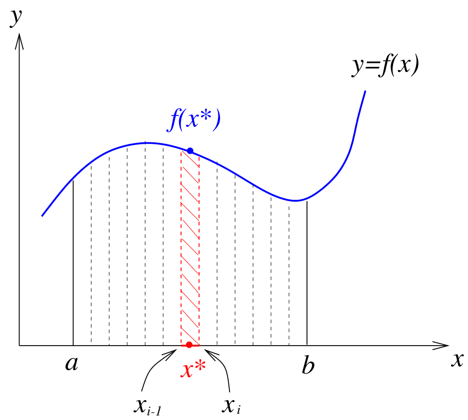
Chapter 14
Multiple integrations

Contents

Review

Single integral: Recall that if a function of one variable is *defined* and *bounded*, then the definite integral of the function is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\triangle x_i, \qquad \text{(Riemann sum)}$$

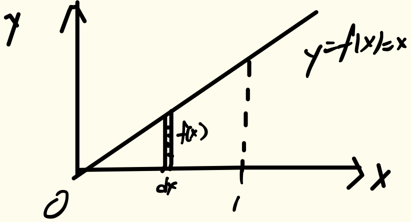


where $[a,b]$ is divided into intervals by points $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, $x_i^* \in [x_{i-1}, x_i]$, and $\triangle x_i = x_i - x_{i-1}$.

Note: as $n \to \infty$, $\triangle x_i \to 0$ ($\because$ $\triangle x_i = (b-a)/n$)

$$\int_a^b f(x)\,dx = A, \quad \text{the (signed) area of } f(x) \text{ on } [a,b].$$

$$A = \int_0^1 f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$$



$$y = x$$

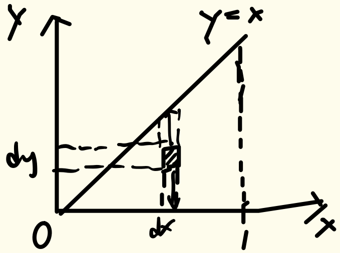
$$\Delta A = f(x)\cdot dx$$

$$A = \int_0^1 f(x)\,dx$$

$$\int_0^1 \int_0^{x} dy\,dx$$

x ← highest pt

0 ← lowest pt



moving y direction first, and you get a vertical strip and move along x

$$= \int_0^1 y\Big|_0^x dx$$

$$= \int_0^1 (x-0)\,dx = \int_0^1 x\,dx = \frac{1}{2}$$

instead of starting with vertical strip, starting with blocks

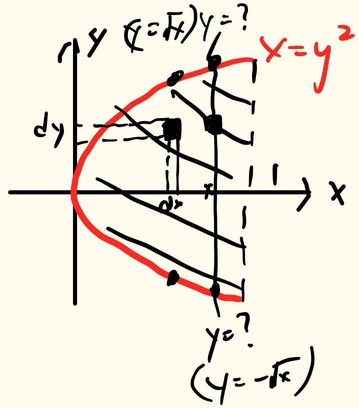
$$dA = dx\cdot dy$$
$$= dy\cdot dx$$

$$\int_0^1 \int_y^1 dx\,dy$$

$$= \int_0^1 x\Big|_y^1 dy = \int_0^1 (1-y)\,dy = y - \frac{y^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\int_{y=0}^{y=1}\int_{y=-\sqrt{x}}^{y=\sqrt{x}} dy\,dx \qquad \text{先}y\text{先}x$$

$$\int_{y=-1}^{y=1}\int_{x=y^2}^{x=1} dx\,dy \qquad \text{先}x\text{先}y$$



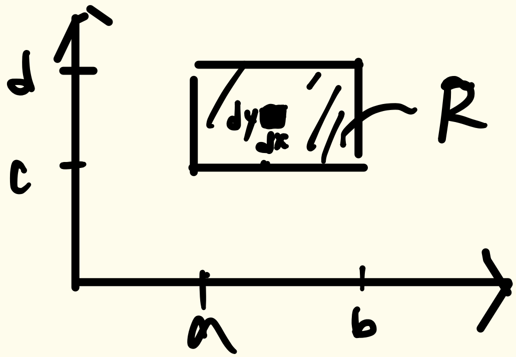
$$\int_0^1 f(x)dx = \int_0^1\int_0^{f(x)} 1\,dy\,dx$$

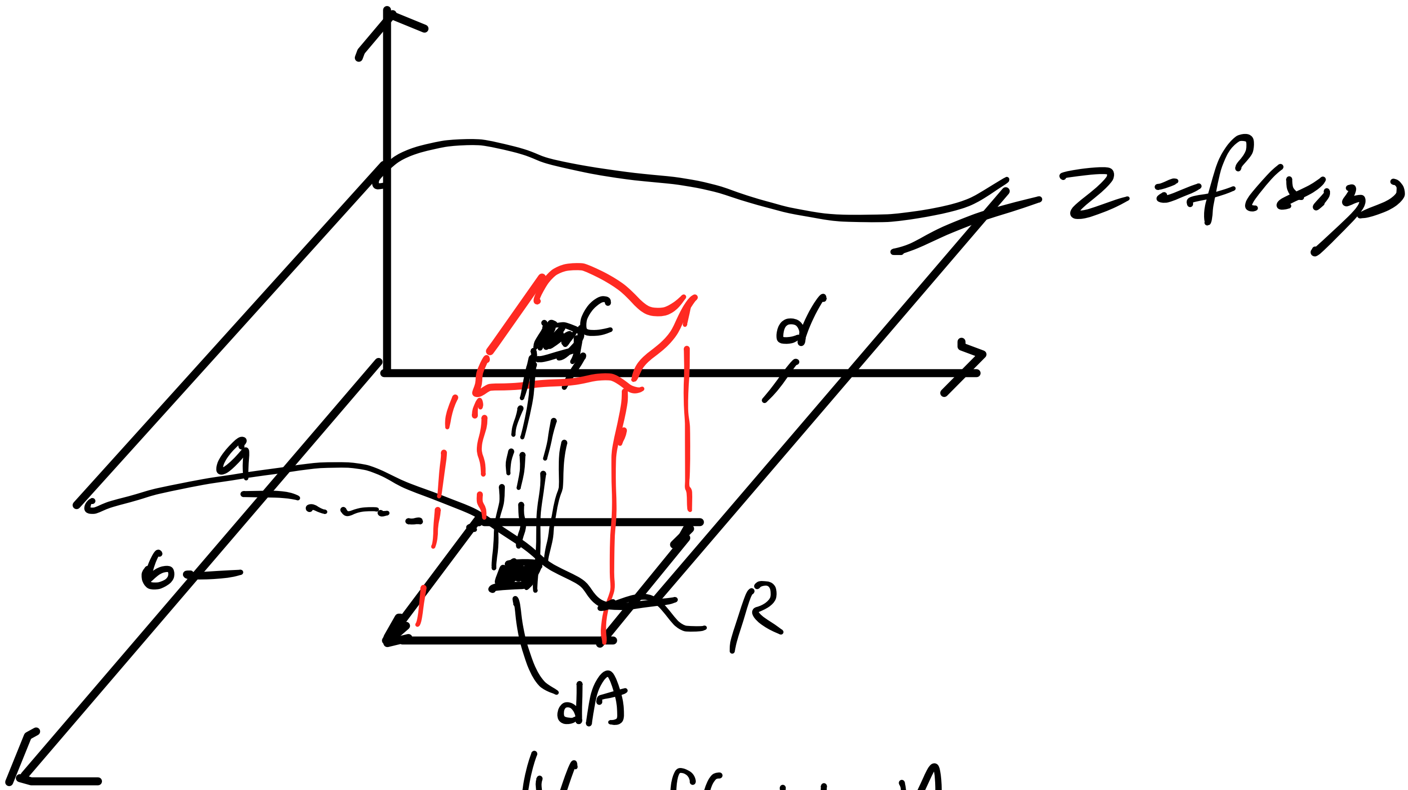
height width

$dA = dx\,dy$

or $= dy\,dx$

$$\iint_R f(x,y)\,dy\,dx = \iint_R \underset{f(x,y)}{\hat{}}\, dx\,dy = \iint_R \underset{f(x,y)}{\hat{}}\, dA$$



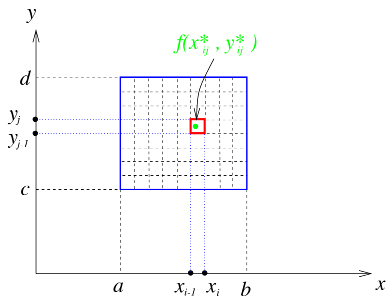


$$dV = f(x,y) \cdot dA$$

$$V = \iint_R f(x,y)\, dA$$

14.1 Double integrals over rectangles

Suppose a function $f(x,y)$ is *defined* and *bounded* in some region $R = [a,b] \times [c,d]$ of the xy-plane that has a finite area. To define the double integral of $f(x,y)$ over R, we first divide R into $m \times n$ rectangles of area $\triangle A_{ij}$ ($i = 1, m$ and $j = 1, n$) and $(x_{ij}^*, y_{ij}^*) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. The product



$$\Delta V_{ij} = f(x_{ij}^*, y_{ij}^*) \cdot \triangle A_{ij}$$

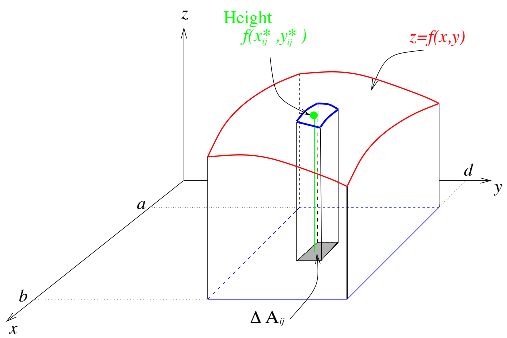
gives a (signed) volume of a rectangular box as indicated. If the limit

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \triangle A_{ij} \quad \text{exists as}$$

$m, n \to \infty$ and $\triangle A_{ij} \to 0$. Then the double integral of f over the rectangle R is

$$V = \iint_R f(x,y)\, dA.$$

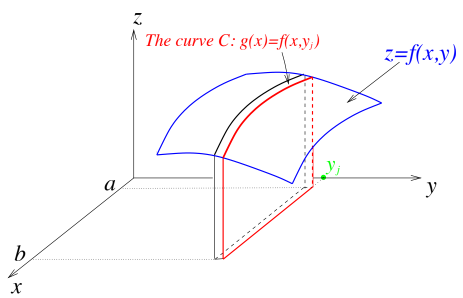
height



To evaluation $\displaystyle\iint_R f(x,y)\, dA$ by considering the cross-section at $y = y_j$:

For the case $R = [a,b] \times [c,d]$. Note that for $y_j \in [c,d]$, we can consider the *partial integral with respect to x* (holding y fixed), i.e.

$$\int_a^b f(x, y_j)\, dx = \int_a^b g(x)\, dx = A(y_j)$$



is the area under the curve C whose equation $z = g(x) = f(x, y_j)$, where y_j is held constant and $x \in [a,b]$. So the volume

$$V = \int_c^d A(y)\, dy = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

Similarly, $V = \displaystyle\int_a^b \int_c^d f(x,y)\, dy\, dx$ by considering the cross-section at $x = x_i$.

Note: If we view R as a flat object (a flat disk) whose density is not uniform throughout. The density at a point (x,y) can be specified by a function $f(x,y)$ (called the density function). Construct a small rectangle centered at (x,y) and let ΔM and ΔA be the mass and area of the portion of the disk enclosed by this rectangle. Then $\Delta M = f(x,y) \times \Delta A$ (Mass = density × area).

$$\therefore \text{ Total mass of the disk } M = \iint_R f(x,y)\, dA.$$

Properties of multiple integrals

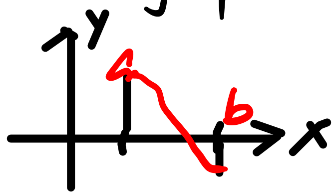
$$\iint\limits_R f(x,y)\,dA = 0 \quad \text{if } R \text{ has a zero area}$$

$$\iint\limits_R f(x,y)\,dA \geqslant 0 \quad \text{if} \quad f(x,y) \geqslant 0$$

above x,y plane

$$\iint\limits_R f(x,y)\,dA \leqslant 0 \quad \text{if} \quad f(x,y) \leqslant 0$$

below x,y plane

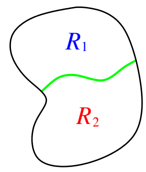


$$\left|\iint\limits_R f(x,y)\,dA\right| \leqslant \iint\limits_R |f(x,y)|\,dA$$

$$\iint\limits_R cf(x,y)\,dA = c\iint\limits_R f(x,y)\,dA$$

$$\iint\limits_R [f(x,y) \pm g(x,y)]\,dA = \iint\limits_R f\,dA \pm \iint\limits_R g\,dA$$

$$\iint\limits_R f\,dA = \iint\limits_{R_1} f\,dA + \iint\limits_{R_2} f\,dA, \quad \text{where} \quad R = R_1 + R_2$$



Ex. 1.1

$$\int_0^2\int_0^1 xe^{x+y}\,dx\,dy = \int_0^2 e^y\left[\int_0^1 xe^x\,dx\right]dy$$

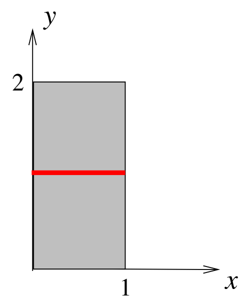
f(x,y)

$$= \int_0^2 e^y[xe^x - e^x]\Big|_0^1\,dy$$

$$= \int_0^2 e^y(1)\,dy$$

$$= e^2 - 1.$$

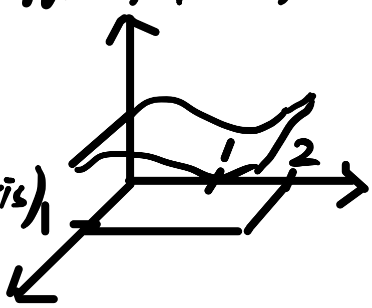
$R = \begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 2 \end{cases}$

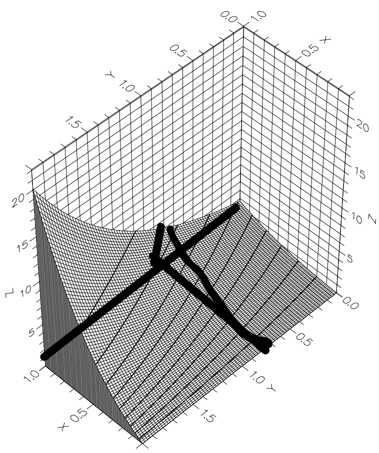


(curves on xy plane)

x: from x=0 to x=1

y: from y=0 to y=1 (points on the y-axis)





Even though it is not always necessary to sketch solids to find their volumes, you are encouraged to sketch them whenever possible. When we encounter triple integrals over three-dimensional regions later in this chapter it will usually be necessary to sketch the regions. Get as much practice as you can.

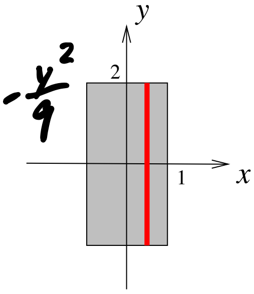
Ex. 1.2 Find the volume of the solid lying under the elliptic paraboloid $x^2/4 + y^2/9 + z = 1$ and above the square $R = [-1, 1] \times [-2, 2]$.

x y

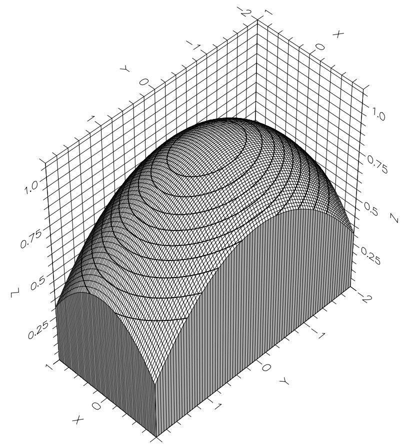
$$\frac{x^2}{4} + \frac{y^2}{9} + z = 1$$

$$z = 1 - \frac{x^2}{4} - \frac{y^2}{9}$$

Let $z = f(x, y) = 1 - \dfrac{x^2}{4} - \dfrac{y^2}{9}$



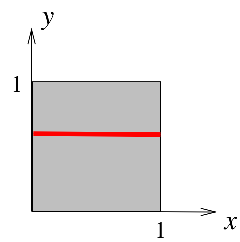
$$\begin{aligned} V = \int_{-1}^{1}\int_{-2}^{2} f(x,y)\,dy\,dx &= \int_{-1}^{1}\left[\int_{-2}^{2}\left(1 - \frac{x^2}{4} - \frac{y^2}{9}\right)dy\right]dx \\ &= \int_{-1}^{1}\left[y - \frac{x^2}{4}y - \frac{y^3}{27}\right]\Bigg|_{-2}^{2} dx \\ &= \int_{-1}^{1}\left[\frac{92}{27} - x^2\right]dx \\ &= \left[\frac{92}{27}x - \frac{x^3}{3}\right]\Bigg|_{-1}^{1} \\ &= \frac{166}{27}. \end{aligned}$$



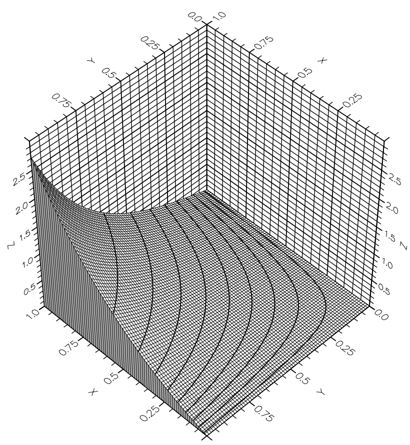
or

$$V = \int\int f(x,y)\,dx\,dy$$

Ex. 1.3 Find $\displaystyle\iint_R xye^{xy^2}\,dA$, where $R = [0, 1] \times [0, 1]$.



$$\begin{aligned} \int_0^1\int_0^1 xye^{xy^2}\,dx\,dy &= \int_0^1 y\left(\int_0^1 xe^{xy^2}\,dx\right)dy \\ &= \int_0^1 y\left[\frac{1}{y^2}e^{xy^2}\cdot x\Big|_0^1 - \int_0^1 \frac{1}{y^2}e^{xy^2}\,dx\right]dy \\ &= \int_0^1 y\left[\frac{1}{y^2}e^{y^2} - \frac{1}{y^2}\frac{1}{y^2}e^{xy^2}\Big|_0^1\right]dy \\ &= \int_0^1 \left[\frac{1}{y}e^{y^2} - \frac{1}{y^3}e^{y^2} - \frac{1}{y^3}\right]dy. \end{aligned}$$



Alternatively,

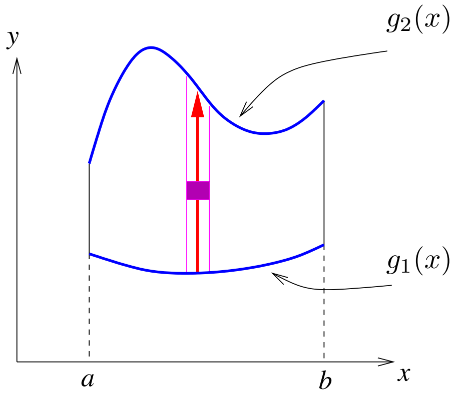
$$\int_0^1\int_0^1 xye^{xy^2}\,dy\,dx = \int_0^1 \left[\frac{1}{2}e^{xy^2}\right]\Bigg|_0^1 dx$$

$$= \frac{1}{2}\int_0^1 (e^x - 1)\,dx = \frac{1}{2}e - 1$$

14.2 Double integrals over general regions

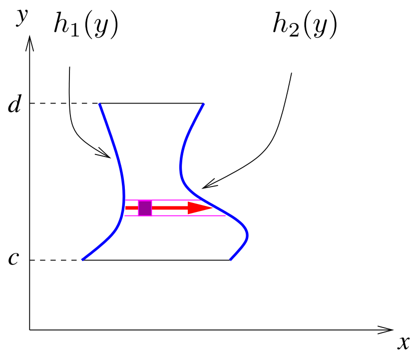
A plane region R is said to be of **Type I** if it lies between the graphs of two continuous function of x, that is if $R = \{(x,y)\,|\,a \leqslant x \leqslant b,\;\; g_1(x) \leqslant y \leqslant g_2(x)\}$, then

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$



Type II: If $R = \{(x,y)\,|\,c \leqslant y \leqslant d,\;\; h_1(y) \leqslant x \leqslant h_2(y)\}$, then

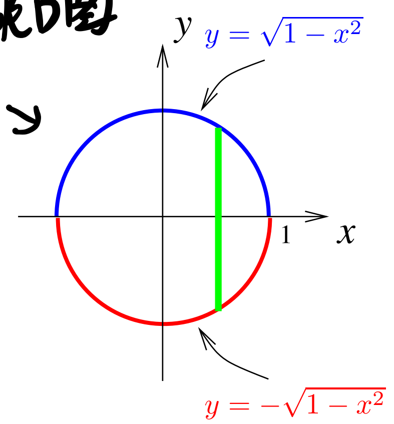
$$\iint\limits_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

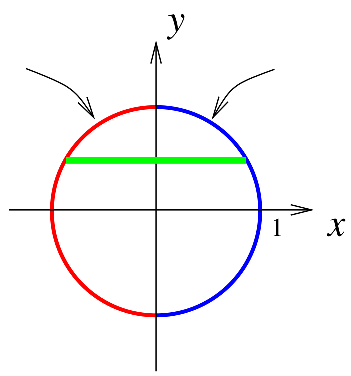


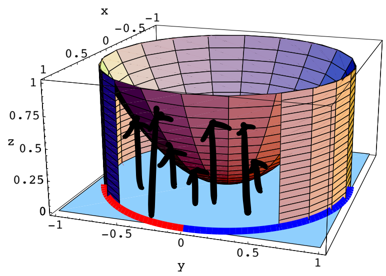
Ex. 2.1 Find $\displaystyle\iint\limits_R (x^2+y^2)\,dA$, where R is the region enclosed by the circle $x^2+y^2=1$.

$$\begin{aligned}
\iint\limits_R (x^2+y^2)\,dA &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2+y^2)\,dydx && \text{(Type I)}\\
&= \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2+y^2)\,dxdy && \text{(Type II)}\\
&= \frac{\pi}{2}.
\end{aligned}$$

要写成图







碗出面的 volume

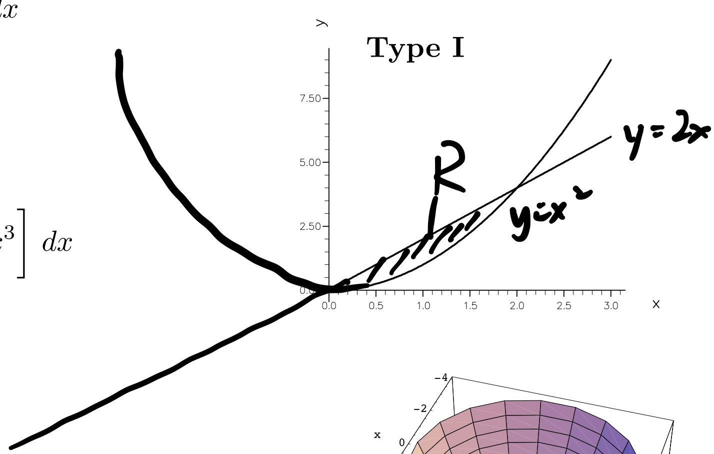
Ex. 2.2 Find $\displaystyle\iint_R (x^2+y^2)\,dA$, where R is the region bounded by the line $y=2x$ and the parabola $y=x^2$.

$$\iint_R (x^2+y^2)\,dA = \int_0^2 \int_{x^2}^{2x} (x^2+y^2)\,dy\,dx$$

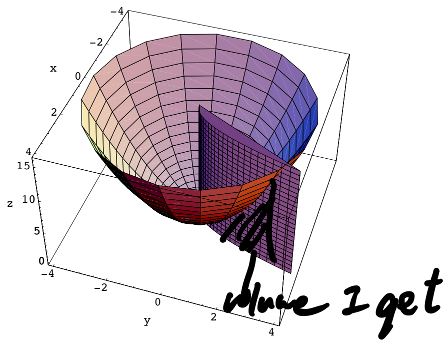
$$= \int_0^2 \left[x^2y+\frac{y^3}{3}\right]\Bigg|_{x^2}^{2x} dx$$

$$= \int_0^2 \left[-\frac{x^6}{3}-x^4+\frac{14}{3}x^3\right] dx$$

$$= \frac{216}{35}.$$



$$= \int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (x^2+y^2)\,dx\,dy$$

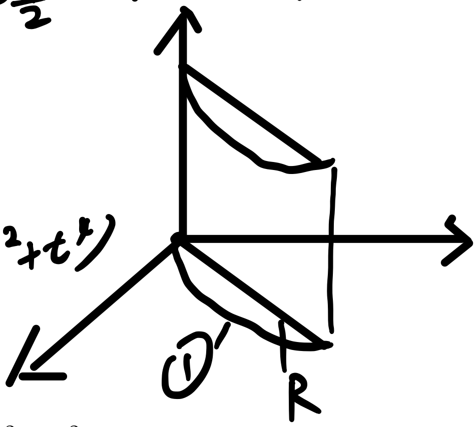


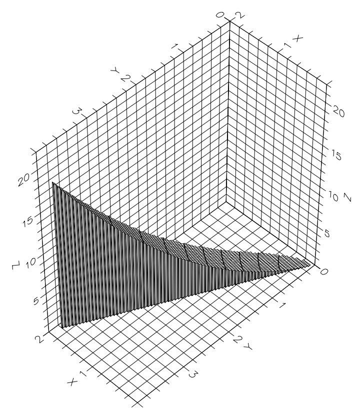
Parametric curve for ①,

①: $y=x^2$

$x=t,\ y=t^2,\ z=t^2+t^4$

$r_1(t)=(t,t^2,t^2+t^4)$





Alternatively,

$$\iint_R (x^2+y^2)\,dA = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2+y^2)\,dx\,dy$$

$$= \int_0^4 \left[\frac{x^3}{3}+y^2x\right]\Bigg|_{y/2}^{\sqrt{y}} dy$$

$$= \int_0^4 \left(\frac{y^{\frac{3}{2}}}{3}+y^{\frac{5}{2}}-\frac{y^3}{24}-\frac{y^3}{2}\right) dy$$

$$= \frac{216}{35}.$$

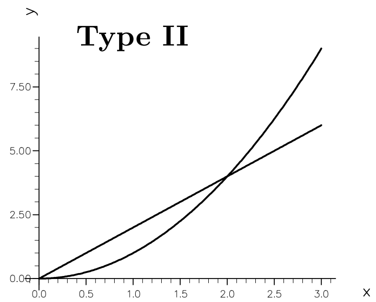
② $y=2x$

$x=t,\ y=2t,$

$z=(t)^2+(2t)^2$

$=5t^2$

$r_2(t)=(t,2t,5t^2)$



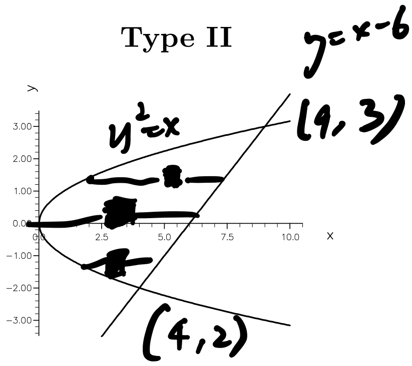
Ex. 2.3 Find $\displaystyle\iint_R 4y^3\,dA$, where R is bounded by $y = x - 6$, $y^2 = x$.

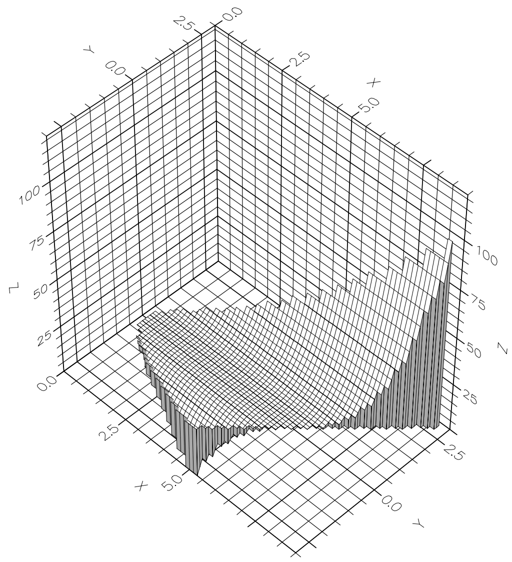
$$\iint_R 4y^3\,dA = \int_{-2}^{3}\int_{y^2}^{y+6} 4y^3\,dx\,dy$$

$$= \int_{-2}^{3} (4y^4 + 24y^3 - 4y^5)\,dy$$

$$= \frac{500}{3}.$$

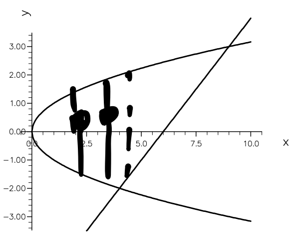
Type II

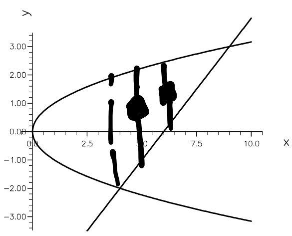




If we had expressed R as type I region, then we would have obtained

$$\iint_R 4y^3\,dA = \int_0^4\int_{-\sqrt{x}}^{\sqrt{x}} 4y^3\,dy\,dx + \int_4^9\int_{x-6}^{\sqrt{x}} 4y^3\,dy\,dx.$$

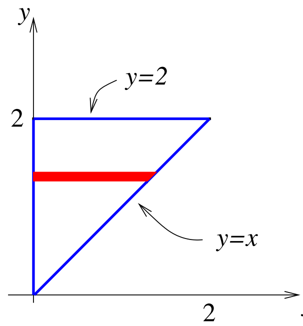




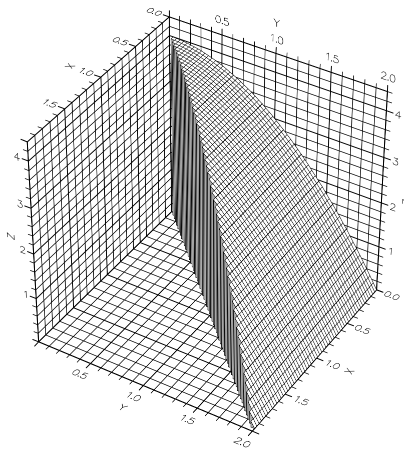
Ex. 2.4 Find the volume of the solid in the first octant that is bounded by the xy-plane, the yz-plane, the plane $y = x$ and the surface $z = 4 - y^2$.

Note that when $z = 0$, $y = \pm 2$.

$$\therefore \; V = \iint_R (4 - y^2)\, dA = \int_0^2 \int_0^y (4 - y^2)\, dx\, dy$$
$$= \int_0^2 (4 - y^2) x \Big|_0^y \, dy$$
$$= \int_0^2 (4 - y^2) y \, dy = 4$$



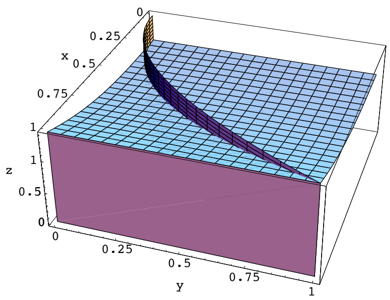
Alternatively, $V = \int_0^2 \int_x^2 (4 - y^2)\, dy\, dx$

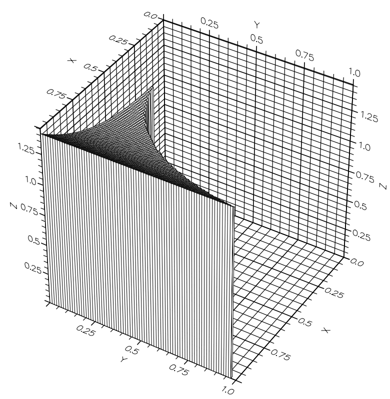


Ex. 2.5 Evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 (x^3 + 1)^{1/2}\, dx dy$$

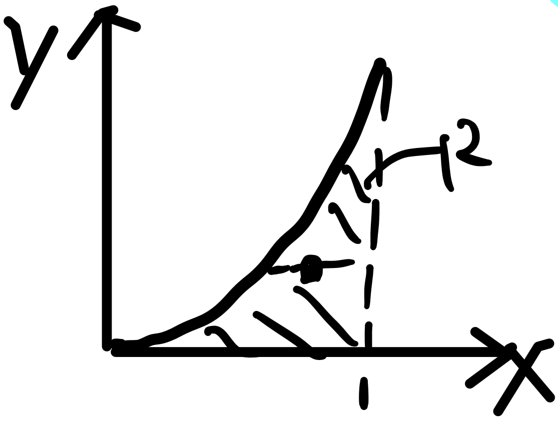
by reversing the order of integration.





Q.5 For x: $x=\sqrt{y}$ to $x=1$ (curves) on xy-plane

For y: $y=0$ to $y=1$ (points) on y-axis



$x=\sqrt{y}$

$y=x^2$

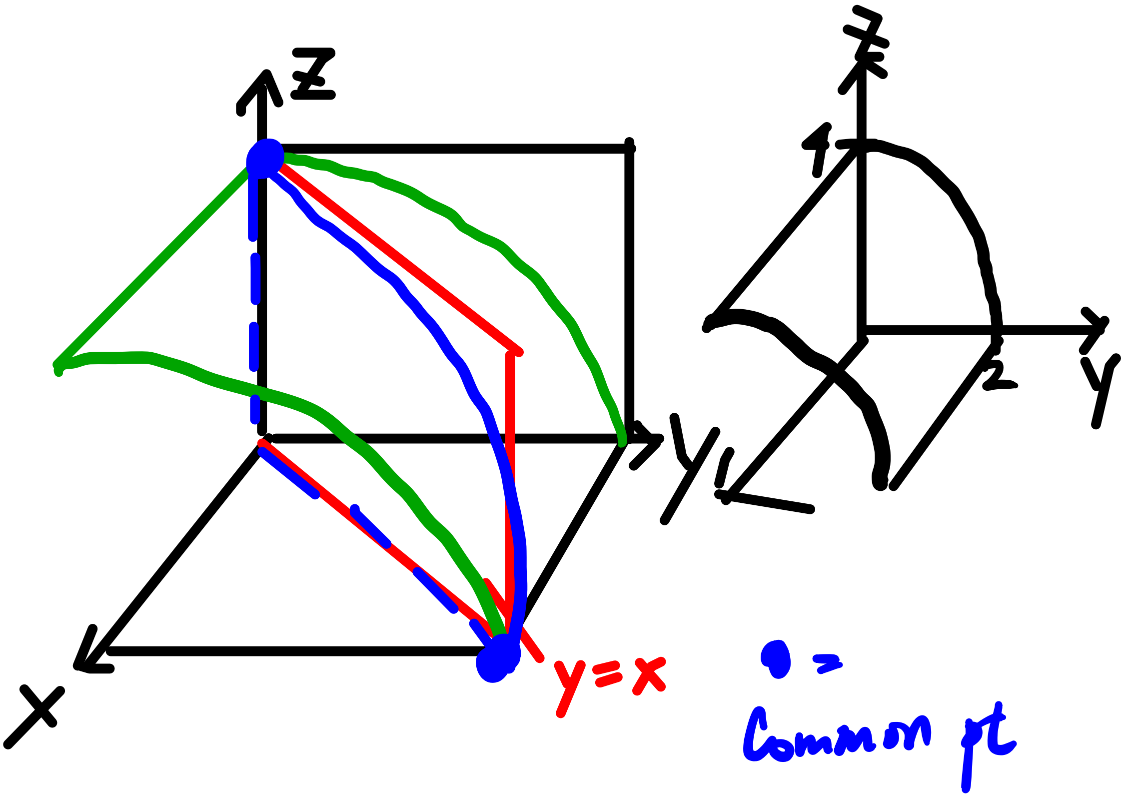
$$\int_0^1\int_0^{x^2}(x^3+1)^{\frac{1}{2}}\,dy\,dx$$

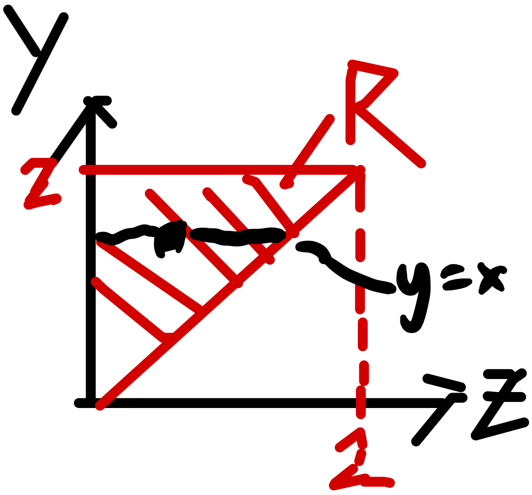
Integrand 不变

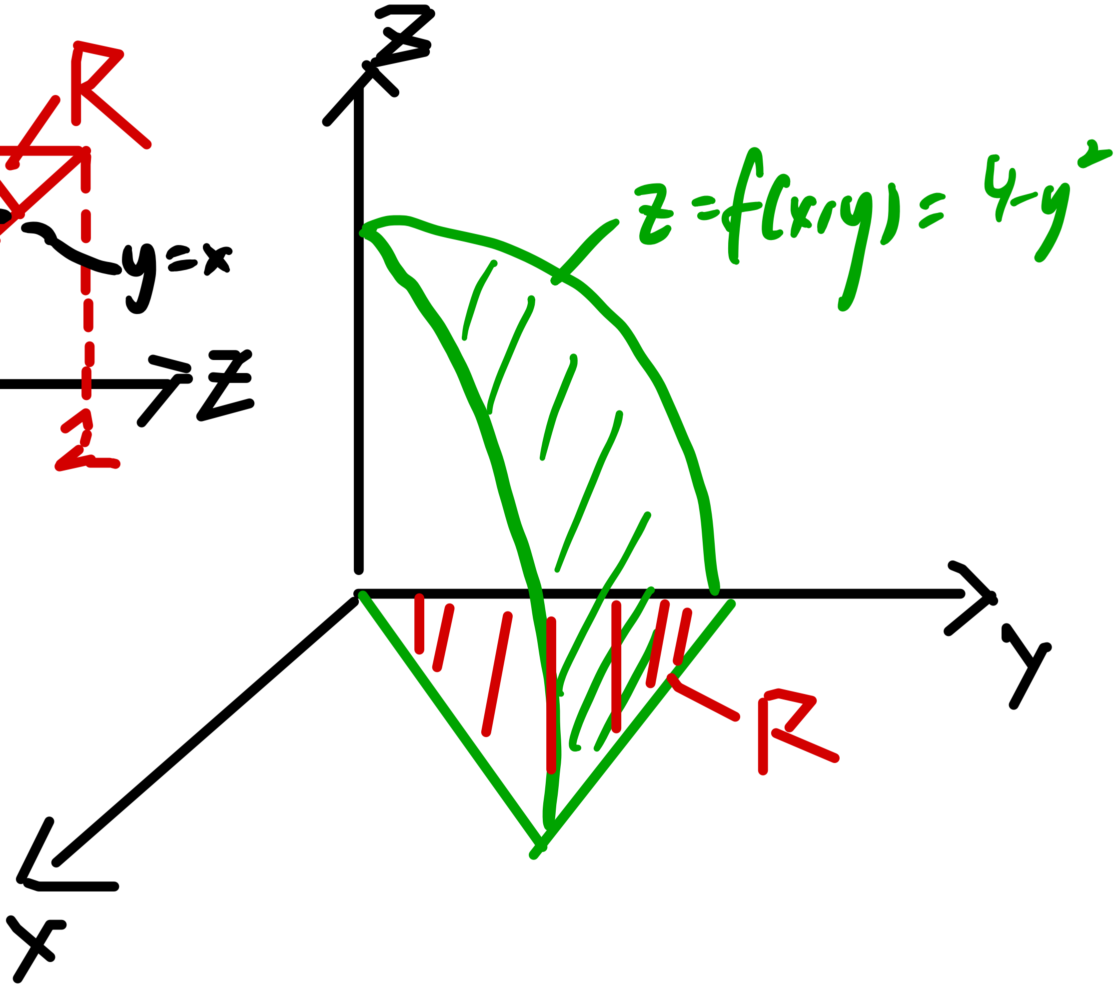
$$=\int_0^1 (x^3+1)^{\frac{1}{2}}x^2\,dx$$

$$=\frac{2}{9}\left(2^{\frac{3}{2}}-1\right)$$

a.4







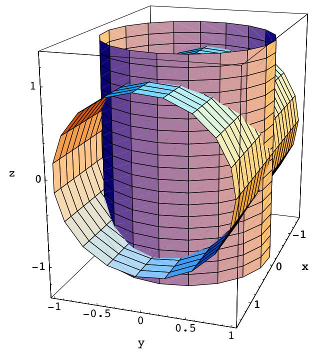
Q.4 Find the volume $\iint_R f(x,y)\, dA$

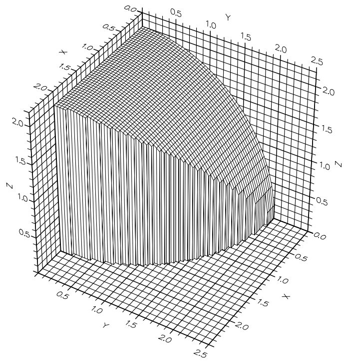
$$= \int_0^2 \int_0^y (4-y^2)\, dx\, dy$$

$$= \int_0^2 \int_x^2 (4-y^2)\, dy\, dx$$

Ex. 2.6 Find the volume bounded by the cylinders

$x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.





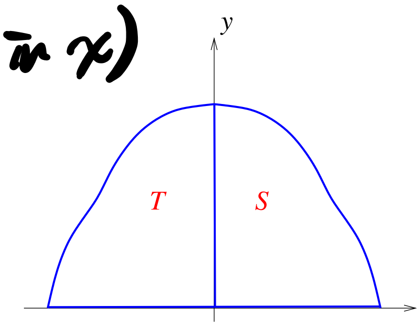
Ans: $V = \dfrac{16}{3}a^3$.

Ex. 2.7 Suppose $\displaystyle\iint_S f(x,y)\,dA = 3$. Find

(even in x)

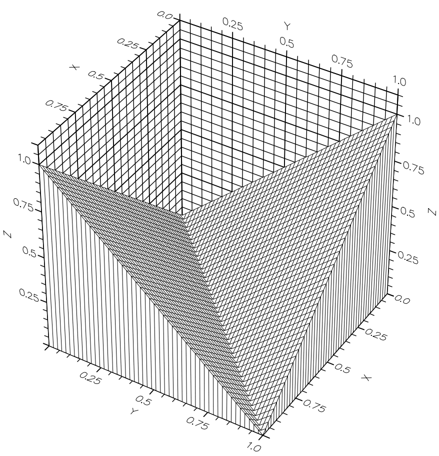
(a) $\displaystyle\iint_R f(x,y)\,dA$ if $f(-x,y) = f(x,y)$. → 6

(b) $\displaystyle\iint_R f(x,y)\,dA$ if $f(-x,y) = -f(x,y)$. → 0



R=S+T and it is symmetric about y-axis.

Ex. 2.8 Find $\displaystyle\int_0^1\int_0^1 |x-y|\,dy\,dx$



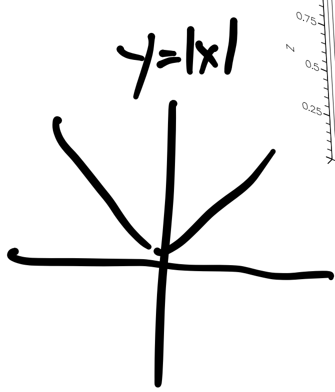
Ans: 1/3.

$\displaystyle\int_{-1}^{1} |x|\,dx$

一样

$\displaystyle = 2\int_0^1 x\,dx$

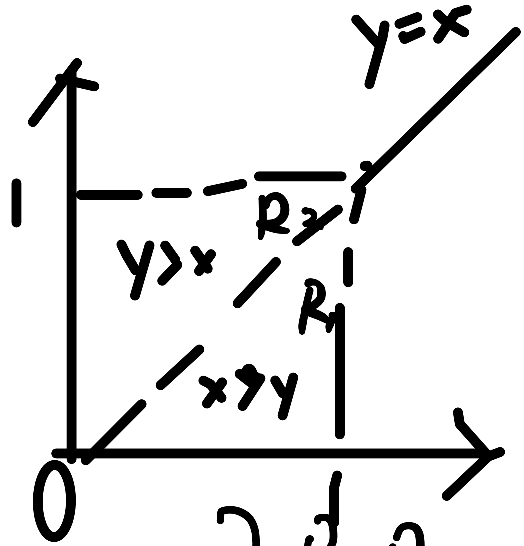
$\displaystyle = \int_{-1}^{0} -x\,dx + \int_0^1 x\,dx$



Ex 1.8

$$\int_0^1\int_0^1 |x-y|\,dy\,dx$$

R



$R = R_1 + R_2$

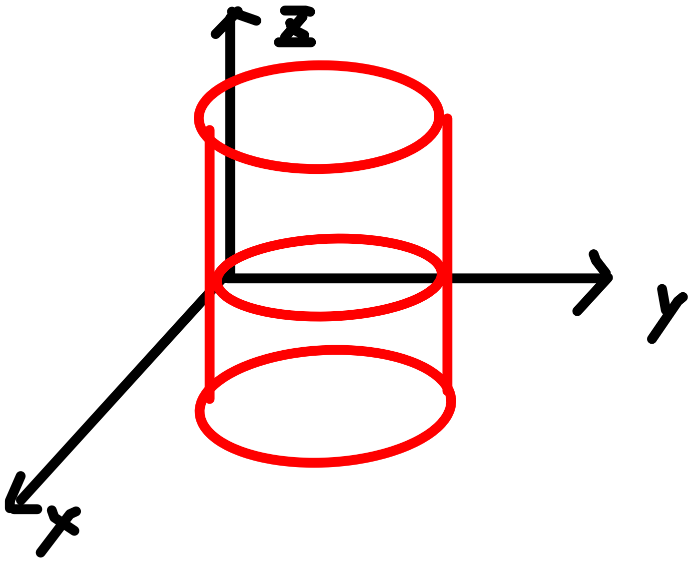
$$f(x,y) = |x-y| = \begin{cases} x-y & \text{if } x-y>0 \ (R_1) \\ -(x-y) & \text{if } y-x>0 \ (R_2) \end{cases}$$

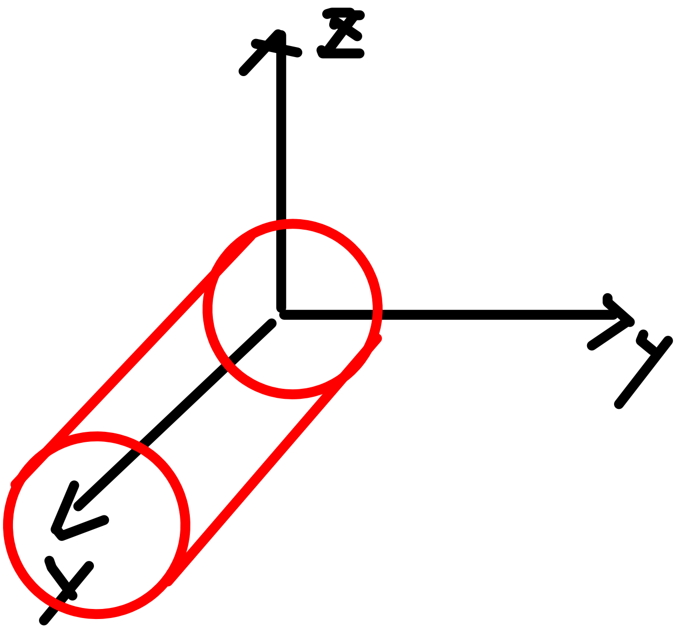
$$\int_0^1\int_0^x (x-y)\,dy\,dx + \int_0^1\int_x^1 -(x-y)\,dy\,dx$$

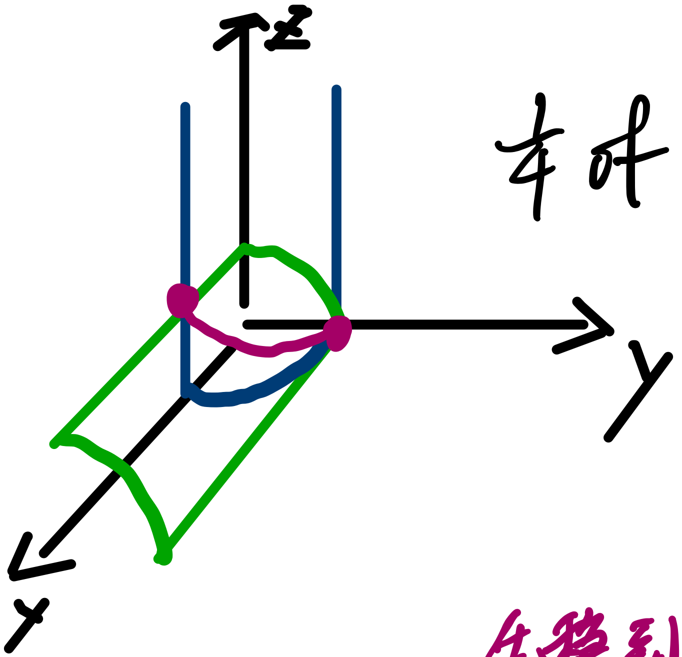
將 region split 開兩塊去計

$$\iint_R f(x,y)\,dA \qquad dA = dx\,dy = dy\,dx$$

2.6

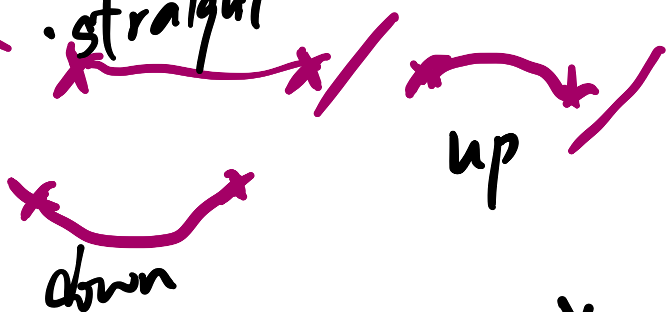


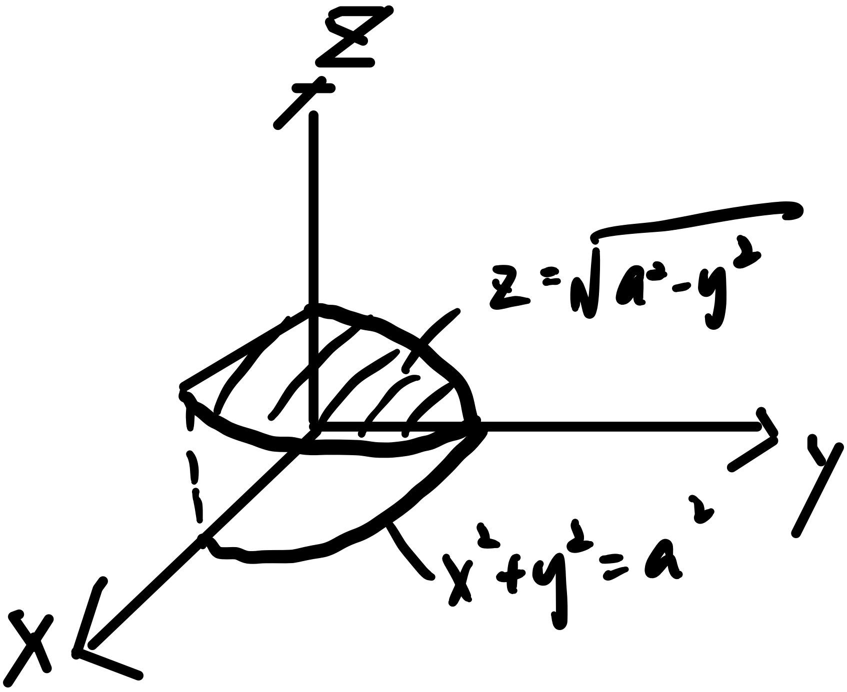


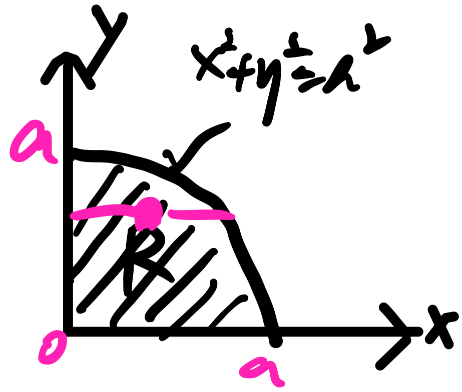


半 of cylinder

先搵到 common pt，再決定係 straight





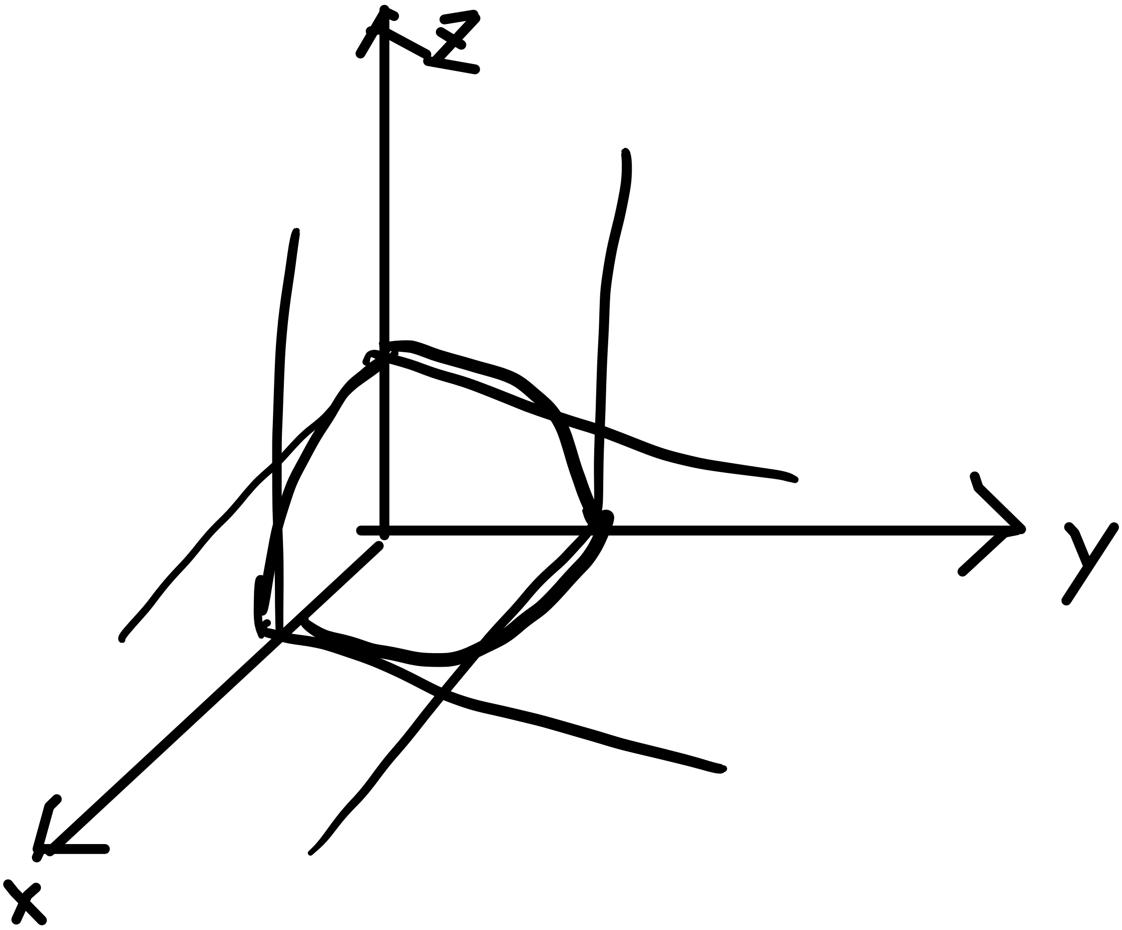


$$\int_0^a\int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-y^2}\,dx\,dy$$

$$=\frac{2}{3}a^2$$

Total volume $= 8\times\frac{2}{3}a^2 = \frac{16a^2}{3}$

★ 要學埋 3 cylinder intersection:


z
y
x

14.4 Double integrals in Polar coordinates

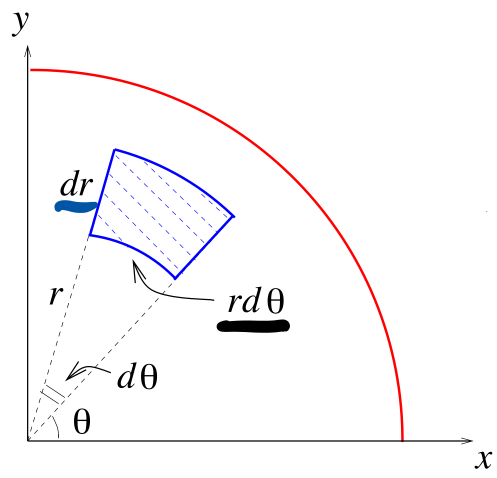
Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dy\,dx,$$

$$= \iint_{R^*} f(r\cos\theta, r\sin\theta)\,r dr d\theta,$$

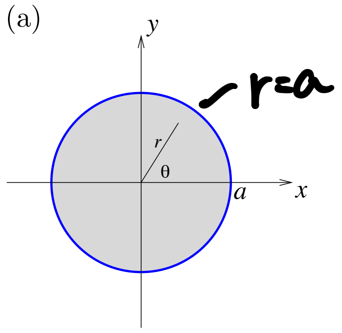
where R^* is description of R in polar coordinates.

$dA_{r\theta} = r\,d\theta\,dr$

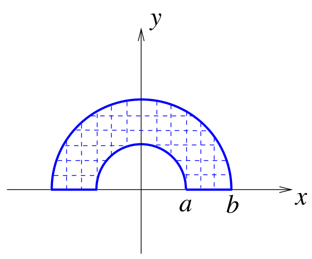


How to describe region in polar coordinates

(a)



(b)



(a)

$$D = \left\{(x,y)\,\middle|\, x \in [-a,a],\quad y \in \left[-\sqrt{a^2-x^2}, \sqrt{a^2-x^2}\right]\right\}$$

$$D = \{(r,\theta)\,|\, r \in [0,a],\quad \theta \in [0,2\pi]\}$$

$$\int_0^{2\pi}\int_0^a r\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_0^a d\theta = \int_0^{2\pi} \frac{a^2}{2} d\theta$$

$$= \frac{a^2}{2}\theta\Big|_0^{2\pi} = \pi a^2$$

(b)

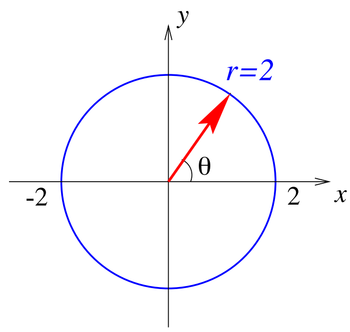
$$D = \{(r,\theta)\,|\, r \in [a,b],\quad \theta \in [0,\pi]\}$$

Ex. 4.1 (a) Find the area bounded by the circle $x^2+y^2=4$.

Using polar coord. $x=r\cos\theta$, $y=r\sin\theta$, then

$$x^2+y^2=4 \quad\Rightarrow\quad r^2=4 \quad\Rightarrow\quad r=2$$

$$\therefore \int_0^{2\pi}\int_0^2 r\,dr\,d\theta = 2\pi\left.\frac{r^2}{2}\right|_0^2 = 4\pi.$$

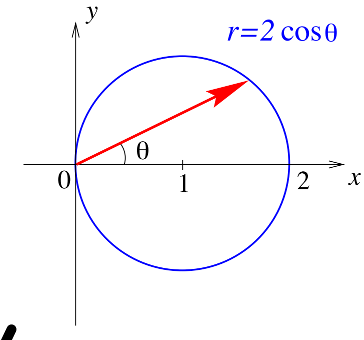


(b) Find the area bounded by the circle $x^2+y^2=2x$.

In polar coord. $r^2=2r\cos\theta \quad\Rightarrow\quad r=2\cos\theta$

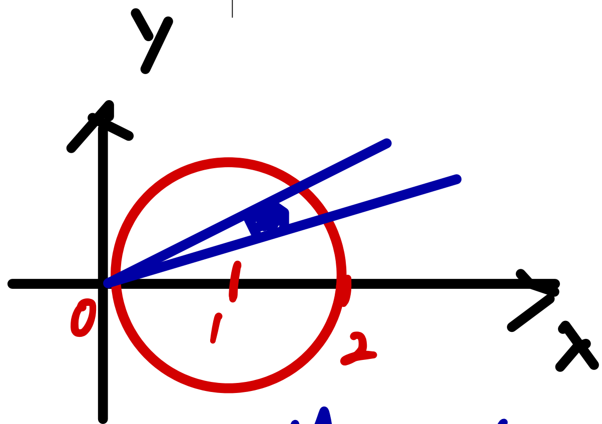
$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta} r\,dr\,d\theta = 2\int_0^{\frac{\pi}{2}}\left.\frac{r^2}{2}\right|_0^{2\cos\theta} d\theta = 4\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta = \pi.$$



$x^2+y^2=2x$
$x=r\cos\theta$
$y=r\sin\theta$

$r^2=2r\cos\theta$
$r=2\cos\theta \quad r\neq 0$

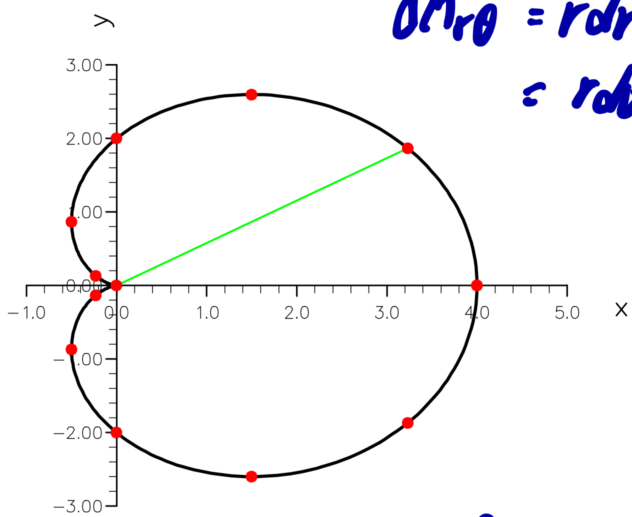


Graph of Polar Equation

Plot the graph: $r=2(1+\cos\theta)$

$\theta°$	$r=2(1+\cos\theta)$
0	4.00
30	3.73
60	3.00
90	2.00
120	1.00
150	0.26
180	0.00
210	0.26
240	1.00
270	2.00
300	3.00
330	3.73
360	4.00

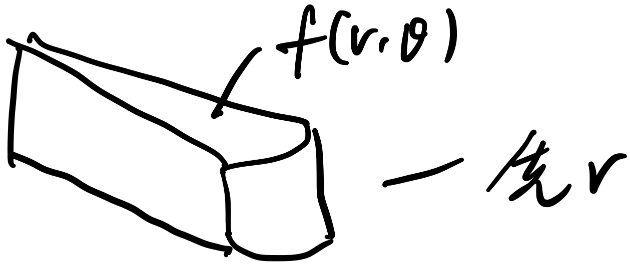
$dA_{r\theta} = r\,dr\,d\theta$
$= r\,d\theta\,dr$

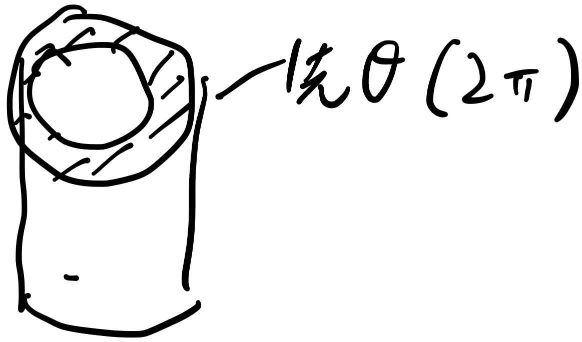


如果先$d\theta$,

$$\int_0^2\int_{-\cos^{-1}\left(\frac{r}{2}\right)}^{\cos^{-1}\left(\frac{r}{2}\right)} r\,d\theta\,dr$$

$r=2\cos\theta$
$\theta=\cos^{-1}\left(\frac{r}{2}\right)$


f(r,θ)
— 先r


先θ (2π)

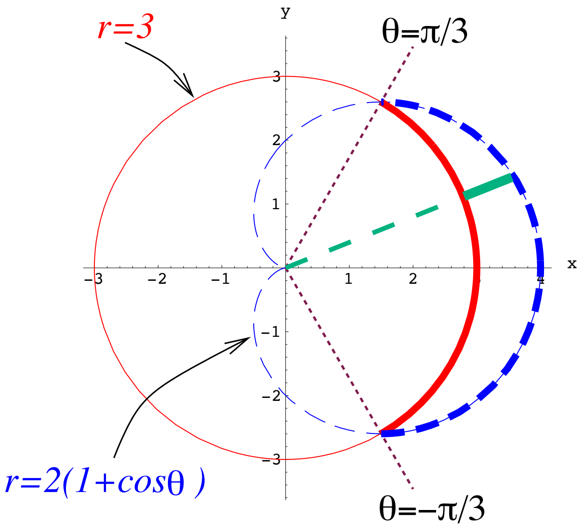
Ex. 4.2 Find the area of the region R which is outside the circle $r = 3$ and inside the cardioid $r = 2(1 + \cos\theta)$.

For points of intersection,

$$3 = 2(1 + \cos\theta)$$

from which it follows that

$$\cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \pm\frac{\pi}{3}.$$



$$\therefore\ A = \iint_R r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_3^{2(1+\cos\theta)} r\,dr\,d\theta$$

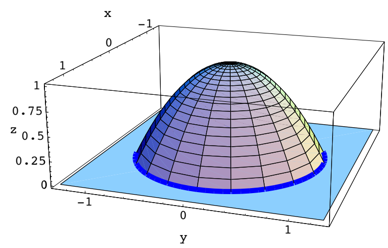
$$= 2\int_0^{\frac{\pi}{3}} \int_3^{2(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} r^2\Big|_3^{2(1+\cos\theta)} d\theta$$

$$= \int_0^{\frac{\pi}{3}} \left[4(1+\cos\theta)^2 - 9\right]\,d\theta$$

Ex. 4.3 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

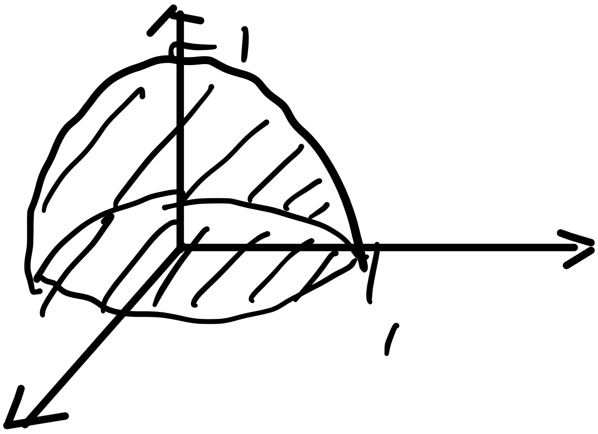
Ans: $V = \pi/2$.

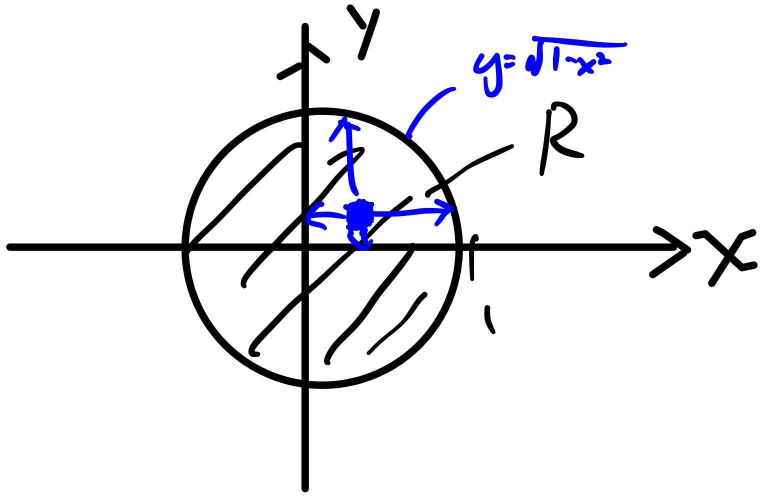


43. $z=0$, $z=1-x^2-y^2$

(inverted rice bowl with max. height=1)

$$\iint_R f(x,y)\,dA$$



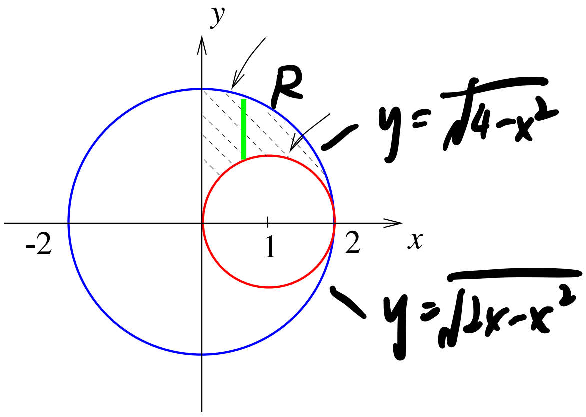


$$4\int_0^1\int_0^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx = 4\int_0^1\left\{\sqrt{1-x^2}-x^2\sqrt{1-x^2}-(1-x^2)^{\frac{3}{2}}\right\}dx$$

唔知点计，所以用 polar

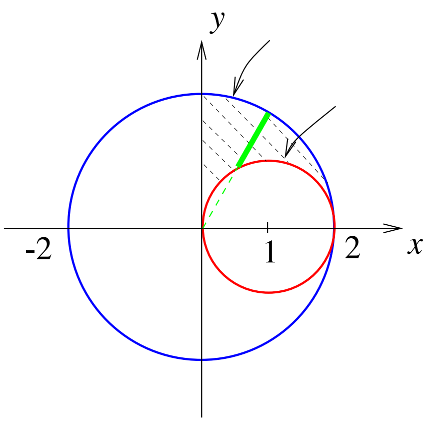
$$\int_0^{2\pi}\int_0^1 (1-r^2)\, r\, dr\, d\theta = \frac{\pi}{2}$$

Ex. 4.4 Find $\displaystyle\iint_R x\,dA$, where R is the region in the first quadrant that lies between the circles $x^2+y^2 = 4$ and $x^2+y^2=2x$.

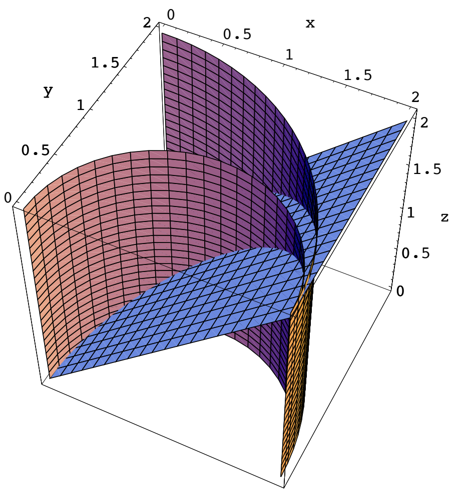


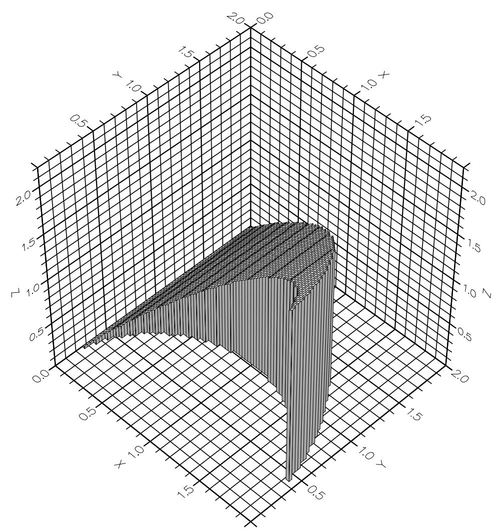
$$\iint_R x\,dA = \int \int \qquad x\,dy\,dx$$

Alternatively, by using polar coord., we have



$$\begin{aligned}
\iint_R x\,dA &= \int_0^{\frac{\pi}{2}}\int_{2\cos\theta}^{2} r\cos\theta\, r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos\theta \left[\int_{2\cos\theta}^{2} r^2\,dr\right] d\theta \\
&= \int_0^{\frac{\pi}{2}} \cos\theta \left.\frac{r^3}{3}\right|_{2\cos\theta}^{2} d\theta \\
&= \frac{1}{3}\int_0^{\frac{\pi}{2}} (8\cos\theta - \cos^4\theta)\,d\theta \\
&= \frac{8}{3} - \frac{8}{12}\left[\cos^3\theta\sin\theta + \frac{3}{2}(\theta + \sin\theta\cos\theta)\right]_0^{\frac{\pi}{2}} \\
&= \frac{16-3\pi}{6}
\end{aligned}$$





Ex 4.4

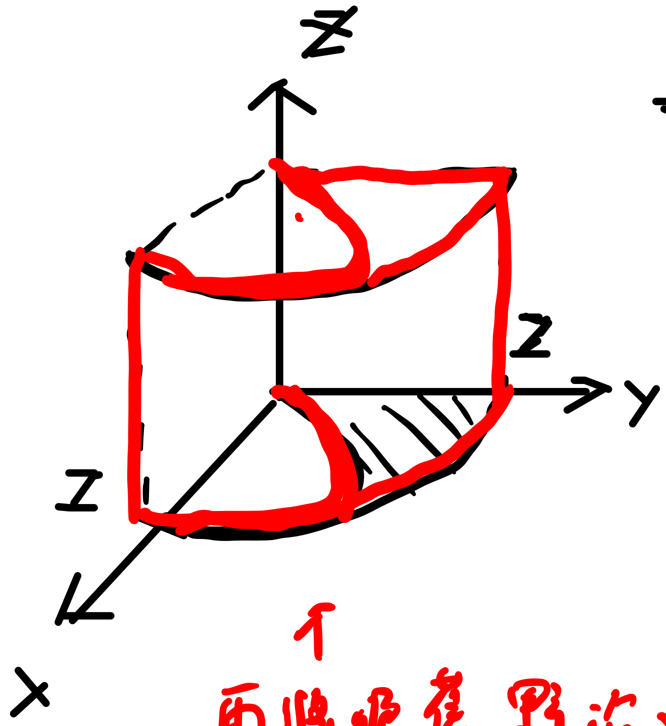
$x^2+y^2=4$

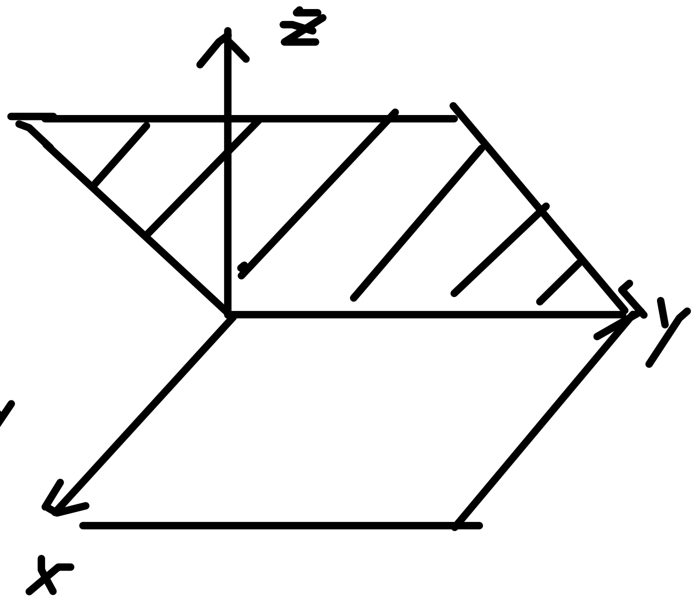
$$\begin{cases} x^2+y^2=2x \\ (x-1)^2+y^2=1 \end{cases}$$

$$\iint_R x \, dA$$

$z=f(x,y)=x$

plane





再將呢舊野沿 x-z plane chop 45°

$$\iint_R x\,dA = \int_0^2 \int_{\sqrt{2x-x^2}}^{\sqrt{4-x^2}} x\,dy\,dx$$

$$= \int_0^2 xy \Big|_{\sqrt{2x-x^2}}^{\sqrt{4-x^2}} dx$$

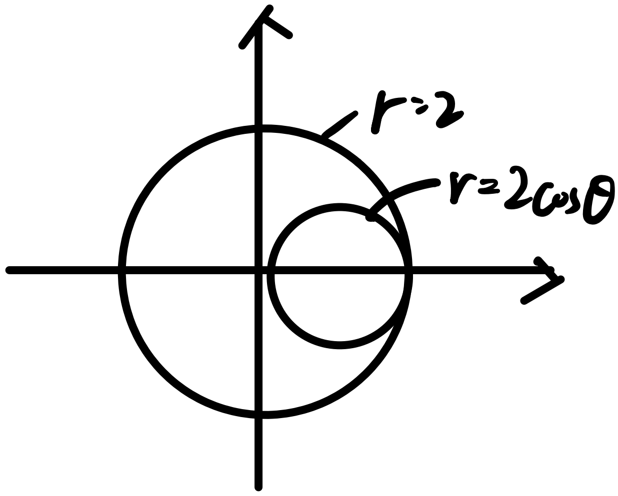
$$= \int_0^2 x\left(\sqrt{4-x^2} - \sqrt{2x-x^2}\right) dx$$

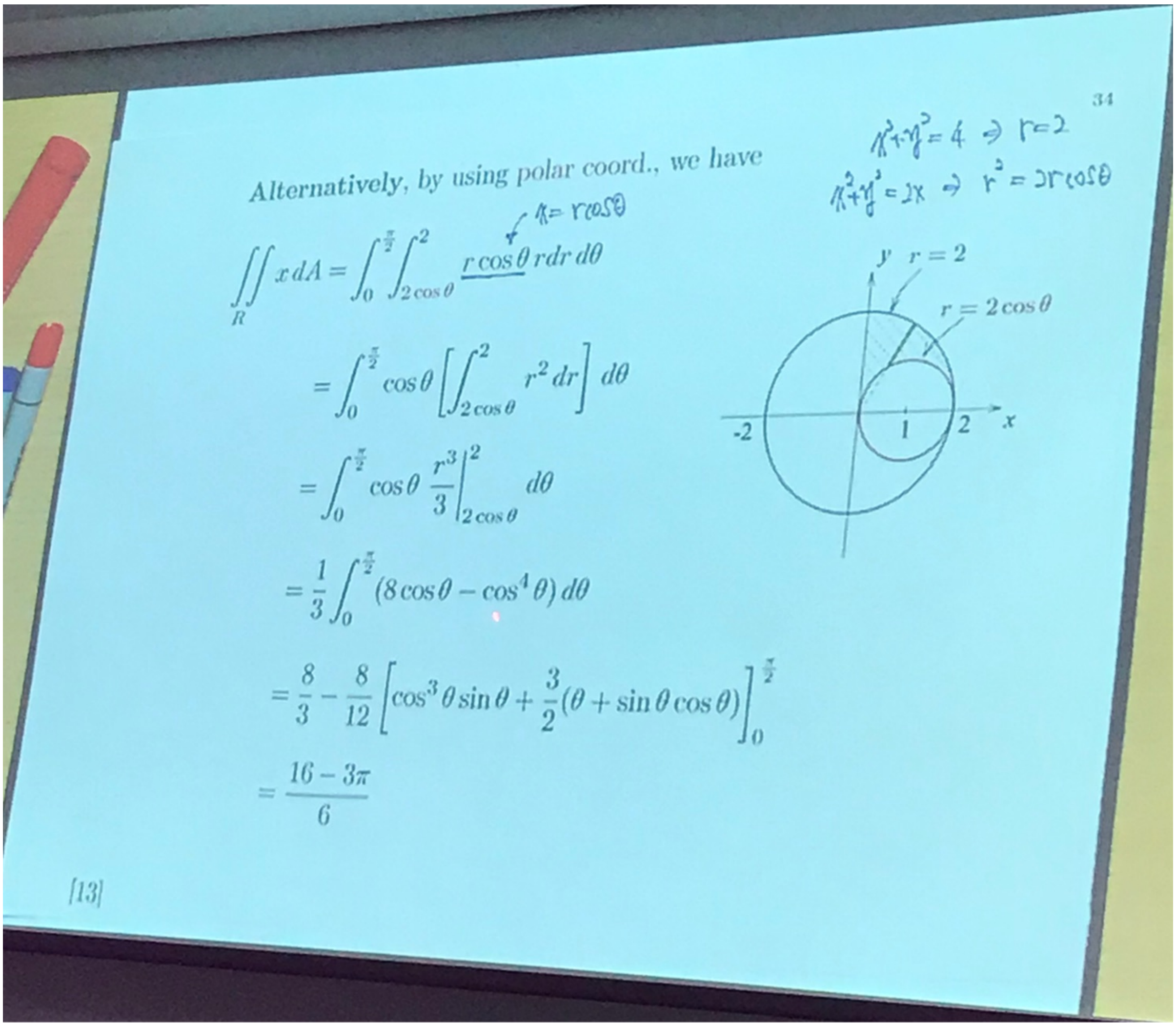
difficult to do

$\therefore$ polar coordinate.

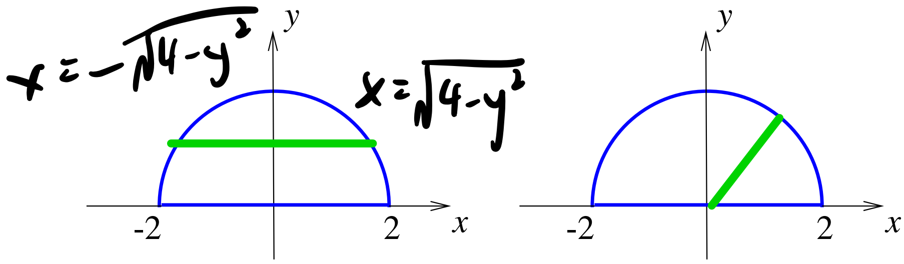
polar coordinate:

$$\int_0^{\frac{\pi}{2}}\int_{2\cos\theta}^{2} r$$





Ex. 4.5 Find $\displaystyle\iint\limits_R x^2y^2\,dA$, where R is bounded by the semi-circle $x^2+y^2=4$ with $y \geqslant 0$.



$$\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2y^2\,dx\,dy$$

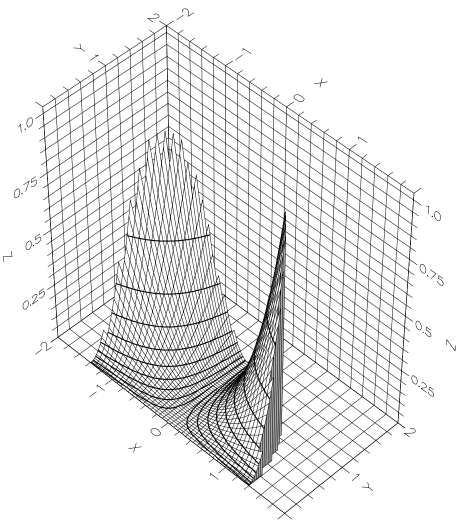
$$= \int_0^\pi\int_0^2 (r^2\cos^2\theta)\,(r^2\sin^2\theta)\,r dr\,d\theta$$

$$= \frac{1}{4}\int_0^\pi\int_0^2 r^5\sin^2 2\theta\,dr\,d\theta$$

$$= \frac{8}{3}\int_0^\pi \sin^2 2\theta\,d\theta$$

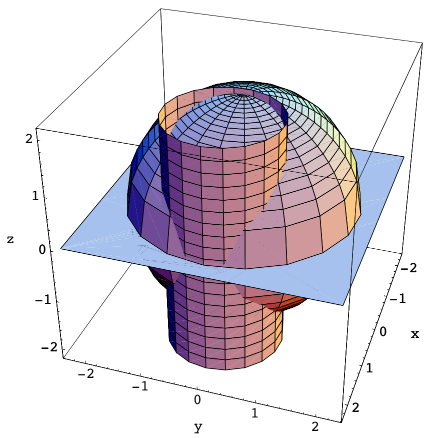
$$= \frac{8}{3}\cdot\frac{1}{2}\int_0^\pi (1-\cos 4\theta)\,d\theta$$

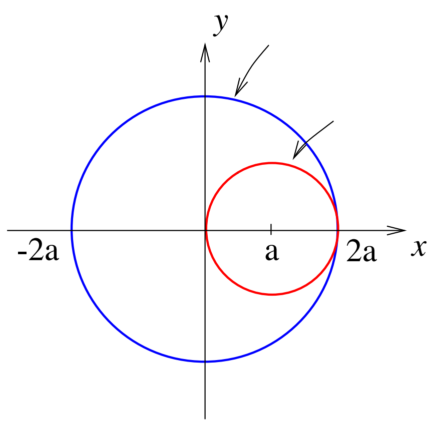
$$= \frac{4}{3}\left[\theta-\frac{1}{4}\sin 4\theta\right]\Bigg|_0^\pi = \frac{4\pi}{3}.$$



Ex. 4.6 Find the volume inside the sphere $x^2+y^2+z^2=4a^2$ and outside the cylinder $x^2+y^2=2ax$.

Ans: $16a^3(3\pi+4)/9$ (see also ex. 6.11 and ex. 7.3)



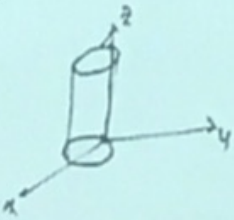


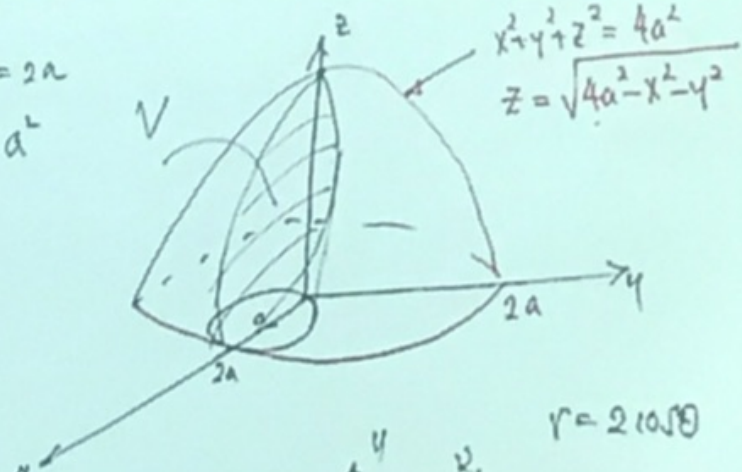
Ex 4,6

x. 4.6 Find the volume inside the sphere $x^2+y^2+z^2=4a^2$ and outside the cylinder $x^2+y^2=2ax$. (see also ex. 6.11 and ex. 7.3)

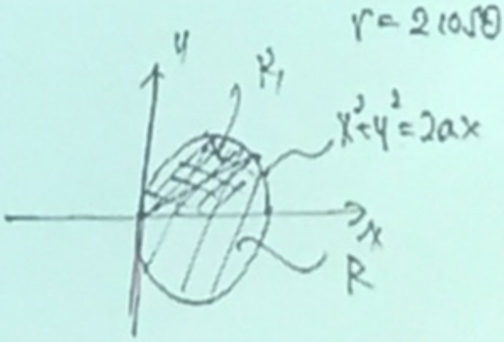
$x^2+y^2+z^2=4a^2$ (0,0,0) with $r=2a$

$x^2+y^2=2ax \Rightarrow (x-a)^2+y^2=a^2$





$$V=\frac{4}{3}\pi(2a)^3-V_1$$



$$4V_1=\iint_{R_1}\sqrt{4a^2-x^2-y^2}\,dx\,dy$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\theta}\sqrt{4a^2-r^2}\;r\,dr\,d\theta$$

Made with Goodnotes

Review

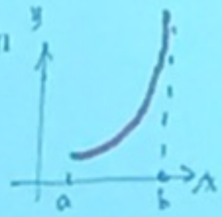
Improper Integral of single variable function: $y = f(x)$.

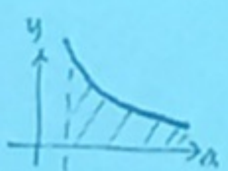
(1) $\displaystyle\int_a^{\boxed{+\infty}} f(x)\,dx = \lim_{\ell \to +\infty} \int_a^{\ell} f(x)\,dx.$

If the limit exists, the improper integral is said to converge. $\displaystyle\lim_{x \to b^-} f(x) = \infty$

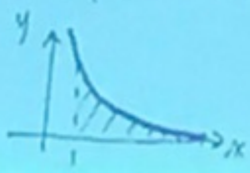
(2) If $f(x)$ is continues on $[a, b)$ but it is 'unbound' at b, then

$$\int_a^b f(x)\,dx = \lim_{\ell \to b^-} \int_a^{\ell} f(x)\,dx.$$





Ex. $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ diverges to ∞.



$\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ converges to 1.

[15]

Ex. 4.7 We define the *improper* integral (over the entire plane $\mathbb{R}^2$)

$$\mathrm{I} = \iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy = \lim_{a\to\infty} \iint\limits_{D_a} e^{-(x^2+y^2)}\, dA,$$

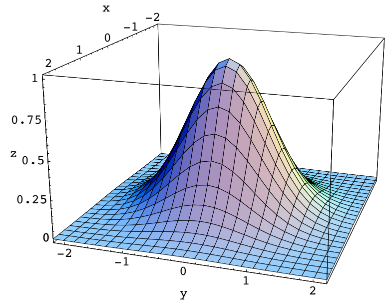
← xyplane 无限延伸

where D_a is the disk with radius a and centre the origin.

Hence show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \pi.$$

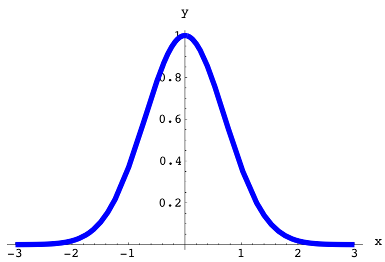
↑ polar 做, 下一阶之Ans



$z = f(x,y) = e^{-(x^2+y^2)}$

Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

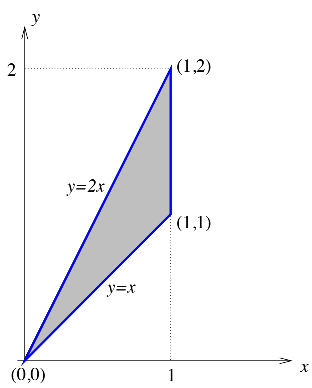


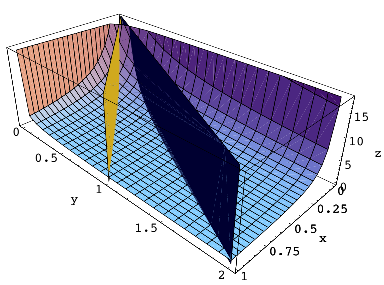
$y = f(x) = e^{-x^2}$

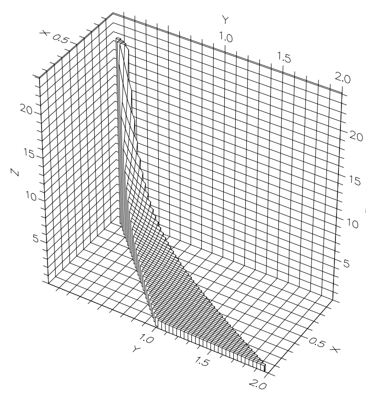
Ex. 4.8 $\displaystyle\iint\limits_R \frac{1}{x\sqrt{y}}\, dA$ over the triangle R with vertices $(0,0)$, $(1,1)$, and $(1,2)$.

The integral is *improper* because the integrand is *unbound* as (x,y) approaches $(0,0)$, a boundary point of R. Nevertheless, iteration leads to a proper integral (Why!!!).

Ans: $4(\sqrt{2}-1)$.

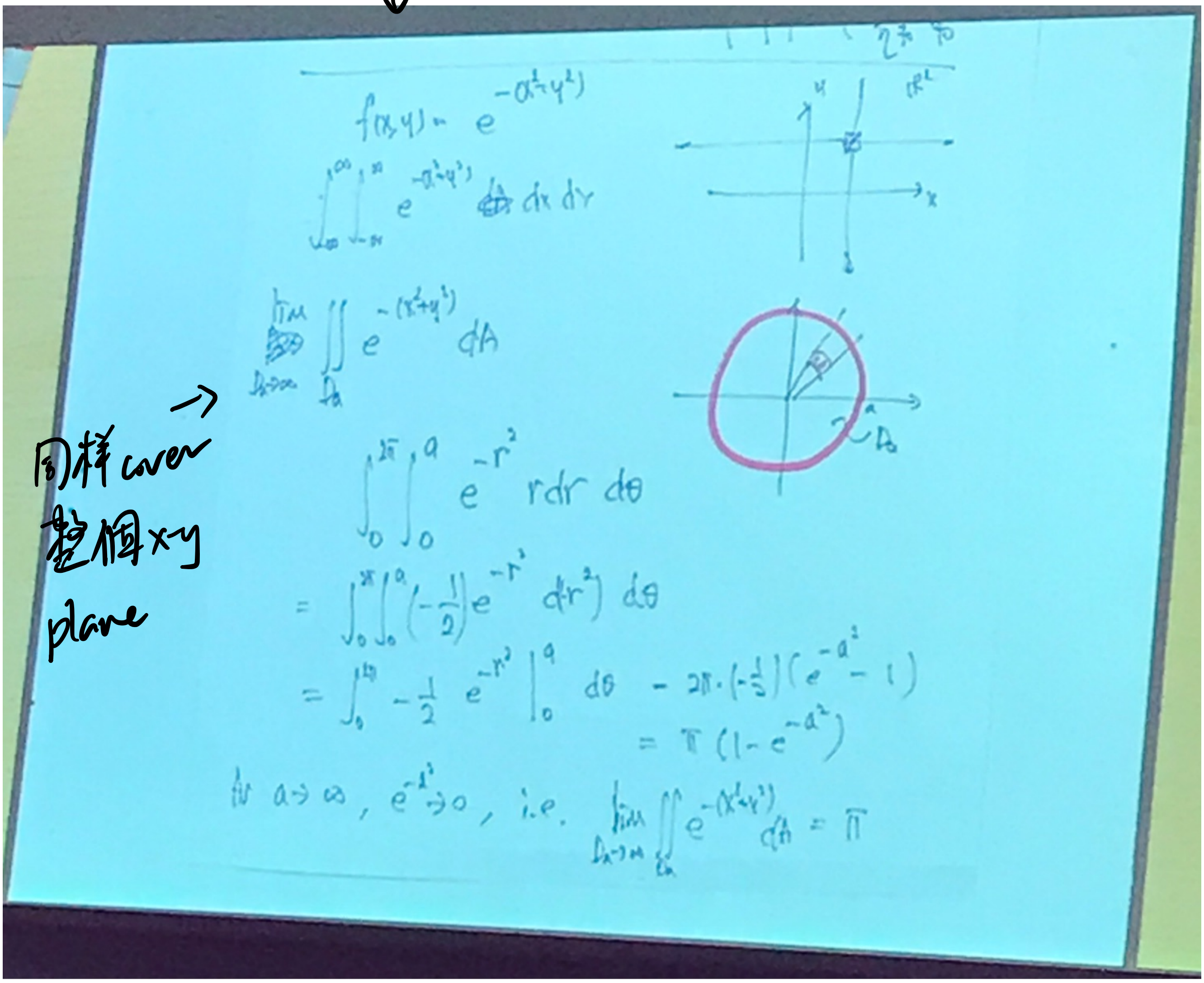






Ex 4.7. (A),

用 polar 做



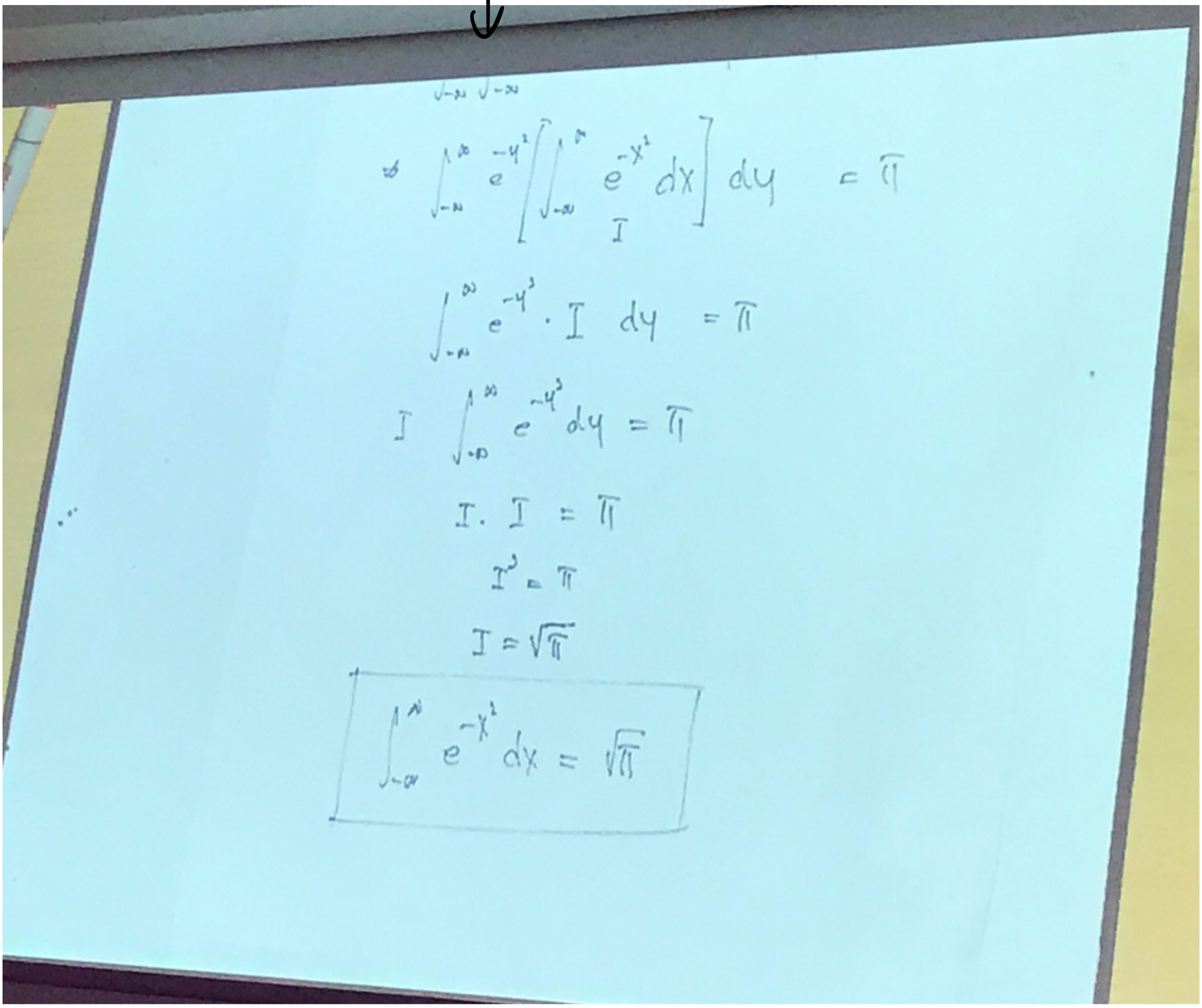
同样 cover 整個 x-y plane

Ex 4.7 (b).

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-y^2}\cdot e^{-x^2}\,dx\,dy$$

$\downarrow$



$$\boxed{\text{Volume}} = \iint_R f(x,y)\, dA_{xy}$$

height.

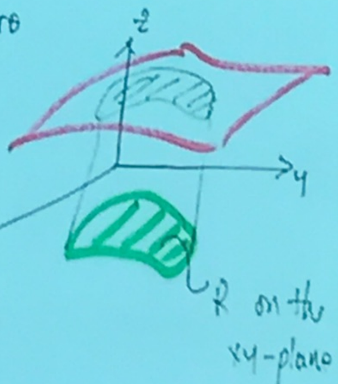
(r,θ)

$$\text{Area} = \iint_R 1\, dA_{r\theta} \qquad dA_{r\theta} = r\,dr\,d\theta = r\,d\theta\,dr$$

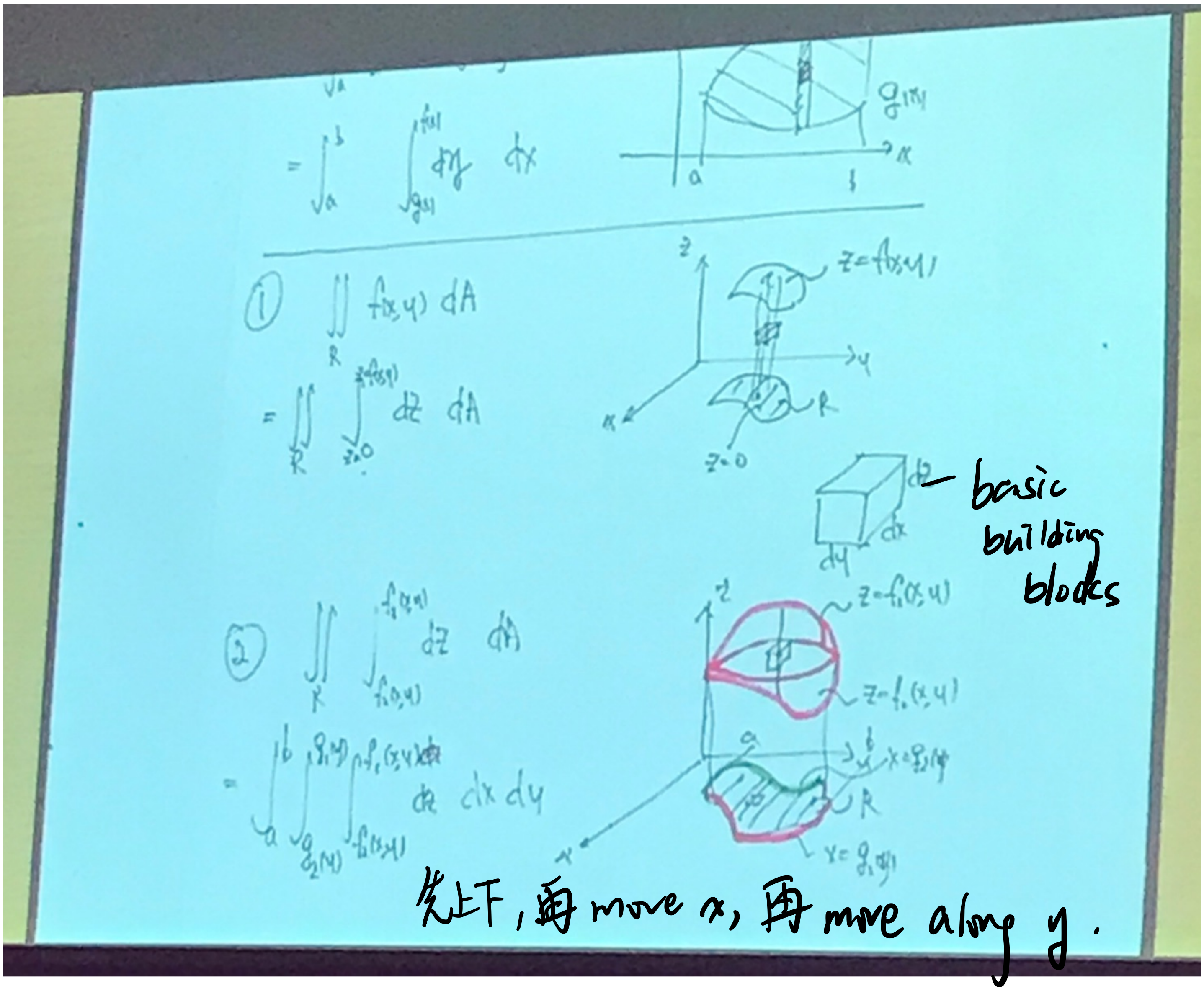
$$\boxed{\text{Volume}} = \iint_R f(r,\theta)\, dA_{r\theta}$$

height.

This volume is bounded by two surfaces (upper and lower)
Upper = roof = $z = f(x,y)$
Lower = xy-plane, $z = 0$

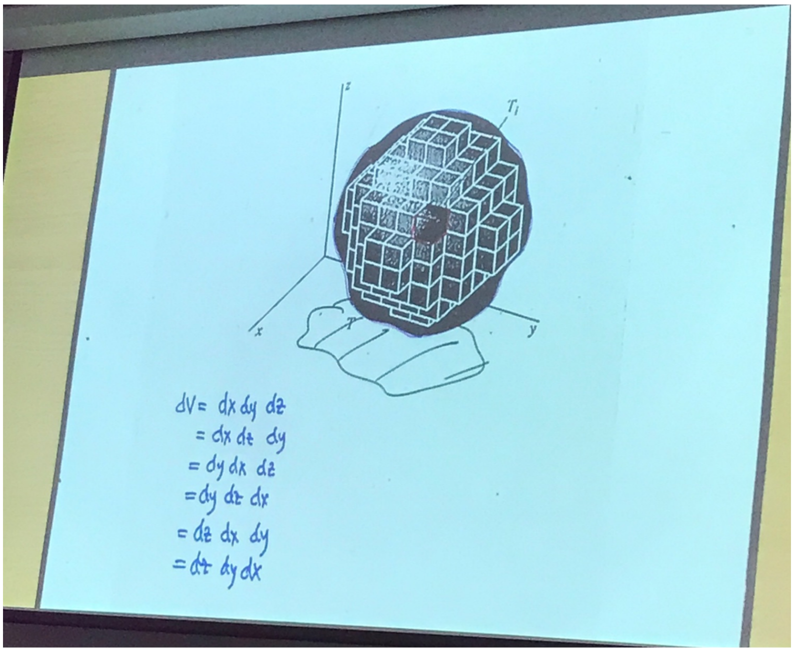


Triple integral



— basic building blocks

先上下，再 move x，再 move along y.



14.5 Triple integrals

Suppose $f(x, y, z)$ is defined on a rectangular box

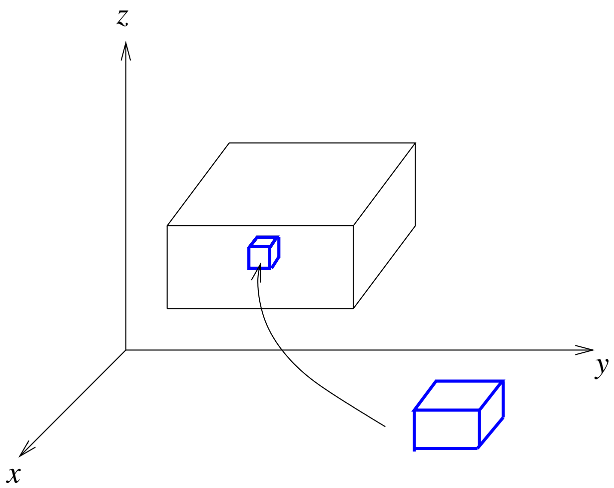
$B = \{(x, y, z) \mid x \in [a, b], y \in [c, d], z \in [e, f]\}$.

The volume B_{ijk} is $\triangle V_{ijk} = \triangle x_i \triangle y_j \triangle z_k$.

Then we can form the Riemann sum

$$\sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\triangle V_{ijk}$$

where $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \in B_{ijk}$.



The triple integral of f over the box B is

$$\iiint\limits_B f(x, y, z)\, dV = \lim_{\substack{\ell, m, n \to \infty \\ \text{or } \triangle x_i, \triangle y_j, \triangle z_k \to 0}} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\,\triangle V_{ijk}$$

if this limit exists.

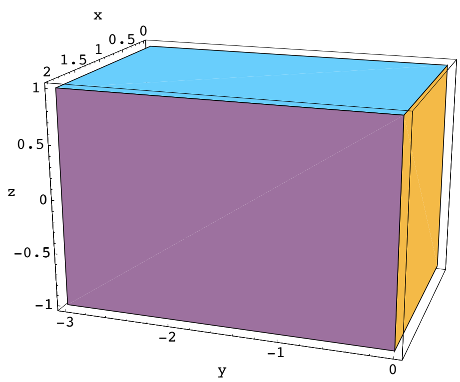
Ex. 5.1 Evaluate the triple integral $\displaystyle\iiint\limits_B (x^2 + yz)\, dV$, where ← 考試唔会出

$B = \{(x, y, z) \mid x \in [0, 2], \quad y \in [-3, 0] \quad \text{and} \quad z \in [-1, 1]\}$.

$$V = \int_0^2 \int_{-3}^0 \int_{-1}^1 (x^2 + yz)\, dz\, dy\, dx$$

$$= \int_0^2 \int_{-3}^0 \left[x^2 z + \frac{1}{2} y z^2\right]\Bigg|_{-1}^{1} dy\, dx$$

$$= \int_0^2 \int_{-3}^0 2x^2\, dy\, dx = \int_0^2 6x^2\, dx = 16.$$



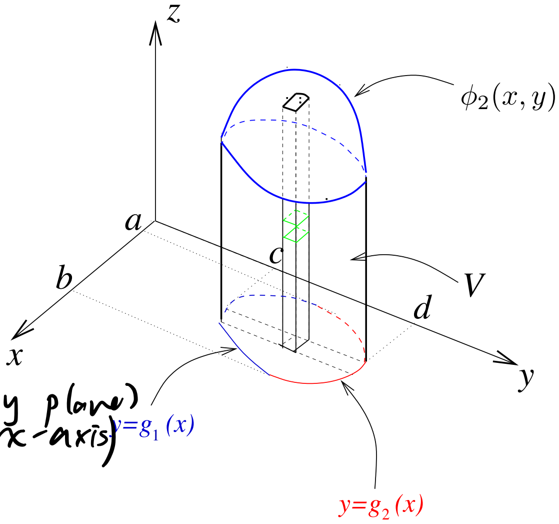
Note:

$$V = \int_{-1}^1 \int_{-3}^0 \int_0^2 (x^2 + yz)\, dx\, dy\, dz$$

$$= \int_{-1}^1 \int_0^2 \int_{-3}^0 (x^2 + yz)\, dy\, dx\, dz.$$

Triple integral over a general bounded region B

$$\iiint\limits_B f(x,y,z)\,dV = \int_{a=x}^{b=x}\int_{g_1(x)=y}^{g_2(x)=y}\int_{0=z}^{\phi_2(x,y)=z} f(x,y,z)\,dz\,dy\,dx.$$



The meaning of the most inner integral is that x and y are held fixed, and therefore $\phi_2(x,y)$ is regarded as constant, while $f(x,y,z)$ is integrated w.r.t. z.

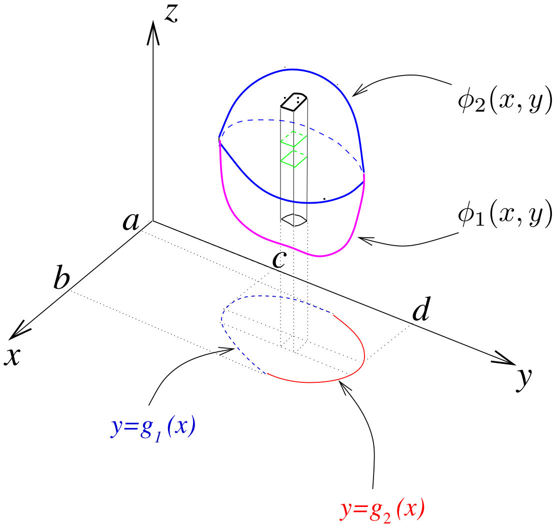
Similarly, the second inner integral is that x is held fixed, and therefore, $g_1(x)$ and $g_2(x)$ are regarded as constants.

z: $z=0$ to $z=\phi_2(x,y)$ (surfaces)
y: $y=g_1(x)$ to $y=g_2(x)$ (curves on xy plane)
x: $x=a$ to $x=b$ (points on x-axis)

Suppose now that V is the region bounded above by the surface $z=\phi_2(x,y)$ and below by $z=\phi_1(x,y)$. In this case, the limits on the first integration w.r.t. z are $\phi_1(x,y)$ and $\phi_2(x,y)$ since every column starts on the surface $z=\phi_1(x,y)$ and ends on the surface $z=\phi_2(x,y)$.

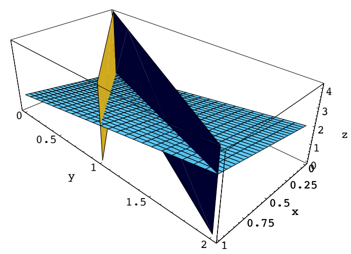
Therefore,

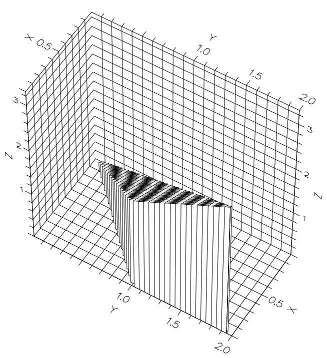
$$\iiint\limits_B f(x,y,z)\,dV = \int_a^b\int_{g_1(x)}^{g_2(x)}\int_{\phi_1(x,y)}^{\phi_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$



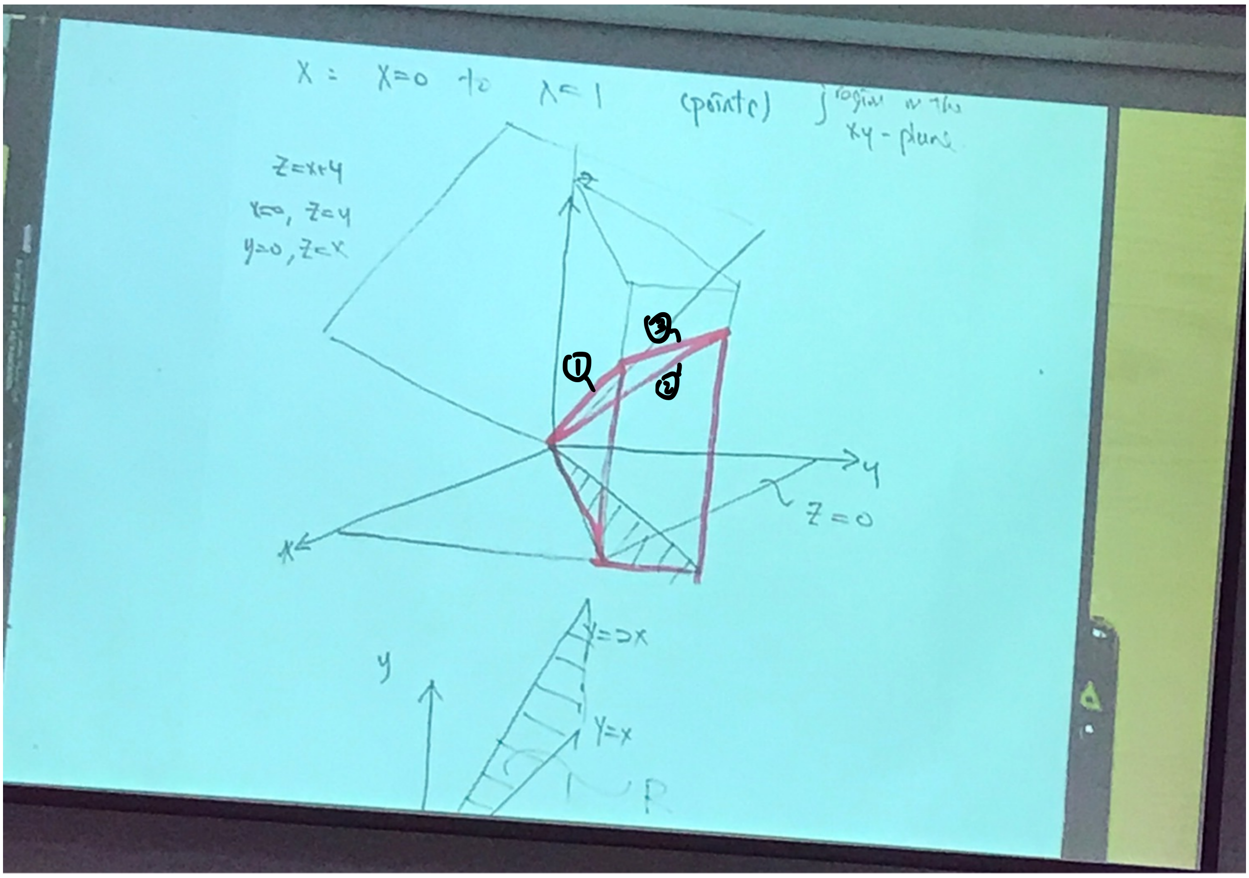
Ex. 5.2

$$\int_0^1\int_x^{2x}\int_0^{x+y} 2xy\,dz\,dy\,dx$$
$$=\int_0^1\int_x^{2x} 2xyz\Big|_0^{x+y}\,dy\,dx$$
$$=\int_0^1\int_x^{2x}(2x^2y+2xy^2)\,dy\,dx$$
$$=\int_0^1\left(x^2y^2+\frac{2}{3}xy^3\right)\Big|_x^{2x}\,dx$$
$$=\int_0^1\frac{23}{3}x^4\,dx=\frac{23}{15}.$$





5.2



Want to find parametric representation of ①, ②, ③

① $\begin{cases} y=x \\ z=x+y \end{cases}$ let $x=t,\ y=t,\ z=2t$

$\underline{r}_1(t) = \langle t, t, 2t \rangle$

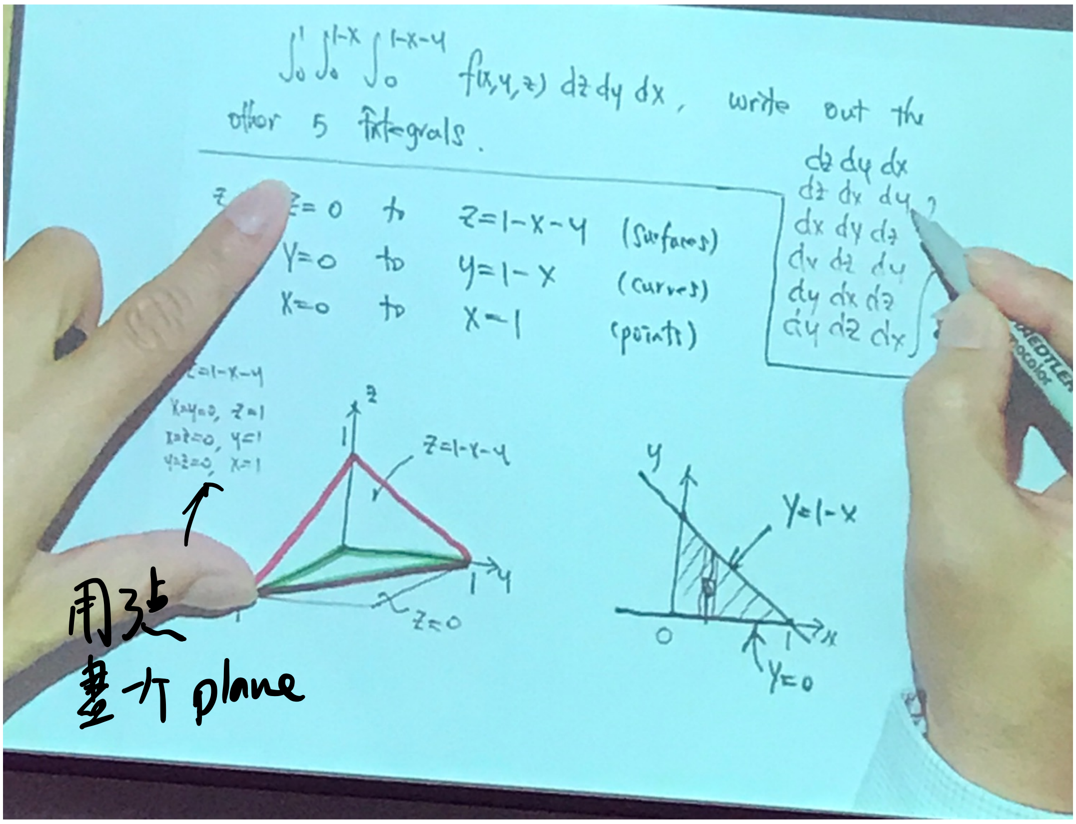
③ $\begin{cases} x=1 \\ z=x+y \end{cases}$ Let $y=t,\ x=1,\ z=t+1$

$\underline{r}_3(t) = \langle 1, t, t+1 \rangle$

② $\begin{cases} y=2x \\ z=x+y \end{cases}$ let $y=t,\ y=2t,\ z=3t$

$\underline{r}_2(t) = \langle t, 2t, 3t \rangle$

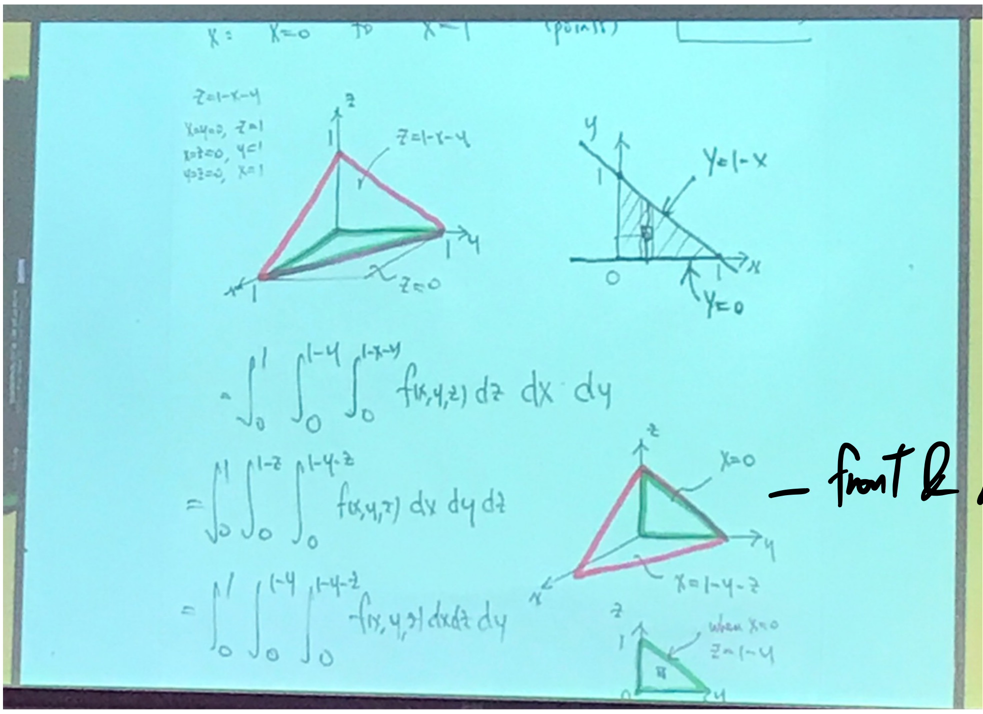
肯定有一题係：



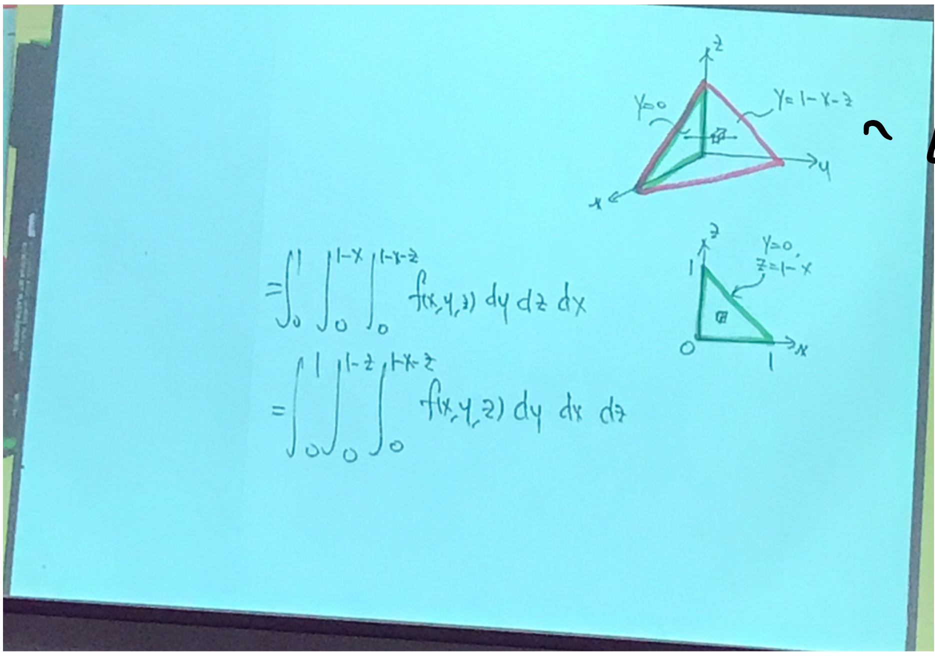
$$\int_0^1\int_0^{1-y}\int_0^{1-x-y} f(x,y,z)\,dz\,dx\,dy$$

$$\int_0^1\int_0^{1-z}\int_0^{1-y-z} f(x,y,z)\,dx\,dy\,dz$$

$$\int_0^1\int_0^{1-y}\int_0^{1-y-z} f(x,y,z)\,dx\,dz\,dy$$



— front & back for dx



~ Left & right for dy

Ex. 5.3 Sketch the solid whose volume is given by the iterated integral and evaluate the integral

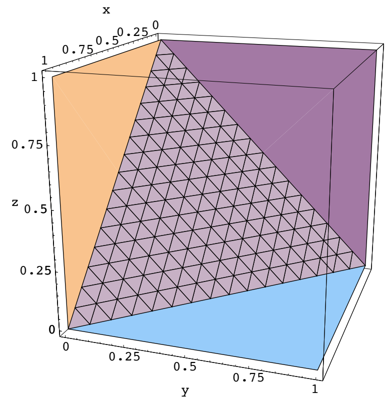
$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dzdydx.$$

$$= \int_0^1 \int_0^{1-x} \left. \frac{z^2}{2} \right|_{z=0}^{z=1-x-y} dydx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y)^2\, dydx$$

$$= \frac{1}{2} \int_0^1 \left. \left[-\frac{(1-x-y)^3}{3} \right] \right|_0^{1-x} dx$$

$$= \frac{1}{6} \int_0^1 (1-x)^3\, dx = \frac{1}{6} \left. \left[-\frac{(1-x)^4}{4} \right] \right|_0^1 = \frac{1}{24}$$



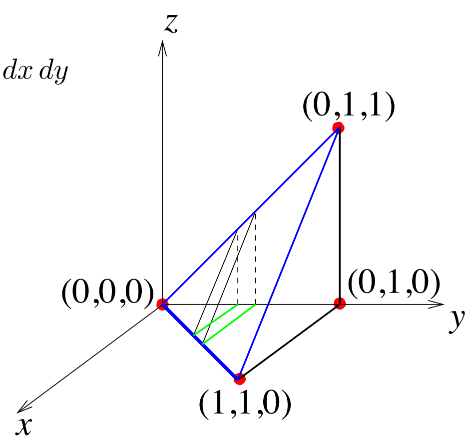
Ex. 5.4 $\displaystyle\iiint_B xz\, dV$, where B is the solid tetrahedron with vertices $(0,0,0)$, $(0,1,0)$, $(1,1,0)$ and $(0,1,1)$.

$$V = \iiint_B xz\, dV = \int_0^1 \int_0^y \int_0^{y-x} xz\, dz\, dx\, dy = \int_0^1 \int_0^y \left. \frac{1}{2} xz^2 \right|_0^{y-x} dx\, dy$$

$$= \int_0^1 \int_0^y (xy^2 - 2x^2y + x^3)\, dx\, dy = \frac{1}{120}.$$

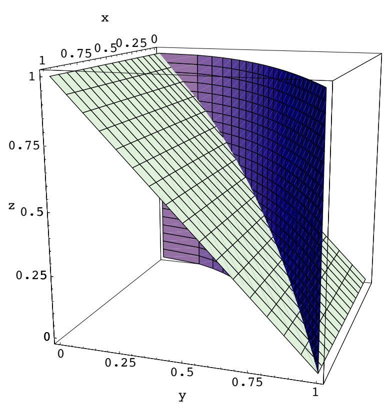
or

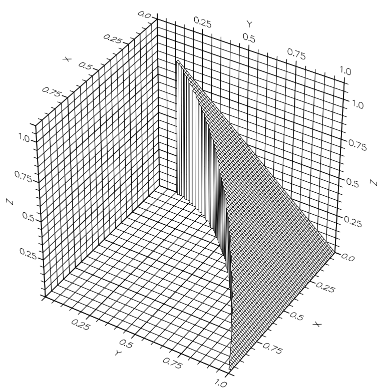
$$V = \int \int \int \quad xz\, dz\, dy\, dx$$



Ex. 5.5 Rewrite the integral $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx$ as an equivalent iterated integral in the order as shown below

$$\int \quad \int \quad \int \quad f(x,y,z)\, dz\, dx\, dy.$$

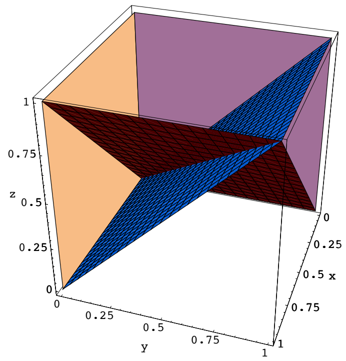


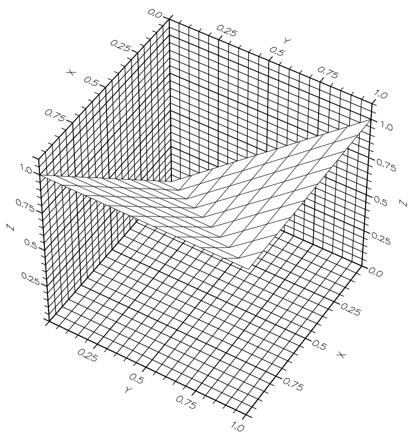


Ex. 5.6 Rewrite the integral $\displaystyle\int_0^1\int_y^1\int_0^z f(x,y,z)\,dx\,dz\,dy$ as an equivalent iterated integral in the order as shown below

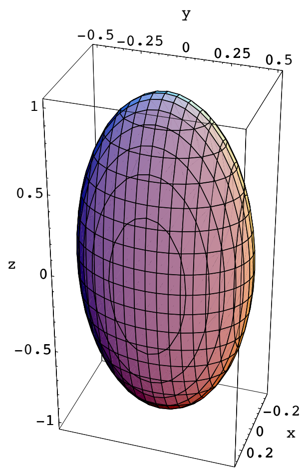
$$\int \quad \int \quad \int \qquad f(x,y,z)\,dy\,dz\,dx.$$

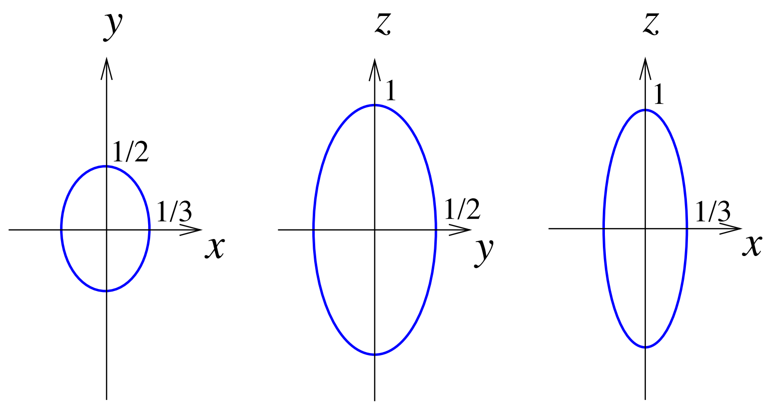
Refer to Nangont





Ex. 5.7 Express the integral $\displaystyle\iiint_B f(x,y,z)\ dV$ as an iterated integral in *six* different ways, where B is the solid bounded by the surface $9x^2+4y^2+z^2=1$.





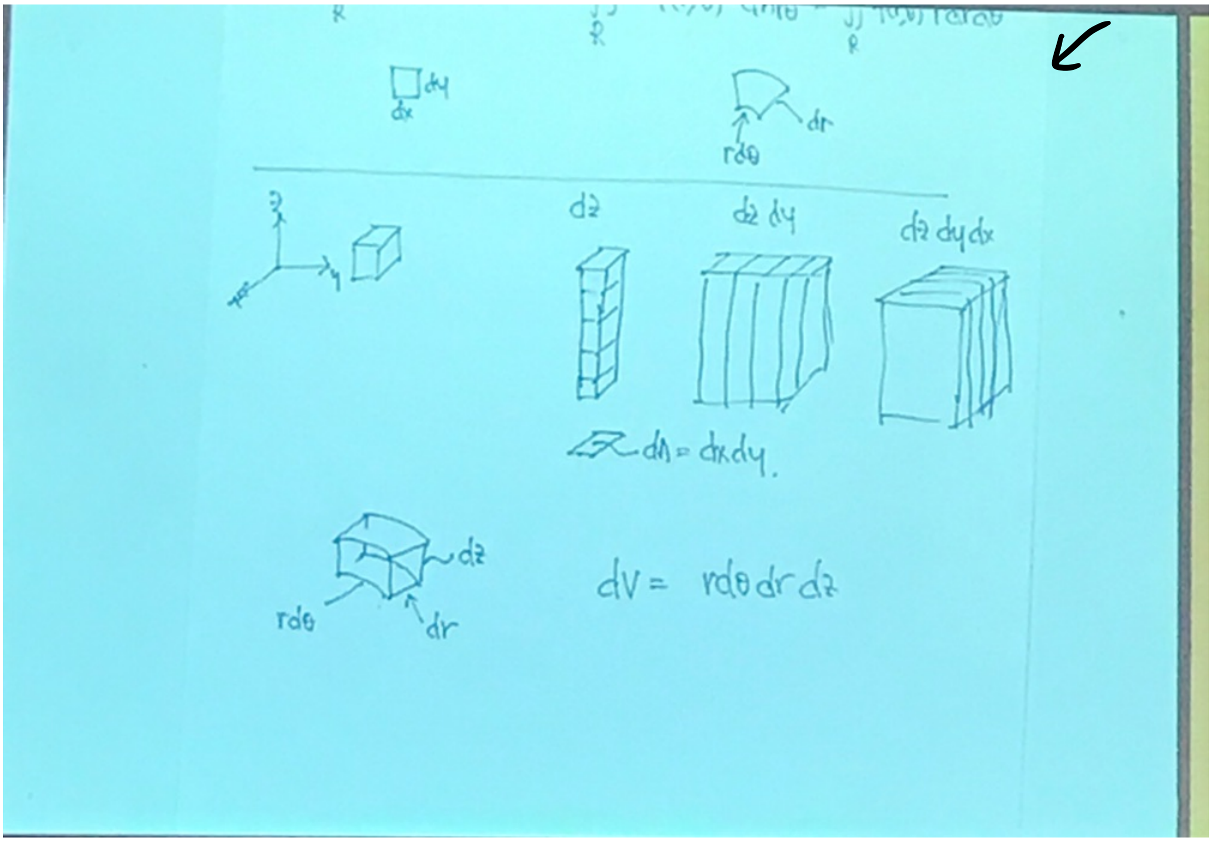
D_1 D_2 D_3

If D_1, D_2 and D_3 are the projections of B on the xy-, yz-, and xz-plane, then

$$D_1=\{(x,y)\,|\,9x^2+4y^2\leqslant 1\}$$
$$D_2=\{(y,z)\,|\,4y^2+z^2\leqslant 1\}$$
$$D_3=\{(x,z)\,|\,9x^2+z^2\leqslant 1\}$$

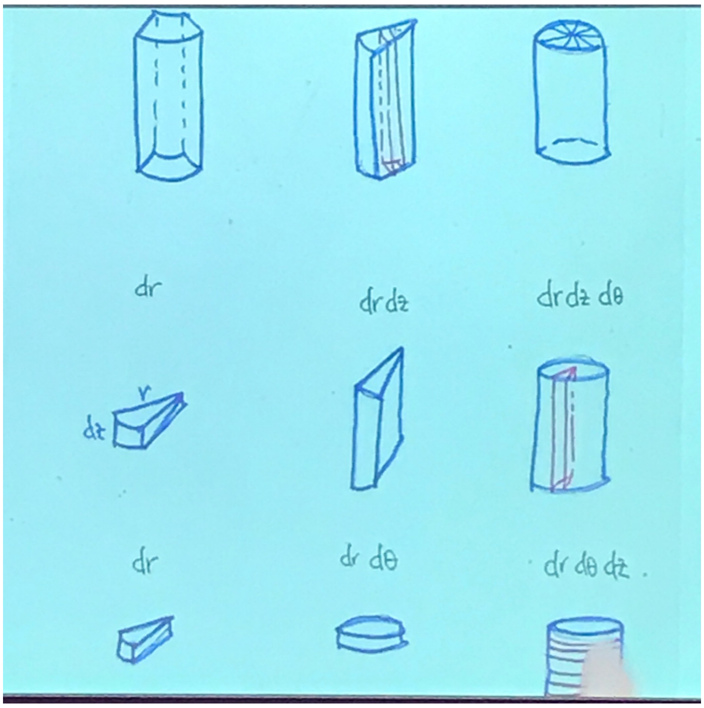
$$\iiint_B f(x,y,z)\,dV=\int \quad \int \quad \int \qquad f(x,y,z)\,dz\,dy\,dx$$

2D In polar/Cartesian



dz dzdr dzdrdθ

rdzdrdθ



14.3 Jacobians

14.6 Change of variables in Multiple integrals

In one-dimensional calculus we often use a change of variable (a substitution) to simplify an integral. Recall the change of variable formula for functions of one variable asserts that if $x = g(u)$ is on $[a, b]$, then

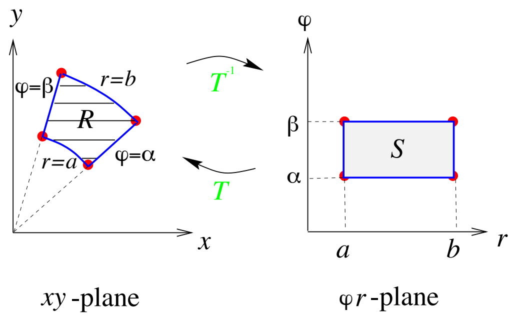
$$\int_b^a f(x)\,dx = \int_c^d f(g(u))g'(u)\,du,$$

where $a = g(c)$ and $b = g(d)$. Note that the change-of-variables process introduces an additional factor $g'(u)$ into the integrand.

A change of variables can also be useful in double (or triple) integrals. We have already seen one example of this: conversion to polar coordinate.

$$\iint\limits_R f(x,y)\,dA_{xy} = \iint\limits_S f(r\cos\theta, r\sin\theta)\,r dr d\theta = \iint\limits_S f(r\cos\theta, r\sin\theta)\,r\,dA_{r\theta},$$

where S is the region in the $r\theta$-plane that corresponds to the region R in the xy-plane.



$$T(r,\theta) = (x,y) = (r\cos\theta, r\sin\theta)$$

The transformation is one-to-one

(i) every point in S gets mapped to a point in R,

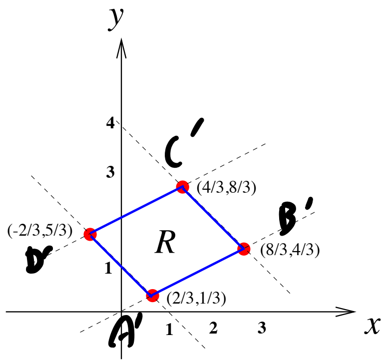
(ii) every point in R is the image of a point in S, and

(iii) different points in S get mapped to different points in R.

Finding a change of variables to simplify a region

Ex. 6.1 Let R be the region bounded by the lines

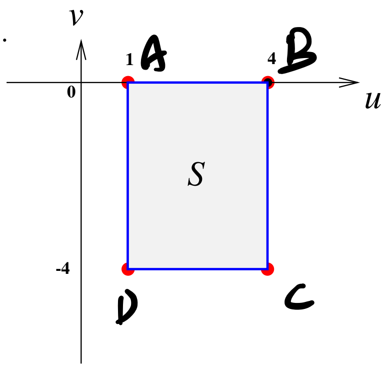
$$x - 2y = 0$$
$$x - 2y = -4$$
$$x + y = 4$$
$$x + y = 1$$



as shown. Find a transformation T from a region S to R such that S is a rectangular region (with sides parallel to the u- and v-axis). $T(S) = R$ or $S = T^{-1}(R)$

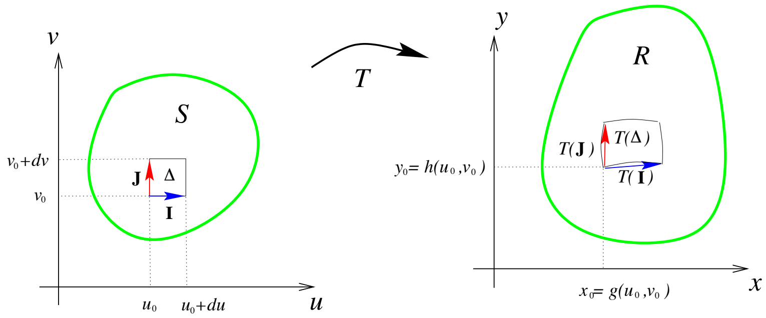
Let $u = x + y$, $v = x - 2y$, then $T(u, v) = (x, y) = \left(\frac{1}{3}(2u + v),\ \frac{1}{3}(u - v)\right)$.

$$v = 0$$
$$v = -4$$
$$u = 4$$
$$u = 1.$$



Note that the transformation T maps the vertices of the region S onto the vertices of the region R.

For $\displaystyle\iint_R f(x,y)\,dxdy$, if we make the change of variables $x = g(u,v)$, $y = h(u,v)$, then first consider the function $T(u,v) = (g(u,v), h(u,v)) = (x,y)$ and assume $T(S) = R$.



Take a small rectangle $\triangle$ (with edges parallel to the u, v axis) at (u_0, v_0) with area $dA = dudv$. Let

$\mathbf{I}$ be the vector from (u_0, v_0) to $(u_0 + du, v_0)$ and

$\mathbf{J}$ be the vector from (u_0, v_0) to $(u_0, v_0 + dv)$. Then

T "takes" $\mathbf{I}$ to the vector $T(\mathbf{I})$ from $(g(u_0, v_0), h(u_0, v_0))$ to $(g(u_0 + du, v_0), h(u_0 + du, v_0))$. Now

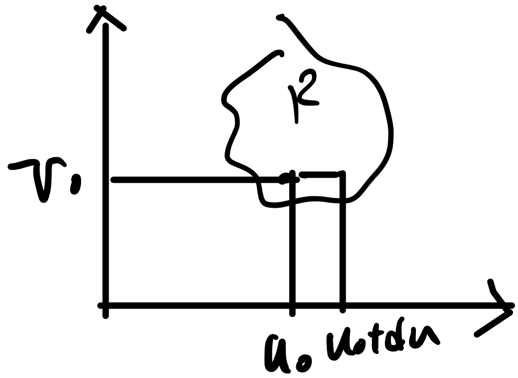
$$\begin{aligned}
T(\mathbf{I}) &= (g(u_0+du, v_0) - g(u_0, v_0),\ h(u_0+du, v_0) - h(u_0, v_0)) \\
&= \left(\frac{g(u_0+du, v_0) - g(u_0, v_0)}{du}, \frac{h(u_0+du, v_0) - h(u_0, v_0)}{du}\right) du \\
&\approx \left(\frac{\partial g}{\partial u}(u_0, v_0),\ \frac{\partial h}{\partial u}(u_0, v_0)\right) du \\
&= \frac{\partial T}{\partial u}(u_0, v_0)\, du.
\end{aligned}$$

Similarly, $T(\mathbf{J}) = \left(\dfrac{\partial g}{\partial v}(u_0, v_0),\ \dfrac{\partial h}{\partial v}(u_0, v_0)\right) dv = \dfrac{\partial T}{\partial v}(u_0, v_0)\, dv$. Then the area of $T(\triangle)$ is

$$dxdy \approx \left\| \left(\frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v}\right) dudv \right\| = \left| \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} dudv \right| = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right| dudv$$

Definition: Jacobian

The **Jacobian** of T is $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$.

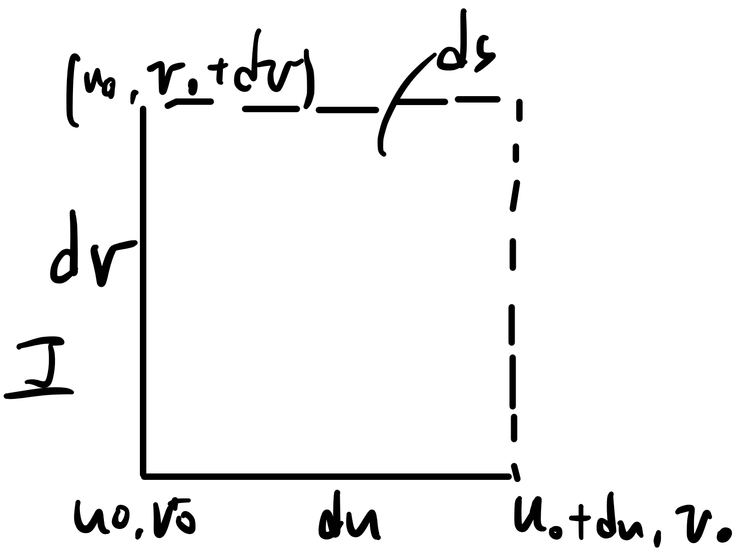


$$T(u,v) = (x,y) = (g(u,v), h(u,v))$$

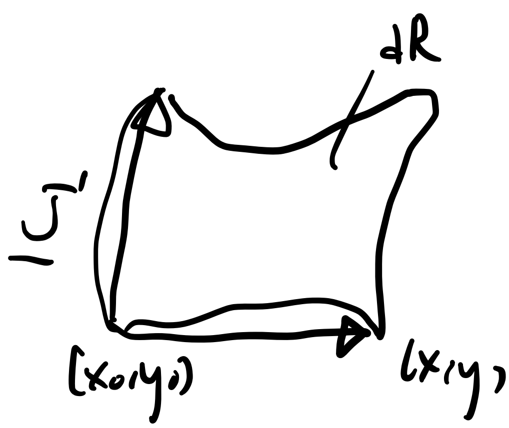
In particular,

$$T(u,v) = (x_0, y_0)$$

$$T(u_0+du, v_0) = (x,y) = (g(u_0+du, v_0), h(u_0+du, v_0))$$



$\underline{\underline{I}}$



$\underline{I'}$

$$\underline{I'} = \left(\frac{g(u_0+du, v_0) - g(u_0, v_0)}{du}, \frac{h(u_0+du, v_0) - h(u_0, v_0)}{du} \right)$$

$$\cong \left(\frac{\partial g}{\partial u}, \frac{\partial h}{\partial u} \right) du$$

Made with Goodnotes

$$\underline{J}' = \left(\frac{\partial g}{\partial v}, \frac{\partial h}{\partial v}\right) dv$$

$$= \left(\frac{\partial g}{\partial u}\frac{\partial h}{\partial v} - \frac{\partial g}{\partial v}\frac{\partial h}{\partial u}\right) du\,dv$$

$$\| \underline{I}' \times \underline{J}' \| = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial h}{\partial u} \\ \frac{\partial g}{\partial v} & \frac{\partial h}{\partial v} \end{vmatrix} du\,dv$$

$$= J\, du\,dv$$

$$dR = J\, dS$$

Computation Formula $\det A \det B = \det(AB)$

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial x}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial x}{\partial v}\dfrac{\partial v}{\partial x} & \dfrac{\partial x}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial x}{\partial v}\dfrac{\partial v}{\partial y} \\ \dfrac{\partial y}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial y}{\partial v}\dfrac{\partial v}{\partial x} & \dfrac{\partial y}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial y}{\partial v}\dfrac{\partial v}{\partial y} \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial y} \\ \dfrac{\partial y}{\partial x} & \dfrac{\partial y}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1\,.
\end{aligned}$$

Theorem

If $x(u,v)$ and $y(u,v)$ have continuous first partial derivatives, and that

$$\frac{\partial(x,y)}{\partial(u,v)} \neq 0 \qquad \text{at} \qquad (u,v) \qquad \text{(one-to-one map)}.$$

Then

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}$$

Change of variable formula for two variables

$$\iint\limits_{R=T(S)} f(x,y)\,dxdy = \iint\limits_{S} f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv.$$

For functions of three variables

$$\iiint\limits_{R=T(S)} f(x,y,z)\,dxdydz = \iiint\limits_{S} f(T(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw,$$

where $T(u,v,w) = (x,y,z)$ is the change of variable function and the **Jacobian** of T is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

Similar statement holds true for functions of n variables.

The examples below show how a change of variables can simplify the integration process. The simplification can occur in various ways. You can make a change of variables to simplify either the region R or the integrand $f(x,y)$, or both.

Ex. 6.2 For polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, then

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

i.e. $dxdy \Rightarrow r\,drd\theta$

For spherical coordinates $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$, then

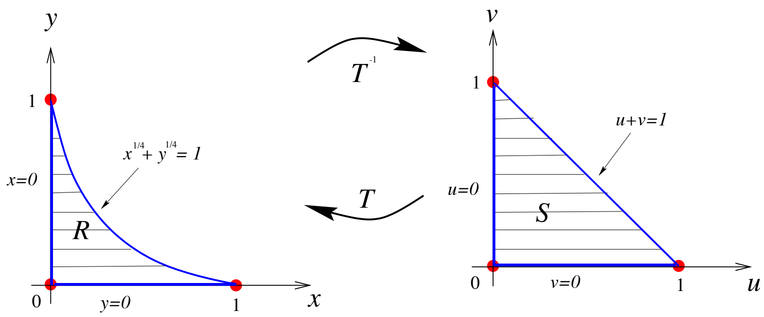
$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$$
$$= \rho^2\sin\phi$$

i.e. $dxdydz \Rightarrow \rho^2\sin\theta\,d\rho d\theta d\phi$

Ex. 6.3 Find the area bounded by $\sqrt[4]{x} + \sqrt[4]{y} = 1$ and the x and y axes.

This integral would be tedious to evaluate directly because the region R is not 'simple'. So instead we find a suitable transformation of variables. Let

Let $u = \sqrt[4]{x}$, $v = \sqrt[4]{y}$, then $x = u^4$, $y = v^4$ and $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 4u^3 & 0 \\ 0 & 4v^3 \end{vmatrix} = 16u^3v^3$



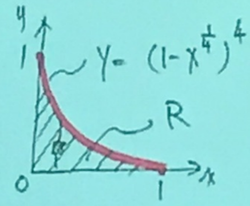
$$\text{Area} = \iint_R dxdy = \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \int_0^1\int_0^{1-v} \underbrace{16u^3v^3}_{J}\,dudv = 4\int_0^1 (1-v)^4 v^3\,dv$$

$$= \frac{1}{70}$$

Ex. 6.3 Find the area bounded by $\sqrt[4]{x}+\sqrt[4]{y}=1$ and the x and y axes.

$x^{\frac{1}{4}}+y^{\frac{1}{4}}=1$ $\qquad$ $y^{\frac{1}{4}} = 1-x^{\frac{1}{4}}$

$y=\left(1-x^{\frac{1}{4}}\right)^4$

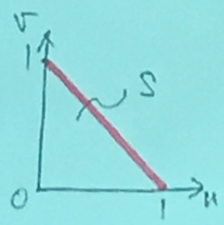


$$A = \int_0^1\int_0^{(1-x^{\frac{1}{4}})^4} dy\, dx$$

$$= \int_0^1 \left(1-x^{\frac{1}{4}}\right)^4 dx \qquad x^{\frac{1}{4}}=\sin^2\theta \qquad (1-\sin^2\theta)^4 = \cos^8\theta$$

Let $u=x^{\frac{1}{4}}$, $v=y^{\frac{1}{4}}$, $x^{\frac{1}{4}}+y^{\frac{1}{4}}=1 \Rightarrow u+v=1$

$$\iint_R dA_{xy} = \iint_S J\, dA_{uv}$$



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}$$

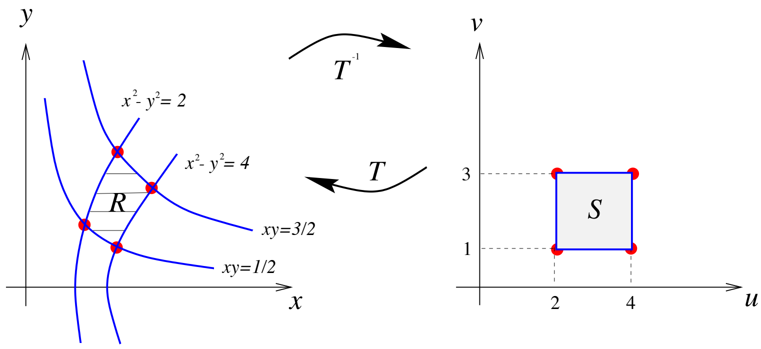
$x=f(u,v)$
$y=g(u,v)$

$u=h(x,y)$
$v=k(x,y)$

Ex. 6.4 Find $\displaystyle\iint_A (x^2+y^2)\,dxdy$, where $A = \left[(x,y)\,\middle|\, x,y>0,\quad 2\leqslant x^2-y^2\leqslant 4,\quad \frac{1}{2}\leqslant xy\leqslant \frac{3}{2}\right]$

$S = [\,(u, v)\mid 2 \le u \le 4,\ \frac{1}{2} \le v \le \frac{3}{2}\,]$

The change of the variables is motivated by the occurrence of the expressions x^2-y^2 and xy in the equations of the boundary.



Let $u=x^2-y^2$, $v=2xy$, then $(x^2+y^2)^2=(x^2-y^2)^2+4x^2y^2=u^2+v^2$ and

$$\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}=\frac{1}{\begin{vmatrix}2x & -2y\\ 2y & 2x\end{vmatrix}}=\frac{1}{4(x^2+y^2)}=\frac{1}{4\sqrt{u^2+v^2}}.$$

So $\displaystyle\iint_R (x^2+y^2)\,dxdy =$ $\displaystyle\int_{v=1}^{3}\int_{u=2}^{4}\sqrt{u^2+v^2}\,\frac{1}{4\sqrt{u^2+v^2}}\,du\,dv$

$= \frac{1}{4}(4-2)(3-1) = 1$

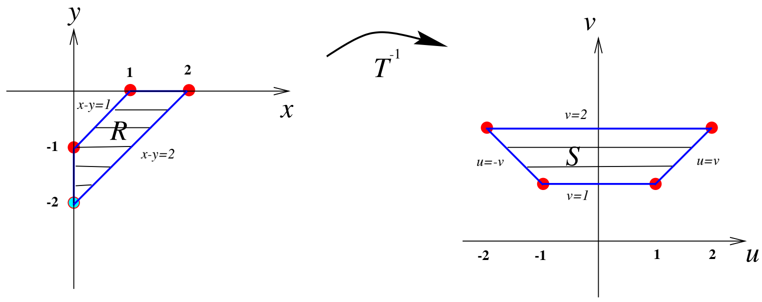
Ex. 6.5 Evaluate the integral $\iint\limits_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$.

Since it is not easy to integrate $e^{(x+y)/(x-y)}$, we make a change of variables suggested by a form of the integrand. In particular, let

$$u = x + y, \qquad v = x - y.$$

These equations define a transformation T^{-1} from the xy-plane to the uv-plane.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}} = -\frac{1}{2}.$$



The sides of R lie on the lines

$$y = 0, \qquad x - y = 2, \qquad x = 0, \qquad x - y = 1$$

and the image lines in the uv-plane are

$$u = v, \qquad v = 2, \qquad u = -v, \qquad v = 1.$$

$$\begin{aligned} \therefore \quad \iint\limits_R e^{(x+y)/(x-y)}\,dA &= \iint\limits_S e^{u/v} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv \\ &= \int_1^2 \int_{-v}^{v} e^{u/v} \left(\frac{1}{2}\right) dudv \\ &= \end{aligned}$$

Change of variables in triple integrals

Ex. 6.6 Find the volume of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

Let $x=au$, $y=bv$ and $z=cw$, then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \Rightarrow \qquad u^2+v^2+w^2=1 \quad \text{(sphere)}$$

and

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}=abc.$$

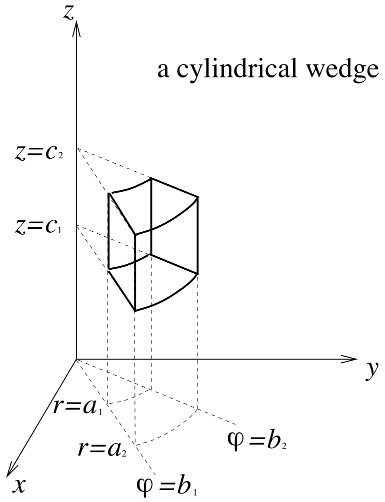
Now S is the region in uvw-space enclosed by the sphere $u^2+v^2+w^2=1$, so

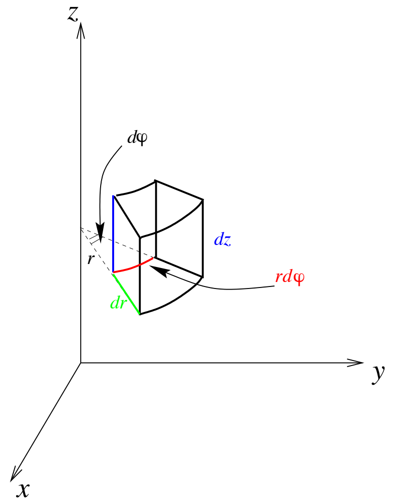
$$\iiint_R dV=\iiint_S abc\,dV=abc\,\frac{4}{3}\pi(1)^3=\frac{4}{3}\pi abc.$$

Triple integrals in cylindrical & spherical coordinates

Cylindrical: Cylindrical coordinates are suited to problems with axial symmetry (around the z-axis).

$$V_c=\iiint_V f\,dV=\iiint_{V(x,y,z)} f(x,y,z)\,dx\,dy\,dz=\iiint_{V(r,\theta,z)} f(r,\theta,z)\,r\,dr\,d\theta\,dz.$$





For cylindrical coordinates $x = r\sin\theta$, $y = r\cos\theta$ and $z = z$, then

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

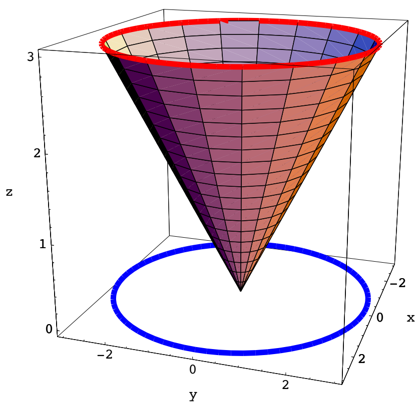
i.e. $dxdydz \Rightarrow r\,drd\theta dz$.

Ex. 6.7 Find the volume of a circular cone with altitude h and with a base of radius a.

Equation of the cone:

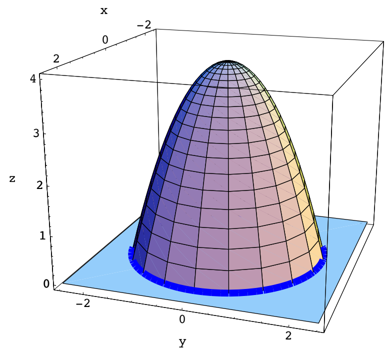
$$z = \frac{h}{a}r$$

where $0 \leqslant r \leqslant a$, $0 \leqslant z \leqslant h$.



$$\begin{aligned} V &= \iint\limits_R \left[\int_{\frac{h}{a}r}^{h} dz\right] dA \\ &= \int_0^{2\pi}\int_0^a\int_{\frac{h}{a}r}^{h} r\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^a rh\left(1-\frac{r}{a}\right) dr\,d\theta \\ &= h\int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^3}{3a}\right)\bigg|_0^a d\theta \\ &= h\frac{1}{6}a^2 2\pi = \frac{1}{3}\pi a^2 h \end{aligned}$$

Ex. 6.8 Find the volume of the region bounded by the paraboloid $z = 4 - x^2 - y^2$ and above the xy-plane.



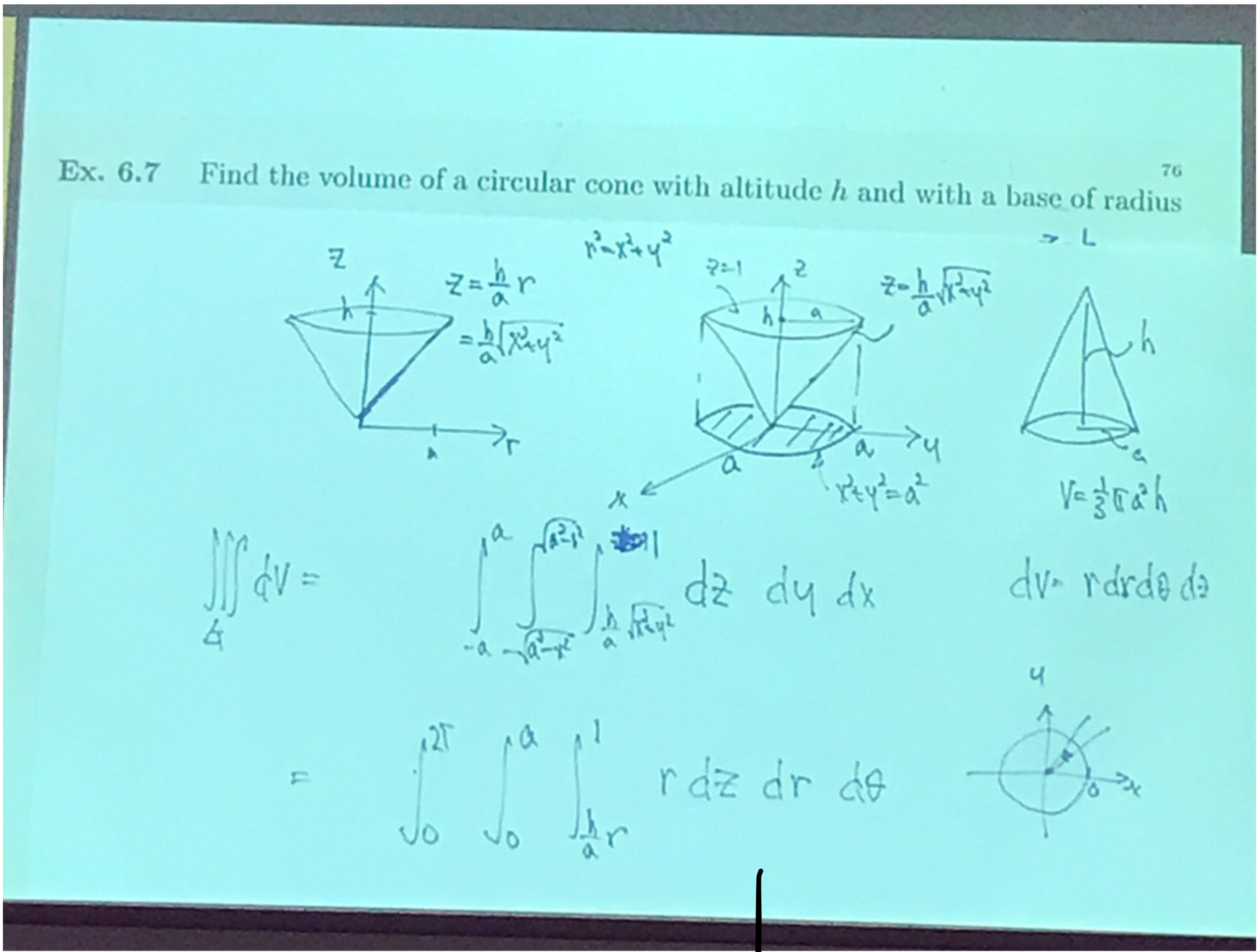
$$V = \iiint\limits_B dV = \iint\limits_R \int_0^{z(x,y)} dzdA = \iint\limits_R z(x,y)\,dA$$

$$= \iint\limits_R (4-x^2-y^2)\,dx\,dy$$

係呢到轉 polar

$$= \int_0^{2\pi}\int_0^2 (4-r^2)\,rdr\,d\theta = \int_0^{2\pi}\left(2r^2-\frac{r^4}{4}\right)\bigg|_0^2 d\theta = 8\pi$$

6.7



cylindrical coordinate system

Exercises for students

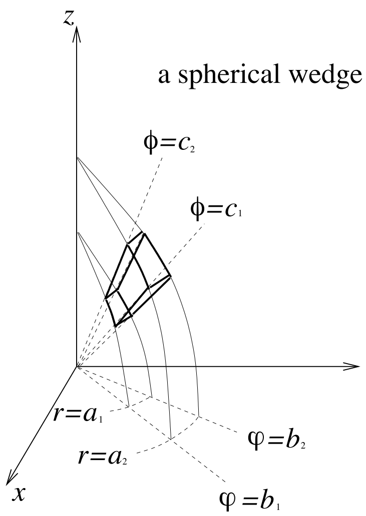
Sketch the solid whose volume is given by the integral and evaluate the integral

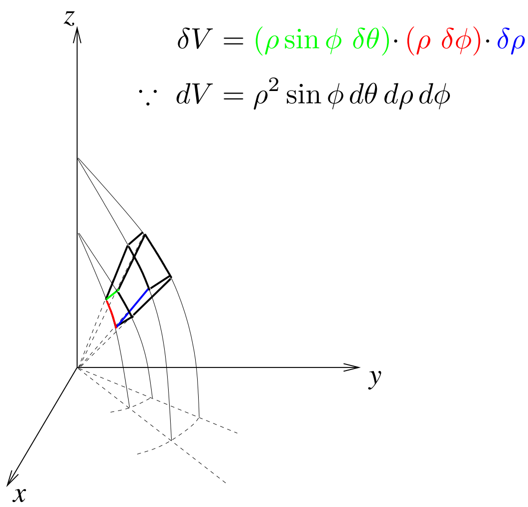
(a) $\displaystyle\int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} r\,dz\,dr\,d\theta$

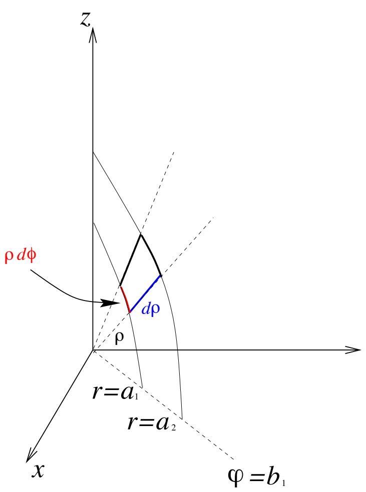
(b) $\displaystyle\int_1^{3}\int_0^{\frac{\pi}{2}}\int_r^{3} r\,dz\,d\theta\,dr$

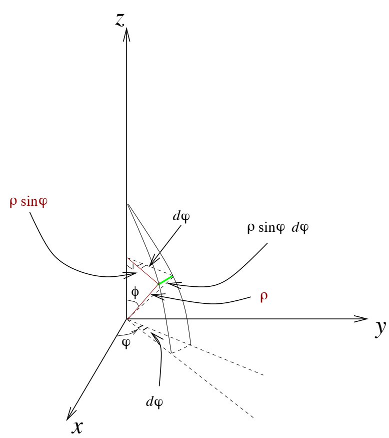
Spherical: Spherical coordinates are suited to problems involving spherical symmetry, and in particular, to regions bounded by spheres centred at the origin, circular cones with axes along the z-axis, vertical planes containing the z-axis.

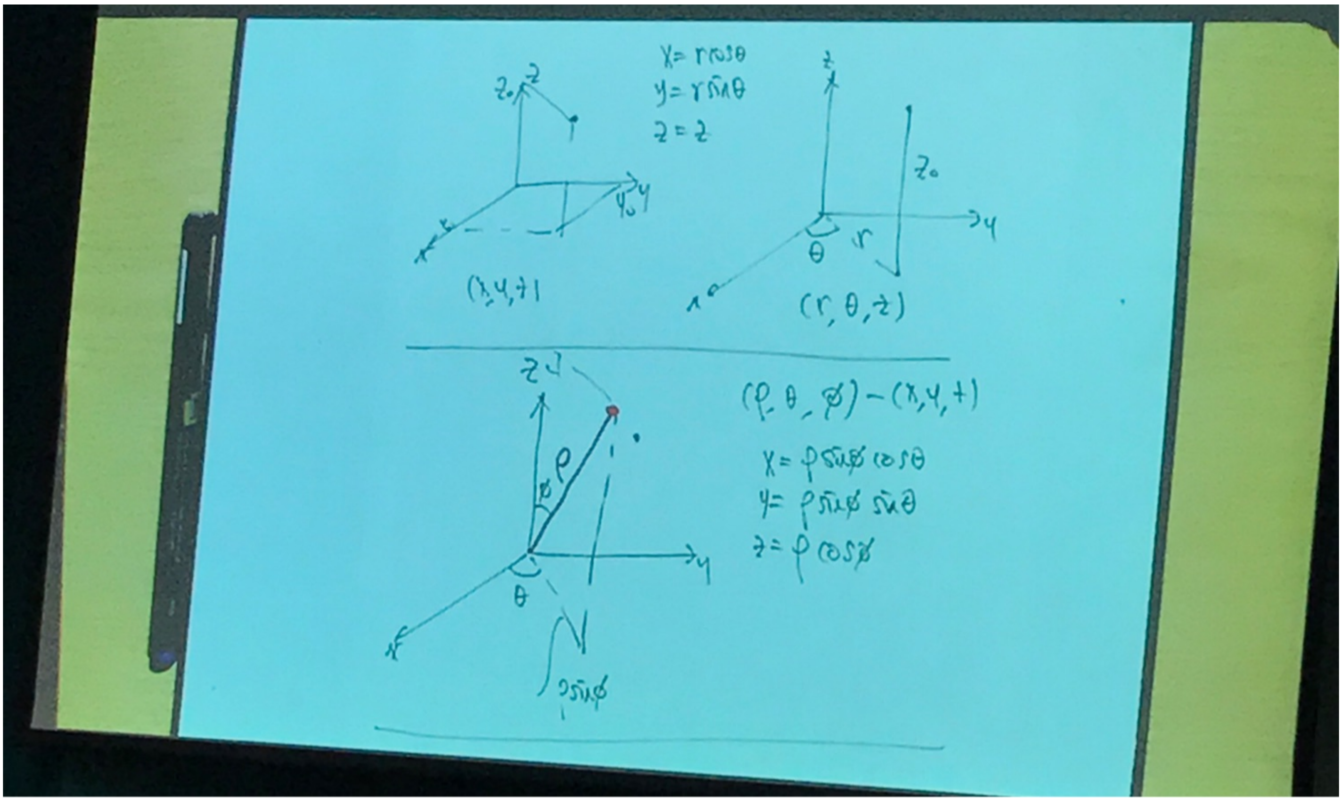
$$V_s = \iiint\limits_V f\,dV = \iiint\limits_{V(\rho,\theta,\phi)} f(\rho,\theta,\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$$

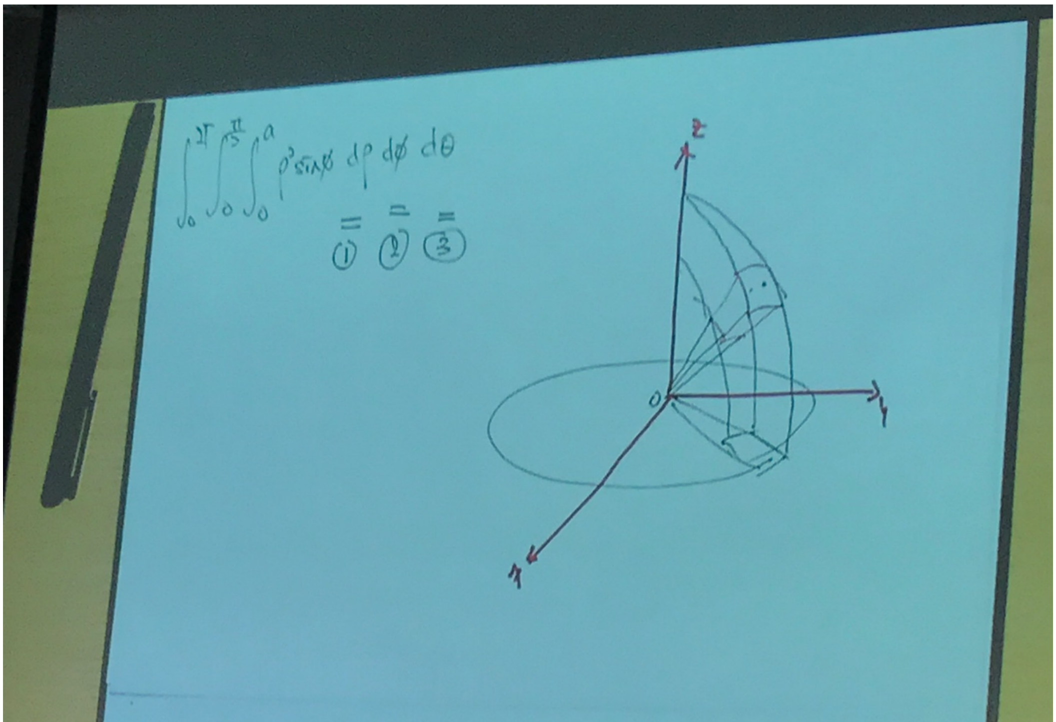









x = r cos θ
y = r sin θ
z = z
(x, y, z)
(r, θ, z)
(ρ, θ, φ) – (x, y, z)
x = ρ sin φ cos θ
y = ρ sin φ sin θ
z = ρ cos φ


$\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{a} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$
① ② ③

$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^a \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

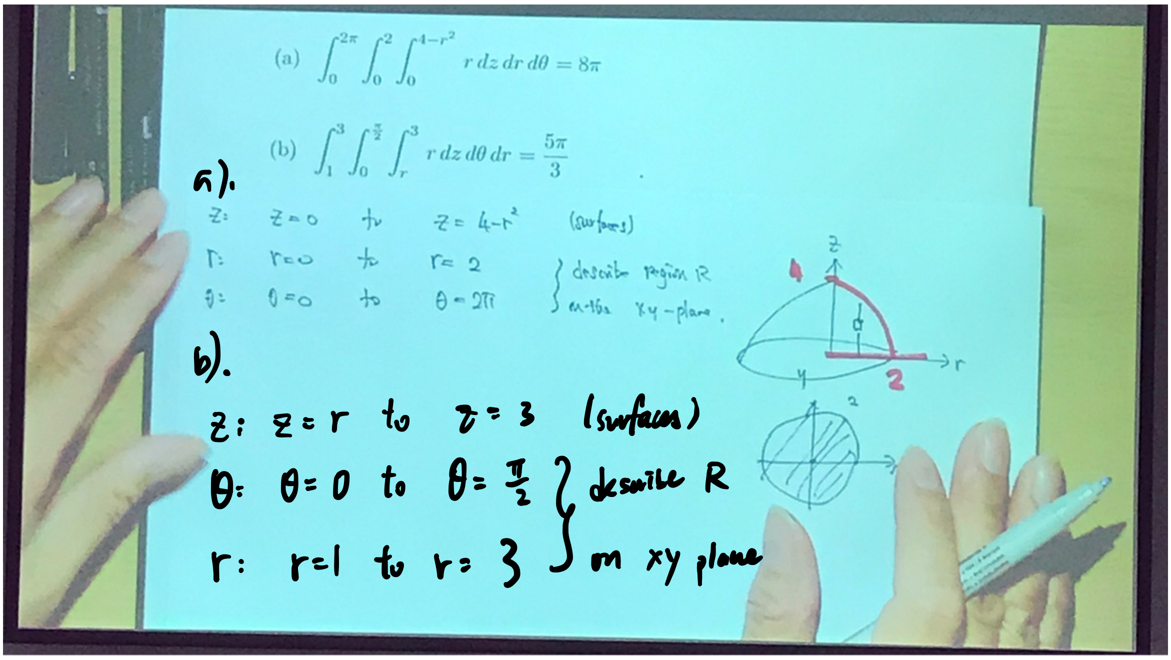
$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \frac{\rho^3}{3}\Big|_0^a \sin\phi \, d\theta \, d\phi$$

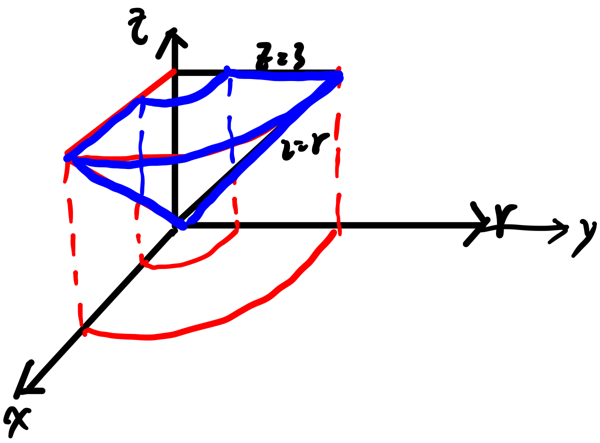
$$= \frac{a^3}{3}\int_0^{\frac{\pi}{2}} \sin\phi \, \theta\Big|_0^{2\pi} \, d\phi$$

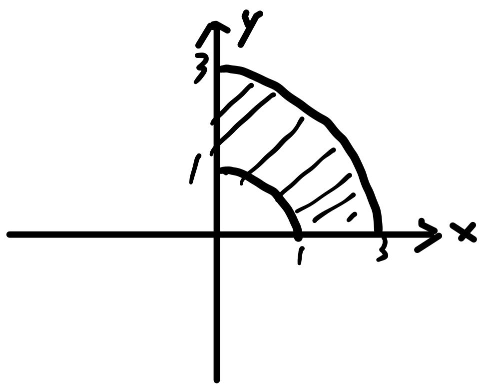
$$= \frac{2\pi a^3}{3}\int_0^{\frac{\pi}{2}} \sin\phi \, d\phi$$

$$= \frac{2\pi a^3}{3} \cos\phi\Big|_0^{\frac{\pi}{2}} = \frac{2\pi a^3}{3} \times 2$$

Ex for student (a).







For spherical coordinates $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$ and $z = \rho\cos\phi$, then

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} = \rho^2\sin\phi$$

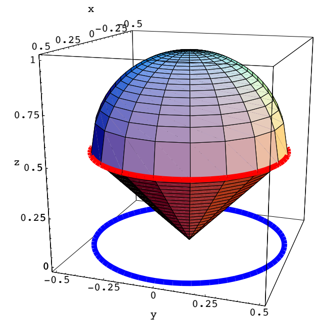
i.e. $dxdydz = \rho^2 \sin\phi \, d\rho d\theta d\phi$.

Ex. 6.9 Find the volume of the sphere with radius a.

$$\begin{aligned} \int\int\int \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi &= \int \sin\phi \, d\phi \cdot \int d\theta \cdot \int \rho^2 \, d\rho \\ &= (2)\cdot(2\pi)\cdot\left(\frac{a^3}{3}\right) = \frac{4}{3}\pi a^3. \end{aligned}$$

The examples below show the choice of coordinate system can greatly affect the difficulty of computation of a multiple integral.

Ex. 6.10 Find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and between the sphere $x^2+y^2+z^2 = z$.



The equation of sphere can be written as

$$x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \Rightarrow \quad \text{centre } \left(0,0,\frac{1}{2}\right), \text{ radius } = \frac{1}{2}.$$

In spherical coord. $x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta \quad$ and $\quad z = \rho\cos\phi$.

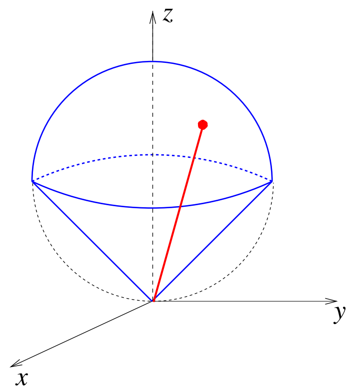
Equation of the sphere: $\rho^2 = \rho\cos\phi \quad \Rightarrow \quad \rho = \cos\phi$

Equation of the cone: $\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta} = \rho\sin\phi$

$$\cos\phi = \sin\phi \quad \Rightarrow \quad \phi = \pi/4.$$

$$\therefore B = \{(\rho,\theta,\phi)\,|\,\theta \in [\qquad\qquad],\ \phi \in [\qquad\qquad],\ \rho \in [\qquad\qquad]\}$$

$$\begin{aligned} V = \iiint_B dV &= \int\quad\int\quad\int \qquad \rho^2\sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int \quad d\theta \quad \int \sin\phi \left[\frac{\rho^3}{3}\right] \qquad d\phi \\ &= \frac{2\pi}{3}\int \quad \sin\phi\cos^3\phi\, d\phi \\ &= \frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8}. \end{aligned}$$

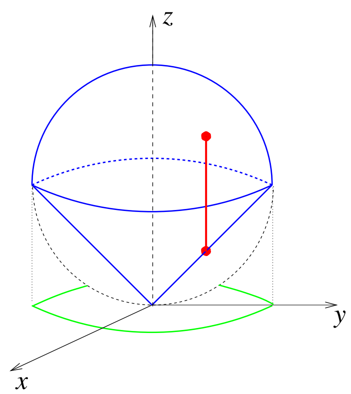


Alternatively, *in cylindrical coord.* $x = r\cos\theta, \quad y = r\sin\theta \quad$ and $\quad z = z$.

Equation of the cone: $z = r$.

Equation of the sphere: $z^2 - z = -r^2 \quad \Rightarrow \quad z = \frac{1}{2} \pm \sqrt{\frac{1}{4}-r^2}$

$$\begin{aligned} V &= \int\quad\int\quad\int \qquad r\, dz\, dr\, d\theta \\ &= 2\pi\int \quad rz\Big|_r^{\frac{1}{2}+\sqrt{\frac{1}{4}-r^2}} \quad dr \\ &= 2\pi\int \quad \left(\frac{1}{2}r + r\sqrt{\frac{1}{4}-r^2} - r^2\right) dr \\ &= 2\pi\left[\frac{1}{4}r^2 - \frac{r^3}{3} - \frac{1}{3}\left(\frac{1}{4}-r^2\right)^{3/2}\right]\Bigg|_0^{\frac{1}{2}} \\ &= \frac{\pi}{8} \end{aligned}$$



Ex. 6.11 Find the volume of the solid that the cylinder $r = a\cos\theta$ cut out of the sphere of radius a centered at the origin. (see also ex. 7.3)

Equation of sphere: $x^2 + y^2 + z^2 = a^2 \quad \Rightarrow \quad z^2 + r^2 = a^2$ (Cylindrical)

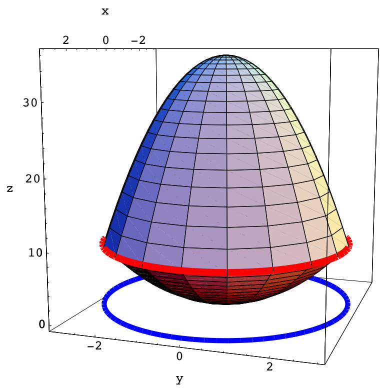
Equation of cylinder: $r = a\cos\theta \quad \Rightarrow \quad r^2 = ar\cos\theta \quad \Rightarrow \quad x^2 + y^2 = ax$ (in Cartesian)

$$B = \left\{ (r, \theta, z) \,\middle|\, z \in \left[-\sqrt{a^2 - r^2}, \sqrt{a^2 - r^2}\right],\ r \in [0, a\cos\theta],\ \theta \in [-\pi/2, \pi/2] \right\}$$

$$\begin{aligned}
V = \iiint_B dV &= \int \int \int \quad r\,dz\,dr\,d\theta \\
&= 4 \int \int \int \quad r\,dz\,dr\,d\theta \\
&= 4 \int \int \quad r\sqrt{a^2 - r^2}\,dr\,d\theta \\
&= -\frac{4}{3} \int \quad (a^3 \sin^3\theta - a^3)\,d\theta \\
&= \frac{2}{9} a^3 (3\pi - 4)
\end{aligned}$$

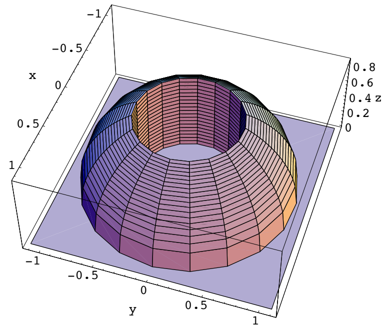
Ex. 6.12 Find the volume of the region B bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - 3x^2 - 3y^2$.

Ans. $V = 162\pi$



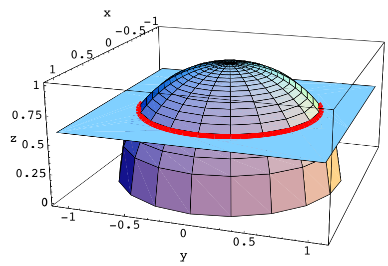
Ex. 6.13 Find the volume of the region B bounded below by the xy-plane, inside the sphere $r^2+z^2=a^2$ and outside the cylinder $r=a/2$.

Ans. $V=\sqrt{3}\pi a^3/4$.



Ex. 6.14 Find the volume of the region B bounded below by the plane $z=b$ and above by the sphere $\rho=a$ $(b\leqslant a)$.

Ans. $V=\pi[2a^3+b^3-3a^2b]/3$.



Ex. 6.15 Show that $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}\;e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz=2\pi$

[Hint: In spherical coord.]

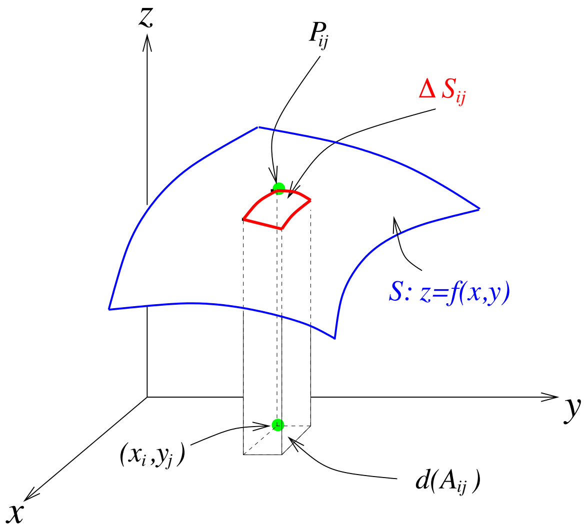
14.7 Applications

Surface area

Let S be a surface with equation $z = f(x,y)$, where f has continuous partial derivatives. Assume $f(x,y) \geqslant 0$ and the domain $R = [a,b] \times [c,d]$ of f is a rectangle, and $x_i \in [a,b]$ and $y_j \in [c,d]$.

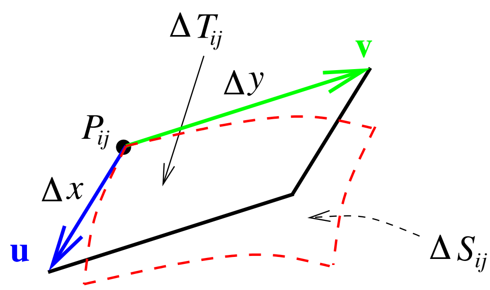
The tangent plane to S at P_{ij} is an approximation to S near P_{ij}.

$\therefore$ Area $S \simeq \sum_{i=1}^{m} \sum_{j=1}^{n} \triangle T_{ij}$ as $m,n \to \infty$.



Let $\mathbf{u}$ and $\mathbf{v}$ be the vectors that start at P_{ij} and lie along the sides of the parallelogram with area $\triangle T_{ij}$. Then

$$\triangle T_{ij} = \|\mathbf{u} \wedge \mathbf{v}\|$$



Note

$$\mathbf{u} = \triangle x_i\,\mathbf{i} + f_x(x_i,y_j)\triangle x_i\,\mathbf{k}$$

$$\mathbf{v} = \triangle y_j\,\mathbf{j} + f_y(x_i,y_j)\triangle y_j\,\mathbf{k}$$

and

$$\mathbf{u} \wedge \mathbf{v} = [-f_x(x_i,y_j)\,\mathbf{i} - f_y(x_i,y_j)\,\mathbf{j} + \mathbf{k}]\,\triangle x_i\ \triangle y_j$$

$$\triangle T_{ij} = \|\mathbf{u} \wedge \mathbf{v}\| = \left[f_x^2(x_i,y_j) + f_y^2(x_i,y_j) + 1\right]^{\frac{1}{2}}\ \triangle A_{ij}$$

$$\therefore\quad A(S) = \sum_{i=1}^{m}\sum_{j=1}^{n} \sqrt{f_x^2 + f_y^2 + 1}\,\triangle A_{ij} \quad \text{as} \quad m,n \to \infty \quad \text{and} \quad \triangle x_i, \triangle y_j \to 0$$

$$= \iint_R \sqrt{f_x^2 + f_y^2 + 1}\,dA$$

Surface integrals in parametric form

A surface S is represented by $\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}$, then

$$A(S) = \iint_R \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA.$$

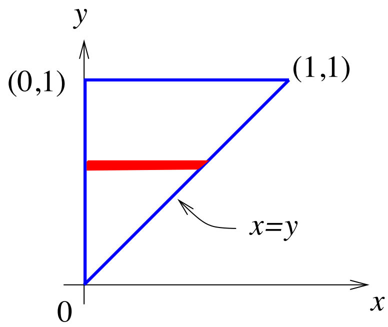
Ex. 7.1 Find the area of the surface $z = x + y^2$ that lies above the triangle with vertices $(0,0)$, $(1,1)$ and $(0,1)$.

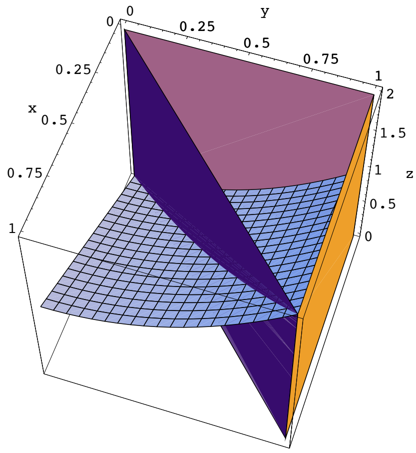
$$S = \iint_R \sqrt{z_x^2 + z_y^2 + 1}\, dA = \int_0^1 \int_0^y \sqrt{1 + 4y^2 + 1}\, dx\, dy$$

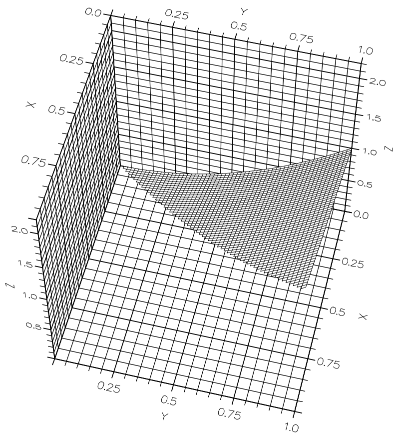
$$= \int_0^1 y\sqrt{2 + 4y^2}\, dy$$

$$= \frac{2}{24}(2 + 4y^2)^{\frac{3}{2}}\Big|_0^1$$

$$= \frac{1}{6}(3\sqrt{6} - \sqrt{2}).$$







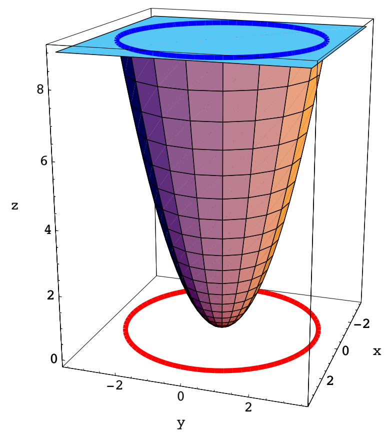
Ex. 7.2 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

$$S = \iint_R \sqrt{z_x^2 + z_y^2 + 1}\ dA = \iint_R \sqrt{(2x)^2 + (2y)^2 + 1}\ dA$$

$$= \iint_R \sqrt{4(x^2 + y^2) + 1}\ dA$$

Converting to polar coordinates, we obtain

$$S = \int \quad \int \quad \sqrt{4r^2 + 1}\ r dr d\theta$$



This example illustrated the formula for change of variables from rectangular coord. (x, y) to polar coord. (r, θ).

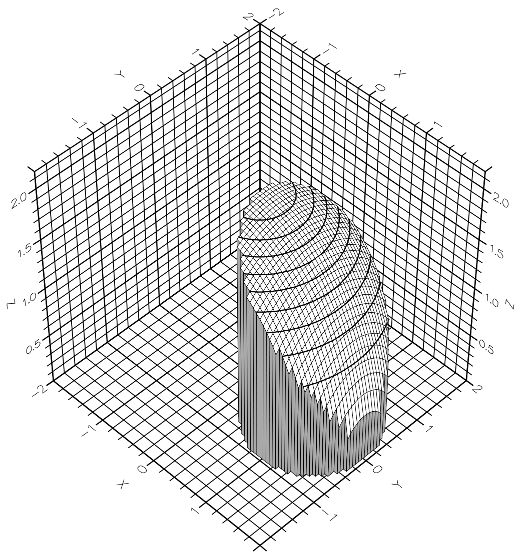
Ex. 7.3 Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies within the cylinder $x^2 + y^2 = ax$ and above the xy-plane.

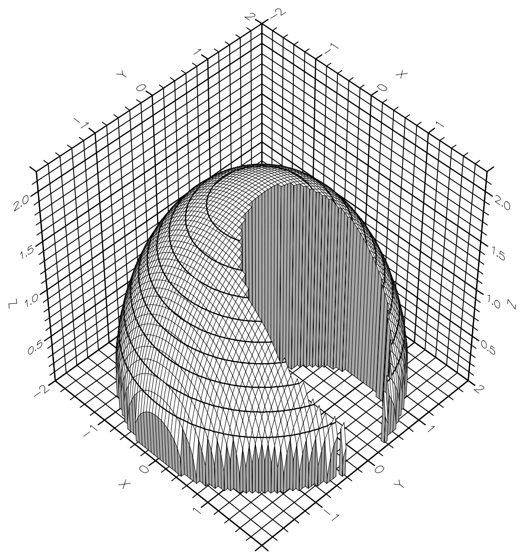
$$x^2 + y^2 = ax \quad \Rightarrow \quad \left(x - \frac{1}{2}a\right)^2 + y^2 = \left(\frac{1}{2}a\right)^2$$

and in polar coord. $\quad r = a\cos\theta$.

$$\begin{aligned} z &= \sqrt{a^2 - x^2 - y^2} \\ z_x &= -x(a^2 - x^2 - y^2)^{-\frac{1}{2}} \\ z_y &= -y(a^2 - x^2 - y^2)^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} S &= \iint_R \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1}\, dA \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{a\cos\theta} \sqrt{\frac{r^2}{a^2 - r^2} + 1}\; r dr\, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{a\cos\theta} \frac{ar}{\sqrt{a^2 - r^2}}\, dr\, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[-a\sqrt{a^2 - r^2}\right]\Bigg|_0^{a\cos\theta} d\theta \\ &= 2a^2 \int_0^{\frac{\pi}{2}} \left[1 - \sqrt{1 - \cos^2\theta}\right] d\theta \\ &= a^2(\pi - 2). \end{aligned}$$





This example illustrated the formula for change of variables from rectangular coord. (x, y) to polar coord. (r, θ).

Exercises for students

Ex. 1 Find the volume of the region B bounded below by the surface $z = x^2 + y^2$ and above by the plane $z = 2y$.

Ex. 2 Find the volume of the solid that lies under the cone $z = \sqrt{x^2 + y^2}$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Ex. 3 Rewrite the integral $\int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} f(x, y, z)\, dz\, dy\, dx$ as an iterated integral in the order $dx\, dy\, dz$.

Ex. 4 Sketch the region G in the first octant of 3-space that has finite volume and is bounded by surfaces $z = 0$, $x = 0$, $y = 2x$ and $z = 4 - y^2$. Write *six* different iterations of the triple integral of $f(x, y, z)$ over G.

Ex. 5 Convert the following cylindrical integral to equivalent iterated integrals in (a) cartesian coordinates and (b) spherical coordinates:

$$\int_0^{2\pi} \int_0^1 \int_0^{\sqrt{9-r^2}} r\, dz\, dr\, d\theta.$$

Evaluate the easiest of the three iterated integrals.

Ex. 6 Evaluate

$$\int_0^6 \int_{-2y}^{1-2y} y^3 (x + 2y)^2 e^{(x+2y)^3}\, dx\, dy$$

by making a suitable change of variables.

Ex. 7 Evaluate

$$\iint_D \frac{xy}{y^2 - x^2}\, dA,$$

where D is the region in the first quadrant bounded by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 4$ and the ellipses $x^2/4 + y^2 = 1$, $x^2/16 + y^2/4 = 1$. (Hint: Sketch the region D, and use it to make an appropriate change of variables.)

Ex. 8 Calculate $\iint_D \ln \sqrt{x^2 + y^2}\, dA$, where D is the unit disk $x^2 + y^2 \leqslant 1$.

Exercises for students (Ans)

Ex. 1 $\frac{4}{3} \times \frac{3}{8}\pi = \frac{1}{2}\pi$

Ex. 2 $\frac{32}{9}$

Ex. 3 $\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$

Ex. 5 $2\pi\left(9 - \frac{16\sqrt{2}}{3}\right)$

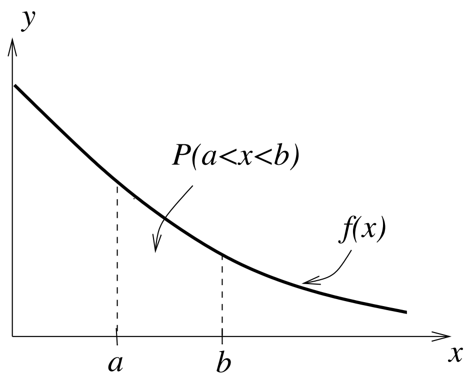
Ex. 6 $108(e-1)$

Ex. 7 $-\frac{3}{5}\ln 4$

Ex. 8 $-\frac{\pi}{2}$

Probability Density Functions (Optional)

The life span x of an electronic component selected at random from a manufacturer's stock is a quantity that cannot be predicted with certainty. The **probability** that the life span x lies in some interval $a \leqslant x \leqslant b$ is denoted by $P(a \leqslant x \leqslant b)$ and is defined to be the fraction of all the components manufactured by the company that can be expected to have life spans in this range. A probability of the form $P(a \leqslant x \leqslant b)$ can be interpreted as the area under a certain curve and evaluated as a definite integral. The curve is the graph of a function $f(x)$ known as the **probability density function** for x. More precisely, a probability density function for the variable x is a positive continuous function $f(x)$ with the property that $P(a \leqslant x \leqslant b)$ is the area under its graph between $x = a$ and $x = b$. The situation is illustrated in figure below.

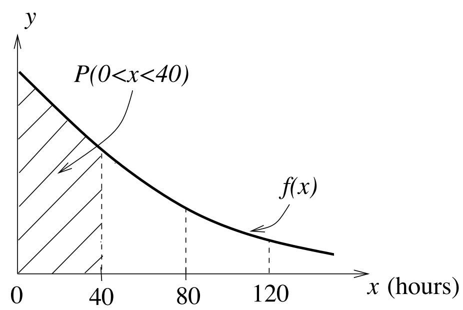


Probability density functions are constructed by statisticians from experimental data and theoretical considerations using techniques explained in most probability and statistics texts.

A possible probability density function for the life span of a light bulb is sketched below

$$P(0 \leqslant x \leqslant 40) = \int_0^{40} f(x)dx$$

Its shape reflects the fact that most bulbs burn out relatively quickly. For example, the probability that a bulb will fail within the first 40 hours is represented by the area under the curve between $x = 0$ and $x = 40$.



Joint Probability Density Functions

Recall that a probability density function for a single variable x is a nonnegative function $f(x)$ such that the probability that x is between a and b is given by the formula

$$P(a \leqslant x \leqslant b) = \int_a^b f(x)\,dx$$

In geometric terms, the probability $P(a \leqslant x \leqslant b)$ is the area under the graph of f from $x = a$ to $x = b$.

In situations involving two variables, you compute probabilities by evaluating double integrals of a two-variable density function. In particular, you integrate a **joint probability density function**, which is a non-negative function $f(x, y)$ such that the probability that x is between a and b and y is between c and d is given by the formula

$$\begin{aligned} P(a \leqslant x \leqslant b \text{ and } c \leqslant y \leqslant d) &= \int_c^d \int_a^b f(x, y)\,dx\,dy \\ &= \int_a^b \int_c^d f(x, y)\,dy\,dx. \end{aligned}$$

More generally, the probability that the ordered pair (x, y) lies in a region R is given by the formula

$$P[(x, y) \text{ in } R] = \iint_R f(x, y)\,dA$$

In geometric terms, the probability that (x, y) is in R is the volume under the graph of f above the region R.

The techniques for constructing joint probability density functions from experimental data are beyond the scope of this book and are discussed in most probability and statistics texts.

Ex. 7.4 Smoke detectors manufactured by a certain firm contain two independent circuits, one manufactured at the firm's HK plant and the other at the firm's plant in China. Reliability studies suggest that if x denotes the life span (in years) of a randomly selected circuit from the HK plant and y the life span (in years) of a randomly selected circuit from the China plant, the joint probability density function for x and y is $f(x, y) = e^{-x}e^{-y}$. If the smoke detector will operate as long as either of its circuits is operating, find the probability that a randomly selected smoke detector will fail within 1 year.

Solution

Since the smoke detector will operate as long as either of its circuits is operating, it will fail within 1 year if and only if *both* of its circuits fail within 1 year. The desired probability is therefore the probability that both $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. The points (x, y) for which both these inequalities hold form the square R. The corresponding probability is the double integral of the density function f over this region R. That is,

$$\begin{aligned} P(0 \leqslant x \leqslant 1 \text{ and } 0 \leqslant y \leqslant 1) &= \int_0^1 \int_0^1 e^{-x}e^{-y}\,dx\,dy \\ &= \int_0^1 \left(-e^{-x}e^{-y}\Big|_{x=0}^{x=1} \right) dy = \int_0^1 -(e^{-1}-1)e^{-y}\,dy \\ &= (e^{-1}-1)e^{-y}\Big|_0^1 = (e^{-1}-1)^2 \\ &\simeq 0.3996 \end{aligned}$$

Ex. 7.5 Suppose x denotes the time (in minutes) that a person sits in the waiting room of a certain dentist and y the time (in minutes) that a person stands in line at a certain bank. You have an appointment with the dentist, after which you are planning to cash a check at the bank. If $f(x, y)$ is the joint probability density function for x and y, write down a double integral that gives the probability that your *total* waiting time will be no more that 20 minutes.

Solution

Ex. 7.6 Suppose x denotes the time (in minutes) that a person stands in line at a certain bank and y the duration (in minutes) of a routine transaction at the teller's window. You arrive at the bank to deposit a check. If the joint probability density function for x and y is $f(x, y) = \frac{1}{8}e^{-x/4}e^{-y/2}$, find the probability that you will complete your business at the bank within 8 minutes.

Ex. 7.7 When studying the spread of an epidemic, we assume that the probability that an infected individual will spread the disease to an uninfected individual is a function of the distance between them. Consider a circular city of radius 10 miles in which the population is uniformly distributed. For an uninfected individual at a fixed point $A(x_0, y_0)$, assume that the probability function is given by

$$f(P, A) = \frac{1}{20}[20 - d(P, A)]$$

where $d(P, A)$ denotes the distance between P and A.

(a) Suppose the exposure of a person to the disease is the sum of the probabilities of catching the disease from all members of the population. Assume that the infected people are uniformly distributed throughout the city, with k infected individuals per square mile. Find a double integral that represents the exposure of a person residing at A.

(b) Evaluate the integral for the case in which A is the center of the city and for the case in which A is located on the edge of the city. Where would you prefer to live?

Table of indefinite integrals

(1) (a) $\displaystyle\int x^n \, dx$

(b) $\displaystyle\int e^{ax} \, dx$

(c) $\displaystyle\int \ln x \, dx$

(2) (a) $\displaystyle\int x\sqrt{x^2+1} \, dx$

(b) $\displaystyle\int x\sqrt{4-x^2} \, dx$

(c) $\displaystyle\int \frac{x}{\sqrt{4-x^2}} \, dx$

(3) (a) $\displaystyle\int xe^x \, dx$

(b) $\displaystyle\int x^2 e^x \, dx$

(c) $\displaystyle\int x^2 e^{2x^3} \, dx$

(4) (a) $\displaystyle\int \cos n\theta \, d\theta$ and $\displaystyle\int \sin n\theta \, d\theta$

(b) $\displaystyle\int \cos^2 n\theta \, d\theta$ and $\displaystyle\int \sin^2 n\theta \, d\theta$

(c) $\displaystyle\int (1+\cos n\theta)^2 \, d\theta$ and $\displaystyle\int (1+\sin n\theta)^2 \, d\theta$

(5) (a) $\displaystyle\int \sin^2 n\theta \cos n\theta \, d\theta$ and $\displaystyle\int \cos^2 n\theta \sin n\theta \, d\theta$

(b) $\displaystyle\int \cos^3 n\theta \, d\theta$ and $\displaystyle\int \sin^3 n\theta \, d\theta$

(c) $\displaystyle\int \cos^4 n\theta \, d\theta$ and $\displaystyle\int \sin^4 n \, d\theta.$

Solutions

1 (a) $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

(b) $\displaystyle\int e^{ax}\,dx = \frac{1}{a}\int e^{ax}\,d(ax) = \frac{1}{a}e^{ax} + C$

(c) $\displaystyle\int \ln x\,dx = \int 1\cdot\ln x\,dx = x\ln x - \int x\frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C$

2 (a) $$\int x\sqrt{x^2+1}\,dx = \frac{1}{2}\int (x^2+1)^{\frac{1}{2}}\,d(x^2) = \frac{1}{2}\int (x^2+1)^{\frac{1}{2}}\,d(x^2+1)$$
$$= \frac{1}{2}\cdot\frac{2(x^2+1)^{\frac{3}{2}}}{3} + C = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$$

(b) $$\int x\sqrt{4-x^2}\,dx = \frac{1}{2}\int (4-x^2)^{\frac{1}{2}}\,d(x^2) = -\frac{1}{2}\int (4-x^2)^{\frac{1}{2}}\,d(4-x^2)$$
$$= -\frac{1}{2}\cdot\frac{2(4-x^2)^{\frac{3}{2}}}{3} + C = -\frac{1}{3}(4-x^2)^{\frac{3}{2}} + C$$

(c) $$\int \frac{x}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\int (4-x^2)^{-\frac{1}{2}}\,d(4-x^2) = -\frac{1}{2}\cdot\frac{(4-x^2)^{\frac{1}{2}}}{1/2} + C = -(4-x^2)^{\frac{1}{2}} + C$$

3 (a) $\displaystyle\int xe^x\,dx = xe^x - \int 1\cdot e^x\,dx = xe^x - e^x + C$

(b) $$\int x^2e^x\,dx = x^2e^x - \int 2xe^x\,dx = x^2e^x - 2\int xe^x\,dx = x^2e^x - 2(xe^x - e^x) + C$$
$$= (x^2 - 2x + 2)e^x + C$$

(c) $$\int x^2e^{2x^3}\,dx = \frac{1}{3}\int e^{2x^3}\,d(x^3) = \frac{1}{3}\cdot\frac{1}{2}\int e^{2x^3}\,d(2x^3) = \frac{1}{6}e^{2x^3} + C$$

4 (a) (i) $\displaystyle\int \cos n\theta\,d\theta = \frac{1}{n}\int \cos n\theta\,d(n\theta) = \frac{1}{n}\sin n\theta + C$

(ii) $\displaystyle\int \sin n\theta\,d\theta = \frac{1}{n}\int \sin n\theta\,d(n\theta) = -\frac{1}{n}\cos n\theta + C$

(b) **Note** that $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, $\quad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

(i) $\displaystyle\int \cos^2 n\theta\,d\theta = \frac{1}{2}\int (1+\cos 2n\theta)\,d\theta = \frac{1}{2}\left(\theta + \frac{1}{2n}\sin 2n\theta\right) + C$

(ii) $\displaystyle\int \sin^2 n\theta\,d\theta = \frac{1}{2}\int (1-\cos 2n\theta)\,d\theta = \frac{1}{2}\left(\theta - \frac{1}{2n}\sin 2n\theta\right) + C$

(c) (i)

$$\begin{aligned}\int (1+\cos n\theta)^2\,d\theta &= \int (1+2\cos n\theta+\cos^2 n\theta)\,d\theta \\ &= \int d\theta + 2\int \cos n\theta\,d\theta + \int \cos^2 n\theta\,d\theta \\ &= \theta + \frac{2}{n}\sin n\theta + \frac{1}{2}\left(\theta + \frac{1}{2n}\sin 2n\theta\right) + C \\ &\qquad \text{(from 4a(i))} \qquad \text{(from 4b(i))} \\ &= \frac{3}{2}\theta + \frac{2}{n}\sin n\theta + \frac{1}{4n}\sin 2n\theta + C\end{aligned}$$

(ii)

$$\begin{aligned}\int (1+\sin n\theta)^2\,d\theta &= \int (1+2\sin n\theta+\sin^2 n\theta)\,d\theta \\ &= \theta - \frac{2}{n}\cos n\theta + \frac{1}{2}\left(\theta - \frac{1}{2n}\sin 2n\theta\right) + C \\ &\qquad \text{(from 4a(ii))} \qquad \text{(from 4b(ii))} \\ &= \frac{3}{2}\theta - \frac{2}{n}\cos n\theta - \frac{1}{4n}\sin 2n\theta + C\end{aligned}$$

5 (a) (i) $\displaystyle\int \sin^2 n\theta\cos n\theta\,d\theta = \frac{1}{n}\int \sin^2 n\theta\,d(\sin n\theta) = \frac{1}{n}\cdot\frac{\sin^3 n\theta}{3} + C = \frac{1}{3n}\sin^3 n\theta + C$

(ii) $\displaystyle\int \cos^2 n\theta\sin n\theta\,d\theta = -\frac{1}{n}\int \cos^2 n\theta\,d(\cos n\theta) = -\frac{1}{n}\cdot\frac{\cos^3 n\theta}{3} + C = -\frac{1}{3n}\cos^3 n\theta + C$

(b) (i)

$$\begin{aligned}\int \cos^3 n\theta\,d\theta &= \frac{1}{n}\int \cos^2 n\theta\,d(\sin n\theta) = \frac{1}{n}\int (1-\sin^2 n\theta)\,d(\sin n\theta) \\ &= \frac{1}{n}\left(\sin n\theta - \frac{\sin^3 n\theta}{3}\right) + C\end{aligned}$$

(ii)

$$\begin{aligned}\int \sin^3 n\theta\,d\theta &= -\frac{1}{n}\int \sin^2 n\theta\,d(\cos n\theta) = -\frac{1}{n}\int (1-\cos^2 n\theta)\,d(\cos n\theta) \\ &= -\frac{1}{n}\left(\cos n\theta - \frac{\cos^3 n\theta}{3}\right) + C\end{aligned}$$

(c) (i) **Note** that

$$
\begin{aligned}
\cos^4 n\theta &= \cos^2 n\theta \cdot \cos^2 n\theta \\
&= \frac{1}{2}(1+\cos 2n\theta) \cdot \frac{1}{2}(1+\cos 2n\theta) \\
&= \frac{1}{4}(1 + 2\cos 2n\theta + \cos^2 2n\theta) \\
&= \frac{1}{4}\left[1 + 2\cos 2n\theta + \frac{1}{2}(1+\cos 4n\theta)\right] \\
&= \frac{1}{4}\left(\frac{3}{2} + 2\cos 2n\theta + \frac{1}{2}\cos 4n\theta\right) \\
\therefore \quad \int \cos^4 n\theta \, d\theta &= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2n\theta + \frac{1}{2}\cos 4n\theta\right) d\theta \\
&= \frac{1}{4}\left(\frac{3}{2}\theta + \frac{2}{2n}\sin 2n\theta + \frac{1}{2}\cdot\frac{1}{4n}\sin 4n\theta\right) + C \\
&= \frac{1}{4}\left(\frac{3}{2}\theta + \frac{1}{n}\sin 2n\theta + \frac{1}{8n}\sin 4n\theta\right) + C
\end{aligned}
$$

(ii) **Note** that

$$
\begin{aligned}
\sin^4 n\theta &= \sin^2 n\theta \cdot \sin^2 n\theta \\
&= \frac{1}{2}(1-\cos 2n\theta) \cdot \frac{1}{2}(1-\cos 2n\theta) \\
&= \frac{1}{4}(1 - 2\cos 2n\theta + \cos^2 2n\theta) \\
&= \frac{1}{4}\left[1 - 2\cos 2n\theta + \frac{1}{2}(1+\cos 4n\theta)\right] \\
&= \frac{1}{4}\left(\frac{3}{2} - 2\cos 2n\theta + \frac{1}{2}\cos 4n\theta\right) \\
\therefore \quad \int \sin^4 n\theta \, d\theta &= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2n\theta + \frac{1}{2}\cos 4n\theta\right) d\theta \\
&= \frac{1}{4}\left(\frac{3}{2}\theta - \frac{2}{2n}\sin 2n\theta + \frac{1}{2}\cdot\frac{1}{4n}\sin 4n\theta\right) + C \\
&= \frac{1}{4}\left(\frac{3}{2}\theta - \frac{1}{n}\sin 2n\theta + \frac{1}{8n}\sin 4n\theta\right) + C
\end{aligned}
$$