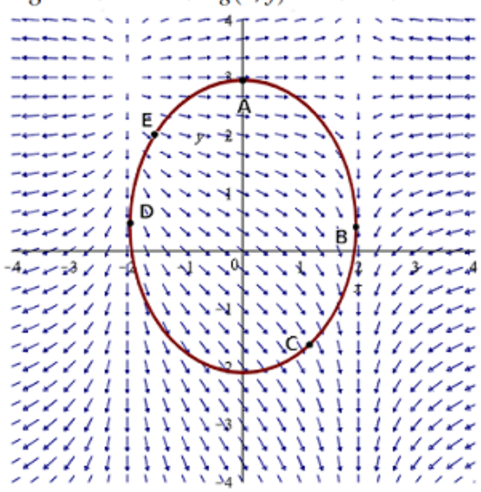
MATH2023
Tutorial 8
April 11, 2019 Tutorial Notes by Ivan So

1 Review

- A **vector field** $F : \mathbb{R}^n \to \mathbb{R}^n$ which assign every point in the concerned space a *vector*.
- **Conservative vector field**:
  - The *gradient operator* assigns a function into a vector field. I.e. for a function $\phi : \mathbb{R}^n \to \mathbb{R}$, the gradient operator map the function into an n-dimensional vector field.
  - In such a case, we say the field is **conservative** and the function ϕ is the **potential function**.
  - e.g. Q1(g) from sample midterm

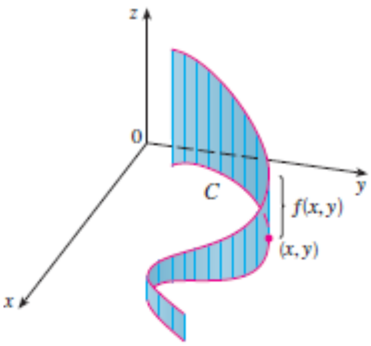
(g) Let $f(x,y)$ be a C^1 function. The diagram below is the plot of the vector field ∇f. The ellipse in the diagram is a level set $g(x,y) = c$ of another C^1 function g.



- The **line integral of function** along a curve C $\int_C f(\mathbf{x})ds := \lim_{n\to\infty} \sum_{i=1}^n f(\mathbf{x}_i^*)\Delta s_i$ for sampling points $\mathbf{x}_i^* \in C$. Explicitly, if C is parametrized by $\mathbf{r}(t)$ with $a \le t \le b$, then

$$\int_C f(\mathbf{x})ds = \int_a^b f(\mathbf{r}(t)) \left\| \frac{d\mathbf{r}}{dt} \right\| dt.$$

Pictorially, its meaning



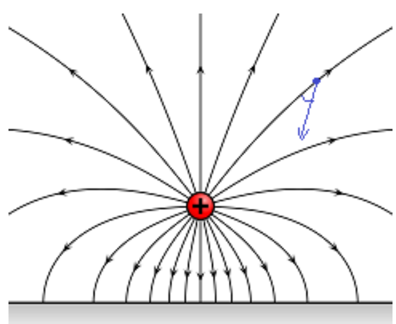
Analogy:

- Single variable calculus: definite integral = area under curve.
- Multivariable calculus: line integral of function = area under the function along the curve.

- The **line integral of a vector field F** along a curve C (parametrized by $\mathbf{r}(t)$ with $a \leq t \leq b$) is defined as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \sum_{i=1}^{n} (\mathbf{F})_{x_i} dx_i = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

Physical realization: Work done of moving a charge in electric field



- **Theorem**(Fundamental Theorem for Line Integral): If the concerned vector field is conservative, given by ∇f, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

if the parameter t of the path satisfies $a \leq t \leq b$.

- **Corollary 1**: The line integral of a conservative field is path independent.
- **Corollary 2** : The line integral of a conservative field over a closed path (a path in which starting point is the end point) is zero.

- *Notation*:
  - If C is a curve obtained by connecting C_1 and C_2, then $\int_{C_1} + \int_{C_2} = \int_C$.
  - Path integral is direction sensitive. $\int_{-C}$ represent the integral of the same path go in the opposite direction (i.e. from t evaluate from b to a instead of a to b).
  - The integral over a closed path is denoted by $\oint_C$.
  - The *positive orientation* of a closed path is the path going in *counterclockwise* direction.
- Some useful theorems:
  1. (Does not matter on domain) $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path $C \Leftrightarrow \int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.
  2. On open connected domain, path independence $\Leftrightarrow$ conservative.
  3. If $\mathbf{F} = \langle P, Q \rangle$, then conservative $\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.
- **Theorem** (Green's Theorem): Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ be any vector field (not necessarily conservative) with P, Q having continuous derivatives, then

$$\oint_{C=\partial D} \mathbf{F} \cdot d\mathbf{r} = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

  - **Corollary**: (Partial inverse of 3, Theorem D in lecture) On open simply connected region [no hole], if $\mathbf{F} = \langle P, Q \rangle$, then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \Rightarrow$ conservative.

  Comment:
  1. Line integral is not always easy to compute, this theorem provides an alternative, the area integral which could be easier to compute in some instances.
  2. This is the specialization of the later introduced **Stoke's Theorem**.

2 Problems

1. True or False.
   (a) $\int_C f(x, y) ds = -\int_{-C} f(x, y) ds$.
   False. Recall the definition of line integral along a given curve C is $\lim_{n\to\infty} \sum_{i=1}^n f(\mathbf{x}_i^*)\Delta s_i$. In contrast to Δx_i is single variable calculus, Δs_i is a quantity *which is always greater than zero*. Therefore the direction of integration along the curve **DOES NOT MATTER**.
   (b) $\mathbf{F} = \langle P, Q \rangle$, then $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \Rightarrow$ conservative.
   False. The use of Green's Theorem's corollary requires the domain being simply connected.

(c) The set $\{(x,y) : x, y \geq 0\}$ is a open connected subset of $\mathbb{R}^2$.
False. It is connected but not open because open subset cannot contain boundary by definition.

(d) The set $\{(x,y) : x \neq 0\} \cup \{(0,0)\}$ is a simply connected subset of $\mathbb{R}^2$.
True. Connected, since $(0,0)$ serve as a "bridge" between $\{(x,y) : x > 0\}$ and $\{(x,y) : x < 0\}$. There are two cases two check for simply connectedness.

- Case 1: The closed path lies inside either of $\{(x,y) : x > 0\}$ or $\{(x,y) : x < 0\}$. The claim is obvious.
- Case 2: The closed path lies in both $\{(x,y) : x > 0\}$ and $\{(x,y) : x < 0\}$. In that case the closed path must passes through $(0,0)$ and one can check they shrink to $(0,0)$ without leaving the domain.

2. Write $\int_C (2x + 9z)ds$ with C parametrized by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$ in terms of t.
Solution:
$$\int_C f(x,y)ds = \int_a^b f(\mathbf{r(t)})|\mathbf{r}'(t)|dt.$$
For our consideration, $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. So
$$\int_C (2x + 9z)ds = \int_0^1 (2t + 9t^3)\sqrt{1 + 4t^2 + 9t^4}dt.$$

3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = \sin x\mathbf{i} + \cos y\mathbf{j} + xz\mathbf{k}$ and C is parametrized by $\mathbf{r}(t) = \langle t^3, -t^2, t \rangle$ with $0 \leq t \leq 1$.
Solution:
$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)dt \\ &= \int_0^1 \langle \sin t^3, \cos(-t^2), t^4 \rangle \cdot \langle 3t^2, -2t, 1 \rangle dt \\ &= (1 - \cos 1) - \sin 1 + 1/5\end{aligned}$$

* Check if my evaluation is correct.

4. If the domain is the whole $\mathbb{R}^2$, determine whether the field $\mathbf{F}(x,y) = (2x - 3y)\mathbf{i} + (-3x + 4y - 8)\mathbf{j}$ is conservative.
If conservative, evaluate the line integral over the path C parametrized by $\mathbf{r}(t) = \langle t, t^2 \rangle$, $0 \leq t \leq 1$ with the Fundamental Theorem of Line Integral.
Solution: $\mathbb{R}^2$ is simply connected, so we can use the Corollary of Green's theorem to check the conservativeness of a vector field. For the given $\mathbf{F}$, $P_y = -3 = Q_x$. Therefore $\mathbf{F}$ is conservative.
Now comes to the problem of looking for the potential function. If ϕ is the potential function, then
$$\nabla\phi = \langle \phi_x, \phi_y \rangle = \langle 2x - 3y, -3x + 4y - 8 \rangle.$$

Integrating both sides of the first component gives:

$$\phi = x^2 - 3xy + g(y)$$

the existence of $g(y)$ is due to the fact that $g(y)$ vanishes under x-partial derivative and gives the given derivative ϕ_x. On the othe hand, differentiate the found y, we have

$$\phi_y = -3x + g'(y) = -3x + 4y - 8 \Rightarrow g(y) = 2y^2 - 8y + C.$$

Therefore

$$\phi(x, y) = x^2 - 3xy + 2y^2 - 8y + C$$

So, from the Fundamental Theorem of Line Integral, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \phi(1, 1) - \phi(0, 0) = -8.$$

5. If the domain is the whole $\mathbb{R}^2$, determine whether the field $\mathbf{F}(x, y) = e^x \sin y\mathbf{i} + e^{-x} \cos y\mathbf{j}$ is conservative.
   If conservative, evaluate the line integral over the path C parametrized by $\mathbf{r}(t) = \langle t, t^2 \rangle$, $0 \leq t \leq 1$ with the Fundamental Theorem of Line Integral.
   Solution: $Q_x = -e^{-x} \cos y$, $P_y = e^x \cos y \neq Q_x$. By the contrapositive of Theorem 3, $\mathbf{F}$ is not conservative (since conservative will implies $P_y = Q_x$, given $P_y \neq Q_x$, conservative of field will lead to contradiction by the theorem).
   *Notice that the properties of the domain is irrelevant here.

6. Evaluate $\oint_C xydx + x^2y^3dy$ with C being the triangle with vertices $(0, 0), (1, 0)$ and $(1, 2)$ using Green's theorem. Check your answer with the classical method.
   Solution: $Q_x - P_y = 2xy^3 - x$. From Green's theorem,

$$\begin{aligned} \oint_C xydx + x^2y^3dy &= \int_0^1 \int_0^{2x} (2xy^3 - x)dydx \\ &= \int_0^1 (2x^2 - 8x^5)dx \\ &= -2/3. \end{aligned}$$

   And I will leave the classical checking for you.

7. Use Green's theorem to prove the 2-dimensional change of variable formula (which will be useful later)

$$\int\int_R dxdy = \int\int_S \det \frac{\partial(x, y)}{\partial(u, v)} dudv.$$

   where R is the domain of integration in terms of x, y-variables. S is for u, v. (Assume the interchangeability of second order derivative of x, y with respect to u, v)
   Solution: The Jacobian matrix is defined as

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}.$$

Therefore,

$$
\begin{aligned}
\int\int_R dxdy &= \oint_{\partial R} xdy \quad \text{(Green's theorem)} \\
&= \oint_{\partial S} x(u,v)dy(u,v) \quad \text{(rewrite the integral in terms of } u,v) \\
&= \oint_{\partial S} x(u,v)y_u du + x(u,v)y_v dvdv \quad \text{(total differential expansion)} \\
&= \int\int_S \left(\frac{\partial}{\partial u}(xy_v) - \frac{\partial}{\partial v}(xy_u)\right) dudv \quad \text{(Green's theorem with } u,v) \\
&= \int\int_S \left(x_u y_v + xy_{vu} - x_v y_u - xy_{uv}\right) dudv \\
&= \int\int_S \det\frac{\partial(x,y)}{\partial(u,v)} dudv
\end{aligned}
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