GSE Math Camp Day 2: Calculus

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Math Camp Materials

All materials for the first week are hosted at:
`http://github.com/klintkanopka/gsemathcamp`

Outline

Motivation

Calculus is the study of (infinitesimal) change, and will be used in education research most often for optimization during model fitting. Understanding these concepts can help make courses like 400B, 430A, and 430B more clear. These ideas are also a prerequisite for the Econ 102 sequence, the Polisci 450 sequence, and any study of machine learning.

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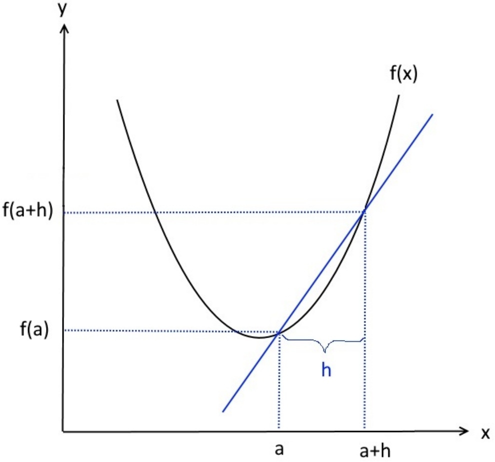
- Derivative: can be thought of as the slope of a tangent line at a point on a function. The derivative tells you the rate of change at a point. It is denoted by $\frac{dy}{dx}$ or f'(x) or y'. The value of the derivative may change for different values of x. In the graph below, we see that the derivative of function f(x) is positive, then 0, then negative (from left to right).

Intuition



Formal Definition

For a function, $f(x)$, we define the derivative, $f'(x)$, as follows:

$$f'(x) = \lim_{\Delta h \to 0} \frac{f(x+h) - f(x)}{h}$$

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7. Log rules: $\frac{d}{dx} ln\ x = \frac{1}{x}$ and $\frac{d}{dx} e^{cx} = ce^{cx}$

Practice

Find the derivative of each function with respect to x.

1. $f(x) = 7x^3 - 3$
2. $g(x) = 4x(5 - x^2)$
3. $h(x) = \frac{1}{x^4}$
4. $f(x) = \ln x^2$
5. $g(x) = \ln(x^2 + 1)$
6. $h(x) = 6x^6 - 5x^5 + 4x^4 - 3x^3 + 2x^2 - x + 1$

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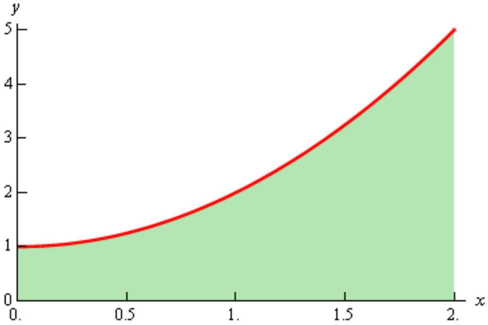

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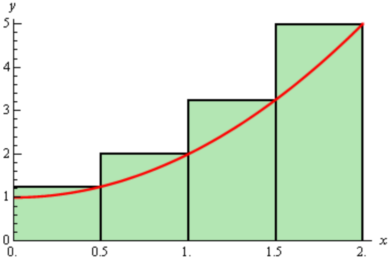

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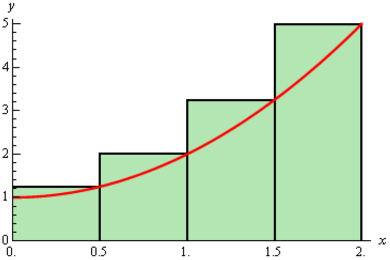

$$\begin{aligned} Area(S) &= \frac{1}{2}f\left(\frac{1}{2}\right) + \frac{1}{2}f(1) + \frac{1}{2}f\left(\frac{3}{2}\right) + \frac{1}{2}f(2) \\ &= \frac{1}{2}\left(\frac{5}{4}\right) + \frac{1}{2}(2) + \frac{1}{2}\left(\frac{13}{4}\right) + \frac{1}{2}(5) \\ &= 5.75 \end{aligned}$$

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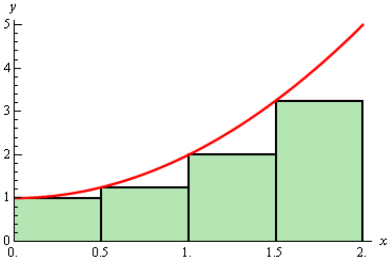

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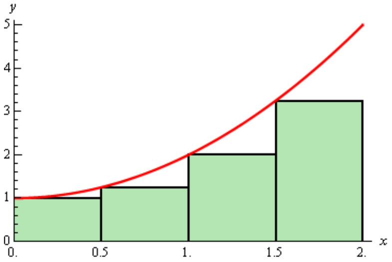

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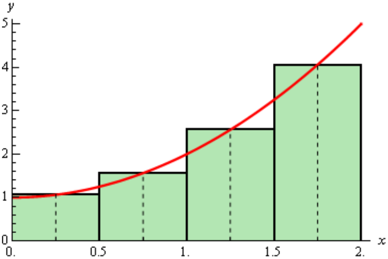

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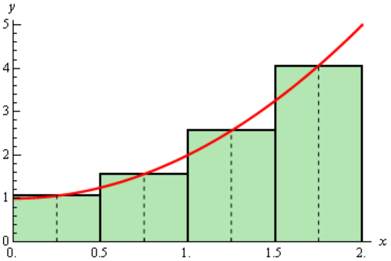

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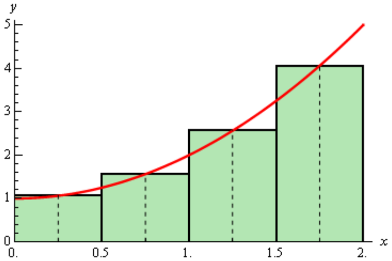

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Which other shapes might we try?

Doing Better

While we can use shapes like rectangles, our estimate of the area will be imprecise. One thing we might want to do is divide up our region into more parts. For example, using rectangles:

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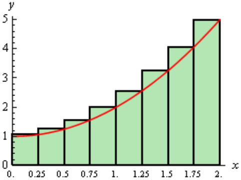

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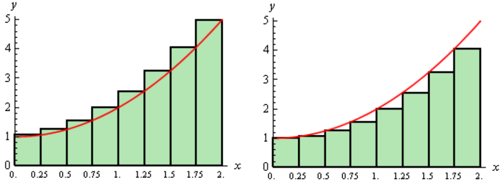

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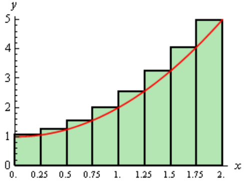

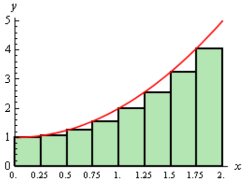

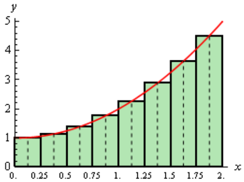

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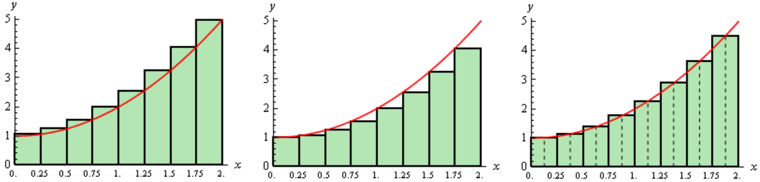

While dividing the region may get us closer to the true answer, it is still not exact!

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IMPORTANT: The derivative of a function can have many antiderivatives. With any antiderivative, we need to add a constant c, unless we are told that the antiderivative passes through a particular point. The correct solution to our example problem, in the absence of any additional information, is $3x^2 + c$

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Definite Integrals

We return to our initial problem of trying to calculate the area under a curve. Using an approximation method (i.e., one of the rectangle methods), we get close, but we want an exact answer.

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- How above making the base of the rectangles infinitesimally small? We can use limits:

$$Area(S) = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \underbrace{f(x_i)}_{height} \underbrace{\Delta x}_{base}$$

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The Riemann Integral is often referred to as the definite integral in calculus textbooks. Formally, $\int_a^b f(x)dx$ denotes the integral of function f from a to b (or the area under curve $f(x)$ from $x = a$ to $x = b$).

Fundamental Theorem of Calculus

Suppose f is continuous on $[a, b]$,

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FTC II provides a way to calculate a definite integral:

1. Find the indefinite integral $F(x)$
2. Evaluate $F(b) - F(a)$

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