

## Exercise 2: Free-Fall Collapse of a Homogeneous Sphere

Analytic solution:

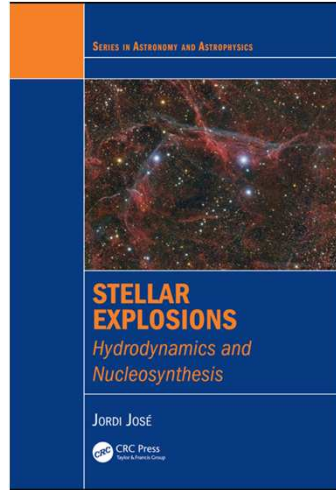
$$\left[ \frac{8\pi G \rho_o}{3} \right]^{1/2} (t - t_o) = \left[ 1 - \frac{r}{r_o} \right]^{1/2} \left[ \frac{r}{r_o} \right]^{1/2} + \arcsin \left( 1 - \frac{r}{r_o} \right)^{1/2}$$

Initial conditions: at  $t = t_o$ ,  $u_o = 0 \text{ cm s}^{-1}$

No artificial viscosity;  $P = 0$

Physical magnitude	Notation	Units of Measure
Lagrangian mass	$m_i$	g
Star's age	$t^n$	s
Velocity	$u_i$	cm s <sup>-1</sup>
Luminosity	$L_i$	erg s <sup>-1</sup>
Radius	$r_i$	cm
Temperature	$T_{i+1/2}$	K
Density	$\rho_{i+1/2}$	g cm <sup>-3</sup>
Specific volume	$V_{i+1/2}$	cm <sup>3</sup> g <sup>-1</sup>
Pressure	$P_{i+1/2}$	dyne cm <sup>-2</sup>
Internal energy	$E_{i+1/2}$	erg g <sup>-1</sup>
Energy generation rate	$\epsilon_{i+1/2}$	erg g s <sup>-1</sup>
Opacity	$\kappa_{i+1/2}$	cm <sup>2</sup> g <sup>-1</sup>

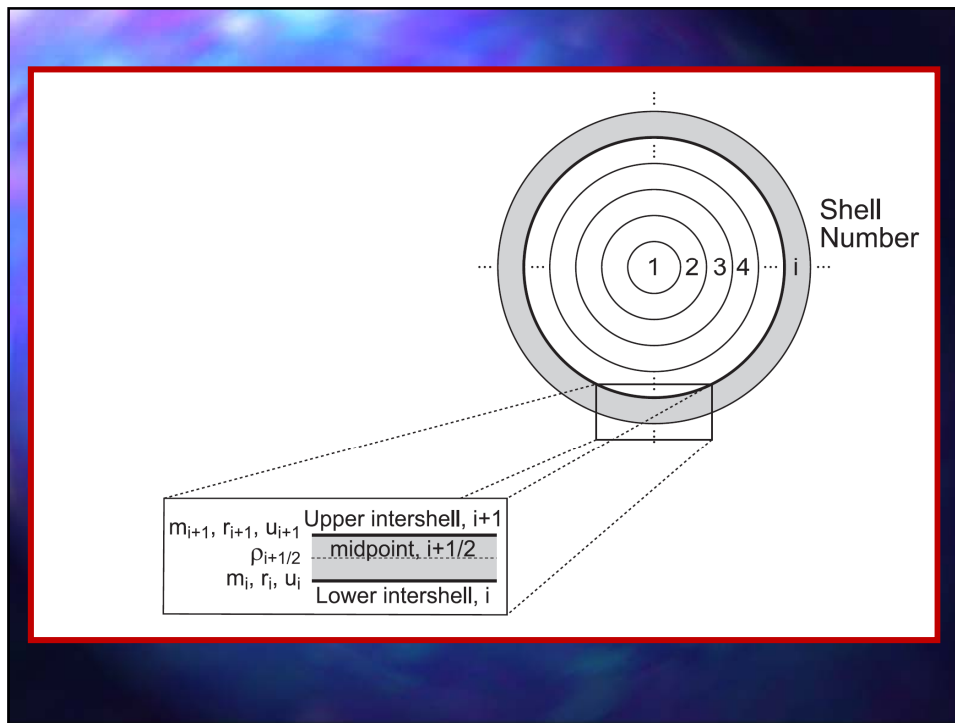
We will consider the **set of differential eqs.** governing, for instance, a given fluid element (*stellar astrophysics, atmospheric transport, sea currents...*)



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### Differential Equations

- Conservation of mass

$$\frac{1}{\rho} = \frac{4}{3}\pi \frac{\partial r^3}{\partial m}$$

- Conservation of momentum ( $P = q = 0$ )

$$\frac{\partial u}{\partial t} = -G \frac{m}{r^2}$$

- Lagrangian velocity

$$\frac{\partial r}{\partial t} = u$$



## Discretization

- Conservation of mass

$$\frac{1}{\rho_{i+1/2}} = \frac{4}{3}\pi \frac{r_{i+1}^3 - r_i^3}{m_{i+1} - m_i}$$

- Conservation of momentum

$$\frac{u_{i+1}^{n+1} - u_{i+1}^n}{\Delta t} = (1 - \beta) \left( \frac{-Gm_{i+1}}{r_{i+1}^2} \right)^n + \beta \left( \frac{-Gm_{i+1}}{r_{i+1}^2} \right)^{n+1}$$

- Lagrangian velocity

$$\frac{r_{i+1}^{n+1} - r_{i+1}^n}{\Delta t} = (1 - \beta)u_{i+1}^n + \beta u_{i+1}^{n+1}$$

## Initial Model, Boundary Conditions, and Scaling

If the  $N$  concentric shells contain the same mass,  $\Delta m \equiv M_{tot}/N$ , the interior mass variable  $m_i$  is simply given by:

$$m_{i+1} = i \Delta m \quad (i = 1, N).$$

Note that  $m_{N+1} \equiv M_{tot}$ , by construction, with  $M_{tot} = \frac{4}{3}\pi R_o^3 \rho_o$  for a homogeneous sphere of initial radius  $R_o$  and density  $\rho_o$ .

If the computational domain extends all the way from the center of the sphere to its surface<sup>34</sup>, the radius and interior mass variable at the first intershell trivially become  $r_1 = m_1 = 0$ . Moreover, for a homogenous sphere,  $\rho_{i+1/2} \equiv \rho_o$  ( $i=1, N$ ), by definition, and assuming an initial static configuration,  $u_i = 0$  ( $i=1, N+1$ ) at  $t = 0$ .

Once boundary conditions are applied to the innermost shell, mass conservation equation becomes

$$r_2 = \left( \frac{3m_2}{4\pi\rho_{3/2}} \right)^{1/3}$$

for  $r_1 = m_1 = 0$ , while the radii of the subsequent intershells can be obtained through

$$r_{i+1} = \left( \frac{3(m_{i+1} - m_i)}{4\pi\rho_{i+1/2}} + r_i^3 \right)^{1/3} = \left( \frac{3\Delta m}{4\pi\rho_{i+1/2}} + r_i^3 \right)^{1/3} \quad (i = 2, N).$$

We will assume *Lagrangian* formulation; physical variables will be *rescaled* to suitable new variables:

$$W = \ln V = \ln (1/\rho)$$

$$R = \ln r$$

$$Q = 1 - m_{\text{int}}/m_{\text{Total}}$$

## Equations for the Innermost, Intermediate and Surface Shells

### Equations for the Innermost Shell

The finite difference equations for the innermost shell of the sphere, with the corresponding boundary conditions, can be written in compact form as a function  $C$  that depends on a number of unknowns. Let's start with the mass conservation equation: for  $m_1 = r_1 = u_1 = 0$ , it can be written as

$$C^1 = \frac{1}{(\rho_{3/2})^{n+1}} - \frac{4}{3}\pi \frac{(r_2^3)^{n+1}}{m_2} = C^1(r_2, \rho_{3/2}) = 0.$$

Momentum conservation and the equation for the Lagrangian velocity can, in turn, be expressed as:

$$C^2 = \frac{u_2^{n+1} - u_2^n}{\Delta t} - (1 - \beta) \left( \frac{-Gm_2}{r_2^2} \right)^n - \beta \left( \frac{-Gm_2}{r_2^2} \right)^{n+1} =$$

$$C^2(u_2, r_2) = 0,$$

$$C^3 = \frac{r_2^{n+1} - r_2^n}{\Delta t} - (1 - \beta)u_2^n - \beta u_2^{n+1} = C^3(u_2, r_2) = 0.$$

Globally, this set of equations can be written as a function of just 3 unknowns,  $u_2$ ,  $r_2$ , and  $\rho_{3/2}$ , such that

$$C^j = C^j(\rho_{3/2}, u_2, r_2) = 0 \quad (j = 1, 3).$$

### Equations for the Intermediate Shells

The same procedure is then applied to the  $N-2$  intermediate shells ( $i=2, N-1$ ), in the form:

$$F_i^1 = \frac{1}{(\rho_{i+1/2})^{n+1}} - \frac{4}{3}\pi \frac{(r_{i+1}^3)^{n+1} - (r_i^3)^{n+1}}{m_{i+1} - m_i} = F_i^1(r_{i+1}, r_i, \rho_{i+1/2}) = 0,$$

$$F_i^2 = \frac{u_{i+1}^{n+1} - u_{i+1}^n}{\Delta t} - (1 - \beta) \left( \frac{-Gm_{i+1}}{r_{i+1}^2} \right)^n - \beta \left( \frac{-Gm_{i+1}}{r_{i+1}^2} \right)^{n+1} =$$

$$F_i^2(u_{i+1}, r_{i+1}) = 0,$$

$$F_i^3 = \frac{r_{i+1}^{n+1} - r_{i+1}^n}{\Delta t} - (1 - \beta)u_{i+1}^n - \beta u_{i+1}^{n+1} = F_i^3(u_{i+1}, r_{i+1}) = 0,$$

or globally,

$$F_i^j = F_i^j(\rho_{i+1/2}, r_i, r_{i+1}, u_{i+1}) = 0 \quad (i = 2, N-1; j = 1, 3).$$

### Equations for the Outermost Shell

Finally, for the outermost shell ( $i = N$ ), we have

$$S^1 = \frac{1}{(\rho_{N+1/2})^{n+1}} - \frac{4}{3}\pi \frac{(r_{N+1}^3)^{n+1} - (r_N^3)^{n+1}}{m_{N+1} - m_N} = S^1(r_{N+1}, r_N, \rho_{N+1/2}) = 0,$$

$$S^2 = \frac{u_{N+1}^{n+1} - u_{N+1}^n}{\Delta t} - (1 - \beta) \left( \frac{-Gm_{N+1}}{r_{N+1}^2} \right)^n - \beta \left( \frac{-Gm_{N+1}}{r_{N+1}^2} \right)^{n+1} =$$

$$S^2(u_{N+1}, r_{N+1}) = 0,$$

$$S^3 = \frac{r_{N+1}^{n+1} - r_{N+1}^n}{\Delta t} - (1 - \beta)u_{N+1}^n - \beta u_{N+1}^{n+1} = S^3(u_{N+1}, r_{N+1}) = 0,$$

which can be written as:

$$S^j = S^j(\rho_{N+1/2}, r_N, r_{N+1}, u_{N+1}) = 0 \quad (j = 1, 3).$$

## Linearization

Let  $x^0$  be a vector containing the *exact* values of the physical variables of the problem,  $r$ ,  $\rho$ , and  $u$ , at  $t^0 = 0$  (i.e., initial model), or, in general, at a given time,  $t^n$ . Let  $x^1$  be the corresponding vector after one step,  $t^1 = t^0 + \Delta t$ . For small enough values of the time-step,  $\Delta t$ , all physical variables would have scarcely varied from their values at  $t^0$  (i.e.,  $x^1 \sim x^0$ ). Therefore, let's consider, as a first approximate guess, that  $x^1 \equiv x^0$ . In general, such a choice would not yield the exact values of the variables at  $t^1$ , so that  $C^j(x^1) \neq 0$ ,  $F_i^j(x^1) \neq 0$ , and  $S_j(x^1) \neq 0$ . Nevertheless, since  $x^1 \sim x^0$ , one can think of a set of corrections,  $\delta x$ , that added to the first guess values,  $x^1 = x^0 + \delta x$ , will actually satisfy  $C^j(x^1) = 0$ ,  $F_i^j(x^1) = 0$ , and  $S^j(x^1) = 0$ . For small corrections, the whole set of structure equations can be written in the form

$$\begin{aligned} C^j(x^1) &= C^j(x^0) + \delta C^j = 0 \\ F_i^j(x^1) &= F_i^j(x^0) + \delta F_i^j = 0 \\ S^j(x^1) &= S^j(x^0) + \delta S^j = 0, \end{aligned} \tag{1.152}$$

$$\begin{aligned} C^j + \frac{\partial C^j}{\partial \rho_{3/2}} \delta \rho_{3/2} + \frac{\partial C^j}{\partial u_2} \delta u_2 + \frac{\partial C^j}{\partial r_2} \delta r_2 &= 0 \\ F_i^j + \frac{\partial F_i^j}{\partial \rho_{i+1/2}} \delta \rho_{i+1/2} + \frac{\partial F_i^j}{\partial r_i} \delta r_i + \frac{\partial F_i^j}{\partial r_{i+1}} \delta r_{i+1} + \frac{\partial F_i^j}{\partial u_{i+1}} \delta u_{i+1} &= 0 \\ (j = 1, 3; i = 2, N-1) \\ S^j + \frac{\partial S^j}{\partial \rho_{N+1/2}} \delta \rho_{N+1/2} + \frac{\partial S^j}{\partial r_N} \delta r_N + \frac{\partial S^j}{\partial r_{N+1}} \delta r_{N+1} + \frac{\partial S^j}{\partial u_{N+1}} \delta u_{N+1} &= 0 \\ (j = 1, 3), \end{aligned}$$



6. Example:  $N=4$

Boundary conditions:  $r_1 = m_1 = u_1 = 0$

Initial model:  $\textcircled{5}$ ,  $m_k$  ( $k=2, N+1$ ) known values

Unknowns:  $r_2, u_2, r_3, u_3, r_4, u_4, r_5, u_5$   
 $s_1, s_2, s_3, s_4$

Equations:

$$\left. \begin{aligned} \frac{\partial C^1}{\partial r_2} \delta r_2 + \frac{\partial C^1}{\partial u_2} \delta u_2 + \frac{\partial C^1}{\partial s_1} \delta s_1 &= -C^1 \\ \frac{\partial C^2}{\partial r_2} \delta r_2 + \frac{\partial C^2}{\partial u_2} \delta u_2 + \frac{\partial C^2}{\partial s_1} \delta s_1 &= -C^2 \\ \frac{\partial C^3}{\partial r_2} \delta r_2 + \frac{\partial C^3}{\partial u_2} \delta u_2 + \frac{\partial C^3}{\partial s_1} \delta s_1 &= -C^3 \end{aligned} \right\} i=1 \text{ (Center)}$$

$$\left. \begin{aligned} \frac{\partial F_2^1}{\partial r_2} \delta r_2 + \frac{\partial F_2^1}{\partial r_3} \delta r_3 + \frac{\partial F_2^1}{\partial s_2} \delta s_2 + \frac{\partial F_2^1}{\partial u_3} \delta u_3 &= -F_2^1 \\ \frac{\partial F_2^2}{\partial r_2} \delta r_2 + \frac{\partial F_2^2}{\partial r_3} \delta r_3 + \frac{\partial F_2^2}{\partial s_2} \delta s_2 + \frac{\partial F_2^2}{\partial u_3} \delta u_3 &= -F_2^2 \\ \frac{\partial F_2^3}{\partial r_2} \delta r_2 + \frac{\partial F_2^3}{\partial r_3} \delta r_3 + \frac{\partial F_2^3}{\partial s_2} \delta s_2 + \frac{\partial F_2^3}{\partial u_3} \delta u_3 &= -F_2^3 \end{aligned} \right\} i=2$$

$$\left. \begin{aligned} \frac{\partial F_3^1}{\partial r_3} \delta r_3 + \frac{\partial F_3^1}{\partial r_4} \delta r_4 + \frac{\partial F_3^1}{\partial s_3} \delta s_3 + \frac{\partial F_3^1}{\partial u_4} \delta u_4 &= -F_3^1 \\ \frac{\partial F_3^2}{\partial r_3} \delta r_3 + \frac{\partial F_3^2}{\partial r_4} \delta r_4 + \frac{\partial F_3^2}{\partial s_3} \delta s_3 + \frac{\partial F_3^2}{\partial u_4} \delta u_4 &= -F_3^2 \\ \frac{\partial F_3^3}{\partial r_3} \delta r_3 + \frac{\partial F_3^3}{\partial r_4} \delta r_4 + \frac{\partial F_3^3}{\partial s_3} \delta s_3 + \frac{\partial F_3^3}{\partial u_4} \delta u_4 &= -F_3^3 \end{aligned} \right\} i=3$$

$$\left. \begin{aligned} \frac{\partial S^1}{\partial r_4} \delta r_4 + \frac{\partial S^1}{\partial r_5} \delta r_5 + \frac{\partial S^1}{\partial \beta_4} \delta \beta_4 + \frac{\partial S^1}{\partial u_r} \delta u_r &= -S^1 \\ \frac{\partial S^2}{\partial r_4} \delta r_4 + \frac{\partial S^2}{\partial r_5} \delta r_5 + \frac{\partial S^2}{\partial \beta_4} \delta \beta_4 + \frac{\partial S^2}{\partial u_r} \delta u_r &= -S^2 \\ \frac{\partial S^3}{\partial r_4} \delta r_4 + \frac{\partial S^3}{\partial r_5} \delta r_5 + \frac{\partial S^3}{\partial \beta_4} \delta \beta_4 + \frac{\partial S^3}{\partial u_r} \delta u_r &= -S^3 \end{aligned} \right\} \begin{array}{l} i=4 \\ \text{(Surface)} \end{array}$$

$$\frac{\partial \psi}{\partial s_i} \rightarrow \begin{bmatrix} \beta_1 & \beta_2 & u_2 & \beta_2 & \beta_3 & u_3 & \beta_3 & \beta_4 & u_4 & \beta_4 & \beta_5 & u_5 \\ \frac{\partial C^1}{\partial \beta_1} & \frac{\partial C^1}{\partial \beta_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial C^2}{\partial \beta_2} & \frac{\partial C^2}{\partial u_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial C^3}{\partial \beta_2} & \frac{\partial C^3}{\partial u_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial F_2^1}{\partial \beta_2} & 0 & \frac{\partial F_2^1}{\partial \beta_3} & \frac{\partial F_2^1}{\partial u_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial F_2^2}{\partial \beta_3} & \frac{\partial F_2^2}{\partial u_3} & \frac{\partial F_2^2}{\partial u_4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial F_3^1}{\partial \beta_3} & \frac{\partial F_3^1}{\partial u_3} & \frac{\partial F_3^1}{\partial u_4} & \frac{\partial F_3^1}{\partial \beta_4} & \frac{\partial F_3^1}{\partial u_4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial F_3^2}{\partial \beta_4} & \frac{\partial F_3^2}{\partial u_4} & \frac{\partial F_3^2}{\partial u_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial F_4^1}{\partial \beta_4} & \frac{\partial F_4^1}{\partial u_4} & \frac{\partial F_4^1}{\partial u_5} & \frac{\partial S^1}{\partial \beta_4} & \frac{\partial S^1}{\partial u_r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial S^2}{\partial \beta_4} & \frac{\partial S^2}{\partial u_r} & \frac{\partial S^2}{\partial u_5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial S^3}{\partial \beta_4} & \frac{\partial S^3}{\partial u_r} & \frac{\partial S^3}{\partial u_5} \end{bmatrix} \times \begin{bmatrix} \delta \beta_1 \\ \delta \beta_2 \\ \delta u_2 \\ \delta \beta_3 \\ \delta u_3 \\ \delta \beta_4 \\ \delta u_4 \\ \delta \beta_5 \\ \delta u_5 \end{bmatrix} = \begin{bmatrix} -C^1 \\ -C^2 \\ -C^3 \\ -F_2^1 \\ -F_2^2 \\ -F_3^1 \\ -F_3^2 \\ -F_4^1 \\ -F_4^2 \\ -S^1 \\ -S^2 \\ -S^3 \end{bmatrix}$$

$\mathbf{A} \quad 12 \times 12$        $\mathbf{B} \quad 12 \times 1$        $\mathbf{C} \quad 12 \times 1$

$\rightarrow \mathbf{B} = \mathbf{A}^{-1} \mathbf{C}$  (requires matrix inversion,  $\mathbf{A}^{-1}$ )  
 $12 = \# \text{ Eqs.} \times \# \text{ Shells}$   
 $j=3 \quad \times \quad N=4$

$\frac{\partial Y}{\partial} \rightarrow$ 

	$r_2$	$u_2$	$s_1$	$r_3$	$u_3$	$s_2$	$r_4$	$u_4$	$s_3$	$r_5$	$u_5$	$s_4$
$\frac{\partial C_1}{\partial r_2}$	0	0	$\frac{\partial C_1}{\partial s_1}$	0	0	0	0	0	0	0	0	0
$\frac{\partial C_2}{\partial r_2}$	$\frac{\partial C_2}{\partial u_2}$	0	0	0	0	0	0	0	0	0	0	0
$\frac{\partial C_3}{\partial r_2}$	$\frac{\partial C_3}{\partial u_2}$	0	0	0	0	0	0	0	0	0	0	0
$\frac{\partial F_1}{\partial r_2}$	0	0	0	$\frac{\partial F_1}{\partial r_3}$	$\frac{\partial F_1}{\partial s_2}$	0	0	0	0	0	0	0
0	0	0	0	$\frac{\partial F_2}{\partial r_3}$	$\frac{\partial F_2}{\partial u_3}$	0	0	0	0	0	0	0
0	0	0	0	$\frac{\partial F_3}{\partial r_3}$	0	0	$\frac{\partial F_3}{\partial s_2}$	0	0	0	0	0
0	0	0	0	0	0	$\frac{\partial F_4}{\partial r_4}$	$\frac{\partial F_4}{\partial u_4}$	0	0	0	0	0
0	0	0	0	0	0	$\frac{\partial F_5}{\partial r_4}$	$\frac{\partial F_5}{\partial u_4}$	0	0	0	0	0
0	0	0	0	0	0	0	0	$\frac{\partial s^1}{\partial r_5}$	0	$\frac{\partial s^1}{\partial u_5}$	$\frac{\partial s^1}{\partial s_4}$	0
0	0	0	0	0	0	0	0	$\frac{\partial s^2}{\partial r_5}$	$\frac{\partial s^2}{\partial u_5}$	$\frac{\partial s^2}{\partial s_4}$	0	0
0	0	0	0	0	0	0	0	$\frac{\partial s^3}{\partial r_5}$	$\frac{\partial s^3}{\partial u_5}$	$\frac{\partial s^3}{\partial s_4}$	0	0

 $\times$ 

$\delta r_2$
$\delta u_2$
$\delta s_1$
$\delta r_3$
$\delta u_3$
$\delta s_2$
$\delta r_4$
$\delta u_4$
$\delta s_3$
$\delta r_5$
$\delta u_5$
$\delta s_4$

 $=$ 

$-C^1$
$-C^2$
$-C^3$
$-F_2^1$
$-F_3^1$
$-F_2^2$
$-F_3^2$
$-F_3^3$
$-s^1$
$-s^2$
$-s^3$

$12 \times 12$   $\times$   $12 \times 1$   $=$   $12 \times 1$

$\rightarrow B = A^{-1} \cdot C$  (requires matrix inversion,  $A^{-1}$ )  
 $12 = \# \text{ Eqs.} \times \# \text{ Shells}$   
 $j=3 \times N=4$

Starting with  $x^1 = x^0 \rightarrow$   
 We evaluate all the partial derivatives in A and eqs. in C

$\frac{\partial Y}{\partial} \rightarrow$ 

	$r_2$	$u_2$	$s_1$	$r_3$	$u_3$	$s_2$	$r_4$	$u_4$	$s_3$	$r_5$	$u_5$	$s_4$
$\frac{\partial C_1}{\partial r_2}$	0	0	$\frac{\partial C_1}{\partial s_1}$	0	0	0	0	0	0	0	0	0
$\frac{\partial C_2}{\partial r_2}$	$\frac{\partial C_2}{\partial u_2}$	0	0	0	0	0	0	0	0	0	0	0
$\frac{\partial C_3}{\partial r_2}$	$\frac{\partial C_3}{\partial u_2}$	0	0	0	0	0	0	0	0	0	0	0
$\frac{\partial F_1}{\partial r_2}$	0	0	0	$\frac{\partial F_1}{\partial r_3}$	$\frac{\partial F_1}{\partial s_2}$	0	0	0	0	0	0	0
0	0	0	0	$\frac{\partial F_2}{\partial r_3}$	$\frac{\partial F_2}{\partial u_3}$	0	0	0	0	0	0	0
0	0	0	0	$\frac{\partial F_3}{\partial r_3}$	0	0	$\frac{\partial F_3}{\partial s_2}$	0	0	0	0	0
0	0	0	0	0	0	$\frac{\partial F_4}{\partial r_4}$	$\frac{\partial F_4}{\partial u_4}$	0	0	0	0	0
0	0	0	0	0	0	$\frac{\partial F_5}{\partial r_4}$	$\frac{\partial F_5}{\partial u_4}$	0	0	0	0	0
0	0	0	0	0	0	0	0	$\frac{\partial s^1}{\partial r_5}$	0	$\frac{\partial s^1}{\partial u_5}$	$\frac{\partial s^1}{\partial s_4}$	0
0	0	0	0	0	0	0	0	$\frac{\partial s^2}{\partial r_5}$	$\frac{\partial s^2}{\partial u_5}$	$\frac{\partial s^2}{\partial s_4}$	0	0
0	0	0	0	0	0	0	0	$\frac{\partial s^3}{\partial r_5}$	$\frac{\partial s^3}{\partial u_5}$	$\frac{\partial s^3}{\partial s_4}$	0	0

 $\times$ 

$\delta r_2$
$\delta u_2$
$\delta s_1$
$\delta r_3$
$\delta u_3$
$\delta s_2$
$\delta r_4$
$\delta u_4$
$\delta s_3$
$\delta r_5$
$\delta u_5$
$\delta s_4$

 $=$ 

$-C^1$
$-C^2$
$-C^3$
$-F_2^1$
$-F_3^1$
$-F_2^2$
$-F_3^2$
$-F_3^3$
$-s^1$
$-s^2$
$-s^3$

$12 \times 12$   $\times$   $12 \times 1$   $=$   $12 \times 1$

$\rightarrow B = A^{-1} \cdot C$  (requires matrix inversion,  $A^{-1}$ )  
 $12 = \# \text{ Eqs.} \times \# \text{ Shells}$   
 $j=3 \times N=4$

Solve for  $\delta X$  ( $B = A^{-1} C$ ) and evaluate  $x^1 = x^0 + \delta x$



The procedure is **iterated** until a given **accuracy criterion** is satisfied ( $\delta x < \varepsilon$ , c.f. corrections are smaller than a given quantity). **Extrapolation** from the last two converged models is recommended as the first guess for the next time step.

