MATH2023
Tutorial 11
May 5, 2019 Tutorial Notes by Ivan So

1 Review

- The **triple integral** is defined as

$$\int\int\int_D f(x,y,z)dV := \lim_{n\to\infty}\sum_{i,j,k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_i.$$

  Interpretation: See triple integration as the calculation of "mass" of an object. Think of the function f as the "density" function, then $f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_i$ is the mass of part of the object. Then $\sum_{i,j,k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_i$ will approximate the "mass" of the whole object.
  Other examples:

  1. See f as the charge density of an object, then $\int\int\int_D f dV$ calculate the amount of charge of the object D.
  2. See f as the function representing the density of air molecules, then $\int\int\int_D f dV$ calculate the number of molecules in region D.

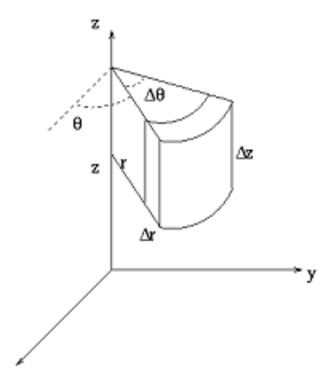
- The **Fubini's theorem** applies in 3D (less technically, the order of integration can be changed).

- Alternative coordinate systems:

  – **Cylindrical Coordinates**: We redefine our coordinate system by

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

  The *differential volume* is given by $dV = rdrd\theta dz$. Pictorially, the following represent the object with volume dV:



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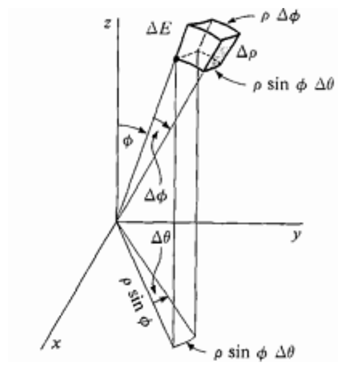
– **Spherical Coordinates**: We redefine our coordinate system by

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

The *differential volume* is given by $dV = \rho^2 \sin\phi d\rho d\phi d\theta$. Pictorially, the following represent the object with volume dV:



– Change of variable in general: Change of variable can be done by inserting **Jacaobian** in the integration. Explicitly,

$$\int \cdots \int_R f(\mathbf{x}) dR = \int \cdots \int_D f(\mathbf{x}(\mathbf{y})) \det \frac{\partial \mathbf{x}}{\partial \mathbf{y}} d^n \mathbf{y}.$$

Simply, for our consideration:

* In 2D: We can think of change of variable as "no longer integrating over a flat surface", i.e. a surface integral:

$$\int\int_{S_{x,y}} f(x,y) dS = \int\int_{R_{u,v}} f(x(u,v), y(u,v)) \underbrace{|\mathbf{r}_u \times \mathbf{r}_v|}_{\text{Jacobian}} dudv.$$

* In 3D: We approximate the differential volume with **parallelepiped** (recall tutor 1). Given the approximation process.

$$\begin{aligned} &\int\int\int_{D_{x,y,z}} f(x,y,z) dD \\ = &\int\int\int_{R_{u,v,w}} f(x(u,v,w), y(u,v,w), z(u,v,w)) \underbrace{|(\mathbf{r}_u \times \mathbf{r}_v) \cdot \mathbf{r}_w|}_{\text{Jacobian}} dudvdw \end{aligned}$$

Remark: The absolute sign of Jacobian guarantee it is always non-negative.

2 Problems

1. Rewrite the integral
$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)dydzdx$$
in other five different orders.

2. Find the center of mass of the solid defined by $0 \le x, y, z \le a$ where the density is $\rho(x,y,z) = x^2 + y^2 + z^2$.

3. Sketch the solid whose volume is given by the integral
$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dxdzdy.$$

4. Evaluate the integral

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dzdydx.$$

5. Find the volume of the intersection solid of perpendicular cylinders.

6. Sketch the solid whose volume is given by the integral

$$\int_{0}^{\pi/3}\int_{0}^{\pi/6}\int_{0}^{3}\rho^2\sin\phi\,d\rho d\theta d\phi.$$

7. Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xydzdydx.$$

8. Rewrite the integral $\int\int\int_R xydV$ under the change of coordinates $x = v + w^2,\ y = w + u^2,\ z = u + v^3$.

9. Evaluate $\int\int_R \sin(9x^2 + 4y^2)dA$, where R is the region with lying inside the ellipse $9x^2 + 4y^2 = 1$ in the first quadrant.

2 Problems

1. Rewrite the integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\,dy\,dz\,dx$$

in other five different orders.

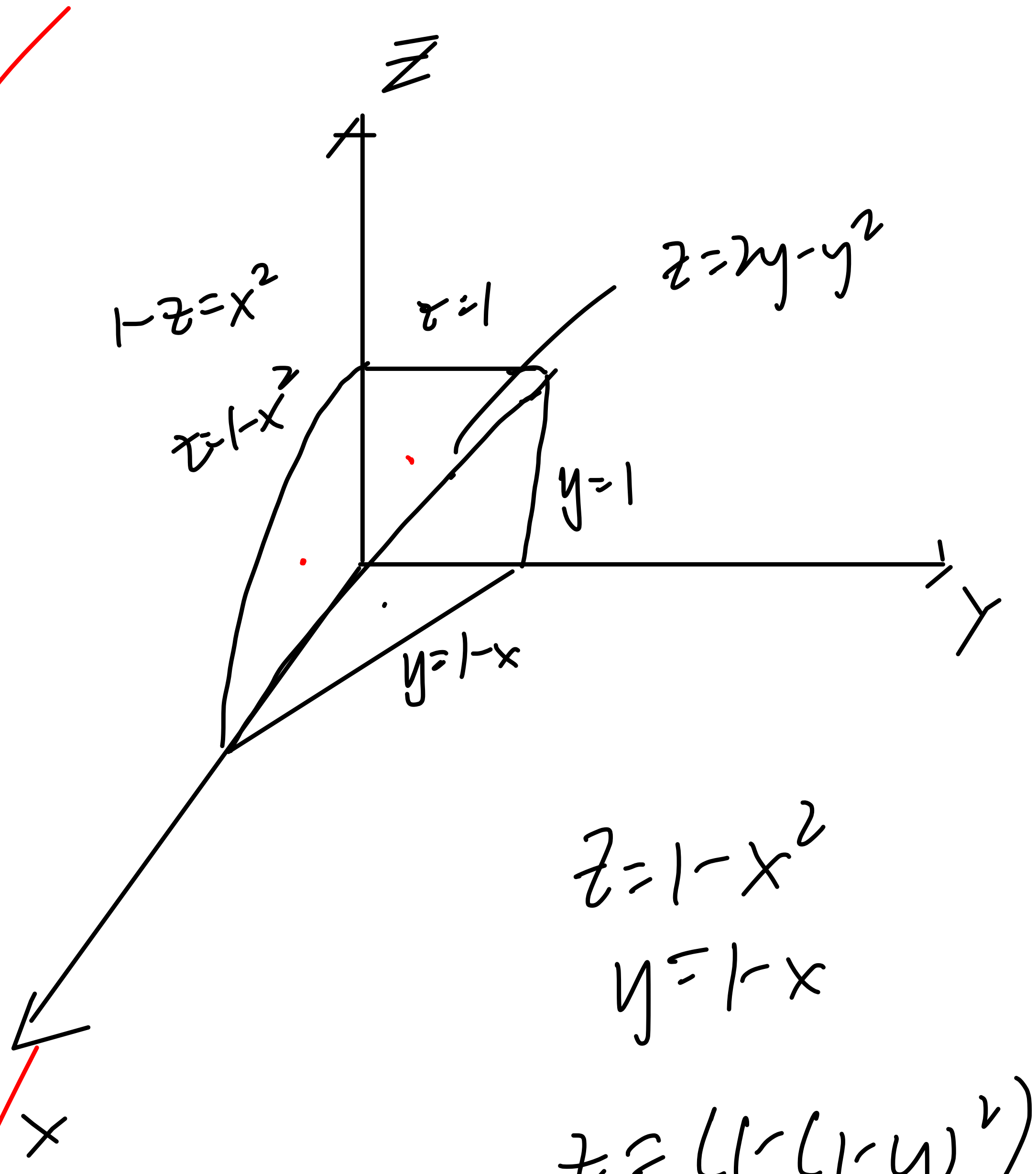
$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} dy\,dx\,dz$$

$$\int_0^1 \int_0^1 \int_0^{1-y} dx\,dy\,dz$$

$$\int_0^1 \int_0^1 \int_0^{1-y} dx\,dz\,dy$$

$$\int_0^1 \int_0^{1-y} \int_0^{1-x^2} dz\,dx\,dy$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x^2} dz\,dy\,dx$$



$z=1-x^2$

$y=1-x$

$z=(1-(1-y)^2)$

$z=1-(1-2y+y^2)$

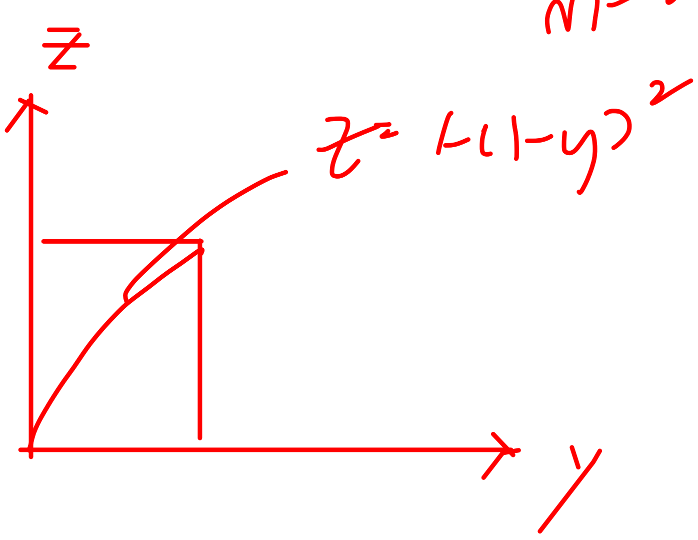
$z=2y-y^2$

$1-z=(1-y)^2$

$\sqrt{1-z}=1-y$

$1-\sqrt{1-z}=y$

$1-(1-y)^2=z$



$$\int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} dx\,dy\,dz$$

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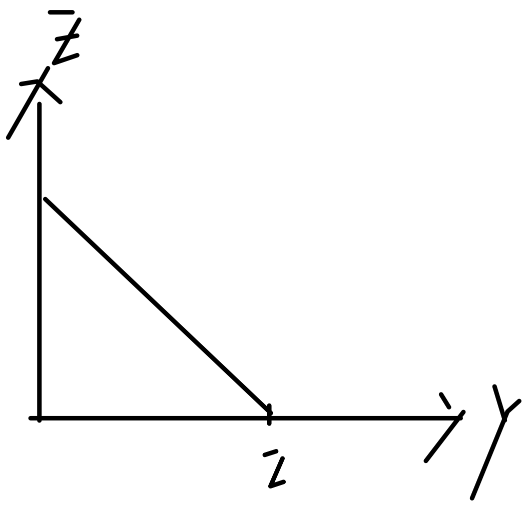
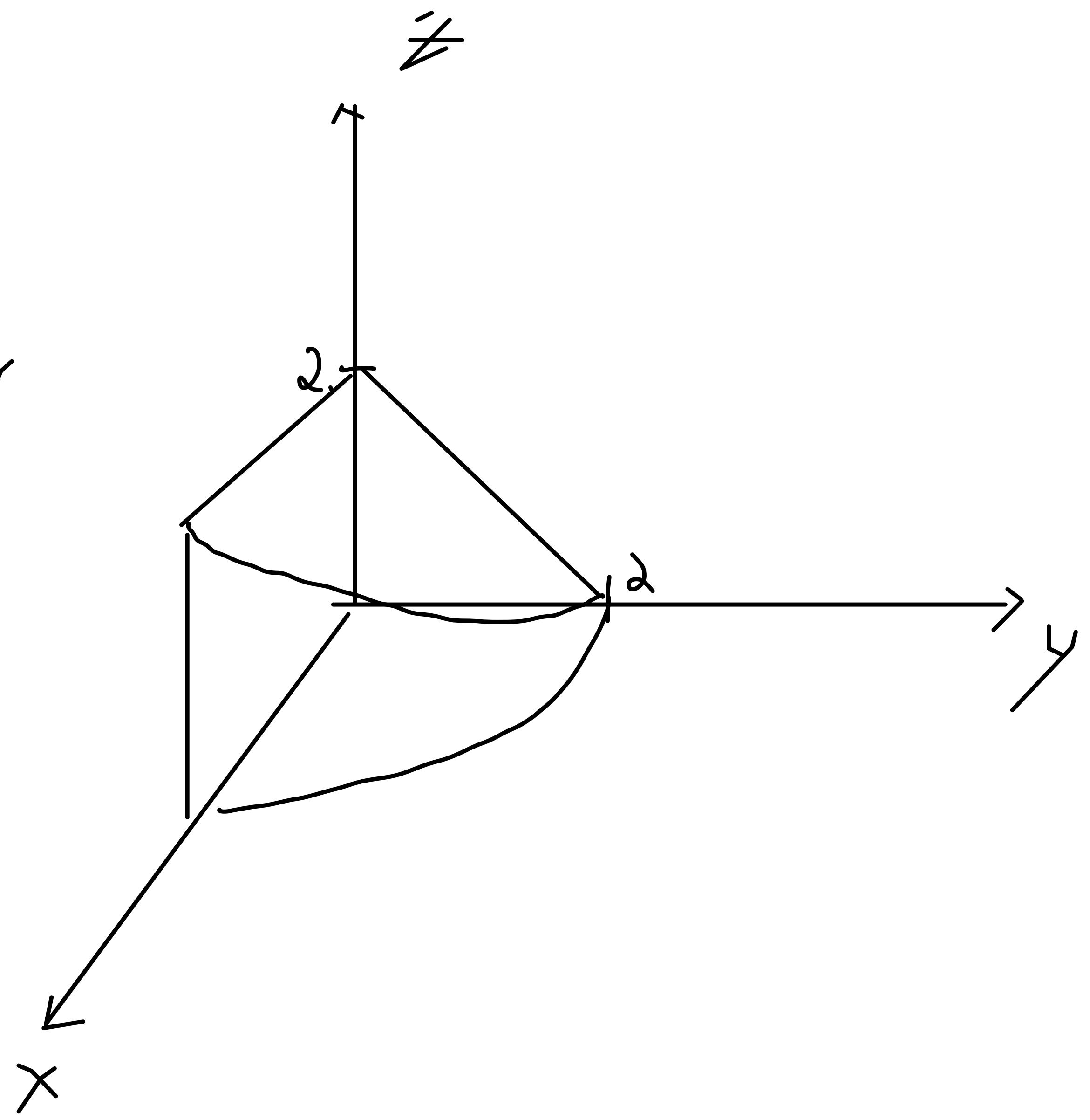
2. Find the center of mass of the solid defined by $0 \leq x, y, z \leq a$ where the density is $\rho(x, y, z) = x^2 + y^2 + z^2$.

3. Sketch the solid whose volume is given by the integral

$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\,dz\,dy.$$





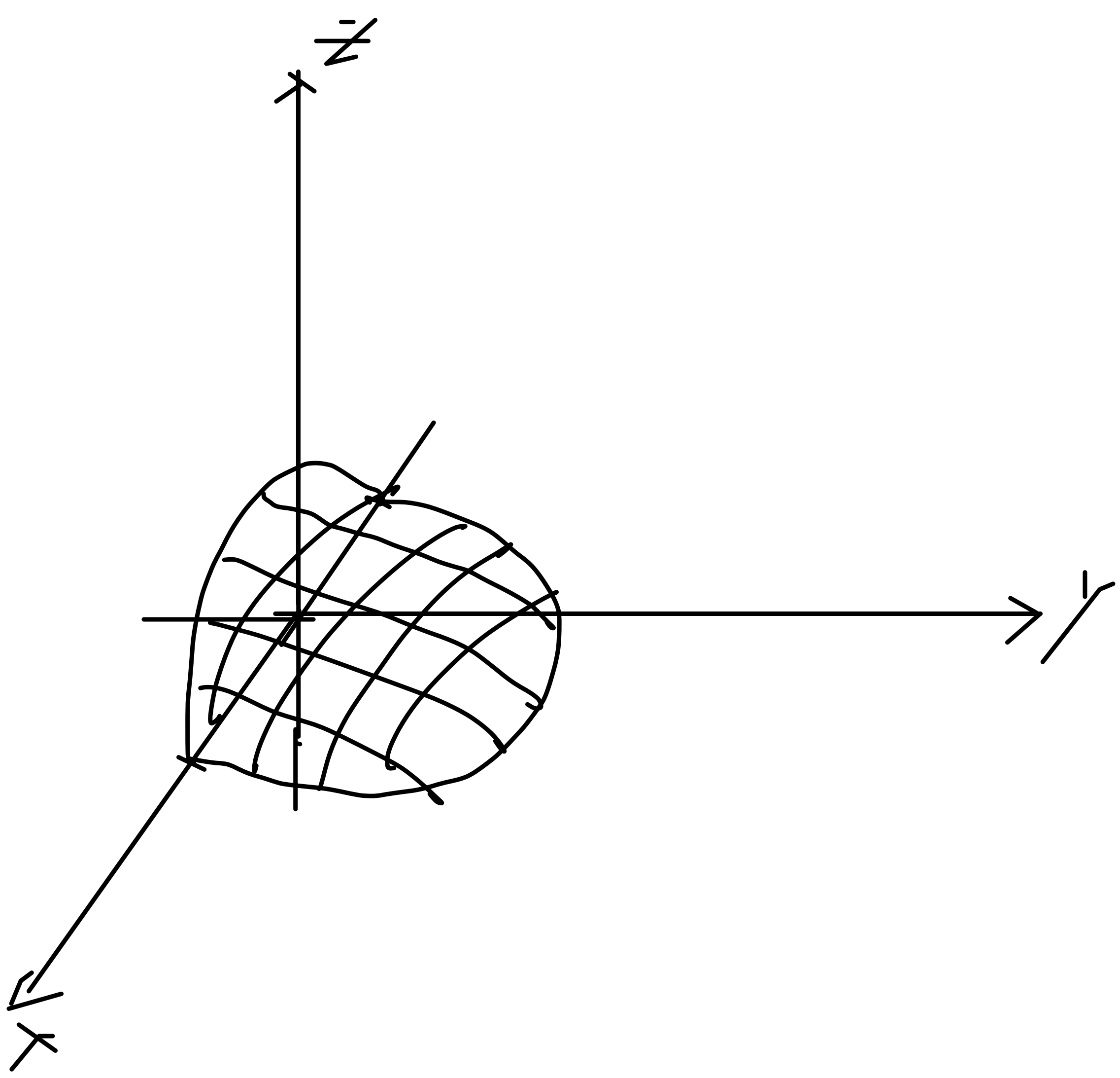
4. Evaluate the integral

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dy\,dx.$$

$\vec{r}(r,z,\theta\rangle = \langle$

$$\int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^2 \quad dz \;dr\,d\theta$$

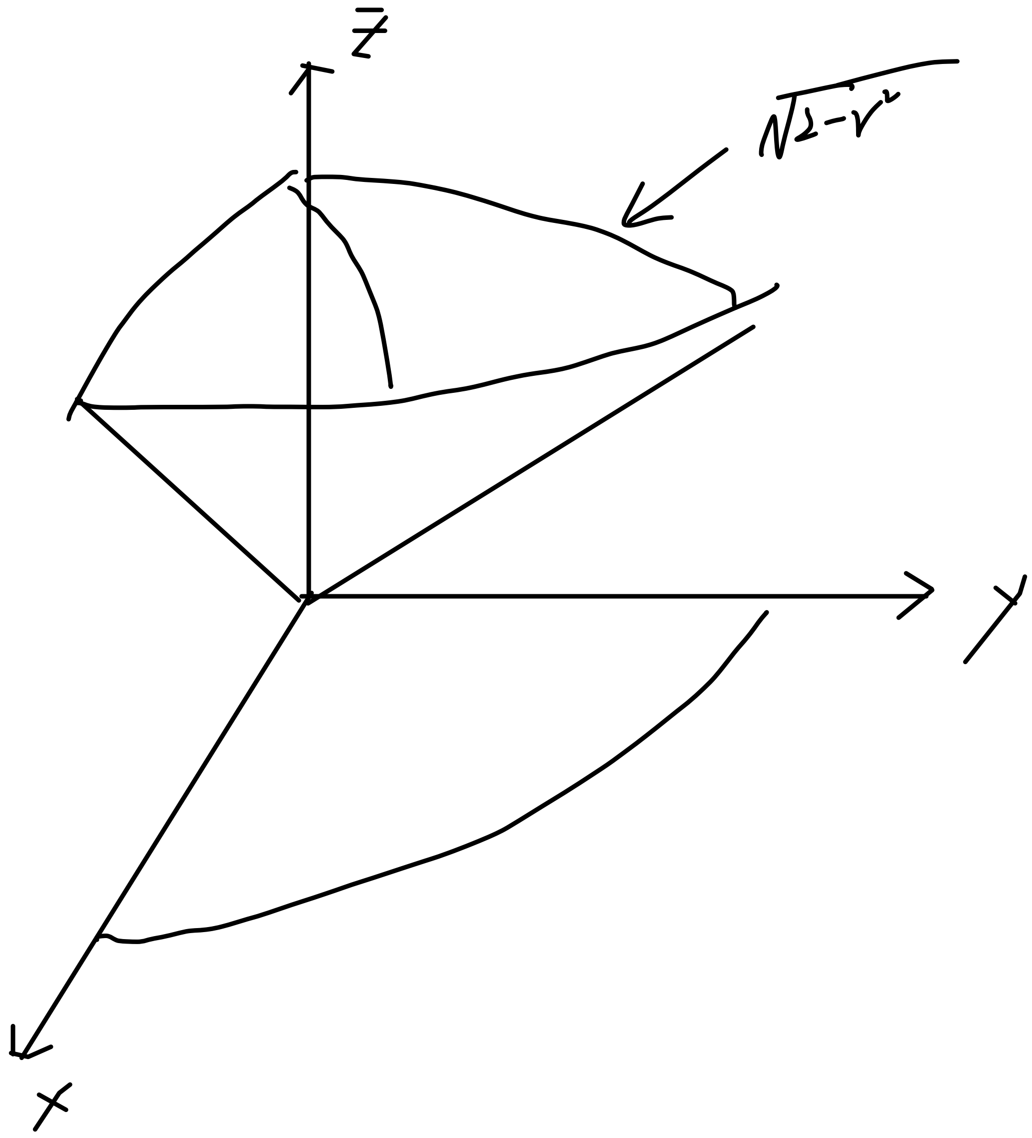
$$= \int_0^{2\pi}\int_0^3 r^2(9-r^2)\,dr\;d\theta$$



5. Find the volume of the intersection solid of perpendicular cylinders.

7. Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx.$$



$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{2}} \qquad d\rho\, d\varphi\, d\theta$$