MATH2013 Multivariable Calculus

From the textbook Calculus - Several Variables (5th) by R. Adams, Addison/Wesley/Longman.

*** At least try to do the underlined ones, the others are recommended exercises.**

Homework 1

(*Total: 21 questions*)

Ex. 10.1

6 Show that the triangle with vertices $(1,2,3)$, $(4,0,5)$, and $(3,6,4)$ has a right angle.

14 Describe (and sketch if possible) the set of points in $\mathbb{R}^3$ which satisfy the equation $z = x$.

22 Describe (and sketch if possible) the set of points in $\mathbb{R}^3$ which satisfy the inequality $z \geqslant \sqrt{x^2+y^2}$.

Ex. 10.2

2 Calculate the following for the given vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} = \mathbf{i} - \mathbf{j} \qquad \text{and} \qquad \mathbf{v} = \mathbf{j} + 2\mathbf{k}.$$

(a) $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, $2\mathbf{u}-3\mathbf{v}$,

(b) the lengths $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$,

(c) unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ in the directions of $\mathbf{u}$ and $\mathbf{v}$, respectively,

(d) the dot product $\mathbf{u}\cdot\mathbf{v}$,

(e) the angle between $\mathbf{u}$ and $\mathbf{v}$,

(f) the scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$,

(g) the vector projection of $\mathbf{v}$ along $\mathbf{u}$.

10 A straight river 500m wide flows due east at a constant speed of 3 km/h. If you can row your boat at speed of 5 km/h in still water, in what direction should you head if you wish to row from point A on the south shore to point B on the north shore directly north of A? How long will the trip take?

16 If a vector $\mathbf{u}$ in $\mathbb{R}^3$ makes angles α, β, and γ with the coordinate axes, show that

$$\hat{\mathbf{u}} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

is a unit vector in the direction of $\mathbf{u}$. Hence show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

19 If $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of two points, P_1 and P_2, and λ is a real number, show that

$$\mathbf{r} = (1-\lambda)\mathbf{r}_1 + \lambda\mathbf{r}_2$$

is the position vector of a point P on the straight line joining P_1 and P_2. Where is P if $\lambda = 1/2$? if $\lambda = 2/3$? if $\lambda = -1$? if $\lambda = 2$?

20 Let $\mathbf{a}$ be a nonzero vector. Describe the set of all points in 3-space whose position vectors $\mathbf{r}$ satisfy $\mathbf{a}\cdot\mathbf{r} = 0$.

24 Let $\mathbf{u} = 2\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$, $\mathbf{v} = \mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}$, and $\mathbf{w} = 2\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}$. Find two unit vectors each of which makes equal angles with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

27 Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors.

(a) Show that $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$.

(b) Show that $\mathbf{u}\cdot\mathbf{v} \leqslant \|\mathbf{u}\|\,\|\mathbf{v}\|$.

(c) Deduce from (a) and (b) that $\|\mathbf{u}+\mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|$.

29 Let $\mathbf{u} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$, $\mathbf{v} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$, and $\mathbf{w} = \mathbf{k}$.

(a) Show that $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\| = 1$ and $\mathbf{u}\bullet\mathbf{v} = \mathbf{u}\bullet\mathbf{w} = 0$. The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ a mutually perpendicular unit vectors, and as such are said to constitute an **orthonormal basis** for $\mathbb{R}^3$.

(b) If $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, show by direct calculation that

$$\mathbf{r} = (\mathbf{r}\bullet\mathbf{u})\mathbf{u} + (\mathbf{r}\bullet\mathbf{v})\mathbf{v} + (\mathbf{r}\bullet\mathbf{w})\mathbf{w}.$$

33 Given constants r, s and t, with $r \neq 0$ and $s \neq 0$, and given a vector $\mathbf{a}$ satisfying $\|\mathbf{a}\|^2 > 4rst$, solve the system of equations

$$r\mathbf{x} + s\mathbf{y} = \mathbf{a}$$
$$\mathbf{x}\cdot\mathbf{y} = t$$

for the unknown vectors $\mathbf{x}$ and $\mathbf{y}$.

Ex. 10.3

14 **(Volume of a tetrahedron)** A **tetrahedron** is a pyramid with a triangular base and three other triangular faces. It has four vertices and six edges. Like any pyramid or cone, its volume is equal to $\frac{1}{3}AH$, where A is the area of the base and H is the height measured perpendicular

to the base. If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors coinciding with the three edges of a tetrahedron that meet at one vertex, show that the tetrahedron has volume given by

$$\text{Volume } = \frac{1}{6}\|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})\| = \frac{1}{6}\left|\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right|.$$

20 If $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 0$ but $\mathbf{v}\times\mathbf{w} \neq \mathbf{0}$, show that there are constants λ and μ such that

$$\mathbf{u} = \lambda\mathbf{v} + \mu\mathbf{w}.$$

26 Find all vectors $\mathbf{x}$ that satisfy the equation

$$(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times \mathbf{x} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}.$$

28. What condition must be satisfied by the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ to guarantee that the equation $\mathbf{a}\times\mathbf{x} = \mathbf{b}$ has a solution for $\mathbf{x}$? Is the solution unique?

Ex. 10.4

8 Find equation of the plane satisfying the given conditions. Passing through the line of intersection of the planes $2x + 3y - z = 0$ and $x - 4y + 2z = -5$, and passing through the point $(-2, 0, -1)$.

18 Find equations of the line specified in vector and scalar parametric forms and in standard form. Through $(2, -1, -1)$ and parallel to each of the two planes $x + y = 0$ and $x - y + 2z = 0$.

28 Find the distance from the origin to the line $x + y + z = 0$, $2x - y - 5z = 1$.

Lecture Note Ex 3.15 (p13),

Exercises for students -Qu 6 (p29)

Homework 2

(Total: 12 questions)

Ex. 11.1

12 Find the velocity, speed and acceleration at time t of the particle whose position is $\mathbf{r}(t)$. Describe the path of the particle.

$$\mathbf{r} = at\cos\omega t\,\mathbf{i} + at\sin\omega t\,\mathbf{j} + b\ln t\,\mathbf{k}$$

24 If the position and velocity vectors of a moving particle are always perpendicular show the path of the particle lies on a sphere.

29 Write the Product Rule for $\frac{d}{dt}(\mathbf{u}\times(\mathbf{v}\times\mathbf{w}))$.

32 Expand and simplify: $\frac{d}{dt}((\mathbf{u}\times\mathbf{u}')\cdot(\mathbf{u}'\times\mathbf{u}''))$.

Ex. 11.3

In exercises 2 and 4, find the required parametrization of the first quadrant part of the circular arc $x^2 + y^2 = a^2$.

2 In terms of the x-coordinate, oriented clockwise.

4 In terms of arc length measured from $(0, a)$, oriented clockwise.

16 Describe the parametric curve $\mathcal{C}$ given by

$$x = a\cos t\sin t, \qquad y = a\sin^2 t, \qquad z = bt.$$

What is the length of $\mathcal{C}$ between $t = 0$ and $t = T > 0$?

18 Describe the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the elliptic cylinder $x^2 + 2z^2 = 1$. Find the total length of this intersection curve.

27 Let $\mathbf{r} = \mathbf{r}_1(t)$, $(a \leqslant t \leqslant b)$ and $\mathbf{r} = \mathbf{r}_2(u)$, $(c \leqslant u \leqslant d)$, be two parametrizations of the same curve C, each one-to-one on its domain and each giving C the same orientation (so that $\mathbf{r}_1(a) = \mathbf{r}_2(c)$ and $\mathbf{r}_1(b) = \mathbf{r}_2(d)$. Then for each t in $[a, b]$ there is a unique $u = u(t)$ such that $\mathbf{r}_2(u(t)) = \mathbf{r}_1(t)$. Show that

$$\int_a^b \left\|\frac{d}{dt}\mathbf{r}_1(t)\right\| dt = \int_c^d \left\|\frac{d}{du}\mathbf{r}_2(u)\right\| du,$$

and thus that the length of C is independent of parametrization.

Extra questions

1 Verify the formula for the arc length element is cylindrical coordinates,

$$ds = \sqrt{\left(\frac{dr}{dt}\right)^2 + (r(t))^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt,$$

given in this section.

2 Verify the formula for the arc length element in spherical coordinates,

$$ds = \sqrt{\left(\frac{d\rho}{dt}\right)^2 + (\rho(t))^2\left(\frac{d\phi}{dt}\right)^2 + (\rho(t)\sin\phi(t))^2\left(\frac{d\theta}{dt}\right)^2}\,dt,$$

given in this section.

3 Reparametrize the curve $\mathbf{r} = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j} + b\cos 2t\,\mathbf{k}$ $(0 \leqslant t \leqslant \pi/2)$ in the same orientation in terms of the arc length meausred from the point when $t = 0$.