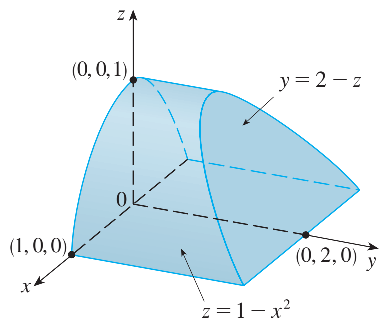
MATH 2023 • Multivariable Calculus
Problem Set #10 • Divergence Theorem

1. (★) Use the Divergence Theorem to find the outward flux $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}} \, dS$ for each of the following $\mathbf{F}$ and S:
   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the surface of any square cube of length b.
   (b) $\mathbf{F} = x^3\mathbf{i} + 3yz^2\mathbf{j} + (3y^2z + x^2)\mathbf{k}$ and S is the sphere with radius $a > 0$ centered at the origin.
   (c) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S is the boundary surface of the cylinder D defined by $x^2 + y^2 \leq 1$ and $0 \leq z \leq 4$.

2. (★) Evaluate $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}} \, dS$ where

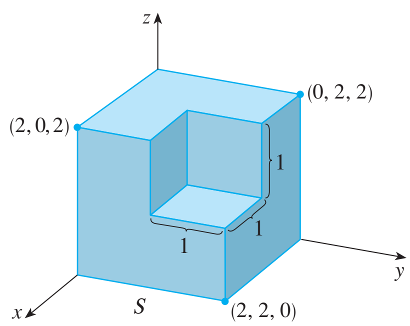
$$\mathbf{F} = xy\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\mathbf{k}$$

   and S is the surface boundary of the region D defined by $z \leq 1 - x^2$, $z \geq 0$, $y \geq 0$ and $y \leq 2 - z$. See the figure below:



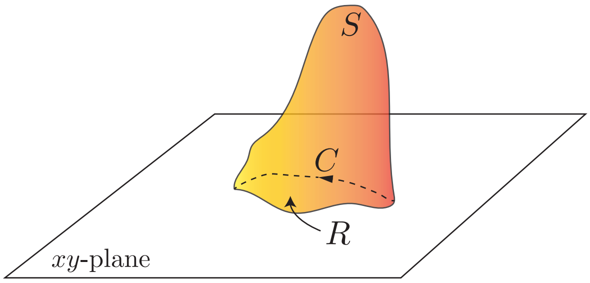
   Comment on why it is preferable to use the Divergence Theorem instead of computing the surface flux directly.

3. (★) Let D be the solid square cube of length 2 with one corner unit cube removed. See the figure below.



   Evaluate the outward flux $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}} \, dS$ where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Comment on why it is preferable to use the Divergence Theorem instead of computing the flux directly.

4. (★★) Let C be an arbitrary simple closed curve on the xy-plane in the three dimensional space, and S is any surface *above* the xy-plane with boundary curve C. See the figure below.



Using the Divergence Theorem, show that:

$$\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = c \times \text{area of the region on the } xy\text{-plane enclosed by } C.$$

Here a, b and c are all constants.

5. (★★) Suppose $f(x,y,z)$ is a C^2 function on $\mathbb{R}^3$ such that $\nabla^2 f(x,y,z) = 0$ on $\mathbb{R}^3$. Here $\nabla^2 f$ means the Laplacian of f, i.e. $\nabla^2 f = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}$.
   (a) Show that:
   $$\oiint_S f\nabla f \cdot \hat{\mathbf{n}}\, dS = \iiint_D |\nabla f|^2\, dV$$
   for any closed oriented surface S enclosing the solid region D.
   (b) If, furthermore, assume that $f(x,y,z) = 0$ for any (x,y,z) on S, what can you say about $f(x,y,z)$ for any (x,y,z) in D?

6. (★★) Suppose S is a closed oriented level surface $f(x,y,z) = c$ of a C^2 function f. Denote D to be the solid enclosed by S. Show that:
$$\oiint_S |\nabla f|\, dS = \pm \iiint_D \nabla^2 f\, dV$$
where $\pm$ depends on whether ∇f points inward or outward on the surface S.

7. (★★) Given two C^2 functions $u(x,y,z)$ and $v(x,y,z)$ defined on $\mathbb{R}^3$. Let S be a closed oriented surface and D is the solid enclosed by S.
   (a) Rewrite $\nabla \cdot (u\nabla v - v\nabla u)$ using **curl**, **grad** and **div**.
   (b) Show that
   $$\oiint_S (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}}\, dS = \iiint_D \left(u\nabla^2 v - v\nabla^2 u\right) dV$$
   (c) Assume further that $\nabla u(x,y,z) \cdot \hat{\mathbf{n}} = \nabla v(x,y,z) \cdot \hat{\mathbf{n}} = 0$ for any (x,y,z) on S, show that
   $$\iiint_D u\nabla^2 v\, dV = \iiint_D v\nabla^2 u\, dV.$$
   [FYI: Using the language in Linear Algebra or Functional Analysis, this result asserts that the Laplace operator ∇^2 is *self-adjoint* with respect to the L^2-inner product on C^2 functions under the boundary condition $D_{\hat{\mathbf{n}}} = 0$.]

1. (★) Use the Divergence Theorem to find the outward flux $\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}_{\text{out}}\,dS$ for each of the following $\mathbf{F}$ and S:

(a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the surface of any square cube of length b.

(b) $\mathbf{F} = x^3\mathbf{i} + 3yz^2\mathbf{j} + (3y^2z + x^2)\mathbf{k}$ and S is the sphere with radius $a > 0$ centered at the origin.

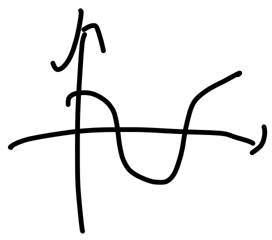
(c) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S is the boundary surface of the cylinder D defined by $x^2 + y^2 \leq 1$ and $0 \leq z \leq 4$.

$$\iiint 3\,dV = 3b^3$$

b). $$\iiint 3x^2 + 3z^2 + 3y^2\,dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 3\rho^2\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 3\frac{a^5}{5}\sin\varphi\,d\varphi\,d\theta$$



$$= \frac{3a^5}{5}\left[-\cos\varphi\right]_0^{\pi}\cdot 2\pi$$

$$= \frac{3a^5}{5}(2\pi)(2).$$

(c) $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ and S is the boundary surface of the cylinder D defined by $x^2 + y^2 \leq 1$ and $0 \leq z \leq 4$.

$$\iiint_E 2x + 2y + 2z \, dV$$

$\times$ $2r\cos\theta + 2r\sin\theta$

$$= \int_0^{2\pi}\int_0^1\int_0^4 2r^2 + 2rz \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^1 (8r^2 + 16r \, dr)$$

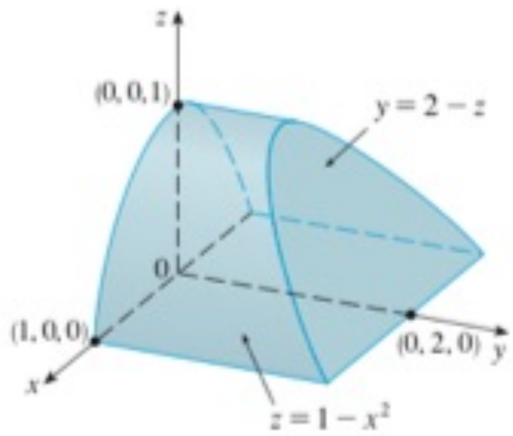
$$= 16\pi\left(\frac{1}{3} + 1\right)$$

$$= \frac{4}{3} \cdot 16\pi.$$

2. (★) Evaluate $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}} \, dS$ where

$$\mathbf{F} = xy\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\mathbf{k}$$

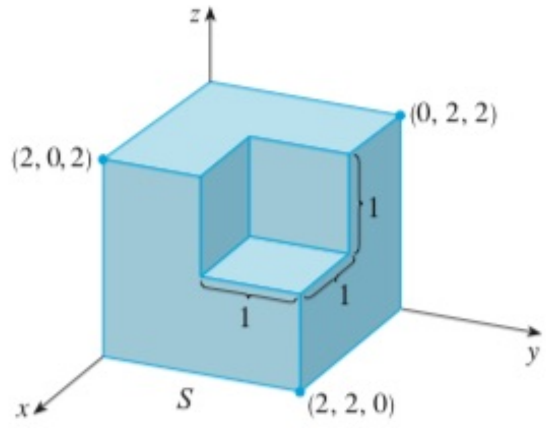
and S is the surface boundary of the region D defined by $z \leq 1 - x^2$, $z \geq 0$, $y \geq 0$ and $y \leq 2 - z$. See the figure below:



$$\iiint_E y + 2y \, dV$$

$$= \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y \, dy \, dz \, dx$$

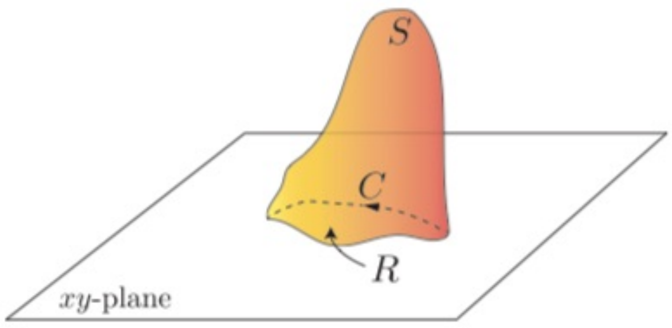
3. (★) Let D be the solid square cube of length 2 with one corner unit cube removed. See the figure below.



Evaluate the outward flux $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}_{\text{out}}\, dS$ where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Comment on why it is preferable to use the Divergence Theorem instead of computing the flux directly.

$$\iiint 3\, dV = 3(2^3 - 1)$$

4. (★★) Let C be an arbitrary simple closed curve on the xy-plane in the three dimensional space, and S is any surface *above* the xy-plane with boundary curve C. See the figure below.



Using the Divergence Theorem, show that:

$$\iint_S (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) \cdot \hat{\mathbf{n}}\, dS = c \times \text{area of the region on the } xy\text{-plane enclosed by } C.$$

$$\iiint \nabla \cdot F \, dV - \iint_{\text{[illegible]}}$$

$$= - \iint_R c \cdot (-1) \, dS$$

5. (★★) Suppose $f(x,y,z)$ is a C^2 function on $\mathbb{R}^3$ such that $\nabla^2 f(x,y,z) = 0$ on $\mathbb{R}^3$. Here $\nabla^2 f$ means the Laplacian of f, i.e. $\nabla^2 f = \nabla \cdot \nabla f = f_{xx} + f_{yy} + f_{zz}$.

(a) Show that:

$$\oiint_S f \nabla f \cdot \hat{\mathbf{n}}\, dS = \iiint_D |\nabla f|^2 \, dV$$

for any closed oriented surface S enclosing the solid region D.

(b) If, furthermore, assume that $f(x,y,z) = 0$ for any (x,y,z) on S, what can you say about $f(x,y,z)$ for any (x,y,z) in D?

??

a). $\iiint_D \nabla \cdot (f \nabla f) \, dV$

$= \nabla f \cdot \nabla f + f \cdot \nabla^2 f$

$= \iiint_D |\nabla f|^2 \, dV$

b).

6. (★★) Suppose S is a closed oriented level surface $f(x, y, z) = c$ of a C^2 function f. Denote D to be the solid enclosed by S. Show that:

$$\oiint_S |\nabla f| \, dS = \pm \iiint_D \overset{f_{xx}}{\nabla^2 f} \, dV$$

where $\pm$ depends on whether ∇f points inward or outward on the surface S.

$$\nabla \cdot |\nabla f|$$

$$= \nabla \sqrt{f_x^2 + f_y^2 + f_z^2}$$

$$\frac{\partial}{\partial x} \sqrt{f_x^2 + f_y^2 + f_z^2}$$

$$\frac{1}{2}\left(f_x^2 + f_y^2 + f_z^2\right)^{-\frac{1}{2}} \cdot (2 f_{xx})$$

$$= \frac{f_{xx}}{\sqrt{f_x^2 + f_y^2 + f_z^2}}$$

Made with Goodnotes

7. (★★) Given two C^2 functions $u(x,y,z)$ and $v(x,y,z)$ defined on $\mathbb{R}^3$. Let S be a closed oriented surface and D is the solid enclosed by S.

(a) Rewrite $\nabla \cdot (u\nabla v - v\nabla u)$ using **curl**, **grad** and **div**.

(b) Show that

$$\oiint_S (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}}\, dS = \iiint_D (u\nabla^2 v - v\nabla^2 u)\ dV$$

(c) Assume further that $\nabla u(x,y,z) \cdot \hat{\mathbf{n}} = \nabla v(x,y,z) \cdot \hat{\mathbf{n}} = 0$ for any (x,y,z) on S, show that

$$\iiint_D u\nabla^2 v\, dV = \iiint_D v\nabla^2 u\, dV.$$

[FYI: Using the language in Linear Algebra or Functional Analysis, this result asserts that the Laplace operator ∇^2 is *self-adjoint* with respect to the L^2-inner product on C^2 functions under the boundary condition $D_{\hat{\mathbf{n}}} = 0$.]

a). $\mathrm{Div}\left(u\,\mathrm{grad}(v) - v\,\mathrm{grad}(u)\right)$

$\nabla \cdot (u\nabla v - v\nabla u)$

$\frac{\partial}{\partial x}(u\nabla v - v\nabla u) + \frac{\partial}{\partial y}(u\nabla v - v\nabla u) + \frac{\partial}{\partial z}(u\nabla v - v\nabla u)$

$u_x \nabla v_x - v_x \nabla u_x$