MATH2023
Tutorial 3
February 25, 2019 Tutorial Notes by Ivan So

1 Review

- **Multivariable function** is defined as the map $f : \mathbb{R}^n \to \mathbb{R}$.
- The **domain** of a multivariable function is a subset $D \subset \mathbb{R}^n$ on which the function is defined. The **range** is the set $\{f(\mathbf{x}) | \mathbf{x} \in D\}$.
- The **graph** of a *two* variable function is the set $\{(x, y, f(x, y)) | (x, y) \in D\}$ for the domain D of f.
- **Level curve** of a function f of two variables are the curves with satisfying $f(x, y) = k$.
- The **limit** L of f at $\mathbf{x}_0 \in \mathbb{R}^n$ is the value in which for any $\epsilon > 0$, there exist $\delta_\epsilon > 0$ such that $\|\mathbf{x} - \mathbf{x}_0\| < \delta_\epsilon \implies |f(\mathbf{x}) - L| < \epsilon$. (in simple language, that is the value that f approach as (x, y) approach (a, b)). Notationally,
$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L.$$
  - You are reminded limit is **NOT** direct function evaluation.
  - Limit is **NOT** always exist.
  - If the limit exist, then **MUST** be unique. So it cannot be *path dependent*.
  - Evaluation can be done with *squeeze theorem* or *polar coordinates*.
- A mutivariable function is **continuous** at $\mathbf{x}_0$ if it satisfies $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$.
- The **partial derivative** of $f : \mathbb{R}^n \to \mathbb{R}$ in the i-th coordinate at $\mathbf{x}_0$ is defined by
$$\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}_0} := \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{e}_i) - f(\mathbf{x}_0)}{h}.$$
Other notations: f_{x_i}. One can regard *unrelated variables* as constants in taking partial derivative. **Remark**: Partial derivative is **NOT** always commutative. They are commutative only if both f_{xy} and f_{yx} are continuous (Clairaut).

2 Problems

1. True or False
   (a) $f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$. false: $\because f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right)$

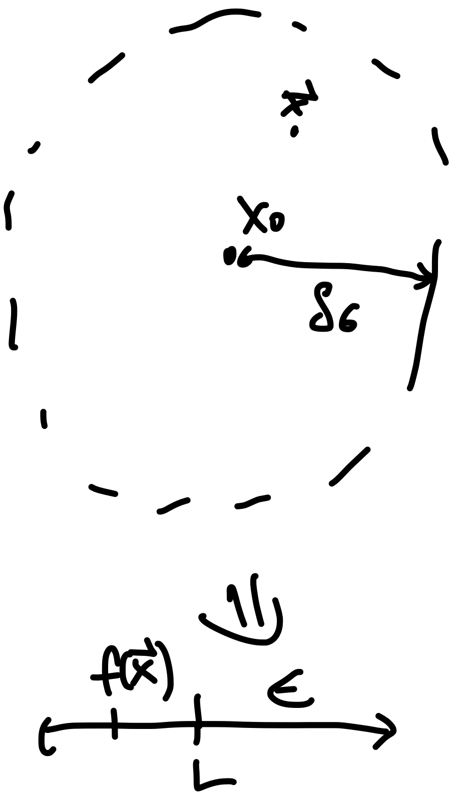
Counter e.g. $f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & \text{when } (x,y) \neq (0,0) \\ 0, & \text{when } (x,y) = 0 \end{cases}$

If they are continuous, this is true.

For interest:

$$\lim_{\vec{x}\to\vec{x_0}} f(\vec{x}) = L \iff \forall \epsilon > 0, \exists \delta_\epsilon \text{ s.t. } \|x - x_0\| < \delta_\epsilon$$

$$\Rightarrow |f(\vec{x}) - L| < \epsilon$$



False : $f_y = \lim_{y \to b} \frac{f(a,y) - f(a,b)}{y-b}$

(b) $f_y(a,b) = \lim_{y\to b} \frac{f(a,b)-f(a,y)}{y-b}$.

(c) There exists a function f with continuous second-order partial derivatives such that $f_x(x,y) = x + y^2$ and $f_y(x,y) = x - y^2$.

False $f = \frac{x^2}{2} + y^2x + C(y)$

$f_y = 2xy + c'(y)$, but no x.

(d) If $f(x,y) \to L$ as $(x,y) \to (a,b)$ for every straight line, then $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

False:

2. Determine the set of points at which the function

$$f(x,y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous.

3. Determine the x-partial derivative of the function

$$f(x,y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

at $(0,0)$.

4. Draw the contour of $f(x,y) = ye^x$ at several levels.

c). $f_x = x + y^2$

$\int f_x\,dx = \frac{x^2}{2} + y^2 x + g(y)$

$\boxed{?}$

assumption:

$f_y = 2xy + g'(y) = x - y^2$

$g'(y) = x - y^2 - 2xy$

g depends on x,

contradiction to original assumption

$\Rightarrow$ False

d). FALSE $\because$ we can always consider non-linear part.

Counter e.g. $f(x,y) = \frac{y^2}{x}$

$\forall$ Linear path, $y = Cx$, $C \in \mathbb{R}$

$$\lim_{\substack{y=Cx \\ (x,y)\to 0}} \frac{y^2}{x} = \lim_{x\to 0} \frac{C^2x^2}{x} = \lim_{x\to 0} C^2 x = 0$$

Consider the path $y = \sqrt{x}$,

then $\lim_{\substack{y=\sqrt{x} \\ (x,y)\to 0}} \frac{x}{x} = 1 \neq 0$

2. Determine the set of points at which the function

$$f(x,y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous.

For $(x,y) \neq (0,0)$ $g(x,y) = \frac{x^2y^2}{2x^2+y^2}$ $(x,y) \neq (0,0) \Rightarrow$ $g(x,y)$ defined everywhere

For $g(x,y)$, it must be continuous

$\because$ denomenator $\neq 0$, both 分子 分母 皆為 polynomial.

For $(x,y) = (0,0)$. Try evaluate limit from $y=x$

$$\lim_{\substack{y=x \\ (x,y)\to(0,0)}} \frac{x^2y^3}{2x^2+y^2} = \lim_{x\to 0} \frac{x^5}{3x^2} = 0 \neq 1$$

not continuous at $(x,y) = (0,0)$.

Domain of continuity : $\mathbb{R}^2 / \{0,0\}$.

3. Determine the x-partial derivative of the function

$$f(x,y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$$

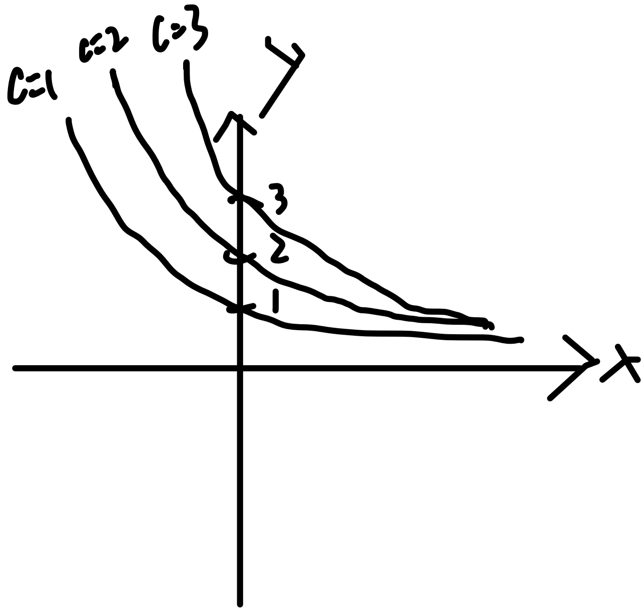
at $(0,0)$.

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-1}{h}$$

$=$ not defined.

4. Draw the contour of $f(x, y) = ye^x$ at several levels.

$y = Ce^{-x}$



5. Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

6. Evaluate $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$.

7. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2|y|+y^2|x|}{x^2+|y|}$.

8. Find the partial derivative with respect to one of the resistors in a 3-parallel resistor system.

9. The temperature at a location in the Northern Hemisphere T depends on the longitude x, latitude y, and time t, so we can write $T = f(x, y, t)$ for a differentiable function f. Lets measure time in hours from the beginning of January.
What are the physical meaning of the three partial derivatives?

5. Evaluate $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

$$\lim_{(r,\theta)\to(0,0)} \frac{r\cos\theta \, r\sin\theta}{r^2\cos\theta + r^2\sin\theta}$$

$$\frac{r^2\cos\theta\sin\theta}{r^2}$$

$$\lim_{(r,\theta)\to(0,0)} = \frac{r^2\sin\theta\cos\theta}{r^2}$$

$$= \sin\theta\cos\theta$$

$$= \frac{1}{2}\sin(2\theta)$$

$$= 0$$

6. Evaluate $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4}$.

Consider line $z = y = x \to$ 1st

and $z = y = -x \to$ 2nd

1st path $f(x,y,z) = \dfrac{x^2+x^3}{2x^2+x^4} = \dfrac{1}{2}$.

2nd path $f(x,y,z) = \dfrac{-x^2}{2x^2+x^4} = -\dfrac{1}{2}$

$\Rightarrow$ not exist

7. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2|y|+y^2|x|}{x^2+|y|}$.

for (x,y) "close" to the origin, $\Rightarrow |x|, |y| < 1$

$\Rightarrow \quad |x|, |y| < 1$

$$0 \le \frac{x^2|y| + y^2|x|}{x^2+|y|} \le \frac{x^2|y| + y^2|x|}{x^2+y^2}$$

squeeze thm.

Polar :

$$\Rightarrow \frac{r^3\cos^2\theta\,|\sin\theta| + r^3\sin^2\theta\,|\cos\theta|}{r^2}$$

$\lim_{r\to 0}$: 0. 上下约 r

Conclusion: $\lim_{(x,y)\to(0,0)} \frac{x^2|y|+y^2|x|}{x^2+|y|} = 0$