Problem 1

Identify the following surfaces

(a) $\mathbf{r} \cdot \hat{\mathbf{u}} = 0$.

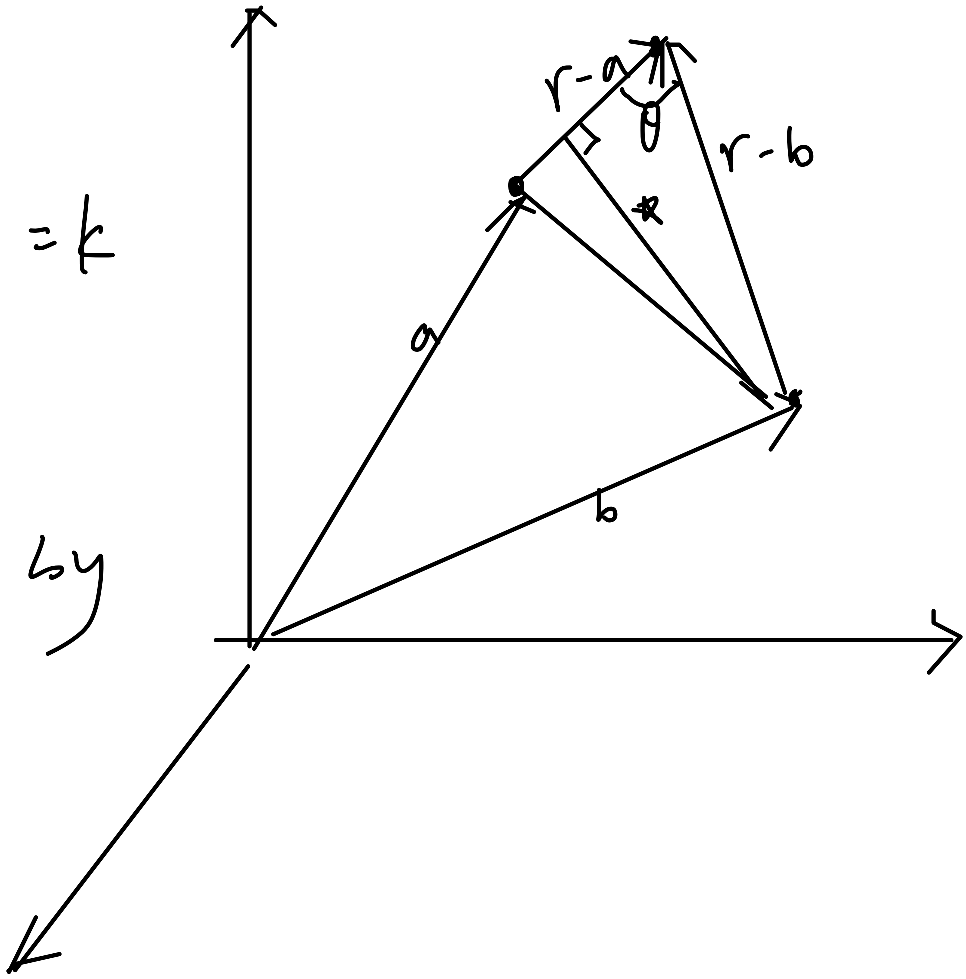
(b) $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = k$.

(c) $\|\mathbf{r} - (\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}\| = k$. [Hint: What are the vectors $(\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$ and $\mathbf{r} - (\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$?]

Here k is fixed scalar, $\mathbf{a}$, $\mathbf{b}$ are fixed 3D vectors and $\hat{\mathbf{u}}$ is a fixed 3D unit vector and $\mathbf{r} = (x, y, z)$.

a). $\hat{u}$ = normal vector, a plane.

b).



$|r-a||r-b|\cos\theta = k$

~~3D line.~~

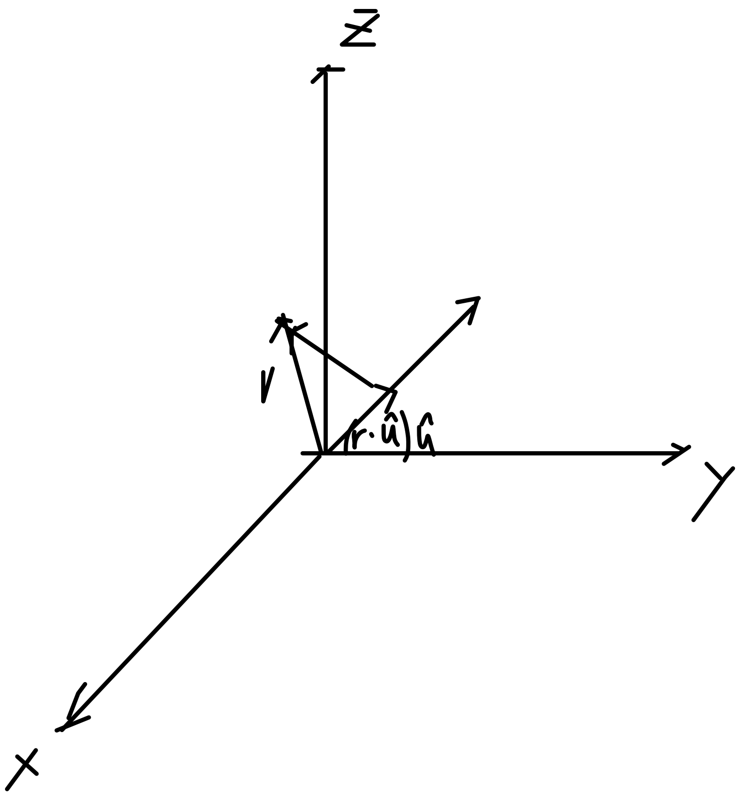
Area spanned by

$(r-a)$, $\vec{r}-\vec{b}$

(c) $\|\mathbf{r} - (\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}\| = k$. [Hint: What are the vectors $(\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$ and $\mathbf{r} - (\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}$?]

Here k is fixed scalar, $\mathbf{a}$, $\mathbf{b}$ are fixed 3D vectors and $\hat{\mathbf{u}}$ is a fixed 3D unit vector and $\mathbf{r} = (x, y, z)$.

a 3D line,



Problem 2

(a) Find the velocity, speed and acceleration at time t of the particle whose position is $\mathbf{r}(t)$. Describe the path of the particle.

$$\mathbf{r} = at\cos\omega t\,\mathbf{i} + at\sin\omega t\,\mathbf{j} + b\ln t\,\mathbf{k}$$

helix

(b) Find the required parametrization of the first quadrant part of the circular arc $x^2 + y^2 = a^2$ in terms of arc length measured from $(0, a)$, oriented clockwise.

(c) Let C be the curve $x^{2/3} + y^{2/3} = a^{2/3}$ on the xy-plane, find the parametric equation of the curve C. Hence find the tangent line to the curve C at $(a, 0)$.

a). $r'(t) = \langle a(t(-w\sin wt) + \cos wt)\bar{i} +$

$a(t(w\cos wt) + \sin wt)\bar{j} +$

$b(\frac{1}{t})k\rangle$

$|r'(t)| = \sqrt{a^2(-wt\sin wt + \cos wt)^2 + a^2(wt\cos wt + \sin wt)^2 + b^2(\frac{1}{t^2})}$

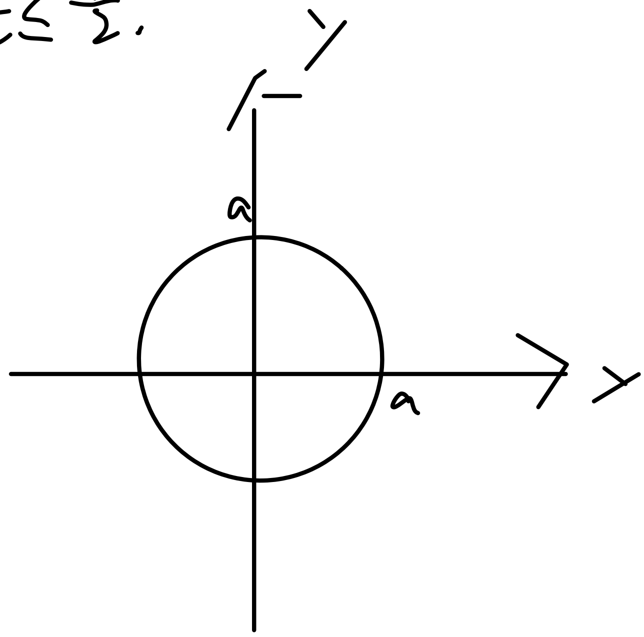
$= \sqrt{a^2(w^2t^2\sin^2 wt + \cos^2 wt - 2wt\sin wt\cos wt + w^2t^2\cos^2 wt + t\sin^2 wt - 2wt\sin wt\cos wt) + \frac{b^2}{t^2}}$

$$= \sqrt{a^2(w^2t^2+1) + \frac{b^2}{t^2}}$$

$$= \sqrt{a^2w^2t^2 + a^2 + \frac{b^2}{t^2}}$$

(b) Find the required parametrization of the first quadrant part of the circular arc $x^2 + y^2 = a^2$ in terms of arc length measured from $(0, a)$, oriented clockwise.

$x = a\cos t,\ y = a\sin t,\ 0 \le t \le \frac{\pi}{2}.$



$s = \theta a.$

(c) Let C be the curve $x^{2/3} + y^{2/3} = a^{2/3}$ on the xy-plane, find the parametric equation of the curve C. Hence find the tangent line to the curve C at $(a, 0)$.

let $x = a\cos^3 t$, $y = a\sin^3 t$

$\vec{r}(t) = a\cos^3 t\, i + a\sin^3 t\, j$.

When $(x, y) = (a, 0)$, $t = 0$.

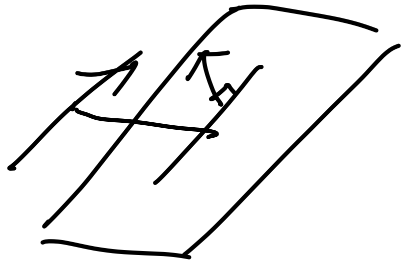
$\vec{r}'(t) = -3a\cos^2 t \sin t\, i + 3a\sin^2 t\cos t\, j$ -

$\vec{r}'(0) = 0i + 0j$.

No tangent line.

Made with Goodnotes

(b) Show that the line $x = -1 + t$, $y = 3 + 2t$, $z = -t$ and the plane $2x - 2y - 2z + 3 = 0$ are parallel, and find the distance between them.



$\vec{n} = \langle 2, -2, -2 \rangle$

$\vec{v} = \langle 1, 2, -1 \rangle$

$\vec{n} \cdot \vec{v} = 2 - 4 + 2 = 0.$

let $t = 0$, $(x, y, z) = (-1, 3, \cancel{-1})$ 0 !

$$D = \left| \frac{-2(-1) - 2(3) - 2(-1) + 3}{\sqrt{2^2 + 2^2 + 2^2}} \right|$$

$$= \left| \frac{-2 - 6 - 2 + 3}{\sqrt{12}} \right|$$

$$= \left| \frac{-7}{\sqrt{12}} \right|$$

$$= \frac{7}{\sqrt{12}}$$

Made with Goodnotes

Problem 4

(a) Find the parametric equation of the curve of intersection C between the plane $z = 2y + 3$ and the surface $z = x^2 + y^2$. Find also the equation of the projection curve of the curve of intersection C onto the xz-plane.

(b) Evaluate $\lim_{(x,y)\to(0,0)} \frac{|x| + |y|}{\sqrt{x^2 + y^2}}$.

$\uparrow$ 1,t

b).

$$\lim_{r\to 0} \frac{\frac{|r\cos\theta| + |r\sin\theta|}{r}}{r}$$

$$= \lim_{r\to 0} \frac{|r\cos\theta| + |r\sin\theta|}{r^2}$$

$$= \lim_{r\to 0} \left(\left|\frac{\cos\theta}{r}\right| + \left|\frac{\sin\theta}{r}\right| \right)$$

does not exist.

Problem 5

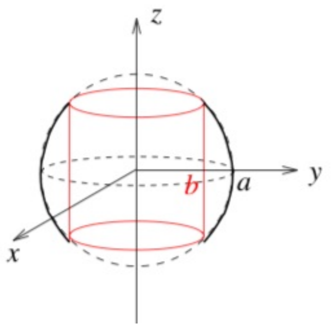
(a) Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

(i) Use the definition of partial derivative to show that $f_x(0,0)$ and $f_y(0,0)$ exist.

(ii) Is the function f continuous at $(0,0)$?

(b) Determine (sketch) the graph of the spherical-coordinate equation $\rho = 2\cos\phi$.

(c) A sphere of radius a is centered at the origin. A hole of radius b is drilled through the sphere, with the axis of the hole lying on the z-axis. Describe the solid region that remains (see Figure) in a (i) cylindrical coordinates; (ii) spherical coordinates.



(a).

$$\frac{\frac{\Delta x (0)}{\Delta x^2}}{\Delta x} \to 0.$$

$$\frac{\frac{\Delta y (0)}{\Delta y^2}}{\Delta y} \to 0.$$

(ii). $\lim_{r \to 0} \frac{r^2 \sin\theta \cos\theta}{r^2}$ not continuous.

Made with Goodnotes

Bonus question

Find the maximum and minimum distances between the point $(1, 1, 1)$ and a point on the curve of intersection of the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = z$. (Note that the curve of intersection is not the origin).

$$x^2+y^2+x^2+y^2 = \sqrt{x^2+y^2}$$

$$2x^2+2y^2 = \sqrt{x^2+y^2}$$

$$(2x^2+2y^2)^2 = x^2+y^2$$

$$4x^4+8x^2y^2+4y^4 = x^2+y^2$$

$$4x^4+8x^2y^2+4y^4-x^2-y^2 = 0 \leftarrow f(x,y)$$

Max $\Rightarrow$ $D = (x-1)^2+(y-1)^2+(\sqrt{x^2+y^2}-1)^2$

subject to $f(x,y)$.

$$\nabla D = \left(2(x-1)+2(\sqrt{x^2+y^2}-1)\left(\frac{2x}{2\sqrt{x^2+y^2}}\right),\right.$$

$$\left.2(y-1)+2(\sqrt{x^2+y^2}-1)\left(\frac{2y}{2\sqrt{x^2+y^2}}\right)\right)$$

$$= \left((2x-2+2\sqrt{x^2+y^2}-2)\left(\frac{x}{\sqrt{x^2+y^2}}\right),\right.$$

$$\left.(2y-2+2\sqrt{x^2+y^2}-2)\left(\frac{y}{\sqrt{x^2+y^2}}\right)\right)$$

$$\nabla f = \langle 16x^3+16xy^2-2x,\ 16x^2y+16y^3-2y\rangle$$

$$\left(2x+2\sqrt{x^2+y^2}-4\right)\left(\frac{x}{\sqrt{x^2+y^2}}\right)=\lambda\left(16x^3+16xy^2-2x\right)$$

$$\left(2y+2\sqrt{x^2+y^2}-4\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right)=\lambda\left(16x^2y+16y^3-2y\right)$$

$$4x^4+8x^2y^2+4y^4-x^2-y^2=0$$

$$\frac{\left(2x+2\sqrt{x^2+y^2}-4\right)x}{\cancel{\sqrt{x^2+y^2}}\left(16x^3+16xy^2-2x\right)}=\frac{\left(2y+2\sqrt{x^2+y^2}-4\right)y}{\cancel{\sqrt{x^2+y^2}}\left(16y^3+16x^2y-2y\right)}$$

$$\frac{2x+2\sqrt{x^2+y^2}-4}{16x^2+16y^2-2}=\frac{\left(2y+2\sqrt{x^2+y^2}-4\right)}{16y^2+16x^2-2}$$

$$2x+\cancel{2\sqrt{x^2+y^2}}\cancel{-4}=2y+\cancel{2\sqrt{x^2+y^2}}\cancel{-4}$$

$$2x=2y$$

$$x=y$$

$$(2x+2\sqrt{2}x-4)\left(\frac{1}{\sqrt{2}}\right)=\lambda(16x^3+16x^3-2x)$$

$$=$$

$$4x^4+8x^4+4x^4-x^2-x^2=0$$

$$16x^4-2x^2=0$$

$x^2=0$ or $16x^2-2=0$

$$16x^2=2$$

$$x^2=\frac{2}{16}$$

$$x=\pm\frac{\sqrt{2}}{4}.$$

$$y=\pm\frac{\sqrt{2}}{4}.$$

For $+\frac{\sqrt{2}}{4}$,

$D^2=1.08578643\leftarrow$ min

For $-\frac{\sqrt{2}}{4}$,

$D^2=3.914213562\leftarrow$ Max