Chapter 2

Spherical Harmonics and Geomagnetic Field Modeling

Spherical harmonics can be viewed as Fourier Series at a dimension higher. This chapter previews spherical harmonics in the context of our application to mathematical modeling of Earth's magnetic field. For further explanation in spherical harmonics, please refer to numerous references in mathematics or Earth physics modelings like Refs. [**?**, **?**].

Geomagnetic field is undoubtedly an important phenomenon that has an impinge on many aspect of biological lives. Some research even indicates certain correlation between magnetic field anomalies with human physiology.

Human being biologically inferior in magnetic field sensing capability, began using magnetic field as a navigation tool only after the invention of compass. Modern application of magnetic field includes satellite position and attitude estimation [1, 2, 3, **?**, **?**]. Thus a precise magnetic field modeling is of prime importance.

The gradual weakening of magnetic field and shifting of the magnetic north pole hinting a possible magnetic polar shift that occurred many times before in Earth's history. Geologic evidence finds that, in fact, we are long overdue of a magnetic polar shift. Given the importance of magnetic field to many life forms of this planet, for example, we are concern of the adaptation of many migratory animals during this change. However, nonetheless, the magnetic field strength is unlikely to degrade to become insignificant during our lifetime.

The familiar picture of the magnetic field extends far into space like the one shown in Fig. 2.1 comprises of both the main field and the magnetosphere. The main field is generated by the dynamo action of the Earth's liquid outer core. The elongated teardrop shape of the outer magnetosphere is formed by pressure of the solar wind (plasma) caused by the streaming electrical particles from the sun. This distortion is quite small near Earth's surface, this not significantly affecting the main field unless in the event of solar eruptions associated with the sunspots. The magnetic field modeling that of our concern here is the main field since it is of more important to our application in navigation.

In this chapter, we first review the normal and associated Legendre polynomials. Then we use these as foundations to solve the Laplace's equation in spherical coordinate to obtain the spherical harmonics to be applied to the geomagnetic field modeling.

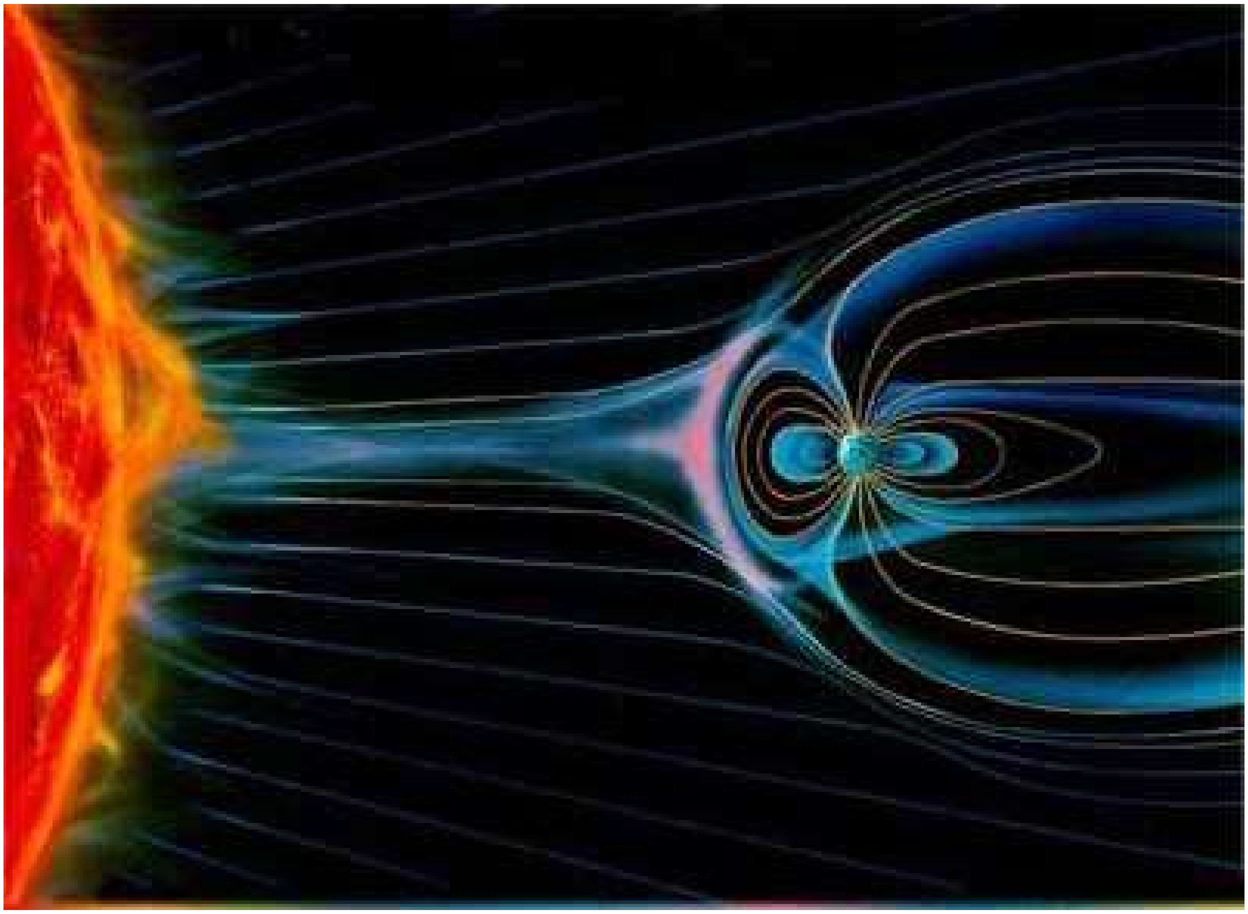


Figure 2.1: Solar wind, as depicted in this artist's illustration, travels from the Sun and envelops the Earth's magnetic field. High-energy pulses of solar wind from sunspot activity ("solar bursts" or "plasma bubbles") travel from the Sun to the Earth at speeds exceeding 500 miles per second. The pulses distort the Earth's magnetic field and produce geomagnetic storms that disrupt the Earth's environment. (Illustration by K. Endo, Nikkei Science, Inc., Japan.)

2.1 Legendre Polynomials

Let us now take a moment examine into the Legendre polynomials and the associated Legendre polynomials. The Legendre polynomials, sometimes also referred to as unassociated Legendre polynomials, are denoted by $P_n(\theta)$ with n being the order. They depend only on the colatitude θ and are solutions to the Legendre differential equations

$$(1-\mu^2)\frac{d^2}{d\mu^2}y - 2\mu\frac{d}{d\mu}y + n(n+1)y = 0 \quad , \tag{2.1}$$

which can also be written as

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dy}{d\mu}\right] + n(n+1)y = 0 \quad . \tag{2.2}$$

Since this is an ordinary second-order differential equation, it has two linearly independent solutions. To simplify arithmetics, it is common to set $\mu = \cos\theta$, where θ is the colatitude

	μ
P_0	1
P_1	μ
P_2	$\frac{1}{2}(3\mu^2 - 1)$
P_3	$\frac{1}{2}(5\mu^3 - 3\mu)$
P_4	$\frac{1}{8}(35\mu^4 - 30\mu^2 + 3)$
P_5	$\frac{1}{8}(63\mu^5 - 70\mu^3 + 15\mu)$
P_6	$\frac{1}{16}(231\mu^6 - 315\mu^4 + 105\mu^2 - 5)$
P_7	$\frac{1}{16}(429\mu^7 - 693\mu^5 + 315\mu^3 - 35\mu)$

Table 2.1: The first few Legendre polynomials in terms of μ.

	$\cos\theta$
P_0	1
P_1	$\cos\theta$
P_2	$\frac{1}{4}(1 + 3\cos 2\theta)$
P_3	$\frac{1}{8}(3\cos\theta + 5\cos 3\theta)$
P_4	$\frac{1}{64}(9 + 20\cos 2\theta + 35\cos 4\theta)$
P_5	$\frac{1}{128}(30\cos\theta + 35\cos 3\theta + 63\cos 5\theta)$
P_6	$\frac{1}{512}(50 + 105\cos 2\theta + 126\cos 3\theta + 231\cos 6\theta)$
P_7	$\frac{1}{1024}(175\cos\theta + 189\cos 3\theta + 231\cos 5\theta + 429\cos 7\theta)$

Table 2.2: The first few Legendre polynomials in terms of θ.

(90°− latitude). In some literature where θ is used to denote latitude instead, then the relation $\mu = \sin\theta$ is used, since sin(colatitude)=cos(latitude). The relation between μ and θ

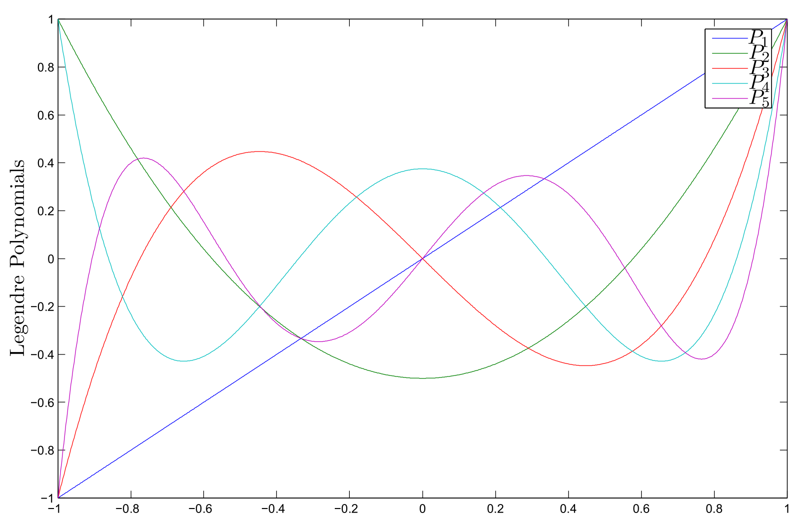


Figure 2.2: Legendre Polynomials

as colatitude is then simply

$$\frac{dy}{d\mu} = \frac{dy}{d(\cos\theta)} = -\frac{1}{\sin\theta}\frac{dy}{d\theta} \quad , \tag{2.3}$$

and

$$\frac{d^2y}{d\mu^2} = \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dy}{d\theta}\right) \tag{2.4a}$$

$$= \frac{1}{\sin^2\theta}\left(\frac{d^2y}{d\theta^2} - \frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}\right) \quad . \tag{2.4b}$$

Now Eqs. (2.1) or (2.1) can be written as

$$\frac{d^2y}{d\theta^2} + \frac{cos\theta}{\sin\theta}\frac{dy}{d\theta} + n(n+1)y = 0 \quad . \tag{2.5}$$

When $n = 0, 1, 2, 3, \ldots$, solutions of the Legendre differential equation are Legendre polynomials and is given by Rodrigue's formula

$$P_n(\mu) = \frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2 - 1)^n \quad . \tag{2.6}$$

The first few Legendre polynomials are given in Table 2.1 as functions of $\mu = \cos\theta$ or Table 2.2 as functions of θ. The Legendre polynomials are required to be *orthonormal*. The orthogonality condition requires that

$$\int_{-1}^{1} P_{pm}(\mu)P_{qm}(\mu)d\mu = \frac{2}{2q+1}\frac{(q+m)!}{(q-m)!}\delta_{pq} \tag{2.7}$$

where δ_{pq} is the Kronecker delta function (see §**??**). For the physical implications of the Legendre polynomials in terms of the colatitude and longitude, please refer to Ref. [**?**, §1.4]. For further explanation on the characteristics of spherical harmonics including their significants, please refer to Ref. [4].

2.2 Associated Legendre Polynomials

The associated Legendre polynomials, $P_n^m(\theta)$, of order n and degree m, on the other hand, are solutions to the associated Legendre differential equations. The associated Legendre differential equation

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dy}{d\mu}\right] + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]y = 0 \quad , \tag{2.8}$$

or in another form

$$(1-\mu^2)\frac{d^2y}{d\mu^2} - 2\mu\frac{dy}{d\mu} + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]y = 0 \quad , \tag{2.9}$$

	μ
P_0^0	1
P_1^0	μ
P_1^1	$(1-\mu^2)^{1/2}$
P_2^0	$\frac{1}{2}(3\mu^2-1)$
P_2^1	$3\mu(1-\mu^2)^{1/2}$
P_2^2	$3(1-\mu^2)$
P_3^0	$\frac{1}{2}\mu(5\mu^2-3)$
P_3^1	$-\frac{3}{2}(1-5\mu^2)(1-\mu^2)^{1/2}$
P_3^2	$15\mu(1-\mu^2)$
P_3^3	$15(1-\mu^2)^{3/2}$
P_4^0	$\frac{1}{8}(35\mu^4-30\mu^2+3)$
P_4^1	$-\frac{5}{2}\mu(3-7\mu^2)(1-\mu^2)^{1/2}$
P_4^2	$\frac{15}{2}(7\mu^2-1)(1-\mu^2)$
P_4^3	$105\mu(1-\mu^2)^{3/2}$
P_4^4	$105(1-\mu^2)^2$
P_5^0	$\frac{1}{8}\mu(63\mu^4-70\mu^2+15)$

Table 2.3: The first few associated Legendre polynomials in terms ofμ.

	$\cos\theta$
P_0^0	1
P_1^0	$\cos\theta$
P_1^1	$\sin\theta$
P_2^0	$\frac{1}{2}(3\cos^2\theta-1)$
P_2^1	$3\sin\theta\cos\theta$
P_2^2	$3\sin^2\theta$
P_3^0	$\frac{1}{2}\cos\theta(5\cos^2\theta-3)$
P_3^1	$\frac{3}{2}(5\cos^2\theta-1)\sin\theta$
P_3^2	$15\cos\theta\sin^2\theta$
P_3^3	$15\sin^3\theta$
P_4^0	$\frac{1}{8}(35\cos^4\theta-30\cos^2\theta+3)$
P_4^1	$\frac{1}{2}\sin\theta(35\cos^3\theta-15\cos\theta)$
P_4^2	$\frac{15}{2}\sin^2\theta(7\cos^2\theta-1)$
P_4^3	$105\sin^3\theta\cos\theta$
P_4^4	$105\sin^4\theta$
P_5^0	

Table 2.4: The first few associated Legendre polynomials in terms of θ.

where in our application, we assume both n and m are integers and m is equal or smaller than n.[1] These are 'special functions' and can be found in numerous handbook of functions like Ref. [?]. When m is positive, and with help from Eq. (2.6), associated Legendre polynomials can be given as

$$P_n^m(\mu) = (1-\mu^2)^{m/2}\frac{d^m}{d\mu^m}P_n(\mu) \tag{2.10a}$$

$$= \frac{(1-\mu^2)^{m/2}}{2^n n!}\frac{d^{n+m}}{d\mu^{n+m}}(\mu^2-1)^n \quad , \tag{2.10b}$$

and when m is negative

$$P_n^{-m}(\mu) = \frac{(n-m)!}{(n+m)!}P_n^m(\mu) \quad . \tag{2.11}$$

Note again that when $m = 0$ they reduce to unassociated Legendre polynomials as in Eq. (2.6).

Table 2.3 shows the first few associated Legendre polynomials as functions of μ and Tab. 2.4 shows them as functions of θ.

One last note is that there are two convention in associated Legendre polynomials; one that with a factor $(-1)^m$ and one that does not. Both conventions appear in literature and often mixed up. Abramowitz and Stegun [?] suggest a differentiation among them as

$$P_n^m = (-1)^m P_{nm} \quad . \tag{2.12}$$

However, care must be exercised here that in most geopotential modeling, the form without the factor $(-1)^m$ is used, so does here. Also, here our P_n^m indicates normalized associated Legendre polynomials but not the $(-1)^m$ factor, as seen from the next section.

2.3 Normalization

Normalization of the associated Legendre polynomials is important in numerical implementation especially at higher order. As seen from Fig. 2.3(a), there are big variation among the polynomials. Normalized associated Legendre polynomials are still solutions of the associated Legendre differential equation. There are a few normalization schemes available, Gauss-Laplace, Schmidt quasi-normalized and fully normalized. In our application, we will only consider the Schmidt quasi-normalized form[2] that was introduced in Ref. [?] since this is the most widely use form in modern geomagnetic literatures

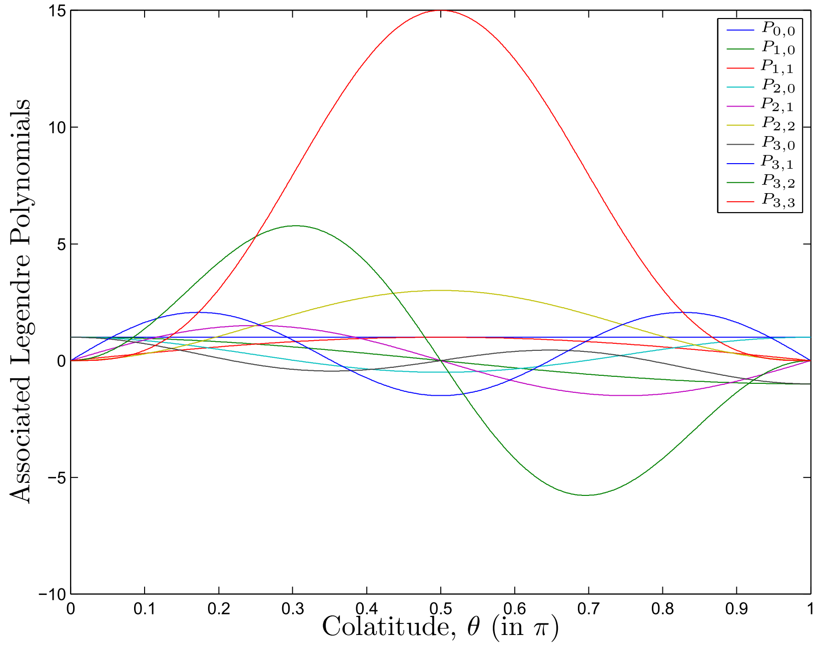
$$P_n^0(\cos\theta) = P_{n,0}(\cos\theta) \tag{2.13a}$$

$$P_n^m(\cos\theta) = \sqrt{\frac{2(n-m)!}{(n+m)!}}P_{n,m}(\cos\theta) \quad . \tag{2.13b}$$

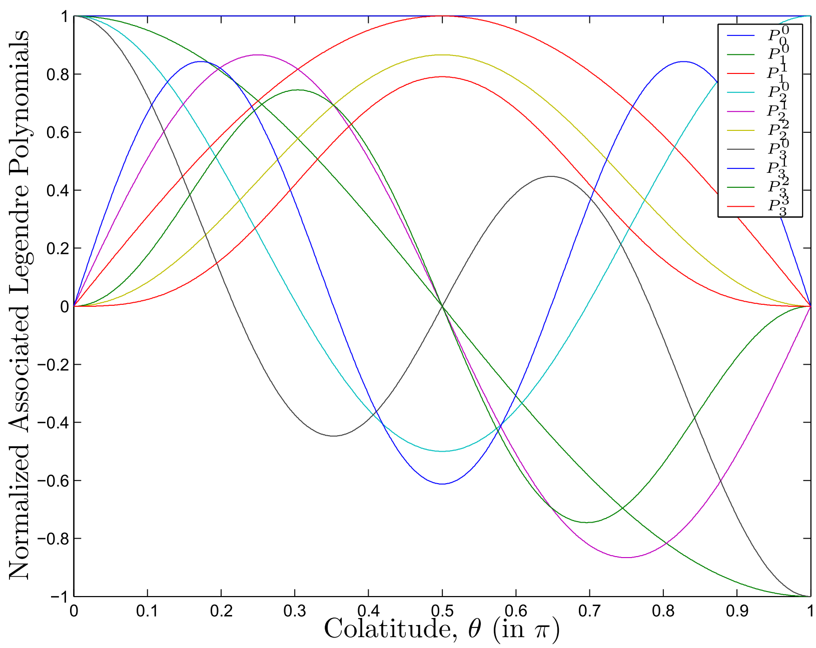
Fig. 2.3(b) shows the same associated Legendre polynomials with Schmidt quasi-normalization scheme, which are numerically much better behaved.

[1] When n is not integer, the solutions are called associated Legendre functions of the first kind.

[2] Not to be confused with the completely normalized formed used by Schmidt [?] especially in gravity analysis.



(a) Associated Legendre polynomials



(b) Associated Legendre polynomials with Schmidt quasi-normalization

Figure 2.3: Figures show the unnormalized and normalized associated Legendre polynomials

2.4 Derivation of Spherical Harmonics

The relation between the familiar cartesian coordinates and spherical coordinates can be referred to Fig. 2.4 where r is the radial distance from the center of the sphere; $\theta \in [0, \pi]$ is the colatitude from the polar axis and $\phi \in [0, 2\pi)$ is the longitude (azimuthal) coordinate.

The conversion between cartesian and spherical coordinates are simply

$$x = r\sin\theta\cos\phi \quad , \quad r = \sqrt{x^2+y^2+z^2} \tag{2.14}$$

$$y = r\sin\theta\sin\phi \quad , \quad \theta = \arctan(\sqrt{x^2+y^2}/z) \tag{2.15}$$

$$z = r\cos\theta \quad , \quad \phi = \arctan(y/x) \quad . \tag{2.16}$$

Laplace's equation, $\nabla^2 U = 0$, where U is a potential function, in cartesian coordinates is given by

$$\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 U}{\partial Z^2} = 0 \quad , \tag{2.17}$$

or in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2} = 0 \quad . \tag{2.18}$$

The solutions to the Laplace's equation are called harmonics or spherical harmonics in the case of spherical coordinates. Let's assume that the variables in U are separable, i.e. $U = R(r)T(\theta)P(\phi)$, then Eq. (2.18) becomes

$$TP\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + RP\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + RT\frac{1}{r^2\sin^2\theta}\frac{\partial^2 P}{\partial\phi^2} = 0 \quad . \tag{2.19}$$

Multiply this equation by $\frac{r^2\sin^2\theta}{RTP}$ to obtain

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{T}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{P}\frac{\partial^2 P}{\partial\phi^2} = 0 \quad . \tag{2.20}$$

The first two terms do not depend on ϕ whereas the last term on the LHS of the equation does not depend on r or θ, so the last term on the LHS must equate to a constant and the first two terms must add up to the negative of this constant. In other words

$$\frac{1}{P}\frac{\partial^2 P}{\partial P^2} = -m^2 \quad , \tag{2.21}$$

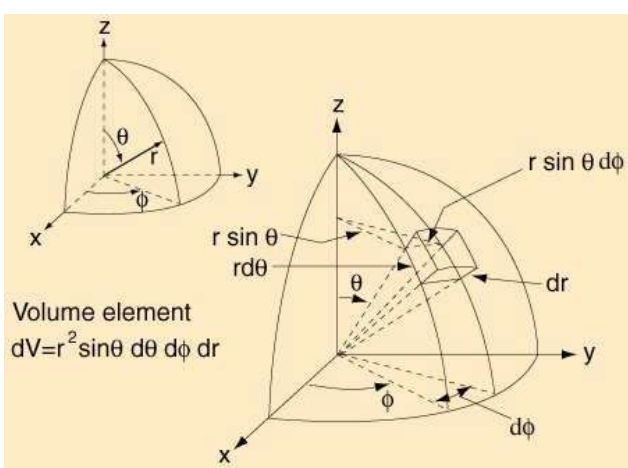


Figure 2.4: Cartesian and Spherical Coordinates

and this is of the familiar ODE form

$$m\frac{\partial^2 x}{\partial t^2} = -kx \quad , \tag{2.22}$$

which is a simple harmonic oscillator and has the characteristic equation of $s^2 + m^2 = 0$, a harmonic function. This is the reason why the solution of Laplace's equations are called harmonic functions. The solution of Eq. (2.21) is simply

$$\begin{aligned} P &= Ae^{m\phi} - Be^{-m\phi} \\ &= A\cos m\phi + B\sin m\phi \quad , \end{aligned} \tag{2.23}$$

where m is an integer. We now look at the first two terms of Eq. (2.20)

$$\frac{\sin^2\theta}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\sin\theta}{T}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) = m^2 \quad , \tag{2.24}$$

and multiply it by $\frac{1}{\sin^2\theta}$,

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{T\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) = \frac{m^2}{\sin^2\theta} \quad . \tag{2.25}$$

Again, the first two terms are decoupled or independent of the other variable, so both must equal to constants,

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = n(n+1) \tag{2.26a}$$

or

$$\frac{1}{R}\left[2r\frac{\partial R}{\partial r} + r^2\frac{\partial^2 R}{\partial r^2}\right] = n(n+1) \quad , \tag{2.26b}$$

with solution

$$R = Cr^n + Dr^{-(n+1)} \quad , \quad n = 0, 1, 2, \ldots \tag{2.27}$$

These solutions are referred to as *solid spherical harmonics.*

Lastly, we deal with the last term of Eq. (2.25),

$$\frac{1}{T\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) = \frac{m^2}{\sin^2\theta} - n(n+1) \quad , \tag{2.28}$$

which is also known as Legendre's[3] function with solution of the form

$$T = K_n^m P_n^m(\cos\theta) \quad , \tag{2.29}$$

where $P_n^m(\cos\theta)$ are the Associated Legendre Polynomials and K_n^m are constants.

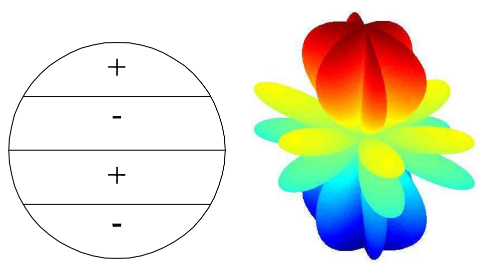
[3]In honor of the French mathematician Adrien-Marie Legendre (1752-1833), a mathematician from Paris. The Legendre polynomials first appeared in a celestial mechanics paper he published in 1784, *Recherches sur la figure des planètes* .

Finally, we put together these solutions to acquire the solution to the Laplace's equation,

$$U = -\frac{1}{a}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left\{\left[C_n^m\left(\frac{a}{r}\right)^{n+1} + C_n^{\prime m}\left(\frac{r}{a}\right)^{n}\right]\cos m\phi + \left[S_n^m\left(\frac{a}{r}\right)^{n+1} + S_n^{\prime m}\left(\frac{r}{a}\right)^{n}\right]\sin m\phi\right\}P_n^m(\cos\theta) \quad , \tag{2.30}$$

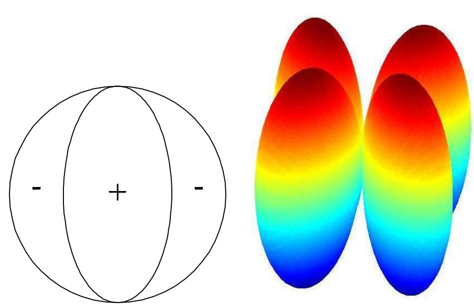
where $P_n^m(\cos\theta)\cos m\phi$ and $P_n^m(\cos\theta)\sin m\phi$ are called spherical harmonics with C_n^m, $C_n^{\prime m}$, S_n^m and $S_n^{\prime m}$ being the spherical harmonics coefficients. The function that put together these terms is called spherical harmonics expansion. The interested reader is also referred to [**?**, pp. 17-31], [4, pp. 251-259] and [**?**, pp. 17-32] for other treatments on the topic of spherical harmonics.

2.5 Characteristics of the Spherical Harmonics



(a) 2D Plot (b) 3D Plot

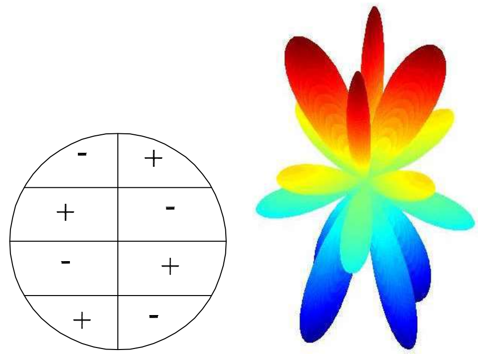
Figure 2.5: Zonal Spherical Harmonics, $m = 0$, $P_{3,0}$



(a) 2D Plot (b) 3D Plot

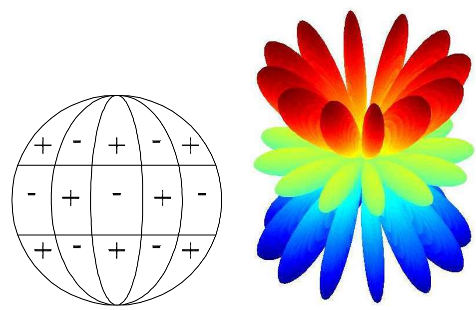
Figure 2.6: Sectorial, Spherical Harmonics, $n = m$, $P_{2,2}$

This section presents some physical characteristics (visual) of spherical harmonics. When $m = 0$, the associated Legendre polynomials reduce to Legendre polynomials. The coefficients when $m = 0$ are called *zonal harmonics* due to their independence from the longitude. When $m < n$, the coefficients are called *tesseral* coefficients. And lastly, when $m = n$, the coefficients are called *sectorial* coefficients. When $n - m$ is even, the spherical harmonics is symmetric about the equator; and antisymmetric about the equator when $n - m$ is odd.



(a) 2D Plot (b) 3D Plot

Figure 2.7: Tesseral, Spherical Harmonics, $n > m > 0$, $P_{4,1}$



(a) 2D Plot (b) 3D Plot

Figure 2.8: Tesseral, Spherical Harmonics, $n > m > 0$, $P_{6,4}$

Also, as notice from Figs. SSSS, the "peaks" and "valleys" becoming more "sharpened" as m increases. Further discussion on characteristics of the spherical harmonics can be found in Refs. [**?**] and [**?**].

2.6 Magnetic Field Modeling with Spherical Harmonics

In this dissertation, we choose to use the World Magnetic Field Model (WMM) by the International Geomagnetic Field Reference (IGRF) augmented with coefficients of the year 2005. This model is suppose to be valid from the year of 2005 until 2010.0. Reference [**?**] has a good explanation of how this model was derived from ground measurements (from observatory stations around the world) and satellite missions like Ørsted and Challenging Minisatellite Payload (CHAMP). The Earth's magnetic field consist of both internal sources and external sources. The internal sources include contribution from the dynamo action of the Earth's outer core fluid (B_m) and the Earth's crust and upper mantle (B_c), while the external sources are generated by the ionosphere and magnetosphere (B_d). However, since B_m contributes over 95% of the magnetic field at the Earth's surface, it is also called "the main field" and only this portion is included in the WMM [**?**]. This model and related software can be obtained from the National Geophysical Data Center[4].

[4]http://www.ngdc.noaa.gov/seg/WMM/

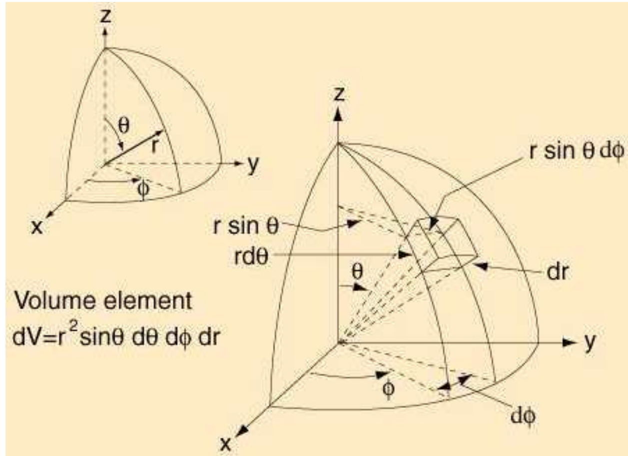


Figure 2.9: Spherical Coordinates

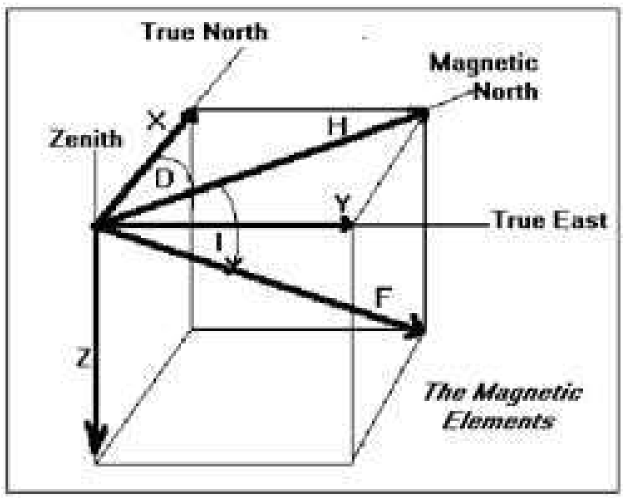


Figure 2.10: Magnetic Field Components

The WMM is modelled with the Laplacian equation in spherical coordinates, Eq. (2.18), which the solutions, Eq. (2.30), are the spherical harmonics. Spherical harmonics are chosen to represent the magnetic field instead of something simpler because of the physical meanings of their coefficients; other forms would be merely just a mathematical fit. The coefficients of the spherical harmonics have physical meaning to the magnetic field due to the fact that the spherical harmonics are solutions of the Maxwell's equation. These coefficients are regarded as Gauss' coefficients for his contribution in the magnetic field research, especially measurements and modeling.

Here we will customize the spherical harmonics to represent the magnetic field. Spherical harmonics are not simply a mathematical fit, it actually represent the physical properties of the magnetic field through its coefficients. So spherical harmonics naturally becomes an ideal modeling for the Earth's magnetic field. In fact, spherical harmonics is used in gravity field modeling too.

Assuming that the electric field generation and the current flowing between Earth and the atmosphere are negligible, Maxwell's equation for the Earth's surface is given by [?]

$$\nabla \times \mathbf{B} = \left[\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right]\mathbf{i} + \left[\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right]\mathbf{j} + \left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right]\mathbf{k} = 0 \quad , \tag{2.31}$$

which reads *the curl of B equals zero*. Since the magnetic field is a potential field, it can also be obtained from the "negative gradient of a scalar potential",

$$\mathbf{B} = -\left[\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right] = -\nabla V \quad . \tag{2.32}$$

The second Maxwell's equation of our concern is

$$\nabla \cdot \mathbf{B} = \left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right] = 0 \quad , \tag{2.33}$$

which reads *the divergence of the field is zero*. Combining these two equations we obtain

$$\nabla \cdot \nabla V = \nabla^2 V = 0 \quad , \tag{2.34}$$

which means the Laplacian of the scalar V is zero.

Equation (2.30) actually consists of two series, one that becomes smaller as r increases (r^{-n}) and one that becomes larger as r increases (r^n). The first series refers to an internal source while the second series refers to an external source. In our application, we only consider the main field, so we retain only the contribution to the magnetic potential field of internal origin, thus eliminating the second series in Eq. (2.30). The parameter a in the equation is simply the radius of the Earth, $R_\oplus$. Finally, we arrive at the spherical harmonics for the Earth's magnetic scalar potential, with U replaces by V,

$$V(r,\theta,\phi) = R_\oplus \sum_{n=1}^{\infty} \left(\frac{R_\oplus}{r}\right)^{n+1} \sum_{m=0}^{n} \left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right] P_n^m(\theta), \tag{2.35}$$

where r is the geocentric radial distance; θ is the colatitude (or coelevation, 90 degrees minus the latitude/elevation); ϕ is the longitude (or azimuth); $R_\oplus$ is the Earth's radius (6371.2 km is widely being used in the magnetic field communities and is also the value adopted by the IGRF), g_n^m and h_n^m are Gauss coefficients and $P_n^m(\theta)$ is the associated Legendre polynomials. This model is said to be of degree n and order m where m is always smaller or equal to n. Notice that the n series starts from one instead of zero because $n = 0$ refers to a magnetic monopole which violates Eq. (2.33). The magnetic field is then obtained by taking the gradient of this scalar potential field with respect to its parameters, $\mathbf{B} = -\nabla V$, [?, pp. 26] [?, pp. 26] [?, pp. 28] [4, pp. 255] [?, pp. 781],

$$B_r(r,\theta,\phi) = -\frac{\partial V}{\partial r} = \sum_{n=1}^{\infty} \left(\frac{R_\oplus}{r}\right)^{n+2} (n+1) \sum_{m=0}^{n} \left\{ g_n^m \cos(m\phi) + h_n^m \sin(m\phi) \right\} P_n^m(\theta) \tag{2.36a}$$

$$B_\theta(r,\theta,\phi) = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\sum_{n=1}^{\infty} \left(\frac{R_\oplus}{r}\right)^{n+2} \sum_{m=0}^{n} \left\{ g_n^m \cos(m\phi) + h_n^m \sin(m\phi) \right\} \frac{dP_n^m(\theta)}{d\theta} \tag{2.36b}$$

$$B_\phi(r,\theta,\phi) = -\frac{1}{r\sin(\theta)}\frac{\partial V}{\partial \phi} = \frac{1}{\sin(\theta)} \sum_{n=1}^{\infty} \left(\frac{R_\oplus}{r}\right)^{n+2} \times$$

$$\sum_{m=0}^{n} m \left\{ g_n^m \sin(m\phi) - h_n^m \cos(m\phi) \right\} P_n^m(\theta). \tag{2.36c}$$

It is easier to present satellite collected data in geocentric inertial or ECI coordinates, which in this case Eq. (2.36a) becomes

$$B_X = (B_r \cos\delta + B_\theta \sin\delta)\cos\alpha - B_\phi \sin\alpha \tag{2.37a}$$
$$B_Y = (B_r \cos\alpha + B_\theta \sin\delta)\sin\alpha + B_\phi \cos\alpha \tag{2.37b}$$
$$B_Z = B_r \sin\delta - B_\theta \cos\delta, \tag{2.37c}$$

where $\delta = 90^\circ - \theta$ (the declination) and α (the right ascension)

$$\alpha = \phi + \alpha_G, \tag{2.38}$$

where α_G is the right ascension of the Greenwich or Sidereal time at Greenwich [**?**, pp. 782]. Sometimes in the magnetic field literature, the X (North), Y (East), Z (Vertical, inward positive) coordinates with reference to an oblate Earth are used,

$$X = -B_\theta \cos\epsilon - B_r \sin\epsilon \tag{2.39a}$$
$$Y = B_\phi \tag{2.39b}$$
$$Z = B_\theta \sin\epsilon - B_r \cos\epsilon, \tag{2.39c}$$

where $\epsilon \equiv \lambda - \delta < 0.2^\circ$, λ is the geodetic latitude. However, the correction term $\sin\epsilon$ are ≤ 100 nT [**?**].

2.7 Recursion

Equations (2.35) are too computationally demanding for repetitive use. Thus we here seek an efficient recursive calculation method. There are many other recursion formulae that exist and available in textbooks or mathematical handbook, such as Ref. [**?**]. However, not all are numerically stable, especially when obtaining high-order polynomials. The recursive scheme that we will employ is the one presented in Ref. [**?**, §3.2.4], which is a stable scheme.

The associated Legendre polynomials obey the following recursion relations

$$(n-m)P_{n,m}(\mu) = (2n-1)\mu P_{n-1,m}(\mu) - (n+m-1)P_{n-2,m}(\mu) \quad , \tag{2.40}$$

From $P_{0,0} = 1$, we can find all $P_{m,m}$ with the recursion

$$P_{m,m}(\mu) = (2m-1)(1-\mu^2)^{1/2} P_{m-1,m-1} \quad . \tag{2.41}$$

The remaining terms can be obtained from

$$P_{m+1,m}(\mu) = (2m+1)\mu P_{m,m}(\mu) \quad , \tag{2.42}$$

and

$$P_{n,m}(\mu) = \frac{1}{n-m}\Big[(2n-1)\mu P_{n-1,m}(\mu) - (n+m-1)P_{n-2,m}(\mu)\Big] \quad , \tag{2.43}$$

for $n > m+1$. In terms of colatitude θ, we can simply replace $\mu = \cos\theta$ and $(1-\mu^2)^{1/2} = \sin\theta$. The diagram below shows this recursion

$$
\begin{array}{ccccccccc}
P_{0,0} & & & & & \searrow & \text{Eq. (2.41)} & & \\
\Downarrow & \searrow & & & & \Downarrow & \text{Eq. (2.42)} & & \\
P_{1,0} & & P_{1,1} & & & \downarrow & \text{Eq. (2.43)} & & \\
\downarrow & & \Downarrow & \searrow & & & & & \\
P_{2,0} & & P_{2,1} & & P_{2,2} & & & & \\
\downarrow & & \downarrow & & \Downarrow & \searrow & & & \\
\vdots & & \vdots & & \vdots & & \ddots & & \\
\downarrow & & \downarrow & & \downarrow & & & \searrow & \\
P_{n,0} & & P_{n,1} & & P_{n,2} & & \cdots & & P_{n,n}
\end{array}
$$

For computation of geomagnetic field in cartesian coordinates $(x,\ y,\ z)$, Cunningham in Ref. [**?**] suggests another scheme to combine the associated Legendre polynomials and the longitude-dependent trigonometry terms into a single recursion, thus further saving computational resources. The addition theorem of cos and sin are used

$$
\cos((m+1)\phi) = \cos(m\phi)\cos(\phi) - \sin(m\phi)\sin(\phi) \tag{2.44a}
$$
$$
\sin((m+1)\phi) = \sin(m\phi)\cos(\phi) + \cos(m\phi)\sin(\phi) \quad . \tag{2.44b}
$$

Now define

$$
V_n^m = \left(\frac{R_\oplus}{r}\right)^{n+1} P_n^m(\cos\theta)\cos m\phi \tag{2.45a}
$$
$$
W_n^m = \left(\frac{R_\oplus}{r}\right)^{n+1} P_n^m(\cos\theta)\sin m\phi \quad . \tag{2.45b}
$$

Now the magnetic scalar potential, Eq. (2.35), can be written as

$$
V(r,\theta,\phi) = R_\oplus \sum_{n=1}^{\infty}\sum_{m=0}^{n}\Big[g_n^m V_n^m + h_n^m W_n^m\Big] \quad . \tag{2.46}
$$

V_n^m and W_n^m satisfy the recursion

$$
V_m^m = (2m-1)\left\{\frac{xR_\oplus}{r^2}V_{m-1}^{m-1} - \frac{yR_\oplus}{r^2}W_{m-1}^{m-1}\right\} \tag{2.47a}
$$
$$
W_m^m = (2m-1)\left\{\frac{xR_\oplus}{r^2}W_{m-1}^{m-1} + \frac{yR_\oplus}{r^2}V_{m-1}^{m-1}\right\} \quad , \tag{2.47b}
$$

and

$$
V_n^m = \left(\frac{2n-1}{n-m}\right)\frac{zR_\oplus}{r^2}V_{n-1}^m - \left(\frac{n+m-1}{n-m}\right)\frac{R_\oplus^2}{r^2}V_{n-2}^m \tag{2.48a}
$$
$$
W_n^m = \left(\frac{2n-1}{n-m}\right)\frac{zR_\oplus}{r^2}W_{n-1}^m - \left(\frac{n+m-1}{n-m}\right)\frac{R_\oplus^2}{r^2}W_{n-2}^m \quad . \tag{2.48b}
$$

Equation (2.48) hold for $n = m + 1$ and even when $V_{m-1,m}$ and $W_{m-1,m}$ are zero. Similarly, the diagram below shows the recursion

$$
\begin{array}{ccccccccc}
V_{0,0}, W_{0,0} & & & & & & \searrow & \text{Eq. SSS} & \\
\downarrow & \searrow & & & & & \downarrow & \text{Eq. SSS} & \\
V_{1,0}, W_{1,0} & & V_{1,1}, W_{1,1} & & & & & & \\
\downarrow & & \downarrow & \searrow & & & & & \\
V_{2,0}, W_{2,0} & & V_{2,1}, W_{2,1} & & V_{2,2}, W_{2,2} & & & & \\
\downarrow & & \downarrow & & \downarrow & \searrow & & & \\
\vdots & & \vdots & & \vdots & & \ddots & & \\
\downarrow & & \downarrow & & \downarrow & & & \searrow & \\
V_{n,0}, W_{n,0} & & V_{n,1}, W_{n,1} & & V_{n,2}, W_{n,2} & & \cdots & & V_{n,n}, W_{n,n}
\end{array}
$$

2.8 The Inverse Square Law

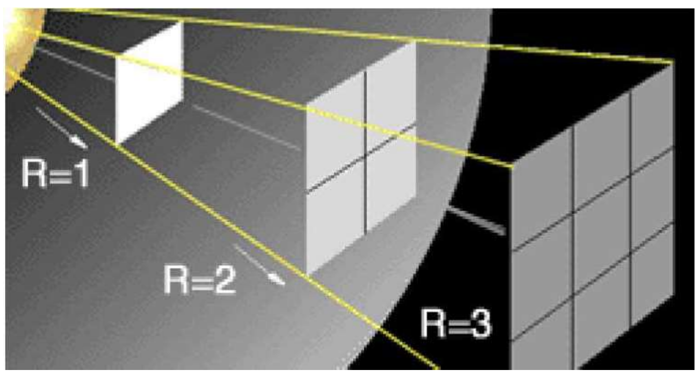


Figure 2.11: The inverse square law

(REDO : This from Basics of Space Flight) Electromagnetic energy decreases as if it were dispersed over the area on an expanding sphere, expressed as $4\pi R^2$ where radius R is the distance the energy has travelled. The amount of energy received at a point on that sphere diminishes as $1/R^2$. This relationship is known as the *inverse-square law* of (electromagnetic) propagation. It accounts for loss of signal strength over space, called *space loss*.

The inverse-square law is significant to the exploration of the universe, because it means that the concentration of electromagnetic radiation decreases very rapidly with increasing distance from the emitter. Whether the emitter is a distant spacecraft with a low-power transmitter or an extremely powerful star, it will deliver only a small amount of electromagnetic energy to a detector on Earth because of the very great distances and the small area that Earth subtends on the huge imaginary sphere.

2.9 Secular Variation Modeling

Secular variation is the slow changes in the main field. This phenomenon has been observed since Gauss' time.

2.10 Units

Standard magnetic field unit is Tesla but Gauss or miliGauss (mG) are widely used. Below are conversion between these units,

From	multiply by	to get
Gauss	10 000	Tesla
nTesla	100	mGauss

2.11 Misc.

The $P_n^m(\theta)$ are calculated as [**?**, pp. 20]

$$R_n^m = \sqrt{n^2 - m^2} \tag{2.49a}$$
$$P_0^0 = 1 \tag{2.49b}$$
$$P_1^0 = \cos(\theta); \qquad P_1^1 = \sin(\theta) \tag{2.49c}$$
$$P_m^m = \sqrt{\frac{2m-1}{2m}} \sin(\theta) P_{m-1}^{m-1} \qquad \text{for } m > 1, n = m \tag{2.49d}$$
$$P_n^m = \left[(2n-1)\cos(\theta) P_{n-1}^m - R_{n-1}^m P_{n-2}^m\right] / R_n^m \qquad \text{for } n > m \tag{2.49e}$$
$$\frac{dP_n^m(\theta)}{d\theta} = \left(n \cos(\theta) P_n^m - R_n^m P_{n-1}^m\right) / \sin(\theta) \quad \text{(except for } \theta = 0, \pi), \tag{2.49f}$$

for the obvious reason that Eq. (2.49f) is undefined at $\theta = 0$ or π rad.