Chapter 2

The Time Value of Money

Contents

2.1 Introduction to the Time Value of Money

- Consider the following scenarios:

Scenario 1

- You have won in a recent lucky draw a cash prize of $10,000. The prize will be forfeited if it is not claimed by January 31, next year. Which of the following options would you choose?

  1. Claim the $10,000 cash prize now.

  2. Claim the $10,000 cash prize one year from now.

- Your Choice: ________ Why? ____

Scenario 2

- You have won in a recent lucky draw a cash prize of $10,000. The terms and conditions provided the following options for you:

  1. You will receive $10,000 cash if you claim the prize now.

  2. You will receive $10,500 cash if you claim the prize one year from now.

- Your Choice: ________ Why? ____

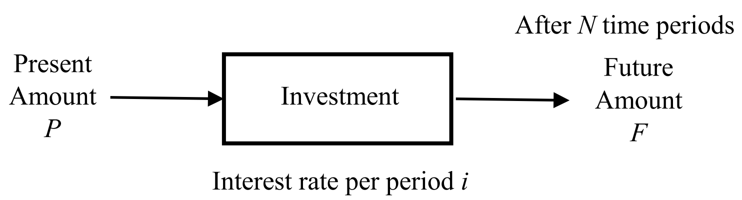
Summary

- The value of money changes with time. It is always better to receive a dollar now than to receive it later (Scenario 1).

- How the value of money changes with time depends on individuals - it will vary from person to person, or from company to company.

Representing Time Value of Money

- The value of money over time may be represented by an "interest rate":



- There are two types of interests:
  1. Simple interests
  2. Compound interests

2.1.1 Simple Interests

- The amount of interest earned is directly proportional to the principal, the number of periods for which the principal is committed, and the interest rate per period.

- Let P = Principal amount
  N = number of periods
  i = simple interest rate per period

- Then total interest $I = P\,N\,i$

- Future amount, $F = P + I$
  $= P + P\,N\,i$
  $= P\,(1 + N\,i\,)$

Example

- Suppose you borrow \$1,000 for 5 years at a simple interest rate of 10% per year. How much do you have to return at the end of 5 years?

- Given P = \$1,000, i = 0.1, N = 5 years.

- F = \$1,000 (1 + 5 × 0.10) = \$1,500

- The year-by-year computations are shown below:

	(1)	(2) = P × 10%	(3) = (1)+(2)
Period (year)	Amount owed at beginning of year	Interest amount for the year	Amount owed at end of year
1	\$1,000	\$100	\$1,100
2	\$1,100	\$100	\$1,200
3	\$1,200	\$100	\$1,300
4	\$1,300	\$100	\$1,400
5	\$1,400	\$100	**\$1,500**

- Hence the amount to be repaid at the end of 5 years = \$1,500.

- Note that the interest payable is constant each year. It is always equal to 10% of the \$1,000 principal.

2.1.2 Compound Interests

- In the case of compound interests, the interest for any period is based on the remaining principal amount plus any accumulated interest up to the beginning of that period.

- An example to illustrate compound interest is given below. We will derive the general formula later.

Example

- Suppose you borrow $1,000 for 5 years at an interest rate of 10% compounded annually. How much do you have to return at the end of 5 years?

- The interests accumulated and the amounts owned at the end of each year are given below:

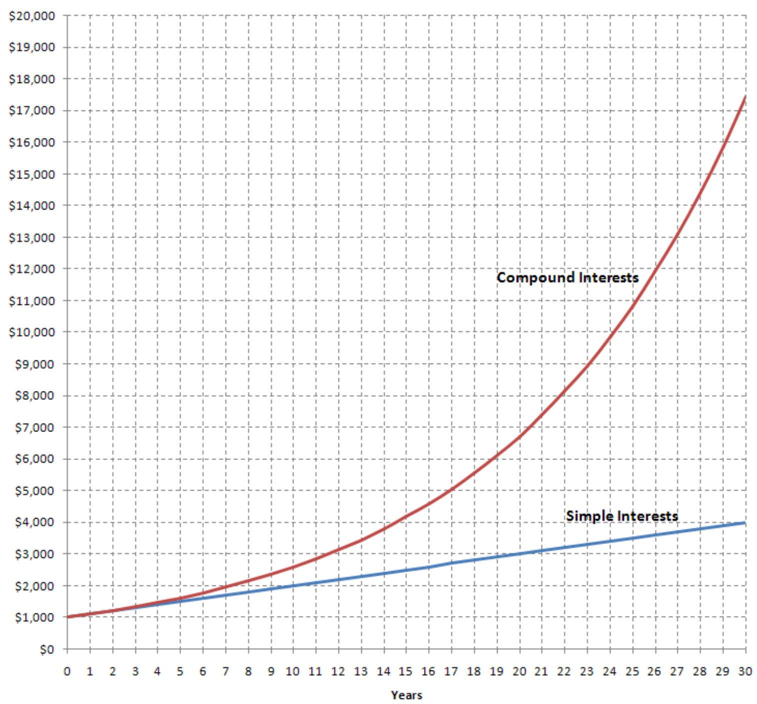
	(1)	(2) = 0.10×(1)	(3)=(1)+(2)
Period (Year)	Amount owned at beginning of year	Interest amount for the year	Amount owned at end of the year
1	1,000.00	100.00	1,100.00
2	1,100.00	110.00	1,210.00
3	1,210.00	121.00	1,331.00
4	1,331.00	133.10	1,464.10
5	1,464.10	146.41	**1,610.51**

- The amount to be repaid at the end of 5 years = $1,610.51

- Note that the interest amount for each year is not constant but increases over each year. This is because it is equal to 10% of the total accumulated amount owned at the beginning of the year.

Comparison of Simple and Compound Interests

- Comparing the value of $1,000 with simple and compound interests at 10% per year for 30 years:



Observations:
1. Simple interest:
   The amount grows linearly

2. Compound interest:
   The amount grows exponentially.

2.2 Equivalent Values of Discrete Cash Flows

2.2.1 Cash flow diagrams

- In cash flow analysis, a useful tool for developing and visualizing cash flow models is the cash flow diagram.

- A **cash flow diagram** is a graphical representation of all the cash flows that occur at different time on a time axis.

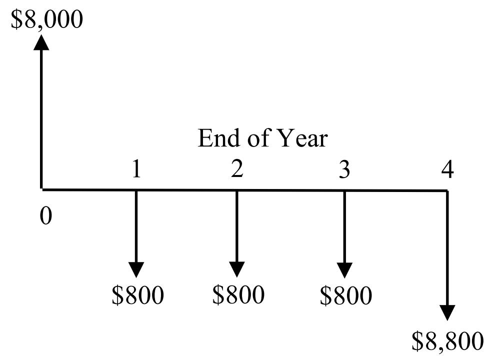
- An upward arrow in the cash flow diagram normally indicates a cash inflow at a specific point in time, while a downward arrow indicates a cash outflow.

Example

- A person needs $8,000 now and borrows the amount from a bank at an interest rate of 10% per year. (All interests are assumed to be compound interests unless otherwise stated).
- Suppose he pays only the interests at the end of each year, and the principal at end of four years, then the cash flows over four years may be computed as follows:

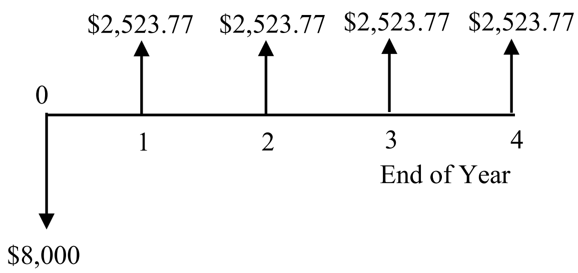
	(*a*)	(*b*) = 0.1× (*a*)	(c)	(*d*)=(*a*)+(*b*)–(*c*)
Year	Amount owned at beginning of year	Interests for the year	Repayment at the end of year	Amount owned at the end of year
1	8,000.00	800.00	800.00	8,000.00
2	8,000.00	800.00	800.00	8,000.00
3	8,000.00	800.00	800.00	8,000.00
4	8,000.00	800.00	8,800.00	0

- The cash flow from the *borrower's point of view* is as follows:



Example

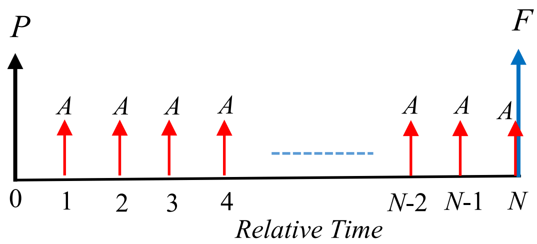
- A person needs $8,000 now and borrows the amount from a bank.
- Suppose the loan is to be repaid with four equal end-of-year payments of $2,523.77 each, the cash flow diagram from the bank's point of view is as follows:



- We will show later on that the four equal amounts is the correct amount needed to pay back the loan when then interest charged by the bank is 10% per year.

2.2.2 Notations for cash flow analysis

- We will use the following notations in discrete cash flow analysis:



i	Time value of money represented by the effective interest rate per time period.
N	Number of periods.
P	A single sum of money at time 0, or The *equivalent value* at a present time 0 of one or more other cash flows.
F	A single sum of money at time *N*, or The *equivalent value* at a future time N of one or more other cash flows.
A	A uniform series of cash flow end of period 1 to end of period *N*, or The equivalent value of such as cash flows of one or more other cash flows.

- We are interested in finding the following relationships:

1. Find the value of *F* when given a *P,* at interest rate *i* for *N* periods.
2. Find the value *P* when given an *F*, at interest rate *i* for *N* periods.

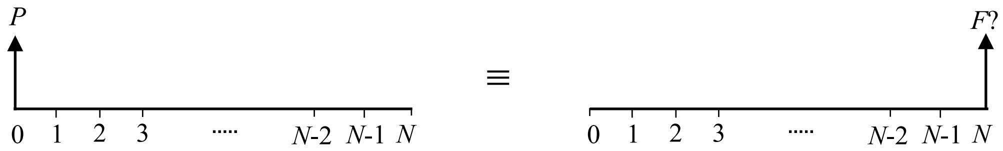
3. Find the value of *P* when given an *A,* at interest rate *i* for *N* periods.
4. Find the value of *F* when given an *A,* at interest rate *i* for *N* periods.

5. Find the value of *A* when given an *F,* at interest rate *i* for *N* periods.
6. Find the value *A* when given a *P,* at interest rate *i* for *N* periods.

2.2.3 Present and Future Equivalent Value of a Single Cash Flow

Finding *F* when Given *P*

- Given a single cash flow P at time 0, what is its **Equivalent Future Value** F at the end of N periods if the interest rate is i per period?



- **Example:** If an amount P is invested at a compound interest rate of i per period, what would be its value after N periods?

- A year-by-year cash flow analysis is shown below:

	(1)	(2) = $i \times$(1)	(3)=(1)+(2)
Period	Amount at beginning of period	Interest earned during period	Amount at end of period
1	P	$i\,P$	$P\,(1+i)$
2	$P\,(1+i)$	$i\,P\,(1+i)$	$P\,(1+i)^2$
3	$P\,(1+i)^2$	$i\,P\,(1+i)^2$	$P\,(1+i)^3$
…	…	…	…
N	$P\,(1+i)^{N-1}$	$i\,P\,(1+i)^{N-1}$	$P\,(1+i)^N$

- The **Future Equivalent Value** F at the end of N periods from now, of a single cash flow P at an effective interest rate i per period is:

$$\boxed{F = P\,(1 + i)^{N}}$$

- The factor $(1 + i)^N$ is called the **<u>Single Payment Compound Amount Factor</u>** and is denoted by the symbol $[F/P, i\%, N]$ which means "find F given P at i% interest per period for N periods".

- We denote the relation between F and P as:

$$\boxed{F = P\,[F/P, i\%, N].}$$

- Values of $[F/P, i\%, N]$ are given in the standard compound interest tables (See Appendix C of the textbook).

Example

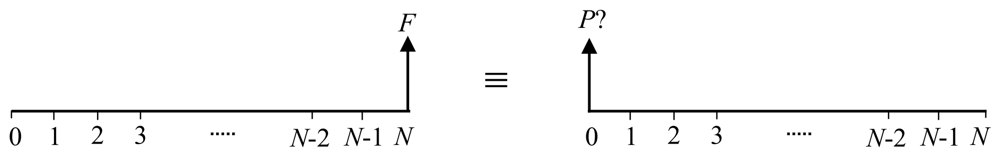
- Suppose you invest \$8,000 in a saving account that earns 10% compound interest per year. What is the amount in the account at the end of 4 years?

- Given P = \$8,000, i=10%, and N = 4 years.
- The amount accumulated at the end of 4 years is equal to:
  $F = 8{,}000\ (1 + 0.10)^4$
  $= 8{,}000\ (1.4641)$
  $= \$\ 11{,}712.80$

- Alternatively, using the compound interest tables (Appendix C of the textbook):
  $F = 8{,}000\ [F/P,\ 10\%,\ 4]$
  $= 8{,}000\ (1.4641)$
  $= \$\ 11{,}712.80$

<u>Finding *P* when Given *F*</u>

- Given a single cash flow F at the end of N periods from now, what is its **Equivalent Present Value** P if the interest rate is i per period?



- **Example**: Suppose you wish to accumulate an amount F at the end of N periods from now, how much must you invest now at a compound interest rate of i per period?

- From the previous case, we know that $F = P\ (1 + i)^N$. Hence $P = F\left(\dfrac{1}{(1+i)^N}\right)$

- The **Present Equivalent Value** P of a single future cash flow F occurring at N periods from *now* at an effective interest rate i per period is:

$$\boxed{P = F\left(\frac{1}{(1+i)^N}\right)}$$

- The factor $\left(\dfrac{1}{(1+i)^N}\right)$ is called the **<u>Single Payment Present Worth Factor</u>**, and is denoted by the symbol $[P/F,\ i\%,\ N]$ which means "find P given F at i% interest per period for N periods".

- We denote the relation between P and F as:

$$\boxed{P = F\ [P/F,\ i\%,\ N].}$$

- Values of $[P/F,\ i\%,\ N]$ are given in the standard compound interest tables.

Example

- An investment is to be worth \$10,000 in six years. If the return on investment 8% per year compounded yearly, how much should be invested today?

- Given F = \$10,000, i=8%, and N = 6 years.

$$
\begin{aligned}
P &= 10{,}000\left(\frac{1}{(1+0.08)^6}\right) \\
&= 10{,}000(0.630170) \\
&= \$6{,}301.70
\end{aligned}
$$

- Alternatively, using the compound interest tables:

P = 10,000 [P/F, 8%, 6]
= 10,000 (0.6302)
= \$ 6,302

- Note the small difference due to rounding errors.

2.2.4 Present and Future Equivalent Values of Uniform Series Cash Flows

Uniform End-of-Period Cash Flows Series

- A uniform end-of-period cash flows series has the following cash flow diagram:

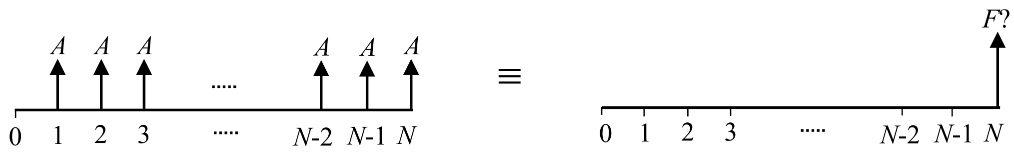
Standard (end-of-period) Uniform Series

- Notes:
  1. The first cash flow of amount *A* occurs at the end of period 1, i.e., at time 1.
  2. The last cash flow of amount *A* occurs at the end of period *N*, i.e., at time *N*.
  3. There are a total of *N* cash flows of amount *A* each.

- The cash flows *A* are also known as A**nnuities.**
- We will also refer to the above *uniform end-of-period cash flows* as a **Standard Uniform Cash Flow Series**.

Finding *F* when Given *A*

- Given a uniform end-of-period cash flows series *A*, what is its **Equivalent Future Value** *F* if the interest rate is *i* per period?



- Note that *F* occurs at the same time as the last cash flow of *A* at time *N*.

- **Example:** Suppose you deposit *N* equal amounts of *A* each, starting at the end of period 1 through end of period *N*, what amount would be accumulated at time *N* if the interest rate is *i* per period?

- *F* is the sum of the future equivalent values at time *N* of each of the individual cash flow in the *A* series from time 1 to *N*.

$F = A\ [F/P,\ i\%,\ N-1] + A\ [F/P,\ i\%,\ N-2] + A\ [F/P,\ i\%,\ N-3] + \ldots + A\ [F/P,\ i\%,\ 1] + A\ [F/P,\ i\%,\ 0]$

$F = A\ [\ (1+i)^{N-1} + (1+i)^{N-2} + (1+i)^{N-3} + \ldots + (1+i)^1 + (1+i)^0\ ]$

$$F = A\left(\frac{(1+i)^{N-1} - \dfrac{1}{(1+i)}}{1 - \dfrac{1}{(1+i)}}\right) = A\left(\frac{(1+i)^N - 1}{i}\right)$$

- The **Future Equivalent Value** F at end of period N of a uniform end of period cash flows beginning at end of period 1 through end of period N is:

$$\boxed{F = A\left(\frac{(1+i)^N - 1}{i}\right)}$$

- The factor $\left(\frac{(1+i)^N - 1}{i}\right)$ is called the **<u>Uniform Series Compound Amount Factor</u>** and is denoted by the symbol [F/A, i%, N].

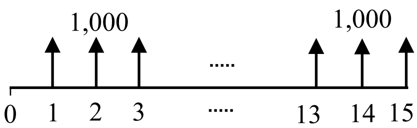
- We denote the relation between F and A as:

$$\boxed{F = A\,[F/A,\ i\%,\ N].}$$

- Values of [F/A, i%, N] are given in the standard compound interest tables.

Example

- 15 equal deposits of \$1,000 each will be made into a bank account paying 5% compound interest per year, the first deposit being one year from now. What is the balance exactly 15 years from now?



- Given A = \$1,000, i=5%, and N = 15 years.

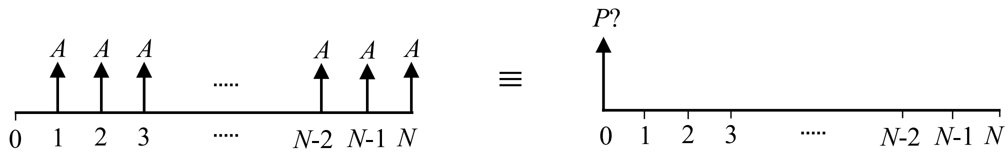
$$F = 1{,}000\left(\frac{(1+0.05)^{15} - 1}{0.05}\right)$$
$$= \$21{,}578.56$$

- Alternatively, using the compound interest tables:

F = 1,000 [F/A, 5%, 15]
= 1,000 (21.578564)
= \$21,578.56

Finding *P* when Given *A*

- Given a uniform end-of-period cash flows series *A,* what is its **Equivalent Present Value** *P* if the interest rate is *i* per period?



- **Example:** Suppose you intend to withdraw *N* equal amounts of *A* each, starting at the end of period 1 from an investment scheme that that earns compound interests at *i* per period, how much money must be invested now?
- We already know how to find *F* given *A*: $F = A\left(\frac{(1+i)^N - 1}{i}\right)$
- We also know how to find *P* given *F*: $P = \left(\frac{1}{(1+i)^N}\right)F$
- By substitution:

$$P = \frac{1}{(1+i)^N}\,A\left(\frac{(1+i)^N - 1}{i}\right) = A\,\left(\frac{(1+i)^N - 1}{i(1+i)^N}\right)$$

- The **Present Equivalent Value** *P* of a uniform end-of-period cash flows beginning at end of period 1 through end of period *N* is:

$$\boxed{P = A\left(\frac{(1+i)^N - 1}{i(1+i)^N}\right)}$$

- The factor $\left(\frac{(1+i)^N - 1}{i(1+i)^N}\right)$ is called the **Uniform Series Present Value Worth Factor** and is denoted by the symbol [*P*/*A*, *i*%, *N*].

- We denote the relation between *P* and *A* as:

$$\boxed{P = A\ [P/A,\ i\%,\ N].}$$

- Values of [*P*/*A*, *i*%, *N*] are given in standard compound interest tables.

Example

- What is the equivalent present value of a series of end-of-year equal incomes valued at \$20,000 each for 5 years if the interest rate is 15% per year?

- Given A = \$20,000, i = 15%, and N = 5 years.

$$P = A\left(\frac{(1+0.15)^5 - 1}{0.15(1+0.15)^5}\right)$$
$$= \$67{,}043.10$$

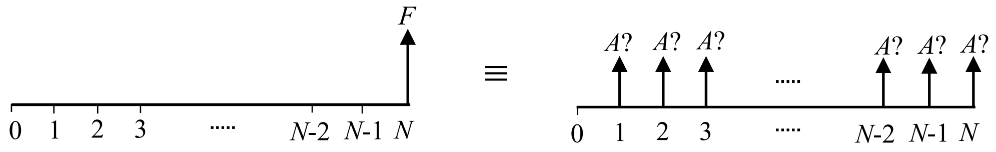
- Alternatively, using the compound interest tables:

$$\begin{aligned} P &= 20{,}000\ [P/A,\ 15\%,\ 5] \\ &= 20{,}000\ (3.352155) \\ &= \$\ 67{,}043.10 \end{aligned}$$

2.2.5 Equivalent Uniform Annual Value of Present and Future Cash Flows

Finding *A* when Given *F*

- Given a single cash flow F at the end of N periods from now, what is its **Equivalent Uniform End-of-Period Cash Flows** A if the interest rate is i per period?



- **Example:** Suppose you intend to accumulate a future amount F at the end of N periods, what equal N end-of-periods amounts must you deposit into an account that pays compound interests at i per period?

- From find F given A case: $$F = A\left(\frac{(1+i)^N - 1}{i}\right)$$

- Hence $$A = F\left(\frac{i}{(1+i)^N - 1}\right)$$

- The Equivalent Uniform End-of-Period cash flows A of a future amount F at end of N periods is

$$\boxed{A = F\left(\frac{i}{(1+i)^N - 1}\right)}$$

- The factor $\left(\frac{i}{(1+i)^N - 1}\right)$ is called the **<u>Sinking Fund Factor</u>** and is denoted by the symbol [A/F, i%, N].

- We denote the relation between A and F as: $A = F$ [A/F, i%, N].

Example

- If you need a lump sum of \$1 million at your retirement 45 years from now, how much must you save per year if the interest rate is 7% per year?

- Given F = \$1,000,000, i = 7%, and N = 45 years.

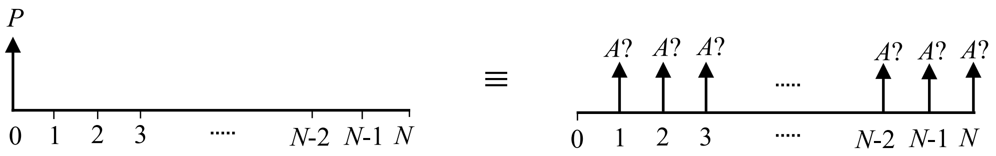
$$A = (1{,}000{,}000)\left(\frac{0.07}{(1+0.07)^{45} - 1}\right) = \$3{,}499.57$$

- Alternatively, using the compound interest tables:

A = 1,000,000 [A/F, 7%, 45]
= 1,000,000 (0.00349957)
= \$3,499.57

Finding *A* when Given *P*

- Given a single cash flow *P* now, what is its **Equivalent Uniform End-of-Period Cash Flow** *A* if the interest rate is *i* per period?



- **Examples:**
  - Suppose you take a *N*-period loan *P* now at interest rate *i* per period, what are the equal amounts *A* you need to pay at end of end of period 1 to end of period *N*?
  - Suppose you invest an amount *P* in a business venture, what equal amounts of *A* each must you receive back at the end of period 1 to end of period *N* to recover your investment at if the required return on investment is *i* per period?

- We already know how to find *P* given *A* that $P = A\left(\frac{(1+i)^N - 1}{i(1+i)^N}\right)$

- Hence: $A = P\left(\frac{i(1+i)^N}{(1+i)^N - 1}\right)$

- The Equivalent Uniform End-of-Period Cash Flows *A* of a present amount *P* is

$$\boxed{A = P\left(\frac{i(1+i)^N}{(1+i)^N - 1}\right)}$$

- The factor $\left(\frac{i(1+i)^N}{(1+i)^N - 1}\right)$ is called the **Capital Recovery Factor** and is denoted by the symbol [*A*/*P*, *i*%, *N*].

- We denote the relation between *A* and *P* as:

$$\boxed{A = P\,[A/P,\, i\%,\, N].}$$

Example

- Consider a loan of \$8,000 to be paid back with 4 equal end-of-year installments? What is the yearly repayment amount if the interest rate is 10%?

- Given P = \$8,000, i=10%, and N = 4 years.

$$\begin{aligned} A &= P\left(\frac{i(1+i)^N}{(1+i)^N - 1}\right) \\ &= P\left(\frac{i}{1-(1+i)^{-N}}\right) \\ &= (8{,}000)\left(\frac{0.1}{1-(1+0.1)^{-4}}\right) \\ &= \$2{,}523.77 \end{aligned}$$

- Alternatively, using the compound interest tables:

$$\begin{aligned} A &= 8{,}000\ [A/P,\ 10\%,\ 4] \\ &= 8{,}000\ (0.315471) \\ &= \$2{,}523.77 \end{aligned}$$

- To verify that the 4 equal annual amounts of \$2,533.77 each do indeed repay the loan after 4 years, we can compute the year-by-year balance as follows:

	(a)	(b) = 0.1× (a)	(c)	(d)=(a)+(b)–(c)
Year	Amount owned at beginning of year	Interests for the year	Repayment at the end of year	Amount owned at the end of year
1	8,000.00	800.00	2523.77	6,276.23
2	6,276.23	627.62	2523.77	4,380.09
3	4,380.09	438.01	2523.77	2,294.33
4	2,294.33	229.43	2523.77	0

2.2.6 Relationships between Interest Factors for Discrete Cash Flows

- Summary of formulas for discrete cash flows with discrete compounding:

Factor	Find	Given	Symbol	Formula
Single Payment Compound Amount	F	P	$[F/P, i\%, N]$	$(1+i)^N$
Single Payment Present Worth	P	F	$[P/F, i\%, N]$	$\frac{1}{(1+i)^N}$
Sinking Fund	A	F	$[A/F, i\%, N]$	$\frac{i}{(1+i)^N-1}$
Uniform Series Compound Amount	F	A	$[F/A, i\%, N]$	$\frac{(1+i)^N-1}{i}$
Capital Recovery	A	P	$[A/P, i\%, N]$	$\frac{i(1+i)^N}{(1+i)^N-1}$
Uniform Series Present Worth	P	A	$[P/A, i\%, N]$	$\frac{(1+i)^N-1}{i(1+i)^N}$

Relations between Interest Formulas

- The six formulas involving P, A and F are not independent. Only two independent ones are needed, and the other four may be derived from these two using the following relations:

1. Reciprocal Rules:

- $[F/P, i\%, N] = \dfrac{1}{[P/F, i\%, N]}$
- $[A/F, i\%, N] = \dfrac{1}{[F/A, i\%, N]}$
- $[A/P, i\%, N] = \dfrac{1}{[P/A, i\%, N]}$

2. Chain Rules:

- $[F/A, i\%, N] = [F/P, i\%, N] \times [P/A, i\%, N]$
- $[F/P, i\%, N] = [F/A, i\%, N] \times [A/P, i\%, N]$

- $[P/A, i\%, N] = [P/F, i\%, N] \times [F/A, i\%, N]$
- $[P/F, i\%, N] = [P/A, i\%, N] \times [A/F, i\%, N]$

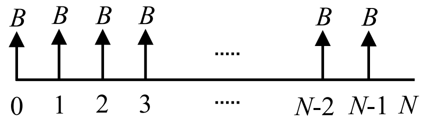
- $[A/F, i\%, N] = [A/P, i\%, N] \times [P/F, i\%, N]$
- $[A/P, i\%, N] = [A/F, i\%, N] \times [F/P, i\%, N]$

3. *A/F* to *A/P*:

- $[A/F, i\%, N] + i = [A/P, i\%, N]$

2.2.7 Uniform Beginning-of-Period Cash Flows

- The six formulas derived in the previous section are for the *uniform **end-of-period** cash flows series*. They are **not applicable** to the uniform **beginning-of-period** cash flows series shown below:



Uniform beginning-of-period cash flows

- Note that:
  1. The first cash flow B occurs at time 0.
  2. The last cash flow B occurs at the beginning of period N, i.e., at time N-1.
  3. Total number of cash flows = N.

- We refer to the above *uniform beginning-of-period cash flows* as a ***B* series**.

- You can deal with these B-series cash flows, i.e., finding their equivalent P, A and F values, by using the formulas for A-series with some modifications.

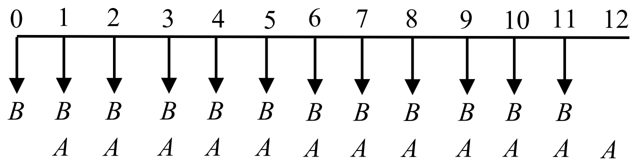
- We illustrate using some examples:

Examples

- You intend to rent a room for 12 months during your overseas exchange program. The landlord asks for a monthly rent of $1,000, payable at the beginning of each month.

  Let B = $1,000 payable at the beginning of each month for 12 months.

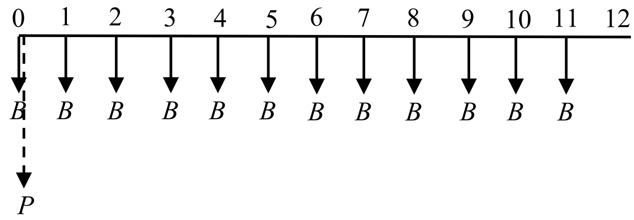
  1. If you wish to pay the rents at the end of each month instead, what amount should you pay if the time value of money to the landlord is 2% per month?



Each A is a B is delayed by one month.

Hence $A = B$ [F/P, 2%, 1]
= $1,000 (1.0200)
= $ 1,020.00

2. If you wish to pay all the rents with one lump sum upon moving in instead, what amount should you pay if the time value of money to the landlord is 2% per month?



P = B at time 0 + Present Value of B's from end-of-month 1 to end-of-month 11

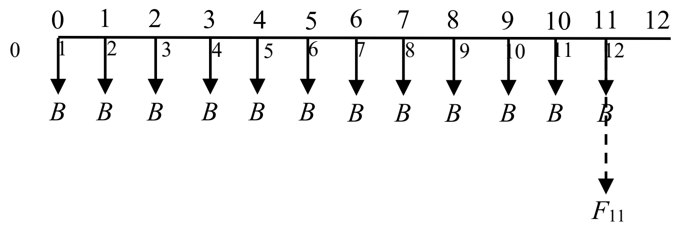
$= B + B\,[P/A, 2\%, 11]$

$= B\,(1 + [P/A, 2\%, 11])$

$= \$1{,}000\,(1 + 9.786848)$

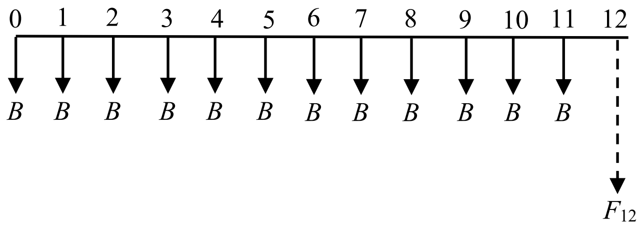
$= \$10{,}786.85$

3. If you wish to pay all the rents with one lump on moving out at the end of 12 months, what amount should you pay if the time value of money to the landlord is 2% per month?



The future value of all cash flows at end-of-month 11 is

$$F_{11} = B\,[F/A, 2\%, \mathbf{12}]$$



But we require F_{12}, which is one period away from F_{11}.

$$\begin{aligned} F_{12} &= F_{11}\,[F/P, 2\%, 1] \\ &= B\,[F/A, 2\%, 12]\,[F/P, 2\%, 1] \\ &= \$1{,}000\,(13.4120897)\,(1.02) \\ &= \$13{,}680.33 \end{aligned}$$

Alternatively, we can first find the P value and then convert it to F value.

$$\begin{aligned} F_{12} &= P\,[F/P, 2\%, 12] \\ &= \$10{,}786.85\,(1.2682418) \quad \text{// From previous example} \\ &= \$13{,}680.33 \end{aligned}$$

2.2.8 Equivalent Values of Uniform Gradient Cash Flows

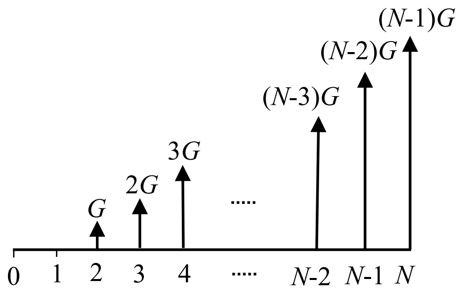
- Consider the case where the cash flows increase by a uniform or constant amount in each period.

Examples

1. The annual maintenance cost of a piece of equipment may increase by a constant amount a year due to more frequent breakdowns as it gets older.

2. The salary of a worker may increase by a constant amount a year according to the employment contract.

- The **Standard Uniform Gradient** cash flows with parameters G and N has the following cash flows:



Notes

- The cash flow at time $k = (k-1)G$ for $k = 2$ to N.

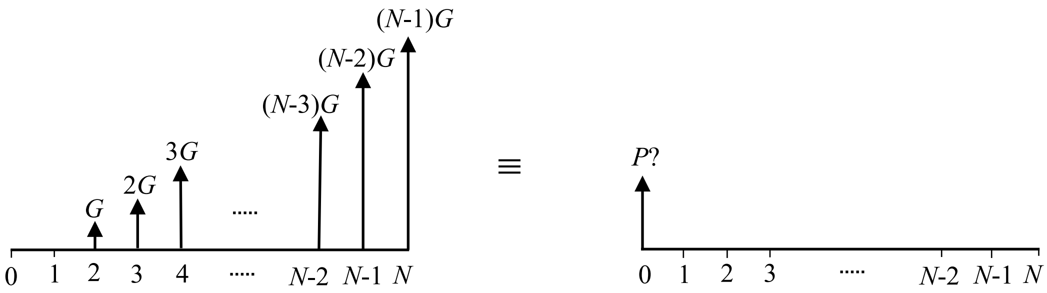
- There are N–1 non-zero cash flows.

- The first non-zero cash flow = G and it occurs at time 2.

- The last cash flow = (N–1) G and it occurs at time N.

<u>Find *P* when given *G*</u>

- Given Standard Uniform Gradient Cash Flows with parameters G and N, what is its **Equivalent Present Value** P if the interest rate is i per period?



- The Present Equivalent Value P of the G-series is the sum of the present equivalent value of the individual cash flows { 0, 0, G, $2G$, $3G$, …, $(N$-$1)G$ }:

$$\begin{aligned} P &= \frac{G}{(1+i)^2} + \frac{2G}{(1+i)^3} + \frac{3G}{(1+i)^4} + \ldots + \frac{(N-2)G}{(1+i)^{N-1}} + \frac{(N-1)G}{(1+i)^N} \\ &= \sum_{k=2}^{N} \frac{(k-1)G}{(1+i)^k} \\ &= G\sum_{k=2}^{N} \frac{(k-1)}{(1+i)^k} \end{aligned}$$

- It can be shown after some algebraic manipulations that

$$P = G\left\{\frac{1}{i}\left(\frac{(1+i)^N - 1}{i(1+i)^N} - \frac{N}{(1+i)^N}\right)\right\} = G\left[\frac{1-(1+Ni)(1+i)^{-N}}{i^2}\right]$$

- Hence the Equivalent Present Value of a Standard Uniform Gradient Cash Flows Series is

$$\boxed{P = G\left[\frac{1-(1+Ni)(1+i)^{-N}}{i^2}\right]}$$

- The factor $\left\{\left[\dfrac{1-(1+Ni)(1+i)^{-N}}{i^2}\right]\right\}$ is called the **<u>Gradient to Present Equivalent Conversion Factor</u>** and is denoted by the symbol [P/G, i%, N].

- We denote the relation between P and G as:

$$\boxed{P = G\,[P/G,\ i\%,\ N].}$$

- The values of the [P/G, i%, N] factor are available in the interest factors tables.

Example

- Find the equivalent present amount of the following cash flows at i=10% per year.

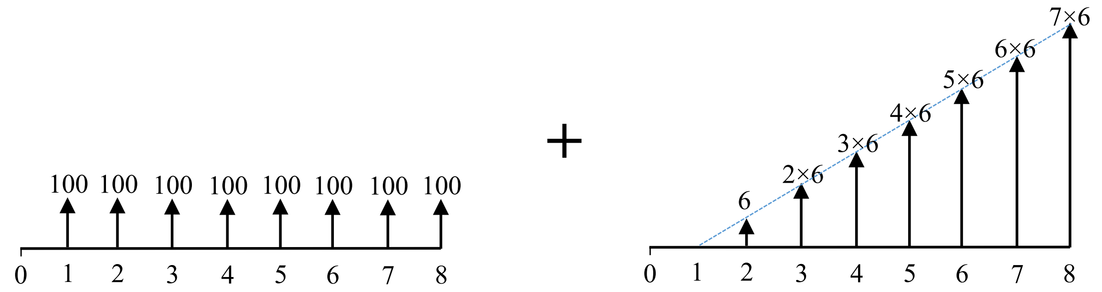
Year:	0	1	2	3	4	5	6	7	8
Amount:	0	\$100	\$106	\$112	\$118	\$124	\$130	\$136	\$142

- The base constant amount of \$100 can be separated from the uniform gradient cash flows:

  \$0, \$0, \$6, \$12, \$18, \$24, \$30, \$36, \$42.

- Hence the original cash flows is equivalent to a two sets of cash flows:

  1. A standard uniform series with A=\$100, N=8 years
  2. A standard uniform gradient series with G=\$6, N=8 years.



- The equivalent present amount is the sum of the equivalent present values of the two cash flows:

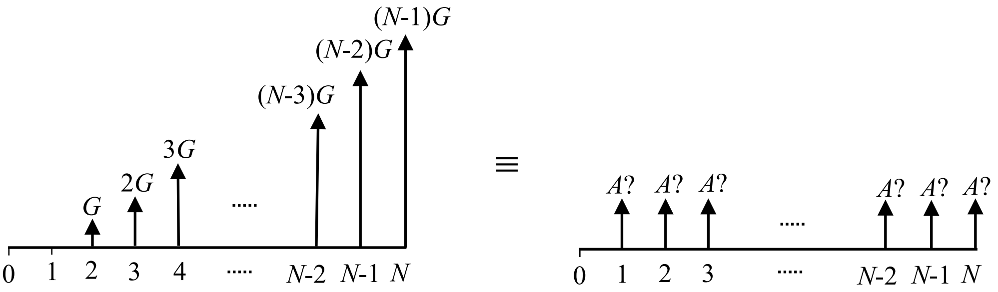
$$
\begin{aligned}
P &= 100\,[P/A, 10\%, 8] + 6\,[P/G, 10\%, 8] \\
&= 100\,(5.334926) + 6\,(16.028672) \\
&= \$\,629.66
\end{aligned}
$$

- Compare the results with the following direct computations using [P/F, i%, k] for k=1 to 8:

$$
\begin{aligned}
P &= \frac{100}{(1+0.1)} + \frac{106}{(1+0.1)^2} + \frac{112}{(1+0.1)^3} + \frac{118}{(1+0.1)^4} + \frac{124}{(1+0.1)^5} + \frac{130}{(1+0.1)^6} + \frac{136}{(1+0.1)^7} + \frac{142}{(1+0.1)^8} \\
&= \$629.66
\end{aligned}
$$

Find *A* when given *G*

- Given a *Standard Uniform Gradient* Cash Flows with parameters *G* and *N*, what is its **Equivalent Uniform End-of-Period Cash Flow** *A* over *N* periods if the interest rate is *i* per period?



- We already know how to find *P* given *G*: $A = P\ [A/P,\ i\%,\ N]$
- We also already know how to find *A* given *P*: $P = G\ [\ P/G,\ i\%,\ \underline{N}]$

- Using the chain rule:

  $A = G\ [P/G,\ i\%,\ \underline{N}]\ [A/P,\ i\%,\ N]$

$$A = G\left\{\frac{1}{i}\left(\frac{(1+i)^N - 1}{i(1+i)^N} - \frac{N}{(1+i)^N}\right)\right\}[A/P, i\%, N]$$
$$= \frac{G}{i}\left\{[P/A, i\% N] - \frac{N}{(1+i)^N}\right\}[A/P, i\%, N]$$
$$= \frac{G}{i}\left\{1 - \frac{Ni(1+i)^N}{(1+i)^N[(1+i)^N - 1]}\right\}$$
$$= \frac{G}{i} - G\left\{\frac{N}{(1+i)^N - 1}\right\}$$
$$= G\left(\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right)$$

- Hence the Equivalent Uniform End-of-Period Cash Flows of a Standard Uniform Gradient cash flow series is:

$$\boxed{A = G\left(\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right)}$$

- The factor $\left(\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right)$ is called the **Gradient to Uniform Series Conversion Factor** and is denoted by the symbol $[A/G,\ i\%,\ N]$.

- We denote the relation between *A* and *G* as:

$$\boxed{A = G\ [A/G,\ i\%,\ N].}$$

- The values of $[A/G,\ i\%,\ N]$ factor are available in the interest factors tables.

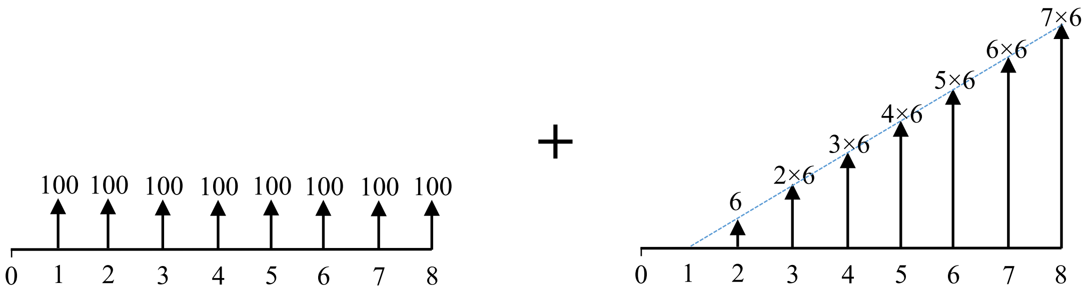
Example

- In the previous example, find the equivalent uniform per period cash flow.

Period:	0	1	2	3	4	5	6	7	8
Amount:	0	\$100	\$106	\$112	\$118	\$124	\$130	\$136	\$142

- Recall that the original cash flows is equivalent to two sets of cash flows as follows:

  1. A standard uniform series with A=\$100, N=8 years
  2. A standard uniform gradient series with G=\$6, N=8 years.



- Therefore the equivalent uniform annual cash flow over 8 years is

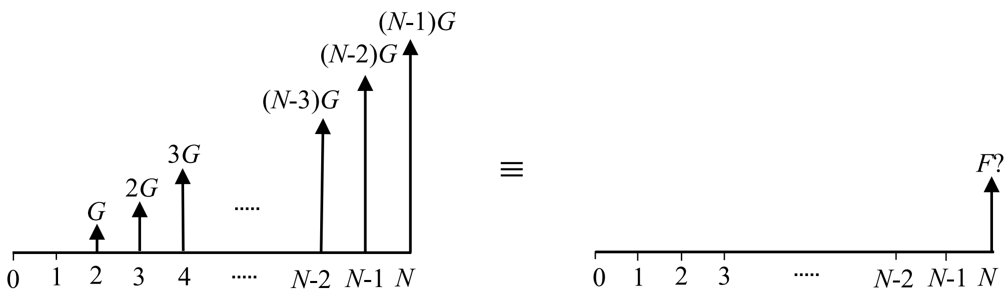
$$
\begin{aligned}
A &= 100 + 6\,[A/G, 10\%, 8] \\
&= 100 + 6\left[\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right] \\
&= 100 + 6\left[\frac{1}{0.1} - \frac{8}{(1+0.1)^8 - 1}\right] \\
&= 100 + 6\,(3.00447859) \\
&= 100 + 18.03 \\
&= \$118.03
\end{aligned}
$$

- Alternatively, since we already know P from the previous example:

$$
\begin{aligned}
A &= P\,[A/P, 10\%, 8] \\
&= 629.66\,(0.187444) \\
&= \$118.03
\end{aligned}
$$

Find F when given G

- Given a Standard Uniform Gradient Cash Flows with parameters G and N, what is its **Equivalent Future Value** F if the interest rate is i per period?



- We already know how to find F given P: $F = P\,[F/P,\ i\%,\ N]$
- We also already know how to find P given G: $P = G\,[P/G,\ i\%,\ N]$

- Using the chain rule:

$$F = G\,[P/G,\ i\%,\ N]\,[F/P,\ i\%,\ N]$$

$$F = G\left\{\frac{1}{i}\left(\frac{(1+i)^N-1}{i(1+i)^N}-\frac{N}{(1+i)^N}\right)\right\}(1+i)^N$$

$$= G\left\{\frac{1}{i}\left(\frac{(1+i)^N-1}{i}-N\right)\right\}$$

$$= \frac{G}{i}\left([F/A, i\%, N]-N\right)$$

- Hence the Equivalent Future Value at time N of a Standard Uniform Gradient Cash Flows Series is

$$\boxed{F = G\left\{\frac{1}{i}\left(\frac{(1+i)^N-1}{i}-N\right)\right\}}$$

- Hence the factor $[F/G, i\%, N] = \left\{\frac{1}{i}\left(\frac{(1+i)^N-1}{i}-N\right)\right\}$

- The values of the factor $[F/G,\ i\%,\ N]$ are not listed in the interests table, but may be computed using values of $[F/A,\ i\%,\ N]$ from the table.

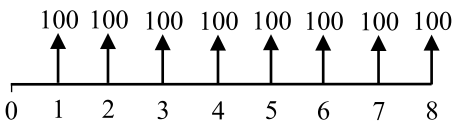
$$[F/G, i\%, N] = \frac{1}{i}\left([F/A, i\%, N]-N\right).$$

Example

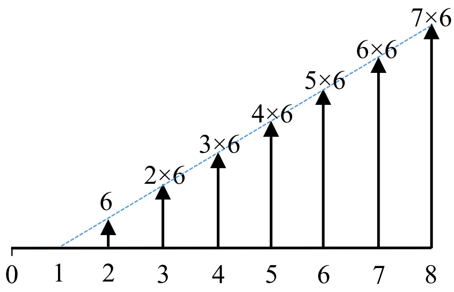
- In the previous example, find the equivalent future amount of the following cash flows at i=10% per year.

Year:	0	1	2	3	4	5	6	7	8
Amount:	0	\$100	\$106	\$112	\$118	\$124	\$130	\$136	\$142

- The equivalent future value is the sum of the equivalent future values of the two cash flows:



+



$$
\begin{aligned}
F &= 100\,[F/A,\ 10\%,\ 8] + 6\,[F/G,\ 10\%,\ 8] \\
&= 100\,[F/A,\ 10\%,\ 8] + \frac{6}{0.1}([F/A,\ 10\%,8] - 8) \\
&= 100\,(11.435888) + \frac{6}{0.1}(11.435888 - 8) \\
&= \$1{,}349.74
\end{aligned}
$$

- Compare the results using many $[F/P,\ 10\%,\ 8 - k]$ for k = 1 to 8:

$$
\begin{aligned}
F &= 100\,(1 + 0.1)^7 + 106\,(1 + 0.1)^6 + 112\,(1 + 0.1)^5 + \ldots + 142 \\
&= \$1{,}349.74
\end{aligned}
$$

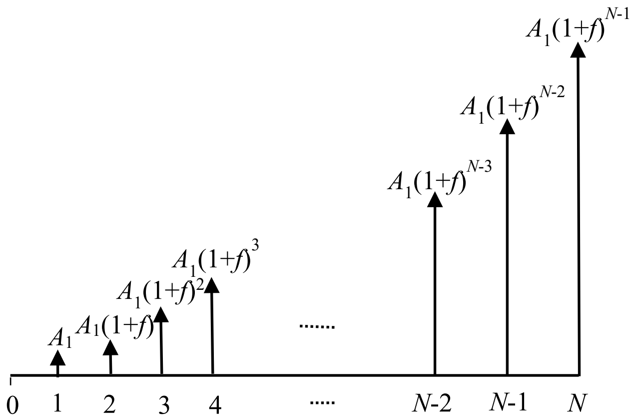
2.2.9 Equivalent Values of Geometric Series Cash Flows

- Consider the case where the cash flows change by *a constant fraction* (percentage) over the preceding period.

Examples

1. The salary of a worker may increase by a constant percentage a year according to the employment contract.
2. A cost component of a project might increase at a constant percentage rate annually.

- The **Standard Geometric Series** cash flows with parameters A_1, f and N has the following cash flows:



  - A_1 = cash flow at time 1.
  - f = inter-period fractional change.
  - Cash flow at time $k = A_1(1+f)^{k-1}$, for $k = 1$ to N.

Find P when given (A_1, f)

- Given a Standard Geometric Series Cash Flows over N periods with parameters A_1 and f, what is its **Equivalent Present Value** P if the interest rate is i per period?

- P is the total equivalent present values of the individual cash flows in the geometric cash flow series:

$$
\begin{aligned}
P &= \sum_{k=1}^{N} A_1(1+f)^{k-1}[P/F, i\%, k] \\
&= \sum_{k=1}^{N} \frac{A_1(1+f)^{k-1}}{(1+i)^k} \\
&= \frac{A_1}{(1+i)} \sum_{k=0}^{N-1} \left(\frac{1+f}{1+i}\right)^k \\
&= \frac{A_1}{(1+i)} \sum_{k=0}^{N-1} x^k \qquad \text{where } x = \frac{(1+f)}{(1+i)}
\end{aligned}
$$

.

- By the sum of a geometric series:

$$\sum_{k=0}^{N-1} x^k = \begin{cases} \dfrac{(1-x^N)}{(1-x)} & x \neq 1 \text{ or } f \neq i \\ N & x = 1 \text{ or } f = i \end{cases}$$

- Hence

$$P = \begin{cases} \dfrac{A_1}{(1+i)} \dfrac{(1-x^N)}{(1-x)} & f \neq i \\ \dfrac{A_1}{(1+i)} N & f = i \end{cases}$$

- Substituting back $x = \dfrac{(1+f)}{(1+i)}$:

$$P = \begin{cases} \dfrac{A_1[1-(1+i)^{-N}(1+f)^N]}{(i-f)} & f \neq i \\ \dfrac{A_1 N}{(1+i)} & f = i \end{cases}$$

Or

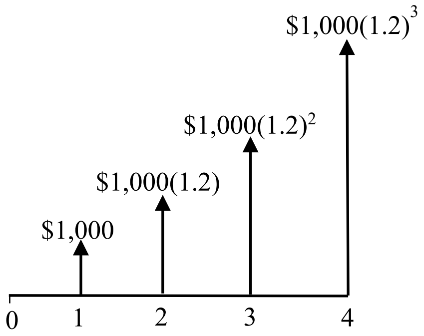
$$P = \begin{cases} \dfrac{A_1[1-[P/F, i\%, N][F/P, f\%, N]]}{i-f} & f \neq i \\ A_1 N[P/F, i\%, 1] & f = i \end{cases}$$

Find *F* and *A* when given (A_1, *f*)

- Given a Standard Geometric Series Cash Flow, we can find its Equivalent Future Value *F* at time *N*, and Equivalent Uniform Annual Value *A* over *N* periods by first computing the *P* values. Then
  - $F = P\ [F/P, i\%, N]$
  - $A = P\ [A/P, i\%, N]$

Example

- Find the equivalent present value of the following cash flows if the interest rate is 25% per year.



- Given $i = 0.25, f = 0.20, A_1 = \$1000, N = 4$ years

- Since $i \neq f$

$$P = \frac{A_1[1-(1+i)^{-N}(1+f)^N]}{(i-f)}$$
$$= \frac{(1{,}000)[1-(1+0.25)^{-4}(1+0.20)^4]}{(0.25-0.20)}$$
$$= \$3{,}013.07$$

- Alternatively:

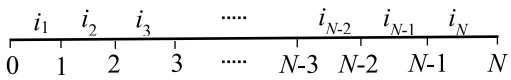
$$P = \frac{1{,}000[1-[P/F,25\%,4][F/P,20\%,4]]}{(0.25-0.20)}$$
$$= \frac{1{,}000}{0.05}\left[1-(0.4096)(2.0736)\right]$$
$$= \$3{,}013.07$$

2.3 Time-Dependent Interest Rates

- We would like to look at the situation when the interest rate is not constant, but is different in different time periods.

<u>Find *F* when given *P* with Time-Dependent Interest Rates</u>

- Let the interest rate in period k be equal to i_k for $k = 1,\ldots,N$.



- Given an initial investment P at time 0:
  - The equivalent value at end of period 1 is $F_1 = P\,(1+i_1)$
  - The equivalent value at the end of period 2 is $F_2 = F_1\,(1+i_2) = P\,(1+i_1)\,(1+i_2)$.
  - The equivalent value at the end of period 3 is $F_3 = F_2\,(1+i_3) = P\,(1+i_1)\,(1+i_2)\,(1+i_3)$.
  - The equivalent value at the end of period N is $F_N = P\,(1+i_1)\,(1+i_2)\,(1+i_3)\,\ldots\,(1+i_N)$.
- Hence the Future Value of P at the end of N periods with time-dependent interest rates is

$$F = P\prod_{k=1}^{N}(1+i_k)$$

Example

- If \$10,000 is deposited into an investment fund that earns interests equal to 5% in the first year, 6% in the second year, and 7% in the third year, what is the final value at the end of 3 years?

  Given P = \$10,000, i_1 = 5%, i_2 = 6%, and i_3 = 7%.

  F = \$ 10,000 (1+0.05) (1+0.06) (1+0.07)
  = \$ 11,909.10

<u>Find *P* when given *F* with Time-Dependent Interest Rates</u>

- Let the interest rate in period k be equal to i_k for $k = 1,\ldots,N$.

- Given a single cash flow F at time N, the equivalent present value P at time zero is

$$P = \frac{F}{\prod_{k=1}^{N}(1+i_k)}.$$

Example

- An investment fund that earns interests equal to 5% in the first year, 6% in the second year, and 7% in the third year. If \$10,000 is to be accumulated at the end of 3 years, what amount must be deposited into the fund now?

Given F = \$10,000

Interest rates: $i_1 = 5\%$, $i_2 = 6\%$, and $i_3 = 7\%$.

$$P = \frac{10{,}000}{(1+0.05)(1+0.06)(1+0.07)}$$
$$= \$8{,}396.94$$

2.4 Multiple Compounding Per Period

2.4.1 Nominal and Effective Interest Rates

- Suppose you invested \$1,000 in an account that pays interests at a rate stated as follows:

"8% per year, compounded quarterly"

- Here the interest rate is specified as an annual percentage, but the interests are compounded not once a year, but every quarter.

- **What interest rates are you actually paying?**

Nominal Interest Rates

- When an interest rates is specified as an annual rate but the compounding period is not once a year, it is known as a **nominal** interest rate.

Effective or Real Interest Rates

- So how much interests will you be earning after one year, and what is the real or effective interest rate?

- 8% **per year** compounded **quarterly** is the same as 2% per **quarter** compounded **quarterly.**

- We apply the single payment compound amount formula with i=2% per period, and each period = 1 quarter or 3 months:

Initial investment		= \$1,000.00
Value at end of 3 months	=1,000 (1 + 0.02)	= \$1,020.00
Value at end of 6 months	=1,020 (1 + 0.02)	= \$1,040.40
Value at end of 9 months	=1,040.40 (1 + 0.02)	= \$1,061.208
Value at end of 12 months	=1,061.208 (1 + 0.02)	= \$1,082.43216

- Since an initial sum of \$1,000 grows to \$1,082.43 after 1 year the *equivalent **interest rate*** that would earn the same amount of interest is:

$$i = \left(\frac{1082.43216 - 1000.00}{1000.00} \right) 100\% = 8.243\%$$

- Hence a nominal rate of 8% per year compounded quarterly is equivalent to an effective annual rate of 8.243% per year.

- What if the interest rate is 8% per year compounded **monthly**? Would the effective annual rate be:

  - < 8.243% ?
  - = 8.243% ?
  - > 8.243% ?

2.4.2 Relation between Nominal and Effective Interest Rates

- Let
  r = the nominal interest rate per year.
  M = number of compounding periods per year.

- The interest rate per compounding period is r/M.

- If an amount P is invested at time zero, its value at the end of one year is

$$F = P(1+\frac{r}{M})^M$$

- The effective annual interest rate

$$i = \frac{F-P}{P} = \left[1+\frac{r}{M}\right]^M - 1$$

- Hence the **Effective Annual Interest Rate** when the nominal interest rate is r per year compounded M times per year is

$$i = \left[1+\frac{r}{M}\right]^M - 1$$

Notes:

1. $M > 1 \quad \Rightarrow \quad i > r$

   If compounding is more than once a year, then effective rate is higher than the nominal rate.

2. $M = 1 \quad \Rightarrow \quad i = r$

   If compounding is exactly once a year, then effective rate is equal to nominal rate.

3. $M < 1 \Rightarrow i < r$

   If compounding is less than once a year (e.g., M=1/2 means compound every 2 years), then effective rate is lower than the nominal rate.

Example

- Suppose the interest charged by a credit card company for outstanding balance rolled to the following month is charged at a rate of 24% per year compounded monthly, what is the effective annual interest rate?

- Here r = 24% per year compounded monthly. Hence M=12.

- Therefore effective annual interest rate is

$$\begin{aligned} i &= (1 + 0.24/12)^{12} - 1 \\ &= (1 + 0.02)^{12} - 1 \\ &= 0.26824 \text{ or } \underline{\mathbf{26.824\%}} \end{aligned}$$

Example

- What is the effective annual interest rate if the nominal rate is 18% per year compounded (1) semi-annually, (2) quarterly, and (3) monthly?

(1) Semi-annual compounding:

$$\text{Effective annual interest rate } = \left(1+\frac{0.18}{2}\right)^2 - 1 = 0.1881 \text{ or } \underline{18.81\% \text{ per year}}$$

(2) Quarterly compounding:

$$\text{Effective annual interest rate } = \left(1+\frac{0.18}{4}\right)^4 - 1 = 0.1925 \text{ or } \underline{19.25\% \text{ per year}}$$

(3) Monthly compounding:

$$\text{Effective annual interest rate } = \left(1+\frac{0.18}{12}\right)^{12} - 1 = 0.1956 = \underline{19.56\% \text{ per year}}$$

- Notice that i increases as M increases.

Effective Interest Rates based on Other Time Periods

- An effective interest rate may also be expressed based on other time periods, other than annual.

Example

- In the previous example:

  (1) 18% per year compounded *semi-annually* is equivalent to 18%/2 = 9% per six months compounded six-monthly. The *effective semi-annual* interest rate is 9%.

  (2) 18% per year compounded *quarterly* is equivalent to 18%/4 = 4.5% per quarter compounded quarterly. The *effective quarterly* interest rate is 4.5%.

  (3) 18% per year compounded *monthly* is equivalent to 18%/12 = 1.5% per month compounded monthly. The *effective monthly* interest rate is 1.5%.

Summary

	Nominal annual rate	Effective rate	Effective annual rate
1	18% compounded semi-annually	9% per six months	18.81%
2	18% compounded quarter	4.5% per quarter	19.25%
3	18% compounded monthly	1.5% per month	19.56%

2.4.3 Solving Problems with Unknown Interest Rates

Example

- Suppose you are interested in buying a used car at $75,000, and the dealer proposed that you make a $5,000 down payment now and pay the balance in equal end-of-month payments of $1,948.20 each over a 48-month period.

i. What is the dealer's effective interest rate?

- Given P = $75,000 – $5,000 = $70,000
  A = $1,948.20
  N = 48 months

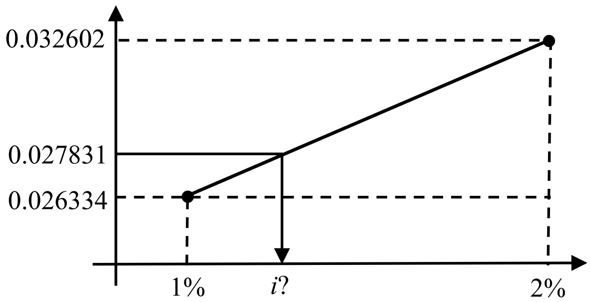
- We need to find i such that

$$\$1{,}948.20 = \$70{,}000\,[A/P,\, i,\, 48]$$

Or $[A/P, i, 48] = 0.027831$

- Solve using interest tables and linear interpolation:

  - $[A/P, 1\%, 48] = 0.026334$
  - $[A/P, \text{?}, 48] = 0.027831$
  - $[A/P, 2\%, 48] = 0.032602$



By similar triangles:

$$\frac{i-1\%}{2\%-1\%} \approx \frac{(0.027800-0.026334)}{(0.032602-0.026334)}$$

$$i \approx 1\% + \frac{(0.027800-0.026334)}{(0.032602-0.026334)}(2\%-1\%)$$
$$= 1.25\% \text{ per month}$$

- Hence interest rate of the dealer = 1.25% per month compounded monthly.
- Nominal interest rate =1.25 × 12 = 15% per year compounded monthly.
- Effective annual interest rate = $(1 + 0.0125)^{12} - 1 = 0.16075$ or 16.075% per year.

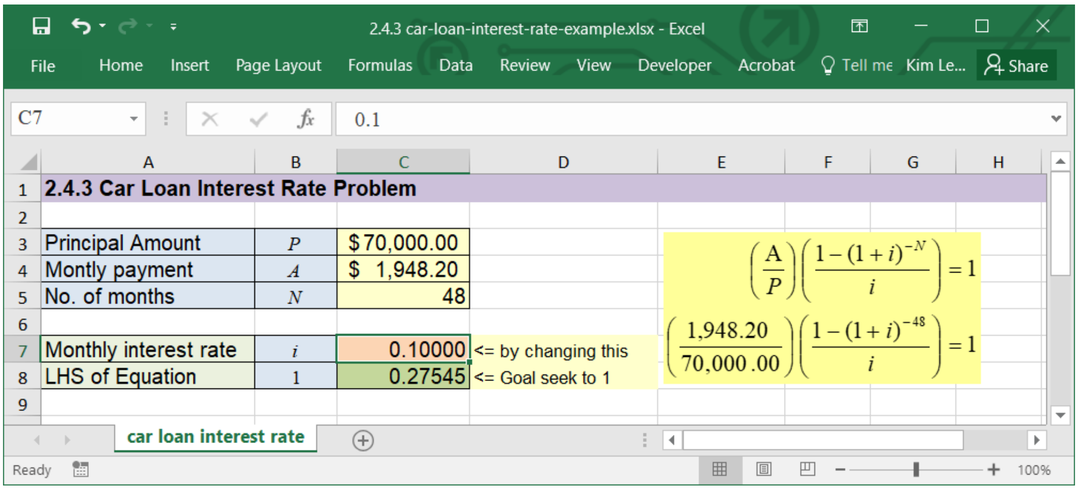
Using Excel Goal Seek to solve Equations

- Excel Goal Seek can be used to solve an equation with one unknown.
- Equation to solve:

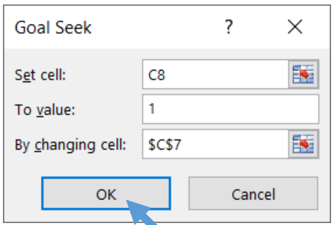
$$[A/P,i,48] = \frac{i(1+i)^{48}}{(1+i)^{48}-1} = \frac{1{,}948.20}{70{,}000.00}$$

$$\text{Or} \quad \left(\frac{1.948.20}{70{,}000.00}\right)\left(\frac{1-(1+i)^{-48}}{i}\right) = 1$$

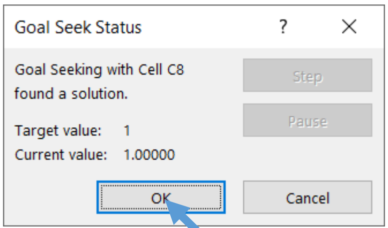
- Set up the equation to solve in Excel using one input cell for the unknown variable (assigned a guess value) and one output (target) cell for LHS of the equation.



- Execute: Data | What-If Analysis | Goal Seek…

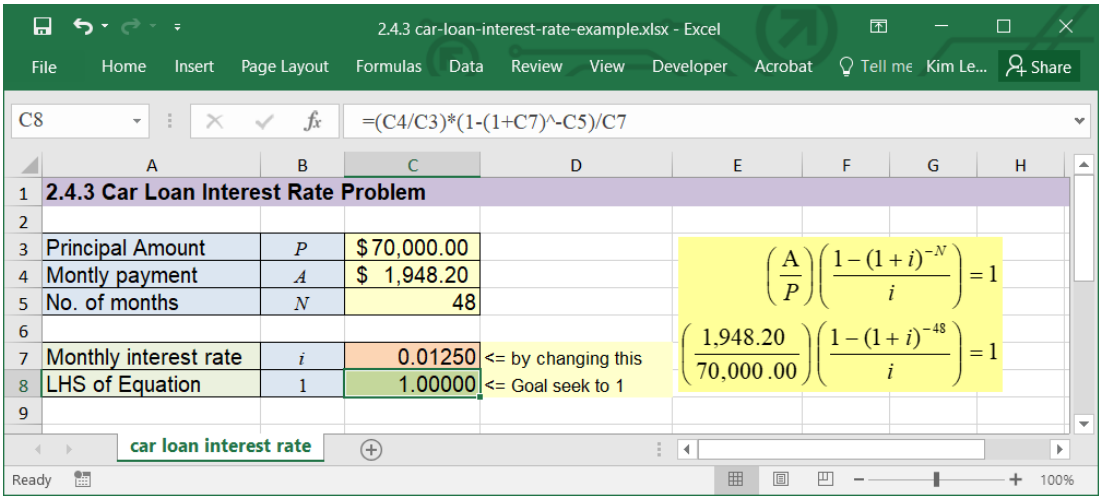


Click



Click

- Result:



Using Python to solve equations:

In [1]:
```
# 2.4.3_find_interest_rate_using_equation_solver.ipynb
""" 2.4.3 Finding interest rate using an equation solver """
from scipy.optimize import root
```

In [2]:
```
# Parameters and function to solve
P = 70_000
A = 1948.20
N = 48
# The function whose root we want to find
func = lambda r : (A/P)*((1-(1+r)**-N)/r)-1
```

In [3]:
```
# Find the root of the function using default solver hybr
guess = 0.1
solution = root(func, x0=guess, options={'xtol': 1E-10})
if solution.success:
    print(f"Interest Rate = {solution.x[0]:.8f}")
else:
    print(solution.message)
```

```
Interest Rate = 0.01250112
```

In [4]:
```
```

Example (continued)

ii. Instead of using the dealer's financing, you decided to make a down payment of $5,000, and borrow the rest from a bank at a nominal interest rate of 12% compounded monthly. What would be your monthly payment to pay off the loan in 4 years?

- Normal interest rate = 12% per year compounded monthly.
- Hence interest rate is i = 12%/12 = 1% per month compounded monthly.
- N = 48 months

$$A = P\,[A/P, 1\%, 48] = (75{,}000 - 5{,}000)\,(0.026334) = \$\ \underline{\mathbf{1{,}843.37}}$$

- Hence you will have to pay $1,843.37 at the end of each month to pay off the loan in four years.

2.4.4 Solving Problems with Multiple Compounding in a year

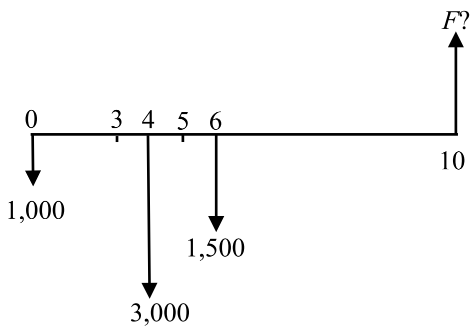
Solution Strategies

1. If cash flows occur more than once per year, use r/M as the interest rate, and $1/M$ year as the time unit.

2. If cash flows occur only annually, the effective annual interest rate can be used and use 1 year as the time unit.

Example

- If you deposit \$1,000 now, \$3,000 four years from now, and \$1,500 six years from now at an interest rate of 12% per year compounded semi-annually, how much money will you have in your account 10 years from now?



- Notice that compounding is semi-annual, but cash flows occur only on the year.

Method 1

- Let each period = ½ year. i = 6% per period (six-month)

$$
\begin{aligned}
F &= 1{,}000\,[F/P, 6\%, \mathbf{20}] + 3{,}000\,[F/P, 6\%, \mathbf{12}] + 1{,}500\,[F/P, 6\%, \mathbf{8}] \\
&= 1{,}000\,(1 + 0.06)^{20} + 3{,}000\,(1 + 0.06)^{12} + 1{,}500\,(1 + 0.06)^{8} \\
&= \$11{,}634.50
\end{aligned}
$$

Method 2

- Since cash flows only occur on the year, let each period = 1 year.

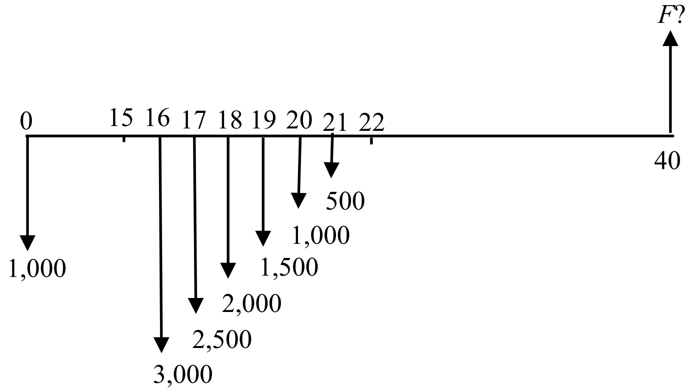
- Effective $i = \left(1 + \dfrac{0.12}{2}\right)^2 - 1 = 0.1236$ per year.

$$
\begin{aligned}
F &= 1{,}000\,[F/P, 12.36\%, \mathbf{10}] + 3{,}000\,[F/P, 12.36\%, \mathbf{6}] + 1{,}500\,[F/P, 12.36\%, \mathbf{4}] \\
&= 1{,}000\,(1 + 0.1236)^{10} + 3{,}000\,(1 + 0.1236)^{6} + 1{,}500\,(1 + 0.1236)^{4} \\
&= \$11{,}634.50
\end{aligned}
$$

Example

- If you deposit \$1,000 now, \$3,000 four years from now, followed by five quarterly deposits decreasing by \$500 per quarter at a rate of 12% per year compounded quarterly, how much money will you have in your account 10 years from now?

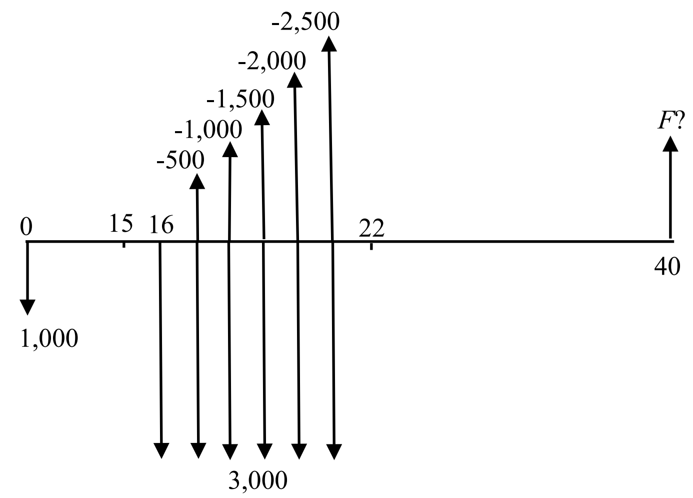
- Let 1 period = 1 quarter (3 months). The cash flow diagram is as follows:



- i = 12%/4 = 3% per quarter.

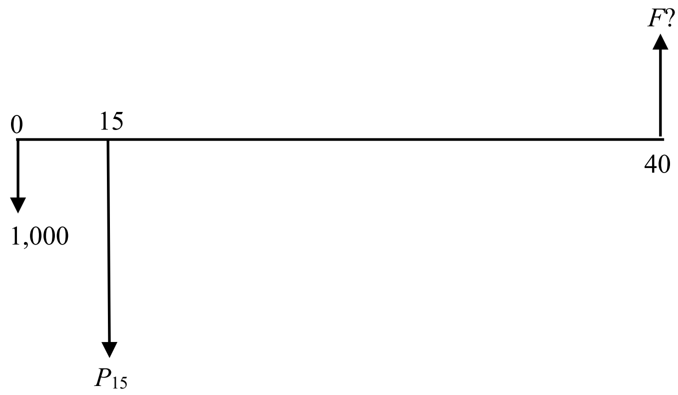
- The cash flows from EoQ 16 through EoQ 21 is equivalent to:

  1. A uniform quarterly cash flows with A = \$3,000, N=6 with time 0 at EoQ 15.
  2. A uniform gradient cash flows with G = -\$500, N=6, with time 0 at EoQ 15.



- The equivalent value of these two cash flows evaluated at EoQ 15

$$P_{15} = 3{,}000\,[P/A, 3\%, 6] - 500\,[P/G, 3\%, 6]$$
$$= \$9{,}713.60$$



- Future value of all the cash flows at end of 10 years (40 quarters)

$$F = P_{15}\ [F/P,\ 3\%,\ 40 - 15] + 1{,}000\ [F/P,\ 3\%,\ 40]$$
$$= 9{,}713.60\ [F/P,\ 3\%,\ 25] + 1{,}000\ [F/P,\ 3\%,\ 40]$$
$$= 9{,}713.60\ [F/P,\ 3\%,\ 25] + 1{,}000\ [F/P,\ 3\%,\ 40]$$
$$= \$23{,}600$$

- Compare the results with the direct formula:

$$F = 1{,}000\ (1.03)^{40} + 3{,}000\ (1.03)^{24} + 2{,}500\ (1.03)^{23} + 2{,}000\ (1.03)^{22} + 1{,}500\ (1.03)^{21} + 1{,}000\ (1.03)^{20} + 500\ (1.03)^{19}$$
$$= \$23{,}600$$

2.5 Continuous Compounding of Discrete Cash Flows

- We assume that the cash flows occur at discrete intervals but the compounding is continuous throughout the interval.

Find *F* when given *P* under Continuous Compounding

- Suppose an amount P_0 is invested at time zero, at a nominal interest rate = r, compounded M times a year.

- The value at the end of 1 year is

$$F_1 = P_0\left(1+\frac{r}{M}\right)^M .$$

- Let $q = M/r$, then

$$F_1 = P_0\left(1+\frac{1}{q}\right)^{rq} = P_0\left[\left(1+\frac{1}{q}\right)^q\right]^r .$$

- Continuous compounding occurs when the number compounding periods per year M is infinity.

- Let $M \to \infty$. Then $q = M/r \to \infty$, we have

$$F_1 = P_0\left[\lim_{q\to\infty}\left(1+\frac{1}{q}\right)^q\right]^r$$

- Applying $\lim_{q\to\infty}\left(1+\frac{1}{q}\right)^q = e = 2.71828...$, the value of P_o at the end of 1 year under continuous compounding at nominal rate r per year is:

$$F_1 = P_0\, e^r.$$

- The value of P_o the end of 2 years under continuous compounding at rate r is

$$F_2 = F_1\, e^r = P_0\, e^{2r}.$$

- The value of P_o the end of N years under continuous compounding at rate r is

$$F_N = P_0\, e^{Nr}.$$

- We denote the factor e^{Nr} by the symbol $[F/P,\underline{r}\%,N]$, where $\underline{r}\%$ means continuous compounding at $r\%$ per period.

- Hence $F = P\,[F/P,\underline{r}\%,N]$

Effective Interest Rate under Continuous Compounding

- From the analysis above, the value of P_0 invested now for one year under continuous compounding at rate $\underline{r}$% per year, is

$$F_1 = P_0\, e^r.$$

- This is equivalent to an effective (discrete compounding) interest rate of

$$i_e = \frac{P_0 e^r - P_0}{P_0} = e^r - 1$$

- Hence we can use all the discrete compounding interest formulas by replacing the discrete compounding rate i in the formulas with $e^r - 1$. That is

$$[X/Y, \underline{r}, N] = [X/Y, e^r - 1, N] \qquad \text{for all } X, Y \in \{ P, A, F, G \}$$

Summary of Formulas for Continuous Compounding, Discrete Cash Flows

- The formulas for interests and annuity under continuous compounding of discrete cash flows are:

	Factor under continuous compounding	Equivalent factor under discrete compounding	Formula
1	$[P/F, \underline{r}\%, N]$	$[P/F, e^r - 1, N]$	e^{-rN}
2	$[F/P, \underline{r}\%, N]$	$[F/P, e^r - 1, N]$	e^{rN}
3	$[P/A, \underline{r}\%, N]$	$[P/A, e^r - 1, N]$	$\dfrac{e^{rN} - 1}{e^{rN}(e^r - 1)}$
4	$[A/P, \underline{r}\%, N]$	$[A/P, e^r - 1, N]$	$\dfrac{e^{rN}(e^r - 1)}{e^{rN} - 1}$
5	$[F/A, \underline{r}\%, N]$	$[F/A, e^r - 1, N]$	$\dfrac{e^{rN} - 1}{e^r - 1}$
6	$[A/F, \underline{r}\%, N]$	$[A/F, e^r - 1, N]$	$\dfrac{e^r - 1}{e^{rN} - 1}$
7	$[P/G, \underline{r}\%, N]$	$[P/G, e^r - 1, N]$	$\dfrac{1 - (1 + N(e^r - 1))(1 + (e^r - 1))^{-N}}{(e^r - 1)^2}$
8	$[A/G, \underline{r}\%, N]$	$[A/G, e^r - 1, N]$	$\dfrac{1}{e^r - 1} - \dfrac{N}{e^{rN} - 1}$
9	$[F/G, \underline{r}\%, N]$	$[F/G, e^r - 1, N]$	$\left(\dfrac{1 - (1 + N(e^r - 1))(1 + (e^r - 1))^{-N}}{(e^r - 1)^2}\right) e^{rN} = \dfrac{[F/A, \underline{r}\%, N] - N}{e^r - 1}$

Interest and Annuity Tables for Continuous Compounding

- Values of the following factors are available in the interests tables:
  - $[F/P, \underline{r}\%, N]$
  - $[P/F, \underline{r}\%, N]$
  - $[F/A, \underline{r}\%, N]$
  - $[P/A, \underline{r}\%, N]$

Example

- Consider a loan of $1,000. What equivalent uniform end-of-year payments must be made for 10 years if the nominal interest rate is 10% per year compound continuously?

  $\underline{r} = 10\%$ per year compounded continuously
  N = 10 years

- Using Continuous compounding interest table:

$$
\begin{aligned}
A &= 1{,}000\ [A/P, \underline{10}\%, 10] && \text{// } A/P \text{ factors are not available in continuous} \\
&= 1{,}000\ \frac{1}{[P/A, \underline{10}\%, 10]} && \text{// compounding interest tables} \\
&= 1{,}000 / 6.0104 && \text{// From Appendix D of text book} \\
&= \$166.38
\end{aligned}
$$

- Using continuous compounding interest formula directly:

$$
\begin{aligned}
A &= P\ \frac{e^{rN}(e^{r} - 1)}{e^{rN} - 1} \\
&= 1{,}000\ \frac{e^{0.1\times 10}(e^{0.1} - 1)}{e^{0.1\times 10} - 1} \\
&= \$166.38
\end{aligned}
$$

- Using the discrete compounding interest formula with $i_e = e^{0.1} - 1 = 0.10517$:

$$
\begin{aligned}
A &= 1{,}000\ [A/P, 10.517\%, 10] \\
&= 1{,}000 \left[\frac{0.10517\,(1+0.10517)^{10}}{(1+0.10517)^{10} - 1} \right] \\
&= 1{,}000\ (0.16638) \\
&= \$166.38
\end{aligned}
$$

Example

- In the previous example, what is the repayment amount if it is to be made at the end of every six months instead?

- In this case
  $\underline{r} = 10/2 = 5\%$ per six-month compounded continuously
  N = 20 six-month periods.

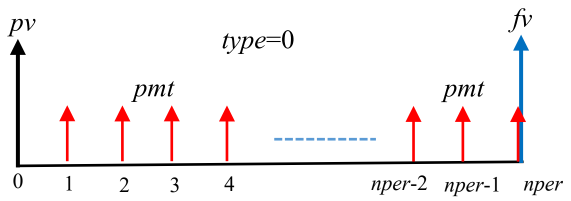
$$
\begin{aligned}
A &= 1{,}000\ [A/P, \underline{5}\%, 20] \\
&= 1{,}000\ [A/P, e^{0.05} - 1, 20] \\
&= 1{,}000 \left[\frac{e^{0.05\times 20}(e^{0.05} - 1)}{e^{0.05\times 20} - 1} \right] \\
&= \$81.11
\end{aligned}
$$

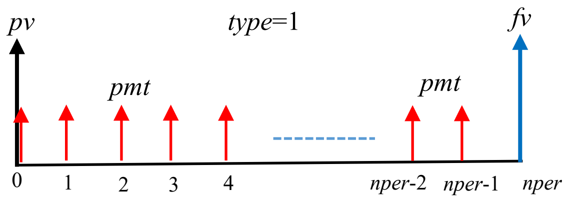
- Note that the six-monthly payment amount \$81.11 is not equal to half of the yearly payment amount (\$166.38) in the previous example. Why?

2.6 Excel Financial Functions

2.6.1 Functions pv(), pmt(), fv(), rate(), nper()

- The five Excel functions pv(), pmt(), fv(), rate(), and nper() compute the value of one variable, given the values of the other four variables so that the **equivalent value of the cash flows is zero.**





Inputs (up to any four):

rate = interest rate per period
nper = total number of periods
pv = a single cash flow at time zero
pmt = uniform cash flow per period
fv = a single cash flow at end of period *nper periods*

Parameters:

$$type = \begin{cases} 0 & \text{uniform end of periods cash flows (default)} \\ 1 & \text{uniform beginning of period cash flows} \end{cases}$$

Outputs for uniform end-of-period cash flows:

- $PV\,(rate, nper, pmt, fv\ [,0]\ )$

  returns $\quad -(\ pmt\ [P/A, rate, nper] + fv\ [P/F, rate, nper]\ )$

- $PMT\,(rate, nper, pv, fv\ [,0]\ )$

  returns $\quad -(\ pv\ [A/P, rate, nper] + fv\ [A/F, rate, nper]\ )$

- $FV(rate, nper, pmt, pv\ [,0]\ )$

  returns $\quad -(\ pv\ [F/P, rate, nper] + pmt\ [F/A, rate, nper]\ )$

- $RATE(nper, pmt, pv, fv\ ,\ 0\ , guess)$

  returns r such that $\quad pv + pmt\ [P/A, r, nper] + fv\ [P/F, r, nper] = 0$

- $NPER(rate, pmt, pv, fv)$

  returns N such that $\quad pv + pmt\ [P/A, rate, N] + fv\ [P/F, rate, N] = 0$

- Take note of the **negative sign** in front of the quantity returned by PV, PMT and FV.

2.6.2 Generating Interest Factors using Excel Financial Functions

Discrete Compounding, Discrete Cash Flows

- The factors in the Interest Factors tables for **discrete compounding** can be generated via Excel functions by using **negative one-dollar** cash flows as inputs:

[*P*/*A*, *i*, *N*]   =PV(*i*, *N*, -1, 0, 0)

[*P*/*F*, *i*, *N*]   =PV(*i*, *N*, 0, -1, 0)

[*A*/*P*, *i*, *N*]   =PMT(*i*, *N*, -1, 0, 0)

[*A*/*F*, *i*, *N*]   =PMT(*i*, *N*, 0, -1, 0)

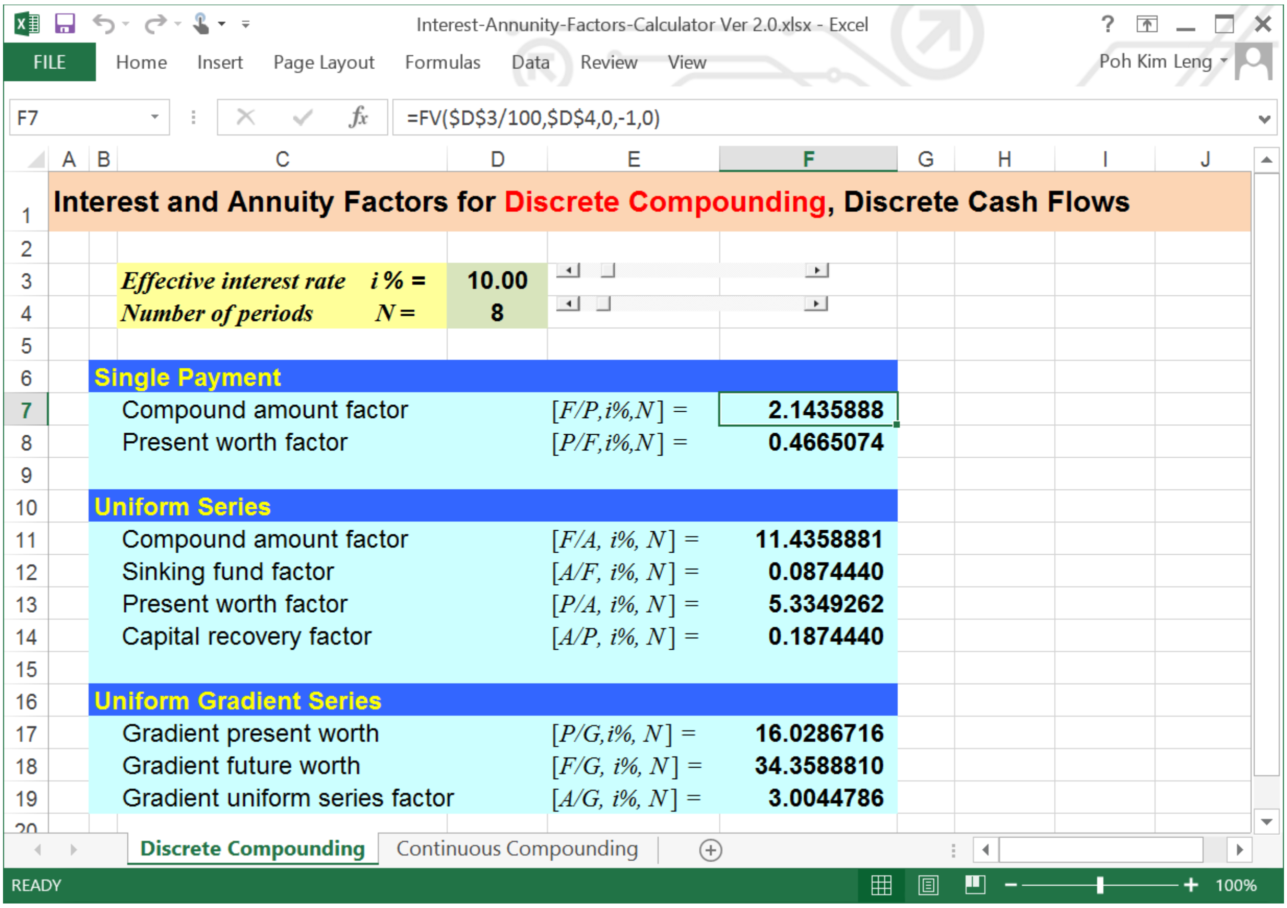
[*F*/*A*, *i*, *N*]   =FV(*i*, *N*, -1, 0, 0)

[*F*/*P*, *i*, *N*]   =FV(*i*, *N*, 0, -1, 0)

[*F*/*G*, *i*, *N*]   =(FV(*i*, *N*, -1, 0, 0) – N) / *i*

[*P*/*G*, *i*, *N*]   = [*P*/*F*, *i*, *N*] * [*F*/*G*, *i*, *N*]   // multiple 2 factors from above

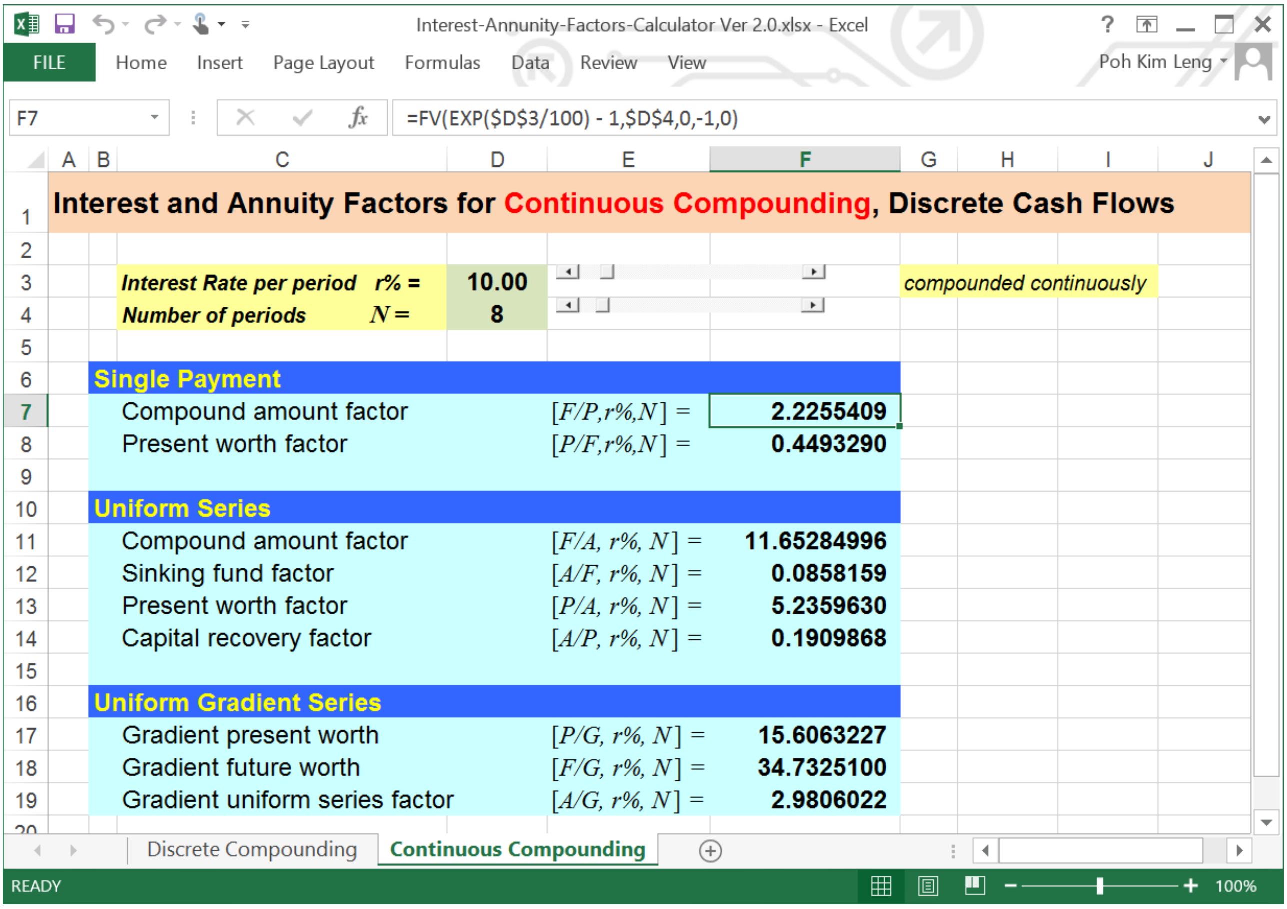
[*A*/*G*, *i*, *N*]   = [*A*/*F*, *i*, *N*] * [*F*/*G*, *i*, *N*]   // multiple 2 factors from above



Continuous Compounding, Discrete Cash Flows

- The factors in the Interest Factors tables for **continuous compounding** at $\underline{r}$ can be generated indirectly via Excel functions for discrete compounding by first computing the equivalent effective interests rate $i = exp(r) - 1$.

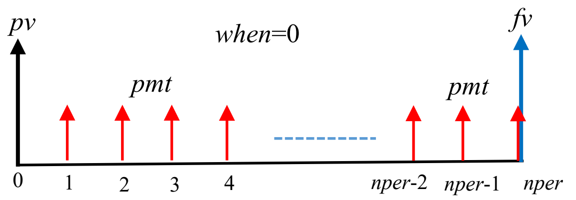
$[P/A, \underline{r}, N]$	=PV($exp(r) - 1$, N, -1, 0, 0)
$[P/F, \underline{r}, N]$	=PV($exp(r) - 1$, N, 0, -1, 0)
$[A/P, \underline{r}, N]$	=PMT($exp(r) - 1$, N, -1, 0, 0)
$[A/F, \underline{r}, N]$	=PMT($exp(r) - 1$, N, 0, -1, 0)
$[F/A, \underline{r}, N]$	=FV($exp(r) - 1$, N, -1, 0, 0)
$[F/P, \underline{r}, N]$	=FV($exp(r) - 1$, N, 0, -1, 0)
$[F/G, \underline{r}, N]$	=(FV($exp(r) - 1$, N, -1, 0, 0) – N) / ($exp(r) - 1$)
$[P/G, \underline{r}, N]$	= $[P/F, \underline{r}, N] * [F/G, \underline{r}, N]$ // multiply 2 factors from above
$[A/G, \underline{r}, N]$	= $[A/F, \underline{r}, N] * [F/G, \underline{r}, N]$ // multiply 2 factors from above

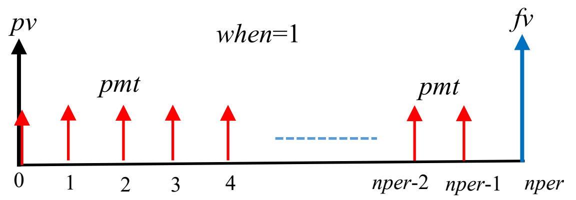


2.7 Python Numpy_Financial Functions

2.7.1 pv(), pmt(), fv(), rate() and nper() functions

The five numpy_financial functions `pv()`, `pmt()`, `fv()`, `rate()` and `nper()` are similar to their corresponding Excel functions with the same names. They also compute the value of one variable, given the values of the other 4 variables so that the ***overall equivalent value of the cash flows is zero***.





Inputs (up to any four of the following):

rate = interest rate per period
nper = total number of periods
pv = a single cash flow at time zero
pmt = uniform cash flow per period
fv = a single cash flow at end of *nper periods*

Parameter:

$$when = \begin{cases} 0 & \text{uniform end of periods cash flows (default)} \\ 1 & \text{uniform beginning of period cash flows} \end{cases}$$

<u>Uniform end-of-period cash flows</u>

```
import numpy_finanical as npf
```

- *npf.pv* (*rate*, *nper*, *pmt, fv*)

  returns $-(\,pmt\ [P/A, rate, nper] + fv\ [P/F, rate, nper]\,)$

- *npf.pmt* (*rate*, *nper*, *pv*, *fv*)

  returns $-(\,pv\ [A/P, rate, nper] + fv\ [A/F, rate, nper]\,)$

- *npf.fv* (*rate*, *nper*, *pmt*, *pv*)

  returns $-(\,pv\ [F/P, rate, nper] + pmt\ [F/A, rate, nper]\,)$

- *npf.rate* (*nper*, *pmt*, *pv*, *fv*, *guess*, *tolerance*)

  returns ***r*** such that $pv + pmt\ [P/A, r, nper] + fv\ [P/F, r, nper] = 0$

- *npf.nper* (*rate*, *pmt*, *pv*, *fv*)

  returns ***N*** such that $pv + pmt\ [P/A, rate, N] + fv\ [P/F, rate, N] = 0$

- Take note of the **negative signs** in front of the quantity returned by PV, PMT and FV.

2.7.2 Computing Interest Factors

- The interest factors for both discrete and continuous compounding can be computed directly using numpy_finanical functions with negative one dollar inputs:

Discrete Compounding:

$[P/A, i, N]$ = `npf.pv(i,N,-1,0)`

$[P/F, i, N]$ = `npf.pv(i,N,0,-1)`

$[A/P, i, N]$ = `npf.pmt(i,N,-1)`

$[A/F, i, N]$ = `npf.pmt(i,N,0,-1)`

$[F/A, i, N]$ = `npf.fv(i,N,-1,0)`

$[F/P, i, N]$ = `npf.fv(i,N,0,-1)`

$[F/G, i, N]$ = `(npf.fv(i,N,-1,0)-N)/i`

$[P/G, i, N] = [P/F, i, N] * [F/G, i, N]$ // multiple 2 factors from above

$[A/G, i, N] = [A/F, i, N] * [F/G, i, N]$ // multiple 2 factors from above

Continuous Compounding:

- Use the functions for discrete compounding with $i = e^r - 1$

2.8 EngFinancialPy: A computational toolbox for financial analysis

- EngFinancialPy is Python module containing classes and functions for financial analysis and decision-making. It builds upon popular Python packages commonly used for numerical and computation, statistical and data analysis, and data visualization.

- The following classes from EngFinancialPy may be used to compute annuity and interest factors for discrete cash flows under discrete and continuous compounding:

  1. IntFactor
  2. GeomCashFlows

- The following class may be used to plot cash flow diagrams:

  - CF_diagram

- See EngFinancialPy Documentations for details and examples.

Readings

1. Chapter 4 of Sullivan *et al* (2020).

2. EngfinancialPy Documentations.