

APSTA-GE 2094

APSY-GE 2524

Modern Approaches in Measurement: Lecture 2

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Announcements

- PS0 is due tomorrow night @ 11.59p!
 - Submit a rendered .pdf on Gradescope
 - Final late deadline Monday night (2/2 @ 11.59p)
 - 10 point late penalty per day
- PS1 is out!
 - Due in two weeks (2.13 @ 11.59p)
 - Files posted on the Problem Sets tab
 - Again, submit on Gradescope
 - This is longer than PS0, and you can complete a little more than half of it after today
- Had some people pop into office hours yesterday and it was fun! Those are Wednesdays 2-3p in the second floor lobby of Kimball Hall

Check-In

- PollEv.com/klintkanopka

Validity

Validity

- On the most basic level, the idea that we have confidence we are measuring the thing we think we are measuring
- Starts with the definition of a *construct*, or the unobservable thing we care about measuring
- Requires some combination of theory and some amount of empirical study to get at

Validity

1. Take 5-10 minutes and in groups of 3 ± 1 , discuss the differences between Cronbach and Meehl's conception of "construct validity," Kane's idea of validity, and Boorsboom's idea of validity
2. What are the pros/cons of each approach?
3. Where might each approach succeed or fail?
4. What approach do you agree with most?

Cronbach and Meehl

- Construct validity is derived from the nomological network
- Core idea: Since we cannot directly observe the latent construct of interest, we have to collect a dense network of evidence through theoretical arguments and correlations to support our ideas of validity
- The idea of "construct validity" and other "types" of validity is extremely antiquated

Kane

- Validity is associated with the interpretation assigned to test scores rather than with the test scores or the test
- Interpretive arguments lay out how a test score will be interpreted and what evidence needs to be brought to bear to evaluate the argument
- Core idea: Validity is highly contextual and changes to instruments or populations can result in changes to the construct being measured

Boorsboom

- A test is valid for measuring an attribute if (a) the attribute exists and (b) variations in the attribute causally produce variation in the measurement outcomes
- Focuses on causal reasoning
- Core idea: Validity should be a function of an instrument, not a score interpretation

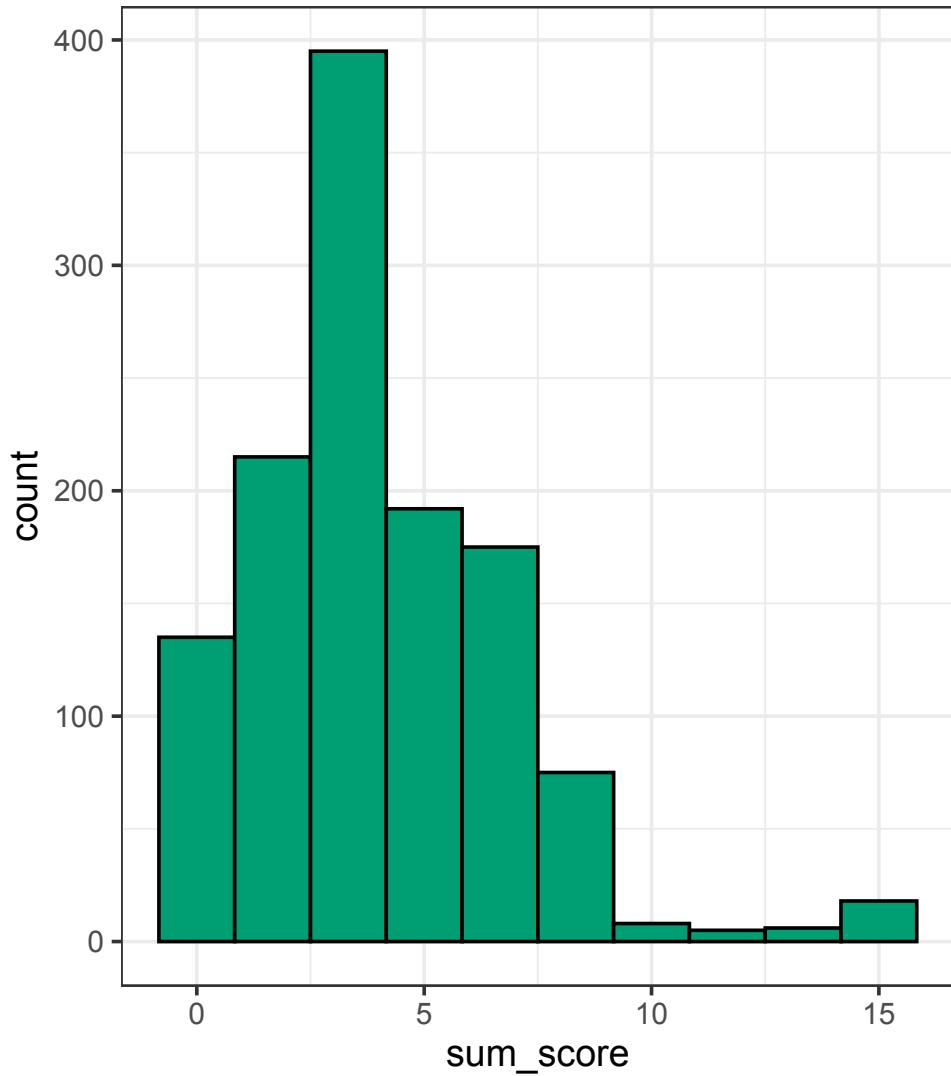
Classical Test Theory Practice

More Item Response Data

- The dataset `animalfights_clean.rds` (downloadable [here](#)) is a pre-cleaned version of the data from PS0.
- In a group of 3 ± 1 take ten minutes to:
 1. Compute the ability (sum score) for each respondent and plot a distribution
 2. Compute the difficulty (*p*-value) for each item and plot them in order of difficulty
 3. Compute the discrimination (item-total correlation) for each item and plot them in order
 4. Check the item `d_king_cobra` for DIF by gender

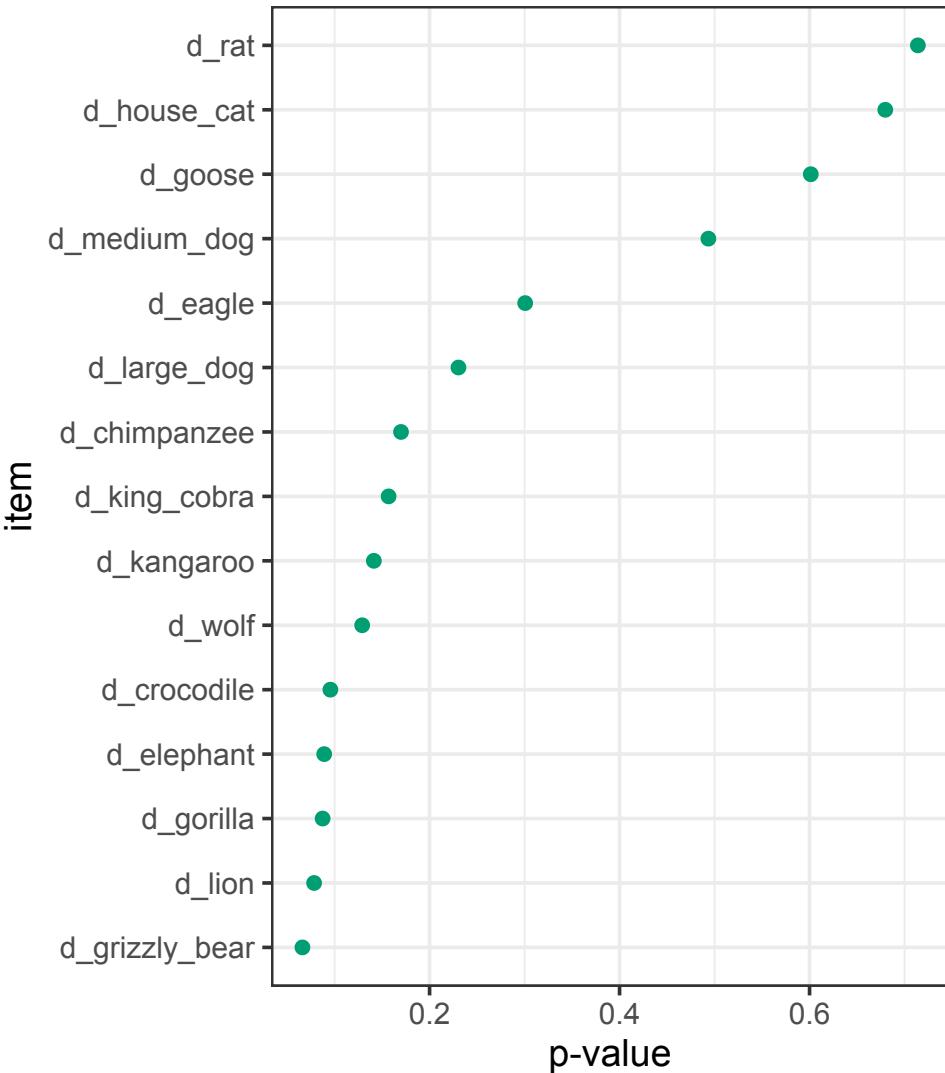
Sum Score

```
1 d <- readRDS('animalfights_clean.rds')
2
3 d_long <- d |>
4   pivot_longer(
5     cols = starts_with('d_'),
6     names_to = 'item',
7     values_to = 'resp'
8   )
9
10 sum_scores <- d_long |>
11   group_by(id) |>
12   summarize(sum_score = sum(resp))
13
14 ggplot(sum_scores, aes(x = sum_score)) +
15   geom_histogram(
16     bins = 10,
17     color = 'black',
18     fill = okabeito_colors(3)
19   ) +
20   theme_bw()
```



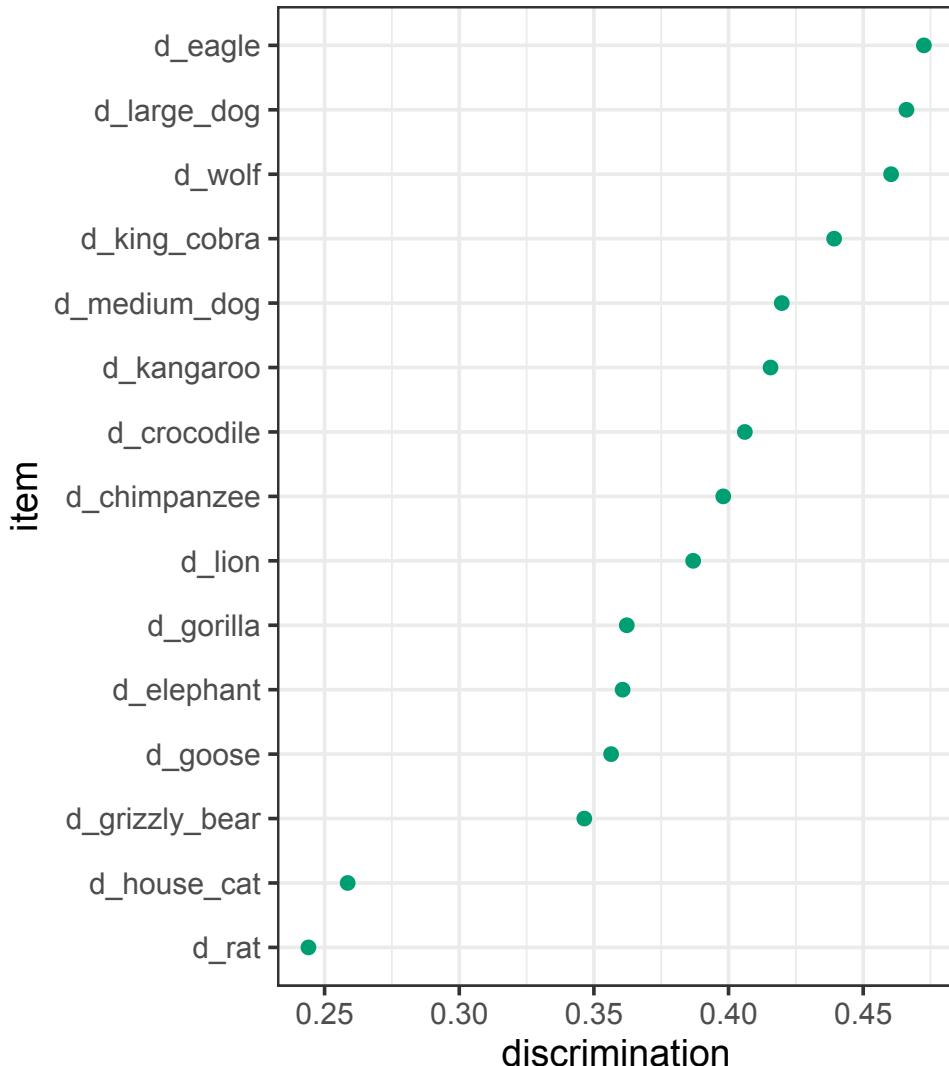
p-value

```
1  diff <- d_long |>
2    group_by(item) |>
3    summarize(p = mean(resp))
4
5  ggplot(diff, aes(x = p, y = reorder(item, p)))
6    geom_point(
7      color = okabeito_colors(3)
8    ) +
9    labs(x = 'p-value', y = 'item') +
10   theme_bw()
```



Item-Total Correlation

```
1 disc <- left_join(d_long, sum_scores, by = 'id'
2   mutate(adjusted_sum_score = sum_score - resp
3   group_by(item) |>
4   summarize(a = cor(resp, adjusted_sum_score))
5
6 ggplot(disc, aes(x = a, y = reorder(item, a)))
7   geom_point(
8     color = okabeito_colors(3)
9   ) +
10  labs(x = 'discrimination', y = 'item') +
11  theme_bw()
```



Differential Item Functioning

- Regression:

```
1 d_long |>
2   left_join(sum_scores, by = 'id') |>
3   mutate(adjusted_sum_score = sum_score - resp) |>
4   filter(item == 'd_king_cobra') |>
5   lm(resp ~ gender + adjusted_sum_score, data= _) |>
6   summary()
```

- Output:

```
1 Residuals:
2      Min       1Q     Median       3Q      Max
3 -0.66917 -0.20893 -0.09386  0.05007  0.99254
4
5 Coefficients:
6                               Estimate Std. Error t value Pr(>|t|)
7 (Intercept)           -0.107599   0.017667  -6.091 1.51e-09 ***
8 genderM              0.086401   0.018991   4.550 5.91e-06 ***
9 adjusted_sum_score   0.057531   0.003625  15.870 < 2e-16 ***
10 ---
11 Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Break

Dimensionality Reduction

Dimensionality Reduction (Conceptually)

- We have data with a bunch of variables (columns)
- We want to approximate the data with fewer variables (columns)
- This is called constructing a *low rank approximation*
 - The *rank* of a matrix is the number of linearly independent columns
 - Cognitively, it's easier to think about and interpret a smaller number of variables
- **Why is this a good idea?**
 - You might have measured a bunch of stuff in order to take a guess at a thing you couldn't really measure, but you don't know how to weight the variables you did measure!
 - Your variables may be redundant measures of some latent construct
 - Maybe you want to look for groups of variables or people



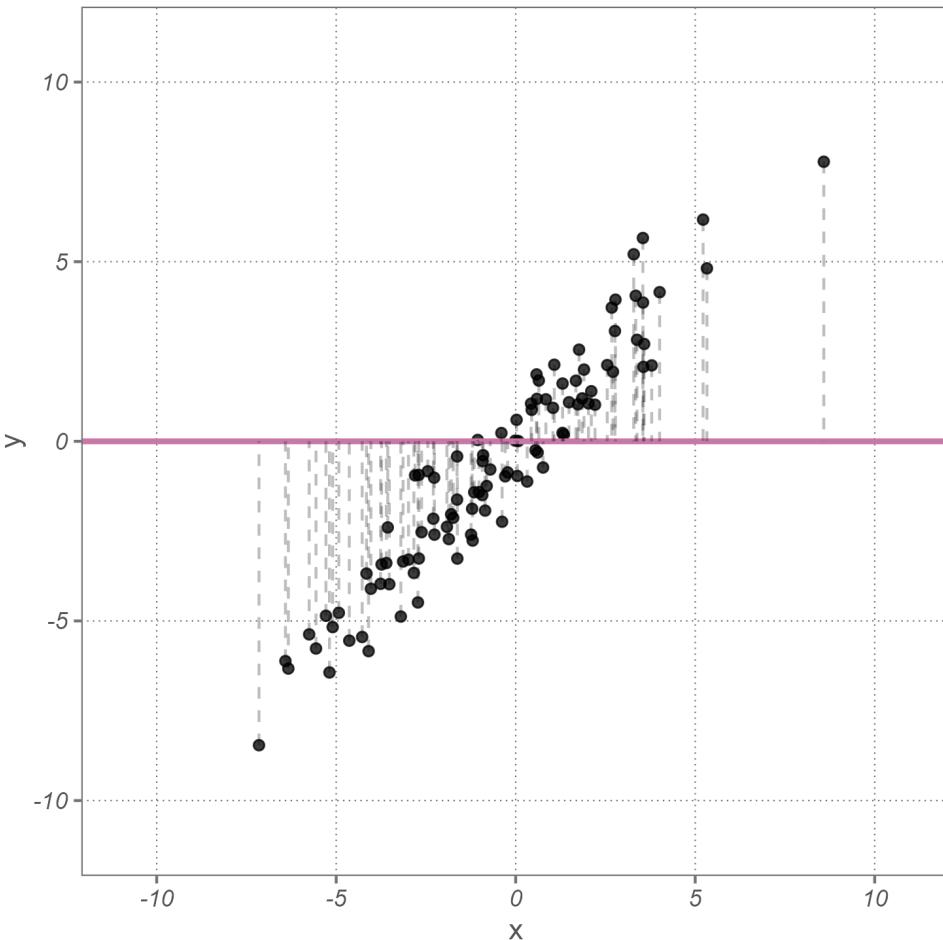
Principal Components Analysis (PCA)

PCA Overview

- Goal: Summarize your data with fewer variables than you have
- Process: Construct a *principal component* (kind of a new variable) as a linear combination of the variables you have that minimizes the *reconstruction error*. Then repeat
 - Reconstruction error is the *orthogonal* distance from each point to the line
- This is usually done with the Singular Value Decomposition
- Note that the task is similar to regression, but you minimize the orthogonal distance from each observation to the line, not the distance from your outcome y to the line!

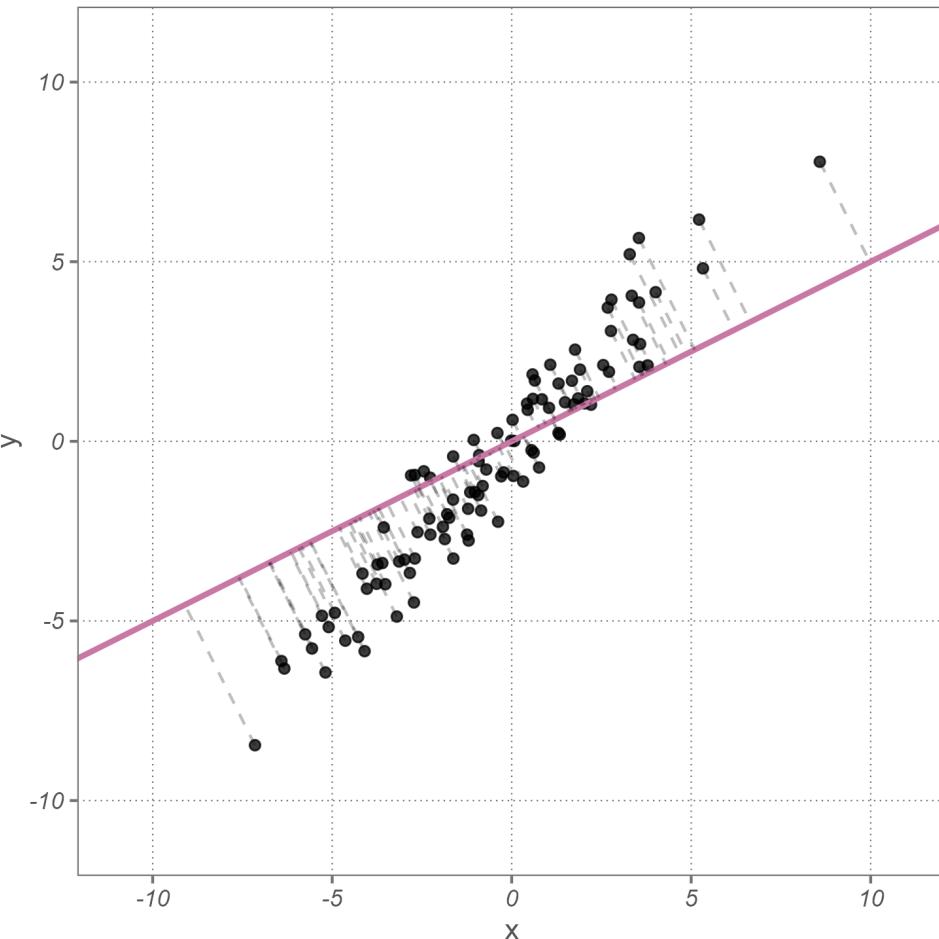
PCA

- The *reconstruction error* is the sum of the squared orthogonal distances from each point to the pink line
- Represented by the dashed lines



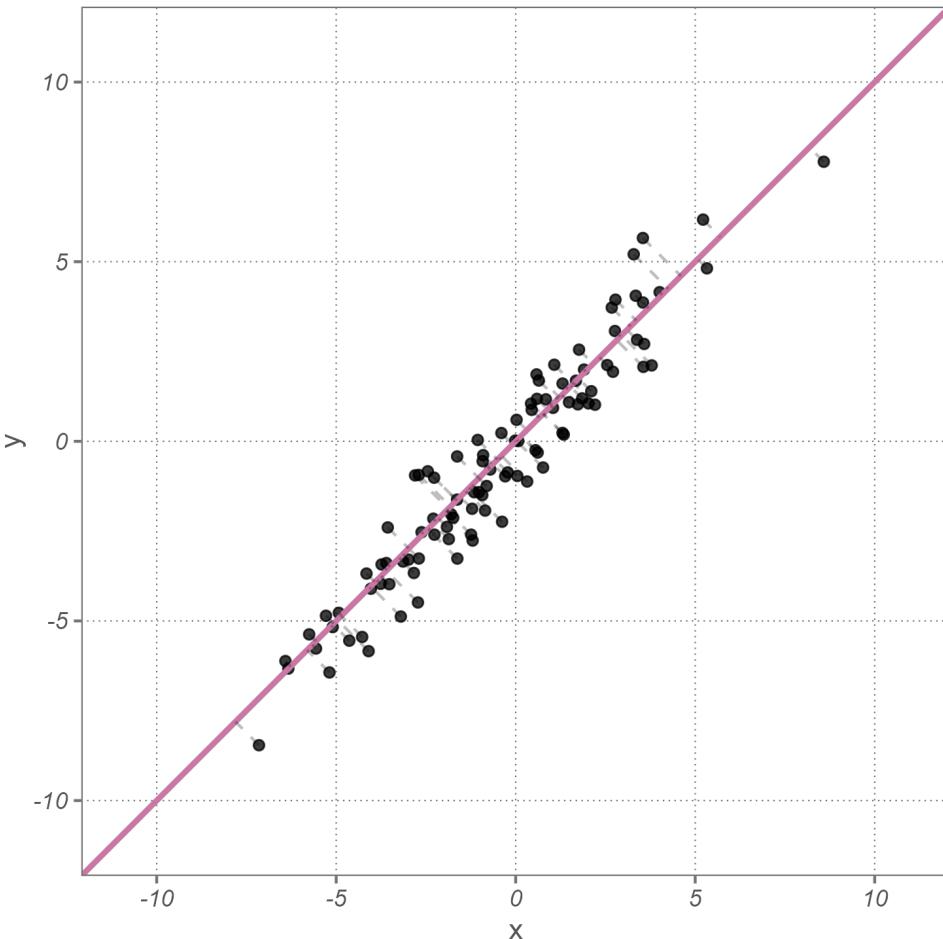
PCA

- The *reconstruction error* is the sum of the squared orthogonal distances from each point to the pink line
- Represented by the dashed lines
- As the line rotates to align with the points, the distances shrink and the projections of each point onto the line spread out



PCA

- The *reconstruction error* is the sum of the squared orthogonal distances from each point to the pink line
- Represented by the dashed lines
- As the line rotates to align with the points, the distances shrink and the projections of each point onto the line spread out
- The first PC does two things:
 1. **Minimizes** the reconstruction error
 2. **Maximizes** the explained variance
- Projections onto the first PC are a single number that retains the most information about the original data



Estimating Principal Components

```
1 library(FactoMineR)
2 library(factoextra)
3
4 # isolate item responses and estimate pca
5 resp <- select(d, starts_with('d_'))
6 pca <- PCA(resp, ncp=10, graph=FALSE)
7
8 # extract eigenvalue information
9 pca_eig <- data.frame(pca$eig) |>
10   rownames_to_column(var='dimension')
11
12 # extract dimensions/loading
13 pca_dim <- data.frame(pca$var$coord) |>
14   rownames_to_column(var='item')
15
16 # extract individual scores/projections
17 pca_resp <- data.frame(pca$ind$coord) |>
18   rownames_to_column(var='person')
```

Eigenvalues: How many PCs are useful?

| dimension | eigenvalue | percentage.of.variance | cumulative.percentage.of.variance |
|-----------|------------|------------------------|-----------------------------------|
| comp 1 | 4.2525175 | 28.350117 | 28.35012 |
| comp 2 | 2.7113610 | 18.075740 | 46.42586 |
| comp 3 | 0.9289215 | 6.192810 | 52.61867 |
| comp 4 | 0.8510230 | 5.673486 | 58.29215 |
| comp 5 | 0.7637384 | 5.091589 | 63.38374 |
| comp 6 | 0.7161497 | 4.774331 | 68.15807 |
| comp 7 | 0.6734326 | 4.489551 | 72.64762 |

Option 1: The Kaiser Criterion

- Select all dimensions with an eigenvalue ≥ 1

| dimension | eigenvalue | percentage.of.variance | cumulative.percentage.of.variance |
|-----------|------------|------------------------|-----------------------------------|
| comp 1 | 4.2525175 | 28.350117 | 28.35012 |
| comp 2 | 2.7113610 | 18.075740 | 46.42586 |
| comp 3 | 0.9289215 | 6.192810 | 52.61867 |
| comp 4 | 0.8510230 | 5.673486 | 58.29215 |

Option 2: Cumulative Variance

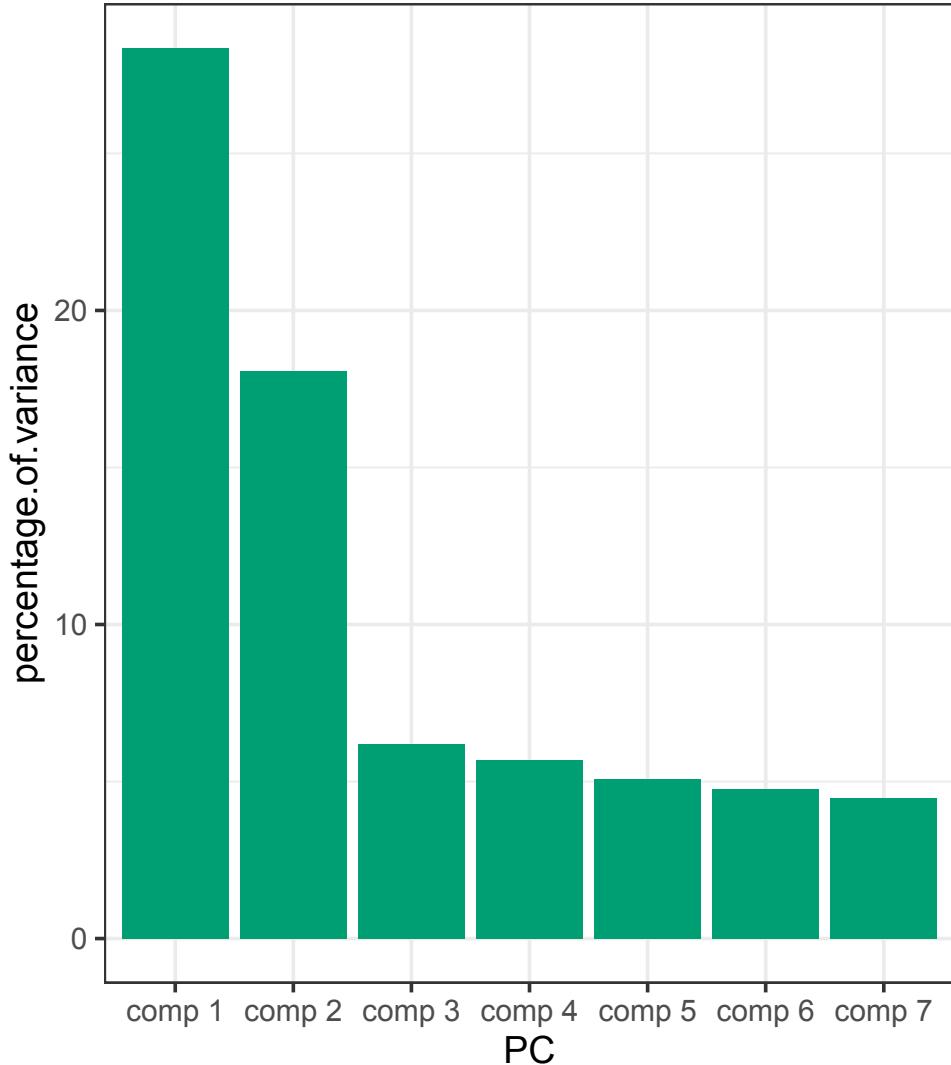
- Pick some cumulative variance that you want (often 50% or 75%)
- Retain dimensions until you exceed that threshold

| dimension | eigenvalue | percentage.of.variance | cumulative.percentage.of.variance |
|-----------|------------|------------------------|-----------------------------------|
| comp 1 | 4.2525175 | 28.350117 | 28.35012 |
| comp 2 | 2.7113610 | 18.075740 | 46.42586 |
| comp 3 | 0.9289215 | 6.192810 | 52.61867 |
| comp 4 | 0.8510230 | 5.673486 | 58.29215 |

Option 3: The Elbow Plot

- Plot the variance explained by each dimension
- Drop dimensions that don't contribute much more over the previous dimension
- This means go up to the "elbow"

```
1 pca_eig |>
2   head(7) |>
3   ggplot(aes(
4     x = dimension,
5     y = percentage.of.variance
6   )) +
7   geom_col(fill = okabeito_colors(3)) +
8   labs(x = 'PC') +
9   theme_bw()
```



Dimensions and Loadings

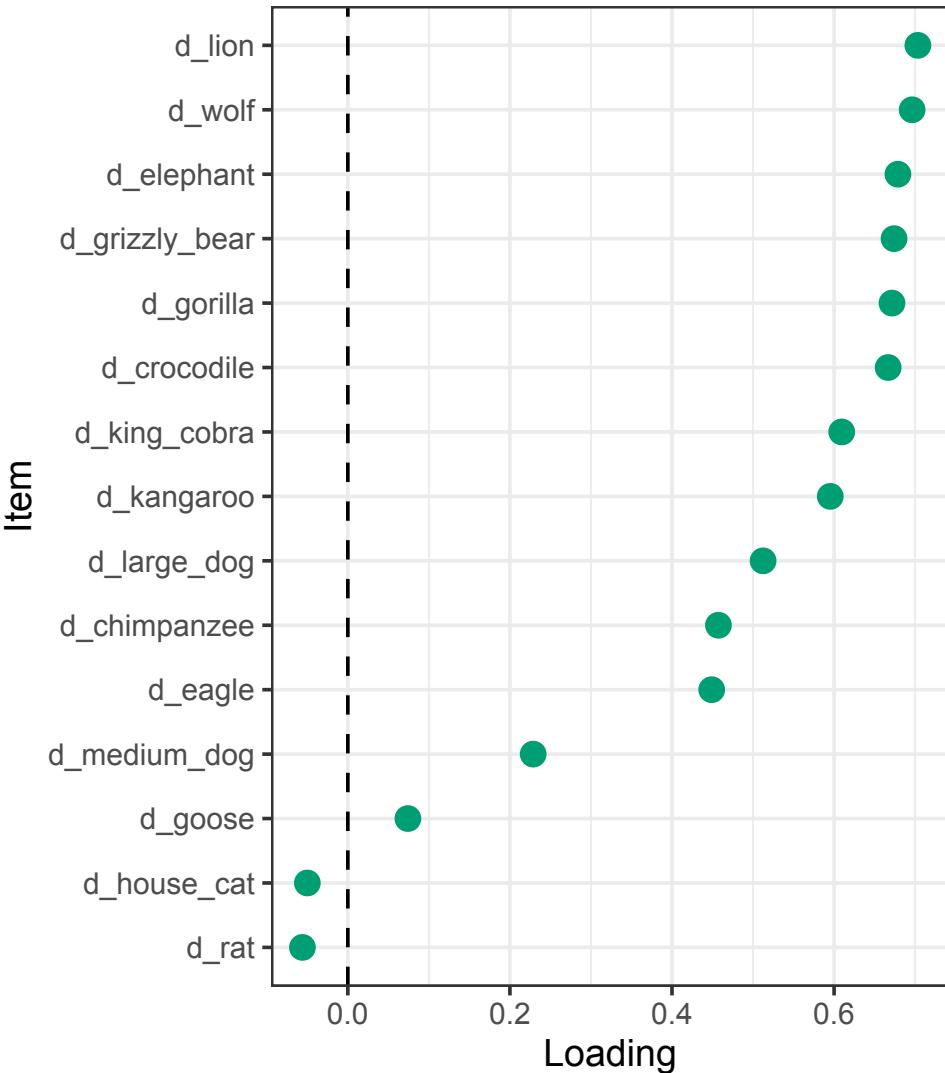
| item | Dim.1 | Dim.2 | Dim.3 |
|----------------|------------|------------|------------|
| d_rat | -0.0560027 | 0.7478180 | 0.3220408 |
| d_house_cat | -0.0499744 | 0.7728305 | 0.2701365 |
| d_medium_dog | 0.2287211 | 0.6610715 | -0.0932625 |
| d_large_dog | 0.5124123 | 0.2998211 | -0.3943514 |
| d_kangaroo | 0.5953089 | 0.0527450 | -0.1915674 |
| d_eagle | 0.4489964 | 0.4131870 | -0.3342900 |
| d_grizzly_bear | 0.6740091 | -0.2427908 | 0.3365680 |

Individuals

| person | Dim.1 | Dim.2 | Dim.3 |
|--------|-----------|------------|------------|
| 1 | 0.277625 | 1.7352156 | -0.7241404 |
| 2 | -1.362983 | 0.4078412 | 1.0116342 |
| 3 | -1.141138 | 1.2108529 | 0.8180881 |
| 4 | -1.384454 | -1.5588112 | 0.0452277 |
| 5 | 1.669899 | 2.1264304 | -0.3689416 |

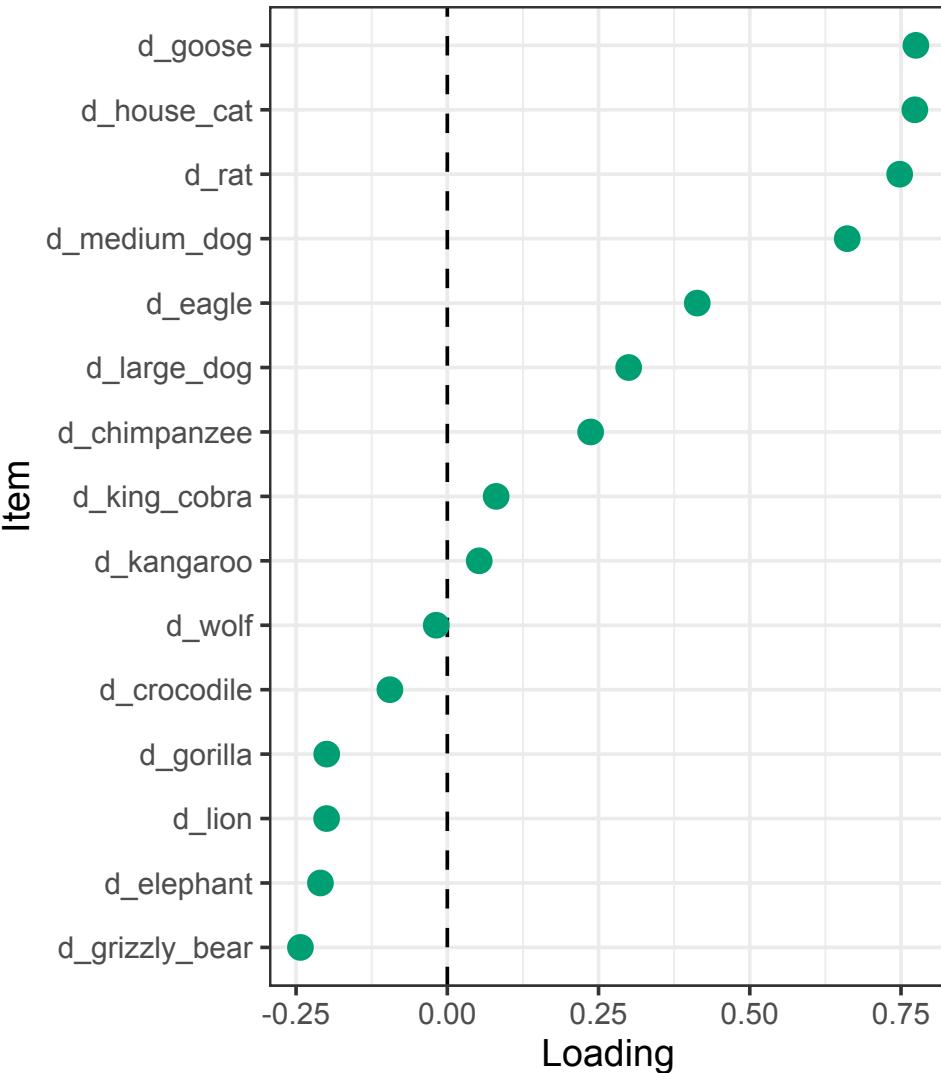
Dimension 1

```
1 ggplot(pca_dim,
2         aes(x = Dim.1,
3               y = reorder(item, Dim.1))) +
4   geom_vline(aes(xintercept = 0), lty = 2) +
5   geom_point(size = 3,
6               color = okabeito_colors(3)) +
7   labs(x = 'Loading', y = 'Item') +
8   theme_bw()
```



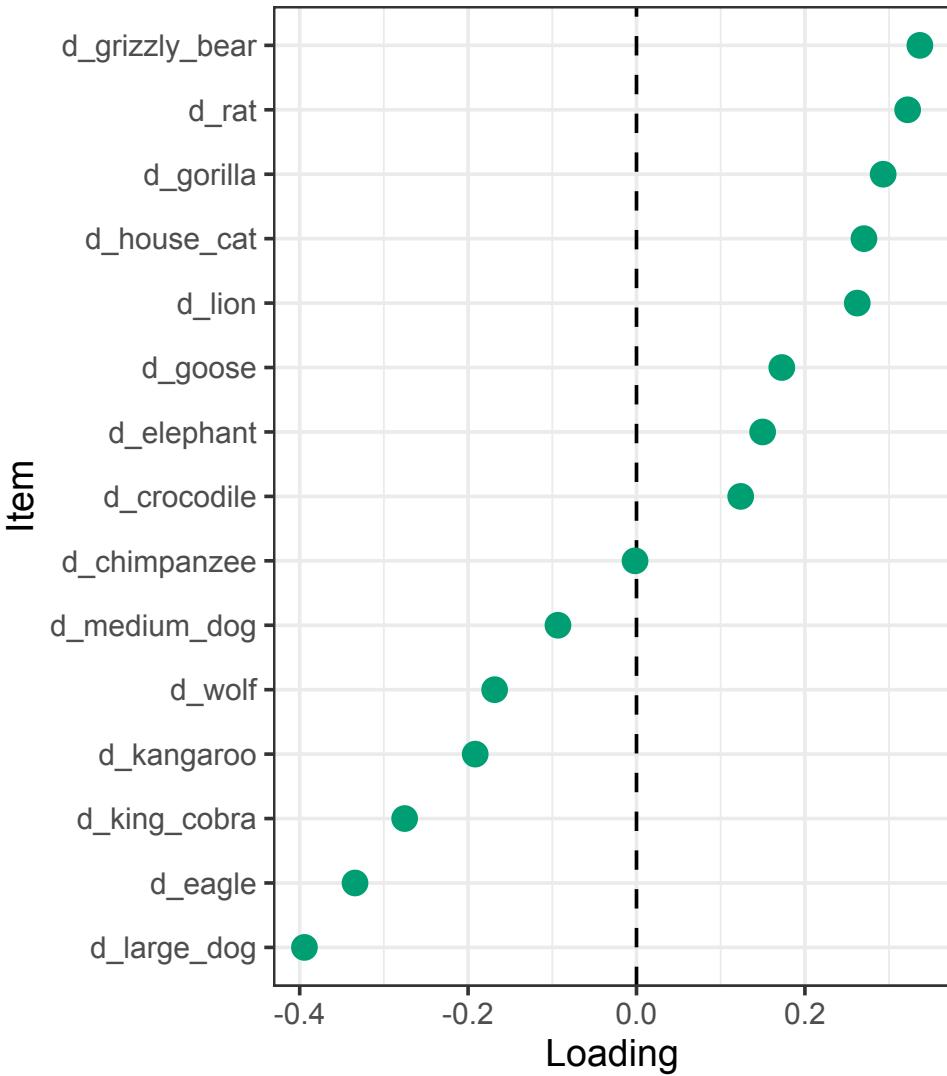
Dimension 2

```
1 ggplot(pca_dim,
2         aes(x = Dim.2,
3               y = reorder(item, Dim.2))) +
4   geom_vline(aes(xintercept = 0), lty = 2) +
5   geom_point(size = 3,
6               color = okabeito_colors(3)) +
7   labs(x = 'Loading', y = 'Item') +
8   theme_bw()
```



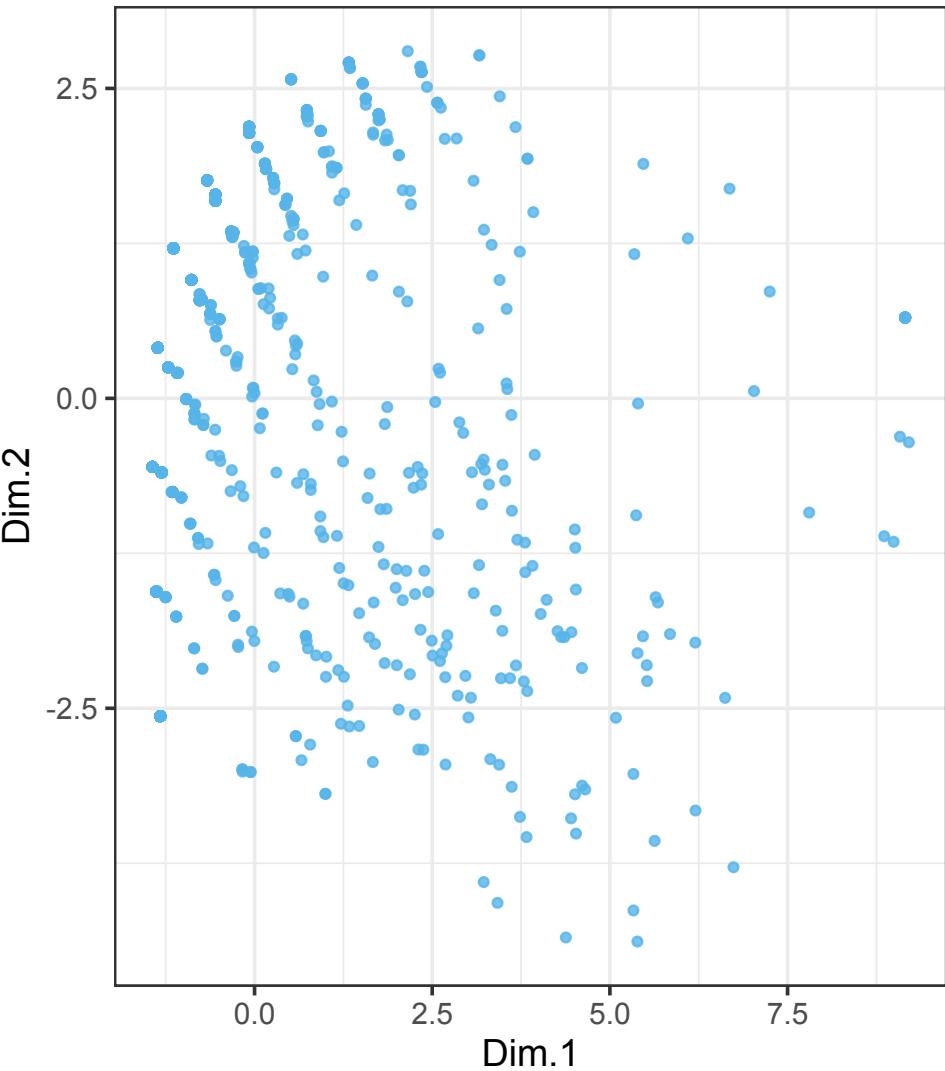
Dimension 3

```
1 ggplot(pca_dim,
2         aes(x = Dim.3,
3               y = reorder(item, Dim.3))) +
4   geom_vline(aes(xintercept = 0), lty = 2) +
5   geom_point(size = 3,
6               color = okabeito_colors(3)) +
7   labs(x = 'Loading', y = 'Item') +
8   theme_bw()
```



Individuals

```
1 ggplot(pca_resp,
2         aes(x = Dim.1, y = Dim.2)) +
3         geom_point(alpha = 0.8, size = 1,
4                     color = okabeito_colors(2)) +
5         theme_bw()
```



PCA Summary

- Constructs successive **orthogonal** dimensions
- Each dimension maximizes the amount of explained residual variance
- Because dimensions are orthogonal, individual locations along these dimensions are uncorrelated **by construction**
- Variable loadings on dimensions can be used to interpret and describe the dimensions
- Individual projections onto these dimensions can be interpreted as scores along those dimensions

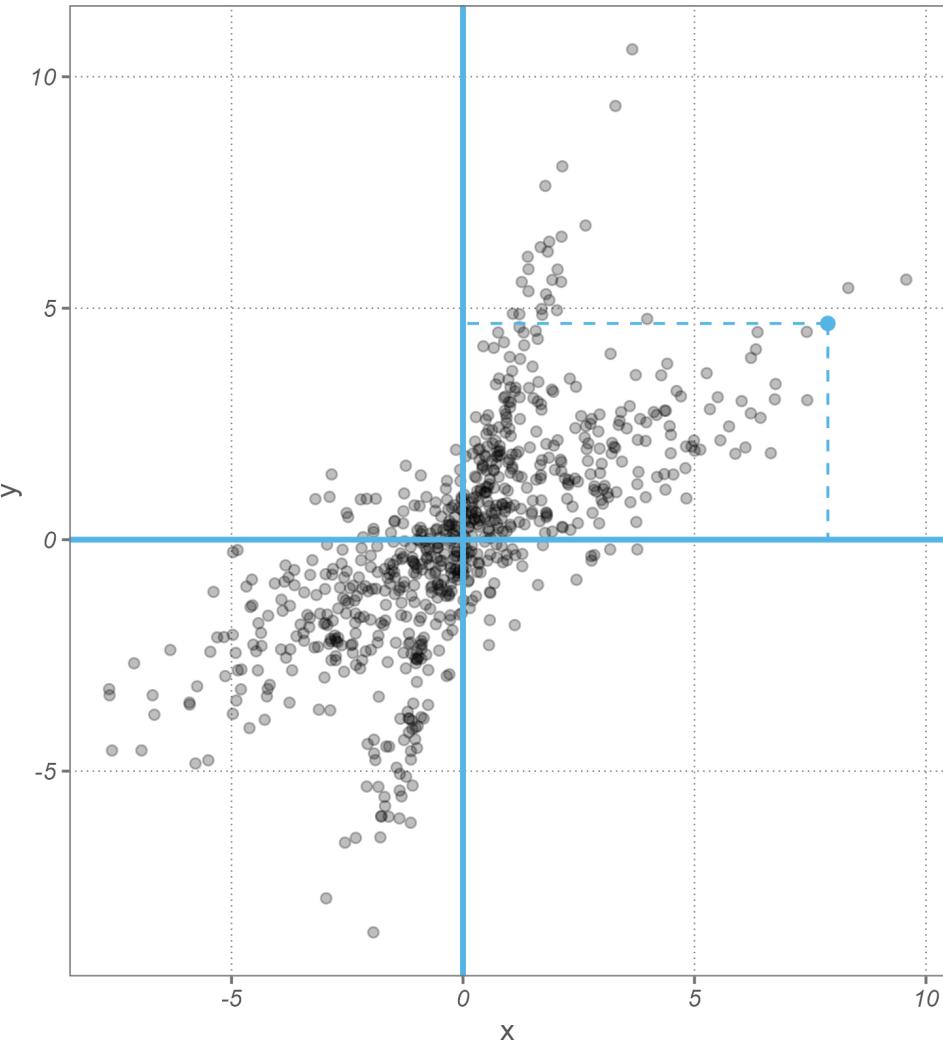
Factor Analysis

(Exploratory) Factor Analysis

- Conceptually different from PCA
- Factor analysis is a latent variable model that seeks to estimate item level associations (loadings) onto latent variables
- The core idea is that levels of the latent variable cause individual item responses, which are observed with error
- "Factor Analysis" is really two things:
 - **Confirmatory Factor Analysis (CFA)** is another name for structural equation modeling (SEM) with item responses
 - **Exploratory Factor Analysis (EFA)** is a dimensionality reduction technique that is conceptually different from PCA but *sometimes* mathematically identical to PCA
- If you allow for correlated factors, EFA is **not** equivalent to PCA

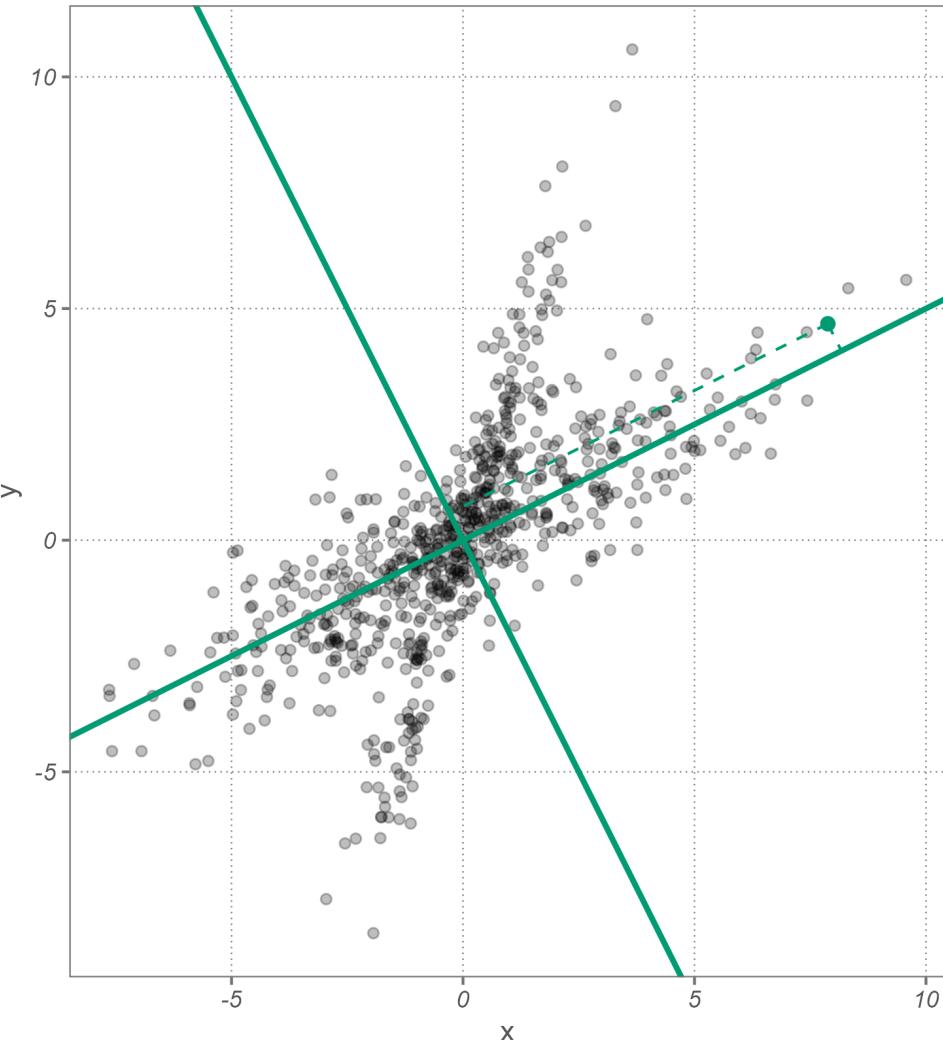
EFA

- Similar to PCA, EFA constructs dimensions that are linear combinations of observed variables and projects points onto them
- EFA models are not identified; there are an infinite number of solutions that describe data equally well (factor indeterminacy problem)



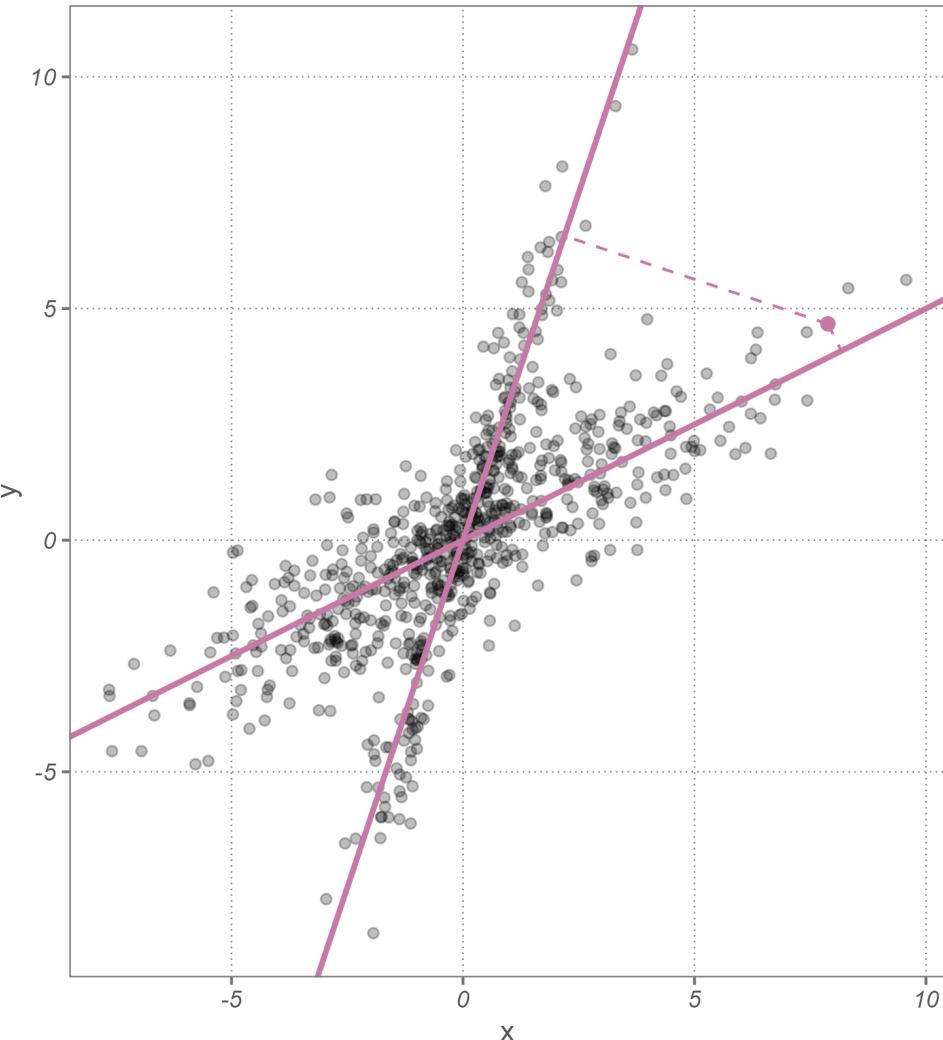
EFA

- Similar to PCA, EFA constructs dimensions that are linear combinations of observed variables and projects points onto them
- EFA models are not identified; there are an infinite number of solutions that describe data equally well (factor indeterminacy problem)
- EFA leverages *rotations* to solve this that try to pick dimensions to satisfy some sort of logical criterion. Varimax rotation is equivalent to PCA



EFA

- Similar to PCA, EFA constructs dimensions that are linear combinations of observed variables and projects points onto them
- EFA models are not identified; there are an infinite number of solutions that describe data equally well (factor indeterminacy problem)
- EFA leverages *rotations* to solve this that try to pick dimensions to satisfy some sort of logical criterion. **Varimax** rotation is equivalent to PCA
- EFA also allows for *oblique* rotations that produce correlated factors. **Promax** is the most commonly used one



Estimating EFA

```
1 library(psych)
2
3 # Check the documentation - it's intense!
4 ?fa
5
6 efa_1 <- fa(resp, nfactors = 1, rotate = 'oblimin')
7 efa_2 <- fa(resp, nfactors = 2, rotate = 'oblimin')
8 efa_3 <- fa(resp, nfactors = 3, rotate = 'oblimin')
```

One Factor Solution

```
1 efa_1$Vaccounted
```

```
1                               MR1
2 SS loadings    3.6045189
3 Proportion Var 0.2403013
```

Two Factor Solution

```
1 efa_2$Vaccounted  
  
1                          MR1          MR2  
2 SS loadings           3.6227900 2.1692315  
3 Proportion Var       0.2415193 0.1446154  
4 Cumulative Var      0.2415193 0.3861348  
5 Proportion Explained 0.6254794 0.3745206  
6 Cumulative Proportion 0.6254794 1.0000000
```

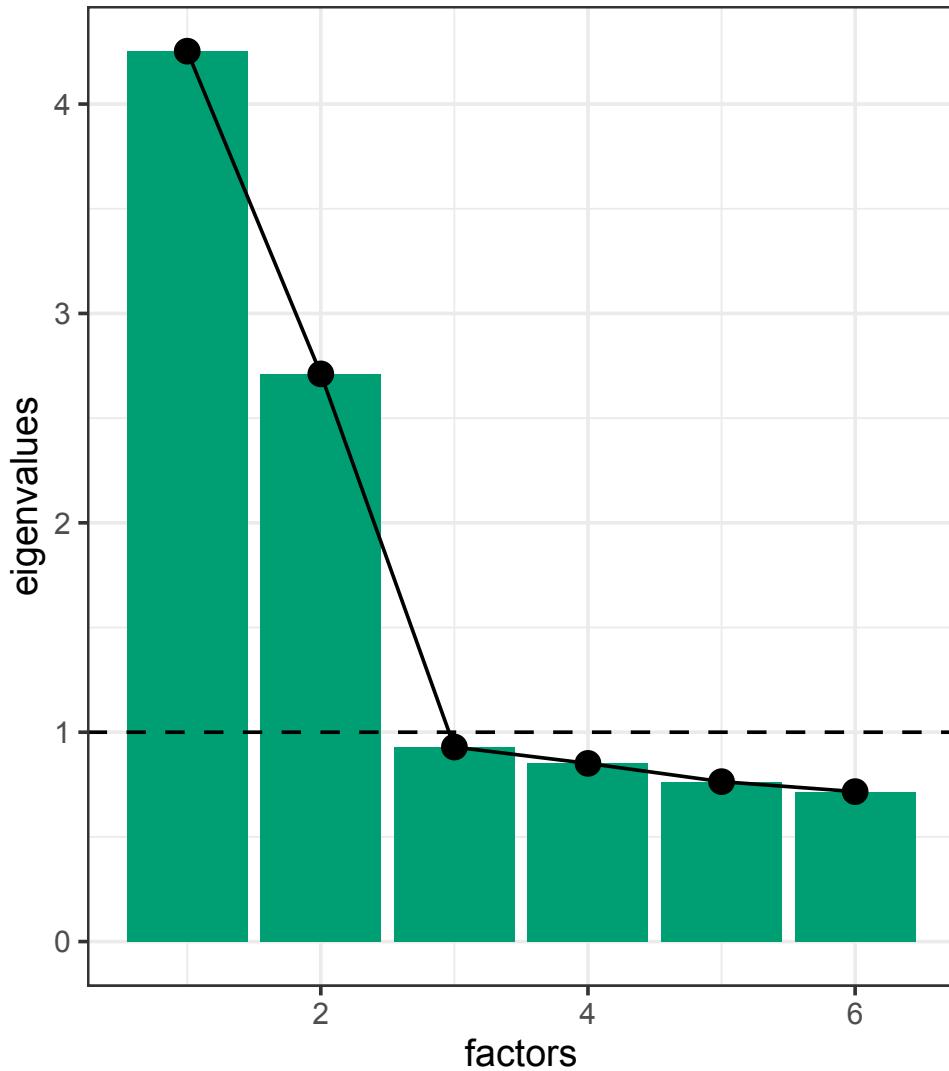
Three Factor Solution

```
1 efa_3$Vaccounted
```

| | MR1 | MR2 | MR3 |
|-------------------------|-----------|-----------|-----------|
| 2 SS loadings | 2.5972061 | 1.8689791 | 1.7516499 |
| 3 Proportion Var | 0.1731471 | 0.1245986 | 0.1167767 |
| 4 Cumulative Var | 0.1731471 | 0.2977457 | 0.4145223 |
| 5 Proportion Explained | 0.4177026 | 0.3005836 | 0.2817138 |
| 6 Cumulative Proportion | 0.4177026 | 0.7182862 | 1.0000000 |

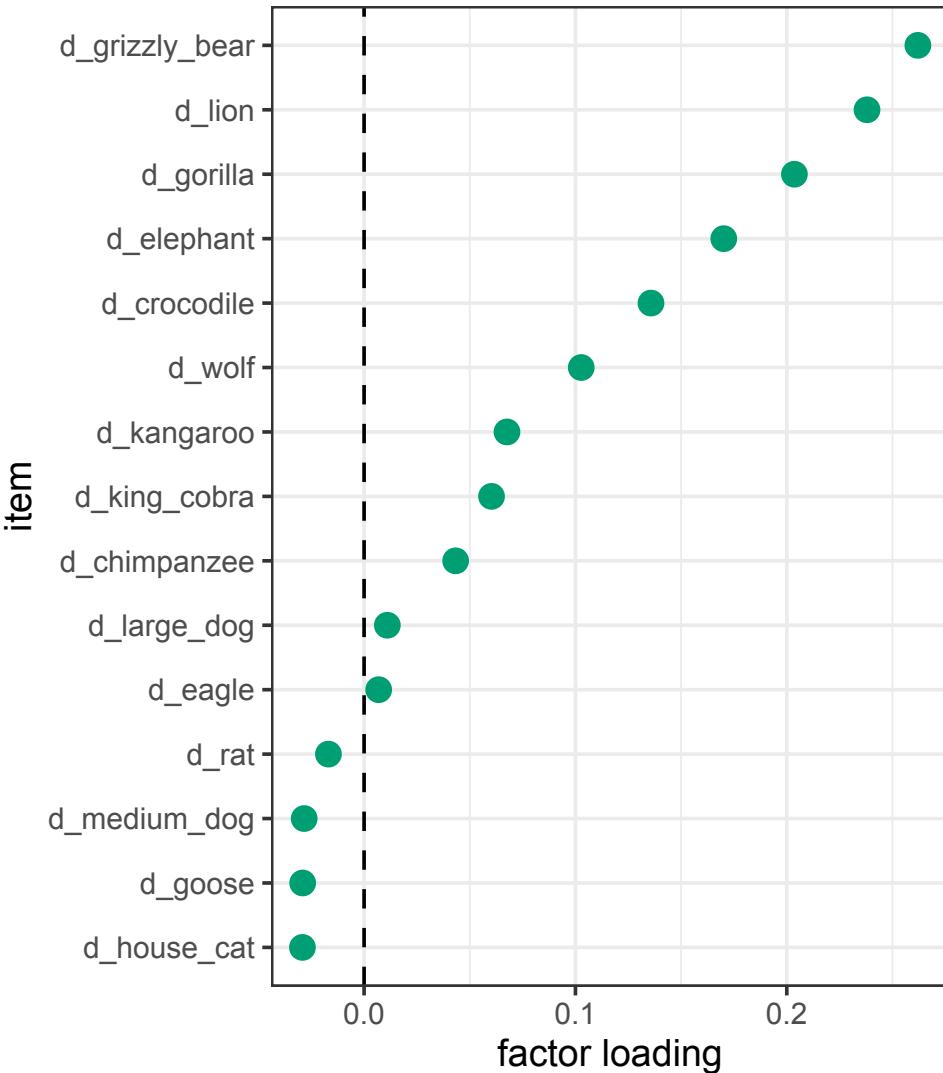
Selecting Number of Factors

```
1 fa_eigenvalues <- data.frame(factors = 1:6,
2   eigenvalues = efa_3$e.values[1:6])
3
4 ## Selecting Number of Factors
5
6 ggplot(fa_eigenvalues,
7   aes(x = factors, y = eigenvalues)) +
8   geom_col(fill = okabeito_colors(3)) +
9   geom_point(size = 3) +
10  geom_line() +
11  geom_hline(aes(yintercept = 1), lty = 2) +
12  theme_bw()
```



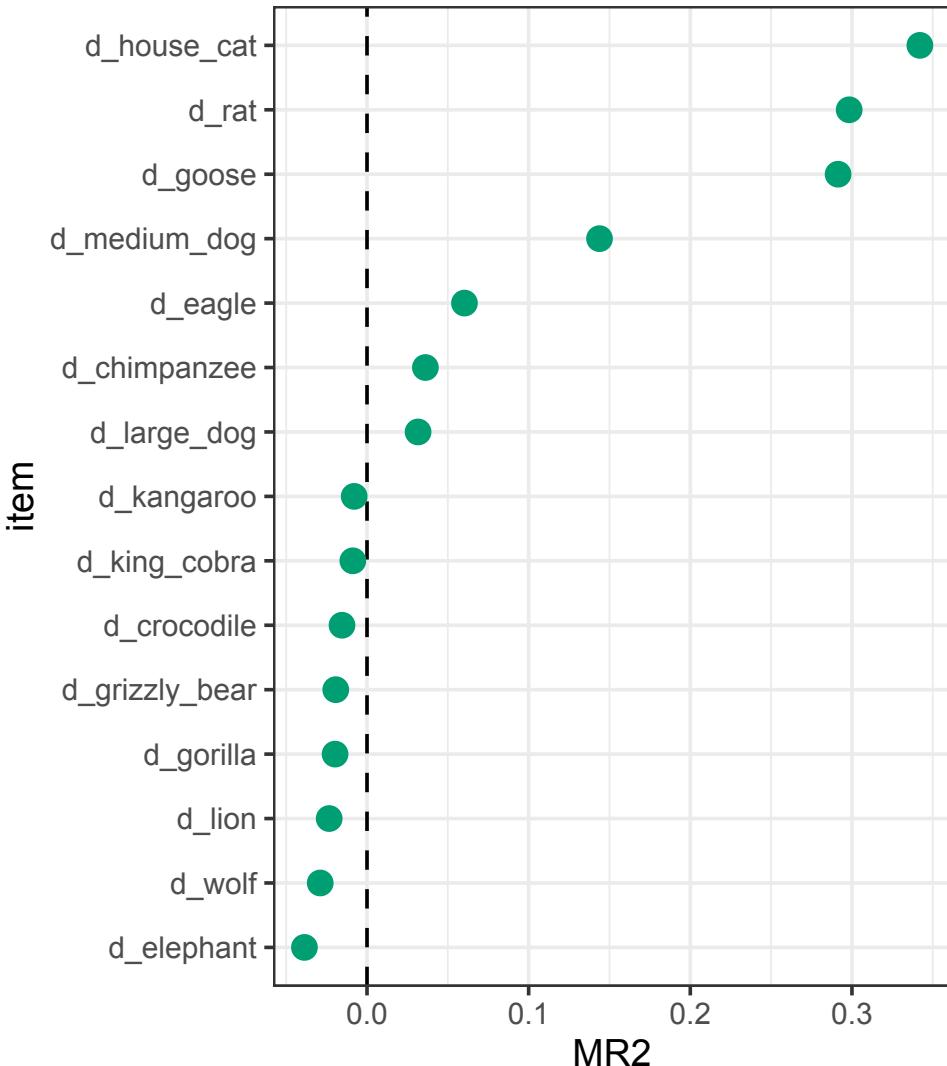
Factor 1 Loadings

```
1 ggplot(efa,
2       aes(x = MR1,
3              y = reorder(item, MR1))) +
4   geom_vline(aes(xintercept = 0), lty = 2) +
5   geom_point(size = 3,
6              color = okabeito_colors(3)) +
7   labs(y = 'item', x = 'factor loading') +
8   theme_bw()
```



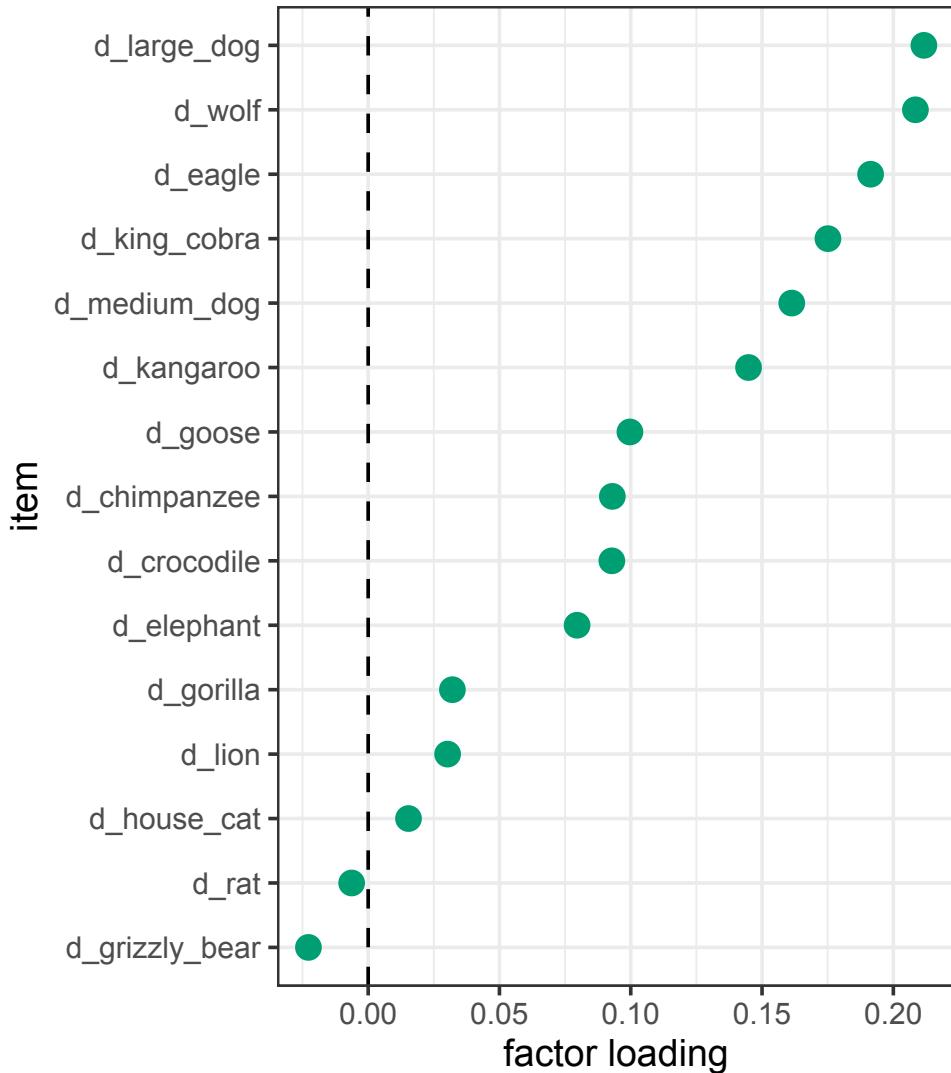
Factor 2 Loadings

```
1 ggplot(efa,
2       aes(x = MR2,
3              y = reorder(item, MR2))) +
4   geom_vline(aes(xintercept = 0), lty = 2) +
5   geom_point(size = 3,
6              color = okabeito_colors(3)) +
7   labs(y = 'item', x = 'factor loading') +
8   theme_bw()
```



Factor 3 Loadings

```
1 ggplot(efa,
2       aes(x = MR3,
3              y = reorder(item, MR3))) +
4   geom_vline(aes(xintercept = 0), lty = 2) +
5   geom_point(size = 3,
6              color = okabeito_colors(3)) +
7   labs(y = 'item', x = 'factor loading') +
8   theme_bw()
```



Individual Scores

```
1 efa_3$scores  
  
1           MR1          MR2          MR3  
2 [1,] -0.285443975 8.387273e-01 0.603740923  
3 [2,] -0.448881256 4.989128e-01 -0.636824429  
4 [3,] -0.505534260 7.864999e-01 -0.314300349  
5 [4,] -0.327039053 -8.288798e-01 -0.873253278  
6 [5,]  0.248132302 9.532238e-01 1.131932390
```

Factor Analysis Summary

- Constructs latent factors from correlation matrix
- Rotates factors to allow for correlated factors (dimensions)
- Factor loadings can be used to interpreted and describe the factors
- Individual scores on these factors can be recovered
- Generally, the use and interpretation are the same as in PCA

Wrap Up

PCA vs. Factor Analysis

- Variables
 - PCA is a model of observed variables
 - Factor analysis is a model of latent variables
- Data
 - PCA focuses on reconstructing the diagonals of the covariance matrix (individual variances)
 - Factor analysis focuses on reconstructing the off-diagonals of the covariance matrix (between-item covariances)
 - PCA produces orthogonal dimensions
 - Factor analysis can produce correlated factors
- Important to note that PCA and factor analysis usually (but not always) produce different results
- Often factor analysis produces more interpretable factors with an oblique rotation that allows for correlated factors

Problem Set 1

- Posted!
- Two weeks to complete
- You won't be able to finish all of it until after next week's lecture!
- Topics covered:
 - Comparing and contrasting PCA and Factor Analysis (with real data)
 - Comparing and contrasting CTT and Item Response Theory (with real data)
- Data will come from the Item Response Warehouse (IRW)
 - Website: <https://itemresponsewarehouse.org/>
 - Paper: <https://link.springer.com/article/10.3758/s13428-025-02796-y>
- Last year, students said this assignment took a long time. I didn't make it shorter, so start soon!

Check-Out

- PollEv.com/klintkanopka