

Last Time Method of Lagrange Multiplier :

Solve Constrained Optimization Problem

$$\rightarrow \max / \min : f(x, y, z)$$

$$\rightarrow \text{subject to } g(x, y, z) = k$$

① :  $\begin{cases} \nabla f = \lambda \nabla g \\ g = k \end{cases}$    
 Lagrange Multiplier  $\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

4 equations in 4 unknowns ( $x, y, z, \lambda$ )

② If  $g$  defines a bounded closed domain

By Extreme Value Theorem  $\Rightarrow$  absolute max/min exist.

Compare the values of  $f(x, y, z)$  at the solutions.

Ex Maximize  $x + 2y + 3z$  subject to  $x^2 + y^2 + z^2 = 1$ .

$$\textcircled{1} : \begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases} \quad \begin{aligned} f(x, y, z) &= x + 2y + 3z \\ g(x, y, z) &= x^2 + y^2 + z^2 \end{aligned}$$

$$\nabla f = \langle 1, 2, 3 \rangle$$

$$\nabla g = \langle 2x, 2y, 2z \rangle$$

$$\begin{cases} 1 = 2\lambda x \\ 2 = 2\lambda y \\ 3 = 2\lambda z \end{cases} \Rightarrow \frac{1}{x} = \frac{2}{y} = \frac{3}{z} = 2\lambda$$

or  $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

$$\begin{aligned} y &= 2x \\ z &= 3x \end{aligned}$$

$$g = 1 \Leftrightarrow x^2 + 4x^2 + 9x^2 = 1 \Rightarrow x^2 = \frac{1}{14} \quad x = \pm \frac{1}{\sqrt{14}}$$

$$y = \pm \frac{2}{\sqrt{14}}, z = \pm \frac{3}{\sqrt{14}}$$

$$f(x, y, z) = \pm \frac{1}{\sqrt{14}} \pm \frac{4}{\sqrt{14}} \pm \frac{9}{\sqrt{14}} = \pm \sqrt{14}.$$

$\Rightarrow \left( \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)$  gives maximum.

Really is just asking which <sup>unit</sup> direction  $\vec{u} = \langle x, y, z \rangle$

such that  $\nabla f \cdot \vec{u}$  is maximized?

$$\downarrow$$
$$\langle 1, 2, 3 \rangle$$

Ex     $f(x,y) = 2x + 3y$     subject to     $\sqrt{x} + \sqrt{y} = 5$

$$\nabla g(x,y)$$

$$\nabla f = \langle 2, 3 \rangle$$

$$\nabla g = \left\langle \frac{1}{2\sqrt{x}}, \frac{1}{2\sqrt{y}} \right\rangle$$

$$\nabla f = \lambda \nabla g \Leftrightarrow \begin{cases} 2 = \frac{\lambda}{2\sqrt{x}} \\ 3 = \frac{\lambda}{2\sqrt{y}} \end{cases} \Rightarrow 4\sqrt{x} = \lambda \Rightarrow 4\sqrt{x} = 6\sqrt{y} \Rightarrow 16x = 36y$$

$$g=5 : \sqrt{x} + \sqrt{\frac{16x}{36}} = 5$$

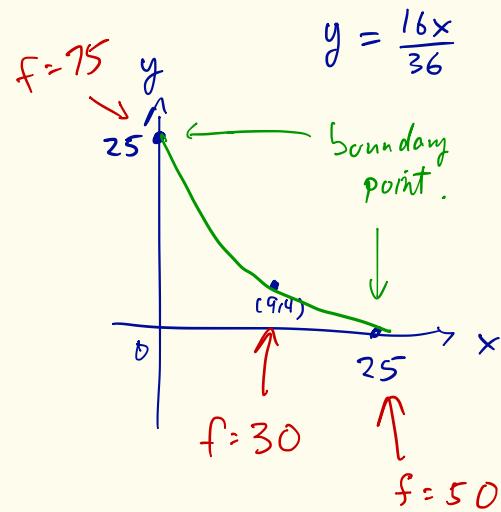
$$\sqrt{x} + \frac{2}{3}\sqrt{x} = 5$$

$$\frac{5}{3}\sqrt{x} = 5$$

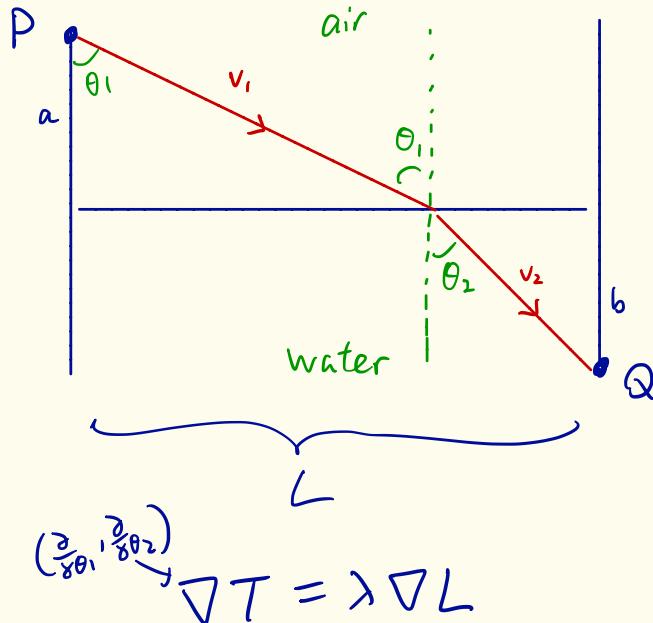
$$\sqrt{x} = 3$$

$$x = 9 \Rightarrow y = 4.$$

$(x,y) = (9,4)$ . is minimum.



## Snell's Law



minimize the time needed to travel from P to Q

$$T = \frac{\text{distance}}{\text{speed}} = \frac{a}{v_1 \cos \theta_1} + \frac{b}{v_2 \cos \theta_2} \quad \leftarrow f$$

$$L = a \tan \theta_1 + b \tan \theta_2 \quad \leftarrow g$$

$$\frac{\partial}{\partial \theta} \frac{1}{\cos \theta} = \frac{\sin \theta}{\cos^2 \theta}$$

$$\left\{ \begin{array}{l} \frac{\partial}{\partial \theta_1} : \frac{a}{v_1} \frac{\sin \theta_1}{\cos \theta_1} = \lambda \frac{a}{\cos \theta_1} \\ \frac{\partial}{\partial \theta_2} : \frac{b}{v_2} \frac{\sin \theta_2}{\cos \theta_2} = \lambda \frac{b}{\cos \theta_2} \end{array} \right.$$

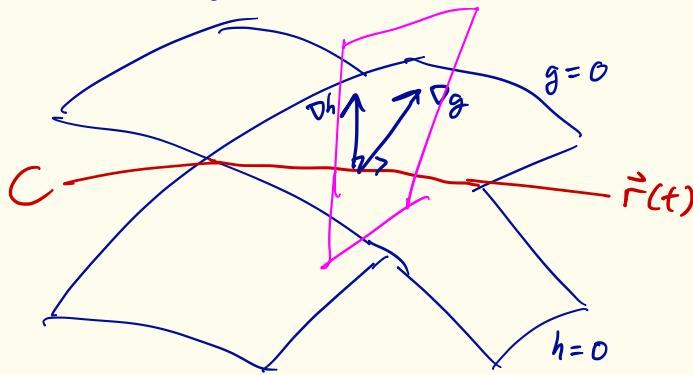
$\iff \lambda = \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}$

Snell's Law

## 2 Constraints

Max.  $f(x, y, z)$  subject to

$$\begin{cases} g(x, y, z) = 0 \\ h(x, y, z) = 0 \end{cases}$$



$\vec{r}(t)$  lies on both  $g=0$  &  $h=0$

$$\Leftrightarrow \vec{r}'(t) \perp \nabla g \quad \text{and} \quad \vec{r}'(t) \perp \nabla h$$

$$\max f(\vec{r}(t)) \Rightarrow \nabla f \cdot \vec{r}'(t) = 0 \Leftrightarrow \nabla f \perp \vec{r}'(t)$$

$\Leftrightarrow \nabla f$  lies in the plane spanned by  $\nabla g$  and  $\nabla h$ .

$$\Leftrightarrow \nabla f = \lambda \nabla g + \mu \nabla h$$

Lagrange Multipliers.

## 2 Constraints

Max.  $f(x, y, z)$  subject to

$$\begin{cases} g(x, y, z) = 0 \\ h(x, y, z) = 0 \end{cases}$$

$$\textcircled{1} \quad \begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ g = 0 \\ h = 0 \end{cases}$$

5 equations in 5 unknowns  $(x, y, z, \lambda, \mu)$

# Lagrange Multipliers Method in General

max.  $f(x_1, x_2, \dots, x_n)$

subject to  $\begin{cases} g_1(x_1, \dots, x_n) = 0 \\ g_2(x_1, \dots, x_n) = 0 \\ \vdots \\ g_k(x_1, \dots, x_n) = 0 \end{cases}$

$$\nabla = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$$

$$\begin{cases} \nabla f = \lambda_1 \nabla g_1 + \dots + \lambda_k \nabla g_k \\ g_1 = 0 \\ \vdots \\ \lambda_k = 0 \end{cases}$$

$n+k$  equations in  $n+k$  unknowns,

$$x_1, \dots, x_n, \lambda_1, \dots, \lambda_k.$$

## Meaning of Lagrange Multiplier ?

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \Leftrightarrow \nabla L = 0 \quad \text{critical pts.}$$

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y).$$

Budget - Constraint :

$$\text{max. Revenue } R(x, y) \quad \begin{matrix} x - \text{apple} \\ y - \text{banana.} \end{matrix}$$

$$\text{subject to } g(x, y) = C$$

Using Lagrange multiplier  $\Rightarrow \text{max} \Leftrightarrow (x^*, y^*, \lambda^*)$

$$\text{Let } M^* = R(x^*, y^*)$$

Treat  $c$  as variable.

If we change  $c$ , max will change!  
 $M^*(c)$

Fact  $\frac{dM^*(c)}{dc} = \lambda^*(c)$

In Economics :  $\lambda^*$  = "shadow price"  
= Change in revenue by loosening  
the constraint by \$1.

PF  $M^*(c) = \hat{\mathcal{L}}(x^*(c), y^*(c), \lambda^*(c), c)$   $(\frac{\partial \hat{\mathcal{L}}}{\partial c} = \lambda)$   
 $\hat{\mathcal{L}} = f(x, y) - \lambda(g(x, y) - c)$ .

$$\frac{dM^*(c)}{dc} = \frac{\partial \hat{\mathcal{L}}}{\partial x} \frac{dx}{dc} + \frac{\partial \hat{\mathcal{L}}}{\partial y} \frac{dy}{dc} + \frac{\partial \hat{\mathcal{L}}}{\partial \lambda} \frac{d\lambda}{dc} + \frac{\partial \hat{\mathcal{L}}}{\partial c} \Rightarrow \lambda^*(c).$$

$\circ \quad \circ \quad \circ \quad \text{by } \nabla \mathcal{L} = 0$

## Part II Integrations

$$\int_a^b f(x) dx \Rightarrow \boxed{1}$$

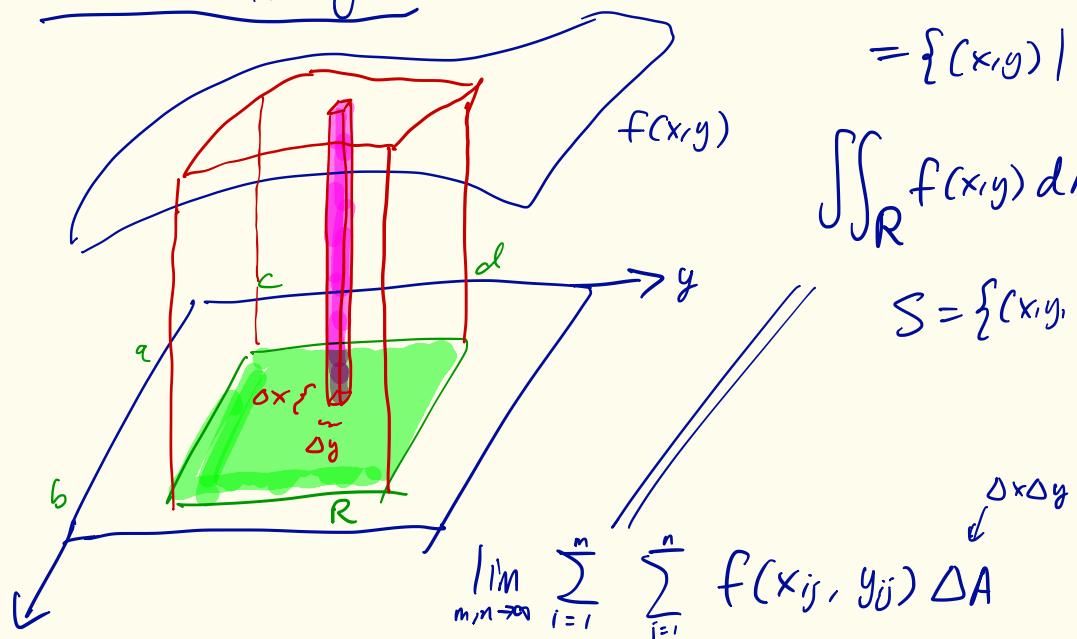
Double Integral defined on

$$R = [a, b] \times [c, d]$$

$$= \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

$$\iint_R f(x, y) dA = \text{Volume of } S$$

$$S = \{(x, y, z) \mid 0 \leq z \leq f(x, y) \mid (x, y) \in R\}$$



$$\int_c^d \int_a^b f(x, y) dx dy$$

## Fubini's Theorem (3033)

If  $f$  is continuous on  $a \leq x \leq b, c \leq y \leq d$ , then

$$\iint_R f \, dA = \int_c^d \int_a^b f \, dx \, dy = \int_a^b \int_c^d f \, dy \, dx.$$

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Properties Linearity:  $\iint_R (f + g) \, dA = \iint_R f \, dA + \iint_R g \, dA$

$$\iint_R c f \, dA = c \iint_R f \, dA \quad c \in \mathbb{R}$$

If  $f \geq g$ , then  $\iint_R f \, dA \geq \iint_R g \, dA$ .

$$\underline{\text{Ex}} \quad R = [0, 3] \times [0, 4]$$

$$\iint_R 4 \, dA = \begin{array}{c} \text{Diagram of a cube with edges 3 and 4, volume } 3 \times 4 \times 4 = 48. \\ \text{Volume of cube} \end{array}$$

$$\iint_R (4-y) \, dA = \begin{array}{c} \text{Diagram of a triangular prism with base } 4 \times 3, \text{ height } 4, \text{ volume } \frac{4 \times 4}{2} \times 3 = 24. \\ \text{Volume of triangular prism} \end{array}$$

$$\iint_R (2+x) \, dA = \begin{array}{c} \text{Diagram of a triangular prism with base } 4 \times 3, \text{ height } 5, \text{ volume } \frac{7 \times 3}{2} \times 4 = 42. \\ \text{Volume of triangular prism} \end{array}$$

$$\underline{\text{Ex}} \quad \iint_R (x^2 - 4y) dA \quad R = [1, 3] \times [2, 4].$$

$$\begin{aligned} \int_2^4 \left( \int_1^3 (x^2 - 4y) dx \right) dy &= \int_2^4 \left( \frac{x^3}{3} - 4yx \right) \Big|_1^3 dy \\ &\quad \text{y is constant here.} \\ &= \int_2^4 \left( 9 - 12y - \frac{1}{3} + 4y \right) dy \\ &= \int_2^4 \left( \frac{26}{3} - 8y \right) dy. \end{aligned}$$

$$\begin{aligned} \int_1^3 \left( \int_2^4 (x^2 - 4y) dy \right) dx &= \int_1^3 \left( x^2 y - 2y^2 \right) \Big|_2^4 dx \\ &= \int_1^3 (4x^2 - 32 - 2x^2 + 8) dx = \int_1^3 2x^2 - 24 dx \\ &= \left. \frac{2x^3}{3} - 24x \right|_1^3 = -\frac{92}{3} \dots \end{aligned}$$

same.

$$\underline{\text{Ex}} \quad \iint_R y \sin(xy) dA \quad \text{for } R = [1, 2] \times [0, \pi]$$

Choose easiest order to integrate!

$\int dy$  hard

$\int dx$  easy!

$$\int_0^\pi \int_1^2 y \sin(xy) dx dy$$

$$\int_1^2 \int_0^\pi y \sin xy dy dx$$

$$= \int_0^\pi y \cdot \left. \frac{-\cos xy}{y} \right|_1^2 dy$$

↑  
Integration by parts!

$$= \int_0^\pi (\cos y - \cos 2y) dy$$

Harder!

$$= \left. \sin y - \frac{\sin 2y}{2} \right|_0^\pi = 0.$$