EXERCISES 16.4

In Exercises 1–4, use the Divergence Theorem to calculate the flux of the given vector field out of the sphere $\mathcal{S}$ with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$.

1. $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$ **2.** $\mathbf{F} = ye^z\mathbf{i} + x^2e^z\mathbf{j} + xy\mathbf{k}$

3. $\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$

4. $\mathbf{F} = x^3\mathbf{i} + 3yz^2\mathbf{j} + (3y^2z + x^2)\mathbf{k}$

In Exercises 5–8, evaluate the flux of $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ outward across the boundary of the given solid region.

5. The ball $(x - 2)^2 + y^2 + (z - 3)^2 \le 9$

6. The solid ellipsoid $x^2 + y^2 + 4(z - 1)^2 \le 4$

7. The tetrahedron $x + y + z \le 3$, $x \ge 0$, $y \ge 0$, $z \ge 0$

8. The cylinder $x^2 + y^2 \le 2y$, $0 \le z \le 4$

9. Let A be the area of a region D forming part of the surface of the sphere of radius R centred at the origin, and let V be the volume of the solid cone C consisting of all points on line segments joining the centre of the sphere to points in D. Show that $V = \frac{1}{3}AR$ by applying the Divergence Theorem to $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

10. Let $\phi(x, y, z) = xy + z^2$. Find the flux of $\nabla\phi$ upward through the triangular planar surface $\mathcal{S}$ with vertices at $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

11. A conical domain with vertex $(0, 0, b)$ and axis along the z-axis has as base a disk of radius a in the xy-plane. Find the flux of

$$\mathbf{F} = (x + y^2)\mathbf{i} + (3x^2y + y^3 - x^3)\mathbf{j} + (z + 1)\mathbf{k}$$

upward through the conical part of the surface of the domain.

12. Find the flux of $\mathbf{F} = (y + xz)\mathbf{i} + (y + yz)\mathbf{j} - (2x + z^2)\mathbf{k}$ upward through the first octant part of the sphere $x^2 + y^2 + z^2 = a^2$.

13. Let D be the region $x^2 + y^2 + z^2 \le 4a^2$, $x^2 + y^2 \ge a^2$. The surface $\mathcal{S}$ of D consists of a cylindrical part, $\mathcal{S}_1$, and a spherical part, $\mathcal{S}_2$. Evaluate the flux of

$$\mathbf{F} = (x + yz)\mathbf{i} + (y - xz)\mathbf{j} + (z - e^x \sin y)\mathbf{k}$$

out of D through (a) the whole surface $\mathcal{S}$, (b) the surface $\mathcal{S}_1$, and (c) the surface $\mathcal{S}_2$.

↑

why not my drawing?

14. Evaluate $\displaystyle\iint_{\mathcal{S}} (3xz^2\mathbf{i} - x\mathbf{j} - y\mathbf{k}) \bullet \hat{\mathbf{N}}\, dS$, where $\mathcal{S}$ is that part of the cylinder $y^2 + z^2 = 1$ that lies in the first octant and between the planes $x = 0$ and $x = 1$.

15. A solid region R has volume V and centroid at the point $(\bar{x}, \bar{y}, \bar{z})$. Find the flux of

$$\mathbf{F} = (x^2 - x - 2y)\mathbf{i} + (2y^2 + 3y - z)\mathbf{j} - (z^2 - 4z + xy)\mathbf{k}$$

out of R through its surface.

16. The plane $x + y + z = 0$ divides the cube $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$ into two parts. Let the lower part (with one vertex at $(-1, -1, -1)$) be D. Sketch D. Note that it has seven faces, one of which is hexagonal. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ out of D through each of its faces.

17. Let $\mathbf{F} = (x^2 + y + 2 + z^2)\mathbf{i} + (e^{x^2} + y^2)\mathbf{j} + (3 + x)\mathbf{k}$. Let $a > 0$, and let $\mathcal{S}$ be the part of the spherical surface $x^2 + y^2 + z^2 = 2az + 3a^2$ that is above the xy-plane. Find the flux of $\mathbf{F}$ outward across $\mathcal{S}$.

18. A pile of wet sand having total volume 5π covers the disk $x^2 + y^2 \le 1$, $z = 0$. The momentum of water vapour is given by $\mathbf{F} = \mathbf{grad}\,\phi + \mu\,\mathbf{curl}\,\mathbf{G}$, where $\phi = x^2 - y^2 + z^2$ is the water concentration, $\mathbf{G} = \frac{1}{3}(-y^3\mathbf{i} + x^3\mathbf{j} + z^3\mathbf{k})$, and μ is a constant. Find the flux of $\mathbf{F}$ upward through the top surface of the sand pile.

In Exercises 19–29, D is a three-dimensional domain satisfying the conditions of the Divergence Theorem, and $\mathcal{S}$ is its surface. $\hat{\mathbf{N}}$ is the unit outward (from D) normal field on $\mathcal{S}$. The functions ϕ and ψ are smooth scalar fields on D. Also, $\partial\phi/\partial n$ denotes the first directional derivative of ϕ in the direction of $\hat{\mathbf{N}}$ at any point on $\mathcal{S}$:

$$\frac{\partial\phi}{\partial n} = \nabla\phi \bullet \hat{\mathbf{N}}.$$

19. Show that $\displaystyle\oiint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\, dS = 0$, where $\mathbf{F}$ is an arbitrary smooth vector field.

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20. Show that the volume V of D is given by

$$V = \frac{1}{3} \oiint_{\mathcal{S}} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \bullet \hat{\mathbf{N}} \, dS.$$

21. If D has volume V, show that

$$\bar{\mathbf{r}} = \frac{1}{2V} \oiint_{\mathcal{S}} (x^2 + y^2 + z^2) \hat{\mathbf{N}} \, dS$$

is the position vector of the centre of gravity of D.

22. Show that $\oiint_{\mathcal{S}} \nabla\phi \times \hat{\mathbf{N}} \, dS = 0$.

23. If $\mathbf{F}$ is a smooth vector field on D, show that

$$\iiint_D \phi \, \mathbf{div}\, \mathbf{F} \, dV + \iiint_D \nabla\phi \bullet \mathbf{F} \, dV = \oiint_{\mathcal{S}} \phi \mathbf{F} \bullet \hat{\mathbf{N}} \, dS.$$

Hint: Use Theorem 3(b) from Section 16.2.

Properties of the Laplacian operator

24. If $\nabla^2\phi = 0$ in D and $\phi(x, y, z) = 0$ on $\mathcal{S}$, show that $\phi(x, y, z) = 0$ in D. *Hint:* Let $\mathbf{F} = \nabla\phi$ in Exercise 23.

25. **(Uniqueness for the Dirichlet problem)** The Dirichlet problem for the Laplacian operator is the boundary-value problem

$$\begin{cases} \nabla^2 u(x, y, z) = f(x, y, z) & \text{on } D \\ u(x, y, z) = g(x, y, z) & \text{on } \mathcal{S}, \end{cases}$$

where f and g are given functions defined on D and $\mathcal{S}$, respectively. Show that this problem can have at most one solution $u(x, y, z)$. *Hint:* Suppose there are two solutions, u and v, and apply Exercise 24 to their difference $\phi = u - v$.

26. **(The Neumann problem)** If $\nabla^2\phi = 0$ in D and $\partial\phi/\partial n = 0$ on $\mathcal{S}$, show that $\nabla\phi(x, y, z) = 0$ on D. The Neumann problem for the Laplacian operator is the boundary-value problem

$$\begin{cases} \nabla^2 u(x, y, z) = f(x, y, z) & \text{on } D \\ \dfrac{\partial}{\partial n} u(x, y, z) = g(x, y, z) & \text{on } \mathcal{S}, \end{cases}$$

where f and g are given functions defined on D and $\mathcal{S}$, respectively. Show that, if D is connected, then any two solutions of the Neumann problem must differ by a constant on D.

27. Verify that $\iiint_D \nabla^2\phi \, dV = \oiint_{\mathcal{S}} \dfrac{\partial\phi}{\partial n} \, dS$.

28. Verify that

$$\iiint_D \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) dV = \oiint_{\mathcal{S}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS.$$

29. By applying the Divergence Theorem to $\mathbf{F} = \phi\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector, show that

$$\iiint_D \nabla\phi \, dV = \oiint_{\mathcal{S}} \phi\hat{\mathbf{N}} \, dS.$$

30. Let P_0 be a fixed point, and for each $\epsilon > 0$ let D_ϵ be a domain with boundary $\mathcal{S}_\epsilon$ satisfying the conditions of the Divergence Theorem. Suppose that the maximum distance from P_0 to points P in D_ϵ approaches zero as $\epsilon \to 0+$. If D_ϵ has volume $\text{vol}(D_\epsilon)$, show that

$$\lim_{\epsilon \to 0+} \frac{1}{\text{vol}(D_\epsilon)} \oiint_{\mathcal{S}_\epsilon} \mathbf{F} \bullet \hat{\mathbf{N}} \, dS = \mathbf{div}\, \mathbf{F}(P_0).$$

This generalizes Theorem 1 of Section 16.1.

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In Exercises 1–4, use the Divergence Theorem to calculate the flux of the given vector field out of the sphere $\mathcal{S}$ with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$.

1. $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$
2. $\mathbf{F} = ye^z\mathbf{i} + x^2e^z\mathbf{j} + xy\mathbf{k}$
3. $\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$
4. $\mathbf{F} = x^3\mathbf{i} + 3yz^2\mathbf{j} + (3y^2z + x^2)\mathbf{k}$

1. $$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_E 1-2+4\,dV$$
$$= 3\iiint_E 1\,dV$$
$$= 3\left(\frac{4}{3}\pi a^3\right)$$
$$= 4\pi a^3$$

2. $$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_E 0\,dV = 0$$

3. $$\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_E 2x + 2y + 1\,dV$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} (2\rho\sin\varphi\cos\theta + 2\rho\sin\varphi\sin\theta + 1)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 2\rho^3\sin^2\varphi\cos\theta + 2\rho^3\sin^2\varphi\sin\theta + \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}\left[\frac{\rho^4}{2}\sin^2\varphi\cos\theta + \frac{\rho^4}{2}\sin^2\varphi\sin\theta + \frac{\rho^3}{3}\sin\varphi\right]_0^a d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{a^4}{2}\sin^2\varphi\cos\theta + \frac{a^4}{2}\sin^2\varphi\sin\theta + \frac{a^3}{3}\sin\varphi\,d\varphi\,d\theta$$

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$$= \int_0^{2\pi}\int_0^{\pi} \frac{a^4}{2}\left(\frac{1}{2} - \frac{\cos 2\varphi}{2}\right)(\sin\theta + \cos\theta) + \frac{a^3}{3}\sin\varphi \, d\varphi \, d\theta$$

$$\int_0^{2\pi}\int_0^{\pi} \frac{a^4}{4}(\sin\theta + \cos\theta) - \frac{a^4}{4}(\sin\theta + \cos\theta)(\cos 2\varphi) + \frac{a^3}{3}\sin\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \frac{a^4}{4}(\sin\theta + \cos\theta)\pi - \frac{a^4}{4}(\sin\theta + \cos\theta)\left[\frac{\sin 2\varphi}{2}\right]_0^{\pi} + \frac{a^3}{3}\left[-\cos\varphi\right]_0^{\pi} d\theta$$

$$= \int_0^{2\pi} \frac{a^4}{4}(\sin\theta + \cos\theta)\pi + \frac{2a^3}{3} \, d\theta$$

$$= \int_0^{2\pi} \frac{a^4\pi}{4}\sin\theta + \frac{a^4\pi}{4}\cos\theta + \frac{2a^3}{3} \, d\theta$$

$$= \frac{a^4\pi}{4}\left[-\cos\theta\right]_0^{2\pi} + \frac{a^4\pi}{4}\left[\sin\theta\right]_0^{2\pi} + \frac{4a^3}{3}\pi$$

$$= \frac{4a^3}{3}\pi.$$

Made with Goodnotes

In Exercises 1–4, use the Divergence Theorem to calculate the flux of the given vector field out of the sphere $\mathcal{S}$ with equation $x^2 + y^2 + z^2 = a^2$, where $a > 0$.

1. $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}$

2. $\mathbf{F} = ye^z\mathbf{i} + x^2e^z\mathbf{j} + xy\mathbf{k}$

3. $\mathbf{F} = (x^2 + y^2)\mathbf{i} + (y^2 - z^2)\mathbf{j} + z\mathbf{k}$

4. $\mathbf{F} = x^3\mathbf{i} + 3yz^2\mathbf{j} + (3y^2z + x^2)\mathbf{k}$

4. $$\iiint_E 3x^2 + 3z^2 + 3y^2 \, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 3\rho^2(\rho^2 \sin\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 3\rho^4 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{3a^5}{5} \sin\varphi\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \frac{3a^5}{5} [-\cos\varphi]_0^{\pi}\, d\theta$$

$$= 2\pi \cdot 2 \cdot \frac{3a^5}{5}$$

$$= \frac{12a^5}{5}\pi$$

Made with Goodnotes

In Exercises 5–8, evaluate the flux of $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ outward across the boundary of the given solid region.

5. The ball $(x-2)^2 + y^2 + (z-3)^2 \le 9$

6. The solid ellipsoid $x^2 + y^2 + 4(z-1)^2 \le 4$

7. The tetrahedron $x + y + z \le 3$, $x \ge 0$, $y \ge 0$, $z \ge 0$

8. The cylinder $x^2 + y^2 \le 2y$, $0 \le z \le 4$

9. Let A be the area of a region D forming part of the surface of the sphere of radius R centred at the origin, and let V be the volume of the solid cone C consisting of all points on line segments joining the centre of the sphere to points in D. Show that $V = \frac{1}{3}AR$ by applying the Divergence Theorem to $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

5. $\vec{r}(\rho,\varphi,\theta) = \langle \rho\sin\varphi\cos\theta + 2,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi + 3 \rangle$

$0 \le \rho \le 3,\quad 0 \le \varphi \le \pi,\quad 0 \le \theta \le 2\pi$

$$\iiint_E 2x + 2y + 2z \, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{3} (2\rho\sin\varphi\cos\theta + 4 + 2\rho\sin\varphi\sin\theta + 2\rho\cos\varphi + 6)\, \rho^2\sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{3} 2\rho^3\sin^2\varphi\cos\theta + 10\rho^2\sin\varphi + 2\rho^3\sin^2\varphi\sin\theta + 2\rho^3\sin\varphi\cos\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{2(3)^4}{4}\sin^2\varphi\cos\theta + \frac{10(3)^3}{3}\sin\varphi + \frac{3^4}{2}\sin^2\varphi\sin\theta + \frac{3^4}{2}\sin\varphi\cos\varphi \, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{81}{2}\left(\frac{1}{2} - \frac{\cos 2\varphi}{2}\right)\cos\theta + 90\sin\varphi + \frac{81}{2}\sin\theta\left(\frac{1}{2} - \frac{\cos 2\varphi}{2}\right) + \frac{81}{2}\sin\varphi\cos\varphi \, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \frac{81}{4}\pi\cos\theta - \frac{81}{4}\cos\theta\left[\frac{\sin 2\varphi}{4}\right]_0^{\pi} + 90\left[-\cos\varphi\right]_0^{\pi} + \frac{81}{4}\pi\sin\theta - \frac{81}{4}\sin\theta\left[\frac{\sin 2\varphi}{4}\right]_0^{\pi} d\theta$$

$$= \int_0^{2\pi} \frac{81\pi}{4}\cos\theta + 180 + \frac{81}{4}\pi\sin\theta \, d\theta$$

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$$= \int_0^{2\pi} \frac{81\pi}{4}\cos\theta + 180 + \frac{81}{4}\pi\sin\theta \, d\theta$$

$$= \frac{81\pi}{4}\left[\sin\theta\right]_0^{2\pi} + 180 \cdot 2\pi + \frac{81}{4}\pi\left[-\cos\theta\right]_0^{2\pi}$$

$$= 360\pi.$$

Made with Goodnotes

In Exercises 5–8, evaluate the flux of $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ outward across the boundary of the given solid region.

5. The ball $(x-2)^2 + y^2 + (z-3)^2 \le 9$

6. The solid ellipsoid $\underset{4}{x^2} + \underset{4}{y^2} + \underset{4}{4(z-1)^2} \le 4$

$$\frac{x^2}{2^2} + \frac{y^2}{2^2} + (z-1)^2 \le 1$$

6. $$\vec{r}(\rho,\varphi,\theta) = \langle 2\rho\sin\varphi\cos\theta,\ 2\rho\sin\varphi\sin\theta,\ \rho\cos\varphi + 1 \rangle$$

$$0 \le \rho \le 1 \quad 0 \le \varphi \le \pi \quad 0 \le \theta \le 2\pi$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \left(4\rho\sin\varphi\cos\theta + 4\rho\sin\varphi\sin\theta + 2\rho\cos\varphi + 2\right) \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}$$

$$\begin{vmatrix} x_\rho & y_\rho & z_\rho \\ x_\varphi & y_\varphi & z_\varphi \\ x_\theta & y_\theta & z_\theta \end{vmatrix} = \begin{vmatrix} 2\sin\varphi\cos\theta & 2\sin\varphi\sin\theta & \cos\varphi \\ 2\rho\cos\varphi\cos\theta & 2\rho\cos\varphi\sin\theta & -\rho\sin\varphi \\ -2\rho\sin\varphi\sin\theta & 2\rho\sin\varphi\cos\theta & 0 \end{vmatrix}$$

$$\langle 2\sin\varphi\cos\theta(2\rho^2\sin^2\varphi\cos\theta),\ 2\sin\varphi\sin\theta(2\rho^2\sin^2\varphi\sin\theta),\ \cos\varphi(4\rho^2\sin\varphi\cos\varphi) \rangle$$

$$= \langle 4\rho^2\sin^3\varphi\cos^2\theta,\ 4\rho^2\sin^3\varphi\sin^2\theta,\ 4\rho^2\sin\varphi\cos^2\varphi \rangle$$

$$|\ | = \sqrt{16\rho^4\sin^6\varphi\cos^4\theta + 16\rho^4\sin^6\varphi\sin^4\theta + 16\rho^4\sin^2\varphi\cos^4\varphi}$$

$$= 4\rho^2\sqrt{\sin^6\varphi(\sin^4\theta + \cos^4\theta) + \sin^2\varphi\cos^4\varphi}$$

$$= 4\rho^2\sqrt{\sin^2\varphi(\sin^4\varphi(\sin^4\theta + \cos^4\theta) + \cos^4\varphi)}$$

$$= 4\rho^2\sqrt{\sin^2\varphi(\sin^4\varphi((\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta) + \cos^4\varphi)}$$

$$4\rho^2\sin\varphi\sqrt{(\sin^4\varphi - 2\sin^4\varphi\sin^2\theta\cos^2\theta) + \cos^4\varphi)}$$

$$4\rho^2\sin\varphi\sqrt{1 - 2\sin^2\varphi\cos^2\varphi - 2\sin^4\varphi\sin^2\theta\cos^2\theta}$$

Made with Goodnotes

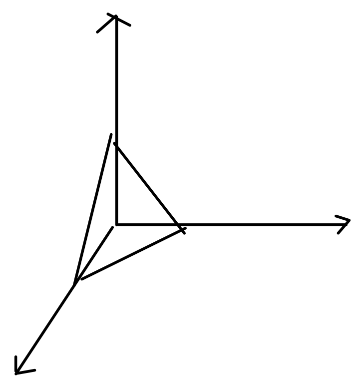
In Exercises 5–8, evaluate the flux of $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ outward across the boundary of the given solid region.

5. The ball $(x-2)^2 + y^2 + (z-3)^2 \le 9$

6. The solid ellipsoid $x^2 + y^2 + 4(z-1)^2 \le 4$

7. The tetrahedron $x + y + z \le 3$, $x \ge 0$, $y \ge 0$, $z \ge 0$

8. The cylinder $x^2 + y^2 \le 2y$, $0 \le z \le 4$



7. $$\int_0^3\int_0^{3-y}\int_0^{3-x-y} 2x+2y+2z \; dz\,dx\,dy$$

$$= \int_0^3\int_0^{3-y} 2x(3-x-y)+2y(3-x-y)+(3-x-y)(3-x-y)\;dx\,dy$$

$$= \int_0^3\int_0^{3-y} 6x-2x^2-2xy+6y-2xy-2y^2+9-3x-3y-3x+x^2+xy-3y+xy+y^2\;dx\,dy$$

$$= \int_0^3\int_0^{3-y} -x^2-2xy-y^2+9\;dx\,dy$$

$$= \int_0^3 \left[-\frac{x^3}{3} - x^2y - y^2x + 9x\right]_0^{3-y} dy$$

$$= \int_0^3 -\frac{(3-y)^3}{3} - (3-y)^2y - y^2(3-y) + 9(3-y)\;dy$$

$$= \int_0^3 -\frac{(27+3\cdot3^2(-y)+3\cdot3(-y)^2-y^3)}{3} - (9-6y+y^2)y - 3y^2+y^3+27-9y\;dy$$

$$= \int_0^3 -\frac{(27-27y+9y^2-y^3)}{3} - 9y+6y^2-y^3-3y^2+y^3+27-9y\;dy$$

$$= \int_0^3 -9+9y-3y^2+\frac{1}{3}y^3-9y+6y^2-y^3-3y^2+y^3+27-9y\;dy$$

$$= \int_0^3 \frac{1}{3}y^3-9y+18\;dy$$

$$= \left[\frac{y^4}{12}-\frac{9y^2}{2}+18y\right]_0^3 = \frac{81}{12}-\frac{81}{2}+54$$

Made with Goodnotes

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5. The ball $(x-2)^2 + y^2 + (z-3)^2 \le 9$
6. The solid ellipsoid $x^2 + y^2 + 4(z-1)^2 \le 4$
7. The tetrahedron $x + y + z \le 3$, $x \ge 0$, $y \ge 0$, $z \ge 0$
8. The cylinder $x^2 + y^2 \le 2y$, $0 \le z \le 4$

$r^2 \le 2r\sin\theta$

$r \le 2\sin\theta$

$\vec{r}(z,r,\theta) = \langle r\cos\theta, r\sin\theta, z \rangle$

$0 \le r \le 2\sin\theta$

$0 \le \theta \le \pi$

$0 \le z \le 4$

$$\iiint_E 2x + 2y + 2z \, dV$$

$$\begin{array}{r} 96 \\ +16 \\ \hline 2 \end{array}$$

$$\int_0^{\pi}\int_0^{2\sin\theta}\int_0^4 2r^2\cos\theta + 2r^2\sin\theta + 2zr \, dz\, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{2\sin\theta} 8r^2\cos\theta + 8r^2\sin\theta + 16r \, dr\, d\theta$$

$$= \int_0^{\pi} \frac{8(8\sin^3\theta)\cos\theta}{3} + \frac{8(8\sin^4\theta)}{3} + 8(4\sin^2\theta)\, d\theta$$

$$= \int_0^{\pi} \frac{64}{3}\sin^3\theta\cos\theta + \frac{64}{3}\left(\frac{1}{4} - \frac{\cos 2\theta}{2} + \frac{\cos^2 2\theta}{4}\right) + 32 - 32\cos 2\theta \, d\theta$$

$$= \int_0^{\pi} \frac{16}{3} - \frac{32}{3}\cos 2\theta + \frac{16}{3}\left(\frac{1}{2} + \frac{\cos 4\theta}{2}\right) + 32 - 32\cos 2\theta \, d\theta$$

$$= \int_0^{\pi} \frac{112}{3} - \frac{32}{3}\cos 2\theta - 32\cos 2\theta + \frac{8}{3} + \frac{8}{3}\cos 4\theta \, d\theta$$

$$= 40\pi - \frac{32}{3}\left[\frac{\sin 2\theta}{2}\right]_0^{\pi} - 32\left[\frac{\sin 2\theta}{2}\right]_0^{\pi} + \frac{8}{3}\left[\frac{\sin 4\theta}{4}\right]_0^{\pi}$$

$$= 40\pi.$$

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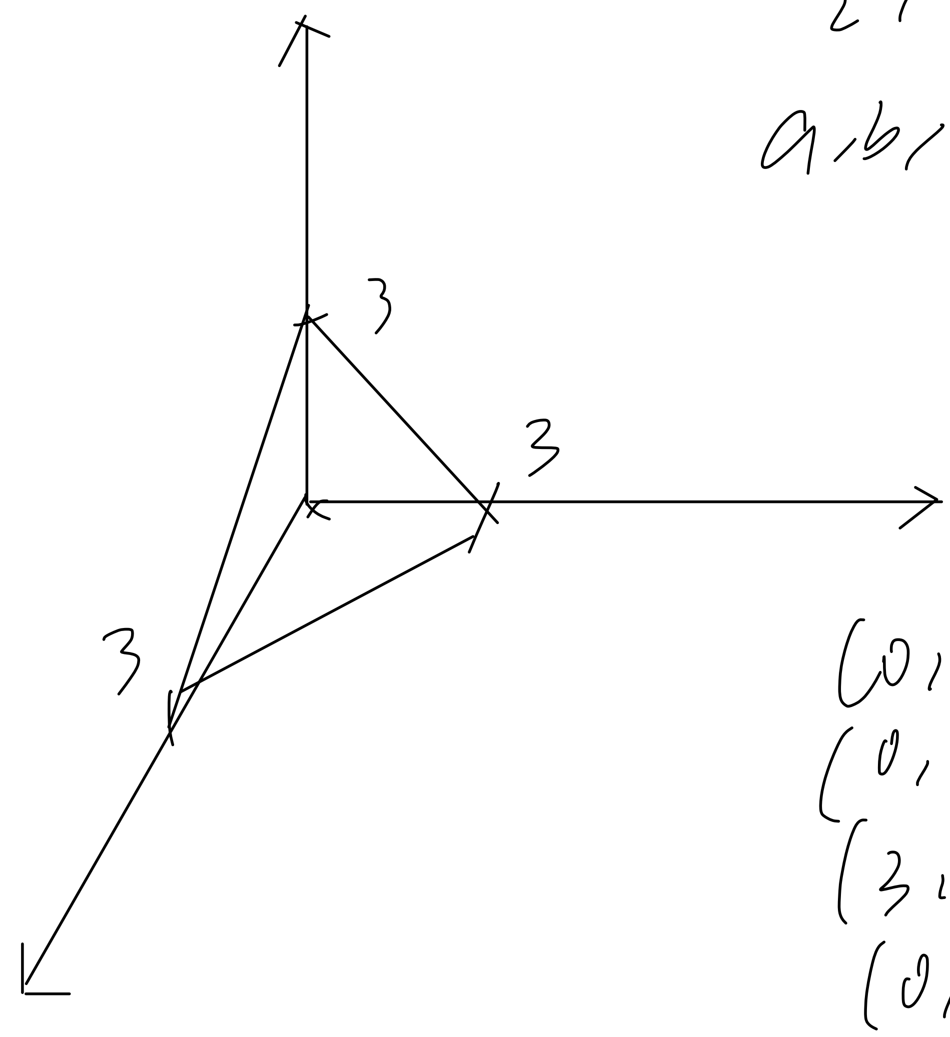
$$\left(x-\frac{3}{2}\right)^2+\left(y-\frac{3}{2}\right)^2+\left(z-\frac{3}{2}\right)^2=\sqrt{\left(\frac{3}{2}\right)^2\times 3}.$$

$\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)$

a, b, c



$(0,0,0)$
$(0,0,3)$
$(3,0,0)$
$(0,3,0)$

$$\sqrt{a^2+b^2+c^2}=\sqrt{a^2+b^2+(c-3)^2}$$

$$c^2 = c^2-6c+9$$

$$0 = -6c+9$$

$$-9 = -6c$$

$$c = \frac{3}{2}$$

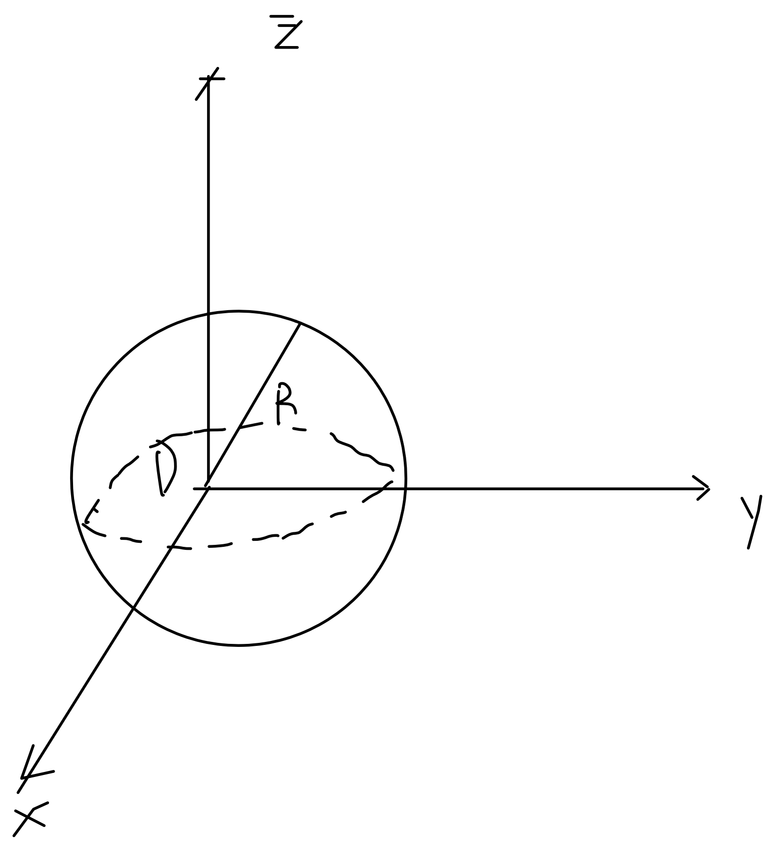
Made with Goodnotes

9. Let A be the area of a region D forming part of the surface of the sphere of radius R centred at the origin, and let V be the volume of the solid cone C consisting of all points on line segments joining the centre of the sphere to points in D. Show that $V = \frac{1}{3}AR$ by applying the Divergence Theorem to $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

????

10. Let $\phi(x, y, z) = xy + z^2$. Find the flux of $\nabla\phi$ upward through the triangular planar surface $\mathcal{S}$ with vertices at $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

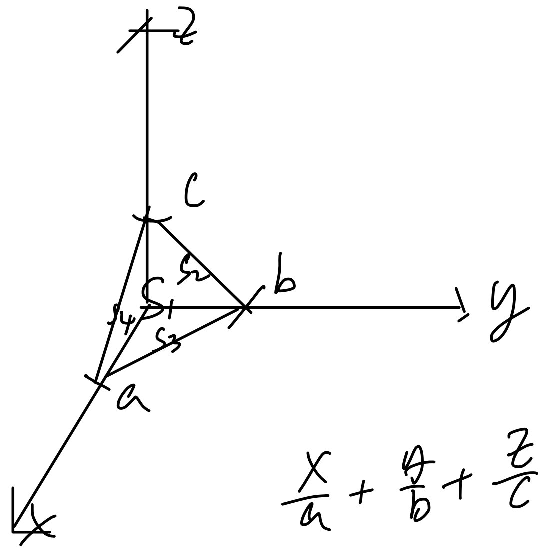
9.



Made with Goodnotes

10. Let $\phi(x, y, z) = xy + z^2$. Find the flux of $\nabla\phi$ upward through the triangular planar surface $\mathcal{S}$ with vertices at $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

$\langle y, x, 2z \rangle$



$$\iiint_{\triangle} \nabla\cdot\nabla\phi \, dV = \underbrace{\iint_{S_1} \nabla\phi\cdot\hat{n}\, dS}_{Q} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} \nabla\phi\cdot\hat{n}\, dS$$

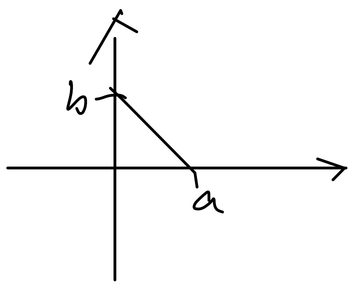
$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\frac{z}{c} = 1 - \frac{x}{a} - \frac{y}{b}$

$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

$Q =$

$$2\iiint_{\triangle} (d\,V - \iint_{S_2} - \iint_{S_3} - \iint_{S_4} \nabla\phi\cdot\hat{n}\, dS$$



$\frac{y}{b} + \frac{x}{a} = 1$

$ay + bx = ab$

$bx = ab - a$

$$2\int_0^b \int_0^{\frac{a(b-y)}{b}} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\, dx\, dy$$

$$= 2\cdot \int_0^b \int_0^{\frac{a(b-y)}{b}} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dx\, dy$$

$$= 2c\int_0^b \frac{a(b-y)}{b} - \frac{a^2(b-y)^2}{2b^2 a} - \frac{y}{b}\left(\frac{a(b-y)}{b}\right) dy$$

$$= 2c\int_0^b a - \frac{a}{b}y - \frac{a}{2b^2}(b^2 - 2by + y^2) - \frac{ay}{b^2}(b-y)\, dy$$

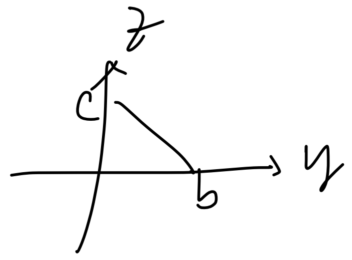
$$= 2c\int_0^b a - \frac{a}{b}y - \frac{a}{2} + \frac{a}{b}y - \frac{ay^2}{2b^2} - \frac{ay}{b} + \frac{ay^2}{b^2}\, dy$$

$$= 2c\int_0^b \frac{a}{2} - \frac{a}{b}y + \frac{ay^2}{2b^2}\, dy = 2c\left[\frac{a}{2}y - \frac{a}{b}\frac{y^2}{2} + \frac{ay^3}{6b^2}\right]_0^b$$

$$= 2c\left(\frac{ab}{2} - \frac{ab}{2} + \frac{ab}{6}\right) = \frac{abc}{3}$$

Made with Goodnotes

$$\iint_{S_2} \nabla\phi\cdot\hat{n}\,dS = \iint_{S_2} -y\,dS$$



$$\frac{z}{c}+\frac{y}{b}=1$$

$$bz+cy=bc$$

$$z=\frac{bc-cy}{b}$$

$$= \int_0^b \int_0^{\frac{bc-cy}{b}} -y\,dz\,dy$$

$$= \int_0^b -y\left(\frac{bc-cy}{b}\right)dy$$

$$= \frac{1}{b}\int_0^b -bcy + cy^2\,dy$$

$$= \frac{1}{b}\left[-bc\frac{y^2}{2} + \frac{cy^3}{3}\right]_0^b$$

$$= \frac{1}{b}\left(-bc\cdot\frac{b^2}{2} + \frac{cb^3}{3}\right)$$

$$= \frac{1}{b}\left(-\frac{cb^3}{2} + \frac{cb^3}{3}\right)$$

$$= \frac{1}{b}\left(-\frac{cb^3}{6}\right)$$

$$= \frac{-cb^2}{6}$$

Made with Goodnotes

$$\iint_{S3} \nabla\phi \cdot \hat{n}\, dS = \iint_{S_3} 2z\, dS = 0$$

$\hat{n} = \langle 0, 0, -1\rangle$

$$\iint_{S4} \nabla\phi \cdot \hat{n}\, dS = \iint_{S4} -x\, dS$$

$\hat{n} = \langle 0, -1, 0\rangle$

$$= \int_0^a \int_0^{\frac{ac-cx}{a}} -x\, dz\, dx$$

$$= \int_0^a -x\left(\frac{ac-cx}{a}\right) dx$$

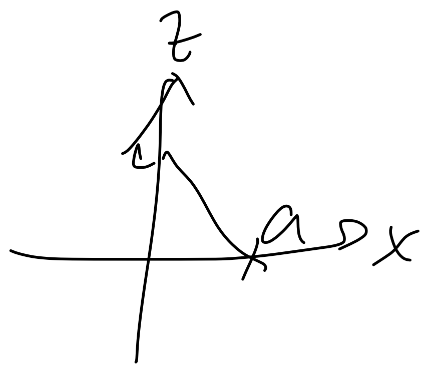
$$= \frac{1}{a}\int_0^a -acx + cx^2\, dx$$

$$= \frac{1}{a}\left[-ac\frac{x^2}{2} + c\frac{x^3}{3}\right]_0^a$$

$$= \frac{1}{a}\left[-\frac{a^3c}{2} + \frac{a^3c}{3}\right]$$

$$= \frac{1}{a}\left(-\frac{a^3c}{6}\right)$$

$$= -\frac{a^2c}{6}$$



$$\frac{z}{c} + \frac{x}{a} = 1$$

$$az + cx = ac$$

$$z = \frac{ac-cx}{a}$$

Made with Goodnotes

Ans: $\frac{2abc}{3} + \frac{b^2c}{6} + \frac{a^2c}{6}$

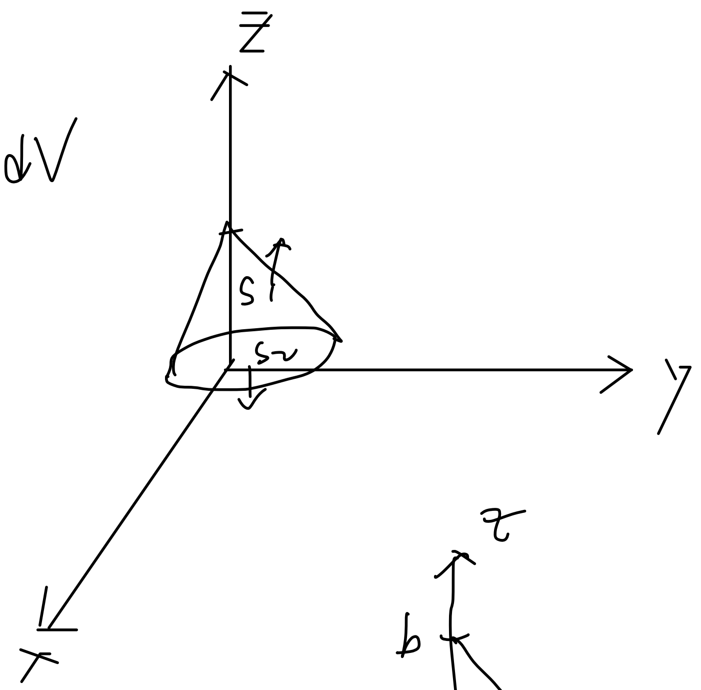
Made with Goodnotes

11. A conical domain with vertex $(0, 0, b)$ and axis along the z-axis has as base a disk of radius a in the xy-plane. Find the flux of

$$\mathbf{F} = (x + y^2)\mathbf{i} + (3x^2y + y^3 - x^3)\mathbf{j} + (z + 1)\mathbf{k}$$

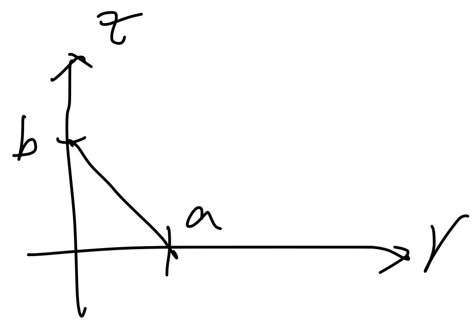
upward through the conical part of the surface of the domain.

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,dS + \iint_{S_2} \vec{F}\cdot\hat{n}\,dS = \iiint_E \nabla\cdot\vec{F}\,dV$$



$$\iiint_E \nabla\cdot\vec{F}\,dV$$

$$= \iiint_E (1+3x^2+3y^2+1)\,dV$$



$$= \iiint_E 3x^2+3y^2+2\,dV$$

$$\frac{z}{b}+\frac{r}{a}=1$$

$$az+br=ab$$

$$az = ab-br$$

$$= \int_0^{2\pi}\int_0^{a}\int_0^{\frac{ab-br}{a}} 3r^3+2r \quad dz\,dr\,d\theta$$

$$= 2\pi\cdot\int_0^a (3r^3+2r)\left(\frac{ab-br}{a}\right)dr$$

$$\frac{2\pi}{a}$$

$$= 2a\pi\int_0^a 3abr^3 - 3br^4 + 2abr - 2br^2\,dr$$

$$= 2a\pi\left[\frac{3abr^4}{4} - \frac{3br^5}{5} + abr^2 - \frac{2br^3}{3}\right]_0^a$$

$$= 2a\pi\left(\frac{3a^5b}{4} - \frac{3a^5b}{5} + a^3b - \frac{2a^3b}{3}\right) \quad \frac{3a^4b\pi}{10} + \frac{2a^2b\pi}{3}$$

$$= 2a\pi\left(\frac{3a^5b}{20} + \frac{1a^3b}{3}\right) = \cancel{\frac{3a^6b\pi}{10}} + \cancel{\frac{2a^4b\pi}{3}}$$

Made with Goodnotes

$$\iint_{S_2} \vec{F} \cdot \hat{n}\, dS = \iint_{S_2} \vec{F} \cdot \langle 0, 0, -1 \rangle\, dS$$

$$= \iint_{S_2} -(z+1)\, dS$$

$$= -\iint_{S_2} dS$$

$$= -\pi a^2$$

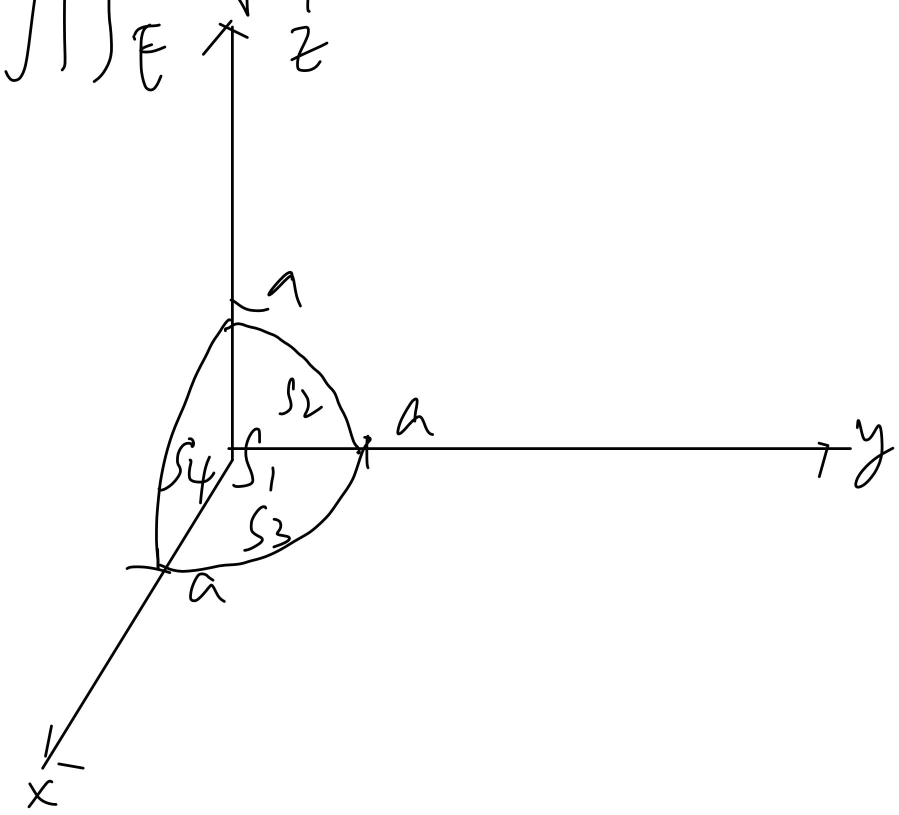
$$\iint_{S_1} \vec{F} \cdot \hat{n}\, dS = \iiint_G \nabla \cdot \vec{F}\, dV - \iint_{S_2} \vec{F} \cdot \hat{n}\, dS$$

$$= \frac{3a^6 b \pi}{10} + \frac{2a^4 b \pi}{3} + \pi a^2.$$

Made with Goodnotes

12. Find the flux of $\mathbf{F} = (y + xz)\mathbf{i} + (y + yz)\mathbf{j} - (2x + z^2)\mathbf{k}$ upward through the first octant part of the sphere $x^2 + y^2 + z^2 = a^2$.

$$\iint_{S_1} \vec{F} \cdot \hat{n}\, dS + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} = \iiint_E \nabla \cdot \vec{F}\, dV$$



$$\iiint_E \nabla \cdot \vec{F}\, dV$$

$$= \iiint_E z + 1 + z - 2z\, dV$$

$$= \iiint_E 1\, dV$$

$$= \frac{4}{3}\pi a^3 \div 8$$

$$= \frac{\pi a^3}{6}$$

Made with Goodnotes

$$\iint_{S_2} \vec{F} \cdot \hat{n} \, dS = \iint_{S_2} \vec{F} \cdot \langle -1, 0, 0 \rangle \, dS$$

$$= \iint_{S_2} -y - xz \, dS$$

$$= \iint_{S_2} -y \, dS$$

$$= \int_0^{\frac{\pi}{2}} \int_0^a -r^2 \cos\theta \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} -\frac{a^3}{3} \cos\theta \, d\theta$$

$$= -\frac{a^3}{3} \left[\sin\theta \right]_0^{\frac{\pi}{2}}$$

$$= -\frac{a^3}{3}$$

Made with Goodnotes

$$\iint_{S_3} \vec{F} \cdot \hat{n}\, dS = \iint_{S_3} \vec{F} \cdot \langle 0, 0, -1 \rangle \, dS$$

$$= \iint_{S_3} 2x + z^2 \, dS$$

$$= 2\iint_{S_3} x \, dS$$

$$= 2 \int_0^{\frac{\pi}{2}} \int_0^{a} r^2 \cos\theta \, dr \, d\theta$$

$$= 2 \int_0^{\frac{\pi}{2}} \frac{a^3}{3} \cos\theta \, d\theta$$

$$= 2 \cdot \frac{a^3}{3} \left[\sin\theta \right]_0^{\frac{\pi}{2}}$$

$$= \frac{2a^3}{3}$$

Made with Goodnotes

$$\iint_{S4} \vec{F} \cdot \hat{n} \, dS = \iint_{S4} \vec{F} \cdot \langle 0, -1, 0 \rangle \, dS$$

$$= \iint_{S4} -y - yz \, dS$$

$$= 0$$

$$\iint_{S1} \vec{F} \cdot \hat{n} \, dS = \iiint_E \nabla \cdot \vec{F} \, dV - \iint_{S2} \vec{F} \cdot \hat{n} \, dS - \iint_{S3} \vec{F} \cdot \hat{n} \, dS$$

$$= \frac{\pi a^3}{6} + \frac{a^3}{3} - \frac{2a^3}{3}$$

$$= \frac{\pi a^3}{6} - \frac{a^3}{3}.$$

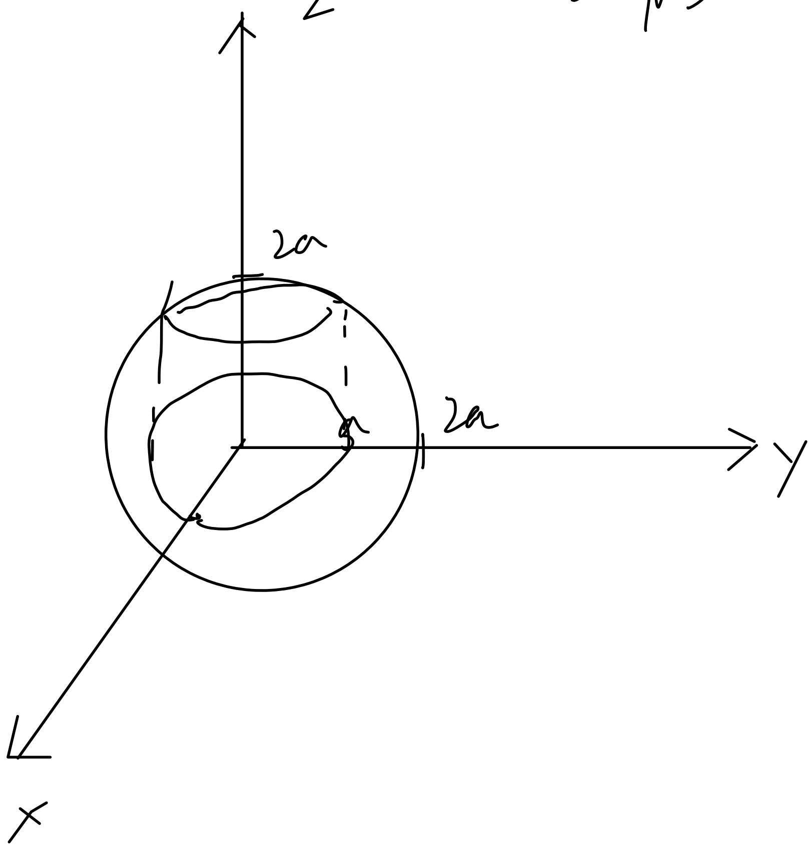
Made with Goodnotes

13. Let D be the region $x^2+y^2+z^2 \le 4a^2$, $x^2+y^2 \ge a^2$. The surface $\mathcal{S}$ of D consists of a cylindrical part, $\mathcal{S}_1$, and a spherical part, $\mathcal{S}_2$. Evaluate the flux of

$$\mathbf{F} = (x+yz)\mathbf{i} + (y-xz)\mathbf{j} + (z-e^x \sin y)\mathbf{k}$$

out of D through (a) the whole surface $\mathcal{S}$, (b) the surface $\mathcal{S}_1$, and (c) the surface $\mathcal{S}_2$.

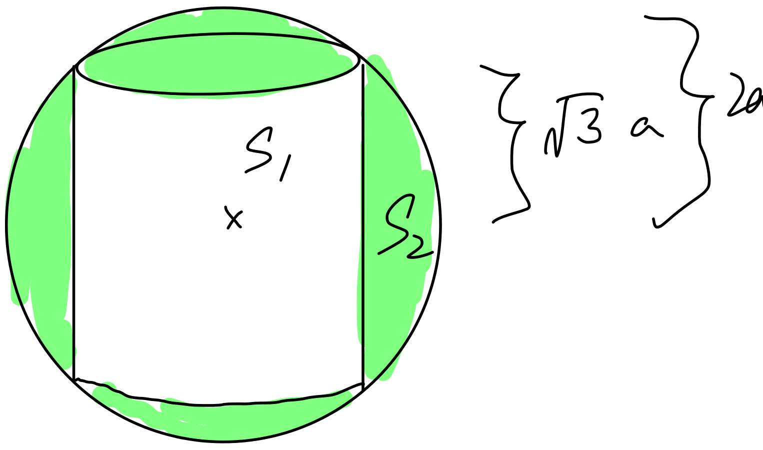
$$a^2+z^2 \le 4a^2$$
$$z^2 \le 3a^2$$
$$z = \sqrt{3}a.$$



a)

$$\iint_S \vec{F}\cdot\hat{n}\,dS$$

$$= \iiint \nabla\cdot\vec{F}\,dV - \iiint \nabla\cdot\vec{F}\,dV$$



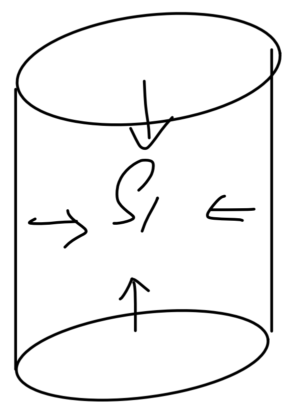
$$= 3\left(\frac{4}{3}\pi a^3 - a^2\pi(2\sqrt{3})\right)$$

$$= 4\pi a^3 - 6\sqrt{3}a^2\pi.$$

Made with Goodnotes

b).

Question = inward.



$$= -\iint_{S_1} \vec{F}\cdot\hat{n}\, dS \quad , \hat{n} = \text{outward.}$$

$$= -\iiint_E \nabla\cdot\vec{F}\, dV$$

$$= -3\iiint_E dV$$

$$= -3\left(\pi a^2 \cdot 2\sqrt{3}\right)$$

$$= -6\sqrt{3}\,\pi a^2$$

c).
$$\iint_{S_2} \vec{F}\cdot\hat{n}\, dS = 3\iiint_E dV$$

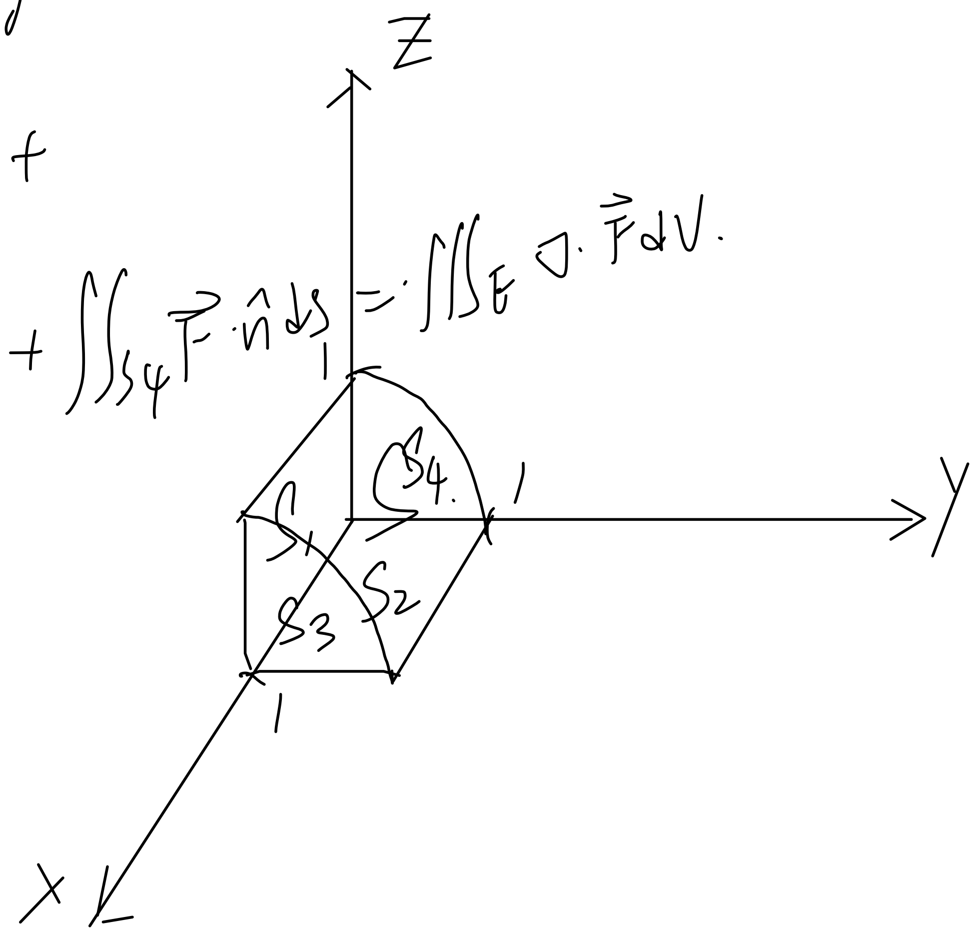
$$= 4\pi a^3.$$

Made with Goodnotes

14. Evaluate $\iint_{\mathcal{S}} (3xz^2\mathbf{i} - x\mathbf{j} - y\mathbf{k}) \bullet \hat{\mathbf{N}}\, dS$, where $\mathcal{S}$ is that part of the cylinder $y^2 + z^2 = 1$ that lies in the first octant and between the planes $x = 0$ and $x = 1$.

Let $\vec{F} = \langle 3xz^2, -x, -y \rangle$

$$\iint_S \vec{F}\cdot\hat{n}\,dS + \iint_{S_1} \vec{F}\cdot\hat{n}\,dS + \iint_{S_2} \vec{F}\cdot\hat{n}\,dS + \iint_{S_3} \vec{F}\cdot\hat{n}\,dS + \iint_{S_4} \vec{F}\cdot\hat{n}\,dS = \iiint_E \nabla\cdot\vec{F}\,dV.$$



$$\iiint_E \nabla\cdot\vec{F}\,dV$$

For $\iiint_E \nabla\cdot\vec{F}\,dV$,

$$\iiint_E 3z^2\,dV$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1\int_0^1 3r^3\sin^2\theta\,dx\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 3r^3\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{3}{4}\sin^2\theta\,d\theta \qquad = \frac{3\pi}{16}.$$

$$= \frac{3}{4}\int_0^{\frac{\pi}{2}} \frac{1}{2} - \frac{\cos 2\theta}{2}\,d\theta = \frac{3}{4}\left(\frac{\pi}{4}\right) - \frac{3}{4}\left[\frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}$$

Made with Goodnotes

$$\iint_{S_1} \vec{F} \cdot \hat{n}\, dS = \iint_{S_1} \vec{F} \cdot \langle 0, -1, 0 \rangle\, dS$$

$$= \iint_{S_1} x\, dS$$

$$= \int_0^1 \int_0^1 x\, dz\, dx$$

$$= \frac{1}{2}.$$

$$\iint_{S_2} \vec{F} \cdot \hat{n}\, dS = \iint_{S_2} \vec{F} \cdot \langle 0, 0, -1 \rangle\, dS$$

$$= \iint_{S_2} y\, dS$$

$$= \int_0^1 \int_0^1 y\, dx\, dy$$

$$= \frac{1}{2}.$$

Made with Goodnotes

$$\iint_S \vec{F} \cdot \hat{n}\, dS = \frac{3\pi}{16} - 1$$

$$= \frac{3\pi - 16}{16} \quad \times$$

$S_3 \neq S_4$!

Made with Goodnotes

15. A solid region R has volume V and centroid at the point $(\bar{x}, \bar{y}, \bar{z})$. Find the flux of

$$\mathbf{F} = (x^2 - x - 2y)\mathbf{i} + (2y^2 + 3y - z)\mathbf{j} - (z^2 - 4z + xy)\mathbf{k}$$

out of R through its surface.

$$\iiint_E 2x - 1 + 4y + 3 - 2z + 4 \, dV$$

$$= \iiint_E 2x + 4y - 2z + 6 \, dV$$

$$= (2\bar{x} + 4\bar{y} - 2\bar{z})V \quad + 6V$$

Made with Goodnotes

16. The plane $x + y + z = 0$ divides the cube $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$ into two parts. Let the lower part (with one vertex at $(-1, -1, -1)$) be D. Sketch D. Note that it has seven faces, one of which is hexagonal. Find the flux of $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ out of D through each of its faces.

$(0, 0, 0)$

$(-1, 1, 0)$

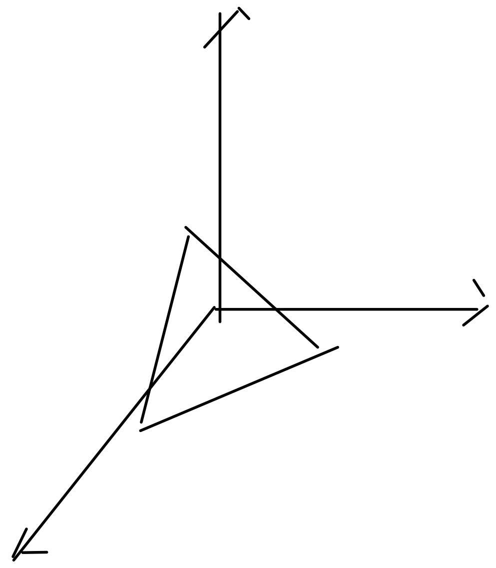
$(1, -1, 0)$.

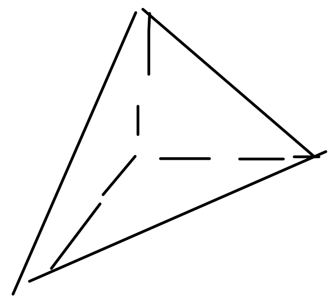
$(-1, 0, 1)$

$(1, 0, -1)$

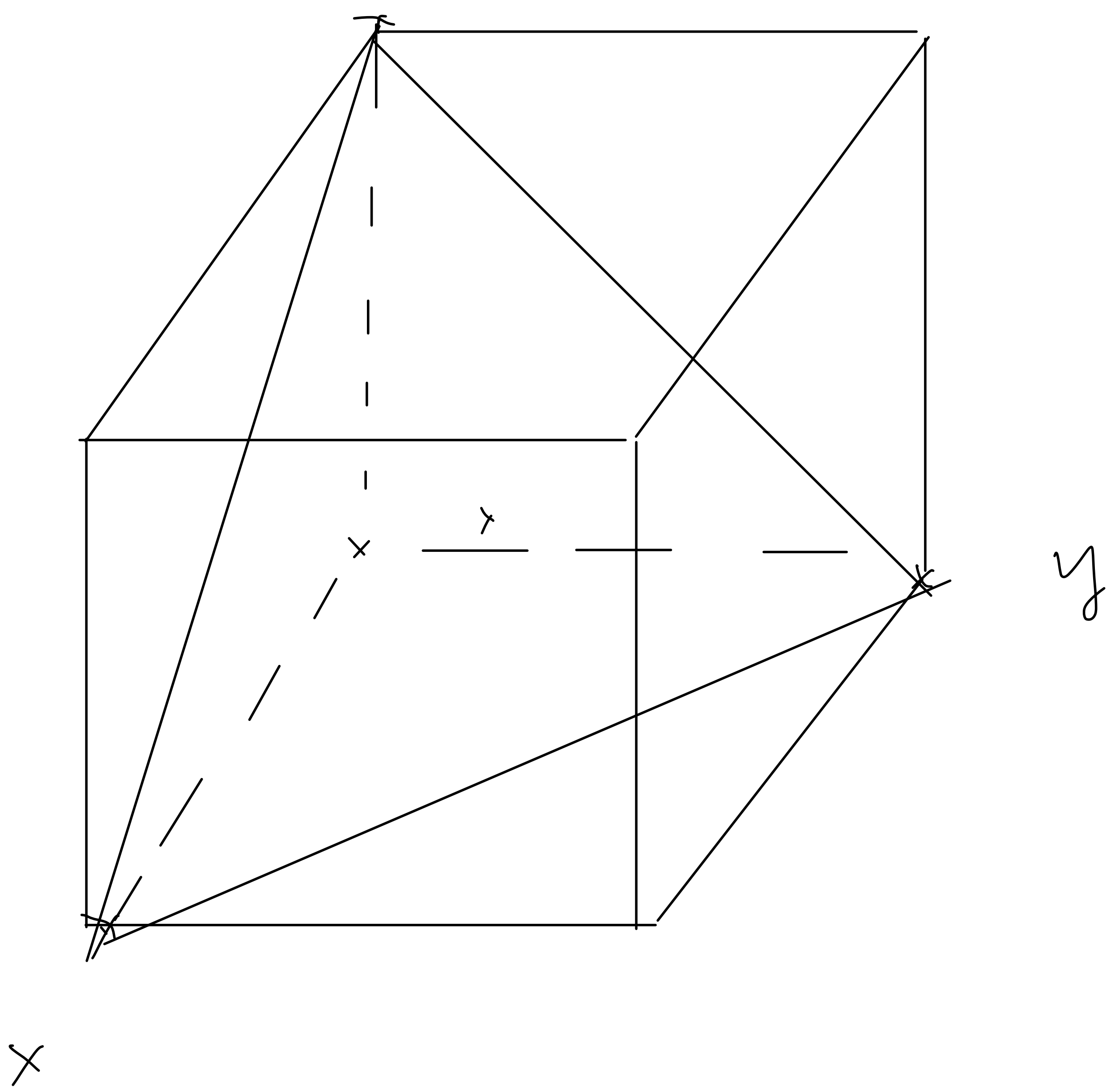
$z = -x - y$.

When $z = 0$,
$x = -y$.





z. ??



Made with Goodnotes

17. Let $\mathbf{F} = (x^2 + y + 2 + z^2)\mathbf{i} + (e^{x^2} + y^2)\mathbf{j} + (3 + x)\mathbf{k}$. Let $a > 0$, and let $\mathcal{S}$ be the part of the spherical surface $x^2 + y^2 + z^2 = 2az + 3a^2$ that is above the xy-plane. Find the flux of $\mathbf{F}$ outward across $\mathcal{S}$.

$$x^2+y^2+z^2-2az+a^2-a^2=3a^2$$

$$x^2+y^2+(z-a)^2=4a^2$$

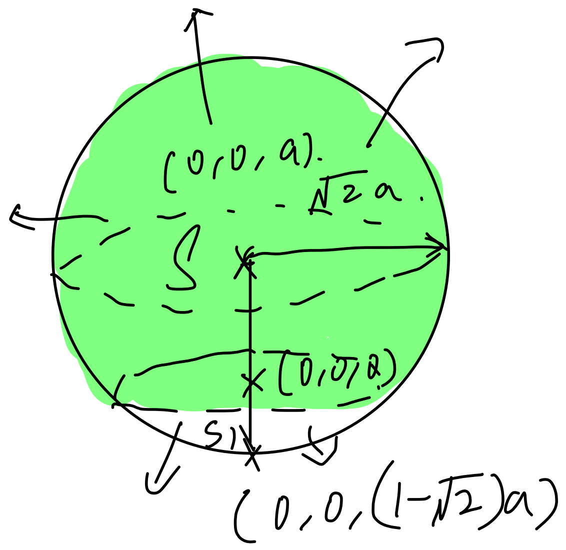
Sub $z=0$: $x^2+y^2=3a^2$

$$x^2+y^2+z^2-2az+a^2=2a^2$$

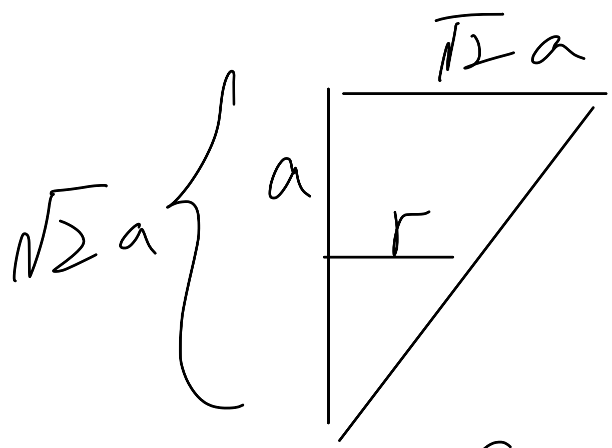
$$x^2+y^2+(z-a)^2=2a^2, \quad z \geq 0.$$

$$\vec{F}(\rho,\varphi,\theta) = \langle \rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi + a \rangle$$

$$0 \leq \rho \leq \sqrt{2}a$$



$$\iint_S \vec{F}\cdot\hat{n}\,dS + \iint_{S_1} \vec{F}\cdot\hat{n}\,dS = \iiint_E \nabla\cdot\vec{F}\,dV.$$



$$\frac{r}{\sqrt{2}a} = \frac{(\sqrt{2}-1)a}{\sqrt{2}a}$$

$$\sqrt{2}r = \sqrt{2}(\sqrt{2}-1)$$

$$\sqrt{2}r = 2-\sqrt{2}$$

$$r = \frac{2-\sqrt{2}}{\sqrt{2}}$$

$$r = \frac{2\sqrt{2}-2}{2}$$

$$r = \sqrt{2}-1$$

Made with Goodnotes

For $\iiint_E \nabla \cdot \vec{F}\, dV$

$$= \iiint_E 2x + 2y \, dV.$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\sqrt{2}a} (2\rho \sin\varphi \cos\theta + 2\rho \sin\varphi \sin\theta)(\rho^2 \sin\varphi) \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\sqrt{2}a} 2\rho^3 \sin^2\varphi \cos\theta + 2\rho^3 \sin^2\varphi \sin\theta \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{2(\sqrt{2})^4 a^4}{4} \sin^2\varphi \cos\theta + \frac{2(\sqrt{2})^4 a^4}{4} \sin^2\varphi \sin\theta \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 4a^4 \sin^2\varphi \cos\theta + 4a^4 \sin^2\varphi \sin\theta \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 4a^4\left(\frac{1}{2} - \frac{\cos 2\varphi}{2}\right)\cos\theta + 4a^4\left(\frac{1}{2} - \frac{\cos 2\varphi}{2}\right)\sin\theta \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 2a^4\cos\theta - 2a^4\cos\theta\cos 2\varphi + 2a^4\sin\theta - 2a^4\cos 2\varphi \sin\theta \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} 2a^4\pi\cos\theta - \left[2a^4\cos\theta \frac{\sin 2\varphi}{2}\right]_0^{\pi} + 2a^4\pi\sin\theta - \left[2a^4 \frac{\sin 2\varphi}{2}\right]_0^{\pi} d\theta$$

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$$= \int_0^{2\pi} 2a^4\pi\cos\theta + 2a^4\pi\sin\theta \, d\theta$$

$$= 2a^4\pi\left[\sin\theta\right]_0^{2\pi} + 2a^4\pi\left[-\cos\theta\right]_0^{2\pi}$$

$$= 0.$$

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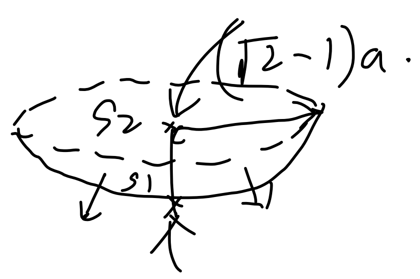
FTr $\iint_{S_1} \vec{F}\cdot\hat{n}\,dS + \iint_{S_2} \vec{F}\cdot\hat{n}\,dS = \iiint_E \nabla\cdot\vec{F}\,dV$

FTr $\iiint_E \nabla\cdot\vec{F}\,dV$,



$\iiint_E 2x+2y\,dV$

$\int_0^{2\pi}\int_0^{(\sqrt{2}-1)a}\int_{(1-\sqrt{2})a}^{0} 2r^2\cos\theta + 2r^2\sin\theta\,dz\,dr\,d\theta$

$= (1-\sqrt{2})a \int_0^{2\pi} \left[\frac{2r^3}{3}\cos\theta + \frac{2r^3}{3}\sin\theta\right]_0^{(\sqrt{2}-1)a} d\theta$

$= (1-\sqrt{2})a \left(\int_0^{2\pi} \frac{2a^3}{3}(\sqrt{2}-1)^3\cos\theta + \frac{2a^3}{3}(\sqrt{2}-1)^3\sin\theta\,d\theta\right)$

$= (1-\sqrt{2})a \left(\frac{2a^3}{3}(\sqrt{2}-1)^3\left[\sin\theta\right]_0^{2\pi} + \frac{2a^3}{3}(\sqrt{2}-1)^3\left[-\cos\theta\right]_0^{2\pi}\right)$

$= 0$.

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$$\iint_{S_2\uparrow} \vec{F}\cdot\hat{n}\,dS$$

$$= \iint_{S_2} \vec{F} \cdot \langle 0,0,1\rangle \, dS$$

$$= \iint_{S_2} 3+x \, dS$$

$$= \int_0^{2\pi}\int_0^{(\sqrt{2}-1)a} 3r + r^2\cos\theta \;\; dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{3r^2}{2} + \frac{r^3}{3}\cos\theta\right]_0^{(\sqrt{2}-1)a} d\theta$$

$$= \int_0^{2\pi} \frac{3a^2}{2}(\sqrt{2}-1)^2 + \frac{a^3}{3}(\sqrt{2}-1)^3\cos\theta \;\; d\theta$$

$$= \frac{3a^2}{2}(\sqrt{2}-1)^2\cdot 2\pi$$

$$= 3\pi a^2(\sqrt{2}-1)^2.$$

$$\iint_{\substack{S_1\\ \swarrow\downarrow\searrow}} \vec{F}\cdot\hat{n}\,dS = -3\pi a^2(\sqrt{2}-1)^2$$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = 3\pi a^2(\sqrt{2}-1)^2$$

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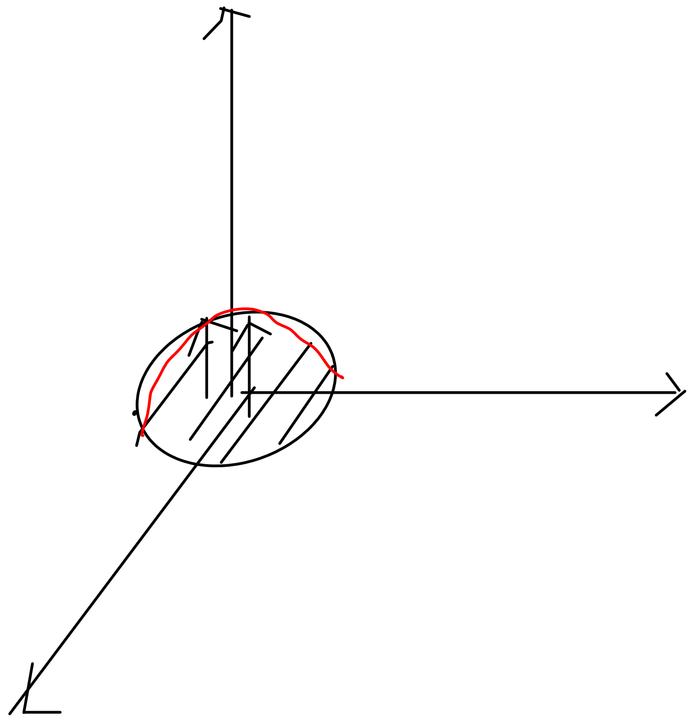
18. A pile of wet sand having total volume 5π covers the disk $x^2 + y^2 \leq 1$, $z = 0$. The momentum of water vapour is given by $\mathbf{F} = \mathbf{grad}\,\phi + \mu\,\mathbf{curl}\,\mathbf{G}$, where $\phi = x^2 - y^2 + z^2$ is the water concentration, $\mathbf{G} = \frac{1}{3}(-y^3\mathbf{i} + x^3\mathbf{j} + z^3\mathbf{k})$, and μ is a constant. Find the flux of $\mathbf{F}$ upward through the top surface of the sand pile.



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In Exercises 19–29, D is a three-dimensional domain satisfying the conditions of the Divergence Theorem, and $\mathcal{S}$ is its surface. $\hat{\mathbf{N}}$ is the unit outward (from D) normal field on $\mathcal{S}$. The functions ϕ and ψ are smooth scalar fields on D. Also, $\partial\phi/\partial n$ denotes the first directional derivative of ϕ in the direction of $\hat{\mathbf{N}}$ at any point on $\mathcal{S}$:

$$\frac{\partial \phi}{\partial n} = \nabla\phi \bullet \hat{\mathbf{N}}.$$

19. Show that $\oiint_{\mathcal{S}} \mathbf{curl}\,\mathbf{F} \bullet \hat{\mathbf{N}}\,dS = 0$, where $\mathbf{F}$ is an arbitrary smooth vector field.

$$\oiint_S \nabla\times F\cdot\hat{N}\,dS = \iiint_D \nabla\cdot(\nabla\times\vec{F})\,dV$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0.$$

$R_y - Q_z,\ P_z - R_x,\ Q_x - P_y.$

$R_{xy} - Q_{xz} + P_{zy} - R_{xy} + Q_{xz} - P_{yx}.$

$= 0.$

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20. Show that the volume V of D is given by

$$V = \frac{1}{3} \oiint_{\mathcal{S}} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \bullet \hat{\mathbf{N}}\, dS.$$

21. If D has volume V, show that

$$\bar{\mathbf{r}} = \frac{1}{2V} \oiint_{\mathcal{S}} (x^2 + y^2 + z^2)\hat{\mathbf{N}}\, dS$$

is the position vector of the centre of gravity of D.

22. Show that $\oiint_{\mathcal{S}} \nabla\phi \times \hat{\mathbf{N}}\, dS = 0.$

23. If $\mathbf{F}$ is a smooth vector field on D, show that

$$\iiint_D \phi \,\mathbf{div}\,\mathbf{F}\, dV + \iiint_D \nabla\phi \bullet \mathbf{F}\, dV = \oiint_{\mathcal{S}} \phi\mathbf{F} \bullet \hat{\mathbf{N}}\, dS.$$

Hint: Use Theorem 3(b) from Section 16.2.

$\iiint_D 1\, dV$ let $\vec{F} = \langle x, 0, 0 \rangle$

$= \oiint_S x\hat{i} \cdot \hat{n}\, dS = \oiint_S y\hat{j} \cdot \hat{n}\, dS =$

$\oiint_S z k \cdot \hat{n}\, dS$

$$3V = \oiint_S (x\hat{i} + y\hat{j} + zk) \cdot \hat{n}\, dS$$

$$V = \frac{1}{3} \oiint_S (x\hat{i} + y\hat{j} + zk) \cdot \hat{n}\, dS.$$

Made with Goodnotes

21. If D has volume V, show that

$$\bar{\mathbf{r}} = \frac{1}{2V} \oiint_{\mathcal{S}} (x^2 + y^2 + z^2)\hat{\mathbf{N}}\, dS$$

is the position vector of the centre of gravity of D.

$\vec{r_x} =$ $\qquad$ Let $\vec{F} = \langle \frac{1}{2}x^2, 0, 0 \rangle$

$$\frac{\iiint_E x \, dV}{\iiint_E \sigma \, dV}$$

$$= \frac{\oiint_S \frac{1}{2} x^2 \vec{i} \cdot \hat{N} \, dS}{V}$$

$$= \frac{1}{2V} \oiint x^2 \vec{i} \cdot \hat{N} \, dS.$$

r_y, r_z 如此~~解~~推。

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22. Show that $\oiint_{\mathcal{S}} \nabla\phi \times \hat{\mathbf{N}}\, dS = 0.$?

$$\oiint_S \nabla\phi \times \hat{N}\, dS$$

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23. If $\mathbf{F}$ is a smooth vector field on D, show that

$$\iiint_D \phi \operatorname{div} \mathbf{F}\, dV + \iiint_D \nabla\phi \bullet \mathbf{F}\, dV = \oiint_{\mathcal{S}} \phi \mathbf{F} \bullet \hat{\mathbf{N}}\, dS.$$

Hint: Use Theorem 3(b) from Section 16.2.

$$\iiint_D \phi\, \nabla\cdot F\, dV + \iiint_D \nabla\phi\, F\, dV$$

$$= \iiint_D \nabla\cdot(\phi\vec{F})\, dV$$

$$= \oiint \phi\vec{F}\cdot\hat{N}\, dS \qquad \vec{F} = \nabla\phi$$

$$\oiint_S \phi\nabla\phi\cdot\hat{N}\, dS = \iiint_D \nabla\phi\cdot\vec{F}\, dV + \iiint_D \phi\,\nabla\cdot F\, dV$$

$$= \iiint_D \nabla\phi\cdot\nabla\phi\, dV + \iiint_D \phi\,\nabla^2\phi\, dV$$

$$= \iiint_D |\nabla\phi|^2\, dV + 0.$$

Since $\oiint_S \phi\nabla\phi\cdot\hat{N}\, dS = 0$,

And $|\nabla\phi|^2$ is non-negative,

the only possible value for $\nabla\phi$ is 0.

$\nabla\phi = 0$, ϕ = constant function.

And due to continuity of function,

If $\phi(x,y,z) = 0$ in S, $\phi(x,y,z) = 0$ in D.

Made with Goodnotes

Properties of the Laplacian operator

24. If $\nabla^2\phi = 0$ in D and $\phi(x, y, z) = 0$ on $\mathcal{S}$, show that $\phi(x, y, z) = 0$ in D. *Hint:* Let $\mathbf{F} = \nabla\phi$ in Exercise 23.

25. **(Uniqueness for the Dirichlet problem)** The Dirichlet problem for the Laplacian operator is the boundary-value problem

$$\begin{cases} \nabla^2 u(x, y, z) = f(x, y, z) & \text{on } D \\ u(x, y, z) = g(x, y, z) & \text{on } \mathcal{S}, \end{cases}$$

where f and g are given functions defined on D and $\mathcal{S}$, respectively. Show that this problem can have at most one solution $u(x, y, z)$. *Hint:* Suppose there are two solutions, u and v, and apply Exercise 24 to their difference $\phi = u - v$.

24.

$$\oiint_S \nabla\phi \cdot \hat{n}\, dS = \iiint_D \nabla \cdot \nabla\phi\, \hat{n}\, dS$$

$$= \iiint_0 \nabla^2\phi\, \hat{n}\, dS$$

$$= 0$$

Made with Goodnotes

25. (Uniqueness for the Dirichlet problem) The Dirichlet problem for the Laplacian operator is the boundary-value problem

$$\begin{cases} \nabla^2 u(x, y, z) = f(x, y, z) & \text{on } D \\ u(x, y, z) = g(x, y, z) & \text{on } \mathcal{S}, \end{cases}$$

where f and g are given functions defined on D and $\mathcal{S}$, respectively. Show that this problem can have at most one solution $u(x, y, z)$. *Hint:* Suppose there are two solutions, u and v, and apply Exercise 24 to their difference $\phi = u - v$.

$$f(u(x,y,z)) = \begin{cases} \nabla^2 u = 0 \text{ on } D \\ u = 0 \text{ on } S \end{cases} \Big/ \begin{cases} \nabla^2 v = 0 \text{ on } D \\ v = 0 \text{ on } S. \end{cases}$$

$$\phi = u - v$$

$$\nabla \phi = \nabla u - \nabla v$$

$$\nabla^2 \phi = \nabla^2 u - \nabla^2 v$$

$$\nabla^2 \phi = 0 - 0$$

$$\therefore \nabla^2 \phi = 0 \text{ in } D$$

$$\phi = 0 - 0 = 0 \text{ on } S$$

using the result for Ex 24. $\phi(x,y,z) = 0$ in D.

$$u(x,y,z) - v(x,y,z) = 0$$

$$u(x,y,z) = v(x,y,z)$$

This prob. can have at most one solution $u(x,y,z)$.

Made with Goodnotes

26. (The Neumann problem) If $\nabla^2\phi = 0$ in D and $\partial\phi/\partial n = 0$ on $\mathcal{S}$, show that $\nabla\phi(x, y, z) = 0$ on D. The Neumann problem for the Laplacian operator is the boundary-value problem

$$\begin{cases} \nabla^2 u(x, y, z) = f(x, y, z) & \text{on } D \\ \dfrac{\partial}{\partial n} u(x, y, z) = g(x, y, z) & \text{on } \mathcal{S}, \end{cases}$$

where f and g are given functions defined on D and $\mathcal{S}$, respectively. Show that, if D is connected, then any two solutions of the Neumann problem must differ by a constant on D.

$$\nabla^2\phi = 0 \text{ in } D$$

$$\frac{\partial\phi}{\partial n} = 0 \text{ on } S.$$

Let $\vec{F} = \nabla\phi$

$$\oint\!\!\oint_S \phi_n \nabla\phi \, dS = \iiint_D \nabla\phi_n \cdot \nabla\phi \, dV + \iiint_D \phi_n \nabla^2\phi \, dV$$

$$0 = \iiint_D \nabla\phi_n \cdot \nabla\phi \, dV$$

$$\nabla\phi_n \cdot \nabla\phi = 0$$

Since $\phi_n = 0$, $\nabla\phi_n = \vec{0}$. ???

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27. Verify that $\iiint_D \nabla^2 \phi \, dV = \oiint_{\mathcal{S}} \frac{\partial \phi}{\partial n} \, dS.$

28. Verify that

$$\iiint_D \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi \right) dV = \oiint_{\mathcal{S}} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dS.$$

$$\iiint_D \nabla^2 \phi \, dV = \iiint_D \nabla \cdot \nabla \phi \, dS$$

$$= \oiint_S \nabla \phi \cdot \hat{n} \, dS$$

$$= \oiint_S \phi_x [n_i] + \phi_y [n_j] + \phi_z [n_k] \, dS$$

???

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29. By applying the Divergence Theorem to $\mathbf{F} = \phi\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector, show that

$$\iiint_D \nabla\phi \, dV = \oiint_S \phi\hat{\mathbf{N}} \, dS.$$

$\vec{c} = \langle a, b, c \rangle$

$\vec{F} \cdot \hat{N} dS$

$\vec{F} \cdot \phi c.$

$$\oiint_S \phi\vec{c} \cdot \hat{N} \, dS = \iiint_D \nabla \cdot (\phi\vec{c}) \, dV$$

$$= \iiint_D \nabla\phi \cdot \vec{c} \, dV + \iiint_D \phi(\nabla \cdot \vec{c}) \, dV$$

$$= \iiint_D \nabla\phi \cdot \vec{c} \, dV$$

$$\oiint_S \phi a\hat{N}_i + \phi_b \hat{N}_j + \phi_c \hat{N}_z \, dS = \iiint_D a\phi_x + b\phi_y + c\phi_z \, dV.$$

$$\oiint_S \phi\hat{N} \, dS \quad = \quad \iiint_D \nabla\phi \, dV \quad \cdot \langle a, b, c \rangle$$

?

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30. Let P_0 be a fixed point, and for each $\epsilon > 0$ let D_ϵ be a domain with boundary $\mathcal{S}_\epsilon$ satisfying the conditions of the Divergence Theorem. Suppose that the maximum distance from P_0 to points P in D_ϵ approaches zero as $\epsilon \to 0+$. If D_ϵ has volume $\operatorname{vol}(D_\epsilon)$, show that

$$\lim_{\epsilon \to 0+} \frac{1}{\operatorname{vol}(D_\epsilon)} \oiint_{\mathcal{S}_\epsilon} \mathbf{F} \bullet \hat{\mathbf{N}}\, dS = \mathbf{div}\, \mathbf{F}(P_0).$$

This generalizes Theorem 1 of Section 16.1.

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