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PROBLEMS

1. **(a)** How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers?
   (b) Repeat part (a) under the assumption that no letter or number can be repeated in a single license plate.
2. How many outcome sequences are possible when a die is rolled four times, where we say, for instance, that the outcome is 3, 4, 3, 1 if the first roll landed on 3, the second on 4, the third on 3, and the fourth on 1?
3. Twenty workers are to be assigned to 20 different jobs, one to each job. How many different assignments are possible?
4. John, Jim, Jay, and Jack have formed a band consisting of 4 instruments. If each of the boys can play all 4 instruments, how many different arrangements are possible? What if John and Jim can play all 4 instruments, but Jay and Jack can each play only piano and drums?
5. For years, telephone area codes in the United States and Canada consisted of a sequence of three digits. The first digit was an integer between 2 and 9, the second digit was either 0 or 1, and the third digit was any integer from 1 to 9. How many area codes were possible? How many area codes starting with a 4 were possible?
6. A well-known nursery rhyme starts as follows:

   "As I was going to St. Ives
   I met a man with 7 wives.
   Each wife had 7 sacks.
   Each sack had 7 cats.
   Each cat had 7 kittens..."

   How many kittens did the traveler meet?
7. **(a)** In how many ways can 3 boys and 3 girls sit in a row?
   (b) In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?
   (c) In how many ways if only the boys must sit together?
   (d) In how many ways if no two people of the same sex are allowed to sit together?
8. How many different letter arrangements can be made from the letters
   (a) Fluke?
   (b) Propose?
   (c) Mississippi?
   (d) Arrange?
9. A child has 12 blocks, of which 6 are black, 4 are red, 1 is white, and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?
10. In how many ways can 8 people be seated in a row if
    (a) there are no restrictions on the seating arrangement?
    (b) persons A and B must sit next to each other?
    (c) there are 4 men and 4 women and no 2 men or 2 women can sit next to each other?
    (d) there are 5 men and they must sit next to each other?
    (e) there are 4 married couples and each couple must sit together?
11. In how many ways can 3 novels, 2 mathematics books, and 1 chemistry book be arranged on a bookshelf if
    (a) the books can be arranged in any order?
    (b) the mathematics books must be together and the novels must be together?
    (c) the novels must be together, but the other books can be arranged in any order?
12. Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a class of 30. How many different outcomes are possible if
    (a) a student can receive any number of awards?
    (b) each student can receive at most 1 award?
13. Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?
14. How many 5-card poker hands are there?
15. A dance class consists of 22 students, of which 10 are women and 12 are men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?
16. A student has to sell 2 books from a collection of 6 math, 7 science, and 4 economics books. How many choices are possible if
    (a) both books are to be on the same subject?
    (b) the books are to be on different subjects?
17. Seven different gifts are to be distributed among 10 children. How many distinct results are possible if no child is to receive more than one gift?
18. A committee of 7, consisting of 2 Republicans, 2 Democrats, and 3 Independents, is to be chosen from a group of 5 Republicans, 6 Democrats, and 4 Independents. How many committees are possible?
19. From a group of 8 women and 6 men, a committee consisting of 3 men and 3 women is to be formed. How many different committees are possible if
    (a) 2 of the men refuse to serve together?
    (b) 2 of the women refuse to serve together?
    (c) 1 man and 1 woman refuse to serve together?

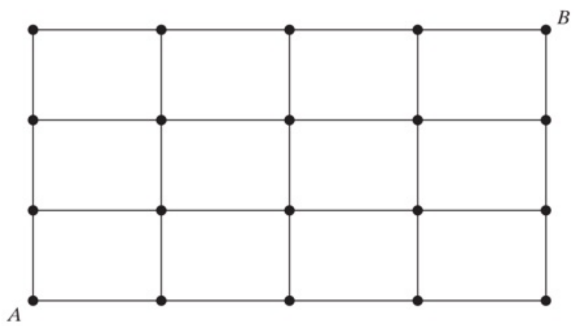
20. A person has 8 friends, of whom 5 will be invited to a party.

(a) How many choices are there if 2 of the friends are feuding and will not attend together?

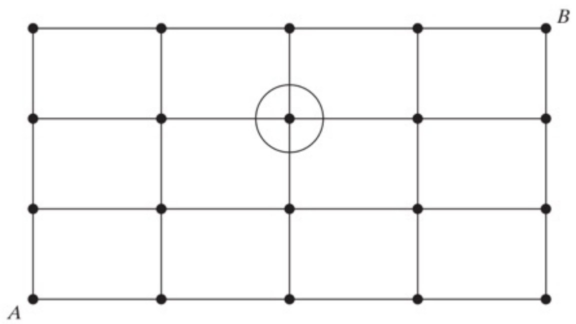
(b) How many choices if 2 of the friends will only attend together?

21. Consider the grid of points shown here. Suppose that, starting at the point labeled A, you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B is reached. How many different paths from A to B are possible?

Hint: Note that to reach B from A, you must take 4 steps to the right and 3 steps upward.



22. In Problem 21, how many different paths are there from A to B that go through the point circled in the following lattice?



23. A psychology laboratory conducting dream research contains 3 rooms, with 2 beds in each room. If 3 sets of identical twins are to be assigned to these 6 beds so that each set of twins sleeps in different beds in the same room, how many assignments are possible?

24. Expand $(3x^2 + y)^5$.

25. The game of bridge is played by 4 players, each of whom is dealt 13 cards. How many bridge deals are possible?

26. Expand $(x_1 + 2x_2 + 3x_3)^4$.

27. If 12 people are to be divided into 3 committees of respective sizes 3, 4, and 5, how many divisions are possible?

28. If 8 new teachers are to be divided among 4 schools, how many divisions are possible? What if each school must receive 2 teachers?

29. Ten weight lifters are competing in a team weight-lifting contest. Of the lifters, 3 are from the United States, 4 are from Russia, 2 are from China, and 1 is from Canada. If the scoring takes account of the countries that the lifters represent, but not their individual identities, how many different outcomes are possible from the point of view of scores? How many different outcomes correspond to results in which the United States has 1 competitor in the top three and 2 in the bottom three?

30. Delegates from 10 countries, including Russia, France, England, and the United States, are to be seated in a row. How many different seating arrangements are possible if the French and English delegates are to be seated next to each other and the Russian and U.S. delegates are not to be next to each other?

***31.** If 8 identical blackboards are to be divided among 4 schools, how many divisions are possible? How many if each school must receive at least 1 blackboard?

***32.** An elevator starts at the basement with 8 people (not including the elevator operator) and discharges them all by the time it reaches the top floor, number 6. In how many ways could the operator have perceived the people leaving the elevator if all people look alike to him? What if the 8 people consisted of 5 men and 3 women and the operator could tell a man from a woman?

***33.** We have 20 thousand dollars that must be invested among 4 possible opportunities. Each investment must be integral in units of 1 thousand dollars, and there are minimal investments that need to be made if one is to invest in these opportunities. The minimal investments are 2, 2, 3, and 4 thousand dollars. How many different investment strategies are available if

(a) an investment must be made in each opportunity?

(b) investments must be made in at least 3 of the 4 opportunities?

Eqt 4.1: $\binom{n-1}{r} + \binom{n-1}{r-1} = \binom{n}{r}$

THEORETICAL EXERCISES

1. Prove the generalized version of the basic counting principle.

2. Two experiments are to be performed. The first can result in any one of m possible outcomes. If the first experiment results in outcome i, then the second experiment can result in any of n_i possible outcomes, $i = 1, 2, \ldots, m$. What is the number of possible outcomes of the two experiments?

3. In how many ways can r objects be selected from a set of n objects if the order of selection is considered relevant?

4. There are $\binom{n}{r}$ different linear arrangements of n balls of which r are black and $n - r$ are white. Give a combinatorial explanation of this fact.

5. Determine the number of vectors $(x_1, \ldots, x_n)$, such that each x_i is either 0 or 1 and

$$\sum_{i=1}^{n} x_i \geq k$$

6. How many vectors $x_1, \ldots, x_k$ are there for which each x_i is a positive integer such that $1 \leq x_i \leq n$ and $x_1 < x_2 < \cdots < x_k$?

7. Give an analytic proof of Equation (4.1).

8. Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

Hint: Consider a group of n men and m women. How many groups of size r are possible?

9. Use Theoretical Exercise 8 to prove that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

10. From a group of n people, suppose that we want to choose a committee of k, $k \leq n$, one of whom is to be designated as chairperson.

(a) By focusing first on the choice of the committee and then on the choice of the chair, argue that there are $\binom{n}{k}k$ possible choices.

(b) By focusing first on the choice of the nonchair committee members and then on the choice of the chair, argue that there are $\binom{n}{k-1}(n - k + 1)$ possible choices.

(c) By focusing first on the choice of the chair and then on the choice of the other committee members, argue that there are $n\binom{n-1}{k-1}$ possible choices.

(d) Conclude from parts (a), (b), and (c) that

$$k\binom{n}{k} = (n-k+1)\binom{n}{k-1} = n\binom{n-1}{k-1}$$

(e) Use the factorial definition of $\binom{m}{r}$ to verify the identity in part (d).

11. The following identity is known as Fermat's combinatorial identity:

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1} \qquad n \geq k$$

Give a combinatorial argument (no computations are needed) to establish this identity.
Hint: Consider the set of numbers 1 through n. How many subsets of size k have i as their highest-numbered member?

12. Consider the following combinatorial identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$$

(a) Present a combinatorial argument for this identity by considering a set of n people and determining, in two ways, the number of possible selections of a committee of any size and a chairperson for the committee.
Hint:
(i) How many possible selections are there of a committee of size k and its chairperson?
(ii) How many possible selections are there of a chairperson and the other committee members?

(b) Verify the following identity for $n = 1, 2, 3, 4, 5$:

$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2}n(n+1)$$

For a combinatorial proof of the preceding, consider a set of n people and argue that both sides of the identity represent the number of different selections of a committee, its chairperson, and its secretary (possibly the same as the chairperson).

Hint:

(i) How many different selections result in the committee containing exactly k people?

(ii) How many different selections are there in which the chairperson and the secretary are the same? (ANSWER: $n2^{n-1}$.)

(iii) How many different selections result in the chairperson and the secretary being different? $n(n-1)2^{n-2}$

(c) Now argue that

$$\sum_{k=1}^{n} \binom{n}{k} k^3 = 2^{n-3} n^2 (n+3)$$

13. Show that, for $n > 0$,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

Hint: Use the binomial theorem.

14. From a set of n people, a committee of size j is to be chosen, and from this committee, a subcommittee of size i, $i \le j$, is also to be chosen.

(a) Derive a combinatorial identity by computing, in two ways, the number of possible choices of the committee and subcommittee—first by supposing that the committee is chosen first and then the subcommittee is chosen, and second by supposing that the subcommittee is chosen first and then the remaining members of the committee are chosen.

(b) Use part (a) to prove the following combinatorial identity:

$$\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \binom{n}{i} 2^{n-i} \quad i \le n$$

(c) Use part (a) and Theoretical Exercise 13 to show that

$$\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} (-1)^{n-j} = 0 \quad i < n$$

15. Let $H_k(n)$ be the number of vectors $x_1, \ldots, x_k$ for which each x_i is a positive integer satisfying $1 \le x_i \le n$ and $x_1 \le x_2 \le \cdots \le x_k$.

(a) Without any computations, argue that

$$H_1(n) = n$$
$$H_k(n) = \sum_{j=1}^{n} H_{k-1}(j) \quad k > 1$$

Hint: How many vectors are there in which $x_k = j$?

(b) Use the preceding recursion to compute $H_3(5)$.
Hint: First compute $H_2(n)$ for $n = 1, 2, 3, 4, 5$.

16. Consider a tournament of n contestants in which the outcome is an ordering of these contestants, with ties allowed. That is, the outcome partitions the players into groups, with the first group consisting of the players that tied for first place, the next group being those that tied for the next-best position, and so on. Let $N(n)$ denote the number of different possible outcomes. For instance, $N(2) = 3$, since, in a tournament with 2 contestants, player 1 could be uniquely first, player 2 could be uniquely first, or they could tie for first.

(a) List all the possible outcomes when $n = 3$.

(b) With $N(0)$ defined to equal 1, argue, without any computations, that

$$N(n) = \sum_{i=1}^{n} \binom{n}{i} N(n-i)$$

Hint: How many outcomes are there in which i players tie for last place?

(c) Show that the formula of part (b) is equivalent to the following:

$$N(n) = \sum_{i=0}^{n-1} \binom{n}{i} N(i)$$

(d) Use the recursion to find $N(3)$ and $N(4)$.

17. Present a combinatorial explanation of why $\binom{n}{r} = \binom{n}{r, n-r}$.

18. Argue that

$$\binom{n}{n_1, n_2, \ldots, n_r} = \binom{n-1}{n_1 - 1, n_2, \ldots, n_r} + \binom{n-1}{n_1, n_2 - 1, \ldots, n_r} + \cdots + \binom{n-1}{n_1, n_2, \ldots, n_r - 1}$$

Hint: Use an argument similar to the one used to establish Equation (4.1).

19. Prove the multinomial theorem.

***20.** In how many ways can n identical balls be distributed into r urns so that the ith urn contains at least m_i balls, for each $i = 1, \ldots, r$? Assume that $n \geq \sum_{i=1}^{r} m_i$.

***21.** Argue that there are exactly $\binom{r}{k}\binom{n-1}{n-r+k}$ solutions of

$$x_1 + x_2 + \cdots + x_r = n$$

for which exactly k of the x_i are equal to 0.

***22.** Consider a function $f(x_1, \ldots, x_n)$ of n variables. How many different partial derivatives of order r does f possess?

***23.** Determine the number of vectors $(x_1, \ldots, x_n)$ such that each x_i is a nonnegative integer and

$$\sum_{i=1}^{n} x_i \leq k$$

SELF-TEST PROBLEMS AND EXERCISES

1. How many different linear arrangements are there of the letters A, B, C, D, E, F for which
 (a) A and B are next to each other?
 (b) A is before B?
 (c) A is before B and B is before C?
 (d) A is before B and C is before D?
 (e) A and B are next to each other and C and D are also next to each other?
 (f) E is not last in line?

2. If 4 Americans, 3 French people, and 3 British people are to be seated in a row, how many seating arrangements are possible when people of the same nationality must sit next to each other?

3. A president, treasurer, and secretary, all different, are to be chosen from a club consisting of 10 people. How many different choices of officers are possible if
 (a) there are no restrictions?
 (b) A and B will not serve together?
 (c) C and D will serve together or not at all?
 (d) E must be an officer?
 (e) F will serve only if he is president?

4. A student is to answer 7 out of 10 questions in an examination. How many choices has she? How many if she must answer at least 3 of the first 5 questions?

5. In how many ways can a man divide 7 gifts among his 3 children if the eldest is to receive 3 gifts and the others 2 each?

6. How many different 7-place license plates are possible when 3 of the entries are letters and 4 are digits? Assume that repetition of letters and numbers is allowed and that there is no restriction on where the letters or numbers can be placed.

7. Give a combinatorial explanation of the identity

$$\binom{n}{r} = \binom{n}{n-r}$$

8. Consider n-digit numbers where each digit is one of the 10 integers $0, 1, \ldots, 9$. How many such numbers are there for which
 (a) no two consecutive digits are equal?
 (b) 0 appears as a digit a total of i times, $i = 0, \ldots, n$?

9. Consider three classes, each consisting of n students. From this group of $3n$ students, a group of 3 students is to be chosen.
 (a) How many choices are possible?
 (b) How many choices are there in which all 3 students are in the same class?
 (c) How many choices are there in which 2 of the 3 students are in the same class and the other student is in a different class?
 (d) How many choices are there in which all 3 students are in different classes?
 (e) Using the results of parts (a) through (d), write a combinatorial identity.

10. How many 5-digit numbers can be formed from the integers $1, 2, \ldots, 9$ if no digit can appear more than twice? (For instance, 41434 is not allowed.)

11. From 10 married couples, we want to select a group of 6 people that is not allowed to contain a married couple.
 (a) How many choices are there?
 (b) How many choices are there if the group must also consist of 3 men and 3 women?

12. A committee of 6 people is to be chosen from a group consisting of 7 men and 8 women. If the committee must consist of at least 3 women and at least 2 men, how many different committees are possible?

***13.** An art collection on auction consisted of 4 Dalis, 5 van Goghs, and 6 Picassos. At the auction were 5 art collectors. If a reporter noted only the number of Dalis, van Goghs, and Picassos acquired by each collector, how many different results could have been recorded if all of the works were sold?

***14.** Determine the number of vectors $(x_1, \ldots, x_n)$ such that each x_i is a positive integer and

$$\sum_{i=1}^{n} x_i \leq k$$

where $k \geq n$.

15. A total of n students are enrolled in a review course for the actuarial examination in probability. The posted results of the examination will list the names of those who passed, in decreasing order of their scores. For instance, the posted result will be "Brown, Cho" if Brown and Cho are the only ones to pass, with Brown receiving the higher score. Assuming that all scores are distinct (no ties), how many posted results are possible?

16. How many subsets of size 4 of the set $S = \{1, 2, \ldots, 20\}$ contain at least one of the elements 1, 2, 3, 4, 5?

17. Give an analytic verification of

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}, \quad 1 \leq k \leq n$$

Now, give a combinatorial argument for this identity.

18. In a certain community, there are 3 families consisting of a single parent and 1 child, 3 families consisting of a single parent and 2 children, 5 families consisting of 2 parents and a single child, 7 families consisting of 2 parents and 2 children, and 6 families consisting of 2 parents and 3 children. If a parent and child from the same family are to be chosen, how many possible choices are there?

19. If there are no restrictions on where the digits and letters are placed, how many 8-place license plates consisting of 5 letters and 3 digits are possible if no repetitions of letters or digits are allowed. What if the 3 digits must be consecutive?

Q13 Sol: **1.13.** (number of solutions of $x_1 + \cdots + x_5 = 4$)(number of solutions of $x_1 + \cdots + x_5 = 5$)(number of solutions of $x_1 + \cdots + x_5 = 6$) $= \binom{8}{4}\binom{9}{4}\binom{10}{4}$.

$10 \times 9 \times 8 \times 26^5 \times$

$3! \times 6!$

1. (a) How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers?
   (b) Repeat part (a) under the assumption that no letter or number can be repeated in a single license plate.
2. How many outcome sequences are possible when a die is rolled four times, where we say, for instance, that the outcome is 3, 4, 3, 1 if the first roll landed on 3, the second on 4, the third on 3, and the fourth on 1?
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1a.26^2×10^5=6760000
1b.26×25×10×9×8×7×6=19656000
2.4!/2!=12 6^4
3.20!
4.4!=24,2×2=4
5.8×2×9=648,2×9=18 2×9
144

6. A well-known nursery rhyme starts as follows:

"As I was going to St. Ives
I met a man with 7 wives.
Each wife had 7 sacks.
Each sack had 7 cats.
Each cat had 7 kittens..."
How many kittens did the traveler meet?

7. (a) In how many ways can 3 boys and 3 girls sit in a row?
   (b) In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?
   (c) In how many ways if only the boys must sit together?
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8. How many different letter arrangements can be made from the letters
   (a) Fluke?
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   (c) Mississippi?
   (d) Arrange?

9. A child has 12 blocks, of which 6 are black, 4 are red, 1 is white, and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

6. $7 \times 7 \times 7 \times 7 = 2401$ ✓
7a. $6! = 720$ ✓
7b. $3! \times 3! \times 2 = 36 \times 2 = 72$ ✓
7c. $4! \times 3! = 144$ ✓
7d. $3! \times 3! \times 2 = 72$ ✓

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- **(a)** Fluke?
- **(b)** Propose?
- **(c)** Mississippi?
- **(d)** Arrange?

9. A child has 12 blocks, of which 6 are black, 4 are red, 1 is white, and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

8 a). $5!$ ✓

b). $\dfrac{7!}{2!2!}$ ✓

c). $\dfrac{11!}{4!4!2!}$ ✓

d). $\dfrac{7!}{2!2!}$ ✓

9. $\dfrac{12!}{6!4!}$ ✓

10. In how many ways can 8 people be seated in a row if

(a) there are no restrictions on the seating arrangement?

(b) persons A and B must sit next to each other?

(c) there are 4 men and 4 women and no 2 men or 2 women can sit next to each other?

(d) there are 5 men and they must sit next to each other?

(e) there are 4 married couples and each couple must sit together?

11. In how many ways can 3 novels, 2 mathematics books, and 1 chemistry book be arranged on a bookshelf if

(a) the books can be arranged in any order?

(b) the mathematics books must be together and the novels must be together?

(c) the novels must be together, but the other books can be arranged in any order?

12. Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a class of 30. How many different outcomes are possible if

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14. How many 5-card poker hands are there?

15. A dance class consists of 22 students, of which 10 are women and 12 are men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

16. A student has to sell 2 books from a collection of 6 math, 7 science, and 4 economics books. How many choices are possible if

(a) both books are to be on the same subject?

(b) the books are to be on different subjects?

17. Seven different gifts are to be distributed among 10 children. How many distinct results are possible if no child is to receive more than one gift?

18. A committee of 7, consisting of 2 Republicans, 2 Democrats, and 3 Independents, is to be chosen from a group of 5 Republicans, 6 Democrats, and 4 Independents. How many committees are possible?

19. From a group of 8 women and 6 men, a committee consisting of 3 men and 3 women is to be formed. How many different committees are possible if

(a) 2 of the men refuse to serve together?

(b) 2 of the women refuse to serve together?

(c) 1 man and 1 woman refuse to serve together?

10 a). $8!$

b). $7! \times 2!$

c). $4! \times 4! \times 2$

d). $5! \times 4!$

e). $4! \times 2^4$

11. a). $\frac{6!}{3!2!}$ $\times$ $6!$

b). $3!3!2!$

c). $4!3!$

12 a). $x_0 + x_1 + x_2 + \cdots + x_{30} = 5$

$x_i \geq 0$

$\binom{30+5-1}{5-1}$ X 30^5

b). $x_0 + x_1 + x_2 + \cdots + x_{30} = 5$

$x_i \leq 1$

0000.... 11111 $\binom{30}{5}$

13. 5 ABCDE

$\sum_{i=1}^{19} i = \frac{20 \times 19}{2} = 190$

AB

AC 4

AD

AE

BC

BD 3

BE

CD 2

CE

DE 1

14. How many 5-card poker hands are there?

15. A dance class consists of 22 students, of which 10 are women and 12 are men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

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(b) 2 of the women refuse to serve together?

(c) 1 man and 1 woman refuse to serve together?

19 a). $\binom{8}{3}\binom{6}{3} - \binom{4}{1}\binom{8}{3}$

$1120 - 224 = 896$

b). $\binom{8}{3}\binom{6}{3} - \binom{6}{1}\binom{6}{3}$

$1,000$

c). $\binom{8}{3}\binom{6}{3} - \binom{7}{2}\binom{5}{2}$

910

14.

17 D D 12 17

$\binom{52}{5}$

15. $\binom{10}{5}\binom{12}{5}$

16. a). $\binom{6}{2} + \binom{7}{2} + \binom{4}{2}$

b). MS $\binom{6}{1}\binom{7}{1} +$

ME $\binom{6}{1}\binom{4}{1} +$

SE $\binom{7}{1}\binom{4}{1}$

17. $\binom{10}{7}$

$10 \times 9 \times 8 \times 7$ $\frac{10!}{3!}$

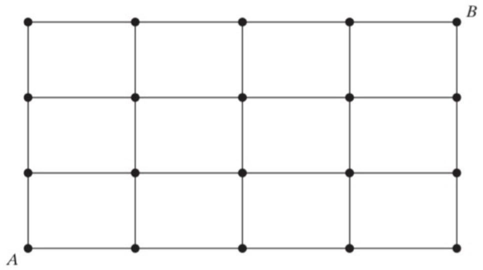
18. $\binom{5}{2}\binom{6}{2}\binom{4}{3}$

20. A person has 8 friends, of whom 5 will be invited to a party.

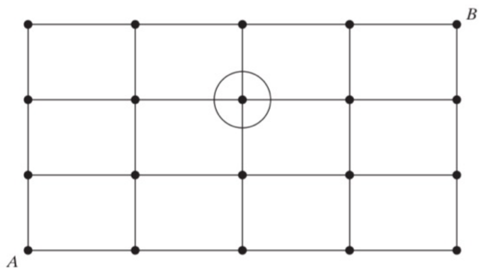
(a) How many choices are there if 2 of the friends are feuding and will not attend together?

(b) How many choices if 2 of the friends will only attend together?

21. Consider the grid of points shown here. Suppose that, starting at the point labeled A, you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B is reached. How many different paths from A to B are possible?
Hint: Note that to reach B from A, you must take 4 steps to the right and 3 steps upward.



22. In Problem 21, how many different paths are there from A to B that go through the point circled in the following lattice?



23. A psychology laboratory conducting dream research contains 3 rooms, with 2 beds in each room. If 3 sets of identical twins are to be assigned to these 6 beds so that each set of twins sleeps in different beds in the same room, how many assignments are possible?

24. Expand $(3x^2 + y)^5$.

25. The game of bridge is played by 4 players, each of whom is dealt 13 cards. How many bridge deals are possible?

26. Expand $(x_1 + 2x_2 + 3x_3)^4$.

27. If 12 people are to be divided into 3 committees of respective sizes 3, 4, and 5, how many divisions are possible?

28. If 8 new teachers are to be divided among 4 schools, how many divisions are possible? What if each school must receive 2 teachers?

29. Ten weight lifters are competing in a team weight-lifting contest. Of the lifters, 3 are from the United States, 4 are from Russia, 2 are from China, and 1 is from Canada. If the scoring takes account of the countries that the lifters represent, but not their individual identities, how many different outcomes are possible from the point of view of scores? How many different outcomes correspond to results in which the United States has 1 competitor in the top three and 2 in the bottom three?

30. Delegates from 10 countries, including Russia, France, England, and the United States, are to be seated in a row. How many different seating arrangements are possible if the French and English delegates are to be seated next to each other and the Russian and U.S. delegates are not to be next to each other?

***31.** If 8 identical blackboards are to be divided among 4 schools, how many divisions are possible? How many if each school must receive at least 1 blackboard?

***32.** An elevator starts at the basement with 8 people (not including the elevator operator) and discharges them all by the time it reaches the top floor, number 6. In how many ways could the operator have perceived the people leaving the elevator if all people look alike to him? What if the 8 people consisted of 5 men and 3 women and the operator could tell a man from a woman?

***33.** We have 20 thousand dollars that must be invested among 4 possible opportunities. Each investment must be integral in units of 1 thousand dollars, and there are minimal investments that need to be made if one is to invest in these opportunities. The minimal investments are 2, 2, 3, and 4 thousand dollars. How many different investment strategies are available if

(a) an investment must be made in each opportunity?

(b) investments must be made in at least 3 of the 4 opportunities?

20a). $\binom{8}{5} - \binom{6}{3}$ 36

They will attend +

b). $\binom{6}{3} + \binom{6}{5}$ they will not.

21. $\binom{7}{3}$

22. $\binom{4}{2}\binom{3}{2}$ 18

23. $6! - 3! - 3 \times 2! \times 2!$

$3! \times 2^3$ 702 × 48.

24. ~~skip~~

25. ~~???~~

26. $\cancel{\binom{4}{x_1 x_2 x_3}} (x_1 + 2x_2 + 3x_3)$

004 013 022
040 103 202
400 031 220
130
301
310

27. If 12 people are to be divided into 3 committees of respective sizes 3, 4, and 5, how many divisions are possible?

28. If 8 new teachers are to be divided among 4 schools, how many divisions are possible? What if each school must receive 2 teachers?

29. Ten weight lifters are competing in a team weight-lifting contest. Of the lifters, 3 are from the United States, 4 are from Russia, 2 are from China, and 1 is from Canada. If the scoring takes account of the countries that the lifters represent, but not their individual identities, how many different outcomes are possible from the point of view of scores? How many different outcomes correspond to results in which the United States has 1 competitor in the top three and 2 in the bottom three?

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***33.** We have 20 thousand dollars that must be invested among 4 possible opportunities. Each investment must be integral in units of 1 thousand dollars, and there are minimal investments that need to be made if one is to invest in these opportunities. The minimal investments are 2, 2, 3, and 4 thousand dollars. How many different investment strategies are available if

(a) an investment must be made in each opportunity?

(b) investments must be made in at least 3 of the 4 opportunities?

6
5
4
3
2
1
0

27. $\dfrac{12!}{3!\,4!\,5!}$

28. $x_1+x_2+x_3+x_4=8$

4^8

$0 \le x_i \le 8$

$\binom{8}{2\,2\,2\,2}$

$y_i = x_i+1$

$y_1-1+y_2-1+\cdots=8$

$y_1+y_2+y_3+y_4=8+4$

$\cancel{\binom{8+4-1}{4-1}}$

29. $\dfrac{10!}{3!\,4!\,2!}$

1 | 3

0 | 4

2 | 3

$\cancel{\dfrac{7!}{2!\,4!\,1!}} \times \binom{3}{1}\binom{3}{2}$

30. R F E U, +6

$\boxed{FE}$, RU ×

$\boxed{RU}\boxed{FE}$

$2!\times 9! - 2!\times 2!\times 8!$

32. $x_1+x_2+x_3+x_4+x_5+x_6=8$

$\binom{8+6-1}{6-1}$

$\binom{5+6-1}{6-1}\binom{3+6-1}{6-1}$

31. $x_1+x_2+x_3+x_4=8$

$\binom{8-1}{4-1}$ × $\binom{8+4-1}{4-1}$

$x_i \ge 1$

$\binom{8+4-1}{4-1}$ × $\binom{8-1}{4-1}$

*33. We have 20 thousand dollars that must be invested among 4 possible opportunities. Each investment must be integral in units of 1 thousand dollars, and there are minimal investments that need to be made if one is to invest in these opportunities. The minimal investments are 2, 2, 3, and 4 thousand dollars. How many different investment strategies are available if

(a) an investment must be made in each opportunity?

(b) investments must be made in at least 3 of the 4 opportunities?

33a).

$x_1+x_2+x_3+x_4=20$

$XX|XX|XXX|X\,XX$

$\cancel{X}XXXXXXXXX$

$\binom{9+4-1}{4-1}$

b). ≥ 3

$= 3 + 4.$

$\binom{9+4-1}{4-1}$

① ② ③ ④

不① $\binom{11+3-1}{3-1}+$

不② $\binom{11+3-1}{3-1}+$

不③ $\binom{12+3-1}{3-1}+$

不④ $\binom{13+3-1}{3-1}$

1. Prove the generalized version of the basic counting principle.

2. Two experiments are to be performed. The first can result in any one of m possible outcomes. If the first experiment results in outcome i, then the second experiment can result in any of n_i possible outcomes, $i = 1, 2, \ldots, m$. What is the number of possible outcomes of the two experiments?

3. In how many ways can r objects be selected from a set of n objects if the order of selection is considered relevant?

4. There are $\binom{n}{r}$ different linear arrangements of n balls of which r are black and $n - r$ are white. Give a combinatorial explanation of this fact.

5. Determine the number of vectors $(x_1, \ldots, x_n)$, such that each x_i is either 0 or 1 and

$$\sum_{i=1}^{n} x_i \geq k$$

6. How many vectors $x_1, \ldots, x_k$ are there for which each x_i is a positive integer such that $1 \leq x_i \leq n$ and $x_1 < x_2 < \cdots < x_k$?

7. Give an analytic proof of Equation (4.1).

8. Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

Hint: Consider a group of n men and m women. How many groups of size r are possible?

9. Use Theoretical Exercise 8 to prove that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

10. From a group of n people, suppose that we want to choose a committee of $k, k \leq n$, one of whom is to be designated as chairperson.

(a) By focusing first on the choice of the committee and then on the choice of the chair, argue that there are $\binom{n}{k} k$ possible choices.

(b) By focusing first on the choice of the nonchair committee members and then on the choice of the chair, argue that there are $\binom{n}{k-1}(n - k + 1)$ possible choices.

(c) By focusing first on the choice of the chair and then on the choice of the other committee members, argue that there are $n\binom{n-1}{k-1}$ possible choices.

(d) Conclude from parts (a), (b), and (c) that

$$k\binom{n}{k} = (n-k+1)\binom{n}{k-1} = n\binom{n-1}{k-1}$$

(e) Use the factorial definition of $\binom{m}{r}$ to verify the identity in part (d).

11. The following identity is known as Fermat's combinatorial identity:

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1} \qquad n \geq k$$

Give a combinatorial argument (no computations are needed) to establish this identity.
Hint: Consider the set of numbers 1 through n. How many subsets of size k have i as their highest-numbered member?

12. Consider the following combinatorial identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$$

(a) Present a combinatorial argument for this identity by considering a set of n people and determining, in two ways, the number of possible selections of a committee of any size and a chairperson for the committee.
Hint:
(i) How many possible selections are there of a committee of size k and its chairperson?
(ii) How many possible selections are there of a chairperson and the other committee members?

(b) Verify the following identity for $n = 1, 2, 3, 4, 5$:

$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2} n(n+1)$$

1. make a list :

$(1, 1, \ldots, 1)$

$\vdots$

$(1, 1, \ldots, n_r) \quad \ldots \quad (n_1, n_2, \ldots, n_r)$

2. Two experiments are to be performed. The first can result in any one of m possible outcomes. If the first experiment results in outcome i, then the second experiment can result in any of n_i possible outcomes, $i = 1, 2, \ldots, m$. What is the number of possible outcomes of the two experiments?

3. In how many ways can r objects be selected from a set of n objects if the order of selection is considered relevant?

$$\sum_{i=1}^{m} \cancel{m} n_i \qquad \sum_{i=1}^{m} n_i$$

3. $$\frac{n!}{(n-r)!}$$

4. There are $\binom{n}{r}$ different linear arrangements of n balls of which r are black and $n-r$ are white. Give a combinatorial explanation of this fact.

5. Determine the number of vectors $(x_1, \ldots, x_n)$, such that each x_i is either 0 or 1 and

$$\sum_{i=1}^{n} x_i \geq k$$

6. How many vectors $x_1, \ldots, x_k$ are there for which each x_i is a positive integer such that $1 \leq x_i \leq n$ and $x_1 < x_2 < \cdots < x_k$?

4. $\dfrac{n!}{(n-r)!\,r!}$ $\therefore$ permutation of n balls, dividing r! black balls, (n−r)! white balls.

5. $x_1 + \cdots + x_n \geq k, \quad 0 \leq k \leq n$

$x_1 + \cdots + x_n = n, \quad 1 = \binom{n}{0}$

$x_1 + \cdots + x_n = n-1, \quad \binom{n}{1}$

$x_1 + \cdots + x_n = n-2, \quad \binom{n}{2}$

$x_1 + \cdots + x_n = k+1, \quad \binom{n}{k+1}$

$\sum_{i=k}^{n} \binom{n}{i}$

6-

6. How many vectors $x_1, \ldots, x_k$ are there for which each x_i is a positive integer such that $1 \le x_i \le n$ and $x_1 < x_2 < \cdots < x_k$?

7. Give an analytic proof of Equation (4.1).

8. Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

Hint: Consider a group of n men and m women. How many groups of size r are possible?

9. Use Theoretical Exercise 8 to prove that

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

10. From a group of n people, suppose that we want to choose a committee of $k, k \le n$, one of whom is to be designated as chairperson.

(a) By focusing first on the choice of the committee and then on the choice of the chair, argue that there are $\binom{n}{k}k$ possible choices.

(b) By focusing first on the choice of the nonchair committee members and then on

$n > k$.

6. ? $\binom{n}{k}$

7. ?

8. $\binom{n+m}{r}$ = From n+m objects choose r objects.

$\frac{(n+m)!}{(n+m-r)!\,r!}$

Case 1: all r objects are from m. $\binom{n}{0}\binom{m}{r}$

Case 2: 1 from n, r-1 from m.

$$\binom{n}{0}\binom{m}{r}$$

$$= \frac{n!}{n!\,0!}\frac{m!}{(m-r)!\,r!} + \frac{n!}{(n-1)!\,1!}\frac{m!}{(m-r+1)!\,(r-1)!}$$

$$(n!\,m!)\left(\frac{1}{\qquad}\right.$$

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

$n,\ m=n,\ r=n$

$$\binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0}$$

$$= \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

$$= \sum_{k=0}^{n}\binom{n}{k}^2$$

10. From a group of n people, suppose that we want to choose a committee of $k, k \leq n$, one of whom is to be designated as chairperson.

(a) By focusing first on the choice of the committee and then on the choice of the chair, argue that there are $\binom{n}{k}k$ possible choices.

(b) By focusing first on the choice of the nonchair committee members and then on the choice of the chair, argue that there are $\binom{n}{k-1}(n-k+1)$ possible choices.

(c) By focusing first on the choice of the chair and then on the choice of the other committee members, argue that there are $n\binom{n-1}{k-1}$ possible choices.

(d) Conclude from parts (a), (b), and (c) that

$$k\binom{n}{k} = (n-k+1)\binom{n}{k-1} = n\binom{n-1}{k-1}$$

(e) Use the factorial definition of $\binom{m}{r}$ to verify the identity in part (d).

e). $k \dfrac{n!}{(n-k)!\,k!}$

$= \dfrac{n!}{(n-k)!\,(k-1)!}$

$\dfrac{n!}{(n-k+1)!\,(k-1)!}(n-k+1)$

$= \dfrac{n!}{(n-k)!\,(k-1)!}$

$n\dfrac{(n-1)!}{(k-1)!\,(n-1-k+1)!}$

$= \dfrac{n!}{(k-1)!\,(n-k)!}$ ✓

10a). ✓

b). $\binom{n}{k-1}$

$\to \binom{n-k+1}{1}$.

c). $\binom{n}{1}\binom{n-1}{k-1}$

11. The following identity is known as Fermat's combinatorial identity:

$$\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1} \qquad n \geq k$$

Give a combinatorial argument (no computations are needed) to establish this identity.
Hint: Consider the set of numbers 1 through n. How many subsets of size k have i as their highest-numbered member?

$1 \cdots n$

size 1 have $1 \cdots n$ as highest no. n

size 2 have 1 as 0

1, 2   2 as 1 $\sum_{i=1}^{n} \binom{i-1}{2-1}$

1, 3   3 as 2

2, 3

12. Consider the following combinatorial identity:

$$\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$$

(a) Present a combinatorial argument for this identity by considering a set of n people and determining, in two ways, the number of possible selections of a committee of any size and a chairperson for the committee.

Hint:

(i) How many possible selections are there of a committee of size k and its chairperson?

(ii) How many possible selections are there of a chairperson and the other committee members?

(b) Verify the following identity for $n = 1, 2, 3, 4, 5$:

$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2}n(n + 1)$$

a). i). $k\binom{n}{k}$ ii) $n \cdot \binom{n-1}{k-1}$

i). $\sum_{k=1}^{n} k\binom{n}{k}$ ii) $n \sum_{k=1}^{n} \binom{n-1}{k-1}$

$$= n \cdot \left(\binom{n-1}{1-1} + \binom{n-1}{2-1} + \cdots + \binom{n-1}{n-1} \right)$$

$$= n\left(\binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1} \right)$$

$$= n(2^{n-1}).$$

For a combinatorial proof of the preceding, consider a set of n people and argue that both sides of the identity represent the number of different selections of a committee, its chairperson, and its secretary (possibly the same as the chairperson).

Hint:

(i) How many different selections result in the committee containing exactly k people?

(ii) How many different selections are there in which the chairperson and the secretary are the same? (ANSWER: $n2^{n-1}$.)

(iii) How many different selections result in the chairperson and the secretary being different?

(c) Now argue that

$$\sum_{k=1}^{n} \binom{n}{k} k^3 = 2^{n-3} n^2 (n + 3)$$

$$\sum_{k=1}^{n} \binom{n}{k} k^2 = 2^{n-2} n (n+1)$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

b) (iii). C.S, different

$$2 \sum_{k=1}^{n} \binom{n}{k} \binom{k}{2} = \sum_{k=1}^{n} \binom{n}{k} k(k-1)$$

$$= \sum_{k=1}^{n} \binom{n}{k} k^2 - \sum_{k=1}^{n} \binom{n}{k} k + \sum_{k=1}^{n} \binom{n}{k} k$$

$$= \binom{n}{1} 1^2 + \binom{n}{2} 2^2 + \binom{n}{3} 3^2 + \cdots + \binom{n}{n} n^2$$

$$= \frac{n!}{(n-1)!\,1!} \cdot 1^2 + \frac{n!}{(n-2)!\,2!} 2^2 + \cdots + \frac{n!}{0!\,n!} n^2$$

$$= n + (n)(n-1)2 + n(n-1)(n-2)3 + \cdots + n^2$$

$$= n(1 + 2(n-1) + 3(n-1)(n-2) + \cdots + n)$$

13. Show that, for $n > 0$,

$$\sum_{i=0}^{n}(-1)^i \binom{n}{i} = 0$$

Hint: Use the binomial theorem.

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k} x^k y^{n-k}$$

$=$

$x = -1$,

$y = 1$.

$$(-1+1)^n = 0.$$

14. From a set of n people, a committee of size j is to be chosen, and from this committee, a subcommittee of size $i, i \leq j$, is also to be chosen.

(a) Derive a combinatorial identity by computing, in two ways, the number of possible choices of the committee and subcommittee—first by supposing that the committee is chosen first and then the subcommittee is chosen, and second by supposing that the subcommittee is chosen first and then the remaining members of the committee are chosen.

(b) Use part (a) to prove the following combinatorial identity:

$$\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \binom{n}{i} 2^{n-i} \quad i \leq n$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

(c) Use part (a) and Theoretical Exercise 13 to show that

$$\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} (-1)^{n-j} = 0 \quad i < n$$

1

a). $\binom{n}{j}\binom{j}{i} = \binom{n}{i}\binom{n-i}{j-i}$

b). $\sum_{j=i}^{n} \binom{n-i}{j-i} = \binom{n-i}{i-i} + \binom{n-i}{i+1-i} + \binom{n-i}{i+2-i} + \binom{n-i}{n-1-i} + \binom{n-i}{n-i}$

$= 1 + n-i + \binom{n-i}{2} + \dots + \binom{n-i}{n-i}$

$= 2^{n-i}$

c).

a).

$$\binom{n}{j}\binom{j}{i} = \binom{n}{i}\binom{n-i}{j-i}$$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$$

$$\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \binom{n}{i} 2^{n-i}$$

$$\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} (-1)^{n-j} = 0$$

$$\sum_{j=i}^{n} \binom{n}{i}\binom{n-i}{j-i} (-1)^{n-j}$$

$$\sum_{j-i=0}^{n} \binom{n}{i}\binom{n-i}{j-i} (-1)^{n-j}$$

$$\sum_{j-i=0}^{n}$$

15. Let $H_k(n)$ be the number of vectors $x_1, \dots, x_k$ for which each x_i is a positive integer satisfying $1 \le x_i \le n$ and $x_1 \le x_2 \le \cdots \le x_k$.

(a) Without any computations, argue that

$$H_1(n) = n$$

$$H_k(n) = \sum_{j=1}^{n} H_{k-1}(j) \quad k > 1$$

Hint: How many vectors are there in which $x_k = j$?

(b) Use the preceding recursion to compute $H_3(5)$.
Hint: First compute $H_2(n)$ for $n = 1, 2, 3, 4, 5$.

15a). $H_1(n) = n$ integer, only x_1.

$H_2(n) = \binom{n}{2}$ $\qquad x_1 \le x_2$.

$= \sum_{j=1}^{n} H_1(j)$

$= H_1(1) + H_1(2) + \cdots + H_1(n)$

$= 1 + 2 + \cdots + n$.

$= \frac{n(n+1)}{2}$

16. Consider a tournament of n contestants in which the outcome is an ordering of these contestants, with ties allowed. That is, the outcome partitions the players into groups, with the first group consisting of the players that tied for first place, the next group being those that tied for the next-best position, and so on. Let $N(n)$ denote the number of different possible outcomes. For instance, $N(2) = 3$, since, in a tournament with 2 contestants, player 1 could be uniquely first, player 2 could be uniquely first, or they could tie for first.

(a) List all the possible outcomes when $n = 3$.

(b) With $N(0)$ defined to equal 1, argue, without any computations, that

$$N(n) = \sum_{i=1}^{n} \binom{n}{i} N(n-i)$$

Hint: How many outcomes are there in which i players tie for last place?

(c) Show that the formula of part (b) is equivalent to the following:

$$N(n) = \sum_{i=0}^{n-1} \binom{n}{i} N(i)$$

(d) Use the recursion to find $N(3)$ and $N(4)$.

$N(3) = \sum_{i=1}^{3} \binom{3}{i} N(3-i)$

$\binom{3}{1} N(2) + \binom{3}{2} N(1) + \binom{3}{0} N(0)$

$3(3) \quad + \quad 3 \quad + 1$

17. Present a combinatorial explanation of why $\binom{n}{r} = \binom{n}{r, n-r}$.

n choose r = n choose r and

n-r choose n-r

18. Argue that

$$\binom{n}{n_1, n_2, \ldots, n_r} = \binom{n-1}{n_1 - 1, n_2, \ldots, n_r} + \binom{n-1}{n_1, n_2 - 1, \ldots, n_r} + \cdots + \binom{n-1}{n_1, n_2, \ldots, n_r - 1}$$

Hint: Use an argument similar to the one used to establish Equation (4.1).

19. Prove the multinomial theorem.

***20.** In how many ways can n identical balls be distributed into r urns so that the ith urn contains at least m_i balls, for each $i = 1, \ldots, r$? Assume that $n \geq \sum_{i=1}^{r} m_i$.

18.

$$\frac{(n-1)!}{(n_1-1)!\,n_2!\cdots n_r!} + \frac{(n-1)!}{n_1!\,(n_2-1)!\,n_3!\cdots} + \cdots$$

19.

$$\left(\begin{array}{c} n - \sum_{i=1}^{r} m_i \\ 1, 2, 3, \ldots, r \end{array} \right) \quad x_i \geq 0.$$

$r_i \geq m_i$

$$\binom{n - \sum_{i=1}^{r} m_i + r - 1}{r - 1}$$

*21. Argue that there are exactly $\binom{r}{k}\binom{n-1}{n-r+k}$ solutions of

$$x_1 + x_2 + \cdots + x_r = n$$

for which exactly k of the x_i are equal to 0.

*22. Consider a function $f(x_1, \ldots, x_n)$ of n variables. How many different partial derivatives of order r does f possess? X

*23. Determine the number of vectors $(x_1, \ldots, x_n)$ such that each x_i is a nonnegative integer and

$$\sum_{i=1}^{n} x_i \le k$$

21. If k of $x_i = 0$, $r-k$ terms of x_i should equal n.

others term >0. $\binom{n-1}{r-k-1} = \binom{n-1}{n-1-(r-k-1)}$

$= \binom{n-1}{n-1-r+k+1}$

$= \binom{n-1}{n-r+k}$

23: $x_i \geq 0$, $\sum_{i=1}^{n} x_i \le k$

$\sum_{i=1}^{n} x_1 = k$. $\Rightarrow \binom{k+n-1}{n-1}$

$k-1 \Rightarrow \binom{k-1+n-1}{n-1}$

$\vdots$

$\binom{0+n-1}{n-1}$

$\sum_{i=0}^{k} \binom{i+n-1}{\cancel{n-1}}$

$\sum_{i=0}^{k} \binom{i+n-1}{i}$

1. How many different linear arrangements are there of the letters A, B, C, D, E, F for which
(a) A and B are next to each other?
(b) A is before B?
(c) A is before B and B is before C?
(d) A is before B and C is before D?
(e) A and B are next to each other and C and D are also next to each other?
(f) E is not last in line?

2. If 4 Americans, 3 French people, and 3 British people are to be seated in a row, how many seating arrangements are possible when people of the same nationality must sit next to each other?

3. A president, treasurer, and secretary, all different, are to be chosen from a club consisting of 10 people. How many different choices of officers are possible if
(a) there are no restrictions?
(b) *A* and *B* will not serve together?
(c) *C* and *D* will serve together or not at all?
(d) *E* must be an officer?
(e) *F* will serve only if he is president?

4. A student is to answer 7 out of 10 questions in an examination. How many choices has she? How many if she must answer at least 3 of the first 5 questions?

5. In how many ways can a man divide 7 gifts among his 3 children if the eldest is to receive 3 gifts and the others 2 each?

6. How many different 7-place license plates are possible when 3 of the entries are letters and 4 are digits? Assume that repetition of letters and numbers is allowed and that there is no restriction on where the letters or numbers can be placed.

7. Give a combinatorial explanation of the identity

$$\binom{n}{r} = \binom{n}{n-r}$$

8. Consider n-digit numbers where each digit is one of the 10 integers $0, 1, \ldots, 9$. How many such numbers are there for which
(a) no two consecutive digits are equal?
(b) 0 appears as a digit a total of i times, $i = 0, \ldots, n$?

9. Consider three classes, each consisting of n students. From this group of $3n$ students, a group of 3 students is to be chosen.
(a) How many choices are possible?
(b) How many choices are there in which all 3 students are in the same class?
(c) How many choices are there in which 2 of the 3 students are in the same class and the other student is in a different class?
(d) How many choices are there in which all 3 students are in different classes?
(e) Using the results of parts (a) through (d), write a combinatorial identity.

10. How many 5-digit numbers can be formed from the integers $1, 2, \ldots, 9$ if no digit can appear more than twice? (For instance, 41434 is not allowed.)

11. From 10 married couples, we want to select a group of 6 people that is not allowed to contain a married couple.
(a) How many choices are there?
(b) How many choices are there if the group must also consist of 3 men and 3 women?

1\. a) $2! \times 5!$

b). $\frac{6!}{2!}$

c). $\frac{6!}{3!}$

d). $\frac{6!}{2!2!}$

e). $2! \times 2! \times 4!$

f). $6! - 5!$

2\. $4!\,3!\,3!\,3!$

3\. a). $\binom{10}{3}3!$

b). $\binom{10}{3}3! - \binom{8}{1}3!$

c). $\binom{8}{1}3! + \binom{8}{3}3!$

d). $\binom{9}{2}3!$

e). $\binom{9}{2}2! + \binom{9}{3}3!$

4. A student is to answer 7 out of 10 questions in an examination. How many choices has she? How many if she must answer at least 3 of the first 5 questions?

5. In how many ways can a man divide 7 gifts among his 3 children if the eldest is to receive 3 gifts and the others 2 each?

6. How many different 7-place license plates are possible when 3 of the entries are letters and 4 are digits? Assume that repetition of letters and numbers is allowed and that there is no restriction on where the letters or numbers can be placed.

4 $\binom{10}{7}$, $\underbrace{\binom{5}{3}\binom{5}{4}}_{=3} + \underbrace{\binom{5}{4}\binom{5}{3}}_{=4} + \underbrace{\binom{5}{5}\binom{5}{2}}_{=5}$

5. $\binom{7}{3\ 2\ 2}$

6. $26^3\,10^4 \times 7! \times \binom{7}{3}$

7. Give a combinatorial explanation of the identity

$$\binom{n}{r} = \binom{n}{n-r}$$

8. Consider n-digit numbers where each digit is one of the 10 integers $0, 1, \ldots, 9$. How many such numbers are there for which

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11. From 10 married couples, we want to select a group of 6 people that is not allowed to contain a married couple.

(a) How many choices are there?

(b) How many choices are there if the group must also consist of 3 men and 3 women?

8a). $10 \times 9^{n-1}$

b). $i=0: \; 9^n$

$i=1: \; \binom{n}{1} 9^{n-1}$

$i=2: \; \binom{n}{2} 9^{n-2}$

$$\sum_{i=0}^{n} \binom{n}{i} 9^{n-i}$$

9a). $\binom{3n}{3}$

b). $\binom{n}{3} \times 3$

c). $\binom{n}{2} \cancel{\binom{2n}{1} \times 3}$

d). $\binom{n}{1}\binom{n}{1}\cancel{\binom{n}{1}}$

选一个class没有 选一个class有2

$$\binom{3}{1}\binom{2}{1}\binom{n}{2}\binom{n}{1}$$

$$\binom{3n}{3} = 3\binom{n}{3} + 3\binom{n}{2}\binom{2n}{1} + n^3$$

10. How many 5-digit numbers can be formed from the integers 1, 2, . . . , 9 if no digit can appear more than twice? (For instance, 41434 is not allowed.)

$>2.$

11. From 10 married couples, we want to select a group of 6 people that is not allowed to contain a married couple.

(a) How many choices are there?

(b) How many choices are there if the group must also consist of 3 men and 3 women?

11223

10.

221

2111

11111

$$\binom{9}{3}\binom{3}{1}\frac{5!}{2!\,2!} +$$

$$\binom{9}{4}\binom{4}{1}\frac{5!}{2!} +$$

$$\binom{9}{5}\,5!$$

52920

11a). $A_1B_1\ A_2B_2\ A_3B_3 \ldots\ A_{10}B_{10}.$

6人中不能有 $A_1B_1 / A_2B_2 / \ldots A_{10}B_{10}.$

cannot

18

$20\times19\times18\times17\times16\cancel{\times15}$

13440

b). $10\times9\times8\times7\cancel{\times6\times5}$

10 9

12. A committee of 6 people is to be chosen from a group consisting of 7 men and 8 women. If the committee must consist of at least 3 women and at least 2 men, how many different committees are possible?

***13.** An art collection on auction consisted of 4 Dalis, 5 van Goghs, and 6 Picassos. At the auction were 5 art collectors. If a reporter noted only the number of Dalis, van Goghs, and Picassos acquired by each collector, how many different results could have been recorded if all of the works were sold?

$\binom{8}{4}\binom{9}{4}\binom{10}{4}$

12. $\geq 3W$ $\geq 2M$

3W3M $\binom{7}{3}\binom{8}{3}$ +

4W2M $\binom{8}{4}\binom{7}{2}$

13. 4D 5V 6P

$\boxed{2+6+9+6=\boxed{23}}$

5P

5V

$\boxed{\begin{matrix}4D\,1V\\4D\,1P\end{matrix}}\times 3$

4V1D

4V1P

4P1D

4P1V

$\boxed{\begin{matrix}3D\,2V\\3D\,2P\\3D\,1V\,1P\end{matrix}}\times 3$

3V

$\boxed{2D\,2V\,1P}\times 3\times 2$

***14.** Determine the number of vectors $(x_1, \ldots, x_n)$ such that each x_i is a positive integer and

$$\sum_{i=1}^{n} x_i \leq k$$

where $k \geq n$.

15. A total of n students are enrolled in a review course for the actuarial examination in probability. The posted results of the examination will list the names of those who passed, in decreasing order of their scores. For instance, the posted result will be "Brown, Cho" if Brown and Cho are the only ones to pass, with Brown receiving the higher score. Assuming that all scores are distinct (no ties), how many posted results are possible?

16. How many subsets of size 4 of the set $S = \{1, 2, \ldots, 20\}$ contain at least one of the elements $1, 2, 3, 4, 5$?

17. Give an analytic verification of

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}, \quad 1 \leq k \leq n$$

Now, give a combinatorial argument for this identity.

18. In a certain community, there are 3 families consisting of a single parent and 1 child, 3 families consisting of a single parent and 2 children, 5 families consisting of 2 parents and a single child, 7 families consisting of 2 parents and 2 children, and 6 families consisting of 2 parents and 3 children. If a parent and child from the same family are to be chosen, how many possible choices are there?

19. If there are no restrictions on where the digits and letters are placed, how many 8-place license plates consisting of 5 letters and 3 digits are possible if no repetitions of letters or digits are allowed. What if the 3 digits must be consecutive?

14.

$1+1+\cdots+1 = n$

$1+1+\cdots+2 = n+1$

$1+3+\cdots+? = k$

$-)\ 1+1+\cdots+1 = n$

$x_1 + x_2 + \cdots + x_n = k-n$ $\quad \binom{k}{n}$

$x_i \geq 0 \qquad \binom{k-n+n-1}{n-1} = \binom{k-1}{n-1}$

15. n 人而有 K个 pass.

$\sum_{k=0}^{n} \binom{n}{k} k!$ ✗

16. all − no contains.

$\binom{20}{4} - \binom{15}{4}$ ✓

17. Give an analytic verification of

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}, \quad 1 \le k \le n$$

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19. If there are no restrictions on where the digits and letters are placed, how many 8-place license plates consisting of 5 letters and 3 digits are possible if no repetitions of letters or digits are allowed. What if the 3 digits must be consecutive?

17. n抽2 = k抽2 ~~+~~

n-k抽2 +

k抽1，n-k抽1

18. $3 + 3\times\binom{1}{1}\binom{2}{1} + 5\times\binom{2}{1}\binom{1}{1} + 7\times\binom{2}{1}\binom{2}{1} +$

$6\times\binom{2}{1}\binom{3}{1}$

19. $\underbrace{26\times25\times24\times23\times22\times10\times9}\times8\times\cancel{8!}\ \binom{8}{3}$

$\times 3!\ \cancel{\times 6!}$

$6!$