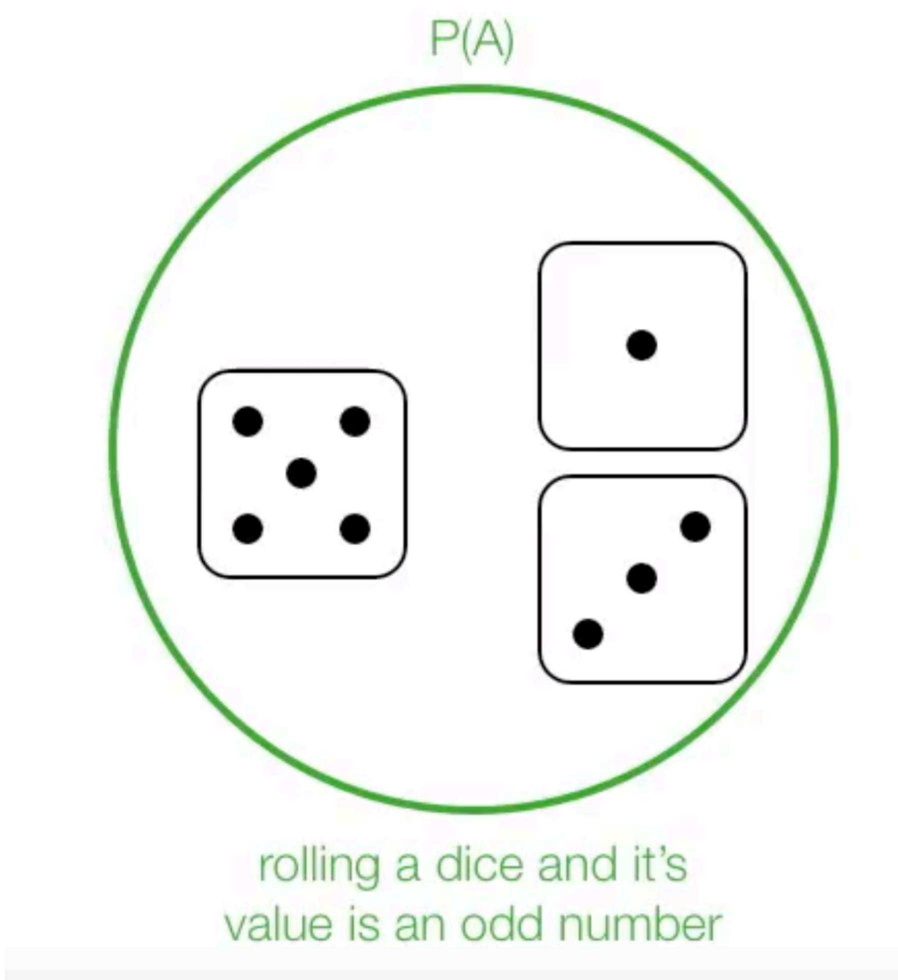
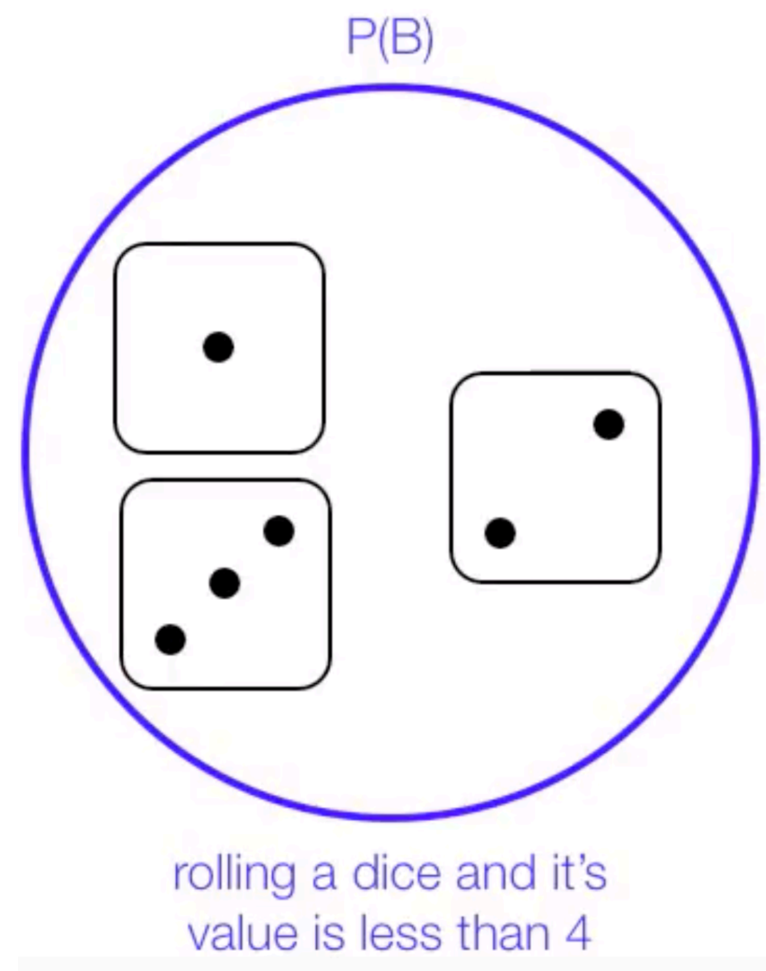
Chapter 3 Conditional Probability and Independence

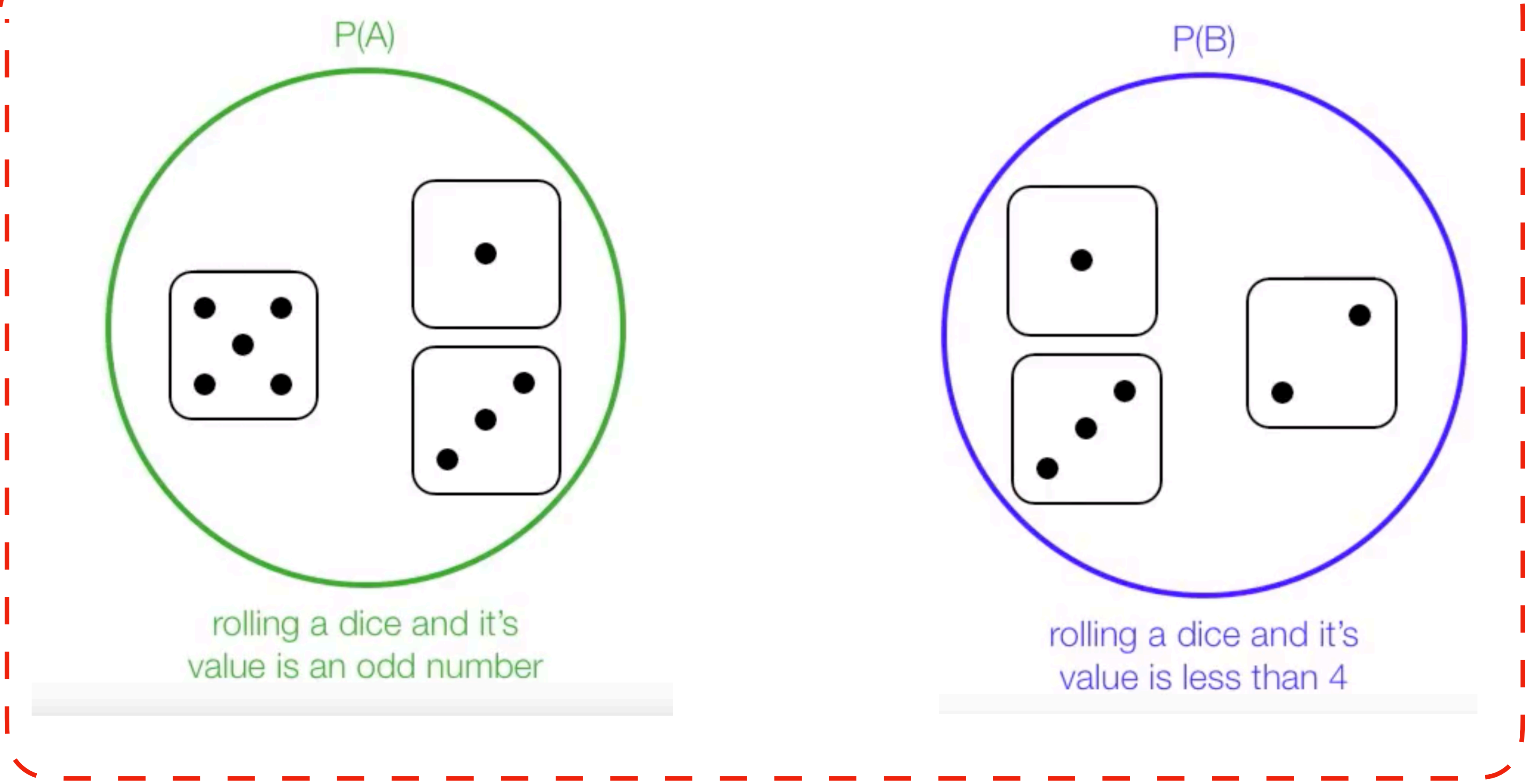




3.1 Introduction

In many problems, we are interested in event B. However, we have some partial information, namely, that an event A has occurred. How to make use of this information?

Example



3.1 Introduction

In calculating $P(B)$, there are occasions that we can consider it under different cases. How do we compute the probability of B under these cases? How do we combine them to give $P(B)$?

Example

Randomly choose a dice and roll it

event $B = \{4 \text{ is observed}\}$

Case 1: 6-faced die is chosen

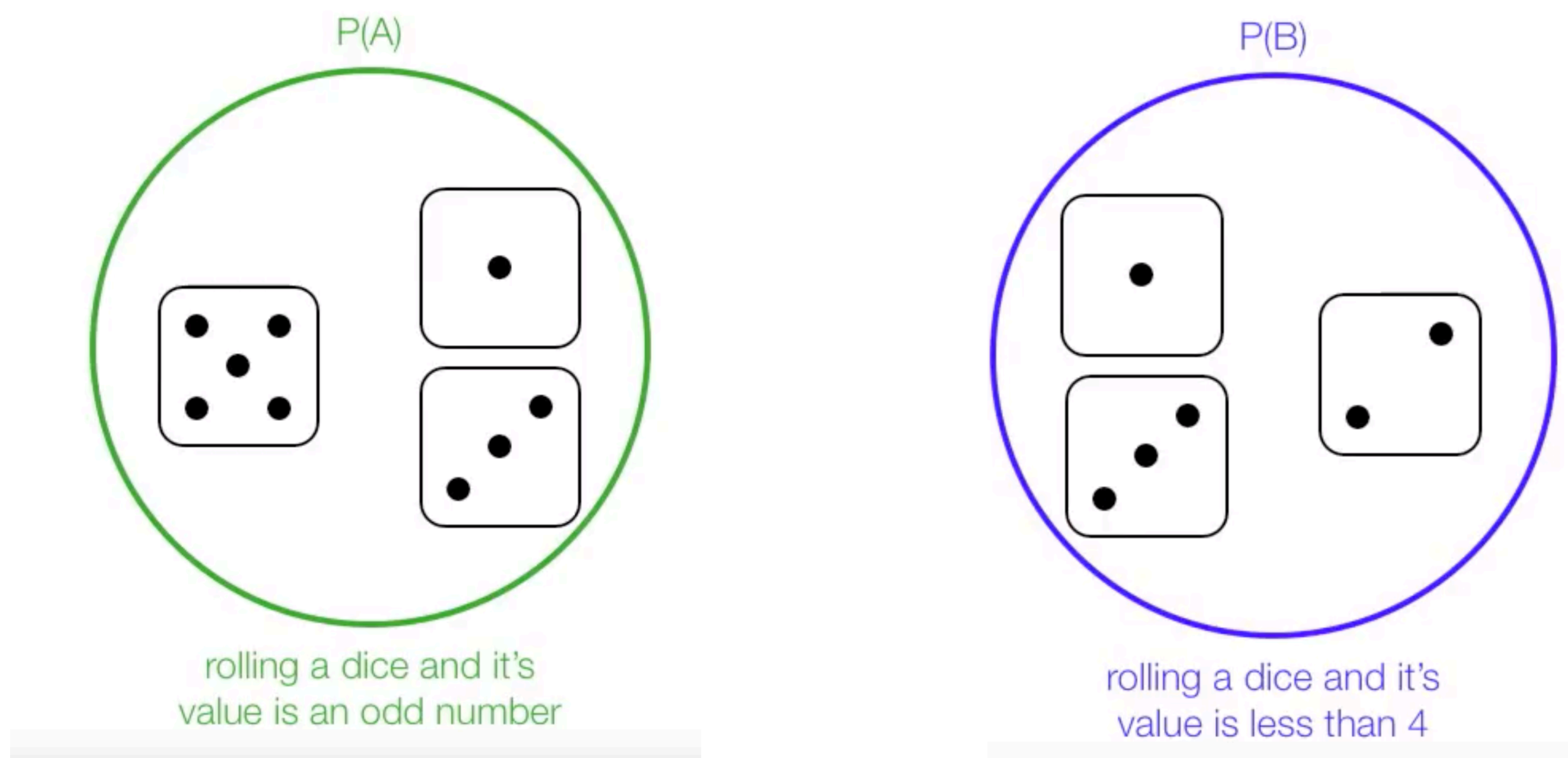
Case 2: 4-faced die is chosen

Case 3: 10-faced die is chosen

Case 4: 12-faced die is chosen

3.2 Conditional Probability

Example



The probability of B when we know that event A occurs?

Solution

If event A occurs, then there are 3 possible outcomes {1, 3, 5} and each outcome occurs with equal probability.
Conditioned on "A occurs", the sample space now becomes {1,3,5}.
Event B includes 1 and 3. Therefore, probability of B conditioned on A is 2/3.

3.2 Conditional Probability

Example Suppose that 2 dice are tossed, and all 36 outcomes are equally likely to occur. Suppose further that we observe that the first die is a 3. Then, given this information, what is the probability that the sum of the 2 dice equals 8?

Solution We reason as follows: Given that the initial die is a 3, it follows that there can be at most 6 possible outcomes of our experiment, namely, $(3,1)$, $(3,2)$, $(3,3)$, $(3,4)$, $(3,5)$, and $(3,6)$. Since each of these outcomes originally had the same probability of occurring, the outcomes should still have equal probabilities. That is, given that the first die is a 3, the (conditional) probability of each of the outcomes $(3,1)$, $(3,2)$, $(3,3)$, $(3,4)$, $(3,5)$, and $(3,6)$ is $\frac{1}{6}$, whereas the (conditional) probability of the other 30 points in the sample space is 0. Hence, the desired probability is $\frac{1}{6}$. $\square$

3.2 Conditional Probability

Definition

Let E be the event that the sum of the dice is 8, and let F be the event that the first die is 3. Then the probability just obtained is called *conditional probability that E occurs given that F has occurred.* It is usually denoted by

$$P(E|F).$$

Generally, we can talk about $P(E|F)$ for all events E and F. It can be derived in the same manner: If F occurs, then in order for E to occur as well, it is necessary that the outcome is in both EF. Now, as we know, F has occurred, it follows that F becomes our new or reduced sample space since anything in F^c will not occur. Hence the probability of EF given F shall equal the probability of EF *relative to* the probability of F.

We sometimes write $E \cap F$ as EF

3.2 Conditional Probability

Theorem (Conditional probability). *If* $P(F) > 0$, *then*

$$P(E|F) = \frac{P(EF)}{P(F)}.$$

Remark Especially, when the sample space S has equally likely outcomes, $P(E|F)$ is just the proportion of EF divided by the proportion of F.

$$P(E|F) = \frac{\text{number of outcomes in } E \cap F}{\text{number of outcomes in } F}$$

3.2 Conditional Probability

Example A student is taking a one-hour-time exam. Suppose the probability that the student will finish the exam in less than x hours is $\frac{x}{2}$, for all $0 \leq x \leq 1$. Given that the student is still working after 0.75 hours, what is the conditional probability that the full hour is used?

Solution Let L_x be the event that the student finishes the exam in less than x hours, $0 \leq x \leq 1$, and let F be the event that the student does not finish in less than 1 hour,

$$P(F) = P(L_1^c) = 1 - P(L_1) = 0.5.$$

Also note that the event that the student is still working at time 0.75 is the complement of the event $L_{0.75}$, the desired probability is

$$P(F|L_{0.75}^c) = \frac{P(FL_{0.75}^c)}{P(L_{0.75}^c)} = \frac{P(F)}{1 - P(L_{0.75})} = \frac{0.5}{0.625} = 0.8$$

3.2 Conditional Probability

Example A coin is flipped twice. Assume that all 4 possible outcomes in the sample space $S = \{(H,H),(H,T),(T,H),(T,T)\}$ are equally likely. What is the conditional probability both flips give heads, given that (a) the first flip give heads; (b) at least one flip gives heads?

Solution Let $B = \{(H,H)\}$ be the event that both flips give heads; let $F = \{(H,H),(H,T)\}$ be the event that the first flip gives the heads; and let $A = \{(H,H),(H,T),(T,H)\}$ be the event that at least one flip gives the heads. The probability for (a) can be obtained from

$$P(B|F) = \frac{P(BF)}{P(F)} = \frac{P(B)}{P(F)} = \frac{1/4}{2/4} = \frac{1}{2}.$$

For (b), we have

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(B)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

3.2 Conditional Probability

Example A bin contains 25 light bulbs, of which 5 are good and function at least 30 days, 10 are partially defective and will fail in the second day of use, while the rest are totally defective and won't light up at all. Given that a randomly chosen bulb initially lights up, what is the probability that it will still be working after one week?

Solution

Let G be event that the randomly chosen bulb is in good condition, T the event that the randomly chosen bulb is totally defective.

Given that the chosen bulb lights up initially, that is, given that T^c occurs. So the conditional probability required is

$$P(G|T^c) = \frac{P(GT^c)}{P(T^c)} = \frac{5/25}{15/25} = 1/3.$$

3.2 Conditional Probability

(Multiplication Rule)
Suppose that $P(E) > 0$, then

$$P(EF) = P(E|F)P(F).$$

$E \cap F$

In words, the above equation states that the probability that both E and F occur is equal to the probability that F occurs multiplied by the conditional probability of E given that F occurred. This equation is very useful in computing the probability of the intersection of events.

3.2 Conditional Probability

Example Celine is undecided as to whether to take a French course or a chemistry course. She estimates that her probability of receiving an $\boldsymbol{A}$ grade would be $\frac{1}{2}$ in a French course, and $\frac{2}{3}$ in a chemistry course. If Celine decides to base her decision on the flip of a fair coin, what is the probability that she gets an $\boldsymbol{A}$ in chemistry.

Solution Let C be the event that Celine takes chemistry and A denote the event that she receives an $\boldsymbol{A}$ in whatever course she takes, then the desired probability is $P(CA)$. This can be computed by

$$P(CA) = P(C)P(A|C) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}.$$

3.2 Conditional Probability

Example Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls, one at a time, from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

meaning that we do not put the selected ball back to the urn

Solution

3.2 Conditional Probability

Example Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls, one at a time, from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

Solution

Let R_1 and R_2 denote respectively the events that the first and second ball drawn are red. We want to compute $P(R_1R_2)$.
As $P(R_1) = 8/12$ and $P(R_2|R_1) = 7/11$,

$$P(R_1R_2) = P(R_1)P(R_2|R_1) = 8/12 \cdot 7/11 = 14/33.$$

3.2 Conditional Probability

(General Multiplication Rule)
Let $A_1, A_2, \ldots, A_n$ be n events, then

$A_1 \cap A_2 \cap \cdots \cap A_n$

$$\begin{aligned} & P(A_1 A_2 \cdots A_n) \\ = \; & P(A_1) P(A_2 | A_1) P(A_3 | A_1 A_2) \cdots P(A_n | A_1 A_2 \cdots A_{n-1}). \end{aligned}$$

Proof. (Sketch)

$$RHS = P(A_1) \frac{P(A_1 A_2)}{P(A_1)} \frac{P(A_1 A_2 A_3)}{P(A_1 A_2)} \cdots \frac{P(A_1 \cdots A_n)}{P(A_1 \cdots A_{n-1})}$$

3.2 Conditional Probability

Example

Three cards are selected successively at random and removed without replacement from a standard deck of 52 playing cards. Calculate the probability of receiving, in order, a king, a queen, a jacket.

Solution

Define events A={the first selected card is King}, B={the second selected card is queen}, C={the third selected card is jacket}.

We need to calculate $P(A \cap B \cap C)$.

$$P(A \cap B \cap C) = P(A)P(B|A)P\big(C|(A \cap B)\big)$$

$$= \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} = 0.000483$$

3.2 Conditional Probability

Example

A box of fuses contains 20 fuses, of which 5 are defective. If three of the fuses are selected randomly and removed from the box in succession without replacement, calculate the probability that all three fuses are defective.

Solution

3.3 Bayes' Theorem

Total Probability

Theorem

Let A and B be any two events, then

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c).$$

Proof.

$$\begin{aligned} P(B) &= P(B \cap (A \cup A^c)) \\ &= P(BA \cup BA^c) \\ &= P(BA) + P(BA^c) \\ &= P(B|A)P(A) + P(B|A^c)P(A^c), \end{aligned}$$

3.3 Bayes' Theorem

Total Probability

Example In answering a question on a multiple-choice test, a student either knows the answer or guesses the answer at random. Let p be the probability that the student knows the answer, and $1-p$ the probability that he doesn't. Suppose there are m alternatives in the question.

(a) What is the probability that he answered it correctly?

(b) What is the probability that the student knew the answer given that he answered it correctly?

Solution Let C denote the event that he answered it correctly, K that he knew the answer. So $P(K) = p$, and $P(K^c) = 1 - p$. Also, $P(C|K) = 1$, and $P(C|K^c) = 1/m$.

(a)

$$\begin{aligned} P(C) &= P(C|K)P(K) + P(C|K^c)P(K^c) \\ &= 1 \cdot p + 1/m \cdot (1-p). \end{aligned}$$

(b) We are asked $P(K|C)$. From $P(K|C) = P(KC)/P(C)$, we need to compute $P(CK)$.

However, $P(CK) = P(C|K)P(K) = p$, so

$$P(K|C) = \frac{P(C|K)P(K)}{P(C)} = \frac{p}{p + (1-p)/m} = \frac{mp}{1 + (m-1)p}.$$

3.3 Bayes' Theorem

Total Probability

Example An insurance company believes that people can be divided into two classes: those who are accident prone and those who are not. Their statistics show that an accident-prone person will have an accident at some time within a fixed 1-year period with probability 0.4, whereas the probability is only 0.2 for a non-accident-prone person. If we assume that 30 percent of the population is accident prone, what is the probability that a new policyholder will have an accident within a year of purchasing a policy.

Solution Let A_1 be the event that the policyholder will have an accident within a year of purchase; and let A be the event that the policyholder is accident prone. From the question, we know $P(A_1|A) = 0.4$ and $P(A_1|A^c) = 0.2$. We also know that $P(A) = 0.3$. Hence, the desired probability, $P(A_1)$, is given by

$$P(A_1) = P(A_1|A)P(A) + P(A_1|A^c)P(A^c) = 0.4 \times 0.3 + 0.2 \times 0.7 = 0.26$$

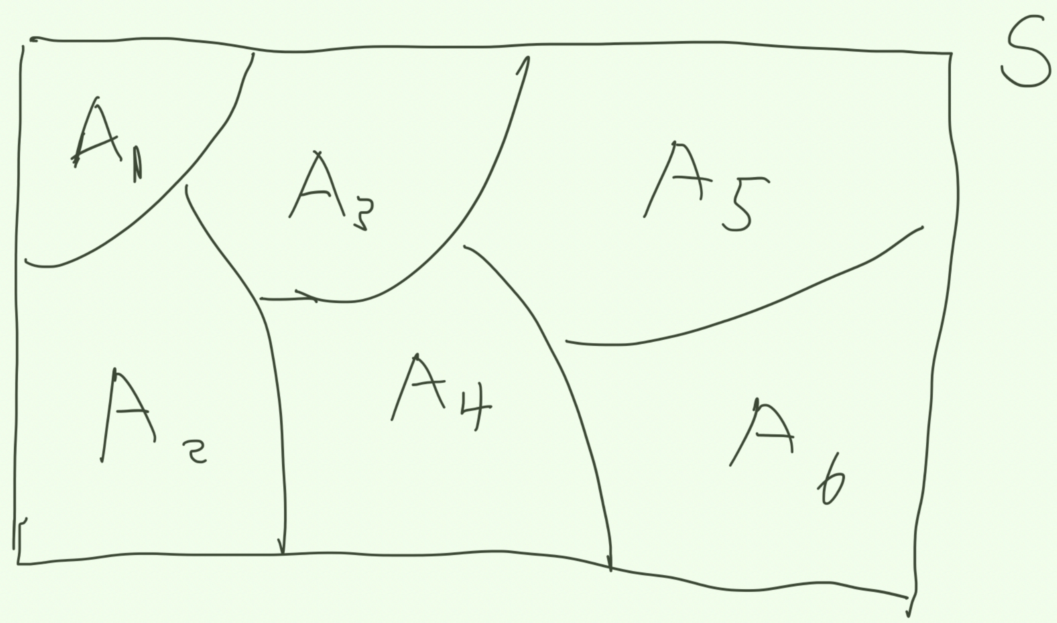
3.3 Bayes' Theorem

Total Probability

Definition *We say that $A_1, A_2, \ldots, A_n$ **partitions** the sample space S if:*

(a) *They are "mutually exclusive", meaning $A_i \cap A_j = \emptyset$, for all $i \neq j$.*

(b) *They are "* exhaustive *", meaning $\cup_{i=1}^{n} A_i = S$.*



3.3 Bayes' Theorem

Total Probability

Law of Total Probability *Suppose the events $A_1, A_2, \ldots, A_n$ partitions the sample space. Assume further that $P(A_i) > 0$ for $1 \le i \le n$. Let B be any event, then*

$$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n).$$

Proof Observe that $B = B \cap S = B \cap (\cup_{i=1}^n A_i) = \cup_{i=1}^n BA_i$
and that $BA_1, \ldots, BA_n$ are mutually exclusive.
Therefore,

$$\begin{aligned} P(B) &= P(\cup_{i=1}^n BA_i) \\ &= P(BA_1) + P(BA_2) + \cdots + P(BA_n) \\ &= P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n). \end{aligned}$$

3.3 Bayes' Theorem

Bayes' formula *Suppose the events $A_1, A_2, \ldots, A_n$ partitions the sample space. Assume further that $P(A_i) > 0$ for $1 \leq i \leq n$. Let B be any event, then for any $1 \leq i \leq n$,*

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \cdots + P(B|A_n)P(A_n)}.$$

Proof

$$P(A_i|B) = \frac{P(A_iB)}{P(B)} = \frac{P(B|A_i)P(A_i)}{P(B)}$$

and use the law of total probability for the denominator

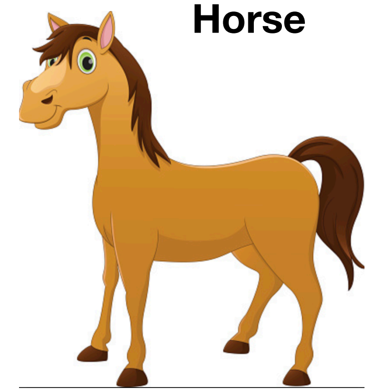
3.3 Bayes' Theorem

Mule

I 'saw' a mule in the village today.

I am pretty sure that my eyes fooled my brain. Because my brain told me that there are only two animals in my village: horse and donkey.

Horse



Donkey

So, it must be a horse or a donkey. How do I determine which one it is?

30%

70%

*I know there are 30% horses and 70% donkeys in the village. Therefore, I have a **'prior belief'** that I have 30% chance to encounter a horse and 70% to encounter a donkey.*

Suppose the a horse is mistaken as a mule with probability 0.4 and a donkey is mistaken as a mule with probability 0.8.

By Bayes' Theorem

'posterior belief'

P(horse | saw a mule)=0.176

P(donkey | saw a mule)=0.823

3.3 Bayes' Theorem

Example A blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested. If 0.5 percent of the population actually has the disease, what is the probability a person has the disease given that the test result is positive?

Solution Let D be the event that the tested person has the disease and E the event that the test result is positive. The desired probability $P(D|E)$ is obtained by

$$P(D|E) = \frac{P(DE)}{P(E)} = \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)} = \frac{95}{294} \approx 0.323$$

3.3 Bayes' Theorem

Example A plane is missing, and it is presumed that it was equally likely to have gone down in any of 3 possible regions. Let $1-\beta_i$ be the probability that the plane will be found upon a search of the ith region when the plane is indeed there, for $i = 1, 2, 3$. What is the conditional probability that the plane is in the ith region, given that a search of region 1 is unsuccessful, $i = 1, 2, 3$?

Solution Let $R_i, i = 1, 2, 3$, be the event that the plane is in region i; and let E be the event that a search of region 1 is unsuccessful. From Bayes' formula we obtain

$$P(R_1|E) = \frac{P(ER_1)}{P(E)} = \frac{P(E|R_1)P(R_1)}{\sum_{i=1}^{3} P(E|R_i)P(R_i)}$$
$$= \frac{\beta_1 \times \frac{1}{3}}{\beta_1 \times \frac{1}{3} + 1 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{\beta_1}{\beta_1 + 2}.$$

For $j = 2, 3$, we have

$$P(R_j|E) = \frac{P(E|R_j)P(R_j)}{P(E)} = \frac{1 \times \frac{1}{3}}{\beta_1 \times \frac{1}{3} + 1 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{1}{\beta_1 + 2}.$$

3.3 Bayes' Theorem

Example

In a certain assembly plant, three machines, I, II and III, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. **What is the probability that it is defective?**

Given the fact that the selected machine is defective, what is the probability that it is type I?

Solution

3.4 Independence

Definition *Two events A and B are said to be* ***independent*** *if*

$$P(AB) = P(A)P(B).$$

They are said to be **dependent** if

$$P(AB) \neq P(A)P(B).$$

Motivation: Suppose $P(B) > 0$. Then,

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A).$$

That is, A is independent of B if knowledge that B has occurred does NOT change the probability that A occurs.

3.4 Independence

Example A card is selected at random from an ordinary deck of 52 playing cards. If E is the event that the selected card is an ace and F is the event that it is spade ♠, then E and F are independent.

Proof. This follows from the definition of independence since $P(EF) = \frac{1}{52}$, whereas $P(E) = \frac{4}{52}$ and $P(F) = \frac{13}{52}$. $\square$

Example Two coins are flipped, and all 4 outcomes are assumed to be equally likely. If E is the event that the first coin lands heads and F the event that the second lands tails, then E and F are independent.

Proof This also follows from the definition of independence easily, since

$$P(EF) = P(\{H,T\}) = \frac{1}{4};$$

$$P(E) = P(\{(H,H),(H,T)\}) = \frac{1}{2};$$

$$P(F) = P(\{(H,T),(T,T)\}) = \frac{1}{2}.$$

3.4 Independence

Example Two coins are flipped, and all 4 outcomes are assumed to be equally likely. If E is the event that the first coin lands heads and F the event that only one coin lands head then E and F are independent ?

Proof

3.4 Independence

Example Suppose that we toss 2 fair dice. Let E_1 be the event that the sum of the dice is 6 and F denote the event that first die equals 4. Then are E_1 and F independent?

$$P(E_1F) = P(\{4,2\}) = \frac{1}{36},$$

whereas

$$P(E_1)P(F) = \frac{5}{36} \times \frac{1}{6} = \frac{5}{216}.$$

Hence, E_1 and F are not independent.

Remark the reason why E_1 and F are dependent in the above example can be explained intuitively in words. Note that since we are interested in the possibility to get 6 with 2 dice we shall be happy to get any of $1, 2, 3, 4, 5$ for the first die, for then we shall still have a chance to get a total of 6. However, if the first die gives 6 already, then there is no chance to get a total 6 with 2 dice. That means, the *chance* to get a total 6 indeed depends on the outcome of the first die.

3.4 Independence

Example Suppose that we toss 2 fair dice. Let E_1 be the event that the sum of the dice is 7 and F denote the event that first die equals 4. Then are E_1 and F independent?

Solution

3.4 Independence

Theorem *If A and B are independent, then so are*

(i) A and B^c; *(ii) A^c and B;* *(iii) A^c and B^c.*

Proof (i) Since

$$\begin{aligned} P(A) &= P(AB) + P(AB^c) \\ &= P(A)P(B) + P(AB^c) \quad \text{as } A, B \text{ are independent} \end{aligned}$$

hence

$$\begin{aligned} P(AB^c) &= P(A) - P(A)P(B) \\ &= P(A)[1 - P(B)] = P(A)P(B^c). \end{aligned}$$

(ii) Similarly for the parts (ii) and (iii).

3.4 Independence

Example Two fair dice are thrown. Let A be the event that the sum of the dice is 7; B the event that first die is 4; and C the event that the second die is 3.
$A = \{(1,6),(2,6),(3,4),(4,3),(5,2),(6,1)\}$.
$B = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}$.
$C = \{(1,3),(2,3),(3,3),(4,3),(5,3),(6,3)\}$.
$AB = \{(4,3)\}$
$AC = \{(4,3)\}$
$BC = \{(4,3)\}$ and
$ABC = \{(4,3)\}$
Then,

$$P(AB) = P(A)P(B)$$

and

$$P(AC) = P(A)P(C)$$

that is A and B are independent; also A and C are independent.
However,

$$P(ABC) = \frac{1}{36} \neq \frac{1}{216} = \frac{1}{6} \times \frac{1}{36} = P(A)P(BC).$$

3.4 Independence

Definition *Three events A, B and C are said to be independent if the following 4 conditions hold:*

$$
\begin{aligned}
P(ABC) &= P(A)P(B)P(C) \qquad (1)\\
P(AB) &= P(A)P(B) \qquad (2)\\
P(AC) &= P(A)P(C) \qquad (3)\\
P(BC) &= P(B)P(C) \qquad (4)
\end{aligned}
$$

Remark Second condition implies A and B are independent;
Third condition implies A and C are independent; and
Fourth condition implies B and C are independent
That is, A, B and C are pairwise independent.

3.4 Independence

Theorem It should be noted that if A, B and C are independent, then A is independent of any event formed from B and C.

(i) A is independent of $B \cup C$.

(ii) A is independent of $B \cap C$.

Proof

$$
\begin{aligned}
P(A(B \cup C)) &= P(AB \cup AC) \\
&= P(AB) + P(AC) - P(AB \cap AC) \\
&= P(A)P(B) + P(A)P(C) - P(ABC) \\
&= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) \\
&= P(A)[P(B) + P(C) - P(B)P(C)] \\
&= P(A)[P(B) + P(C) - P(BC)] \\
&= P(A)P(B \cup C),
\end{aligned}
$$

that is, A and $B \cup C$ are independent.

3.4 Independence

Definition *Events $A_1, A_2, \ldots, A_n$ are said to be independent if, for every sub-collection of events $A_{i_1}, \ldots, A_{i_r}$, we have*

$$P(A_{i_1} \cdots A_{i_r}) = P(A_{i_1}) \cdots P(A_{i_r}).$$

For $n = 4$, that is, 4 events A_1, A_2, A_3 and A_4 are independent if we can verify (i), (ii) and (iii) below:

(i) $r = 4$:

$$P(A_1A_2A_3A_4) = P(A_1)P(A_2)P(A_3)P(A_4)$$

(ii) $r = 3$: there are 4 conditions,

$$\begin{aligned} P(A_1A_2A_3) &= P(A_1)P(A_2)P(A_3) \\ P(A_1A_2A_4) &= P(A_1)P(A_2)P(A_4) \\ P(A_1A_3A_4) &= P(A_1)P(A_3)P(A_4) \\ P(A_2A_3A_4) &= P(A_2)P(A_3)P(A_4) \end{aligned}$$

(iii) $r = 2$: there are 6 conditions,

$$\begin{aligned} P(A_1A_2) &= P(A_1)P(A_2) \\ P(A_1A_3) &= P(A_1)P(A_3) \\ P(A_1A_4) &= P(A_1)P(A_4) \\ P(A_2A_3) &= P(A_2)P(A_3) \\ P(A_2A_4) &= P(A_2)P(A_4) \\ P(A_3A_4) &= P(A_3)P(A_4) \end{aligned}$$

3.4 Independence

Example We are given a loaded coin with probability of getting a head $= p$; with probability of getting a tail $= 1 - p$.

This loaded coin is tossed n times independently. What is the probability of getting

(i) at least 1 head in these n tosses?

(ii) exactly k heads in these n tosses?

Solution

Let A_i be the event that the ith toss results in a head.

(i) Probability required is

$$\begin{aligned} P(A_1 \cup \cdots \cup A_n) &= 1 - P((A_1 \cup \cdots \cup A_n)^c) = 1 - P(A_1^c A_2^c \cdots A_n^c) \\ &= 1 - P(A_1^c)P(A_2^c) \cdots P(A_n^c) = 1 - (1-p)^n. \end{aligned}$$

(ii) Probability of first k heads follows by $n - k$ tails is

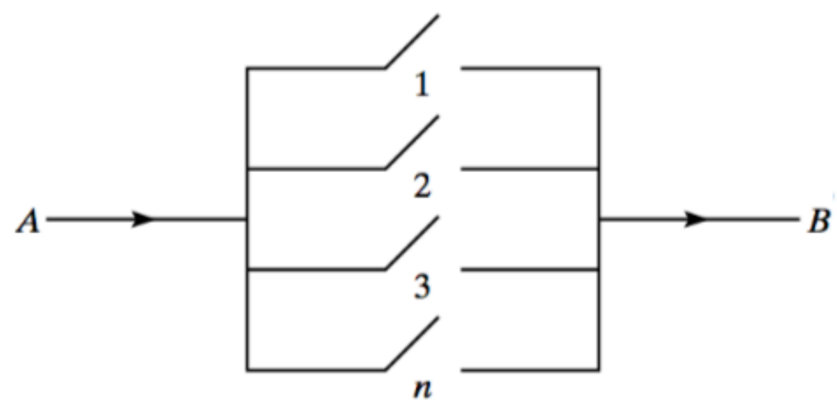
$$P(A_1 \cdots A_k A_{k+1}^c \cdots A_n^c) = P(A_1) \cdots P(A_k) P(A_{k+1}^c) \cdots P(A_n^c) = p^k (1-p)^{n-k}.$$

In order to have exactly k heads, there are $\frac{n!}{k!(n-k)!} = \binom{n}{k}$ ways of having k heads and $n - k$ tails.

Hence the probability required is $\binom{n}{k} p^k (1-p)^{n-k}$.

3.4 Independence

Example A system composed of n separate components is said to be a parallel system if it functions when at least one of the components functions, see the figure below For



such a system, if component i, independent of other components, functions with probability $p_i, i = 1, \cdots, n$, what is the probability that the system functions?

Solution

Let A_i be the event that component i functions. Then

$$\begin{aligned} P(\text{system functions}) &= 1 - P(\text{system does not function}) \\ &= 1 - P(\text{all components do not function}) \quad = 1 - P\Big(\bigcap_{i=1}^{n} A_i^c\Big) \\ &= 1 - \prod_{i=1}^{n}(1-p_i). \end{aligned}$$