The Chaotic Pendulum

Luna Greenberg and *Hamza Yasin*

11017146 and *<Hamza put your student ID here>*

School of Physics and Astronomy

University of Manchester

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Abstract

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1 Introduction

2 Theory

Chaos can generally be seen through

- Sensitivity to initial conditions,
- Topological mixing, and
- Dense periodic orbits.

There are multiple maps and plots that can be made to examine these prperties. The most common of these is the Poincaré section. This is a plot of the phase space of the system, with the position of the pendulum on the x-axis and the velocity on the y-axis. The Poincaré section is a useful tool for examining the periodic orbits of the system, and the general method of using the state space of the system is useful in analyzing the other two properties, topological mixing and initial condition sensitivity. Other tools include the bifurcation diagram, which shows the states of the system as a function of the driving force, as well as the Poincaré plot, which is a useful tool in examining the underlying structure of the system. [1]

3 Methodology

When simulating a chaotic system, it is important to use a numerical method that is very precise and to use a small enough time step to ensure that the system is accurately represented. The Euler method is a simple method that can be used to simulate the system, but it is not very accurate. The Runge-Kutta method is a more accurate method that can be used to simulate the system, and although it may be more computationally expensive, the Runge-Kutta method is a good choice for simulating a chaotic system, specifically RK4 [1]. In this project, RK4 was used to calculate the steps of angular velocity using the equation of motion for the forced, damped pendulum, and Euler's method was used to calculate the angle from the angular velocity, as the angular velocities are calculated discretely, there is no way to take a half step which is required for RK4. The equation of motion for this system is given by [2]

$$\frac{d^2\theta}{dt^2} = -\frac{g}{R}\sin(\theta) - \frac{b}{M}\frac{d\theta}{dt} + F_d\sin(\Omega_d t) \tag{1}$$

where θ is the angle of the pendulum, t is the time since the initial condition, g is the acceleration due to gravity, R is the length of the pendulum, b is the damping coefficient, M is the mass of the pendulum, F_d is the driving force, and Ω_d is the frequency of the driving force. This is a second order differential equation, requiring a mesh of 2d initial conditions. For this analysis, we assume g, R, b, and M to all be constant, and vary the driving force, F_d, and the frequency of the driving force, Ω_d. The system will be analyzed using Poincaré sections [3], bifurcation diagrams [4], Poincaré plots [5], and the Lyapunov exponent [6], as well as a qualitative analysis of the phase space evolution of the system.

3.1 Poincaré Sections

Poincaré sections for this project were constructed by analyzing individual initial conditions and plotting the position of the pendulum against the velocity of the pendulum until the system reached a periodic orbit. The Poincaré section was then plotted, and the periodic orbit was analyzed. This was repeated for multiple initial conditions. Determining what makes a periodic orbit in a chaotic system with numerical methods is not completely obvious - the error in the numerical method was tracked for each initial condition, and the periodic orbit was determined to be when the point returned to within the error of the initial condition. The error was determined by the step size of the Runge-Kutta method and Euler method and then propagated forward through the equation of motion.

The Poincaré section will look different depending on whether or not the system is chaotic. If the system is not chaotic, which is theoretically what should be recovered in the case of a small driving force, the Poincaré section will look like an ellipse or a circle. If the system is chaotic, the Poincaré section will look like a dense cloud of points, visually not appearing to have any structure. We can perform an ellipse regression test on the poincare map and use the residuals to determine whether or not the system is chaotic.

3.2 Bifurcation Diagrams

Bifurcation diagrams were constructed by analyzing the system of the pendulum as a function of the driving force. The system was simulated for a range of driving forces, and the system of the pendulum was plotted for many initial conditions over a long time frame. The bifurcation diagram was then analyzed for periodic orbits and chaotic behavior. Bifurcation diagrams were created for both the angle of the system and the angular velocity, and were made using a 2d histogram on a log scale of counts

A dense bifurcation diagram is indicative of a chaotic system, while a sparse bifurcation diagram is indicative of a non-chaotic system. The system may become chaotic at a certain driving force, and the bifurcation diagram can be used to determine the driving force at which it does.

3.3 Poincaré Plots

Poincaré plots were constructed by plotting the position of the pendulum at a given point on the x axis and the next position of the pendulum on the y axis to get a sense of the underlying structure of the system. Structure in the Poincare plot can be indicative of certain patterns of behavior and are often used to determine the governing structure of the system. In our case, we can use the distribution of points to determine whether or not the system is exhibiting chaotic behavior - if the system is chaotic, the Poincaré plot will be dense, though there may be a general structure that can give insight into the system.

3.4 Lyapunov Exponent

The Lyapunov exponent is a measure of the sensitivity to initial conditions of a system. It measures the rate at which adjacent points in the phase space diverge from each other. A chaotic system will have a positive Lyapunov exponent, while a non-chaotic system will not fit the exponential growth of the Lyapunov exponent very well, which can be tested with regression.

3.5 Phase Space Analysis

The phase space is a 2d plot of the angle of the pendulum on the x-axis and the angular velocity of the pendulum on the y-axis. The phase space is a useful tool for analyzing the behavior of the system, and can be used qualitatively to assess sensitivity to initial conditions and topological mixing. A full GUI was created for analyzing the system qualitatively, however the format of the lab report means that only certain snapshots can be shown, and a link to mp4 animations will be provided, as well as a link to the project on GitHub, which can be run with python to show the GUI.

4 Results

4.1 Poincaré Sections

4.2 Bifurcation Diagrams

4.3 Poincaré Plots

4.4 Lyapunov Exponent

4.5 Phase Space Analysis

5 Conclusions

5.1 Poincaré Sections

5.2 Bifurcation Diagrams

5.3 Poincaré Plots

5.4 Lyapunov Exponent

5.5 Phase Space Analysis

5.6 Wholistic Analysis

References

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