Last Time

Limit : $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

Limit does not exist if you can find 2 diffrent paths such that the limit along those paths are not equal.

Exist : Squeeze Thm / Polar Coordinate

$$0 \le |f(x,y)| \le g(x,y)$$

$$\downarrow$$

$$0$$

$$x^2+y^2=r^2$$

$$\lim_{(x,y)\to(0,0)} \Longrightarrow \lim_{r\to 0^+}$$

$$\lim_{(x,y)\to(3,2)} f(x,y). \quad \rightsquigarrow \quad \lim_{(x,y)\to(0,0)} f(x+3,\, y+2).$$

Ex $$\lim_{(x,y)\to(2,1)} \frac{x-2}{(x-2)^2+(y-1)^2} \iff \lim_{(x,y)\to(0,0)} \frac{x}{x^2+y^2}$$

If $f(x,y)$ is continuous near (a,b), then

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

① polynomials: $x^2+yx+2y^3$

② e^x, $\sin x$, $\cos x$, $|x|$ ✓

③ $\ln x$, $\tan x$, $\sqrt[n]{x}$ in the domain.

④ $+$, $-$, $\times$

⑤ $\frac{f(x,y)}{g(x,y)} \leftarrow \neq 0$ $\quad f, g$ continuous

⑥ $f \circ g$ domain.

f_x , $D_x f$, $\frac{\partial f}{\partial x}$: derivative along x-direction

keep y fixed.

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

again: If f is continuos e.g. ① — ⑥

f_x is just usual differentiation by treating y as a constant.

Clairaut's Theorem (Mixed Partials Theorem)

If $f(x,y)$ is defined on & near (a,b)

and f_{xy} exists and continuous at (a,b)

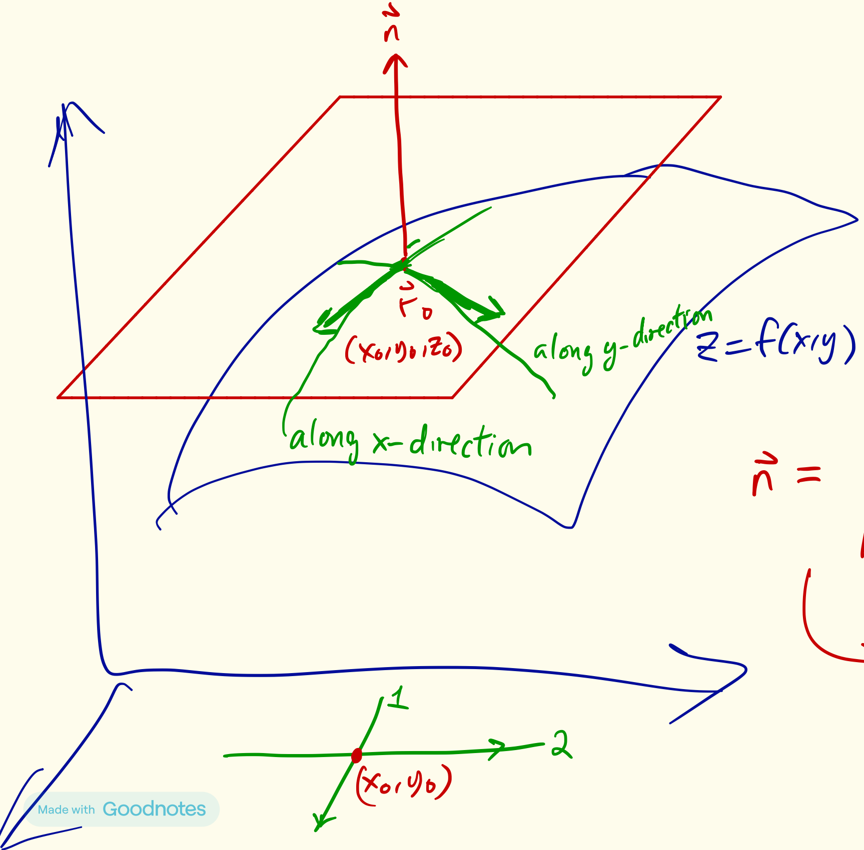
then f_{yx} exists and

$f_{xy}(a,b) = f_{yx}(a,b)$.

$\sqrt{(x-a)^2+(y-b)^2} < \varepsilon$

Tangent Plane of $z=f(x,y)$

→ point + normal vector.



① : $\vec{r}(t) = \langle t, y_0, f(t,y_0) \rangle$

$\vec{r}\,'(t) = \langle 1, 0, f_x \rangle$

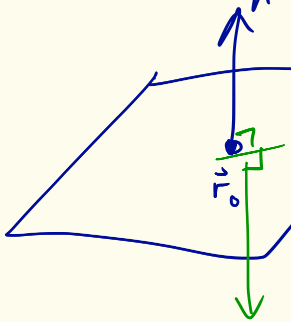
② $\vec{r}(t) = \langle 0, 1, f_y \rangle$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \langle -f_x, -f_y, 1 \rangle$$

→ eq. tangent plane!

$\vec{n} = \langle -f_x, -f_y, 1 \rangle$

upward pointing.



$\vec{r}_0$

downward pointing.

$\langle f_x, f_y, -1 \rangle$

Ex $\quad z = \underbrace{x^2 + 4y^2}_{f(x,y)}$ at $(1,1,5)$ find tangent plane.

$f_x = 2x \quad , \quad f_y = 8y \quad , \quad \vec{n} = \langle -2x, -8y, 1 \rangle$

$= \langle -2, -8, 1 \rangle$

at $(1,1,5)$.

$\vec{r}_0$

$$\boxed{-2x - 8y + z = -5}$$

$$\boxed{2x + 8y - z = 5}$$

Ex $\quad z = \sin xy^2 \quad$ at $(\pi, 1, 0) = \vec{r}_0$

$$\vec{n} = \langle f_x , f_y , -1 \rangle$$

$$= \langle y^2 \cos xy^2 , 2xy \cos xy^2 , -1 \rangle$$

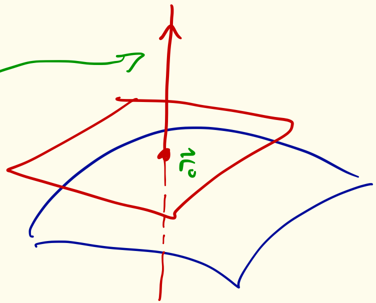
$$= \langle -1 , -2\pi , -1 \rangle \text{ at } \vec{r}_0$$

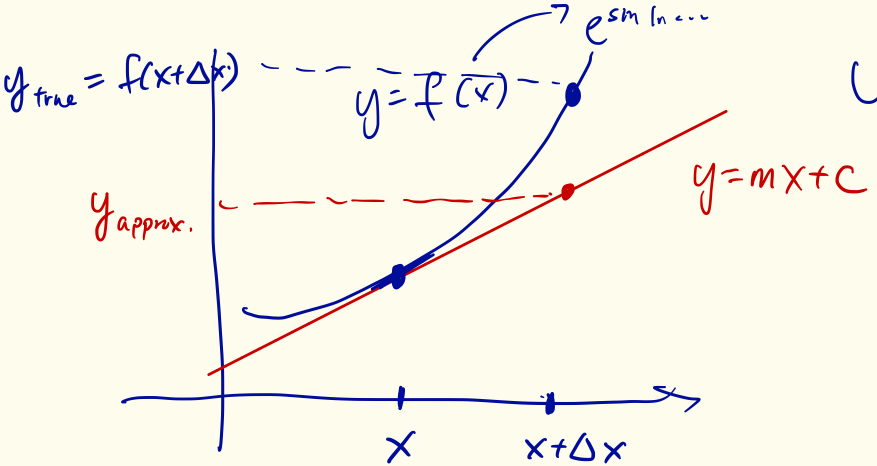
$$-x - 2\pi y - z = -3\pi$$

$$x + 2\pi y + z = 3\pi$$

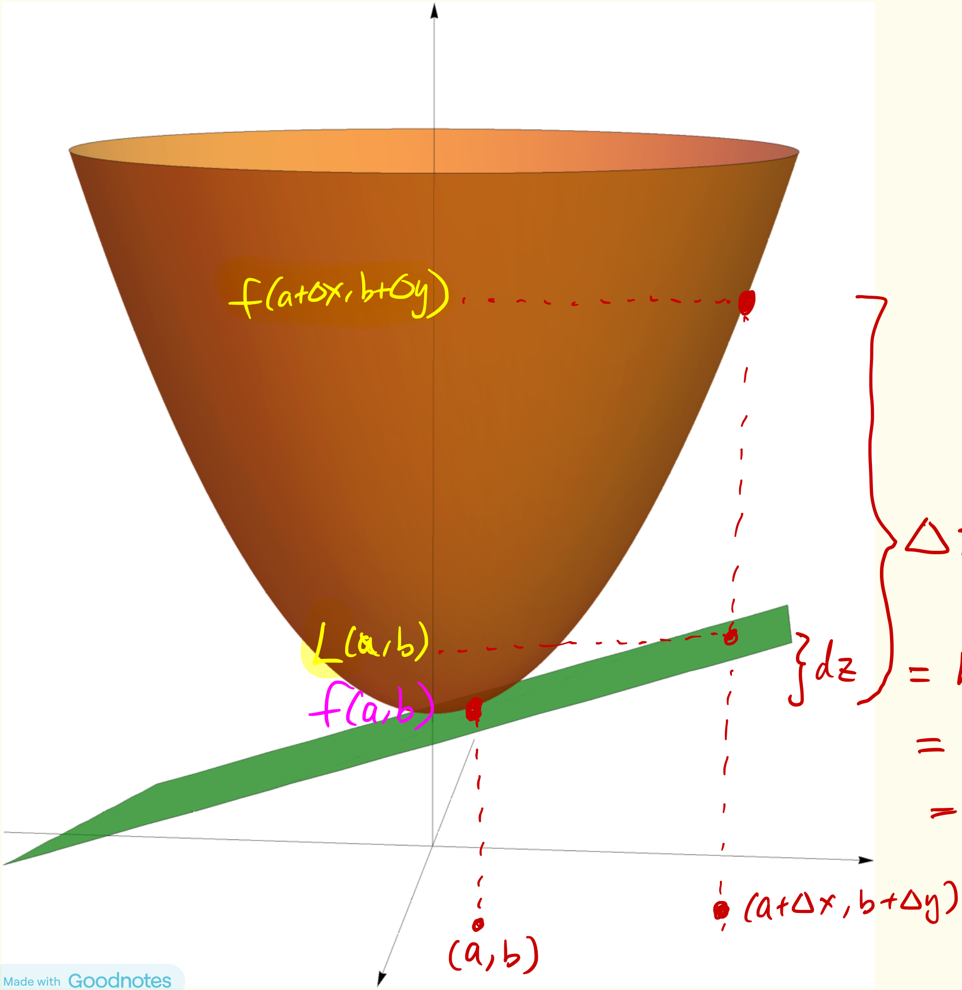
Normal Line of surface :

$$\begin{cases} \vec{r}_0 \\ \vec{v} = \vec{n} . \end{cases}$$





Useful for approximation



tangent plane : $\langle -f_x, -f_y, 1 \rangle$

$$z = f_x(x-a) + f_y(y-b) + f(a,b)$$

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$$

= height of tangent plane

$$= f_x(x-a) + f_y(y-b)$$

$$= f_x \underbrace{dx}_{=\Delta x} + f_y \underbrace{dy}_{=\Delta y}$$

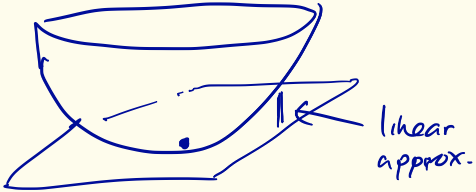
Total Differential:

$$dz = f_x\, dx + f_y\, dy.$$

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

$\boxed{\text{Chain Rule}}$

$$= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$



Def If f_x, f_y exist near (a,b) and continuous at (a,b), then f is differentiable at (a,b).

MOST functions in this course is differentiable

"If f has a good linear approximation" on the domain.

$\Longrightarrow$ f is differentiable.

(CounterEx)

Ex

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

f_x exists : $\displaystyle\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$ at $(0,0)$.

$f_y = 0$ $\Longrightarrow$ tangent plane : $z = 0$.

"f is not differentiable"

Chain Rule

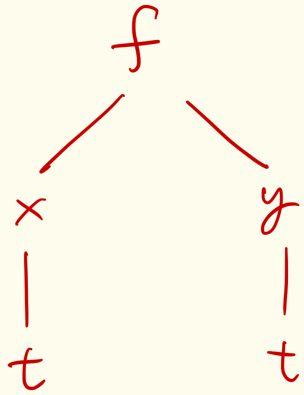
$z = f(x, y)$ differentiable function
"nice"

$x = g(t)$
$y = h(t)$ $\rightsquigarrow$ $z = f(x(t), y(t))$ also a function in t.

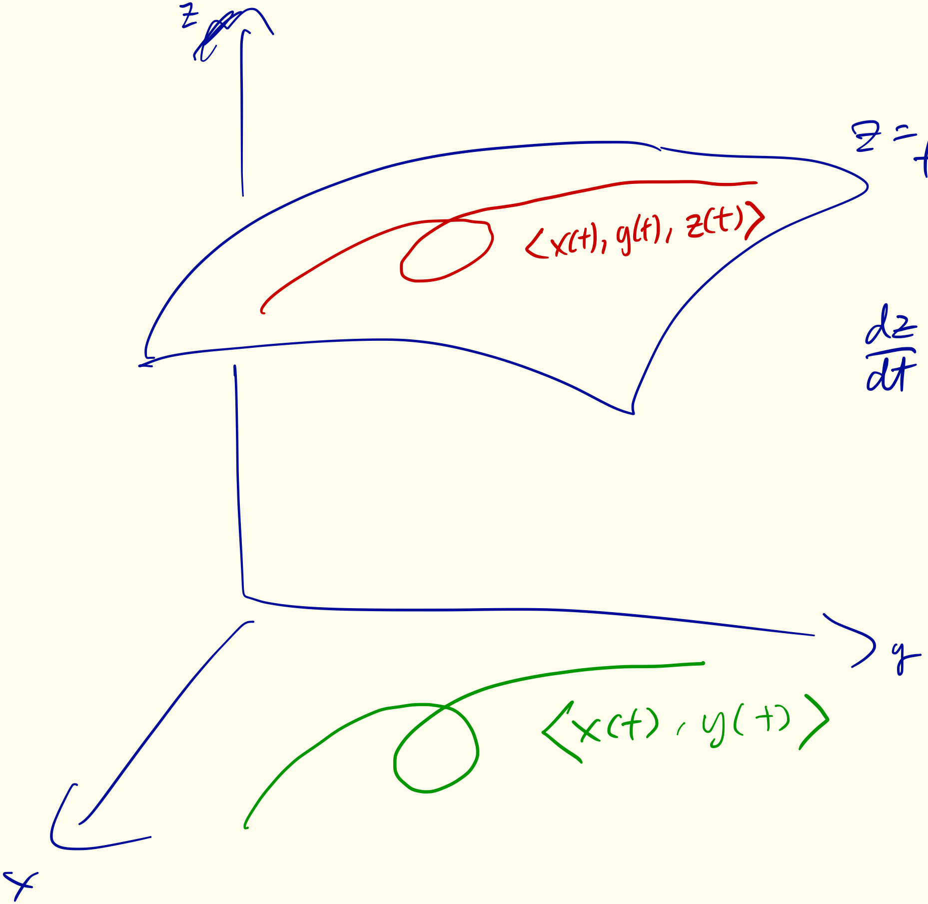
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

↑
add up all the paths on the tree diagram.

Tree Diagram



$$z = f(x, y)$$



$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Ex $z = x^2y + 3x^3y^4$

$x = 2\sin t$
$y = \cos t$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 9x^2y^4)\, 2\cos t + (x^2 + 12x^3y^3)(-\sin t)$$

= in terms of t.

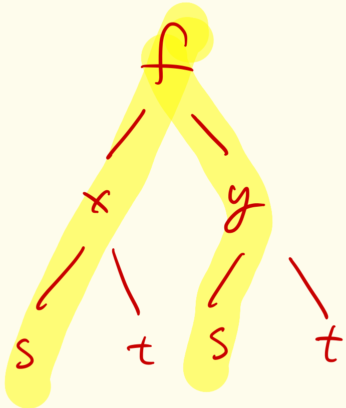
at $t=0$? $\longrightarrow$ $x = 0$, $y = 1$

$= 0$

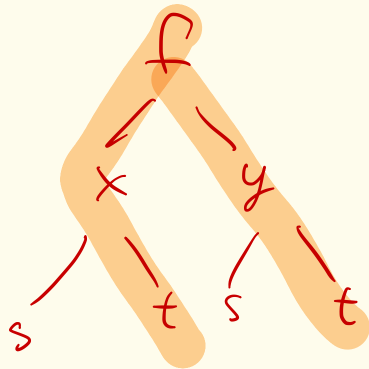
$$z = f(x, y)$$

$$x = g(s, t)$$

$$y = h(s, t)$$



$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$



$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$