A Solution Manual for:
A First Course In Probability: Seventh Edition
by Sheldon M. Ross.

John L. Weatherwax*

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Introduction

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Miscellaneous Problems

The Crazy Passenger Problem

The following is known as the "crazy passenger problem" and is stated as follows. A line of 100 airline passengers is waiting to board the plane. They each hold a ticket to one of the 100 seats on that flight. (For convenience, let's say that the k-th passenger in line has a ticket for the seat number k.) Unfortunately, the first person in line is *crazy*, and will ignore the seat number on their ticket, picking a random seat to occupy. All the other passengers are quite normal, and will go to their proper seat unless it is already occupied. If it is occupied, they will then find a free seat to sit in, at random. What is the probability that the last (100th) person to board the plane will sit in their proper seat (#100)?

If one tries to solve this problem with conditional probability it becomes very difficult. We begin by considering the following cases if the first passenger sits in seat number 1, then all

*wax@alum.mit.edu

the remaining passengers will be in their correct seats and certainly the #100'th will also. If he sits in the last seat #100, then certainly the last passenger cannot sit there (in fact he will end up in seat #1). If he sits in any of the 98 seats *between* seats #1 and #100, say seat k, then all the passengers with seat numbers $2, 3, \ldots, k-1$ will have empty seats and be able to sit in their respective seats. When the passenger with seat number k enters he will have as possible seating choices seat #1, one of the seats $k+1, k+2, \ldots, 99$, or seat #100. Thus the options available to this passenger are the *same* options available to the first passenger. That is if he sits in seat #1 the remaining passengers with seat labels $k+1, k+2, \ldots, 100$ can sit in their assigned seats and passenger #100 can sit in his seat, or he can sit in seat #100 in which case the passenger #100 is blocked, or finally he can sit in one of the seats between seat k and seat #99. The only difference is that this k-th passenger has fewer choices for the "middle" seats. This k passenger effectively becomes a new "crazy" passenger.

From this argument we begin to see a recursive structure. To fully specify this recursive structure lets generalize this problem a bit an assume that there are N total seats (rather than just 100). Thus at each stage of placing a k-th crazy passenger we can choose from

- seat #1 and the last or N-th passenger will then be able to sit in their assigned seat, since all intermediate passenger's seats are unoccupied.
- seat # N and the last or N-th passenger will be unable to sit in their assigned seat.
- any seat before the N-th and after the k-th. Where the k-th passenger's seat is taken by a crazy passenger from the previous step. In this case there are $N-1-(k+1)+1 = N-k-1$ "middle" seat choices.

If we let $p(n,1)$ be the probability that given one crazy passenger and n total seats to select from that the last passenger sits in his seat. From the argument above we have a recursive structure give by

$$\begin{aligned} p(N,1) &= \frac{1}{N}(1) + \frac{1}{N}(0) + \frac{1}{N}\sum_{k=2}^{N-1} p(N-k,1) \\ &= \frac{1}{N} + \frac{1}{N}\sum_{k=2}^{N-1} p(N-k,1)\,. \end{aligned}$$

where the first term is where the first passenger picks the first seat (where the N will sit correctly with probability one), the second term is when the first passenger sits in the N-th seat (where the N will sit correctly with probability zero), and the remaining terms represent the first passenger sitting at position k, which will then require repeating this problem with the k-th passenger choosing among $N-k+1$ seats.

To solve this recursion relation we consider some special cases and then apply the principle of mathematical induction to prove it. Lets take $N = 2$. Then there are only two possible arraignments of passengers $(1,2)$ and $(2,1)$ of which one (the first) corresponds to the second passenger sitting in his assigned seat. This gives

$$p(2,1) = \frac{1}{2}\,.$$

If $N = 3$, then from the $3! = 6$ possible choices for seating arraignments

$$(1,2,3)\ (1,3,2)\ (2,3,1)\ (2,1,3)\ (3,1,2)\ (3,2,1)$$

Only

$$(1,2,3)\ (2,1,3)\ (3,2,1)$$

correspond to admissible seating arraignments for this problem so we see that

$$p(3,1) = \frac{3}{6} = \frac{1}{2}\,.$$

If we hypothesis that $p(N,1) = \frac{1}{2}$ for all N, placing this assumption into the recursive formulation above gives

$$p(N,1) = \frac{1}{N} + \frac{1}{N}\sum_{k=2}^{N-1}\frac{1}{2} = \frac{1}{2}\,.$$

Verifying that indeed this constant value satisfies our recursion relationship.

Chapter 1 (Combinatorial Analysis)

Chapter 1: Problems

Problem 1 (counting license plates)

Part (a): In each of the first two places we can put any of the 26 letters giving 26^2 possible letter combinations for the first two characters. Since the five other characters in the license plate must be numbers, we have 10^5 possible five digit letters their specification giving a total of

$$26^2 \cdot 10^5 = 67600000\,,$$

total license plates.

Part (b): If we can't repeat a letter or a number in the specification of a license plate then the number of license plates becomes

$$26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 = 468000\,,$$

total license plates.

Problem 2 (counting die rolls)

We have six possible outcomes for each of the die rolls giving $6^4 = 1296$ possible total outcomes for all four rolls.

Problem 3 (assigning workers to jobs)

Since each job is different and each worker is unique we have 20! different pairings.

Problem 4 (creating a band)

If each boy can play each instrument we can have $4! = 24$ ordering. If Jay and Jack can play only two instruments then we will assign the instruments they play first with 2! possible orderings. The other two boys can be assigned the remaining instruments in 2! ways and thus we have

$$2! \cdot 2! = 4\,,$$

possible unique band assignments.

Problem 5 (counting telephone area codes)

In the first specification of this problem we can have $9 - 2 + 1 = 8$ possible choices for the first digit in an area code. For the second digit there are two possible choices. For the third digit there are 9 possible choices. So in total we have

$$8 \cdot 2 \cdot 9 = 144\,,$$

possible area codes. In the second specification of this problem, if we must start our area codes with the digit "four" we will only have $2 \cdot 9 = 18$ area codes.

Problem 6 (counting kittens)

The traveler would meet $7^4 = 2401$ kittens.

Problem 7 (arranging boys and girls)

Part (a): Since we assume that each person is unique, the total number of ordering is given by $6! = 720$.

Part (b): We have 3! orderings of each group of the three boys and girls. Since we can put these groups of boys and girls in 2! different ways (either the boys first or the girls first) we have

$$(2!) \cdot (3!) \cdot (3!) = 2 \cdot 6 \cdot 6 = 72\,,$$

possible orderings.

Part (c): If the boys must sit together we have $3! = 6$ ways to arrange the block of boys. This block of boys can be placed either at the ends or in between any of the individual 3! orderings of the girls. This gives four locations where our block of boys can be placed we have

$$4 \cdot (3!) \cdot (3!) = 144\,,$$

possible orderings.

Part (d): The only way that no two people of the same sex can sit together is to have the two groups interleaved. Now there are 3! ways to arrange each group of girls and boys, and to interleave we have two different choices for interleaving. For example with three boys and girls we could have

$$g_1 b_1 g_2 b_2 g_3 b_3 \quad \text{vs.} \quad b_1 g_1 b_2 g_2 b_3 g_3\,,$$

thus we have

$$2 \cdot 3! \cdot 3! = 2 \cdot 6^2 = 72\,,$$

possible arrangements.

Problem 8 (counting arrangements of letters)

Part (a): Since "Fluke" has five unique letters we have $5! = 120$ possible arrangements.

Part (b): Since "Propose" has seven letters of which four (the "o"'s and the "p"'s) repeat we have

$$\frac{7!}{2! \cdot 2!} = 1260\,,$$

arrangements.

Part (c): Now "Mississippi" has eleven characters with the "i" repeated four times, the "s" repeated four times and the "p" repeated two times, so we have

$$\frac{11!}{4! \cdot 4! \cdot 2!} = 34650\,,$$

possible rearranges.

Part (d): "Arrange" has seven characters with two repeated so it has

$$\frac{7!}{2!} = 2520\,,$$

different arrangements.

Problem 9 (counting colored blocks)

Assuming each block is unique we have 12! arrangements, but since the six black and the four red blocks are not distinguishable we have

$$\frac{12!}{6! \cdot 4!} = 27720\,,$$

possible arrangements.

Problem 10 (seating people in a row)

Part (a): We have $8! = 40320$ possible seating arrangements.

Part (b): We have 6! ways to place the people (not including A and B). We have 2! ways to order A and B. Once the pair of A and B is determined, they can be placed in between any ordering of the other six. For example, any of the "x"'s in the expression below could be replaced with the A B pair

$$x\, P_1\, x\, P_2\, x\, P_3\, x\, P_4\, x\, P_5 x\, P_6\, x\,.$$

Giving seven possible locations for the A,B pair. Thus the total number of orderings is given by

$$2! \cdot 6! \cdot 7 = 10800 \, .$$

Part (c): To place the men and women according to the given rules, the men and women must be interleaved. We have 4! ways to arrange the men and 4! ways to arrange the women. We can start our sequence of eight people with a woman or a man (giving two possible choices). We thus have

$$2 \cdot 4! \cdot 4! = 1152 \, ,$$

possible arrangements.

Part (d): Since the five men must sit next to each other their ordering can be specified in $5! = 120$ ways. This block of men can be placed in between any of the three women, or at the end of the block of women, who can be ordered in 3! ways. Since there are four positions we can place the block of men we have

$$5! \cdot 4 \cdot 3! = 2880 \, ,$$

possible arrangements.

Part (e): The four couple have 2! orderings within each pair, and then 4! orderings of the pairs giving a total of

$$(2!)^4 \cdot 4! = 384 \, ,$$

total orderings.

Problem 11 (counting arrangements of books)

Part (a): We have $(3 + 2 + 1)! = 6! = 720$ arrangements.

Part (b): The mathematics books can be arranged in 2! ways and the novels in 3! ways. Then the block ordering of mathematics, novels, and chemistry books can be arranged in 3! ways resulting in

$$(3!) \cdot (2!) \cdot (3!) = 72 \, ,$$

possible arrangements.

Part (c): The number of ways to arrange the novels is given by $3! = 6$ and the other three books can be arranged in 3! ways with the blocks of novels in any of the four positions in between giving

$$4 \cdot (3!) \cdot (3!) = 144 \, ,$$

possible arrangements.

Problem 12 (counting awards)

Part (a): We have 30 students to choose from for the first award, and 30 students to choose from for the second award, etc. So the total number of different outcomes is given by

$$30^5 = 24300000$$

Part (b): We have 30 students to choose from for the first award, 29 students to choose from for the second award, etc. So the total number of different outcomes is given by

$$30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 = 17100720$$

Problem 13 (counting handshakes)

With 20 people the number of pairs is given by

$$\binom{20}{2} = 190\,.$$

Problem 14 (counting poker hands)

A deck of cards has four suits with thirteen cards each giving in total 52 cards. From these 52 cards we need to select five to form a poker hand thus we have

$$\binom{52}{5} = 2598960\,,$$

unique poker hands.

Problem 15 (pairings in dancing)

We must first choose five women from ten in $\binom{10}{5}$ possible ways, and five men from 12 in $\binom{12}{5}$ ways. Once these groups are chosen then we have 5! pairings of the men and women. Thus in total we will have

$$\binom{10}{5}\binom{12}{5} 5! = 252 \cdot 792 \cdot 120 = 23950080\,,$$

possible pairings.

Problem 16 (forced selling of books)

Part (a): We have to select a subject from three choices. If we choose math we have $\left(\begin{array}{c} 6 \\ 2 \end{array}\right) = 15$ choices of books to sell. If we choose science we have $\left(\begin{array}{c} 7 \\ 2 \end{array}\right) = 21$ choices of books to sell. If we choose economics we have $\left(\begin{array}{c} 4 \\ 2 \end{array}\right) = 6$ choices of books to sell. Since each choice is mutually exclusive in total we have $15 + 21 + 6 = 42$, possible choices.

Part (b): We must pick two subjects from $\left(\begin{array}{c} 3 \\ 2 \end{array}\right) = 3$ choices. If we denote the letter "M" for the choice math the letter "S" for the choice science, and the letter "E" for the choice economics then the three choices are

$$(M, S) \quad (M, E) \quad (S, E) \,.$$

For each of the choices above we have $6 \cdot 7 + 6 \cdot 4 + 7 \cdot 4 = 94$ total choices.

Problem 17 (distributing gifts)

We can choose seven children to give gifts to in $\left(\begin{array}{c} 10 \\ 7 \end{array}\right)$ ways. Once we have chosen the seven children, the gifts can be distributed in 7! ways. This gives a total of

$$\left(\begin{array}{c} 10 \\ 7 \end{array}\right) \cdot 7! = 604800 \,,$$

possible gift distributions.

Problem 18 (selecting political parties)

We can choose two Republicans from the five total in $\left(\begin{array}{c} 5 \\ 2 \end{array}\right)$ ways, we can choose two Democrats from the six in $\left(\begin{array}{c} 6 \\ 2 \end{array}\right)$ ways, and finally we can choose three Independents from the four in $\left(\begin{array}{c} 4 \\ 3 \end{array}\right)$ ways. In total, we will have

$$\left(\begin{array}{c} 5 \\ 2 \end{array}\right) \cdot \left(\begin{array}{c} 6 \\ 2 \end{array}\right) \cdot \left(\begin{array}{c} 4 \\ 3 \end{array}\right) = 600 \,,$$

different committees.

Problem 19 (counting committee's with constraints)

Part (a): We select three men from six in $\binom{6}{3}$, but since two men won't serve together we need to compute the number of these pairings of three men that have the two that won't serve together. The number of committees we can form (with these two together) is given by

$$\binom{2}{2} \cdot \binom{4}{1} = 4\,.$$

So we have

$$\binom{6}{3} - 4 = 16\,,$$

possible groups of three men. Since we can choose $\binom{8}{3} = 56$ different groups of women, we have in total $16 \cdot 56 = 896$ possible committees.

Part (b): If two women refuse to serve together, then we will have $\binom{2}{2} \cdot \binom{6}{1}$ groups with these two women in them from the $\binom{8}{3}$ ways to draw three women from eight. Thus we have

$$\binom{8}{3} - \binom{2}{2} \cdot \binom{6}{1} = 56 - 6 = 50\,,$$

possible groupings of woman. We can select three men from six in $\binom{6}{3} = 20$ ways. In total then we have $50 \cdot 20 = 1000$ committees.

Part (c): We have $\binom{8}{3} \cdot \binom{6}{3}$ total committees, and

$$\binom{1}{1} \cdot \binom{7}{2} \cdot \binom{1}{1} \cdot \binom{5}{2} = 210\,,$$

committees containing the man and women who refuse to serve together. So we have

$$\binom{8}{3} \cdot \binom{6}{3} - \binom{1}{1} \cdot \binom{7}{2} \cdot \binom{1}{1} \cdot \binom{5}{2} = 1120 - 210 = 910\,,$$

total committees.

Problem 20 (counting the number of possible parties)

Part (a): There are a total of $\left(\begin{array}{c} 8 \\ 5 \end{array}\right)$ possible groups of friends that could attend (assuming no feuds). We have $\left(\begin{array}{c} 2 \\ 2 \end{array}\right) \cdot \left(\begin{array}{c} 6 \\ 3 \end{array}\right)$ sets with our two feuding friends in them, giving

$$\left(\begin{array}{c} 8 \\ 5 \end{array}\right) - \left(\begin{array}{c} 2 \\ 2 \end{array}\right) \cdot \left(\begin{array}{c} 6 \\ 3 \end{array}\right) = 36$$

possible groups of friends

Part (b): If two fiends must attend together we have that $\left(\begin{array}{c} 2 \\ 2 \end{array}\right) \left(\begin{array}{c} 6 \\ 3 \end{array}\right)$ if the *do* attend the party together and $\left(\begin{array}{c} 6 \\ 5 \end{array}\right)$ if they *don't* attend at all, giving a total of

$$\left(\begin{array}{c} 2 \\ 2 \end{array}\right) \left(\begin{array}{c} 6 \\ 3 \end{array}\right) + \left(\begin{array}{c} 6 \\ 5 \end{array}\right) = 26 \,.$$

Problem 21 (number of paths on a grid)

From the hint given that we must take four steps to the right and three steps up, we can think of any possible path as an arraignment of the letters "U" for up and "R" for right. For example the string

$$UUURRRR \,,$$

would first step up three times and then right four times. Thus our problem becomes one of counting the number of unique arraignments of three "U"'s and four "R"'s, which is given by

$$\frac{7!}{4! \cdot 3!} = 35 \,.$$

Problem 22 (paths on a grid through a specific point)

One can think of the problem of going through a specific point (say P) as counting the number of paths from the start A to P and then counting the number of paths from P to the end B. To go from A to P (where P occupies the $(2, 2)$ position in our grid) we are looking for the number of possible unique arraignments of two "U"'s and two "R"'s, which is given by

$$\frac{4!}{2! \cdot 2!} = 6 \,,$$

possible paths. The number of paths from the point P to the point B is equivalent to the number of different arraignments of two "R"'s and one "U" which is given by

$$\frac{3!}{2! \cdot 1!} = 3 \,.$$

From the basic principle of counting then we have $6 \cdot 3 = 18$ total paths.

Problem 23 (assignments to beds)

Assuming that twins sleeping in different bed in the same room counts as a different arraignment, we have $(2!) \cdot (2!) \cdot (2!) = 8$ possible assignments of each set of twins to a room. Since there are 3! ways to assign the pair of twins to individual rooms we have $6 \cdot 8 = 48$ possible assignments.

Problem 24 (practice with the binomial expansion)

This is given by

$$(3x^2 + y)^5 = \sum_{k=0}^{5} \binom{5}{k} (3x^2)^k y^{5-k} .$$

Problem 25 (bridge hands)

We have 52! unique permutations, but since the different arraignments of cards within a given hand do not matter we have

$$\frac{52!}{(13!)^4} ,$$

possible bridge hands.

Problem 26 (practice with the multinomial expansion)

This is given by the multinomial expansion

$$(x_1 + 2x_2 + 3x_3)^4 = \sum_{n_1+n_2+n_3=4} \binom{4}{n_1\,, n_2\,, n_3} x_1^{n_1} (2x_2)^{n_2} (3x_3)^{n_3}$$

The number of terms in the above summation is given by

$$\binom{4+3-1}{3-1} = \binom{6}{2} = \frac{6 \cdot 5}{2} = 15 .$$

Problem 27 (counting committees)

This is given by the multinomial coefficient

$$\left(\begin{array}{c} 12 \\ 3\,,4\,,5 \end{array} \right) = 27720$$

Problem 28 (divisions of teachers)

If we decide to send n_1 teachers to school one and n_2 teachers to school two, etc. then the total number of unique assignments of (n_1, n_2, n_3, n_4) number of teachers to the four schools is given by

$$\left(\begin{array}{c} 8 \\ n_1\,,n_2\,,n_3\,,n_4 \end{array} \right) .$$

Since we want the total number of divisions, we must sum this result for all possible combinations of n_i, or

$$\sum_{n_1+n_2+n_3+n_4=8} \left(\begin{array}{c} 8 \\ n_1\,,n_2\,,n_3\,,n_4 \end{array} \right) = (1+1+1+1)^8 = 65536\,,$$

possible divisions.

If each school must receive two in each school, then we are looking for

$$\left(\begin{array}{c} 8 \\ 2\,,2\,,2\,,2 \end{array} \right) = \frac{8!}{(2!)^4} = 2520\,,$$

orderings.

Problem 29 (dividing weight lifters)

We have 10! possible permutations of all weight lifters but the permutations of individual countries (contained within this number) are irrelevant. Thus we can have

$$\frac{10!}{3! \cdot 4! \cdot 2! \cdot 1!} = \left(\begin{array}{c} 10 \\ 3\,,4\,,2\,,1 \end{array} \right) = 12600\,,$$

possible divisions. If the united states has one competitor in the top three and two in the bottom three. We have $\left(\begin{array}{c} 3 \\ 1 \end{array} \right)$ possible positions for the US member in the first three positions and $\left(\begin{array}{c} 3 \\ 2 \end{array} \right)$ possible positions for the two US members in the bottom three positions, giving a total of

$$\left(\begin{array}{c} 3 \\ 1 \end{array} \right) \left(\begin{array}{c} 3 \\ 2 \end{array} \right) = 3 \cdot 3 = 9\,,$$

combinations of US members in the positions specified. We also have to place the other countries participants in the remaining $10 - 3 = 7$ positions. This can be done in $\left(\begin{array}{c} 7 \\ 4, 2, 1 \end{array} \right) = \frac{7!}{4! \cdot 2! \cdot 1!} = 105$ ways. So in total then we have $9 \cdot 105 = 945$ ways to position the participants.

Problem 30 (seating delegates in a row)

If the French and English delegates are to be seated next to each other, they can be can be placed in 2! ways. Then this pair constitutes a new "object" which we can place anywhere among the remaining eight people, i.e. there are 9! arraignments of the eight remaining people and the French and English pair. Thus we have $2 \cdot 9! = 725760$ possible combinations. Since in some of these the Russian and US delegates are next to each other, this number over counts the true number we are looking for by $2 \cdot 28! = 161280$ (the first two is for the number of arrangements of the French and English pair). Combining these two criterion we have

$$2 \cdot (9!) - 4 \cdot (8!) = 564480 .$$

Problem 31 (distributing blackboards)

Let x_i be the number of black boards given to school i, where $i = 1, 2, 3, 4$. Then we must have $\sum_i x_i = 8$, with $x_i \geq 0$. The number of solutions to an equation like this is given by

$$\left(\begin{array}{c} 8+4-1 \\ 4-1 \end{array} \right) = \left(\begin{array}{c} 11 \\ 3 \end{array} \right) = 165 .$$

If each school must have at least one blackboard then the constraints change to $x_i \geq 1$ and the number of such equations is give by

$$\left(\begin{array}{c} 8-1 \\ 4-1 \end{array} \right) = \left(\begin{array}{c} 7 \\ 3 \end{array} \right) = 35 .$$

Problem 32 (distributing people)

Assuming that the elevator operator can only tell the number of people getting off at each floor, we let x_i equal the number of people getting off at floor i, where $i = 1, 2, 3, 4, 5, 6$. Then the constraint that all people are off at the sixth floor means that $\sum_i x_i = 8$, with $x_i \geq 0$. This has

$$\left(\begin{array}{c} n+r-1 \\ r-1 \end{array} \right) = \left(\begin{array}{c} 8+6-1 \\ 6-1 \end{array} \right) = \left(\begin{array}{c} 13 \\ 5 \end{array} \right) = 1287 ,$$

possible distribution people. If we have five men and three women, let m_i and w_i be the number of men and women that get off at floor i. We can solve this problem as the combination of two problems. That of tracking the men that get off on floor i and that of tracking the women that get off on floor i. Thus we must have

$$\begin{aligned}\sum_{i=1}^{6} m_i &= 5 \quad m_i \geq 0 \\ \sum_{i=1}^{6} w_i &= 3 \quad w_i \geq 0\,.\end{aligned}$$

The number of solutions to the first equation is given by

$$\binom{5+6-1}{6-1} = \binom{10}{5} = 252\,,$$

while the number of solutions to the second equation is given by

$$\binom{3+6-1}{6-1} = \binom{8}{5} = 56\,.$$

So in total then (since each number is exclusive) we have $252 \cdot 56 = 14114$ possible elevator situations.

Problem 33 (possible investment strategies)

Let x_i be the number of investments made in opportunity i. Then we must have

$$\sum_{i=1}^{4} x_i = 20$$

with constraints that $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$. Writing this equation as

$$x_1 + x_2 + x_3 + x_4 = 20$$

we can subtract the lower bound of each variable to get

$$(x_1 - 2) + (x_2 - 2) + (x_3 - 3) + (x_4 - 4) = 20 - 2 - 2 - 3 - 4 = 9\,.$$

Then defining $v_1 = x_1 - 2$, $v_2 = x_2 - 2$, $v_3 = x_3 - 3$, and $v_4 = x_4 - 4$, then our equation becomes $v_1 + v_2 + v_3 + v_4 = 9$, with the constraint that $v_i \geq 0$. The number of solutions to equations such as these is given by

$$\binom{9+4-1}{4-1} = \binom{12}{3} = 220\,.$$

Part (b): First we pick the three investments from the four possible in $\binom{4}{3} = 4$ possible ways. The four choices are denoted in table 1, where a one denotes that we invest in that option. Then investment choice number one requires the equation $v_2+v_3+v_4 = 20-2-3-4 =$

choice	$v_1 = x_1 - 2 \geq 0$	$v_2 = x_2 - 2 \geq 0$	$v_3 = x_3 - 3 \geq 0$	$v_4 = x_4 - 4 \geq 0$
1	0	1	1	1
2	1	0	1	1
3	1	1	0	1
4	1	1	1	0

Table 1: All possible choices of three investments.

11, and has $\left(\begin{array}{c} 11+3-1 \\ 3-1 \end{array}\right) = \left(\begin{array}{c} 13 \\ 2 \end{array}\right) = 78$ possible solutions. Investment choice number two requires the equation $v_1 + v_3 + v_4 = 20 - 2 - 3 - 4 = 11$, and again has $\left(\begin{array}{c} 11+3-1 \\ 3-1 \end{array}\right) = \left(\begin{array}{c} 13 \\ 2 \end{array}\right) = 78$ possible solutions. Investment choice number three requires the equation $v_1 + v_2 + v_4 = 20 - 2 - 2 - 4 = 12$, and has $\left(\begin{array}{c} 12+3-1 \\ 3-1 \end{array}\right) = \left(\begin{array}{c} 14 \\ 2 \end{array}\right) = 91$ possible solutions. Finally, investment choice number four requires the equation $v_1 + v_2 + v_3 = 20 - 2 - 2 - 3 = 13$, and has $\left(\begin{array}{c} 13+3-1 \\ 3-1 \end{array}\right) = \left(\begin{array}{c} 15 \\ 2 \end{array}\right) = 105$ possible solutions. Of course we could also invest in all four opportunities which has the same number of possibilities as in part (a) or 220. Then in total since we can do any of these choices we have $220 + 105 + 91 + 78 + 78 = 572$ choices.

Chapter 1: Theoretical Exercises

Problem 1 (the generalized counting principle)

This can be proved by recursively applying the basic principle of counting.

Problem 2 (counting dependent experimental outcomes)

We have m choices for the outcome of the first experiment. If the first experiment returns i as an outcome, then there are n_i possible outcomes for the second experiment. Thus if the experiment returns "one" we have n_1 possible outcomes, if it returns "two" we have n_2 possible outcomes, etc. To count the number of possible experimental outcomes we can envision a tree like structure representing the totality of possible outcomes, where we have m branches leaving the root node indicating the m possible outcomes from the first experiment. From the first of these branches we have n_1 additional branches representing the outcome of the second experiment when the first experimental outcome was a one. From the second branch we have n_2 additional branches representing the outcome of the second experiment when the first experimental outcome was a two. We can continue this process, with the m-th branch from the root node having n_m leaves representing the outcome of the second experiment when the first experimental outcome was a m. Counting all of these outcomes we have

$$n_1 n_2 n_3 \cdots n_m$$

total experimental outcomes.

Problem 3 (selecting r objects from n)

To select r objects from n, we will have n choices for the first object, $n-1$ choices for the second object, $n-2$ choices for the third object, etc. Continuing we will have $n-r+1$ choices for the selection of the r-th object. Giving a total of $n(n-1)(n-2)\cdots(n-r+1)$ total choices if the order of selection matters. If it does not then we must divide by the number of ways to rearrange the r selected objects i.e. $r!$ giving

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!},$$

possible ways to select r objects from n when the order of selection of the r object does not matter.

Problem 4 (combinatorial explanation of $\left(\begin{array}{c} n \\ k \end{array}\right)$)

If all balls are distinguishable then there are $n!$ ways to arrange all the balls. With in this arrangement there are $r!$ ways to uniquely arrange the black balls and $(n-r)!$ ways to uniquely arranging the white balls. These arraignments don't represent new patterns since the balls with the same color are in fact indistinguishable. Dividing by these repeated patterns gives

$$\frac{n!}{r!(n-r)!},$$

gives the unique number of permutations.

Problem 5 (the number of binary vectors who's sum is greater than k)

To have the sum evaluate to exactly k, we must select at k components from the vector x to have the value one. Since there are n components in the vector x, this can be done in $\left(\begin{array}{c} n \\ k \end{array}\right)$ ways. To have the sum exactly equal $k+1$ we must select $k+1$ components from x to have a value one. This can be done in $\left(\begin{array}{c} n \\ k+1 \end{array}\right)$ ways. Continuing this pattern we see that the number of binary vectors x that satisfy

$$\sum_{i=1}^{n} x_i \geq k$$

is given by

$$\sum_{l=k}^{n} \binom{n}{l} = \binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2} + \ldots + \binom{n}{k+1} + \binom{n}{k}.$$

Problem 6 (counting the number of increasing vectors)

If the first component x_1 were to equal n, then there is no possible vector that satisfies the inequality $x_1 < x_2 < x_3 < \ldots < x_k$ constraint. If the first component x_1 equals $n-1$ then again there are no vectors that satisfy the constraint. The first largest value that the component x_1 can take on and still result in a complete vector satisfying the inequality constraints is when $x_1 = n-k+1$ For that value of x_1, the other components are determined and are given by $x_2 = n-k+2$, $x_3 = n-k+3$, up to the value for x_k where $x_k = n$. This assignment provides *one* vector that satisfies the constraints. If $x_1 = n-k$, then we can construct an inequality satisfying vector x by assigning the $k-1$ other components x_2, x_3, up to x_k by assigning the integers $n-k+1\,, n-k+2\,, \ldots n-1\,, n$ to the $k-1$ components. This can be done in $\binom{k}{1}$ ways. Continuing if $x_1 = n-k-1$, then we can obtain a valid vector x by assign the integers $n-k\,, n-k+1\,, \ldots n-1\,, n$ to the $k-1$ other components of x. This can be seen as an equivalent problem to that of specifying two blanks from $n-(n-k)+1 = k+1$ spots and can be done in $\binom{k+1}{2}$ ways. Continuing to decrease the value of the x_1 component, we finally come to the case where we have n locations open for assignment with k assignments to be made (or equivalently $n-k$ blanks to be assigned) since this can be done in $\binom{n}{n-k}$ ways. Thus the total number of vectors is given by

$$1 + \binom{k}{1} + \binom{k+1}{2} + \binom{k+2}{3} + \ldots + \binom{n-1}{n-k-1} + \binom{n}{n-k}.$$

Problem 7 (choosing r from n by drawing subsets of size $r-1$)

Equation 4.1 from the book is given by

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Considering the right hand side of this expression, we have

$$\begin{aligned}
\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(n-1-r+1)!(r-1)!} + \frac{(n-1)!}{(n-1-r)!r!} \\
&= \frac{(n-1)!}{(n-r)!(r-1)!} + \frac{(n-1)!}{(n-1-r)!r!} \\
&= \frac{n!}{(n-r)!r!}\left(\frac{r}{n} + \frac{n-r}{n}\right) \\
&= \binom{n}{r},
\end{aligned}$$

and the result is proven.

Problem 8 (selecting r people from from n men and m women)

We desire to prove

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \ldots + \binom{n}{r}\binom{m}{0}.$$

We can do this in a combinatorial way by considering subgroups of size r from a group of n men and m women. The left hand side of the above represents one way of obtaining this identity. Another way to count the number of subsets of size r is to consider the number of possible groups can be found by considering a subproblem of how many men chosen to be included in the subset of size r. This number can range from zero men to r men. When we have a subset of size r with zero men we must have all women. This can be done in $\binom{n}{0}\binom{m}{r}$ ways. If we select one man and $r-1$ women the number of subsets that meet this criterion is given by $\binom{n}{1}\binom{m}{r-1}$. Continuing this logic for all possible subset of the men we have the right hand side of the above expression.

Problem 9 (selecting n from $2n$)

From problem 8 we have that when $m = n$ and $r = n$ that

$$\binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \ldots + \binom{n}{n}\binom{n}{0}.$$

Using the fact that $\binom{n}{k} = \binom{n}{n-k}$ the above is becomes

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2,$$

which is the desired result.

Problem 10 (committee's with a chair)

Part (a): We can select a committee with k members in $\binom{n}{k}$ ways. Selecting a chairperson from the k committee members gives

$$k \binom{n}{k}$$

possible choices.

Part (b): If we choose the non chairperson members first this can be done in $\binom{n}{k-1}$ ways. We then choose the chairperson based on the remaining $n-k+1$ people. Combining these two we have

$$(n-k+1)\binom{n}{k-1}$$

possible choices.

Part (c): We can first pick the chair of our committee in n ways and then pick $k-1$ committee members in $\binom{n-1}{k-1}$. Combining the two we have

$$n\binom{n-1}{k-1},$$

possible choices.

Part (d): Since all expressions count the same thing they must be equal and we have

$$k\binom{n}{k} = (n-k+1)\binom{n}{k-1} = n\binom{n-1}{k-1}.$$

Part (e): We have

$$\begin{aligned}
k\binom{n}{k} &= k\frac{n!}{(n-k)!k!} \\
&= \frac{n!}{(n-k)!(k-1)!} \\
&= \frac{n!(n-k+1)}{(n-k+1)!(k-1)!} \\
&= (n-k+1)\binom{n}{k-1}
\end{aligned}$$

Factoring out n instead we have

$$\begin{aligned}
k\binom{n}{k} &= k\frac{n!}{(n-k)!k!} \\
&= n\frac{(n-1)!}{(n-1-(k-1))!(k-1)!} \\
&= n\binom{n-1}{k-1}
\end{aligned}$$

Problem 11 (Fermat's combinatorial identity)

We desire to prove the so called Fermat's combinatorial identity

$$\begin{aligned}
\binom{n}{k} &= \sum_{i=k}^{n}\binom{i-1}{k-1} \\
&= \binom{k-1}{k-1} + \binom{k}{k-1} + \cdots + \binom{n-2}{k-1} + \binom{n-1}{k-1}\,.
\end{aligned}$$

Following the hint, consider the integers $1, 2, \cdots, n$. Then consider subsets of size k from n elements as a sum over i where we consider i to be the largest entry in all the given subsets of size k. The smallest i can be is k of which there are $\binom{k-1}{k-1}$ subsets where when we add the element k we get a complete subset of size k. The next subset would have $k+1$ as the largest element of which there are $\binom{k}{k-1}$ of these. There are $\binom{k+1}{k-1}$ subsets with $k+2$ as the largest element etc. Finally, we will have $\binom{n-1}{k-1}$ sets with n the largest element. Summing all of these subsets up gives $\binom{n}{k}$.

Problem 12 (moments of the binomial coefficients)

Part (a): Consider n people from which we want to count the total number of committees of any size with a chairman. For a committee of size $k=1$ we have $1\cdot\binom{n}{1} = n$ possible choices. For a committee of size $k=2$ we have $\binom{n}{2}$ subsets of two people and two choices for the person who is the chair. This gives $2\binom{n}{2}$ possible choices. For a committee of size $k=3$ we have $3\binom{n}{3}$, etc. Summing all of these possible choices we find that the total number of committees with a chair is

$$\sum_{k=1}^{n} k\binom{n}{k}\,.$$

Another way to count the total number of all committees with a chair, is to consider first selecting the chairperson from which we have n choices and then considering all possible subsets of size $n-1$ (which is 2^{n-1}) from which to construct the remaining committee members. The product then gives $n2^{n-1}$.

Part (b): Consider again n people where now we want to count the total number of committees of size k with a chairperson and a secretary. We can select all subsets of size k in $\left(\begin{array}{c} n \\ k \end{array}\right)$ ways. Given a subset of size k, there are k choices for the chairperson and k choices for the secretary giving $k^2 \left(\begin{array}{c} n \\ k \end{array}\right)$ committees of size k with a chair and a secretary. The total number of these is then given by summing this result or

$$\sum_{k=1}^{n} k^2 \left(\begin{array}{c} n \\ k \end{array}\right) .$$

Now consider first selecting the chair which can be done in n ways. Then selecting the secretary which can either be the chair or one of the $n-1$ other people. If we select the chair and the secretary to be the same person we have $n-1$ people to choose from to represent the committee. All possible subsets from as set of $n-1$ elements is given by 2^{n-1}, giving in total $n2^{n-1}$ possible committees with the chair and the secretary the same person. If we select a different person for the secretary this chair/secretary selection can be done in $n(n-1)$ ways and then we look for all subsets of a set with $n-2$ elements (i.e. 2^{n-2}) so in total we have $n(n-1)2^{n-2}$. Combining these we obtain

$$n2^{n-1} + n(n-1)2^{n-2} = n2^{n-2}(2+n-1) = n(n+1)2^{n-2} .$$

Equating the two we have

$$\sum_{k=1}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) k^2 = 2^{n-2} n(n+1) .$$

Part (c): Consider now selecting all committees with a chair a secretary and a stenographer, where each can be the same person. Then following the results of Part (b) this total number is given by $\sum_{k=1}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) k^3$. Now consider the following situations and a count of how many cases they provide.

- If the same person is the chair, the secretary, and the stenographer, then this combination gives $n2^{n-1}$ total committees.
- If the same person is the chair and the secretary, but not the stenographer, then this combination gives $n(n-1)2^{n-2}$ total committees.
- If the same person is the chair and the stenographer, but not the secretary, then this combination gives $n(n-1)2^{n-2}$ total committees.
- If the same person is the secretary and the stenographer, but not the chair, then this combination gives $n(n-1)2^{n-2}$ total committees.

- Finally, if no person has more than one job, then this combination gives $n(n-1)(n-2)2^{n-3}$ total committees.

Adding all of these possible combinations up we find that

$$n(n-1)(n-2)2^{n-3} + 3n(n-1)2^{n-2} + n2^{n-1} = n^2(n+3)2^{n-3}.$$

Problem 13 (an alternating series of binomial coefficients)

From the binomial theorem we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

If we select $x = -1$ and $y = 1$ then $x + y = 0$ and the sum above becomes

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k,$$

as we were asked to prove.

Problem 14 (committees and subcommittees)

Part (a): Pick the committee of size j in $\binom{n}{j}$ ways. The subcommittee of size i from these j can be selected in $\binom{j}{i}$ ways, giving a total of $\binom{j}{i}\binom{n}{j}$ committees and subcommittee. Now assume that we pick the subcommittee first. This can be done in $\binom{n}{i}$ ways. We then pick the committee in $\binom{n-i}{j-i}$ ways resulting in a total $\binom{n}{i}\binom{n-i}{j-i}$.

Part (b): I think that the lower index on this sum should start at i (the smallest subcommittee size). If so then we have

$$\begin{aligned}
\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} &= \sum_{j=i}^{n} \binom{n}{i}\binom{n-i}{j-i} \\
&= \binom{n}{i} \sum_{j=i}^{n} \binom{n-i}{j-i} \\
&= \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} = \binom{n}{i} 2^{n-i}.
\end{aligned}$$

Part (c): Consider the following manipulations of a binomial like sum

$$
\begin{aligned}
\sum_{j=i}^{n} \binom{n}{j} \binom{j}{i} x^{j-i} y^{n-i-(j-i)} &= \sum_{j=i}^{n} \binom{n}{i} \binom{n-i}{j-i} x^{j-i} y^{n-j} \\
&= \binom{n}{i} \sum_{j=i}^{n} \binom{n-i}{j-i} x^{j-i} y^{n-j} \\
&= \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} x^{j} y^{n-(j+i)} \\
&= \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} x^{j} y^{n-i-j} \\
&= \binom{n}{i} (x+y)^{n-i} .
\end{aligned}
$$

In summary we have shown that

$$
\sum_{j=i}^{n} \binom{n}{j} \binom{j}{i} x^{j-i} y^{n-j} = \binom{n}{i} (x+y)^{n-i} \quad \text{for} \quad i \leq n
$$

Now let $x = 1$ and $y = -1$ so that $x + y = 0$ and using these values in the above we have

$$
\sum_{j=i}^{n} \binom{n}{j} \binom{j}{i} (-1)^{n-j} = 0 \quad \text{for} \quad i \leq n .
$$

Problem 15 (the number of ordered vectors)

As stated in the problem we will let $H_k(n)$ be the number of vectors with components $x_1, x_2, \cdots, x_k$ for which each x_i is a positive integer such that $1 \leq x_i \leq n$ and the x_i are ordered i.e. $x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_n$

Part (a): Now $H_1(n)$ is the number of vectors with one component (with the restriction on its value of $1 \leq x_1 \leq n$). Thus there are n choices for x_1 so $H_1(n) = n$.

We can compute $H_k(n)$ by considering how many vectors there can be when the last component i.e. x_k has value of j. This would be the expression $H_{k-1}(j)$, since we know the value of the k-th component. Since j can range from 1 to n the total number of vectors with k components (i.e. $H_k(n)$) is given by the sum of all the previous $H_{k-1}(j)$. That is

$$
H_k(n) = \sum_{j=1}^{n} H_{k-1}(j) .
$$

Part (b): We desire to compute $H_3(5)$. To do so we first note that from the formula above the points at level k (the subscript) depends on the values of H at level $k-1$. To evaluate this expression when $n = 5$, we need to evaluate $H_k(n)$ for $k = 1$ and $k = 2$. We have that

$$\begin{aligned}
H_1(n) &= n \\
H_2(n) &= \sum_{j=1}^{n} H_1(j) = \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \\
H_3(n) &= \sum_{j=1}^{n} H_2(j) = \sum_{j=1}^{n} \frac{j(j+1)}{2} .
\end{aligned}$$

Thus we can compute the first few values of $H_2(\cdot)$ as

$$\begin{aligned}
H_2(1) &= 1 \\
H_2(2) &= 3 \\
H_2(3) &= 6 \\
H_2(4) &= 10 \\
H_2(5) &= 15 .
\end{aligned}$$

So that we find that

$$\begin{aligned}
H_3(5) &= H_2(1) + H_2(2) + H_2(3) + H_2(4) + H_2(5) \\
&= 1 + 3 + 6 + 10 + 15 = 35 .
\end{aligned}$$

Problem 16 (the number of tied tournaments)

Part (a): See Table 2 for the enumerations used in computing $N(3)$. We have denoted A, B, and C by the people all in the first place.

Part (b): To argue the given sum, we consider how many outcomes there are when i-players tie for last place. To determine this we have to choose the i players from n that will tie (which can be done in $\left(\begin{array}{c} n \\ i \end{array} \right)$ ways). We then have to distributed the remaining $n - i$ players in winning combinations (with ties allowed). This can be done recursively in $N(n - i)$ ways. Summing up all of these terms we find that

$$N(n) = \sum_{i=1}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) N(n - i) .$$

Part (c): In the above expression let $j = n - i$, then our limits on the sum above change as follows

$$\begin{aligned}
i = 1 &\rightarrow j = n - 1 \quad \text{and} \\
i = n &\rightarrow j = 0 ,
\end{aligned}$$

so that the above sum for $N(n)$ becomes

$$N(n) = \sum_{j=0}^{n-1} \left(\begin{array}{c} n \\ j \end{array} \right) N(j) .$$

First Place	Second Place	Third Place
A,B,C		
A,B	C	
A,C	B	
C,B	A	
A	B,C	
B	C,A	
C	A,B	
A	B	C
B	C	A
C	A	B
A	C	B
$\vdots$	$\vdots$	$\vdots$
B	A	C
C	B	A

Table 2: Here we have enumerated many of the possible ties that can happen with three people. The first row corresponds to all three in first place. The next three rows corresponds to two people in first place and the other in second place. The third row corresponds to two people in second place and one in first. The remaining rows correspond to one person in each position. The ellipses ($\vdots$) denotes thirteen possible outcomes.

Part (d): For the specific case of $N(3)$ we find that

$$\begin{aligned}
N(3) &= \sum_{j=0}^{2} \binom{3}{j} N(j) \\
&= \binom{3}{0} N(0) + \binom{3}{1} N(1) + \binom{3}{2} N(2) \\
&= N(0) + 3N(1) + 3N(2) = 1 + 3(1) + 3(3) = 13\,.
\end{aligned}$$

We also find for $N(4)$ that

$$\begin{aligned}
N(4) &= \sum_{j=0}^{3} \binom{4}{j} N(j) \\
&= \binom{4}{0} N(0) + \binom{4}{1} N(1) + \binom{4}{2} N(2) + \binom{4}{3} N(3) \\
&= N(0) + 4N(1) + \frac{3 \cdot 4}{2} N(2) + 4N(3) = 1 + 4(1) + 6(3) + 4(13) = 75\,.
\end{aligned}$$

Problem 17 (why the binomial equals the multinomial)

The expression $\binom{n}{r}$ is the number of ways to choose r objects from n, leaving another group of $n-r$ objects. The expression $\binom{n}{r, n-r}$ is the number of divisions of n distinct objects into two groups of size r and of size $n-r$ respectively. As these are the same thing the numbers are equivalent.

Problem 18 (a decomposition of the multinomial coefficient)

To compute $\binom{n}{n_1, n_2, n_3, \cdots, n_r}$ we consider fixing one particular object from the n. Then this object can end up in any of the r individual groups. If it appears in the first one then we have $\binom{n-1}{n_1-1, n_2, n_3, \cdots, n_r}$, possible arrangements for the other objects. If it appears in the second group then the remaining objects can be distributed in $\binom{n-1}{n_1, n_2-1, n_3, \cdots, n_r}$ ways, etc. Repeating this argument for all of the r groups we see that the original multinomial coefficient can be written as sums of these individual multinomial terms as

$$\begin{aligned}\binom{n}{n_1, n_2, n_3, \cdots, n_r} &= \binom{n-1}{n_1-1, n_2, n_3, \cdots, n_r} \\ &+ \binom{n-1}{n_1, n_2-1, n_3, \cdots, n_r} \\ &+ \cdots \\ &+ \binom{n-1}{n_1, n_2, n_3, \cdots, n_r-1}.\end{aligned}$$

Problem 19 (the multinomial theorem)

The multinomial therm is

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{n_1+n_2+\cdots+n_r=n} \binom{n}{n_1, n_2, \cdots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r},$$

which can be proved by recognizing that the product of $(x_1 + x_2 + \cdots + x_r)^n$ will contain products of the type $x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$, and recognizing that the number of such terms, i.e. the coefficient in front of this term is a count of the number of times we can select n_1 of the variable x_1's, and n_2 of the variable x_2, etc from the n variable choices. Since this number equals the multinomial coefficient we have proven the multinomial theorem.

Problem 20 (the number of ways to fill bounded urns)

Let x_i be the number of balls in the ith urn. We must have $x_i \geq m_i$ and we are distributing the n balls so that $\sum_{i=1}^r x_i = n$. To solve this problem lets shift our variables so that each must be greater than or equal to zero. Our constraint then becomes (by subtracting the lower bound on x_i)

$$\sum_{i=1}^{r}(x_i - m_i) = n - \sum_{i=1}^{r} m_i\,.$$

This expression motivates us to define $v_i = x_i - m_i$. Then $v_i \geq 0$ so we are looking for the number of solutions to the equation

$$\sum_{i=1}^{r} v_i = n - \sum_{i=1}^{r} m_i\,,$$

where v_i must be greater than or equal to zero. This number is given by

$$\left(\begin{array}{c} n - \sum_{i=1}^{r} m_i + r - 1 \\ r - 1 \end{array}\right).$$

Problem 21 (k zeros in an integer equation)

To find the number of solutions to

$$x_1 + x_2 + \cdots + x_r = n\,,$$

where exactly k of the x_r's are zero, we can select k of the x_i's to be zero in $\left(\begin{array}{c} r \\ k \end{array}\right)$ ways and then count the number of solutions with positive (greater than or equal to one solutions) for the remaining $r - k$ variables. The number of solutions to the remaining equation is $\left(\begin{array}{c} n-1 \\ r-k-1 \end{array}\right)$ ways so that the total number is the product of the two or

$$\left(\begin{array}{c} r \\ k \end{array}\right)\left(\begin{array}{c} n-1 \\ r-k-1 \end{array}\right).$$

Problem 22 (the number of partial derivatives)

Let n_i be the number of derivatives taken of the x_ith variable. Then a total order of n derivatives requires that these componentwise derivatives satisfy $\sum_{i=1}^{n} n_i = n$, with $n_i \geq 0$. The number of such is given by

$$\left(\begin{array}{c} n+n-1 \\ n-1 \end{array}\right) = \left(\begin{array}{c} 2n-1 \\ n-1 \end{array}\right).$$

Problem 23 (counting discrete wedges)

We require that $x_i \geq 1$ and that they sum to a value less than k, i.e.

$$\sum_{i=1}^{n} x_i \leq k\,.$$

To count the number of solutions to this equation consider the number of equations with $x_i \geq 1$ and $\sum_{i=1}^n x_i = \hat{k}$, which is

$$\left(\begin{array}{c} \hat{k} - 1 \\ n - 1 \end{array} \right)$$

so to calculate the number of equations to the requested problem we add these up for all $\hat{k} < k$. The number of solutions is given by

$$\sum_{\hat{k}=n}^{k} \left(\begin{array}{c} \hat{k} - 1 \\ n - 1 \end{array} \right) \quad \text{with} \quad k > n \,.$$

Chapter 1: Self-Test Problems and Exercises

Problem 1 (counting arrangements of letters)

Part (a): Consider the pair of A with B as one object. Now there are two orderings of this "fused" object i.e. AB and BA. The remaining letters can be placed in 4! orderings and once an ordering is specified the fused A/B block can be in any of the five locations around the permutation of the letters $CDEF$. Thus we have $2 \cdot 4! \cdot 5 = 240$ total orderings.

Part (b): We want to enforce that A must be before B. Lets begin to construct a valid sequence of characters by first placing the other letters $CDEF$, which can be done in $4! = 24$ possible ways. Now consider an arbitrary permutation of $CDEF$ such as $DFCE$. Then if we place A in the left most position (such as as in $ADFCE$), we see that there are five possible locations for the letter B. For example we can have $ABDFCE$, $ADBFCE$, $ADFBCE$, $ADFCBE$, or $ADFCEB$. If A is located in the second position from the left (as in $DAFCE$) then there are four possible locations for B. Continuing this logic we see that we have a total of $5+4+3+2+1 = \frac{5(5+1)}{2} = 15$ possible ways to place A and B such that they are ordered with A before B in each permutation. Thus in total we have $15 \cdot 4! = 360$ total orderings.

Part (c): Lets solve this problem by placing A, then placing B and then placing C. Now we can place these characters at any of the six possible character locations. To explicitly specify their locations lets let the integer variables n_0, n_1, n_2, and n_3 denote the number of blanks (from our total of six) that are before the A, between the A and the B, between the B and the C, and after the C. By construction we must have each n_i satisfy

$$n_i \geq 0 \quad \text{for} \quad i = 0, 1, 2, 3 \,.$$

In addition the sum of the n_i's plus the three spaces occupied by A, B, and C must add to six or

$$n_0 + n_1 + n_2 + n_3 + 3 = 6 \,,$$

or equivalently

$$n_0 + n_1 + n_2 + n_3 = 3 \,.$$

The number of solutions to such integer equalities is discussed in the book. Specifically, there are

$$\binom{3+4-1}{4-1} = \binom{6}{3} = 20\,,$$

such solutions. For each of these solutions, we have $3! = 6$ ways to place the three other letters giving a total of $6 \cdot 20 = 120$ arraignments.

Part (d): For this problem A must be before B and C must be before D. Let begin to construct a valid ordering by placing the letters E and F first. This can be done in two ways EF or FE. Next lets place the letters A and B, which if A is located at the left most position as in AEF, then B has three possible choices. As in Part (b) from this problem there are a total of $3+2+1=6$ ways to place A and B such that A comes before B. Following the same logic as in Part (b) above when we place C and D there are $5+4+3+2+1=15$ possible placements. In total then we have $15 \cdot 6 \cdot 2 = 180$ possible orderings.

Part (e): There are $2!$ ways of arranging A and B, $2!$ ways of arranging C and D, and $2!$ ways of arranging the remaining letters E and F. Lets us first place the blocks of letters consisting of the pair A and B which can be placed in any of the positions around E and F. There are three such positions. Next lets us place the block of letters consisting of C and D which can be placed in any of the four positions (between the E, F individual letters, or the A and B block). This gives a total number of arrangements of

$$2! \cdot 2! \cdot 2! \cdot 3 \cdot 4 = 96\,.$$

Part (f): E can be placed in any of five choices, first, second, third, fourth or fifth. Then the remaining blocks can be placed in $5!$ ways to get in total $5(5!) = 600$ arrangement's.

Problem 2 (counting seatings of people)

We have $4!$ arrangements of the Americans, $3!$ arrangements of the French, and $3!$ arrangements of the Britch and then $3!$ arrangements of these groups giving

$$4! \cdot 3! \cdot 3! \cdot 3!\,,$$

possible arringments.

Problem 3 (counting presidents)

Part (a): With no restrictions we must select three people from ten. This can be done in $\binom{10}{3}$ ways. Then with these three people there are $3!$ ways to specify which person is the president, the treasurer, etc. Thus in total we have

$$\binom{10}{3} \cdot 3! = \frac{10!}{7!} = 720\,,$$

possible choices.

Part (b): If A and B will not searve together we can construct the total number of choices by considering clubs consisting of instances with A included but no B, B included by no A, and finally neither A or B included. This can be represented as

$$1 \cdot \binom{8}{2} + 1 \cdot \binom{8}{2} + \cdot \binom{8}{3} = 112 \,.$$

This result needs to again be multipled by 3! as in Part (a) of this problem. When we do so we find we obtain 672.

Part (c): In the same way as in Part (b) of this problem lets count first the number of clubs with C and D in them and second the number of clubs without C and D in them. This number is

$$\binom{8}{1} + \binom{8}{3} = 64 \,.$$

Again multiplying by 3! we find a total number of $3! \cdot 64 = 384$ clubs.

Part (d): For E to be an officer means that E must be selected as a club member. The number of other members that can be selected is given by $\binom{9}{2} = 36$. Again multiplying this by 3! gives a total of 216 clubs.

Part (e): If for F to serve F must be a president we have two cases. The first is where F serves and is the president and the second where F does not serve. When F is the president we have two permutations for the jobs of the other two selected members. When F does not serve, we have $3! = 6$ possible permutions in assigning titles amoung the selected people. In total then we have

$$2\binom{9}{2} + 6\binom{9}{3} = 576 \,,$$

possible clubs.

Problem 4 (anwsering questions)

She must select seven questions from ten, which can be done in $\binom{10}{7} = 120$ ways. If she must select at least three from the first five then she can choose to anwser three, four or all five of the questions. Counting each of these choices in tern, we find that she has

$$\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = 110 \,.$$

possible ways.

Problem 5 (dividing gifts)

We have $\left(\begin{array}{c} 7 \\ 3 \end{array}\right)$ ways to select three gifts for the first child, then $\left(\begin{array}{c} 4 \\ 2 \end{array}\right)$ ways to select two gifts for the second, and finally $\left(\begin{array}{c} 2 \\ 2 \end{array}\right)$ for the third child. Giving a total of

$$\left(\begin{array}{c} 7 \\ 3 \end{array}\right) \cdot \left(\begin{array}{c} 4 \\ 2 \end{array}\right) \cdot \left(\begin{array}{c} 2 \\ 2 \end{array}\right) = 210 \,,$$

arrangements.

Problem 6 (license plates)

We can pick the location of the three letters in $\left(\begin{array}{c} 7 \\ 3 \end{array}\right)$ ways. Once these positions are selected we have 26^3 different combinations of letters that can be placed in the three spots. From the four remaining slots we can place 10^4 different digits giving in total

$$\left(\begin{array}{c} 7 \\ 3 \end{array}\right) \cdot 26^3 \cdot 10^4 \,,$$

possible seven place license plates.

Problem 7 (a simple combinatorial argument)

Remember that the expression $\left(\begin{array}{c} n \\ r \end{array}\right)$ counts the number of ways we can select r items from n. Notice that once we have specified a particular selection of r items, by construction we have also specified a particular selection of $n - r$ items, i.e. the remaining ones that are unselected. Since for each specification of r items we have an equivalent selection of $n - r$ items, the number of both i.e. $\left(\begin{array}{c} n \\ r \end{array}\right)$ and $\left(\begin{array}{c} n \\ n-r \end{array}\right)$ must be equal.

Problem 8 (counting n-digit numbers)

Part (a): To have no to consecutive digits equal, we can select the first digit in one of ten possible ways. The next digit in one of nine possible ways (we can't use the digit we selected for the first position). For the third digit we have three possible choices, etc. Thus in total we have

$$10 \cdot 9 \cdot 9 \cdots 9 = 10 \cdot 9^{n-1} \,,$$

possible digits.

Part (b): We now want to count the number of n-digit numbers where the digit 0 appears i times. Lets pick the locations where we want to place the zeros. This can be done in $\left(\begin{array}{c} n \\ i \end{array} \right)$ ways. We then have nine choices for the other digits to place in the other $n - i$ locations. This gives 9^{n-i} possible enoumerations for non-zero digits. In total then we have

$$\left(\begin{array}{c} n \\ i \end{array} \right) 9^{n-i} \,,$$

n digit numbers with i zeros in them.

Problem 9 (selecting three students from three classes)

Part (a): To choose three students from $3n$ total students can be done in $\left(\begin{array}{c} 3n \\ 3 \end{array} \right)$ ways.

Part (b): To pick three students from the same class we must first pick the class to draw the student from. This can be done in $\left(\begin{array}{c} 3 \\ 1 \end{array} \right) = 3$ ways. Once the class has been picked we have to pick the three students in from the n in that class. This can be done in $\left(\begin{array}{c} n \\ 3 \end{array} \right)$ ways. Thus in total we have

$$3 \left(\begin{array}{c} n \\ 3 \end{array} \right) \,,$$

possible selections of three students all from one class.

Part (c): To get two students in the same class and another in a different class, we must first pick the class from which to draw the two students from. This can be done in $\left(\begin{array}{c} 3 \\ 1 \end{array} \right) = 3$ ways. Next we pick the other class from which to draw the singleton student from. Since there are two possible classes to select this student from this can be done in two ways. Once both of these classes are selected we pick the individual two and one students from their respective classes in $\left(\begin{array}{c} n \\ 2 \end{array} \right)$ and $\left(\begin{array}{c} n \\ 1 \end{array} \right)$ ways respectively. Thus in total we have

$$3 \cdot 2 \cdot \left(\begin{array}{c} n \\ 2 \end{array} \right) \left(\begin{array}{c} n \\ 1 \end{array} \right) = 6n \frac{n(n-1)}{2} = 3n^2(n-1) \,,$$

ways.

Part (d): Three students (all from a different class) can be picked in $\left(\begin{array}{c} n \\ 1 \end{array} \right)^3 = n^3$ ways.

Part (e): As an identity we have then that

$$\left(\begin{array}{c} 3n \\ 3 \end{array} \right) = 3 \left(\begin{array}{c} n \\ 3 \end{array} \right) + 3n^2(n-1) + n^3 \,.$$

We can check that this expression is correct by expanding each side. Expanding the left hand side we find that

$$\binom{3n}{3} = \frac{3n!}{3!(3n-3)!} = \frac{3n(3n-1)(3n-2)}{6} = \frac{9n^3}{2} - \frac{9n^2}{2} + n\,.$$

While expanding the right hand side we find that

$$\begin{aligned}
3\binom{n}{3} + 3n^2(n-1) + n^3 &= 3\frac{n!}{3!(n-3)!} + 3n^3 - 3n^2 + n^3 \\
&= \frac{n(n-1)(n-2)}{2} + 4n^3 - 3n^2 \\
&= \frac{n(n^2-3n+2)}{2} + 4n^3 - 3n^2 \\
&= \frac{n^3}{2} - \frac{3n^2}{2} + n + 4n^3 - 3n^2 \\
&= \frac{9n^3}{2} - \frac{9n^2}{2} + n\,,
\end{aligned}$$

which is the same, showing the equivalence.

Problem 10 (counting five digit numbers with no triple counts)

Lets first enumerate the number of five digit numbers that can be constructed with no repeated digits. Since we have nine choices for the first digit, eight choices for the second digit, seven choices for the third digit etc. We find the number of five digit numbers with no repeated digits given by $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = \frac{9!}{4!} = 15120$.

Now lets count the number of five digit numbers where *one* of the digits $1, 2, 3, \cdots, 9$ repeats. We can pick the digit that will repeat in nine ways and select its position in the five digits in $\binom{5}{2}$ ways. To fill the remaining three digit location can be done in $8 \cdot 7 \cdot 6$ ways. This gives in total

$$9 \cdot \binom{5}{2} \cdot 8 \cdot 7 \cdot 6 = 30240\,.$$

Lets now count the number five digit numbers with two repeated digits. To compute this we might argue as follows. We can select the first digit and its location in $9 \cdot \binom{5}{2}$ ways. We can select the second repeated digit and its location in $8 \cdot \binom{3}{2}$ ways. The final digit can be selected in seven ways, giving in total

$$9\binom{5}{2} \cdot 8\binom{3}{2} \cdot 7 = 15120\,.$$

We note, however, that this analysis (as it stands) double counts the true number of five digits numbers with two repeated digits. This is because in first selecting the first digit from nine classes and then selecting the second digit from eight choices the total two digits chosen can actually be selected in the opposite order but placed in same spots from among our five digits. Thus we have to divide the above number by two giving

$$\frac{15120}{2} = 7560\,.$$

So in total we have by summing up all these mutually exclusive events we find that the total number of five digit numbers allowing repeated digits is given by

$$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 + 9 \binom{5}{2} \cdot 8 \cdot 7 \cdot 6 + \frac{1}{2} \cdot 9 \cdot \binom{5}{2} 8 \binom{3}{2} \cdot 7 = 52920\,.$$

Problem 11 (counting first round winners)

Lets consider a simple case first and then generalize this result. Consider some symbolic players denoted by A, B, C, D, E, F. Then we can construct a pairing of players by first selecting three players and then ordering the remaining three players with respect to the first chosen three. For example, lets first select the players B, E, and F. Then if we want A to play E, C to play F, and D to play B we can represent this graphically by the following

$$\begin{array}{c} B\,E\,F \\ D\,A\,C\,, \end{array}$$

where the players in a given fixed column play each other. From this we can select three different winners by selecting who wins each match. This can be done in 2^3 total ways. Since we have two possible choices for the winner of the first match, two possible choices for the winner of the second match, and finally two possible choices for the winner of the third match. Thus two generalize this procedure to $2n$ people we must first select n players from the $2n$ to for the "template" first row. This can be done in $\binom{2n}{n}$ ways. We then must select one of the $n!$ orderings of the remaining n players to form matches with. Finally, we must select winners of each match in 2^n ways. In total we would then conclude that we have

$$\binom{2n}{n} \cdot n! \cdot 2^n = \frac{(2n)!}{n!} \cdot 2^n\,,$$

total first round results. The problem with this is that it will double count the total number of pairings. It will count the pairs AB and BA as distinct. To remove this over counting we need to divide by the total number of ordered n pairs. This number is 2^n. When we divide by this we find that the total number of first round results is given by

$$\frac{(2n)!}{n!}\,.$$

Problem 12 (selecting committees)

Since we must select a total of six people consisting of at least three women and two men, we could select a committee with four women and two mean *or* a committee with three woman and three men. The number of ways of selecting this first type of committee is given by $\binom{8}{4}\binom{7}{2}$. The number of ways to select the second type of committee is given by $\binom{8}{3}\binom{7}{3}$. So the total number of ways to select a committee of six people is

$$\binom{8}{4}\binom{7}{2} + \binom{8}{3}\binom{7}{3}$$

Problem 13 (the number of different art sales)

Let D_i be the number of Dalis collected/bought by the i-th collector, G_i be the number of van Goghs collected by the i-th collector, and finally P_i the number of Picassos' collected by the i-th collector when $i = 1, 2, 3, 4, 5$. Then since all paintings are sold we have the following constraints on D_i, G_i, and P_i,

$$\sum_{i=1}^{5} D_i = 4\,, \quad \sum_{i=1}^{5} G_i = 5\,, \quad \sum_{i=1}^{5} P_i = 6\,.$$

Along with the requirements that $D_i \geq 0$, $G_i \geq 0$, and $P_i \geq 0$. Remembering that the number of solutions to an equation like

$$x_1 + x_2 + \cdot + x_r = n\,,$$

when $x_i \geq 0$ is given by $\binom{n+r-1}{r-1}$. Thus the number of solutions to the first equation above is given by $\binom{4+5-1}{5-1} = \binom{8}{4} = 70$, the number of solutions to the second equation is given by $\binom{5+5-1}{5-1} = \binom{9}{4} = 126$, and finally the number of solutions to the third equation is given by $\binom{6+5-1}{5-1} = \binom{10}{4} = 210$. Thus the total number of solutions is given by the product of these three numbers. We find that

$$\binom{8}{4}\binom{9}{4}\binom{10}{4} = 1852200\,,$$

See the Matlab file `chap_1_st_13.m` for these calculations.

Problem 14 (counting vectors that sum to less than k)

We want to evaluate the number of solutions to $\sum_{i=1}^{n} x_i \leq k$ for $k \geq n$, and x_i a positive integer. Now since the smallest value that $\sum_{i=1}^{n} x_i$ can be under these conditions is given when $x_i = 1$ for all i and gives a resulting sum of n. Now we note that for this problem the sum $\sum_{i=1}^{n} x_i$ take on any value greater than n up to and including k. Consider the number of solutions to $\sum_{i=1}^{n} x_i = j$ when j is fixed such that $n \le j \le k$. This number is given by $\left(\begin{array}{c} j-1 \\ n-1 \end{array} \right)$. So the total number of solutions is given by summing this expression over j for j ranging from n to k. We then find the total number of vectors $(x_1, x_2, \cdots, x_n)$ such that each x_i is a positive integer and $\sum_{i=1}^{n} x_i \le k$ is given by

$$\sum_{j=n}^{k} \left(\begin{array}{c} j-1 \\ n-1 \end{array} \right).$$

Problem 15 (all possible passing students)

With n total students, lets assume that k people pass the test. These k students can be selected in $\left(\begin{array}{c} n \\ k \end{array} \right)$ ways. All possible orderings or rankings of these k people is given by $k!$ so that the we have

$$\left(\begin{array}{c} n \\ k \end{array} \right) k!,$$

different possible orderings when k people pass the test. Then the total number of possible test postings is given by

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) k!.$$

Problem 16 (subsets that contain at least one number)

There are $\left(\begin{array}{c} 20 \\ 4 \end{array} \right)$ subsets of size four. The number of subsets that contain at least one of the elements $1, 2, 3, 4, 5$ is the *complement* of the number of subsets that don't contain any of the elements $1, 2, 3, 4, 5$. This number is $\left(\begin{array}{c} 15 \\ 4 \end{array} \right)$, so the total number of subsets that contain at least one of $1, 2, 3, 4, 5$ is given by

$$\left(\begin{array}{c} 20 \\ 4 \end{array} \right) - \left(\begin{array}{c} 15 \\ 4 \end{array} \right) = 4845 - 1365 = 3480.$$

Problem 17 (a simple combinatorial identity)

To show that

$$\left(\begin{array}{c} n \\ 2 \end{array} \right) = \left(\begin{array}{c} k \\ 2 \end{array} \right) + k(n-k) + \left(\begin{array}{c} n-k \\ 2 \end{array} \right) \quad \text{for} \quad 1 \le k \le n,$$

is true, begin by expanding the right hand side (RHS) of this expression. Using the definition of the binomial coefficients we obtain

$$\begin{aligned}
\text{RHS} &= \frac{k!}{2!(k-2)!} + k(n-k) + \frac{(n-k)!}{2!(n-k-2)!} \\
&= \frac{k(k-1)}{2} + k(n-k) + \frac{(n-k)(n-k-1)}{2} \\
&= \frac{1}{2}\left(k^2 - k + kn - k^2 + n^2 - nk - n - kn + k^2 + k\right) \\
&= \frac{1}{2}\left(n^2 - n\right) .
\end{aligned}$$

Which we can recognize as equivalent to $\binom{n}{2}$ since from its definition we have that

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2} .$$

proving the desired equivalence. A combinatorial argument for this expression can be given in the following way. The left hand side $\binom{n}{2}$ represents the number of ways to select two items from n. Now for any k (with $1 \leq k \leq n$) we can think about the entire set of n items as being divided into two parts. The first part will have k items and the second part will have the remaining $n-k$ items. Then by considering all possible halves the two items selected could come from will yield the decomposition shown on the right hand side of the above. For example, we can draw our two items from the initial k in the first half in $\binom{k}{2}$ ways, from the second half (which has $n-k$ elements) in $\binom{n-k}{2}$ ways, or by drawing one element from the set with k elements and another element from the set with $n-k$ elements, in $k(n-k)$ ways. Summing all of these terms together gives

$$\binom{k}{2} + k(n-k) + \binom{n-k}{2} \quad \text{for} \quad 1 \leq k \leq n ,$$

as an equivalent expression for $\binom{n}{2}$.

Chapter 2 (Axioms of Probability)

Chapter 2: Problems

Problem 1 (the sample space)

The sample space consists of the possible experimental outcomes, which in this case is given by

$$\{(R,R),(R,G),(R,B),(G,R),(G,G),(G,B),(B,R),(B,G),(B,B)\}\,.$$

If the first marble is not replaced then our sample space loses all "paired" terms in the above (i.e. terms like (R,R)) and it becomes

$$\{(R,G),(R,B),(G,R),(G,B),(B,R),(B,G)\}\,.$$

Problem 2 (the sample space of continually rolling a die)

The sample space consists of all possible die rolls to obtain a six. For example we have

$$\{(6),(1,6),(2,6),(3,6),(4,6),(5,6),(1,1,6),(1,2,6),\cdots,(2,1,6),(2,2,6)\cdots\}$$

The points in E_n are all sequences of rolls with n elements in them, so that $\cup_1^\infty E_n$ is all possible sequences ending with a six. Since a six must happen eventually, we have $(\cup_1^\infty E_n)^c = \phi$.

Problem 8 (mutually exclusive events)

Since A and B are mutually exclusive then $P(A\cup B) = P(A) + P(B)$.

Part (a): To calculate the probability that either A or B occurs we evaluate $P(A\cup B) = P(A) + P(B) = 0.3 + 0.5 = 0.8$

Part (b): To calculate the probability that A occurs but B does not we want to evaluate $P(A\backslash B)$. This can be done by considering

$$P(A\cup B) = P(B\cup(A\backslash B)) = P(B) + P(A\backslash B)\,,$$

where the last equality is due to the fact that B and $A\backslash B$ are mutually independent. Using what we found from part (a) $P(A\cup B) = P(A) + P(B)$, the above gives

$$P(A\backslash B) = P(A) + P(B) - P(B) = P(A) = 0.3\,.$$

Part (c): To calculate the probability that both A and B occurs we want to evaluate $P(A \cap B)$, which can be found by using

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Using what we know in the above we have that

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.8 - 0.3 - 0.5 = 0,$$

Problem 9 (accepting credit cards)

Let A be the event that a person carries the American Express card and B be the event that a person carries the VISA card. Then we want to evaluate $P(\cup B)$, the probability that a person carries the American Express card or the person carries the VISA card. This can be calculated as

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.24 + 0.64 - 0.11 = 0.77.$$

Problem 10 (wearing rings and necklaces)

Let $P(A)$ be the probability that a student wears a ring. Let $P(B)$ be the probability that a student wears a necklace. Then from the information given we have that

$$\begin{aligned} P(A) &= 0.2 \\ P(B) &= 0.3 \\ P((A \cup B)^c) &= 0.3. \end{aligned}$$

Part (a): We desire to calculate for this subproblem $P(A \cup B)$, which is given by

$$P(A \cup B) = 1 - P((A \cup B)^c) = 1 - 0.6 = 0.4,$$

Part (b): We desire to calculate for this subproblem $P(AB)$, which can be calculated by using the inclusion/exclusion identity for two sets which is

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

so solving for $P(AB)$ in the above we find that

$$P(AB) = P(A) + P(B) - P(A \cup B) = 0.2 + 0.3 - 0.4 = 0.1.$$

Problem 11 (smoking cigarettes v.s cigars)

Let A be the event that a male smokes cigarettes and let B be the event that a male smokes cigars. Then the data given is that $P(A) = 0.28$, $P(B) = 0.07$, and $P(AB) = 0.05$.

Part (a): We desire to calculate for this subproblem $P((A \cup B)^c)$, which is given by (using the inclusion/exclusion identity for two sets)

$$\begin{aligned} P((A \cup B)^c) &= 1 - P(A \cup B) \\ &= 1 - (P(A) + P(B) - P(AB)) \\ &= 1 - 0.28 - 0.07 + 0.05 = 0.7 \,. \end{aligned}$$

Part (b): We desire to calculate for this subproblem $P(B \cap A^c)$ We will compute this from the identity

$$P(B) = P((B \cap A^c) \cup (B \cap A)) = P(B \cap A^c) + P(B \cap A) \,,$$

since the events $B \cap A^c$ and $B \cap A$ are mutually exclusive. With this identity we see that the event that we desire the probability of $(B \cap A^c)$ is given by

$$P(B \cap A^c) = P(B) - P(A \cap B) = 0.07 - 0.05 = 0.02 \,.$$

Problem 12 (language probabilities)

Let S be the event that a student is in a Spanish class, let F be the event that a student is in a French class and let G be the event that a student is in a German class. From the data given we have that

$$\begin{aligned} P(S) &= 0.28 \,, \quad P(F) = 0.26 \,, \quad P(G) = 0.16 \\ P(S \cap F) &= 0.12 \,, \quad P(S \cap G) = 0.04 \,, \quad P(F \cap G) = 0.06 \\ P(S \cap F \cap G) &= 0.02 \,. \end{aligned}$$

Part (a): We desire to compute

$$P(\neg(S \cup F \cup G)) = 1 - P(S \cup F \cup G) \,.$$

Define the event A to be $A = S \cup F \cup G$, then we will use the inclusion/exclusion identity for three sets which expresses $P(A) = P(S \cup F \cup G)$ in terms of set intersections as

$$\begin{aligned} P(A) &= P(S) + P(F) + P(G) - P(S \cap F) - P(S \cap G) - P(F \cap G) + P(S \cap F \cap G) \\ &= 0.28 + 0.26 + 0.16 - 0.12 - 0.04 - 0.06 + 0.02 = 0.5 \,. \end{aligned}$$

So that we have that $P(\neg(S \cup F \cup G)) = 1 - 0.5 = 0.5$.

Part (b): Using the definitions of the events above for this subproblem we want to compute

$$P(S \cap (\neg F) \cap (\neg G)) \,, \quad P((\neg S) \cap F \cap (\neg G)) \,, \quad P((\neg S) \cap (\neg F) \cap G) \,.$$

As these are all of the same form, lets first consider $P(S \cap (\neg F) \cap (\neg G))$, which equals $P(S \cap (\neg(F \cup G)))$. Now decomposing S into two disjoint sets $S \cap (\neg(F \cup G))$ and $S \cap (F \cup G)$ we see that $P(S)$ can be written as

$$P(S) = P(S \cap (\neg(F \cup G))) + P(S \cap (F \cup G)) \,.$$

Now since we know $P(S)$ if we knew $P(S\cap(F\cup G))$ we can compute the desired probability. Distributing the intersection in $S\cap(F\cup G)$, we see that we can write this set as

$$S\cap(F\cup G) = (S\cap F)\cup(S\cap G)\,.$$

So that $P(S\cap(F\cup G))$ can be computed (using the inclusion/exclusion identity) as

$$\begin{aligned}
P(S\cap(F\cup G)) &= P((S\cap F)\cup(S\cap G)) \\
&= P(S\cap F) + P(S\cap G) - P((S\cap F)\cap(S\cap G)) \\
&= P(S\cap F) + P(S\cap G) - P(S\cap F\cap G) \\
&= 0.12 + 0.04 - 0.02 = 0.14\,.
\end{aligned}$$

Thus

$$\begin{aligned}
P(S\cap(\neg(F\cup G))) &= P(S) - P(S\cap(F\cup G)) \\
&= 0.28 - 0.14 = 0.14\,.
\end{aligned}$$

In the same way we find that

$$\begin{aligned}
P((\neg S)\cap F\cap(\neg G)) &= P(F) - P(F\cap(S\cup G)) \\
&= P(F) - (P(F\cap S) + P(F\cap G) - P(F\cap S\cap G) \\
&= 0.26 - 0.12 - 0.06 + 0.02 = 0.1\,.
\end{aligned}$$

and that

$$\begin{aligned}
P((\neg S)\cap(\neg F)\cap G) &= P(G) - P(G\cap(S\cup F)) \\
&= P(G) - (P(G\cap S) + P(G\cap F) - P(S\cap F\cap G) \\
&= 0.16 - 0.04 - 0.06 + 0.02 = 0.08\,.
\end{aligned}$$

With all of these intermediate results we can compute that the probability that a student is taking exactly one language class is given by the sum of the probabilities of the three events introduced at the start of this subproblem. We find that this sum is given by

$$0.14 + 0.1 + 0.08 = 0.32\,.$$

Part (c): If two students are chosen randomly the probability that at least one of them is taking a language class is the complement of the probability that neither is taking a language class. From part a of this problem we know that fifty students are not taking a language class, from the one hundred students at the school. Therefore the probability that we select two students *both* not in a language class is given by

$$\frac{\binom{50}{2}}{\binom{100}{2}} = \frac{1225}{4950} = \frac{49}{198}\,,$$

thus the probability of drawing two students at least one of which is in a language class is given by

$$1 - \frac{49}{198} = \frac{149}{198}\,.$$

Problem 13 (the number of paper readers)

Before we begin to solve this problem lets take the given probabilities of *intersections* of events and convert them into probabilities of *unions* of events. Then if we need these values later in the problem we will have them. This can be done with the inclusion-exclusion identity. For two general sets A and B the inclusion-exclusion identity is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) .$$

Using this we can evaluate the probabilities of union of events.

$$\begin{aligned}
P(\text{II} \cup \text{III}) &= P(\text{II}) + P(\text{III}) - P(\text{II} \cap \text{III}) = 0.3 + 0.05 - 0.04 = 0.31 \\
P(\text{I} \cup \text{II}) &= P(\text{I}) + P(\text{II}) - P(\text{I} \cap \text{II}) = 0.1 + 0.3 - 0.08 = 0.32 \\
P(\text{I} \cup \text{III}) &= P(\text{I}) + P(\text{III}) - P(\text{I} \cap \text{III}) = 0.1 + 0.05 - 0.02 = 0.13 \\
P(\text{I} \cup \text{II} \cup \text{III}) &= P(\text{I}) + P(\text{II}) + P(\text{III}) - P(\text{I} \cap \text{II}) - P(\text{I} \cap \text{III}) \\
&- P(\text{II} \cap \text{III}) + P(\text{I} \cap \text{II} \cap \text{III}) \\
&= 0.1 + 0.3 + 0.05 - 0.08 - 0.02 - 0.04 + 0.01 = 0.32 .
\end{aligned}$$

We will now use these results in the following whereever needed.

Part (a): The requested proportion of people who read only one paper can be represented from three disjoint probabilities/proportions:

1. $P(\text{I} \cap \neg\text{II} \cap \neg\text{III})$ which represents the proportion of people who only read paper I.
2. $P(\neg\text{I} \cap \text{II} \cap \neg\text{III})$ which represents the proportion of people who only read paper II.
3. $P(\neg\text{I} \cap \neg\text{II} \cap \text{III})$ which represents the proportion of people who only read paper III.

The sum of these three probabilities will be the total number of people who read only one newspaper. To compute the first probability ($P(\text{I} \cap \neg\text{II} \cap \neg\text{III})$) we begin by noting that

$$P(\text{I} \cap \neg\text{II} \cap \neg\text{III}) + P(\text{I} \cap \neg(\neg\text{II} \cap \neg\text{III})) = P(\text{I}) ,$$

which is true since we can obtain the event I by intersecting it with two sets that union to the entire sample space i.e. $\neg\text{II} \cap \neg\text{III}$, and its negation $\neg(\neg\text{II} \cap \neg\text{III})$. With this expression we can evaluate our desired probability $P(\text{I} \cap \neg\text{II} \cap \neg\text{III})$ using the above. Simple subtraction gives

$$\begin{aligned}
P(\text{I} \cap \neg\text{II} \cap \neg\text{III}) &= P(\text{I}) - P(\text{I} \cap \neg(\neg\text{II} \cap \neg\text{III})) \\
&= P(\text{I}) - P(\text{I} \cap (\text{II} \cup \text{III})) \\
&= P(\text{I}) - P((\text{I} \cap \text{II}) \cup (\text{I} \cap \text{III})) .
\end{aligned}$$

Where the last two equations follows from the first by some simple set theory. Since the problem statement gives the probabilities of the events $\text{I} \cap \text{II}$ and $\text{I} \cap \text{III}$, to be able to further evaluate the right hand side of the expression above requires the ability to compute probabilities of unions of such sets. This can be done with the inclusion-exclusion identity which for two general sets A and B is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Thus the above desired probability then becomes

$$\begin{aligned} P(\mathrm{I} \cap \neg\mathrm{II} \cap \neg\mathrm{III}) &= P(\mathrm{I}) - P(\mathrm{I} \cap \mathrm{II}) - P(\mathrm{I} \cap \mathrm{III}) + P((\mathrm{I} \cap \mathrm{II}) \cap (\mathrm{I} \cap \mathrm{III})) \\ &= P(\mathrm{I}) - P(\mathrm{I} \cap \mathrm{II}) - P(\mathrm{I} \cap \mathrm{III}) + P(\mathrm{I} \cap \mathrm{II} \cap \mathrm{III}) \\ &= 0.1 - 0.08 - 0.02 + 0.01 = 0.01\,, \end{aligned}$$

using the numbers provided. For the probability $P(\neg\mathrm{I} \cap \mathrm{II} \cap \neg\mathrm{III})$ of we can use the work earlier with the substitutions

$$\begin{aligned} \mathrm{I} &\to \mathrm{II} \\ \mathrm{II} &\to \mathrm{I}\,. \end{aligned}$$

Since in the first probability we computed the event *not* negated is event I, while in the second probability this is event II. This substitution gives

$$\begin{aligned} P(\neg\mathrm{I} \cap \mathrm{II} \cap \neg\mathrm{III}) &= P(\mathrm{II}) - P(\mathrm{II} \cap \mathrm{I}) - P(\mathrm{II} \cap \mathrm{III}) + P(\mathrm{II} \cap \mathrm{I} \cap \mathrm{III}) \\ &= 0.3 - 0.08 - 0.04 + 0.01 = 0.19\,, \end{aligned}$$

For the probability $P(\neg\mathrm{I} \cap \neg\mathrm{II} \cap \mathrm{III})$ of we can use the work earlier with the substitutions

$$\begin{aligned} \mathrm{I} &\to \mathrm{III} \\ \mathrm{III} &\to \mathrm{I}\,. \end{aligned}$$

To give

$$\begin{aligned} P(\neg\mathrm{I} \cap \neg\mathrm{II} \cap \mathrm{III}) &= P(\mathrm{III}) - P(\mathrm{III} \cap \mathrm{II}) - P(\mathrm{III} \cap \mathrm{I}) + P(\mathrm{I} \cap \mathrm{II} \cap \mathrm{III}) \\ &= 0.05 - 0.04 - 0.02 + 0.01 = 0.00\,. \end{aligned}$$

Finally the number of people who read only one newspaper is given by

$$0.01 + 0.19 + 0.00 = 0.2\,,$$

so the number of people who read only one newspaper is given by $0.2 \times 10^5 = 20{,}000$.

Part (b): The requested proportion of people who read at least two newspapers can be represented from three disjoint probabilities/proportions:

1. $P(\mathrm{I} \cap \mathrm{II} \cap \neg\mathrm{III})$
2. $P(\mathrm{I} \cap \neg\mathrm{II} \cap \mathrm{III})$
3. $P(\neg\mathrm{I} \cap \mathrm{II} \cap \mathrm{III})$
4. $P(\mathrm{I} \cap \mathrm{II} \cap \mathrm{III})$

We can compute each in the following ways. For the first probability we note that

$$\begin{aligned}
P(\neg \mathrm{I} \cap \mathrm{II} \cap \mathrm{III}) + P(\mathrm{I} \cap \mathrm{II} \cap \mathrm{III}) &= P(\mathrm{II} \cap \mathrm{III}) \\
&= P(\mathrm{II}) + P(\mathrm{III}) - P(\mathrm{II} \cup \mathrm{III}) \\
&= 0.3 + 0.5 - 0.31 = 0.04\,.
\end{aligned}$$

So that $P(\neg \mathrm{I} \cap \mathrm{II} \cap \mathrm{III}) = 0.04 - P(\mathrm{I} \cap \mathrm{II} \cap \mathrm{III}) = 0.04 - 0.01 = 0.03$. Using this we find that

$$\begin{aligned}
P(\mathrm{I} \cap \neg \mathrm{II} \cap \mathrm{III}) &= P(\mathrm{I} \cap \mathrm{III}) - P(\mathrm{I} \cap \mathrm{II} \cap \mathrm{III}) \\
&= P(\mathrm{I}) + P(\mathrm{III}) - P(\mathrm{I} \cup \mathrm{III}) - P(\mathrm{I} \cap \mathrm{II} \cap \mathrm{III}) \\
&= 0.1 + 0.5 - 0.13 - 0.01 = 0.01\,,
\end{aligned}$$

and that

$$\begin{aligned}
P(\mathrm{I} \cap \mathrm{II} \cap \neg \mathrm{III}) &= P(\mathrm{I} \cap \mathrm{II}) - P(\mathrm{I} \cap \mathrm{II} \cap \mathrm{III}) \\
&= P(\mathrm{I}) + P(\mathrm{II}) - P(\mathrm{I} \cup \mathrm{II}) - P(\mathrm{I} \cap \mathrm{II} \cap \mathrm{III}) \\
&= 0.1 + 0.3 - 0.32 - 0.01 = 0.07\,.
\end{aligned}$$

We also have $P(\mathrm{I} \cap \mathrm{II} \cap \mathrm{III}) = 0.01$, from the problem statement. Combining all of this information the total percentage of people that read at least two newspapers is given by

$$0.03 + 0.01 + 0.07 + 0.01 = 0.12\,,$$

so the total number of people is given by $0.12 \times 10^5 = 12000$.

Part (c): For this part we to compute $P((\mathrm{I} \cap \mathrm{II}) \cup (\mathrm{III} \cap \mathrm{II}))$, which gives

$$\begin{aligned}
P((\mathrm{I} \cap \mathrm{II}) \cup (\mathrm{III} \cap \mathrm{II})) &= P(\mathrm{I} \cap \mathrm{II}) + P(\mathrm{III} \cap \mathrm{II}) - P(\mathrm{I} \cap \mathrm{II} \cap \mathrm{III}) \\
&= 0.08 + 0.04 - 0.01 = 0.11\,,
\end{aligned}$$

so the number of people read at least one morning paper and one evening paper is $0.11 \times 10^5 = 11000$.

Part (d): To not read any newspaper we are looking for

$$1 - P(\mathrm{I} \cup \mathrm{II} \cup \mathrm{III}) = 1 - 0.32 = 0.68\,,$$

so the number of people is 68000.

Part (e): To read only one morning paper and one evening paper is expressed as

$$P(\mathrm{I} \cup \mathrm{II} \cup \neg \mathrm{III}) + P(\neg \mathrm{I} \cap \mathrm{II} \cap \mathrm{III})\,.$$

The first expression has been calculated as 0.01, while the second expansion has been calculated as 0.03 giving a total 0.04 giving a total of 40000 people who read I as their morning paper and II as their evening paper or who read III as their morning paper and II as their evening paper. This number excludes the number who read all three papers.

Problem 14 (an inconsistent study)

Following the hint given in the book, we let M denote the set of people who are married, W the set of people who are working professionals, and G the set of people who are college graduates. If we choose a random person and ask what the probability that he/she is either married or working or a graduate we are looking to compute $P(M \cup W \cup G)$. By the inclusion/exclusion theorem we have that the probability of this event is given by

$$\begin{aligned} P(M \cup W \cup G) &= P(M) + P(W) + P(G) \\ &- P(M \cap W) - P(M \cap G) - P(W \cap G) \\ &+ P(M \cap W \cap G) \,. \end{aligned}$$

From the given data each individual event probability can be estimated as

$$P(M) = \frac{470}{1000} \,, \quad P(G) = \frac{525}{1000} \,, \quad P(W) = \frac{312}{1000}$$

and each pairwise event probability can be estimated as

$$P(M \cap G) = \frac{147}{1000} \,, \quad P(M \cap W) = \frac{86}{1000} \,, \quad P(W \cap G) = \frac{42}{1000}$$

Finally the three-way event probability can be estimated as

$$P(M \cap W \cap G) = \frac{25}{1000} \,.$$

Using these numbers in the inclusion/exclusion formula above we find that

$$\begin{aligned} P(M \cup W \cup G) &= 0.47 + 0.525 + 0.312 - 0.147 - 0.086 - 0.042 + 0.025 \\ &= 1.057 > 1 \,, \end{aligned}$$

in contradiction to the rules of probability.

Problem 15 (probabilities of various poker hands)

Part (a): We must count the number of ways to obtain five cards of the same suit. We can first pick the suit in $\binom{4}{1} = 4$ ways afterwhich we must pick five cards in $\binom{13}{5}$ ways. So in total we have

$$4 \binom{13}{5} = 5148 \,,$$

ways to pick cards in a flush giving a probability of

$$\frac{4 \binom{13}{5}}{\binom{52}{5}} = 0.00198 \,.$$

Part (b): We can select the first denomination "a" in thirteen ways with $\binom{4}{2}$ ways to obtain the faces for these two cards. We can select the second denomination "b" in twelve ways with $\binom{4}{1}$ possible faces, the third denomination in eleven ways with four faces, the fourth denomination in ten ways again with four possible faces. The selection of the cards "b", "c", and "d" can be permuted in any of the 3! ways and the same hand results. Thus we have in total for the number of paired hands the following count

$$\frac{13\binom{4}{2}\cdot 12\binom{4}{1}\cdot 11\binom{4}{1}\cdot 10\binom{4}{1}}{3!} = 1098240\,.$$

Giving a probability of 0.42256.

Part (c): To calculate the number of hands with two pairs we have $\binom{13}{1}\binom{4}{2}$ ways to select the "a" pair. Then $\binom{12}{1}\binom{4}{2}$ ways to select the "b" pair. Since first selecting the "a" pair and then the "b" pair results in the same hand as selecting the "b" pair and then the "a" pair this direct product over counts the total number of "a" and "b" pairs by $2! = 2$. Finally, we have $\binom{11}{1}\binom{4}{1}$ ways to pick the last card in the hand. Thus we have

$$\frac{\binom{13}{1}\binom{4}{2}\cdot\binom{12}{1}\binom{4}{2}}{2!}\cdot\binom{11}{1}\binom{4}{1} = 123552\,,$$

total number of hands. Giving a probability of 0.04754.

Part (d): We have $\binom{13}{1}\binom{4}{3}$ ways to pick the "a" triplet. We can then pick "b" in $\binom{12}{1}\cdot 4$ and pick "c" in $\binom{11}{1}\cdot 4$. This combination over counts by two so that the total number of three of a kind hands is given by

$$\binom{13}{1}\cdot\binom{4}{3}\frac{\binom{12}{1}\cdot 4\binom{11}{1}\cdot 4}{2!} = 54912\,,$$

giving a probability of 0.021128.

Part (e): We have $13\cdot\binom{4}{4}$ ways to pick the "a" denomination and twelve ways to pick the second card with a possible four faces, giving in total $13\cdot 12\cdot 4 = 624$ possible hands. This gives a probability of 0.00024.

Problem 16 (poker dice probabilities)

Part (a): Using the results from parts (b)-(g) for this problem our probability of interest is

$$1 - P_b - P_c - P_d - P_e - P_f - P_g \,,$$

where P_i is the probability computed during part "i" of this problem. Using the values provided in the problem we can evaluate the above to 0.0925.

Part (b): So solve this problem we will think of the die's outcome as being a numerical specifications (one through six) of five "slots". In this specification there are 6^5 total outcomes for a trial with the five dice. To determine the number of one pair "hands", we note that we can pick the number in the pair in six ways and their locations from the five bins in $\binom{5}{2}$ ways. Another number in the hand can be chosen from the five remaining numbers and placed in any of the remaining bins in $\binom{3}{1}$ ways. Continuing this line of reasoning for the values and placements of the remaining two die, we have

$$6 \cdot \binom{5}{2} \cdot 5 \binom{3}{1} \cdot 4 \binom{2}{1} \cdot 3 \binom{1}{1} \,,$$

as the number of *ordered* placements of our four distinct numbers. Since the ordered placement of the three different singleton numbers does not matter we must divide this result by 3!, which results in a value of 3600. Then the probability of one pair is given by

$$\frac{3600}{6^5} = 0.4629 \,.$$

Part (c): A roll consisting of two pairs can be obtained by first selecting the numerical value for the first pair. This can be done in six ways. The numerical value for the second pair can be selected in five ways and the numerical value of the third die in four ways. We can place the first pair in bins in $\binom{6}{2}$ different ways, the second pair in bins in $\binom{4}{2}$ ways, and the remaining singleton number in bins in $\binom{1}{1}$ ways. Giving a product representing the number of orderings of two pairs as

$$\left(6 \cdot \binom{6}{2}\right)\left(5 \cdot \binom{4}{2}\right)\left(4 \cdot \binom{1}{1}\right) = 10800 \,.$$

This number represents an *ordered* sequence of three blocks of items. To compute the unordered sequence of three items we need to divide this number by 3!, giving 1800. Combined this gives a probability of obtaining two pair of

$$\frac{1800}{6^5} = 0.2315 \,.$$

Part (c): We can pick the number for the digit that is repeated three times in six ways, another digit in five ways and the final digit in four ways. The number of ways we can place the three die with the same numeric value is given by $\binom{5}{3}$ ways. So the number of permutations of these three numbers is given by

$$6 \cdot 5 \cdot 4 \cdot \binom{5}{3} = 1200\,.$$

This gives a probability of $\frac{1200}{6^5} = 0.154$.

Part (e): Assuming that a full house is five die, three and two of which the same numeric value. The we can choose the two numbers represented on the die in six ways for the first and in five ways for the second. In this case however that the order of the two numbers chosen *does* matter, since if we change the order of the two numbers the number on the triple changes the number on the double changes. Thus the probability of a full houses is given by

$$\frac{6 \cdot 5}{6^5} = 0.00386\,.$$

Part (f): To get four die with the same numeric value we must pick one special number out of six in $\binom{6}{1}$ ways representing the four common die. We then pick one more number from the remaining five in $\binom{5}{1}$ ways representing the number on the lone die. Thus we have $\binom{6}{1} \cdot \binom{5}{1}$ ways to pick the two numbers to use in the selection of this hand. Since the order of the two chosen numbers does *not* matter we need to divide that number by 2! so the count of the number of arraignments is given by

$$\frac{\binom{6}{1} \cdot \binom{5}{1}}{2} = 15\,.$$

This gives a requested probability of $\frac{15}{6^5} = 0.001929$.

Part (g): If all five die are the same then there are one of six possibilities (the six numbers on a die). The total number of possible die throws is $6^5 = 7776$ giving a probability to throw this hand of

$$\frac{6}{6^5} = \frac{1}{6^4} = 0.0007716\,.$$

Problem 17 (randomly placing rooks)

A possible placement of a rook on the chess board can be obtained by specifying the row and column at which we will locate our rook. Since there are eight rows and eight columns there are $8^2 = 64$ possible placements for a given rook. After we place each rook we obviously have one less position where we can place the additional rooks. So the total number of possible locations where we can place eight rooks is given by

$$64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57\,,$$

since the order of placement does not matter we must divide this number by 8! to get

$$\frac{64!}{8!(64-8)!} = \left(\begin{array}{c} 64 \\ 8 \end{array} \right) = 4426165368 \,.$$

The number of locations where eight rooks can be placed who won't be able to capture any of the other is given by

$$8^2 \cdot 7^2 \cdot 6^2 \cdot 5^2 \cdot 4^2 \cdot 3^2 \cdot 2^2 \cdot 1^2 \,,$$

Which can be reasoned as follows. The first rook can be placed in 64 different places. Once this rook is located we cannot place the next rook in the same row or column that the first rook holds. This leaves seven choices for a row and seven choices for a column giving a total of $7^2 = 49$ possible choices. Since the order of these choices does not matter we will need to divide this product by 8! giving a total probability of

$$\frac{\frac{8!^2}{8!}}{\left(\begin{array}{c} 64 \\ 8 \end{array} \right)} = 9.109\, 10^{-6} \,,$$

in agreement with the book.

Problem 18 (randomly drawing blackjack)

The total number of possible two card hands is given by $\left(\begin{array}{c} 52 \\ 2 \end{array} \right)$. We can draw an ace in one of four possible ways i.e. in $\left(\begin{array}{c} 4 \\ 1 \end{array} \right)$ ways. For blackjack the other card must be a ten or a jack or a queen or a king (of any suite) and can be drawn in $\left(\begin{array}{c} 4+4+4+4 \\ 1 \end{array} \right) = \left(\begin{array}{c} 16 \\ 1 \end{array} \right)$ possible ways. Thus the number of possible ways to draw blackjack is given by

$$\frac{\left(\begin{array}{c} 4 \\ 1 \end{array} \right)\left(\begin{array}{c} 16 \\ 1 \end{array} \right)}{\left(\begin{array}{c} 52 \\ 2 \end{array} \right)} = 0.048265 \,.$$

Problem 19 (symmetric dice)

We can solve this problem by considering the disjoint events that both die land on colors given by red, black, yellow, or white. For the first die to land on red will happen with probability 2/6, the same for the second die. Thus the probability that both die land on red is given by

$$\left(\frac{2}{6}\right)^2 \,.$$

Summing up all the probabilities for all the possible colors, we have a total probability of obtaining the same color on both die given by

$$\left(\frac{2}{6}\right)\left(\frac{2}{6}\right)+\left(\frac{2}{6}\right)\left(\frac{2}{6}\right)+\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)+\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)=\frac{5}{18}.$$

Problem 21 (the number of children)

Part (a): Let P_i be the probability that the family chosen has i children. Then we see from the numbers provided that $P_1 = \frac{4}{20} = \frac{1}{5}$, $P_2 = \frac{8}{20} = \frac{2}{5}$, $P_3 = \frac{5}{20} = \frac{1}{4}$, and $P_4 = \frac{1}{20}$, assuming a uniform probability of selecting any given family.

Part (b): We have

$$4(1) + 8(2) + 5(3) + 2(4) + 1(5) = 4 + 16 + 15 + 8 + 5 = 48\,,$$

total children. Then the probability a random child comes from a family with i children is given by (and denoted by P_i) is $P_1 = \frac{4}{48}$, $P_2 = \frac{16}{48}$, $P_3 = \frac{15}{48}$, $P_4 = \frac{8}{48}$, and $P_5 = \frac{5}{48}$.

Problem 22 (shuffling a deck of cards)

To have the ordering exactly the same we must have k tails in a row followed by $n-k$ heads in a row, where $k = 0$ to $k = n$. The probability of getting k tails followed by $n-k$ heads is

$$\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k}=\left(\frac{1}{2}\right)^n$$

Now since each of these outcomes is mutually exclusive to compute the total probability we can sum this result for $k = 0$ to $k = n$ to get

$$\sum_{k=0}^{n}\left(\frac{1}{2}\right)^n=\frac{n+1}{2^n}.$$

Problem 23 (a larger roll than the first)

We begin by constructing the sample space of possible outcomes. These numbers are computed in table 3, where the row corresponds to the outcome of the first die through and the column corresponds to the outcome of the second die through. In each square we have placed a one if the number on the second die is strictly larger than the first. Since each element of our sample space has a probability of $1/36$, by enumeration we find that

$$\frac{15}{36}=\frac{5}{12},$$

is our desired probability.

	1	2	3	4	5	6
1	0	1	1	1	1	1
2	0	0	1	1	1	1
3	0	0	0	1	1	1
4	0	0	0	0	1	1
5	0	0	0	0	0	1
6	0	0	0	0	0	0

Table 3: The elements of the sample space where the second die is strictly larger in value than the first.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Table 4: The possible values for the sum of the values when two die are rolled.

Problem 24 (the probability the sum of the die is i)

As in Problem 23 we can explicitly enumerate these probabilities by counting the number of times each occurrence happens, in Table 4 we have placed the sum of the two die in the center of each square. Then by counting the number of squares where are sum equals each number from two to twelve, we have

$$\begin{aligned}
P_2 &= \frac{1}{36}, \quad P_7 = \frac{6}{36} = \frac{1}{6} \\
P_3 &= \frac{2}{36} = \frac{1}{18}, \quad P_8 = \frac{5}{36} \\
P_4 &= \frac{3}{36} = \frac{1}{12}, \quad P_9 = \frac{4}{36} = \frac{1}{9} \\
P_5 &= \frac{4}{36} = \frac{1}{9}, \quad P_{10} = \frac{3}{36} = \frac{1}{12} \\
P_6 &= \frac{5}{36}, \quad P_{11} = \frac{2}{36} = \frac{1}{18}, \quad P_{12} = \frac{1}{36}.
\end{aligned}$$

Problem 25 (rolling a five before a seven)

A sum of five has a probability of $P_5 = \frac{2}{18} = \frac{1}{9}$ of occurring. A sum of seven has a probability of $P_7 = \frac{1}{6}$ of occurring, so the probability that neither a five or a seven is given by $1 - \frac{1}{9} - \frac{1}{6} = \frac{13}{18}$. Following the hint we let E_n be the event that a five occurs on the

n-th roll and no five or seven occurs on the $n-1$-th rolls up to that point. Then

$$P(E_n) = \left(\frac{13}{18}\right)^{n-1}\frac{1}{9},$$

since we want the probability that a five comes first, this can happen at roll number one ($n = 1$), at roll number two ($n = 2$) or any subsequent roll. Thus the probability that a five comes first is given by

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{13}{18}\right)^{n-1}\frac{1}{9} &= \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{13}{18}\right)^{n} \\ &= \frac{1}{9}\frac{1}{\left(1-\frac{13}{18}\right)} = \frac{2}{5} = 0.4\,.\end{aligned}$$

Problem 26 (winning at craps)

From Problem 24 we have computed the individual probabilities for various sum of two random die. Following the hint, let E_i be the event that the initial die sum to i and that the player wins. We can compute some of these probabilities immediately $P(E_2) = P(E_3) = P(E_{12}) = 0$, and $P(E_7) = P(E_{11}) = 1$. We now need to compute $P(E_i)$ for $i = 4, 5, 6, 8, 9, 10$. Again following the hint define $E_{i,n}$ to be the event that the player initial sum is i and wins on the n-th **subsequent** roll. Then

$$P(E_i) = \sum_{n=1}^{\infty} P(E_{i,n})\,,$$

since if we win, it must be either on the first, or second, or third, etc roll *after the initial roll*. We now need to calculate the $P(E_{i,n})$ probabilities for each n. As an example of this calculation first lets compute $P(E_{4,n})$ which means that we initially roll a sum of four and the player wins on the n-th subsequent roll. We will win if we roll a sum of a four or loose if we roll a sum of a seven, while if roll anything else we continue, so to win when $n = 1$ we see that

$$P(E_{4,1}) = \frac{1+1+1}{36} = \frac{1}{12},$$

since to get a sum of four we can roll pairs consisting of $(1, 3)$, $(2, 2)$, and $(3, 1)$.

To compute $P(E_{4,2})$ the rules of craps state that we will win if a sum of four comes up (with probability $\frac{1}{12}$) and loose if a sum of a seven comes up (with probability $\frac{6}{36} = \frac{1}{6}$) and continue playing if anything else is rolled. This last event (continued play) happens with probability

$$1 - \frac{1}{12} - \frac{1}{6} = \frac{3}{4}.$$

Thus $P(E_{4,2}) = \left(\frac{3}{4}\right)\frac{1}{12} = \frac{1}{16}$. Here the first $\frac{3}{4}$ is the probability we don't roll a four or a seven on the $n = 1$ roll and the second $\frac{1}{12}$ comes from rolling a sum of a four on the second roll (where $n = 2$). In the same way we have for $P(E_{4,3})$ the following

$$P(E_{4,3}) = \left(\frac{3}{4}\right)^2\frac{1}{12}.$$

Here the first two factors of $\frac{3}{4}$ are from the two rolls that "keep us in the game", and the factor of $\frac{1}{12}$, is the roll that allows us to win. Continuing in this in this manner we see that

$$P(E_{4,4}) = \left(\frac{3}{4}\right)^3 \frac{1}{12},$$

and in general we find that

$$P(E_{4,n}) = \left(\frac{3}{4}\right)^{n-1} \frac{1}{12} \quad \text{for} \quad n \geq 1\,.$$

To compute $P(E_{i,n})$ for other i, the derivations just performed, only change in the probabilities required to roll the initial sum. We thus find that for other initial rolls (heavily using the results of Problem 24) that

$$\begin{aligned}
P(E_{5,n}) &= \frac{1}{9}\left(1 - \frac{1}{9} - \frac{1}{6}\right)^{n-1} = \frac{1}{9}\left(\frac{13}{18}\right)^{n-1} \\
P(E_{6,n}) &= \frac{5}{36}\left(1 - \frac{5}{36} - \frac{1}{6}\right)^{n-1} = \frac{5}{36}\left(\frac{25}{36}\right)^{n-1} \\
P(E_{8,n}) &= \frac{5}{36}\left(1 - \frac{5}{36} - \frac{1}{6}\right)^{n-1} = \frac{5}{36}\left(\frac{25}{36}\right)^{n-1} \\
P(E_{9,n}) &= \frac{1}{9}\left(1 - \frac{1}{9} - \frac{1}{6}\right)^{n-1} = \frac{1}{9}\left(\frac{13}{18}\right)^{n-1} \\
P(E_{10,n}) &= \frac{1}{12}\left(1 - \frac{1}{12} - \frac{1}{6}\right)^{n-1} = \frac{1}{12}\left(\frac{3}{4}\right)^{n-1}.
\end{aligned}$$

To compute $P(E_4)$ we need to sum the results above. We have that

$$\begin{aligned}
P(E_4) &= \frac{1}{12}\sum_{n\geq 1}\left(\frac{3}{4}\right)^{n-1} = \frac{1}{12}\sum_{n\geq 0}\left(\frac{3}{4}\right)^{n} \\
&= \frac{1}{12}\frac{1}{\left(1-\frac{3}{4}\right)} = \frac{1}{3}.
\end{aligned}$$

Note that this also gives the probability for $P(E_{10})$. For $P(E_5)$ we find $P(E_5) = \frac{2}{5}$, which also equals $P(E_9)$. For $P(E_6)$ we find that $P(E_6) = \frac{5}{11}$, which also equals $P(E_8)$. Then our probability of winning craps is given by summing all of the above probabilities weighted by the associated priors of rolling the given initial roll. We find by defining I_i to be the event that the initial roll is i and W the event that we win at craps that

$$\begin{aligned}
P(W) &= 0\,P(I_2) + 0\,P(I_3) + \frac{1}{3}P(I_4) + \frac{4}{9}P(I_5) + \frac{5}{9}P(I_6) \\
&+ 1\,P(I_7) + \frac{5}{9}P(I_8) + \frac{4}{9}P(I_9) + \frac{1}{3}P(I_{10}) + 1\,P(I_{11}) + 0\,P(I_{12})\,.
\end{aligned}$$

Using the results of Exercise 25 to evaluate $P(I_i)$ for each i we find that the above summation gives

$$P(W) = \frac{244}{495} = 0.49292\,.$$

These calculations are performed in the Matlab file `chap_2_prob_26.m`.

Problem 27 (drawing the first red ball)

We want the probability that A selects the first red ball. Since A draws first he will select a red ball on the first draw with probability $\frac{3}{10}$. If he does not select a red ball B will draw next and he must not draw a red ball (or the game will stop). The probability that A draws a red ball on the *third* total draw is then

$$P_3 = \left(1 - \frac{3}{10}\right)\left(1 - \frac{3}{9}\right)\left(\frac{3}{8}\right) .$$

Continuing this pattern we see that for A to draw a ball on the *fifth* total draw will happen with probability

$$P_5 = \left(1 - \frac{3}{10}\right)\left(1 - \frac{3}{9}\right)\left(1 - \frac{3}{8}\right)\left(1 - \frac{3}{7}\right)\left(\frac{3}{6}\right) ,$$

and finally on the *seventh* total draw with probability

$$P_7 = \left(1 - \frac{3}{10}\right)\left(1 - \frac{3}{9}\right)\left(1 - \frac{3}{8}\right)\left(1 - \frac{3}{7}\right)\left(1 - \frac{3}{6}\right)\left(1 - \frac{3}{5}\right)\left(\frac{3}{4}\right) .$$

If player A does not get a red ball after seven draws he will not draw a red ball before player B. The total probability that player A draws a red ball first is given by the sum of all these individual probabilities of these mutually exclusive events. In the Matlab code `chap_2_prob_27.m` we evaluate this sum and find the probability that A wins given by

$$P(A) = \frac{7}{12} .$$

So the corresponding probability that B wins is $1 - \frac{7}{12} = \frac{5}{12}$ showing the benefit to being the first "player" in a game like this.

Problem 28 (sampling colored balls from an urn)

Part (a): We want the probability that each ball will be of the same color. This is given by

$$\frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{5+6+8}{3}} = 0.8875 .$$

Part (b): The probability that all three balls are of different colors is given by

$$\frac{\binom{5}{1}\binom{6}{1}\binom{8}{1}}{\binom{19}{3}} = 0.247 .$$

If we replace the ball after drawing it, then the probabilities that each ball is the same color is now given by

$$\left(\frac{5}{19}\right)^3 + \left(\frac{6}{19}\right)^3 + \left(\frac{8}{19}\right)^3 = 0.124\,.$$

while if we want three balls of different colors, then this happens with probability given by

$$\left(\frac{5}{19}\right)\left(\frac{6}{19}\right)\left(\frac{8}{19}\right) = 0.3499\,.$$

Problem 30 (the chess club)

Part (a): For Rebecca and Elise to be paired they must first be selected onto their respected schools chess teams and then be paired in the tournament. Thus if S is the event that the sisters play each other then

$$P(S) = P(R)P(E)P(\text{Paired}|R, E)\,,$$

where R is the event that that Rebecca is selected for her schools chess team and E is the event that Elise is selected for her schools team and Paired is the event that the two sisters play each other. Computing these probabilities we have

$$P(R) = \frac{\binom{1}{1}\binom{7}{3}}{\binom{8}{4}} = \frac{1}{2}\,,$$

and

$$P(E) = \frac{\binom{1}{1}\binom{8}{3}}{\binom{9}{4}} = \frac{4}{9}\,,$$

and finally

$$P(\text{Paired}) = \frac{1 \cdot 3!}{4!} = \frac{1}{4}\,.$$

so that $P(S) = \frac{1}{2} \cdot \frac{4}{9} \cdot \frac{1}{4} = \frac{1}{18}$.

Part (b): The event that Rebecca and Elise are chosen and then do not play each other will occur with a probability of

$$P(R)P(E)P(\text{Paired}^c|R, E) = \frac{1}{2} \cdot \frac{4}{9}\left(1 - \frac{1}{4}\right) = \frac{1}{6}\,.$$

Part (c): For this part we can have either (and these events are mutually exclusive) Rebecca picked to represent her school or Elise picked to represent her school but not both and not neither. Since $\binom{1}{1}\binom{7}{3}$ is the number of ways to choose the team A with Rebecca as a member and $\binom{8}{4}$ are the number of ways to choose team B without having Elise as a member, their product is the number of ways of choosing the first option above. This given a probability of

$$\frac{\binom{1}{1}\binom{7}{3}}{\binom{8}{4}} \cdot \frac{\binom{8}{4}}{\binom{9}{4}} = \frac{5}{18}.$$

In the same way the other probability is given by

$$\frac{\binom{7}{4}}{\binom{8}{4}} \cdot \frac{\binom{1}{1}\binom{8}{3}}{\binom{9}{4}} = \frac{2}{9}.$$

Thus the probability we are after is the sum of the two probabilities above and is given by $\frac{9}{18} = \frac{1}{2}$.

Problem 31 (selecting basketball teams)

Part (a): On the first draw we will certainly get one of the team members. Then on the second draw we must get any team member *but* the one that we just drew. This happens with probability $\frac{2}{3}$. Finally, we must get the team member we have not drawn in the first two draws. This happens with probability $\frac{1}{3}$. In total then, the probability to draw an entire team is given by

$$1 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}.$$

Part (b): The probability the second player plays the same position as the first drawn player is given by $\frac{1}{3}$, while the probability that the third player plays the same position as the first two is given by $\frac{1}{3}$. Thus this event has a probability of

$$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}.$$

Problem 32 (a girl in the i-th position)

We can compute all permutations of the $b + g$ people that have a girl in the i-th spot as follows. We have g choices for the specific girl we place in the i-th spot. Once this girl is selected we have $b + g - 1$ other people to place in the $b + g - 1$ slots around this i-th spot. This can be done in $(b+g-1)!$ ways. So the total number of ways to place a girl at position i is $g(b + g - 1)!$. Thus the probability of finding a girl in the i-th spot is given by

$$\frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}.$$

Problem 33 (a forest of elk)

Warning: for some reason I get a different answer than the result given in the back of the book. Thus if someone finds something wrong with the logic below would they please let me know.

After tagging the initial elk, the proportion of tagged elk is given by $p = \frac{5}{20} = \frac{1}{4}$. If N is the random variable representing the number of previously tagged elk. Then after capturing four more elk the probability that $N = k$ of these elk have been tagged before is given by the *binomial* probability density with parameters $(n, p) = (4, \frac{1}{4})$. Thus the probability that two of the recently caught four have already been tagged is given by

$$P\{N = 2\} = \binom{4}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^{4-2} = \frac{27}{128}.$$

Problem 34 (the probability of a Yarborough)

We must not have a ten, a jack, a queen, or a king in our hand of thirteen cards. The number of ways to select a hand that does not have any of these cards is equivalent to selecting thirteen cards from among a set that does not contain any of the cards mentioned above. Specifically this number is

$$\frac{\binom{52-4-4-4-4}{13}}{\binom{52}{13}} = \frac{\binom{36}{13}}{\binom{52}{13}} = 0.0036\,,$$

a relatively small probability.

Problem 35 (selecting psychiatrists for a conference)

The probability that at least one psychologist is choose is given by considering all selections of sets of psychologists that contain at least one

$$\frac{\binom{30}{2}\binom{24}{1} + \binom{30}{1}\binom{24}{2} + \binom{30}{0}\binom{24}{3}}{\binom{54}{3}} = 0.8363\,.$$

Where in the numerator we have enumerated all possible selections of three people such that at least one psychologist is chosen.

Problem 36 (choosing two identical cards)

Part (a): We have $\left(\begin{array}{c} 52 \\ 2 \end{array}\right)$ possible ways to draw two cards from the 52 total. For us to draw two aces, this can be done in $\left(\begin{array}{c} 4 \\ 2 \end{array}\right)$ ways. Thus our probability is given by

$$\frac{\left(\begin{array}{c} 4 \\ 2 \end{array}\right)}{\left(\begin{array}{c} 52 \\ 2 \end{array}\right)} = 0.00452\,.$$

Part (b): For the two cards to have the same value we can pick the value to represent in thirteen ways and the two cards in $\left(\begin{array}{c} 4 \\ 2 \end{array}\right)$ ways. Thus our probability is given by

$$\frac{13\left(\begin{array}{c} 4 \\ 2 \end{array}\right)}{\left(\begin{array}{c} 52 \\ 2 \end{array}\right)} = 0.0588\,.$$

Problem 37 (solving enough problems on an exam)

Part (a): The student has a probability of $\frac{7}{10} = 0.7$ of answering any question correctly. Then from five questions the probability that she answers k questions correctly is given by a binomial distribution with parameters $(n, p) = (5, 0.7)$ or

$$P\{X = k\} = \left(\begin{array}{c} 5 \\ k \end{array}\right) p^k (1-p)^{5-k}\,.$$

So to answer all five correctly will happen with probability of

$$P\{X = 5\} = \left(\begin{array}{c} 5 \\ 5 \end{array}\right) (0.7)^5 (0.3)^0 = (0.7)^5 = 0.168\,.$$

Part (b): To answer at least four of the questions correctly will happen with probability

$$P\{X = 4\} + P\{X = 5\} = \left(\begin{array}{c} 5 \\ 4 \end{array}\right) (0.7)^4 (0.3)^1 + 0.168 = 0.3601 + 0.168 = 0.5281\,.$$

Problem 38 (two red socks)

We are told that three of the socks are red so that $n-3$ are not red. When we select two socks, the probability that they are both red is given by

$$\frac{3}{n} \cdot \frac{2}{n-1}\,.$$

If we want this to be equal to $\frac{1}{2}$ we must solve for n in the following expression

$$\frac{3}{n} \cdot \frac{2}{n-1} = \frac{1}{2} \quad \Rightarrow \quad n^2 - n = 12\,.$$

Using the quadratic formula this has a solution given by

$$n = \frac{1 \pm \sqrt{1 + 4(1)(12)}}{2(1)} = \frac{1 \pm 7}{2}\,.$$

Taking the positive solution we have that $n = 4$.

Problem 39 (five different hotels)

When the first person checks into the hotel, the next person will check into a different hotel with probability $\frac{4}{5}$. The next person will check into a different hotel with probability $\frac{3}{5}$. Thus the probability that we check into three different hotels is given by

$$\frac{4}{5} \cdot \frac{3}{5} = \frac{12}{25} = 0.48\,.$$

Problem 41 (obtaining a six at least once)

This is the complement of the probability that a six never appears or

$$1 - \left(\frac{5}{6}\right)^4 = 0.5177\,.$$

Problem 42 (double sixes)

The probability that a double six appear at least once is the complement of the probability that a double six never appears. The probability of not seeing a double six is given by $1 - \frac{1}{36} = \frac{35}{36}$, so the probability that a double six appears at least once in n throws is given by

$$1 - \left(\frac{35}{36}\right)^n\,.$$

To make this probability at least $1/2$ we need to have

$$1 - \left(\frac{35}{36}\right)^n \geq \frac{1}{2}\,.$$

which gives when we solve for n

$$n \geq \frac{\ln(\frac{1}{2})}{\ln(\frac{35}{36})} \approx 24.6\,,$$

so we should take $n = 25$.

Problem 43 (the probability you are next to me)

Part (a): The number of ways to arrange N people is $N!$. To count the number of permutation of the other people and the "pair" A and B consider A and B as fused together as one unit (say AB) to be taken with the other $N-2$ people. So in total we have $N-2+1$ things to order. This can be done in $(N-1)!$ ways. Note that for every permutation we also have two orderings of A and B i.e. AB and BA so we have $2(N-1)!$ orderings where A and B are fused together. The the probability we have A and B fused together is given by $\frac{2(N-1)!}{N!} = \frac{2}{N}$.

Part (b): If the people are arraigned in a circle there are $(N-1)!$ unique arraignments of the total people. The number of arrangement as in part (a) is given by $2(N-2+1-1)! = 2(N-2)!$ so our probability is given by

$$\frac{2(N-2)!}{(N-1)!} = \frac{2}{N-1}\,.$$

Problem 44 (people between A and B)

Note that we have 5! orderings of the five individual people.

Part (a): The number of permutations that have one person between A and B can be determined as follows. First pick the person to put between A and B from our three choices C, D, and E. Then pick the ordering of A and B i.e AB or BA. Then considering this AB object as *one* object we have to place it with two other people in 3! ways. Thus the number of orderings with one person between A and B is givne by $3 \cdot 2 \cdot 3!$, giving a probability of this event of

$$\frac{3 \cdot 2 \cdot 3!}{5!} = 0.3\,.$$

Part (b): Following Part (a) we can pick the two people from the three remaining in $\binom{3}{2} = 3$ (ignoring order) ways. Since the people can be ordered in two different ways and A and B on the outside can be ordered in two different ways, we have $3 \cdot 2 \cdot 2 = 12$ ways to create the four person "object" with A and B on the outside. This can ordered with the remaining single person in two ways. Thus our probability is given by

$$\frac{2 \cdot 12}{5!} = \frac{1}{5}\,.$$

Part (c): To have three people between A and B, A and B must be on the ends with $3! = 6$ possible ordering of the remaining people. Thus with two orderings of A and B we have a probability of

$$\frac{2 \cdot 6}{5!} = \frac{1}{10}\,.$$

Problem 45 (trying keys at random)

Part (a): If unsuccessful keys are removed as we try them, then the probability that the k-th attempt opens the door can be computed by recognizing that all attempts up to (but not including) the k-th have resulted in failures. Specifically, if we let N be the random variable denoting the attempt that opens the door we see that

$$\begin{aligned}
P\{N = 1\} &= \frac{1}{n} \\
P\{N = 2\} &= \left(1 - \frac{1}{n}\right)\frac{1}{n-1} \\
P\{N = 3\} &= \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n-1}\right)\frac{1}{n-2} \\
&\vdots \\
P\{N = k\} &= \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n-1}\right)\cdots\left(1 - \frac{1}{n-(k-2)}\right)\frac{1}{n-(k-1)}\,.
\end{aligned}$$

We can check that this result is a valid expression to represent a probability by selecting a value for n and verifying that when we sum the above over k for $1 \le k \le n$ we sum to one. A verification of this can be found in the Matlab file `chap_2_prob_45.m`, along with explicit calculations of the mean and variance of N.

Part (b): If unsuccessful keys are not removed then the probability that the correct key is selected at draw k is a geometric random with parameter $p = 1/n$. Thus our probabilities are given by $P\{N = k\} = (1-p)^{k-1}p$, and we have an expectation and a variance given by

$$\begin{aligned}
E[N] &= \frac{1}{p} = n \\
\text{Var}(N) &= \frac{1-p}{p^2} = n(n-1)\,.
\end{aligned}$$

Chapter 2: Theoretical Exercises

Problem 1 (set identities)

To prove this let $x \in E \cap F$ then by definition $x \in E$ and therefore $x \in E \cup F$. Thus $E \cap F \subset E \cup F$.

Problem 2 (more set identities)

If $E \subset F$ then $x \in E$ implies that $x \in F$. If $y \in F^c$, then this implies that $y \notin F$ which implies that $y \in E$, for if y was in E then it would have to be in F which we know it is not.

Problem 3 (more set identities)

We want to prove that $F = (F \cap E) \cup (F \cap E^c)$. We will do this using the standard proof where we show that each set in the above is a subset of the other. We begin with $x \in F$. Then if $x \in E$, x will certainly be in $F \cap E$, while if $x \notin E$ then x will be in $F \cap E^c$. Thus in either case ($x \in E$ or $x \notin E$) x will be in the set $(F \cap E) \cup (F \cap E^c)$.

If $x \in (F \cap E) \cup (F \cap E^c)$ then x is in either $F \cap E$, $F \cap E^c$, or both by the definition of the union operation. Now x cannot be in both sets or else it would simultaneously be in E and E^c, so x must be in one of the two sets only. Being in either set means that $x \in F$ and we have that the set $(F \cap E) \cup (F \cap E^c)$ is a subset of F. Since each side is a subset of the other we have shown set equality.

To prove that $E \cup F = E \cup (E^c \cap F)$, we will begin by letting $x \in E \cup F$, thus x is an element of E or an element of F or of both. If x is in E at all then it is in the set $E \cup (E^c \cap F)$. If $x \notin E$ then it must be in F to be in $E \cup F$ and it will therefore be in $E^c \cap F$. Again both sides are subsets of the other and we have shown set equality.

Problem 6 (set expressions for various events)

Part (a): This would be given by the set $E \cap F^c \cap G^c$.

Part (b): This would be given by the set $E \cap G \cap F^c$.

Part (c): This would be given by the set $E \cup F \cup G$.

Part (d): This would be given by the set

$$((E \cap F) \cap G^c) \cup ((E \cap G) \cap F^c) \cup ((F \cap G) \cap E^c) \cup (E \cap F \cap G)\,.$$

This expresses the fact that satisfy this criterion by being inside two other events or by being inside three events.

Part (e): This would be given by the set $E \cap F \cap G$.

Part (f): This would be given by the set $(E \cup F \cup G)^c$.

Part (g): This would be given by the set

$$(E \cap F^c \cap G^c) \cup (E^c \cap F \cap G^c) \cup (E^c \cap F^c \cap G)$$

Part (h): At most two occur is the complement of all three taking place, so this would be given by the set $(E \cap F \cap G)^c$. Note that this includes the possibility that none of the events happen.

Part (i): This is a subset of the sets in Part (d) (i.e. without the set $E \cap F \cap G$) and is given by the set

$$((E \cap F) \cap G^c) \cup ((E \cap G) \cap F^c) \cup ((F \cap G) \cap E^c) .$$

Part (j): At most three of them occur must be the entire samples space since we only have three events total.

Problem 7 (set simplifications)

Part (a): We have that $(E \cup F) \cap (E \cup F^c) = E$.

Part (b): For the set

$$(E \cap F) \cap (E^c \cup F) \cap (E \cup F^c)$$

We begin with the set

$$\begin{aligned}(E \cap F) \cap (E^c \cup F) &= ((E \cap F) \cap E^c) \cup (E \cap F \cap F) \\ &= \emptyset \cup (E \cap F) \\ &= E \cap F .\end{aligned}$$

So the above becomes

$$\begin{aligned}(E \cap F) \cap (E \cup F^c) &= ((E \cap F) \cap E) \cup ((E \cap F) \cap F^c) \\ &= (E \cap F) \cup \emptyset \\ &= E \cap F .\end{aligned}$$

Part (c): We find that

$$\begin{aligned}(E \cup F) \cap (F \cup G) &= ((E \cup F) \cap F) \cup ((E \cup F) \cap G) \\ &= F \cup ((E \cap G) \cup (F \cap G)) \\ &= (F \cup (E \cap G)) \cup (F \cup (F \cap G)) \\ &= (F \cup (E \cap G)) \cup F \\ &= F \cup (E \cap G) .\end{aligned}$$

Problem 8 (counting partitions)

Part (b): Following the hint this result can be derived as follows. We select one of the $n + 1$ items in our set of $n + 1$ items to be denoted as special. With this item held out we partition the remaining n items into two sets a set of size k and its complement a set of size $n - k$ (we can take k values from $\{0, 1, 2, \ldots, n\}$). Each of these partitions has n or fewer elements. Specifically, the set of size k has T_k partitions. Lumping our special item with the set of size $n-k$ we obtain a set of size $n-k+1$. Grouped with the set of size k we have a partition of our original set of size $n+1$. Since the number of k subset elements can be chosen in $\left(\begin{array}{c} n \\ k \end{array}\right)$ ways we have

$$1+\sum_{k=1}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) T_k \,,$$

possible partitions of the set $\{1, 2, \ldots, n, n+1\}$. Note that the one in the above formulation represents the $k=0$ set and corresponds to the relatively trivial partition consisting of the entire set itself.

Problem 10

From the inclusion/exclusion principle we have

$$\begin{aligned} P(E \cup F \cup G) &= P(E)+P(F)+P(G)-P(E \cap F)-P(E \cap G)-P(F \cap G) \\ &+ P(E \cap F \cap G) \end{aligned}$$

Now consider the following decompositions of sets into mutually exclusive components

$$\begin{aligned} E \cap F &= (E \cap F \cap G^c) \cup (E \cap F \cap G) \\ E \cap G &= (E \cap G \cap F^c) \cup (E \cap G \cap F) \\ F \cap G &= (F \cap G \cap E^c) \cup (F \cap G \cap E) \,. \end{aligned}$$

Since each set above is mutually exclusive we have that

$$\begin{aligned} P(E \cap F) &= P(E \cap F \cap G^c) + P(E \cap F \cap G) \\ P(E \cap G) &= P(E \cap G \cap F^c) + P(E \cap G \cap F) \\ P(F \cap G) &= P(F \cap G \cap E^c) + P(F \cap G \cap E) \,. \end{aligned}$$

Adding these three sets we have that

$$P(E\cap F)+P(E\cap G)+P(F\cap G) = P(E\cap F\cap G^c)+P(E\cap F\cap F^c)+P(F\cap G\cap E^c)+3P(E\cap F\cap G) \,,$$

which when put into the inclusion/exclusion identity above gives the desired result.

Problem 11 (Bonferroni's inequality)

From the inclusion/exclusion identity for two sets we have

$$P(E \cup F) = P(E) + P(F) - P(EF) \,.$$

Since $P(E \cup F) \le 1$, the above becomes

$$P(E) + P(F) - P(EF) \le 1 \,.$$

or

$$P(EF) \ge P(E) + P(F) - 1 \,,$$

which is known as Bonferroni's inequality. From the numbers given we find that

$$P(EF) \ge 0.9 + 0.8 - 1 = 0.7 \,.$$

Problem 12 (exactly one of E or F occurs)

Exactly one of the events E or F occurs is given by the probability of the set

$$(EF^c) \cup (E^cF) .$$

Since the two sets above are mutually exclusive the probability of this set is given by

$$P(EF^c) + P(E^cF) .$$

Since $E = (EF^c) \cup (EF)$, we then have that $P(E)$ can be expressed as

$$P(E) = P(EF^c) + P(EF) .$$

In the same way we have for $P(F)$ the following

$$P(F) = P(E^cF) + P(EF) .$$

so the above expression for our desired event (exactly one of E or F occurring) using these two expressions for $P(E)$ and $P(F)$ is given by

$$\begin{aligned} P(EF^c) + P(E^cF) &= P(E) - P(EF) + P(F) - P(EF) \\ &= P(E) + P(F) - 2P(EF) , \end{aligned}$$

as requested.

Problem 13 (E and not F)

Since $E = EF \cup EF^c$, and both sets on the right hand side of this equation are mutually exclusive we find that

$$P(E) = P(EF) + P(EF^c) ,$$

or solving for $P(EF^c)$ we find

$$P(EF^c) = P(E) - P(EF) ,$$

as expected.

Problem 15 (drawing k white balls from r total)

This is given by

$$P_k = \frac{\binom{M}{k}\binom{N}{r-k}}{\binom{M+N}{r}} \quad \text{for} \quad k \leq r .$$

Problem 16 (more Bonferroni)

From Bonferroni's inequality for two sets $P(EF) \geq P(E) + P(F) - 1$, when we apply this identity recursivly we see that

$$\begin{aligned}
P(E_1E_2E_3\cdots E_n) &\geq P(E_1) + P(E_2E_3\cdots E_n) - 1 \\
&\geq P(E_1) + P(E_2) + P(E_3E_4\cdots E_n) - 2 \\
&\geq P(E_1) + P(E_2) + P(E_3) + P(E_4\cdots E_n) - 3 \\
&\geq \cdots \\
&\geq P(E_1) + P(E_2) + \cdots + P(E_n) - (n-1)\,.
\end{aligned}$$

That the final term is $n-1$ can be verified to be correct by evaluating this expression for $n = 2$ which yeilds the original Bonferroni inequality.

Problem 19

k-balls will be with drawn if there are $r-1$ red balls in the first $k-1$ draws and the kth draw is the rth red ball. This happens with probability

$$\begin{aligned}
P &= \frac{\binom{n}{r-1}\binom{m}{k-1-(r-1)}}{\binom{n+m}{k-1}} \cdot \frac{\binom{n-(r-1)}{1}}{\binom{n+m-(k-1)}{1}} \\
&= \frac{\binom{n}{r-1}\binom{m}{k-1-(r-1)}}{\binom{n+m}{k-1}} \cdot \left(\frac{n-(r-1)}{n+m-(k-1)}\right)\,.
\end{aligned}$$

Here the first probability is that required to obtain $r-1$ red balls from n and $k-1-(r-1) = k-r$ blue balls from m. The next probability is the one requested to obtain the last kth red ball.

Problem 21 (counting total runs)

Following the example from 5o if we assume that we have an *even* number of total runs i.e. say $2k$, then we have two cases for the distribution of the win and loss runs. The wins and losses runs must be interleaved since we have the same number of each i.e. k, so we can start with a loosing block and end with a winning block or start with a winning block and end with a loosing block as in the following diagram

$$\begin{aligned}
&LL\ldots L, WW\ldots W, L\ldots L, WW\ldots W \\
&WW\ldots W, LL\ldots L, W\ldots W, LL\ldots L\,.
\end{aligned}$$

In either case, the number of wins including all winning streaks i must sum to the total number of wins n and the number of losses in all loosing streaks i must sum to the total number of losses. In equations, using x_i to denote the number of wins in the i-th winning streak and y_i to denote the number of losses in the i-th loosing streak we have that

$$\begin{aligned} x_1 + x_2 + \ldots + x_k &= n \\ y_1 + y_2 + \ldots + y_k &= m \,. \end{aligned}$$

Under the constraint that $x_i \geq 1$ and $y_i \geq 1$ since we are told that we have exactly k wins and losses (and therefore can't remove any of the unknowns. The number of solutions to the first and second equation above are given by

$$\binom{n-1}{k-1} \quad \text{and} \quad \binom{m-1}{k-1} \,.$$

Giving a total count on the number of possible situations where we have k winning streaks and k loosing streaks of

$$2 \cdot \binom{n-1}{k-1} \cdot \binom{m-1}{k-1}$$

Note that the "two" in the above formulation accounts for the two possibilities, i.e. we begin with a winning or loosing streak. Combined this give a probability of

$$\frac{2 \cdot \binom{n-1}{k-1} \cdot \binom{m-1}{k-1}}{\binom{n+m}{n}} \,.$$

If instead we are told that we have a total of $2k+1$ runs as an outcome we could have one more winning streak than loosing streak or corresponding one more loosing streak than winning streak. Assuming that we have one more winning streak than loosing our distribution of wins and looses looks schematically like the following

$$W W \ldots W \,, L L \ldots L \,, W W \ldots W \,, L \ldots L \,, W W \ldots W$$

Then counting the total number of wins and losses with our x_i and y_i variables we must have in this case

$$\begin{aligned} x_1 + x_2 + \ldots + x_k + x_{k+1} &= n \\ y_1 + y_2 + \ldots + y_k &= m \,. \end{aligned}$$

The first equation has $\binom{n-1}{k+1-1} = \binom{n-1}{k}$ solutions and the second has $\binom{m-1}{k-1}$. If instead we have one more loosing streak than winning our distribution of wins and looses looks schematically like the following

$$L L \ldots L \,, W W \ldots W \,, L L \ldots L \,, W \ldots W \,, L L \ldots L$$

Then counting the total number of wins and losses with our x_i and y_i variables we must have in this case

$$\begin{aligned} x_1 + x_2 + \ldots + x_k &= n \\ y_1 + y_2 + \ldots + y_k + y_{k+1} &= m \,. \end{aligned}$$

The first equation has $\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}$ solutions and the second has $\begin{pmatrix} m-1 \\ k+1-1 \end{pmatrix} = \begin{pmatrix} m-1 \\ k \end{pmatrix}$.

Since either of these two mutually exclusive cases can occur the total number is given by

$$\begin{pmatrix} n-1 \\ k \end{pmatrix} \cdot \begin{pmatrix} m-1 \\ k-1 \end{pmatrix} + \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \cdot \begin{pmatrix} m-1 \\ k \end{pmatrix} .$$

Giving a probability of

$$\frac{\begin{pmatrix} n-1 \\ k \end{pmatrix} \cdot \begin{pmatrix} m-1 \\ k-1 \end{pmatrix} + \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \cdot \begin{pmatrix} m-1 \\ k \end{pmatrix}}{\begin{pmatrix} n+m \\ n \end{pmatrix}} .$$

as expected.

Chapter 2: Self-Test Problems and Exercises

Problem 1 (a cafeteria sample space)

Part (a): We have two choices for the entree, three choices for the starch, and four choices for the dessert giving $2 \cdot 3 \cdot 4 = 24$ total outcomes in the sample space.

Part (b): Now we have two choices for the entrees, and three choices for the starch giving six total outcomes.

Part (c): Now we have three choices for the starch and four choices for the desert giving 12 total choices.

Part (d): The event $A \cap B$ means that we pick chicken for the entree and ice cream for the desert, so the three possible outcomes correspond to the three possible starches.

Part (e): We have two choices for an entree and four for a desert giving eight possible choices.

Part (f): This event is a dinner of chicken, rice, and ice cream.

Problem 2 (purchasing suits and ties)

Let S_u, S_h , and T be the events that a person purchases a s**u**it, a s**h**irt, and a **t**ie respectively. Then the problem gives the information that

$$\begin{array}{ccc} P(S_u) = 0.22 & P(S_h) = 0.3 & P(T) = 0.28 \\ P(S_u \cap S_h) = 0.11 & P(S_u \cap T) = 0.14 & P(S_h \cap T) = 0.1 \end{array}$$

and $P(S_u \cap S_h \cap T) = 0.06$.

Part (a): This is the event $P((S_u \cup S_h \cup T)^c)$, which we see is given by

$$\begin{aligned} P((S_u \cup S_h \cup T)^c) &= 1 - P(S_u \cup S_h \cup T) \\ &= 1 - P(S_u) - P(S_h) - P(T) + P(S_u \cap S_h) + P(S_u \cap T) \\ &+ P(S_h \cap T) - P(S_u \cap S_h \cap T) \\ &= 1 - 0.22 - 0.3 - 0.28 + 0.11 + 0.14 + 0.1 - 0.06 = 0.49\,. \end{aligned}$$

Part (b): Exactly one item means that we want to evaluate each of the following three mutually exclusive events

$$P(S_u \cap S_h^c \cap T^c) \quad \text{and} \quad P(S_u^c \cap S_h \cap T^c) \quad \text{and} \quad P(S_u^c \cap S_h^c \cap T)$$

and add the resulting probabilities up. We note that problem thirteen from this chapter was solved in this same way. To compute this probability we will begin by computing the probability that *two* or more items were purchased. This is the event

$$(S_u \cap S_h) \cup (S_u \cap T) \cup (S_h \cap T)\,,$$

which we denote by E_2 for shorthand. Using the inclusion/exclusion identity we have that the probability of the event E_2 is given by

$$\begin{aligned} P(E_2) &= P(S_u \cap S_h) + P(S_u \cap T) + P(S_h \cap T) \\ &- P(S_u \cap S_h \cap S_u \cap T) - P(S_u \cap S_h \cap S_h \cap T) - P(S_u \cap T \cap S_h \cap T) \\ &+ P(S_u \cap S_h \cap S_u \cap T \cap S_h \cap T) \\ &= P(S_u \cap S_h) + P(S_u \cap T) + P(S_h \cap T) \\ &- P(S_u \cap S_h \cap T) - P(S_u \cap S_h \cap T) - P(S_u \cap S_h \cap T) + P(S_u \cap S_h \cap T) \\ &= P(S_u \cap S_h) + P(S_u \cap T) + P(S_h \cap T) - 2P(S_u \cap S_h \cap T) \\ &= 0.11 + 0.14 + 0.1 - 2(0.06) = 0.23\,. \end{aligned}$$

If we let E_0 and E_1 be the events that we purchase no items or one item, then the probability that we purchase exactly one item must satisfy

$$1 = P(E_0) + P(E_1) + P(E_2)\,,$$

which we can solve for $P(E_1)$. We find that

$$P(E_1) = 1 - P(E_0) - P(E_2) = 1 - 0.49 - 0.23 = 0.28\,.$$

Problem 3 (the fourteenth card is an ace)

Since the probability that any one specific card is the fourteenth is $1/52$ and we have four ways of getting an ace in the fourteenth spot we have a probability given by

$$\frac{4}{52} = \frac{1}{13}\,.$$

Another way to solve this problem is to recognized that we have 52! ways of ordering the 52 cards in the deck. Then the number of ways that the fourteenth card can be an ace is given by the fact that we have four choices for the ace in the fourteenth position and then the requirement that we need to place $52 - 1 = 51$ other cards in 51! ways so we have a probability of

$$\frac{4(51!)}{52!} = \frac{4}{52} = \frac{1}{13}.$$

To have the first ace occurs in the fourteenth spot we have to pick thirteen cards to place in the thirteen slots in front of this ace (from the $52 - 4 = 48$ "non" ace cards). This can be done in

$$48 \cdot 47 \cdot 46 \cdots (48 - 13 + 1) = 48 \cdot 47 \cdot 46 \cdots 36,$$

ways. Then we have four choices for the ace to pick in the fourteenth spot, then finally we have to place the remaining $52 - 14 = 38$ cards in 38! ways. Thus our probability is given by

$$\frac{(48 \cdot 47 \cdot 46 \cdots 36) \cdot 4 \cdot (38!)}{52!} = 0.03116.$$

Problem 4 (temperatures)

Let $A = \{t_{LA} = 70\}$ be the event that the temperature in LA is 70. Let $B = \{t_{NY} = 70\}$ be the event that the temperature in NY is 70. Let $C = \{\max(t_{LA}, t_{NY}) = 70\}$ be the event that the max of the two temperatures is 70. Let $D = \{\min(t_{LA}, t_{NY}) = 70\}$ be the event that the min of the two temperatures is 70. We note that $C \cap D = A \cap B$ and $C \cup D = A \cup B$. Then we want to compute $P(D)$. Since

$$P(C \cup D) = P(C) + P(D) - P(C \cap D),$$

by the inclusion/exclusion identity for two sets. We also have

$$\begin{aligned} P(C \cup D) &= P(A \cup B) = P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(C \cup D) \end{aligned}$$

By the relationship $C \cup D = A \cup B$ and the inclusion/exclusion identity for A and B. We can equate these two expressions to obtain

$$P(A) + P(B) - P(C \cap D) = P(C) + P(D) - P(C \cap D),$$

or

$$P(D) = P(A) + P(B) - P(C) = 0.3 + 0.4 - 0.2 = 0.5.$$

Problem 5 (the top four cards)

Part (a): There are 52! arraignments of the cards. Then we have 52 choices for the first card, $52 - 4 = 48$ choices for the second card, $52 - 4 - 4 = 42$ choices for the third card etc. This gives a probability of

$$\frac{52 \cdot 48 \cdot 42 \cdot 38(52 - 4)!}{52!} = 0.613.$$

Part (b): For different suits we have 52! total arraignments and to impose that constraint that the top four all have different suits we have 52 choices for the first and then $52-13 = 39$ choices for the second card, $39-13 = 26$ choices for the third card etc. This gives a probability of

$$\frac{52 \cdot 39 \cdot 26 \cdot (52-4)!}{52!} = 0.1055 \,.$$

Problem 6 (balls of the same color)

We have this probability given by

$$\frac{\binom{3}{1}\binom{4}{1}}{\binom{6}{1}\binom{10}{1}} + \frac{\binom{3}{1}\binom{6}{1}}{\binom{6}{1}\binom{10}{1}} = \frac{1}{2} \,.$$

Where the first term is the probability that the first ball drawn is red and the second term is the probability that the second ball is drawn is black.

Problem 9 (number of elements in various sets)

Both of these claims follow directly from the inclusion-exclusion identity if we assume that every element in our finite universal set S (with n elements) is equally likely and has probability $1/n$.

Problem 14 (Boole's inequality)

We begin by decomposing the countable union of sets A_i

$$A_1 \cup A_2 \cup A_3 \ldots$$

into a countable union of disjoint sets C_j. Define these disjoint sets as

$$\begin{aligned}
C_1 &= A_1 \\
C_2 &= A_2 \backslash A_1 \\
C_3 &= A_3 \backslash (A_1 \cup A_2) \\
C_4 &= A_4 \backslash (A_1 \cup A_2 \cup A_3) \\
&\vdots \\
C_j &= A_j \backslash (A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_{j-1})
\end{aligned}$$

Then by construction

$$A_1 \cup A_2 \cup A_3 \cdots = C_1 \cup C_2 \cup C_3 \cdots \,,$$

and the C_j's are disjoint, so that we have

$$\Pr(A_1 \cup A_2 \cup A_3 \cup \cdots) = \Pr(C_1 \cup C_2 \cup C_3 \cup \cdots) = \sum_j \Pr(C_j)\,.$$

Since $\Pr(C_j) \leq \Pr(A_j)$, for each j, this sum is bounded above by

$$\sum_j \Pr(A_j)\,,$$

Chapter 3 (Conditional Probability and Independence)

Chapter 3: Problems

Problem 3 (hands of bridge)

Warning: For some reason I get a different answer than the result given in the back of the book. Thus if someone finds something wrong with the logic below would they please let me know.

Equation 2.1 in the book is

$$p(E|F) = \frac{p(EF)}{p(F)}\,.$$

Let E be the event that east-west has the three of spades and F be the event that north-south has eight spades. Then

$$P(F) = \frac{\binom{13}{8}\binom{39}{18}}{\binom{52}{26}}\,.$$

This can be reasoned as follows. We have thirteen total spades from which we should pick eight to the give the north-south pair (the rest will go to the east-west pair). We then have 39 other cards (non-spades) from which to pick 18 to make a total of 26 for the north-south pair. This is divided the number of ways to select 26 cards from the 52 total cards. For $P(EF)$ we find that

$$P(EF) = \frac{\binom{12}{8}\binom{39}{18}}{\binom{51}{26}}\,.$$

This can be reasoned as follows. Because we must first place the three of spades at the east position once this is done we have 12 other spades to pick 8 from, to distributed to the north-south pair. The other four will go to the east-west pair. Then we have $39 = 52 - 13$ remaining cards to place, from which 18 need to go to the north-south position. Since the three of spades is held by the east player, we are selecting 26 cards to go to north-south from a total of 51. With these two results we see that

$$\begin{aligned}
p(E|F) &= \frac{p(EF)}{p(F)} = \frac{\binom{12}{8}\binom{39}{18}}{\binom{51}{26}} \cdot \frac{\binom{52}{26}}{\binom{13}{8}\binom{39}{18}} \\
&= \frac{5 \cdot 25 \cdot 52}{13 \cdot 26 \cdot 51} = 0.377\,.
\end{aligned}$$

Problem 4 (at least one six)

This is solved in the same way as in problem number 2. In solving we will let E be the event that at least one of the pair of die lands on a 6 and "$X = i$" be shorthand for the event the sum of the two die is i. Then we desire to compute

$$p(E|X = i) = \frac{P(E, X = i)}{p(X = i)}.$$

We begin by computing $p(X = i)$ for $i = 2, 3, 4, \cdots, 12$. We find that

$$\begin{aligned} p(X = 2) &= \frac{1}{36}, \quad p(X = 8) = \frac{5}{36} \\ p(X = 3) &= \frac{2}{36}, \quad p(X = 9) = \frac{4}{36} \\ p(X = 4) &= \frac{3}{36}, \quad p(X = 10) = \frac{3}{36} \\ p(X = 5) &= \frac{4}{36}, \quad p(X = 11) = \frac{2}{36} \\ p(X = 6) &= \frac{5}{36}, \quad p(X = 12) = \frac{1}{36} \\ p(X = 7) &= \frac{6}{36}. \end{aligned}$$

We next compute $p(E, X = i)$ for $i = 2, 3, 4, \cdots, 12$ we find that

$$\begin{aligned} p(E, X = 2) &= 0, \quad p(E, X = 8) = \frac{2}{36} \\ p(E, X = 3) &= 0, \quad p(E, X = 9) = \frac{2}{36} \\ p(E, X = 4) &= 0, \quad p(E, X = 10) = \frac{2}{36} \\ p(E, X = 5) &= 0, \quad p(E, X = 11) = \frac{2}{36} \\ p(E, X = 6) &= 0, \quad p(E, X = 12) = \frac{1}{36} \\ p(E, X = 7) &= \frac{2}{36}. \end{aligned}$$

Finally computing our conditional probabilities we find that

$$P(E|X = 2) = p(E|X = 3) = p(E|X = 4) = p(E|X = 5) = p(E|X = 6) = 0.$$

and

$$\begin{aligned} p(E|X = 7) &= \frac{1}{3}, \quad p(E|X = 10) = \frac{2}{3} \\ p(E|X = 8) &= \frac{2}{5}, \quad p(E|X = 11) = \frac{2}{2} = 1 \\ p(E|X = 9) &= \frac{1}{2}, \quad p(E|X = 12) = \frac{1}{1} = 1. \end{aligned}$$

Problem 5 (the first two selected are white)

We have that

$$P = \frac{\binom{6}{2}\binom{9}{2}}{\binom{15}{4}}$$

is the probability of drawing two white balls and two black balls independently of the order of the draws. Since we are concerned with the probability of an *ordered* sequence of draws we should enumerate these. Let W by the event that the first two balls are white and B the event that the second two balls are black. Then we desire the probability $P(W \cap B) = P(W)\,P(B|W)$. Now

$$P(W) = \frac{\binom{6}{2}}{\binom{15}{2}} = \frac{15}{105} \approx 0.152$$

and

$$P(B|W) = \frac{\binom{9}{2}}{\binom{13}{2}} = \frac{36}{78} \approx 0.461$$

so that $P(W \cap B) = 0.0659 = \frac{6}{91}$.

Problem 6 (exactly three white balls)

Let F be the event that the first and third drawn balls are white and let E be the event that the sample contains exactly three white balls. Then we desire to compute $P(F|E) = \frac{P(F \cap E)}{P(E)}$. Working the without replacement we have that

$$P(E) = \frac{\binom{8}{3} \cdot \binom{4}{1}}{\binom{12}{4}} = \frac{224}{495}.$$

and $P(F \cap E)$ is the probability that our sample has three white balls and the first and third balls are white. To calculate this we can explicitly enumerate the possibilities in $F \cap E$ as $\{(W, W, W, B), (W, B, W, W)\}$, showing that

$$P(F \cap E) = \frac{2}{\binom{12}{4}}.$$

Given these two results we then have that

$$P(F|E) = \frac{2}{\binom{8}{3} \cdot \binom{4}{1}} = \frac{1}{112}.$$

To work the problem with replacement we have that

$$P(E) = \binom{4}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) = \frac{2^5}{3^4}.$$

As before we can enumerate the sample in $E \cap F$. This set is $\{(W,W,W,B),(W,B,W,W)\}$, and has probabilities given by

$$\left(\frac{2}{3}\right)^3\frac{1}{3} + \left(\frac{2}{3}\right)^3\frac{1}{3} = \frac{2^4}{3^4}.$$

so the probabilities we are after is

$$\frac{\frac{2^4}{3^4}}{\frac{2^5}{3^4}} = \frac{1}{2}.$$

Problem 7 (the king's sister)

The two possible children have a sample space given by

$$\{(M,M),(M,F),(F,M),(F,F)\},$$

each with probability $1/4$. Then if we let E be the event that one child is a male and F be the event that one child is a female and one child is a male, the probability that we want to compute is given by

$$P(F|E) = \frac{P(FE)}{P(E)}.$$

Now

$$P(E) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

and FE consists of the set $\{(M,F),(F,M)\}$ so

$$P(FE) = \frac{1}{2},$$

so that

$$P(F|E) = \frac{1/2}{3/4} = \frac{2}{3}.$$

Problem 8 (two girls)

Let F be the event that both children are girls and E the event that the eldest child is a girl. Now $P(E) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ and the event EF has probability $\frac{1}{4}$. Then

$$P(F|E) = \frac{P(FE)}{P(E)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

Problem 9 (a white ball from urn A)

Let F be the event that the ball chosen from urn A was white. Let E be the event that two while balls were chosen. Then the desired probability is $P(F|E) = \frac{P(FE)}{P(E)}$. Lets first calculate $P(E)$ or the probability that two white balls were chosen. This event can happen in the following mutually exclusive draws

$$(W, W, R)\,, (W, R, W)\,, (R, W, W)\,.$$

We can calculate the probabilities of each of these events

- The first draw will happen with probability $\left(\frac{2}{6}\right)\left(\frac{8}{12}\right)\left(\frac{3}{4}\right) = \frac{1}{6}$
- The second draw will happen with probability $\left(\frac{1}{3}\right)\left(\frac{4}{12}\right)\left(\frac{1}{4}\right) = \frac{1}{36}$
- The third draw will happen with probability $\left(\frac{4}{6}\right)\left(\frac{8}{12}\right)\left(\frac{1}{4}\right) = \frac{1}{9}$

so that

$$P(E) = \frac{1}{6} + \frac{1}{36} + \frac{1}{9} = \frac{11}{36}\,.$$

Now FE consists of only the events $\{(W, W, R), (W, R, W)\}$ since now the first draw must be white. The event FE has probability given by $\frac{1}{6} + \frac{1}{36} = \frac{7}{36}$, so that we find

$$P(F|E) = \frac{7/36}{11/36} = \frac{7}{11} = 0.636\,.$$

Problem 10 (three spades given that we draw two others)

Let F be the event that the first card selected is a spade and E the event that the second and third cards are spades. Then we desire to compute $P(F|E) = \frac{P(FE)}{P(E)}$. Now $P(E)$ is the probability that the second and third cards are spades, which equals the union of two events. The first is event that the first, second, and third cards are spades and the second is the event that the first card is not a spade while the second and third cards are spades. Note that this first event is also FE above. Thus we have

$$P(FE) = \frac{13 \cdot 12 \cdot 11}{52 \cdot 51 \cdot 50}$$

Letting G be the event that the first card is not a spade while the second and third cards are spades, we have that

$$P(G) = \frac{(52 - 13) \cdot 13 \cdot 12}{52 \cdot 51 \cdot 50} = \frac{39 \cdot 13 \cdot 12}{52 \cdot 51 \cdot 50}\,,$$

so

$$P(E) = \frac{39 \cdot 13 \cdot 12}{52 \cdot 51 \cdot 50} + \frac{13 \cdot 12 \cdot 11}{52 \cdot 51 \cdot 50} = \frac{11}{39 + 11} = \frac{11}{50} = 0.22\,.$$

Problem 11 (probabilities on two cards)

We are told to let B be the event that both cards are aces, A_s the event that the ace of spades is chosen and A the event that at least one ace is chosen.

Part (a): We are asked to compute $P(B|A_s)$. Using the definition of conditional probabilities we have that

$$P(B|A_s) = \frac{P(BA_s)}{P(A_s)}.$$

The event BA_s is the event that both cards are aces and one is the ace of spades. This event can be represented by the sample space

$$\{(AD, AS), (AH, AS), (AC, AS)\}.$$

where D, S, H, and C stand for diamonds, spades, hearts, and clubs respectively and the order of these elements in the set above does not matter. So we see that

$$P(BA_s) = \frac{3}{\left(\begin{array}{c} 52 \\ 2 \end{array}\right)}.$$

The event A_s is given by the set $\{AS, *\}$ where $*$ is a wild-card denoting any of the possible fifty-one other cards besides the ace of spades. Thus we see that

$$P(A_s) = \frac{51}{\left(\begin{array}{c} 52 \\ 2 \end{array}\right)}.$$

These together give that

$$P(B|A_s) = \frac{3}{51} = \frac{1}{17}.$$

Part (b): We are asked to compute $P(B|A)$. Using the definition of conditional probabilities we have that

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(B)}{P(A)}.$$

The event B are the hand $\{(AD, AS), (AD, AH), (AD, \cdots)\}$ and has $\left(\begin{array}{c} 4 \\ 2 \end{array}\right)$ elements i.e. from the four total aces select two. So that

$$P(B) = \frac{\left(\begin{array}{c} 4 \\ 2 \end{array}\right)}{\left(\begin{array}{c} 52 \\ 2 \end{array}\right)}.$$

The set A is the event that at least one ace is chosen. This is the complement of the set that no ace is chosen. No ace can be chosen in $\left(\begin{array}{c} 48 \\ 2 \end{array}\right)$ ways so that

$$P(A) = 1 - \frac{\left(\begin{array}{c} 48 \\ 2 \end{array}\right)}{\left(\begin{array}{c} 52 \\ 2 \end{array}\right)} = \frac{\left(\begin{array}{c} 52 \\ 2 \end{array}\right) - \left(\begin{array}{c} 48 \\ 2 \end{array}\right)}{\left(\begin{array}{c} 52 \\ 2 \end{array}\right)}.$$

This gives for $P(B|A)$ the following

$$P(B|A) = \frac{\binom{4}{2}}{\binom{52}{2} - \binom{48}{2}} = \frac{6}{198} = \frac{1}{33}\,.$$

Problem 12 (passing the actuarial exams)

We let E_i be the event that the ith actuarial exam is passed. Then the given probabilities can be expressed as

$$P(E_1) = 0.9\,, \quad P(E_2|E_1) = 0.8\,, \quad P(E_3|E_1, E_2) = 0.7\,.$$

Part (a): The desired probability is given by $P(E_1E_2E_3)$ or conditioning we have

$$P(E_1E_2E_3) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2) = 0.9 \cdot 0.8 \cdot 0.7 = 0.504\,.$$

Part (b): The desired probability is given by $P(E_2^c|(E_1E_2E_3)^c)$ and can be expressed using the set identity

$$(E_1E_2E_3)^c = E_1 \cup (E_1E_2^c) \cup (E_1E_2E_3^c)\,,$$

are the only ways that one can not pass all three tests i.e. one must fail one of the first three tests. Note that these sets are mutually independent. Now

$$P(E_2^c|(E_1E_2E_3)^c) = \frac{P(E_2^c\,(E_1E_2E_3)^c)}{P((E_1E_2E_3)^c)}\,.$$

We know how to compute $P((E_1E_2E_3)^c)$ because it is equal to $1 - P(E_1E_2E_3)$ and we can compute $P(E_1E_2E_3)$. From the above set identity the event $E_2^c\,(E_1E_2E_3)^c$ is composed of only one set, namely $E_1E_2^c$, since if we don't pass the second test we don't take the third test. We now need to evaluate the probability of this event. We find

$$\begin{aligned}
P(E_1E_2^c) &= P(E_2^c|E_1)P(E_1) \\
&= (1 - P(E_2|E_1))P(E_1) \\
&= (1 - 0.8)(0.9) = 0.18\,.
\end{aligned}$$

With this the conditional probability sought is given by $\frac{0.18}{1-0.504} = 0.3629$

Problem 13

Define p by $p \equiv P(E_1E_2E_3E_4)$. Then by conditioning on the events E_1, E_1E_2, and $E_1E_2E_3$ we see that p is given by

$$\begin{aligned}
p &= P(E_1E_2E_3E_4) \\
&= P(E_1)P(E_2E_3E_4|E_1) \\
&= P(E_1)P(E_2|E_1)P(E_3E_4|E_1E_2) \\
&= P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3)\,.
\end{aligned}$$

So we need to compute each probability in this product. We have

$$
\begin{aligned}
P(E_1) &= \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}} \\
P(E_2|E_1) &= \frac{\binom{3}{1}\binom{36}{12}}{\binom{39}{13}} \\
P(E_3|E_1E_2) &= \frac{\binom{2}{1}\binom{24}{12}}{\binom{26}{13}} \\
P(E_4|E_1E_2E_3) &= \frac{\binom{1}{1}\binom{12}{12}}{\binom{13}{13}} = 1\,.
\end{aligned}
$$

so this probability is then given by (when we multiply each of the above expressions)

$$p = 0.1055\,.$$

See the Matlab file `chap_3_prob_13.m` for these calculations.

Problem 14

Part (a): We will compute this as a conditional probability since the number of each colored balls depend on the results from the previous draws. Let B_i be the event that a black ball is selected on the ith draw and W_i the event that a white ball is selected on the ith draw. Then the probability we are looking for is given by

$$
\begin{aligned}
P(B_1B_2W_3W_4) &= P(B_1)P(B_2|B_1)P(W_3|B_1B_2)P(W_4|B_1B_2W_3) \\
&= \left(\frac{7}{5+7}\right)\left(\frac{9}{5+9}\right)\left(\frac{5}{5+11}\right)\left(\frac{7}{7+11}\right) = 0.0455\,.
\end{aligned}
$$

See the Matlab file `chap_3_prob_14.m` for these calculations.

Part (b): The set discussed is given by the $\binom{4}{2} = 6$ sets given by

$$
\begin{array}{ccc}
(B_1, B_2, W_3, W_4)\,, & (B_1, W_2, B_3, W_4)\,, & (B_1, W_2, W_3, B_4) \\
(W_1, B_2, B_3, B_4)\,, & (W_1, B_2, W_3, B_4)\,, & (W_1, W_2, B_3, B_4)\,.
\end{array}
$$

The probabilities of each of these events can be computed as in Part (a) of this problem. The probability requested is then the sum of the probabilities of all these mutually exclusive events.

Problem 15

Let S be the event a woman is a smoker and E the event that a woman has an entopic pregnancy. Then our given information is that $P(E|S) = 2P(E|S^c)$, $P(S) = 0.32$, $P(S^c) = 0.68$, and we want to calculate $P(S|E)$. We have using Bayes' rule that

$$\begin{aligned}
P(S|E) &= \frac{P(E|S)P(S)}{P(E|S)P(S) + P(E|S^c)P(S^c)} \\
&= \frac{P(E|S)(0.32)}{P(E|S)(0.32) + 2P(E|S)(0.68)} \\
&= \frac{0.32}{(0.32) + 2(0.68)} = 0.19048\,.
\end{aligned}$$

Problem 16

Let C be the event of a Cesarean section birth, let S be the event that the baby survives. The facts given in the problem are that

$$P(S) = 0.98\,, \quad P(S^c) = 0.02\,, \quad P(C) = 0.15\,, \quad P(C^c) = 0.85\,, \quad P(S|C) = 0.96\,.$$

We want to calculate $P(S|C^c)$. From $P(S)$ we can compute $P(S|C^c)$ by conditioning on C as

$$P(S) = P(S|C)P(C) + P(S|C^c)P(C^c)\,.$$

Using the information given in the problem into the above we find that

$$0.98 = 0.96(0.15) + P(S|C^c)(0.85)\,,$$

or that $P(S|C^c) = 0.983$.

Problem 17

Let D be the event a family owns a dog, and C the event that a family owns a cat. Then from the numbers given in the problem we have that $P(D) = 0.36$, $P(C) = 0.3$, and $P(C|D) = 0.22$.

Part (a): We are asked to compute $P(CD) = P(C|D)P(D) = 0.22 \cdot 0.36 = 0.0792$.

Part (b): We are asked to compute

$$P(D|C) = \frac{P(C|D)P(D)}{P(C)} = \frac{0.22 \cdot (0.36)}{0.3} = 0.264\,.$$

Problem 18

Let I, L, and C be the event that a random person is an independent, liberal, or a conservative respectfully. Let V be the event that a person voted. Then from the problem we are given that

$$P(I) = 0.46\,, \quad P(L) = 0.3\,, \quad P(C) = 0.24$$

and

$$P(V|I) = 0.35\,, \quad P(V|L) = 0.62\,, \quad P(V|C) = 0.58\,.$$

We want to compute $P(I|V)$, $P(L|V)$, and $P(C|V)$ which by Bayes' rule are given by (for $P(I|V)$ for example)

$$P(I|V) = \frac{P(V|I)P(I)}{P(V)} = \frac{P(V|I)P(I)}{P(V|I)P(I) + P(V|L)P(L) + P(V|C)P(C)}\,.$$

All desired probabilities will need to calculate $P(V)$ which we do (as above) by conditioning on the various types of voters. We find that it is given by

$$\begin{aligned} P(V) &= P(V|I)P(I) + P(V|L)P(L) + P(V|C)P(C) \\ &= 0.35(0.46) + 0.62(0.3) + 0.58(0.24) = 0.4862\,. \end{aligned}$$

Then the requested conditional probabilities are given by

$$\begin{aligned} P(I|V) &= \frac{0.35(0.46)}{0.48} = 0.3311 \\ P(L|V) &= \frac{P(V|L)P(L)}{P(V)} = \frac{0.62(0.3)}{0.48} = 0.3875 \\ P(C|V) &= \frac{P(V|C)P(C)}{P(V)} = \frac{0.58(0.24)}{0.48} = 0.29\,. \end{aligned}$$

Part (d): This is $P(V)$ which from Part (c) we know to be equal to 0.48.

Problem 19

Let M be the event a person who attends the party is male, W the event a person who attends the party is female, and E the event that a person was smoke free for a year. The problem gives

$$P(E|M) = 0.37\,, \quad P(M) = 0.62\,, \quad P(E|W) = 0.48\,, \quad P(W) = 1 - P(M) = 0.38\,.$$

Part (a): We are asked to compute $P(W|E)$ which by Bayes' rule is given by

$$\begin{aligned} P(W|E) &= \frac{P(E|W)P(W)}{P(E)} \\ &= \frac{P(E|W)P(W)}{P(E|W)P(W) + P(E|M)P(M)} \\ &= \frac{0.48(0.38)}{0.48(0.38) + 0.37(0.62)} = 0.442\,. \end{aligned}$$

Part (b): For this part we want to compute $P(E)$ which by conditioning on the sex of the person equals $P(E) = P(E|W)P(W) + P(E|M)P(M) = 0.4118$.

Problem 20

Let F be the event that a student is female. Let C be the event that a student is majoring in computer science. Then we are told that $P(F) = 0.52$, $P(C) = 0.05$, and $P(FC) = 0.02$.

Part (a): We are asked to compute $P(F|C) = \frac{P(FC)}{P(C)} = \frac{0.02}{0.05} = 0.4$.

Part (b): We are asked to compute $P(C|F) = \frac{P(FC)}{P(F)} = \frac{0.02}{0.52} = 0.3846$.

Problem 21

We are given the following joint probabilities

$$
\begin{aligned}
P(W_<, H_<) &= \frac{212}{500} = 0.424 \\
P(W_<, H_>) &= \frac{198}{500} = 0.396 \\
P(W_>, H_<) &= \frac{36}{500} = 0.072 \\
P(W_>, H_>) &= \frac{54}{500} = 0.108\,.
\end{aligned}
$$

Where the notation $W_<$ is the event that the wife makes less than $25,000$, $W_>$ is the event that the wife makes more than $25,000$, $H_<$ and $H_>$ are the events that the husband makes less than or more than $25,000$ respectively.

Part (a): We desire to compute $P(H_<)$, which we can do by considering all possible situations involving the wife. We have

$$
P(H_<) = P(H_<, W_<) + P(H_<, W_>) = \frac{212}{500} + \frac{36}{500} = 0.496\,.
$$

Part (b): We desire to compute $P(W_>|H_>)$ which we do by remembering the definition of conditional probability. We have $P(W_>|H_>) = \frac{P(W_>, H_>)}{P(H_>)}$. Since $P(H_>) = 1 - P(H_<) = 1 - 0.496 = 0.504$ using the above we find that $P(W_>|H_>) = 0.2142 = \frac{3}{14}$.

Part (c): We have

$$
P(W_>|H_<) = \frac{P(W_>, H_<)}{P(H_<)} = \frac{0.072}{0.496} = 0.145 = \frac{9}{62}\,.
$$

Problem 22

Part (a): The probability that no two die land on the same number means that each die must land on a unique number. To count the number of such possible combinations we see that there are six choices for the red die, five choices for the blue die, and then four choices for the yellow die yielding a total of $6 \cdot 5 \cdot 4 = 120$ choices where each die has a different number. There are a total of 6^3 total combinations of all possible die through giving a probability of

$$\frac{120}{6^3} = \frac{5}{9}$$

Part (b): We are asked to compute $P(B < Y < R|E)$ where E is the event that no two die lands on the same number. From Part (a) above we know that the count of the number of rolls that satisfy event E is 120. Now the number of rolls that satisfy the event $B < Y < R$ can be counted in a manner like Problem 6 from Chapter 1. For example, if R shows a roll of three then the only possible valid rolls where $B < Y < R$ for B and Y are $B = 1$ and $Y = 2$. If R shows a four then we have $\left(\begin{array}{c} 3 \\ 2 \end{array} \right) = 3$ possible choices i.e. either

$$(B = 1, Y = 2)\,, \quad (B = 1, Y = 3)\,, \quad (B = 2, Y = 3)\,.$$

for the possible assignments to the two values for the B and Y die. If $R = 5$ we have $\left(\begin{array}{c} 4 \\ 2 \end{array} \right) = 6$ possible assignments to B and Y. Finally, if $R = 6$ we have $\left(\begin{array}{c} 5 \\ 2 \end{array} \right) = 10$ possible assignments to B and Y. Thus we find that

$$P(B < Y < R|E) = \frac{1 + 3 + 6 + 10}{120} = \frac{1}{6}$$

Part (c): We see that

$$P(B < Y < R) = P(B < Y < R|E)P(E) + P(B < Y < R|E^c)P(E^c)\,,$$

Since $P(B < Y < R|E^c) = 0$ from the above we have that

$$P(B < Y < R) = \left(\frac{1}{6}\right)\left(\frac{5}{9}\right) = \frac{5}{54}\,.$$

Problem 23

Part (a): Let W be the event that the ball chosen from urn II is white. Then we should solve this problem by conditioning on the color of the ball drawn from first urn. Specifically

$$P(W) = P(W|B_I = w)P(B_I = w) + P(W|B_I = r)P(B_I = r)\,.$$

Here $B_I = w$ is the event that the ball drawn from the first urn is white and $B_I = r$ is the event that the the drawn ball is red. We know that $P(B_I = w) = \frac{1}{3}$, $P(B_I = r) = \frac{2}{3}$, $P(W|B_I = w) = \frac{2}{3}$, and $P(W|B_I = r) = \frac{1}{3}$. We then have

$$P(W) = \frac{2}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3} = \frac{2+2}{9} = \frac{4}{9}$$

Part (b): Now we are looking for

$$P(B_I = w|W) = \frac{P(W|B_I = w)P(B_I = w)}{P(W)}.$$

Since everything is known in the above we can compute this as

$$P(B_I = w|W) = \frac{\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)}{\frac{4}{9}} = \frac{1}{2}.$$

Problem 24

Part (a): Let E be the event that both balls are gold and F the event that at least one ball is gold. The probability we desire to compute is then $P(E|F)$. Using the definition of conditional probability we have that

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{P(\{G,G\})}{P(\{G,G\},\{G,B\},\{B,G\})} = \frac{1/4}{1/4+1/4+1/4} = \frac{1}{3}$$

Part (b): Since now the balls are mixed together in the urn, the difference between the pair $\{G,B\}$ and $\{B,G\}$ is no longer present. Thus we really have two cases to consider.

- Either both balls are gold or
- One ball is gold and the other is black.

Thus to have a second ball be gold will occur once out of these two choices and our probability is then 1/2.

Problem 25

Let p be the proportion of the people who are over fifty and the number we desire to estimate. Let α_1 denote the proportion of the time a person *under* fifty spends on the streets and α_2 the same proportion for people *over* fifty. Then we claim that the method suggested would measure

$$\frac{\alpha_2}{\alpha_1+\alpha_2}.$$

This can be seen as follows. Since by looking around during the day one would measure approximately $N\alpha_2$ people over the age of fifty and approximately $N\alpha_1$ people under the age of fifty where N is the number of people out during the day. The proportion measured would then be

$$\frac{N\alpha_2}{N\alpha_1 + N\alpha_2} = \frac{\alpha_2}{\alpha_1 + \alpha_2}\,.$$

This will approximately equal p if α_1 and α_2 adequately represent the *population* percentages. These two numbers most certainly wouldn't represent population percentages since many people over fifty might not be out on the streets. Because of this α_2 will be an underestimate of the true population percentage.

Problem 26

From the problem, assuming that CB represents the event that a person is colorblind, we are told that

$$P(CB|M) = 0.05\,, \quad \text{and} \quad P(CB|W) = 0.0025\,.$$

We are asked to compute $P(M|CB)$, which we will do by using the Bayes' rule. We find

$$P(M|CB) = \frac{P(CB|M)P(M)}{P(CB)}\,.$$

We will begin by computing $P(CB)$ by conditioning on the sex of the person. We have

$$\begin{aligned} P(CB) &= P(CB|M)P(M) + P(CB|F)P(F) \\ &= 0.05(0.5) + 0.0025(0.5) = 0.02625\,. \end{aligned}$$

Then using Bayes' rule we find that

$$P(M|CB) = \frac{0.05(0.5)}{0.02625} = 0.9523 = \frac{20}{21}\,.$$

If the population consisted of twice as many males as females we would then have $P(M) = 2P(F)$ giving $P(M) = \frac{2}{3}$ and $P(F) = \frac{1}{3}$ and our calculation becomes

$$P(CB) = 0.05\left(\frac{2}{3}\right) + 0.0025\left(\frac{1}{3}\right) = 0.03416\,.$$

so that

$$P(M|CB) = \frac{0.05(2/3)}{0.03416} = 0.9756 = \frac{40}{41}\,.$$

Problem 27 (counting the number of people in each car)

Since we desire to estimate the number of people in a given car, if we choose the first method we will place too much emphasis on cars that carry a large number of people. For example if we imagine that a large bus of people arrives then on average we will select more people from this bus than from cars that only carry one person. This is the same effect as in the discussion in the book about the number of students counted on various numbers of buses and would not provide an unbiased estimate. The second method suggested would provide an unbiased estimate and would be the preferred method.

Problem 29 (used tennis balls)

Let E_0, E_1, E_2, E_3 be the event that we select 0, 1, 2, or 3 used tennis balls during our first draw consisting of three balls. Then let A be the event that when we draw three balls the second time *none* of the selected balls have been used. The problem asks us to compute $P(A)$, which we can compute $P(A)$ by conditioning on the mutually exclusive events E_i for $i = 0, 1, 2, 3$ as

$$P(A) = \sum_{i=0}^{3} P(A|E_i)P(E_i) \, .$$

Now we can compute the prior probabilities $P(E_i)$ as follows

$$\begin{aligned} P(E_0) &= \frac{\binom{6}{0}\binom{9}{3}}{\binom{15}{3}} \, , \quad P(E_1) = \frac{\binom{6}{1}\binom{9}{2}}{\binom{15}{3}} \\ P(E_2) &= \frac{\binom{6}{2}\binom{9}{1}}{\binom{15}{3}} \, , \quad P(E_3) = \frac{\binom{6}{3}\binom{9}{0}}{\binom{15}{3}} \, . \end{aligned}$$

Where the random variable representing the number of selected used tennis balls is a hypergeometric random variable and we have explicitly enumerated these probabilities above. We can now compute $P(A|E_i)$ for each i. Beginning with $P(A|E_0)$ which we recognize as the probability of event A under the situation where in the first draw of three balls we draw no used balls initially i.e. we draw all new balls. Since event E_0 is assumed to happen with certainty when we go to draw the second of three balls we have 6 new balls and 9 used balls. This gives the probability of event A as

$$P(A|E_0) = \frac{\binom{9}{0}\binom{6}{3}}{\binom{15}{3}} \, .$$

In the same way we can compute the other probabilities. We find that

$$P(A|E_1) = \frac{\binom{8}{0}\binom{7}{3}}{\binom{15}{3}} \, , \quad P(A|E_2) = \frac{\binom{7}{0}\binom{8}{3}}{\binom{15}{3}} \, , \quad P(A|E_3) = \frac{\binom{6}{0}\binom{9}{3}}{\binom{15}{3}} \, .$$

With these results we can calculate $P(A)$. This is done in the Matlab file `chap_3_prob_29.m` where we find that $P(A) \approx 0.0893$.

Problem 30 (boxes with marbles)

Let B be the event that the drawn ball is black and let X_1 (X_2) be the event that we select the first (second) box. Then to calculate $P(B)$ we will condition on the box drawn from as

$$P(B) = P(B|X_1)P(X_1) + P(B|X_2)P(X_2)\,.$$

Now $P(B|X_1) = 1/2$, $P(B|X_2) = 2/3$, $P(X_1) = P(X_2) = 1/2$ so

$$P(B) = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{2}{3}\right) = \frac{7}{12}\,.$$

If we see that the ball is white (i.e. it is not black i.e event B^c has happened) we now want to compute that it was drawn from the first box i.e.

$$P(X_1|B^c) = \frac{P(B^c|X_1)P(X_1)}{P(B^c|X_1)P(X_1) + P(B^c|X_2)P(X_2)} = \frac{3}{5}\,.$$

Problem 31 (Ms. Aquina's holiday)

After Ms. Aquina's tests are completed and the doctor has the results he will flip a coin. If it lands *heads* and the results of the tests are *good* he will call with the good news. If the results of the test are *bad* he will not call. If the coin flip lands *tails* he will not call regardless of the tests outcome. Lets let B denote the event that Ms. Aquina has cancer and the and the doctor has bad news. Let G be the event that Ms. Aquina does not have cancer and the results of the test are good. Finally let C be the event that the doctor calls the house during the holiday.

Part (a): Now the event that the doctor does not call (i.e. C^c) will add support to the hypothesis that Ms. Aquina has cancer (or event B) if and only if it is more likely that the doctor will not call given that she does have cancer. This is the event C^c will cause $\beta \equiv P(B|C^c)$ to be greater than $\alpha \equiv P(B)$ if and only if

$$P(C^c|B) \geq P(C^c|B^c) = P(C^c|G)\,.$$

From a consideration of all possible outcomes we have that

$$P(C^c|B) = 1\,,$$

since if the results of the tests come back negative (and Ms. Aquina has cancer), the doctor will not call regardless of the coin flip. We also have that

$$P(C^c|G) = \frac{1}{2}\,,$$

since if the results of the test are good, the doctor will only call if the coin flip lands heads and not call otherwise. Thus the fact that the doctor does not call adds evidence to the belief that Ms. Aquina has cancer. Logic similar to this is discussed in the book after the example of the bridge championship controversy.

Part (b): We want to explicitly find $\beta = P(B|C^c)$ using Bayes' rule. We find that

$$\beta = \frac{P(C^c|B)P(B)}{P(C^c)} = \frac{1(\alpha)}{(3/4)} = \frac{4}{3}\alpha > \alpha\,.$$

Which explicitly verifies the intuition obtained in Part (a).

Problem 32 (the number of children)

Let C_1, C_2, C_3, C_4 be the events that the family has $1, 2, 3, 4$ children respectively. Let E be the evidence that the chosen child is the eldest in the family.

Part (a): We want to compute

$$P(C_1|E) = \frac{P(E|C_1)P(C_1)}{P(E)}\,.$$

We will begin by computing $P(E)$. We find that

$$P(E) = \sum_{i=1}^{4} P(E|C_i)P(C_i) = 1(0.1) + \frac{1}{2}(0.25) + \frac{1}{3}(0.35) + \frac{1}{4}(0.3) = 0.4167\,,$$

so that $P(C_1|E) = 1(0.1)/0.4167 = 0.24$.

Part (b): We want to compute

$$P(C_4|E) = \frac{P(E|C_4)P(C_4)}{P(E)} = \frac{(0.25)(0.3)}{0.4167} = 0.18\,.$$

These calculations are done in the file `chap_3_prob_32.m`.

Problem 33 (English v.s. American)

Let E (A) be the event that this man is English (American). Also let L be the evidence found on the letter. Then we want to compute $P(E|L)$ which we will do with Bayes' rule. We find (counting the number of vowels in each word) that

$$\begin{aligned} P(E|L) &= \frac{P(L|E)P(E)}{P(L|E)P(E) + P(L|E^c)P(E^c)} \\ &= \frac{(3/6)(0.4)}{(3/6)(0.4) + (2/5)(0.6)} = \frac{5}{11}\,. \end{aligned}$$

Problem 34 (some new interpretation of the evidence)

From Example 3f in the book we had that

$$P(G|C) = \frac{P(GC)}{P(C)} = \frac{P(C|G)P(G)}{P(C|G)P(G) + P(C|G^c)P(G^c)}.$$

But now we are told $P(C|G) = 0.9$, since we are assuming that if we are guilty we will have the given characteristic with 90% certaintly. Thus we now would compute for $P(G|C)$ the following

$$P(G|C) = \frac{0.9(0.6)}{0.9(0.6) + 0.2(0.4)} = \frac{27}{31}.$$

Problem 37 (gambling with a fair coin)

Let F denote the event that the gambler is observing results from a fair coin. Also let O_1, O_2, and O_3 denote the three observations made during our experiment. We will assume that before any observations are made the probability that we have selected the fair coin is $1/2$.

Part (a): We desire to compute $P(F|O_1)$ or the probability we are looking at a fair coin given the first observation. This can be computed using Bayes' theorem. We have

$$\begin{aligned} P(F|O_1) &= \frac{P(O_1|F)P(F)}{P(O_1|F)P(F) + P(O_1|F^c)P(F^c)} \\ &= \frac{\frac{1}{2}\left(\frac{1}{2}\right)}{\frac{1}{2}\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right)} = \frac{1}{3}. \end{aligned}$$

Part (b): With the second observation and using the "posteriori's become priors" during a recursive update we now have

$$\begin{aligned} P(F|O_2, O_1) &= \frac{P(O_2|F, O_1)P(F|O_1)}{P(O_2|F, O_1)P(F|O_1) + P(O_2|F^c, O_1)P(F^c|O_1)} \\ &= \frac{\frac{1}{2}\left(\frac{1}{3}\right)}{\frac{1}{2}\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right)} = \frac{1}{5}. \end{aligned}$$

Part (c): In this case because the two-headed coin cannot land tails we can immediately conclude that we have selected the fair coin. This result can also be obtained using Bayes' theorem as we have in the other two parts of this problem. Specifically we have

$$\begin{aligned} P(F|O_3, O_2, O_1) &= \frac{P(O_3|F, O_2, O_1)P(F|O_2, O_1)}{P(O_3|F, O_2, O_1)P(F|O_2, O_1) + P(O_3|F^c, O_2, O_1)P(F^c|O_2, O_1)} \\ &= \frac{\frac{1}{2}\left(\frac{1}{5}\right)}{\frac{1}{2}\left(\frac{1}{5}\right) + 0} = 1. \end{aligned}$$

Verifying what we know must be true.

Problem 42 (special cakes)

Let R be the event that the special cake will rise correctly. Then from the problem statement we are told that $P(R|A) = 0.98$, $P(R|B) = 0.97$, and $P(R|C) = 0.95$, with the prior information of $P(A) = 0.5$, $P(B) = 0.3$, and $P(C) = 0.2$. Then this problem asks for $P(A|R^c)$. Using Bayes' rule we have

$$P(A|R^c) = \frac{P(R^c|A)P(A)}{P(R^c)},$$

where $P(R^c)$ is given by conditioning on A, B, or C as

$$\begin{aligned} P(R^c) &= P(R^c|A)P(A) + P(R^c|B)P(B) + P(R^c|C)P(C) \\ &= 0.02(0.5) + 0.03(0.3) + 0.05(0.2) = 0.029\,, \end{aligned}$$

so that $P(A|R^c)$ is given by

$$P(A|R^c) = \frac{0.02(0.5)}{0.029} = 0.344\,.$$

Problem 43 (three coins in a box)

Let C_1, C_2, C_3 be the event that the first, second, and third coin is chosen and flipped. Then let H be the event that the flipped coin showed heads. Then we would like to evaluate $P(C_1|H)$. Using Bayes' rule we have

$$P(C_1|H) = \frac{P(H|C_1)P(C_1)}{P(H)}.$$

We compute $P(H)$ first. We find conditioning on the the coin selected that

$$\begin{aligned} P(H) &= \sum_{i=1}^{3} P(H|C_i)P(C_i) = \frac{1}{3}\sum_{i=1}^{3} P(H|C_i) \\ &= \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{3}{4}. \end{aligned}$$

Then $P(C_1|H)$ is given by

$$P(C_1|H) = \frac{1(1/3)}{(3/4)} = \frac{4}{9}.$$

Problem 44 (a prisoners' dilemma)

I will argue that the jailers reasoning is sound. Before asking his question the probability of event A (A is executed) is $P(A) = 1/3$. If prisoner A is told that B (or C) is to be set free then we need to compute $P(A|B^c)$. Where A, B, and C are the events that prisoner A, B, or C is to be executed respectively. Now from Bayes' rule

$$P(A|B^c) = \frac{P(B^c|A)P(A)}{P(B^c)}.$$

We have that $P(B^c)$ is given by

$$P(B^c) = P(B^c|A)P(A) + P(B^c|B)P(B) + P(B^c|C)P(C) = \frac{1}{3} + 0 + \frac{1}{3} = \frac{2}{3}.$$

So the above probability then becomes

$$P(A|B^c) = \frac{1(1/3)}{2/3} = \frac{1}{2} > \frac{1}{3}.$$

Thus the probability that prisoner A will be executed has increased as claimed by the jailer.

Problem 45 (is it the fifth coin?)

Let C_i be the event that the ith coin was selected to be flipped. Since any coin is equally likely we have $P(C_i) = \frac{1}{10}$ for all i. Let H be the event that the flipped coin shows heads, then we want to compute $P(C_5|H)$. From Bayes' rule we have

$$P(C_5|H) = \frac{P(H|C_5)P(C_5)}{P(H)}.$$

We compute $P(H)$ by conditioning on the selected coin C_i we have

$$\begin{aligned}
P(H) &= \sum_{i=1}^{10} P(H|C_i)P(C_i) \\
&= \sum_{i=1}^{10} \frac{i}{10}\left(\frac{1}{10}\right) = \frac{1}{100}\sum_{i=1}^{10} i \\
&= \frac{1}{100}\left(\frac{10(10+1)}{2}\right) = \frac{11}{20}.
\end{aligned}$$

So that

$$P(C_5|H) = \frac{(5/10)(1/10)}{(11/20)} = \frac{1}{11}.$$

Problem 46 (one accident means its more likely that you will have another)

Consider the expression $P(A_2|A_1)$. By the definition of conditional probability this can be expressed as

$$P(A_2|A_1) = \frac{P(A_1, A_2)}{P(A_1)},$$

so the desired expression to show is then equivalent to the following

$$\frac{P(A_1, A_2)}{P(A_1)} > P(A_1)\,,$$

or $P(A_1, A_2) > P(A_1)^2$. Considering first the expression $P(A_1)$ by conditioning on the sex of the policy holder we have

$$P(A_1) = P(A_1|M)P(M) + P(A_1|W)P(W) = p_m\alpha + p_f(1-\alpha)\,.$$

where M is the event the policy holder is male and W is the event that the policy holder is female. In the same way we have for the joint probability $P(A_1, A_2)$ that

$$P(A_1, A_2) = P(A_1, A_2|M)P(M) + P(A_1, A_2|W)P(W)\,.$$

Assuming that A_1 and A_2 are *independent* given the specification of the policy holders sex we have that

$$P(A_1, A_2|M) = P(A_1|M)P(A_2|M)\,,$$

the same expression holds for the event W. Using this in the expression for $P(A_1, A_2)$ above we obtain

$$\begin{aligned} P(A_1, A_2) &= P(A_1|M)P(A_2|M)P(M) + P(A_1|W)P(A_2|W)P(W) \\ &= p_m^2\alpha + p_f^2(1-\alpha)\,. \end{aligned}$$

We now look to see if $P(A_1, A_2) > P(A_1)^2$. Computing the expression $P(A_1, A_2) - P(A_1)^2$, (which we hope to be able to show is always positive) we have that

$$\begin{aligned} P(A_1, A_2) - P(A_1)^2 &= p_m^2\alpha + p_f^2(1-\alpha) - (p_m\alpha + p_f(1-\alpha))^2 \\ &= p_m^2\alpha + p_f^2(1-\alpha) - p_m^2\alpha^2 - 2p_mp_f\alpha(1-\alpha) - p_f^2(1-\alpha)^2 \\ &= p_m^2\alpha(1-\alpha) + p_f^2(1-\alpha)\alpha - 2p_mp_f\alpha(1-\alpha) \\ &= \alpha(1-\alpha)(p_m^2 + p_f^2 - 2p_mp_f) \\ &= \alpha(1-\alpha)(p_m - p_f)^2\,. \end{aligned}$$

Note that this is always positive. Thus we have shown that $P(A_1|A_2) > P(A_1)$. In words, this means that given that we have an accident in the first year this information will increase the probability that we will have an accident in the second year to a value greater than we would have without the knowledge of the accident during year one (A_1).

Problem 47 (the probability on which die was rolled)

Let X be the the random variable that specifies the number on the die roll i.e. the integer $1, 2, 3, \cdots, 6$. Let W be the event that all the balls drawn are white. Then we want to evaluate $P(W)$, which can be computed by conditioning on the value of X. Thus we have

$$P(W) = \sum_{i=1}^{6} P\{W|X=i\}P(X=i)$$

Since $P\{X = i\} = 1/6$ for every i, we need only to compute $P\{W|X = i\}$. We have that

$$
\begin{aligned}
P\{W|X=1\} &= \frac{5}{15} \approx 0.33 \\
P\{W|X=2\} &= \left(\frac{5}{15}\right)\left(\frac{4}{14}\right) \approx 0.095 \\
P\{W|X=3\} &= \left(\frac{5}{15}\right)\left(\frac{4}{14}\right)\left(\frac{3}{13}\right) \approx 0.022 \\
P\{W|X=4\} &= \left(\frac{5}{15}\right)\left(\frac{4}{14}\right)\left(\frac{3}{13}\right)\left(\frac{2}{12}\right) \approx 0.0036 \\
P\{W|X=5\} &= \left(\frac{5}{15}\right)\left(\frac{4}{14}\right)\left(\frac{3}{13}\right)\left(\frac{2}{12}\right)\left(\frac{1}{11}\right) \approx 0.0003 \\
P\{W|X=6\} &= 0
\end{aligned}
$$

Then we have

$$
P(W) = \frac{1}{6}(0.33 + 0.95 + 0.022 + 0.0036 + 0.0003) = 0.0756\,.
$$

If all the balls selected are white then the probability our die showed a three was

$$
P\{X = 3|W\} = \frac{P\{W|X = 3\}P(X = 3)}{P(W)} = 0.048\,.
$$

Problem 48 (which cabinet did we select)

This question is the same as asking what is the probability we select cabinet A given that a silver coin is seen on our draw. Then we want to compute $P(A|S) = \frac{P(S|A)P(A)}{P(S)}$. Now

$$
P(S) = P(S|A)P(A) + P(S|B)P(B) = 1\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4}
$$

Thus

$$
P(A|S) = \frac{1(1/2)}{(3/4)} = \frac{2}{3}\,.
$$

Problem 49 (prostate cancer)

Let C be the event that man has cancer and A (for androgen) the event of taking an elevated PSA measurement. Then in the problem we are given

$$
\begin{aligned}
P(A|C^c) &= 0.135 \\
P(A|C) &= 0.268\,,
\end{aligned}
$$

and in addition we have $P(C) = 0.7$.

Part (a): We want to evaluate $P(C|A)$ or

$$\begin{aligned} P(C|A) &= \frac{P(A|C)P(C)}{P(A)} \\ &= \frac{P(A|C)P(C)}{P(A|C)P(C) + P(A|C^c)P(C^c)} \\ &= \frac{(0.268)(0.7)}{(0.268)(0.7) + (0.135)(0.3)} = 0.822\,. \end{aligned}$$

Part (b): We want to evaluate $P(C|A^c)$ or

$$\begin{aligned} P(C|A^c) &= \frac{P(A^c|C)P(C)}{P(A^c)} \\ &= \frac{(1-0.268)(0.7)}{1-0.228} = 0.633\,. \end{aligned}$$

If the prior probability of cancer changes (i.e. $P(C) = 0.3$) then the above formulas yield

$$\begin{aligned} P(C|A) &= 0.459 \\ P(C|A^c) &= 0.266\,. \end{aligned}$$

Problem 50 (assigning probabilities of risk)

Let G, A, B be the events that a person is of good risk, an average risk, or a bad risk respectively. Then in the problem we are told that (if E denotes the event that an accident occurs)

$$\begin{aligned} P(E|G) &= 0.05 \\ P(E|A) &= 0.15 \\ P(E|B) &= 0.3 \end{aligned}$$

In addition the a priori assumptions on the proportion of people that are good, average and bad risks are given by $P(G) = 0.2$, $P(A) = 0.5$, and $P(B) = 0.3$. Then in this problem we are asked to compute $P(E)$ or the probability that an accident will happen. This can be computed by conditioning on the probability of a person having an accident from among the three types, i.e.

$$\begin{aligned} P(E) &= P(E|G)P(G) + P(E|A)P(A) + P(E|B)P(B) \\ &= 0.05(0.2) + (0.15)(0.5) + (0.3)(0.3) = 0.175\,. \end{aligned}$$

If a person had no accident in a given year we want to compute $P(G|E^c)$ or

$$\begin{aligned} P(G|E^c) &= \frac{P(E^c|G)P(G)}{P(E^c)} = \frac{(1-P(E|G))P(G)}{1-P(E)} \\ &= \frac{(1-0.05)(0.2)}{1-0.175} = \frac{38}{165} \end{aligned}$$

also to compute $P(A|E^c)$ we have

$$
\begin{aligned}
P(A|E^c) &= \frac{P(E^c|A)P(A)}{P(E^c)} = \frac{(1-P(E|A))P(A)}{1-P(E)} \\
&= \frac{(1-0.15)(0.5)}{1-0.175} = \frac{17}{33}
\end{aligned}
$$

Problem 51 (letters of recommendation)

Let R_s, R_m, and R_w be the event that our worker receives a strong, moderate, or weak recommendation respectively. Let J be the event that our applicant gets the job. Then the problem specifies

$$
\begin{aligned}
P(J|R_s) &= 0.8 \\
P(J|R_m) &= 0.4 \\
P(J|R_w) &= 0.1\,,
\end{aligned}
$$

with priors on the type of recommendation given by

$$
\begin{aligned}
P(R_s) &= 0.7 \\
P(R_m) &= 0.2 \\
P(R_w) &= 0.1\,,
\end{aligned}
$$

Part (a): We are asked to compute $P(J)$ which by conditioning on the type of recommendation received is

$$
\begin{aligned}
P(J) &= P(J|R_s)P(R_s) + P(J|R_m)P(R_m) + P(J|R_w)P(R_w) \\
&= 0.8(0.7) + (0.4)(0.2) + (0.1)(0.1) = 0.65 = \frac{13}{20}\,.
\end{aligned}
$$

Part (b): Given the event J is held true then we are asked to compute the following

$$
\begin{aligned}
P(R_s|J) &= \frac{P(J|R_s)P(R_s)}{P(J)} = \frac{(0.8)(0.7)}{(0.65)} = \frac{56}{65} \\
P(R_m|J) &= \frac{P(J|R_m)P(R_m)}{P(J)} = \frac{(0.4)(0.2)}{(0.65)} = \frac{8}{65} \\
P(R_w|J) &= \frac{P(J|R_w)P(R_w)}{P(J)} = \frac{(0.1)(0.1)}{(0.65)} = \frac{1}{65}
\end{aligned}
$$

Note that this last probability can also be calculated as $P(R_w|J) = 1 - P(R_w|J) - P(R_w|J)$.

Part (c): For this we are asked to compute

$$\begin{aligned}
P(R_s|J^c) &= \frac{P(J^c|R_s)P(R_s)}{P(J^c)} = \frac{(1-0.8)(0.7)}{(0.35)} = \frac{2}{5} \\
P(R_m|J^c) &= \frac{P(J^c|R_m)P(R_m)}{P(J^c)} = \frac{(1-0.4)(0.2)}{(0.35)} = \frac{12}{35} \\
P(R_w|J^c) &= \frac{P(J^c|R_w)P(R_w)}{P(J^c)} = \frac{(1-0.1)(0.1)}{(0.35)} = \frac{9}{35} \,.
\end{aligned}$$

Problem 52 (college acceptance)

Let M, T, W, R, F, and S correspond to the events that mail comes on Monday, Tuesday, Wednesday, Thursday, Friday, or Saturday (or later) respectively. Let A be the event that our student is accepted.

Part (a): To compute $P(M)$ we can condition on whether or not the student is accepted as

$$P(M) = P(M|A)P(A) + P(M|A^c)P(A^c) = 0.15(0.6) + 0.05(0.4) = 0.11 \,.$$

Part (b): We desire to compute $P(T|M^c)$. Using the definition of conditional probability we find that (again conditioning $P(T)$ on whether she is accepted or not)

$$\begin{aligned}
P(T|M^c) &= \frac{P(T, M^c)}{P(M^c)} = \frac{P(T)}{1-P(M)} \\
&= \frac{P(T|A)P(A) + P(T|A^c)P(A^c)}{1-P(M)} \\
&= \frac{0.2(0.6) + 0.1(0.4)}{1-0.11} = \frac{16}{89} \,.
\end{aligned}$$

Part (c): We want to calculate $P(A|M^c, T^c, W^c)$. Again using the definition of conditional probability (twice) we have that

$$P(A|M^c, T^c, W^c) = \frac{P(A, M^c, T^c, W^c)}{P(M^c, T^c, W^c)} = \frac{P(M^c, T^c, W^c|A)P(A)}{P(M^c, T^c, W^c)} \,.$$

To evaluate terms like $P(M^c, T^c, W^c|A)$, and $P(M^c, T^c, W^c|A^c)$, lets compute the probability that mail will come on Saturday or later given that she is accepted or not. Using the fact that $P(\cdot|A)$ and $P(\cdot|A^c)$ are both probability densities and must sum to one over their first argument we calculate that

$$\begin{aligned}
P(S|A) &= 1 - 0.15 - 0.2 - 0.25 - 0.15 - 0.1 = 0.15 \\
P(S|A^c) &= 1 - 0.05 - 0.1 - 0.1 - 0.15 - 0.2 = 0.4 \,.
\end{aligned}$$

With this result we can calculate that

$$\begin{aligned}
P(M^c, T^c, W^c|A) &= P(R|A) + P(F|A) + P(S|A) = 0.15 + 0.1 + 0.15 = 0.4 \\
P(M^c, T^c, W^c|A^c) &= P(R|A^c) + P(F|A^c) + P(S|A^c) = 0.15 + 0.2 + 0.4 = 0.75 \,.
\end{aligned}$$

Also we can compute $P(M^c, T^c, W^c)$ by conditioning on whether she is accepted or not. We find

$$\begin{aligned} P(M^c, T^c, W^c) &= P(M^c, T^c, W^c|A)P(A) + P(M^c, T^c, W^c|A^c)P(A^c) \\ &= 0.4(0.6) + 0.75(0.4) = 0.54\,. \end{aligned}$$

Now we finally have all of the components we need to compute what we were asked to. We find that

$$P(A|M^c, T^c, W^c) = \frac{P(M^c, T^c, W^c|A)P(A)}{P(M^c, T^c, W^c)} = \frac{0.4(0.6)}{0.54} = \frac{4}{9}\,.$$

Part (d): We are asked to compute $P(A|R)$ which using Bayes' rule gives

$$P(A|R) = \frac{P(R|A)P(A)}{P(R)}\,.$$

To compute this lets begin by computing $P(R)$ again obtained by conditioning on whether our student is accepted or not. We find

$$P(R) = P(R|A)P(A) + P(R|A^c)P(A^c) = 0.15(0.6) + 0.15(0.4) = 0.15\,.$$

So that our desired probability is given by

$$P(A|R) = \frac{0.15(0.6)}{0.15} = \frac{3}{5}\,.$$

Part (e): We want to calculate $P(A|S)$. Using Bayes' rule gives

$$P(A|S) = \frac{P(S|A)P(A)}{P(S)}\,.$$

To compute this, lets begin by computing $P(S)$ again obtained by conditioning on whether our student is accepted or not. We find

$$P(S) = P(S|A)P(A) + P(S|A^c)P(A^c) = 0.15(0.6) + 0.4(0.4) = 0.25\,.$$

So that our desired probability is given by

$$P(A|S) = \frac{0.15(0.6)}{0.25} = \frac{9}{25}\,.$$

Problem 53 (the functioning of a parallel system)

With n components a parallel system will be working if at least one component is working. Let H_i be the event that the component i for $i = 1, 2, 3, \cdots, n$ is working. Let F be the event that the entire system is functioning. We want to compute $P(H_1|F)$. We have

$$P(H_1|F) = \frac{P(F|H_1)P(H_1)}{P(F)}\,.$$

Now $P(F|H_1) = 1$ since if the first component is working the system is functioning. In addition, $P(F) = 1 - \left(\frac{1}{2}\right)^n$ since to be *not* functioning all components must not be working. Finally $P(H_1) = 1/2$. Thus our probability is

$$P(H_1|F) = \frac{1/2}{1 - (1/2)^n}.$$

Problem 54 (independence of E and F)

Part (a): These two events would be independent. The fact that one person has blue eyes and another unrelated person has blue eyes are in no way related.

Part (b): These two events seem unrelated to each other and would be modeled as independent.

Part (c): As height and weigh are related, I would think that these two events are not independent.

Part (d): Since the United States is in the western hemisphere these two two events are related and they are not independent.

Part (e): Since rain one day would change the probability of rain on other days I would say that these events are related and therefore not independent.

Problem 55 (independence in class)

Let S be a random variable denoting the sex of the randomly selected person. The S can take on the values m for male and f for female. Let C be a random variable representing denoting the class of the chosen student. The C can take on the values f for freshman and s for sophomore. We want to select the number of sophomore girls such that the random variables S and C are independent. Let n denote the number of sophomore girls. Then counting up the number of students that satisfy each requirement we have

$$\begin{aligned}
P(S = m) &= \frac{10}{16 + n} \\
P(S = f) &= \frac{6 + n}{16 + n} \\
P(C = f) &= \frac{10}{16 + n} \\
P(C = s) &= \frac{6 + n}{16 + n}.
\end{aligned}$$

The joint density can also be computed and are given by

$$
\begin{aligned}
P(S=m, C=f) &= \frac{4}{16+n} \\
P(S=m, C=s) &= \frac{6}{16+n} \\
P(S=f, C=f) &= \frac{6}{16+n} \\
P(S=f, C=s) &= \frac{n}{16+n}.
\end{aligned}
$$

Then to be independent we must have $P(C,S) = P(S)P(C)$ for all possible C and S values. Considering the point case where $(S=m, C=f)$ we have that n must satisfy

$$
\begin{aligned}
P(S=m, C=f) &= P(S=m)P(C=f) \\
\frac{4}{16+n} &= \left(\frac{10}{16+n}\right)\left(\frac{10}{16+n}\right)
\end{aligned}
$$

which when we solve for n gives $n=9$. Now one should check that this value of n works for all other equalities that must be true, for example one needs to check that when $n=9$ the following are true

$$
\begin{aligned}
P(S=m, C=s) &= P(S=m)P(C=s) \\
P(S=f, C=f) &= P(S=f)P(C=f) \\
P(S=f, C=s) &= P(S=f)P(C=s).
\end{aligned}
$$

As these can be shown to be true, $n=9$ is the correct answer.

Problem 58 (generating fair flips with a biased coin)

Part (a): Consider pairs of flips. Let E be the event that a pair of flips returns (H,T) and let F be the event that the pair of flips returns (T,H). From the discussion on Page 93 Example 4h the event E will occur first with probability

$$
\frac{P(E)}{P(E)+P(F)}.
$$

Now $P(E) = p(1-p)$ and $P(F) = (1-p)p$, so the probability of obtaining event E and declaring tails before the event F would be

$$
\frac{p(1-p)}{2p(1-p)} = \frac{1}{2}.
$$

In the same way we will have event F before event E with probability $\frac{1}{2}$.

Problem 59 (the first four outcomes)

Part (a): This probability would be p^4.

Part (b): This probability would be $(1-p)p^3$.

Part (c): Given two mutually exclusive events E and F the probability that E occurs before F is given by

$$\frac{P(E)}{P(E)+P(F)}\,.$$

Denoting E by the event that we obtain a T, H, H, H pattern and F the event that we obtain a H, H, H, H pattern the above becomes

$$\frac{p^3(1-p)}{p^4+p^3(1-p)} = \frac{1-p}{p+(1-p)} = 1-p\,.$$

Problem 60 (the color of your eyes)

Since Smith's sister has blue eyes and this is a recessive trait, both of Smith's parents must have the gene for blue eyes. Let R denote the gene for brown eyes and L denote the gene for blue eyes (these are the second letters in the words brown and blue respectively). Then Smith will have a gene makeup possibly given by $(R, R), (R, L), (L, R)$, where the left gene is the one received from his mother and the right gene is the one received from his father.

Part (a): With the gene makeup given above we see that in two cases from three total Smith will have a blue gene. Thus this probability is $2/3$.

Part (b): Since Smith's wife has blue eyes, Smith's child will receive a L gene from his mother. The probability Smith's first child will have blue eyes is then dependent on what gene they receive from Smith. Letting B be the event that Smith's first child has blue eyes (and conditioning on the possible genes Smith could give his child) we have

$$P(B) = 0\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{3}\,.$$

As stated above, this result is obtained by conditioning on the possible gene makeups of Smith. For example let (X, Y) be the notation for the "event" that Smith has a gene make up given by (X, Y) then the above can be written symbolically (in terms of events) as

$$P(B) = P(B|(R,R))P(R,R) + P(B|(R,L))P(R,L) + P(B|(L,R))P(L,R)\,.$$

Evaluating each of the above probabilities gives the result already stated.

Part (c): The fact that the first child has brown eyes makes it more likely that Smith has a genotype, given by (R, R). We compute the probability of this genotype given the event

E (the event that the first child has brown eyes using Bayes' rule as)

$$
\begin{aligned}
P((R,R)|E) &= \frac{P(E|(R,R))P(R,R)}{P(E|(R,R))P(R,R) + P(E|(R,L))P(R,L) + P(E|(L,R))P(L,R)} \\
&= \frac{1\left(\frac{1}{3}\right)}{1\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right)} \\
&= \frac{1}{2}.
\end{aligned}
$$

In the same way we have for the other possible genotypes that

$$
P((R,L)|E) = \frac{\frac{1}{2}\left(\frac{1}{3}\right)}{\frac{2}{3}} = \frac{1}{4} = P((L,R)|E).
$$

Thus the same calculation as in Part (b), but now conditioning on the fact that the first child has brown eyes (event E) gives for a probability of the event B_2 (that the second child we have *blue* eyes)

$$
\begin{aligned}
P(B_2|E) &= P(B_2|(R,R),E)P((R,R)|E) + P(B_2|(R,L),E)P((R,L)|E) + P(B_2|(L,R),E)P((L,R)|E) \\
&= 0\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{4}.
\end{aligned}
$$

This means that the probability that the second child has *brown* eyes is then

$$
1 - P(B_2|E) = \frac{3}{4}.
$$

Problem 61 (more recessive traits)

From the information that the two parents are normal but that they produced an albino child we know that both parents must be carriers of albinoism. Their non-albino child can have any of three possible genotypes each with probability 1/3 given by $(A,A), (A,a), (a,A)$. Lets denote this parent by P_1 and the event that this parent is a carrier for albinoism as C_1. Note that $P(C_1) = 2/3$ and $P(C_1^c) = 1/3$. We are told that the spouse of this person (denoted P_2) is a carrier for albinoism.

Part (a): The probability their first offspring is an albino depends on how likely our first parent is a carrier of albinoism. We have (with E_1 the event that their first child is an albino) that

$$
P(E_1) = P(E_1|C_1)P(C_1) + P(E_1|C_1^c)P(C_1^c).
$$

Now $P(E_1|C_1) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$, since both parents must contribute their albino gene, and $P(E_1|C_1^c) = 0$ so we have that

$$
P(E_1) = \frac{1}{4}\left(\frac{2}{3}\right) = \frac{1}{6}.
$$

Part (b): The fact that the first newborn is not an albino changes the probability that the first parent is a carrier or the value of $P(C_1)$. To calculate this we will use Bayes' rule

$$\begin{aligned}
P(C_1|E_1^c) &= \frac{P(E_1^c|C_1)P(C_1)}{P(E_1^c|C_1)P(C_1) + P(E_1^c|C_1^c)P(C_1^c)} \\
&= \frac{\frac{3}{4}\left(\frac{2}{3}\right)}{\frac{3}{4}\left(\frac{2}{3}\right) + 1\left(\frac{1}{3}\right)} \\
&= \frac{3}{5}.
\end{aligned}$$

so we have that $P(C_1^c|E_1^c) = \frac{2}{5}$, and following the steps in Part (a) we have (with E_2 the event that the couples second child is an albino)

$$\begin{aligned}
P(E_2|E_1^c) &= P(E_2|E_1^c, C_1)P(C_1|E_1^c) + P(E_2|E_1^c, C_1^c)P(C_1^c|E_1^c) \\
&= \frac{1}{4}\left(\frac{3}{5}\right) = \frac{3}{20}.
\end{aligned}$$

Problem 62 (target shooting with Barbara and Dianne)

Let H be the event that the duck is "hit", by either Barbra or Dianne's shot. Let B and D be the events that Barbra (respectively Dianne) hit the target. Then the outcome of the experiment where both Dianne and Barbra fire at the target (assuming that their shots work independently is)

$$\begin{aligned}
P(B^c, D^c) &= (1 - p_1)(1 - p_2) \\
P(B^c, D) &= (1 - p_1)p_2 \\
P(B, D^c) &= p_1(1 - p_2) \\
P(B, D) &= p_1 p_2.
\end{aligned}$$

Part (a): We desire to compute $P(B, D|H)$ which equals

$$P(B, D|H) = \frac{P(B, D, H)}{P(H)} = \frac{P(B, D)}{P(H)}$$

Now $P(H) = (1 - p_1)p_2 + p_1(1 - p_2) + p_1 p_2$ so the above probability becomes

$$\frac{p_1 p_2}{(1 - p_1)p_2 + p_1(1 - p_2) + p_1 p_2} = \frac{p_1 p_2}{p_1 + p_2 - p_1 p_2}.$$

Part (b): We desire to compute $P(B|H)$ which equals

$$P(B|H) = P(B, D|H) + P(B, D^c|H).$$

Since the first term $P(B, D|H)$ has already been computed we only need to compute $P(B, D^c|H)$. As before we find it to be

$$P(B, D^c|H) = \frac{p_1(1 - p_2)}{(1 - p_1)p_2 + p_1(1 - p_2) + p_1 p_2}.$$

So the total result becomes

$$P(B|H) = \frac{p_1p_2 + p_1(1-p_2)}{(1-p_1)p_2 + p_1(1-p_2) + p_1p_2} = \frac{p_1}{p_1 + p_2 - p_1p_2}.$$

Problem 63 (dueling)

For a given trial while dueling we have the following possible outcomes (events) and their associated probabilities

- Event I: A is hit and B is not hit. This happens with probability $p_B(1-p_A)$.
- Event II: A is not hit and B is hit. This happens with probability $p_A(1-p_B)$.
- Event III: A is hit and B is hit. This happens with probability p_Ap_B.
- Event IV: A is not hit and B is not hit. This happens with probability $(1-p_A)(1-p_B)$.

With these definitions we can compute the probabilities of various other events.

Part (a): To solve this we recognize that A is hit if events I and III happen and the dueling continues if event IV happens. We can compute $p(A)$ (the probability that A is hit) by conditioning on the outcome of the first duel. We have

$$p(A) = p(A|I)p(I) + p(A|II)p(II) + p(A|III)p(III) + p(A|IV)p(IV).$$

Now in the case of event IV the dual continues afresh and we see that $p(A|IV) = p(A)$. Using this fact and the definitions of events I-IV we have that the above becomes

$$p(A) = 1 \cdot p_B(1-p_A) + 0 \cdot p_A(1-p_B) + 1 \cdot p_Ap_B + p(A) \cdot (1-p_A)(1-p_B).$$

Now solving for $p(A)$ in the above we find that

$$p(A) = \frac{p_B}{(1-(1-p_A)(1-p_B))}.$$

Part (b): Let D be the event that both duelists are hit. Then to compute this, we can condition on the outcome of the first dual. Using the same arguments as above we find

$$\begin{aligned} p(D) &= p(D|I)p(I) + p(D|II)p(II) + p(D|III)p(III) + p(D|IV)p(IV) \\ &= 0 + 0 + 1 \cdot p_Ap_B + p(D) \cdot (1-p_A)(1-p_B). \end{aligned}$$

On solving for $P(D)$ we have

$$p(D) = \frac{p_Ap_B}{1-(1-p_A)(1-p_B)}.$$

Part (c): Lets begin by computing the probability that the dual ends after one dual. Let G_1 be the event that the game ends with *more than* (or after) one dual. We have, conditioning on the events I-IV that

$$p(G_1) = 0 + 0 + 0 + 1 \cdot (1 - p_A)(1 - p_B) = (1 - p_A)(1 - p_B)\,.$$

Now let G_2 be the event that the game ends with *more than* (or after) two duals. Then

$$p(G_2) = (1 - p_A)(1 - p_B)p(G_1) = (1 - p_A)^2(1 - p_B)^2\,.$$

Generalizing this result we have for the probability that the games ends after n duels is

$$p(G_n) = (1 - p_A)^n(1 - p_B)^n\,.$$

Part (d): Let G_1 be the event that the game ends with more than one dual and let A be the event that A is hit. Then to compute $p(G_1|A^c)$ by conditioning on the first experiment we have

$$\begin{aligned} p(G_1|A^c) &= p(G_1, I|A^c)p(I) + p(G_1, II|A^c)p(II) \\ &+ p(G_1, III|A^c)p(III) + p(G_1, IV|A^c)p(IV) \\ &= 0 + 0 + 0 + p(G_1, IV|A^c)(1 - p_A)(1 - p_B)\,. \end{aligned}$$

So now we need to evaluate $p(G_1, IV|A^c)$, which we do using the definition of conditional probability. We find

$$p(G_1, IV|A^c) = \frac{p(G_1, IV, A^c)}{p(A^c)} = \frac{1}{p(A^c)}\,.$$

Where $p(A^c)$ is the probability that A is not hit *on the first experiment*. This can be computed as

$$\begin{aligned} p(A) &= p_B(1 - p_A) + p_A p_B = p_B \quad \text{so} \\ p(A^c) &= 1 - p_B\,, \end{aligned}$$

and the above is then given by

$$p(G_1|A^c) = \frac{(1 - p_A)(1 - p_B)}{1 - p_B} = 1 - p_A\,.$$

In the same way as before this would generalize to the following (for the event G_n)

$$p(G_n) = (1 - p_A)^n(1 - p_B)^{n-1}$$

Part (e): Let AB be the event that both duelists are hit. Then in the same way as Part (d) above we see that

$$p(G_1, IV|AB) = \frac{p(G_1, IV, AB)}{p(AB)} = \frac{1}{p(AB)}\,.$$

Here $p(AB)$ is the probability that A and B are hit on any given experiment so $p(AB) = p_A p_B$, and

$$p(G_1|AB) = \frac{(1 - p_A)(1 - p_B)}{p_A p_B}$$

and in general

$$p(G_n|AB) = \frac{(1 - p_A)^n(1 - p_B)^n}{p_A p_B}\,.$$

	Woman answers correctly	Woman answers incorrectly
Man answers correctly	p^2	$p(1-p)$
Man answers incorrectly	$(1-p)p$	$(1-p)^2$

Table 5: The possible probabilities of agreement for the couple in Problem 64, Chapter 3. When asked a question four possible outcomes can occur, corresponding to the correctness of the mans (woman's) answer. The first row corresponds to the times when the husband answers the question correctly, the second row to the times when the husband answers the question incorrectly. In the same way, the first column corresponds to the times when the wife is correct and second column to the times when the wife is incorrect.

Problem 64 (game show strategies)

Part (a): Since each person has probability p of getting the correct answer, either one selected to represent the couple will answer correctly with probability p.

Part (b): To compute the probability that the couple answers correctly under this strategy we will condition our probability on the "agreement" matrix in Table 5, i.e. the possible combinations of outcomes the couple may encounter when asked a question that they both answer. Lets define E be the event that the couple answers correctly, and let C_m (C_w) be the events that the man (women) answers the question correctly. We find that

$$\begin{aligned} P(E) &= P(E|C_m, C_w)P(C_m, C_w) + P(E|C_m, C_w^c)P(C_m, C_w^c) \\ &+ P(E|C_m^c, C_w)P(C_m^c, C_w) + P(E|C_m^c, C_w^c)P(C_m^c, C_w^c) . \end{aligned}$$

Now $P(E|C_m^c, C_w^c) = 0$ since both the man and the woman agree but they both answer the question incorrectly. In that case the couple would return the incorrect answer to the question. In the same way we have that $P(E|C_m, C_w) = 1$. Following the strategy of flipping a coin when the couple answers disagree we note that $P(E|C_m, C_w^c) = P(E|C_m^c, C_w) = 1/2$, so that the above probability when using this strategy becomes

$$P(E) = 1 \cdot p^2 + \frac{1}{2}p(1-p) + \frac{1}{2}(1-p)p = p ,$$

where in computing this result we have used the joint probabilities found in Table 5 to evaluate terms like $P(C_m, C_w^c)$. Note that this result is the same as in Part (a) of this problem showing that there is no benefit to using this strategy.

Problem 65 (how accurate are we when we agree/disagree)

Part (a): We want to compute (using the notation from the previous problem)

$$P(E|(C_m, C_w) \cup (C_m^c, C_w^c)) .$$

Defining the event A to be equal to $(C_m, C_w) \cup (C_m^c, C_w^c)$. We see that this is equal to

$$P(E|(C_m, C_w) \cup (C_m^c, C_w^c)) = \frac{P(E, A)}{P(A)} = \frac{p^2}{p^2 + (1-p)^2} = \frac{0.36}{0.36 + 0.16} = \frac{9}{13} .$$

Part (b): We want to compute $P(E|(C_m^c, C_w)\cup(C_m, C_w^c))$, but in the second strategy above if the couple disagrees they flip a fair coin to decide. Thus this probability is equal to 1/2.

Problem 70 (hemophilia and the queen)

Let C be the event that the queen is a carrier of the gene for hemophilia. We are told that $P(C) = 0.5$. Let H_i be the event that the i-th prince has hemophilia. The we observe the event $H_1^c H_2^c H_3^c$ and we want to compute $P(C|H_1^c H_2^c H_3^c)$. Using Bayes' rule we have that

$$P(C|H_1^c H_2^c H_3^c) = \frac{P(H_1^c H_2^c H_3^c|C)P(C)}{P(H_1^c H_2^c H_3^c|C)P(C) + P(H_1^c H_2^c H_3^c|C^c)P(C^c)}.$$

Now

$$P(H_1^c H_2^c H_3^c|C) = P(H_1^c|C)P(H_2^c|C)P(H_3^c|C).$$

By the independence of the birth of the princes. Now $P(H_i^c|C) = 0.5$ so that the above is given by

$$P(H_1^c H_2^c H_3^c|C) = (0.5)^3 = \frac{1}{8}.$$

Also $P(H_1^c H_2^c H_3^c|C^c) = 1$ so the above probability becomes

$$P(C|H_1^c H_2^c H_3^c) = \frac{(0.5)^3(0.5)}{(0.5)^3(0.5) + 1(0.5)} = \frac{1}{9}.$$

In the next part of this problem (below) we will need the complement of this probability or

$$P(C^c|H_1^c H_2^c H_3^c) = 1 - P(C|H_1^c H_2^c H_3^c) = \frac{8}{9}.$$

If the queen has a fourth prince, then we want to compute $P(H_4|H_1^c H_2^c H_3^c)$. Let A be the event $H_1^c H_2^c H_3^c$ (so that we don't have to keep writing this) then conditioning on whether the queen is a carrier, we see that the probability we seek is given by

$$\begin{aligned} P(H_4|A) &= P(H_4|C, A)P(C|A) + P(H_4|C^c, A)P(C^c|A) \\ &= P(H_4|C)P(C|A) + P(H_4|C^c)P(C^c|A) \\ &= \frac{1}{2}\left(\frac{1}{9}\right) = \frac{1}{18}. \end{aligned}$$

Problem 71 (winning the western division)

We are asked to compute the probabilities that each of the given team wins the western division. We will assume that the team with the largest total number of wins will be the division winner. We are also told that each team is equally likely to win each game it plays. We can take this information to mean that each team wins each game it plays with probability 1/2. We begin to solve this problem, by considering the three games that the Atlanta Braves play against the San Diego Padres. In Table 6 we enumerate all of the possible outcomes,

Probability	Win	Loss	Total Wins	Total Losses
$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$	0	3	87	75
$3\left(\frac{1}{2}\right)^3 = \frac{3}{8}$	1	2	88	74
$3\left(\frac{1}{2}\right)^3 = \frac{3}{8}$	2	1	89	73
$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$	3	0	90	72

Table 6: The win/loss record for the Atlanta Braves each of the four total possible outcomes when they play the San Diego Padres.

S.F.G. Total Wins	S.F.G. Total Losses	L.A.D. Total Wins	L.A.D. Total Losses
86	76	89	73
87	75	88	74
88	74	87	75
89	73	86	76

Table 7: The total win/loss record for both the San Francisco Giants (S.F.G) and the Los Angeles Dodgers (L.A.D.). The first row corresponds to the San Francisco Giants winning *no* games while the Los Angeles Dodgers win *three* games. The number of wins going to the San Francisco Giants increases as we move down the rows of the table, until we reach the third row where the Giants have won three games and the Dodgers none.

i.e. the total number of wins or losses that can occur to the Atlanta Braves during these three games, along with the probability that each occurs.

We can construct the same type of a table for the San Francisco Giants when they play the Los Angeles Dodgers. In Table 7 we list all of the possible total win/loss records for both the San Francisco Giants and the Los Angeles Dogers. Since the probabilities are the same as listed in Table 6 the table does not explicitly enumerate these probabilities.

From these results (and assuming that the the team with the most wins will win the division) we can construct a table which represents for each of the possible wins/losses combination above, which team will be the division winner. Define the events B, G, and D to be the events that the Braves, Giants, and Los Angles Dodgers win the western division. Then in Table 8 we summarize the results of the two tables above where for the first row assumes that the Atlanta Braves win *none* of their games and the last row assumes that the Atlanta Braves win *all* of their games. In the same way the first column corresponds to the case when the San Francisco Giants win *none* of their games and the last column corresponds to the case when they win *all* of their games.

In anytime that two teams tie each team has a 1/2 of a chance of winning the tie-breaking game that they play next. Using this result and the probabilities derived above we can

	1/8	3/8	3/8	1/8
1/8	D	D	G	G
3/8	D	B/D	B/G	G
3/8	B/D	B	B	B/G
1/8	B	B	B	B

Table 8: The possible division winners depending on the outcome of the three games that each team must play. The rows (from top to bottom) correspond to the Atlanta Braves winning more and more games (from the three that they play). The columns (from left to right) correspond to the San Francisco Giants winning more and more games (from the three they play). Note that as the Giants win more games the Dogers must loose more games. Ties are determined by the presence of two symbols at a given location.

evaluate the individual probabilities that each team wins. We find that

$$\begin{aligned}
P(D) &= \frac{1}{8}\left(\frac{1}{8}+\frac{3}{8}\right)+\frac{3}{8}\left(\frac{1}{8}+\frac{1}{2}\cdot\frac{3}{8}\right)+\frac{3}{8}\left(\frac{1}{2}\cdot\frac{1}{8}\right)=\frac{13}{64}\\
P(G) &= \frac{1}{8}\left(\frac{3}{8}+\frac{1}{8}\right)+\frac{3}{8}\left(\frac{1}{2}\cdot\frac{3}{8}+\frac{1}{8}\right)+\frac{3}{8}\left(\frac{1}{2}\cdot\frac{1}{8}\right)=\frac{13}{64}\\
P(B) &= \frac{3}{8}\left(\frac{1}{2}\cdot\frac{3}{8}+\frac{1}{2}\cdot\frac{3}{8}\right)+\frac{3}{8}\left(\frac{1}{2}\cdot\frac{1}{8}+\frac{3}{8}+\frac{3}{8}+\frac{1}{2}\cdot\frac{1}{8}\right)+\frac{1}{8}(1)=\frac{19}{32}
\end{aligned}$$

Note that these probabilities to add to one as they should. The calculations for this problems are `chap_3_prob_71.m`.

Problem 76

If E and F are mutually exclusive events in an experiment, then $P(E\cup F)=P(E)+P(F)$. We desire to compute the probability that E occurs before F, which we will denote by p. To compute p we condition on the three mutually exclusive events E, F, or $(E\cup F)^c$. This last event are all the outcomes not in E or F. Letting the event A be the event that E occurs before F, we have that

$$p=P(A|E)P(E)+P(A|F)P(F)+P(A|(E\cup F)^c)P((E\cup F)^c)\,.$$

Now

$$\begin{aligned}
P(A|E) &= 1\\
P(A|F) &= 0\\
P(A|(E\cup F)^c) &= p\,,
\end{aligned}$$

since if neither E or F happen the next experiment will have E before F (and thus event A with probability p). Thus we have that

$$\begin{aligned}
p &= P(E)+pP((E\cup F)^c)\\
&= P(E)+p(1-P(E\cup F))\\
&= P(E)+p(1-P(E)-P(F))\,.
\end{aligned}$$

Solving for p gives

$$p = \frac{P(E)}{P(E) + P(F)}\,,$$

as we were to show.

Chapter 3: Theoretical Exercises

Problem 1 (conditioning on more information)

We have

$$P(A \cap B|A) = \frac{P(A \cap B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}\,.$$

and

$$P(A \cap B|A \cup B) = \frac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)}\,.$$

But since $A \cup B \supset A$, the probabilities $P(A \cup B) \geq P(A)$, so

$$\frac{P(A \cap B)}{P(A)} \geq \frac{P(A \cap B)}{P(A \cup B)}$$

giving

$$P(A \cap B|A) \geq P(A \cap B|A \cup B)\,,$$

the desired result.

Problem 2 (a school community)

Using the definition of conditional probability we can compute

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}\,.$$

since $A \subset B$. In words $P(A|B)$ is the amount of A in B. For $P(A|\neg B)$ we have

$$P(A|\neg B) = \frac{P(A \cap \neg B)}{P(\neg B)} = \frac{P(\phi)}{P(\neg B)} = 0\,.$$

Since if $A \subset B$ then $A \cap \neg B$ is empty or in words given that $\neg B$ occurred and $A \subset B$, A cannot have occurred and therefore has zero probability. For $P(B|A)$ we have

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1\,,$$

or in words since A occurs and B contains A, B must have occurred giving probability one. For $P(B|\neg A)$ we have

$$P(B|\neg A) = \frac{P(B \cap \neg A)}{P(\neg A)}\,,$$

which cannot be simplified further.

Problem 3 (biased selection of the first born)

We define n_1 to be the number of families with one child, n_2 the number of families with two children, and in general n_k to be the number of families with k children. In this problem we want to compare two different methods for selecting children. In the first method, M_1, we pick one of the m families and then randomly choose a child from that family. In the second method, M_2, we directly pick one of the $\sum_{i=1}^{k} i n_i$ children randomly. Let E be the event that a first born child is chosen. Then the question seeks to prove that

$$P(E|M_1) > P(E|M_2) .$$

We will solve this problem by conditioning no the number of families with i children. For example under M_1 we have (dropping the conditioning on M_1 for notational simplicity) that

$$P(E) = \sum_{i=1}^{k} P(E|F_i)P(F_i) ,$$

where F_i is the event that the chosen family has i children. This later probability is given by

$$P(F_i) = \frac{n_i}{m} ,$$

for we have n_i families with i children from m total families. Also

$$P(E|F_i) = \frac{1}{i} ,$$

since the event F_i means that our chosen family has i children and the event E means that we select the first born, which can be done in $\frac{1}{i}$ ways. In total then we have under M_1 the following for $P(E)$

$$P(E) = \sum_{i=1}^{k} P(E|F_i)P(F_i) = \sum_{i=1}^{k} \frac{1}{i}\left(\frac{n_i}{m}\right) = \frac{1}{m}\sum_{i=1}^{k}\frac{n_i}{i} .$$

Now under the second method again $P(E) = \sum_{i=1}^{k} P(E|F_i)P(F_i)$ but under the second method $P(F_i)$ is the probability we have selected a family with i children and is given by

$$\frac{i n_i}{\sum_{i=1}^{k} i n_i} ,$$

since $i n_i$ is are the number of children from families with i children and the denominator is the total number of children. Now $P(E|F_i)$ is still the probability of having selected a family with ith children we select the first born child. This is

$$\frac{n_i}{i n_i} = \frac{1}{i} ,$$

since we have $i n_i$ total children from the families with i children and n_i of them are first born. Thus under the second method we have

$$P(E) = \sum_{i=1}^{k}\left(\frac{1}{i}\right)\left(\frac{i n_i}{\sum_{l=1}^{k} l n_l}\right) = \frac{1}{\left(\sum_{l=1}^{k} l n_l\right)}\sum_{i=1}^{k} n_i .$$

Then our claim that $P(E|M_1) > P(E|M_2)$ is equivalent to the statement that

$$\frac{1}{m}\sum_{i=1}^{k}\frac{n_i}{i} \geq \frac{\sum_{i=1}^{k} n_i}{\sum_{i=1}^{k} i n_i}$$

or remembering that $m = \sum_{i=1}^{k} n_i$ that

$$\left(\sum_{i=1}^{k} i n_i\right)\left(\sum_{j=1}^{k}\frac{n_j}{j}\right) \geq \left(\sum_{i=1}^{k} n_i\right)\left(\sum_{j=1}^{k} n_j\right).$$

Problem 7 (extinct fish)

Part (a): We desire to compute P_w the probability that the last ball drawn is white. This probability will be

$$P_w = \frac{n}{n+m},$$

because we have n white balls that can be selected from $n+m$ total balls that can be placed in the last spot.

Part (b): Let R be the event that the red fish species are the *first* species to become extinct. Then following the hint we write $P(R)$ as

$$P(R) = P(R|G_l)P(G_l) + P(R|B_l)P(B_l).$$

Here G_l is the event that the green fish species are the *last* fish species to become extinct and B_l the event that the blue fish species are the *last* fish species to become extinct. Now we conclude that

$$P(G_l) = \frac{g}{r+b+g},$$

and

$$P(B_l) = \frac{b}{r+b+g}.$$

We can see these by considering the blue fish as an example. If the blue fish are the last ones extinct then we have b possible blue fish to select from the $r+b+g$ total number of fish to be the last fish. Now we need to compute the conditional probabilities $P(R|G_l)$. This can be thought of as the event that the red fish go extinct and then the blue fish. This is the same type of experiment as in Part (a) of this problem in that we must have a blue fish go extinct (i.e. a draw with a blue fish last). This can happen with probability

$$\frac{b}{r+b+g-1},$$

where the denominator is one less than $r+b+g$ since the last fish drawn must be a green fish by the required conditioning. In the same way we have that

$$P(R|B_l) = \frac{g}{r+b+g-1}.$$

So that the total probability $P(R)$ is then given by

$$\begin{aligned}
P(R) &= \left(\frac{b}{r+b+g-1}\right)\left(\frac{g}{r+b+g}\right) + \left(\frac{g}{r+b+g-1}\right)\left(\frac{b}{r+b+g}\right) \\
&= \frac{2bg}{(r+b+g-1)(r+b+g)} \,.
\end{aligned}$$

Problem 8 (some inequalities)

Part (a): If $P(A|C) > P(B|C)$ and $P(A|C^c) > P(B|C^c)$, then consider $P(A)$ which by conditioning on C and C^c becomes

$$\begin{aligned}
P(A) &= P(A|C)P(C) + P(A|C^c)P(C^c) \\
&> P(B|C)P(C) + P(B|C^c)P(C^c) = P(B) \,.
\end{aligned}$$

Where the second line follows from the given inequalities.

Part (b): Following the hint, let C be the event that the sum of the pair of die is 10, A the event that the first die lands on a 6 and B the event that the second die lands a 6. Then $P(A|C) = \frac{1}{3}$, and $P(A|C^c) = \frac{5}{36-3} = \frac{5}{33}$. So that $P(A|C) > P(A|C^c)$ as expected. Now $P(B|C)$ and $P(B|C^c)$ will have the same probabilities as for A. Finally, we see that $P(A \cap B|C) = 0$, while $P(A \cap B|C^c) = \frac{1}{33} > 0$. So we have found an example where $P(A \cap B|C) < P(A \cap B|C^c)$ and a counter example has been found.

Problem 10 (pairwise independence does not imply independence)

Let $A_{i,j}$ be the event that person i and j have the same birthday. We desire to show that these events are pairwise independent. That is the two events $A_{i,j}$ and $A_{r,s}$ are independent but the totality of all $\left(\begin{array}{c} n \\ 2 \end{array}\right)$ events are not independent. Now

$$P(A_{i,j}) = P(A_{r,s}) = \frac{1}{365} \,,$$

since for the specification of either one persons birthday the probability that the other person will have that birthday is $1/365$. Now

$$P(A_{i,j} \cap A_{r,s}) = P(A_{i,j}|A_{r,s})P(A_{r,s}) = \left(\frac{1}{365}\right)\left(\frac{1}{365}\right) = \frac{1}{365^2} \,.$$

This is because $P(A_{i,j}|A_{r,s}) = P(A_{i,j})$ i.e. the fact that people r and s have the same birthday has no effect on whether the event $A_{i,j}$ is true. This is true even if one of the people in the pairs (i,j) and (r,s) is the same. When we consider the intersection of *all* the sets $A_{i,j}$, the situation changes. This is because the event $\cap_{(i,j)} A_{i,j}$ (where the intersection is over all pairs (i, j)) is the event that *every* pair of people have the same birthday, i.e. that everyone has the same birthday. This will happen with probability

$$\left(\frac{1}{365}\right)^{n-1},$$

while if the events $A_{i,j}$ were independent the required probability would be

$$\prod_{(i,j)} P(A_{i,j}) = \left(\frac{1}{365}\right)^{\binom{n}{2}} = \left(\frac{1}{365}\right)^{\frac{n(n-1)}{2}}.$$

Since $\binom{n}{2} \neq n - 1$, these two results are not equal and the totality of events $A_{i,j}$ are not independent.

Problem 12 (an infinite sequence of flips)

Let a_i be the probability that the ith coin lands heads. The consider the random variable N, specifying the location where the first head occurs. This problem then is like a geometric random variable where we want to determine the first time a success occurs. Then we have for a distribution of $P\{N\}$ the following

$$P\{N = n\} = a_n \prod_{i=1}^{n-1} (1 - a_i).$$

This states that the first $n - 1$ flips must land tails and the last flip (the nth) then lands heads. Then when we add this probability up for $n = 1, 2, 3, \cdots, \infty$ i.e.

$$\sum_{n=1}^{\infty} \left[a_n \prod_{i=1}^{n-1} (1 - a_i) \right],$$

is the probability that a head occurs *somewhere* in the infinite sequence of flips. The other possibility would be for a head to *never* appear. This will happen with a probability of

$$\prod_{i=1}^{\infty} (1 - a_i).$$

Together these two expressions consist of all possible outcomes and therefore must sum to one. Thus we have proven the identity

$$\sum_{n=1}^{\infty} \left[a_n \prod_{i=1}^{n-1} (1 - a_i) \right] + \prod_{i=1}^{\infty} (1 - a_i) = 1,$$

or the desired result.

Problem 13 (winning by flipping)

Let $P_{n,m}$ be the probability that A who starts the game accumulates n head before B accumulates m heads. We can evaluate this probability by conditioning on the outcome of the first flip made by A. If this flip lands heads, then A has to get $n-1$ more flips before B's obtains m. If this flip lands tails then B obtains control of the coin and will receive m flips before A receives n with probability $P_{m,n}$ by the implicit symmetry in the problem. Thus A will accumulate the correct number of heads with probability $1 - P_{m,n}$. Putting these two outcomes together (since they are the mutually exclusive and exhaustive) we have

$$P_{n,m} = pP_{n-1,m} + (1-p)(1-P_{m,n}),$$

or the desired result.

Problem 14 (gambling against the rich)

Let P_i be the probability you eventually go broke when your initial fortune is i. Then conditioning on the result of the first wager we see that P_i satisfies the following difference equation

$$P_i = pP_{i+1} + (1-p)P_{i-1}.$$

This simply states that the probability you go broke when you have a fortune of i is p times P_{i+1} if you win the first wager (since if you win the first wager you now have $i+1$ as your fortune) *plus* $1-p$ times P_{i-1} if you loose the first wager (since if you loose the first wager you will have $i-1$ as your fortune). To solve this difference equation we recognize that its solution must be given in terms of a constant raised to the ith power i.e. α^i. Using the anzats that $P_i = \alpha^i$ and inserting this into the above equation we find that α must satisfy the following

$$p\alpha^2 - \alpha + (1-p) = 0.$$

Using the quadratic equation to solve this equation for α we find α given by

$$\begin{aligned}
\alpha &= \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} \\
&= \frac{1 \pm \sqrt{(2p-1)^2}}{2p} \\
&= \frac{1 \pm (2p-1)}{2p}.
\end{aligned}$$

Taking the plus sign gives $\alpha^+ = 1$, while taking the minus sign in the above gives $\alpha^- = \frac{q}{p}$. Now the general solution to this difference equation is then given by

$$P_i = C_1 + C_2 \left(\frac{q}{p}\right)^i \quad \text{for} \quad i \geq 0.$$

Problem 16 (the probability of an even number of successes)

Let P_n be the probability that n Bernoulli trials result in an even number of successes. Then the given difference equation can be obtained by conditioning on the result of the first trial as follows. If the first trial is a success then we have $n-1$ trials to go and to obtain an even *total* number of tosses we want the number of successes in this $n-1$ trials to be *odd* This occurs with probability $1-P_{n-1}$. If the first trial is a failure then we have $n-1$ trials to go and to obtain an even total number of tosses we want the number of successes in this $n-1$ trials to be *even* This occurs with probability P_{n-1}. Thus we find that

$$P_n = p(1-P_{n-1}) + (1-p)P_{n-1} \quad \text{for} \quad n \geq 1\,.$$

Some special point cases are easily computed. We have by assumption that $P_0 = 1$, and $P_1 = q$ since with only one trial, this trial must be a failure to get a total even number of successes. Given this difference equation and a potential solution we can verify that this solution satisfies our equation and therefore know that it is a solution. One can easily check that the given P_n satisfies $P_0 = 1$ and $P_1 = q$. In addition, for the given assumed solution we have that

$$P_{n-1} = \frac{1+(1-2p)^{n-1}}{2}\,,$$

From which we find (using this expression in the right hand side of the difference equation above)

$$\begin{aligned}
p(1-P_{n-1}) + (1-p)P_{n-1} &= p + (1-2p)P_{n-1} \\
&= p + (1-2p)\left(\frac{1+(1-2p)^{n-1}}{2}\right) \\
&= p + \frac{1-2p}{2} + \frac{(1-2p)^n}{2} \\
&= \frac{1}{2} + \frac{(1-2p)^n}{2} = P_n\,.
\end{aligned}$$

Showing that P_n is a solution the given difference equation.

Problem 24 (round robin tournaments)

In this problem we specify an integer k and then ask whether it is possible for every set of k players to have there exist a member from the *other* $n-k$ players that beat these k players when competing against these k. To show that this is possible if the given inequality is true, we follow the hint. In the hint we enumerate the $\left(\begin{array}{c} n \\ k \end{array}\right)$ sets of k players and let B_i be the event that *no* of the other $n-k$ contestant beats every one of the k players in the i set of k. Then $P(\cup_i B_i)$ is the probability that at least one of the subsets of size k has no external player that beats everyone. Then $1 - P(\cup_i B_i)$ is the probability that every subset of size k has an external player that beats everyone. Since this is the event we want to be possible we desire that

$$1 - P(\cup_i B_i) > 0\,,$$

or equivalently

$$P(\cup_i B_i) < 1\,.$$

Now Boole's inequality states that $P(\cup_i B_i) \leq \sum_i P(B_i)$, so if we pick our k such that $\sum_i P(B_i) < 1$, we will necessarily have $P(\cup_i B_i) < 1$ possible. Thus we will focus on ensuring that $\sum_i P(B_i) < 1$.

Lets now focus on evaluating $P(B_i)$. Since this is the probability that no contestant from outside the ith cluster beats all players inside, we can evaluate it by considering a particular player outside the k member set. Denote the other player by X. Then X would beat all k members with probability $\left(\frac{1}{2}\right)^k$, and thus with probability $1 - \left(\frac{1}{2}\right)^k$ does *not* beat all players in this set. As the set B_i, requires that all $n-k$ players *not* beat the k players in this ith set, each of the $n-k$ exterior players must fail at beating the k players and we have

$$P(B_i) = \left(1 - \left(\frac{1}{2}\right)^k\right)^{n-k}\,.$$

Now $P(B_i)$ is in fact independent of i (there is no reason it should depend on the particular subset of players) we can factor this result out of the sum above and simply multiply by the number of terms in the sum which is $\left(\begin{array}{c} n \\ k \end{array}\right)$ giving the requirement for possibility of

$$\left(\begin{array}{c} n \\ k \end{array}\right)\left(1 - \left(\frac{1}{2}\right)^k\right)^{n-k} < 1\,,$$

as was desired to be shown.

Chapter 3: Self-Test Problems and Exercises

Problem 25

Now following the hint we have

$$P(E|E\cup F) = P(E|E\cup F, F)P(F) + P(E|E\cup F, \neg F)P(\neg F)\,.$$

But $P(E|E\cup F, F) = P(E|F)$, since $E\cup F \supset F$, and $P(E|E\cup F, \neg F) = P(E|E\cap\neg F) = 1$, so the above becomes

$$P(E|E\cup F) = P(E|F)P(F) + (1 - P(F))\,.$$

Dividing by $P(E|F)$ we have

$$\frac{P(E|E\cup F)}{P(E|F)} = P(F) + \frac{1-P(F)}{P(E|F)}\,.$$

Since $P(E|F) \leq 1$ we have that $\frac{1-P(F)}{P(E|F)} \geq 1 - P(F)$ and the above then becomes

$$\frac{P(E|E\cup F)}{P(E|F)} \geq P(F) + (1 - P(F)) = 1$$

giving the desired result of $P(E|E \cup F) \geq P(E|F)$. In words this says that the probability that E occurs given E *or* F occurs must be larger than if we just know that only F occurs.

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Table 9: The possible values for the product of two die when two die are rolled.

Chapter 4 (Random Variables)

Chapter 4: Problems

Problem 2 (the product of two die)

We begin by constructing the sample space of possible outcomes. These numbers are computed in table 9, where the row corresponds to the first die and the column corresponds to the second die. In each square we have placed the product of the two die. Each pair has probability of 1/36, so by enumeration we find that

$$
\begin{aligned}
P\{X=1\} &= \frac{1}{36}, \quad P\{X=2\} = \frac{2}{36} \\
P\{X=3\} &= \frac{2}{36}, \quad P\{X=4\} = \frac{3}{36} \\
P\{X=5\} &= \frac{2}{36}, \quad P\{X=6\} = \frac{2}{36} \\
P\{X=8\} &= \frac{2}{36}, \quad P\{X=9\} = \frac{1}{36} \\
P\{X=10\} &= \frac{2}{36}, \quad P\{X=12\} = \frac{4}{36} \\
P\{X=15\} &= \frac{2}{36}, \quad P\{X=16\} = \frac{1}{36} \\
P\{X=18\} &= \frac{2}{36}, \quad P\{X=20\} = \frac{1}{36} \\
P\{X=24\} &= \frac{2}{36}, \quad P\{X=25\} = \frac{1}{36} \\
P\{X=30\} &= \frac{2}{36}, \quad P\{X=36\} = \frac{1}{36},
\end{aligned}
$$

with any other integer having zero probability.

	1	2	3	4	5	6
1	(1,1,2,0)	(2,1,3,-1)	(3,1,4,-2)	(4,1,5,-3)	(5,1,6,-4)	(6,1,7,-5)
2	(2,1,3,1)	(2,2,4,0)	(3,2,5,-1)	(4,2,6,-2)	(5,2,7,-3)	(6,2,8,-4)
3	(3,1,4,2)	(3,2,5,1)	(3,3,6,0)	(4,3,7,-1)	(5,3,8,-2)	(6,3,9,-3)
4	(4,1,5,3)	(4,2,6,2)	(4,3,7,1)	(4,4,8,0)	(5,4,9,-1)	(6,4,10,-2)
5	(5,1,6,4)	(5,2,7,3)	(5,3,8,2)	(5,4,9,1)	(5,5,10,0)	(6,5,11,-1)
6	(6,1,7,5)	(6,2,8,4)	(6,3,9,3)	(6,4,10,2)	(6,5,11,1)	(6,6,12,0)

Table 10: The possible values for the maximum, minimum, sum, and first minus second die observed when two die are rolled.

Problem 7 (the functions of two die)

In table 10 we construct a table of all possible outcomes associated with the two die rolls. In that table the row corresponds to the first die and the column corresponds to the second die. Then for each part of the problem we find that

Part (a): $X \in \{1, 2, 3, 4, 5, 6\}$.

Part (b): $X \in \{1, 2, 3, 4, 5, 6\}$.

Part (c): $X \in \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Part (d): $X \in \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$.

Problem 8 (probabilities on die)

The solution to this problem involves counting up the number of times that X equals the given value and then dividing by $6^2 = 36$. For each part we have the following

Part (a): From table 10, for this part we find that

$$\begin{aligned} P\{X=1\} &= \frac{1}{36}, \quad P\{X=2\} = \frac{3}{12}, \quad P\{X=3\} = \frac{5}{36}, \\ P\{X=4\} &= \frac{7}{36}, \quad P\{X=5\} = \frac{1}{4}, \quad P\{X=6\} = \frac{11}{36} \end{aligned}$$

Part (b): From table 10, for this part we find that

$$\begin{aligned} P\{X=1\} &= \frac{11}{36}, \quad P\{X=2\} = \frac{1}{4}, \quad P\{X=3\} = \frac{7}{36}, \\ P\{X=4\} &= \frac{1}{12}, \quad P\{X=5\} = \frac{7}{36}, \quad P\{X=6\} = \frac{1}{36} \end{aligned}$$

Part (c): From table 10, for this part we find that

$$\begin{aligned}
P\{X=2\} &= \frac{1}{36}\,, \quad P\{X=3\} = \frac{1}{18}\,, \quad P\{X=4\} = \frac{1}{12}\,, \\
P\{X=5\} &= \frac{1}{9}\,, \quad P\{X=6\} = \frac{5}{36}\,, \quad P\{X=7\} = \frac{1}{6}\,, \\
P\{X=8\} &= \frac{5}{36}\,, \quad P\{X=9\} = \frac{1}{9}\,, \quad P\{X=10\} = \frac{1}{12}\,, \\
P\{X=11\} &= \frac{1}{18} \quad P\{X=12\} = \frac{1}{36}\,.
\end{aligned}$$

Part (d): From table 10, for this part we find that

$$\begin{aligned}
P\{X=-5\} &= \frac{1}{36}\,, \quad P\{X=-4\} = \frac{1}{18}\,, \quad P\{X=-3\} = \frac{1}{9}\,, \\
P\{X=-2\} &= \frac{1}{9}\,, \quad P\{X=-1\} = \frac{5}{36}\,, \quad P\{X=0\} = \frac{1}{6}\,, \\
P\{X=1\} &= \frac{5}{36}\,, \quad P\{X=2\} = \frac{1}{9}\,, \quad P\{X=3\} = \frac{1}{12}\,, \\
P\{X=4\} &= \frac{1}{18}\,, \quad P\{X=5\} = \frac{1}{36}\,.
\end{aligned}$$

Problem 25 (events registered with probability p)

We can solve this problem by conditioning on the number of true events (from the original Poisson random variable N) that occur. We begin by letting M be the number of events counted by our "filtered" Poisson random variable. Then we want to show that M is another Poisson random variable with parameter λp. To do so consider the probability that M has counted j "filtered events", by conditioning on the number of observed events from the original Poisson random variable. We find

$$P\{M=j\} = \sum_{n=0}^{\infty} P\{M=j|N=n\} \left(\frac{e^{-\lambda}\lambda^n}{n!} \right)$$

The conditional probability in this sum can be computed using the acceptance rule defined above. For if we have n original events the number of derived events is a binomial random variable with parameters (n,p). Specifically then we have

$$P\{M=j|N=n\} = \begin{cases} \left(\begin{array}{c} n \\ j \end{array} \right) p^j (1-p)^{n-j} & j \le n \\ 0 & j > n\,. \end{cases}$$

Putting this result into the original expression for $P\{M=j\}$ we find that

$$P\{M=j\} = \sum_{n=j}^{\infty} \left(\begin{array}{c} n \\ j \end{array} \right) p^j (1-p)^{n-j} \left(\frac{e^{-\lambda}\lambda^n}{n!} \right)$$

To evaluate this we note that $\binom{n}{j}\frac{1}{n!} = \frac{1}{j!(n-j)!}$, so that the above simplifies as following

$$
\begin{aligned}
P\{M = j\} &= \frac{e^{-\lambda}p^j}{j!}\sum_{n=j}^{\infty}\frac{1}{(n-j)!}(1-p)^{n-j}\lambda^n \\
&= \frac{e^{-\lambda}p^j}{j!}\sum_{n=j}^{\infty}\frac{1}{(n-j)!}(1-p)^{n-j}(\lambda)^j\lambda^{n-j} \\
&= \frac{e^{-\lambda}(p\lambda)^j}{j!}\sum_{n=j}^{\infty}\frac{((1-p)\lambda)^{n-j}}{(n-j)!} \\
&= \frac{e^{-\lambda}(p\lambda)^j}{j!}\sum_{n=0}^{\infty}\frac{((1-p)\lambda)^{n}}{n!} \\
&= \frac{e^{-\lambda}(p\lambda)^j}{j!}e^{(1-p)\lambda} = e^{-p\lambda}\frac{(p\lambda)^j}{j!},
\end{aligned}
$$

from which we can see M is a Poisson random variable with parameter λp as claimed.

Problem 29 (a machine that breaks down)

Under the first strategy we would check the first possibility and if needed check the second possibility. This has an expected cost of

$$C_1 + R_1,$$

if the first possibility is true (which happens with probability p) and

$$C_1 + C_2 + R_2,$$

if the second possibility is true (which happens with probability $1-p$). Here I am explicitly assuming that if the first check is a failure we must then check the second possibility (at a cost C_2) before repair (at a cost of R_2). Another assumption would be that if the first check is a failure then we know that the second cause is the real one and we don't have to check for it. This results in a cost of $C_1 + R_2$ rather than $C_1 + C_2 + R_2$. The first assumption seems more consistent with the problem formulation and will be the one used. Thus under the first strategy we have an expected cost of

$$p(C_1 + R_1) + (1-p)(C_1 + C_2 + R_2),$$

so our expected cost becomes

$$C_1 + pR_1 + (1-p)(C_2 + R_2) = C_1 + C_2 + R_2 + p(R_1 - C_2 - R_2).$$

Now under the second strategy we would first check the second possibility and if needed check the first possibility. This first action has an expected cost of

$$C_2 + R_2,$$

if the second possibility is true cause (this happens with probability $1-p$) and

$$C_2 + C_1 + R_1\,,$$

if the first possibility is true (which happens with probability p). This gives an expected cost when using the second strategy of

$$(1-p)(C_2+R_2) + p(C_2+C_1+R_1) = C_2 + R_2 + p(C_1+R_1-R_2)\,.$$

The expected cost under strategy number one will be less than the expected cost under strategy number if

$$C_1 + C_2 + R_2 + p(R_1 - C_2 - R_2) < C_2 + R_2 + p(C_1 + R_1 - R_2)\,.$$

When we solve for p the above simplifies to

$$p > \frac{C_1}{C_1 + C_2}\,.$$

As the threshold value to use for the different strategies. This result has the intuitive understanding in that if p is "significantly" large (meaning the break is more likely to be caused by the first possibility) we should check the first possibility first. While if p is not significantly large we should check the second possibility first.

Problem 30 (the St. Petersburg paradox)

The probability that the first tail appears on the nth flip means that the $n-1$ heads must first appear and then a tail. This gives a probability of

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^{n}\,.$$

Then the expected value of our winnings is given by

$$\sum_{n=1}^{\infty} 2^n \left(\frac{1}{2}\right)^{n} = \sum_{n=1}^{\infty} 1 = +\infty\,.$$

Part (a): If a person payed 10^6 to play this game he would only "win" if the first tail appeared on toss greater than or equal to n^* where $n^* \geq \log_2(10^6) = 6\log_2(10) = 6\frac{\ln(10)}{\ln(2)} = 19.931$, or $n^* = 20$. In that case this event would occur with probability

$$\sum_{k=n^*}^{\infty} \left(\frac{1}{2}\right)^{k} = \left(\frac{1}{2}\right)^{n^*} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} = \left(\frac{1}{2}\right)^{n^*-1}\,,$$

since $\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 2$. With $n^* = 20$ we see that this probability is given by 9.5367×10^{-7} a rather small number. Thus many would not be willing to play under these conditions.

Part (b): In this case, if we play k games then we will definitely "win" if the first tail appears on a flip n^* (or greater) where n^* solves

$$-k\,10^6 + 2^{n^*} > 0\,,$$

or

$$n^* > 6\log_2(10) + \log_2(k) = 19.931 + \log_2(k)\,.$$

Since this target n^* grows logarithmically with k one would expect that enough random experiments were ran that eventually a very high paying result would appear. Thus many would be willing to pay this game.

Problem 31 (scoring your guess)

Since the meterologist truly believes that it will rain with probability p^* if he quotes a probability p, then the expected score he will recieve is given by

$$E[S;p] = p^*(1-(1-p)^2) + (1-p^*)(1-p^2)\,.$$

We want to pick a value of p such that we maximize this expression. To do so, consider the derivative of this expression set equal to zero and solve for the value of p. We find that

$$\frac{dE[S;p]}{dp} = p^*(2(1-p)) + (1-p^*)(-2p) = 0\,.$$

solving for p we find that $p = p^*$. Taking the second derivative of this expression we find that

$$\frac{d^2E[S;p]}{dp^2} = -2p^* - 2(1-p^*) = -2 < 0\,,$$

showing that $p = p^*$ is a maximum. This is a nice reason for using this metric, since it behoves the meteriologist to quote the probability of rain that he truly believes is true.

Problem 32 (testing diseased people)

We have one hundred people which we break up into ten groups of ten for the purposes of testing for a disease. For each group we will test the entire group of people with one test. This test will be "positive" (meaning at least one person has the disease) with probability $1-(0.9)^{10}$. Since 0.9^{10} is the probability that all people are normal and the complement of this probability is the probability that at least one person has the disease. Then the expected number of tests for each group of ten is then

$$1 + 0((0.9)^{10}) + 10(1-(0.9)^{10}) = 11 - 10(0.9)^{10} = 7.51\,.$$

Where the first 1 is because we will certainly test the pooled people and the remaining to expressions represent the case where the entire pooled test result comes back negative (no more tests needed) and the case where the entire pooled test result comes back positive (meaning we have ten individual tests to then do).

Problem 33 (the number of papers to purchase)

Let b be the variable denoting the number of papers bought and N the *random* variable denoting the number of papers demanded. Finally, let the random variable P be the newsboys' profits. Then with these definitions the newsboys' profits is given by

$$P = -10b + 15\min(N, b) \quad \text{for} \quad b \geq 1\,,$$

This is because if we only buy b papers we can only sell a maximum of b papers independent of what the demand N is. Then to calculate the expected profit we have that

$$\begin{aligned} E[P] &= -10b + 15E[\min(N, b)] \\ &= -10b + 15\sum_{n=0}^{10} \min(n, b) \left(\begin{array}{c} 10 \\ n \end{array} \right) \left(\frac{1}{3}\right)^n \left(\frac{2}{3}\right)^{10-n} . \end{aligned}$$

To evaluate the optimal number of papers to buy we can plot this as a function of b for $1 \leq b \leq 15$. In the Matlab file `chap_3_prob_33.m`, where this function is computed and plotted. See Figure **??**, for a figure of the produced plot. There one can see that the maximum expected profit occurs when we order $b = 3$ newspapers. The expected profit in that case is given by 8.36.

Problem 35 (a game with marbles)

Part (a): Define W to be the random variable expression the winnings obtained when one plays the proposed game. The expected value of W is then given by

$$E[W] = 1.1P_{\text{sc}} - 1.0P_{\text{dc}}$$

where the notation "sc" means that the two drawn marbles are of the same color and the notation "dc" means that the two drawn marbles are of different colors. Now to calculate each of these probabilities we introduce the four possible events that can happen when we draw to marbles: RR, BB, RB, and BR. As an example the notation RB denotes the event that we first draw a red marble and then second draw a black marble. With this notation we see that P_{sc} is given by

$$\begin{aligned} P_{\text{sc}} &= P\{RR\} + P\{BB\} \\ &= \frac{5}{10}\left(\frac{4}{9}\right) + \frac{5}{10}\left(\frac{4}{9}\right) = \frac{4}{9} . \end{aligned}$$

while P_{dc} is given by

$$\begin{aligned} P_{\text{dc}} &= P\{RB\} + P\{BR\} \\ &= \frac{5}{10}\left(\frac{5}{9}\right) + \frac{5}{10}\left(\frac{5}{9}\right) = \frac{5}{9} . \end{aligned}$$

With these two results the expected profit is then given by

$$1.1\left(\frac{4}{9}\right) - 1.0\left(\frac{5}{9}\right) = -\frac{1}{15} .$$

Part (b): The variance of the amount one wins can be computed by the standard expression for variance in term of expectations. Specifically we have

$$\text{Var}(W) = E[W^2] - E[W]^2 .$$

Now using the results from Part (a) above we see that

$$E[W^2] = \frac{4}{9}(1.1)^2 + \frac{5}{9}(-1.0)^2 = \frac{82}{75} .$$

so that

$$\text{Var}(W) = \frac{82}{75} - \left(\frac{1}{15}\right)^2 = \frac{49}{45} \approx 1.08 .$$

Problem 38 (evaluating expectations and variances)

Part (a): We find, expanding the quadratic and using the linearity property of expectations that

$$E[(2+X)^2] = E[4 + 4X + X^2] = 4 + 4E[X] + E[X^2] .$$

In terms of the variance, $E[X^2]$ is given by $E[X^2] = \text{Var}(X) + E[X]^2$, both terms of which we know from the problem statement. Using this the above becomes

$$E[(2+X)^2] = 4 + 4(1) + (5 + 1^2) = 14 .$$

Part (b): We find, using properties of the variance that

$$\text{Var}(4 + 3X) = \text{Var}(3X) = 9\text{Var}(X) = 9 \cdot 5 = 45 .$$

Exercise 39 (drawing two white balls in four draws)

The probability of drawing a white ball is $3/6 = 1/2$. Thus if we consider event that we draw a white ball a success, the probability requested is that in four trials, two are found to be successes. This is equal to a binomial distribution with $n = 4$ and $p = 1/2$, thus our desired probability is given by

$$\left(\begin{array}{c} 4 \\ 2 \end{array}\right) \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{4-2} = \frac{6}{4 \cdot 4} = \frac{3}{8} .$$

Problem 40 (guessing on a multiple choice exam)

With three possible answers possible for each question we have a $1/3$ chance of guessing any specific question correctly. Then the probability that the student gets four or more correct by guessing would be the required sum of a binomial distribution. Specifically we have

$$\left(\begin{array}{c} 5 \\ 4 \end{array}\right) \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^1 + \left(\begin{array}{c} 5 \\ 5 \end{array}\right) \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^0 = \frac{11}{243} .$$

Where the first term is the probability the student guess four questions (from five) correctly and the second term is the probability that the student guesses all five questions correctly.

Problem 41 (proof of extrasensory perception)

Randomly guessing the man would get seven correct answers (out of ten) with probability

$$\binom{10}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 = 0.11718\,.$$

Problem 48 (defective disks)

For this problem lets take the gaurrentee that the company provides to mean that a package will be considered "defective" if it has *more than* one defective disk. The probability that more than one disk in a pack is defective (P_d) is given by

$$P_d = 1 - \binom{10}{0}(0.01)^0(0.99)^{10} - \binom{10}{1}(0.01)^1(0.99)^9 \approx 0.0042\,,$$

since $\binom{10}{0}(0.01)^0(0.99)^{10}$ is the probability that *no* disks are defective in the package of ten disks, and $\binom{10}{1}(0.01)^1(0.99)^9$ is the probability that one of the ten disks is defective. If a customer buys three packs of disks the probability that he returns exactly one pack is the probability that from his three packs one package is defective. This is given by a binomial distribution with parameters $n = 3$ and $p = 0.0042$. We find this to be

$$\binom{3}{1}(0.0042)^1(1 - 0.0042)^2 = 0.0126\,.$$

Problem 49 (flipping coins)

We are told in the problem statement that the event the first coin C_1, lands heads happens with probability 0.4, while the event that the second coin C_2 lands heads happens with probability 0.7.

Part (a): Let E be the event that exactly seven of the ten flips land on heads then conditioning on the initially drawn coin (either C_1 or C_2) we have

$$P(E) = P(E|C_1)P(C_1) + P(E|C_2)P(C_2)\,.$$

Now we can evaluate each of these conditional probabilities as

$$\begin{aligned} P(E|C_1) &= \begin{pmatrix} 10 \\ 7 \end{pmatrix} (0.4)^7 (0.6)^3 = 0.0424 \\ P(E|C_2) &= \begin{pmatrix} 10 \\ 7 \end{pmatrix} (0.7)^7 (0.3)^3 = 0.2668 \,. \end{aligned}$$

So $P(E)$ is given by (assuming uniform probabilities on the coin we initially select)

$$P(E) = 0.5 \cdot 0.0424 + 0.5 \cdot 0.2668 = 0.1546 \,.$$

Part (b): If we are told that the first three of the ten flips are heads then we desire to compute what is the conditional probability that exactly seven of the ten flips land on heads. To compute this let A be the event that the first three flips are heads. Then we want to compute $P(E|A)$, which we can do by conditioning on the initial coin selected, i.e.

$$P(E|A) = P(E|A, C_1)P(C_1) + P(E|A, C_2)P(C_2) \,.$$

Now as before we find that

$$\begin{aligned} P(E|A, C_1) &= \begin{pmatrix} 7 \\ 4 \end{pmatrix} (0.4)^4 (0.6)^3 = 0.1935 \\ P(E|A, C_2) &= \begin{pmatrix} 7 \\ 4 \end{pmatrix} (0.7)^4 (0.3)^3 = 0.2268 \,. \end{aligned}$$

So the above probability is given by

$$P(E|A) = 0.5 \cdot 0.1935 + 0.5 \cdot 0.2668 = 0.2102 \,.$$

Problem 62 (the probability that no wife sits next to her husband)

From Problem 66, the probability that couple i is selected next to each other is given by $\frac{2}{2n-1} = \frac{1}{n-1/2}$. Then we can approximate the probability that the total number of couples sitting together is a Poisson distribution with parameter $\lambda = n\frac{1}{n-1/2} = \frac{2n}{2n-1}$. Thus the probability that no wife sits next to her husband is given by evaluating a Poisson distribution with count equal to zero and $\lambda = \frac{2n}{2n-1}$ or

$$\exp\{-\frac{2n}{2n-1}\} \,.$$

When $n = 10$ this expression is $\exp\{-\frac{20}{19}\} \approx 0.349$. The exact formula is computed in example 5n from Chapter 2, where the exact probability is given as 0.3395 showing that our approximation is rather close.

Problem 65 (the diseased)

Part (a): Since the probability that the number of soldiers with the given disease is a binomial distribution with parameters $(n, p) = (500, \frac{1}{10^3})$, we can approximate this distribution with a Poisson distribution with rate $\lambda = 500\frac{1}{10^3} = 0.5$. Then the required probability is given by

$$P\{N \geq 1\} = 1 - P\{N = 0\} = 1 - e^{-0.5} \approx 0.3934\,.$$

Part (b): We are now looking for

$$\begin{aligned}
P\{N \geq 2 | N > 0\} &= \frac{P\{N \geq 2\,, N > 0\}}{P\{N > 0\}} \\
&= \frac{1 - P\{N < 0\}}{P\{N > 0\}} \\
&\approx \frac{1 - e^{-0.5}(1 + 0.5)}{0.3934} \\
&= 0.2293\,.
\end{aligned}$$

Part (c): If Jones knows that he has the disease then the news that the test result comes back positive is not informative to him. Therefore he believes that the distribution of the number of men with the disease is binomial with parameters $(n, p) = (499, \frac{1}{10^3})$. As such, it can be approximated with a Poisson distribution with parameter $\lambda = np = \frac{499}{10^3} = 0.499$. Then to him the probability that more than one person has the disease is given by

$$P\{N \geq 2 | N > 0\} = 1 - P\{N < 1\} = 1 - e^{-0.499} \approx 0.3928\,.$$

Part (d): We desire to compute the probability that any of the $500 - i$ remaining people have the disease that is (with the number N the total number of people with the disease) let E be the event that the people $1, 2, 3, \cdots i - 1$ do not have the disease while i does the probability we desire is then

$$P\{N \geq 2 | E\} = \frac{P\{N \geq 2\,, E\}}{P\{E\}}\,.$$

Now the probability $P\{E\} = (1 - p)^i p$, since E is a geometric random variable. Now $P\{N \geq 2\,, E\}$ is the probability that since person i has the disease that at least one more person has the disease in the $M - i$ additional people (here $M = 500$) and is given by

$$\sum_{k=1}^{M-i} \binom{M-i}{k} p^k (1-p)^{M-i-k}$$

so this probability (the entire conditional probability) is then

$$P\{N \geq 2 | E\} = \frac{\sum_{k=1}^{M-i} \binom{M-i}{k} p^k (1-p)^{M-i-k}}{(1-p)^i p}\,,$$

which becomes (when we put the numbers for this problem in the expression above) the following

$$P\{N \geq 2|E\} = \frac{\sum_{k=1}^{500-i} \left(\begin{array}{c} 500-i \\ k \end{array} \right) \left(\frac{1}{10^3}\right)^k \left(1 - \frac{1}{10^3}\right)^{500-i-k}}{\left(1 - \frac{1}{10^3}\right)^i \left(\frac{1}{10^3}\right)} .$$

Problem 66 (seating couples next to each other)

Part (a): There are $(2n-1)!$ different possible seating orders around a circular table when each person is considered unique. For couple i to be seated next to each other, consider this couple as one unit, then we have in total now

$$2n - 2 + 1 = 2n - 1 ,$$

unique "items" to place around our table. Here an item can be an individual person or the ith couple considered as one unit. Specifically we have taken the total $2n$ people and subtracted the specific ith couple (of two people) and put back the couple considered as one unit (the plus one). Thus there are $(2n-1-1)! = (2n-2)!$ rotational orderings of the remaining $2n-2$ people and the "fused" couple. Since there are an additional ordering of the individual people in the pair, we have a total of $2(2n-2)!$ orderings where couple i is together. Thus our probability is given by

$$P(C_i) = \frac{2(2n-2)!}{(2n-1)!} = \frac{2}{2n-1} .$$

Part (b): To compute $P(C_j|C_i)$ when $j \neq i$ we note that it is equal to

$$\frac{P(C_j, C_i)}{P(C_i)} .$$

Here $P(C_j, C_i)$ is the joint probability where both couple i and couple j are together. Since we have evaluated $P(C_i)$ in Part a of this problem we will now evaluate $P(C_j, C_i)$ in the same way as earlier. With couple i and j considered as individual units, the number of "items" we have to distribute around our table is given by

$$2n - 2 + 1 - 2 + 1 = 2n - 2 .$$

Here as before we subtract the individual people in the couple and then add back in a "fused" couple considered as one unit. Thus the number of unique permutations of these items around our table is given by $4(2n-2-1)! = 4(2n-3)!$. The factor of four is for the different orderings of the husband and wife in each fused pair. Thus our our joint probability is then given by

$$P(C_j, C_i) = \frac{4(2n-3)!}{(2n-1)!} = \frac{2}{(2n-1)(n-1)} ,$$

so that our conditional probability $P(C_j|C_i)$ is given by

$$P(C_j|C_i) = \frac{2/(2n-1)(n-1)}{2/(2n-1)} = \frac{1}{n-1} .$$

Part (c): When n is large we want to approximate $1 - P(C_1 \cup C_2 \cup \ldots \cup C_n)$, which is given by

$$\begin{aligned}
1 - P(C_1 \cup C_2 \cup \ldots \cup C_n) &= 1 - \left(\sum_{i=1}^{n} P(C_i) - \sum_{i<j} P(C_i, C_j) + \cdots \right) \\
&= 1 - \left(\frac{2n}{2n-1} - \sum_{i<j} P(C_j|C_i) P(C_i) + \cdots \right) \\
&= 1 - \left(\frac{2n}{2n-1} - \binom{n}{2} \frac{2}{(2n-1)(n-1)} + \cdots \right)
\end{aligned}$$

But since $P(C_j|C_i) = \frac{1}{n-1} \approx \frac{1}{n-1/2} = P(C_j)$, when n is very large. Thus while the events C_i and C_j are not independent, their dependence is weak for large n. Thus by the Poisson paradigm we can expect the number of couples sitting together to have a Poisson approximation with rate $\lambda = n\left(\frac{2}{2n-1}\right) \approx 1$. Thus the probability that no married couple sits next to each other is $P\{N = 0\} = e^{-1}$.

Chapter 4: Theoretical Exercises

Problem 10 (an expectation with a binomial random variable)

If X is a binomial random variable with parameters (n, p) then

$$\begin{aligned}
E\left[\frac{1}{X+1}\right] &= \sum_{k=0}^{n} \left(\frac{1}{k+1}\right) P\{X = k\} \\
&= \sum_{k=0}^{n} \left(\frac{1}{k+1}\right) \binom{n}{k} p^k (1-p)^{n-k} .
\end{aligned}$$

Factoring out $1/(n+1)$ we obtain

$$E\left[\frac{1}{X+1}\right] = \frac{1}{n+1} \sum_{k=0}^{n} \left(\frac{n+1}{k+1}\right) \binom{n}{k} p^k (1-p)^{n-k} .$$

This result is beneficial since if we now consider the fraction and the n choose k term we see that

$$\left(\frac{n+1}{k+1}\right) \binom{n}{k} = \left(\frac{n+1}{k+1}\right) \frac{n!}{k!(n-k)!} = \frac{(n+1)!}{(k+1)!(n-k)!} = \binom{n+1}{k+1} .$$

This substitution turns our summation into the following

$$E\left[\frac{1}{X+1}\right] = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k+1} p^k (1-p)^{n-k} .$$

the following manipulations allow us to evaluate this summation. We have

$$
\begin{aligned}
E\left[\frac{1}{X+1}\right] &= \frac{1}{p(n+1)}\sum_{k=0}^{n}\binom{n+1}{k+1}p^{k+1}(1-p)^{n+1-(k+1)} \\
&= \frac{1}{p(n+1)}\sum_{k=1}^{n+1}\binom{n+1}{k}p^{k}(1-p)^{n+1-k} \\
&= \frac{1}{p(n+1)}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}p^{k}(1-p)^{n+1-k} - (1-p)^{n+1}\right] \\
&= \frac{1}{p(n+1)}(1-(1-p)^{n+1}) \\
&= \frac{1-(1-p)^{n+1}}{p(n+1)},
\end{aligned}
$$

as we were to show.

Problem 11 (each sequence of k successes is equally likely)

Each specific instance of k success and $n-k$ failures has probability $p^k(1-p)^{n-k}$. Since each success occurs with probability p each failure occurs with probability $1-p$. As each arraignment has the same number of p's and $1-p$'s each has the *same* probability.

Problem 13 (maximum likelihood estimation with a binomial random variable)

Since X is a binomial random variable with parameters (n,p) we have that

$$
P\{X=k\} = \binom{n}{k}p^k(1-p)^{n-k}.
$$

Then the p that maximizes this expression is given by taking the derivative of the above (with respect to p) setting the resulting expression equal to zero and solving for p. We find that this derivative is given by

$$
\frac{d}{dp}P\{X=k\} = \binom{n}{k}kp^{k-1}(1-p)^{n-k} + \binom{n}{k}p^k(1-p)^{n-k-1}(n-k)(-1).
$$

Which when set equal to zero and solve for p we find that $p=\frac{k}{n}$, or the empirical counting estimate of the probability of success.

Problem 16 (the location of the maximum of the Poisson distribution)

Since X is a Poisson random variable the probability mass function for X is given by

$$
P\{X=i\} = \frac{e^{-\lambda}\lambda^i}{i!}.
$$

Following the hint we compute the requested fraction. We find that

$$\frac{P\{X=i\}}{P\{X=i-1\}} = \left(\frac{e^{-\lambda}\lambda^i}{i!}\right)\left(\frac{(i-1)!}{e^{-\lambda}\lambda^{i-1}}\right) = \frac{\lambda}{i}.$$

Now from the above expression if $i < \lambda$ then the "lambda" fraction $\frac{\lambda}{i} > 1$, meaning that the probabilities satisfy $P\{X=i\} > P\{X=i-1\}$ which implies that $P\{X=i\}$ is increasing for these values of i. On the other hand if $i > \lambda$ then $\frac{\lambda}{i} < 1$ we $P\{X=i\} < P\{X=i-1\}$ and $P\{X=i\}$ is decreasing for these values of i. Thus when $i < \lambda$, our probability $P\{X=i\}$ is increasing, while when $i > \lambda$, our probability $P\{X=i\}$ is decreasing. From this we see that the maximum of $P\{X=i\}$ is then when i is the largest integer still less than or equal to λ.

Problem 17 (the probability of an even Poisson sample)

Since X is a Poisson random variable the probability mass function for X is given by

$$P\{X=i\} = \frac{e^{-\lambda}\lambda^i}{i!}.$$

To help solve this problem it is helpful to recall that a binomial random variable with parameters (n,p) can be approximated by a Poisson random variable with $\lambda = np$, and that this approximation improves as $n \to \infty$. To begin then, let E denote the event that X is even. Then to evaluate the expression $P\{E\}$ we will use the fact that a binomial random variable can be approximated by a Poisson random variable. When we consider X to be a binomial random variable we have from theoretical Exercise 15 in this chapter that

$$P\{E\} = \frac{1}{2}(1+(q-p)^n).$$

Using the Poisson approximation to the binomial we will have that $p = \lambda/n$ and $q = 1-p = 1-\lambda/n$, so the above expression becomes

$$P\{E\} = \frac{1}{2}\left(1+\left(1-\frac{2\lambda}{n}\right)^n\right).$$

Taking n to infinity (as required to make the binomial approximation by the Poisson distribution exact) and remembering that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x,$$

the probability $P\{E\}$ above goes to

$$P\{E\} = \frac{1}{2}\left(1+e^{-2\lambda}\right),$$

as we were to show.

Part (b): To directly evaluate this probability consider the summation representation of the requested probability, i.e.

$$\begin{aligned} P\{E\} &= \sum_{i=0,2,4,\cdots}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \\ &= e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^{2i}}{(2i)!}\,. \end{aligned}$$

When we look at this it looks like the Taylor expansion of $\cos(\lambda)$ but without the required alternating $(-1)^i$ factor. This observation might trigger the recollection that the above series is in fact the Taylor expansion of the $\cosh(\lambda)$ function. This can be seen from the definition of the cosh function which is

$$\cosh(\lambda) = \frac{e^{\lambda}+e^{-\lambda}}{2}\,,$$

when one Taylor expands the exponentials on the right hand side of the above expression. Thus the above probability for $P\{E\}$ is given by

$$e^{-\lambda}\left(\frac{1}{2}(e^{\lambda}+e^{-\lambda})\right) = \frac{1}{2}(1+e^{-2\lambda})\,,$$

as claimed.

Problem 26 (an integral expression for the CDF of a Poisson random variable)

We will begin by evaluating $\int_{\lambda}^{\infty} e^{-x}x^n dx$. To perform repeated integration by parts we remember the integration by parts "formula" $udv = uv - vdu$, and in the following we will let u be the polynomial in x and dv the exponential. To start this translates into letting $u = x^n$ and $dv = e^{-x}$, and we have

$$\begin{aligned} \int_{\lambda}^{\infty} e^{-x}x^n dx &= -x^n e^{-x}\Big|_{\lambda}^{\infty} + \int_{\lambda}^{\infty} nx^{n-1}e^{-x}dx \\ &= \lambda^n e^{-\lambda} + n\int_{\lambda}^{\infty} x^{n-1}e^{-x}dx \\ &= \lambda^n e^{-\lambda} + n\left[-x^{n-1}e^{-x}\Big|_{\lambda}^{\infty} + \int_{\lambda}^{\infty}(n-1)x^{n-2}e^{-x}dx\right] \\ &= \lambda^n e^{-\lambda} + n\lambda^{n-1}e^{-\lambda} + n(n-1)\int_{\lambda}^{\infty} x^{n-2}e^{-x}dx\,. \end{aligned}$$

Continuing to perform one more integration by parts (so that we can fully see the pattern) we have that this last integral given by

$$\begin{aligned} \int_{\lambda}^{\infty} x^{n-2}e^{-x}dx &= -x^{n-2}e^{-x}\Big|_{\lambda}^{\infty} + \int_{\lambda}^{\infty}(n-2)x^{n-3}e^{-x}dx \\ &= \lambda^{n-2}e^{-\lambda} + (n-2)\int_{\lambda}^{\infty} x^{n-3}e^{-x}dx\,. \end{aligned}$$

Then we have for our total integral the following

$$\begin{aligned}\int_\lambda^\infty e^{-x}x^n dx &= \lambda^n e^{-\lambda} + n\lambda^{n-1}e^{-\lambda} + n(n-1)\lambda^{n-2}e^{-\lambda} \\ &+ n(n-1)(n-2)\int_\lambda^\infty x^{n-3}e^{-x}dx\,.\end{aligned}$$

Using mathematical induction the total pattern can be seen as

$$\begin{aligned}\int_\lambda^\infty e^{-x}x^n dx &= \lambda^n e^{-\lambda} + n\lambda^{n-1}e^{-\lambda} + n(n-1)\lambda^{n-2}e^{-\lambda} + \cdots \\ &+ n(n-1)(n-2)\cdots(n-k)\int_\lambda^\infty x^{n-k}e^{-x}dx \\ &= \lambda^n e^{-\lambda} + n\lambda^{n-1}e^{-\lambda} + n(n-1)\lambda^{n-2}e^{-\lambda} + \cdots + n!\int_\lambda^\infty e^{-x}dx \\ &= \lambda^n e^{-\lambda} + n\lambda^{n-1}e^{-\lambda} + n(n-1)\lambda^{n-2}e^{-\lambda} + \cdots + n!e^{-x}\,.\end{aligned}$$

When we divide this sum by $n!$ we find it is given by

$$\frac{\lambda^n}{n!}e^{-\lambda} + \frac{\lambda^{n-1}}{(n-1)!}e^{-\lambda} + \frac{\lambda^{n-2}}{(n-2)!}e^{-\lambda} + \cdots + \lambda e^{-\lambda} + e^{-\lambda}$$

or the left hand side of the expression given in the problem statement i.e.

$$\sum_{i=0}^n \frac{e^{-\lambda}\lambda^i}{i!}\,,$$

as we were to show.

Problem 29 (ratios of hypergeometric probabilities)

For a Hypergeometric random variable we have

$$P(i) = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}} \quad \text{for} \quad i = 0, 1, \cdots, m\,.$$

So that the requested ratio is given by

$$
\begin{aligned}
\frac{P(k+1)}{P(k)} &= \frac{\binom{m}{k+1}\binom{N-m}{n-k-1}}{\binom{N}{n}} \cdot \frac{\binom{N}{n}}{\binom{m}{k}\binom{N-m}{n-k}} \\
&= \frac{\binom{m}{k+1}}{\binom{m}{k}} \frac{\binom{N-m}{n-k-1}}{\binom{N-m}{n-k}} \\
&= \frac{\frac{m!}{(k+1)!(m-k-1)!} \cdot \frac{(N-m)!}{(n-k-1)!(N-m-n+k+1)!}}{\frac{m!}{k!(m-k)!} \cdot \frac{(N-m)!}{(n-k)!(N-m-n+k)!}} \\
&= \frac{k!(m-k)!}{(k+1)!(m-k-1)!} \cdot \frac{(n-k)!(N-m-n+k)!}{(n-k-1)!(N-m-n+k+1)!} \\
&= \frac{(m-k)(n-k)}{(k+1)(N-m-n+k+1)} .
\end{aligned}
$$

Chapter 5 (Continuous Random Variables)

Chapter 5: Problems

Problem 1 (normalizing a continuous random variable)

Part (a): The integral of the f must evaluate to one, which requires

$$\begin{aligned}\int_{-1}^{1} c(1-x^2)dx &= 2c\int_0^1 (1-x^2)dx \\ &= 2c\left(x - \frac{x^3}{3}\right|_0^1 = 2c\left(1-\frac{1}{3}\right) = \frac{4c}{3}\,.\end{aligned}$$

For this to equal one, we must have $c = \frac{3}{4}$.

Part (b): The cumulative distribution is given by

$$\begin{aligned}F(x) &= \int_{-1}^{x} \frac{3}{4}(1-\xi^2)d\xi \\ &= \frac{3}{4}\left(\xi - \frac{\xi^3}{3}\right|_{-1}^{x} \\ &= \frac{3}{4}\left(x - \frac{x^3}{3}\right) + \frac{1}{2} \quad \text{for} \quad -1 \le x \le 1\,.\end{aligned}$$

Problem 2 (how long can our system function?)

We must first evalaute the constant in our distribution function. Specifically to be a probability density we must have

$$\int_0^\infty cxe^{-x/2}dx = 1\,.$$

Integrating by parts we find that

$$\begin{aligned}\int_0^\infty cxe^{-x/2}dx &= c\left[\frac{xe^{-x/2}}{(-1/2)}\right|_0^\infty - \frac{1}{(-1/2)}\int_0^\infty e^{-x/2}dx\Bigg] \\ &= c\left[2\int_0^\infty e^{-x/2}dx\right] \\ &= 2c\,\frac{e^{-x/2}}{(-1/2)}\Bigg|_0^\infty = -4c(0-1) = 4c\,.\end{aligned}$$

So for this to equal one we must have $c = 1/4$. Then the probability that our system last at least five months is given by

$$
\begin{aligned}
\int_5^\infty \frac{1}{4} x e^{-x/2} dx &= \frac{1}{4}\left[\frac{x e^{-x/2}}{(-1/2)}\Big|_5^\infty - \int_5^\infty \frac{e^{-x/2}}{(-1/2)} dx \right] \\
&= \frac{1}{4}\left[0 + 10 e^{-5/2} + 2\int_5^\infty e^{-x/2} dx \right] \\
&= \cdots = \frac{7}{2} e^{-5/2} .
\end{aligned}
$$

Problem 3 (possible density functions)

Even with a value of C specified a problem with this function f is that it is negative for some values of x. Specifically f will be *zero* when $x(2 - x^2) = 0$, which happens when $x = 0$ or $x = \pm\sqrt{2} = \pm 1.4142$. With these zeros found we see that if x is less than $\sqrt{2}$ then $x(2-x^2)$ is positive, however if x is greater than $\sqrt{2}$ (but still less than $5/2$) the expression $x(2-x^2)$ is negative. Thus whatever the sign of c, $f(x)$ will be negative for some region of the interval. Since f cannot be negative this functional form cannot be a probability density function.

For the second function this f is zero when $x(2 - x) = 0$, which happens when $x = 0$ and $x = 2$. Since $2 < 5/2 = 2.5$. This f will also change signs regardless of the constant C as x crosses the value 2. Since f takes on both positive and negative signed values it can't be a distribution function.

Problem 4 (the lifetime of electronics)

Part (a): The requested probability is given by

$$
P\{X > 20\} = \int_{20}^\infty \frac{10}{x^2} dx = \frac{1}{2} .
$$

Part (b): The requested cumulative distribution function is given by

$$
F(x) = \int_{10}^\infty \frac{10}{\xi^2} d\xi = \frac{10\xi^{-1}}{(-1)}\Big|_{10}^x = 1 - \frac{10}{x} \quad \text{for} \quad 10 \le x .
$$

Part (c): To function for at least fifteen hours will happen with probability $1 - F(15) = 1 - (1 - \frac{10}{15}) = \frac{2}{3}$. To have three of six such devices function for at least fifteen hours is given by sums of binomial probability density functions. Specifically we have this probability given by

$$
\sum_{k=3}^{6} \binom{6}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{6-k} ,
$$

which we recognized as the "complement" of the binomial cumulative distribution function. To evaluate this we can use the Matlab command `binocdf(2,6,2/3)`. See the Matlab file `chap_5_prob_4.m` for these calculations and we find that the above equals 0.8999. In performing this analysis we are assuming independence of the devices.

Problem 11 (picking a point on a line)

An interpretation of this statement is that a point is picked randomly on a line segment of length L would be that the point "X" is selected from a uniform distribution over the interval $[0, L]$. Then the question asks us to find

$$P\left\{\frac{\min(X, L-X)}{\max(X, L-X)} < \frac{1}{4}\right\}.$$

This probability can be evaluated by integrating over the appropriate region. Formally we have the above equal to

$$\int_E p(x)dx$$

where $p(x)$ is the uniform probability density for our problem, i.e. $\frac{1}{L}$ and the set "E" is $x \in [0, L]$ and satisfying the inequality above, i.e.

$$\min(x, L-x) \leq \frac{1}{4}\max(x, L-x).$$

Plotting the functions $\max(x, L-x)$, and $\min(x, L-x)$ in Figure 1, we see that the regions of X where we should compute the integral above are restricted to the two ends of the segment. Specifically, the integral above becomes,

$$\int_0^{l_1} p(x)dx + \int_{l_2}^{L} p(x)dx.$$

since the region $\min(x, L-x) < \frac{1}{4}\max(x, L-x)$ in satisfied in the region $[0, l_1]$ and $[l_2, L]$ only. Here l_1 is the solution to

$$\min(x, L-x) = \frac{1}{4}\max(x, L-x) \quad \text{when} \quad x < L-x,$$

i.e. we need to solve

$$x = \frac{1}{4}(L-x)$$

which has as its solution $x = \frac{L}{5}$. For l_2 we must solve

$$\min(x, L-x) = \frac{1}{4}\max(x, L-x) \quad \text{when} \quad L-x < x,$$

i.e. we need to solve

$$L-x = \frac{1}{4}x,$$

which has as its solution $x = \frac{4}{5}L$. With these two limits we have for our probability

$$\int_0^{\frac{L}{5}} \frac{1}{L}dx + \int_{\frac{4}{5}L}^{L} \frac{1}{L}dx = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}.$$

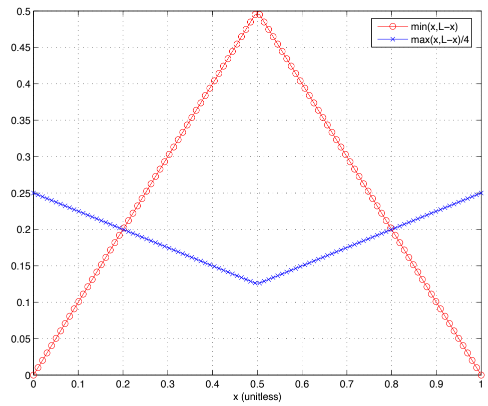

Figure 1: A graphical view of the region of x's over which the integral for this problem should be computed.

Problem 17 (the expected number of points scored)

We desire to calculate $E[P(D)]$, where $P(D)$ is the points scored when the distance to the target is D. This becomes

$$\begin{aligned} E[P(D)] &= \int_0^{10} P(D) f(D) dD \\ &= \frac{1}{10}\int_0^{10} P(D) dD \\ &= \frac{1}{10}\left(\int_0^1 10 dD + \int_1^3 5 dD + \int_3^5 3 dD + \int_5^{10} 0 dD\right) \\ &= \frac{1}{10}(10 + 5(2) + 3(2)) = \frac{26}{10} = 2.6\,. \end{aligned}$$

Problem 18 (variable limits on a normal random variable)

Since X is a normal random variable we can evaluate the given probability $P\{X > 9\}$ as

$$\begin{aligned} P\{X > 9\} &= P\{\frac{X-5}{\sigma} > \frac{9-5}{\sigma}\} \\ &= P\left\{Z > \frac{4}{\sigma}\right\} \\ &= 1 - P\left\{Z < \frac{4}{\sigma}\right\} \\ &= 1 - \Phi(\frac{4}{\sigma}) = 0.2\,, \end{aligned}$$

so solving for $\Phi(4/\sigma)$ we have that $\Phi(4/\sigma) = 0.8$, which can be inverted by using the Matlab command `norminv` and we calculate that

$$\frac{4}{\sigma} = \Phi^{-1}(0.8) = 0.8416\,.$$

which then implies that $\sigma = 4.7527$, so $\text{Var}(X) = \sigma^2 \approx 22.58$.

Problem 19 (more limits on normal random variables)

Since X is a normal random variable we can evaluate the given probability $P\{X > c\}$ as

$$\begin{aligned} P\{X > c\} &= P\{\frac{X-12}{2} > \frac{c-12}{2}\} \\ &= P\left\{Z > \frac{c-12}{2}\right\} \\ &= 1 - P\left\{Z < \frac{c-12}{2}\right\} \\ &= 1 - \Phi(\frac{c-12}{2}) = 0.1\,, \end{aligned}$$

so solving for $\Phi(\frac{c-12}{2}$ we have that $\Phi(\frac{c-12}{2}) = 0.9$, which can be inverted by using the Matlab command `norminv` and we calculate that

$$\frac{c-12}{2} = \Phi^{-1}(0.9) = 1.28\,,$$

which then implies that $c = 30.757$.

Problem 20 (the expected number of people in favor of a proposition)

Now the number of people who favor the proposed rise in taxes is a binomial random variable with parameters $(n, p) = (100, 0.65)$. Using the normal approximation to the binomial, we have a normal with a mean of $np = 100(0.65) = 65$, and a variance of $\sigma^2 = np(1-p) = 100(0.65)(0.35) = 22.75$, so the probabilities desired are given as

Part (a):

$$\begin{aligned} P\{N \geq 50\} &= P\{N > 49.5\} \\ &= P\left\{\frac{N-65}{\sqrt{22.75}} > \frac{49.5-65}{4.76}\right\} \\ &= P\{Z > -3.249\} \\ &= 1 - \Phi(-3.249)\,. \end{aligned}$$

Where in the first equality we have used the "continuity approximation". Using the Matlab command `normcdf(x)` to evaluate the function $\Phi(x)$ we have the above equal to ≈ 0.9994.

Part (b):

$$\begin{aligned}
P\{60 \le N \le 70\} &= P\{59.5 < N < 70.5\} \\
&= P\left\{ \frac{59.5 - 65}{\sqrt{22.75}} < Z < \frac{70.5 - 65}{\sqrt{22.75}} \right\} \\
&= P\{-1.155 < Z < 1.155\} \\
&= \Phi(1.155) - \Phi(-1.155) \approx 0.7519 \,.
\end{aligned}$$

Part (c):

$$\begin{aligned}
P\{N < 75\} &= P\{N < 74.5\} \\
&= P\left\{ Z < \frac{74.5 - 65}{4.76} \right\} \\
&= P\{Z < 1.99\} \\
&= \Phi(1.99) \approx 0.9767 \,.
\end{aligned}$$

Problem 21 (percentage of men with height greater than six feet two inches)

We desire to compute $P\{X > 6 \cdot 12 + 2\}$, where X is the random variable expressing height (measured in inches) of a 25-year old man. This probability can be computed by converting to the standard normal in the usual way. We have

$$\begin{aligned}
P\{X > 6 \cdot 12 + 2\} &= P\left\{ \frac{X - 71}{\sqrt{6.25}} > \frac{3}{\sqrt{6.25}} \right\} \\
&= P\left\{ Z > \frac{3}{\sqrt{6.25}} \right\} \\
&= 1 - P\left\{ Z < \frac{3}{\sqrt{6.25}} \right\} \\
&= 1 - \Phi(1.2) \approx 0.1151 \,.
\end{aligned}$$

For the second part of this problem we are looking for

$$P\{X > 6 \cdot 12 + 5 | X > 6 \cdot 12\} \,.$$

Again this can be computed by converting to a standard normal, after first considering the joint density. We have

$$\begin{aligned}
P\{X > 6 \cdot 12 + 5 | X > 6 \cdot 12\} &= \frac{P\{X > 77, X > 72\}}{P\{X > 72\}} \\
&= \frac{P\{X > 77\}}{P\{X > 72\}} \\
&= \frac{1 - P\left\{ Z < \frac{6}{\sqrt{6.25}} \right\}}{1 - P\left\{ Z < \frac{1}{\sqrt{6.25}} \right\}} \\
&= \frac{1 - \Phi(\frac{6}{\sqrt{6.25}})}{1 - \Phi(\frac{1}{\sqrt{6.25}})} \approx 0.0238 \,.
\end{aligned}$$

Some of the calculations for this problem can be found in the file `chap_5_prob_21.m`.

Problem 22 (number of defective products)

Part (a): Lets calculate the percentage that are *acceptable* if we let the variable X be the width of our normally distributed slot this percentage is given by

$$P\{0.895 < X < 0.905\} = P\{X < 0.905\} - P\{X < 0.895\}\,.$$

Each of these individual cumulative probabilities can be calculated by transforming to the standard normal, in the usual way. We have that the above is equal to (since the population mean is 0.9 and the population standard deviation is 0.003)

$$\begin{aligned}
& P\left\{\frac{X-0.9}{0.003} < \frac{0.905-0.9}{0.003}\right\} - P\left\{\frac{X-0.9}{0.003} < \frac{0.895-0.9}{0.003}\right\} \\
= \ & \Phi\left(1.667\right) - \Phi\left(-1.667\right) = 0.904\,.
\end{aligned}$$

So that the probability (or percentage) of defective forgings is one minus this number (times 100 to convert to percentages). This is $0.095 \times 100 = 9.5$.

Part (b): This question is asking to find the value of σ such that

$$P\{0.895 < X < 0.905\} = \frac{99}{100}\,.$$

Since these limits on X are symmetric about $X = 0.9$ we can simplify this probability by using

$$P\{0.895 < X < 0.905\} = 1 - 2P\{X < 0.895\} = 1 - 2P\left\{\frac{X-0.9}{\sigma} < \frac{0.905-0.9}{\sigma}\right\}$$

We thus have to solve for σ in

$$1 - 2\Phi(\frac{-0.005}{\sigma}) = 0.99$$

or inverting the Φ function and solving for σ we have

$$\sigma = \frac{-0.005}{\Phi^{-1}(0.005)}\,.$$

Using the Matlab command `norminv` to evaluate the above we have $\sigma = 0.0019$. See the Matlab file `chap_5_prob_22.m` for these calculations.

Problem 23 (probabilities on the number of 5's to appear)

The probability that one six occurs is $p = 1/6$ so the total number of sixes rolled is a binomial random variable. We can approximate this density as a Gaussian with a mean given by $np = \frac{1000}{6} \approx 166.6$ and a variance of $\sigma^2 = np(1-p) = 138.8$. Then the desired probabilities are

$$
\begin{aligned}
P\{150 \leq N \leq 200\} &= P\{149.5 < N < 200.5\} \\
&= P\{-1.45 < Z < 2.877\} \\
&= \Phi(2.87) - \Phi(-1.45) \approx 0.9253\,.
\end{aligned}
$$

If we are told that a six appears two hundred times then the probability that a five will appear on the other rolls is $1/5$ and it must appear on one of the $1000 - 200 = 800$ other rolls. Thus we can approximate the binomial random variable (with parameters $(n, p) = (800, 1/5)$) with a normal with mean $np = \frac{800}{5} = 160$ and variance $\sigma^2 = np(1-p) = 128$. So the requested probability is

$$
\begin{aligned}
P\{N < 500\} &= P\{N < 149.5\} \\
&= P\left\{Z < \frac{149.5 - 160}{\sqrt{128}}\right\} \\
&= P\{Z < -0.928\} \\
&= \Phi(-0.928) \approx 0.1767\,.
\end{aligned}
$$

Problem 24 (probability of enough long living batteries)

If each chips lifetime is denoted by the random variable X (assumed Gaussian with the given mean and variance), then each chip will have a lifetime less than $1.8\,10^6$ hours with probability given by

$$
\begin{aligned}
P\{X < 1.8\,10^6\} &= P\left\{\frac{X - 1.4\,10^6}{3\,10^5} < \frac{(1.8 - 1.4)\,10^6}{3\,10^5}\right\} \\
&= P\left\{Z < \frac{4}{3}\right\} = \Phi(4/3) \approx 0.9088\,.
\end{aligned}
$$

With this probability, the number N, in a batch of 100 that will have a lifetime less than $1.8\,10^6$ is a binomial random variable with parameters $(n, p) = (100, 0.9088)$. Therefore, the probability that a batch will contain at least 20 is given by

$$
P\{N \geq 20\} = \sum_{n=20}^{100} \left(\begin{array}{c} 100 \\ n \end{array}\right) (0.908)^n (1 - 0.908)^{100-n}\,.
$$

Rather than evaluate this exactly we can approximate this binomial random variable N with a Gaussian random variable with a mean given by $\mu = np = 100(0.908) = 90.87$, and a variance given by $\sigma^2 = np(1-p) = 8.28$ (equivalently $\sigma = 2.87$). Then the probability that a given batch of 100 has at least 20 that have lifetime less than $1.8\,10^6$ hours is given by

$$\begin{aligned} P\{N \geq 20\} &= P\{N \geq 19.5\} \\ &= P\left\{\frac{N - 90.87}{2.87} \geq \frac{19.5 - 90.87}{2.87}\right\} \\ &\approx P\{Z \geq -24.9\} \\ &= 1 - P\{Z \leq -24.9\} \\ &= 1 - \Phi(-24.9) \approx 1\,. \end{aligned}$$

Where in the first line above we have used the continuity correction required when we approximate a discrete density by a continuous one, and in the third line above we use our Gaussian approximation to the binomial distribution.

Problem 25 (the probability of enough acceptable items)

The number N of acceptable items is a binomial random variable so we can approximate it with a Gaussian with mean $\mu = pn = 150(0.95) = 142.5$, and a variance of $\sigma^2 = np(1-p) = 7.125$. From the variance we have a standard deviation of $\sigma \approx 2.669$. Thus the desired probability is given by

$$\begin{aligned} P\{N \geq 140\} &= P\{N \geq 139.5\} \\ &= P\left\{\frac{N - 142.5}{2.669} \geq \frac{139.5 - 142.5}{2.669}\right\} \\ &\approx P\{Z \geq -1.127\} \\ &= 1 - P\{Z \leq -1.127\} \\ &= 1 - \Phi(-1.127) \approx 0.8701\,. \end{aligned}$$

Where in the first line above we have used the continuity correction required when we approximate a discrete density by a continuous one, and in the third line above we use our Gaussian approximation to the binomial distribution. We note that we solved this problem in terms of the number of items that are *acceptable*. An equivalent formulation could easily be done in terms of the number that are *unacceptable* by using the complementary probability $q \equiv 1 - p = 1 - 0.95 = 0.05$.

Problem 26 (calculating the probability of error)

Let N be the random variable that represents the number of heads that result when we flip our coin 1000 times. Then N is distributed as binomial random variable with a probability of success p that depends on whether we are considering the biased or unbiased (fair) coin. It the coin is actually *fair* we will make an error in our assessment of its type if N is greater than 525 according to the statement of this problem. Thus the probability that we reach a false conclusion is given by

$$P\{N \geq 525\}\,.$$

To compute this probability we will use the normal approximation to the binomial distribution. In this case the normal to use to approximate this binomial distribution has a mean given by $\mu = np = 1000(0.5) = 500$ and a variance given by $\sigma^2 = np(1-p) = 250$ since we know we are looking at the fair coin where $p = 0.5$. To evaluate this probability we have

$$\begin{aligned}
P\{N \geq 525\} &= P\{N \geq 524.5\} \\
&= P\left\{\frac{N-500}{\sqrt{250}} \geq \frac{524.5-500}{\sqrt{250}}\right\} \\
&\approx P\{Z \geq 1.54\} \\
&= 1 - P\{Z \leq 1.54\} \\
&= 1 - \Phi(1.54) \approx 0.0606\,.
\end{aligned}$$

Where in the first line above we have used the continuity correction required when we approximate a discrete density by a continuous one, and in the third line above we use our Gaussian approximation to the binomial distribution. In the case where the coin is actually biased our probability of obtaining a head becomes $p = 0.55$ and we will reach a false conclusion in this case when

$$P\{N < 525\}\,.$$

To compute this probability we will use the normal approximation to the binomial distribution. In this case the normal to use to approximate this binomial distribution has a mean given by $\mu = np = 1000(0.55) = 550$ and a variance given by $\sigma^2 = np(1-p) = 247.5$. To evaluate this probability we have

$$\begin{aligned}
P\{N < 525\} &= P\{N < 524.5\} \\
&= P\left\{\frac{N-550}{\sqrt{247.5}} < \frac{524.5-550}{\sqrt{247.5}}\right\} \\
&\approx P\{Z < -1.62\} \\
&= \Phi(-1.62) \approx 0.0525\,.
\end{aligned}$$

Where in the first line above we have used the continuity correction required when we approximate a discrete density by a continuous one, and in the third line above we use our Gaussian approximation to the binomial distribution.

Problem 28 (the number of left handed students)

The number of students that are left handed (denoted by N) is a Binomial random variable with parameters $(n, p) = (200, 0.12)$. From the normal approximation to the binomial we can approximate this distribution with a Gaussian with mean $\mu = pn = 200(0.12) = 24$, and a variance of $\sigma^2 = np(1-p) = 21.120$. From the variance we have a standard deviation of

$\sigma \approx 4.59$. Thus the desired probability is given by

$$
\begin{aligned}
P\{N > 20\} &= P\{N > 19.5\} \\
&= P\left\{\frac{N-24}{4.59} > \frac{19.5-24}{4.59}\right\} \\
&\approx P\{Z > -0.9792\} \\
&= 1 - P\{Z \le -0.9792\} \\
&= 1 - \Phi(-0.9792) \approx 0.8363\,.
\end{aligned}
$$

Where in the second line above we have used the continuity correction that improves our accuracy when we approximate a discrete density by a continuous one, and in the third line above we use our Gaussian approximation to the binomial distribution. These calculations can be find in the file `chap_5_prob_28.m`.

Problem 29 (a simple model of stock movement)

If we count each time the stock rises in value as a "success", we see that the movement of the stock for one timestep is a Bernoulli random variable with parameter p. So after n timesteps the number of rises is a binomial random variable with parameters (n, p). The price of the security after n timesteps where we have k "successes" will then be given by $su^k d^{n-k}$. The probability we are looking for then is given by

$$
\begin{aligned}
P\{su^k d^{n-k} \ge 1.3\,s\} &= P\{u^k d^{n-k} \ge 1.3\} \\
&= P\left\{\left(\frac{u}{d}\right)^k \ge \frac{1.3}{d^n}\right\} \\
&= P\left\{k \ge \frac{\ln(\frac{1.3}{d^n})}{\ln(\frac{u}{d})}\right\} \\
&= P\left\{k \ge \frac{\ln(1.3) - n\ln(d)}{\ln(u) - \ln(d)}\right\}\,.
\end{aligned}
$$

Using the numbers given in the problem i.e. $d = 0.99$ $u = 1.012$, and $n = 1000$, we have that

$$
\frac{\ln(1.3) - n\ln(d)}{\ln(u) - \ln(d)} \approx 469.2\,.
$$

To approximate the above probability we can use the Gaussian approximation to the binomial distribution, which would have a mean given by $np = 0.52(1000) = 520$ and a variance given by $np(1-p) = 249.6$, so using this approximation the above probability then becomes

$$
\begin{aligned}
P\{k \ge 469.2\} &= P\{k \ge 470\} \\
&= P\{k > 469.5\} \\
&= P\{Z > \frac{469.5 - 520}{15.7}\} \\
&= P\{Z > -3.21\} \\
&= 1 - P\{Z < -3.21\} \\
&= 1 - \Phi(-3.23) \approx 0.9994\,.
\end{aligned}
$$

Problem 30 (priors on the type of region)

Let E be the event that we make an error in our classification of the given pixel. Then we can make an error in two symmetric ways. The first is that we classify the pixel as black when it should be classified as white. The second is where we classify the pixel as white when it should be black. Thus we can compute $P(E)$ by conditioning on the true type of the pixel i.e. whether it is B (for black) or W (for white). We have

$$P(E) = P(E|B)P(B) + P(E|W)P(W)\,.$$

Since we are told that the prior probability that the pixel is black is given by α, the prior probability that the pixel is W is then given by $1-\alpha$ and the above becomes

$$P(E) = P(E|B)\alpha + P(E|W)(1-\alpha)\,.$$

The problem then asks for the value of α such that the error in making each type of error is the same, we desire to pick α such that

$$P(E|B)\alpha = P(E|W)(1-\alpha)\,,$$

or upon solving for α we find that

$$\alpha = \frac{P(E|W)}{P(E|W)+P(E|B)}\,.$$

We now need to evaluate $P(E|W)$ and $P(E|B)$. Now $P(E|W)$ is the probability that we classify this pixel as *black* given that it is white. If we classify the pixel with a value of 5 as black, then all points with pixel value greater than 5 would also be classified as black and $P(E|W)$ is then given by

$$P(E|W) = \int_5^\infty \mathcal{N}(x;4,4)dx = \int_{(5-4)/2}^\infty \mathcal{N}(z;0,1)dz = 1-\Phi(1/2) = 0.3085\,.$$

Where $\mathcal{N}(x;\mu,\sigma^2)$ is an expression for the normal probability density function with mean μ and variance σ^2. In the same way we have that

$$P(E|B) = \int_{-\infty}^5 \mathcal{N}(x;6,9)dx = \int_{-\infty}^{(5-6)/3} \mathcal{N}(z;0,1)dz = \Phi(-1/3) = 0.3694\,.$$

Thus with these two expressions α becomes

$$\alpha = \frac{1-\Phi(1/2)}{(1-\Phi(1/2))+\Phi(-1/3)} = 0.4551\,.$$

Problem 31 (the optimal location of a fire station)

Part (a): If x (the location of the fire) is uniformly distributed in $[0, A)$ then we would like to select a (the location of the fire station) such that

$$F(a) \equiv E[\,|X-a|\,]\,,$$

is a minimum. We will compute this by breaking the integral involved in the definition of the expectation into regions where $x - a$ is negative and positive. We find that

$$\begin{aligned}
E[\,|X-a|\,] &= \int_0^A |x-a|\frac{1}{A}dx \\
&= -\frac{1}{A}\int_0^a (x-a)dx + \frac{1}{A}\int_a^A (x-a)dx \\
&= -\frac{1}{A}\left.\frac{(x-a)^2}{2}\right|_0^a + \frac{1}{A}\left.\frac{(x-a)^2}{2}\right|_a^A \\
&= -\frac{1}{A}\left(0 - \frac{a^2}{2}\right) + \frac{1}{A}\left(\frac{(A-a)^2}{2} - 0\right) \\
&= \frac{a^2}{2A} + \frac{(A-a)^2}{2A}\,.
\end{aligned}$$

To find the a that minimizes this we compute $F'(a)$ and set this equal to zero. Taking the derivative and setting this equal to zero we find that

$$F'(a) = \frac{a}{A} + \frac{2(A-a)(-1)}{2A} = 0\,.$$

Which gives a solution a^* given by $a^* = \frac{A}{2}$. A second derivative of our function F shows that $F''(a) = \frac{2}{A} > 0$ showing that the point $a^* = A/2$ is indeed a minimum.

Part (b): The problem formulation is the same as in part (a) but since the distribution of the location of fires is now an exponential we now want to minimize

$$F(a) \equiv E[\,|X-a|\,] = \int_0^\infty |x-a|\lambda e^{-\lambda x}dx\,.$$

We will compute this by breaking the integral involved in the definition of the expectation into regions where $x - a$ is negative and positive. We find that

$$\begin{aligned}
E[\,|X-a|\,] &= \int_0^\infty |x-a|\lambda e^{-\lambda x}dx \\
&= -\int_0^a (x-a)\lambda e^{-\lambda x}dx + \int_a^\infty (x-a)\lambda e^{-\lambda x}dx \\
&= -\lambda\left(\left.\frac{(x-a)}{-\lambda}e^{-\lambda x}\right|_0^a + \frac{1}{\lambda}\int_0^a e^{-\lambda x}dx\right) \\
&\quad +\lambda\left(\left.\frac{(x-a)}{-\lambda}e^{-\lambda x}\right|_a^\infty + \frac{1}{\lambda}\int_a^\infty e^{-\lambda x}dx\right) \\
&= -\lambda\left(\frac{-a}{\lambda} - \left.\frac{1}{\lambda^2}e^{-\lambda x}\right|_0^a\right) + \lambda\left(0 - \left.\frac{1}{\lambda^2}e^{-\lambda x}\right|_a^\infty\right) \\
&= a + \frac{1}{\lambda}(e^{-\lambda a} - 1) - \frac{1}{\lambda}(-e^{-\lambda a}) \\
&= a + \frac{1 + 2e^{-\lambda a}}{\lambda}\,.
\end{aligned}$$

To find the a that minimizes this we compute $F'(a)$ and set this equal to zero. Taking the derivative we find that

$$F'(a) = 1 - 2e^{-\lambda a} = 0\,.$$

Which gives a solution a^* given by $a^* = \frac{\ln(2)}{\lambda}$. A second derivative of our function F shows that $F''(a) = 2\lambda e^{-\lambda a} > 0$ showing that the point $a^* = \frac{\ln(2)}{\lambda}$ is indeed a minimum.

Problem 32 (probability of repair times)

Part (a): We desire to compute $P\{T > 2\}$ which is given by

$$P\{T > 2\} = \int_2^\infty \frac{1}{2} e^{-1/2\,t} dt\,.$$

To evaluate this let $v = \frac{t}{2}$, giving $dv = \frac{dt}{2}$, from which the above becomes

$$P\{T > 2\} = \int_1^\infty e^{-v} dv = -e^{-v}\Big|_1^\infty = e^{-1}\,.$$

Part (b): The probability we are after is given by $P\{T > 10|T > 9\}$ which equals $P\{T > 10 - 9\} = P\{T > 1\}$ by the memoryless property of the exponential distribution. This is given by

$$P\{T > 1\} = 1 - P\{T < 1\} = 1 - (1 - e^{-1/2}) = e^{-1/2}\,.$$

Chapter 5: Theoretical Exercises

Problem 10 (points of inflection of the Gaussian

We are told that $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\}$. And points of inflection are given by $f''(x) = 0$. To calculate $f''(x)$ we need $f'(x)$. We find

$$f'(x) \approx -\left(\frac{x-\mu}{\sigma^2}\right) \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\}\,.$$

So that the second derivative is given by

$$f''(x) \approx -\frac{1}{\sigma^2} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\} + \left(\frac{(x-\mu)^2}{\sigma^2}\right) \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\}\,.$$

Setting $f''(x)$ equal to zero we find that this requires x satisfy

$$\exp\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\} \left[-1 + \frac{(x-\mu)^2}{\sigma^2}\right] = 0, .$$

or $(x-\mu)^2 = \sigma^2$. Which has as solutions $x = \mu \pm \sigma$.

Problem 11 ($E[X^2]$ of an exponential random variable)

Theoretical Exercise number 5 states that

$$E[X^n] = \int_0^\infty n x^{n-1} P\{X > x\} dx \,.$$

For an exponential random variable we have our cumulative distribution function given by

$$P\{X \leq x\} = 1 - e^{-\lambda x} \,.$$

so that $P\{X > x\} = e^{-\lambda x}$, and thus our expectation becomes

$$E[X^n] = \int_0^\infty n x^{n-1} e^{-\lambda x} dx$$

Now if $n = 2$ we find that this expression becomes in this case

$$\begin{aligned}
E[X^2] &= \int_0^\infty 2x e^{-\lambda x} dx \\
&= 2 \int_0^\infty x e^{-\lambda x} dx \\
&= 2 \left[\left. \frac{x e^{-\lambda x}}{-\lambda} \right|_0^\infty + \frac{1}{\lambda} \int_0^\infty e^{-\lambda x} dx \right] \\
&= \frac{2}{\lambda} \left[\left. \frac{e^{-\lambda x}}{-\lambda} \right|_0^\infty \right] = \frac{2}{\lambda^2} \,,
\end{aligned}$$

as expected.

Problem 12 (the median of a continuous random variable)

Part (a): When X is uniformly distributed over (a, b) the median is the value m that solves

$$\int_a^m \frac{dx}{b-a} = \int_m^b \frac{dx}{b-a} \,.$$

Integrating both sides gives that $m - a = b - m$, which has a solution of $m = \frac{a+b}{2}$.

Part (b): When X is a normal random variable with parameters (μ, σ^2) we find that m must satisfy

$$\int_{-\infty}^m \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\} dx = \int_m^\infty \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\} dx \,.$$

To evaluate the integral on both sides of this expression we let $v = \frac{x-\mu}{\sigma}$, so that $dv = \frac{dx}{\sigma}$ and each integral becomes

$$\begin{aligned}
\int_{-\infty}^{\frac{m-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{v^2}{2}\} dv &= \int_{\frac{m-\mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} \exp\{-\frac{v^2}{2}\} dv \\
&= 1 - \int_{-\infty}^{\frac{m-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{v^2}{2}\} dv \,.
\end{aligned}$$

Remembering the definition of the cumulative distribution function $\Phi(\cdot)$ as

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy \,,$$

we see that the above can be written as $\Phi(\frac{m-\mu}{\sigma}) = 1 - \Phi(\frac{m-\mu}{\sigma})$, so that

$$2\Phi(\frac{m-\mu}{\sigma}) = 1 \quad \text{or} \quad \Phi(\frac{m-\mu}{\sigma}) = \frac{1}{2}$$

Thus we have $m = \mu + \sigma\Phi^{-1}(1/2)$, since we can compute Φ^{-1} using the Matlab function `norminv`, we find that $\Phi^{-1}(1/2) = 0$, which intern implys that $m = \mu$.

Part (c): If X is an exponential random variable with rate λ then m must satisfy

$$\int_0^m \lambda e^{-\lambda x} dx = \int_m^\infty \lambda e^{-\lambda x} dx = 1 - \int_0^m \lambda e^{-\lambda x} dx \,.$$

Introducing the cumulative distribution function for the exponential distribution (given by $F(x) = \int_0^m \lambda e^{-\lambda x} dx$) the above equation can be seen to be $F(m) = 1 - F(m)$ or $F(m) = \frac{1}{2}$. So in general the median m is given by $m = F^{-1}(1/2)$ where F is the cumulative distribution function. For the exponential random variable this expression gives

$$1 - e^{-\lambda m} = \frac{1}{2} \quad \text{or} \quad m = \frac{\ln(2)}{\lambda} \,.$$

Problem 14 (if X is an exponential random variable then cX is)

If X is an exponential random variable with parameter λ, then defining $Y = cX$ the distribution function for Y is given by

$$\begin{aligned} F_Y(a) &= P\{Y \leq a\} \\ &= P\{cX \leq a\} \\ &= P\left\{X \leq \frac{a}{c}\right\} \\ &= F_X(\frac{a}{c}) \,. \end{aligned}$$

So, taking the derivative of the above expression, to obtain the density function for Y we see that

$$\begin{aligned} f_Y(a) &= \frac{dF_Y}{da} \\ &= \frac{d}{da} F_X(\frac{a}{c}) \\ &= F_X'(\frac{a}{c})\frac{1}{c} \\ &= \frac{1}{c} f_X(\frac{a}{c}) \end{aligned}$$

But since X is an exponential random variable with parameters λ we have that

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

so that we have for $f_Y(y)$ the following

$$f_Y(y) = \frac{1}{c} \begin{cases} \lambda e^{-\lambda \frac{y}{c}} & \frac{y}{c} \geq 0 \\ 0 & \frac{y}{c} < 0 \end{cases}$$

or

$$f_Y(y) = \begin{cases} \frac{\lambda}{c} e^{-\frac{\lambda}{c} y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

showing that Y is another exponential random variable with parameter $\frac{\lambda}{c}$.

Problem 18 (the expectation of X^k when X is exponential)

If X is exponential with mean $1/\lambda$ then $f(x) = \lambda e^{-\lambda x}$ so that

$$E[X^k] = \int_0^\infty \lambda x^k e^{-\lambda x} dx = \lambda \int_0^\infty x^k e^{-\lambda x} dx \,.$$

To transform to the gamma integral, let $v = \lambda x$, so that $dv = \lambda dx$ and the above integral becomes

$$\lambda \int_0^\infty \frac{v^k}{\lambda^k} e^{-v} \frac{dv}{\lambda} = \lambda^{-k} \int_0^\infty v^k e^{-v} dv \,.$$

Remembering the definition of the Γ function as $\int_0^\infty v^k e^{-v} dv \equiv \Gamma(k+1)$ and that when k is an integer $\Gamma(k+1) = k!$, we see that the above integral is equal to $k!$ and we have that

$$E[X^k] = \frac{k!}{\lambda^k} \,,$$

as required.

Problem 19 (the variance of a gamma random variable)

If X is a gamma random variable then

$$f(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)} \,,$$

when $x \geq 0$ and is zero otherwise. To compute the variance we require $E[X^2]$ which is given by

$$\begin{aligned} E[X^2] &= \int_0^\infty x^2 f(x) dx \\ &= \int_0^\infty x^2 \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)} dx \\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha + 1} e^{-\lambda x} dx \,. \end{aligned}$$

To evaluate the above integral, let $v = \lambda x$ so that $dv = \lambda dx$ then the above becomes

$$\frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{v^{\alpha+1}}{\lambda^{\alpha+1}} e^{-v} \frac{dv}{\lambda} = \frac{\lambda^\alpha}{\lambda^{\alpha+2}\Gamma(\alpha)} \int_0^\infty v^{\alpha+1} e^{-v} dv = \frac{\Gamma(\alpha+2)}{\lambda^2 \Gamma(\alpha)} \,.$$

Where we have used the definition of the gamma function in the above. If we "factor" the gamma function as

$$\Gamma(\alpha+2) = (\alpha+1)\Gamma(\alpha+1) = (\alpha+1)\alpha\Gamma(\alpha) \,,$$

we see that

$$E[X^2] = \frac{\alpha(\alpha+1)}{\lambda^2} \,,$$

when X is a gamma random variable with parameters (α, λ). Since $E[X] = \frac{\alpha}{\lambda}$ we can compute $\text{Var}(X) = E[X^2] - E[X]^2$ as

$$\text{Var}(X) = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2} \,,$$

as claimed.

Problem 20 (the gamma function at $1/2$)

We want to consider $\Gamma(1/2)$ which is defined as

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x} dx \,.$$

Since the argument of the exponential is the square of the term $x^{1/2}$ this observation might motivate the substitution $y = \sqrt{x}$. Following the hint let $y = \sqrt{2x}$, so that

$$dy = \frac{1}{\sqrt{2x}} dx \,.$$

So that with this substitution $\Gamma(1/2)$ becomes

$$\Gamma(1/2) = \int_0^\infty \sqrt{2}\, dy\, e^{-y^2/2} = \sqrt{2} \int_0^\infty e^{-y^2/2} dy \,.$$

Now from the normalization of the standard Gaussian we know that

$$\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \exp\{-\frac{y^2}{2}\} dy = 1 \,,$$

which easily transforms (by integrating only over the positive real numbers) into

$$2 \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp\{-\frac{y^2}{2}\} dy = 1 \,.$$

Finally manipulating this into the specific integral required to evaluate $\Gamma(1/2)$ we find that

$$\sqrt{2} \int_0^\infty \exp\{-\frac{y^2}{2}\} dy = \sqrt{\pi} \,,$$

which shows that $\Gamma(1/2) = \sqrt{\pi}$ as requested.

Problem 21 (the hazard rate function for the gamma random variable)

The hazard rate function for a random variable T that has a density function $f(t)$ and distribution function $F(t)$ is given by

$$\lambda(t) = \frac{f(t)}{1 - F(t)}.$$

For a gamma distribution with parameters (α, λ) we know our $f(t)$ is given by

$$f(t) = \begin{cases} \frac{\lambda e^{-\lambda t}(\lambda t)^{\alpha-1}}{\Gamma(\alpha)} & t \geq 0 \\ 0 & t < 0 \,. \end{cases}$$

Lets begin by calculating the cumulative density function for a gamma random variable with parameters (α, λ). We find that

$$F(t) = \int_0^t f(\xi) d\xi = \int_0^t \frac{\lambda e^{-\lambda \xi}(\lambda \xi)^{\alpha-1}}{\Gamma(\alpha)} d\xi \,,$$

which cannot be simplified further. We then have that

$$\begin{aligned} 1 - F(t) &= \int_0^\infty f(\xi) d\xi - \int_0^t f(\xi) d\xi \\ &= \int_t^\infty f(\xi) d\xi \\ &= \int_t^\infty \frac{\lambda e^{-\lambda \xi}(\lambda \xi)^{\alpha-1}}{\Gamma(\alpha)} d\xi \,, \end{aligned}$$

which also cannot be simplified further. Thus our hazard rate is given by

$$\begin{aligned} \lambda(t) &= \frac{\frac{\lambda e^{-\lambda t}(\lambda t)^{\alpha-1}}{\Gamma(\alpha)}}{\int_t^\infty \frac{\lambda e^{-\lambda \xi}(\lambda \xi)^{\alpha-1}}{\Gamma(\alpha)} d\xi} \\ &= \frac{t^{\alpha-1} e^{-\lambda t}}{\int_t^\infty \xi^{\alpha-1} e^{-\lambda \xi} d\xi} \\ &= \frac{1}{\int_t^\infty \left(\frac{\xi}{t}\right)^{\alpha-1} e^{-\lambda(\xi - t)} d\xi} \,. \end{aligned}$$

To try and simplify this further let $v = \frac{\xi}{t}$ so that $dv = \frac{d\xi}{t}$, and the above becomes

$$\lambda(t) = \frac{1}{\int_1^\infty v^{\alpha-1} e^{-\lambda t(v-1)} t dv} = \frac{1}{t e^{\lambda t} \int_1^\infty v^{\alpha-1} e^{-\lambda t v} dv} \,.$$

Which is one expression for the hazard rate for a gamma random variable. We can try and reduce the integral in the bottom of the above fraction to that of the "upper incomplete gamma function" by making the substitution $y = \lambda t v$ so that $dy = \lambda t dv$ and obtaining

$$\begin{aligned} \lambda(t) &= \frac{1}{t e^{\lambda t} \int_{\lambda t}^\infty \frac{y^{\alpha-1}}{(\lambda t)^{\alpha-1}} e^{-y} \frac{dy}{\lambda t}} \\ &= \frac{(\lambda t)^\alpha}{t e^{\lambda t}} \frac{1}{\int_{\lambda t}^\infty y^{\alpha-1} e^{-y} dy} \\ &= \frac{(\lambda t)^\alpha}{t e^{\lambda t}} \frac{1}{\Gamma(\alpha, \lambda t)} \,. \end{aligned}$$

Where we have introduced the **upper incomplete gamma function** whos definition is given by

$$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt\,.$$

Problem 27 (modality of the beta distribution)

The beta distribution with parameters (a,b) has a probability density function given by

$$f(x) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1} \quad \text{for} \quad 0 \le x \le 1\,.$$

Part (a): Our mode of this distribution will equal either the endpoints of our interval i.e. $x=0$ or $x=1$ or the location where the first derivative of $f(x)$ vanishes. Computing this derivative the expression $\frac{df}{dx}=0$ implies

$$\begin{aligned}\frac{df}{dx}(x) &= (a-1)x^{a-2}(1-x)^{b-1} + (b-1)x^{a-1}(1-x)^{b-2}(-1) = 0 \\ &\Rightarrow x^{a-2}(1-x)^{b-2}\left[(2-a-b)x + (a-1)\right] = 0\,,\end{aligned}$$

which can be solved for the x^* that makes this an equality and gives

$$x^* = \frac{a-1}{a+b-2} \quad \text{assuming} \quad a+b-2 \neq 0\,.$$

In this case to guarantee that this is a *maximum* we should check that the second derivative of f at the value of $\frac{a-1}{a+b-2}$ is indeed *negative*. This second derivative is computed in the Mathematica file `chap_5_te_27.nb` where it is shown to be negative for the given domains of a and b. To guarantee that this value is *interior* to the interval $(0,1)$ we should verify that

$$0 < \frac{a-1}{a+b-2} < 1$$

which since $a+b-2>0$ is equivalent to

$$0 < a-1 < a+b-2\,.$$

or from the first inequality we have that $a>1$ and from the second inequality $(a-1 < a+b-2)$ we have that $b>1$ verifying that our point x^* is in the interior of this interval and our distribution is unimodal as was asked.

Part (b): Now the case when $a=b=1$ is covered below, so lets consider $a=1$. From the requirement $a+b<2$ we must have $b<1$ and our density function in this case is given by

$$f(x) = \frac{(1-x)^{b-1}}{B(1,b)}\,.$$

This has a derivative given by

$$f'(x) = \frac{(1-b)(1-x)^{b-2}}{B(1,b)}\,,$$

and is *positive* over the entire interval since $b < 1$. Because the derivative is positive over the entire domain the distribution is unimodal and the single mode will occur at the right most limit i.e. $x = 1$. Now if $b = 1$ in the same way we have $a < 1$ and our density function is given by

$$f(x) = \frac{x^{a-1}}{B(a,1)} .$$

Which has a derivative given by

$$f'(x) = \frac{(a-1)x^{a-2}}{B(a,1)} ,$$

and is *negative* because $a < 1$. Because the derivative is negative over the entire domain the distribution is unimodal and the unique mode will occur at the left most limit of our domain i.e. $x = 0$. Finally, we consider the case where $a < 1$, $b < 1$ and neither equal to one. In this case from the derivative above our minimum or maximum is given by $\frac{a-1}{a+b-2}$ which for the domain of a and b given here is *positive* implying that the point x^* is a minimum. Thus we have two local maximums at the endpoints $x = 0$ and $x = 1$. One can also show (in the same way as above) that for this domain of a and b the point x^* is in the interior of the interval.

Part (c): If $a = b = 1$, then the density function for the beta distribution becomes (since $\text{Beta}(1,1) \equiv B(1,1) = 1$) is

$$f(x) = 1 ,$$

and we have the density of the uniform distribution, which is "flat" and has all points modes.

Problem 28 ($Y = F(X)$ is a uniform random variable)

If $Y = F(X)$ then the distribution function of Y is given by

$$\begin{aligned} F_Y(a) &= P\{Y \le a\} \\ &= P\{F(X) \le a\} \\ &= P\{X \le F^{-1}(a)\} \\ &= F(F^{-1}(a)) = a . \end{aligned}$$

Thus $f_Y(a) = \frac{dF_Y}{da} = 1$, showing that Y is a uniform random variable.

Problem 29 (the probability density function for $Y = aX + b$)

We begin by computing the cumulative distribution function of the random variable Y as

$$\begin{aligned} F_Y(y) &= P\{Y \le y\} \\ &= P\{aX + b \le y\} \\ &= P\{X \le \frac{y-b}{a}\} \\ &= F_X(\frac{y-b}{a}) . \end{aligned}$$

Taking the derivative to obtain the distribution function for Y we find that

$$f_Y(y) = \frac{dF_Y}{dy} = F_X'(\frac{y-b}{a})\frac{1}{a} = \frac{1}{a}f_X(\frac{y-b}{a}).$$

Problem 30 (the probability density function for the lognormal distribution)

We begin by computing the cumulative distribution function of the random variable Y as

$$\begin{aligned} F_Y(a) &= P\{Y \le a\} \\ &= P\{e^X \le a\} \\ &= P\{X \le \log(a)\} \\ &= F_X(\log(a)). \end{aligned}$$

Since X is a normal random variable with mean μ and variance σ^2 it has a cumulative distribution function given by

$$F_X(a) = \Phi\left(\frac{a-\mu}{\sigma}\right)$$

so that the cumulative distribution function for Y becomes

$$F_Y(a) = \Phi\left(\frac{\log(a)-\mu}{\sigma}\right).$$

The density function for the random variable Y is given by the derivative of the cumulative distribution function thus we have

$$f_Y(a) = \frac{F_Y(a)}{da} = \Phi'\left(\frac{\log(a)-\mu}{\sigma}\right)\left(\frac{1}{\sigma}\right)\left(\frac{1}{a}\right).$$

Since $\Phi'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ we have for the probability density function for a lognormal random variable given by

$$f_Y(a) = \frac{1}{\sqrt{2\pi}\sigma a}\exp\left\{-\frac{1}{2}\frac{(\log(a)-\mu)^2}{\sigma}\right\}.$$

Problem 31 (Legendre's theorem on relatively primeness)

Part (a): If k is the greatest common divisor of *both* X and Y then k must divide the random variable X and the random variable Y. In addition, X/k and Y/k must be relatively prime i.e. have no common factors. Now to show the given probability we first argue that that k will divide X with probability $1/k$ (approximately) and divide Y with probability $1/k$ (approximately). This can be reasoned heuristically by considering the case where X and Y are drawn from say $1, 2, \ldots, 10$. Then if $k = 2$ the numbers five numbers $2, 4, 6, 8, 10$ are all divisible by 2 and so the probability 2 will divide a random number from this set is $5/10 = 1/2$. If $k = 3$ then the three numbers $3, 6, 9$ are all divisible by 3 and so the probability 3 will divide a random number from this set is $3/10 \approx 1/3$. In the same way when $k = 4$

the probability that 4 will divide one of the numbers in our set is $2/10 = 1/5 \approx 1/4$. These approximations become exact as N goes to infinity. Finally, X/k and Y/k will be relatively prime with probability Q_1. Letting $E_{X,k}$ to be event that X is divisible by k, $E_{Y,k}$ the event that Y is divisible by k, and $E_{X/k,Y/k}$ the event that X/k and Y/k are relatively prime we have that

$$\begin{aligned} Q_k &= P\{D = k\} \\ &= P\{E_{X,k}\}P\{E_{Y,k}\}P\{E_{X/k,Y/k}\} \\ &= \left(\frac{1}{k}\right)\left(\frac{1}{k}\right)Q_1 \,. \end{aligned}$$

which is the desired results.

Part (b): From above we have that $Q_k = Q_1/(k^2)$, so summing both sides for $k = 1, 2, 3, \cdots$ gives (since $\sum_k Q_k = 1$, i.e. the greatest common divisor must be one of the numbers $1, 2, 3, \vdots$)

$$1 = Q_1 \sum_{k=1}^{\infty} \frac{1}{k^2} \,,$$

which gives the desired result of

$$Q_1 = \frac{1}{\sum_{k=1}^{\infty} \frac{1}{k^2}} \,.$$

Since $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$ the above expression for Q_1 becomes

$$Q_1 = \frac{1}{\frac{\pi^2}{6}} = \frac{6}{\pi^2} \,.$$

Part (c): Now Q_1 is the probability that X and Y are relatively prime will be true if $P_1 = 2$ is not a divisor of X and Y. The probability that P_1 is not a divisor of X is $1/P_1$ and the same for Y. So the probability that P_1 is a divisor for *both* X and Y is $(1/P_1)^2$. The probability that P_1 is *not* a divisor of both will happen with probability $1 - (1/P_1)^2$. The same logic applies for P_2 giving that the probability that X and Y don't have P_2 as a factor is $1 - (1/P_2)^2$. Since for X and Y be be relatively prime they cannot have any P_i as a joint factor, and thus we are looking for the conjunction of each of the individual probabilities. This is that P_1 is not a divisor, that P_2 is not a divisor, etc. This requires the product of all of these terms giving for Q_1 that

$$Q_1 = \prod_{i=1}^{\infty}\left(1 - \frac{1}{P_i^2}\right) = \prod_{i=1}^{\infty}\left(\frac{P_i^2 - 1}{P_i^2}\right) \,.$$

Problem 32 (the P.D.F. for $Y = g(X)$, when g is decreasing)

Theorem 7.1 expresses how to obtain the probability density function for Y when $Y = g(X)$ and the probability density function for X is known. To prove this result in the case when

$g(\cdot)$ is decreasing lets compute the cumulative distribution function for Y i.e.

$$\begin{aligned} F_Y(y) &= P\{Y \leq y\} \\ &= P\{g(X) \leq y\} \end{aligned}$$

By plotting a typical decreasing function $g(x)$ we see that the set above is given by the set of x values such that $x \geq g^{-1}(y)$ and the above expression becomes

$$F_Y(y) = \int_{g^{-1}(y)}^{\infty} f(x)dx\,.$$

Talking the derivative of this expression with respect to y we obtain

$$F_Y'(y) = f(g^{-1}(y))(-1)\frac{dg^{-1}(y)}{dy}\,.$$

Since $\frac{dg^{-1}(y)}{dy}$ is negative

$$(-1)\frac{dg^{-1}(y)}{dy} = \left|\frac{dg^{-1}(y)}{dy}\right|\,,$$

and using this in the above the theorem in this case is proven.

Chapter 5: Self-Test Problems and Exercises

Problem 1 (playing times for basketball)

Part (a): The probability that the players plays over fifteen minute is given by

$$\begin{aligned} \int_{15}^{40} f(x)dx &= \int_{15}^{20} 0.025dx + \int_{20}^{30} 0.05dx + \int_{30}^{40} 0.025dx \\ &= 0.025 \cdot (5) + 0.05 \cdot (10) + 0.025 \cdot (10) = 0.875\,. \end{aligned}$$

Part (b): The probability that the players plays between 20 and 35 minute is given by

$$\begin{aligned} \int_{20}^{35} f(x)dx &= \int_{20}^{30} 0.05dx + \int_{30}^{35} 0.025dx \\ &= 0.05 \cdot (10) + 0.025 \cdot (5) = 0.625\,. \end{aligned}$$

Part (c): The probability that the players plays less than 30 minutes is given by

$$\begin{aligned} \int_{10}^{30} f(x)dx &= \int_{10}^{20} 0.025dx + \int_{20}^{30} 0.05dx \\ &= 0.025 \cdot (10) + 0.05 \cdot (10) = 0.75\,. \end{aligned}$$

Part (d): The probability that the players plays more than 36 minutes is given by

$$\int_{36}^{40} f(x)dx = 0.025 \cdot (4) = 0.1\,.$$

Problem 2 (a power law probability density)

Part (a): Our random variable must normalize so that $\int f(x)dx = 1$, or

$$\int_0^1 cx^n dx = c\frac{x^{n+1}}{n+1}\bigg|_0^1 = \frac{c}{n+1}.$$

so that from the above we see that $c = n+1$. Our probability density function is then given by

$$f(x) = \begin{cases} (n+1)x^n & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Part (b): This expression is then given by

$$P\{X > x\} = \int_x^1 (n+1)\xi^n d\xi = \xi^{n+1}\big|_x^1 = 1 - x^{n+1} \quad \text{for} \quad 0 < x < 1.$$

Thus we have

$$P\{X > x\} = \begin{cases} 1 & x < 0 \\ 1 - x^{n+1} & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$

Problem 5 (a discrete uniform random variable)

We want to prove that $X = \text{int}(nU) + 1$ is a uniform random variable. To prove this first fix n, then $X = i$ is true if and only if

$$\text{Int}(nU) + 1 = i \quad \text{for} \quad i = 1, 2, 3, \cdots, n.$$

or

$$\text{Int}(nU) = i - 1.$$

or

$$\frac{i-1}{n} \le U < \frac{i}{n} \quad \text{for} \quad i = 1, 2, 3, \cdots, n$$

Thus the probability that $X = i$ is equal to

$$P\{X = i\} = \int_{\frac{i-1}{n}}^{\frac{i}{n}} 1 d\xi = \frac{i}{n} - \frac{i-1}{n} = \frac{1}{n} \quad \text{for} \quad i = 1, 2, 3, \cdots, n.$$

Problem 6 (bidding on a contract)

Assume we select a bid price b. Then our profit will be $b - 100$ if get the contract and zero if we don't get the contract. Thus our profit is a random variable that depends on the bid received by the competing company u. Our profit is then given by (here P is for *profit*)

$$P(b) = \begin{cases} 0 & b > u \\ b - 100 & b < u \end{cases}$$

Lets compute the expected profit

$$
\begin{aligned}
E[P(b)] &= \int_{70}^{b} 0 \cdot \frac{1}{140-70} d\xi + \int_{b}^{140} (b-100) \cdot \frac{1}{140-70} d\xi \\
&= \frac{(b-100)(140-b)}{70} = \frac{240b - b^2 - 14000}{70} \,.
\end{aligned}
$$

Then to find the maximum of the expected profit we take the derivative of the above expression with respect to b, setting that expression equal to zero and solve for b. The derivative set equal to zero is given by

$$
\frac{dE[P(b)]}{db} = \frac{1}{70}(240 - 2b) = 0 \,.
$$

Which has $b = 120$ as a solution. Since $\frac{d^2E[P(b)]}{db^2} = -\frac{2}{70} < 0$, this value of b is indeed a maximum of the function $P(b)$. Using this value of b our expected profit is given by $\frac{400}{70} = \frac{40}{7}$.

Problem 10 (the lifetime of automobile tires)

Part (a): We want to compute $P\{X \geq 40000\}$, which we do by converting to a standard normal. We find

$$
\begin{aligned}
P\{X \geq 40000\} &= P\left\{\frac{X - 34000}{4000} \geq 1.5\right\} \\
&= 1 - P\{Z < 1.5\} = 1 - \Phi(1.5) = 0.0668 \,.
\end{aligned}
$$

Part (b): We want to compute $P\{30000 \leq X \leq 35000\}$, which we do by converting to a standard normal. We find

$$
\begin{aligned}
P\{30000 \leq X \leq 35000\} &= P\left\{\frac{30000 - 34000}{4000} \leq Z \leq \frac{35000 - 34000}{4000}\right\} \\
&= P\{-1 \leq Z \leq 0.25\} \approx 0.4401 \,.
\end{aligned}
$$

Part (c): We want to compute

$$
P\{X \geq 40000 | X \geq 30000\} = \frac{P\{X \geq 40000, X \geq 30000\}}{P\{X \geq 30000\}} = \frac{P\{X \geq 40000\}}{P\{X \geq 30000\}} \,.
$$

We again do this by converting to a standard normal. We find

$$
\begin{aligned}
\frac{P\{X \geq 40000\}}{P\{X \geq 30000\}} &= \frac{P\left\{Z \geq \frac{40000-34000}{4000}\right\}}{P\left\{Z \geq \frac{30000-34000}{4000}\right\}} \\
&= \frac{1 - \Phi(1.5)}{1 - \Phi(-1.0)} = 0.0794 \,.
\end{aligned}
$$

All of these calculations can be found in the Matlab file `chap_5_st_10.m`.

Problem 11 (the annual rainfall in Cleveland)

Part (a): Let X be the random variable denting the anual rainfall in Cleveland. Then we want to evaluate $P\{X \geq 44\}$. Which we can do by converting to a standard normal. We find that

$$\begin{aligned} P\{X \geq 44\} &= P\left\{\frac{X - 40.2}{8.4} \geq \frac{44 - 40.2}{8.4}\right\} \\ &= 1 - \Phi(0.452) = 0.3255\,. \end{aligned}$$

Part (b): Following the assumptions stated for this problem lets begin by calculating $P(A_i)$ for $i = 1, 2, \ldots, 7$. Assuming independence each is equal to the value calculated in part (a) of this problem. Lets denote that common value by p. Then the random variable representing the number of years where the rainfall exceeds 44 inches (in a seven year time frame) is a Binomial random variable with parameters $(n, p) = (7, 0.3255)$. Thus the probability that three of the next seven years will have more than 44 inches of rain is given by

$$\binom{7}{3} p^3 (1-p)^4 = 0.2498\,.$$

These calculations are performed in the Matlab file `chap_5_st_11.m`.

Problem 14 (hazard rates)

Part (a): We have

$$P\{X > 2\} = 1 - P\{X \leq 2\} = 1 - (1 - e^{-2^2}) = e^{-2^2} = e^{-4}\,.$$

Part (b): We find

$$\begin{aligned} P\{1 < X < 3\} &= P\{X \leq 3\} - P\{X < 1\} \\ &= (1 - e^{-9}) - (1 - e^{-1}) = e^{-1} - e^{-9}\,. \end{aligned}$$

Part (c): The hazard rate function is defined as

$$\lambda(x) = \frac{f(x)}{1 - F(x)}\,.$$

Where f is the density function and F is the distribution function. We find for this problem that

$$f(t) = \frac{dF}{dx} = \frac{d}{dx}(1 - e^{-x^2}) = 2xe^{-x^2}\,.$$

so $\lambda(x)$ is given by

$$\lambda(x) = \frac{2xe^{-x^2}}{1 - (1 - e^{-x^2})} = 2x\,.$$

Part (d): The expectation is given by (using integration by parts to evaluate the first integral)

$$\begin{aligned} E[X] &= \int_0^\infty x f(x) dx = \int_0^\infty x(2xe^{-x^2}) dx \\ &= 2\left(\left. \frac{xe^{-x^2}}{-2} \right|_0^\infty + \frac{1}{2}\int_0^\infty e^{-x^2} dx \right) \\ &= \int_0^\infty e^{-x^2} dx \,. \end{aligned}$$

From the unit normaliztaion of the standard Gaussian $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-s^2/2} ds = 1$ we can compute the value of the above integral. Using this expression we find that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$ thus

$$E[X] = \frac{\sqrt{\pi}}{2} \,.$$

Part (d): The variance is given by $\mathrm{Var}(X) = E[X^2] - E[X]^2$ so first computing the expectation of X^2 we have that

$$\begin{aligned} E[X^2] &= \int_0^\infty x^2 f(x) dx = \int_0^\infty x^2(2xe^{-x^2}) dx \\ &= 2\left(\left. \frac{x^2 e^{-x^2}}{-2} \right|_0^\infty + \frac{1}{2}\int_0^\infty 2xe^{-x^2} dx \right) \\ &= 2\int_0^\infty xe^{-x^2} dx = 2\left(\left. \frac{e^{-x^2}}{-2} \right|_0^\infty \right) = 1 \,. \end{aligned}$$

Thus

$$\mathrm{Var}(X) = 1 - \frac{\pi}{4} = \frac{4-\pi}{4} \,.$$

Chapter 6 (Jointly Distributed Random Variables)

Chapter 6: Theoretical Exercises

Problem 33 (the P.D.F. of the ratio of normals is a Cauchy distribution)

As stated in the problem, let X_1 and X_2 be distributed as standard normal random variables (i.e. they have mean 0 and variance 1). Then we want the distribution of the variable X_1/X_2. To this end define the random variables U and V as $U = X_1/X_2$ and $V = X_2$. The distribution function of U is then what we are after. From the definition of U and V in terms of X_1 and X_2 we see that $X_1 = UV$ and $X_2 = V$. To solve this problem we will derive the joint distribution function for U and V and then marginalize out V giving the distribution function for U, alone. Now from Theorem $2-4$ on page 45 of Schaums probability and statistics outline the distribution of the joint random variable (U, V), in term of the joint random variable (X_1, X_2) is given by

$$g(u, v) = f(x_1, x_2) \left| \frac{\partial(x_1, x_2)}{\partial(u, v)} \right| .$$

Now

$$\left| \frac{\partial(x_1, x_2)}{\partial(u, v)} \right| = \left| \begin{array}{cc} v & u \\ 0 & 1 \end{array} \right| = |v| ,$$

so that

$$g(u, v) = f(x_1, x_2)|v| = p(x_1)\, p(x_2)|x_2| ,$$

as $f(x_1, x_2) = p(x_1)p(x_2)$ since X_1 and X_2 are assumed independent. Now using the fact that the distribution of X_1 and X_2 are standard normals we get

$$g(u, v) = \frac{1}{2\pi} \exp(-\frac{1}{2}\,(u\,v)^2) \exp(-\frac{1}{2}\,v^2)\,|v| .$$

Marginalizing out the variable V we get

$$g(u) = \int_{-\infty}^{\infty} g(u, v) dv = \frac{1}{\pi} \int_0^{\infty} v\, e^{-\frac{1}{2}(1+u^2)v^2} dv$$

To evaluate this integral let $\eta = \sqrt{\frac{1+u^2}{2}}\, v$, and after performing the integration we then find that

$$g(u) = \frac{1}{1 + u^2} .$$

Which is the distribution function for a Cauchy random variable.

Chapter 7 (Properties of Expectations)

Chapter 7: Problems

Problem 1 (expected winnings with coins and dice)

If we roll a heads then we win twice the digits on the die roll. If we roll a tail then we win 1/2 the digit on the die. Now we have a 1/2 chance of getting a head (or a tail) and a 1/6 chance of getting any individual number on the die. Thus the expected winnings are given by

$$\frac{1}{2}\cdot\frac{1}{6}\left(\frac{1}{2}\cdot 1\right)+\frac{1}{2}\cdot\frac{1}{6}\left(\frac{1}{2}\cdot 2\right)+\cdots+\frac{1}{2}\cdot\frac{1}{6}(2\cdot 1)+\frac{1}{2}\cdot\frac{1}{6}(2\cdot 2)+\cdots$$

or factoring out the 1/2 and the 1/6 we obtain

$$\frac{1}{2}\cdot\frac{1}{6}\left(\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\frac{4}{2}+\frac{5}{2}+\frac{6}{2}+2+2\cdot 2+2\cdot 3+2\cdot 4+2\cdot 5+2\cdot 6\right)$$

which equals

$$\frac{105}{24}\,.$$

Problem 2

Part (a): We have six choices for a suspect, six choices for a weapon and nine choices for a room giving in total $6\cdot 6\cdot 9=324$ possible combinations.

Part (b): Now let the random variables S, W, and R be the number of suspects, weapons, and rooms that the player receives and let X be the number of solutions possible after observing S, W, and R. Then X is given by

$$X=(6-S)(6-W)(9-R)\,.$$

Part (c): Now we must have

$$S+W+R=3 \quad \text{with} \quad 0\leq S\leq 3\,, \quad 0\leq W\leq 3\,, \quad 0\leq R\leq 3$$

Each specification of these three numbers (S,W,R) occurs with a uniform probability given by

$$\frac{1}{\binom{3+3-1}{3-1}}=\frac{1}{\binom{5}{2}}=\frac{1}{10}\,,$$

using the results given in Chapter 1. Thus the expectation of X is given by

$$
\begin{aligned}
E[X] &= \frac{1}{10}\sum_S\sum_W\sum_R(6-S)(6-W)(9-R) \\
&= \frac{1}{10}\sum_{s=0}^{3}(6-s)\sum_{w=0}^{3}(6-w)\sum_{r=0}^{3}(9-r) \\
&= \frac{1}{10}\left[6\sum_{W+R=3}(6-W)(9-R)+5\sum_{W+R=2}(6-W)(9-R)\right. \\
&+ \left. 4\sum_{W+R=1}(6-W)(9-R)+3\sum_{W+R=0}(6-W)(9-R)\right] \\
&= 190.4\,.
\end{aligned}
$$

Problem 3

We have by definition

$$
\begin{aligned}
E[|X-Y|^\alpha] &= \int\int|x-y|^\alpha f_{X,Y}(x,y)dxdy \\
&= \int\int|x-y|^\alpha dxdy\,.
\end{aligned}
$$

Since the area of region of the $x-y$ plane where $y > x$ is equal to the area of the $x-y$ plane where $y < x$, we can compute the above integral by doubling the integration domain $y < x$ to give

$$
\begin{aligned}
2\int_{x=0}^{1}\int_{y=0}^{x}(x-y)^\alpha dydx &= 2\int_{x=0}^{1}(-1)\frac{(x-y)^{\alpha+1}}{\alpha+1}|_0^x dx \\
&= \frac{2}{\alpha+1}\int_{x=0}^{1}x^{\alpha+1}dx \\
&= \frac{2}{\alpha+1}\frac{x^{\alpha+2}}{\alpha+2}|_0^1 \\
&= \frac{2}{(\alpha+1)(\alpha+2)}\,.
\end{aligned}
$$

Problem 18 (counting matched cards)

Let A_i be the event that when we turn over card i it matches the required cards face. For example A_1 is the event that turning over card one reveals an ace, A_2 is the event that turning over the second card reveals a duce etc. The the number of matched cards N is give by the sum of these indicator random variable as

$$
N = \sum_{i=1}^{52} A_i\,.
$$

Taking the expectation of this result and using linearity requires us to evaluate $E[A_i] = P(A_i)$. For card i the probability that when we turn it over it matches the expected face is given by

$$P(A_i) = \frac{4}{52},$$

since there are four suites that could match a given face. Thus we then have for the expected number of matching cards that

$$E[N] = \sum_{i=1}^{52} E[A_i] = \sum_{i=1}^{52} P(A_i) = 52 \cdot \frac{4}{52} = 4\,.$$

Problem 21 (more birthday problems)

Let $A_{i,j,k}$ be an indicator random variable if persons i, j, and k have the same birthday *and no one else does*. Then if we let N denote the random variable representing the number of groups of three people all of whom have the same birthday we see that N is given by a sum of these random variables as

$$N = \sum_{i<j<k} A_{i,j,k}\,.$$

Then taking the expectation of the above expression we have

$$E[N] = \sum_{i<j<k} E[A_{i,j,k}]\,.$$

Now there are $\binom{100}{3}$ terms in the above sum (since there are one hundred total people and our sum involves all subsets of three people), and the probability of each event $A_{i,j,k}$ happening is given by

$$\begin{aligned} P(A_{i,j,k}) &= \frac{1}{365^2}\left(1 - \frac{1}{365}\right)^{100-3} \\ &= \frac{1}{365^2}\left(\frac{364}{365}\right)^{97} \end{aligned}$$

since person j and person k's birthdays must match that of person i, and the remaining 97 people must have different birthdays (the problem explicitly states we are looking for the expected number days that are the birthday of *exactly* three people and not more). Thus the total expectation of the number of groups of three people that have the same birthday is then given by

$$E[N] = \binom{100}{3} \frac{1}{365^2}\left(\frac{364}{365}\right)^{97} = 0.93014\,,$$

in agreement with the back of the book.

Problem 22 (number of times to roll a fair die to get all six sides)

This is exactly like the coupon collecting problem where we have six coupons with a probability of obtaining any one of them given by $1/6$. Then this problem is equivalent to determining the expected number of coupons we need to collect before we get a complete set. From Example 2i from the book we have the expected number of rolls X to be given by

$$E[X] = N\left[1 + \frac{1}{2} + \cdots + \frac{1}{N-1} + \frac{1}{N}\right]$$

when $N = 6$ this becomes

$$E[X] = 6\left[1 + \frac{1}{2} + \cdots + \frac{1}{5} + \frac{1}{6}\right] = 14.7\,.$$

Problem 30 (a squared expectation)

We find, by expanding the quadratic and using independence, that

$$E[(X-Y)^2] = E[X^2 - 2XY + Y^2] = E[X^2] - 2E[X]E[Y] + E[Y^2]\,.$$

In terms of the variance $E[X^2]$ is given by $E[X^2] = \text{Var}(X) + E[X]^2$ so the above becomes

$$\begin{aligned} E[(X-Y)^2] &= \text{Var}(X) + E[X]^2 - 2E[X]E[Y] + \text{Var}(Y) + E[Y]^2 \\ &= \sigma^2 + \mu^2 - 2\mu^2 + \sigma^2 + \mu^2 = 2\sigma^2\,. \end{aligned}$$

Problem 33 (evaluating expectations and variances)

Part (a): We find, expanding the quadratic and using the linearity property of expectations that

$$E[(2+X)^2] = E[4 + 4X + X^2] = 4 + 4E[X] + E[X^2]\,.$$

In terms of the variance, $E[X^2]$ is given by $E[X^2] = \text{Var}(X) + E[X]^2$, both terms of which we know from the problem statement. Using this the above becomes

$$E[(2+X)^2] = 4 + 4(1) + (5 + 1^2) = 14\,.$$

Part (b): We find, using properties of the variance that

$$\text{Var}(4 + 3X) = \text{Var}(3X) = 9\text{Var}(X) = 9 \cdot 5 = 45\,.$$

Problem 48 (conditional expectation of die rolling)

Part (a): The probability that the first six is rolled on the nth roll is given by a geometric random variable with parameter $p = 1/6$. Thus the expected number of rolls to get a six is given by

$$E[X] = \frac{1}{p} = 6\,.$$

Part (b): We want to evaluate $E[X|Y=1]$. Since in this expectation we are told that the first roll of our dice results in a five we have that

$$E[X|Y=1] = 1 + E[X] = 1 + \frac{1}{p} = 1 + 6 = 7\,,$$

since after the first roll we again have that the number of rolls to get the first six is a geometric random variable with $p = 1/6$.

Part (c): We want to evaluate $E[X|Y=5]$, which means that the first five happens on the fifth roll. Thus the rolls $1, 2, 3, 4$ all have a probability of $1/5$ to show a six. After the fifth roll, there are again six possible outcomes of the die so the probability of obtaining a six is given by $1/6$. Defining the event A to be the event that we do not roll a six in any of the first four rolls (and implicitly given that the first five happens on the fifth roll) we see that

$$P(A) = \left(\frac{4}{5}\right)^4 = 0.4096\,,$$

since with probability of $1/5$ we will roll a six and with probability $4/5$ we will not roll a six. With this definition and using the definition of expectation we find that

$$\begin{aligned} E[X|Y=5] &= 1\left(\frac{1}{5}\right) + 2\left(\frac{4}{5}\right)\frac{1}{5} + 3\left(\frac{4}{5}\right)^2\frac{1}{5} + 4\left(\frac{4}{5}\right)^3\frac{1}{5} \\ &+ \sum_{k=6}^{\infty} k\left(P(A)\left(\frac{5}{6}\right)^{k-6}\frac{1}{6}\right)\,. \end{aligned}$$

We will evaluate this last sum numerically. This is done in the Matlab file `chap_7_prob_48.m`, where we find that

$$[X|Y=5] = 5.8192\,,$$

in agreement with the book.

Problem 50 (compute $E[X^2|Y=y]$)

By definition, the requested expectation is given by

$$E[X^2|Y=y] = \int_0^{\infty} x^2 f(x|Y=y)dx\,.$$

Lets begin by computing $f(x|Y=y)$, using the definition of this density in terms of the joint density

$$f(x|y) = \frac{f(x,y)}{f(y)}\,.$$

Since we are given $f(x,y)$ we begin by first computing $f(y)$. We find that

$$\begin{aligned}
f(y) &= \int_0^\infty f(x,y)dx = \int_0^\infty \frac{e^{-x/y}e^{-y}}{y}dx \\
&= \frac{e^{-y}}{y}\int_0^\infty e^{-x/y}dx = \frac{e^{-y}}{y}(-y)\left. e^{-x/y}\right|_0^\infty \\
&= e^{-y}.
\end{aligned}$$

So that $f(x|y)$ is given by

$$f(x|y) = \frac{e^{-x/y}e^{-y}}{y}e^{y} = \frac{e^{-x/y}}{y}.$$

With this expression we can evaluate our expectation above. We have (using integration by parts several times)

$$\begin{aligned}
E[X^2|Y=y] &= \int_0^\infty x^2\frac{e^{-x/y}}{y}dx \\
&= \frac{1}{y}\int_0^\infty x^2 e^{-x/y}dx \\
&= \frac{1}{y}\left(x^2(-y)e^{-x/y}\Big|_0^\infty - \int_0^\infty 2x(-y)e^{-x/y}dx \right) \\
&= 2\int_0^\infty xe^{-x/y}dx \\
&= 2\left(x(-y)e^{-x/y}\Big|_0^\infty - \int_0^\infty (-y)e^{-x/y}dx \right) \\
&= 2y\int_0^\infty e^{-x/y}dx \\
&= 2y(-y)\left. e^{-x/y}\right|_0^\infty = 2y^2.
\end{aligned}$$

Problem 51 (compute $E[X^3|Y=y]$)

By definition, the requested expectation is given by

$$E[X^3|Y=y] = \int x^3 f(x|Y=y)dx.$$

Lets begin by computing $f(x|Y=y)$, using the definition of this density in terms of the joint density

$$f(x|y) = \frac{f(x,y)}{f(y)}.$$

Since we are given $f(x,y)$ we begin by first computing $f(y)$. We find that

$$f(y) = \int_0^y f(x,y)dx = \int_0^y \frac{e^{-y}}{y}dx = e^{-y}.$$

So that $f(x|y)$ is given by

$$f(x|y) = \frac{e^{-y}}{y}e^{y} = \frac{1}{y}.$$

With this expression we can evaluate our expectation above. We have

$$E[X^3|Y=y] = \int_0^y x^3 \frac{1}{y} dx = \frac{1}{y} \left. \frac{x^4}{4} \right|_0^y = \frac{y^3}{4}.$$

Problem 52 (the average weight)

Let W denote the random variable representing the weight of a person selected from the total population. Then we can compute $E[W]$ by conditioning on the subgroups. Letting G_i denote the event we are drawing from subgroup i, we have

$$E[W] = \sum_{i=1}^{r} E[W|G_i]P[G_i] = \sum_{i=1}^{r} w_i p_i .$$

Problem 53 (the time to escape)

Let T be the random variable denoting the number of days until the prisoner reaches freedom. We can evaluate $E[T]$ by conditioning on the door selected. If we denote D_i be the event the prisoner selects door i then we have

$$E[T] = E[T|D_1]P(D_1) + E[T|D_2]P(D_2) + E[T|D_3]P(D_3) .$$

Each of the above expressions can be evaluated. For example if the prisoner selects the first door then after two days he will be right back where he started and thus has in expectation $E[T]$ more days left. Thus

$$E[T|D_1] = 2 + E[T] .$$

Using logic like this we see that $E[T]$ can be expressed as

$$\begin{aligned} E[T] &= E[T|D_1]P(D_1) + E[T|D_2]P(D_2) + E[T|D_3]P(D_3) \\ &= (2 + E[T])(0.5) + (4 + E[T])(0.3) + (1)(0.2) . \end{aligned}$$

Solving the above expression for $E[T]$ we find that $E[T] = 12$.

Problem 58 (flipping a biased coin until a head and a tail appears)

Part (a): We reason as follows if the first flip lands heads then we will continue to flip until a tail appears at which point we stop. If the first flip lands tails we will continue to flip until a head appears. In both cases the number of flips required until we obtain our desired outcome (a head and a tail) is a geometric random variable. Thus computing the desired expectation is easy once we condition on the result of the first flip. Let H denote the event that the first flip lands heads then with N denoting the random variable denoting the number of flips until both a head and a tail occurs we have

$$E[N] = E[N|H]P\{H\} + E[N|H^c]P\{H^c\} .$$

Since $P\{H\} = p$ and $P\{H^c\} = 1 - p$ the above becomes

$$E[N] = pE[N|H] + (1-p)E[N|H^c] .$$

Now we can compute $E[N|H]$ and $E[N|H^c]$. Now $E[N|H]$ is one plus the expected number of flips required to obtain a tail. The expected number of flips required to obtain a tail is the expectation of a geometric random variable with probability of succuss $1 - p$ and thus we have that

$$E[N|H] = 1 + \frac{1}{1-p} .$$

The addition of the one in the above expression is due to the fact that we were required to performed one flip to determining what the first flip was. In the same way we have

$$E[N|H^c] = 1 + \frac{1}{p} .$$

With these two sub-results we have that $E[N]$ is given by

$$E[N] = p + \frac{p}{1-p} + (1-p) + \frac{1-p}{p} = 1 + \frac{p}{1-p} + \frac{1-p}{p} .$$

Part (b): We can reason this probability as follows. Since once the outcome of the first coin flip is observed we repeatedly flip our coin as many times as needed to obtain the opposite face we see that we will end our experiment on a head only if the first coin flip is a *tail.* Since this happens with probability $1 - p$ this must also be the probability that the last flip lands heads.

Chapter 7: Theoretical Exercises

Problem 6 (the integral of the complement of the distribution function)

We desire to prove that

$$E[X] = \int_0^\infty P\{X > t\}dt .$$

Following the hint in the book define the random variable $X(t)$ as

$$X(t) = \begin{cases} 1 & \text{if} \quad t < X \\ 0 & \text{if} \quad t \geq X \end{cases}$$

Then integrating the variable $X(t)$ we see that

$$\int_0^\infty X(t)dt = \int_0^X 1dt = X .$$

Thus taking the expectation of both sides we have

$$E[X] = E\left[\int_0^\infty X(t)dt\right] .$$

This allows us to use the assumed identity that we can pass the expectation inside the integration as

$$E\left[\int_0^\infty X(t)dt\right] = \int_0^\infty E[X(t)]dt\,,$$

so applying this identity to the expression we have for $E[X]$ above we see that $E[X] = \int_0^\infty E[X(t)]dt$. From the definition of $X(t)$ we have that $E[X(t)] = P\{X > t\}$ and we then finally obtain the fact that

$$E[X] = \int_0^\infty P\{X > t\}dt\,,$$

as we were asked to prove.

Problem 10 (the expectation of a sum of random variables)

We begin by defining $R(k)$ to be

$$R(k) \equiv E\left[\frac{\sum_{i=1}^k X_i}{\sum_{i=1}^n X_i}\right]\,.$$

Then we see that $R(k)$ satisfies a recursive expression given by

$$R(k) - R(k-1) = E\left[\frac{X_k}{\sum_{i=1}^n X_i}\right] \quad \text{for} \quad 2 \le k \le n\,.$$

To further simplify this we would like to evaluate the expectation on the right hand side of the above. Now by the assumed independence of all X_i's the expectation on the right handside of the above is *independent* of k, and is a constant C. Thus it can be evaluated by considering

$$\begin{aligned} 1 &= E\left[\frac{\sum_{k=1}^n X_k}{\sum_{i=1}^n X_i}\right] \\ &= \sum_{k=1}^n E\left[\frac{X_k}{\sum_{i=1}^n X_i}\right] \\ &= nC\,. \end{aligned}$$

Which when we solve for C gives $C = 1/n$ or in terms of the original expectations

$$E\left[\frac{X_k}{\sum_{i=1}^n X_i}\right] = \frac{1}{n} \quad \text{for} \quad 1 \le k \le n\,.$$

Thus using our recursive expression $R(k) = R(k-1) + 1/n$, we see that since

$$R(1) = E\left[\frac{X_1}{\sum_{i=1}^n X_i}\right] = \frac{1}{n}\,,$$

that

$$R(2) = \frac{1}{n} + \frac{1}{n} = \frac{2}{n}\,.$$

Continuing our iterations in this way we find that

$$R(k) = E\left[\frac{\sum_{i=1}^k X_i}{\sum_{i=1}^n X_i}\right] = \frac{k}{n} \quad \text{for} \quad 1 \le k \le n\,.$$

Problem 13 (record values)

Part (a): Let R_j be an indicator random variable denoting whether or not the j-th random variable (from n) is a record value. This is that $R_j = 1$ if and only if X_j is a record value i.e. $X_j \geq X_i$ for all $1 \leq i \leq j$, and X_j is zero otherwise. Then the number N of record values is given by summing up these indicator

$$N = \sum_{j=1}^{n} R_j \,.$$

Taking the expectation of this expression we find that

$$E[N] = \sum_{j=1}^{n} E[R_j] = \sum_{j=1}^{n} P\{R_j\} \,.$$

Now $P\{R_j\}$ is the probability that X_j is the maximum from among all X_i samples where $1 \leq i \leq j$. Since each X_i is equally likely to be the maximum we have that

$$P\{R_j\} = P\{X_j = \max_{1 \leq i \leq j}(X_i)\} = \frac{1}{j} \,,$$

and the expected number of record values is given by

$$E[N] = \sum_{j=1}^{n} \frac{1}{j} \,,$$

as claimed.

Part (b): From the discussion in the text if N is a random variable denoting the number of record values that occur then we have

$$\binom{N}{2} = \sum_{i<j} R_i R_j \,.$$

Thus taking the expectation and expanding the expression $\binom{N}{2}$ in the above we have

$$E[N^2 - N] = E\left[2 \sum_{i<j} R_i R_j\right] = 2 \sum_{i<j} P(R_i, R_j) \,.$$

Now $P(R_i, R_j)$ is the probability that X_i and X_j are record values. Since there is no constraint on R_j if R_i is a record value this probability is given by

$$P(R_i, R_j) = \frac{1}{j}\frac{1}{i} \,.$$

Thus we have that

$$\begin{aligned}
E[N^2] &= E[N] + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{j}\frac{1}{i} \\
&= \sum_{j=1}^{n} \frac{1}{j} + 2 \sum_{i=1}^{n-1} \frac{1}{i} \sum_{j=i+1}^{n} \frac{1}{j} \,,
\end{aligned}$$

so that the variance is given by

$$
\begin{aligned}
\mathrm{Var}(N) &= E[N^2] - E[N]^2 \\
&= \sum_{j=1}^{n} \frac{1}{j} + 2\sum_{i=1}^{n-1} \frac{1}{i} \sum_{j=i+1}^{n} \frac{1}{j} - \left(\sum_{j=1}^{n} \frac{1}{j} \right)^2 \\
&= \sum_{j=1}^{n} \frac{1}{j} + \left(2\sum_{i=1}^{n-1} \frac{1}{i} \sum_{j=i+1}^{n} \frac{1}{j} \right) - \sum_{j=1}^{n} \frac{1}{j^2} - \left(2\sum_{i=1}^{n-1} \frac{1}{i} \sum_{j=i+1}^{n} \frac{1}{j} \right) \\
&= \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{n} \frac{1}{j^2} .
\end{aligned}
$$

where we have used the fact that $(\sum_i a_i)^2 = \sum_i a_i^2 + 2\sum_{i<j} a_i a_j$, thus

$$
\mathrm{Var}(N) = \sum_{j=1}^{n} \frac{1}{j} - \frac{1}{j^2} = \sum_{j=1}^{n} \frac{j-1}{j^2} ,
$$

as claimed.

Problem 15

Part (a): Define X_i to be an indicator random variable such that if trial i is a success then $X_i = 1$ otherwise $X_i = 0$. Then if X is a random variable representing the number of successes from all n trials we have that

$$
X = \sum_i X_i ,
$$

taking the expectation of both sides we find that $E[X] = \sum_i E[X_i] = \sum_i P_i$. Thus an expression for the mean μ is given by

$$
\mu = \sum_i P_i .
$$

Part (b): Using the result from the book we have that

$$
\binom{X}{2} = \sum_{i<j} X_i X_j ,
$$

so that taking the expectation of the above gives

$$
E[\binom{X}{2}] = \frac{1}{2} E[X^2 - X] = \sum_{i<j} E[X_i X_j] .
$$

But the expecation of $X_i X_j$ is given by (using independence of the trials X_i and X_j) $E[X_i X_j] = P\{X_i X_j\} = P\{X_i\}P\{X_j\}$. Thus the above expecation becomes

$$
E[X^2] = E[X] + 2\sum_{i<j} P_i P_j = \mu + 2\sum_{i=1}^{n-1} P_i \sum_{j=i+1}^{n} P_j .
$$

From which we can compute the variance of X as

$$\begin{aligned}
\mathrm{Var}(X) &= E[X^2] - E[X]^2 \\
&= \mu + 2\sum_{i=1}^{n-1} P_i \sum_{j=i+1}^{n} P_j - \left(\sum_{i=1}^{n} P_i\right)^2 \\
&= \mu + 2\sum_{i=1}^{n-1} P_i \sum_{j=i+1}^{n} P_j - \sum_{i=1}^{n} P_i^2 - 2\sum_{i=1}^{n-1} P_i \sum_{j=i+1}^{n} P_j \\
&= \sum_{i=1}^{n} P_i(1-P_i)\,.
\end{aligned}$$

To find the values of P_i that maximize this variance we use the method of Lagrange multipliers. Consider the following Lagrangian

$$L = \sum_{i=1}^{n} P_i(1-P_i) + \lambda\left(\sum_{i=1}^{n} P_i - 1\right)\,.$$

Taking the derivatives of this expression with respect to P_i and λ gives

$$\begin{aligned}
\frac{\partial L}{\partial P_i} &= 1 - P_i - P_i + \lambda \quad \text{for} \quad 1 \le i \le n \\
\frac{\partial L}{\partial \lambda} &= \sum_{i=1}^{n} P_i - 1\,.
\end{aligned}$$

The first equation gives for P_i (in terms of λ) the expression that $P_i = \frac{1+\lambda}{2}$ which when put into the second constraint gives

$$\lambda = \frac{2}{n} - 1 = \frac{2-n}{n}\,.$$

Which means that

$$P_i = \frac{1}{n}\,.$$

To determine if this maximizes or minimizes the functional $\mathrm{Var}(X)$ we need to consider the second derivative of the $\mathrm{Var}(X)$ expression, i.e.

$$\frac{\partial^2 \mathrm{Var}(X)}{\partial P_i \partial P_j} = -2\delta_{ij}\,,$$

with δ_{ij} the Kronecker delta. Thus the matrix of second derivatives is negative definite implying that our solutions $P_i = \frac{1}{n}$ will *maximize* the variance.

Part (c): To select a choice of P_i's that minimizes this variance we note that $\mathrm{Var}(X) = 0$ if $P_i = 0$ or $P_i = 1$ for every i. In this case the random variable X is a constant.

Chapter 8 (Limit Theorems)

Chapter 8: Problems

Problem 1 (bounding the probability we are between two numbers)

We are told that $\mu = 20$ and $\sigma^2 = 20$ so that

$$P\{0 < X < 40\} = P\{-20 < X - 20 < 20\} = 1 - P\{|X - 20| > 20\}.$$

Now by Chebyshev's inequality

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2},$$

we know that

$$P\{|X - 20| > 20\} \leq \frac{20}{20^2} = 0.05.$$

This implys that (negating both sides that)

$$-P\{|X - 20| > 20\} > -0.05,$$

so that $1 - P\{|X - 20| > 20\} > 0.95$. In summary then we have that $P\{0 < X < 40\} > 0.95$.

Problem 2 (distribution of test scores)

We are told, that if X is the students score in taking this test then $E[X] = 75$.

Part (a): Then by Markov's inequality we have

$$P\{X \geq 85\} \leq \frac{E[X]}{85} = \frac{75}{85} = \frac{15}{17}.$$

If we also know the variance of X is given by $\mathrm{Var}X = 25$, then we can use the one-sided Markov inequality given by

$$P\{X - \mu \geq a\} \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

With $\mu = 75$, $a = 10$, $\sigma^2 = 25$ this becomes

$$P\{X \geq 85\} \leq \frac{25}{25 + 10^2} = \frac{1}{5}.$$

Part (b): Using Chernoff's inequality given by

$$P\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2},$$

we have (since we want $5k = 10$ or $k = 2$) that

$$P\{|X - 75| \geq 2 \times 5\} \leq \frac{1}{2^2} = 0.25\,,$$

Thus

$$P\{|X - 75| \leq 10\} = 1 - P\{|X - 75| \geq 10\} = 1 - \frac{1}{4} = \frac{3}{4}\,.$$

Part (c): We desire to compute

$$P\{75 - 5 \leq \frac{1}{n}\sum_{i=1}^{n} x_i \leq 75 + 5\} = P\{|\frac{1}{n}\sum_{i=1}^{n} x_i - 75| \leq 5\}$$

Defining $X = \sum_{i=1}^{n} X_i$, we have that $\mu = E[X] = 75$ and $\text{Var}(X) = \frac{1}{n^2} \times n\text{Var}(X) = \frac{25}{n}$. So to use Chernoff' inequality on this problem we desire a k such that $k\left(\frac{5}{\sqrt{n}}\right) = 5$ so $k = \sqrt{n}$ and then Chernoff's bound gives

$$P\{|\frac{1}{n}\sum_{i=1}^{n} x_i - 75| > 5\} \leq \frac{1}{n}\,.$$

So to make $P\{|\frac{1}{n}\sum_{i=1}^{n} x_i - 75| > 5\} \leq 0.1$ we must take

$$\frac{1}{n} \leq 0.1 \Rightarrow n \geq 10\,.$$

Problem 3 (an example with the central limit theorem)

We want to compute n such that

$$P\{\left|\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - 75}{5/\sqrt{n}}\right| \leq \frac{5}{5/\sqrt{n}}\} \geq 0.9\,.$$

Now by the central limit theorem the expression

$$\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - 75}{5/\sqrt{n}}\,,$$

we have that the above can be written (first removing the absolute values)

$$\begin{aligned} P\{\left|\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - 75}{5/\sqrt{n}}\right| \leq \sqrt{n}\} &= 1 - 2P\{\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - 75}{5/\sqrt{n}} \leq \sqrt{n}\} \\ &= 1 - 2\Phi(-\sqrt{n})\,. \end{aligned}$$

Setting this equal to 0.9 gives $\Phi(-\sqrt{n}) = 0.05$, or when we solve for n we obtain

$$n > (-\Phi^{-1}(0.05))^2 = 2.7055\,.$$

In the file `chap_8_prob_3.m` we use the Matlab command `norminv` to compute this value. We see that we should take $n \geq 3$.

Problem 4 (sums of Poisson random variables)

Part (a): The Markov inequality is $P\{X \geq a\} \leq \frac{E[X]}{a}$, so if $X = \sum_{i=1}^{20} X_i$ then $E[X] = \sum_{i=1}^{20} E[X_i] = 20$, and the Markov inequality becomes in this case

$$P\{X \geq 15\} \leq \frac{20}{15} = \frac{4}{3}\,.$$

Note that since all probabilities must be less than one, this bound is not informative.

Part (b): We desire to compute (using the central limit theorem) $P\{\sum_{i=1}^{20} X_i > 15\}$. Thus the desired probability is given by (since $\sigma = \sqrt{\mathrm{Var}(X_i)} = 1$)

$$\begin{aligned} P\{\frac{\sum_{i=1}^{20} X_i - 20}{\sqrt{20}} > \frac{15-20}{\sqrt{20}}\} &= 1 - P\{Z < -\frac{5}{\sqrt{20}}\} \\ &= 0.8682\,. \end{aligned}$$

This calculation can be found in `chap_8_prob_4.m`.

Problem 5 (rounding to integers)

Let $R = \sum_{i=1}^{50} R_i$ be the approximate sum where each R_i is the rounded variable and let $X = \sum_{i=1}^{50} X_i$ be the exact sum. We desire to compute $P\{|X - R| > 3\}$, which can be simplified to give

$$\begin{aligned} P\{|X - R| > 3\} &= P\left\{\left|\sum_{i=1}^{50} X_i - \sum_{i=1}^{50} R_i\right| > 3\right\} \\ &= P\left\{\left|\sum_{i=1}^{50}(X_i - R_i)\right| > 3\right\}\,. \end{aligned}$$

Now $X_i - R_i$ are independent uniform random variables between $[-0.5, 0.5]$ so the above can be evaluated using the central limit theorem. For this sum of random variables the mean of the individual random variables $X_i - R_i$ is zero while the standard deviation σ is given by

$$\sigma^2 = \frac{(0.5 - (-0.5))^2}{12} = \frac{1}{12}\,.$$

Thus by the central limit theorem we have that

$$\begin{aligned} P\left\{\left|\sum_{i=1}^{50}(X_i - R_i)\right| > 3\right\} &= P\left\{\left|\frac{\sum_{i=1}^{50}(X_i - R_i)}{50/\sqrt{12}}\right| > \frac{3}{50/\sqrt{12}}\right\} \\ &= 2P\left\{\frac{\sum_{i=1}^{50}(X_i - R_i)}{50/\sqrt{12}} < \frac{-3}{50/\sqrt{12}}\right\} \\ &= 2\Phi(\frac{-3}{50/\sqrt{12}}) = 0.8353\,. \end{aligned}$$

This calculation can be found in `chap_8_prob_5.m`.

Problem 6 (rolling a die until our sum exceeds 800)

The sum of n die rolls is given by $X = \sum_{i=1}^{n} X_i$ with X_i a random variable taking values $1, 2, 3, 4, 5, 6$ all with probability of $1/6$. Then

$$\mu = E[\sum_{i=1}^{n} E[X_i] = nE[X_i] = \frac{n}{6}(1+2+3+4+5+6) = \frac{7}{2}n$$

In addition, because of the independence of our X_i we have that $\mathrm{Var}(X) = n\mathrm{Var}(X_i)$. For the individual random variables X_i we have that $\mathrm{Var}(X_i) = E[X_i^2] - E[X_i]^2$. For die we have

$$E[X_i^2] = \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}\,.$$

so that our variance is given by

$$\mathrm{Var}(X_i) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = 2.916\,.$$

Now the probability we want to calculate is given by $P\{X > 300\}$, which we can maniuplate into a form where we can apply the central limit theorm. We have

$$P\left\{\frac{X - \frac{7n}{2}}{\sqrt{2.916}\sqrt{n}} > \frac{300 - \frac{7n}{2}}{\sqrt{2.916}\sqrt{n}}\right\}$$

Now if $n = 80$ we have the above given by

$$P\left\{\frac{X - \frac{7}{2}\cdot 80}{\sqrt{2.916}\sqrt{80}} > \frac{300 - \frac{7}{2}\cdot 80}{\sqrt{2.916}\sqrt{80}}\right\} = 1 - P\{Z < 1.309\} = 1 - \Phi(1.309) = 0.0953\,.$$

Problem 7 (working bulbs)

The total lifetime of all the bulbs is given by

$$T = \sum_{i=1}^{100} X_i\,,$$

where X_i is an exponential random variable with mean five hours. Then since the random variable T is the sum of independent identically distributed random variables we can use the central limit theorm to derive estimates about T. For example we know that

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}}\,,$$

is approximatly a standard normal. Thus to evaluate (since $\sigma^2 = 25$) we have that

$$\begin{aligned} P\{T > 525\} &= P\left\{\frac{T - 100(5)}{10(5)} > \frac{525 - 500}{50}\right\} \\ &= 1 - P\{Z < 1/2\} \\ &= 1 - \Phi(0.5) = 1 - 0.6915 = 0.3085\,. \end{aligned}$$

Problem 8 (working bulbs with replacement times)

Our expression for the total time that there is a working bulb in problem 7 without any replacment time is given by

$$T = \sum_{i=1}^{100} X_i \,.$$

If there is a random time required to replace each bulb then we our random variable T must now include this randomness and becomes

$$T = \sum_{i=1}^{100} X_i + \sum_{i=1}^{99} U_i \,.$$

Again we desire to evaluate $P\{T \leq 550\}$. To evaluate this let

$$T = \sum_{i=1}^{99} (X_i + U_i) + X_{100} \,,$$

which motivates us to define the random variables V_i as

$$V_i = \begin{cases} X_i + U_i & i = 1, \cdots, 99 \\ X_{100} & i = 100 \end{cases}$$

Then $T = \sum_{i=1}^{100} V_i$ and the V_i's are all independent. Below we will introduce the variables μ_i and σ_i to be the mean and the standard deviation respectivly of the random variable V_i. Taking the expectation of T we find

$$\begin{aligned} E[T] &= \sum_{i=1}^{100} E[V_i] = \sum_{i=1}^{99} (E[X_i] + E[U_i]) + E[X_{100}] \\ &= 100 \cdot 5 + 99 \left(\frac{1}{4} \right) = 524.75 \,. \end{aligned}$$

In the same way the variance of this summation is also given by

$$\begin{aligned} \mathrm{Var}(T) &= \sum_{i=1}^{99} (\mathrm{Var}(X_i) + \mathrm{Var}(U_i)) + \mathrm{Var}(X_{100}) \\ &= 100 \cdot 5 + 99 \cdot \frac{1}{4} \left(\frac{1}{12} \right) = 502.0625 \,. \end{aligned}$$

By the central limit theorm we have that

$$P\left\{ \sum_{i=1}^{100} V_i \leq 550 \right\} = P\left\{ \frac{\sum_{i=1}^{100} (V_i - \mu_i)}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \leq \frac{550 - \sum_{i=1}^{100} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \right\} \,.$$

Where the variables μ_i and σ_i the means and standard deviations of the variables V_i. Calculating the expression on the right handside of the inequality above i.e.

$$\frac{550 - \sum_{i=1}^{100} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \,,$$

we find it equal to $\frac{550-524.75}{\sqrt{502.0625}} = 1.1269$. Therefore we see that

$$P\left\{\sum_{i=1}^{100} V_i \leq 550\right\} \approx \Phi(1.1269) = 0.8701\,,$$

using the Matlab function `normcdf`.

Problem 9 (how large n needs to be)

Warning: This result does not match the back of the book. If anyone can find anything incorrect with this problem please let me know.

A gamma random variable with parameters $(n, 1)$ is equivalent to a sum of n exponential random variables each with parameter $\lambda = 1$. i.e. $X = \sum_{i=1}^n X_i$, with each X_i an exponential random variable with $\lambda = 1$. This result is discussed in Example 3b Page 282 Chapter 6 in the book. Then the requested problem seems equivalent to computing n such that

$$P\left\{\left|\frac{\sum_{i=1}^n X_i}{n} - 1\right| > 0.01\right\} < 0.01\,.$$

which we will do by converting this into an expression that looks like the central limit theorem and then evaluate. Recognizing that X is a sum of exponential with parameters $\lambda = 1$, we have that

$$\mu = E[X] = E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n \frac{1}{\lambda} = n\,.$$

In the same way since $\text{Var}(X_i) = \frac{1}{\lambda^2} = 1$, we have that

$$\sigma^2 = \text{Var}(X) = \sum_{i=1}^n \text{Var}(X_i) = n\,.$$

Then the central limit theorem applied to the random variable X claims that as $n \to \infty$, we have

$$P\left\{\left|\frac{\sum_{i=1}^n X_i - n}{\sqrt{n}}\right| < a\right\} = \Phi(a) - \Phi(-a)\,.$$

or taking the requested probabilistic statement and converting it we find that

$$\begin{aligned}
P\left\{\left|\frac{\sum_{i=1}^n X_i}{n} - 1\right| > 0.01\right\} &= 1 - P\left\{\left|\frac{\sum_{i=1}^n X_i}{n} - 1\right| \leq 0.01\right\} \\
&= 1 - P\left\{\left|\frac{\sum_{i=1}^n X_i - n}{n}\right| \leq 0.01\right\} \\
&= 1 - P\left\{\left|\frac{\sum_{i=1}^n X_i - n}{\sqrt{n}}\right| \leq 0.01\sqrt{n}\right\} \\
&\approx 1 - (\Phi(0.01\sqrt{n}) - \Phi(-0.01\sqrt{n}))\,.
\end{aligned}$$

From the following identity on the cumulative distribution of a normal random variable we have that $\Phi(x) - \Phi(-x) = 1 - 2\Phi(-x)$, so that the above equals

$$1 - (1 - 2\Phi(-0.01\sqrt{n})) = 2\Phi(-0.01\sqrt{n}) \,.$$

To have this be less that 0.01 requires a value of n such that

$$2\Phi(-0.01\sqrt{n}) \leq 0.01 \,.$$

Solving for n then gives $n \geq (-100\Phi^{-1}(0.005))^2 = (257.58)^2$.

Problem 11 (a simple stock model)

Given the recurrence relationship $Y_n = Y_{n-1} + X_n$ for $n \geq 1$, with $Y_0 = 100$, we see that a solution to this is given by

$$Y_n = \sum_{k=1}^{n} X_k + Y_0 \,.$$

If we assume that the X_k's are independent identically distributed random variables with mean 0 and variance σ^2, we are asked to evaluate

$$P\{Y_{10} > 105\} \,.$$

Which we will do by transforming this problem into something that looks like an application of the central limit theorem. We find that

$$\begin{aligned}
P\{Y_{10} > 105\} &= P\{\sum_{k=1}^{10} X_k > 5\} \\
&= P\left\{\frac{\sum_{k=1}^{10} X_k - 10 \cdot (0)}{\sqrt{10}} > \frac{5 - 10 \cdot (0)}{\sqrt{10}}\right\} \\
&= 1 - P\left\{\frac{\sum_{k=1}^{10} X_k - 10 \cdot (0)}{\sqrt{10}} < \frac{5}{\sqrt{10}}\right\} \\
&\approx 1 - \Phi(\frac{5}{\sqrt{10}}) = 0.0569 \,.
\end{aligned}$$

Problem 19 (expectations of functions of random variables)

For each of the various parts we will apply Jensen's inequality $E[f(X)] \geq f(E[X])$ which requires $f(x)$ to be convex i.e. $f''(x) \geq 0$. Now since we are told that $E[X] = 25$ we can compute the following.

Part (a): For the function $f(x) = x^3$, we have that $f''(x) = 6x \geq 0$ since we are told that X is a nonnegative random variable. Thus Jensen's inequality gives

$$E[X^3] \geq 25^3 = 15625 \,.$$

Part (b): For the function $f(x) = \sqrt{x}$, we have that $f'(x) = \frac{1}{2\sqrt{x}}$, and $f''(x) = -\frac{1}{4\sqrt{x}} < 0$. Thus $f(x)$ is not a convex function but $-f(x)$ is. Applying Jensen's inequality to $-f(x)$ gives $E[-\sqrt{X}] \geq -\sqrt{25} = -5$ or

$$E[\sqrt{X}] \leq 5\,.$$

Part (c): For the function $f(x) = \log(x)$, we have that $f'(x) = \frac{1}{x}$, and $f''(x) = -\frac{1}{x^2} < 0$. Thus $f(x)$ is not a convex function but $-f(x)$ is. Applying Jensen's inequality to $-f(x)$ gives $E[-\log(X)] \geq -\log(25)$ or

$$E[\log(X)] \leq \log(25)\,.$$

Part (d): For the function $f(x) = e^{-x}$, we have that $f''(x) = e^{-x} > 0$. Thus $f(x)$ is a convex function. Applying Jensen's inequality to $f(x)$ gives

$$E[e^{-X}] \geq e^{E[X]} = e^{25}\,.$$

Chapter 8: Theoretical Exercises

Problem 1 (an alternate Chebyshev inequality)

Now the Chebyshev inequality is given by

$$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}\,.$$

Defining $k = \sigma\kappa$ the above becomes

$$P\{|X - \mu| \geq \sigma\kappa\} \leq \frac{\sigma^2}{\sigma^2\kappa^2} = \frac{1}{\kappa^2}\,,$$

which is the desired inequality.

Problem 12 (an upper bound on the complemetary error function)

From the definition of the normal density we have that

$$P\{X > a\} = \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx\,,$$

which we can simplify by the following change of variable. Let $v = x - a$ (then $dv = dx$) and the above becomes

$$\begin{aligned} P\{X > a\} &= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-(v+a)^2/2} dv \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-(v^2+2va+a^2)/2} dv \\ &= \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{v^2}{2}} e^{-va} dv \\ &\leq \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{v^2}{2}} dv \,, \end{aligned}$$

since $e^{-va} \leq 1$ for all $v \in [0, \infty)$ and $a > 0$. Now because of the identity

$$\int_0^\infty e^{-\frac{v^2}{2}} dv = \sqrt{\frac{\pi}{2}} \,,$$

we see that the above becomes

$$P\{X > a\} \leq \frac{1}{2} e^{-\frac{a^2}{2}} \,.$$

Problem 13 (a problem with expectations)

We are assuming that if $E[X] < 0$ and $\theta \neq 0$ such that $E[e^{\theta X}] = 1$, and want to show that $\theta > 0$. To do this recall Jensen's inequality which for a convex function f and an arbitrary random variable Y is given by

$$E[f(Y)] \geq f(E[Y]) \,.$$

If we let the random variable $Y = e^{\theta X}$ and the function $f(y) = -\ln(y)$, then Jensen's inequality becomes (since this function f is convex)

$$-E[\theta X] \geq -\ln(E[e^{\theta X}]) \,,$$

or using the information from the problem we have

$$\theta E[X] \leq \ln(1) = 0 \,.$$

Now since $E[X] < 0$ by dividing by this expression we have $\theta > 0$ as was to be shown.

Chapter 9 (Additional Topics in Probability)

Chapter 9: Problems

Problem 2 (helping Al cross the highway)

At the point where Al wants to cross the highway the number of cars that cross is a Poisson process with rate $\lambda = 3$, the probability that k cars appear in t time is given by

$$P\{N = k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!} .$$

Thus Al will have no problem in the case when *no* cars come during her crossing. If her crossing time takes s second this will happen with probability

$$P\{N = 0\} = e^{-\lambda s} = e^{-3s} .$$

Note that this is the density function for a Poisson random variable (or the cumulative distribution function of a Poisson random variable with $n = 0$). This expression is tabulated for $s = 2, 5, 10, 20$ seconds in `chap_9_prob_2.m`.

Problem 3 (helping a nimble Al cross the highway)

Following the results from Problem 2, Al will cross unhurt, with probability

$$P\{N = 0\} + P\{N = 1\} = e^{-\lambda s} + e^{-\lambda s}(\lambda s) = e^{-3s} + 3se^{-3s} .$$

Note that this is the cumulative distribution function for a Poisson random variable. This expression is tabulated for $s = 5, 10, 20, 30$ seconds in `chap_9_prob_3.m`.

Chapter 10 (Simulation)

Chapter 10: Problems

Problem 2 (simulating a specified random variable)

Assuming our random variable has a density given by

$$f(x) = \begin{cases} e^{2x} & -\infty < x < 0 \\ e^{-2x} & 0 < x < \infty \end{cases}$$

Lets compute the cumulative distribution $F(x)$ for this density function. This is needed if we simulate from f using the inverse transformation method. We find that

$$\begin{aligned} F(x) &= \int_{-\infty}^{x} e^{2\xi} d\xi \quad \text{for} \quad -\infty < x < 0 \\ &= \left. \frac{e^{2\xi}}{2} \right|_{-\infty}^{x} = \frac{1}{2} e^{2x} \,. \end{aligned}$$

and that

$$\begin{aligned} F(x) &= \frac{1}{2} + \int_{0}^{x} e^{-2\xi} d\xi \quad \text{for} \quad 0 < x < \infty \\ &= \frac{1}{2} + \left. \frac{e^{-2\xi}}{(-2)} \right|_{0}^{x} = 1 - \frac{1}{2} e^{-2x} \,. \end{aligned}$$

Then to simulate from the density $f(\cdot)$ we require the inverse of this cumulative probability density function. Since our F is given in terms of two different domains we will compute this inverse function in the same way. If $0 < y < \frac{1}{2}$, then the equation we need to invert i.e. $y = F(x)$ is equivalent to

$$y = \frac{1}{2} e^{2x} \quad \text{or} \quad x = \frac{1}{2} \ln(2y) \quad \text{for} \quad 0 < y < \frac{1}{2}$$

While if $\frac{1}{2} < y < 1$ then $y = F(x)$ is equivalent to

$$y = 1 - \frac{1}{2} e^{-2x} \,,$$

or by solving for x we find that

$$x = -\frac{1}{2} \ln(2(1-y)) \quad \text{for} \quad \frac{1}{2} < y < 1 \,.$$

Thus combining these two results we find that

$$F^{-1}(y) = \begin{cases} \frac{1}{2} \ln(2y) & 0 < y < \frac{1}{2} \\ -\frac{1}{2} \ln(2(1-y)) & \frac{1}{2} < y < 1 \end{cases}$$

Thus our simulation method would repeatedly generate uniform random variables $U \in (0, 1)$ and apply $F^{-1}(U)$ (defined above) to them computing the corresponding y's. These y's are guaranteed to be derived from our density function f.