

Berkeley Math Circle
Int II/Pre-Adv – HW #2

Instructor: Erik Herrera

Distributed: 2022/06/14

This assignment contains 5 pages (including this cover page) and 4 questions. Please give a complete proof for each question.

1. Consider an infinite sequence of fair coin tosses. We saw in lecture that the probability of getting at least one head is 1.
 - (a) What is the probability of getting k tails then getting 2 heads in a row?
 - (b) what is the probability that you get two heads in a row somewhere in an infinite sequence of coin tosses?

2. Let A_1 be the set of all numbers of the form

$$\frac{1}{n}$$

where n is a positive integer.

- (a) Show that A_1 is countable. That is, show there is a way to order all the elements in A_1 .
- (b) What is the length of A_1 as a subset of $[0, 1]$?

3. Let A_1 be the set from the previous question. Let A_2 be the set of all numbers of the form

$$\frac{1}{\sqrt{n}}$$

where n is a positive integer.

Show that $A_1 \cup A_2$ is countable. What does this imply about its length as a subset of $[0, 1]$?

4. **Bonus:** Let A_m be the set of all numbers of the form

$$\frac{1}{\sqrt[m]{n}}$$

where n is a positive integer. Let $X = A_1 \cup A_2 \cup A_3 \cup \dots$. That is, X is the set of all reciprocals of all roots of positive integers. Show that X is countable. What does this imply about the length of X ?