Example 1 A communication system consists of n identical antennas lined up in a linear order. The system can recieve signal if no two consecutive attenas are defective. If a system of n antennas with m of them are defective. What is the probability that the system can recieve signals?
Ans. For a special case,n=4,m=2, we can enumerate all possible configurations.We let 0 denote a defective antennas, 1 denote a working antennas.

用1與0代表,分別有 0011, 0101, 0110, 1001, 1010, 1100

— = 有連續2个0

$\Rightarrow \frac{3}{6} = \frac{1}{2}$

$$\frac{\text{\# of configurations work}}{\text{\# of all possible configuration}}$$

Chapter 1

Combinatorial Analysis

1.1 Introduction

Many problems in probability theory can be solved simply by counting the number of different ways that a certain event can occur. Effective methods for counting would then be useful in our study of probability. The mathematical theory of counting is formally known as *combinatorial analysis*.

1.2 Basic Principle of Counting

The basic principle of counting is a simple but useful result for all our work.

(**The Basic Principle of Counting**)
Suppose that two experiments are to be performed. If

- experiment 1 can result in any one of m possible outcomes; and
- experiment 2 can result in any one of n possible outcomes;

then together there are mn possible outcomes of the two experiments.

Proof. Make a list:

$$\begin{array}{cccc}
(1,1) & (1,2) & \dots & (1, n) \\
(2,1) & (2,2) & \dots & (2, n) \\
\vdots & \vdots & \vdots & \vdots \\
(m,1) & (m,2) & \dots & (m, n)
\end{array}$$

□

Theorem(Basic Principle of Counting) suppose that 2 experiement are to be conducted. The 1st experiment has m possible outcomes. For every outcome of experiment 1,there are n possible outcomes for experiment 2, Then,there are mn possible outcomes for the 2 experiemtns.

Example 1.1. A small community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?

Solution:
By regarding the choice of the woman as the outcome of the first experiment and the subsequent choice of one of her children as the outcome of the second experiment, we see from the basic principle that there are $10 \times 3 = 30$ possible choices.

Example 1.2. In a class of 40 students, we choose a president and a vice president. There are

$$40 \times 39 = 1560$$

possible choices.

For more than two experiments, we have

(**The Generalized Basic Principle of Counting**)
Suppose that r experiments are to be performed. If

- experiment 1 results in n_1 possible outcomes;
- experiment 2 results in n_2 possible outcomes;
- $\cdots$
- experiment r results in n_r possible outcomes;

then together there are $n_1 n_2 \cdots n_r$ possible outcomes of the r experiments.

Example 1.3. How many different 7-place license plates are possible if the first 3 places for letters and the final 4 places by numbers?

Solution:
First 3 places each has 26 ways, and final 4 places each has 10 ways. Therefore, the total possible number of ways is

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 175,760,000.$$

Example 1.4. How many functions defined on a domain with n elements are possible if each functional value is either 0 or 1?

Solution:
Let's label the elements of the domain: $1, 2, \ldots, n$. Since $f(i)$ is either 0 or 1 for each $i = 1, 2, \ldots, n$, it follows that there are

$$2 \times 2 \times \cdots \times 2 = 2^n$$

possible functions.

Example 1.5. In Example 1.3, how many license plates would be possible if repetition among letters or numbers were prohibited?

Solution:
In this case, there would be $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$ possible license plates.

1.3 Permutations

How many different arrangements of the letters a, b and c are possible?

Solution:
Direct enumeration: 6 possible arrangements. They are

$$abc, acb, bac, bca, cab, \text{ and } cba.$$

Why 6?

$$6 = 3 \times 2 \times 1.$$

(**General Principle**)
Suppose there are n (distinct) objects, then the total number of different arrangements is

$$n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = n!$$

with the convention that

$$0! = 1.$$

Example 1.6. Seating arrangement in a row: 9 people sitting in a row. There are $9! = 362,880$ ways.

Example 1.7. 6 men and 4 women in a class. Students ranked according to test result (assume no students obtained the same score).

(a) How many different rankings are possible?

Solution:

$$10! = 3,628,800.$$

(b) Ranking men among themselves, ranking women among themselves.

Solution:
There will be

$$6! \times 4! = 720 \cdot 24 = 17280$$

different rankings.

We shall now determine the number of permutations of a set of n objects when some of them are indistinguishable from one another. Consider the following example.

Example 1.8. How many different letter arrangements can be formed using the letters M I S S?

如果问 distinct arrangement of MISS $= \frac{4!}{2!}$

Solution:
Trick: Make the S's distinct, call them S_1 and S_2.

$$\begin{array}{cc} MIS_1S_2 & MIS_2S_1 \\ MS_1IS_2 & MS_2IS_1 \\ MS_1S_2I & MS_2S_1I \\ \cdots & \cdots \end{array}$$

If they were all different, there would be 4! ways.
We double counted by 2! ways, so the actual number of different arrangements is

$$\frac{4!}{2!} = 12.$$

(**General Principle**)
For n objects of which n_1 are alike, n_2 are alike, $\ldots$, n_r are alike, there are

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

different permutations of the n objects.

Example 1.9. How many ways to rearrange Mississippi?

Solution:
11 letters of which 1 M, 4 I's, 4 S's and 2 P's.
There are

$$\frac{11!}{1!4!4!2!} = 34,650.$$

Example 1.10 (Seating in circle). 10 people sitting around a round dining table. It is the relative positions that really matters – who is on your left, on your right. No. of seating arrangements is

$$\frac{10!}{10} = 9!.$$

Generally, for n people sitting in a circle, there are

$$\frac{n!}{n} = (n-1)!$$

possible arrangements.

Example 1.11 (Making necklaces). n different pearls string in a necklace. Number of ways of stringing the pearls is

$$\frac{(n-1)!}{2}.$$

1.4 Combinations

In how many ways can we choose 3 items from 5 items: A, B, C, D and E?

Solution:
5 ways to choose first item,
4 ways to choose second item, and
3 ways to choose third item
So the number of ways (in this order) is $5 \cdot 4 \cdot 3$.
However,

$$ABC, ACB, BAC, BCA, CAB \text{ and } CBA$$

will be considered as the same group.
So the number of different groups (order not important) is

$$\frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1}.$$

Generally, if there are n distinct objects, of which we choose a group of r items,

$$
\begin{aligned}
&\text{Number of possible groups}\\
&= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\\
&= \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}\\
&\qquad \times \frac{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}\\
&= \frac{n!}{r!(n-r)!}.
\end{aligned}
$$

Remark 1. (i) **Notation:**
Number of ways of choosing r items from n items: ${}_nC_r$ or $\binom{n}{r}$.

(ii) For $r = 0, 1, \ldots, n$,

$$\binom{n}{r} = \binom{n}{n-r}.$$

(iii)

$$\binom{n}{0} = \binom{n}{n} = 1.$$

(iv) **Convention:**
When n is a nonnegative integer, and $r < 0$ or $r > n$, take

$$\binom{n}{r} = 0.$$

Example 1.12. A committee of 3 is to be formed from a group of 20 people.

1. How many possible committees can be formed?
2. Suppose further that, two guys: Peter and Paul refuse to serve in the same committee. How many possible committees can be formed with the restriction that these two guys don't serve together?

Solution:

1. No. of different committees that can be formed = $\binom{20}{3} = 1140$.

2. Two possible cases:

   Case 1. Both of them are not in the committee.
   Ways to do that $= \binom{18}{3} = 816$.

   Case 2. One of them in.
   Ways to form $= \binom{2}{1}\binom{18}{2} = 306$.

   Total $= 816 + 306 = 1122$.

   Alternative solution: (sketch)

$$\binom{20}{3} - \binom{2}{2}\binom{18}{1} = 1140 - 18 = 1122.$$

Example 1.13. Consider a set of n antennas of which m are defective and $n-m$ are functional and assume that all of the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

Solution:
Line up the $n-m$ functional antennas among themselves. In order that no two defectives are consecutive, we insert each defective antenna into one of the $n-m+1$ spaces between the functional antennas. Hence, there are $\binom{n-m+1}{m}$ possible orderings in which there is at least one functional antenna between any two defective ones.

Some Useful Combinatorial Identities

For $1 \le r \le n$,

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Proof.
Algebraic:

$$\begin{aligned} \text{RHS} &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} \\ &= \frac{(n-1)!}{r!(n-r)!}\left[r + (n-r)\right] = \frac{n!}{r!(n-r)!}. \end{aligned}$$

Combinatorial:
Consider the cases where the first object (i) is chosen, (ii) not chosen:

$$\binom{1}{1}\cdot\binom{n-1}{r-1}+\binom{1}{0}\cdot\binom{n-1}{r}.$$

$\square$

(**The Binomial Theorem**)
Let n be a nonnegative integer, then

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}.$$

In view of the binomial theorem, $\binom{n}{k}$ is often referred to as the binomial coefficient.

Example 1.14. How many subsets are there of a set consisting of n elements?

Solution:
Since there are $\binom{n}{k}$ subsets of size k, the desired answer is

$$\sum_{k=0}^{n}\binom{n}{k}=(1+1)^n=2^n.$$

This result could also have been obtained by considering whether each element in the set is being chosen or not (to be part of a subset). As there are 2^n possible assignments, the result follows.

Example 1.15.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0.$$

Solution:
Let $x=-1$, $y=1$ in the binomial theorem.

Example 1.16.

$$\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots=\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots.$$

Solution:
Start with the previous example. Move the negative terms to the right hand side of the equation.

1.5 Multinomial Coefficients

A set of n distinct items is to be divided into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. How many different divisions are possible?

Solution:
Note that there are $\binom{n}{n_1}$ possible choices for the first group; for each choice of the first group, there are $\binom{n-n_1}{n_2}$ possible choices for the second group; for each choice of the first two groups, there are $\binom{n-n_1-n_2}{n_3}$ possible choices for the third group; and so on. It then follows from the generalized version of the basic counting principle that there are

$$
\begin{aligned}
&\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \cdots \times \binom{n-n_1-n_2-\cdots-n_{r-1}}{n_r} \\
&= \frac{n!}{(n-n_1)!n_1!} \times \frac{(n-n_1)!}{(n-n_1-n_2)!n_2!} \times \cdots \times \frac{(n-n_1-n_2-\cdots-n_{r-1})!}{0!n_r!} \\
&= \frac{n!}{n_1!n_2!\cdots n_r!}
\end{aligned}
$$

possible divisions.
(**Notation**)
If $n_1 + n_2 + \cdots + n_r = n$, we define $\binom{n}{n_1, n_2, \ldots, n_r}$ by

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}.$$

Thus $\binom{n}{n_1, n_2, \ldots, n_r}$ represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$.

Example 1.17. A police department in a small city consists of 10 officers. If the department policy is to have 5 of the officers patrolling the streets, 2 of the officers working full time at the station, and 3 of the officers on reserve at the station, how many different divisions of the 10 officers into the 3 groups are possible?

Solution:
There are $\frac{10!}{5!2!3!} = 2520$ possible divisions.

Example 1.18. Ten children are to be divided into an A team and a B team of 5 each. The A team will play in one league and the B team in another. How many different divisions are possible?

Solution:
There are $\frac{10!}{5!5!} = 252$ possible divisions.

Example 1.19. In order to play a game of basketball, 10 children at a playground divide themselves into two teams of 5 each. How many different divisions are possible?

Solution:
Note that this example is different from the previous one because now the order of the two teams is irrelevant. That is, there is no A and B team, but just a division consisting of 2 groups of 5 each. Hence, the desired answer is

$$\frac{10!/(5!5!)}{2!} = 126.$$

(The Multinomial Theorem)

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{(n_1,\ldots,n_r):n_1+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

Note that the sum is over all nonnegative integer-valued vectors $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$.

Example 1.20.

$$\begin{aligned}(x_1 + x_2 + x_3)^2 &= \binom{2}{2,0,0} x_1^2 x_2^0 x_3^0 + \binom{2}{0,2,0} x_1^0 x_2^2 x_3^0 + \binom{2}{0,0,2} x_1^0 x_2^0 x_3^2 \\ &\quad + \binom{2}{1,1,0} x_1^1 x_2^1 x_3^0 + \binom{2}{1,0,1} x_1^1 x_2^0 x_3^1 + \binom{2}{0,1,1} x_1^0 x_2^1 x_3^1 \\ &= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3.\end{aligned}$$

Remark 2. The preceding result shows that there is a one-to-one correspondence between the set of the possible tournament results and the set of permutations of $1, \ldots, n$. For more details on how such a correspondence can be constructed, refer to [Ross].

1.6 The Number Of Integer Solutions Of Equations

Proposition 1.21. *There are* $\binom{n-1}{r-1}$ *distinct positive integer-valued vectors* $(x_1, x_2, \ldots, x_r)$ *that satisfies the equation*

$$x_1 + x_2 + \cdots + x_r = n,$$

where $x_i > 0$ *for* $i = 1, \ldots, r$.

Proof. Consider n distinct objects. We want to divide them into r nonempty groups. To do so, we can select $r-1$ of the $n-1$ spaces between the n objects as our dividing points.

$$\mathrm{X} \wedge \mathrm{X} \wedge \mathrm{X} \wedge \cdots \wedge \mathrm{X} \wedge \mathrm{X} \wedge \mathrm{X}$$

In the diagram there are $n-1$ spaces in between the n objects represented by X. We choose $r-1$ of them to divide the n objects into r nonempty groups.

So there are $\binom{n-1}{r-1}$ distinct positive integer-valued vectors $(x_1, x_2, \ldots, x_r)$ that satisfies the equation. $\square$

Proposition 1.22. *There are* $\binom{n+r-1}{r-1}$ *distinct non-negative integer-valued vectors* $(x_1, x_2, \ldots, x_r)$ *that satisfies the equation*

$$x_1 + x_2 + \cdots + x_r = n,$$

where $x_i \geq 0$ *for* $i = 1, \ldots, r$.

Proof. Let $y_i = x_i + 1$, then $y_i > 0$ and the number of non-negative solutions of

$$x_1 + x_2 + \cdots + x_r = n$$

is the same as the number of positive solutions of

$$(y_1 - 1) + (y_2 - 1) + \cdots + (y_r - 1) = n$$

i.e.,

$$y_1 + y_2 + \cdots + y_r = n + r,$$

which is $\binom{n+r-1}{r-1}$. $\square$

Example 1.23. How many distinct nonnegative integer-valued solutions of $x_1 + x_2 = 3$ are possible?

Solution:
There are $\binom{3+2-1}{2-1} = 4$ such solutions: $(0,3),(1,2),(2,1),(3,0)$.

Example 1.24. An investor has 20 thousand dollars to invest among 4 possible investments. Each investment must be in units of a thousand dollars. If the total 20 thousand is to be invested, how many different investment strategies are possible? What if not all the money need be invested?

Solution:
If we let $x_i, i = 1,2,3,4$, denote the number of thousands invested in investment i, then, when all is to be invested, x_1, x_2, x_3, x_4 are integers satisfying equation

$$x_1 + x_2 + x_3 + x_4 = 20, \quad x_i \geq 0$$

Hence, by Proposition 1.22, there are $\binom{23}{3} = 1771$ possible investment strategies. If not all of the money need be invested, then if we let x_5 denote the amount kept in reserve, a strategy is a nonnegative integer-valued vector x_1, x_2, x_3, x_4, x_5 satisfying the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

Hence, by Proposition 1.22, there are now $\binom{24}{4} = 10,626$ possible strategies.

Example 1.25. How many terms are there in the multinomial expansion of $(x_1 + x_2 + \cdots + x_r)^n$?

Solution:

$$(x_1 + x_2 + \cdots + x_r)^n = \sum \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

where the sum is over all nonnegative integer-valued $(n_1, \ldots, n_r)$ such that $n_1 + \cdots + n_r = n$. Hence, by Proposition 1.22, there are $\binom{n+r-1}{r-1}$ such terms.

Example 1.26. Let us consider again Example 1.13. Imagine that the defective items are lined up among themselves and the functional ones are now to be put in position. Let us denote x_1 as the number of functional items to the left of the first defective, x_2 as the number of functional items between the first two defectives, and so on. That is, schematically, we have

$$x_1 \quad 0 \quad x_2 \quad 0 \quad \cdots \quad 0 \quad x_m \quad 0 \quad x_{m+1}$$

Now, there will be at least one functional item between any pair of defectives as long as $x_i > 0$, $i = 2, \ldots, m$. Hence, the number of outcomes satisfying the condition is the number of vectors $x_1, \ldots, x_{m+1}$ that satisfy the equation

$$x_1 + \cdots + x_{m+1} = n - m, \quad x_1 \geq 0, \quad x_{m+1} \geq 0, \quad x_i > 0, \quad i = 2, \ldots, m.$$

Let $y_1 = x_1 + 1$, $y_i = x_i$ for $i = 2, \ldots, m$ and $y_{m+1} = x_{m+1} + 1$, we see that this number is equal to the number of positive vectors $(y_1, \ldots, y_{m+1})$ that satisfy the equation

$$y_1 + y_2 + \cdots + y_{m+1} = n - m + 2.$$

By Proposition 1.21, there are $\binom{n-m+1}{m}$ such outcomes.

Suppose now that we are interested in the number of outcomes in which each pair of defective items is separated by at least 2 functional items. By the same reasoning as that applied previously, this would equal the number of vectors satisfying the equation

$$x_1 + \cdots + x_{m+1} = n - m, \quad x_1 \geq 0, \quad x_{m+1} \geq 0, \quad x_i \geq 2, \quad i = 2, \ldots, m.$$

Let $y_1 = x_1 + 1$, $y_i = x_i - 1$ for $i = 2, \ldots, m$ and $y_{m+1} = x_{m+1} + 1$, we see that this is the same as the number of positive solutions of the equation

$$y_1 + y_2 + \cdots + y_{m+1} = n - 2m + 3.$$

By Proposition 1.21, there are $\binom{n-2m+2}{m}$ such outcomes.