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which can be generalized to give

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i) - \sum_{i < j} P(A_i A_j) + \sum_{i < j < k} P(A_i A_j A_k) + \cdots + (-1)^{n+1} P(A_1 \cdots A_n)$$

If  $S$  is finite and each one point set is assumed to have equal probability, then

$$P(A) = \frac{|A|}{|S|}$$

where  $|E|$  denotes the number of outcomes in the event  $E$ .

$P(A)$  can be interpreted either as a long-run relative frequency or as a measure of one's degree of belief.

### PROBLEMS

1. A box contains 3 marbles: 1 red, 1 green, and 1 blue. Consider an experiment that consists of taking 1 marble from the box and then replacing it in the box and drawing a second marble from the box. Describe the sample space. Repeat when the second marble is drawn without replacing the first marble.
  2. In an experiment, die is rolled continually until a 6 appears, at which point the experiment stops. What is the sample space of this experiment? Let  $E_n$  denote the event that  $n$  rolls are necessary to complete the experiment. What points of the sample space are contained in  $E_n$ ? What is  $\left(\bigcup_1^\infty E_n\right)^c$ ?
  3. Two dice are thrown. Let  $E$  be the event that the sum of the dice is odd, let  $F$  be the event that at least one of the dice lands on 1, and let  $G$  be the event that the sum is 5. Describe the events  $EF$ ,  $E \cup F$ ,  $FG$ ,  $EF^c$ , and  $EFG$ .
  4.  $A$ ,  $B$ , and  $C$  take turns flipping a coin. The first one to get a head wins. The sample space of this experiment can be defined by
- $$S = \left\{ 1, 01, 001, 0001, \dots, 0000 \dots \right\}$$
- (a) Interpret the sample space.
  - (b) Define the following events in terms of  $S$ :
    - (i)  $A$  wins =  $A$ .
    - (ii)  $B$  wins =  $B$ .
    - (iii)  $(A \cup B)^c$ .

Assume that  $A$  flips first, then  $B$ , then  $C$ , then  $A$ , and so on.
  5. A system is comprised of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector  $(x_1, x_2, x_3, x_4, x_5)$ , where  $x_i$  is equal to 1 if component  $i$  is working and is equal to 0 if component  $i$  is failed.
  - (a) How many outcomes are in the sample space of this experiment?
  - (b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let  $W$  be the event that the system will work. Specify all the outcomes in  $W$ .
  - (c) Let  $A$  be the event that components 4 and 5 are both failed. How many outcomes are contained in the event  $A$ ?
  - (d) Write out all the outcomes in the event  $AW$ .
  6. A hospital administrator codes incoming patients suffering gunshot wounds according to whether they have insurance (coding 1 if they do and 0 if they do not) and according to their condition, which is rated as good (g), fair (f), or serious (s). Consider an experiment that consists of the coding of such a patient.
    - (a) Give the sample space of this experiment.
    - (b) Let  $A$  be the event that the patient is in serious condition. Specify the outcomes in  $A$ .
    - (c) Let  $B$  be the event that the patient is uninsured. Specify the outcomes in  $B$ .
    - (d) Give all the outcomes in the event  $B^c \cup A$ .
  7. Consider an experiment that consists of determining the type of job—either blue-collar or white-collar—and the political affiliation—Republican, Democratic, or Independent—of the 15 members of an adult soccer team. How many outcomes are
    - (a) in the sample space?
    - (b) in the event that at least one of the team members is a blue-collar worker?



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- (c) in the event that none of the team members considers himself or herself an Independent?
8. Suppose that  $A$  and  $B$  are mutually exclusive events for which  $P(A) = .3$  and  $P(B) = .5$ . What is the probability that
- either  $A$  or  $B$  occurs?
  - $A$  occurs but  $B$  does not?
  - both  $A$  and  $B$  occur?
9. A retail establishment accepts either the American Express or the VISA credit card. A total of 24 percent of its customers carry an American Express card, 61 percent carry a VISA card, and 11 percent carry both cards. What percentage of its customers carry a credit card that the establishment will accept?
10. Sixty percent of the students at a certain school wear neither a ring nor a necklace. Twenty percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing
- a ring or a necklace?
  - a ring and a necklace?
11. A total of 28 percent of American males smoke cigarettes, 7 percent smoke cigars, and 5 percent smoke both cigars and cigarettes.
- What percentage of males smokes neither cigars nor cigarettes?
  - What percentage smokes cigars but not cigarettes?
12. An elementary school is offering 3 language classes: one in Spanish, one in French, and one in German. The classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students that are in both Spanish and French, 4 that are in both Spanish and German, and 6 that are in both French and German. In addition, there are 2 students taking all 3 classes.
- If a student is chosen randomly, what is the probability that he or she is not in any of the language classes?
  - If a student is chosen randomly, what is the probability that he or she is taking exactly one language class?
  - If 2 students are chosen randomly, what is the probability that at least 1 is taking a language class?
13. A certain town with a population of 100,000 has 3 newspapers: I, II, and III. The proportions of townspeople who read these papers are as follows:
- I: 10 percent    I and II: 8 percent    I and II and III: 1 percent  
II: 30 percent    I and III: 2 percent  
III: 5 percent    II and III: 4 percent

(The list tells us, for instance, that 8000 people read newspapers I and II.)

- Find the number of people who read only one newspaper.
- How many people read at least two newspapers?
- If I and III are morning papers and II is an evening paper, how many people read at least one morning paper plus an evening paper?
- How many people do not read any newspapers?
- How many people read only one morning paper and one evening paper?

14. The following data were given in a study of a group of 1000 subscribers to a certain magazine: In reference to job, marital status, and education, there were 312 professionals, 470 married persons, 525 college graduates, 42 professional college graduates, 147 married college graduates, 86 married professionals, and 25 married professional college graduates. Show that the numbers reported in the study must be incorrect.

*Hint:* Let  $M$ ,  $W$ , and  $G$  denote, respectively, the set of professionals, married persons, and college graduates. Assume that one of the 1000 persons is chosen at random, and use Proposition 4.4 to show that if the given numbers are correct, then  $P(M \cup W \cup G) > 1$ .

15. If it is assumed that all  $\binom{52}{5}$  poker hands are equally likely, what is the probability of being dealt
- a flush? (A hand is said to be a flush if all 5 cards are of the same suit.)
  - one pair? (This occurs when the cards have denominations  $a, a, b, c, d$ , where  $a, b, c$ , and  $d$  are all distinct.)
  - two pairs? (This occurs when the cards have denominations  $a, a, b, b, c$ , where  $a, b$ , and  $c$  are all distinct.)
  - three of a kind? (This occurs when the cards have denominations  $a, a, a, b, c$ , where  $a, b$ , and  $c$  are all distinct.)
  - four of a kind? (This occurs when the cards have denominations  $a, a, a, a, b$ .)

16. Poker dice is played by simultaneously rolling 5 dice. Show that
- $P\{\text{no two alike}\} = .0926$ ;
  - $P\{\text{one pair}\} = .4630$ ;
  - $P\{\text{two pair}\} = .2315$ ;
  - $P\{\text{three alike}\} = .1543$ ;
  - $P\{\text{full house}\} = .0386$ ;
  - $P\{\text{four alike}\} = .0193$ ;
  - $P\{\text{five alike}\} = .0008$ .

17. If 8 rooks (castles) are randomly placed on a chessboard, compute the probability that none of the rooks can capture any of the others. That is,



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compute the probability that no row or file contains more than one rook.

18. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?
19. Two symmetric dice have both had two of their sides painted red, two painted black, one painted yellow, and the other painted white. When this pair of dice is rolled, what is the probability that both dice land with the same color face up?
20. Suppose that you are playing blackjack against a dealer. In a freshly shuffled deck, what is the probability that neither you nor the dealer is dealt a blackjack?
21. A small community organization consists of 20 families, of which 4 have one child, 8 have two children, 5 have three children, 2 have four children, and 1 has five children.
  - (a) If one of these families is chosen at random, what is the probability it has  $i$  children,  $i = 1, 2, 3, 4, 5$ ?
  - (b) If one of the children is randomly chosen, what is the probability that child comes from a family having  $i$  children,  $i = 1, 2, 3, 4, 5$ ?
22. Consider the following technique for shuffling a deck of  $n$  cards: For any initial ordering of the cards, go through the deck one card at a time and at each card, flip a fair coin. If the coin comes up heads, then leave the card where it is; if the coin comes up tails, then move that card to the end of the deck. After the coin has been flipped  $n$  times, say that one round has been completed. For instance, if  $n = 4$  and the initial ordering is 1, 2, 3, 4, then if the successive flips result in the outcome h, t, t, h, then the ordering at the end of the round is 1, 4, 2, 3. Assuming that all possible outcomes of the sequence of  $n$  coin flips are equally likely, what is the probability that the ordering after one round is the same as the initial ordering?
23. A pair of fair dice is rolled. What is the probability that the second die lands on a higher value than does the first?
24. If two dice are rolled, what is the probability that the sum of the upturned faces equals  $i$ ? Find it for  $i = 2, 3, \dots, 11, 12$ .
25. A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first.
 

*Hint:* Let  $E_n$  denote the event that a 5 occurs on the  $n$ th roll and no 5 or 7 occurs on the first  $n - 1$  rolls. Compute  $P(E_n)$  and argue that  $\sum_{n=1}^{\infty} P(E_n)$  is the desired probability.
26. The game of craps is played as follows: A player rolls two dice. If the sum of the dice is either a 2,

3, or 12, the player loses; if the sum is either a 7 or an 11, the player wins. If the outcome is anything else, the player continues to roll the dice until she rolls either the initial outcome or a 7. If the 7 comes first, the player loses, whereas if the initial outcome reoccurs before the 7 appears, the player wins. Compute the probability of a player winning at craps.

*Hint:* Let  $E_i$  denote the event that the initial outcome is  $i$  and the player wins. The desired probability is  $\sum_{i=2}^{12} P(E_i)$ . To compute  $P(E_i)$ , define the events  $E_{i,n}$  to be the event that the initial sum is  $i$  and the player wins on the  $n$ th roll. Argue that

$$P(E_i) = \sum_{n=1}^{\infty} P(E_{i,n}).$$

27. An urn contains 3 red and 7 black balls. Players  $A$  and  $B$  withdraw balls from the urn consecutively until a red ball is selected. Find the probability that  $A$  selects the red ball. ( $A$  draws the first ball, then  $B$ , and so on. There is no replacement of the balls drawn.)
28. An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be (a) of the same color? (b) of different colors? Repeat under the assumption that whenever a ball is selected, its color is noted and it is then replaced in the urn before the next selection. This is known as *sampling with replacement*.
29. An urn contains  $n$  white and  $m$  black balls, where  $n$  and  $m$  are positive numbers.
  - (a) If two balls are randomly withdrawn, what is the probability that they are the same color?
  - (b) If a ball is randomly withdrawn and then replaced before the second one is drawn, what is the probability that the withdrawn balls are the same color?
  - (c) Show that the probability in part (b) is always larger than the one in part (a).
30. The chess clubs of two schools consist of, respectively, 8 and 9 players. Four members from each club are randomly chosen to participate in a contest between the two schools. The chosen players from one team are then randomly paired with those from the other team, and each pairing plays a game of chess. Suppose that Rebecca and her sister Elise are on the chess clubs at different schools. What is the probability that
  - (a) Rebecca and Elise will be paired?
  - (b) Rebecca and Elise will be chosen to represent their schools but will not play each other?
  - (c) either Rebecca or Elise will be chosen to represent her school?

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31. A 3-person basketball team consists of a guard, a forward, and a center.
  - (a) If a person is chosen at random from each of three different such teams, what is the probability of selecting a complete team?
  - (b) What is the probability that all 3 players selected play the same position?
32. A group of individuals containing  $b$  boys and  $g$  girls is lined up in random order; that is, each of the  $(b+g)!$  permutations is assumed to be equally likely. What is the probability that the person in the  $i$ th position,  $1 \leq i \leq b+g$ , is a girl?
33. A forest contains 20 elk, of which 5 are captured, tagged, and then released. A certain time later, 4 of the 20 elk are captured. What is the probability that 2 of these 4 have been tagged? What assumptions are you making?
34. The second Earl of Yarborough is reported to have bet at odds of 1000 to 1 that a bridge hand of 13 cards would contain at least one card that is ten or higher. (By *ten or higher* we mean that a card is either a ten, a jack, a queen, a king, or an ace.) Nowadays, we call a hand that has no cards higher than 9 a *Yarborough*. What is the probability that a randomly selected bridge hand is a *Yarborough*?
35. Seven balls are randomly withdrawn from an urn that contains 12 red, 16 blue, and 18 green balls. Find the probability that
  - (a) 3 red, 2 blue, and 2 green balls are withdrawn;
  - (b) at least 2 red balls are withdrawn;
  - (c) all withdrawn balls are the same color;
  - (d) either exactly 3 red balls or exactly 3 blue balls are withdrawn.
36. Two cards are chosen at random from a deck of 52 playing cards. What is the probability that they
  - (a) are both aces?
  - (b) have the same value?
37. An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them. If a student has figured out how to do 7 of the problems, what is the probability that he or she will answer correctly
  - (a) all 5 problems?
  - (b) at least 4 of the problems?
38. There are  $n$  socks, 3 of which are red, in a drawer. What is the value of  $n$  if, when 2 of the socks are chosen randomly, the probability that they are both red is  $\frac{1}{2}$ ?
39. There are 5 hotels in a certain town. If 3 people check into hotels in a day, what is the probability that they each check into a different hotel? What assumptions are you making?
40. A town contains 4 people who repair televisions. If 4 sets break down, what is the probability that exactly  $i$  of the repairers are called? Solve the problem for  $i = 1, 2, 3, 4$ . What assumptions are you making?
41. If a die is rolled 4 times, what is the probability that 6 comes up at least once?
42. Two dice are thrown  $n$  times in succession. Compute the probability that double 6 appears at least once. How large need  $n$  be to make this probability at least  $\frac{1}{2}$ ?
43. (a) If  $N$  people, including  $A$  and  $B$ , are randomly arranged in a line, what is the probability that  $A$  and  $B$  are next to each other?  
 (b) What would the probability be if the people were randomly arranged in a circle?
44. Five people, designated as  $A, B, C, D, E$ , are arranged in linear order. Assuming that each possible order is equally likely, what is the probability that
  - (a) there is exactly one person between  $A$  and  $B$ ?
  - (b) there are exactly two people between  $A$  and  $B$ ?
  - (c) there are three people between  $A$  and  $B$ ?
45. A woman has  $n$  keys, of which one will open her door.
  - (a) If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her  $k$ th try?
  - (b) What if she does not discard previously tried keys?
46. How many people have to be in a room in order that the probability that at least two of them celebrate their birthday in the same month is at least  $\frac{1}{2}$ ? Assume that all possible monthly outcomes are equally likely.
47. If there are 12 strangers in a room, what is the probability that no two of them celebrate their birthday in the same month?
48. Given 20 people, what is the probability that, among the 12 months in the year, there are 4 months containing exactly 2 birthdays and 4 containing exactly 3 birthdays?
49. A group of 6 men and 6 women is randomly divided into 2 groups of size 6 each. What is the probability that both groups will have the same number of men?
50. In a hand of bridge, find the probability that you have 5 spades and your partner has the remaining 8.
51. Suppose that  $n$  balls are randomly distributed into  $N$  compartments. Find the probability that  $m$  balls will fall into the first compartment. Assume that all  $N^n$  arrangements are equally likely.

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52. A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be  
 (a) no complete pair?  
 (b) exactly 1 complete pair?
53. If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.
54. Compute the probability that a bridge hand is void in at least one suit. Note that the answer is not

$$\frac{\binom{4}{1} \binom{39}{13}}{\binom{52}{13}}$$

(Why not?)

*Hint:* Use Proposition 4.4.

55. Compute the probability that a hand of 13 cards contains  
 (a) the ace and king of at least one suit;  
 (b) all 4 of at least 1 of the 13 denominations.
56. Two players play the following game: Player *A* chooses one of the three spinners pictured in Figure 2.6, and then player *B* chooses one of the remaining two spinners. Both players then spin their spinner, and the one that lands on the higher number is declared the winner. Assuming that each spinner is equally likely to land in any of its 3 regions, would you rather be player *A* or player *B*? Explain your answer!

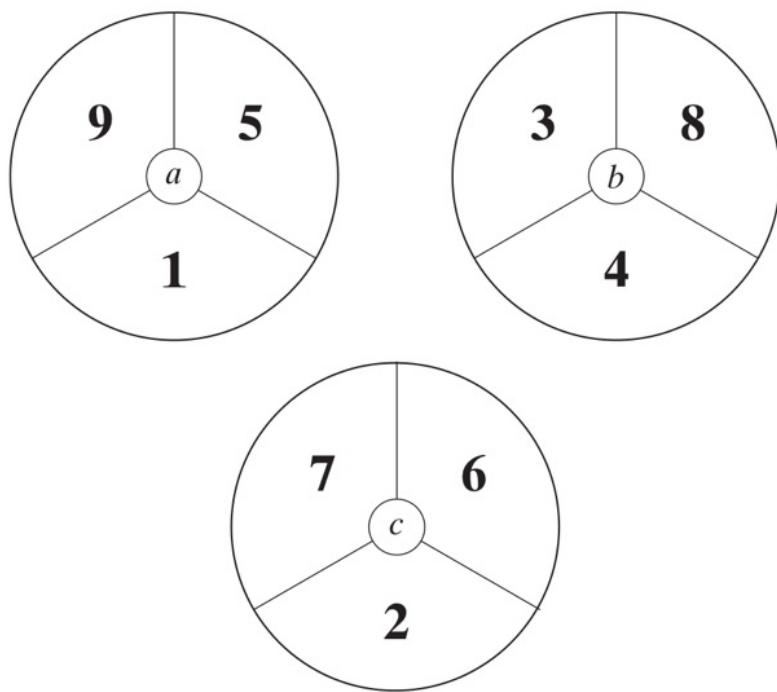


FIGURE 2.6: Spinners

## THEORETICAL EXERCISES

Prove the following relations:

1.  $EF \subset E \subset E \cup F$ .
2. If  $E \subset F$ , then  $F^c \subset E^c$ .
3.  $F = FE \cup FE^c$  and  $E \cup F = E \cup E^c F$ .

4.  $\left(\bigcup_{i=1}^{\infty} E_i\right)F = \bigcup_{i=1}^{\infty} E_i F$  and  
 $\left(\bigcap_{i=1}^{\infty} E_i\right) \cup F = \bigcap_{i=1}^{\infty} (E_i \cup F)$ .

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5. For any sequence of events  $E_1, E_2, \dots$ , define a new sequence  $F_1, F_2, \dots$  of disjoint events (that is, events such that  $F_i F_j = \emptyset$  whenever  $i \neq j$ ) such that for all  $n \geq 1$ ,

$$\bigcup_1^n F_i = \bigcup_1^n E_i$$

6. Let  $E, F$ , and  $G$  be three events. Find expressions for the events so that, of  $E, F$ , and  $G$ ,
- (a) only  $E$  occurs;
  - (b) both  $E$  and  $G$ , but not  $F$ , occur;
  - (c) at least one of the events occurs;
  - (d) at least two of the events occur;
  - (e) all three events occur;
  - (f) none of the events occurs;
  - (g) at most one of the events occurs;
  - (h) at most two of the events occur;
  - (i) exactly two of the events occur;
  - (j) at most three of the events occur.
7. Find the simplest expression for the following events:
- (a)  $(E \cup F)(E \cup F^c)$ ;
  - (b)  $(E \cup F)(E^c \cup F)(E \cup F^c)$ ;
  - (c)  $(E \cup F)(F \cup G)$ .
8. Let  $S$  be a given set. If, for some  $k > 0$ ,  $S_1, S_2, \dots, S_k$  are mutually exclusive nonempty subsets of  $S$  such that  $\bigcup_{i=1}^k S_i = S$ , then we call the set  $\{S_1, S_2, \dots, S_k\}$  a *partition* of  $S$ . Let  $T_n$  denote the number of different partitions of  $\{1, 2, \dots, n\}$ . Thus,  $T_1 = 1$  (the only partition being  $S_1 = \{1\}$ ) and  $T_2 = 2$  (the two partitions being  $\{\{1, 2\}\}, \{\{1\}, \{2\}\}$ ).
- (a) Show, by computing all partitions, that  $T_3 = 5, T_4 = 15$ .
  - (b) Show that

$$T_{n+1} = 1 + \sum_{k=1}^n \binom{n}{k} T_k$$

and use this equation to compute  $T_{10}$ .

*Hint:* One way of choosing a partition of  $n + 1$  items is to call one of the items *special*. Then we obtain different partitions by first choosing  $k, k = 0, 1, \dots, n$ , then a subset of size  $n - k$  of the non-special items, and then any of the  $T_k$  partitions of the remaining  $k$  nonspecial items. By adding the special item to the subset of size  $n - k$ , we obtain a partition of all  $n + 1$  items.

9. Suppose that an experiment is performed  $n$  times. For any event  $E$  of the sample space, let  $n(E)$  denote the number of times that event  $E$  occurs and define  $f(E) = n(E)/n$ . Show that  $f(\cdot)$  satisfies Axioms 1, 2, and 3.

10. Prove that  $P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E^c FG) - P(EF^c G) - P(EFG^c) - 2P(EFG)$ .

11. If  $P(E) = .9$  and  $P(F) = .8$ , show that  $P(EF) \geq .7$ . In general, prove Bonferroni's inequality, namely,

$$P(EF) \geq P(E) + P(F) - 1$$

12. Show that the probability that exactly one of the events  $E$  or  $F$  occurs equals  $P(E) + P(F) - 2P(EF)$ .

13. Prove that  $P(EF^c) = P(E) - P(EF)$ .

14. Prove Proposition 4.4 by mathematical induction.

15. An urn contains  $M$  white and  $N$  black balls. If a random sample of size  $r$  is chosen, what is the probability that it contains exactly  $k$  white balls?

16. Use induction to generalize Bonferroni's inequality to  $n$  events. That is, show that

$$P(E_1 E_2 \cdots E_n) \geq P(E_1) + \cdots + P(E_n) - (n - 1)$$

17. Consider the matching problem, Example 5m, and define  $A_N$  to be the number of ways in which the  $N$  men can select their hats so that no man selects his own. Argue that

$$A_N = (N - 1)(A_{N-1} + A_{N-2})$$

This formula, along with the boundary conditions  $A_1 = 0, A_2 = 1$ , can then be solved for  $A_N$ , and the desired probability of no matches would be  $A_N/N!$ .

*Hint:* After the first man selects a hat that is not his own, there remain  $N - 1$  men to select among a set of  $N - 1$  hats that does not contain the hat of one of these men. Thus, there is one extra man and one extra hat. Argue that we can get no matches either with the extra man selecting the extra hat or with the extra man not selecting the extra hat.

18. Let  $f_n$  denote the number of ways of tossing a coin  $n$  times such that successive heads never appear. Argue that

$$f_n = f_{n-1} + f_{n-2} \quad n \geq 2, \text{ where } f_0 = 1, f_1 = 2$$

*Hint:* How many outcomes are there that start with a head, and how many start with a tail? If  $P_n$  denotes the probability that successive heads never appear when a coin is tossed  $n$  times, find  $P_n$  (in terms of  $f_n$ ) when all possible outcomes of the  $n$  tosses are assumed equally likely. Compute  $P_{10}$ .

19. An urn contains  $n$  red and  $m$  blue balls. They are withdrawn one at a time until a total of  $r, r \leq n$ ,



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red balls have been withdrawn. Find the probability that a total of  $k$  balls are withdrawn.

*Hint:* A total of  $k$  balls will be withdrawn if there are  $r - 1$  red balls in the first  $k - 1$  withdrawals and the  $k$ th withdrawal is a red ball.

20. Consider an experiment whose sample space consists of a countably infinite number of points. Show that not all points can be equally likely. Can all points have a positive probability of occurring?
- \*21. Consider Example 5o, which is concerned with the number of runs of wins obtained when  $n$  wins and  $m$  losses are randomly permuted. Now

consider the total number of runs—that is, win runs plus loss runs—and show that

$$\begin{aligned} P\{2k \text{ runs}\} &= 2 \frac{\binom{m-1}{k-1} \binom{n-1}{k-1}}{\binom{m+n}{n}} \\ P\{2k+1 \text{ runs}\} &= \frac{\binom{m-1}{k-1} \binom{n-1}{k} + \binom{m-1}{k} \binom{n-1}{k-1}}{\binom{m+n}{n}} \end{aligned}$$

### SELF-TEST PROBLEMS AND EXERCISES

1. A cafeteria offers a three-course meal consisting of an entree, a starch, and a dessert. The possible choices are given in the following table:

| Course  | Choices                                    |
|---------|--|
| Entree  | Chicken or roast beef                      |
| Starch  | Pasta or rice or potatoes                  |
| Dessert | Ice cream or Jello or apple pie or a peach |

A person is to choose one course from each category.

- (a) How many outcomes are in the sample space?
- (b) Let  $A$  be the event that ice cream is chosen. How many outcomes are in  $A$ ?
- (c) Let  $B$  be the event that chicken is chosen. How many outcomes are in  $B$ ?
- (d) List all the outcomes in the event  $AB$ .
- (e) Let  $C$  be the event that rice is chosen. How many outcomes are in  $C$ ?
- (f) List all the outcomes in the event  $ABC$ .
2. A customer visiting the suit department of a certain store will purchase a suit with probability .22, a shirt with probability .30, and a tie with probability .28. The customer will purchase both a suit and a shirt with probability .11, both a suit and a tie with probability .14, and both a shirt and a tie with probability .10. A customer will purchase all 3 items with probability .06. What is the probability that a customer purchases
  - (a) none of these items?
  - (b) exactly 1 of these items?
3. A deck of cards is dealt out. What is the probability that the 14th card dealt is an ace? What is the probability that the first ace occurs on the 14th card?

4. Let  $A$  denote the event that the midtown temperature in Los Angeles is  $70^{\circ}\text{F}$ , and let  $B$  denote the event that the midtown temperature in New York is  $70^{\circ}\text{F}$ . Also, let  $C$  denote the event that the maximum of the midtown temperatures in New York and in Los Angeles is  $70^{\circ}\text{F}$ . If  $P(A) = .3$ ,  $P(B) = .4$ , and  $P(C) = .2$ , find the probability that the minimum of the two midtown temperatures is  $70^{\circ}\text{F}$ .

5. An ordinary deck of 52 cards is shuffled. What is the probability that the top four cards have
  - (a) different denominations?
  - (b) different suits?
6. Urn  $A$  contains 3 red and 3 black balls, whereas urn  $B$  contains 4 red and 6 black balls. If a ball is randomly selected from each urn, what is the probability that the balls will be the same color?
7. In a state lottery, a player must choose 8 of the numbers from 1 to 40. The lottery commission then performs an experiment that selects 8 of these 40 numbers. Assuming that the choice of the lottery commission is equally likely to be any of the  $\binom{40}{8}$  combinations, what is the probability that a player has
  - (a) all 8 of the numbers selected by the lottery commission?
  - (b) 7 of the numbers selected by the lottery commission?
  - (c) at least 6 of the numbers selected by the lottery commission?
8. From a group of 3 freshmen, 4 sophomores, 4 juniors, and 3 seniors a committee of size 4 is randomly selected. Find the probability that the committee will consist of
  - (a) 1 from each class;
  - (b) 2 sophomores and 2 juniors;
  - (c) only sophomores or juniors.



## Self -Test Problems and Exercises 57

9. For a finite set  $A$ , let  $N(A)$  denote the number of elements in  $A$ .

(a) Show that

$$N(A \cup B) = N(A) + N(B) - N(AB)$$

(b) More generally, show that

$$\begin{aligned} N\left(\bigcup_{i=1}^n A_i\right) &= \sum_i N(A_i) - \sum_{i < j} \sum N(A_i A_j) \\ &\quad + \dots + (-1)^{n+1} N(A_1 \dots A_n) \end{aligned}$$

10. Consider an experiment that consists of six horses, numbered 1 through 6, running a race, and suppose that the sample space consists of the  $6!$  possible orders in which the horses finish. Let  $A$  be the event that the number-1 horse is among the top three finishers, and let  $B$  be the event that the number-2 horse comes in second. How many outcomes are in the event  $A \cup B$ ?
11. A 5-card hand is dealt from a well-shuffled deck of 52 playing cards. What is the probability that the hand contains at least one card from each of the four suits?
12. A basketball team consists of 6 frontcourt and 4 backcourt players. If players are divided into roommates at random, what is the probability that there will be exactly two roommate pairs made up of a backcourt and a frontcourt player?
13. Suppose that a person chooses a letter at random from R E S E R V E and then chooses one at random from V E R T I C A L. What is the probability that the same letter is chosen?
14. Prove Boole's inequality:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

15. Show that if  $P(A_i) = 1$  for all  $i \geq 1$ , then

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1.$$

16. Let  $T_k(n)$  denote the number of partitions of the set  $\{1, \dots, n\}$  into  $k$  nonempty subsets, where  $1 \leq k \leq n$ . (See Theoretical Exercise 8 for the definition of a partition.) Argue that

$$T_k(n) = kT_k(n-1) + T_{k-1}(n-1)$$

*Hint:* In how many partitions is  $\{1\}$  a subset, and in how many is 1 an element of a subset that contains other elements?

17. Five balls are randomly chosen, without replacement, from an urn that contains 5 red, 6 white, and 7 blue balls. Find the probability that at least one ball of each color is chosen.
18. Four red, 8 blue, and 5 green balls are randomly arranged in a line.
- (a) What is the probability that the first 5 balls are blue?
  - (b) What is the probability that none of the first 5 balls are blue?
  - (c) What is the probability that the final 3 balls are differently colored?
  - (d) What is the probability that all the red balls are together?
19. Ten cards are randomly chosen from a deck of 52 cards that consists of 13 cards of each of 4 different suits. Each of the selected cards is put in one of 4 piles, depending on the suit of the card.
- (a) What is the probability that the largest pile has 4 cards, the next largest has 3, the next largest has 2, and the smallest has 1 card?
  - (b) What is the probability that two of the piles have 3 cards, one has 4 cards, and one has no cards?
20. Balls are randomly removed from an urn initially containing 20 red and 10 blue balls. What is the probability that all of the red balls are removed before all of the blue ones have been removed?

1. A box contains 3 marbles: 1 red, 1 green, and 1 blue. Consider an experiment that consists of taking 1 marble from the box and then replacing it in the box and drawing a second marble from the box. Describe the sample space. Repeat when the second marble is drawn without replacing the first marble.

2. In an experiment, die is rolled continually until a 6 appears, at which point the experiment stops. What is the sample space of this experiment? Let  $E_n$  denote the event that  $n$  rolls are necessary to complete the experiment. What points of the sample space are contained in  $E_n$ ? What is  $(\bigcup_1^{\infty} E_n)^c$ ?

3. Two dice are thrown. Let  $E$  be the event that the sum of the dice is odd, let  $F$  be the event that at least one of the dice lands on 1, and let  $G$  be the event that the sum is 5. Describe the events  $EF, E \cup F, FG, EF^c$ , and  $EFG$ .

4.  $A, B$ , and  $C$  take turns flipping a coin. The first one to get a head wins. The sample space of this experiment can be defined by

$$S = \left\{ 1, 01, 001, 0001, \dots, 0000\dots \right\}$$

- (a) Interpret the sample space.  
(b) Define the following events in terms of  $S$ :  
(i)  $A$  wins =  $A$ .  
(ii)  $B$  wins =  $B$ .  
(iii)  $(A \cup B)^c$ .  
Assume that  $A$  flips first, then  $B$ , then  $C$ , then  $A$ , and so on.

5. A system is comprised of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of

1.  $S: \left\{ \begin{array}{lll} RR & GR & BR \\ RG & GG & BG \\ RB & GB & BB \end{array} \right\}$

$S: \left\{ \begin{array}{lll} RG & GR & BR \\ RB & GB & BG \end{array} \right\}$

2.  $\left\{ 6, 16, 26, 36, 46, \dots \right\}$

$56, 116, \dots \dots \dots \right\}$

$E_n: n-1 \text{ non } 6,$   
 $n \text{ die } \rightarrow 6.$

$$\left( \bigcup_1^{\infty} E_n \right)^c$$

$\overbrace{E_1 + E_2 + E_3 + \dots + E_{\infty}}^C$   
= Event that no 6 happen.

3. Two dice are thrown. Let  $E$  be the event that the sum of the dice is odd, let  $F$  be the event that at least one of the dice lands on 1, and let  $G$  be the event that the sum is 5. Describe the events  $EF$ ,  $E \cup F$ ,  $FG$ ,  $EF^c$ , and  $EFG$ .

4.  $A$ ,  $B$ , and  $C$  take turns flipping a coin. The first one to get a head wins. The sample space of this experiment can be defined by

$$S = \left\{ 1, 01, 001, 0001, \dots, 0000 \dots \right.$$

- (a) Interpret the sample space.  
(b) Define the following events in terms of  $S$ :  
(i)  $A$  wins =  $A$ .  
(ii)  $B$  wins =  $B$ .  
(iii)  $(A \cup B)^c$ .

Assume that  $A$  flips first, then  $B$ , then  $C$ , then  $A$ , and so on.

3.  $E \bar{F} : 16 \quad 12 \quad 14$   
 $\quad \quad 21 \quad 41 \quad 61$

$E \cup \bar{F} : 16 \quad 12 \quad 14 \quad 21 \quad 41 \quad 61 \quad 32 \quad 34 \quad 36$   
 $\quad \quad 52 \quad 54 \quad 56 \quad 23 \quad 43 \quad 63 \quad 25 \quad 45 \quad 65$   
 $\quad \quad 11 \quad 13 \quad 15 \quad 31 \quad 51$

$F \bar{G} : 14 \quad 41$

$E \bar{F}^c : 32 \quad 34 \quad 36 \quad 52 \quad 54 \quad 56$   
 $\quad \quad 23 \quad 43 \quad 63 \quad 25 \quad 45 \quad 65$

$E \bar{F} G : 41 \quad 14$

4. A, B, and C take turns flipping a coin. The first one to get a head wins. The sample space of this experiment can be defined by

$$S = \left\{ 1, 01, 001, 0001, \dots, 0000\dots \right.$$

- (a) Interpret the sample space.  
 (b) Define the following events in terms of S:  
 (i) A wins = A.  
 (ii) B wins = B.  
 (iii)  $(A \cup B)^c$ .

Assume that A flips first, then B, then C, then A, and so on.

4a), A wins: 1, 001, 0001, 00001, ...

B wins: 01, 0001, ...

C wins: 001, 00001, ...

5).  $A \in S$

(ii)  $B \in S$

$$\begin{aligned} \text{(iii). } (A \cup B)^c &= A^c \cap B^c \\ &= C \in S. \end{aligned}$$

Final answer =  
in term of S

5. A system is comprised of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector  $(x_1, x_2, x_3, x_4, x_5)$ , where  $x_i$  is equal to 1 if component  $i$  is working and is equal to 0 if component  $i$  is failed.

- (a) How many outcomes are in the sample space of this experiment?
- (b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let  $W$  be the event that the system will work. Specify all the outcomes in  $W$ .
- (c) Let  $A$  be the event that components 4 and 5 are both failed. How many outcomes are contained in the event  $A$ ?
- (d) Write out all the outcomes in the event  $AW$ .

5a).  $2^5$

b).  $W = \{(1, 1, 0, 0, 0), (1, 1, 1, 1, 0), (0, 0, 1, 1, 0), (1, 0, 1, 0, 1)\}$

c).  $2^3$

(d).  $AW = \{(1, 1, 0, 0, 0)\}$

6. A hospital administrator codes incoming patients suffering gunshot wounds according to whether they have insurance (coding 1 if they do and 0 if they do not) and according to their condition, which is rated as good (g), fair (f), or serious (s). Consider an experiment that consists of the coding of such a patient.

- (a) Give the sample space of this experiment.
- (b) Let  $A$  be the event that the patient is in serious condition. Specify the outcomes in  $A$ .
- (c) Let  $B$  be the event that the patient is uninsured. Specify the outcomes in  $B$ .
- (d) Give all the outcomes in the event  $B^c \cup A$ .

$\begin{matrix} (g) \\ 1 \\ f \\ (s) \end{matrix}$

b).  $\begin{matrix} (s) \\ 0 \\ f \\ (s) \end{matrix}$

c).  $\begin{matrix} (g) \\ 1 \\ f \\ (s) \end{matrix}$

$$d). B^c \cup A = S - (B - A \cap B)$$

$$= S - (B - Os)$$

$$= S - (0g, 0f)$$

$$= 1g \quad 1f \quad (s)$$



7. Consider an experiment that consists of determining the type of job—either blue-collar or white-collar—and the political affiliation—Republican, Democratic, or Independent—of the 15 members of an adult soccer team. How many outcomes are
- (a) in the sample space?
  - (b) in the event that at least one of the team members is a blue-collar worker?

a)  $6^{15}$

b).  $6^{15} - \text{no blue coll}$

$6^{15} - 3^{15}$

- (c) in the event that none of the team members considers himself or herself an Independent?

c)  $4^{15}$

8. Suppose that  $A$  and  $B$  are mutually exclusive events for which  $P(A) = .3$  and  $P(B) = .5$ . What is the probability that

- (a) either  $A$  or  $B$  occurs?
- (b)  $A$  occurs but  $B$  does not?
- (c) both  $A$  and  $B$  occur?

9. A retail establishment accepts either the American Express or the VISA credit card. A total of 24 percent of its customers carry an American Express card, 61 percent carry a VISA card, and 11 percent carry both cards. What percentage of its customers carry a credit card that the establishment will accept? 14

$$P(A) = .3 + .5 = .8$$

b). 0.3

c). 0

9.  $P(\text{American Express} \cup \text{VISA})$

$$= 0.24 + 0.61 - 0.11$$

$$= 0.74$$

10. Sixty percent of the students at a certain school wear neither a ring nor a necklace. Twenty percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing  
(a) a ring or a necklace?  
(b) a ring and a necklace?



$$1^{\text{st}} \quad P(\text{Ring}^c \cap \text{necklace}^c) = 0.6$$

$$P(\text{Ring}) = 0.2$$

$$P(\text{necklace}) = 0.3$$

a). 0.4

b).  $0.2 + 0.3 - x = 0.4$

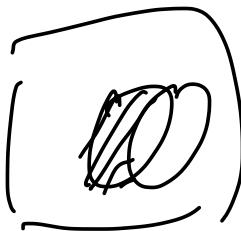
$$x = 0.1$$

11. A total of 28 percent of American males smoke cigarettes, 7 percent smoke cigars, and 5 percent smoke both cigars and cigarettes.

- (a) What percentage of males smokes neither cigars nor cigarettes?  
 (b) What percentage smokes cigars but not cigarettes?

12. An elementary school is offering 3 language classes: one in Spanish, one in French, and one in German. The classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students that are in both Spanish and French, 4 that are in both Spanish and German, and 6 that are in both French and German. In addition, there are 2 students taking all 3 classes.

- (a) If a student is chosen randomly, what is the probability that he or she is not in any of the language classes?  
 (b) If a student is chosen randomly, what is the probability that he or she is taking exactly one language class?  
 (c) If 2 students are chosen randomly, what is the probability that at least 1 is taking a language class?



$$11a). P(\text{cigars}^C \cap \text{cigarettes}^C) = 1 - P(\text{cigars} \cup \text{cigarettes})$$

$$= 1 - (0.28 + 0.07 - 0.05)$$

$$= 1 - 0.3 = 0.7$$

$$b). P(\text{cigars} \cap \text{cigarettes}^C)$$

$$= P(\text{cigars}) - P(\text{cigars} \cap \text{cigarettes})$$

$$= 0.07 - 0.05 = 0.02$$

12. An elementary school is offering 3 language classes: one in Spanish, one in French, and one in German. The classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students that are in both Spanish and French, 4 that are in both Spanish and German, and 6 that are in both French and German. In addition, there are 2 students taking all 3 classes.

- (a) If a student is chosen randomly, what is the probability that he or she is not in any of the language classes?
- (b) If a student is chosen randomly, what is the probability that he or she is taking exactly one language class?
- (c) If 2 students are chosen randomly, what is the probability that at least 1 is taking a language class?

Q. 71

1 take

2 take

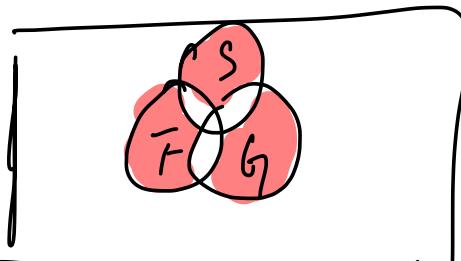
$$28 + 26 + 16 - 12 - 4 - 6 + 2$$

$$12 + 14 + 10 = 36$$

$$12 + 14 + 10 = 36$$

=

120).



Spanish  $\cup$  French  $\cup$  German :

$$28 + 26 + 16 - 12 - 4 - 6 + 2$$

$$= 50 \Rightarrow \text{Ans}$$

$$b) 28 - 12 - 4 + 2 + 26 - 12 - 6 + 2 +$$

$$16 - 4 - 6 + 2$$

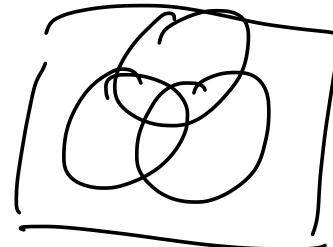
$$= 14 + 10 + 8 = 32 \Rightarrow \text{Ans}$$

13. A certain town with a population of 100,000 has 3 newspapers: I, II, and III. The proportions of townspeople who read these papers are as follows:

I: 10 percent    I and II: 8 percent    I and II and III: 1 percent  
 II: 30 percent    I and III: 2 percent  
 III: 5 percent    II and III: 4 percent

(The list tells us, for instance, that 8000 people read newspapers I and II.)

- (a) Find the number of people who read only one newspaper.
- (b) How many people read at least two newspapers?
- (c) If I and III are morning papers and II is an evening paper, how many people read at least one morning paper plus an evening paper?
- (d) How many people do not read any newspapers?
- (e) How many people read only one morning paper and one evening paper?



$$(3a) - | + | \quad + 0 = 100,000 \times 0.12 = 20,000$$

$$b) - 7,2 : 2 \text{ newspaper} + 3 \\ 0.072 = 0.12 + 0.01 \\ 0.072 - 0.01 = 0.12$$

$$c) - 7/ | m + | \bar{E} \\ | m + | \bar{E} \quad 2m + | \bar{E} \quad 10 \\ | \bar{E} + | \bar{E} \\ 8 + 4 - 1 = 11 \\ \Rightarrow 0.12$$

$$\text{d) } 10 + 30 + 5 - 8 - 2 - 4 + 1 \\ = 32$$

$$1 - v \cdot 32 = \cancel{v \cdot 68} \times 100,000.$$

$$\text{e) } \cancel{2.11} \times 100,000$$

|  
~~1000~~

14. The following data were given in a study of a group of 1000 subscribers to a certain magazine: In reference to job, marital status, and education, there were 312 professionals, 470 married persons, 525 college graduates, 42 professional college graduates, 147 married college graduates, 86 married professionals, and 25 married professional college graduates. Show that the numbers reported in the study must be incorrect.

*Hint:* Let  $M$ ,  $W$ , and  $G$  denote, respectively, the set of professionals, married persons, and college graduates. Assume that one of the 1000 persons is chosen at random, and use Proposition 4.4 to show that if the given numbers are correct, then  $P(M \cup W \cup G) > 1$ .

$$P(M) = \frac{312}{1000}$$

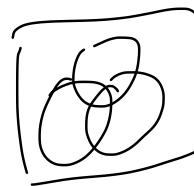
$$P(M \cap G) = \frac{42}{1000}$$

$$P(W) = \frac{470}{1000}$$

$$P(W \cap G) = \frac{147}{1000}$$

$$P(G) = \frac{525}{1000}$$

$$P(W \cap M) = \frac{86}{1000}$$



$$P(W \cap M \cap G) = \frac{25}{1000}$$

86

$$\frac{312}{1000} + \frac{470}{1000} + \frac{525}{1000} - \frac{147}{1000} - \frac{86}{1000} - \frac{25}{1000} = 1.062 > 1.000$$

X 86 !

$x \cup y \cup z = 1$ .

15. If it is assumed that all  $\binom{52}{5}$  poker hands are equally likely, what is the probability of being dealt
- a flush? (A hand is said to be a flush if all 5 cards are of the same suit.)
  - one pair? (This occurs when the cards have denominations  $a, a, b, c, d$ , where  $a, b, c$ , and  $d$  are all distinct.)
  - two pairs? (This occurs when the cards have denominations  $a, a, b, b, c$ , where  $a, b$ , and  $c$  are all distinct.)
  - three of a kind? (This occurs when the cards have denominations  $a, a, a, b, c$ , where  $a, b$ , and  $c$  are all distinct.)
  - four of a kind? (This occurs when the cards have denominations  $a, a, a, a, b$ .)

16. Poker dice is played by simultaneously rolling 5 dice. Show that

- $P\{\text{no two alike}\} = .0926$ ;
- $P\{\text{one pair}\} = .4630$ ;
- $P\{\text{two pair}\} = .2315$ ;
- $P\{\text{three alike}\} = .1543$ ;
- $P\{\text{full house}\} = .0386$ ;
- $P\{\text{four alike}\} = .0193$ ;
- $P\{\text{five alike}\} = .0008$ .

B

ABC -

$$(15^5), \quad \left(\frac{4}{1}\right) \left(\frac{13}{1}\right) + \left(\frac{5^2}{2}\right)$$

$$(15^5), \quad \left(\frac{4}{1}\right) \left(\frac{13}{1}\right)$$

c).

l).

l).

$$\frac{52 \times 48 \times 44 \times 40 \times 1^2}{A^5}$$

$$\frac{52 \times 48 \times 44}{A^3} \times \frac{9}{B^2} \times \frac{6}{C^2}$$

A B C

$$\frac{52 \times 48 \times 44 \times 3 \times 2}{A^5}$$

$$\frac{52 \times 48 \times 3 \times 2 \times 1}{A^5}$$

16. Poker dice is played by simultaneously rolling 5 dice. Show that

- (a)  $P\{\text{no two alike}\} = .0926$ ;
- (b)  $P\{\text{one pair}\} = .4630$ ;
- (c)  $P\{\text{two pair}\} = .2315$ ;
- (d)  $P\{\text{three alike}\} = .1543$ ;
- (e)  $P\{\text{full house}\} = .0386$ ;
- (f)  $P\{\text{four alike}\} = .0193$ ;
- (g)  $P\{\text{five alike}\} = .0008$ .

a). no two alike.

ABCDE.

$$\frac{6 \times 5 \times 4 \times 3 \times 2}{6^5} = 0.0926$$

b). one pair

$$\binom{6}{1} \binom{5}{3} \times 4$$

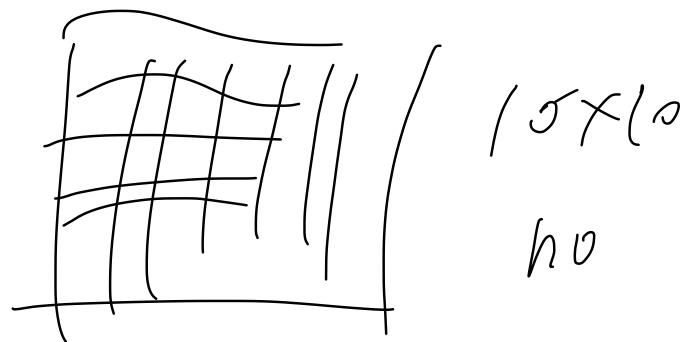
$$6 \times \cancel{(\cancel{5})} \times \cancel{(\cancel{4})} \times \cancel{(\cancel{3})}$$

17. If 8 rooks (castles) are randomly placed on a chessboard, compute the probability that none of the rooks can capture any of the others. That is,

compute the probability that no row or file contains more than one rook.

18. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?

17 -



$$15 \times 10$$

no

$$A_1 : (a_1, b_1)$$

$$A_2 : (a_2, b_2) \dots A_8 : (a_8, b_8)$$

$$\underbrace{\binom{8}{2} \times \binom{8}{2}}_{\cancel{8!} \times 8!} \times 8! \times 8!$$

1  
2  
3  
4  
5  
6  
7  
8

18. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?
19. Two symmetric dice have both had two of their sides painted red, two painted black, one painted yellow, and the other painted white. When this pair of dice is rolled, what is the probability that both dice land with the same color face up?

$$18. \quad 2 \times \frac{4}{52} \times \frac{16}{51} = 0.0241$$

$$19. \quad \begin{aligned} & \frac{1}{6} \times \frac{2}{6} + \\ & \frac{2}{6} \times \frac{2}{6} + \\ & \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} = 0.2778 \end{aligned}$$

20. Suppose that you are playing blackjack against a dealer. In a freshly shuffled deck, what is the probability that neither you nor the dealer is dealt a blackjack?

21. A small community organization consists of 20 families, of which 4 have one child, 8 have two children, 5 have three children, 2 have four children, and 1 has five children.

(a) If one of these families is chosen at random, what is the probability it has  $i$  children,  $i = 1, 2, 3, 4, 5$ ?

(b) If one of the children is randomly chosen, what is the probability that child comes from a family having  $i$  children,  $i = 1, 2, 3, 4, 5$ ?

22. Consider the following technique for shuffling a deck of  $n$  cards: For any initial ordering of the cards, go through the deck one card at a time and at each card, flip a fair coin. If the coin comes up heads, then leave the card where it is; if the coin comes up tails, then move that card to the end of the deck. After the coin has been flipped  $n$  times, say that one round has been completed. For instance, if  $n = 4$  and the initial ordering is 1, 2, 3, 4, then if the successive flips result in the outcome  $h, t, t, h$ , then the ordering at the end of the round is 1, 4, 2, 3. Assuming that all possible outcomes of the sequence of  $n$  coin flips are equally likely, what is the probability that the ordering after one round is the same as the initial ordering?

23. A pair of fair dice is rolled. What is the probability that the second die lands on a higher value than does the first?

24. If two dice are rolled, what is the probability that the sum of the upturned faces equals  $i$ ? Find it for  $i = 2, 3, \dots, 11, 12$ .

25. A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first.

*Hint:* Let  $E_n$  denote the event that a 5 occurs on the  $n$ th roll and no 5 or 7 occurs on the first  $n - 1$  rolls. Compute  $P(E_n)$  and argue that  $\sum_{n=1}^{\infty} P(E_n)$  is the desired probability.

26. The game of craps is played as follows: A player rolls two dice. If the sum of the dice is either a 2,

$$\frac{x AJ}{\binom{52}{2}} \times \frac{x AJ}{\binom{50}{2}}$$

$$20. P(A \cap B) = 1 - (P(A) + P(B) - P(A \cup B))$$

$$\left(1 - \frac{4 \times 16}{52 \times 51}\right) \times \left(1 - \cancel{\frac{4}{50} \times \frac{16}{49}}\right) \\ = 0.8975$$

$$21(a), T=1: \frac{4}{20}, \frac{8}{20}, \frac{5}{20}, \frac{2}{20}$$

(b). Total children >

$$4 + 16 + 15 + 8 + 5$$

, Pf.

$$\frac{4}{48}, \frac{16}{48}, \frac{15}{48}, \cancel{\frac{8}{48}}, \frac{5}{48}$$

22.  $(n-1)$  all head \*

$$n \text{ head / tail}) + ((n-1) \text{ all tail}).$$

$$\left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{2}\right)^{n-1} \cdot 2 \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n-2}$$

23. A pair of fair dice is rolled. What is the probability that the second die lands on a higher value than does the first?

24. If two dice are rolled, what is the probability that the sum of the upturned faces equals  $i$ ? Find it for  $i = 2, 3, \dots, 11, 12$ .

25. A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first.

*Hint:* Let  $E_n$  denote the event that a 5 occurs on the  $n$ th roll and no 5 or 7 occurs on the first  $n - 1$  rolls. Compute  $P(E_n)$  and argue that  $\sum_{n=1}^{\infty} P(E_n)$  is the desired probability.

26. The game of craps is played as follows: A player rolls two dice. If the sum of the dice is either a 2,

$$23. \quad \begin{array}{c} 12 \\ 13 \\ 14 \\ 15 \\ 16 \end{array} \quad \begin{array}{c} 23 \\ 24 \\ 25 \\ 26 \end{array} \quad \frac{5+4+3+2+1}{36} = \frac{15}{36}.$$

$$24. \quad \frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$$

$$25. \quad \sum_{n=1}^{\infty} \left( \frac{26}{36} \right)^{n-1} \times \frac{4}{36}$$

$$\frac{4}{36} \cdot \sum_{n=1}^{\infty} \left( \frac{26}{36} \right)^{n-1}$$

$$\frac{4}{36} \left( 1 + \frac{26}{36} + \left( \frac{26}{36} \right)^2 + \left( \frac{26}{36} \right)^3 + \dots \right)$$

$$= \frac{4}{36} \left( \frac{1}{1 - \frac{26}{36}} \right)$$

$$\approx 0.4$$

26. The game of craps is played as follows: A player rolls two dice. If the sum of the dice is either a 2, 3, or 12, the player loses; if the sum is either a 7 or an 11, the player wins. If the outcome is anything else, the player continues to roll the dice until she rolls either the initial outcome or a 7. If the 7 comes first, the player loses, whereas if the initial outcome reoccurs before the 7 appears, the player wins. Compute the probability of a player winning at craps.

*Hint:* Let  $E_i$  denote the event that the initial outcome is  $i$  and the player wins. The desired probability is  $\sum_{i=2}^{12} P(E_i)$ . To compute  $P(E_i)$ , define the events  $E_{i,n}$  to be the event that the initial sum is  $i$  and the player wins on the  $n$ th roll. Argue that

$$P(E_i) = \sum_{n=1}^{\infty} P(E_{i,n}).$$

$$\sum_{i=2}^{12} \sum_{n=1}^{\infty} P(E_{i,n}) = \\ \left( \frac{3}{36} + \frac{4}{36} + \frac{5}{36} \right) \times 2 \\ \sim 0.6667.$$

For  $i=2, 3, 12$ ,  $P(E_{i,n}) = 0$ .  
 $4, 5, 6, 8, 9, 10, 11$ ,  $\boxed{7, 11} \therefore 1$ .

$$n=4. \quad \frac{3}{36}$$

$$n=2 \quad \frac{3}{36} \times \frac{8}{36}$$

$$n=3 \quad \frac{3}{36} \times \frac{28}{36} \sim \frac{8}{36}$$

$$n=4 \quad \frac{3}{36} \times \left(\frac{28}{36}\right)^2 \times \frac{8}{36}$$

...

$$\frac{\frac{3}{36} \times \frac{8}{36}}{1 - \frac{28}{36}} = \frac{1}{12}.$$

$$n=5. \quad \frac{\frac{4}{36} \times \frac{8}{36}}{1 - \frac{28}{36}}$$

$n=1$

7. An urn contains 3 red and 7 black balls. Players A and B withdraw balls from the urn consecutively until a red ball is selected. Find the probability that A selects the red ball. (A draws the first ball, then B, and so on. There is no replacement of the balls drawn.)

8. An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be (a) of the same color? (b) of different colors? Repeat under the assumption that whenever a ball is selected, its color is noted and it is then replaced in the urn before the next selection. This is known as *sampling with replacement*.

9. An urn contains  $n$  white and  $m$  black balls, where  $n$  and  $m$  are positive numbers.

- (a) If two balls are randomly withdrawn, what is the probability that they are the same color?
- (b) If a ball is randomly withdrawn and then replaced before the second one is drawn, what is the probability that the withdrawn balls are the same color?
- (c) Show that the probability in part (b) is always larger than the one in part (a).

10. The chess clubs of two schools consist of, respectively, 8 and 9 players. Four members from each club are randomly chosen to participate in a contest between the two schools. The chosen players from one team are then randomly paired with those from the other team, and each pairing plays a game of chess. Suppose that Rebecca and her sister Elise are on the chess clubs at different schools. What is the probability that

- (a) Rebecca and Elise will be paired?
- (b) Rebecca and Elise will be chosen to represent their schools but will not play each other?
- (c) either Rebecca or Elise will be chosen to represent her school?

$$a). \left(\frac{1}{19}\right)^3 + \left(\frac{6}{19}\right)^3 + \left(\frac{8}{19}\right)^3$$

$$b). \frac{5}{19} \times \frac{6}{19} \times \frac{8}{19} \times 3! \approx 0.2099$$

$$a) - \frac{\binom{n}{2}}{\binom{n+m}{2}} + \frac{\binom{m}{2}}{\binom{n+m}{2}} = \frac{\frac{n(n-1)}{2} + \frac{m(m-1)}{2}}{(n+m)(n+m-1)}$$

$$b) . \frac{n}{n+m} + \frac{m}{n+m} = 1$$

$$P. \quad \frac{3}{10} + \frac{7}{10} \times \frac{6}{9} \times \frac{3}{8} + \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6}$$

$$T \quad \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3} +$$

$$\frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} \times \frac{1}{4} \times \frac{3}{3}$$

$$= 0.5917$$

30. The chess clubs of two schools consist of, respectively, 8 and 9 players. Four members from each club are randomly chosen to participate in a contest between the two schools. The chosen players from one team are then randomly paired with those from the other team, and each pairing plays a game of chess. Suppose that Rebecca and her sister Elise are on the chess clubs at different schools.

What is the probability that

- (a) Rebecca and Elise will be paired?
- (b) Rebecca and Elise will be chosen to represent their schools but will not play each other?
- (c) either Rebecca or Elise will be chosen to represent her school?

30(a).



$$\frac{4}{8} \times \frac{4}{9}$$

$$P(E \cap R \text{ paired}) =$$

$$P(E \text{ chosen}) \times P(R \text{ paired} | E \text{ chosen})$$

$$P(E \cap R \text{ paired}) = \frac{4}{8} \times \frac{4}{9} \times \frac{1}{4} \times \frac{1}{4} = 0.0133$$

$$\frac{4}{8} \times \frac{4}{9} \times \frac{3}{4} \times \frac{3}{4} = 0.125$$

$$\frac{4}{8} \times \frac{5}{9} + \frac{4}{8} \times \frac{4}{9} = 0.5$$

31. A 3-person basketball team consists of a guard, a forward, and a center.
- (a) If a person is chosen at random from each of three different such teams, what is the probability of selecting a complete team?
- (b) What is the probability that all 3 players selected play the same position?
32. A group of individuals containing  $b$  boys and  $g$  girls is lined up in random order; that is, each of the  $(b+g)!$  permutations is assumed to be equally likely. What is the probability that the person in the  $i$ th position,  $1 \leq i \leq b+g$ , is a girl?
33. A forest contains 20 elk, of which 5 are captured, tagged, and then released. A certain time later, 4 of the 20 elk are captured. What is the probability that 2 of these 4 have been tagged? What assumptions are you making?
34. The second Earl of Yarborough is reported to have bet at odds of 1000 to 1 that a bridge hand of 13 cards would contain at least one card that is ten or higher. (By *ten or higher* we mean that a card is either a ten, a jack, a queen, a king, or an ace.) Nowadays, we call a hand that has no cards higher than 9 a *Yarborough*. What is the probability that a randomly selected bridge hand is a Yarborough?
35. Seven balls are randomly withdrawn from an urn that contains 12 red, 16 blue, and 18 green balls. Find the probability that
- (a) 3 red, 2 blue, and 2 green balls are withdrawn;
- (b) at least 2 red balls are withdrawn;
- (c) all withdrawn balls are the same color;
- (d) either exactly 3 red balls or exactly 3 blue balls are withdrawn.
36. Two cards are chosen at random from a deck of 52 playing cards. What is the probability that they
- (a) are both aces?
- (b) have the same value?
37. An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them. If a student has figured out how to do 7 of the problems, what is the probability that he or she will answer correctly
- (a) all 5 problems?
- (b) at least 4 of the problems?

38. There are  $n$  socks, 3 of which are red, in a drawer. What is the value of  $n$  if, when 2 of the socks are chosen randomly, the probability that they are both red is  $\frac{1}{2}$ ?
39. There are 5 hotels in a certain town. If 3 people check into hotels in a day, what is the probability that they each check into a different hotel? What assumptions are you making?
40. A town contains 4 people who repair televisions. If 4 sets break down, what is the probability that

$$a). GFC \\ \left(\frac{1}{3}\right)^3 \times 1! \\ b). GFG + FGF + GGF = 3 \times \left(\frac{1}{3}\right)^3$$

$$32. \frac{g}{b+g}, \quad \textcircled{B}.$$

$$33. P(\text{capture}) \\ = P(\text{capture} | \text{tagged})P(\text{tagged}) \\ = \binom{4}{2} \times \frac{4}{20} \times \frac{3}{19} \times \frac{15}{18} \times \frac{14}{17}$$

34. ??

$$35a). \binom{12}{3} \binom{16}{2} \binom{18}{2} \\ \binom{12+16+18}{7}$$

$$b). 32 \\ 1 - \text{C2- (or 1).} \\ 1 - \frac{\binom{16+18}{7}}{\binom{12+16+18}{7}} = \frac{\binom{12}{1} \binom{16+18}{6}}{\binom{12+16+18}{7}}$$

$$37. \frac{\binom{12}{2}}{\binom{12+16+18}{7}} + \frac{\binom{16}{2}}{\binom{12+16+18}{7}} + \frac{\binom{18}{2}}{\binom{12+16+18}{7}}$$

$$38. \frac{\binom{2}{3} \binom{16+18}{4}}{\binom{12+16+18}{7}} + \frac{\binom{16}{3} \binom{12+18}{4}}{\binom{12+16+18}{7}} - \frac{\binom{12}{3} \binom{16}{3} \binom{18}{1}}{\binom{12+16+18}{7}}$$

$$36. a). \frac{\binom{4}{2}}{\binom{5+2}{2}} \quad b). \frac{\binom{4}{2} \times 13}{\binom{5+2}{2}}$$

$$37a). \frac{\binom{7}{3}}{\binom{10}{5}} \quad b). \frac{\binom{7}{4} \binom{3}{1}}{\binom{10}{5}} + \frac{\binom{7}{3}}{\binom{10}{5}}$$

$$38. \frac{\cancel{4} \times 3}{\cancel{5}^3}$$

$$36. \frac{\binom{3}{2}}{\binom{n}{2}} = \frac{1}{4}$$

$$\binom{3}{2} = \frac{1}{2} \frac{n(n-1)}{2}$$

$$6 = \frac{n(n-1)}{2}$$

$$12 = n^2 - n$$

$$0 = n^2 - n - 12$$

$$0 = (n+4)(n-3)$$

$$n = 3,$$

40. A town contains 4 people who repair televisions. If 4 sets break down, what is the probability that exactly  $i$  of the repairers are called? Solve the problem for  $i = 1, 2, 3, 4$ . What assumptions are you making?

$$121: \left(\frac{1}{4}\right)^4, i=2$$

AAA  
 AABBB  
 AB  
 CCC  
 BBBB  
 CCC  
 BBBB  
 CCC  
 BBBB

AAA  
 AABBB  
 AB  
 CCC

$$46. 1 - \frac{n!}{(n-k)!} = \frac{1}{2}$$

All fall to have same.

41. If a die is rolled 4 times, what is the probability that 6 comes up at least once?

42. Two dice are thrown  $n$  times in succession. Compute the probability that double 6 appears at least once. How large need  $n$  be to make this probability at least  $\frac{1}{2}$ ?

43. (a) If  $N$  people, including  $A$  and  $B$ , are randomly arranged in a line, what is the probability that  $A$  and  $B$  are next to each other?

- (b) What would the probability be if the people were randomly arranged in a circle?

44. Five people, designated as  $A, B, C, D, E$ , are arranged in linear order. Assuming that each possible order is equally likely, what is the probability that

- (a) there is exactly one person between  $A$  and  $B$ ?  
 (b) there are exactly two people between  $A$  and  $B$ ?  
 (c) there are three people between  $A$  and  $B$ ?

45. A woman has  $n$  keys, of which one will open her door.

- (a) If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her  $k$ th try?  
 (b) What if she does not discard previously tried keys?

46. How many people have to be in a room in order that the probability that at least two of them celebrate their birthday in the same month is at least  $\frac{1}{2}$ ? Assume that all possible monthly outcomes are equally likely.

47. If there are 12 strangers in a room, what is the probability that no two of them celebrate their birthday in the same month?

48. Given 20 people, what is the probability that, among the 12 months in the year, there are 4 months containing exactly 2 birthdays and 4 containing exactly 3 birthdays?

49. A group of 6 men and 6 women is randomly divided into 2 groups of size 6 each. What is the probability that both groups will have the same number of men?

50. In a hand of bridge, find the probability that you have 5 spades and your partner has the remaining 8.

51. Suppose that  $n$  balls are randomly distributed into  $N$  compartments. Find the probability that  $m$  balls will fall into the first compartment. Assume that all  $N^n$  arrangements are equally likely.

$$6 \quad 4 \quad 3$$

$n \sim 1 \quad n \sim 2 \quad \dots \quad k_1 \sim k \quad \dots$   
 $n-k_1 \quad n-k_2 \quad \dots$

41.  $\frac{1}{1}$

$1 - \underline{\underline{0}}$

47.  $\frac{1}{12}$

$$1 - \left(\frac{5}{6}\right)^4$$

$$1 - \left(\frac{25}{36}\right)^n \geq \frac{1}{2}$$

$$\left(\frac{25}{36}\right)^n \leq \frac{1}{2}$$

$$n \log\left(\frac{25}{36}\right) \leq \log\frac{1}{2}$$

$$n \geq \frac{\log 0.5}{\log\left(\frac{25}{36}\right)}$$

$$n \geq 24.605$$

$$n = 25.$$

43.  $\frac{2! \times (n-1)!}{n!} = \frac{2}{n}$

(a)  $\frac{2! \times (n-v)!}{(n-1)!} = \frac{2}{n-1}$

44(a),  $\frac{3! \times 2 \times 3}{5!}$

$A \times B -$   
 $- A \times B -$   
 $- A \times B$

b7,  $\frac{3! \times 2! \times 2 \times 2}{5!}$

$- A \times B -$

c).  $\frac{2 \times 3!}{5!}$

45. a),  $\frac{n-1}{n} \times \frac{n-2}{n-1} \times \dots \times \frac{n-k+1}{n-k+2} \times \frac{1}{n-k+1} = \frac{1}{n}$

b),  $\binom{n}{k}$

49. A group of 6 men and 6 women is randomly divided into 2 groups of size 6 each. What is the probability that both groups will have the same number of men?

50. In a hand of bridge, find the probability that you have 5 spades and your partner has the remaining 8.

51. Suppose that  $n$  balls are randomly distributed into  $N$  compartments. Find the probability that  $m$  balls will fall into the first compartment. Assume that all  $N^n$  arrangements are equally likely.

$$49. \quad \frac{3}{12} \times \frac{3}{6}$$

$n - m$

$m$

$n - m - 1$   $\cancel{\text{E}}$ .

51.

52. A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be

- (a) no complete pair?  
 (b) exactly 1 complete pair?

53. If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.

54. Compute the probability that a bridge hand is void in at least one suit. Note that the answer is not
52. a)
- $\frac{\binom{10}{8}}{\binom{20}{8}}$
- 
- b)
- $\frac{\binom{9}{6}}{\binom{20}{6}}$

$$\frac{\binom{4}{1} \binom{39}{13}}{\binom{52}{13}}$$

53.

there will be

- (a) no complete pair?
- (b) exactly 1 complete pair?

53. If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.

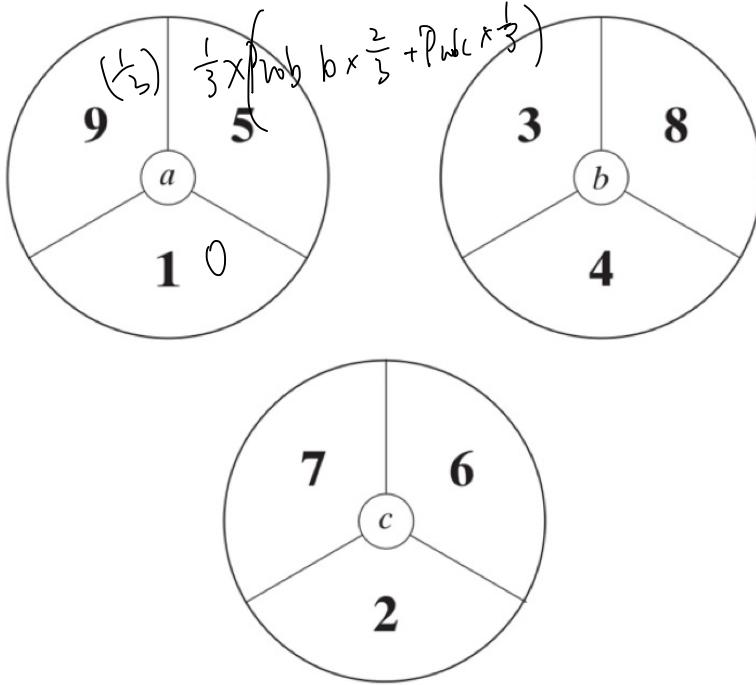
54. Compute the probability that a bridge hand is void in at least one suit. Note that the answer is not

$$\frac{\binom{4}{1} \binom{39}{13}}{\binom{52}{13}}$$

55. Compute the probability that a hand of 13 cards contains

- (a) the ace and king of at least one suit;
- (b) all 4 of at least 1 of the 13 denominations.

56. Two players play the following game: Player *A* chooses one of the three spinners pictured in Figure 2.6, and then player *B* chooses one of the remaining two spinners. Both players then spin their spinner, and the one that lands on the higher number is declared the winner. Assuming that each spinner is equally likely to land in any of its 3 regions, would you rather be player *A* or player *B*? Explain your answer!



55. ??

56. -

**THEORETICAL EXERCISES**

---

Prove the following relations:

1.  $EF \subset E \subset E \cup F.$  ✓

2. If  $E \subset F$ , then  $F^c \subset E^c.$  ✓

3.  $F = FE \cup FE^c$  and  $E \cup F = E \cup E^c F.$

4.  $\left( \bigcup_{i=1}^{\infty} E_i \right) F = \bigcup_{i=1}^{\infty} E_i F$  and  
 $\left( \bigcap_{i=1}^{\infty} E_i \right) \cup F = \bigcap_{i=1}^{\infty} (E_i \cup F).$



?  $\bigcup_{i=1}^{\infty} E_i \cap F = F \left( \sum_{i=1}^{\infty} E_i \right) = \sum$

$E_1 \cap F \cup E_2 \cap F \cup E_3 \cap F \dots$

$(E_1 \cup F) \cap (E_2 \cup F) \cap (E_3 \cup F) \dots$

5. For any sequence of events  $E_1, E_2, \dots$ , define a new sequence  $F_1, F_2, \dots$  of disjoint events (that is, events such that  $F_i F_j = \emptyset$  whenever  $i \neq j$ ) such that for all  $n \geq 1$ ,

$$\bigcup_1^n F_i = \bigcup_1^n E_i = S - \bigcup_1^n E_i^c$$

5.

$= F_1$

$$= F_2$$

6. Let  $E, F$ , and  $G$  be three events. Find expressions for the events so that, of  $E, F$ , and  $G$ ,

- (a) only  $E$  occurs;
- (b) both  $E$  and  $G$ , but not  $F$ , occur;
- (c) at least one of the events occurs;
- (d) at least two of the events occur;
- (e) all three events occur;
- (f) none of the events occurs;
- (g) at most one of the events occurs;
- (h) at most two of the events occur;
- (i) exactly two of the events occur;
- (j) at most three of the events occur.

$\downarrow$

$\vdots$

$$= F_n$$

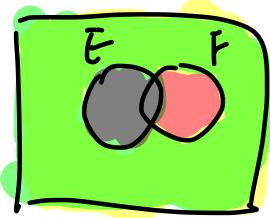
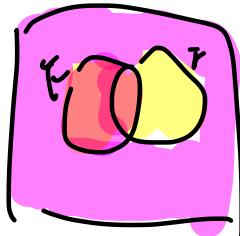
6.  $E, E \cap G \cap F^c$ , (c)?, (d)?, e.  $E \cap F \cap G$

f).  $E^c \cap F^c \cap G^c$  (g). ?? h) ?? (i). ?? (j) ??

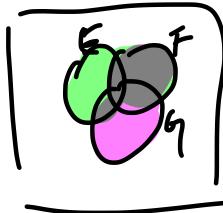
7. Find the simplest expression for the following events:

- (a)  $(E \cup F)(E \cup F^c)$ ;  $E$
- (b)  $(E \cup F)(E^c \cup F)(E \cup F^c)$ ;  $F$
- (c)  $(E \cup F)(F \cup G)$ .

b)-



c).



$$F \cup (E \cap G)$$

8. Let  $S$  be a given set. If, for some  $k > 0$ ,  $S_1, S_2, \dots, S_k$  are mutually exclusive nonempty subsets of  $S$  such that  $\bigcup_{i=1}^k S_i = S$ , then we call the set  $\{S_1, S_2, \dots, S_k\}$  a *partition* of  $S$ . Let  $T_n$  denote the number of different partitions of  $\{1, 2, \dots, n\}$ . Thus,  $T_1 = 1$  (the only partition being  $S_1 = \{1\}$ ) and  $T_2 = 2$  (the two partitions being  $\{\{1, 2\}\}, \{\{1\}, \{2\}\}$ ).

- (a) Show, by computing all partitions, that  $T_3 = 5$ ,  $T_4 = 15$ .
- (b) Show that

$$T_{n+1} = 1 + \sum_{k=1}^n \binom{n}{k} T_k$$

and use this equation to compute  $T_{10}$ .

*Hint:* One way of choosing a partition of  $n + 1$  items is to call one of the items *special*. Then we obtain different partitions by first choosing  $k$ ,  $k = 0, 1, \dots, n$ , then a subset of size  $n - k$  of the non-special items, and then any of the  $T_k$  partitions of the remaining  $k$  nonspecial items. By adding the special item to the subset of size  $n - k$ , we obtain a partition of all  $n + 1$  items.

(a).  $\left\{ \left\{ \{1, 2, 3\} \right\}, \left\{ \{1, 2\}, \{3\} \right\}, \left\{ \{1\}, \{2, 3\} \right\}, \left\{ \{1\}, \{2\}, \{3\} \right\}, \left\{ \{1, 3\}, \{2\} \right\} \right\}$

$$\tilde{T}_3 = 5$$

|                  |              |
|------------------|--------------|
| $\{2, 3, 4\}$    | $1, 2, 3, 4$ |
| $\{1, 2, 3\}, 4$ | $1, 2, 3, 4$ |
| $\{1, 2, 3, 4\}$ | $1, 3, 4, 2$ |
| $\{1, 2, 3\}, 4$ | $1, 4, 2, 3$ |
| $\{1, 2, 4\}, 3$ | $2, 1, 4, 3$ |
| $\{1, 3, 4\}, 2$ | $2, 1, 3, 4$ |
| $\{1, 2, 3\}, 4$ | $3, 1, 2, 4$ |
| $\{1, 2, 4\}, 3$ |              |

b).

Suppose that an experiment is performed  $n$  times.  
 For any event  $E$  of the sample space, let  $n(E)$  denote the number of times that event  $E$  occurs and define  $f(E) = n(E)/n$ . Show that  $f(\cdot)$  satisfies Axioms 1, 2, and 3.

probit ... .

7 7

$$0 \leq \frac{n(E)}{n} \leq 1$$

1. all occur =  $n$   
 no occur = 0.

2  $\frac{n(S)}{n} = 1$   
 all occur.

3.  $\frac{n(\bigcup_{i=1}^{\infty} E_i)}{n} = \sum_{i=1}^{\infty} \frac{n(E_i)}{n}$

$$n\left(\bigcup_{i=1}^{\infty} E_i\right) = n(E_1) + n(E_2) + \dots = \sum_{i=1}^{\infty} n(E_i)$$

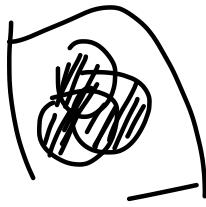
$$P(E) + P(F) - P(EF) \leq 1$$

11.  $P(E \cup F) = P(E) + P(F) - P(EF)$   $P(E) + P(F) - P(EF) \leq 1$

$$P(E \cup F) + P(FF) = 0.9 + 0.8$$

$$P(E \cup F) + P(FF) = 1.7$$

$$0 \leq P(E \cup F) \leq 1, P(FF) \geq 0 \}$$



10.

???

$$P(EF) \geq P(E) + P(F) - 1$$

12. Show that the probability that exactly one of the events  $E$  or  $F$  occurs equals  $P(E) + P(F) - 2P(EF)$ .

13. Prove that  $P(EF^c) = P(E) - P(EF)$ .

14. Prove Proposition 4.4 by mathematical induction.

15. An urn contains  $M$  white and  $N$  black balls. If a random sample of size  $r$  is chosen, what is the probability that it contains exactly  $k$  white balls?

12. Show that the probability that exactly one of the events  $E$  or  $F$  occurs equals  $P(E) + P(F) - 2P(EF)$ .

13. Prove that  $P(EF^c) = P(E) - P(EF)$ .

14. Prove Proposition 4.4 by mathematical induction.

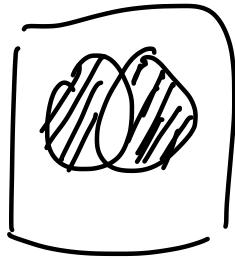
??

15. An urn contains  $M$  white and  $N$  black balls. If a random sample of size  $r$  is chosen, what is the probability that it contains exactly  $k$  white balls?

16. Use induction to generalize Bonferroni's inequality to  $n$  events. That is, show that

$$P(E_1 E_2 \dots E_n) \geq P(E_1) + \dots + P(E_n) - (n-1)$$

12.



Exactly 1 occur =

$$(P(E) - P(E \cap F)) + (P(F) - P(E \cap F))$$

= ...

13. graph.

14. ?

15.



$$\frac{\binom{M}{k} \binom{N}{r-k}}{\binom{M+N}{r}}$$

6. Use induction to generalize Bonferroni's inequality to  $n$  events. That is, show that

$$P(E_1 E_2 \dots E_n) \geq P(E_1) + \dots + P(E_n) - (n-1)$$

7. Consider the matching problem, Example 5m, and define  $A_N$  to be the number of ways in which the  $N$  men can select their hats so that no man selects his own. Argue that

$$A_N = (N-1)(A_{N-1} + A_{N-2})$$

This formula, along with the boundary conditions  $A_1 = 0, A_2 = 1$ , can then be solved for  $A_N$ , and the desired probability of no matches would be  $A_N/N!$ .

*Hint:* After the first man selects a hat that is not his own, there remain  $N-1$  men to select among a set of  $N-1$  hats that does not contain the hat of one of these men. Thus, there is one extra man and one extra hat. Argue that we can get no matches either with the extra man selecting the extra hat or with the extra man not selecting the extra hat.

$$16. \quad P(E_1) \geq P(E_1) - (1-1) \text{ True.}$$

$$\begin{aligned} P(E_1 E_2 \dots E_{n-1}) &\geq P(E_1) + \dots + P(E_{n-1}) - (n-2) \text{ True.} \\ P(\bar{E}_1 \bar{E}_2 \dots \bar{E}_n) &\geq P(E_1) + \dots + P(E_{n-1}) + P(E_n) - P(E_1 \bar{E}_2 \bar{E}_3 \dots \bar{E}_{n-1} \bar{E}_n) \\ &\quad - (n-1) \end{aligned}$$

$$\text{where } P(\bar{E}_1 \bar{E}_2 \dots \bar{E}_n) \leq 1$$

$$\geq P(E_1) \text{ proved.}$$

7. Consider the matching problem, Example 5m, and define  $A_N$  to be the number of ways in which the  $N$  men can select their hats so that no man selects his own. Argue that

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*Hint:* After the first man selects a hat that is not his own, there remain  $N - 1$  men to select among a set of  $N - 1$  hats that does not contain the hat of one of these men. Thus, there is one extra man and one extra hat. Argue that we can get no matches either with the extra man selecting the extra hat or with the extra man not selecting the extra hat.

?

Remain  $n-1$  men  
select  $n-1$  hats

$$\binom{N-1}{1}$$

18. Let  $f_n$  denote the number of ways of tossing a coin  $n$  times such that successive heads never appear. Argue that

$$f_n = f_{n-1} + f_{n-2} \quad n \geq 2, \text{ where } f_0 = 1, f_1 = 2$$

*Hint:* How many outcomes are there that start with a head, and how many start with a tail? If  $P_n$  denotes the probability that successive heads never appear when a coin is tossed  $n$  times, find  $P_n$  (in terms of  $f_n$ ) when all possible outcomes of the  $n$  tosses are assumed equally likely. Compute  $P_{10}$ .

|   |    |     |
|---|----|-----|
| H | HT | HTH |
| T | TH | HTT |
|   | TT | THT |

Start with head =  $H \frac{2^{n-1}}{2^{n-1}}$

1, 1, 3, 5, ...

start with head,  $2^{n-1} +$  start with tail

$1 \times 2^{n-2} \times \dots$   
???,

19. An urn contains  $n$  red and  $m$  blue balls. They are withdrawn one at a time until a total of  $r$ ,  $r \leq n$ ,

red balls have been withdrawn. Find the probability that a total of  $k$  balls are withdrawn.

*Hint:* A total of  $k$  balls will be withdrawn if there are  $r - 1$  red balls in the first  $k - 1$  withdrawals and the  $k$ th withdrawal is a red ball.

20. Consider an experiment whose sample space consists of a countably infinite number of points. Show that not all points can be equally likely. Can all points have a positive probability of occurring?

- \*21. Consider Example 5o, which is concerned with the number of runs of wins obtained when  $n$  wins and  $m$  losses are randomly permuted. Now

$$19. P(k \text{ balls withdrawn})$$

$$= P(\text{kth withdrawal} = \text{red})$$

$$P(R_k) = \frac{n-r+1}{n+m-r} \times \binom{n}{r-1}$$

$\cap$  r-1 red in k-1 withdrawal

$$\binom{n}{k-1}$$

20. if all points equally likely

$$\text{if } P(E_1) = P(E_2) = \dots = P(E_n) \leq 1$$

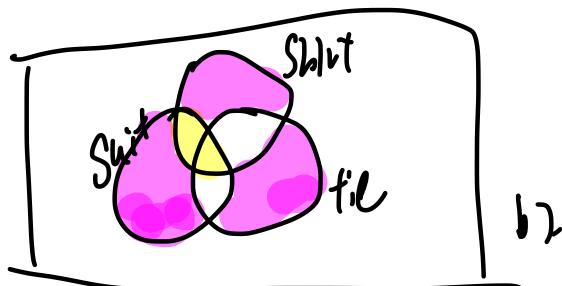
$$\text{then } P(E_1) + P(E_2) + \dots + P(E_n) = np \leq 1$$

1. A cafeteria offers a three-course meal consisting of an entree, a starch, and a dessert. The possible choices are given in the following table:

| Course  | Choices                                    |
|---------|--|
| Entree  | Chicken or roast beef                      |
| Starch  | Pasta or rice or potatoes                  |
| Dessert | Ice cream or Jello or apple pie or a peach |

A person is to choose one course from each category.

- (a) How many outcomes are in the sample space?
  - (b) Let  $A$  be the event that ice cream is chosen. How many outcomes are in  $A$ ?
  - (c) Let  $B$  be the event that chicken is chosen. How many outcomes are in  $B$ ?
  - (d) List all the outcomes in the event  $AB$ .
  - (e) Let  $C$  be the event that rice is chosen. How many outcomes are in  $C$ ?
  - (f) List all the outcomes in the event  $ABC$ .
2. A customer visiting the suit department of a certain store will purchase a suit with probability .22, a shirt with probability .30, and a tie with probability .28. The customer will purchase both a suit and a shirt with probability .11, both a suit and a tie with probability .14, and both a shirt and a tie with probability .10. A customer will purchase all 3 items with probability .06. What is the probability that a customer purchases
- (a) none of these items?
  - (b) exactly 1 of these items?
3. A deck of cards is dealt out. What is the probability that the 14th card dealt is an ace? What is the probability that the first ace occurs on the 14th card?



$$0.51 - \left( \text{shirt} \cap \text{tie} + \text{suit} \cap \text{shirt} + \text{suit} \cap \text{tie} - 2(\text{shirt} \cap \text{suit} \cap \text{tie}) \right)$$

$$\approx 0.51 - (0.11 + 0.14 + 0.1 - 2 \times 0.06) = 0.28$$

$$2 \times 3 \times 4$$

$$16) \quad 2 \times 3$$

$$1c) \quad 3 \times 4$$

$$d) \quad CPI, CRPI, CPPI$$

$$e) \quad 2 \times 4$$

$$f) \quad CPI$$

$$2 a). \text{none} =$$

$$1 - (0.22 + 0.3 + 0.28 - 0.11 - 0.14 - 0.1 + 0.06) \\ \leq 1 - 0.51 = 0.49$$

3. A deck of cards is dealt out. What is the probability that the 14th card dealt is an ace? What is the probability that the first ace occurs on the 14th card?

-Mr7.

前有 0 张

有 1 张

有 2 张

有 3 张

$$\left[ \frac{\binom{4}{1} \times \frac{4}{52-13}}{\binom{48}{13}} \right] +$$

$$\frac{\binom{4}{1} \binom{48}{12} \times \frac{1}{52-13}}{\binom{52}{13}} +$$

$$\frac{\binom{4}{2} \binom{48}{11} \times \frac{1}{52-13}}{\binom{52}{13}} +$$

$$\frac{\binom{4}{3} \binom{48}{10} \times \frac{1}{52-13}}{\binom{52}{13}}$$

4. Let  $A$  denote the event that the midtown temperature in Los Angeles is  $70^{\circ}\text{F}$ , and let  $B$  denote the event that the midtown temperature in New York is  $70^{\circ}\text{F}$ . Also, let  $C$  denote the event that the maximum of the midtown temperatures in New York and in Los Angeles is  $70^{\circ}\text{F}$ . If  $P(A) = .3$ ,  $P(B) = .4$ , and  $P(C) = .2$ , find the probability that the minimum of the two midtown temperatures is  $70^{\circ}\text{F}$ .

5. An ordinary deck of 52 cards is shuffled. What is the probability that the top four cards have  
 (a) different denominations?  
 (b) different suits?

6. Urn  $A$  contains 3 red and 3 black balls, whereas urn  $B$  contains 4 red and 6 black balls. If a ball is randomly selected from each urn, what is the probability that the balls will be the same color?

7. In a state lottery, a player must choose 8 of the numbers from 1 to 40. The lottery commission then performs an experiment that selects 8 of these 40 numbers. Assuming that the choice of the lottery commission is equally likely to be any of the  $\binom{40}{8}$  combinations, what is the probability that a player has

- (a) all 8 of the numbers selected by the lottery commission?  
 (b) 7 of the numbers selected by the lottery commission?  
 (c) at least 6 of the numbers selected by the lottery commission?

8. From a group of 3 freshmen, 4 sophomores, 4 juniors, and 3 seniors a committee of size 4 is randomly selected. Find the probability that the committee will consist of  
 (a) 1 from each class;  
 (b) 2 sophomores and 2 juniors;  
 (c) only sophomores or juniors.

4. Max =

$$P(\leq A)P(\leq B) + P(A)P(\leq B) = 0.2$$

$$0.4 P(\leq A) + 0.3 P(\leq B) = 0.2$$

$$\min: \begin{cases} 1 - P(\geq A) \\ 1 - P(\geq B) \end{cases} = 0.5$$

$$P(B) P(\geq A) + P(A) P(\geq B)$$

$$0.2 = 0.4 - 0.4 P(\geq A) + 0.3 - 0.3 P(\geq B)$$

$$0.4 P(\geq A) = 0.5$$

$$+ 0.3 P(\geq B)$$

$$0.4 P(\geq A) + 0.3 P(\geq B)$$

$$\approx 0.5$$

5. An ordinary deck of 52 cards is shuffled. What is the probability that the top four cards have  
 (a) different denominations?  $\frac{5}{4}$   
 (b) different suits?  $\frac{1}{4}$
6. Urn A contains 3 red and 3 black balls, whereas urn B contains 4 red and 6 black balls. If a ball is randomly selected from each urn, what is the probability that the balls will be the same color?

5a).

$$\frac{12}{52} \times \frac{31}{51} \times \frac{26}{50} \times \frac{13}{49}$$

b).  $\frac{3}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49}$  ???

1.  $\frac{3}{6} \times \frac{4}{10} + \frac{3}{6} \times \frac{6}{10}$

7. In a state lottery, a player must choose 8 of the numbers from 1 to 40. The lottery commission then performs an experiment that selects 8 of these 40 numbers. Assuming that the choice of the lottery commission is equally likely to be any of the  $\binom{40}{8}$  combinations, what is the probability that a player has

- (a) all 8 of the numbers selected by the lottery commission?
- (b) 7 of the numbers selected by the lottery commission?
- (c) at least 6 of the numbers selected by the lottery commission?

8. From a group of 3 freshmen, 4 sophomores, 4 juniors, and 3 seniors a committee of size 4 is randomly selected. Find the probability that the committee will consist of

- (a) 1 from each class;
- (b) 2 sophomores and 2 juniors;
- (c) only sophomores or juniors.

7a).

$$\frac{\binom{12}{8}}{\binom{32}{8}}, \quad \frac{\binom{8}{1} \binom{32}{1}}{\binom{40}{8}},$$

$$\frac{\binom{8}{6} \binom{32}{2}}{\binom{40}{8}} + \text{previous}$$

8a).

$$\frac{\binom{3}{1} \binom{4}{1} \binom{4}{1} \binom{3}{1}}{\binom{14}{4}}$$

$$\frac{\binom{4}{2} \binom{4}{2}}{\binom{14}{4}}$$

4 -

$$\frac{2}{\binom{14}{4}}$$

9. For a finite set  $A$ , let  $N(A)$  denote the number of elements in  $A$ .

(a) Show that

$$N(A \cup B) = N(A) + N(B) - N(AB)$$

(b) More generally, show that

$$\begin{aligned} N\left(\bigcup_{i=1}^n A_i\right) &= \sum_i N(A_i) - \sum_{i < j} \sum N(A_i A_j) \\ &\quad + \cdots + (-1)^{n+1} N(A_1 \cdots A_n) \end{aligned}$$

10. Consider an experiment that consists of six horses, numbered 1 through 6, running a race, and suppose that the sample space consists of the  $6!$  possible orders in which the horses finish. Let  $A$  be the event that the number-1 horse is among the top three finishers, and let  $B$  be the event that the number-2 horse comes in second. How many outcomes are in the event  $A \cup B$ ?

1a) .. divide n,  $\lim n \rightarrow \infty$ , pwb it axiom  
b). same

[0.]  $A + B - AB$   
 $\nearrow \uparrow \uparrow$   
 $5! \times 3 \quad 5! \quad 4! \times 2$

11. A 5-card hand is dealt from a well-shuffled deck of 52 playing cards. What is the probability that the hand contains at least one card from each of the four suits?

12. A basketball team consists of 6 frontcourt and 4 backcourt players. If players are divided into roommates at random, what is the probability that there will be exactly two roommate pairs made up of a backcourt and a frontcourt player?

13. Suppose that a person chooses a letter at random from R E S E R V E and then chooses one at random from V E R T I C A L. What is the probability that the same letter is chosen?

14. Prove Boole's inequality:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

11.

$$\frac{13^4 \times 4!}{\binom{52}{5}}$$

12.

|     |                                    |
|-----|------------------------------------|
| B F | $\binom{4}{1} \binom{6}{1} \times$ |
| B F | $\binom{3}{1} \binom{5}{1} \times$ |
| B B | $\binom{4}{1} \binom{3}{1} \times$ |
| B B | $\binom{2}{1} \binom{1}{1} \times$ |
| F F | $\binom{2}{1} \binom{1}{1}$        |

(OK)

$$\binom{10}{2} \binom{8}{2} \binom{6}{2}$$

$$\binom{4}{2} \binom{2}{2}$$

$$\approx 0.1524.$$

13. Suppose that a person chooses a letter at random from R, E, S, E, R, V, E and then chooses one at random from V, E, R, T, I, C, A, L. What is the probability that the same letter is chosen?

$$R: \frac{2}{7} \times \frac{1}{8} = 0.10 + 1$$

$$E: \frac{3}{7} \times \frac{1}{8}$$

$$V: \frac{1}{7} \times \frac{1}{8}$$

14. Prove Boole's inequality:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i) - \sum_{1 \leq i_1 < i_2 \leq n} P(A_{i_1} A_{i_2}) + (-1)^{r+1}$$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) - \sum_{1 \leq i_1 < i_2 \leq \infty} P(A_{i_1} A_{i_2})$$

Since second term

$\leq$  ,  
larger than the  
following

15. Show that if  $P(A_i) = 1$  for all  $i \geq 1$ , then

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1.$$

$$A_i = S$$

$$P(A_1 \cap A_2) = 1$$

$$P(A_1 \cap A_2 \cap \dots \cap A_n \dots)$$

$$= P(S \cap S \cap \dots \cap S \dots)$$

$$= 1.$$

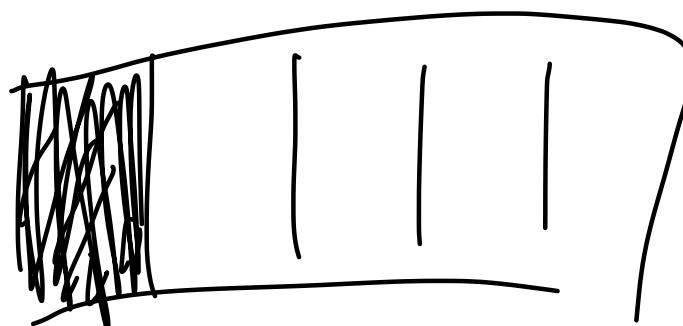
16. Let  $T_k(n)$  denote the number of partitions of the set  $\{1, \dots, n\}$  into  $k$  nonempty subsets, where  $1 \leq k \leq n$ . (See Theoretical Exercise 8 for the definition of a partition.) Argue that

$$T_k(n) = kT_k(n-1) + T_{k-1}(n-1)$$

*Hint:* In how many partitions is  $\{1\}$  a subset, and in how many is 1 an element of a subset that contains other elements?

17. Five balls are randomly chosen, without replacement, from an urn that contains 5 red, 6 white, and 7 blue balls. Find the probability that at least one ball of each color is chosen.

16.



$$k=5,$$

$$1, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \binom{4}{4}$$

$$\begin{aligned} &1, (2, \\ &(2), 13, 14, \\ &, 234, 15 \end{aligned}$$

17.

RWB

$$\frac{\binom{5}{1} \binom{6}{1} \binom{7}{1} \binom{15}{2}}{\binom{18}{5}}$$

18. Four red, 8 blue, and 5 green balls are randomly arranged in a line.

- (a) What is the probability that the first 5 balls are blue?
- (b) What is the probability that none of the first 5 balls are blue?
- (c) What is the probability that the final 3 balls are differently colored.
- (d) What is the probability that all the red balls are together?

$$a). \frac{\binom{8}{5}}{\binom{17}{5}}$$

$$b). \frac{\binom{9}{5}}{\binom{17}{5}}$$

$$c). \frac{\binom{4}{1}\binom{6}{1}\binom{5}{1}}{\binom{17}{3}}$$

$$d). \frac{\binom{13}{5}}{\binom{17}{4}\binom{13}{8}}$$

19. Ten cards are randomly chosen from a deck of 52 cards that consists of 13 cards of each of 4 different suits. Each of the selected cards is put in one of 4 piles, depending on the suit of the card.

- (a) What is the probability that the largest pile has 4 cards, the next largest has 3, the next largest has 2, and the smallest has 1 card?
- (b) What is the probability that two of the piles have 3 cards, one has 4 cards, and one has no cards?

20. Balls are randomly removed from an urn initially containing 20 red and 10 blue balls. What is the probability that all of the red balls are removed before all of the blue ones have been removed?

$$19a). \frac{\binom{13}{4} \binom{13}{3} \binom{13}{2} \binom{13}{1}}{\binom{52}{4} \binom{48}{3} \binom{45}{2} \binom{44}{1}} \times 4!$$

$$b). \frac{\binom{13}{3} \binom{13}{2} \binom{13}{1}}{\binom{52}{3} \binom{49}{2} \binom{46}{1}}$$

$$20. \frac{1}{\binom{30}{20}}$$