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1.7). P(goes through (4,4) $\cap$ goes through (2,2))

= P(2 ups and 2 rights to (2,2)) $\times$

P(2 ups and 2 rights to (4,4))

$$= \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 \times \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2$$

$$= \frac{64}{729} \approx 0.0878$$

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1 iii). P(goes through point (4,4))

$= P($ 4 ups and 4 rights $)$

$= \binom{8}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^4 = 0.1707$ or $\frac{1120}{6561}$.

P(avoids point (2,2) | goes through point (4,4))

$= \dfrac{P(\text{avoids point } (2,2) \cap \text{goes through point } (4,4))}{\frac{1120}{6561}}$

$= \dfrac{\frac{1120}{6561} - \frac{64}{729}}{\frac{1120}{6561}} = 0.4857$ or $\frac{17}{35}$ //

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1 (iii). $P(\text{goes through } (66, 34))$

$= P(\text{66 rights and 34 ups})$

$= P(X = 66)$ where X is the number of stepping right.

$X \sim Bin(100, \frac{2}{3})$.

$\approx P(65.5 < X < 66.5)$

$= P\left(\dfrac{65.5 - \frac{200}{3}}{\sqrt{\frac{200}{9}}} < Z < \dfrac{66.5 - \frac{200}{3}}{\sqrt{\frac{200}{9}}}\right)$

$= P(-0.24748737 < Z < -0.03535339)$

$\approx P(-0.25 < Z < -0.04)$

$= \Phi(0.25) - \Phi(0.04)$

$= 0.5987 - 0.5160 = 0.0827$ //

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$$\int_5^{\infty} f_X(x)\, dx = 1$$

$$\int_5^{\infty} \frac{1}{x^2} dx = \frac{1}{c}$$

$$\left[-x^{-1}\right]_5^{\infty} = \frac{1}{c}$$

$$\frac{1}{5} = \frac{1}{c}$$

$$c = 5.$$

Q2 i). $P(X = 10) = \frac{5}{100} = 0.05$

ii). let a be 25% th quantile of X:

$$\int_5^{a} \frac{5}{x^2} dx = 0.25$$

$$\left[\cdot x^{-1}\right]_5^{a} = 0.05$$

$$-\frac{1}{a} + \frac{1}{5} = 0.05$$

$$-\frac{1}{a} = -0.15$$

$$\frac{1}{a} = 0.15$$

$$a = \frac{20}{3}$$

$\therefore$ 25% th quantile of X is $\frac{20}{3}$.

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2 iii).
$$E[x^2 e^{-x}] = \int_5^\infty x^2 e^{-x} \left(\frac{5}{x^2}\right) dx$$

$$= 5 \int_5^\infty e^{-x} dx$$

$$= 5 \left[-e^{-x}\right]_5^\infty$$

$$= 5e^{-5}$$

$$\approx 0.0337$$

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3 i). $f_X(x) = \begin{cases} \frac{1}{3}, & -2 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$

$$P(X \le x) = F_X(x) = \begin{cases} 0, & \text{if } x < -2 \\ \frac{x+2}{3}, & \text{if } -2 < x < 1 \\ 1, & \text{if } x \geq 1 \end{cases}$$

$Y_1 = |X|.$

$Y_1 = X$ if $x \geq 0$

$Y_1 = -X$ if $x < 0$

If $x \geq 0$, $F_Y(y) = P(Y_1 \le y) = P(X \le y) = F_X(y)$

If $x < 0$, $F_Y(y) = P(-Y_1 \le y) = P(-X \le y) = P(X \geq -y)$

$= 1 - P(X \le -y) = 1 - F_X(-y)$

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$$\therefore\ F_Y(y) = \begin{cases} 0 & , \text{ if } \quad y < -2 \\ 1 - \frac{2-y}{3} & , \text{ if } \quad -2 < y < 0 \\ \frac{y+2}{3} & , \text{ if } \quad 0 \leq y < 1 \\ 1 & , \text{ if } \quad y \geq 1 \end{cases}$$

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3 ii). Let $g(x) = e^{2x}$,

with $Y = g(x)$, $g^{-1}(y) = \frac{\ln y}{2}$.

$F_{Y_2}(y) = P(g(x) \le y) = P(X \le g^{-1}(y)) = F_X(g^{-1}(y))$

$= F_X\left(\frac{\ln y}{2}\right)$.

$$f_{Y_2}(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| , & -2 < y < 1 \\ 0 , & \text{otherwise.} \end{cases}$$

$$f_{Y_2}(y) = \begin{cases} \frac{1}{3} \left(\frac{1}{2}\right)\left(\frac{1}{y}\right) , & -2 < y < 1 \\ 0 , & \text{otherwise} \end{cases}$$

$$f_{Y_2}(y) = \begin{cases} \frac{1}{6y} , & -2 < y < 1 \\ 0 , & \text{otherwise} \end{cases}$$

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4. i). Yes, $\because f_{X,Y}(x,y) = 2e^{-x-y} = 2e^{-x}e^{-y}$

for $(x,y) \in A$.

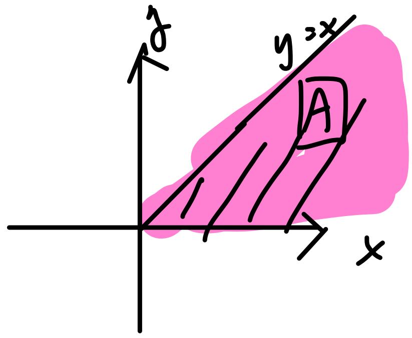
Since $f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$

where $f_X(x) = 2e^{-x}$

$f_Y(y) = e^{-y}$ for $(x,y) \in A$

$\therefore$ X and Y are independent.

ii).



$$f_X(x) = \int_0^x 2e^{-(x+y)}\,dy$$

$$f_X(x) = 2\int_0^x e^{-x}e^{-y}\,dy$$

$$f_X(x) = 2\left[-e^{-x}e^{-y}\right]_0^x$$

$$f_X(x) = 2\left(-e^{-x}e^{-x} + e^{-x}\right)$$

$$f_X(x) = \begin{cases} 2\left(e^{-x} - e^{-2x}\right) & \text{for } 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

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iii. $P(0.3 < Y \leq 0.6 \mid X = 0.4)$

$P(X = 0.4) = 2(e^{-0.4} - e^{-0.8})$

$f_Y(y) = \int_y^{\infty} 2e^{-x}e^{-y}\,dx$

$f_Y(y) = 2\left[-e^{-y}e^{-x}\right]_y^{\infty}$

$f_Y(y) = 2(e^{-y}e^{-y})$

$$f_Y(y) = \begin{cases} 2e^{-2y}, & 0 < y < x \\ 0, & \text{otherwise} \end{cases}$$

$P(0.3 < Y \leq 0.6 \cap X = 0.4) = P(0.3 < Y \leq 0.4)$

$= \int_{0.3}^{0.4} 2e^{-2y}\,dy = \left[-e^{-2y}\right]_{0.3}^{0.4}$

$= -e^{-0.8} + e^{-0.6}$

$\therefore$ Required probability $= \dfrac{e^{-0.6} - e^{-0.8}}{2(e^{-0.4} - e^{-0.8})} \approx 0.2251$

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Q5 a) $f_{U,V}(u,v) = f_{X,Y}(x,y)\,|J(x,y)|^{-1}$

where $x = a(u,v)$ and $y = b(u,v)$.

$$J(x,y) = \begin{vmatrix} \frac{\partial xy}{\partial x} & \frac{\partial xy}{\partial y} \\ \frac{\partial \frac{x}{y}}{\partial x} & \frac{\partial \frac{x}{y}}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix}$$

$$= -\frac{x}{y} - \frac{x}{y} = -\frac{2x}{y}.$$

$$f_{U,V}(u,v) = \begin{cases} \frac{2}{x^2y^3}\left(\frac{y}{2x}\right) & x \geq 1,\ y \geq 1. \\ 0 & \text{, otherwise} \end{cases}$$

$$f_{U,V}(u,v) = \begin{cases} \frac{1}{x^3y^2}, & x \geq 1,\ y \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

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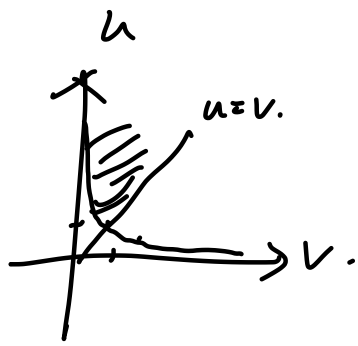
since $u=xy$, $v=\frac{x}{y}$, $x=\sqrt{uv}$, $y=\sqrt{\frac{u}{v}}$

$$f_{U,V}(u,v)=\begin{cases}\dfrac{1}{(uv)^{\frac{3}{2}}\,\frac{u}{v}}, & \sqrt{uv}\geq 1,\ \sqrt{\frac{u}{v}}\geq 1\\ 0, & \text{otherwise}\end{cases}$$

$$f_{U,V}(u,v)=\begin{cases}\dfrac{1}{u^{\frac{5}{2}}\,v^{\frac{1}{2}}}, & uv\geq 1,\ \frac{u}{v}\geq 1\\ 0, & \text{otherwise.}\end{cases}$$

b). $u\geq\frac{1}{v}$, $u\geq v$

$$f_V(v)=\int_1^{\infty}\frac{1}{u^{5/2}\,v^{1/2}}\,du$$



$$=\frac{1}{v^{1/2}}\int_1^{\infty}\frac{1}{u^{5/2}}\,du$$

$$=\frac{1}{v^{1/2}}\left[\frac{1}{-5/2+1}u^{-5/2+1}\right]_1^{\infty}$$

$$=\frac{1}{v^{1/2}}\left(0-\frac{1}{-\frac{3}{2}}(1)\right)$$

$$=\frac{1}{v^{1/2}}\left(\frac{2}{3}\right)=\frac{2}{3v^{1/2}}$$

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Q5 b7. $f_V(v) = \begin{cases} \dfrac{2}{3v^{1/2}}, & \frac{1}{4} < v < 4, \quad 4 \geq 1 \\ 0, & \text{otherwise.} \end{cases}$

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Q6 i).

$$M_X(t) = E[e^{tX}] = \sum_x e^{tx} p_X(x)$$

$$= \sum_{k=-\infty}^{\infty} e^{tk} p_X(k) = \left(\frac{1}{2} + \frac{e^{-t}+e^{t}}{4}\right)^2$$

$$= \frac{1}{4} + \frac{e^{-t}+e^{t}}{4} + \frac{(e^{-t}+e^{t})^2}{16}$$

$$= \frac{1}{4} + \frac{e^{-t}+e^{t}}{4} + \frac{e^{-2t}+e^{2t}+2}{16}$$

$$= \frac{3}{8} + \frac{1}{4}e^{-t} + \frac{1}{4}e^{t} + \frac{1}{16}e^{-2t} + \frac{1}{16}e^{2t}.$$

$\therefore$ P.M.F =

X	$p(X=x)$
$X=-2$	$\frac{1}{16}$
$X=-1$	$\frac{1}{4}$
$X=0$	$\frac{3}{8}$
$X=1$	$\frac{1}{4}$
$X=2$	$\frac{1}{16}$

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Q 6) ii). $M_{X+Y}(t) = M_X(t)\, M_Y(t)$.

$$M_{X+Y}(t) = \frac{1}{(1-t)^{5/2}}, \quad t<1.$$

$$E[(X+Y)^2] = M_{X+Y}^{(2)}(0),$$

where $M_{X+Y}^{(2)}(0) = \frac{d}{dt^2} M_{X+Y}(t)\Big|_{t=0}$.

$$\frac{d}{dt^2} (1-t)^{-5/2}$$

$$= \frac{d}{dt} \left(-\frac{5}{2}\right)(-1)(1-t)^{-7/2}$$

$$= \frac{d}{dt} \frac{5}{2} (1-t)^{-7/2}$$

$$= \frac{35}{4}(1-t)^{-9/2}. \quad - (*)$$

putting $t=0$ into equation $(*)$, we have:

$$\frac{35}{4}(1-0)^{-9/2} = \frac{35}{4}, \therefore E[(X+Y)^2] = \frac{35}{4}.$$

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Q7. Let X_i be the indicator random variable that $X_i = 1$ if ith pair consist of a man and a woman and $X_i = 0$ if otherwise.

Let X be the random variable of number of pairs that consist of a man and a woman.

$$X = \sum_{i=1}^{10} X_i$$

$$E[X] = E\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} E[X_i] = \sum_{i=1}^{10} P(X_i = 1)$$

$P(X_i = 1) = P($a man and a woman are selected$)$

$$= \frac{\binom{5}{1}\binom{10}{1}}{\binom{20}{2}} = \frac{5}{19}$$

$$E[X] = \frac{50}{19}$$

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$$E[X_i X_j] = P(X_i = 1, X_j = 1) = P(X_j = 1 | X_i = 1) P(X_i = 1)$$

$$= \frac{\binom{4}{1}\binom{9}{1}}{\binom{18}{2}} \frac{\binom{5}{1}\binom{19}{1}}{\binom{20}{2}} = \frac{20}{323}.$$

$$Cov(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j)$$

$$= \frac{20}{323} - \frac{50}{19}\left(\frac{50}{19}\right)$$

$$= -6.86328852.$$

$$Var\left(\sum_{i=1}^{10} X_i\right) = \sum_{i=1}^{10} Var(X_i) + 2\sum_{1 \le i < j \le n} cov(X_i, X_j)$$

$$Var\left(\sum_{i=1}^{10} X_i\right) = \frac{50}{19} + 2(-6.86328852)$$

$$= -11.09499756.$$

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Q8. $\rho(Y,Z) = \dfrac{cov(Y,Z)}{\sqrt{Var(Y)Var(Z)}}$

For $cov(Y,Z)$,

$$Cov(Y,Z) = Cov(a + bZ^2 + cZ^3, Z)$$

$$= cov(a, Z) + cov(bZ^2, Z) + cov(cZ^3, Z)$$

For $cov(bZ^2, Z) = E(bZ^3) - E(bZ^2)E(Z)$

For $E[Z^3]$, $E(Z^3) = M_Z^{(3)}(0) = \frac{d}{dt^3} M_Z(t)\Big|_{t=0}$.

Where $M_Z(t) = E(e^{tz}) = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$

$= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-t)^2/2} dz = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z^2-2tz)/2} dz$

$= \frac{e^{t^2}}{2}$.

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$$\frac{d}{dt^3} \frac{e^{t^2}}{2} = \frac{d}{dt^2} \frac{1}{2} e^{t^2} \cdot 2t = \frac{d}{dt^2} te^{t^2}$$

$$= \frac{d}{dt}\left(t e^{t^2}(2t) + e^{t^2}\right) = \frac{d}{dt}\left(2t^2e^{t^2} + e^{t^2}\right)$$

$$= 2t^2e^{t^2}(2t) + 2e^{t^2}(2t) + e^{t^2}(2t)$$

$$= 4t^3e^{t^2} + 4te^{t^2} + 2te^{t^2}$$

$M_Z^{(3)}(0) = 0$, $M_Z^{(2)}(0) = 0$.

$\therefore$ $\text{Cov}(bZ^2, Z) = 0$, $\text{cov}(cZ^3, Z) = 0$.

$\text{Cov}(a, Z) = E(aZ) - E(a)E(Z) = 0$.

$\text{Cov}(Y, Z) = 0$.

$\rho(Y, Z) = 0$.