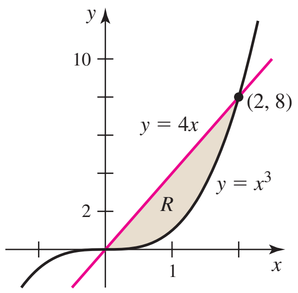
MATH 2023 • Multivariable Calculus
Problem Set #5 • Double Integrals

1. (★) Set-up the lower and upper bounds of each double integral below using **both** $dxdy$ and $dydx$ orders. Compute the integral using **both** orders and verify that they give the same value.

   (a) $\iint_R 2xy\,dA$ where R is the region as shown below:

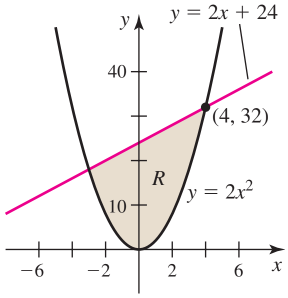


Solution:

$$\begin{aligned}
\iint_R 2xy\,dA &= \int_{y=0}^{y=8}\int_{x=\frac{y}{4}}^{x=y^{1/3}} 2xy\,dxdy \\
&= \int_{y=0}^{y=8} \left[x^2y\right]_{x=\frac{y}{4}}^{x=y^{1/3}}\,dy \\
&= \int_{y=0}^{y=8} \left(y^{2/3}\cdot y - \frac{y^2}{16}\cdot y\right)\,dy \\
&= \int_{y=0}^{y=8} \left(y^{5/3} - \frac{y^3}{16}\right)\,dy \\
&= 32
\end{aligned}$$

$$\begin{aligned}
\iint_R 2xy\,dA &= \int_{x=0}^{x=2}\int_{y=x^3}^{y=4x} 2xy\,dydx \\
&= \int_{x=0}^{x=2} \left[xy^2\right]_{y=x^3}^{y=4x}\,dx \\
&= \int_{x=0}^{x=2} \left(x\cdot 16x^2 - x\cdot x^6\right)\,dx \\
&= \int_{x=0}^{x=2} \left(16x^3 - x^7\right)\,dx \\
&= 32
\end{aligned}$$

(b) $\iint_R 1\,dA$ where R is the region bounded between $y = 2x + 24$ and $y = 2x^2$:



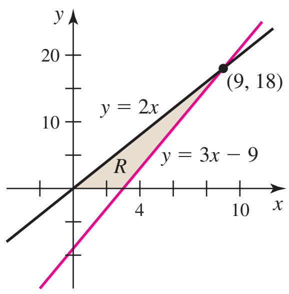
Solution: (Answer only)

$$dxdy \text{ order: } \int_{y=0}^{y=18}\int_{x=-\sqrt{\frac{y}{2}}}^{x=\sqrt{\frac{y}{2}}} 1\,dxdy + \int_{y=18}^{y=32}\int_{x=\frac{y-24}{2}}^{x=\sqrt{\frac{y}{2}}} 1\,dxdy$$

$$dydx \text{ order: } \int_{x=-3}^{x=4}\int_{y=2x^2}^{y=2x+24} 1\,dydx$$

$$\text{Answer: } \frac{343}{3}$$

(c) $\iint_R x^2\,dA$ where R is the region bounded between $y = 2x$, $y = 3x - 9$ and the x-axis:



Solution: (Answer only)

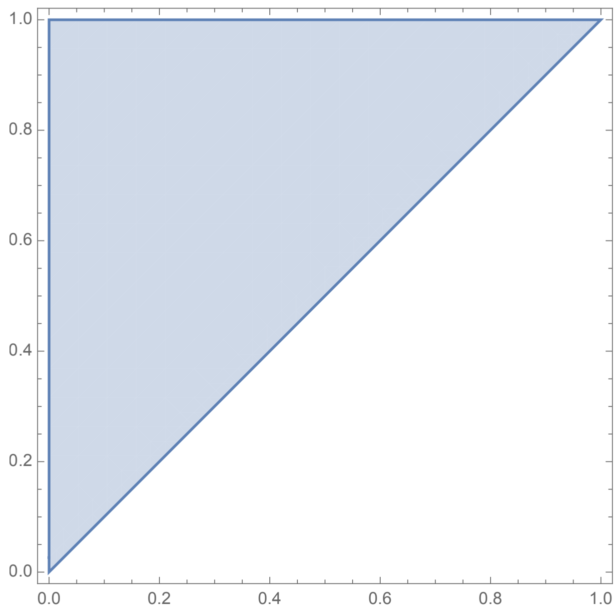
$$dxdy \text{ order: } \int_{y=0}^{y=18}\int_{x=\frac{y}{2}}^{x=\frac{y+9}{3}} x^2\,dxdy$$

$$dydx \text{ order: } \int_{x=0}^{x=3}\int_{y=0}^{y=2x} x^2\,dydx + \int_{x=3}^{x=9}\int_{y=3x-9}^{y=2x} x^2\,dydx$$

$$\text{Answer: } \frac{1053}{2}$$

2. (★) Evaluate the integral $\iint_T \sqrt{a^2 - y^2}\, dA$ where T is the triangle with vertices $(0,0)$, $(0,a)$ and (a,a). Set-up the integral in both $dxdy$ and $dydx$ orders, and choose the *easier* one to compute.

Solution:



Below is the case where $a \geq 0$. The case where $a < 0$ is similar (but different).

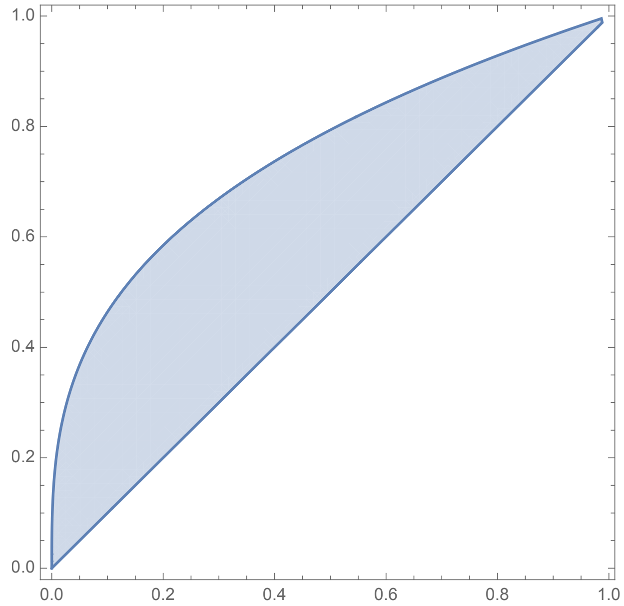
$$\begin{aligned}\iint_T \sqrt{a^2 - y^2}\, dA &= \int_{x=0}^{x=a} \int_{y=x}^{y=a} \sqrt{a^2 - y^2}\, dydx \\ &= \int_{y=0}^{y=a} \int_{x=0}^{x=y} \sqrt{a^2 - y^2}\, dxdy\end{aligned}$$

Easier to integrate using $dxdy$-order:

$$\begin{aligned}\int_{y=0}^{y=a} \int_{x=0}^{x=y} \sqrt{a^2 - y^2}\, dxdy &= \int_{y=0}^{y=a} \left[x\sqrt{a^2 - y^2} \right]_{x=0}^{x=y} dy \\ &= \int_{y=0}^{y=a} y\sqrt{a^2 - y^2}\, dy \\ &= -\frac{1}{2} \int_{y=0}^{y=a} \sqrt{a^2 - y^2}\, d(a^2 - y^2) \\ &= -\frac{1}{2} \left[\frac{2}{3} \left(a^2 - y^2\right)^{3/2} \right]_{y=0}^{y=a} \\ &= \frac{1}{3} a^3\end{aligned}$$

3. (★★) Consider the integral $\int_0^1 \int_x^{x^{1/3}} \sqrt{1-y^4}\, dydx$. It is almost impossible to compute the inner integral. Try to switch the order of integration to evaluate it. [Hint: You should first sketch the region of integration.]

Solution:

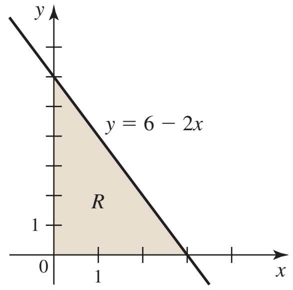


According to the region represented by the integral, switching the integration order gives:

$$\begin{aligned}
\int_0^1 \int_x^{x^{1/3}} \sqrt{1-y^4}\, dydx &= \int_0^1 \int_{x=y^3}^{x=y} \sqrt{1-y^4}\, dxdy \\
&= \int_0^1 \left[x\sqrt{1-y^4} \right]_{x=y^3}^{x=y} dy \\
&= \int_0^1 \left(y\sqrt{1-y^4} - y^3\sqrt{1-y^4} \right) dy \\
&= \int_0^1 y\sqrt{1-y^4}\, dy - \int_0^1 y^3\sqrt{1-y^4}\, dy \\
&= \frac{\pi}{8} - \frac{1}{6}
\end{aligned}$$

Remark: Let $u = y^2$ for the first integral, and let $v = 1 - y^4$ for the second integral (detail omitted).

4. (★★) Evaluate the integrals $\iint_R \frac{1}{3-x}\,dA$ and $\iint_R \frac{1}{y-6}\,dA$. Try to avoid *integration-by-parts* if possible.

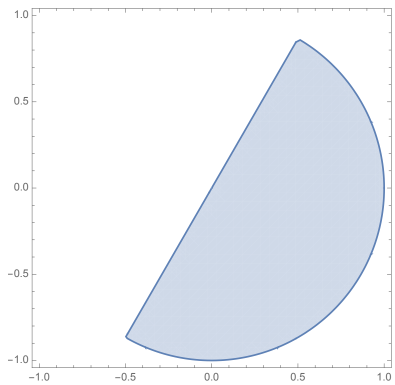


Solution:

$$\begin{aligned}
\iint_R \frac{1}{3-x}\,dA &= \int_{x=0}^{x=3} \int_{y=0}^{y=6-2x} \frac{1}{3-x}\,dydx \\
&= \int_{x=0}^{x=3} \left[\frac{y}{3-x}\right]_{y=0}^{y=6-2x} dx \\
&= \int_{x=0}^{x=3} \frac{6-2x}{3-x}\,dx \\
&= \int_{x=0}^{x=3} \frac{2(3-x)}{(3-x)}\,dx \\
&= \int_{x=0}^{x=3} 2\,dx = 6 \\
\iint_R \frac{1}{y-6}\,dA &= \int_{y=0}^{y=6} \int_{x=0}^{x=\frac{6-y}{2}} \frac{1}{y-6}\,dxdy \\
&= \int_{y=0}^{y=6} \left[\frac{x}{y-6}\right]_{x=0}^{x=\frac{6-y}{2}} dy \\
&= \int_{y=0}^{y=6} -\frac{1}{2}\,dy = -3.
\end{aligned}$$

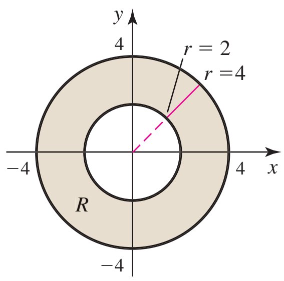
5. (★) Evaluate $\iint_R (x+y)\, dA$ using polar coordinates where R is the region in the first quadrant lying inside the disk $x^2+y^2 \le a^2$ and under the line $y=\sqrt{3}x$.

Solution: (Sketch only)



$$\iint_R (x+y)\, dA = \int_{\theta=-\frac{2\pi}{3}}^{\theta=\frac{\pi}{3}} \int_{r=0}^{r=a} (r\cos\theta + r\sin\theta)\, r dr d\theta$$
$$= \int_{\theta=-\frac{2\pi}{3}}^{\theta=\frac{\pi}{3}} \int_{r=0}^{r=a} r^2(\cos\theta+\sin\theta)\, dr d\theta = \frac{\sqrt{3}-1}{3}a^3$$

6. (★) Consider the annular region R below. Express the integral $\iint_R (x^2+y^2)\, dA$ in **both** rectangular and polar coordinates. Choose the *easier* system to compute the integral.



Solution: See Solutions to Worksheet #11 Q5 and Worksheet #12 Q2. Obviously it is easier to compute it using polar coordinates.

7. (★★) Evaluate each of the following integrals:

(a) $\int_0^{2\pi}\int_0^1 e^{-x^2}\sin y\,dxdy$

Solution: Since it is impossible to integrate e^{-x^2}, we try to switch the order of integration to see if there is any luck. The bounds are all constants, so we simply switch the integral signs:

$$\begin{aligned}\int_0^{2\pi}\int_0^1 e^{-x^2}\sin y\,dxdy &= \int_0^1\int_0^{2\pi} e^{-x^2}\sin y\,dydx\\ &= \int_0^1\left[-e^{-x^2}\cos y\right]_{y=0}^{y=2\pi}dx\\ &= \int_0^1\left(-e^{-x^2}+e^{-x^2}\right)dx\\ &= \int_0^1 0\,dx = 0\end{aligned}$$

(b) $\int_{-1}^{0}\int_0^{\sqrt{y+1}}\left(x-\frac{x^3}{3}\right)^{5/2}dxdy$

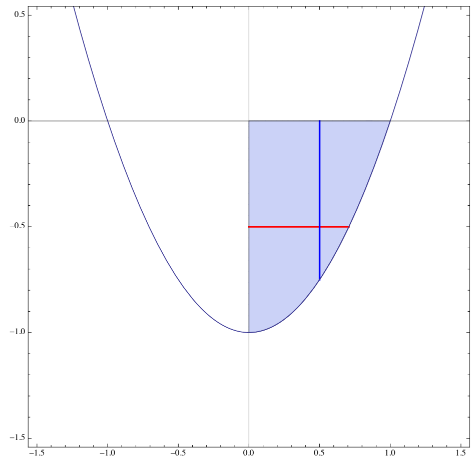
Solution: It is very difficult to integrate the given function by dx, we try to switch the order of integration to see if there is any luck. Note that the bounds involve variables, so we have to sketch the diagram of the domain first. In the $dxdy$-order, the x-sample strips enter the region from $x=0$ and leaves from $x=\sqrt{y+1}$ (i.e. part of the parabola $y=x^2-1$), see the red strip in the diagram.

To switch order, we draw a sample blue strip in the diagram, which enters the region at $y=x^2-1$ and leaves at $y=0$. The minimum value of x is 0 and the maximum of x is 1. Therefore, after switching dx and dy, we get:

$$\begin{aligned}\int_{-1}^{0}\int_0^{\sqrt{y+1}}\left(x-\frac{x^3}{3}\right)^{5/2}dxdy &= \int_0^1\int_{x^2-1}^{0}\left(x-\frac{x^3}{3}\right)^{5/2}dydx\\ &= \int_0^1\left[\left(x-\frac{x^3}{3}\right)^{5/2}y\right]_{y=x^2-1}^{y=0}dx\\ &= \int_0^1\left(x-\frac{x^3}{3}\right)^{5/2}(1-x^2)\,dx\\ &= \int_{u=0}^{u=2/3}u^{5/2}\,du\end{aligned}$$

where we let $u=x-\frac{x^3}{3}$, then $du=(1-x^2)dx$. Keep going:

$$\int_{u=0}^{u=2/3}u^{5/2}\,du=\left[\frac{2}{7}u^{7/2}\right]_{u=0}^{u=2/3}=\frac{2}{7}\left(\frac{2}{3}\right)^{7/2}.$$



(c) $\displaystyle\iint_Q \frac{1}{(1+2x^2+2y^2)^3}\,dA$ where Q is the entire first quadrant of the xy-plane

Solution: The term x^2+y^2 appears in the integrand, so it is typically better to be done in polar coordinates. The first quadrant Q is represented in polar coordinates by $0 \leq r < \infty$ and $0 \leq \theta \leq 2\pi$. Therefore,

$$\begin{aligned}\iint_Q \frac{1}{(1+2x^2+2y^2)^3}\,dA &= \int_0^{\frac{\pi}{2}}\int_0^{\infty} \frac{1}{(1+2r^2)^3}\,rdrd\theta \\ &= \int_0^{\frac{\pi}{2}}\left[-\frac{1}{8(1+2r^2)^2}\right]_{r=0}^{r=\infty} d\theta \\ &= \int_0^{\frac{\pi}{2}}\left(-0+\frac{1}{8}\right)d\theta = \frac{\pi}{2}\cdot\frac{1}{8} \\ &= \frac{\pi}{16}.\end{aligned}$$

(d) $\displaystyle\iint_Q (x^2-y+1)\,dA$ where Q is the region $\{(x,y) \mid x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 4\}$

Solution: The domain Q is a sector, so it's again better be done by polar coordinates. In polar coordinates, the domain Q is represented by $0 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$. Therefore,

$$\begin{aligned}\iint_Q (x^2-y+1)\,dA &= \int_0^{\pi/2}\int_0^2 (r^2\cos^2\theta - r\sin\theta + 1)\,rdrd\theta \\ &= \int_0^{\pi/2}\int_0^2 (r^3\cos^2\theta - r^2\sin\theta + r)\,drd\theta \\ &= 2\pi - \frac{8}{3}\end{aligned}$$

The Half/Double-Angle Formula should be used to compute the integral for $\cos^2\theta$.

(e) $\displaystyle\iint_{\mathbb{R}^2} (x^2+y^2)e^{-(x^4+2x^2y^2+y^4)}\,dA$

Solution: It is useful to note that $x^4+2x^2y^2+y^4 = (x^2)^2+2(x^2)(y^2)+(y^2)^2 = (x^2+y^2)^2$, and so it suggests that the integral had better be done by polar coordinates. The domain $\mathbb{R}^2$ can be expressed in polar coordinates as $0 \le r < \infty$ and $0 \le \theta \le 2\pi$:

$$\begin{aligned}
\iint_{\mathbb{R}^2} (x^2+y^2)e^{-(x^4+2x^2y^2+y^4)}\,dA &= \int_0^{2\pi}\int_0^{\infty} r^2 e^{-(r^2)^2}\, r dr d\theta \\
&= \int_0^{2\pi}\int_0^{\infty} r^3 e^{-r^4}\,dr d\theta \\
&= \int_0^{2\pi} \left[-\frac{e^{-r^4}}{4}\right]_{r=0}^{r=\infty} d\theta \\
&= \int_0^{2\pi} \left(-0+\frac{1}{4}\right) d\theta \\
&= 2\pi\cdot\frac{1}{4} = \frac{\pi}{2}.
\end{aligned}$$

8. (★★) Some single-variable integrals are "notoriously" difficult to compute. One example is $\int e^{-x^2} dx$ despite the fact that this integral is of central importance in mathematics (pure/applied), physics, statistics and engineering. However, some of these difficult integrals can be evaluated via double integral methods.

   This problem investigates another well-known integral which has no closed-form antiderivative:
   $$\int \frac{\log(1-x)}{x}dx.$$

   The goal of this problem is to show that this integral over $0 \le x \le 1$ can be written as an infinite series.

   Consider the function
   $$f(x,y) = \frac{1}{1-xy}, \qquad 0 \le x \le 1, \quad 0 \le y \le 1.$$

   It is defined almost everywhere on the rectangle $0 \le x \le 1$ and $0 \le y \le 1$ (we say 'almost' because it's undefined only at $(x,y) = (1,1)$, but this single point is negligible).

(a) Show that: $\displaystyle\int_0^1 \frac{1}{1-xy}dy = -\frac{\log(1-x)}{x}$.

Solution: Let $u(y) = xy$ regarding x is a constant and y is the variable. Then $du = xdy$. When $y = 0$, $u = 0$; when $y = 1$, $u = x$. Therefore, by u-substitution:

$$\begin{aligned}\int_0^1 \frac{1}{1-xy}dy &= \int_0^x \frac{1}{x(1-u)}du \\ &= \frac{1}{x}\int_0^x \frac{1}{1-u}du \\ &= \frac{1}{x}\left[-\log(1-u)\right]_{u=0}^{u=x} \\ &= -\frac{1}{x}(\log(1-x) - \log(1-0)) = -\frac{\log(1-x)}{x}.\end{aligned}$$

(b) Note that $|xy| < 1$ except for the negligible point $(x, y) = (1, 1)$, so the function $f(x, y)$ can be expressed as a geometric series:

$$\frac{1}{1-xy} = 1 + (xy) + (xy)^2 + (xy)^3 + \dots$$

Using this geometric series, show that

$$\int_0^1\int_0^1 \frac{1}{1-xy}dydx = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \sum_{k=1}^{\infty}\frac{1}{k^2}.$$

Solution: By geometric series expansion:

$$\frac{1}{1-xy} = 1 + (xy) + (xy)^2 + (xy)^3 + \dots$$

Integrate both sides:

$$\begin{aligned}&\int_0^1\int_0^1 \frac{1}{1-xy}dydx \\ &= \int_0^1\int_0^1 (1 + xy + x^2y^2 + x^3y^3 + \dots)dydx \\ &= \int_0^1\int_0^1 dydx + \int_0^1\int_0^1 xydydx + \int_0^1\int_0^1 x^2y^2dydx + \int_0^1\int_0^1 x^3y^3dydx + \dots \\ &= \int_0^1 [y]_{y=0}^{y=1}dx + \int_0^1 \left[x\cdot\frac{y^2}{2}\right]_{y=0}^{y=1}dx + \int_0^1 \left[x^2\cdot\frac{y^3}{3}\right]_{y=0}^{y=1}dx + \int_0^1 \left[x^3\cdot\frac{y^4}{4}\right]_{y=0}^{y=1}dx + \dots \\ &= \int_0^1 dx + \int_0^1 \frac{x}{2}dx + \int_0^1 \frac{x^2}{3}dx + \int_0^1 \frac{x^3}{4}dx + \dots \\ &= [x]_{x=0}^{x=1} + \left[\frac{1}{2}\cdot\frac{x^2}{2}\right]_{x=0}^{x=1} + \left[\frac{1}{3}\cdot\frac{x^3}{3}\right]_{x=0}^{x=1} + \left[\frac{1}{4}\cdot\frac{x^4}{4}\right]_{x=0}^{x=1} + \dots \\ &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \\ &= \sum_{k=1}^{\infty}\frac{1}{k^2}.\end{aligned}$$

(c) Using (a) and (b), show that

$$-\int_0^1 \frac{\log(1-x)}{x}dx = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Solution: Simply combine the results obtained in (a) and (b):

$$\begin{aligned}
\int_0^1 \left(\int_0^1 \frac{1}{1-xy}dy\right) dx &= \sum_{k=1}^{\infty} \frac{1}{k^2} && \text{(from (b))} \\
\int_0^1 \left(-\frac{\log(1-x)}{x}\right) dx &= \sum_{k=1}^{\infty} \frac{1}{k^2} && \text{(from (a))} \\
-\int_0^1 \frac{\log(1-x)}{x}dx &= \sum_{k=1}^{\infty} \frac{1}{k^2}.
\end{aligned}$$

(d) Using the above approach, *mutatis mutandis*, show that for any $0 \le z \le 1$, we have:

$$-\int_0^z \frac{\log(1-x)}{x}dx = \sum_{k=1}^{\infty} \frac{z^k}{k^2}.$$

[Remark: *Mutatis mutandis* is a Latin phrase meaning "changing only those things which need to be changed".]

Solution: Replace the outer $\int_0^1$ by $\int_0^z$ (while keeping the inner $\int_0^1$ unchanged):

$$\begin{aligned}
&\int_0^z \int_0^1 \frac{1}{1-xy}dydx \\
&= \int_0^z \int_0^1 (1 + xy + x^2y^2 + x^3y^3 + \ldots)dydx \\
&= \int_0^z \int_0^1 dydx + \int_0^z \int_0^1 xydydx + \int_0^z \int_0^1 x^2y^2dydx + \int_0^z \int_0^1 x^3y^3dydx + \ldots \\
&= \int_0^z [y]_{y=0}^{y=1} dx + \int_0^z \left[x \cdot \frac{y^2}{2}\right]_{y=0}^{y=1} dx + \int_0^z \left[x^2 \cdot \frac{y^3}{3}\right]_{y=0}^{y=1} dx + \int_0^z \left[x^3 \cdot \frac{y^4}{4}\right]_{y=0}^{y=1} dx + \ldots \\
&= \int_0^z dx + \int_0^z \frac{x}{2}dx + \int_0^z \frac{x^2}{3}dx + \int_0^z \frac{x^3}{4}dx + \ldots \\
&= [x]_{x=0}^{x=z} + \left[\frac{1}{2} \cdot \frac{x^2}{2}\right]_{x=0}^{x=z} + \left[\frac{1}{3} \cdot \frac{x^3}{3}\right]_{x=0}^{x=z} + \left[\frac{1}{4} \cdot \frac{x^4}{4}\right]_{x=0}^{x=z} + \ldots \\
&= z + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \frac{z^4}{4^2} + \ldots \\
&= \sum_{k=1}^{\infty} \frac{z^k}{k^2}.
\end{aligned}$$

From (a), we get:

$$-\int_0^z \frac{\log(1-x)}{x}dx = \sum_{k=1}^{\infty} \frac{z^k}{k^2}.$$

9. (★★★) The purpose of this problem is to use double integrals to derive a somewhat surprising result in electrostatics, that is the electric force exerted on a charged particle by an infinite sheet of uniformly distributed charges is *independent* of how far the particle and the sheet are apart from each other.

The paragraphs below describe the physical set-up of the problem. Although it may be possible to proceed to the problem without knowing the physics background, it is strongly recommend to read through the paragraphs below so as to understand the motivation of this problem.

According to the Coulomb's Law, the electric force $\mathbf{F}$ exerted **on** a point particle with charge Q located at (x_0, y_0, z_0), **by** a point particle with charge q located at (x, y, z), is given by:

$$\mathbf{F} = \frac{qQ}{4\pi\varepsilon_0} \frac{(x_0 - x)\mathbf{i} + (y_0 - y)\mathbf{j} + (z_0 - z)\mathbf{k}}{\left((x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2\right)^{3/2}}$$

where ε_0 is positive constant (depending on the medium).

The Coulomb's Law is also called the Inverse Square Law because one can easily verify that the magnitude of the force satisfies:

$$|\mathbf{F}| = \frac{qQ}{4\pi\varepsilon_0 d^2}$$

where $d = \sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}$ is the distance between the two particles.

If there is a sequence of *discrete* charged particles located at (x_1, y_1, z_1), (x_2, y_2, z_2), $\ldots$, each with charge q, then the resultant electric force exerted on a particle with charge Q located at (x_0, y_0, z_0), is given by the vector sum of all forces:

$$\mathbf{F} = \sum_{i=1}^{\infty} \frac{qQ}{4\pi\varepsilon_0} \frac{(x_0 - x_i)\mathbf{i} + (y_0 - y_i)\mathbf{j} + (z_0 - z_i)\mathbf{k}}{\left((x_0 - x_i)^2 + (y_0 - y_i)^2 + (z_0 - z_i)^2\right)^{3/2}}.$$

This is called the Principle of Superposition by physicists.

Now given there is an infinite sheet of uniformly distributed charges on the xy-plane, and for each small area element dA on the xy-plane, the amount of charges is given by $\sigma\, dA$, where σ is a constant that represents the area density of charges. Suppose there is a particle with charge Q located above the xy-plane at $(0, 0, z_0)$, i.e. $z_0 > 0$. For simplicity, call this the Q-particle.

Now regard a small area element located at $(x, y, 0)$ on the xy-plane as a charged "particle" with charge $q = \sigma\, dA$, then the force exerted on the Q-particle by this area element is given by substituting $(x, y, z) = (x, y, 0)$ and $(x_0, y_0, z_0) = (0, 0, z_0)$:

$$\frac{Q(\sigma\, dA)}{4\pi\varepsilon_0} \frac{(0 - x)\mathbf{i} + (0 - y)\mathbf{j} + (z_0 - 0)\mathbf{k}}{\left((0 - x)^2 + (0 - y)^2 + (z_0 - 0)^2\right)^{3/2}} = \frac{Q\sigma}{4\pi\varepsilon_0} \frac{-x\mathbf{i} - y\mathbf{j} + z_0\mathbf{k}}{\left(x^2 + y^2 + z_0^2\right)^{3/2}}\, dA.$$

Therefore, by the Principle of Superposition, the resultant electric force exerted on the Q-particle by the sheet of charges is given by this double integral over the entire xy-plane (i.e. $\mathbb{R}^2$):

$$\mathbf{F}_{\text{resultant}} = \iint_{\mathbb{R}^2} \frac{Q\sigma}{4\pi\varepsilon_0} \frac{-x\mathbf{i} - y\mathbf{j} + z_0\mathbf{k}}{\left(x^2 + y^2 + z_0^2\right)^{3/2}}\, dA.$$

Here integrating a vector simply means integrating each component of the vector treating $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as "constants".

(a) Show that $\mathbf{i}$ and $\mathbf{j}$-components of $\mathbf{F}_{\text{resultant}}$ are zero.

Solution: Since $x^2 + y^2 = r^2$, we use polar coordinates again to simplify our calculations.

$$\begin{aligned}
&\mathbf{F}_{\text{resultant}} \\
&= \int_0^{2\pi}\int_0^{\infty} \frac{Q\sigma}{4\pi\varepsilon_0} \frac{-r\cos\theta\,\mathbf{i} - r\sin\theta\,\mathbf{j} + z_0\mathbf{k}}{(r^2+z_0^2)^{3/2}}\, r dr d\theta \\
&= \frac{Q\sigma}{4\pi\varepsilon_0}\left[\left(\int_0^{2\pi}\int_0^{\infty} \frac{-r^2\cos\theta\, drd\theta}{(r^2+z_0^2)^{3/2}}\right)\mathbf{i} + \left(\int_0^{2\pi}\int_0^{\infty} \frac{-r^2\sin\theta\, drd\theta}{(r^2+z_0^2)^{3/2}}\right)\mathbf{j}\right. \\
&\qquad\qquad \left. + \left(\int_0^{2\pi}\int_0^{\infty} \frac{z_0 r\, drd\theta}{(r^2+z_0^2)^{3/2}}\right)\mathbf{k}\right]
\end{aligned}$$

Note that the integral for the $\mathbf{i}$-component has constant bounds and the integrand can be decomposed into a product of an r-function and a θ-function, i.e.

$$\int_0^{2\pi}\int_0^{\infty} \frac{-r^2\cos\theta\, drd\theta}{(r^2+z_0^2)^{3/2}} = -\lim_{R\to\infty}\left(\int_0^{2\pi}\cos\theta\, d\theta\right)\left(\int_0^{R} \frac{r^2}{(r^2+z_0^2)^{3/2}} dr\right)$$

Since $\int_0^{2\pi}\cos\theta\, d\theta = 0$, the $\mathbf{i}$-component is zero. Similar argument applies to the $\mathbf{j}$-component.

Alternatively, you may argue that the integrand of the $\mathbf{i}$-component:

$$-\frac{x}{(x^2+y^2+z_0^2)^{3/2}}$$

is an odd function of x, and the domain $\mathbb{R}^2$ is symmetric about the axis $x = 0$. The positive contribution on the right of the axis exactly cancels out the negative contribution on the left. Similar for the $\mathbf{j}$-component.

(b) Derive that:

$$\mathbf{F}_{\text{resultant}} = \frac{Q\sigma}{2\varepsilon_0}\mathbf{k}.$$

[Remark: The result in (b) asserts that the resultant force on the Q-particle does *not* depend on how far it is from the infinite sheet! Believe it or not?]

Solution:

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\infty} \frac{z_0 r\, drd\theta}{(r^2+z_0^2)^{3/2}} &= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\infty} \frac{z_0 r}{(r^2+z_0^2)^{3/2}} dr\right) \\
&= 2\pi\left[-\frac{z_0}{(r^2+z_0^2)^{1/2}}\right]_{r=0}^{r=\infty} \\
&= 2\pi\left(-0 + \frac{z_0}{(0+z_0^2)^{1/2}}\right) = 2\pi
\end{aligned}$$

Combine the results with (a), we obtained:

$$\mathbf{F}_{\text{resultant}} = \frac{Q\sigma}{4\pi\varepsilon_0}\, 2\pi\,\mathbf{k} = \frac{Q\sigma}{2\varepsilon_0}\,\mathbf{k}.$$