MATH 2023 – Multivariable Calculus

Lecture #14 Worksheet April 2, 2019

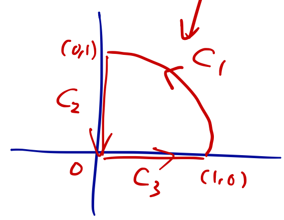
Problem 1. Find the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for

$$\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$$

along the quarter circle in the first quadrant with counterclockwise orientation.

$x(t)$ $y(t)$

$\langle \cos t, \sin t \rangle$



$$\int_{C_1} \vec{F} \cdot d\vec{r}$$

$dx = x'(t)dt$

$$= \int_{C_1} x^2 dx - xy\, dy \quad dy = y'(t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 t(-\sin t\, dt) - (\cos t)(\sin t)(\cos t\, dt)$$

$$= \int_0^{\frac{\pi}{2}} -2\cos^2 t \sin t\, dt$$

$$= \frac{2}{3}\cos^3 t \Big|_0^{\frac{\pi}{2}} = -\frac{2}{3}$$

$$\int_{C_2} x^2 \underbrace{dx}_{=0} - \underbrace{x}_{=0}y\, dy = 0 \quad \left(\begin{cases} x = 0 \\ y = 1 - t \end{cases} \quad 0 \le t \le 1 \iff dx = 0 \right)$$

$$\int_{C_3} x^2 dx - xy \underbrace{dy}_{=0} \quad \left(\begin{cases} x = t \\ y = 0 \end{cases} \quad 0 \le t \le 1 \iff dy = 0 \right)$$

$$\| \quad \int_0^1 t^2 dt = \frac{1}{3}$$

$$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = -\frac{2}{3} + 0 + \frac{1}{3} = -\frac{1}{3}$$

Problem 2. Let

$$\mathbf{F}(x,y) = \langle \overset{P}{3+2xy}, \overset{Q}{x^2-3y^2} \rangle$$

(a) Show that this is a conservative vector field

(b) Hence find $f(x,y)$ such that $\nabla f = \mathbf{F}$.

(c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for

$$\mathbf{r}(t) = \langle e^t \sin t, e^t \cos t \rangle, \qquad 0 \le t \le 2\pi.$$

a) $\frac{\partial P}{\partial y} = 2x$, $\frac{\partial Q}{\partial x} = 2x$, $D = \mathbb{R}^2$.

By Thm D, conservative ✓.

b) $3x + x^2y - y^3$

$\cos(0) = 1$. $3x + x^2y - y^3 + C$

c) $f(\vec{r}(2\pi)) - f(\vec{r}(0))$

$(\cdots + \cancel{C}) - (\cdots + \cancel{C})$

$= f(0, e^{2\pi}) - f(0, 1)$

$= 1 - e^{6\pi}$

Problem 3. Let $\mathbf{F}(x, y) = \langle y^2, 2xy + e^{3z}, 3ye^{3z}\rangle$

we dont know formula yet ?

(a) Show that this is a conservative vector field.

$\Updownarrow$

$\text{curl } \vec{F} = \vec{0}$

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $C : \mathbf{r}(t) = \langle t, t^2, t^3\rangle$ from $t = 0$ to $t = 1$.

Can still find $\nabla f = \vec{F}$

$$f = xy^2 + ye^{3z} \quad //$$

b) $$\int_C \vec{F} \cdot d\vec{r} = f(1,1,1) - f(0,0,0)$$
$$= 1 + e^3$$

Problem 4. Let $\mathbf{F}(x,y) = \langle \cos(x+2y), 2\cos(x+2y) \rangle$. Find curves C_1 and C_2 that are not closed, such that

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0, \quad \text{and} \quad \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$$

$\vec{F}$ is conservative : $f = \sin(x+2y)$ //

$$\int_C F \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

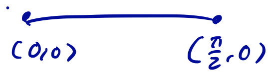
$$\begin{cases} f(0,0) = 0 \\ f(2\pi,0) = 0 \end{cases}$$

C_1 :

✓

$$\begin{cases} f(0,0) = 0 \\ F(\frac{\pi}{2},0) = 1 \end{cases}$$

C_2 .



✓.