**23.** $f(x, y) = \dfrac{x - y}{x + y}$

Remark:
如果其中一個 level curve 的 domain 有限制，其 family 也有限制.

**23.** $f(x, y) = \dfrac{x - y}{x + y} = C$, a family of straight lines through the origin, but not including the origin.

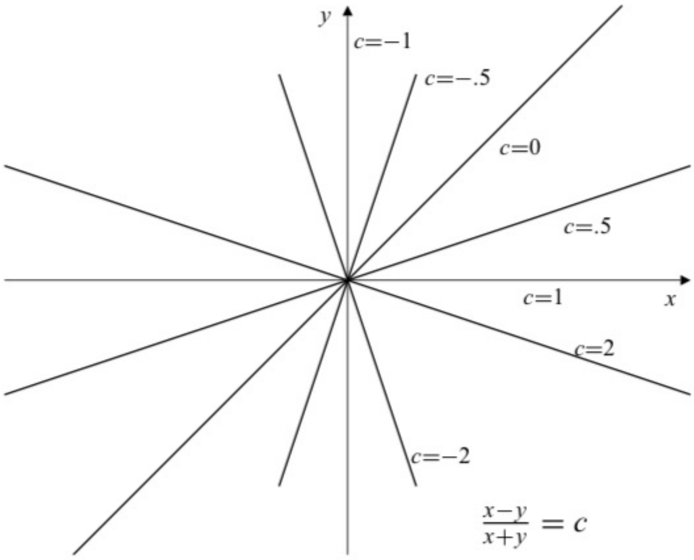



Fig. 12.1.23

---

**24.** $f(x, y) = \dfrac{y}{x^2 + y^2}$

→ Techique:
Sub $f(x,y) = C$,
Completing squares

**24.** $f(x, y) = \dfrac{y}{x^2 + y^2} = C$.

This is the family $x^2 + \left(y - \frac{1}{2C}\right)^2 = \frac{1}{4C^2}$ of circles passing through the origin and having centres on the $y$-axis. The origin itself is, however, not on any of the level curves.

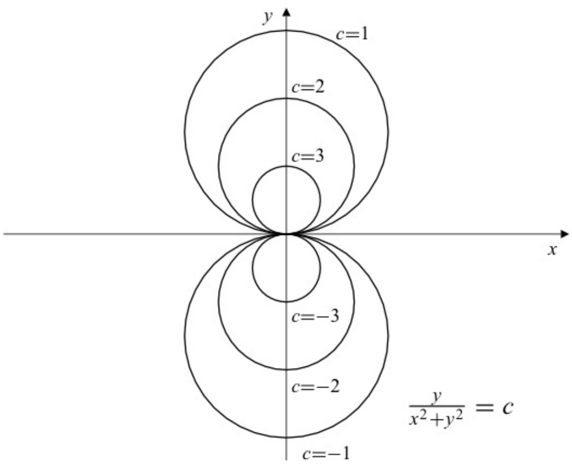



Fig. 12.1.24

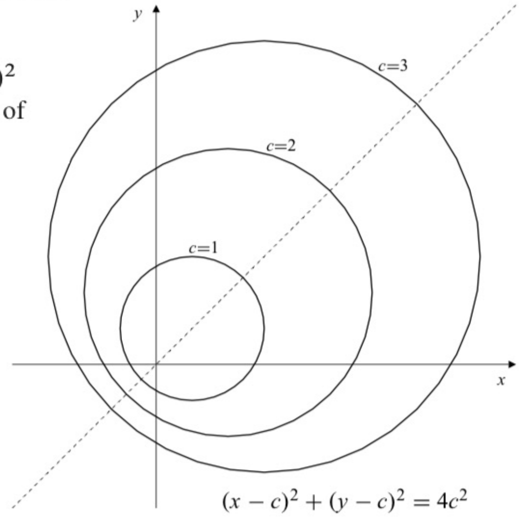
**34.** If we assume $z \geq 0$, the equation $4z^2 = (x - z)^2 + (y - z)^2$ defines $z$ as a function of $x$ and $y$. Sketch some level curves of this function. Describe its graph.

Technique: 先代 c.

$4c^2 = (x-c)^2 + (y-c)^2$,

畫個，再想像，代 z 的 volume.

**34.** $4z^2 = (x - z)^2 + (y - z)^2$.
If $z = c > 0$, we have $(x - c)^2 + (y - c)^2 = 4c^2$, which is a circle in the plane $z = c$, with centre $(c, c, c)$ and radius $2c$.




$(x - c)^2 + (y - c)^2 = 4c^2$

Fig. 12.1.34

The graph of the function $z = z(x, y) \geq 0$ defined by the given equation is (the upper half of) an elliptic cone with axis along the line $x = y = z$, and circular cross-sections in horizontal planes.

Made with Goodnotes

## EXERCISES 12.1

Specify the domains of the functions in Exercises 1–10.

**1.** $f(x, y) = \dfrac{x+y}{x-y}$

**2.** $f(x, y) = \sqrt{xy}$

**3.** $f(x, y) = \dfrac{x}{x^2+y^2}$

**4.** $f(x, y) = \dfrac{xy}{x^2-y^2}$

**5.** $f(x, y) = \sqrt{4x^2+9y^2-36}$

**6.** $f(x, y) = \dfrac{1}{\sqrt{x^2-y^2}}$

**7.** $f(x, y) = \ln(1+xy)$

**8.** $f(x, y) = \sin^{-1}(x+y)$

**9.** $f(x, y, z) = \dfrac{xyz}{x^2+y^2+z^2}$

**10.** $f(x, y, z) = \dfrac{e^{xyz}}{\sqrt{xyz}}$

Sketch the graphs of the functions in Exercises 11–18.

**11.** $f(x, y) = x, \quad (0 \le x \le 2, \quad 0 \le y \le 3)$

**12.** $f(x, y) = \sin x, \quad (0 \le x \le 2\pi, \quad 0 \le y \le 1)$

**13.** $f(x, y) = y^2, \quad (-1 \le x \le 1, \quad -1 \le y \le 1)$

**14.** $f(x, y) = 4 - x^2 - y^2, \quad (x^2+y^2 \le 4,\ x \ge 0,\ y \ge 0)$

**15.** $f(x, y) = \sqrt{x^2+y^2}$

**16.** $f(x, y) = 4 - x^2$

**17.** $f(x, y) = |x| + |y|$

**18.** $f(x, y) = 6 - x - 2y$

Sketch some of the level curves of the functions in Exercises 19–26.

**19.** $f(x, y) = x - y$

**20.** $f(x, y) = x^2 + 2y^2$

**21.** $f(x, y) = xy$

**22.** $f(x, y) = \dfrac{x^2}{y}$

**23.** $f(x, y) = \dfrac{x-y}{x+y}$

**24.** $f(x, y) = \dfrac{y}{x^2+y^2}$

**25.** $f(x, y) = xe^{-y}$

**26.** $f(x, y) = \sqrt{\dfrac{1}{y} - x^2}$

Exercises 27–28 refer to Figure 12.11, which shows contours of a hilly region with heights given in metres.

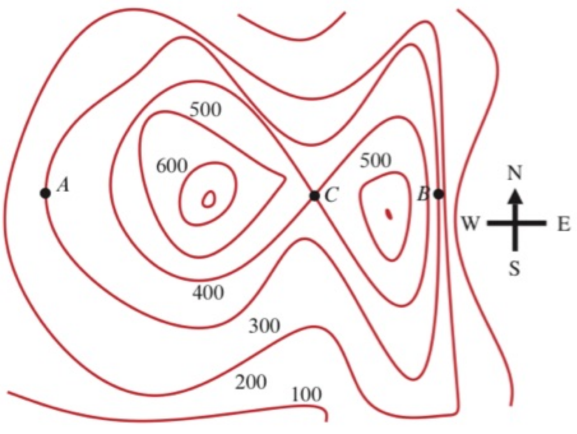



Figure 12.11

**27.** At which of the points $A$ or $B$ is the landscape steeper? How do you know?

**28.** Describe the topography of the region near point $C$.

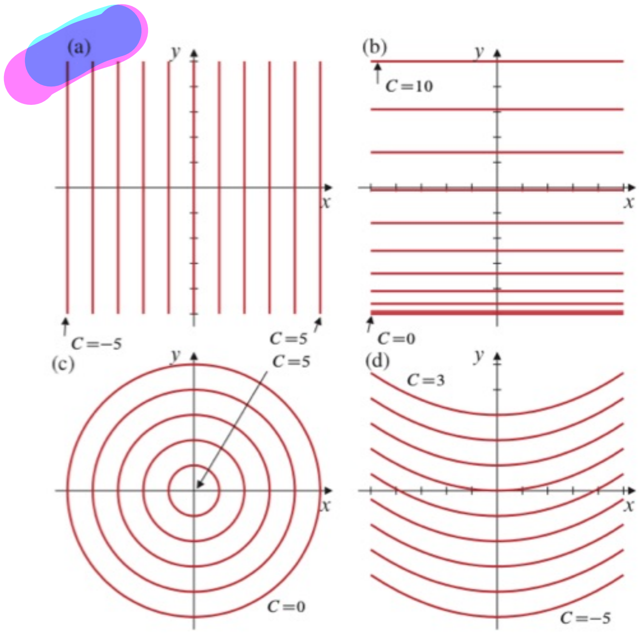



Figure 12.12

Describe the graphs of the functions $f(x, y)$ for which families of level curves $f(x, y) = C$ are shown in the figures referred to in Exercises 29–32. Assume that each family corresponds to equally spaced values of $C$ and that the behaviour of the family is representative of all such families for the function.

**29.** See Figure 12.12(a).

**30.** See Figure 12.12(b).

**31.** See Figure 12.12(c).

**32.** See Figure 12.12(d).

**33.** Are the curves $y = (x - C)^2$ level curves of a function $f(x, y)$? What property must a family of curves in a region of the $xy$-plane have to be the family of level curves of a function defined in the region?

**34.** If we assume $z \ge 0$, the equation $4z^2 = (x-z)^2 + (y-z)^2$ defines $z$ as a function of $x$ and $y$. Sketch some level curves of this function. Describe its graph.

**35.** Find $f(x, y)$ if each level curve $f(x, y) = C$ is a circle centred at the origin and having radius
(a) $C$ (b) $C^2$ (c) $\sqrt{C}$ (d) $\ln C$.

**36.** Find $f(x, y, z)$ if for each constant $C$ the level surface $f(x, y, z) = C$ is a plane having intercepts $C^3$, $2C^3$, and $3C^3$ on the $x$-axis, the $y$-axis, and the $z$-axis, respectively.

Describe the level surfaces of the functions specified in Exercises 37–41.

**37.** $f(x, y, z) = x^2 + y^2 + z^2$

**38.** $f(x, y, z) = x + 2y + 3z$

**39.** $f(x, y, z) = x^2 + y^2$

**40.** $f(x, y, z) = \dfrac{x^2+y^2}{z^2}$

**41.** $f(x, y, z) = |x| + |y| + |z|$

**42.** Describe the "level hypersurfaces" of the function

$$f(x, y, z, t) = x^2 + y^2 + z^2 + t^2.$$

Specify the domains of the functions in Exercises 1–10.

**1.** $f(x, y) = \dfrac{x+y}{x-y}$

**2.** $f(x, y) = \sqrt{xy}$

**6.** $f(x, y) = \dfrac{1}{\sqrt{x^2 - y^2}}$

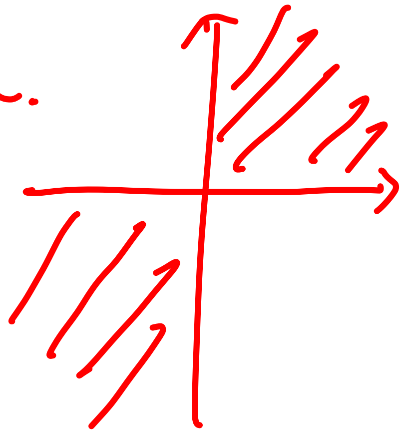
**7.** $f(x, y) = \ln(1 + xy)$

**8.** $f(x, y) = \sin^{-1}(x + y)$

**9.** $f(x, y, z) = \dfrac{xyz}{x^2 + y^2 + z^2}$

**10.** $f(x, y, z) = \dfrac{e^{xyz}}{\sqrt{xyz}}$

2.



1. $x - y \neq 0$
   $x \neq y$.

2. $xy \geq 0$

6. $x^2 - y^2 \geq 0$
   $x^2 \geq y^2$
   $x \geq y$

7. $1 + xy \geq 0$
   $xy \geq -1$

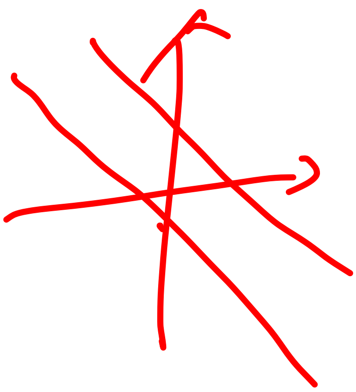
$xy = 1$
$y = \frac{1}{x}$.

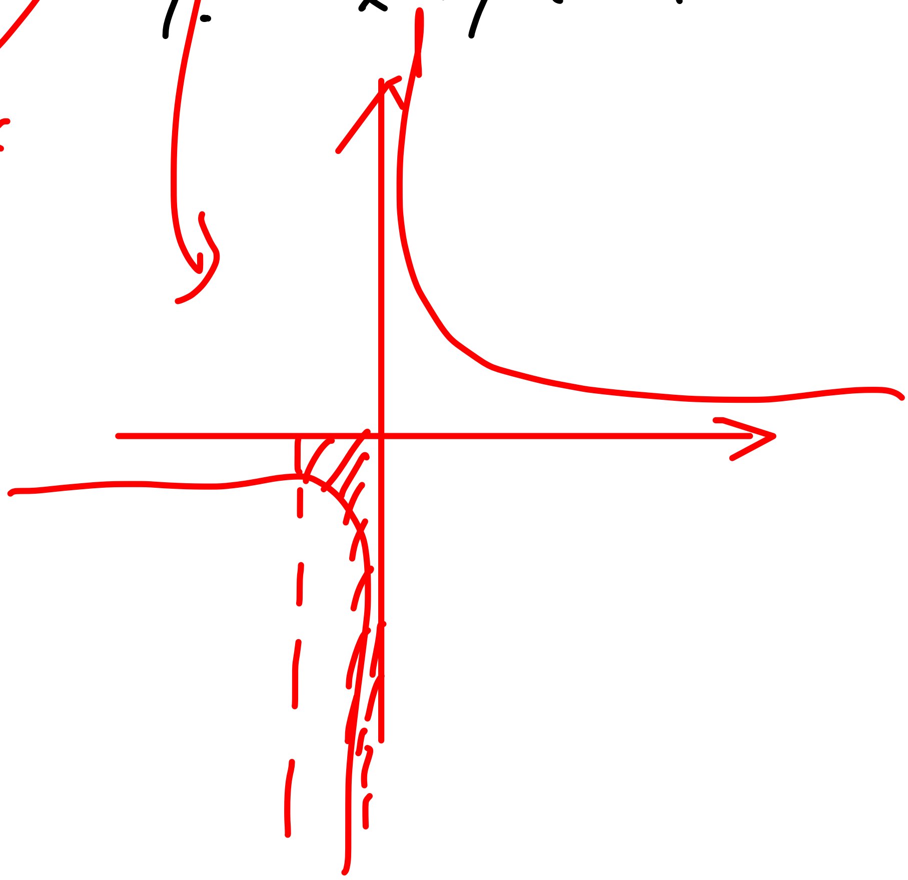
8. $-1 \leq x + y \leq 1$
   $x + y \geq -1$
   $y \geq -1 - x$

9. $x^2 + y^2 + z^2 \neq 0$

10. $xyz > 0$

8.





Specify the domains of the functions in Exercises 1–10. → Specify 要文字 describe.

**1.** $f(x, y) = \dfrac{x + y}{x - y}$

**2.** $f(x, y) = \sqrt{xy}$

**6.** $f(x, y) = \dfrac{1}{\sqrt{x^2 - y^2}}$

**7.** $f(x, y) = \ln(1 + xy)$

**8.** $f(x, y) = \sin^{-1}(x + y)$

**9.** $f(x, y, z) = \dfrac{xyz}{x^2 + y^2 + z^2}$

**10.** $f(x, y, z) = \dfrac{e^{xyz}}{\sqrt{xyz}}$

1. $\{x,y\} \in \mathbb{R}^2 / \{x=y\}$

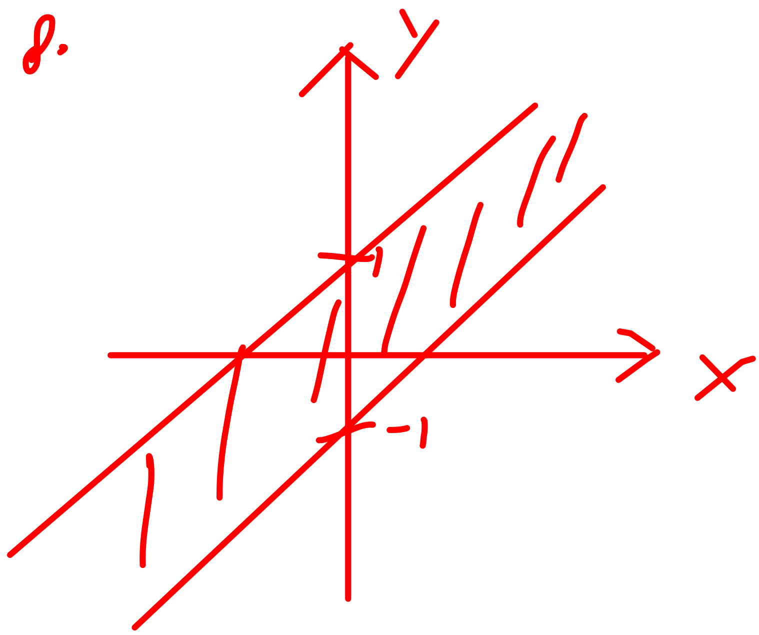
2. $\{x,y\} \in xy \geq 0$, first and third quadrant.

6. $x^2 - y^2 \geq 0$

$|x| \geq |y|$

8. $-1 \leq x+y \leq 1$

8.



**9.** $f(x, y, z) = \dfrac{xyz}{x^2 + y^2 + z^2}$

**10.** $f(x, y, z) = \dfrac{e^{xyz}}{\sqrt{xyz}}$

10.

9. $x, y, z \in \mathbb{R}^3 / \{0, 0, 0\}$

10. $x, y, z \geq 0$, four quadrant.

**3.** $f(x, y) = \dfrac{x}{x^2 + y^2}$

**4.** $f(x, y) = \dfrac{xy}{x^2 - y^2}$

**5.** $f(x, y) = \sqrt{4x^2 + 9y^2 - 36}$

3. $x^2 + y^2 \neq 0$ $\quad \{x, y\} \in \mathbb{R}^2 \setminus \{0, 0\}$

4. $x^2 - y^2 \neq 0 \Rightarrow x \neq \pm y.$

5. $4x^2 + 9y^2 - 36 \geq 0$

$$\Downarrow$$

$4x^2 + 9y^2 \geq 36$

outside the ellipse.

Sketch the graphs of the functions in Exercises 11–18.

**11.** $f(x, y) = x, \qquad (0 \le x \le 2, \quad 0 \le y \le 3)$

**12.** $f(x, y) = \sin x, \qquad (0 \le x \le 2\pi, \quad 0 \le y \le 1)$

**13.** $f(x, y) = y^2, \qquad (-1 \le x \le 1, \quad -1 \le y \le 1)$

**14.** $f(x, y) = 4 - x^2 - y^2, \quad (x^2 + y^2 \le 4,\ x \ge 0,\ y \ge 0)$

**15.** $f(x, y) = \sqrt{x^2 + y^2}$

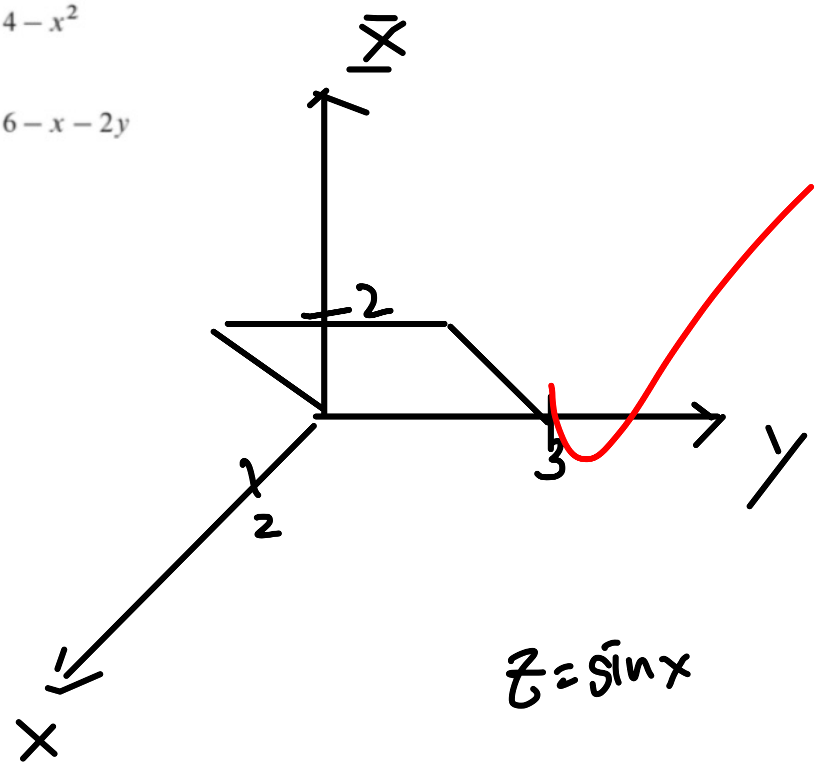
**16.** $f(x, y) = 4 - x^2$

**17.** $f(x, y) = |x| + |y|$

**18.** $f(x, y) = 6 - x - 2y$

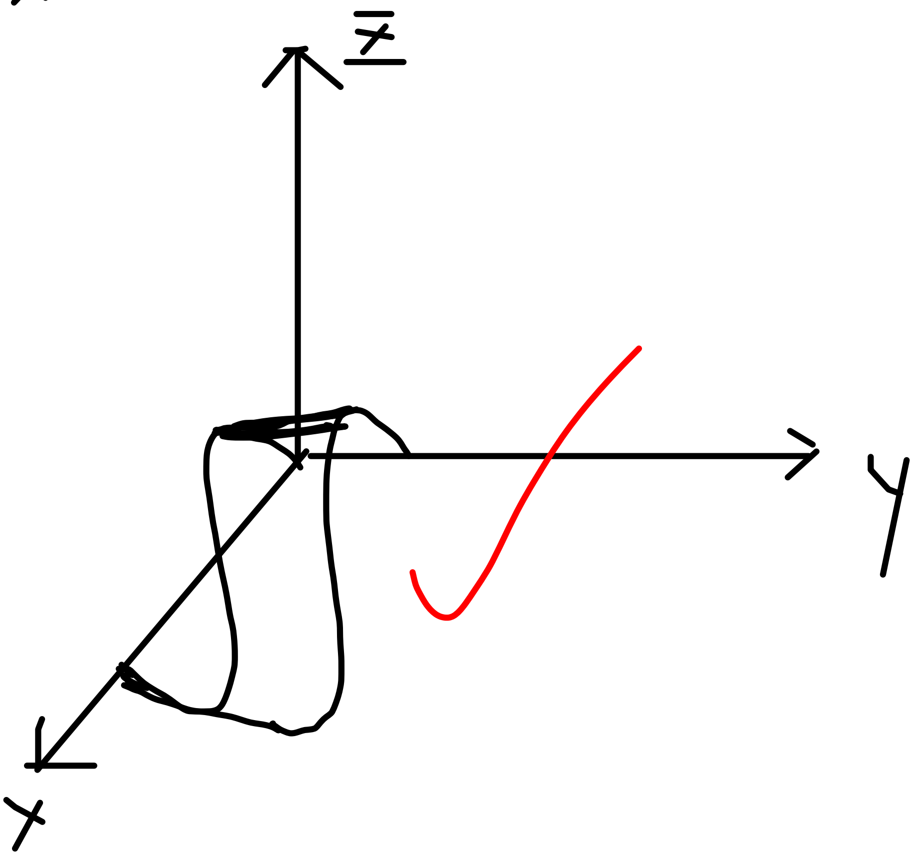
$z = x$

11.




$z = \sin x$

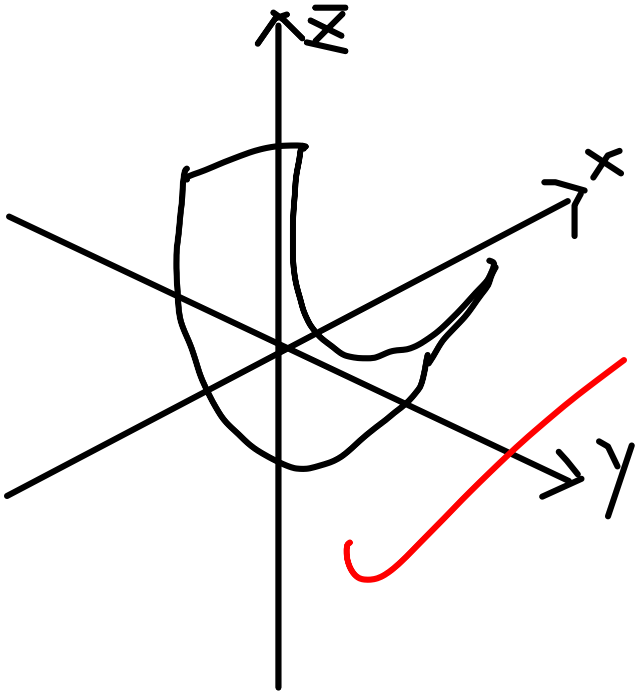
12.



**13.** $f(x, y) = y^2, \qquad (-1 \le x \le 1, \quad -1 \le y \le 1)$

**14.** $f(x, y) = 4 - x^2 - y^2, \quad (x^2 + y^2 \le 4,\ x \ge 0,\ y \ge 0)$

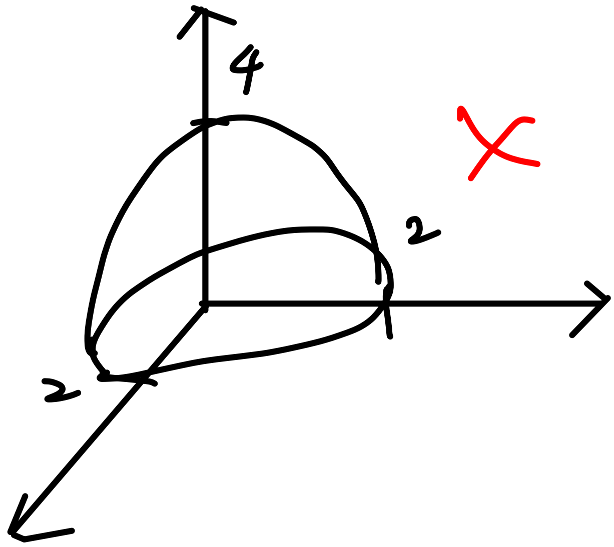
13

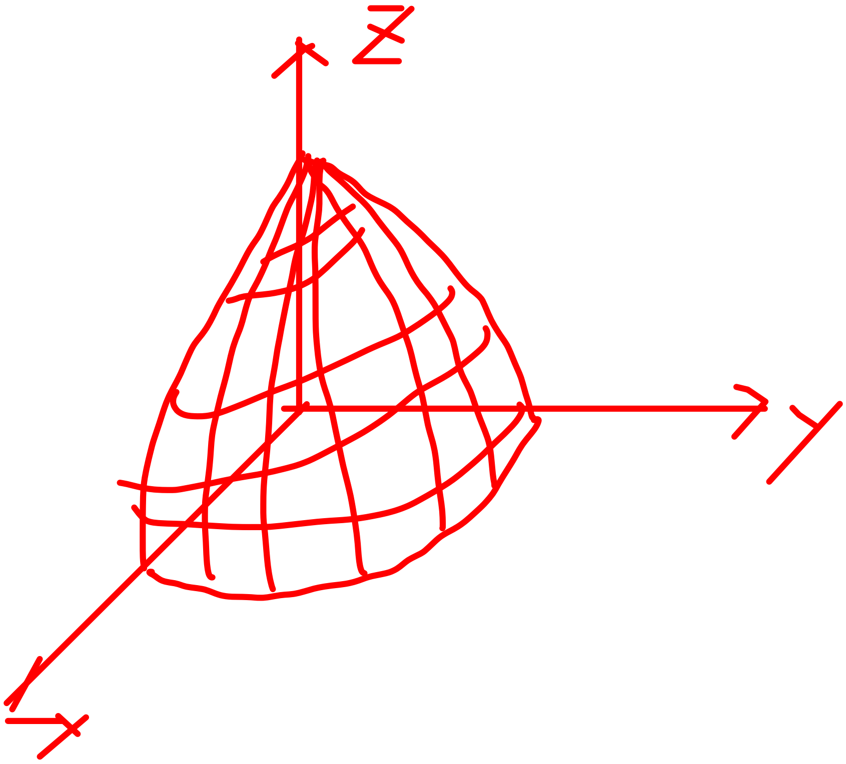



14. $4-(x^2+y^2)$

$x \geq 0,\ y \geq 0.$

only $\frac{1}{4}$ of sphere.

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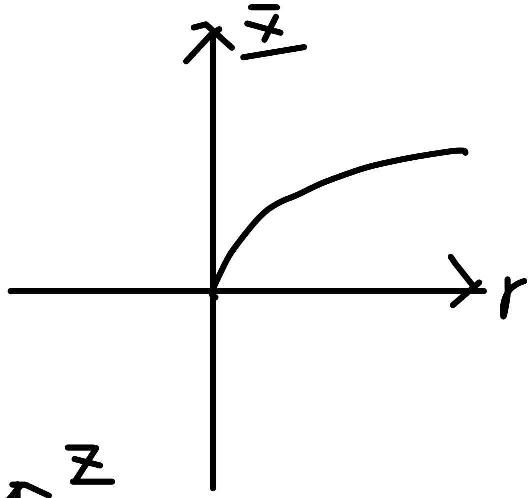
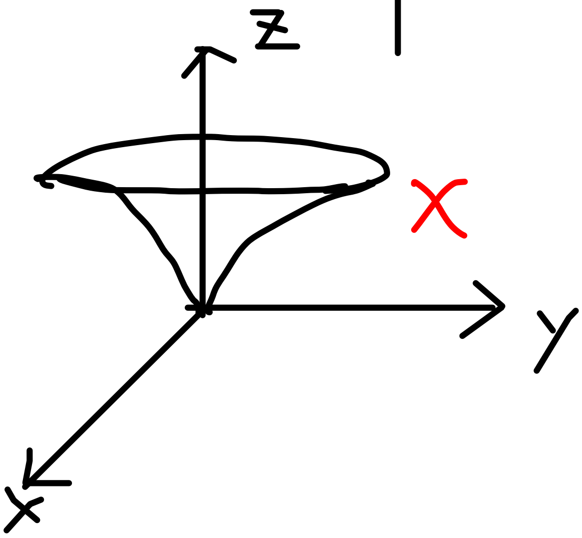
**15.** $f(x, y) = \sqrt{x^2 + y^2}$ **16.** $f(x, y) = 4 - x^2$

$z^2 = x^2 + y^2$

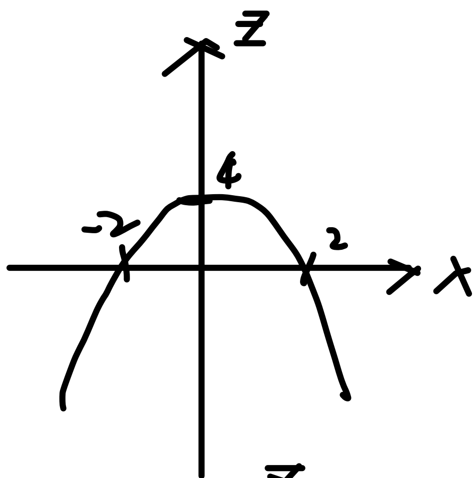
$z = \sqrt{r}$

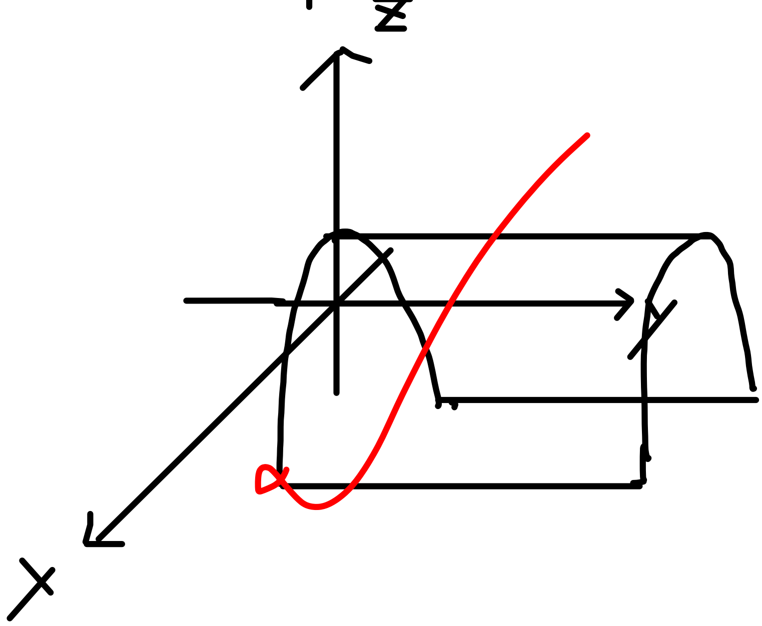



$z = r.$

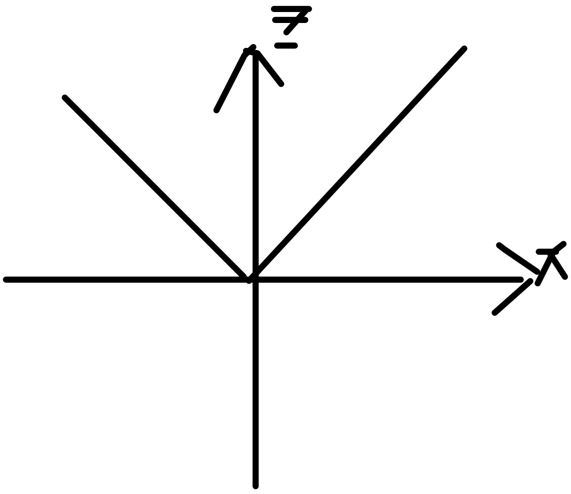


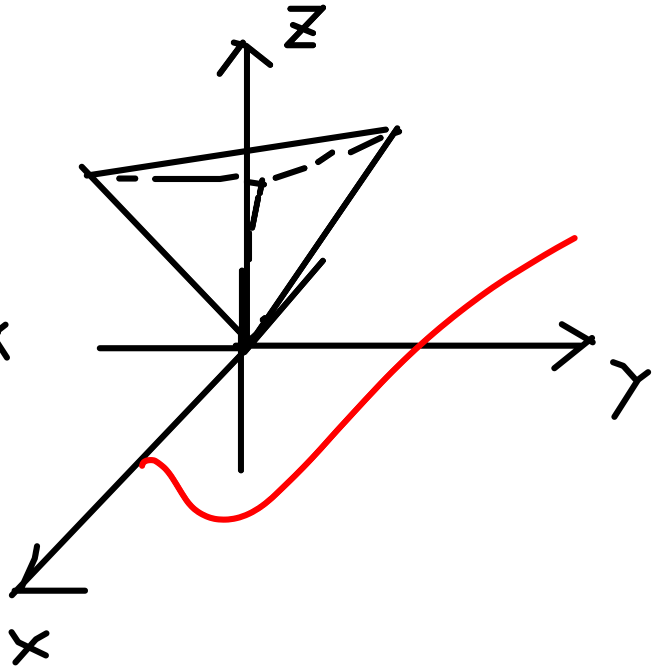

16. $z = 4 - x^2$



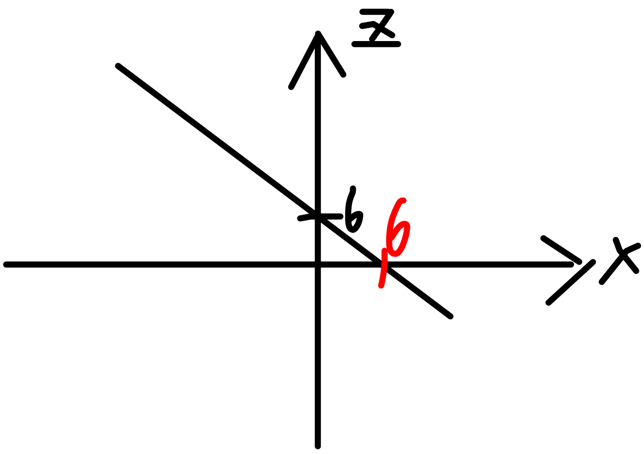



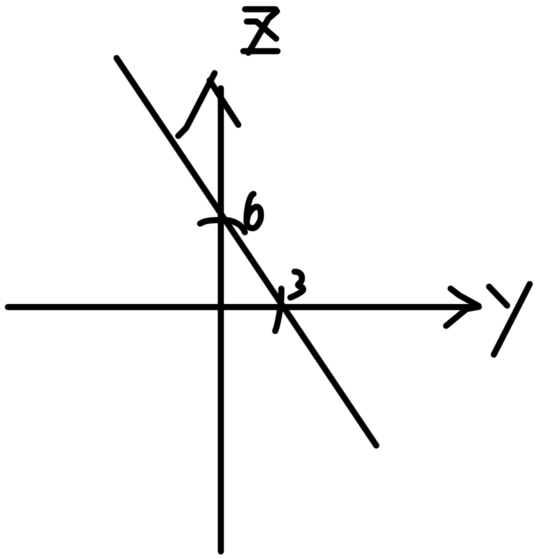
**17.** $f(x, y) = |x| + |y|$ **18.** $f(x, y) = 6 - x - 2y$



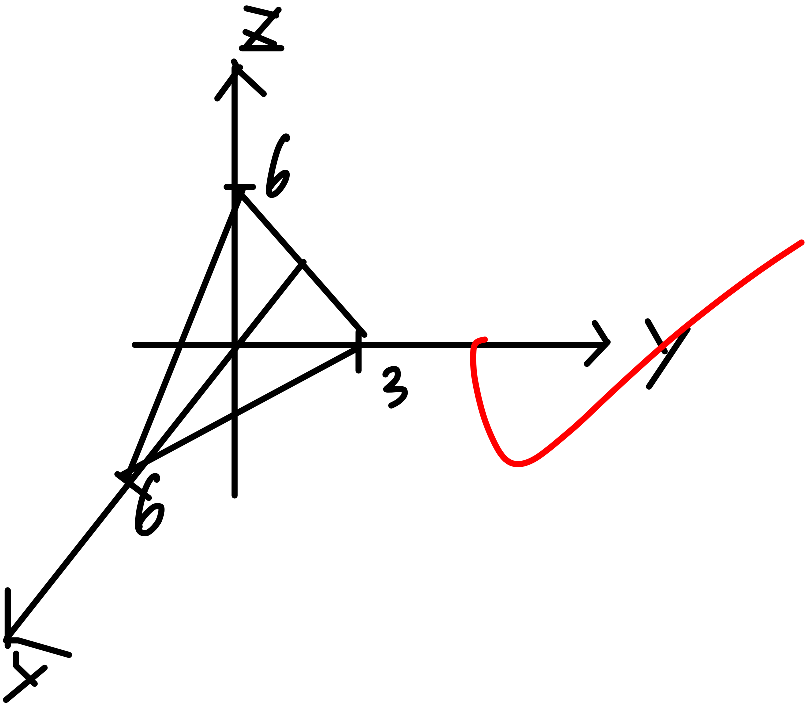




18. $z = 6 - x - 2y$.




$x + 2y + z = 6$

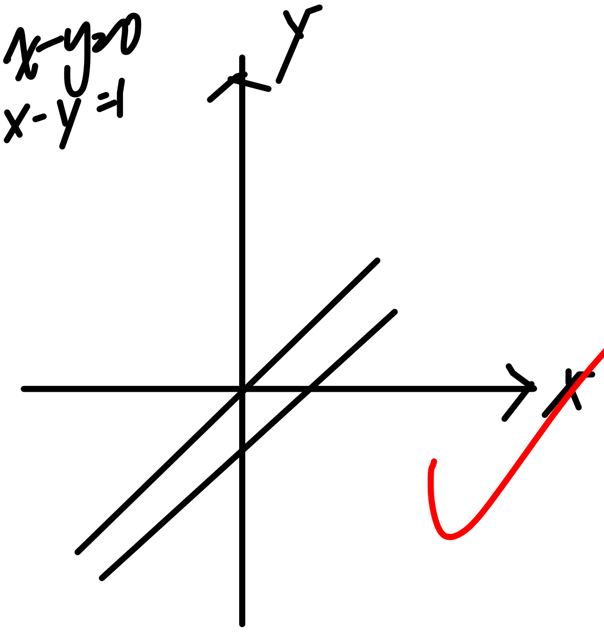


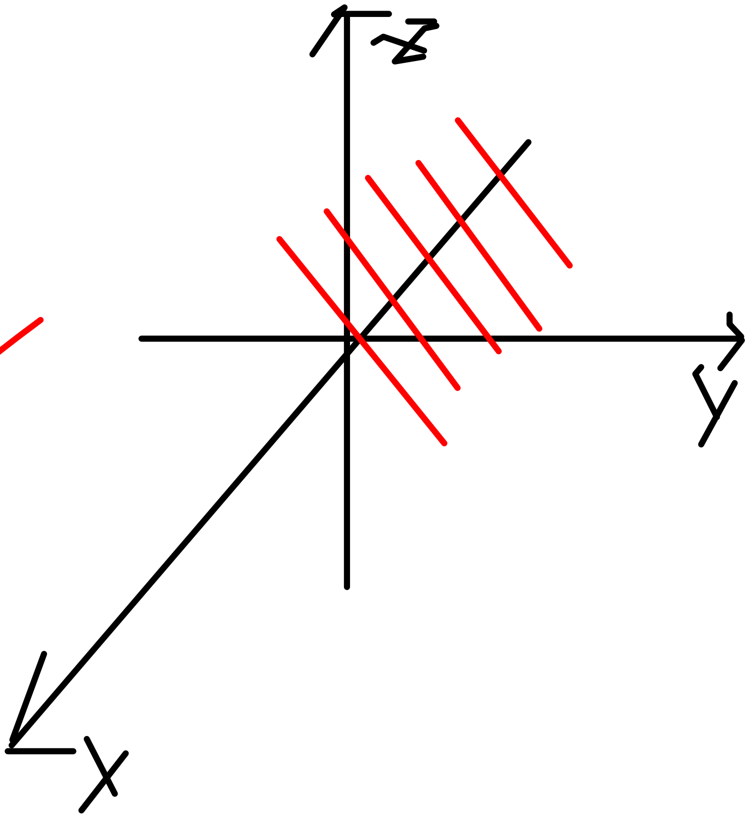
**19.** $f(x, y) = x - y$

**20.** $f(x, y) = x^2 + 2y^2$

$x - y = 0$

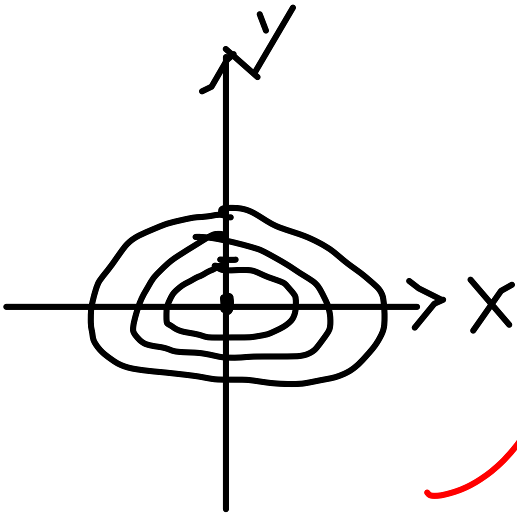
$x - y = 1$



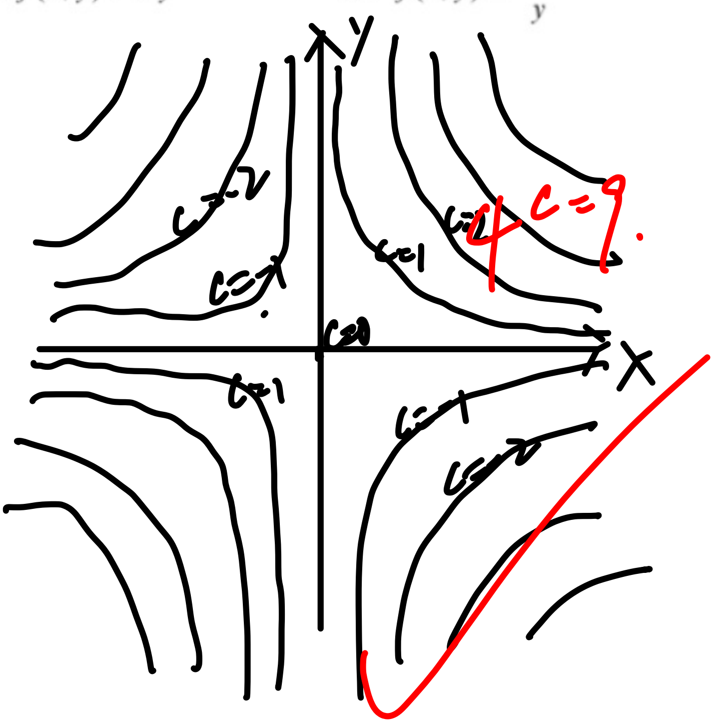

$x^2 + \left(\frac{1}{\sqrt{2}}\right)^2 y^2 = 0$

20



**21.** $f(x, y) = xy$ **22.** $f(x, y) = \dfrac{x^2}{y}$

21.




$0 = xy.$

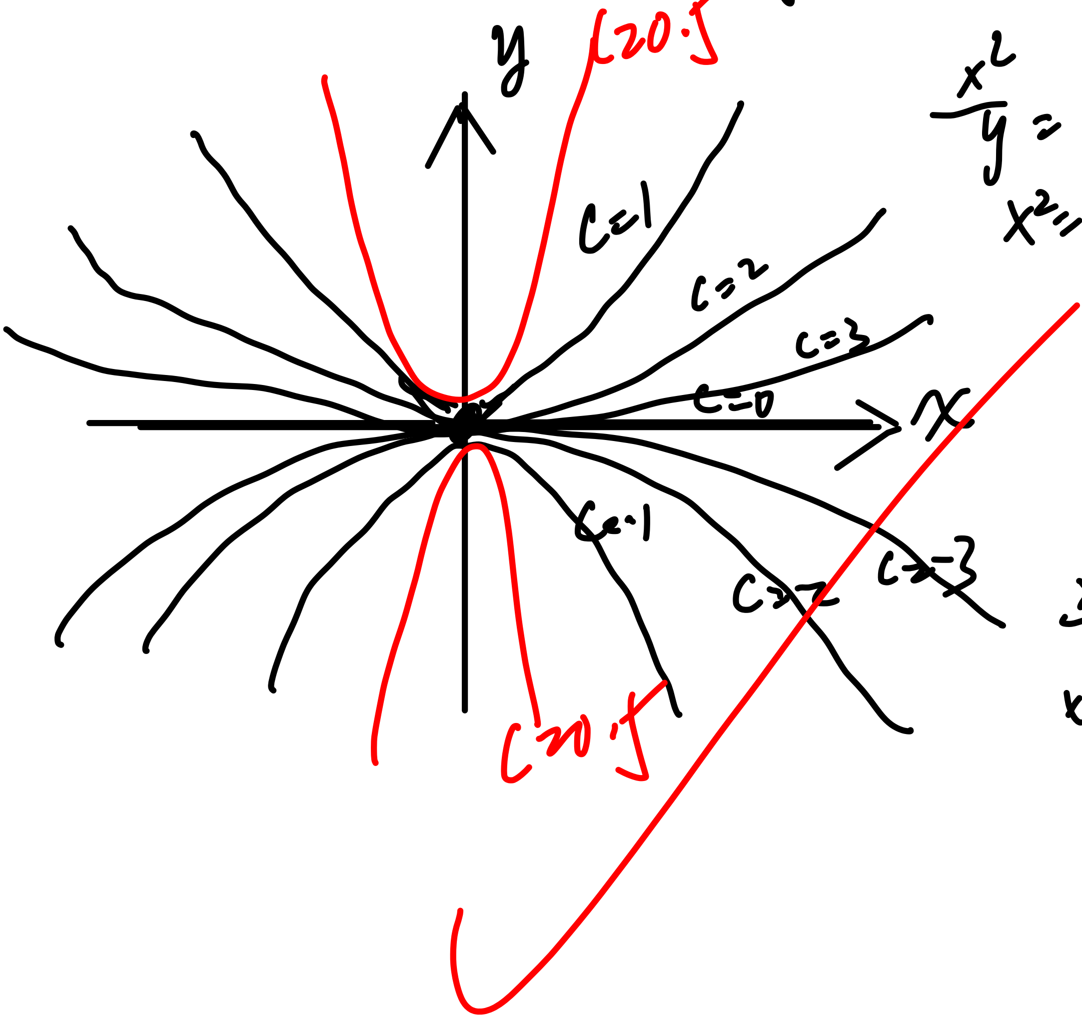
$1 = xy$ $-1 = xy$

$\frac{1}{x} = y$ $-\frac{1}{x} = y.$

| | | | | |
|---|---|---|---|---|
| x | -2 | -1 | 1 | 2 |
| y | $\frac{1}{2}$ | 1 | -1 | $-\frac{1}{2}$ |

22.

$\frac{x^2}{y} = 0$




$\frac{x^2}{y} = 1$

$x^2 = y.$

$\frac{x^2}{y} = 2$

$x^2 = 2y$

$y = \frac{x^2}{2}$

$\frac{x^2}{y} = -1$

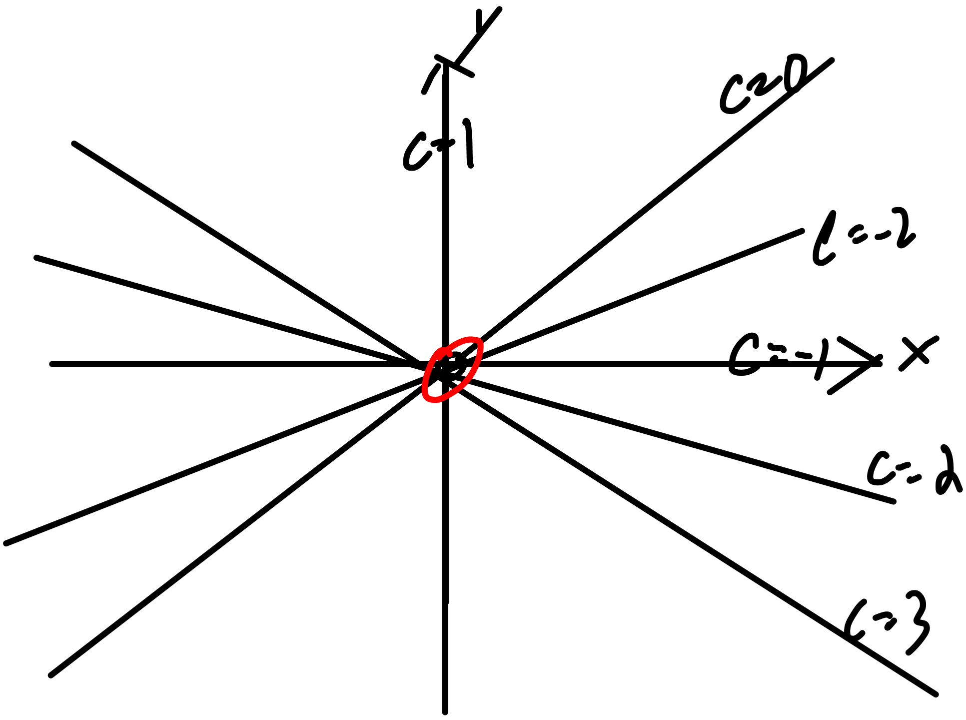
$x^2 = -y$

$y = -x^2$

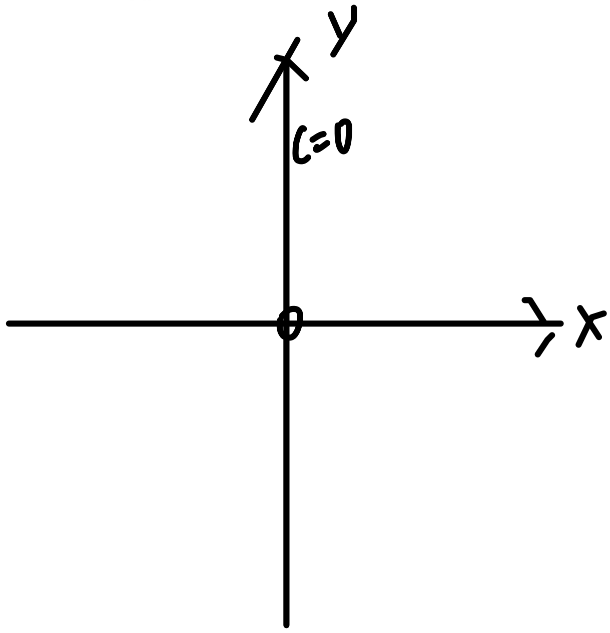
**23.** $f(x, y) = \dfrac{x - y}{x + y}$

**24.** $f(x, y) = \dfrac{y}{x^2 + y^2}$

23.



**24.** $f(x, y) = \dfrac{y}{x^2 + y^2}$




$y = x^2 + y^2$

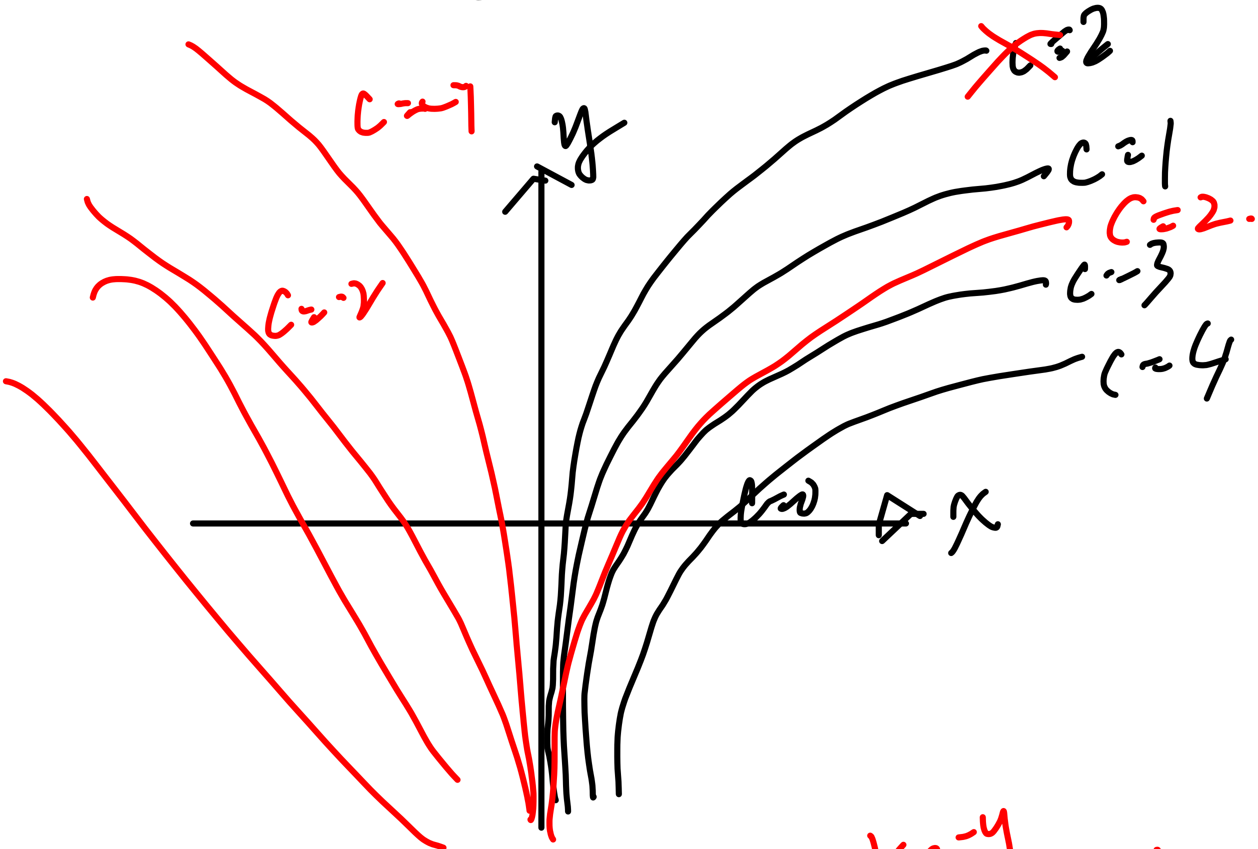
$0 = x^2 + y^2 - y$

$x^2 + (y-1)$

$x^2 + y^2 \neq 0$

**25.** $f(x, y) = xe^{-y}$

$$\frac{x}{e^y} = 0$$




$$xe^{-y} = -1$$

$$\frac{x}{e^y} = -1$$

$$x = -e^y$$

$$-x = e^y$$

$$\ln(-x) = y$$

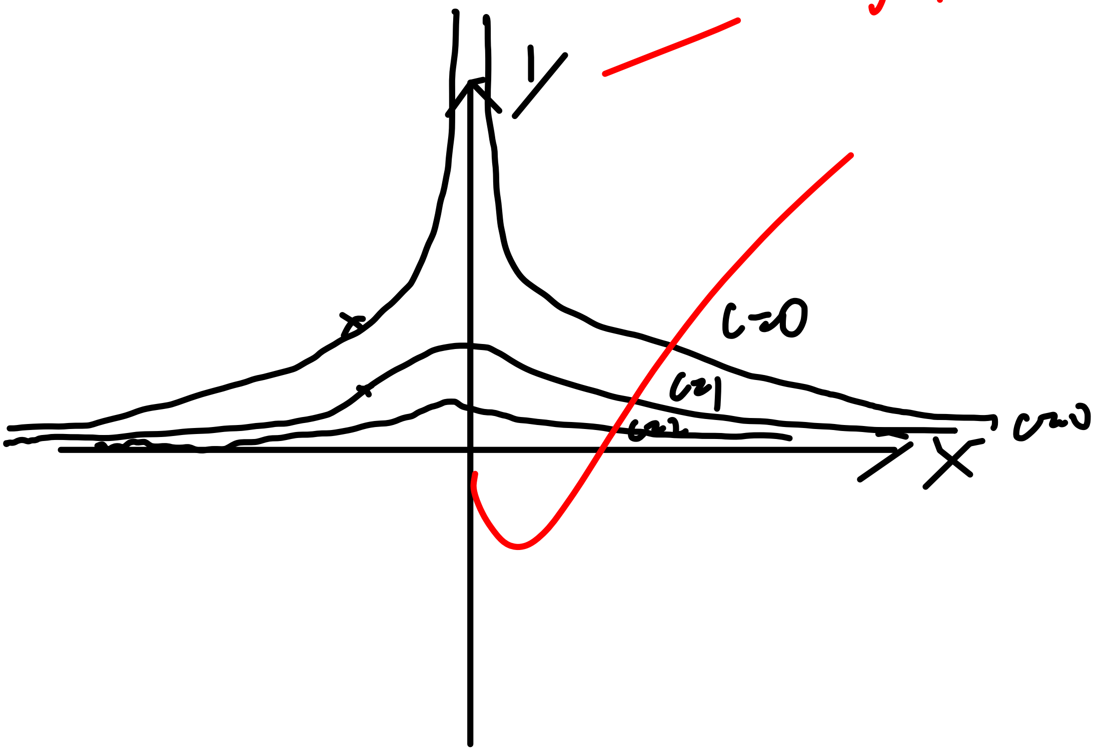
**26.** $f(x, y) = \sqrt{\frac{1}{y} - x^2}$

$$\sqrt{\frac{1}{y} - x^2} = c$$

$$\frac{1}{y} - x^2 = c^2$$

$$y = \frac{1}{x^2 + c^2}$$

$$y = \frac{1}{x^2}$$



Exercises 27–28 refer to Figure 12.11, which shows contours of a hilly region with heights given in metres.

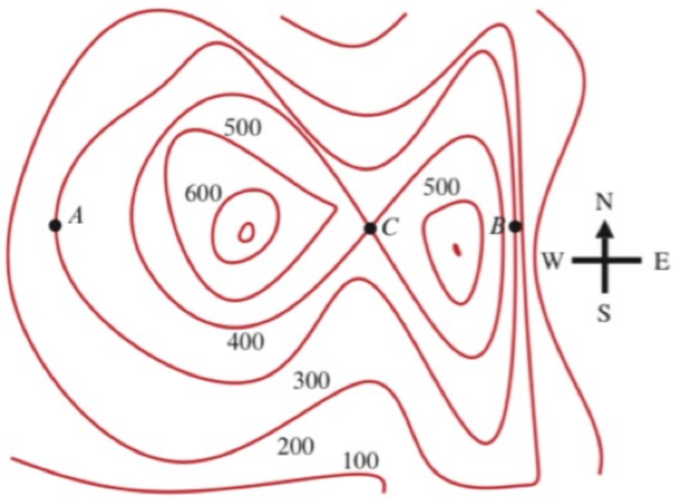



Figure 12.11

**27.** At which of the points $A$ or $B$ is the landscape steeper? How do you know?

B. For same distance, shorter time to travel.

**28.** Describe the topography of the region near point $C$.

It is saddle point, travel east-west direction = local min

concave up

travel north-south direction = concave down

local max

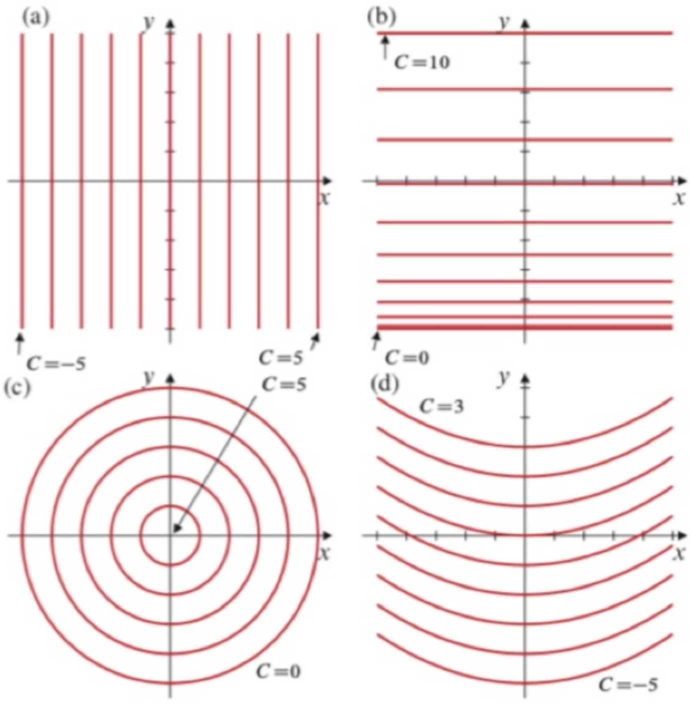
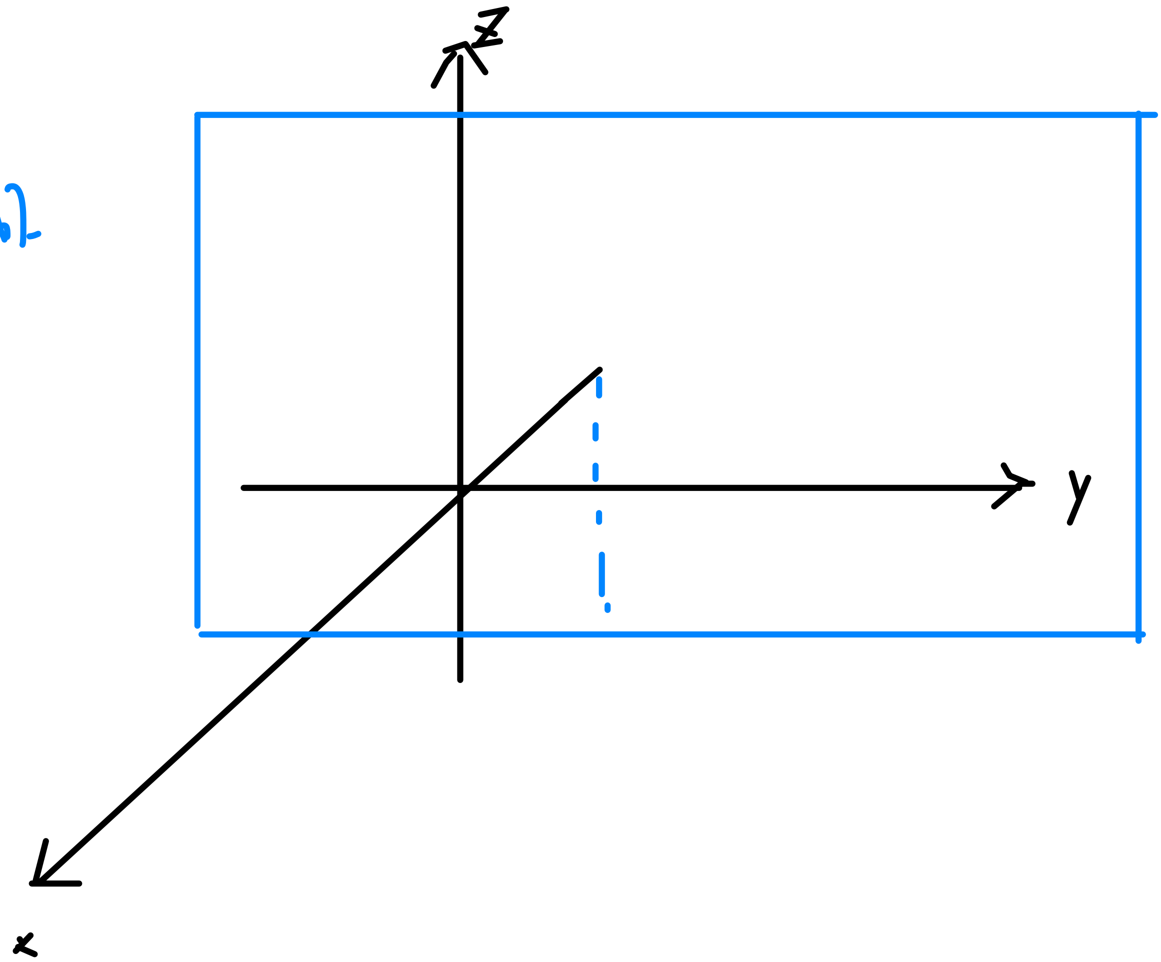


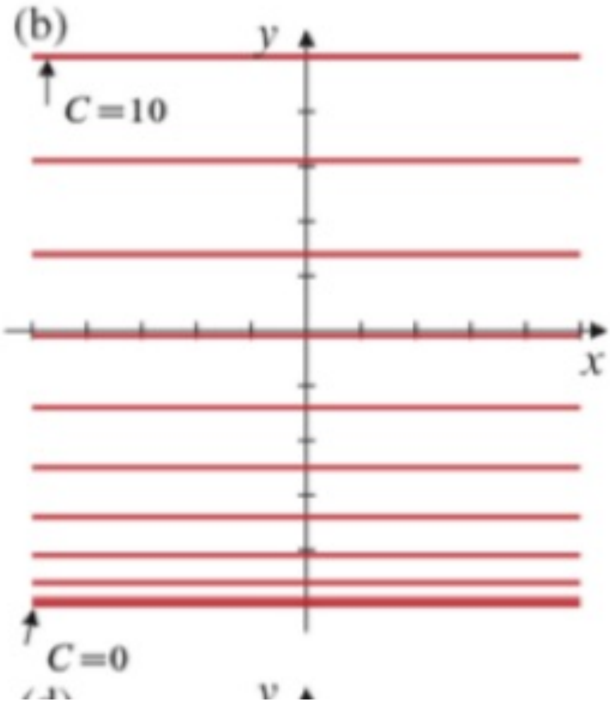

Figure 12.12

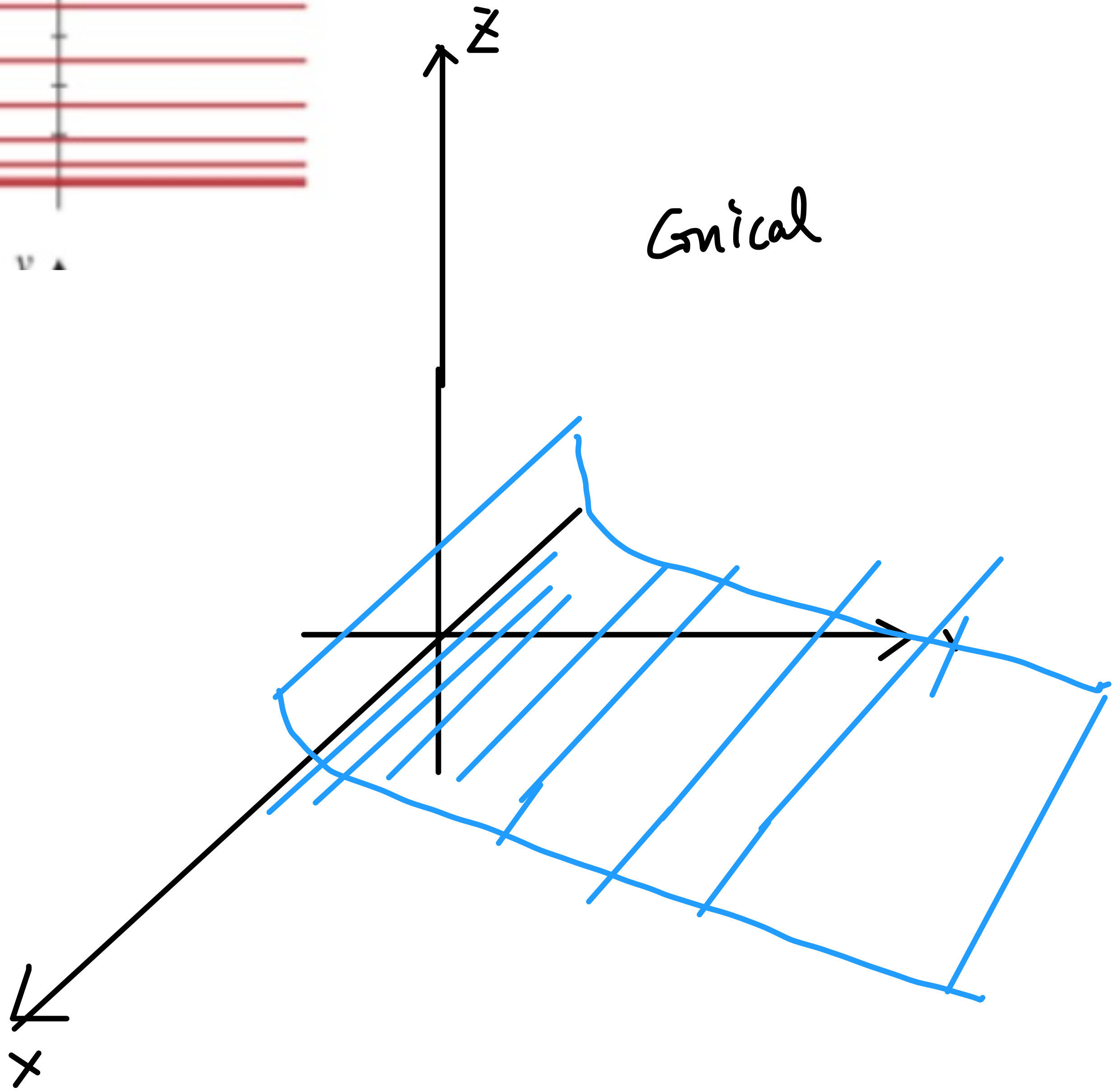
Describe the graphs of the functions $f(x, y)$ for which families of level curves $f(x, y) = C$ are shown in the figures referred to in Exercises 29–32. Assume that each family corresponds to equally spaced values of $C$ and that the behaviour of the family is representative of all such families for the function.

a ?

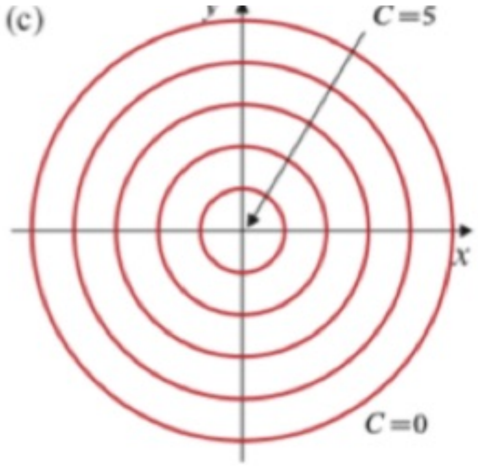
a)

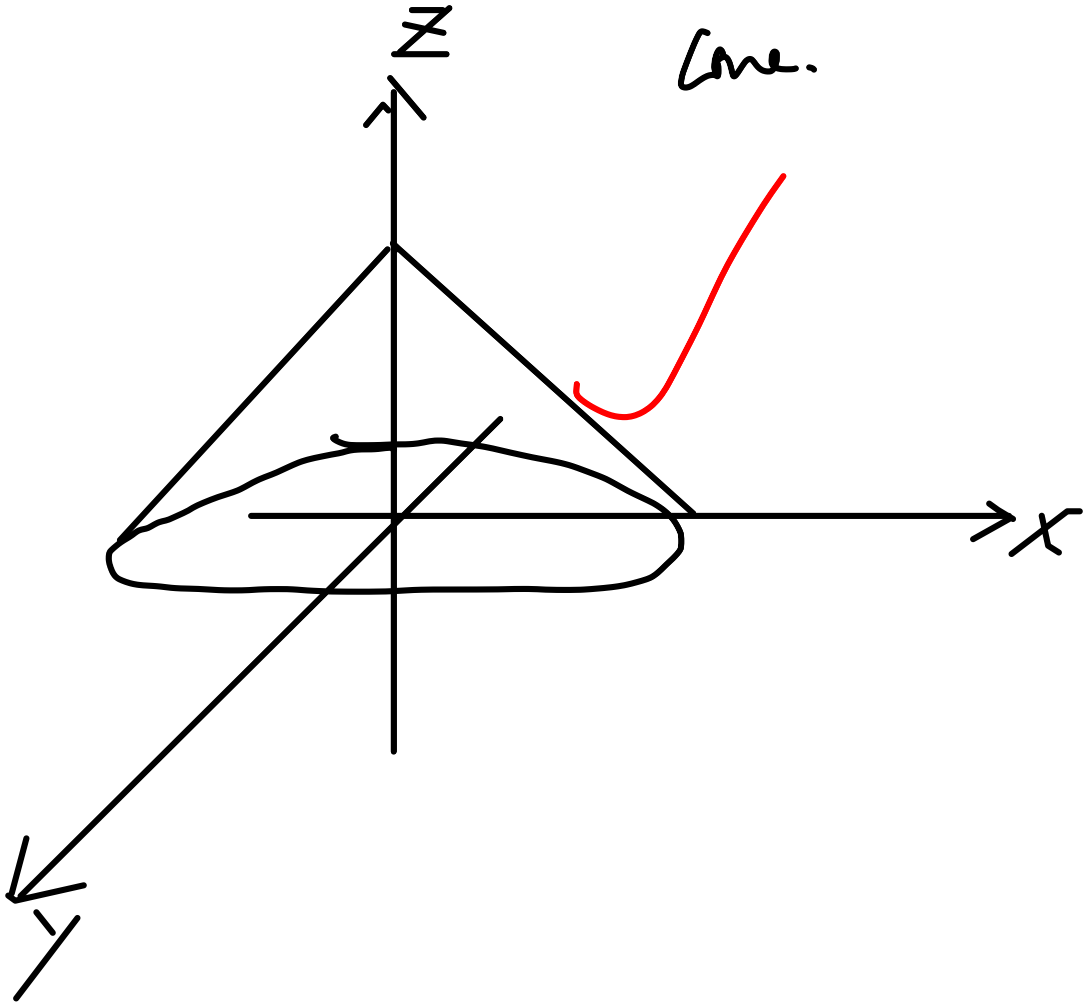




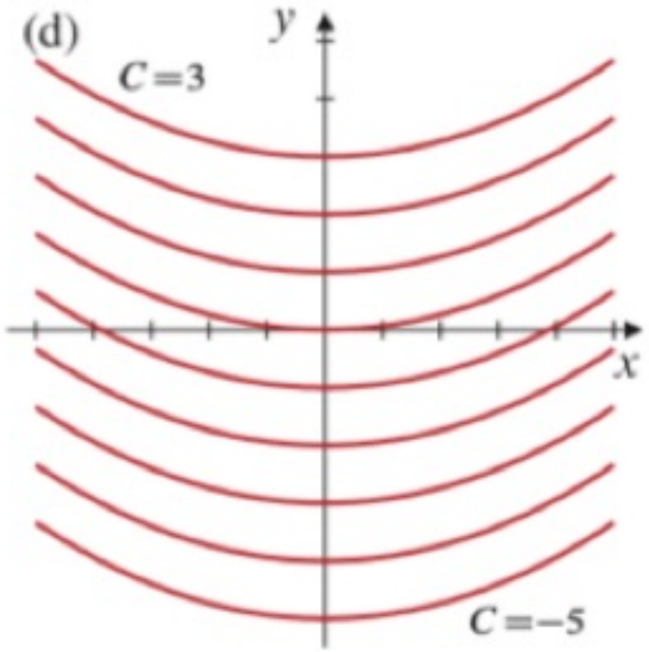



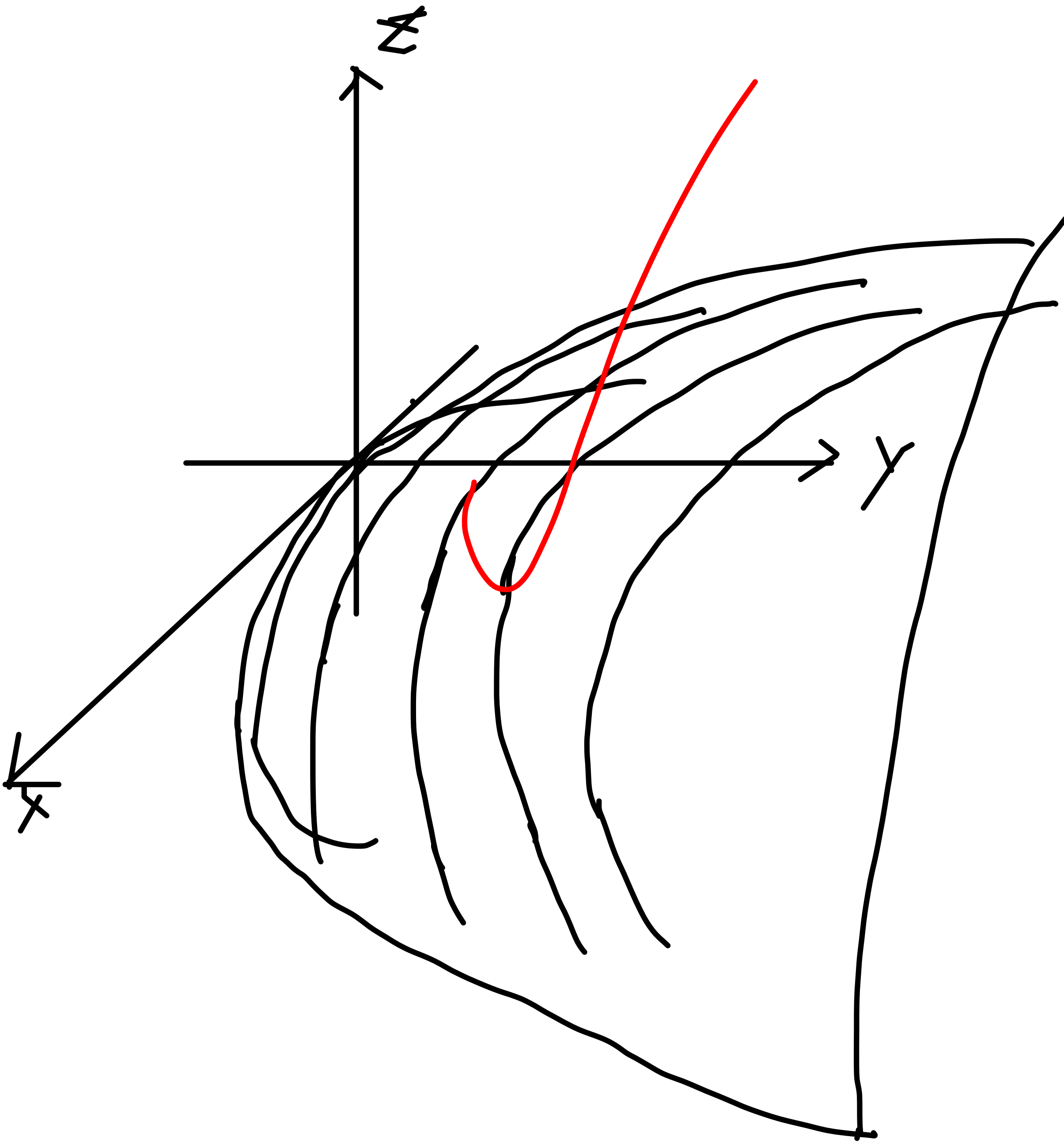

Gnical




Cone.


(d)
y
C=3
x
C=−5


z
y
x

**33.** Are the curves $y = (x - C)^2$ level curves of a function $f(x, y)$? What property must a family of curves in a region of the $xy$-plane have to be the family of level curves of a function defined in the region?

$$y = x^2 - 2Cx + C^2$$

$$y - x^2 + 2Cx = C^2$$

**34.** If we assume $z \geq 0$, the equation $4z^2 = (x - z)^2 + (y - z)^2$ defines $z$ as a function of $x$ and $y$. Sketch some level curves of this function. Describe its graph.

$$4z^2 = x^2 - 2xz + z^2 + y^2 - 2yz + z^2$$

$$2z^2 = x^2 - 2xz + y^2 - 2yz$$

$$=$$

this function. Describe its graph.

**35.** Find $f(x, y)$ if each level curve $f(x, y) = C$ is a circle centred at the origin and having radius
(a) $C$ (b) $C^2$ (c) $\sqrt{C}$ (d) $\ln C$.

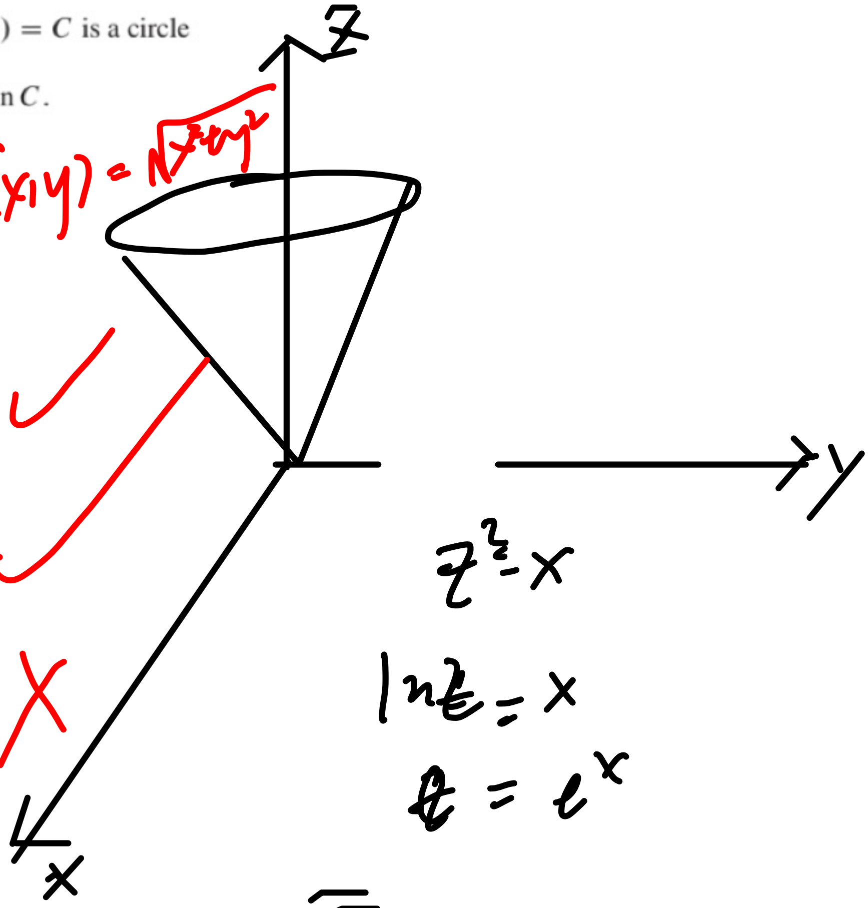
a). $x^2+y^2=z^2$ $f(x,y)=\sqrt{x^2+y^2}$

b). $x^2+y^2=z^4$ ✓

c). $x^2+y^2=z$ ✓

d). $x^2+y^2=e^{2z}$ X

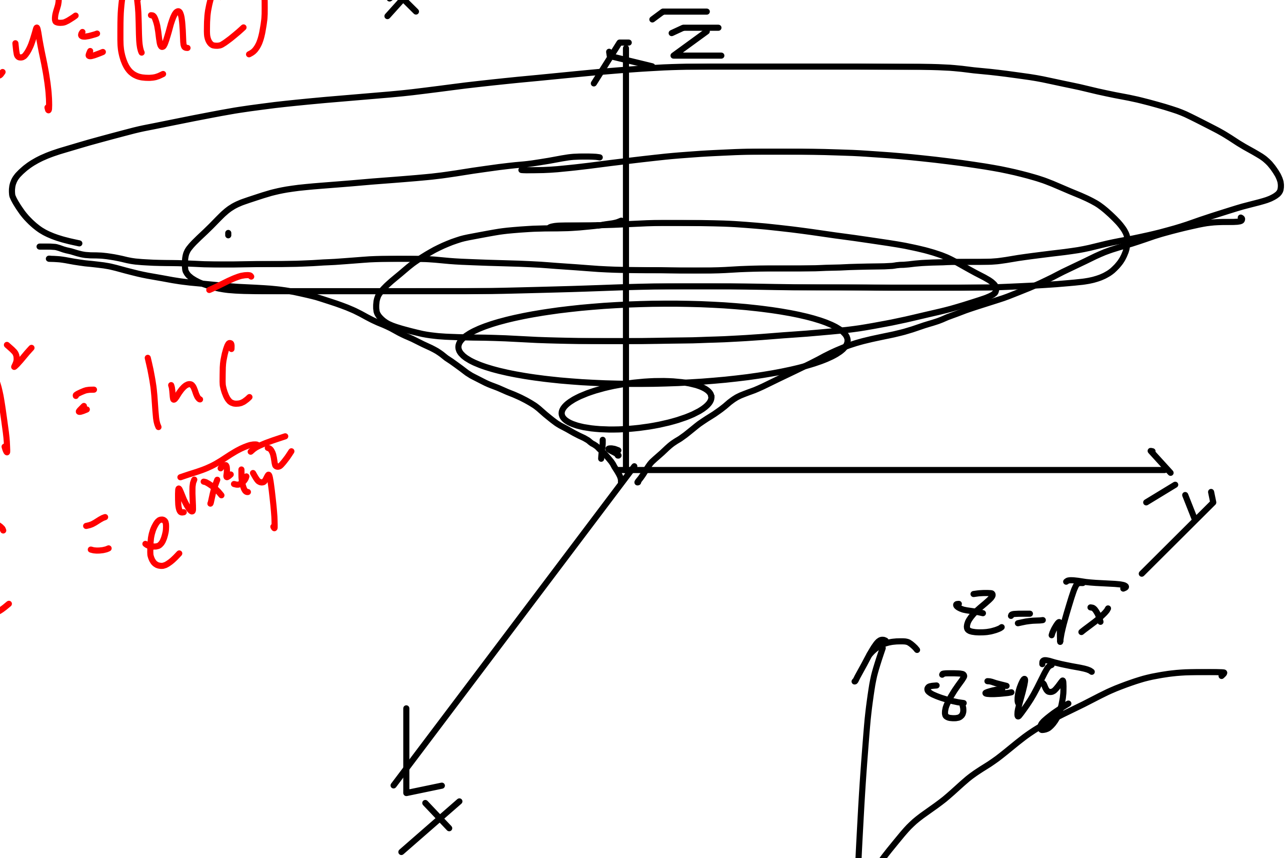
$x^2+y^2=(\ln C)^2$




$z^2=x$

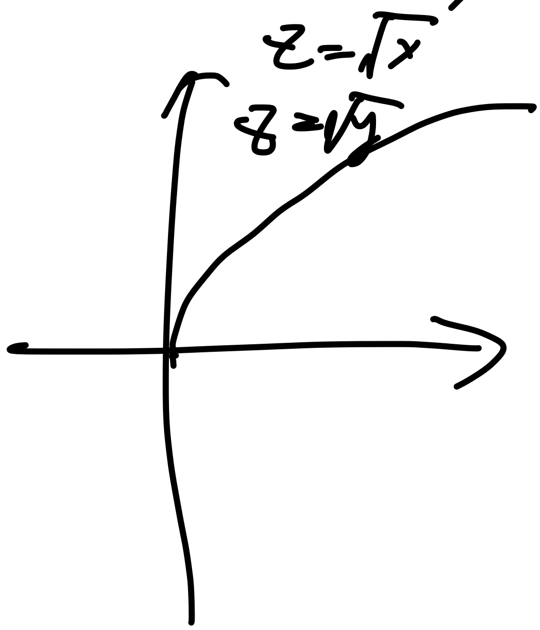
$\ln z=x$

$z=e^x$




$\sqrt{x^2+y^2}=\ln C$

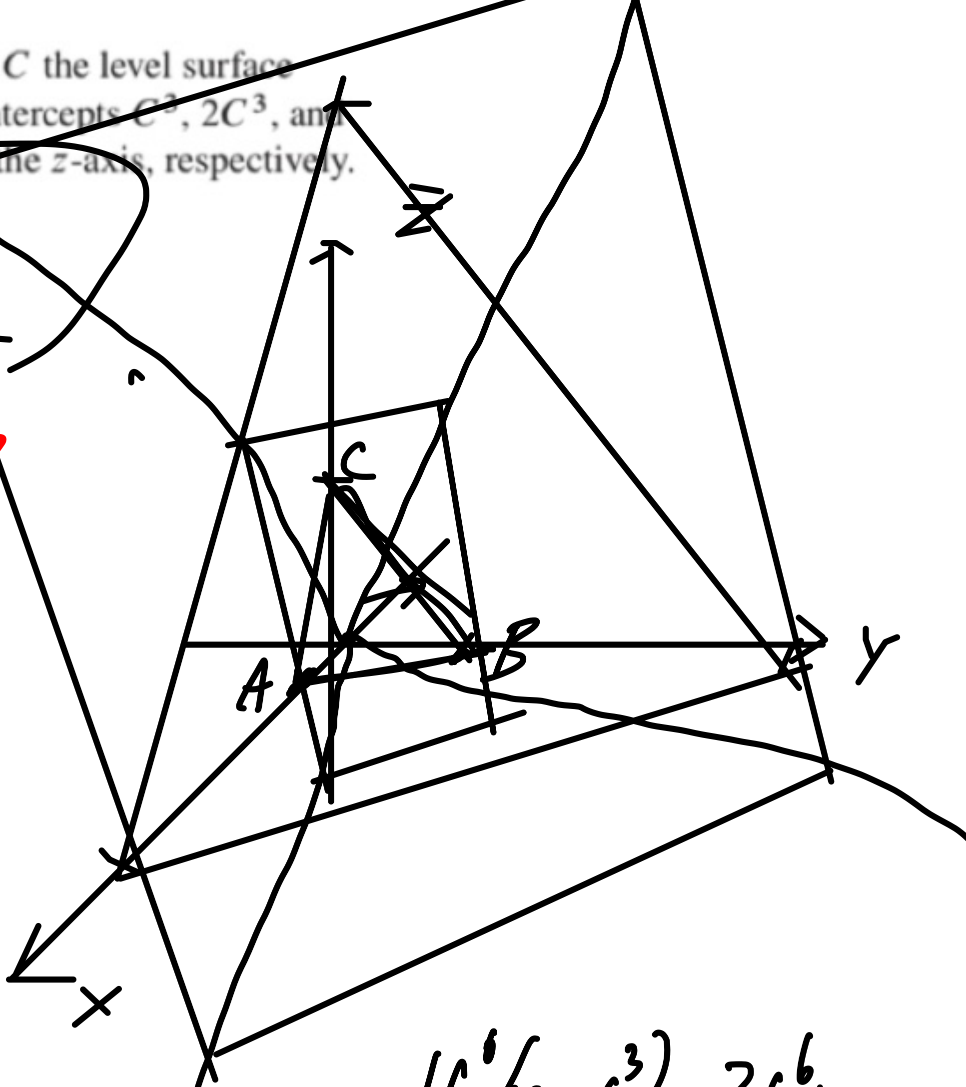
$C=e^{\sqrt{x^2+y^2}}$



**36.** Find $f(x, y, z)$ if for each constant $C$ the level surface $f(x, y, z) = C$ is a plane having intercepts $C^3$, $2C^3$, and $3C^3$ on the $x$-axis, the $y$-axis, and the $z$-axis, respectively.

$6x + 3y + 2z = z^3$

$x + \frac{1}{2}y + \frac{1}{3}z = z^3$




$(C^3, 0, 0)$

$(0, 2C^3, 0)$

$(0, 0, 3C^3)$

$\vec{BA} \times \vec{BC}$

$\vec{BA} = \langle C^3, -2C^3, 0 \rangle$

$\vec{BC} = \langle 0, -2C^3, 3C^3 \rangle$

$\vec{BA} \times \vec{BC} = \begin{vmatrix} i & j & k \\ C^3 & -2C^3 & 0 \\ 0 & -2C^3 & 3C^3 \end{vmatrix} = \langle -6C^6, -3C^6, -2C^6 \rangle$

$= C^6 \langle -6, -3, -2 \rangle$

$-6C^6(x - C^3) - 3C^6 y - 2C^6 z = 0$

$-6C^6 x - 3C^6 y - 2C^6 z = -6C^9$

$\boxed{6x + 3y + 2z = C^3}$

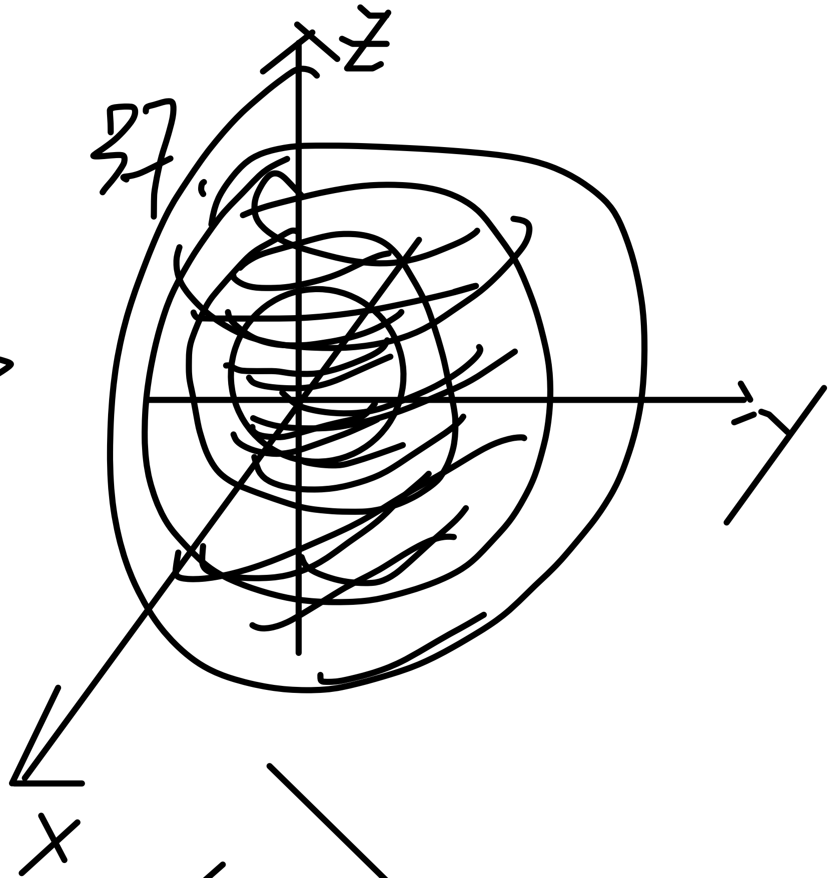
Describe the level surfaces of the functions specified in Exercises 37–41.

**37.** $f(x, y, z) = x^2 + y^2 + z^2$ sphere.

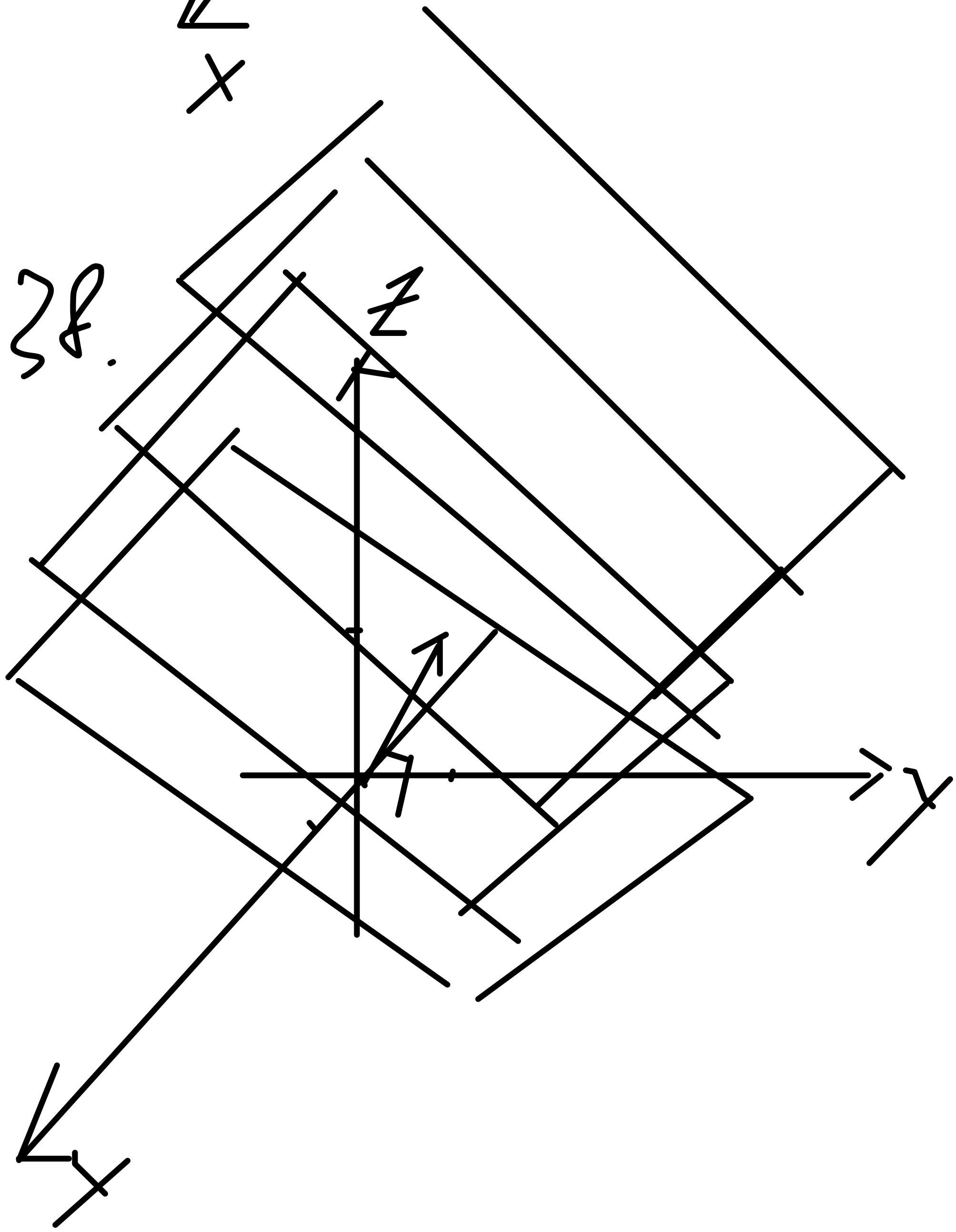
**38.** $f(x, y, z) = x + 2y + 3z$ plane

$\vec{n} = \langle 1, 2, 3 \rangle$

37.

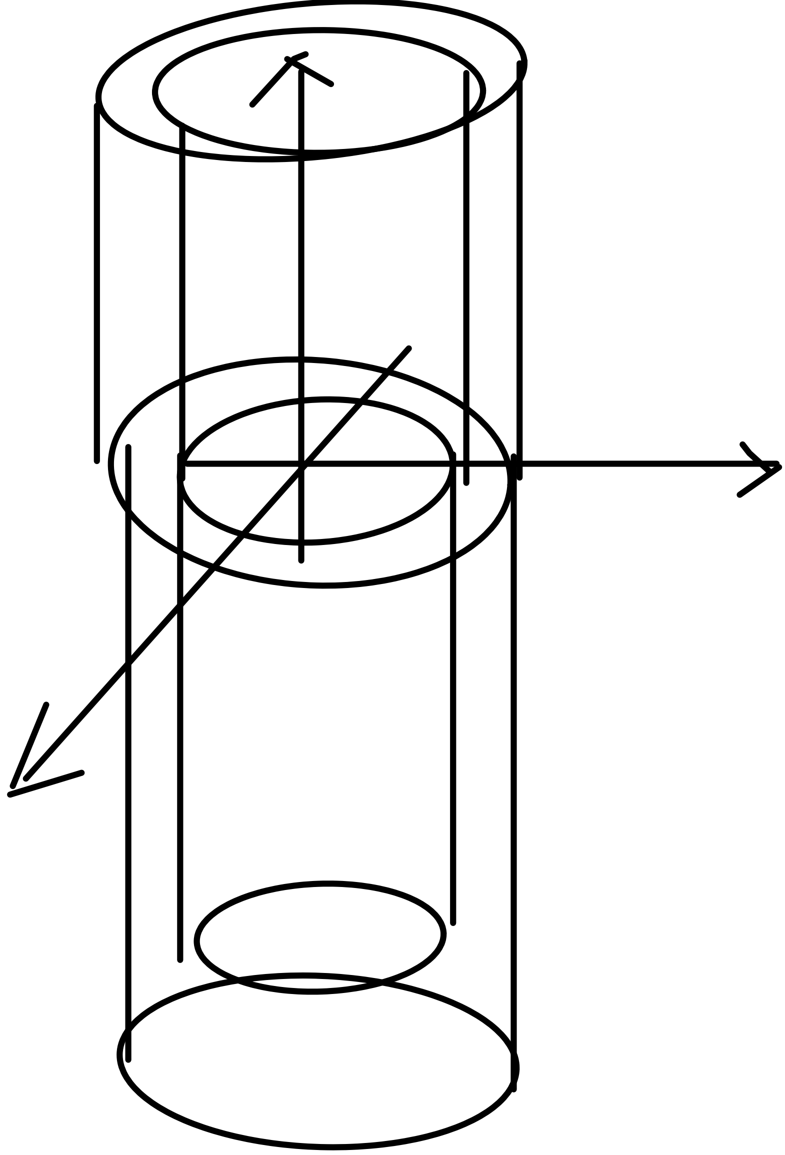



38.



**39.** $f(x, y, z) = x^2 + y^2$

**40.** $f(x, y, z) = \dfrac{x^2 + y^2}{z^2}$



**40.** $f(x, y, z) = \dfrac{x^2 + y^2}{z^2}$

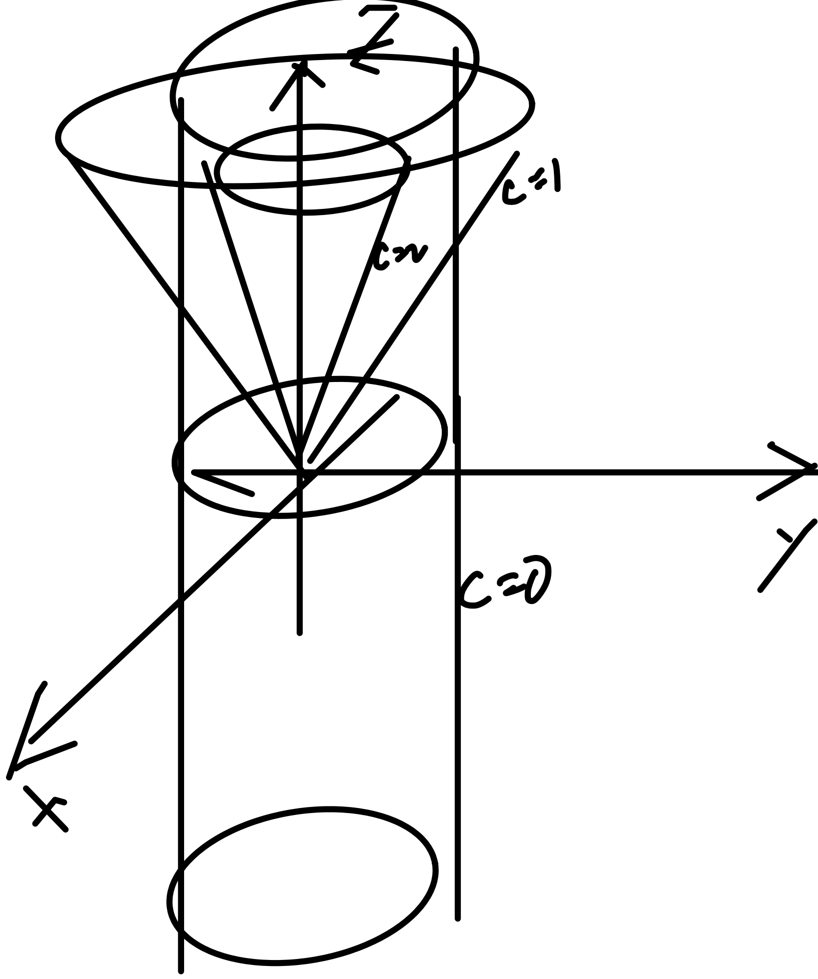
$$\frac{x^2+y^2}{z^2} = 0$$

$$x^2+y^2 = 0$$

$$x^2+y^2 = z^2$$

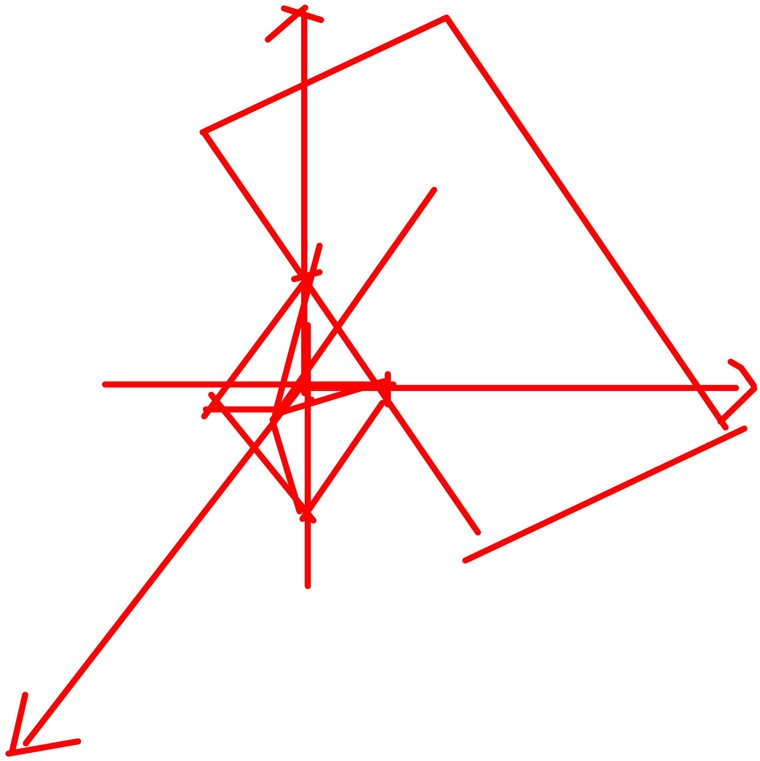
$$x^2+y^2 = 2z^2$$

$=$



**41.** $f(x, y, z) = |x| + |y| + |z|$

$|x| + |y| + |z| = c$



**42.** Describe the "level hypersurfaces" of the function

$c$

$$f(x, y, z, t) = x^2 + y^2 + z^2 + t^2.$$

$x^2+y^2+z^2 = c - t^2$