# Markov Chain Monte Carlo

ISC 5225 Midterm

Oct 13, 2022

1. **Transformation Method for Normal Distribution**

Suppose we want to use the transformation method to sample from the normal distribution $p(x; \mu, \sigma^2)$,

$$p(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \tag{1}$$

with mean $\mu = 0$, and variance $\sigma^2 = 1$. The cumulative distribution function corresponding to this unit normal distribution is,

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\, dz = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right], \tag{2}$$

where $\text{erf}(x)$ is the error function, cannot be analytically inverted to find $x = F^{-1}(u)$ for use in the transformation method.

Interestingly, a reasonably accurate approximation for $F^{-1}(\cdot)$ is given by:

$$x = F^{-1}(u) \approx \begin{cases} \dfrac{a_0 + a_1 t}{1 + b_1 t + b_2 t^2} - t & \text{for } u \leq 0.5 \text{ with } t = \sqrt{-2\log(u)} \\ t - \dfrac{a_0 + a_1 t}{1 + b_1 t + b_2 t^2} & \text{for } u > 0.5 \text{ with } t = \sqrt{-2\log(1-u)} \end{cases} \tag{3}$$

with constants $a_0 = 2.30753$, $a_1 = 0.27061$, $b_1 = 0.99229$, and $b_2 = 0.04481$.[1]

(a) Write a program to sample from $p(x; \mu = 0, \sigma^2 = 1)$ using this approximation. Generate $n = 10^4$ samples from this distribution.

(b) Compare the histogram of these samples with the PDF, eqn (1)

(c) Compute the mean and standard deviation of these samples.

**Points**: (a) 20, (b) 10, (c) 5

[1] e.g. say, $u = 0.4 \leq 0.5$, then set $t = \sqrt{-2\log(0.4)}$, and substitute into the top branch and return $x = (a_0 + a_1 t)/(1 + b_1 t + b_2 t^2) - t$. Follow the lower branch if $u > 0.5$.

2. **Importance Sampling**

Consider the integral,

$$I = \int_{-\infty}^{\infty} \frac{\theta}{1+\theta^2} e^{-(\theta-1)^2/2}\, d\theta. \tag{4}$$

For this question, you are encouraged to use the sampler developed in the previous question.[2]

(a) Use importance sampling with $n = 10^4$ samples drawn from the normal distribution, $\pi(\theta) = p(\theta; \mu = 0, \sigma^2 = 4)$ to evaluate the integral.

(b) Estimate the error $\sigma_I$ in the integral without resorting to multiple runs.

**Points**: (a) 20, (b) 10

[2]To sample $x'$ from a normal distribution with mean $\mu$ and variance $\sigma^2$, set $x' = \sigma x + \mu$, where $x$ is sampled from a unit normal in Q1. You may use a built-in Gaussian random number generator as a fallback.

3. **Bayesian Inference**

Suppose the number of major hurricanes $k$ in the North Atlantic per year follows a Poisson distribution, $p(k|\lambda) = (\lambda^k e^{-\lambda})/k!$, with an average rate of $\lambda$ per year.

The number of such hurricanes over a $n = 5$ year period between 2017-2021 was $k_1 = 6$, $k_2 = 2$, $k_3 = 3$, $k_4 = 6$, $k_5 = 4$. If we assume that these observations are independent of each other, then the combined likelihood is given by,

$$p(\mathbf{k}|\lambda) = \prod_{i=1}^{5} p(k_i|\lambda), \tag{5}$$

where $\mathbf{k} = [k_1, k_2, k_3, k_4, k_5]$ is the set of $n = 5$ observations. Our goal is to estimate the posterior distribution $p(\lambda|\mathbf{k})$ from these observations by using Bayesian inference:

$$p(\lambda|\mathbf{k}) \propto p(\mathbf{k}|\lambda)\, p(\lambda). \tag{6}$$

Suppose we assume that the prior distribution $p(\lambda)$ is a Gamma distribution,

$$p(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}$$

with parameters $a = 5$ and $b = 0.5$, and $\Gamma(\cdot)$ is the Gamma function. Since Gamma and Poisson are conjugate distributions, it can be shown that the posterior distribution,

$$p(\lambda|\mathbf{k}) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$

is also a Gamma distribution with parameters $\alpha = a + \sum_i k_i$ and $\beta = n + b = 5 + b$.

(a) Plot the prior and posterior distributions, and comment on difference between them.[3]

(b) Suppose the prior $p(\lambda)$ distribution was uniform $U[0, 20]$. Describe a step-by-step approach for estimating the posterior distribution. You are not required to implement this part, but if you do, I will give you 10 bonus points.

**Points**: (a) 15, (b) 10 (+10 bonus).

[3] Hint: You are not required to do any Monte Carlo sampling for this problem.