Fall Semester	Foundations of Computational Mathematics I	2022

HW 1

Due Date Sep 23rd, 2022
Points 100 pts

1. Derive the relative and absolute condition number of a function at the point x. Based on our notation from the class, note that the change of data δd is equivalent to the change in the function value, i.e $f(x)$. In particular, let's assume we perturb x by $h > 0$, where $\delta d = f(x+h) - f(x)$.
   (a) Show that $K_{rel}(x) = \left|\dfrac{xf'(x)}{f(x)}\right|$
   (b) Find $K_{abs}(x)$
   (c) In particular, let's consider the function
   $$f(x) = \frac{\beta + x}{\beta - x}$$
   Find the relative and absolute condition number for evaluating $f(x)$ by using (a)-(b). Then, compute the condition numbers around $x = 1$ and $x = 100$ when $\beta = 1.01$. Discuss your results.

2. Consider the roots of the following quadratic equation with a single parameter $\beta > 1$
   $$p(x) = x^2 + 2\beta x + 1 = 0$$
   We can define the vector-valued function that maps β to the two roots $x_+(\beta)$ and $x_-(\beta)$
   $$G : \mathbb{R} \to \mathbb{R}^2, \beta \mapsto \begin{bmatrix} x_+(\beta) \\ x_-(\beta) \end{bmatrix} = \begin{bmatrix} -\beta + \sqrt{\beta^2 - 1} \\ -\beta - \sqrt{\beta^2 - 1} \end{bmatrix}$$
   (a) What happens to the roots as $\beta \to \infty$, please describe with full details.
   (b) For $\beta > 1$, consider the derivatives with respect to β of each of the roots and derive and approximate the relative condition number for the vector of roots $G(\beta)$ when β is perturbed slightly. (Note that since G is a vector-valued function, you can use the vector standard Euclidean 2-norm such as
   $$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to \|v\|_2 = \sqrt{v_1^2 + v_2^2}$$
   to measure the size of the solution and error vectors in $\mathbb{R}^2$. The absolute value can be used as the norm of the scalars β and $\delta\beta$ in $\mathbb{R}$.
   (c) Analyze and discuss. What is the relative condition number when β is away from 1? What is the relative condition number when $\beta \to \infty$ and how does this related to (a)? What is the relative condition number if β approaches one, and why do we observe this?

3. Consider the floating point system with $\beta = 10$ and $t = 3$. The associated floating point arithmetic is

$$x \boxed{*} y = fl(x \, * \, y)$$

Let x and y be two floating point numbers with $x < y$ and consider computing their average $\alpha = (x+y)/2$ by utilizing following three algorithms:

- $\alpha_1 = ((x+y)/2.0)$
- $\alpha_2 = ((x/2.0) + (y/2.0))$
- $\alpha_3 = (x + ((y-x)/2.0))$

Note that the parentheses indicate the order of the floating point operations. Here, let $x = 5.01$ and $y = 5.02$. First evaluate α_1, α_2, and α_3 in the specified floating point system. Then, explain and discuss the results.

4. (Preliminaries) First, prove that $\|\cdot\|_2$ is a vector norm, then show $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$ for any vectors x and y