Chapter 4 Basic Decision Analysis

"When you have collected all the facts and fears and made your decision, turn off all your fears and go ahead!"

George S. Patton

4.1 Introduction to the Party Problem

- We will learn about various basic concepts in Decision Analysis using a simple yet rich example called the Party Problem.

- The Party Problem was first introduced by Ronald Howard. It may be stated as follows:

  - Kim wants to decide on the location to hold a party tomorrow.

  - Ideally, she would like to hold the party outdoors in the garden, but she is concerned about the weather.

  - If the party is held outdoors, it would be a disaster if it rains.

  - On the other hand, if she holds the party indoors, it would be weather-proof, but she would not enjoy the party as much as she would if the party was outdoors.

  - She could also “compromise” by holding the party partially outdoors under the porch. She would be partially “shielded” from bad weather, but would not enjoy it as much if it were fully outdoors.

  - Considering all her choices, the uncertainty about the weather, and her happiness for different outcomes, where should she hold her birthday party?

- In summary, there is one decision to be made subject to one uncertainty:

  - **Decision:** Party location: Indoors, Porch, or Outdoors.

  - **Uncertainty:** Weather: Sunny or Rainy.

- Note that the event **Weather** needs to pass the clarity test. The two outcomes sunny and rainy need to be precisely defined to avoid ambiguity.

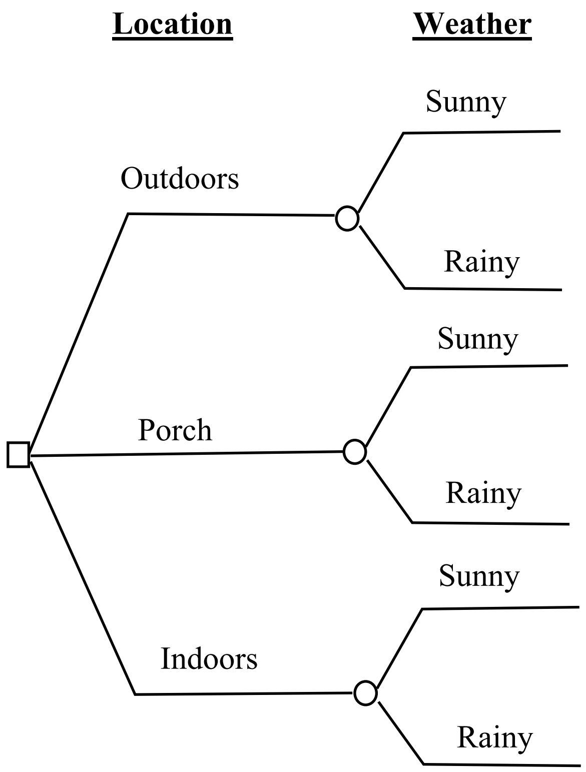
4.2 Solving the Party Problem

4.2.1 The Decision Tree

- The first step in solving a decision problem is to structure it and use an appropriate tool to represent the decision model.

- We use the classical **Decision Tree** to represent the decision model here. A more advanced tool called Influence Diagram will be introduced in Chapter 5.

- The Decision Tree for the Party Problem:



- There are two types of nodes in a decision tree:

  1. **Decision nodes** are indicated by squares. These are points on the tree where the decision maker is free to make a choice, i.e., pick one of the branches.

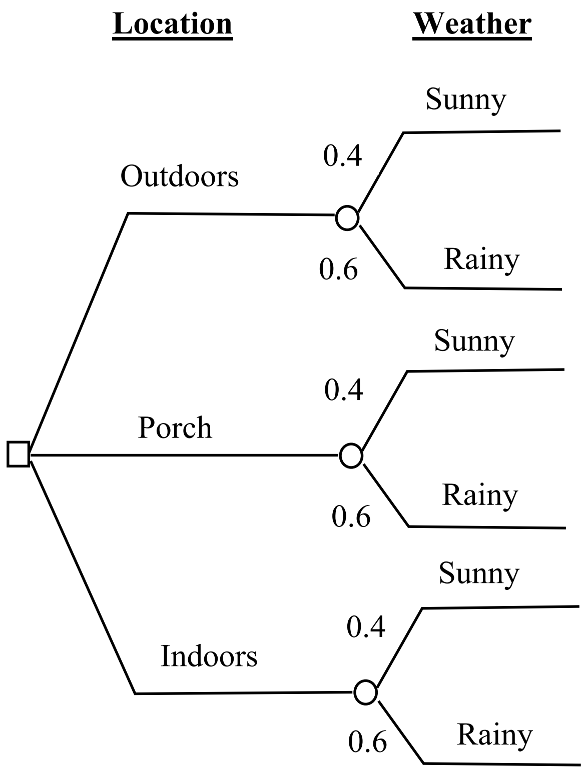
  2. **Chance nodes** are indicated by circles. These are points on the tree where the decision maker does not have control over the outcome that is to happen.

4.2.2 Assigning Probabilities to the Decision Tree

- After reviewing all available information about the weather tomorrow, we assign the following probabilities for the two weather outcomes:

  - Probability that the weather tomorrow will be sunny = 0.4
  - Probability that the weather tomorrow will be rainy = 0.6

- The Decision Tree with Probabilities:



- We may solve the Party Problem using the following methods:

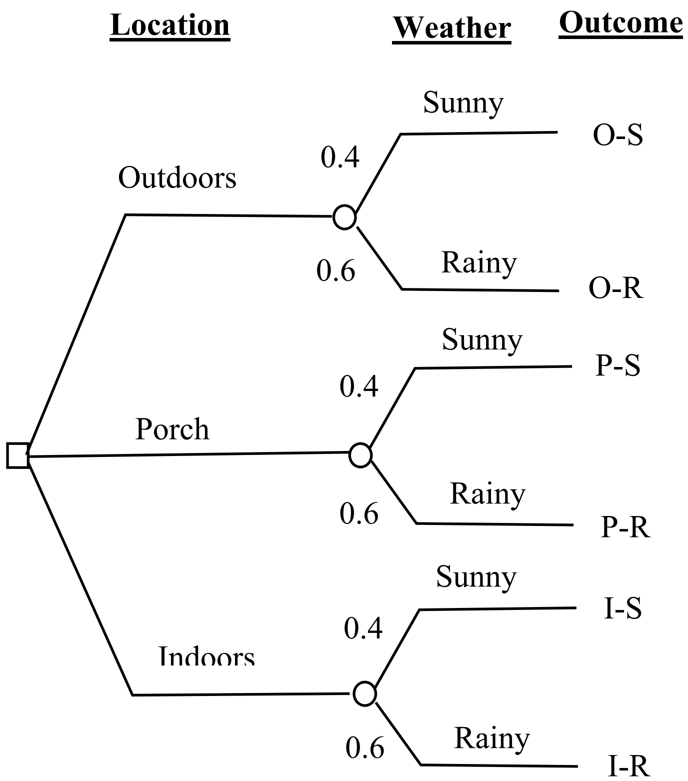
  1. Directly assess the utility values of all outcomes. This method relies only on decision theory based on the five rules of actional thought.

  2. Assign equivalent dollar values to all outcomes and then access their utility values.

  3. Assign equivalent dollar values to outcomes and use a utility function over dollar values that was independently accessed.

4.2.3 Direct Assessment of Utility Values

- We note that there are six possible outcomes whose utility values must be assessed. We label the outcomes as follows:



- By the order rules, we may rank the outcomes by their desirability or preferences:

Notation	Outcome	Ordering
O-S	Outdoors-Sunny	Best
P-S	Porch-Sunny	
I-R	Indoors-Rainy	
I-S	Indoors-Sunny	
P-R	Porch-Rainy	
O-R	Outdoors-Rainy	Worst

- To determine the utility values of the outcomes, we let
  - u(Best) = u(O-S) = 1
  - u(Worst) = u(O-R) = 0

- Using the *Continuity Rule*, we may determine the utility of the other four intermediate outcomes, namely u(P-S), u(I-R), u(I-S), and u(P-R).

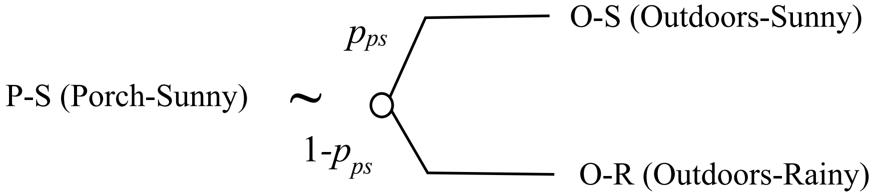
- Note that based on the ranking of the desirability of the outcomes:

$$1 = u(\text{O-S}) \geq u(\text{P-S}) \geq u(\text{I-R}) \geq u(\text{I-S}) \geq u(\text{P-R}) \geq u(\text{O-R}) = 0$$

Assessing the Utility of Porch-Sunny Outcome

- The *Continuity Rule* is applied as follows:

- Kim is asked at what value of p_{ps} would she be indifferent between being in the Porch-Sunny (P-S) state *for sure* and facing a risky situation where with probability p_{ps} she would be "upgraded" to best outcome Outdoors-Sunny (O-S), and with probability $(1 - p_{ps})$ she would be "downgraded" to the worst outcome Outdoors-Rainy (O-R).



- ***Definition**:* The value of p_{ps} is known as the **Preference Probability** for the Porch-Sunny (P-S) outcome with respect to the outcome O-S (best) and the outcome O-R (worst).

- At the point of indifference, the expected utility of both sides must be equal. Hence

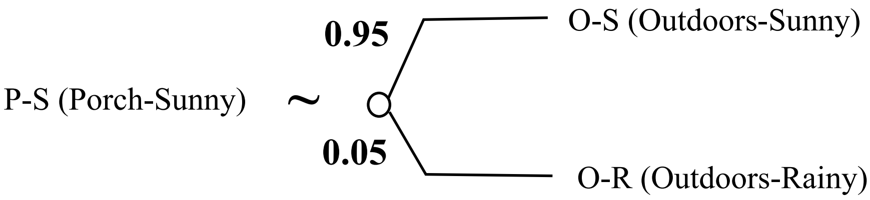
$$u(\text{P-S}) = p_{ps}\, u(\text{O-S}) + (1 - p_{ps})\, u(\text{O-R})$$

- Using the boundary values u(O-S) = 1 and u(O-R) = 0, we obtain

$$u(\text{P-S}) = p_{ps}.$$

- Hence the **Utility of an Outcome** is equal to its **Preference Probability** with respect to the best outcome (with utility=1) and the worst outcome (with utility=0).

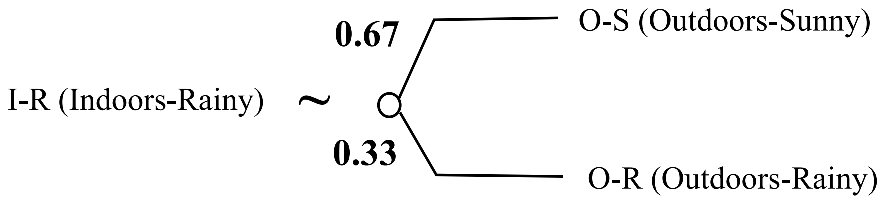
- Suppose after thinking about her preference for the Porch-Sunny outcome with respect to the best outcome (O-S) and worst outcome (O-R), Kim indicates that $p_{ps} = 0.95$, that is:



$$\Rightarrow u(\text{P-S}) = 0.95$$

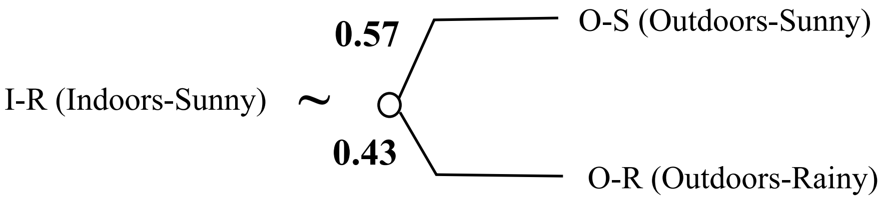
- Then u(P-S) = 0.95

- The utilities for all the other outcomes can also be determined similarly:
- For I-R (Indoors-Rainy) outcome:



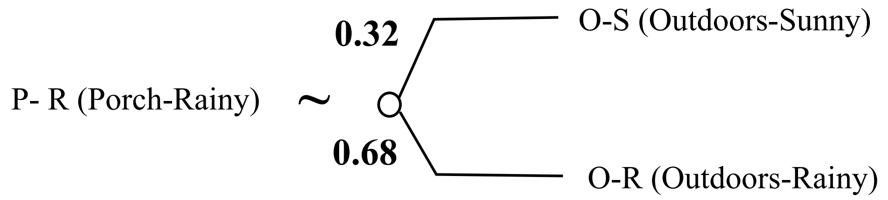
$$\Rightarrow u(\text{I-R}) = 0.67$$

- For I-S (Indoors-Sunny) outcome:



$$\Rightarrow u(\text{I-S}) = 0.57$$

- For P-R (Porch-Rainy) outcome:



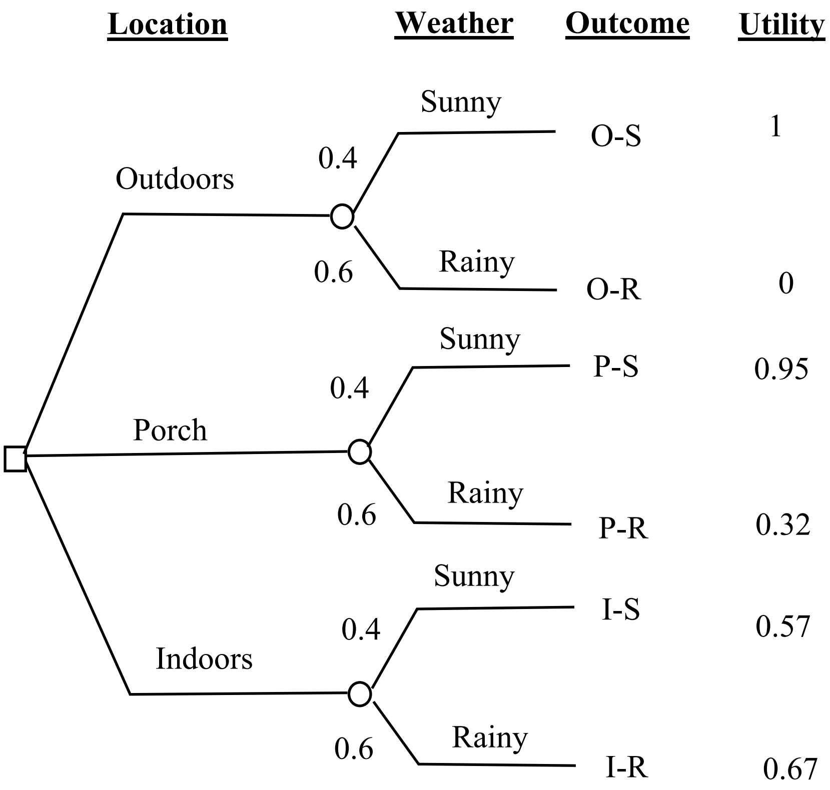
$$\Rightarrow u(\text{P-R}) = 0.32$$

- Summary of Results:

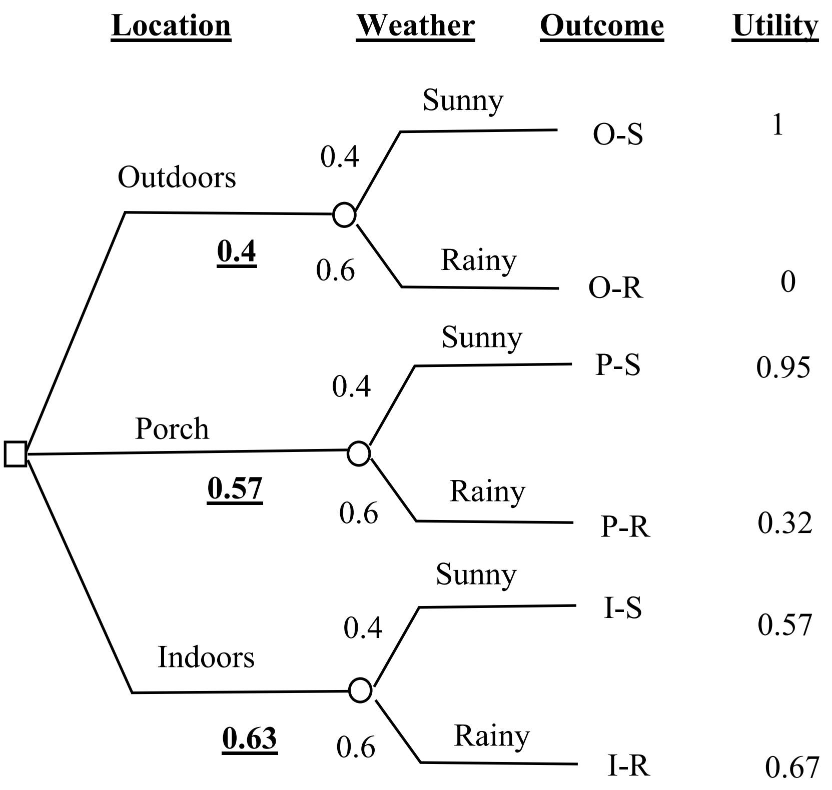
Notation	Outcome	Utility
O-S	Outdoors-Sunny	1
P-S	Porch-Sunny	0.95
I-R	Indoors-Rainy	0.67
I-S	Indoors-Sunny	0.57
P-R	Porch-Rainy	0.32
O-R	Outdoors-Rainy	0

Computing Expected Utility of Alternatives

- We insert the utility values for the outcomes in the decision tree:



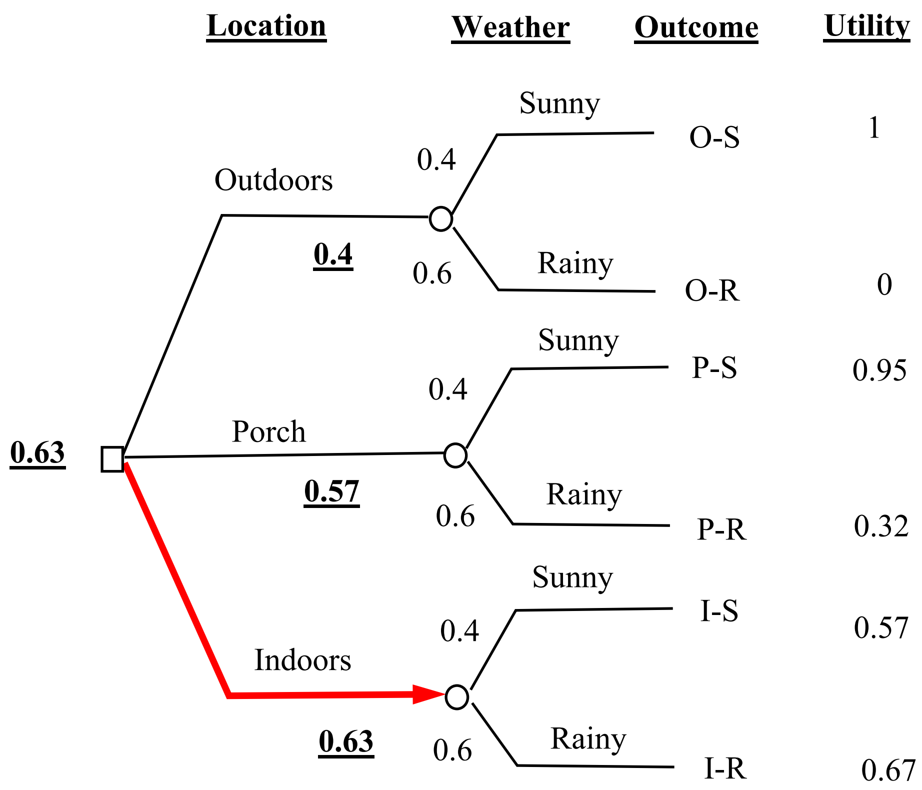
- We compute the expected utility values for each alternative as follows:



- The Expected Utilities for the three alternatives are computed:

  - EU(Outdoors) = 0.4 (1) + 0.6 (0) = 0.40
  - EU(Porch) = 0.4 (0.95) + 0.6 (0.32) = 0.57
  - EU(Indoors) = 0.4 (0.57) + 0.6 (0.67) = 0.63

- The best alternative is the one with the highest expected utility.



- Hence Kim's **best decision** is to hold the party **Indoors** which has a Maximum Expected Utility of 0.63.

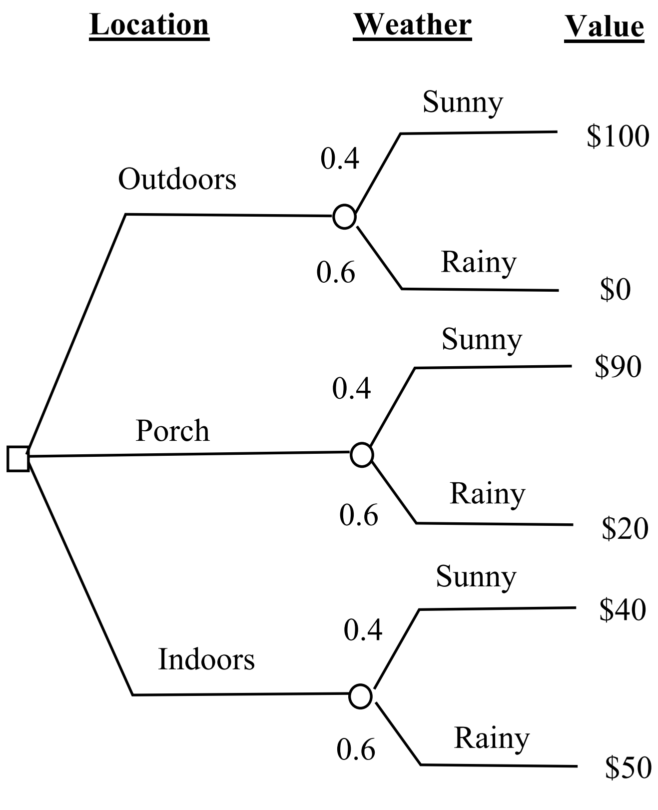
4.2.4 Assigning Equivalent Dollar Values to Outcomes

- Instead of directly determining the utility of each outcome, another way to solve the Party Problem is to first determine the **Equivalent Monetary** or **Dollar Values** for each outcome, and then convert these dollar values to their utility values.

- Suppose Kim assigns the following equivalent dollar values for each outcome:

Outcome	Equivalent Dollar Value
Outdoors-Sunny	\$100
Porch-Sunny	\$90
Indoors-Rainy	\$50
Indoors-Sunny	\$40
Porch-Rainy	\$20
Outdoors-Rainy	\$0

- We may insert these equivalent dollar values at the outcome nodes of the decision tree:

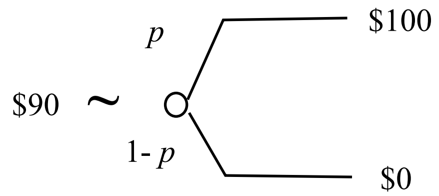


- Before we can roll back the decision tree, we need to determine Kim's utility values of each of the outcomes.

- This can be assessed using the *Continuity Rule*.

- Since the best outcome is equivalent to \$100, and the worst outcome is equivalent to \$0, we let $u(\$100) = 1$ and $u(\$0) = 0$.

- To access the utility of \$90, Kim is asked for what value of p would she be indifferent between receiving \$90 for sure vs. a risky deal with probability p of receiving \$100 and probability $1 - p$ of receiving \$0.



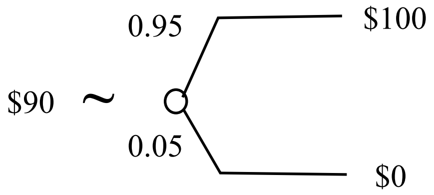
- Suppose Kim assign $p = 0.95$, then

$$u(\$90) = 0.95\, u(\$100) + (1 - 0.95)\, u(\$0)$$

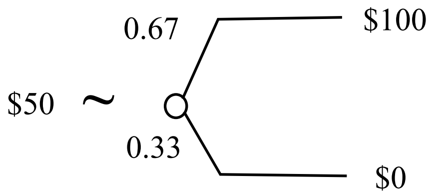
- Using the boundary conditions $u(\$100) = 1$ and $u(\$0) = 0$

$$u(\$90) = 0.95$$

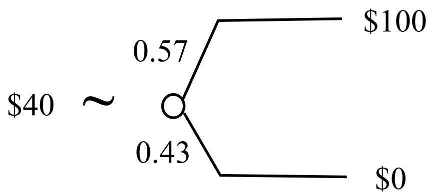
- The following results are obtained for all the dollar values:



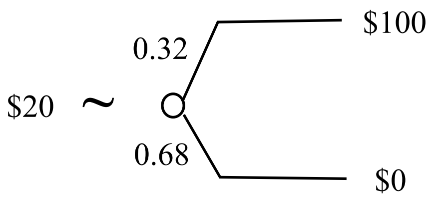
$\Rightarrow \; u(\$90) = 0.95$



$\Rightarrow \; u(\$50) = 0.67$



$\Rightarrow \; u(\$40) = 0.57$

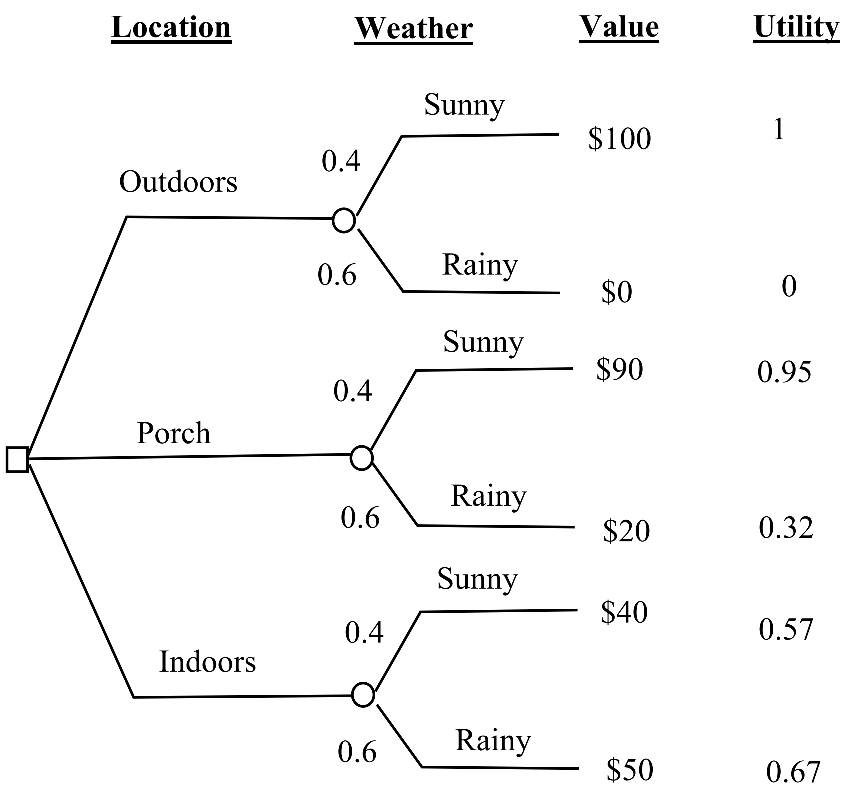


$\Rightarrow \; u(\$20) = 0.32$

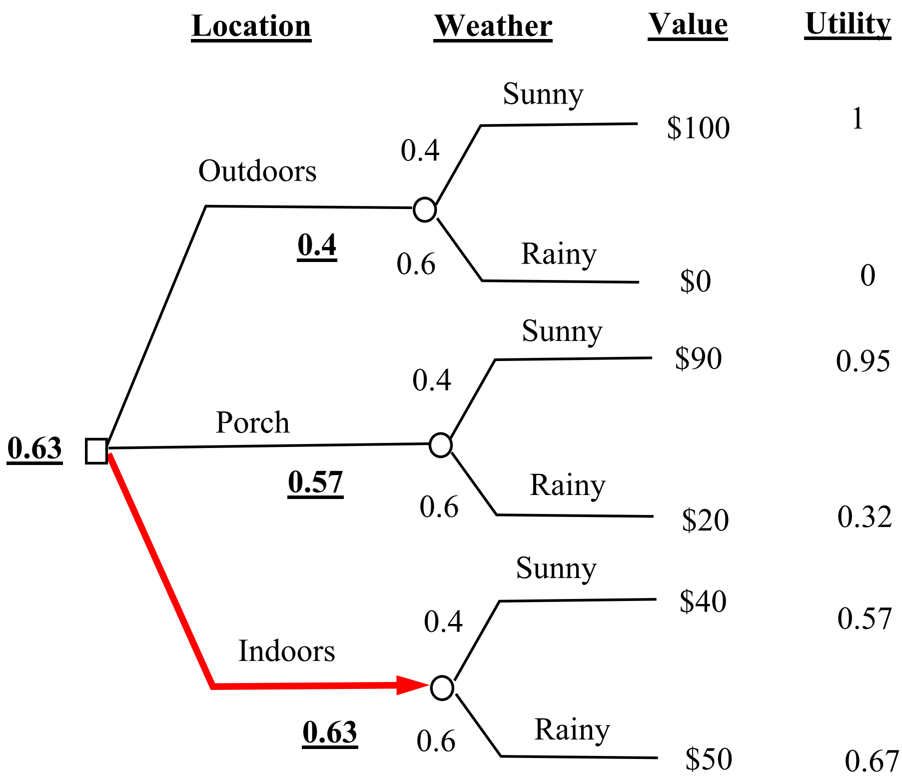
- Note that these utility values must be consistent with the values that were obtained directly in the previous section from the outcomes:

Outcome	Equivalent Dollar Value	Utility
Outdoors-Sunny	$100	1.00
Porch-Sunny	$90	0.95
Indoors-Rainy	$50	0.67
Indoors-Sunny	$40	0.57
Porch-Rainy	$20	0.32
Outdoors-Rainy	$0	0.00

- Including the utilities of the dollar values to the decision tree:



- Rolling back the tree using the utilities

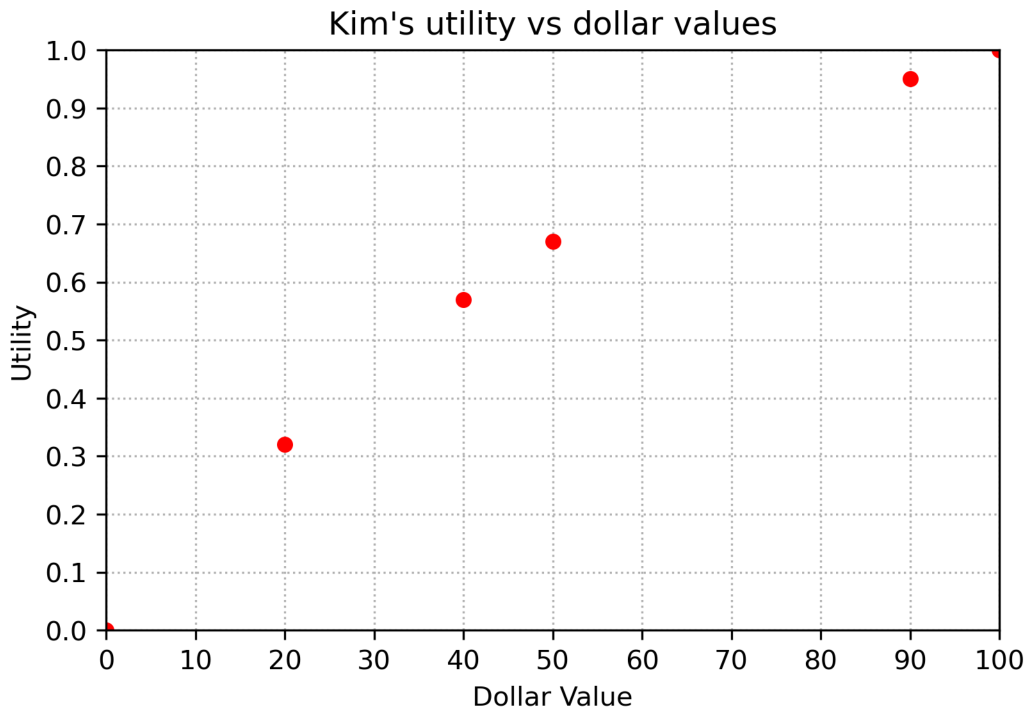


- Kim's best decision is again "Indoors" with a maximum expected utility of 0.63.

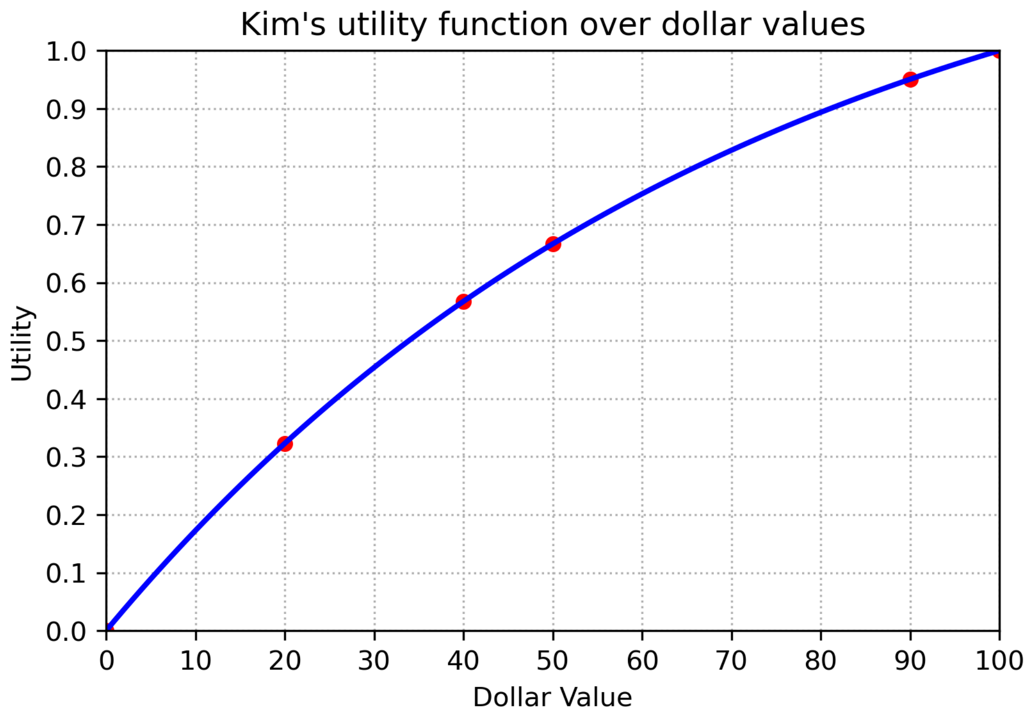
4.2.5 Using Utility Function over Dollar Values

Constructing Kim's Utility Function over Dollar Values

- If we plot Kim's utility values against the equivalent dollar values, we obtain the following:



- By the *Continuity Rule*, a utility value exists for all x between 0 and 100. Hence we can construct Kim's utility function by joining up the given points.



Notes

- In this example, we first determine the equivalent dollar values for each outcome and then determine the utility of these dollar values to construct the utility function. This process can be quite tedious if the number of outcomes is large.

- In practice, we normally directly and independently determine the utility function over a range of dollar values by assuming some functional form (e.g. exponential, quadratic) for the utility function, and then estimate the parameters of the assumed function. See Chapter 6 for details.

Procedure for Solving a Decision Problem with a Utility Function over Dollar Values

- The steps are as follows:

  1. Draw a decision tree and fill in the probabilities.

  2. For each outcome, assign an equivalent dollar value.

  3. Determine the utility of each outcome using the utility function over Dollar Values which may be assessed independently.

  4. Roll back the decision tree from the leaves to the root node as follows:

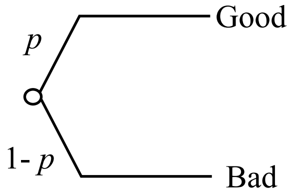
     a. At each chance node, compute the expected utility.

     b. At each decision node, select the branch with the maximum expected utility.

  5. The best decision is the one with maximum expected utility.

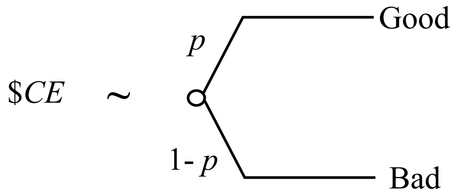
4.2.6 Certainty Equivalent of Risky Deals

- Suppose a person is faced with an uncertain or risky prospect such that with probability p, he may receive a good outcome, and with probability $(1 - p)$, a bad outcome.



- He can insure himself against the bad outcome by trading the risky deal for a **sure sum of money.**

- The **Certainty Equivalent** (*CE*) of an uncertain prospect is the amount in which a person is just *indifferent* between *receiving it for sure* or *retaining the risky prospec*t and facing the uncertain outcomes for good or for bad.



- The *CE* of a deal is also the owner's **Personal Indifferent Selling Price** (PISP) of the deal.

- *CE* of a deal can be determined by first finding the expected utility of the uncertain deal, and then finding the equivalent dollar value that has the same utility. That is

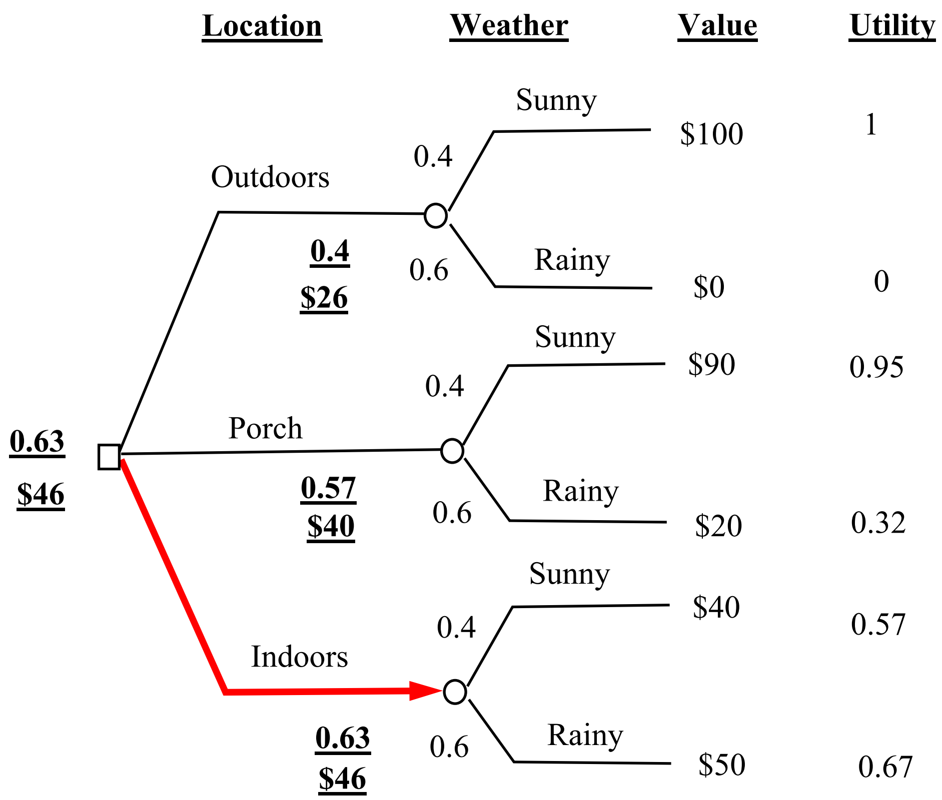
$$u(CE \text{ of a Deal}) = \text{Expected Utility of Deal}$$

$$CE = u^{-1}(\text{Expected utility of Deal})$$

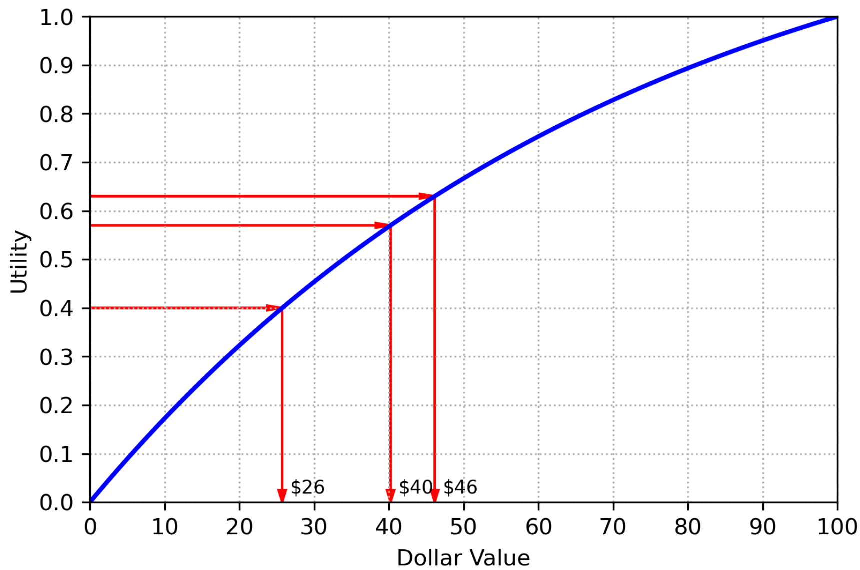
- Hence to find the *CE* of a deal, we first compute its expected utility and then take its inverse to convert it back into an equivalent dollar value.

Finding Certainty the Equivalent of each Alternative in the Party Problem

- The certainty equivalent of each choice in the Party Problem may be computed as follows:



- The certainty equivalents of each alternative are determined as follows:



1. Outdoors: Expected utility = 0.4.
   Certainty equivalent = $u^{-1}(0.4)$ = \$26

2. Porch: Expected utility = 0.57.
   Certainty equivalent = $u^{-1}(0.57)$ = \$40

3. Indoors: Expected utility = 0.63.
   Certainty equivalent = $u^{-1}(0.63)$ = \$46

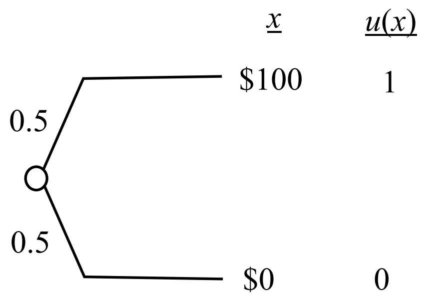
- Note that the best decision is also the one with the highest certainty equivalent. Why?

4.2.7 Using the Utility Function for Other Decision Problems

- A utility function is normally assessed independently and may be used in all decision situations faced by the decision maker.

- For example, we can help Kim determine her personal indifferent selling price for the coin tossing game with pay-offs \$0 and \$100.

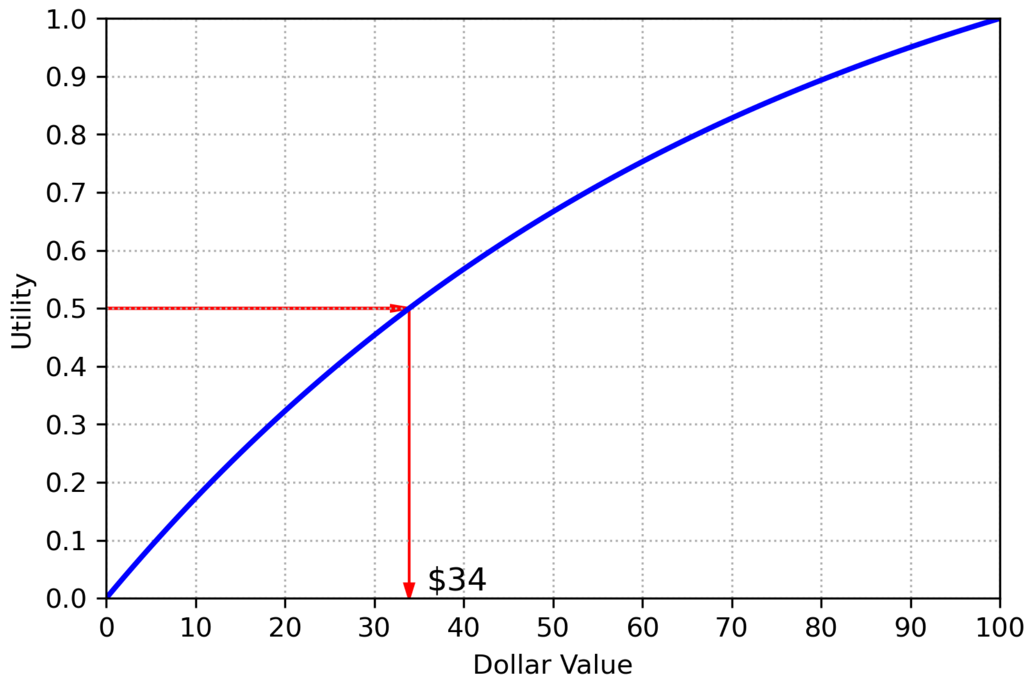
- Kim's uncertain outcomes for the game may be represented by the following tree:



- Kim's expected utility for the game = 0.5 (1) + 0.5 (0) = 05.

- If Kim owns the game, then her personal indifferent selling price is the certainty equivalent of the deal which can be determined as follows:

- Kim's personal indifferent selling price for the \$0-\$100 coin tossing game = $u^{-1}(0.5)$ = \$34.



4.2.8 A Utility Function is Unique Up to a Positive Linear Transformation

- Suppose we have a utility function $u_1(x)$ where x is the equivalent dollar value.

- Using $u_1(x)$, we may find the optimal decision as well as the certainty equivalents for each alternative.

- Consider the utility function $u_2(x) = a + b\, u_1(x)$ where a and b are constants, and $b > 0$. $u_2(x)$ is a positive linear transformation of $u_1(x)$, and vice-versa.

- We will show that the same optimal decision and certainty equivalents for all alternatives will be obtained if $u_2(x)$ is used in place of $u_1(x)$.

- First, the optimal decision is preserved because:

  (a) Using $u_1(x)$, the Expected Utility is

$$EU_1 = \sum_x p(x) u_1(x)$$

  (b) Using $u_2(x)$, the Expected Utility is

$$\begin{aligned} EU_2 &= \sum_x p(x) u_2(x) \\ &= \sum_x p(x)[a + bu_1(x)] \\ &= a\sum_x p(x) + b\sum_x p(x)u_1(x) \\ &= a + bEU_1 \end{aligned}$$

  Since $b > 0$, it follows that the alternative with the maximum expected utility is preserved when we switched the utility function from $u_1(x)$ to $u_2(x)$.

- The certainty equivalents for all alternatives are preserved because:

  (a) Using $u_1(x)$, the certainty equivalent is $CE_1 = u_1^{-1}(EU_1) \Rightarrow u_1(CE_1) = EU_1$.

(b) Using $u_2(x)$, the certainty equivalent is $CE_2 = u_2^{-1}(EU_2) \Rightarrow$

$$\begin{aligned} u_2(CE_2) &= EU_2 \\ &= a + bEU_1 \\ &= a + bu_1(CE_1) \\ &= u_2(CE_1) \end{aligned}$$

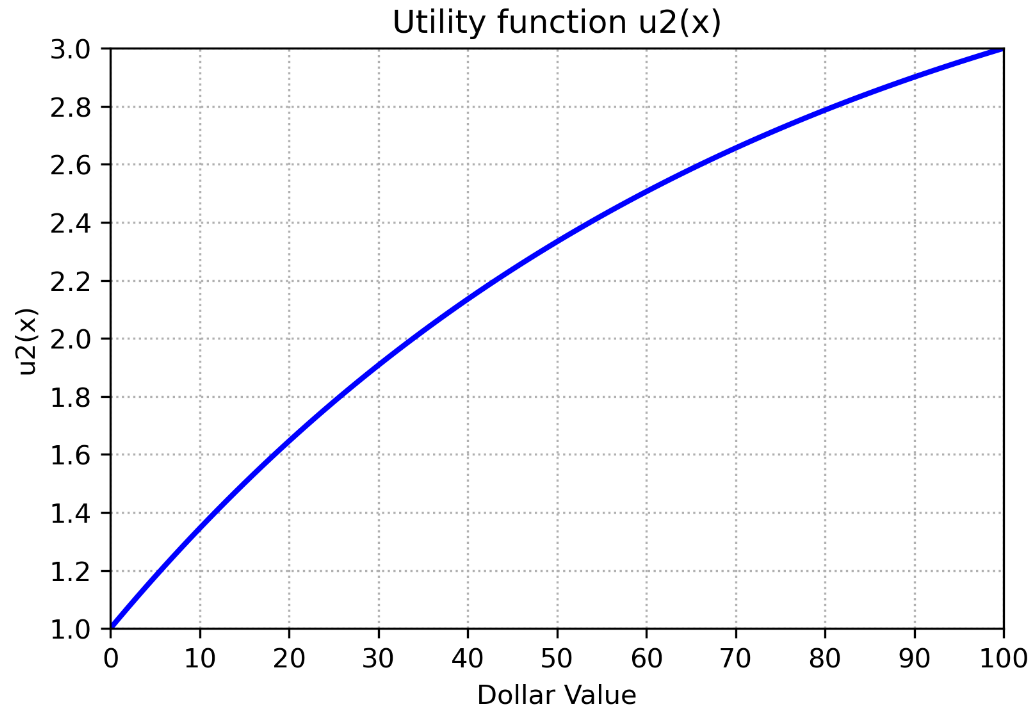
$\Rightarrow CE_2 = CE_1$.

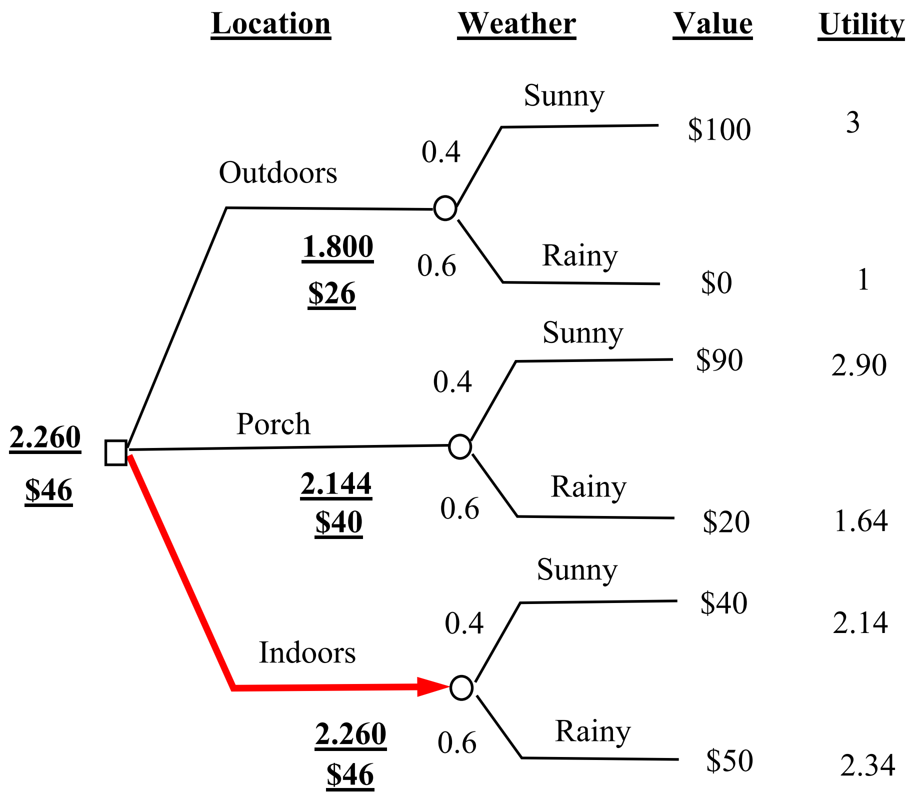
  Hence the certainty equivalents are preserved.

- Conclusion: Since the optimal decision and certainty equivalents are preserved when we transform $u_1(x)$ to $u_2(x)$, and vice versa, it follows that a utility function is **unique** only up to a **positive linear transformation.**

Example

- In the Party Problem, suppose we transform Kim's original utility function $u_1(x)$ to $u_2(x) = 1 + 2\,u_1(x)$:





- Notice that the optimal decision and certainty equivalents for all alternatives are preserved.

- The only difference is that now, the range of possible utility values is from 1 to 3, the best outcome for Kim (Outdoors-Sunny) has a utility value of 3 while the worst outcome (Outdoors-Rainy) has a utility value of 1.

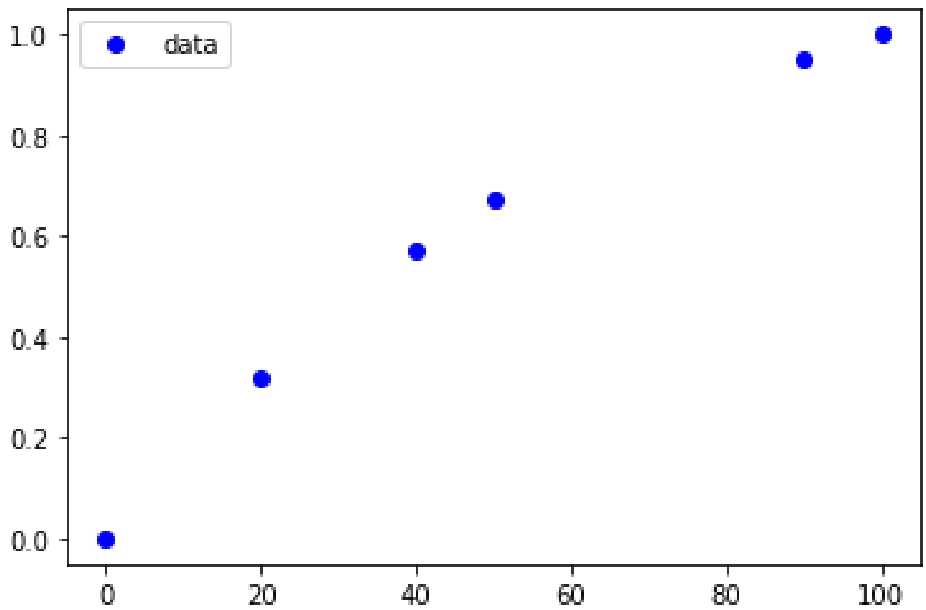
4.2.9 Fitting an Equation to Kim's Utility Function

- We can fit an equation representing Kim's utility function to the data using Python function: `scipy.optimize.curve_fit`.

In [1]:
```
""" Fitting an equation to Kim's Utility Function """
import numpy as np
from scipy.optimize import curve_fit
import matplotlib.pyplot as plt
```

In [2]:
```
# Data to fit
xdata = [0, 20, 40, 50, 90, 100]
udata = [0, 0.32, 0.57, 0.67, 0.95, 1]
```

In [3]:
```
# Visualize the data
fig, ax = plt.subplots()
ax.plot(xdata, udata, 'bo', label='data')
ax.legend()
plt.show()
```



- An exponential function looks like a good fit. Let's try it.

In [4]:
```
# Function to fit: u(x) = a - b exp(-x/r)
# To ensure u(0) = 0 is satisfied, we must have a = b.
# Hence u(x) = a (1 - exp(-x/r))
func = lambda x, r, a: a*(1 - np.exp(-x/r))
```

In [5]:
```
# Fit the function to data
popt, pcov = curve_fit(func, xdata, udata)
print(f"\nFitted parameters: {popt}")
```

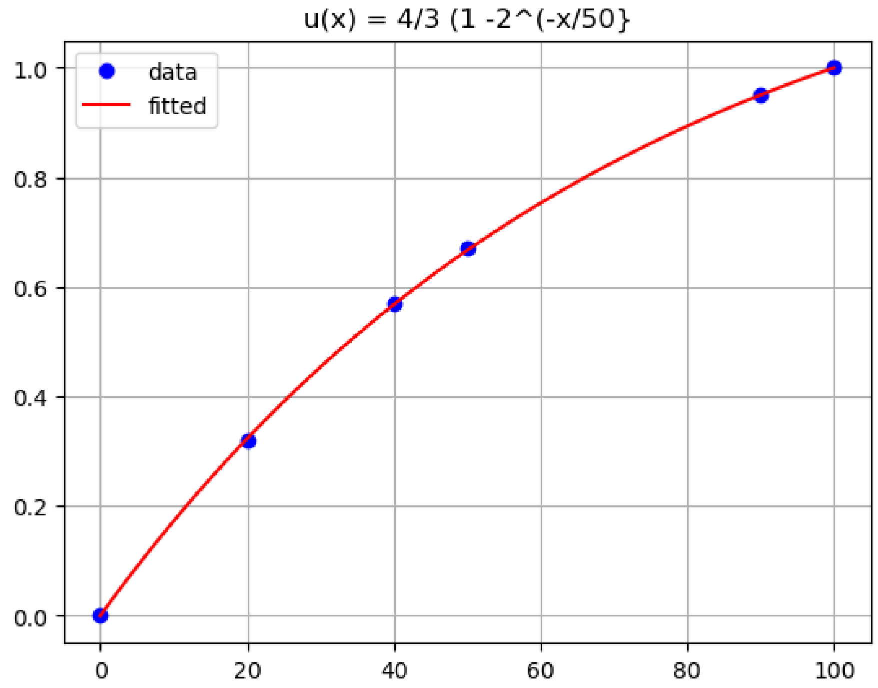
```
Fitted parameters: [71.62551915  1.32915241]
```

In [6]:
```
# Compare fitted results with the data
print("  x   data  fitted")
for x, u in zip(xdata, udata):
print(f"{x:3}   {u:.2f}  {func(x, *popt):.2f}")
```

```
  x   data  fitted
  0   0.00  0.00
 20   0.32  0.32
 40   0.57  0.57
 50   0.67  0.67
 90   0.95  0.95
100   1.00  1.00
```

In [7]:
```
# The fitted utility function can also be written as
#        u(x) = (4/3)*(1 - 2**(-x/50))

# Visualize the results
fig, ax = plt.subplots()
ax.plot(xdata, udata, 'bo', label='data')
x = np.linspace(0, 100, 101)
ufit = (4/3)*(1 - 2**(-x/50))
ax.plot(x, ufit, 'r-', label='fitted')
ax.set_title('u(x) = 4/3 (1 -2^(-x/50}')
ax.legend()
ax.grid()
plt.show()
```



- Hence Kim's utility function can be represented by

$$u(x) = \frac{4}{3}\left[1 - 2^{-x/50}\right]$$

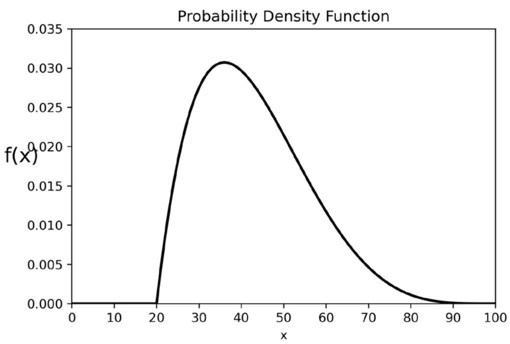
or

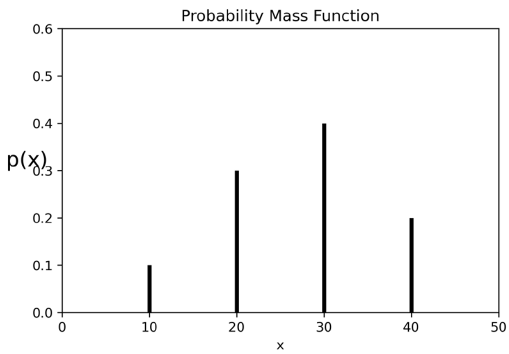
$$\mathrm{u}(x) = \frac{4}{3}\left[1 - e^{-x\ln(2)/50}\right]$$

4.3 Risk Profiles

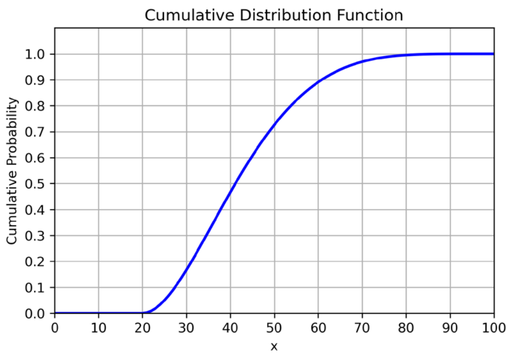
- The risk associated with taking a decision alternative can be described by its **Risk Profile**.
- A Risk Profile of an alternative is a plot of the probabilities associated with all the possible outcomes of the alternative. It not only indicates the probability associated with each possible outcome but also how spread out these outcomes are.
- A risk profile may be represented in any of the following forms:

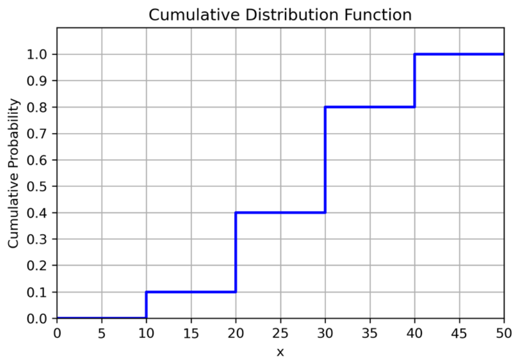
1. **Probability Density Function** (PDF) if the outcome values are continuous, or **Probability Mass Function** (PMF) if the outcome values are discrete.





2. **Cumulative Distribution Function** (CDF) of the outcome values.

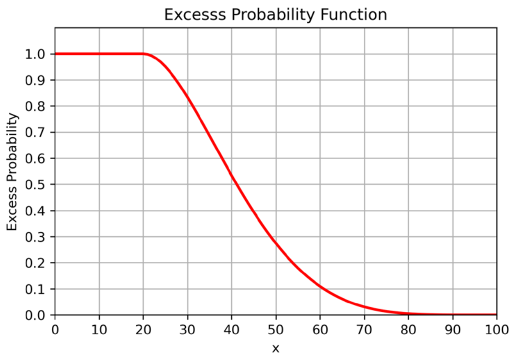


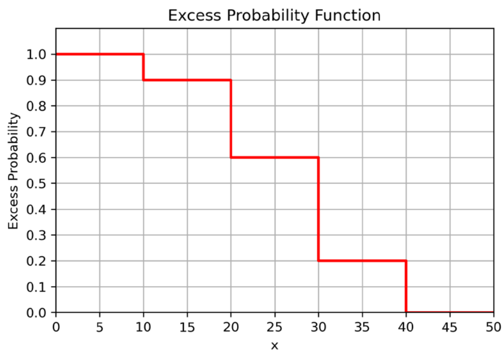


$$CDF = F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$CDF = F(x) = \sum_{t \le x} p(t)$$

3. **Excess Probability Function** (EPF) of the value of outcomes.



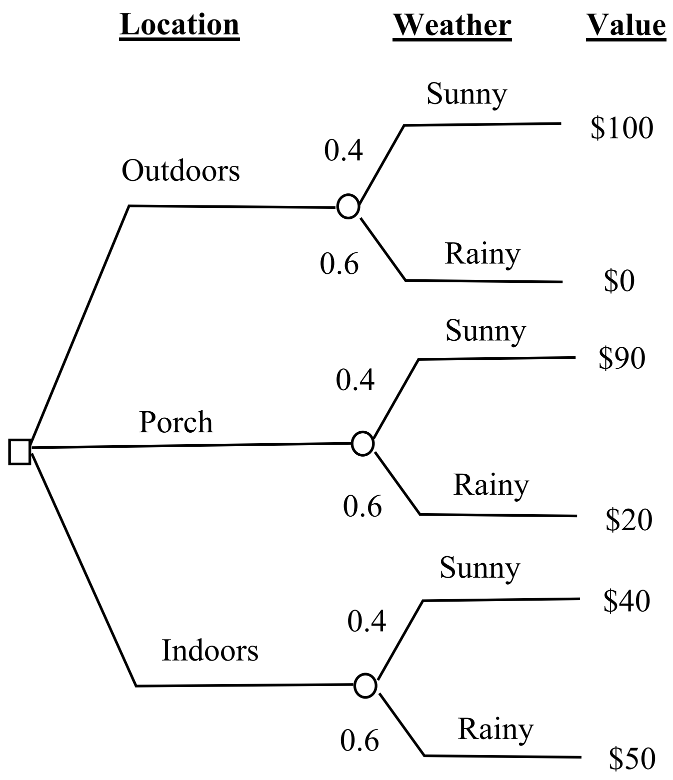


$$EPF = G(x) = \int_{x}^{+\infty} f_x(t)\,dt = 1 - F(x)$$

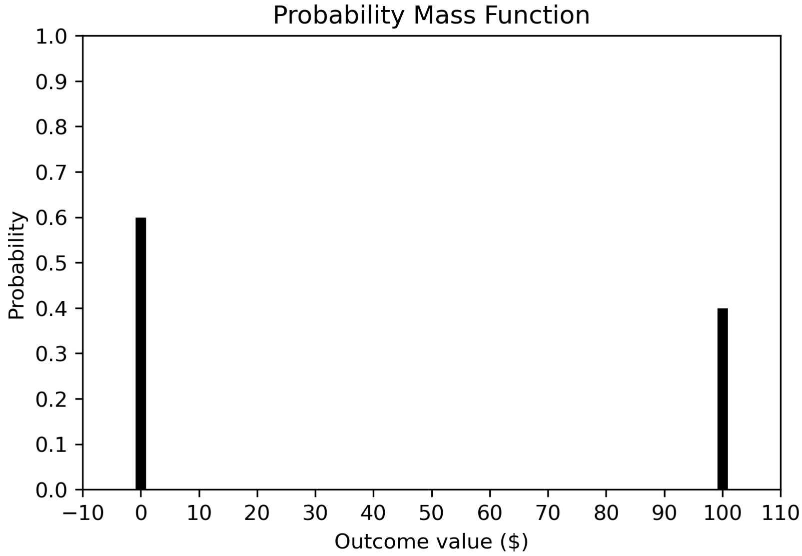
$$EPF = G(x) = \sum_{t \ge x} p(t)$$

Example

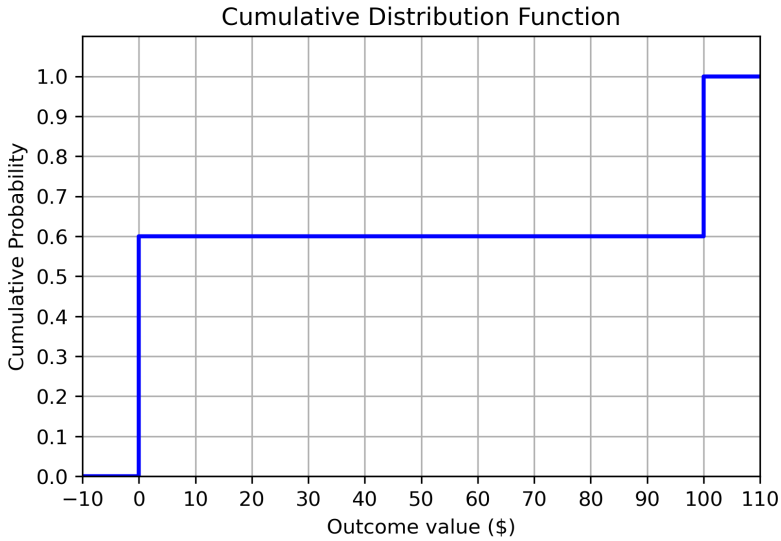
Kim's Party Problem:



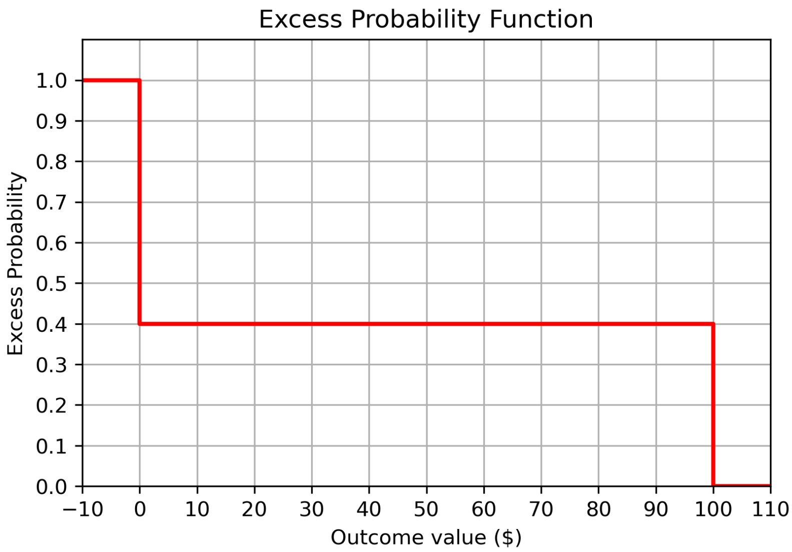
Risk Profiles for Outdoors Alternative



Probability Distribution for Outdoors

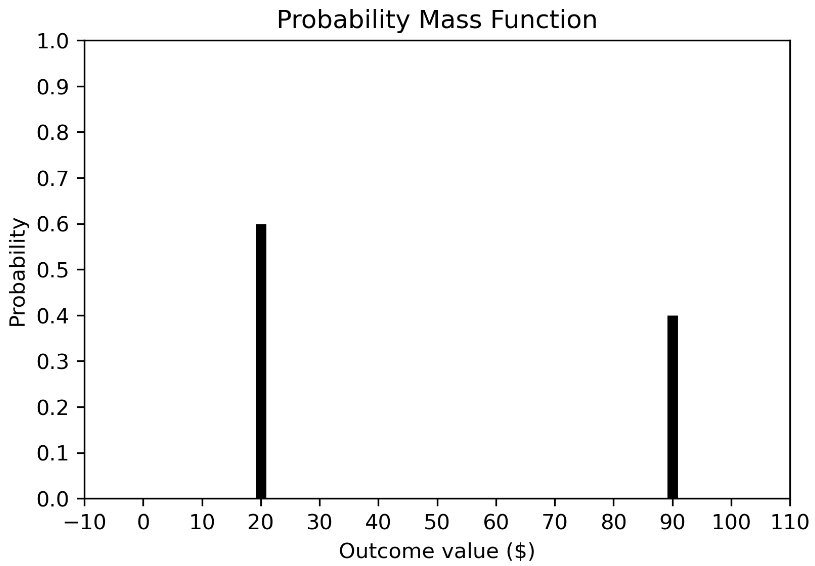


Cumulative Distribution Function for Outdoors

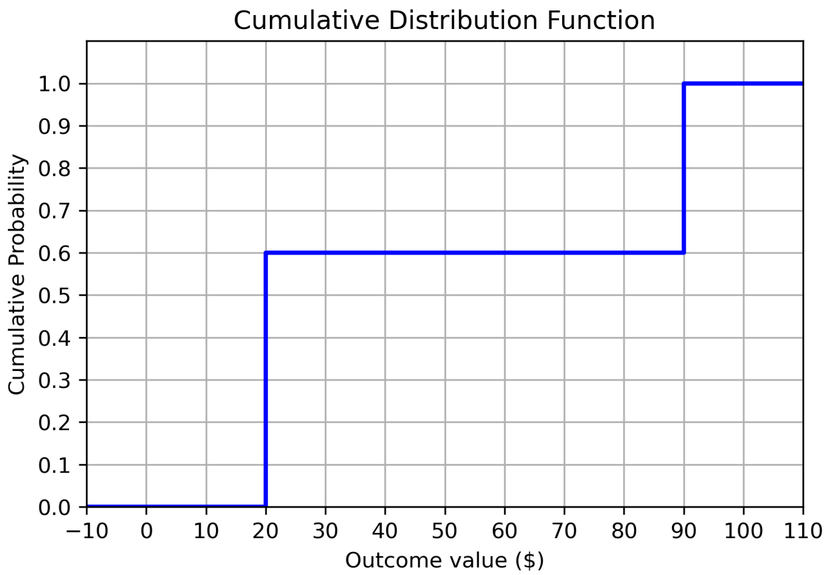


Excess Probability Function for Outdoors

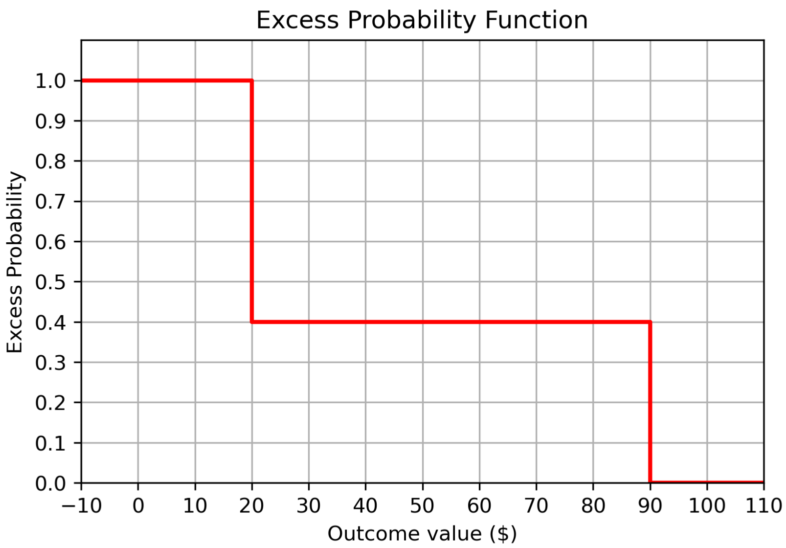
Risk Profiles for Porch Alternative



Probability Distribution for Porch

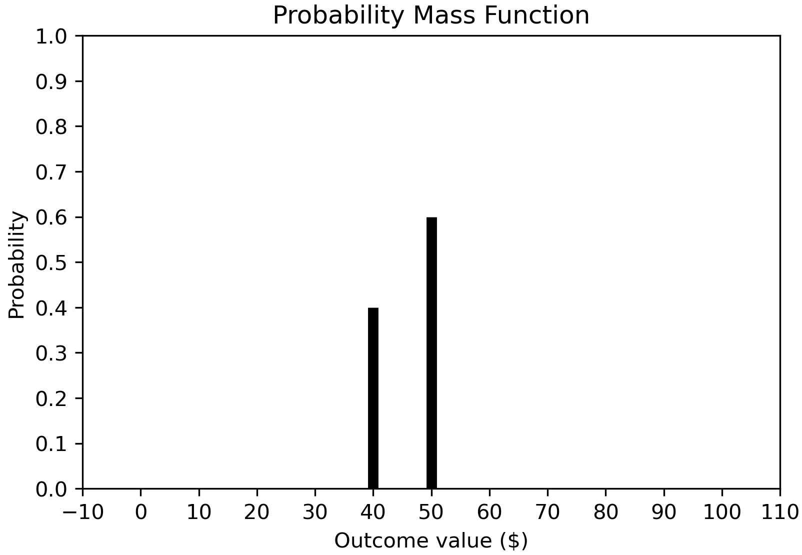


Cumulative Distribution Function for Porch

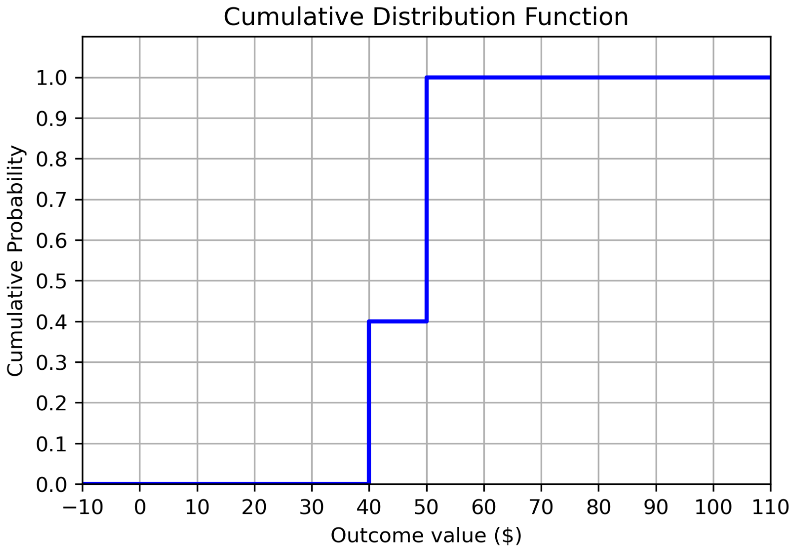


Excess Probability Function for Porch

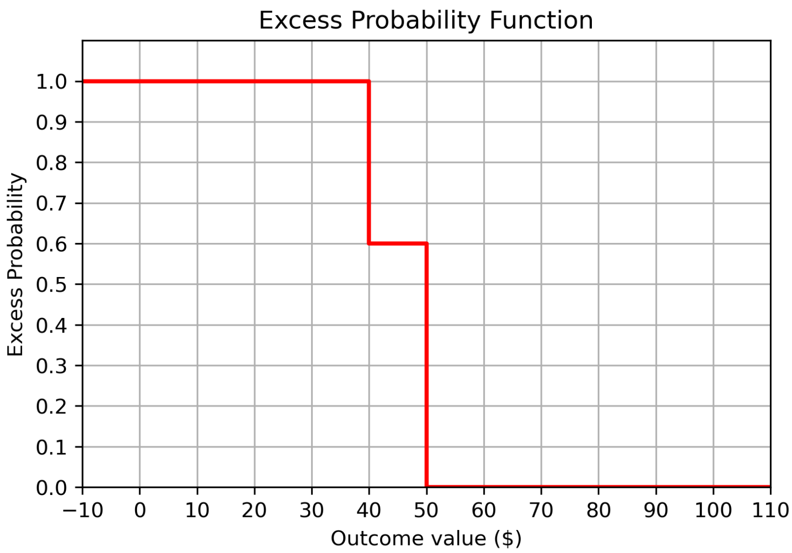
Risk Profiles for Indoors Alternative



Probability Distribution for Indoors



Cumulative Distribution Function for Indoors

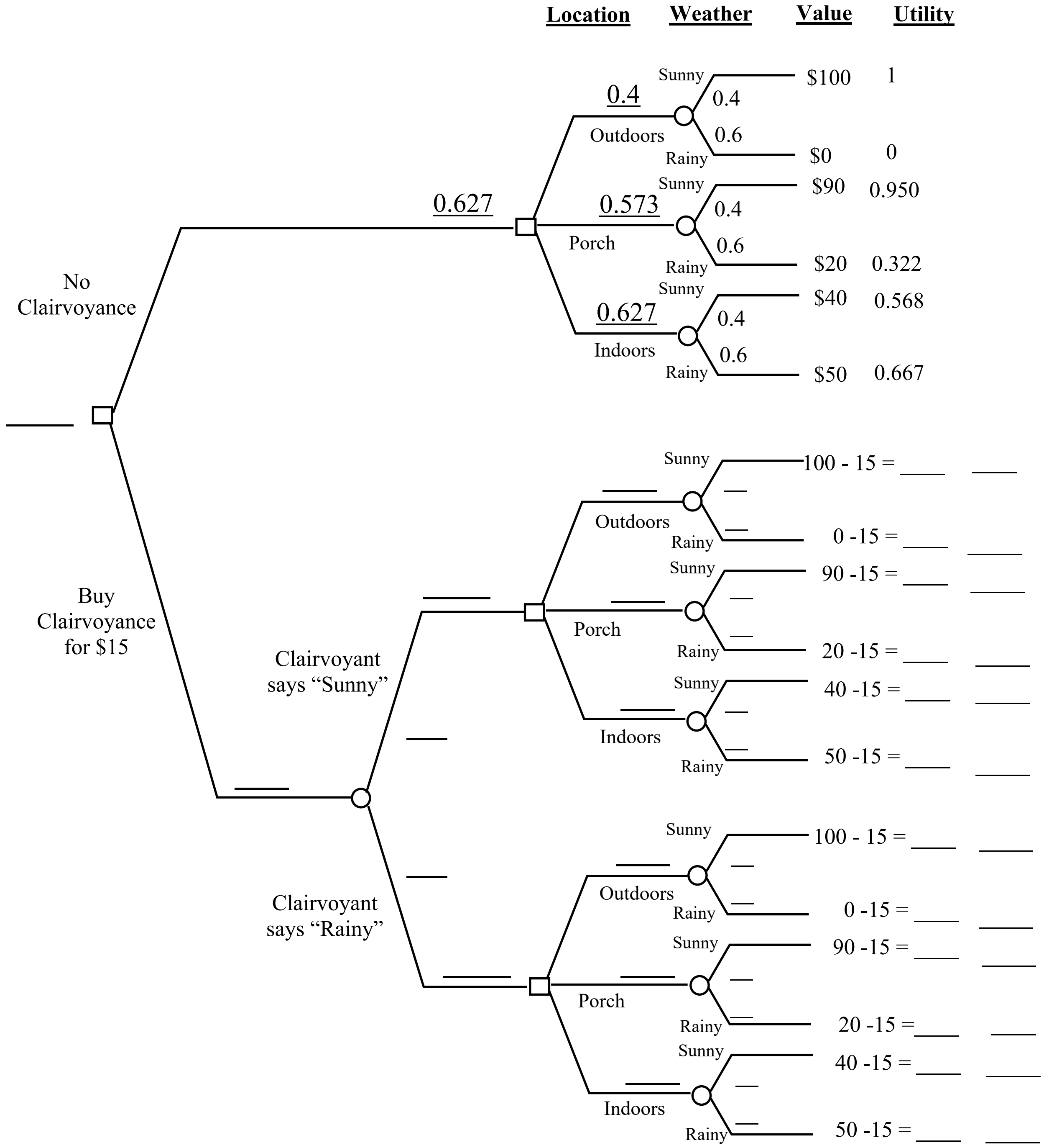


Excess Probability Function for Outdoors

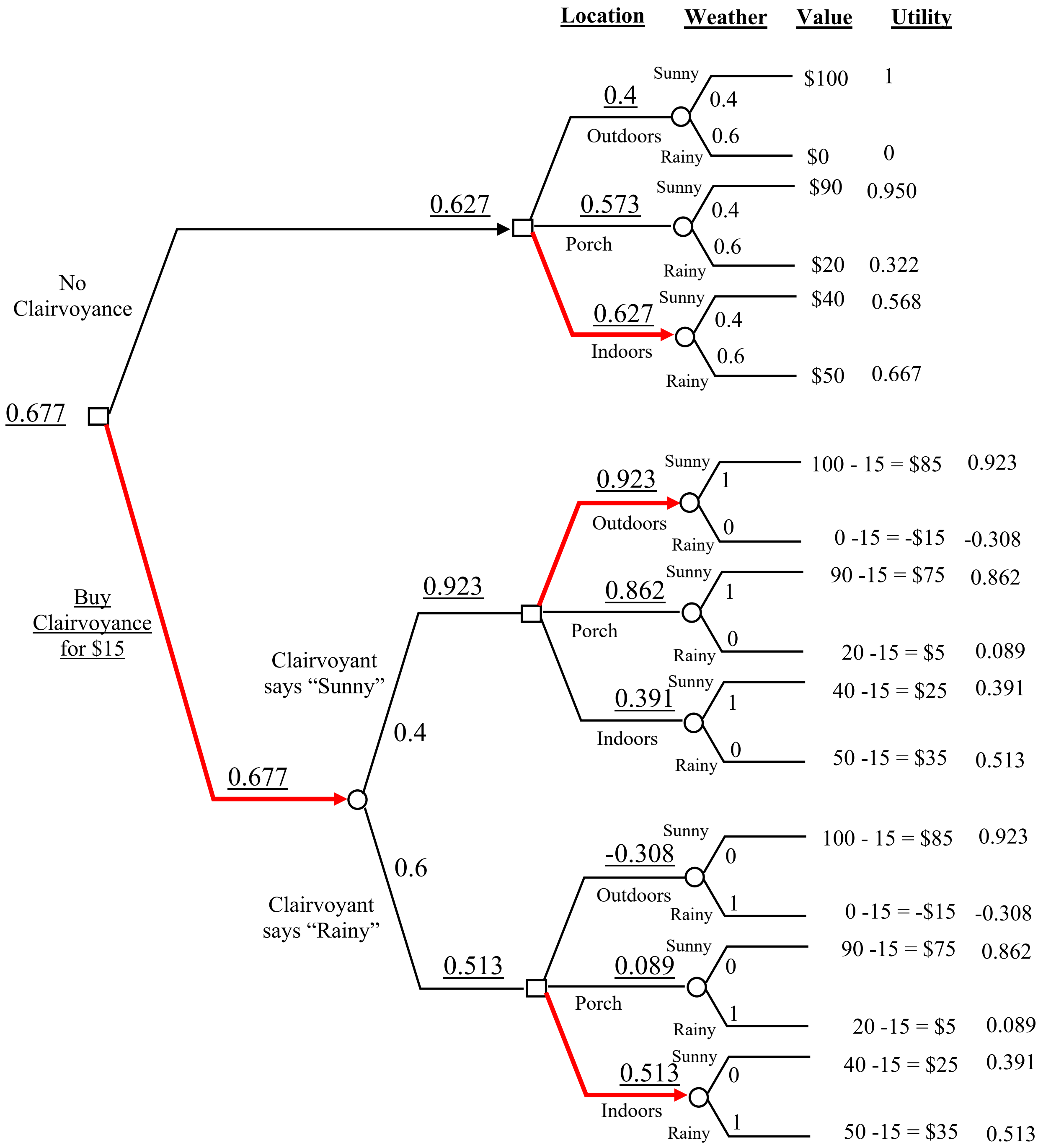
4.4 Value of Information Analysis

4.4.1 The Value of Clairvoyance on Weather

- Suppose that a clairvoyant offers to tell Kim whether the weather will be sunny or rainy tomorrow if she pays a fee of $15.
- Should Kim pay $15 to the clairvoyant for the information about the weather tomorrow which is guaranteed to be true?
- This problem of whether to "buy" this perfect information or not is a decision to be made. It may be represented by extending the decision tree as follows:



Model with the decision to buy clairvoyance on the weather or not for $15 before deciding on the party location



Optimal Decision Policy:

- Kim should buy the perfect information from the clairvoyant for $15.
- If the clairvoyant says “Sunny”:
  Select “Outdoors” as the party location
- Else if the clairvoyant says “Rainy”:
  Select “Indoors” as the party location

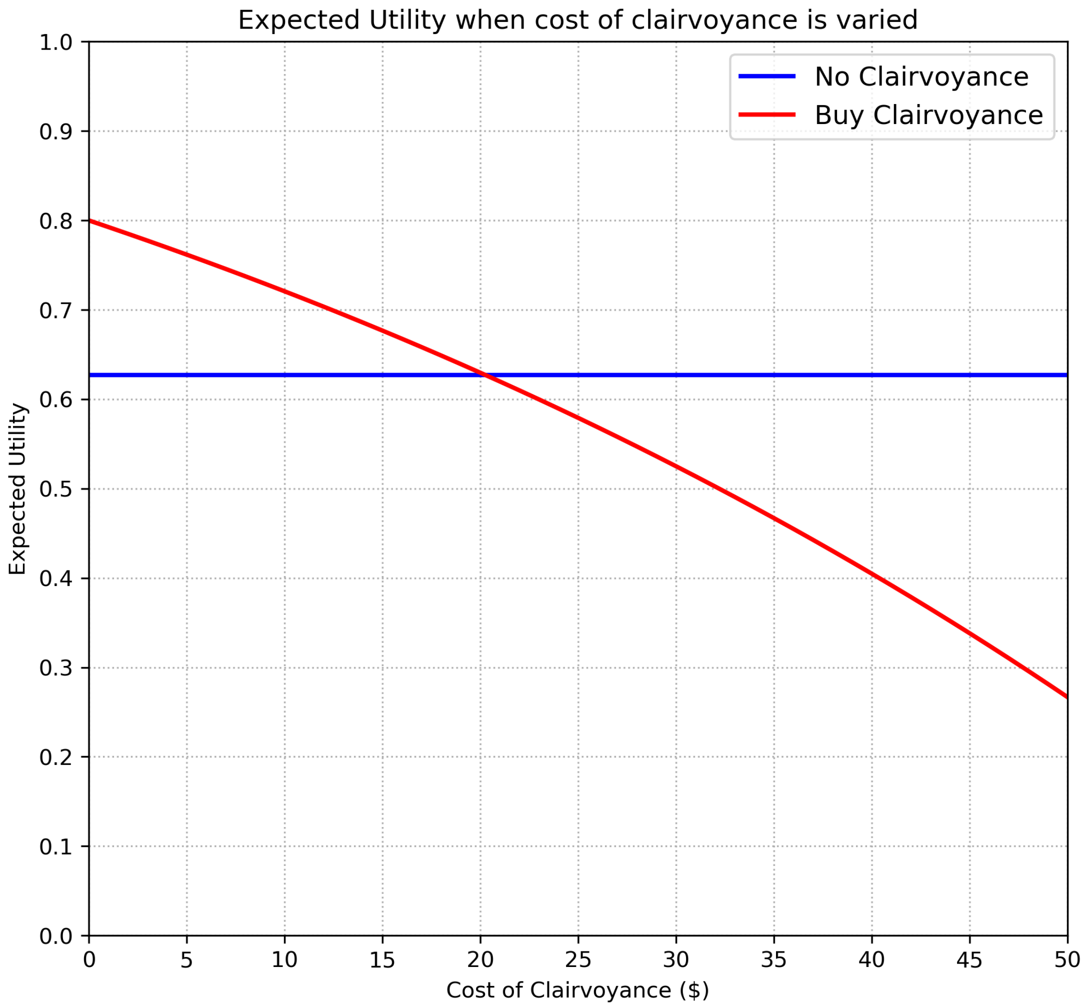
Impact of Changing the Cost of Clairvoyance

- For $15, Kim should pay the clairvoyant. What if the fee is increased? Would she still be willing to pay for the information?

- We consider the impact of changing the cost of clairvoyance on Kim's decision.

- We determine, by resolving the decision tree repeatedly, the expected utilities for "No clairvoyance" and "Buy Clairvoyance for $*x*, for values of *x* in the range of $0 to $50.

- The following results are obtained:

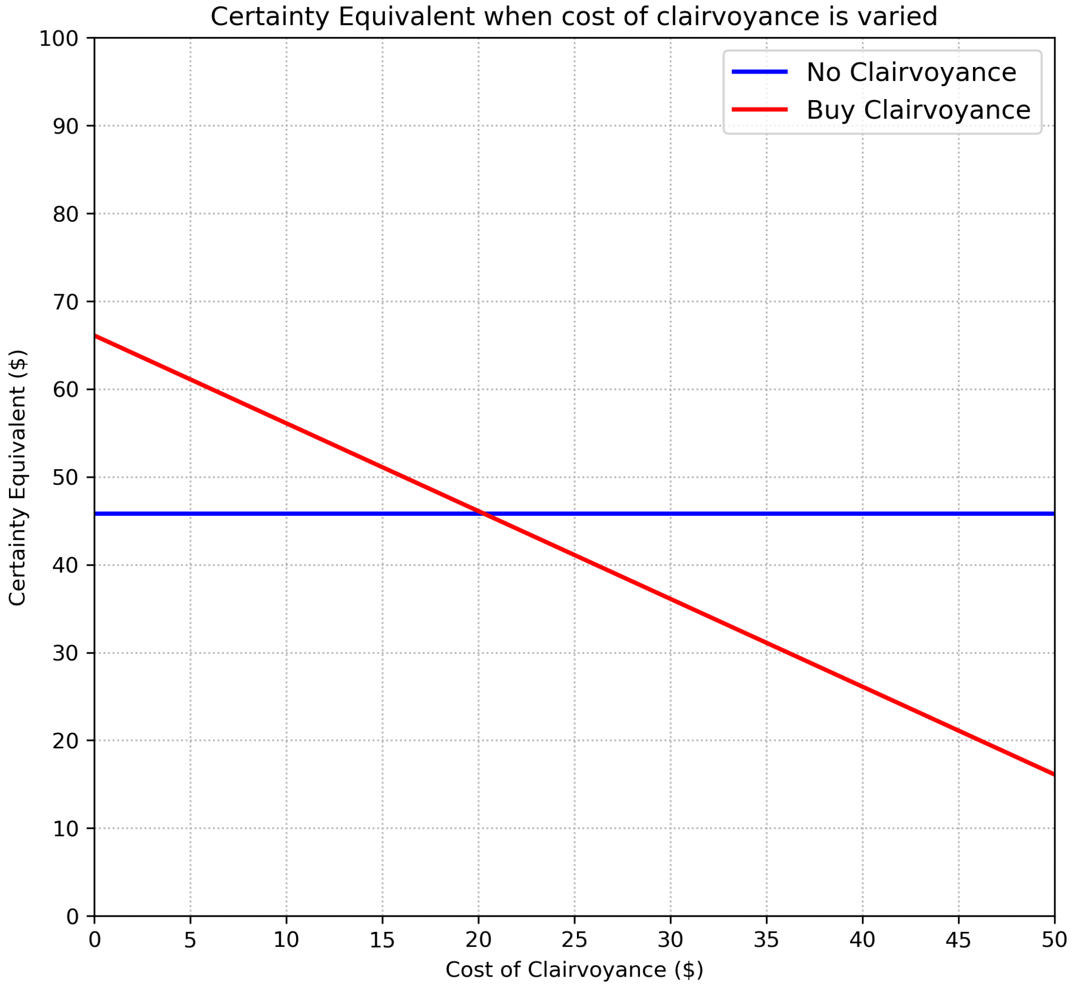


- Kim is indifferent between "Buy clairvoyance" and "No clairvoyance" when the cost of clairvoyance is about $20.

- Hence the breakeven price of clairvoyance on the weather is about $20.

- **Definition**: The **Value of Clairvoyance** or **Expected Value of Perfect Information** on an uncertain variable is the cost of clairvoyance at which the decision maker is just indifferent between buying and not buying the information.

- We can also find the value of clairvoyance on the weather by plotting the certainty equivalents of "Buy clairvoyance" and "No clairvoyance" for different values of the cost of clairvoyance:



Interpretation of Value of Information

- The value of clairvoyance or expected value of perfect information represents the maximum amount one should be willing to pay for the *perfect* information.

- The value of clairvoyance provides a *benchmark* against which to compare any information-gathering scheme that may be proposed.

- If the cost of the scheme exceeds the value of clairvoyance, then there is no need to examine the scheme in any further detail.

- The expected value of perfect information or clairvoyance for Kim is $20. Thus, no other sources of information about the weather could be worth more than $20 to her.

4.4.2 Expected Value of Imperfect Information

- Suppose, instead of clairvoyance, Kim was offered the service of a rain detector which will indicate either “Rainy” or “Sunny” with true detection rates of 80% for both states.

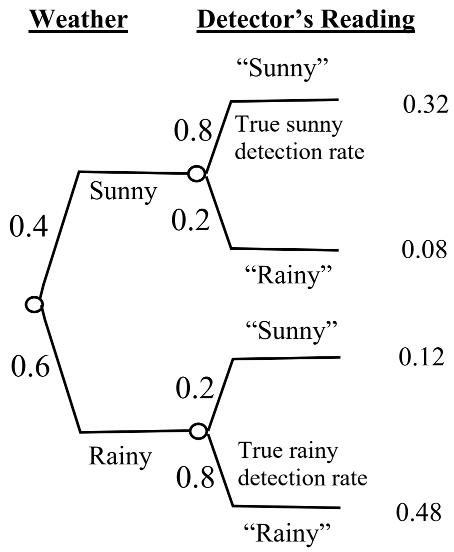
- That is, if the actual weather is going to be sunny, it will read “sunny” with probability of 0.8, and if the actual weather is going to be rainy, it will read “rainy” with a probability of 0.8.

- The fee for using the detector is $12, a 20% discount on the $15 asking price of the clairvoyant. Should Kim pay $12 to use the detector which has 80% true-detection rates?

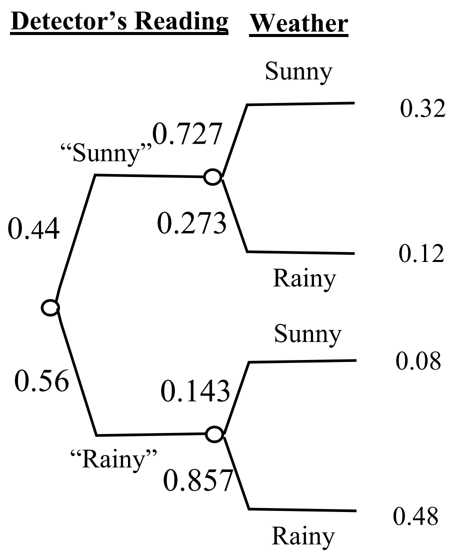
- Recall that the Value of Clairvoyance for the weather was $20. We should first check if the fee charged for imperfect information exceeds the value of clairvoyance. If so, we need not proceed any further.

- Since $12 < $20, there is a possibility that it is worth paying $12 for the imperfect information provided by the detector.

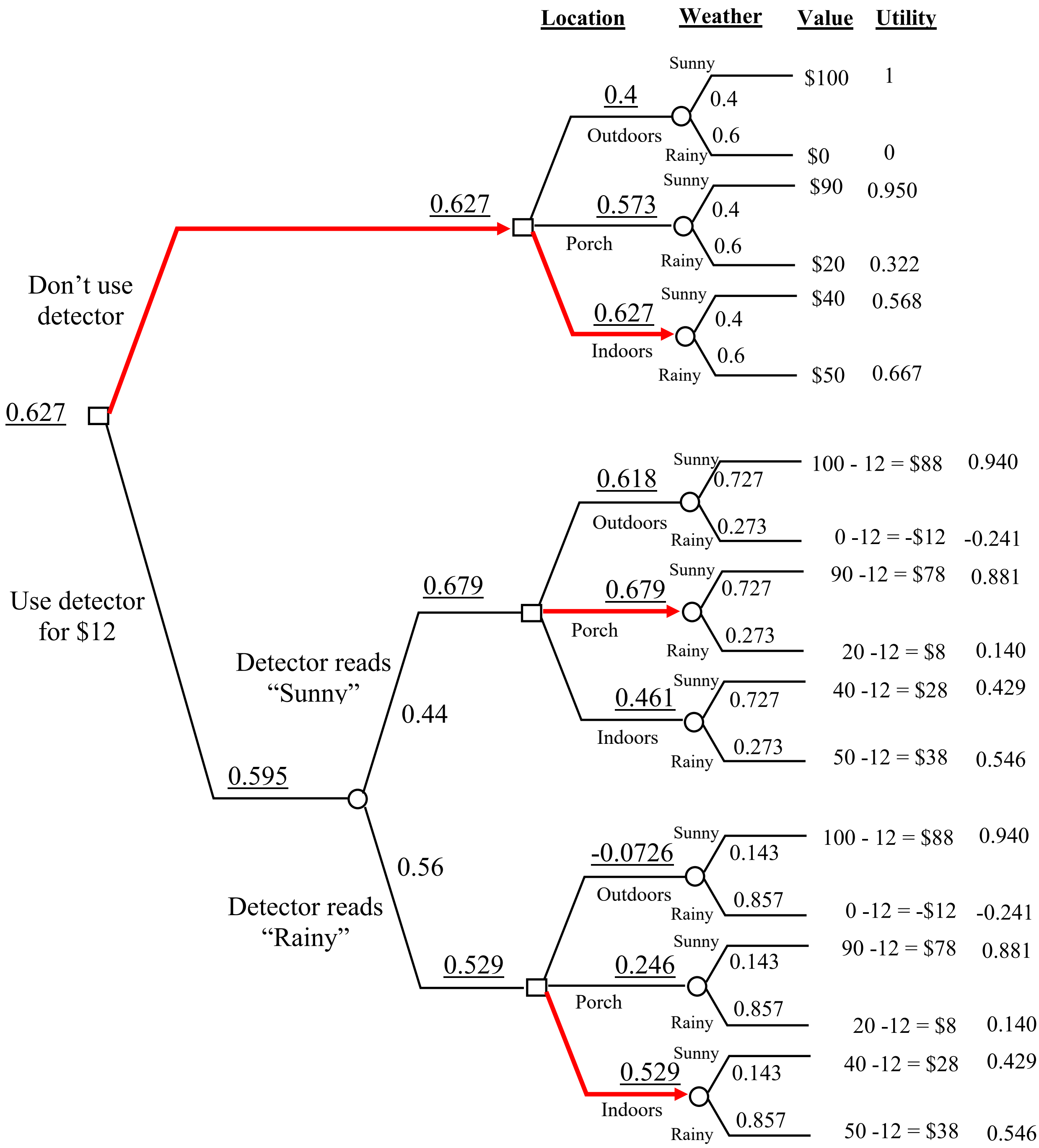
- The probability tree representing the performance of the detector is as follows:



- By flipping the tree, we obtain a tree that will give the probability of actual weather given what the rain detector said:



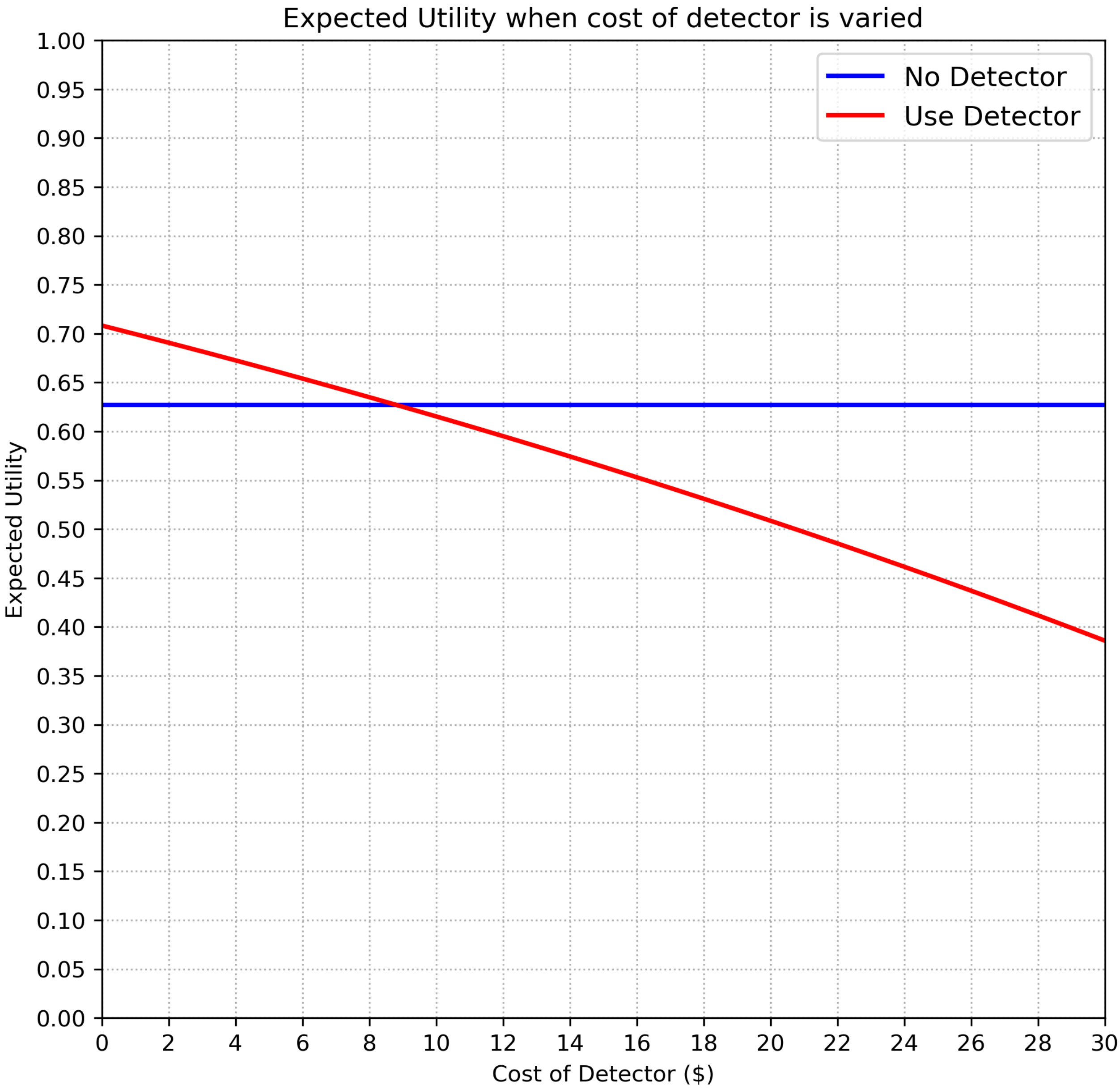
- The decision tree that includes the decision on whether to use the detector for $12 or not is:



- Note how the conditional probabilities are transferred from the "flipped tree" into the above tree.
- Conclusion: It is not worth paying $12 for the use of the rain detector.
- However, if the fee for the detector is lowered, it is possible that the decision might be switched in favor of it.
- Like in the perfect information case, we are interested in the amount for which the decision maker is indifferent between the two alternatives.

Impact of Changing the Cost of Detector (Imperfect Information)

- Comparing the expected utility and certainty equivalent for "Use rain detector" with "No detector" when the cost of the detector is varied from \$0 to \$30.

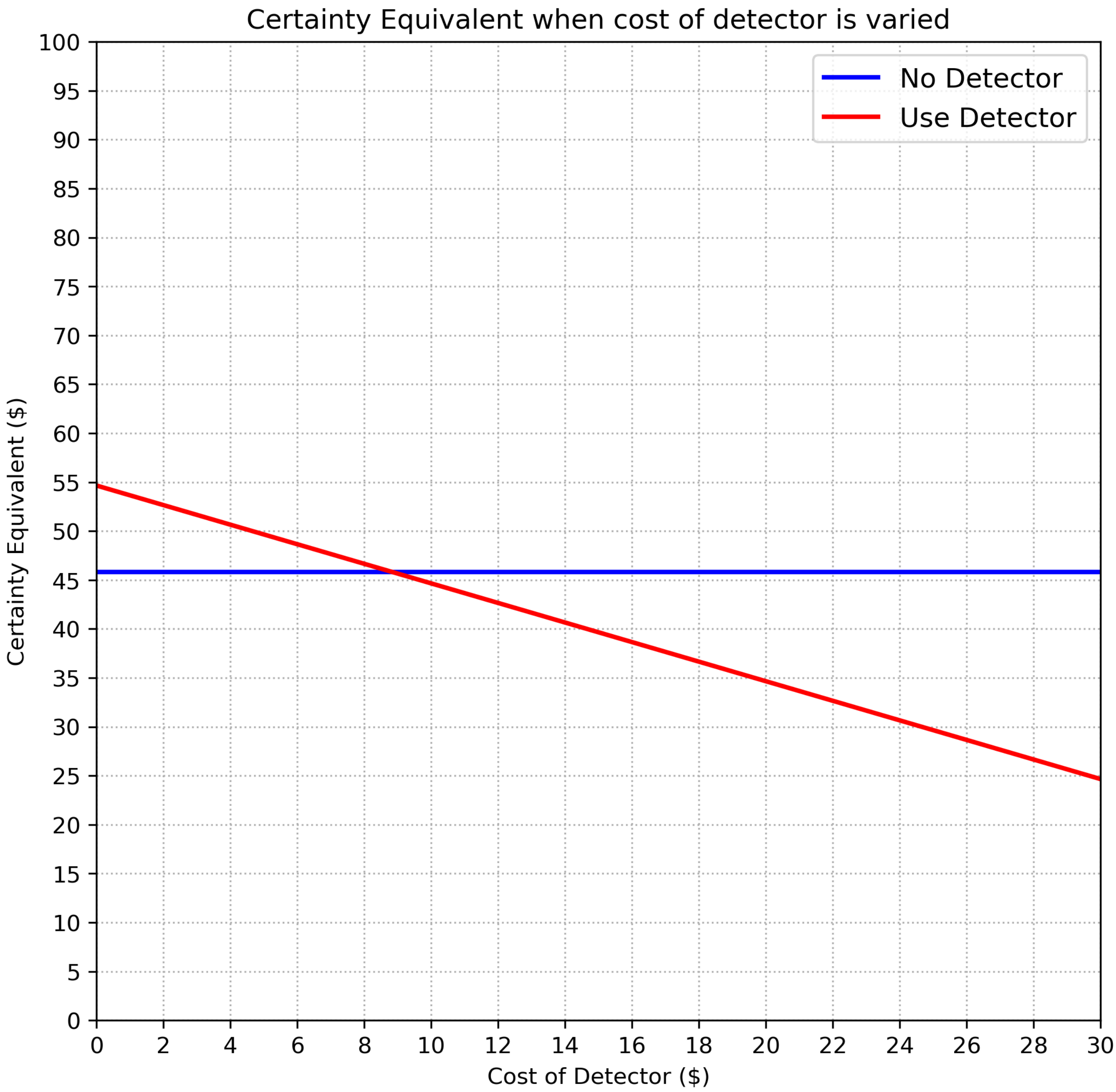


- Kim is indifferent between "Use detector" and "No detector" when the cost of the detector is about \$8.80

- Hence the Expected Value of Imperfect Information provided by the detector is \$8.80.

- **Definition: The Expected Value of (Imperfect) Information (EVI)** of an uncertain event is the cost of the information at which the decision maker is just indifferent between buying and not buying the imperfect information.

- We can also find the expected value of information provided by the rain detector by plotting the certainty equivalents of "Use detector" and "No detector" for different values of the cost of the detector.



Upper and lower bounds for value of information

- It can be shown that the expected value of information for any uncertain event via any form of the information-gathering process is always non-negative. That is, it is either zero or positive.

- For any uncertain event, the expected value of information is bounded above by its value of clairvoyance.

$$0 \leq \text{Expected Value of Imperfect Information} \leq \text{Expected Value of Perfect Information}$$

4.5 Risk Neutral Decision Maker

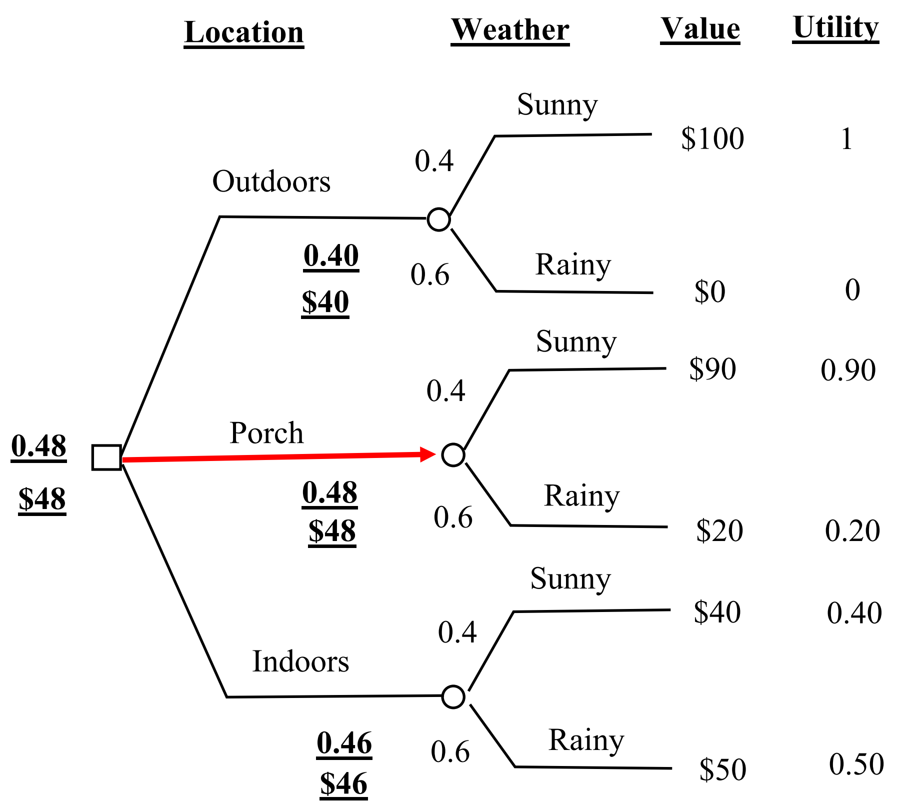
4.5.1 Jane's Party Problem

- Jane faces the same problem in deciding where to hold a party.

- Jane's assignments of dollar values are the same as those of Kim's, but her utilities (preference probabilities) are different.

Prospect	Jane's dollar value	Jane's Utility
Outdoors - Sunny	\$100	1.00
Porch - Sunny	\$90	0.90
Indoors - Rainy	\$50	0.50
Indoors - Sunny	\$40	0.40
Porch - Rainy	\$20	0.20
Outdoors - Rainy	\$0	0.00

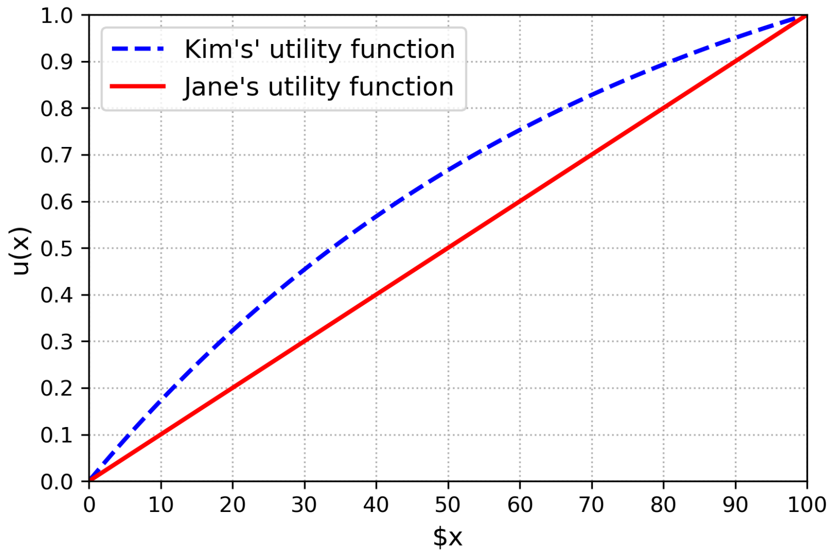
- Jane's decision tree is as shown below:



- The optimal decision for Jane is to hold the party on the porch.

- Notice that this is different from Kim's optimal choice which is indoors. This is due to the difference in preferences between the two.

- Comparing Jane's and Kim's utility curves:



- Jane has a linear utility function.

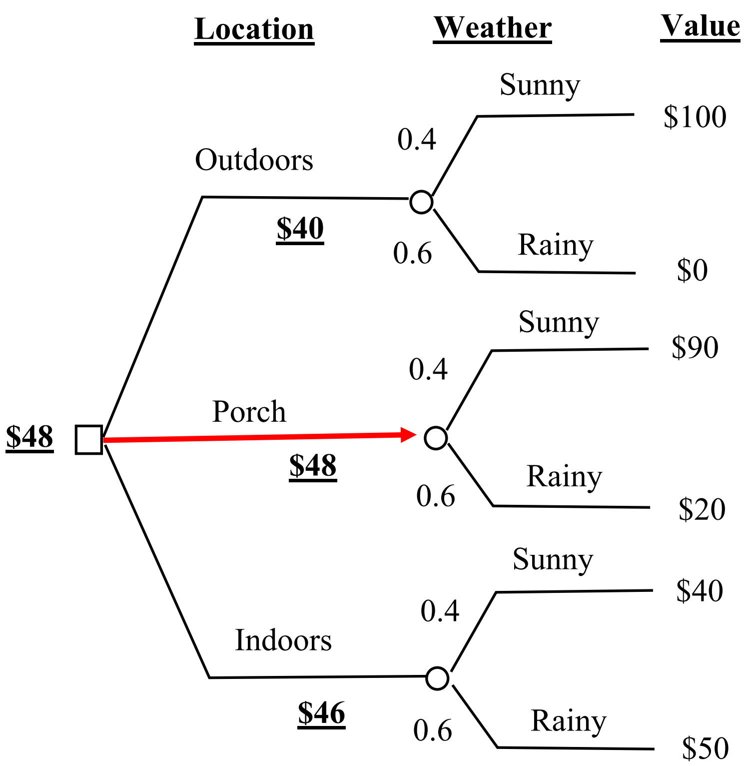
- A decision maker whose utility function is linear is said to be **risk neutral**.

- When the decision maker is risk neutral:

  - There is no need to use a utility function. Just work on the dollar values on the decision tree. This is equivalent to using the function $u(x) = x$.

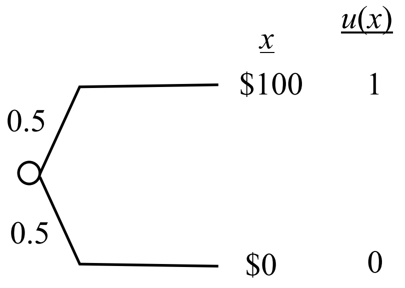
  - The certainty equivalent of any deal or sub-deal will be equal to the expected value.

Decision Tree for Jane's Party Problem



Jane's Selling Price for the Coin Tossing Game

- What is Jane's certainty equivalent or personal indifferent selling price for the coin tossing game (\$100 or nothing)? How is it compared with Kim's? What can you say about their attitudes toward risk-taking?



Jane: Expected utility = 0.5 (1) + 0.5 (0) = 0.5
Certainty equivalent = $u^{-1}(0.5)$ = \$50.

Kim: Expected utility = 0.5 (1) + 0.5 (0) = 0.5
Certainty equivalent = $u^{-1}(0.5)$ = \$34.

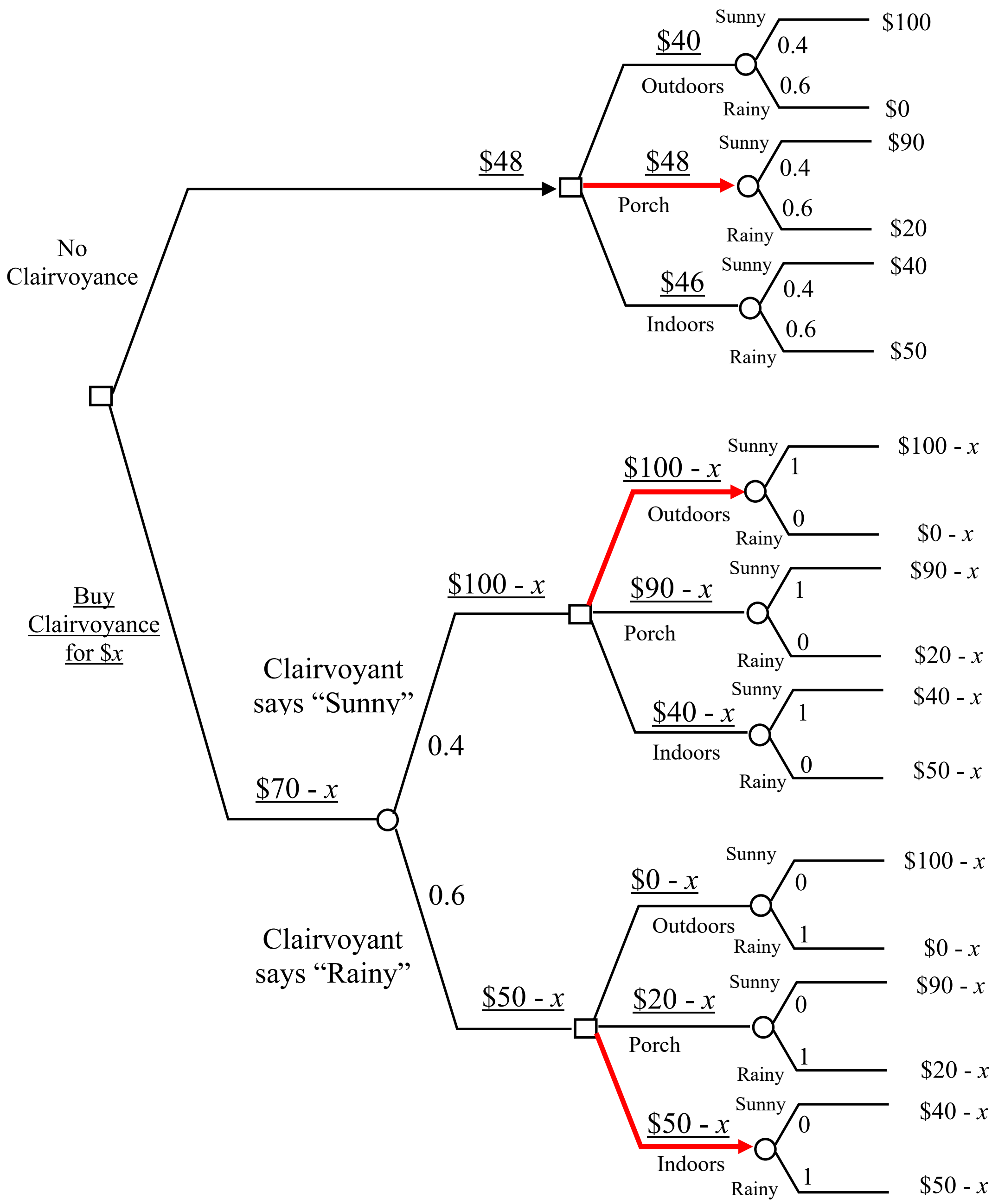
- Kim is more averse to risk than Jane. She is willing to sell off the deal at a lower price than Jane.

4.5.2 Value of Clairvoyance for Risk-Neutral Case

Finding Jane's value of clairvoyance on weather

- There are two methods for the risk-neutral case:
  1. Expected value indifference method
  2. Difference of Expected Values method

Method 1 (Expected Value Indifference Method)



- When Jane is indifferent between no information and $x-clairvoyance:

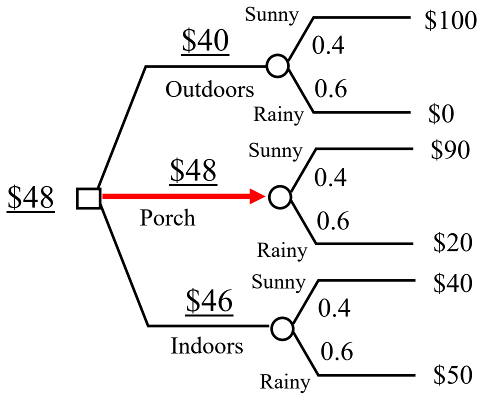
$$\$70 - x = \$48 \Rightarrow x = \$70 - \$48 = \$22$$

- Hence Value of Clairvoyance = $22.00

Method 2 (Difference of Expected Values Method)

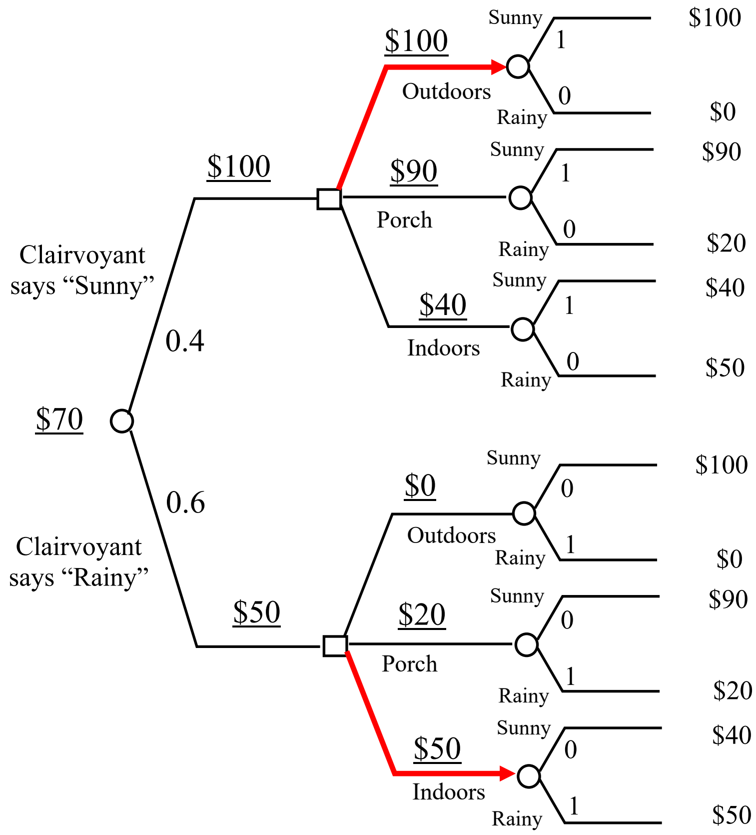
- The value of clairvoyance for the risk-neutral case can be simply computed by taking the difference between the expected value for the case when there is FREE information and the expected value for the base case without information.

Base model with no clairvoyance



Expected Value with no information = $48

Decision model with free clairvoyance



Expected value with free clairvoyance = $70

Jane's value of clairvoyance for weather is $70 - $48 = $22.

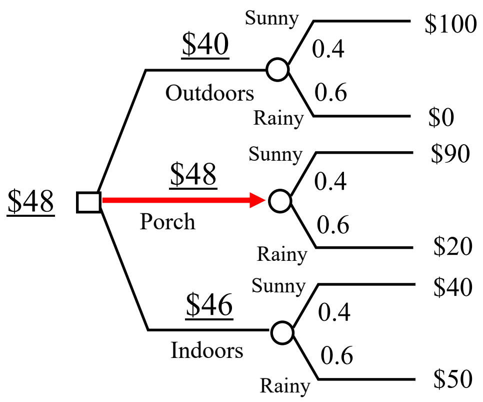
- We have thus found that Kim and Jane have different values of clairvoyance for the weather.
- These differences arise solely from differences in taste (preference), not from differences in structure or information.

4.5.3 Expected Value of Imperfect Information for Risk-Neutral Case

Jane's Expected Value of Imperfect Information for Rain Detector

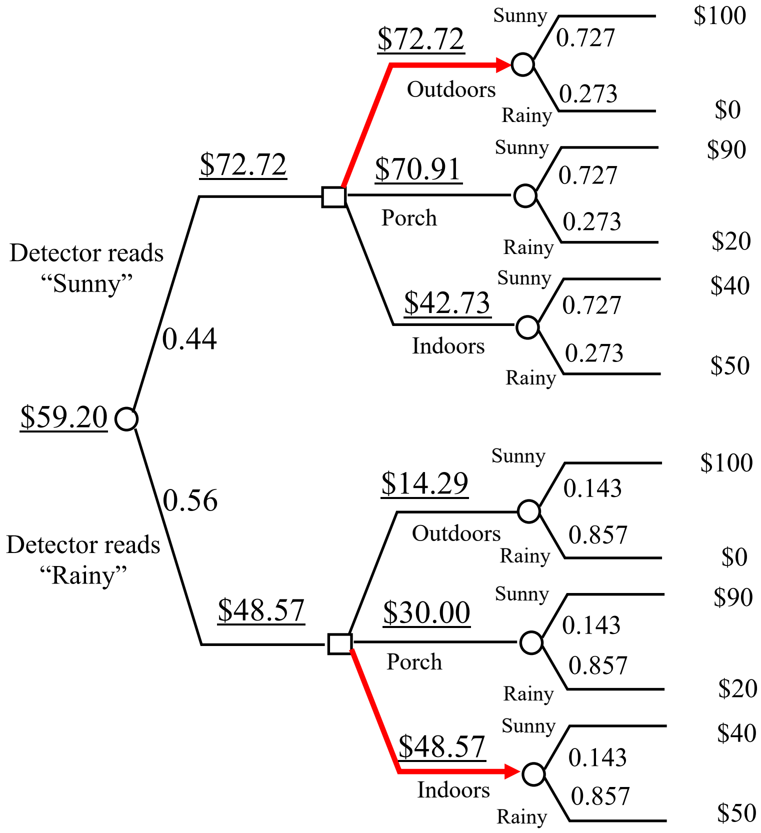
- The expected value of imperfect information for a risk-neutral decision-maker can be computed by taking the difference between the expected value for the case when there is FREE (imperfect) information and the expected value for the base case with no information.

Base model with no information



Expected Value with no information = $48.00

Decision model with free use of rain detector



Expected Value with free use of rain detector= $59.20

- Jane's expected value of information for the rain detector is $59.20 - $48.00 = $11.20
- Again, we have found that Kim and Jane have different expected values of imperfect information for the same rain detector (true detection rates = 80%).
- These differences arise solely from differences in preference, not from differences in structure or information.

4.6 Sensitivity Analysis

- Sensitivity analysis can reveal important insights into the nature of a decision problem by determining how optimal the decision policy would change when one or more numbers that constitute the decision basis are changed or varied over a range of values. It also helps determine whether additional effort should be expended in increasing the accuracy of the numbers we have used.

4.6.1 Sensitivity of Kim's Decision to the Probability of Sunshine

- Suppose Kim is concerned about how sensitive her decision is to the probability of sunshine she assigns; she currently assigned 0.4.

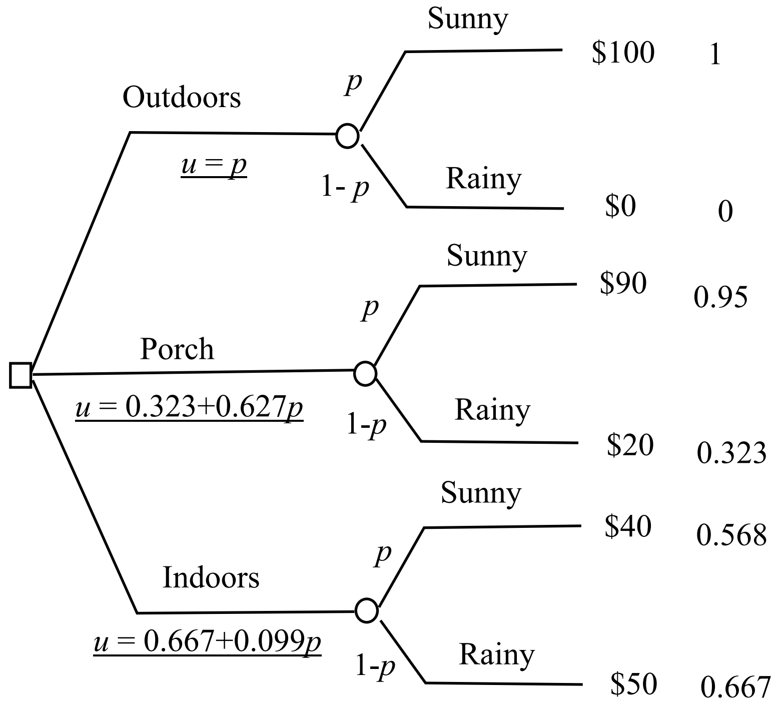
- She wants to know whether a small change in this probability will cause changes of alternative and /or substantial changes in the value of the party to her.

- She may also be interested in a sensitivity analysis to this probability because she expects to receive additional information and wants to know how she should adapt her party strategy in the face of a new probability of sunshine.

Sensitivity of Kim's Expected Utility to Probability of Sunshine

- Let p be the probability of sunshine.

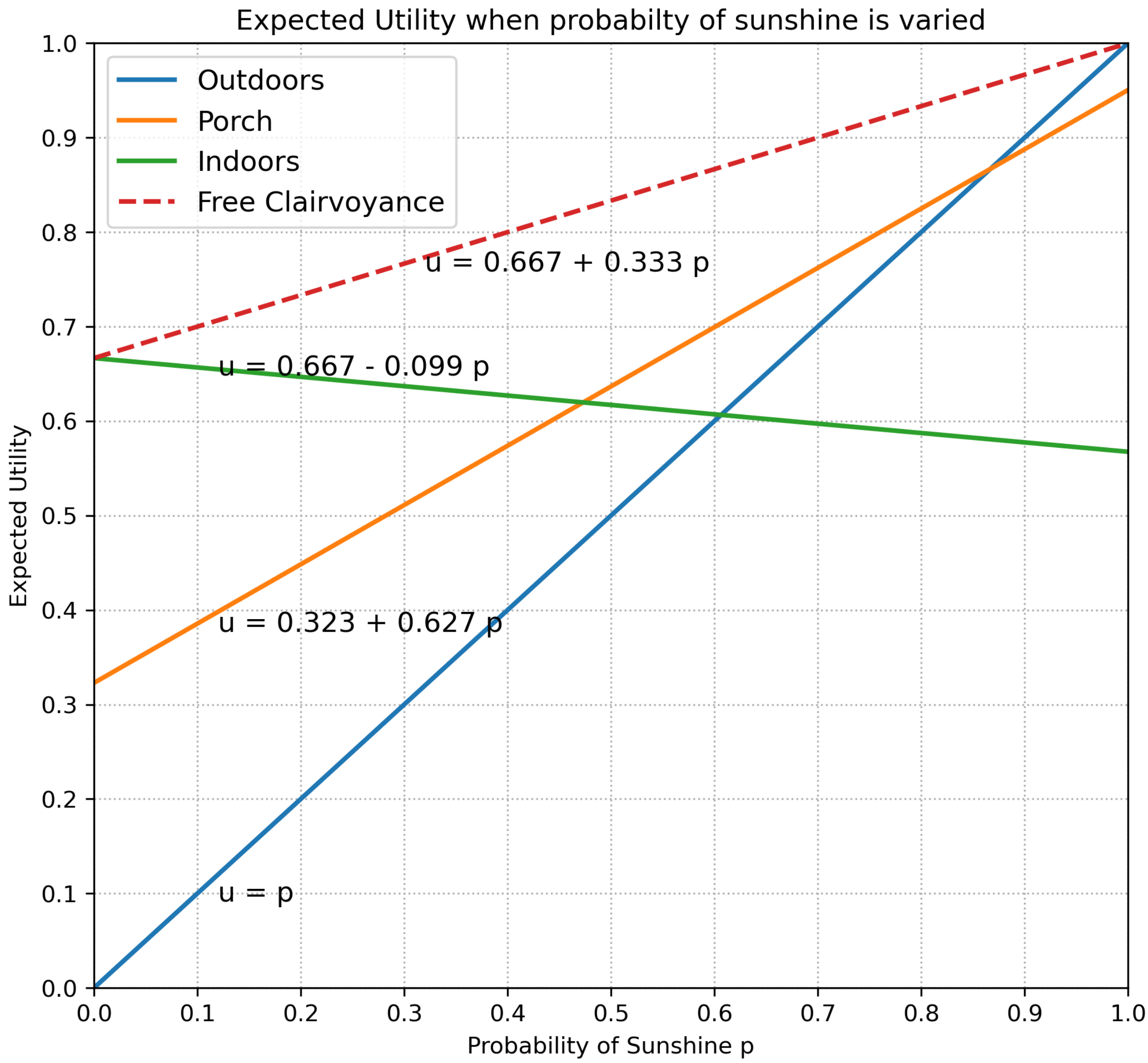
- The decision tree is:



- The expected utilities for the three alternatives as a function of p are:

EU(Outdoors)	$= p$
EU(Porch)	$= 0.950p + 0.323(1-p) = 0.323 + 0.627p$
EU(Indoors)	$= 0.586p + 0.667(1-p) = 0.667 - 0.099p$

- We plot the expected utility of each alternative as shown (**Rainbow Diagram**)



- The best alternative to follow for a given value of p is the one with the highest expected utility. The best alternative when $p = 0$ is the "Indoors" alternative. The indoors line remains highest until it crosses the porch line at $p = 0.47$. Then the porch line is highest until it crosses the outdoors line at $p = 0.87$.

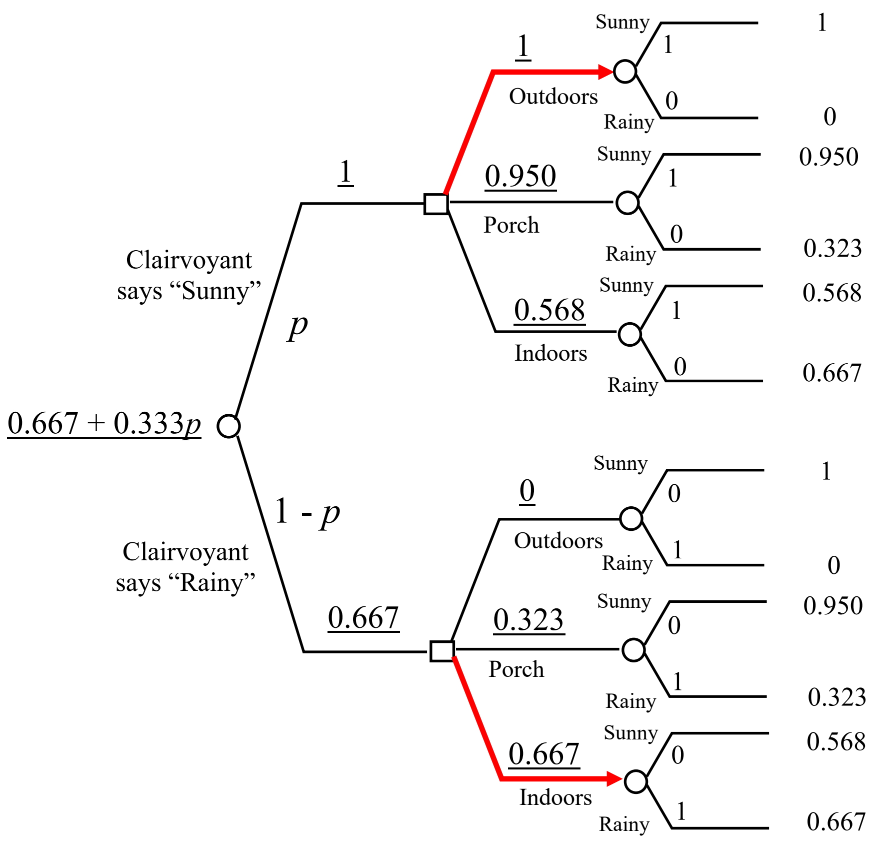
- In summary:

Probability of sunshine, p	**Best decision**
$0 \leq p \leq 0.47$	Indoors
$0.47 \leq p \leq 0.87$	Porch
$0.87 \leq p \leq 1$	Outdoors

- Note that the value of the probability of sunshine that Kim originally assigned, 0.4, leads her to follow the indoors alternative. If her assignment of the probability of sunshine increases to 0.47, she should move to the porch. Only when she is quite sure (p=0.87) of sunshine would she move her party outdoors.

Sensitivity of Kim's Expected Utility for Free Clairvoyance to Probability of Sunshine

- Decision model with free clairvoyance when the probability of sunshine is p:



- Expected utility with free clairvoyance on the weather =

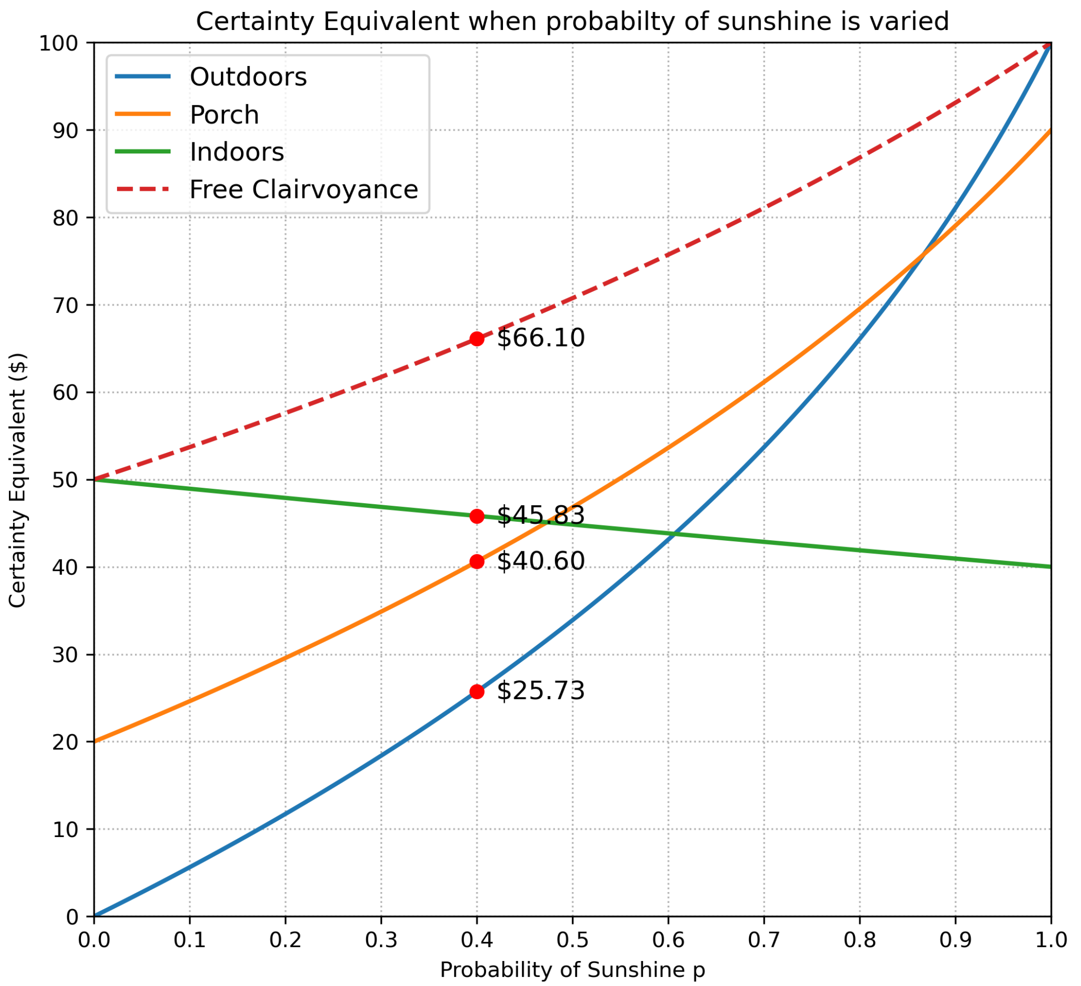
$$p\,(1) + (1-p)(0.667) = 0.667 + 0.333\,p$$

- This is the top dotted line on the rainbow diagram.

Sensitivity of Kim's Certainty Equivalent to Probability of Sunshine

- We may compute the certainty equivalent for each alternative as a function of the probability of sunshine p and plot the rainbow diagram in terms of certainty equivalent.

	Expected Utility			**Certainty Equivalent**			Optimal
p	Outdoors	Porch	Indoors	Outdoors	Porch	Indoors	Decision
0.00	0.0000	0.3229	0.6667	0.000	20.000	50.000	Indoors
0.05	0.0500	0.3542	0.6617	2.757	22.276	49.466	Indoors
0.10	0.1000	0.3856	0.6568	5.624	24.625	48.935	Indoors
0.15	0.1500	0.4170	0.6518	8.609	27.054	48.409	Indoors
0.20	0.2000	0.4484	0.6468	11.723	29.567	47.886	Indoors
0.25	0.2500	0.4798	0.6419	14.978	32.172	47.367	Indoors
0.30	0.3000	0.5111	0.6369	18.387	34.873	46.852	Indoors
0.35	0.3500	0.5425	0.6320	21.964	37.680	46.340	Indoors
0.40	0.4000	0.5739	0.6270	25.729	40.601	45.832	Indoors
0.45	0.4500	0.6053	0.6221	29.700	43.645	45.328	Indoors
0.50	0.5000	0.6366	0.6171	33.904	46.823	44.827	Porch
0.55	0.5500	0.6680	0.6121	38.367	50.147	44.329	Porch
0.60	0.6000	0.6994	0.6072	43.125	53.632	43.835	Porch
0.65	0.6500	0.7308	0.6022	48.219	57.294	43.345	Porch
0.70	0.7000	0.7622	0.5973	53.700	61.152	42.857	Porch
0.75	0.7500	0.7935	0.5923	59.632	65.228	42.373	Porch
0.80	0.8000	0.8249	0.5874	66.096	69.548	41.892	Porch
0.85	0.8500	0.8563	0.5824	73.197	74.143	41.414	Porch
0.90	0.9000	0.8877	0.5774	81.074	79.052	40.940	Outdoors
0.95	0.9500	0.9191	0.5725	89.918	84.318	40.468	Outdoors
1.00	1.0000	0.9504	0.5675	100.000	90.000	40.000	Outdoors



4.6.2 Sensitivity of Kim's Decision to Her Attitude to Risk

- Kim's utility curve can be approximated with the exponential function:

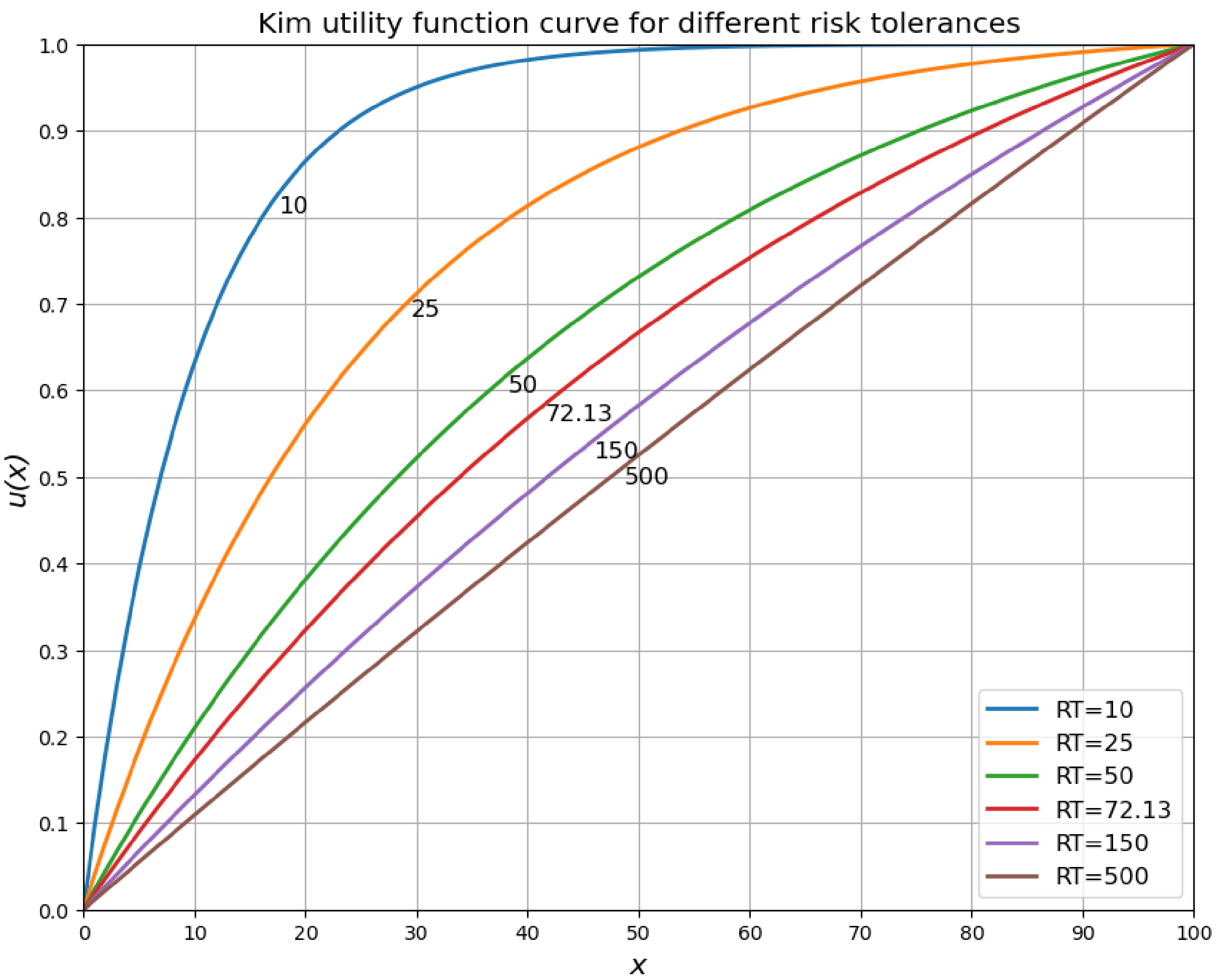
$$u(x) = 1.33\left(1 - e^{\frac{-x}{72.13}}\right)$$

  where x is in dollars, and $u(0) = 0$ and $u(100) = 1$.

- Kim's risk attitude and degree of risk aversion are measured by the coefficient of \$72.13 in the exponent of the equation. It is called the **Risk Tolerance** (RT) and we will study it in detail later (Chapter 6).

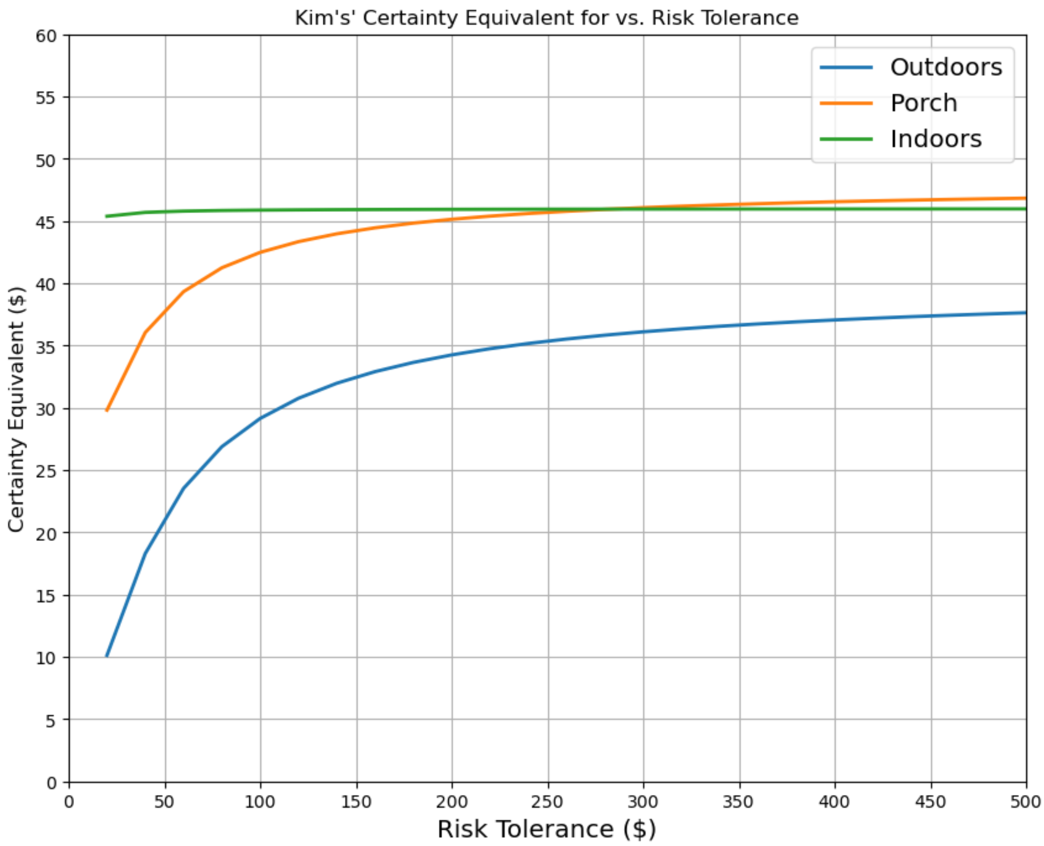
- We would like to know how Kim's optimal decision will change when we vary this parameter in the utility function.

- However, each time we change the risk tolerance in the utility function, we need to rescale the utility function so that the boundary conditions $u(0) = 0$ and $u(100) = 1$ are always satisfied.



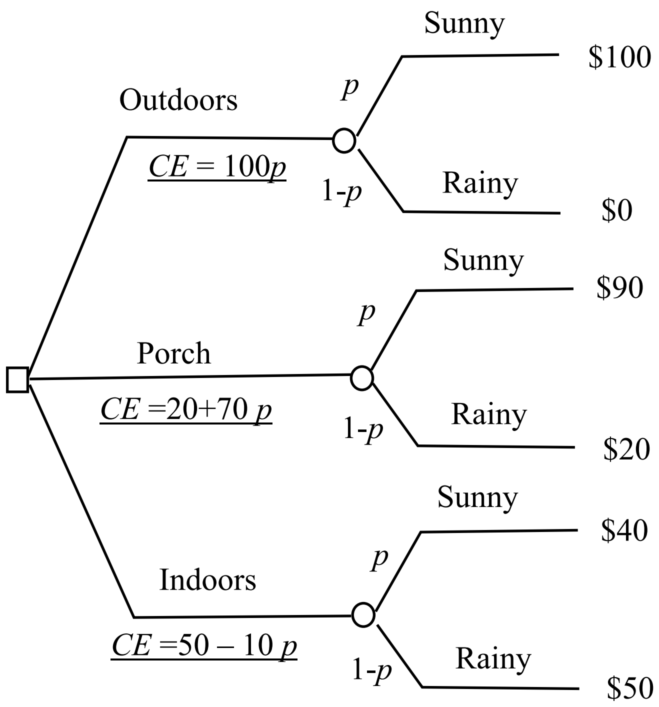
- The following table gives the expected utilities and certainty equivalents for the three alternatives as the risk tolerance is varied from $20 to $500.

	Expected Utility			Certainty Equivalent			Optimal
RT ($)	Outdoors	Porch	Indoors	Outdoors	Porch	Indoors	Decision
20	0.4000	0.7801	0.9027	10.13	29.82	45.39	Indoors
40	0.4000	0.6470	0.7418	18.30	36.05	45.70	Indoors
60	0.4000	0.5928	0.6582	23.53	39.33	45.80	Indoors
80	0.4000	0.5646	0.6114	26.88	41.25	45.85	Indoors
100	0.4000	0.5476	0.5821	29.15	42.49	45.88	Indoors
120	0.4000	0.5362	0.5622	30.77	43.34	45.90	Indoors
140	0.4000	0.5281	0.5478	31.97	43.97	45.91	Indoors
160	0.4000	0.5220	0.5369	32.91	44.46	45.92	Indoors
180	0.4000	0.5173	0.5284	33.65	44.83	45.93	Indoors
200	0.4000	0.5135	0.5216	34.25	45.14	45.94	Indoors
220	0.4000	0.5104	0.5160	34.75	45.39	45.95	Indoors
240	0.4000	0.5079	0.5113	35.17	45.60	45.95	Indoors
260	0.4000	0.5057	0.5074	35.53	45.78	45.95	Indoors
280	0.4000	0.5038	0.5040	35.84	45.94	45.96	Indoors
300	0.4000	0.5022	0.5011	36.10	46.07	45.96	Porch
320	0.4000	0.5008	0.4985	36.34	46.19	45.96	Porch
340	0.4000	0.4996	0.4963	36.55	46.30	45.96	Porch
360	0.4000	0.4985	0.4942	36.74	46.39	45.97	Porch
380	0.4000	0.4975	0.4924	36.91	46.47	45.97	Porch
400	0.4000	0.4966	0.4908	37.06	46.55	45.97	Porch
420	0.4000	0.4958	0.4893	37.19	46.62	45.97	Porch
440	0.4000	0.4951	0.4880	37.32	46.68	45.97	Porch
460	0.4000	0.4945	0.4868	37.43	46.74	45.97	Porch
480	0.4000	0.4938	0.4857	37.54	46.79	45.97	Porch
500	0.4000	0.4933	0.4846	37.64	46.84	45.98	Porch

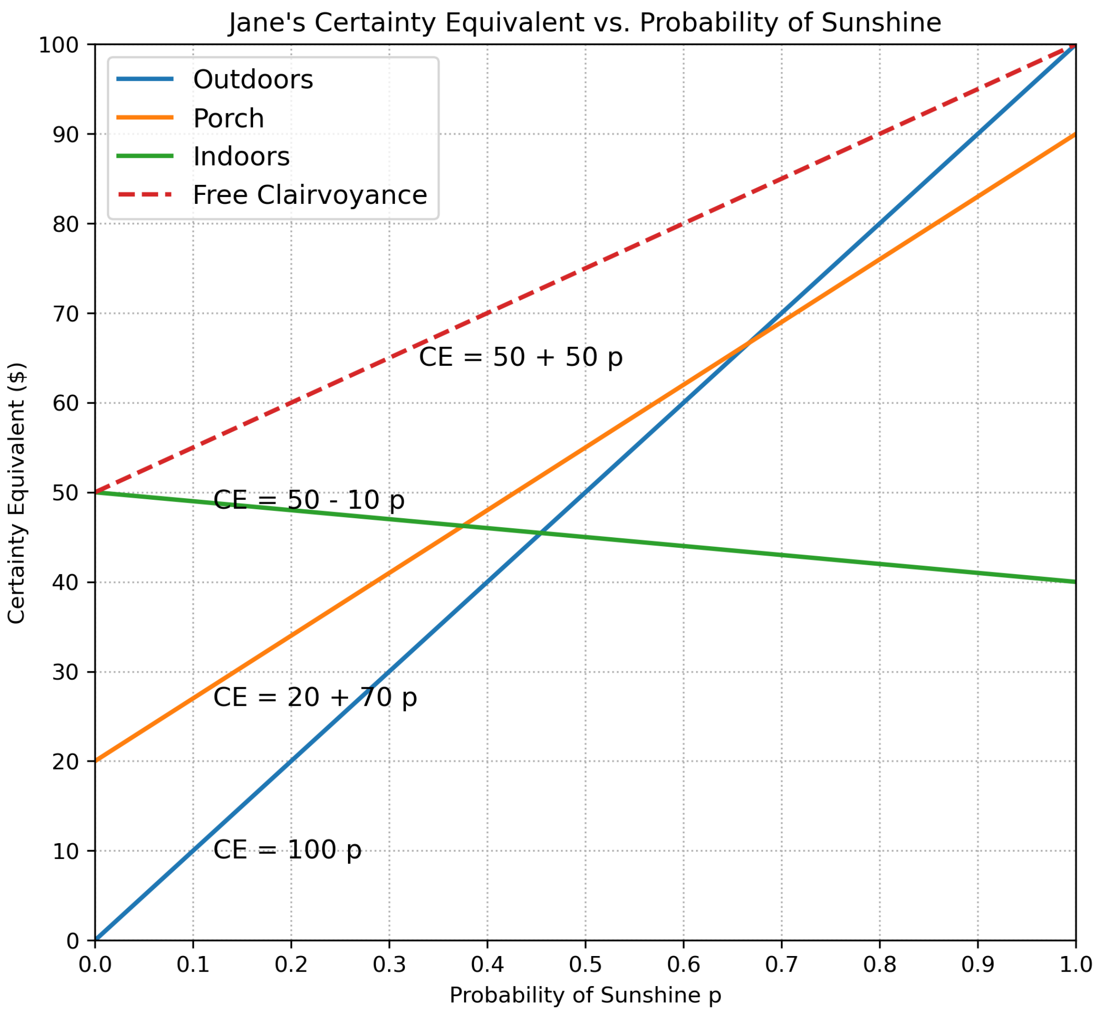


4.6.3 Sensitivity Analysis for Jane

- We can a perform similar analysis for Jane who is risk neutral.
- The analysis can be done directly in dollar values with CE = EV.



- Jane's rainbow diagram is plotted as shown:



- Comparing Kim's and Jane's decision thresholds

Best decision	Probability of sunshine, p	
	Kim	**Jane**
Indoors	$0 \leq p \leq 0.47$	$0 \leq p \leq 0.375$
Porch	$0.47 \leq p \leq 0.87$	$0.375 \leq p \leq 0.667$
Outdoors	$0.87 \leq p \leq 1$	$0.667 \leq p \leq 1$

- From the thresholds, we note that Kim is more "averse to risk" than Jane.

4.7 Stochastic Dominance Analysis

- When the decision maker is not risk-neutral, we require his/her utility function to find the best alternative.

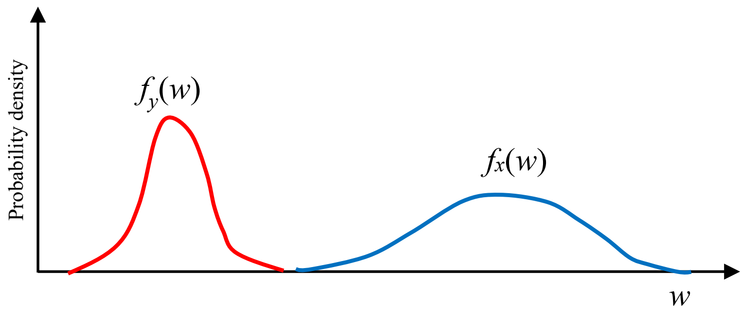
- However, in some none risk neutral situations, when the required utility function is not available, it may still be possible to make a rational choice between two alternatives by comparing their risk profiles using stochastic dominance analysis.

4.7.1 Deterministic Dominance

- Consider a decision problem with two alternatives *X* and *Y*.

- Let the risk profiles for *X* and *Y* be denoted by the probability distributions $f_x(w)$ and $f_y(w)$, respectively where $a \leq w \leq b$.

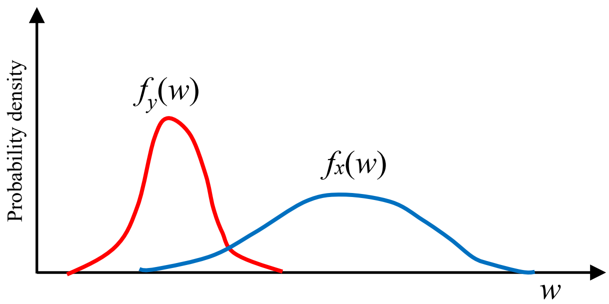
- Suppose we plot the two risk profiles and they appear as follows:



Deterministic Dominance

Definition: Deterministic Dominance

- Alternative *X* **Deterministically Dominates** alternative *Y* if and only if the minimum possible value of *X* is greater than the maximum possible value of *Y*. Under this situation, alternative *X* is preferred to alternative *Y* regardless of the probability distributions of *X* and *Y*.



No Deterministic Dominance

- When the risk profiles for *X* and *Y* overlap, there is no deterministic dominance between *X* and *Y* as some outcome values of *X* are less than some outcome values of *Y*.

- Under this situation, we cannot choose between *X* and *Y* using their expected values if the decision maker is not risk neutral. However, we can check for possible stochastic dominance between *X* and *Y*.

4.7.2 First-order Stochastic Dominance

- Let $F_x(w)$ and $F_y(w)$ be the *Cumulative Distribution Functions* (CDF) of the outcome values of alternatives X and Y, respectively:

$$F_x(w) = \int_a^w f_x(t)dt \qquad F_y(w) = \int_a^w f_y(t)dt \qquad a \le w \le b.$$

- Let $G_x(w)$ and $G_y(w)$ be the *Excess Probability Functions* (EPF) of the outcomes of alternatives of X *and* Y, respectively:

$$G_x(w) = 1 - F_x(w) = \int_w^b f_x(t)dt \qquad G_y(w) = 1 - F_y(w) = \int_w^b f_y(t)dt \qquad a \le w \le b.$$

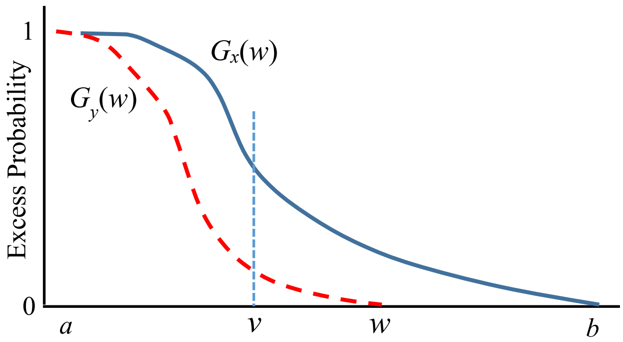
<u>**Definition (First-order Stochastic Dominance)**</u>

- The risk profile for X *First-Order Stochastically Dominates* that of Y (denoted as $X >_{1SD} Y$) if and only if

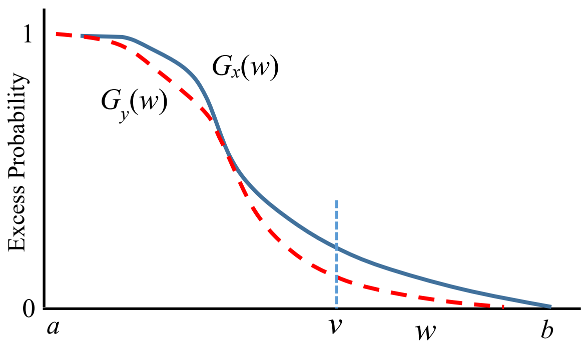
$$G_x(w) \ge G_y(w) \qquad \text{for all } w,\ a \le w \le b \qquad (1)$$

and $$G_x(w) > G_y(w) \qquad \text{for some } w,\ a \le w \le b \qquad (2)$$

- That is, the EPF for X is always either on or above the EPF for Y and the two EPF are not identical.



X* first-order stochastically dominates *Y



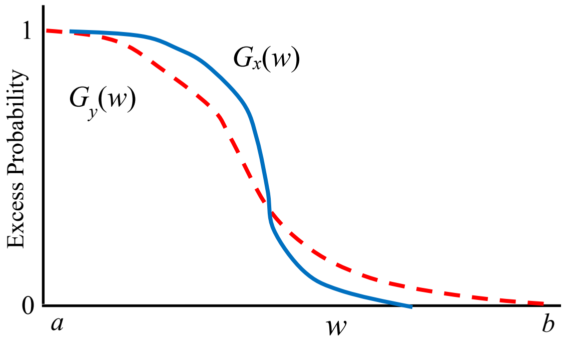
X* first-order stochastically dominates *Y

Interpretation of first-order stochastic dominance

- Given any desired outcome v, the probability of achieving at least v (i.e., the upside potential) for alternative X is either greater than or equal to that for alternative Y.

No first-order stochastic dominance

- When the two risk profiles cross each other, there is no first-order stochastic dominance.

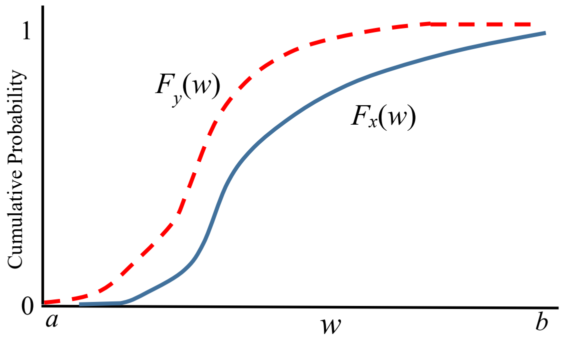


No first-order stochastic dominance between *X* and *Y*

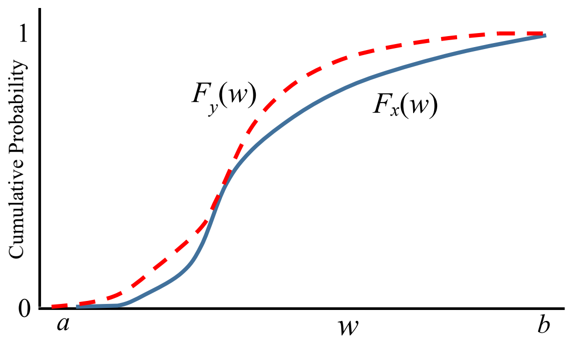
Stochastic Dominance Based on Cumulative Distribution Functions

- Since $G_x(w) = 1 - F_x(w)$ and $G_y(w) = 1 - F_y(w)$, it follows that we can also compare the CDFs of X and Y to determine if there is first order stochastic dominance between them.

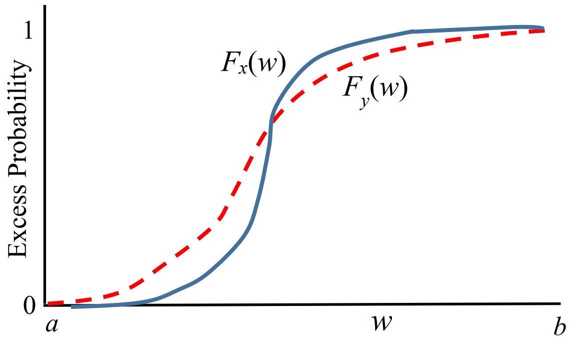
- In this case, the CDF of X must be always either below or touching the CDF for Y, and the two graphs are not identical.



X first-order stochastically dominates Y



X first-order stochastically dominates Y



No first-order stochastic dominance between X and Y

4.7.3 Rational Choice under First-Order Stochastic Dominance

- Stochastic dominance may allow us to make rational choices even if the exact utility function of the decision-maker is unknown.

<u>Preference ordering under first-order stochastic dominance</u>

- If the risk profile for X first-order stochastically dominates that for Y, and the utility function $u(w)$ is non-decreasing in w, then

  (a) $E_x[u(w)] > E_y[u(w)]$
  where $E_x[u(w)] = \int_a^b f_x(w)u(w)\,dw$ and $E_y[u(w)] = \int_a^b f_y(w)u(w)\,dw$ are the expected utilities of X and Y respectively; and

  (b) Alternative X is preferred to alternative Y.

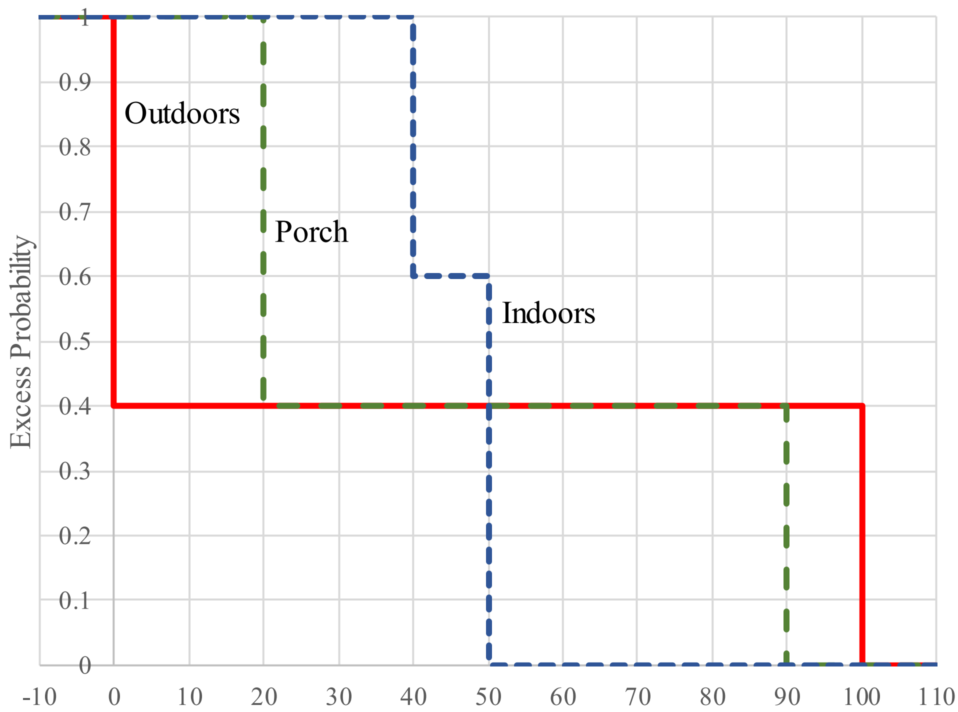
Proof: See Appendix A to this chapter.

- Since all real people's utility functions are non-decreasing, it follows that first-order stochastic dominance is practically applicable under **all classes of risk attitude** and is independent of the actual utility function (so long as it is non-decreasing).

- Hence this criterion can always be applied to eliminate alternatives without actually knowing the decision maker's utility function.

Example (Kim's Party Problem)

- The risk profiles for the three alternatives are shown below:



- As all three excess probability distributions cross each other, there is no first-order stochastic dominance among the three alternatives.

- Hence none of the alternatives can be eliminated based on first-order stochastic dominance.

4.7.4 Second-Order Stochastic Dominance

- In first-order stochastic dominance, the EPFs and CDFs must not cross each other. If they cross each other, we can check for **second-order** stochastic dominance by considering the cumulative areas under their EPFs or CDFs.

- Let $D_{x\text{-}y}(w)$ denotes the difference in the *Cumulative Areas* up to w under the EPFs of X and Y:

$$D_{x-y}(w) = \int_a^w \left[G_x(t) - G_y(t)\right] dt$$

<u>Definition (Second-Order Stochastic Dominance)</u>

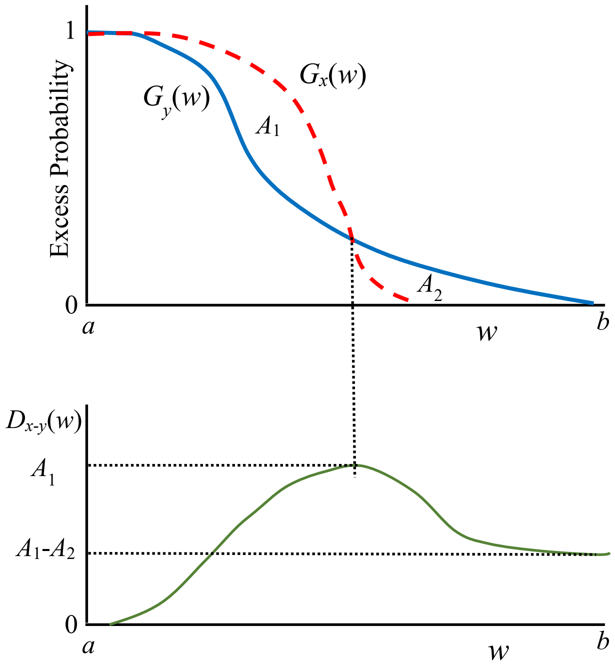
- The risk profile for X *Second-Order Stochastically Dominates* that of Y (denoted as $X >_{2SD} Y$) if and only if

$$D_{x-y}(w) = \int_a^w \left[G_x(t) - G_y(t)\right] dt \geq 0 \qquad \text{for all } w,\ \ a \leq w \leq b \qquad (1)$$

and $$D_{x-y}(w) = \int_a^w \left[G_x(t) - G_y(t)\right] dt > 0 \qquad \text{for some } w,\ \ a \leq w \leq b \qquad (2)$$

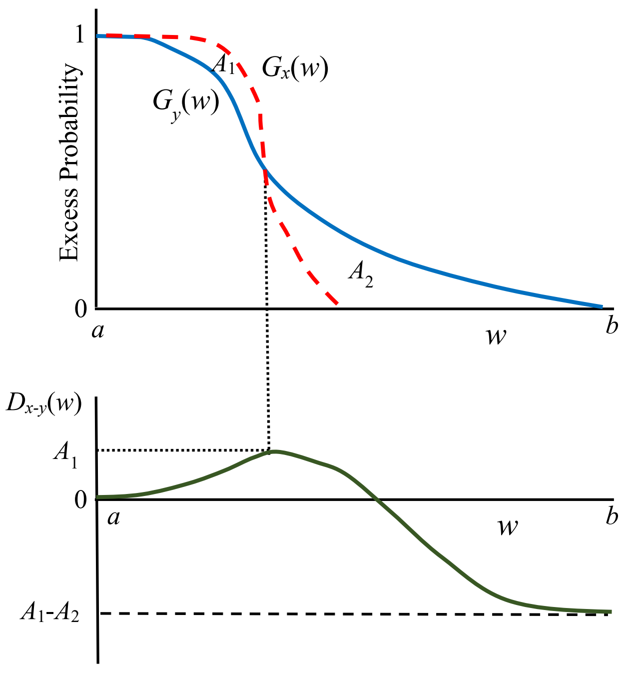
- That is, for all possible outcomes w, the difference in the cumulative area under the EPFs between X and Y up to w, must not be negative and the two EPFs are not identical.

Example (Second order stochastic dominance)



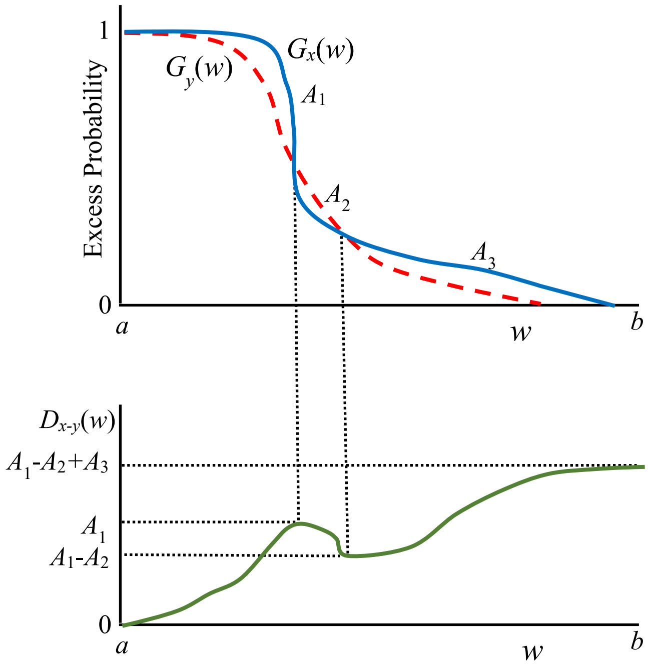
X* second-order stochastically dominates *Y

Example (No second-order stochastic dominance)



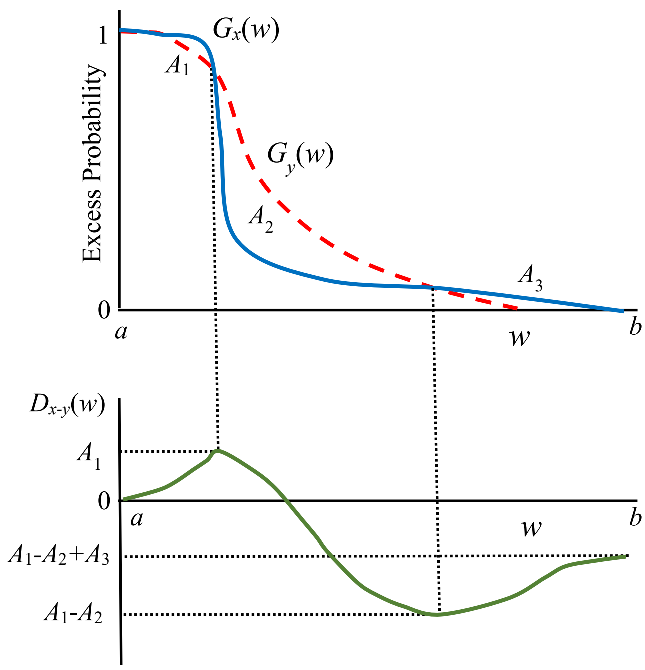
No Second-Order Stochastic Dominance between *X* and *Y*

Example (Second-order stochastic dominance)



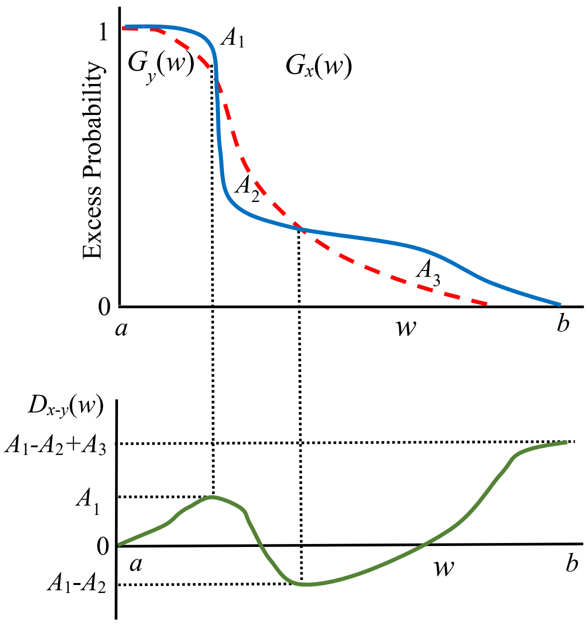
X* second-order stochastically dominates *Y

Example (No second-order stochastic dominance)



No Second-Order Stochastic Dominance between *X* and *Y*

Example (No second-order stochastic dominance)



No Second-Order Stochastic Dominance between *X* and *Y*

- Note that

  Deterministic Dominance
  $\Rightarrow$ First-Order Stochastic Dominance
  $\Rightarrow$ Second-Order Stochastic Dominance

  but the converses of the above are all not true.

4.7.5 Rational Choice under Second-Order Stochastic Dominance

- If the risk profile for *X* second-order stochastically dominates that for *Y*, and the utility function $u(w)$ is non-decreasing and concave in w (i.e., the decision-maker is risk-averse), then

  (a) $E_x[u(w)] > E_y[u(w)]$
  where $E_x[u(w)] = \int_a^b f_x(w)u(w)\,dw$ and $E_y[u(w)] = \int_a^b f_y(w)u(w)\,dw$ are the expected utilities of *X* and *Y* respectively; and

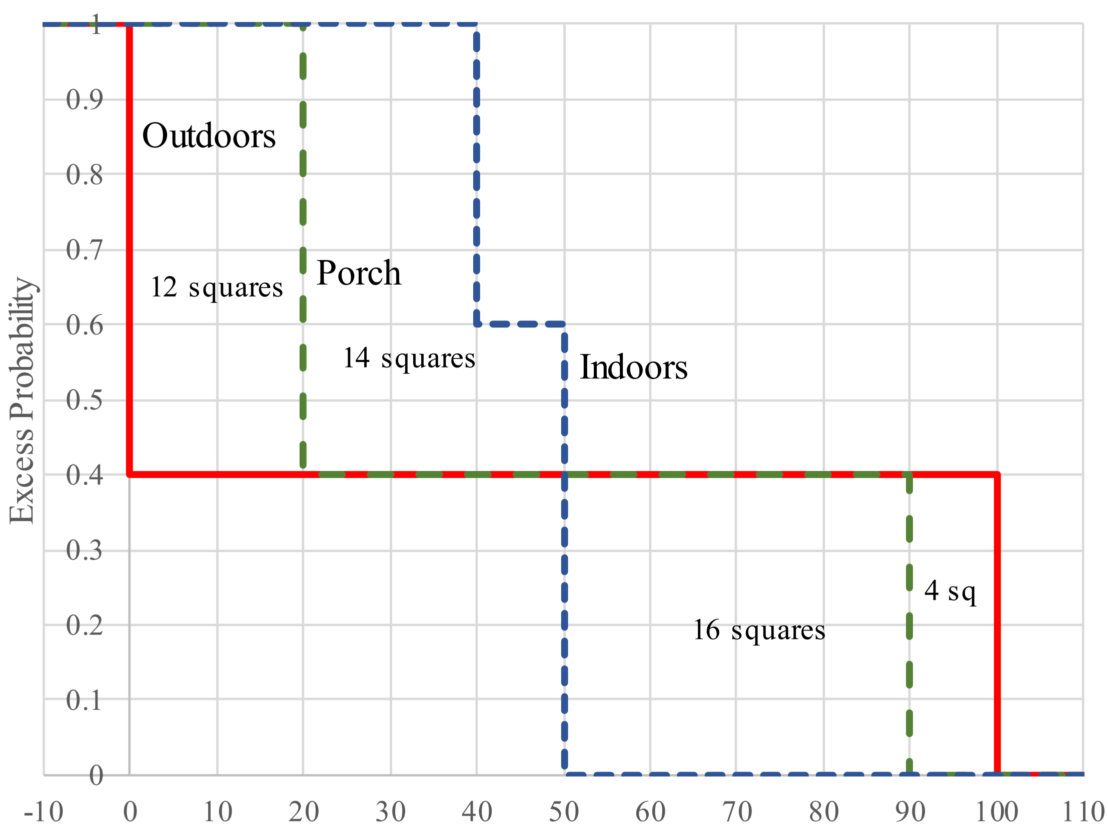
  (b) Alternative *X* is preferred to alternative *Y*.

Proof: See Appendix B to this chapter.

- Second-order stochastic dominance allows us to make a rational choice between two alternatives only when the decision maker is risk averse. When the decision maker is risk-seeking, the result is inconclusive.

Example: Kim's Party Problem

- The risk profiles for the three alternatives are shown below:



- We had earlier concluded that there is no first-order dominance among the three alternatives. We shall now check for second-order stochastic dominance.

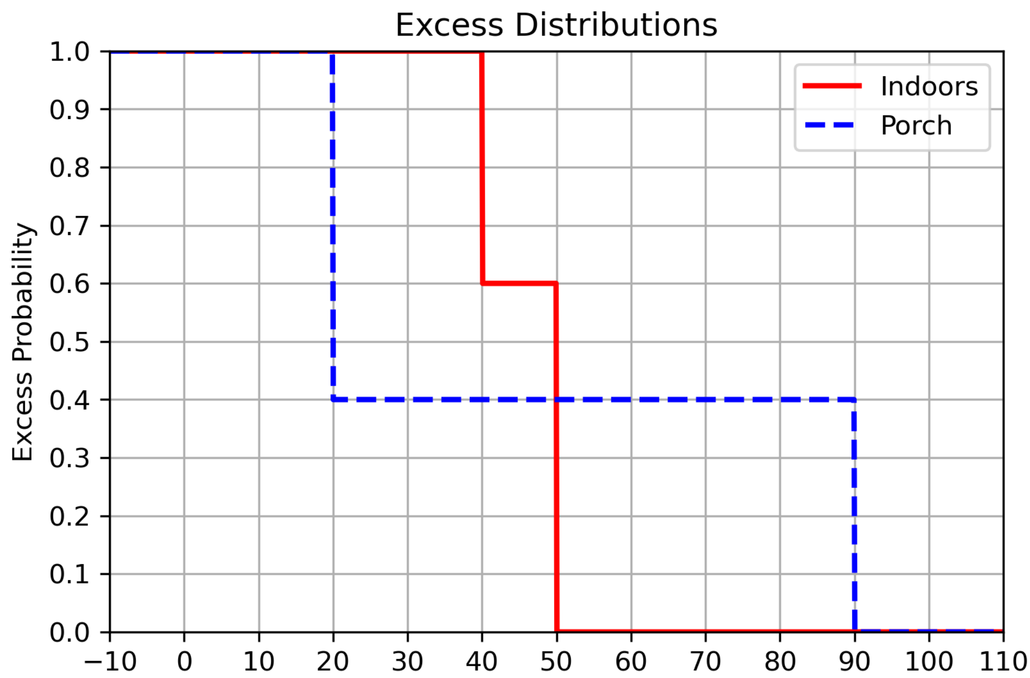
- From the risk profiles, we can conclude that out of the total of 6 pairs of alternatives to check for 2SD, we just need to check the following three pairs:

  1. Does Indoors 2SD Porch?

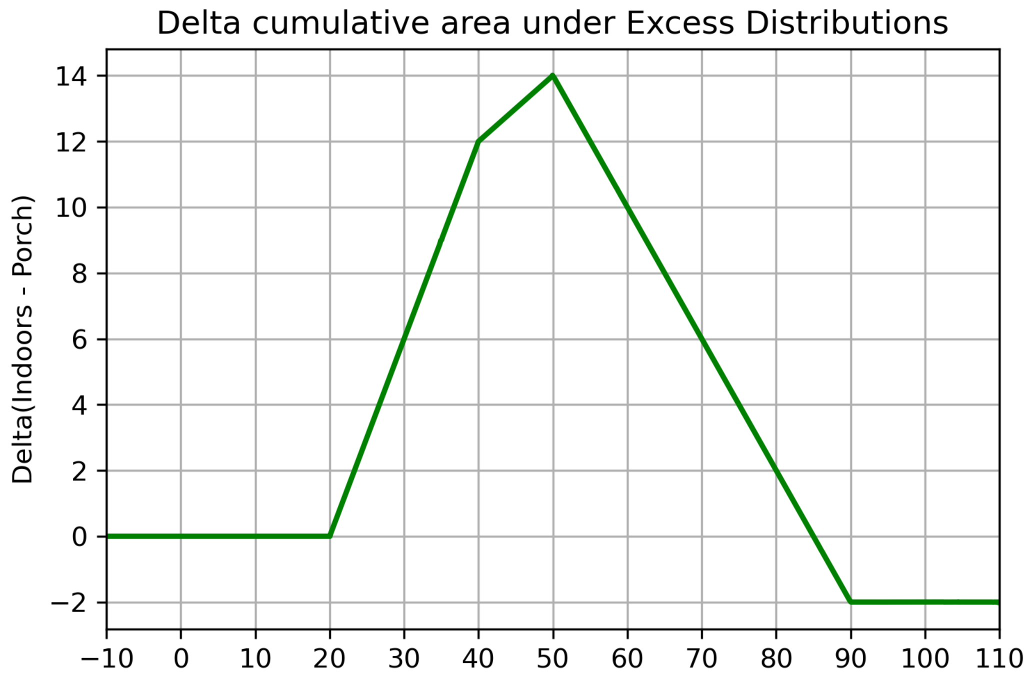
  2. Does Indoors 2SD Outdoors?

  3. Does Porch 2SD Outdoors?

1. Checking if Indoors 2SD Porch:

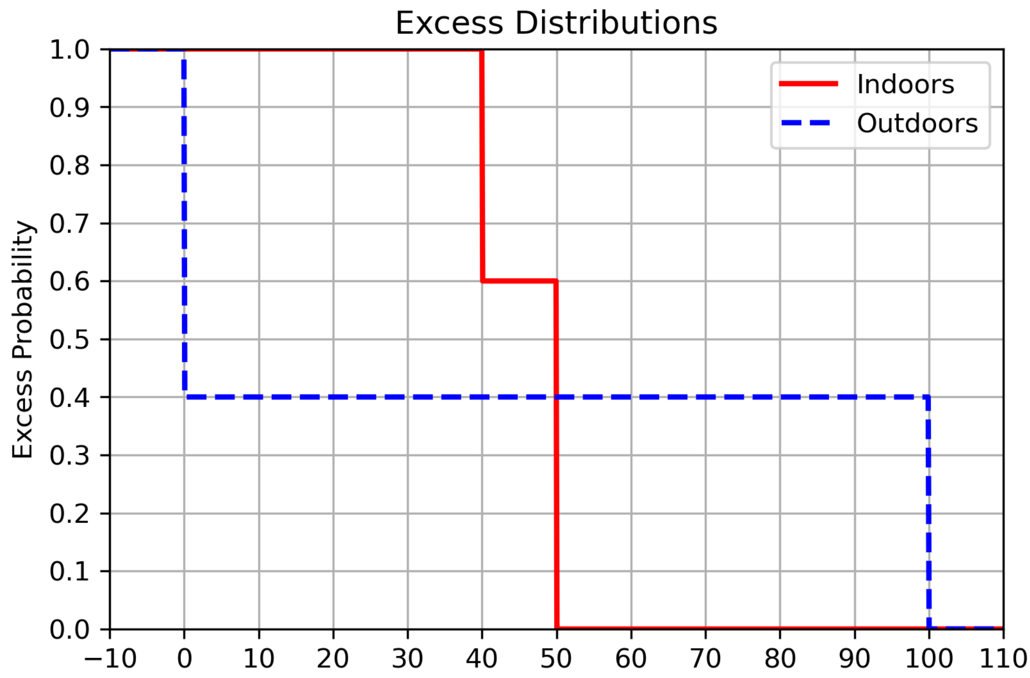


- Differences in cumulative areas under risk profiles:

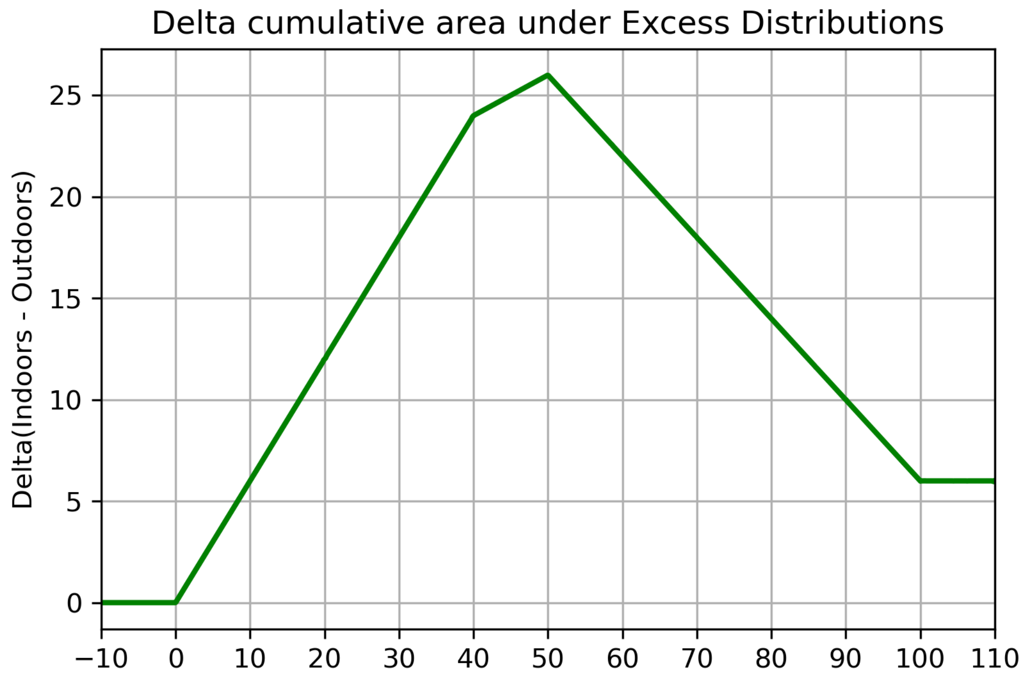


- Conclusion: Indoors does not 2SD Porch.

2. Checking if Indoors 2SD Outdoors:

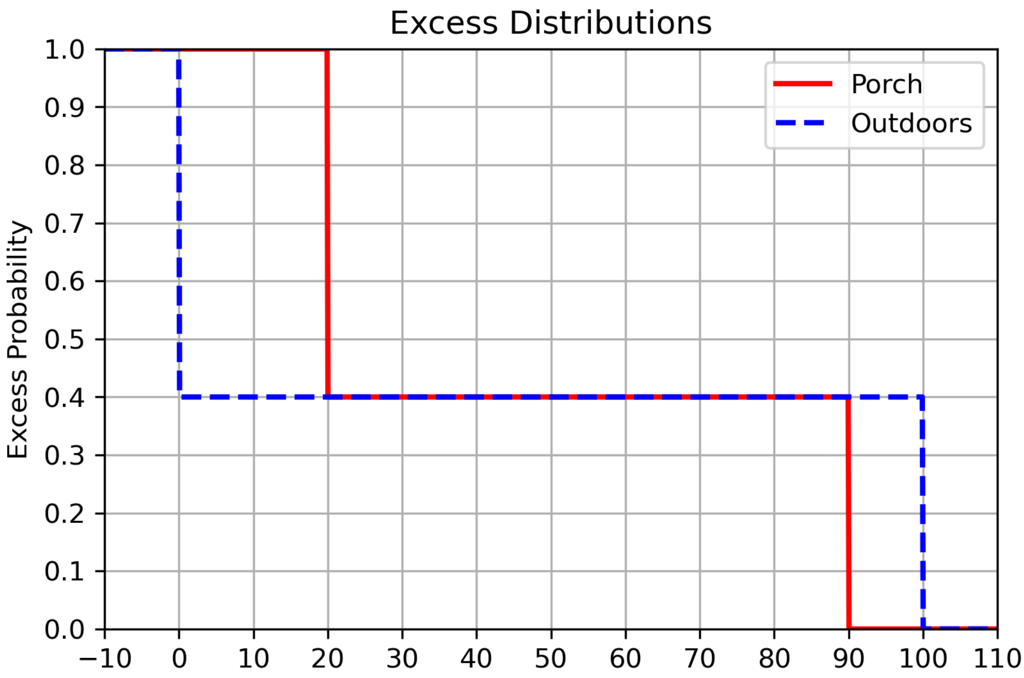


- Differences in cumulative areas under risk profiles:

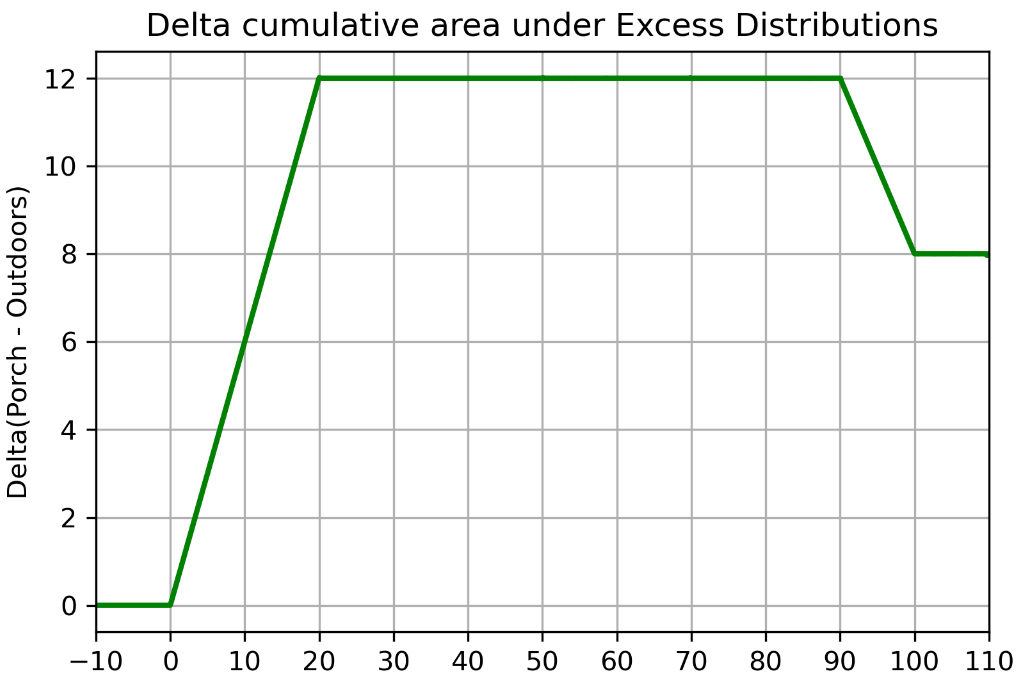


- Conclusion: Indoors 2SD Outdoors

3. Checking if Porch 2SD Outdoors:



- Differences in cumulative areas under risk profiles:



- Conclusion: Porch 2SD Outdoors

- Summary of Results:

    1. Porch second-order stochastic dominates Outdoors.

    2. Indoors second-order stochastic dominates Outdoors.

- Hence Kim will never select the Outdoors alternative (at p=0.4) as long as she is risk averse.

- The resolution between Indoors and Porch would require Kim's actual utility function.

Summary

- Stochastic dominance analysis involves the comparison of the risk profiles of the alternatives.

- It may allow us to eliminate an alternative that is found to be dominated by another alternative under the right conditions.

- A rational choice can be made between two alternatives when there is first-order stochastic dominance between their risk profiles and the decision maker's utility function is non-decreasing in the value outcomes, i.e., the decision maker prefers more money to less.

- A rational choice can be made between two alternatives when there is second-order stochastic dominance between their profiles and the decision maker's utility function is non-decreasing concave in the value outcomes, i.e. the decision maker is risk averse.

4.8 Decision Modeling and Analysis with Excel + Add-on

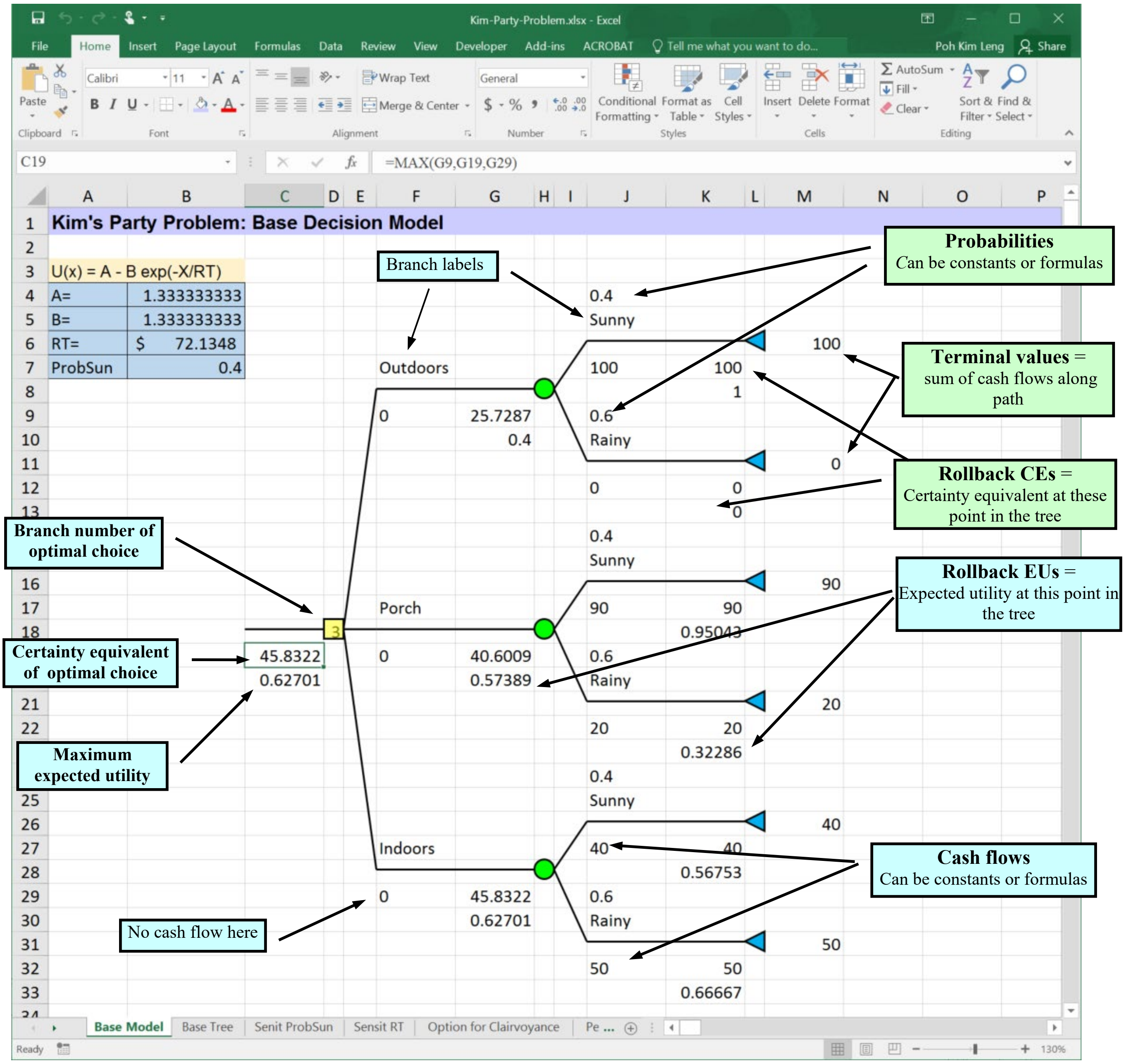
4.8.1 TreePlan

- Website: http://www.treeplan.com

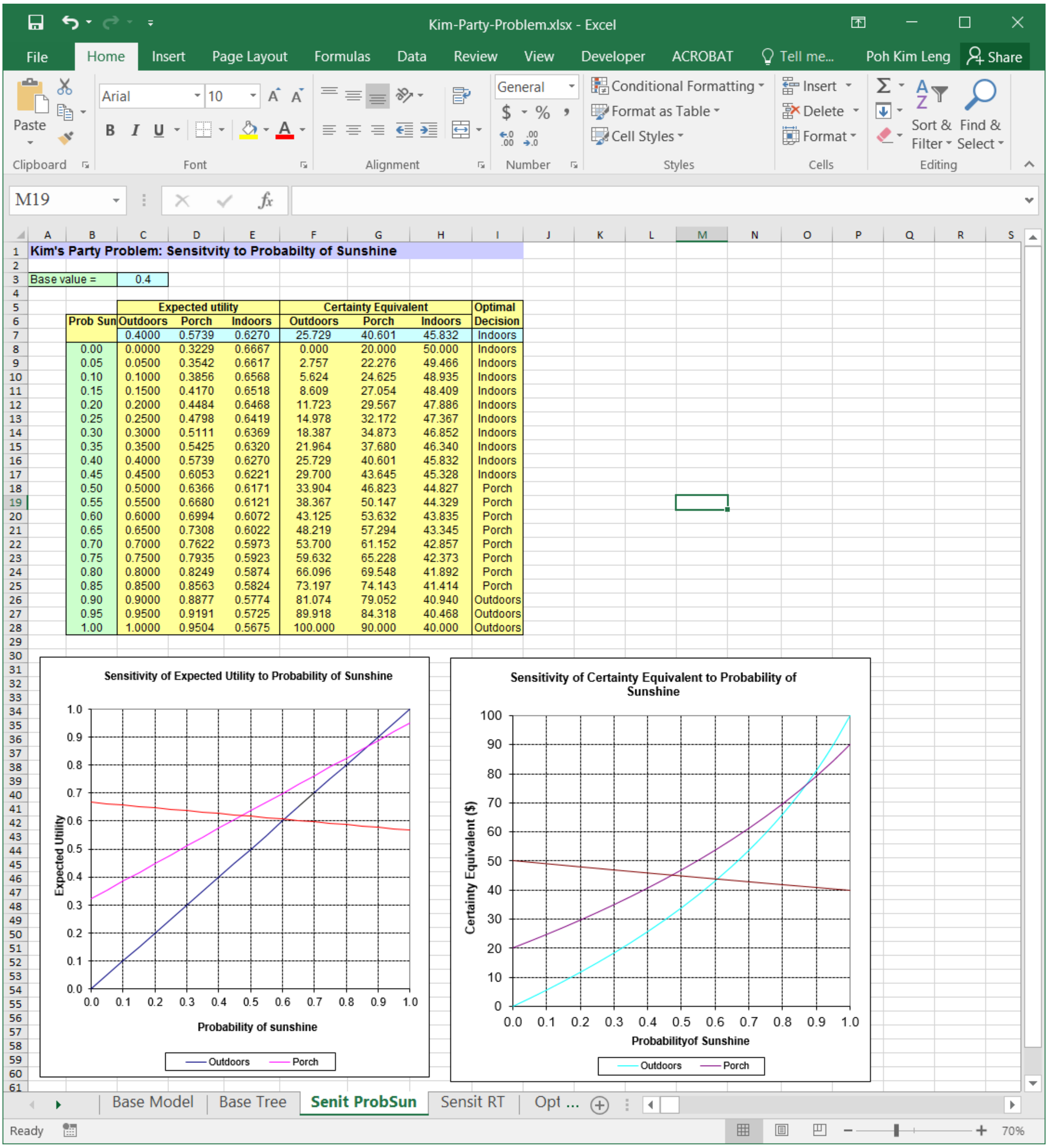
- Main Features:
  1. Provides tools for creating and modifying decision trees.
  2. Endpoint payoffs are cumulative sum of local payoffs along the branches.
  3. Supports built in exponential utility function $u(x) = A - B\, e^{-x/R}$.
  4. Automatic decision tree rollback and displays both expected utility values and certainty equivalents at nodes.

- Main drawbacks:
  1. No sensitivity analysis tools for plotting rainbow diagrams. Users must use Excel Table to create tables on their own and plot using Excel Chart.
  2. No automatic risk profile generation.
  3. No automatic expected value of perfect information computations.
  4. No automatic tree flipping.

- Modeling Kim's Party Problem using Excel + Treeplan:



- Using Excel Table function to generate a one-way sensitivity table for the probability of sunshine, and then using Excel Chart to plot rainbow diagrams.

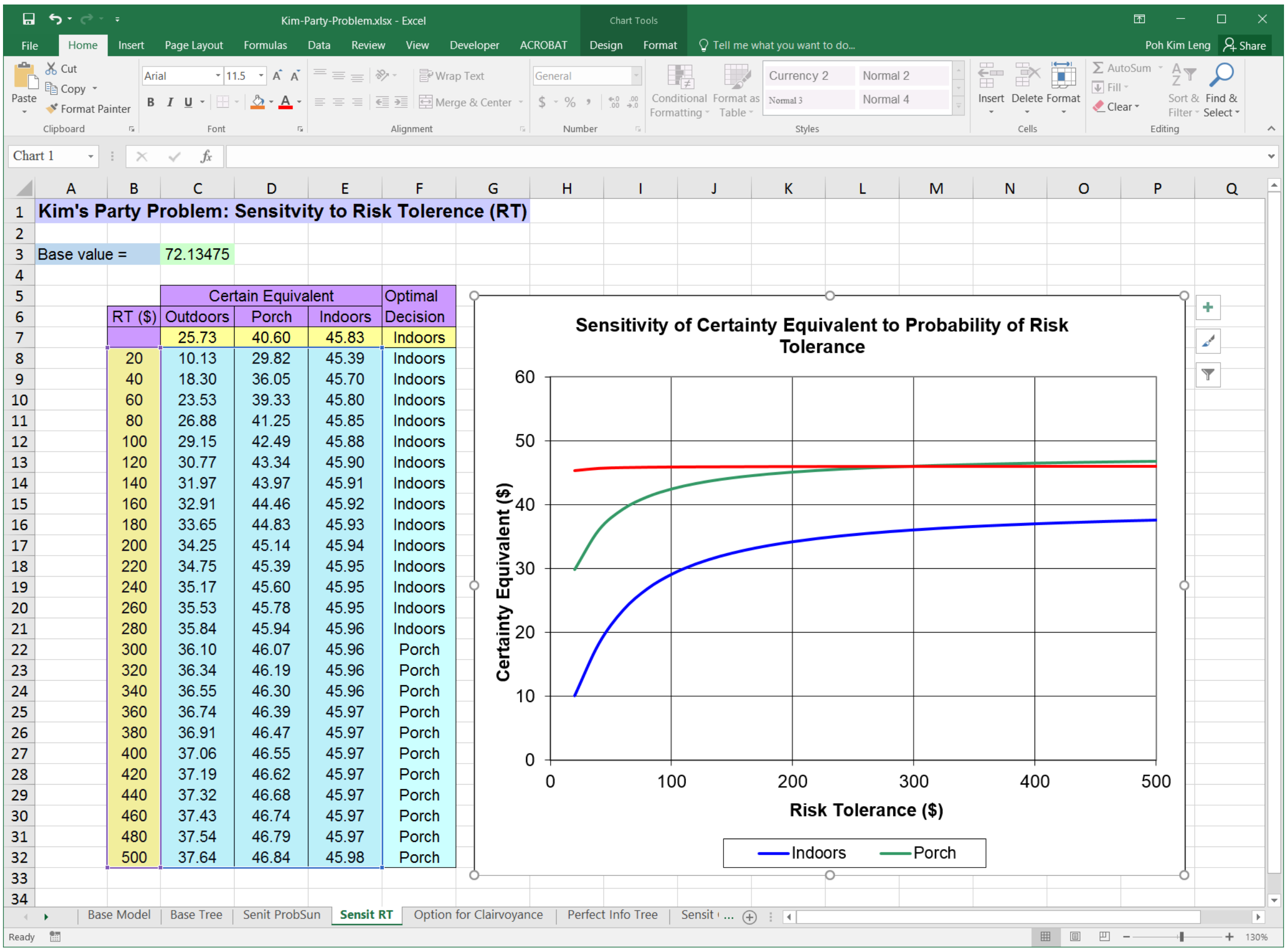


Kim's Party Problem: Sensitvity to Probabilty of Sunshine

Base value = 0.4

	Expected utility			Certainty Equivalent			Optimal
Prob Sun	Outdoors	Porch	Indoors	Outdoors	Porch	Indoors	Decision
	0.4000	0.5739	0.6270	25.729	40.601	45.832	Indoors
0.00	0.0000	0.3229	0.6667	0.000	20.000	50.000	Indoors
0.05	0.0500	0.3542	0.6617	2.757	22.276	49.466	Indoors
0.10	0.1000	0.3856	0.6568	5.624	24.625	48.935	Indoors
0.15	0.1500	0.4170	0.6518	8.609	27.054	48.409	Indoors
0.20	0.2000	0.4484	0.6468	11.723	29.567	47.886	Indoors
0.25	0.2500	0.4798	0.6419	14.978	32.172	47.367	Indoors
0.30	0.3000	0.5111	0.6369	18.387	34.873	46.852	Indoors
0.35	0.3500	0.5425	0.6320	21.964	37.680	46.340	Indoors
0.40	0.4000	0.5739	0.6270	25.729	40.601	45.832	Indoors
0.45	0.4500	0.6053	0.6221	29.700	43.645	45.328	Indoors
0.50	0.5000	0.6366	0.6171	33.904	46.823	44.827	Porch
0.55	0.5500	0.6680	0.6121	38.367	50.147	44.329	Porch
0.60	0.6000	0.6994	0.6072	43.125	53.632	43.835	Porch
0.65	0.6500	0.7308	0.6022	48.219	57.294	43.345	Porch
0.70	0.7000	0.7622	0.5973	53.700	61.152	42.857	Porch
0.75	0.7500	0.7935	0.5923	59.632	65.228	42.373	Porch
0.80	0.8000	0.8249	0.5874	66.096	69.548	41.892	Porch
0.85	0.8500	0.8563	0.5824	73.197	74.143	41.414	Porch
0.90	0.9000	0.8877	0.5774	81.074	79.052	40.940	Outdoors
0.95	0.9500	0.9191	0.5725	89.918	84.318	40.468	Outdoors
1.00	1.0000	0.9504	0.5675	100.000	90.000	40.000	Outdoors

- Using Excel Table function to generate a one-way sensitivity table for Risk Tolerance, and then using Excel Chart to plot rainbow diagrams.



Kim's Party Problem: Sensitvity to Risk Tolerence (RT)

Base value = 72.13475

	Certain Equivalent			Optimal
RT ($)	Outdoors	Porch	Indoors	Decision
	25.73	40.60	45.83	Indoors
20	10.13	29.82	45.39	Indoors
40	18.30	36.05	45.70	Indoors
60	23.53	39.33	45.80	Indoors
80	26.88	41.25	45.85	Indoors
100	29.15	42.49	45.88	Indoors
120	30.77	43.34	45.90	Indoors
140	31.97	43.97	45.91	Indoors
160	32.91	44.46	45.92	Indoors
180	33.65	44.83	45.93	Indoors
200	34.25	45.14	45.94	Indoors
220	34.75	45.39	45.95	Indoors
240	35.17	45.60	45.95	Indoors
260	35.53	45.78	45.95	Indoors
280	35.84	45.94	45.96	Indoors
300	36.10	46.07	45.96	Porch
320	36.34	46.19	45.96	Porch
340	36.55	46.30	45.96	Porch
360	36.74	46.39	45.97	Porch
380	36.91	46.47	45.97	Porch
400	37.06	46.55	45.97	Porch
420	37.19	46.62	45.97	Porch
440	37.32	46.68	45.97	Porch
460	37.43	46.74	45.97	Porch
480	37.54	46.79	45.97	Porch
500	37.64	46.84	45.98	Porch

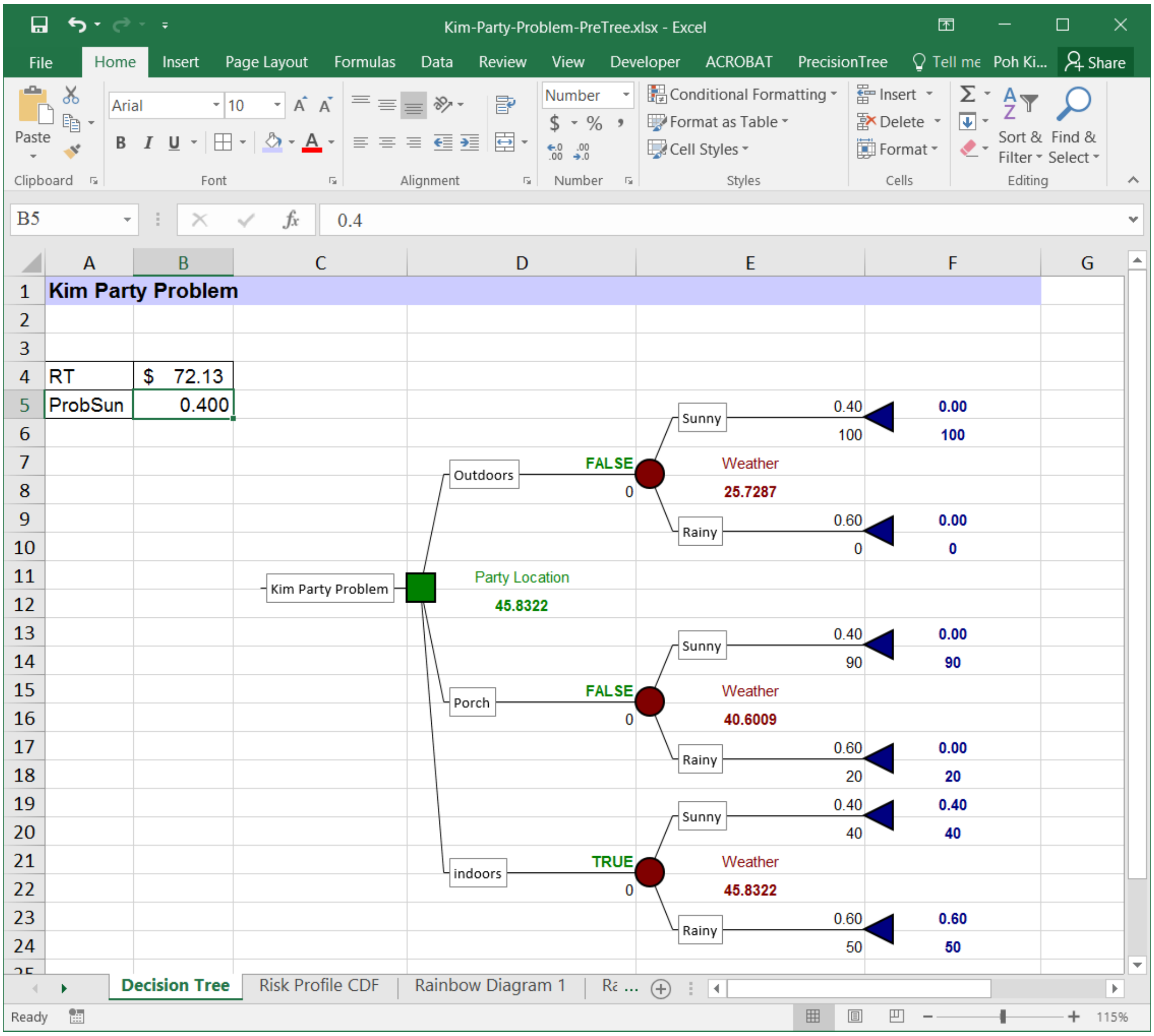
4.8.2 PrecisionTree

- Website: http://www.palisade.com/precisiontree

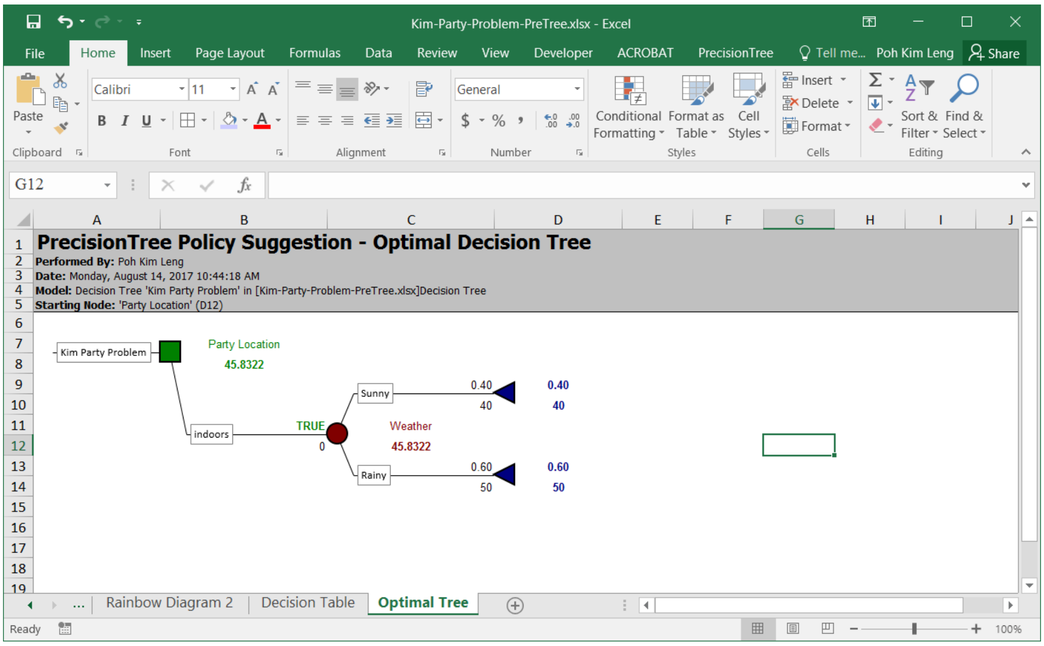
- Main Features:
  1. Provides tools for creating and modifying decision trees.
  2. Options for endpoint payoffs computations.
  3. Supports built-in exponential utility function.
  4. Automatic tree rollback and displays both expected utility values and certainty equivalents at nodes.
  5. Automatic generation of risk profiles.
  6. Provides one-way and two-way sensitivity analysis tools (rainbow diagrams generation)
  7. Tools for flipping trees.

- Main drawbacks:
  1. No automatic expected value of perfect information computation.

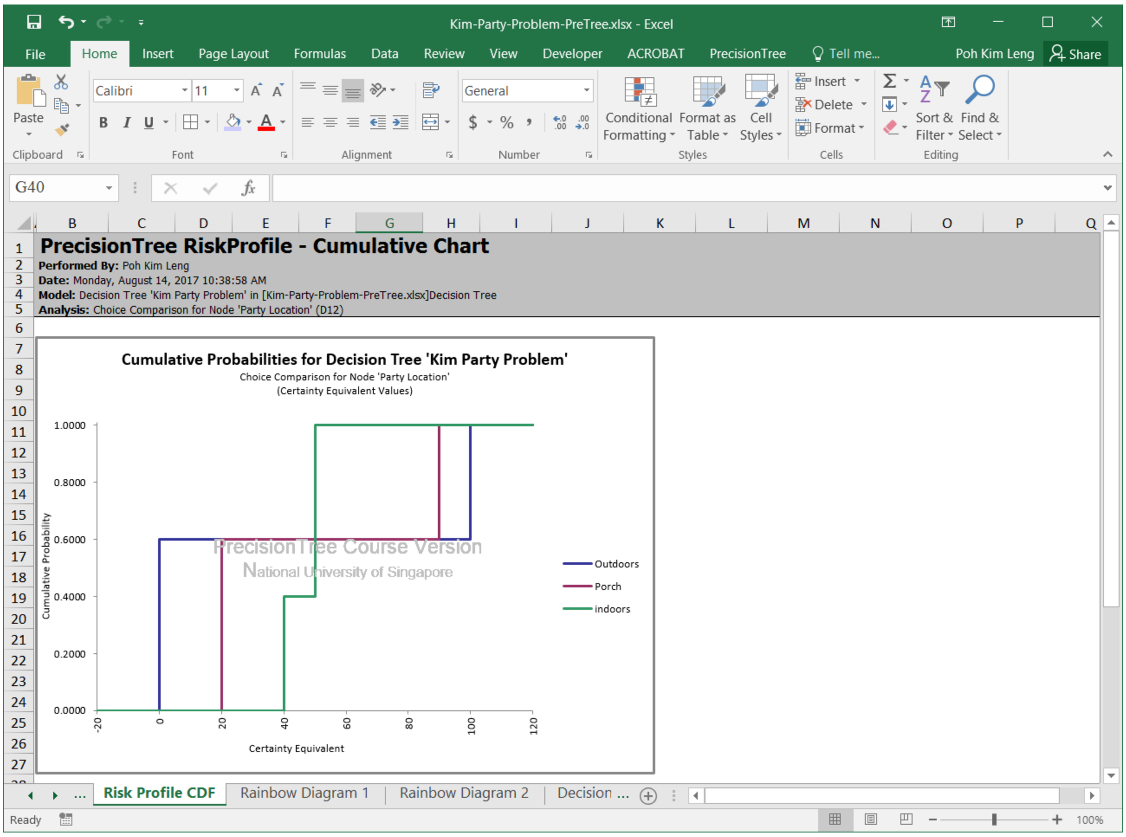
- Modeling Kim's Party Problem using Excel + PrecisionTree:



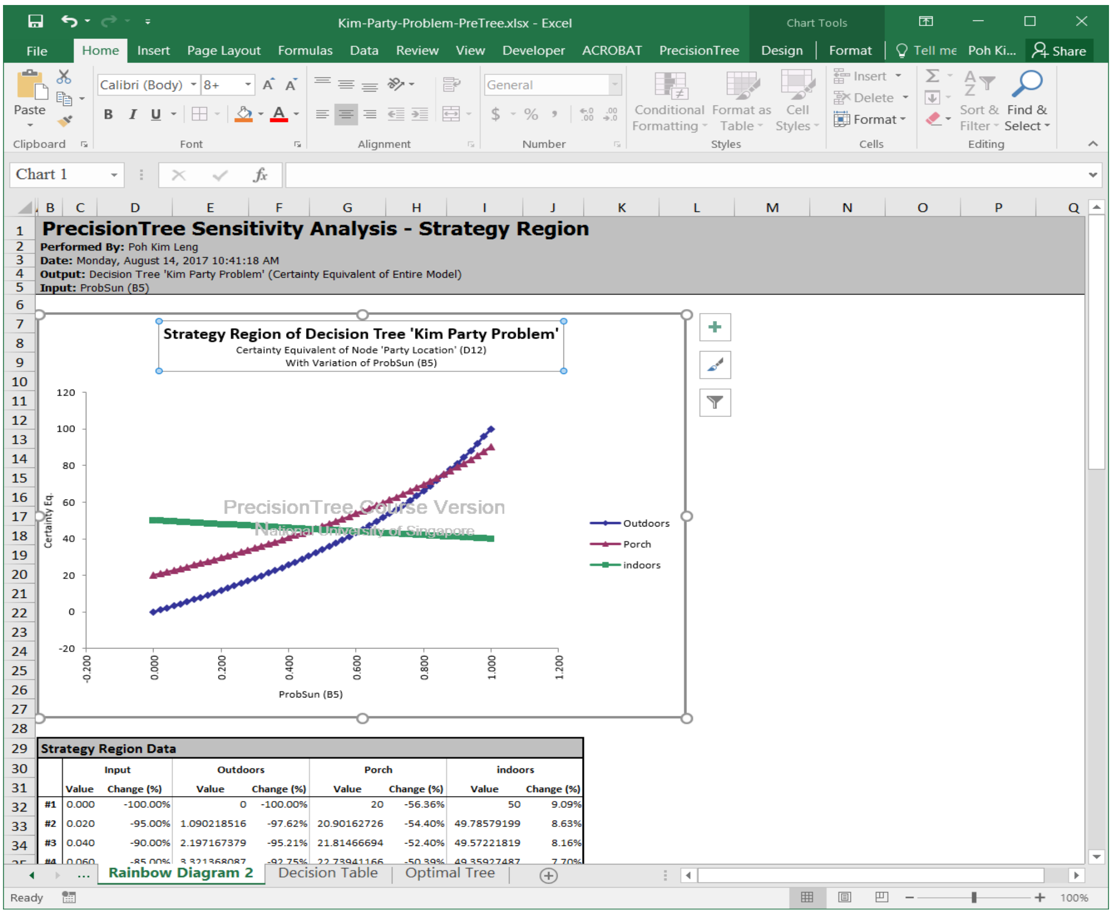
- Optimal Decision Tree.



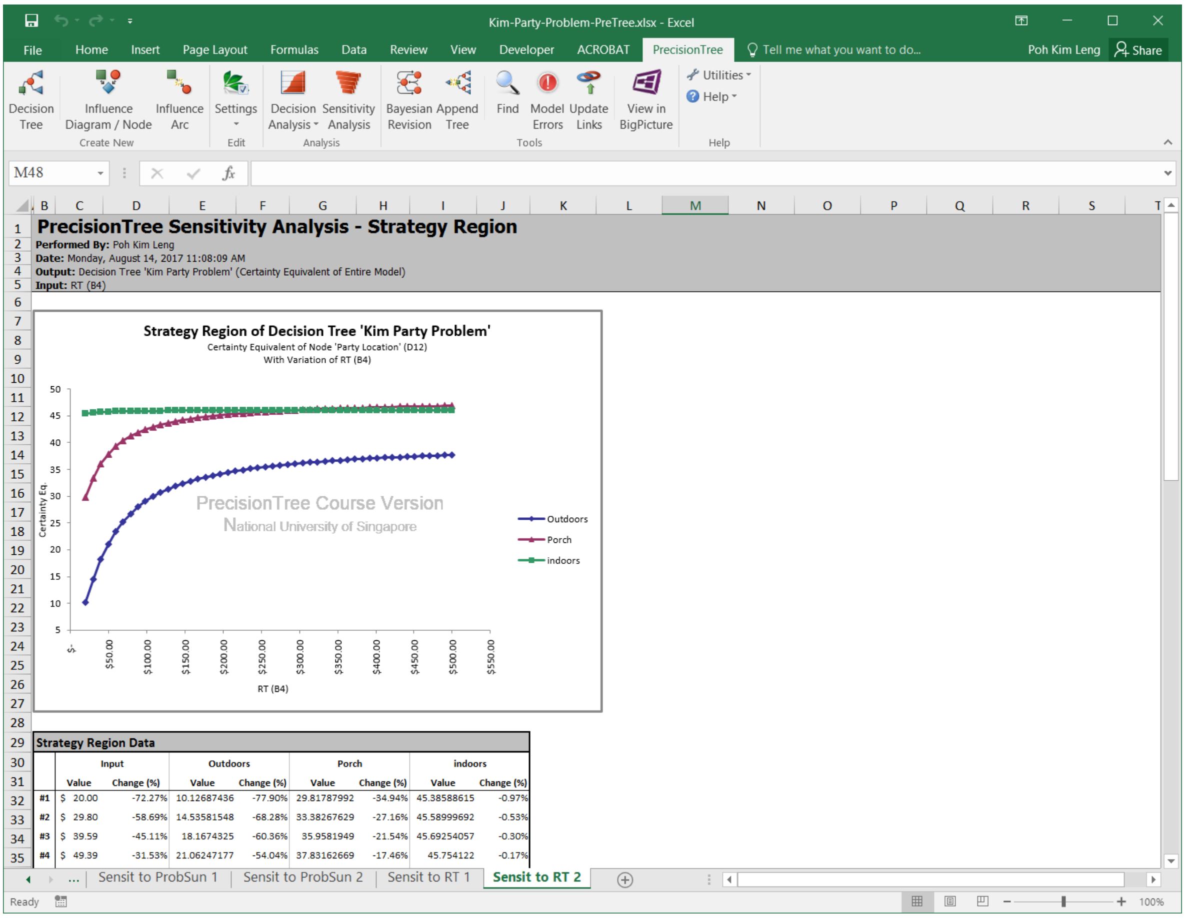
- Risk Profiles automatically generated



- Rainbow Diagram for Sensitivity to Probability of Sunshine automatically generated.



- Rainbow Diagram for Sensitivity to Risk Tolerance automatically generated.



Acknowledgment

The Party Problem example and the characters used in this course were adopted from Professor Ronald A. Howard's Decision Analysis class in the Department of Management Science and Engineering at Stanford University.

Exercises

P4.1 (Clement and Reilly 2001, Problem 3.22, p 103)

To be, or not to be, that is the question:
Whether 'tis nobler in the mind to suffer
The slings and arrows of outrageous fortune
Or to take arms against a sea of troubles,
And by opposing end them. To die – to sleep –
No more; and by a sleep to say we end
The heartache, and the thousand natural shocks
That flesh is heir to. 'Tis a consummation
Devoutly to be wished. To die – to sleep.
To sleep – perchance to dream: ay, there's the rub!
For in that sleep of death what dreams may come
When we have shuffled off this mortal coil,
Must give us pause. There's the respect
That makes calamity of so long life.
For who would bear the whips and scorns of time,
The oppressor's wrong, the proud man's contumely,
The pangs of despised love, the law's delay,
The insolence of office, and the spurns
That patient merit of the unworthy takes,
When he himself might his quietus make
When a bare bodkin? Who would these fardels bear,
To grunt and sweat under a weary life,
But that the dread of something after death –
The undiscovered country, from whose bourn
No traveler returns – puzzles the will,
And makes us rather bear those ills we have
Than fly to others that we know not of?

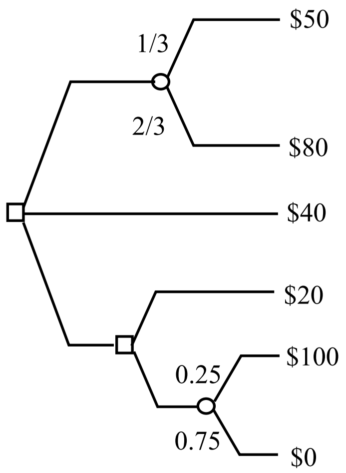
-- *Hamlet*, Act III, Scene 1

Describe Hamlet's decision. What are his choices? What risk does he perceive? Construct a decision tree for Hamlet.

P4.2 Kim has the following preference probabilities for deals with \$100 as the best outcome and \$0 as the worst outcome.

Value (\$)	Preference probability
0	0
10	0.17
20	0.32
40	0.57
50	0.67
80	0.89
90	0.95
100	1

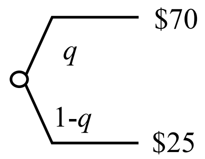
What is Kim's certainty equivalent for the following opportunity?



P4.3 Connie has the preference probabilities listed below:

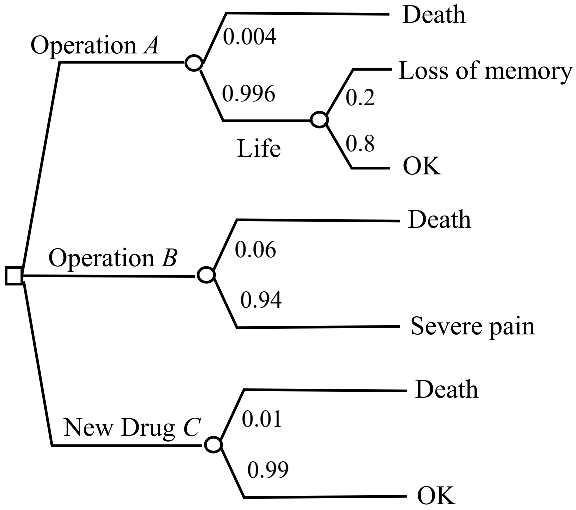
Value (\$)	Preference probability
0	0
25	0.3
40	0.5
70	0.8
100	1.0

Connie is offering a deal to her friend Sam, who has \$40 to spend. The dollar values on the deal are shown below, but the probabilities have yet to be determined.

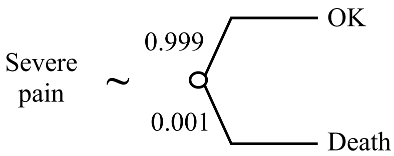


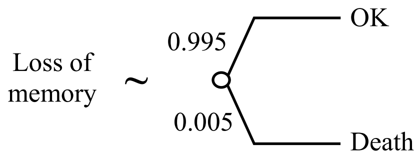
Using only the information available (and without worrying about whether Sam will want to buy the deal), what is the value of q for which Connie is indifferent to selling or not selling the deal for \$40?

P4.4 Dr. Tan follows the rules of actional thought. He is an eminent brain surgeon. One day, he has a patient, Mr. Goh, who is unconscious. In considering medical procedures to use for Mr. Goh, he creates the following tree about state-of-the-art medical technology and its impact on patients.



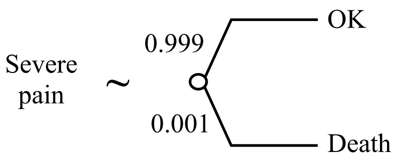
Since Mr. Goh is unconscious, Dr. Tan cannot assess his preference over the possible outcomes. One of the nurses suggests that Dr. Tan use the preference of a typical patient as he sees appropriate from his experience. The preference ordering Dr. Tan lays out is as follows, from the most preferred to the least preferred: OK, Severe pain, Loss of memory, Death. He pulls from his medical records a file on a recent patient in a very similar situation as Mr. Goh; he intends to use this patient's preference in the preliminary analysis for Mr. Goh's case.

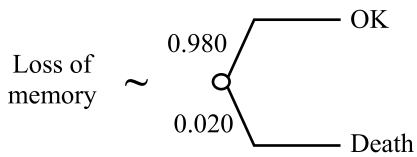




(***a***) According to the above information, what is Dr. Tan's best decision?

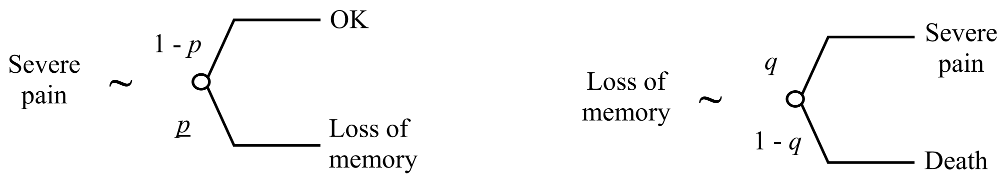
(***b***) Just as Dr. Tan is thinking about how to improve this decision, Mr. Goh regains consciousness. Mr. Goh, who also considers the rules of actional thought as his norm for decision-making, starts reviewing Dr. Tan's decision process. He agrees with Dr. Tan's ordering and assessments, except that he feels memory is less valuable to him than Dr. Tan supposed. The following is Mr. Goh's modified preference assessment.





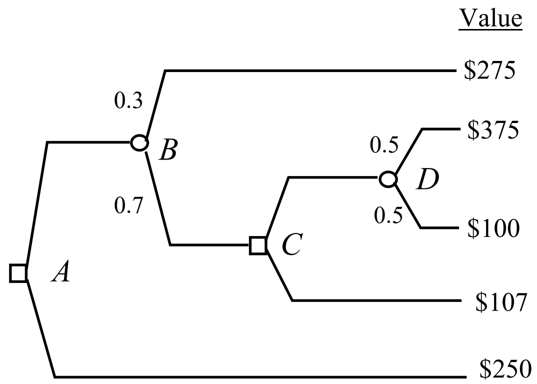
What is Mr. Goh's best decision?

(*c*) Suppose Mr. Goh gives the following preference probabilities instead of those above.



where $p = 0.05$ and $q = 0.981$. Is the above information sufficient for you to recommend a decision? If so, give your recommendation; if not, explain.

P4.5 Jeanne who is risk neutral faced the following decision problem where the dollar values are winnings.



(*a*) What is Jeanne's certainty equivalent for decision *A*?

(*b*) What is the most Jeanne should pay for clairvoyance on *D* before making decision *A*?

(*c*) What is the most Jeanne should pay for clairvoyance on *D* that will be available between decisions *A* and *C*?

(*d*) What is the most Jeanne should pay for clairvoyance on *B* before making decision *A*?

P4.6 Dorothy loves to spend her week-long vacation hiking in Yosemite National Park. But in certain environmental conditions, the streams of Yosemite Valley are contaminated by an organism that will make her very sick. She can't bring enough water with her to avoid drinking stream water, so if the streams are contaminated, she will certainly get sick.

Dorothy determines that getting sick from contaminated water on a hike is like losing \$2000. Going hiking and not getting sick is worth \$500 to Dorothy. Dorothy is risk neutral.

(*a*) Dorothy is considering going on a hiking trip in the winter. She believes that the probability of the streams being contaminated at this time of the year is 0.02. What does she decide? What is her certain equivalent for this choice?

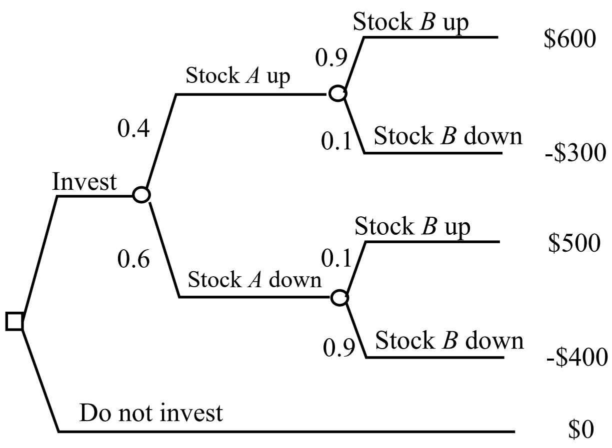
(*b*) What is the value of perfect information on the presence of contaminated water?

To reduce the chance of getting sick, Dorothy could go hiking during the warmer seasons, either Spring or Summer. Unfortunately, the park entrance in the Spring costs \$20; however, she feels that the chance that the streams are contaminated is only 0.005. Alternatively, she could wait until the middle of Summer to take her vacation, when the probability of contaminated streams is 0.0001. But Summer is the peak tourist season, and park admission is \$30.

(*c*) If she has the opportunity to take either a Spring or a Summer hiking trip, which trip should she take? Assume that she can take only one, not both.

(*d*) Regardless of your answer for part (*c*), assumes she waits until Summer to take her hiking vacation. Right before she leaves, she is offered clairvoyance on contaminated streams. (Her only alternative at that time is to stay home, which is worth \$0.) What is the most she should pay for this information?

P4.7 Al, who is risk neutral, faces the following investment decision on stocks *A* and *B* whose performance he believes to be related. If he invests, he must invest in both stocks.



(*a*) Find Al's certainty equivalent for the deal.

(*b*) Find Al's value of clairvoyance on the performance of Stock *A*.

(*c*) Find Al's value of clairvoyance on the performance of Stock *B*.

(*d*) Find Al's value of clairvoyance on both the performance of Stocks *A* and *B* together.

P4.8 Roy has prostate cancer. He has three alternatives: surgery, chemotherapy, and playing a lot of golf. Prostate cancer is a very bad disease, so he will die soon, regardless of what he does. He likes to play golf. If he chooses golf (and has no medical treatment), he will live one year in pain, but be able to enjoy his golf game during that time, then he will die. If he undergoes chemotherapy, he will suffer tremendous nausea for six months and then he will either live for one good year or die right away. If he has prostate surgery he will either live two good years or die on the operating table.

After interviewing his doctor and finding out as well as he can how each of these will feel to him, he decides that he can order these prospects and he assigns the preference probabilities given in the left column.

1.0	Live two good years.
0.9	Live six months nauseated and then one year okay.
0.7	Live one year in pain.
0.2	Live six months nauseated.
0.0	Die now.

He says he wants to make his decision based on the probabilities assigned by his doctor, who says that there is a 60% chance that the surgery will give a better outcome and a 60% chance that chemotherapy will give a better outcome.

(*a*) Which alternative should Roy choose?

(*b*) If it is possible, calculate the value of perfect information (clairvoyance) on whether the surgery will be a success. If not, explain why and indicate what additional assessment(s) would make it possible to perform this calculation.

(*c*) Why might Roy want to know the value of perfect information (clairvoyance)?

Appendix A Proof for Rational Choice under First Order Stochastic Dominance and Non-decreasing Utility Functions

Let X and Y be two random variables with EPFs $G_x(w)$ and $G_y(w)$ respectively, and CDFs $F_x(w)$ and $F_y(w)$ respectively, where $F_x(w) = 1 – G_x(w)$ and $F_y(w) = 1 – G_y(w)$.

We provide here the proof for the case where the domains of X and Y are bounded within $[a, b]$ that is $F_x(a) = F_y(a) = 0$, and $F_x(b) = F_y(b) = 1$. The proof may be extended to the case with unbounded limits.

We want to show that if X first-order stochastically dominates Y, i.e., $G_x(w) \geq G_y(w)\ \forall\ w \in [a, b]$ and $G_x(w) > G_y(w)\ \exists w \in [a, b]$, or equivalently, $F_x(w) \leq F_y(w)\ \forall w \in [a, b]$ and $F_x(\mathrm{w}) < F_y(w)\ \exists\ w \in [a, b]$, then $E_x[u(w)] > E_y[u(w)]$ for any strictly non-decreasing function $u(w)$.

The expected utility for X is

$E_x[u(w)] = \int_a^b f_x(w)u(w)dw$ where $f_x(w)$ is the PDF for X.

Integrating by parts, and noting that $\frac{dF_x(w)}{dw} = f_x(w)$, we obtain

$$E_x[u(w)] = \left[u(w)F_x(w)\right]_a^b - \int_a^b F_x(w)u'(w)dw$$

$$= u(b) - \int_a^b F_x(w)u'(w)dw \tag{1}$$

Similarly, the expected utility for Y is

$$E_y[u(w)] = u(b) - \int_a^b F_y(w)u'(w)dw \tag{2}$$

(1) and (2) $\Rightarrow$ $$E_x[u(w)] - E_y[u(w)] = -\int_a^b [F_x(w) - F_y(w)]u'(w)dw$$

We note that by first-order stochastic dominance, the first term in the integral is always non-positive but is strictly negative for at least some w, while the second term $u'(w)$ is always positive for strictly non-decreasing function.

Hence $E_x[u(w)] - E_y[u(w)] > 0$ as required.

■

Appendix B Proof for Rational Choice under Second-Order Stochastic Dominance and Non-decreasing Concave Utility Functions

Using the notations from Appendix A, we want to show that if X second order stochastically dominates Y, i.e., $\int_a^w G_x(t) - G_y(t)dt \geq 0 \ \ \forall w \in [a,b]$ and $\int_a^w G_x(t) - G_y(t)dt > 0 \ \ \exists w \in [a,b]$ or equivalently, $\int_a^w F_x(t) - F_y(t)dt \leq 0 \ \ \forall w \in [a,b]$ and $\int_a^w F_x(t) - F_y(t)dt < 0 \ \ \exists w \in [a,b]$,
then $E_x[u(w)] > E_y[u(w)]$ for any non-decreasing strictly concave (risk-averse) function $u(w)$.

Let $\phi_x(w) = \int_a^w F_x(t)dt$ and $\phi_y(w) = \int_a^w F_y(t)dt$ denote the cumulative areas under the CDFs of X and Y up to level w, respectively.

The conditions for second order stochastic dominance may be written as
$\phi_x(w) - \phi_y(w) \leq 0 \ \ \forall w \in [a,b]$ and $\phi_x(w) - \phi_y(w) < 0 \ \ \exists w \in [a,b]$.

The expected utility for X is $E_x[u(w)] = \int_a^b f_x(w)u(w)dw$.

Integrating by parts,

$$E_x[u(w)] = \int_a^b f_x(w)u(w)dw = \left[u(w)F_x(w)\right]_a^b - \int_a^b F_x(w)u'(w)dw$$

$$= u(b) - \int_a^b F_x(w)u'(w)dw$$

Integrating by parts one more time,

$$E_x[u(w)] = u(b) - \left[\phi_x(w)u'(w)\right]_a^b + \int_a^b \phi_x(w)u''(w)dw$$

$$= u(b) - \phi_x(b)u'(b) + \int_a^b \phi_x(w)u''(w)dw \tag{3}$$

Similarly, the expected utility for Y is

$$E_y[u(w)] = u(b) - \phi_y(b)u'(b) + \int_a^b \phi_y(w)u''(w)dw \tag{4}$$

(3) and (4) $\Rightarrow$ $E_x[u(w)] - E_y[u(w)] = -\left[\phi_x(b) - \phi_y(b)\right]u'(b) + \int_a^b \left[\phi_x(w) - \phi_y(w)\right]u''(w)dw$

We note that $\left[\phi_x(b) - \phi_y(b)\right] \leq 0$, $\left[\phi_x(w) - \phi_y(w)\right] \leq 0 \ \ \forall w$ but $\left[\phi_x(w) - \phi_y(w)\right] < 0 \ \ \exists w$. Also $u'(b) > 0$ and $u''(w) < 0 \, \forall w$ for a strictly non-decreasing concave function. Hence $E_x[u(w)] - E_y[u(w)] > 0$ as required.

■