MATH 2023 – Multivariable Calculus

Lecture #08 Worksheet ♯ March 5, 2019

Problem 1. Find the maximum, minimum and saddle points of the following functions:

① $\nabla f = 0$
② D

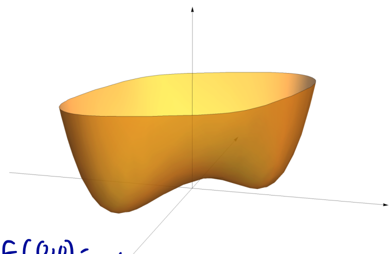
$$f(x,y) = x^4 + y^4 - 4xy + 1$$

$\nabla f = \langle \underbrace{4x^3 - 4y}_{f_x}, \underbrace{4y^3 - 4x}_{f_y} \rangle \Rightarrow \begin{cases} x^3 = y \\ y^3 = x \end{cases} \Rightarrow (x,y) = (0,0)$
$= (1,1)$
$= (-1,-1)$

$D = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix}$

$D(0,0) = -16 < 0$ saddle

$D(\pm1,\pm1) = 128 > 0$, $12x^2 > 0$ local min.



$f(0,0) = \dots$

$$f(x,y) = x^2 + y^2 + x^{-2}y^{-2}$$

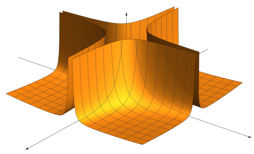
$\nabla f = \langle 2x - 2x^{-3}y^{-2}, 2y - 2x^{-2}y^{-3} \rangle = \vec{0} \Rightarrow \begin{cases} x = \frac{1}{x^3y^2} \\ y = \frac{1}{x^2y^3} \end{cases} \Rightarrow \begin{cases} x^4y^2 = 1 \\ x^2y^4 = 1 \end{cases} \Rightarrow (x,y) = (\pm1, \pm1)$

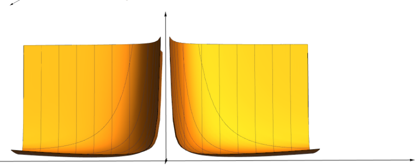
(Note: (0,0) is also critical point if (0,0) is in the domain)

$D = \begin{vmatrix} 8 & \pm4 \\ \pm4 & 8 \end{vmatrix} > 0$

$f_{yy} = f_{xx} = 2 + 6x^{-4}y^{-2} = 8$ always

$f_{xy} = f_{yx} = 4x^{-3}y^{-3} = \pm4$

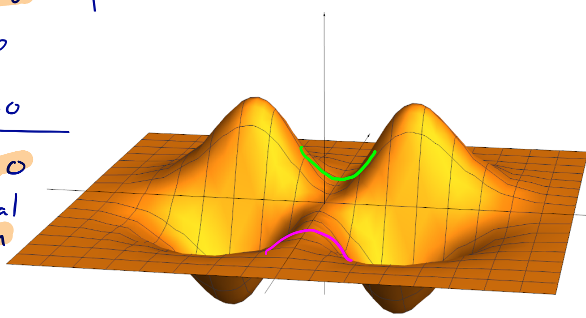




$$f(x,y) = x^2 y e^{-x^2-y^2}$$

$f_x = 2xy(1-x^2)e^{-x^2-y^2}$
$f_y = x^2(1-2y^2)e^{-x^2-y^2}$
critical point at $(0,0)$ and $(\pm1, \pm\frac{1}{\sqrt2})$

	$(0,y)$	$(\pm1, \frac{1}{\sqrt2})$	$(\pm1, -\frac{1}{\sqrt2})$
$f_{xx} = 2y(1-5x^2+2x^4)e^{-x^2-y^2}$	0	<0	>0
$f_{xy} = 2x(1-x^2)(1-2y^2)e^{-x^2-y^2}$	0	0	0
$f_{yy} = -2x^2y(3-2y^2)e^{-x^2-y^2}$	0	<0	>0
D	0	>0	>0
	no conclusion?	local max	local min



*** critical point $(0,y)$
$D = 0$
When $y > 0$, $f(x,y) > 0$ near $(0,y)$
$\Rightarrow f(0,y) = 0$ is min.
$y < 0$, $f(0,y) = 0$ is max

saddle point
(e.g. not max/min along $y = x$)

$z = 4 - x - 2y$

Problem 2. Find the shortest distance from $(1, 0, -2)$ to the plane $x + 2y + z = 4$ using calculus. Verify the result using the distance formula.

?

$\|(x,y,z) - (1,0,-2)\| = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$

same as minimizing $(x-1)^2 + y^2 + (z+2)^2$

$= (x-1)^2 + y^2 + (6-x-2y)^2 = f(x,y)$

$\nabla f : \langle 2(x-1) + 2(6-x-2y)(-1), \quad 2y + 2(6-x-2y)(-2) \rangle$

$2x - 2 - 12 + 2x + 4y \qquad 2y - 24 + 4x + 8y$

$= \langle 4x + 4y - 14, \quad 4x + 10y - 24 \rangle = \langle 0, 0 \rangle$

$6y = 10 \Rightarrow y = \frac{5}{3} = \frac{10}{6}$

$x = \frac{14 - \frac{20}{3}}{4} = \frac{11}{6}$

$z = 4 - x - 2y = \frac{24}{6} - \frac{11}{6} - \frac{20}{6} = -\frac{7}{6}.$

distance $= \sqrt{(\frac{5}{6})^2 + (\frac{10}{6})^2 + (\frac{5}{6})^2} = \sqrt{\frac{150}{36}} = \frac{5\sqrt{6}}{6}$

$\parallel$

distance formula: $\left| \frac{1 + 0 - 2 - 4}{\sqrt{1^2 + 2^2 + 1^2}} \right| = \frac{5}{\sqrt{6}}$

Problem 3. Find the absolute maximum and minimum of $f(x,y) = x^2 - 2xy + 2y$ on

- the rectangle $R = \{(x,y) : 0 \le x \le 3, 0 \le y \le 2\}$.

① critical points:

$\nabla f : \langle 2x-2y, -2x+2 \rangle$

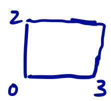
$= \langle 0,0 \rangle$ when $(x,y) = (1,1)$

~~(local min.)~~ saddle!

$f(1,1) = 1$.

$D = \begin{vmatrix} 2 & -2 \\ -2 & 0 \end{vmatrix} = -4$

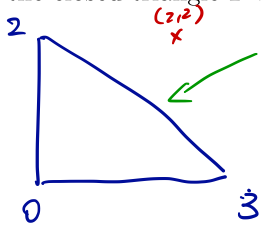
②



			max		min	
$x=0$	$\Longrightarrow 2y$	:	4	(0,2)	0	(0,0)
$x=3$	$\Rightarrow 9-4y$		9	(3,0)	1	(3,2)
$y=0$	$\Rightarrow x^2$	:	9	(3,0)	0	(0,0)
$y=2$	$\Rightarrow x^2-4x+4 = (x-2)^2$		4	(0,2)	0	(2,2)

max $= 9$ at $(3,0)$, min $= 0$ at $(0,0)$ and $(2,2)$

- the closed triangle T bounded by $(0,0), (0,2), (3,0)$



$2x+3y=6 \Longrightarrow y = \frac{6-2x}{3}$

$x^2 - 2x\left(\frac{6-2x}{3}\right) + 2\left(\frac{6-2x}{3}\right)$

$= \frac{7x^2}{3} - \frac{16x}{3} + 4$

$f'(x) = \frac{14x}{3} - \frac{16x}{3} = 0 \Longrightarrow x = \frac{8}{7} \Rightarrow y = \frac{26}{21}$

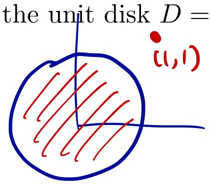
$f\left(\frac{8}{7}, \frac{26}{21}\right) = \frac{20}{21}$ not min nor max.

max: $(3,0)$

min: $(0,0)$

- the unit disk $D = \{(x,y) : x^2 + y^2 \le 1\}$

D:



$\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ into $f(x,y)$

$f(t) = \cos^2 t - 2\cos t \sin t + 2\sin t$

$f'(t) = -2\cos t \sin t - 2\cos^2 t + 2\sin^2 t + 2\cos t = 0$

$\sin t = \sqrt{1-\cos^2 t}$ Let $\lambda = \cos t$

$= -2\lambda\sqrt{1-\lambda^2} - (2\lambda^2 + 2(1-\lambda^2) + 2\lambda) = 0$ $(\ )^2$

$\Longrightarrow$ poly of degree 4!

solved by software only ☹

	$f(t)$	
$t = 0°$	1	
34.56°	0.88	
130.52°	2.93	max
248.05°	−2.41	min.