Berkeley Math Circle

Pre-Adv – HW #2

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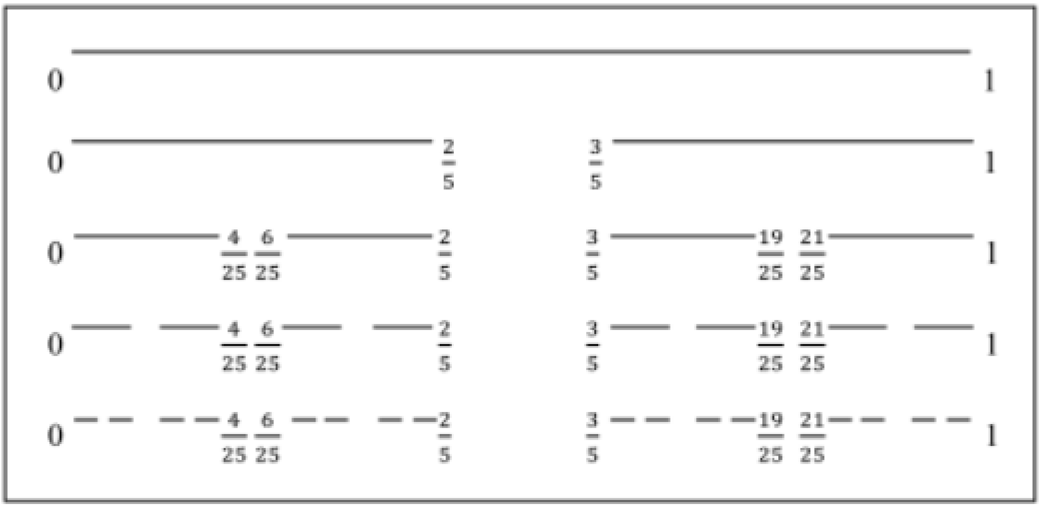
This assignment contains 5 pages (including this cover page) and 4 questions. Please give a complete proof for each question.

1. In lecture, we made the Cantor set by removing the middle third of a segment and repeating this process. Let us do the something similar, but this time we will remove the middle fifth.

   Start with the interval $[0, 1]$. Remove the middle fifth of this interval. For the two remaining subintervals, remove each of their middle fifths. For each of the four remaining subintervals remove each of their middle fifths. Continue this indefinitely. (See diagram below)



   What is the length of the final set?

2. In lecture, we saw that the set of infinite sequences of coin tosses was uncountable.

   Show that the interval $[0, 1]$ is uncountable. That is, show that any list of the numbers between 0 and 1 is incomplete.

3. Suppose we have a countable sequence of sets $A_1, A_2, A_3 \ldots$. If each set A_i contains countably many elements, is it possible for the union

$$A_1 \cup A_2 \cup A_3 \cup \ldots$$

to be uncountable?

4. **Bonus:**

In lecture, we made the Cantor set by removing the middle third of a segment and repeating this process. Let us do the something similar, but this time the portions we will remove will get smaller.

Start with the interval $[0, 1]$. Remove the middle fourth of this interval. For the two remaining subintervals, remove a an interval of length $1/16$. For each of the four remaining subintervals remove an interval of length $1/64$. Continue this indefinitely. (See diagram below).

At step k we remove intervals of length $1/4^k$.

(a) What is the length of the set after the first n steps?

(b) What is the length of the final set after the process is complete?