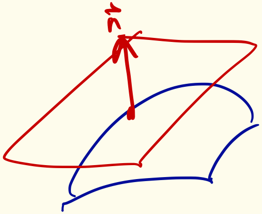
Last Time

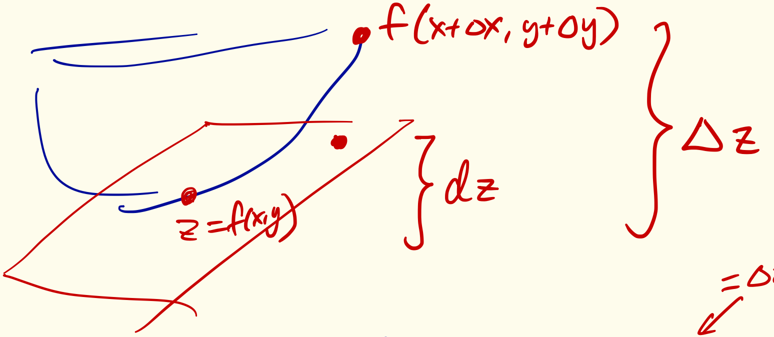
Tangent Plane $z=f(x,y)$



$\vec{n} = \langle f_x, f_y, -1 \rangle$ downward

or

$\langle -f_x, -f_y, +1 \rangle$ upward.



$$dz = f_x \underbrace{dx}_{=\Delta x} + f_y \underbrace{dy}_{=\Delta y}.$$

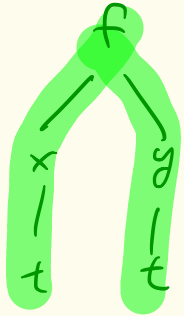
$$\Rightarrow \frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

Chain Rule

Chain Rule

$z = f(x, y)$, $x(t)$, $y(t)$

depends on t

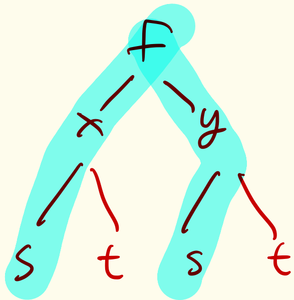


then $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$

$f(x, y)$

$x(s, t)$

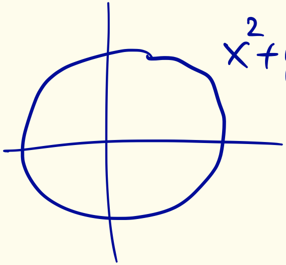
$y(s, t)$



$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}.$$

Implicit Differentiation

$$F(x,y) = 0$$



$x^2+y^2=1$

how y changes with respect to x?

$$\frac{dy}{dx} = ?$$

Chain Rule

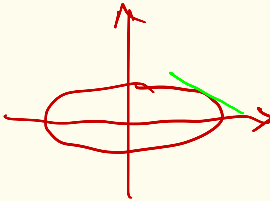
$$\frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

(Ellipse)

Ex $x^2+4y^2=1$

$$\implies x^2+4y^2-1=0$$

$$\frac{dy}{dx} = -\frac{2x}{8y} = -\frac{1}{4}\frac{x}{y}$$



Ex $\quad xy^2 = \sin y \rightsquigarrow \underbrace{xy^2 - \sin y}_{F(x,y)} = 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{y^2}{2xy - \cos y}$$

Similarly : $F(x, y, z) = 0$

How does z change w.r.t. x and y ?

z — dependent

x, y — independent variable.

$$\frac{\partial}{\partial x}: \quad \frac{\partial F}{\partial x} \underbrace{\frac{\partial x}{\partial x}}_{=1} + \frac{\partial F}{\partial y} \underbrace{\frac{\partial y}{\partial x}}_{=0} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \iff \frac{dx}{dy} = -\frac{F_y}{F_x}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

$\boxed{\text{Warning}}$ does not always hold in multivariable functions!

Ex $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \frac{\partial x}{\partial r} = \cos\theta$

$$\frac{\partial r}{\partial x} = \left(r = \frac{x}{\cos\theta} \rightsquigarrow \frac{\partial r}{\partial x} = \frac{1}{\cos\theta} \quad \times \right.$$

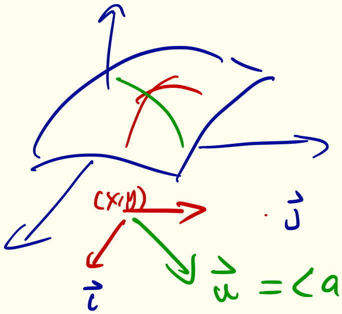
because θ depends on x in polar coord!)

$r = \sqrt{x^2 + y^2}$

$\frac{y}{x} = \tan\theta \implies \theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta.$$

Partial derivatives:



$\vec{u} = \langle a, b \rangle$ unit vector

Directional Derivatives

Def $D_{\vec{u}} f(x,y) = \frac{d}{dt} f(x+at, \ y+bt) \Big|_{t=0}$.

$$\left(\begin{aligned} D_{\vec{i}} f &= f_x \\ D_{\vec{j}} f &= f_y \end{aligned} \right)$$

Thm $D_{\vec{u}} f(x,y) = f_x \cdot a + f_y \cdot b$

$= \nabla f \cdot \vec{u}$

$\nabla f = \langle f_x, f_y \rangle$ gradient vector.

Pf By chain rule: $f(\overset{x+at}{x(t)}, \overset{y+bt}{y(t)})$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} = f_x \cdot a + f_y \cdot b.$$

Ex $\quad f(x,y) = e^{x^2 y}$

$\vec{u} = \langle 3,4 \rangle$ $\qquad$ What is $D_{\vec{u}} f$ at P ?

$P = (1,0)$

$D_{\vec{u}}$ is always using unit direction! $\qquad \vec{u} = \langle 3,4 \rangle$

$$\frac{\vec{u}}{|\vec{u}|} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$$\nabla f = \langle f_x, f_y \rangle$$

$$= \langle e^{x^2 y} \cdot 2xy, e^{x^2 y} x^2 \rangle$$

$$= \langle 0, 1 \rangle \quad \text{at } P$$

$$D_{\vec{u}} f(1,0) = \langle 0,1 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \frac{4}{5} \mathbin{/\!/}$$

Thm Maximum of $|D_{\vec{u}} f|$ occurs as the same direction as ∇f.

Pf $D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f| \overbrace{|\vec{u}|}^{1} \cos\theta$

$\Rightarrow$ max when $|\cos\theta| = 1$

$\Rightarrow \theta = 0^\circ$ or 180°.

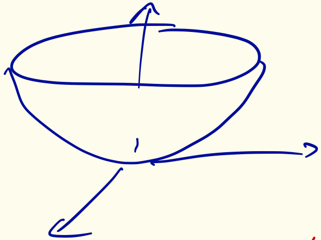
$\vec{u} \parallel \nabla f$.

$f(x, y, z)$:

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

Ex $f(x,y) = x^2 + 4y^2$

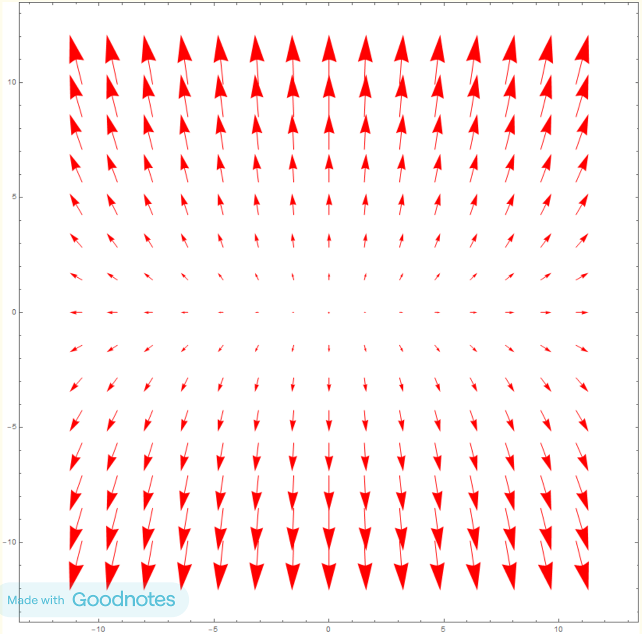
$$\nabla f = \langle f_x, f_y \rangle$$
$$= \langle 2x, 8y \rangle.$$

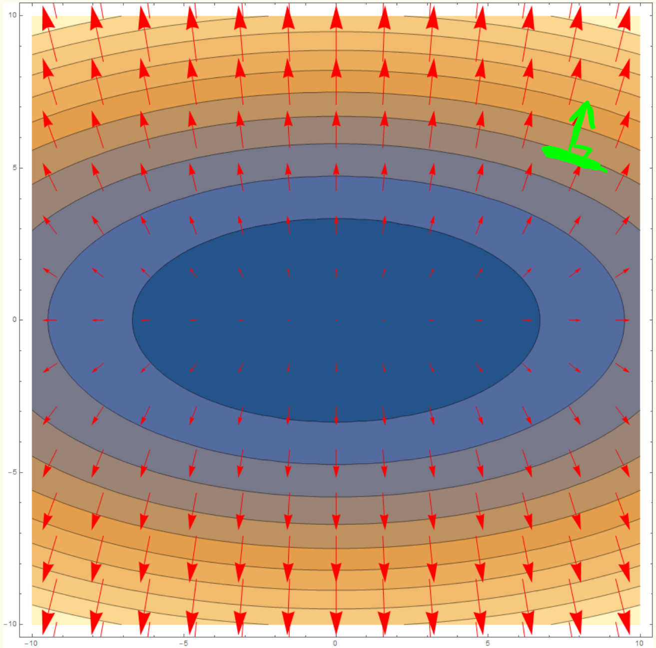


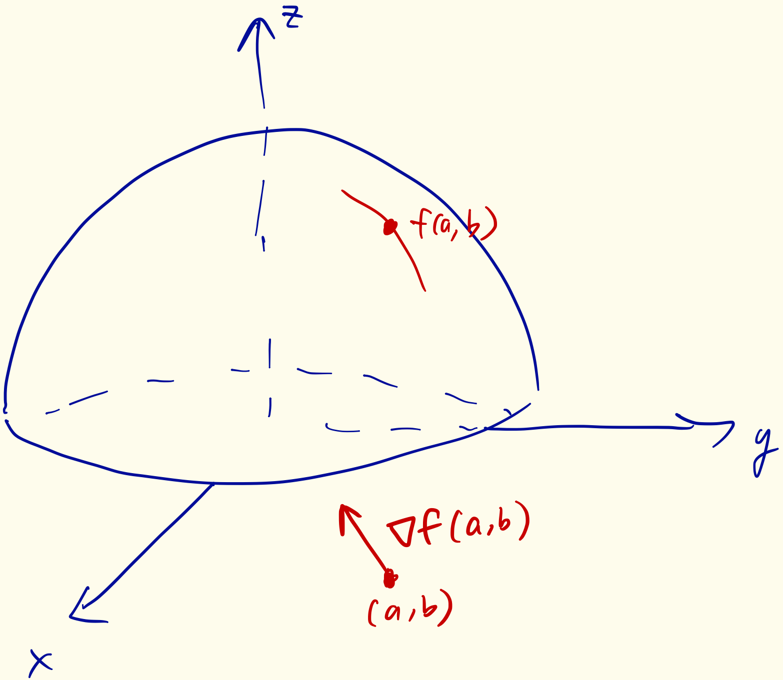
(1) : ∇f points towards greatest ascent

(2) : ∇f perpendicular to level curves.

$\boxed{\text{Gradient Vector Field}}$







$D_{\vec{u}}$ is max when $\vec{u} \parallel \nabla f$.

∇f : greatest ascent

$-\nabla f$: greatest descent.

Level curves

$f(x,y) = k \leftarrow$ const. height.

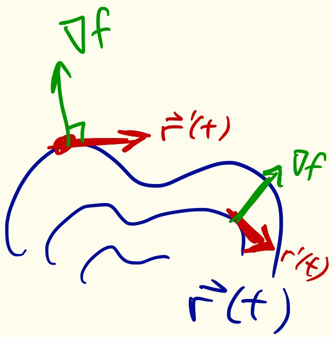
$\vec{r}(t)$ such that $f(x(t), y(t)) = k$

$\vec{r}(t) = \langle x(t), y(t) \rangle$

Chain Rule

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

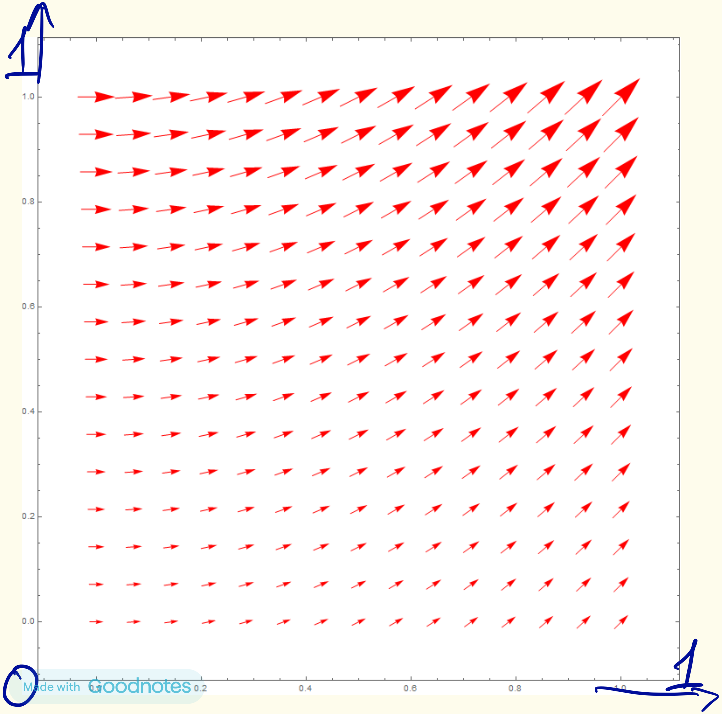
$$= \underbrace{\nabla f}_{=\langle f_x, f_y\rangle} \cdot \underbrace{\vec{r}\,'(t)}_{=\langle x', y'\rangle} = 0$$

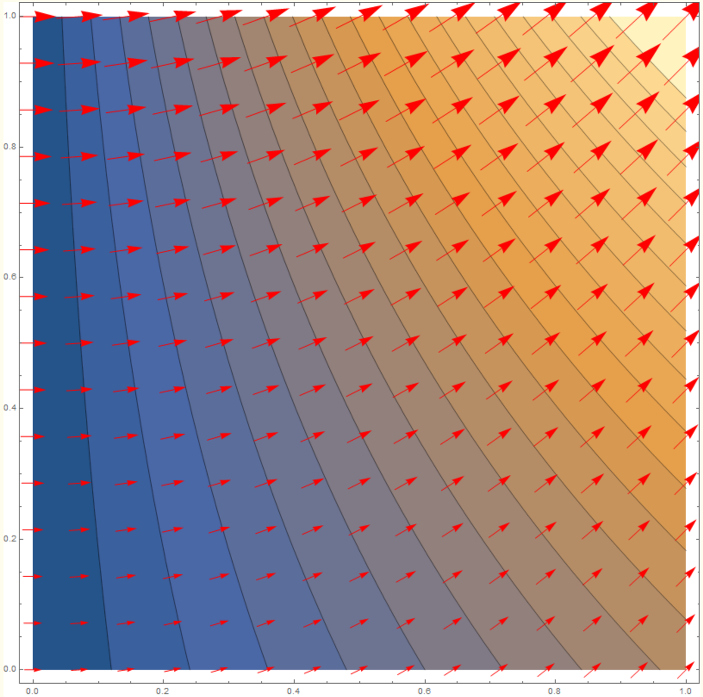


∇f is $\perp$ to level curves //.

Ex $f(x,y) = xe^y$

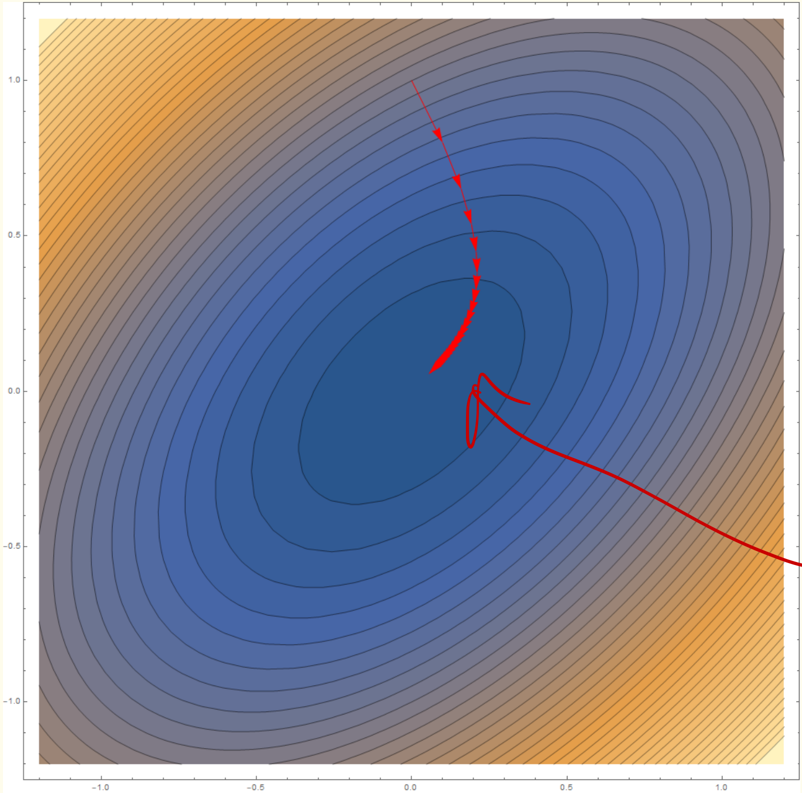
$$\nabla f = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$





<u>Ex</u> $f(x,y) = x^2 - xy + y^2$

$\nabla f = \langle 2x - y,\ 2y - x \rangle$



$-\nabla f$

$\nabla f(0,1)$
$= \langle -1, 2 \rangle$

$-\nabla f = \langle 1, -2 \rangle$

Integral Curve.

Integral Curve:

$\vec{r}(t)$ such that $\vec{r}\,'(t) = \nabla f$

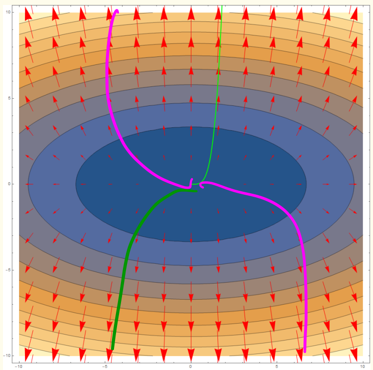
$$\begin{cases} x'(t) = f_x \\ y'(t) = f_y \end{cases}$$

$\boxed{\text{ODE}}$

Ex $f(x,y) = x^2 + 4y^2$, $\nabla f = \langle 2x, 8y \rangle$

Integral Curve: $\begin{cases} x' = 2x \\ y' = 8y \end{cases}$

e.g. $\begin{cases} x(t) = e^{2t} \\ y(t) = e^{8t} \end{cases} \Rightarrow y = x^4$



Tangent Planes

$$z = f(x,y)$$

$$\vec{n} = \langle f_x, f_y, -1 \rangle$$

$$F(x,y,z) = f(x,y) - z = 0$$

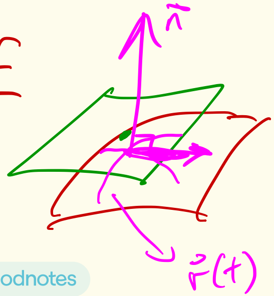
then $\vec{n} = \nabla F$!

In general

Level surfaces $f(x,y,z) = 0$

then $\vec{n}$ of tangent plane $= \nabla f$.

Pf



For every curve $\vec{r}(t)$ on f,

$$f(x(t), y(t), z(t)) = 0$$

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

$$= \nabla f \cdot \vec{r}'(t)$$

Ex $\quad yz = \ln(x+z)$

Find tangent plane at $(0,0,1)$

$$F(x,y,z) = yz - \ln(x+z) = 0$$

$$\nabla F = \langle F_x , F_y , F_z \rangle$$

$$= \left\langle \frac{-1}{x+z} , z , y - \frac{1}{x+z} \right\rangle$$

at $(0,0,1) = \langle -1 , 1 , -1 \rangle = \vec{n}$

$$\boxed{-x + y - z = -1}$$

$0 \qquad 1$