Question 1

20 pts

A system can function for a random amount of time X. If the density of X is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

(i) What is the probability that the system functions for at least 5 months?
(ii) Find $\mathbb{E}(X)$.

Hint: keep exactly 4 decimal places after the decimal point.

(i) 0.2872

$f(x) \geq 5$

(ii) 4.

$$\int_0^\infty Cxe^{-\frac{x}{2}}\,dx = 1$$

$u \quad v'$

$u' = 1$

$v = -2e^{-\frac{x}{2}}$

$$\left[-2xe^{-\frac{x}{2}}\right]_0^\infty + \int_0^\infty 2e^{-\frac{x}{2}}\,dx = \frac{1}{c}$$

$$\int_0^\infty e^{-\frac{x}{2}}\,dx = \frac{1}{2c}$$

$$\left[-2e^{-\frac{x}{2}}\right]_0^\infty = \frac{1}{2c}$$

$$0 - (-2) = \frac{1}{2c}$$

$$2 = \frac{1}{2c}$$

$$4c = 1$$

$$C = \frac{1}{4}.$$

$$f(x) = \begin{cases} \frac{1}{4} x e^{-\frac{x}{2}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$$P(X \geq 5) = \int_5^\infty \frac{1}{4} x e^{-\frac{x}{2}} dx$$

$$= \left[-2x e^{-\frac{x}{2}} \right]_5^\infty + \int_5^\infty 2 e^{-\frac{x}{2}} dx$$

$$= 10 e^{-\frac{5}{2}} + 2 \left[-2 e^{-\frac{x}{2}} \right]_5^\infty$$

$$= 10 e^{-\frac{5}{2}} + 2 \left(2 e^{-\frac{5}{2}} \right)$$

$$= \frac{14}{4} e^{-\frac{5}{2}}$$

$$E(X) = \int_0^\infty \frac{1}{4} x^2 e^{-\frac{x}{2}} \, dx$$

$u \quad v'$

$u' = 2x$

$v = -2e^{-\frac{x}{2}}$

$$= \left[-2x^2 e^{-\frac{x}{2}}\right]_0^\infty + \int_0^\infty x e^{-\frac{x}{2}} \, dx$$

$u \quad v'$

$u' = 1$

$v = -2e^{-\frac{x}{2}}$

$$\cancel{\left[-2xe^{-\frac{x}{2}}\right]_0^\infty} + \int_0^\infty 2e^{-\frac{x}{2}} \, dx$$

$$2\left[-2e^{-\frac{x}{2}}\right]_0^\infty$$

$$= -4\left[-1\right]$$

$$= 4.$$

Question 2

10 pts

A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is arandom variable with probability density function

$$f(x) = \begin{cases} 5(1-x)^4, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

What must the capacity of the tank be so that the probability of the supply's being exhausted in a given week is 0.01?

Hint: keep exactly 4 decimal places after the decimal point.

$P(X \le a) \ge 0.99$

$\int_0^a 5(1-x)^4 dx = 1-(1-a)^5$

let $u = 1-x$,
$du = -dx$.

$-\int_1^{1-a} 5u^4 du = 5\int_{1-a}^{1} u^4 du$

$= 5\left[\frac{u^5}{5}\right]_{1-a}^{1}$

$= 1-(1-a)^5 \ge 0.99$

$0.01 \ge (1-a)^5$

$\frac{\log 0.01}{5} \ge \log(1-a)$

$\log(1-a) \le -0.4$

$1-a \le 10^{-0.4}$

$\le a$

Question 3 $X = \frac{2}{3}$ $\frac{2}{3}$ $\frac{3}{3}$ b a **20 pts**

A point is chosen at random on a line segment of length 1. Let X be the ratio of the shorter to the longer segment.
(i) Find $\mathbb{E}(X)$
(ii) Find $P(X < \frac{1}{4})$.

$\frac{-x}{1+x}$

Hint: keep exactly 4 decimal places after the decimal point.

(i) ~~0.5~~ 0.3862

(ii) 0.4

$f(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{otherwise.} \end{cases}$

$- \int_1^{0.5} \frac{1-x}{x} dx$

$\int_{0.5}^1 \frac{1}{x} - 1 \, dx$

$\int_0^{0.5} \frac{x}{1-x}$

X be the point. dist $= x$.
other part $= 1 - x$.

$\int_0^{0.5} \frac{x}{1-x} dx +$

ratio =

$x < 0.5$ $\int_{0.5}^1 \frac{1-x}{x} dx$

$\frac{x}{1-x} < \frac{1}{4}$

$x < \frac{1}{5}$ $\frac{2}{5}$

$x > 0.5$ $\int_{0.5}^1 \frac{1}{x} - 1 \, dx$

$= 2\ln|0.5| - 0.5 \quad 1$

$\frac{x}{1-x} <$

$\int_0^{0.5} \frac{x}{1-x} dx + \int_{0.5}^1 \frac{1-x}{x} dx$

$f_X(x) =$ $E(X) = \int x f(x) dx$

$\frac{a}{b},\ a<b,\ <1$

$=$

$\text{ratio} = \frac{x}{1-x}$

$f_X(x) \cdot$

$$\int_0^{0.5} \frac{x^2}{1-x} dx + \int_{0.5}^{1} 1-x \, dx$$

$u = 1-x$

$$= -\int_1^{0.5} \frac{(1-u)^2}{u} du + \left[x - \frac{x^2}{2}\right]_{0.5}^{1}$$

$$= \int_{0.5}^{1} \frac{1-2u+u^2}{u} du + (1-0.5) - (0.5 - 0.125)$$

$$= \int_{0.5}^{1} \frac{1}{u} du - 2\int_{0.5}^{1} du + \int_{0.5}^{1} u \, du + 0.125$$

$$= [\ln|u|]_{0.5}^{1} - 2(0.5) + \left[\frac{u^2}{2}\right]_{0.5}^{1} + 0.125$$

$$= -\ln(0.5) - 1 + \frac{1}{2} - \frac{0.5^2}{2} + 0.125 = \ln 2 - 0.5$$

$$\int_0^{0.5} \frac{x}{1-x}\,dx + \int_{0.5}^{1} \frac{1-x}{x}\,dx$$

$\Downarrow$ $u = 1-x$

$$= \int_{0.5}^{1} \frac{1-u}{u}\,du + \int_{0.5}^{1} \frac{1-x}{x}\,dx$$

$$= 2\int_{0.5}^{1} \frac{1}{x} - 1\,dx$$

$$= 2\left(\left[\ln|x| - x\right]_{0.5}^{1}\right)$$

$$= 2\left(-1 - (\ln 0.5 - 0.5)\right)$$

$$= 0.3862$$

$$P(X=x) = \frac{r}{1-r} \text{ if } r<x$$
$$\frac{1-r}{r} \text{ if } r>x$$

$$Y = \frac{X}{1-X}$$

$$P(Y<t)$$

$$\frac{x}{1-x} < \frac{1}{4} \qquad\qquad \frac{1-x}{x} < \frac{1}{4}$$

$$P(X \le x) = F_X(x) = \begin{cases} \dfrac{r}{1-r} & \text{if } 0 < r < \\ \dfrac{1-r}{r} & \text{if} \end{cases}$$

$r \sim U(0,1)$

x | $1-x$ $\quad \frac{1}{4}$.

x | $1-x$ $\quad \frac{1}{4}$.

$$P\left(X < \tfrac{1}{4}\right) = P\left(\frac{r}{1-r} < \frac{1}{4}\right) = P\left(r < \tfrac{1}{5}\right)$$

$$P\left(\frac{1-r}{r} < \frac{1}{4}\right) = P\left(r > \tfrac{4}{5}\right)$$

Solution: Assume that X is a random variable that is equal to the distance from the right hand hand of the line. Then if we assume that X is uniformly distributed over $(0, L)$. The probability that we are interested in is that:

$$
\begin{aligned}
& P\left\{min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) < \frac{1}{4}\right\} \\
= \; & 1 - P\left\{min\left(\frac{X}{L-X}, \frac{L-X}{X}\right) \geq \frac{1}{4}\right\} \\
= \; & 1 - P\left\{\frac{X}{L-X} \geq \frac{1}{4} \text{ and } \frac{L-X}{X} \geq \frac{1}{4}\right\} \\
= \; & 1 - P\left\{X \geq \frac{L}{5}, \text{ and } X \leq \frac{4L}{5}\right\} \\
= \; & 1 - P\left\{\frac{L}{5} \leq X \leq \frac{4L}{5}\right\} \\
= \; & 1 - \int_{\frac{L}{5}}^{\frac{4L}{5}} \frac{1}{L} dx = 1 - \frac{3}{5} = \frac{2}{5}.
\end{aligned}
$$

$X = \min\left(\frac{R}{1-R}, \frac{1-R}{R}\right)$ where $0 < x < 1$

$$
\begin{aligned}
P(X \leq x) &= 1 - P\left(\min\left(\frac{R}{1-R}, \frac{1-R}{R}\right) \geq x\right) \\
&= 1 - P\left(\frac{R}{1-R} \geq x \text{ and } \frac{1-R}{R} \geq x\right) \\
&= 1 - P\left(R \geq x - xR \text{ and } 1-R \geq xR\right) \\
&= 1 - P\left((x+1)R \geq x \text{ and } 1 \geq (x+1)R\right) \\
&= 1 - P\left(R \geq \frac{x}{x+1} \text{ and } R \leq \frac{1}{x+1}\right) \\
&= 1 - P\left(\frac{x}{x+1} \leq R \leq \frac{1}{x+1}\right) \\
&= 1 - \int_{\frac{x}{x+1}}^{\frac{1}{x+1}} dR = 1 - \left(\frac{1}{x+1} - \frac{x}{x+1}\right) = \frac{x-1}{x+1}
\end{aligned}
$$

$P(x \geq x) =$

$$E(X) = \int_0^1 \frac{1-x}{x+1} dx$$

$=$

$$\int_0^1 \frac{1-x}{x+1}\,dx$$

$$= \int_0^1 \frac{1}{x+1} - \frac{x}{x+1}\,dx$$

$$= \ln|x+1| - \int_0^1 \frac{x}{x+1}\,dx \qquad \text{let } u = x+1,\ du = dx$$

$$= \left[\ln(x+1)\right]_0^1 - \int_1^2 \frac{u-1}{u}\,du$$

$$= \ln 2 - \int_1^2 1 + \int_1^2 \frac{1}{u}\,du$$

$$= \ln 2 - [u]_1^2 + [\ln u]_1^2$$

$$= \ln 2 - 2 + 1 + \ln 2$$

$$= 2\ln 2 - 1$$

Question 4

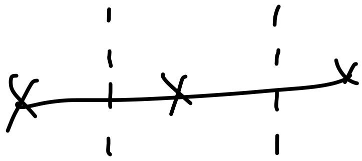
20 pts

A bus travels between the two cities A and B, which are 100 miles apart. If the bus has a breaking down, the distance from the breakdown to city A has a uniform distribution over $(0, 100)$. There is a bus service station in city A, in B, and in the center of the route between A and B. Let Y be the distance from the breakdown to the closet service station. Calculate E(Y).

Hint: keep exactly 4 decimal places after the decimal point.

E(Y)= 12.5

If $Y < 25$



Question 5 20 pts

Let X be a uniform $(0,1)$ random variable. For any given positive integer n, compute $\mathbb{E}(X^n)$

Hint: keep exactly 4 decimal places after the decimal point.

When n=1: 2.

When n=6: ~~0.1428~~ 0.2857

$$P_X(X) = \begin{cases} 1 & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$Y = X^6$, $F_Y(y) = P(Y \le y)$ $\quad 0 < y < 1$

$= P(X^6 \le y) = P(-\sqrt[6]{y} \le X \le \sqrt[6]{y})$

$$F_Y(y) = \int_{-\sqrt[6]{y}}^{\sqrt[6]{y}} dx = 2\sqrt[6]{y}$$

$2y^{\frac{1}{6}}$

$$f_Y(y) = \frac{1}{3}y^{-\frac{5}{6}}$$

$$\int_0^1 \frac{1}{3}y^{\frac{1}{6}}\,dy = \frac{1}{3}\left[\frac{6}{7}y^{\frac{7}{6}}\right]_0^1$$

$$= \frac{1}{3}\left(\frac{6}{7}\right) =$$

Question 6

25 pts

If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

(a) $P(X > 5)$; (b) $P(4 < X < 16)$; (c) $P(X < 8)$;
(d) $P(X < 20)$; (e) $P(X > 16)$.

Hint: keep exactly 4 decimal places after the decimal point.

Hint: use the standard normal table.

(a) 0.7967

(b) 0.6826

(c) 0.3707

(d) 0.9525

(e) 0.1587

-0.81

a). $P(Z > \frac{5-10}{6})$

$P(Z > -0.833)$

b). $P(\frac{4-10}{6} < Z < \frac{16-10}{6})$

$= P(-1 < Z < 1)$

c). $P(Z < \frac{8-10}{6})$

$P(Z < -0.33)$

d). $P(Z < \frac{20-10}{6})$

$= P(Z < 1.67)$

e). $P(Z > 1)$.

Question 7

10 pts

Suppose that X is a normal random variable with mean 5. If $P(X > 9) = 0.2005$ approximately what is $\mathrm{Var}(X)$?

Hint: keep exactly 4 decimal places after the decimal point.

Hint: use the standard normal table.

Var(X)= 22.6757

$P(Z > \frac{9-5}{\sigma}) = 0.2005$

$\frac{9-5}{\sigma} = 0.84$

$=$

Question 8

10 pts

Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that $P(X > c) = 0.1003$

Hint: keep exactly 4 decimal places after the decimal point.

Hint: use the standard normal table.

c= 14.56

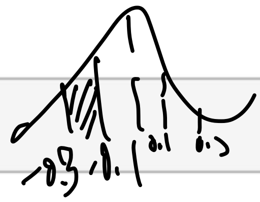
$P(Z > \frac{c-12}{2}) = 0.1003$

$\frac{c-12}{2} = 1.28$

$=$

Question 9

20 pts



$>0.1 = 0.4602$

$>0.3 = 0.3821$

Use normal approximation of Binomial distribution to answer the following question: If a fair coin is flipped 100 times,

(i): what is the probability of observing 49 Heads?

(ii): what is the probability of observing strictly more than 57 Heads?

$Bin(100, 0.5)$

Hint: keep exactly 4 decimal places after the decimal point.

Hint: use the standard normal table.

$\frac{X-np}{\sqrt{npq}} \sim Z.$

(i) 0.0781

(ii) 0.0668

$P(X>57)$

$P(Z > \frac{57.5-50}{5})$

$\frac{48.5-50}{\sqrt{25}} < Z < \frac{49.5-50}{5}$

$-0.3 < Z < -0.1$

Question 10

10 pts

If X is an exponential random variable with parameter $\lambda = 1$, compute the probability density function of the random variable Y defined by $Y = \log X$.

- $f_Y(y) = e^{y-e^y}$, $y \in (0, \infty)$
- $f_Y(y) = e^{y-e^y}$, $y \in (-\infty, \infty)$
- $f_Y(y) = ye^{-e^y}$, $y \in (0, \infty)$
- $f_Y(y) = e^{-e^y}$, $y \in (0, \infty)$
- $f_Y(y) = e^{-e^y}$, $y \in (-\infty, \infty)$

$X \sim Exp$

$\begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

$Y = \log X,$

$F_Y(y) = P(Y \geq y) = P(\log X \geq y) =$

$P(X \geq e^y)$

$\int_0^{e^y} e^{-x} dx$

$$\int_0^{e^y} e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_0^{e^y}$$

$$F_Y(y) = -e^{-e^y} + 1$$

$$f_Y(y) = -e^{-e^y} \frac{d}{dy}(-e^y)$$

$$= e^{-e^y}(e^y)$$

$$= e^{-e^y + y}$$

$$e^{y-e^y}$$