Chapter 9 Multiple Criteria Decision Making I
Analytic Hierarchy Process

孙子曰：兵者，国之大事，死生之地，存亡之道，不可不察也。故经之以五事，校之以计，而索其情：一曰道，二曰天，三曰地，四曰将，五曰法。

Sun Tze said: War is a matter of vital importance to the State; the province of life or death; the road to survival or ruin. It is mandatory that it be thoroughly studied. Therefore, appraise it in terms of the five fundamental factors and make comparisons. The factors are: (1) Moral influence; (2) Weather; (3) Terrain; (4) Command; and (5) Doctrine.

Multiple Criteria Decision Making in Sun Tze's Art of War

9.1 Introduction

- The **Analytic Hierarchy Process** (AHP) is a Multiple Criteria Decision Making method developed by Thomas L. Saaty.

- It is an easy-to-use decision-making tool for problem solving and decision making in complex environment that involves consideration of multiple and possibly conflicting criteria as well as in situations that require judgements on qualitative factors.

- AHP is based on four steps:

    1. **Decompositions:** A complex problem is decomposed into a hierarchy with a goal or objective at the top, each lower level consisting of a few elements; element is also, in turn, decomposed and so on. The available alternatives are usually at the lowest level of the hierarchy.

    2. **Prioritization:** The impact of the elements at each level of the hierarchy on it parent element is assessed through pairwise comparisons on a ratio scale. This produces a series of pairwise comparison matrixes which are individually evaluated to produce the elements' local weights.

    3. **Synthesis**: the priorities in Step 2 are aggregated together through the *Principle of Hierarchic Composition* to compute the overall assessments or *Global Weights* of the alternatives.

    4. **Sensitivity Analysis**: The impact on decision outcomes to changes in the importance of the elements in the hierarchy is determined via one-way sensitivity analysis to gain managerial insight about the problem.

9.2 Prioritization

9.2.1 Priority or Preference Weights

- We first concentrate on Step 2 of the AHP which is the prioritization of elements at each level of the model.

- Suppose we want to assess the **relative importance** or **priority** on a set of elements or items. These elements can be the criteria, sub-criteria, or alternatives in the AHP model.

- We express the degrees of importance or priority of each items using a set of weights which are usually *normalized* (i.e., they add up to exactly 1 or 100%).

- More formally, suppose we have n items to compare, we seek a vector

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

such that $\sum_{i=1}^{n} w_i = 1$, where w_i expresses the importance, priority or preference weight for item i.

- We will simply refer to the w_i's as the ***weights***.

9.2.2 Method of Pairwise Comparison

Pairwise Comparison Matrix

- Suppose we have n items to compare. We first create a pairwise comparison reciprocal matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = [a_{ij}]$$

such that each element $a_{ij} = \dfrac{w_i}{w_j}$ for all i, j.

Valid Pairwise Comparison Matrix

- Matrix $\boldsymbol{A}$ is valid if and only if it satisfies the followings properties:

  1. The entries of A must be *positive*, *i.e.,* $a_{ij} > 0$ all i, j.
  2. The matrix $\boldsymbol{A}$ is a *reciprocal matrix* with $a_{ij} = \dfrac{1}{a_{ji}}$ for all i, j.
  3. The diagonal elements of $\boldsymbol{A}$ are always one, i.e. $a_{ii} = 1$, for all i.

Example

- Consider 3 elements or items x_1, x_2, x_3 which we wish to compare and prioritize.
- Suppose we assess that importance of x_2 is about *twice* that of x_1, the importance of x_1 is about *thrice* that of x_3, and the importance of x_2 is about *five times* that of x_3, i.e.,

$$\frac{w_2}{w_1} \approx 2; \ \frac{w_1}{w_3} \approx 3; \ \frac{w_2}{w_3} \approx 5$$

- Then $a_{21} = 2$, $a_{13} = 3$, and $a_{23} = 5$.
- The pairwise comparison matrix for x_1, x_2 and x_3 is as follows:

$$A = \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 2 & 1 & 5 \\ \frac{1}{3} & \frac{1}{5} & 1 \end{bmatrix}$$

- We call the pairwise comparison matrix the A-matrix.

Input requirements to construct the *A*-matrix

- In general, given n items, $n(n-1)/2$ number of comparisons are needed.

Perfectly Consistent *A*-matrix

- A pairwise comparison matrix is said to be ***Perfectly Consistent*** if and only if
  1. It is a valid A-matrix, and
  2. $a_{ij} = \dfrac{a_{ik}}{a_{jk}} = a_{ik}a_{kj}$ for all i, j, and k.

Example

- The following matrix is perfectly consistent:

$$\begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 2 & 1 & 6 \\ \frac{1}{3} & \frac{1}{6} & 1 \end{bmatrix}$$

since $a_{12} = 1/2$, $a_{13} = 3$, and $a_{23} = 6 = a_{13}/a_{12}$.

- The following matrix is valid but not perfectly consistent:

$$\begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 2 & 1 & 5 \\ \frac{1}{3} & \frac{1}{5} & 1 \end{bmatrix}$$

since $a_{12} = 1/2$, $a_{13} = 3$, and $a_{23} = 5 \neq a_{13}/a_{12} = 6$.

Relation between *A* and *w*

- Given a perfectly consistent *A*-matrix, if we post-multiply ***A*** by the column vector $\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$,

  we get

$$\mathbf{A}w = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_{n1}} \\ \vdots & \vdots & & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & \frac{w_n}{w_{n1}} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} n\,w_1 \\ n\,w_2 \\ \vdots \\ n\,w_n \end{bmatrix} = n\,w$$

$$\Rightarrow \quad \boldsymbol{A}\,\boldsymbol{w} = n\,\boldsymbol{w}$$

$$\Rightarrow \quad (\boldsymbol{A} - n\,\mathbf{I})\,\boldsymbol{w} = 0.$$

- This gives us a relationship between ***A*** and ***w***.

9.2.3 Computing Priority Weights from a Pairwise Comparison Matrix

Computing *w* when *A* is perfectly consistent.

- If ***A*** is a *Perfectly Consistent* pairwise comparison matrix, then ***w*** may be computed by normalizing any column *j* of ***A***.

i.e. $w_i = \dfrac{a_{ij}}{\sum_{k=1}^{n} a_{kj}}$ for $i = 1$ to n.

Example

- Given $A = \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 2 & 1 & 6 \\ \frac{1}{3} & \frac{1}{6} & 1 \end{bmatrix}$ which is a perfectly consistent matrix.

- Normalizing the first column, we obtain $w = \begin{bmatrix} \frac{1}{1+2+1/3} \\ \frac{2}{1+2+1/3} \\ \frac{1/3}{1+2+1/3} \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix}$

- The same result is obtained by normalizing column 2 or column 3.

Computing *w* when *A* is not perfectly consistent.

- In reality, ***A*** is usually imperfect since it is based on human expert's judgments. That is, $a_{ij} \neq a_{ik} a_{kj}$ for some *i*, *j*, *k*.

- The best estimate for ***w*** can be estimated from the relation $(\boldsymbol{A} - \lambda\, \mathbf{I})\, \boldsymbol{w} = 0$ where λ is a constant that is approximately equal to *n*.

- But from linear algebra, λ and ***w*** are the eigenvalue and eigenvector of ***A*** respectively.

- It can be shown that a positive reciprocal matrix has only *one real dominant* eigenvalue which shall be denoted as λ_{max}.

Methods for finding λ_{max} and *w* given *A*

1. Linear Algebra method (exact but solution may be numerical).

2. Numerical method (very good solution within known tolerance limits).

3. Approximation methods (not exact, but fast and good enough; errors not controllable)

9.2.4 Linear Algebra Method

- We illustrate the Linear Algebra method with a numerical example:

- Given matrix $\boldsymbol{A} = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ 3 & 1 & 3 \\ 2 & \frac{1}{3} & 1 \end{bmatrix}$ which is valid but not perfectly consistent:

- The relationship between $\boldsymbol{\lambda}$ and $\boldsymbol{w}$ is the equation $(\boldsymbol{A} - \lambda\,\mathbf{I})\,\boldsymbol{w} = \mathbf{0}$, i.e.

$$\begin{aligned} (1-\lambda)w_1 + \tfrac{1}{3}w_2 + \tfrac{1}{2}w_3 &= 0 \\ 3w_1 + (1-\lambda)w_2 + 3w_3 &= 0 \\ 2w_1 + \tfrac{1}{3}w_2 + (1-\lambda)w_3 &= 0 \end{aligned}$$

- A trivial solution is obviously $w_1 = w_2 = w_3 = 0$, and λ is any number.

- For non- trivial solutions, the determinant of the coefficients matrix $(\boldsymbol{A} - \lambda\,\mathbf{I})$ must be zero:

$$\mathrm{Det}(\boldsymbol{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & \frac{1}{3} & \frac{1}{2} \\ 3 & 1-\lambda & 3 \\ 2 & \frac{1}{3} & 1-\lambda \end{vmatrix} = 0$$

- Use cofactors expansion, we obtain a cubic equation in λ:

$$(1-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ \frac{1}{3} & 1-\lambda \end{vmatrix} - \frac{1}{3}\begin{vmatrix} 3 & 3 \\ 2 & 1-\lambda \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 3 & 1-\lambda \\ 2 & \frac{1}{3} \end{vmatrix} = 0$$

$\Rightarrow$ $(1-\lambda)[(1-\lambda)^2 - 1] - (1/3)[3(1-\lambda) - 6] + (1/2)[1 - 2(1-\lambda)] = 0$

$\Rightarrow$ $(1-\lambda)(1-2\lambda + \lambda^2 - 1) + (1 + \lambda) + (1/2)(2\lambda - 1) = 0$

$\Rightarrow$ $(1-\lambda)(\lambda^2 - 2\lambda) + 1 + \lambda + \lambda - 0.5 = 0$

$\Rightarrow$ $-\lambda^3 + 3\,\lambda^2 + 0.5 = 0$

$\Rightarrow$ $\lambda = 3.0536$ // 2 other complex roots ignored

- We can try to find $\boldsymbol{w}$ by substituting $\lambda = 3.0536$ back into $(\boldsymbol{A} - \lambda\,\mathbf{I})\,\boldsymbol{w} = \mathbf{0}$, and solve the following set of equations:

$$\begin{aligned} -2.053622w_1 + \tfrac{1}{3}w_2 + \tfrac{1}{2}w_3 &= 0 \\ 3w_1 - 2.053622w_2 + 3w_3 &= 0 \\ 2w_1 + \tfrac{1}{3}w_2 - 2.053622w_3 &= 0 \end{aligned}$$

- However, the 3 equations are not linearly independent. We may drop any of the 3 equations and add the normalizing equation $w_1 + w_2 + w_3 = 1$ to obtain 3 linearly independent equations.

- After replacing the first equation by the normalization equation, we solve the following 3 equations:

$$\begin{aligned} 3w_1 - 2.053622w_2 + 3w_3 &= 0 \\ 2w_1 + \tfrac{1}{3}w_2 - 2.053622w_3 &= 0 \\ w_1 + w_2 + w_3 &= 1 \end{aligned}$$

- In matrix notations:

$$\begin{bmatrix} 3 & -2.053622 & 3 \\ 2 & 1/3 & -2.053622 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Solution:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3 & -2.053622 & 3 \\ 2 & 1/3 & -2.053622 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.15706 \\ 0.59363 \\ 0.25931 \end{bmatrix}$$

- Hence the priority weights are:

w_1 = 0.1571
w_2 = 0.5936
w_3 = 0.2493

- The steps for computing λ_{max} and ***w*** using linear algebra method:

1. Given a valid pairwise comparison matrix ***A*** of size *n*.
2. Determine λ by solving $|\boldsymbol{A} - \lambda \mathbf{I}| = \mathbf{0}$ // Can use numerical method
3. Form a set of *n* linear equations: $(\boldsymbol{A} - \lambda \mathbf{I})\, \boldsymbol{w} = \mathbf{0}$
4. Replace any equation in Step 3 with $\sum_{j=1}^{n} w_j = 1$
5. Determine ***w*** by solving the set of *n* linear equations.

Weights in Distributive and Ideal forms

- Weights are said to be expressed in **Normalized** or **Distributive Form** if they add up to 1.0 or 100%. It tells us how to allocate importance or priorities among the items. The weights in the previous example are expressed in distributive form.

- If we divide all the priority weights by the largest weight, we obtain the weights expressed in **Idealized Form:** It tells us the importance or performance of each item relative to the best (or ideal) item which is allocated a weight 1.0.

- In the above example, the weights expressed in ideal form are:

w_2 = 1
w_3 = 0.420
w_1 = 0.265

9.2.5 The Scale for Pairwise Comparison and Consistency Measures

- Saaty recommended a 9-point scale whose validity is supported by some empirical studies. 9 is also the maximum number of concepts a person can reason without any background noise. (Miller's magic number 7±2).

Saaty's Intensity of Importance Scale

Intensity of Importance	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
3	Weak importance of one over another	Experience and judgment slightly favor one activity over another
5	Essential or strong importance	Experience and judgment strongly favor one activity over another
7	Demonstrated importance	An activity is strongly favored and its dominance demonstrated in practice
9	Absolute importance	The evidence favoring one activity over another is of the highest possible order of affirmation
2,4,6,8	Intermediate values between the two adjacent judgments	When compromise is needed
Reciprocals of above non-zero numbers	If activity *i* has one of the above non-zero numbers assigned to it when compared with activity *j*, then has the reciprocal when compared with *i*	

The Consistency Index (CI)

- In practice, we cannot ensure that the ***A*** matrix is perfectly consistent, i.e., for all i, j and k, $a_{ij} = a_{ik} / a_{jk}$.

- If ***A*** were perfectly consistent, we can find the values of ***w*** by simply normalizing any column j of ***A***, i.e.,

$$w_i = \frac{a_{ij}}{\sum_{k=1}^{n} a_{kj}} \quad \forall i = 1,2,\ldots,n. \qquad \text{and} \qquad \lambda_{max} = n \quad \text{exactly.}$$

- Unfortunately, some inconsistency in judgments cannot be totally avoided. However, we must ensure that the ***A***-matrix does not contain too much inconsistencies.

- It can be shown that for any positive reciprocal matrix of any size n, $\lambda_{max} \geq n$, and $(\lambda_{max} - n)$ increases with the amount of inconsistency in ***A***.

- Saaty defined the **Consistency Index** (CI) of ***A*** to be:

$$CI = \frac{\lambda_{\max} - n}{n-1}$$

(Note that actually CI is a measure of inconsistency and not of consistency)

Example

- For the matrix $\boldsymbol{A} = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ 3 & 1 & 3 \\ 2 & \frac{1}{3} & 1 \end{bmatrix}$, we have found that $\lambda = 3.0536$.

- Hence $CI = \dfrac{\lambda_{\max} - n}{n-1} = \dfrac{3.0536 - 3}{3-1} = 0.0268$

- The matrix $\boldsymbol{A}$ has a CI of 0.0268. But how do we know if this is acceptable?

The Consistency Ratio and the 10% Rule

- Consider a positive reciprocal matrix of size $n > 2$ whose entries are randomly selected from the 9-point scale. This can be done using Monte Carlo simulation.

- These matrices are those with the highest inconsistency.

- The average CI values for these matrices are called **Random Indices** (RI) and are given below:

Size of matrix	Random Index (RI)
3	0.58
4	0.90
5	1.12
6	1.24
7	1.32
8	1.41
9	1.45
10	1.49
11	1.51
12	1.54
13	1.56
14	1.57
15	1.58

- The ***Consistency Ratio*** (CR) of $\boldsymbol{A}$ is defined to be:

$$\text{CR} = \frac{\text{CI of A}}{\text{RI for size } n}$$

The 10% Rule of Practice:

- A matrix with C.R. ≤ 0.1 is typically considered acceptable.

- If CR > 0.1, there is a need to reassess some of the entries to reduce the amount of inconsistencies.

Example

- For the matrix $A = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ 3 & 1 & 3 \\ 2 & \frac{1}{3} & 1 \end{bmatrix}$, we have found that CI =0.0268.

- Hence $\text{CR} = \dfrac{\text{CI of A}}{\text{RI for size 3}} = \dfrac{0.0268}{0.58} = 0.0463 < 0.1$

  $\Rightarrow$ Inconsistency is within acceptable limit of 0.1.

Transitivity Relation in Pairwise Comparisons

- Given three items: x_i, x_j and x_k to be compared.

- Suppose $x_i \succ x_j$ and $x_j \succ x_k$. If $x_i \succ x_k$ then we said that the preference or importance of the items satisfy the transitivity property.

- Transitivity implies that in the pairwise comparison matrix, $a_{ij} > 1$ and $a_{jk} > 1 \Rightarrow a_{ik} > 1$.

- Transitivity is a requirement in utility theory, but not in AHP. However, ensuring transitivity in a pairwise comparison matrix is likely to achieve good consistency ratio.

Example

- Given the matrix:

$$\begin{bmatrix} 1 & 3 & 1/3 \\ 1/3 & 1 & 5 \\ 3 & 1/5 & 1 \end{bmatrix}, \text{ we obtain } \mathbf{w} = \begin{bmatrix} 0.33014 \\ 0.39142 \\ 0.27845 \end{bmatrix}, \ \lambda = 4.838,\ CI = 0.919,\ CR = 1.584 >>> 0.1$$

- We have
  $a_{12} = 3 > 1 \Rightarrow x_1$ is preferred to x_2
  $a_{23} = 5 > 1 \Rightarrow x_2$ is preferred to x_3
  but $a_{31} = 3 > 1 \Rightarrow x_3$ is preferred to x_1

- Hence transitivity is violated and the *CR* of the matrix is very high.

- Suppose the violation was unintentional and a_{31} was supposed to be 1/3.

- Fixing entries a_{31} and a_{13} so that transitivity is satisfied:

$$\begin{bmatrix} 1 & 3 & 3 \\ 1/3 & 1 & 5 \\ 1/3 & 1/5 & 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0.56660 \\ 0.32296 \\ 0.11045 \end{bmatrix}, \qquad \lambda = 3.295,\ CI = 0.14739,\ CR = 0.254 > 0.1$$

- The *CR* has dramatically improved from 1.584 to 0.254 by just satisfying transitivity, but the new CR is still not within acceptable limit. Some of the entries in the matrix should be reassessed until CR < 0.1.

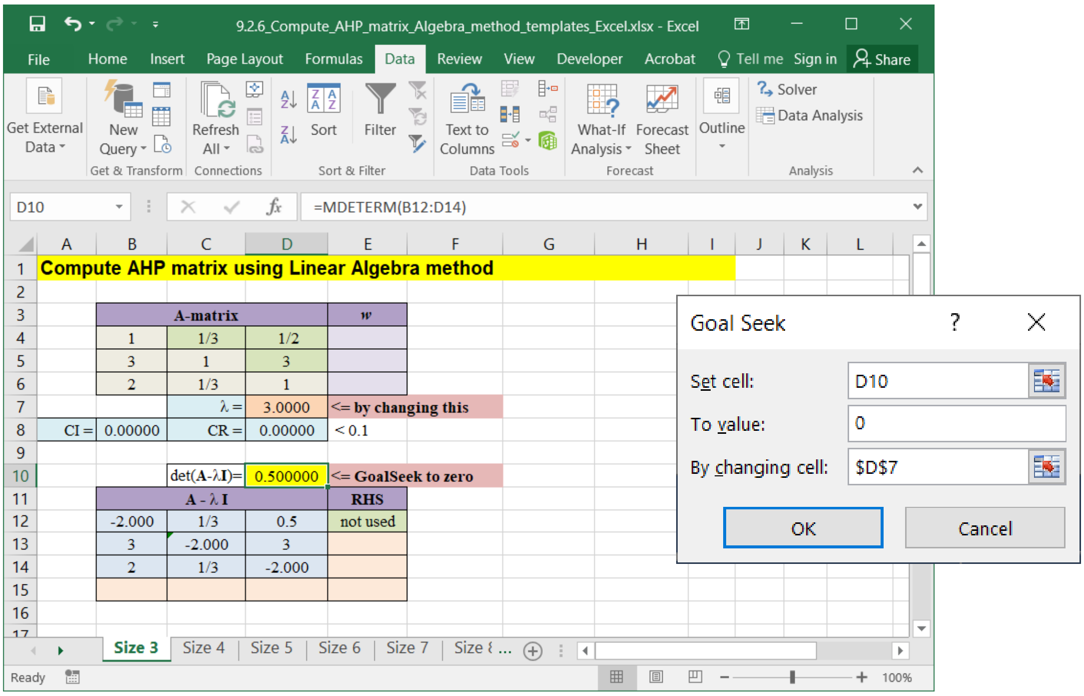
9.2.6 Using Computing Tools to Compute w and λ_{max}

1. Using only Excel Worksheet Functions

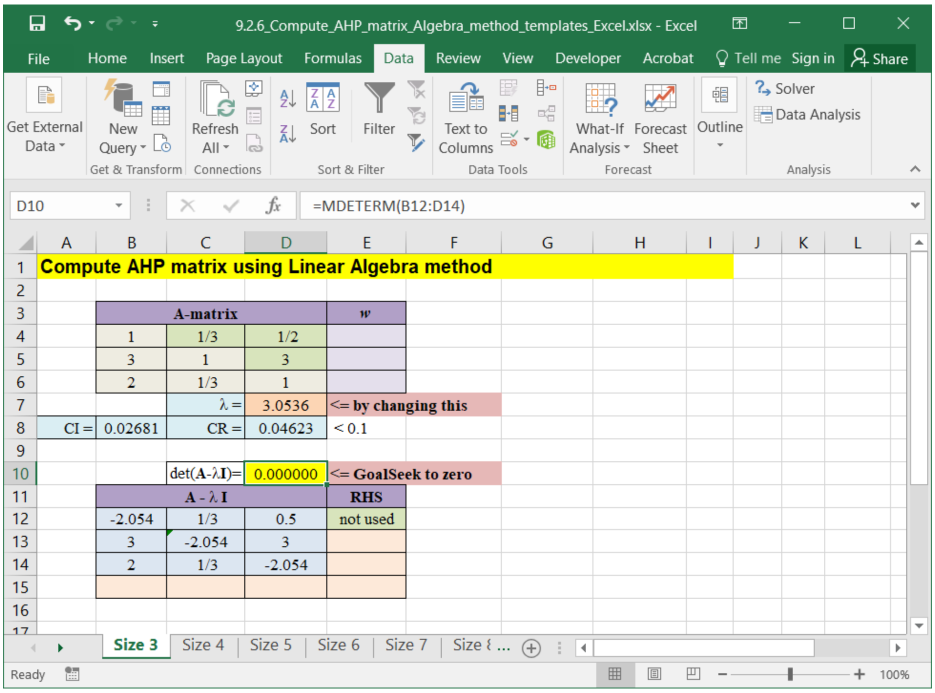
- Excel does not have a built-in function to compute eigenvalues and eigenvectors of matrices. The following process uses only Excel Worksheet Functions: Goal Seek to find λ, and then MMULT and MINVRSE functions to find $\boldsymbol{w}$. No VBA code required.

Step 1: Set up the pairwise comparison matrix $\boldsymbol{A}$ and compute the $\boldsymbol{A} - \lambda\,\mathbf{I}$ matrix with a guess value for λ.

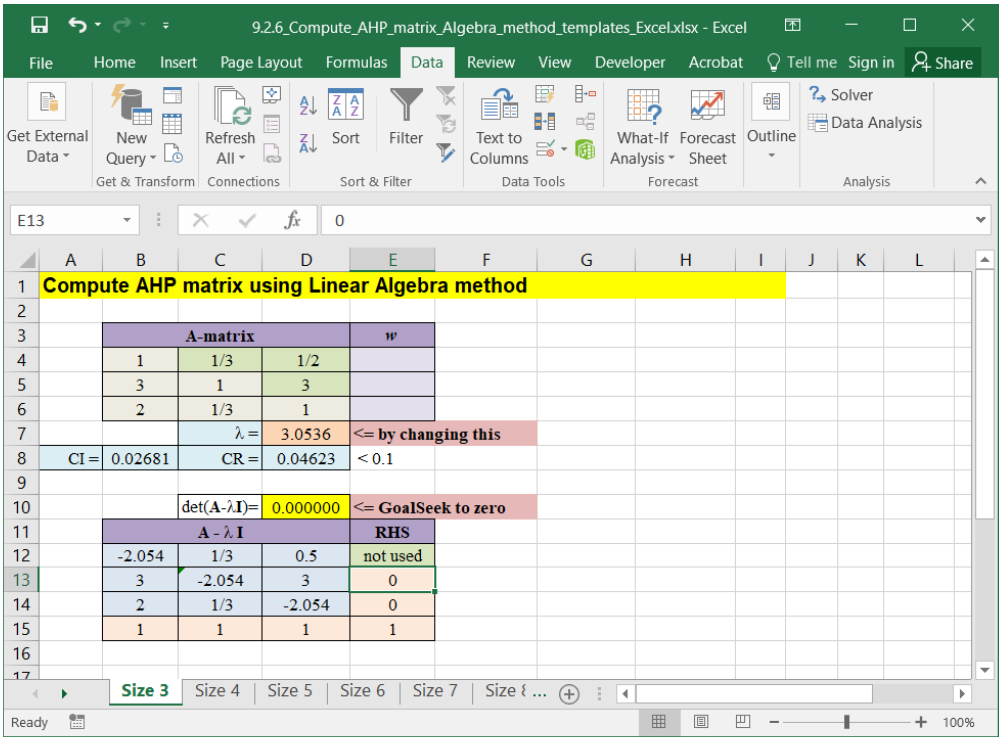
Step 2: Use Goal Seek to find λ by solving the equation $|\,\boldsymbol{A} - \lambda\,\mathbf{I}\,| = 0$.



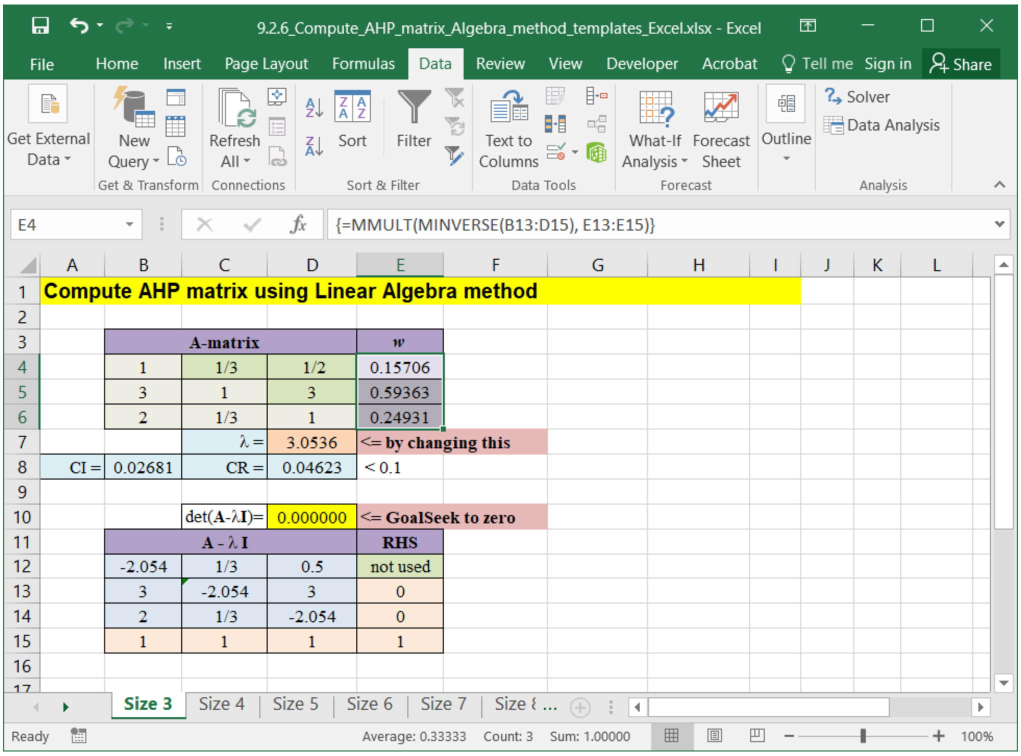
- Solution obtained:



Step 3: Set up the set of linear equations to solve by adding a row of all ones, and right-hand size vector of zeros.



Step 4: Compute $\boldsymbol{w}$ using Excel Worksheet functions MMULT and MINVERSE:



Step 5: Compute CI and CR using λ found in Step 2.

- Results:

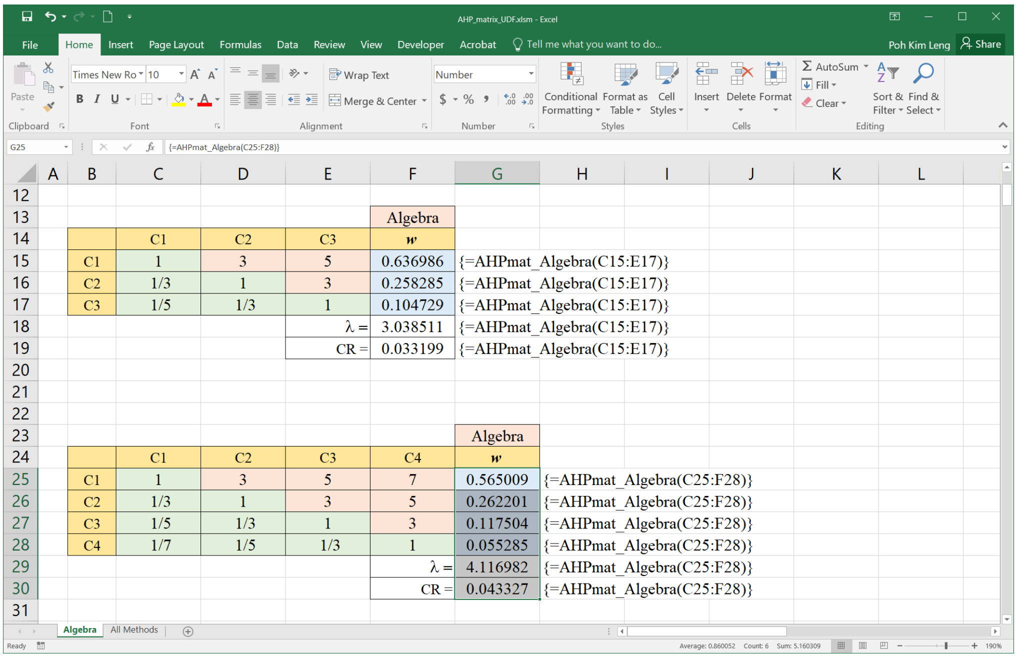
  $w_1 = 0.157056, \quad w_2 = 0.59363, \quad w_3 = 0.24931$

  $\lambda_{\max} = 3.0536, \quad CI = 0.02681, \quad CR = 0.04623 < 0.1$

2. Using Excel User Defined Function

- An Excel User Defined Function (UDF) can be used to hide all the messy workings above. It takes a only single argument matrix A and return a column array $[w, \lambda_{max}, CR]^T$ of length $N+2$.

- The numerical bisection search method is used to solve the determinant equation to find λ_{max}. You can download the VBA source code from CodeHub.



	C1	C2	C3	Algebra *w*	
C1	1	3	5	0.636986	{=AHPmat_Algebra(C15:E17)}
C2	1/3	1	3	0.258285	{=AHPmat_Algebra(C15:E17)}
C3	1/5	1/3	1	0.104729	{=AHPmat_Algebra(C15:E17)}
			λ =	3.038511	{=AHPmat_Algebra(C15:E17)}
			CR =	0.033199	{=AHPmat_Algebra(C15:E17)}

	C1	C2	C3	C4	Algebra *w*	
C1	1	3	5	7	0.565009	{=AHPmat_Algebra(C25:F28)}
C2	1/3	1	3	5	0.262201	{=AHPmat_Algebra(C25:F28)}
C3	1/5	1/3	1	3	0.117504	{=AHPmat_Algebra(C25:F28)}
C4	1/7	1/5	1/3	1	0.055285	{=AHPmat_Algebra(C25:F28)}
				λ =	4.116982	{=AHPmat_Algebra(C25:F28)}
				CR =	0.043327	{=AHPmat_Algebra(C25:F28)}

3. Using Python (Linear Algebra method)

In [1]:
```
""" Compute AHP matrix using Linear Algebra method """
import numpy as np
from scipy.optimize import root
```

In [2]:
```
A = np.array([[1, 1/3, 1/2],
              [3,   1,   3 ],
              [2, 1/3,   1 ]])

n, _ = A.shape    # Number of elements
I = np.eye(n)     # Identity matrix
```

In [3]:
```
# Find lambda_max by finding root of the determinant equation
eq = lambda y: np.linalg.det(A-I*y)
sol = root(eq, x0=n, options={'xtol':1e-12})
lambda_max = sol.x[0]
print(f"lambda_max = {lambda_max:.6f}")
```

```
lambda_max = 3.053622
```

In [4]:
```
# Find w by solving a set of linear equations M w = b
M = A - I*lambda_max    # M = A - lambda_max I for first n-1 rows
M[n-1] = np.ones(n)     # Replace the last row with [1, 1..., 1]
b = np.append(np.zeros(n-1), [1])  # b = [0, 0, ..., 1]
w = np.linalg.solve(M,b)
print(f"w = {w}")
```

```
w = [0.15705579 0.59363369 0.24931053]
```

In [5]:
```python
# Compute CI and CR
CI = (lambda_max-n)/(n-1)
CR = CI/0.58     # RI = 0.58 for n = 3
print(f"CI= {CI:.6f}, CR= {CR:.6f}")
```

```
CI= 0.026811, CR= 0.046225
```

4. Using Python numpy.linalg.eig() function

- You may also use the `numpy.linalg.eig()` function to first compute all the eigenvalues and eigenvectors (reals and complex's), and then extract out the dominant real eigenvalue and its corresponding eigenvector.

In [1]:
```python
""" Compute AHP matrix using np.linalg.eig function """
import numpy as np
```

In [2]:
```python
A = np.array([[1, 1/3, 1/2],
              [3,  1,   3 ],
              [2, 1/3,  1 ]])
```

In [3]:
```python
# Compute all the eigenvalues and eigenvectors
eigVal, eigVec = np.linalg.eig(A)

# Get the dominant real eigenvalue and its eigenvector
lambda_max, w = max([(val.real, vec.real) for val, vec
                    in zip(eigVal, eigVec.T) if np.isreal(val)])
w = w/w.sum()  # Normalize w. Can idealize it also.
print(f"lambda_max = {lambda_max:.6f}")
print(f"w = {w}")
```

```
lambda_max = 3.053622
w = [0.15705579 0.59363369 0.24931053]
```

In [4]:
```python
n, _ = A.shape
CI = (lambda_max - n)/(n - 1)
CR = CI/0.58    # RI for size 3 is 0.58
print(f"CI = {CI:.6f}, CR = {CR:.6f}")
```

```
CI = 0.026811, CR = 0.046225
```

9.2.7 Approximation Methods for Computing w and λ_{max}

- Approximation methods can be used to find approximate values of λ_{max} and w very quickly with little computing resources.

- Two easy-to-use approximation methods are:
  1. Row Geometric Mean Method
  2. Column Normalization Method

1. Row Geometric Mean (RGM) Approximation Method

Procedure:

1. Given a valid pairwise comparison matrix A, compute the geometric mean of each row of A.

2. Normalize the numbers obtain in Step 1 to obtain an approximate w.

3. For each row i, compute an approximate value of λ by finding the product of the i^{th} row of A and the column vector w, and then dividing by w_i.

4. The best approximate value of λ is obtained by averaging all the values of λ found in Step 3.

Example

- Consider matrix $A = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ 3 & 1 & 3 \\ 2 & \frac{1}{3} & 1 \end{bmatrix}$

			Row geometric mean	Normalized weights	Approximate λ using row i
1	1/3	1/2	$\sqrt[3]{(1 \cdot \frac{1}{3} \cdot \frac{1}{2})} = 0.55032$	**0.1571**	3.0536
3	1	3	$\sqrt[3]{(3 \cdot 1 \cdot 3)} = 2.08008$	**0.5936**	3.0536
2	1/3	1	$\sqrt[3]{(2 \cdot \frac{1}{3} \cdot 1)} = 0.87358$	**0.2493**	3.0536
			Total 3.50398	1.0000	λ=3.0536

- Normalizing the row geometric means:

  $w_1 =$ 0.55032 / 3.50398 = 0.1571
  $w_2 =$ 2.08008 / 3.50398 = 0.5936
  $w_3 =$ 0.87358 / 3.50398 = 0.2493

- Computing an approximate value of λ using each row:

  Row 1: $\lambda \approx$ [(1)(0.1571) + (1/3)(0.5936) + (1/2)(0.2493)] / 0.1571 = 3.0536
  Row 2: $\lambda \approx$ [(3)(0.1571) + (1)(0.5936) + (3)(0.2493)] / 0.5936 = 3.0536
  Row 3: $\lambda \approx$ [(2)(0.1571) + (1/3)(0.5936) + (1)(0.2493)] / 0.2493 = 3.0536

- Results: $w \approx$ [0.1571, 0.5936, 0.2493]
  $\lambda \approx 3.0536$
  CI = (3.0536 – 3)/2 = 0.0268
  CR = 0.0268/0.58 = 0.0462

Justification of the Row Geometric Mean Method

- Given a perfectly consistent A-matrix:

$$\begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_n} \\ \vdots & \vdots & & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & \frac{w_n}{w_n} \end{bmatrix}$$

- The Geometric Mean of the i^{th} row of $\boldsymbol{A}$ is

$$\mu_i = \sqrt[n]{\frac{w_i}{w_1}\frac{w_i}{w_2}\cdots\frac{w_i}{w_n}} = \frac{w_i}{\sqrt[n]{w_1 w_2 \cdots w_n}} = Kw_i \qquad \text{for } i = 1 \text{ to } n$$

  where K is a constant.

- By normalizing the μ_i's, we eliminate K and retrieve the values of w_i's.

- Hence if $\boldsymbol{A}$ is perfectly consistent, then this method gives the exact weights, and when $\boldsymbol{A}$ is not perfectly consistent, it can give a good approximation of the weights.

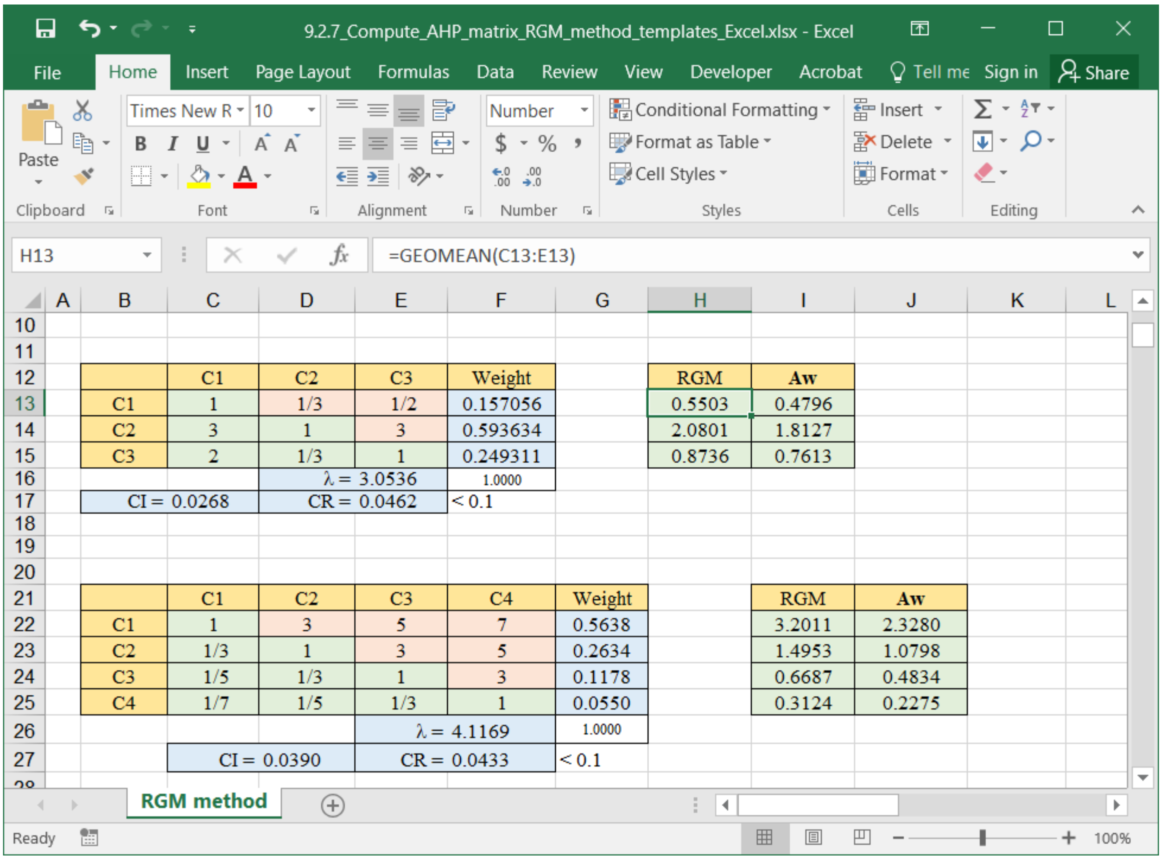
- To find λ, first we observe that if $\boldsymbol{w}$ is exact, then $\mathbf{Aw} = \lambda\mathbf{w} = \begin{bmatrix} \lambda w_1 \\ \lambda w_2 \\ \vdots \\ \lambda w_n \end{bmatrix}$.

  Hence, by dividing any i^{th} entry of $\mathbf{A}\boldsymbol{w} = \lambda\boldsymbol{w}$ by w_i, we will get the exact value of λ.

  However, when $\boldsymbol{w}$ is not exact, we use can use all the rows of $\boldsymbol{A}$ and the approximate $\boldsymbol{w}$ to obtain n approximate values of λ, and then take their average.

  Hence $\qquad \lambda_{\max} \approx \frac{1}{n}\sum_{i=1}^{n}\lambda_i \qquad$ where $\lambda_i = \dfrac{\sum_{j=1}^{n} a_{ij} w_j}{w_i}$ for $i = 1$ to n.

Using only Excel Worksheet Functions (no VBA)



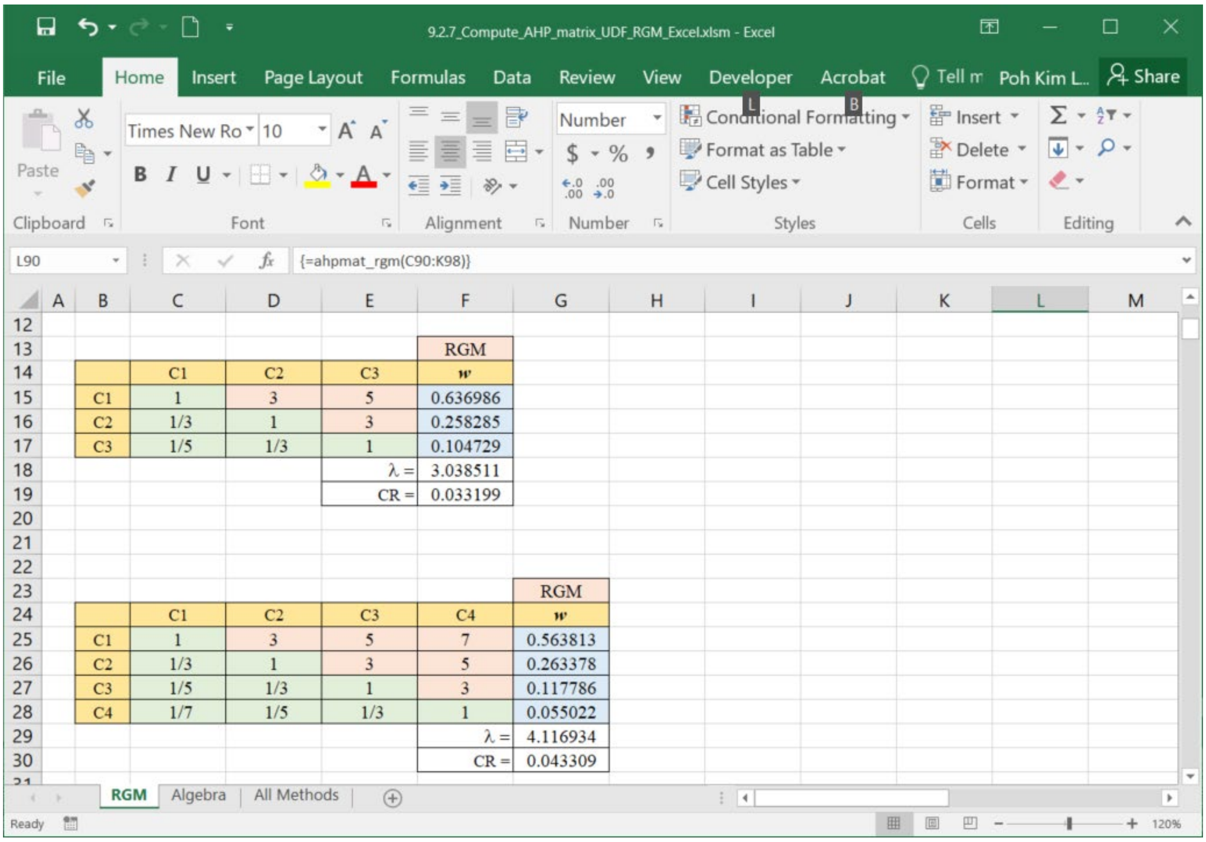
	C1	C2	C3	Weight
C1	1	1/3	1/2	0.157056
C2	3	1	3	0.593634
C3	2	1/3	1	0.249311
		λ = 3.0536		1.0000
CI = 0.0268		CR = 0.0462		< 0.1

RGM	Aw
0.5503	0.4796
2.0801	1.8127
0.8736	0.7613

	C1	C2	C3	C4	Weight
C1	1	3	5	7	0.5638
C2	1/3	1	3	5	0.2634
C3	1/5	1/3	1	3	0.1178
C4	1/7	1/5	1/3	1	0.0550
			λ = 4.1169		1.0000
	CI = 0.0390		CR = 0.0433		< 0.1

RGM	Aw
3.2011	2.3280
1.4953	1.0798
0.6687	0.4834
0.3124	0.2275

Using Excel User Defined Function (VBA)



	C1	C2	C3	RGM w
C1	1	3	5	0.636986
C2	1/3	1	3	0.258285
C3	1/5	1/3	1	0.104729
			λ =	3.038511
			CR =	0.033199

	C1	C2	C3	C4	RGM w
C1	1	3	5	7	0.563813
C2	1/3	1	3	5	0.263378
C3	1/5	1/3	1	3	0.117786
C4	1/7	1/5	1/3	1	0.055022
				λ =	4.116934
				CR =	0.043309

Using Python

In [1]:
```python
""" Compute AHP matrix using Row Geometric Mean approximation method """
import numpy as np
from scipy.stats import gmean
```

In [2]:
```python
A = np.array([[ 1, 1/3, 1/2],
              [ 3,   1,   3 ],
              [ 2, 1/3,   1 ]] )
```

In [3]:
```python
# Compute the geometric mean of each row, then normalize it.
rgm = gmean(A, axis=1)
w = rgm/rgm.sum()
print(f"w = {w}")
```

```
w = [0.15705579 0.59363369 0.24931053]
```

In [4]:
```python
# Estimate lambda_max using all rows
lambda_max = (np.dot(A,w)/w).mean()
n, _ = A.shape
CI = (lambda_max-n)/(n-1)
CR = CI/0.58
print(f"lambda_max= {lambda_max:.6f}, CI= {CI:.6f}, CR= {CR:.6f}")
```

```
lambda_max= 3.053622, CI= 0.026811, CR= 0.046225
```

2. The Column Normalization Approximation Method

Procedure:

1. Given a valid pairwise comparison matrix ***A***, normalize each column of ***A***.
2. Compute the average across each row of the matrix to obtain an approximate ***w***.
3. For each row i, compute an approximate value of λ by finding the product of the i^{th} row of ***A*** and the column vector ***w***, and then dividing by w_i.
4. An approximate value of λ is obtained by averaging all the values of λ in Step 3.

Example

- Consider matrix $\boldsymbol{A} = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ 3 & 1 & 3 \\ 2 & \frac{1}{3} & 1 \end{bmatrix}$

			Normalized columns			Row Average	Approximate λ using row i
			Column 1	Column 2	Column 3		
1	1/3	1/2	0.1667	0.2000	0.1111	**0.1593**	3.0226
3	1	3	0.5000	0.6000	0.6667	**0.5889**	3.0942
2	1/3	1	0.3333	0.2000	0.2222	**0.2518**	3.0449
6.000	1.667	4.500				1.0000	λ=**3.0539**

- Computing approximate values of λ using each row:

  Row 1: $\lambda \approx$ [(1)(0.1593) + (1/3)(0.5889) + (1/2)(0.2518)] / 0.1593 = 3.0226
  Row 2: $\lambda \approx$ [(3)(0.1593) + (1)(0.5889) + (3)(0.2518)] / 0.5889 = 3.0942
  Row 3: $\lambda \approx$ [(2)(0.1593) + (1/3)(0.5889) + (1)(0.2518)] / 0.2518 = 3.0449

- Results: $\boldsymbol{w} \approx$ [0.1593, 0.5889, 0.2518]
  $\lambda \approx 3.0539$
  CI = (3.0539 – 3)/2 = 0.0270
  CR = 0.0270/0.58 = 0.0465

Justification of the Column Normalization Method

- Given a perfectly consistent *A*-matrix:

$$\begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \cdots & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \cdots & \frac{w_2}{w_n} \\ \vdots & \vdots & & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \cdots & \frac{w_n}{w_n} \end{bmatrix}$$

- By normalizing each column, we obtain the following matrix

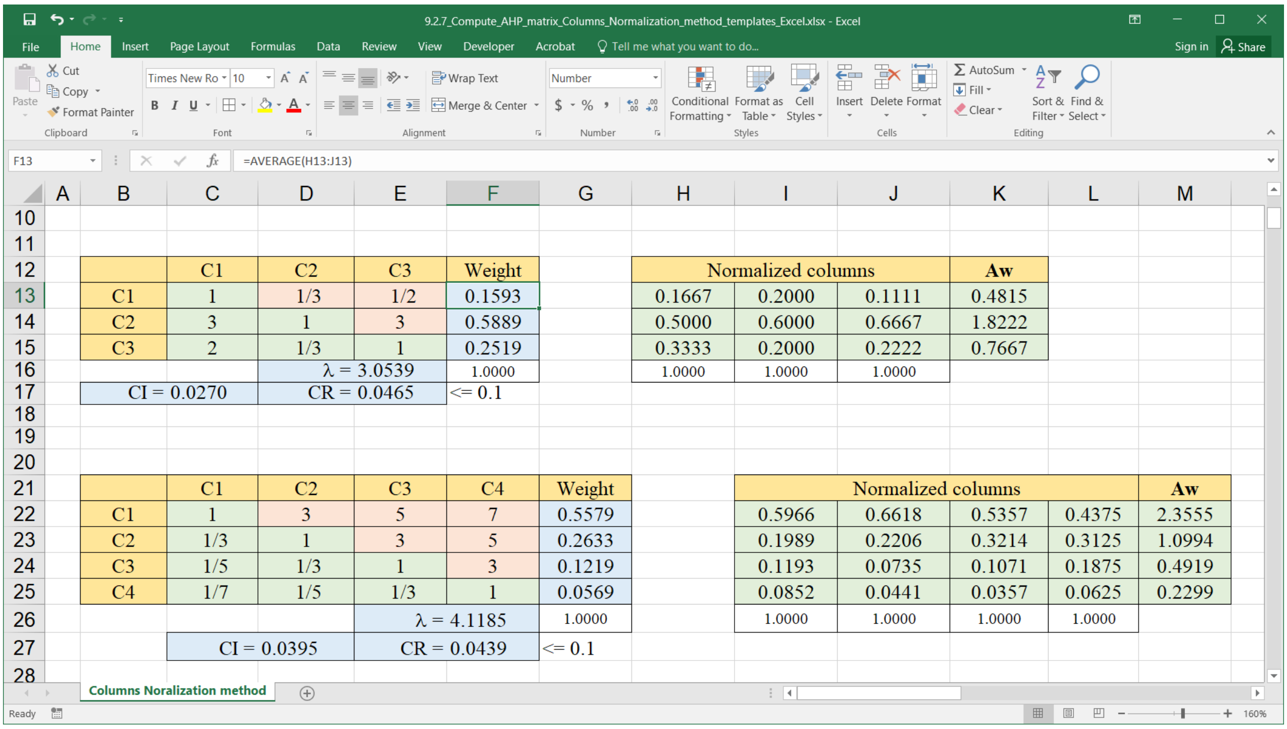
$$\begin{bmatrix} w_1 & w_1 & \cdots & w_1 \\ w_2 & w_2 & \cdots & w_2 \\ \vdots & \vdots & & \vdots \\ w_n & w_n & \cdots & w_n \end{bmatrix}$$

- Hence if $\boldsymbol{A}$ is perfectly consistent, the exact weights may be obtained by normalizing any columns of $\boldsymbol{A}$.

- However, when A is not perfectly consistent, each normalized columns will be slightly different, and by averaging across the rows, we obtain a good approximation for the weights.

- The step for computing an approximate value of λ is the same as in the RGM method.

Using only Excel Worksheet Functions



F13 =AVERAGE(H13:J13)

	C1	C2	C3	Weight
C1	1	1/3	1/2	0.1593
C2	3	1	3	0.5889
C3	2	1/3	1	0.2519
		λ = 3.0539		1.0000
CI = 0.0270		CR = 0.0465		<= 0.1

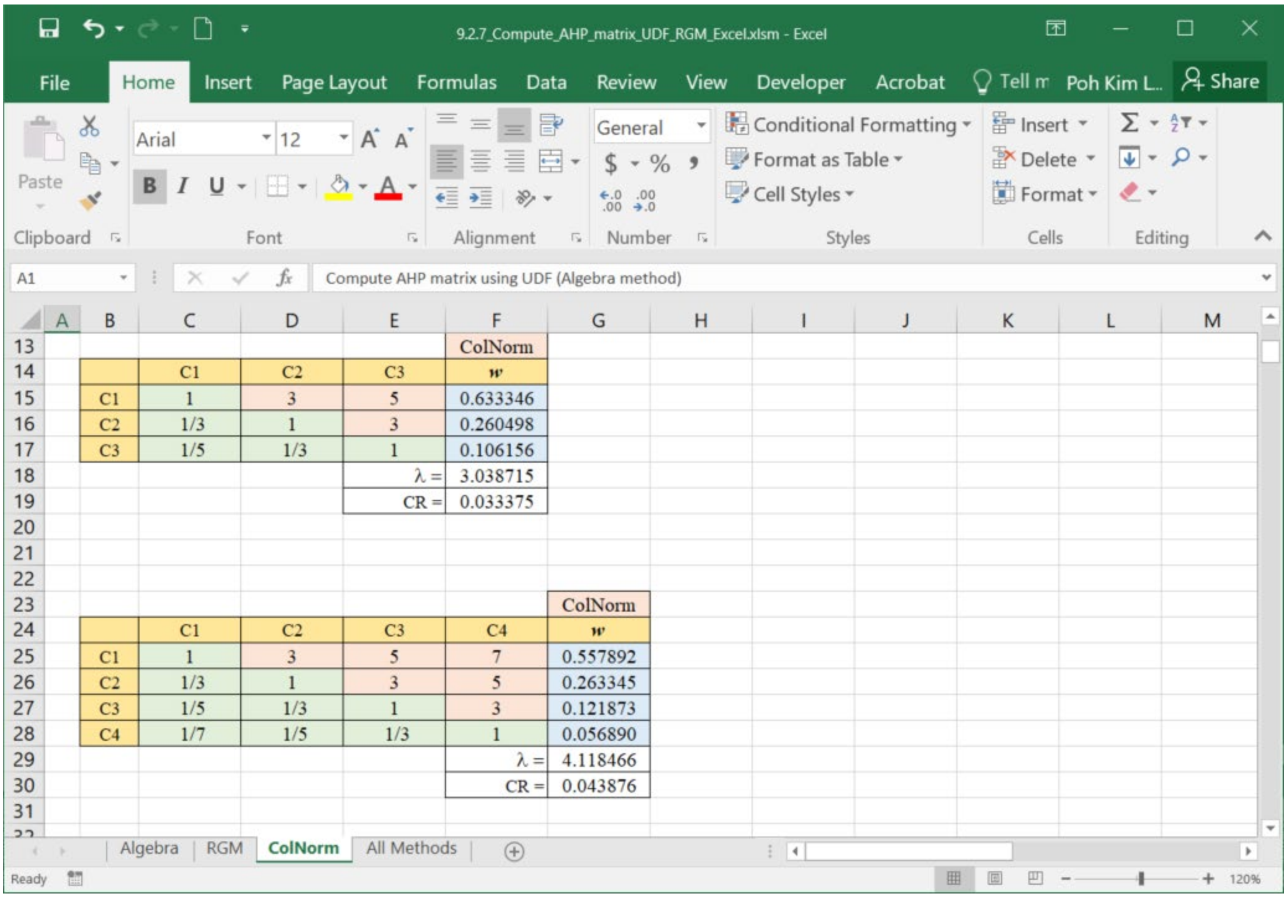
Normalized columns			Aw
0.1667	0.2000	0.1111	0.4815
0.5000	0.6000	0.6667	1.8222
0.3333	0.2000	0.2222	0.7667
1.0000	1.0000	1.0000	

	C1	C2	C3	C4	Weight
C1	1	3	5	7	0.5579
C2	1/3	1	3	5	0.2633
C3	1/5	1/3	1	3	0.1219
C4	1/7	1/5	1/3	1	0.0569
			λ = 4.1185		1.0000
	CI = 0.0395		CR = 0.0439		<= 0.1

Normalized columns				Aw
0.5966	0.6618	0.5357	0.4375	2.3555
0.1989	0.2206	0.3214	0.3125	1.0994
0.1193	0.0735	0.1071	0.1875	0.4919
0.0852	0.0441	0.0357	0.0625	0.2299
1.0000	1.0000	1.0000	1.0000	

Columns Noralization method

Using Excel User Defined Function



A1 Compute AHP matrix using UDF (Algebra method)

	C1	C2	C3	ColNorm w
C1	1	3	5	0.633346
C2	1/3	1	3	0.260498
C3	1/5	1/3	1	0.106156
			λ =	3.038715
			CR =	0.033375

	C1	C2	C3	C4	ColNorm w
C1	1	3	5	7	0.557892
C2	1/3	1	3	5	0.263345
C3	1/5	1/3	1	3	0.121873
C4	1/7	1/5	1/3	1	0.056890
				λ =	4.118466
				CR =	0.043876

Algebra | RGM | ColNorm | All Methods

Using Python

In [1]:
```
""" Compute AHP matrix using Column Normalization approximation method """
""" This method is fast but not very accurate; Use RGM method instead  """
import numpy as np
```

In [2]:
```
A = np.array([[1, 1/3, 1/2],
              [3,  1,   3 ],
              [2, 1/3,  1 ]])
```

In [3]:
```
# Normalise each column of A, then average across each row.
w = (A/A.sum(axis=0)).mean(axis=1)
print(f"w = {w}")
```

```
w = [0.15925926 0.58888889 0.25185185]
```

In [4]:
```
# Estimate lambda_max using all rows
lambda_max = (np.dot(A,w)/w).mean()
n, _ = A.shape
CI = (lambda_max-n)/(n-1)
CR = CI/0.58
print(f"lambda_max= {lambda_max:.6f}, CI = {CI:.6f}, CR= {CR:.6f}")
```

```
lambda_max= 3.053904, CI = 0.026952, CR= 0.046469
```

Comparison of Approximation Methods

	Exact Analytical Method	Geometric Mean Method	Column Normalization Method
w_1	0.15705579	0.15705579	0.15925926
w_2	0.59363369	0.59363369	0.58888889
w_3	0.24931053	0.24931053	0.25185185
λ_{max}	3.053622	3.053622	3.053904

Notes:

- Row Geometric Mean Method provides better results than the Column Normalization Method.
- RGM method is very good for small matrix size, but the errors will increase as the size increases.
- Uuse the RGM method if you have to use an approximation method.
- Use the Column Normalization method only if you do not have a scientific calculator to find the n^{th} roots.

9.2.8 Numerical Method for Computing w and λ_{max}

Power Iterations method

- The Power Iterations method can be used to find the dominant eigenvalue and its's corresponding eigenvector by numerical iterations until the solution converges to within some tolerance limits.

Algorithm:

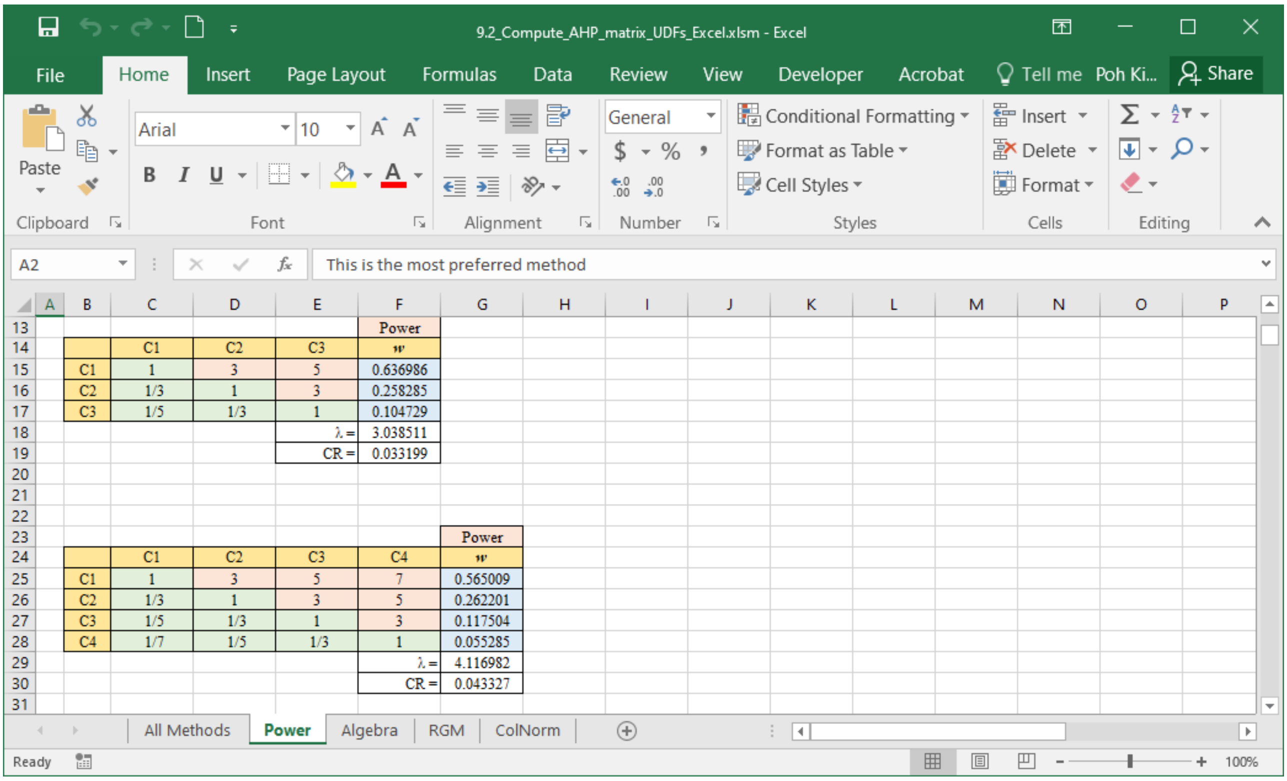
Given a valid matrix $\boldsymbol{A}$.

Let $\boldsymbol{w}_0$ = initial guess by an approximation method.
Iteration $k = 0$

While $k \le$ max_iterations
  $\boldsymbol{w}_{k+1} = \boldsymbol{A}\, w_k$
  $\boldsymbol{w}_{k+1} = \boldsymbol{w}_{k+1}$ / sum($\boldsymbol{w}_{k+1}$)   // normalize it
  If | $\boldsymbol{w}_{k+1}$ - $\boldsymbol{w}_k$ | < tolerance:
    $\boldsymbol{w}_k = \boldsymbol{w}_{k+1}$ // take the last value before breaking out
    break
  End If
  $\boldsymbol{w}_k = \boldsymbol{w}_{k+1}$
  $k = k+1$
End While

$\boldsymbol{\lambda} = \boldsymbol{A}\boldsymbol{w}_k / \boldsymbol{w}_k$ // component wise
$\lambda_{max} = \text{mean}(\boldsymbol{\lambda})$
$CI = (\lambda_{max} - n) / (n - 1)$
$CR = CI/RI_n$

Using Excel User-Defined Function



	C1	C2	C3	Power w
C1	1	3	5	0.636986
C2	1/3	1	3	0.258285
C3	1/5	1/3	1	0.104729
			λ =	3.038511
			CR =	0.033199

	C1	C2	C3	C4	Power w
C1	1	3	5	7	0.565009
C2	1/3	1	3	5	0.262201
C3	1/5	1/3	1	3	0.117504
C4	1/7	1/5	1/3	1	0.055285
				λ =	4.116982
				CR =	0.043327

Using Python

In [1]:
```
""" Compute AHP matrix using Power Iterations method """
import numpy as np
from scipy.stats import gmean
```

In [2]:
```
A = np.array([[1, 1/3, 1/2],
              [3,  1,   3 ],
              [2, 1/3,  1 ]])
```

In [3]:
```
# Initial solution:
#   Use RGM approximation. Faster convergence
gm = gmean(A, axis=1)
w = gm/gm.sum()
#   Use column normalization. More iterations needed
# w = (A/A. sum(axis=0)).mean(axis=1)
```

In [4]:
```
# Perform Power Iterations
max_iter= 1000000
epsilon = 1.E-16
for iter in range(max_iter):
    w1 = np.dot(A,w)     # w(k+1) = A w(k)
    w1 = w1/w1.sum()     # normalize w(k+1)
    if all(np.absolute(w1-w) < epsilon):
        w = w1
        print(f"Tolerance {epsilon} achieved at iter #{iter}")
        break
    w = w1
print(f"w = {w}")
```

```
Tolerance 1e-16 achieved at iter #2
w = [0.15705579 0.59363369 0.24931053]
```

In [5]:
```
# Estimate lambda_max using all rows
lambda_max = (np.dot(A,w)/w).mean()
n, _ = A.shape
CI = (lambda_max-n)/(n-1)
CR =  CI/0.58
print(f"lambda_max= {lambda_max:.6f}, CI= {CI:.6f}, CR= {CR:.6f}")
```

```
lambda_max= 3.053622, CI= 0.026811, CR= 0.046225
```

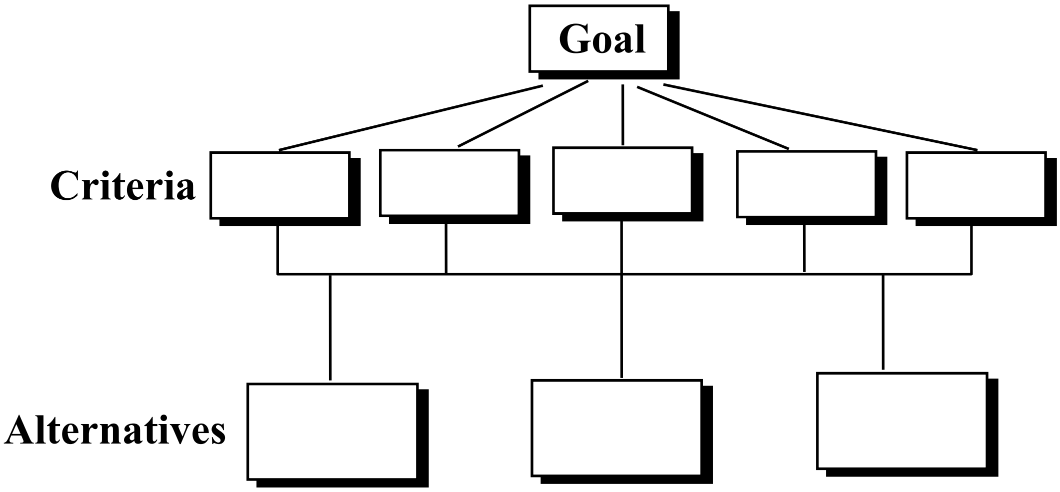
9.3 Modeling and Solving a AHP Model

9.3.1 Case Study: Job Selection Problem

- We illustrate the rest of the steps for AHP via a case study on job selection.

- Suppose a recent NUS graduate has three job offers. How should she make her choice using AHP?

Constructing the Hierarchy:



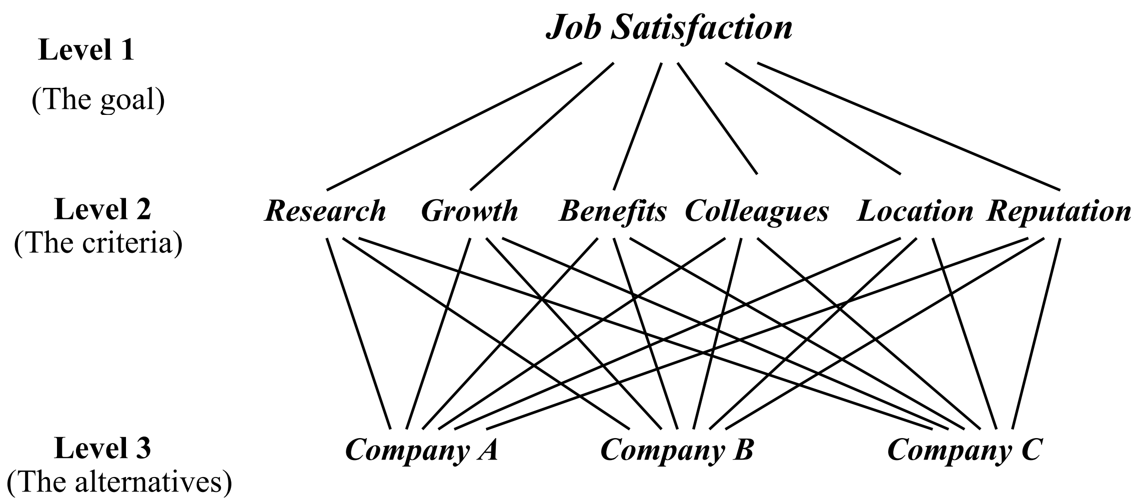
Goal: Job Satisfaction

The Hierarchy consists of three levels:

Level 1: Goal

Level 2: Criteria that contribute towards achievement of the goal in level 1.

Level 3: The alternatives under consideration.



Performing Judgments and Computing the Local Weights

- Perform pairwise comparison of level-2 elements with respect to level 1, i.e., evaluate the contributions of each of the six criteria towards achieving the goal "Job Satisfaction".

	Research	Growth	Benefits	Colleagues	Location	Reputation
Research	1	1	1	4	1	1/2
Growth		1	2	4	1	1/2
Benefits			1	5	3	1/2
Colleagues				1	1/3	1/3
Location					1	1
Reputation						1

- Solve for λ_{max} and $\boldsymbol{w}$.
  λ_{max}= 6.4203 CI =0.08407 CR=0.06780 < 10%
  $\boldsymbol{w}$ = [0.158408 0.189247 0.197997 0.048310 0.150245 0.255792]

- That is with respect to Job satisfaction, the contributions from the various criteria are:

	Criterion	Weight
1	Research	15.8 %
2	Growth	18.9 %
3	Benefits	19.8 %
4	Colleagues	4.8 %
5	Location	15.0 %
6	Reputation	25.6 %

- Next, we perform pairwise comparison of level-3 elements (i.e., the alternative jobs) with respect to each of the six criteria in level 2. This will result in six 3×3 *A*-matrices.

- Comparison of alternatives w.r.t. "Research":

	A	B	C
A	1	1/4	1/2
B		1	3
C			1

λ_{max} = 3.01829, CI =0.00915, CR =0.01577 < 0.1, $\boldsymbol{w}$ = [0.13650, 0.62501, 0.23849]

- Comparison of alternatives w.r.t. "Growth":

	A	B	C
A	1	1/4	1/5
B		1	1/2
C			1

λ_{max} = 3.0246, CI =0.012298, CR=0.0212 < 0.1, $\boldsymbol{w}$ = [0.09739, 0.33307, 0.56954]

- Comparison of alternatives w.r.t. “Benefits”:

	A	B	C
A	1	3	1/3
B		1	1/7
C			1

$\lambda_{max} = 3.0070$, CI = 0.003511, CR =0.00605 < 0.1, $\boldsymbol{w}$ = [0.2426 , 0.08794, 0.6694]

- Comparison of alternatives w.r.t. “Colleagues”:

	A	B	C
A	1	1/3	5
B		1	7
C			1

$\lambda_{max} = 3.0649$, CI = 0.032444, CR = 0.05594 < 0.1, $\boldsymbol{w}$ = [0.27895, 0.64912, 0.07193]

- Comparison of alternatives w.r.t. “Location”:

	A	B	C
A	1	1	7
B		1	7
C			1

$\lambda_{max} = 3$, CI = 0, CR = 0 < 0.1, $\boldsymbol{w}$= [0.46667, 0.46667, 0.06667]

- Comparison of alternatives w.r.t. “Reputation”:

	A	B	C
A	1	7	9
B		1	2
C			1

λ_{max}= 3.0217, CI = 0.01086, CR = 0.01873 < 0.1, $\boldsymbol{w}$ = [0.79276, 0.13122, 0.07602]

Computing the Global Weights of the Alternatives

- The results so far may be summarized as follows:

	Criterion	Criterion's weight	Alternative	Alt's local weights w.r.t. criterion
1	Research	**0.158408**		
			Company A	0.13650
			Company B	0.62501
			Company C	0.23849
2	Growth	**0.189247**		
			Company A	0.09739
			Company B	0.33307
			Company C	0.56954
3	Benefits	**0.197997**		
			Company A	0.24264
			Company B	0.08795
			Company C	0.66942
4	Colleagues	**0.048310**		
			Company A	0.27905
			Company B	0.64912
			Company C	0.07193
5	Location	**0.150245**		
			Company A	0.46667
			Company B	0.46667
			Company C	0.06667
6	Reputation	**0.255792**		
			Company A	0.79276
			Company B	0.13122
			Company C	0.07602

- The Global Weight of an alternative is equal to criterion-weighted sum of its local weights.

- In Matrix notations:

$$\begin{bmatrix} w_A \\ w_B \\ w_C \end{bmatrix} = \begin{bmatrix} 0.13650 & 0.09739 & 0.24264 & 0.27895 & 0.46667 & 0.79276 \\ 0.62501 & 0.33307 & 0.08795 & 0.64912 & 0.46667 & 0.13122 \\ 0.23849 & 0.56954 & 0.66942 & 0.07193 & 0.06667 & 0.07602 \end{bmatrix} \begin{bmatrix} 0.158408 \\ 0.189247 \\ 0.197997 \\ 0.048310 \\ 0.150245 \\ 0.255792 \end{bmatrix} = \begin{bmatrix} 0.3745 \\ 0.3145 \\ 0.3110 \end{bmatrix}$$

- The Global weights of the three alternatives are:

Alternative	Global weight
Company A	0.3745
Company B	0.3145
Company C	0.3110

Final Decision

- Choose Company A which has the highest global weight of 0.3745.
- Notice that Company B and C are close behind.

9.3.2 Sensitivity Analysis

- Sensitivity Analysis can be performed to determine the impact on the global weights of the alternatives and hence their rankings if the weight of a criterion (i.e. its priority) is changed while keeping the weights of all the other criteria in the same relative proportion to their respective base values.

Job Selection Problem:

- Suppose we denote the weights of the 6 criteria by $w_1, w_2, \ldots, w_6$, respectively.

- The base values of the weights are:

$w_1 = 0.158408$
$w_2 = 0.189247$
$w_3 = 0.197997$
$w_4 = 0.048310$
$w_5 = 0.150245$
$w_6 = 0.255792$

- The current value of w_1 is 0.158408. Suppose we wish to investigate the impact of w_1 by allowing it to vary from $p = 0$ to 1, then the other five criterion weights are adjusted as all the weights must always add up to one.

- The weights are computed as follows:

Let $S = \sum_{i=2}^{6} w_i$ = Sum(0.189247, 0.197997, 0.048310, 0.150245, 0.255792) = 0.841592

$w'_1 = p$
$w'_2 = (1-p)\ 0.189247 / S = 0.22487\ (1-p)$
$w'_3 = (1-p)\ 0.197997 / S = 0.23527\ (1-p)$
$w'_4 = (1-p)\ 0.048310 / S = 0.05740\ (1-p)$
$w'_5 = (1-p)\ 0.150245 / S = 0.17852\ (1-p)$
$w'_6 = (1-p)\ 0.255792 / S = 0.30394\ (1-p)$

- The alternatives' global weights are then computed using the adjusted criteria weights:

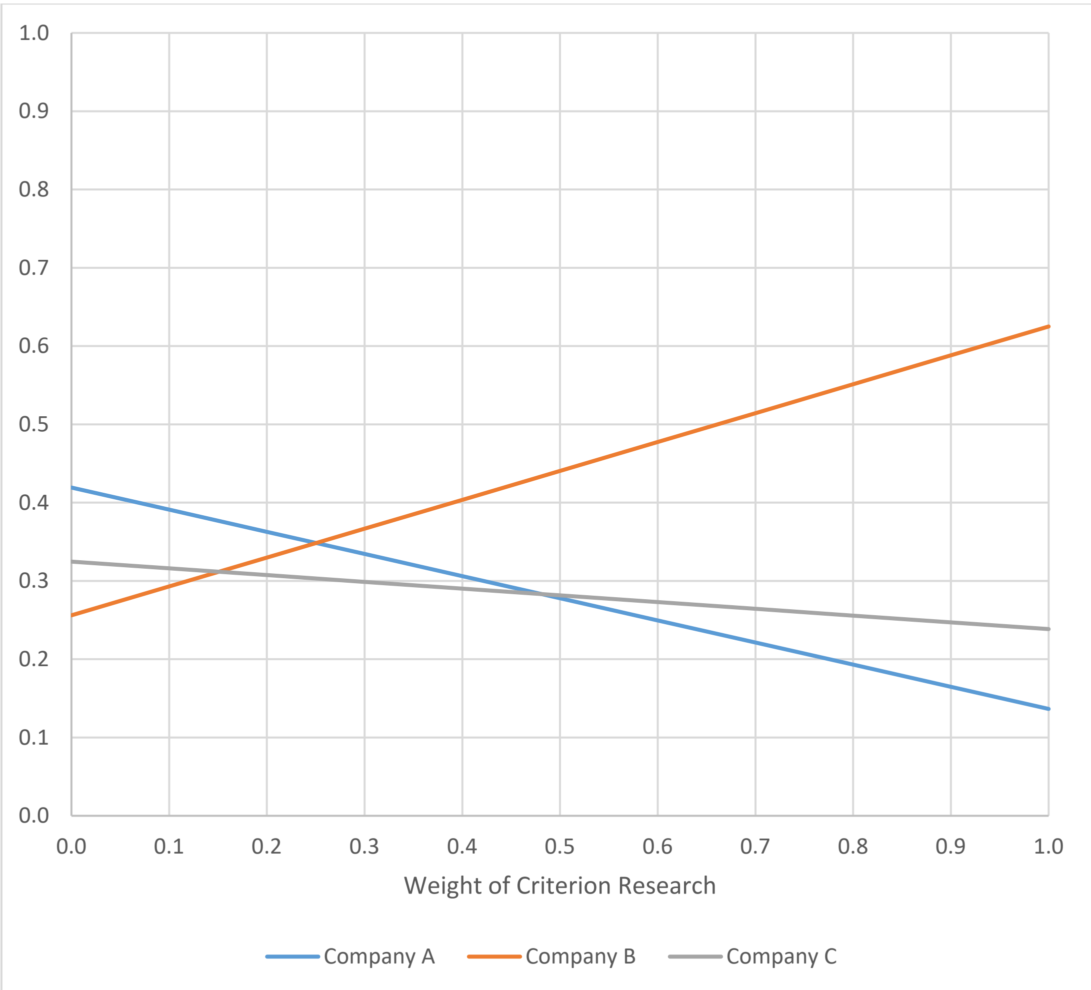
$$\begin{bmatrix} w_A \\ w_B \\ w_C \end{bmatrix} = \begin{bmatrix} 0.1365 & 0.0974 & 0.2426 & 0.2790 & 0.4667 & 0.7928 \\ 0.6250 & 0.3331 & 0.0880 & 0.6491 & 0.4667 & 0.1312 \\ 0.2385 & 0.5695 & 0.6694 & 0.0719 & 0.0666 & 0.0760 \end{bmatrix} \begin{bmatrix} w'_1 \\ w'_2 \\ w'_3 \\ w'_4 \\ w'_5 \\ w'_6 \end{bmatrix}$$

- The results of varying the value of each of the criterion weight one-at-a-time from 0 to 1 (step 0.1) are given below.

Impact of changing weight of criterion Research

Weight for Criterion Research	Global Weight of Alternative		
	Company A	Company B	Company C
0.0	**0.4193**	0.2561	0.3247
0.1	**0.3910**	0.2930	0.3160
0.2	**0.3627**	0.3298	0.3074
0.3	0.3344	**0.3667**	0.2988
0.4	0.3062	**0.4036**	0.2902
0.5	0.2779	**0.4405**	0.2816
0.6	0.2496	**0.4774**	0.2730
0.7	0.2213	**0.5143**	0.2643
0.8	0.1931	**0.5512**	0.2557
0.9	0.1648	**0.5881**	0.2471
1.0	0.1365	**0.625**	0.2385

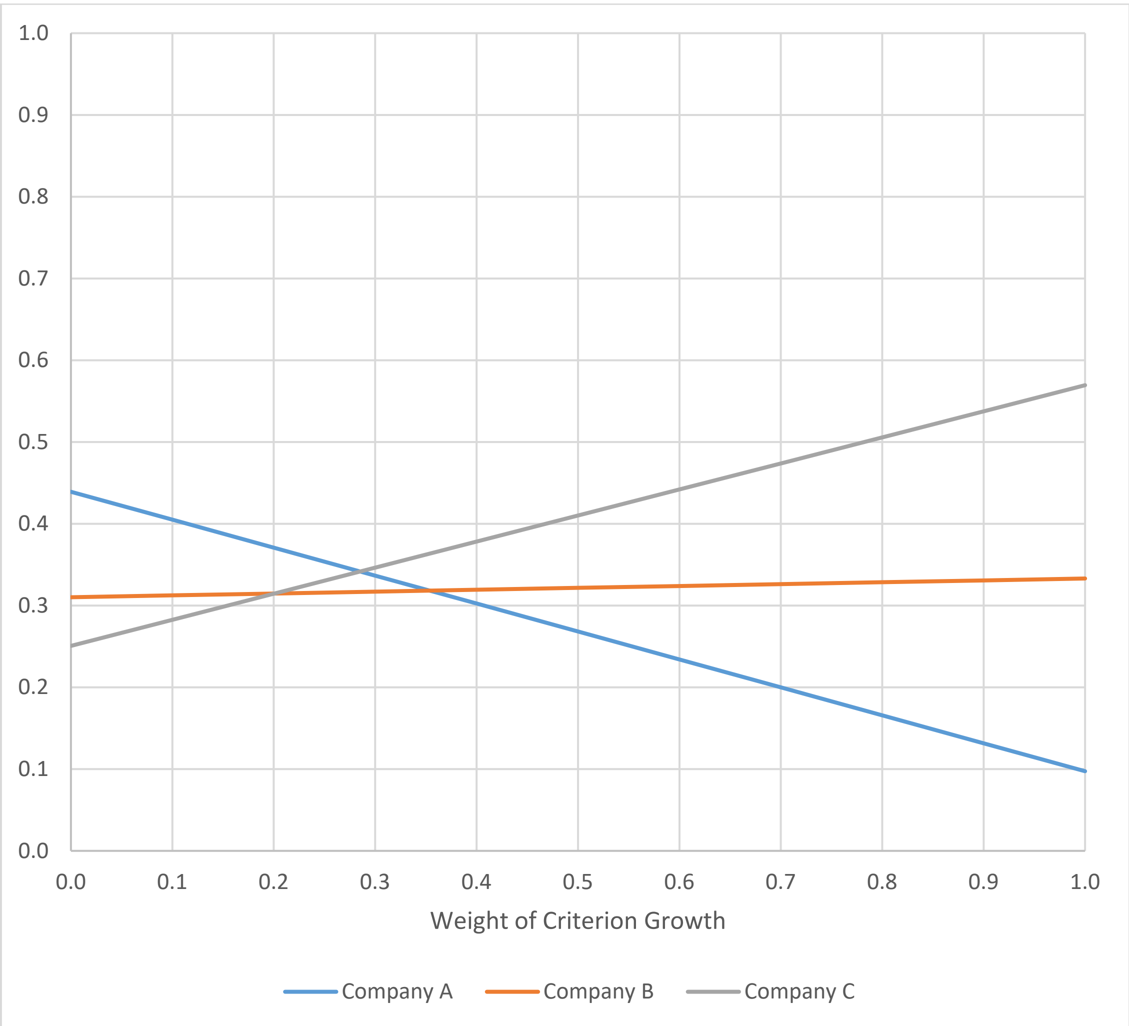
Rainbow Diagram on the Impact of Changing weight of Criterion Research



Impact of changing weight of criterion Growth

Weight for Criterion Growth	Global Weight of Alternative		
	Company A	Company B	Company C
0.0	**0.4392**	0.3102	0.2507
0.1	**0.4050**	0.3125	0.2826
0.2	**0.3708**	0.3148	0.3144
0.3	0.3366	0.3170	**0.3463**
0.4	0.3025	0.3193	**0.3782**
0.5	0.2683	0.3216	**0.4101**
0.6	0.2341	0.3239	**0.4420**
0.7	0.1999	0.3262	**0.4739**
0.8	0.1658	0.3285	**0.5057**
0.9	0.1316	0.3308	**0.5376**
1.0	0.0974	0.3331	**0.5695**

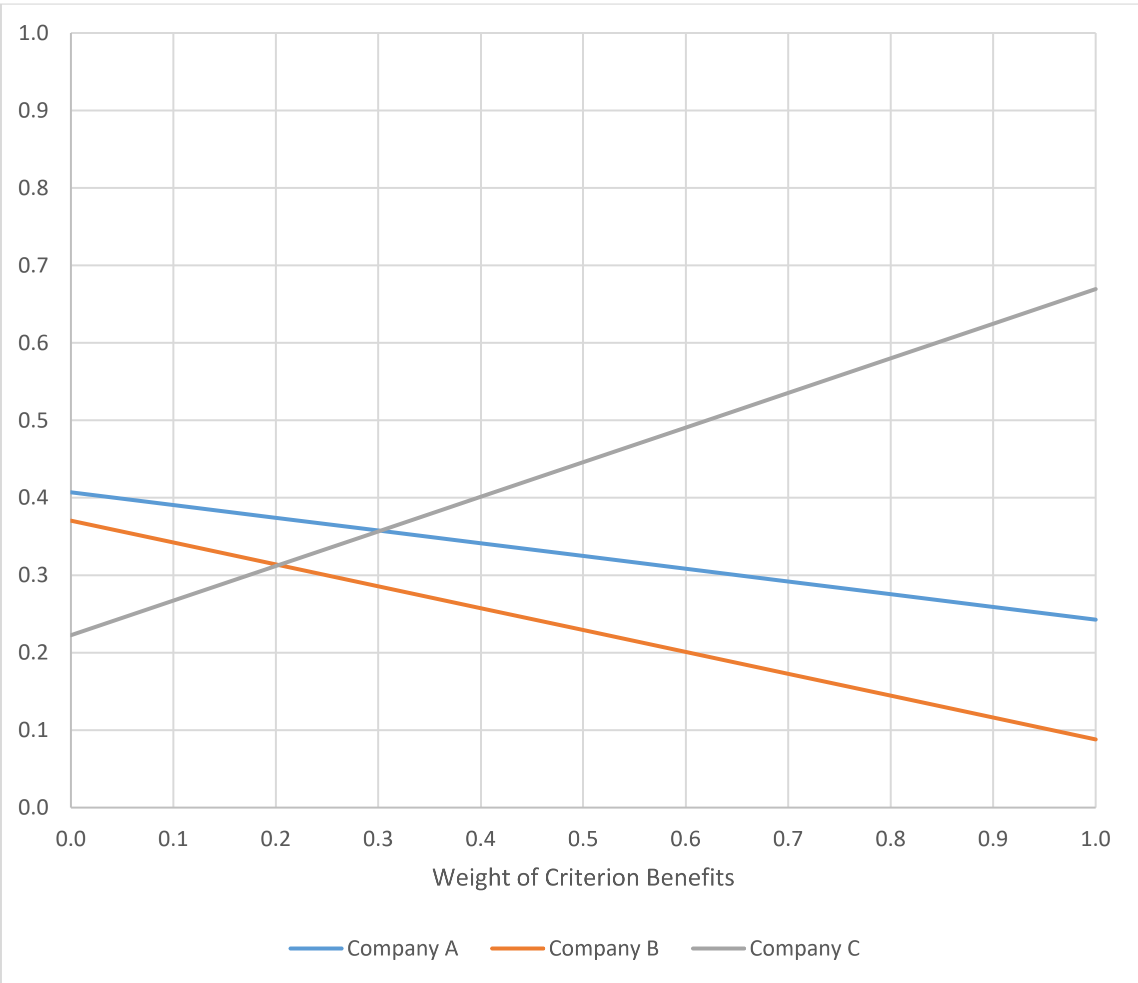
Rainbow Diagram on the Impact of Changing weight of Criterion Growth



Impact of changing weight of criterion Benefits

Weight for Criterion Benefits	Global Weight of Alternative		
	Company A	Company B	Company C
0.0	**0.4070**	0.3704	0.2225
0.1	**0.3906**	0.3422	0.2672
0.2	**0.3742**	0.3139	0.3119
0.3	**0.3577**	0.2857	0.3566
0.4	0.3413	0.2575	**0.4013**
0.5	0.3248	0.2292	**0.4460**
0.6	0.3084	0.2010	**0.4907**
0.7	0.2919	0.1727	**0.5353**
0.8	0.2755	0.1445	**0.5800**
0.9	0.2590	0.1162	**0.6247**
1.0	0.2426	0.0880	**0.6694**

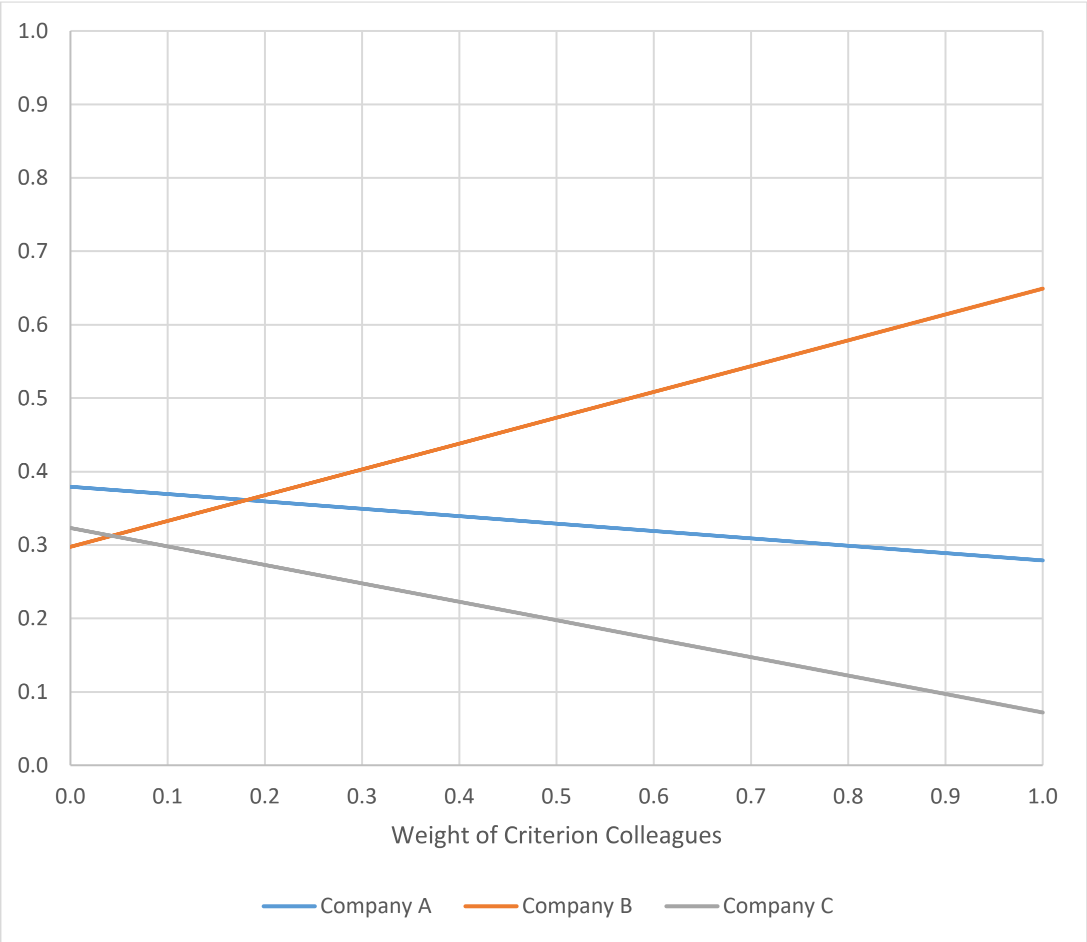
Rainbow Diagram on the Impact of Changing weight of Criterion Benefits



Impact of changing weight of criterion Colleagues

Weight for Criterion Colleagues	Global Weight of Alternative		
	Company A	Company B	Company C
0.0	**0.3793**	0.2975	0.3232
0.1	**0.3693**	0.3327	0.2980
0.2	0.3593	**0.3678**	0.2729
0.3	0.3492	**0.4030**	0.2478
0.4	0.3392	**0.4382**	0.2227
0.5	0.3292	**0.4733**	0.1975
0.6	0.3191	**0.5085**	0.1724
0.7	0.3091	**0.5436**	0.1473
0.8	0.2991	**0.5788**	0.1222
0.9	0.2890	**0.6139**	0.0970
1.0	0.2790	**0.6491**	0.0719

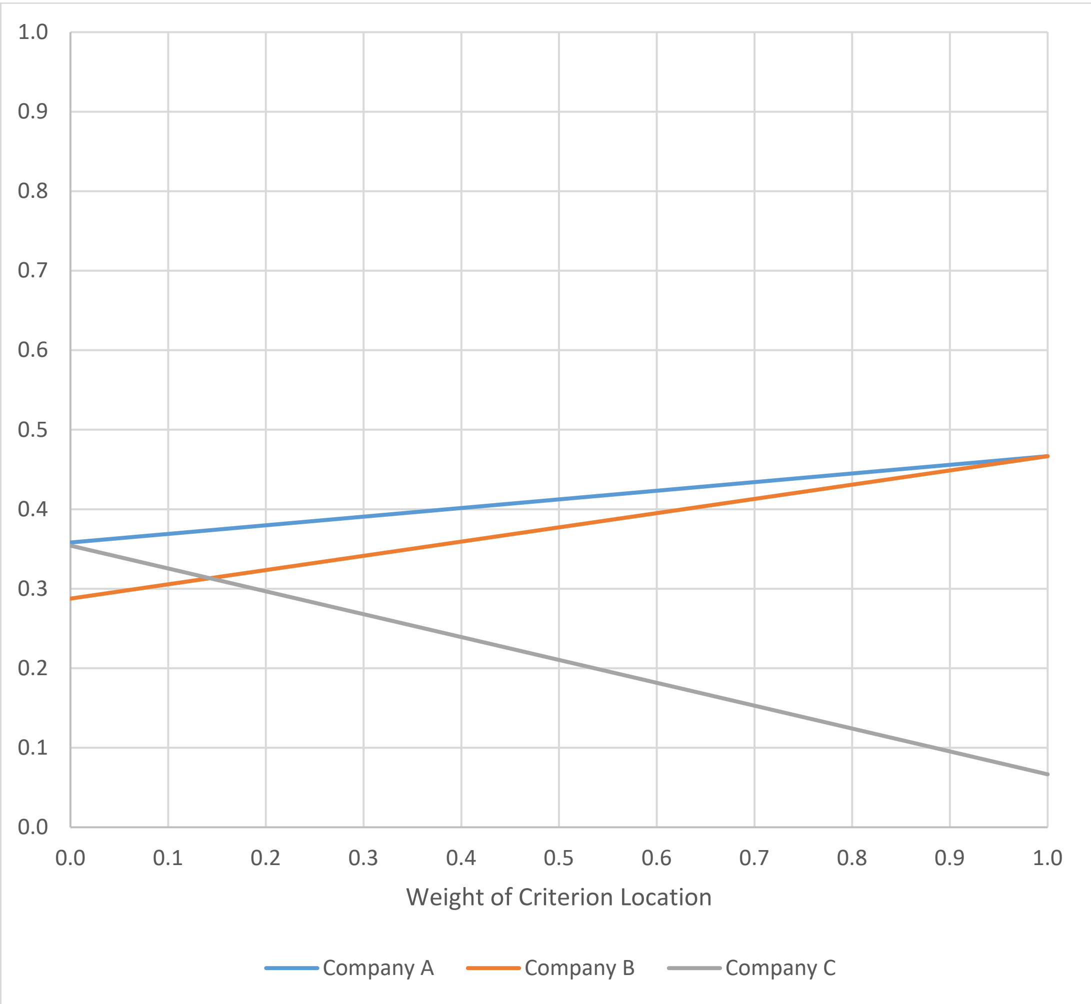
Rainbow Diagram on the Impact of Changing weight of Criterion Colleagues



Impact of changing weight of criterion Location

Weight for Criterion Location	Global Weight of Alternative		
	Company A	Company B	Company C
0.0	**0.3582**	0.2876	0.3542
0.1	**0.3690**	0.3055	0.3255
0.2	**0.3799**	0.3234	0.2967
0.3	**0.3907**	0.3413	0.2679
0.4	**0.4016**	0.3592	0.2392
0.5	**0.4124**	0.3771	0.2104
0.6	**0.4233**	0.3951	0.1817
0.7	**0.4341**	0.4130	0.1529
0.8	**0.4450**	0.4309	0.1241
0.9	**0.4558**	0.4488	0.0954
1.0	**0.4667**	0.4667	0.0666

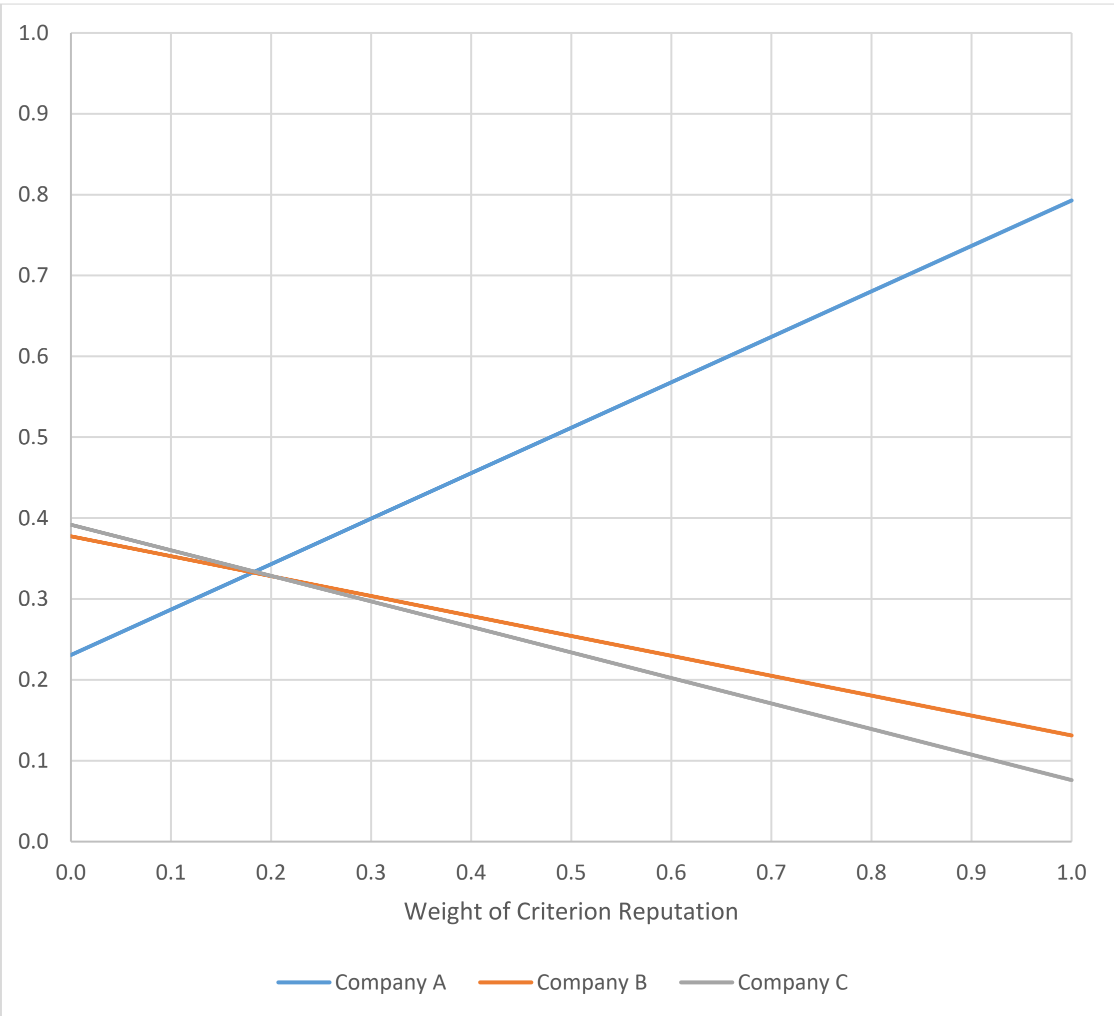
Rainbow Diagram on the Impact of Changing weight of Criterion Location



Impact of changing weight of criterion Reputation

Weight for Criterion Reputation	Global Weight of Alternative		
	Company A	Company B	Company C
0.0	0.2307	0.3775	**0.3918**
0.1	0.2869	0.3529	**0.3602**
0.2	0.3431	0.3282	**0.3286**
0.3	**0.3993**	0.3036	0.2971
0.4	**0.4555**	0.2790	0.2655
0.5	**0.5117**	0.2544	0.2339
0.6	**0.5680**	0.2297	0.2023
0.7	**0.6242**	0.2051	0.1707
0.8	**0.6804**	0.1805	0.1392
0.9	**0.7366**	0.1558	0.1076
1.0	**0.7928**	0.1312	0.0760

Rainbow Diagram on the Impact of Changing weight of Criterion Reputation



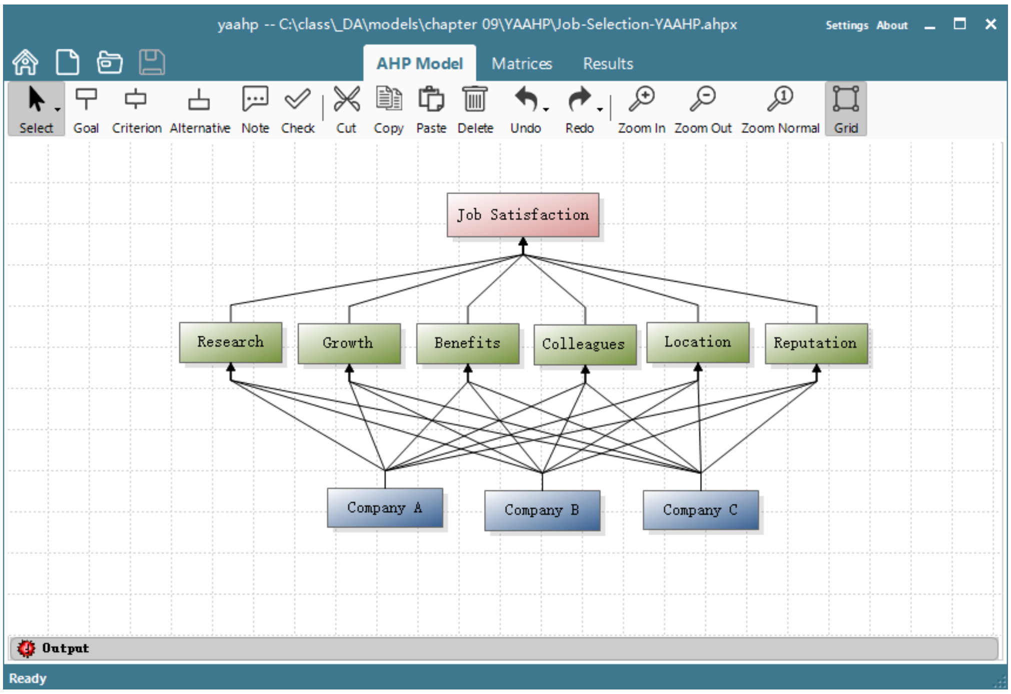
9.4 Using Computer Software to Solve AHP Models

9.4.1 YAAHP (Yet Another AHP)

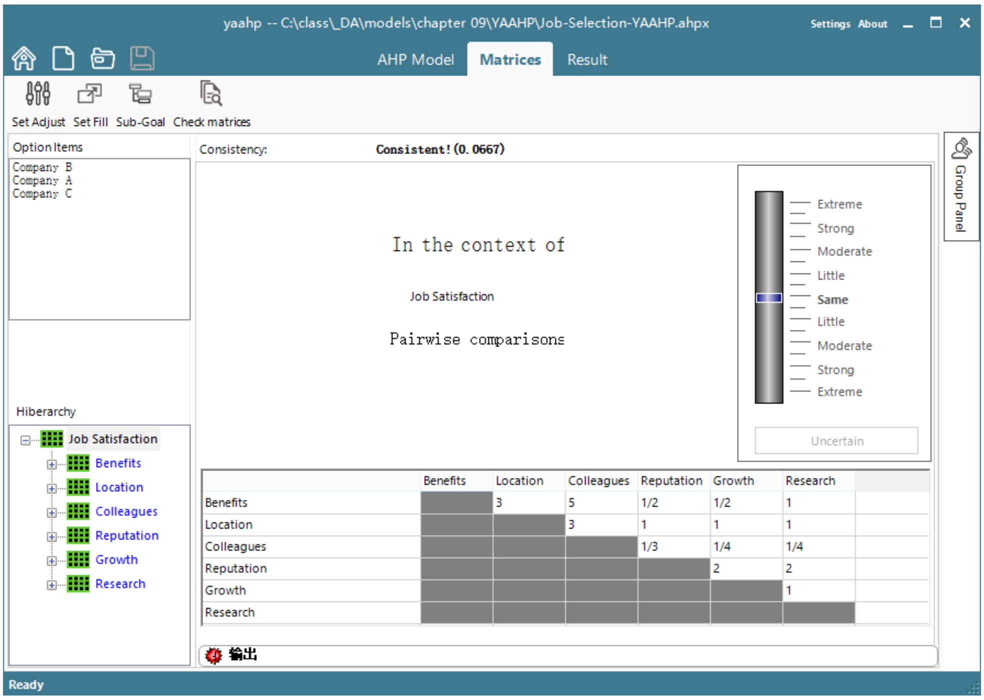
- YAAHP solves standard AHP models and Fuzzy Comprehensive Evaluation Models.
  - Chinese version: http://www.metadecsn.com/download/

- Note: Trial versions from 11 onward do not support sensitivity analysis.

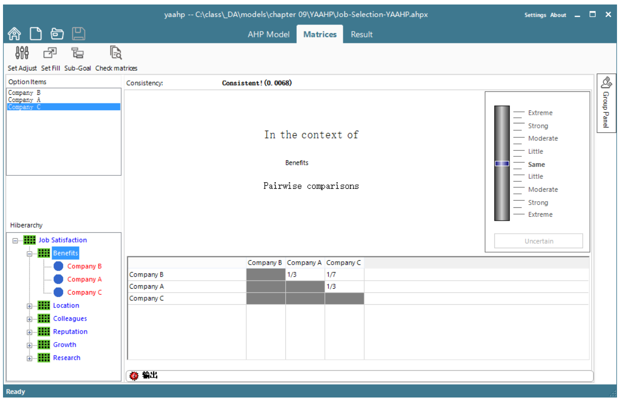
- **<u>Hierarchy Modeling Interface</u>**

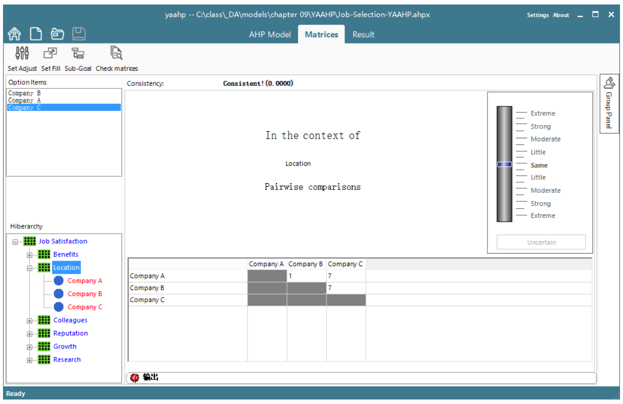


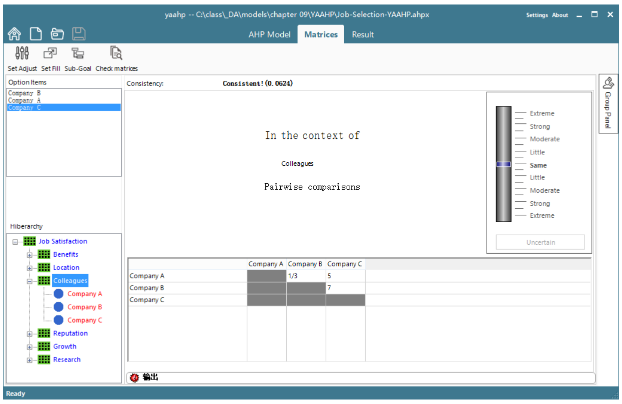
<u>Pairwise Comprison of the Crieria</u>

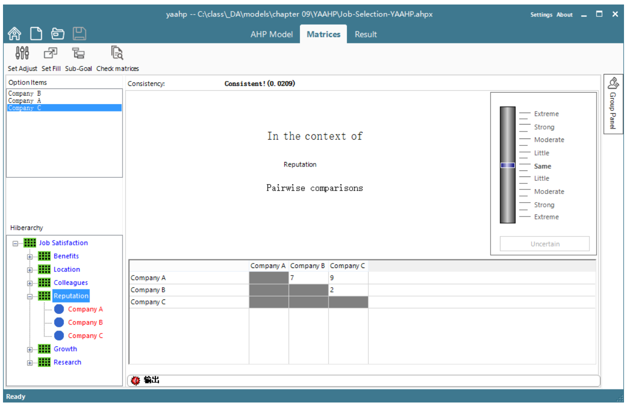


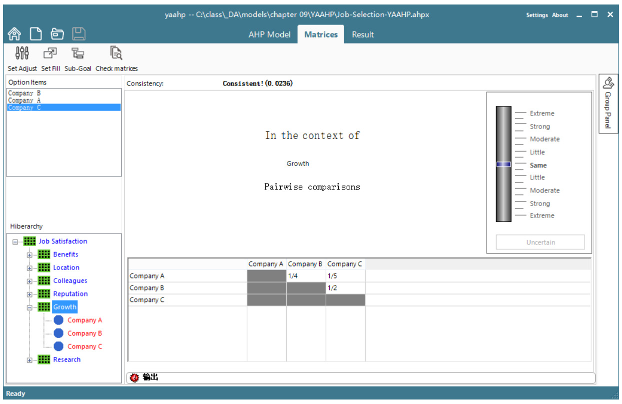
<u>Pairwise comprision of alternatives w.r.t. each criterion</u>

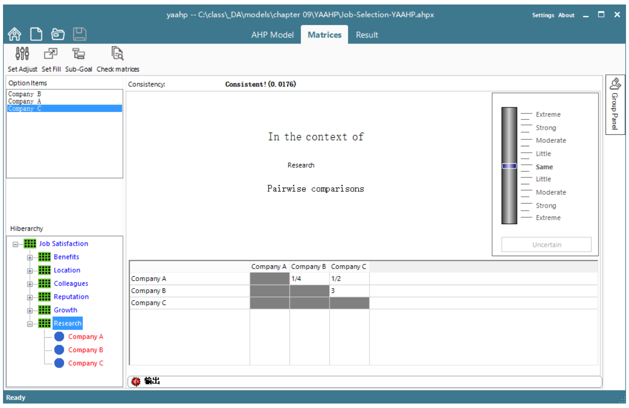




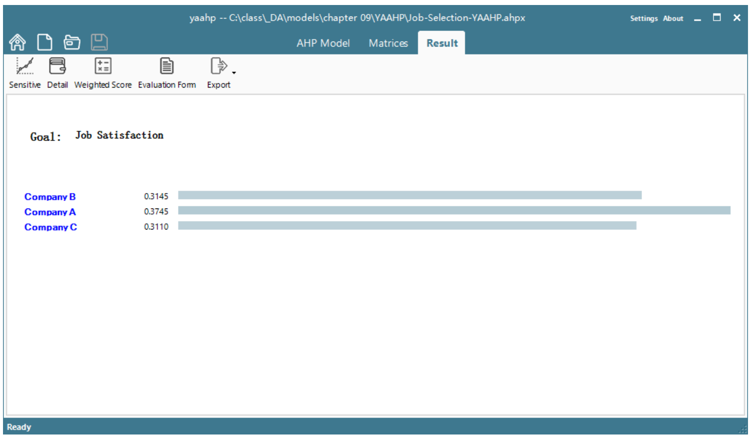




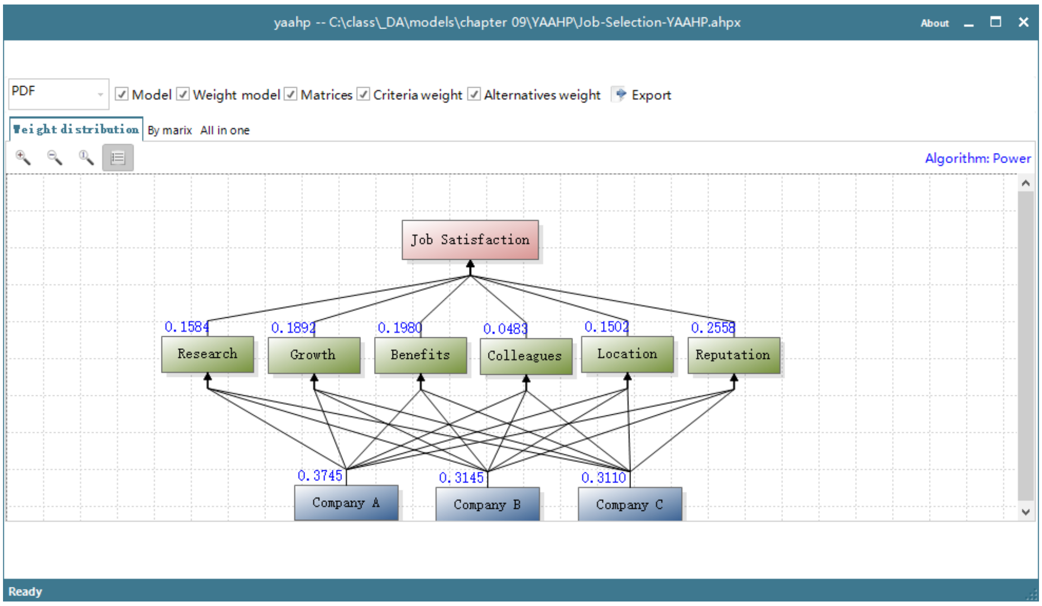




Results



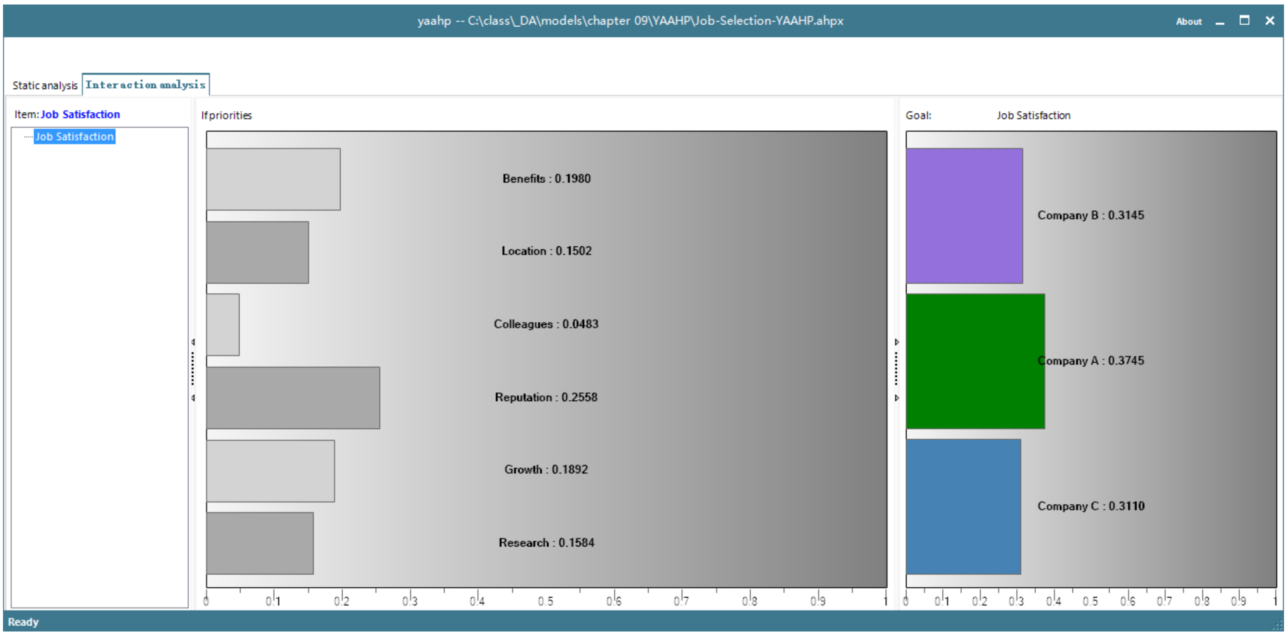
Summary of Results



Sensitivity Analysis with YAAHP

Interactive Dynamic Sensitivity

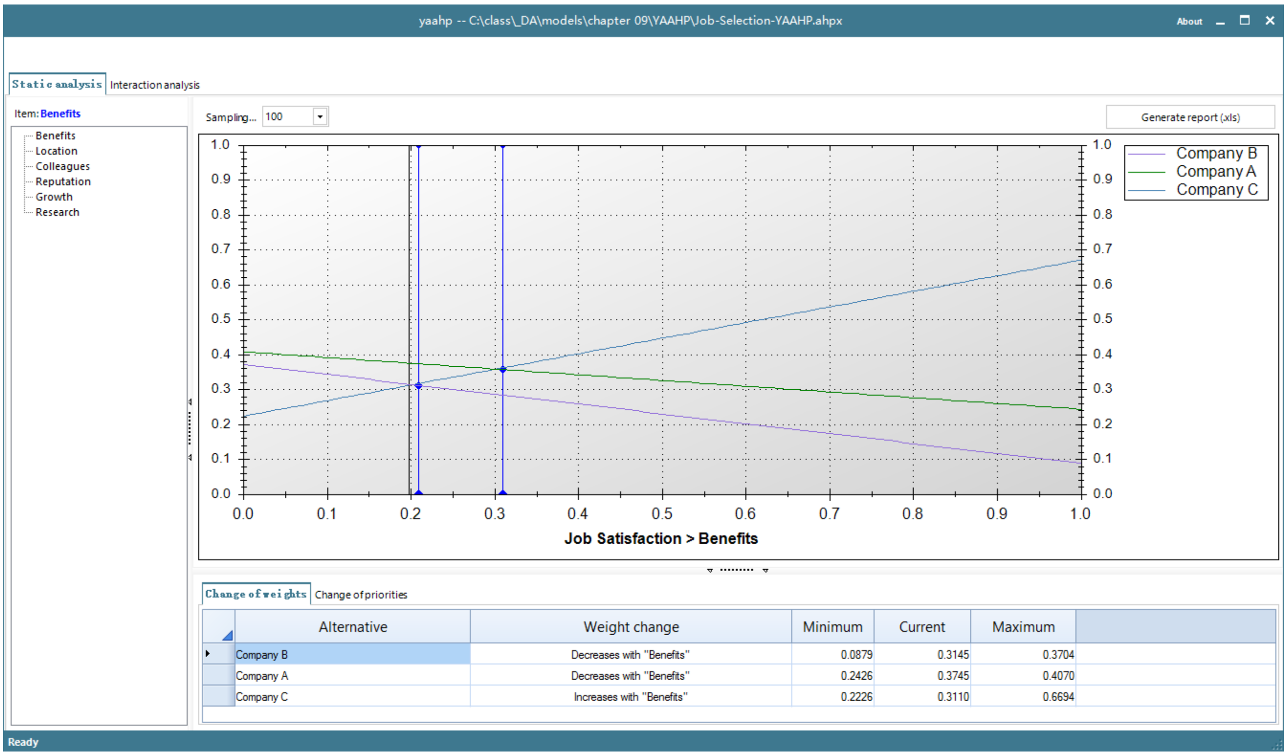
- Dynamic sensitivity enables us to see the change in global weight of the alternatives as we increase or decrease the priority weight of the criteria.



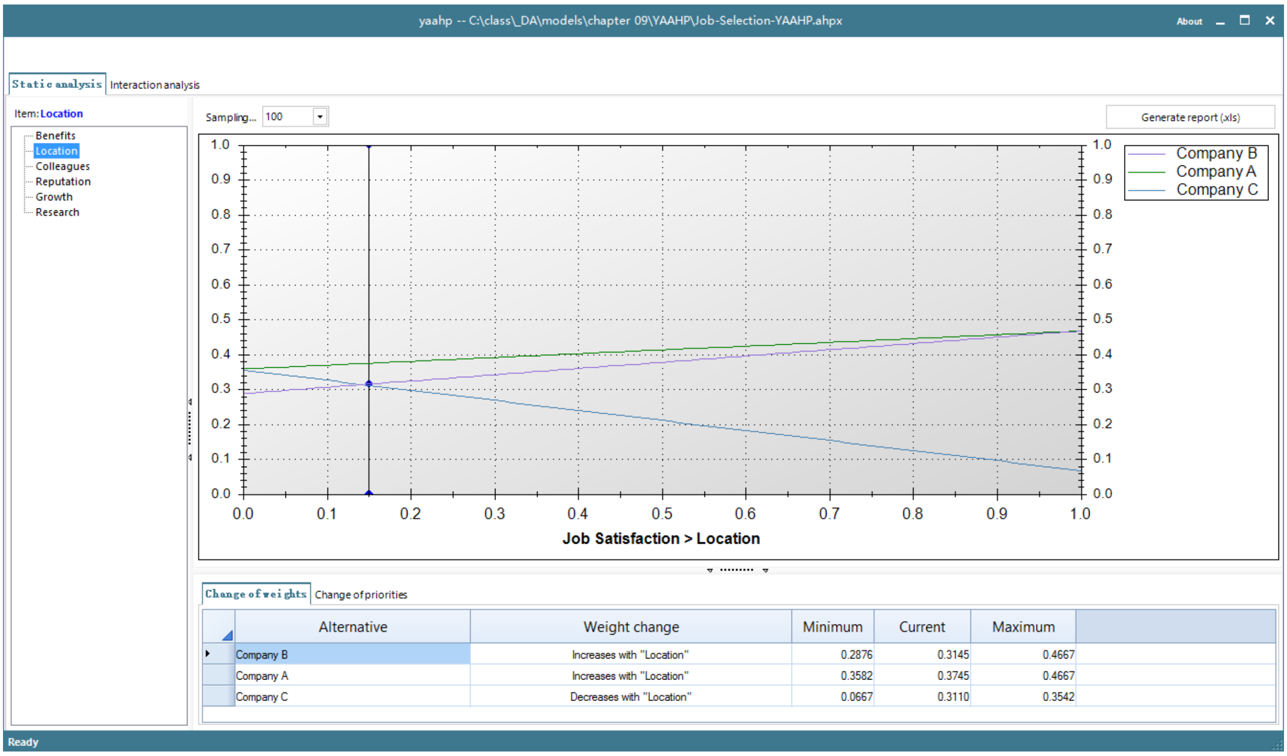
Rainbow Diagrams on Change in Criterion Weight

- These diagrams how the global weights of the alternatives change as the weight of a criterion changes from 0 to 1.0, while keeping the weights of the other criteria in the same relative proportion as their base values.

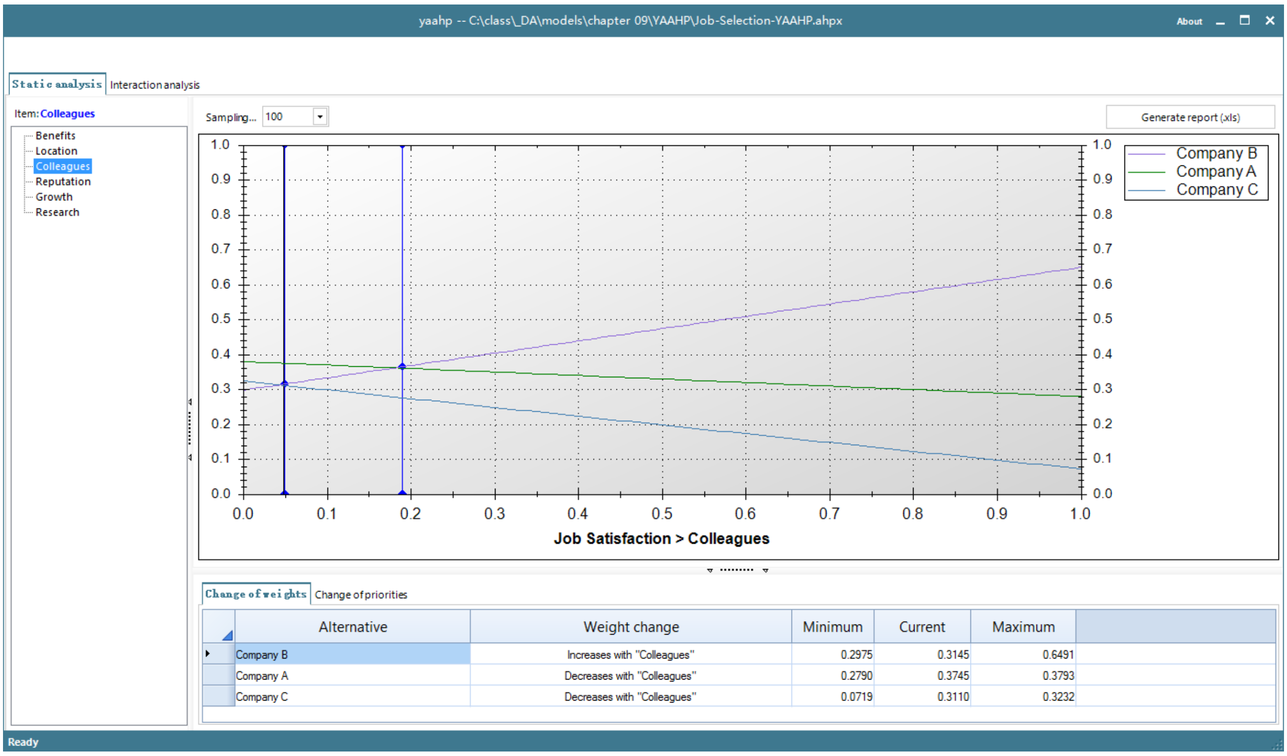
Rainbow Diagram for alternative global weights when Benefit weight is varied from 0 to 1



Rainbow Diagram or alternative global weights when Location weight is varied from 0 to 1

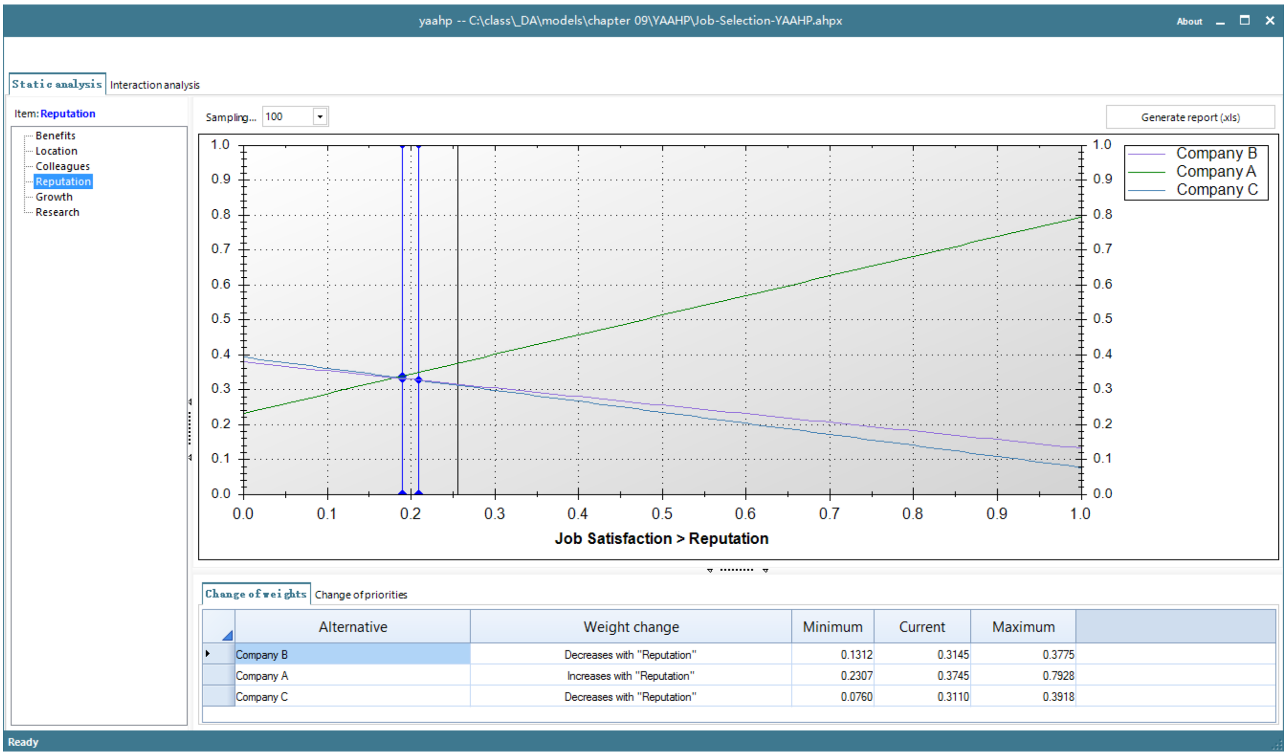


Rainbow Diagram for alternative global weights when Colleagues weight is varied from 0 to 1



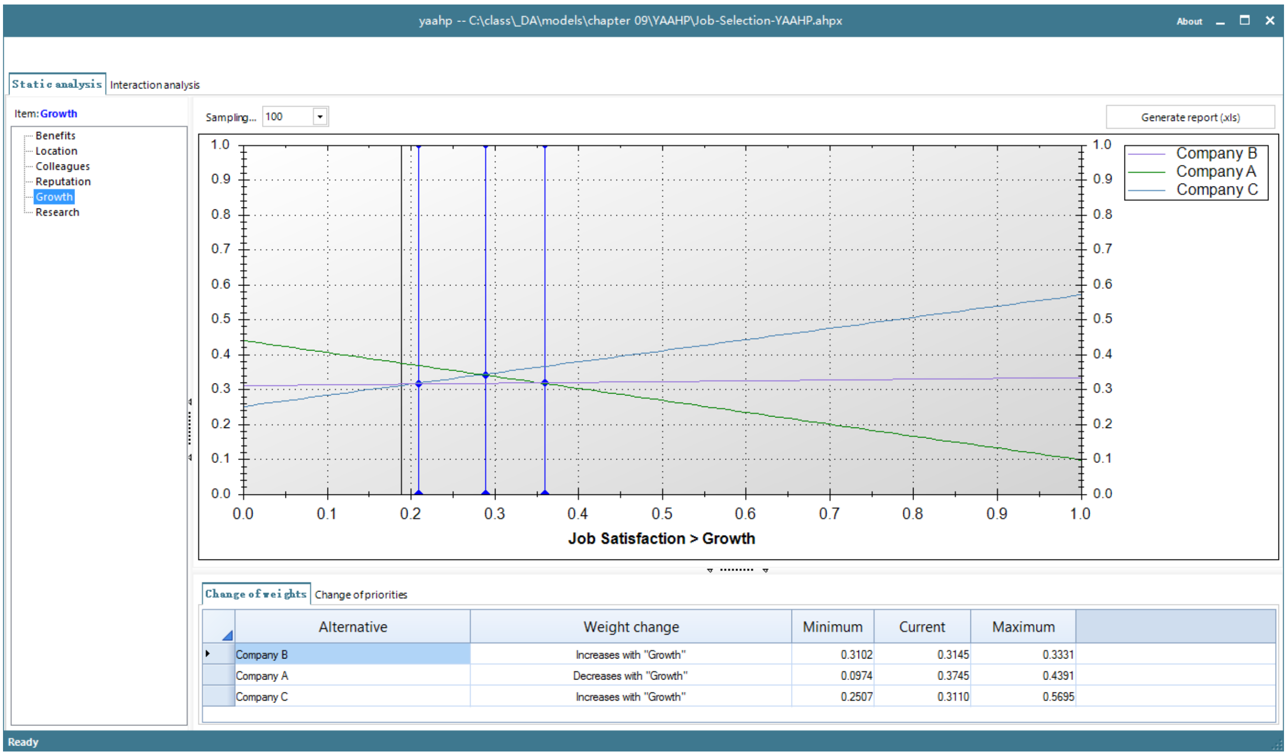
Alternative	Weight change	Minimum	Current	Maximum
Company B	Increases with "Colleagues"	0.2975	0.3145	0.6491
Company A	Decreases with "Colleagues"	0.2790	0.3745	0.3793
Company C	Decreases with "Colleagues"	0.0719	0.3110	0.3232

Rainbow Diagram for alternative global weights when Reputation wt is varied from 0 to 1

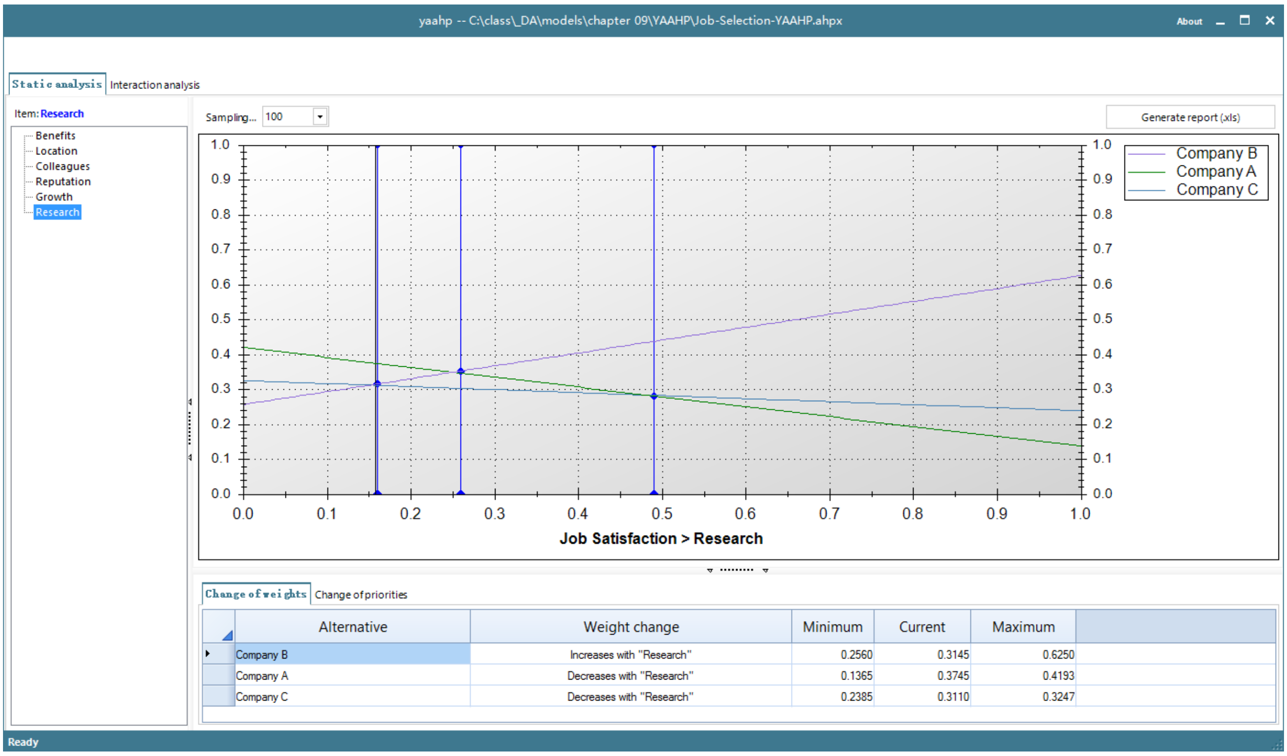


Alternative	Weight change	Minimum	Current	Maximum
Company B	Decreases with "Reputation"	0.1312	0.3145	0.3775
Company A	Increases with "Reputation"	0.2307	0.3745	0.7928
Company C	Decreases with "Reputation"	0.0760	0.3110	0.3918

Rainbow Diagram for or alternative global weights when Growth weight is varied from 0 to 1

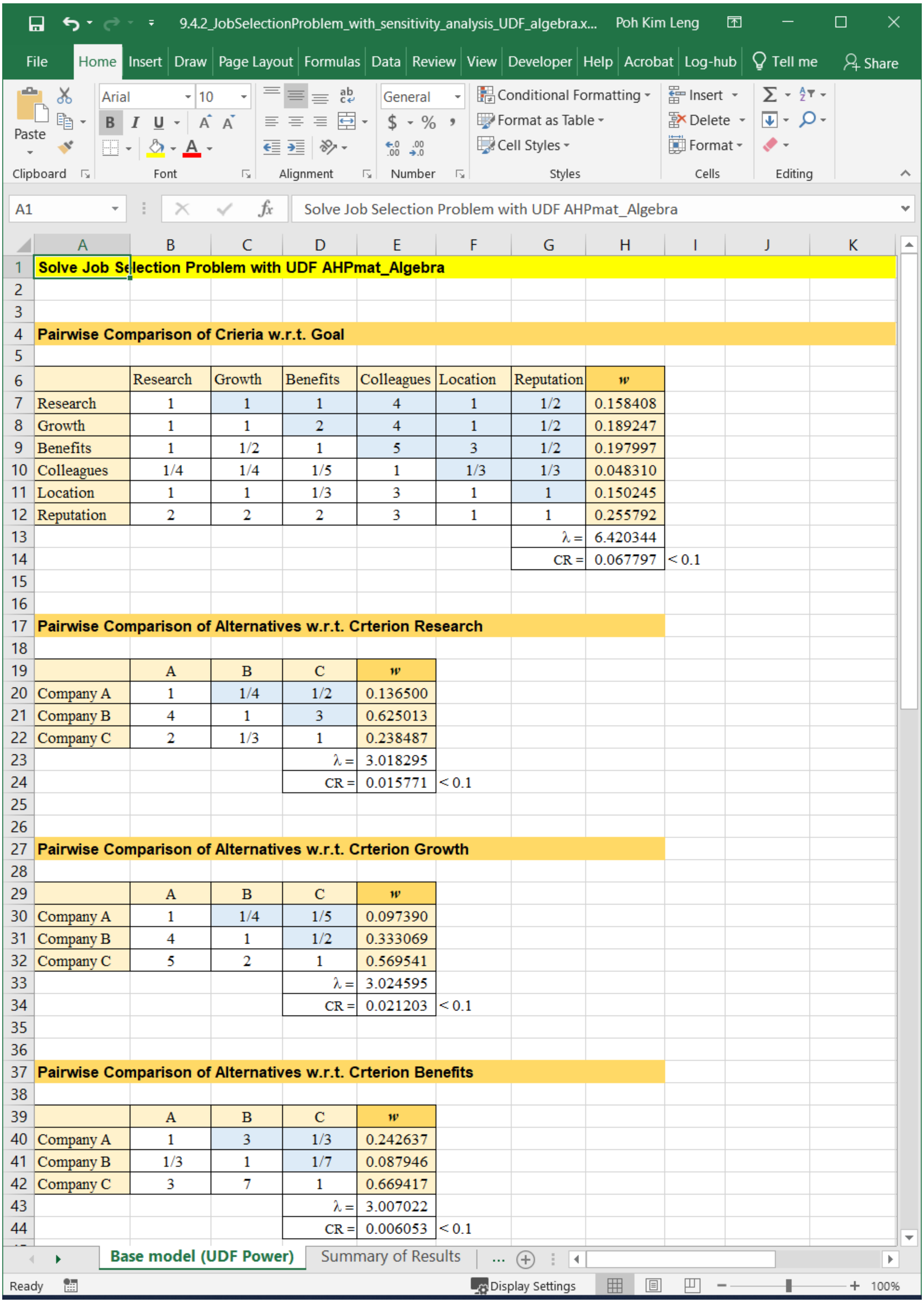


Rainbow Diagram for or alternative global weights hen Research weight is varied from 0 to 1



9.4.2 Using Excel with User-defined Functions

- You can use Excel UDFs to compute the weights, λ, and CR for each matrices, and then compute the global weights using worksheet functions. Sensitivity analysis can also be performed.



Solve Job Selection Problem with UDF AHPmat_Algebra

Pairwise Comparison of Crieria w.r.t. Goal

	Research	Growth	Benefits	Colleagues	Location	Reputation	*w*	
Research	1	1	1	4	1	1/2	0.158408	
Growth	1	1	2	4	1	1/2	0.189247	
Benefits	1	1/2	1	5	3	1/2	0.197997	
Colleagues	1/4	1/4	1/5	1	1/3	1/3	0.048310	
Location	1	1	1/3	3	1	1	0.150245	
Reputation	2	2	2	3	1	1	0.255792	
						λ =	6.420344	
						CR =	0.067797	< 0.1

Pairwise Comparison of Alternatives w.r.t. Crterion Research

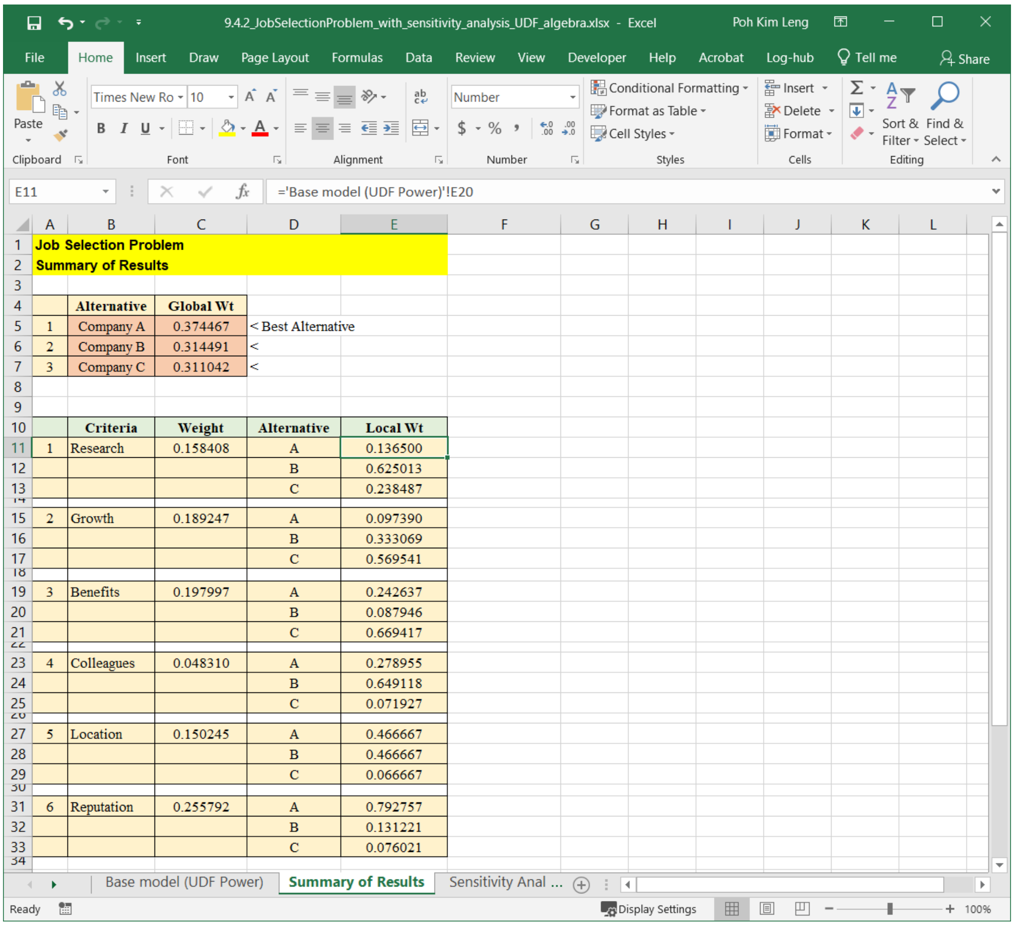
	A	B	C	*w*	
Company A	1	1/4	1/2	0.136500	
Company B	4	1	3	0.625013	
Company C	2	1/3	1	0.238487	
			λ =	3.018295	
			CR =	0.015771	< 0.1

Pairwise Comparison of Alternatives w.r.t. Crterion Growth

	A	B	C	*w*	
Company A	1	1/4	1/5	0.097390	
Company B	4	1	1/2	0.333069	
Company C	5	2	1	0.569541	
			λ =	3.024595	
			CR =	0.021203	< 0.1

Pairwise Comparison of Alternatives w.r.t. Crterion Benefits

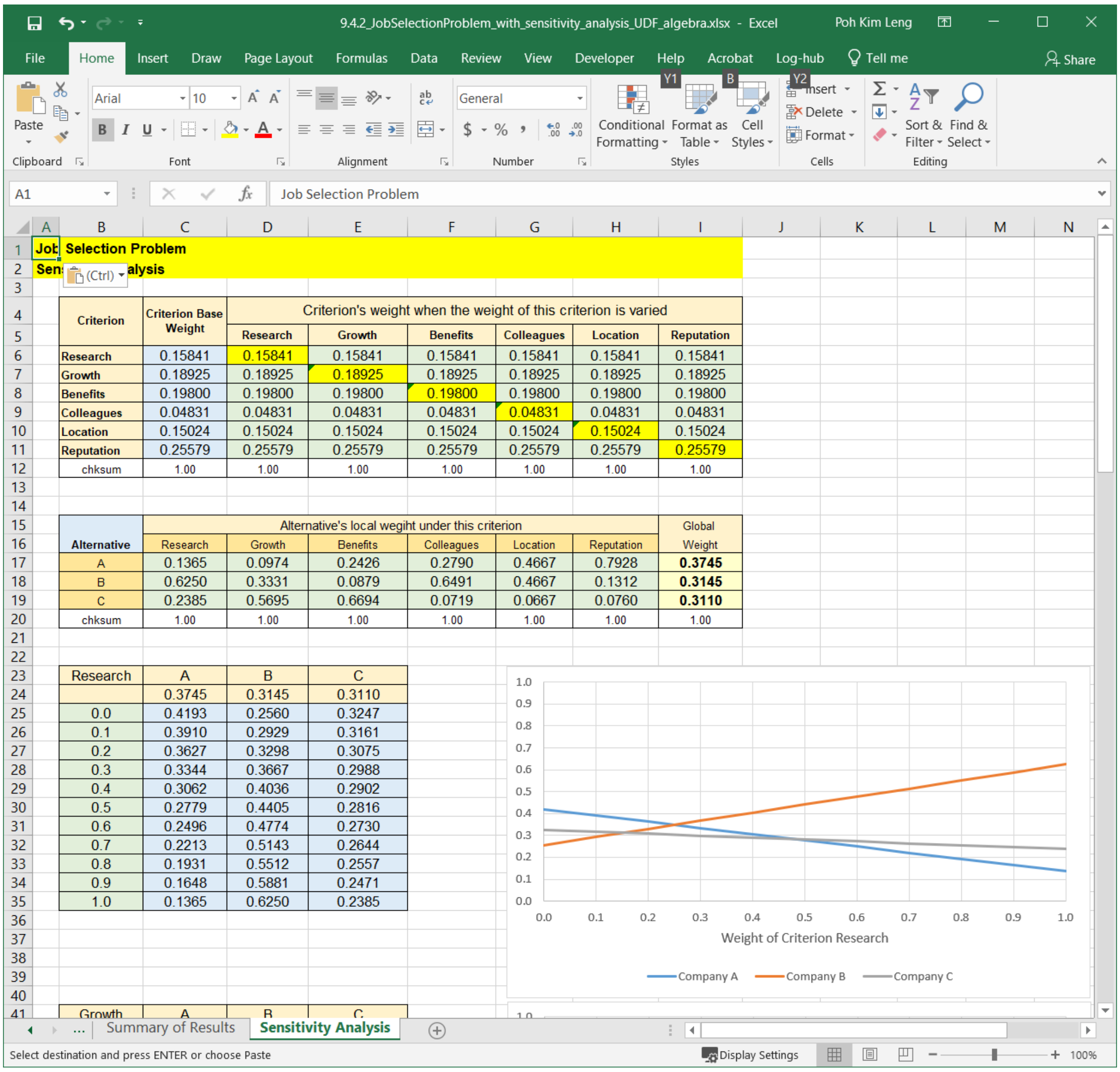
	A	B	C	*w*	
Company A	1	3	1/3	0.242637	
Company B	1/3	1	1/7	0.087946	
Company C	3	7	1	0.669417	
			λ =	3.007022	
			CR =	0.006053	< 0.1



Job Selection Problem
Summary of Results

	Alternative	Global Wt	
1	Company A	0.374467	< Best Alternative
2	Company B	0.314491	<
3	Company C	0.311042	<

	Criteria	Weight	Alternative	Local Wt
1	Research	0.158408	A	0.136500
			B	0.625013
			C	0.238487
2	Growth	0.189247	A	0.097390
			B	0.333069
			C	0.569541
3	Benefits	0.197997	A	0.242637
			B	0.087946
			C	0.669417
4	Colleagues	0.048310	A	0.278955
			B	0.649118
			C	0.071927
5	Location	0.150245	A	0.466667
			B	0.466667
			C	0.066667
6	Reputation	0.255792	A	0.792757
			B	0.131221
			C	0.076021



Job Selection Problem
Sen (Ctrl) alysis

Criterion	Criterion Base Weight	Criterion's weight when the weight of this criterion is varied					
		Research	Growth	Benefits	Colleagues	Location	Reputation
Research	0.15841	0.15841	0.15841	0.15841	0.15841	0.15841	0.15841
Growth	0.18925	0.18925	0.18925	0.18925	0.18925	0.18925	0.18925
Benefits	0.19800	0.19800	0.19800	0.19800	0.19800	0.19800	0.19800
Colleagues	0.04831	0.04831	0.04831	0.04831	0.04831	0.04831	0.04831
Location	0.15024	0.15024	0.15024	0.15024	0.15024	0.15024	0.15024
Reputation	0.25579	0.25579	0.25579	0.25579	0.25579	0.25579	0.25579
chksum	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Alternative	Alternative's local weight under this criterion						Global Weight
	Research	Growth	Benefits	Colleagues	Location	Reputation	
A	0.1365	0.0974	0.2426	0.2790	0.4667	0.7928	**0.3745**
B	0.6250	0.3331	0.0879	0.6491	0.4667	0.1312	**0.3145**
C	0.2385	0.5695	0.6694	0.0719	0.0667	0.0760	**0.3110**
chksum	1.00	1.00	1.00	1.00	1.00	1.00	1.00

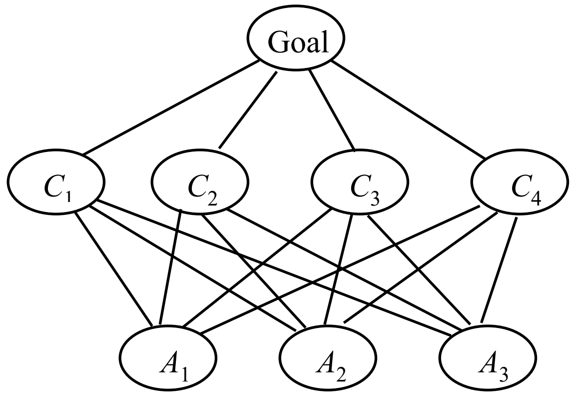
Research	A	B	C
	0.3745	0.3145	0.3110
0.0	0.4193	0.2560	0.3247
0.1	0.3910	0.2929	0.3161
0.2	0.3627	0.3298	0.3075
0.3	0.3344	0.3667	0.2988
0.4	0.3062	0.4036	0.2902
0.5	0.2779	0.4405	0.2816
0.6	0.2496	0.4774	0.2730
0.7	0.2213	0.5143	0.2644
0.8	0.1931	0.5512	0.2557
0.9	0.1648	0.5881	0.2471
1.0	0.1365	0.6250	0.2385

Growth	A	B	C

9.5 AHP Models with Complex Hierarchies

9.5.1 Models with more than 3 levels

- So far, we have seen the use of a 3-level complete hierarchy where the second level signifies the criteria for evaluation, and the third or lowest level denotes the alternative to be evaluated.



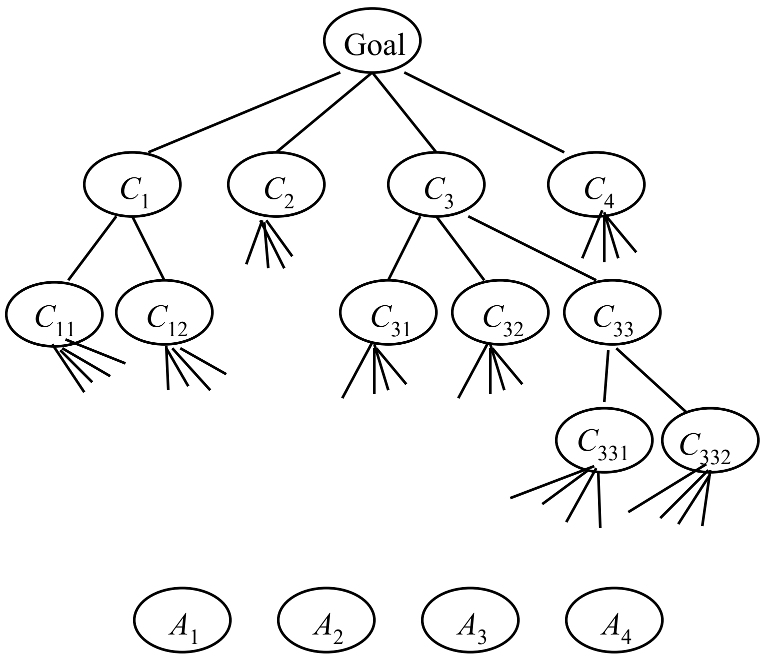
- We call such a hierarchy a **Simple 3-Level Hierarchy.** It is the simplest form of hierarchy for performing multiple criteria evaluation of alternatives.

Multi-Level Tree Hierarchies

- An AHP model may have main criteria which may be decomposed into sub-criteria, and some of the sub-criteria may have further decomposed, etc.

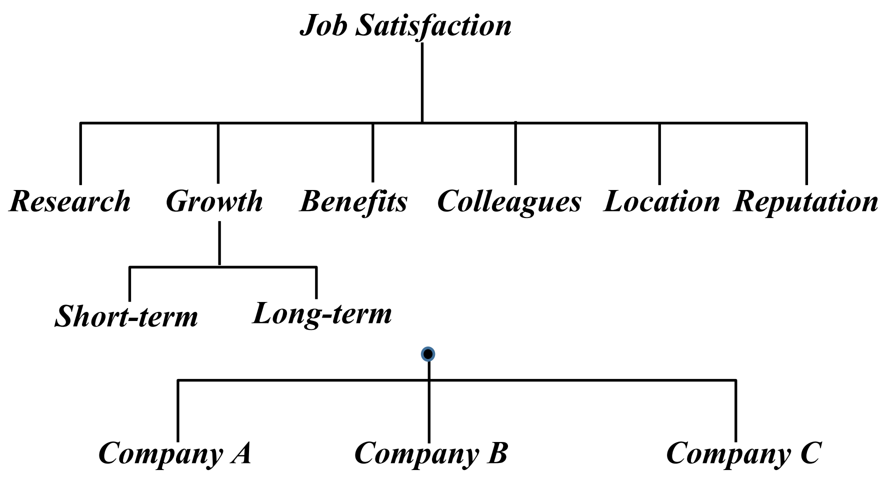
- The hierarchy does not need to be balanced in the depth of sub-trees.

- The criteria that are directly above the alternatives are called leaf-criteria.



9.5.2 Case Study: Job Selection Problem with Growth Sub-criteria

- In the previous Job Selection Problem, suppose that the Growth criterion has two sub-criteria: short-term growth and long-term growth:



- We pairwise compare the 6 criteria w.r.t. the Goal and obtain the same weights as before.

- The two sub-criteria are judged w.r.t. their parent (Growth):

	Short-term growth	Long-term growth
Short-term growth	1	1/3
Long-term growth	3	1

$\boldsymbol{w}$ = [0.25, 0.75]

- We pairwise compare the 3 alternatives w.r.t. the seven leaf criteria and/or sub-criteria:

- The results for the criteria Research, Benefits, Colleagues, Location, and Reputations are as before.

- The results for the sub-criteria short-term growth and long-term growth are as follows:

Comparison of alternatives w.r.t. “Short-Term Growth”:

	A	B	C
A	1	1/3	1/7
B		1	1/3
C			1

λ_{max} = 3.0070, CI= 0.003511, CR.=0.006053 < 0.1, $\boldsymbol{w}$ = [0.08795, 0.24264, 0.66942]

Comparison of alternatives w.r.t. “Long-Term Growth”:

	A	B	C
A	1	3	5
B		1	2
C			1

λ_{max} = 3.0037, CI =0.001847, CR =0.003815 < 0.1, $\boldsymbol{w}$ = [0.64833, 0.22965, 0.12202]

- The global weights for the alternatives are computed as follows:

	Criteria	Sub-criteria (if any)	Global weights of leaf criteria	Alt's local weight w.r.t Leaf Criterion
1	Research (0.158408)		0.158408	
				Job A (0.13650)
				Job B (0.62501)
				Job C (0.23849)
2	Growth (0.189247)			
2.1		Short-term (0.25)	0.047312	
				Job A (0.08795)
				Job B (0.24264)
				Job C (0.66942)
2.1		Long-term (0.75)	0.141935	
				Job A (0.64833)
				Job B (0.22965)
				Job C (0.12201)
3	Benefits (0.197997)		0.197997	
				Job A (0.24264)
				Job B (0.08795)
				Job C (0.66942)
4	Colleagues (0.048310)		0.048310	
				Job A (0.27895)
				Job B (0.64912)
				Job C (0.07193)
5	Location (0.150245)		0.150245	
				Job A (0.46667)
				Job B (0.46667)
				Job C (0.06667)
6	Reputation (0.255792)		0.255792	
				Job A (0.79276)
				Job B (0.13122)
				Job C (0.07602)

- Global weight for criterion "Short-Term Growth" = $0.25 \times 0.189247 = 0.047312$
- Global weight for criterion "Long-Term Growth" = $0.75 \times 0.189247 = 0.141935$

- The Global Weight of an alternative is equal to the leaf criterion-weighted sum of its local weights. In Matrix notations:

$$\begin{bmatrix} w_A \\ w_B \\ w_C \end{bmatrix} = \begin{bmatrix} 0.13650 & 0.08795 & 0.64833 & 0.24264 & 0.27895 & 0.46667 & 0.79276 \\ 0.62501 & 0.24264 & 0.22965 & 0.08795 & 0.64912 & 0.46667 & 0.13122 \\ 0.23849 & 0.66942 & 0.12202 & 0.66942 & 0.07193 & 0.06667 & 0.07602 \end{bmatrix} \begin{bmatrix} 0.158408 \\ 0.047312 \\ 0.141935 \\ 0.197997 \\ 0.048310 \\ 0.150245 \\ 0.255792 \end{bmatrix} = \begin{bmatrix} 0.4522 \\ 0.2955 \\ 0.2523 \end{bmatrix}$$

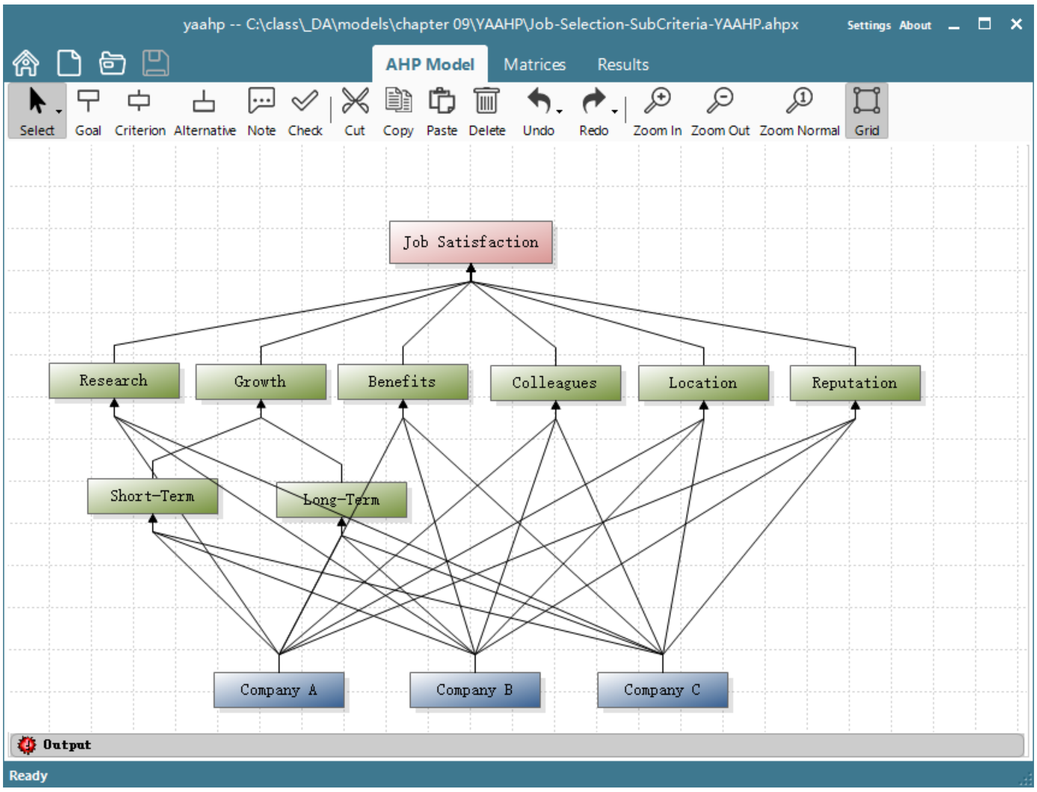
- The results are as follows:

Alternative	Global weight
Job A	**0.4522**
Job B	0.2955
Job C	0.2523

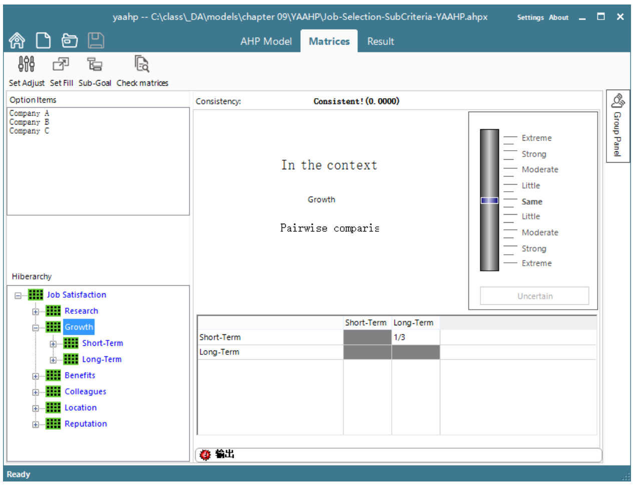
- Hence choose Job A since it has the highest global weight.

9.5.3 Using YAAHP Application Software

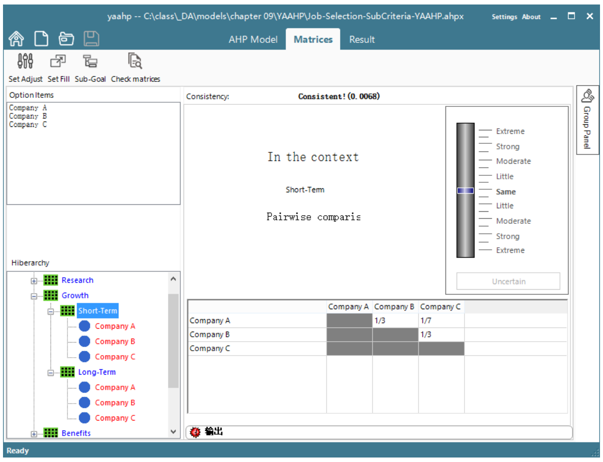
<u>The Hierarchy with Sub Criteria</u>

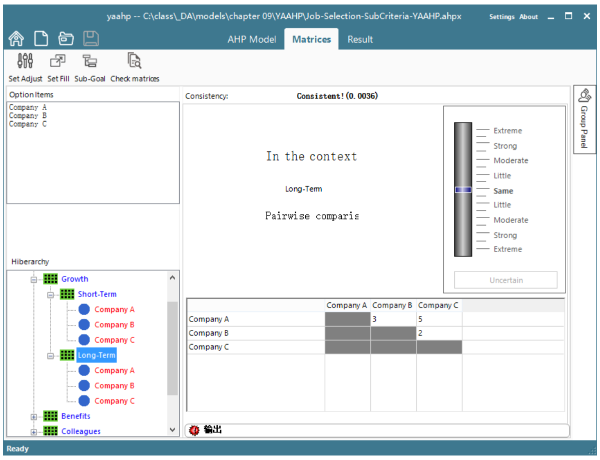


<u>Pairwise comparison of sub-criteria with respective Growth</u>

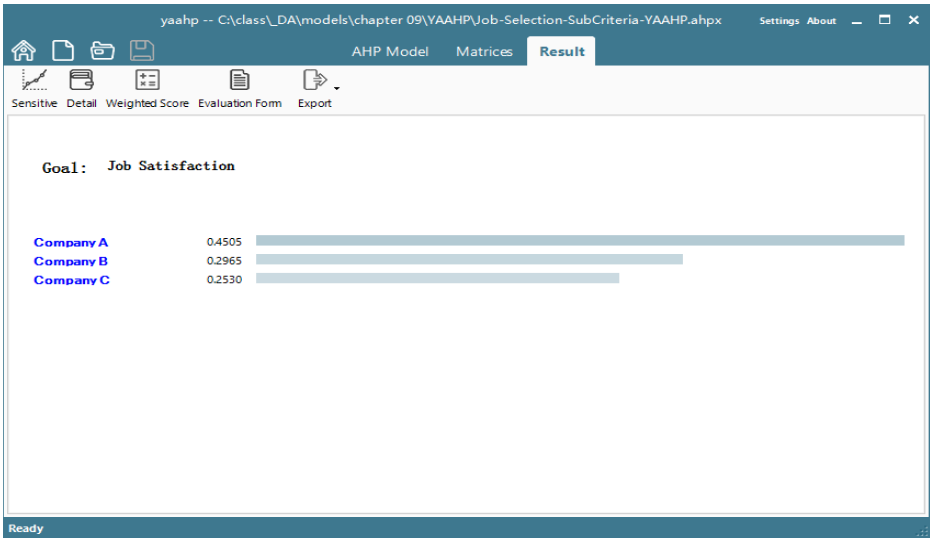


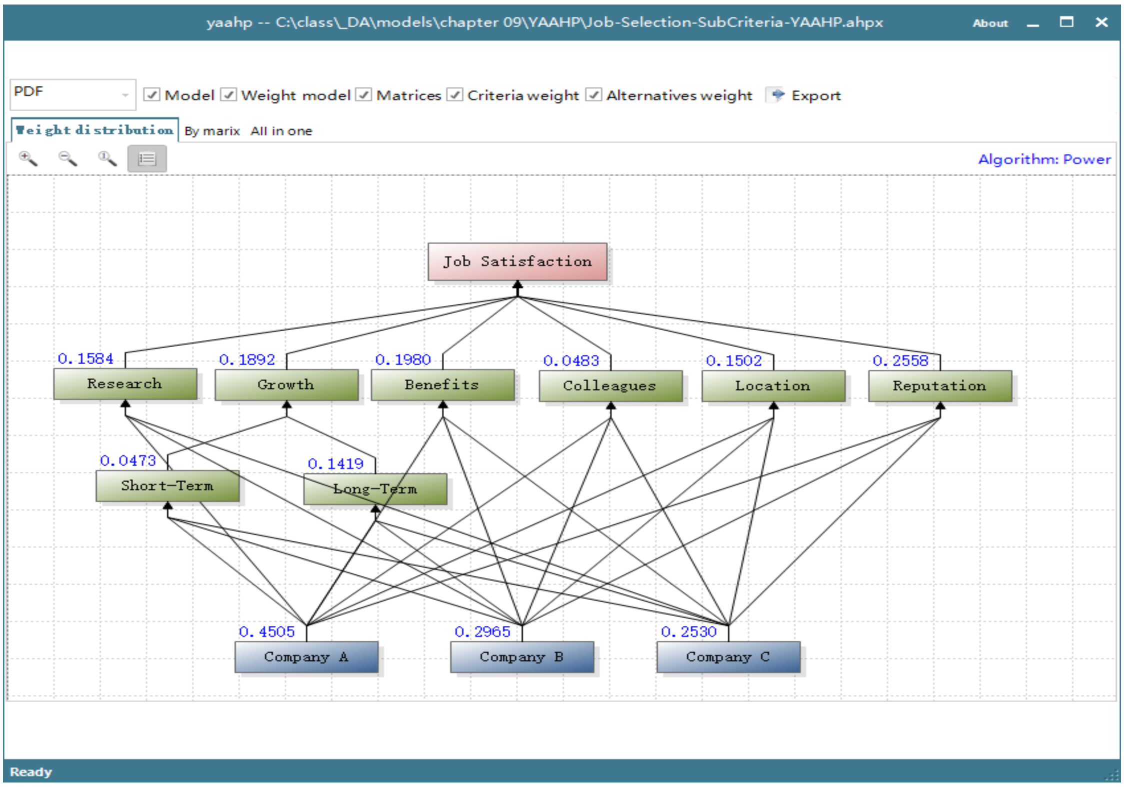
Pairwise Comparison of Alternatives w.r.t. Sub-criteria



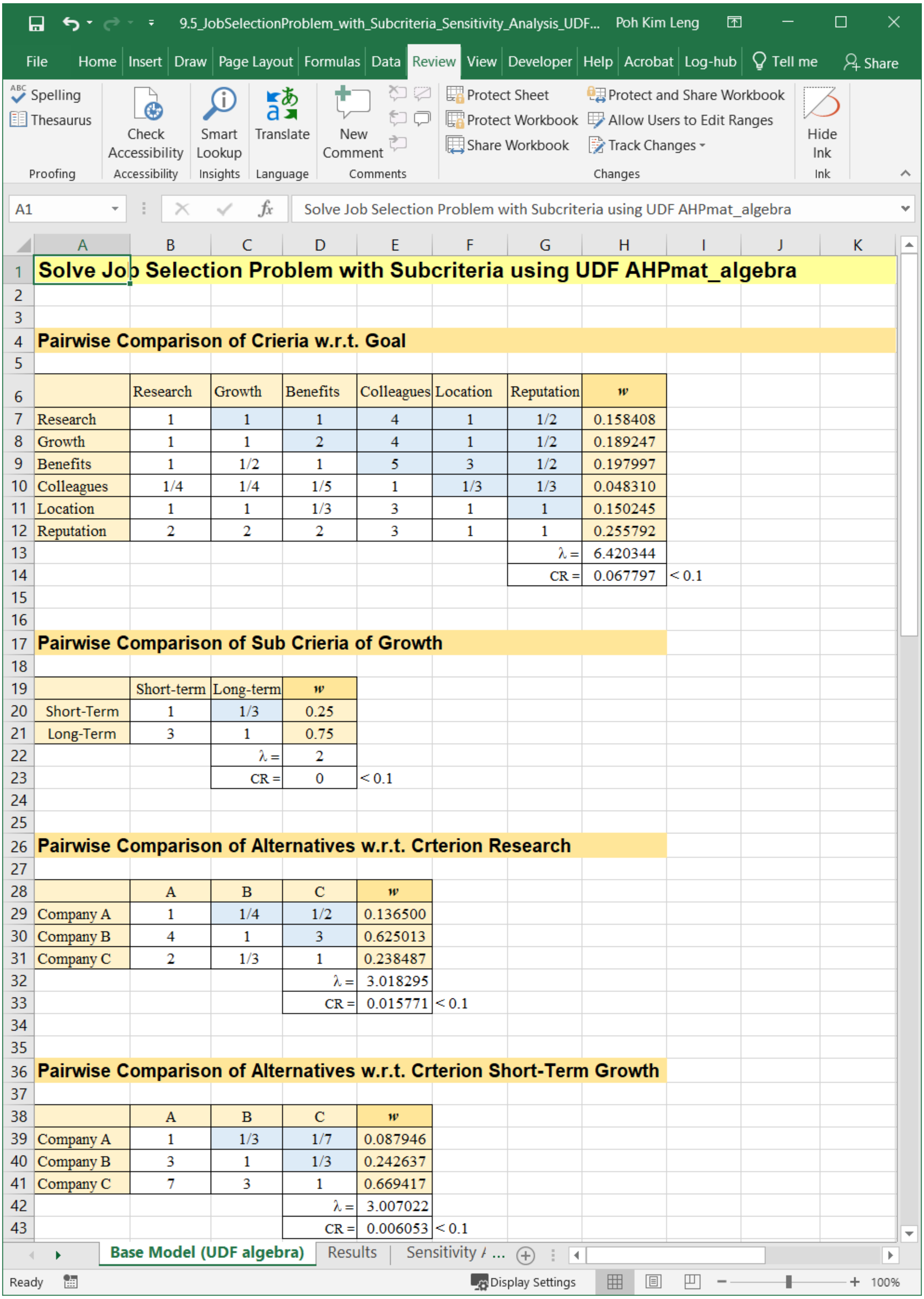


Results





9.5.4 Using Excel with User-Defined Functions



Solve Job Selection Problem with Subcriteria using UDF AHPmat_algebra

Pairwise Comparison of Crieria w.r.t. Goal

	Research	Growth	Benefits	Colleagues	Location	Reputation	*w*	
Research	1	1	1	4	1	1/2	0.158408	
Growth	1	1	2	4	1	1/2	0.189247	
Benefits	1	1/2	1	5	3	1/2	0.197997	
Colleagues	1/4	1/4	1/5	1	1/3	1/3	0.048310	
Location	1	1	1/3	3	1	1	0.150245	
Reputation	2	2	2	3	1	1	0.255792	
						λ =	6.420344	
						CR =	0.067797	< 0.1

Pairwise Comparison of Sub Crieria of Growth

	Short-term	Long-term	*w*	
Short-Term	1	1/3	0.25	
Long-Term	3	1	0.75	
		λ =	2	
		CR =	0	< 0.1

Pairwise Comparison of Alternatives w.r.t. Crterion Research

	A	B	C	*w*	
Company A	1	1/4	1/2	0.136500	
Company B	4	1	3	0.625013	
Company C	2	1/3	1	0.238487	
			λ =	3.018295	
			CR =	0.015771	< 0.1

Pairwise Comparison of Alternatives w.r.t. Crterion Short-Term Growth

	A	B	C	*w*	
Company A	1	1/3	1/7	0.087946	
Company B	3	1	1/3	0.242637	
Company C	7	3	1	0.669417	
			λ =	3.007022	
			CR =	0.006053	< 0.1

Solve Job Selection Problem with Subcriteria

Summary of Results

	Alternative	Global Weight	
1	Company A	0.45222	< - Best Alternative
2	Company B	0.29553	-
3	Company C	0.25225	-

	Main Criteria	Main Criteria Weight	Sub-Criteria	Sub-Criteria Local Weight	Leaf Criteria Global Weight	Alternative	Alternative Local Weights
1	Research	0.158408			0.158408	Company A	0.136500
						Company B	0.625013
						Company C	0.238487
2	Growth	0.189247					
			Short-term	0.250000	0.047312	Company A	0.087946
						Company B	0.242637
						Company C	0.669417
			Long-term	0.750000	0.141935	Company A	0.648329
						Company B	0.229651
						Company C	0.122020
3	Benefits	0.197997			0.197997	Company A	0.242637
						Company B	0.087946
						Company C	0.669417
4	Colleagues	0.048310			0.048310	Company A	0.278955
						Company B	0.649118
						Company C	0.071927
5	Location	0.150245			0.150245	Company A	0.466667
						Company B	0.466667
						Company C	0.066667
6	Reputation	0.255792			0.255792	Company A	0.792757
						Company B	0.131221
						Company C	0.076021

Base Model (UDF algebra) | Results | Sensitivity Ana ...

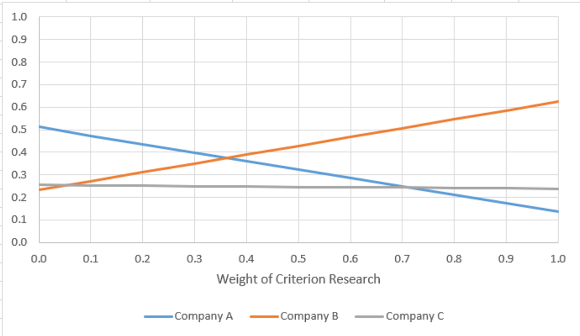
9.5_JobSelectionProblem_with_Subcriteria_Sensitivity_Analysis_UDF_algebra.xlsx - Excel

A1 | Solve Job Selection Problem with Subcriteria

Rainbow Diagram for Criterion Research

Leaf weights	0.158408	0.047312	0.141935	0.197997	0.048310	0.150245	0.255792	1.00

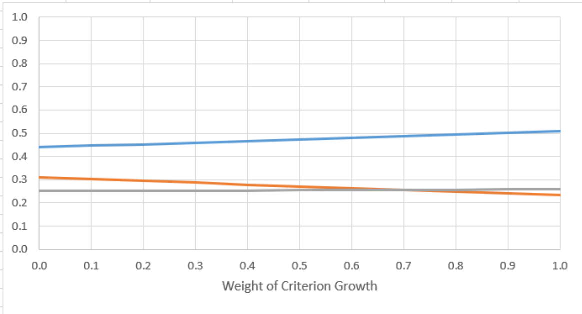
	A	B	C
	0.4522	0.2955	0.2522
0.0	0.5116	0.2335	0.2548
0.1	0.4741	0.2727	0.2532
0.2	0.4366	0.3118	0.2516
0.3	0.3991	0.3510	0.2499
0.4	0.3616	0.3901	0.2483
0.5	0.3241	0.4293	0.2467
0.6	0.2866	0.4684	0.2450
0.7	0.2490	0.5076	0.2434
0.8	0.2115	0.5467	0.2418
0.9	0.1740	0.5859	0.2401
1.0	0.1365	0.6250	0.2385



Rainbow Diagram for Criterion Growth

Leaf weights	0.158408	0.047312	0.141935	0.197997	0.048310	0.150245	0.255792	1.00

	A	B	C
	0.4522	0.2955	0.2522
0.0	0.4391	0.3102	0.2507
0.1	0.4461	0.3024	0.2515
0.2	0.4530	0.2947	0.2523
0.3	0.4599	0.2870	0.2532
0.4	0.4668	0.2793	0.2540
0.5	0.4737	0.2715	0.2548
0.6	0.4806	0.2638	0.2556
0.7	0.4875	0.2561	0.2564
0.8	0.4944	0.2483	0.2572
0.9	0.5013	0.2406	0.2581
1.0	0.5082	0.2329	0.2589



Results | Sensitivity Analysis

9.6 The Ratings Method in AHP

9.6.1 Evaluating Alternatives with Ratings

- Suppose you would like to evaluate or rank a very large number of alternatives or subjects, e.g., evaluating employees in an organization for raises.

- If the standard AHP is used, a very large number of pairwise comparisons would be required and it would not be very practical to do so.

- The rating approach in AHP allows for the evaluation of a large number of alternatives by first setting up an evaluation or rating system.

- In the rating approach, a hierarchy is developed in the usual way down to the level of criteria or sub-criteria.

- The criteria and sub-criteria are prioritized in the usual way and their weights are expressed in **Normalized** or **Distributive Form**.

- Each of the leaf criteria or sub-criteria is then given set of **Intensity** or **Performance Ratings** with respect to that criterion.

  Examples: *Excellent, Good, Average, Poor, Very poor.*
  Very high, High, Average, Low, *Very low*.

- The type and number of intensity/performance ratings for each criterion may be different.

- The performance ratings for each criterion are then prioritized by pairwise comparisons to determine their relative importance with respect to the criterion they are measuring. The weights of these performance ratings are expressed in **Idealized Form**.

- The alternatives or candidates are independently evaluated one at a time in terms of rating intensities for each of the criteria.

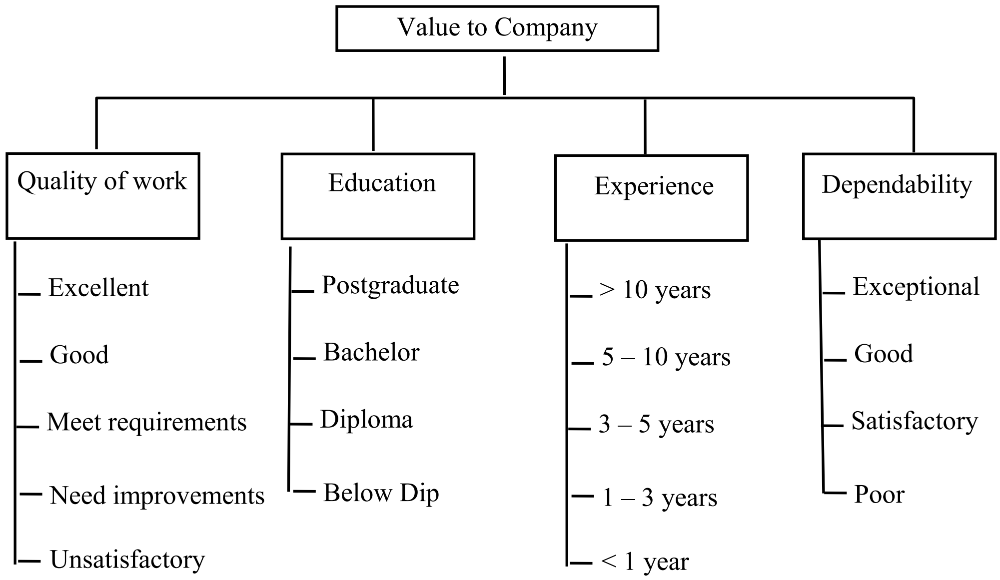
- The global weight for each alternative is then evaluated in the usual additive weighted sum manner.

9.6.2 Case Study: Evaluation of Employees

- A firm would like to evaluate the value of its employees to the company.

- The company consider these four criteria contributing to the goal:
  1. Quality of Work
  2. Education
  3. Experience
  4. Dependability

- Each of the 4 criteria is measured using performance ratings as shown below:



- The four criteria are pairwise compared with respect to the Goal.

	Quality of work	Education	Experience	Dependability	Local Weight
Quality of work	1	5	7	3	0.565009
Education	1/5	1	3	1/3	0.117504
Experience	1/7	3	1	1/5	0.055285
Dependability	1/3	3	5	1	0.262201

$\lambda = 4.117$ CI = 0.03899 CR = 0.04333 < 0.1 $\boldsymbol{w} = [0.565009, 0.117504, 0.055285, 0.262201]$

- The weights for the four criteria are: 0.565009, 0.117504, 0.055285, and 0.262201, respectively.

- The managers then pairwise compare the performance ratings or intensities according to the priority with respect to their parent criterion or sub-criterion.

- For examples, the managers might ask:

  (a) With respect to "Education", how important to the company is an employee with a post-graduate degree is with respect to an employee with only a bachelor degree.

  The assessments would be different if education is applied to waiters in a restaurant than to researchers in a technology driven company.

(***b***) With respect to "Dependability", how important to the company is an employee who is rated "exceptional" compared with an employee who is rated "average"?

The assessments would be different if dependability is applied to waiters in a restaurant than to captains of a passenger airplane.

- For the criterion "Quality of Work", the following matrix was obtained:

Criterion: "Quality of Work"	Excellent	Good	Meet Requirements	Need Improvements	Unsatisfactory	Local weight
Excellent	1	1	5	7	9	0.428747
Good	1	1	3	5	7	0.337852
Meet Requirements	1/5	1/3	1	3	5	0.136310
Need Improvements	1/7	1/5	1/3	1	2	0.060049
Unsatisfactory	1/9	1/7	1/5	1/2	1	0.037042

$\lambda = 5.1356$, CI = 0.03390, CR = 0.03027 < 0.1 $\quad$ $\boldsymbol{w}$ = [0.428747, 0.337852, 0.136310, 0.060049. 0.037042]

- The weights of the ratings **idealized form**:

Rating under Quality of Work	Weight in Ideal Form
Excellent	1
Good	0.787997
Meet Requirements	0.317927
Need Improvements	0.140058
Unsatisfactory	0.086395

- For the criterion "Education", the following matrix was obtained:

Criterion: "Education"	Postgraduate	Bachelor	Diploma	Below Dip	Local weight
Postgraduate	1	3	5	9	0.573455
Bachelor	1/3	1	3	7	0.271227
Diploma	1/5	1/3	1	3	0.110233
Below Dip	1/9	1/7	1/3	1	0.045086

$\lambda = 4.0876$, CI = 0.02921, CR = 0.0325 < 0.1

- The weights of the ratings **idealized form**:

Rating under Education	Weight in Ideal Form
Postgraduate	1
Bachelor	0.472971
Diploma	0.192225
Below Dip	0.078621

- For the criterion “Experience”, the following matrix was obtained:

Criterion: “Experience”	> 10 years	5 - 10 years	3 - 5 years	1 - 3 years	< 1 year	Local weight
> 10 years	1	1	3	5	7	0.393131
5 - 10 years	1	1	2	3	5	0.304538
3 - 5 years	1/3	1/2	1	2	3	0.153971
1 - 3 years	1/5	1/3	1/2	1	3	0.099081
< 1 year	1/7	1/5	1/3	1/3	1	0.049280

$\lambda = 5.0872$, CI = 0.02181, CR = 0.01947 < 0.1

- The weights of the ratings **idealized form**:

Rating under Experience	Weight in Ideal Form
> 10 years	1
5 - 10 years	0.774648
3 - 5 years	0.391653
1 - 3 years	0.252030
< 1 year	0.125353

- For the criterion “dependability”, the following matrix was obtained:

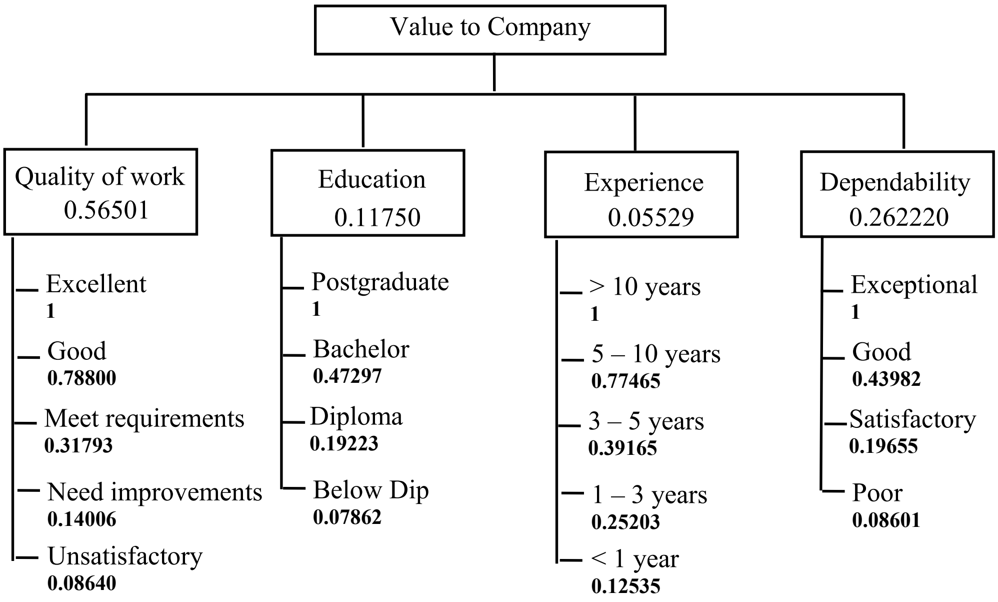
Criterion: “Dependability”	Exceptional	Good	Satisfactory	Poor	Normalized weight
Exceptional	1	3	5	9	0.58059
Good	1/3	1	3	5	0.25536
Satisfactory	1/5	1/3	1	3	0.11411
Poor	1/9	1/5	1/3	1	0.04994

$\lambda = 4.0763$, CI = 0.02543, CR = 0.02836 < 0.1

- The weights of the ratings **idealized form**:

Rating under Dependability	Weight in Ideal Form
Exceptional	1
Good	0.43983
Satisfactory	0.19655
Poor	0.08601

- The hierarchy with criteria weights and ratings' weights:



- Finally, the managers rate each individual employee by assigning the intensity rating that applies to him or her under each criterion.

	Candidate	0.56501	0.11750	0.05529	0.26220
		Rating for Quality	Rating for Education	Rating for Experience	Rating for Dependability
1	John Lim	Good	Postgraduate	3 - 5 years	Satisfactory
2	Tan Ah Huay	Meet Requirements	Diploma	> 10 years	Good
3	Chow Ah Beng	Need Improvements	Below Dip	1 -3 years	Poor
4	Mary Lau	Good	Bachelor	< 1 year	Exceptional
5	Harry Lee	Excellent	Diploma	5 – 10 years	Good
6					
7					

- The Overall Rating of each employee with respect to the Goal is the criteria-weighted sum of their individual ratings.

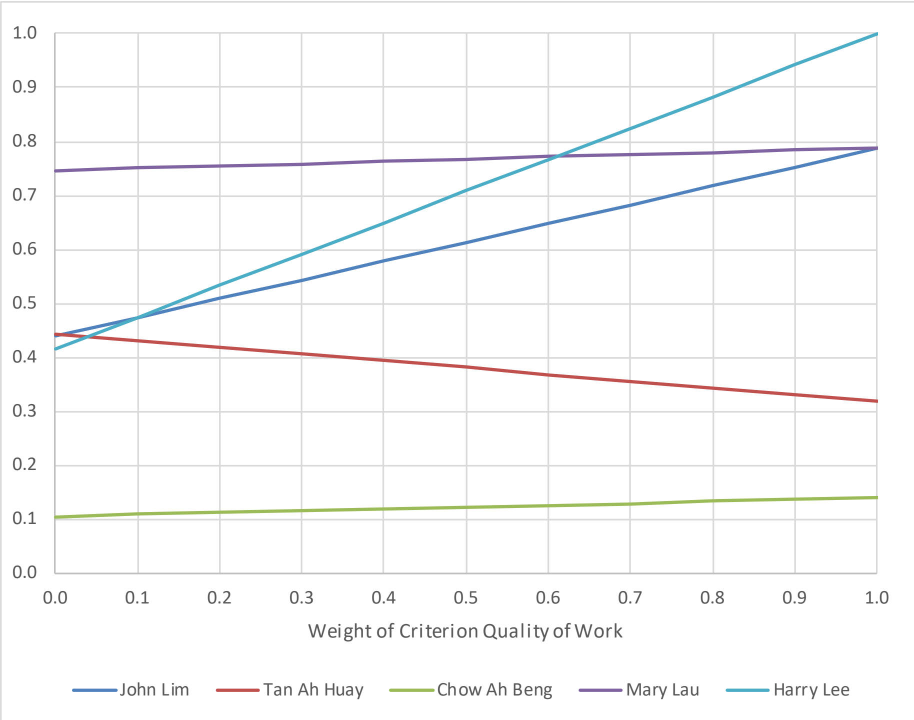
	Candidate	0.56501	0.11750	0.05529	0.26220	1.0000
		Rating for Quality	Rating for Education	Rating for Experience	Rating for Dependability	**Overall Rating**
1	John Lim	0.78800	1.00000	0.39165	0.19655	0.63592
2	Tan Ah Huay	0.31793	0.19223	1.00000	0.43982	0.37283
3	Chow Ah Beng	0.14006	0.07862	0.25203	0.08601	0.12486
4	Mary Lau	0.78800	0.47297	0.12535	1.00000	0.76993
5	Harry Lee	1.00000	0.19223	0.77465	0.43982	0.74575
6						
7						

9.6.3 Sensitivity Analysis

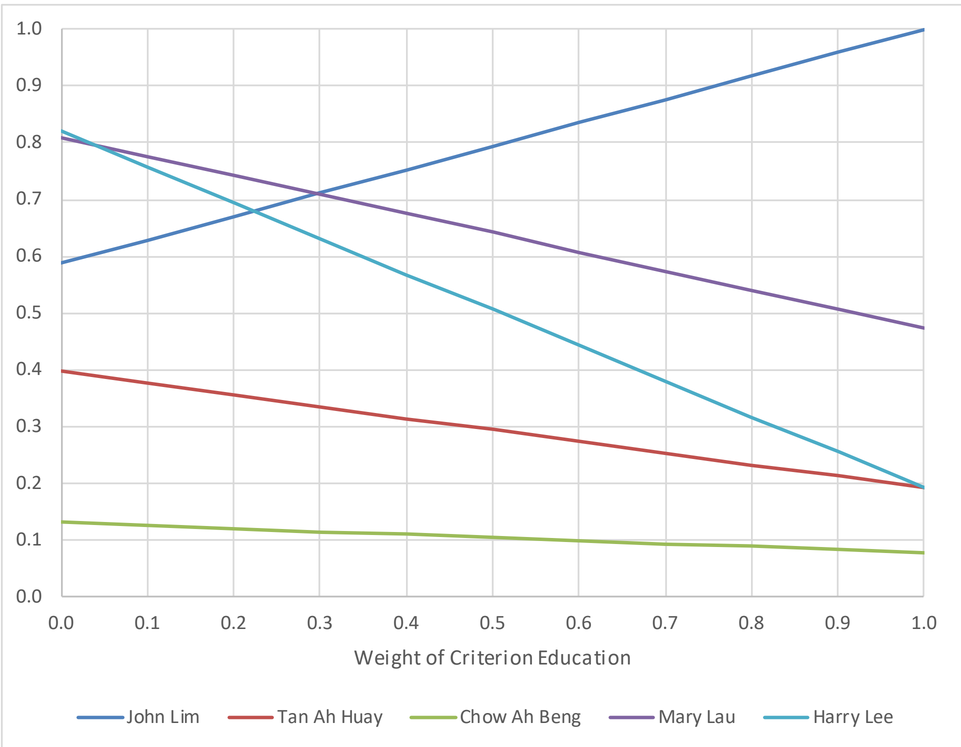
- Similar to the standard AHP, sensitivity analysis can be performed here by varying the weight of each criterion, one-at-a-time, from 0 to 1 while keeping the weights of the other criteria in the same relative proportion as in the base case.

- Rainbow diagrams are plotted for the overall ratings of the alternatives or candidates.

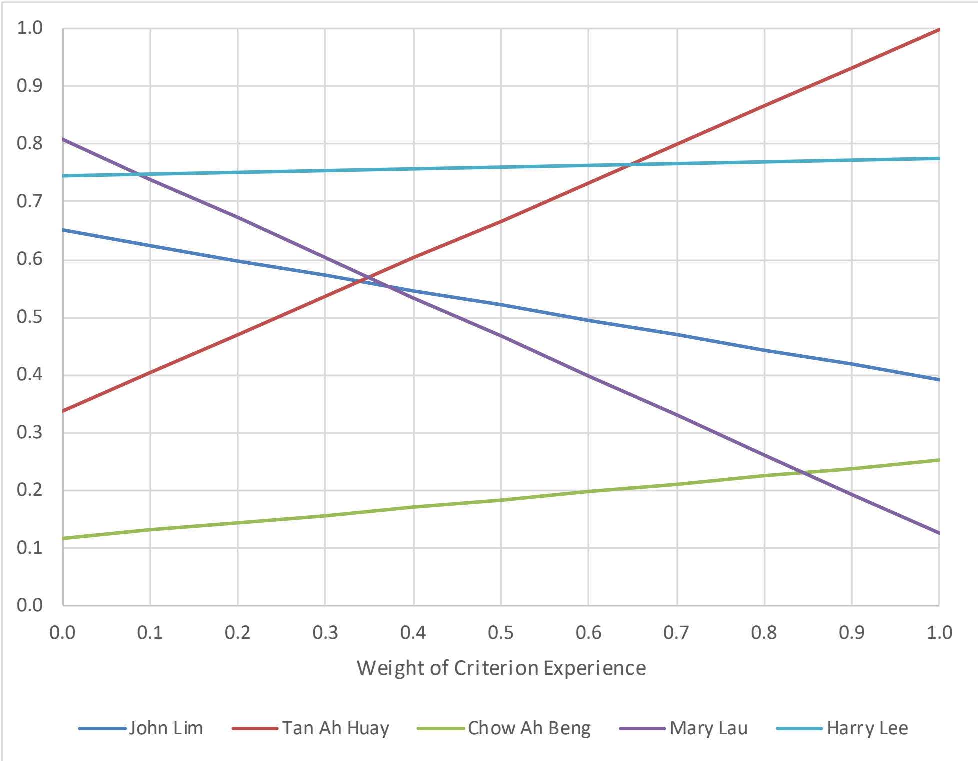
Rainbow when the priority weight of criterion Quality is varied from 0 to 1.



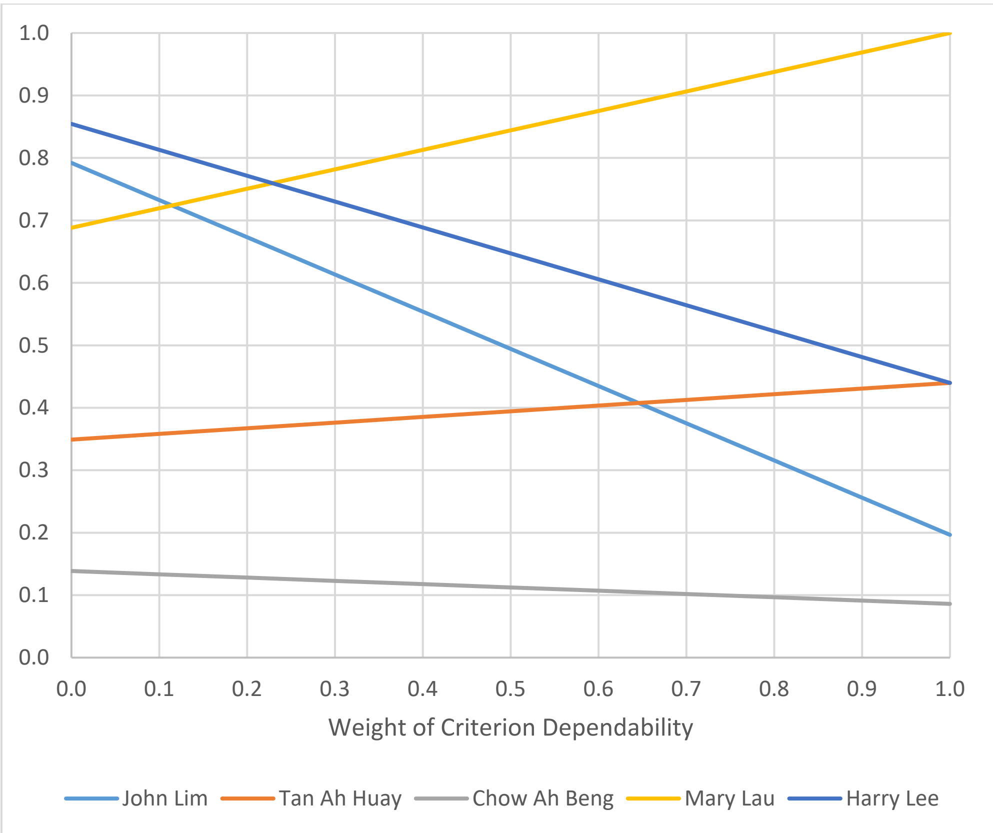
Rainbow when the priority weight of criterion Education is varied from 0 to 1



Rainbow when the priority weight of criterion Experience is varied from 0 to 1



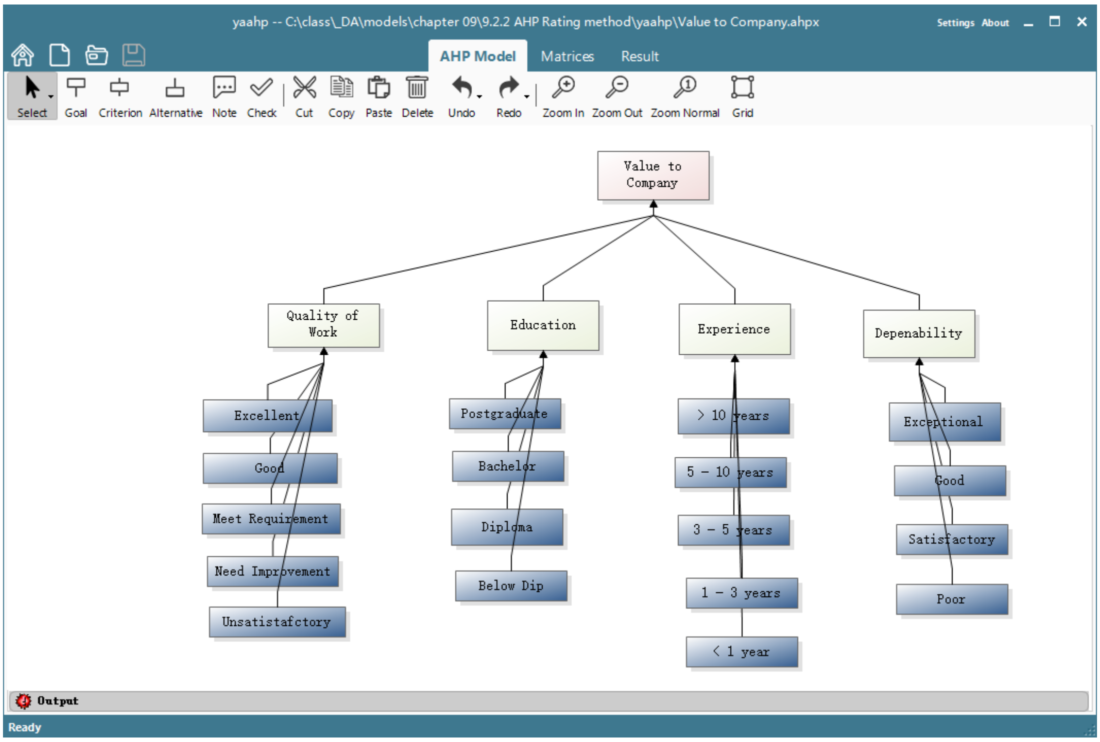
Rainbow when the priority weight of criterion Dependability is varied from 0 to 1



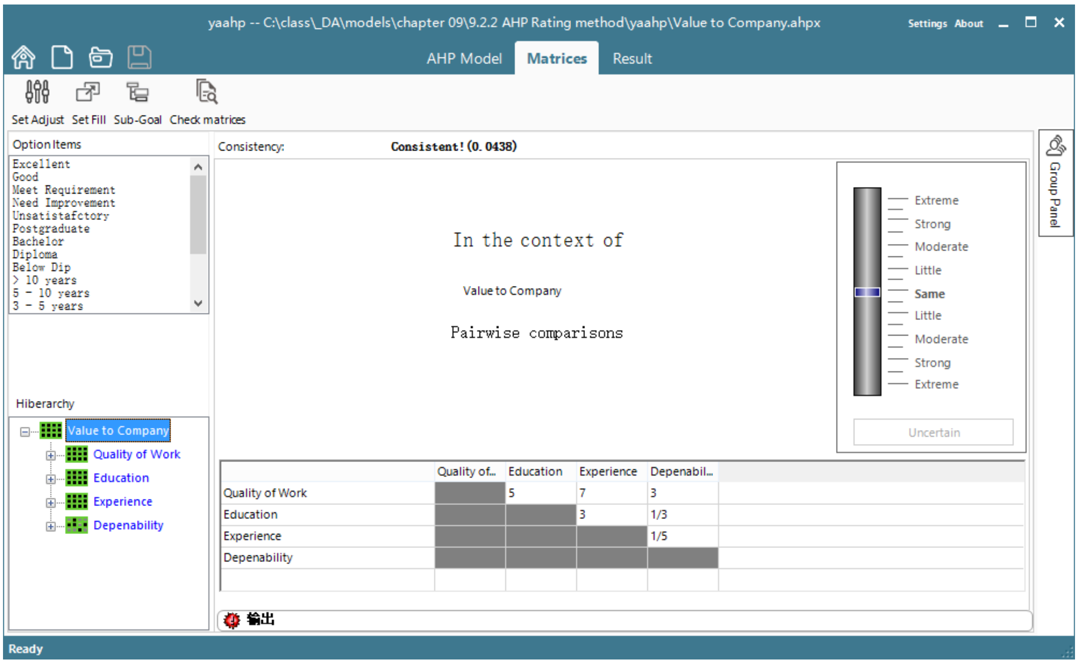
9.6.4 Using YAAHP

- YAAHP does not directly support the Rating Method in AHP. However, it is possible to use it in the creation of the evaluation model, determination of priority weights, and then transfer results to Excel for the evaluation step.

1. Create a Tree Hierarchy with the Rating Intensities as independent "alternatives" under each criterion.

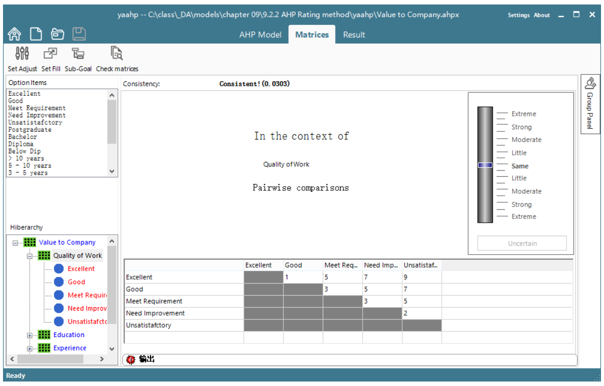


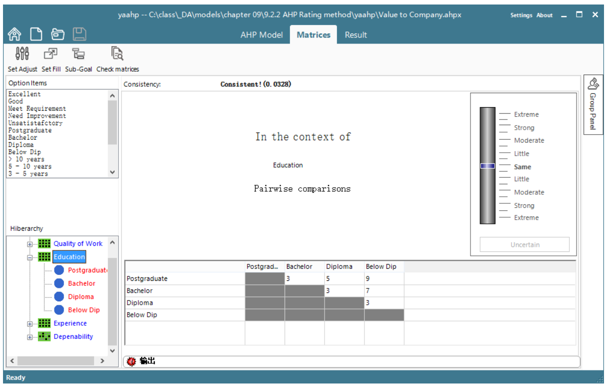
2. Pairwise compare the criteria and obtain their priority weights.

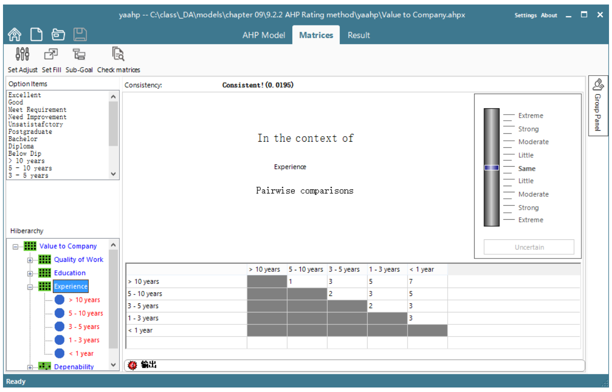


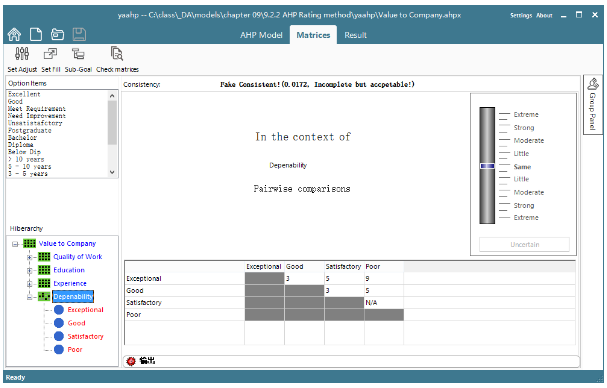
	Quality of...	Education	Experience	Depenabil...
Quality of Work		5	7	3
Education			3	1/3
Experience				1/5
Depenability				

3. Pairwise compare the “Rating Intensities” under each criterion.

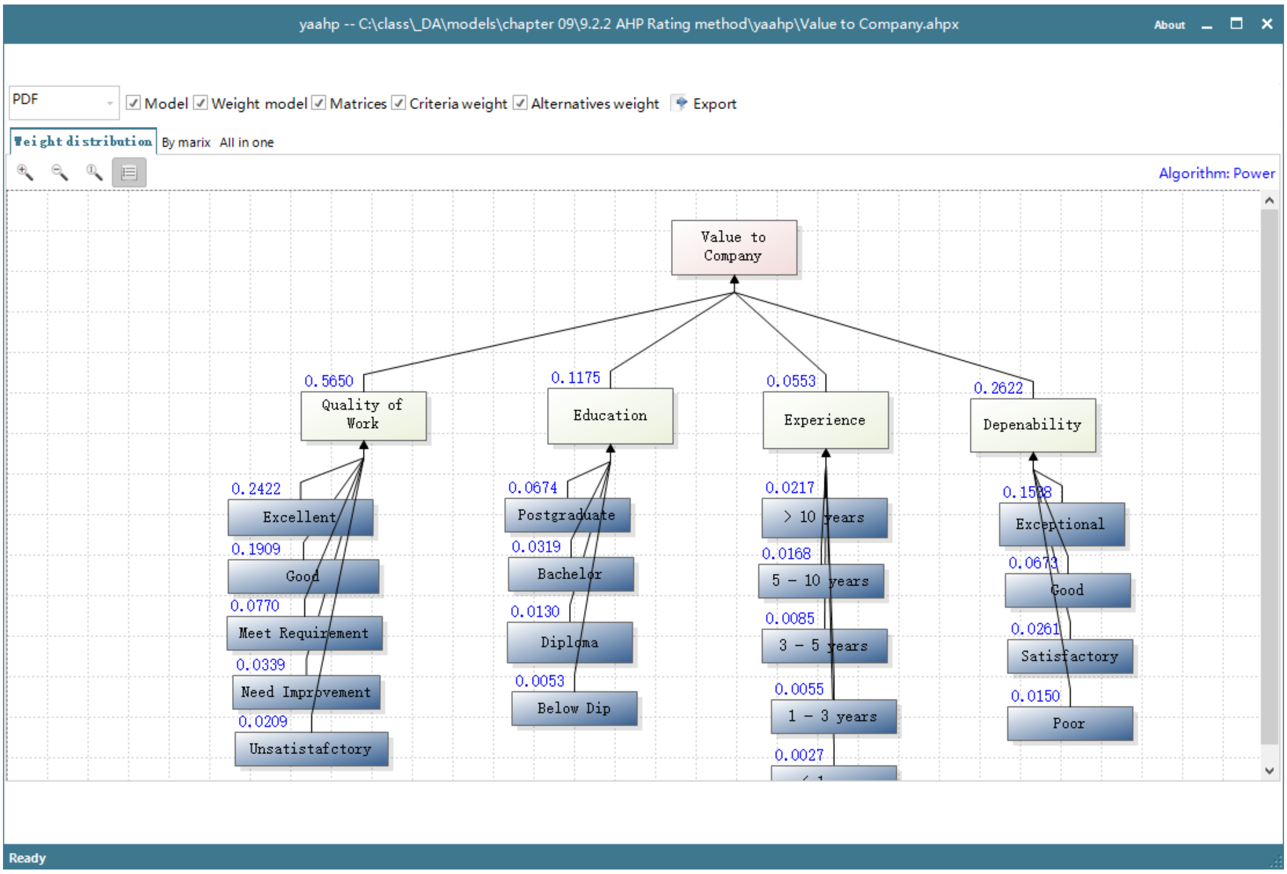




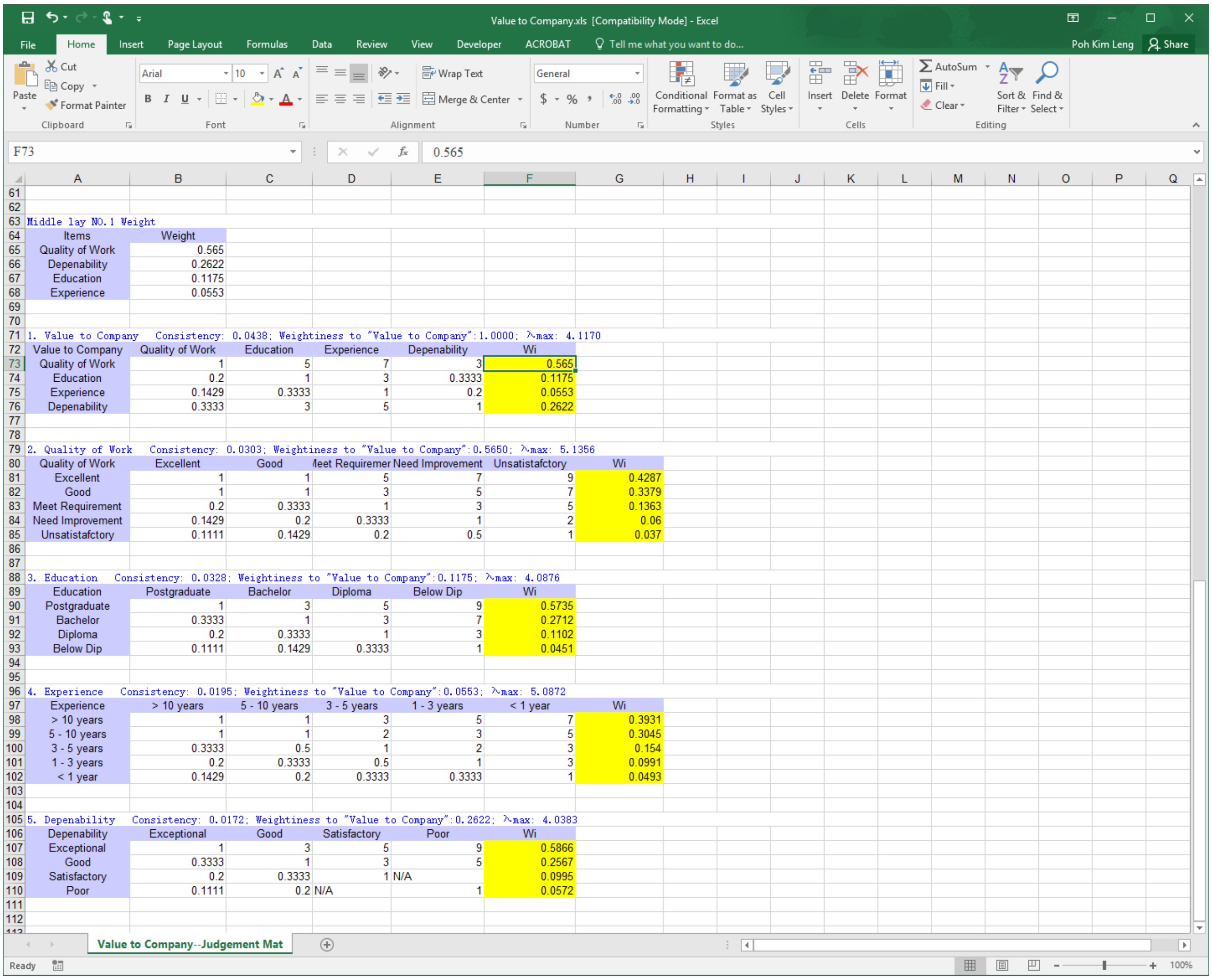




4. Evaluate the Model.



5. Export Results to Excel


Middle lay NO.1 Weight

Items	Weight
Quality of Work	0.565
Depenability	0.2622
Education	0.1175
Experience	0.0553

1. Value to Company Consistency: 0.0438; Weightiness to "Value to Company":1.0000; λmax: 4.1170

Value to Company	Quality of Work	Education	Experience	Depenability	Wi
Quality of Work	1	5	7	3	0.565
Education	0.2	1	3	0.3333	0.1175
Experience	0.1429	0.3333	1	0.2	0.0553
Depenability	0.3333	3	5	1	0.2622

2. Quality of Work Consistency: 0.0303; Weightiness to "Value to Company":0.5650; λmax: 5.1356

Quality of Work	Excellent	Good	Meet Requiremer	Need Improvement	Unsatistafctory	Wi
Excellent	1	1	5	7	9	0.4287
Good	1	1	3	5	7	0.3379
Meet Requirement	0.2	0.3333	1	3	5	0.1363
Need Improvement	0.1429	0.2	0.3333	1	2	0.06
Unsatistafctory	0.1111	0.1429	0.2	0.5	1	0.037

3. Education Consistency: 0.0328; Weightiness to "Value to Company":0.1175; λmax: 4.0876

Education	Postgraduate	Bachelor	Diploma	Below Dip	Wi
Postgraduate	1	3	5	9	0.5735
Bachelor	0.3333	1	3	7	0.2712
Diploma	0.2	0.3333	1	3	0.1102
Below Dip	0.1111	0.1429	0.3333	1	0.0451

4. Experience Consistency: 0.0195; Weightiness to "Value to Company":0.0553; λmax: 5.0872

Experience	> 10 years	5 - 10 years	3 - 5 years	1 - 3 years	< 1 year	Wi
> 10 years	1	1	3	5	7	0.3931
5 - 10 years	1	1	2	3	5	0.3045
3 - 5 years	0.3333	0.5	1	2	3	0.154
1 - 3 years	0.2	0.3333	0.5	1	3	0.0991
< 1 year	0.1429	0.2	0.3333	0.3333	1	0.0493

5. Depenability Consistency: 0.0172; Weightiness to "Value to Company":0.2622; λmax: 4.0383

Depenability	Exceptional	Good	Satisfactory	Poor	Wi
Exceptional	1	3	5	9	0.5866
Good	0.3333	1	3	5	0.2567
Satisfactory	0.2	0.3333	1	N/A	0.0995
Poor	0.1111	0.2	N/A	1	0.0572

6. Convert the Rating Weights in distributive form (highlighted in yellow) to Ideal form.

7. Complete the rest of the steps for the rating method in Excel.

9.7 Advanced Topic: Rank Preservation and Reversal in AHP

9.7.1 The Rank Reversal Problem

- *Belton & Gear* (1983) found that AHP does not satisfy "independence of irrelevant alternatives".

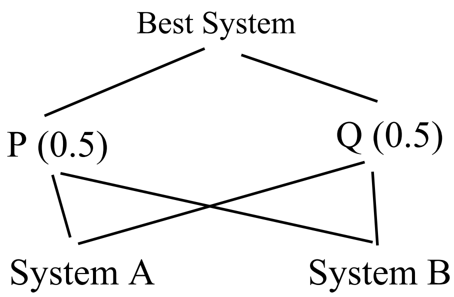
- It can be shown that using AHP, adding a new alternative to a set of original alternatives may change the ranking order of the original set of alternatives even when the additional alternative is a dominated or irreverent one.

- This has since been known as the "**Rank Reversal Problem**" of AHP and is one of the most serious challenges to the AHP.

Example

- Consider a simple systems evaluation problem with two criteria P and Q (equal weight) and two candidates Systems A and B.

- Hierarchy:



- Pairwise comparison matrices:

P	A	B	Weight
A	1	5	0.8333
B	1/5	1	0.1667

Q	A	B	Weight
A	1	1/3	0.2500
B	3	1	0.7500

- Global Weights:

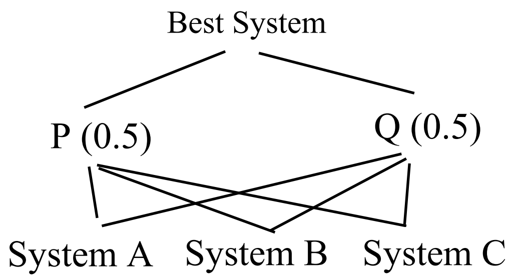
  1. System A = 0.5 (0.8333) + 0.5 (0.2500) = **<u>0.5417</u>**

  2. System B = 0.5 (0.1667) + 0.5 (0.7500) = 0.4583

- Hence System A is preferred to System B.

- Suppose that another vendor came along, and we have to add a third system C into the evaluation process:

- Hierarchy:



- Pairwise comparison matrices:

P	A	B	C	Weight
A	1	5	1	0.4545
B	1/5	1	1/5	0.0909
C	1	5	1	0.4545

Q	A	B	C	Weight
A	1	1/3	2	0.2222
B	3	1	6	0.6667
C	1/2	1/6	1	0.1111

- Notes:
  1. The sub-matrixes for Systems A and B remain the same. Hence, the relative performance between Systems A and B under the two criteria have not changed.

  2. System C is dominated by System A. System C is equally as good as A under criterion P but is worse off than A under criterion Q.

- Global Weights:
  1. System A = 0.5 (0.4545) + 0.5 (0.2222) = 0.3384
  2. System B = 0.5 (0.0909) + 0.5 (0.6667) = **0.3788**
  3. System C = 0.5 (0.4545) + 0.5 (0.1111) = 0.2828

- System B is now preferred to System A, i.e., the introduction of System C has caused Systems A and B to globally rank reverse even though we did not revise any of their relative weights under the two criteria.

Example: US Presidential Elections 1991

- Initially, Bush was leading Clinton in popularity polls. However, the entry of Ross Perot into the election at a later stage took votes away from Bush. Clinton eventually won the election.

Rank Reversal may be exploited

Example

- In the systems selection example, System C could have been deliberately introduced by the vendor for System B so as to win the contract.

9.7.2 The Distributive and Ideal Modes of AHP

- According to Saaty, the ability of AHP to adjust rank is a desirable feature that should be allowed in some real world situations.

- See Saaty (1994, 2000) for some of the justifications.

Why Does Rank-Reversal Occur?

- Rank reversal is not unique to AHP as it is not because of the eigenvector computations, because of the 9-point scale, nor because of inconsistencies in judgments.

- Rank reversal can take place with any technique that decomposes and synthesizes in a relative fashion, regardless of whether it uses pairwise comparisons, eigenvector calculations, or demands perfect consistency.

- Rank reversal occurs because of an abundance or dilution effect (or what has also been called a substitution effect). Value or worth is, more often than not, affected by relative abundance or scarcity.

When Should Rank Reversal be Allowed?

1. If the system is ***closed*** where a fixed amount of resources to be distributed, then rank reversal should be allowed.

2. If the system is ***open*** where resources can be added or removed then rank reversal should not happen.

Distributive Mode of AHP

- In the ***Distributive Mode****,* the criteria weights and the alternatives' weights under each criterion are normalized so that they sum to one.
- This is the original or standard AHP. This is the default method unless otherwise stated.

Ideal Mode of AHP

- In the ***Ideal Mode*** the alternatives' weights under each criterion are expressed in **ideal form**, i.e., divided by the maximum weight in that column.
- *The criteria weights remain normalized.*
- The Ideal Mode can ***prevent*** rank-reversal of original alternatives if a dominated alternative is added. However, if a non-dominated alternative is added, rank reversal may still occur under ideal mode. See for example, Exercise P9.4.
- The rating approach is a special case of the Ideal Mode. Rank reversal will not occur.

Software Support

- **YAAHP** supports only the distributive mode.

Example (Using Ideal Mode)

- In the previous example, rank reversal occurred under the standard or Distributive Mode.
- We will show that rank reversal does not occur under the Ideal Mode for this particular problem.

Comparison of System A and System B

- Pairwise comparison matrices:

P	A	B	Normalized	Ideal
A	1	5	0.8333	1.0000
B	1/5	1	0.1667	0.2000

Q	A	B	Normalized	Ideal
A	1	1/3	0.2500	0.3333
B	3	1	0.7500	1.0000

- Computing Global Weights under Ideal Mode:
  1. System A = 0.5 (1.0000) + 0.5 (0.3333) = 0.6667
  2. System B = 0.5 (0.2000) + 0.5 (1.0000) = 0.6000

- Note that the criteria weights are normalized (0.5, 0.5).

- Normalized Global Weights under Ideal Mode:
  1. System A = **0.5263**
  2. System B = 0.4737

- System A is preferred to System B.

With System C Included

- Pairwise comparison matrices:

P	A	B	C	Normalized	Ideal
A	1	5	1	0.4545	1.0000
B	1/5	1	1/5	0.0909	0.2000
C	1	5	1	0.4545	1.0000

Q	A	B	C	Normalized	Ideal
A	1	1/3	2	0.2222	0.3333
B	3	1	6	0.6667	1.0000
C	1/2	1/6	1	0.1111	0.1667

- Computing Global Weights under Ideal Mode:
  1. System A = 0.5 (1.0000) + 0.5 (0.3333) = 0.6667
  2. System B = 0.5 (0.2000) + 0.5 (1.0000) = 0.6000
  3. System C = 0.5 (1.0000) + 0.5 (0.1667) = 0.5833

- Note that the criteria weights remain normalized (0.5, 0.5).

- Normalized Global Weights under Ideal Mode:
  1. System A = **0.3604**
  2. System B = 0.3243
  3. System C = 0.3153

- System A is still preferred to System B. Rank Reversal between A and B did not occur.

References

1. T.L. Saaty, *The Analytic Hierarchy Process: Planning, Priority Setting, Resource Allocation*, McGraw Hill, 1980 (2nd Edition by RWS Publication, 1990).

2. E. H. Forman and Mary Ann Selly, *Decision by Objectives: How to convince others that you are right*. World Scientific, 2001. Available for download from Expert Choice website.

3. V. Belton and T. Gear. On a Shortcoming of Saaty's method of Analytic Hierarchies. *Omega* **11**(3):228-230, 1983.

4. T.L. Saaty, How to make a decision: The analytic hierarchy process. *Interfaces* **24**(6):19-43, 1994. (This paper introduced the Ideal Mode and provided guidelines on choosing between distributive and ideal modes).

5. I. Millet and T.L. Saaty. On the relativity of relative measures: Accommodating both rank preservation and rank reversals in the AHP. *European Journal of Operational Research* **121**(1): 205-212, 2000. (This paper updated on the guideline for choosing between distributive and ideal modes).

6. T.L. Saaty, *Decision Making for Leaders*: *The Analytical Hierarchy Process for Decisions in a Complex World*, Wadsworth, 1982. (2nd edition 1995 by RWS Publications).

7. T.L. Saaty and L.G. Vargas, *The Logic of Priorities: Applications in Business, Energy, Health, and Transportation*, Kluwer-Nijhoff, 1982.

8. T.L. Saaty and K.P. Kearns, *Analytical Planning: The Organization of Systems*, Pergamon Press, 1985.

9. T.L. Saaty and L.G. Vargas, *Prediction, Projection and Forecasting: Applications of the Analytical Hierarchy Process in Economics, Finance, Politics, Games and Sports*, Kluwer Academic Publishers, 1990.

10. T.L. Saaty, *Fundamentals of Decision Making and Priority Theory with the Analytical Hierarchy Process*, RWS Publications, Pittsburgh, 1994.

Exercises

P9.1 Your department wants to purchase a new personal computer. Three objectives are important in determining which computer should be purchased: Cost, user-friendliness, and software availability. The pairwise comparison matrix for these objectives is as follows:

	Cost	User-friendliness	Software availability
Cost	1	1/4	1/5
User-friendliness		1	1/2
Software availability			1

Three computers are being considered for purchase. The performance of each computer with regard to each objective is indicated by the following pairwise comparison matrices.

For Cost:

	Computer 1	Computer 2	Computer 3
Computer 1	1	3	5
Computer 2		1	2
Computer 3			1

For user-friendliness:

	Computer 1	Computer 2	Computer 3
Computer 1	1	1/3	1/2
Computer 2		1	5
Computer 3			1

For software availability:

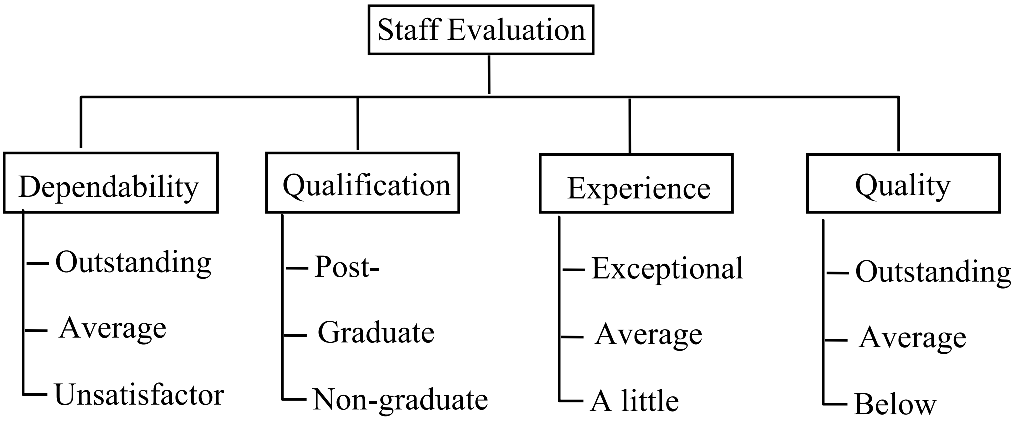
	Computer 1	Computer 2	Computer 3
Computer 1	1	1/3	1/7
Computer 2		1	1/5
Computer 3			1

(***a***) If the AHP method is followed, which computer should be purchased?

(***b***) Check the pairwise comparison matrices for consistency.

(***c***) Does all preference relations in the pairwise matrices satisfy the transitivity property?

P9.2 A company is evaluating its employees for raises and has decided on the four main criteria: Dependability, Qualification, Experience, and Quality (of work). Employees will be evaluated under each criterion using the ratings or standards as shown in the hierarchy below:



The prioritization pairwise comparison matrix for the main criteria is as follows:

	Dependability	Qualification	Experience	Quality
Dependability	1	2	3	4
Qualification		1	2	3
Experience			1	2
Quality				1

For each of the criteria, the pairwise comparison matrix for its intensities or ratings is as follows:

Dependability:

	Outstanding	Average	Unsatisfactory
Outstanding	1	3	7
Average		1	3
Unsatisfactory			1

Qualification:

	Postgraduate	Graduate	Non-graduate
Postgraduate	1	3	5
Graduate		1	3
Non-graduate			1

Experience:

	Exceptional	Average	Little
Exceptional	1	5	9
Average		1	3
Little			1

Quality:

	Outstanding	Average	Below average
Outstanding	1	5	9
Average		1	3
Below average			1

(*a*) Set up a staff evaluation system for the company using the Rating Method of AHP.

(*b*) John and Bill have been assessed by their supervisors as follows:

Employee	**Assessment for Criterion**			
	Dependability	**Qualification**	**Experience**	**Quality**
John Chen	Average	Graduate	Average	Outstanding
Bill Zhang	Outstanding	Non-graduate	Exceptional	Average

Should John be given a higher pay rise than Bill? Why?

P9.3 In determining where to invest some money, two criteria – expected rate of return and degree of risk – are being considered equally important. Two investments (1 and 2) have the following pairwise comparison matrices:

Expected Return:

	Investment 1	Investment 2
Investment 1	1	1/2
Investment 2		1

Degree of Risk:

	Investment 1	Investment 2
Investment 1	1	3
Investment 2		1

(*a*) How should the two investments be ranked?

(*b*) Now suppose another investment (investment 3) is available. Suppose the pairwise comparison matrices for these investments are as follows:

Expected Return:

	Investment 1	Investment 2	Investment 3
Investment 1	1	1/2	4
Investment 2		1	8
Investment 3			1

Degree of Risk:

	Investment 1	Investment 2	Investment 3
Investment 1	1	3	1/2
Investment 2		1	1/6
Investment 3			1

How should the three investments be ranked now?

(*c*) Comment on the pairwise comparison matrices and the rankings found in Part (*a*) and Part (*b*).

(*d*) Try the Ideal Mode of AHP and observe if rank reversal can be prevented for this problem.