Fourier Analysis

Jarod Klion

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Periodic Functions

Definition

A function $f(x)$ is called periodic if, for some constant $p \neq 0$, $f(x+p) = f(x)$ for all x. The smallest positive p is typically referred to as the period of a function.

By definition, periodic functions exhibit repetitive behavior. For a real-valued function, an entire graph can be made from repeating one specific portion at regular intervals.

Examples:

- $\sin x$ and $\cos x$ have period 2π
- $\sin(nx)$ and $\cos(nx)$ have period $\frac{2\pi}{n}$
- For $L \neq 0$, $\sin(\frac{2\pi x}{L})$ and $\cos(\frac{2\pi x}{L})$ have period L

The most "basic" periodic functions are sine and cosine.

Infinite Dimensions

Recall

A vector space is a set V with **vector addition** and **scalar multiplication** satisfying the eight "vector" axioms.

The definition of a vector space sets no limit on the dimension, so it is possible to have an infinite-dimensional vector space, which is often the case in a so-called **function space**.
Main takeaways:

- There still exists an orthonormal basis
- An inner product still exists - $\langle f(x), g(x)\rangle = \int_a^b f(x)g(x)dx$

The ideas of periodic functions and infinite-dimensional vector spaces will form the basis of understanding for Fourier series.

Fourier Series

For any periodic, piece-wise continuous function $f(x)$ with period $2L$ defined on the interval $[-L, L]$, there exists a Fourier series expansion involving an infinite series of sines and cosines:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \tag{1}$$

where a_0, a_n, and b_n are known as the Fourier coefficients of $f(x)$ and are calculated by the formulas:

$$a_0 = \frac{1}{2\pi}\int_{-L}^{L} f(x)dx \qquad a_n = \frac{1}{\pi}\int_{-L}^{L} f(x) cos\left(\frac{n\pi x}{L}\right) dx \tag{2}$$

$$b_0 = 0 \qquad b_n = \frac{1}{\pi}\int_{-L}^{L} f(x) cos\left(\frac{n\pi x}{L}\right) dx$$

Example

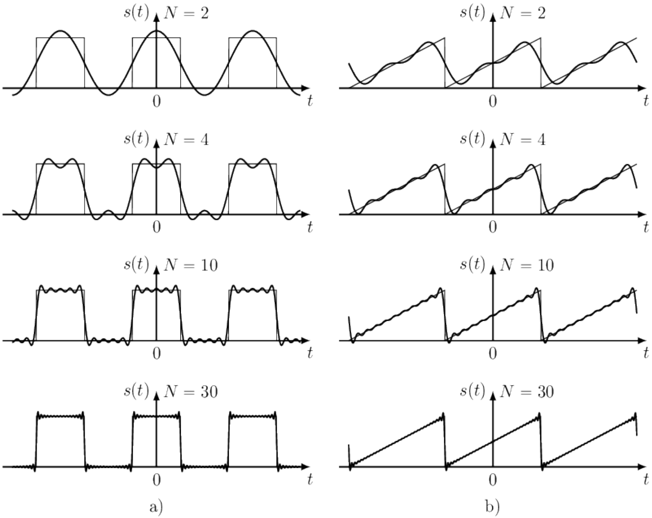

Figure: Fourier series approximations for: a — square wave; b — sawtooth wave

A Journey to the Fourier Transform

With little regard for mathematical rigor, we will introduce a new function F(ω) based on the Fourier series coefficients and define $\omega = \frac{n\pi}{L}$:

$$\hat{f}(\omega) = \sqrt{\tfrac{\pi}{2}}(a_n - ib_n) = \tfrac{1}{\sqrt{2\pi}}\left[\int_{-L}^{L} f(x)\cos(\omega x)dx - i\int_{-L}^{L} f(x)\sin(\omega x)dx\right]$$

Allowing $f(x)$ to take any value from $-\infty$ to ∞ and some simplifying of the above yields:

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \tag{3}$$

$\hat{f}(\omega)$ is known as the Fourier transform of f(x), and it retains all of the essential information of $f(x)$.

Visualizing 2D FT

- Images are two-dimensional, so let's extend the idea of the FT to 2D in order to apply it to the image:

$$\hat{f}(\vec{\omega}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\vec{x}) e^{-i(\vec{\omega}\cdot\vec{x})} d\vec{x}$$

- The FT breaks down a gray-scale image into a combination of sinusoidal waves, with each pixel now having a coordinate (ω_x, ω_y) representing the x-frequency and y-frequency contribution.
- The low frequencies of the image are concentrated around the center, while the higher frequencies are near the corners.

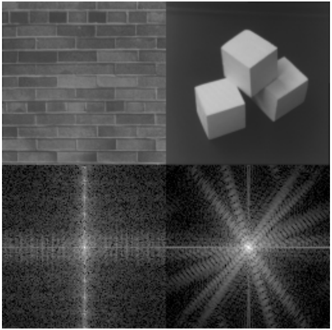

Figure: Original images on top and their corresponding Fourier transform magnitudes, $|\hat{f}(\omega)|$

Edge Detection

- Make use of a high-pass filter on the transformed image to allow only frequencies higher than some user-defined threshold
- High frequencies depict a sudden change of contrast between two pixels.
- Removes most of the low frequency tonal information, leaving a very dark image

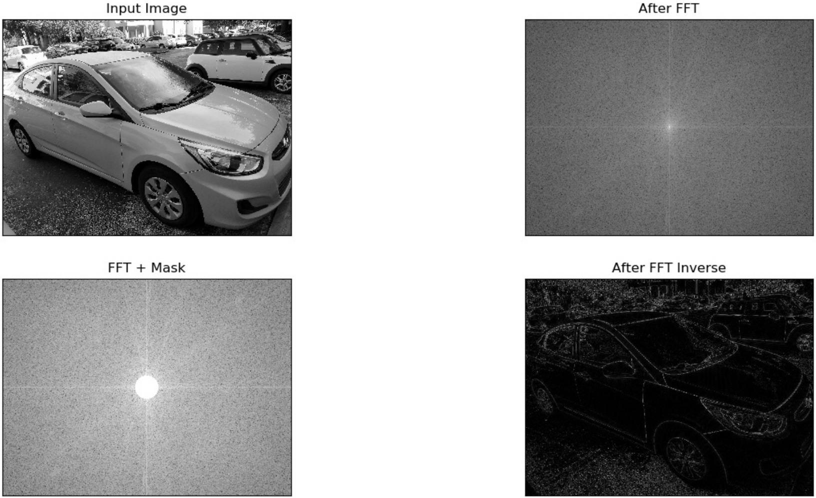

Figure: 4 stages of a possible high-pass filter process

Noise Filtering

- Noise creates many sudden contrast changes between pixels (high frequency)
- Cut off all frequencies greater than a user-defined one using a low-pass filter
- Has the effect of "smoothing" out the image
- Actual sharp edges, like walls or pillars, will be affected, too, so use carefully

Low-Pass Filter

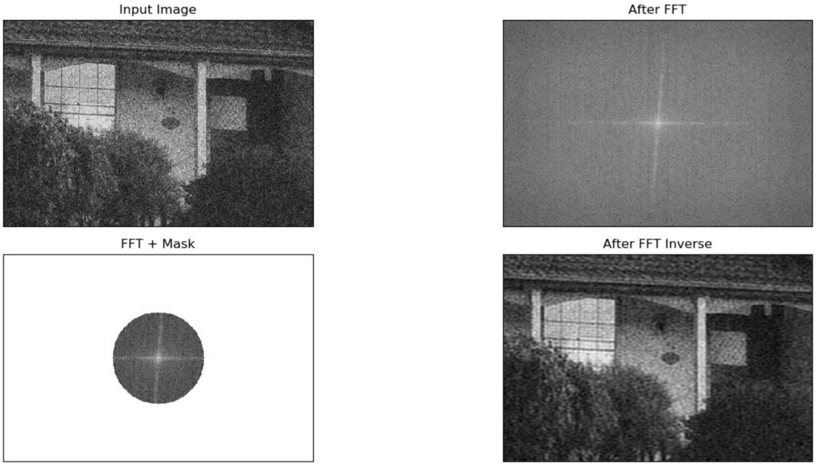

Figure: 4 stages of a possible low-pass filter process

Band-Pass Filter

- A combination of HPF and LPF
- Two concentric rings in which only those frequency values greater than the high-value band and lower than the low-value band can pass

Figure: Side-by-side comparison of a HPF and BPF

References

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