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THE THIEM METHOD FOR DETERMINING MEABILITY OF WATER-BEARING MATERIALS

AND ITS APPLICATION TO THE DETERMINATION OF SPECIFIC YIELD

RESULTS OF INVESTIGATIONS IN
THE PLATTE RIVER VALLEY, NEBRASKA

BY

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Prepared in cooperation with the Conservation and Survey
Division of the University of Nebraska

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THE THIEM METHOD FOR DETERMINING PERMEABILITY OF WATER-BEARING MATERIALS AND ITS APPLICATION TO THE DETERMINATION OF SPECIFIC YIELD

By Leland K. Wenzel

ABSTRACT

The Thiem method for determining permeability of water-bearing materials [1] consists of pumping a well, or, where the ground water is confined under pressure, allowing the well to flow and observing the decline of the water table or piezometric surface in nearby observation wells. The coefficient of permeability is computed by the formula

$$P = \frac{527.7\ q \log_{10} \frac{a_1}{a}}{m\ (s - s_1)}$$

where P is the coefficient of permeability; q is the rate of pumping, in gallons a minute; a and a_1 are respective distances of two observation wells from the pumped well, in feet; m, for artesian conditions, is the vertical thickness of the water-bearing bed, in feet; m, for water-table conditions, is the average vertical thickness, at a_1 and a, of the saturated part of the water-bearing bed, in feet; and s and s_1 are the draw-downs at the two observation wells, in feet. This formula is mathematically developed by assuming ideal geologic and ground-water conditions, such as a uniform permeability, a uniform thickness of water-bearing bed, a horizontal water table or piezometric surface, and a cone of depression that has reached equilibrium in form. As these conditions are rarely approached, the applicability of the formula and hence of the method has been regarded as questionable.

Two rather elaborate pumping tests were made in 1931 near Grand Island, Nebr., to ascertain the accuracy of the Thiem method and to investigate the possibilities of determining specific yield by a pumping test. The behavior of the ground water was observed over a large area around the pumped wells by measuring the fluctuation of the water table in 81 observation wells during the period of pumping and after pumping was stopped. A study of the data obtained from these tests indicates that the Thiem method is applicable to conditions that are found in nature. However, to obtain consistent and accurate determinations of permeability it is necessary to employ an arbitrary procedure in computing the coefficient. The draw-down of the water table at any distance from the discharging well should be taken as the average of the draw-down at that distance up-gradient and down-gradient from the well. In Thiem's formula only results for the draw-down of the water table that are obtained from the part of the cone of depression that has reached approximate equilibrium in form can be used. The part of the cone that has reached approximate equilibrium is determined by fre-

[1] Thiem, G., Hydrologische Methoden, Leipzig, 1906.

quent measurements of the draw-down during the period of pumping. If the discharging well fails to penetrate through the water-bearing bed, the draw-down of the water table close to the well should not be used, because of irregularities in the cone of depression. Moreover, there are usually near the well some changes in the permeability of the water-bearing material resulting from the development of the well. In the first test described in this report the cone of depression reached approximate equilibrium in form out to about 200 feet from the pumped well after 48 hours of pumping and was affected by irregular conditions near the well as far as 40 feet from the well. Hence the draw-downs that were used for computations of permeability were selected from that part of the cone between 40 and 200 feet from the pumped well. In the second test pumping was stopped several times, and the cone of depression did not reach approximate equilibrium in form.

Computations were made to determine the specific yield of the water-bearing materials from the data obtained in the pumping tests. The results show that the specific yield can be readily determined by this method. Samples of the material were analyzed in the laboratory for specific yield, and the results obtained compared favorably with those determined by the pumping method.

INTRODUCTION

INVESTIGATION IN THE PLATTE VALLEY, NEBR.

An investigation of the ground-water resources of Nebraska has for some time been in progress under the supervision of G. E. Condra, director of the Conservation and Survey Division of the University of Nebraska. At the request of Dr. Condra, a ground-water investigation of that part of the Platte River Valley lying between Chapman and Gothenburg, Nebr., was undertaken July 1, 1930, as a cooperative project between the Conservation and Survey Division and the United States Geological Survey, under the general supervision of O. E. Meinzer, geologist in charge, division of ground water in the Geological Survey. The writer was assigned to this cooperative project and began work July 12, 1930.

The investigation has for its purpose the determination of the source, quantity, and availability of the ground water, with a view to accomplishing maximum recovery and utilization. Field work has been carried on continuously since the project was begun, and comprehensive data have been collected concerning the occurrence and behavior of the ground water in that part of the valley. The area is one in which there is rather intensive irrigation by ground water, and it was found that determinations of permeability and specific yield of the water-bearing materials should be made in order to obtain a quantitative estimate of the ground-water supply. It was decided to use the Thiem method for the determination of permeability and the pumping method for the determination of specific yield.

Neither of these methods had been adequately verified by experiments for accuracy and practicability. Hence it was necessary to make rather elaborate tests that would determine the reliability of these methods and at the same time yield the actual figures for permeability and specific yield. Accordingly, two tests of these methods were made in the summer of 1931 near Grand Island, Nebr.

The pumping method for determining specific yield was outlined by Meinzer [2] independent of the Thiem method for determining permeability. However, the data necessary for the determination of specific yield and permeability by these methods can be obtained from one pumping test with a very small amount of additional effort. Where the ground water is confined under pressure the Thiem method for determining permeability may be used, but the pumping method for determining specific yield fails because there is usually no unwatering of the water-bearing materials. Hence the method for determining specific yield is strictly a pumping method, but the Thiem method applies also to areas where wells discharge water under artesian pressure. In the area where the tests described in this report were made the water-bearing materials consist of unconsolidated sand and gravel. As the upper surface of the zone of saturation lies several feet below the top of the water-bearing materials, the ground water is not confined under pressure, and both the Thiem method for determining permeability and the pumping method for determining specific yield could be used.

The behavior of the ground water near the pumped wells was observed in detail during and after the period of pumping, and these observations provided an opportunity to determine the effect of differences between theoretical and observed conditions on the computations for permeability and specific yield by these methods. The results obtained from the tests, a review of Thiem's development of the formula for permeability, and a theoretical review of the formula are incorporated in this report. Another report, now in preparation, will give the other data obtained in the Platte River investigation and the conclusions that were reached as to the ground-water conditions in that valley.

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[2] Meinzer, O. E., Outline of methods for estimating ground-water supplies: U. S. Geol. Survey Water-Supply Paper 638, p. 136, 1932.

dean of the Graduate School, University of Wisconsin, and G. Thiem, consulting hydrologist, Leipzig, Germany, for their criticism of the manuscript; and to O. E. Meinzer, geologist in charge of the division of ground water of the Geological Survey, for his advice and suggestions throughout the investigation and for his review of the manuscript.

HYDROLOGIC PROPERTIES OF WATER-BEARING FORMATIONS

Most ground-water investigations are concerned with the quantity of water that is available for use by man. Perhaps the greatest difficulty in the determination of this quantity lies in the variability in the texture and hence in the hydrologic properties of the water-bearing materials. The hydrologic properties vary greatly, even with apparently slight differences in texture. Hence the ordinary geologic descriptions are quite inadequate for hydrologic investigations, and quantitative descriptions based on laboratory determinations have become essential.

The two hydrologic properties of greatest significance are permeability and specific yield. Mechanical analyses and determinations of porosity and moisture equivalent are useful chiefly as indirect means of determining these two essential hydrologic properties.

About 1843 Poiseuille [3] discovered the law of flow through capillary tubes—namely, that the rate of flow is proportional to the hydraulic gradient. Later Darcy [4] verified this law and demonstrated its application to water percolating through the capillary interstices of sand and other porous media. He expressed this law by means of the formula $v=\frac{Kp'}{h}$, in which v is the velocity of the water through a column of permeable material, p' the difference in head at the ends of the column, h the length of the column, and K a constant that depends upon the character of the material, especially on the size of the grains. Because it is usually more essential to determine the quantity of water flowing through a certain cross section of permeable material than to determine the velocity through the material, Darcy's formula is sometimes expressed as

$$Q=PIA \qquad (1)$$

in which Q is the quantity of water discharged in a unit of time, P the constant, which depends upon the texture of the material, I the hydraulic gradient, and A the cross-sectional area through which the water percolates. This formula serves as a basis for determining the quantities of ground water that percolate from areas of recharge to

[3] Poiseuille, J., Recherches expérimentales sur le mouvement des liquides dans les tubes de très petits diamètres: Acad. sci. Paris Mém. sav. étrang., vol. 9, p. 433, 1846.

[4] Darcy, H., Les fontaines publiques de la ville de Dijon, Paris, 1856.

areas of discharge, and consequently it is used for determining the safe yields of ground-water supplies.

The constant P in equation 1 is the most difficult factor to determine. The hydraulic gradient of an area can be obtained from contour maps of the water table or piezometric surface,[5] and the cross-sectional area of the water-bearing material can be approximately determined from the logs of wells penetrating the material. The constant P has been designated by different names and has been expressed in various units. According to the present usage of the United States Geological Survey, it is called the "coefficient of permeability", defined as the rate of flow, in gallons a day, through a square foot of cross section, under a hydraulic gradient of 100 percent, at a temperature of 60° F.[6] In field terms the coefficient of permeability may be expressed as the number of gallons a day at 60° F. that is conducted laterally through each mile of the water-bearing bed under investigation (measured at right angles to the direction of flow), for each foot of thickness of bed and for each foot per mile of hydraulic gradient.[6] Coefficients of permeability range widely. Fine sand is in general less permeable than coarse sand and therefore transmits less water through equal cross-sectional areas under the same hydraulic gradient. Clay may contain more water per unit volume than sand or gravel, but the permeability of a clayey material is generally low, and therefore the quantity of water transmitted through it is usually much less than is transmitted through sand and gravel. Coefficients of permeability ranging from 0.005 for clay to more than 20,000 for sand and gravel have been determined in the hydrologic laboratory of the United States Geological Survey.

The permeabilities of water-bearing materials may be determined by laboratory tests of samples of the materials or by determinations of ground-water velocities in the field. Hazen [7] and Slichter [8] have studied the rate of flow of water through sand and have developed formulas which essentially include the determination of the permeability of the sand. King [9] has reviewed the results of the investigators

[5] The upper surface of the zone of saturation in ordinary soil or rock is called the "water table." If a well is sunk it remains empty until it enters a saturated permeable bed—that is, until it enters the zone of saturation. Then water flows into the well. If the rock through which the well passes is all permeable the first water that is struck will stand in the well at about the level of the top of the zone of saturation—that is, at about the level of the water table. If the rock overlying the bed in which water is struck is impermeable the water is generally under pressure that will raise it in the well to some point above the level at which it was struck. In such a place there is no water table, and the imaginary surface to which the water rises under its full head is called the "piezometric surface."

[6] Stearns, N. D., Laboratory tests on physical properties of water-bearing materials: U. S. Geol. Survey Water-Supply Paper 596, p. 148, 1928.

[7] Hazen, Allen, Some physical properties of sands and gravels: Massachusetts State Board of Health 24th Ann. Rept., p. 553, 1892.

[8] Slichter, C. S., The motions of underground waters: U. S. Geol. Survey Water-Supply Paper 67, p. 26, 1902.

[9] King, F. H., Principles and conditions of the movements of ground water: U. S. Geol. Survey 19th Ann. Rept., pt. 2, pp. 178-204, 1898.

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on the flow of water through porous media and has described an apparatus for measuring the flow in the laboratory.[10] In the Geological Survey the permeability of water-bearing materials is now determined in the laboratory by means of apparatus devised by Meinzer.[11] The coefficient of permeability of a water-bearing material is determined directly by measuring the rates of flow of water through a sample of the material with known cross section and thickness under observed differences of head. The laboratory methods are open to the criticism that the coefficients of permeability of the samples tested may differ widely from the average coefficient of the material as found in nature. The material that is tested in the laboratory must necessarily be removed from the ground, and as a result, especially with the more unconsolidated material, the soil particles do not remain in their original arrangement. Moreover, the coefficients of permeability determined in the laboratory necessarily apply only to very small samples, and unless a great number of samples are tested an average coefficient for a large area cannot be determined. These statements do not imply that laboratory determinations are not significant; they are intended merely to point out some of the inherent difficulties involved in such tests and to emphasize the importance of carefully and thoroughly investigating a method such as the Thiem method, which determines permeability in the field over a large area and without disturbing the water-bearing material.

A method for determining the natural velocities of ground waters, patterned after the method of the German hydrologist A. Thiem, was developed by Slichter.[12] Several small wells are driven into the water-bearing materials in such a manner that the water moves from one well toward one or more of the other wells. A salt is introduced into the up-gradient well and is allowed to move down-gradient with the ground water to the other wells, where its arrival is detected electrically. The rate of movement of the salt and hence the rate of movement of the ground water is computed from the elapsed time between the introduction of the salt in the central well and its detection in a well located down-gradient. The quantity of water flowing through a given cross-sectional area of the water-bearing material is computed by the formula

$$Q = pAv \qquad (2)$$

where Q=quantity of water;

p=porosity of the water-bearing material;

A=cross-sectional area;

v=average velocity of the ground water.

[10] King, F. H., op. cit., p. 228.

[11] Stearns, N. D., op. cit., p. 144.

[12] Slichter, C. S., op. cit., p. 48.

The coefficient of permeability of the water-bearing material is computed by equating equations 1 and 2:

$$pAv = PIA \text{ and } P = \frac{pAv}{IA} = \frac{pv}{I} \quad (3)$$

There are difficulties in the use of this method. The method is not satisfactorily adaptable to localities where the ground water has low velocity, because the salt solution, whose specific gravity is higher than that of the natural water, sinks rather rapidly and may not reach the down-gradient wells. In using this method in such a locality, the wells are located comparatively close to one another—usually about 4 feet apart. Under these conditions errors in determining the velocity of the ground water are often introduced by failure to sink the wells exactly plumb, by the diffusion of the salt solution, and by increase in the hydraulic gradient caused by the rise of water in the up-gradient well at the time the salt is introduced.

The specific yield of a water-bearing formation is defined by Meinzer as the ratio of (1) the volume of water which, after being saturated, it will yield by gravity to (2) its own volume.[13] It is a measure of the quantity of water that a formation will yield when it is drained by lowering of the water table. Thus if 100 cubic feet of saturated water-bearing material when drained will supply 20 cubic feet of water, the specific yield of the material is said to be 20 percent.

The practical use of the specific yield is obvious. The quantity of water that a saturated material will furnish from storage depends upon its specific yield. To estimate the water supply obtainable from a material for each foot that the water table is lowered, or to estimate the available supply represented by each foot of rise in the water table during periods of recharge, it is necessary to determine the specific yield.

Meinzer [14] gives seven more or less distinct methods of determining specific yield—namely, (1) saturating samples in the laboratory and allowing them to drain; (2) saturating in the field a considerable body of material situated above the water table and above the capillary fringe and allowing it to drain downward naturally; (3) collecting samples immediately above the capillary fringe after the water table has gone down an appreciable distance, as it commonly does in summer and autumn; (4) ascertaining the volume of sediments drained by heavy pumping, a record being kept of the quantity of water that is pumped; (5) ascertaining the volume of sediments saturated by a measured amount of seepage from one or more streams; (6) making indirect determinations in the laboratory with small

[13] Meinzer, O. E., Outline of ground-water hydrology: U. S. Geol. Survey Water-Supply Paper 494, p. 28, 1923.

[14] Meinzer, O. E., Methods for estimating ground-water supplies: U. S. Geol. Survey Water-Supply Paper 638, p. 113, 1932.

samples by the application of centrifugal force; and (7) making mechanical analyses and determinations of porosity and estimating therefrom the specific retention and the specific yield.

Much work has been done on the determination of specific yield by able investigators, but the methods just enumerated are still not thoroughly developed. The method of determining specific yield from pumping tests probably is the least developed of all.

OUTLINE OF THIEM AND PUMPING-TEST METHODS

PERMEABILITY

The Thiem method is very simple in principle. It consists of pumping a well that penetrates water-bearing material, the permeability of which is to be determined, and observing the decline of the water table or piezometric surface around the pumped well. Ground water obeys the law of fluids in that it always flows away from a point of high pressure toward one of low pressure. In other words, it flows in the direction of the hydraulic gradient. When a well is pumped some water inevitably is taken out of storage from the well and from the material surrounding it. This reduces the pressure, creates a hydraulic gradient toward the well, and causes ground water to flow into the well. If the water-bearing formation has a water table, considerable ground water may have to be removed from storage before a gradient will be developed that is steep enough to make the water flow toward the well at the rate that it is pumped and thus establish approximate equilibrium. If the formation is filled with water under pressure only a comparatively small amount of water has to be removed from storage in order to give the required gradient, and hence the draw-down will be more rapid and approximate equilibrium will be more quickly established.

When, with a constant rate of pumping, equilibrium is established, water is no longer removed from storage around the well but flows to the well as rapidly as it is withdrawn. If before pumping begins the water table or piezometric surface in a homogeneous formation is horizontal, water percolates toward the pumped well equally from all directions, and the same quantity of water percolates toward the pumped well through each of the indefinite series of concentric cylindrical sections around the pumped well. Because the areas of the large cylinders through which the water percolates are greater than the areas of the smaller cylinders, the velocity of the ground water passing through them is proportionally less and the hydraulic gradients are proportionally smaller.

According to equation 1 the discharge through any of the concentric cylindrical sections of water-bearing material, Q, is equal to PiA, and the permeability of the material, P, equals $\frac{Q}{iA}$. The symbol i is used in this report to represent the hydraulic gradient at a point on the cone of depression around a well that is discharging water, and the symbol I is used to represent the normal hydraulic gradient that the water table or piezometric surface possesses when the well is idle. The two symbols are interchangeable in equation 1, their use depending upon whether the water table or piezometric surface is cone-shaped or is approximately a plane. As previously explained, after approximate equilibrium has been reached the discharge through all concentric cylindrical sections of water-bearing material is the same, and the total discharge is equal to the quantity of water being pumped from the well. The hydraulic gradient at a given distance from the pumped well can be determined from the slope of the water table or piezometric surface. For artesian conditions the area of the cylindrical section through which the ground water percolates at that distance from the pumped well is equal to $2\pi xm$, if x is the distance from the pumped well and m is the thickness of the water-bearing material. For water-table conditions the area is equal to $2\pi x(m-s)$, where s is the draw-down at the distance x from the pumped well. Thus the permeability of the water-bearing material can be computed by substituting these figures in the equation $P=\frac{Q}{iA}$.

In 1906 G. Thiem,[15] son of the German hydrologist A. Thiem, published the results of his work in connection with the determination of additional water supply for the city of Prague and its suburbs. In this investigation he used what has since been known as the "Thiem method" for determining permeability and sunk 10 sets of wells, each set including 1 well that was pumped and 2 observation wells. The observation wells were placed in line with the pumped well but in any convenient direction regardless of the natural hydraulic gradient. A formula was developed for computing the permeability from the data obtained from the pumped well and the two observation wells.

SPECIFIC YIELD

The determination of specific yield by the pumping method is based on the withdrawal of water from storage during the period of pumping. Water is taken from storage until an approximate equilibrium is reached. Thus for a time the flow through successive concentric cylindrical sections around the pumped well will not be equal—that is, the flow through a large cylinder will be less than the flow through a small cylinder, because a part of the ground water that percolates through the small cylinder is derived from storage between the two cylinders. The volume of material between any two cylinders that is unwatered in a given time can, of course, be computed from the

[15] Thiem, G., Hydrologische Methoden, Leipzig, 1906.

draw-down of the water table as shown by successive measurements of the depth to water in observation wells. The average hydraulic gradient that causes the water to percolate toward the pumped well can be determined from the same records of depth to water, provided the altitude of the tops of the observation wells is known. The total quantity of water that percolates through each of the two cylinders in the given time is computed by use of the formula $Q=PiA$. The difference between these quantities represents the volume of ground water taken from storage between the two cylinders. The specific yield of the water-bearing material is then determined by dividing this volume by the volume of material unwatered in the same time.

DEVELOPMENT OF THIEM'S FORMULA

Thiem's formula for computing the coefficient of permeability may be written in the convenient form

$$P=\frac{527.7q\ \log_{10}\frac{a_1}{a}}{m(s-s_1)} \qquad (4)$$

in which P=the coefficient of permeability as defined on page 5;

q=rate of pumping, in gallons a minute;

a and a_1=distances of two observation wells from the pumped well, in feet;

m (for artesian conditions)=vertical thickness of water-bearing bed, in feet;

m (for water-table conditions)=average vertical thickness (at a_1 and a) of the saturated part of the water-bearing bed, in feet;

s and s_1=draw-downs at the two observation wells, in feet.

Thiem assumed a region where the water table or piezometric surface had an initial slope or hydraulic gradient before pumping began. His final formula did not contain a factor involving this slope, and he concluded that an initial slope of the water surface had no effect on the coefficient of permeability as computed by his formula. A review of the development of his formula indicates that during the development his original system of oblique coordinates was changed to a system of rectangular coordinates, which eliminated the factor involving the hydraulic gradient, and the resultant formula theoretically pertains only to regions where the water table or piezometric surface is horizontal. The following is Thiem's development, with some added interpretation, for water-table conditions:

A water-bearing bed of uniform permeability is assumed to rest on a relatively impervious formation, as indicated in figure 1. Water moves through the bed under a normal hydraulic gradient that is parallel to the slope of the underlying impervious bed. In this ground-water stream there is a well equipped with a pump extending to the bottom of the water-bearing material, and two observation wells are placed in line with the pumped well. The pump is operated at a uniform rate during a period in which the water table declines and takes a form somewhat similar to an inverted cone around the pumped well. The draw-down (decline of the water surface) in each observation well, the distances of these wells from the pumped well, the rate of discharge of the pumped well, and the thickness of the water-bearing bed are measured. The coefficient of permeability is computed by substituting these measurements in Thiem's formula.

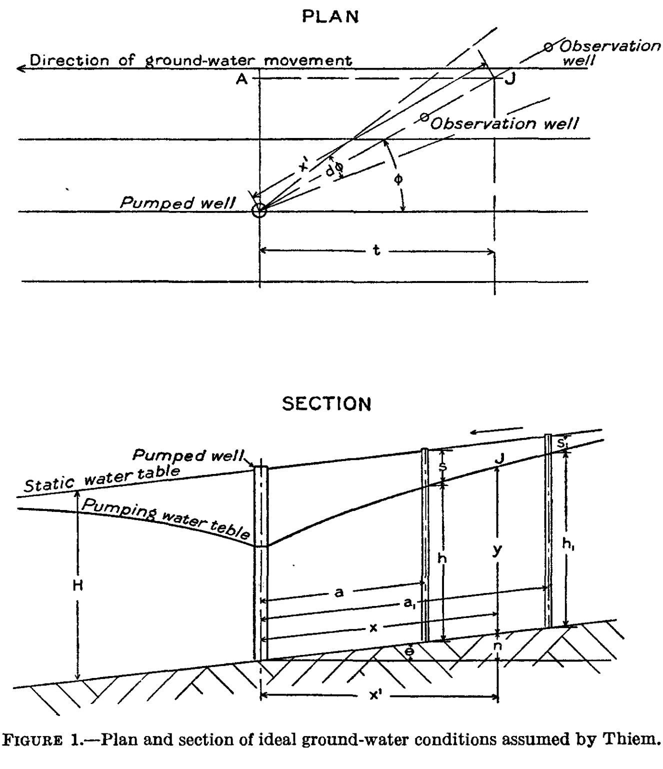


FIGURE 1.—Plan and section of ideal ground-water conditions assumed by Thiem.

The following symbols, in addition to those previously given, are used in the development of the formula:

I=natural hydraulic gradient;

A=cross-sectional area in square feet—that is, area of any designated cylindrical section through which the water percolates on its way to the pumped well;

H=thickness, in feet, of the saturated part of the water-bearing bed in the undisturbed condition of the water table;

Q=rate of pumping, in gallons a day;

x and y=oblique coordinates of a point, J, on the cone of depression, with reference to the point of intersection of the impermeable bottom of the formation with the axis of the well as the origin;

i=hydraulic gradient, in feet per foot, at any point, J, on the cone of depression;

x^1=distance, in feet, of point J from the pumped well;

ϕ=the angle that a line through the pumped well and the two observation wells makes with the uninfluenced direction of ground-water movement;

Θ=the angle of inclination of the impermeable bottom;

t=distance, in feet, of the projection of point J from the pumped well measured along the uninfluenced direction of the movement of the ground water.

In the ground plan, figure 1, assume a small sector with an angle $d\phi$ whose apex is at the axis of the well and whose sides form an angle ϕ with the uninfluenced direction of ground-water flow. At the point J, at a distance x from the well, the flow dQ passes through the sector $d\phi$. By using Darcy's equation, $Q=PiA$ (1), it is possible to compute the flow dQ through the sector $d\phi$ at the point J.

The length of the arc at $J=x \cos \Theta d\phi$. The vertical thickness of the saturated water-bearing material is y. So the area through which the flow dQ passes is equal to $xy \cos \Theta d\phi$.

The hydraulic gradient, i, at any point on the cone of depression is equal to the rate of change of the coordinates. These coordinates must be at right angles, so the horizontal coordinate is x^1 and the vertical coordinate is $(n+y)$.

$$x^1=x \cos \Theta \qquad (5)$$

and

$$n=x \sin \Theta \qquad (6)$$

thus

$$i=\frac{d(n+y)}{dx^1}=\frac{d(x \sin \Theta+y)}{d(x \cos \Theta)} \qquad (7)$$

By substituting in Darcy's equation,

$$dQ=\frac{Pd(x \sin \Theta+y)xy \cos \Theta d\phi}{d(x \cos \Theta)} \qquad (8)$$

From the plan and section (fig. 1)

$$t=x \cos \Theta \cos \phi \qquad (9)$$

The natural hydraulic gradient is equal to $\frac{n}{t}$, hence

$$t=\frac{n}{I} \qquad (10)$$

By equating (9) and (10),

$$n=xI \cos \Theta \cos \phi \qquad (11)$$

$$n=x \sin \Theta \qquad (12)$$

Equating (11) and (12), we get

$$x \sin \Theta = xI \cos \Theta \cos \phi \quad \text{(13)}$$

and

$$\sin \Theta = I \cos \Theta \cos \phi \quad \text{(14)}$$

When both sides of equation 14 are squared,

$$\sin^2 \Theta = I^2 \cos^2 \Theta \cos^2 \phi \quad \text{(15)}$$

$1-\sin^2 \Theta$ may be substituted for $\cos^2 \Theta$; thus equation 15 becomes

$$\sin^2 \Theta = I^2(1-\sin^2 \Theta) \cos^2 \phi \quad \text{(16)}$$

$$= I^2 \cos^2 \Theta - I^2 \sin^2 \Theta \cos^2 \phi \quad \text{(17)}$$

and

$$I^2 \cos^2 \phi = \sin^2 \Theta + I^2 \sin^2 \Theta \cos^2 \phi \quad \text{(18)}$$

$$= \sin^2 \Theta (1 + I^2 \cos^2 \phi) \quad \text{(19)}$$

Thus

$$\frac{\sin^2 \Theta}{I^2 \cos^2 \phi} = \frac{1}{1 + I^2 \cos^2 \phi} \quad \text{(20)}$$

and, by taking the square root,

$$\frac{\sin \Theta}{I \cos \phi} = \frac{1}{\sqrt{1 + I^2 \cos^2 \phi}} \quad \text{(21)}$$

Equation 14 may be written

$$\frac{\sin \Theta}{I \cos \phi} = \cos \Theta \quad \text{(22)}$$

and by equating 21 and 22,

$$\cos \Theta = \frac{1}{\sqrt{1 + I^2 \cos^2 \phi}} \quad \text{(23)}$$

$1-\cos^2 \Theta$ may be substituted for $\sin^2 \Theta$; thus equation 15 may be written

$$1 - \cos^2 \Theta = I^2 \cos^2 \Theta \cos^2 \phi \quad \text{(24)}$$

$$1 = \cos^2 \Theta + I^2 \cos^2 \Theta \cos^2 \phi \quad \text{(25)}$$

$$1 = \cos^2 \Theta \; (I^2 \cos^2 \phi + 1) \quad \text{(26)}$$

and

$$\frac{1}{I^2 \cos^2 \phi + 1} = \cos^2 \Theta \quad \text{(27)}$$

By taking the square root,

$$\frac{1}{\sqrt{I^2 \cos^2 \phi + 1}} = \cos \Theta \quad \text{(28)}$$

and by multiplying both sides of equation 28 by $\cos \phi$ I,

$$\frac{\cos \phi I}{\sqrt{I^2 \cos^2 \phi + 1}} = \cos \Theta \cos \phi I \quad \text{(29)}$$

Equation 14 may be substituted in equation 29, thus:

$$\frac{\cos \phi I}{\sqrt{I^2 \cos^2 \phi + 1}} = \sin \Theta \quad \text{(30)}$$

Equations 28 and 30 may be substituted in equation 8, thus:

$$dQ=\frac{\left[Pd\left(\frac{x \cos \phi I}{\sqrt{I^2 \cos^2 \phi+1}}+y\right)\right][xyd\phi]}{\left[d\left(\frac{x}{\sqrt{I^2 \cos^2 \phi+1}}\right)\right][\sqrt{I^2 \cos^2 \phi+1}]} \tag{31}$$

Thiem states at this point that I^2 is very small and therefore can be assumed to be zero, thus introducing a small error. If $I^2=0$, equation 31 becomes

$$dQ=\frac{Pd(x \cos \phi I+y)xyd\phi}{dx} \tag{32}$$

$$=Pxyd\phi\left(\cos \phi I+\frac{dy}{dx}\right) \tag{33}$$

$$=PxyI \cos \phi d\phi+Pxy \frac{dy}{dx} d\phi \tag{34}$$

In order to integrate equation 34, Thiem changed from an oblique system of coordinates to a rectangular system of coordinates. Thus $I=0$, $n=0$, and $\Theta=0$, and Thiem's final equation will pertain only to horizontal water-table conditions. Equation 34 then becomes

$$dQ=Pxy \frac{dy}{dx} d\phi \tag{35}$$

By integrating with respect to ϕ and Q,

$$\int dQ=Pxy\frac{dy}{dx}\int_0^{2\pi} d\phi \tag{36}$$

and

$$Q=2\pi Pxy\frac{dy}{dx} \tag{37}$$

If the equation is now further integrated with respect to x and y,

$$\int\frac{dx}{x}=\frac{2\pi P}{Q}\int ydy \tag{38}$$

$$\log_e x=\frac{2\pi Py^2}{2Q}+C \tag{39}$$

and

$$y^2=\frac{Q \log_e x}{\pi P}+C \tag{40}$$

This is Thiem's general equation. If equation 37 is integrated between limits $x=a$, $x=a_1$, and $y=h$, $y=h_1$ (fig. 1, section) equation 40 is developed into a more practical form. From equation 37

$$\int_a^{a_1}\frac{dx}{x}=\frac{2\pi P}{Q}\int_h^{h_1} ydy \tag{41}$$

$$\left[\log_e x\right]_a^{a_1} = \frac{2\pi P}{Q}\left[\frac{y^2}{2}\right]_h^{h_1} \text{ - - - - - - - - - - - - - - - - } (42)$$

$$\log_e a_1 - \log_e a = \frac{2\pi P}{Q}\left(\frac{h_1^2}{2} - \frac{h^2}{2}\right) \text{ - - - - - - - - - - - - - - } (43)$$

$$\log_e a_1 - \log_e a = \frac{\pi P(h_1^2 - h^2)}{Q} \text{ - - - - - - - - - - - - - - - } (44)$$

and

$$P = \frac{Q(\log_e a_1 - \log_e a)}{\pi(h_1^2 - h^2)} \text{ - - - - - - - - - - - - - - - - - } (45)$$

$$h_1^2 - h^2 = (h_1 + h)(h_1 - h) \text{ - } (46)$$

and $(h_1 - h)$ is equal to the difference of draw-downs $(s - s_1)$. Thus

$$h_1^2 - h^2 = (h_1 + h)(s - s_1) \text{ - - - - - - - - - - - - - - - - } (47)$$

and equation 45 becomes

$$P = \frac{Q\ (\log_e a_1 - \log_e a)}{\pi(h_1 + h)(s - s_1)} \text{ - - - - - - - - - - - - - - - - - } (48)$$

This is Thiem's final equation and applies to regions where the ground water is not confined below an impermeable bed. Thiem developed a formula that applies to artesian conditions in the same manner. His artesian formula differs from equation 48 only in that $(h_1 + h)$ is replaced by $2m$. If m is used as defined in this paper (p. 10), the equation for both water-table and artesian conditions may be expressed

$$P = \frac{Q\ (\log_e a_1 - \log_e a)}{2\pi m(s - s_1)} \text{ - - - - - - - - - - - - - - - - - } (49)$$

Equation 49 includes factors involving natural logarithms—that is, logarithms with base e. It is developed to the more convenient form given on page 10 in the following manner:

From equation 49

$$P = \frac{Q \log_e \frac{a_1}{a}}{2\pi m(s - s_1)} \text{ - - - - - - - - - - - - - - - - - - - } (50)$$

$\log_e x = 2.30259 \log_{10} x$

thus

$$P = \frac{2.30259\, Q\, \log_{10} \frac{a_1}{a}}{2\pi m(s - s_1)} \text{ - - - - - - - - - - - - - - - - - } (51)$$

If the rate of pumping is expressed in gallons a minute, the equation becomes

$$P = \frac{2.30259\left(1{,}440q \log_{10}\frac{a_1}{a}\right)}{2\pi m(s - s_1)} = \frac{527.7q \log_{10}\frac{a_1}{a}}{m(s - s_1)} \text{ - - - - - - - - } (4)$$

which is Thiem's formula in modified form, for convenient use in the United States.

Thiem's formula may be developed more simply by starting with the assumption of a horizontal water table or piezometric surface (fig. 2). For water-table conditions the demonstration is as follows:

$$Q = PiA \qquad (1)$$

i, at any point on the cone of depression, is equal to the slope, or $\frac{dy}{dx}$, and the increment area through which the flow dQ moves is equal to $xyd\phi$. Therefore

$$dQ = P\frac{dy}{dx}xyd\phi \qquad (52)$$

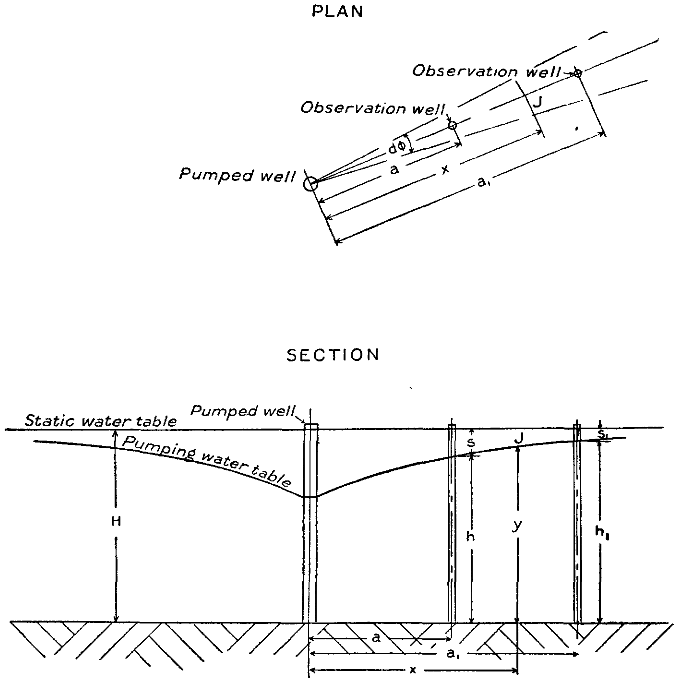


FIGURE 2.—Plan and section showing assumed ground-water conditions for the development of the formula from horizontal water table.

With horizontal conditions, x and y are independent of ϕ; hence ϕ may be integrated independently of x and y, and equation 52 becomes

$$Q = P\frac{dy}{dx}xy\Big[\phi\Big]_0^{2\pi} \qquad (53)$$

and

$$Q = 2\pi Pxy\frac{dy}{dx} \qquad (54)$$

By integrating x and y,

$$\int \frac{dx}{x} = \frac{2\pi P}{Q} \int y dy \text{ --------------------- (55)}$$

$$\log_e x = \frac{2\pi P}{2Q} y^2 + C \text{ --------------------- (56)}$$

and

$$y^2 = \frac{Q \log_e x}{\pi P} + C \text{ --------------------- (57)}$$

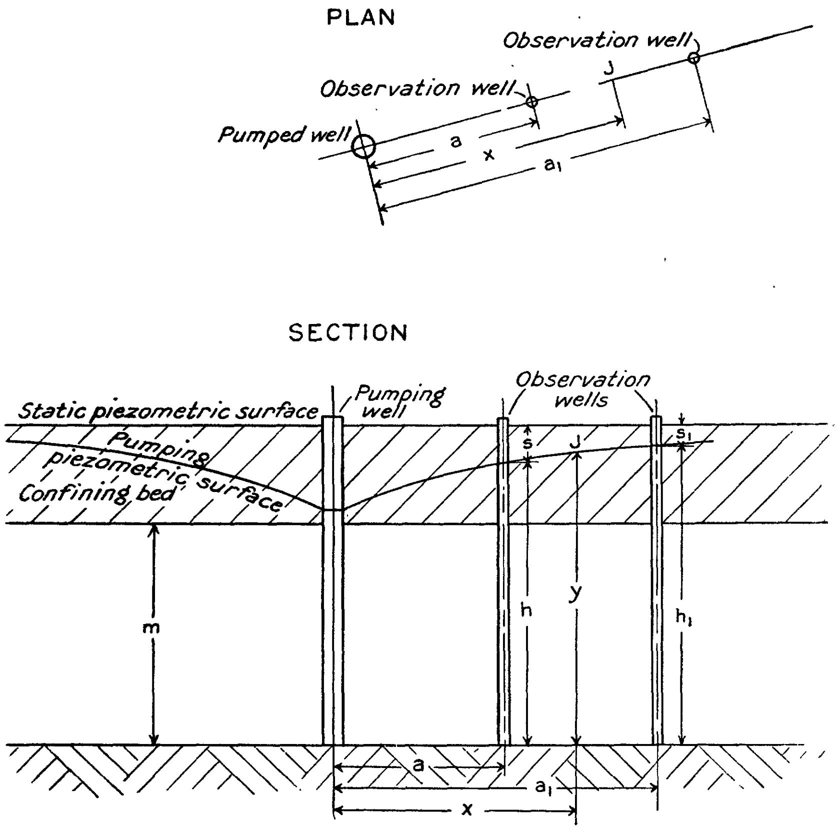


FIGURE 3.—Plan and section showing assumed ground-water conditions for the development of the formula from horizontal artesian conditions.

Equation 57 is identical with equation 40 obtained by Thiem. The formula for computing the coefficient of permeability may be developed from equation 57 in the same manner as it was from equation 40.

Thiem's formula for artesian conditions can be developed in a manner similar to the one that applies to water-table conditions (fig. 3).

The demonstration is as follows, again beginning with

$$Q=PiA \qquad (1)$$

The hydraulic gradient at any point on the cone of depression is equal to $\frac{dy}{dx}$. The total area through which the flow, Q, passes is $2\pi xm$, where the water bed is horizontal. Therefore

$$Q=2\pi Pxm\frac{dy}{dx} \qquad (58)$$

and

$$\frac{dx}{x}=\frac{2\pi Pmdy}{Q} \qquad (59)$$

By integrating,

$$\int_a^{a_1}\frac{dx}{x}=\frac{2\pi mP}{Q}\int_h^{h_1}dy \qquad (60)$$

$$\log_e a_1-\log_e a=\frac{2\pi mP}{Q}(h_1-h) \qquad (61)$$

and

$$P=\frac{Q(\log_e a_1-\log_e a)}{2\pi m(h_1-h)}=\frac{Q\log_e\frac{a_1}{a}}{2\pi m(s-s_1)} \qquad (62)$$

Equation 62 is developed to equation 4 in the same way that equation 4 is obtained from equation 50.

CONFIRMATION OF THIEM'S FORMULA FROM OTHER WORK DONE IN THE UNITED STATES

The theoretical work by Slichter and by Turneaure and Russell, done several years before Thiem's paper was published, is briefly described below, for the purpose of showing that Thiem's formula can be deduced from their results. Thus Thiem's formula is given essential confirmation by these eminent hydrologists in the United States, for the particular conditions to which it applies—namely, a homogeneous water-bearing material, an original horizontal water table or piezometric surface, and an original uniform thickness of the saturated part of the water-bearing formation. Thiem's formula has essentially been derived by the others, but because of differences in their use of symbols and in the final form of their formulas, these formulas have not generally been recognized as being different manners of expressing identical conclusions. Thiem's outstanding contribution was in the application that he made of his formula for determining permeability.

The following discussion outlines the development of Slichter's formula and points out its relation to Thiem's formula for artesian conditions.[16] Slichter starts with the assumption of a homogeneous water-bearing material overlain and underlain by impervious material, and an artesian well that completely penetrates the material.

[16] Slichter, C. S., Theoretical investigation of the motion of ground waters: U. S. Geol. Survey 19th Ann. Rept., pt. 2, p. 359, 1899.

The following nomenclature and units are changed somewhat from that of Slichter to correspond to the nomenclature and units used previously in this paper:

m, thickness of water-bearing material in feet (Slichter's a):

K, a constant defined by Slichter as "the quantity of water that would be transmitted in unit time through a cylinder of stone of unit length and cross section, under unit difference in head at the ends";

v, velocity of the ground water, in feet per day;

r, radius of the well, in feet;

Q, rate of discharge of the pumped well, in gallons a day;

h, amount of lowering of water in well by pumping, in feet;

J, a point on the cone of depression;

x, distance of point J from the axis of the well, in feet;

Z, pressure at point J, in feet of water (Slichter's p);

R, distance, in feet, from the wall of the well at which the pressure may be assumed to be equal to its normal value (that is, $Z=0$ when $x=R+r$).

The velocity at point J at distance x from the axis is given by

$$v=\frac{KdZ}{dx} \tag{63}$$

The velocity varies inversely with the distance from the axis of the well, so

$$v=\frac{c}{x} \tag{64}$$

in which c is a constant to be determined. After equating 63 and 64,

$$\frac{cdx}{x}=KdZ \tag{65}$$

From which

$$c \log_e x=KZ+C_1 \tag{66}$$

When $x=r$, $Z=h$, and when $Z=0$, $x=R+r$; thus

$$c \log_e (R+r)=C_1 \tag{67}$$

and

$$c \log_e r-Kh=C_1 \tag{68}$$

By equating 67 and 68,

$$c \log_e (R+r)=c \log_e r-Kh \tag{69}$$

Therefore

$$c=\frac{Kh}{\log_e\left(\frac{R+r}{r}\right)} \tag{70}$$

and by substituting equation 70 in 64,

$$v=\frac{Kh}{x \log_e\left(\frac{R+r}{r}\right)} \tag{71}$$

The velocity at the wall of the well is found by placing $x=r$. An expression for the total amount of water flowing into the well in unit time is obtained by multiplying the velocity at the wall of the well by $2\pi rm$. Therefore

$$Q=\frac{2\pi hKm}{\log_e\left(\frac{R+r}{r}\right)} \qquad (72)$$

At this point Slichter solved equation 72 for Q, having determined K by means of a previously developed formula that depends upon the laboratory analysis of the water-bearing material for the effective size of the sand grains. However, equation 72 can be converted to Thiem's formula for permeability by proper substitution. Solving equation 72 for K, we get

$$K=\frac{Q\log_e\left(\frac{R+r}{r}\right)}{2\pi hm} \qquad (73)$$

K, as previously defined, is really a coefficient of permeability. Therefore the symbol P may be substituted for K, giving

$$P=\frac{Q\log_e\left(\frac{R+r}{r}\right)}{2\pi hm} \qquad (74)$$

$(R+r)$ corresponds to the distance a_1 in Thiem's formula, and r corresponds to the distance a. Equation 74 can then be written

$$P=\frac{Q\log_e\frac{a_1}{a}}{2\pi hm} \qquad (75)$$

The term h is equal to the draw-down at the pumped well, at the distance r from the axis of the well. The draw-down at the distance $(R+r)$ was assumed by Slichter to be zero. Therefore, h represents the difference in draw-downs between the two points on the cone of depression r and $(R+r)$ and is equivalent to Thiem's term $(s-s_1)$. Substituting in equation 75, we have

$$P=\frac{Q\log_e\frac{a_1}{a}}{2\pi m(s-s_1)}=\frac{Q(\log_e a_1-\log_e a)}{2\pi m(s-s_1)} \qquad (76)$$

Thiem's final formula for artesian conditions is identical with formula 76.

Turneaure and Russell [17] published the development of a formula which is similar to the simple development from horizontal water-table conditions given on page 16. Using Darcy's law as a basis for their development, they arrived at essentially the same equation as 57, with the exception that the factor p (porosity of the water-

[17] Turneaure, F. E., and Russell, H. L., Public water supplies, 1st ed., p. 269, John Wiley & Sons, Inc., 1901.

bearing material) is included. As given by Turneaure and Russell, the equation is

$$Q \log_e x = \pi K p y^2 + c \text{ ------------------ } (77)$$

At this point c is evaluated by substituting $x=r$ (radius of the well) and $y=h$ (saturated thickness of water-bearing material at the wall of the well). Then

$$c = Q \log_e r - \pi K p h^2 \text{ ------------------ } (78)$$

and by substituting in equation 77

$$y^2 = \frac{Q \log_e \frac{x}{r}}{\pi K p} + h^2 \text{ ------------------ } (79)$$

If in equation 79 the value of x is taken to be R, or the distance from the axis of the well at which the change in water level is inappreciable, the corresponding value of y will be H, the original depth of water, and equation 79 will become

$$H^2 = \left(\frac{Q}{\pi K p}\right)\left(\log_e \frac{R}{r}\right) + h^2 \text{ ------------------ } (80)$$

and

$$Q = \frac{(H^2 - h^2)(\pi K p)}{\log_e \frac{R}{r}} \text{ ------------------ } (81)$$

The product Kp corresponds to Thiem's coefficient of permeability, P. Hence

$$Q = \frac{(H^2 - h^2)\pi P}{\log_e \frac{R}{r}} \text{ ------------------ } (82)$$

In equation 82, H and h represent the thicknesses of the saturated part of the water-bearing bed at R and r, respectively. In Thiem's formula h_1 and h represent the thicknesses of the saturated part of the water-bearing bed at a_1 and a, respectively. The characters used in Thiem's formula may be substituted in equation 82, and that equation then becomes

$$Q = \frac{(h_1^2 - h^2)\pi P}{\log_e \frac{a_1}{a}} \text{ ------------------ } (83)$$

and

$$P = \frac{Q \log_e \frac{a_1}{a}}{\pi (h_1^2 - h^2)} = \frac{Q\ (\log_e a_1 - \log_e a)}{\pi (h_1 + h)(s - s_1)} \text{ ------------ } (84)$$

This equation is Thiem's final formula 48, for computing the coefficient of permeability from water-table conditions.

As shown above, there is little difference between the formulas of Slichter, Turneaure and Russell, and Thiem. The principal variance occurs in that Thiem determined the coefficient of permeability, whereas the others determined the quantity of water entering the well and obtained the coefficient of permeability from laboratory analyses of the water-bearing material. Thiem's formula includes the draw-down of the water level in observation wells at two definite and measurable points on the cone of depression, but the formulas of the others contain the draw-down at the more indefinite points r and R. The draw-down at the wall of the well at a distance r from the axis of the well has usually been taken to be the water level in the well while pumping was in progress. This sometimes introduces a large error, because a part of the draw-down in the pumped well is caused by the loss of head of the water as it enters the well. Moreover, the texture of water-bearing material, if it is sand or gravel, is likely to be disturbed for several feet around a pumped well by the development of the well, and therefore the effective diameter r may be considerably larger than the nominal diameter of the well.

The formulas of Slichter and of Turneaure and Russell include the determination of the radius of the cone of depression, R. Slichter assumed this distance to be 600 feet, and Turneaure and Russell determined it with a formula derived by the following reasoning: "Assuming that all the water in the circle of influence flows into the well, the width of the strip of the ground-water stream tributary to the well will be $2R$, and the original cross section of this portion of the ground-water stream is $2RH$." Then from formula 1, $Q=PI(2RH)$ and $R=\frac{Q}{2PIH}$. By substituting the value of Q from equation 82 the formula, after reduction, becomes

$$R=\frac{\pi(H^2-h^2)}{2IH\log_e\frac{R}{r}} \qquad (85)$$

This formula involves the draw-down in the pumped well and the radius of the well, and therefore it is subject to the difficulties previously enumerated in ascertaining these items. It is certain that R is rather difficult to determine, and under some conditions it has been known to exceed 5,000 feet.

Recently the results of laboratory experiments on the flow of water through sand by Wyckoff, Botset, and Muskat [18] were published. They constructed a small apparatus in which the ground-water conditions around a pumped well were reproduced. They observed the draw-downs of the water table and piezometric surface at several distances from the well under various rates of flow. A formula was prepared from the data obtained from these experiments, by which the flow into a well could be computed from a knowledge

[18] Wyckoff, R. D., Botset, H. G., and Muskat, M., Flow of liquids through porous media under the action of gravity: Physics, vol. 3, no. 2, pp. 90–113, August 1932.

of the permeability of the water-bearing material and the saturated thicknesses of the material at two points on the cone of depression. The formula with the nomenclature altered to correspond to usage in this report is

$$Q=\frac{\pi K\rho g(h_1^2-h^2)}{\log_e \frac{a_1}{a}} \qquad (86)$$

in which K is a coefficient of permeability, ρ is the density of the fluid, g is the acceleration of gravity, and h and h_1 are the fluid pressures at the respective distances a and a_1 from the pumped well. Equation 86 may be written

$$K\rho g=\frac{Q \log_e \frac{a_1}{a}}{\pi(h_1^2-h^2)} \qquad (87)$$

The density of water is essentially 1, and therefore ρ may be regarded as equal to 1. Kg in equation 87 is therefore equivalent to the coefficient of permeability, P, as contained in Thiem's formula. The fluid pressure is probably equivalent for most conditions found in nature to the saturated thickness of the water-bearing materials. Therefore

$$P=\frac{Q \log_e \frac{a_1}{a}}{\pi(h^2_1-h^2)} \qquad (88)$$

which may then be reduced to equation 48 (p. 15).

FORMULA FOR DETERMINING THE CONE OF DEPRESSION

A formula can be developed for the cone of depression from Thiem's formula, provided the conditions are the same as those assumed in developing Thiem's formula for artesian conditions—the water-bearing material is homogeneous and of uniform thickness, the ground water is confined between horizontal impermeable formations, and the piezometric surface is horizontal before pumping is started (fig. 3).

From Thiem's modified formula (equation 4, p. 15)

$$s=\frac{527.7q \log_{10}\frac{a_1}{a}}{mP}+s_1 \qquad (89)$$

Let the quantity $\frac{527.7q}{mP}$ be represented by B, a constant. Then

$$s=B \log_{10}\frac{a_1}{a}+s_1 \qquad (90)$$

which is the equation for the cone of depression pertaining to artesian conditions. The draw-down, s, at any point on the cone of depression can be computed by substituting the corresponding figure for a, the distance from the pumped well at which the draw-down occurs.

The equation for the cone of depression pertaining to water-table conditions is developed as follows from equation 4:

$$P=\frac{527.7q \log_{10}\frac{a_1}{a}}{m(s-s_1)}=\frac{1055.4q \log_{10}\frac{a_1}{a}}{(h_1^2-h^2)} \qquad (91)$$

or

$$h^2=h_1^2-\frac{1055.4q \log_{10}\frac{a_1}{a}}{P} \qquad (92)$$

Let the quantity $\frac{1055.4q}{P}$ be represented by F, a constant. Then

$$h^2=h_1^2-F \log_{10}\frac{a_1}{a} \qquad (93)$$

h^2 is equal to $(H-s)^2$ (fig. 2); therefore

$$(H-s)^2=h_1^2-F \log_{10}\frac{a_1}{a} \qquad (94)$$

and

$$s=H-\sqrt{h_1^2-F \log_{10}\frac{a_1}{a}} \qquad (95)$$

The draw-down, s, at any point on the cone of depression can be computed by substituting the corresponding figure for a, the distance from the pumped well at which the draw-down occurs.

The slope of the cone of depression at any point may also be computed from Thiem's formula. Starting with the formula as stated in equation 49 (p. 15),

$$s-s_1=\frac{Q(\log_e a_1-\log_e a)}{2\pi mP} \qquad (96)$$

Let the quantity $\frac{Q}{2\pi mP}$ be represented by E, a constant. Then

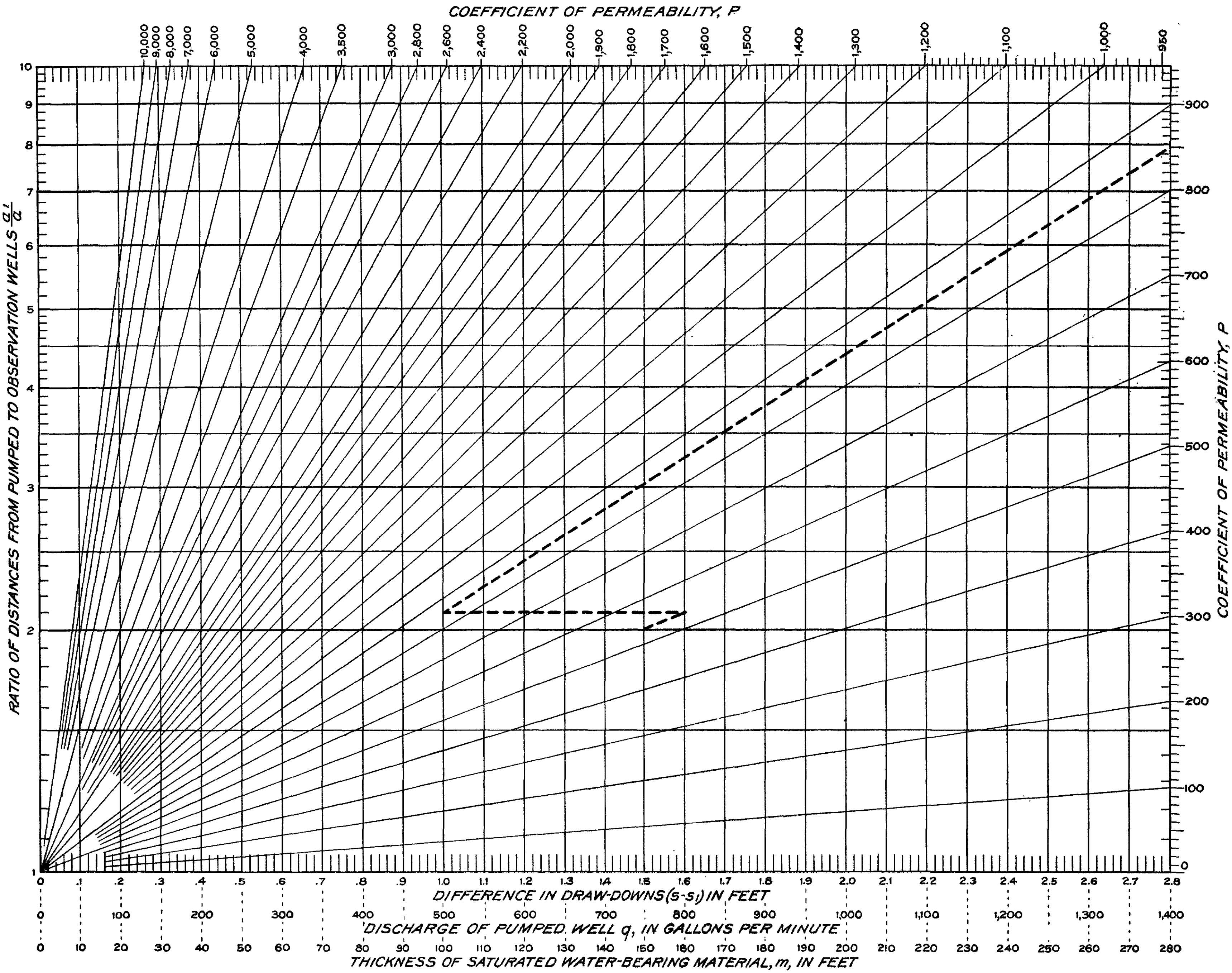
$$s-s_1=E(\log_e a_1-\log_e a) \qquad (97)$$

If a_1 is a fixed point in any given pumping test, then $\log_e a_1$ is a constant in that test. Therefore the quantity $E \log_e a_1$ is also a constant and may be represented by L. Then

$$s-s_1=L-E \log_e a \qquad (98)$$

and

$$s=L+s_1-E \log_e a \qquad (99)$$



GRAPH FOR COMPUTING COEFFICIENTS OF PERMEABILITY.

Subtracting both sides of equation 99 from h, we have

$$h-s=E\log_e a-L+h-s_1 \text{ --------------- } (100)$$

By substituting the general factors $(h-s)=y$ and $a=x$ in equation 100 (fig. 3), we get

$$y=E\log_e x-L+h-s_1 \text{ ---------------- } (101)$$

and by differentiating with respect to y and x,

$$dy=\frac{Edx}{x} \text{ ---------------------- } (102)$$

The slope of the cone of depression is equal to dy/dx. Therefore,

$$\frac{dy}{dx}=i=\frac{E}{x} \text{ --------------------- } (103)$$

and

$$i=\frac{Q}{2\pi Pmx} \text{ -------------------- } (104)$$

In a similar manner it can be shown that for water-table conditions the slope of the cone of depression at any distance, x, from the pumped well can be computed by the formula

$$i=\frac{Q}{2\pi Pxy} \text{ --------------------- } (105)$$

This formula differs from equation 104 for artesian conditions only in that the thickness of the water-bearing formation, m, is replaced by the thickness of the saturated water-bearing material, y.

The slope of the cone of depression, i, at any point may be computed for both water table and artesian conditions by substituting for x, the distance of the point from the pumped well.

GRAPHIC SOLUTION OF THIEM'S FORMULA

By the use of the graph presented in plate 1 the coefficient of permeability may be determined without the usual computations, if the factors contained in Thiem's formula are known. This graph is particularly useful for determining the effect on the computed coefficient of permeability of changes in the factors in Thiem's formula, and it provides a rapid method for calculating the permeability for several regions and comparing the ground-water conditions of those regions. The graphic solution for permeability is made in the following manner:

Locate the point of intersection of the horizontal line corresponding to $\frac{a_1}{a}$ and of the vertical line corresponding to $(s-s_1)$; move inward or outward along the radial line through this point to the intersection of the line with the vertical line representing the discharge of the well, *q;* move horizontally from this point to the right or left to the vertical line representing the saturated thickness of the formation; move radially outward from this point to either the upper or the right margin of the diagram, where the coefficient of permeability, *P*, is read. As an example, the graphic solution corresponding to the conditions

$$\frac{a_1}{a}=2,\ s-s_1=1.5,\ q=800,\ \text{and}\ m=100$$

is shown in plate 1 by a heavy dashed line. The coefficient of permeability so determined is approximately 850.

PUMPING TESTS IN NEBRASKA

Two pumping tests were made near Grand Island, Nebr., during the summer of 1931 on the farm of Fred Meyer, about 4 miles east of Grand Island, in the NW¼ sec. 17, T. 11 N., R. 8 W. This location was selected after a thorough inventory of existing irrigation wells in the vicinity, as the one that most nearly approached the ideal conditions desired for the pumping tests. The irrigation well used for test 1 was in a pasture just west of a large field of corn (well 83, fig. 4). The land near the well was rather flat, although the field of corn was slightly higher than the pasture. There was a dry slough about 800 feet west of the well, but as no drainage had entered it for some time preceding the pumping tests, it probably did not affect the normal level of the ground water. Throughout the area covered by figure 4 the water table ranged only from 2 to 10 feet below the land surface, and hence the sinking of observation wells was not difficult. It is probable that the water table was lowered somewhat during the period of tests by drafts made on the zone of saturation by plants, but the amount of lowering was small, as indicated by the small decline of the water table in those wells located farthest from the pumped well. There were three irrigation wells within a mile of the test wells, but none of them were operated during the tests or for several days before the tests were begun.

Before the pumping tests were made a test hole was drilled near observation well 76 to determine the thickness of the water-bearing materials. Sand and gravel showing a great range in size and some clay were penetrated to a depth of about 110 feet, where bedrock was struck. The hole was continued into the bedrock to a depth of 143 feet below the ground surface. Later a well was drilled about 25 feet south of the existing irrigation well for the second pumping test, and samples of the water-bearing materials penetrated were sent to the hydrologic laboratory of the United States Geological Survey for determinations of porosity, moisture equivalent, and permeability and a mechanical analysis (table 1). A log of the materials encoun-

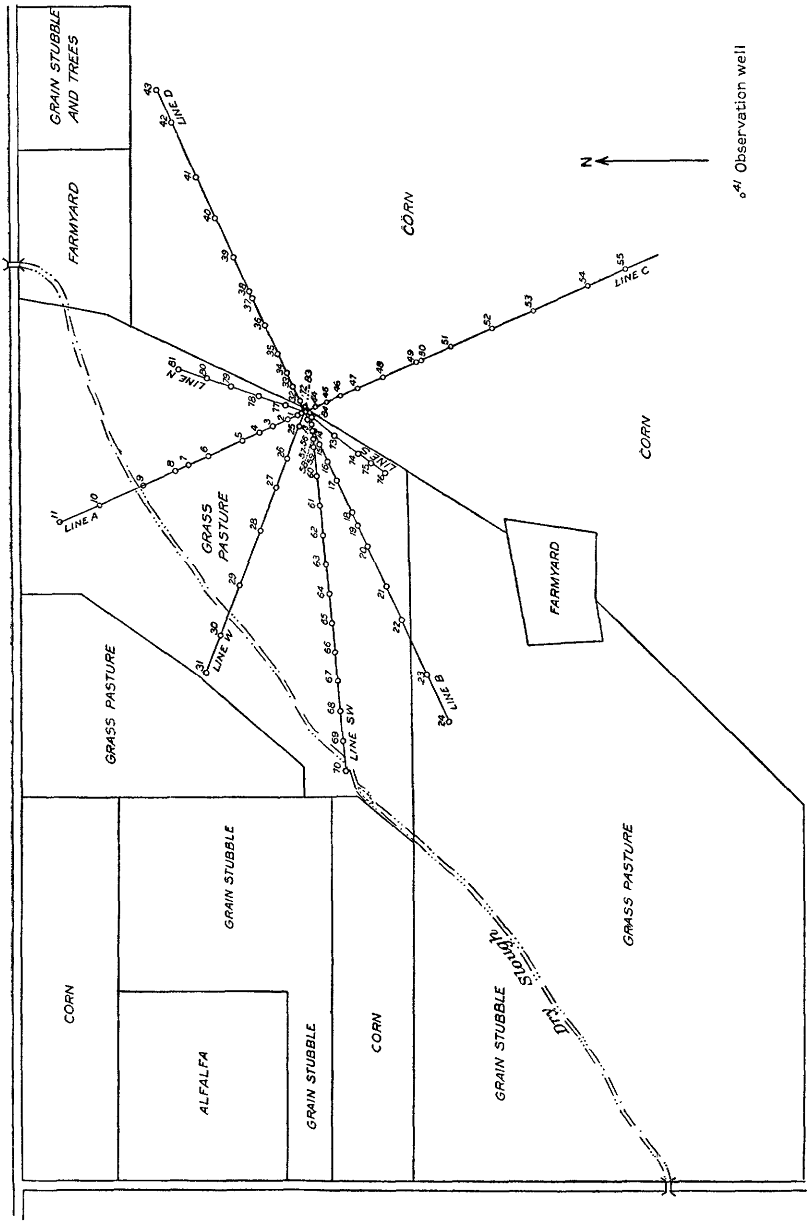


FIGURE 4.—Map showing location of wells used in pumping tests.

tered in this well (84, fig. 4) is given in table 2. This well was 12 inches in diameter and was drilled to a depth of 105 feet; the lower

48 feet and the upper 24 feet of the casing were perforated. The existing irrigation well used for the first pumping test was 24 inches in diameter and 40 feet deep, and all the casing was perforated.

TABLE 1.—*Physical properties of samples of alluvium taken from well 84, near Grand Island, Nebr.*

[Determined in the hydrologic laboratory of the U. S. Geological Survey by V. C. Fishel]

Depth (feet)	Mechanical analysis (percent by weight)							
	Larger than 2.0 mm	2.0-1.0 mm	1.00-0.50 mm	0.50-0.25 mm	0.25-0.125 mm	0.125-0.062 mm	0.062-0.005 mm	Less than 0.005 mm
6 to 10	29.7	16.9	18.9	17.1	15.4	1.3	0.4	0.2
10 to 16	14.1	17.9	31.2	30.4	5.5	.3	.2	.1
16 to 20	16.8	15.2	25.8	29.4	10.5	1.6	.5	.1
20 to 25	18.6	18.8	21.3	24.8	13.8	1.9	.6	.2
25 to 30	7.5	17.2	25.0	30.0	16.0	3.4	.8	.3
30 to 39	36.4	20.8	21.4	15.0	4.7	.8	.5	.1
39 to 40	3.4	3.6	1.8	4.7	26.0	14.0	31.5	13.6
40 to 42	15.9	11.0	20.1	33.4	15.4	2.6	.4	.2
42 to 46	15.4	15.2	20.2	19.5	16.4	7.0	4.5	1.5
46 to 51	17.3	10.7	13.1	29.4	24.4	3.2	1.0	.4
51 to 55	39.6	12.8	9.5	15.7	13.5	4.7	2.5	1.0
55 to 61	27.4	14.9	16.3	22.4	11.8	3.7	2.1	1.0
61 to 66	20.6	19.6	19.7	19.1	9.7	4.3	4.4	.9
66 to 71	18.1	18.0	17.7	23.7	14.0	3.3	3.0	1.9
71 to 78	[1] 79.3	3.5	3.9	6.3	4.0	1.5	1.0	.3
78 to 86	14.3	11.9	18.2	25.1	18.7	6.5	3.0	1.7
86 to 92	36.2	10.3	14.6	17.1	11.6	4.3	3.4	2.3
92 to 99	15.1	10.4	22.8	31.1	13.9	3.0	2.5	1.0
99 to 105	25.8	13.3	13.7	21.9	14.3	5.2	3.5	2.0

Depth (feet)	Apparent specific gravity	Porosity (percent)	Moisture equivalent		Coefficient of permeability
			Percent by weight	Percent by volume	
6 to 10	1.90	27.1	1.4	2.6	480
10 to 16	1.84	30.9	1.5	2.7	1,685
16 to 20	1.80	32.3	1.1	2.0	1,460
20 to 25	1.89	28.5	1.4	2.6	1,095
25 to 30	1.83	31.0	1.4	2.6	1,095
30 to 39	1.81	30.6	1.0	1.9	4,350
39 to 40	1.56	40.3	17.4	27.1	2
40 to 42	1.83	31.2	1.5	2.7	925
42 to 46	1.92	26.3	1.6	3.0	150
46 to 51	1.84	30.2	1.6	3.0	350
51 to 55	1.94	26.2	1.7	3.3	780
55 to 61	1.92	25.6	1.6	3.0	730
61 to 66	1.92	25.0	1.4	2.8	2,095
66 to 71	1.94	26.3	1.6	3.0	1,050
71 to 78	2.02	22.8			2,185
78 to 86	1.88	41.8	1.6	3.1	220
86 to 92	1.97	21.5	1.9	3.9	495
92 to 99	1.86	29.9	1.2	2.1	430
99 to 105	1.90	27.6	1.5	2.9	285

[1] 76.0 percent larger than 5 mm.

TABLE 2.—*Log of well 84, drilled for second pumping test*

	Thickness	Depth		Thickness	Depth
	Feet	*Feet*		*Feet*	*Feet*
Top soil	1	1	Sand and gravel	16	71
Sand and gravel	38	39	Coarse gravel	7	78
Clay	1	40	Sand and gravel	8	86
Sand and small gravel	10	50	Fine sand and gravel	6	92
Sand, gravel, and clay	5	55	Medium sand	13	105

A. LINE SW OF 1-INCH OBSERVATION WELLS.

B. MEASURING THE DEPTH TO THE WATER TABLE.

A. WEIR FOR MEASURING THE DISCHARGE OF THE PUMPED WELL.

B. PUMPING ARRANGEMENT IN SECOND TEST.

A transit was set up over well 83, and six radiating lines of wells were laid out. Lines C and D were projections of lines A and B, and line W bisected the 90° angle formed by the intersecting lines A and B. Lines N and S only approximately bisected the angles formed by the intersections of lines A and D and lines B and C, because the topographic features were such that actual bisections would have been difficult. Line SW was laid out from well 84 (pl. 2, *A*).

More than 80 observation wells were sunk, most of them relatively close to the pumped wells, where the decline of the water table during pumping would be the greatest. Some of the observation wells were 1 inch in diameter and were fitted with 18-inch screen drive points. These wells were driven into the saturated sand and gravel to such depths that the water table during pumping would not drop below the bottoms of the wells. Several observation wells 3 inches in diameter were fitted with drilling bits at their lower ends and were jetted down with a drilling rig. Holes in the bits allowed water to enter the wells freely. The diameters and depths of the observation wells are recorded in table 3.

TABLE 3.—*Location, diameter, depth, and altitude of wells used in the pumping tests*

Well no.	Line	Diameter	Depth of well below measuring point	Distance of measuring point above land surface	Altitude of measuring point	Distance from pumped well 83	Distance from pumped well 84
		Inches	*Feet*	*Feet*	*Feet*	*Feet*	*Feet*
1	A	3	21.4	0.7	1,814.34	24.9	42.3
2	A	3	10.3	.1	1,815.66	59.9	74.6
3	A	3	10.1	0	1,815.26	114.4	127.9
4	A	3	10.3	.1	1,814.63	164.2	177.2
5	A	1	11.5	1.4	1,815.83	229.0	241.5
6	A	1	6.5	.5	1,812.35	354.1	366.4
7	A	3	10.2	.2	1,815.39	429.3	441.2
8	A	1	11.4	1.2	1,815.52	478.9	490.4
9	A	3	10.3	.1	1,814.97	604.0	616.1
10	A	3	10.3	.1	1,814.73	754.6	766.5
11	A	3	10.5	.4	1,814.05	903.8	916.2
13	B	3	21.4	.2	1,814.84	29.9	14.2
14	B	3	10.9	.3	1,815.17	70.0	49.6
15	B	1	11.5	1.3	1,816.10	120.0	98.9
16	B	3	9.9	.1	1,815.67	184.9	163.1
17	B	3	9.9	.2	1,815.46	254.7	233.0
18	B	3	10.2	.3	1,815.08	375.3	353.8
19	B	3	10.3	.5	1,815.86	424.6	402.8
20	B	3	10.2	.4	1,816.30	499.7	477.7
21	B	3	9.9	.3	1,816.32	649.7	627.6
22	B	3	9.6	.3	1,816.39	775.3	752.9
23	B	1	11.5	1.9	1,816.95	974.3	951.8
24	B	3	10.6	.1	1,817.12	1,149.3	1,127.0
25	W	3	10.3	.3	1,815.39	49.7	48.8
26	W	3	10.3	.3	1,814.78	170.0	164.3
27	W	3	10.2	.3	1,815.33	270.0	264.0
28	W	3	10.6	.1	1,813.46	430.0	423.4
29	W	3	10.1	.3	1,815.68	625.0	618.0
30	W	3	10.7	.3	1,815.39	804.5	797.6
31	W	3	10.1	.1	1,815.06	939.7	932.5
32	D	1	21.5	1.2	1,820.42	40.1	63.1
33	D	1	16.5	1.0	1,819.17	95.1	117.7
34	D	1	16.5	1.2	1,818.99	144.7	166.9
35	D	1	16.5	1.2	1,818.93	214.3	236.7

TABLE 3.—*Location, diameter, depth, and altitude of wells used in the pumping tests*—Continued

Well no.	Line	Diameter	Depth of well below measuring point	Distance of measuring point above land surface	Altitude of measuring point	Distance from pumped well 83	Distance from pumped well 84
		Inches	*Feet*	*Feet*	*Feet*	*Feet*	*Feet*
36	D	1	16.5	0.9	1,818.31	323.8	345.8
37	D	1	16.5	.9	1,818.27	423.2	445.0
38	D	1	16.5	1.1	1,818.33	448.2	470.2
39	D	1	16.5	1.0	1,818.05	572.9	594.4
40	D	1	16.5	1.2	1,818.83	722.7	744.2
41	D	1	12.7	1.8	1,818.08	872.2	893.6
42	D	1			1,817.19	1,072.5	1,094.2
43	D	1	12.7	.5	1,816.73	1,197.0	1,218.3
44	C	1	23.0	1.0	1,820.12	39.3	35.0
45	C	1	17.2	.5	1,818.37	80.5	71.9
46	C	1	11.1	.5	1,818.02	130.3	120.2
47	C	1	12.7	.5	1,817.37	195.6	185.0
48	C	1	12.6	.8	1,817.90	285.6	274.7
49	C	1	11.4	.4	1,818.36	410.2	398.7
50	C	1	12.3	.5	1,818.30	425.2	413.9
51	C	1	12.4	.4	1,818.80	535.4	524.2
52	C	1	12.7	.7	1,819.45	685.3	673.9
53	C	1	12.6	.4	1,818.04	834.6	822.8
54	C	1	12.7	.5	1,817.99	1,034.7	1,022.9
55	C	1	12.6	.6	1,817.37	1,174.9	1,162.8
56	SW	1	11.0	.5	1,814.97	46.7	25.9
57	SW	1	10.5	.9	1,815.98	69.5	50.7
58	SW	1	10.6	.8	1,815.29	93.6	75.8
59	SW	1	11.0	.7	1,816.18	118.0	100.8
60	SW	1	11.0	.8	1,815.88	216.9	200.7
61	SW	1	10.6	.9	1,816.14	316.6	300.7
62	SW	1	10.9	.9	1,816.19	416.5	400.8
63	SW	1	10.7	.8	1,816.47	516.5	500.9
64	SW	1	10.8	2.1	1,816.93	616.5	600.9
65	SW	1	10.9	2.4	1,817.55	716.5	701.0
66	SW	1	12.5	2.6	1,817.42	816.6	801.2
67	SW	1	10.9	2.4	1,818.39	916.6	901.2
68	SW	1	11.1	2.2	1,816.54	1,016.6	1,001.3
69	SW	1	6.1	1.3	1,815.37	1,116.9	1,101.5
70	SW	1	11.4	2.0	1,817.83	1,217.1	1,201.8
71	A	1			1,812.89	2.6	26.0
72	A	1			1,814.74	12.3	32.2
73	S	1	11.1	.9	1,815.94	130.1	105.3
74	S	1	12.0	.7	1,815.90	225.2	200.3
75	S	1	12.6	.5	1,816.05	279.9	255.1
76	S	1	13.0	2.3	1,817.74	382.7	356.0
77	N	1	6.1	.8	1,813.08	63.2	87.1
78	N	1	12.8	1.7	1,815.32	160.0	183.5
79	N	1	13.0	.7	1,815.48	261.5	285.2
80	N	1	12.3	1.2	1,816.13	342.0	365.7
81	N	1	11.8	1.3	1,816.41	445.8	469.0
82	SW	1	27.0	0	1,815.49	34.8	2.0
83		24	39.5	−.5	1,812.66	0	24.8
84		12	102.0	−3.0	[1] 1,814.90	24.8	0

[1] First test. Altitude for second test 1,812.35 feet.

Each observation well was pumped with a pitcher pump until the water discharged was clear, indicating that the ground water had free access to the well and that the water level in the well showed the level of the water table outside the well. Definite points were established at each well from which measurements of depth to the water level could be made, and the distance of these measuring points above the land surface was recorded. To determine the altitude of the measuring points, instrumental levels were run to all the observation wells and to the two pumped wells (table 3).

Both pumping tests were started early in the morning. During the day preceding each of the tests several measurements were made of the depth to water in the observation wells, in order to determine the static level of the water table, and a few minutes before pumping began additional measurements were made as a check on the measurements of the day before. The measurements were made with a steel tape graduated in hundredths of a foot. The end of the tape was loaded with a swiveled weight, so that the tape would hang plumb, and the lower foot or so of the tape was coated with blue carpenter's chalk, so that the depth of immersion of the tape into the water could be plainly seen. The period of pumping in the first test was about 48 hours, and the average rate of pumping 540 gallons a minute. During the second pumping test the pump was stopped several times because of trouble with the 50-horsepower gasoline engine that was used to drive it. In order to make the two tests as comparable as possible, pumping in the second test was continued a few hours longer than the 48-hour period, so that the total quantity of water pumped was about equal to the quantity pumped during the first test. Records of pumping time are given in table 4.

TABLE 4.—*Record of pumping time*

Well 83 (test 1)	
Started	July 29, 1931, 6:05 a. m.
Stopped	July 31, 1931, 6:04 a. m.
Well 84 (test 2)	
Started	Sept. 9, 1931, 8:05 a. m.
Stopped	11:18 a. m.
Started	11:35 a. m.
Stopped	12:35 p. m.
Started	12.37 p. m.
Stopped	2:00 p. m.
Started	3:38 p. m.
Stopped	5:55 p. m.
Started	6:31 p. m.
Stopped	Sept. 10, 1931, 4:26 a. m.
Started	6:03 a. m.
Stopped	8:57 a. m.
Well 84 (test 2)—Continued	
Started	Sept. 10, 1931, 9:32 a. m.
Stopped	9:36 a. m.
Started	9:38 a. m.
Stopped	9:39 a. m.
Started	9:40 a. m.
Stopped	9:48 a. m.
Started	9:51 a. m.
Stopped	11:17 a. m.
Started	11:19 a. m.
Stopped	11:49 a. m.
Started	11:55 a. m.
Stopped	12:06 p. m.
Started	12:11 p. m.
Stopped	Sept. 11, 1931, 10:28 a. m.
Started	10:34 a. m.
Stopped	2:05 p. m.

Twelve men were employed during the tests to make measurements of the depth to water in the observation wells, to measure the discharge of the pumped wells, and to operate the power unit. The men worked on alternate shifts of 6 hours during the first test and 8 hours during the second test. During each shift three men made measurements of depth to water in the observation wells (pl. 2, *B*). These measurements were continued throughout the night with the aid of lanterns. The water from the pumped wells was discharged into a stilling basin about 25 feet east of the wells (pl. 3, *B*), from which it flowed over a rectangular weir and through a small canal across the field of corn (pl. 3, *A*). The water was then used to irrigate the corn grown in the eastern part of the field.

After the completion of each pumping test measurements of depth to water in the observation wells were continued for at least 24 hours, so that the recovery of the water table could be determined. The measurements of depth to water made during pumping and after pumping had stopped, which number about 9,500, are on file and may be consulted in the office of the United States Geological Survey at Washington.

RESULTS OBTAINED FROM THE PUMPING TESTS

A considerable amount of study has been devoted to the records obtained in the pumping tests above described in an effort to determine the best procedure to be used in future tests for determining permeability by the pumping method. The present tests involved more time and expense than could ordinarily be spent on a field determination of the permeability of a water-bearing formation, and it was with the view of reducing the complexity of the tests that this study was made. The conclusions given below, under the heading "Computations of coefficients of permeability", show that satisfactory results can be obtained by less elaborate tests if certain facts developed in these tests are kept in mind.

The data collected in the tests are so adequate that a detailed study could be made of the behavior of the ground water in the vicinity of a pumped well, both during pumping and after pumping stopped. However, the writer has been able to make only a rather cursory examination of the whole mass of data, and hence the results here presented are not all that could be obtained if a more intensive study were made.

DRAW-DOWN CURVES

To obtain the draw-down of the water table at any time it is only necessary to subtract the depth to the water level before pumping started from the depth at that particular time. The altitude of the water level at any time can be obtained by subtracting the draw-down at that time from the altitude of the normal water level. A continuous curve representing the decline of the water level in a well during the period of pumping is called a "draw-down curve."

Draw-down curves were plotted for many of the observation wells, chiefly for the first pumping test, because the curves for the second test show irregularities caused by interruptions in pumping. The draw-down curves for the first test are remarkably regular. A smooth curve could be drawn through most of the points, and the very few points that plotted far from the curve were obviously caused by errors in making the measurements. The equation for the draw-down curve is not known, but the general form of the curve is shown by typical curves given in plate 4 for wells on line *A*. As that line was approximately at right angles to the natural direction of ground-water movement, the altitude of the static water level was nearly the same in all the wells.

The slope of the draw-down curve indicates the rate at which the water was withdrawn from storage in the sand and gravel. The form of the draw-down curves of the wells close to the pumped well indicates that a large volume of water was withdrawn from storage immediately after pumping began, the amount withdrawn being in general inversely proportional to the distance from the pumped well. The draw-down curves of the wells comparatively far from the pumped well indicate that there was no withdrawal of water from storage at those distances from the pumped well for several hours after pumping began and that the maximum rate of withdrawal of water was not reached for some time after the first water was withdrawn. Obviously this lag was caused by the fact that all the water necessary to supply the pumped well was at first obtained from the sand and gravel nearby. The draw-down curves given in plate 4 show that the water levels in the observation wells were still declining after 48 hours of pumping. In other words, the cone of depression around the pumped well had not reached a condition of equilibrium.

The draw-down curves of observation wells near the pumped well reflected unavoidable changes in the rate of pumping. The pump was stopped several times during the second pumping test, and these shut-downs caused the water levels in nearby observation wells to rise. Typical draw-down curves for the second test, given in plate 5, indicate that water continued to percolate toward the pumped well during the periods of interruption in pumping, and that this water began to refill the sand and gravel that had been unwatered during the periods of pumping. Of course, as soon as pumping was resumed the water table was again lowered. However, this lowering was resumed from a new level—the level caused by the rise of the water table during the period in which there was no pumping—and plate 5 shows that sometimes it took several hours to lower the water level to the point where it stood before pumping was stopped. Only the wells comparatively close to the pumped well showed a rise of water level during the interruptions in pumping. The wells farther away showed a continuous decline of the water table, thus indicating that the water which caused the rise of the water levels in wells close to the pumped well came, at least in part, from storage in the area farther from the pumped well. There were several interruptions of pumping in the second test, and consequently considerable water was taken out of storage at some distance from the pumped well and stored temporarily in the sand and gravel close to the pumped well. This process tended to reduce the draw-down in the wells close to the pumped well, and therefore the coefficients of permeability computed from draw-downs observed in the second test are believed to be greater than the true permeability of the material. For this reason more study has been devoted to the first test, in which pumping was carried on at a nearly constant rate. It is probable that the inequalities in the cone of depression in the second test would have disappeared if pumping had been continued without interruption for several more days.

RECOVERY CURVES

A recovery curve is a continuous curve representing the movement of the water level in an observation well after pumping has stopped. Depending upon the location of the well, the movement after pumping has stopped may be either an immediate rise of the water level or a decline that is eventually followed by a rise. Measurements of the depth to the water level in the observation wells were continued in both tests after pumping had stopped in order to determine the rate and amount of recovery of the water table. Several typical recovery curves are shown in plates 4 and 5 as continuations of the draw-down curves. The recovery curves are usually smoother than the draw-down curves because of the absence of irregularities caused by interruptions in pumping or variations in the rate of pumping. The recovery curves show that in observation wells close to the pumped well the recovery was most rapid immediately after pumping had stopped, whereas in observation wells comparatively far from the pumped well the recovery was most rapid several hours after pumping had stopped.

After pumping stopped, water continued to percolate toward the pumped well under the hydraulic gradient set up during the period of pumping, but instead of being discharged by the well it refilled the interstices in the sand and gravel that had been unwatered by the pumping. As the unwatered sand and gravel was gradually refilled the hydraulic gradient toward the well decreased, and the flow toward the well decreased proportionally. Thus the rate of recovery became progressively slower. The water level in wells comparatively far from the pumped well declined for several hours after pumping stopped. In these areas water continued to be taken from storage to supply the water that refilled the sediments around the pumped well. Of course, in time there was a general equalization of water levels over the entire region, and the water table assumed a form similar to that it had before pumping began. However, the ultimate level of the water table was a little lower than before pumping began, because water had been permanently removed from the zone of saturation during the period of pumping.

The rate of recovery of the water table is in general inversely proportional to the distance from the pumped well. However, this is true only for a short time after pumping ceases. Even though the water table close to the pumped well initially has a greater amount to recover, the rate of rise after a certain time is the same as the rate of rise of the water table at greater distances. After pumping is stopped the water table close to the well rises until the possible amount of recovery remaining—that is, the remaining draw-down—is equal to the remaining draw-down at distances farther from the pumped well. This is specifically shown in table 5. After 48 hours of pumping the decline of the water level in well 1, 24.9 feet from the pumped well, was 4.03 feet, and the decline in well 2, 59.9 feet from the pumped well, was 2.81 feet. After 2 hours of recovery the remaining draw-down in each well was about 1.68 feet, and the rates of recovery for the next 22 hours were the same in the two wells. After 48 hours of pumping the draw-down in well 3, 114.4 feet from the pumped well, was 2.03 feet—about half of the draw-down of well 1. After 12 hours of recovery the remaining draw-down in all three wells was 0.77 foot, and the rates of recovery from that time on were the same. If the measurements of depth to water had been continued longer, the indicated rate of recovery in the wells farther from the pumped well would eventually have been nearly the same as the rate in wells 1, 2, and 3.

TABLE 5.—*Draw-down of the water table during test 1, at several times after pumping stopped*

Well no.	Distance from pumped well (feet)	Draw-down (feet) at time indicated (hours) after pumping stopped												
		0	2	4	6	8	10	12	14	16	18	20	22	24
1	24.9	4.03	1.67	1.32	1.12	0.98	0.86	0.77	0.70	0.64	0.59	0.55	0.51	0.48
2	59.9	2.81	1.69	1.33	1.13	.99	.87	.77	.70	.64	.59	.55	.51	.47
3	114.4	2.03	1.49	1.25	1.17	.94	.85	.77	.70	.64	.59	.55	.51	.48
4	164.2	1.62	1.30	1.11	.98	.88	.80	.72	.67	.60	.56	.52	.49	.46
5	229.0	1.14	1.01	.91	.83	.76	.70	.64	.59	.55	.51	.48	.45	.43
6	354.1	.65	.64	.60	.56	.53	.50	.48	.45	.43	.41	.39	.37	.35
7	429.3	.52	.51	.49	.47	.45	.43	.41	.40	.38	.36	.35	.33	.32
8	478.9	.44	.44	.44	.43	.42	.41	.39	.38	.36	.35	.33	.32	.30
9	604.0	.26	.27	.28	.28	.28	.28	.28	.27	.27	.26	.26	.25	.24
10	754.6	.15	.16	.16	.17	.17	.17	.17	.17	.18	.18	.18	.18	.17
11	903.8	.11	.11	.11	.12	.12	.12	.12	.13	13	.13	.13	.14	.14

Table 5 also illustrates the decline of the water table after pumping ceases in wells comparatively far from the pumped well. Recovery started almost at once in wells 1 to 7, but in well 8 there was a lag of a few hours, and it was 6 hours before the water level reached a point 0.01 foot above its level at the time when pumping stopped. In well 9 there was an actual decline of 0.02 foot during the first 4 hours of recovery, and it was not until 14 hours after pumping had stopped that there was any recovery from this low level. In well 10 there was a decline for 22 hours after pumping had stopped, and in well 11 the water level was apparently still declining after 24 hours. This lag is also shown by the recovery curves in plate 4.

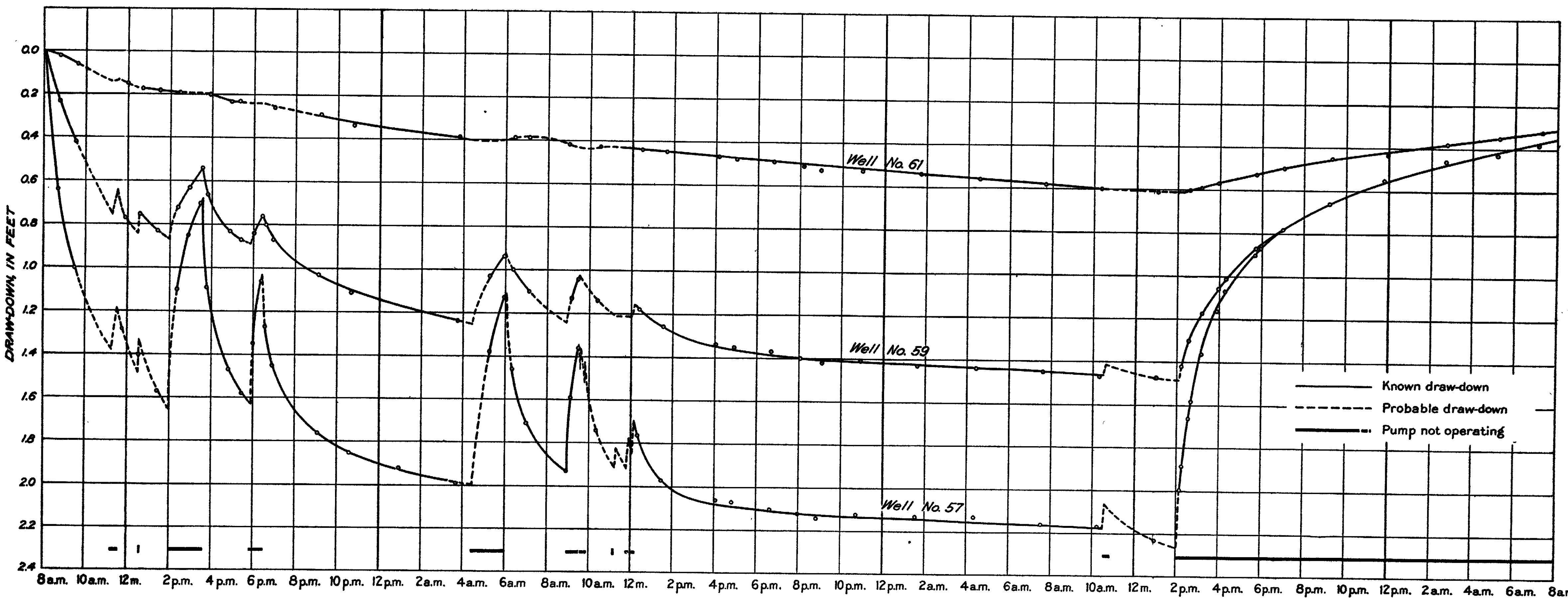
The writer has devoted some time in an attempt to develop an equation for the recovery curves and their relation to the permeability of the water-bearing material. It would seem that the rate of recovery of the water level in an observation well is dependent on the quantity of water pumped, the draw-down of the water level at the time pumping stopped, the distance of the observation well from the pumped well, the initial hydraulic gradient, the thickness of the water-bearing formation, and the permeability of the formation. No equation was found that could be used for the draw-down curves of all the observation wells. However, the following general equation is suitable for many of the curves:

$$R=\frac{D}{\frac{K}{T^n}+1}$$

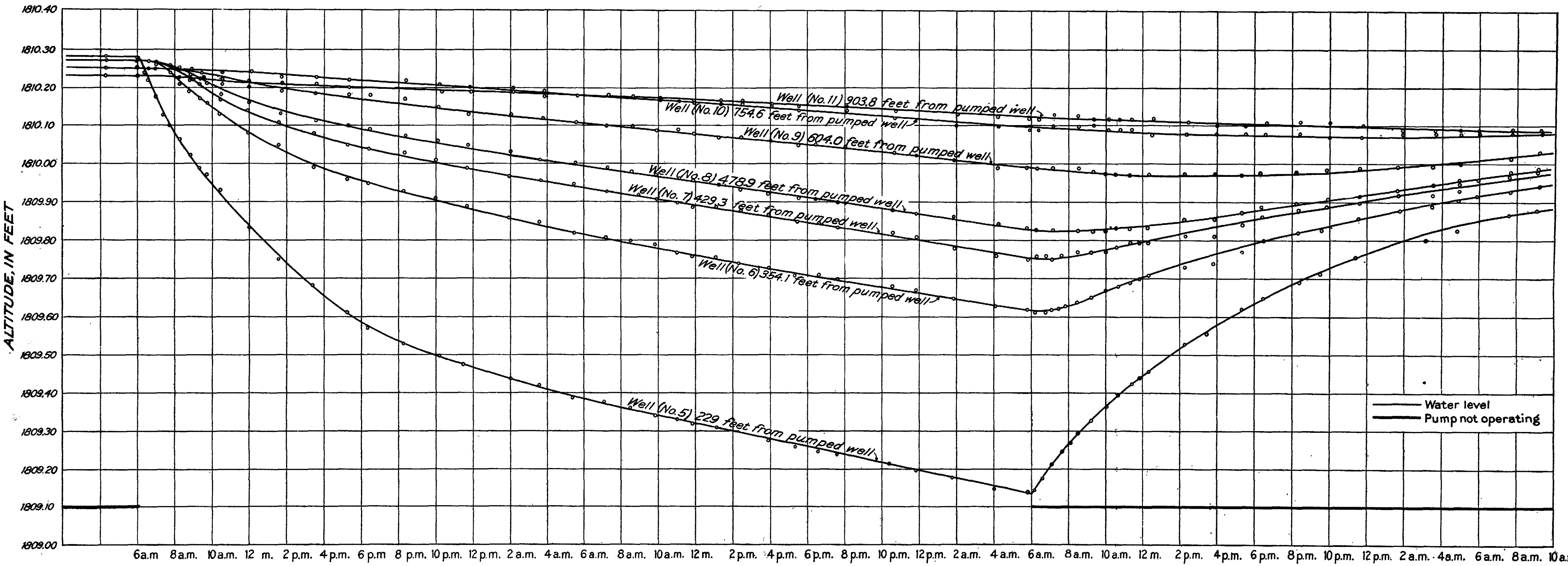
where R is the recovery of the water level, in feet; D is the draw-down from static water level at the time pumping stopped, in feet; K is a coefficient for each particular well; T is the elapsed period of recovery, in hours; and n is an exponent. K and n contain the varying distance factor as well as several other constants enumerated above. The formula is based on the assumption that when T equals 0, R equals 0; and when T equals infinity, R equals D. Of course, if pumping were carried on over an extended period during which there were no recharge, R would probably never equal D, because of the permanent withdrawal of water from storage. In figure 5 the recovery curve of well 5 is plotted as determined from the theoretical equation, and the actual field measurements are also indicated.

CONES OF DEPRESSION

Soon after pumping begins the water table around a pumped well assumes a form which is comparable to an inverted cone, although it is not a true cone. Where the water-bearing material is homogeneous, the so-called "cone of depression" will be circular if the initial water table is horizontal, but elliptical if the initial water table is sloping. The form of the cone of depression at any time can be shown by either profiles or contours on the water table. Profiles at different angles with the direction of initial slope of the water table may differ widely in form. The profiles in figure 6 are based on the draw-downs in wells on lines B and D, which have nearly the same direction as the initial slope of the water table. The development of the cone is shown by the several profiles, and it is interesting to note the rate at which the radius of the cone increased with the period of pumping. The profile of the cone after 2 hours of pumping shows that most of



TYPICAL DRAW-DOWN AND RECOVERY CURVES FOR TEST 2.



TYPICAL DRAW-DOWN AND RECOVERY CURVES FOR TEST 1.

1 2 3 4 5 6

100 0 100 800 FEET

Contour interval 0.10 foot

CONTOURS ON THE WATER TABLE BEFORE PUMPING AND AT SEVERAL TIMES AFTER PUMPING BEGAN.

1, Before pumping began; 2, second hour; 3, sixth hour; 4, twelfth hour; 5, twenty-fourth hour; 6, forty-eighth hour,

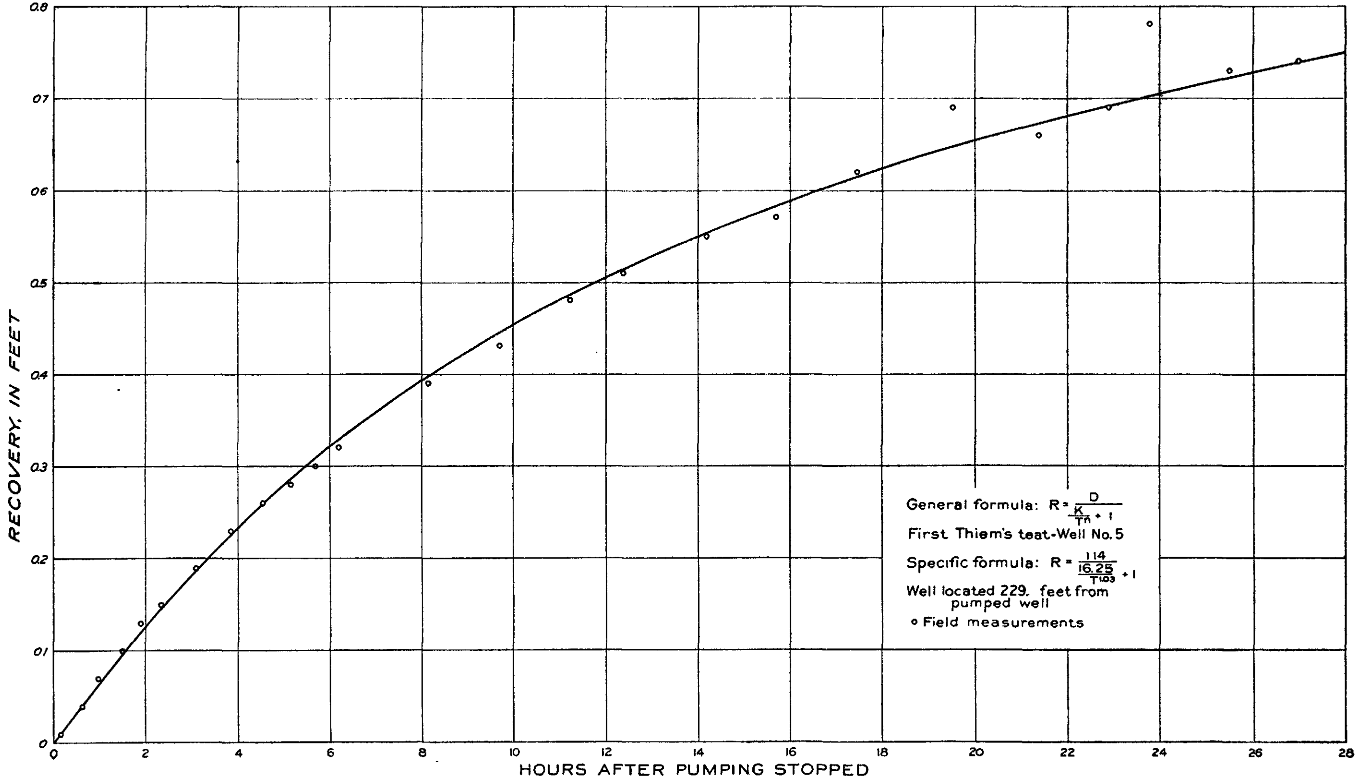


FIGURE 5.—Computed recovery curve for well 5, test 1.

the water pumped came from the unwatered sediments within a distance of 430 to 540 feet from the pumped well, because there was no draw-down of the water table beyond those distances. After 6 hours of pumping the radius of the cone had increased to about 800 feet, and after 12 hours of pumping there were small but measurable draw-downs at 1,200 feet from the pumped well. Further pumping undoubtedly increased the radius of the cone beyond the most distant observation wells.

It is probable that the rate at which the radius of the cone of depression develops depends somewhat upon the specific yield of the water-bearing sand and gravel. Most of the water that is discharged by the pumped well during the first hours of pumping is taken directly from the sediments around the pumped well. The amount of water taken from storage in this manner is indicated by the draw-down of the water table, which provides a method for determining the specific yield of the sediments (p. 9). If the water-bearing materials have a low specific yield, the radius of the cone of depression will probably develop more rapidly than if the materials have a high specific yield, for if the specific yield is low, there is less water available in the sediments, and hence the effect of pumping will be transmitted outward from the pumped well at a more rapid rate. The actual difference in the rate at which the radius of the cone of depression will develop in materials having different specific yields depends, of course, not alone on the specific yield but also on the relation between the specific yield and the permeability of the materials.

The slope of the cone of depression is steeper up-gradient from the pumped well than down-gradient (fig. 6). This indicates that if the permeability is the same, less water is percolating to the well from the down-gradient side. The slope of the cone down-gradient from the well becomes progressively less than the slope at the corresponding distance up-gradient, until at some distance down-gradient from the well the water table is horizontal. This point is called the ground-water divide. All water below this divide percolates away from the pumped well, and all water above this divide percolates toward the well. The ground-water divide moves down-gradient as the pumping period is increased and the cone of depression becomes larger. The ground-water divide in test 1, as indicated in figure 6, was about 280 feet down-gradient from the pumped well after 2 hours of pumping. The divide gradually moved to about 360 feet below the pumped well after 6 hours of pumping, 440 feet after 12 hours, 500 feet after 24 hours, 560 feet after 36 hours, and about 600 feet after 48 hours.

Contours on the water table before pumping began and at several times after pumping began are shown in plate 6. The limits of the area included in these maps are somewhat less than the distance of the farthest observation well from the pumped well, because there were

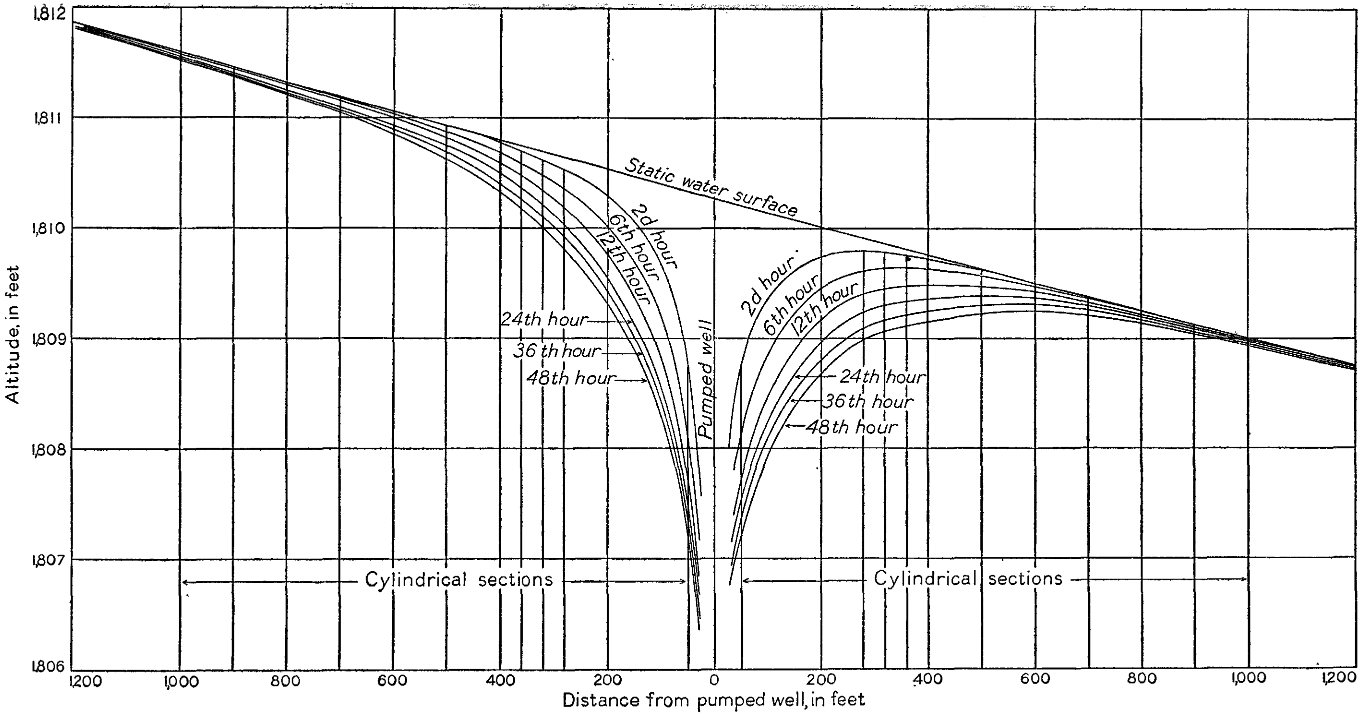


FIGURE 6.—Profiles of the cone of depression at several times after pumping began and location of cylindrical sections used for computing specific yield.

too few observation wells to furnish data for contours over a larger area. The contours of the original water table were drawn readily by direct interpolation between the altitudes of water levels in the observation wells, but to draw the contours of the water table at given times after pumping began it was necessary to plot a profile of the cone of depression for each of the several lines and to determine the distance of each contour from the pumped well by inspection of the profile. A certain amount of judgment had to be used in tracing the contours between the several lines of wells.

The map showing contours on the water table before pumping began (map 1, pl. 6) indicates that the initial slope of the water table was nearly in the direction of the lines B and D and that this slope was very uniform, averaging about 6.9 feet to a mile. The contour interval selected was 0.1 foot for all the maps. These maps were drawn to illustrate the development of the cone of depression and especially the movement of the ground-water divide. The cone of depression is well defined close to the pumped well on map 2, showing contours on the water table after 2 hours of pumping, although there are some inequalities which are caused by the steep slope of the cone. The contours are nearly circular close to the well but gradually become elliptical at greater distances up to the ground-water divide. The divide as shown on these maps is a semielliptical line between the water which eventually enters the pumped well and that which percolates on down-gradient. A draw-down of the water table at some distance from the pumped well does not necessarily indicate that the water at that distance lies within the ground-water divide. It may indicate only that the draw-down of the water table created a slope of the water table that tended to move the water from its normal path, but this water may not reach the pumped well. The contours in map 2 show there was a draw-down of the water table for several hundred feet down-gradient from the ground-water divide. Also maps 2 and 3 show that there was a draw-down of the water table on lines A and C beyond the ground-water divide. The ground-water divide has been represented on maps 2 to 6 as the only line normal to the contours on the water table. Its position can be drawn in from the points of inflection of the contour lines. The steady decline of the ground-water divide down-gradient can be observed in maps 2 to 6, as well as by the profiles of the cone of depression in figure 6. The contour maps show in addition the lateral development of the divide and the increase in cross-sectional area through which water percolates to the pumped well.

The development of the cone of depression can probably best be seen by discussing one contour—for example, the 1,809.5-foot contour—on map 2 and observing the position of this line on the contour maps for subsequent times. After 2 hours of pumping this line is somewhat elliptical, for it crosses line B at a point 85 feet up-gradient from the pumped well and line D at a point 120 feet down-gradient from the pumped well. After 6 hours of pumping this contour is still closed but includes a larger area. Its intersection with line B has advanced up-gradient to a point about 130 feet from the pumped well, and its intersection with line D has declined down-gradient to a point about 210 feet from the pumped well. After 12 hours of pumping this contour is no longer closed, and during the remaining 36 hours of pumping it diverges still more on the down-gradient side of the well, while its intersection with line B steadily moves up-gradient. The development of the cone close to the pumped well as the period of pumping lengthens can readily be seen by observing the increase in area included in the first closed contour, which is the same for maps 3 to 6.

After 2 hours of pumping the contours 300 to 400 feet up-gradient from the pumped well were affected but little by the pumping. As pumping continued the lines became closer together (indicating an increase in the slope of the cone of depression), and after 48 hours they had a noticeable curvature.

The movement of the ground water was, of course, always normal to the contours on the water table. Continuous lines drawn normal to the contours are called "lines of flow" of the ground water, and they trace the path of movement of a particle of water. The lines of flow on map 1, plate 6, are approximately parallel, but on maps 2 to 6 the lines of flow included between the ground-water divide and the pumped well converge toward the well. The distances through which particles of water moved in reaching the pumped well differed considerably. Thus a particle of water 400 feet from the pumped well on line B traveled to the pumped well through a much shorter path than a particle of water at the same distance from the pumped well on line A (map 6). Moreover, the particle of water on line B had to move with a greater velocity than the particle on line A, because the average hydraulic gradient along its path was greater. Consequently, the quantity of water that passed through a unit area on line B was much greater than the quantity that passed through a similar area on line A.

COMPUTATION OF COEFFICIENTS OF PERMEABILITY

Thiem's equation 4 was used for computing coefficients of permeability, and as the draw-downs at only two points on the cone of depression are required for the computation of a coefficient, the data obtained in the pumping tests are sufficient for a great many computations. Two general methods were used. In one method coefficients were computed by using the draw-downs that were measured in the observation wells, and in the other method coefficients were computed by using the interpolated draw-downs at selected distances from the pumped well, which were obtained from profiles of the cone of depression.

The coefficients of permeability computed by using the difference in draw-down at any two points on the cone of depression would be equal if the form of the observed cone of depression was the same as that of the theoretical cone of depression obtained by Thiem's formula. However, the cones of depression in both pumping tests were not identical with the theoretical cone, and the computed permeability ranged through wide limits. Computations of permeability were made by using the draw-downs in all possible combinations of observation wells on line A, after 48 hours of pumping, in test 1 (table 6). The coefficients of permeability thus computed ranged from 535 to 5,630. The equation used was

$$P=\frac{527.7\times 540\times \log\frac{a_1}{a}}{m(s-s_1)}$$

in which the symbols are those given on page 10.

TABLE 6.—*Coefficients of permeability computed by Thiem's formula for all possible combinations of observation wells on line A*

Well no.	71	72	1	2	3	4	5	6	7	8	9	10
72	535	------	------	------	------	------	------	------	------	------	------	------
1	615	840	------	------	------	------	------	------	------	------	------	------
2	668	885	919	------	------	------	------	------	------	------	------	------
3	690	927	974	1,058	------	------	------	------	------	------	------	------
4	729	949	1,000	1,079	1,112	------	------	------	------	------	------	------
5	746	941	976	1,014	983	866	------	------	------	------	------	------
6	767	956	998	1,038	1,030	1,045	1,188	------	------	------	------	------
7	785	984	1,027	1,096	1,098	1,093	1,262	1,830	------	------	------	------
8	778	998	1,043	1,102	1,130	1,152	1,315	1,786	1,760	------	------	------
9	810	1,018	1,070	1,141	1,180	1,192	1,374	1,700	1,630	1,605	------	------
10	835	1,052	1,113	1,196	1,255	1,292	1,500	1,880	1,895	1,950	2,505	------
11	845	1,090	1,160	1,260	1,345	1,420	1,660	2,150	2,255	2,320	3,320	5,630

An inspection of table 6 shows that the computed coefficients were the smallest when both of the observation wells selected were close to the pumped well and the largest when both were far from the pumped well. This indicates that the difference in draw-down of wells close to the pumped well was great in comparison to the difference in draw-down of wells farther from the pumped well. In other words, the cone of depression had developed very little at distances far from the pumped well. These wide variations make the determination of the most nearly correct coefficient almost an impossibility from computations of this kind. The coefficients computed from combinations of observation wells on other lines gave similar results.

Coefficients could be computed, of course, from draw-downs at any time after pumping started. If the difference in draw-down changed during the period of pumping, the computed coefficient would likewise change. In order to observe the effect of the length of the period of pumping on the computed permeability, coefficients were computed for several combinations of wells on line A and for several periods of time after pumping began (table 7). In general, the computed coefficients became smaller as the period of pumping increased, because the difference in draw-down became larger. However, the range in coefficients was smallest when the observation wells were selected close to the pumped well. At all times the coefficients became larger as the farthest observation well was selected farther from the pumped well, indicating that the form of the cone of depression was not equal to that of the theoretical cone.

TABLE 7.—*Coefficients of permeability computed by Thiem's formula for several combinations of observation wells on line A and for several periods of pumping*

Well nos.	Coefficients at different times (hours) after pumping began					
	2	6	12	24	36	48
1 and 2	960	868	879	935	918	919
1 and 3	1,110	958	948	1,003	981	974
1 and 4	1,180	1,008	974	1,025	1,003	1,000
1 and 5	1,268	1,037	987	1,008	982	976
1 and 6	1,413	1,132	1,040	1,040	1,010	998
1 and 7	1,510	1,198	1,090	1,076	1,043	1,027
1 and 8	1,550	1,225	1,107	1,095	1,060	1,043

Another computation was made by using the draw-downs obtained from profiles of the cone of depression. The draw-downs at several distances on lines A, B, C, and D were determined after 48 hours of pumping, and coefficients were computed by using $a=50$ feet and a_1 equal to several distances (table 8). The computed coefficients of permeability vary in about the same manner as those shown in table 6, but the variation is considerably less. The greatest variation occurred on line B, but this was only from 823 to 1,180. On lines C and D the coefficients computed for $a_1=75$ feet were larger than the coefficients obtained for $a_1=200$ feet, but on lines A and B the opposite was true. This fact suggests that the difference in the variation of the coefficients may be caused by the initial slope of the water table, because lines C and D are extensions of lines A and B, respectively. In general, this suggestion is confirmed by averaging the coefficients computed for several distances on line A with those computed for the same distances on line C, and similarly averaging the coefficients computed for lines B and D (table 8). The averages for lines A and C are very nearly the same as the averages at corresponding distances on lines B and D. The differences in the coefficients could be due to the fact that the wells penetrated materials of varying permeability, but it does not seem probable that the materials would be distributed areally in such a manner that all wells on one line penetrated a material of one permeability and all wells on another line penetrated a material of another permeability.

TABLE 8.—*Coefficients of permeability computed by Thiem's formula, using $a=50$ feet*

a_1 (feet)	Coefficients					
	Line A	Line B	Line C	Line D	Average of lines A and C	Average of lines B and D
75	957	823	1,032	1,200	995	1,012
100	994	856	980	1,208	987	1,032
150	1,055	888	952	1,114	1,004	1,001
200	1,025	914	970	1,102	996	1,008
250	1,000	929	987	1,087	994	1,008
300	1,008	943	1,002	1,090	1,005	1,017
350	1,022	958	1,018	1,100	1,020	1,029
400	1,050	978	1,030	1,104	1,040	1,042
450	1,074	994	1,050	1,115	1,062	1,055
500	1,096	1,020	1,075	1,132	1,081	1,076
550	1,112	1,040	1,100	1,152	1,106	1,096
600	1,130	1,067	1,122	1,168	1,126	1,118
650	1,140	1,086	1,147	1,182	1,144	1,134
700	1,156	1,109	1,170	1,198	1,163	1,154
750	1,171	1,130	1,180	1,216	1,176	1,173
800	1,192	1,147	1,210	1,237	1,201	1,192
850	1,215	1,169	1,229	1,260	1,222	1,215
900	1,232	1,180	1,224	1,280	1,228	1,230

As the variations in the coefficients of permeability are caused by differences between observed and theoretical draw-downs it is well to examine the observed draw-downs and to determine the manner and amount of their deviation from theoretical draw-downs. The observed draw-downs of the water table as taken from profiles of the cone of depression after 48 hours of pumping in test 1 for several distances and directions from the pumped well are given in table 9. The draw-downs decrease regularly with the distance from the pumped well, but the draw-downs at equal distances from the pumped well are not equal. However, as would be expected from the previous averaging of the coefficients, the average of the draw-downs on lines A and C at equal distances from the pumped well are very nearly equal to the average draw-downs on lines B and D at the same distances from the pumped well.

The computations of coefficients of permeability have indicated that the computed coefficients become larger as the observation wells are selected at greater distances from the pumped well. As the differences in draw-down are substituted in the denominator of Thiem's formula, the reason for this increase in the coefficients is that the difference in draw-downs is relatively too small. This, in turn, indicates that the cone of depression has not reached a condition of equilibrium. The draw-downs at several distances on line A were averaged with the draw-downs at corresponding distances on line C, and the differences in draw-down ($s-s_1$) were computed, s being taken as equal to the average of the draw-downs on lines A and C at 40 feet from the

TABLE 9.—*Draw-down of water table, in feet, after 48 hours of pumping during test 1, for several distances and directions from the pumped well*

Distance from pumped well (feet)	Draw-down					
	Line A	Line B	Line C	Line D	Average of lines A and C	Average of lines B and D
50	3.06	3.14	2.98	2.93	3.02	3.08
75	2.52	2.505	2.48	2.50	2.52	2.50
100	2.17	2.11	2.08	2.20	2.13	2.16
150	1.74	1.57	1.52	1.68	1.63	1.63
200	1.35	1.22	1.17	1.33	1.26	1.28
250	1.03	.95	.92	1.06	.96	1.01
300	.82	.74	.73	.86	.78	.80
350	.66	.575	.57	.70	.62	.64
400	.57	.455	.44	.56	.51	.52
450	.49	.36	.35	.445	.42	.41
500	.41	.29	.28	.37	.35	.33
550	.34	.23	.23	.305	.29	.27
600	.28	.195	.195	.245	.24	.23
650	.22	.16	.16	.195	.19	.18
700	.18	.14	.14	.155	.16	.15
750	.15	.115	.12	.13	.14	.13
800	.125	.085	.10	.11	.11	.10
850	.12	.075	.075	.10	.10	.09
900	.105	.065	.06	.085	.08	.08
950		.06	.05	.08		.07
1,000		.05	.04	.065		.06
1,050		.05	.025	.065		.06

pumped well. These differences in draw-down are given in table 10. They increase as the distance from the pumped well increases. The difference in draw-downs between 40 and 60 feet from the pumped well decreased as the period of pumping increased, and for distances up to 200 feet the difference was practically the same for the final 24 hours of pumping. For distances greater than 200 feet the difference in draw-downs increased throughout the period of pumping.

TABLE 10.—*Differences in average draw-down on lines A and C, in feet*

Distances from pumped well (feet)	Difference at different times (hours) after pumping began						
	2	6	12	24	36	48	[1] 48+
40 and 60	0.56	0.55	0.54	0.53	0.53	0.53	0.53
40 and 80	.88	.95	.94	.91	.92	.91	.90
40 and 100	1.08	1.21	1.23	1.20	1.20	1.19	1.19
40 and 120	1.24	1.41	1.45	1.43	1.42	1.44	1.42
40 and 140	1.37	1.57	1.64	1.61	1.61	1.63	1.62
40 and 160	1.46	1.70	1.78	1.76	1.77	1.79	1.79
40 and 180	1.53	1.81	1.92	1.91	1.92	1.93	1.95
40 and 200	1.59	1.91	2.04	2.04	2.05	2.07	2.08
40 and 240	1.67	2.04	2.23	2.25	2.26	2.29	2.32
40 and 280	1.72	2.15	2.36	2.41	2.43	2.46	2.52
40 and 320	1.76	2.23	2.47	2.53	2.55	2.60	2.69
40 and 360	1.79	2.29	2.55	2.64	2.67	2.71	2.84
40 and 400	1.81	2.33	2.61	2.71	2.75	2.80	2.98
40 and 500	1.83	2.39	2.72	2.85	2.92	2.99	3.27
40 and 600	1.84	2.42	2.78	2.93	3.02	3.11	3.51
40 and 700	1.84	2.43	2.81	2.97	3.08	3.19	3.71
40 and 800	1.84	2.43	2.82	3.00	3.12	3.23	3.88
40 and 900	1.84	2.43	2.83	3.01	3.13	3.25	4.03
40 and 1,000	1.84	2.43	2.84	3.03	3.15	3.28	4.12

[1] Theoretical difference in draw-down after the cone of depression reaches equilibrium.

The data given in table 10 indicate that after 48 hours of pumping the cone of depression had reached approximate equilibrium in form for a distance of about 200 feet from the pumped well. The cone of depression is said to have reached "approximate" equilibrium, because as the water table was lowered the cross-sectional area through which the water percolated was decreased, and the form of the cone was changed slightly to compensate for this decrease. An inspection of table 10 shows that the cone had reached approximate equilibrium at 160 feet from the pumped well after only 12 hours of pumping. This means that at these distances from the pumped well the coefficients of permeability computed between 12 to 48 hours after pumping began would be approximately equal to the coefficients computed after a much longer period of pumping. With the equation developed for the cone of depression, the theoretical draw-downs at various distances from the pumped well were computed, and the differences in draw-downs were obtained. These are given in the last column in table 10 and are the theoretical differences in draw-down that would occur after the whole cone of depression had reached a condition of equilibrium. The table shows that for distances up to about 200 feet from the pumped well the theoretical and observed draw-downs are practically equal, but that for greater distances the theoretical difference is larger than the observed difference. Hence the coefficients of permeability computed from the differences in draw-down up to 200 feet would be nearly equal, but the computed coefficients would increase beyond that distance.

It has been shown that the observed draw-down of the water table does not equal the theoretical draw-down at all points on the cone of depression; hence values for the coefficient of permeability may vary widely when computed directly from the difference in draw-down of two observation wells. However, the observed draw-downs on one side of the pumped well when averaged with the observed draw-downs at corresponding distances on the opposite side of the well approached or equaled the theoretical draw-downs at those distances. Coefficients of permeability computed from the draw-downs averaged in this manner were nearly equal for that part of the cone of depression that had reached approximate equilibrium in form—that is, for distances from about 40 to 200 feet from the pumped well. This indicates that the pumping method can be used in the field and that it will yield consistent results provided the observation wells are located within that part of the cone of depression that has reached approximate equilibrium and on a straight line through the pumped well—preferably on a line along the natural hydraulic gradient—and on both sides of the pumped well.

The coefficients of permeability computed from various combinations of the draw-downs of the water table between the limits of 40 and 200 feet from the pumped well in test 1 differ but little and average about 997 (table 11). In table 1 are given the coefficients of permeability of samples of the sand and gravel taken from well 84 during the process of drilling, as determined in the hydrologic laboratory of the United States Geological Survey. These coefficients reach a maximum of 4,350, but their average, weighted as to thickness, is about 1,200.

TABLE 11.—*Final computation of coefficients of permeability from the average draw-downs on lines A and C where the cone of depression had reached approximate equilibrium in form*

a_1 (feet)	a (feet)	$\log\frac{a_1}{a}$	m (feet)	$s-s_1$ (feet)	P
60	40	0.176	96.94	0.53	976
80	40	.301	97.14	.91	970
100	40	.398	97.28	1.19	980
120	40	.477	97.40	1.44	969
140	40	.544	97.50	1.63	975
160	40	.602	97.58	1.79	982
180	40	.653	97.65	1.93	987
200	40	.699	97.71	2.07	985
80	60	.125	97.40	.38	962
100	60	.222	97.54	.66	983
120	60	.301	97.67	.91	965
140	60	.368	97.76	1.10	975
160	60	.426	97.84	1.26	985
180	60	.477	97.91	1.40	992
200	60	.523	97.98	1.53	994
100	80	.097	97.73	.28	1,010
120	80	.176	97.85	.53	967
140	80	.243	97.95	.72	982
160	80	.301	98.03	.88	994
180	80	.352	98.10	1.02	1,002
200	80	.398	98.16	1.15	1,005
120	100	.079	98.00	.25	919
140	100	.146	98.09	.44	964
160	100	.204	98.17	.60	987
180	100	.255	98.24	.74	1,000
200	100	.301	98.30	.87	1,003
140	120	.067	98.22	.19	1,023
160	120	.125	98.30	.35	1,035
180	120	.176	98.37	.49	1,040
200	120	.222	98.43	.62	1,043
160	140	.058	98.39	.16	1,050
180	140	.111	98.46	.30	1,071
200	140	.155	98.52	.43	1,043
180	160	.051	98.54	.14	1,053
200	160	.097	98.60	.27	1,038
200	180	.045	98.68	.13	1,000

Average, 997.

Two methods for determining the most probable coefficient of permeability from the observed data are suggested by C. E. Van Orstrand, geophysicist of the United States Geological Survey. These methods as explained by Mr. Van Orstrand are as follows:

The value of P is to be determined from the equation

$$C+Q\frac{\log_e x}{\pi P}=y^2 \text{-------------------- (40)}$$

One method of obtaining a close approximation to the value of P consists in assuming at least three values of P, preferably at equal intervals and over a range so great that the true value of P falls between the extremes. From each observed value of y we thus obtain a value of C as shown in table 12. The average value of C, 8,409.15, and the corresponding value of P, 975, are then substituted in formula 40, to obtain y_c, the computed value of y.

TABLE 12.—*Computation of v^2 for $P=975$ and $Q=777,600$ gallons a day*

x (feet)	$\frac{Q}{\pi}\log_e x$	C	y_o	y_c	v
40	913, 063. 35	8, 410. 54	96. 68	96. 67	+0. 01
60	1, 013, 421. 91	8, 410. 37	97. 21	97. 20	+. 01
80	1, 084, 630. 29	8, 411. 37	97. 59	97. 58	+. 01
100	1, 139, 861. 41	8, 409. 45	97. 87	97. 87	. 00
120	1, 184, 988. 85	8, 412. 16	98. 12	98. 10	+. 02
140	1, 223, 143. 71	8, 410. 35	98. 31	98. 30	+. 01
160	1, 256, 194. 76	8, 407. 94	98. 47	98. 48	−. 01
180	1, 285, 349. 88	8, 405. 62	98. 61	98. 63	−. 02
200	1, 311, 428. 35	8, 404. 54	98. 74	98. 76	−. 02
Average		8, 409. 15	Σv^2		. 0017

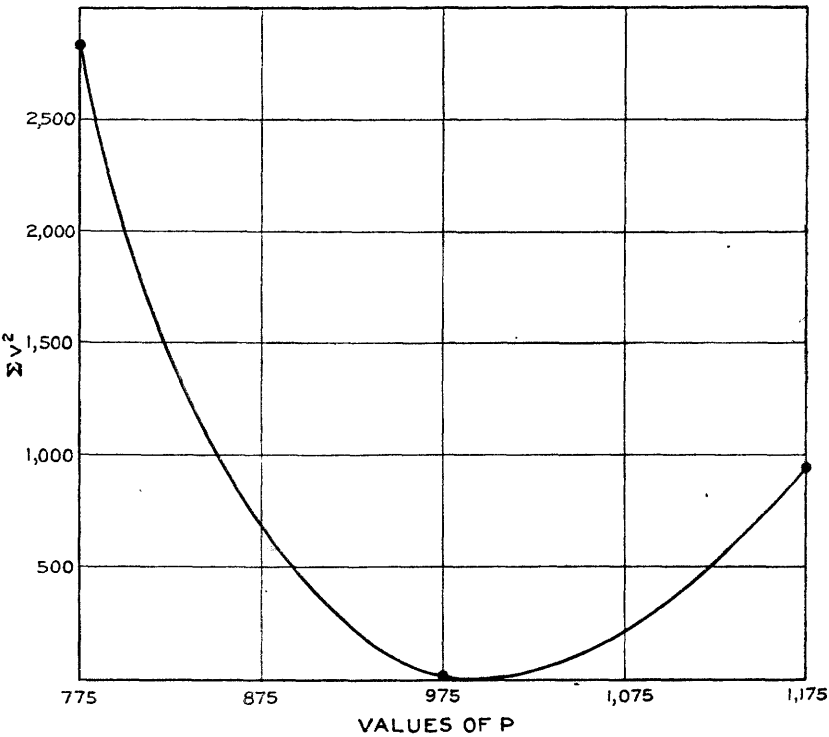


FIGURE 7.—Relation of Σv^2 to P.

The differences (v) between the observed and computed values, the residuals, are tabulated in the last column, and we find $\Sigma v^2=0.0017$. Proceeding in a similar manner for the values $P=775$ and 1,175, we find for Σv^2 the respective values 0.2835 and 0.0948. Plotting all the values as shown in figure 7, we see at once that the value of P

that makes Σv^2 a minimum is slightly less than 1,000. This, in accordance with the principle of least squares, is the most probable value of P.

Another method of procedure arrives at the correct result in a more direct manner: Let us write down an observation equation (40) for each observed value of y as given in table 12.

TABLE 13.—*Observation equations for y^2*

Observation equations	v for y^2	y_o	v for y	
$C+\ \ 913,063\ P^1=9,347$	0	96.68	0	−0.01
$C+1,013,422\ P^1=9,450$	+.01	97.21	.00	.00
$C+1,084,630\ P^1=9,524$	+.02	97.59	+.01	+.01
$C+1,139,861\ P^1=9,579$	+.01	97.87	.00	.00
$C+1,184,989\ P^1=9,628$	+.04	98.12	+.02	+.02
$C+1,223,144\ P^1=9,665$	+.02	98.31	+.01	+.01
$C+1,256,195\ P^1=9,696$	−.01	98.47	+.00	.00
$C+1,285,350\ P^1=9,724$	−.03	98.61	−.02	−.01
$C+1,311,428\ P^1=9,750$	−.03	98.74	−.02	−.01
$P^1=\frac{1}{P}$	Σv^2	----------	.0014	.0009

Combining these equations by the method of least squares, we have the normal equations.

$$\begin{aligned} 9C+ \quad 10{,}412{,}082P^1 &= 86{,}363 \\ 10{,}412{,}082C+12{,}186{,}687{,}602{,}840P^1 &= 100{,}055{,}976{,}372 \end{aligned}$$

the solution of which gives $P=980.083$, $C=8{,}415.381$. The residuals corresponding to these values of P and C are given in the next to the last column of table 13. These values are slightly erroneous, owing to the fact that y^2 instead of y was used in evaluating P and C.

To obtain the correct values, let us compute the partial differential coefficients in the equation

$$dy=\frac{\partial y}{\partial P}dP+\frac{\partial y}{\partial C}dC.$$

As dy is here the error in y—that is, v—we write the observation equations in the form

$$\frac{dC}{2y}-\frac{Q \log_e x}{2\pi y P^2}dP=dy=v.$$

Substituting the appropriate values from tables 12 and 13, we have

$$\begin{aligned} 0.005172dC-0.004916dP &= \ \ 0.00 \\ 0.005144dC-0.005426dP &= \ \ 0.00 \\ 0.005123dC-0.005785dP &= +0.01 \\ 0.005109dC-0.006062dP &= \ \ 0.00 \\ 0.005096dC-0.006286dP &= +0.02 \\ 0.005086dC-0.006476dP &= +0.01 \\ 0.005078dC-0.006640dP &= \ \ 0.00 \\ 0.005070dC-0.006785dP &= -0.02 \\ 0.005064dC-0.006913dP &= -0.02 \end{aligned}$$

The solution of these equations gives $dP=6.782$, $dC=8.162$; hence

$$P=980.083+6.782=986.87$$
$$C=8{,}415.381+8.162=8{,}423.54$$

The substitution of these constants in equation 40 gives the values in the last column of table 13. As shown in the table, the value of Σv^2 has been reduced from 0.0014 to 0.0009 by the last solution. Assuming that the work has been done correctly and that the numbers have been rounded off correctly, a further reduction in the value of Σv^2 is impossible.

DIFFERENCES BETWEEN FIELD AND THEORETICAL CONDITIONS

The conditions found in the field rarely approach closely the theoretical considerations from which Thiem's formula was developed. The water-bearing formations usually differ from place to place in thickness and in permeability. Wells usually do not extend to the bottom of the formation, and the bottom is not always parallel to the water table or piezometric surface. Most water tables and piezometric surfaces are not horizontal, and few pumping tests are continued until the cone of depression reaches approximate equilibrium in form over a large area.

The errors introduced into Thiem's formula by the differences between field and theoretical conditions can be minimized. The thickness of the water-bearing material can be obtained from as many well logs as possible, and an average value used for m. Of course, the value used in Thiem's formula is not important if the computed coefficient of permeability is to be used to determine the quantity of water that percolates through some cross section of the material in which the factor of thickness appears, because in that case m cancels. The effect of variations in the permeability of the water-bearing material can be lessened by selecting an average value for the computed coefficient by one of the methods previously explained. Other differences—such as pumped wells that do not extend through the formation, water tables that are not horizontal, and cones of depression that have not reached equilibrium in form—cause part of the cone of depression to differ from the theoretical cone. The effect of these differences can be minimized by substituting in Thiem's formula only the draw-down of the water table from that part of the cone of depression which corresponds with the theoretical cone.

The reason why the observed cone of depression differs from the theoretical cone may be made clear by a review of the behavior of the water table during the first pumping test. As soon as the pump began discharging water from the well a hydraulic gradient from all directions was established toward the well, and the water table was lowered. The lowering of the water table unwatered a considerable volume of sediments, the water in them gradually draining down to the new water table and eventually entering the well. Thus most of the water which the pumped well discharged was obtained from the unwatered sediments. The decline in the water table close to the pumped well was large at first, and then as a gradient adequate to discharge the 540 gallons of water a minute required for the well was formed, the water table was lowered more slowly. Farther from the pumped well the water table lowered less rapidly, because (1) water was being supplied to the pumped well from sources closer to the well and (2) the hydraulic gradient necessary to force the required amount of water through a unit area of the sediments was smaller. Gradually the water table around the pumped well assumed a slope that was large enough to cause equal quantities of water to flow toward the pumped well without withdrawing large amounts of water from storage. Thus the slope of the cone of the depression for distances up to 200 feet from the well reached essential equilibrium in form after 48 hours of pumping. Beyond the distance of 200 feet, however, the slope of the water table was less than necessary to transmit 540 gallons of water a minute through cylindrical sections of the sediments. If pumping had been continued the water table at distances greater than 200 feet from the pumped well would have continued to decline until the necessary hydraulic gradient had been developed. To develop the necessary hydraulic gradient at distances comparatively far from the pumped well, a very large quantity of water must be taken from storage. Of course, most of the water taken from storage must be discharged by the pumped well, and hence the time required for the cone of depression to reach equilibrium depends to some extent upon the specific yield of the water-bearing materials. The slope of the cone of depression necessary to force 540 gallons a minute through cylindrical sections is developed at increasing distances from the pumped well only by the decline of the water table at those distances. Thus it is evident that in order for the cone to maintain its slope nearer the pumped well, the water table must continue to decline there, but at a diminishing rate. This decline in turn unwaters more sediments, and the development of the cone proceeds at a still slower rate.

The fact that the cone of depression reaches essential equilibrium near the pumped well comparatively soon after pumping begins provides the opportunity to use Thiem's formula. Fortunately the difference in draw-down, $s-s_1$, is substituted in Thiem's equation, and so long as this difference is constant, the coefficients will be equal. Little error is introduced by the increase in absolute draw-down as pumping is continued. Thus, though the cone of depression has not reached equilibrium it is still possible to use the formula. It is interesting to observe the difference between the theoretical and observed draw-downs shown in table 10. The differences are practically equal for distances up to 200 feet from the pumped well, but for distances farther from the pumped well the theoretical difference is larger. The theoretical difference in draw-downs of the water table at 40 feet and 1,000 feet from the pumped well is 4.12 feet, and the observed difference after 48 hours of pumping was 3.28 feet. This indicates that with further pumping the water table at 40 feet and 1,000 feet from the pumped well would lower in such amounts that the difference between the declines would be increased by 0.84 foot. There is no way of ascertaining the net decline of the water table at either of these distances from the pumped well, as the net decline depends upon the length of the period of pumping.

The pumped well used in test 1 did not extend through the water-bearing sand and gravel, and it seems probable that this influenced the cone of depression close to the pumped well. As shown in table 6, the coefficients of permeability computed by using the draw-downs in wells 71 and 72, 2.6 feet and 12.3 feet, respectively, from the center of the pumped well, were comparatively small, indicating that the draw-downs near the pumped well were relatively great. It seems probable that these comparatively large draw-downs were an effect of the well's failing to penetrate the entire thickness of the formation, because the form of the cone of depression reached essential equilibrium from 40 feet to 200 feet from the pumped well, and the draw-downs were of such magnitude that the computed coefficients of permeability were practically equal, whereas the water table within 40 feet of the pumped well reached essential equilibrium but did not correspond to the theoretical cone. A part of these large draw-downs may also have been caused by changes in the permeability of the formation due to the rearrangement of the sand and gravel during the development of the well.

It is unfortunate that there were interruptions during the period of the second test, for these interruptions so changed the normal draw-downs in the observation wells that the computations of permeability from the data obtained in this test are of doubtful value. Some computations were made, and the coefficients averaged about 1,300 for wells 56 to 61 on line SW. Because there were interruptions in pumping it would be expected that the draw-downs would be smaller and consequently the coefficients larger. It is difficult to make an intensive study of this test, because only line SW extended through pumped well 84. The draw-down measurements made during these tests indicate the behavior of the water table at several distances from the pumped well when pumping is not continuous, and the measurements of depth to the water table made after the pumping stopped are valuable for determining the rate and amount of the recovery of the water table.

DETERMINATION OF SPECIFIC YIELD BY THE PUMPING METHOD

The data obtained during the first pumping test are adequate for detailed study of the use of the pumping method for determining the specific yield of water-bearing materials. This method was suggested by Meinzer [19] and essentially consists of determining the ratio of (1) the quantities of ground water that in a given time are taken from storage between concentric cylindrical sections around the pumped well to (2) the volumes of sediments between the cylinders that are unwatered in that time. A preliminary report by the writer on this method, with reference to the pumping tests described in this paper, has been published.[20] As the quantities of water taken from storage are determined by ascertaining the difference in the quantities of water that percolate through the cylinders in a given time, the specific yield may be expressed by the equation

$$y=\frac{100(Y_1-Y)}{V} \qquad (106)$$

where y is the specific yield, Y_1 is the quantity of ground water, in cubic feet, that percolates through the smaller cylinder, Y is the quantity of ground water, in cubic feet, that percolates through the larger cylinder, and V is the volume of water-bearing material, in cubic feet, that is unwatered between the cylinders.

The volumes of water-bearing material that were unwatered between concentric cylindrical sections around the pumped well during certain periods of pumping were computed by the formula

$$V=\pi(a^2-a_1^2)\frac{(s+s_1)}{2} \qquad (107)$$

where V is the volume of unwatered material, in cubic feet; a is the radius of the large cylinder, in feet; a_1 is the radius of the small cylinder, in feet; and s and s_1 are the draw-downs of the water table, in feet, at the distances a and a_1, respectively, from the pumped well. The draw-down at any given distance from the pumped well was taken as the average draw-down at that distance on all lines. The volumes of material that were unwatered during 6, 12, 24, 36, and 48 hours of pumping were determined by computing the volumes unwatered between 22 cylinders whose radii ranged from 10 feet to 900 feet, but only the volumes of water-bearing material that were unwatered between several selected concentric cylindrical sections around the pumped well are given in this report (table 14).

[19] Meinzer, O. E., Outline of methods for estimating ground-water supplies: U. S. Geol. Survey Water-Supply Paper 638, p. 136, 1932.

[20] Wenzel, L. K., Specific yield determined from a Thiem's pumping test: Am. Geophys. Union Trans., 1933, pp. 475–477.

TABLE 14.—*Volumes of water-bearing material, in cubic feet, that were unwatered between concentric cylindrical sections around the pumped well for several periods of pumping*

Radii of cylinders (feet)	Volume during different periods of pumping (hours)				
	0–6	0–12	0–24	0–36	0–48
50 and 280	160, 500	227, 700	275, 600	308, 300	336, 400
50 and 320	179, 800	260, 100	319, 300	361, 400	397, 000
50 and 360	196, 000	289, 200	359, 500	411, 400	455, 200
50 and 400	209, 400	315, 000	395, 800	458, 200	510, 600
50 and 500	234, 200	367, 300	474, 300	561, 500	632, 400
50 and 600	248, 900	401, 000	532, 100	640, 000	731, 000
50 and 700	259, 100	421, 600	575, 000	699, 300	806, 700
50 and 800	263, 800	437, 000	608, 000	745, 200	865, 600
50 and 900	263, 800	449, 000	634, 800	785, 300	916, 300

The quantities of water that were taken from storage between the concentric cylindrical sections were determined by computing the quantity of water that percolated through the smaller cylinder and subtracting from it the quantity of ground water that percolated through the larger cylinder. The quantities of water that percolated through each cylinder in a given time were computed by the formula

$$Y=\frac{2\pi PiahT}{7.48\times 24} \qquad (108)$$

where Y is the quantity of ground water, in cubic feet; P is the coefficient of permeability; i is the average hydraulic gradient, in feet per foot, at the distance a from the pumped well; h is the average thickness, in feet, of the saturated water-bearing material at the distance a; and T is the period of pumping, in hours. The quantities of water that percolated through several selected cylindrical sections in short periods were computed, and then a summation of these quantities was made (table 15). The computations were first made for short periods of time in order to decrease the error introduced by changes of the hydraulic gradient, especially in the first few hours of pumping. The periods of pumping given in table 15 were arbitrarily selected in order to show the change in specific yield with the period of draining.

TABLE 15.—*Computed quantities of ground water, in cubic feet, that percolated through several concentric cylindrical sections around the pumped well for several periods of pumping*

Radius of cylinder (feet)	Quantity during different periods of pumping (hours)				
	0–6	0–12	0–24	0–36	0–48
50	23, 800	51, 400	104, 000	155, 400	206, 800
280	8, 000	22, 300	57, 900	97, 700	139, 600
320	7, 200	20, 300	53, 000	89, 800	128, 200
360	5, 700	16, 700	45, 200	79, 300	116, 000
400	4, 700	14, 000	39, 600	70, 400	105, 200
500	2, 000	8, 100	26, 300	51, 300	80, 800
600	1, 400	5, 600	19, 500	36, 400	60, 200
700	1, 200	4, 800	12, 800	23, 600	38, 800
800	700	2, 700	8, 800	18, 100	29, 400
900	0	1, 200	5, 800	12, 800	20, 300

The cylinder with a radius of 50 feet was chosen as the control cylinder, and the quantities of water that were taken from storage were computed by determining the differences in the quantities of water that percolated through that cylinder and the quantities that percolated through larger cylinders (table 16). Specific yield was then determined by dividing the quantities of ground water taken from storage in certain periods of time (table 16) by the volumes of water-bearing material that were unwatered in the corresponding periods (table 14). The results are given in table 17.

TABLE 16.—*Quantities of ground water, in cubic feet, taken from storage between several concentric cylindrical sections around the pumped well and for several periods of pumping*

Radii of cylinders (feet)	Quantity during different periods of pumping (hours)				
	0–6	0–12	0–24	0–36	0–48
50 and 280	15,800	29,100	46,100	57,700	67,200
50 and 320	16,600	31,100	51,000	65,600	78,600
50 and 360	18,100	34,700	58,800	76,100	90,800
50 and 400	19,100	37,400	64,400	85,000	101,600
50 and 500	21,800	43,300	77,700	104,100	126,000
50 and 600	22,400	45,800	84,500	119,000	146,600
50 and 700	22,600	46,600	91,200	131,800	168,000
50 and 800	23,100	48,700	95,200	137,300	177,400
50 and 900	23,800	50,200	98,200	142,600	186,500

TABLE 17.—*Specific yield as computed for several concentric cylindrical sections and for several periods of pumping*

Radii of cylinders (feet)	Specific yield during different periods of pumping (hours)				
	0–6	0–12	0–24	0–36	0–48
50 and 280	9.8	12.8	16.7	18.7	20.0
50 and 320	9.2	11.9	16.0	18.2	19.8
50 and 360	9.2	12.0	16.3	18.5	19.9
50 and 400	9.1	11.9	16.3	18.5	19.9
50 and 500	9.3	11.8	16.4	18.5	19.9
50 and 600	9.0	11.4	15.9	18.6	20.0
50 and 700	8.7	11.1	15.9	18.8	20.8
50 and 800	8.8	11.1	15.7	18.4	20.5
50 and 900	-------	11.2	15.5	18.2	20.3
Average	9.2	11.7	16.1	18.5	20.1

As an illustration, the specific yield will be computed for the volume of sediments unwatered during 48 hours of pumping between concentric cylindrical sections with radii of 50 and 280 feet. The average slope of the cone of depression during the last 12 hours of pumping was 2.59 percent at 50 feet from the pumped well and 0.369 percent at 280 feet from the well. The quantity of water that percolated through the 50-foot cylinder from 36 to 48 hours after pumping began was as follows:

$$Y_1=\frac{2\times 3.1416\times 975\times 0.0259\times 50\times 96.92\times 12}{7.48\times 24}=51{,}400 \text{ cubic feet}$$

This quantity added to the 155,400 cubic feet (determined in the same manner) that percolated through this cylinder in the preceding 36 hours gives a total of 206,800 cubic feet ground water that percolated through the 50-foot cylinder in 48 hours. The quantity of water that percolated through the 280-foot cylinder in the last 12 hours of pumping was as follows:

$$Y=\frac{2\times3.1416\times975\times0.00369\times280\times99.42\times12}{7.48\times24}=41{,}900 \text{ cubic feet}$$

This quantity added to the 97,700 cubic feet that percolated through this cylinder in the preceding 36 hours gives a total of 139,600 cubic feet of ground water that percolated through the 280-foot cylinder in 48 hours. The quantity of ground water taken from storage between the 50-foot and 280-foot cylinders equals 206,800 minus 139,600, or 67,200 cubic feet; the volume of material that was unwatered between these cylinders was 336,400 cubic feet (table 14). Hence the specific yield is computed as follows:

$$\frac{67{,}200\times100}{336{,}400}=20.0$$

The computed specific yield becomes larger as the pumping increases (table 17), the reason being that all the water in the material does not drain out of it immediately. Investigations have shown that a sample of material after being saturated will yield a very large percentage of its water within a few hours, though it may continue to yield small amounts for several years. When pumping first starts a comparatively large volume of material is unwatered, partly because only a small percentage of the water contained in the interstices of the sediments immediately drains down to the water table. As pumping is continued more water gradually drains out of the unwatered sediments, and hence the specific yield computed from the first few hours of pumping is relatively small. The specific yield computed from the volume of water-bearing material unwatered in the last few hours of pumping would be larger, because of the addition of water that percolated down from the material previously unwatered. The average values for specific yield (table 17) plotted against the periods of pumping fall on a smooth curve. By extending this curve the conclusion is reached that the true specific yield lies between 22 and 23.

A sample of the water-bearing material that was unwatered during this test was examined in the hydrologic laboratory of the United States Geological Survey. The porosity was found to be 27.1 and the moisture equivalent 2.6 (table 1). If the moisture equivalent is used roughly for specific retention, as is sometimes done, the specific yield of the sample is computed to be 24.5; if Piper's relation between moisture equivalent and specific retention [21] is used, giving a specific retention of about 5 for materials with a moisture equivalent of 2.6, the specific yield is computed to be 22.1. Both of these values compare favorably with the value determined from this pumping test.

[21] Piper, A. M., Notes on the relation between the moisture equivalent and the specific yield of water-bearing materials: Am. Geophys. Union Trans., 1933, pp. 481–487.

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