(t)\times\mathbf{F}(t).$$

Solution: Use the product rule:

$$\begin{aligned}
\frac{d}{dt}\mathbf{L} &= \frac{d}{dt}\left(\mathbf{r}(t)\times m\mathbf{r}'(t)\right) \\
&= \underbrace{\mathbf{r}'(t)\times m\mathbf{r}'(t)}_{\mathbf{r}' \parallel m\mathbf{r}'} + \mathbf{r}(t)\times m\mathbf{r}''(t) \\
&= \mathbf{0} + \mathbf{r}(t)\times\mathbf{F}(t) && \text{(Newton's Second Law)} \\
&= \mathbf{r}(t)\times\mathbf{F}(t)
\end{aligned}$$

(b) When $\mathbf{L}(t)$ is a constant vector, we say that the angular momentum is *conserved*. According to the result in (a), under what condition on $\mathbf{r}(t)$ and $\mathbf{F}(t)$ will the angular momentum be conserved? Also, give one example in physics that this condition is satisfied.

Solution: In order for $\mathbf{L}(t)$ to be conserved, we need $\frac{d\mathbf{L}}{dt} = \mathbf{0}$. According to (a), it happens when $\mathbf{r}(t) \times \mathbf{F}(t) = \mathbf{0}$, or equivalently, $\mathbf{r}(t)$ is parallel to $\mathbf{F}(t)$ at any t.

There are many situations in physics that $\mathbf{F}$ is parallel to $\mathbf{r}$. For instance, the gravitational force exerted by the Sun on the Earth is a clear example: Pick the origin to be the center of the Sun, the gravitational force field due to the Sun is radially symmetric, in a sense that the gravity $\mathbf{F}$ due to the Sun on the Earth is always in the opposition direction of the position vector $\mathbf{r}$. Precisely, physicists assert that the gravitational force satisfies the inverse-square law:

$$\mathbf{F} = -\frac{GMm}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}$$

where G is a constant, M is the mass of the Sun and m is the mass of the Earth. In this case, $\mathbf{r} \times \mathbf{F} = \mathbf{0}$ and so $\mathbf{L}$ is conserved. (Further remark: from class, we proved that if $\mathbf{L}$ is conserved, then $\mathbf{r}(t)$ travels on a single plane – that explains why the Earth rotates around the Sun on a fixed plane.)

8. (★★) Consider two point particles with masses m_1 and m_2, and their trajectories are $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ respectively. Denote $\mathbf{F}(t)$ to be the force exerted on the m_1-particle by the m_2-particle at time t. By Newton's Third Law, the force exerted on the m_2-particle by the m_1-particle at time t (i.e. the reverse force) is given by $-\mathbf{F}(t)$. Assume there are no other forces exerted on any of these particles.

(a) Consider the following vector:

$$\mathbf{C}(t) := \frac{m_1\mathbf{r}_1(t) + m_2\mathbf{r}_2(t)}{m_1 + m_2}.$$

In physics, this vector is pointing at the center of mass of the two particles. Show that $\mathbf{C}''(t) = \mathbf{0}$ for any t using Newton's Second and Third Laws.

Solution: By Newtow's Laws, we have $\mathbf{F}(t) = m_1\mathbf{r}_1''(t)$ and $-\mathbf{F}(t) = m_2\mathbf{r}_2''(t)$. Using these, we get:

$$\begin{aligned}
\mathbf{C}'(t) &= \frac{m_1\mathbf{r}_1'(t) + m_2\mathbf{r}_2'(t)}{m_1 + m_2} \\
\mathbf{C}''(t) &= \frac{m_1\mathbf{r}_1''(t) + m_2\mathbf{r}_2''(t)}{m_1 + m_2} \\
&= \frac{\mathbf{F}(t) + (-\mathbf{F}(t))}{m_1 + m_2} = \mathbf{0}.
\end{aligned}$$

(b) Hence, show that there exist two constant vectors $\mathbf{r}_0$ and $\mathbf{v}$ such that

$$\frac{m_1\mathbf{r}_1(t) + m_2\mathbf{r}_2(t)}{m_1 + m_2} = \mathbf{r}_0 + t\mathbf{v}.$$

[Question: What is the physical significance of this result?]

Solution: Using (a), we get $\mathbf{C}''(t) = \mathbf{0}$. Since $\mathbf{C}''(t) = \frac{d}{dt}\mathbf{C}'(t)$, we can deduce:

$$\mathbf{C}'(t) = \mathbf{c}_1$$

for some constant vector $\mathbf{c}_1$. By integration, we get:

$$\mathbf{C}(t) = \int \mathbf{c}_1 \, dt = \mathbf{c}_1 \cdot t + \mathbf{c}_2$$

where $\mathbf{c}_2$ is any constant vector. The required result follows from relabelling the constant vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ by $\mathbf{v}$ and $\mathbf{r}_0$ respectively.
Since the parametric equation of the form $\mathbf{r}_0 + t\mathbf{v}$ represents a straight-line, this results assert that the center of masses is travelling along a straight path with constant velocity.

9. (★) For each of the following curves, first reparametrize it by arc-length and then compute its curvature function $\kappa(s)$:

(a) $\mathbf{r}_1(t) = (R\cos\omega t)\,\mathbf{i} + (R\sin\omega t)\,\mathbf{j}, \quad 0 \le t \le \frac{2\pi}{\omega}$.

Solution: Note that we say $0 \le t \le \frac{2\pi}{\omega}$. It implicitly infers that $\omega > 0$. However, R can be negative! Yet we can ignore the case $R = 0$ (since it would give a "point" rather than a curve).
First compute:

$$\begin{aligned}
\mathbf{r}_1'(t) &= (-R\omega\sin\omega t)\,\mathbf{i} + (R\omega\cos\omega t)\,\mathbf{j} \\
\left|\mathbf{r}_1'(t)\right| &= \sqrt{R^2\omega^2(\sin^2\omega t + \cos^2\omega t)} = |R|\,\omega \\
s &= \int_0^t \left|\mathbf{r}_1'(\tau)\right| \, d\tau = \int_0^t |R|\,\omega \, d\tau = |R|\,\omega t \\
t &= \frac{s}{|R|\,\omega}
\end{aligned}$$

An arc-length parametrization for $\mathbf{r}_1$ is given by:

$$\begin{aligned}
\mathbf{r}_1(s) &= \left(R\cos\left(\omega \cdot \frac{s}{|R|\,\omega}\right)\right)\,\mathbf{i} + \left(R\sin\left(\omega \cdot \frac{s}{|R|\,\omega}\right)\right)\,\mathbf{j} \\
&= \left(R\cos\frac{s}{|R|}\right)\,\mathbf{i} + \left(R\sin\frac{s}{|R|}\right)\,\mathbf{j}
\end{aligned}$$

When $t = 0$, $s = |R|\,\omega t = 0$. When $t = \frac{2\pi}{\omega}$, $s = |R|\,\omega \cdot \frac{2\pi}{\omega} = 2\pi\,|R|$. Therefore, the range for the parameter s is $0 \le s \le 2\pi\,|R|$.

To compute its curvature, we differentiate $\mathbf{r}_1(s)$ twice with respect to s:

$$\mathbf{r}_1'(s) = \left(-\frac{R}{|R|}\sin\frac{s}{|R|}\right)\mathbf{i} + \left(\frac{R}{|R|}\cos\frac{s}{|R|}\right)\mathbf{j}$$
$$\mathbf{r}_1''(s) = \left(-\frac{R}{|R|^2}\cos\frac{s}{|R|}\right)\mathbf{i} + \left(-\frac{R}{|R|^2}\sin\frac{s}{|R|}\right)\mathbf{j}$$
$$\kappa_1(s) = |\mathbf{r}_1''(s)| = \left|-\frac{R}{|R|^2}\right| = \frac{1}{|R|}.$$

(b) $\mathbf{r}_2(t) = \langle 1,2,3\rangle + (\ln t)\,\langle 1,0,-1\rangle, \quad 0 < t < \infty$

Solution: Straight-forward computations show: $|\mathbf{r}_2'(t)| = \frac{\sqrt{2}}{t}$.

However, the integral $\int_0^t |\mathbf{r}_2'(\tau)|\, d\tau = \int_0^t \frac{\sqrt{2}}{\tau}\, d\tau$ does not converge. Instead, we set:

$$s = \int_1^t |\mathbf{r}_2'(\tau)|\, d\tau = \int_1^t \frac{\sqrt{2}}{\tau}\, d\tau = \sqrt{2}\ln t.$$

Solving for t in terms of s, we get: $\ln t = \frac{s}{\sqrt{2}}$. Therefore, the arc-length parametrization of $\mathbf{r}_2$ is given by:

$$\mathbf{r}_2(s) = \langle 1,2,3\rangle + (\ln t)\,\langle 1,0,-1\rangle = \langle 1,2,3\rangle + \frac{s}{\sqrt{2}}\,\langle 1,0,-1\rangle.$$

From $s = \sqrt{2}\ln t$, the range for s is given by $-\infty < s < \infty$.
It is clear that $\mathbf{r}_2''(s) = \mathbf{0}$, we have $\kappa_2(s) = |\mathbf{r}_2''(s)| = 0$.

(c) $\mathbf{r}_3(t) = (\cos^3 t)\,\mathbf{i} + (\sin^3 t)\,\mathbf{j}, \quad 0 \le t \le \frac{\pi}{2}$.

Solution: Straight-forward computations give:

$$\begin{aligned}
\mathbf{r}_3'(t) &= (-3\cos^2 t\sin t)\,\mathbf{i} + (3\sin^2 t\cos t)\,\mathbf{j} \\
|\mathbf{r}_3'(t)| &= \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t} \\
&= 3\sqrt{\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)} \\
&= 3\sqrt{\cos^2 t\sin^2 t} = 3\cos t\sin t \\
s &= \int_0^t |\mathbf{r}'(\tau)|\, d\tau = \int_0^t 3\cos\tau\sin\tau\, d\tau \\
&= \int_{\tau=0}^{\tau=t} 3\sin\tau\, d(\sin\tau) = \left[\frac{3\sin^2\tau}{2}\right]_{\tau=0}^{\tau=t} = \frac{3}{2}\sin^2 t
\end{aligned}$$

Solving for t, we get $t = \sin^{-1}\sqrt{\frac{2s}{3}}$, hence an arc-length parametrization of $\mathbf{r}_3$ is:

$$\mathbf{r}_3(s) = \left(\cos\sin^{-1}\sqrt{\frac{2s}{3}}\right)^3\mathbf{i} + \left(\sin\sin^{-1}\sqrt{\frac{2s}{3}}\right)^3\mathbf{j}, \quad 0 \le s \le \frac{3}{2}.$$

Although the above express gives an arc-length parametrization of the curve, it is very tedious to differentiate, let alone finding its curvature. Using the fact that $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$ and $\sin(\sin^{-1}(x)) = x$, we can simplify $\mathbf{r}_3(s)$ as:

$$\begin{aligned}\mathbf{r}_3(s) &= \left(\sqrt{1-\left(\sqrt{\frac{2s}{3}}\right)^2}\right)^3 \mathbf{i} + \left(\sqrt{\frac{2s}{3}}\right)^3 \mathbf{j}\\ &= \left(1-\frac{2s}{3}\right)^{3/2} \mathbf{i} + \left(\frac{2s}{3}\right)^{3/2} \mathbf{j}\end{aligned}$$

which is much easier to work with. By straight-forward differentiations, we get:

$$\begin{aligned}\mathbf{r}_3'(s) &= -\left(1-\frac{2s}{3}\right)^{1/2} \mathbf{i} + \left(\frac{2s}{3}\right)^{1/2} \mathbf{j}\\ \mathbf{r}_3''(s) &= \frac{1}{3}\left(1-\frac{2s}{3}\right)^{-1/2} \mathbf{i} + (6s)^{-1/2} \mathbf{j}\\ \kappa_3(s) &= \left|\mathbf{r}_3'(s)\right| = \frac{1}{\sqrt{2s(3-2s)}}\end{aligned}$$

Give an example of a path whose arc-length parametrization cannot be explicitly found even with computer softwares.

Solution: While the arc-length (and its arc-length parametrization) of most curves appeared in textbook problems can be found explicitly, it is not the case in general. If one randomly writes down a curve $\mathbf{r}(t)$, there is more than 90% chance that you can't find the explicit expression of its arc-length, let alone its arc-length parametrization. It is because $|\mathbf{r}'(\tau)|$ involves a square-root, and the integral of a square-root is very difficult to compute even with computer softwares. One "notorious" example is the ellipse: $\mathbf{r}(t) = (a\cos t)\ \mathbf{i} + (b\sin t)\ \mathbf{j}$, $0 \le t \le 2\pi$, where $a \neq b$, for which we have $|\mathbf{r}'(t)| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$. However, it is impossible to find an explicit expression for:

$$s = \int_0^t \sqrt{a^2\sin^2\tau + b^2\cos^2\tau}\, d\tau.$$

Even if one can express it as a infinite series in t, the next step: solving t in terms of s is impossible to carry out.

10. (★★) Suppose

$$\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} + \frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\mathbf{j} + t\mathbf{k}$$

represents the path of a race-car climbing up a hill from $(0,0,0)$ at $t = 0$. A truck, on the other hand, drives slowly in unit speed from $(0,0,0)$ at time $t = 0$ along the same path and direction as the race-car. Find a parametrization which represents the path of the truck.

Solution: The path of the truck is exactly the arc-length parametrization of $\mathbf{r}(t)$ such that $s = 0$ when $t = 0$. It has been done in class. See the worksheet for Lecture #02.

11. (★★★) We define the curvature of a path by $\kappa(s) = |\mathbf{r}''(s)|$ where $\mathbf{r}(s)$ is the arc-length parametrization of the path. However, the arc-length parametrization $\mathbf{r}(s)$ is often difficult to find explicitly. The purpose of this exercise is to derive an equivalent formula for the curvature which does not require finding an arc-length parametrization.

Given a path $\mathbf{r}(t)$, we let $\mathbf{r}(s)$ be its arc-length parametrization so that s and t are related by:

$$s = \int_0^t \left|\mathbf{r}'(\tau)\right| \, d\tau.$$

(a) Show, using the chain rule, that:

$$\mathbf{r}'(t) = \mathbf{r}'(s)\,\frac{ds}{dt}$$
$$\mathbf{r}''(t) = \mathbf{r}''(s)\left(\frac{ds}{dt}\right)^2 + \mathbf{r}'(s)\,\frac{d^2s}{dt^2}$$

Solution: By the chain rule, we have:

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\,\frac{ds}{dt} = \mathbf{r}'(s)\,\frac{ds}{dt} \tag{1}$$
$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}\left(\mathbf{r}'(s)\,\frac{ds}{dt}\right) \qquad \text{from (1)}$$
$$= \frac{d\mathbf{r}'(s)}{dt}\,\frac{ds}{dt} + \mathbf{r}'(s)\,\frac{d^2s}{dt^2} \tag{2}$$

By the chain rule again, we get:

$$\frac{d\mathbf{r}'(s)}{dt} = \frac{d\mathbf{r}'(s)}{ds}\,\frac{ds}{dt} = \mathbf{r}''(s)\,\frac{ds}{dt}$$

Substitute this back to (2) , we obtain:

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{r}''(s)\left(\frac{ds}{dt}\right)^2 + \mathbf{r}'(s)\,\frac{d^2s}{dt^2} \tag{3}$$

(b) Show that:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^3 \mathbf{r}'(s) \times \mathbf{r}''(s)$$

Solution: Taking the cross product of (1) and (3) yields:

$$\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2} = \left(\frac{ds}{dt}\right)^3 \mathbf{r}'(s) \times \mathbf{r}''(s) + \underbrace{\frac{d^2 s}{dt^2}\frac{ds}{dt}\,\mathbf{r}'(s) \times \mathbf{r}'(s)}_{=\mathbf{0}} = \left(\frac{ds}{dt}\right)^3 \mathbf{r}'(s) \times \mathbf{r}''(s).$$

(c) Using (a) and (b), show that the curvature, which is defined as $\kappa(s) := |\mathbf{r}''(s)|$, can be expressed in terms of t as:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Although it looks more complicated, this formula does not require the procedure of finding arc-length parametrization.

Solution: As $\mathbf{r}(s)$ travels a constant speed, in class we showed $\mathbf{r}'(s)$ and $\mathbf{r}''(s)$ are two orthogonal vectors. Therefore we have

$$\left|\mathbf{r}'(s) \times \mathbf{r}''(s)\right| = \underbrace{\left|\mathbf{r}'(s)\right|}_{1}\underbrace{\left|\mathbf{r}''(s)\right|}_{\kappa(s)} \sin\frac{\pi}{2} = \kappa(s).$$

Taking the magnitude on both sides of the result obtained (b), we get:

$$\left|\frac{d\mathbf{r}}{dt} \times \frac{d^2\mathbf{r}}{dt^2}\right| = \kappa \left|\frac{ds}{dt}\right|^3.$$

Therefore, we get:

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{\left|\frac{ds}{dt}\right|^3}.$$

The proof can be easily completed by the definition of $s(t)$ and the Fundamental Theorem of Calculus:

$$s = \int_0^t \left|\mathbf{r}'(\tau)\right| \, d\tau$$

$$\frac{ds}{dt} = \left|\mathbf{r}'(t)\right|$$

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #2 • Multivariable Functions, Partial Derivatives

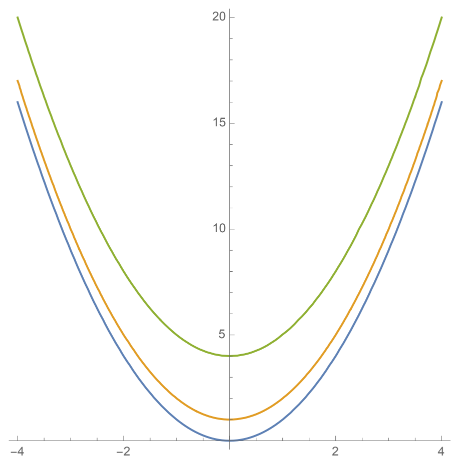
1. (★) Let $f(x, y) = \sqrt{y - x^2}$

   (a) What is the (largest possible) domain of f?

   Solution: The domain is $\{(x, y) : y \geq x^2\}$, the region above the parabola $y = x^2$ (including the parabola) in $\mathbb{R}^2$.

   (b) Sketch the level sets $f = 0$, $f = 1$ and $f = 2$ in the same diagram.

   Solution:



2. (★) Let

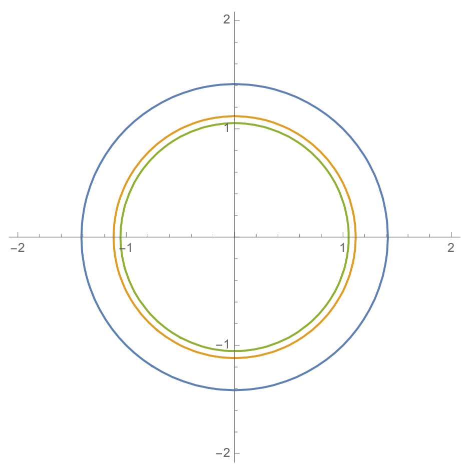
$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2 - 1}}$$

   (a) What is the (largest possible) domain of f?

   Solution: The domain is $\{(x, y) : x^2 + y^2 > 1\}$, the region outside the unit circle $x^2 + y^2 = 1$, excluding the circle itself.

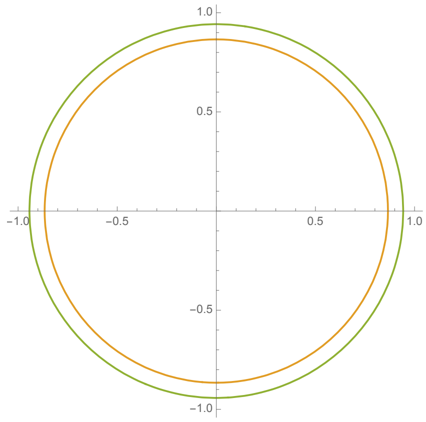
   (b) Sketch the level sets $f = 1$, $f = 2$ and $f = 3$ in the same diagram.

   Solution:



(c) Repeat (a) and (b) for the function $g(x,y) = \dfrac{1}{\sqrt{1-x^2-y^2}}$.

Solution: The domain is $\{(x,y) : x^2+y^2 < 1\}$, the region inside the unit circle $x^2+y^2=1$, excluding the circle itself.



3. (★) Compute all the first and second partial derivatives of the following functions. For the second partials f_{xy} and f_{yx}, compute both and verify that they are indeed the same.

(a) $f(x,y) = y^{2015} + 2x^2 + 2xy$

Solution: (Answer only)

$$f_x = 4x + 2y \qquad f_y = 2x + 2015y^{2014}$$
$$f_{xx} = 4 \qquad f_{xy} = 2$$
$$f_{yx} = 2 \qquad f_{yy} = 2015 \times 2014y^{2013}$$

(b) $f(x,y) = e^{x^2y}$

Solution: (Answer only)

$$f_x = 2e^{x^2y}xy \qquad f_y = e^{x^2y}x^2$$
$$f_{xx} = 2e^{x^2y}y + 4e^{x^2y}x^2y^2 \qquad f_{xy} = 2e^{x^2y}x + 2e^{x^2y}x^3y$$
$$f_{yx} = 2e^{x^2y}x + 2e^{x^2y}x^3y \qquad f_{yy} = x^4e^{x^2y}$$

(c) $f(x,y) = \frac{x}{x^2+y^2}$

Solution: (Answer only)

$$f_x = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad f_y = -\frac{2xy}{(x^2+y^2)^2}$$
$$f_{xx} = \frac{2\left(x^3-3xy^2\right)}{(x^2+y^2)^3} \qquad f_{xy} = -\frac{2y\left(y^2-3x^2\right)}{(x^2+y^2)^3}$$
$$f_{yx} = -\frac{2y\left(y^2-3x^2\right)}{(x^2+y^2)^3} \qquad f_{yy} = -\frac{2x\left(x^2-3y^2\right)}{(x^2+y^2)^3}$$

(d) $f(x, y) = x \ln(x^2 + y^2)$

Solution: (Answer only)

$$f_x = \frac{2x^2}{x^2+y^2} + \log\left(x^2+y^2\right) \qquad f_y = \frac{2xy}{x^2+y^2}$$
$$f_{xx} = \frac{2\left(x^3+3xy^2\right)}{\left(x^2+y^2\right)^2} \qquad f_{xy} = \frac{2y\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$$
$$f_{yx} = \frac{2y\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2} \qquad f_{yy} = \frac{2x\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$$

4. (★★) Compute the first partial derivative $\frac{\partial f}{\partial x}$ of the following functions (where $x, y > 0$).

(a) $f(x, y) = e^{x^y}$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} e^{(x^y)} = \frac{\partial}{\partial (x^y)} e^{(x^y)} \cdot \frac{\partial}{\partial x} x^y$$
$$= e^{(x^y)} \cdot y x^{y-1} = y x^{y-1} e^{x^y}$$

(b) $f(x, y) = e^{y^x}$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} e^{(y^x)} = \frac{\partial}{\partial (y^x)} e^{(y^x)} \cdot \frac{\partial}{\partial x} y^x$$
$$= e^{(y^x)} \cdot y^x \ln y = y e^{y^x} \ln y$$

(c) $f(x, y) = x^{e^y}$

Solution: (Answer only)

$$\frac{\partial f}{\partial x} = e^y x^{e^y - 1}$$

(d) $f(x, y) = y^{e^x}$

Solution: (Answer only)

$$\frac{\partial f}{\partial x} = y^{e^x} e^x \ln y$$

(e) $f(x, y) = x^{y^e}$

Solution: (Answer only)

$$\frac{\partial f}{\partial x} = y^e x^{y^e - 1}$$

(f) $f(x,y) = y^{x^e}$

Solution: (Answer only)

$$\frac{\partial f}{\partial x} = ex^{e-1}y^{x^e}\ln y$$

5. (★) Compute both the third-order derivatives h_{xyy} and h_{yyx} of the following function, and verify that they are indeed the same.

$$h(x,y,z) = \cos(x^2 + y^3 z).$$

Solution:

$$\begin{aligned}
h_x &= -2x\sin\left(x^2+y^3z\right)\\
h_{xy} &= \left(h_x\right)_y = -6xy^2z\cos\left(x^2+y^3z\right)\\
h_{xyy} &= \left(h_{xy}\right)_y = 18xy^4z^2\sin\left(x^2+y^3z\right) - 12xyz\cos\left(x^2+y^3z\right)
\end{aligned}$$

$$\begin{aligned}
h_y &= -3y^2z\sin\left(x^2+y^3z\right)\\
h_{yy} &= \left(h_y\right)_y = -6yz\sin\left(x^2+y^3z\right) - 9y^4z^2\cos\left(x^2+y^3z\right)\\
h_{yyx} &= \left(h_{yy}\right)_x = 18xy^4z^2\sin\left(x^2+y^3z\right) - 12xyz\cos\left(x^2+y^3z\right)
\end{aligned}$$

6. (★★) Find the second derivative $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ of each function $f(x,y)$ below. [Hint: There is a smart way to compute each of them.]

(a)

$$f(x,y) = \sin(x+y)\ \cos(x-y)$$

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \cos(x+y)\cos(x-y) + \sin(x+y)\sin(x-y)\\
&= \cos(x+y-(x-y))\\
&= \cos(2y)
\end{aligned}$$

Here we have used the compound-angle formula for cos. Since this does not depend on x we have $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 0$.

(b)

$$f(x,y) = \cos(xy) + \left(\frac{\sin^{2016} y + \cos^{2014} y}{\sin^2 \log(y^4+1) + 2015} \right)^{\frac{1}{2015}}.$$

Solution: Since the second term does not depend on y, we can switch the order of the partial derivatives taken and obtain:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(-y\sin(xy)+0) = -\sin(xy) - xy\cos(xy)$$

(c)

$$f(x,y) = \frac{e^{x+y}+e^{x-y}}{e^{x+y}-e^{x-y}}$$

Solution: By factoring out e^x in both numerator and denominator we obtain:

$$f(x,y) = \frac{e^x e^y + e^x e^{-y}}{e^x e^y - e^x e^{-y}} = \frac{e^x(e^y+e^{-y})}{e^x(e^y-e^{-y})} = \frac{e^y+e^{-y}}{e^y-e^{-y}}$$

Therefore it makes sense to switch the order of partial differentiation and calculate

$$\frac{\partial f}{\partial x} = 0$$

Thus we also have $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0.$

7. (★★) Suppose that $f(x,y)$ is a function such that $\frac{\partial^2 f}{\partial x \partial y} \equiv 0$. Show that f can be decomposed into the form:

$$f(x,y) = F(x) + G(y)$$

where $F(x)$ and $G(y)$ are some single-variable functions.

Solution: Given that:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = 0,$$

we know $\frac{\partial f}{\partial y}$ is independent of x, and depends only on y. Therefore, it can be written as:

$$\frac{\partial f}{\partial y} = g(y)$$

for an arbitrary differentiable function $g(y)$.

Now, we obtained that the y-derivative of f is $g(y)$. To find the function f, we can integrate $g(y)$ with respect to y:

$$f(x,y) = \int g(y)\, dy + F(x).$$

As $\frac{\partial}{\partial y}$ is a partial derivative, the integration "constant" is not really a constant but is a quantity not depending on y. In other words, the integration "constant" is a function $F(x)$ of x.

Since $\int g(y)\,dy$ is also an arbitrary function of y, for simplicity we relabel it as $G(y)$. Therefore, we get $f(x,y) = F(x) + G(y)$.

8. (★★) Let $u(x,y,z,t)$ be the temperature at the point (x,y,z) at the time t. Combining with several important laws in thermodynamics, including the Fourier's Law and conservation of energy, it can be derived (detail omitted) that the temperature function $u(x,y,z,t)$ satisfies the following equation:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

where k is a positive constant depending only on the medium. This equation is known as the **heat equation**.

The study of the heat equation is an important topic in physics, engineering and mathematics (both pure and applied). Through solving the heat equation with an initial condition $u(x,y,z,0) = g(x,y,z)$, it predicts how heat diffuses for a given an initial heat profile $g(x,y,z)$ at time $t = 0$.

Your task in this problem is to verify that the following given function is a solution to the heat equation:

$$\varphi(x,y,z,t) = \frac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(-\frac{x^2+y^2+z^2}{4kt}\right).$$

This particular solution φ represents the heat diffusion with highly concentrated heat source at the origin $(0,0,0)$ at time $t = 0$. As time goes by, the temperature profile becomes more and more uniformly distributed. (In physics, this solution is also closely related to the *Dirac delta function*.)

By following the outline below, show that φ satisfies the heat equation:

(a) Show that:

$$\ln \varphi(x,y,z,t) = -\ln(4\pi k)^{\frac{3}{2}} - \frac{3}{2}\ln t - \frac{x^2+y^2+z^2}{4kt}.$$

Solution: Take ln on the function φ:

$$\begin{aligned}
\ln \varphi(x,y,z,t) &= \ln\left(\frac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(-\frac{x^2+y^2+z^2}{4kt}\right)\right) \\
&= \ln\left(\frac{1}{(4\pi k)^{3/2}t^{3/2}}\right) + \ln\exp\left(-\frac{x^2+y^2+z^2}{4kt}\right) \\
&= -\ln((4\pi k)^{3/2}t^{3/2}) - \frac{x^2+y^2+z^2}{4kt} && (\ln\exp u = u) \\
&= -\ln(4\pi k)^{3/2} - \ln t^{3/2} - \frac{x^2+y^2+z^2}{4kt} \\
&= -\ln(4\pi k)^{3/2} - \frac{3}{2}\ln t - \frac{x^2+y^2+z^2}{4kt}
\end{aligned}$$

(b) Using (a), show that:

$$\frac{\partial \varphi}{\partial t} = \left(\frac{x^2+y^2+z^2}{4kt^2} - \frac{3}{2t}\right)\varphi.$$

Solution: Differentiating both sides of (a) with respect to t:

$$\begin{aligned}
\frac{\partial}{\partial t}\ln\varphi &= \frac{\partial}{\partial t}\left(-\ln(4\pi k)^{3/2} - \frac{3}{2}\ln t - \frac{x^2+y^2+z^2}{4kt}\right)\\
\frac{d}{d\varphi}\ln\varphi\cdot\frac{\partial\varphi}{\partial t} &= 0 - \frac{3}{2t} - \frac{x^2+y^2+z^2}{4k}\frac{\partial}{\partial t}\frac{1}{t}\\
\frac{1}{\varphi}\frac{\partial\varphi}{\partial t} &= -\frac{3}{2t} - \frac{x^2+y^2+z^2}{4k}\left(-\frac{1}{t^2}\right)\\
&= \frac{x^2+y^2+z^2}{4kt^2} - \frac{3}{2t}.\\
\frac{\partial\varphi}{\partial t} &= \left(\frac{x^2+y^2+z^2}{4kt^2} - \frac{3}{2t}\right)\varphi.
\end{aligned}$$

(c) Using (a) again, show that:

$$\frac{\partial\varphi}{\partial x} = -\frac{x\varphi}{2kt} \quad \text{and} \quad \frac{\partial^2\varphi}{\partial x^2} = \frac{1}{2kt}\left(\frac{x^2}{2kt} - 1\right)\varphi.$$

Solution: Differentiating both sides of (a) with respect to x:

$$\begin{aligned}
\frac{\partial}{\partial x}\ln\varphi &= \frac{\partial}{\partial x}\left(-\ln(4\pi k)^{3/2} - \frac{3}{2}\ln t - \frac{x^2+y^2+z^2}{4kt}\right)\\
\frac{d}{d\varphi}\ln\varphi\cdot\frac{\partial\varphi}{\partial x} &= 0 + 0 - \frac{\partial}{\partial x}\left(\frac{x^2+y^2+z^2}{4kt}\right)\\
\frac{1}{\varphi}\frac{\partial\varphi}{\partial x} &= -\frac{2x+0+0}{4kt} = -\frac{x}{2kt}\\
\frac{\partial\varphi}{\partial x} &= -\frac{x\varphi}{2kt}.
\end{aligned}$$

Differentiate both sides of the above result by x:

$$\begin{aligned}
\frac{\partial^2\varphi}{\partial x^2} &= -\frac{\partial}{\partial x}\left(\frac{x\varphi}{2kt}\right)\\
&= -\frac{1}{2kt}(\varphi + x\varphi_x)\\
&= -\frac{1}{2kt}\left(\varphi - x\cdot\frac{x\varphi}{2kt}\right) && \text{(from our result of } \tfrac{\partial\varphi}{\partial x}\text{)}\\
&= \frac{1}{2kt}\left(\frac{x^2\varphi}{2kt} - \varphi\right)\\
&= \frac{1}{2kt}\left(\frac{x^2}{2kt} - 1\right)\varphi && \text{(factor out } \varphi\text{).}
\end{aligned}$$

(d) Hence, verify that φ satisfies the heat equation: $\varphi_t = k(\varphi_{xx} + \varphi_{yy} + \varphi_{zz})$.

Solution: Similar calculation as in (c) shows:

$$\frac{\partial^2 \varphi}{\partial y^2} = \frac{1}{2kt}\left(\frac{y^2}{2kt} - 1\right)\varphi \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial z^2} = \frac{1}{2kt}\left(\frac{z^2}{2kt} - 1\right)\varphi.$$

Therefore,

$$\begin{aligned}
&k(\varphi_{xx} + \varphi_{yy} + \varphi_{zz}) \\
&= k \cdot \left\{\frac{1}{2kt}\left(\frac{x^2}{2kt} - 1\right)\varphi + \frac{1}{2kt}\left(\frac{y^2}{2kt} - 1\right)\varphi + \frac{1}{2kt}\left(\frac{z^2}{2kt} - 1\right)\varphi\right\} \\
&= \frac{1}{2t}\left(\frac{x^2}{2kt} - 1\right)\varphi + \frac{1}{2t}\left(\frac{y^2}{2kt} - 1\right)\varphi + \frac{1}{2kt}\left(\frac{z^2}{2t} - 1\right)\varphi \\
&= \left(\frac{x^2}{4kt^2} - \frac{1}{2t} + \frac{y^2}{4kt^2} - \frac{1}{2t} + \frac{z^2}{4kt^2} - \frac{1}{2t}\right)\varphi \\
&= \left(\frac{x^2 + y^2 + z^2}{4kt^2} - \frac{3}{2t}\right)\varphi \\
&= \varphi_t \qquad \text{(from part (b)).}
\end{aligned}$$

(e) (Optional) Show that

$$\lim_{t \to 0^+} \varphi(x, y, z, t) = \begin{cases} \infty & \text{if } (x, y, z) = (0, 0, 0) \\ 0 & \text{if } (x, y, z) \neq (0, 0, 0) \end{cases}$$

Solution: Since we are taking $t \to 0^+$, we can regard $t > 0$.
If $(x, y, z) = (0, 0, 0)$, then

$$\varphi(x, y, z, t) = \varphi(0, 0, 0, t) = \frac{1}{(4\pi kt)^{\frac{3}{2}}} \underbrace{\exp\left(-\frac{0^2 + 0^2 + 0^2}{4kt}\right)}_{=1} = \frac{1}{(4\pi kt)^{\frac{3}{2}}}.$$

$$\lim_{t \to 0^+} \varphi(0, 0, 0, t) = \lim_{t \to 0^+} \frac{1}{(4\pi kt)^{\frac{3}{2}}} = \infty.$$

If $(x, y, z) \neq (0, 0, 0)$, then $x^2 + y^2 + z^2 \neq 0$. For simplicity, denote:

$$A := -\frac{x^2 + y^2 + z^2}{4k} \neq 0$$

then

$$\varphi(x, y, z, t) = \frac{1}{(4\pi kt)^{\frac{3}{2}}} e^{-\frac{A}{t}} = \frac{e^{-\frac{A}{t}}}{(4\pi k)^{3/2} t^{3/2}}.$$

When taking $t \to 0^+$, we regard (x, y, z) as constants. As $e^{-\frac{A}{t}}$ goes to 0 much faster than $t^{3/2}$ does, we can conclude that:

$$\lim_{t \to 0^+} \frac{e^{-\frac{A}{t}}}{t^{3/2}} = 0.$$

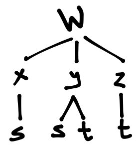
Therefore, it concludes that $\lim\limits_{t \to 0^+} \varphi(x, y, z, t) = 0$ when $(x, y, z) \neq 0$.

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #3 • Chain Rule, Directional Derivatives, Gradients

1. (★) Suppose $w = f(x, y, z)$ where $x = g(s)$, $y = h(s, t)$ and $z = k(t)$. Assume all functions involved are C^1. Draw the tree diagram to showcase the relations between w, x, y, z, s and t. Hence, write down the chain rule for calculating the partial derivatives: $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$. Use the symbols ∂ and d appropriately.

Solution:

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$



2. (★) Recall that the rectangular-polar coordinates conversion rules are given as follows:

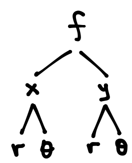
$$x = r\cos\theta$$
$$y = r\sin\theta$$

A function $f(x, y)$ is said to be **rotationally/radially symmetric** if $\frac{\partial f}{\partial \theta} = 0$, i.e. when regarded as a function of (r, θ), it depends only the radial variable r but not the angular variable θ. For instance, $f(x, y) = x^2 + y^2$ is rotationally symmetric since $f(r, \theta) = r^2$. Using the chain rule, show that f is rotationally symmetric if and only if:

$$y\frac{\partial f}{\partial x} = x\frac{\partial f}{\partial y}.$$

Solution: By the chain rule:

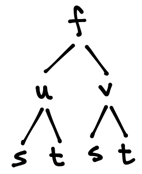
$$\begin{aligned}\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \frac{\partial f}{\partial x}\frac{\partial}{\partial \theta}(r\cos\theta) + \frac{\partial f}{\partial y}\frac{\partial}{\partial \theta}(r\sin\theta) \\ &= \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta) \\ &= -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.\end{aligned}$$



Therefore, $\frac{\partial f}{\partial \theta} = 0$ if and only if $y\frac{\partial f}{\partial x} = x\frac{\partial f}{\partial y}$.

3. (★) Suppose $f(u,v)$ is a C^2 function, and $u = s^2 - t$ and $v = s + t^2$. Express the second partial derivative $\dfrac{\partial^2 f}{\partial s \partial t}$ in terms of f_{uu}, f_{uv}, f_{vv}, s and t.

Solution:



$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial t} \\
&= f_u \cdot (-1) + f_v \cdot 2t = -f_u + 2t f_v \\
\frac{\partial^2 f}{\partial s \partial t} &= \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial t}\right) \\
&= \frac{\partial}{\partial s}\left(-f_u + 2t f_v\right) \\
&= -\left(\frac{\partial f_u}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f_u}{\partial v}\frac{\partial v}{\partial s}\right) + 2t\left(\frac{\partial f_v}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f_v}{\partial v}\frac{\partial v}{\partial s}\right) \\
&= -\left(f_{uu} \cdot 2s + f_{uv} \cdot 1\right) + 2t\left(f_{vu} \cdot 2s + f_{vv} \cdot 1\right) \\
&= -2s f_{uu} + (\underset{4st}{\cancel{4t}} - 1) f_{uv} + 2t f_{vv}.
\end{aligned}$$

Here we have used the fact that f is C^2, and so $f_{uv} = f_{vu}$.

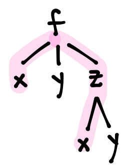
4. (★) Let $f(x,y,z)$ be a C^1 function of three variables, and z be a C^1 function of (x,y) such that

$$f(x,y,z(x,y)) = 0.$$

Using the chain rule, show that:

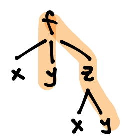
$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z}.$$

Solution:



$$\begin{aligned}
\frac{\partial}{\partial x} f(x,y,z(x,y)) &= \frac{\partial}{\partial x} 0 = 0 \\
\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} &= 0 \\
\frac{\partial z}{\partial x} &= -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{f_x}{f_z}
\end{aligned}$$

Similarly,



$$\begin{aligned}
\frac{\partial}{\partial y} f(x,y,z(x,y)) &= \frac{\partial}{\partial y} 0 = 0 \\
\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} &= 0 \\
\frac{\partial z}{\partial y} &= -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = -\frac{f_y}{f_z}
\end{aligned}$$

5. (★★) Let $f(x,y)$ be a C^1 function. Consider two parametric curves $\mathbf{r}_1(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j}$ and $\mathbf{r}_2(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j}$ which satisfy:

$$\mathbf{r}_1(0) = \mathbf{r}_2(0) \quad \text{and} \quad \mathbf{r}_1'(0) = \mathbf{r}_2'(0).$$

(a) Show that

$$\left.\frac{d}{dt}\right|_{t=0} f(x_1(t), y_1(t)) = \left.\frac{d}{dt}\right|_{t=0} f(x_2(t), y_2(t)).$$

Solution:
Using the chain rule, we get:

$$\begin{aligned}\frac{d}{dt} f(x_1(t), y_1(t)) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= \frac{\partial f}{\partial x} x_1'(t) + \frac{\partial f}{\partial y} y_1'(t) \quad \text{(since } x = x_1 \text{ and } y = y_1 \text{ in this case)}\end{aligned}$$

At $t = 0$, we get:

$$\left.\frac{d}{dt}\right|_{t=0} f(x_1(t), y_1(t)) = \left.\frac{\partial f}{\partial x}\right|_{(x_1(0), y_1(0))} x_1'(0) + \left.\frac{\partial f}{\partial y}\right|_{(x_1(0), y_1(0))} y_1'(0)$$

Similarly, one can also show:

$$\left.\frac{d}{dt}\right|_{t=0} f(x_2(t), y_2(t)) = \left.\frac{\partial f}{\partial x}\right|_{(x_2(0), y_2(0))} x_2'(0) + \left.\frac{\partial f}{\partial y}\right|_{(x_2(0), y_2(0))} y_2'(0)$$

It is given from the problem that:

$$\mathbf{r}_1(0) = \mathbf{r}_2(0) \quad \text{and} \quad \mathbf{r}_1'(0) = \mathbf{r}_2'(0).$$

Therefore, we have:

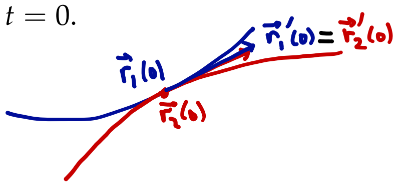
$$\begin{aligned} x_1(0) &= x_2(0) & \qquad x_1'(0) &= x_2'(0) \\ y_1(0) &= y_2(0) & \qquad y_1'(0) &= y_2'(0) \end{aligned}$$

and so:

$$\left.\frac{d}{dt}\right|_{t=0} f(x_1(t), y_1(t)) = \left.\frac{d}{dt}\right|_{t=0} f(x_2(t), y_2(t)).$$

(b) Give a geometric interpretation of the above result.

Solution: This result shows that any two parametric curves with the same position and velocity at $t = 0$ will give the same rate of change of a function f along these two curves at $t = 0$.



6. (★★) The wave equation is an important partial differential equation which governs the propagation of waves. Let $u(x, y, z, t)$ be the displacement of the wave at position (x, y, z) at time t. It can be shown by several physical laws (such as the Hooke's Law) that u satisfies:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{1}$$

where c is a constant (which is the wave speed).

In one (spatial) dimension, the wave equation can be stated as:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{2}$$

It turns out that the chain rule of several variables has a nice application on solving the one dimensional wave equation. The following exercise guides you to show that if $u(x, t)$ is a solution to the one dimensional wave equation, then it must take the form $u(x, t) = F(x - ct) + G(x + ct)$ where F and G are arbitrary differentiable functions of single variable.

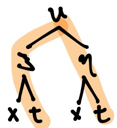
Let $u(x, t)$ solve the one dimensional wave equation (2).

(a) Define $\xi = x - ct$ and $\eta = x + ct$. Regard u as a function of ξ and η, and ξ and η are functions of x and t. Using the chain rule of multivariable functions, show that:

$$u_t = c(u_\eta - u_\xi) \quad \text{and} \quad u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}).$$

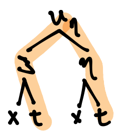
Solution: By chain rule:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} \\
&= u_\xi \cdot (x - ct)_t + u_\eta \cdot (x + ct)_t \\
&= u_\xi \cdot (-c) + u_\eta \cdot c \\
&= c(u_\eta - u_\xi). \\
u_{tt} &= \frac{\partial u_t}{\partial t} \\
&= c\frac{\partial}{\partial t}(u_\eta - u_\xi).
\end{aligned}$$

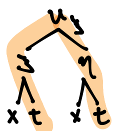


Since u_η and u_ξ are also functions of (ξ, η), the chain rule applies to u_η and u_ξ in the same way as it does to u:

$$\begin{aligned}
\frac{\partial u_\eta}{\partial t} &= \frac{\partial u_\eta}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u_\eta}{\partial \eta}\frac{\partial \eta}{\partial t} \\
&= u_{\eta\xi} \cdot (x - ct)_t + u_{\eta\eta} \cdot (x + ct)_t \\
&= -cu_{\eta\xi} + cu_{\eta\eta}.
\end{aligned}$$



$$\begin{aligned}
\frac{\partial u_\xi}{\partial t} &= \frac{\partial u_\xi}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u_\xi}{\partial \eta}\frac{\partial \eta}{\partial t} \\
&= u_{\xi\xi} \cdot (x - ct)_t + u_{\xi\eta} \cdot (x + ct)_t \\
&= -cu_{\xi\xi} + cu_{\xi\eta}.
\end{aligned}$$



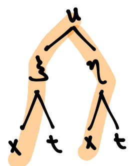
Substitute these two results into u_{tt}, we have:

$$\begin{aligned}
u_{tt} &= c\frac{\partial}{\partial t}(u_\eta - u_\xi) \\
&= c\left(\frac{\partial u_\eta}{\partial t} - \frac{\partial u_\xi}{\partial t}\right) \\
&= c(-cu_{\eta\xi} + cu_{\eta\eta} + cu_{\xi\xi} - cu_{\xi\eta}) \\
&= c^2(u_{\xi\xi} - u_{\xi\eta} - u_{\eta\xi} + u_{\eta\eta}) \\
&= c^2(u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta}).
\end{aligned}$$

(b) Using the chain rule agin, show that

$$u_x = u_\xi + u_\eta \quad \text{and} \quad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Solution: Apply the chain rule again:



$$\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
&= u_\xi (x - ct)_x + u_\eta (x + ct)_x \\
&= u_\xi + u_\eta. \\
u_{xx} &= (u_x)_x \\
&= (u_\xi + u_\eta)_x \\
&= (u_\xi)_x + (u_\eta)_x \\
&= (u_\xi)_\xi \xi_x + (u_\xi)_\eta \eta_x + (u_\eta)_\xi \xi_x + (u_\eta)_\eta \eta_x \\
&= u_{\xi\xi}(x - ct)_x + u_{\xi\eta}(x + ct)_x + u_{\eta\xi}(x - ct)_x + u_{\eta\eta}(x + ct)_x \\
&= u_{\xi\xi} + u_{\xi\eta} + u_{\eta\xi} + u_{\eta\eta} \\
&= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.
\end{aligned}$$

(c) Combining the results of (a), (b) and the wave equation, show that $u_{\xi\eta} = 0$.

Solution: Results in (a) and (b) show:

$$u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}), \qquad u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Substitute them into the wave equation $u_{tt} = c^2 u_{xx}$, we have:

$$\begin{aligned}
c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) &= c^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \\
u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\
-2u_{\xi\eta} &= 2u_{\xi\eta} \\
0 &= 2u_{\xi\eta} + 2u_{\xi\eta} = 4u_{\xi\eta} \\
u_{\xi\eta} &= 0.
\end{aligned}$$

(d) Finally, deduce that u, as a function of ξ and η, must be in the form of:

$$u(\xi, \eta) = F(\xi) + G(\eta)$$

where F and G are arbitrary functions. Hence, in terms of the original variables x and t, u must take the form $u(x,t) = F(x - ct) + G(x + ct)$.

Solution: Finally, we have $u_{\xi\eta} = 0$. Therefore,

$$\frac{\partial u_\xi}{\partial \eta} = 0.$$

Thus u_ξ is independent of η, so u_ξ is a function of ξ only. Let $f(\xi)$ be this function and so

$$u_\xi = f(\xi), \quad \text{or equivalently,} \quad \frac{\partial u}{\partial \xi} = f(\xi)$$

Integrating both sides by ξ, we get:

$$u = \int f(\xi) d\xi + \text{'integration constant'}.$$

Note that the integration 'constant' is no longer a constant, but a quantity not depending on the integration variable ξ. In other words, the 'integration constant' now becomes a function of η.
Denote this function by $G(\eta)$, then:

$$u(\xi, \eta) = \int f(\xi) d\xi + G(\eta).$$

Since $f(\xi)$ is an arbitrary function of ξ, $\int f(\xi) d\xi$ is arbitrary too. We rewrite $\int f(\xi) d\xi$ as $F(\xi)$, where F is an arbitrary function of ξ. Finally, we have:

$$u(\xi, \eta) = F(\xi) + G(\eta).$$

Since $\xi = x - ct$ and $\eta = x + ct$, in terms of the (x,t)-variables:

$$u(x,t) = F(x - ct) + G(x + ct).$$

FYI: The general solution $u(x,t) = F(x - ct) + G(x + ct)$ of the wave equation describes the superposition of two waves – one has a graph given by the function F and it shifts to the right by c unit lengths per unit time, another has a graph given by the function G and it shifts to the left by c unit lengths per unit time. The results in this exercise show that all solutions to the one-dimensional wave equation are superposition of these waves.

7. (★★★) In many physics, geometry and engineering applications, it is often more convenient to use polar or sphereical coordinates since many physical quantities are rotationally symmetric.

The conversion rule of rectangular and polar coordinates is given by:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Let u be a function of x and y. Since (x, y) can be converted into (r, θ), we can also regard u as a function of (r, θ). The chain rule can be used to derive some conversion formulae between u_x, u_y and u_r, u_θ.

An important operator in physics, geometry and engineering is called the **Laplacian**. In two dimensions, it is defined as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}.$$

In this exercise, we will show that $\nabla^2 u$ can be expressed in polar form as:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

The polar form of the Laplacian is often used when dealing with rotationally symmetric functions, i.e. a function u which does not depend on θ but only on r. For such functions, their Laplacian is simply:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r.$$

(a) Use the fact that $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$, show that:

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}.$$

Solution: Using the fact that $r^2 = x^2 + y^2$, by differentiating both sides with respect to x, we get:

$$\begin{aligned} \frac{\partial}{\partial x} r^2 &= \frac{\partial}{\partial x}(x^2 + y^2) \\ 2r\frac{\partial r}{\partial x} &= 2x \Longrightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \end{aligned}$$

Similarly, by differentiating both sides with respect to y, we get:

$$\begin{aligned} \frac{\partial}{\partial y} r^2 &= \frac{\partial}{\partial x}(x^2 + y^2) \\ 2r\frac{\partial r}{\partial y} &= 2y \Longrightarrow \frac{\partial r}{\partial y} = \frac{y}{r} \end{aligned}$$

To find $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$, we consider the fact that $\tan\theta = \frac{y}{x}$. Differentiate both sides with respect to x, we get:

$$\begin{aligned} \frac{\partial}{\partial x}\tan\theta &= \frac{\partial}{\partial x}\left(\frac{y}{x}\right) \\ \sec^2\theta\,\frac{\partial \theta}{\partial x} &= -\frac{y}{x^2} \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{x^2\sec^2\theta} \end{aligned}$$

Since $\sec^2\theta = 1 + \tan^2\theta$, and recall that $\tan\theta = \frac{y}{x}$, we have $\sec^2\theta = 1 + \frac{y^2}{x^2}$. Therefore, after simplication, we can obtain:

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2\sec^2\theta} = -\frac{y}{x^2\cdot\left(1 + \frac{y^2}{x^2}\right)} = -\frac{y}{x^2 + y^2} = -\frac{y}{r^2}.$$

Similarly, differentiating both sides of $\tan\theta = \frac{y}{x}$ by y, we get:

$$\begin{aligned}\frac{\partial}{\partial y}\tan\theta &= \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{x}\\ \sec^2\theta\,\frac{\partial\theta}{\partial y} &= \frac{1}{x}\\ \frac{\partial\theta}{\partial y} &= \frac{1}{x\sec^2\theta}\end{aligned}$$

We have previously derived that $\sec^2\theta = 1+\frac{y^2}{x^2}$, and so:

$$\frac{\partial\theta}{\partial y} = \frac{1}{x\left(1+\frac{y^2}{x^2}\right)} = \frac{1}{x\left(\frac{x^2+y^2}{x^2}\right)} = \frac{1}{\frac{r^2}{x}} = \frac{x}{r^2}.$$

(b) Regard u as a function of (r,θ), and (r,θ) are functions of (x,y). Sketch a tree diagram to showcase these relations. Using the chain rule, show that:

$$\begin{aligned}u_x &= \frac{xu_r}{r} - \frac{yu_\theta}{r^2},\\ u_y &= \frac{yu_r}{r} + \frac{xu_\theta}{r^2}.\end{aligned}$$

Solution:

$$\begin{aligned}u_x &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x}\\ &= u_r\cdot\frac{x}{r} + u_\theta\cdot\left(-\frac{y}{r^2}\right) && \text{from (a)}\\ &= \frac{xu_r}{r} - \frac{yu_\theta}{r^2}\\ u_y &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y}\\ &= u_r\cdot\frac{y}{r} + u_\theta\cdot\frac{x}{r^2} && \text{from (a)}\\ &= \frac{yu_r}{r} + \frac{xu_\theta}{r^2}\end{aligned}$$

(c) Using quotient and product rules, show that:

$$u_{xx} = \frac{u_r}{r} + \frac{xu_{rx}}{r} - \frac{x^2u_r}{r^3} - \frac{yu_{\theta x}}{r^2} + \frac{2xyu_\theta}{r^4}$$
$$u_{yy} = \frac{u_r}{r} + \frac{yu_{ry}}{r} - \frac{y^2u_r}{r^3} + \frac{xu_{\theta y}}{r^2} - \frac{2xyu_\theta}{r^4}$$

Solution:

$$\begin{aligned}
u_{xx} &= \frac{\partial u_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xu_r}{r} - \frac{yu_\theta}{r^2}\right) && \text{from (b)}\\
&= \frac{r\cdot\frac{\partial}{\partial x}(xu_r) - xu_r\frac{\partial r}{\partial x}}{r^2} - \frac{r^2\cdot\frac{\partial}{\partial x}(yu_\theta) - yu_\theta\cdot\frac{\partial}{\partial x}r^2}{r^4} && \text{quotient rule}\\
&= \frac{r(u_r + xu_{rx}) - xu_r\cdot\frac{x}{r}}{r^2} - \frac{r^2\cdot yu_{\theta x} - yu_\theta\cdot 2r\cdot\frac{\partial r}{\partial x}}{r^4} && \text{product rule}\\
&= \frac{u_r}{r} + \frac{xu_{rx}}{r} - \frac{x^2u_r}{r^3} - \frac{yu_{\theta x}}{r^2} + \frac{2ryu_\theta\cdot\frac{x}{r}}{r^4} && \text{break it down}
\end{aligned}$$

which is exactly what we need to show after simplifying the last term. Similarly for u_{yy}:

$$\begin{aligned}
u_{yy} &= \frac{\partial}{\partial y}u_y = \frac{\partial}{\partial y}\left(\frac{yu_r}{r} + \frac{xu_\theta}{r^2}\right)\\
&= \frac{r\frac{\partial}{\partial y}(yu_r) - yu_r\frac{\partial r}{\partial y}}{r^2} + \frac{r^2\frac{\partial}{\partial y}(xu_\theta) - xu_\theta\cdot\frac{\partial}{\partial y}r^2}{r^4}\\
&= \frac{r\left(u_r + yu_{ry}\right) - yu_r\cdot\frac{y}{r}}{r^2} + \frac{r^2xu_{\theta y} - xu_\theta\cdot 2r\cdot\frac{\partial r}{\partial y}}{r^4}\\
&= \frac{u_r}{r} + \frac{yu_{ry}}{r} - \frac{y^2u_r}{r^3} + \frac{xu_{\theta y}}{r^2} - \frac{2rxu_\theta\cdot\frac{y}{r}}{r^4}
\end{aligned}$$

as required (after simplifying the last term).

(d) Since u_r and u_θ are functions of (r,θ), and (r,θ) are functions of (x,y), they share the same tree diagram as u in part (b), and hence we have

$$u_{rx} = \frac{\partial u_r}{\partial x} = \frac{\partial u_r}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u_r}{\partial \theta}\frac{\partial \theta}{\partial x}$$

and similar for other second derivatives u_{ry}, $u_{\theta x}$ and $u_{\theta y}$. Show that:

$$xu_{rx} + yu_{ry} = ru_{rr}$$
$$xu_{\theta y} - yu_{\theta x} = u_{\theta\theta}$$

Solution: We apply the chain rule for each of u_{rx}, u_{ry}, $u_{\theta x}$ and $u_{\theta y}$:

$$\begin{aligned}
u_{rx} &= \frac{\partial u_r}{\partial x} = \frac{\partial u_r}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u_r}{\partial \theta}\frac{\partial \theta}{\partial x}\\
&= u_{rr}\cdot\frac{x}{r} + u_{r\theta}\cdot\left(-\frac{y}{r^2}\right) && \text{from (a)}
\end{aligned}$$

Note that $\frac{\partial u_r}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial r^2} = u_{rr}$. Similar for $u_{r\theta}$.
Similarly,

$$\begin{aligned} u_{ry} &= \frac{\partial u_r}{\partial y} = \frac{\partial u_r}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u_r}{\partial \theta}\frac{\partial \theta}{\partial y} \\ &= u_{rr}\cdot\frac{y}{r} + u_{r\theta}\cdot\frac{x}{r^2} \end{aligned}$$

Combining these two results, we then have:

$$\begin{aligned} xu_{rx} + yu_{ry} &= x\left(u_{rr}\cdot\frac{x}{r} - u_{r\theta}\cdot\frac{y}{r^2}\right) + y\left(u_{rr}\cdot\frac{y}{r} + u_{r\theta}\cdot\frac{x}{r^2}\right) \\ &= u_{rr}\cdot\frac{x^2}{r} - u_{r\theta}\cdot\frac{xy}{r^2} + u_{rr}\cdot\frac{y^2}{r} + u_{r\theta}\cdot\frac{xy}{r^2} \\ &= u_{rr}\cdot\frac{x^2+y^2}{r} && \text{cancellation} \\ &= u_{rr}\cdot\frac{r^2}{r} = ru_{rr} \end{aligned}$$

The second identity can be proven in a similar way:

$$\begin{aligned} u_{\theta y} &= \frac{\partial}{\partial y}u_\theta = \frac{\partial u_\theta}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u_\theta}{\partial \theta}\frac{\partial \theta}{\partial y} \\ &= u_{\theta r}\cdot\frac{y}{r} + u_{\theta\theta}\cdot\frac{x}{r^2} \\ u_{\theta x} &= \frac{\partial}{\partial x}u_\theta = \frac{\partial u_\theta}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u_\theta}{\partial \theta}\frac{\partial \theta}{\partial x} \\ &= u_{\theta r}\cdot\frac{x}{r} + u_{\theta\theta}\cdot\left(-\frac{y}{r^2}\right) \end{aligned}$$

Therefore, we can show:

$$\begin{aligned} xu_{\theta y} - yu_{\theta x} &= x\left(u_{\theta r}\cdot\frac{y}{r} + u_{\theta\theta}\cdot\frac{x}{r^2}\right) - y\left(u_{\theta r}\cdot\frac{x}{r} + u_{\theta\theta}\cdot\left(-\frac{y}{r^2}\right)\right) \\ &= u_{\theta r}\cdot\frac{xy}{r} + u_{\theta\theta}\cdot\frac{x^2}{r^2} - u_{\theta r}\cdot\frac{xy}{r} + u_{\theta\theta}\cdot\frac{y^2}{r} \\ &= u_{\theta\theta}\cdot\frac{x^2+y^2}{r^2} \\ &= u_{\theta\theta}\cdot\frac{r^2}{r^2} = u_{\theta\theta} \end{aligned}$$

as required.

(e) Combining the results proved in previous parts, show that:

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Solution: OK! Finally, we can combine everything together:

$$\begin{aligned}
&u_{xx} + u_{yy} \\
&= \frac{u_r}{r} + \frac{xu_{rx}}{r} - \frac{x^2u_r}{r^3} - \frac{yu_{\theta x}}{r^2} + \frac{2xyu_\theta}{r^4} \\
&\quad + \frac{u_r}{r} + \frac{yu_{ry}}{r} - \frac{y^2u_r}{r^3} + \frac{xu_{\theta y}}{r^2} - \frac{2xyu_\theta}{r^4} && \text{from (c)} \\
&= \frac{2u_r}{r} + \frac{xu_{rx} + yu_{ry}}{r} - \frac{(x^2+y^2)u_r}{r^3} \\
&\quad + \frac{xu_{\theta y} - yu_{\theta x}}{r^2} && \text{factorization, tidy up} \\
&= \frac{2u_r}{r} + \frac{ru_{rr}}{r} - \frac{r^2u_r}{r^3} + \frac{u_{\theta\theta}}{r^2} && \text{from (d)} \\
&= \frac{2u_r}{r} + u_{rr} - \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} \\
&= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}
\end{aligned}$$

which is exactly as required! Cheers!

8. (★) Compute the directional derivative of the following functions at the given point P in the direction of the given vector $\mathbf{v}$. Moreover, find the unit direction $\mathbf{u}$ along which the function increases most rapidly.

(a) $f(x,y) = x^2 - y^2$, $P(-1,-3)$, $\mathbf{v} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$.

Solution:

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = 2x\,\mathbf{i} - 2y\,\mathbf{j} \\
\nabla f(P) &= \nabla f(-1,-3) = 2(-1)\,\mathbf{i} - 2(-3)\,\mathbf{j} = -2\mathbf{i} + 6\mathbf{j} \\
|\mathbf{v}| &= \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = 1 \\
\hat{\mathbf{v}} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{v} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} \\
D_{\hat{\mathbf{v}}}f(P) &= \nabla f(P) \cdot \hat{\mathbf{v}} = (-2\mathbf{i} + 6\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = -6
\end{aligned}$$

Since $D_{\mathbf{u}}f(P) = \nabla f(P) \cdot \mathbf{u} = |\nabla f(P)|\,|\mathbf{u}|\cos\theta = |\nabla f(P)|\cos\theta$ for any unit vector $\mathbf{u}$. Here θ is the angle between $\nabla f(P)$ and $\mathbf{u}$. It is the largest when $\theta = 0$ (i.e. $\cos\theta = 1$). Therefore, $D_{\mathbf{u}}f(P)$ achieves its maximum when $\nabla f(P)$ and $\mathbf{u}$ are parallel. Therefore, the unit direction $\mathbf{u}$ along which the function increases most rapidly is given by:

$$\mathbf{u} = \frac{\nabla f(P)}{|\nabla f(P)|} = \frac{-2\mathbf{i} + 6\mathbf{j}}{\sqrt{40}}.$$

(b) $g(x,y) = e^{-x-y}$, $P(\ln 2, \ln 3)$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$.

Solution:

$$\begin{aligned}
\nabla g &= -e^{-x-y}\mathbf{i} - e^{-x-y}\mathbf{j} \\
\nabla g(P) &= -e^{-\ln 2 - \ln 3}\mathbf{i} - e^{-\ln 2 - \ln 3}\mathbf{j} = -\frac{1}{6}\mathbf{i} - \frac{1}{6}\mathbf{j} \\
\hat{\mathbf{v}} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} \\
D_{\hat{\mathbf{v}}} g(P) &= \nabla g(P) \cdot \hat{\mathbf{v}} = \left(-\frac{1}{6}\mathbf{i} - \frac{1}{6}\mathbf{j}\right) \cdot \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = -\frac{2}{6\sqrt{2}} \\
\mathbf{u} &= \frac{\nabla g(P)}{|\nabla g(P)|} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}
\end{aligned}$$

(c) $h(x,y) = e^{xy}$, $P(1,0)$, $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j}$.

Solution: (Answer only)

$$D_{\hat{\mathbf{v}}} h(P) = \frac{12}{13}, \quad \mathbf{u} = \mathbf{j}.$$

(d) $F(x,y,z) = xy + yz + zx + 4$, $P(2,-2,1)$, $\mathbf{v} = -\mathbf{j} - \mathbf{k}$.

Solution: (Answer only)

$$D_{\hat{\mathbf{v}}} F(P) = -\frac{3}{\sqrt{2}}, \quad \mathbf{u} = \frac{-\mathbf{i} + 3\mathbf{j}}{\sqrt{10}}.$$

(e) $G(x,y,z) = e^{xyz} - 1$, $P(0,1,-1)$, $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution: (Answer only)

$$\nabla_{\hat{\mathbf{v}}} G(P) = \frac{2}{3}, \quad \mathbf{u} = -\mathbf{i}.$$

9. (★) For each surface and the given point P, find the value a such that P lies on the surface, and then find an equation of the tangent plane to the surface at the point P:

(a) $x^2 + y + z = 3$, $P(2,0,a)$

Solution: To solve for a, we put $(x,y,z) = (2,0,a)$ into the given equation:

$$2^2 + 0 + a = 3 \Longrightarrow a = -1$$

To find the equation of the tangent plane at P, we need a normal vector to the surface:

$$\nabla(x^2 + y + z - 3) = 2x\mathbf{i} + \mathbf{j} + \mathbf{k}$$

Therefore, $\mathbf{n} = (2x\mathbf{i} + \mathbf{j} + \mathbf{k})|_P = 2(2)\mathbf{i} + \mathbf{j} + \mathbf{k} = \langle 4, 1, 1 \rangle$. Equation of the tangent plane at P is: $4x + 1y + 1z = 4(2) + 1(0) + 1(-1)$. After simplification:

$$4x + y + z = 7.$$

(b) $xy \sin z = 1, \ P(a, 2, \pi/6)$

Solution: (Answer only)

$$a = 1$$
$$\nabla (xy \sin z - 1)|_P = \mathbf{i} + \frac{1}{2}\mathbf{j} + \sqrt{3}\mathbf{k}$$
$$x + \frac{1}{2}y + \sqrt{3}z = 2 + \frac{\pi}{2}.$$

(c) $yze^{xz} = 8, \quad P(0, a, 4)$

Solution: (Answer only)

$$a = 2$$
$$\nabla (yze^{xz} - 8)|_P = 32\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$
$$16x + 2y + z = 8$$

(d) $z = e^{xy}, \quad P(1, 0, a)$

Solution: (Answer only)

$$a = 1$$
$$\nabla(z - e^{xy})|_P = -\mathbf{j} + \mathbf{k}$$
$$-y + z = 1$$

(e) $z = \ln(1 + xy), \quad P(1, 2, a)$.

Solution:

$$a = \ln 3$$
$$\nabla(z - \ln(1 + xy))|_P = -\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \mathbf{k}$$
$$-\frac{2}{3}x - \frac{1}{3}y + z = -\frac{4}{3} + \ln 3.$$

10. (★) Let

$$V(x,y,z) = -\frac{GMm}{\sqrt{x^2+y^2+z^2}}$$

where G, M and m are constants. Define $\mathbf{F}(x,y,z) = -\nabla V(x,y,z)$.

(a) Verify that:

$$\mathbf{F}(x,y,z) = -GMm\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}.$$

Solution:

$$\begin{aligned}
\frac{\partial V}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{GMm}{\sqrt{x^2+y^2+z^2}}\right) \\
&= -GMm\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-1/2} \\
&= -GMm\cdot\left(-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}\frac{\partial}{\partial x}(x^2+y^2+z^2)\right) \\
&= GMm\cdot\frac{1}{2}\cdot\frac{2x}{(x^2+y^2+z^2)^{3/2}} \\
&= GMm\cdot\frac{x}{(x^2+y^2+z^2)^{3/2}}
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
\frac{\partial V}{\partial y} &= GMm\cdot\frac{y}{(x^2+y^2+z^2)^{3/2}} \\
\frac{\partial V}{\partial z} &= GMm\cdot\frac{z}{(x^2+y^2+z^2)^{3/2}}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\nabla V &= \frac{\partial V}{\partial x}\mathbf{i}+\frac{\partial V}{\partial y}\mathbf{j}+\frac{\partial V}{\partial z}\mathbf{k} \\
&= GMm\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}
\end{aligned}$$

as required. Note that $\mathbf{F} = -\nabla V$.

(b) Show that $|\mathbf{F}(x,y,z)|$ is inversely proportional to the squared distance from (x,y,z) to the origin in $\mathbb{R}^3$.

Solution:

$$
\begin{aligned}
|\mathbf{F}(x,y,z)| &= \left| -GMm \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}} \right| \\
&= GMm \frac{1}{(x^2+y^2+z^2)^{3/2}} |x\mathbf{i}+y\mathbf{j}+z\mathbf{k}| \\
&= GMm \frac{(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^{3/2}} = GMm \frac{1}{x^2+y^2+z^2}
\end{aligned}
$$

Note that the distance from (x,y,z) to the origin is given by $\rho = \sqrt{x^2+y^2+z^2}$. Hence, we have:

$$|\mathbf{F}(x,y,z)| = \frac{GMm}{\rho^2}$$

as required.

11. (★★) Consider the function

$$f(x,y) = \cos(x+y)$$

as well as the plane Π given by the equation

$$x - y = 0\,.$$

The intersection of the graph of f with Π is a curve C. Find the slope of the tangent line to C at the point (π,π) using directional derivatives. [Hint: First sketch a diagram of the graph, the plane and the curve.]

Solution: The intersection between the plane Π and the xy-plane is the straight-line $y = x$. There are two unit vectors in the x-y-plane parallel to Π, given by $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ and its negative $-\mathbf{u} = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$. The directional derivative of f in the direction of u is now equal to

$$
\begin{aligned}
\nabla f \cdot \mathbf{u} &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \mathbf{u} \\
&= -\frac{1}{\sqrt{2}} \sin(x+y) - \frac{1}{\sqrt{2}} \sin(x+y) \\
&= -2\sqrt{2}\sin(x+y)
\end{aligned}
$$

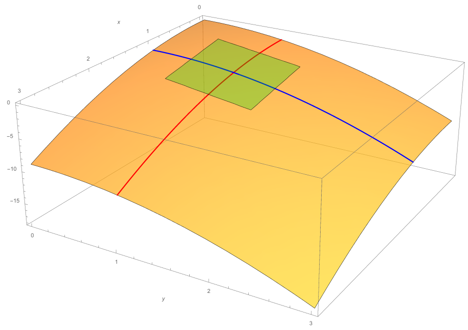
So the slope of C at (π,π) is therefore equal to 0, by evaluating this equation at $(x,y) = (\pi,\pi)$.

Alternatively one can also work with $-\mathbf{u}$ instead of $\mathbf{u}$ and one would obtain the negative of the above solution, which is also 0 in our situation.

12. (★★) One approach for finding the normal vector of the tangent plane at a given point (x_0,y_0) to a graph $z = f(x,y)$ is by writing the graph equation as a level surface $z - f(x,y) = 0$ of a three-variable function $g(x,y,z) := z - f(x,y)$. Then, the gradient $\nabla g = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$ at point $(x_0,y_0,f(x_0,y_0))$ is perpendicular to the level surface $\{g = 0\}$,

and so we can take it to be a normal vector of the tangent plane as long as $\nabla g \neq \mathbf{0}$ at $(x_0, y_0, f(x_0, y_0))$.

In fact, it is also possible to show the normal vector is $\left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$ using a *purely* two-variable argument instead of *going up one higher dimension*.



(a) Consider a given function $f(x, y)$, and a given point (x_0, y_0). Find a parametrization:

$$\mathbf{r}_1(t) = ?\mathbf{i} + ?\mathbf{j} + ?\mathbf{k}$$

of the curve on the graph $z = f(x, y)$ travelling in the x-direction while keeping y fixed at y_0 (i.e. the red curve in the diagram). Hence, find the tangent vector of the curve $\mathbf{r}_1(t)$ at the point $(x_0, y_0, f(x_0, y_0))$. Label this tangent vector by $\mathbf{T}_1$.

Solution: The projection of the red curve on the xy-plane is a straight line passing through (x_0, y_0) and is parallel to the x-axis, i.e. vector $\mathbf{i}$. Therefore, the (x, y)-coordinates of the path is given by:

$$\begin{aligned} x(t) &= x_0 + t \\ y(t) &= y_0 \end{aligned}$$

The z-coordinate of the red curve is determined by the function f, i.e. $z(t) = f(x(t),\ y(t)) = f(x_0 + t,\ y_0)$. Therefore, the parametrization of the red curve is:

$$\mathbf{r}_1(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = (x_0 + t)\mathbf{i} + y_0\mathbf{j} + f(x_0 + t,\ y_0)\ \mathbf{k}.$$

When $t = 0$, the "particle" is at the point $(x_0, y_0, f(x_0, y_0))$. Therefore, the tangent vector $\mathbf{T}_1$ to the curve at this point is given by:

$$\mathbf{T}_1 = \mathbf{r}_1'(0) = \mathbf{i} + 0\mathbf{j} + \left.\frac{d}{dt} f(x_0 + t,\ y_0)\right|_{t=0} \mathbf{k}.$$

To compute $\frac{d}{dt} f(x_0 + t,\ y_0)$, we apply the chain rule:

$$\begin{aligned} \frac{d}{dt} f(x_0 + t,\ y_0) &= \frac{\partial f}{\partial x}(x_0 + t,\ y_0)\ \frac{d(x_0 + t)}{dt} + \frac{\partial f}{\partial y}(x_0 + t,\ y_0)\ \frac{d(y_0)}{dt} \\ &= \frac{\partial f}{\partial x}(x_0 + t,\ y_0). \end{aligned}$$

Evaluate at $t = 0$, we get:

$$\left.\frac{d}{dt} f(x_0 + t,\ y_0)\right|_{t=0} = \frac{\partial f}{\partial x}(x_0,\ y_0).$$

Therefore, we get:

$$\mathbf{T}_1 = \mathbf{i} + \frac{\partial f}{\partial x}(x_0,\ y_0)\ \mathbf{k}.$$

(b) Find a parametrization $\mathbf{r}_2(t)$ of the curve on the graph $z = f(x, y)$ travelling in the y-direction while keeping x fixed at x_0 (i.e. the blue curve in the diagram). Hence, find the tangent vector of $\mathbf{r}_2(t)$ at the point $(x_0, y_0, f(x_0, y_0))$. Label this tangent vector by $\mathbf{T}_2$.

Solution: Similar to (a), the parametrization is given by:

$$\mathbf{r}_2(t) = x_0\mathbf{i} + (y_0 + t)\mathbf{j} + f(x_0, y_0 + t)\,\mathbf{k}$$

and the tangent vector at $(x_0, y_0, f(x_0, y_0))$ is:

$$\mathbf{T}_2 = \mathbf{r}_2'(0) = \mathbf{j} + \frac{\partial f}{\partial y}(x_0,\ y_0)\,\mathbf{k}$$

(c) Since both $\mathbf{T}_1$ and $\mathbf{T}_2$ are tangent vectors to the graph, they are parallel to the tangent plane. Therefore, the normal vector to the tangent plane must be perpendicular to both $\mathbf{T}_1$ and $\mathbf{T}_2$. Using this fact, show that the normal vector to the tangent plane is given by

$$\left\langle -\frac{\partial f}{\partial x}(x_0, y_0), -\frac{\partial f}{\partial y}(x_0, y_0), 1 \right\rangle.$$

Solution: Since the normal vector $\mathbf{n}$ to the tangent plane is orthogonal to the tangent vectors $\mathbf{T}_1$ and $\mathbf{T}_2$, it can be computed by taking cross product: $\mathbf{n} = \mathbf{T}_1 \times \mathbf{T}_2$ which will yield the vector stated in the problem.

Optional

13. The spherical coordinates (ρ, θ, ϕ) is another important coordinate system in $\mathbb{R}^3$. We will learn that in later chapters. The conversion rules between spherical and rectangular coordinates are given by:

$$\begin{aligned} x &= \rho \sin\phi \cos\theta \\ y &= \rho \sin\phi \sin\theta \\ z &= \rho \cos\phi \end{aligned}$$

Given a C^2 function $f(x, y, z)$, it can be regarded as a function of (ρ, θ, ϕ) as well under the above conversion rule. Show that the Laplacian $\nabla^2 f := f_{xx} + f_{yy} + f_{zz}$ can be expressed in spherical coordinates as:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}.$$

[Note: It is a very time consuming exercise. It took me 4 hours to do it when I was an undergraduate.]

1. (★) Find all local extrema (a.k.a. critical points) of the following functions $f(x, y)$. Determine the nature (a local minimum, a local maximum or a saddle) of each of them using the Second Derivative Test whenever possible. If the Second Derivative Test is inconclusive, use some other methods to determine its nature.

   (a) $f(x, y) = 4 + x^3 + y^3 - 3xy$

Solution: To find critical points, we solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, i.e.

$$3x^2 - 3y = 0$$
$$3y^2 - 3x = 0$$

After simplification: $x^2 = y$ and $y^2 = x$. Substitute $y = x^2$ into $y^2 = x$, we get:

$$\left(x^2\right)^2 = x \Longrightarrow x^4 - x = 0 \Longrightarrow x(x^3 - 1) = 0$$

Therefore, we have $x = 0$ or $x^3 = 1$. The latter case gives $x = 1$.
Since $y = x^2$, we get $y = 0$ when $x = 0$; whereas $y = 1$ when $x = 1$. There are two critical points:

$$(x, y) = (0, 0), (1, 1).$$

Next, compute the second derivatives:

$$f_{xx} = 6x \qquad f_{xy} = -3$$
$$f_{yx} = -3 \qquad f_{yy} = 6y$$

P	$f_{xx}(P)$	$f_{yy}(P)$	$f_{xy}(P)$	$\left(f_{xx}f_{yy} - f_{xy}^2\right)(P)$	P is:
$(0, 0)$	0	0	-3	-9	saddle
$(1, 1)$	6	6	-3	$(6)(6) - (-3)^2 > 0$	local minimum

   (b) $f(x, y) = x^2 + 4y^2 - 2x^2y + 4$

Solution: (Answer only)

Critical Point	f_{xx}	$f_{xx}f_{yy} - f_{xy}^2$	Nature
$(0, 0)$	2	16	local minimum
$(-\sqrt{2}, \frac{1}{2})$	0	-32	saddle
$(\sqrt{2}, \frac{1}{2})$	0	-32	saddle

   (c) $f(x, y) = \sin x \, \cos y$

Solution: To solve for critical points, we solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, i.e.

$$\cos x \, \cos y = 0 \quad \text{and} \quad -\sin x \, \sin y = 0$$

From the first equation, we get two cases: Case A: $\cos x = 0$, or Case B: $\cos y = 0$.
Case A: Since $\cos x = 0$, we have $x = \frac{\pi}{2} + k\pi$ where k is any integer. Then $\sin x = \pm 1$ (depending on whether k is even or odd), and the second equation

$\sin x \ \sin y = 0$ then tells us that $\sin y = 0$, and so $y = m\pi$ where m is any integer. Critical points obtained in this case are:

$$(x, y) = \left(\frac{\pi}{2} + k\pi,\ m\pi\right), \quad \text{where } m \text{ and } k \text{ are any integers.}$$

<u>Case B</u>: From $\cos y = 0$, we get $y = \frac{\pi}{2} + n\pi$ where n is any integer, then substitute it into the second equation, we get $\sin x \cdot (\pm 1) = 0$ where $\pm$ depends on whether k is even or odd. Therefore, $x = q\pi$ where q is any integer. Critical points obtained in this case are:

$$(x, y) = (q\pi,\ \frac{\pi}{2} + n\pi) \quad \text{where } n \text{ and } q \text{ are any integers.}$$

To apply the Second Derivative Test, we first compute:

$$\begin{aligned} f_{xx} &= -\sin x \ \cos y \\ f_{xy} &= -\cos x \ \sin y \\ f_{yy} &= -\sin x \ \cos y \\ f_{xx} f_{yy} - f_{xy}^2 &= \sin^2 x \ \cos^2 y - \cos^2 x \ \sin^2 y \end{aligned}$$

P		f_{xx}	f_{yy}	f_{xy}	$\left(f_{xx}f_{yy} - f_{xy}^2\right)$	P is:
$(\frac{\pi}{2} + k\pi, m\pi)$	k odd, m odd	-1	-1	0	1	local max
$(\frac{\pi}{2} + k\pi, m\pi)$	k odd, m even	1	1	0	1	local min
$(\frac{\pi}{2} + k\pi, m\pi)$	k even, m odd	1	1	0	1	local min
$(\frac{\pi}{2} + k\pi, m\pi)$	k even, m even	-1	-1	0	1	local max
$(q\pi,\ \frac{\pi}{2} + n\pi)$	any integers q, n	0	0	± 1	-1	saddle

(d) $f(x, y) = x^4 + y^4$

Solution: By solving $\nabla f = 4x^3\mathbf{i} + 4y^3\mathbf{j} = \mathbf{0}$, we get $(0, 0)$ as the only critical point of f. However, one can check that $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(0, 0)$, and so the Second Derivative Test does not conclude anything.
However, this function $f(x, y) = x^4 + y^4$ is always non-negative. Therefore, $f(x, y) \geq f(0, 0) = 0$ for any (x, y). In other words, $(0, 0)$ is a local minimum point (in fact a global minimum).

2. (★★) Give an example of a C^2 function $f(x,y)$ such that:

   (a) $(0,0)$ is a saddle point and $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(0,0)$.

   Solution: One of many examples: $f(x,y) = x^4 - y^4$. Along the x-axis, the graph is $f(x,0) = x^4$ which is concave up. However, along the y-axis, the graph is $f(0,y) = -y^4$ which is concave down. It is straight-forward to verify that $f_{xx}(0,0) = f_{yy}(0,0) = f_{xy}(0,0) = 0$.

   (b) $(0,0)$ is a local minimum, $f_{xx}f_{yy} - f_{xy}^2 = 0$, $f_{xx} > 0$ and $f_{yy} = 0$ at $(0,0)$.

   Solution: One of many examples: $f(x,y) = x^2 + y^4$. It must be non-negative, and $f(0,0) = 0$. Therefore, the origin must be a minimum point (in fact the absolute minimum). It is straight-forward to verify that $f_{xx}(0,0) = 2$ whereas $f_{yy}(0,0) = f_{xy}(0,0) = 0$.

   (c) $(0,0)$ is a local maximum, $f_{xx}f_{yy} - f_{xy}^2 = 0$, $f_{xx} = 0$ and $f_{yy} < 0$ at $(0,0)$.

   Solution: One of many examples: $f(x,y) = -x^4 - y^2$.

3. (★) Using Lagrange's Multipliers, find the maximum and minimum values of the given function subject to the given constraint:

   (a) $f(x,y) = x + 2y$ subject to $x^2 + y^2 = 4$

   Solution: Let $g(x,y) = x^2 + y^2$, then the constraint is the level-set $g(x,y) = 4$. The Lagrange's Multiplier system is:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} & 1 &= \lambda \cdot 2x \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} & 2 &= \lambda \cdot 2y \\ g(x,y) &= 4 & x^2 + y^2 &= 4 \end{aligned}$$

   From the first equation, we must have $\lambda \neq 0$ (otherwise if $\lambda = 0$, then $1 = 0$). Therefore, by rearranging the first and second equations, we get:

$$\frac{1}{\lambda} = 2x \quad \text{and} \quad \frac{1}{\lambda} = y$$

   Therefore, $2x = y$ and combine with the third equation, we get:

$$x^2 + (2x)^2 = 4 \implies 5x^2 = 4 \implies x^2 = \frac{4}{5} \implies x = \sqrt{\frac{4}{5}} \text{ or } -\sqrt{\frac{4}{5}}.$$

   Since $y = 2x$, the candidate points are:

$$(x,y) = (\sqrt{4/5},\ 2\sqrt{4/5}), (-\sqrt{4/5},\ -2\sqrt{4/5}).$$

Substitute them into $f(x, y)$, we get:

$$f\left(\sqrt{\frac{4}{5}}, 2\sqrt{\frac{4}{5}}\right) = 5\sqrt{\frac{4}{5}}$$
$$f\left(-\sqrt{\frac{4}{5}}, -2\sqrt{\frac{4}{5}}\right) = -5\sqrt{\frac{4}{5}}$$

The former is the maximum, the latter is the minimum.

(b) $f(x, y) = x - y$ subject to $x^2 + y^2 = 20 + 3xy$

Solution: Note that the constraint is not yet in the level-set form. By rearrangement, we get:

$$x^2 + y^2 - 3xy = 20$$

Therefore, we let $g(x, y) = x^2 + y^2 - 3xy$ and then the constraint becomes the level-set $g(x, y) = 20$. We need to solve the Lagrange's Multiplier system $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 20$:

$$\begin{aligned} 1 &= \lambda(2x - 3y) \\ -1 &= \lambda(2y - 3x) \\ x^2 + y^2 - 3xy &= 20 \end{aligned}$$

Solutions are: $(x, y) = (-2, 2)$, $(2, -2)$. Substitute them into f, we get:

$$f(-2, 2) = -4, \text{ and } f(2, -2) = 4.$$

The maximum is 4 and the minimum is -4.

(c) $f(x, y, z) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$

Solution: Let $g(x, y, z) = x^2 + 2y^2 + 4z^2$, then the constraint is $g(x, y, z) = 9$. The Lagrange's Multiplier system is:

$$\begin{aligned} yz &= \lambda \cdot 2x \\ xz &= \lambda \cdot 4y \\ xy &= \lambda \cdot 8z \\ x^2 + 2y^2 + 4z^2 &= 9 \end{aligned}$$

The strategy of solving this system is as follows: Suppose the third equation $xy = \lambda \cdot 8z \neq 0$, then by dividing the first equation by the third one gives:

$$\frac{yz}{xy} = \frac{2\lambda x}{8\lambda z} \implies \frac{z}{x} = \frac{x}{4z} \implies 4z^2 = x^2.$$

Also, divide the second equation by the third one gives:

$$\frac{xz}{xy} = \frac{4y\lambda}{8z\lambda} \implies \frac{z}{y} = \frac{y}{2z} \implies y^2 = 2z^2$$

Then, substitute $x^2 = 4z^2$ and $y^2 = 2z^2$ into the fourth equation gives: $4z^2 + 4z^2 + 4z^2 = 9$. This gives the solutions for z, which can be used to determine the values of x and y.
Next, we need to find the solutions for the case when the third equation is zero, i.e. $xy = \lambda \cdot 8z = 0$.
We skip the detail of computations here. Below are the results:

(x, y, z)	$f(x, y, z)$
$(\pm 3, 0, 0)$	0
$(0, 0, \pm 3/2)$	0
$(0, \pm 3/\sqrt{2}, 0)$	0
$(\pm\sqrt{3}, \pm\sqrt{3/2}, \pm\sqrt{3}/2)$	$\pm\frac{3}{2}\sqrt{\frac{3}{2}}$

→ If $x=3$, $x^2 \neq 4z^2$!
why??

The maximum is $\frac{3}{2}\sqrt{\frac{3}{2}}$ and the minimum is $-\frac{3}{2}\sqrt{\frac{3}{2}}$.

(d) $f(x_1, x_2, \ldots, x_n) = x_1 + 2x_2 + \ldots + nx_n$ subject to $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$

Solution: Let $g(x_1, \ldots, x_n) = x_1^2 + \ldots + x_n^2$, then the constraint becomes the level-set $g = 1$. The Lagrange's Multiplier system is:

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \lambda \frac{\partial g}{\partial x_1} & 1 &= 2\lambda x_1 \\ \frac{\partial f}{\partial x_2} &= \lambda \frac{\partial g}{\partial x_2} & 2 &= 2\lambda x_2 \\ &\vdots & &\vdots \\ \frac{\partial f}{\partial x_n} &= \lambda \frac{\partial g}{\partial x_n} & n &= 2\lambda x_n \\ g(x_1, \ldots, x_n) &= 1 & x_1^2 + \ldots + x_n^2 &= 1 \end{aligned}$$

It is a very large system. However, it is helpful to note that λ cannot be zero (otherwise from the first equation we would have $1 = 0$). By rearrangement, we can get:

$$x_1 = \frac{1}{2\lambda},\ x_2 = \frac{2}{2\lambda},\ \ldots,\ x_n = \frac{n}{2\lambda}.$$

Substitute them into the constraint equation, we get:

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{2}{2\lambda}\right)^2 + \ldots + \left(\frac{n}{2\lambda}\right)^2 = 1$$

After simplification, we get:

$$\frac{1^2 + 2^2 + \ldots + n^2}{(2\lambda)^2} = 1 \implies 2\lambda = \pm\sqrt{1^2 + 2^2 + \ldots + n^2}.$$

Therefore, the candidate points are: $x_1 = \frac{1}{2\lambda}$, $x_2 = \frac{2}{2\lambda}$, $\ldots$, $x_n = \frac{n}{2\lambda}$ where $2\lambda = \pm\sqrt{1^2 + 2^2 + \ldots + n^2}$.

$$f(x_1, \ldots, x_n) = \frac{1}{2\lambda} + 2 \cdot \frac{2}{2\lambda} + \ldots + n \cdot \frac{n}{2\lambda} = \pm\frac{1^2 + 2^2 + \ldots + n^2}{\sqrt{1^2 + 2^2 + \ldots + n^2}}$$

The minimum is $-\frac{1^2+2^2+\ldots+n^2}{\sqrt{1^2+2^2+\ldots+n^2}}$ and the maximum is $\frac{1^2+2^2+\ldots+n^2}{\sqrt{1^2+2^2+\ldots+n^2}}$.

4. (★★) The rationale behind Lagrange's Multipliers method is that if a function f achieves its maximum or minimum on the constraint $g = c$ at points P, then the level sets of f and g at P must be tangent to each other, and so $\nabla f(P)$ is parallel to $\nabla g(P)$. Therefore, we solve the system:

$$\nabla f(P) = \lambda \nabla g(P) \quad \text{and} \quad g(P) = c$$

to locate all such P's.

However, another way to determine whether $\nabla f(P)$ and $\nabla g(P)$ are parallel is by their cross product:

$$\nabla f(P) \parallel \nabla g(P) \quad \text{if and only if} \quad \nabla f(P) \times \nabla g(P) = \mathbf{0}.$$

By solving the vector equation $\nabla f(P) \times \nabla g(P) = \mathbf{0}$ for P (instead of using Lagrange's Multiplier), try to redo Problems #3(a)(b)(c).

What is the limitation of this method when compared to Lagrange's Multiplier?

Solution: Take (a) as an example, in which we have $f(x, y) = x + 2y$ and $g(x, y) = x^2 + y^2$. By direct computation, their gradients are:

$$\nabla f = \mathbf{i} + 2\mathbf{j}, \quad \nabla g = 2x\mathbf{i} + 2y\mathbf{j}$$

and their cross product is given by $\nabla f \times \nabla g = (-4x + 2y)\mathbf{k}$. By setting $\nabla f \times \nabla g = \mathbf{0}$, we get $-4x + 2y = 0$, and so: $y = 2x$.

Using the constraint equation $x^2 + y^2 = 4$, we have $x^2 + (2x)^2 = 4$, whose solutions are $x = \pm\sqrt{4/5}$. Therefore, using $y = 2x$, we get:

$$(x, y) = (\sqrt{4/5},\ 2\sqrt{4/5}), (-\sqrt{4/5},\ -2\sqrt{4/5})$$

which are exactly the same as in #3(a).

Similar approaches can be applied to solve #3(b)(c).

However, since **cross product is not defined** in dimensions higher than 3, we cannot apply this approach to maximize/minimize a function for 4 variables or higher.

5. (★) A closed rectangular water tank is to be made with three different materials. The top part will be made by a thin material which costs \$1 per cm^2. The four sides will use stronger material which costs \$2 per cm^2. To support the weight of water, the bottom of the tank has to be made with very durable and strengthened material which costs \$5 per cm^2.

Suppose the volume of the water tank is to be 96cm^3 . What dimensions of the tank will *minimize* the cost of construction?

Solution: (Sketch) Let l be the length, w be the width and h be the height.

$$\begin{aligned} \text{minimize:} \quad & f(l, w, h) = \underbrace{wl}_{\text{top}} + \underbrace{2(2hl) + 2(2wh)}_{\text{sides}} + \underbrace{5(wl)}_{\text{base}} = 6wl + 4hl + 4wh \\ \text{subject to:} \quad & g(l, w, h) = lwh = 96 \end{aligned}$$

Answer: $(l, w, h) = (4, 4, 6)$ is the only solution to the Lagrange's Multiplier system. It must give the minimum, since the maximum does not exist. Given any fixed volume, we can always construct a rectangular cube with arbitrarily large surface area.

6. (★) Find the point(s) on the cone:

$$z = \sqrt{x^2 + y^2}$$

that is/are closest to the point $(1, 3, 1)$.

Solution: (Sketch) To ease computations, we rewrite the cone equation as:

$$z^2 = x^2 + y^2, \text{ or equivalently } z^2 - x^2 - y^2 = 0.$$

Let $g(x, y, z) = z^2 - x^2 - y^2$, then $g = 0$ is the constraint.

$$\begin{aligned} &\text{minimize:} && f(x, y, z) = (x-1)^2 + (y-3)^2 + (z-1)^2 \\ &\text{subject to:} && g(x, y, z) = z^2 - x^2 - y^2 = 0 \end{aligned}$$

Answer: $(x, y, z) = \left(\frac{10+\sqrt{10}}{20}, \frac{3(10+\sqrt{10})}{20}, \frac{1+\sqrt{10}}{2}\right), \quad \left(\frac{10-\sqrt{10}}{20}, \frac{3(10-\sqrt{10}}{20}, \frac{1-\sqrt{10}}{2}\right).$

The second point is rejected since the original cone equation $z = \sqrt{x^2 + y^2}$ requires that $z \geq 0$. It is straight-forward to check that:

$$f\left(\frac{10+\sqrt{10}}{20}, \frac{3(10+\sqrt{10})}{20}, \frac{1+\sqrt{10}}{2}\right) = \frac{11}{2} - \sqrt{10}$$

Therefore, the required point is $\left(\frac{10+\sqrt{10}}{20}, \frac{3(10+\sqrt{10})}{20}, \frac{1+\sqrt{10}}{2}\right)$ and the distance from this point to $(1, 3, 1)$ is given by $\sqrt{\frac{11}{2} - \sqrt{10}}$.

7. (★★) Consider the surface given by the equation

$$x^2 y^2 z^2 = 1.$$

(a) Show that for any point (a, b, c) on the surface, its tangent plane does not contain the origin.

Solution: Let $g(x, y, z) = x^2 y^2 z^2$, then the surface is a level-set $g(x, y, z) = 1$. To find the normal vector to the surface at (a, b, c), we compute:

$$\nabla g(x, y, z) = 2xy^2z^2\mathbf{i} + 2x^2yz^2\mathbf{j} + 2x^2y^2z\mathbf{k}$$

Therefore, the normal vector to the required tangent plane can be taken to be:

$$\mathbf{n} = ab^2c^2\mathbf{i} + a^2bc^2\mathbf{j} + a^2b^2c\mathbf{k}$$

As the plane passes through (a,b,c), the equation of the tangent is then given by:

$$ab^2c^2x + a^2bc^2y + a^2b^2cz = ab^2c^2 \cdot a + a^2bc^2 \cdot b + a^2b^2c \cdot c$$

As (a,b,c) lies on the surface $x^2y^2z^2$, we can assume that all of a, b and c are non-zero (otherwise if any of them are zero, we would get $0 = 1$). After simplification, we get:

$$ab^2c^2x + a^2bc^2y + a^2b^2cz = 3a^2b^2c^2 \implies \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3.$$

By substituting $(x,y,z) = (0,0,0)$, we get LHS $= 0$ whereas RHS $= 3$. Therefore, $(0,0,0)$ is not contained on the tangent plane.

(b) Find all points (a,b,c) on the surface such that the tangent plane at these points are closest to the origin.

Solution: By Q5 in Problem Set #1, the distance between the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$ and the origin $(0,0,0)$ is given by:

$$d = \frac{3}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

Since (a,b,c) lies on the given surface, they are subject to the constraint $a^2b^2c^2 = 1$. Therefore, the distance is also given by:

$$d = \frac{3}{\sqrt{\frac{b^2c^2+a^2c^2+b^2c^2}{a^2b^2c^2}}} = \frac{3}{\sqrt{b^2c^2 + a^2c^2 + b^2c^2}}$$

To minimize d is equivalent to maximize $a^2b^2 + b^2c^2 + c^2a^2$. Let:

$$f(a,b,c) = a^2b^2 + b^2c^2 + c^2a^2 \qquad \text{(to be maximized)}$$
$$g(a,b,c) = a^2b^2c^2 = 1 \qquad \text{(constraint)}$$

Solving Lagrange's Multiplier system $\nabla f = \lambda \nabla g$ and $g = 1$, we can get these candidate points:

(a,b,c)	$f(a,b,c)$
$(-1,-1,-1)$	3
$(-1,-1,1)$	3
$(-1,1,-1)$	3
$(-1,1,1)$	3
$(1,-1,-1)$	3
$(1,-1,1)$	3
$(1,1,-1)$	3
$(1,1,1)$	3

In all of the above cases, we have $d = \frac{3}{\sqrt{3}} = \sqrt{3}$. Therefore, they are all the maximum points for f and hence are also points on the surface which their tangent plane closest to the origin.

8. (★★) Suppose

$$f(x,y) = 2x^2 + xy - 8x - y + 6$$

Let T be the triangular region (boundary included) in the xy-plane with vertices $(0,0)$, $(0,3)$ and $(3,0)$.

(a) Find all the interior critical point(s) of f in the region T.

Solution: $\nabla f = \langle 4x + y - 8,\ x - 1\rangle$. Set $\nabla f = \mathbf{0}$, we get $x = 1$ and $y = 4$. The only possible interior critical point for f is $(x, y) = (1, 4)$, which is beyond the range of T. Therefore, there is no interior critical point in the region T.

(b) Find the maximum and minimum values of $f(x, y)$ when (x, y) is restricted on the vertical side of T, i.e. the line segment joining $(0,0)$ and $(0,3)$.

Solution: The vertical side of T is given by the equation $x = 0$. Substitute this into the function, we get:

$$f(0, y) = -y + 6$$

which is a strictly decreasing function of y. Therefore, on the vertical side, the maximum and minimum of f are attained at the end points $(0,0)$ and $(0,3)$. Since $f(0,0) = 6$ and $f(0,3) = 3$. The maximum value of f is 6 and minimum value is 3 when restricted on this vertical side.

(c) Find the maximum and minimum values of $f(x, y)$ when (x, y) is restricted on the horizontal side of T, i.e. the line segment joining $(0,0)$ and $(3,0)$.

Solution: The horizontal side has equation $y = 0$. Regarding x as the variable, the function is given by:

$$f(x, 0) = 2x^2 - 8x + 6.$$

Since $\frac{d}{dx} f(x, 0) = 4x - 8$, the only critical point is $x = 2$. Compute f at this critical point and the end-points of the horizontal sides, we get: $f(2,0) = -2$, $f(0,0) = 6$ and $f(3,0) = 0$. Therefore the max of f is 6 and the min is -2 when restricted on the horizontal side.

(d) Write the equation of the line joining the vertices $(0,3)$ and $(3,0)$. Hence find the maximum and minimum values of $f(x, y)$ when (x, y) is restricted to the hypotenuse of T.

Solution: The equation of the hypotenuse is $x + y = 3$. Let $g(x, y) = x + y$, then the hypotenuse is the level set $g = 3$. Using Lagrange's Multiplier:

$$\begin{aligned} f_x &= \lambda g_x & \qquad 4x + y - 8 &= \lambda \\ f_y &= \lambda g_y & x - 1 &= \lambda \\ g(x,y) &= 3 & x + y &= 3 \end{aligned}$$

From the first two equations, we get $4x + y - 8 = x - 1$ (Be flexible! There is no need to divide cases for this system!) Hence, $3x + y = 7$. Combining with the third equation $x + y = 3$, we get $x = 2$ and $y = 1$. The only critical point on the hypotenuse is $(x, y) = (2, 1)$. Check the values of f at the critical point and end-points, we get:

$$f(2,1) = -1, \ f(3.0) = 0, \ f(0,3) = 3.$$

Therefore, the maximum of f is 3 and the minimum is -1 when restricted to the hypotenuse.

(e) Determine the absolute maximum and minimum of $f(x,y)$ over the domain T.

Solution: Since f has no interior critical point in the region T, its absolute max and min must take place on the boundary of the region, i.e. the three sides of the triangle. From previous parts, we got:

side	max	min
vertical	6	3
horizontal	6	-2
hypotenuse	3	-1

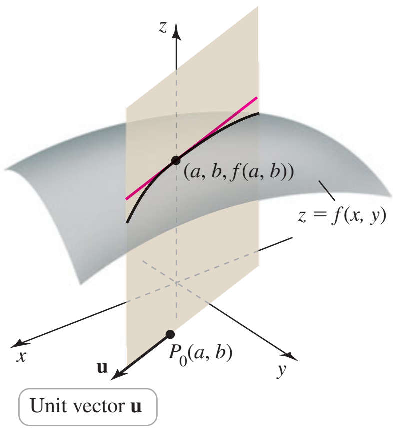
Therefore, the absolute maximum of f is 6 and the absolute minimum is -2 over the region T.

9. (★★) This purpose of this problem is to explain why the Second Derivative Test works for two-variable functions. Let $f(x,y)$ be a C^2 function with a critical point $P_0(a,b)$. Given any unit direction $\hat{\mathbf{u}} = u_1\mathbf{i} + u_2\mathbf{j}$, the path:

$$x = a + tu_1, \quad y = b + tu_2$$

is a straight-line passing through (a,b) in the direction of $\hat{\mathbf{u}}$. As such, the intersection curve of the vertical plane shown in the diagram below can be expressed as:

$$z = f(a + tu_1, b + tu_2)$$



Therefore, the first derivative $\left.\frac{dz}{dt}\right|_{t=0}$ measures the slope of tangent at $(a,b,f(a,b))$, whereas the second derivative $\left.\frac{d^2z}{dt^2}\right|_{t=0}$ indicates whether the curve is concave up or down around the point $(a,b,f(a,b))$.

(a) Using the chain rule, show that:

$$\frac{dz}{dt} = f_x\, u_1 + f_y\, u_2$$
$$\frac{d^2z}{dt^2} = f_{xx}u_1^2 + 2f_{xy}u_1u_2 + f_{yy}u_2^2$$

Solution: Given that $z = f(a + tu_1, b + tu_2)$, then f is a function of (x, y), and (x, y) is a function of t. Precisely, we have: $x = a + tu_1$ and $y = b + tu_2$ (note that a, b, u_1 and u_2 are constants). Using the chain rule, we get:

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\
&= f_x\,\frac{d}{dt}(a + tu_1) + f_y\,\frac{d}{dt}(b + tu_2) \\
&= f_x\, u_1 + f_y\, u_2 \\
\frac{d^2z}{dt^2} &= \frac{d}{dt}\left(\frac{dz}{dt}\right) \\
&= \frac{d}{dt}\left(f_x\, u_1 + f_y\, u_2\right) && \text{(from above)} \\
&= \frac{df_x}{dt}\, u_1 + \frac{df_y}{dt}\, u_2 \\
&= \left(\frac{\partial f_x}{\partial x}\frac{dx}{dt} + \frac{\partial f_x}{\partial y}\frac{dy}{dt}\right) u_1 \\
&\quad + \left(\frac{\partial f_y}{\partial x}\frac{dx}{dt} + \frac{\partial f_y}{\partial y}\frac{dy}{dt}\right) u_2 \\
&= \left(f_{xx}\, u_1 + f_{xy}\, u_2\right) u_1 && \text{(recall that } x = a + tu_1 \\
&\quad + \left(f_{yx}\, u_1 + f_{yy}\, u_2\right) u_2 && \text{and } y = b + tu_2\text{)} \\
&= f_{xx}u_1^2 + f_{xy}u_2u_1 + f_{yx}u_1u_2 + f_{yy}u_2^2 \\
&= f_{xx}u_1^2 + 2f_{xy}u_1u_2 + f_{yy}u_2^2 && \text{(Mixed Partials Theorem)}
\end{aligned}$$

(b) By *completing-the-square*, show further that:

$$\frac{d^2z}{dt^2} = \begin{cases} f_{xx}\left[\left(u_1 + \frac{f_{xy}}{f_{xx}}u_2\right)^2 + \left(\frac{f_{xx}f_{yy}-f_{xy}^2}{f_{xx}^2}\right)u_2^2\right] & \text{if } f_{xx} \neq 0 \\ f_{yy}\left[\left(\frac{f_{xy}}{f_{yy}}u_1 + u_2\right)^2 + \left(\frac{f_{xx}f_{yy}-f_{xy}^2}{f_{yy}^2}\right)u_1^2\right] & \text{if } f_{yy} \neq 0 \\ 2f_{xy}u_1u_2 & \text{if } f_{xx} = f_{yy} = 0 \end{cases}$$

Solution: If $f_{xx} \neq 0$, then:

$$\begin{aligned} \frac{d^2z}{dt^2} &= f_{xx}u_1^2 + 2f_{xy}u_1u_2 + f_{yy}u_2^2 \\ &= f_{xx}\left[u_1^2 + \frac{2f_{xy}}{f_{xx}}u_1u_2 + \frac{f_{yy}}{f_{xx}}u_2^2\right] \\ &= f_{xx}\left[u_1^2 + \frac{2f_{xy}}{f_{xx}}u_1u_2 + \left(\frac{f_{xy}}{f_{xx}}u_2\right)^2 - \left(\frac{f_{xy}}{f_{xx}}u_2\right)^2 + \frac{f_{yy}}{f_{xx}}u_2^2\right] \\ &= f_{xx}\left[\left(u_1 + \frac{f_{xy}}{f_{xx}}u_2\right)^2 + \left(-\frac{f_{xy}^2}{f_{xx}^2} + \frac{f_{yy}}{f_{xx}}\right)u_2^2\right] \\ &= f_{xx}\left[\left(u_1 + \frac{f_{xy}}{f_{xx}}u_2\right)^2 + \left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}^2}\right)u_2^2\right] \end{aligned}$$

For the case $f_{yy} \neq 0$, we can obtain the result using symmetry argument: by switching x with y, and u_1 with u_2.

The last case is trivial (since $\frac{d^2z}{dt^2} = f_{xx}u_1^2 + 2f_{xy}u_1u_2 + f_{yy}u_2^2$).

(c) The nature of the point $(a, b, f(a, b))$ can be determined by the following argument:

i. If $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$, what can you say about $\frac{d^2z}{dt^2}$? Using this observation, conclude the nature of the point $(a, b, f(a, b))$.

Solution: In this case, we have:

$$\frac{d^2z}{dt^2} = \underbrace{f_{xx}}_{+}\left[\underbrace{\left(u_1 + \frac{f_{xy}}{f_{xx}}u_2\right)^2}_{+\text{ or }0} + \underbrace{\left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}^2}\right)}_{+}\underbrace{u_2^2}_{+\text{ or }0}\right]$$

and therefore $\frac{d^2z}{dt^2} \geq 0$ for any u_1 and u_2. The only situation with $\frac{d^2z}{dt^2} = 0$ is that both $u_1 + \frac{f_{xy}}{f_{xx}}u_2 = 0$ and $u_2 = 0$. However, it would imply $u_1 = u_2 = 0$, which contradicts to the fact that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit (hence non-zero) vector.

Therefore, $\frac{d^2z}{dt^2} > 0$ for any unit direction $\mathbf{u}$. The graph $z = f(x, y)$ is concave up in **all** directions $\mathbf{u}$, and so the critical point must be a **local minimum**.

ii. How about if $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} < 0$?

Solution: In this case we have:

$$\frac{d^2z}{dt^2} = \underbrace{f_{xx}}_{-} \left[\underbrace{\left(u_1 + \frac{f_{xy}}{f_{xx}} u_2 \right)^2}_{+\text{ or }0} + \underbrace{\left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}^2} \right)}_{+} \underbrace{u_2^2}_{+\text{ or }0} \right]$$

and therefore $\frac{d^2z}{dt^2} \leq 0$. By similar discussion as in (i), the only situation in which $\frac{d^2z}{dt^2} = 0$ is that $u_1 = u_2 = 0$ (that is impossible for a unit vector $\mathbf{u}$). Therefore, $\frac{d^2z}{dt^2} < 0$ for any unit vector $\mathbf{u}$. The graph $z = f(x, y)$ is concave down in all directions $\mathbf{u}$, and so the critical point must be a local maximum.

iii. How about if $f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} = 0$?

Solution: This is not possible. Otherwise we would have $-f_{xy}^2 > 0$.

iv. What if $f_{xx}f_{yy} - f_{xy}^2 < 0$?

Solution: In the case $f_{xx} \neq 0$, we have:

$$\frac{d^2z}{dt^2} = f_{xx} \left[\underbrace{\left(u_1 + \frac{f_{xy}}{f_{xx}} u_2 \right)^2}_{+\text{ or }0} + \underbrace{\left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{xx}^2} \right) u_2^2}_{-\text{ or }0} \right]$$

and in the case $f_{yy} \neq 0$, we have:

$$\frac{d^2z}{dt^2} = f_{yy} \left[\underbrace{\left(\frac{f_{xy}}{f_{yy}} u_1 + u_2 \right)^2}_{+\text{ or }0} + \underbrace{\left(\frac{f_{xx}f_{yy} - f_{xy}^2}{f_{yy}^2} \right) u_1^2}_{-\text{ or }0} \right]$$

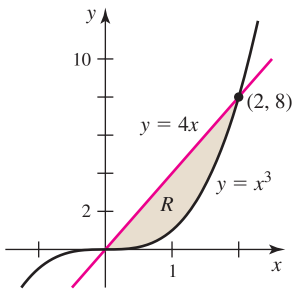
In either case, whether $\frac{d^2z}{dt^2}$ is positive or negative depends on the choice of u_1 and u_2. In other words, the graph $z = f(x, y)$ is concave up in some directions and concave down in some other directions. The critical point is a saddle.

In the last case where $f_{xx} = f_{yy} = 0$, we then have $\frac{d^2z}{dt^2} = 2f_{xy}u_1u_2$. Therefore, whether it is positive or negative depends only the choice of $\mathbf{u}$. The critical point is again a saddle.

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #5 • Double Integrals

1. (★) Set-up the lower and upper bounds of each double integral below using **both** $dxdy$ and $dydx$ orders. Compute the integral using **both** orders and verify that they give the same value.

   (a) $\iint_R 2xy\,dA$ where R is the region as shown below:

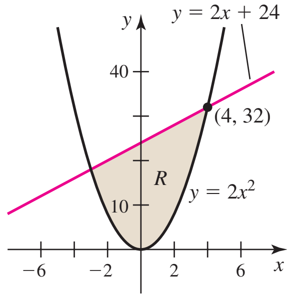


Solution:

$$\begin{aligned}
\iint_R 2xy\,dA &= \int_{y=0}^{y=8}\int_{x=\frac{y}{4}}^{x=y^{1/3}} 2xy\,dxdy \\
&= \int_{y=0}^{y=8} \left[x^2y\right]_{x=\frac{y}{4}}^{x=y^{1/3}}\,dy \\
&= \int_{y=0}^{y=8} \left(y^{2/3}\cdot y - \frac{y^2}{16}\cdot y\right)\,dy \\
&= \int_{y=0}^{y=8} \left(y^{5/3} - \frac{y^3}{16}\right)\,dy \\
&= 32
\end{aligned}$$

$$\begin{aligned}
\iint_R 2xy\,dA &= \int_{x=0}^{x=2}\int_{y=x^3}^{y=4x} 2xy\,dydx \\
&= \int_{x=0}^{x=2} \left[xy^2\right]_{y=x^3}^{y=4x}\,dx \\
&= \int_{x=0}^{x=2} \left(x\cdot 16x^2 - x\cdot x^6\right)\,dx \\
&= \int_{x=0}^{x=2} \left(16x^3 - x^7\right)\,dx \\
&= 32
\end{aligned}$$

(b) $\iint_R 1\,dA$ where R is the region bounded between $y = 2x + 24$ and $y = 2x^2$:



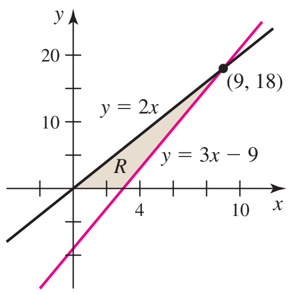
Solution: (Answer only)

$$dxdy \text{ order: } \int_{y=0}^{y=18} \int_{x=-\sqrt{\frac{y}{2}}}^{x=\sqrt{\frac{y}{2}}} 1\,dxdy + \int_{y=18}^{y=32} \int_{x=\frac{y-24}{2}}^{x=\sqrt{\frac{y}{2}}} 1\,dxdy$$

$$dydx \text{ order: } \int_{x=-3}^{x=4} \int_{y=2x^2}^{y=2x+24} 1\,dydx$$

$$\text{Answer: } \frac{343}{3}$$

(c) $\iint_R x^2\,dA$ where R is the region bounded between $y = 2x$, $y = 3x - 9$ and the x-axis:



Solution: (Answer only)

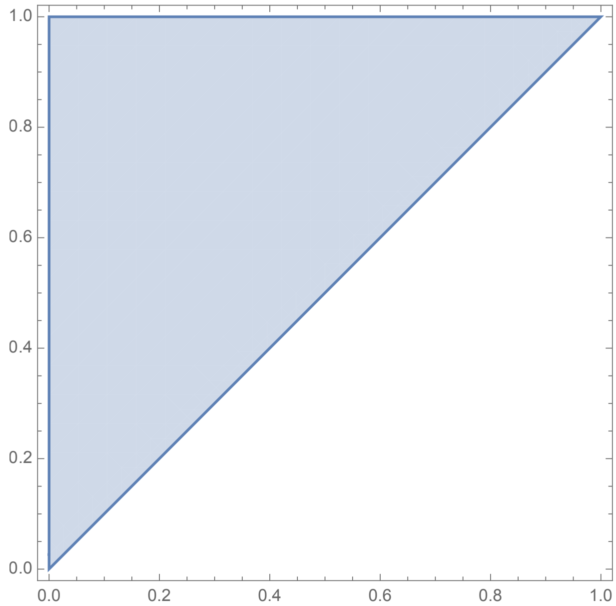
$$dxdy \text{ order: } \int_{y=0}^{y=18} \int_{x=\frac{y}{2}}^{x=\frac{y+9}{3}} x^2\,dxdy$$

$$dydx \text{ order: } \int_{x=0}^{x=3} \int_{y=0}^{y=2x} x^2\,dydx + \int_{x=3}^{x=9} \int_{y=3x-9}^{y=2x} x^2\,dydx$$

$$\text{Answer: } \frac{1053}{2}$$

2. (★) Evaluate the integral $\iint_T \sqrt{a^2 - y^2}\, dA$ where T is the triangle with vertices $(0,0)$, $(0,a)$ and (a,a). Set-up the integral in both $dxdy$ and $dydx$ orders, and choose the *easier* one to compute.

Solution:



Below is the case where $a \geq 0$. The case where $a < 0$ is similar (but different).

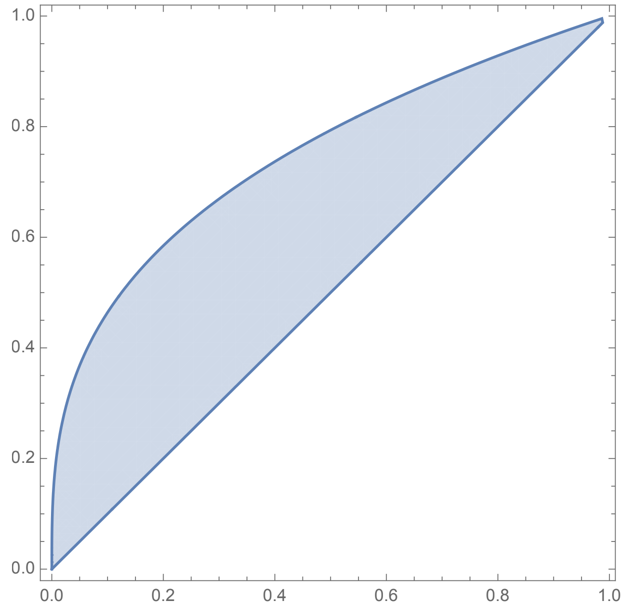
$$\begin{aligned}\iint_T \sqrt{a^2 - y^2}\, dA &= \int_{x=0}^{x=a} \int_{y=x}^{y=a} \sqrt{a^2 - y^2}\, dydx \\ &= \int_{y=0}^{y=a} \int_{x=0}^{x=y} \sqrt{a^2 - y^2}\, dxdy\end{aligned}$$

Easier to integrate using $dxdy$-order:

$$\begin{aligned}\int_{y=0}^{y=a} \int_{x=0}^{x=y} \sqrt{a^2 - y^2}\, dxdy &= \int_{y=0}^{y=a} \left[x\sqrt{a^2 - y^2} \right]_{x=0}^{x=y} dy \\ &= \int_{y=0}^{y=a} y\sqrt{a^2 - y^2}\, dy \\ &= -\frac{1}{2} \int_{y=0}^{y=a} \sqrt{a^2 - y^2}\, d(a^2 - y^2) \\ &= -\frac{1}{2} \left[\frac{2}{3} \left(a^2 - y^2\right)^{3/2} \right]_{y=0}^{y=a} \\ &= \frac{1}{3} a^3\end{aligned}$$

3. (★★) Consider the integral $\int_0^1 \int_x^{x^{1/3}} \sqrt{1-y^4}\,dydx$. It is almost impossible to compute the inner integral. Try to switch the order of integration to evaluate it. [Hint: You should first sketch the region of integration.]

Solution:

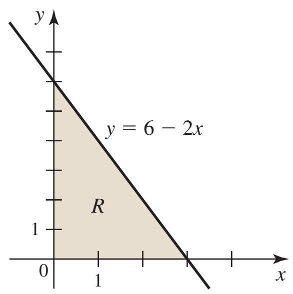


According to the region represented by the integral, switching the integration order gives:

$$\begin{aligned}
\int_0^1 \int_x^{x^{1/3}} \sqrt{1-y^4}\,dydx &= \int_0^1 \int_{x=y^3}^{x=y} \sqrt{1-y^4}\,dxdy \\
&= \int_0^1 \left[x\sqrt{1-y^4}\right]_{x=y^3}^{x=y} dy \\
&= \int_0^1 \left(y\sqrt{1-y^4} - y^3\sqrt{1-y^4}\right) dy \\
&= \int_0^1 y\sqrt{1-y^4}\,dy - \int_0^1 y^3\sqrt{1-y^4}\,dy \\
&= \frac{\pi}{8} - \frac{1}{6}
\end{aligned}$$

Remark: Let $u = y^2$ for the first integral, and let $v = 1 - y^4$ for the second integral (detail omitted).

4. (★★) Evaluate the integrals $\iint_R \frac{1}{3-x}\,dA$ and $\iint_R \frac{1}{y-6}\,dA$. Try to avoid *integration-by-parts* if possible.

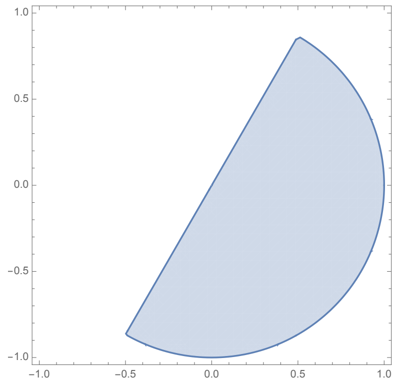


Solution:

$$\begin{aligned}
\iint_R \frac{1}{3-x}\,dA &= \int_{x=0}^{x=3}\int_{y=0}^{y=6-2x} \frac{1}{3-x}\,dydx \\
&= \int_{x=0}^{x=3} \left[\frac{y}{3-x}\right]_{y=0}^{y=6-2x} dx \\
&= \int_{x=0}^{x=3} \frac{6-2x}{3-x}\,dx \\
&= \int_{x=0}^{x=3} \frac{2(3-x)}{(3-x)}\,dx \\
&= \int_{x=0}^{x=3} 2\,dx = 6 \\
\iint_R \frac{1}{y-6}\,dA &= \int_{y=0}^{y=6}\int_{x=0}^{x=\frac{6-y}{2}} \frac{1}{y-6}\,dxdy \\
&= \int_{y=0}^{y=6} \left[\frac{x}{y-6}\right]_{x=0}^{x=\frac{6-y}{2}} dy \\
&= \int_{y=0}^{y=6} -\frac{1}{2}\,dy = -3.
\end{aligned}$$

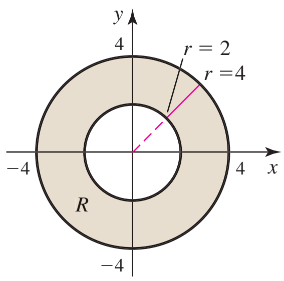
5. (★) Evaluate $\iint_R (x+y)\, dA$ using polar coordinates where R is the region in the first quadrant lying inside the disk $x^2+y^2 \le a^2$ and under the line $y=\sqrt{3}x$.

Solution: (Sketch only)



$$\iint_R (x+y)\, dA = \int_{\theta=-\frac{2\pi}{3}}^{\theta=\frac{\pi}{3}} \int_{r=0}^{r=a} (r\cos\theta + r\sin\theta)\, r dr d\theta$$
$$= \int_{\theta=-\frac{2\pi}{3}}^{\theta=\frac{\pi}{3}} \int_{r=0}^{r=a} r^2(\cos\theta+\sin\theta)\, dr d\theta = \frac{\sqrt{3}-1}{3}a^3$$

6. (★) Consider the annular region R below. Express the integral $\iint_R (x^2+y^2)\, dA$ in **both** rectangular and polar coordinates. Choose the *easier* system to compute the integral.



Solution: See Solutions to Worksheet #11 Q5 and Worksheet #12 Q2. Obviously it is easier to compute it using polar coordinates.

7. (★★) Evaluate each of the following integrals:

(a) $\displaystyle\int_0^{2\pi}\int_0^1 e^{-x^2}\sin y\,dxdy$

Solution: Since it is impossible to integrate e^{-x^2}, we try to switch the order of integration to see if there is any luck. The bounds are all constants, so we simply switch the integral signs:

$$\begin{aligned}\int_0^{2\pi}\int_0^1 e^{-x^2}\sin y\,dxdy &= \int_0^1\int_0^{2\pi} e^{-x^2}\sin y\,dydx\\ &= \int_0^1 \left[-e^{-x^2}\cos y\right]_{y=0}^{y=2\pi} dx\\ &= \int_0^1 \left(-e^{-x^2}+e^{-x^2}\right) dx\\ &= \int_0^1 0\,dx = 0\end{aligned}$$

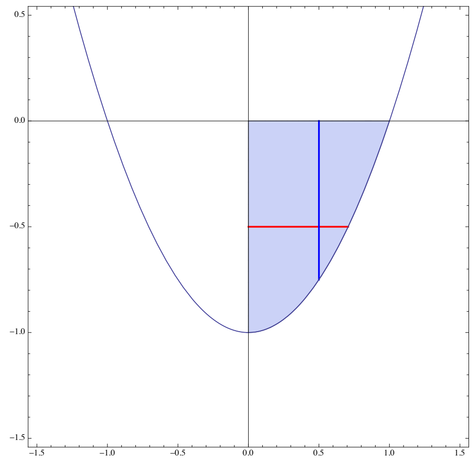
(b) $\displaystyle\int_{-1}^0\int_0^{\sqrt{y+1}} \left(x-\frac{x^3}{3}\right)^{5/2} dxdy$

Solution: It is very difficult to integrate the given function by dx, we try to switch the order of integration to see if there is any luck. Note that the bounds involve variables, so we have to sketch the diagram of the domain first. In the $dxdy$-order, the x-sample strips enter the region from $x=0$ and leaves from $x=\sqrt{y+1}$ (i.e. part of the parabola $y=x^2-1$), see the red strip in the diagram.
To switch order, we draw a sample blue strip in the diagram, which enters the region at $y=x^2-1$ and leaves at $y=0$. The minimum value of x is 0 and the maximum of x is 1. Therefore, after switching dx and dy, we get:

$$\begin{aligned}\int_{-1}^0\int_0^{\sqrt{y+1}} \left(x-\frac{x^3}{3}\right)^{5/2} dxdy &= \int_0^1\int_{x^2-1}^0 \left(x-\frac{x^3}{3}\right)^{5/2} dydx\\ &= \int_0^1 \left[\left(x-\frac{x^3}{3}\right)^{5/2} y\right]_{y=x^2-1}^{y=0} dx\\ &= \int_0^1 \left(x-\frac{x^3}{3}\right)^{5/2} (1-x^2)\,dx\\ &= \int_{u=0}^{u=2/3} u^{5/2}\,du\end{aligned}$$

where we let $u=x-\frac{x^3}{3}$, then $du=(1-x^2)dx$. Keep going:

$$\int_{u=0}^{u=2/3} u^{5/2}\,du = \left[\frac{2}{7}u^{7/2}\right]_{u=0}^{u=2/3} = \frac{2}{7}\left(\frac{2}{3}\right)^{7/2}.$$



(c) $\displaystyle\iint_Q \frac{1}{(1+2x^2+2y^2)^3}\,dA$ where Q is the entire first quadrant of the xy-plane

Solution: The term x^2+y^2 appears in the integrand, so it is typically better to be done in polar coordinates. The first quadrant Q is represented in polar coordinates by $0 \le r < \infty$ and $0 \le \theta \le 2\pi$. Therefore,

$$\begin{aligned}
\iint_Q \frac{1}{(1+2x^2+2y^2)^3}\,dA &= \int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{1}{(1+2r^2)^3}\,r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{8(1+2r^2)^2}\right]_{r=0}^{r=\infty} d\theta \\
&= \int_0^{\frac{\pi}{2}} \left(-0+\frac{1}{8}\right) d\theta = \frac{\pi}{2}\cdot\frac{1}{8} \\
&= \frac{\pi}{16}.
\end{aligned}$$

(d) $\displaystyle\iint_Q (x^2-y+1)\,dA$ where Q is the region $\{(x,y) \mid x \ge 0,\ y \ge 0,\ x^2+y^2 \le 4\}$

Solution: The domain Q is a sector, so it's again better be done by polar coordinates. In polar coordinates, the domain Q is represented by $0 \le r \le 2$ and $0 \le \theta \le \frac{\pi}{2}$. Therefore,

$$\begin{aligned}
\iint_Q (x^2-y+1)\,dA &= \int_0^{\pi/2}\int_0^2 (r^2\cos^2\theta - r\sin\theta + 1)\,r\,dr\,d\theta \\
&= \int_0^{\pi/2}\int_0^2 (r^3\cos^2\theta - r^2\sin\theta + r)\,dr\,d\theta \\
&= 2\pi - \frac{8}{3}
\end{aligned}$$

The Half/Double-Angle Formula should be used to compute the integral for $\cos^2\theta$.

(e) $\displaystyle\iint_{\mathbb{R}^2} (x^2+y^2)e^{-(x^4+2x^2y^2+y^4)}\,dA$

Solution: It is useful to note that $x^4+2x^2y^2+y^4 = (x^2)^2+2(x^2)(y^2)+(y^2)^2 = (x^2+y^2)^2$, and so it suggests that the integral had better be done by polar coordinates. The domain $\mathbb{R}^2$ can be expressed in polar coordinates as $0 \le r < \infty$ and $0 \le \theta \le 2\pi$:

$$\begin{aligned}\iint_{\mathbb{R}^2} (x^2+y^2)e^{-(x^4+2x^2y^2+y^4)}\,dA &= \int_0^{2\pi}\int_0^{\infty} r^2 e^{-(r^2)^2}\,rdrd\theta \\ &= \int_0^{2\pi}\int_0^{\infty} r^3 e^{-r^4}\,drd\theta \\ &= \int_0^{2\pi}\left[-\frac{e^{-r^4}}{4}\right]_{r=0}^{r=\infty} d\theta \\ &= \int_0^{2\pi}\left(-0+\frac{1}{4}\right) d\theta \\ &= 2\pi\cdot\frac{1}{4} = \frac{\pi}{2}.\end{aligned}$$

8. (★★) Some single-variable integrals are "notoriously" difficult to compute. One example is $\int e^{-x^2}dx$ despite the fact that this integral is of central importance in mathematics (pure/applied), physics, statistics and engineering. However, some of these difficult integrals can be evaluated via double integral methods.

   This problem investigates another well-known integral which has no closed-form antiderivative:

$$\int \frac{\log(1-x)}{x}dx.$$

   The goal of this problem is to show that this integral over $0 \le x \le 1$ can be written as an infinite series.

   Consider the function

$$f(x,y) = \frac{1}{1-xy}, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

   It is defined almost everywhere on the rectangle $0 \le x \le 1$ and $0 \le y \le 1$ (we say 'almost' because it's undefined only at $(x,y) = (1,1)$, but this single point is negligible).

(a) Show that: $\displaystyle\int_0^1 \frac{1}{1-xy}dy = -\frac{\log(1-x)}{x}$.

Solution: Let $u(y) = xy$ regarding x is a constant and y is the variable. Then $du = xdy$. When $y = 0$, $u = 0$; when $y = 1$, $u = x$. Therefore, by u-substitution:

$$\begin{aligned}\int_0^1 \frac{1}{1-xy}dy &= \int_0^x \frac{1}{x(1-u)}du \\ &= \frac{1}{x}\int_0^x \frac{1}{1-u}du \\ &= \frac{1}{x}\left[-\log(1-u)\right]_{u=0}^{u=x} \\ &= -\frac{1}{x}(\log(1-x) - \log(1-0)) = -\frac{\log(1-x)}{x}.\end{aligned}$$

(b) Note that $|xy| < 1$ except for the negligible point $(x,y) = (1,1)$, so the function $f(x,y)$ can be expressed as a geometric series:

$$\frac{1}{1-xy} = 1 + (xy) + (xy)^2 + (xy)^3 + \ldots$$

Using this geometric series, show that

$$\int_0^1\int_0^1 \frac{1}{1-xy}dydx = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \sum_{k=1}^{\infty}\frac{1}{k^2}.$$

Solution: By geometric series expansion:

$$\frac{1}{1-xy} = 1 + (xy) + (xy)^2 + (xy)^3 + \ldots$$

Integrate both sides:

$$\begin{aligned}&\int_0^1\int_0^1 \frac{1}{1-xy}dydx \\ &= \int_0^1\int_0^1 (1 + xy + x^2y^2 + x^3y^3 + \ldots)dydx \\ &= \int_0^1\int_0^1 dydx + \int_0^1\int_0^1 xydydx + \int_0^1\int_0^1 x^2y^2dydx + \int_0^1\int_0^1 x^3y^3dydx + \ldots \\ &= \int_0^1 [y]_{y=0}^{y=1}dx + \int_0^1\left[x\cdot\frac{y^2}{2}\right]_{y=0}^{y=1}dx + \int_0^1\left[x^2\cdot\frac{y^3}{3}\right]_{y=0}^{y=1}dx + \int_0^1\left[x^3\cdot\frac{y^4}{4}\right]_{y=0}^{y=1}dx + \ldots \\ &= \int_0^1 dx + \int_0^1\frac{x}{2}dx + \int_0^1\frac{x^2}{3}dx + \int_0^1\frac{x^3}{4}dx + \ldots \\ &= [x]_{x=0}^{x=1} + \left[\frac{1}{2}\cdot\frac{x^2}{2}\right]_{x=0}^{x=1} + \left[\frac{1}{3}\cdot\frac{x^3}{3}\right]_{x=0}^{x=1} + \left[\frac{1}{4}\cdot\frac{x^4}{4}\right]_{x=0}^{x=1} + \ldots \\ &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \\ &= \sum_{k=1}^{\infty}\frac{1}{k^2}.\end{aligned}$$

(c) Using (a) and (b), show that

$$-\int_0^1 \frac{\log(1-x)}{x}dx = \sum_{k=1}^{\infty}\frac{1}{k^2}.$$

Solution: Simply combine the results obtained in (a) and (b):

$$\int_0^1\left(\int_0^1 \frac{1}{1-xy}dy\right)dx = \sum_{k=1}^{\infty}\frac{1}{k^2} \qquad \text{(from (b))}$$

$$\int_0^1\left(-\frac{\log(1-x)}{x}\right)dx = \sum_{k=1}^{\infty}\frac{1}{k^2} \qquad \text{(from (a))}$$

$$-\int_0^1 \frac{\log(1-x)}{x}dx = \sum_{k=1}^{\infty}\frac{1}{k^2}.$$

(d) Using the above approach, *mutatis mutandis*, show that for any $0 \le z \le 1$, we have:

$$-\int_0^z \frac{\log(1-x)}{x}dx = \sum_{k=1}^{\infty}\frac{z^k}{k^2}.$$

[Remark: *Mutatis mutandis* is a Latin phrase meaning “changing only those things which need to be changed”.]

Solution: Replace the outer $\int_0^1$ by $\int_0^z$ (while keeping the inner $\int_0^1$ unchanged):

$$\begin{aligned}
&\int_0^z\int_0^1 \frac{1}{1-xy}dydx\\
&= \int_0^z\int_0^1 (1+xy+x^2y^2+x^3y^3+\ldots)dydx\\
&= \int_0^z\int_0^1 dydx + \int_0^z\int_0^1 xydydx + \int_0^z\int_0^1 x^2y^2dydx + \int_0^z\int_0^1 x^3y^3dydx + \ldots\\
&= \int_0^z [y]_{y=0}^{y=1}dx + \int_0^z\left[x\cdot\frac{y^2}{2}\right]_{y=0}^{y=1}dx + \int_0^z\left[x^2\cdot\frac{y^3}{3}\right]_{y=0}^{y=1}dx + \int_0^z\left[x^3\cdot\frac{y^4}{4}\right]_{y=0}^{y=1}dx + \ldots\\
&= \int_0^z dx + \int_0^z \frac{x}{2}dx + \int_0^z\frac{x^2}{3}dx + \int_0^z\frac{x^3}{4}dx + \ldots\\
&= [x]_{x=0}^{x=z} + \left[\frac{1}{2}\cdot\frac{x^2}{2}\right]_{x=0}^{x=z} + \left[\frac{1}{3}\cdot\frac{x^3}{3}\right]_{x=0}^{x=z} + \left[\frac{1}{4}\cdot\frac{x^4}{4}\right]_{x=0}^{x=z} + \ldots\\
&= z + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \frac{z^4}{4^2} + \ldots\\
&= \sum_{k=1}^{\infty}\frac{z^k}{k^2}.
\end{aligned}$$

From (a), we get:

$$-\int_0^z \frac{\log(1-x)}{x}dx = \sum_{k=1}^{\infty}\frac{z^k}{k^2}.$$

9. (★★★) The purpose of this problem is to use double integrals to derive a somewhat surprising result in electrostatics, that is the electric force exerted on a charged particle by an infinite sheet of uniformly distributed charges is *independent* of how far the particle and the sheet are apart from each other.

The paragraphs below describe the physical set-up of the problem. Although it may be possible to proceed to the problem without knowing the physics background, it is strongly recommend to read through the paragraphs below so as to understand the motivation of this problem.

According to the Coulomb's Law, the electric force $\mathbf{F}$ exerted **on** a point particle with charge Q located at (x_0, y_0, z_0), **by** a point particle with charge q located at (x, y, z), is given by:

$$\mathbf{F} = \frac{qQ}{4\pi\varepsilon_0} \frac{(x_0 - x)\mathbf{i} + (y_0 - y)\mathbf{j} + (z_0 - z)\mathbf{k}}{\left((x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2\right)^{3/2}}$$

where ε_0 is positive constant (depending on the medium).

The Coulomb's Law is also called the Inverse Square Law because one can easily verify that the magnitude of the force satisfies:

$$|\mathbf{F}| = \frac{qQ}{4\pi\varepsilon_0 d^2}$$

where $d = \sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}$ is the distance between the two particles.

If there is a sequence of *discrete* charged particles located at (x_1, y_1, z_1), (x_2, y_2, z_2), $\ldots$, each with charge q, then the resultant electric force exerted on a particle with charge Q located at (x_0, y_0, z_0), is given by the vector sum of all forces:

$$\mathbf{F} = \sum_{i=1}^{\infty} \frac{qQ}{4\pi\varepsilon_0} \frac{(x_0 - x_i)\mathbf{i} + (y_0 - y_i)\mathbf{j} + (z_0 - z_i)\mathbf{k}}{\left((x_0 - x_i)^2 + (y_0 - y_i)^2 + (z_0 - z_i)^2\right)^{3/2}}.$$

This is called the Principle of Superposition by physicists.

Now given there is an infinite sheet of uniformly distributed charges on the xy-plane, and for each small area element dA on the xy-plane, the amount of charges is given by $\sigma\, dA$, where σ is a constant that represents the area density of charges. Suppose there is a particle with charge Q located above the xy-plane at $(0, 0, z_0)$, i.e. $z_0 > 0$. For simplicity, call this the Q-particle.

Now regard a small area element located at $(x, y, 0)$ on the xy-plane as a charged "particle" with charge $q = \sigma\, dA$, then the force exerted on the Q-particle by this area element is given by substituting $(x, y, z) = (x, y, 0)$ and $(x_0, y_0, z_0) = (0, 0, z_0)$:

$$\frac{Q(\sigma\, dA)}{4\pi\varepsilon_0} \frac{(0 - x)\mathbf{i} + (0 - y)\mathbf{j} + (z_0 - 0)\mathbf{k}}{\left((0 - x)^2 + (0 - y)^2 + (z_0 - 0)^2\right)^{3/2}} = \frac{Q\sigma}{4\pi\varepsilon_0} \frac{-x\mathbf{i} - y\mathbf{j} + z_0\mathbf{k}}{\left(x^2 + y^2 + z_0^2\right)^{3/2}}\, dA.$$

Therefore, by the Principle of Superposition, the resultant electric force exerted on the Q-particle by the sheet of charges is given by this double integral over the entire xy-plane (i.e. $\mathbb{R}^2$):

$$\mathbf{F}_{\text{resultant}} = \iint_{\mathbb{R}^2} \frac{Q\sigma}{4\pi\varepsilon_0} \frac{-x\mathbf{i} - y\mathbf{j} + z_0\mathbf{k}}{\left(x^2 + y^2 + z_0^2\right)^{3/2}}\, dA.$$

Here integrating a vector simply means integrating each component of the vector treating $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as "constants".

(a) Show that **i** and **j**-components of $\mathbf{F}_{\text{resultant}}$ are zero.

Solution: Since $x^2 + y^2 = r^2$, we use polar coordinates again to simplify our calculations.

$$\begin{aligned}
&\mathbf{F}_{\text{resultant}} \\
&= \int_0^{2\pi}\int_0^{\infty} \frac{Q\sigma}{4\pi\varepsilon_0} \frac{-r\cos\theta\,\mathbf{i} - r\sin\theta\,\mathbf{j} + z_0\mathbf{k}}{(r^2+z_0^2)^{3/2}}\, r dr d\theta \\
&= \frac{Q\sigma}{4\pi\varepsilon_0}\left[\left(\int_0^{2\pi}\int_0^{\infty} \frac{-r^2\cos\theta\, dr d\theta}{(r^2+z_0^2)^{3/2}}\right)\mathbf{i} + \left(\int_0^{2\pi}\int_0^{\infty} \frac{-r^2\sin\theta\, dr d\theta}{(r^2+z_0^2)^{3/2}}\right)\mathbf{j}\right. \\
&\qquad\qquad \left. + \left(\int_0^{2\pi}\int_0^{\infty} \frac{z_0 r\, dr d\theta}{(r^2+z_0^2)^{3/2}}\right)\mathbf{k}\right]
\end{aligned}$$

Note that the integral for the **i**-component has constant bounds and the integrand can be decomposed into a product of an r-function and a θ-function, i.e.

$$\int_0^{2\pi}\int_0^{\infty} \frac{-r^2\cos\theta\, dr d\theta}{(r^2+z_0^2)^{3/2}} = -\lim_{R\to\infty}\left(\int_0^{2\pi}\cos\theta\, d\theta\right)\left(\int_0^{R}\frac{r^2}{(r^2+z_0^2)^{3/2}}dr\right)$$

Since $\int_0^{2\pi}\cos\theta\, d\theta = 0$, the **i**-component is zero. Similar argument applies to the **j**-component.
Alternatively, you may argue that the integrand of the **i**-component:

$$-\frac{x}{(x^2+y^2+z_0^2)^{3/2}}$$

is an odd function of x, and the domain $\mathbb{R}^2$ is symmetric about the axis $x = 0$. The positive contribution on the right of the axis exactly cancels out the negative contribution on the left. Similar for the **j**-component.

(b) Derive that:

$$\mathbf{F}_{\text{resultant}} = \frac{Q\sigma}{2\varepsilon_0}\mathbf{k}.$$

[Remark: The result in (b) asserts that the resultant force on the Q-particle does *not* depend on how far it is from the infinite sheet! Believe it or not?]

Solution:

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\infty} \frac{z_0 r\, dr d\theta}{(r^2+z_0^2)^{3/2}} &= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\infty}\frac{z_0 r}{(r^2+z_0^2)^{3/2}}dr\right) \\
&= 2\pi\left[-\frac{z_0}{(r^2+z_0^2)^{1/2}}\right]_{r=0}^{r=\infty} \\
&= 2\pi\left(-0 + \frac{z_0}{(0+z_0^2)^{1/2}}\right) = 2\pi
\end{aligned}$$

Combine the results with (a), we obtained:

$$\mathbf{F}_{\text{resultant}} = \frac{Q\sigma}{4\pi\varepsilon_0}2\pi\,\mathbf{k} = \frac{Q\sigma}{2\varepsilon_0}\mathbf{k}.$$

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #6 • Triple Integrals

1. (★) Consider the triple integral:

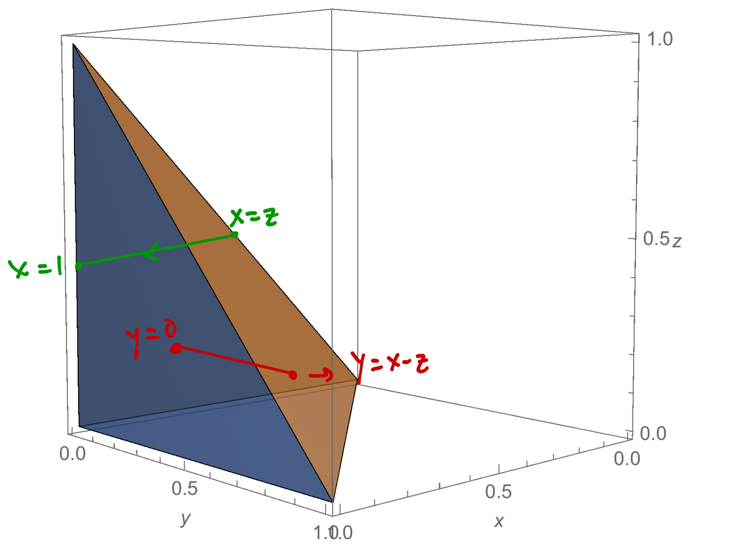
On xy-plane : $x=y$
On yz-plane: $y=-z$
On $x=1$: $y+z=1$
On xz-plane: $x=z$

$y = x - z$
$x - y - z = 0$

$x=z \leadsto x=1$ shadow → on xz-plane

$$\int_0^1 \int_z^1 \int_0^{x-z} f(x,y,z)\, dydxdz.$$

(a) Sketch the solid described by the integral.



(b) Express the integral using each of the other five orders, i.e. $dydzdx$, $dxdydz$, $dxdzdy$, $dzdxdy$ and $dzdydx$.

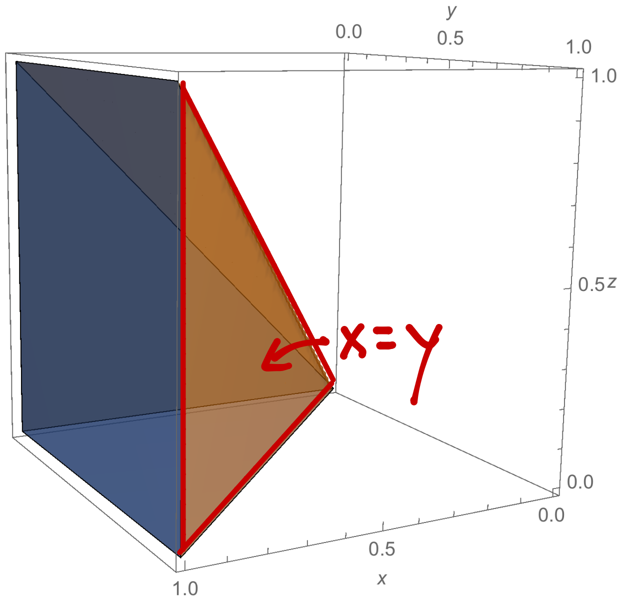
Solution: (Answer only)

$$\begin{aligned}
\int_0^1 \int_z^1 \int_0^{x-z} f(x,y,z)\, dydxdz &= \int_0^1 \int_0^x \int_0^{x-z} f(x,y,z)\, dydzdx \\
&= \int_0^1 \int_0^{1-z} \int_{y+z}^1 f(x,y,z)\, dxdydz \\
&= \int_0^1 \int_0^{1-y} \int_{y+z}^1 f(x,y,z)\, dxdzdy \\
&= \int_0^1 \int_y^1 \int_0^{x-y} f(x,y,z)\, dzdxdy \\
&= \int_0^1 \int_0^x \int_0^{x-y} f(x,y,z)\, dzdydx
\end{aligned}$$

2. (★★) Consider the triple integral:

$$\int_0^1 \int_z^1 \int_0^x e^{x^3}\, dydxdz.$$

(a) Sketch the solid described by the integral.

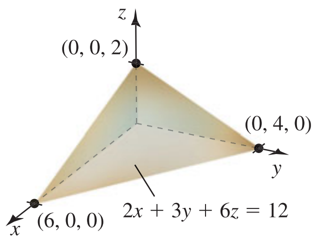


(b) Pick a *good* order of integration and compute the integral *by hand*.

Solution: We use $dzdydx$-order since the integrand e^{x^3} depends only on x. That way we should be able to compute the inner- and middle integral easily. [Note: $dydzdx$-order should works as well.]
Use z as the "pillar" variable, so that (x, y) are the base variables. The triple integral can be rewritten as:

$$\begin{aligned}
\int_0^1 \int_0^x \int_0^x e^{x^3}\, dzdydx &= \int_0^1 \int_0^x xe^{x^3}\, dydx \\
&= \int_0^1 x^2 e^{x^3}\, dx \\
&= \left[\frac{1}{3} e^{x^3}\right]_{x=0}^{x=1} \\
&= \frac{1}{3}(e-1).
\end{aligned}$$

3. (★★) Consider the right tetrahedron solid T in the first octant bounded by the xy-, yz-, xz-planes and the plane Π with vertices $(6,0,0)$, $(0,4,0)$ and $(0,0,2)$.



(a) Show that the equation of the plane Π is given by $2x+3y+6z=12$.

Solution: Straight-forward.

(b) Evaluate the following triple integral:

$$\iiint_T \left(\frac{1}{12-3y-6z}+\frac{1}{12-2x-6z}+\frac{1}{12-2x-3y}\right)\,dV.$$

Please do the computations *by hand*. Pick carefully the orders of integration to simplify your computations.

Solution: Denote T_{yz} the projection of T on the yz-plane. Similar for T_{xy} and T_{xz}. We split the integral into three and use different order of integration for each of them:

$$\begin{aligned}\iiint_T \frac{1}{12-3y-6z}\,dV &= \iint_{T_{yz}}\int_{x=0}^{x=\frac{1}{2}(12-3y-6z)} \frac{1}{12-3y-6z}\,dxdydz\\ &= \iint_{T_{yz}} \frac{1}{2}\,dA = \frac{1}{2}\text{ area of } T_{yz} = \frac{1}{2}\,\frac{4\times 2}{2} = 2\end{aligned}$$

$$\begin{aligned}\iiint_T \frac{1}{12-2x-6z}\,dV &= \iint_{T_{xz}}\int_{y=0}^{y=\frac{1}{3}(12-2x-6z)} \frac{1}{12-2x-6z}\,dydxdz\\ &= \iint_{T_{xz}} \frac{1}{3}\,dxdz = \frac{1}{3}\,\frac{6\times 2}{2} = 2\end{aligned}$$

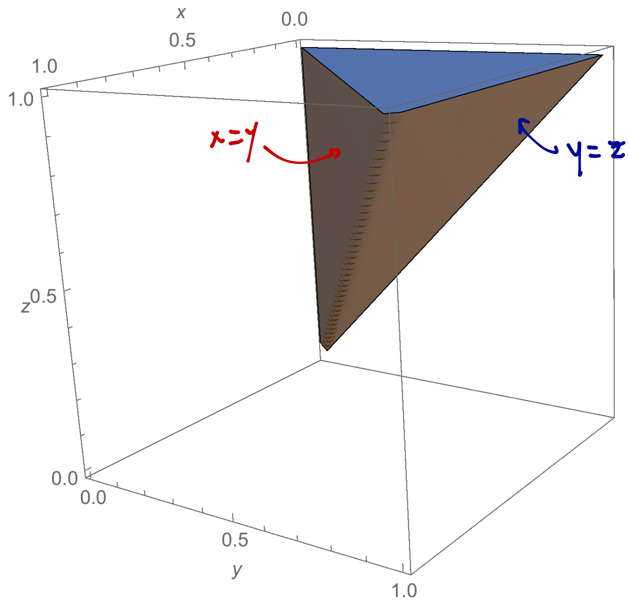
$$\begin{aligned}\iiint_T \frac{1}{12-2x-3y}\,dV &= \iint_{T_{xy}}\int_{z=0}^{z=\frac{1}{6}(12-2x-3y)} \frac{1}{12-2x-3y}\,dzdxdy\\ &= \iint_{T_{xy}} \frac{1}{6}\,dxdy = \frac{1}{6}\,\frac{6\times 4}{2} = 2\end{aligned}$$

Therefore,

$$\iiint_T \left(\frac{1}{12-3y-6z}+\frac{1}{12-2x-6z}+\frac{1}{12-2x-3y}\right)\,dV = 2+2+2=6.$$

4. (★★) Let a be a positive constant. Given that $f(x)$ is a continuous function of x, show that:
$$\int_0^a \int_0^z \int_0^y f(x)\,dxdydz = \int_0^a \frac{(a-x)^2}{2} f(x)\,dx$$

Solution: The triple integral represents the following solid:



Note that the integrand $f(x)$ depends only on x. We switch the order of integration to: $dzdydx$ (so that one can compute the inner and middle integrals without any problem):

$$\begin{aligned}
\int_0^a \int_0^z \int_0^y f(x)\,dxdydz &= \int_0^a \int_x^a \int_y^a f(x)\,dzdydx \\
&= \int_0^a \int_x^a f(x)\,(a-y)\,dydx \\
&= \int_0^a \left[-f(x)\cdot\frac{(a-y)^2}{2}\right]_{y=x}^{y=a} dx \\
&= \int_0^a f(x)\cdot\frac{(a-x)^2}{2}\,dx
\end{aligned}$$

5. (★) Evaluate $\iiint_D (x^2+y^2)\,dV$ over the solid D which lies above the cone $z = c\sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 = a^2$.

Solution: The cone can be expressed in spherical coordinates as:

$$\underbrace{\rho\cos\phi}_{z} = \underbrace{c\rho\sin\phi}_{c\sqrt{x^2+y^2}} \Longrightarrow \phi = \tan^{-1}\frac{1}{c}.$$

Hence, the solid D can be expressed in spherical coordinates as:

$$0 \le \rho \le a, \quad 0 \le \phi \le \tan^{-1}\frac{1}{c}, \quad 0 \le \theta \le 2\pi.$$

Therefore, we have:

$$\begin{aligned}
\iiint_D (x^2+y^2)\,dV &= \int_0^{2\pi}\int_0^{\tan^{-1}\frac{1}{c}}\int_0^a \underbrace{\rho^2\sin^2\phi}_{x^2+y^2}\cdot\underbrace{\rho^2\sin\phi\,d\rho d\phi d\theta}_{dV} \\
&= \int_0^{2\pi}\int_0^{\tan^{-1}\frac{1}{c}}\int_0^a \rho^4\sin^3\phi\,d\rho d\phi d\theta \\
&= \frac{2\pi a^5}{5}\int_0^{\tan^{-1}\frac{1}{c}} \sin^3\phi\,d\phi \\
&= \frac{2\pi a^5}{5}\int_0^{\tan^{-1}\frac{1}{c}} \left(\cos^2\phi - 1\right) d(\cos\phi) \\
&= \frac{2\pi a^5}{5}\left[\frac{\cos^3\phi}{3} - \cos\phi\right]_0^{\tan^{-1}\frac{1}{c}} \\
&= \frac{2\pi a^5}{5}\left(\frac{1}{3}\cos^3\tan^{-1}\frac{1}{c} - \cos\tan^{-1}\frac{1}{c} + \frac{2}{3}\right)
\end{aligned}$$

You may use the fact that $\cos\tan^{-1}x = \frac{1}{\sqrt{1+x^2}}$ to simplify the final answer, but it is not necessary.

6. (★) Find the volume of the solid bounded by the xy-plane, the cone $z = 2a - \sqrt{x^2+y^2}$ and the cylinder $x^2+y^2 = 2ay$.

Solution: (Sketch only) In cylindrical coordinates, the cone is given by $z = 2a - r$ and the cylinder is $r^2 = 2ar\sin\theta$, or equivalently, $r = 2a\sin\theta$. Therefore,

$$\text{volume} = \iiint_D 1\,dV = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2a\sin\theta}\int_{z=0}^{z=2a-r} 1\,r\,dzdrd\theta = \frac{2}{9}(9\pi - 16)a^3$$

Here we presented the case when $a > 0$. The other case is similar (yet different).

7. (★★) Let $\phi(x, y, z) = \dfrac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(-\dfrac{x^2+y^2+z^2}{4kt}\right)$ where $t > 0$. Show that for each fixed $t > 0$, we have:

$$\iiint_{\mathbb{R}^3} \phi(x, y, z)\, dV = 1.$$

Solution: The appearance of the term $x^2 + y^2 + z^2$ suggests it may be best to use spherical coordinates, since $x^2 + y^2 + z^2 = \rho^2$. The bounds for the whole space $\mathbb{R}^3$ is:

$$0 \le \rho < \infty, \quad 0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

$$\begin{aligned}
\iiint_{\mathbb{R}^3} \phi(x, y, z)\, dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{1}{(4\pi kt)^{3/2}} \exp\left(-\frac{\rho^2}{4kt}\right) \rho^2 \sin\phi \, d\rho d\phi d\theta \\
&= \frac{1}{(4\pi kt)^{3/2}} \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_0^{\infty} \rho^2 e^{-\rho^2/4kt}\, d\rho\right) \\
&= \frac{1}{(4\pi kt)^{3/2}} \cdot 2\pi \cdot 2 \cdot \left(\int_0^{\infty} \rho^2 e^{-\rho^2/4kt}\, d\rho\right)
\end{aligned}$$

It comes down to computing $\int_0^{\infty} \rho^2 e^{-\rho^2/4kt}\, d\rho$. Note that

$$\frac{d}{d\rho}\left(e^{-\rho^2/4kt}\right) = e^{-\rho^2/4kt} \cdot \left(-\frac{2\rho}{4kt}\right) = -\frac{\rho}{2kt} \cdot e^{-\rho^2/4kt}.$$

$$\begin{aligned}
\int_0^{\infty} \rho^2 e^{-\rho^2/4kt}\, d\rho &= -2kt \int_0^{\infty} \rho \cdot \underbrace{\left(-\frac{\rho}{2kt} e^{-\rho^2/4kt}\right) d\rho}_{d\left(e^{-\rho^2/4kt}\right)} \\
&= -2kt \left\{ \left[\rho e^{-\rho^2/4kt}\right]_{\rho=0}^{\rho\to\infty} - \int_0^{\infty} e^{-\rho^2/4kt} d\rho \right\} \\
&= -2kt \left\{ [0-0] - \sqrt{4kt} \int_0^{\infty} e^{-(\rho/\sqrt{4kt})^2}\, d\left(\frac{\rho}{\sqrt{4kt}}\right) \right\} \\
&= 2kt\sqrt{4kt} \int_0^{\infty} e^{-u^2}\, du = 4k^{3/2}t^{3/2} \cdot \frac{\sqrt{\pi}}{2}.
\end{aligned}$$

The last two steps follows from integration by substitutions (let $u = \rho/\sqrt{4kt}$) and Lecture Notes P.64.

Combining these results, we get:

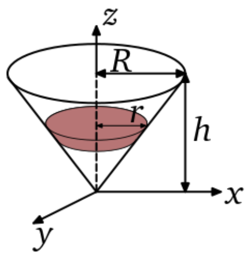
$$\iiint_{\mathbb{R}^3} \phi(x, y, z)\, dV = \frac{1}{(4\pi kt)^{3/2}} \cdot 2\pi \cdot 2 \cdot 4k^{3/2}t^{3/2} \cdot \frac{\sqrt{\pi}}{2} = 1.$$

Alternatively, one can also breakdown $\phi(x, y, z)$ into:

$$\frac{1}{(4\pi kt)^{3/2}} e^{-x^2/4kt} e^{-y^2/4kt} e^{-z^2/4kt}$$

and set up the integral using rectangular coordinates.

8. (★★) Consider a right circular solid cone (denoted by K) with radius R, height h, mass m and uniform density δ.



The moment of inertia about the z-axis of the solid is defined to be:

$$I_z := \iiint_K D_z(x,y,z)^2 \, \delta dV$$

where $D_z(x,y,z)$ is the perpendicular distance between the point (x,y,z) and the z-axis.

(a) *Set up*, but do not evaluate, the integral I_z using each of the following coordinates:
   i. rectangular coordinates
   ii. cylindrical coordinates
   iii. spherical coordinates

Solution: $D_z(x,y,z)$ is the distance from (x,y,z) to the z-axis, which is also the distance from (x,y,z) to $(0,0,z)$ – draw a picture to convince yourself on that! Therefore, $D_z(x,y,z) = \sqrt{x^2+y^2}$.
The equation of the cone is given by:

$$z = \frac{h}{R}r \qquad \text{(cylindrical)}$$
$$z = \frac{h}{R}\sqrt{x^2+y^2} \qquad \text{(rectangular)}$$
$$\varphi = \tan^{-1}\frac{R}{h} \qquad \text{(spherical)}$$

The equation of the flat top of the cone is given by:

$$z = h \qquad \text{(both cylindrical and rectangular)}$$
$$\rho = h\sec\theta \qquad \text{(spherical)}$$

$$I_z = \int_{y=-R}^{y=R} \int_{x=-\sqrt{R^2-y^2}}^{x=\sqrt{R^2-y^2}} \int_{z=\frac{h}{R}\sqrt{x^2+y^2}}^{z=h} \delta(x^2+y^2)\, dzdxdy$$

$$I_z = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=R} \int_{z=\frac{h}{R}r}^{z=h} \delta r^2 \, rdzdrd\theta$$
$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=R} \int_{z=\frac{h}{R}r}^{z=h} \delta r^3 \, dzdrd\theta$$

$$I_z = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\tan^{-1}\frac{R}{h}} \int_{\rho=0}^{\rho=h\sec\theta} \delta(\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta)\, \rho^2\sin\varphi d\rho d\varphi d\theta$$
$$= \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\tan^{-1}\frac{R}{h}} \int_{\rho=0}^{\rho=h\sec\theta} \delta\rho^4\sin^3\varphi \, d\rho d\varphi d\theta$$

(b) Rank the ease of computations of the above coordinate systems for evaluating the integral I_z, then compute I_z using the easiest coordinate system. Express your final answer in terms of the mass m, not the density δ.

Solution: From the easiest to the hardest: cylindrical, spherical, rectangular. Using rectangular coordinates would involve some difficult trig substitution. Using spherical coordinates will amount to integrating $\sec^5\theta$

$$\begin{aligned} I_z &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R}\int_{z=\frac{h}{R}r}^{z=h} \delta r^3\, dz dr d\theta \\ &= \frac{\delta\pi R^4 h}{10} \\ &= \frac{\pi R^4 h}{10}\,\frac{m}{\frac{1}{3}\pi R^2 h} \\ &= \frac{3}{10} mR^2 \end{aligned}$$

9. (★★) Given a solid T with mass m and uniform density δ, the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is defined to be:

$$\bar{x} = \frac{\iiint_T x\ \delta dV}{\iiint_T \delta dV}, \quad \bar{y} = \frac{\iiint_T y\ \delta dV}{\iiint_T \delta dV}, \quad \bar{z} = \frac{\iiint_T z\ \delta dV}{\iiint_T \delta dV}$$

The moment of inertia of T about the z-axis is defined as:

$$I_z := \iiint_T D_z(x,y,z)^2\ \delta dV$$

where $D_z(x,y,z)$ is the perpendicular distance between the point (x,y,z) and the z-axis. Now consider the axis L passing through the center of mass $(\bar{x}, \bar{y}, \bar{z})$ and parallel to the z-axis. The moment of inertia of the solid about the axis L is defined as:

$$I_{\text{cm}} := \iiint_T D_L(x,y,z)^2\ \delta dV$$

where $D_L(x,y,z)$ is the perpendicular distance between the point (x,y,z) and the axis L. Prove the following result (which is called the Parallel Axis Theorem):

$$I_z = I_{\text{cm}} + md^2$$

where d is the distance between the z-axis and the axis L.

Solution: As in Problem 2, $D_z(x,y,z)^2 = x^2 + y^2$. Therefore,

$$I_z = \iiint_T \delta(x^2+y^2)\ dV$$

$D_L(x,y,z)$ is the distance from (x,y,z) to the axis L. Since L is a vertical line passing through $(\bar{x}, \bar{y}, \bar{z})$, the x- and y-coordinates of every point on L must be $\bar{x}$ and $\bar{y}$. The distance $D_L(x,y,z)$ is measured between the points (x,y,z) and $(\bar{x}, \bar{y}, z)$, i.e. the perpendicular distance. Therefore, $D_L(x,y,z)^2 = (x-\bar{x})^2 + (y-\bar{y})^2$.

The distance d between the two vertical axes (z-axis and L) is the distance between any two points at the same altitude. In other words, $d^2 = \bar{x}^2 + \bar{y}^2$.

Consider $I_{cm} + md^2$:

$$\begin{aligned} I_{cm} + md^2 &= \iiint_T \delta\left((x-\bar{x})^2 + (y-\bar{y})^2\right)\, dV + m(\bar{x}^2 + \bar{y}^2) \\ &= \iiint_T \delta\left(x^2 - 2\bar{x}x + \bar{x}^2 + y^2 - 2\bar{y}y + \bar{y}^2\right)\, dV + m(\bar{x}^2 + \bar{y}^2) \\ &= \iiint_T \delta(x^2+y^2)\, dV - 2\iiint_T \delta\left(\bar{x}x + \bar{y}y\right)\, dV + \iiint_T \delta\left(\bar{x}^2 + \bar{y}^2\right)\, dV + m(\bar{x}^2 + \bar{y}^2) \end{aligned}$$

Note that $\bar{x}$ and $\bar{y}$ are constants, we get:

$$\begin{aligned} I_{cm} + md^2 &= I_z - 2\bar{x}\iiint_T \delta x\, dV - 2\bar{y}\iiint_T \delta y\, dV + \left(\bar{x}^2 + \bar{y}^2\right)\iiint_T \delta dV + m(\bar{x}^2 + \bar{y}^2) \\ &= I_z - 2\bar{x}\cdot m\bar{x} - 2\bar{y}\cdot m\bar{y} + m(\bar{x}^2 + \bar{y}^2) + m(\bar{x}^2 + \bar{y}^2) \\ &= I_z - 2m(\bar{x}^2 + \bar{y}^2) + m(\bar{x}^2 + \bar{y}^2) + m(\bar{x}^2 + \bar{y}^2) = I_z. \end{aligned}$$

Here we have used the fact that $m = \iiint_T \delta\, dV$ and the definition of $\bar{x}$ and $\bar{y}$.

10. (★) The change-of-variable formula for the volume element dV is given by:

$$dxdydz = \left|\det \frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw. \qquad (*)$$

(a) Using (*), verify that:

$$dxdydz = \rho^2 \sin\phi\, d\rho d\phi d\theta.$$

Solution: It suffices to show:

$$\left|\det \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}\right| = \rho^2 \sin\phi.$$

Using the conversion rules $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$ (here we used MATH convention), we get:

$$\begin{aligned} \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{bmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \end{bmatrix} \\ &= \begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix} \end{aligned}$$

Then by direct computations, we get:

$$\begin{aligned} \det \frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \rho^2\cos^2\phi\sin\phi\cos^2\theta + \rho^2\sin^3\phi\sin^2\theta \\ &\quad - (-\rho^2\cos^2\phi\sin\phi\sin^2\theta - \rho^2\sin^3\phi\cos^2\theta) \\ &= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi = \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi. \end{aligned}$$

(b) Let $u = 2x$, $v = 3y$ and $w = 5z$. Using (*), express $dxdydz$ in terms of $dudvdw$.

Solution: By rearrangement, we get $x = \frac{u}{2}$, $y = \frac{v}{3}$ and $z = \frac{w}{5}$

$$\begin{aligned}\det \frac{\partial(x,y,z)}{\partial(u,v,w)} &= \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} \\ &= \det \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} = \frac{1}{2 \times 3 \times 5} = \frac{1}{30}.\end{aligned}$$

Therefore,

$$dxdydz = \left| \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw = \frac{1}{30} dudvdw.$$

11. (★★★) Consider a solid sphere with radius R centered at the origin in $\mathbb{R}^3$ which carries a uniform distribution of charges with density δ. Each volume element dV in the sphere can be regarded as a particle with charge $\delta\, dV$.

Fix a particle with charge q at $(0, 0, z_0)$ where $z_0 > R$, i.e. outside the sphere, and call it the q-particle. As in the previous Problem Set, the electric force exerted on the q-particle by a charged element $\delta\, dV$ at (x, y, z) in the solid sphere is given by the Coulomb's Law (in vector form):

$$d\mathbf{F} = \frac{q\,\delta\, dV}{4\pi\varepsilon_0} \frac{(0-x)\mathbf{i} + (0-y)\mathbf{j} + (z_0 - z)\mathbf{k}}{\left((0-x)^2 + (0-y)^2 + (z_0 - z)^2\right)^{3/2}}$$

Similar to the previous Problem Set, the Principle of Superposition asserts that the resultant force exerted on the q-particle by the whole sphere is given by "summing-up", i.e. integrating, each the force element $d\mathbf{F}$ over the sphere:

$$\mathbf{F}_{\text{resultant}} = \iiint_{\text{sphere}} d\mathbf{F}.$$

(a) Show that:

$$\mathbf{F}_{\text{resultant}} = \left(\int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{q\delta}{4\pi\varepsilon_0} \frac{\rho^2 \sin\varphi \cdot (z_0 - \rho\cos\varphi)}{(\rho^2 - 2\rho z_0 \cos\varphi + z_0^2)^{3/2}}\, d\rho d\varphi d\theta \right) \mathbf{k}$$

Solution: Use spherical coordinates:

$$\begin{aligned}&\mathbf{F}_{\text{resultant}} \\ &= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} d\mathbf{F} \\ &= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{q\,\delta\, dV}{4\pi\varepsilon_0} \frac{-x\mathbf{i} - y\mathbf{j} - (z - z_0)\mathbf{k}}{\left(x^2 + y^2 + (z - z_0)^2\right)^{3/2}} \\ &= \frac{q\,\delta}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{-\rho\sin\phi\cos\theta\,\mathbf{i} - \rho\sin\phi\sin\theta\,\mathbf{j} - (\rho\cos\phi - z_0)\,\mathbf{k}}{(\rho^2\sin^2\phi + (\rho\cos\phi - z_0)^2)^{3/2}} \cdot \rho^2 \sin\phi\, d\rho d\phi d\theta\end{aligned}$$

Note that the **i** and **j** components are zero since:

$$\int_0^{2\pi} \sin\theta\, d\theta = \int_0^{2\pi} \cos\theta\, d\theta = 0.$$

After simplification of the **k**-component, one can obtain the required result.

(b) Try to compute the above integral, either by software or by hand, and show that:

$$\mathbf{F}_{\text{resultant}} = \frac{q\delta R^3}{3\varepsilon_0 z_0^2}\mathbf{k} = \frac{qQ}{4\pi\varepsilon_0 z_0^2}\mathbf{k}$$

where Q is the total amount of charges in the sphere.

[Remark 1: This result shows that the resultant force exerted on the q-particle by the charged sphere will be the same if one replaces it by a particle at the origin with the same amount of charges.]

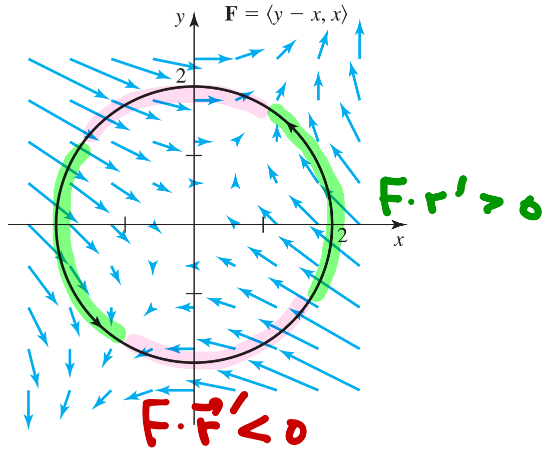
[Remark 2: Using the Gauss's Law for Electricity, the above result can be obtained easily by considering the surface flux of $\mathbf{F}_{\text{resultant}}$. We will discuss that later, and will derive the Gauss's Law using the Divergence Theorem (assuming Coulomb's Law).]

Solution: Compute the integral in (a). Being a human being in the 21th Century, you should do it using Mathematica or WolframAlpha. Don't waste your time doing it by hand (unless you are required to in later E&M course).

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #7 • Line Integrals, Conservative Vector Fields, Curl Operator

Do not use the Green's Theorem in any problem in this set.

1. (★) Let $\mathbf{F} = (y-x)\mathbf{i} + x\mathbf{j}$ on $\mathbb{R}^2$, and C be the counter-clockwise circular path with radius 2 centered at the origin. See the figure below:



(a) On the above figure, highlight the portion of the path C at which $\mathbf{F}\cdot\mathbf{r}' > 0$.

(b) On the above figure, highlight (with another color) the portion of the path C at which $\mathbf{F}\cdot\ \mathbf{r}' < 0$.

(c) Calculate the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ from the definition. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

Solution: First parametrize the path:

$$\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

That is, we have $x = 2\cos t$ and $y = 2\sin t$.

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi} ((y-x)\mathbf{i} + x\mathbf{j})\cdot((-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j})\ dt \\ &= \int_0^{2\pi} -2(y-x)\sin t + 2x\cos t\,dt \\ &= \int_0^{2\pi} -2(2\sin t - 2\cos t)\sin t + 2(2\cos t)\cdot\cos t\,dt \\ &= \int_0^{2\pi} -4\sin^2 t + 4\sin t\cos t + 4\cos^2 t\,dt\end{aligned}$$

Recall from single-variable calculus that $\int_0^{2\pi} \sin^2 t\,dt = \int_0^{2\pi} \cos^2 t\,dt$, so we have:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} 4\sin t\cos t\,dt = \int_0^{2\pi} 2\sin 2t\,dt = [-\cos 2t]_0^{2\pi} = 0.$$

We cannot argue whether or not $\mathbf{F}$ is conservative by just showing $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for ONE closed curve – we need $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for ALL closed curves.

(d) Calculate $\nabla \times \mathbf{F}$, i.e. the curl of $\mathbf{F}$. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

Solution:

$$\begin{aligned}\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-x & x & 0 \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y-x & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y-x & x \end{vmatrix} \mathbf{k} \\ &= 0\mathbf{i} - 0\mathbf{j} + \left(\frac{\partial}{\partial x} x - \frac{\partial}{\partial y}(y-x) \right) \mathbf{k} \\ &= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.\end{aligned}$$

Since $\mathbf{F}$ is defined everywhere, the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is simply-connected. By Curl Test, we conclude that $\mathbf{F}$ is conservative.

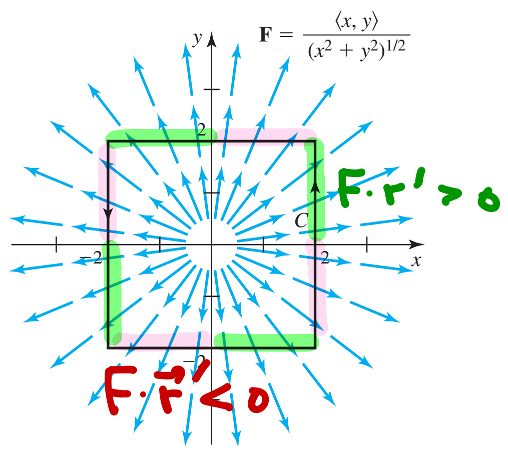
(e) Find a potential function f such that $\mathbf{F} = \nabla f$, or show that such an f does not exist. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

Solution: By inspection, it is not difficult to see that:

$$\mathbf{F} = \nabla \left(xy - \frac{x^2}{2} \right).$$

Therefore, $f(x, y)$ can be taken to be $xy - \frac{x^2}{2}$, and so $\mathbf{F}$ is conservative (by definition).

2. ($\bigstar$) Let $\mathbf{F} = \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\mathbf{j}$, and C be the counter-clockwise square path with vertices $(2, -2)$, $(2, 2)$, $(-2, 2)$ and $(-2, -2)$. See the figure below:



Do (a)-(e) of Problem #1 with this $\mathbf{F}$ and C instead.

Solution: Part (c): To calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ from the definition, we split the path C into four segments:

From $(2,2)$ to $(-2,2)$: $\mathbf{r}_1(t) = \langle 2,2\rangle + t(\langle -2,2\rangle - \langle 2,2\rangle) = \langle 2-4t, 2\rangle, \qquad 0 \le t \le 1.$

$$\begin{aligned}
\int_0^1 \mathbf{F} \cdot \mathbf{r}_1'(t)\,dt &= \int_0^1 \frac{(2-4t)\mathbf{i} + 2\mathbf{j}}{\sqrt{(2-4t)^2 + 2^2}} \cdot (-4\mathbf{i} + 0\mathbf{j})\,dt \\
&= \int_0^1 \frac{-4(2-4t)}{\sqrt{(2-4t)^2+2^2}}\,dt \\
&= \int_{-2}^{2} \frac{u}{\sqrt{u^2+4}}\,dt && \text{(Let } u = 2-4t\text{)} \\
&= 0 && \text{(Odd function!)}
\end{aligned}$$

From $(-2,2)$ to $(-2,-2)$: $\mathbf{r}_2(t) = \langle -2, 2-4t\rangle, \quad 0 \le t \le 1.$

$$\begin{aligned}
\int_0^1 \mathbf{F} \cdot \mathbf{r}_2'(t)\,dt &= \int_0^1 \frac{-2\mathbf{i} + (2-4t)\mathbf{j}}{\sqrt{2^2 + (2-4t)^2}} \cdot (0\mathbf{i} - 4\mathbf{j})\,dt \\
&= \int_0^1 \frac{-4(2-4t)}{\sqrt{2^2+(2-4t)^2}}\,dt \\
&= 0
\end{aligned}$$

Note the integral is the same as the previous one.

From $(-2,-2)$ to $(2,-2)$: $\mathbf{r}_3(t) = \langle -2+4t, -2\rangle, \quad 0 \le t \le 1.$

$$\begin{aligned}
\int_0^1 \mathbf{F} \cdot \mathbf{r}_3'(t)\,dt &= \int_0^1 \frac{(-2+4t)\mathbf{i} - 2\mathbf{j}}{\sqrt{(-2+4t)^2 + 2^2}} \cdot (4\mathbf{i} + 0\mathbf{j})\,dt \\
&= \int_0^1 \frac{4(-2+4t)}{\sqrt{(-2+4t)^2+2^2}}\,dt \\
&= \int_0^1 \frac{-4(2-4t)}{\sqrt{2^2+(2-4t)^2}}\,dt \\
&= 0
\end{aligned}$$

From $(2,-2)$ to $(2,2)$: $\mathbf{r}_4(t) = \langle 2, -2+4t\rangle, \quad 0 \le t \le 1.$

$$\begin{aligned}
\int_0^1 \mathbf{F} \cdot \mathbf{r}_4'(t)\,dt &= \int_0^1 \frac{2\mathbf{i} + (-2+4t)\mathbf{j}}{\sqrt{2^2 + (-2+4t)^2}} \cdot (0\mathbf{i} + 4\mathbf{j})\,dt \\
&= \int_0^1 \frac{4(-2+4t)}{\sqrt{2^2+(-2+4t)^2}}\,dt \\
&= 0
\end{aligned}$$

Finally, adding up the above line segments, we get:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0+0+0+0 = 0.$$

Note that we can't use this result alone to argue if $\mathbf{F}$ is conservative, as we have just shown that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for ONE closed curve C (but not for ALL closed curves C).

For Part (d):

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2}} & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix} \mathbf{k} \\
&= 0\mathbf{i} - 0\mathbf{j} + \left(\frac{\partial}{\partial x} \frac{y}{\sqrt{x^2+y^2}} - \frac{\partial}{\partial y} \frac{x}{\sqrt{x^2+y^2}} \right) \mathbf{k} \\
&= \left\{ \left(-\frac{y}{2}(x^2+y^2)^{-3/2} \cdot 2x \right) - \left(-\frac{x}{2}(x^2+y^2)^{-3/2} \cdot 2y \right) \right\} \mathbf{k} \\
&= 0\mathbf{k} = \mathbf{0}
\end{aligned}$$

Although $\nabla \times \mathbf{F} = \mathbf{0}$, the domain of $\mathbf{F}$ is $\mathbb{R}^2 \backslash \{(0,0)\}$ which is NOT simply-connected. The Curl Test cannot be used here, and so this result alone cannot conclude on whether $\mathbf{F}$ is conservative.

For Part (e): We can verify that:

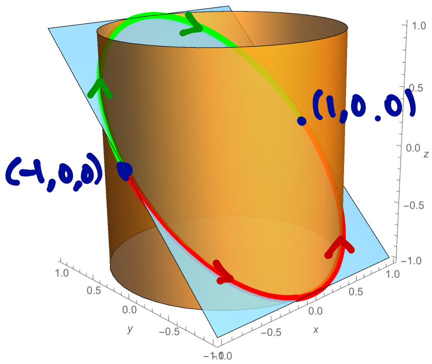
$$\mathbf{F}(x,y) = \nabla \left(\sqrt{x^2+y^2} \right).$$

Therefore, one can take the potential function $f(x,y) = \sqrt{x^2+y^2}$, and so $\mathbf{F}$ is conservative from definition.

3. (★) Let C be the curve of intersection of the cylinder $x^2+y^2=1$ and the plane $z=y$.

(a) Sketch the cylinder, the plane and the curve C in the same diagram.

Solution:



(b) Let $\mathbf{F} = y\mathbf{i} + z\mathbf{j} - x\mathbf{k}$. Calculate the line integral $\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ where Γ is a portion of C from $(-1,0,0)$ to $(1,0,0)$. There are two possible such Γ's. Do both.

Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

Solution: We need to first parametrize the path Γ. There are two such possible paths, namely the counter-clockwise path and clockwise path (when looking from the top).

For the counter-clockwise path, the curve lies on the cylinder $x^2 + y^2 = 1$ and therefore projects down to the unit circle centered at the origin on the xy-plane. This unit circle is parametrized by $x = \cos t$ and $y = \sin t$. Therefore, the red path also has x and y coordinates given by $x = \cos t$ and $y = \sin t$. Furthermore, the red path lies on the plane $z = y$, and so we have $z = \sin t$. To sum up, the parametric equation for the red path is:

$$\mathbf{r}_1(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin t)\mathbf{k}, \quad \pi \leq t \leq 2\pi.$$

The bounds for t are chosen so that $\mathbf{r}_1(\pi) = \langle -1, 0, 0\rangle$ and $\mathbf{r}_1(2\pi) = \langle 1, 0, 0\rangle$, which are the coordinates of the starting and ending points of the red path.

$$\begin{aligned}
&\int_{\text{red path}} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{\pi}^{2\pi} (y\mathbf{i} + z\mathbf{j} - x\mathbf{k}) \cdot \mathbf{r}_1'(t)\, dt \\
&= \int_{\pi}^{2\pi} ((\sin t)\mathbf{i} + (\sin t)\mathbf{j} - (\cos t)\mathbf{k}) \cdot ((-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\cos t)\mathbf{k})\, dt \\
&= \int_{\pi}^{2\pi} (-\sin^2 t + \sin t \cos t - \cos^2 t)\, dt = \int_{\pi}^{2\pi} (-1 + \sin t \cos t)\, dt \\
&= \left[-t - \frac{1}{2}\cos^2 t \right]_{\pi}^{2\pi} = -\pi
\end{aligned}$$

For the clockwise path, we can parametrize it by replacing all t's in $\mathbf{r}_1(t)$ by $-t$, i.e.:

$$\begin{aligned}
\mathbf{r}_2(t) &= (\cos(-t))\mathbf{i} + (\sin(-t))\mathbf{j} + (\sin(-t))\mathbf{k} \\
&= (\cos t)\mathbf{i} - (\sin t)\mathbf{j} - (\sin t)\mathbf{k}
\end{aligned}$$

In order to give starting point $(-1, 0, 0)$ and ending point $(1, 0, 0)$, we can set the bounds for t to be $\pi \leq 2\pi$, then $\mathbf{r}_2(\pi) = \langle -1, 0, 0\rangle$ and $\mathbf{r}_2(2\pi) = \langle 1, 0, 0\rangle$.

$$\begin{aligned}
&\int_{\text{green path}} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_{\pi}^{2\pi} ((-\sin t)\mathbf{i} + (-\sin t)\mathbf{j} - (\cos t)\mathbf{k}) \cdot ((-\sin t)\mathbf{i} + (-\cos t)\mathbf{j} + (-\cos t)\mathbf{k})\, dt \\
&= \int_{\pi}^{2\pi} (1 + \sin t \cos t)\, dt = \left[t - \frac{1}{2}\cos^2 t \right]_{\pi}^{2\pi} = \pi
\end{aligned}$$

Since we can find two different paths with the same starting and ending points so that the line integral of $\mathbf{F}$ over them are not equal, we conclude that $\mathbf{F}$ is not conservative.

(c) Find a potential function f such that $\mathbf{F} = \nabla f$, or show that such an f does not exist. Is the result *alone* sufficient to determine whether $\mathbf{F}$ is conservative or not?

Solution: Set up:

$$\frac{\partial f}{\partial x} = y$$
$$\frac{\partial f}{\partial y} = z$$
$$\frac{\partial f}{\partial z} = -x$$

Integrating the first equation gives:

$$f(x,y,z) = xy + g(y,z)$$

Then:

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}(y,z)$$

Combining with the second equation, we get:

$$z = x + \frac{\partial g}{\partial y}(y,z) \implies z - \frac{\partial g}{\partial y}(y,z) = x.$$

Now that the RHS is a function of x while the LHS is a function of y and z. It is a contradiction. Therefore, such an f does not exist and so $\mathbf{F}$ is not conservative by definition.

Alternatively, we can also check that:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & -x \end{vmatrix} = -\mathbf{i} + \mathbf{j} - \mathbf{k} \neq \mathbf{0}$$

Conservative vector field must have zero curl. Now that curl of $\mathbf{F}$ is non-zero, the vector field $\mathbf{F}$ is not conservative.

4. (★) Determine whether or not each of the following vector fields is conservative or not. If so, find its potential function f such that $\mathbf{F} = \nabla f$.

(a) $\mathbf{F} = (e^{-y} - ze^{-x})\mathbf{i} + (e^{-z} - xe^{-y})\mathbf{j} + (e^{-x} - ye^{-z})\mathbf{k}$

Solution: Set up:

$$\frac{\partial f}{\partial x} = e^{-y} - ze^{-x}$$
$$\frac{\partial f}{\partial y} = e^{-z} - xe^{-y}$$
$$\frac{\partial f}{\partial z} = e^{-x} - ye^{-z}$$

By integrating the first equation, we get:

$$f(x,y,z) = xe^{-y} + ze^{-x} + g(y,z) \quad \text{where } g(y,z) \text{ is an arbitrary function}$$

Then, by differentiation we get $\frac{\partial f}{\partial y} = -xe^{-y} + \frac{\partial g}{\partial y}$, and combined with the second equation, we must have:

$$\frac{\partial g}{\partial y} = e^{-z}.$$

Integrating this, we get $g(y,z) = ye^{-z} + h(z)$ for some arbitrary function $h(z)$, and so

$$f(x,y,z) = xe^{-y} + ze^{-x} + ye^{-z} + h(z).$$

Again by differentiating, we get: $\frac{\partial f}{\partial z} = e^{-x} - ye^{-z} + h'(z)$. Combine with the third equation, we get $h'(z) = 0$ and so h is a constant.
It can be easily verified that $\nabla\left(xe^{-y} + ze^{-x} + ye^{-z} + C\right) = \mathbf{F}$, so $\mathbf{F}$ is conservative with potential funciton $f(x,y,z) = xe^{-y} + ze^{-x} + ye^{-z} + C$.

(b) $\mathbf{F} = (x^2 - xy)\mathbf{i} + (y^2 - xy)\mathbf{j}$

Solution: Set up:

$$\frac{\partial f}{\partial x} = x^2 - xy$$
$$\frac{\partial f}{\partial y} = y^2 - xy$$

Integrating the first equation we get:

$$f(x,y) = \frac{x^3}{3} - \frac{x^2 y}{2} + g(y)$$

By differentiation, we get $\frac{\partial f}{\partial y} = -\frac{x^2}{2} + g'(y)$. Combining with the second equation, we must have

$$y^2 - xy = -\frac{x^2}{2} + g'(y).$$

However, that would imply

$$\frac{x^2}{2} = g'(y) - y^2 + xy.$$

LHS is a function of x only, while RHS depends on both x and y. Therefore, such a function f cannot exist and therefore $\mathbf{F}$ is not conservative.

Alternatively, one can show $\mathbf{F}$ is not conservative by showing:

$$\nabla \times \mathbf{F} = (x - y)\mathbf{k}.$$

Therefore, $\nabla \times \mathbf{F}$ is non-zero, and so $\mathbf{F}$ is not conservative.

5. (★) Determine the values of A and B for which the vector field below is conservative:

$$\mathbf{F}(x,y,z) = Ax\ln z\,\mathbf{i} + By^2z\,\mathbf{j} + \left(\frac{x^2}{z} + y^3\right)\mathbf{k},$$

where the domain of $\mathbf{F}$ is the upper-half space $\{(x,y,z) : z > 0\}$.

For each such pair of A and B, find the potential function f for the vector field.

Solution: Note that the domain of $\mathbf{F}$ is the upper-half space, which *is* simply-connected! Therefore, we have:

$$\mathbf{F} \text{ is conservative } \iff \nabla \times \mathbf{F} = \mathbf{0}$$

By straight-forward computations (omitted here), we get:

$$\nabla \times \mathbf{F} = (3-B)y^2\mathbf{i} + \frac{(A-2)x}{z}\mathbf{j} + 0\mathbf{k}$$

Therefore, $\mathbf{F}$ is conservative if and only if $A = 2$ and $B = 3$.

For this pair of A and B, we solve the equation $\mathbf{F} = \nabla f$ for f:

$$\frac{\partial f}{\partial x} = 2x\ln z$$
$$\frac{\partial f}{\partial y} = 3y^2z$$
$$\frac{\partial f}{\partial z} = \frac{x^2}{z} + y^3$$

Integrating the first equation, we get:

$$f(x,y,z) = x^2\ln z + g(y,z).$$

Then, we have $\frac{\partial f}{\partial z} = \frac{x^2}{z} + \frac{\partial g}{\partial z}$, and by comparison with the third equation, we get $\frac{\partial g}{\partial z} = y^3$, and so $g(y,z) = y^3z + h(y)$. Substitute back into f, it comes down to solving h:

$$f(x,y,z) = x^2\ln z + y^3z + h(y)$$

By considering $\frac{\partial f}{\partial y} = 3y^2z + h'(y)$ and the second equation, we conclude $h'(y) = 0$ and so $h(y) = C$. The potential function for the vector field is: $f(x,y,z) = x^2\ln z + y^3z + C$ where C is any real constant.

6. (★★) Consider the path C:

$$\mathbf{r}(t) = \left(\cos^{2M} t\right)\mathbf{i} + \left(\sin^N t\right)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi.$$

Here M is the age of the Earth, and N is the age of the Universe. Assume both M and N are positive finite integers.

Evaluate the line integral:

$$\int_C \left(e^{-y} - ze^{-x}\right)\,dx + \left(e^{-z} - xe^{-y}\right)\,dy + \left(e^{-x} - ye^{-z}\right)\,dz$$

Solution: The vector field represented by this line integral is:

$$\mathbf{F} = (e^{-y} - ze^{-x})\mathbf{i} + (e^{-z} - xe^{-y})\mathbf{j} + (e^{-x} - ye^{-z})\mathbf{k}$$

It appeared in Problem #4(a) in which we showed it is conservative with a potential function $f(x, y, z) = xe^{-y} + ze^{-x} + ye^{-z}$. Therefore, to compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, we simply need to find the starting and ending points of the path C:

$$\mathbf{r}(0) = \mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \langle 1, 0, 0 \rangle$$
$$\mathbf{r}(\pi) = \mathbf{i} + 0\mathbf{j} + \pi\mathbf{k} = \langle 1, 0, \pi \rangle$$

By Fundamental Theorem of Line Integrals, we get:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 0, \pi) - f(1, 0, 0) = (e^0 + \pi e^{-1} + 0) - (e^0 + 0 + 0) = \pi e^{-1}$$

Alternatively, given that $\mathbf{F}$ is a conservative vector field, one can calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ by choosing an easier path joining the same endpoints as C. Clearly, a straight-line path L is an easier path.

From above, the starting and ending points of C are $(1, 0, 0)$ and $(1, 0, \pi)$ respectively. The straight-line L can be parametrized by:

$$\mathbf{r}_L(t) = \langle 1, 0, 0 \rangle + t(\langle 1, 0, \pi \rangle - \langle 1, 0, 0 \rangle) = \langle 1, 0, t\pi \rangle, \qquad 0 \le t \le 1.$$

$$\begin{aligned}
&\int_L \mathbf{F} \cdot d\mathbf{r} = \int_L \mathbf{F} \cdot \mathbf{r}_L'(t)\, dt \\
&= \int_0^1 \left((e^{-y} - ze^{-x})\mathbf{i} + (e^{-z} - xe^{-y})\mathbf{j} + (e^{-x} - ye^{-z})\mathbf{k} \right) \cdot \mathbf{r}_L'(t)\, dt \\
&= \int_0^1 \left((e^{-y} - ze^{-x})\mathbf{i} + (e^{-z} - xe^{-y})\mathbf{j} + (e^{-x} - ye^{-z})\mathbf{k} \right) \cdot (1\mathbf{i} + 0\mathbf{j} + \pi\mathbf{k})\, dt \\
&= \int_0^1 \pi(e^{-x} - ye^{-z})\, dt = \int_0^1 \pi(e^{-1} - 0)\, dt = \pi e^{-1}
\end{aligned}$$

Note that along the straight-line L, we have $x = 1$, $y = 0$ and $z = t\pi$.

Finally, since $\mathbf{F}$ is conservative by Problem #4(a), we have $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_L \mathbf{F} \cdot d\mathbf{r} = \pi e^{-1}$.

7. (★★) Given a conservative vector field $\mathbf{F}$ in $\mathbb{R}^3$, the potential *energy* of $\mathbf{F}$ is a scalar-valued function $V(x, y, z)$ such that $\mathbf{F} = -\nabla V$. Suppose $\mathbf{r}(t)$ is the path of a particle with mass m traveling in accordance to the Newton's Second Law $\mathbf{F}(\mathbf{r}(t)) = m\mathbf{r}''(t)$. Then its kinetic energy is defined to be:

$$\text{KE} = \frac{1}{2} m \left| \mathbf{r}'(t) \right|^2.$$

The total (kinetic + potential) energy of the particle at time t is therefore given by:

$$E(t) := \frac{1}{2} m \left| \mathbf{r}'(t) \right|^2 + V(\mathbf{r}(t)).$$

Show that the total energy is conserved, i.e. $E'(t) = 0$ for all time t.

[Hint: the only fact you need to know about Physics is the Newton's Second Law given above. It is purely a math problem.]

Solution: The key idea is to write $|\mathbf{r}'(t)|^2$ as $\mathbf{r}'(t) \cdot \mathbf{r}'(t)$. Also, it's essential to observe that along the path, the potential energy V is first of all a function of x, y and z, and (x, y, z) are functions of t. Therefore, one can use the chain rule to find $\frac{dV}{dt}$.

$$\begin{aligned}
\frac{dE}{dt} &= \frac{d}{dt}\left(\frac{1}{2}m\mathbf{r}'(t) \cdot \mathbf{r}'(t)\right) + \frac{dV}{dt} \\
&= \frac{1}{2}m \underbrace{\left(\mathbf{r}''(t) \cdot \mathbf{r}'(t) + \mathbf{r}'(t) \cdot \mathbf{r}''(t)\right)}_{\text{product rule}} + \underbrace{\left(\frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}\right)}_{\text{chain rule}} \\
&= m\mathbf{r}''(t) \cdot \mathbf{r}'(t) + \underbrace{\left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right) \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}\right)}_{\text{a good trick to learn}} \\
&= \mathbf{F} \cdot \mathbf{r}'(t) + \nabla V \cdot \mathbf{r}'(t) \\
&= -\nabla V \cdot \mathbf{r}'(t) + \nabla V \cdot \mathbf{r}'(t) \\
&= 0.
\end{aligned}$$

8. (★★) Denote $\mathbf{e}_\rho = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$ and $\mathbf{e}_r = \dfrac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}$, which are the unit radial vector fields in $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

   (a) Show that if $\mathbf{F}(x, y, z) = f(\rho)\mathbf{e}_\rho$ where f is a function depending only on $\rho = \sqrt{x^2 + y^2 + z^2}$, then $\nabla \times \mathbf{F} = \mathbf{0}$ on the domain of $\mathbf{F}$. Is this result alone sufficient to claim that $\mathbf{F}$ is conservative?

Solution: Using $\rho^2 = x^2 + y^2 + z^2$, one can differentiate both sides by x, y and z individually and show:

$$\frac{\partial \rho}{\partial x} = \frac{x}{\rho}, \quad \frac{\partial \rho}{\partial y} = \frac{y}{\rho}, \quad \frac{\partial \rho}{\partial z} = \frac{z}{\rho}.$$

Now consider the vector field $\mathbf{F} = f(\rho)\mathbf{e}_\rho$ whose components are given by:

$$\mathbf{F} = f(\rho)\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho} = \frac{f(\rho)}{\rho}x\mathbf{i} + \frac{f(\rho)}{\rho}y\mathbf{j} + \frac{f(\rho)}{\rho}z\mathbf{k}$$

Then,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{f(\rho)}{\rho}x & \frac{f(\rho)}{\rho}y & \frac{f(\rho)}{\rho}z \end{vmatrix}$$

We are going to compute the $\mathbf{i}$-component as an example (the $\mathbf{j}$- and $\mathbf{k}$-components are similar). The $\mathbf{i}$-component is given by:

$$\begin{aligned} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{f(\rho)}{\rho}y & \frac{f(\rho)}{\rho}z \end{vmatrix} \mathbf{i} &= \left\{ \frac{\partial}{\partial y}\left(\frac{f(\rho)}{\rho}z\right) - \frac{\partial}{\partial z}\left(\frac{f(\rho)}{\rho}y\right) \right\} \mathbf{i} \\ &= \left\{ \frac{d}{d\rho}\left(\frac{f(\rho)}{\rho}\right)\frac{\partial\rho}{\partial y}\cdot z - \frac{d}{d\rho}\left(\frac{f(\rho)}{\rho}\right)\frac{\partial\rho}{\partial z}\cdot y \right\} \mathbf{i}. \end{aligned}$$

Here we have used the chain rule on $\frac{\partial}{\partial y}\left(\frac{f(\rho)}{\rho}\right)$ and $\frac{\partial}{\partial z}\left(\frac{f(\rho)}{\rho}\right)$, as $\frac{f(\rho)}{\rho}$ is a function of ρ, and ρ is a function of (x, y, z). Note that:

$$\begin{aligned} &\frac{d}{d\rho}\left(\frac{f(\rho)}{\rho}\right)\frac{\partial\rho}{\partial y}\cdot z - \frac{d}{d\rho}\left(\frac{f(\rho)}{\rho}\right)\frac{\partial\rho}{\partial z}\cdot y \\ &= \frac{d}{d\rho}\left(\frac{f(\rho)}{\rho}\right)\cdot\frac{y}{\rho}\cdot z - \frac{d}{d\rho}\left(\frac{f(\rho)}{\rho}\right)\cdot\frac{z}{\rho}\cdot y \\ &= 0. \end{aligned}$$

Therefore, the $\mathbf{i}$-component is zero. Similarly one can also show that the $\mathbf{j}$- and $\mathbf{k}$-components are zero.

Assuming f is C^1 – it should have been stated in the problem, the domain of $\mathbf{F}$ is $\mathbb{R}^3\backslash\{(0,0,0)\}$ since $\mathbf{e}_\rho$ is undefined only at the origin. Therefore, the domain of $\mathbf{F}$ is simply-connected, and now that we showed $\nabla \times \mathbf{F} = \mathbf{0}$. By curl test, we conclude that $\mathbf{F}$ is conservative.

(b) Show that if $\mathbf{G}(x, y) = g(r)\mathbf{e}_r$ where g is a function depending only on $r = \sqrt{x^2 + y^2}$, then $\nabla \times \mathbf{G} = \mathbf{0}$ on the domain of $\mathbf{G}$. Is this result alone sufficient to claim that $\mathbf{G}$ is conservative?

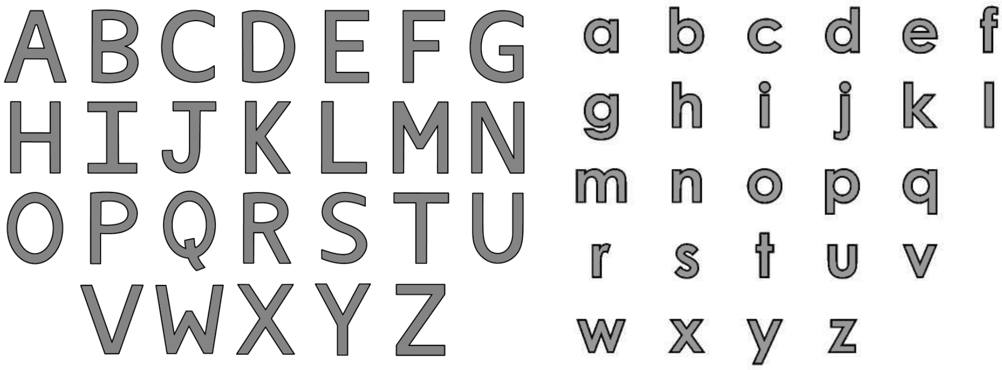
Solution: The way to show $\nabla \times \mathbf{G} = \mathbf{0}$ is very similar to part (a). Here we need to know:

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \text{and} \quad \mathbf{G} = \frac{g(r)}{r}(x\mathbf{i} + y\mathbf{j})$$

One can then calculate the curl of $\mathbf{G}$ using the chain rule.

However, this result alone cannot conclude whether $\mathbf{G}$ is conservative. Since $\mathbf{e}_r$ is undefined at $(0, 0)$ but $\mathbb{R}^2\backslash\{(0,0)\}$ is NOT simply-connected. The curl test does not apply here.

9. (★) Regard each English letter as a solid region in $\mathbb{R}^2$. Which capital letters are simply-connected? Which small letters are simply-connected?



Solution:

Simply-connected capital letters:	C E F G H I J K L M N S T U V W X Y Z
Simply-connected small letters:	c f h k l m n r s t u v w x y z

Note that the small i and j are not connected, and hence not simply-connected.

10. (★★) The notation $\mathbb{R}^3\backslash X$ means the xyz-space $\mathbb{R}^3$ with the set X removed. Determine whether $\mathbb{R}^3\backslash X$ is simply-connected when X is each of the following:

(a) X is the origin

(b) X is the entire y-axis

(c) X is the positive y-axis

(d) X is the solid sphere $x^2 + y^2 + z^2 \leq 1$

(e) X is the surface sphere $x^2 + y^2 + z^2 = 1$

(f) X is the solid cylinder $x^2 + y^2 \leq 1$

(g) X is the solid half-cylinder $x^2 + y^2 \leq 1$ and $z \geq 0$.

(h) X is the surface cylinder $x^2 + y^2 = 1$

(i) X is the surface half-cylinder $x^2 + y^2 = 1$ and $z \geq 0$

(j) X is a solid torus

(k) X is a surface torus

(l) X is a simple closed curve

Give an example of a proper subset X of $\mathbb{R}^3$ such that both X and $\mathbb{R}^3\backslash X$ are simply-connected. [Note: "proper" means X cannot be empty, and cannot be the whole $\mathbb{R}^3$.]

Solution: $\mathbb{R}^3\backslash X$ is simply-connected for those X's in: (a)(c)(d)(g)(i), whereas $\mathbb{R}^3\backslash X$ is not simply-connected for those X's in: (b)(e)(f)(h)(j)(k)(l).

Here is one of many examples of X so that both X and $\mathbb{R}^3\backslash X$ are both simply-connected: When X is the upper-half space $\{(x, y, z) : z > 0\}$. Then $\mathbb{R}^3\backslash X$ is the lower-half space.

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #8 • Green's Theorem

1. (★) Use the Green's Theorem to evaluate

$$\oint_C (4y^2 + e^{x^2})\,dx - (2x + e^{y^2})\,dy$$

where C is each of the following (assume C is counter-clockwise oriented):

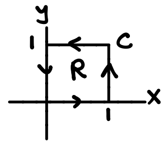
(a) the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$

Solution: The line integral is associated with the vector field $\mathbf{F} = (4y^2 + e^{x^2})\mathbf{i} - (2x + e^{y^2})\mathbf{j}$. By direct computation, we get:

$$\nabla \times \mathbf{F} = -2(1+4y)\mathbf{k} \Longrightarrow (\nabla \times \mathbf{F}) \cdot \mathbf{k} = -2(1+4y).$$

By Green's Theorem:

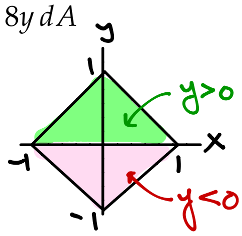
$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \int_0^1 (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dxdy \\ &= \int_0^1 \int_0^1 -2(1+4y)\,dxdy \\ &= -6\end{aligned}$$



(b) the square with vertices $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$

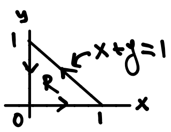
Solution: Denote the solid square by R, then by Green's Theorem:

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA \\ &= \iint_R -2(1+4y)\,dA = -\iint_R 2\,dA - \iint_R 8y\,dA \\ &= -2\,\text{Area}(R) - \iint_R 8y\,dA \\ &= -4 - \iint_R 8y\,dA\end{aligned}$$



Since y is an odd function and the region R is symmetric about the x-axis, we have $\iint_R 8y\,dA = 0$. Hence $\oint_C \mathbf{F} \cdot d\mathbf{r} = -4$.

(c) the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$



Solution: The hypotenuse of the triangle is given by equation $y = 1 - x$. Using the Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dydx = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} -2(1+4y)\,dydx = -\frac{7}{3}.$$

(d) the unit circle $x^2 + y^2 = 1$

Solution: The enclosed region is a unit circle. It is best to use polar coordinates.

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi}\int_0^1 (\nabla\times\mathbf{F})\cdot\mathbf{k}\, r dr d\theta \\ &= \int_0^{2\pi}\int_0^1 -2(1+4y)\, r dr d\theta \\ &= \int_0^{2\pi}\int_0^1 -2(1+4r\sin\theta) r\, dr d\theta \\ &= -2\pi.\end{aligned}$$

2. (★★) The purpose of this problem is to explore a line integral for computing areas.

(a) Let C be a simple closed curve in $\mathbb{R}^2$ and the area enclosed by C is denoted by A. Show that:

$$A = \frac{1}{2}\oint_C -y\,dx + x\,dy$$

Solution:

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\oint_C \langle -y, x, 0\rangle \cdot d\mathbf{r} = \frac{1}{2}\iint_R (\nabla\times\langle -y,x,0\rangle)\cdot\mathbf{k}\,dA.$$

Here R is the region enclosed by C. By direct computations, we get:

$$\nabla\times\langle -y,x,0\rangle = 2\mathbf{k}.$$

Hence

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\iint_R 2\,dA = \text{Area of } R.$$

(b) Let E be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a, b > 0$. Find the area bounded by E using the result of (a).

Solution: The ellipse can be parametrized by:

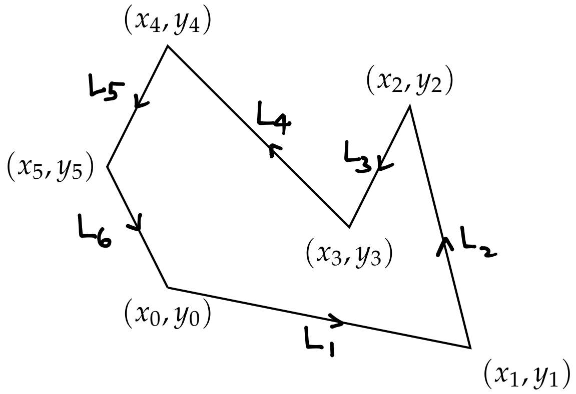
$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (b\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

In other words, $x = a\cos t$ and $y = b\sin t$.

$$\begin{aligned}A &= \frac{1}{2}\oint_C -y\,dx + x\,dy \\ &= \frac{1}{2}\int_{t=0}^{t=2\pi} -\underbrace{(b\sin t)}_{y}\, d(\underbrace{a\cos t}_{x}) + \underbrace{(a\cos t)}_{x}\, d(\underbrace{b\sin t}_{y}) \\ &= \frac{1}{2}\int_{t=0}^{t=2\pi} ab\sin^2 t\,dt + ab\cos^2 t\,dt \\ &= \frac{1}{2}\int_{t=0}^{t=2\pi} ab\,dt = ab\pi.\end{aligned}$$

(c) Let P be a n-sided polygon with vertices (x_0, y_0), (x_1, y_1), $\ldots$, (x_{n-1}, y_{n-1}). See the figure below for an example when $n = 6$. For convenience, we denote $(x_n, y_n) = (x_0, y_0)$. Using (a), show that the area $A(P)$ bounded by the polygon P is given by:

$$A(P) = \frac{1}{2}\sum_{i=1}^{n}(x_{i-1}y_i - x_i y_{i-1}).$$



Solution: Denote L_i to be the straight-line from (x_{i-1}, y_{i-1}) to (x_i, y_i), which is parametrized by:

$$\mathbf{r}_i(t) = \underbrace{\langle x_{i-1}, y_{i-1}\rangle}_{\text{starting point}} + t\underbrace{\langle x_i - x_{i-1}, y_i - y_{i-1}\rangle}_{\text{direction}}, \quad 0 \le t \le 1.$$

Then on L_i, we have $x = x_{i-1} + t(x_i - x_{i-1})$ and $y = y_{i-1} + t(y_i - y_{i-1})$, and so:

$$dx = (x_i - x_{i-1})\,dt \quad \text{and} \quad dy = (y_i - y_{i-1})\,dt.$$

The polygon can then be represented as the directed path $L_1 + L_2 + \ldots + L_n$, or simply $\sum_{i=1}^{n} L_i$. By (a), we have:

$$\begin{aligned}
A(P) &= \frac{1}{2}\oint_{\sum_{i=1}^{n} L_i} -y\,dx + x\,dy = \frac{1}{2}\sum_{i=1}^{n}\int_{L_i} -y\,dx + x\,dy \\
&= \frac{1}{2}\sum_{i=1}^{n}\int_{t=0}^{t=1} -\underbrace{(y_{i-1} + t(y_i - y_{i-1}))\,(x_i - x_{i-1})\,dt}_{y\,dx} \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\int_{t=0}^{t=1}\underbrace{(x_{i-1} + t(x_i - x_{i-1}))\,(y_i - y_{i-1})\,dt}_{x\,dy} \\
&= \frac{1}{2}\sum_{i=1}^{n}\left[-(x_i - x_{i-1})\left(y_{i-1}t + \frac{(y_i - y_{i-1})t^2}{2}\right)\right]_{t=0}^{t=1} \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\left[(y_i - y_{i-1})\left(x_{i-1}t + \frac{(x_i - x_{i-1})t^2}{2}\right)\right]_{t=0}^{t=1} \\
&= -\frac{1}{2}\sum_{i=1}^{n}(x_i - x_{i-1})\cdot\frac{y_i + y_{i-1}}{2} + \frac{1}{2}\sum_{i=1}^{n}(y_i - y_{i-1})\cdot\frac{x_i + x_{i-1}}{2}
\end{aligned}$$

which yields the desired result after simplifications.

3. (★★) Consider the following system of differential equations:

$$\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y)$$

where f and g are C^1 on $\mathbb{R}^2$. Given that $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} > 0$ on $\mathbb{R}^2$, show that the system cannot have a non-constant periodic solution. We say a solution $(x(t), y(t))$ is periodic if there exists $T > 0$ such that $(x(0), y(0)) = (x(T), y(T))$.

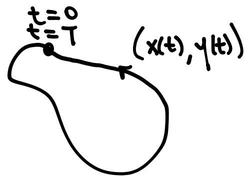
Hint: Proof by contradiction. Apply Green's Theorem on $\mathbf{F} = -g(x,y)\mathbf{i} + f(x,y)\mathbf{j}$.

Solution: Suppose the system has a non-constant periodic solution $(x(t), y(t))$. Let $T > 0$ be the first time such that $(x(T), y(T)) = (x(0), y(0))$, then the curve:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \quad 0 \le t \le T$$

is a simple closed curve in $\mathbb{R}^2$. Denote this simple closed curve by C and let R be the region enclosed by C. Apply Green's Theorem on the vector field $\mathbf{F} = -g(x,y)\mathbf{i} + f(x,y)\mathbf{j}$ over C:

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA \\ &= \iint_R \left(\frac{\partial f}{\partial x} - \left(-\frac{\partial g}{\partial y}\right)\right) \mathbf{k} \cdot \mathbf{k}\, dA \\ &= \iint_R \underbrace{\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)}_{\text{given}>0} dA > 0\end{aligned}$$



On the other hand, the line integral can be shown to be zero:

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_{t=0}^{t=T} \underbrace{\langle -g(x,y), f(x,y) \rangle}_{\mathbf{F}} \cdot \underbrace{\langle x'(t), y'(t) \rangle}_{\mathbf{r}'(t)}\, dt \\ &= \int_{t=0}^{t=T} -g(x,y)\, x'(t) + f(x,y)\, y'(t)\, dt.\end{aligned}$$

From the given differential equations, we have

$$-g(x,y)\, x'(t) + f(x,y)\, y'(t) = -g(x,y)\, f(x,y) + f(x,y)\, g(x,y) = 0.$$

Therefore, we have:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

which contradicts to the previous result, so the system cannot have non-constant periodic solution.

[FYI: This result is called the Bendixson-Dulac's Theorem. First established in 1901 by Ivar Bendixson. This short proof using Green's Theorem is later discovered by Henri Dulac in 1933.]

4. (★★) Consider the vector field $\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ which is defined at every point on $\mathbb{R}^2$ except the origin.

(a) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$ at every point in $\mathbb{R}^2$ except the origin.

Solution: Straight-forward.

(b) Show, by direct computation, that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ is non-zero where C is the unit circle, counter-clockwise oriented, with centered at the origin.

Solution: The unit circle is parametrized by $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$. Hence,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{t=2\pi} \underbrace{\left(-\frac{\sin t}{\cos^2 t + \sin^2 t}\mathbf{i} + \frac{\cos t}{\cos^2 t + \sin^2 t}\mathbf{j}\right)}_{\mathbf{F}} \cdot \underbrace{(-(\sin t)\mathbf{i} + (\cos t)\mathbf{j})}_{\mathbf{r}'(t)}\, dt$$

$$= \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\cos^2 t + \sin^2 t}\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

(c) The following students are confused about the above vector field $\mathbf{F}$ in relation to some facts and theorems stated in class. Pretend that you are a teaching assistant of this course, point out their misconceptions.

i. Student A said, "Given that $\nabla \times \mathbf{F} = \mathbf{0}$, the Curl Test asserts that $\mathbf{F}$ is conservative and so the closed-path line integral in (b) should be zero. How come the answer for (b) is non-zero???!!!"

Solution: The domain of $\mathbf{F}$ is $\mathbb{R}^2 \backslash \{(0,0)\}$ which is NOT simply-connected. The curl test cannot be used here.

ii. Student B said, "Given that $\nabla \times \mathbf{F} = \mathbf{0}$, the Green's Theorem asserts that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA = \iint_R \mathbf{0} \cdot \mathbf{k}\, dA = 0$$

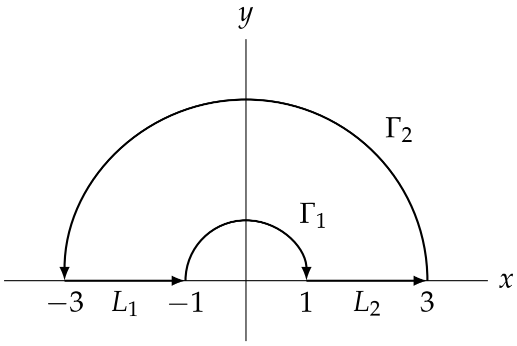
for any closed-path C. Why can the answer in (b) be non-zero???!!!" "

Solution: The unit circle C encloses the origin at which $\mathbf{F}$ is not defined. Green's Theorem cannot be used for this curve C.

iii. Student C said, "It can be verified that $\mathbf{F} = \nabla\left(\tan^{-1}\frac{y}{x}\right)$ and so $\mathbf{F}$ is conservative with potential function $f(x, y) = \tan^{-1}\frac{y}{x}$. Any line integral of a conservative vector field over a closed curve must be zero. How come can the closed-path integral in (b) be non-zero???!!!"

Solution: The domain of the potential function f needs to be the same as that of a vector field $\mathbf{F}$. In our case, the domain of $\mathbf{F}$ is $\mathbb{R}^2 \backslash \{(0,0)\}$ whereas the domain of $\tan^{-1}\frac{y}{x}$ is $\mathbb{R}^2 \backslash \{\text{y-axis}\}$. Hence, $\tan^{-1}\frac{y}{x}$ cannot be regarded as the (global) potential function of $\mathbf{F}$. We cannot show $\mathbf{F}$ is conservative in this way.

5. (★★) In the figure shown below, Γ_1 and Γ_2 are circular arcs centered at the origin. L_1 and L_2 are straight-lines. Consider the closed path $C = L_1 + \Gamma_1 + L_2 + \Gamma_2$.



Compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ of each vector field below using the Green's Theorem in an *appropriate* way:

(a) $\mathbf{F} = y^3\mathbf{i} - x^3\mathbf{j}$

Solution: The vector field is C^1 everywhere in $\mathbb{R}^2$. No problem to apply Green's Theorem. Direct computations show:

$$\nabla \times \mathbf{F} = -3(x^2 + y^2)\mathbf{k} \Longrightarrow (\nabla \times \mathbf{F}) \cdot \mathbf{k} = -3(x^2 + y^2).$$

By Green's Theorem:

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA \\ &= \int_0^{\pi} \int_1^3 -3(x^2 + y^2)\, r\, dr d\theta \\ &= -3 \int_0^{\pi} \int_1^3 r^3\, dr d\theta = -60\pi.\end{aligned}$$

(b) $\mathbf{F} = -\dfrac{y-3}{(x-3)^2 + (y-3)^2}\mathbf{i} + \dfrac{x-3}{(x-3)^2 + (y-3)^2}\mathbf{j}$

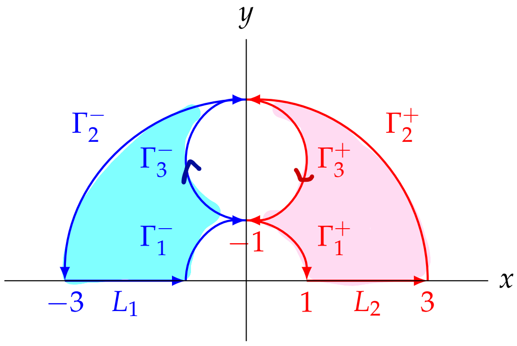
Solution: By somewhat lengthy computations, one can verify that $\nabla \times \mathbf{F} = \mathbf{0}$. The domain of $\mathbf{F}$ is $\mathbb{R}^2 \backslash \{(3,3)\}$. Fortunately, the closed path above does not enclose $(3,3)$ – no problem to use Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \underbrace{(\nabla \times \mathbf{F})}_{=\mathbf{0}} \cdot \mathbf{k}\, dA = 0$$

(c) $\mathbf{F} = -\dfrac{y-2}{x^2 + (y-2)^2}\mathbf{i} + \dfrac{x}{x^2 + (y-2)^2}\mathbf{j}$

Solution: By another somewhat lengthy computations, we get $\nabla \times \mathbf{F} = \mathbf{0}$. The domain of $\mathbf{F}$ is $\mathbb{R}^2 \backslash \{(0,2)\}$. However, the closed path C encloses this bad point $(0,2)$ – we can't use Green's Theorem directly.

To handle this path, we construct a "hole" with radius 1 centered at $(0,2)$. Denote the boundary of the hole by $\Gamma_3 = \Gamma_3^+ + \Gamma_3^-$ as shown in the figure below. Note that Γ_3 is a **clockwise** circle.



Consider the red and blue paths individually. Each of the red and blue path does not enclose the bad point $(0,2)$, we can apply Green's Theorem without problem:

$$\int_{\Gamma_2^+} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_3^+} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_1^+} \mathbf{F}\cdot d\mathbf{r} + \int_{L_2} \mathbf{F}\cdot d\mathbf{r} = \iint_{R^+} \underbrace{(\nabla\times\mathbf{F})}_{=\mathbf{0}}\cdot\mathbf{k}\,dA = 0$$

$$\int_{\Gamma_2^-} \mathbf{F}\cdot d\mathbf{r} + \int_{L_1} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_1^-} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_3^-} \mathbf{F}\cdot d\mathbf{r} = \iint_{R^-} \underbrace{(\nabla\times\mathbf{F})}_{=\mathbf{0}}\cdot\mathbf{k}\,dA = 0$$

Summing up and use the fact that $\Gamma_i = \Gamma_i^- + \Gamma_i^+$ (for $i = 1,2,3$), we get:

$$\underbrace{\int_{\Gamma_2} \mathbf{F}\cdot d\mathbf{r} + \int_{L_1} \mathbf{F}\cdot d\mathbf{r} + \int_{\Gamma_1} \mathbf{F}\cdot d\mathbf{r} + \int_{L_2} \mathbf{F}\cdot d\mathbf{r}}_{C=\Gamma_2+L_1+\Gamma_1+L_2} + \oint_{\Gamma_3} \mathbf{F}\cdot d\mathbf{r} = 0.$$

Hence,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} + \oint_{\Gamma_3} \mathbf{F}\cdot d\mathbf{r} = 0$$

To find $\oint_C \mathbf{F}\cdot d\mathbf{r}$, it suffices to find $\oint_{\Gamma_3} \mathbf{F}\cdot d\mathbf{r}$. Note that Γ_3 is **clockwise**, it is parametrized by:

$$\mathbf{r}(t) = (0+\cos(-t))\mathbf{i} + (2+\sin(-t))\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Then, $x = \cos t$, $y = 2 - \sin t$, and so $x^2 + (y-2)^2 = 1$.

$$\begin{aligned}\oint_{\Gamma_3} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \left(-\frac{(2-\sin t)-2}{1}\mathbf{i} + \frac{\cos t}{1}\mathbf{j}\right)\cdot((-\sin t)\mathbf{i} - (\cos t)\mathbf{j})\ dt \\ &= \int_0^{2\pi} (-\sin^2 - \cos^2 t)\,dt = -2\pi.\end{aligned}$$

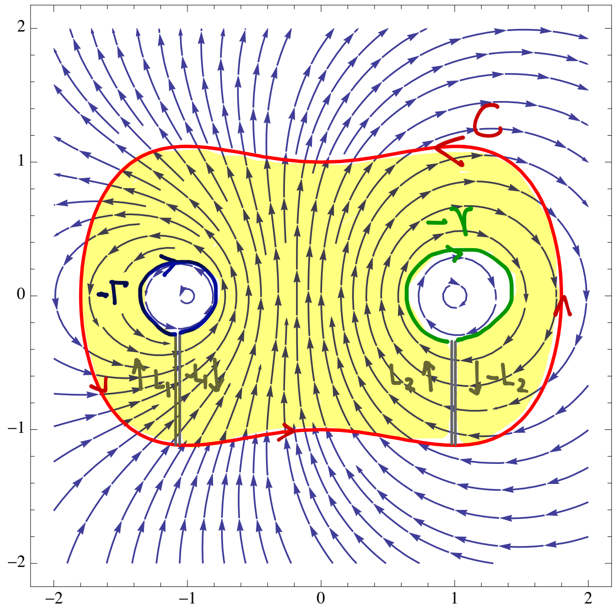
Therefore,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = -\int_{\Gamma_3} \mathbf{F}\cdot d\mathbf{r} = 2\pi.$$

6. (★★) Consider the flow of fluid (shown in blue in the figure below) which is represented by the vector field:

$$\mathbf{F} = \left(-\frac{y}{(x+1)^2+y^2} + \frac{2y}{(x-1)^2+y^2}\right)\mathbf{i} + \left(\frac{x+1}{(x+1)^2+y^2} - \frac{2(x-1)}{(x-1)^2+y^2}\right)\mathbf{j}$$

C is an arbitrary simple closed curve (red in the figure) which encloses all points at which $\mathbf{F}$ is not defined.



(a) At which point(s) the vector field $\mathbf{F}$ is/are *not* defined? Is the domain of $\mathbf{F}$ simply-connected?

Solution: $\mathbf{F}$ is NOT defined at $(-1,0)$ and $(1,0)$. The domain of $\mathbf{F}$ is

$$\mathbb{R}^2\backslash\{(-1,0),(1,0)\}$$

which is NOT simply-connected.

(b) Verify that $\nabla \times \mathbf{F} = \mathbf{0}$ at every point in $\mathbb{R}^2$ where $\mathbf{F}$ is defined.

Solution: Straight-forward, but quite lengthy.

(c) Show that from the definition of line integrals:

i. $\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 2\pi$ for any counter-clockwise circle Γ centered at $(-1,0)$ with radius less than 2.

Solution: Γ is parametrized by:

$$\mathbf{r}(t) = \langle -1+\varepsilon\cos t, \varepsilon\sin t\rangle, \quad 0 \le t \le 2\pi.$$

On this path, the vector field is given by:

$$\mathbf{F} = \left\langle -\frac{\varepsilon\sin t}{\varepsilon^2} + \frac{2\varepsilon\sin t}{(\varepsilon\cos t-2)^2+\varepsilon^2\sin^2 t}, \frac{\varepsilon\cos t}{\varepsilon^2} - \frac{2(\varepsilon\cos t-2)}{(\varepsilon\cos t-2)^2+\varepsilon^2\sin^2 t}\right\rangle.$$

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -\varepsilon \sin t,\, \varepsilon \cos t\rangle \\
\mathbf{F}\cdot\mathbf{r}'(t) &= 1 - \frac{2\varepsilon^2 \sin^2 t}{(\varepsilon\cos t - 2)^2 + \varepsilon^2 \sin^2 t} - \frac{2\varepsilon\cos t(\varepsilon\cos t - 2)}{(\varepsilon\cos t - 2)^2 + \varepsilon^2 \sin^2 t} \\
&= 1 - \frac{2\varepsilon^2 - 4\varepsilon\cos t}{(\varepsilon\cos t - 2)^2 + \varepsilon^2 \sin^2 t} \\
&= 1 - \frac{2\varepsilon^2 - 4\varepsilon\cos t}{\varepsilon^2 - 4\varepsilon\cos t + 4} \\
\oint_\Gamma \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} 1 - \frac{2\varepsilon^2 - 4\varepsilon\cos t}{\varepsilon^2 - 4\varepsilon\cos t + 4}\, dt \\
&= 2\pi.
\end{aligned}$$

Mathematica was used to compute this difficult integral.

ii. $\oint_\gamma \mathbf{F}\cdot d\mathbf{r} = -4\pi$ for any counter-clockwise circle γ centered at $(1,0)$ with radius less than 2.

Solution: Similar to (i). Parametrize the path by $\mathbf{r}(t) = \langle 1+\varepsilon\cos t, \varepsilon\sin t\rangle$.

(d) Using the above results, show that:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = -2\pi$$

for any simple closed curve C in $\mathbb{R}^2$ that encloses all points at which $\mathbf{F}$ is not defined.

Solution: C encloses points at which $\mathbf{F}$ is not defined. We need to drill two circular holes centered at $(-1,0)$ and $(1,0)$. Then, apply Green's Theorem on the closed path $C + L_1 - \Gamma - L_1 + L_2 - \gamma - L_2$, which does not enclose $(-1,0)$ and $(1,0)$, we get:

$$\begin{aligned}
&\oint_{C+L_1-\Gamma-L_1+L_2-\gamma-L_2} \mathbf{F}\cdot d\mathbf{r} \\
&= \oint_C \mathbf{F}\cdot d\mathbf{r} + \int_{L_1} \mathbf{F}\cdot d\mathbf{r} - \oint_\gamma \mathbf{F}\cdot d\mathbf{r} - \int_{L_1} \mathbf{F}\cdot d\mathbf{r} + \int_{L_2} \mathbf{F}\cdot d\mathbf{r} - \oint_\gamma \mathbf{F}\cdot d\mathbf{r} - \int_{L_2} \mathbf{F}\cdot d\mathbf{r} \\
&= \iint_R \underbrace{(\nabla\times\mathbf{F})}_{=\mathbf{0}} \cdot \mathbf{k}\, dA = 0
\end{aligned}$$

After cancellations, we get:

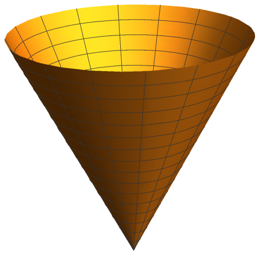
$$\oint_C \mathbf{F}\cdot d\mathbf{r} - \oint_\gamma \mathbf{F}\cdot d\mathbf{r} - \oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$$

From (c), we conclude that:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_\gamma \mathbf{F}\cdot d\mathbf{r} + \oint_\Gamma \mathbf{F}\cdot d\mathbf{r} = -4\pi + 2\pi = -2\pi.$$

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #9 • Surface Integrals, Stokes' Theorem

1. (★) Consider the right circular cone surface (just the *shell*, and the flat top is *not* included) with base radius R and height h, and with z-axis as the central axis and the origin as the vertex. See the figure below):



Suppose the cone has uniform surface density σ and its total mass is m.

(a) Write down a parametrization $\mathbf{r}(u, v)$ of the cone, and indicate the range of the parameters. It is OK to use other letters for the pair of parameters.

(b) Find the surface area of the cone.

(c) Find the moment of inertia $I_z := \iint_S (x^2 + y^2)\sigma\, dS$. about the z-axis. Express your final answer in terms of the mass m.

(d) Compute the surface flux of the vector field $\mathbf{F} = \mathbf{i}$ through the cone. Choose $\hat{\mathbf{n}}$ to be the upward unit normal. Do not use Stokes' Theorem in this problem.

Solution: There are at least two ways to parametrize the surface cone.

Using Cylindrical Coordinates: The cone surface is represented by $z = \frac{h}{R}r$ (use similar triangles to figure out the ratio of sides)

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \frac{h}{R}r\mathbf{k}, \quad 0 \le r \le R,\ 0 \le \theta \le 2\pi.$$

By direct computations (omitted here), one can get:

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \left(-\frac{h}{R}r\cos\theta\right)\mathbf{i} + \left(-\frac{h}{R}r\sin\theta\right)\mathbf{j} + r\mathbf{k}$$
$$\left|\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\right| = \sqrt{\frac{h^2r^2}{R^2} + r^2} = r\sqrt{1 + \frac{h^2}{R^2}}$$

Therefore,

$$\begin{aligned}\text{surface area} &= \iint_S dS = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=R}\left|\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\right| drd\theta \\ &= \int_0^{2\pi}\int_0^R r\sqrt{1+\frac{h^2}{R^2}}\, drd\theta = \pi R^2\sqrt{1+\frac{h^2}{R^2}}\end{aligned}$$

$$\begin{aligned}
I_z &= \iint_S \delta r^2 dS = \int_0^{2\pi} \int_{r=0}^{r=R} \delta r^3 \sqrt{1 + \frac{h^2}{R^2}} \, drd\theta \\
&= \delta \cdot \frac{\pi R^4}{2} \sqrt{1 + \frac{h^2}{R^2}} \\
&= \frac{m}{\pi R^2 \sqrt{1 + \frac{h^2}{R^2}}} \cdot \frac{\pi R^4}{2} \\
&= \frac{mR^2}{2}
\end{aligned}$$

$$\begin{aligned}
\iint_S \mathbf{i} \cdot \hat{\mathbf{n}} \, dS &= \int_0^{2\pi} \int_{r=0}^{r=R} \mathbf{i} \cdot \left(\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right) \, drd\theta \\
&= \int_0^{2\pi} \int_0^R -\frac{h}{R} r \cos\theta \, drd\theta \\
&= -\frac{h}{R} \left(\int_0^{2\pi} \cos\theta \, d\theta \right) \left(\int_0^R r \, dr \right) = 0
\end{aligned}$$

Using Spherical Coordinates: The surface cone is represented by the equation $\varphi = \tan^{-1} \frac{R}{h}$. For simplicity, denote $\alpha = \tan^{-1} \frac{R}{h}$, then the cone is represented by $\varphi = \alpha$ and so:

$$\mathbf{r}(\rho, \theta) = (\rho \sin\alpha \cos\theta)\mathbf{i} + (\rho \sin\alpha \sin\theta)\mathbf{j} + (\rho \cos\alpha)\mathbf{k}, \quad 0 \le \rho \le \sqrt{h^2 + R^2},\ 0 \le \theta \le 2\pi.$$

Direct computations give:

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \rho} \times \frac{\partial \mathbf{r}}{\partial \theta} &= (-\rho \sin\alpha \cos\alpha \cos\theta)\mathbf{i} + (-\rho \sin\alpha \cos\alpha \sin\theta)\mathbf{j} + (\rho \sin^2\alpha)\mathbf{k} \\
\left| \frac{\partial \mathbf{r}}{\partial \rho} \times \frac{\partial \mathbf{r}}{\partial \theta} \right| &= \rho \sin\alpha
\end{aligned}$$

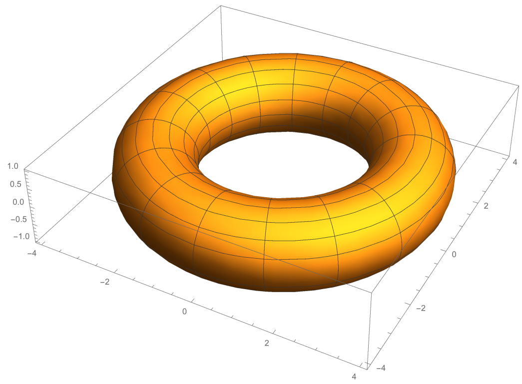
$$\begin{aligned}
\text{surface area} &= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=\sqrt{h^2+R^2}} \rho \sin\alpha d\rho d\theta \\
&= \pi(h^2 + R^2) \sin\alpha = \pi(h^2 + R^2) \sin(\tan^{-1} \frac{R}{h}) \\
&= \pi(h^2 + R^2) \cdot \frac{R}{\sqrt{h^2 + R^2}} = \pi R \sqrt{h^2 + R^2} \\
I_z &= \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=\sqrt{h^2+R^2}} \delta \rho^2 \sin^2\alpha \cdot \rho \sin\alpha \, d\rho d\theta \\
&= \delta \cdot \frac{(h^2 + R^2)^2}{4} \cdot 2\pi \sin^3\alpha \\
&= \frac{m}{\pi R \sqrt{h^2 + R^2}} \cdot \frac{(h^2 + R^2)^2}{4} \cdot 2\pi \cdot \left(\frac{R}{\sqrt{h^2 + R^2}} \right)^3 = \frac{mR^2}{2}
\end{aligned}$$

$$\begin{aligned}\iint_S \mathbf{i}\cdot\hat{\mathbf{n}}\,dS &= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{h^2+R^2}} \mathbf{i}\cdot\left(\frac{\partial\mathbf{r}}{\partial\rho}\times\frac{\partial\mathbf{r}}{\partial\theta}\right)\,d\rho d\theta \\ &= \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{h^2+R^2}} (-\rho\sin\alpha\cos\alpha\cos\theta)\,d\rho d\theta \\ &= -\left(\int_0^{2\pi}\cos\theta\,d\theta\right)\left(\int_0^{\sqrt{h^2+R^2}}\rho\,d\rho\right)\cdot\sin\alpha\cos\alpha = 0.\end{aligned}$$

2. (★) Consider the parametrization of a torus (i.e. donut):

$$\mathbf{r}(u,v) = ((R+a\cos u)\cos v)\,\mathbf{i} + ((R+a\cos u)\sin v)\,\mathbf{j} + (a\sin u)\,\mathbf{k}$$

where $0\le u\le 2\pi$ and $0\le v\le 2\pi$. Here R and a are constants such that $R>a>0$.



Suppose the torus has uniform surface density σ and its total mass is m.

(a) Find the surface area of the torus.

(b) Find the moment of inertia $I_z := \oiint_S (x^2+y^2)\sigma\,dS$ about the z-axis. Express your final answer in terms of m.

(c) Compute the surface flux of the vector field $\mathbf{F}=\mathbf{k}$ through the torus. Choose $\hat{\mathbf{n}}$ to be the outward unit normal. Do not use Stokes' Theorem in this problem.

Solution:

$$\begin{aligned}&\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\\ &= (-a(R+a\cos u)\cos u\cos v)\mathbf{i} + (-a(R+a\cos u)\cos u\sin v)\mathbf{j}\\ &\quad + (-a(R+a\cos u)\sin u)\mathbf{k}\end{aligned}$$

$$\left|\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v}\right| = a(R+a\cos u)$$

$$
\begin{aligned}
\text{surface area} &= \int_0^{2\pi}\int_0^{2\pi} \left|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right| \, dudv \\
&= \int_0^{2\pi}\int_0^{2\pi} a(R+a\cos u)\,dudv = 4\pi^2 aR \\
I_z &= \int_0^{2\pi}\int_0^{2\pi} \delta(R+a\cos u)^2\cdot a(R+a\cos u)\,dudv \\
&= \delta\cdot 2\pi^2 aR(3a^2+2R^2) \\
&= \frac{m}{4\pi^2 aR}\cdot 2\pi^2 aR(3a^2+2R^2) \\
&= \frac{m}{2}(3a^2+2R^2) \\
\iint_T \mathbf{k}\cdot\hat{\mathbf{n}}\,dS &= \int_0^{2\pi}\int_0^{2\pi} \mathbf{k}\cdot\left(\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right)\,dudv \\
&= \int_0^{2\pi}\int_0^{2\pi} -a(R+a\cos u)\sin u\,dudv = 0.
\end{aligned}
$$

3. (★★) In Chapter 2, we claimed without proof that $\nabla f(P)$ is perpendicular to the level surface $f=c$ at P (we proved the case of level *curves* only). In this problem, we are going to complete the proof for level surfaces.

Let $f(x,y,z)$ be a C^1 function, and S be the level surface $f(x,y,z)=c$. Consider a parametrization $\mathbf{r}(u,v)$ for S, then if one can show $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are both perpendicular to ∇f, then we are done because the normal vector $\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}$ to the surface S will then be parallel to ∇f. By considering $f(\mathbf{r}(u,v))=c$, show that $\nabla f\cdot\frac{\partial \mathbf{r}}{\partial u}=0$.

[The fact that $\nabla f\cdot\frac{\partial \mathbf{r}}{\partial v}=0$ can be shown in a similar way.]

Solution: Since $\mathbf{r}(u,v)=x(u,v)\mathbf{i}+y(u,v)\mathbf{j}+z(u,v)\mathbf{k}$ represents a point on the level surface $f(x,y,z)=c$, we have

$$f(x(u,v),y(u,v),z(u,v))=c \quad \text{or in short} \quad f(\mathbf{r}(u,v))=c.$$

By chain rule,

$$
\begin{aligned}
f(\mathbf{r}(u,v)) &= c \\
\Longrightarrow \frac{\partial}{\partial u} f(\mathbf{r}(u,v)) &= \frac{\partial c}{\partial u} \\
\Longrightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial u} &= 0 \\
\Longrightarrow \left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial x}{\partial u}\mathbf{i}+\frac{\partial y}{\partial u}\mathbf{j}+\frac{\partial z}{\partial u}\mathbf{k}\right) &= 0 \\
\Longrightarrow \nabla f\cdot\frac{\partial \mathbf{r}}{\partial u} &= 0
\end{aligned}
$$

Therefore, ∇f is perpendicular to the tangent vector $\frac{\partial \mathbf{r}}{\partial u}$. Similarly, one can show ∇f is perpendicular to the tangent vector $\frac{\partial \mathbf{r}}{\partial v}$. It concludes that ∇f is a normal vector to the level surface $f=c$.

4. (★) Suppose S is a level surface $f(x, y, z) = c$ of a C^1 function f. Show that:

$$\iint_S \nabla f \cdot \hat{\mathbf{n}}\, dS = \pm \iint_S |\nabla f|\, dS$$

where $\pm$ depends on the choice of unit normal $\hat{\mathbf{n}}$.

Solution: Since S is a level surface $f = c$, its unit normal vector is given by:

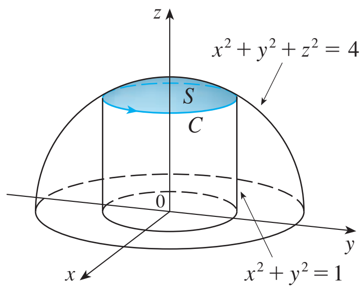
$$\hat{\mathbf{n}} = \pm \frac{\nabla f}{|\nabla f|}.$$

Therefore, we have:

$$\iint_S \nabla f \cdot \hat{\mathbf{n}}\, dS = \pm \iint_S \nabla f \cdot \frac{\nabla f}{|\nabla f|}\, dS = \pm \iint_S \frac{|\nabla f|^2}{|\nabla f|}\, dS = \pm \iint_S |\nabla f|\, dS.$$

Here we have used the fact that $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.

5. (★) Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. Denote C to be the boundary of S with orientation indicated in the diagram below:



(a) Write down the parametrizations of both the surface S and the curve C. For the surface S, choose a *suitable* coordinate system so that the parameters have constant bounds.

Solution:

$$\mathbf{r}(\phi, \theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}; \quad 0 \le \phi \le \frac{\pi}{6},\ 0 \le \theta \le 2\pi$$

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + \sqrt{3}\mathbf{k}; \quad 0 \le t \le 2\pi$$

(b) Consider the vector field $\mathbf{F}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$. Compute both $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$ directly. Verify that they are equal.

Solution: Along C, we have $x = \cos t$, $y = \sin t$ and $z = \sqrt{3}$.

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^{2\pi} \left(\sqrt{3}\cos t\,\mathbf{i} + \sqrt{3}\sin t\,\mathbf{j} + \cos t\,\sin t\,\mathbf{k}\right) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\, dt \\ &= \int_0^{2\pi} 0\, dt = 0.\end{aligned}$$

To compute $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$, we first compute:

$$\nabla \times \mathbf{F} = (x - y)\mathbf{i} + (x - y)\mathbf{j}.$$

Using the parametrization $\mathbf{r}(\phi, \theta)$, we get:

$$\begin{aligned}\hat{\mathbf{n}}\, dS &= \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta}\, d\phi d\theta \\ &= \left\{(4\sin^2\phi\cos\theta)\,\mathbf{i} + (4\sin^2\phi\sin\theta)\,\mathbf{j} + (4\sin\theta\cos\theta)\,\mathbf{k}\right\} d\phi d\theta\end{aligned}$$

Therefore,

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS &= \int_0^{2\pi}\int_0^{\pi} (x - y)(4\sin^2\phi)(\cos\theta + \sin\theta)\, d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} \underbrace{2\sin\phi(\cos\theta - \sin\theta)}_{x-y}\; 4\sin^2\phi\; (\cos\theta + \sin\theta)\, d\phi d\theta \\ &= 8\left(\int_0^{2\pi} (\cos^2\theta - \sin^2\theta)\, d\theta\right)\left(\int_0^{\pi} \sin^3\phi\, d\phi\right) \\ &= 0\end{aligned}$$

6. (★) Let C be the simple closed curve given parametrized by:

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin 2t)\mathbf{k}, \quad 0 \le t \le 2\pi.$$

(a) Show that the curve lies on the surface $z = 2xy$.

Solution: Use the identity $\sin 2t = 2\sin t\,\cos t$.

(b) Use the Stokes' Theorem to evaluate the line integral:

$$\oint_C e^{x^2} dx + yz dy + \frac{x^2}{2} dz.$$

[Why is it difficult to compute this line integral *directly*?]

Solution: The line integral can be written as $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F} = e^{x^2}\mathbf{i} + yz\mathbf{j} + \frac{x^2}{2}\mathbf{k}$$

By straight-forward computations, we get:

$$\nabla \times \mathbf{F} = -y\mathbf{i} - x\mathbf{j} + 0\mathbf{k}.$$

Let Σ be the surface $z = 2xy$ enclosed by C. It can be parametrized by cylindrical coordinates:

$$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \underbrace{(2r^2\sin\theta\cos\theta)}_{z=2xy}\mathbf{k}, \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

Using Stokes' Theorem, we have:

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_\Sigma (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS \\
&= \int_0^{2\pi}\int_0^1 \underbrace{(-y\mathbf{i} - x\mathbf{j})}_{\nabla\times\mathbf{F}} \cdot \underbrace{\frac{\nabla(z-2xy)}{|\nabla(z-2xy)|}}_{\hat{\mathbf{n}}} \underbrace{\left|\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\right| dr d\theta}_{dS} \\
&= \int_0^{2\pi}\int_0^1 (-y\mathbf{i} - x\mathbf{j}) \cdot \frac{-2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}}{\sqrt{1+4x^2+4y^2}}\, r\sqrt{1+4r^2}\, dr d\theta \\
&= \int_0^{2\pi}\int_0^1 2(x^2+y^2)\, \frac{1}{\sqrt{1+4r^2}} \cdot r \cdot \sqrt{1+4r^2}\, dr d\theta \\
&= \int_0^{2\pi}\int_0^1 2r^3\, dr d\theta = \pi.
\end{aligned}$$

7. (★) Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$. Show that the line integral

$$\oint_C 2y dx + 3z dy - x dz$$

depends only on the area of the region enclosed by C on the above given plane and the orientation of C, but not on the position or shape of C.

Solution: The line integral can be written as $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F} = 2y\mathbf{i} + 3z\mathbf{j} - x\mathbf{k}.$$

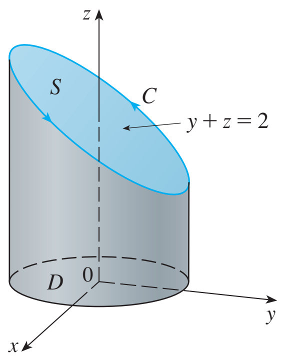
By calculation, we get: $\nabla \times \mathbf{F} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Let Σ be the region enclosed by C on the plane $2x + 2y + z = 2$, then the unit normal to the plane is given by:

$$\hat{\mathbf{n}} = \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{9}}$$

Stokes' Theorem asserts that:

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \pm \iint_\Sigma (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS \qquad (\pm \text{ depends on orientation of } C) \\
&= \pm \iint_\Sigma (-3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \cdot \frac{2\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{9}}\, dS \\
&= \mp \frac{6}{\sqrt{9}} \iint_\Sigma 1\, dS = \mp 2 \times \text{area of } \Sigma
\end{aligned}$$

8. (★★) Consider the curve of intersection C of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$, with orientation shown in the diagram below. The surface S is the planar region enclosed by C, and its projection onto the xy-plane is denoted by D.



(a) Using a suitable coordinate system, write down a parametrization of S such that the parameters have constant bounds.

Solution: Use cylindrical coordinates:

$$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \underbrace{(2 - r\sin\theta)}_{z=2-y}\mathbf{k}$$

where $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

(b) Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}$ is given by:

$$\mathbf{F}(x,y,z) = -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$$

Solution:

$$\nabla \times \mathbf{F} = (1 + 2y)\mathbf{k}$$

Using Stokes' Theorem (note that $\mathbf{F}$ is defined and C^1 everywhere in $\mathbb{R}^3$), we get:

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS \\
&= \int_0^{2\pi}\int_0^1 (1 + 2y)\mathbf{k} \cdot \frac{\mathbf{j} + \mathbf{k}}{\sqrt{2}} \left|\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta}\right| dr d\theta \\
&= \int_0^{2\pi}\int_0^1 \frac{1 + 2r\sin\theta}{\sqrt{2}} \left|r\mathbf{j} + r\mathbf{k}\right| dr d\theta \\
&= \int_0^{2\pi}\int_0^1 \frac{1 + 2r\sin\theta}{\sqrt{2}} \sqrt{2}r\, dr d\theta \\
&= \int_0^{2\pi}\int_0^1 (r + 2r^2\sin\theta)\, dr d\theta \\
&= \pi
\end{aligned}$$

(c) Let $\mathbf{G}(x,y,z) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} + \frac{1}{z+1}\mathbf{k}$.

i. Verify that $\nabla \times \mathbf{G} = \mathbf{0}$ at every point in the domain of $\mathbf{G}$. Does this result determine that $\mathbf{G}$ is conservative or not?

Solution: Showing $\nabla \times \mathbf{G} = \mathbf{0}$ is straight-forward. Note that the domain of $\mathbf{G}$ is $\mathbb{R}^3$ removing the z-axis and the horizontal plane $z = -1$. It is not a simply-connected domain. We cannot conclude that $\mathbf{G}$ is conservative or not based the result $\nabla \times \mathbf{G} = \mathbf{0}$.

ii. Denote Γ to be the projection of C onto the xy-plane. Using the Stokes' Theorem in an *appropriate* way, show that:

$$\oint_C \mathbf{G} \cdot d\mathbf{r} = \oint_\Gamma \mathbf{G} \cdot d\mathbf{r}.$$

Solution: Similar to Worksheet #22, Q3.

iii. Evaluate $\oint_C \mathbf{G} \cdot d\mathbf{r}$ using the above result.

Solution: Γ can be parametrized by $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + 0\mathbf{k}$, where $0 \le t \le 2\pi$. We can compute the line integral $\oint_\Gamma \mathbf{G} \cdot d\mathbf{r}$ directly.

$$\begin{aligned}
\oint_\Gamma \mathbf{G} \cdot d\mathbf{r} &= \int_0^{2\pi} \left(-\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} + \frac{1}{z+1}\mathbf{k}\right) \cdot \underbrace{((-\sin t)\mathbf{i} + (\cos t)\mathbf{j})}_{\mathbf{r}'(t)}\,dt \\
&= \int_0^{2\pi} \left(-\frac{\sin t}{1}\mathbf{i} + \frac{\cos t}{1}\mathbf{j} + \frac{1}{0+1}\mathbf{k}\right) \cdot (-(\sin t)\mathbf{i} + (\cos t)\mathbf{j})\,dt \\
&= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = 2\pi.
\end{aligned}$$

9. (★) Two of the four Maxwell's Equations (Faraday's and Ampère's Laws) assert that:

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the current, and c, μ_0 and ε_0 are positive constants. Using Stokes' Theorem, show that for any (stationary) simply-connected orientable surface S with boundary C, we have:

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{1}{c}\frac{\partial}{\partial t}\iint_S \mathbf{B} \cdot \hat{\mathbf{n}}\,dS$$
$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 \iint_S \mathbf{J} \cdot \hat{\mathbf{n}}\,dS + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\iint_S \mathbf{E} \cdot \hat{\mathbf{n}}\,dS$$

[You don't need to know any physics to do this problem.]

Solution:

$$\begin{aligned} -\frac{1}{c}\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot\hat{\mathbf{n}}\,dS &= -\frac{1}{c}\iint_S \frac{\partial}{\partial t}\mathbf{B}\cdot\hat{\mathbf{n}}\,dS \\ &= \iint_S (\nabla\times\mathbf{E})\cdot\hat{\mathbf{n}}\,dS && \text{(Given)} \\ &= \oint_C \mathbf{E}\cdot d\mathbf{r} && \text{(Stokes' Theorem)} \end{aligned}$$

Similarly,

$$\begin{aligned} \mu_0\iint_S \mathbf{J}\cdot\hat{\mathbf{n}}\,dS + \mu_0\varepsilon_0\frac{\partial}{\partial t}\iint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,dS &= \iint_S \left(\mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\cdot\hat{\mathbf{n}}\,dS \\ &= \iint_S (\nabla\times\mathbf{B})\cdot\hat{\mathbf{n}}\,dS \\ &= \oint_C \mathbf{B}\cdot d\mathbf{r} \end{aligned}$$

10. (★★) Let $\mathbf{F}(x,y,z) = \langle 0, -\frac{z}{2}, \frac{y}{2}\rangle$.

(a) Show that $\nabla\times\mathbf{F} = \mathbf{i}$.

Solution: Straight-forward

(b) Find vector fields $\mathbf{G}$ and $\mathbf{H}$ such that $\nabla\times\mathbf{G} = \mathbf{j}$ and $\nabla\times\mathbf{H} = \mathbf{k}$.

Solution: Mimic the vector field $\mathbf{F}$ in part (a), and some trial-and-errors:

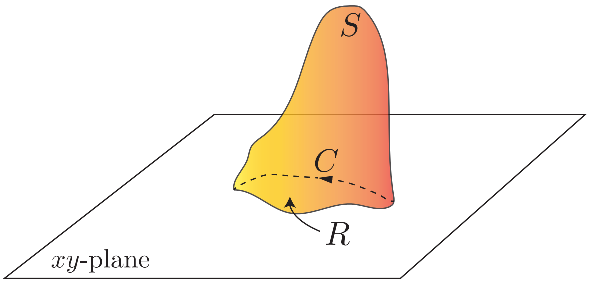
$$\begin{aligned} \nabla\times\left\langle \frac{z}{2}, 0, -\frac{x}{2}\right\rangle &= \mathbf{j} \\ \nabla\times\left\langle -\frac{y}{2}, \frac{x}{2}, 0\right\rangle &= \mathbf{k} \end{aligned}$$

Let $\mathbf{G} = \langle \frac{z}{2}, 0, -\frac{x}{2}\rangle$ and $\mathbf{H} = \langle -\frac{y}{2}, \frac{x}{2}, 0\rangle$.
[Note that the problem only requires us to find one such $\mathbf{G}$ and one such $\mathbf{H}$. In fact it can be shown that all other possible $\mathbf{G}$'s are given by:

$$\left\langle \frac{z}{2}, 0, -\frac{x}{2}\right\rangle + \nabla f$$

where f is any C^2 scalar functions defined everywhere in $\mathbb{R}^3$. Similar for $\mathbf{H}$.]

(c) Let C be an arbitrary simple closed curve on the xy-plane in the three dimensional space, and S is any surface *above* the xy-plane with boundary curve C. See the figure below.



Show that:

$$\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = c \times \text{area of the region on the } xy\text{-plane enclosed by } C.$$

Here a, b and c are all constants.

Solution: Using (a) and (b), we get:

$$a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = a(\nabla \times \mathbf{F}) + b(\nabla \times \mathbf{G}) + c(\nabla \times \mathbf{H}) = \nabla \times (a\mathbf{F} + b\mathbf{G} + c\mathbf{H}).$$

Using Stokes' Theorem, we get:

$$\begin{aligned}
\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS &= \iint_S (\nabla \times (a\mathbf{F} + b\mathbf{G} + c\mathbf{H})) \cdot \hat{\mathbf{n}}\, dS \\
&= \oint_C (a\mathbf{F} + b\mathbf{G} + c\mathbf{H}) \cdot d\mathbf{r} && \text{(Stokes')} \\
&= \iint_R (\nabla \times (a\mathbf{F} + b\mathbf{G} + c\mathbf{H})) \cdot \mathbf{k}\, dA && \text{(Green's)} \\
&= \iint_R (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \mathbf{k}\, dA \\
&= \iint_R c\, dA = c \times \text{area of } R
\end{aligned}$$

(d) Using the results of (a), (b), and the Stokes' Theorem, redo Problems #1(d) and #2(c).

Solution: For #1(d), use the fact that $\nabla \times \left\langle 0, -\frac{z}{2}, \frac{y}{2} \right\rangle = \mathbf{i}$ and Stokes' Theorem:

$$\begin{aligned}
\iint_{\text{cone}} \mathbf{i} \cdot \hat{\mathbf{n}}\, dS &= \iint_{\text{cone}} \left(\nabla \times \left\langle 0, -\frac{z}{2}, \frac{y}{2} \right\rangle\right) \cdot \hat{\mathbf{n}}\, dS \\
&= \oint_{\text{boundary circle}} \left\langle 0, -\frac{z}{2}, \frac{y}{2} \right\rangle \cdot d\mathbf{r} && \text{(Stokes')}
\end{aligned}$$

The boundary circle of the cone is parametrized by:

$$\mathbf{r}(t) = (R\cos t)\mathbf{i} + (R\sin t)\mathbf{j} + h\mathbf{k}, \quad 0 \le t \le 2\pi.$$

Therefore,

$$\oint_{\text{boundary circle}} \left\langle 0, -\frac{z}{2}, \frac{y}{2} \right\rangle \cdot d\mathbf{r} = \int_0^{2\pi} \left\langle 0, -\frac{h}{2}, \frac{R}{2}\sin t \right\rangle \cdot \langle -R\sin t, R\cos t, 0\rangle \, dt$$

$$= \int_0^{2\pi} -\frac{hR}{2}\cos t \, dt = 0.$$

#2(c) is to calculate the flux of $\mathbf{k}$ through the surface torus. From part (c) we know that $\mathbf{k} = \nabla \times \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle$. Therefore, $\mathbf{k}$ is a solenoidal vector field. From Worksheet #22 Q2 we can use the Stokes' Theorem (after cutting the torus along two circles so that it becomes simply-connected) to conclude that the surface flux is zero.

Optional – about the Gauss-Bonnet's Theorem

11. Given a oriented surface S with parametrization $\mathbf{r}(u, v)$, we denote:

$$E = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u} \qquad F = \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \qquad G = \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial v}$$

$$e = \frac{\partial^2 \mathbf{r}}{\partial u^2} \cdot \hat{\mathbf{n}} \qquad f = \frac{\partial^2 \mathbf{r}}{\partial u \partial v} \cdot \hat{\mathbf{n}} \qquad g = \frac{\partial^2 \mathbf{r}}{\partial v^2} \cdot \hat{\mathbf{n}}$$

The *Gauss curvature* at the point $\mathbf{r}(u, v)$ is defined to be:

$$K(u, v) := \frac{eg - f^2}{EG - F^2}.$$

The geometric intuition behind the Gauss curvature *may* be covered in MATH 4223. In Differential Geometry, there is a beautiful theorem – the Gauss-Bonnet's Theorem – which asserts that if S is closed, oriented and smooth (without corners), then:

$$\oiint_S K \, dS = 4\pi(1 - \text{ number of holes of } S)$$

Therefore, if S is a sphere, the above surface integral should be 4π as there is no hole. If S is a torus (which has one hole), the above surface integral should be 0. Verify this theorem for the sphere and torus, by parametrizing them and compute the above integral directly over the sphere and the torus.

As an optional problem, you may use computer softwares to ease your calculations.

MATH 2023 • Spring 2015-16 • Multivariable Calculus
Problem Set #10 • Divergence Theorem

1. (★) Use the Divergence Theorem to find the outward flux $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}} \, dS$ for each of the following $\mathbf{F}$ and S:

(a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the surface of any square cube of length b.

Solution: It is easy to see that $\nabla \cdot \mathbf{F} = 3$. The solid D enclosed by S is the solid square cube of length b. Divergence Theorem shows:

$$\begin{aligned} \oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}} \, dS &= \iiint_D \nabla \cdot \mathbf{F} \, dV \\ &= \iiint_D 3 \, dV = 3 \times \text{volume of } D \\ &= 3b^2. \end{aligned}$$

(b) $\mathbf{F} = x^3\mathbf{i} + 3yz^2\mathbf{j} + (3y^2z + x^2)\mathbf{k}$ and S is the sphere with radius $a > 0$ centered at the origin.

Solution: $\nabla \cdot \mathbf{F} = 3(x^2 + y^2 + z^2) = 3\rho^2$. The solid D enclosed by S is the solid sphere with radius a centered at the origin, i.e. $D = \{\rho \leq a\}$.

$$\begin{aligned} \oiint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS &= \iiint_D \nabla \cdot \mathbf{F} \, dV \\ &= \int_0^{2\pi} \int_0^{\pi} \int_0^{a} 3\rho^2 \cdot \rho^2 \sin\phi \, d\rho d\phi d\theta \\ &= \frac{12\pi a^5}{5}. \end{aligned}$$

(c) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S is the boundary surface of the cylinder D defined by $x^2 + y^2 \leq 1$ and $0 \leq z \leq 4$.

Solution: $\nabla \cdot \mathbf{F} = 2(x + y + z)$. The solid D is described by inequalities $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 4$ in cylindrical coordinates:

$$\begin{aligned} \oiint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS &= \int_0^{2\pi} \int_0^{1} \int_0^{4} \nabla \cdot \mathbf{F} \, r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^{1} \int_0^{4} 2(r\cos\theta + r\sin\theta + z) \, r dz dr d\theta \\ &= 16\pi \end{aligned}$$

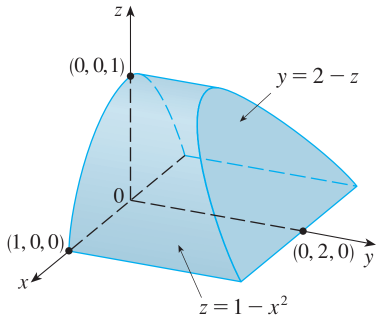
Remark: To simplify the computations, it is good to keep in mind that:

$$\int_0^{2\pi} \cos\theta \, d\theta = \int_0^{2\pi} \sin\theta \, d\theta = 0.$$

2. (★) Evaluate $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}} \, dS$ where

$$\mathbf{F} = xy\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\mathbf{k}$$

and S is the surface boundary of the region D defined by $z \leq 1 - x^2$, $z \geq 0$, $y \geq 0$ and $y \leq 2 - z$. See the figure below:



Comment on why it is preferable to use the Divergence Theorem instead of computing the surface flux directly.

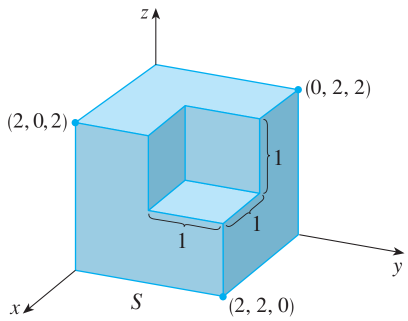
Solution:

$$\nabla \cdot \mathbf{F} = 3y$$

$$\begin{aligned} \oiint_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dS &= \iiint_D \nabla \cdot \mathbf{F} \, dV \\ &= \int_{x=-1}^{x=1} \int_{z=0}^{z=1-x^2} \int_{y=0}^{y=2-z} 3y \, dy dz dx \\ &= \frac{184}{35} \end{aligned}$$

Easier to use Divergence Theorem as the surface S has 4 faces. To compute the surface flux directly we would need to split the surface flux into 4 parts and parametrize them individually.

3. (★) Let D be the solid square cube of length 2 with one corner unit cube removed. See the figure below.



Evaluate the outward flux $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}}\, dS$ where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Comment on why it is preferable to use the Divergence Theorem instead of computing the flux directly.

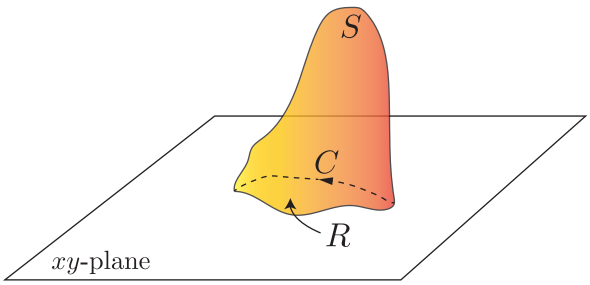
Solution:

$$\nabla \cdot \mathbf{F} = 3$$

$$\begin{aligned} \oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \iiint_D \nabla \cdot \mathbf{F}\, dV \\ &= \iiint_D 3\, dV = 3 \times \text{volume of } D \\ &= 3(2^3 - 1^3) = 21. \end{aligned}$$

The surface S has 9 faces!!! Without the Divergence Theorem, we will need to compute the surface flux by split it into 9 parts!

4. (★★) Let C be an arbitrary simple closed curve on the xy-plane in the three dimensional space, and S is any surface *above* the xy-plane with boundary curve C. See the figure below.



Using the Divergence Theorem, show that:

$$\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\; dS = c \times \text{area of the region on the } xy\text{-plane enclosed by } C.$$

Here a, b and c are all constants.

Solution:

$$\nabla \cdot (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial c}{\partial z} = 0 + 0 + 0 = 0.$$

However, note that S is not a closed surface, but $S \cup R$ is closed. Apply the Divergence Theorem on $S \cup R$ instead:

$$\oiint_{S \cup R} (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = \iiint_{\text{solid enclosed}} \underbrace{\nabla \cdot \mathbf{F}}_{0}\, dV = 0.$$

$$\iint_R (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = \iint_R (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot (-\mathbf{k})\, dS = -\iiint_R c\, dS = -c \times \text{area}(R)$$

Since:

$$\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS + \iint_R (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S \cup R} (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = 0$$

we conclude that:

$$\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = -\iint_R (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = c \times \text{area of } R.$$

5. (★★) Suppose $f(x,y,z)$ is a C^2 function on $\mathbb{R}^3$ such that $\nabla^2 f(x,y,z) = 0$ on $\mathbb{R}^3$. Here $\nabla^2 f$ means the Laplacian of f, i.e. $\nabla^2 f = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}$.

(a) Show that:

$$\oiint_S f \nabla f \cdot \hat{\mathbf{n}}\, dS = \iiint_D |\nabla f|^2\, dV$$

for any closed oriented surface S enclosing the solid region D.

Solution:

$$\begin{aligned}\oiint_S f \nabla f \cdot \hat{\mathbf{n}}\, dS &= \iiint_D \nabla \cdot (f \nabla f)\, dV \\ &= \iiint_D (\nabla f \cdot \nabla f) + f \nabla \cdot \nabla f\, dV \\ &= \iiint_D |\nabla f|^2 + f \underbrace{\nabla^2 f}_{=0}\, dV \\ &= \iiint_D |\nabla f|^2\, dV\end{aligned}$$

(b) If, furthermore, assume that $f(x,y,z) = 0$ for any (x,y,z) on S, what can you say about $f(x,y,z)$ for any (x,y,z) in D?

Solution: If $f = 0$ on S, then the surface integral:

$$\oiint_S f \nabla f \cdot \hat{\mathbf{n}}\, dS = \oiint_S 0\, \nabla f \cdot \hat{\mathbf{n}}\, dS = 0.$$

Then from (a), we get:

$$\iiint_D |\nabla f|^2\, dV = 0$$

Since $|\nabla f|^2 \geq 0$, the only chance that the above integral is zero is that $\nabla f = \mathbf{0}$ at every point in D. This means f is a constant function is D. By continuity, this constant must match with the value of f on the boundary S, hence $f \equiv 0$ in D.

6. (★★) Suppose S is a closed oriented level surface $f(x,y,z) = c$ of a C^2 function f. Denote D to be the solid enclosed by S. Show that:

$$\oiint_S |\nabla f| \, dS = \pm \iiint_D \nabla^2 f \, dV$$

where $\pm$ depends on whether ∇f points inward or outward on the surface S.

Solution: Note that S is the level surface $f = c$. Hence $\hat{\mathbf{n}} = \pm \frac{\nabla f}{|\nabla f|}$.

$$\begin{aligned}
\iiint_D \nabla^2 f \, dV &= \iiint_D \nabla \cdot \nabla f \, dV \\
&= \oiint_S \nabla f \cdot \hat{\mathbf{n}} \, dS \\
&= \pm \oiint_S \nabla f \cdot \frac{\nabla f}{|\nabla f|} \, dS \\
&= \pm \oiint_S \frac{|\nabla f|^2}{|\nabla f|} \, dS = \pm \oiint_S |\nabla f| \, dS
\end{aligned}$$

7. (★★) Given two C^2 functions $u(x,y,z)$ and $v(x,y,z)$ defined on $\mathbb{R}^3$. Let S be a closed oriented surface and D is the solid enclosed by S.

(a) Rewrite $\nabla \cdot (u\nabla v - v\nabla u)$ using **curl**, **grad** and **div**.

Solution:

$$\operatorname{div}(u \operatorname{grad}(v) - v \operatorname{grad}(u)).$$

(b) Show that

$$\oiint_S (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}} \, dS = \iiint_D \left(u\nabla^2 v - v\nabla^2 u\right) \, dV$$

Solution:

$$\begin{aligned}
\oiint_S (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}} \, dS &= \iiint_D \nabla \cdot (u\nabla v - v\nabla u) \, dV \\
&= \iiint_D (\nabla u \cdot \nabla v + u\nabla \cdot \nabla v - \nabla v \cdot \nabla u - v\nabla \cdot \nabla u) \, dV \\
&= \iiint_D (u\nabla^2 v - v\nabla^2 u) \, dV
\end{aligned}$$

(c) Assume further that $\nabla u(x,y,z) \cdot \hat{\mathbf{n}} = \nabla v(x,y,z) \cdot \hat{\mathbf{n}} = 0$ for any (x,y,z) on S, show that

$$\iiint_D u\nabla^2 v \, dV = \iiint_D v\nabla^2 u \, dV.$$

Solution: Simply apply the result of (b) using the given conditions that $\nabla u \cdot \hat{\mathbf{n}} = \nabla v \cdot \hat{\mathbf{n}} = 0$ on S.