Fall Semester	Foundations of Computational Mathematics I	2022

HW 3

Due Date Nov 16th, 2022
Points 100 pts

1. Consider the system of equations $Ax = b$, where A is a nonsingular lower triangular matrix, i.e

$$A = D - L,$$

where D is a diagonal and nonsingular, and L is strictly lower triangular matrix.

   (a) Show that (forward) Gauss-Seidel will converge to $x = A^{-1}b$ in a finite number of steps (in exact arithmetic) for any initial guess x_0 and give a tight upper bound on the number of steps required.

   (b) Also could you show the same results for the (backward) Gauss-Seidel?

2. When attempting to solve $Ax = b$ where A is known to be nonsingular via an iterative method, we have seen various theorems that give sufficient conditions on A to guarantee the convergence of various iterative methods. It is not always easy to verify these conditions for a given matrix A.
Let P and Q be two permutation matrices. Rather than solving $Ax = b$, we could solve

$$(PAQ)(Q^T x) = Pb$$

using an iterative method. Sometimes it is possible to examine A and choose P and(or) Q so that it is easy to apply one of our sufficient condition theorems. The Gauss-Seidel and Jacobi iterative methods did not converge for linear systems with the matrix

$$A = \begin{bmatrix} 3 & 7 & -1 \\ 7 & 4 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Why? Can you choose P and Q that the permuted system converges for one or both of Gauss-Seidel and Jacobi?

3. When solving $Ax = b$ or equivalently the associated quadratic definite minimization problem using Conjugate Gradient, we have

$$x_{k+1} = x_0 + \alpha_0 p_0 + \cdots + \alpha_k p_k,$$

where the p_j are A-orthogonal vectors. It can be shown that

$$\text{span}(p_0, \ldots, p_k) = \text{span}(r_0, Ar_0, \ldots, A^k r_0)$$

where $r_0 = b - Ax_0$ and x_0 is the initial guess for the solution $x^* = A^{-1}b$. Therefore,

$$x_{k+1} = x_0 + \gamma_0 r_0 + \gamma_1 Ar_0 + \cdots \gamma_k A^k r_0 = x_0 - P_k(A) r_0$$

where $P_k(A) = \gamma_0 I + \gamma_1 A + \cdots + \gamma_k A^k$ is matrix that is called a matrix polynomial evaluated at A. (A space whose span can be defined by a matrix polynomial is called a Krylov space). Denote $d_j = A^j r_0$ for $j = 0, 1, \ldots$, and determine the relationship between the coefficients $\alpha_0, \ldots, \alpha_k$, and the coefficients $\gamma_0, \ldots, \gamma_k$

4. Determine the necessary and sufficient conditions for $x = A^{-1}b$ to be a fixed point of

$$x_{k+1} = Gx_k + f$$