Homework 1

Jarod Klion

September 10, 2021

Problem 1(2.1)

Consider $a * b := ab + a + b, \quad$ where $a, b \in \mathbb{R}\backslash\{-1\}$

(a) To show that $(\mathbb{R}\backslash\{-1\}, *)$ is an Abelian group, we must prove it satisfies five properties:

- **Closure:** Check if $a * b = -1$ ever to ensure it's always in the group
  $a * b = ab + a + b = -1$
  $a(b+1) + b + 1 = 0$
  $(a+1)(b+1) = 0$, so either a or b = -1, which is impossible since they are an element of the reals except -1
- **Associative:**$(a * b) * c = (a * b)c + (a * b) + c$
  $\rightarrow (ab + a + b)c + ab + a + b + c = abc + ac + bc + ab + a + b + c$
  $\rightarrow a(bc + b + c) + bc + a + b + c = (bc + b + c)a + a + (bc + b + c)$
  $\rightarrow (b * c)a + (b * c) + a = a * (b * c)$
- **Neutral (Identity) Element:** We will find the identity element, I, such that $a * I = a$
  $\rightarrow aI + a + I = a = I(a+1) = 0$, so I = 0, since $a \neq -1$
- **Inverse Element:** We will find the inverse, a^{-1}, of an element such that $a * a^{-1} = 0$
  $\rightarrow a * a^{-1} = aa^{-1} + a + a^{-1} = 0$
  $\rightarrow a^{-1}(a+1) + a = 0$
  $\rightarrow a^{-1} = -\frac{a}{a+1}$
- **Commutative:** To be an Abelian group specifically, this property must hold true.
  $a * b = ab + a + b$
  $b * a = ba + b + a$
  $\therefore$ From the associative property, we can conclude that $a * b = b * a$

(b) $3 * x * x = 15$
$3 * (x * x) = 3 * (x^2 + 2x) = 3(x^2 + 2x) + 3 + x^2 + 2x$
$3x^2 + 6x + 3 + x^2 + 2x = 4x^2 + 8x + 3$
$4x^2 + 8x + 3 = 15 \rightarrow x^2 + 2x - 3 =$
$(x+3)(x-1) = 0$
$x = 1, -3$

Problem 2 (2.4)

(a) $\begin{bmatrix}1 & 2\\4 & 5\\7 & 8\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix}$, [3,2]x[3,3] NOT POSSIBLE

(b) $$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix} = \begin{bmatrix}1+3 & 1+2 & 2+3\\4+6 & 4+5 & 5+6\\7+9 & 7+8 & 8+9\end{bmatrix} = \begin{bmatrix}4 & 3 & 5\\10 & 9 & 11\\16 & 15 & 17\end{bmatrix}$$

(c) $$\begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix} = \begin{bmatrix}1+4 & 2+5 & 3+6\\4+7 & 5+8 & 6+9\\1+7 & 2+8 & 3+9\end{bmatrix} = \begin{bmatrix}5 & 7 & 9\\11 & 13 & 15\\8 & 10 & 12\end{bmatrix}$$

(d) $$\begin{bmatrix}1 & 2 & 1 & 2\\4 & 1 & -1 & -4\end{bmatrix}\begin{bmatrix}0 & 3\\1 & -1\\2 & 1\\5 & 2\end{bmatrix} = \begin{bmatrix}0+2+2+10 & 3-2+1+4\\0+1-2-20 & 12-1-1-8\end{bmatrix} = \begin{bmatrix}14 & 6\\-21 & 2\end{bmatrix}$$

(e) $$\begin{bmatrix}0 & 3\\1 & -1\\2 & 1\\5 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & 1 & 2\\4 & 1 & -1 & -4\end{bmatrix} = \begin{bmatrix}0+12 & 0+3 & 0-3 & 0-12\\1-4 & 2-1 & 1+1 & 2+4\\2+4 & 4+1 & 2-1 & 4-4\\5+8 & 10+2 & 5-2 & 10-8\end{bmatrix} = \begin{bmatrix}12 & 3 & -3 & -12\\-3 & 1 & 2 & 6\\6 & 5 & 1 & 0\\13 & 12 & 3 & 2\end{bmatrix}$$

Problem 3 (2.5)

(a) $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 2 & 5 & -7 & -5 \\ 2 & -1 & 1 & 3 \\ 5 & 2 & -4 & 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ 6 \end{bmatrix}$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right] \xrightarrow[R_4-5R_1]{R_2-2R_1, R_3-2R_1} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -3 & 3 & 5 & 2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & -\frac{5}{3} & -1 & -\frac{4}{3} \\ 0 & -3 & 3 & 5 & 2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right] \xrightarrow[R_4+3R_2]{R_1-R_2, R_3+3R_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & \frac{7}{3} \\ 0 & 1 & -\frac{5}{3} & -1 & -\frac{4}{3} \\ 0 & 0 & -2 & 2 & 2 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{2}R_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & \frac{7}{3} \\ 0 & 1 & -\frac{5}{3} & -1 & -\frac{4}{3} \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & -4 & 4 & -3 \end{array}\right] \xrightarrow[R_4+4R_3]{R_1-\frac{2}{3}R_3, R_2+\frac{5}{3}R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{3} & \frac{5}{3} \\ 0 & 1 & 0 & -\frac{8}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The system is inconsistent since $0 \neq 1$

(b) $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -3 & 0 \\ 2 & -1 & 0 & 1 & -1 \\ -1 & 2 & 0 & -2 & -1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 6 \\ 5 \\ -1 \end{bmatrix}$

$$\longrightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right] \xrightarrow[R_4+R_1]{R_2-R_1, R_3-2R_1} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 2 & 0 & -3 & -1 & 3 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -3/2 & -1/2 & 3/2 \\ 0 & 1 & 0 & 1 & -3 & -1 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{array}\right] \xrightarrow[R_4-R_2]{R_1+R_2, R_3-R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -3/2 & 1/2 & 9/2 \\ 0 & 1 & 0 & -3/2 & -1/2 & 3/2 \\ 0 & 0 & 0 & 5/2 & -5/2 & -5/2 \\ 0 & 0 & 0 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

$$\xrightarrow[R_4+\frac{1}{5}R_3]{\frac{2}{5}R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -3/2 & 1/2 & 9/2 \\ 0 & 1 & 0 & -3/2 & -1/2 & 3/2 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2+\frac{3}{2}R_3]{R_1+\frac{3}{2}R_3} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left\{ \mathbf{x} \in \mathbb{R}^5 : \mathbf{x} = \lambda_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

Problem 4 (2.6)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\longrightarrow \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{-R_3} \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow[R_2 - R_3]{R_1 - R_3} \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

$$\left\{ \mathbf{x} \in \mathbb{R}^6 : \mathbf{x} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}$$

Problem 5 (2.7)

$\mathbf{A}\mathbf{x} = 12\mathbf{x}$

$\mathbf{A} = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}$, and $\sum_{i=1}^{3} x_i = 1$

$[\mathbf{A} - 12\mathbf{I}]\mathbf{x} = 0 \Rightarrow \begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} -6 & 4 & 3 \\ 0 & -8 & 12 \\ 0 & 8 & -12 \end{bmatrix}$

$\xrightarrow{R_3+R_2} \begin{bmatrix} -6 & 4 & 3 \\ 0 & -8 & 12 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+\frac{1}{2}R_2} \begin{bmatrix} -6 & 0 & 9 \\ 0 & -8 & 12 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[-\frac{1}{4}R_2]{-\frac{1}{6}R_1} \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}$

From this system of equations, we get that: $x_1 = \frac{3}{2}x_3 = x_2$. However, we need to consider the other constraint that $x_1 + x_2 + x_3 = 1$. Substituting the values from the system into this constraint, we get: $\frac{3}{2}x_3 + \frac{3}{2}x_3 + x_3 = 1$, or $x_3 = \frac{1}{4}$.

$\therefore \mathbf{x} = \begin{pmatrix} \frac{3}{8} \\ \frac{3}{8} \\ \frac{1}{4} \end{pmatrix}$

Problem 6 (2.9)

(a) Yes.$\mathbf{A} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$, which is closed under addition and scalar multiplication with a 0 vector, so it's a subspace.

(b) Yes, for similar reasons as above but with one less column vector in the span.

(c) No. Solutions of the possible inhomogenous systems are not subspaces of $\mathbb{R}^3$ since they can be empty.

(d) No. It is trivial to see that the set is not closed under scalar multiplication that would lead to fractions in ξ_2

Problem 7 (2.10)

(a) $x_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, x_3 = \begin{pmatrix} 3 \\ -3 \\ 8 \end{pmatrix}$

$$\to \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix} \xrightarrow{-R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & -1 & 3 \\ 2 & -1 & 3 \\ 3 & -2 & 8 \end{bmatrix}$$

$$\xrightarrow[R_3-3R_1]{R_2-2R_1} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -3 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_3-R_2]{R_1+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\xrightarrow[R_2+3R_3]{\frac{1}{2}R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There is a pivot in every column, so the vectors are linearly independent.

(b) $x_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

$$\to \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[R_3-R_1]{R_2-2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_4-R_2, R_5-R_2]{R_1-R_2, R_3+R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[R_4+R_3, R_5+R_3]{R_1+R_3, R_2-2R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There is a pivot in every column, so the vectors are linearly independent.