Sample Midterm Q7

7. Let $\mathbf{c}(t)$ be parametric curve in $\mathbb{R}^3$ such that $|\mathbf{c}(t)| = 1$ at all time t. Define another parametric curve $\mathbf{r}(t)$ by:

$$\mathbf{r}(t) = f(t)\mathbf{c}(t).$$

(a) Show that $\mathbf{c}(t)$ and $\mathbf{c}'(t)$ are orthogonal at all time t.

(b) Using (a), or otherwise, show that $|\mathbf{r}'(t)| = \sqrt{f(t)^2\,|\mathbf{c}'(t)|^2 + f'(t)^2}$.

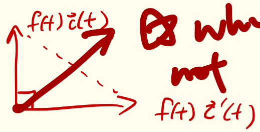
(c) Using (b), or otherwise, find an arc-length parametrization $\mathbf{q}(s)$ of the following curve:

$$\mathbf{q}(t) = \overbrace{\frac{3}{5}e^{-t}}^{f(t)}\,[\overbrace{(\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{k}}^{c(t)}]$$

(a) $\vec{c} \perp \vec{c}\,' \iff \vec{c} \cdot \vec{c}\,' = 0$

$|\vec{c}(t)| = 1 \implies \vec{c}(t) \cdot \vec{c}(t) = 1 \xRightarrow{\frac{d}{dt}} 2\vec{c}(t) \cdot \vec{c}\,'(t) = 0$

(b) $|\vec{r}\,'(t)| = |f(t)\,\vec{c}\,'(t) + f'(t)\,\vec{c}(t)|.$



why not

by Pythagorus $= \sqrt{|f(t)\vec{c}\,'(t)|^2 + |f'(t)\cancel{\vec{c}(t)}|^2}$

$c' = \langle \cos t, 0, -\sin t \rangle$

$|c'| = 1$

(c) by part b) $|q'(t)| = \sqrt{\left(\frac{3}{5}e^{-t}\right)^2 + \left(-\frac{3}{5}e^{-t}\right)^2}$

① $= \sqrt{\frac{18}{25}}\, e^{-t}$

$s(t) = \int_0^t |q'(u)|\,du = \int_0^t \sqrt{\frac{18}{25}}\, e^{-u}\,du = \sqrt{\frac{18}{25}}\,(1 - e^{-t})$

② $t = -\ln\left(1 - \frac{s}{\sqrt{18/25}}\right)$

③ Substitute : $\frac{3}{5}\left(1 - \frac{s}{\sqrt{18/25}}\right)\left\langle \sin(\quad), 0, \cos(\quad) \right\rangle$

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PS#4Q7

7. (★★) Consider the surface given by the equation

$$x^2y^2z^2 = 1.$$

(a) Show that for any point (a, b, c) on the surface, its tangent plane does not contain the origin.

(b) Find all points (a, b, c) on the surface such that the tangent plane at these points are closest to the origin.

(a) $\nabla F = \langle 2xy^2z^2, 2x^2yz^2, 2x^2y^2z \rangle$ at (a,b,c) $\to a^2b^2c^2 = 1$

$= \langle 2ab^2c^2, 2a^2bc^2, 2a^2b^2c \rangle$

$= \langle \frac{2}{a}, \frac{2}{b}, \frac{2}{c} \rangle$

tangent plane : $\frac{2}{a}x + \frac{2}{b}y + \frac{2}{c}z = d = 6$

(a,b,c) lies on the plane!

does not contain $(0,0,0)$!

Tutorial 3

3. Determine the x-partial derivative of the function

$$f(x,y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

at $(0,0)$. ← special! First principle.

$$\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-1}{h} \to \infty \ !$$

No limit.

$f_x(0,0)$ <u>does not</u> exist.

19. What condition must the constants a, b, and c satisfy to guarantee that $\lim_{(x,y)\to(0,0)} xy/(ax^2 + bxy + cy^2)$ exists? Prove your answer.

$$\begin{cases} x=0 : \text{If } c \neq 0, \ f(x,y) \to 0. \\ y=0 : \text{If } a \neq 0, \ f(x,y) \to 0 \\ x=y : \dfrac{x^2}{ax^2+bx^2+cx^2} = \dfrac{1}{a+b+c} \neq 0 \end{cases}$$

no limit !!

When $a=c=0$: $f(x,y) = \dfrac{xy}{bxy} = \dfrac{1}{b}$ ✓.

$\therefore$ final sol. = ?

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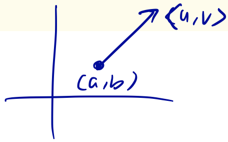
J. Fung's HW#3



17 Let $\mathbf{u} = u\,\mathbf{i} + v\,\mathbf{j}$ be a unit vector, and let

$$f_{\mathbf{u}}(t) = f(a + tu,\, b + tv)$$

be the single-variable function obtained by restricting the domain of $f(x, y)$ to points of the straight line through (a, b) parallel to $\mathbf{u}$. a If $f_{\mathbf{u}}(t)$ is continuous at $t = 0$ for every unit vector $\mathbf{u}$, does it follow that f is continuous at (a, b)? b Conversely, does the continuity of f at (a, b) guarantee the continuity of $f_{\mathbf{u}}(t)$ at $t = 0$? Justify your answers.

Ex $\quad f(x,y) = \dfrac{xy^2}{x^2+y^4}$ has 0 as limit for all paths $y = mx$

but $= \frac{1}{2}$ when $x = y^2$

a is wrong

b true.

PS#3Q6 Assume $u(\xi,\eta)$ such that ξ = xi , η = eta

(c) $\Rightarrow$ $u_{\xi\eta} = 0.$

(d) Finally, deduce that u, as a function of ξ and η, must be in the form of:

$$u(\xi, \eta) = F(\xi) + G(\eta)$$

where F and G are arbitrary functions.

$$(u_\xi)_\eta = 0$$

- u_ξ does not depend on η.

$$u_\xi = f(\xi)$$

- $u = \cancel{\int f(\xi)} + C(\eta)$

$F(\xi)$

such that $F'(\xi) = f(\xi)$.

PS#2 Q8 (Simplified)

8. (★★) $\varphi(x,y,z,t) = \dfrac{1}{(4\pi kt)^{\frac{3}{2}}} \exp\left(-\dfrac{x^2+y^2+z^2}{4kt}\right)$.

verify that φ satisfies the heat equation: $\varphi_t = k(\varphi_{xx} + \varphi_{yy} + \varphi_{zz})$.

(a) Show that: $\ln\varphi(x,y,z,t) = -\ln(4\pi k)^{\frac{3}{2}} - \dfrac{3}{2}\ln t - \dfrac{x^2+y^2+z^2}{4kt}$.

(b) Using (a), show that: $\dfrac{\partial\varphi}{\partial t} = \left(\dfrac{x^2+y^2+z^2}{4kt^2} - \dfrac{3}{2t}\right)\varphi$.

(c) Using (a) again, show that:

$$\frac{\partial\varphi}{\partial x} = -\frac{x\varphi}{2kt} \quad \text{and} \quad \frac{\partial^2\varphi}{\partial x^2} = \frac{1}{2kt}\left(\frac{x^2}{2kt} - 1\right)\varphi.$$

$$\ln\varphi = \cdots$$

$$\frac{\partial}{\partial t}\ln\varphi = \frac{1}{\varphi}\frac{\partial\varphi}{\partial t} = \frac{\partial}{\partial t}(\cdots)$$

WebWork HW4 Q3

(1 point)

Calculate the derivatives using implicit differentiation:

$\frac{\partial U}{\partial T}$ and $\frac{\partial T}{\partial U}$ of $(TU - V)^2 \ln(W - UV) = 1$ at $(T, U, V, W) = (7, 1, 3, 6)$

$\left.\frac{\partial U}{\partial T}\right|_{(7,1,3,6)} =$

$\left.\frac{\partial T}{\partial U}\right|_{(7,1,3,6)} =$

$$\frac{\partial U}{\partial T} = -\left.\frac{F_T}{F_u}\right|_{7136}$$

$$\frac{\partial T}{\partial u} = -\left.\frac{F_u}{F_T}\right|_{7136}$$

$$F = (TU - V)^2 \ln(W - UV) - 1 .$$

Jimmy Fung HW #4

21 The temperature $T(x, y)$ at points of the xy-plane is given by $T(x, y) = x^2 - 2y^2$.

(a) Draw a contour diagram for T showing some isotherms (curves of constant temperature).

(b) In what direction should an ant at position $(2, -1)$ move if it wishes to cool off as quickly as possible?

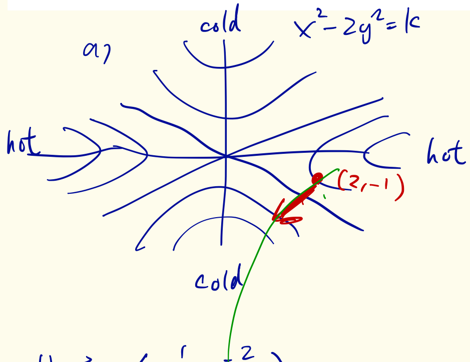
greatest descent

(c) If an ant moves in that direction at speed k (units distance per unit time), at what rate does it experience the decrease of temperature?

(d) At what rate would the ant experience the decrease of temperature if it moves from $(2, -1)$ at speed k in the direction of the vector $-\mathbf{i} - 2\mathbf{j}$?

(e) Along what curve through $(2, -1)$ should the ant move in order to continue to experience maximum rate of cooling?

a) $x^2 - 2y^2 = k$



b) $-\nabla f = -\langle 2x, -4y \rangle$

$= \langle -2x, 4y \rangle$ at $(2,-1)$

$= \langle -4, -4 \rangle \Rightarrow \langle -\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle$

c) : $D_{\vec{u}} f$ at $\vec{u}$

$= -\nabla f \cdot \vec{u}$

$= -\langle -4, -4 \rangle \cdot \langle -\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle$

$= \frac{-8}{\sqrt{2}}$

d) $\vec{u} = \langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \rangle$

$\nabla f \cdot \vec{u}$.

e) $\vec{r}\,' \parallel -\nabla f$

$$\begin{cases} x' = -2x \\ y' = 4y \end{cases} \Rightarrow \begin{cases} x = 2e^{-2t} \\ y = -1e^{4t} \end{cases} \Rightarrow x^2 = \frac{-4}{y}$$

$(2,-1)$ at $t=0$

$y x^2 = -4$ //

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WebWork

(1 point) Does the function

$+\infty$

$$f(x, y) = \frac{x^2}{2} + 6y^3 + 7y^2 - 3x$$

$-\infty$

have a global maximum and global minimum? If it does, identify the value of the maximum and minimum. If it does not, be sure that you are able to explain why.

why y^3 就 no max ?

Global maximum? none

*(Enter the value of the global maximum, or **none** if there is no global maximum.)*

Global minimum? none

*(Enter the value of the global minimum, or **none** if there is no global minimum.)*

WebWork 5-5

Please explain more on the classification of maximum/minimum/saddle point, when the second derivative test gives no conclusion? For example the webwork 5 problem 5, if we slightly modify f(x,y) and consider f(x,y)=x^3+y^3-3y-2, the critical points (0,1) and (0,-1). Thank you.

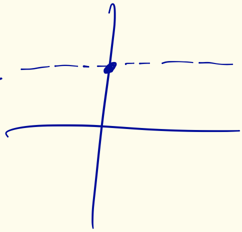
Reply

$\nabla f = \langle 3x^2, 3y^2-3 \rangle$

critical point:
(0,1)
(0,-1)

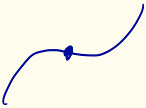
$D = \begin{vmatrix} 6x & 0 \\ 0 & 6y \end{vmatrix} = 36xy = 0$ at $\nearrow$

at (0,1) : I can choose path $y=1$



$f(x,1) = x^3 - 4$

as $x \to 0$



not max nor min

$\Rightarrow$ saddle.

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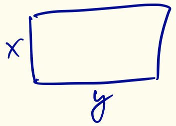
Tutorial #5

7. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.



max. $A = xy$

$2x + 2y = p$

$\nabla A = \lambda \nabla g$

$$y = \lambda 2$$
$$\|$$
$$x = \lambda 2$$

J. Fung HW5

g ||

22 Find the maximum and minimum values of $xy + z^2$ on the ball $x^2 + y^2 + z^2 \leqslant 1$. Use Lagrange multipliers to treat the boundary case.

① $\nabla f = \langle y, x, 2z \rangle \implies (0,0,0)$

② on boundary : $\nabla f = \lambda \nabla g \leftarrow (2x, 2y, 2z)$

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ 2z = 2\lambda z \end{cases} \implies z = 0 \text{ or } \lambda = 1$$

$$\begin{matrix} y^2 = 2\lambda xy \\ x^2 = 2\lambda xy \end{matrix} \implies x^2 = y^2 \implies x = \pm y$$

$z = 0$, $x = \pm y \implies \left(\frac{\pm 1}{\sqrt{2}}, \frac{\pm 1}{\sqrt{2}}, 0\right)$

$\lambda = 1$, $\begin{cases} y = 2x \\ y = \pm x \end{cases} \Rightarrow (0, 0, \pm 1)$

③ Check Values of these 6 points.

PS#4Q4

$$\nabla f = \lambda \nabla g \iff \nabla f \parallel \nabla g$$

4. $\nabla f(P) \parallel \nabla g(P)$ if and only if $\nabla f(P) \times \nabla g(P) = \mathbf{0}$.

By solving the vector equation $\nabla f(P) \times \nabla g(P) = \mathbf{0}$ for P (instead of using Lagrange's Multiplier), try to redo Problems #3(a)(b)(c).

3. (c) $f(x, y, z) = xyz$ subject to $x^2 + 2y^2 + 4z^2 = 9$ g

What is the limitation of this method when compared to Lagrange's Multiplier?

cross only 3

$$\nabla f = \langle yz, xz, xy \rangle$$

$$\nabla g = \langle 2x, 4y, 8z \rangle$$

$$\nabla f \times \nabla g = \begin{vmatrix} i & j & k \\ yz & xz & xy \\ 2x & 4y & 8z \end{vmatrix} = \langle 8xz^2 - 4xy^2,\ 2x^2y - 8yz^2,\ 4y^2z - 2x^2z \rangle = 0$$

$$\begin{cases} 8xz^2 = 4xy^2 \\ 2x^2y = 8yz^2 \\ 4y^2z = 2x^2z \end{cases} \iff \begin{array}{l} x = 2z \\ y = \frac{3}{2}z \end{array}$$

$$x^2 + 2y^2 + 4z^2 = 9$$

$$\Rightarrow z = \ldots$$

PS #3 Q13

13. The spherical coordinates (ρ, θ, ϕ) is another important coordinate system in $\mathbb{R}^3$. We will learn that in later chapters. The conversion rules between spherical and rectangular coordinates are given by:

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

Given a C^2 function $f(x, y, z)$, it can be regarded as a function of (ρ, θ, ϕ) as well under the above conversion rule. Show that the Laplacian $\nabla^2 f := f_{xx} + f_{yy} + f_{zz}$ can be expressed in spherical coordinates as:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}.$$

[Note: It is a very time consuming exercise. It took me 4 ~~hours~~ to do it when I was an undergraduate.]

5 min.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \iff \begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \end{array}$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\to \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$u_y = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}$$

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

$$r = \rho \sin\phi$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \text{and} \quad \begin{cases} z = \rho\cos\phi \\ r = \rho\sin\phi \end{cases}$$

$$\underbrace{u_{xx} + u_{yy}} + u_{zz}$$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}$$

$$u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi} + \frac{1}{(\rho\sin\phi)^2}u_{\theta\theta} + \frac{1}{r}u_r$$

$$\frac{1}{r}u_r = \frac{u_\rho \sin\phi}{\rho\sin\phi} + \frac{u_\phi \frac{\cos\phi}{\rho}}{\rho\sin\phi}$$

$$u_{\rho\rho} + \frac{1}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi} + \frac{1}{\rho^2\sin^2\phi}u_{\theta\theta} + \frac{1}{\rho}u_\rho + \frac{\cos\phi}{\rho^2\sin\phi}u_\phi .$$

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