Markov Chain Monte Carlo

Homework

Error Propagation and Importance Sampling

(i) Error Propagation

Suppose you conduct two independent experiments to measure a quantity z, and obtain the estimates, $z_1 = 6 \pm 2$ and $z_2 = 4 \pm 1$. The two measurements do not arise from one experiment repeated twice, but rather from two completely different experimental setups.

Your goal is to combine these two estimates to come with an improved consensus estimate.

(a) Suppose your first attempt is the simple mean $\bar{z} = (z_1 + z_2)/2$. Find $\bar{z}$ and the corresponding error $\sigma_{\bar{z}}$ using the standard propagation of error formula.

(b) You think about the problem, and notice that z_2 has a smaller error bar than z_1. You wonder if you should consider a weighted mean,

$$\bar{z}_w = w z_1 + (1 - w) z_2,$$

where $0 \leq w \leq 1$ is a weight. Find the corresponding error $\sigma_{\bar{z}_w}$ using the standard propagation of error formula, as a function of w.

(c) Find the value of w that minimizes the error $\sigma_{\bar{z}_w}$. For this example, show that the weights $w_1 = w$ and $w_2 = (1 - w)$ associated with z_1 and z_2 are inversely proportional to $\sigma_{z_1}^2$ and $\sigma_{z_2}^2$, respectively.

(d) What is the corresponding $\bar{z}_w$ and $\sigma_{\bar{z}_w}$?

(ii) Importance Sampling: Midterm 2015

Consider the integral,

$$I = \int_0^\infty x^{1/2} e^{-x}\, dx.$$

(a) Develop an expression and program to sample from the exponential distribution, $\pi(x) = e^{-x}, x \geq 0$ (10 points).

(b) Use importance sampling with 10^3 points to sample from $\pi(x)$, and estimate the integral.

- Describe the formulae you use, and your strategy in sufficient detail. (10 points)
- Report the estimated error, based on a single simulation run. (15 points)

(iii) Transformation Method and Importance Sampling

Consider evaluating the integral,

$$I = \int_0^\pi f(x)\, dx = \int_0^\pi \frac{dx}{x^2 + \cos^2 x},$$

by drawing points from the distribution $\pi(x) \sim e^{-ax}$, with $a > 0$.

(a) Normalize $\pi(x)$ so that $\int_0^\pi \pi(x)\, dx = 1$.

(b) Use the transformation method to sample from $\pi(x)$, given uniformly distributed random numbers $u \sim U[0,1]$. Test your formula for $a = 1$ by plotting a normalized histogram and $\pi(x, a = 1)$.

(c) We can rewrite the integral (for importance sampling) as,

$$I = \int_0^\pi \frac{f(x)}{\pi(x)} \pi(x)\, dx.$$

The resulting variance of the integral $\sigma_I^2 = \sigma_{f/\pi}^2/n$, where n is the number of Monte Carlo points. With $n = 10^5$, vary a between 0.05 and 2.0 and make a plot of $\sigma_I(a)$. From the plot, estimate the value of a (to within ± 0.1) which minimizes $\sigma_I(a)$.