Remark:

**17.** In what directions at the point $(2, 0)$ does the function $f(x, y) = xy$ have rate of change $-1$? Are there directions in which the rate is $-3$? How about $-2$?

→ 先求 $\nabla f(2,0)$, 得出 $= 2\mathbf{j}$

For rate of change $= -1$

$$\begin{cases} (0\mathbf{i} + 2\mathbf{j}) \cdot (x\mathbf{i} + y\mathbf{j}) = -1 \\ x^2 + y^2 = 1 \end{cases}$$

兩條件一齊求先得,
因此 $\boxed{-3}$ 沒有 sol.

Remark

**18.** In what directions at the point $(a, b, c)$ does the function $f(x, y, z) = x^2 + y^2 - z^2$ increase at half of its maximal rate at that point?

maximum rate $= |\nabla f(a,b,c)|$

Technique:
Integral curve

**21.** The temperature $T(x, y)$ at points of the $xy$-plane is given by $T(x, y) = x^2 - 2y^2$.

(a) Draw a contour diagram for $T$ showing some isotherms (curves of constant temperature).

(b) In what direction should an ant at position $(2, -1)$ move if it wishes to cool off as quickly as possible?

(c) If the ant moves in that direction at speed $k$ (units distance per unit time), at what rate does it experience the decrease of temperature?

(d) At what rate would the ant experience the decrease of temperature if it moved from $(2, -1)$ at speed $k$ in the direction of the vector $-\mathbf{i} - 2\mathbf{j}$?

(e) Along what curve through $(2, -1)$ should the ant move in order to continue to experience maximum rate of cooling?

e) To continue to experience maximum rate of cooling, the ant should crawl along the curve $x = x(t)$, $y = y(t)$, which is everywhere tangent to $\nabla T(x, y)$. Thus we want

$$\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = \lambda(2x\mathbf{i} - 4y\mathbf{j}).$$

Thus $\frac{1}{y}\frac{dy}{dt} = -\frac{2}{x}\frac{dx}{dt}$, from which we obtain, on integration,

$$\ln|y(t)| = -2\ln|x(t)| + \ln|C|,$$

or $yx^2 = C$. Since the curve passes through $(2, -1)$, we have $yx^2 = -4$. Thus, the ant should crawl along the path $y = -4/x^2$.

① 寫出关係
② 利用 $\lambda$ 寫成呢个 form
③ 兩边一齐 in

Technique：题型

**26.** Find a vector tangent to the curve of intersection of the two cylinders $x^2 + y^2 = 2$ and $y^2 + z^2 = 2$ at the point $(1, -1, 1)$.

**27.** Repeat Exercise 26 for the surfaces $x + y + z = 6$ and $x^2 + y^2 + z^2 = 14$ and the point $(1, 2, 3)$.

① Grad 1

**26.** At $(1, -1, 1)$ the surface $x^2 + y^2 = 2$ has normal

$$\mathbf{n}_1 = \nabla(x^2 + y^2)\Big|_{(1,-1,1)} = 2\mathbf{i} - 2\mathbf{j},$$

② Grad 2

and $y^2 + z^2 = 2$ has normal

$$\mathbf{n}_2 = \nabla(y^2 + z^2)\Big|_{(1,-1,1)} = -2\mathbf{j} + 2\mathbf{k}.$$

A vector tangent to the curve of intersection of the two surfaces at $(1, -1, 1)$ must be perpendicular to both these normals. Since

③ Cross 之

$$(\mathbf{i} - \mathbf{j}) \times (-\mathbf{j} + \mathbf{k}) = -(\mathbf{i} + \mathbf{j} + \mathbf{k}),$$

the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$, or any scalar multiple of this vector, is tangent to the curve at the given point.

---

题型

**28.** The temperature in 3-space is given by

主function →

$$T(x, y, z) = x^2 - y^2 + z^2 + xz^2.$$

玩野 function, 穿 curve of intersection

At time $t = 0$ a fly passes through the point $(1, 1, 2)$, flying along the curve of intersection of the surfaces $z = 3x^2 - y^2$ and $2x^2 + 2y^2 - z^2 = 0$. If the fly's speed is $\underline{\underline{7}}$, what rate of temperature change does it experience at $t = 0$?

Made with Goodnotes

題型:

**28.** The temperature in 3-space is given by

$$T(x, y, z) = x^2 - y^2 + z^2 + xz^2.$$ 主 function

At time $t = 0$ a fly passes through the point $(1, 1, 2)$, flying along the curve of intersection of the surfaces $z = 3x^2 - y^2$ and $2x^2 + 2y^2 - z^2 = 0$. If the fly's speed is 7, what rate of temperature change does it experience at $t = 0$?

A, B function, 要 walk along curve of intersection

**28.** A vector tangent to the path of the fly at $(1, 1, 2)$ is given by

① tangent vector: $\nabla g_1 \times \nabla g_2$

$$\begin{aligned}
\mathbf{v} &= \nabla(3x^2 - y^2 - z) \times \nabla(2x^2 + 2y^2 - z^2)\Big|_{(1,1,2)} \\
&= (6x\mathbf{i} - 2y\mathbf{j} - \mathbf{k}) \times (4x\mathbf{i} + 4y\mathbf{j} - 2z\mathbf{k})\Big|_{(1,1,2)} \\
&= (6\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \times (4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}) \\
&= 4\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -2 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 4(3\mathbf{i} + 5\mathbf{j} + 8\mathbf{k}).
\end{aligned}$$

② 計 $\nabla f$

$(1, 1, 2)$ given by

$$\begin{aligned}
\nabla T(1, 1, 2) &= (2x + z^2)\mathbf{i} - 2y\mathbf{j} + 2z(1 + x)\mathbf{k}\Big|_{(1,1,2)} \\
&= 6\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}.
\end{aligned}$$

③ $D_u f \times$ speed

Thus the fly, passing through $(1, 1, 2)$ with speed 7, experiences temperature changing at rate

$$\begin{aligned}
7 \times \frac{\mathbf{v}}{|\mathbf{v}|} \bullet \nabla T(1, 1, 2) &= 7\frac{3\mathbf{i} + 5\mathbf{j} + 8\mathbf{k}}{\sqrt{98}} \bullet (6\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}) \\
&= \frac{1}{\sqrt{2}}(18 - 10 + 64) = \frac{72}{\sqrt{2}}.
\end{aligned}$$

Proof this statement →

**29.** If $f(x, y, z)$ is differentiable at the point $(a, b, c)$ and $\nabla f(a, b, c) \neq \mathbf{0}$, then $\nabla f(a, b, c)$ is normal to the level surface of $f$ which passes through $(a, b, c)$.

The proof is very similar to that of Theorem 6 of Section 3.7, modified to include the extra variable. The angle $\theta$ between $\nabla f(a, b, c)$ and the secant vector from $(a, b, c)$ to a neighbouring point $(a + h, b + k, c + \ell)$ on the level surface of $f$ passing through $(a, b, c)$ satisfies

dot product ☆

$$\begin{aligned}
\cos\theta &= \frac{\nabla f(a, b, c) \bullet (h\mathbf{i} + k\mathbf{j} + \ell\mathbf{k})}{|\nabla f(a, b, c)|\sqrt{h^2 + k^2 + \ell^2}} \\
&= \frac{hf_1(a, b, c) + kf_2(a, b, c) + \ell f_3(a, b, c)}{|\nabla f(a, b, c)|\sqrt{h^2 + k^2 + \ell^2}} \\
&= \frac{-1}{|\nabla f(a, b, c)|\sqrt{h^2 + k^2 + \ell^2}}\Big[f(a + h, b + k, c + \ell) \\
&\qquad - f(a, b, c) - hf_1(a, b, c) - kf_2(a, b, c) - \ell f_3(a, b, c)\Big] \\
&\to 0 \quad \text{as } (h, k, \ell) \to (0, 0, 0)
\end{aligned}$$

because $f$ is differentiable at $(a, b, c)$. Thus $\theta \to \dfrac{\pi}{2}$, and $\nabla f(a, b, c)$ is normal to the level surface of $f$ through $(a, b, c)$.

Made with Goodnotes

Remark

**37.** Let $f(x, y) = \begin{cases} 2x^2y/(x^4 + y^2), & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$
Use the definition of directional derivative as a limit (Definition 7) to show that $D_{\mathbf{u}} f(0, 0)$ exists for every unit vector $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ in the plane. Specifically, show that $D_{\mathbf{u}} f(0, 0) = 0$ if $v = 0$, and $D_{\mathbf{u}} f(0, 0) = 2u^2/v$ if $v \neq 0$. However, as was shown in Example 4 in Section 12.2, $f(x, y)$ has no limit as $(x, y) \to (0, 0)$, so it is not continuous there. Even if a function has directional derivatives in all directions at a point, it may not be continuous at that point.

**37.** $f(x, y) = \begin{cases} \dfrac{2x^2y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$

Let $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ be a unit vector. If $v \neq 0$, then

$$\begin{aligned} D_{\mathbf{u}} f(0, 0) &= \lim_{h \to 0+} \frac{1}{h} \frac{2(h^2u^2)(hv)}{h^4u^4 + h^2v^2} \\ &= \lim_{h \to 0+} \frac{2u^2v}{h^2u^4 + v^2} = \frac{2u^2}{v}. \end{aligned}$$

因为v不可以0,
所以要分case!

If $v = 0$, then $u = \pm 1$ and

$$D_{\mathbf{u}} f(0, 0) = \lim_{h \to 0+} = \frac{1}{h} \frac{0}{h^2} = 0.$$

Thus $f$ has a directional derivative in every direction at the origin even though it is not continuous there.

## EXERCISES 12.7

In Exercises 1–6, find:

(a) the gradient of the given function at the point indicated, ✓

(b) an equation of the plane tangent to the graph of the given function at the point whose $x$ and $y$ coordinates are given, and

(c) an equation of the straight line tangent, at the given point, to the level curve of the given function passing through that point.

**1.** $f(x, y) = x^2 - y^2$ at $(2, -1)$

**2.** $f(x, y) = \dfrac{x - y}{x + y}$ at $(1, 1)$

**3.** $f(x, y) = \dfrac{x}{x^2 + y^2}$ at $(1, 2)$

**4.** $f(x, y) = e^{xy}$ at $(2, 0)$

**5.** $f(x, y) = \ln(x^2 + y^2)$ at $(1, -2)$

**6.** $f(x, y) = \sqrt{1 + xy^2}$ at $(2, -2)$

In Exercises 7–9, find an equation of the tangent plane to the level surface of the given function that passes through the given point.

**7.** $f(x, y, z) = x^2y + y^2z + z^2x$ at $(1, -1, 1)$

**8.** $f(x, y, z) = \cos(x + 2y + 3z)$ at $\left(\dfrac{\pi}{2}, \pi, \pi\right)$

**9.** $f(x, y, z) = y\,e^{-x^2} \sin z$ at $(0, 1, \pi/3)$

In Exercises 10–13, find the rate of change of the given function at the given point in the specified direction.

**10.** $f(x, y) = 3x - 4y$ at $(0, 2)$ in the direction of the vector $-2\mathbf{i}$

**11.** $f(x, y) = x^2y$ at $(-1, -1)$ in the direction of the vector $\mathbf{i} + 2\mathbf{j}$

**12.** $f(x, y) = \dfrac{x}{1 + y}$ at $(0, 0)$ in the direction of the vector $\mathbf{i} - \mathbf{j}$

**13.** $f(x, y) = x^2 + y^2$ at $(1, -2)$ in the direction making a (positive) angle of $60°$ with the positive $x$-axis

**14.** Let $f(x, y) = \ln |\mathbf{r}|$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. Show that $\nabla f = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$.

**15.** Let $f(x, y, z) = |\mathbf{r}|^{-n}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $\nabla f = \dfrac{-n\mathbf{r}}{|\mathbf{r}|^{n+2}}$.

**16.** Show that, in terms of polar coordinates $(r, \theta)$ (where $x = r\cos\theta$ and $y = r\sin\theta$), the gradient of a function $f(r, \theta)$ is given by

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}},$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction of the position vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\hat{\boldsymbol{\theta}}$ is a unit vector at right angles to $\hat{\mathbf{r}}$ in the direction of increasing $\theta$.

**17.** In what directions at the point $(2, 0)$ does the function $f(x, y) = xy$ have rate of change $-1$? Are there directions in which the rate is $-3$? How about $-2$?

**18.** In what directions at the point $(a, b, c)$ does the function $f(x, y, z) = x^2 + y^2 - z^2$ increase at half of its maximal rate at that point?

**19.** Find $\nabla f(a, b)$ for the differentiable function $f(x, y)$ given the directional derivatives

$$D_{(\mathbf{i}+\mathbf{j})/\sqrt{2}} f(a, b) = 3\sqrt{2} \text{ and } D_{(3\mathbf{i}-4\mathbf{j})/5} f(a, b) = 5.$$

**20.** If $f(x, y)$ is differentiable at $(a, b)$, what condition should angles $\phi_1$ and $\phi_2$ satisfy in order that the gradient $\nabla f(a, b)$ can be determined from the values of the directional derivatives $D_{\phi_1} f(a, b)$ and $D_{\phi_2} f(a, b)$?

**21.** The temperature $T(x, y)$ at points of the $xy$-plane is given by $T(x, y) = x^2 - 2y^2$.

(a) Draw a contour diagram for $T$ showing some isotherms (curves of constant temperature).

(b) In what direction should an ant at position $(2, -1)$ move if it wishes to cool off as quickly as possible?

(c) If the ant moves in that direction at speed $k$ (units distance per unit time), at what rate does it experience the decrease of temperature?

(d) At what rate would the ant experience the decrease of temperature if it moved from $(2, -1)$ at speed $k$ in the direction of the vector $-\mathbf{i} - 2\mathbf{j}$?

(e) Along what curve through $(2, -1)$ should the ant move in order to continue to experience maximum rate of cooling?

**22.** Find an equation of the curve in the $xy$-plane that passes through the point $(1, 1)$ and intersects all level curves of the function $f(x, y) = x^4 + y^2$ at right angles.

**23.** Find an equation of the curve in the $xy$-plane that passes through the point $(2, -1)$ and that intersects every curve with equation of the form $x^2y^3 = K$ at right angles.

**24.** Find the second directional derivative of $e^{-x^2-y^2}$ at the point $(a, b) \neq (0, 0)$ in the direction directly away from the origin.

**25.** Find the second directional derivative of $f(x, y, z) = xyz$ at $(2, 3, 1)$ in the direction of the vector $\mathbf{i} - \mathbf{j} - \mathbf{k}$.

**26.** Find a vector tangent to the curve of intersection of the two cylinders $x^2 + y^2 = 2$ and $y^2 + z^2 = 2$ at the point $(1, -1, 1)$.

**27.** Repeat Exercise 26 for the surfaces $x + y + z = 6$ and $x^2 + y^2 + z^2 = 14$ and the point $(1, 2, 3)$.

**28.** The temperature in 3-space is given by

$$T(x, y, z) = x^2 - y^2 + z^2 + xz^2.$$

At time $t = 0$ a fly passes through the point $(1, 1, 2)$, flying along the curve of intersection of the surfaces $z = 3x^2 - y^2$ and $2x^2 + 2y^2 - z^2 = 0$. If the fly's speed is 7, what rate of temperature change does it experience at $t = 0$?

**29.** State and prove a version of Theorem 6 for a function of three variables.

**30.** What is the level surface of $f(x, y, z) = \cos(x + 2y + 3z)$ that passes through $(\pi, \pi, \pi)$? What is the tangent plane to that level surface at that point? (Compare this exercise with Exercise 8 above.)

**31.** If $\nabla f(x, y) = 0$ throughout the disk $x^2 + y^2 < r^2$, prove that $f(x, y)$ is constant throughout the disk.

**32.** Theorem 6 implies that the level curve of $f(x, y)$ passing through $(a, b)$ is smooth (has a tangent line) at $(a, b)$ provided $f$ is differentiable at $(a, b)$ and satisfies $\nabla f(a, b) \neq \mathbf{0}$. Show that the level curve need not be smooth at $(a, b)$ if $\nabla f(a, b) = \mathbf{0}$. (*Hint:* Consider $f(x, y) = y^3 - x^2$ at $(0, 0)$.)

**33.** If $\mathbf{v}$ is a nonzero vector, express $D_{\mathbf{v}}(D_{\mathbf{v}}f)$ in terms of the components of $\mathbf{v}$ and the second partials of $f$. What is the interpretation of this quantity for a moving observer?

**34.** An observer moves so that his position, velocity, and acceleration at time $t$ are given by the formulas $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, $\mathbf{v}(t) = d\mathbf{r}/dt$, and $\mathbf{a}(t) = d\mathbf{v}/dt$. If the temperature in the vicinity of the observer depends only on position, $T = T(x, y, z)$, express the second time derivative of temperature as measured by the observer in terms of $D_{\mathbf{v}}$ and $D_{\mathbf{a}}$.

**35.** Repeat Exercise 34 but with $T$ depending explicitly on time as well as position: $T = T(x, y, z, t)$.

**36.** Let $f(x, y) = \begin{cases} \dfrac{\sin(xy)}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

(a) Calculate $\nabla f(0, 0)$.

(b) Use the definition of directional derivative to calculate $D_{\mathbf{u}}f(0, 0)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

(c) Is $f(x, y)$ differentiable at $(0, 0)$? Why?

**37.** Let $f(x, y) = \begin{cases} 2x^2y/(x^4 + y^2), & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

Use the definition of directional derivative as a limit (Definition 7) to show that $D_{\mathbf{u}}f(0, 0)$ exists for every unit vector $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ in the plane. Specifically, show that $D_{\mathbf{u}}f(0, 0) = 0$ if $v = 0$, and $D_{\mathbf{u}}f(0, 0) = 2u^2/v$ if $v \neq 0$. However, as was shown in Example 4 in Section 12.2, $f(x, y)$ has no limit as $(x, y) \to (0, 0)$, so it is not continuous there. Even if a function has directional derivatives in all directions at a point, it may not be continuous at that point.

In Exercises 1–6, find:

(a) the gradient of the given function at the point indicated,

(b) an equation of the plane tangent to the graph of the given function at the point whose $x$ and $y$ coordinates are given, and

(c) an equation of the straight line tangent, at the given point, to the level curve of the given function passing through that point.

?

$\nabla f$ = tangent of level curves.

**1.** $f(x, y) = x^2 - y^2$ at $(2, -1)$

**2.** $f(x, y) = \dfrac{x-y}{x+y}$ at $(1, 1)$

**3.** $f(x, y) = \dfrac{x}{x^2+y^2}$ at $(1, 2)$

**4.** $f(x, y) = e^{xy}$ at $(2, 0)$

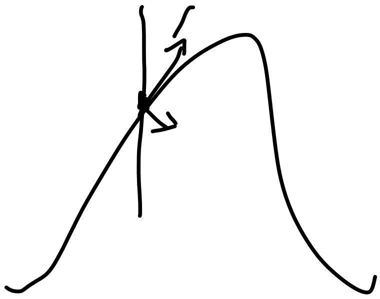
**5.** $f(x, y) = \ln(x^2 + y^2)$ at $(1, -2)$

**6.** $f(x, y) = \sqrt{1 + xy^2}$ at $(2, -2)$

1(a). $\nabla f = \langle 2x, -2y \rangle$ $\quad f(2,-1) = 3$

$\nabla f(2,-1) = \langle 4, 2 \rangle$

b). $4x + 2y - z = 3$



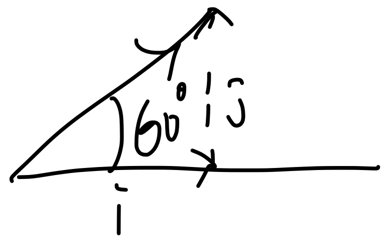
c). $\vec{v} = \langle 4, 2 \rangle$

$P_0 = (2, -1)$

$2 + 4t = x$

$-1 + 2t = y$

**13.** $f(x, y) = x^2 + y^2$ at $(1, -2)$ in the direction making a (positive) angle of $60°$ with the positive $x$-axis




$$\begin{array}{ccccc} & 0 & \frac{\pi}{6} & \frac{\pi}{4} & \frac{\pi}{3} \\ \sin & 0 & \frac{\sqrt{1}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{2} \end{array}$$

$\frac{\sqrt{4}}{2}$.

$$\cos 60°\, \hat{i} + \sin 60°\, \hat{j}$$

$$\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$$

**14.** Let $f(x, y) = \ln|\mathbf{r}|$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. Show that $\nabla f = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$.

$$\ln\left(\sqrt{x^2+y^2}\right)$$

$$= \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{(x^2+y^2)}$$

$$= \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}$$

$$\nabla f = \frac{r}{|r|^2}$$

Made with Goodnotes

**16.** Show that, in terms of polar coordinates $(r, \theta)$ (where $x = r\cos\theta$ and $y = r\sin\theta$), the gradient of a function $f(r, \theta)$ is given by

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}},$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction of the position vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\hat{\boldsymbol{\theta}}$ is a unit vector at right angles to $\hat{\mathbf{r}}$ in the direction of increasing $\theta$.

**17.** In what directions at the point $(2, 0)$ does the function $f(x, y) = xy$ have rate of change $-1$? Are there directions in which the rate is $-3$? How about $-2$?

$$\nabla f = \langle y, x \rangle$$

$-1$ : $\nabla f(2,0) = \langle 0, 2 \rangle$ $\qquad \hat{u} = xi + yj$

$$\nabla f \cdot \hat{u} = -1 \qquad \hat{u} = -\tfrac{1}{2}j$$

$$2y = -1$$

$$y = -\tfrac{1}{2}$$

$-3$ $\qquad \nabla f \cdot \hat{u} = -3 \qquad \hat{u} = -\tfrac{3}{2}j$

$$2(y) = -3$$

$$y = -\tfrac{3}{2}$$

$-2$ $\qquad \hat{u} = -j$

**18.** In what directions at the point $(a, b, c)$ does the function $f(x, y, z) = x^2 + y^2 - z^2$ increase at half of its maximal rate at that point?

$$\nabla f = \langle 2x, 2y, -2z \rangle$$

$$\nabla f = \langle 2a, 2b, -2c \rangle \text{ at } (a, b, c)$$

$$\frac{1}{2} \nabla f \cdot u =$$

**20.** If $f(x, y)$ is differentiable at $(a, b)$, what condition should angles $\phi_1$ and $\phi_2$ satisfy in order that the gradient $\nabla f(a, b)$ can be determined from the values of the directional derivatives $D_{\phi_1} f(a, b)$ and $D_{\phi_2} f(a, b)$?

**20.** Given the values $D_{\phi_1} f(a, b)$ and $D_{\phi_2} f(a, b)$, we can solve the equations

$$\begin{aligned} f_1(a, b) \cos\phi_1 + f_2(a, b) \sin\phi_1 &= D_{\phi_1} f(a, b) \\ f_1(a, b) \cos\phi_2 + f_2(a, b) \sin\phi_2 &= D_{\phi_2} f(a, b) \end{aligned}$$

for unique values of $f_1(a, b)$ and $f_2(a, b)$ (and hence determine $\nabla f(a, b)$ uniquely), provided the coefficients satisfy

$$0 \neq \begin{vmatrix} \cos\phi_1 & \sin\phi_1 \\ \cos\phi_2 & \sin\phi_2 \end{vmatrix} = \sin(\phi_2 - \phi_1).$$

Thus $\phi_1$ and $\phi_2$ must not differ by an integer multiple of $\pi$.

**21.** The temperature $T(x, y)$ at points of the $xy$-plane is given by $T(x, y) = x^2 - 2y^2$.

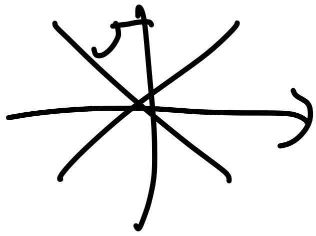
(a) Draw a contour diagram for $T$ showing some isotherms (curves of constant temperature).

(b) In what direction should an ant at position $(2, -1)$ move if it wishes to cool off as quickly as possible?

(c) If the ant moves in that direction at speed $k$ (units distance per unit time), at what rate does it experience the decrease of temperature?

(d) At what rate would the ant experience the decrease of temperature if it moved from $(2, -1)$ at speed $k$ in the direction of the vector $-\mathbf{i} - 2\mathbf{j}$?

(e) Along what curve through $(2, -1)$ should the ant move in order to continue to experience maximum rate of cooling?



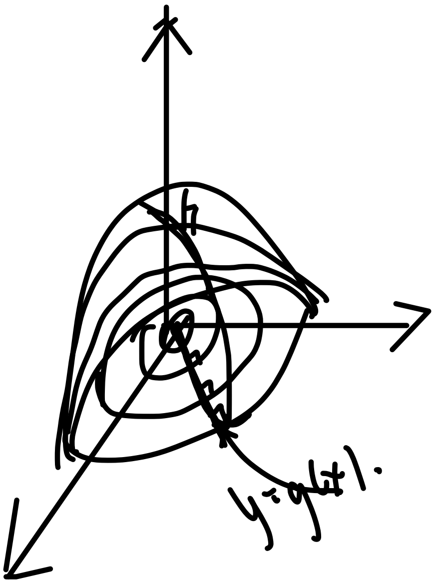
**22.** Find an equation of the curve in the $xy$-plane that passes through the point $(1, 1)$ and intersects all level curves of the function $f(x, y) = x^4 + y^2$ at right angles.

$$\nabla f = \langle 4x^3, 2y \rangle$$

$$\left\langle \frac{dx}{dt} \hat{i}, \frac{dy}{dt} \hat{j} \right\rangle = \lambda \langle 4x^3 \hat{i}, 2y \hat{j} \rangle$$

$$\frac{dx}{dt} \frac{1}{4x^3} = \frac{dy}{dt} \frac{1}{2y}$$

$$\frac{dx}{dt} \frac{1}{x^3} = \frac{dy}{dt} \frac{2}{y}$$




$$\frac{1}{-3+1} x^{-3+1} = 2\ln|y| + C$$

$$-\frac{1}{2} x^{-2} = 2\ln|y| + C$$

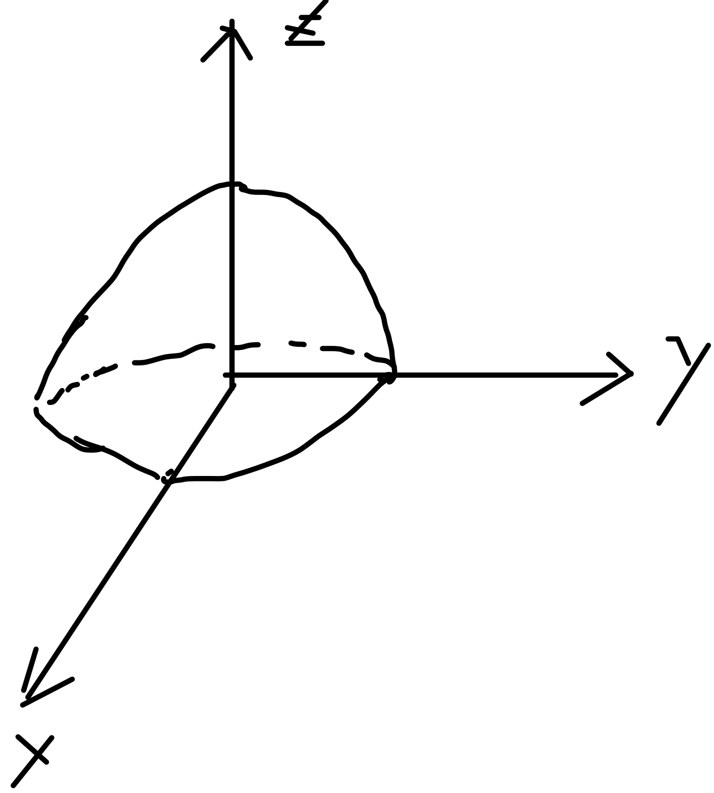
$$-\frac{1}{2} = C$$

$$-\frac{1}{2} x^{-2} = 2\ln|y| - \frac{1}{2}$$

$$x^{-2} = -4\ln|y| + 1$$

$$=$$

Made with Goodnotes


z
y
x

**23.** Find an equation of the curve in the $xy$-plane that passes through the point $(2, -1)$ and that intersects every curve with equation of the form $x^2y^3 = K$ at right angles.

**24.** Find the second directional derivative of $e^{-x^2-y^2}$ at the point $(a, b) \neq (0, 0)$ in the direction directly away from the origin.

**25.** Find the second directional derivative of $f(x, y, z) = xyz$ at $(2, 3, 1)$ in the direction of the vector $\mathbf{i} - \mathbf{j} - \mathbf{k}$.

24.

**26.** Find a vector tangent to the curve of intersection of the two cylinders $x^2 + y^2 = 2$ and $y^2 + z^2 = 2$ at the point $(1, -1, 1)$.

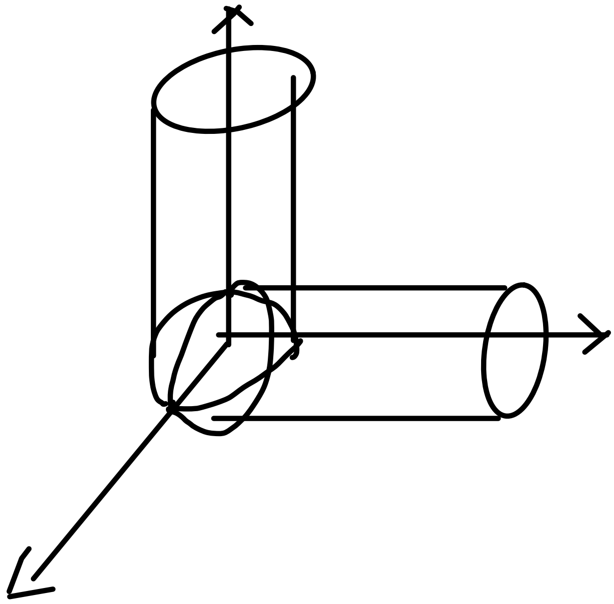
**27.** Repeat Exercise 26 for the surfaces $x + y + z = 6$ and $x^2 + y^2 + z^2 = 14$ and the point $(1, 2, 3)$.

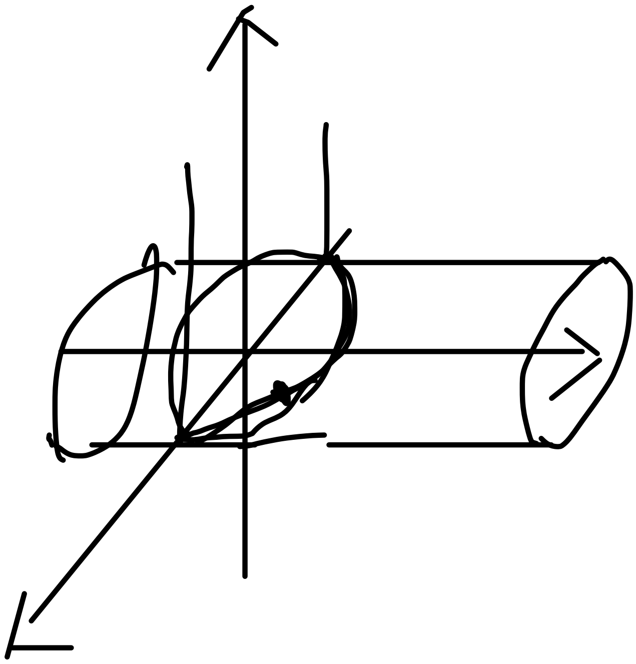
$y^2 = 2 - x^2$

$y^2 = 2 - z^2$

$2 - x^2 = 2 - z^2$

$x^2 = z^2$





**28.** The temperature in 3-space is given by

$$T(x, y, z) = x^2 - y^2 + z^2 + xz^2.$$

At time $t = 0$ a fly passes through the point $(1, 1, 2)$, flying along the curve of intersection of the surfaces $z = 3x^2 - y^2$ and $2x^2 + 2y^2 - z^2 = 0$. If the fly's speed is 7, what rate of temperature change does it experience at $t = 0$?

**30.** What is the level surface of $f(x, y, z) = \cos(x + 2y + 3z)$ that passes through $(\pi, \pi, \pi)$? What is the tangent plane to that level surface at that point? (Compare this exercise with Exercise 8 above.)

**31.** If $\nabla f(x, y) = 0$ throughout the disk $x^2 + y^2 < r^2$, prove that $f(x, y)$ is constant throughout the disk.

**32.** Theorem 6 implies that the level curve of $f(x, y)$ passing through $(a, b)$ is smooth (has a tangent line) at $(a, b)$ provided $f$ is differentiable at $(a, b)$ and satisfies $\nabla f(a, b) \neq \mathbf{0}$. Show that the level curve need not be smooth at $(a, b)$ if $\nabla f(a, b) = \mathbf{0}$. (*Hint:* Consider $f(x, y) = y^3 - x^2$ at $(0, 0)$.)

**33.** If $\mathbf{v}$ is a nonzero vector, express $D_{\mathbf{v}}(D_{\mathbf{v}} f)$ in terms of the components of $\mathbf{v}$ and the second partials of $f$. What is the interpretation of this quantity for a moving observer?

**34.** An observer moves so that his position, velocity, and acceleration at time $t$ are given by the formulas $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, $\mathbf{v}(t) = d\mathbf{r}/dt$, and $\mathbf{a}(t) = d\mathbf{v}/dt$. If the temperature in the vicinity of the observer depends only on position, $T = T(x, y, z)$, express the second time derivative of temperature as measured by the observer in terms of $D_{\mathbf{v}}$ and $D_{\mathbf{a}}$.

**35.** Repeat Exercise 34 but with $T$ depending explicitly on time as well as position: $T = T(x, y, z, t)$.

**36.** Let $f(x, y) = \begin{cases} \dfrac{\sin(xy)}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

(a) Calculate $\nabla f(0, 0)$.

(b) Use the definition of directional derivative to calculate $D_{\mathbf{u}} f(0, 0)$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$.

(c) Is $f(x, y)$ differentiable at $(0, 0)$? Why?

**37.** Let $f(x, y) = \begin{cases} 2x^2 y/(x^4 + y^2), & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

Use the definition of directional derivative as a limit (Definition 7) to show that $D_{\mathbf{u}} f(0, 0)$ exists for every unit vector $\mathbf{u} = u\mathbf{i} + v\mathbf{j}$ in the plane. Specifically, show that $D_{\mathbf{u}} f(0, 0) = 0$ if $v = 0$, and $D_{\mathbf{u}} f(0, 0) = 2u^2/v$ if $v \neq 0$. However, as was shown in Example 4 in Section 12.2, $f(x, y)$ has no limit as $(x, y) \to (0, 0)$, so it is not continuous there. Even if a function has directional derivatives in all directions at a point, it may not be continuous at that point.

36. $\displaystyle \lim_{h \to 0^+} \frac{f(x + \frac{1}{\sqrt{2}}h,\ y + \frac{1}{\sqrt{2}}h) - f(x,y)}{h}$

$$= \frac{1}{h} \frac{\sin\left(\frac{h^2}{2}\right)}{\sqrt{\frac{1}{2}h^2 + \frac{1}{2}h^2}}$$

$$= \frac{1}{h^2} \sin\left(\frac{h^2}{2}\right)$$

$$= \frac{\cos\left(\frac{h^2}{2}\right)(h)}{2h}$$

$$= \frac{\cos\left(\frac{h^2}{2}\right)}{2} = \frac{1}{2}.$$