HW 2

2.11.) $\lambda_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + \lambda_2 \begin{bmatrix}1\\2\\3\end{bmatrix} + \lambda_3 \begin{bmatrix}2\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\-2\\5\end{bmatrix}$

$$\left[\begin{array}{ccc|c}1&1&2&1\\1&2&-1&-2\\1&3&1&5\end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c}1&1&2&1\\0&1&-3&-3\\0&2&-1&4\end{array}\right] \xrightarrow[R_3-2R_2]{R_1-R_2} \left[\begin{array}{ccc|c}1&0&5&4\\0&1&-3&-3\\0&0&5&10\end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_3} \left[\begin{array}{ccc|c}1&0&5&4\\0&1&-3&-3\\0&0&1&2\end{array}\right] \xrightarrow[R_2+3R_3]{R_1-5R_3} \left[\begin{array}{ccc|c}1&0&0&-6\\0&1&0&3\\0&0&1&2\end{array}\right] \Rightarrow \boxed{\vec{y} = \begin{bmatrix}1\\-2\\5\end{bmatrix} = -6\vec{x}_1 + 3\vec{x}_2 + 2\vec{x}_3}$$

2.12.) $U_1 = \text{Span}\left\{\begin{bmatrix}1\\1\\-3\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\0\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}\right\}, \quad U_2 = \left\{\begin{bmatrix}-1\\-2\\2\\1\end{bmatrix}, \begin{bmatrix}2\\-2\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\6\\-2\\-1\end{bmatrix}\right\} \Rightarrow \begin{array}{l} v_3 = \frac{1}{3}(v_1 - 2v_2) \\ v_6 = -v_4 - 2v_5 \end{array}$ so don't consider them

$V = a_1v_1 + a_2v_2 = a_4v_4 + a_5v_5 \Rightarrow a_1v_1 + a_2v_2 - a_4v_4 + a_5v_5 = 0$

$$\rightarrow \begin{bmatrix}1&2&1&-2\\1&-1&2&2\\-3&0&-2&0\\1&-1&-1&0\end{bmatrix} \xrightarrow[R_4-R_1]{R_2-R_1,\ R_3+3R_1} \begin{bmatrix}1&2&1&-2\\0&-3&1&4\\0&6&1&-6\\0&-3&-2&2\end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix}1&2&1&-2\\0&1&-\frac{1}{3}&-4/3\\0&6&1&-6\\0&-3&-2&2\end{bmatrix} \xrightarrow[R_4+3R_2]{R_1-2R_2,\ R_3-6R_2} \begin{bmatrix}1&0&5/3&2/3\\0&1&-1/3&-4/3\\0&0&3&2\\0&0&-3&-2\end{bmatrix}$$

$$\xrightarrow[R_4+R_3]{\frac{1}{3}R_3} \begin{bmatrix}1&0&5/3&2/3\\0&1&-1/3&-4/3\\0&0&1&2/3\\0&0&0&0\end{bmatrix} \xrightarrow[R_2+\frac{1}{3}R_3]{R_1-\frac{5}{3}R_3} \begin{bmatrix}1&0&0&-4/9\\0&1&0&-10/9\\0&0&1&2/3\\0&0&0&0\end{bmatrix} \begin{array}{l} a_1 = \frac{4}{9}a_5 \\ a_2 = 10/9\, a_5 \\ a_4 = -\frac{2}{3}a_5 \end{array} \Rightarrow \begin{bmatrix}a_1\\a_2\\a_4\\a_5\end{bmatrix} = \begin{bmatrix}4\\10\\-6\\9\end{bmatrix}$$

$$\rightarrow \boxed{V = 4v_1 + 10v_2 = -6v_4 + 9v_5 = \quad U_1 \cap U_2 = \left\{\begin{bmatrix}24\\-6\\-12\\-6\end{bmatrix}\right\}}$$

2.13.) $A_1 = \begin{bmatrix}1&0&1\\1&-2&-1\\2&1&3\\1&0&1\end{bmatrix}, \quad A_2 = \begin{bmatrix}3&-3&0\\1&2&3\\7&-5&2\\3&-1&2\end{bmatrix} \xrightarrow[R_3-7R_1,\ R_4-3R_1]{\frac{1}{3}R_1,\ R_2-R_1} \begin{bmatrix}1&-1&0\\0&3&3\\0&2&2\\0&2&2\end{bmatrix} \xrightarrow[R_3-R_2,\ R_4-R_2]{\frac{1}{3}R_2,\ R_1+R_2} \begin{bmatrix}1&0&1\\0&1&1\\0&0&0\\0&0&0\end{bmatrix}$

a.)
b.) $\xrightarrow[R_4-R_1]{R_2-R_1,\ R_3-2R_1} \begin{bmatrix}1&0&1\\0&-2&-2\\0&1&1\\0&0&0\end{bmatrix} \xrightarrow[R_3-R_2]{-\frac{1}{2}R_2} \begin{bmatrix}1&0&1\\0&1&1\\0&0&0\\0&0&0\end{bmatrix} \Rightarrow \boxed{U_1 = \text{Span}\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix}\right\},\ \dim(U_1) = 1\ ;\ U_2 = \text{span}\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix}\right\},\ \dim(U_2) = 1}$

c.) $\boxed{U_1 \cap U_2 = \left\{\begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}}$ since $U_1 = U_2$

2.14.) $U_1 = \text{Span}\left\{\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\3\\1\end{bmatrix}\right\}, \quad U_2 = \text{Span}\left\{\begin{bmatrix}3\\1\\7\\3\end{bmatrix}, \begin{bmatrix}-3\\2\\-5\\-1\end{bmatrix}, \begin{bmatrix}0\\3\\2\\2\end{bmatrix}\right\} \quad \begin{array}{l} v_3 = v_1 + v_2 \\ v_6 = v_4 + v_5 \\ V = a_1v_1 + a_2v_2 = a_4v_4 + a_5v_5 \end{array}$

a.) From above, $\boxed{R_K(A_1) = 2 = \dim(U_1)}$ & $\boxed{R_K(A_2) = 2 = \dim(U_2)}$

b.) pivots in column 1 & 2 of A_1 & A_2, so:

$\boxed{\text{basis } U_1 = \text{span}\left\{\begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\-2\\1\\0\end{bmatrix}\right\}}$, and $\boxed{\text{basis } U_2 = \text{span}\left\{\begin{bmatrix}3\\1\\7\\3\end{bmatrix}, \begin{bmatrix}-3\\2\\-5\\-1\end{bmatrix}\right\}}$

c.) $U_1 \cap U_2 = \begin{bmatrix}1&0&-3&3\\1&-2&-1&-2\\2&1&-7&5\\1&0&-3&1\end{bmatrix} \xrightarrow[R_3-2R_1,\ R_4-R_1]{R_2-R_1} \begin{bmatrix}1&0&-3&3\\0&-2&2&-5\\0&1&-1&-1\\0&0&0&-2\end{bmatrix} \xrightarrow[R_2+2R_3]{R_3 \leftrightarrow R_2} \begin{bmatrix}1&0&-3&3\\0&1&-1&-1\\0&0&0&-7\\0&0&0&2\end{bmatrix} \xrightarrow[R_2+R_3,\ R_4-2R_3]{-\frac{1}{7}R_3,\ R_1-3R_3} \begin{bmatrix}1&0&-3&0\\0&1&-1&0\\0&0&0&1\\0&0&0&0\end{bmatrix} \begin{array}{l} a_1 = 3a_4 \\ a_2 = a_4 \\ a_5 = 0 \\ a_4 \text{ free} \end{array}$

$\vec{a} = \begin{bmatrix}a_1\\a_2\\a_4\\a_5\end{bmatrix} = \begin{bmatrix}3\\1\\1\\0\end{bmatrix} \Rightarrow V = 3v_1 + v_2 = v_4 = \begin{bmatrix}3\\1\\7\\3\end{bmatrix}$, so $\boxed{U_1 \cap U_2 = \left\{\begin{bmatrix}3\\1\\7\\3\end{bmatrix}\right\}}$

2.16c.) $x \mapsto \Phi(x) = \cos(x)$

$\Phi(\lambda x) = \cos(\lambda x) \neq \lambda\cos(x)$, so <u>NOT</u> linear

d.) $x \mapsto \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix} x$

$$\left(\lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \psi \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) \mapsto \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix}\left(\lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \psi \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = \lambda \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \psi \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$\therefore$ This is a linear transformation as are all matrix transforms

e.) This is the 2x2 rotation matrix, which is also linear similar to **(d)**

2.17.) $$\Phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 - 3x_2 \\ 2x_1 + 3x_2 + x_3 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ -3 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \boxed{A_\Phi = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}}$$

$$\boxed{\mathrm{Im}(\Phi) = \mathrm{span}\left\{\begin{bmatrix} 3 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\};\ \mathrm{Ker}(\Phi) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\};\ \dim(\mathrm{Im}(\Phi)) = 3,\ \dim(\ker(\Phi)) = 0}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix} \xrightarrow[R_4 - 2R_1]{R_2 - 3R_1,\ R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -4 & -1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[R_3 + 4R_4,\ R_4 \leftrightarrow R_2]{R_1 - R_4,\ R_2 + R_4} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix} \xrightarrow[R_4 + 3R_3]{R_1 - 2R_3,\ R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \boxed{rk(A) = 3}$$

2.19.) $A_\Phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

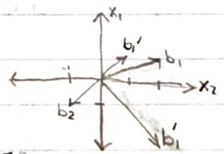
a.) $\mathrm{Im}(\Phi) = \left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ $\mathrm{Ker}(\Phi) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

b.) $B = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \Rightarrow P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ $\det(P) = 1\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1$ $\mathrm{Adj}(P) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & -2 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

$P^{-1} = \frac{1}{\det(P)}\mathrm{Adj}(P) = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$

$$\tilde{A}_\Phi = P^{-1} A_\Phi P = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 1 \\ 3 & 4 & 1 \end{bmatrix} \Rightarrow \boxed{\tilde{A}_\Phi = \begin{bmatrix} 6 & 9 & 1 \\ -3 & -5 & 0 \\ -1 & -1 & 0 \end{bmatrix}}$$

2.20.) $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $b_1' = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $b_2' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



a.) both B & B' have 2 linearly independent vectors, so they are bases of $\mathbb{R}^2$

b.) $[B|B'] \rightarrow [I_n|P] \Rightarrow$ $$\left[\begin{array}{cc|cc} 2 & -1 & 2 & 1 \\ 1 & -1 & -2 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & -1 & -2 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} & -3 & \frac{1}{2} \end{array}\right] \xrightarrow[R_1 - R_2]{\cdot 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 4 & 0 \\ 0 & 1 & 6 & -1 \end{array}\right]$$

$$\boxed{P = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}}$$

2.20c.) $C_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$, $C_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

i.) $C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix} \Rightarrow \det(C) = 1\begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 1+3 = \boxed{4, \text{ so the vectors are linearly independent}}$

ii.) $C' = (c_1', c_2', c_3') = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \boxed{P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}}$, since converting to standard basis

$a_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{matrix} a_1 = 1 \\ a_2 = 2 \\ a_3 = -1 \end{matrix} \Rightarrow C_1 = c_1' + 2c_2' - c_3'$

d.) $\Phi(\vec{b}_1 + \vec{b}_2) = C_2 + C_3 = \Phi(\vec{b}_1) + \Phi(\vec{b}_2)$ ①

$\Phi(\vec{b}_1 - \vec{b}_2) = 2C_1 - C_2 + 3C_3 = \Phi(\vec{b}_1) - \Phi(\vec{b}_2)$ ②

①+②: $\Phi(\vec{b}_1) = C_1 + 2C_3$

①−②: $\Phi(\vec{b}_2) = -C_1 + C_2 - C_3$

$\boxed{A_\Phi = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}}$

e.) $A' = T^{-1}AS = P_2 A_\Phi P_1$ since we need $\underbrace{B' \rightarrow B}_{P_1} \underbrace{\rightarrow C}_{A} \underbrace{\rightarrow C'}_{P_2}$

$A' = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 6 & -1 \\ 2 & 1 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix} = A'}$

f.) $[x]_{B'} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

i.) $[x]_B = P_1 [x]_{B'} = \begin{bmatrix} 4 & 0 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} 8 \\ 9 \end{bmatrix}}$

ii.) $A_\Phi x_B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix}$

iii.) $P_2 \begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}$

iv.) $A'[x]_B = \begin{bmatrix} 0 & 2 \\ -10 & 3 \\ 12 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 12 \end{bmatrix}$

9.) Let $\vec{x}, \vec{y} \in \text{Span}(B)$. Then $\exists c_1, \dots, c_n \in \mathbb{R}$ & $k_1, \dots, k_n \in \mathbb{R}$ such that,

$\vec{x} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$ & $\vec{y} = k_1\vec{v}_1 + \dots + k_n\vec{v}_n$

Closed under addition: $\vec{x} + \vec{y} = (c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n) + (k_1\vec{v}_1 + k_2\vec{v}_2 + \dots + k_n\vec{v}_n)$

$= (c_1+k_1)\vec{v}_1 + (c_2+k_2)\vec{v}_2 + \dots + (c_n+k_n)\vec{v}_n \in \text{Span}(B)$, so the span is closed under add.

Closed under scalar multi.: $k\vec{u} = k(c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n)$

$= kc_1\vec{v}_1 + kc_2\vec{v}_2 + \dots + kc_n\vec{v}_n$

$= (kc_1)\vec{v}_1 + (kc_2)\vec{v}_2 + \dots (kc_n)\vec{v}_n \in \text{Span}(B)$, so span is closed under scalar multiplication

$\therefore$ Span(B) is a subspace of V