ISC 5228
Markov Chain Monte Carlo

Bivariate Distributions: Skill and Luck

1 Motivation

Which game requires more skill: poker or chess?

Why did VHS, and not Betamax, succeed so wildly?

Person X bought stock Y a few years years ago is now a multimillionaire; how much of that was skill?

I find the tangled relationship between skill, luck, and success (or selection) endlessly fascinating. If you are interested, the references at the end, provide additional social context, and a starting point for deeper exploration.

In this lab, we shall explore some of these ideas using an instructive (but oversimplified) model.

2 Introduction

For the purposes of this lab, we shall assume that **skill** encompasses attributes that can be trained and mastered by diligent practice. **Luck**, on the other hand, is random, arbitrary, and external. It encompasses factors completely outside an individual's control. We shall assume that luck and skill are independent.

As an example of these definitions from one of the sources listed at the end, consider the set of Hollywood aspirants (success = stardom). Here, perhaps, acting ability = skill, while beauty = luck (McArdle, 2012).[1]

The goal of the lab is to explore two important topics +(i) correlations induced by selection, and (ii) the "paradox of skill".

The paradox of skill is a weird phenomenon. As a discipline professionalizes (think sports or investing) skill levels improve. The difference between the top and the worst "players" decreases. As the spread of skill becomes tighter, luck becomes ever more important.

3 Static Model and Selection-Induced Correlations

Let us model the distribution of skill (x_1) and luck (x_2) in a population by a bivariate normal distribution with mean zero, variance $\sigma^2_{x_1} = \sigma^2_{x_2} = 1$, and correlation coefficient $\rho = 0$.

$$\mathcal{N}(x_1, x_2) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x_1^2 + x_2^2)\right) = \mathcal{N}(x_1)\mathcal{N}(x_2). \tag{1}$$

Thus, by construction, x_1 and x_2 are independent variables.

[1] Luck need not be a physical attribute. In sports, for example, skill = fitness/acumen, and luck = odd bounce or a bad referee call.

E1: Generate N = 5000 points (x_1, x_2) from this bivariate distribution (eqn 1) and visualize them using a scatter plot. Plot histograms to show the marginal distributions of x_1 and x_2.

This is the pool of aspirants. Let us prescribe a simple model for success: the algebraic sum of luck and skill exceed a threshold. For concreteness, let us stipulate that for success,

$$x_1 + x_2 \geq 1. \tag{2}$$

E2: For the samples generated in E1, show/highlight only the points $x_1 + x_2 \geq 1$. These points are drawn from the conditional PDF $\pi(x_1, x_2 | x_1 + x_2 \geq 1)$. This distribution is no longer bi-variate normal. **Note**: You are not expected to find this conditional distribution analytically.

Compute the Pearson correlation coefficient for x_1 and x_2 (you may use a built-in function).

Estimate the variance in skill for the original and successful populations. Show that the selection process narrows the spread in skill.

Observe the negative correlation between skill and luck among the successful.

This has interesting real-world implications in a variety of settings where the combination of two independent attributes determines success. For restaurants, these two attributes may be quality of food and ambience. If you only care about great-tasting food, perhaps you should choose the crowded place that looks decrepit instead of the place next door that has great ambience.

E3: Repeat E2 for a more stringent selection criteria:

$$x_1 + x_2 \geq 2. \tag{3}$$

Comment on the effect of the stringency of the criteria on the correlation and variance.

4 Repeated Game Model and the Paradox of Skill

The calculations above were static. Let us consider a dynamic game, where the criterion for success is applied repeatedly and is made stricter over time.

The game initially starts with a large number (say $N = 100,000$) of players. Suppose that the innate skill level (x_1) of these players is fixed and normally distributed (mean zero and unit standard deviation).

Suppose these players play a game over and over again. In each game, normally distributed luck (x_2) is independently assigned to the players. That is, a fresh $x_2 \sim \mathcal{N}(\mu = 0, \sigma = 1)$ is drawn for each player during each game. The threshold for success increases gradually for later games. In game number i, the threshold of success is,

$$x_1 + x_2 \geq 1 + \mu_i, \tag{4}$$

where μ_i is the mean skill level $\langle x_1 \rangle$ of participants in the i^{th} game. Thus, in the first game ($i = 1$), the mean skill level $\mu_1 = \langle x_1 \rangle \approx 0$.

Players who make the cut (eqn 4) in game i survive, and move on to the next game $(i+1)$. Note that the skill level is held fixed throughout; x_1 is sampled only once at the beginning.

The game ends when only 0.1% or less (100 individuals here) of the original players are left.

E4: Write a program to model this game.

As a function of the game number, plot the number of surviving players.

Also plot the mean and variance of the level of skill among the survivors as a function of the game number.

4.1 Math Behind the Paradox of Skill

For independent variables, it can be shown that the variance of the result $x_1 + x_2$ is the sum of the variance of x_1 and x_2.

$$\begin{aligned}\text{var}(x_1 + x_2) &= \text{var}(x_1) + \text{var}(x_2)\\ \text{var(outcome)} &= \text{var(skill)} + \text{var(luck)}\end{aligned}$$

As Mauboussin and Callahan (2013) put it,

> If the variance in the skill distribution is shrinking and the variance in luck is stable, luck plays a growing role in shaping results. An absolute improvement in skill, when combined with a relative decline in the range of skill, means that luck is more important than ever. This concept is called the "paradox of skill."

5 References

(i) Megan McArdle, "When Correlation Is Not Causation, But Something Much More Screwy", The Atlantic, **2012** (link)

(ii) Michael J. Mauboussin and Dan Callahan, "Alpha and the Paradox of Skill", Credit Suisse , **2013** (link)

(iii) Robert H. Frank, "Why Luck Matters More Than You Might Think", **2016** (link)