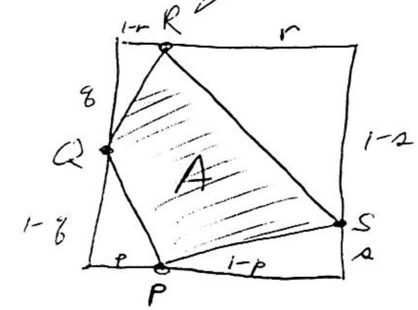
Problem: Take a unit square and choose points randomly and uniformly on each side of the square: P on one side, Q, on another, R, and S ditto. What is the probability that the quadrilateral PQRS will have area greater than 1/2?

(1)

Draw a picture



$A < 1/2$
$A = 1/2$ Almost surely won't occur!
$A > 1/2$

label points and lengths

Formula for shaded region?
Easier to get ~~area~~ of Triangle and
ASK $Pr(\underbrace{\text{Area of triangles}}_{A_T}) < 1/2$

Turn into an algebraic problem

$$A_T < 1/2$$

is the same as

$$\frac{1}{2}p(1-q) + \frac{1}{2}q(1-r) + \frac{1}{2}r(1-s) + \frac{1}{2}s(1-p) < 1/2$$

$$p(1-q) + q(1-r) + r(1-s) + s(1-p) < 1$$

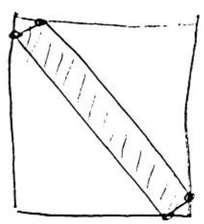
Unclear where to go from here
Dead End?

(2)

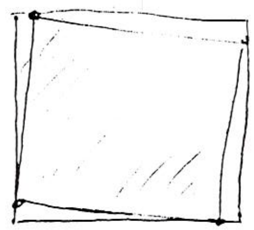
Wild idea: Would the corresponding problem for a triangle* be easier? Not even clear how it would be stated.

*Equilateral or Isoc. Rt.?

What is the range of possible values for A (or A_T) (Is A always 1/2, No!)



Small A
Could be as small as you like

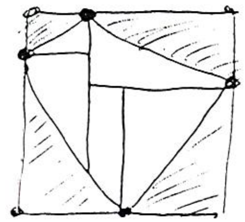


Large A
Could be made as close to 1 as you like

Which is more likely?

Could outer triangle be arranged so ~~that~~ they are the interior quadrilaterals and the interior quad. is cut up into triangles?

(3)



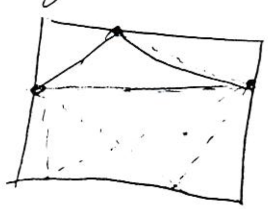
"Dissect" the interior

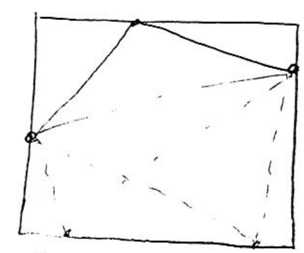
No way to ~~break~~ it into ~~4~~ right triangles

Also not clear how to make a quadrilateral out of the outer triangles

If this had worked, every interior quad. with area > 1/2 could have been matched with one having area < 1/2 so that Pr. = 1/2

Leave ~~three~~ points fixed and let one vary

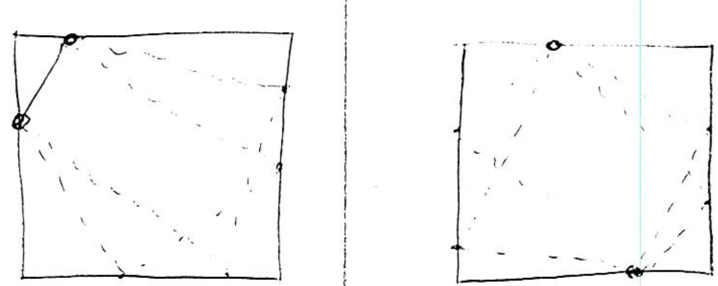




Varying the point on this side varies A somewhat, but other 3 points determine size quite a bit.

(4)

Vary two points, leaving the others fixed

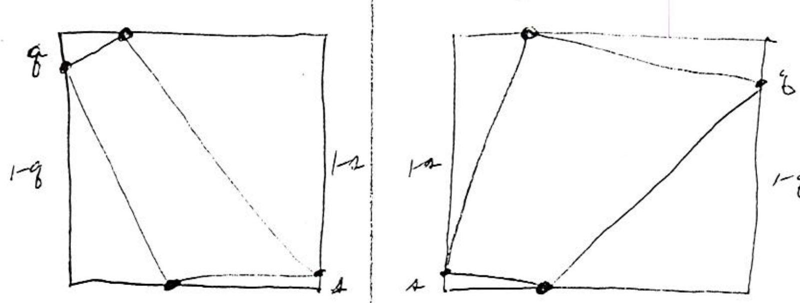


fixed points adjacent

fixed points opposite

Flash!

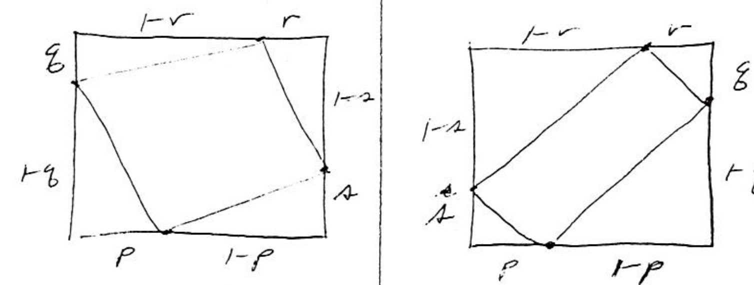
Small A $\xrightarrow{?}$ Large A



Exchange side points and it <u>looks</u> as though the small A case turns into a large A case

(5)

Check Details



Add Areas of triangles A_T and A_T' (left) (right)

$$A_T = \frac{1}{2}p(1-q) + \frac{1}{2}q(1-r) + \frac{1}{2}r(1-s) + \frac{1}{2}s(1-p)$$

$$A_T' = \frac{1}{2}ps + \frac{1}{2}(1-s)(1-r) + \frac{1}{2}rq + \frac{1}{2}(1-q)(1-p)$$

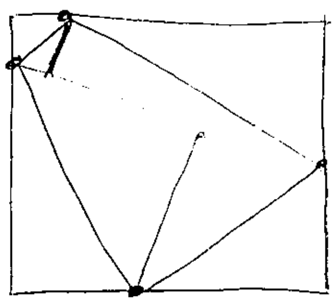
$$A_T + A_T' = \left[\frac{1}{2}(ps) + \frac{1}{2}s(1-p)\right] + \left[\frac{1}{2}(1-s)(1-r) + \frac{1}{2}(r)(1-s)\right]$$
$$+ \left[\frac{1}{2}rq + \frac{1}{2}q(1-r)\right] + \left[\frac{1}{2}(1-q)(1-p) + \frac{1}{2}p(1-q)\right]$$
$$= \frac{1}{2}s\left(p+(1-p)\right) + \frac{1}{2}(1-s)\left((1-r)+r\right)$$
$$+ \frac{1}{2}q\left(r+(1-r)\right) + \frac{1}{2}(1-q)\left((1-p)+p\right)$$
$$= \frac{1}{2}s + \frac{1}{2}(1-s) + \frac{1}{2}q + \frac{1}{2}(1-q)$$
$$= \frac{1}{2} + \frac{1}{2} = 1$$

$$A_T + A_T' = \frac{1}{2}$$

$\text{Prob}(A_T < \frac{1}{2}) = \frac{1}{2}$ Done!

6

Postscript: Dissection might be possible?



Inner quad into 4 Rt triangles

But do they fit in the exterior?