MATH2023
Tutorial 11
May 13, 2019 Tutorial Notes by Ivan So

1 Review

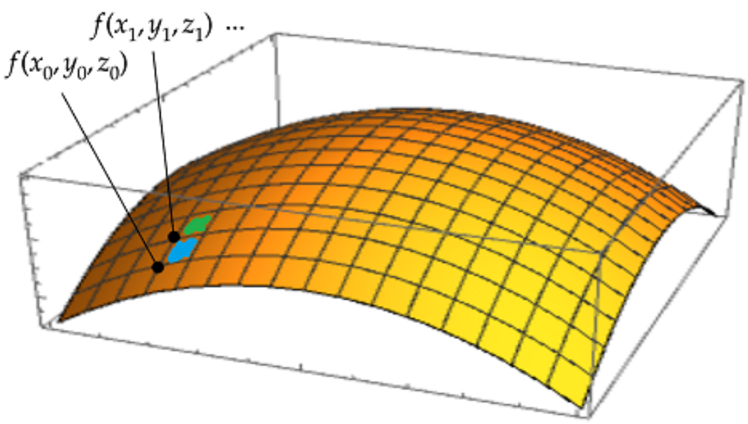
- The **surface area** of S parametrized by $\mathbf{r}(u,v)$ is given by

$$A(S) = \int\int_D |\mathbf{r}_u \times \mathbf{r}_v| dA.$$

- The **surface integral for function** f over S is defined by

$$\int_S f dS = \int\int_D f(x(u,v), y(u,v), z(u,v)) |\mathbf{r}_u \times \mathbf{r}_v| dv du.$$

Interpretation: We are summing the value of function a point multiplied by a differential area of a small patch.

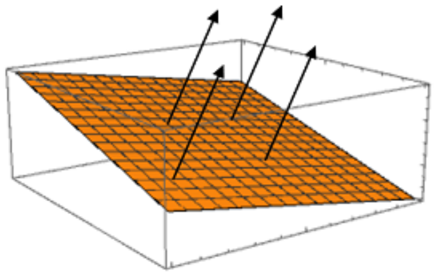


- The **surface integral** for *vector field* over S is defined by

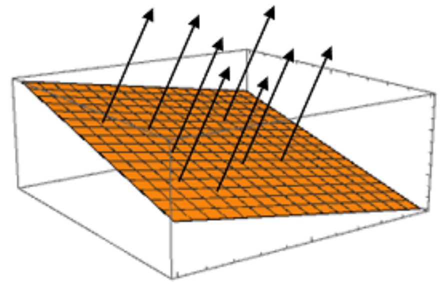
$$\int_S \mathbf{F} \cdot \mathbf{n} dS = \int\int_D \langle P(u,v), Q(u,v), R(u,v) \rangle \cdot \langle n_1(u,v), n_2(u,v), n_3(u,v) \rangle du dv$$

for normal vector of S $\mathbf{n}$ (**very important**: $\mathbf{n}$ necessary to be unit in length).

Interpretation: The measure of flux of vector field (component of the vector field parallel to the normal vector) through a surface (you may non-rigorously think of it as counting the number of vector field line passing through the surface).



fewer flux



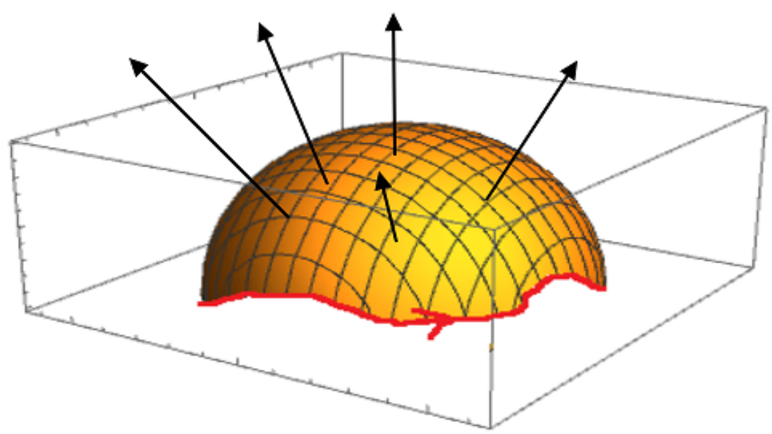
more flux

- **Stoke's Theorem**: The line integral over a closed curve is:

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA$$

where S is a surface with ∂S as boundary and $\mathbf{n}$ is the unit normal vector of the surface of obeying the *positive orientation* (satisfying the right hand grip rule).

Interpretation: Stoke's theorem said the loop integral of a vector field can be calculated by measuring the flux of the *curl* through the surface with the concerned curve as the boundary.



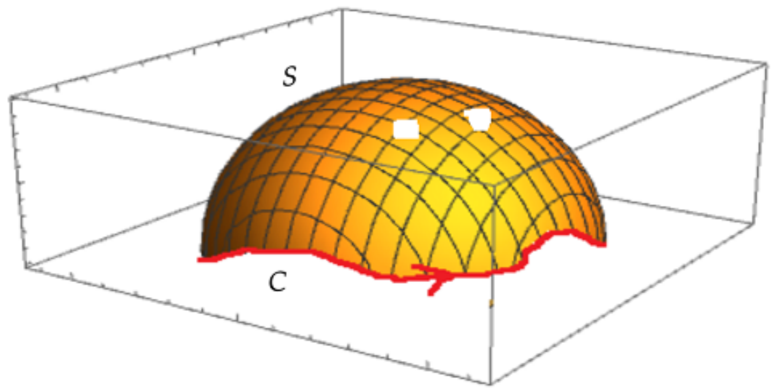
The arrows are the vector field lines of the curl of the vector field.

2 Problems

1. **True or False**

   (a) The constant vector field $\mathbf{F}(x, y, z) = \langle 1, -1/2, -1/2 \rangle$ has a non-vanishing flux through the surface $x + y + z = 1$.
   False. Normal of surface: $\mathbf{n} = \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle$. So $\mathbf{F} \cdot \mathbf{n} = 0$, vanishing.

(b) In the following diagram, $\oint_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$ is still satisfied, where S is a surface with holes.



False. Intuitively: if we got another surface with the holes covered, the contribution will be different in the surface integral.
More mathematically: The boundary of the surface is not only the curve, the curve at the boundary of the wholes are also counted as the boundaries (see generalized Stoke's Theorem).

2. Evaluate $\int\int_S (x^2 z + y^2 z) dS$, where S is the upper hemisphere with radius 2.
Solution: Using the parametrization $\mathbf{r}(u, v) = \langle r_0 \sin u \cos v, r_0 \sin u \sin v, r_0 \cos u \rangle$, then $|\mathbf{r}_u \times \mathbf{r}_v| = r_0^2 \sin u$ (refer to tutorial 10). Then,

$$\int\int_S (x^2 z + y^2 z) dS = \int_0^{2\pi} \int_0^{\pi/2} [(r_0 \sin u \cos v)^2 r_0 \cos u + (r_0 \sin u \sin v)^2 r_0 \cos u] r_0^2 \sin u du dv$$

3. Let $\mathbf{F}(x, y, z) = \mathbf{r}/|\mathbf{r}|^3$, where $\mathbf{r} = \langle x, y, z \rangle$. Show that the flux through the surface of the sphere is independent of the radius.
Solution: Using the spherical parametrization (denote it by $\mathbf{R}(u, v)$ in this question to avoid confusion), then $|\mathbf{R}_u \times \mathbf{R}_v| = r_0^2 \sin u$, where r_0 is the radius of the sphere, then the evaluation of the surface integral is as follows:

$$\begin{aligned}
\int\int_S \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{2\pi} \int_0^{\pi} \frac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} r_0^2 \sin u du dv \\
&= \int_0^{2\pi} \int_0^{\pi} \sin u du dv \quad (\mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2 = r_0^2) \\
&= 4\pi \quad (\text{independent of } r_0)
\end{aligned}$$

4. Find the center of mass of a hemisphere shell assuming uniform density.
Solution: Suppose σ is the constant density of mass, then again use the parametrization

of sphere with radius r_0: $\mathbf{r}(u,v) = \langle r_0 \sin u \cos v, r_0 \sin u \sin v, r_0 \cos u \rangle$, then $|\mathbf{r}_u \times \mathbf{r}_v| = r_0^2 \sin u$. Then, then center of mass is given by

$$\begin{aligned}
\bar{\mathbf{r}} &= \int\int_S \mathbf{r} dm \Big/ \int\int_S dm \quad \text{(integrate the center of mass coordinates all at once)} \\
&= \int_0^{2\pi}\int_0^{\pi/2} \mathbf{r}\sigma|\mathbf{r}_u \times \mathbf{r}_v| dudv \Big/ \int_0^{2\pi}\int_0^{\pi/2} \sigma|\mathbf{r}_u \times \mathbf{r}_v| dudv \\
&= \int\int_S \mathbf{r}\sigma dS \Big/ \int\int_S \sigma dS \\
&= \int_0^{2\pi}\int_0^{\pi/2} \sigma r_0^3 \langle (\sin u)^2 \cos v, (\sin u)^2 \sin v, \sin u \cos u \rangle dudv \Big/ \sigma(2\pi r_0^2) \\
&= \langle 0, 0, r_0/2 \rangle
\end{aligned}$$

5. Use Stoke's Theorem to evaluate the close loop integral $\oint_F \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = \langle x + y^2, y + z^2, z + x^2 \rangle$ over C, where C is the triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$, where the loop is running counterclockwise if we view from infinitely far away in the positive quadrant.
Solution: Parametrization of the concerned plane: $\mathbf{r}(u,v) = \langle u, v, 1-u-v \rangle$. $\mathbf{r}_u \times \mathbf{r}_v = \langle 1,1,1 \rangle$. Notice that $|\mathbf{r}_u \times \mathbf{r}_v|\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$ (given $\mathbf{n}$ pointing in the appropriate direction up to sign). So we can calculate the integral through Stoke's Theorem

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int\int_R (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA \\
&= \int_0^1 \int_0^{1-u} (-2(1-u-v) - 2u - 2v) dvdu \\
&= -1
\end{aligned}$$

6. Let C be the closed simple curve lies in the plane $x + y + z = 1$. Show that the line integral

$$\oint_C zdx - 2xdy + 3ydz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.
Solution:

$$\nabla \times \mathbf{F} = \langle 3, 1, -2 \rangle.$$

A parametrization of the plane is $\mathbf{r}(u,v) = \langle u, v, 1-u-v \rangle$. Meanwhile, unit normal vector of the plane is $\mathbf{n} = \frac{1}{\sqrt{3}}\langle 1,1,1 \rangle$. From Stoke's Theorem,

$$\begin{aligned}
\oint_C zdx - 2xdy + 3ydz &= \int\int_{D(C)} \langle 3, 1, -2 \rangle \cdot \cdot \mathbf{n} |\mathbf{r}_u \times \mathbf{r}_v| dA \\
&= M \int\int_{D(C)} |\mathbf{r}_u \times \mathbf{r}_v| dA
\end{aligned}$$

where M is certain constant. Notice that the integral multiplied ot M is the area enclosed on the plane. This proved the claim.

7. Evaluate

$$\oint_C (y + \sin x)dx + (z^2 + \cos y)dy + x^3 dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$ for $0 \leq t \leq 2\pi$.
Solution: Notice that $z = 2xy$. Meanwhile,

$$\nabla \times \mathbf{F} = \langle -2z, -3x^2, -1 \rangle.$$

Positive orientation normal vector of the surface: $\mathbf{n} = \langle -z_x, -z_y, 1 \rangle = \langle -2y, -2x, 1 \rangle$. So from Stoke's Theorem, we have

$$\begin{aligned}
\oint_C zdx - 2xdy + 3ydz &= \int_0^1 \int_0^1 \langle -2z, -3x^2, -1 \rangle \cdot \langle -2y, -2x, 1 \rangle dydx \\
&= \int_0^1 \int_0^1 \langle -4xy, -3x^2, -1 \rangle \cdot \langle -2y, -2x, 1 \rangle dydx \\
&= \int_0^1 \int_0^1 (8xy^2 + 6x^3 - 1) dydx \\
&= \int_0^1 \left(4y^2 + \frac{3}{2} - 1 \right) dx \\
&= \frac{4}{3} + \frac{1}{2} = \frac{15}{6}.
\end{aligned}$$