This is a mess notes of some useful equations for attitude and orbital mechanics simulation. For detailed derivations, assumptions, please refer to the referenced materials.

1 Notations and Conventions

Description	Notation	Comment
Body angular rate	$\boldsymbol{\omega} = \boldsymbol{\omega}^B_{B/I} = \boldsymbol{\omega}^B_{I\to B}$	Angular rotation of the body frame with respect to the inertial frame expressed in the body frame.
Attitude quaternion	$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}_{vec} \\ q_{sca} \end{bmatrix} = \boldsymbol{q}_{B/I} = \boldsymbol{q}_{I\to B}$	Scalar-last right-handed unit quaternion representing attitude transformation from inertial to body frame.
Position vector	$\boldsymbol{p}^I_{V/I} = \boldsymbol{p}^I_{I\to V}$	A position vector from the origin of the I frame to the Vehicle, expressed in the I frame. Similarly to a velocity vector. Note that strictly speaking, a frame has no origin.

2 Common Attitude Functions

$$[\boldsymbol{a}\times] \equiv \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \qquad 3\times 3$$

$$\Xi \equiv \begin{bmatrix} q_{sca} I_3 + [\boldsymbol{q}_{vec}\times] \\ -\boldsymbol{q}^T_{vec} \end{bmatrix} \qquad 4\times 3$$

$$\Psi \equiv \begin{bmatrix} q_{sca} I_3 - [\boldsymbol{q}_{vec}\times] \\ -\boldsymbol{q}^T_{vec} \end{bmatrix} \qquad 4\times 3$$

$$\Omega \equiv \begin{bmatrix} -[\boldsymbol{\omega}\times] & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^T & 0 \end{bmatrix} \qquad 4\times 4$$

$$\bar{\Omega} \equiv \begin{bmatrix} \cos\left(\frac{1}{2}\|\boldsymbol{\omega}\|\Delta t\right) I_3 - [\boldsymbol{\psi}\times] & \boldsymbol{\psi} \\ -\boldsymbol{\psi}^T & \cos\left(\frac{1}{2}\|\boldsymbol{\omega}\|\Delta t\right) \end{bmatrix} \qquad 4\times 4$$

$$\boldsymbol{\psi} \equiv \frac{\sin\left(\frac{1}{2}\|\boldsymbol{\omega}\|\Delta t\right)\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|} \qquad 3\times 1$$

$$\Gamma \equiv \begin{bmatrix} [\boldsymbol{n}\times] & \boldsymbol{n} \\ -\boldsymbol{n}^T & 0 \end{bmatrix} \qquad 4\times 4$$

Note that

- $\Omega^T(\boldsymbol{b}) = -\Omega(\boldsymbol{b})$ and $\Gamma^T(\boldsymbol{b}) = -\Gamma(\boldsymbol{b})$
- Ω and Γ can be functions to other 3×1 vectors.

3 Attitude Parametrization

$$\begin{aligned} \boldsymbol{q} &= \begin{bmatrix} \boldsymbol{q}_{vec} \\ q_{sca} \end{bmatrix} \\ &= \begin{bmatrix} \hat{\boldsymbol{e}} \sin\left(\frac{\vartheta}{2}\right) \\ \cos\left(\frac{\vartheta}{2}\right) \end{bmatrix} \end{aligned}$$

$$A = \Xi^T(\boldsymbol{q})\,\Psi(\boldsymbol{q})$$

4 Attitude Kinematics

$$\dot{A} = -[\boldsymbol{\omega}\times]\,A$$

$$\begin{aligned} A_{k+1} &= \Phi_k A_k \\ \Phi_k &= I_3 - [\boldsymbol{\omega}\times]\frac{\sin(\|\boldsymbol{\omega}\|\,\Delta t)}{\|\boldsymbol{\omega}\|} + [\boldsymbol{\omega}\times]^2 \frac{1-\cos(\|\boldsymbol{\omega}\|\,\Delta t)}{\|\boldsymbol{\omega}\|^2} \end{aligned}$$

$$\dot{\boldsymbol{q}} = \frac{1}{2}\Xi(\boldsymbol{q})\,\boldsymbol{\omega}$$

$$\boldsymbol{q}_{k+1} = \bar{\Omega}(\boldsymbol{\omega}_k, \Delta t)\,\boldsymbol{q}_k$$

Assuming $\|\boldsymbol{\omega}\|\,\Delta t$ small or below Nyquist's interval limit

$$A_{k+1} = (I_3 - \Delta t\,[\boldsymbol{\omega}\times])\,A_k$$

$$\boldsymbol{q}_{k+1} = \left[I_3 + \frac{\Delta t}{2}\Omega(\boldsymbol{\omega})\right]\boldsymbol{q}_k$$

5 Attitude Dynamics

$$J\dot{\boldsymbol{\omega}} = \boldsymbol{\tau} - [\boldsymbol{\omega}\times]\,(J\boldsymbol{\omega})$$

J is the moment of inertia matrix, often principal axes of inertia: J_x, J_y, J_z . For example, TRMM's inertial matrix

$$J = \begin{bmatrix} \quad \end{bmatrix}$$

6 Attitude Determination

TRIAD

Given $\boldsymbol{b}_1 = A\boldsymbol{n}_1$ and $\boldsymbol{b}_2 = A\boldsymbol{n}_2$, form

$$\begin{array}{lll} \boldsymbol{c}_1 = \boldsymbol{n}_1 \quad , & \boldsymbol{c}_2 = \frac{\boldsymbol{n}_1\times\boldsymbol{n}_2}{\|\boldsymbol{n}_1\times\boldsymbol{n}_2\|} \quad , & \boldsymbol{c}_3 = \boldsymbol{c}_1\times\boldsymbol{c}_2 \\ \boldsymbol{d}_1 = \boldsymbol{b}_1 \quad , & \boldsymbol{d}_2 = \frac{\boldsymbol{b}_1\times\boldsymbol{b}_2}{\|\boldsymbol{b}_1\times\boldsymbol{b}_2\|} \quad , & \boldsymbol{d}_3 = \boldsymbol{d}_1\times\boldsymbol{d}_2 \end{array}$$

Form C and D

$$C = \left[\begin{array}{ccc} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \boldsymbol{c}_3 \end{array} \right]$$
$$D = \left[\begin{array}{ccc} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 \end{array} \right]$$

Attitude matrix A is calculated by

$$D = AC$$
$$C = DC^T$$

Davenport q-Method

$$K = -\sum_{i=1}^{N} w_i \Omega\left(\boldsymbol{b}_i\right)\Gamma\left(\boldsymbol{n}_i\right)$$

Solve this eigenvalue/eigenvector problem

$$K\boldsymbol{q} = \lambda\boldsymbol{q}$$

There are 4 solutions, solution is the eigenvector corresponding to the largest eigenvalue.

QUEST Measurement Model

The error covariance

$$P = \left[-\sum_{i=1}^{N} \sigma_i^2 \left[A\boldsymbol{n}_i \times\right]\right]^{-1}$$

Can replace with $\tilde{\boldsymbol{b}}_i = A\boldsymbol{n}_i$, good enough as approximation error is in the second-order.

7 Attitude Estimation

Murrel's Form

Gyro model

$$\begin{array}{llllll} \boldsymbol{\omega} & = \tilde{\boldsymbol{\omega}} - \boldsymbol{\beta} - \boldsymbol{\eta}_v & , & E\left\{\boldsymbol{\eta}_v\right\} = \mathbf{0} & , & E\left\{\boldsymbol{\eta}_v\boldsymbol{\eta}_v^T\right\} = \sigma_v^2 I_3 \\ \dot{\boldsymbol{\beta}} & = \boldsymbol{\eta}_u & , & E\left\{\boldsymbol{\eta}_u\right\} = \mathbf{0} & , & E\left\{\boldsymbol{\eta}_u\boldsymbol{\eta}_u^T\right\} = \sigma_u^2 I_3 \end{array}$$

Example parameters $\sigma_v = \sqrt{10} \times 10^{-10}\text{rad/sec}^{3/2}$ and $\sigma_u = \sqrt{10} \times 10^{-7}\text{rad/sec}^{1/2}$
Attitude observations for a single sensor

$$\tilde{\boldsymbol{b}}_i = A\left(\boldsymbol{q}\right)\boldsymbol{n}_i + \boldsymbol{\nu}_i$$

$$\Delta\hat{\boldsymbol{x}} = \left[\begin{array}{c} \delta\boldsymbol{\alpha} \\ \delta\hat{\boldsymbol{\beta}} \end{array} \right]$$

where $\delta\boldsymbol{q}_{vec} \approx \delta\boldsymbol{\alpha}/2$
State Update

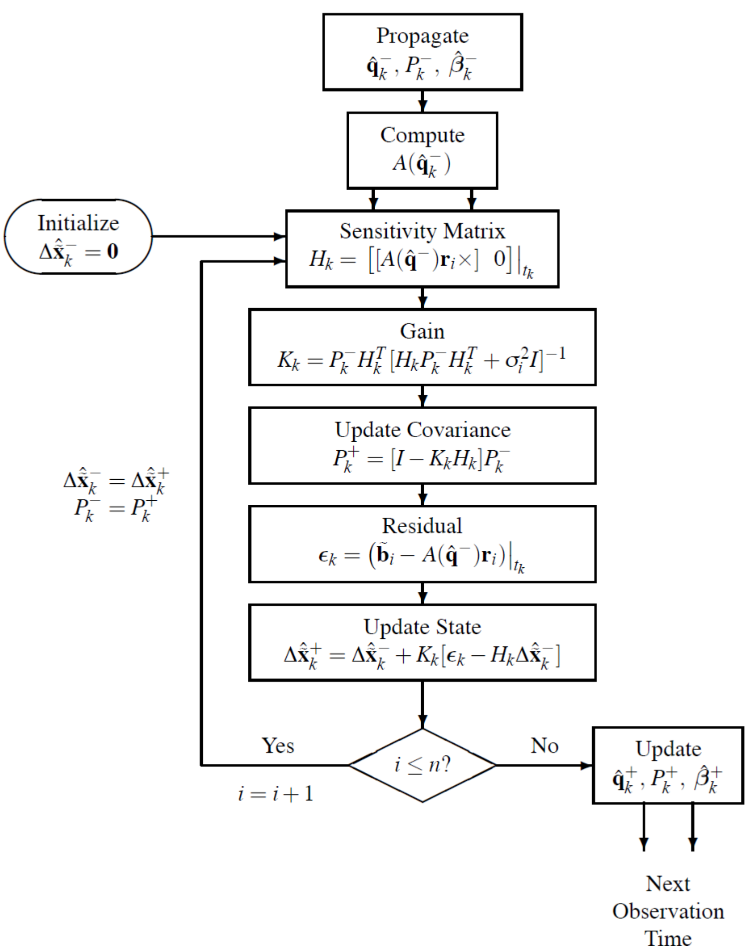
$$
\begin{aligned}
\hat{\boldsymbol{q}}_{k+1}^{+} &= \hat{\boldsymbol{q}}_{k+1}^{-} + \frac{1}{2}\Xi\left(\hat{\boldsymbol{q}}_{k+1}^{-}\right)\delta\boldsymbol{\alpha}_{k+1}^{+} \\
\hat{\boldsymbol{\beta}}_{k+1}^{+} &= \hat{\boldsymbol{\beta}}_{k+1}^{-} + \delta\hat{\boldsymbol{\beta}}_{k+1}^{+} \\
\hat{\boldsymbol{\omega}}_{k+1} &= \tilde{\boldsymbol{\omega}}_{k+1} - \hat{\boldsymbol{\beta}}_{k+1}^{+}
\end{aligned}
$$

State Propagation

$$
\begin{aligned}
\hat{\boldsymbol{q}}_{k+1}^{-} &= \bar{\Omega}\left(\hat{\boldsymbol{\omega}}_{k}^{+}, \Delta t\right)\hat{\boldsymbol{q}}_{k}^{+} \\
\hat{\boldsymbol{\beta}}_{k+1}^{-} &= \hat{\boldsymbol{\beta}}_{k}^{+}
\end{aligned}
$$

Covariance propagation

$$
\begin{aligned}
P_{k+1}^{-} &= \Phi_k P_k^{+}\Phi_k^T + \Gamma_k Q_k \Gamma_k^T \\
\Gamma_k &= \begin{bmatrix} -I_3 & 0 \\ 0 & I_3 \end{bmatrix} \\
\Phi_k &= \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \\
\Phi_{11} &= I_3 - [\hat{\boldsymbol{\omega}}\times]\frac{\sin\left(\|\hat{\boldsymbol{\omega}}\|\Delta t\right)}{\|\hat{\boldsymbol{\omega}}\|} + [\hat{\boldsymbol{\omega}}\times]^2\frac{I - \cos\left(\|\hat{\boldsymbol{\omega}}\|\Delta t\right)}{\|\hat{\boldsymbol{\omega}}\|^2} \\
\Phi_{12} &= [\hat{\boldsymbol{\omega}}\times]\frac{I - \cos\left(\|\hat{\boldsymbol{\omega}}\|\Delta t\right)}{\|\hat{\boldsymbol{\omega}}\|^2} - I_3\Delta t - [\hat{\boldsymbol{\omega}}\times]^2\frac{\|\hat{\boldsymbol{\omega}}\|\Delta t - \sin\left(\|\hat{\boldsymbol{\omega}}\|\Delta t\right)}{\|\hat{\boldsymbol{\omega}}\|^3} \\
\Phi_{21} &= 0_3 \\
\Phi_{22} &= I_3
\end{aligned}
$$



NASA Orion's

8 Orbital Mechanics

$$\ddot{\boldsymbol{r}}^I = -\frac{\mu}{\|\boldsymbol{r}^I\|^2}\boldsymbol{r}^I + \boldsymbol{a}^I_{SRP}$$

9 State Estimation

Error-State Kalman Filter

Divided-Difference-2 Sigma-Point Kalman Filter

Steady-State Kalman Filter

Closed-Loop Linear Covariance Analysis

References

1. Crassidis, John. An Overview of Spacecraft Attitude Determination and Estimation. NASA Engineering and Safety Center (NESC) Academy. https://nescacademy.nasa.gov/video/bdeb764e048940a6b2ae05c3cfdf5d261d

2. Markley, Landis & Crassidis, John. (2014). Fundamentals of Spacecraft Attitude Determination and Control. 10.1007/978-1-4939-0802-8.

3. Crassidis, John & Junkins, John. (2004). Optimal Estimation of Dynamic Systems. 10.1201/b11154.