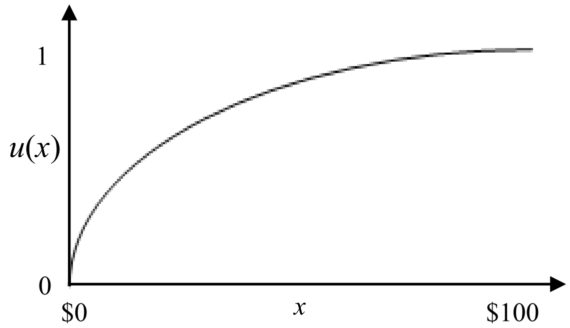
Chapter 6 Advanced Decision Analysis: Modeling Risk Preferences

"Take calculated risks. That is quite different from being rash."
Gen George S. Patton

6.1. Decision Making under General Risk Preferences

6.1.1 Wealth Utility Function

- Recall that in the Party Problem, we derived the following utility function for Kim:

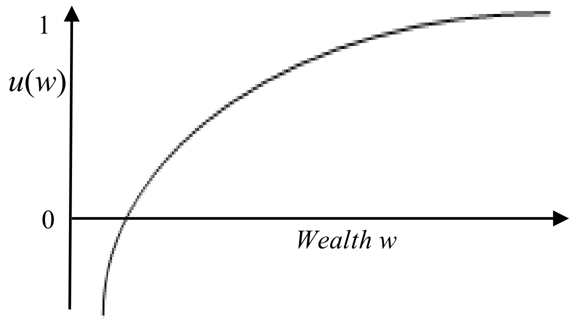


- Here, values on the *x*-axis that vary from \$0 to \$100 are the equivalent dollar amounts added to Kim's present wealth. We have not considered the person's state of wealth in the utility function.

- In general, a gain of an amount, say \$100 provides less satisfaction to a wealthy person compared to a poor person.

- Hence we must take into account the wealth of a person in the computation of utilities.

- Utility functions should therefore be constructed based on the wealth of a person.



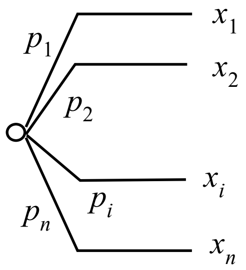
Wealth utility function

- We called the above the **Wealth Utility Function** of a person.

- We should always use the wealth utility function for decision analysis unless we have more information about the risk taking preference of the decision maker.

6.1.2 Certainty Equivalent or Selling Price of Risky Asset

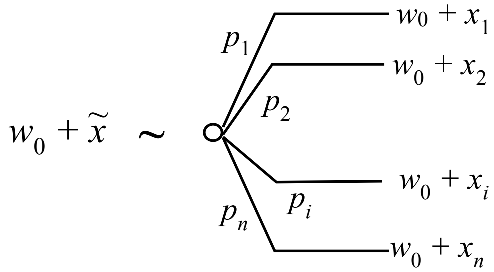
- Consider a person whose current wealth is w_o and who owns a risky asset represented by the following deal:



- Here, x_i (i =1 to n) is the amount added to his wealth if the i^{th} outcome (with probability p_i) occurs. Some of these outcomes may be good, bad, positive, or negative.

- The **Personal Indifferent Selling Price** or **Certainty Equivalent** of a risky asset is the minimum sure amount of money a person is willing to receive to eliminate all risk associated with the risky asset.

- More specifically, it is the sure amount of money $\tilde{x}$ added to his/her current wealth w_o that makes him/her indifferent between facing the risk versus receiving the sure or risk-free amount $w_0 + \tilde{x}$.



- Equating the utility of selling to the expected utility of not selling the deal and facing the consequences:

$$u(w_o + \tilde{x}) = \sum_{i=1}^{n} p_i \, u(w_o + x_i)$$

or

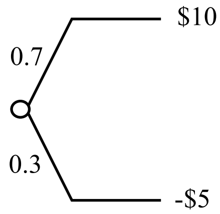
$$\tilde{x} = u^{-1}\left(\sum_{i=1}^{n} p_i \, u(w_o + x_i) \right) - w_o$$

- Hence if we know the *wealth utility function* and *current wealth* of a person, we can directly compute his/her personal indifferent selling price of any risky asset.

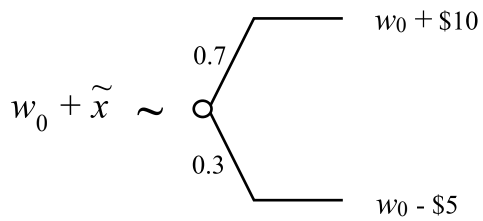
Example

- Consider a decision maker with the wealth utility function $u(w) = \begin{cases} \sqrt{w} & w \geq 0 \\ -\sqrt{-w} & w < 0 \end{cases}$

  where w is in dollars and owns the following risky asset:



- Let $\tilde{x}$ be the decision maker's Personal Indifferent Selling Price or Certainty Equivalent.



i. When w_o = \$10: $u(10 + \tilde{x}) = 0.7\, u(10 + 10) + 0.3\, u(10 - 5)$

$u(10 + \tilde{x}) = 0.7\, u(20) + 0.3\, u(5)$

Assuming that $\tilde{x} \geq -10$: $\sqrt{10 + \tilde{x}} = 0.7\sqrt{20} + 0.3\sqrt{5}$

$\tilde{x} = 4.450000$

ii. When w_o = \$100: $u(100 + \tilde{x}) = 0.7\, u(100 + 10) + 0.3\, u(100 - 5)$

$u(100 + \tilde{x}) = 0.7\, u(110) + 0.3\, u(95)$

Assuming that $\tilde{x} \geq -100$: $\sqrt{100 + \tilde{x}} = 0.7\sqrt{110} + 0.3\sqrt{95}$

$\tilde{x} = 5.384601$

iii. When w_o = \$1,000: $u(1000 + \tilde{x}) = 0.7\, u(1000 + 10) + 0.3\, u(1000 - 5)$

$u(1000 + \tilde{x}) = 0.7\, u(1010) + 0.3\, u(995)$

Assuming that $\tilde{x} \geq -1{,}000$: $\sqrt{1000 + \tilde{x}} = 0.7\sqrt{1010} + 0.3\sqrt{995}$

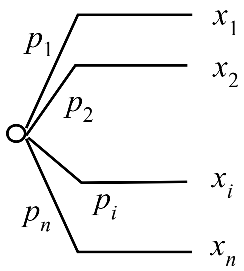
$\tilde{x} = 5.488217$

- Hence a person's Personal Indifferent Selling Price or Certainty Equivalent of a risky asset depends on his/her utility function and initial wealth.

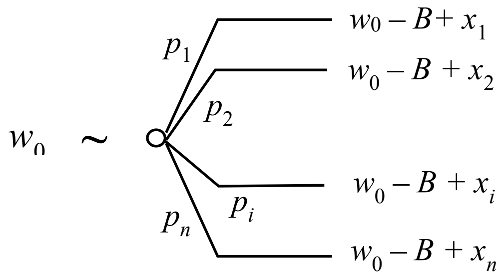
6.1.3 Buying Price of Risky Asset

- The **Personal Indifferent Buying Price** is the maximum sure amount of money a person is willing to pay to own a risky asset or deal and faces its uncertain outcomes.

- Consider the following risky asset or deal X which is offered to a decision maker.



- If B is the decision maker's Personal Indifferent Buying Price, then he/she would just be indifferent between staying status quo with current wealth w_o and taking the risk in receiving one of the possible outcomes $\{w_o - B + x_i\}$ with probability p_i.



- Equating the utility of not buying and the expected utility of buying the deal and facing the consequences:

$$u(w_o) = \sum_{i=1}^{n} p_i u(w_o - B + x_i)$$

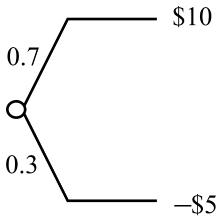
- In general, it is not possible to express the personal indifferent buying price B in a closed form unlike in the case of the personal indifferent selling price. But the value of B can be found by solving the above equation using numerical methods.

- Also, in general, $B \neq \tilde{x}$. That is, a person's Personal Indifferent Buying Price is not equal to the Certainty Equivalent or Personal Indifferent Selling Price for the same risky asset.

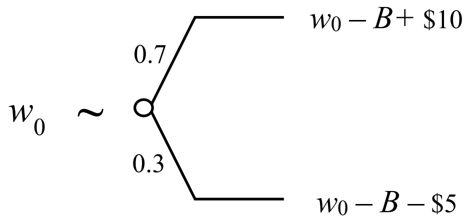
Example

- Consider a decision maker with the wealth utility function $u(w) = \begin{cases} \sqrt{w} & w \geq 0 \\ -\sqrt{-w} & w < 0 \end{cases}$

  where w is in dollars and is offered the following risky asset:



- Let B = Personal Indifferent Buying Price for the deal:



i. When w_o = \$10: $u(10) = 0.7\, u(10 - B + 10) + 0.3\, u(10 - B - 5)$
$u(10) = 0.7\, u(20 - B) + 0.3\, u(5 - B)$

Assuming that $B \leq 5$: $\sqrt{10} = 0.7\sqrt{20 - B} + 0.3\sqrt{5 - B}$

Using an equation solver: $B = 3.726958$

ii. When w_o = \$100: $u(100) = 0.7\, u(100 - B + 10) + 0.3\, u(100 - B - 5)$
$u(100) = 0.7\, u(110 - B) + 0.3\, u(95 - B)$

Assuming that $B \leq 95$: $\sqrt{100} = 0.7\sqrt{110 - B} + 0.3\sqrt{95 - B}$

Using an equation solver: $B = 5.378193$

iii. When w_o = \$1,000: $u(1000) = 0.7\, u(1000 - B + 10) + 0.3\, u(1000 - B - 5)$
$u(1000) = 0.7\, u(1010 - B) + 0.3\, u(995 - B)$

Assuming that $B \leq 995$: $\sqrt{1000} = 0.7\sqrt{1010 - B} + 0.3\sqrt{995 - B}$

Using an equation solver: $B = 5.488152$

- Hence a person's personal indifferent buying price of a risky asset depends on his/her initial wealth.

Comparison of PISP and PIBP

- Comparing the personal different buying price and personal indifferent selling price for the same initial wealth:

Initial wealth $	Personal Indifferent Selling Price $	Personal Indifferent Buying Price $
10	4.450000	3.726958
100	5.384601	5.378193
1,000	5.488217	5.488152

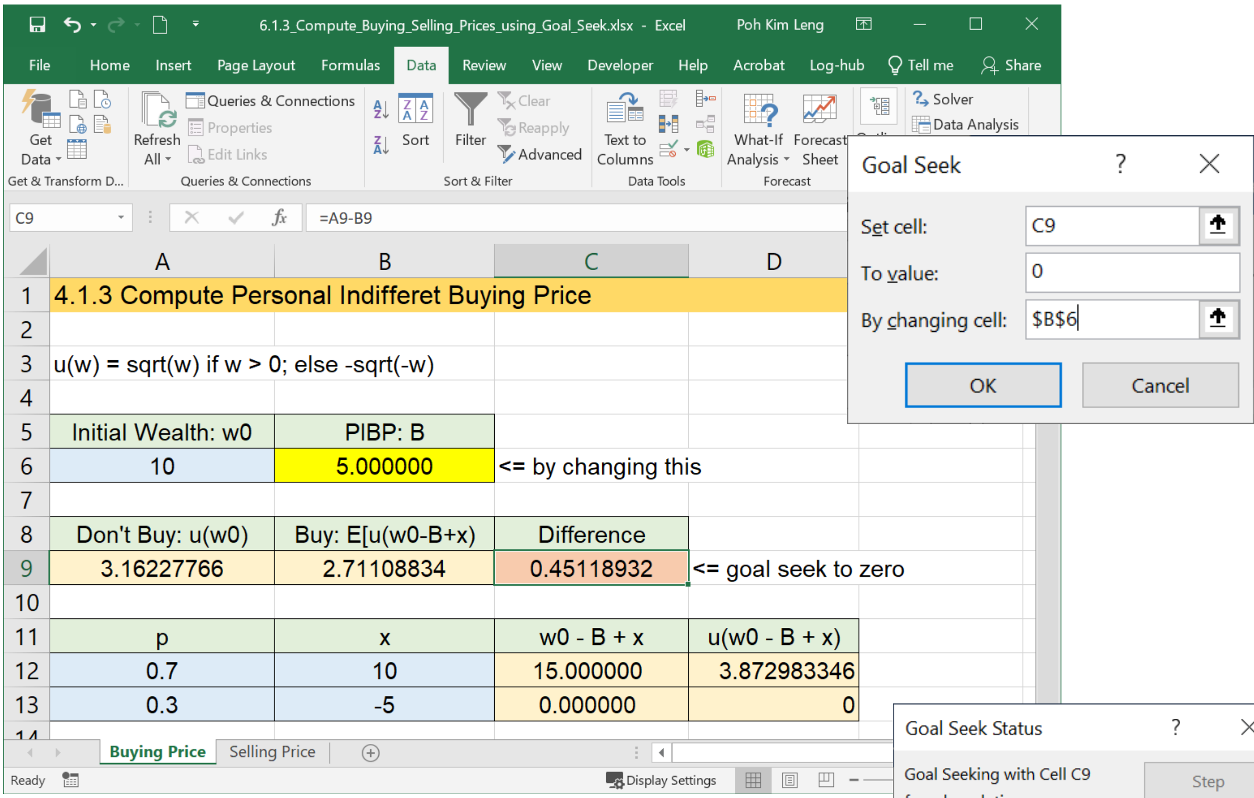
- Note that all the above figures will change if the decision maker has a different utility function.

Solving non-linear equations:

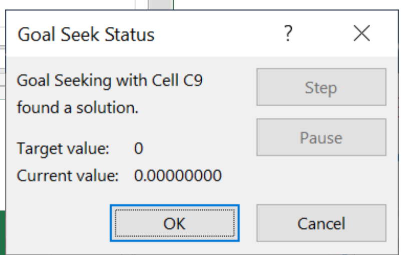
1. Using Excel Goal Seek Function:

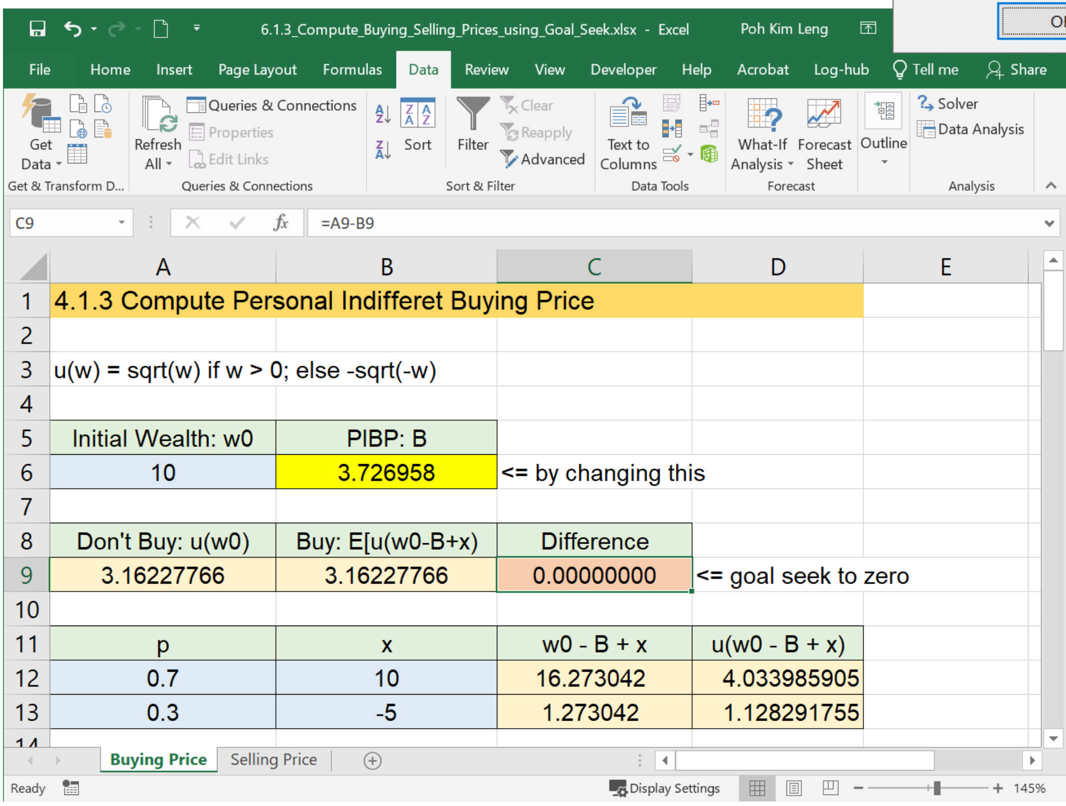
- The Goal Seek function allows you to work backward in Excel to determine what is the value of a particular input cell that will make the value of an output cell equal to a certain value.

Data -> What-if Analysis -> Goal Seek:



Solution:





2. Python scipy.optimize.root() function

This function finds the root of a vector function.

In [1]:
```
""" Compute PIBP and PISP using root function """
# 6.1.3_Compute_Buying_Selling_Prices_using_root_function.py
import numpy as np
from scipy.optimize import root
```

In [2]:
```
# The Deal
p = np.array([0.7, 0.3])
x = np.array([ 10,  -5])
```

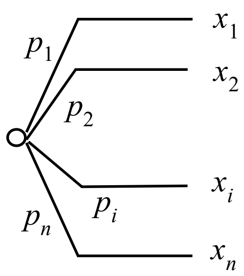
In [3]:
```
# Define the Equations to Solve:
# Sell vs Don't sell: u(w0+s) = E[u(w0+x)]
def sell_func(s, w0, p, x):
    return wuf(w0+s) - np.dot(p, wuf_vec(w0+x))

# Don't buy vs Buy: u(w0) = E[u(w0-B+x)]
def buy_func(b, w0, p, x):
    return wuf(w0) - np.dot(p, wuf_vec(w0-b+x))
```

In [4]:
```
# Find PISP and PIBP for different values of w0
guess = 5
for w0 in [10, 100, 1000, 10000]:
    print(f"\nw0 = {w0}:")

    sol_sell = root(sell_func, x0=guess, args=(w0, p, x),
                    method='hybr', options={'xtol':1E-10})
    if not sol_sell.success:
        print(f"  {sol_sell.message}, Best solution found:")
    print(f"  PIBP = {sol_sell.x[0]:.6f}")

    sol_buy = root(buy_func, x0=guess, args=(w0, p, x),
                   method='hybr', options={'xtol':1E-10})
    if not sol_buy.success:
        print(f"  {sol_buy.message}. Best solution found:")
    print(f"  PIBP = {sol_buy.x[0]:.6f}")
```

```
w0 = 10:
  PISP = 4.450000
  PIBP = 3.726958

w0 = 100:
  PISP = 5.384601
  PIBP = 5.378193

w0 = 1000:
  PISP = 5.488217
  PIBP = 5.488152

w0 = 10000:
  PISP = 5.498819
  PIBP = 5.498818
```

6.1.4 Risk Premium of Risky Asset

- Given the risky asset or deal *X*:



- The **Expected Monetary or Dollar Value** of deal *X* is.

$$EV = \bar{x} = \sum_{i=1}^{n} p_i x_i$$

- Note that EV does not depend on the decision maker's utility function. It depends only on the probabilities and outcome values of *X*.
- Let w_o be the current wealth of the decision maker.
- Recall that his/her **Certainty Equivalent** or **Personal Indifferent Selling Price** for deal *X* is

$$CE = \tilde{x} = u^{-1}\left(\sum_{i=1}^{n} p_i u(w_o + x_i)\right) - w_o .$$

Definition (Risk Premium)

- The decision maker's **Risk Premium** (π) of a risky deal is the difference between the expected value (EV) of the deal and his certainty equivalent (CE) of the deal. That is

$$\begin{aligned}\pi &= EV - CE \\ &= \bar{x} - \tilde{x} \\ &= \bar{x} - \left[u^{-1}\left(\sum_{i=1}^{n} p_i u(w_o + x_i)\right) - w_o\right] \\ &= w_0 + \bar{x} - u^{-1}\left(\sum_{i=1}^{n} p_i u(w_o + x_i)\right)\end{aligned}$$

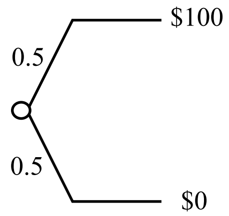
- If we rearrange the terms in the above equation, we obtain

$$u(w_0 + \bar{x} - \pi) = \sum_{i=1}^{n} p_i u(w_o + x_i)$$

- The risk premium may be interpreted as the maximum amount that the decision maker is willing to forgo out of the expected dollar value to avoid any risk associated with the deal.
- Risk premium can be positive, zero or negative.

Example

- Suppose you currently own the risky deal:



- The expected dollar value is $50.

- Suppose you are indifferent between receiving $40 for sure or going with the risky deal, then
  - Your certainty equivalent or selling price for the deal is $40.
  - Your risk premium is $50 – $40 = $10.
  - This means that you are willing to forgo $10 out of the expected value of $50 to avoid the risk of receiving $0.
  - But you also give up the opportunity of receiving $100.

- Now, suppose your certainty equivalent is $60.
  - Your risk premium is $50 – $60 = –$10.
  - In this case, you are willing to bear more risk than in the previous case, and you must be paid $10 above the expected value of $50 before you would be willing to give up the risk.

6.1.5 Risk Attitudes

- In everyday life, we observe that people have different attitudes toward taking risks.

  - Some people are *Risk Averse* in that they are willing to avoid taking risks by selling risky deals at lower values.

  - On the other hand, some people are *Risk Takers* in that they prefer to take risks and would only sell away risky deals at high prices

- Hence there are different types of risk attitudes.

Definitions (Risk Averse, Risk Seeking, and Risk Neutral)

1. A decision maker is **Risk Averse** within a range of his/her wealth, if for any risky deal, his/her risk premium is **positive**.

2. A decision maker is **Risk Seeking** with a range of his/her wealth, if for any risky deal, his/her risk premium is **negative**.

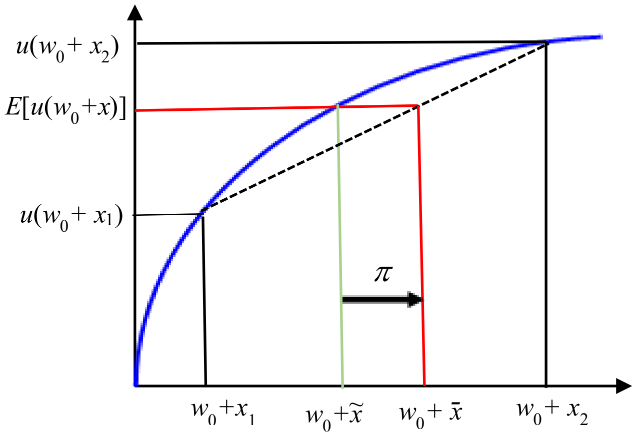
3. A decision maker is **Risk Neutral** with a range of his/her wealth, if for any risky deal, his/her risk premium is **zero**.

- We would like to know what the decision maker's wealth utility function will be like under different risk attitudes.

- Without loss of generality, we consider deals with only two outcomes:

Risk-Averse Case:

- Risk premium π must be **Positive**

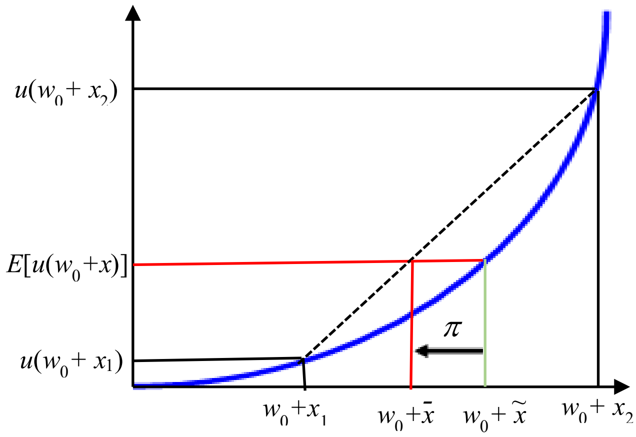


- For π to be positive, given any two outcomes x_1 and x_2, the wealth utility function curve must be above the cord between the two points.

- Hence the wealth utility function must be an increasing **Concave Function**.

Risk-Seeking Case:

- Risk premium π must be **Negative**

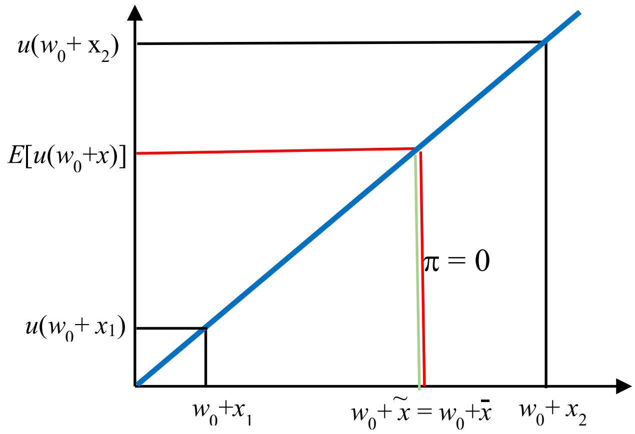


- For π to be negative, given any two outcomes x_1 and x_2, the wealth utility function curve must be below the cord between the two points.

- Hence the wealth utility function must be an increasing **Convex Function**.

Risk-Neutral Case:

- Risk premium π must be **Zero.**



- For π to be zero, given any two outcomes x_1 and x_2, the wealth utility function curve must be a straight line between the two points.

- Hence the wealth utility function must be an increasing **Linear Function**.

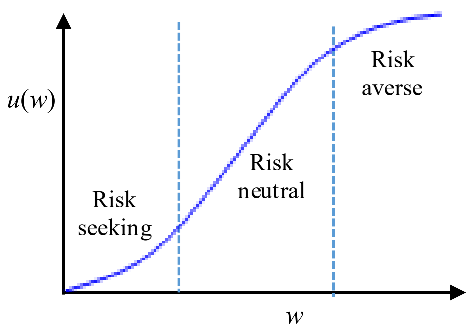
Summary: Risk Premium, Curvature of $u(w)$, and Risk Attitudes:

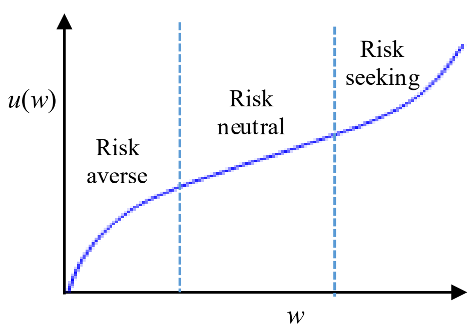
1. $\pi > 0$ $\leftrightarrow$ Concave utility function $\leftrightarrow$ Risk averse
2. $\pi < 0$ $\leftrightarrow$ Convex utility function $\leftrightarrow$ Risk seeking
3. $\pi = 0$ $\leftrightarrow$ Linear utility function $\leftrightarrow$ Risk neutral

Risk Attitudes may change over a range of wealth

- A person can exhibit changing risk attitudes over a spectrum of wealth levels.

- For example, the utility function of a person might exhibit the following shapes:





6.1.6 Measure of Absolute Risk Aversion

- In the previous section, we categorized the risk attitude of a person as risk-averse, risk-seeking, or risk-neutral according to the signs of the risk premiums or the curvature of his/her utility function.

1.	$\pi > 0$	$\leftrightarrow$	Concave utility function	$\leftrightarrow$	Risk averse
2.	$\pi < 0$	$\leftrightarrow$	Convex utility function	$\leftrightarrow$	Risk seeking
3.	$\pi = 0$	$\leftrightarrow$	Linear utility function	$\leftrightarrow$	Risk neutral

- Risk Attitudes only classify risk taking preferences into three categories. We need a more precise way of measuring a person's degree of risk aversion or tolerance to risk on a continuous scale.

- Arrow and Pratt have proposed such a measure for absolute risk aversion.

- Let w_o be the initial wealth of a person who owns the risky deal with payoffs x that follows the probability density function $f(x)$.

- Let $a \le x \le b$.

- The mean and variance of x are $\bar{x} = \int_a^b x\, f(x)\, dx$ and $\sigma^2 = \int_a^b (x - \bar{x})^2\, f(x)\, dx$, respectively.

- Suppose $\tilde{x}$ is the selling price or certainty equivalent for the deal, then

$$u(w_0 + \tilde{x}) = \int_a^b u(w_0 + x) f(x) dx \tag{1}$$

- Using the first-order approximation for $u(w_0 + \tilde{x})$ around $(w_0 + \bar{x})$:

$$\begin{aligned} u(w_0 + \tilde{x}) &\approx u(w_0 + \bar{x}) + (w_0 + \tilde{x} - (w_0 + \bar{x}))u'(w_0 + \bar{x}) \\ &= u(w_0 + \bar{x}) + (\tilde{x} - \bar{x})u'(w_0 + \bar{x}) \end{aligned} \tag{2}$$

- Using the second order approximation for $u(w_o + x)$:

$$u(w_0 + x) = u(w_0 + \bar{x}) + (x - \bar{x})u'(w_0 + \bar{x}) + ((x - \bar{x})^2 / 2!)u''(w_0 + \bar{x}) \tag{3}$$

- Substituting equation (3) into the RHS of equation (1):

$$u(w_o + \tilde{x}) = u(w_o + \bar{x}) + \frac{\sigma^2}{2} u''(w_o + \bar{x}) \tag{4}$$

- From equations (2) and (4), we get

$$\bar{x} - \tilde{x} \approx \frac{1}{2}\sigma^2 \left[\frac{-u''(w_o + \bar{x})}{u'(w_o + \bar{x})} \right] \tag{5}$$

- Hence the ***Risk Premium*** can be approximated by

$$\pi = \bar{x} - \tilde{x} \approx \frac{1}{2}\sigma^2 \left[\frac{-u''(w_o + \bar{x})}{u'(w_o + \bar{x})} \right]$$

- We observe that the risk premium π depends on two terms:

  1. σ^2 = variance of x, reflecting the level of risk embodied in the risky deal.

  2. $\left(\frac{-u''}{u'} \right)$ reflecting the nature of the utility function, which is specific to the decision maker's risk attitude. This term is called the **Degree of Absolute Risk Aversion.**

- The above explains why individuals in objectively identical situations might have different risk premiums or certainty equivalents.

- Indeed, if two individuals have the same wealth and hold the same risky deal, that is, if $\bar{x}$ and σ^2 are the same), they can have different risk premiums because their degree of absolute risk aversion, measured by $\left(\frac{-u''}{u'} \right)$ differs.

Definition: Arrow-Pratt Degree of Absolute Risk Aversion

- The **Arrow-Pratt degree of Absolute Risk Aversion** of a decision maker is defined as:

$$r_A(w) = \frac{-u''(w)}{u'(w)}$$

  where $u(w)$ is the decision maker's wealth utility function.

- Note that $r_A(w)$ is in unit of $\$^{-1}$

Definition: Risk Tolerance

- The **Risk Tolerance** of a decision maker is defined as the inverse of his Arrow-Pratt degree of absolute risk aversion, i.e.

$$\begin{aligned} \rho(w) &= \frac{1}{r_A(w)} \\ &= \frac{u'(w)}{-u''(w)} \end{aligned}$$

  where $u(w)$ is the decision maker's wealth utility function.

- Note that $\rho(w)$ is in units of \$.
- We will also denote risk tolerance as $r_T(w)$.

Some properties of $r_A(w)$ and $\rho(w)$

- We note that since the decision maker always prefers more to less, we have $u'(w) > 0$ for all w.
- Hence the sign of $r_A(w)$ or $\rho(w)$ is solely determined by $-u''(w)$.

1. If $r_A(w)$ or $\rho(w) > 0$ then $u''(w) < 0$ $\Rightarrow$ $u(w)$ is concave $\Rightarrow$ risk averse
2. If $r_A(w)$ or $\rho(w) < 0$ then $u''(w) > 0$ $\Rightarrow$ $u(w)$ is convex $\Rightarrow$ risk seeking
3. If $r_A(w) = 0$ or $\rho(w) = \infty$ then $u''(w) = 0$ $\Rightarrow$ $u(w)$ is linear $\Rightarrow$ risk neutral

- The converses of all of the above are also true. Hence

1. $r_A(w) > 0 \Leftrightarrow \rho(w) > 0$ $\Leftrightarrow$ $u(w)$ is concave $\Leftrightarrow$ risk averse
2. $r_A(w) < 0 \Leftrightarrow \rho(w) < 0$ $\Leftrightarrow$ $u(w)$ is convex $\Leftrightarrow$ risk seeking
3. $r_A(w) = 0 \Leftrightarrow \rho(w) = \infty$ $\Leftrightarrow$ $u(w)$ is linear $\Leftrightarrow$ risk neutral

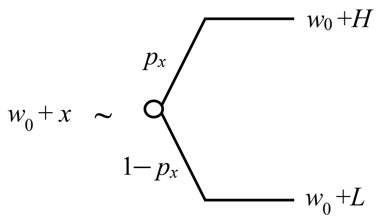
6.1.7 Eliciting General Utility Functions

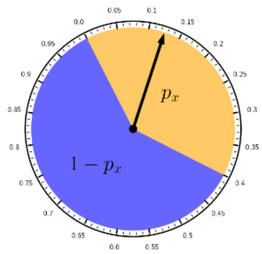
- There are two approaches to eliciting general utility functions. Both rely on reference deals but are different in the parameters being varied.

- In all cases, we assume that the utility function is to be assessed over the range of values $\$L \le x \le \$H$ with the boundary conditions: $u(w_o + L) = 0$ and $u(w_o + H) = 1$.

6.1.7.1 The Preference Probability Approach

- The steps are as follows:
  1. Pick a value of x such that $\$L \le x \le \$H$.
  2. Determine the preference probability p_x, such that the decision maker is indifferent between receiving $\$x$ and the risky deal with probability p_x for $\$H$ and $1 - p_x$ for $\$L$.



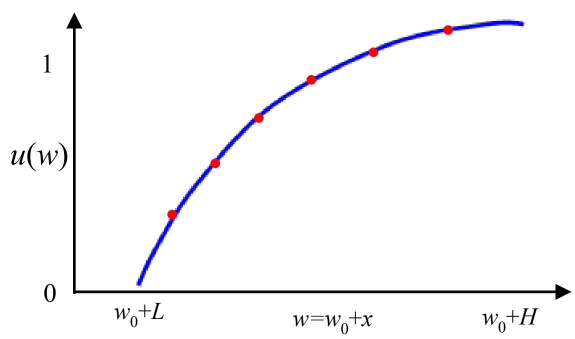


Probability Wheel

  3. Since $u(w_o + L) = 0$ and $u(w_o + H) = 1$, it follows that
$$u(w_o + x) = p_x\, u(w_o + H) + (1 - p_x)\, u(w_o + L) = p_x.$$

  4. If the probability wheel is used:
     - Set the probability wheel to p for outcome $\$H$ and $1 - p$ for outcome $\$L$.
     - Ask the decision maker to choose between receiving $\$x$ for sure or the risky deal represented by the probability wheel.
       - If the decision maker prefers $\$x$, increase the value of p on the wheel and repeat.
       - If the decision maker prefers the deal, decrease the value of p and repeat.
       - If the decision maker is indifferent between the two alternatives, then $p_x = p$.

  5. Repeat for other values of x between $\$L$ and $\$H$ and plot p_x versus $w_o + x$ to obtain the general utility function.

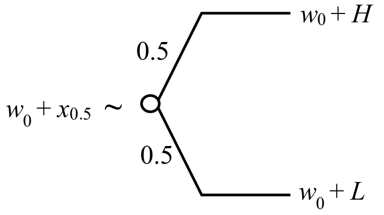


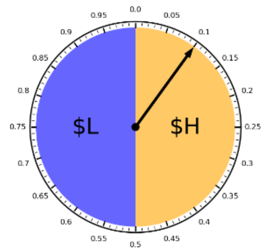
6.1.7.2 The Mid-Point Certainty Equivalent Approach

- In the Preference Probability approach, we fix x, determine $u(w_o+x)$ and then vary x.

- In this method, we construct a series of reference deals whose expected utility are 0.5 and find their certainty equivalents. The steps are as follows:

1. Determine the decision maker's certainty equivalent $w_o+x_{0.5}$ for the deal:



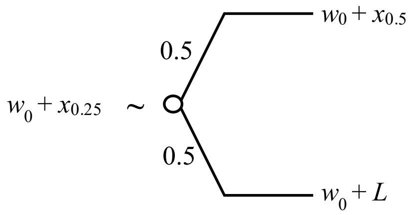


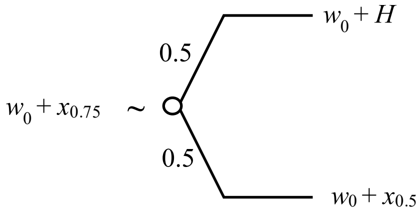
Probability Wheel

2. Since $u(w_o + L) = 0$ and $u(w_o + H) = 1$, it follows that
   $u(w_o + x_{0.5}) = 0.5\ u(w_o + H) + 0.5\ u(w_o + L) = 0.5.$

3. If the probability wheel is used:
   - Set the probability wheel to equal areas for outcomes \$$H$ and \$$L$.
   - Select a value of x between \$$L$ and \$$H$.
   - Ask the decision maker to choose between receiving \$$x$ for sure or the risky deal represented by the wheel.
     - If the decision maker prefers x, reduce the value of x and repeat.
     - If the decision maker prefers the deal, increase the value of x and repeat.
     - If the decision maker is indifferent between the two alternatives, then $WCE = w_o+x$.

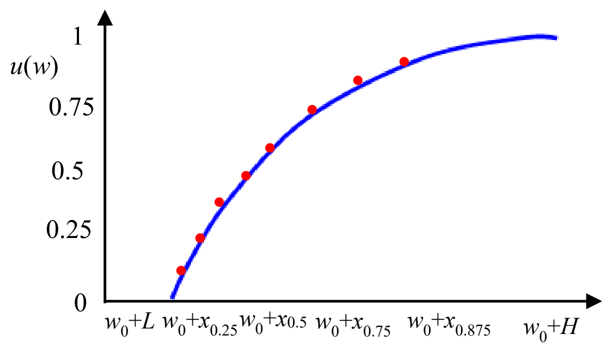
4. Repeat the above using the following reference deals to find $w_o+x_{0.25}$ and $w_o+x_{0.75}$.





5. This will provide us with $x_{0.25}$ such that $u(w_o + x_{0.25}) = 0.25$ and $x_{0.75}$ such that $u(w_o + x_{0.75}) = 0.75$

6. Values of $x_{0.125}$, such that $u(w_o + x_{0.125}) = 0.125$; $x_{0.375}$, such that $u(w_o + x_{0.375}) = 0.375$; $x_{0.625}$, such that $u(w_o+x_{0.625}) = 0.625$ and $x_{0.875}$, such that $u(w_o + x_{0.875}) = 0.875$, may also be similarly determined.



6.2. Decision Making under Constant Absolute Risk Aversion

6.2.1 Utility Functions under Constant Absolute Risk Aversion (CARA)

- The degree of absolute risk aversion generally depends on wealth w and is given by the Arrow-Pratt measure:

$$r_A(w) = \frac{-u''(w)}{u'(w)}$$

- We also require that $u(w)$ be an increasing function, i.e., $u'(w) > 0$ for all w.

- Consider the situation when the **degree of absolute risk aversion is constant** for all wealth w.

- Let $r_A(w)$ be constant and equal to r for all w. Then

$$\frac{-u''(w)}{u'(w)} = r$$

$$\Rightarrow \qquad u''(w) + ru'(w) = 0$$

- The solution to the second-order differential equation satisfying $u'(w) > 0$ depends on the sign and value of r.

1. When $r > 0$ (risk-averse case), the solution is

   $u(w) = a - be^{-rw}$ where a and $b > 0$ are constants

   or $u(w) = a - be^{-w/\rho}$ where $\rho = \frac{1}{r}$ is the constant risk tolerance

2. When $r < 0$ (risk-seeking case), the solution is

   $u(w) = a + be^{-rw}$ where a and $b > 0$ are constants

   or $u(w) = a + be^{-w/\rho}$ where $\rho = \frac{1}{r}$ is the constant risk tolerance

3. When $r = 0$ (risk-neutral case), the solution is

   $u(w) = a + bw$ where a and $b > 0$ are constants

- Hence when the decision maker's degree of absolute risk aversion is constant, his/her wealth utility function is either exponential (Case 1 and Case 2) or linear (Case 3) in form.

- Consider now the converse. What if the decision maker's utility is either exponential or linear in form?

    1. When the wealth utility function is exponential in form:

    $$u(w) = a \pm be^{-rw}, \quad u'(w) = \pm b(-r)e^{-rw}, \quad u''(w) = \pm br^2 e^{-rw}$$

    $$\Rightarrow \qquad r_A(w) = \frac{-u''(w)}{u'(w)} = r$$

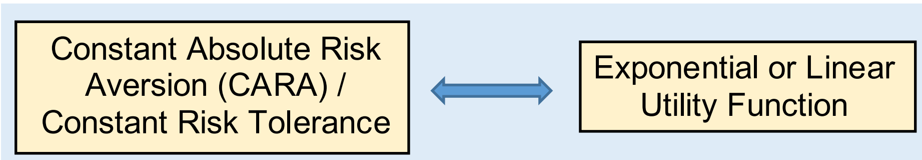
    2. When the wealth utility function is linear in form:

    $$u(w) = a + bw, \quad u'(w) = b, \quad u''(w) = 0$$

    $$\Rightarrow \qquad r_A(w) = \frac{-u''(w)}{u'(w)} = 0$$

- Hence when the decision maker's wealth utility function is either exponential or linear in form, his/her degree of absolute risk aversion is constant.

Summary



- Constant degree of absolute risk aversion is necessary and sufficient for the wealth utility function to be either linear or exponential in form:

    1. A decision maker has a non-zero constant degree of absolute risk aversion if and only if he/she has an exponential wealth utility function. In this case, he/she can be either risk-averse or risk- seeking in attitude.

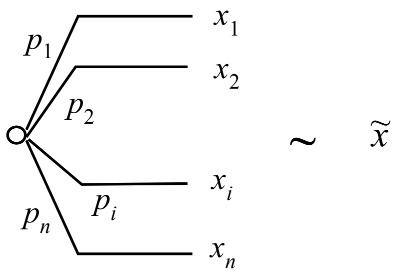
    $r_A(w)$ = constant $\neq 0$ $\Leftrightarrow$ Exponential Wealth Utility Function

    2. A decision maker has zero degree of absolute risk aversion if and only if he/she has a linear wealth utility function. In this case, he/she is risk-neutral.

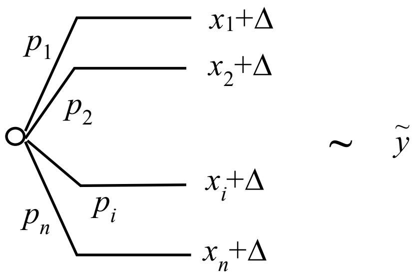
    $r_A(w) = 0$ $\Leftrightarrow$ Linear Utility Function $\Leftrightarrow$ Risk-Neutral

6.2.2 The Delta Property

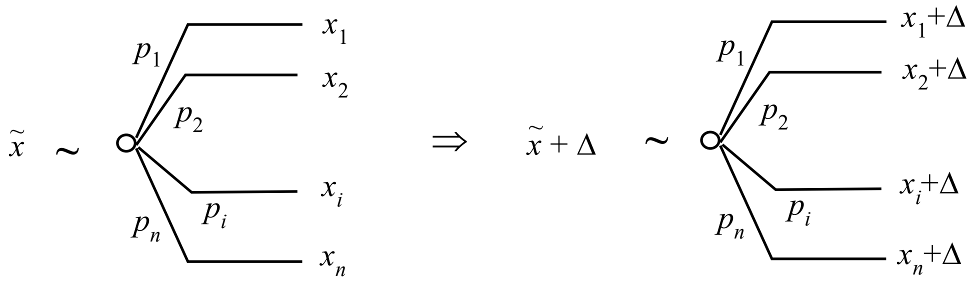
- Suppose a person owns a risky asset or deal X given below:



- Let $\tilde{x}$ be his/her certainty equivalent of X.
- Suppose somebody came along and offer to "top up" all the payoffs in the deal by an amount of Δ dollars. The new deal Y, called the Δ-shifted deal of X can be represented as follows:



- Let $\tilde{y}$ be the person's certainty equivalent for Y.
- Now, if the certainty equivalent of the Δ-shifted deal is $\tilde{y} = \tilde{x} + \Delta$ for any value of Δ, then we say the risk preference of the decision maker satisfies the **Delta Property**.
- That is, Delta Property holds if the following is true for all Δ:



- In words, a person's risk preference satisfies the ***Delta Property*** if his/her certainty equivalent for any risky asset is changed by Δ whenever all the payoffs of the assets are changed by any arbitrary amount Δ.
- Mathematically, Delta Property holds if

$$u(w_0 + \tilde{x}) = \sum_{i=1}^{n} p_i u(w_0 + x_i) \quad \Rightarrow \quad \forall\Delta \;\; u(w_0 + \tilde{x} + \Delta) = \sum_{i=1}^{n} p_i u(w_0 + x_i + \Delta)$$

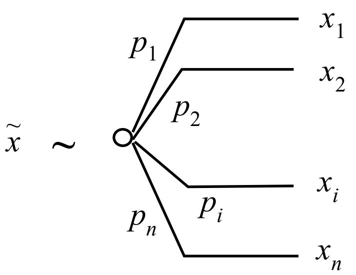
6.2.3 Delta Property is Equivalent to Constant Absolute Risk Aversion

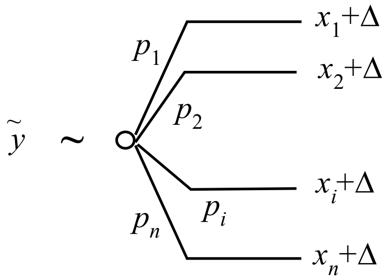
6.2.3.1 Delta Property Implies Constant Absolute Risk Aversion

- If a decision maker satisfies the delta property, what can we say about his/her degree of absolute risk aversion and hence his/her utility function?

- First, we would like to show that delta property implies constant absolute risk aversion.

- Given deals *X* and *Y*:





$$u(w_0+\tilde{x}) = \sum_{i=1}^{n} p_i u(w_0+x_i)$$

$$\bar{x} = \sum_{i=1}^{n} p_i x_i$$

$$\sigma^2 = \sum_{i=1}^{n} p_i (x_i-\bar{x})^2$$

$$\pi = \bar{x} - \tilde{x}$$

$$u(w_o+\tilde{y}) = \sum_{i=1}^{n} p_i u(w_0+x_i+\Delta)$$

$$\bar{y} = \bar{x} + \Delta$$

$$\sigma_{+\Delta}^2 = \sigma^2$$

$$\pi_{+\Delta} = \bar{y} - \tilde{y}$$

- Suppose the decision maker has the Δ property, then $\tilde{y} = \tilde{x} + \Delta$.

- Now the risk premium for the Δ-shifted deal is $\pi_{+\Delta} = \bar{y} - \tilde{y} = (\bar{x}+\Delta) - (\tilde{x}+\Delta) = \bar{x} - \tilde{x} = \pi$.

- This means that under delta property, the risk premium is constant under a Δ-shift.
- Since the variance is also constant under a Δ-shift, it follows from $\pi = \frac{1}{2}\sigma^2\left[\frac{-u''(w_0+\bar{x})}{u'(w_0+\bar{x})}\right]$ that the term in the square brackets must also be constant. Hence the decision maker has constant absolute risk aversion.

- Conclusion:

  Delta Property $\Rightarrow$ Constant Absolute Risk Aversion (CARA)

  $\Rightarrow$ Linear or Exponential Utility function

6.2.3.2 Constant Absolute Risk Aversion implies Delta Property

- We shall now show that if a decision maker has constant absolute risk aversion (CARA), then he/she follows the delta property.
- Since CARA implies that the utility function is either linear or exponential, we just need to check these two cases.

Case 1: Let $u(w) = a + bw$ where $b > 0$

$$u(w_0 + \tilde{x}) = \sum_{i=1}^{n} p_i u(w_0 + x_i)$$

$$a + b(w_0 + \tilde{x}) = \sum_{i=1}^{n} p_i (a + b(w_0 + x_i))$$

- Adding the quantity ($b\ \Delta$) to both sides:

$$a + b(w_0 + \tilde{x} + \Delta) = \sum_{i=1}^{n} p_i (a + b(w_0 + x_i + \Delta))$$

$$u(w_o + \tilde{x} + \Delta) = \sum_{i=1}^{n} p_i u(w_o + x_i + \Delta)$$

$\Rightarrow$ Delta Property

Case 2: Let $u(w) = a - be^{-w/\rho}$ where $b > 0$ for risk-averse and $b < 0$ for risk-seeking.

$$u(w_o + \tilde{x}) = \sum_{i=1}^{n} p_i u(w_o + x_i)$$

$$a - be^{-(w_0+\tilde{x})/\rho} = \sum_{i=1}^{n} p_i (a - be^{-(w_0+x_i)/\rho})$$

$$-be^{-(w_0+\tilde{x})/\rho} = \sum_{i=1}^{n} p_i (-be^{-(w_0+x_i)/\rho})$$

- Multiplying both sides by $e^{-\Delta/\rho}$ and adding constant a back to both sides

$$a - be^{-\Delta/\rho} e^{-(w_0+\tilde{x})/\rho} = a + e^{-\Delta/\rho} \sum_{i=1}^{n} p_i (-be^{-(w_0+x_i)/\rho})$$

$$a - be^{-(w_0+\tilde{x}+\Delta)/\rho} = \sum_{i=1}^{n} p_i (a - be^{-(w_0+x_i+\Delta)/\rho})$$

$$u(w_o + \tilde{x} + \Delta) = \sum_{i=1}^{n} p_i u(w_o + x_i + \Delta)$$

$\Rightarrow$ Delta Property

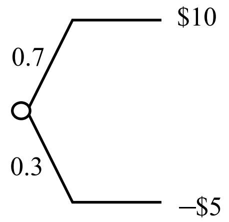
- Hence (CARA) $\Rightarrow$ Delta Property

- Since the converse was already proven, we may conclude:

Constant Absolute Risk Aversion (CARA) $\Leftrightarrow$ Delta Property

Example

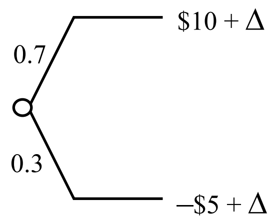
- Consider a decision maker with the wealth utility function $u(w) = 1 - 2^{-w/50}$ where w is in dollars and he/she owns the following risky asset:



- Let his/her current wealth be w_0 and certainty equivalent be $\tilde{x}$.
- We can find $\tilde{x}$ as follows:

$$
\begin{gathered}
u(w_0 + \tilde{x}) = 0.7u(w_0 + 10) + 0.3u(w_0 - 5) \\
1 - 2^{-(w_0+\tilde{x})/50} = 0.7\,(1 - 2^{-(w_0+10)/50}) + 0.3\,(1 - 2^{-(w_0-5)/50}) \\
2^{-(w_0+\tilde{x})/50} = 0.7\,(2^{-(w_0+10)/50}) + 0.3\,(2^{-(w_0-5)/50}) \\
2^{-\tilde{x}/50} = 0.7\,(2^{-10/50}) + 0.3\,(2^{5/50}) \\
2^{-\tilde{x}/50} = 0.9309174 \\
\tilde{x} = \$5.163744
\end{gathered}
$$

- Suppose all the payoff for the deal are increased by an arbitrary amount Δ, let his/her certainty new equivalent be $\tilde{y}$.



- We may find $\tilde{y}$ as follows:

$$
\begin{gathered}
u(w_0 + \tilde{y}) = 0.7u(w_0 + 10 + \Delta) + 0.3u(w_0 - 5 + \Delta) \\
1 - 2^{-(w_0+\tilde{y})/50} = 0.7\,(1 - 2^{-(w_0+10+\Delta)/50}) + 0.3\,(1 - 2^{-(w_0-5+\Delta)/50}) \\
2^{-(w_0+\tilde{y})/50} = 0.7\,(2^{-(w_0+10+\Delta)/50}) + 0.3\,(2^{-(w_0-5+\Delta)/50}) \\
2^{-\tilde{y}/50} = 0.7\,(2^{-(10+\Delta)/50}) + 0.3\,(2^{-(-5+\Delta)/50}) \\
2^{-\tilde{y}/50} = 2^{-\Delta/50}\left[0.7\,(2^{-10/50}) + 0.3\,(2^{5/50})\right] \\
2^{(-\tilde{y}+\Delta)/50} = 0.9309174 \\
\tilde{y} = 5.163744 + \Delta \\
\tilde{y} = \tilde{x} + \Delta
\end{gathered}
$$

- We note that for any arbitrary Δ, the decision maker's certainty equivalent (PISP) shifted by Δ.

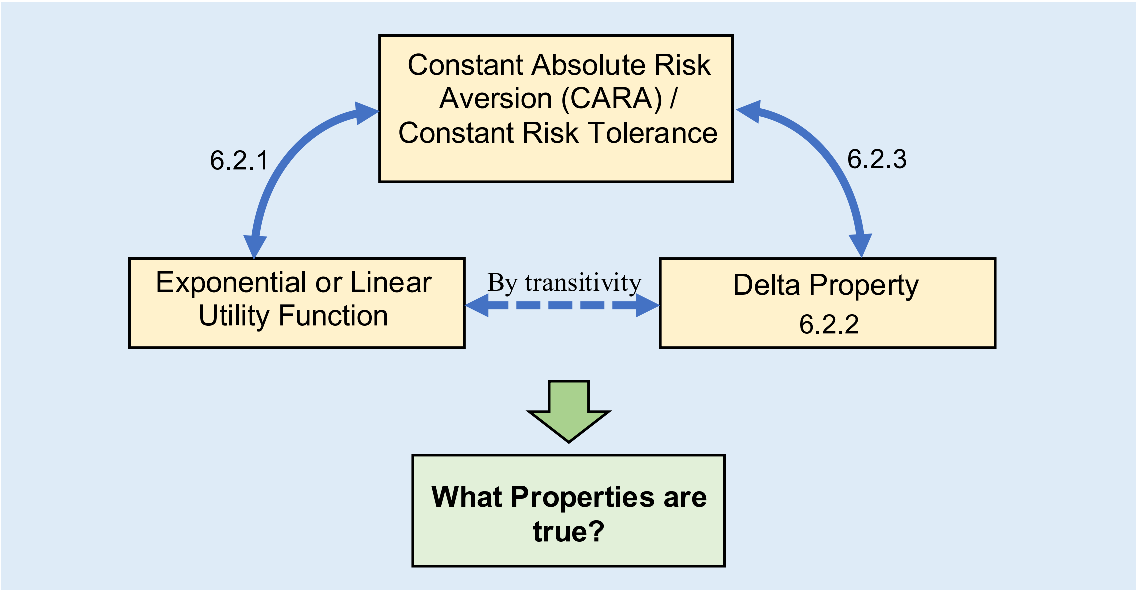
- Furthermore, we note that for both deals, the certainty equivalents do not depend on w_0. We will discuss this important property in the next section.

Summary

- The followings are equivalent:
  1. The decision maker has Constant Absolute Risk Aversion
  2. The decision maker has Constant Risk Tolerance
  3. The decision maker's Utility Function is either Linear or Exponential
  4. The decision maker follows the Delta Property

Question

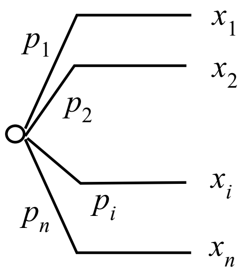
- Given a decision maker with any of the above conditions, are there special properties that can be exploited to help simplify the decision analysis process?



6.2.4 Delta Property Makes Certainty Equivalent Independent of Wealth

- We shall show that Delta Property makes a person's certainty equivalent independent of his/her wealth.

- Let a person with delta property and current wealth w_o owns the following risky asset X:



- If $\tilde{x}_0$ is his/her current certainty equivalent, then

$$u(w_0 + \tilde{x}_0) = \sum_{i=1}^{n} p_i \, u(w_0 + x_i) . \tag{1}$$

- Suppose his/her wealth is changed from w_o to any arbitrary level w_1. Let his/her certainty equivalent under wealth w_1 be $\tilde{x}_1$. Then

$$u(w_1 + \tilde{x}_1) = \sum_{i=1}^{n} p_i \, u(w_1 + x_i) . \tag{2}$$

- Equation (1) is equivalent to

$$u(w_0 + \tilde{x}_0 + w_1 - w_1) = \sum_{i=1}^{n} p_i \, u(w_0 + x_i + w_1 - w_1)$$

$$\Rightarrow \quad u(w_1 + \tilde{x}_0 - (w_1 - w_0)) = \sum_{i=1}^{n} p_i \, u(w_1 + x_i - (w_1 - w_0)) \tag{3}$$

- Invoking the delta property on (3) with $\Delta = (w_1 - w_0)$, we obtain

$$u(w_1 + \tilde{x}_0) = \sum_{i=1}^{n} p_i \, u(w_1 + x_i) \tag{4}$$

(2) and (4) $\Rightarrow \tilde{x}_0 = \tilde{x}_1$

- Hence the certainty equivalent is preserved under an arbitrary change of wealth.

- Therefore, when a person satisfies the delta property, his/her certainty equivalent for any deal is independent of his/her wealth.

- Hence in all analyses involving a decision maker with delta property, we do not need to consider the wealth. We may conveniently let $w_o = 0$ without affecting the answers. This is called the **zero-wealth effect.**

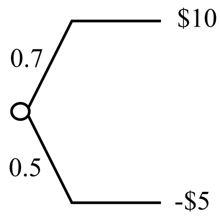
- We may also use a utility function $u(x)$ where x is profits/losses.

Example

- Let the decision marker's wealth utility function be $u(w) = 1 - 2^{-w/50}$ where w is in dollars.

- This is equivalent to the exponential function $u(w) = 1 - e^{-w\ln(2)/50}$. Hence delta property applies.

- Consider the risky deal:



- Let his/her current wealth be w_o.

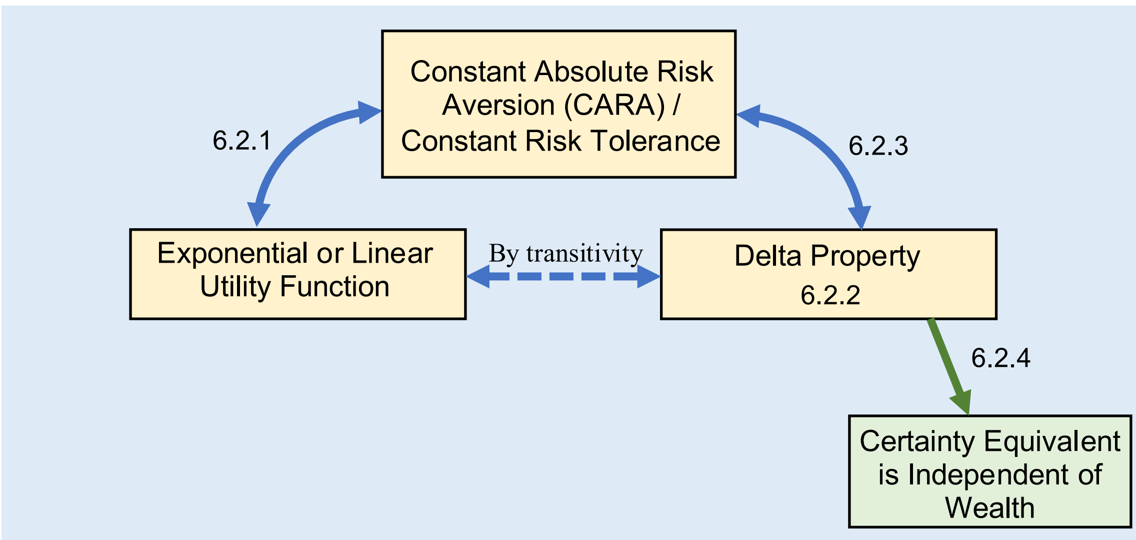
- We may find his/her certainty equivalent $\tilde{x}$ as follows:

$$u(w_0 + \tilde{x}) = 0.7u(w_0 + 10) + 0.3u(w_0 - 5)$$
$$1 - 2^{-(w_0+\tilde{x})/50} = 0.7\,(1 - 2^{-(w_0+10)/50}) + 0.3\,(1 - 2^{-(w_0-5)/50})$$
$$2^{-(w_0+\tilde{x})/50} = 0.7\,(2^{-(w_0+10)/50}) + 0.3\,(2^{-(w_0-5)/50})$$
$$2^{\widetilde{-x}/50} = 0.7\,(2^{-10/50}) + 0.3\,(2^{5/50})$$
$$2^{-\tilde{x}/50} = 0.9309174$$
$$\tilde{x} = \$5.163744$$

- We observe that the certainty equivalent does not depend on the initial wealth w_o.

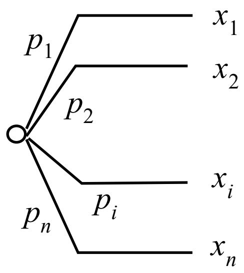
- We could have just set $w_o = 0$ and used the utility function $u(x) = 1 - 2^{-x/50}$ where x is the profit/loss and obtained the same results.

Summary



Delta Property makes Buying Price = Certainty Equivalent (Selling Price)

- Let a decision maker with delta property and current wealth w_o be offered the following risky asset X:



- Suppose his/her buying price is b, then

$$u(w_o) = \sum_{i=1}^{n} p_i u(w_o - b + x_i) \qquad (1)$$

- Suppose he/she owns the deal, and his/her selling price is $\tilde{x}$, then

$$u(w_o + \tilde{x}) = \sum_{i=1}^{n} p_i u(w_o + x_i) \qquad (2)$$

- Applying the delta property to (2) with $\Delta = -b$, we have

$$u(w_o + \tilde{x} - b) = \sum_{i=1}^{n} p_i u(w_o - b + x_i) \qquad (3)$$

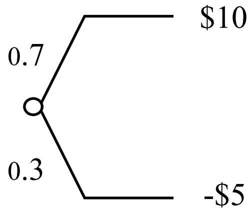
- (1) and (3) $\Rightarrow$ $\tilde{x} = b$.

- Hence Delta Property $\Rightarrow$

  Personal Indifferent Selling Price = Personal Indifferent Buying Price

Example

- Consider the previous example with the utility function $u(w) = 1 - 2^{-w/50}$ and the following deal:



- Let the current wealth be w_0.

- We have shown in the previous example that PISP = $\tilde{x} = \$5.163744$ and is independent of w_0.

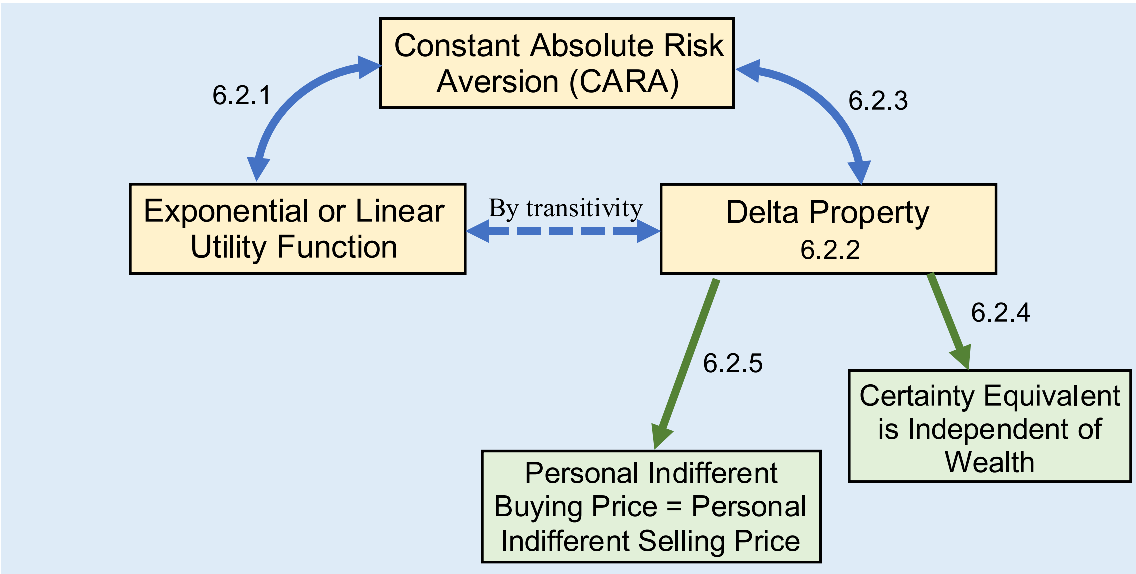
- Let the Personal Indifferent Buying Price of the same deal be B.

- We may find B as follows:

$$u(w_0) = 0.7u(w_0 - B + 10) + 0.3u(w_0 - B - 5)$$
$$1 - 2^{-w_0/50} = 0.7\,(1 - 2^{-(w_0 - B + 10)/50}) + 0.3\,(1 - 2^{-(w_0 - B - 5)/50})$$
$$2^{-w_0/50} = 0.7\,(2^{-(w_0 - B + 10)/50}) + 0.3\,(2^{-(w_0 - B - 5)/50})$$
$$1 = 0.7\,(2^{-(-B+10)/50}) + 0.3\,(2^{-(-B-5)/50})$$
$$2^{-B/50} = 0.7\,(2^{-10/50}) + 0.3\,(2^{5)/50})$$
$$2^{-B/50} = 0.9309174$$
$$B = \$5.163744$$
$$B = \tilde{x}$$

- We observed that Personal Indifferent Buying Price = Personal Indifferent Selling Price and they are both independent of w_0.

Summary



6.2.5 Value of Information Computation under Delta Property

- We learn in Chapter 4 that for a *risk-neutral* decision maker, we can compute the expected value of (perfect or imperfect) information or any uncertain variable by first computing the expected value of the optimal decision policy with FREE information and then subtracting from it the expected value of the optimal decision policy without information.

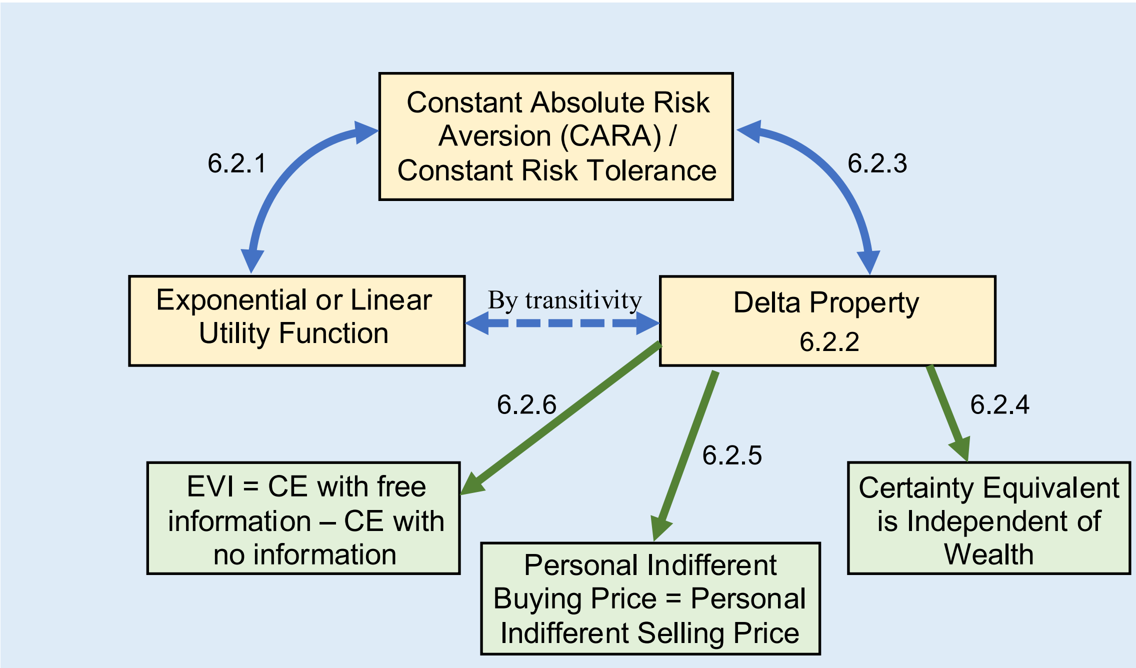
- That is, the Expected Value of Information for a risk-neutral person

  = EV with FREE information – EV with NO information.

- It can be shown that we may use the same computational procedure if the decision maker satisfies the *delta property,* but we use the difference of the Certainty Equivalents instead. See Appendix A for the proof.

- Note that if the delta property is not satisfied, then the break-even value method must be used to determine the expected value of perfect and imperfect information.

Summary



6.2.6 Summary of Relations

1. If the decision maker is **Risk Neutral,** then

   - He/she has
     1. Linear utility function
     2. Degree of Absolute Risk Aversion = 0
     3. Risk Tolerance = $\pm\infty$
     4. Use expected dollar values to make decisions and to determine personal indifferent buying and selling prices.

   - Given any risky deal:
     1. CE is independent of Wealth
     2. CE = EV = Personal Indifferent Selling Price = Personal Indifferent Buying Price
     3. Delta Property holds
     4. EVI = EV with free information – EV with no information

2. If the decision maker has the **Exponential Utility function**, then
   - He/she
     1. Has Constant Degree of Absolute Risk Aversion $\neq 0$
     2. Has Constant Risk Tolerance $\neq \pm\infty$
     3. May be either risk-averse or risk-seeking in attitude
   - Given any risky deal:
     1. CE is independent of Wealth
     2. Personal Indifferent Buying Price = Personal Indifferent Selling Price (CE) $\neq$ EV
     3. Delta Property holds
     4. EVI = CE with free information – CE with no information

3. If the decision maker's risk preference is generally unknown and does not fall into (1) or (2) above:

   - He/she
     1. May not be risk neutral
     2. May not have a constant degree of risk aversion or risk tolerance
     3. May not follow the delta property
     4. Has a general wealth utility function that is neither linear nor exponential in form.
     5. Must use the wealth utility function to make decisions and to determine personal indifferent buying and selling prices.
   - Given any risky deal:
     1. Buying Price $\neq$ Selling Price (CE) $\neq$ EV
     2. EVI = break-even cost of information (use a rainbow diagram to solve)

- Note that if a person is just known to have the Delta Property, then he either has either
  1. Exponential utility function and can be either risk averse or risk seeking in attitude, OR
  2. Linear utility function and is risk neutral.

6.3. Eliciting the Exponential Utility Function

6.3.1 The Exponential Utility Function

- The general exponential utility function is

$$u(x) = a - b\,e^{-x/\rho}$$

where ρ is the **Risk Tolerance**
a and b are constants that depend on the boundary conditions.

- Note that when $\rho > 0$ (risk averse) we must have $b > 0$ to ensure that $u'(x) > 0$
$\rho < 0$ (risk seeking) we must have $b < 0$ to ensure that $u'(x) > 0$

- Recall from Chapter 4 that a utility function is unique only up to a positive linear transportation. Hence the actual values of the two constants a and b will not affect the optimal decision.

- However, it is often useful to set the utility function values to a range between 0 and 1 at specific values of x.

- Suppose the values of x in the decision problem are somewhere between \$$L$ and \$$H$, and we would like to set the boundary conditions:

$u(L) = 0$ and $u(H) = 1$,

then solving the two simultaneous equations:

$$u(L) = a - b\,e^{-L/\rho} = 0 \qquad (1)$$

$$u(H) = a - b\,e^{-H/\rho} = 1 \qquad (2)$$

$$\Rightarrow \quad a = \frac{1}{1 - e^{-(H-L)/\rho}} \quad \text{and} \quad b = \frac{e^{L/\rho}}{1 - e^{-(H-L)/\rho}} = a e^{L/\rho}.$$

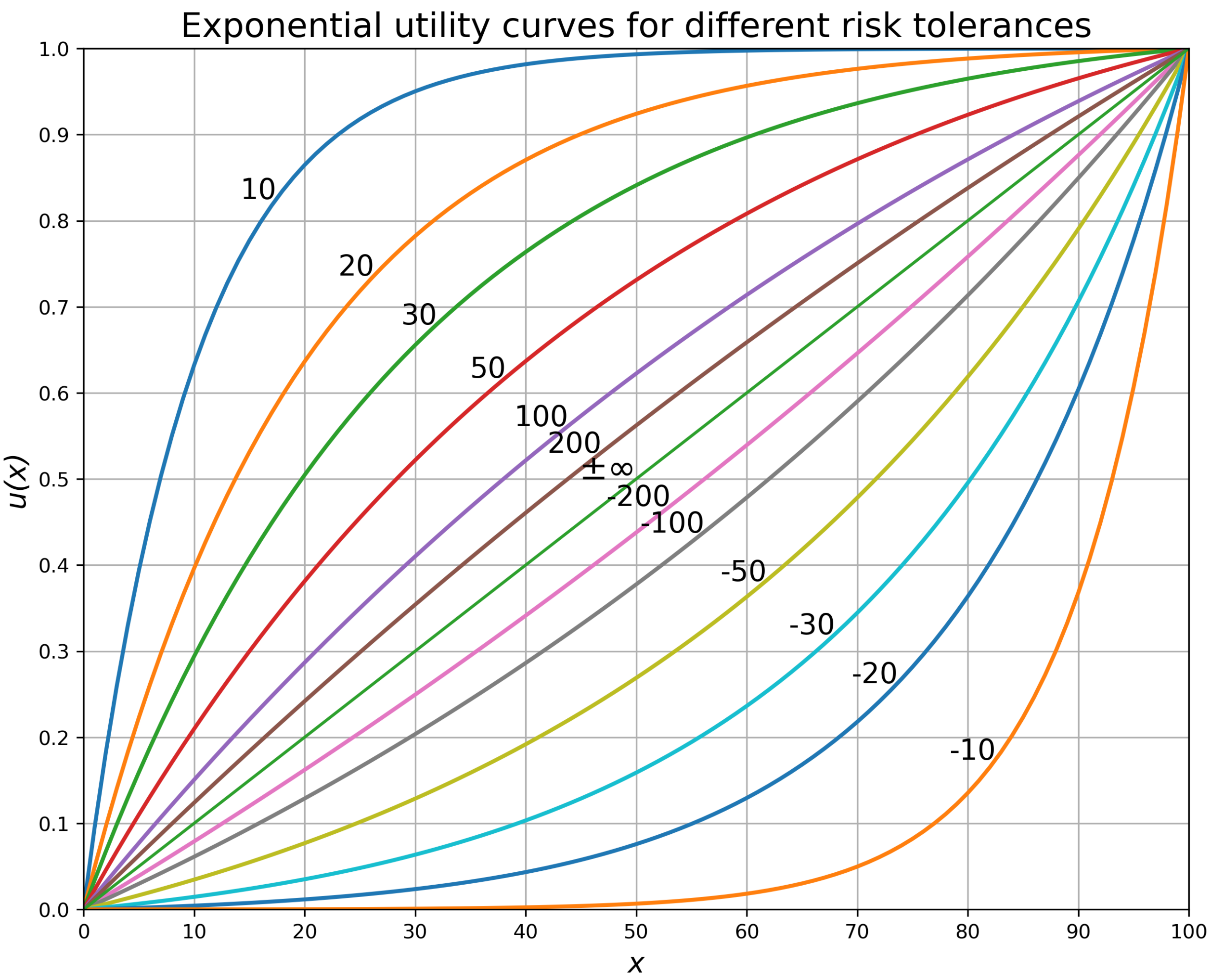
- Hence the exponential utility function with the boundaries conditions $u(L) = 0$ and $u(H) = 1$ is

$$u(x) = \frac{1 - e^{-(x-L)/\rho}}{1 - e^{-(H-L)/\rho}}$$

- To find the Certainty Equivalent at utility value = U

$$CE(U) = u^{-1}(U) = -\rho \ln\left(\frac{a - U}{b}\right)$$

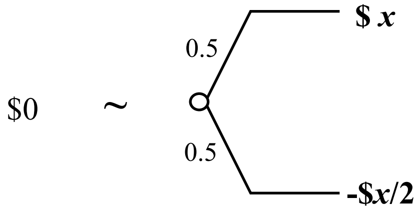
- The utility curves for $u(x)$ for $L = 0$, $H = 100$, and various values of ρ are shown below:

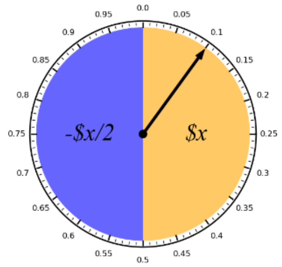


6.3.2 Assessing Risk Tolerance: 50-50 Direct Method

- Let the utility function be $u(x) = a - be^{-x/\rho}$ where ρ is the risk tolerance and *a*, *b* are constants.

- The decision maker is asked to choose between accepting or rejecting a risky deal which has a 50% chance of **winning $x** and a 50% chance of **losing $x / 2**:





Probability wheel

- If the value of *x* is very large, a risk-averse decision-maker would probably reject the deal and avoid taking risks. On the other hand, if the value of *x* is low enough, the decision maker would probably accept the deal and take the risk.

- If x^* is the value of *x* such that the decision maker is just indifferent between accepting and not accepting the deal, then it can be shown that his/her risk tolerance is **approximately equals** to x^*.

- It can be shown that the risk tolerance is $\rho = 1.039\, x^*$

Proof

- At the point of difference between accepting and not accepting the deal:

$$u(0) = \frac{1}{2}u(x) + \frac{1}{2}u\left(-\frac{x}{2}\right)$$

$$a - b = \frac{1}{2}\left[a - b\, e^{\frac{-x}{\rho}}\right] + \frac{1}{2}\left[a - b\, e^{\frac{x}{2\rho}}\right]$$

$$2 = e^{\frac{-x}{\rho}} + e^{\frac{x}{2\rho}}$$

- Let $y = e^{\frac{x}{2\rho}}$, then $y^3 - 2y^2 + 1 = 0 \;\Rightarrow\; (y-1)(y^2 - y - 1) = 0$

- The solutions are $y = 1$ or $y = \dfrac{1 \pm \sqrt{5}}{2}$.

- Only solution $y = \dfrac{1+\sqrt{5}}{2}$ is useful.

- Hence $y = \dfrac{1+\sqrt{5}}{2} = e^{\frac{x}{2\rho}} \quad \Rightarrow \quad \rho = \dfrac{x}{2\ln(\frac{1+\sqrt{5}}{2})} = 1.039\, x$

- Therefore, $\boldsymbol{\rho \approx x}$ (with a 3.9% error)

How to find x^*?

- In practice, to find the value of x^*, we use the probability wheel set to equal areas for both colors and pose a series of probing questions to the decision maker by increasing/decreasing the value of x until he/she is just indifferent between accepting and rejecting the deal with equal chances of receiving \$$x$ or losing \$$x/2$.

Iterative Procedure for Finding Risk Tolerance (Risk Averse Case)

1. Set X to some value.
2. Repeat
3. Ask the decision maker if he/she would accept or reject the deal which has a 50% chance of winning \$$X$ or losing \$$X/2$, or if he/she is indifferent between accepting and rejecting.
4. If the decision maker accepted the deal, then increase X, but keep it below previous values for which the decision maker had rejected.
5. Else if the decision maker rejected the deal, then decrease X, but keep it above previous values for which the decision maker had accepted.
6. Until the decision maker is indifferent between accepting and rejecting.
7. Risk Tolerance = 1.039 X.

Example (Risk-Averse Case)

- If a decision maker is risk-averse and indicates that he is indifferent between accepting and not accepting the deal with a 50-50 chance of either **winning $20,000** or **losing $10,000**, then $x^* =$ $20,000.

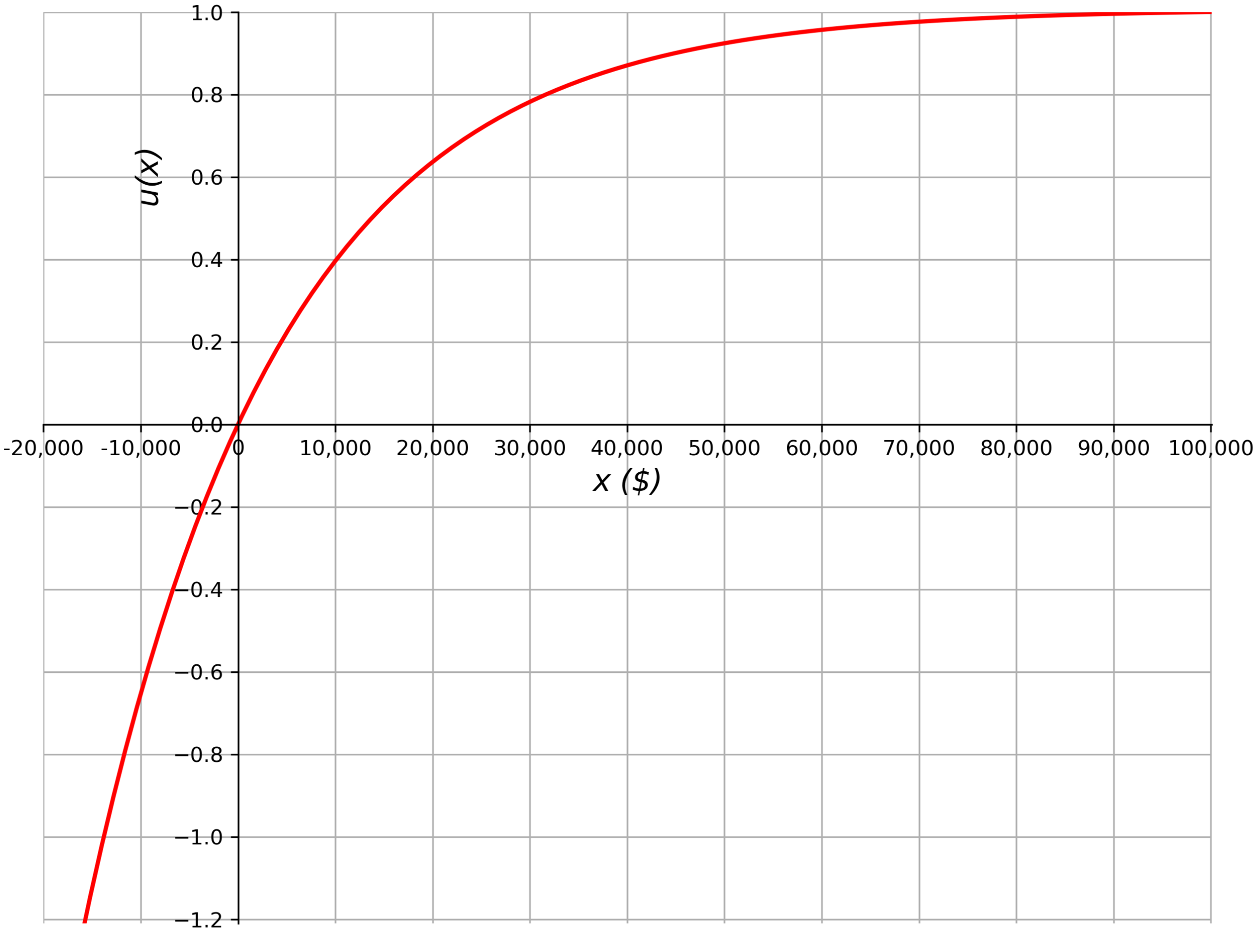
- His utility function is (approximately)

$$u(x) = a - be^{-x/20{,}000}$$ where x is in dollars, and $b > 0$.

- If we fit the boundary conditions $u(0) = 0$ and $u(100{,}000) = 1$, then

$a = 1.0067837$ and $b = 1.0067837$

- Hence $u(x) = 1.0067837\,(1 - e^{-x/20{,}000})$



Example (Risk Seeking Case)

- When the decision maker is risk-seeking, his risk tolerance is negative.

- Suppose he indicates that he is indifferent between accepting and not accepting the game with a 50-50 chance of either **winning \$10,000** or **losing \$20,000**, then x^* = - \$20,000.

- His utility function is (approximately)

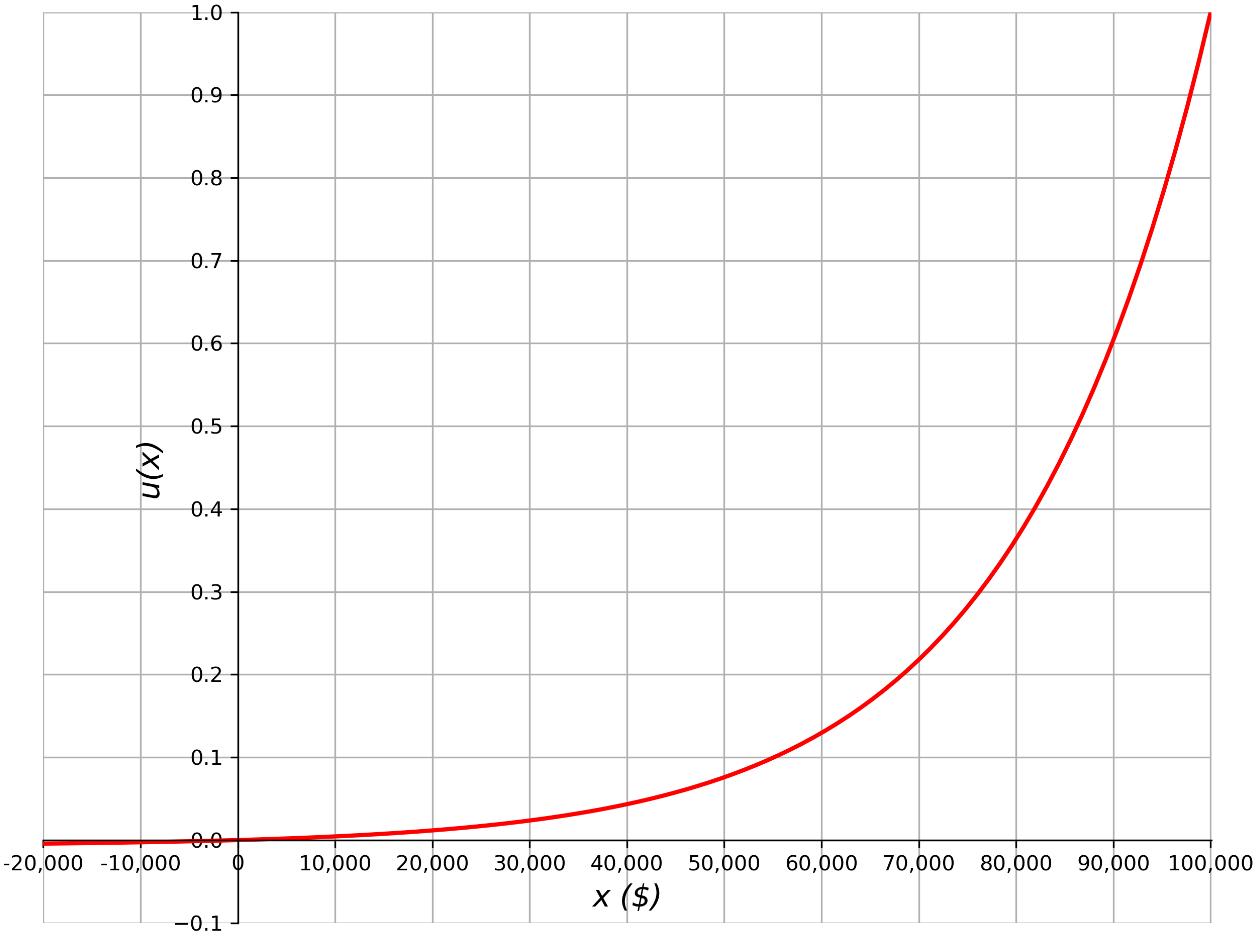
$$u(x) = a - be^{-x/-20{,}000}$$
$$= a - be^{x/20{,}000}$$

where x is in dollars, and $b < 0$

- If we fit the boundary conditions $u(0) = 0$ and $u(100{,}000) = 1$, then

$$a = -0.0067837 \quad \text{and} \quad b = -0.0067837$$

- Hence $u(x) = 0.0067837\,(-1 + e^{x/20{,}000})$



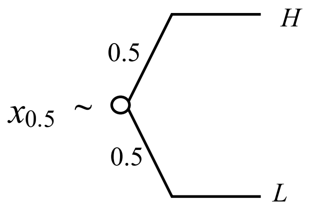
Major Disadvantage

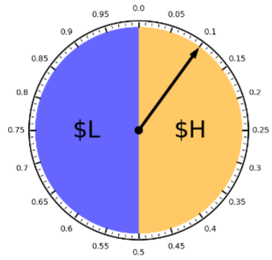
- This method requires posting deals with outcomes having negative values. The decision maker may not be comfortable working with these values.

- The following method requires only deals with non-negative outcomes.

6.3.3 Assessing Risk Tolerance: 50-50 Certainty Equivalent Method

- Suppose the range of values for which the utility function is to be assessed is from $\$L$ to $\$H$, then the decision maker is asked for what value of $x_{0.5}$ in dollars would he be indifferent between accepting $\$ x_{0.5}$ for sure and deal with an equal chance of winning either $\$H$ or $\$L$.





Probability Wheel

- Since we desire $u(L) = 0$ and $u(H) = 1$, it follows that $u(x_{0.5}) = 0.5$.

- The risk tolerance ρ is then the solution to the equation:

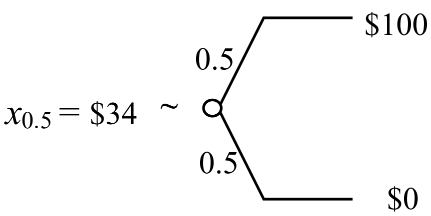
$$\frac{1-e^{-(x_{0.5}-L)/\rho}}{1-e^{-(H-L)/\rho}} = 0.5$$

- An equation solver is needed to determine the risk tolerance.

- In practice, to find the value of $x_{0.5}$, we use the probability wheel and set it to equal areas for both colors. A value of x is offered. If the decision maker chooses $\$x$ over the wheel, the value of x is decreased. If the decision maker chooses the wheel instead, the value of x is increased. The process is repeated until the decision maker is indifferent between $\$x$ and the wheel.

Example (Risk-Averse Decision Maker Case)

- If L=\$0, H=\$100, and the decision maker indicates $x_{0.5}$ = \$34.



- We need to solve $$\frac{1-e^{-(34-0)/\rho}}{1-e^{-(100-0)/\rho}} = 0.5$$

- Since $x_{0.5}$ = \$34 < EV = \$50, it follows that the decision maker is risk averse.

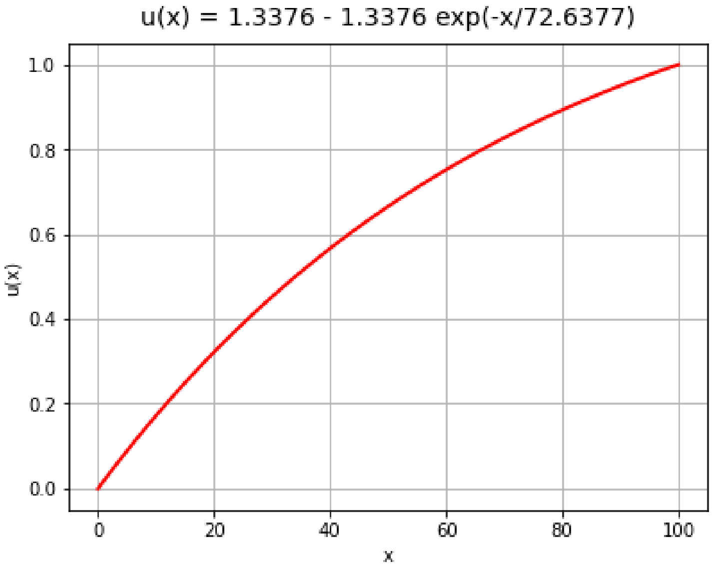
- Hence if an equation solver such as Excel Goal Seek or Python scipy.optimize.root() function is used, set the guess for the risk tolerance to a **positive value**.

- Using any equation solver: $\rho = \$72.64$

- Hence the utility function is

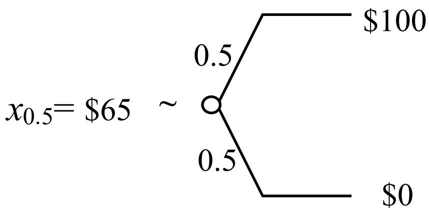
$$u(x) = \frac{1-e^{-x/72.64}}{1-e^{-100/72.64}} = 1.33(1-e^{-x/72.64})$$

with the boundary conditions $u(0) = 0$, $u(100) = 1$.



Example (Risk Seeking Decision Maker Case)

- If L=\$0, H=\$100. Suppose the decision maker indicates $x_{0.5}$ = \$65.



- We need to solve: $$\frac{1-e^{-(65-0)/\rho}}{1-e^{-(100-0)/\rho}} = 0.5$$

- Since $x_{0.5}$ = \$65 > EV = \$50, it follows that the decision maker is risk seeking.

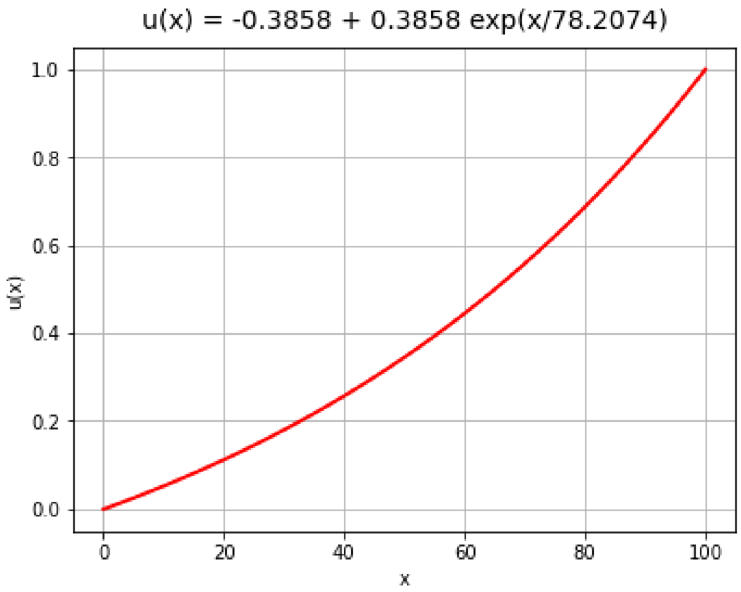
- Hence if an equation solver such as Excel Goal Seek or Python scipy.optimize.root() function is used, set the guess for the risk tolerance to a **negative value.**

- Using any equation solver: ρ = - \$78.21

- Hence the utility function is

$$u(x) = \frac{1-e^{-x/-78.14}}{1-e^{-100/-78.14}} = -0.38533\,(1-e^{x/78.14}) = 0.38533(\,e^{x/78.14} - 1)$$

with the boundary conditions $u(0) = 0$, $u(100) = 1$.



Advantages

- Does not require reference deal with negative payoffs.
- Same probing questions for both risk-averse and risk-seeking attitudes.

Disadvantage

- Requires an equation solver.

6.4. Value of Information Analysis with Delta Property

6.4.1 Expected Value of Information with Delta Property

- When the decision maker exhibits the delta property, the value of perfect or imperfect information can be computed easily by taking the difference of the Certainty Equivalents of two decision models:

  1. Model 1: Base model with no information
  2. Model 2: Model with free perfect or imperfect information.

- **Expected Value of Information = CE with Free Information – CE with no information.**

- Note that if the decision maker does not have the delta property (i.e., is neither risk neutral nor has the exponential utility function), then the break-even method must be used to determine the expected value of perfect and imperfect information.

6.4.2 Value of Information Analysis for the Party Problem

- Consider again Kim's Party Problem.

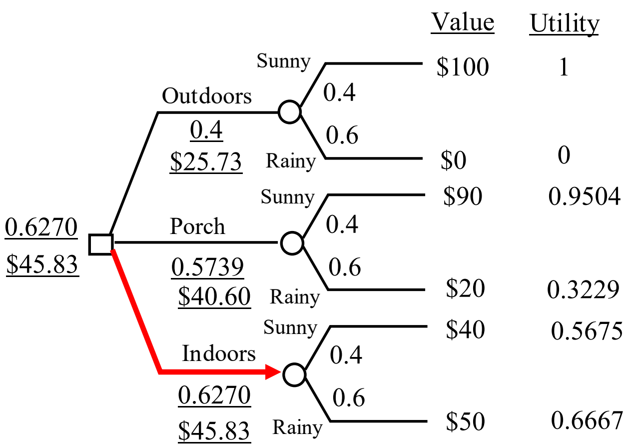
- A check on Kim's utility curve reveals that it fits well to an exponential utility function of the form:

$$u(x) = \frac{4}{3}\left[1 - \left(\tfrac{1}{2}\right)^{x/50}\right] = \frac{4}{3}\left[1 - e^{-x\ln 2/50}\right].$$

- Kim satisfies the delta property with constant risk tolerance = 50/ln(2) = $72.135

- First, we consider the base model with no information:

Base model with no information

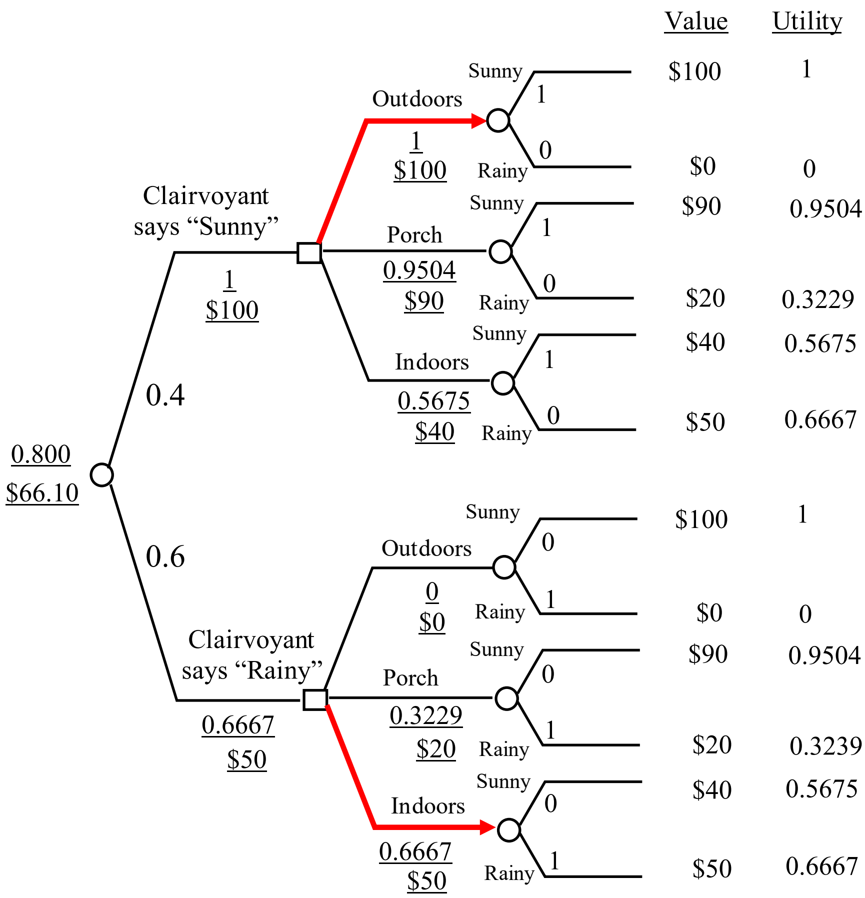


Expected utility of optimal decision = 0.6270

Certainty equivalent = $u^{-1}(0.6270)$ = \$45.83

- Next, we consider the model with free perfect information on the weather:

Decision model with free clairvoyance



Expected utility of optimal decision policy = 0.800

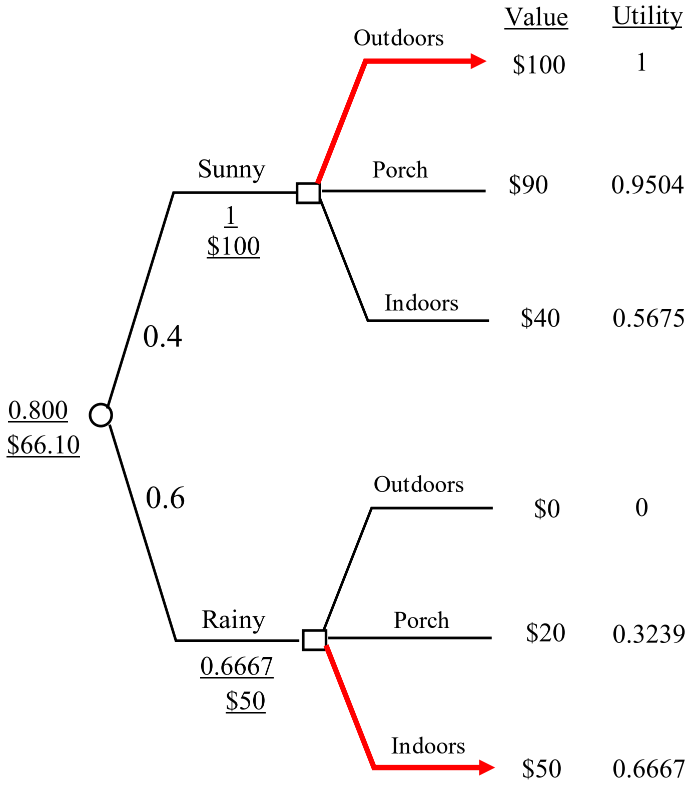
Certainty equivalent = $u^{-1}(0.800)$ = $66.10

- Hence the expected value of perfect information on weather = $66.10 – $45.83 = $20.27 which is the same as that found previously in Chapter 4 using the break-even cost method.

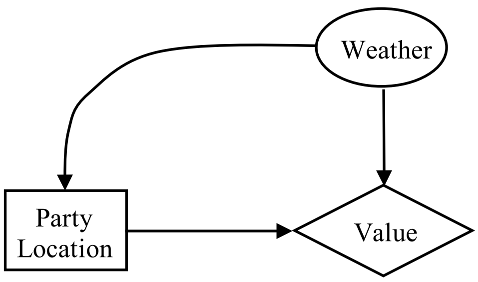
Simplified decision tree for the model with free clairvoyance

- The simplified decision tree assumes that the decision-maker "observes" the outcome of the weather before picking the party location.

Simplified decision model with free clairvoyance



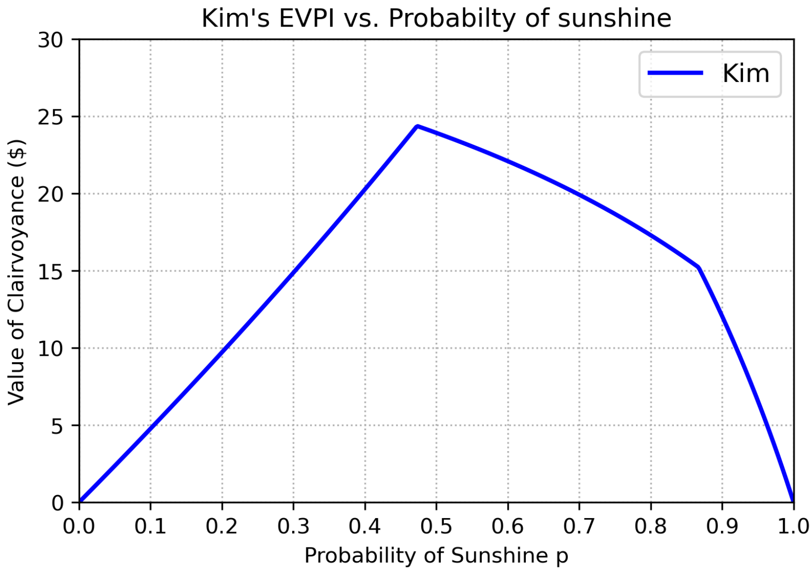
- The simplified decision model with free perfect information is consistent with the influence diagram with an information arc from "Weather" to "Party Location":



6.4.3 Impact of Changing the Probability of Sunshine on the Value of Clairvoyance

Kim Party Problem

- If we compute Kim's value of clairvoyance for values of probability of sunshine p between 0 and 1, we obtain the following:

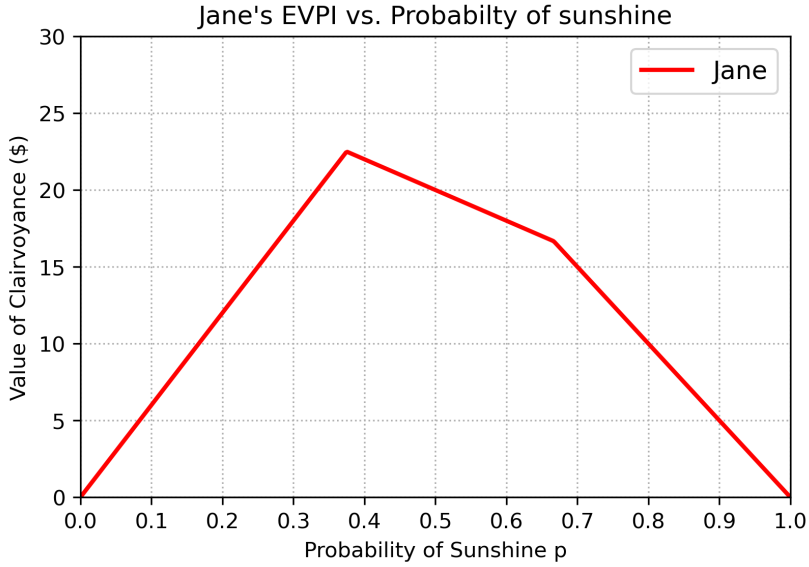


Observations:

- The value of clairvoyance is $20.27 at the base value of $p = 0.4$.
- The value of clairvoyance is zero at $p = 0$ and $p = 1$ because at these two values, Kim has no uncertainty about the weather.
- As p increases from 0, the value of clairvoyance increases, reaching a peak of about $24.36 at $p = 0.47$.
- When p exceeds 0.47, the value of clairvoyance falls, and falls even more rapidly when p exceeds 0.87.

Jane Party Problem

- If we compute Jane's value of clairvoyance for values of probability of sunshine p between 0 and 1, we obtain the following:



Observations:

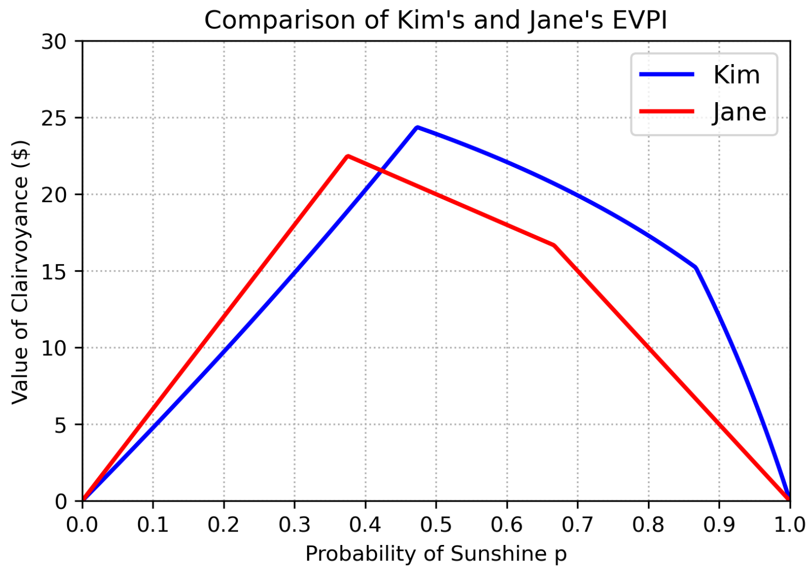
- The value of clairvoyance is \$22 at the base value of $p = 0.4$.

- The value of clairvoyance is zero at $p = 0$ and $p = 1$ because at these two values, Jane has no uncertainty about the weather.

- As p increases from 0, the value of clairvoyance increases, reaching a peak of about \$22.5 at $p = 0.375$.

- When p exceeds 0.375, the value of clairvoyance falls and falls even more rapidly when p exceeds 0.667.

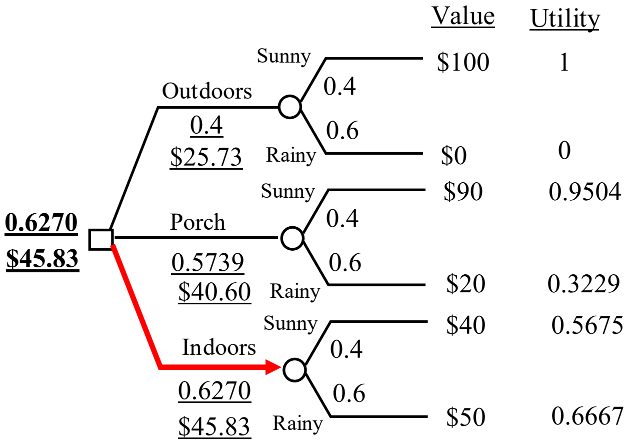
Comparison of Kim's and Jane's Value of Clairvoyance for different values of *p*.



6.4.4 Expected Value of Imperfect Information with Delta Property

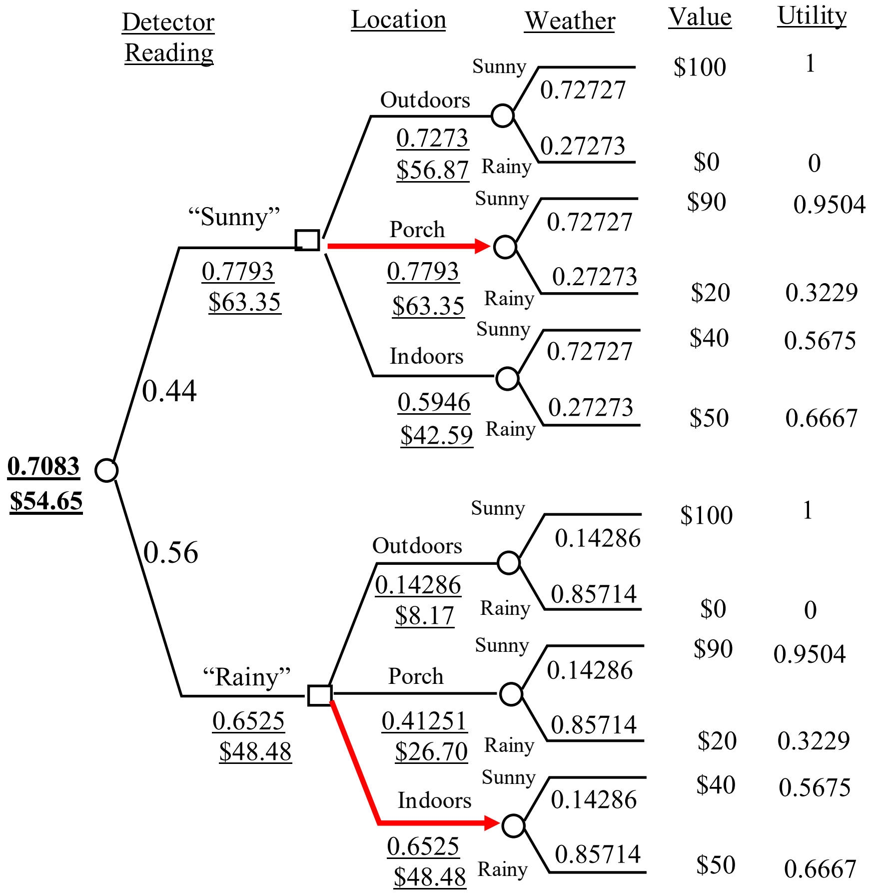
Kim's Party Problem

- First, we consider the base model with no information, i.e., no rain detector is used:



Expected utility of optimal decision = 0.6270
Certainty equivalent = $u^{-1}(0.6270)$ = $45.83

- Next, we consider the Decision Model with free use of the rain detector:



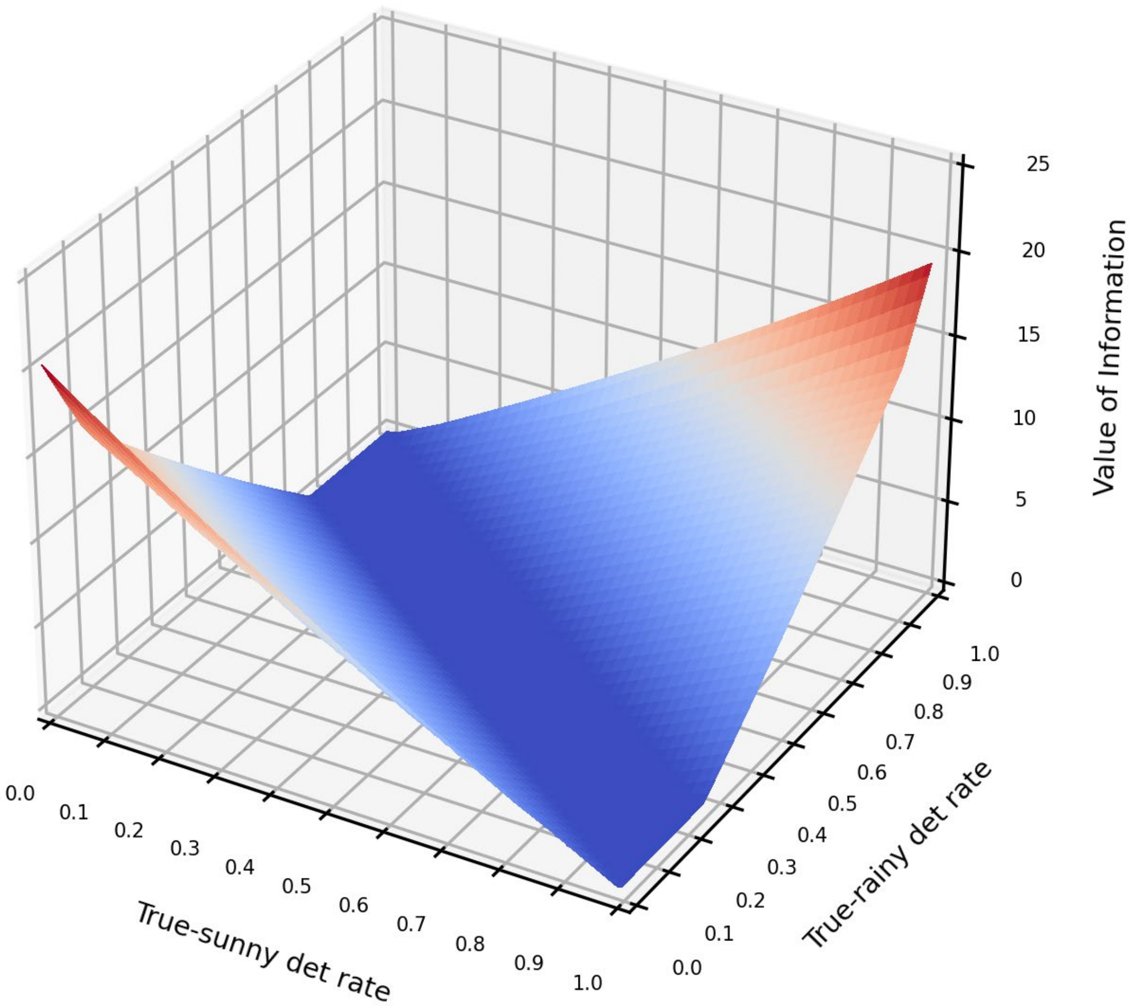
Expected utility under optimal decision policy = 0.7083
Certainty equivalent = $u^{-1}(0.7083)$ = $54.65

- The expected value of imperfect information on the weather provided by the rain detector = $54.65 – $45.83 = $8.82
- Hence Kim should not pay more than $8.82 for using this imperfect detector.

6.4.5 Impact of Changing the True-Detection Rates on the Value of Information

Two-Way Sensitivity Analysis

- If we compute Kim's value of information (with the probability of sunny weather fixed at 0.4) on the use of the rain detector by varying the true sunny detection rate and true rainy detection rate simultaneously, the following two-way rainbow diagram is obtained:



Observations:

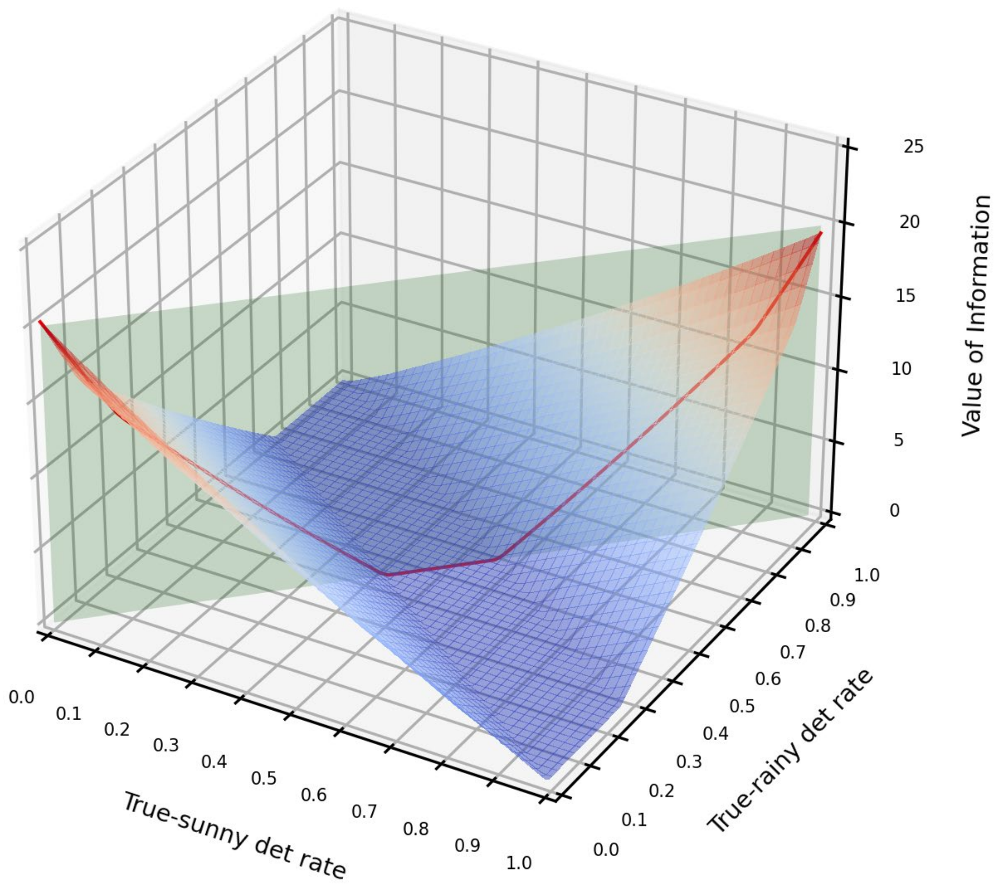
- The expected value of information is high when the true-detection rates are both very high and near (1, 1) or when both are very low and near (0, 0).

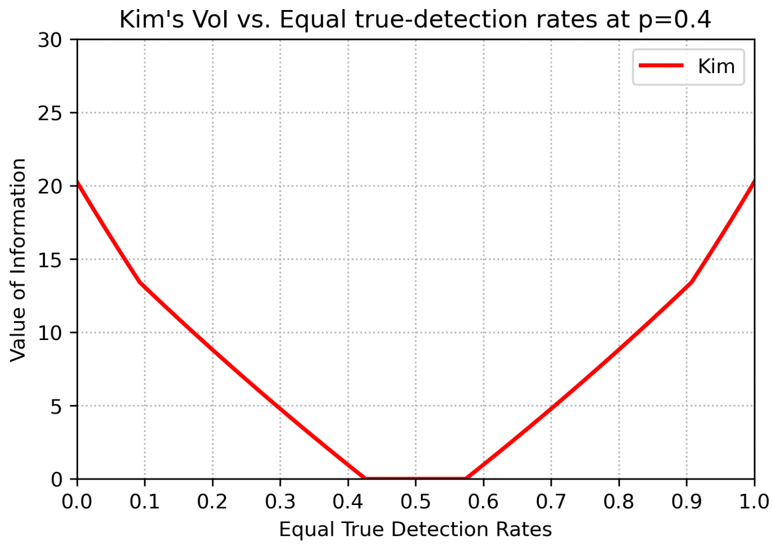
- The expected value of information is low when one of the true detection rates is high and the other is low.

- There is a strip in the valley where the expected value of information is zero or near zero.

One-Way Sensitivity Analysis with Equal Detection Rates

- Consider the special case when the two true detection rates are always equal:



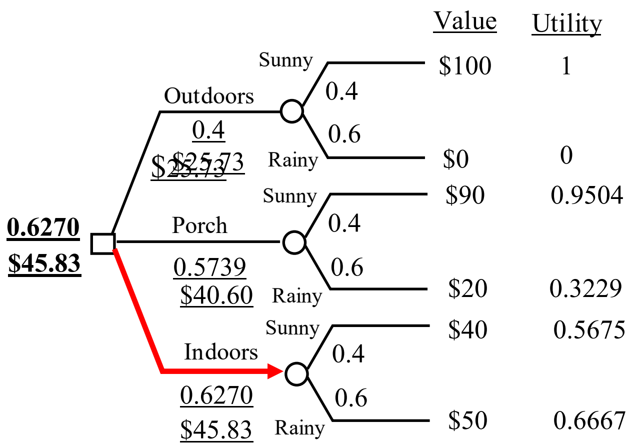


- We observe that the expected value of information is maximum when the true-detection rates are either both 100% (perfect) or 0% (negatively perfect) and is zero when at values around 50%.

- What is the significance of having a **zero value of information** when the true-detection rates are around 0.5?

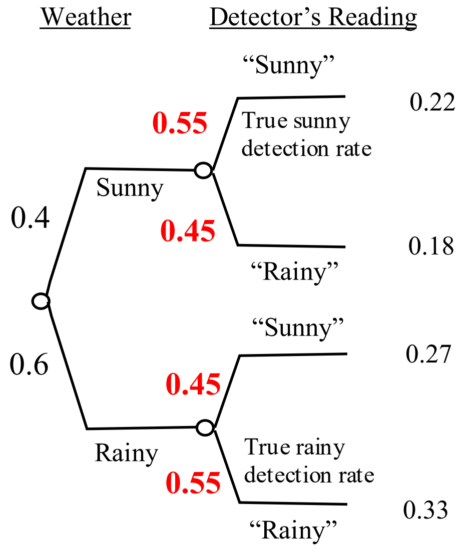
Consider the case when Probability of Sunny = 0.4 and True-Detection Rates = 0.55

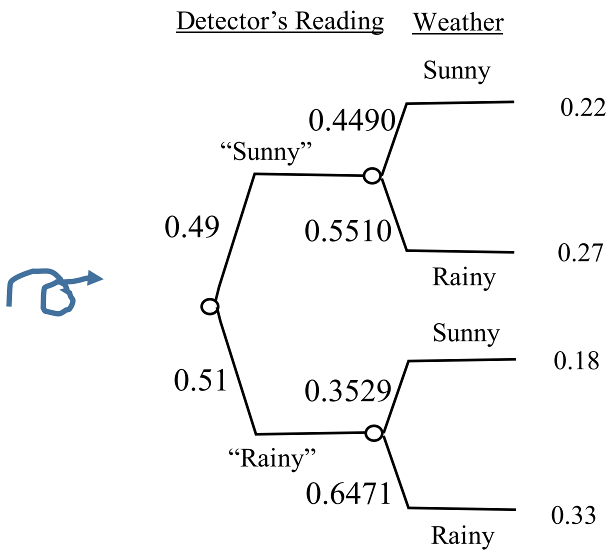
Decision Model with no Information



- The decision is to go indoors with certainty equivalent = $45.83

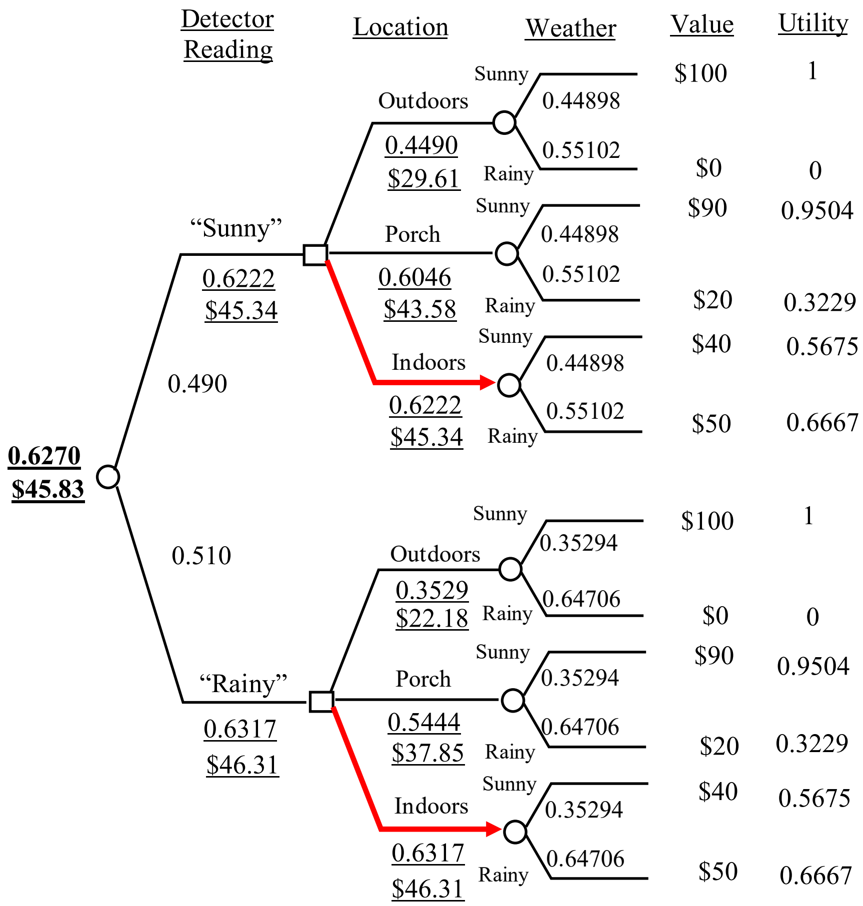
Flipping the Tree for Detector Readings





- Notice that the posterior probability for sunny does not change much from the prior probability for both "sunny" and "rainy" readings. In fact, when the detection rates are all 50%, posterior probability = prior probability.

Decision Model with Free Use of Detector
(Probability of Sunny = 0.4 and True-detection rates = 55%)



- Certainty Equivalent with use of detector = $45.83.
- Expected Value of Information for the 55%-rate detector (when the probability of sunshine is 0.4)

  = $45.83 - $45.83

  = $0.
- Notice that the decision is always to have the party indoors which is the same as the base decision without information, for all possible detector readings.

6.5. Time Preferences

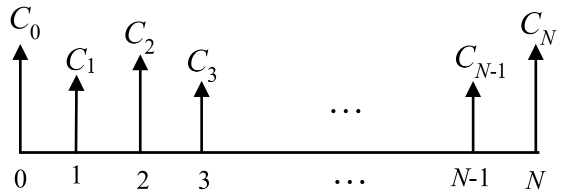
6.5.1 The Time Value of Money

- In decision analysis, we often encounter situations that involve cash inflows or outflows at different points in time. The *Time Value of Money* must be taken into consideration.

- The **Present Value** of receiving cash C_k for sure k years from now at an *effective risk-free annual interest rate* r_f, is

$$PV_k(C_k, r_f) = \frac{C_k}{(1+r_f)^k}$$

- In general, suppose a project involves a series of deterministic cash flows over N years as shown by the cash flow diagram below:



- Note
  1. C_k is the cash flow at the end of year k, for $k = 0,\ldots,N$.

  2. A positive value of C_k indicates a cash inflow, while a negative value indicates a cash outflow.

- The **Net Present Value** (*NPV*) or **Present Worth** (*PW*) of the N-period cash flows series at an effective annual interest rate of r_f is

$$NPV = C_o + \frac{C_1}{(1+r_f)} + \frac{C_2}{(1+r_f)^2} + \ldots + \frac{C_N}{(1+r_f)^N} = \sum_{k=0}^{N} \frac{C_k}{(1+r_f)^k}$$

Decision Criteria based on NPV (Deterministic Cash Flows)

- A single project is economically feasible if the NPV computed at an interest rate known as the *MARR* is non-negative, i.e., NPV ≥ 0.

- Given a set of mutually exclusive investment alternatives the alternative with the maximum NPV computed at an interest rate known as the *MARR*, is the most preferred.

- The *MARR* (*Minimum Attractive Rate of Return*) is normally set to be at least equal the *WACC* (*Weighted Average Cost of Capital*).

Example

- Consider two mutually exclusive alternatives which have the following cash flows:

Alternative	Cash flow at End of Year				
	0	1	2	3	4
Project A	-5,000	2,500	2,000	1,500	1,000
Project B	-5,000	1,500	1,500	1,500	2,500

- If the *MARR* or *WACC* is 10% per year, which alternative is preferred?

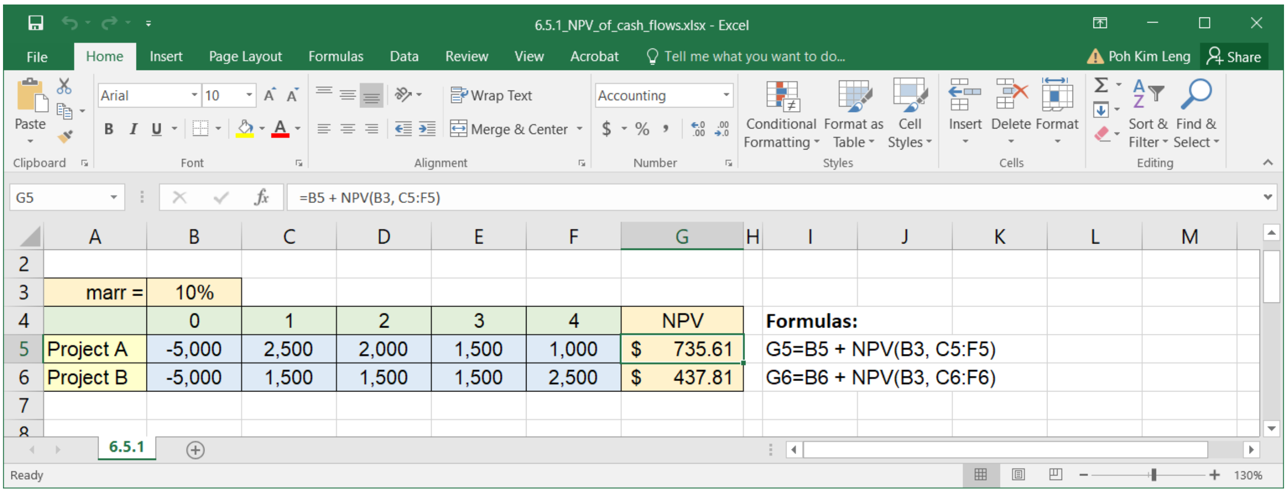
- The NPV for the two alternatives are:

$$NPV(A) = -5000 + \frac{2500}{(1+0.1)} + \frac{2000}{(1+0.1)^2} + \frac{1500}{(1+0.1)^3} + \frac{1000}{(1+0.1)^4} = \$735.61 > 0$$

$$NPV(B) = -5000 + \frac{1500}{(1+0.1)} + \frac{1500}{(1+0.1)^2} + \frac{1500}{(1+0.1)^3} + \frac{2500}{(1+0.1)^4} = \$437.81 > 0$$

- Hence Project A is preferred to Project B.

Excel:



Python:

In [1]:
```python
""" Compute NPV of Cash flows """
import numpy_financial as npf
```

In [2]:
```python
marr = 0.1
Project_A = [ -5000, 2500, 2000, 1500, 1000 ]
Project_B = [ -5000, 1500, 1500, 1500, 2500 ]
```

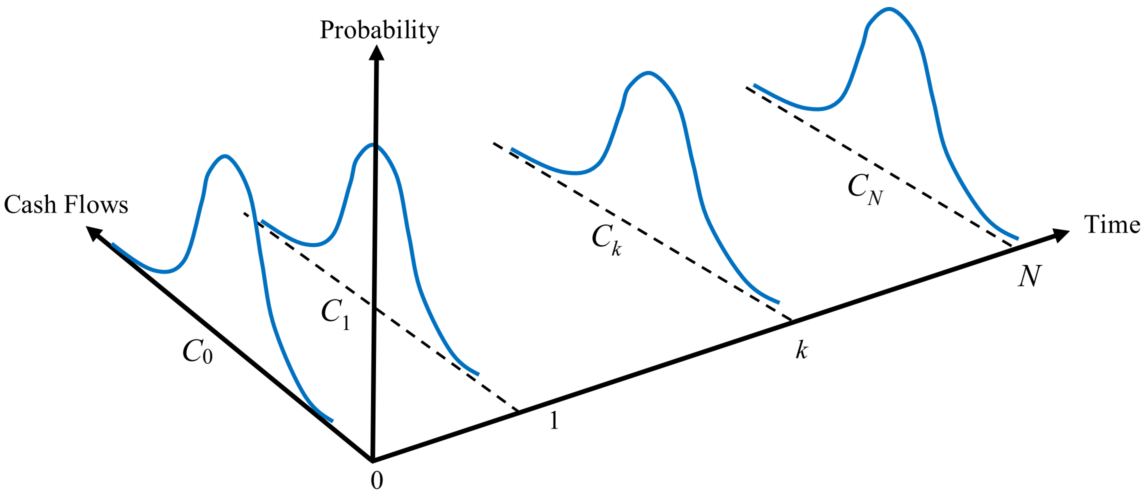
In [3]:
```python
NPV_A = npf.npv(marr, Project_A)
NPV_B = npf.npv(marr, Project_B)
print(f"NPV(A) = ${NPV_A:,.2f}")
print(f"NPV(B) = ${NPV_B:,.2f}")
```

```
NPV(A) = $735.61
NPV(B) = $437.81
```

6.5.2 Uncertain Cash Flow over Time

- Suppose we encounter a series of **uncertain** end-of-year cash flows, how should we account for both the time value of money and uncertainty?

- Each cash flow C_k at the end of year k is now a random variable subject to a probability distribution.



- Given an investment with such cash flows, how do we decide on it?

Separation of Risk and Time Preferences

Utility of Net Present Value

- We define the **Utility of Net Present Value** of a N-period cash flows { C_k for k=0 to N } as

$$u(NPV(C_o, C_1, \ldots, C_N)) = u\left(\sum_{k=0}^{N} \frac{C_k}{(1+r_f)^k}\right)$$

where $u(x)$ is a utility function for cash flows occurring at the present.
r_f is the effective risk-free interest rate.

- Here, the time value of money is first accounted for by discounting the cash flows using the *risk-free interest rate* r_f, and then the utility value is computed using the utility function for cash flows occurring at the present.

- When the cash flows are uncertain, the **Expected Utility of Net Present Value** is computed:

$$EU[NPV] = \sum_{C_0 .. C_N} p(C_o, C_1, \ldots, C_N) u\left(\sum_{k=0}^{N} \frac{C_k}{(1+r_f)^k})\right)$$

where $p(C_0, C_1, \ldots, C_N)$ is the joint probability distribution for the cash flows.

- The above computation can be done easily by computing the NPV of cash flows at the end-points on a decision tree and then rowing back the expected utilities to find the optimal decision policy.

Certainty Present Equivalent:

- If we invert the $EU[NPV]$ back to dollar values, we will obtain the **Certainty Present Equivalent**:

  $$CPE = u^{-1}(EU[NPV])$$

- Hence if an investor owns the asset, then CPE represents his current *Personal Indifferent Selling Price* of the asset.

- If the investor has the delta property (i.e., is either risk neutral or has the exponential utility function), then the CPE also represents his current buying price for the asset.

Example

- A 2-year risky investment requires an initial cash outflow of $1,000.

- Cash flows at the end of year 1 are uncertain:

	Cash flow at End of Year 1		
Cash flow	$580	$500	$460
Probability	0.2	0.5	0.3

- Cash flows at the end of year 2 are uncertain and depend on the preceding year's cash flow:

	Cash flow at EoY2 given that cash flow at EoY1=$580		
Cash flow	$1,000	$960	$800
Conditional Probability	0.3	0.3	0.4

	Cash flow at EoY2 given that cash flow at EoY1=$500		
Cash flow	$770	$720	$680
Conditional Probability	0.1	0.8	0.1

	Cash flow at EoY2 given that cash flow at EoY1=$460		
Cash flow	$760	$650	$600
Conditional Probability	0.3	0.2	0.5

- If the company's $MARR$ = 5%, determine if the company should go ahead with the investment by comparing it to the alternative investment that offers a risk-free effective annual return of $MARR$ for 2 years when the company is

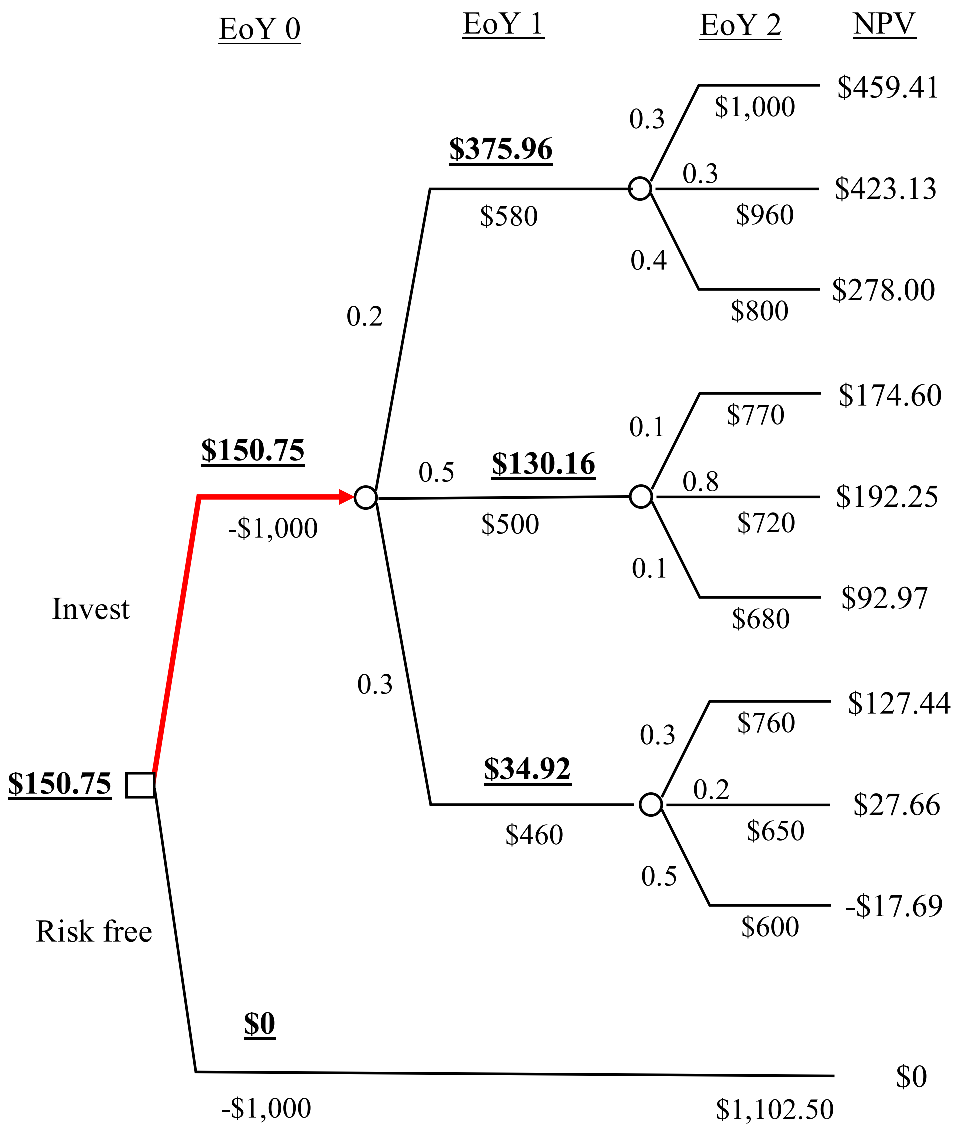
  (***a***) Risk Neutral

  (***b***) Risk averse with constant risk tolerance = $500.

  (***c***) Over what range of value of risk tolerance would the company invest in the risky project?

(***a***) Company is Risk Neutral.

- Decision Tree:



- Note that for the risk-free investment:
  - Cash flow received at EoY 2 = 1,000 $(1+0.05)^2$ = $1,102.5
  - Hence NPV = $0.

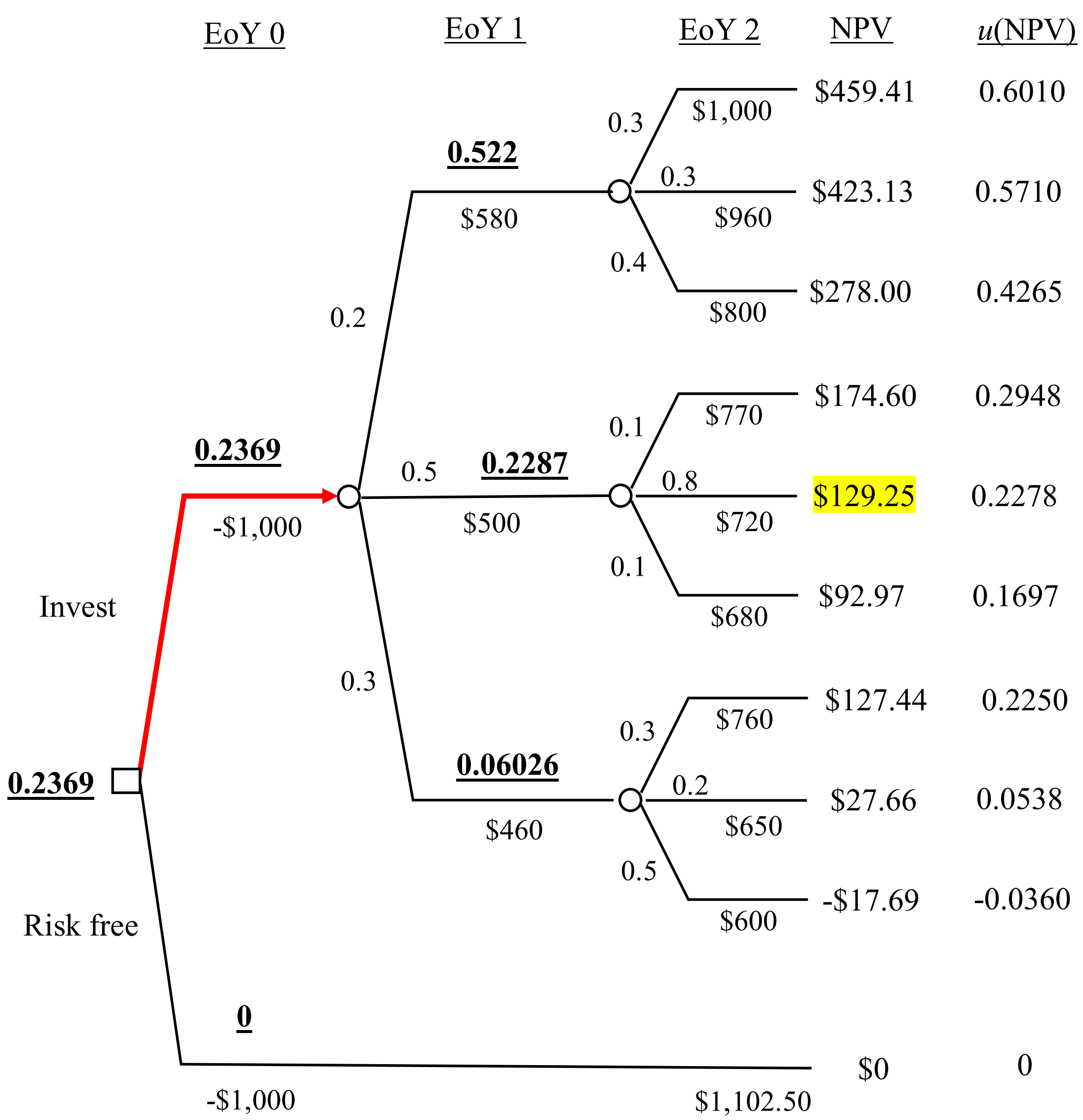
- From the decision tree analysis:
  - Expected NPV of risky investment = $150.75
  - Expected NPV of risk-free investment = $0.

- Hence the company should invest in the risky project.

(*b*) Company has constant risk tolerance = \$500.

- Let the utility function be $u(x) = 1 - e^{-x/500}$.
  (Do you know why we did not bother to find the exact value of the scaling constants *A* and *B* for the utility function?)

- Decision Tree with expected utility values:



- From the Decision Tree Analysis:
  - *EU*(Invest) = 0.2369,
  - Certainty Present Equivalent of investment = **\$135.15**

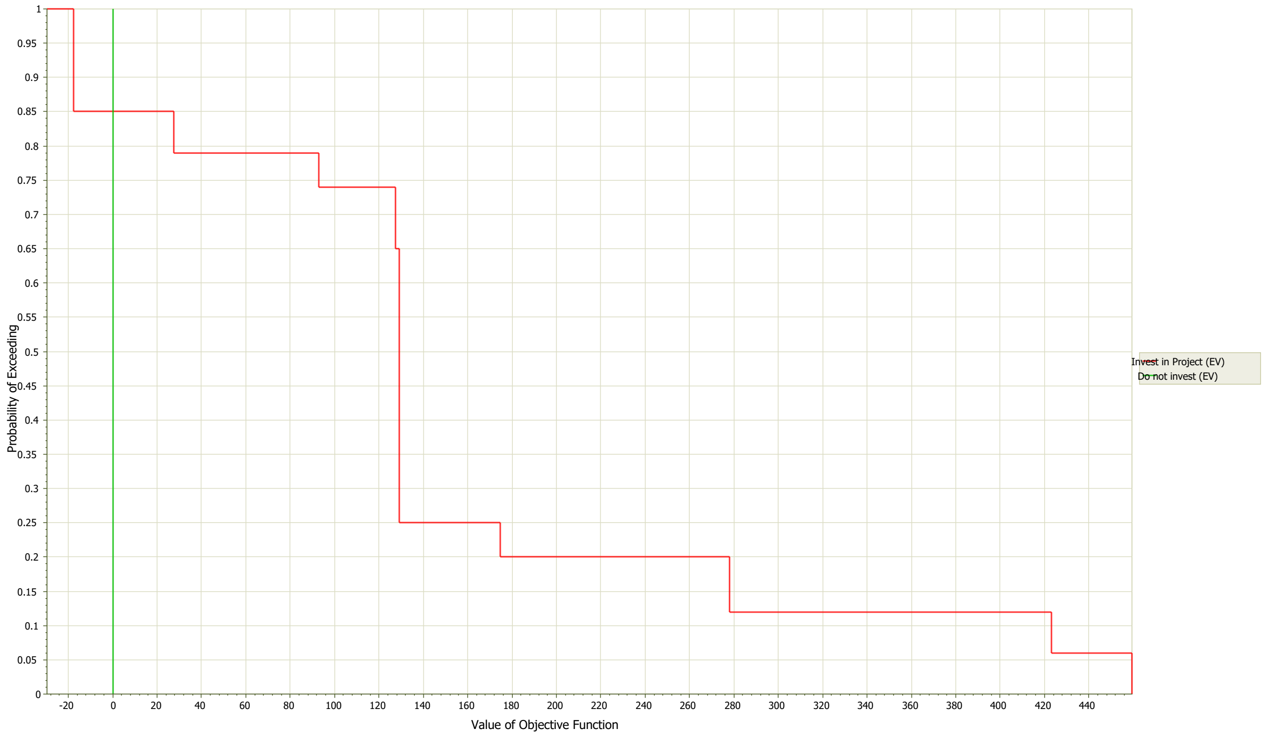
  - *EU*(Risk-free alternative) = 0
  - Certainty Present Equivalent of risk-free alternative = \$0

- Hence the company should go ahead with the risky investment.

- If the company already owns the investment, it would value it at \$135.15 and be willing to sell it at any price above it.

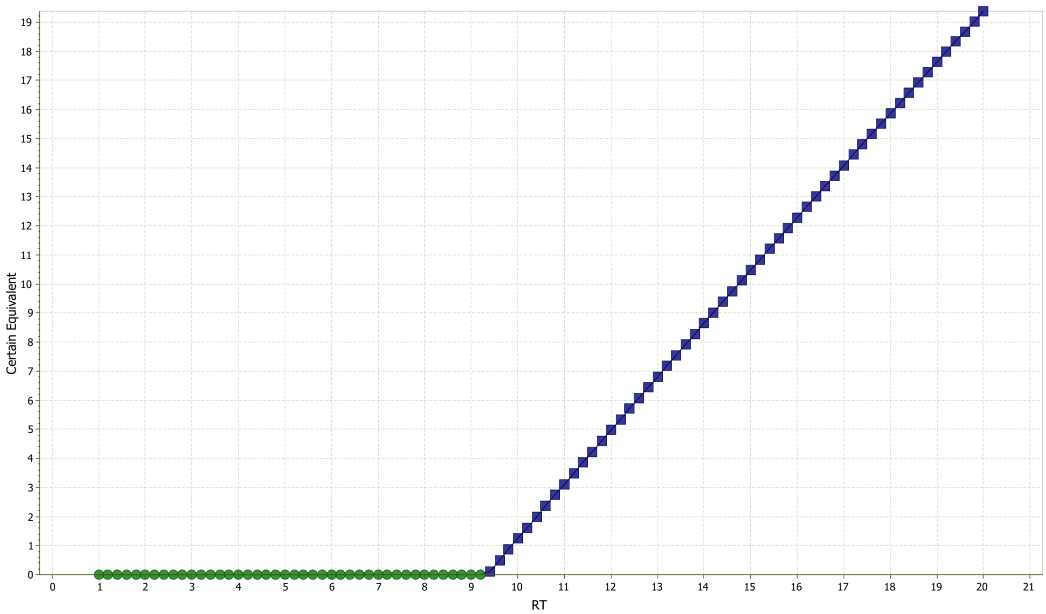
(*c*) Sensitivity Analysis on Risk Tolerance

Risk Profile for the Investment



- There is no first or second-order stochastic dominance between the two alternatives. Hence there might be decision reversal at some risk tolerance.

- Rainbow Diagram for alternatives' NPV when risk tolerance is varied from \$1 to \$20:



- The company should invest in the risky project for \$9.40 $\leq$ RT $\leq \infty$.

References

1. K. Arrow (1965). *Aspects of the theory of risk-bearing*, Yrjo Jahnsson Saatio, Helsinki.

2. J. Pratt (1964). Risk aversion in the small and in the large, *Econometrica* 32:133-136.

3. K. Arrow (1971). *Essays in the theory of risk bearing*, Markham.

4. W.G. Sullivan, E.M. Wicks and C.P. Koelling (2020). *Engineering Economy*, 17^{th} Edition, Pearson.

5. K.L. Poh. IE5003 Cost Analysis & Engineering Economy (2016) Lecture Notes.

Exercises

P6.1 For each of the following utility functions, determine its risk tolerance and degree of absolute risk aversion.

(a) The quadratic utility function: $u(w) = w - \beta w^2$

(*b*) The logarithmic utility function: $u(w) = \ln w$

(*c*) The power utility function: $u(w) = \text{sgn}(\beta)\, w^{\beta}$

Note:

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

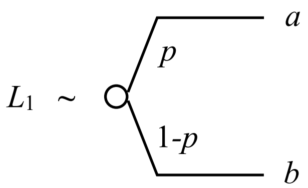
P6.2 John has the utility function $u(x) = 1 - 3^{-x/50}$ over the range of $x = -\$50$ to $\$5000$.

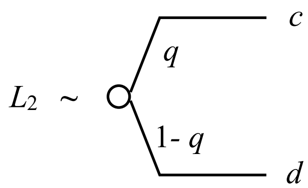
(*a*) What is John's risk attitude?

(*b*) What is John's degree of absolute risk aversion?

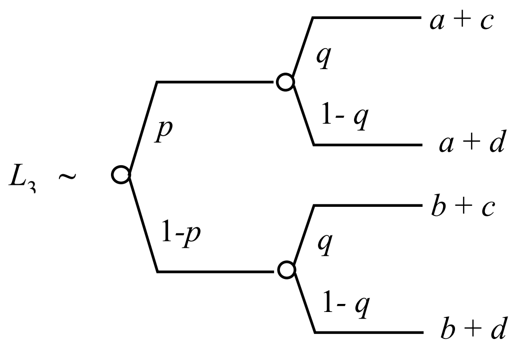
(*c*) At what probability (p) of winning \$50 versus losing \$50 with $(1 - p)$ probability is John indifferent between having this deal and not having this deal?

P6.3 Jim follows the *delta property* and owns two independent deals L_1 and L_2,



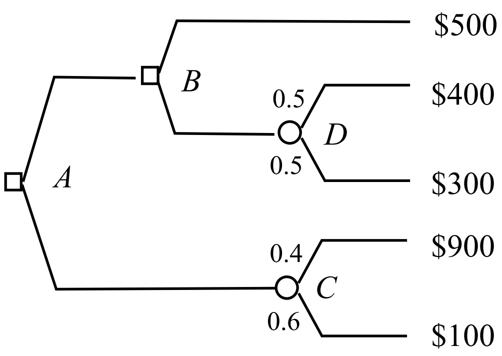


where a, b, c, and d are prospects in dollars. Let L_3 be the following compound deal:



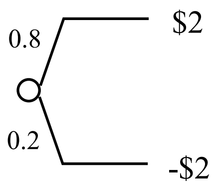
Show that the certainty equivalent of L_3 is the sum of the certain equivalents of L_1 and L_2.

P6.4 George is faced with the following decision problem:



If George has a constant risk tolerance of $1,000 for dollar amounts between -$1,000 and $2,500, what is his *preference probability* for decision ***A*** with respect to the outcomes $2,500 and -$1,000?

P6. 5 Susan follows the delta property. She is indifferent between accepting and rejecting the following free deal:

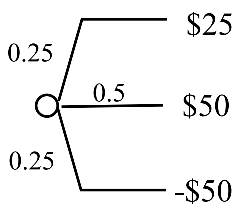


(***a***) What is Susan's risk tolerance?

(***b***) What is Susan's risk attitude?

(***c***) What's Susan utility function such that $u(\$0) = 0$ and $u(\$5) = 1$.

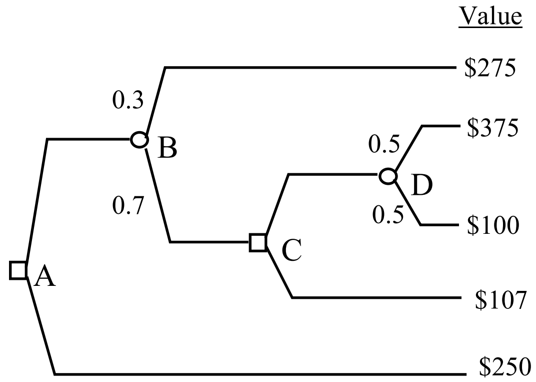
P6.6 Susan has the wealth utility function $u(w) = \frac{w^2}{2000}$, where $w \geq 0$, is in dollars. Her current wealth is worth $200, and she faces the following deal:



(***a***) What is Susan's personal indifference selling price for this deal?

(***b***) What is Susan's personal indifference buying price for this deal?

P6.7 Consider decision *A* (below) where the dollar values are winnings and the probabilities are the assessments of the deal's owner.



(***a***) Suppose Rex, whose utility function for dollars is $u(x) = 1 - e^{-\frac{x}{5000}}$ owns decision *A*.

i. What would be his certainty equivalent for decision *A*?

ii. What is the most Rex should pay for clairvoyance on *D* before making decision *A*?

iii. What is the most Rex should pay for clairvoyance on *D* before making decision *C* but after making decision *A*?

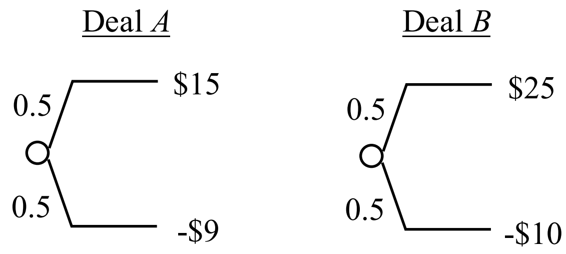
iv. What is the most Rex should pay for clairvoyance on *B* before making decision *A*?

(***b***) Paulina's utility function for dollars of total wealth is $u(x) = \dfrac{x^2}{10{,}000}$. Her current wealth is $1,000. Suppose she owned decision *A*.

i. What would be her certainty equivalent for decision *A*?

ii. Should she pay $15 for clairvoyance on *D* before making decision *A*?

P6.8 Les is considering giving away deals *A* and *B*.



(***a***) Stan is risk neutral and has only $20 in total wealth.

i. Suppose Les offers Deal *A*. What is Stan's certainty equivalent for Deal A? Should he accept Deal *A*?

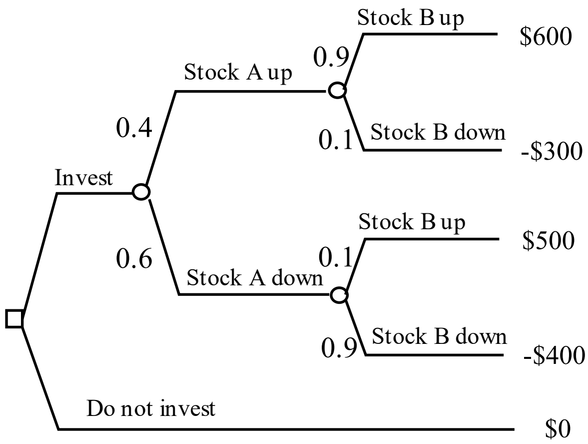
ii. Suppose that Les offers Stan Deal *B* instead. What is Stan's certainty equivalent for Deal B? Should he accept deal *B*?

iii. Now suppose that Les offers Stan both Deal *A* and Deal *B* as a bundle. What is Stan's certainty equivalent for possessing both deals?

(***b***) Burke has the utility function $u(x) = ln\ x$, where x is *total wealth*. Burke also has an initial total wealth of $20 when he meets Les. Answer questions (*i*), (*ii*), and (*iii*) above for Burke.

(***c***) How does the sum of Stan's certainty equivalents for Deals *A* and *B* compare to his certainty equivalent for the bundle? How about Burke? Explain.

P6.9 Kay faces the decision problem as shown below. Her utility function is $u(x) = 2 - 9^{\frac{-x}{1000}}$.



(***a***) What is the utility value for each alternative (Invest versus Not Invest)?

(***b***) What is the expected value of the dollar measures for each alternative?

(***c***) What is Kay's best decision in this circumstance? What is her certainty equivalent for the deal?

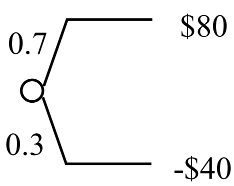
(***d***) Which of the first two answers, (*a*) or (*b*), did you use to answer part (*c*)? Why? What is wrong with using the other one?

(***e***) Should Kay pay \$10 for clairvoyance on the performance of Stock *A*?

(***f***) Should Kay pay \$10 for clairvoyance on the performance of Stock *B*?

(***g***) Find Kay's value of clairvoyance on both the performance of Stocks *A* and *B* together.

P6.10 Daniel's wealth utility function is $u(w) = \begin{cases} w+10, & w \geq 0 \\ 6w+10, & w < 0 \end{cases}$, where w is in dollars. He owns \$20 and the following deal:



Mad Dog has the utility function $u(x) = 2^{\frac{x}{50}}$ and he currently has \$75.

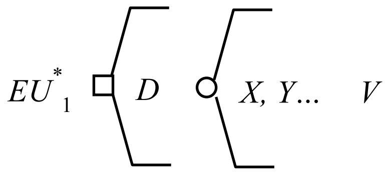
(***a***) What is Daniel's certainty equivalent for the deal?

(***b***) How much money can you make by buying Daniel's deal and selling it to Mad Dog?

Appendix A Proof for Value of Information Computation Procedure under Delta Property or Constant Absolute Risk Aversion

- We want to prove that Delta Property ⇒ EVI = CE with free info – CE with no info.

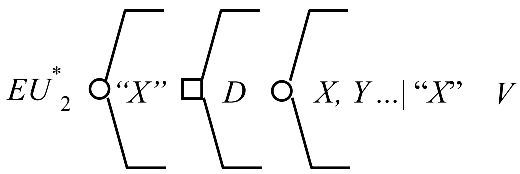
- Consider a decision problem as shown below:



- Let the maximum expected utility be EU^*_1, and the corresponding certainty equivalent be CE_1.

- Hence $u(CE_1) = EU^*_1$.

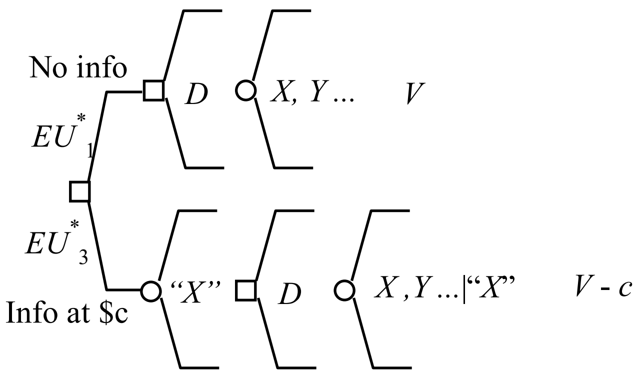
- Suppose we are interested in the value of information on X. Consider the decision model with free information on X:



- Let the maximum expected utility be EU^*_2, and the corresponding certainty equivalent be CE_2.

- Hence $u(CE_2) = EU^*_2$

- Suppose we use the indifference method to compute the VoI for X, and let c = cost of info:



- The difference between "Free info" and "Info at $$c$", is just a Δ-shift with $\Delta = -c$.

- Delta property ⇒ EU["Info at $$c$"] = $EU^*_3 = u(CE_2 - c)$

- The VoI for X is the value of c at the point of indifference between "Free info" and "Info at $$c$".

- Hence $u(CE_2 - VoI(X)) = u(CE_1)$
  ⇒ $CE_2 - VoI(X) = CE_1$
  ⇒ $VoI(X) = CE_2 - CE_1$
  ⇒ $VoI(X)$ = CE with free info – CE with no info.