HW 3

3.1.) $\langle \vec{x},\vec{y}\rangle := x_1y_1 - (x_1y_2 + x_2y_1) + 2(x_2y_2)$ is inner product?

Must be positive definite, symmetric bilinear mapping

1.) <u>Bilinear</u>: $\langle \lambda\vec{x}+\gamma\vec{y},\vec{z}\rangle = (\lambda x+\gamma y)_1 z_1 - [(\lambda x+\gamma y)_1 z_2 + (\lambda x+\gamma y)_2 z_1] + 2((\lambda x+\gamma y)_2 z_2)$

$= (\lambda x_1 z_1 - (\lambda x_1 z_2 + \lambda x_2 z_1) + 2\lambda x_2 z_2) + (\gamma y_1 z_1 - (\gamma y_1 z_2 + \gamma y_2 z_1) + 2\gamma y_2 z_2)$

$= \lambda\langle \vec{x},\vec{z}\rangle + \gamma\langle \vec{y},\vec{z}\rangle$

$\langle \vec{x}, \lambda\vec{y}+\gamma\vec{z}\rangle = x_1(\lambda y_1+\gamma z_1) - [x_1(\lambda y_2+\gamma z_2) + x_2(\lambda y_1+\gamma z_1)] + 2x_2(\lambda y_2+\gamma z_2)$

$= \lambda x_1 y_1 - \lambda(x_1y_2 + x_2y_1) + 2\lambda x_2 y_2 + \gamma x_1 z_1 - \gamma(x_1z_2 + x_2z_1) + 2\gamma x_2 z_2$

$= \lambda\langle x,y\rangle + \gamma(x,z)$, so bilinear ✓

2.) <u>Symmetric</u>: $\langle y,x\rangle = y_1x_1 - (y_1x_2 + y_2x_1) + 2(y_2x_2) = \langle x,y\rangle$, so symmetric ✓

3.) <u>Pos. Def.</u>: $\langle x,x\rangle = x_1^2 - (2x_1x_2) + 2x_2^2 = (x_1-x_2)^2 + x_2^2$ is sum of squares of real numbers, so $\langle x,x\rangle > 0$

$\langle 0,0\rangle = (0-0)^2 + 0 = 0$, so positive definite ✓

$\therefore \langle\cdot,\cdot\rangle$ <u>is</u> an inner product

3.2.) $\langle \vec{x},\vec{y}\rangle = x^T\begin{bmatrix}2&0\\1&2\end{bmatrix}y$ is <u>NOT</u> an inner product since A isn't symmetric

Addt. $\langle \vec{x},\vec{x}\rangle = [x_1\ x_2]\begin{bmatrix}2&0\\1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = [x_1\ x_2]\begin{bmatrix}2x_1\\x_1+2x_2\end{bmatrix} = 2x_1^2 + x_1x_2 + 2x_2^2$ is not always > 0

3.4.) $\vec{x} = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{y} = \begin{bmatrix}-1\\-1\end{bmatrix}$

a.) $\langle \vec{x},\vec{y}\rangle := x^Ty$

$\|x\| = \sqrt{[1\ 2]\begin{bmatrix}1\\2\end{bmatrix}} = \sqrt{5}$, $\|y\| = \sqrt{[-1\ -1]\begin{bmatrix}-1\\-1\end{bmatrix}} = \sqrt{2}$, $\langle x,y\rangle = [1\ 2]\begin{bmatrix}-1\\-1\end{bmatrix} = -3$

$\omega = \cos^{-1}\left(\frac{\langle x,y\rangle}{\|x\|\|y\|}\right) = \cos^{-1}\left(\frac{-3}{\sqrt{10}}\right) = \boxed{2.82 \text{ rad} = 161.6^\circ}$

b.) $\langle x,y\rangle = x^T\begin{bmatrix}2&1\\1&3\end{bmatrix}y$

$\|x\| = \sqrt{[1\ 2]\begin{bmatrix}2&1\\1&3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}} = \sqrt{[1\ 2]\begin{bmatrix}4\\7\end{bmatrix}} = \sqrt{18}$; $\|y\| = \sqrt{[-1\ -1]\begin{bmatrix}2&1\\1&3\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix}} = \sqrt{[-1\ -1]\begin{bmatrix}-3\\-4\end{bmatrix}} = \sqrt{7}$

$\langle x,y\rangle = [1\ 2]\begin{bmatrix}2&1\\1&3\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix} = [1\ 2]\begin{bmatrix}-3\\-4\end{bmatrix} = -11$

$\omega = \cos^{-1}\left(\frac{-11}{3\sqrt{14}}\right) = \boxed{2.941 \text{ rad} = 168.5^\circ}$

3.8.) $b_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $b_2 = \begin{bmatrix}-1\\2\\0\end{bmatrix}$

$u_1 = b_1 = \begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow u_1u_1^T = \begin{bmatrix}1\\1\\1\end{bmatrix}[1\ 1\ 1] = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$ & $\|u_1\|^2 = u_1^Tu_1 = [1\ 1\ 1]\begin{bmatrix}1\\1\\1\end{bmatrix} = 3$

$u_2 = b_2 - \pi_{span[u_1]}b_2 = b_2 - \frac{u_1u_1^T}{\|u_1\|^2}b_2 = \begin{bmatrix}-1\\2\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}-1\\2\\0\end{bmatrix} = \begin{bmatrix}-1\\2\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-4/3\\5/3\\-1/3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}-4\\5\\-1\end{bmatrix}$

$c_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$, $c_2 = \frac{1}{\sqrt{42}}\begin{bmatrix}-4\\5\\-1\end{bmatrix} \Rightarrow \boxed{C = (c_1,c_2) = \left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{42}}\begin{bmatrix}-4\\5\\-1\end{bmatrix}\right\}}$

HW 3

3.9a.) $\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n}$

For $\vec{x} = (x_1, x_2, \dots, x_n)$ & $y = (1, 1, \dots, 1)$

$\langle x, y \rangle = \sum_{i=1}^{n} x_i$, $\|x\| = \sqrt{\sum x_i^2}$, $\|y\| = \sqrt{n}$

Given that $\sum_{i=1}^{n} x_i = 1$, $1 \leq \sqrt{\sum x_i^2}\,\sqrt{n}$

$\rightarrow \frac{1}{\sqrt{n}} \leq \sqrt{\sum_{i=1}^{n} x_i^2}$ & square both sides: $\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n}$ QED