Homework 2

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October 21, 2022

1. Following is often referred to as the equivalence of the norms. For any square matrix $A \in \mathbb{R}^{n\times n}$, prove the following relations:

   (a) $\frac{1}{n}K_2(A) \leq K_1(A) \leq nK_2(A)$

   We will start by proving some equivalences for vector norms using the Cauchy-Schwarz inequality and then translating it to matrix norms and subsequently condition numbers:

$$\|x\|_2^2 = \sum_{i=1}^{n} x_i^2 \leq (\sum_{i=1}^{n} |x_i|)(\sum_{i=1}^{n} |x_i|) = \|x\|_1^2 \tag{1}$$

   Proving an equivalence between the one- and two-norm:

$$\begin{aligned}
\|x\|_1 &= \sum_{i=1}^{n} |x_i| * 1 \\
&\leq \sqrt{\sum_{i=1}^{n} |x_i|^2 * \sum_{i=1}^{n} 1^2} \\
&\leq \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} |1|^2\right)^{1/2} = \sqrt{n}\|x\|_2 \\
\Rightarrow \|x\|_1 &\leq \sqrt{n}\|x\|_2 \qquad (2)
\end{aligned}$$

   Proving equivalence between the one- and ∞-norm:

$$\begin{aligned}
\|x\|_1 = \sum_{i=1}^{n} |x_i| \leq \sum_{i=1}^{n} |\max_{j\leq n} x_j| \leq n \max_{1\leq j\leq n} x_j = n\|x\|_\infty \\
\Rightarrow \frac{1}{n}\|x\|_1 \leq \|x\|_\infty \qquad (3)
\end{aligned}$$

   Proving equivalence between the two- and ∞-norm:

$$\begin{aligned}
\|x\|_2 = \sum_{i=1}^{n} |x_i|^2 \geq \max_i |x_i| = \|x\|_\infty \\
\|x\|_\infty \leq \|x\|_2 \qquad (4) \\
\|x\|_2^2 = \sum_{i=1}^{n} |x_i|^2 \leq \text{n} \max_i |x_i|^2 = n\|x\|_\infty^2 \\
\|x\|_2 \leq \sqrt{n}\|x\|_\infty \qquad (5)
\end{aligned}$$

Putting together the norm equivalences we've shown so far, we can show the norm equivalence between $\|x\|_1$ and $\|x\|_\infty$ to be:

$$\frac{1}{n}\|x\|_1 \leq \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_1$$
$$\frac{1}{n}\|x\|_1 \leq \|x\|_\infty \leq n\|x\|_1 \tag{6}$$

Now, some matrix norm equivalences using inequalities 1 & 2:

$$\|Ax\|_2 \leq \|Ax\|_1 \leq \|A\|_1\|x\|_1 \leq \|A\|_1\sqrt{n}\|x\|_2$$
$$\Rightarrow \frac{1}{\sqrt{n}}\|A\|_2 \leq \|A\|_1 \tag{7}$$
$$\|Ax\|_1 \leq \sqrt{n}\|Ax\|_2 \leq \sqrt{n}\|A\|_2\|x\|_2 \leq \sqrt{n}\|A\|_2\|x\|_1$$
$$\Rightarrow \|A\|_1 \leq \sqrt{n}\|A\|_2 \tag{8}$$

Since the condition number of a matrix is defined as $K(A) = \|A\|\|A^{-1}\|$, follow a similar process as above and multiply the norms to get the equivalence:

$$\frac{1}{n}K_2(A) \leq K_1(A) \leq nK_2(A) \text{ QED} \tag{9}$$

(b) $\frac{1}{n^2}K_1(A) \leq K_{\text{inf}}(A) \leq n^2K_1(A)$

Translating inequality 6 into matrix norms and the subsequent condition numbers, we can see that we arrive at the desired expression:

$$\frac{1}{n^2}K_1(A) \leq K_\infty(A) \leq n^2K_1(A) \tag{10}$$

2. Let A be a sparse matrix of order n. Prove that the computational cost of the LU factorization of A is given by,

$$\frac{1}{2}\sum_{k=1}^{n} l_k(A)(l_k(A)+3) \text{ flops},$$

where $l_k(A)$ is the number of active rows at the k-th step of the factorization (i.e, the number of rows of A with $i > k$ and $a_{ik} \neq 0$, and having accounted for all nonzero entries.