Counting Regions in the Plane and More[1]

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Berkeley Math Circle, Intermediate II/Pre-Advanced Groups

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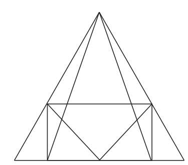
1. Overarching Problem

Problem 1 (Regions in a Circle). The n vertices of a polygons are arranged on a circle so that no three diagonals intersect in the same point. How many regions inside the circle are formed this way by all the segments connecting the n points?

Problem 2 (Preparation). Attacking Problem 1 for any n is hard. What is a good way to start on problems like this? What is your conjecture for the final answer?

2. Warm Up

Problem 3 (Warm-up). How many $\triangle$'s are in the picture?



Problem 4 (Thinking Deeper) How many *different* ways can be used to count here? Which do you prefer? Which way is "most generalizable" to other problems?

3. Special vs. Generic Positions

Problem 5 (Understanding the Conditions). Why did Problem 1 say that no three diagonals should intersect in a point? If we allow for three or more segments to intersect in a point, will this change the answer? Will it increase it? Decrease it?

Definition 1 (General Position). Two lines in the plane are said to be in *general position* if they intersect in a point. Three (or more line) lines in the plane are said to be in *general position* if any two of them intersect, but no three (or more) lines intersect in the same point.

Problem 6 (Special Configurations). In the plane, when shall we say that two lines are in a *special position*? How about three lines in the plane in a *special position*? Draw all different configurations of three lines in the plane in a *special* position. Now draw all different configurations of three lines in the plane in a *general* position. How many special and how many generic configurations did you get?

[1]Some problems and pictures are taken from "A Decade of the Berkeley Math Circle – The American Experience," volume I, edited by Zvezdelina Stankova and Tom Rike, published by the American Mathematical Society in the MSRI Mathematical Circles Library.

Problem 7 (Pentagons and Hexagons). In a regular pentagon connect any two vertices. Are the *diagonals* in special or in generic position? How about a regular hexagon? Count in each case the number of regions into which the polygon is cut up by its diagonals.

4. Solving Predecessor Problems

Problem 1 restricted us to looking *inside* a circle? This might be why the problem is so hard! Let's look at the whole plane by temporarily eliminating the circle.

Problem 8 (Lines in Generic Position). Draw n lines in the plane so that no three intersect in the same point and no two are parallel. Into how many regions do these lines divide the plane?

Problem 9 (Games in a Tournament). n teams participated in a basketball tournament. Each team played every other team exactly once. How many games were played in total?

Problem 10 (Counting Diagonals). How many diagonals does the n-gon in our overarching problem have? What is the relation between the last three problems?

5. Poking Around and Extending the Problems

Problem 11 (Debates in a Championship). In a debate championship with n teams, each debate is done between 3 teams. How many debates can possible happen during this tournament? How about if each debate involves 4 teams?

Problem 12 (Diagonal Intersections). How many intersections of diagonals does the n-gon in our overarching problem have? What is the connection between the last two problems?

6. The Final Attack on the Overarching Problem

Problem 13 (Adding a New Segment). In the set-up of Problem 1, erase all segments, but leave the n points and the circle. Start all over the segments one at a time, in any order. Recall that the segments can be either sides or diagonals of the n-gon. Suppose a newly added segment intersects k (already drawn) diagonals. How many more regions inside the circle has this new segment added? Is there an elegant way to phrase the answer without mentioning the number k?

Problem 14 (Putting All Together). Use any of our results so far to calculate the total number of regions inside the circle. Is there an elegant way to phrase the answer? Did you check it for correctness for $n = 1, 2, \ldots, 6$? What does your answer predict for $n = 7$? Check it by brute force, if you don't believe it.

7. More Homework for the Die-Hards

Problem 15 (Summing Up). Prove the following formulas for any $n \geq 1$:

(a) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}\cdot$

(b) $1 + 3 + 5 + 7 + \cdots + (2n-1) = n^2$.

(c) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}\cdot$

Problem 16 (Off to Space). Generalize the overarching problem to 3 dimensions. The n vertices of a polyhedron are arranged on a sphere so that when among the planes connecting three of the points, no three intersect in the same line. How many regions inside the sphere are formed this way by all these planes?

Problem 17 (Polygons). Count the number of regions inside a (convex) n-gon made by connecting any two of its vertices if:

(a) no three diagonals intersect at the same point;
(b) the n-gon is *regular*.

Problem 18 (Binomial Theorem). The number of ways to choose k objects out of n objects is denoted by the *binomial coefficient* $\binom{n}{k}$ (read "n choose k"). Prove that this binomial coefficient is calculated by the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ for any } n, k \geq 0,$$

where $0!$ is defined to be 1 in order to make the formula work when n or k is 0. When $n < k$ or one of the n or k is < 0, then the binomial coefficient $\binom{n}{k} = 0$.

Problem 19 (Ultimate Summations). Prove the following formulas:

(a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$ for all $n \geq 0$.

(b) $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots + \binom{n}{n-1} = 2^{n-1}$ for n odd.[2]

Problem 20 (Coincidence?) The answer 31 in the case of $n = 6$ points in the overarching Problem 1 broke the "pattern" of powers 2^k and made us rethink what is going on. The answer to the warm-up Problem 2 was also 31. Is this a coincidence, or is there a way to transform one problem into the other in this case?

[2]For n even, the analogous sum $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots + \binom{n}{n-2} + \binom{n}{n}$ is a lot harder to calculate, but it turns out that parts of this particular sum is connected to our overarching problem.