HKUST MATH 102

Midterm One Examination

Multivariable and Vector Calculus

30 Oct 2007

Answer ALL 5 questions

Time allowed – 120 minutes

Directions – This is a closed book examination. No talking or whispering are allowed. Work must be shown to receive points. An answer alone is not enough. Please write neatly. Answers which are illegible for the grader cannot be given credit.

Note that you can work on *both* sides of the paper and do not detach pages from this exam packet or unstaple the packet.

Student Name: ____________________

Student Number: ____________________

Tutorial Session: ____________________

Question No.	Marks
1	/20
2	/20
3	/20
4	/20
5	/20
Bonus	/5
Total	/100

Problem 1 (20 points) Your Score: ____

Identify the following surfaces

(a) $\mathbf{r} \cdot \widehat{\mathbf{u}} = 0.$

(b) $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) = k.$

(c) $\|\mathbf{r} - (\mathbf{r} \cdot \widehat{\mathbf{u}})\widehat{\mathbf{u}}\| = k.$ [Hint: What are the vectors $(\mathbf{r} \cdot \widehat{\mathbf{u}})\widehat{\mathbf{u}}$ and $\mathbf{r} - (\mathbf{r} \cdot \widehat{\mathbf{u}})\widehat{\mathbf{u}}$?]

Here k is fixed scalar, $\mathbf{a}$, $\mathbf{b}$ are fixed 3D vectors and $\widehat{\mathbf{u}}$ is a fixed 3D unit vector and $\mathbf{r} = (x, y, z)$.

Solution:

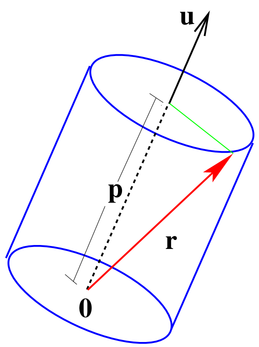
(a) Let $\widehat{\mathbf{u}} = (u_1, u_2, u_3)$, then $u_1 x + u_2 y + u_3 z = 0$, this is a plane with its normal in the direction of $\mathbf{u}$ and passing through the origin.

(b)

$$\begin{aligned} (\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{b}) &= k \\ \mathbf{r} \cdot \mathbf{r} - (\mathbf{a} + \mathbf{b}) \cdot \mathbf{r} + \mathbf{a} \cdot \mathbf{b} &= k \\ \|\mathbf{r} - \frac{\mathbf{a} + \mathbf{b}}{2}\|^2 &= k - \mathbf{a} \cdot \mathbf{b} + \frac{\|\mathbf{a} + \mathbf{b}\|^2}{4} \\ &= k + \frac{\|\mathbf{a} - \mathbf{b}\|^2}{4} \end{aligned}$$

$\therefore$ It is a sphere center at $(\mathbf{a} + \mathbf{b})/2$ with radius $\left[k + \|\mathbf{a} - \mathbf{b}\|^2/4\right]^{1/2}$.

(c) Note that $\mathbf{p} = (\mathbf{r} \cdot \widehat{\mathbf{u}})\widehat{\mathbf{u}}$ is the vector component of $\mathbf{r}$ onto the vector $\mathbf{u}$ and $\mathbf{r} - (\mathbf{r} \cdot \widehat{\mathbf{u}})\widehat{\mathbf{u}}$ is the vector component of $\mathbf{r}$ orthogonal to $\mathbf{u}$ and the norm of this vector is a constant k. Therefore, we can conclude that it is a circular cylinder of radius k with its axis parallel to $\mathbf{u}$.



Problem 2 (20 points) Your Score:

(a) Find the velocity, speed and acceleration at time t of the particle whose position is $\mathbf{r}(t)$. Describe the path of the particle.

$$\mathbf{r} = at\,\cos\omega t\,\mathbf{i} + at\,\sin\omega t\,\mathbf{j} + b\,\ln t\,\mathbf{k}$$

(b) Find the required parametrization of the first quadrant part of the circular arc $x^2 + y^2 = a^2$ in terms of arc length measured from $(0, a)$, oriented clockwise.

(c) Let C be the curve $x^{2/3} + y^{2/3} = a^{2/3}$ on the xy-plane, find the parametric equation of the curve C. Hence find the tangent line to the curve C at $(a, 0)$.

Solution:

(a) Position: $\mathbf{r}(t) = at\cos\omega t\,\mathbf{i} + at\sin\omega t\,\mathbf{j} + b\ln t\,\mathbf{k}$

Velocity: $\mathbf{v} = \mathbf{r}'(t) = a(\cos\omega t - \omega t\sin\omega t)\,\mathbf{i} + a(\sin\omega t + \omega t\cos\omega t)\,\mathbf{j} + (b/t)\,\mathbf{k}$

Acceleration: $\mathbf{a} = \mathbf{r}''(t) = -a\omega(2\sin\omega t + \omega t\cos\omega t)\,\mathbf{i} + a\omega(2\cos\omega t - \omega t\sin\omega t)\,\mathbf{j} - (b/t^2)\,\mathbf{k}$

Speed: $\|\mathbf{v}\| = \left[a^2(1 + \omega^2 t^2) + b^2/t^2\right]^{1/2}$

Let

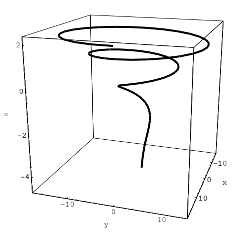
$$x = at\cos\omega t \tag{1}$$
$$y = at\sin\omega t \tag{2}$$
$$z = b\ln t \quad \Rightarrow \quad t = e^{z/b} \tag{3}$$

From (1) and (2)

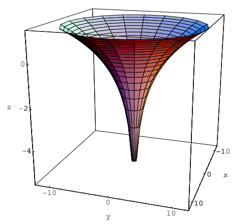
$$\frac{x^2}{a^2t^2} + \frac{y^2}{a^2t^2} = 1$$
$$x^2 + y^2 = a^2t^2 = a^2e^{2z/b}$$

$\therefore$ Path: a spiral on the surface (a"cone" - see figure below)

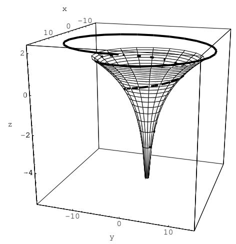
$$x^2 + y^2 = a^2e^{2z/b}.$$



Taking $a = 2\pi$ and $b = 2$



The surface $x^2 + y^2 = a^2e^{2z/b}$



The surface and the curve

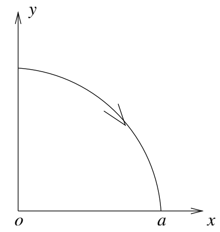
Problem 2 (20 points) Your Score:

(b) On the first quadrant part of the circle $x^2 + y^2 = a^2$, we have

$$\mathbf{r} = \mathbf{r}(\theta) = a\sin\theta\,\mathbf{i} + a\cos\theta\,\mathbf{j}, \qquad (0 \leqslant \theta \leqslant \pi/2)$$

The arc length $s = \theta a$, hence

$$\mathbf{r}(s) = a\sin\frac{s}{a}\,\mathbf{i} + a\cos\frac{s}{a}\,\mathbf{j},$$



(c) Let $x = a\cos^3\theta, \quad y = a\sin^3\theta$, then

$$\mathbf{r}(\theta) = a\cos^3\theta\,\mathbf{i} + a\sin^3\theta\,\mathbf{j}$$

then

$$\mathbf{r}'(\theta) = -3a\cos^2\theta\sin\theta\,\mathbf{i} + 3a\sin^2\theta\cos\theta\,\mathbf{j}.$$

At $(a, 0)$, $\theta = 0$ and $\mathbf{r}'(0) = \mathbf{0}$.

$\therefore$ The curve is not differentiable at $(a, 0)$, hence, there is no tangent line.

Problem 3 (20 points) Your Score:

(a) Assume **a**, **b**, **c**, **x** and **y** are three dimensional vectors and if

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a}) = [\mathbf{x}\cdot\mathbf{y}]^2.$$

Use suffix notation to find **x** and **y** in terms of the vectors **a**, **b** and **c**.

(b) Show that the line $x=-1+t$, $y=3+2t$, $z=-t$ and the plane $2x-2y-2z+3=0$ are parallel, and find the distance between them.

Solution:

(a) Note that from question (1a), we have $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \varepsilon_{ijk}a_ib_jc_k$.

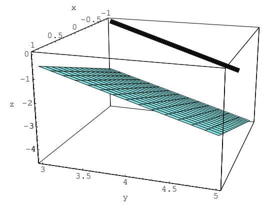
$$\begin{aligned}
\text{L.H.S.} &= (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})\\
&= (\mathbf{a}\times\mathbf{b})_i\cdot[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})]_i\\
&= \varepsilon_{ijk}a_jb_k\ \varepsilon_{ipq}(\mathbf{b}\times\mathbf{c})_p(\mathbf{c}\times\mathbf{a})_q\\
&= \varepsilon_{ijk}\varepsilon_{ipq}\ a_jb_k\ \varepsilon_{pmn}b_mc_n\ \varepsilon_{qrs}c_ra_s\\
&= (\delta_{jp}\delta_{kq}-\delta_{jq}\delta_{kp})\ \varepsilon_{pmn}\varepsilon_{qrs}\ a_jb_kb_mc_nc_ra_s\\
&= (\varepsilon_{jmn}\varepsilon_{krs}-\varepsilon_{kmn}\varepsilon_{jrs})\ a_jb_kb_mc_nc_ra_s\\
&= (\varepsilon_{jmn}a_jb_mc_n)(\varepsilon_{krs}b_kc_ra_s)-(\varepsilon_{kmn}b_kb_mc_n)(\varepsilon_{jrs}a_jc_ra_s)\\
&= [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})][\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]-[\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})][\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})]\\
&= [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2
\end{aligned}$$

$\therefore\ \mathbf{x}=\mathbf{a}$ and $\mathbf{y}=\mathbf{b}\times\mathbf{c}$.

Problem 3 (20 points) Your Score:

(b) The vector equation of the line is

$$\mathbf{r}(t) = (-1,3,0)+(1,2,-1)t = \mathbf{r}_1+\mathbf{v}t$$



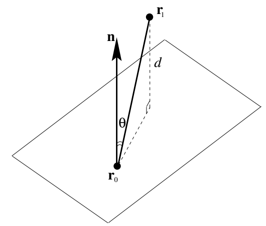
The given line is parallel to the plane if **v** is perpendicular to the normal vector **n** of the plane, i.e.

$$\mathbf{v}\cdot\mathbf{n} = (1,2,-1)\cdot(2,-2,-2) = 2-4+2=0$$

$\therefore$ **v** is perpendicular to **n**.

Now the problem becomes to find the distance between any point on the line and the given plane, i.e.

$$\begin{aligned}
d &= \|\mathbf{r}_1-\mathbf{r}_0\|\ |\cos\theta|\\
&= \|\mathbf{r}_1-\mathbf{r}_0\|\cdot\|\widehat{\mathbf{n}}\|\ |\cos\theta|\\
&= |(\mathbf{r}_1-\mathbf{r}_0)\cdot\widehat{\mathbf{n}}|
\end{aligned}$$



To find a point $\mathbf{r}_0$ on the plane, let $x=y=0$, then $z=3/2$, so

$$\begin{aligned}
d &= \left|((-1,3,0)-(0,0,3/2))\cdot(2,-2,-2)/\sqrt{12}\right|\\
&= |(-1,3,-3/2)\cdot(2,-2,2)|\,/\sqrt{12}\\
&= |-2-6+3|\,/\sqrt{12}\\
&= 5/\sqrt{12}
\end{aligned}$$

Problem 4 (20 points) Your Score:

(a) Find the parametric equation of the curve of intersection C between the plane $z = 2y + 3$ and the surface $z = x^2 + y^2$. Find also the equation of the projection curve of the curve of intersection C onto the xz-plane.

(b) Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{|x| + |y|}{\sqrt{x^2 + y^2}}$.

Solution:

(a) For intersection, we have

$$\begin{aligned} x^2 + y^2 &= 2y + 3 \\ x^2 + y^2 - 2y &= 3 \\ x^2 + (y-1)^2 &= 4 \qquad \text{(This is a circle.)} \end{aligned}$$

Let

$$\begin{aligned} x(\theta) &= 2\cos\theta \\ y(\theta) &= 2\sin\theta + 1, \qquad 0 \leqslant \theta < 2\pi \\ z(\theta) &= 4\sin\theta + 5 \end{aligned}$$

or $\mathbf{r}(\theta) = 2\cos\theta\,\mathbf{i} + (2\sin\theta + 1)\,\mathbf{j} + (4\sin\theta + 5)\,\mathbf{k}, \quad 0 \leqslant \theta < 2\pi.$

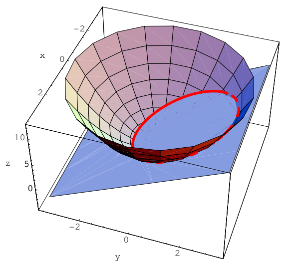
The projection curve has to be independent of y, therefore from

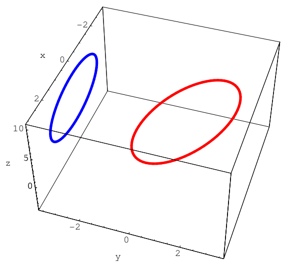
$$\begin{aligned} x &= 2\cos\theta \\ z &= 4\sin\theta + 5 \end{aligned}$$

we have

$$\frac{x^2}{4} + \frac{(z-5)^2}{25} = 1.$$

It is an ellipse.





Problem 4 (20 points) Your Score:

(b) We can see that the limit does not exist by looking at the limit along the paths $x = 0$ or $y = 0$ and $x = y$. On the one hand

$$\lim_{(0,y)\to(0,0)} \frac{|x| + |y|}{\sqrt{x^2 + y^2}} = \frac{|y|}{\sqrt{y^2}} = 1,$$

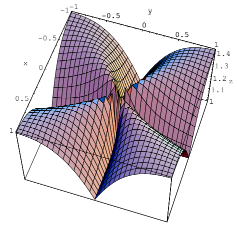
or

$$\lim_{(x,0)\to(0,0)} \frac{|x| + |y|}{\sqrt{x^2 + y^2}} = \frac{|x|}{\sqrt{x^2}} = 1,$$

while

$$\lim_{y=x,\ x\to 0} \frac{|x| + |y|}{\sqrt{x^2 + y^2}} = \frac{2\,|x|}{\sqrt{2x^2}} = \sqrt{2}.$$

Two different limits along two different paths, hence the limit does not exist.



Problem 5 (20 points) Your Score:

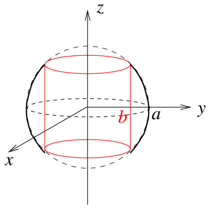
(a) Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

(i) Use the definition of partial derivative to show that $f_x(0,0)$ and $f_y(0,0)$ exist.

(ii) Is the function f continuous at $(0,0)$?

(b) Determine (sketch) the graph of the spherical-coordinate equation $\rho = 2\cos\phi$.

(c) A sphere of radius a is centered at the origin. A hole of radius b is drilled through the sphere, with the axis of the hole lying on the z-axis. Describe the solid region that remains (see Figure) in a (i) cylindrical coordinates; (ii) spherical coordinates.

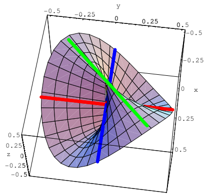


Solution:

(a) (i)

$$f_x(0,0) = \lim_{\triangle x \to 0} \frac{f(\triangle x, 0) - f(0,0)}{\triangle x} = \lim_{\triangle x \to 0} \frac{0-0}{\triangle x} = 0 \qquad \text{(blue)}$$

$$f_y(0,0) = \lim_{\triangle y \to 0} \frac{f(0, \triangle y) - f(0,0)}{\triangle y} = \lim_{\triangle y \to 0} \frac{0-0}{\triangle y} = 0. \qquad \text{(red)}$$



(ii) Take the path along $y = 0$, $x \neq 0$ (blue), then

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{x^2} = 0.$$

Take the path along $y = x$ (green), then

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}.$$

So different limits along different paths, hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

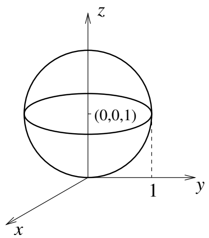
$\therefore$ $f(x,y)$ is not continuous at $(0,0)$.

(b) Multiplying by ρ gives

$$\rho^2 = 2\rho\cos\phi;$$

then substituting $\rho^2 = x^2+y^2+z^2$ and $z = \rho\cos\phi$ yields

$$x^2+y^2+z^2 = 2z$$



as the rectangular-coordinate equation of the graph. Completing the square in z now gives

$$x^2+y^2+(z-1)^2 = 1,$$

so the graph is a sphere with center $(0,0,1)$ and radius 1. It is tangent to the xy-plane at the origin (as shown in the figure).

Problem 5 (20 points) Your Score:

(c) Note that the equaation for sphere: $x^2+y^2+z^2 = a^2$, and for cylinder: $x^2+y^2 = b^2$.

(i) In cylindrical coordinates:

$$r^2 + z^2 = a^2 \quad \Rightarrow \quad z = \pm\sqrt{a^2 - r^2} \qquad \text{(sphere)}$$

$$r = b \qquad \text{(cylinder)}$$

$$\therefore \left\{ (r,\theta,z) \,\middle|\, b \leqslant r \leqslant a, \quad 0 \leqslant \theta \leqslant 2\pi, \quad z = \pm\sqrt{a^2-r^2} \right\}.$$

(ii) In spherical coordinates: ($x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$)

$$\rho = a \qquad \text{(sphere)}$$

$$\rho\sin\phi = b \qquad \text{(cylinder)}$$

$$\therefore \left\{ (\rho,\theta,\phi) \,\middle|\, \frac{b}{\sin\phi} \leqslant \rho \leqslant a, \quad 0 \leqslant \theta \leqslant 2\pi, \quad \sin^{-1}\left(\frac{b}{a}\right) \leqslant \phi \leqslant \pi - \sin^{-1}\left(\frac{b}{a}\right) \right\}.$$

Note that $\sin\phi = \dfrac{b}{a} \quad \Rightarrow \quad \phi = \sin^{-1}\left(\dfrac{b}{a}\right)$.

Bonus question (5 points) Your Score:

Find the maximum and minimum distances between the point $(1,1,1)$ and a point on the curve of intersection of the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=z$. (Note that the curve of intersection is not the origin).

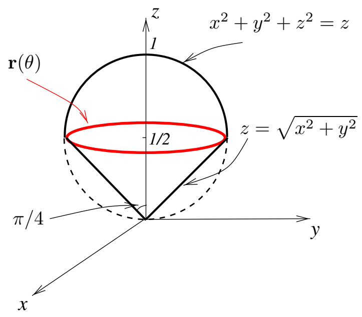
Solution:

First, find the curve of intersection.

$$\begin{aligned} z^2+z^2&=z\\ 2z^2-z&=0\\ \Rightarrow \quad z=0 \quad &\text{or} \quad z=\frac{1}{2}. \end{aligned}$$

If $z=\frac{1}{2}$, $x^2+y^2=\left(\frac{1}{2}\right)^2$.

(Note that $z\neq 0$.)



$\therefore$ The parametric form of the curve of intersection is

$$\mathbf{r}(\theta)=\frac{1}{2}\cos\theta\,\mathbf{i}+\frac{1}{2}\sin\theta\,\mathbf{j}+\frac{1}{2}\mathbf{k}, \qquad 0\leqslant\theta<2\pi.$$

$\therefore$ Let $\theta=\theta_0$ be the required point on the curve, then

$$\begin{aligned} D=d^2&=\left(\frac{1}{2}\cos\theta-1\right)^2+\left(\frac{1}{2}\sin\theta-1\right)^2+\left(\frac{1}{2}-1\right)^2\\ &=\frac{1}{4}\cos^2\theta-\cos\theta+1+\frac{1}{4}\sin^2\theta-\sin\theta+1+\frac{1}{4}\\ &=\frac{5}{2}-\cos\theta-\sin\theta\\ \frac{dD}{d\theta}&=\sin\theta-\cos\theta\\ \frac{d^2D}{d\theta^2}&=\cos\theta+\sin\theta. \end{aligned}$$

For critical points, $\frac{dD}{d\theta}=0$, i.e. $\tan\theta_0=1 \qquad \Rightarrow \qquad \theta_0=\frac{\pi}{4}$ or $\frac{5\pi}{4}$.

$$d_{\min}=\sqrt{\frac{5}{2}-\cos\frac{\pi}{4}-\sin\frac{\pi}{4}}=\sqrt{\frac{5}{2}-\sqrt{2}} \qquad (D''>0)$$

$$d_{\max}=\sqrt{\frac{5}{2}+\cos\frac{5\pi}{4}-\sin\frac{5\pi}{4}}=\sqrt{\frac{5}{2}+\sqrt{2}} \qquad (D''<0)$$