

# The Chaotic Pendulum

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## **Abstract**

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# 1 Introduction

## 2 Theory

Chaos can generally be seen through

- Sensitivity to initial conditions,
- Topological mixing, and
- Dense periodic orbits.

There are multiple maps and plots that can be made to examine these properties. The most common of these is the Poincaré section. This is a plot of the phase space of the system, with the position of the pendulum on the x-axis and the velocity on the y-axis. The Poincaré section is a useful tool for examining the periodic orbits of the system, and the general method of using the state space of the system is useful in analyzing the other two properties, topological mixing and initial condition sensitivity. Other tools include the bifurcation diagram, which shows the states of the system as a function of the driving force, as well as the Poincaré plot, which is a useful tool in examining the underlying structure of the system. [1]

## 3 Methodology

When simulating a chaotic system, it is important to use a numerical method that is very precise and to use a small enough time step to ensure that the system is accurately represented. The Euler method is a simple method that can be used to simulate the system, but it is not very accurate. The Runge-Kutta method is a more accurate method that can be used to simulate the system, and although it may be more computationally expensive, the Runge-Kutta method is a good choice for simulating a chaotic system, specifically RK4 [1]. In this project, RK4 was used to calculate the steps of angular velocity using the equation of motion for the forced, damped pendulum, and Euler's method was used to calculate the angle from the angular velocity, as the angular velocities are calculated discretely, there is no way to take a half step which is required for RK4. The equation of motion for this system is given by [2]

$$\frac{d^2\theta}{dt^2} = -\frac{g}{R}\sin(\theta) - \frac{b}{M}\frac{d\theta}{dt} + F_d\sin(\Omega_d t) \quad (1)$$

where  $\theta$  is the angle of the pendulum,  $t$  is the time since the initial condition,  $g$  is the acceleration due to gravity,  $R$  is the length of the pendulum,  $b$  is the damping coefficient,  $M$  is the mass of the pendulum,  $F_d$  is the driving force, and  $\Omega_d$  is the frequency of the driving force. This is a second order differential equation, requiring a mesh of 2d initial conditions. For this analysis, we assume  $g$ ,  $R$ ,  $b$ , and  $M$  to all be constant, and vary the driving force,  $F_d$ , and the frequency of the driving force,  $\Omega_d$ . The system will be analyzed using Poincaré sections [3], bifurcation

diagrams [4], Poincaré plots [5], and the Lyapunov exponent [6], as well as a qualitative analysis of the phase space evolution of the system.

### 3.1 Poincaré Sections

Poincaré sections for this project were constructed by analyzing individual initial conditions and plotting the position of the pendulum against the velocity of the pendulum until the system reached a periodic orbit. The Poincaré section was then plotted, and the periodic orbit was analyzed. This was repeated for multiple initial conditions. Determining what makes a periodic orbit in a chaotic system with numerical methods is not completely obvious - the error in the numerical method was tracked for each initial condition, and the periodic orbit was determined to be when the point returned to within the error of the initial condition. The error was determined by the step size of the Runge-Kutta method and Euler method and then propagated forward through the equation of motion.

### 3.2 Bifurcation Diagrams

Bifurcation diagrams were constructed by analyzing the system of the pendulum as a function of the driving force. The system was simulated for a range of driving forces, and the system of the pendulum was plotted for many initial conditions over a long time frame. The bifurcation diagram was then analyzed for periodic orbits and chaotic behavior. Bifurcation diagrams were created for both the angle of the system and the angular velocity, and were made using a 2d histogram on a log scale of counts

### 3.3 Poincaré Plots

Poincaré plots were constructed by plotting the position of the pendulum at a given point on the x axis and the next position of the pendulum on the y axis to get a sense of the underlying structure of the system. Structure in the Poincare plot can be indicative of certain patterns of behavior and are often used to determine the governing structure of the system. In our case, we can use the distribution of points to determine whether or not the system is exhibiting chaotic behavior - if the system is chaotic, the Poincaré plot will be dense, though there may be a general structure that can give insight into the system.

### 3.4 Lyapunov Exponent

The Lyapunov exponent is a measure of the sensitivity to initial conditions of a system. The Lyapunov exponent is calculated by taking the derivative of the system with respect to the initial

condition and then taking the logarithm of the absolute value of the derivative. The Lyapunov exponent is then the average of this value over time. The Lyapunov exponent is a useful tool for determining the sensitivity to initial conditions of a system, and can be used to determine whether or not a system is chaotic.

### **3.5 Phase Space Analysis**

The phase space is a 2d plot of the angle of the pendulum on the x-axis and the angular velocity of the pendulum on the y-axis. The phase space is a useful tool for analyzing the behavior of the system, and can be used qualitatively to assess sensitivity to initial conditions and topological mixing. A full GUI was created for analyzing the system qualitatively, however the format of the lab report means that only certain snapshots can be shown, and a link to mp4 animations will be provided, as well as a link to the project on GitHub, which can be run with python to show the GUI.

## 4 Results

### 4.1 Poincaré Sections

### 4.2 Bifurcation Diagrams

### 4.3 Poincaré Plots

### 4.4 Lyapunov Exponent

### 4.5 Phase Space Analysis

## 5 Conclusions

### 5.1 Poincaré Sections

### 5.2 Bifurcation Diagrams

### 5.3 Poincaré Plots

### 5.4 Lyapunov Exponent

### 5.5 Phase Space Analysis

### 5.6 Wholistic Analysis

## References

- [1] S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering*. Westview Press, 2000.
- [2] S.-Y. Kim and K. Lee, “Multiple transitions to chaos in a damped parametrically forced pendulum,” *Phys. Rev. E*, vol. 53, pp. 1579–1586, Feb 1996.
- [3] G. Teschl, *Ordinary Differential Equations and Dynamical Systems*. Graduate Studies in Mathematics, AMER MATHEMATICAL SOCIETY, 2024.
- [4] J. D. Crawford, “Introduction to bifurcation theory,” *Rev. Mod. Phys.*, vol. 63, pp. 991–1037, Oct 1991.
- [5] P. W. Kamen, H. Krum, and A. M. Tonkin, “Poincare plot of heart rate variability allows quantitative display of parasympathetic nervous activity in humans,” *Clinical science*, vol. 91, no. 2, pp. 201–208, 1996.
- [6] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, “Determining lyapunov exponents from a time series,” *Physica D: Nonlinear Phenomena*, vol. 16, no. 3, pp. 285–317, 1985.