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A DETERMINISTIC METHOD FOR ESTIMATING ATTITUDE FROM ONLY MAGNETOMETER DATA*

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Abstract

The paper presents a new deterministic algorithm to estimate spacecraft attitude using only magnetometer data. The algorithm exploits the dynamical equations of motion to propagate attitude and thus requires knowledge of both external and internal torques, except in the limiting case of a spacecraft rotating with constant angular velocity. In this case the resultant transcendental equations were reduced to a simple quadratic equation. The least-squares criterion was used to select the physical root.

The constant-angular-velocity version of the algorithm was tested using low-frequency telemetry data for the Earth Radiation Budget Satellite (ERBS) under normal conditions. It was demonstrated that observed small oscillations of body-fixed components of the angular velocity vector near their mean values result in minor attitude determination errors. More significant attitude determination errors come from analog-to-digital conversion of magnetometer data with the quantization increment of ~6.44 milligauss (mG), taking into account that these measurements are used to compute the second derivatives of the body-fixed components of the geomagnetic field with respect to time.

To test the general version of the algorithm the satellite dynamics under contingency conditions was simulated in different regimes such as 1) torque-free rotation of the spacecraft with and without a momentum wheel, 2) free rotation of the spacecraft perturbed by gravity gradient and magnetic torques (dominant for the ERBS) and 3) magnetic despin of the spacecraft in the B-dot mode. In all three cases the body-fixed components of the magnetic field were then discretized to simulate telemetry data. In addition to the ERBS, this part of tests included simulation of a real emergency situation for the Relay Mirror Experiment satellite. In the latter case magnetometer measurements are transfered with a much smaller quantization increment of ~0.5 mG than for ERBS. It was found that the resultant transcendental equations usually have several solutions depending on specifics of the spacecraft dynamics. For free-torque motion the conservation of the energy and the angular momentum was used to select the physical solution. The reasults of the simulation showed that magnetometer measurements with a quantization increment of ~0.5 mG make it possible to determine the attitude of the spacecraft with the maximum error of 3 degrees. On the other hand, errors in the attitude determination of the ERBS are expected to be about 10 degrees under similar circumstances.

The algorithm has been used to determine the attitude of the uncontrolled RME satellite using real telemetry data.

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1. Introduction

The idea of developing an attitude determination system using only three-axis magnetometer measurements has been attracting attention for many years, despite its relatively low accuracy. The light weight and low cost.of such a system are usually considered as its main advantages. For a spacecraft in low-attitude Earth orbit, Kalman filtering has been proven to be an effective tool to derive the attitude from magnetometer measurements with a 2-degree (deg) accuracy.[1,2]

The paper is intended to develop an attitude determination algorithm using only magnetometer measurements under contingency conditions which led to loss of a control of spacecraft. Our initial studies[3,4] was inspired by studies of the attitude motion of the Earth Radiation Budget Satellite (ERBS) during July 2, 1987, control anomaly,[5,6] when a hydrazine thruster-controlled yaw inversion maneuver resulted in a 2.1 degree per second (deg/sec) attitude spin. As a result a high-speed rotation of a spacecraft and the large pitch and roll angles experienced during the tumble, both Sun sensors and horizon scanners became unreliable. In addition, the design of the ERBS Attitude Determination System (ADS) could not accommodate gyro-rate data giving rise to saturation of gyro output. This left a three-axis magnetometor as the only reliable source for course attitude determination.

The ERBS anomaly is probably not the best example of a contingency situation to which the developed method can be successfully applied. Both Sun sensor and scanner data during the anomaly provided at least partial information concerning the attitude. After the spacecraft was inertially despin using the hydrazine thruster, gyro data also became available, making it possible to determine the attitude of the spacecraft by combining these data with magnetometer magnetometer measurements. In addition, a large quantization increment of ~6.44 mG significantly reduced the accuracy of the method.

A much more practical impact is expected to be achieved for the Relay Mirror Experiment (RME) satellite.[7] The control over this spacecraft was lost after the failure with the second horizon sensor. As a result, magnetometer measurements became the only source of information about spinning rates and the attitude of the going-out-of-control spacecraft. In addition, digitization errors in magnetometer data turned out to be much smaller (0.5 mG, instead of 6.44 mG for the ERBS).

The main advantage of the developed algorithm is that it provides an attitude solution for the dynamical equations for unknown initial conditions. The approximate solution found by means of this deterministic scheme can be then refined using the Kalman filtering,[1,2] for example.

2. Reducing the number of unknown variables by means of the first derivative of geomagnetic field with respect to time

Let $\vec{\boldsymbol{B}}^B$ and $\vec{\boldsymbol{B}}^I$ be the vectors of geomagnetic field measured in the body-fixed and inertial frames, respectively. They are related in a usual way:

$$\mathbf{A}\vec{\boldsymbol{B}}^I = \vec{\boldsymbol{B}}^B , \tag{1a}$$

where $\mathbf{A}$ is an unknown attitude matrix. The second vector equation can be obtained by relating time derivatives $\dot{\vec{\boldsymbol{B}}}^B$ and $\dot{\vec{\boldsymbol{B}}}^I$ of the two vectors, namely,

$$\mathbf{A}\dot{\vec{\boldsymbol{B}}}^I = \dot{\vec{\boldsymbol{B}}}^B + \vec{\boldsymbol{\omega}}^B \times \vec{\boldsymbol{B}}^B , \tag{1b}$$

where $\vec{\boldsymbol{\omega}}^B$ is an angular velocity vector measured in the body-fixed frame. If spacecraft rates are known, then the spacecraft attitude can be determined simply using the TRIAD method.[1] The main purpose of this paper is however to determine simultaneously both attitude and rates. As a result of two constraints:

$$|\vec{\boldsymbol{B}}^B| = |\vec{\boldsymbol{B}}^I| \tag{2}$$

and

$$(\vec{\boldsymbol{B}}^B \bullet \dot{\vec{\boldsymbol{B}}}^B) = (\vec{\boldsymbol{B}}^I \bullet \dot{\vec{\boldsymbol{B}}}^I) , \tag{3}$$

a pair of vector equations (1a) and (1b) for six unknown variables (three attitude angles and three rates) gives only four independent scalar equations, allowing one to reduce the number of unknown quantities to 2. One of the remaining unknown quantities is the projection, ω_1, of the angular velocity vector on the direction of the geomagnetic field.

To define the remaining unknown quantity, it is convenient to introduce two sets of mutually orthogonal vectors

$$\vec{\boldsymbol{U}}_1^I \equiv \vec{\boldsymbol{B}}^I , \vec{\boldsymbol{U}}_2^I \equiv \hat{\boldsymbol{U}}_1^I \times \dot{\vec{\boldsymbol{B}}}^I , \ \vec{\boldsymbol{U}}_3^I \equiv \hat{\boldsymbol{U}}_1^I \times \vec{\boldsymbol{U}}_2^I , \tag{4a}$$

and

$$\vec{\boldsymbol{D}}_1^B \equiv \vec{\boldsymbol{B}}^I , \ \vec{\boldsymbol{D}}_2^B \equiv \hat{\boldsymbol{D}}_1^B \times \dot{\vec{\boldsymbol{B}}}^B , \ \vec{\boldsymbol{D}}_3^B \equiv \hat{\boldsymbol{D}}_1^B \times \vec{\boldsymbol{D}}_2^B , \tag{4b}$$

where $\hat{\boldsymbol{U}}_1^I \equiv \vec{\boldsymbol{B}}^I / \mathrm{B}$ and $\hat{\boldsymbol{D}}_1^B \equiv \vec{\boldsymbol{B}}^B / \mathrm{B}$ with $\mathrm{B} \equiv |\vec{\boldsymbol{B}}^B|$. (Note that normalization of vectors $\vec{\mathbf{U}}_{\mathrm{k}}^{\mathrm{I}}$ and $\vec{\mathbf{D}}_{\mathrm{k}}^{\mathrm{B}}$ (k=2,3) was changed compared with Refs. .)

Since $\vec{\boldsymbol{D}}_1^B = \mathbf{A}\vec{\boldsymbol{U}}_1^I$, the vectors $\vec{\boldsymbol{U}}_k^I$ and $\vec{\boldsymbol{D}}_k^B$ (k=2,3) lie in the same plane perpendicular to the vector $\vec{\boldsymbol{B}}^B$, and therefore a pair of the unit vectors $\mathbf{A}\hat{\boldsymbol{U}}_k^I \equiv \mathbf{A}\vec{\boldsymbol{U}}_k^I / |\vec{\boldsymbol{U}}_k^I|$ (k=2,3) can be obtained from the known vectors $\hat{\boldsymbol{D}}_k^B \equiv \vec{\boldsymbol{D}}_k^B / |\vec{\boldsymbol{D}}_k^B|$ (k=2,3) by a rotation about $\vec{\boldsymbol{B}}^B$ over an angle Φ, that is,

$$\mathbf{A}\hat{\boldsymbol{U}}_2^I = \cos\Phi\, \hat{\boldsymbol{D}}_2^B + \sin\Phi\, \hat{\boldsymbol{D}}_3^B \tag{5a}$$

and

$$\mathbf{A}\hat{\boldsymbol{U}}_3^I = -\sin\Phi\, \hat{\boldsymbol{D}}_2^B + \cos\Phi\, \hat{\boldsymbol{D}}_3^B. \tag{5b}$$

Eqs.(5a) and (5b) imply that the attitude matrix $\mathbf{A}$ belongs to a one-dimensional family of orthogonal matrices

$$\mathbf{A}(\Phi) \equiv [\hat{\boldsymbol{D}}_1^B, \hat{\boldsymbol{D}}_2^B, \hat{\boldsymbol{D}}_3^B]\ \mathbf{T}_1(\Phi)\, [\hat{\boldsymbol{U}}_1^I, \hat{\boldsymbol{U}}_2^I, \hat{\boldsymbol{U}}_3^I]^{\mathrm{T}}, \tag{6}$$

where

$$\mathbf{T}_1(\Phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\Phi & -\sin\Phi \\ 0 & \sin\Phi & \cos\Phi \end{bmatrix} \tag{7}$$

By analogy the angular velocity vector $\vec{\boldsymbol{\omega}}^B$ belongs to a two-dimensional family of vectors

$$\vec{\boldsymbol{\omega}}^B(\Phi, \omega_1) \equiv \omega_1\, \hat{\boldsymbol{D}}_1^B + \vec{\boldsymbol{\omega}}_\perp^B(\Phi) \tag{8}$$

where the vector

$$\vec{\boldsymbol{\omega}}_\perp^B(\Phi) \equiv \omega_2(\Phi)\, \hat{\boldsymbol{D}}_2^B + \omega_3(\Phi)\, \hat{\boldsymbol{D}}_3^B \tag{9}$$

is implicitly defined via the equation:

$$\mathbf{A}(\Phi)\dot{\vec{\boldsymbol{B}}}^I = \dot{\vec{\boldsymbol{B}}}^B + [\vec{\boldsymbol{\omega}}_\perp^B(\Phi)\ \vec{\boldsymbol{B}}^B]. \tag{10}$$

In fact, substituting Eq. (9) into Eq. (10), one finds

$$\mathbf{A}(\Phi)\dot{\vec{\boldsymbol{B}}}^I = \dot{\vec{\boldsymbol{B}}}^B - \mathrm{B}\,[\,\omega_2(\Phi)\,\hat{\boldsymbol{D}}_3^B - \omega_3(\Phi)\,\hat{\boldsymbol{D}}_2^B\,], \tag{11}$$

and hence

$$\mathbf{A}(\Phi)\vec{\boldsymbol{U}}_2^I = [\,|\vec{\boldsymbol{D}}_2^B| + \omega_2(\Phi)]\,\hat{\boldsymbol{D}}_2^B + \omega_3(\Phi)\,\hat{\boldsymbol{D}}_3^B \tag{12a}$$

and

$$\mathbf{A}(\Phi)\vec{\boldsymbol{U}}_3^I = -\,\omega_3(\Phi)\,\hat{\boldsymbol{D}}_2^B + [\,|\vec{\boldsymbol{D}}_2^B| + \omega_2(\Phi)]\,\hat{\boldsymbol{D}}_3^B\ . \tag{12b}$$

To make these expressions consistent with Eqs. (5a) and (5b), one simply needs to put

$$\omega_2(\Phi) = |\vec{\boldsymbol{U}}_2^I|\cos\Phi - |\vec{\boldsymbol{D}}_2^B| \tag{13a}$$

and

$$\omega_3(\Phi) = |\vec{\boldsymbol{U}}_2^I|\sin\Phi, \tag{13b}$$

where we also took into account that $|\vec{\boldsymbol{U}}_2^I| = |\vec{\boldsymbol{U}}_3^I|$. Therefore the projection of the angular velocity vector $\vec{\boldsymbol{\omega}}^B$ on the plane perpendicular to the geomagnetic field lies on the circle of radius $|\vec{\boldsymbol{U}}_2^I|$, with the center lying at the end of the vector $-\vec{\boldsymbol{D}}_2^B$. (The plane of the circle goes through the origin of the body-fixed frame perpendicular to $\vec{\boldsymbol{B}}^B$.)

3. Simulating the second derivative of geomagnetic field with respect to time

To find two unknown quantities Φ and ω_1, we model a residual of the second time derivative of geomagnetic field with respect to time using the dynamical equation of motion, namely, we put

$$\vec{\boldsymbol{\Lambda}}(\Phi,\omega_1) \equiv \vec{\boldsymbol{\Lambda}}_0^0(\Phi) + \vec{\boldsymbol{\Lambda}}_1^0(\Phi)\,\omega_1 + \dot{\vec{\boldsymbol{\omega}}}^B[\mathbf{A}(\Phi),\vec{\boldsymbol{\omega}}^B(\Phi,\omega_1)] \times \vec{\boldsymbol{B}}^B, \tag{14}$$

where

$$\vec{\boldsymbol{\Lambda}}_0^0(\Phi) \equiv \ddot{\vec{\boldsymbol{B}}}^B - \mathbf{A}(\Phi)\ddot{\vec{\boldsymbol{B}}}^I + 2\,\vec{\boldsymbol{\omega}}_\perp^B(\Phi) \times \dot{\vec{\boldsymbol{B}}}^B - |\vec{\boldsymbol{\omega}}_\perp^B(\Phi)|^2\,\vec{\boldsymbol{B}}^B, \tag{15a}$$

(with $\ddot{\vec{\boldsymbol{B}}}^B$ and $\ddot{\vec{\boldsymbol{B}}}^I$ standing for the second time derivatives of geomagnetic field measured in the body-fixed and inertial frames, respectively), and

$$\vec{\Lambda}_1^0(\Phi) \equiv \hat{\boldsymbol{D}}_1^B \times \vec{\boldsymbol{H}}^0(\Phi), \tag{15b}$$

with

$$\vec{\boldsymbol{H}}^0(\Phi) \equiv \mathbf{A}(\Phi)\,\dot{\vec{\boldsymbol{B}}}^I + \dot{\vec{\boldsymbol{B}}}^B, \tag{16}$$

The angular acceleration vector $\dot{\vec{\boldsymbol{\omega}}}^B[\mathbf{A},\vec{\boldsymbol{\omega}}^B]$ is expressed in terms of an attitude-dependent torque $\vec{\boldsymbol{N}}^B[\mathbf{A}]$, a spacecraft tensor of inertia $\mathbf{I}$, a wheel momentum $\vec{\boldsymbol{h}}$, and its time derivative $\dot{\vec{\boldsymbol{h}}}$:

$$\dot{\vec{\boldsymbol{\omega}}}^B[A,\vec{\boldsymbol{\omega}}^B] = \mathbf{I}^{-1}\left\{\ \vec{\boldsymbol{N}}^B[\mathbf{A}] - \dot{\vec{\boldsymbol{h}}} + \vec{\boldsymbol{\omega}}^B \times \left(\mathbf{I}\vec{\boldsymbol{\omega}}^B + \vec{\boldsymbol{h}}\right)\ \right\}. \tag{17}$$

By expressing this vector in terms of the quantities Φ and ω_1, one finds

$$\dot{\vec{\boldsymbol{\omega}}}^B[\mathbf{A}(\Phi),\vec{\boldsymbol{\omega}}^B(\Phi,\omega_1)] = \vec{\boldsymbol{\Omega}}_0(\Phi) + \omega_1\vec{\boldsymbol{\Omega}}_1(\Phi) + \omega_1^2\,\vec{\boldsymbol{\Omega}}_2\,, \tag{18}$$

where

$$\vec{\boldsymbol{\Omega}}_0(\Phi) \equiv -\mathbf{I}^{-1}\left\{\vec{\boldsymbol{N}}^B[\mathbf{A}(\Phi)] - \dot{\vec{\boldsymbol{h}}} + \vec{\boldsymbol{\omega}}_\perp^B(\Phi)\times\left(\mathbf{I}\,\vec{\boldsymbol{\omega}}_\perp^B(\Phi) + \vec{\boldsymbol{h}}\right)\right\}, \tag{19a}$$

$$\vec{\boldsymbol{\Omega}}_1(\Phi) \equiv -\mathbf{I}^{-1}\left[\hat{\boldsymbol{D}}_1^B\times\left(\mathbf{I}\,\vec{\boldsymbol{\omega}}_\perp^B(\Phi) + \vec{\boldsymbol{h}}\right) + \vec{\boldsymbol{\omega}}_\perp^B(\Phi)\times\mathbf{I}\,\hat{\boldsymbol{D}}_1^B\right], \tag{19b}$$

$$\vec{\boldsymbol{\Omega}}_2 \equiv \mathbf{I}^{-1}[\mathbf{I}\,\hat{\boldsymbol{D}}_1^B\times\hat{\boldsymbol{D}}_1^B]. \tag{19c}$$

(A possible dependence of the torque on spacecraft rates can be easily taken into account, which may however lead to more complicated solutions, if Eq. (18) is inapplicable anymore.) Eq. (14) thus takes the form:

$$\vec{\Lambda}(\Phi,\omega_1) \equiv \vec{\Lambda}_0(\Phi) + \vec{\Lambda}_1(\Phi)\,\omega_1 + \vec{\Lambda}_2\,\omega_1^2, \tag{20}$$

where

$$\vec{\Lambda}_0(\Phi) \equiv \vec{\Lambda}_0^0(\Phi) + \vec{\boldsymbol{\Omega}}_0(\Phi)\times\vec{\boldsymbol{B}}^B, \tag{21a}$$

$$\vec{\Lambda}_1(\Phi) \equiv \hat{\boldsymbol{D}}_1^B \times \vec{\boldsymbol{H}}(\Phi), \tag{21b}$$

$$\vec{\Lambda}_2 \equiv \vec{\boldsymbol{\Omega}}_2 \times \vec{\boldsymbol{B}}^B, \tag{21c}$$

and

$$\vec{\boldsymbol{H}}(\Phi) \equiv \vec{\boldsymbol{H}}^{0} - \mathrm{B}\ \vec{\boldsymbol{\Omega}}_{1}(\Phi). \tag{22}$$

The loss function $|\vec{\boldsymbol{\Lambda}}(\Phi,\omega_1)|^2$ can be represented as a quartic polynomial in ω_1:

$$|\vec{\boldsymbol{\Lambda}}(\Phi,\omega_1)|^2 = Q_4[\omega_1;\Phi], \tag{23}$$

with the first four coefficients to be the known periodic functions of Φ:

$$\mathrm{Q}_{4,0}(\Phi) \equiv |\vec{\boldsymbol{\Lambda}}_0(\Phi)|^2, \tag{24a}$$
$$\mathrm{Q}_{4,1}(\Phi) \equiv 2\left(\vec{\boldsymbol{\Lambda}}_0(\Phi)\bullet\vec{\boldsymbol{\Lambda}}_1(\Phi)\right), \tag{24b}$$
$$\mathrm{Q}_{4,2}(\Phi) \equiv |\vec{\boldsymbol{\Lambda}}_1(\Phi)|^2, \tag{24c}$$
$$\mathrm{Q}_{4,3}(\Phi) \equiv 2\left(\vec{\boldsymbol{\Lambda}}_2\bullet\vec{\boldsymbol{\Lambda}}_1(\Phi)\right), \tag{24d}$$

whereas the coefficient of the quartic term is a constant:

$$\mathrm{Q}_{4,4} \equiv |\vec{\boldsymbol{\Lambda}}_2|^2. \tag{24e}$$

The attitude/rate solution sought for is associated with one of the minima of $\mathrm{Q}_4[\omega_1;\Phi]$ with respect to ω_1 and Φ, though, as our studies[1-5] showed, the absolute minimum does not necessarily gives the correct root.

Minimization of the polynomial $\mathrm{Q}_4[\omega_1;\Phi]$ with respect to ω_1 at a fixed value of Φ can be done analytically. In fact, at each value of Φ the polynomial has either a single minimum or two minima separated by a maximum which are given by real roots $\omega_1^*(\Phi)$ of the cubic polynomial:

$$\mathrm{Q}_3[\omega_1;\Phi] \equiv \frac{\partial \mathrm{Q}_4}{\partial \omega_1}\ . \tag{25}$$

The roots associated with the minima are selected by requiring that

$$\mathrm{W}_2\,[\,\omega_1^*(\Phi),\,\Phi] > 0, \tag{26}$$

where

$$\mathrm{Q}_2[\omega_1;\Phi] \equiv \frac{\partial \mathrm{Q}_3}{\partial \omega_1}\ . \tag{27}$$

The function $Q_4[\omega_1^*(\Phi), \Phi]$ is then minimized with respect to Φ for of the selected root. Since the function is periodic in Φ, at least one minimum always exists.

The main advantage of the minimization procedure is that it always has a solution, but this can be achieved only via rather time-consuming search for minima of $Q_4[\omega_1^*(\Phi), \Phi]$ in Φ. In Section 5 we show how the search for observed values of Φ and ω_1 can be simplified under assumption that the minimum sought for coincides with a zero of the loss function $|\vec{\Lambda}(\Phi, \omega_1)|^2$.

4. An Alternative Root Search Algorithm Using a Pair of Coupled Trnanscendental Equations

If our modeling were sufficiently accurate, then the observed values of Φ and ω_1 would nulify the residual $\vec{\boldsymbol{\Lambda}}(\Phi, \omega_1)$ and hence could be found by solving the vector equation:

$$\vec{\boldsymbol{\Lambda}}(\Phi^*, \omega_1^*) = \vec{\boldsymbol{0}} \ . \tag{28}$$

By projecting Eq. (27) onto three mutually perpendicular axes, one comes to three scalar equations for two unknown quantities. Let us show that the projection on the field direction, $\vec{\boldsymbol{B}}^B$, vanishes for any values of Φ and ω_1:

$$(\vec{\boldsymbol{\Lambda}}(\Phi, \omega_1) \bullet \vec{\boldsymbol{B}}^B) \equiv 0, \tag{29}$$

so that the two axes used to construct the equations for Φ and ω_1 must significantly deviate from $\vec{\boldsymbol{B}}^B$. In fact, it directly follows from Eqs. (15b), (21b), and (21c), that both vectors $\vec{\boldsymbol{\Lambda}}_1(\Phi)$ and $\vec{\boldsymbol{\Lambda}}_2$ in Eq. (20) are perpendicular to $\vec{\boldsymbol{B}}^B$ for all Φ. To prove that $\vec{\boldsymbol{\Lambda}}_0^0(\Phi)$ and hence $\vec{\boldsymbol{\Lambda}}_0(\Phi)$ are perpendicular to $\vec{\boldsymbol{B}}^B$, one should take into account that, according to Eqs. (13a) and (13b),

$$|\vec{\boldsymbol{\omega}}_\perp^B(\Phi)|^2 = |\vec{\boldsymbol{U}}_2^I|^2 + |\vec{\boldsymbol{D}}_2^B| - 2\,|\vec{\boldsymbol{U}}_2^I|\,|\vec{\boldsymbol{D}}_2^B| \cos\Phi \tag{30}$$

so that

$$(\vec{\boldsymbol{B}}^B \bullet \vec{\boldsymbol{\Lambda}}_0^0(\Phi)) \equiv (\vec{\boldsymbol{B}}^B \bullet \ddot{\vec{\boldsymbol{B}}}^B) - (\vec{\boldsymbol{B}}^I \bullet \ddot{\vec{\boldsymbol{B}}}^I) - |\vec{\boldsymbol{U}}_2^I|^2 + |\vec{\boldsymbol{D}}_2^B|. \tag{31}$$

By making use of Eq. (3), one can then verify that

$$|\vec{\boldsymbol{U}}_2^I|^2 - |\vec{\boldsymbol{D}}_2^B| = |\dot{\vec{\boldsymbol{B}}}^I|^2 - |\dot{\vec{\boldsymbol{B}}}^B|^2, \tag{32}$$

and hence the vector $\vec{\boldsymbol{\Lambda}}_0^0(\Phi)$ is indeed perpendicular to $\vec{\boldsymbol{B}}^B$ at any Φ, as a direct consequence of the constraint

$$(\vec{\boldsymbol{B}}^B \bullet \ddot{\vec{\boldsymbol{B}}}^B) + |\dot{\vec{\boldsymbol{B}}}^B|^2 = (\vec{\boldsymbol{B}}^I \bullet \ddot{\vec{\boldsymbol{B}}}^I) + |\dot{\vec{\boldsymbol{B}}}^I|^2. \tag{33}$$

In a series of publications,[1-5] the author discussed a variety of different choices for the two axes used to construct a pair of coupled equations for Φ and ω_1. It should be however pointed out that different numerical algoriths will give the same solutions only if the loss function vanishes at its minimum. The larger the minimum value is (due to various errors), the more significant discrepancies between different solutions may be.

5. Algebraic Solution

Let us now show that, with the appropriate choice of the axes, the sought-for value of the angle Φ is given by one of roots of an 8th-order polynomial in an auxiliary variable

$$\zeta(\Phi) \equiv \tan(\Phi/2) \tag{34}$$

For this reason we direct the two axes perpendicular to $\vec{\boldsymbol{B}}^B$ along the vectors (19c) and (22). By projecting vector (20) onto direction of the vector $\vec{\boldsymbol{\Omega}}_2$ and taking into account (21c) one finds that

$$\omega_1^* = -(\vec{\boldsymbol{\Lambda}}_0(\Phi^*) \bullet \vec{\boldsymbol{\Omega}}_2) \,/\, \left(\vec{\boldsymbol{\Lambda}}_1(\Phi^*) \bullet \vec{\boldsymbol{\Omega}}_2\right). \tag{35}$$

Substituting this expression into Eq. (20) at $\Phi=\Phi^*$ and equating the scalar product $\vec{\boldsymbol{\Lambda}}(\Phi^*, \omega_1^*)$ and $\vec{\boldsymbol{H}}(\Phi)$ to zero gives

$$(\vec{\boldsymbol{H}}(\Phi^*) \bullet \vec{\boldsymbol{\Lambda}}_0(\Phi^*)) + \mathrm{B}\,(\vec{\boldsymbol{\Lambda}}_0(\Phi^*) \bullet \vec{\boldsymbol{\Omega}}_2)^2 \,/\, (\vec{\boldsymbol{\Lambda}}_1(\Phi^*) \bullet \vec{\boldsymbol{\Omega}}_2) = 0, \tag{36}$$

where we also made use of the identity

$$(\vec{\boldsymbol{H}}(\Phi) \bullet \vec{\boldsymbol{\Lambda}}_2) \equiv \mathrm{B}\left(\vec{\boldsymbol{\Lambda}}_1(\Phi) \bullet \vec{\boldsymbol{\Omega}}_2\right). \tag{37}$$

Our next step is to show that all possible solutions of Eq. (36) are described by real roots of ane 8th-order polynomial. In fact, an analysis of Eqs. (7), (9), (13a), and (13b) shows that

$$\mathbf{T}_1(\Phi) = c^2(\Phi)\, \mathbf{T}_1^{(2)}[\zeta(\Phi)], \tag{38}$$

$$\vec{\boldsymbol{\omega}}_{\perp}^{B}(\Phi) = c^2(\Phi)\; \vec{\boldsymbol{\omega}}_{\perp}^{(2)}\left[\zeta(\Phi)\right], \tag{39}$$

where

$$c\,(\Phi) \equiv \cos\,(\,\Phi\,/\,2\,)\,, \tag{40}$$

$$\mathbf{T}_1^{(2)}[\zeta] \equiv \begin{bmatrix} 1+\zeta^2 & 0 & 0 \\ 0 & 1-\zeta^2 & -2\zeta \\ 0 & 2\zeta & 1-\zeta^2 \end{bmatrix}, \tag{41}$$

$$\vec{\boldsymbol{\omega}}_{\perp}^{(2)}\left[\zeta\right] \equiv |\vec{\boldsymbol{U}}_2^{I}|\,(1 - \zeta^2)\;\hat{\boldsymbol{D}}_2^{B} \;-\; (1+\zeta^2)\;\vec{\boldsymbol{D}}_2^{B} \;+ 2\zeta\,|\vec{\boldsymbol{U}}_2^{I}|\;\hat{\boldsymbol{D}}_3^{B}\;, \tag{42}$$

We also assume that the total torque can be represented as

$$\vec{\boldsymbol{N}}^{B}[\mathbf{A}] = \vec{\boldsymbol{N}}_0^{B} + \sum_{i,j} \vec{\boldsymbol{N}}_{1;ij}^{B}\, A_{ij} + \sum_{i,j}\sum_{i',j'} \vec{\boldsymbol{N}}_{2;ij,i'j'}^{B}\, A_{ij} A_{i'j'}\;, \tag{43}$$

where A_{ij} are elements of the attitude matrix $\mathbf{A}$ and $\vec{\boldsymbol{N}}_0^{B}$, $\vec{\boldsymbol{N}}_{1;ij}^{B}$, $\vec{\boldsymbol{N}}_{2;ij,i'j'}^{B}$ are some coefficients. In this paper the first and last terms are represented by magnetic and gravity torques; a linear term would appear if one takes into account aerodynamic and solar-pressure torques, which are currently neglected. Substituting Eqs. (6), (38), and (41) into Eq. (43) thus gives

$$\vec{\boldsymbol{N}}^{B}[\mathbf{A}(\Phi)] = c^2(\Phi)\, \vec{\boldsymbol{N}}^{(4)}[\zeta(\Phi)]\;, \tag{44}$$

where $\vec{\boldsymbol{N}}^{(4)}[\zeta]$ is a fourth-order polynomial in ζ. One thus concludes that

$$\left(\vec{\boldsymbol{\Lambda}}_1(\Phi) \bullet \vec{\boldsymbol{\Omega}}_2\right) = c^2(\Phi)\;\Pi_2\left[\zeta(\Phi)\right], \tag{45a}$$

$$\left(\vec{\boldsymbol{\Lambda}}_0(\Phi) \bullet \vec{\boldsymbol{\Omega}}_2\right) = c^4(\Phi)\;\Pi_4\left[\zeta(\Phi)\right], \tag{45b}$$

$$\left(\vec{\boldsymbol{\Lambda}}_0(\Phi)\bullet\vec{\boldsymbol{H}}(\Phi)\right) = c^6(\Phi)\ \Pi_6[\zeta(\Phi)], \tag{45c}$$

where $\Pi_n[\zeta]$ are polynomials of the n^{th}-order in ζ. (Note that the notation for this polynomials is changed, compared with Ref.). Eq. (37) can be thus written as

$$P_8[\zeta^*] = 0, \tag{46}$$

where

$$P_8[\zeta] \equiv \Pi_2[\zeta]\Pi_6[\zeta] + B\Pi_4^2[\zeta]. \tag{47}$$

If the geomagnetic field approaches one of the spacecraft principal axes of inertia, the vector $\vec{\boldsymbol{\Omega}}_2$ vanishes nullifying all 9 coefficients of polynomial (47). Since the second term in the left-hand side of Eq. (36) tends to zero in this case, the solution sought for can be found among real roots of the polynomial $\Pi_6[\zeta]$, which is equivalent to the requirement that the vectors $\vec{\boldsymbol{H}}(\Phi^*)$ and $\vec{\boldsymbol{\Lambda}}_0(\Phi^*)$ must be perpendicular to each other. This implies that the vectors $\vec{\boldsymbol{\Lambda}}_0(\Phi^*)$ and $\vec{\boldsymbol{\Lambda}}_1(\Phi^*)$, being both perpendicular to the same pair of the vectors $\vec{\boldsymbol{H}}(\Phi^*)$ and $\vec{\boldsymbol{B}}^B$, are either parallel or antiparallel to each other, as expected from Eq. (28) at $\vec{\boldsymbol{\Omega}}_2 = \vec{\boldsymbol{0}}$. (If the vector $\vec{\boldsymbol{H}}(\Phi^*)$ is directed along the geomagnetic field then both vectors, $\vec{\boldsymbol{\Lambda}}_0(\Phi^*)$ and $\vec{\boldsymbol{\Lambda}}_1(\Phi^*)$ vanish.) One comes to a similar conslusion by analyzing loss function $|\vec{\boldsymbol{\Lambda}}(\Phi,\omega_1)|^2$, which turns into a quadratic polynomial of ω_1, so that the function $\omega_1^*(\Phi)$ can be found analytically:

$$\omega_1^*(\Phi) = -\left(\vec{\boldsymbol{\Lambda}}_0(\Phi)\bullet\vec{\boldsymbol{\Lambda}}_1(\Phi)\right) / |\vec{\boldsymbol{\Lambda}}_1(\Phi)|^2, \tag{48}$$

and hence

$$\left|\vec{\boldsymbol{\Lambda}}(\Phi,\omega_1^*(\Phi))\right|^2 = |\vec{\boldsymbol{\Lambda}}_0(\Phi)|^2 - \left(\vec{\boldsymbol{\Lambda}}_0(\Phi)\bullet\vec{\boldsymbol{\Lambda}}_1(\Phi)\right)^2 / |\vec{\boldsymbol{\Lambda}}_1(\Phi)|^2. \tag{49}$$

The loss function may vanish if and only if $\vec{\boldsymbol{\Lambda}}_0(\Phi^*)$ and $\vec{\boldsymbol{\Lambda}}_1(\Phi^*)$ are collinear vectors.

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