The Chaotic Pendulum

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Second year computational project report

April 2024

Abstract

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1 Introduction

2 Theory

Chaos can generally be seen through

- Sensitivity to initial conditions,
- Topological mixing, and
- Dense periodic orbits.

There are multiple maps and plots that can be made to examine these prperties. The most common of these is the Poincaré section. This is a plot of the phase space of the system, with position on the x-axis and velocity on the y-axis. The Poincaré section is a useful tool for examining the periodic orbits of the system, and the general method of using the state space of the system is useful in analyzing the other two properties, topological mixing and initial condition sensitivity. Other tools include the bifurcation diagram, which shows the states of the system as a function of the driving force, as well as the Poincaré plot, which is a useful tool in examining the underlying structure of the system. [1]

3 Methodology

For simulating a system such as the chaotic pendulum, care must be taken to ensure the numerical methods used are very precise, and to use a small enough time step to ensure that the system is accurately represented. Euler's method is simple and can be used to simulate the system, but it is not very accurate in contrast to the Runge-Kutta method, which uses half steps in the integration to give a better stepwise estimate. Although more computationally expensive, the Runge-Kutta method is a good choice for this project, specifically RK4 [1]. RK4 was used to calculate the steps of angular velocity using the equation of motion for the forced, damped pendulum, and Euler's method was used to calculate the angle from the angular velocity, as since the angular velocities are calculated discretely with RK4 there is no way to take the half step which is required for another RK4 iteration in a way that would provide a benefit to the numerical computation. The equation of motion for this system [2] is given by

$$\frac{d^2\theta}{dt^2} = -\frac{g}{R}\sin(\theta) - \frac{b}{M}\frac{d\theta}{dt} + F_d \sin(\Omega_d t) \tag{1}$$

where θ is the angle of the pendulum, t is the time since the initial condition, g is the acceleration due to gravity, R is the length of the pendulum, b is the damping coefficient, M is the mass of the pendulum, F_d is the driving force, and Ω_d is the frequency of the driving force. This is a second order differential equation, requiring a mesh of 2d initial conditions. For this analysis, we assume $g = 9.81\text{ms}^{-2}$, $R = 10\text{m}$, $b = 0.1\text{kgs}^{-1}$, and $M = 10\text{kg}$ to all be constant, and vary the driving force, F_d, and the frequency of the driving force, Ω_d. The system will be analyzed using Poincaré sections [3], bifurcation diagrams [4], Poincaré plots [5], and the Lyapunov exponent [6], as well as a qualitative analysis of the phase space evolution of the system.

3.1 Poincaré Sections

Poincaré sections for this project were constructed by analyzing individual initial conditions and plotting the position of the pendulum against the velocity of the pendulum until the system reached a periodic orbit. Determining what makes a periodic orbit in a chaotic system with numerical methods is not completely obvious - the error in the numerical method was tracked for each initial condition, and the periodic orbit was determined to be when the point returned to within the error of the initial condition. The error was determined by the step size of the Runge-Kutta method and Euler method and then propagated forward through the equation of motion.

The Poincaré section will look different depending on whether or not the system is chaotic. If the system is not chaotic, which is theoretically what should be recovered in the case of a small driving force, the Poincaré section will look like an ellipse or a circle. If the system is chaotic, the Poincaré section will look like a dense cloud of points, visually not appearing to have any structure. We can use qualitative analysis of the structure of the graph and whether or not such graphs are calculable in the given time frame to determine the chaotic behavior of the system, since a chaotic system should have dense periodic orbits that don't closely resemble ellipses.

3.2 Bifurcation Diagrams

Bifurcation diagrams were constructed by analyzing the system of the pendulum as a function of the driving force. The system was simulated for a range of driving forces, and the system of the pendulum was plotted for many initial conditions over a long time frame. The bifurcation diagram was then analyzed for periodic orbits and chaotic behavior. Bifurcation diagrams were created for both the angle of the system and the angular velocity, and were made using a 2d histogram on a log scale of counts

A spread bifurcation diagram is indicative of a chaotic system, while a rigid bifurcation diagram is indicative of a non-chaotic system. The system may become chaotic at a certain driving force, and the bifurcation diagram can be used to determine the driving force at which it does.

3.3 Poincaré Plots

Poincaré plots were constructed by plotting the position of the pendulum at a given point on the x axis and the next position of the pendulum on the y axis to get a sense of the underlying structure of the system. Structure in the Poincare plot can be indicative of certain patterns of behavior and are often used to determine the governing structure of the system, though the Poincaré plot is not as useful for determining the chaotic behavior of the system as other methods, and would require more in-depth analysis to reach conclusions of chaos.

3.4 Lyapunov Exponent

The Lyapunov exponent is a measure of the sensitivity to initial conditions of a system. It measures the rate at which adjacent points in the phase space diverge from each other. The Lyapunov exponent applies locally to two adjacent points in phase space, but may not reflect the overall behavior for the system for cases such as this one where the system is modular. It may therefore be more useful to analyze the difference graphs from which the Lyapunov exponent would be constructed.

3.5 Phase Space Analysis

The phase space is a 2d plot of the angle of the pendulum on the x-axis and the angular velocity of the pendulum on the y-axis. The phase space is a useful tool for analyzing the behavior of the system, and can be used qualitatively to assess sensitivity to initial conditions and topological mixing. A full GUI was created for analyzing the system qualitatively, however the format of the lab report means that only certain snapshots can be shown, and a link to mp4 animations will be provided, as well as a link to the project on GitHub, which can be run with python to show the GUI.

4 Results

4.1 Poincaré Sections

The forces and frequencies analyzed for the Poincaré sections were chosen qualitatively based on the bifurcation diagrams. Below are three groups of Poincaré sections for driving forces of 0.5, 1.0, and 2.0 Newtons, organized into columns of uniform driving frequency.

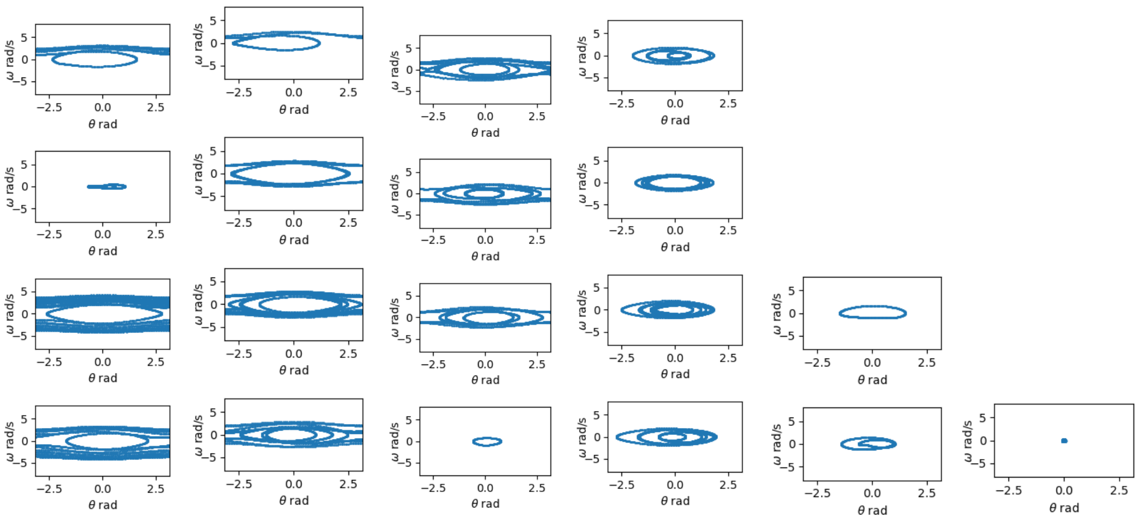

Figure 1: Poincaré sections of the chaotic pendulum for a driving force of a half Newton, as a function of the driving frequency Ω_d=0.15, 0.3, 0.5, 1.0, 1.5, and 3.0 (from left to right).

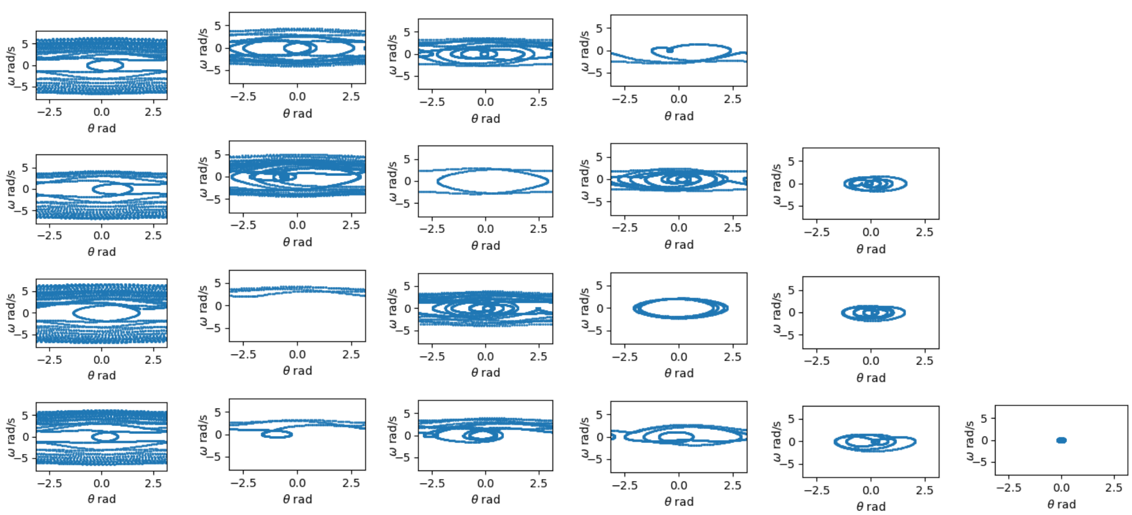

Figure 2: Poincaré sections of the chaotic pendulum for a driving force of one Newton, as a function of the driving frequency Ω_d=0.15, 0.3, 0.5, 1.0, 1.5, and 3.0 (from left to right).

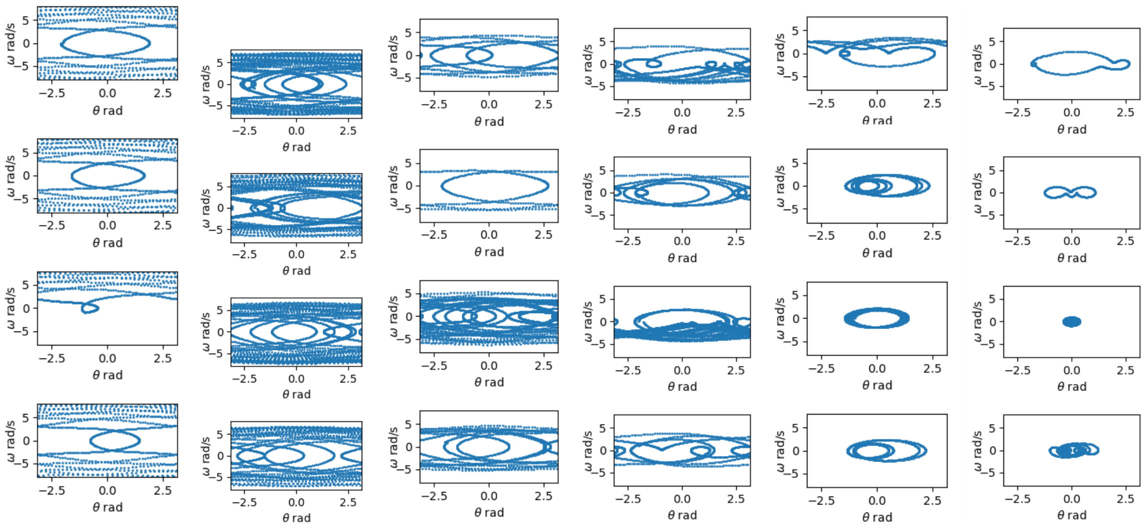

Figure 3: Poincaré sections of the chaotic pendulum for a driving force of two Newtons, as a function of the driving frequency Ω_d=0.15, 0.3, 0.5, 1.0, 1.5, and 3.0 (from left to right).

Two main qualitative observations can be made from the Poincaré sections - the first is that at higher values of the driving frequencies, the iterator has a harder time finding 4 periodic orbits, indicating that the periodic orbits are less dense at those values, and thus the system is less chaotic. The second qualitative observation is that the system is more chaotic at higher driving forces, as can be seen by the density of the points and lack of ellipse structure in the Poincaré sections for higher forces that is well observed at lower force values. These qualitative differences help demonstrate the chaotic behavior of the system at higher force and lower frequency values.

4.2 Bifurcation Diagrams

Below are a selection of bifurcation diagrams from the dynamic program that was created to analyze the chaotic pendulum. Due to the amount of data points involved and the limitations of scatter plots on performance, a log-scaled 2d histogram was used to represent the data. The values of the driving frequency were chosen based on qualitative analysis of the system.

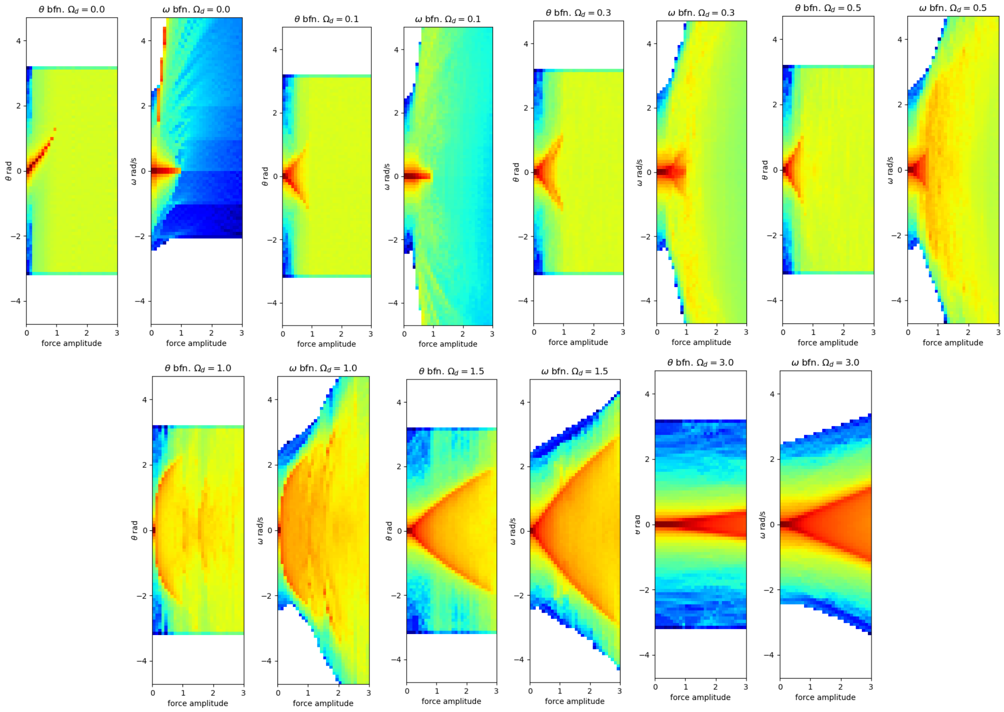

Figure 4: Bifurcation diagrams of the angle and angular velocity of the pendulum as a function of the driving force

Using these bifurcation diagrams, we observed the system to generally be highly chaotic above a driving force of one Newton, and to generally be moderately to lightly chaotic below a driving force of one Newton, becoming non-chaotic closer to zero. This result was used to inform the selection of driving forces for the Poincaré sections. The chaos of the system can be seen through the bifurcation diagrams based on their spread, and as can be seen in every case the diagram becomes more spread for higher force values, with the most rapid spreading for values of the driving frequency between 0.15 and 1.5 Hz. Below this range the frequency is so low that the force becomes either dominant in the motion of the pendulums or negligible depending on its magnitude, and above the range the force is acting too quickly for many chaotic effects to occur, and a dense collection of points can be seen at a smaller spread of angles and angular velocities.

4.3 Poincaré Plots

Poincaré plots were established for forces and frequencies of interest. They were constructed as scatter plots to get an idea of the spread of the data without diving too deep into the distribution of the data.

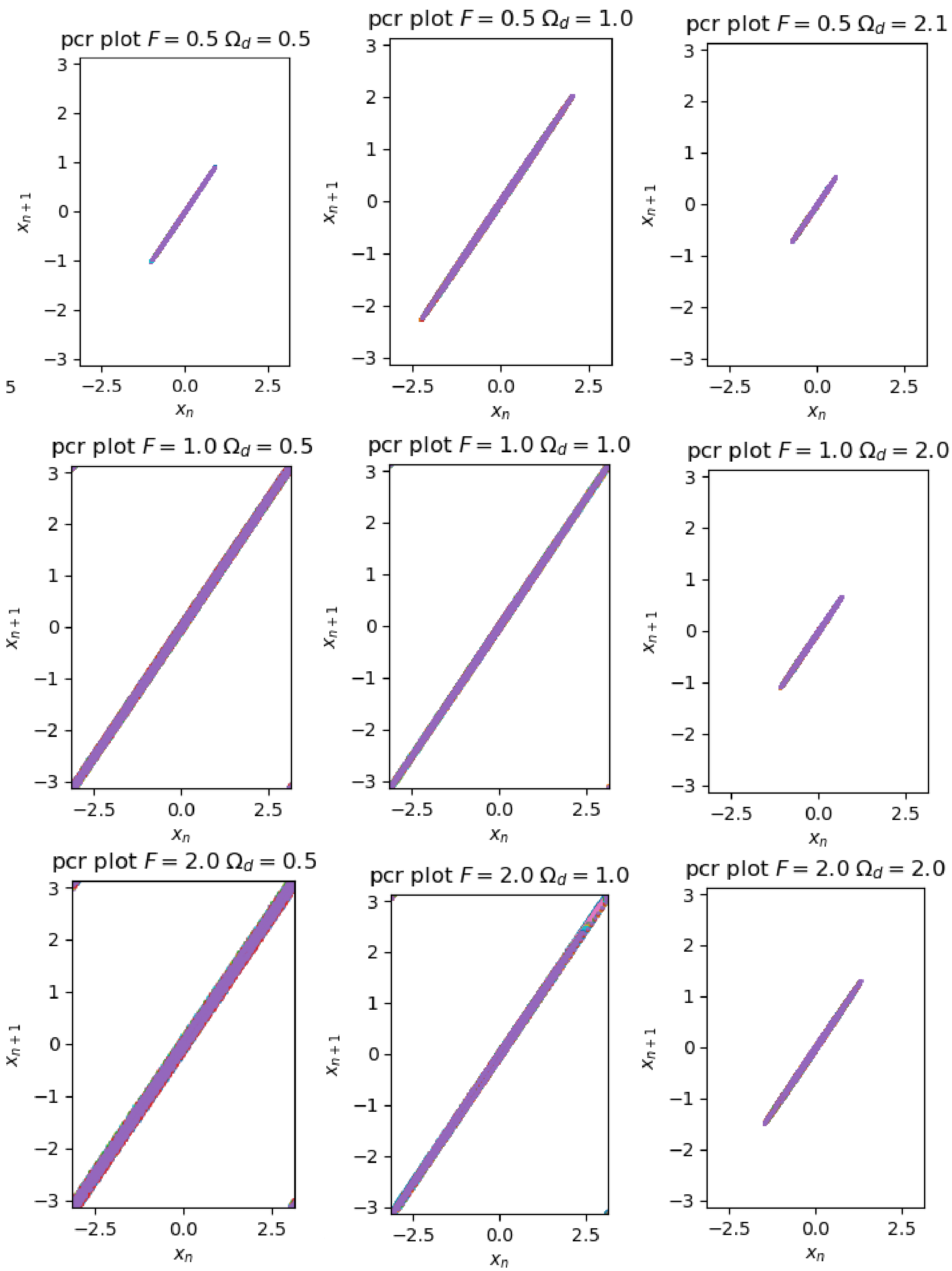

Figure 5: Poincaré plots of the chaotic pendulum for different driving forces and driving frequencies.

It can be seen that the Poincaré plots are the most spread for higher values of the force and lower values of the frequency, which can be an indicator of chaotic behavior. What's most interesting about the Poincaré plots is that they all follow a similar structure, a line of some width from the bottom left to the top right of the graph. This tells us that for each angle, the system will either move to the left or to the right with some relatively predictable velocity. These graphs aren't very useful for determining the chaotic behavior of the system, but they are useful for determining the underlying structure of the system, and showing that chaos does not necessarily mean that the system is unpredictable.

4.4 Lyapunov Exponent

While the Lyapunov Exponent could be calculated numerically, we thought it more appropriate to use a qualitative analysis of the system and the difference data to assess whether or not the system was chaotic and its sensitivity to initial conditions. The reason for this is that the graph that would be used to determine the Lyapunov exponent contains far more information about the behavior of different initial conditions and about the growth or decay of the differences than a single number would. Due to space concerns on the GUI the force and frequency were not included in the titles of the graphs, but they follow the same layout with the same values as the Poincaré plots

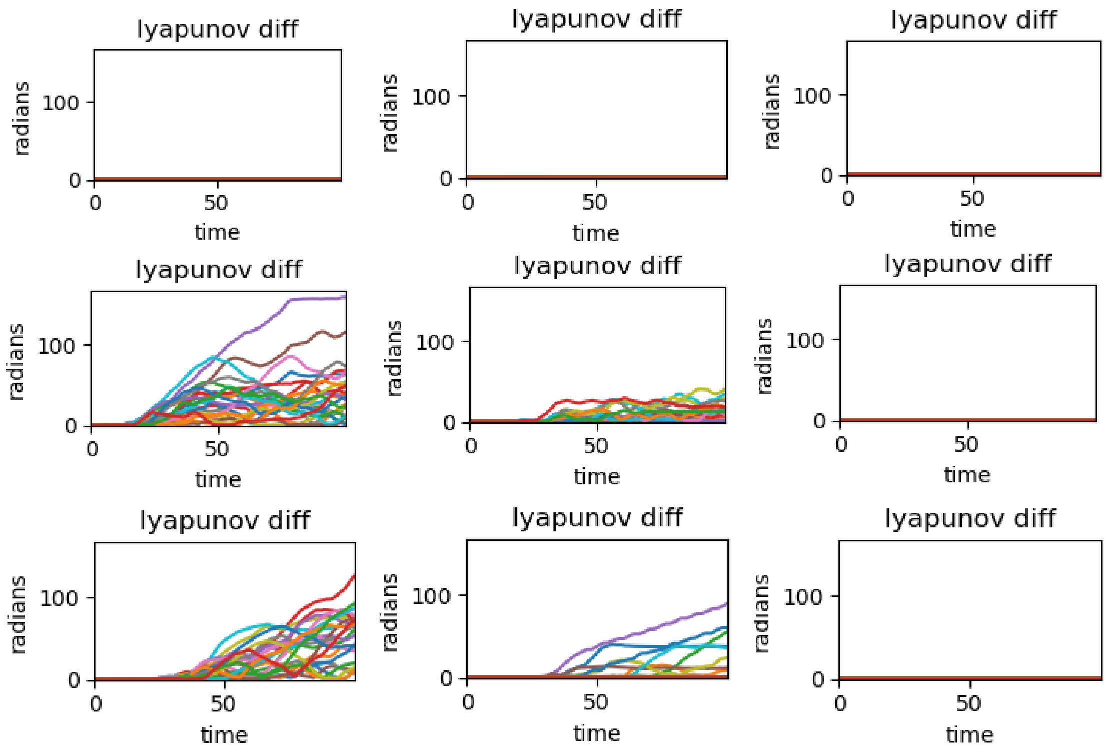

Figure 6: Poincaré plots of the chaotic pendulum for different driving forces and driving frequencies.

The Lyapunov exponent graphs are a much better qualitative indicator of chaos than the Poincaré plots for the same force and frequency magnitudes. The graphs that aren't all within one rotation (2π) of each other show an extreme sensitivity to initial conditions, which tends to happen for the lower frequency and higher force values. The graphs themselves aren't particularly exponential over large scales, which becomes even more relevant when you consider that these are the un-modded difference graphs - if you were to observe the experiment physically where $2\pi = 0$, it becomes impossible to see an exponential difference in initial conditions. The choice to use qualitative analysis with regard to the Lyapunov exponent becomes apparent - the point at which local exponential differences become outdone by global patterns is arbitrary, and quantitative analysis would have little use.

4.5 Phase Space Analysis

5 Conclusions

References

[1] S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering.* Westview Press, 2000.

[2] S.-Y. Kim and K. Lee, "Multiple transitions to chaos in a damped parametrically forced pendulum," *Phys. Rev. E*, vol. 53, pp. 1579–1586, Feb 1996.

[3] G. Teschl, *Ordinary Differential Equations and Dynamical Systems.* Graduate Studies in Mathematics, AMER MATHEMATICAL SOCIETY, 2024.

[4] J. D. Crawford, "Introduction to bifurcation theory," *Rev. Mod. Phys.*, vol. 63, pp. 991–1037, Oct 1991.

[5] P. W. Kamen, H. Krum, and A. M. Tonkin, "Poincare plot of heart rate variability allows quantitative display of parasympathetic nervous activity in humans," *Clinical science*, vol. 91, no. 2, pp. 201–208, 1996.

[6] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, "Determining lyapunov exponents from a time series," *Physica D: Nonlinear Phenomena*, vol. 16, no. 3, pp. 285–317, 1985.