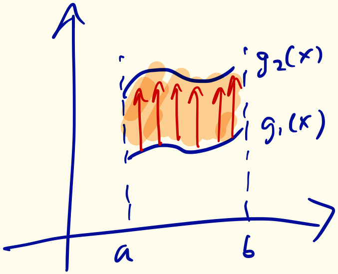
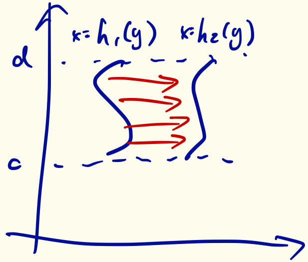
Last Time

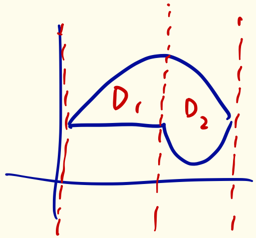
$$\iint_D f(x,y)\,dA$$



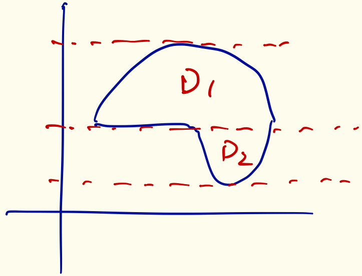


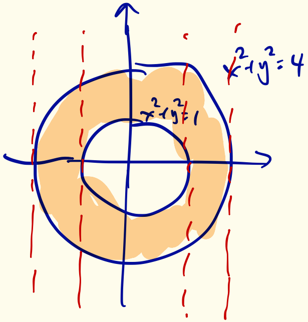
$$\int_a^b \underbrace{\underbrace{\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy}_{\text{x is const here}}}_{\text{a function in x}}\,dx$$

$$\int_c^d \underbrace{\underbrace{\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx}_{\text{y is const here}}}_{\text{a function in y}}\,dy$$



$$\iint_D dA = \iint_{D_1} dA + \iint_{D_2} dA$$



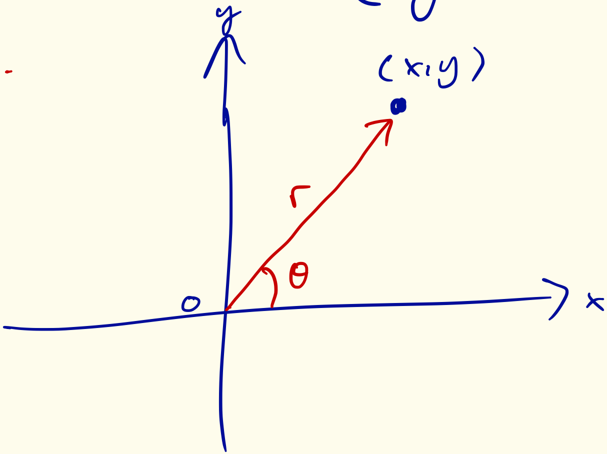


4 regions ?!

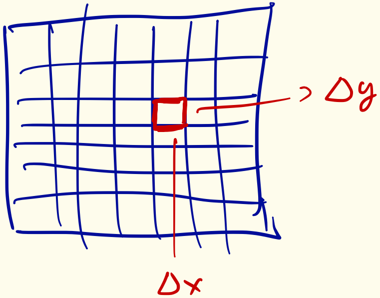
BAD.

$\Longrightarrow$ Polar Coordinate!

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

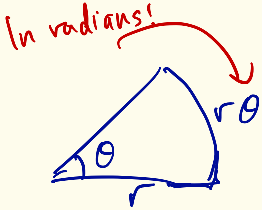


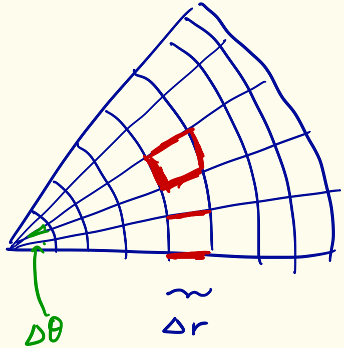
Recall

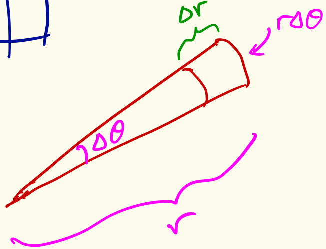


$$\Delta A = \Delta x \Delta y$$

$$\sum \Delta A = \sum \Delta x \Delta y \rightsquigarrow \int_A dx\,dy$$







length × width

$\downarrow \qquad \downarrow$

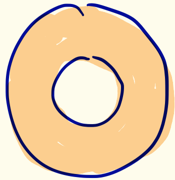
$\Delta r \qquad r\Delta\theta$

$$\sum \Delta A = \sum r \Delta r \Delta\theta \rightsquigarrow \boxed{\int_A r\,dr\,d\theta}$$

Thm

$$\iint_R f(x,y)\,dA = \int_{\cdot}^{\cdot\cdot}\int_{\cdot\cdot}^{\cdot\cdot} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Ex $\iint_D 3x+4y^2 \, dA$, D bounded by $\begin{cases} x^2+y^2=4 \\ x^2+y^2=1 \end{cases}$



$$\int_0^{2\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr\, d\theta$$

$\int_0^{2\pi}\cos\theta\, d\theta = 0$

$\int_0^{2\pi}\sin^2\theta = \pi$

$$\int_0^{2\pi} 1-\cos^2\theta$$

$$2\pi - \int_0^{2\pi}\frac{1+\cos 2\theta}{2}$$

$$= 2\pi - \frac{1}{2}(2\pi) - \frac{1}{2}\left[\frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

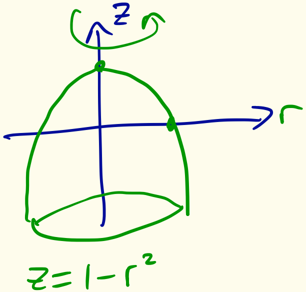
$$= \pi - \frac{1}{4}(0) = \pi$$

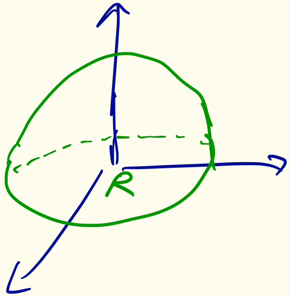
$$\pi\int_1^2 4r^3\, dr$$

$$= \pi r^4 \Big|_1^2$$

$$= 15\pi$$

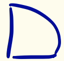
Ex Volume of Solid under $z = 1 - x^2 - y^2$ over x-y plane.





$$\iint_R 1 - x^2 - y^2 \, dA \implies \int_0^{2\pi}\int_0^1 (1-r^2)\, r\, dr\, d\theta$$

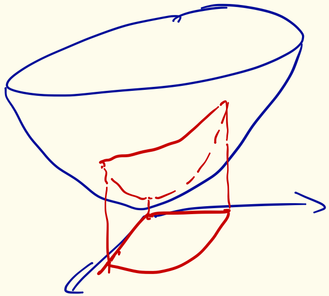
Ex Volume of $z = x^2 + 2y^2$ over quarter unit circle



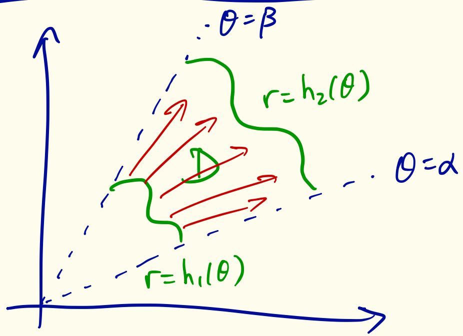
$$\iint_R x^2 + 2y^2 \, dA = \int_0^{\frac{\pi}{2}}\int_0^1 (r^2 + r^2\sin^2\theta)\, r\, dr\, d\theta$$

$\downarrow$

$x^2 + y^2 + y^2$



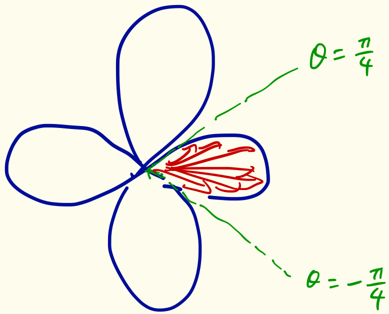
General Regions



always this order

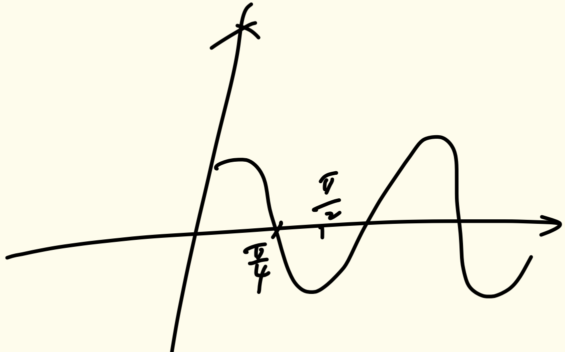
$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,\underline{r\,dr\,d\theta}$$

Ex Area bounded by $r = \cos 2\theta$.



$$= 4 \times \text{Area}(\bigcirc)$$

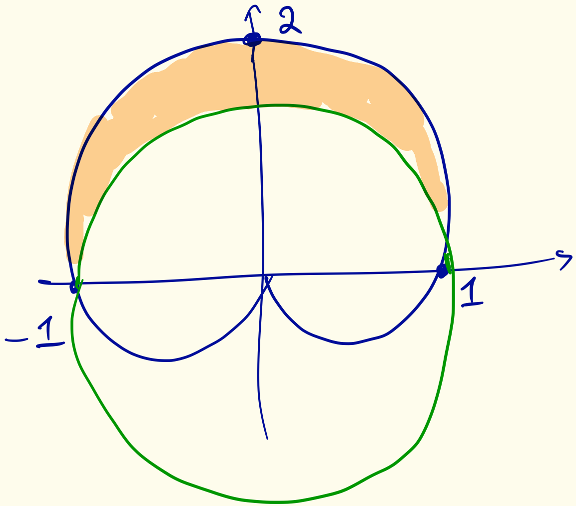
$$= 4 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{\cos 2\theta} 1 \cdot r \, dr \, d\theta .$$



$$= 4 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{r^2}{2} \Big|_{0}^{\cos 2\theta} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos^2 2\theta \, d\theta \qquad = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \cos 4\theta) \, d\theta \quad = \frac{\pi}{2} .$$

Ex Area bounded by $r = 1 + \sin\theta$ and outside unit circle $r=1$

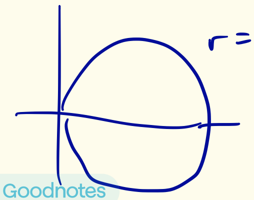


$$\int_0^{\pi}\int_1^{1+\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi} \frac{r^2}{2}\Big|_1^{1+\sin\theta} d\theta$$

$$= \int_0^{\pi} \frac{\sin^2\theta + 2\sin\theta}{2}\, d\theta$$

$$= \dots$$

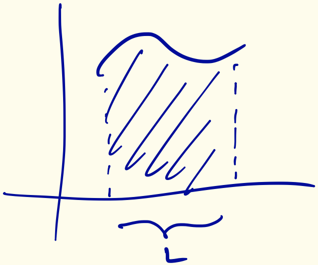


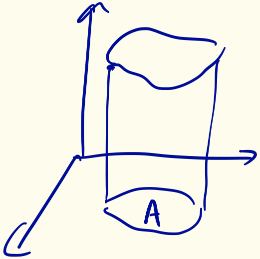
$r = 2\cos\theta \iff (x-1)^2 + y^2 = 1$

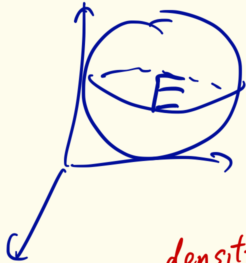
Rem Always use radians!!!

Triple Integration (Details Later)

"density = $\frac{\text{mass}}{\text{volume}}$





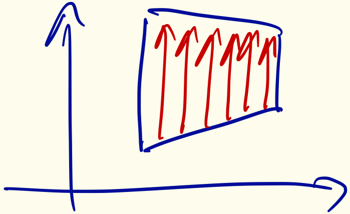


density

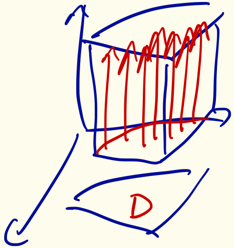
$$\int f(x)\,dx \rightsquigarrow \text{Area} \qquad \iint_D f(x,y)\,dA \rightsquigarrow \text{Volume} \qquad \iiint f(x,y,z)\,dV \rightsquigarrow \text{Mass}$$

$$\int \underline{1}\,dx \rightsquigarrow \text{Length} \qquad \iint_D 1\,dA \rightsquigarrow \text{Area} \qquad \iiint_E 1\,dV \rightsquigarrow \text{Volume}$$

$$\iint_D dA$$



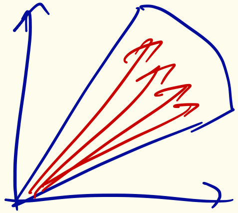
$$\iiint_E dV$$



$$= \iint_D \left(\int dz \right) dA$$

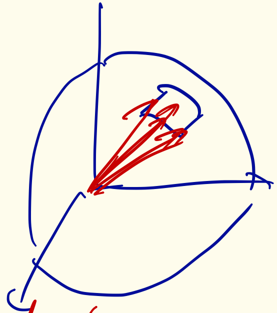
Polar

$$\iint_D dA$$

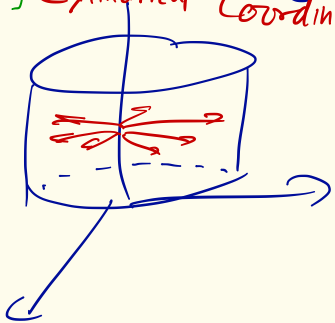


Spherical Coordinates

$$\iiint_E dV$$



Cylindrical Coordinates.



(Polar in xy
Keep z)