

A Problem Solving Course

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Preface

This book is designed for use in a university course for students who are not at the Putnam Fellow level, but who aspire to compete in the Putnam. If you belong to the faculty of a college or university, you may request a pdf with solutions to many of the problems in this book. Do this by sending me proof in the form of a URL listing you as faculty that includes an email address that matches the email address of your message. Send it to kenneth_levasseur@uml.edu.

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Chapter 1

Format for a Problem Solving Class

This chapter contains notes on the way I teach a problem solving class. I'm sure that there are plenty of variations that work at least as well, and I hope this will provide the reader with ideas for their own system. This was originally written before the COVID pandemic. One semester was spent in a virtual format, which didn't go so well, but some revisions have been made since then.

The course runs in the Fall, culminating in the William Lowell Putnam Competition on the first Saturday in December. We meet once a week for 2 hours and 45 minutes. The first 13 weeks of the semester are before the Putnam, and we meet once after the Putnam to debrief.

1.1 Before the first class

Students in my class are required to compete in the Putnam but in a typical year there are some students who are not in class who are interested in the Putnam. Since it's never too early to start preparing, I post warm-up problems, most recently on Perusall. [I had used Piazza in the past but their cost of access increased.] As students register for the class, I add them to the list, which also includes the non-registered students. I post problems that are not quite up to the difficulty of Putnam problems - in fact many are way easier. However, this lets students rediscover some basic concepts. Solutions are available to students even if they register late, and so these problems serve as a starting point to the course.

1.2 The first class

In the first class, we review two basic proof tools, mathematical induction and the pigeonhole principle. I suggest starting the class by reviewing these two topics through examples. I try to be as careful as possible in writing the solutions in a logically coherent form, setting the tone for what I will expect to see from students. The choice of Fermat's Little Theorem for the induction proof also anticipates a later class in which we do Number Theory. I don't dwell on these examples any longer than necessary, shooting for a total time of around 30 minutes.

I normally divide the class into groups of 2, 3 or 4 to work on problems. Exactly how this is done depends on class size, and other factors. The first

problem, dealing with the sum of the first n odd positive integers, is a warm-up that I expect most students to have seen it already. I instruct the class to work on this one first with the expectation that most groups will complete it within ten minutes. I circulate among the groups to help anyone who is struggling with the first problem. The rest of the problems are assigned to different groups. The groups are encouraged to put solutions on the board when they are ready. When a few problems have been written out, we take a break from solving to review the solutions. Groups that have completed a problem are assigned a new one and we continue until the period is almost over. Normally a few problems are left unsolved and they are assigned for homework.

1.3 Subsequent classes

Starting with the second class, students are instructed to read the introductory text of a *chapter* before the class and to work on selected problems at the end of that chapter. The readings are posted on Perusall and their reading activity and work on the problems is graded. In class meetings, we discuss the problems and students work together on more of the problems at the end of the chapter. Students should be aware that not every solution to problems that are posed in a given week use the technique that is highlighted in that week. Additionally, time allotted to discuss any progress on past problems.

Most other classes have roughly the same format as the second one.

One tip that you might keep in mind is that you should enough board space (preferably slate) to accomodate solutions from all groups who are going to post a solution to a problem. In my case, if I have N students, I'd ideally like at least $6 \times \frac{N}{3} = 2N$ board feet in the room. It's not uncommon to have classrooms with hardly any board space, so being proactive as far as this is concerned is worth while. One semester, I was given a high tech classroom with electronic whiteboards. That was not good!

1.4 Some “Prerequisite” Topics

With the assumption that we have to start somewhere, I give students a list, certainly not complete, of topics that expect students to have a good grasp of to participate in the class. This list is far from complete, but highlights topics that appear in many of the problems we encounter. Most of the topics from calculus are not on the list, but are also assumed.

List 1.4.1 Prerequisites

- Polynomial operations and properties.
- Properties of the complex numbers
- Geometric series, both finite and infinite. Alternating series.
- Matrix Algebra
- Basic Geometry
- The definition of the trigonometric functions as circular functions, basic trigonometric identities. Laws of sine and cosine.

There are several other topics could be listed here but I do spend a short amount of time reviewing them, such as induction and modular arithmetic.

Topics that are not addressed but could be added in the future:

- Discrete Calculus
- Convexity, Jensen's Inequality
- Symmetry

One area that seems off limits to the Putnam is statistics.

Chapter 2

Induction and Pigeonholes

Mathematical induction and the pigeonhole principle are two common proof techniques that students should have seen. They are good warm-up topic for the first class.

2.1 Mathematical Induction

We illustrate a basic proof by induction by proving Fermat's Little Theorem.

Theorem 2.1.1 Fermat's Little Theorem. *Let p be a prime number, and n an integer. Then $n^p - n$ is divisible by p .*

Proof. The Theorem is clearly true for the case of $p = 2$ since any integer and its square have the same odd/even parity and their difference must be even. We now assume that p is odd. The theorem is clearly true for $n = 0$. This forms the basis for an induction proof for positive values of n . Assume for some $n \geq 0$, that $n^p - n$ is divisible by p . Now consider $(n + 1)^p - (n + 1)$.

$$\begin{aligned}(n + 1)^p - (n + 1) &= \left(n^p + \sum_{k=1}^{p-1} \binom{p}{k} n^{p-k} + 1 \right) - n - 1 \\ &= (n^p - n) + \sum_{k=1}^{p-1} \binom{p}{k} n^{p-k}\end{aligned}$$

The first term is divisible by p by the induction hypothesis, and the second is also divisible by p because $\binom{p}{k}$ is divisible by p for $k = 1, 2, \dots, p - 1$. Therefore, the theorem is true for all positive values of n . Finally, given the truth of the theorem for positive integers, it is clearly true for negative values. ■

There may appear to be gaps in the proof from the point of view of a typical undergraduate course. Saying as we did at the end that something is clearly true may not be acceptable to some instructors in some courses. However, in the context of preparing for the Putnam, knowing how basic your arguments need to be is important. In this author's opinion the negative case is sufficiently obvious to skip the details. The same is true of the statement that $\binom{p}{k}$ is divisible by p for $k = 1, 2, \dots, p - 1$.

Our next example will be about connected undirected planar graphs.

List 2.1.2 What do these words mean?

Undirected	Edges have no direction assigned, normally represented as a two element subset of the vertex set.
Connected	For any two vertices, there is a sequence of edges that provide a path between them.
Planar	It is possible to draw the graph on a plane so that no two edges cross.

Figure 3 is an example of one such graph.

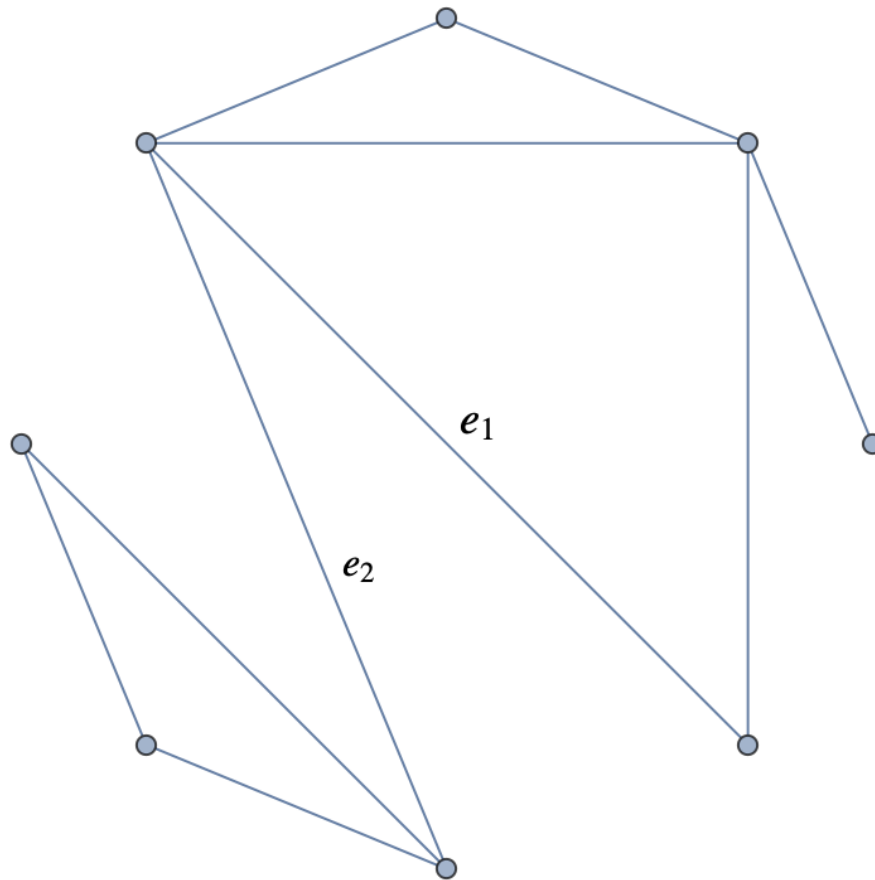


Figure 2.1.3 An example of a connected undirected planar graph

Example 2.1.4 Euler's Formula. Euler's Formula states that if G is a connected undirected planar graph, with v vertices, e edges and r regions, then $v + r - e = 2$

We will prove this by induction on the number of edges in the graph. Let $p(n)$ be "The statement is true for all such graphs with n edges, where $n \geq 0$."

Basis: If a connected undirected planar graph has zero edges, it must be that the graph has only one vertex and the plane is "divided" into only one region. Therefore $v = 1$, $e = 0$, and $r = 1$; and $v + r - e = 1 + 1 - 0 = 2$.

Induction. Assume Euler's formula is true for all connected undirected planar graphs with k edges where $k \leq n$. Assume we such a graph, G , with

$n + 1$ edges. We remove one of the edges from G , making it an undirected planar graph with n edges. However it may no longer be connected. We need to consider two possible cases.

Case 1: The graph with the removed edge is still connected. In the example above, if we remove e_1 , we have this situation. Since we have a connected undirected planar graph with n edges, the induction hypothesis applied with $k = n$, and so Euler's formula is true. Now, consider returning the edge we had removed. In so doing, the number of edges increases by 1, the number of vertices stays the same, and the number of regions increases by 1 since the returned edge divides a region into two parts. Therefore, the net change in the expression $v + r - e$ is $0 + 1 - 1 = 0$ and so it is still equal to 2.

Case 2: The graph with the removed edge has two components. In the example above, if we remove e_2 , we have this situation. The vertices in each of the two components are connected. Assume the components have k_1 and k_2 edges, where $k_1 + k_2 = n$. Furthermore, assume the numbers of vertices and regions in the first component is v_1 and r_1 , while the second component has v_2 and r_2 for its vertices and regions. By the induction hypothesis,

$$v_1 + r_1 - k_1 = 2 \text{ and } v_2 + r_2 - k_2 = 2$$

Now when we bring back the edge we had removed, the original graph had $v_1 + v_2$ vertices and $k_1 + k_2 + 1 = n + 1$ edges. The number of regions is $r_1 + r_2 - 1$ because the infinite regions of the two components is now one region. Collecting this information, we have

$$\begin{aligned} v + r - e &= (v_1 + v_2) + (r_1 + r_2 - 1) - (k_1 + k_2 + 1) \\ &= (v_1 + r_1 - k_1) + (v_2 + r_2 - k_2) - 2 \\ &= 2 + 2 - 2 = 2 \end{aligned}$$

Therefore Euler's formula is true for this case. \square

Example 2.1.5 Recursive Thinking. Consider the sequence of matrices $A(n)$ defined by $A(1) = [1]$ and for $n \geq 1$, $A(n)$ is an $(n - 1) \times (n - 1)$ identity matrix with a column of 1's appended to it, and then a row of 1's appended.

For example, $A(3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. What is the determinant of $A(n)$?

To solve this problem, we can expand the determinant along the first row. This gives us two minors to evaluate, one of which is the determinant of $A(n - 1)$. The other minor is a matrix that takes a form like this one, in the case where

we are expanding $|A(5)|$: $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

By shifting the bottom row up, one row at a time, with $n - 2$ shifts, we get an upper triangular matrix with all 1's in the diagonal, and so the minor is $(-1)^{(n-2)}$. The cofactor corresponding to this minor will have a sign $(-1)^{n+1}$. Therefore the value of the cofactor will be $(-1)^{n+1} \cdot (-1)^{(n-2)} = -1$ and so we have

$$|A(n)| = |A(n - 1)| - 1.$$

Since $|A(1)| = 1$, we conclude that $|A(n)| = 2 - n$.

Occasionally, an assist from technology will jump-start a solution. In this example, it might not be necessary. However, here is a verification that our solution was correct, or if you hadn't seen the recursion, it might serve as a

hint. Change the value of n to see how the determinant depends on it.

```
n=5
A=Matrix([[i==j or i==n-1 or j==n-1 for i in range(n)] for j
          in range(n)] )
[A,det(A)]
```

```
[[1 0 0 0 1]
 [0 1 0 0 1]
 [0 0 1 0 1]
 [0 0 0 1 1]
 [1 1 1 1 1], -3]
```

□

2.2 The Pigeonhole Principle



Figure 2.2.1 More pigeons than pigeonholes. Image by en:User:McKay CC-BY-SA-3.0 via Wikimedia Commons.

Informally, the Pigeonhole Principle says that if you have more pigeons than you have pigeonholes, then at least one pair of pigeons have to share the same pigeonhole. The trick to using this simple idea is to identify the pigeons and pigeonholes in a problem. Here is a relatively simple example. A clue that we can use the Pigeonhole Principle problem in this case is that we are asked to prove the existence of two objects that share a property.

Example 2.2.2 Prove that every set of 10 two-digit integer numbers has two disjoint subsets with the same sum of elements. (Subsets are pigeons, possible sums are holes)

We want two subsets with the same sum, so consider the possible sums to be pigeonholes, and now picture subsets of our ten integer set to be the pigeons.

We are not given a specific set of integers, but we can identify the range of possible sums of elements in their subsets. The smallest sum is 10 while the largest proper subset is $91 + 92 + \cdots + 99 = 855$. That gives us 846 pigeonholes. There are $2^{10} - 2 = 1022 > 845$ non-empty proper subsets of a ten element set. Therefore, if we imagine each subset to roost with its sum, we are sure there there must be two subsets with the same sum. Now those two subsets may not be disjoint, but if we simply remove elements in their intersection, we are done. \square

2.3 Problems

1. Find and prove a formula for the sum of the first n consecutive odd positive integers. For example, if $n = 4$ then $1 + 3 + 5 + 7 = 16$.

Hint. Look at the first few examples.

2. Prove that in a room with n people, at least two people know exactly the same number of people. Assume knowing is a symmetric relation: If Paul knows Pat, then Pat knows Paul.

Hint. It is impossible to have someone know $n - 1$ people and someone else know nobody in the room.

3. Let S be any set of 18 distinct integers chosen from the arithmetic progression $1, 4, 7, \dots, 100$. Prove that there must be two integers in S whose sum is 101.

Hint. Count the number of pairs that can add up to 101.

4. Choose 51 positive integers from 1 to 100. Prove that one of them is a multiple of another.

Hint. Each integer has a maximal odd factor.

5. Show that a $2^n \times 2^n$ square with a corner tile removed can be covered without overlaps by L-shaped figures (each figure contains 3 tiles). (If you feel adventurous, how about an $n \times n$ square for arbitrary n ?)
6. Given nine points inside the unit square, with no three colinear, prove that some three of them form a triangle whose area does not exceed $1/8$.
7. Prove that every positive integers is the sum of distinct nonconsecutive Fibonacci numbers.
8. Inside a circle of radius 4 are chosen 61 points. Show that among them there are two at distance at most $\sqrt{2}$ from each other.
9. Show that there is a positive term of the Fibonacci sequence that is divisible by 1000. Recall that the Fibonacci sequence is defined by $F_0 = 0$, $F_1 = 1$ and for $n > 1$, $F_n = F_{n-1} + F_{n-2}$. Note: The starting values of 0 and 1 are often replaced with 1 and 1, but for this problem it's slightly easier to structure a proof.

Hint. Start with two pairs of consecutive terms that have identical remainders mod 1000.

10. You have coins C_1, C_2, \dots, C_n . For each k , C_k is biased so that, when tossed, it has probability $1/(2k + 1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n .
11. Prove that any positive integer can be represented as $\pm 1^2 \pm 2^2 \pm 3^2 \pm \cdots \pm n^2$ for some positive integer n and some choice of signs.
12. A fair coin is tossed repeatedly until there is a run of an odd number of heads followed by a tail. Determine the expected number of tosses.

- Hint.** Identify beginnings of the process that either end it or “reset” it.
13. Show that every set of n integers has a nonempty subset such that the sum of its elements is divisible by n .
 14. Show that for any polynomial $f(x)$ over the integers with degree $d < n$, we have $\sum_{k=0}^n (-1)^k \binom{n}{k} f(k) = 0$.
 15. Given a sequence of integers x_1, x_2, \dots, x_n whose sum is 1, prove that exactly one of the cyclic shifts

$$\begin{array}{c}
 x_1, x_2, \dots, x_n \\
 x_2, \dots, x_n, x_1 \\
 \vdots \\
 x_n, x_1, \dots, x_{n-1}
 \end{array}$$

has all of its partial sums positive. (By a partial sum we mean the sum of the first k terms, $k \leq n$.)

Chapter 3

Tricks of the Trade

In this chapter we highlight some “tricks of the trade” that can help you piece together a solution to many problems. Naturally, no single trick is likely to be needed in a given competition, but all of them have been played a part in solving some problems in past competitions.

3.1 Telescoping Sums

Telescoping sums are occasionally embedded in more challenging problems than this one.

Example 3.1.1 Let $f(n) = \sum_{k=1}^n \frac{1}{\sqrt{k} + \sqrt{k+1}}$. Evaluate $f(9999)$.

Rationalizing the denominator reveals a **telescoping sum** here. For any nonnegative k ,

$$\begin{aligned} \frac{1}{\sqrt{k} + \sqrt{k+1}} &= \frac{\sqrt{k+1} - \sqrt{k}}{(\sqrt{k} + \sqrt{k+1})(\sqrt{k+1} - \sqrt{k})} \\ &= \frac{\sqrt{k+1} - \sqrt{k}}{k+1-k} \\ &= \sqrt{k+1} - \sqrt{k} \end{aligned} .$$

So $f(9999) = \sum_{k=1}^{9999} (\sqrt{k+1} - \sqrt{k}) = \sqrt{9999+1} - \sqrt{1} = 100 - 1 = 99$. (Solution by Tung Nguyen) \square

3.2 Completing the Product

The elementary trick of completing the square, given a quadratic and linear term, can be generalized. Given three terms, completing the product of two binomials can play a part in a solution. This occurred in the 2018 Putnam, problem A1.

Find all ordered pairs (a, b) of positive integers for which

$$\frac{1}{a} + \frac{1}{b} = \frac{3}{2018}.$$

Elementary algebra leads to

$$3ab - 2018(a + b) = 0$$

By distributing the 2018 we have three terms. Multiplying by 3 and adding 2018^2 to both sides completes a product of two binomial factors on the the left side:

$$(3a - 2018)(3b - 2018) = 2018^2.$$

On year numbers.. Problem posers often like to involve the current year in at least one problem in a competition; so it's a good idea to be familiar with the current year's factorization.

From the Putnam Archive:

With this equation, we can identify solutions the original equation in the positive integers. Each of the factors is congruent to 1 (mod 3). There are 6 positive factors of $2018^2 = 2^2 \cdot 1009^2$ that are congruent to 1 (mod 3): 1, 2^2 , 1009, $2^2 \cdot 1009$, 1009^2 , $2^2 \cdot 1009^2$. These lead to the 6 possible pairs: $(a, b) = (673, 1358114)$, $(674, 340033)$, $(1009, 2018)$, $(2018, 1009)$, $(340033, 674)$, and $(1358114, 673)$.

As for negative factors, the ones that are congruent to 1 (mod 3) are -2 , $-2 \cdot 1009$, $-2 \cdot 1009^2$. However, all of these lead to pairs where $a \leq 0$ or $b \leq 0$.

3.3 Trig Substitution

When real numbers are known to have absolute value less than or equal to 1, consider equating them with cosines or sines. Here is an example from the 2000 Putnam, Problem B4.

Example 3.3.1 2000 Putnam, B4. Let $f : [1, 1] \rightarrow \mathbb{R}$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x . Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

Solution: Substitution of 1 and -1 for x produces $f(1) = f(-1) = 0$.

The restriction on x and the expression $2x^2 - 1$ may remind one of the identity $2\cos^2 x - 1 = \cos 2x$; and so the substitution $x = \cos t$ is a reasonable step leading to $f(\cos 2t) = 2\cos t f(\cos t)$.

The next step in several published solutions is to define a second function, g , by

$$g(t) = \frac{f(\cos t)}{\sin t}, \text{ where } t \neq \pi k, k \in \mathbb{Z}$$

A “meta exercise” is to provide a motivation for this definition of g .

$$g(2t) = \frac{f(\cos 2t)}{\sin 2t} = \frac{2\cos t f(\cos t)}{2\sin t \cos t} = g(t)$$

Combining this with the periodicity of g tells us that

$$g\left(1 + \frac{n\pi}{2^k}\right) = g(2^{k+1} + 2\pi n) = g(2^{k+1}) = g(1)$$

The continuity of g in its domain and the density of $\{1 + \frac{n\pi}{2^k} \mid n, k \in \mathbb{Z}\}$ implies that g must be constant.

Returning our attention to f we have $f(\cos t) = c \sin t$ for some constant c , which implies $f(x) = c\sqrt{1-x^2}$ for $x \in (-1, 1)$. Finally, this tells us that f must be even. However, when we turn to the original functional equation, $xf(x) = f(2x^2 - 1)$ we have an odd function on the left and an even function on the right. Our only possibility is the desired conclusion. \square

3.4 Generalization

Many mainstream topics in the mathematics curriculum can be generalized. One example is the binomial expansion. The expansion of $(x + y)^n$ for non-negative integers is in every curriculum, but its generalization for non-integral exponents is not.

Example 3.4.1 Fractional Binomial Expansion. The power series expansion about $x = 0$ of an expression such as $\sqrt{1 - x}$ certainly can be derived using basic calculus tools, but an alternate approach can be taken by (correctly) assuming that the binomial expansion theorem applies for the exponent $\frac{1}{2}$. If m is a positive integer, $\binom{m}{n} = 0$ for $n > m$, but a reasonable generalization of $\binom{\alpha}{n}$ for a non-integer α is

$$\frac{\prod_{k=0}^{n-1} (\alpha - k)}{n!}$$

which will never equal zero for non-integral values of α . Thus we have

$$\sqrt{1 - x} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-x)^n$$

The first two terms of this sum are 1 and $-\frac{1}{2}x$. We will proceed to simplify the general form of the coefficient of x^n for $n > 1$.

$$\begin{aligned} \binom{\frac{1}{2}}{n} (-1)^n &= \frac{\frac{1}{2} \cdot \frac{-1}{2} \cdot \frac{-3}{2} \cdots (\frac{1}{2} - (n-1))}{n!} (-1)^n \\ \text{multiply fractions by 2} &= \frac{1 \cdot -1 \cdot -3 \cdots (-2n+3)}{2^n n!} (-1)^n \\ \text{multiply negatives by factors of -1} &= -\frac{1 \cdot 1 \cdot 3 \cdots (2n-3)}{2^n n!} \\ \text{fill in even and higher factors of } (2n!) &= -\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n-3) \cdot (2n-2) \cdot (2n-1) \cdot (2n)}{2^n \cdot 2^n \cdot (n!)^2 \cdot (2n-1)} \\ \text{(put things together)} &= -\frac{(2n)!}{4^n \cdot (n!)^2 \cdot (2n-1)} \end{aligned}$$

This general formula actually works for all n , including 0 and 1. Thus,

$$\sqrt{1 - x} = - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n \cdot (n!)^2 \cdot (2n-1)} x^n$$

□

Remark 3.4.2 The Pochhammer symbol is the notation $(x)_n$, where n is a non-negative integer. It is commonly used to represent the “falling factorial” expression $x \cdot (x-1) \cdot (x-2) \cdots (x-(n-1))$. This gives a more concise way to express the expansion we have just examined as $\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} (-x)^n$. However, you still need to get into the weeds and manipulate the expression in the end.

A different sense in which generalization applies to problem solving is when a problem is more easily solved by first generalizing the context of the problem. This problem from the 1982 Putnam is a good example.

Example 3.4.3 A Putnam Integral. Evaluate $\int_0^{\infty} \frac{\tan^{-1}(\pi x) - \tan^{-1}(x)}{x} dx$.

The integral can be evaluated by generalizing it first to a function: $F(y) = \int_0^{\infty} \frac{\tan^{-1}(yx) - \tan^{-1}(x)}{x} dx$. We then can differentiate with respect to y to get a

simple expression of $F(y)$. After applying the obvious condition $F(1) = 0$, we can substitute π for y to get our final answer. We leave it as an exercise for you to get the final answer of $\frac{1}{2}\pi \ln(\pi)$. The fact that we can differentiate inside an integral as is suggested here is referred to as the **Leibniz Integration Rule**. \square

3.5 Exercises

1. Let $a, b, c \in [0, 1]$. Prove that

$$\sqrt{abc} + \sqrt{(1a)(1b)(1c)} \leq 1$$

Hint. Start with proving $\sqrt{ab} + \sqrt{(1a)(1b)} \leq 1$

2. Evaluate $\int x^3 \sqrt{4 - 9x^2} dx$
3. For any triangle ABC , prove that

$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$$

4. Let $f(x) = \frac{1}{\sqrt{1+x}}$. Estimate the value of $f(\frac{1}{2})$ with a fractional binomial expansion so that the error is less than $\frac{1}{100}$.
5. Complete the derivation of the value of the integral in [Example 3.4.3](#).
6. Evaluate $\int_0^1 \frac{t^3 - 1}{\log(t)} dt$.

Chapter 4

Counting and Indirect Proofs

4.1 Counting Two Ways

Most students are familiar with formula for a binomial coefficient, $\binom{n}{k}$. Here, we will derive the formula by two-way counting.

Example 4.1.1 The binomial coefficient $\binom{n}{k}$ represents the the number of k element subsets of an n element set , $0 \leq k \leq n$.

We will count the number of ways to permute k objects from a set of n elements two ways. First, we can first choose one of the k element subsets of of the set, and second, choose one of the $k!$ permutations of those elements. By the rule of products, we can do this $\binom{n}{k} \cdot k!$. A second approach is to select the the elements from our set to go into positions 1 through k in the permutation immediately. The number of ways of doing this is $n \cdot (n-1) \cdots (n-(k-1))$. The two expressions count the same number and so they are equal. We equate them and then solve for the value of the binomial coefficient: $\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$. \square

4.2 Indirect Proofs

Example 4.2.1 No Least Positive Real Number. Prove that there is no least positive real number.

Assume that there is a least positive real number, which we'll call L . Certainly, we can assume that $L < 1$ since we know of many positive real number that satisfy this inequality. If we multiply the inequality by L , we find that $L^2 < L$. Since L^2 is a positive real number we have a contradiction that L is least. Hence no such least number can exist. \square

Example 4.2.2 The Harmonic Series. Students often struggle with the idea that although the terms of the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, converge to zero, the series itself diverges. Encountering this fact in a calculus class in which proof may not be emphasized is often the problem. Many of the dozens of proofs that the harmonic series diverges are indirect. Here is one such example.

Assume the series converges and its limit is H .

$$\begin{aligned} H &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2k-1} + \frac{1}{2k} + \cdots \\ &> \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \cdots + \frac{1}{2k} + \frac{1}{2k} + \cdots \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{k} + \cdots \\ &= H \end{aligned}$$

The contradiction we arrive at, $H > H$, implies that the series does indeed diverge. \square

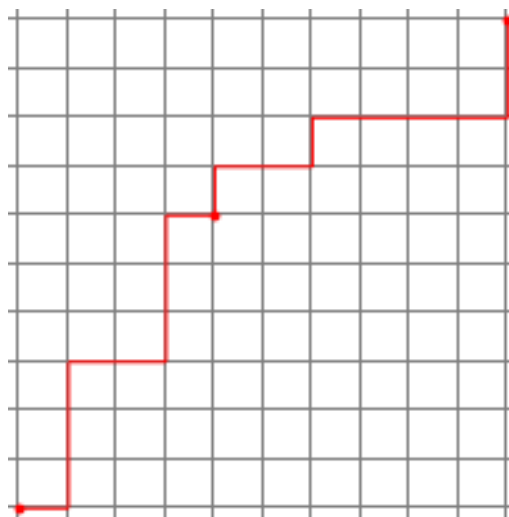
4.3 Problems

- Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S , then so is ab). Let T and U be disjoint subsets of S whose union is S . Given that the product of any three (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U , show that at least one of the two subsets T , U is closed under multiplication.
- Let's say I have a group of $n + 1$ people who want to see a show. Assume they all have different ages. I have three tickets into the theatre: a backstage pass and two regular (but distinguishable) tickets. I have to give out the tickets according to the following two rules:
 - The backstage pass must go to the oldest person who gets a ticket.
 - The person who gets the backstage pass can't get either of the other two tickets, but the two normal tickets can both go to the same person.

How many ways can I give away the tickets? There are two ways to count. Find both and equate them.

Hint. You get two different results depending on whether you select who will be oldest first, or you decide what three people will get tickets first.

- Lattice Paths.** Consider paths such as the one in the grid below that start at the bottom left, $(0, 0)$, and reach the top right, $(10, 10)$. These paths can only go up and to the right. How many paths are there in all? How many of those paths pass through the point $(4, 6)$? Generalize both of your results.

**Figure 4.3.1** A Lattice Path

Hint. Think of each path as a sequence of instructions to go right (R) and up (U).

4. Prove that if $n \geq 1$, then $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$

Hint. Consider the lattice paths from $(0,0)$ to (n,n) passing through any point on the diagonal $i + j = n$.

5. Prove that the number of odd coefficients in each row of Pascal's triangle is a power of 2.
6. Every point of three-dimensional space is colored red, green, or blue. Prove that one of the colors attains all distances, meaning that any positive real number represents the distance between two points of this color.

Hint. If false, there are “non-attainable” distances for each color. Select one for each color. Call them r , g and b , with $r \geq g \geq b$. Select any red point, call it R , and consider the surface of the sphere of radius r centered at R . This is a blue/green sphere. Now select any green point, G on the surface. If we consider G to be a pole of the blue/green sphere, the intersection of the blue/green sphere with the sphere of radius g centered at G is a parallel on the blue/green sphere that is blue. Given our assumptions about the unattainable distances, there must be points on the blue parallel that are b units apart, which is a contradiction.

7. The union of nine planar surfaces, each of area equal to 1, has a total area equal to 5. Prove that the overlap of some two of these surfaces has an area greater than or equal to $\frac{1}{9}$.
8. Prove that there is no polynomial with integer coefficients $P(x)$ with the property that $P(7) = 5$ and $P(15) = 9$.
9. How many positive integers not exceeding 2019 are multiples of 3 or 4 but not 5? You would count the numbers 3, 12, and 16, but not 15 or 20.

solution!—>

Chapter 5

Calculus

Calculus is always represented in the Putnam, but not always in ways that students expect.

5.1 Examples

Example 5.1.1 Putnam 2002 A1. The first problem in the 2002 Putnam was

Let k be a fixed positive integer. The n -th derivative of $\frac{1}{x^k-1}$ has the form $\frac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$.

One solution published solution from [10] follows. By differentiating $P_n(x)/(x^k-1)^{n+1}$, we find that $P_{n+1}(x) = (x^k-1)P'_n(x) - (n+1)kx^{k-1}P_n(x)$; substituting $x = 1$ yields $P_{n+1}(1) = -(n+1)kP_n(1)$. Since $P_0(1) = 1$, an easy induction gives $P_n(1) = (-k)^n n!$ for all $n \geq 0$. \square

Example 5.1.2 Riemann Sums and a challenging integral. Typical students often have more expertise in calculus than any other subject, but there are often gaps in their understanding of the topic. One of them is Riemann sums and their role in integration. A common textbook example is to start with a right Riemann sum of $\int_0^1 x^5 dx$, $\sum_{k=1}^n \left(\frac{k}{n}\right)^5 \cdot \frac{1}{n}$ and maybe disguise it to some extent. Then ask for the limit as n goes to infinity. Naturally the answer is simply the integral, $\frac{1}{6}$. The first part of solving B-1 in the 1976 Putnam was to identify a slightly more disguised example.

Evaluate

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \left(\left\lfloor \frac{2n}{k} \right\rfloor - 2 \left\lfloor \frac{n}{k} \right\rfloor \right)$$

In this case, the function we are integrating is a bit more complicated, and integration is a second issue that isn't so straightforward. The function is $f(x) = \left\lfloor \frac{2}{x} \right\rfloor - 2 \left\lfloor \frac{1}{x} \right\rfloor$.

Evaluating the integral is non-trivial, but here is an overview. When working with floor/ceiling functions, one trick is to assume some condition that lets you work with an equality or inequality to infer more information. In this case,

we assume that $2\lfloor \frac{1}{x} \rfloor = 2k$ for some positive integer k .

$$\begin{aligned} 2\lfloor \frac{1}{x} \rfloor = 2k &\Leftrightarrow \lfloor \frac{1}{x} \rfloor = k \\ &\Leftrightarrow k \leq \frac{1}{x} < k+1 \\ &\Leftrightarrow \frac{1}{k+1} < x \leq \frac{1}{k} \end{aligned}$$

Therefore,

$$\begin{aligned} k \leq \frac{1}{x} < k+1 &\Leftrightarrow 2k \leq \frac{2}{x} < 2k+2 \\ &\Leftrightarrow \lfloor \frac{2}{x} \rfloor = 2k \text{ or } 2k+1. \end{aligned}$$

If $\frac{1}{k+1} < x \leq \frac{2}{2k+1}$, we have $\lfloor \frac{2}{x} \rfloor = 2k+1$; and if $\frac{2}{2k+1} < x \leq \frac{1}{k}$, we have $\lfloor \frac{2}{x} \rfloor = 2k$. Therefore, on the interval $(\frac{1}{k+1}, \frac{1}{k}]$,

$$f(x) = \begin{cases} 1 & \text{if } \frac{1}{k+1} < x \leq \frac{2}{2k+1} \\ 0 & \text{if } \frac{2}{2k+1} < x \leq \frac{1}{k} \end{cases}$$

and $\int_{\frac{1}{k+1}}^{\frac{1}{k}} f(x)dx = \frac{2}{2k+1} - \frac{1}{k+1}$

Finally, we have

$$\begin{aligned} \int_0^1 f(x)dx &= \sum_{k=1}^{\infty} \left(\frac{2}{2k+1} - \frac{1}{k+1} \right) \\ &= 2 \sum_{k=1}^{\infty} \left(\frac{1}{2k+1} - \frac{1}{2k+2} \right) \\ &= 2 \left(\left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} \right) - \left(1 - \frac{1}{2} \right) \right) \\ &= 2 \ln(2) - 1 \\ &= \ln(4) - 1 \end{aligned}$$

□

5.2 Exercises

1. Let $a, b, c, d, e \in \mathbb{R}$ such that $a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{e}{5} = 0$. Show that the polynomial $a + bx + cx^2 + dx^3 + ex^4$ has at least one real zero. Note: In this context, e isn't the number we call e , it's just an arbitrary real number.
2. Prove that $\sum_{k=1}^n \frac{1}{k} > \ln(n+1)$ for all $n \geq 1$.
3. Generalizing the first part of this problem will lead to a characterization of the other parts.
 - (a) Find a partition $\{t_0 = 1, t_1, t_2 = 2\}$ of $[1, 2]$ that maximizes the lower Riemann sum that estimates the value of $\int_1^2 \frac{1}{x} dx$ among all other partitions of size 2.
 - (b) Find a partition $\{t_0 = 1, t_1, t_2, t_3 = 2\}$ of $[1, 2]$ that maximizes the

lower Riemann sum that estimates the value of $\int_1^2 \frac{1}{x} dx$ among all other partitions of size 3.

- (c) Find a partition $\{t_0 = 1, t_1, t_2, \dots, t_n = 2\}$ of $[1, 2]$ that maximizes the lower Riemann sum that estimates the value of $\int_1^2 \frac{1}{x} dx$ among all other partitions of size n .

4. Determine the value of $\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{k^2 + n^2}$

Hint. This is a limit of Riemann sums in disguise!

5. Let $p(x)$ be a polynomial of even degree with a positive leading coefficient. Suppose $p(x) \geq p''(x)$ for every x . Show that $p(x) \geq 0$ for every x .
6. If f is a differentiable function on $[0, 1]$ such that $f(0) = f(1) = 0$, and if $\int_0^1 |f'(x)| dx = 1$, what can be said about $f(\frac{1}{2})$? For example, is it possible that $f(\frac{1}{2}) = 0$?, or 1?, or $\frac{1}{2}$?, or -1?, or 2? Note: *Differentiable* can be replaced with *Absolutely Continuous*, but this is a somewhat more advanced property that isn't likely to be used in the Putnam (I could be wrong!)
7. Evaluate $\int_0^{\pi/2} (\sin^2(\sin(x)) + \cos^2(\cos(x))) dx$.
8. For a positive constant k , consider the region \mathcal{R} bounded by the line $y = kx$ and the parabola $y = x^2$. There is only one choice of k such that the solids obtained by revolving \mathcal{R} about the x -axis and the y -axis have the same volume. Find the value of this special choice of k .
9. For which real numbers c is $\frac{e^x + e^{-x}}{2} \leq e^{cx^2}$ for all real x ?
10. Let $f(x)$ be a function such that $f(1) = 1$ and for $x \geq 1$

$$f'(x) = \frac{1}{x^2 + f(x)^2}.$$

Prove that $\lim_{x \rightarrow \infty} f(x)$ exists and is less than $1 + \frac{\pi}{4}$.

11. Evaluate the infinite product $\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}$.

Hint. Look at the “partial products.”

12. The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. What is the rate of change (in mm/hr) of the distance between the tips of the hands at one o'clock?
13. Prove the following or provide a counterexample: If $\{f_n\}$ is a sequence of functions defined on the interval $[0, 1]$ and that for each x in the interval $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, then $\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 f(x) dx$.
14. Evaluate $\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\frac{(2n)!}{n!n^n} \right)$.
- Hint.** Think Riemann sum.
15. Evaluate $\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\prod_{k=1}^n \frac{n^2 + k^2}{n^2} \right)$.

Chapter 6

Number Theory

The starting point for number theory is the set of positive integers. After developing basic arithmetic properties of this set, number theory goes in countless directions and is one of the richest of all mathematical subjects. Here, we touch on just a few key ideas.

6.1 Primes

A prime are positive integer greater than one that are only divisible by one and itself. A fundamental theorem in number theory states that every positive integer greater than one is uniquely factorable as a product of one or more primes.

You should be familiar with the fact that there are an infinite number of primes, and if pressed, should be able to prove it. There is only one even prime, 2. If we divide an odd prime by 4, the remainder will either be 1 or 3; so every odd prime has the form $4k + 1$ or $4k + 3$. Here is a proof that there are an infinite number of primes of the second form. Suppose, to the contrary, that there are only a finite number of primes of the form $4k + 3$, and that they are $p_1 = 4k_1 + 3$, $p_2 = 4k_2 + 3, \dots, p_r = 4k_r + 3$. Consider the integer $Q = 4(p_1 \cdot p_2 \cdot \dots \cdot p_r) + 3$. We observe that Q is an odd integer that is not divisible by any of the p_i . It can be factored into a product of primes that are not in the set of p_i but the factors cannot all be of the form $4k + 1$ and so there is yet another prime of the form $4k + 3$ that was not accounted for in our list of such primes. This contradiction implies that the set of these integer must be infinite.

We had just proven an instance of a more general theorem called *Dirichlet's Theorem*, which states that if $4k + 3$ is replaced with $\alpha k + \beta$ where $\gcd(\alpha, \beta) = 1$ we get the same conclusion, that there are an infinite number of primes of that form.

6.2 Residues

The residues of integers modulo n , $n \geq 2$ are the remainders upon dividing integers by n . We start with a simple use of residues.

Example 6.2.1 Show that $n^2 + 1$ is divisible by 7 for no positive integer n .

$$\begin{aligned} 7 \mid (n^2 + 1) &\Leftrightarrow n^2 \equiv -1 \pmod{7} \\ &\Leftrightarrow n^2 \equiv 6 \pmod{7} \end{aligned}$$

We can easily verify that for any n , the residue of $n^2 \pmod{7}$ is never 6. \square

Example 6.2.2 Chinese Remainder Theorem. Let a, b be integers and m, n positive integers such that $\gcd(m, n) = 1$. Then there is a unique integer, x , in the set $\{0, 1, 2, \dots, mn - 1\}$ such that $x \equiv a \pmod{m}$ and $x \equiv b \pmod{n}$.

In order for an integer to satisfy the first congruence, we need $x = a + m \cdot q_1$ for some integer q_1 . Substitution into the second congruence gives us

$$a + m \cdot q_1 \equiv b \pmod{n} \Rightarrow m \cdot q_1 \equiv b - a \pmod{n}$$

We know that since m and n are relatively prime, there exist integers s and t such that

$$m \cdot s + n \cdot t = 1 \Rightarrow m \cdot s \equiv 1 \pmod{n}$$

and so

$$q_1 \equiv s(b-a) \pmod{n} \Rightarrow x = a + m \cdot (s(b-a) + nq_2) = x = a + m \cdot s \cdot (b-a) + (m \cdot n)q_2$$

for some integer q_2 . This set of solutions constitutes one residue class mod $m \cdot n$ and so there is a unique solution in the desired set. \square

6.3 Euler's Theorem

Euler's Theorem generalizes Fermat's Little Theorem.

Example 6.3.1 Compute $5^{159} \pmod{21}$.

One solution to this problem is to observe that $5^{159} \equiv 2 \pmod{3}$ by applying Fermat's little theorem with $p = 3$; and then switching to calculations mod 7, find that $5^{159} \equiv 6 \pmod{7}$. We can then use the procedure outlined in the proof of the Chinese Remainder Theorem to find that $5^{159} \equiv 20 \pmod{21}$. Although this solution fine, it can be determined more efficiently with Euler's Theorem. \square

Theorem 6.3.2 Euler's Theorem. Let n be a positive integer greater than 1 and a a positive integer that is relatively prime to n . Then $a^{\phi(n)} \equiv 1 \pmod{n}$

Proof. Consider the set U_n of all positive integers less than n that are relatively prime to n . The cardinality of this set is $\phi(n)$. We note that the function f on U_n defined by $f(b) = a \cdot b \pmod{n}$ is a permutation of U_n . Let $X = \prod_{x \in U_n} x$. Then

$$\begin{aligned} X &= \prod_{x \in U_n} x = \prod_{x \in U_n} f(x) \\ &\equiv \prod_{x \in U_n} a \cdot x \pmod{n} \\ &\equiv a^{\phi(n)} \prod_{x \in U_n} x \pmod{n} \\ &= a^{\phi(n)} \cdot X \end{aligned}$$

Multiplying both sides of the equivalence $X \equiv a^{\phi(n)} \cdot X \pmod{n}$ by the mod n inverse of X gets us the desired result. \blacksquare

Example 6.3.3 Revisiting the previous example, we can compute the value of $\phi(21)$ based on the exercise on [Euler's Phi Function](#): $\phi(21) = \phi(3) \cdot \phi(7) = 2 \cdot 6 = 12$. This lets us reduce 5^{159} to $5^3 \pmod{21}$, which reduces further to 20. \square

6.4 Problems

Before starting these problems, you might want to review the statement of [Fermat's Little Theorem](#).

- We say that two positive rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are *close* if $bc - ad = \pm 1$.
 - Prove that if $\frac{a}{b}$ and $\frac{c}{d}$ are close, then $\frac{a+c}{b+d}$ is close to both $\frac{a}{b}$ and $\frac{c}{d}$.
 - Find five rational numbers that are close to $\frac{2}{7}$.
 - Find a connection between pairs of close rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$, and the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
- Prove that some positive integral power of 17 ends in 0001 (base 10).
- Show, without using a calculator, that $2^9 + 2^{99}$ is divisible by 100.
- Find a Pythagorean triple that includes the number 2021 as a side length.
Hint. 2025 is a perfect square.
- Euler's Phi Function.** Let $\varphi(k)$ be the number of integers in the set $\{1, 2, \dots, k-1\}$ that are relatively prime to k .
 - Find and prove a formula for $\varphi(p^n)$ if $n \geq 1$ and p is prime.
 - Prove that if a and b are relatively prime positive integers greater than 1, then $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$.
 - Determine the value of $\varphi(100!)$.
- For which positive integers n is there a sum of n consecutive integers that is a perfect square?
- Show that the equation $x^2 - y^2 = 2xyz$ has no solutions in the positive integers. **Hint:**
Hint. Consider a prime divisor of xy .
- In [Example 5.1.2](#), we showed an example of how to deal with floor/ceiling functions. Here is a problem from the 1983 Putnam that can be solved by starting with a similar approach. We quote the problem verbatim, where $[x]$ was used for the floor function.
Let $f(n) = n + [\sqrt{n}]$ where $[x]$ is the largest integer less than or equal to x . Prove that, for every positive integer m , the sequence

$$m, f(m), f(f(m)), f(f(f(m))), \dots$$

contains at least one square of an integer.

Hint. Assume $m = k^2 + j$ where $0 \leq j \leq 2k$.

- Consider the sequence T defined by $T_1 = 2$ and $T_{n+1} = T_n^2 - T_n + 1$ when $n \geq 1$. Prove that if $m \neq n$ then T_m and T_n have no common factor greater than 1.

Hint. Look closely at the first few terms of the sequence. A plausible

formula may be suggested.

old9 Find all integral solutions to $|p^r - q^s| = 1$, where p and q are primes and r and s are positive integers greater than or equal to 2. Prove that there are no other solutions than the ones you list.

10. What is the sum of the digits of the sum of the digits of the sum of the digits of 4444^{4444} ?

11. Let n be an integer greater than or equal to four. Each of the numbers x_1, x_2, \dots, x_n equal 1 or -1, and

$$x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + \cdots + x_{n-3}x_{n-2}x_{n-1}x_n + x_{n-2}x_{n-1}x_nx_1 + x_{n-1}x_nx_1x_2 + x_nx_1x_2x_3 = 0.$$

Prove that n is divisible by 4.

Hint. Use odd/even parity twice.

12. Find all positive integers n such that $n!$ ends in exactly 1000 zeros.

Hint. Count the factors of 5 in $n!$.

13. Prove that among any three distinct integers we can find two, say a and b , such that the number $a^3b - ab^3$ is a multiple of 10.

14. Prove that for any positive integer n there exists an n -digit number divisible by 2^n and containing only the digits 2 and 3.

16. (2019 Putnam B-1) Denote by \mathbb{Z}^2 the set of all points (x, y) in the plane with integer coordinates. For each integer $n \geq 0$, let P_n be the subset of \mathbb{Z}^2 consisting of the point $(0, 0)$ together with all points (x, y) such that $x^2 + y^2 = 2^k$ for some integer $k \leq n$. Determine, as a function of n , the number of four-point subsets of P_n whose elements are the vertices of a square.

17. Does there exist a positive integer with the property that the sum of the (base 10) digits of its square is equal to 100? If so, find one. What if 100 is replaced with 2019?

Hint. Before launching into a search of whether a number that we seek exists, it is worth considering the problem mod 9 since the sum of the digits of a positive integer is congruent mod 9 to the integer itself. Controlling the digits of a square is easiest with certain special numbers. One such type number is $10^k - d$ where k is a positive integer and d is a digit.

Chapter 7

Inequalities

As with most topics, we could spend a whole semester on inequalities. I normally concentrate on the relationship between the arithmetic, geometric and harmonic means, the Cauchy-Bunyakovsky-Schwarz inequality, and Young's inequality in a single class.

7.1 Means

There are many means on a finite number of nonnegative real numbers. Here we focus on a few of the most common. The *arithmetic mean* of n numbers, or what we often call their average, is their sum divided by n . The *geometric mean* is computed by taking n^{th} root of their product. We establish the relationship between these two means in the following theorem.

Theorem 7.1.1 The AM/GM inequality. Assume $n \geq 2$. If a_i , $i = 1, 2, \dots, n$ are non-negative real numbers, then

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

with equality if and only if all the a_i are equal.

Proof. A proof by induction, but a somewhat unconventional one taken from [1]. If $n \geq 2$ then let $P(n)$ the inequality for n numbers. Instead of going forward one step at a time starting at 2, we start at 2 and prove the two general implications $P(n) \Rightarrow P(n-1)$ and $P(n) \Rightarrow P(2n)$. After proving the basis, $n = 2$, we can apply these two implications in several ways; for example, we can infer the truth for values 4, 3, 8, 7, 6, 5, 16, 15, 14, \dots , 9, 32, 31, 30 \dots

We leave the proof for $n = 2$ as an exercise. Then we assume that for some

$n \geq 2$, the inequality is true and then assume we have $2n$ numbers.

$$\begin{aligned} \prod_{k=1}^{2n} a_k &= \left(\prod_{k=1}^n a_k \right) \left(\prod_{k=n+1}^{2n} a_k \right) \\ &\leq \left(\sum_{k=1}^n \frac{a_k}{n} \right)^n \left(\sum_{k=n+1}^{2n} \frac{a_k}{n} \right)^n \quad \text{by } P(n), \text{ twice} \\ &\leq \left(\frac{\sum_{k=1}^{2n} \frac{a_k}{n}}{2} \right)^{2n} \quad \text{by } P(2) \\ &= \left(\frac{\sum_{k=1}^{2n} a_k}{2n} \right)^{2n} \end{aligned}$$

Now if $n \geq 3$ we assume $P(n)$ is true and we have $n-1$ numbers. We append the arithmetic mean of these numbers, $A = \sum_{k=1}^{n-1} \frac{a_k}{n-1}$, to our collection and apply $P(n)$:

$$\begin{aligned} \left(\prod_{k=1}^{n-1} a_k \right) A &\leq \left(\frac{\left(\sum_{k=1}^{n-1} a_k \right) + A}{n} \right)^n \\ &= \left(\frac{(n-1)A + A}{n} \right)^n = A^n \end{aligned}$$

dividing by A , we get $\left(\prod_{k=1}^{n-1} a_k \right) \leq A^{n-1} = \left(\frac{\sum_{k=1}^{n-1} a_k}{n-1} \right)^{n-1}$. ■

The AM/GM inequality can be used in optimization. Here is an example.

Example 7.1.2 Problem: If x and y are positive real numbers such that $x + 2y = 1$, what is the largest possible value of x^2y ?

A first attempt at this problem might be to write this:

$$x^2y \leq \left(\frac{x + x + y}{3} \right)^3$$

However, it's $x + 2y$, not $2x + y$ that is invariant. The objective function needs to be adjusted with the right constants.

$$\begin{aligned} x^2y &= \frac{1}{4} \cdot x \cdot x \cdot (4y) \\ &\leq \frac{1}{4} \left(\frac{2(x + 2y)}{3} \right)^3 \\ &= \frac{1}{4} \left(\frac{2}{3} \right)^3 = \frac{2}{27} \end{aligned}$$

with equality when $x = 4y$, which implies that the maximum is attained when $y = \frac{1}{6}$ and $x = \frac{2}{3}$. □

In the exercises we examine the harmonic mean and how it is related to the arithmetic and geometric means.

7.2 The CBS Inequality

The Cauchy-Bunyakovsky-Schwarz, or CBS, inequality has both discrete and continuous versions. The real discrete version follows.

Theorem 7.2.1 Cauchy-Bunyakovsky-Schwarz Inequality. *If x and y are vectors in \mathbb{R}^n , then*

$$\left(\sum_{i=1}^n x_i y_i\right)^2 \leq \left(\sum_{i=1}^n x_i^2\right) \left(\sum_{i=1}^n y_i^2\right) \quad (7.2.1)$$

with equality if and only if the two vectors are scalar multiples of one another.

Proof. There are many proofs of this inequality. Here is a brief one from [1]. We observe that the inequality can be expressed more compactly using norm and inner product notation as $(x, y)^2 \leq \|x\|^2 \|y\|^2$.

We can assume that both x and y are nonzero vectors, for otherwise the inequality is obvious. Let α be a real variable, and consider the square of the norm of the vector $\alpha x + y$. If x and y are linearly independent, then $\|\alpha x + y\| > 0$. We expand the square of this norm:

$$\begin{aligned} \|\alpha x + y\|^2 &= (\alpha x + y, \alpha x + y) \\ &= \alpha^2 \|x\|^2 + 2\alpha(x, y) + \|y\|^2 \end{aligned}$$

This quadratic polynomial in α has no real roots and so the discriminant, $4(x, y)^2 - 4\|x\|^2 \|y\|^2 < 0$, which implies CBS.

If x and y are linearly dependent, we get equality since the discriminant would equal zero. ■

Example 7.2.2 From [15], This illustrates the “1-Trick.” Show that for each real sequence x_1, x_2, \dots, x_n one has

$$x_1 + x_2 + \dots + x_n \leq \sqrt{n} (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

As the title suggest, we can derive this quite simply by taking $y_k = 1$ for all k in (7.2.1). □

7.3 Young’s Inequality

Given positive numbers a and b whose reciprocals add up to one, the product of two positive real numbers is less than or equal to a weighted sum of p^{th} and q^{th} powers of a and b .

Theorem 7.3.1 Young’s inequality. *If a and b are nonnegative real numbers and p and q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$a \cdot b \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. The proof follows from the following more general inequality, where $f(x) = x^{p-1}$. ■

Theorem 7.3.2 *If f is a strictly increasing function on the positive real numbers, $f(0) = 0$, and a and b are positive real numbers, then*

$$ab \leq \int_0^a f(x) dx + \int_0^b f^{-1}(y) dy$$

Proof. If $b = f(a)$, then the rectangle $[0, a] \times [0, b]$ is divided into two regions, the region below the curve $y = f(x)$, and the region to the left of $x = f^{-1}(y)$. The sum of the two integrals is exactly equal to the area, $a \cdot b$. If $f(a) > b$, the region whose area $\int_0^b f^{-1}(y) dy$ computes is contained within the rectangle, but the region whose area $\int_0^a f(x) dx$ computes spills outside the rectangle, making the inequality a strict one. This case is illustrated by Figure 7.3.3. In the other case, where $f(a) < b$, the excess area that makes the inequality a strict one is accounted for by $\int_0^b f^{-1}(y) dy$. ■

This interactive SageMath expression illustrates the proof above for the case where $f(x) = x^{p-1}$, $p > 1$.

```
@interact()
def young(a=(2,(1/4,3)),b=(3/2,(1/4,3)),p=(7/4,(1/8,5,1/8))):
    pl=plot(x^(p-1),(x,0,a),fill='min',
            fillcolor='red',ticks=[[[]],[[]])
    pl+=plot(x^(p-1),(x,0,b^(1/(p-1))),fill='max',
            fillcolor='blue')
    pl+=text('a='+str(n(a,3)),[a,-0.1])
    pl+=text('b='+str(n(b,3)),[-0.1, b])
    pl+=text('separating_curve_is_$y=x^{p-1}$',[1.3*a,0.5*b])
    show(pl,aspect_ratio=1)
```

The output from the SageMath expression above is a dynamic version of this image:

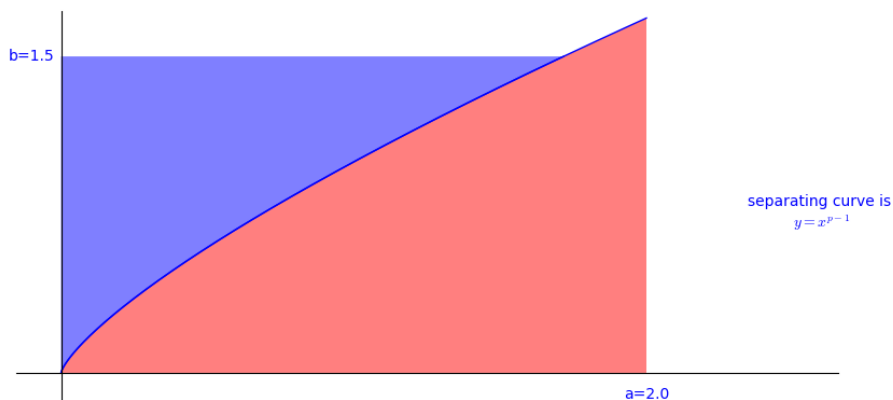


Figure 7.3.3 A visualization of the derivation of Young's Inequality

7.4 Problems

- Let a and b be positive real numbers. Prove that $\sqrt{ab} \leq \frac{a+b}{2}$ with equality if and only if $a = b$.
- How does the **harmonic mean** of two nonnegative real numbers, a and b , $\frac{2}{\frac{1}{a} + \frac{1}{b}}$, compare with their geometric and arithmetic means?
Hint. Select two simple numbers and compute their harmonic, geometric and arithmetic means. If there is going to be consistency, you know what it should be. Then the hard work is to prove it!
- Let a and b be positive real numbers. Identify and prove the relationship between their **Quadratic Mean**, $\sqrt{\frac{a^2+b^2}{2}}$, and their other means.

4. An express delivery service restricts the size of packages it will accept. Packages cannot exceed 18 inches in length plus girth, i. e., length + $2 \times$ width + $2 \times$ height ≤ 18 . Find the maximum volume of an acceptable package.
5. Let a_1, a_2, \dots, a_n be positive, with sum 1. Show that $\sum_{i=1}^n a_i^2 \geq \frac{1}{n}$ and $\sum_{i=1}^n a_i^4 \geq \frac{1}{n^3}$.
Hint. Extend the “one trick.”
6. (a) (a) Compute the determinant $D = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ two ways to derive the identity $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$.
 (b) Prove that if x, y , and z are distinct real numbers, then $\sqrt[3]{x-y} + \sqrt[3]{y-z} + \sqrt[3]{z-x} \neq 0$.
7. Let $x_1 + x_2 + x_3 = \frac{\pi}{2}$, where the x_i are positive. Show that $\sin x_1 \sin x_2 \sin x_3 \leq \frac{1}{8}$.
8. Let f be a continuous and monotonically increasing function such that $f(0) = 0$ and $f(1) = 1$. Prove that

$$f(0.1) + f(0.2) + \dots + f(0.9) + f^{-1}(0.1) + f^{-1}(0.2) + \dots + f^{-1}(0.9) \leq 9.9.$$
9. Let T be a tetrahedron with three mutually perpendicular edges of lengths a, b , and c . Let l be the sum of the length of the six edges of T . What is the maximum possible volume of T ?
10. Let a_1, a_2, \dots, a_n be positive real numbers and let s be their sum. Show that $(1 + a_1)(1 + a_2) \dots (1 + a_n) \leq 1 + \frac{s}{1!} + \frac{s^2}{2!} + \dots + \frac{s^n}{n!}$.
11. For positive numbers x and y , prove that $x^x + y^y \geq x^y + y^x$, with equality if and only if $x = y$.
12. Find the maximum of the function $f(x, y, z) = 5x - 6y + 7z$ on the ellipsoid $2x^2 + 3y^2 + 4z^2 = 1$.
13. **The Rearrangement Inequality.** Let $\{a_k\}$ and $\{b_k\}$ are two non-decreasing sequences of n real numbers. Let σ be a permutation of the indices 1 through n . Prove that

$$\sum_{k=1}^n a_k b_k \geq \sum_{k=1}^n a_k b_{\sigma(k)}.$$

14. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

Hint. Assume $a \geq b \geq c$ and note that this implies $\frac{1}{b+c} \geq \frac{1}{a+c} \geq \frac{1}{a+b}$.

Chapter 8

Sequences and Series

8.1 The Fibonacci sequence

The Fibonacci sequence is defined by $f_0 = 1$, $f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-2} + f_{n-1}$. Warm-up Prove that any two consecutive Fibonacci numbers are coprime. Note: the terms “coprime” and “relatively prime” mean the same thing, have no common factor other than 1.

8.2 Generating Functions

Students should know how to derive the power series representation of a function, but might not be familiar with the inverse process of identifying a function, given its power series. This is a particularly powerful tool for deriving a formula from a combinatorial sequence. We will illustrate this technique by deriving a formula for the number of different binary trees with n vertices.

Definition 8.2.1 Binary Tree.

- (1) A tree consisting of no vertices (the empty tree) is a binary tree
- (2) A vertex together with two subtrees that are both binary trees is a binary tree. The subtrees are called the left and right subtrees of the binary tree.

◇

Let $B(n)$ be the number of different binary trees of size n (n vertices), $n \geq 0$. By our definition of a binary tree, $B(0) = 1$. Our first step in developing a formula for $B(n)$ will be to derive a formula for its generating function, $G(z) = \sum_{n=0}^{\infty} B(n)z^n$

Consider any positive integer $n + 1$, $n \geq 0$. A binary tree of size $n + 1$ has two subtrees, the sizes of which add up to n . The possibilities can be broken down into $n + 1$ cases:

Case 0: Left subtree has size 0; right subtree has size n .

Case 1: Left subtree has size 1; right subtree has size $n - 1$.

⋮

Case k : Left subtree has size k ; right subtree has size $n - k$.

⋮

Case n : Left subtree has size n ; right subtree has size 0.

We can count the number of possibilities by multiplying the number of ways that the left subtree can be filled, $B(k)$, by the number of ways that the right subtree can be filled, $B(n-k)$. Since the sum of these products equals $B(n+1)$, we obtain the recurrence relation for $n \geq 0$:

$$\begin{aligned} B(n+1) &= B(0)B(n) + B(1)B(n-1) + \cdots + B(n)B(0) \\ &= \sum_{k=0}^n B(k)B(n-k) \end{aligned}$$

Think of each side of the recurrence relation as term of a sequence and take the generating function of both sides:

$$\sum_{n=0}^{\infty} B(n+1)z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n B(k)B(n-k) \right) z^n \quad (8.2.1)$$

The left side of (8.2.1) is $\frac{G(z)-B(0)}{z} = \frac{G(z)-1}{z}$. The right side is the square of $G(z)$. Therefore,

$$\frac{G(z)-1}{z} = G(z)^2 \Rightarrow zG(z)^2 - G(z) + 1 = 0$$

Using the quadratic equation we find two solutions:

$$G_1(z) = \frac{1 + \sqrt{1-4z}}{2z} \quad \text{and} \quad (8.2.2)$$

$$G_2(z) = \frac{1 - \sqrt{1-4z}}{2z} \quad (8.2.3)$$

It is apparent that $G_1(z)$ is not what we are looking for because it has a singularity at $z = 0$ while the $G(z)$ we are looking for satisfies $G(0) = 1$. $G_2(z)$ does satisfy the condition at zero and so we will expand it to extract its coefficients. We can compute the first few coefficients numerically to verify that we are on the right track

$$G_2(x) = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + \cdots \quad (8.2.4)$$

It's easy to verify that these coefficient satisfy our recurrence relation. So now we will develop a general expression for $B(n)$ using the generalized binomial theorem.

$$\begin{aligned} G(z) &= \frac{1 - \sqrt{1-4z}}{2z} \\ &= \frac{1 - \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (-4z)^n}{2z} \\ &= \frac{-1}{2} \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-4)^n z^{n-1} \\ &= \sum_{n=0}^{\infty} \frac{-1}{2} \binom{\frac{1}{2}}{n+1} (-4)^{n+1} z^n \end{aligned}$$

We leave it as an exercise to verify that

$$B(n) = \frac{-1}{2} \binom{\frac{1}{2}}{n+1} (-4)^{n+1} = \frac{1}{n+1} \binom{2n}{n}$$

This sequence of numbers is often called the **Catalan numbers**. For more information on the Catalan numbers, see the entry A000108 in The [On-Line Encyclopedia of Integer Sequences](#).

8.3 Problems

1. Consider products of pairs of Fibonacci numbers that are two positions apart in the sequence. What do you see? Describe the pattern and write a general formula. Use your observation to examine the ratios of consecutive Fibonacci numbers. What happens in the “long run?”
2. Find the sum $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n!$.
3. Evaluate $\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$
4. $\lfloor \sqrt{44} \rfloor = 6$ and $\lfloor \sqrt{4444} \rfloor = 66$. Generalize and prove.
5. Express $\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1}-2^{k+1})(3^k-2^k)}$ as a rational number.
6. Prove that $\sum_{k=1}^n \frac{1}{\sqrt{k}} < 2\sqrt{n}$
7. Show that if the sequence $\{a_n\}$ is monotonically decreasing and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} n(a_n - a_{n+1})$ converges and the two sums are equal.
8. Prove that if H_n is the n^{th} partial sum of the harmonic series, then

$$\sum_{n=1}^{\infty} H_n x^n = \frac{1}{1-x} \ln \frac{1}{1-x}.$$

9. The sequence a_0, a_1, a_2, \dots satisfies $a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$ for all nonnegative integers m and n with $m \geq n$. If $a_1 = 1$, determine a_n .
Hint. Compute a few values to get a reasonable guess.
10. Let p be a prime number and s be a positive integer. Show that for any $i \in \{0, 1, \dots, p^s - 1\}$,

$$\binom{p^s - 1}{i} \equiv (-1)^i \pmod{p}.$$

Hint. Consider generating functions

11.
 - (a) How many ways can you tile a $2 \times n$ rectangle, $n \geq 1$, using 1×2 tiles?
 - (b) How many ways can you tile a $3 \times 2n$ rectangle, $n \geq 1$, using 1×2 tiles?
12. Let $p(x) = x^2 - 3x + 2$. Show that for any positive integer n there exist unique numbers a_n and b_n such that the polynomial $q_n(x) = x^n - a_n x - b_n$ is divisible by $p(x)$. Hint: If true, then for all $n \geq 1$, $q_{n+1}(x) - xq_n(x)$ is divisible by $p(x)$.
13. Let n be a nonnegative integer and let $S(n)$ be the number of solutions to $x + 2y + 2z = n$ in the nonnegative integers. Find a formula for $S(n)$.
14. Evaluate $\sum_{n=1}^{\infty} \frac{n}{n^4 + n^2 + 1}$.

Chapter 9

Polynomials

9.1 Prerequisite Knowledge

The most common way to define a **polynomial** in x is as an algebraic expression of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where n is a nonnegative integer, the a_i are in some predetermined set, and x is not in that set. Alternatively, we can define polynomials recursively.

Definition 9.1.1 Polynomial in x over S . Let S be a set of numbers and x an element outside of S .

- Any nonzero element of S is a polynomial of degree zero.
- Let n be a positive integer and $x \notin S$, $x \neq 0$. A polynomial of degree n in x is any expression that can be written in the form $q(x) \cdot x + b$, where b is an element of S and $q(x)$ is a polynomial of degree $n - 1$ over S .

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Remark 9.1.2 In the remainder of this chapter, if we simply refer to a polynomial, you can assume that it is a polynomial over any field.

What follows is a series of theorems and basic properties relating to polynomials. Proofs are available from many sources. We assume you are familiar with polynomial addition and multiplication, so we take division as a starting point. The following is an analogue to the division property for integers.

Theorem 9.1.3 Division Property for Polynomials. *Let F be a field and let $p(x)$ and $g(x)$ be two polynomials of over F with $g(x) \neq 0$. Then there exist unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $p(x) = g(x)q(x) + r(x)$, where $r(x) = 0$ or $\deg r(x) < \deg g(x)$.*

Theorem 9.1.4 The Remainder Theorem. *Let F be a field, $p(x)$ a polynomial over F , and $a \in F$. Then $p(x) = (x - a)q(x) + p(a)$, where $q(x)$ is some polynomial over F .*

The proof of this next theorem is advanced, but it would be reasonable to expect that you could invoke in the course of solving a problem.

Theorem 9.1.5 The Fundamental Theorem of Algebra. *Every non-constant polynomial in a single variable with complex coefficients has at least one complex root.*

Remark 9.1.6 You could prove by induction that [The Fundamental Theorem of Algebra](#) implies that a polynomial of degree n over the complex numbers can be factored into the product of n monomial factors.

Theorem 9.1.7 Vieta Relations. Suppose that $p(x) = \sum_{k=0}^n a_k x^k$ has n roots: x_1, x_2, \dots, x_n (possibly repeated). The Vieta relations give us a way to equate these roots with the coefficients of $p(x)$. The three most commonly used of these are

$$\begin{aligned} 1. \quad & \prod_{k=1}^n x_k = (-1)^n \frac{a_0}{a_n} \\ 2. \quad & \sum_{1 \leq i < j \leq n} x_i x_j = \frac{a_{n-2}}{a_n} \\ 3. \quad & \sum_{i=1}^n x_i = -\frac{a_{n-1}}{a_n} \end{aligned}$$

Polynomials in an Expression. Although indeterminates are usually simple symbols like x or y , they can be any expression. For example, $y = \frac{1}{x^3} - \frac{1}{x} - 1$ is not a polynomial in x . However, if we treat the expression $\frac{1}{x}$ as an indeterminate, then we can view y as $y = (\frac{1}{x})^3 - \frac{1}{x} - 1$, a polynomial in $\frac{1}{x}$.

Another example is $\cos 2\theta = 2 \cos \theta - 1$, which is a polynomial in $\cos \theta$. In fact, for every positive integer n , $\cos n\theta$ is a polynomial in $\cos \theta$. See [Exercise 9.3.15](#).

9.2 Examples

Example 9.2.1 2019 Putnam, B5. Here is a recent Putnam problem that can be attacked using a variety of properties of polynomials.

Let F_m be the m th Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \dots, 1008$. Find integers j and k such that $p(2019) = F_j - F_k$.

Probably the simplest solution uses differences. Observe that if we consider a related polynomial, $q(x)$, such that $q(n) = F_{2n+1}$ for $n = 0, 1, 2, \dots, 1008$, the first differences, $q(n+1) - q(n)$, $n = 0, 1, 2, \dots, 1007$ are the Fibonacci numbers $F_2, F_4, \dots, F_{2016}$. The second differences are the odd-indexed Fibonacci numbers between the first differences. The third differences are back to the even indexed values, etc. Since the 1008th difference computed from the first 1009 values of $q(x)$ is F_{1009} . Since the degree of $q(x)$ is 1008, all 1008th differences must be F_{1009} . We can compute another diagonal of the difference array for $q(x)$ to see that $q(1009) = F_{1009} + F_{1010} + \dots + F_{2017}$. Using $F_i = F_{i+2} - F_{i+1}$ this last sum reduces to $F_{2019} - F_{1010}$. This provides us with the desired values of j and k .

A second solution, published in a [YouTube video](#) by Prof. Mohamed Omar (Harvey Mudd) (<https://youtu.be/MGljS1W4tzY>), is a variation on an earlier solution published in [\[10\]](#). It uses the Lagrange interpolation formula: given x_0, \dots, x_n and y_0, \dots, y_n , the unique polynomial P of degree at most n satisfying $P(x_i) = y_i$ for $i = 0, \dots, n$ is

$$\sum_{i=0}^n P(x_i) \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

We construct such a polynomial as described in the problem and evaluate it at 2019 to get

$$\begin{aligned} p(2019) &= \sum_{i=0}^{1008} F_{2i+1} \prod_{j \neq i} \frac{2019 - (2j - 1)}{(2i - 1) - (2j - 1)} \\ &= \sum_{i=0}^{1008} F_{2i+1} \prod_{j \neq i} \frac{1009 - j}{i - j} \end{aligned}$$

We observe that the product expressions inside the sum above involve factorials that can be combined into an expression that involves a binomial coefficient.

$$\begin{aligned} \prod_{j \neq i} \frac{1009 - j}{i - j} &= \frac{\prod_{j \neq i} (1009 - j)}{\prod_{j \neq i} (i - j)} \\ &= \frac{1009! / (1009 - i)}{i! (1008 - i)! (-1)^{1008-i}} \\ &= (-1)^{1008-i} \binom{1009}{i} \end{aligned}$$

We now return to the value of $p(2019)$ with the this expression for the product and [Binet's Formula](#) for the Fibonacci numbers.

$$\begin{aligned} p(2019) &= \sum_{i=0}^{1008} \frac{1}{\sqrt{5}} (\alpha^{2i+1} - \beta^{2i+1}) (-1)^{1008-i} \binom{1009}{i} \\ &= \frac{1}{\sqrt{5}} \left(\sum_{i=0}^{1008} \alpha^{2i+1} (-1)^{1008-i} \binom{1009}{i} - \sum_{i=0}^{1008} \beta^{2i+1} (-1)^{1008-i} \binom{1009}{i} \right) \\ &= \frac{1}{\sqrt{5}} \left(\sum_{i=0}^{1008} (-\alpha) (\alpha^2)^i (-1)^{1009-i} \binom{1009}{i} - \sum_{i=0}^{1008} (-\beta) (\beta^2)^i (-1)^{1009-i} \binom{1009}{i} \right) \\ &= \frac{1}{\sqrt{5}} \left(\sum_{i=0}^{1008} (-\alpha) (\alpha^2)^i (-1)^{1009-i} \binom{1009}{i} - \sum_{i=0}^{1008} (-\beta) (\beta^2)^i (-1)^{1009-i} \binom{1009}{i} \right) \\ &= \frac{1}{\sqrt{5}} (-\alpha ((\alpha^2 - 1)^{1009} - (\alpha^2)^{1009}) + \beta ((\beta^2 - 1)^{1009} - (\beta^2)^{1009})) \\ &= \frac{1}{\sqrt{5}} (-\alpha ((\alpha)^{1009} - (\alpha^2)^{1009}) + \beta ((\beta)^{1009} - (\beta^2)^{1009})) \end{aligned}$$

This last expression can be simplified using Binet's formula to deduce that $p(2019) = F_{2019} - F_{1010}$. \square

9.3 Exercises

1. Binet's Formula. Derive Binet's Formula for the Fibonacci numbers:

$$F_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n)$$

Where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the roots of the polynomial $x^2 - x - 1$.

2. Rational Roots of Polynomials over the Integers.

(a) Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial over the

integers with $n \geq 1$, $a_n, a_0 \neq 0$. Prove that if $c/d \in \mathbb{Q}$, $\gcd(c, d) = 1$, is a root of $p(x)$, then $c|a_0$ and $d|a_n$

- (b) Find all rational roots of $p(x) = 36x^3 - 81x^2 + 38x - 5$.
3. Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with $n \geq 1$, $a_n, a_0 \neq 0$. How are the roots of $p(x)$ and the roots of $q(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$ related?
4. Let $p(x)$ be a polynomial over the real numbers.
- (a) Prove that if $z = a + bi$ is a root of $p(x)$, then $\bar{z} = a - bi$ is also a root of $p(x)$.
- (b) Prove that $p(x)$ can be factored into a product of linear and quadratic polynomials with real coefficients.
5. Let $f(x)$ be a polynomial of degree greater than or equal to two with $f(-1) = f(1) = 0$, and $f(x) > 0$ on the interval $(-1, 1)$. Let L_{-1} and L_1 be tangents to the $f(x)$ at -1 and 1 , respectively. What is the area, T , of the triangle formed by L_{-1} , L_1 , and the x -axis? Is T bounded?
6. This exercise can initiate a more general discussion of conditions under which polynomial interpolation is possible
- (a) Given three real numbers $x_0 < x_1 < x_2$, and three more real numbers y_0, y_1, y_2 , not necessarily distinct, does there always exist a unique polynomial, $p(x)$, of degree less than or equal to 2 such that $p(x_0) = y_0$, $p(x_1) = y_1$, and $p(x_2) = y_2$?
- (b) Same as part (a) but replace $p(x_2) = y_2$ with $p'(x_2) = y_2$.
- (c) Same as part (a) but replace $p(x_1) = y_1$ with $p'(x_1) = y_1$.
7. **Reciprocal Polynomials.** A **reciprocal polynomial** is a polynomial whose coefficients are symmetric; that is, if $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$ is a reciprocal polynomial of degree n if $a_n = a_0$, $a_{n-1} = a_1$, etc.
- (a) Prove that if $p(x)$ is a reciprocal polynomial of odd degree, then $p(x) = (x + 1)q(x)$ where $q(x)$ is a reciprocal polynomial of even degree.
- (b) Prove that if $p(x) = a_{2k} x^{2k} + a_{2k-1} x^{2k-1} + \cdots + a_k x^k + \cdots$ is a reciprocal polynomial of even degree, then $p(x) = 0$ can be written in the equivalent form
- $$a_{2k}(x^k + 1/x^k) + a_{2k-1}(x^{k-1} + 1/x^{k-1}) + \cdots + a_k$$
- and that with the substitution $t = x + 1/x$, the equation becomes a polynomial of degree k in t .
- (c) Find all roots of the polynomial $2x^4 + 5x^3 + x^2 + 5x + 2$.
8. Find all polynomials, $p(x)$, with real coefficients satisfying the equation $p(x-1)p(x+1) = p(p(x))$.
9. Find all polynomials $P(x)$ with the property that $P(x)$ is a multiple of its second derivative, $P''(x)$.
10. The zeros of the polynomial $P(x) = x^3 - 10x + 11$ are u , v , and w . Determine the value of $\arctan u + \arctan v + \arctan w$.

Hint. $\tan(s+t) = \frac{\tan s + \tan t}{1 - (\tan s)(\tan t)}$.

11. Find a quadratic polynomial with the property that the sum of the squares of its roots is equal to 113.
12. Let $p(t)$ be any quadratic polynomial. It is easily verified that

$$p(1) + p(4) + p(6) + p(7) = p(2) + p(3) + p(5) + p(8).$$

This problem generalizes this property.

- (a) Partition the set $\{1, 2, 3, \dots, 15, 16\}$ into two sets such that given any cubic polynomial $p(t)$, the sum of the numbers $p(k)$ where k ranges over one of the two sets is the same as the sum where k ranges over the other
- (b) Generalize to polynomials of arbitrary nonnegative degree.

Hint. If you backtrack to the constant and linear cases of this problem, you can look for a pattern that suggests a cubic, and then general, solution.

13. Determine all polynomials $P(x)$ such that $P(x^2 + 1) = P(x)^2 + 1$ and $P(0) = 0$.
14. Let $a, b \in \mathbb{R}$, $a \neq 0$. Prove that if $p(x)$ and $q(x)$ are polynomials of degree n , $n \geq 1$, such that $p(ak + b) = q(x)$, $k = 0, 1, \dots, n$, then $p(ax + b) = q(x)$
15. **Chebyshev Polynomials of the First Kind.** Prove that if n is a nonnegative integer then $\cos n\theta$ is a polynomial in $\cos \theta$. If $\cos n\theta = T_n(\cos \theta)$, then $T_n(x)$ is the n th Chebyshev Polynomial of the First kind.

Chapter 10

Games

Mathematical games are a common type of problem in the Putnam. The two most recent cases have been difficult, A5 in 2017 and B5 in 2014. In analyzing games, it is always assumed that players use optimal strategies, i. e., they make rational moves in their best interests.

10.1 Finite Two Person Games - Nim

Nim is the name associated with a variety of games in which players take turns removing stones, sticks or some other objects from one or more piles, with the winner normally being the last player who can make a valid move.

Here is one of the simplest versions of Nim. We start with a single pile of stones. Let's say 14. A valid move for either player is to remove 1, 2 or 3 stones from the pile. That's all there is to the game. With an initial pile of 14, the first player will always win. Here is why: The first player will remove 2 piles from the pile leaving twelve stones. Now the second player is free to remove 1, 2, or 3 stones. Player 1 will respond by removing however many stones are needed to leave player 2 with eight stones. The long-term strategy for player 1 is to always leave player 2 with a multiple of four stones. Player 2 can never respond in such a way that player 1's strategy can't be continued. Eventually, player 2 will be left with four stones. Clearly player 1 is destined to win. Of course if the starting number of stones is a multiple of 4, it's player 2 who is destined to win using the same strategy.

Warm-up: How does the game change if the moves that are allowed are to remove 1, 2 or 4 stones, but not 3 stones?

10.2 Mind Games

Some games cannot really be played in that they would require an infinite number of moves. Other game-related problems ask a long term strategy, such as the following, where playing the game once is not the point.

Example 10.2.1 Near-Far. In the game Near-Far, Alpha and Beta each select a point in the closed unit disc. Alpha wins if the points are within $1/2$ of each other and Beta wins if they are not. They play the game repeatedly. What is the best strategy for each of them to adopt, and what is the probability of Alpha winning?

It would seem that the best strategy for Beta would be to select a random

point on the edge of the disc. Even if Alpha knows this, the best response is to select a point that “covers” as much of the edge as possible. This can be done by selecting a point $\sqrt{3}/2$ units from the center of the disc, covering an arc of length $\pi/3$. If they both stick to these stratagem, the probability Alpha wins is $\frac{1}{6}$. However, Beta can also select the center of the disc, being sure of winning if Alpha sticks to the strategy described above. Knowing this, Alpha should employ a mixed strategy, also selecting the center on occasion.

The mixed strategies can be summarized with a matrix showing the expected pay-off for Alpha. The C-strategy is to select the center of the disc while the E-strategies are to select a random point on the disc at distances of $\sqrt{3}/2$ and 1 for Alpha and Beta, respectively.

Table 10.2.2 Mixed Strategy Matrix for Near-Far

		Beta	
		C	E
Alpha	C	1	-1
	E	-1	$-\frac{2}{3}$

If Alpha chooses the C-strategy with probability α and Beta chooses C with probability β , then the expected payoff for Alpha will be

$$A(\alpha, \beta) = \frac{7\alpha\beta}{3} - \frac{\alpha}{3} - \frac{\beta}{3} - \frac{2}{3}$$

This function has a saddle point at $\alpha = \beta = \frac{1}{7}$ and its value there is $-\frac{5}{7}$, which implies that the probability Alpha wins is $\frac{1}{7}$. \square

10.3 See also

Problem B2 in the February 2021 Putnam.

10.4 Problems

1. **Nim Piles.** This game starts with three piles of stones. The piles contain 5, 7, and 11 stones. Two players take turns. On each turn, a player must remove at least one stone, and may remove any number of stones provided they all come from the same pile. The goal is to be the last player to take a stone. What player has a definite winning strategy; i. e. will always win if he/she makes the right moves.

Hint. Look closely at these two positions: two piles of three stones, or three piles with three, two and one stone. These are both losing positions for the next player for the same reason.

2. A line of squares is laid out starting with square 1, then square 2 to its right, square 3 further right, etc. The number of squares can be as long as you like. Put some coins in some of the squares. For example. one on square 1, two on square 3, and one on square 6. Take turns moving one coin any positive number of squares to the left. There are no restrictions otherwise. You can jump onto or over other coins, or jump clear off the line. You can have any number of coins on a square. Your aim is to be the person who makes the last move. If players 1 and 2 alternate moves with player 1 going first, who will win?

3. Two players play the following game. They start with two piles of candy, one with 21 and the other with 20 candies. At each turn, a player eats one of the piles and splits the other into two piles. The first player who can't make a legal move loses. Who will win, the first player or the second player?

Hint. If you leave our opponent with two “piles” with one candy each, you win.

4. Two players play the following game. Start with the equation

$$_x^3 + _x^2 + _x + _ = 0$$

The players take turns writing real numbers instead of blanks. The first player wins if the resulting cubic equation has exactly one real solution; otherwise he loses. Does either player have a winning strategy?

5. Consider the following game played by players A and B with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players. Beginning with A, the players take turns discarding one of the remaining cards of their choice and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?
6. Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy.
7. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty 3×3 matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the 3×3 matrix is completed with five 1's and four 0's. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?
8. Alice and Barbara play a game with a pack of $2n$ cards, on each of which is written a positive integer. There is no restriction on the size of integers and whether they repeat or not. The pack is shuffled and the cards laid out in a row, with the numbers facing upwards. Alice starts, and the girls take turns to remove one card from either end of the row, until Barbara picks up the final card. Each girl's score is the sum of the numbers on her chosen cards at the end of the game. High score wins. Is either player sure to win?
9. A game is played as follows. The first player selects an interval $[a, b]$. The second player selects an interval $[c, d] \subset [a, b]$. The first player selects an interval inside $[c, d]$, and so on. The game goes on forever. The first player will win if the intersection of all segments contains a rational number. Is he going to win?
- Hint.** The key is that the rationals are countably infinite while the irrationals are not.
10. Players 2 and 3 alternate removing stones in a nim-type game with player 2 going first and winner being the one to remove the last stone. Player 2 may remove either 1 or 2 stones in any move and Player 3 can remove either 1 or 3 stones in any move. If the pile starts with n stones, for what value of n will Player 2 win the game?

Hint. Player 3 can never remove an even number of stones.

Chapter 11

Geometry

There are three basic approaches to geometric problems:

- (1) Axiomatic approach, where everything is deduced from basic facts, such as congruence tests for triangles (SAS, ASA, and SSS), similarity of triangles, angles in the circle theorem, etc. These problems are good practice, but you are unlikely to see them on the Putnam exam.
- (2) Method of coordinates. Points on the plane are interpreted as coordinates $(x, y) \in \mathbb{R}^2$, or vectors, or complex numbers. Calculations can often be simplified by using basic linear algebra (scalar products, etc.) and knowing geometric interpretations of various algebraic operations (e.g. multiplication of complex numbers). Alternatively, a lot of things can be computed using trig functions.
- (3) Symmetries and transformations. This is a more dynamic approach, where you apply and compose rotations, symmetries, etc.

Often, problems are only formulated using geometric language but the solution uses some counting trick, or combinatorics, etc.

11.1 Examples

This is a useful “lemma” that lets you convert a statement about the angle between two line segments to a statement about distances between endpoints.

Lemma 11.1.1 *Two line segment on the plane AB and CD are perpendicular if and only if $|AC|^2 + |BD|^2 = |AD|^2 + |BC|^2$*

Proof.

(\Rightarrow). Assuming the two segments are perpendicular, we locate the point of intersection, M , of the two lines they determine, as in [Figure 11.1.2](#). In the figure, we assume that M does not lie on either segment, but the equations below still apply in other cases. The right angle at M determines four right triangles and thus

$$\begin{aligned} |AC|^2 &= |MA|^2 + |MC|^2 & |AD|^2 &= |MA|^2 + |MD|^2 \\ |BD|^2 &= |MB|^2 + |MD|^2 & |BC|^2 &= |MB|^2 + |MC|^2 \end{aligned}$$

If we add the equations on the right and also the two on the left, we get the desired conclusion.

(\Leftarrow). Again, let M be the point of intersection of the two extended segments.

In Figure 11.1.3 we depict the case where M lies on both segments. We assume the angle between the segments is not $\pi/2$ and so we assume that $\angle AMC > \pi/2$. Thus, the left sides of the equations on the left are now greater than the right sides, while the left sides of the equations on the right are less than the right sides. Therefore, $|AC|^2 + |BD|^2 > |AD|^2 + |BC|^2$.

Finally, how can this special case be used to prove the case when M does not lie on both segments?

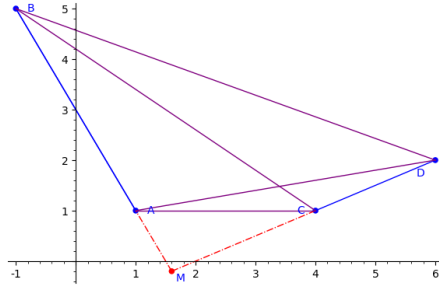


Figure 11.1.2 Two perpendicular segments

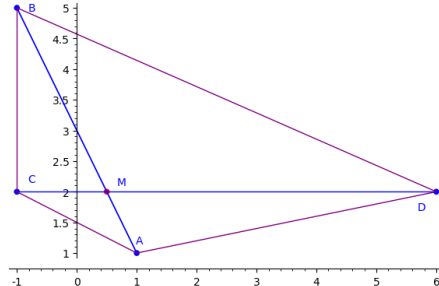


Figure 11.1.3 Two segments that are not perpendicular

■

11.2 Exercises

1. Prove that a central angle subtended by a given circular arc is twice the angle of an inscribed angle for the same arc.
2. Inscribe a rectangle of base b and height h and an isosceles triangle of base b in a circle of radius one as shown. For what value of h do the rectangle and triangle have the same area?
3. What convex quadrilaterals can be inscribed into a circle? There is a name for these quadrilaterals, but your answer should describe them, not just name them.
4. A rectangle with sides a and b , $a < b$ is rotated about its center, as shown in the figure, so that the two rectangles share two vertices. What is the area of the parallelogram that makes up the intersection of the two rectangles?

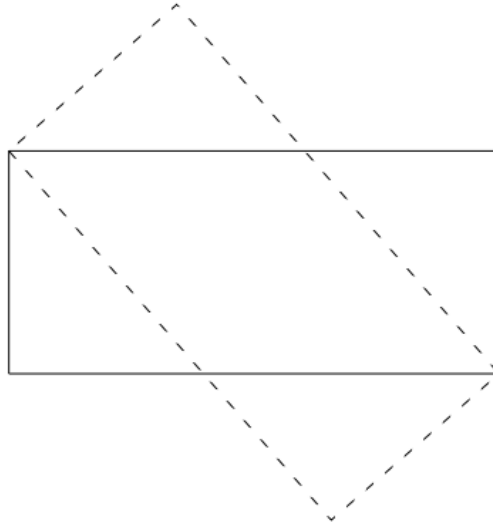


Figure 11.2.1 What is the area of the intersection?

5. Prove that if the lengths of the sides of a triangle form an arithmetic progression, then the radius of the inscribed circle is one third of one of the heights of the triangle.
Hint. Try two-way counting.
6. A piece of paper is in the shape of a rectangle $ABCD$ with $AB = CD = 3$ and $AD = BC = 5$. The paper is folded so that A and C coincide. Find the length of the crease. Generalize your result.
Hint. The line connecting A and C is perpendicular to the fold.
7. On the hyperbola $xy = 1$ consider four points whose x -coordinates are x_1 , x_2 , x_3 , and x_4 . Show that if these points lie on a circle, then $x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 1$.
8. Let ABC be a triangle and let the angle bisector of $\angle A$ intersect the side BC at a point D . Show that $\frac{AB}{AC} = \frac{BD}{CD}$.
9. A trapezoid and a triangle are inscribed a circle. One side of the trapezoid is a diameter of the circle and the sides of the triangle are parallel to sides of the trapezoid. Prove that the trapezoid and triangle have the same area.

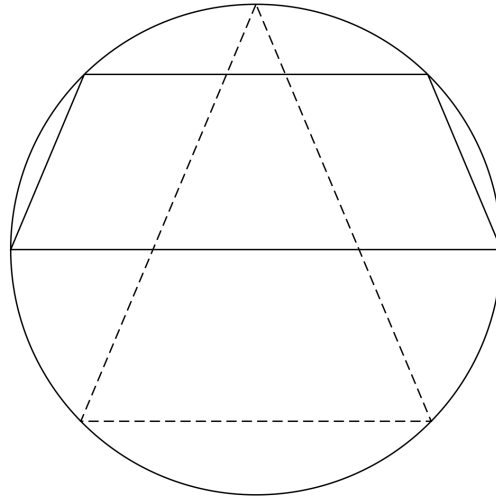


Figure 11.2.2 Do the trapezoid and triangle have equal areas?

Hint. Use coordinates with the circle at the origin and the diameter on the x -axis. You can assume that the radius of the circle is 1.

10. Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?

Hint. Start with a square and divide it into four squares.

11. Given nine lattice points in \mathbb{R}^3 prove that there exist two of them with the property that the midpoint of the segment between them is a lattice point.
12. Consider triangle ABC with the following trisection points,
- P on segment AB closest to B
 - R on segment BC closest to C and
 - Q on segment CA closest to A .

If these points are connected by segments to the opposite vertices of the triangle, is the area of the inner triangle created by the segments related to the area of triangle ABC ?

Chapter 12

Probability

If x is a random variable with outcomes x_1, x_2, x_3, \dots and corresponding probabilities p_1, p_2, p_3, \dots , then the expected value of x is $x_1p_1 + x_2p_2 + x_3p_3 + \dots$.

12.1 Conditional Probability

Example 12.1.1 Family Matters. Assume that the probability that a child is born a boy is exactly $\frac{1}{2}$. Now suppose you are told that a certain family has two children and one of them is a boy. What is probability that the family has two boys? Our intuition may tell us a certain answer, but many of us are wrong. This is a case for conditional probability. \square

Definition 12.1.2 Conditional Probability. Suppose a random event takes place and let H a condition on the event that has a positive probability. The probability that a second condition, A , is met with knowledge that H is true is called the “the probability of A conditioned on H ,” denoted $P(A | H)$. \diamond

Theorem 12.1.3 The Law of Conditional Probability. *Given a random event with a condition, H , that has a positive probability. Let A be a second condition on this event. Then*

$$P(A | H) = \frac{P(A \wedge H)}{P(H)}$$

In the example above, the random event of two children being born in sequence has four equally likely outcomes; BB, BG, GB, GG. We are told that one of the first three has occurred, so $P(H) = \frac{3}{4}$ in this case. The condition that one of the children is boy and both are boys is the same the condition that both are boys, so $P(A \wedge H) = \frac{1}{4}$ in this case, therefore the probability we want is $\frac{1/4}{3/4} = \frac{1}{3}$. This conclusion often surprises people. Did it surprise you?

12.2 Geometric Probability

Some probability problems can be solved using geometric means. Here is an example that illustrates the technique.

Example 12.2.1 Throwing Darts. What is the probability that the sum of two randomly chosen numbers in the interval $[0, 1]$ does not exceed 1 and

their product exceeds $\frac{3}{16}$?

This is a routine in that we can think of a selection of the random numbers as throwing a dart at a square dartboard so that each point on the dartboard is equally likely. This isn't realistic for good dart players, but imagine it anyways! The condition on which we want to assess the probability determines a region on the dartboard. The area of that region divided by the total area of the dartboard, which is 1, is the probability we are looking for. This means the problem reduces to the computation of the area shown in Figure 12.2.2, which can be done with basic calculus in this case.

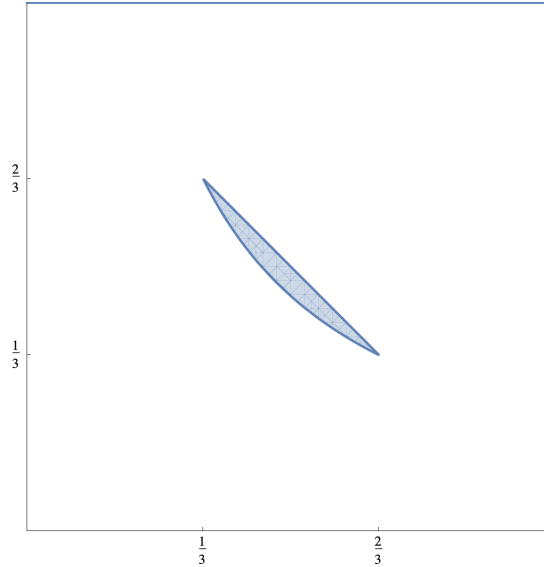


Figure 12.2.2 A target on a square dart board

□

The challenge of geometric probability problems tends to be identification of regions that represent the possible outcomes and the outcomes that are of interest. The dimension can be higher than two.

12.3 Exercises

- Suppose that n integers, $n \geq 3$ are randomly arranged in some list. We might question whether they have ended up in strictly increasing order. To test for this unlikely event, we might examine the first two numbers in the list. Since their positions are random, the probability that they are in increasing order, is $\frac{1}{2}$. If they are in order, we might next make a comparison of the second and third numbers, the third and fourth numbers, etc. Since each pair can be in its current relative position or in the reverse, the probability of each pair being in order is $\frac{1}{2}$. If we continue to the end with increasing orders and the last two integers are in order, then we might conclude that the probability that the whole list is ordered is $\frac{1}{2^{n-1}}$. Is this correct?
Hint. We wouldn't ask if the logic were correct!
- Steph Kuri shoots free throws on a basketball court. She makes the first and misses the second, and thereafter the probability that she makes the next shot is equal to the proportion of shots she has made so far. What is the probability that she makes exactly 50 of her first 100 shots?

3. King Arthur is sick of fights for inheritance and decides to announce the following law. From now on, no family will be allowed to have another child after a boy is born. What will happen to the percentage of males if the law is followed?
4. Four players in a game, N(orth), W(est), S(outh), and E(ast) sit at a round table. N holds a gold coin. One of three things happen, each with probability $1/3$: N passes the coin to his right to W, N passes the coin to his left to E, or N walks away with the coin and the game is over. In the first two cases, the holder of the coin repeats the previous step, passing the coin to the right or left, or keeping the coin. What are the probabilities of winning for the four players?
5. Suppose two teams play a series of games in which the probability that either wins any of the games is $\frac{1}{2}$. If they play a best of seven series, where the winner is the first to win four games, what is the probability that the series lasts seven games?
6. **Simplified NCAA basketball pool.** There are 64 teams who play single elimination tournament, hence 6 rounds, and you have to predict all the winners in all 63 games. Your score is then computed as follows: 32 points for correctly predicting the final winner, 16 points for each correct finalist, and so on, down to 1 point for every correctly predicted winner for the first round. Knowing nothing about any team, you flip fair coins to decide every one of your 63 bets. Compute your expected number of points.
7. Three real numbers a, b, c are randomly (and uniformly) chosen from the interval $[0, 1]$. What is the probability that there exists a triangle with sides a, b, c ?
8. Let p_n be the probability that $c + d$ is a perfect square, where the integers c and d are selected independently at random from the set $\{1, \dots, n\}$. Find $\lim_{n \rightarrow \infty} \sqrt{n}p_n$.
Hint. Start by counting points in the diagonals of the square array of lattice points (c, d) such that $c + d$ is a square. There are two formulae for the number of points: one for diagonals that produce the squares of the numbers 2 through $\lfloor \sqrt{n+1} \rfloor$, and a second for those that produce squares of the the numbers from $\lfloor \sqrt{n+1} \rfloor + 1$ to $\lfloor \sqrt{2n} \rfloor$.
9. Two real numbers x and y are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.
10. Let $\alpha \in [0, 1]$ be an arbitrary number, rational or irrational. The only randomizing device is an unfair coin, with probability $p \in (0, 1)$ of heads. Design a game between Alice and Bob so that Alice's winning probability is exactly α . The game has to end in a finite number of tosses with probability 1.
11. Begin with the set $\{1, 2, \dots, n\}$. Toss a coin n times, once for each member of the set. Keep the elements that scored "Heads" and discard the elements that got "Tails". You now have a certain subset S of the original set. Call this whole process a "step". Now take a step from S . That is, toss a coin for each element of S , and keep those that get "Heads", getting a sub-subset S' , etc. This game halts when the empty set is reached. Let $f(n, k, r)$ be the probability that after k steps, exactly r objects remain.
 - (a) Find a recurrence relation for f , find the generating function for f , and find f itself.

- (b) What is the average number of steps in a complete game?
- 12.** An urn contains a number of colored balls, with the same number of balls in each color. If 20 balls of a new color are added to the urn, the probability of drawing (without replacement) two balls of the same color is not changed. How many balls are in the urn (before the new balls are added)?
- 13.** Suppose that the probability that the length of a telephone call is between t_1 and t_2 minutes, $t_1 < t_2$ is $\int_{t_1}^{t_2} e^{-t} dt$. What is the probability that a phone call lasts less than one minute. What is the probability that a phone call lasts less than two minutes given that it lasts at least one minute?

Chapter 13

Linear Algebra

13.1 Examples

Example 13.1.1 Oddtown. In a town with n people, m clubs have been formed. Every club has an odd number of members, and every two clubs have an even number of members in common. Prove that $m \leq n$.

Let v_i be the membership vector for club i : $(v_i)_j = 1$ and only if person j belongs to club i . We view these m vectors as being in the vector space \mathbb{Z}_2^n of n -tuples of integers mod 2. We will prove that the set of vectors $\{v_1, v_2, \dots, v_m\}$ is linearly independent, which implies that $m \leq n$.

The conditions of problem translate to the following with respect to the dot products of these vectors.

$$\text{Each club has an odd number of members} \Rightarrow v_i \cdot v_i = 1$$

and

$$\text{Any two clubs have an even number of members in common} \Rightarrow v_i \cdot v_j = 0 \text{ if } i \neq j$$

Now, assume $\sum_{i=1}^m c_i v_i = \mathbf{0}$. We will show that each $c_k = 0$ using the linearity of the dot product.

$$\begin{aligned} 0 &= v_k \cdot \mathbf{0} \\ &= v_k \cdot \sum_{i=1}^m c_i v_i \\ &= \sum_{i=1}^m c_i (v_k \cdot v_i) \\ &= c_k (v_k \cdot v_k) \\ &= c_k \end{aligned}$$

Therefore we have derived the inequality. □

13.2 Exercises

1. Let $N = 2^n$ for some positive integer n and $\omega = e^{\frac{2\pi i}{N}}$. Let $U(\omega)$ be the $N \times N$ matrix with $U(\omega)_{i,j} = \omega^{(i-1)(j-1)}$. Compute $U(\omega)U(\omega^{-1})$.

2. Let A and B be $n \times n$ matrices satisfying $A + B = AB$. Show that $AB = BA$.
3. Consider the matrix equation $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^n = \begin{pmatrix} A_n & B_n & C_n \\ D_n & E_n & F_n \\ G_n & H_n & J_n \end{pmatrix}$. Identify formulae for A_n, B_n, \dots, J_n .
4. Let A be a square matrix, $A \neq I$, and suppose there exists a positive integer m such that $A^m = I$. Calculate $\det(I + A + A^2 + \dots + A^{m-1})$.
Hint. The matrix that you want the determinant of is a finite geometric series.
5. Let's call a real 3×3 matrix a "magic square" if all its row-sums and column-sums are equal to 0. Show that all magic squares form a vector space. Find its dimension.
6. Let n be a positive integer, and let v_1, v_2, \dots, v_k , $1 \leq k \leq n$, be linearly independent vectors in \mathbb{R}^n spanning a subspace V_k . Prove that there exist k orthogonal vectors w_1, w_2, \dots, w_k that span V_k . (Note: Two nonzero vectors are orthogonal if their dot product is equal to 0.)
Hint. Use the linearity of the dot product: $x \cdot (ay + z) = a(x \cdot y) + x \cdot z$, where a is a scalar.
7. Let A be the $n \times n$ matrix whose entry in the i -th row and j -th column is $\frac{1}{\min(i,j)}$ for $1 \leq i, j \leq n$. Compute $\det(A)$.
Hint. Expand the determinant along the last row of the matrix.
8. The dashboard of a nuclear power station has several lights. Some lights are on and some are off. There are also several buttons. Pressing each button changes the state of several lights (from on to off and from off to on). It is known that for every set of lights there exists a button connected to an odd number of lights in this set. Show that one can turn off all lights by pressing some buttons.
Hint. Start by creating a matrix, one row for each button and a column for each light.
9. Compute the determinant of the $n \times n$ matrix $[a_{ij}]$ such that $a_{ij} = |i - j|$.
10. Think of a square matrix as placed on a checkerboard, so that the leading diagonal consists entirely of white squares. Then if the signs of all the entries on black squares are changed, prove that the eigenvalues are unchanged.
11. It is well known that all real-valued functions on \mathbb{R} form a vector space. Does the function $\sin x$ belong to the linear span of the functions $1, \cos x, \cos 2x, \cos 3x, \dots$? How about the function $\sin^2 x$?

Chapter 14

Abstract Algebra

Groups and finite fields are among the topics that have made their way into the Putnam lately.

14.1 Groups

Let G be a group. Recall that the order of $g \in G$ is the minimum positive integer k such that g^k is the group identity.

Example 14.1.1 1969 Putnam, altered. Show that a group cannot be the union of two of its proper subgroups.

Assume H and K are subgroups of some group G , and that $x \in H - K$, $y \in K - H$. We know that $x * y$ must be in either H or K . Assume it's in H .

$$x \in H \Rightarrow x^{-1} \in H \Rightarrow x^{-1} * (x * y) = y \in H$$

This contradiction implies that no such pair of subgroups exists. \square

14.2 Rings and Fields

Problem: Assume that $p(x)$ is a polynomial of degree n over a field F that has n distinct roots in F . Let $I = (p(x)) = \{s(x)p(x) \mid s(x) \in F[x]\}$ be the principle ideal generated by $p(x)$. Characterize the factor ring $F[x]/I$.

Assume

$$p(x) = \prod_{k=1}^n (x - \alpha_k),$$

where the α_k 's are distinct elements of F . We can show that $F[x]/I$ is isomorphic to the ring F^n of n -tuples of elements of F with coordinatewise operations over F . Define $\phi : F[x]/I \rightarrow F^n$ by

$$\phi(r(x) + I) = (r(\alpha_1), r(\alpha_2), \dots, r(\alpha_n))$$

We leave it to the reader to verify that ϕ is an isomorphism by checking the following:

- ϕ is well defined bijection.
- For all $r(x) + I, s(x) + I \in F[x]/I$:

$$\phi((r(x) + I) + (s(x) + I)) = \phi(r(x) + I) + \phi(s(x) + I)$$

- For all $r(x) + I, s(x) + I \in F[x]/I$:

$$\phi((r(x) + I) \cdot (s(x) + I)) = \phi(r(x) + I) \cdot \phi(s(x) + I)$$

A quick note on the need ϕ to be well defined. Since the same coset can be generated from different polynomials, and since ϕ is defined in terms of a generating polynomial, we need to be sure that if $s(x)$ and $t(x)$ are different polynomials for which $s(x) + I = t(x) + I$, we get $\phi(s(x) + I) = \phi(t(x) + I)$.

14.3 Exercises

1. Let S be a set which is closed under the binary operation \circ , with the following properties:

- There is an element $e \in S$ such that $a \circ e = e \circ a = a, \forall a \in S$
- $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d) \forall a, b, c, d \in S$.

Prove or disprove:

(a) \circ is associative on S

(b) \circ is commutative on S

2. Let $*$ be an associative binary operation on a set A such that for all $a, b \in A$, $a * b = b * a \Rightarrow a = b$. Prove that for all $a, b, c \in A$, $a * b * c = a * c$.
3. (Putnam 1972) Let S be a set and a binary operation on S satisfying the laws

(i) $x(xy) = y$ for all $x, y \in S$

(ii) $(yx)x = y$ for all $x, y \in S$

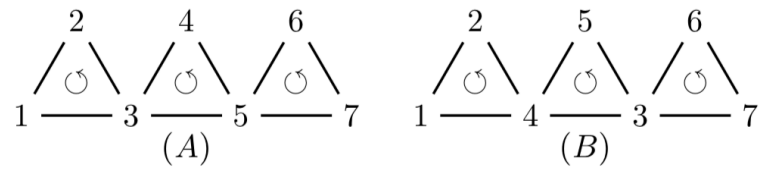
Show that $*$ is commutative but not necessarily associative.

4. Let G be a finite group with identity e . If G contains distinct elements g and h , neither equal to e , such that

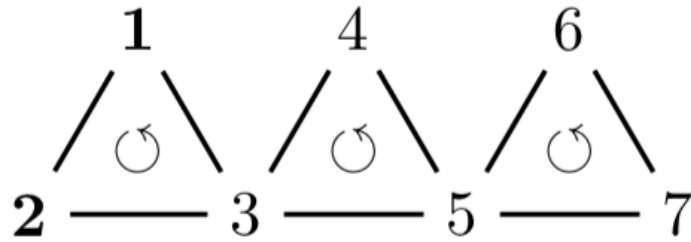
$$g^5 = e \text{ and } ghg^{-1} = h^2,$$

determine the order of h .

5. In abstract algebra, the symmetric group S_n is the group of all permutations on $\{1, 2, \dots, n\}$; i. e., bijections on $\{1, 2, \dots, n\}$. A transposition in S_n is a function τ such that $\tau(i) = j$ and $\tau(j) = i$, where $i \neq j$ and $\tau(k) = k$ for $k \neq i, j$. A fundamental theorem involving permutations (which we will take as given) is that every permutation is a composition of an even number of transpositions or an odd number of transpositions, but not both. Prove that exactly half of the elements of S_n are “even.”
6. S_n is a group of order $n!$. What is the smallest value of n for which there exists an element of order 10?, of order 20?
7. Consider the puzzle below, where one can rotate each of the three triangles. For example, rotating the middle triangle in (A) once gives the configuration in (B).

**Figure 14.3.1** Spinning Triangles Puzzle

Prove that it there is no sequence of rotations that produce the following configuration, starting from (A).

**Figure 14.3.2** Spinning Triangles Puzzle - Impossible Configuration

Chapter 15

2020 UML Problem Solving Competition

A three hour local competition was held on December 5, 2020 in place of the Putnam, which was delayed until February 2021.

15.1 Results

- First Place: Payton Collins, Senior Math major
- Second Place: Nurlan Gasimli, Freshman Math major
- Third Place: Charles Mirabile, Senior CS major
- Second Place among Freshmen/Sophomores: Hunter Marion, Sophomore Math major

15.2 Problems

1. Let R denote the set of points (x, y) satisfying $x^2 - 4|x| + y^2 + 3 \leq 0$. What is the area of R ?
2. Let $f(n)$ be the number of n -letter words that can be formed with the letters A, B, C and such that the letter A occurs an even number of times. For example, when $n = 1$, there are 2 such words, namely B, C so $f(1) = 2$; when $n = 2$, there are 5 such words, namely AA, BB, BC, CB, CC, so $f(2) = 5$. Find, with proof, a simple formula for $f(n)$. (The formula should not involve a summation.)
3. An equilateral triangle in the first quadrant has vertices at the points $(0, 0)$, $(x_1, 5)$, and $(x_2, 12)$. What is the ordered pair (x_1, x_2) ? Show your work.
4. A square matrix A has a "square root" if there exists a matrix B such that $B^2 = A$.
 - (a) Prove that $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has no square root.
 - (b) Determine, with proof, whether $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a square

root.

5. Let n be a positive integer. Find all $(n+1)$ -tuples $(n, x_1, x_2, \dots, x_n)$ where the x_i are positive real numbers and satisfy the equations

$$x_1 + x_2 + \dots + x_n = 9$$

and

$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = 1.$$

6. At the Hard Math Casino there is a wheel with the numbers 1, 2, 3 and 4. When you spin the wheel, a pointer points to one of the numbers, each with probability $\frac{1}{4}$. You are given a “target number” which is a positive integer n . You spin the wheel repeatedly, adding the resulting numbers and win the game if you get a sum of exactly n . Let $p(n)$ be the probability of winning if your target number is n . What is $\lim_{n \rightarrow \infty} p(n)$?

Chapter 16

2022 UML Problem Solving Competition

A two hour local competition was held on 15 April 2022.

16.1 Results

- First Place: Joshua Costantini, Math major

16.2 Problems

1. How many positive integers are there which, in their decimal representation, have strictly decreasing digits? Explain! Note: We are counting integers such as 94310, 7 or 540 and but not 441 or 0.
2. Prove that there are infinitely many number pairs (a, b) such that

$$a + \frac{1}{b} = b + \frac{1}{a}$$

where $a \neq b$. Find the possible values of ab .

3. Show that for all non-negative reals a, b, c ,

$$\sqrt{a + \sqrt{b + \sqrt{c}}} \leq a^{1/2} + b^{1/4} + c^{1/8}$$

When is there equality?

4. Let \mathbf{a} and \mathbf{b} be non zero vectors in \mathbb{R}^3 that are not multiples of one another. Provide a formula (with proof) for a vector bisecting the angle between \mathbf{a} and \mathbf{b} .
5. There are 87 airports in New England. Suppose that from each of these airports a plane takes off and flies to the nearest neighboring airport. Assuming the mutual distances between the airports are all distinct prove that there is no airport at which more than five planes land.
6. Tammy and Sammy play a dice game with three different six-sided dice:

A, B and C. Here are the numbers on the sides of each die:

A 4, 4, 4, 4, 4, 1

B 5, 5, 5, 2, 2, 2

C 6, 3, 3, 3, 3, 3

Tammy selects one of the dice and then Sammy selects one of the two remaining dice. Then they each roll their die once, with the winner being the one with the higher roll. Who has the advantage in this game and why?

Appendix A

Hints to Selected Exercises

Some solutions are partial.

Appendix B

Table of Common Sequences

For your convenience, here are the first few powers of 2, squares, Fibonacci numbers and values of the Euler Phi function.

n	2^n	n^2	F_n	$\phi(n)$
0	1	0	1	0
1	2	1	1	1
2	4	4	2	1
3	8	9	3	2
4	16	16	5	2
5	32	25	8	4
6	64	36	13	2
7	128	49	21	6
8	256	64	34	4
9	512	81	55	6
10	1024	100	89	4
11	2048	121	144	10
12	4096	144	233	4
13	8192	169	377	12
14	16384	196	610	6
15	32768	225	987	8
16	65536	256	1597	8
17	131072	289	2584	16
18	262144	324	4181	6
19	524288	361	6765	18
20	1048576	400	10946	8
21	2097152	441	17711	12
22	4194304	484	28657	10
23	8388608	529	46368	22
24	16777216	576	75025	8
25	33554432	625	121393	20
26	67108864	676	196418	12
27	134217728	729	317811	18
28	268435456	784	514229	12
29	536870912	841	832040	28
30	1073741824	900	1346269	8
31	2147483648	961	2178309	30
32	4294967296	1024	3524578	16
33	8589934592	1089	5702887	20
34	17179869184	1156	9227465	16
35	34359738368	1225	14930352	24
36	68719476736	1296	24157817	12

References

Many of the problems in this book have been used in multiple sources. If a problem was set in a known competition, it is cited in the problem solution. For most other problems, no effort has been made to identify when it was first posed, but most of the problems in this book come from the following list of sources.

The problem sections of Mathematical Association of America journals:

- The American Mathematical Monthly
- Mathematics Magazine
- The College Mathematics Journal

Problems from past exams are used in several cases. A list of the competitions from which I've drawn examples follows.

- MAA Northeast Section Collegiate Math Competition
- Virginia Tech Regional Mathematics Contest
- William Lowell Putnam Mathematics Competition (naturally!)

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Colophon

This book was authored in PreTeXt.