Homework 3

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1. Consider the system of equations $Ax = b$, where A is a nonsingular lower triangular matrix, i.e

$$A = D - L,$$

where D is diagonal and nonsingular, and L is strictly lower triangular matrix.

(a) Show that (forward) Gauss-Seidel will converge to $x = A^{-1}b$ in a finite number of steps (in exact arithmetic) for any initial guess x_0 and give a tight upper bound on the number of steps required.

- In Gauss-Seidel, $x^{k+1} = (D - E)^{-1}Fx^k + (D - E)^{-1}b$. In matrix form, $A = D - (E + F)$ and since A is a nonsingular lower triangular matrix, F is the zero matrix. In this problem, $E = L$, leaving us immediately with $x = (D - L)^{-1}b = A^{-1}b$ in one step for any x_0.

(b) Also, could you show the same results for the (backward) Gauss-Seidel?

- Now, Backward Gauss-Seidel takes the form $x^{k+1} = (D-F)^{-1}Ex^k + (D-F)^{-1}b$. As before, $E = L$ and $F = 0$, leaving us with $x^{k+1} = D^{-1}Lx^k + D^{-1}b$. There does not appear to be any further substitutions that would lead to meaningful simplifications, leaving us with a form much different than for the forward Gauss-Seidel.

2. When attempting to solve $Ax = b$ where A is known to be nonsingular via an iterative method, we have seen various theorems that give sufficient conditions on A to guarantee the convergence of various iterative methods. It is not always easy to verify these conditions for a given matrix A. Let P and Q be two permutation matrices. Rather than solving $Ax = b$, we could solve,

$$(PAQ)(Q^T x) = Pb$$

using an iterative method. Sometimes it is possible to examine A and choose P and(or) Q so that it is easy to apply one of our sufficient condition theorems. The Gauss-Seidel and Jacobi iterative methods did not converge for linear systems with the matrix

$$A = \begin{bmatrix} 3 & 7 & -1 \\ 7 & 4 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Why? Can you choose P and Q that the permuted system converges for one or both of Gauss-Seidel and Jacobi?

- In the non-permuted state, neither Gauss-Seidel or Jacobi converge because the spectral radius of their respective iteration matrix is greater than 1. There are several P and Q that can be chosen to permute the system such that it converges for one or both of Gauss-Seidel and Jacobi. We will apply the theorem that states that both these iterative methods converge iff A is a strictly diagonally dominant matrix by rows. For example, if we take

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

we end up with this permutated system and associated spectral radii for Jacobi and Gauss-Seidel, respectively:

$$PAQ = \begin{bmatrix} 7 & 1 & 4 \\ -1 & 2 & 1 \\ 3 & -1 & 7 \end{bmatrix},$$

$$\rho(B_J) \approx 0.496,$$

$$\rho(B_{GS}) \approx 0.175$$

Since both $\rho(B_J)$ and $\rho(B_{GS})$ are less than 1, Jacobi and Gauss-Seidel are guaranteed to converge.

3. When solving $Ax = b$ or equivalently the associated quadratic definite minimization problem using Conjugate Gradient, we have

$$x_{k+1} = x_0 + \alpha_0 p_0 + \cdots + \alpha_k p_k,$$

where the p_j are $A-$orthogonal vectors. It can be shown that

$$\text{span}(p_0, ..., p_k) = \text{span}(r_0, Ar_0, ..., A^k r_0)$$

where $r_0 = b - Ax_0$ and x_0 is the initial guess for the solution $x^* = A^{-1}b$. Therefore,

$$x_{k+1} = x_0 + \gamma_0 r_0 + \gamma_1 Ar_0 + \cdots + \gamma_k A^k r_0 = x_0 - P_k(A) r_0$$

where $P_k(A) = \gamma_0 I + \gamma_1 A + \cdots + \gamma_k A^k$ is a matrix that is called a matrix polynomial evaluated at A. (A space whose span can be defined by a matrix polynomial is called a Krylov space). denote $d_j = A^j r_0$ for $j = 0, 1, ...,$ and determine the relationship between the coefficients $\alpha_0, \ldots, \alpha_k$, and the coefficients $\gamma_0, \ldots, \gamma_k$.

- To determine the relationship between the coefficients, we will compare the equations on each side:

  $$x_0 + \alpha_0 p_0 + \cdots + \alpha_k p_k = x_0 + \gamma_0 r_0 + \gamma_1 Ar_0 + \cdot + \gamma_k A^k r_0.$$

  Letting $d_j = A^j r_0$ and simplifying, we get a relation between the coefficients:

  $$\alpha_0 p_0 + \cdots + \alpha_k p_k = \gamma_0 d_0 + \gamma_1 d_1 + \cdots + \gamma_k d_k$$

  $$\sum_{j=0}^{k} \alpha_j p_j = \sum_{j=0}^{k} \gamma_j d_j$$

  $$\alpha_n = \gamma_n \frac{d_n}{p_n}$$

4. Determine the necessary and sufficient conditions for $x = A^{-1}b$ to be a fixed point of

$$x_{k+1} = Gx_k + f$$

- Letting $\phi(x) = Gx + f$, we can determine the conditions required for a fixed point to exist. $\phi(x)$ must be a continuous function on a given interval $[a, b]$ and $\phi(x) \in [a, b]$ $\forall x \in [a, b]$ by the theorem presented in class.