Exercise 2: Free-Fall Collapse of a Homogeneous Sphere

Analytic solution:

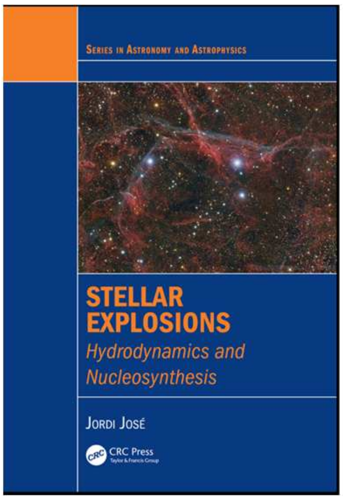
$$\left[\frac{8\pi G\rho_o}{3}\right]^{1/2}(t-t_o)=\left[1-\frac{r}{r_o}\right]^{1/2}\left[\frac{r}{r_o}\right]^{1/2}+\arcsin\left(1-\frac{r}{r_o}\right)^{1/2}$$

Initial conditions: at $t = t_o$, $u_o = 0\ cm\ s^{-1}$

No artificial viscosity; $P = 0$

Physical magnitude	Notation	Units of Measure
Lagrangian mass	m_i	g
Star's age	t^n	s
Velocity	u_i	$\mathrm{cm\,s^{-1}}$
Luminosity	L_i	$\mathrm{erg\,s^{-1}}$
Radius	r_i	cm
Temperature	$T_{i+1/2}$	K
Density	$\rho_{i+1/2}$	$\mathrm{g\,cm^{-3}}$
Specific volume	$V_{i+1/2}$	$\mathrm{cm}^3 g^{-1}$
Pressure	$P_{i+1/2}$	$dyne\,\mathrm{cm}^{-2}$
Internal energy	$E_{i+1/2}$	$\mathrm{erg\,g^{-1}}$
Energy generation rate	$\epsilon_{i+1/2}$	$\mathrm{erg\,g\,s^{-1}}$
Opacity	$\kappa_{i+1/2}$	$\mathrm{cm^2\,g^{-1}}$

We will consider the set of differential eqs. governing, for instance, a given fluid element (*stellar astrophysics*, *atmospheric transport*, *sea currents*...)


Series in Astronomy and Astrophysics
STELLAR EXPLOSIONS
Hydrodynamics and Nucleosynthesis
Jordi José
CRC Press
Taylor & Francis Group

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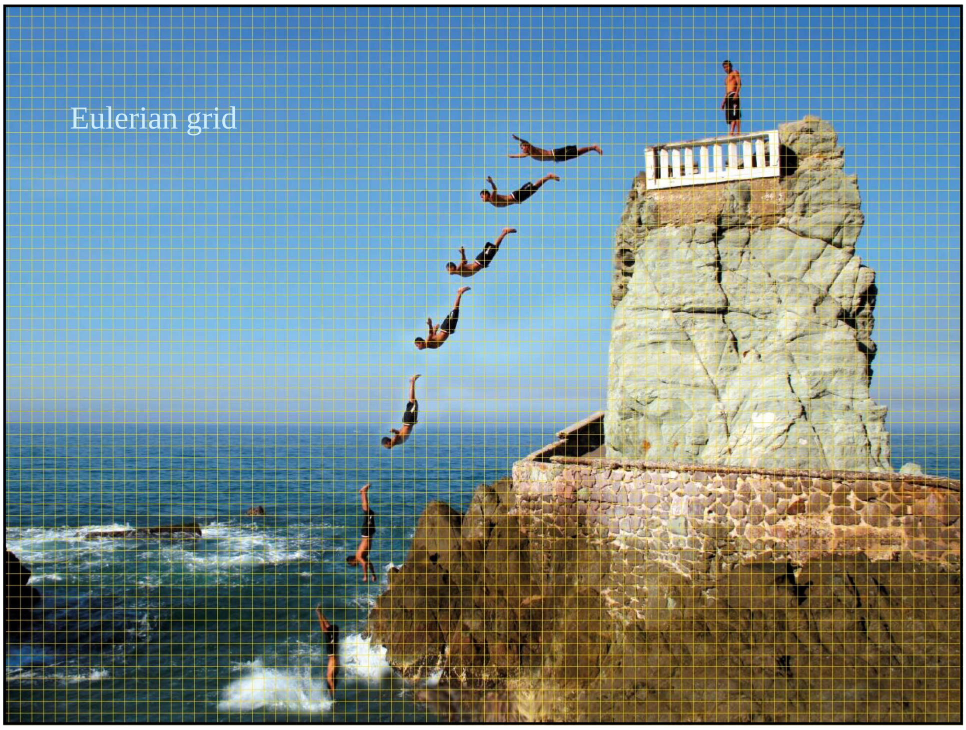
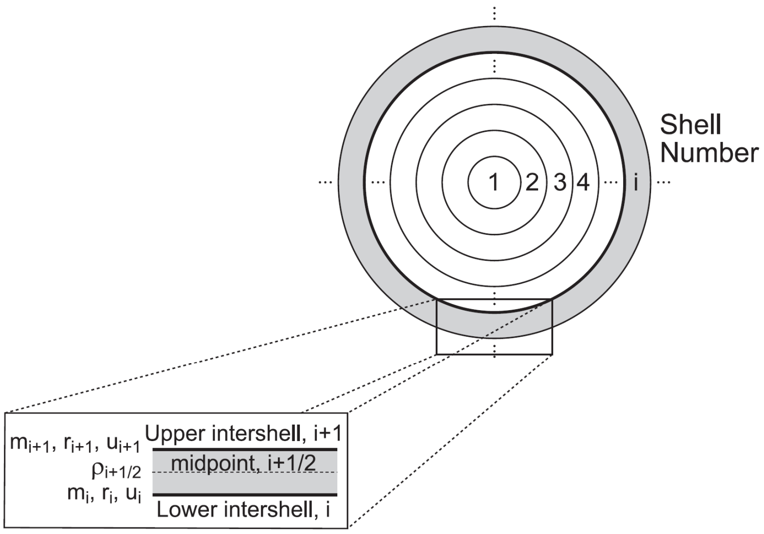

Eulerian grid


Lagrangian grid



Differential Equations

- Conservation of mass

$$\frac{1}{\rho} = \frac{4}{3}\pi\frac{\partial r^3}{\partial m}$$

- Conservation of momentum $(P = q = 0)$

$$\frac{\partial u}{\partial t} = -G\frac{m}{r^2}$$

- Lagrangian velocity

$$\frac{\partial r}{\partial t} = u$$

Discretization

- Conservation of mass

$$\frac{1}{\rho_{i+1/2}} = \frac{4}{3}\pi\frac{r_{i+1}^3 - r_i^3}{m_{i+1} - m_i}$$

- Conservation of momentum

$$\frac{u_{i+1}^{n+1} - u_{i+1}^{n}}{\Delta t} = (1-\beta)\left(\frac{-Gm_{i+1}}{r_{i+1}^2}\right)^n + \beta\left(\frac{-Gm_{i+1}}{r_{i+1}^2}\right)^{n+1}$$

- Lagrangian velocity

$$\frac{r_{i+1}^{n+1} - r_{i+1}^{n}}{\Delta t} = (1-\beta)u_{i+1}^{n} + \beta u_{i+1}^{n+1}$$

Initial Model, Boundary Conditions, and Scaling

If the N concentric shells contain the same mass, $\Delta m \equiv M_{tot}/N$, the interior mass variable m_i is simply given by:

$$m_{i+1} = i\,\Delta m \qquad (i = 1, N)\,.$$

Note that $m_{N+1} \equiv M_{tot}$, by construction, with $M_{tot} = \frac{4}{3}\pi R_o^3 \rho_o$ for a homogeneous sphere of initial radius R_o and density ρ_o.

If the computational domain extends all the way from the center of the sphere to its surface[34], the radius and interior mass variable at the first intershell trivially become $r_1 = m_1 = 0$. Moreover, for a homogenous sphere, $\rho_{i+1/2} \equiv \rho_o$ (i=1, N), by definition, and assuming an initial static configuration, $u_i = 0$ (i=1, N+1) at $t = 0$.

Once boundary conditions are applied to the innermost shell, mass conservation equation becomes

$$r_2 = \left(\frac{3m_2}{4\pi\rho_{3/2}}\right)^{1/3}$$

for $r_1 = m_1 = 0$, while the radii of the subsequent intershells can be obtained through

$$r_{i+1} = \left(\frac{3(m_{i+1} - m_i)}{4\pi\rho_{i+1/2}} + r_i^3\right)^{1/3} =$$

$$\left(\frac{3\Delta m}{4\pi\rho_{i+1/2}} + r_i^3\right)^{1/3} \qquad (i = 2, N)\,.$$

We will assume *Lagrangian* formulation; physical variables will be rescaled to suitable new variables:

$W = \ln V = \ln (1/\rho)$

$R = \ln r$

$Q = 1 - m_{int}/m_{Total}$

Equations for the Innermost, Intermediate and Surface Shells

Equations for the Innermost Shell

The finite difference equations for the innermost shell of the sphere, with the corresponding boundary conditions, can be written in compact form as a function C that depends on a number of unknowns. Let's start with the mass conservation equation: for $m_1 = r_1 = u_1 = 0$, it can be written as

$$C^1 = \frac{1}{(\rho_{3/2})^{n+1}} - \frac{4}{3}\pi\frac{(r_2^3)^{n+1}}{m_2} = C^1(r_2, \rho_{3/2}) = 0\,.$$

Momentum conservation and the equation for the Lagrangian velocity can, in turn, be expressed as:

$$C^2 = \frac{u_2^{n+1} - u_2^n}{\Delta t} - (1-\beta)\left(\frac{-Gm_2}{r_2^2}\right)^n - \beta\left(\frac{-Gm_2}{r_2^2}\right)^{n+1} =$$

$$C^2(u_2, r_2) = 0\,,$$

$$C^3 = \frac{r_2^{n+1} - r_2^n}{\Delta t} - (1-\beta)u_2^n - \beta u_2^{n+1} = C^3(u_2, r_2) = 0\,.$$

Globally, this set of equations can be written as a function of just 3 unknowns, u_2, r_2, and $\rho_{3/2}$, such that

$$C^j = C^j(\rho_{3/2}, u_2, r_2) = 0 \qquad (j = 1, 3)\,.$$

Equations for the Intermediate Shells

The same procedure is then applied to the $N-2$ intermediate shells (i=2, N-1), in the form:

$$F_i^1 = \frac{1}{(\rho_{i+1/2})^{n+1}} - \frac{4}{3}\pi \frac{(r_{i+1}^3)^{n+1} - (r_i^3)^{n+1}}{m_{i+1} - m_i} = F_i^1(r_{i+1}, r_i, \rho_{i+1/2}) = 0\,,$$

$$F_i^2 = \frac{u_{i+1}^{n+1} - u_{i+1}^n}{\Delta t} - (1-\beta)\left(\frac{-Gm_{i+1}}{r_{i+1}^2}\right)^n - \beta\left(\frac{-Gm_{i+1}}{r_{i+1}^2}\right)^{n+1} = F_i^2(u_{i+1}, r_{i+1}) = 0\,,$$

$$F_i^3 = \frac{r_{i+1}^{n+1} - r_{i+1}^n}{\Delta t} - (1-\beta)u_{i+1}^n - \beta u_{i+1}^{n+1} = F_i^3(u_{i+1}, r_{i+1}) = 0\,,$$

or globally,

$$F_i^j = F_i^j(\rho_{i+1/2}, r_i, r_{i+1}, u_{i+1}) = 0 \qquad (i = 2, N-1; j = 1,3)\,.$$

Equations for the Outermost Shell

Finally, for the outermost shell (i = N), we have

$$S^1 = \frac{1}{(\rho_{N+1/2})^{n+1}} - \frac{4}{3}\pi \frac{(r_{N+1}^3)^{n+1} - (r_N^3)^{n+1}}{m_{N+1} - m_N} = S^1(r_{N+1}, r_N, \rho_{N+1/2}) = 0\,,$$

$$S^2 = \frac{u_{N+1}^{n+1} - u_{N+1}^n}{\Delta t} - (1-\beta)\left(\frac{-Gm_{N+1}}{r_{N+1}^2}\right)^n - \beta\left(\frac{-Gm_{N+1}}{r_{N+1}^2}\right)^{n+1} = S^2(u_{N+1}, r_{N+1}) = 0\,,$$

$$S^3 = \frac{r_{N+1}^{n+1} - r_{N+1}^n}{\Delta t} - (1-\beta)u_{N+1}^n - \beta u_{N+1}^{n+1} = S^3(u_{N+1}, r_{N+1}) = 0\,,$$

which can be written as:

$$S^j = S^j(\rho_{N+1/2}, r_N, r_{N+1}, u_{N+1}) = 0 \qquad (j = 1,3)\,.$$

Linearization

Let x^0 be a vector containing the *exact* values of the physical variables of the problem, r, ρ, and u, at $t^0 = 0$ (i.e., initial model), or, in general, at a given time, t^n. Let x^1 be the corresponding vector after one step, $t^1 = t^0 + \Delta t$. For small enough values of the time-step, Δt, all physical variables would have scarcely varied from their values at t^0 (i.e., $x^1 \sim x^0$). Therefore, let's consider, as a first approximate guess, that $x^1 \equiv x^0$. In general, such a choice would not yield the exact values of the variables at t^1, so that $C^j(x^1) \neq 0$, $F_i^j(x^1) \neq 0$, and $S_j(x^1) \neq 0$. Nevertheless, since $x^1 \sim x^0$, one can think of a set of corrections, δx, that added to the first guess values, $x^1 = x^0 + \delta x$, will actually satisfy $C^j(x^1) = 0$, $F_i^j(x^1) = 0$, and $S^j(x^1) = 0$. For small corrections, the whole set of structure equations can be written in the form

$$\begin{aligned} C^j(x^1) &= C^j(x^0) + \delta C^j = 0 \\ F_i^j(x^1) &= F_i^j(x^0) + \delta F_i^j = 0 \\ S^j(x^1) &= S^j(x^0) + \delta S^j = 0\,, \end{aligned} \tag{1.152}$$

$$C^j + \frac{\partial C^j}{\partial \rho_{3/2}}\delta\rho_{3/2} + \frac{\partial C^j}{\partial u_2}\delta u_2 + \frac{\partial C^j}{\partial r_2}\delta r_2 = 0$$

$$F_i^j + \frac{\partial F_i^j}{\partial \rho_{i+1/2}}\delta\rho_{i+1/2} + \frac{\partial F_i^j}{\partial r_i}\delta r_i + \frac{\partial F_i^j}{\partial r_{i+1}}\delta r_{i+1} + \frac{\partial F_i^j}{\partial u_{i+1}}\delta u_{i+1} = 0$$

$$(j = 1, 3; i = 2, N-1)$$

$$S^j + \frac{\partial S^j}{\partial \rho_{N+1/2}}\delta\rho_{N+1/2} + \frac{\partial S^j}{\partial r_N}\delta r_N + \frac{\partial S^j}{\partial r_{N+1}}\delta r_{N+1} + \frac{\partial S^j}{\partial u_{N+1}}\delta u_{N+1} = 0$$

$$(j = 1, 3)\,,$$

6. Example: $N = 4$

Boundary conditions : $r_1 = m_1 = u_1 = 0$

Initial model : ρ , m_k $(k=2, N+1)$ known values

Unknowns : r_2, u_2 , r_3, u_3, r_4, u_4 , r_5, u_5

$\rho_1, \rho_2, \rho_3, \rho_4$

Equations :

$$\left.\begin{aligned}
\frac{\partial C^1}{\partial r_2}\delta r_2 + \frac{\partial C^1}{\partial u_2}\delta u_2 + \frac{\partial C^1}{\partial \rho_1}\delta \rho_1 &= -C^1 \\
\frac{\partial C^2}{\partial r_2}\delta r_2 + \frac{\partial C^2}{\partial u_2}\delta u_2 + \frac{\partial C^2}{\partial \rho_1}\delta \rho_1 &= -C^2 \\
\frac{\partial C^3}{\partial r_2}\delta r_2 + \frac{\partial C^3}{\partial u_2}\delta u_2 + \frac{\partial C^3}{\partial \rho_1}\delta \rho_1 &= -C^3
\end{aligned}\right\} \quad i=1 \quad \text{(Center)}$$

$$\left.\begin{aligned}
\frac{\partial F_2^1}{\partial r_2}\delta r_2 + \frac{\partial F_2^1}{\partial r_3}\delta r_3 + \frac{\partial F_2^1}{\partial \rho_2}\delta \rho_2 + \frac{\partial F_2^1}{\partial u_3}\delta u_3 &= -F_2^1 \\
\frac{\partial F_2^2}{\partial r_2}\delta r_2 + \frac{\partial F_2^2}{\partial r_3}\delta r_3 + \frac{\partial F_2^2}{\partial \rho_2}\delta \rho_2 + \frac{\partial F_2^2}{\partial u_3}\delta u_3 &= -F_2^2 \\
\frac{\partial F_2^3}{\partial r_2}\delta r_2 + \frac{\partial F_2^3}{\partial r_3}\delta r_3 + \frac{\partial F_2^3}{\partial \rho_2}\delta \rho_2 + \frac{\partial F_2^3}{\partial u_3}\delta u_3 &= -F_2^3
\end{aligned}\right\} \quad i=2$$

$$\left.\begin{aligned}
\frac{\partial F_3^1}{\partial r_3}\delta r_3 + \frac{\partial F_3^1}{\partial r_4}\delta r_4 + \frac{\partial F_3^1}{\partial \rho_3}\delta \rho_3 + \frac{\partial F_3^1}{\partial u_4}\delta u_4 &= -F_3^1 \\
\frac{\partial F_3^2}{\partial r_3}\delta r_3 + \frac{\partial F_3^2}{\partial r_4}\delta r_4 + \frac{\partial F_3^2}{\partial \rho_3}\delta \rho_3 + \frac{\partial F_3^2}{\partial u_4}\delta u_4 &= -F_3^2 \\
\frac{\partial F_3^3}{\partial r_3}\delta r_3 + \frac{\partial F_3^3}{\partial r_4}\delta r_4 + \frac{\partial F_3^3}{\partial \rho_3}\delta \rho_3 + \frac{\partial F_3^3}{\partial u_4}\delta u_4 &= -F_3^3
\end{aligned}\right\} \quad i=3$$

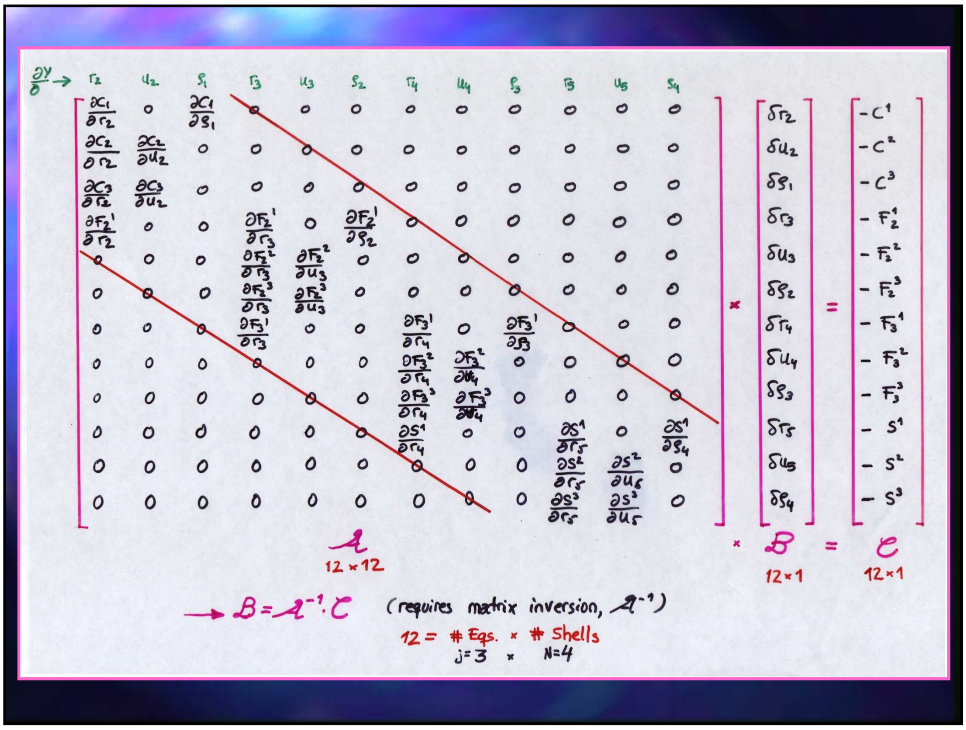
$$
\left.
\begin{aligned}
\frac{\partial S^1}{\partial r_4}\delta r_4 + \frac{\partial S^1}{\partial r_5}\delta r_5 + \frac{\partial S^1}{\partial \rho_4}\delta \rho_4 + \frac{\partial S^1}{\partial u_5}\delta u_5 &= -S^1 \\
\frac{\partial S^2}{\partial r_4}\delta r_4 + \frac{\partial S^2}{\partial r_5}\delta r_5 + \frac{\partial S^2}{\partial \rho_4}\delta \rho_4 + \frac{\partial S^2}{\partial u_5}\delta u_5 &= -S^2 \\
\frac{\partial S^3}{\partial r_4}\delta r_4 + \frac{\partial S^3}{\partial r_5}\delta r_5 + \frac{\partial S^3}{\partial \rho_4}\delta \rho_4 + \frac{\partial S^3}{\partial u_5}\delta u_5 &= --S^3
\end{aligned}
\right\}
\begin{array}{c} i=4 \\ \text{(Surface)} \end{array}
$$

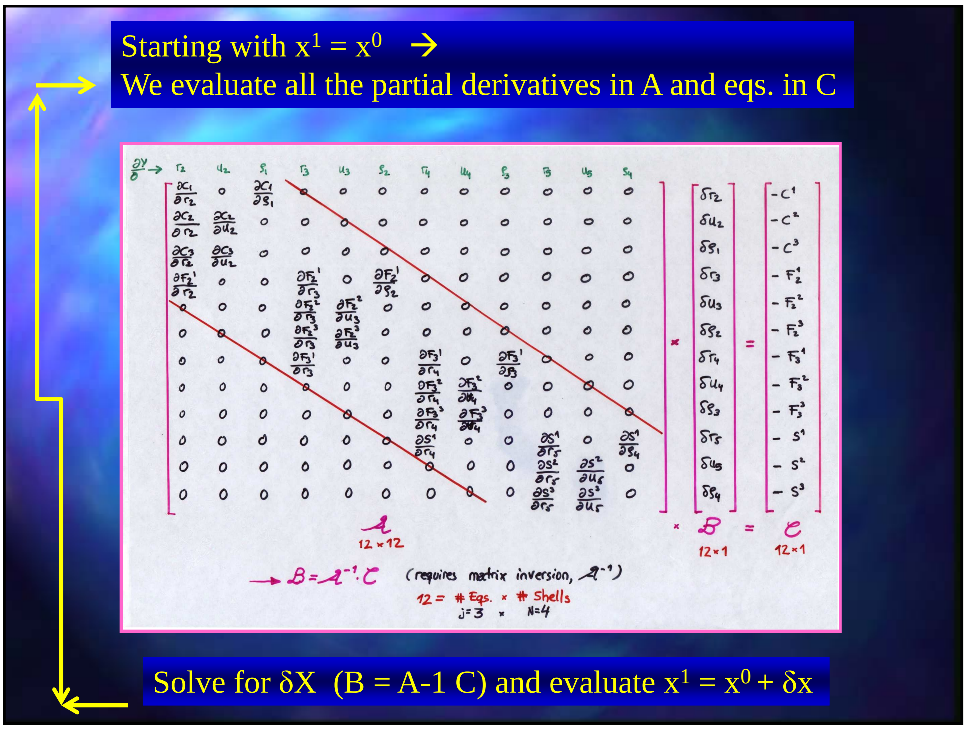
$$
\begin{array}{c}
\frac{\partial y}{\partial} \rightarrow \quad \rho_1 \quad r_2 \quad u_2 \quad \rho_2 \quad r_3 \quad u_3 \quad \rho_3 \quad r_4 \quad u_4 \quad \rho_4 \quad r_5 \quad u_5 \\
\underbrace{\begin{bmatrix}
\frac{\partial C^1}{\partial \rho_1} & \frac{\partial C^1}{\partial r_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\partial C^2}{\partial r_2} & \frac{\partial C^2}{\partial u_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\partial C^3}{\partial r_2} & \frac{\partial C^3}{\partial u_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\partial F_2^1}{\partial r_2} & 0 & \frac{\partial F_2^1}{\partial \rho_2} & \frac{\partial F_2^1}{\partial r_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial F_2^2}{\partial r_3} & \frac{\partial F_2^2}{\partial u_3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial F_2^3}{\partial r_3} & \frac{\partial F_2^3}{\partial u_3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial F_3^1}{\partial r_3} & 0 & \frac{\partial F_3^1}{\partial \rho_3} & \frac{\partial F_3^1}{\partial r_4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial F_3^2}{\partial r_4} & \frac{\partial F_3^2}{\partial u_4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial F_3^3}{\partial r_4} & \frac{\partial F_3^3}{\partial u_4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial S^1}{\partial r_4} & 0 & \frac{\partial S^1}{\partial \rho_4} & \frac{\partial S^1}{\partial r_5} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial S^1}{\partial r_5} & \frac{\partial S^1}{\partial u_5} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial S^1}{\partial r_5} & \frac{\partial S^1}{\partial u_5}
\end{bmatrix}}_{\substack{\mathcal{A} \\ 12 \times 12}}
\times
\underbrace{\begin{bmatrix}
\delta\rho_1 \\ \delta r_2 \\ \delta u_2 \\ \delta\rho_2 \\ \delta r_3 \\ \delta u_3 \\ \delta\rho_3 \\ \delta r_4 \\ \delta u_4 \\ \delta\rho_4 \\ \delta r_5 \\ \delta u_5
\end{bmatrix}}_{\substack{\mathcal{B} \\ 12 \times 1}}
=
\underbrace{\begin{bmatrix}
-C^1 \\ -C^2 \\ -C^3 \\ -F_2^1 \\ -F_2^2 \\ -F_2^3 \\ -F_3^1 \\ -F_3^2 \\ -F_3^3 \\ -S^1 \\ -S^2 \\ -S^3
\end{bmatrix}}_{\substack{\mathcal{G} \\ 12 \times 1}}
\end{array}
$$

$\rightarrow \mathcal{B} = \mathcal{A}^{-1} \cdot \mathcal{G}$ (requires matrix inversion, $\mathcal{A}^{-1}$)

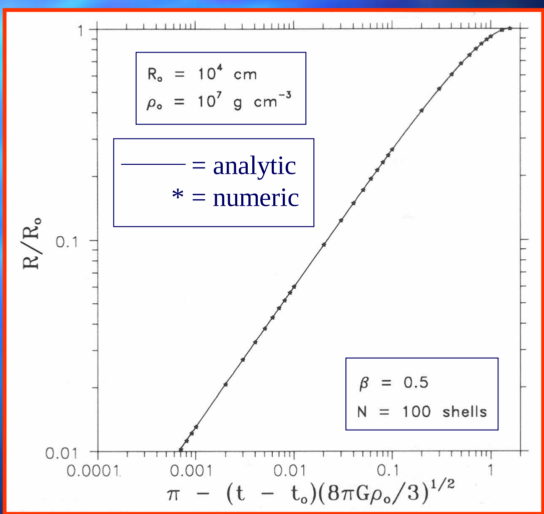
$12 = \#\text{Eqs.} \times \#\text{Shells}$

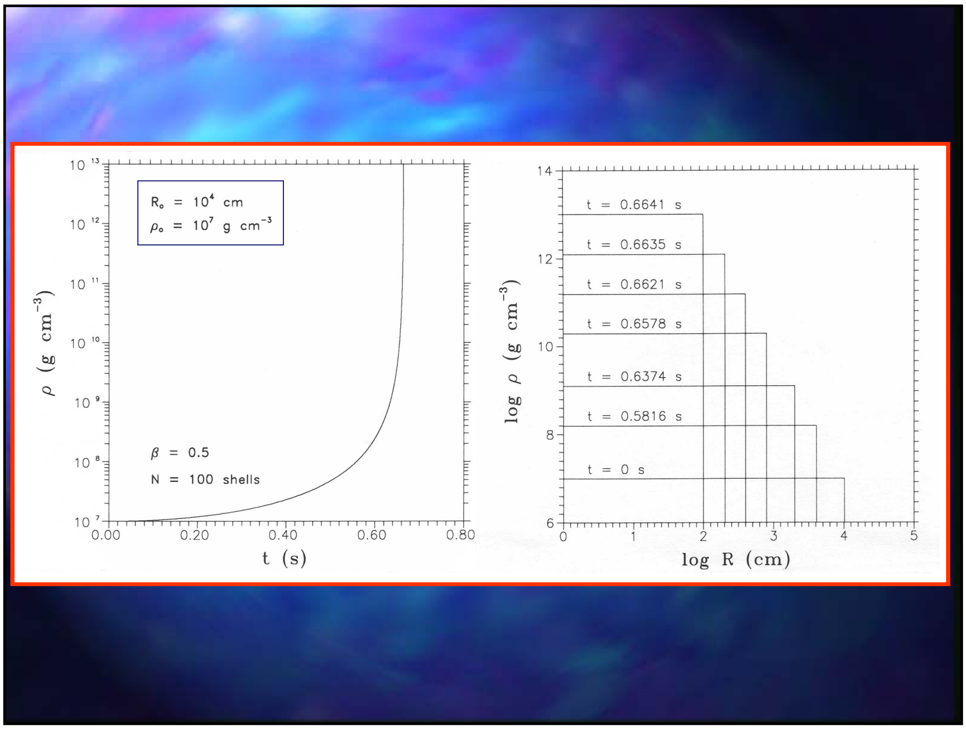
$j = 3 \times N = 4$


12×12
12×1
12×1
B = A⁻¹·C (requires matrix inversion, A⁻¹)
12 = # Eqs. × # Shells
j=3 × N=4


Starting with $x^1 = x^0$ →
We evaluate all the partial derivatives in A and eqs. in C
12×12
12×1
12×1
B = A⁻¹·C (requires matrix inversion, A⁻¹)
12 = # Eqs. × # Shells
j=3 × N=4
Solve for δX (B = A-1 C) and evaluate $x^1 = x^0 + \delta x$

The procedure is iterated until a given accuracy criterion is satisfied ($\delta x < \varepsilon$, c.f. corrections are smaller than a given quantity). Extrapolation from the last two converged models is recommended as the first guess for the next time step.





$R_o = 10^4$ cm
$\rho_o = 10^7$ g cm^{-3}
$\beta = 0.5$
N = 100 shells
ρ (g cm^{-3})
t (s)
t = 0.6641 s
t = 0.6635 s
t = 0.6621 s
t = 0.6578 s
t = 0.6374 s
t = 0.5816 s
t = 0 s
log ρ (g cm^{-3})
log R (cm)