APSTA-GE 2094

APSY-GE 2524

Modern Approaches in Measurement: Lecture 2

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Announcements

- PS0 is due tomorrow night @ 11.59p!
  - Submit a rendered `.pdf` on Gradescope
  - Final late deadline Monday night (2/2 @ 11.59p)
  - 10 point late penalty per day
- PS1 is out!
  - Due in two weeks (2.13 @ 11.59p)
  - Files posted on the Problem Sets tab
  - Again, submit on Gradescope
  - This is longer than PS0, and you can complete a little more than half of it after today
- Had some people pop into office hours yesterday and it was fun! Those are Wednesdays 2-3p in the second floor lobby of Kimball Hall

Check-In

- PollEv.com/klintkanopka

Validity

Validity

- On the most basic level, the idea that we have confidence we are measuring the thing we think we are measuring
- Starts with the definition of a *construct*, or the unobservable thing we care about measuring
- Requires some combination of theory and some amount of empirical study to get at

Validity

1. Take 5-10 minutes and in groups of 3 ± 1, discuss the differences between Cronbach and Meehl's conception of "construct validity," Kane's idea of validity, and Boorsboom's idea of validity
2. What are the pros/cons of each approach?
3. Where might each approach succeed or fail?
4. What approach do you agree with most?

Cronbach and Meehl

- Construct validity is derived from the nomological network
- Core idea: Since we cannot directly observe the latent construct of interest, we have to collect a dense network of evidence through theoretical arguments and correlations to support our ideas of validity
- The idea of "construct validity" and other "types" of validity is extremely antiquated

Kane

- Validity is associated with the interpretation assigned to test scores rather than with the test scores or the test
- Interpretive arguments lay out how a test score will be interpreted and what evidence needs to be brought to bear to evaluate the argument
- Core idea: Validity is highly contextual and changes to instruments or populations can result in changes to the construct being measured

Boorsboom

- A test is valid for measuring an attribute if (a) the attribute exists and (b) variations in the attribute causally produce variation in the measurement outcomes
- Focuses on causal reasoning
- Core idea: Validity should be a function of an instrument, not a score interpretation

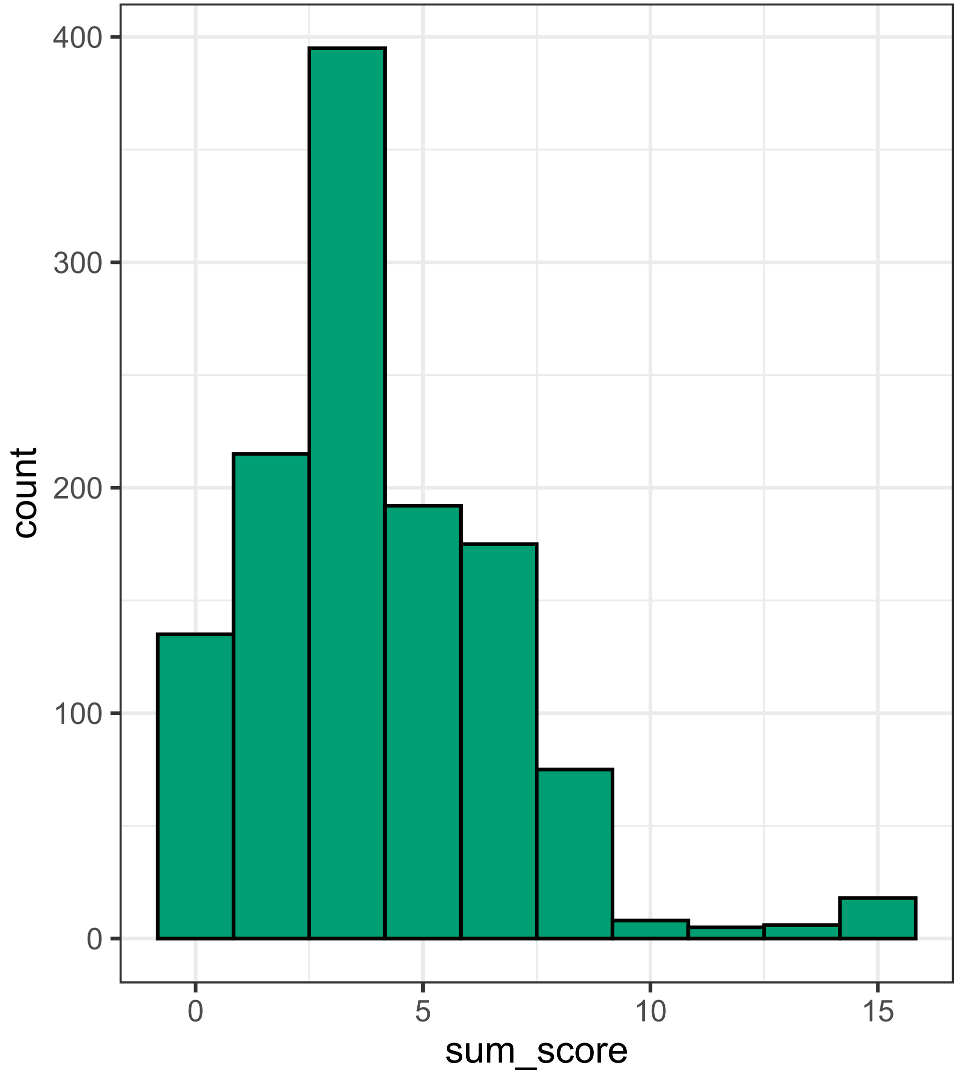
Classical Test Theory Practice

More Item Response Data

- The dataset `animalfights_clean.rds` (downloadable here) is a pre-cleaned version of the data from PS0.
- In a group of 3 ± 1 take ten minutes to:
  1. Compute the ability (sum score) for each respondent and plot a distribution
  2. Compute the difficulty (p-value) for each item and plot them in order of difficulty
  3. Compute the discrimination (item-total correlation) for each item and plot them in order
  4. Check the item `d_king_cobra` for DIF by gender

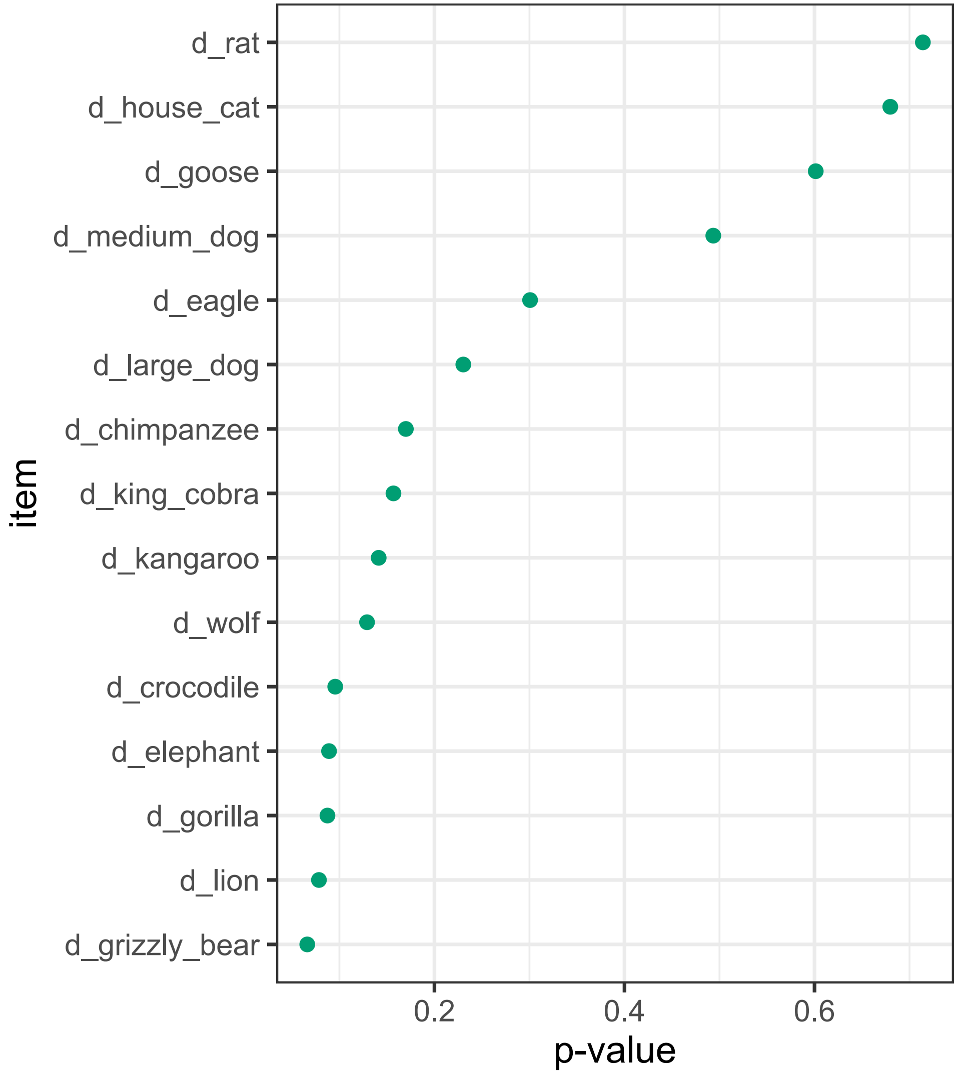
Sum Score

```r
d <- readRDS('animalfights_clean.rds')

d_long <- d |>
  pivot_longer(
    cols = starts_with('d_'),
    names_to = 'item',
    values_to = 'resp'
  )

sum_scores <- d_long |>
  group_by(id) |>
  summarize(sum_score = sum(resp))

ggplot(sum_scores, aes(x = sum_score)) +
  geom_histogram(
    bins = 10,
    color = 'black',
    fill = okabeito_colors(3)
  ) +
  theme_bw()
```



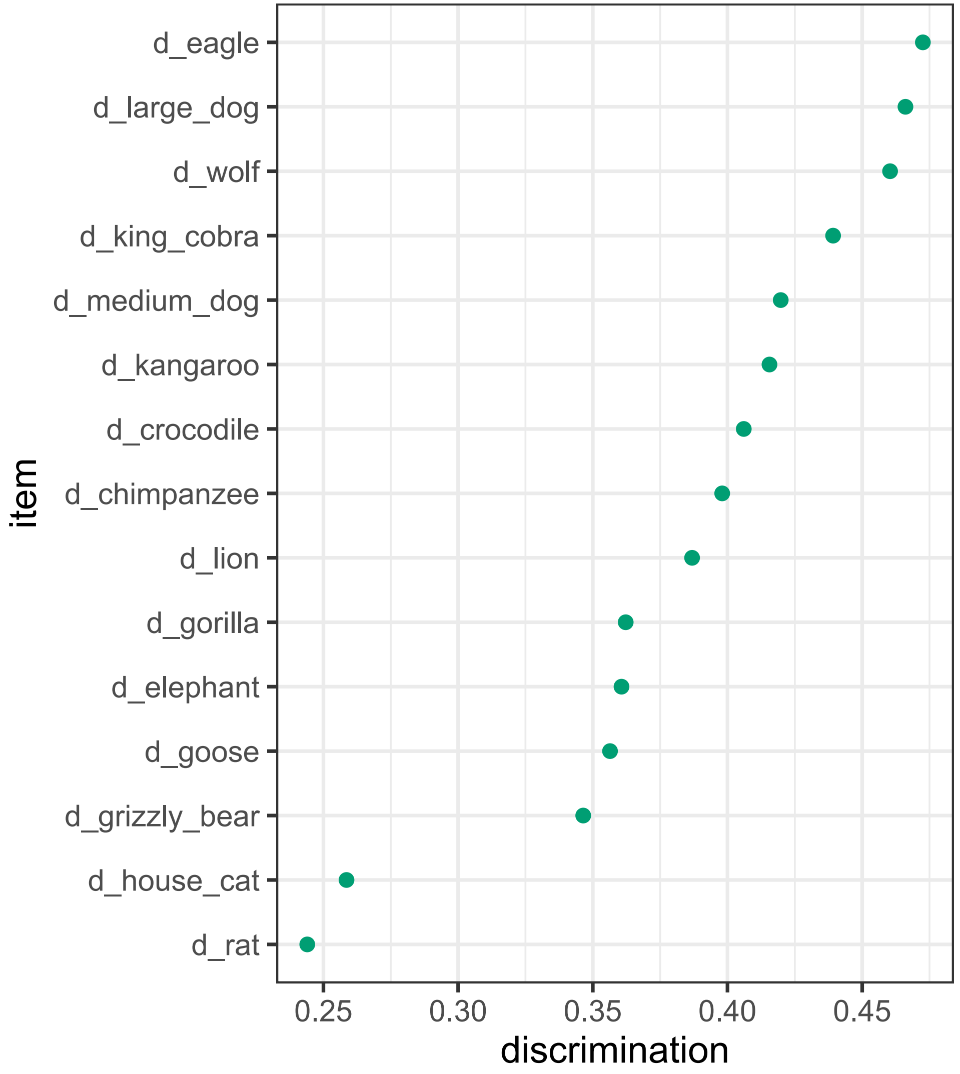
p-value

```r
diff <- d_long |>
  group_by(item) |>
  summarize(p = mean(resp))

ggplot(diff, aes(x = p, y = reorder(item, p))) +
  geom_point(
    color = okabeito_colors(3)
  ) +
  labs(x = 'p-value', y = 'item') +
  theme_bw()
```



Item-Total Correlation

```
disc <- left_join(d_long, sum_scores, by = 'id
  mutate(adjusted_sum_score = sum_score - resp
  group_by(item) |>
  summarize(a = cor(resp, adjusted_sum_score))

ggplot(disc, aes(x = a, y = reorder(item, a)))
  geom_point(
    color = okabeito_colors(3)
  ) +
  labs(x = 'discrimination', y = 'item') +
  theme_bw()
```



Differential Item Functioning

- Regression:

```r
d_long |>
  left_join(sum_scores, by = 'id') |>
  mutate(adjusted_sum_score = sum_score - resp) |>
  filter(item == 'd_king_cobra') |>
  lm(resp ~ gender + adjusted_sum_score, data= _) |>
  summary()
```

- Output:

```
Residuals:
     Min       1Q   Median       3Q      Max 
-0.66917 -0.20893 -0.09386  0.05007  0.99254 

Coefficients:
                    Estimate Std. Error t value Pr(>|t|)    
(Intercept)        -0.107599   0.017667  -6.091 1.51e-09 ***
genderM             0.086401   0.018991   4.550 5.91e-06 ***
adjusted_sum_score  0.057531   0.003625  15.870  < 2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

Break

Dimensionality Reduction

Dimensionality Reduction (Conceptually)

- We have data with a bunch of variables (columns)
- We want to approximate the data with fewer variables (columns)
- This is called constructing a *low rank approximation*
  - The *rank* of a matrix is the number of linearly independent columns
  - Cognitively, it's easier to think about and interpret a smaller number of variables
- **Why is this a good idea?**
  - You might have measured a bunch of stuff in order to take a guess at a thing you couldn't really measure, but you don't know how to weight the variables you did measure!
  - Your variables may be redundant measures of some latent construct
  - Maybe you want to look for groups of variables or people

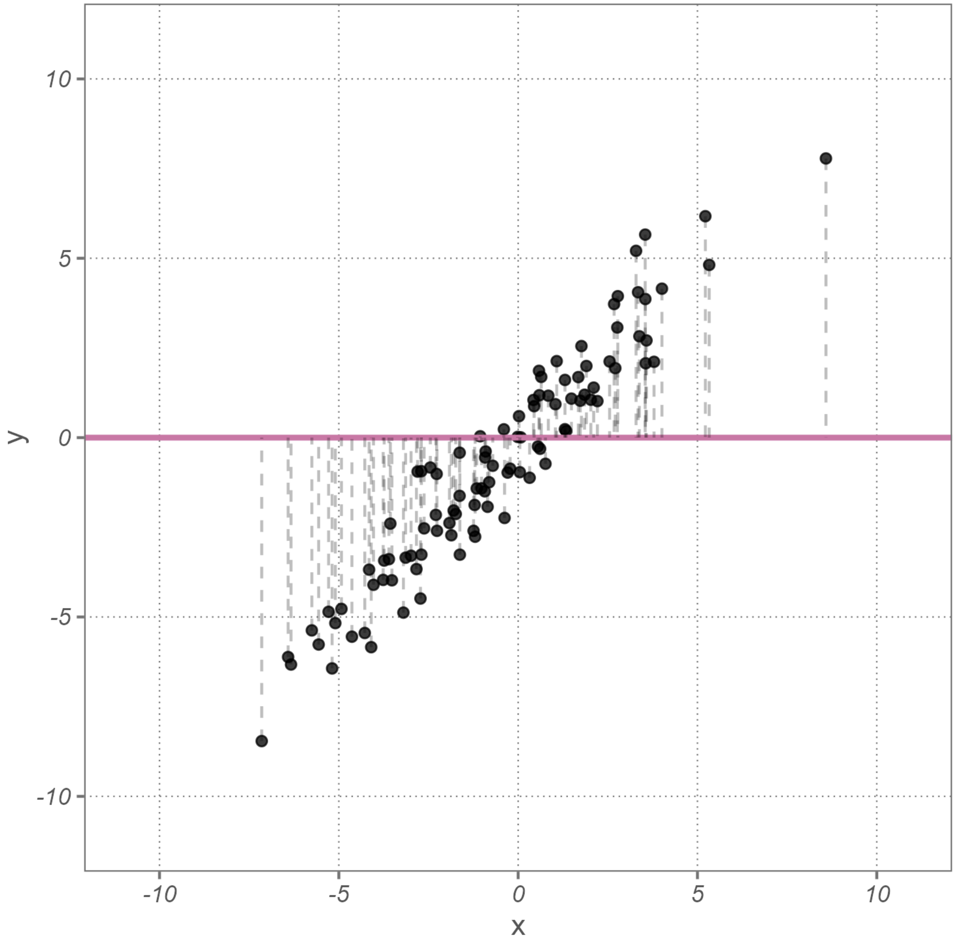
Principal Components Analysis (PCA)

PCA Overview

- Goal: Summarize your data with fewer variables than you have
- Process: Construct a *principal component* (kind of a new variable) as a linear combination of the variables you have that minimizes the *reconstruction error*. Then repeat
  - Reconstruction error is the *orthogonal* distance from each point to the line
- This is usually done with the Singular Value Decomposition
- Note that the task is similar to regression, but you minimize the orthogonal distance from each observation to the line, not the distance from your outcome y to the line!

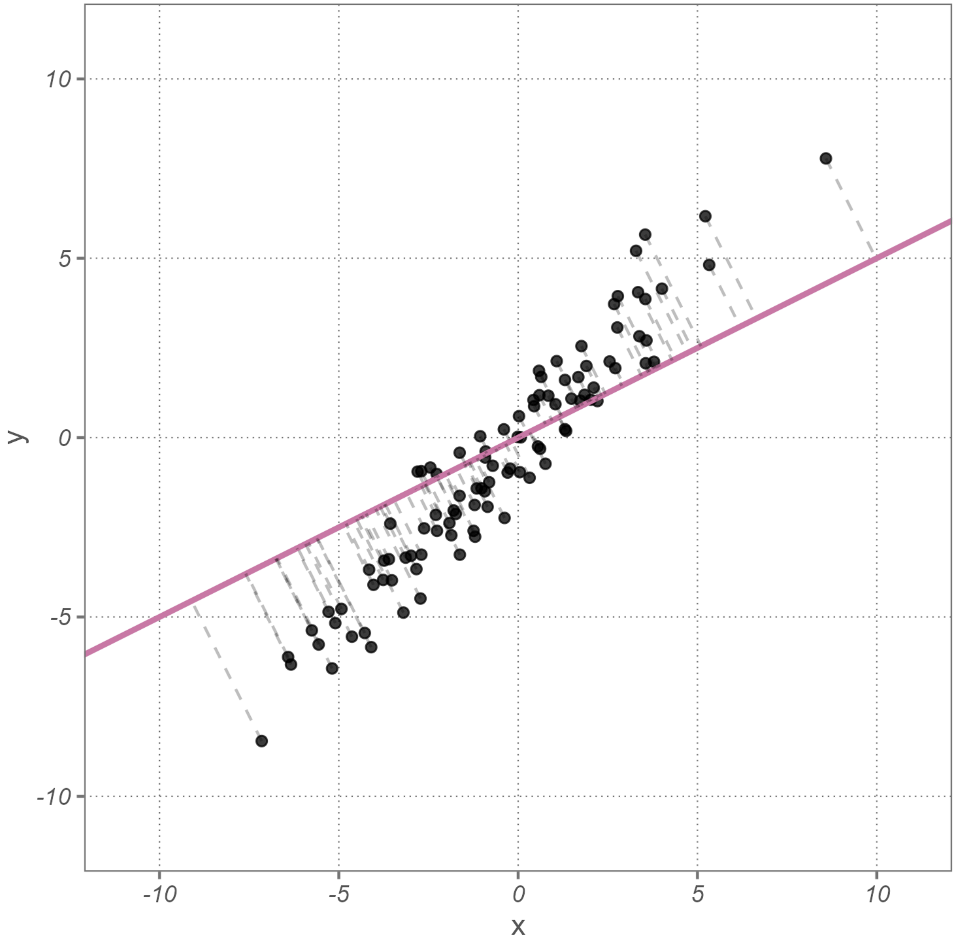
PCA

- The *reconstruction error* is the sum of the squared orthogonal distances from each point to the pink line
- Represented by the dashed lines



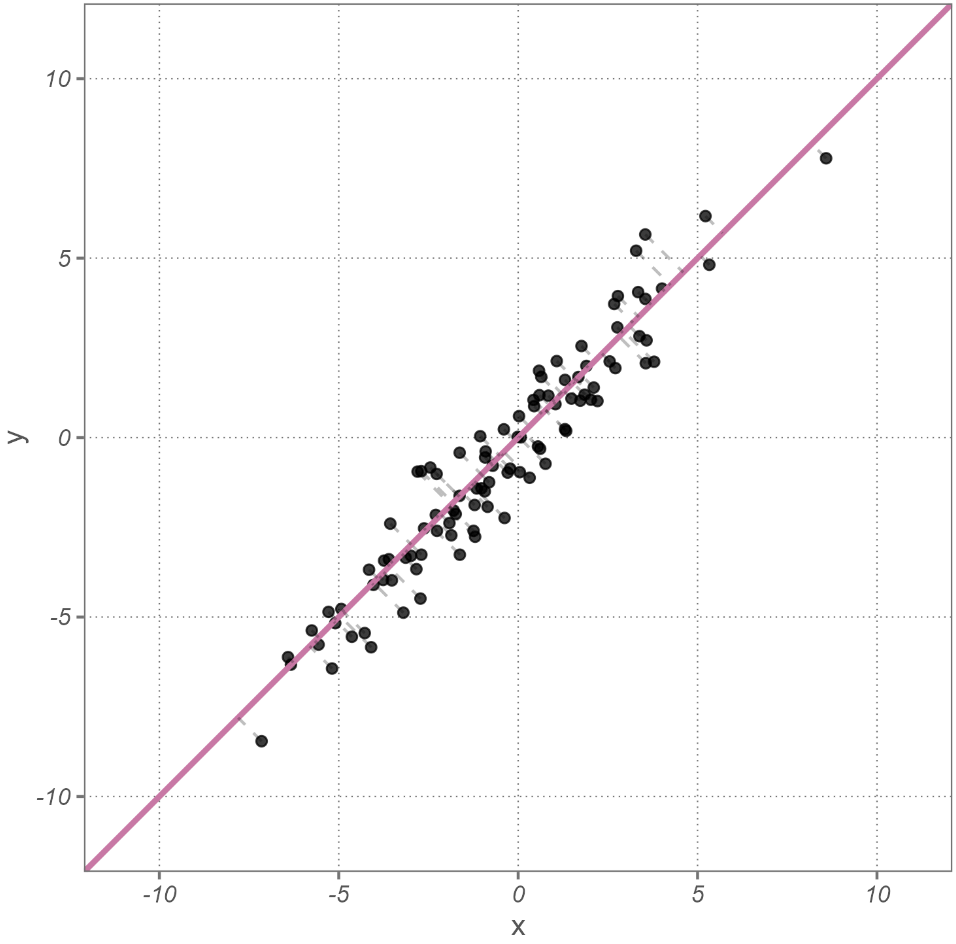
PCA

- The *reconstruction error* is the sum of the squared orthogonal distances from each point to the pink line
- Represented by the dashed lines
- As the line rotates to align with the points, the distances shrink and the projections of each point onto the line spread out



PCA

- The *reconstruction error* is the sum of the squared orthogonal distances from each point to the pink line
- Represented by the dashed lines
- As the line rotates to align with the points, the distances shrink and the projections of each point onto the line spread out
- The first PC does two things:
  1. **Minimizes** the reconstruction error
  2. **Maximizes** the explained variance
- Projections onto the first PC are a single number that retains the most information about the original data



Estimating Principal Components

```r
library(FactoMineR)
library(factoextra)

# isolate item responses and estimate pca
resp <- select(d, starts_with('d_'))
pca <- PCA(resp, ncp=10, graph=FALSE)

# extract eigenvalue information
pca_eig <- data.frame(pca$eig) |>
  rownames_to_column(var='dimension')

# extract dimensions/loadings
pca_dim <- data.frame(pca$var$coord) |>
  rownames_to_column(var='item')

# extract individual scores/projections
pca_resp <- data.frame(pca$ind$coord) |>
  rownames_to_column(var='person')
```

Eigenvalues: How many PCs are useful?

dimension	eigenvalue	percentage.of.variance	cumulative.percentage.of.variance
comp 1	4.2525175	28.350117	28.35012
comp 2	2.7113610	18.075740	46.42586
comp 3	0.9289215	6.192810	52.61867
comp 4	0.8510230	5.673486	58.29215
comp 5	0.7637384	5.091589	63.38374
comp 6	0.7161497	4.774331	68.15807
comp 7	0.6734326	4.489551	72.64762

Option 1: The Kaiser Criterion

- Select all dimensions with an eigenvalue ≥ 1

dimension	eigenvalue	percentage.of.variance	cumulative.percentage.of.variance
comp 1	4.2525175	28.350117	28.35012
comp 2	2.7113610	18.075740	46.42586
comp 3	0.9289215	6.192810	52.61867
comp 4	0.8510230	5.673486	58.29215

Option 2: Cumulative Variance

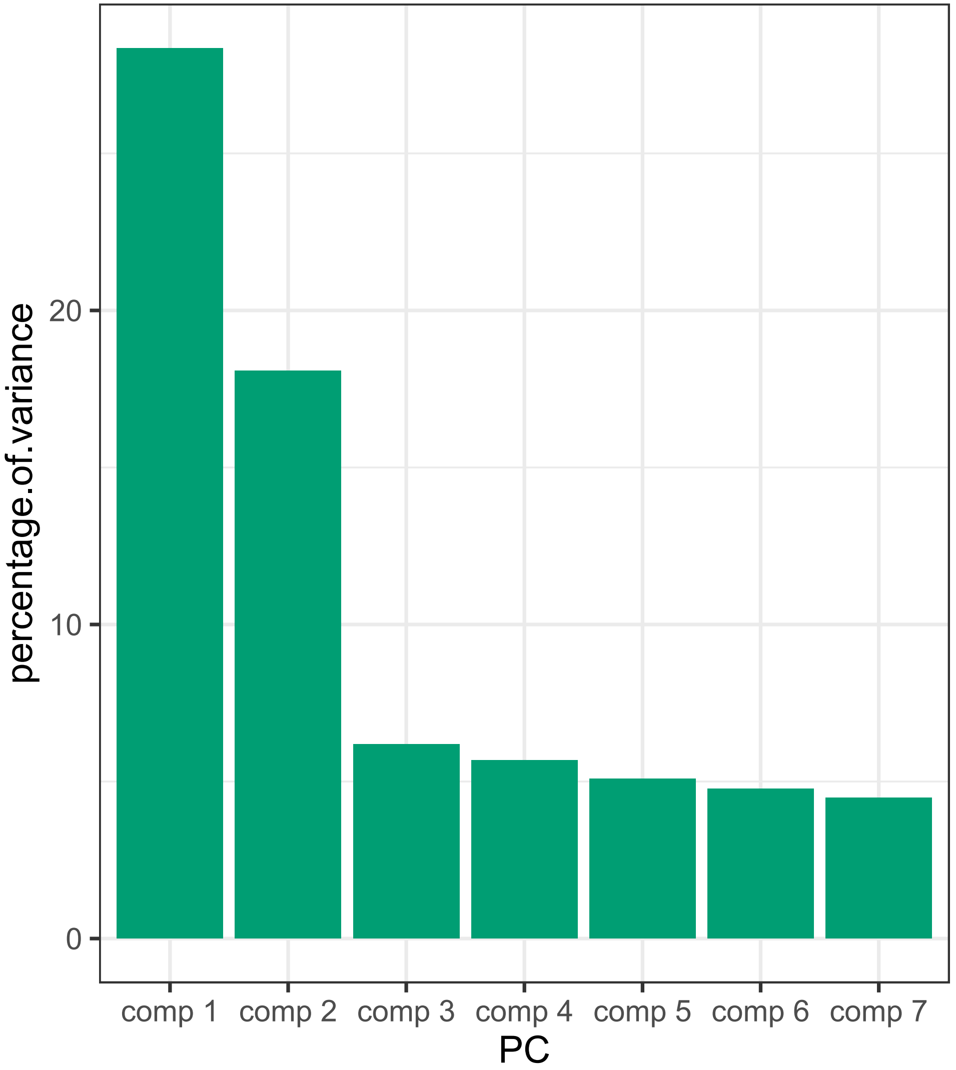
- Pick some cumulative variance that you want (often 50% or 75%)
- Retain dimensions until you exceed that threshold

dimension	eigenvalue	percentage.of.variance	cumulative.percentage.of.variance
comp 1	4.2525175	28.350117	28.35012
comp 2	2.7113610	18.075740	46.42586
comp 3	0.9289215	6.192810	52.61867
comp 4	0.8510230	5.673486	58.29215

Option 3: The Elbow Plot

- Plot the variance explained by each dimension
- Drop dimensions that don't contribute much more over the previous dimension
- This means go up to the "elbow"

```
pca_eig |>
  head(7) |>
  ggplot(aes(
    x = dimension,
    y = percentage.of.variance
  )) +
  geom_col(fill = okabeito_colors(3)) +
  labs(x = 'PC') +
  theme_bw()
```



Dimensions and Loadings

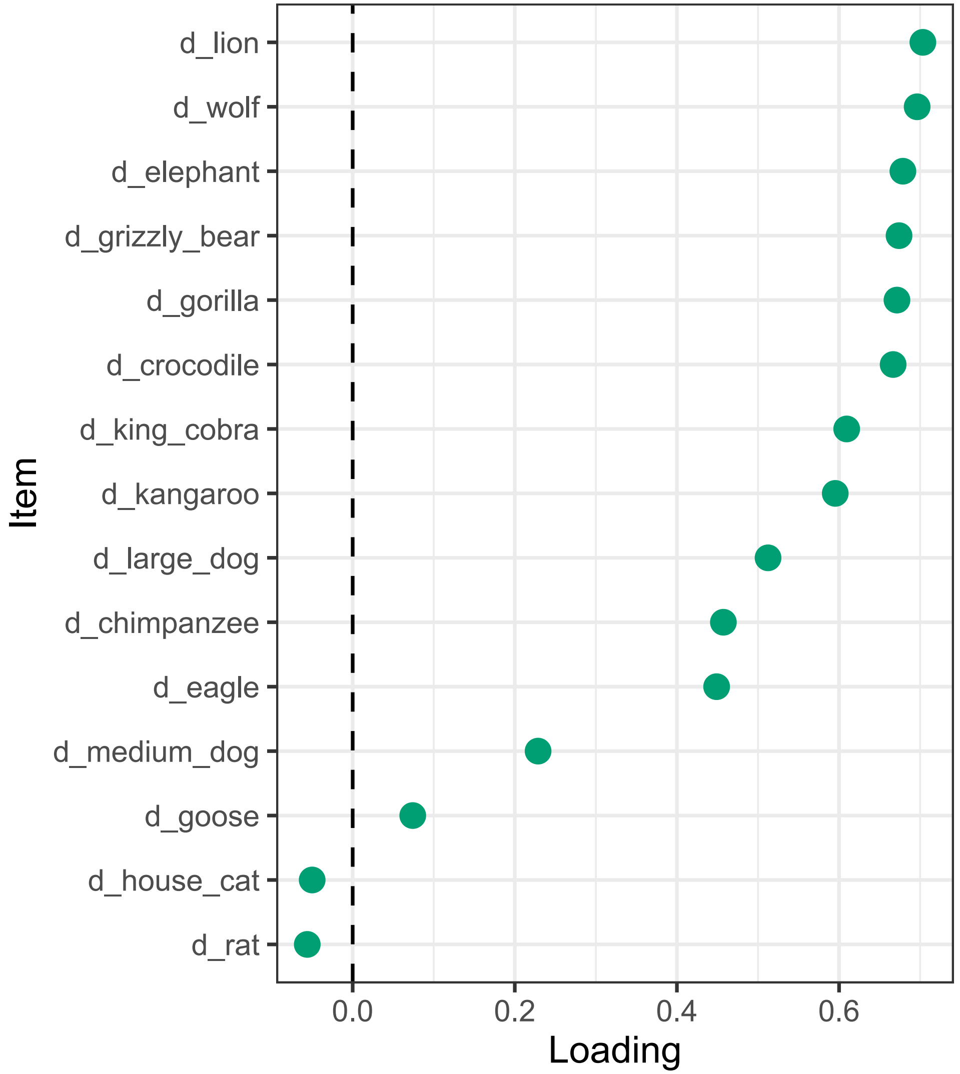
item	Dim.1	Dim.2	Dim.3
d_rat	-0.0560027	0.7478180	0.3220408
d_house_cat	-0.0499744	0.7728305	0.2701365
d_medium_dog	0.2287211	0.6610715	-0.0932625
d_large_dog	0.5124123	0.2998211	-0.3943514
d_kangaroo	0.5953089	0.0527450	-0.1915674
d_eagle	0.4489964	0.4131870	-0.3342900
d_grizzly_bear	0.6740091	-0.2427908	0.3365680

Individuals

person	Dim.1	Dim.2	Dim.3
1	0.277625	1.7352156	-0.7241404
2	-1.362983	0.4078412	1.0116342
3	-1.141138	1.2108529	0.8180881
4	-1.384454	-1.5588112	0.0452277
5	1.669899	2.1264304	-0.3689416

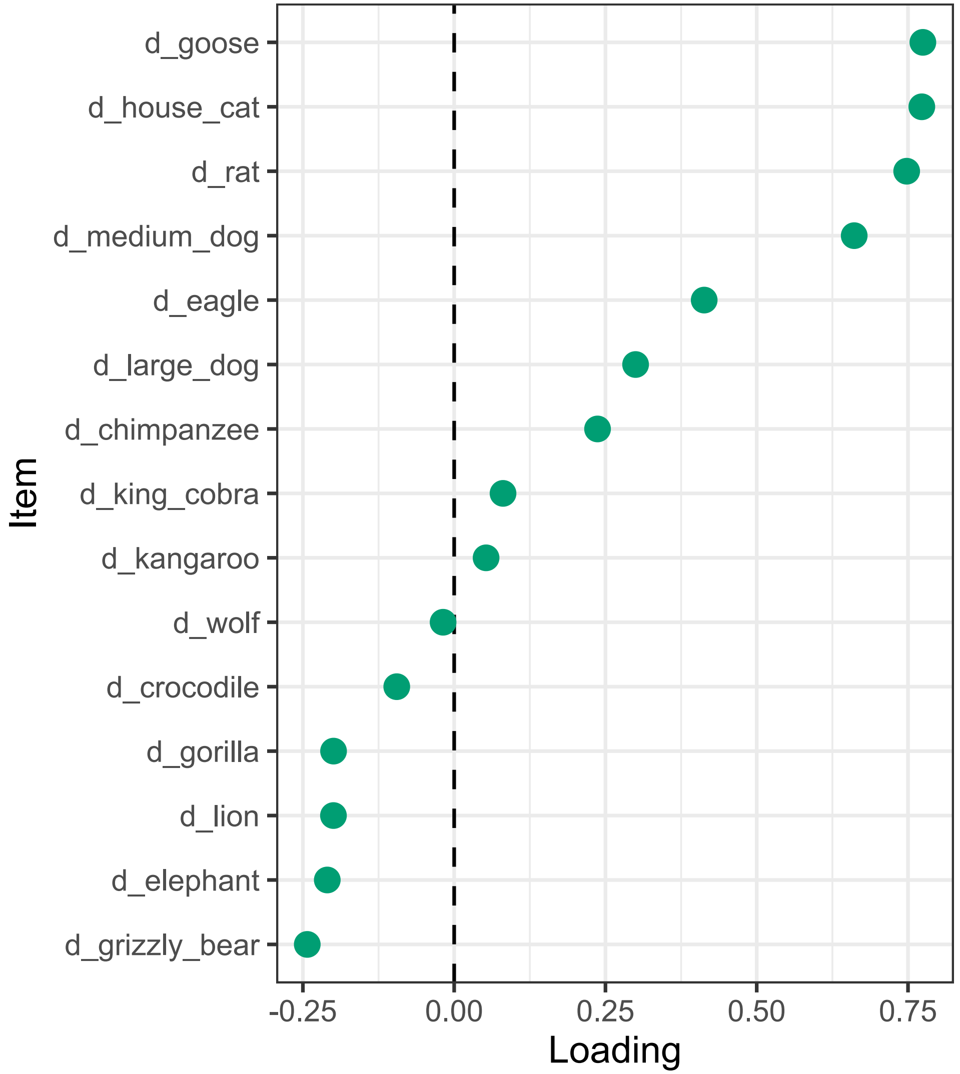
Dimension 1

```r
ggplot(pca_dim,
       aes(x = Dim.1,
           y = reorder(item, Dim.1))) +
  geom_vline(aes(xintercept = 0), lty = 2) +
  geom_point(size = 3,
             color = okabeito_colors(3)) +
  labs(x = 'Loading', y = 'Item') +
  theme_bw()
```



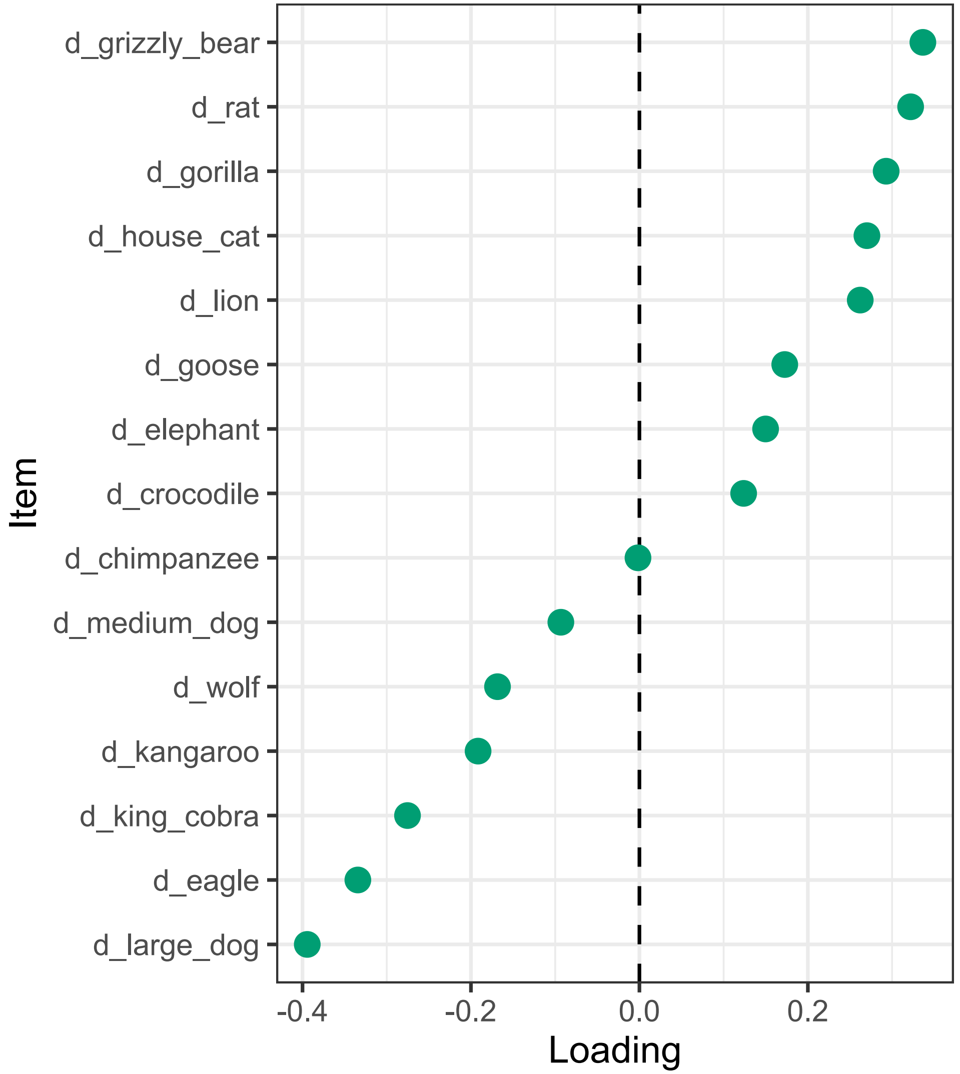
Dimension 2

```r
ggplot(pca_dim,
       aes(x = Dim.2,
           y = reorder(item, Dim.2))) +
  geom_vline(aes(xintercept = 0), lty = 2) +
  geom_point(size = 3,
             color = okabeito_colors(3)) +
  labs(x = 'Loading', y = 'Item') +
  theme_bw()
```



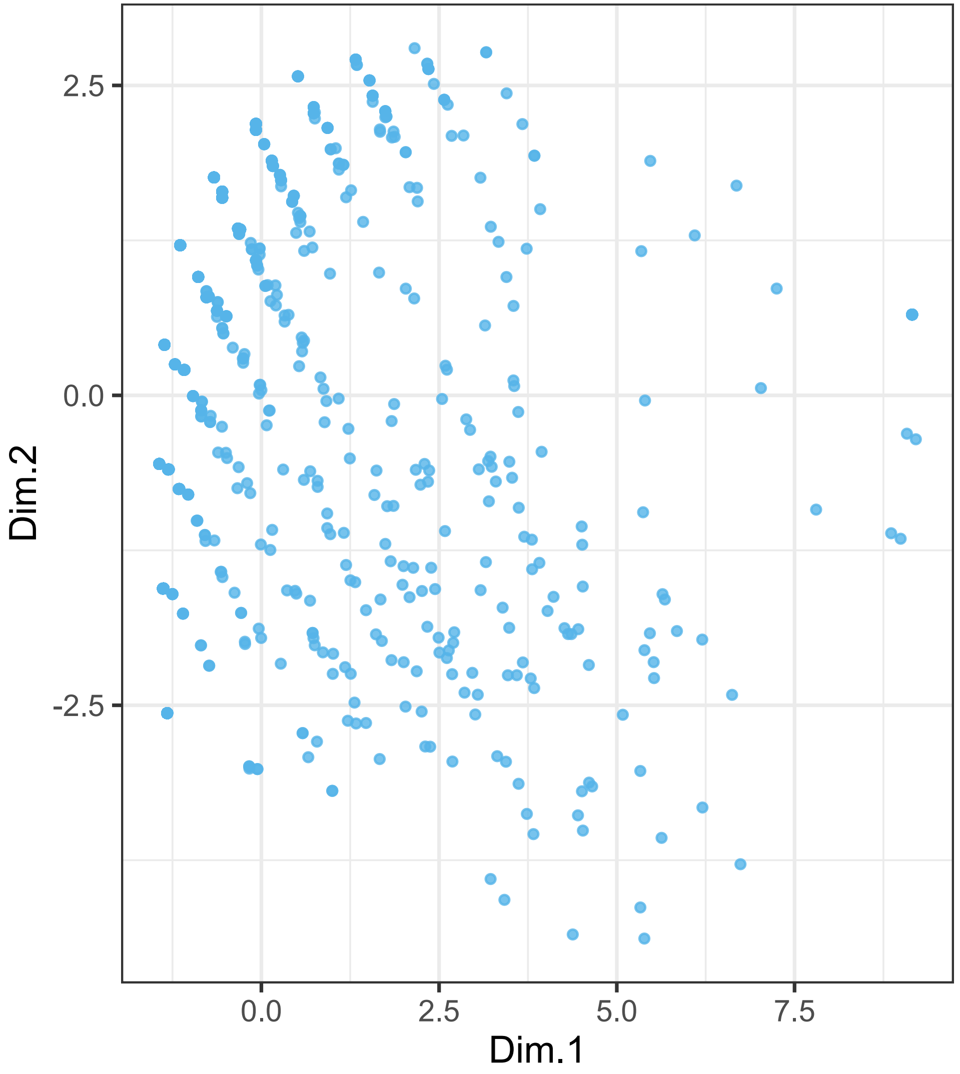
Dimension 3

```r
ggplot(pca_dim,
       aes(x = Dim.3,
           y = reorder(item, Dim.3))) +
  geom_vline(aes(xintercept = 0), lty = 2) +
  geom_point(size = 3,
             color = okabeito_colors(3)) +
  labs(x = 'Loading', y = 'Item') +
  theme_bw()
```



Individuals

```r
ggplot(pca_resp,
         aes(x = Dim.1, y = Dim.2)) +
  geom_point(alpha = 0.8, size = 1,
               color = okabeito_colors(2)) +
  theme_bw()
```



PCA Summary

- Constructs successive **orthogonal** dimensions
- Each dimension maximizes the amount of explained residual variance
- Because dimensions are orthogonal, individual locations along these dimensions are uncorrelated **by construction**
- Variable loadings on dimensions can be used to interpret and describe the dimensions
- Individual projections onto these dimensions can be interpreted as scores along those dimensions

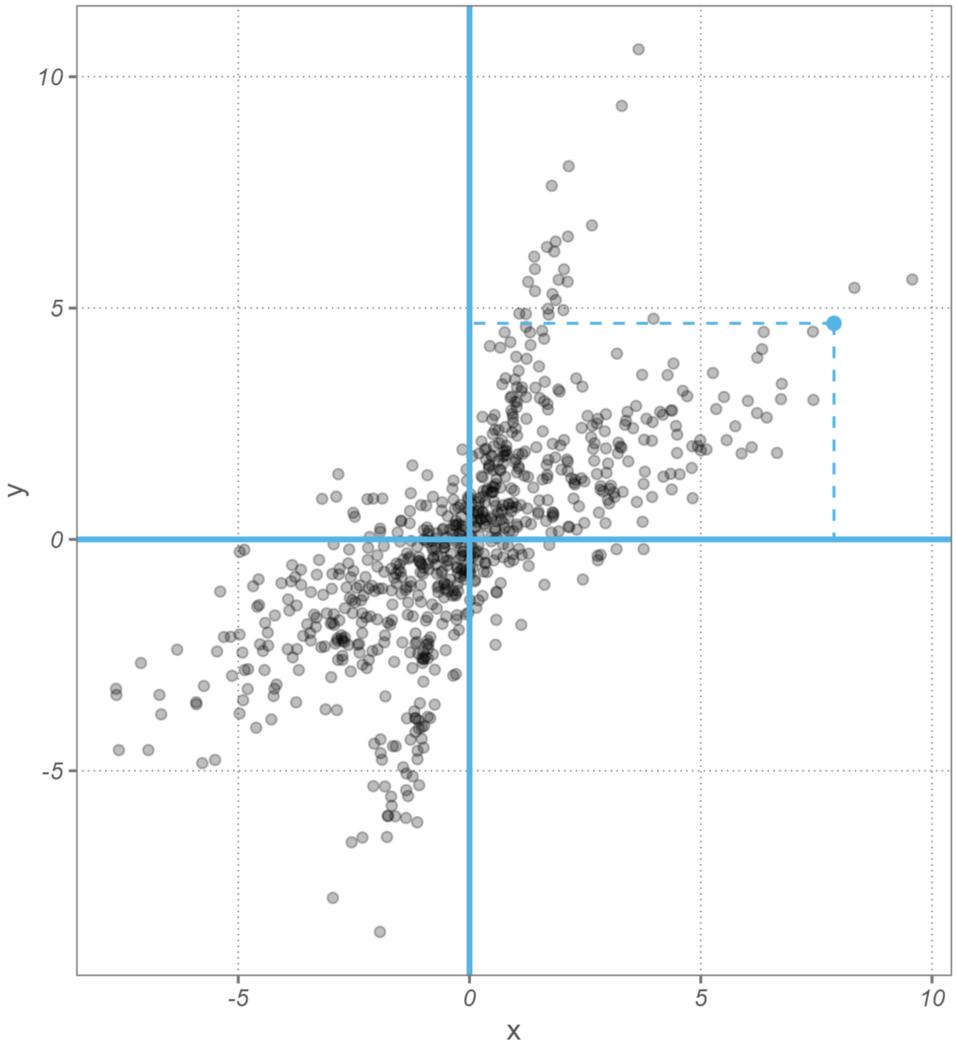
Factor Analysis

(Exploratory) Factor Analysis

- Conceptually different from PCA
- Factor analysis is a latent variable model that seeks to estimate item level associations (loadings) onto latent variables
- The core idea is that levels of the latent variable cause individual item responses, which are observed with error
- "Factor Analysis" is really two things:
  - **Confirmatory Factor Analysis (CFA)** is another name for structural equation modeling (SEM) with item responses
  - **Exploratory Factor Analysis (EFA)** is a dimensionality reduction technique that is conceptually different from PCA but *sometimes* mathematically identical to PCA
- If you allow for correlated factors, EFA is **not** equivalent to PCA

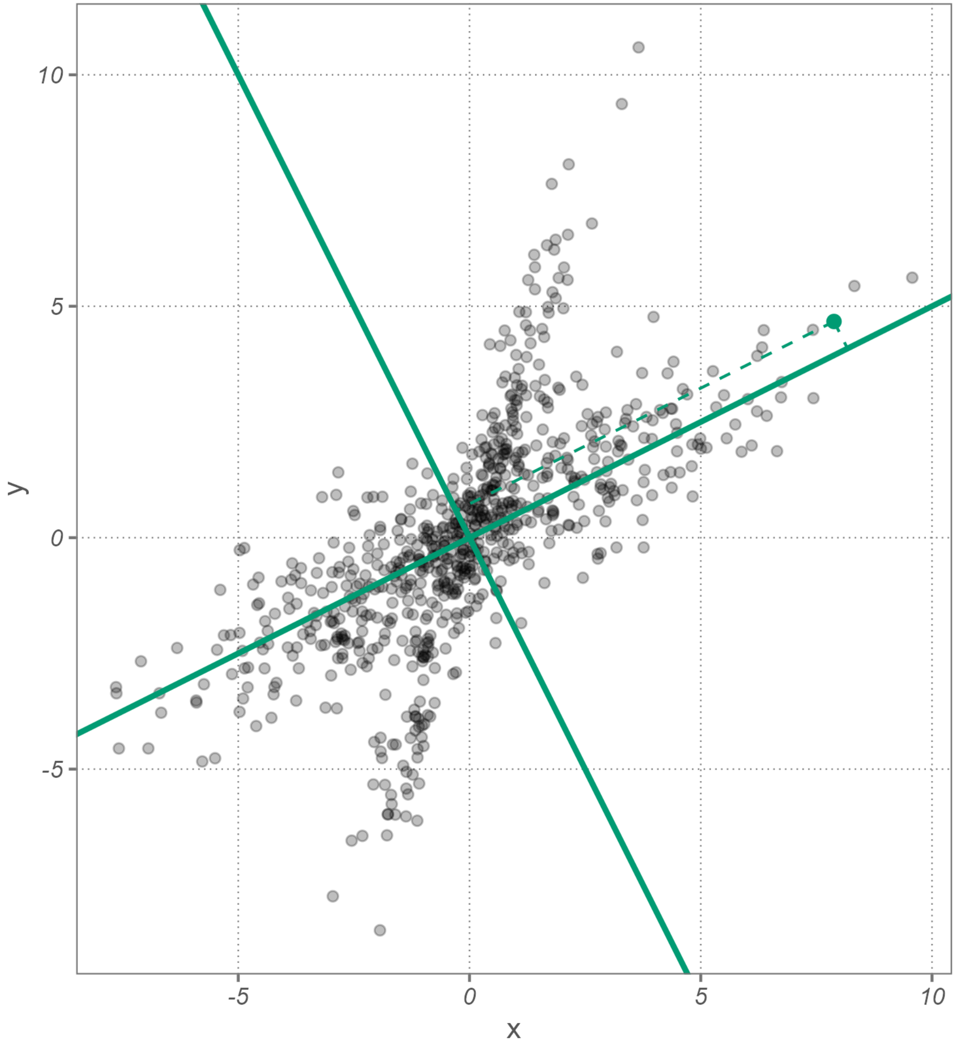
EFA

- Similar to PCA, EFA constructs dimensions that are linear combinations of observed variables and projects points onto them
- EFA models are not identified; there are an infinite number of solutions that describe data equally well (factor indeterminacy problem)



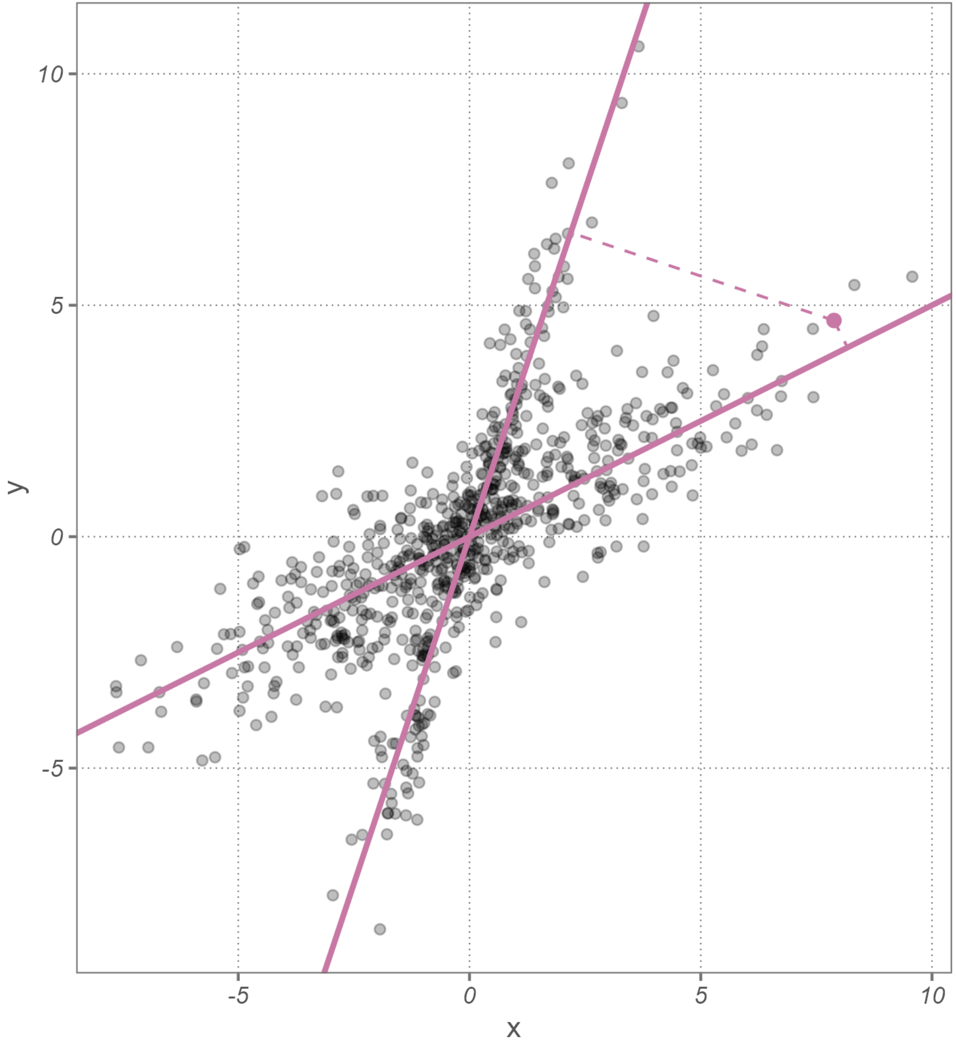
EFA

- Similar to PCA, EFA constructs dimensions that are linear combinations of observed variables and projects points onto them
- EFA models are not identified; there are an infinite number of solutions that describe data equally well (factor indeterminacy problem)
- EFA leverages *rotations* to solve this that try to pick dimensions to satisfy some sort of logical criterion. Varimax rotation is equivalent to PCA



EFA

- Similar to PCA, EFA constructs dimensions that are linear combinations of observed variables and projects points onto them
- EFA models are not identified; there are an infinite number of solutions that describe data equally well (factor indeterminacy problem)
- EFA leverages *rotations* to solve this that try to pick dimensions to satisfy some sort of logical criterion. **Varimax** rotation is equivalent to PCA
- EFA also allows for *oblique* rotations that produce correlated factors. **Promax** is the most commonly used one



Estimating EFA

```r
library(psych)

# Check the documentation - it's intense!
?fa

efa_1 <- fa(resp, nfactors = 1, rotate = 'oblimin')
efa_2 <- fa(resp, nfactors = 2, rotate = 'oblimin')
efa_3 <- fa(resp, nfactors = 3, rotate = 'oblimin')
```

One Factor Solution

```r
efa_1$Vaccounted
```

```
                     MR1
SS loadings    3.6045189
Proportion Var 0.2403013
```

Two Factor Solution

```
efa_2$Vaccounted
```

```
                            MR1       MR2
SS loadings           3.6227900 2.1692315
Proportion Var        0.2415193 0.1446154
Cumulative Var        0.2415193 0.3861348
Proportion Explained  0.6254794 0.3745206
Cumulative Proportion 0.6254794 1.0000000
```

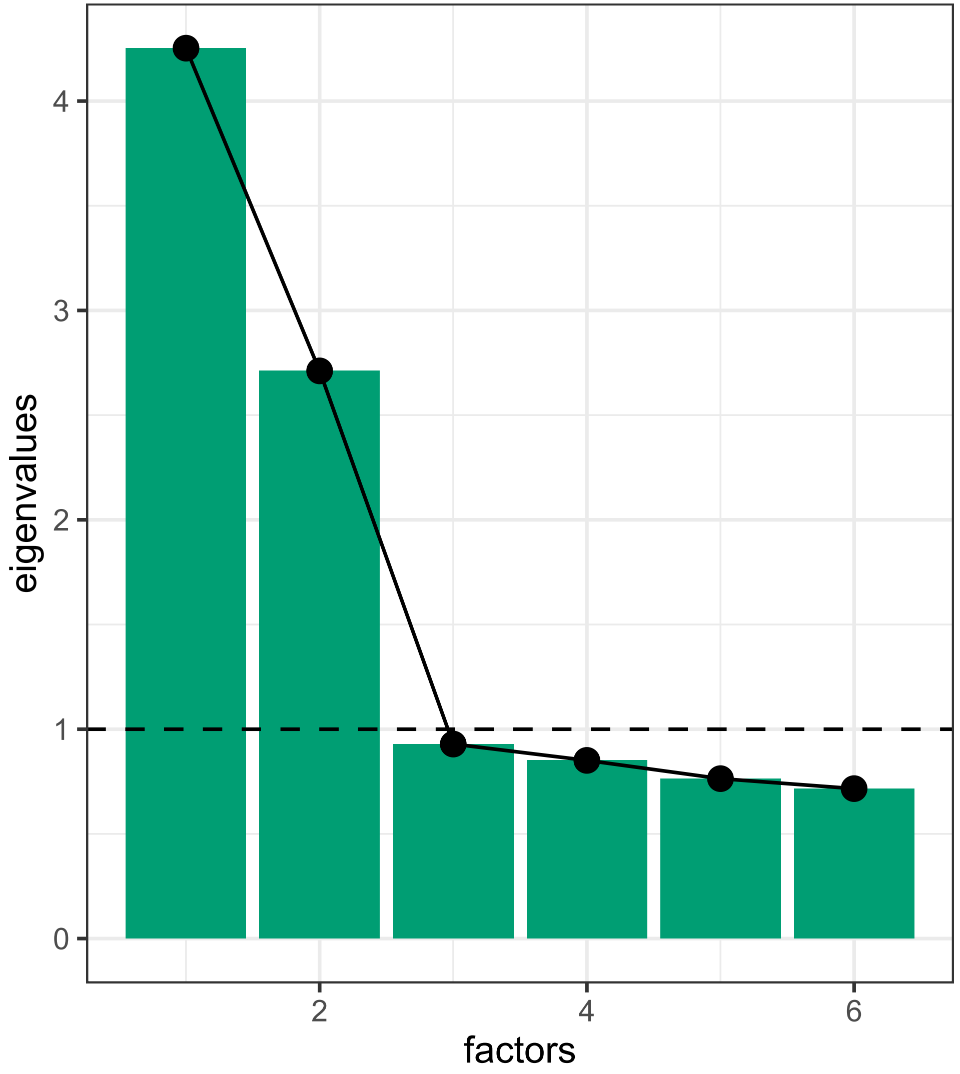
Three Factor Solution

```
efa_3$Vaccounted
```

```
                            MR1       MR2       MR3
SS loadings           2.5972061 1.8689791 1.7516499
Proportion Var        0.1731471 0.1245986 0.1167767
Cumulative Var        0.1731471 0.2977457 0.4145223
Proportion Explained  0.4177026 0.3005836 0.2817138
Cumulative Proportion 0.4177026 0.7182862 1.0000000
```

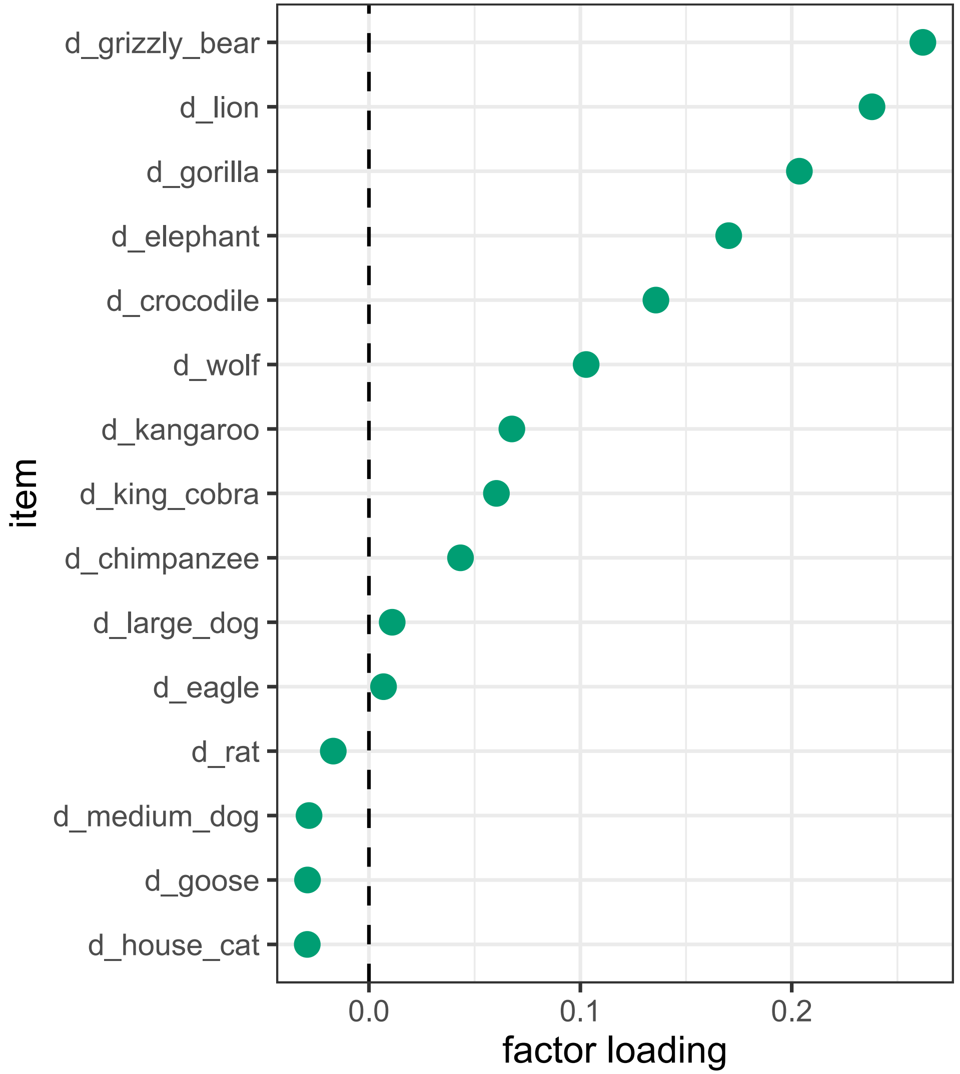
Selecting Number of Factors

```r
fa_eigenvalues <- data.frame(factors = 1:6,
  eigenvalues = efa_3$e.values[1:6])

## Selecting Number of Factors

ggplot(fa_eigenvalues,
       aes(x = factors, y = eigenvalues)) +
  geom_col(fill = okabeito_colors(3)) +
  geom_point(size = 3) +
  geom_line() +
  geom_hline(aes(yintercept = 1), lty = 2) +
  theme_bw()
```



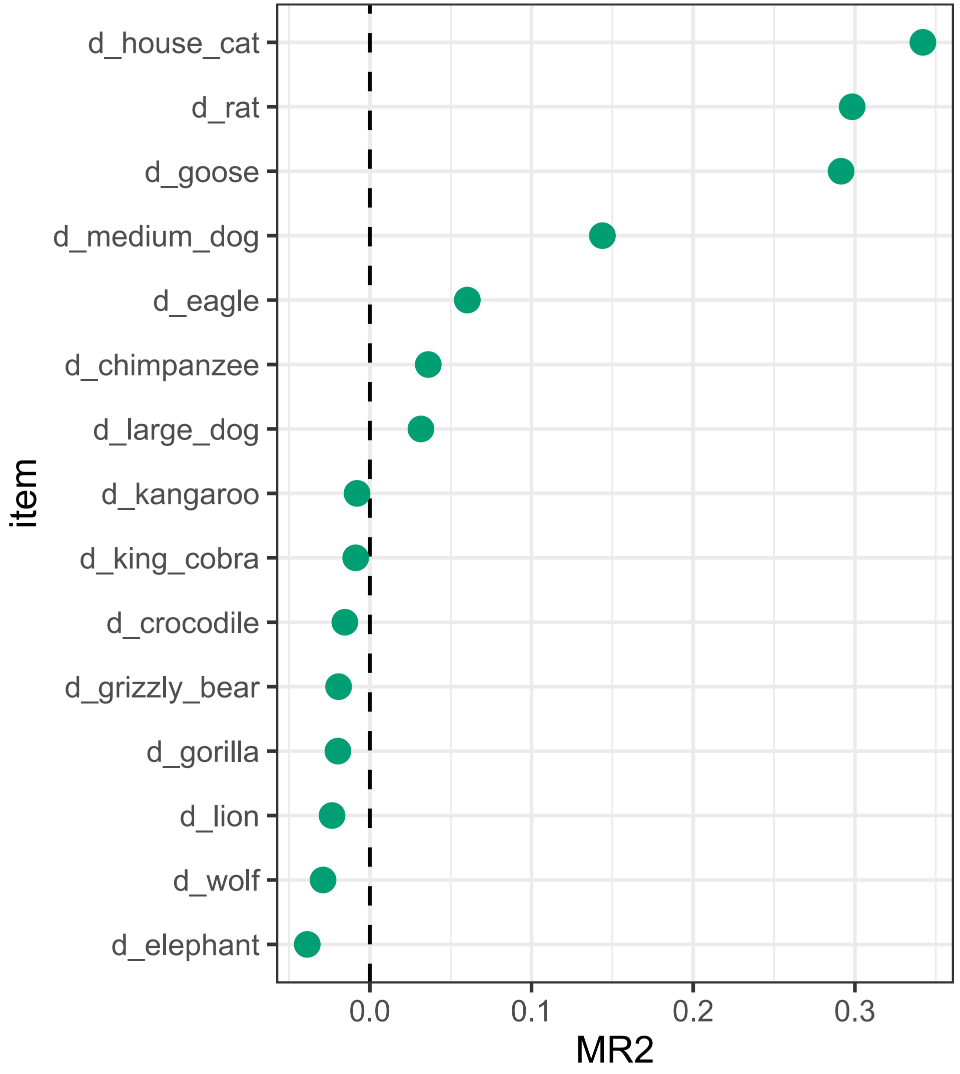
Factor 1 Loadings

```r
ggplot(efa,
       aes(x = MR1,
           y = reorder(item, MR1))) +
  geom_vline(aes(xintercept = 0), lty = 2) +
  geom_point(size = 3,
             color = okabeito_colors(3)) +
  labs(y = 'item', x = 'factor loading') +
  theme_bw()
```



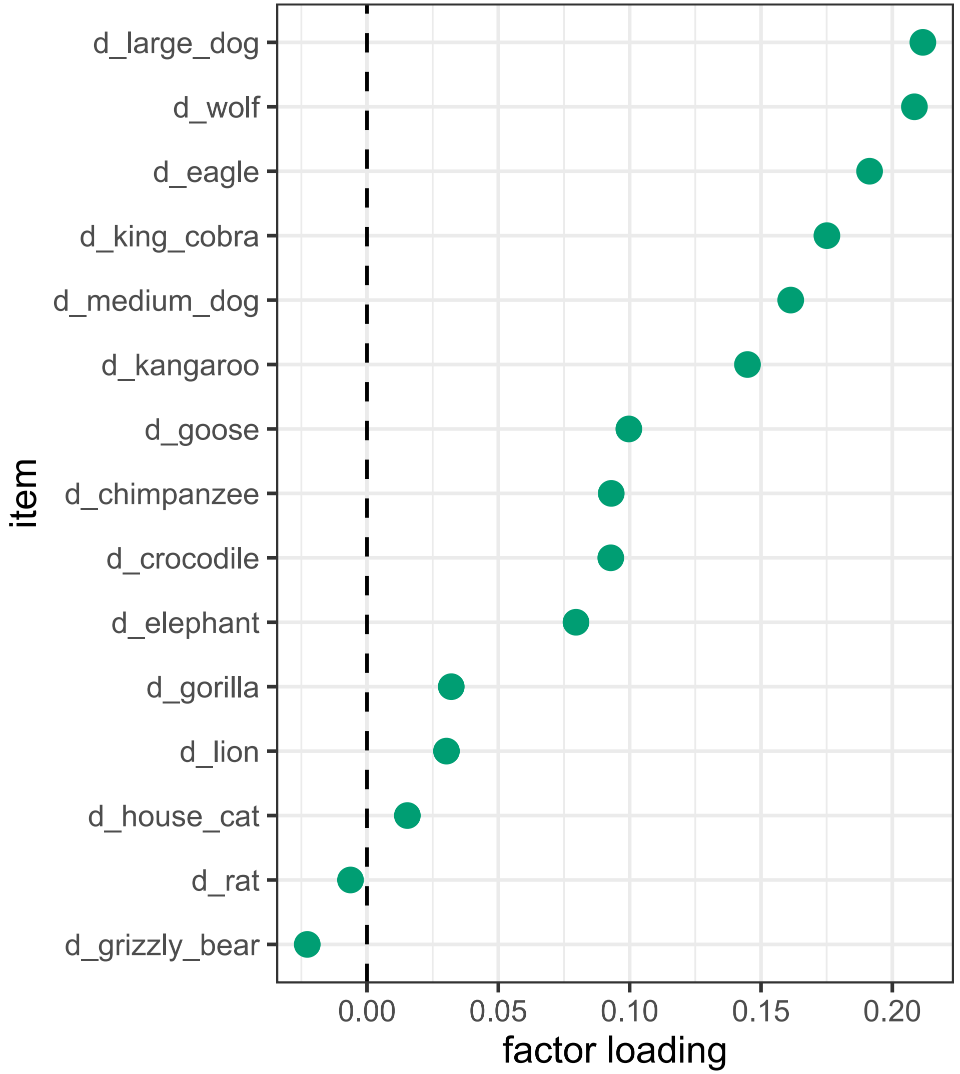
Factor 2 Loadings

```
ggplot(efa,
       aes(x = MR2,
           y = reorder(item, MR2))) +
  geom_vline(aes(xintercept = 0), lty = 2) +
  geom_point(size = 3,
             color = okabeito_colors(3)) +
  labs(y = 'item', x = 'factor loading') +
  theme_bw()
```



Factor 3 Loadings

```r
ggplot(efa,
       aes(x = MR3,
           y = reorder(item, MR3))) +
  geom_vline(aes(xintercept = 0), lty = 2) +
  geom_point(size = 3,
             color = okabeito_colors(3)) +
  labs(y = 'item', x = 'factor loading') +
  theme_bw()
```



Individual Scores

```
efa_3$scores
```

```
                MR1           MR2         MR3
   [1,] -0.285443975  8.387273e-01  0.603740923
   [2,] -0.448881256  4.989128e-01 -0.636824429
   [3,] -0.505534260  7.864999e-01 -0.314300349
   [4,] -0.327039053 -8.288798e-01 -0.873253278
   [5,]  0.248132302  9.532238e-01  1.131932390
```

Factor Analysis Summary

- Constructs latent factors from correlation matrix
- Rotates factors to allow for correlated factors (dimensions)
- Factor loadings can be used to interpreted and describe the factors
- Individual scores on these factors can be recovered
- Generally, the use and interpretation are the same as in PCA

Wrap Up

PCA vs. Factor Analysis

- Variables
  - PCA is a model of observed variables
  - Factor analysis is a model of latent variables
- Data
  - PCA focuses on reconstructing the diagonals of the covariance matrix (individual variances)
  - Factor analysis focuses on reconstructing the off-diagonals of the covariance matrix (between-item covariances)
  - PCA produces orthogonal dimensions
  - Factor analysis can produce correlated factors
- Important to note that PCA and factor analysis usually (but not always) produce different results
- Often factor analysis produces more interpretable factors with an oblique rotation that allows for correlated factors

Problem Set 1

- Posted!
- Two weeks to complete
- You won't be able to finish all of it until after next week's lecture!
- Topics covered:
  - Comparing and contrasting PCA and Factor Analysis (with real data)
  - Comparing and contrasting CTT and Item Response Theory (with real data)
- Data will come from the Item Response Warehouse (IRW)
  - Website: https://itemresponsewarehouse.org/
  - Paper: https://link.springer.com/article/10.3758/s13428-025-02796-y
- Last year, students said this assignment took a long time. I didn't make it shorter, so start soon!

Check-Out

- PollEv.com/klintkanopka