# Chapter 3 Decision Theory

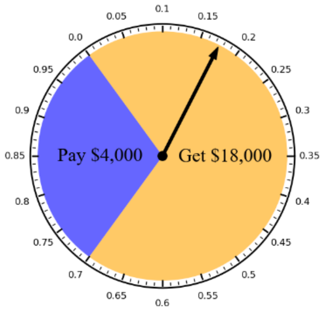
"*Be willing to make decisions. That's the most important quality in a good leader*."

General George S. Patton

## 3.1 Expected Value Criterion: Not the Way to Go

- Suppose you face a situation where you must choose between alternatives *A* and *B* as follows:

  Alternative *A*: Receive \$10,000 for sure.
  Alternative *B*: 70% chance of receiving \$18,000 or
  30% chance of losing \$4,000.




  What is your personal choice?

  Answer:

  Why?

- Let us check your personal choice with the **Expected Monetary (Dollar) Value** (EMV) criterion:

  $EMV(A) = (1)\ 10{,}000 \quad = \$\ 10{,}000$

  $EMV(B) = (0.7)\ 18{,}000 + (0.3)\ (-4{,}000) \quad = \$\ 11{,}400$

- Now, compare your choice with the above. What do you observe?

- Many, but not all, would choose Alternative *A* due to the worry that there is a significant 30% chance of losing \$4,000!

- What can you conclude?

- What's wrong with the expected monetary value criterion?

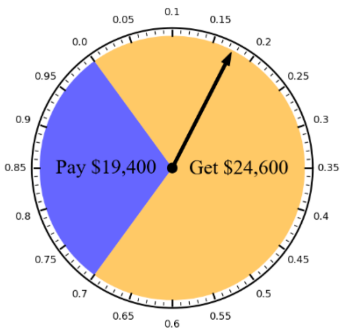
- ***EMV* does not take Risk into account.**

### Same *EMV*, but different Risk

- Compare now Alternative *B* with:

  Alternative *C*: 70% chance of receiving \$24,600
  30% chance of losing \$19,400

  $EMV(C) = 0.7\ (24{,}600) + 0.3(-19{,}400) = \$11{,}400$




- Note that $EMV(B) = EMV(C)$, but are they "equivalent"?

- Alternative *C* seems to be "more risky" than Alternative *B* even though they have the same *EMV*.

## The Petersburg Paradox

- In 1713, Nicolas Bernoulli suggested playing the following game.

  1. An unbiased coin is tossed repeatedly until it lands with Tails for the first time.

  2. The player is paid

     \$2 if Tails comes up on the first toss,
     \$4 if Tails first appears on the second toss,
     \$8 if Tails first appears on the third toss,
     \$16 if Tails first appears on the fourth toss and so forth.

- If you were offered to play this game only ONCE, what is the maximum you would be willing to pay to play it? ____________________________

- If we follow the EMV criterion:

$$\begin{aligned}\text{EMV of one game} &= \sum_{k=1}^{\infty}\left(\tfrac{1}{2}\right)^k (\$2.00)^k \\ &= \left(\tfrac{1}{2}\right)(\$2.00) + \left(\tfrac{1}{4}\right)(\$4.00) + \left(\tfrac{1}{8}\right)(\$8.00) + \cdots \\ &= 1 + 1 + 1 + \cdots = \infty\end{aligned}$$

- The game has an Expected Value of **Infinity.**

- This means that you should be willing to pay up to an infinite amount of money to play the game. But why are people only willing to pay just a few dollars to play the game once?

## Daniel Bernoulli's Solution to the Petersburg Paradox

- The Petersburg Paradox was resolved 25 years later by Nicolas's cousin, Daniel Bernoulli.

- Daniel Bernoulli suggested that one should not act based on the expected monetary reward, but on a different kind of expectation, which he called *moral expectation*. He argued that any gain in wealth brings a utility inversely proportional to the whole wealth of the individual.

- Analytically, the utility $u$ of a gain $\Delta$ in wealth, relative to the current wealth $w$, can be represented as

$$u(w+\Delta) - u(w) = c\frac{\Delta}{w}$$

  where $c$ is some positive constant.

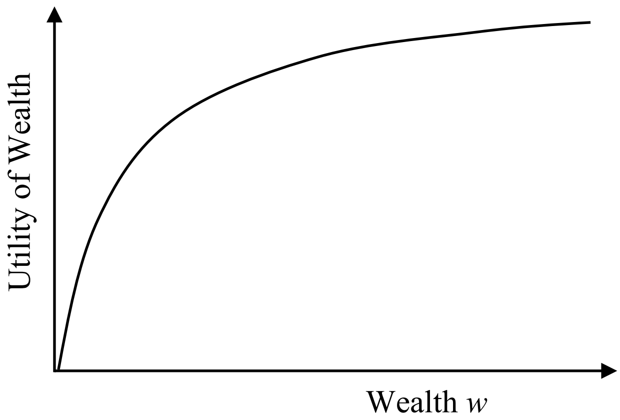
- For $\Delta$ sufficiently small we obtain

$$du(w) = c\frac{dw}{w}.$$

- Solving this differential equation, we obtain

$$u(w) = c\ ln(w) + d$$

where $d$ is a constant of integration.




- Daniel Bernoulli proposed that the game should be valued by computing the **Expected Utility** instead of the expected monetary value. Hence

$$\begin{aligned}\text{Expected Utility} &= \sum_{k=1}^{\infty}\left(\tfrac{1}{2}\right)^k u(w_0 + 2^k) \\ &= \left(\tfrac{1}{2}\right)u(w_0+2) + \left(\tfrac{1}{4}\right)u(w_0+4) + \left(\tfrac{1}{8}\right)u(w_0+8) + \cdots\end{aligned}$$

where $w_o$ is the current or initial wealth of the player.

- If the utility function suggested by Daniel Bernoulli is used, the expected utility of the game will converge to a finite value indicating the actual value of the game to the potential player.

## The Expected Utility Principle

- One should always consider the ***Expected Utility*** and not the expected monetary or dollar values when making decisions under uncertainty.
- In practice, the utility function proposed by Bernoulli represents only one of the many possible forms of utility functions.
- Different functions with different parameters (shapes) can be used to represent different risk-taking behaviors or attitudes of the decision-maker. See Chapter 6 for details.

## 3.2 Decision Theory: Way to Go

### 3.2.1 The Rules of Actional Thought

- How a person should act or decide rationally under uncertainty?

- Answer: By following the following rules or axioms:

  1. The Ordering Rule
  2. The Equivalence or Continuity Rule
  3. The Substitution or Independence Rule
  4. Decomposition Rule
  5. The Choice Rule

- The above five rules form the axioms for ***Decision Theory***.

### Rule 1: The Ordering Rule

- The decision-maker must be able to state his/her preference among all outcomes of any deal.
- That is, for any two outcomes $X$ and $Y$, he/she must be able to state exactly one of the following:

  1. He/she prefers $X$ to $Y$, denoted as $X \succ Y$
  2. He/she prefers $Y$ to $X$, denoted as $Y \succ X$
  3. He/she is indifferent between $X$ and $Y$, denoted as $X \sim Y$

- Furthermore, the ***Transitivity Property*** must be satisfied: That is, if he prefers $X$ to $Y$, and $Y$ to $Z$, then he must prefer $X$ to $Z$.

  Mathematically, $X \succ Y$ and $Y \succ Z \Rightarrow X \succ Z$.

- The Ordering Rule implies that the decision-maker can provide a complete preference ordering of all the outcomes from the best to the worst.

### Justification for the Transitivity Property: The Money Pump Argument

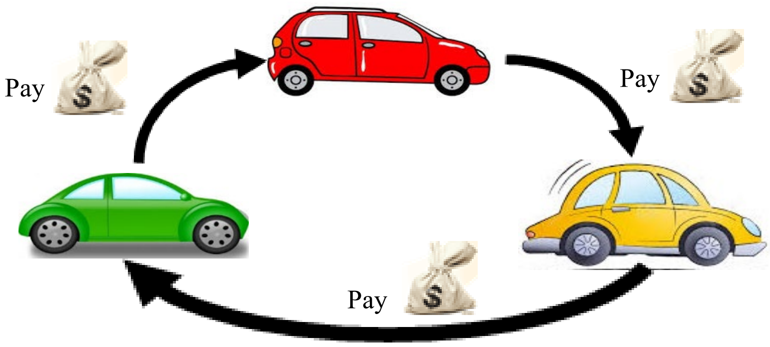
- Suppose a person does not follow the transitivity property.

- Consider three cars, A, B, and C.
- Let his preference which violates the transitivity property be:

  * Car A $\succ$ Car B
  * Car B $\succ$ Car C
  * Car C $\succ$ Car A

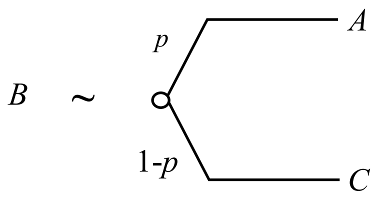
- Suppose he currently owns Car C.

- Then he can be made to switch from Car C to Car B, then from Car B to Car A, and then from Car A to back to the original Car C. Each time he switches, he is willing to pay an amount of money. The cycle can be repeated, turning a person who violates the transitivity property into a **money pump**.




## Rule 2: The Equivalence or Continuity Rule

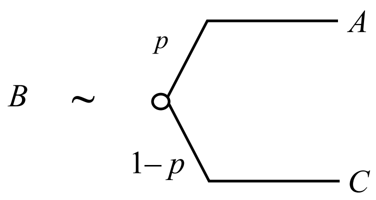
- Given prospects *A*, *B* and *C* such that $A \succ B \succ C$, then there exists *p* where $0 < p < 1$ such that the decision maker will be indifferent between receiving prospect *B for sure* and receiving a deal with a probability *p* for prospect *A* and a probability of $1-p$ for prospect *C*.



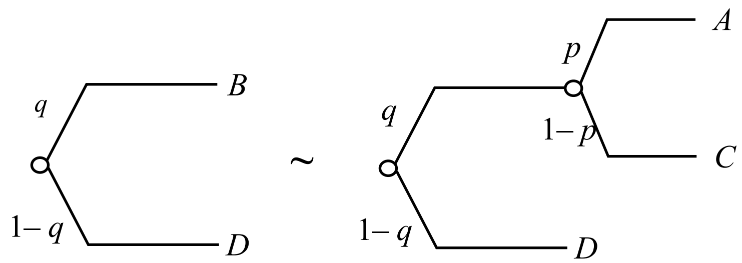

- Given that $A \succ B \succ C$
  - *B* is called the **Certainty Equivalent** of the uncertain deal on the right.
  - *p* is called the **Preference Probability** of Prospect *B* with respect to Prospects *A* and *C*.

## Rule 3: The Substitution Rule

- We can always substitute a deal with its certainty equivalent without affecting preferences.
- For example, suppose the decision maker is indifferent between *B* and the *A*-*C* deal below.

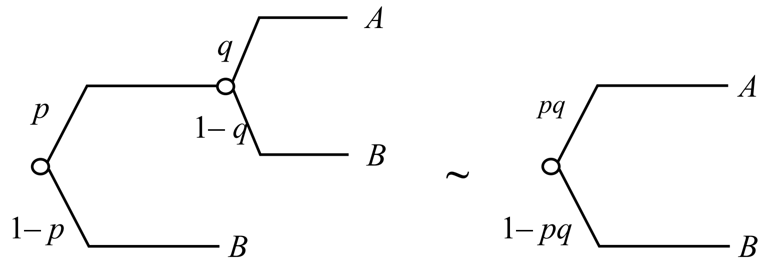



- Then he must be indifferent between the two deals below where *B* is *substituted* for the *A*-*C* deal.



## Rule 4: The Decomposition Rule

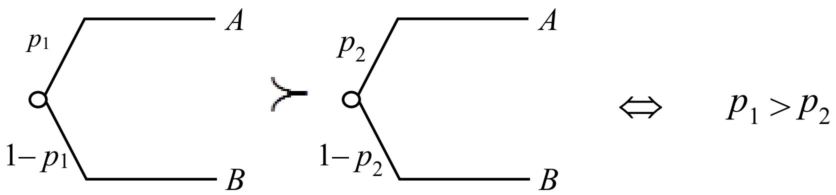
- We can reduce compound deals to simple ones using the rules of probabilities.
- For example, a decision maker should be indifferent between the following two deals:




## Rule 5: The Choice or Monotonicity Rule

- Suppose that a decision maker can choose between two deals $L_1$ and $L_2$ as follows:

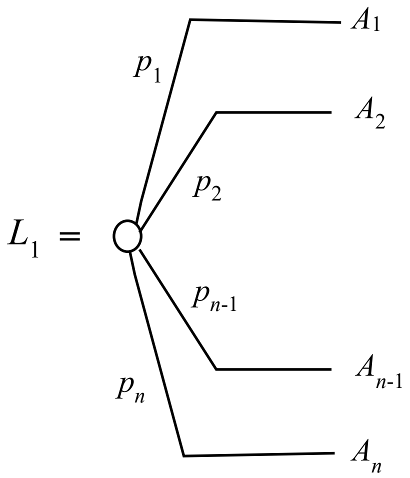

- If the decision maker prefers $A$ to $B$, then he or she must prefer $L_1$ to $L_2$ if and only if $p_1 > p_2$.
- That is, if $A \succ B$ then

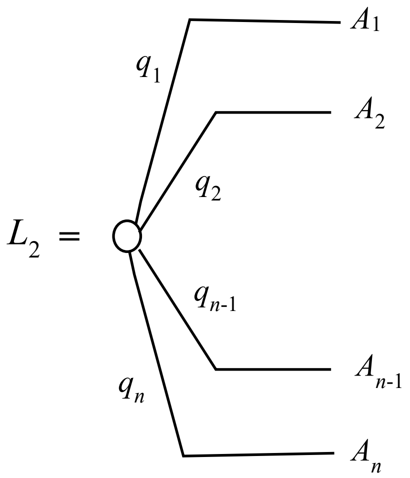



- In other words, the decision maker must prefer the deal that offers the greater chance of receiving the better outcome.

### 3.2.2 Maximum Expected Utility Principle

- Let a decision maker faces the choice between two uncertain deals $L_1$ and $L_2$, with outcomes $A_1$, $A_2$, … , $A_n$ as follows:




- There is no loss of generality in assuming that $L_1$ and $L_2$ have the same set of outcomes $A_1$, $A_2$, … , $A_n$ because we can always assign zero probability to those outcomes that do not exist in either $L_1$ or $L_2$.

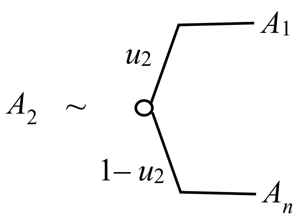
- It is not clear whether $L_1$ or $L_2$ is preferred.

- By the *Ordering Rule*, let $A_1 \succ A_2 \succ \cdots \succ A_n$.

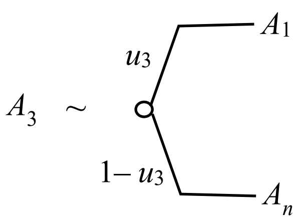
- Again, there is no loss of generality as we can always renumber the subscripts according to the preference order.

- We note that $A_1$ is the most preferred outcome, while $A_n$ is the least preferred outcome.

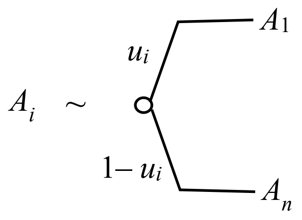
- Since $A_1 \succ A_2 \succ A_n$, it follows by the *Equivalent or Continuity Rule* that there exists a number $u_2$ such that $0 \le u_2 \le 1$ and




- Similarly, since $A_1 \succ A_3 \succ A_n$, it follows by the *Equivalent* or *Continuity Rule* that there exists a number $u_3$ such that $0 \le u_3 \le 1$ and

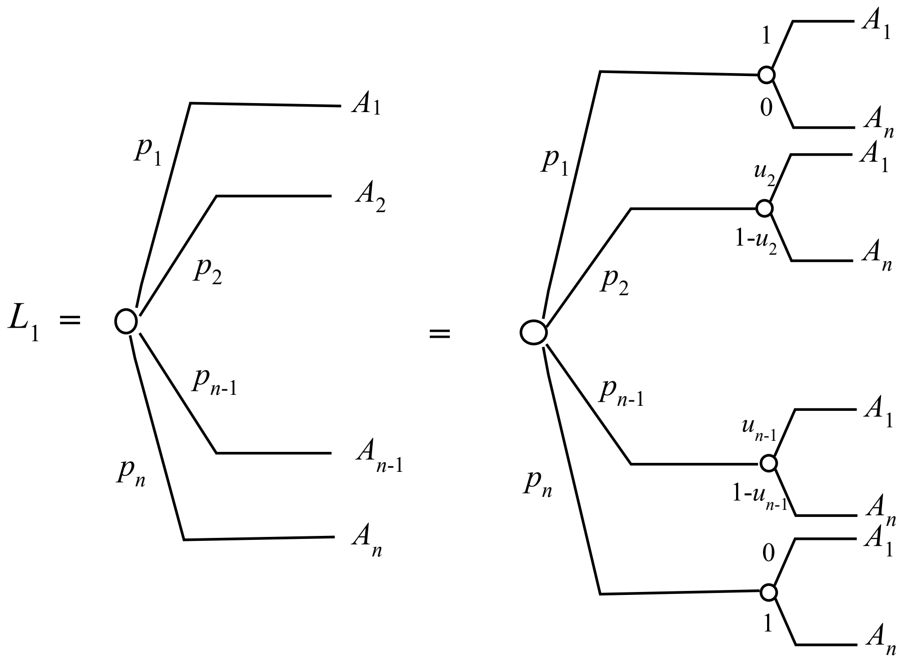


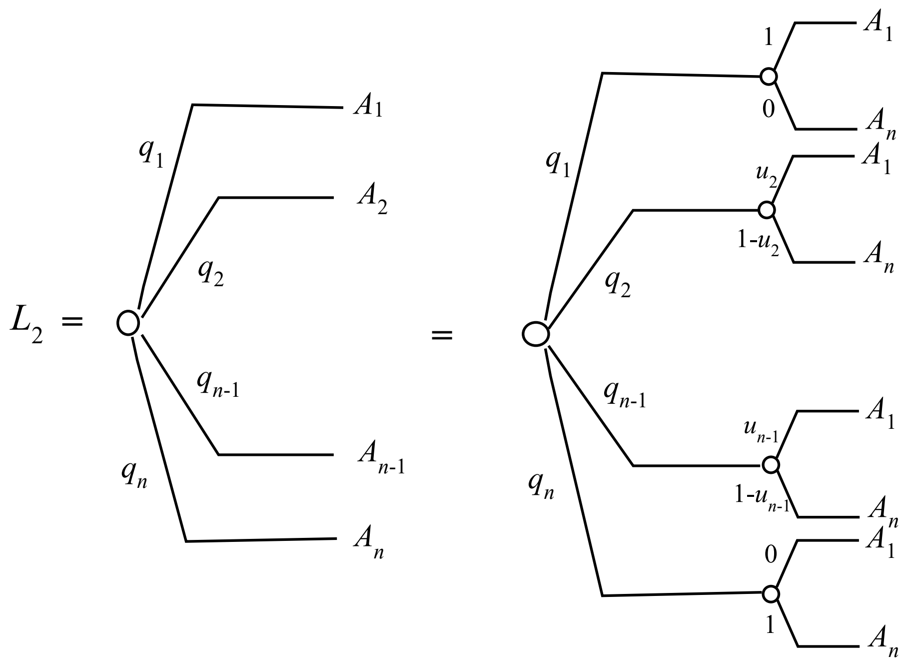
- Hence for each outcome $A_i$ ($i$ =1,…, $n$) there exists a number $u_i$ such that $0 \le u_i \le 1$ and



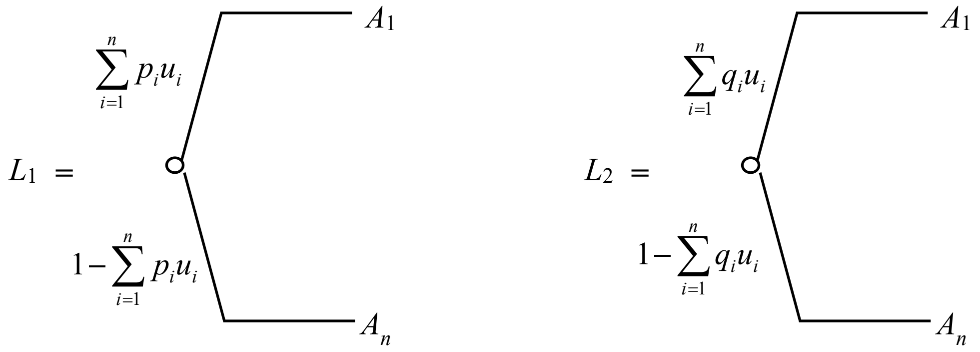

- Note that $u_1 = 1$ and $u_n= 0$. Why?

- By the *Substitution Rule*, we replace each $A_i$ ($i$ = 1,…, $n$) in $L_1$ and $L_2$ with the above constructed $n$ umber of equivalent deals.



- By the *Decomposition Rule*, $L_1$ and $L_2$ may be reduced to equivalent deals with only two outcomes $A_1$ and $A_n$ as follows.




Finally, by the *Choice Rule*, since $A_1 \succ A_n$, the decision maker prefers deal $L_1$ to deal $L_2$ if and only if $\sum_{i=1}^{n} p_i u_i > \sum_{i=1}^{n} q_i u_i$ .

## Utilities and Utility Functions

- We define the quantity $u_i$ ($i = 1,\ldots, n$) as the **Utility** of outcome $A_i$ and the function that returns the values $u_i$ given $A_i$ as a **Utility Function**, i.e. $u(A_i) = u_i$.

- The quantities $\sum_{i=1}^{n} p_i u(A_i)$ and $\sum_{i=1}^{n} q_i u(A_i)$ are known as the **Expected Utilities** for deals $L_1$ and $L_2$, respectively.

- Hence the decision maker must prefer the deal with a **higher Expected Utility**.

## Case of More Than 2 Alternatives

- When there are more than two alternatives, the decision maker should choose the one with **Maximum Expected Utility.**

- Suppose there are $m$ alternatives $L_1$ to $L_m$ each with outcomes $A_1$ to $A_n$. Let $p_{ij}$ be the probability of outcome $A_i$ in alternative $L_j$, for $i = 1$ to $n$, and $j = 1$ to $m$. Then the best alternative is

$$L^* = \arg \operatorname*{Max}_{j} \sum_{i=1}^{n} p_{ij} u(A_i)$$

## Comparing Expected Utility Criterion with Expected Monetary Value Criterion

- The expected utility criterion considers both return and risk whereas the expected monetary value criterion does not consider risk. The alternative with the maximum expected utility is the best taking into account the trade-off between return and risk.

- The best preference trade-off depends on a person's **Risk Attitude**.

- Different types of utility functions represent different attitudes and **Degrees of Aversion** to risk-taking (c/o Chapter 6).

## Exercises

**P3.1** Suppose you can choose either Deal *A* or Deal *B*, and you get to keep whatever you might win.

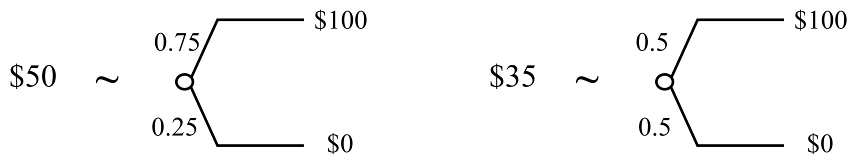
- **Deal *A***: A coin is flipped. When it lands, if the side facing up is Heads, you win \$1000, otherwise nothing.
- **Deal *B***: A die is rolled. If the side facing up is a One, you win \$1000, otherwise nothing.

**(*a*)** Which deal would you choose to own, Deal *A* or Deal *B*? Why?

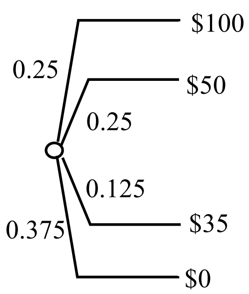
**(*b*)** Suppose now the coin is flipped and the die is rolled. The results are a Tails and a One. Do you think that you have made a good decision? Why or why not?

**(*c*)** If you were given another opportunity to choose between Deal *A* and Deal *B* before flipping the coin and rolling the die again, which would you select?

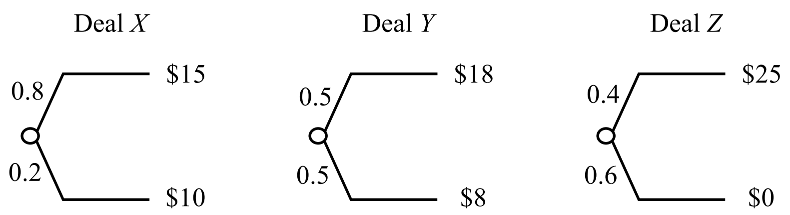
**P3.2** Jo has certainty equivalents for two deals as follows:



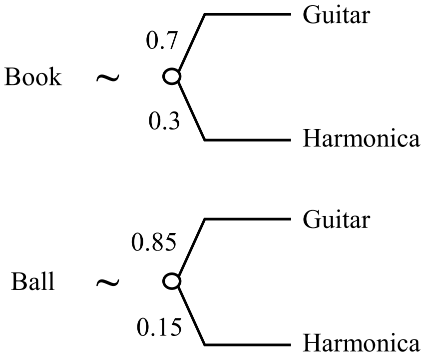

Use the substitution rule to determine Jo's certainty equivalent for the following deal:




**P3.3** John is rushing to the box office to buy a ticket to see a new play. He gets to the counter and realizes that he left his wallet at home. He has only \$3 change, and the ticket costs \$20. A local philanthropist, Mr. Ho, steps up and offers John his choice of three deals for free, which he will resolve immediately. All John cares about is getting a ticket to the play, and there is no time to get more money elsewhere or use other forms of payment. Which deal should John choose?

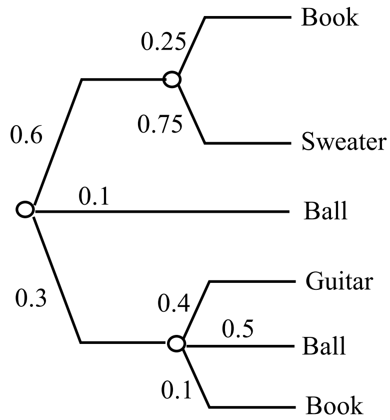


**P3.4** Chris prefers a guitar to a harmonica, and specifies the following equivalence relations:

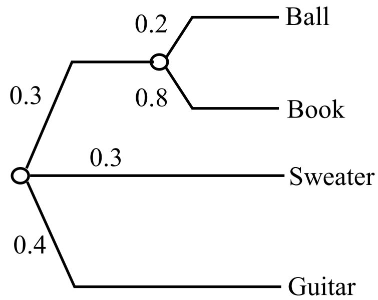



(***a***) What is Chris's preference ordering for a guitar, harmonica, book, sweater, and ball?

(***b***) What is Chris's preference probability (with respect to a hypothetical Guitar-Harmonica deal) that is equivalent to the following deal?




(***c***) Does Chris prefer a book or the following deal? (Hint: first express this deal as a probability of a guitar versus a harmonica.)




(***d***) What can we infer about Chris's preference for four sweaters versus one book, from her statements above?