34. Find the distance from the point $(1, 1, 0)$ to the circular paraboloid with equation $z = x^2 + y^2$.

Solving technique:

Set closest point $Q = (x, y, z)$

Then $\overrightarrow{PQ} = \langle x-1, y-1, z \rangle$

Then $\overrightarrow{PQ} \parallel \nabla f$. $(\overrightarrow{PQ} = t \nabla f)$

$\nabla f = \langle 2x, 2y, -1 \rangle$

$x - 1 = t(2x)$

$y - 1 = t(2y) \Rightarrow t = -\frac{1}{2}$.

$Q: (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \Rightarrow$ distance $= \frac{\sqrt{3}}{2}$.

---

35. Find the distance from the point $(0, 0, 1)$ to the elliptic paraboloid having equation $z = x^2 + 2y^2$.

Remark: 呢種情況要考慮 $x \neq 0$, $y \neq 0$, 及 $x = y = 0$ 的情況，再比較出最短距離

35. If $Q = (X, Y, Z)$ is the point on the surface $z = x^2 + 2y^2$ that is closest to $P = (0, 0, 1)$, then

$$\overrightarrow{PQ} = X\mathbf{i} + Y\mathbf{j} + (Z - 1)\mathbf{k}$$

must be normal to the surface at Q, and hence must be parallel to $\mathbf{n} = 2X\mathbf{i} + 4Y\mathbf{j} - \mathbf{k}$. Hence $\overrightarrow{PQ} = t\mathbf{n}$ for some real number t, so

$$X = 2tX, \qquad Y = 4tY, \qquad Z - 1 = -t.$$

If $X \neq 0$, then $t = 1/2$, so $Y = 0$, $Z = 1/2$, and $X = \sqrt{Z} = 1/\sqrt{2}$. The distance from $(1/\sqrt{2}, 0, 1/2)$ to $(0, 0, 1)$ is $\sqrt{3}/2$ units.
If $Y \neq 0$, then $t = 1/4$, so $X = 0$, $Z = 3/4$, and $Y = \sqrt{Z/2} = \sqrt{3/8}$. The distance from $(0, \sqrt{3/8}, 3/4)$ to $(0, 0, 1)$ is $\sqrt{7}/4$ units.
If $X = Y = 0$, then $Z = 0$ (and $t = 1$). The distance from $(0, 0, 0)$ to $(0, 0, 1)$ is 1 unit.
Since

$$\frac{\sqrt{7}}{4} < \frac{\sqrt{3}}{2} < 1,$$

the closest point to $(0, 0, 1)$ on $z = x^2 + 2y^2$ is $(0, \sqrt{3/8}, 3/4)$, and the distance from $(0, 0, 1)$ to that surface is $\sqrt{7}/4$ units.

$$\lim_{h \to 0} h^2 \sin\left(\frac{1}{h^2}\right).$$

Technique: Squeeze theorem:

$$-1 \leq \sin\left(\frac{1}{h^2}\right) \leq 1$$

$$-h^2 \leq h^2 \sin\left(\frac{1}{h^2}\right) \leq h^2$$

$\lim_{h \to 0} -h^2 = 0$ $\qquad$ $\lim_{h \to 0} h^2 = 0$

$$\Downarrow$$

$$\lim_{h \to 0} h^2 \sin\left(\frac{1}{h^2}\right) = 0 \,//$$

EXERCISES 12.3

In Exercises 1–10, find all the first partial derivatives of the function specified, and evaluate them at the given point.

1. $f(x, y) = x - y + 2, \quad (3, 2)$
2. $f(x, y) = xy + x^2, \quad (2, 0)$
3. $f(x, y, z) = x^3 y^4 z^5, \quad (0, -1, -1)$
4. $g(x, y, z) = \dfrac{xz}{y + z}, \quad (1, 1, 1)$
5. $z = \tan^{-1}\left(\dfrac{y}{x}\right), \quad (-1, 1)$
6. $w = \ln(1 + e^{xyz}), \quad (2, 0, -1)$
7. $f(x, y) = \sin(x\sqrt{y}), \quad \left(\dfrac{\pi}{3}, 4\right)$
8. $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}, \quad (-3, 4)$
9. $w = x^{(y \ln z)}, \quad (e, 2, e)$
10. $g(x_1, x_2, x_3, x_4) = \dfrac{x_1 - x_2^2}{x_3 + x_4^2}, \quad (3, 1, -1, -2)$

In Exercises 11–12, calculate the first partial derivatives of the given functions at $(0, 0)$. You will have to use Definition 4.

11. $f(x, y) = \begin{cases} \dfrac{2x^3 - y^3}{x^2 + 3y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$
12. $f(x, y) = \begin{cases} \dfrac{x^2 - 2y^2}{x - y}, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$

In Exercises 13–22, find equations of the tangent plane and normal line to the graph of the given function at the point with specified values of x and y.

13. $f(x, y) = x^2 - y^2$ at $(-2, 1)$
14. $f(x, y) = \dfrac{x - y}{x + y}$ at $(1, 1)$
15. $f(x, y) = \cos(x/y)$ at $(\pi, 4)$
16. $f(x, y) = e^{xy}$ at $(2, 0)$
17. $f(x, y) = \dfrac{x}{x^2 + y^2}$ at $(1, 2)$
18. $f(x, y) = y\,e^{-x^2}$ at $(0, 1)$
19. $f(x, y) = \ln(x^2 + y^2)$ at $(1, -2)$
20. $f(x, y) = \dfrac{2xy}{x^2 + y^2}$ at $(0, 2)$
21. $f(x, y) = \tan^{-1}(y/x)$ at $(1, -1)$
22. $f(x, y) = \sqrt{1 + x^3 y^2}$ at $(2, 1)$
23. Find the coordinates of all points on the surface with equation $z = x^4 - 4xy^3 + 6y^2 - 2$ where the surface has a horizontal tangent plane.
24. Find all horizontal planes that are tangent to the surface with equation $z = xye^{-(x^2+y^2)/2}$. At what points are they tangent?

In Exercises 25–31, show that the given function satisfies the given partial differential equation.

25. $z = x\,e^y, \quad x\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y}$
26. $z = \dfrac{x + y}{x - y}, \quad x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 0$
27. $z = \sqrt{x^2 + y^2}, \quad x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$
28. $w = x^2 + yz, \quad x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = 2w$
29. $w = \dfrac{1}{x^2 + y^2 + z^2}, \quad x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} + z\dfrac{\partial w}{\partial z} = -2w$
30. $z = f(x^2 + y^2)$, where f is any differentiable function of one variable,
$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0.$$
31. $z = f(x^2 - y^2)$, where f is any differentiable function of one variable,
$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$$

32. Give a formal definition of the three first partial derivatives of the function $f(x, y, z)$.
33. What is an equation of the "tangent hyperplane" to the graph $w = f(x, y, z)$ at $\big(a, b, c, f(a, b, c)\big)$?
34. Find the distance from the point $(1, 1, 0)$ to the circular paraboloid with equation $z = x^2 + y^2$.
35. Find the distance from the point $(0, 0, 1)$ to the elliptic paraboloid having equation $z = x^2 + 2y^2$.
36. Let $f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

    Note that f is not continuous at $(0, 0)$. (See Example 3 of Section 12.2.) Therefore, its graph is not smooth there. Show, however, that $f_1(0, 0)$ and $f_2(0, 0)$ both exist. Hence, the existence of partial derivatives does not imply that a function of several variables is continuous. This is in contrast to the single-variable case.
37. Determine $f_1(0, 0)$ and $f_2(0, 0)$ if they exist, where
$$f(x, y) = \begin{cases} (x^3 + y)\sin\dfrac{1}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$
38. Calculate $f_1(x, y)$ for the function in Exercise 37. Is $f_1(x, y)$ continuous at $(0, 0)$?
39. Let $f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

    Calculate $f_1(x, y)$ and $f_2(x, y)$ at all points (x, y) in the plane. Is f continuous at $(0, 0)$? Are f_1 and f_2 continuous at $(0, 0)$?
40. Let $f(x, y, z) = \begin{cases} \dfrac{xy^2 z}{x^4 + y^4 + z^4}, & \text{if } (x, y, z) \neq (0, 0, 0) \\ 0, & \text{if } (x, y, z) = (0, 0, 0). \end{cases}$

    Find $f_1(0, 0, 0)$, $f_2(0, 0, 0)$, and $f_3(0, 0, 0)$. Is f continuous at $(0, 0, 0)$? Are f_1, f_2, and f_3 continuous at $(0, 0, 0)$?

In Exercises 1–10, find all the first partial derivatives of the function specified, and evaluate them at the given point.

1. $f(x, y) = x - y + 2, \quad (3, 2)$

2. $f(x, y) = xy + x^2, \quad (2, 0)$

$f_x = 1$

$f_y = -1$

2. $f_x\big|_{(2,0)} = y + 2x$

$= 4$

$f_y = x$

$f_y = 2$

$\frac{3z}{4+z}$

3. $f(x, y, z) = x^3 y^4 z^5, \quad (0, -1, -1)$

4. $g(x, y, z) = \dfrac{xz}{y + z}, \quad (1, 1, 1)$

$f_x = 3y^4 z^5 x^2 = 0$

$f_y = 4x^3 y^4 z^5 = 0$

$f_z = 5x^3 y^4 z^4 = 0$

$g_z = \dfrac{(y+z)x - \cancel{2}x}{(y+z)^2}$

$g_x = \dfrac{\cancel{2}\ 1}{4}$

$g_x = \cancel{2} \cdot \frac{1}{4}$

4. $g_x = \dfrac{z}{y+z} = \dfrac{1}{2}$

$g_y = -xz(y+z)^{-2} = \dfrac{-1}{4}$

Made with Goodnotes

5. $z = \tan^{-1}\left(\frac{y}{x}\right)$, $(-1, 1)$

6. $w = \ln(1 + e^{xyz})$, $(2, 0, -1)$

7. $f(x, y) = \sin(x\sqrt{y})$, $\left(\frac{\pi}{3}, 4\right)$

8. $f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$, $(-3, 4)$

5. $f_x = -\frac{\cancel{y}\, x^2}{1+\frac{y^2}{x^2}} = -\frac{1}{2}$

$f_y = \frac{1}{x\left(1+\frac{y^2}{x^2}\right)} = \frac{1}{-2}$

6. $f_x = \frac{yze^{xyz}}{1+e^{xyz}} = 0$

$f_y = \frac{xze^{xyz}}{1+e^{xyz}} = -1$

$f_z = \frac{xye^{xyz}}{1+e^{xyz}} = 0$

7. $f_x = \cos(x\sqrt{y})\sqrt{y}$

$f_y = \cos(x\sqrt{y})\left(\frac{1}{2}y^{-\frac{1}{2}}x\right)$

8. $f_x = -\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}}(2x) = 0.524$

$= -x(x^2+y^2)^{-\frac{3}{2}}$

$f_y = -\frac{1}{2}(x^2+y^2)^{-\frac{3}{2}}(2y)$

$= -y(x^2-y^2)^{-\frac{3}{2}}$

$= -0.8$

9. $w = x^{(y \ln z)}, \quad (e, 2, e)$

10. $g(x_1, x_2, x_3, x_4) = \dfrac{x_1 - x_2^2}{x_3 + x_4^2}, \quad (3, 1, -1, -2)$

x^n $\quad a^x \quad a^x$

$f_x = (y \ln z)\, x^{(y \ln z)-1}$

$= 2 \quad e^{2-1}$

$= 2e.$

$f_y = (x^y)^{\ln z}$

$= (\ln z)(x^y)^{\ln z - 1}$

$x^y \ln x$

$= \ln x \cdot \ln z\, (x^y)^{\ln z}.$

$= e^2.$

$f_z = (x^y)^{\ln z} = (x^y)^{\ln z} \ln(x^y)\, \frac{1}{z}$

$= \frac{1}{z}\, (x^y)^{\ln z}\, y \ln(x)$

$= \frac{1}{e}(e^2)(2)$

$= 2e$

10. $g(x_1, x_2, x_3, x_4) = \frac{x_1 - x_2^2}{x_3 + x_4^2}$, $(3, 1, -1, -2)$

$$\frac{x_1}{x_3+x_4^2} - \frac{x_2^2}{x_3+x_4^2}$$

$$f_{x_1} = \frac{1}{x_3+x_4^2} = \frac{1}{(-1)+4} = \frac{1}{3}.$$

$$f_{x_2} = \frac{-2x_2}{x_3+x_4^2} = \frac{-2(1)}{(-1)+(-2)^2} = \frac{-2}{-1+4} = -\frac{2}{3}$$

$$f_{x_3} = (x_1 - x_2^2)(-1)(x_3+x_4^2)^{-2}$$

$$= \frac{x_2^2 - x_1}{(x_3+x_4^2)^2} \quad -3.$$

$$= \frac{\cancel{-2}\ -2}{(-1+4)^2} = \frac{\cancel{-2}\ -2}{9} = \cancel{-\frac{1}{3}}.$$

$$f_{x_4} = (x_1 - x_2^2)(-1)(x_3+x_4^2)^{-2}(2x_4)$$

$$= (3-1)(-1)(-1+4)^{-2}(2(-2))$$

$$= (2)(-1)(3)^{-2}(-4)$$

$$= \frac{8}{9}.$$

In Exercises 11–12, calculate the first partial derivatives of the given functions at $(0, 0)$. You will have to use Definition 4.

11. $f(x, y) = \begin{cases} \dfrac{2x^3 - y^3}{x^2 + 3y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

$$f_x = \lim_{\substack{h \to 0 \\ y=0}} \frac{\frac{2h^3}{h^2} - 0}{h}.$$

$$= \lim_{\substack{h \to 0 \\ y=0}} 2 = 2.$$

$$f_y = \lim_{\substack{h \to 0 \\ x=0}} \frac{\frac{-h^3}{3h^2} - 0}{h} = -\frac{1}{3}.$$

12. $f(x, y) = \begin{cases} \dfrac{x^2 - 2y^2}{x - y}, & \text{if } x \neq y \\ 0, & \text{if } x = y. \end{cases}$

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^2}{h} - 0}{h}$$

$$= 1$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= 2.$$

Made with Goodnotes

In Exercises 13–22, find equations of the tangent plane and normal line to the graph of the given function at the point with specified values of x and y.

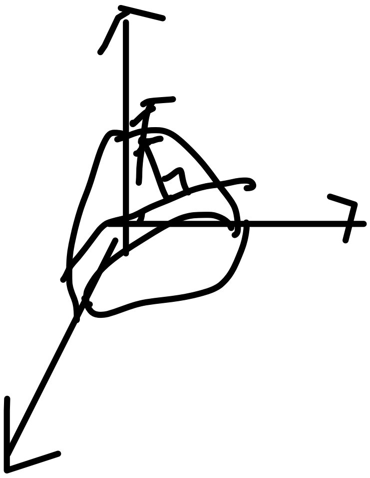
13. $f(x, y) = x^2 - y^2$ at $(-2, 1)$

14. $f(x, y) = \dfrac{x-y}{x+y}$ at $(1, 1)$

When $P(-2,1)$, $z = 3$.

$z = x^2 - y^2$

$0 = x^2 - y^2 - z$



$\nabla f = \langle 2x, -2y, -1 \rangle$

$\nabla f(-2,1) = \langle -4, -2, -1 \rangle$

$-4x - 2y - z = (8 - 2 - 3)$

$4x + 2y + z = -3$

$\vec{n} = \langle 4, 2, 1 \rangle$

normal line:

$\vec{r}(t) = (-2, 1, 3) + (4, 2, 1)t$

14. When $P(1,1)$, $z = 0$.

$0 = \boxed{\dfrac{x-y}{x+y}} - z$

$f_x = \dfrac{(x+y) - (x-y)}{(x+y)^2}$

$f_x = \dfrac{2y}{(x+y)^2} = \dfrac{2}{4} = \dfrac{1}{2}$

$$f_y = \frac{x-y}{x+y} :$$

$$f_y = \frac{(x+y)(-1) - (x-y)}{(x+y)^2}$$

$$= \frac{(-x-y) - (x-y)}{(x+y)^2}$$

$$= \frac{-2x}{(x+y)^2} = \frac{-2}{4} = -\frac{1}{2}.$$

$$\nabla f(1,1,0) = \langle \frac{1}{2}, -\frac{1}{2}, -1 \rangle$$

$$\frac{1}{2}x - \frac{1}{2}y - z = 0$$

$$x - y - 2z = 0$$

$$\vec{r}(t) = \langle 1, -1, -2 \rangle t + \langle 1, 1, 0 \rangle$$

Made with Goodnotes

15. $f(x, y) = \cos(x/y)$ at $(\pi, 4)$

16. $f(x, y) = e^{xy}$ at $(2, 0)$

$$f_x = -\sin\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)$$

$$f_x = -\sin\left(\frac{\pi}{4}\right)\left(\frac{1}{4}\right)$$

$$f_{x(\pi,4)} - \frac{1}{4}\left(\frac{\sqrt{2}}{2}\right)$$

$$f_{x(\pi,4)} - \frac{\sqrt{2}}{8}$$

$$f_y = -\sin\left(\frac{x}{y}\right)\left(-xy^{-2}\right)$$

$$f_y = \frac{x}{y^2}\sin\left(\frac{x}{y}\right)$$

$$f_{y(\pi,4)} = \frac{\pi}{16}\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{16}\left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{\sqrt{2}\pi}{32}$$

$$0 \quad \frac{\pi}{6} \quad \frac{\pi}{4}$$

$$\frac{\sqrt{2}}{2} \quad \frac{\sqrt{1}}{2} \quad \frac{\sqrt{2}}{2}$$

Eqt:

$$-\frac{\sqrt{2}}{8}x + \frac{\sqrt{2}\pi}{32}y - z =$$

$$-\frac{\sqrt{2}}{8}\pi + \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2}$$

$$-\frac{\sqrt{2}}{8}x + \frac{\sqrt{2}\pi}{32}y - z = \frac{\sqrt{2}}{2}$$

$$\frac{1}{8}x - \frac{\pi}{32}y + \frac{\sqrt{2}z}{2} = \frac{1}{2}$$

$$4x - \pi y + 16\sqrt{2}z = 16$$

$$\nabla f\left(\pi, 4, \frac{\sqrt{2}}{2}\right) = \left\langle -\frac{\sqrt{2}}{8}, \frac{\sqrt{2}\pi}{32}, -1 \right\rangle$$

Made with Goodnotes

16. $f(x, y) = e^{xy}$ at $(2, 0)$

17. $f(x, y) = \dfrac{x}{x^2 + y^2}$ at $(1, 2)$

$z = 1.$

$f_x = ye^{xy} = 0$

$f_y = xe^{xy} = 2$

$\nabla f = \langle 0, 2, -1 \rangle \ (\vec{n}).$

$2y - z = -1$

17.

$$f_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2}$$

$3 - 8 - 25(\frac{1}{5})$

$3 - 8 - 5$

$$f_x = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{4-1}{(1+4)^2} = \frac{3}{25}$$

$$f_y = \frac{-x(2y)}{(x^2+y^2)^2} = \frac{-2(1)(2)}{(5)^2} = \frac{-4}{25}$$

$z = \frac{1}{5}.$

$3x - 4y - 25z = -10$

$\nabla f = \langle 3, -4, -25 \rangle$

18. $f(x, y) = y\,e^{-x^2}$ at $(0, 1)$

19. $f(x, y) = \ln(x^2 + y^2)$ at $(1, -2)$

20. $f(x, y) = \dfrac{2xy}{x^2 + y^2}$ at $(0, 2)$

18. $f_x = ye^{-x^2}(-2x) = 0$

$f_y = e^{-x^2} = 1$

$z = 1$

$\nabla f = \langle 0, 1, -1 \rangle$

$y - z = 0$

$y = z$

19. $f_x = \dfrac{2x}{x^2+y^2} = \dfrac{2}{5}$

$f_y = \dfrac{2y}{x^2+y^2} = \dfrac{-4}{5}$

$z = \ln(5)$

$\nabla f = \langle 2, -4, -5 \rangle$

$2x - 4y - 5z = 10 - 5\ln 5$

20. $f(x, y) = \dfrac{2xy}{x^2 + y^2}$ at $(0, 2)$

21. $f(x, y) = \tan^{-1}(y/x)$ at $(1, -1)$

22. $f(x, y) = \sqrt{1 + x^3 y^2}$ at $(2, 1)$

$$20.\quad f_x = \frac{(x^2+y^2)(2y) - (2xy)(2x)}{(x^2+y^2)^2}$$

$$= \frac{(4)(4) - 0}{4^2} = 1.$$

$$f_y = \frac{(x^2+y^2)(2x) - 2xy(2y)}{(x^2+y^2)^2}$$

$$= \frac{4(0) - 2(0)(2)(2)(2)}{4^2}$$

$$= 0$$

$$z = 0.$$

$$\nabla f = \langle 1, 0, -1 \rangle$$

$$x - z = 0$$

$$x = z.$$

21. $f(x, y) = \tan^{-1}(y/x)$ at $(1, -1)$

$$f_x = \frac{1}{1+\frac{y^2}{x^2}}(-yx^{-2})$$

$$0 \quad \frac{\pi}{6} \quad \frac{\pi}{4} \quad \frac{\pi}{3} \quad \frac{\pi}{2}$$

$$0 \quad \frac{\sqrt{3}}{3} \quad \frac{\sqrt{9}}{3} \quad \frac{\sqrt{27}}{3} \quad X.$$

$$= \frac{1}{1+\frac{1}{1}}(-(1))(1)$$

$$= -\frac{1}{2}.$$

$$f_y = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$f_y = \frac{1}{2}.$$

$$\tan\theta = \frac{y}{x}$$

$$=$$

$$z = \tan^{-1}(-1)$$

$$z = -\tan^{-1}(1)$$

$$\nabla f = \langle -1, 1, -2 \rangle$$

$$\frac{\pi}{4} \times 2.$$

$$-x + y - 2z = -2 + 2\tan^{-1}(1)$$

22. $f(x, y) = \sqrt{1 + x^3y^2}$ at $(2, 1)$

$$f_x = \frac{1}{2}(1+x^3y^2)^{-\frac{1}{2}}(3y^2x^2)$$

$$= \frac{3}{2}(1+8)^{-\frac{1}{2}}(4)$$

$$= \frac{6}{3}$$

$$= 2.$$

$$f_y = \frac{1}{2}(1+x^3y^2)^{-\frac{1}{2}}(2yx^3)$$

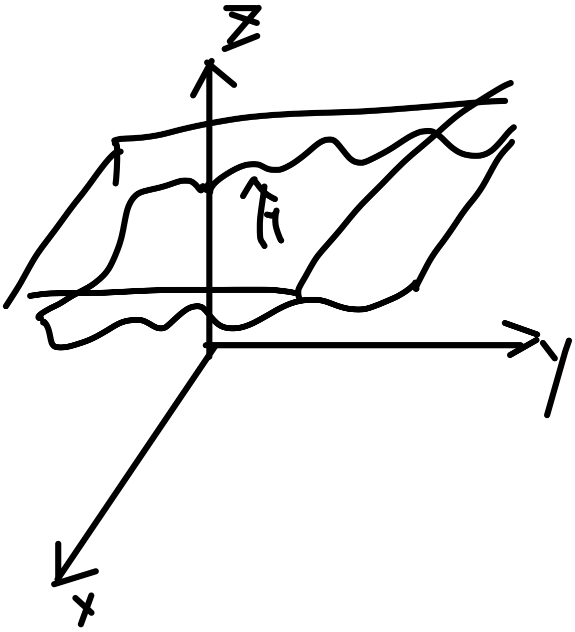
$$= \frac{8}{3}$$

$$z = \sqrt{1+8} = 3.$$

$$\nabla f = \langle 2, \frac{8}{3}, -1 \rangle = \langle 6, 8, -3 \rangle$$

$$6x + 8y - 3z = 11$$

23. Find the coordinates of all points on the surface with equation $z = x^4 - 4xy^3 + 6y^2 - 2$ where the surface has a horizontal tangent plane.



$$\nabla f = (0, 0, -1)$$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

$f_x = 4x^3 - 4y^3 = 0$ $\quad x^3 - y^3 = 0.$

$f_y = -12xy^2 + 12y = 0$

$12y - 12xy^2 = 0$ $\quad 1 - xy = 0.$

$4x^3 - 4y^3 = 12xy^2 + 12y$

抽左 $12y = 0$, 因此另個 sol. ($y = 0$, 捨去 $x = 0$).

$$\begin{cases} x^3 - y^3 = 0. \\ 1 - xy = 0. \end{cases}$$

$\{0, 0\}$.

$xy = 1$ $\quad x^3 - \left(\frac{1}{x}\right)^3$

$y = \frac{1}{x}$ $\quad x^3 - \frac{1}{x^3} = 0$

$x^3 = \frac{1}{x^3}$

$\{1, 1\}$, $\{-1, -1\}$. $\quad y = \pm 1.$ $\quad x^6 = 1$

$x = \pm 1$

24. Find all horizontal planes that are tangent to the surface with equation $z = xye^{-(x^2+y^2)/2}$. At what points are they tangent?

$$f_x = y\left(x \frac{\partial}{\partial x} e^{-(x^2+y^2)/2} + e^{-(x^2+y^2)/2}\right)$$

$$f_x = y\left(x e^{-(x^2+y^2)/2} \frac{\partial}{\partial x}\left(-(x^2+y^2)/2\right) + e^{-(x^2+y^2)/2}\right)$$

$y(1-x^2)=0$. $y=0$ or $1-x^2=0$ $\quad (1-y^2)x=0$

$x^2=1$

$\{\pm 1, 1\}$ $\{\pm 1, 0\}$

$$= e^{-(x^2+y^2)/2}\, y\left(x\left(-\tfrac{1}{2}\right)(2x) + 1\right)$$

$$= e^{-(x^2+y^2)/2}\, y(1-x^2)$$

$$f_y = x\left(y \frac{\partial}{\partial y} e^{-(x^2+y^2)/2} + e^{-(x^2+y^2)/2}\right)$$

$\{1, 1\}$

$$= x\, e^{-(x^2+y^2)/2}(-y \cdot y + 1)$$

$$= x\, e^{-(x^2+y^2)/2}(1-y^2)$$

$\{1, 0\}$

$\{-1, 0\}$

$\{0, 1\}$

$\{0, -1\}$

$$\begin{cases} f_x = 0 \\ f_y = 0. \end{cases}$$

$y=0$ or $1-x^2=0$

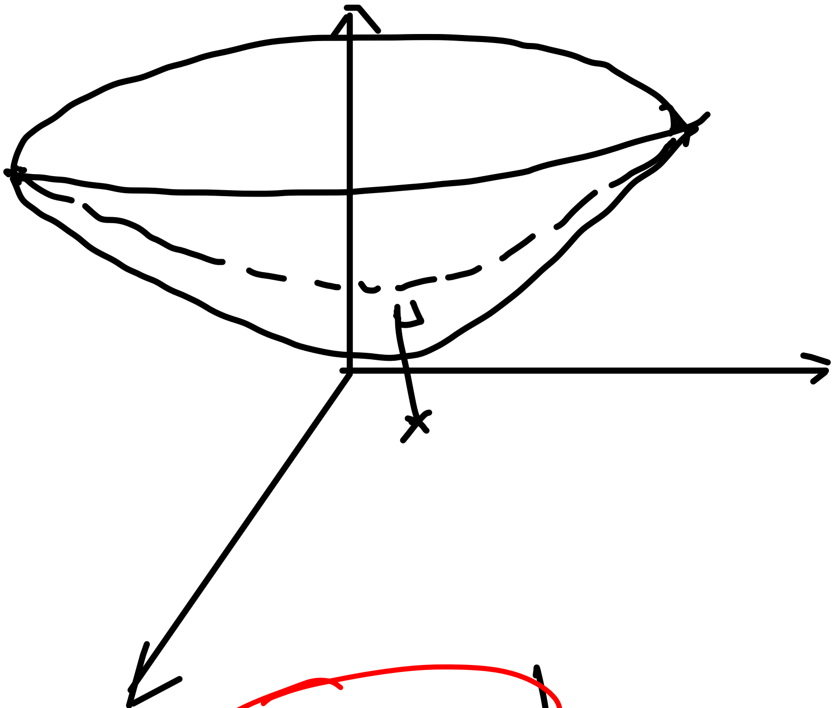
$x = \pm 1$

$x=0$ or $y=\pm 1$

$$\begin{cases} x(1-y^2)=0 \\ y(1-x^2)=0 \end{cases}$$

34. Find the distance from the point (1, 1, 0) to the circular paraboloid with equation $z = x^2 + y^2$.

??



$$|\langle 1,1,0\rangle - \langle x^2, y^2, -z\rangle| \le L$$

$$|\langle 1-x^2, 1-y^2, z\rangle|$$

$$\lim_{x,y,z \to (0,0,0)} \sqrt{(1-x^2)^2+(1-y^2)^2+z^2}$$

$$= \sqrt{1+1+0}$$

$$= \sqrt{2}.$$

36. Let $f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$

Note that f is not continuous at $(0, 0)$. (See Example 3 of Section 12.2.) Therefore, its graph is not smooth there. Show, however, that $f_1(0, 0)$ and $f_2(0, 0)$ both exist. Hence, the existence of partial derivatives does not imply that a function of several variables is continuous. This is in contrast to the single-variable case.

?

$$f_1 = \frac{\frac{2h(0)}{h^2} - 0}{h}$$

$$f_1 = 0.$$

$$f_2 = \frac{\frac{2(0)h}{h^2} - 0}{h}$$

$$f_2 = 0.$$

37. Determine $f_1(0,0)$ and $f_2(0,0)$ if they exist, where

$$f(x,y) = \begin{cases} (x^3+y)\sin\dfrac{1}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

$$f_x = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$f_x = \lim_{h \to 0} \frac{h^3 \sin\frac{1}{h^2}}{h},$$

$$f_x = \lim_{h \to 0} h^2 \sin\frac{1}{h^2} \leftarrow 0. \text{ squeeze thm.}$$

$$f_x = \lim_{h \to 0} \frac{\sin\frac{1}{h^2}}{\frac{1}{h^2}}$$

$$= \lim_{\frac{1}{h^2} \to \infty} \frac{\sin\frac{1}{h^2}}{\frac{1}{h^2}}$$

$$= 1$$

$$f_y = \lim_{h \to 0} \sin\frac{1}{h^2}$$

$f_y = \infty$ does not exist

39. Let $f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & \text{if } (x, y) \neq (0,0) \\ 0, & \text{if } (x, y) = (0,0). \end{cases}$

Calculate $f_1(x, y)$ and $f_2(x, y)$ at all points (x, y) in the plane. Is f continuous at $(0, 0)$? Are f_1 and f_2 continuous at $(0, 0)$?

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2} = r(\cos^3\theta - \sin^3\theta)$$

$$= r(\cos\theta - \sin\theta)$$
$$(\cos^2\theta + \sin^2\theta + \cos\theta\sin\theta)$$
$$= r(\cos\theta - \sin\theta)(1 + \cos\theta\sin\theta)$$
$$= 0.$$

Yes continuous.

$$f_1(x,y) = \frac{3x^2}{2x} = \frac{3}{2}x. \quad f_1(0,0) = 0.$$

$$\lim_{h\to 0} f_x = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \frac{\frac{h^3}{h^2}}{h} = 1$$

$$\lim_{h\to 0} f_y = \lim_{h\to 0} \frac{\frac{-h^3}{h^2}}{h} = -1.$$

Not continuous.

Not.

40. Let $f(x,y,z) = \begin{cases} \dfrac{xy^2z}{x^4+y^4+z^4}, & \text{if } (x,y,z) \neq (0,0,0) \\ 0, & \text{if } (x,y,z) = (0,0,0). \end{cases}$

Find $f_1(0,0,0)$, $f_2(0,0,0)$, and $f_3(0,0,0)$. Is f continuous at $(0,0,0)$? Are f_1, f_2, and f_3 continuous at $(0,0,0)$?

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy^2z}{x^4+y^4+z^4} \quad (x=y=z,\ y=z=0,)$$

① ②.

No such limit.

$$f_1 = -(x^4+y^4+z^4)^{-2}(y^2z)$$

$$f_1 = \lim_{h\to 0} \frac{f(h,0,0)-f(0,0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{0}{h^4}-0}{h}$$

$$= \lim_{h\to 0} \frac{0}{h}$$

$$= 0.$$

chain rule!

$$\cancel{f_1 = \frac{-y^2z}{(x^4+y^4+z^4)^2}}$$

$f_2 = 0$, $f_3 = 0$. they are ~~continuous~~,

$f(x,y,z)$.

$$0 \le \left| \frac{-y^2z}{(x^4+y^4+z^4)} \right| \le y^2z \to 0.$$